General multilevel Monte Carlo methods for pricing discretely monitored Asian options

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Abstract

We describe general multilevel Monte Carlo methods that estimate the price of an Asian option monitored at \( m \) fixed dates. Our algorithms yield an unbiased estimator with standard deviation \( O(\epsilon) \) in \( O(m + \epsilon^{-2}) \) expected time for a variety of processes such as the Black-Scholes model, the CEV model, Merton’s jump-diffusion model, a class of exponential Levy processes and, via the Milstein scheme, processes driven by scalar stochastic differential equations. Using the Euler scheme, our approach estimates the Asian option price with root mean square error \( O(\epsilon) \) in \( O(m + (\ln(\epsilon))^2\epsilon^{-2}) \) expected time for processes driven by multidimensional stochastic differential equations. Preliminary numerical experiments confirm that our approach outperforms the conventional Monte Carlo method by a factor of order \( m \).

Keywords: discretely monitored Asian option, multilevel Monte Carlo method, option pricing, variance reduction

1 Introduction

Asian options are financial derivatives whose payoff depends on the arithmetic average of an underlying during a specific time-period. Asian options are useful to corporations which are exposed to average exchange rates or commodity prices over a certain period of time. Pricing Asian options has been the subject of many studies. Under the Black-Scholes model, the price of a continuously sampled Asian option can be expressed as an infinite series (Linetsky 2004). Transform based methods have been used to value Asian options under Markov processes (Cai, Song and Kou 2015, Cui, Lee and Liu 2018). A convex programming method that computes optimal model-independent bounds on Asian option prices is described in (Kahalé 2017). Monte Carlo methods can price Asian options under various models, but conventional Monte Carlo algorithms have a high computational cost, which motivates the need to improve the efficiency of such methods. Control variate techniques for pricing Asian options with Monte Carlo simulation are given in (Kemna and Vorst 1990, Dingeç and Hörmann 2012, Shiraya and Takahashi 2017). An importance sampling algorithm for pricing Asian options is given in (Glasserman, Heidelberger and Shahabuddin 1999). When the underlying follows a stochastic differential equation (SDE) satisfying certain regularity conditions, the multilevel Monte Carlo method (MLMC) method described in (Giles 2008b) estimates the price of a continuously monitored Asian option with mean square error \( \epsilon^2 \) in \( O((\ln(\epsilon))^2\epsilon^{-2}) \) time using the Euler discretization. This computational cost has been reduced to \( O(\epsilon^{-2}) \) time using the Milstein scheme for scalar SDEs (Giles 2008a, Giles, Debrabant and Rößler 2013) and multi-dimensional SDEs (Giles, Szpruch et al. 2014). For a broad class of exponential Levy processes, the MLMC method described in (Giles and Xia 2017) estimates the price of a continuously monitored Asian option.

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with mean square error $\epsilon^2$ in $O(\epsilon^{-2})$ time. Randomized multilevel Monte Carlo methods that yield unbiased estimators are given in (Rhee and Glynn 2015, Vihola 2017). For a general introduction to Monte Carlo methods, see (Glasserman 2004).

Consider now an Asian option with a given maturity monitored at $m$ fixed dates. Even in the Black-Scholes model, the time required to estimate the option price with variance $O(\epsilon^2)$ is $\Theta(m\epsilon^{-2})$ under the conventional Monte Carlo method, assuming the payoff variance is upper and lower bounded by constants independent of $m$. This is because the simulation of the underlying prices at the $m$ dates takes $\Theta(m)$ time.

This paper describes a general MLMC framework to price an Asian option monitored at $m$ dates. The basic idea behind our approach is to (approximately) simulate the forward prices at only a subset of the $m$ dates at a given iteration. The forward prices at the remaining dates are then approximated by the average of the surrounding forward prices. Our approach does not make any assumptions on the nature of the stochastic process driving the underlying. It however assumes the existence of a linear relationship between the underlying and the forward prices, that the underlying price is square-integrable, and makes certain assumptions on the running time required to simulate, with a given precision, the underlying on a discrete time grid. The latter condition is satisfied in any model where the underlying prices can be simulated exactly at $m'$ fixed dates in $O(m')$ expected time. Using the Milstein scheme, it is also satisfied by processes driven by scalar SDEs. Our approach yields an unbiased estimator with variance $O(\epsilon^2)$ for the Asian option price in $O(m + \epsilon^{-2})$ expected time for a variety of processes such as the Black-Scholes model, the CEV model, Merton’s jump-diffusion model, the variance gamma, NIG and $\alpha$-stable exponential Levy processes and, using the Milstein scheme, processes driven by scalar SDEs. Using the Euler scheme, our approach also estimates the Asian option price with mean square error $O(\epsilon^2)$ in $O(m + (\ln(\epsilon))^2\epsilon^{-2})$ expected time for processes driven by one-dimensional or multidimensional SDEs. We are not aware of any previous Monte Carlo or MLMC method that provably achieves such tradeoffs between the running time and target accuracy, even under the Black-Scholes model. Giles, Debrabant and Rößler (2013) and Giles, Szpruch et al. (2014) mention that their methods can be used to price Asian options monitored at $m$ dates, but do not analyse the performance of their algorithms in terms of $m$. Our paper makes three main contributions:

1. Our MLMC method prices Asian options monitored at $m$ dates and achieves a target accuracy $O(\epsilon)$ in $O(m + \epsilon^{-2})$ or $O(m + (\ln(\epsilon))^2\epsilon^{-2})$ expected time, depending on the assumptions satisfied by the diffusion process. It applies to a wide range of processes, including processes with jumps.

2. When the underlying prices can be simulated exactly at $m'$ fixed dates in $O(m')$ expected time, we compute an explicit upper-bound on the variance of our estimator in terms of the underlying variance at $T$.

3. Our method does not make any assumptions on the dates at which the option is monitored, and does not assume that the weights associated with the monitoring dates have the same sign or order of magnitude, but assumes that the sum of the absolute values of these weights is upper-bounded by a constant independent of $m$.

The rest of the paper is organized as follows. Section 2 describes the modelling framework and recalls the randomized MLMC method. Our algorithms for Asian options pricing are described in Section 3. Preliminary numerical simulations are presented in Section 4. Concluding remarks can be found in Section 5. Omitted proofs are in the appendix.
2 Preliminaries

2.1 The modelling framework

Assume that interest rates are deterministic. Let $T$ be a fixed maturity and $m$ a positive integer. For $1 \leq j \leq m$, denote by $F_j$ the forward price of an underlying calculated at time $t_j$ for maturity $T$, where $t_0 < \cdots < t_m$, with $t_0 = 0$ and $t_m = T$. Let $A = \sum_{j=1}^{m} w_j F_j$ be a linear combination of the forward prices, where the $w_j$’s are signed weights whose absolute values sum up to 1. Consider an Asian option with payoff $f(A)$ at maturity $T$, where $f$ is a $\kappa$-Lipschitz real-valued function of one variable. Such a payoff can model Asian options that arise in a broad range of situations. For instance, the payoff of an average price call with strike $K$ is upper-bounded by a constant independent of $\kappa$. Let $(A_t)_{t \geq 0}$ be a positive martingale under $Q$. The existence of $Q$ can be shown under no-arbitrage conditions (see (Glasserman 2004, §1.2.2)). We also assume that the forward prices $F_j$, $1 \leq j \leq m$, are square-integrable, and that $\kappa$ is upper-bounded by a constant independent of $m$.

2.2 The randomized multilevel Monte Carlo method

Here we recall a randomized multilevel Monte Carlo method given in (Rhee and Glynn 2015) that efficiently calculates the expectation of a random variable $Y$ that is approximated, in the $L^2$-sense, by random variables $Y_l$, $l \geq 0$. We assume that $Y$ and $(Y_l)$, $l \geq 0$, are square-integrable. Let $(p_l)$, $l \geq 0$, be a probability distribution such that $p_l > 0$ for $l \geq 0$. Assume that $N \in \mathbb{N}$ is a random variable independent of $(Y_l : l \geq 0)$ and such that $\Pr(N = l) = p_l$ for $l \geq 0$. Set

$$Z = \frac{Y_N - Y_{N-1}}{p_N}$$

with $Y_{-1} := 0$. If $X$ is a square-integrable random variable, let $||X|| = \sqrt{E(X^2)}$. The following result is due to (Rhee and Glynn 2015) (see also (Vihola 2017, Theorem 2)).

Theorem 2.1 (Rhee and Glynn 2015). Assume that $||Y_l - Y||$ converges to 0 as $l$ goes to infinity. If $\sum_{l=0}^{\infty} ||Y_l - Y_l-1||^2 / p_l$ is finite then $Z$ is square-integrable, $E(Z) = E(Y)$, and

$$||Z||^2 = \sum_{l=0}^{\infty} \frac{||Y_l - Y_{l-1}||^2}{p_l}.$$

For $l \geq 0$, denote by $C_l$ the expected cost of computing $Y_l - Y_{l-1}$, and let $C$ be the expected cost of computing $Z$. Propositions 2.1 and 2.2 below are in the same spirit as results previously obtained in (Giles 2008b, Theorem 3.1) and in (Rhee and Glynn 2015). For completeness, we give their proof in the appendix. Proposition 2.1 shows that, under certain conditions on $Y_l$ and $C_l$, the sequence $(p_l)$, $l \geq 0$, can be chosen so that both $||Z||$ and $C$ are finite.

Proposition 2.1. Assume that $||Y_0||^2 \leq \nu$ and that, for $l \geq 0$,

$$||Y_l - Y||^2 \leq \nu 2^{-\beta l}$$

(2)
and $C_1 \leq c2^l$, where $c$, $\nu$ and $\beta$ are positive constants, with $\beta \in (1, 2]$. If, for $l \geq 0$,

$$p_l = (1 - 2^{-(\beta+1)/2})2^{-(\beta+1)/2},$$

(3) then $Z$ is square-integrable, $E(Z) = E(Y)$, and

$$||Z||^2 \leq \frac{20\nu}{1 - 2^{-(\beta-1)/2}}.$$  

(4)

Furthermore,

$$C \leq \frac{c}{1 - 2^{-(\beta-1)/2}}.$$  

(5)

If we relax (2), Proposition 2.2 shows how to construct a biased estimator $Z_L$ of $Y$, for any positive integer $L$, with expected cost and variance linear in $L$ and a bias that decreases geometrically with $L$.

**Proposition 2.2.** Assume that $||Y_0||^2 \leq \nu$ and that, for $l \geq 0$,

$$||Y_l - Y||^2 \leq \nu2^{-l}$$  

(6)

and $C_l \leq c2^l$, where $\nu$ and $c$ are positive constants. Let $p_l = 2^{-(l+1)}$ for $l \geq 0$. Fix a positive integer $L$ and set

$$Z_L = \frac{Y_N - Y_{N-1}}{p_N}1_{N \leq L}.$$  

Then

$$(E(Z_L - Y))^2 \leq \nu2^{-L},$$  

(7)

and

$$||Z_L||^2 \leq 12\nu(L + 1).$$  

(8)

Furthermore, the expected cost of computing $Z_L$ is at most $cL$.

### 3 Multilevel algorithms for Asian options

We construct multilevel approximations of $A$ in Subsection 3.1 and use them in Subsections 3.2 and 3.3 to build estimators for the Asian option price. The algorithm of Subsection 3.2 assumes that the underlying prices can be simulated exactly, while the algorithms of Subsection 3.3 assume that the underlying prices can be simulated approximately. Set $a = f((\sum_{j=1}^m w_j)F_0)$ and $U = f(A) - a$.

#### 3.1 Multilevel approximations of the arithmetic average

This subsection constructs an increasing sequence of subsets of $\{1, \ldots, m\}$ and shows that $A$ is approximated, with increasing accuracy, by weighted averages of forward prices corresponding to these subsets. For integers $i$ and $j$ with $1 \leq i \leq m$ and $0 \leq j \leq m$, let

$$W(i,j) = \sum_{k=i}^{j} w_k,$$

and

$$W'(i,j) = \sum_{k=i}^{j} |w_k|.$$
By convention, \( W(i, j) = W'(i, j) = 0 \) if \( j < i \). Define the subsets \( J_l \) of \( \{1, \ldots, m\} \), for \( l \geq 0 \), as follows. Set \( L = \lceil \log_2 m \rceil \) and \( J_l = \{1, \ldots, m\} \) for \( l \geq L \). For \( 0 \leq l \leq L - 1 \), let

\[
J_l = \{ j \in \{1, \ldots, m\} : 2^l W'(1, j - 1) < [2^l W'(1, j)] \}.
\] (9)

Note that \( J_0 = \{m\} \). Roughly speaking, \( J_l \) consists of the indices \( j \) where the sequence \( W'(1, j) \) “jumps” over a multiple of \( 2^{-l} \). It is therefore reasonable to expect that the sequence \( (J_l) \), \( l \geq 0 \), is increasing and that the size of \( J_l \) is at most \( 2^l + 1 \).

**Proposition 3.1.** For \( l \geq 0 \),

\[
|J_l| \leq 2^l + 1
\] (10)

and

\[
J_l \subseteq J_{l+1}.
\] (11)

Proposition 3.1 implies that, for \( 0 \leq l \leq L - 1 \),

\[
J_l = \{ j \in J_{l+1} : 2^l W'(1, j - 1) < [2^l W'(1, j)] \}.
\] (12)

For \( l \geq 0 \), define the following trapezoidal approximation of \( A \):

\[
A_l = \sum_{j \in J_l} w_j F_j + \frac{1}{2} \sum_{(i,k) \in P_l} W(i+1,k-1)(F_i + F_k),
\] (13)

where \( P_l \) is the set of pairs of consecutive of elements of the set \( \{0\} \cup J_l \). Thus \( A_l \) is obtained from \( A \) by replacing each \( F_j \) with \( (F_i + F_k)/2 \) for each pair \( (i,k) \in P_l \) and each integer \( j \) with \( i < j < k \). By construction, \( A_l \) is a deterministic linear function of \( F_j \), \( j \in J_l \). Note that \( A_l = A \) for \( l \geq L \). Theorem 3.1 below gives a bound on the \( L^2 \)-distance between \( A_0 \) and \( W(1,m)F_0 \) on one hand, and between \( A_l \) and \( A \) on the other hand.

**Theorem 3.1.** \( ||A_0 - W(1,m)F_0||^2 \leq \text{Var}(F_m) \) and, for \( l \geq 0 \),

\[
||A_l - A||^2 \leq 2^{-2l} \text{Var}(F_m).
\] (14)

Algorithm M below calculates the coefficients \( W(i+1,k-1) \) in (13), for \( 0 \leq l \leq L - 1 \) and \( (i,k) \in P_l \), in \( O(m) \) total time, using the following steps.

1. Calculate recursively \( W(1,j) \) and \( W'(1,j) \) for \( 1 \leq j \leq m \).
2. Construct by backward induction the subsets \( J_l \), for \( 0 \leq l \leq L \), using (12). This takes \( O(m) \) total time because the set \( J_l \) can be constructed and stored as an ordered sequence in \( O(2^l) \) time, for any \( l \in \{0, \ldots, L - 1\} \).
3. For \( l \in \{0, \ldots, L - 1\} \) and each pair \( (i,k) \in P_l \), calculate \( W(i+1,k-1) \) via the relation \( W(i+1,k-1) = W(1,k-1) - W(1,i) \).

### 3.2 The exact simulation case

**Assumption 1 (A1).** There is a constant \( c \) independent of \( m \) such that, for any subset \( J \) of \( \{1, \ldots, m\} \), the expectation of the time required to simulate the vector \( (F_j) \), \( j \in J \), is at most \( c|J| \).

Assumption A1 holds for a variety of processes such as the Black-Scholes model, the CEV model, Merton’s jump-diffusion model, and for several exponential Levy processes such as the variance gamma, NIG and \( \alpha \)-stable processes. Algorithms that simulate processes with jumps are described in (Glasserman 2004, §3.5). Theorem 3.2 below shows how to construct an unbiased estimator of the Asian option price under Assumption A1.
Theorem 3.2. Assume that A1 holds. Let $N \in \mathbb{N}$ be a random variable independent of $(F_j : 1 \leq j \leq m)$ and such that $\Pr(N = l) = p_l$ for non-negative integer $l$, where

$$p_l = (1 - 2^{-3/2})2^{-3l/2}.$$  

For $l \geq 0$, let $U_l = f(A_l) - a$, and set $V = (U_N - U_{N-1})/p_N$, where $U_{-1} := 0$. Then $V$ is square-integrable,

$$E(f(A)) = E(V) + a,$$  

and

$$\text{Var}(V) \leq 70\kappa^2 \text{Var}(F_m).$$  

Furthermore, the expectation of the time required to simulate $V$ is upper-bounded by a constant independent of $m$.

Proof. Since $|J_l| \leq 2^l$, the expectation of the time required to simulate the vector $(F_j), j \in J_l$, is at most $c2^l$. In view of (13), there is a constant $c'$ independent of $m$ such that, for $l \geq 0$, the expectation of the time to simulate $U_l - U_{l-1}$ is at most $c'2^l$. As $|U_0| \leq \kappa|A_0 - W(1, m)F_0|$, Theorem 3.1 implies that $||U_0||^2 \leq \kappa^2\text{Var}(F_m)$. Similarly, as $|U_l - U| \leq \kappa|A_l - A|$ for $l \geq 0$, Theorem 3.1 shows that

$$||U_l - U||^2 \leq \kappa^2 2^{-2l} \text{Var}(F_m).$$

The conditions of Proposition 2.1 are thus met for $Y = U$ and $Y_j = U_l$ for $l \geq 0$, with $\nu = \kappa^2\text{Var}(F_m)$, $\beta = 2$ and $c = c'$. By (3), the expectation of the time required to simulate $V$ is at most $4c'$. Furthermore, $V$ is square-integrable with $E(V) = E(U)$, which yields (16). Similarly, (10) follows from (4). \hfill \Box

Theorem 3.2 shows that $e^{-RT}(V + a)$ is an unbiased estimator of the Asian option price that can be simulated in constant time with variance bounded by a constant independent of $m$. Simulating $|e^{-2}|$ independent copies of $V$ yields an unbiased estimator of the option price with variance $O(e^2)$ in $O(m + e^{-2})$ expected time, including the $O(m)$ preprocessing cost of Algorithm M.

3.3 The approximate simulation case

For $J \subseteq \{1, \ldots, m\}$, let $\mathbb{R}^J$ denote the set of vectors of size $|J|$, indexed by elements of $J$.

Assumption 2 (A2). There are constants $c_1$, $c_2$ and $\beta \in [1, 2]$ such that, for $l \geq 0$ and $J \subseteq \{1, \ldots, m\}$, there is a random vector $\hat{F} = \hat{F}(J, l) \in \mathbb{R}^J$ such that $||\hat{F}_j - F_j||^2 \leq c_22^{-\beta l}$ for any $j \in J$. For $l \geq 1$ and $J \subseteq J \subseteq \{1, \ldots, m\}$, the expected time required to simulate the vector $(\hat{F}(J', l - 1), \hat{F}(J, l))$ is at most $c_1(|J| + 2^l)$.

The first condition in A2 says that for $l \geq 0$ and any subset $J$ of $\{1, \ldots, m\}$, the forward price $F_j$ is approximated by $\hat{F}_j$ with “mean square error” at most $c_22^{-\beta l}$ for any $j \in J$. The second condition in A2 gives an upper bound on the expected time needed to jointly simulate $\hat{F}(J', l - 1)$ and $\hat{F}(J, l)$. (3.4) below shows that A2 holds under certain conditions when the Euler or Milstein scheme is used to approximately simulate forward prices.

Assume now that A2 holds. For $l \geq 0$, let $\hat{F}^l = \hat{F}(J_l, l)$ and

$$\hat{A}_l = \sum_{j \in J_l} w_j \hat{F}_j^l + \frac{1}{2} \sum_{(i,k) \in P_l} W(i + 1, k - 1)(\hat{F}_i^l + \hat{F}_k^l).$$  

(17)

Thus $\hat{A}_l$ is obtained from $A$ by replacing each $F_j$ with $\hat{F}_j^l$ if $j \in J_l$ and by $(\hat{F}_i^l + \hat{F}_k^l)/2$ if $(i,k) \in P_l$ and $i < j < k$. Hence $\hat{A}_l$ is a deterministic linear function of the vector $\hat{F}^l$.  

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Proposition 3.2. If Assumption A2 holds then \( \| \hat{A}_0 - W(1, m) F_0 \| ^2 \leq c_3 \) and \( \| \hat{A}_t - A \| ^2 \leq c_3 2^{-\beta t} \) for \( t \geq 0 \), where \( c_3 = 2(c_2 + \text{Var}(F_m)) \).

Theorem 3.3 below shows how to construct an unbiased estimator of the Asian option price under Assumption A2, with \( \beta > 1 \). The case \( \beta = 1 \) will be considered in Theorem 3.4.

**Theorem 3.3.** Assume that A2 holds with \( \beta > 1 \). Let \( N \in \mathbb{N} \) be a random variable independent of \( (\hat{F}(j, l) : l \geq 0) \) and such that \( \text{Pr}(N = l) = p_l \) for non-negative integer \( l \), where \( p_l \) is given by (3). Let \( \hat{U}_l = f(\hat{A}_l) - a \) for \( l \geq 0 \), and let

\[
\hat{V} = \frac{\hat{U}_N - \hat{U}_{N-1}}{p_N},
\]

where \( \hat{U}_{-1} := 0 \). Then \( \hat{V} \) is square-integrable and

\[
E(f(A)) = E(\hat{V}) + a.
\]

Furthermore, \( \text{Var}(\hat{V}) \) and the expectation of the time required to simulate \( \hat{V} \) are upper-bounded by constants independent of \( m \).

As per the discussion following Theorem 3.2, Theorem 3.3 shows that \( e^{-rT}(\hat{V} + a) \) is an unbiased estimator of the Asian option price that can be simulated in constant time and with variance bounded by a constant independent of \( m \). Independent runs of this estimator yield an unbiased estimator of the option price with variance \( O(\epsilon^2) \) in \( O(m + \epsilon^{-2}) \) expected time.

Theorem 3.4 below constructs a biased estimator of the option price under Assumption A2 with \( \beta = 1 \).

**Theorem 3.4.** Assume that A2 holds with \( \beta = 1 \). Fix \( \epsilon \in (0, 1/2) \) and set \( L = \lceil 2 \log_2(1/\epsilon) \rceil \). Let \( N \in \mathbb{N} \) be a random variable independent of \( (\hat{F}(j, l) : l \geq 0) \) and such that \( \text{Pr}(N = l) = 2^{-(l+1)} \) for \( l \in \mathbb{N} \). Let \( \hat{U}_l = f(\hat{A}_l) - a \) for \( l \geq 0 \), and let

\[
\hat{V} = \frac{\hat{U}_N - \hat{U}_{N-1}}{p_N} 1_{N \leq L},
\]

where \( \hat{U}_{-1} := 0 \). Then \( \hat{V} \) is square-integrable and

\[
(E(\hat{V}) + a - E(f(A)))^2 \leq c_3 \kappa^2 \epsilon^2,
\]

where \( c_3 \) is defined as in Proposition 3.2. Furthermore, there are constants \( c_4 \) and \( c_5 \) independent of \( m \) and of \( \epsilon \) such that \( \text{Var}(\hat{V}) \leq c_4 \ln(1/\epsilon) \) and the expectation of the time required to simulate \( \hat{V} \) is upper-bounded by \( c_5 \ln(1/\epsilon) \).

Under the assumptions of Theorem 3.4 the price of the Asian option can be calculated with \( O(\epsilon^2) \) mean square error in \( O(m + \epsilon^{-2} \ln(1/\epsilon)) \) expected time, where the constants behind the \( O \) notation are independent of \( m \) and of \( \epsilon \), as follows. We simulate \( n \) independent copies of \( \hat{V} \), where \( n = \lceil \ln(1/\epsilon) \epsilon^{-2} \rceil \), and take their average \( \bar{V} \). As \( \text{Var}(\hat{V}) = \text{Var}(\hat{V})/n \), it follows that \( \text{Var}(\bar{V}) \leq c_4 \kappa^2 \epsilon^2 \). Furthermore, since the square function is convex,

\[
(E(\hat{V}) + a - E(f(A)))^2 \leq c_3 \kappa^2 \epsilon^2.
\]

Since the mean square error is the sum of the squared bias and of the variance, we conclude that

\[
\| \hat{V} + a - E(f(A)) \|^2 \leq (c_4 + c_3 \kappa^2) \epsilon^2.
\]

Thus \( e^{-rT}(\hat{V} + a) \) is an estimate of the price \( e^{-rT}E(f(A)) \) with mean square error \( O(\epsilon^2) \). The total expected time to simulate \( \hat{V} \) is \( O(m + \ln^2(\epsilon) \epsilon^{-2}) \), including the cost of Algorithm M.
3.4 The Euler and Milstein schemes

This subsection shows that A2 holds when the forward price follows a continuous diffusion process satisfying certain regularity conditions. Assume that the forward price $F(t)$ for maturity $T$ calculated at $t \in [0, T]$ satisfies the SDE

$$dF(t) = b(F(t), t)dW,$$

where $b$ is a real-valued function on $\mathbb{R}^2$ and $W$ is a one-dimensional Brownian motion under $Q$. For $J \subseteq \{1, \ldots, m\}$, let $\tau_0, \tau_1, \ldots, \tau_n$ be the elements of the time grid

$$G(J, l) = \{j : j \in J\} \cup \{i2^{-l}T : 0 \leq i \leq 2^l\},$$

sorted in increasing order. Note that $n \leq |J| + 2^l$ and the maximum distance $\delta$ between two consecutive elements of $G(J, l)$ is at most $2^{-l}T$. Using the time grid $G(J, l)$, the Euler scheme approximates the forward price path via the sequence $\tilde{F} = \tilde{F}(J, l)$ defined recursively as follows: $\tilde{F}_0 = F_0$ and, for $0 \leq k \leq n - 1$,

$$\tilde{F}_{k+1} = \tilde{F}_k + b(\tilde{F}_k, \tau_k)(\Delta W),$$  \hspace{1cm} (20)

where $\Delta W = W(\tau_{k+1}) - W(\tau_k)$. It follows from (Kloeden and Platen 1992, Theorem 10.6.3) that, under certain regularity conditions on $b$,

$$E(\max_{0 \leq k \leq n} (\tilde{F}_k - F(\tau_k))^2) \leq K_1 \delta,$$  \hspace{1cm} (21)

where $K_1$ is a constant that does not depend on $\delta$. Define $\hat{F} = \hat{F}(J, l) \in \mathbb{R}^J$ as follows. For $j \in J$, set $\hat{F}_j = \tilde{F}_k$, where $k$ is the index such that $\tau_k = t_j$. In other words, $\hat{F}$ is the “restriction” of $\tilde{F}$ to the dates corresponding to $J$. It follows from (21) that $||\hat{F}_j - F_j||^2 \leq K_1 T 2^{-l}$ for $j \in J$. Furthermore, for $l \geq 1$ and $J' \subseteq J \subseteq \{1, \ldots, m\}$, the grid $G(J', l - 1)$ is contained in $G(J, l)$. The vector $(\hat{F}(J', l - 1), \hat{F}(J, l))$ can thus be simulated in at most $c_1(|J| + 2^l)$ time, where $c_1$ is a constant independent of $m$, by first simulating $W$ on the elements of $G(J, l)$ and then using the same $W$ to calculate recursively $\hat{F}(J, l)$ and $\hat{F}(J', l - 1)$ via (20). Thus Assumption A2 holds for these processes with $\beta = 1$ for the Euler scheme.

Similarly, under regularity conditions on $b$, we can calculate $\hat{F}(J, l)$ by computing the sequence $F_\ast^* = F_\ast^*(J, l)$ via the Milstein scheme

$$F_{k+1}^* = F_k^* + b(F_k^*, \tau_k)(\Delta W) + \frac{1}{2}b(F_k^*, \tau_k)b'(F_k^*, \tau_k)((\Delta W)^2 - (\tau_{k+1} - \tau_k)),$$

where $b'$ is the partial derivative of $b$ with respect to its first argument. It then follows from (Kloeden and Platen 1992, Theorem 10.6.3) that, under certain regularity conditions on $b$,

$$E(\max_{0 \leq k \leq n} (F_k^* - F(\tau_k))^2) \leq K_2 \delta^2,$$

where $K_2$ does not depend on $\delta$. By arguments similar to those used in the Euler scheme analysis, we conclude that Assumption A2 holds for the Milstein scheme with $\beta = 2$ for scalar continuous processes satisfying certain regularity conditions. A straightforward generalization of the preceding arguments shows that Assumption A2 holds for the Euler scheme with $\beta = 1$ for multi-dimensional continuous processes satisfying certain regularity conditions.

4 Numerical example

Our numerical experiments assume that the underlying is a stock with no dividends that evolves according to the Black-Scholes model with $S_0 = 50$, $\sigma = 30\%$, $r = 5\%$, and $T = 4$. Except
for the maturity, these parameters are the same as in (Glasserman 2004, Example 4.1.1). We assume that

\[ A = m^{-1}(\sum_{i=1}^{m} S_i), \]

where \( S_i \) is the stock price at \( t_i = iT/m \). In order to compare the MLMC method described in Theorem 3.2 to the standard Monte Carlo method, we first observe that the expected running time required to simulate \( V \) is proportional to \( E(N) \), whereas the cost of simulating a single path in the standard Monte Carlo method is proportional to \( m \). As the efficiency of an unbiased estimator is inversely proportional to the time variance product, we define the variance reduction factor of the MLMC algorithm as follows:

\[ \text{VRF} = \frac{m\sigma_{MC}^2}{E(N)\sigma_{MLMC}^2}, \]

where \( \sigma_{MC} \) (resp. \( \sigma_{MLMC} \)) is the standard deviation of the conventional Monte Carlo (resp. MLMC) estimator. Table 1 gives variance reduction factors of the MLMC method described in Theorem 3.2 for selected values of \( m \) and \( K \). The standard deviations \( \sigma_{MC} \) (resp. \( \sigma_{MLMC} \)) were estimated using \( 10^4 \) independent samples of the conventional Monte Carlo (resp. MLMC) estimator, and \( E(N) \) was estimated by taking the empirical average of \( N \) over these samples. As suggested by Theorem 3.2, the VRFs in Table 1 are roughly proportional to \( m \) for each value of \( K \).

Table 1: Variance reduction factors for the Black-Scholes model with \( S_0 = 50, \sigma = 30\%, r = 5\% \), and \( T = 4 \).

| \( K \) | \( m \) | 125 | 250 | 500 | 1000 |
|------|-------|-----|-----|-----|-----|
| 40   | 17    | 35  | 65  | 138 |
| 50   | 12    | 24  | 49  | 98  |
| 60   | 9     | 18  | 38  | 75  |

5 Concluding remarks

We have described a general MLMC framework to estimate the price of an Asian option monitored at \( m \) dates. We assume the existence of a linear relation between the underlying and forward prices. Our approach yields an unbiased estimator with variance \( O(\epsilon^2) \) in \( O(m + \epsilon^{-2}) \) expected time for a variety of processes such as the Black-Scholes model, the CEV model, Merton’s jump-diffusion model, the variance gamma, NIG and \( \alpha \)-stable exponential Levy processes and, via the Milstein scheme, processes driven by scalar SDEs. Using the Euler scheme, our approach also estimates the Asian option price with mean square error \( O(\epsilon^2) \) in \( O(m + (\ln(\epsilon))\epsilon^{-2}) \) expected time for processes driven by multidimensional SDEs. Our method is simple to implement. Numerical experiments confirm that our approach outperforms the conventional Monte Carlo method by a factor of order \( m \).

A Proof of Proposition 2.1

Since \((x + x')^2 \leq 2(x^2 + x'^2)\) for any real numbers \( x \) and \( x' \), if \( X \) and \( X' \) are square-integrable random variables,

\[ ||X + X'||^2 \leq 2(||X||^2 + ||X'||^2). \] (22)
For \( l \geq 1 \), by applying (22) with \( X = Y_l - Y \) and \( X' = Y_{l-1} - Y \), it follows that
\[
||Y_l - Y_{l-1}||^2 \leq 2(||Y_l - Y||^2 + ||Y_{l-1} - Y||^2).
\] (23)

As observed in (Rhee and Glynn 2015),
\[
C = \sum_{l=0}^{\infty} p_l C_l.
\]

Since \( p_l \leq 2^{-1(\beta+1)l/2} \), it follows that
\[
C \leq c \sum_{l=0}^{\infty} 2^{-1(\beta-1)l/2},
\]
which yields (5). Since \( ||Y_{l-1} - Y||^2 \leq 4\nu 2^{-\beta l} \) by (2), it follows that from (23) that
\[
||Y_l - Y_{l-1}||^2 \leq 10\nu 2^{-\beta l}.
\] (24)

As \( ||Y_0||^2 \leq \nu \), (24) holds also for \( l = 0 \). Thus, as \( p_l \geq 2^{-1(\beta+1)l/2} \),
\[
\sum_{l=0}^{\infty} \frac{||Y_l - Y_{l-1}||^2}{p_l} \leq 20\nu \sum_{l=0}^{\infty} 2^{-1(\beta-1)l/2} = \frac{20\nu}{1 - 2^{-1(\beta-1)/2}}.
\]

By Theorem 2.1 we conclude that \( Z \) is square-integrable with \( E(Z) = E(Y) \), and that (1) holds. \( \square \)

B Proof of Proposition 2.2

First observe that \( Z_L \) coincides with the RHS of (1) if \( Y_l \) is replaced by \( Y_L \) for \( l \geq L \). Thus, by applying Theorem 2.1 to the sequence \( Y_{\min(l,L)} : l \geq 0 \), with \( Y = Y_L \), it follows that \( E(Z_L) = E(Y_L) \), and
\[
||Z_L||^2 = \sum_{l=0}^{L} \frac{||Y_l - Y_{l-1}||^2}{p_l}.
\]

Hence
\[
(E(Z_L - Y))^2 = (E(Y_L - Y))^2 \leq ||Y_L - Y||^2,
\]
which yields (7). On the other hand, for \( l \geq 1 \), as \( ||Y_{l-1} - Y||^2 \leq 2\nu 2^{-l} \) by (3), it follows from (23) that
\[
||Y_l - Y_{l-1}||^2 \leq 6\nu 2^{-l}.
\] (25)

Since \( ||Y_0||^2 \leq \nu \), (25) also holds for \( l = 0 \). Hence,
\[
\sum_{l=0}^{L} \frac{||Y_l - Y_{l-1}||^2}{p_l} \leq 12\nu (L + 1),
\]
which implies (8). Finally, the expected cost of computing \( Z_L \) is \( \sum_{l=0}^{L} p_l C_l \), which is upper-bounded by \( cL \) since \( p_l C_l \leq c/2 \). \( \square \)
C Proof of Proposition 3.1

We first show (10). As this equation clearly holds for \( l \geq L \), we assume that \( 0 \leq l \leq L - 1 \). Let \( j \) and \( j' \) be two elements of \( J_l \), with \( j < j' \). As \( j \leq j' - 1 \),

\[
[2^l W'(1, j)] \leq [2^l W'(1, j' - 1)] \\
\leq 2^l W'(1, j' - 1) \\
< [2^l W'(1, j')],
\]

where the last equation follows from (9). Thus the map \( j \mapsto [2^l W'(1, j)] \) from \( J_l \) to \( \{0, \ldots, 2^l\} \) is strictly increasing. This implies (10).

We now show (11). As this relation is obvious when \( l \geq L - 1 \), assume that \( 0 \leq l \leq L - 2 \). Since \( 2 \lfloor x \rfloor \leq \lfloor 2x \rfloor \) for \( x \in \mathbb{R} \), for any an element \( j \) of \( J_l \),

\[
2^{l+1} W'(1, j - 1) < 2 [2^l W'(1, j)] \\
\leq [2^{l+1} W'(1, j)],
\]

where the first equation follows from (9). Thus, \( j \in J_{l+1} \). This implies (11). \( \square \)

D Proof of Theorem 3.1

We first prove standard properties of martingales in Proposition D.1 below.

**Proposition D.1.** For \( 0 \leq i \leq j \leq k \leq m \),

\[
E(F_i(F_k - F_j)) = 0,
\]

(26)

and

\[
||F_i|| \leq ||F_j||.
\]

(27)

Moreover,

\[
||F_j - F_i||^2 \leq ||F_k||^2 - ||F_i||^2.
\]

(28)

**Proof.** Let \( \mathcal{F} = (\mathcal{F}_i), 0 \leq i \leq m, \) be the natural filtration of the random process \( (F_i), 0 \leq i \leq m \). By the tower law,

\[
E(F_i(F_k - F_j)) = E(E(F_i(F_k - F_j)|\mathcal{F}_j)) \\
= E(F_iE(F_k - F_j|\mathcal{F}_j)) \\
= 0.
\]

The last equation follows from the fact that \( (F_i), 0 \leq i \leq m, \) is a martingale with respect to \( \mathcal{F} \). This implies (26). In particular, \( E(F_i(F_j - F_i)) = 0 \). As \( F_j = (F_j - F_i) + F_i, \) it follows that

\[
||F_j||^2 = ||F_j - F_i||^2 + ||F_i||^2,
\]

which proves (27). Using the inequality \( ||F_j|| \leq ||F_k|| \) implies (28). \( \square \)

Next, we prove the following proposition, which implies that the sum of the absolute values of the weights \( |w_k| \), where integer \( k \) ranges on a sub-interval of \( [1, m] \) not intersecting \( J_l \), is at most \( 2^{-l} \).

**Proposition D.2.** For \( l \geq 0, \) if \( (i, k) \in \mathcal{P}_l \) then \( W'(i + 1, k - 1) \leq 2^{-l} \).
Proof. The desired inequality clearly holds if \( k = i + 1 \). Assume that \( k > i + 1 \). Thus \( l \leq L - 1 \). For any integer \( j \) in \( [i + 1, k - 1] \), since \( j \notin J_l \),

\[
2^l W'(1, j - 1) \geq [2^l W'(1, j)],
\]

and so \([2^l W'(1, j - 1)] = [2^l W'(1, j)]\). Hence

\[
2^l W'(1, k - 1) - 1 \leq [2^l W'(1, k - 1)] = [2^l W'(1, i)] \leq 2^l W'(1, i).
\]

As \( W'(i + 1, k - 1) = W'(1, k - 1) - W'(1, i) \), this completes the proof.

We now prove Theorem 3.1. By (13) and as \( J_0 = \{ m \} \),

\[
A_0 = w_m F_m + \frac{1}{2} W(1, m - 1)(F_0 + F_m).
\]

Hence

\[
A_0 - W(1, m)F_0 = \left( \frac{1}{2} W(1, m - 1) + w_m(F_m - F_0) \right),
\]

and so \( ||A_0 - W(1, m)F_0|| \leq ||F_m - F_0|| \). As \( E(F_m) = F_0 \), this implies the desired bound on \( ||A_0 - W(1, m)F_0||^2 \).

Fix now \( l \geq 0 \). For \( (i, k) \in \mathcal{P}_l \), let

\[
B_i = \sum_{j=i+1}^{k-1} w_j (F_j - F_i)
\]

and

\[
B'_i = \sum_{j=i+1}^{k-1} w_j (F_j - F_k).
\]

Rewriting (13) as

\[
A_l = \sum_{j \in J_l} w_j F_j + \frac{1}{2} \sum_{(i,k) \in \mathcal{P}_l j=i+1} ^{k-1} w_j (F_i + F_k),
\]

and noting that

\[
A = \sum_{j \in J_l} w_j F_j + \sum_{(i,k) \in \mathcal{P}_l j=i+1} ^{k-1} w_j F_j,
\]

it follows that

\[
A - A_l = \frac{1}{2} \sum_{(i,k) \in \mathcal{P}_l} (B_i + B'_i).
\]

Hence, by the triangular inequality,

\[
||A - A_l|| \leq \frac{1}{2} \sum_{(i,k) \in \mathcal{P}_l} ||B_i|| + \frac{1}{2} \sum_{(i,k) \in \mathcal{P}_l} ||B'_i||. \tag{29}
\]

We bound each of the two terms in the RHS of (29) separately. First observe that if \( (i, k) \) and \( (i', k') \) are two distinct elements of \( \mathcal{P}_l \) with \( i < i' \), then

\[
E(B_i B_{i'}) = \sum_{j=i+1}^{k-1} \sum_{j'=i'+1}^{k'-1} w_j w_{j'} E((F_j - F_i)(F_{j'} - F_{i'})) = 0.
\]
where the second equation follows from (26). Thus
\[ \sum_{(i,k) \in \mathcal{P}_l} \left\| B_i \right\|^2 = \sum_{(i,k) \in \mathcal{P}_l} \sum_{l} \left\| B_i \right\|^2. \]

On the other hand, for \((i, k) \in \mathcal{P}_l\), by the triangular inequality,
\[ \left\| B_i \right\| \leq \sum_{j=i+1}^{k-1} |w_j| \left\| F_j - F_i \right\| \leq W'(i + 1, k - 1) \sqrt{\left\| F_k \right\|^2 - \left\| F_i \right\|^2}, \]
where the second equation follows from (28). Using Proposition D.2 it follows that
\[ \sum_{(i,k) \in \mathcal{P}_l} \left\| B_i \right\|^2 \leq 2^{-2l} \sum_{(i,k) \in \mathcal{P}_l} (\left\| F_k \right\|^2 - \left\| F_i \right\|^2) = 2^{-2l} \left( \left\| F_m \right\|^2 - \left\| F_0 \right\|^2 \right) = 2^{-2l} \text{Var}(F_m). \]

We conclude that
\[ \left\| \sum_{(i,k) \in \mathcal{P}_l} B_i \right\| \leq 2^{-l} \text{Std}(F_m). \]

The same upper bound on \( \left\| \sum_{(i,k) \in \mathcal{P}_l} B'_i \right\| \) can be shown in a similar way. Hence
\[ \left\| A - A_l \right\| \leq 2^{-l} \text{Std}(F_m). \]

This concludes the proof.

\[ \square \]

### E Proof of Proposition 3.2

By (13) and (17),
\[ \hat{A}_l - A_l = \sum_{j \in J_l} w_j (\hat{F}^l_j - F_j) + \frac{1}{2} \sum_{(i,k) \in \mathcal{P}_l} W(i + 1, k - 1) ((\hat{F}^l_i - F_i) + (\hat{F}^l_k - F_k)). \]

Hence
\[ \left\| \hat{A}_l - A_l \right\| \leq \sum_{j \in J_l} w_j \left\| \hat{F}^l_j - F_j \right\| + \frac{1}{2} \sum_{(i,k) \in \mathcal{P}_l} W(i + 1, k - 1) \left( \left\| \hat{F}^l_i - F_i \right\| + \left\| \hat{F}^l_k - F_k \right\| \right). \]

As \( \left\| \hat{F}^l_j - F_j \right\| \leq \sqrt{c_2 2^{-\beta l}} \) for \( j \in J_l \) and
\[ \sum_{j \in J_l} w_j + \sum_{(i,k) \in \mathcal{P}_l} W(i + 1, k - 1) = 1, \]

it follows that \( \left\| \hat{A}_l - A_l \right\| \leq \sqrt{c_2 2^{-\beta l}}. \) Together with (14) and (22), this shows that \( \left\| \hat{A}_l - A \right\|^2 \leq c_3 2^{-\beta l}. \) Similarly, as \( \left\| A_0 - W(1, m) F_0 \right\|^2 \leq \text{Var}(F_m), \) we have \( \left\| A_0 - W(1, m) F_0 \right\|^2 \leq c_3. \)

\[ \square \]
Proof of Theorem 3.3

The proof is similar to that of Theorem 3.2. By A2 and (10), the vector \((\hat{F}^{l-1}, \hat{F}^l)\) can be simulated in \(O(2^l)\) expected time for \(l \geq 1\). Hence, by (17), there is a constant \(c'\) independent of \(m\) such that, for \(l \geq 0\), the expected time required to simulate \(\hat{U}_l - \hat{U}_{l-1}\) is at most \(c'2^l\). As \(|\hat{U}_0| \leq \kappa|\hat{A}_0 - W(1, m)F_0|\), Proposition 3.2 implies that \(|\hat{U}_0|^2 \leq c_3\kappa^2\), where \(c_3\) is defined as in Proposition 3.2. Similarly, for \(l \geq 0\), as \(|\hat{U}_l - U| \leq \kappa|\hat{A}_l - A|\), Proposition 3.2 shows that \(||\hat{U}_l - U||_2^2 \leq c_3\kappa^2 - \beta l\).

The conditions of Proposition 2.1 are thus met for \(Y = U\) and \(Y_l = \hat{U}_l\) for \(l \geq 0\), with \(\nu = c_3\kappa^2\) and \(c = c'\). Thus, \(\hat{V}\) is square-integrable with \(E(\hat{V}) = E(U)\). This implies (18). By (1),

\[||\hat{V}||^2 \leq \frac{20c_3\kappa^2}{1 - 2^{-(\beta-1)/2}},\]

and so \(\text{Var}(\hat{V})\) is upper-bounded by a constant independent of \(m\). By (1), the expectation of the time required to simulate \(\hat{V}\) is at most \(c'/(1 - 2^{-(\beta-1)/2})\). This concludes the proof.

Proof of Theorem 3.4

The proof is similar to those of Theorems 3.2 and 3.3. As shown in the proof of Theorem 3.3, there is a constant \(c'\) independent of \(m\) and of \(\epsilon\) such that the expected cost of computing \(\hat{U}_l - \hat{U}_{l-1}\) is at most \(c'2^l\) for \(l \geq 0\). Also, \(||\hat{U}_0||^2 \leq c_3\kappa^2\) and, for \(l \geq 0\),

\[||\hat{U}_l - U||^2 \leq c_3\kappa^2 2^{-l},\]

The conditions of Proposition 2.2 are thus met for \(Y = U\) and \(Y_l = \hat{U}_l\) for \(l \geq 0\), with \(\nu = c_3\kappa^2\) and \(c = c'\). Let \(L = \lceil2\log_2(1/\epsilon)\rceil\). By (17), \(\hat{V}\) is square-integrable and \((E(\hat{V} - U))^2 \leq c_3\kappa^2 \epsilon^2\). This implies (19). Similarly, (8) implies that

\[\text{Var}(\hat{V}) \leq 48c_3\kappa^2 \log_2(1/\epsilon).\]

Furthermore, the expectation of the time required to simulate \(\hat{V}\) is at most \(4c'\log_2(1/\epsilon)\).

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