An Inductive Construction for Many-Valued Coalgebraic Modal Logic

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Abstract—In this paper, we present an abstract framework of many-valued modal logic with the interpretation of atomic propositions and modal operators as predicate lifting over coalgebras for an endofunctor on the category of sets. It generalizes Pattinson’s stratification method for coalgebraic modal logic to the many-valued setting. In contrast to standard techniques of canonical model construction and filtration, this method employs an induction principle to prove the soundness, completeness, and finite model property of the logic. As a consequence, we can lift a restriction on the previous approach [1] that requires the underlying language must have the expressive power to internalize the meta-level truth valuation operations.

I. INTRODUCTION

Many-valued logic and modal logic are both long-standing themes in the research of symbolic logic and its applications since the last century. While the former is the main formalism for reasoning about vague information in terms of (mathematical) fuzzy logic [2], the latter has found diverse applications in software engineering, AI, economics, and philosophy [3]. In past decades, there have been increasing interests in systems integrating these two logics. From the early work motivated by reasoning with incomplete information [4], [5] and multi-expert opinions [6], [7] to more recent studies on mathematical fuzzy logic [8]–[12], such integration not only extends application scopes of the respective formalism, but also raises interesting theoretical issues.

Nowadays, there are many different variants of modal logic because of its extensive application scopes. Hence, it is desirable to have a general framework to encompass a variety of semantics of modal logics. Coalgebraic modal logic, first proposed by Moss [13], provides such a framework. There are two main approaches to coalgebraic modal logic: the relation lifting [13] and the predicate lifting [14], [15]. Venema et al. first studied a sound and complete axiomatization for coalgebraic modal logic through relation lifting [16]. On the other hand, the soundness and completeness of coalgebraic modal logic under predicate lifting were investigated in [15] and [14] using the finite-model construction and the induction on modal ranks respectively. For a more comprehensive survey of coalgebraic modal logic, see [17]. Analogously, many-valued coalgebraic modal logic can provide a uniform framework for a number of many-valued modal logics. Along this direction, Bílková and Dostál studied coalgebraic semantics of many-valued modal logic and proved its Hennessy-Milner property [18], [19]. Also, Schröder and Pattinson developed a coalgebraic semantics over standard Lukasiewicz algebra for fuzzy description logics and fuzzy probabilistic logics [20].

A well-known result in coalgebraic modal logic is that its soundness and completeness can be determined at the one-step level. In [1], it is shown that the same result also hold in the many-valued setting by using the technique of canonical model construction developed in [15], [21]. However, when applied to the many-valued case, the construction requires that the language must have the expressive power of internalizing the meta-level truth valuation operators. Hence, the objective of the paper is to lift the restriction by using the induction method proposed in [14] to prove the soundness, completeness, and finite model property of many-valued coalgebraic modal logics.

This paper is structured as follows. In the next section, we introduce the preliminary concepts and notations used in this paper. In Section III, we present the syntax and semantics of many-valued coalgebraic modal logic $\mathcal{MC}$ and its rank $n$ fragment $\mathcal{L}_n$, and show how to connect them using the technique of induction along the terminal sequence developed in [14]. In Section IV, we define proof systems of $\mathcal{MC}$ and $\mathcal{L}_n$ as consequence relations $\mathcal{L}$ and $\mathcal{L}_n$ respectively, and also establish the connection between them. In Section V, we present one-step logic in the many-valued setting and demonstrate how to prove soundness and completeness of $\mathcal{L}$ by assuming its one-step soundness and completeness. Finally, we prove the finite model property assuming the finiteness of the underlying functor in Section VI and conclude the paper in Section VII.

II. PRELIMINARIES AND NOTATIONS

In many-valued logic, we usually generalized the set of truth values from the Boolean algebra $2$ to an lattice $\mathbb{A}$. The many-valued structure considered in this paper is the (finite) residuated lattice, which provides semantics for a wide class of substructural logics [23].

1In the full paper [22], we also address the relationship between properties of functors for defining coalgebras and the design of proof systems.
Definition 1: We say that $\mathbb{A} = (A, \vee^A, \wedge^A, \rightarrow^A, \odot^A, 0^A, 1^A)$ is a commutative integral Full-Lambek algebra (FL-algebra) (aka residuated integral Full-Lambek algebra) if

- $(A, \vee^A, \wedge^A, 0^A, 1^A)$ is a bounded lattice,
- $(A, \odot^A, 1^A)$ is a commutative monoid,
- We can define ordering $\leq^A$ as $a \leq^A b$ iff $a \wedge^A b = b$ iff $a \vee^A b = a$,
- $\rightarrow^A$ is the residuated implication with respect to $\odot^A$, i.e. for all $a, b, c \in A$, $a \odot^A b \leq^A c$ iff $b \leq^A a \rightarrow^A c$,
- $\cdot \leq^A 1^A$ for all $a \in A$.

We will omit the superscript for each operation on the FL-algebras $\mathbb{A}$ without causing any confusion. When we mention an FL-algebra $\mathbb{A}$, we always use its carrier set.

To introduce the notion of predicate lifting, we assume the familiarity of basic category theory, mainly the definitions of category, functor, and natural transformation [24]. In this paper, we are exclusively concerned with the category of sets, denoted by $\text{Set}$, whose objects are sets and morphisms are functions between sets. We assume that $T : \text{Set} \rightarrow \text{Set}$ is a nontrivial endofunctor, i.e. there exists a set $S$ such that $TS \neq \emptyset$.

Definition 2: A $T$-coalgebra is a pair $(S, \sigma)$ where $S$ is a set and $\sigma : S \rightarrow TS$ is a function.

Recall that for any two sets $X$ and $Y$, a Hom-set $\text{Hom}(X, Y)$ denote the set of all functions (i.e. morphisms) from $X$ to $Y$. In addition, $\text{Hom}(\cdot, Y) : \text{Set} \rightarrow \text{Set}$ is a contravariant functor that sends a set $X$ to $\text{Hom}(X, Y)$ and a function $f : X_1 \rightarrow X_2$ to a function $\text{Hom}(f, Y) : \text{Hom}(X_2, Y) \rightarrow \text{Hom}(X_1, Y)$ such that for any $g \in \text{Hom}(X_2, Y)$, $\text{Hom}(f, Y)(g) = g \circ f$. We also call $\text{Hom}(\cdot, Y)$ a Hom-functor. We adapt the definition of predication lifting proposed in [19]. For simplicity, we only consider the unary case in the rest of this paper. One can easily generalize all the results to $n$-ary cases for any $n \geq 1$.

Definition 3: A predicate lifting for $T$ is defined as a natural transformation
\[
\lambda : \text{Hom}(\cdot, A) \Rightarrow \text{Hom}(T\cdot, A)
\]
with FL-algebra $\mathbb{A}$.

Because a natural transformation is a family of morphisms indexed by objects of the category, we use $\lambda_S$ to denote the morphism
\[
\text{Hom}(S, A) \rightarrow \text{Hom}(TS, A)
\]
for an object $S$ in $\text{Set}$.

In coalgebraic logic, we can regard propositional symbols as nullary modalities. To do that, we first define the product functor $T_P = EV_P \times T$ for a set of propositional symbols $P$, where $EV_P$ is a constant functor which maps an object (i.e. a set) to $\text{Hom}(P, A)$ and a morphism to $\text{id}_{\text{Hom}(P, A)}$ (i.e. the identity function from $\text{Hom}(P, A)$ to itself). The $T_P$-coalgebra is a pair $(S, \sigma_V)$ where $S$ is a set and $\sigma_V : S \rightarrow T_P S$ is defined by
\[
\sigma_V(s) := \langle V(s), \sigma(s) \rangle,
\]
where $V : S \rightarrow \text{Hom}(P, A)$ is a valuation of propositional symbols over $S$ and $\sigma : S \rightarrow TS$ is a $T$-coalgebra. Then, for any propositional symbol $p \in P$, the nullary predicate lifting
\[
\lambda^p : \text{Hom}(\cdot, A^0) \Rightarrow \text{Hom}(T_P\cdot, A)
\]
for the functor $T_P$ is defined as
\[
\lambda^p_S(\nu, \delta) := \nu(p)
\]
for $\nu \in EV_P S$, $\delta \in TS$, and the unique map $!_S : S \rightarrow A^0$. Note that $A^0$ is a singleton set which is the terminal object in $\text{Set}$. Also, we use the predicate lifting $\lambda$ to define
\[
\lambda^p : \text{Hom}(\cdot, A) \Rightarrow \text{Hom}(T_P\cdot, A)
\]
as
\[
\lambda^p_S(X)(\nu, \delta) := \lambda_S(X)(\delta)
\]
for any $X : S \rightarrow A$. From now on, we assume that $\Lambda$ is a set of (unary) predicate liftings and define $\Lambda_P$ as $\Lambda \cup \{\lambda^p | p \in P\}$.

III. SYNTAX AND SEMANTICS

In this section, we present the syntax and semantics of many-valued coalgebraic modal logic.

A. Syntax

Given an FL-algebra $\mathbb{A}$, an endofunctor $T$ over $\text{Set}$, a set of propositional symbols $P$, and a set of predicate liftings $\Lambda$, the alphabet of our language consists of logical connectives $\lor, \land, \rightarrow$, modalities $\forall \lambda$ and $\exists \lambda$, for every $\lambda \in \Lambda$ and $p \in P$, and constant symbols $\bar{c}$ for every $c \in A$ when $A$ is finite and $c \in \{0, 1\}$ when $A$ is infinite. The language $\mathcal{ML}$ is then defined inductively as follows:

\[
\varphi ::= \bar{c} | \varphi \lor \varphi | \varphi \land \varphi | \varphi \rightarrow \varphi | \forall \lambda \varphi | \exists \lambda \varphi.
\]

We abbreviate $(\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$ as $\varphi \leftrightarrow \psi$. Moreover, because we identify the propositional symbol $p$ with the nullary modality $\exists \lambda p$, we usually write $p$ in place of $\exists \lambda p$ in a formula.

Remark 1: We assume that there is a sound and complete multi-valued logic $\text{ML}(\mathbb{A})$ with respect to a FL-algebra $\mathbb{A}$. When $\mathbb{A}$ is a finite FL-algebra, such an $\text{ML}(\mathbb{A})$ necessarily exists (see the appendix in [8]). In this case, $\mathcal{ML}$ contains a constant symbol $\bar{c}$ for any $c \in A$. If $\mathbb{A}$ is infinite, $\text{ML}(\mathbb{A})$ exists in some (e.g. standard BL, Lukasiewicz, Gödel, and product algebras; see [2]) but not necessarily all cases. For the infinite-valued case, $\mathcal{ML}$ only need to contain constants $0$ and $1$.

To use the induction principle, we need to stratify $\mathcal{ML}$ according to modal ranks of formulas.

Definition 4: The rank of a formula $\varphi \in \mathcal{ML}$ is defined inductively as follows:

- $\text{rank}(\bar{c}) = \text{rank}(\exists \lambda \bar{c}) = 0$,
- $\text{rank}(\varphi \lor \psi) = \max\{\text{rank}(\varphi), \text{rank}(\psi)\}$ for $\varphi, \psi \in \{\lor, \land, \rightarrow\}$,
- $\text{rank}(\forall \lambda \varphi) = \text{rank}(\varphi) + 1$.

We define the rank-$n$ language $\mathcal{L}_n$ for each $n \in \omega$ as follows. First, we define the rank-0 language $\mathcal{L}_0$ inductively as
\[
\pi ::= \bar{c} | \pi \lor \pi | \pi \land \pi | \pi \rightarrow \pi | \forall \lambda \pi.
\]
Second, for any nonempty set $\Phi$, let $\Lambda(\Phi)$ denote

$$\{\sigma|\lambda \in \Lambda, \varphi \in \Phi\}.$$ 

and let $\mathcal{L}(\Phi)$ be the smallest set containing $\Phi$ and closed under the operation of $\lor, \land, \&$, and $\rightarrow$. Then, the rank-$n$ language is defined as

$$\mathcal{L}_n = \mathcal{L}(\Lambda(\mathcal{L}_{n-1} \cup \mathcal{L}_{n-1}))$$

for any $n \geq 1$. By construction, we can see that $\mathcal{ML} = \bigcup_{n \in \omega} \mathcal{L}_n$, where $\mathcal{L}_n$ contains all $\varphi \in \mathcal{ML}$ with rank($\varphi$) $\leq n$.

B. Semantics

Next, we develop the semantics for $\mathcal{ML}$. There are two different models which correspond to the layered structure of $\mathcal{ML}$. For the full language $\mathcal{ML}$, the semantic is defined as follows.

**Definition 5:** A T-model for $\mathcal{ML}$ is a $T_P$-coalgebra $\mathcal{C} = \langle S, \sigma_V \rangle$ consisting of a nonempty set of states $S$ and a map $\sigma_V : S \rightarrow T_P S$ defined by $\sigma_V(s) = \langle V(s), \sigma(s) \rangle$, where $V : S \rightarrow \text{Hom}(P, A)$ is a valuation of propositional symbols $P$ over $S$ and $\sigma : S \rightarrow TS$ is a T-coalgebra. We define the semantics $\|\varphi\|_e : S \rightarrow A$ for $\varphi \in \mathcal{ML}$ inductively such that for all $s \in S$

- $\|\varphi\|_e(s) = c \in A$.
- $\|\varphi \land \psi\|_e(s) = \|\varphi\|_e(s) \land^\Lambda \|\psi\|_e(s)$ for $* \in \{\lor, \land, \&\}$ and its corresponding algebraic operation $*$\textsuperscript{A} $\in \{\lor, \land, \&\}$ on $A$.
- $\|\Box \varphi\|_e(s) = \lambda^\Lambda_s(\|\varphi\|_e(s))$, for any predicate lifting $\lambda \in \Lambda$, and
- $\|\Box \varphi\|_e(s) = \lambda^\Lambda_s(\|\varphi\|_e(s))$, for any $p \in P$.

Given a set of formulas $\Gamma \subseteq \mathcal{ML}$, we use $\|\Gamma\|_e(s)$ to denote the set $\{\|\varphi\|_e(s) | \varphi \in \Gamma\}$ for any $s \in S$.

**Definition 6:** Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{ML}$ and let $\mathcal{C} = \langle S, \sigma_V \rangle$ be a T-model. Then, $\Gamma \models_e \varphi$ denotes that for all $s \in S$, $\|\Gamma\|_e(s) = \{1\}$ implies $\|\varphi\|_e(s) = 1$. If $\Gamma \models_e \varphi$ for all T-models $\mathcal{C}$, then we say $\varphi$ is a semantic consequence of $\Gamma$ and denote it by $\Gamma \models \varphi$. When $\Gamma = \emptyset$, we simply abbreviate the notations as $\models_e \varphi$ and $\models \varphi$ respectively.

As many-valued coalgbeic modal logic provides an uniform framework for a variety of many-valued modal logics, the following illustrative example show a typical instance. More concrete instances can be found in [1].

**Example 1:** Let us first recall the definition of the $A$-valued powerset functor $P^A$ that sends a set $X$ to $\text{Hom}(X, A)$ and a function $f : X \rightarrow Y$ to a mapping $P^A f : \text{Hom}(X, A) \rightarrow \text{Hom}(Y, A)$ such that $P^A f(g)(y) = \bigvee_{x \in f(x)} g(x)$ for any $g \in \text{Hom}(X, A)$ and $y \in Y$ [19]. We consider modalities in $A = \{\top, \bot\}$. Then, a $P^A$-model $\mathcal{C} = \langle W, \sigma, V \rangle$ is defined such that $\sigma : W \rightarrow \text{Hom}(W, A)$ is the functional representation of the $A$-valued accessibility relation on $W$. The predicate liftings $\Box, \Diamond : \text{Hom}(W, A) \rightarrow \text{Hom}(P^A W, A)$ are defined by

$$\Box(f)(g) = \bigvee_{x \in W} g(x) \rightarrow f(x)$$

for any $f, g : W \rightarrow A$. Thus, the interpretation of modal formulas is

$$\|\Box \varphi\|_e(w) = \bigvee_{u \in W} (\sigma(w)(u) \rightarrow \|\varphi\|_e(u))$$

and

$$\|\Diamond \varphi\|_e(w) = \bigvee_{u \in W} (\sigma(w)(u) \circ \|\varphi\|_e(u)).$$

Hence, we can instantiate the Kripke semantics originally proposed in [8] with the coalgbeic framework.

In the special case when $P^A$ is the 2-valued power set functor $P$. The definition of predicate liftings $\Box, \Diamond : \text{Hom}(W, A) \rightarrow \text{Hom}(P W, A)$ is reduced to

$$\Box(f)(X) = \bigvee_{x \in X} f(x) \text{ and } \Diamond(f)(X) = \bigvee_{x \in X} f(x)$$

for any $f : W \rightarrow A$ and $X \subseteq W$, and the interpretation of modal formulas is simplified to

$$\|\Box \varphi\|_e(w) = \bigvee_{u \in \sigma(w)} \|\varphi\|_e(u)$$

and

$$\|\Diamond \varphi\|_e(w) = \bigvee_{u \in \sigma(w)} \|\varphi\|_e(u),$$

which is simply the crisp Kripke model for many-valued modal logic proposed in [8].

Besides the general semantics for $\mathcal{ML}$, we can also define the step-$n$ semantic for the rank-$n$ language $\mathcal{L}_n$.

**Definition 7:** Let $1 = \{\bullet\}$ be a singleton set (i.e. a terminal object) in Set. A (pseudo)-terminal sequence based on the functor $T_P$ is defined by induction:

$$T^0_P 1 = \tilde{1} := \text{Hom}(P, A) \times 1, \quad T^n_P 1 = T^0_P (T^{n-1}_P 1) \text{ if } n > 0.$$

Then, for any $n \geq 0$ and $\varphi \in \mathcal{L}_n$, its $n$-step semantics $\|\varphi\|_n : T^0_P 1 \rightarrow A$ is defined as follows: for all $\langle \nu, \delta \rangle \in T^n_P 1$

- $\|\varphi\|_n(\langle \nu, \delta \rangle) := c \in A$,
- $\|\varphi \land \psi\|_n(\langle \nu, \delta \rangle) := \|\varphi\|_n(\langle \nu, \delta \rangle) \land^\Lambda \|\psi\|_n(\langle \nu, \delta \rangle)$ for $* \in \{\lor, \land, \&\}$ and its corresponding algebraic operation $*^\Lambda \in \{\lor, \land, \&\}$ on $A$,
- $\|\Box \varphi\|_n(\langle \nu, \delta \rangle) := \lambda^\Lambda_{\nu}((\|\varphi\|_{n-1}(\langle \nu, \delta \rangle)))$, where $\lambda \in \Lambda$ is a predicate lifting,
- $\|\Box \varphi\|_n(\langle \nu, \delta \rangle) := \nu(p)$.

Note that our definition of terminal sequence is a little unusual because its starting point $T^0_P 1$ is not the terminal object 1 but $\tilde{1}$. Such an unusualness is necessary to ensure the well-definedness of $\|\varphi\|_n$ for any propositional symbol $p \in \mathcal{L}_\Omega$.

As above, for a set of formulas $\Gamma \subseteq \mathcal{L}_n$, we use the notation $\|\Gamma\|_n(t)$ to denote the set $\{\|\varphi\|_n(t) | \varphi \in \Gamma\}$ for any $t \in T^0_P 1$.

**Definition 8:** Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_n$. We say that $\varphi$ is a step-$n$ semantic consequence of $\Gamma$, denoted by $\Gamma \models_n \varphi$, if $\|\Gamma\|_n(t) = \{1\}$ implies $\|\varphi\|_n(t) = 1$ for all $t \in T^0_P 1$. When $\Gamma = \emptyset$, we simply write $\Gamma \models_n \varphi$ as $\models_n \varphi$.
Given a $T_P$-coalgebra $(S, \sigma_V)$, let us define $\sigma_k$ for $k \in \omega$ inductively as follows

$$\sigma_0 : S \to 1, \quad s \mapsto (V(s), \bullet),$$

$$\sigma_k = T_P\sigma_{k-1} \circ \sigma_V.$$

Then, we have the following connection between $\|\cdot\|_{\psi}$ and $\|\cdot\|_n$.

**Theorem 2:** For any $\phi \in \mathcal{L}_n$, we have $\|\psi\|_\psi = \|\phi\|_n \circ \sigma_\phi$.

**Proof:** We prove this theorem with simultaneous induction on $n$ and on the complexity of $\phi \in \mathcal{L}_n$ for $\phi$. We first fix a state $s \in S$.

1) $n = 0$: by induction on the complexity of $\pi \in \mathcal{L}_0$

   a) $\pi = \bar{\varepsilon}$ for $c \in A$: $\|\bar{\varepsilon}\|_c(s) = c = \|\varepsilon\|_0(\sigma_0(s))$

   b) $\pi = \circ_{\lambda_{\psi}}$ for a propositional symbol $\psi$: we have

     $$\|\circ_{\lambda_{\psi}}\|_{v_1}(s) = \lambda_{\psi}(V(s))$$

     $$\quad = V(s)$$

     $$\quad = \|\circ_{\lambda_{\psi}}\|_0(V(s), \bullet))$$

     $$\quad = \|\circ_{\lambda_{\psi}}\|_0(\sigma_0(s))$$

   c) $\pi = \pi_0 * \pi_1$ for $\pi_0, \pi_1 \in \mathcal{L}_0$ and logical connectives $*$: by the inductive hypothesis, we have

     $$\|\pi_0 * \pi_1\|_{v_1}(s) = \|\pi_0\|_{v_1}(s) *^h \|\pi_1\|_{v_1}(s)$$

     $$\quad = \|\pi_0\|_{v_1}(\sigma_\phi(s)) *^h \|\pi_1\|_{v_1}(\sigma_\phi(s))$$

     $$\quad = \|\pi_0\|_{v_1}(\sigma_\phi(s))$$

2) $n \geq 1$: by induction on the complexity of $\phi \in \mathcal{L}_n$

   a) $\phi = \bar{\varepsilon}$ or $\phi = \phi_0 \cdot \phi_1$: the proof is the same as in the case of $n = 0$

   b) $\phi = \circ_{\lambda_{\psi}}$: Recall that $\sigma_\phi(s) = (\text{id}_{\text{Hom}(P,A)} \times T\sigma_{n-1})(\sigma_V(s)) = (V(s), T\sigma_{n-1}(\sigma_\phi(s)))$. Hence,

     $$\|\circ_{\lambda_{\psi}}\|_{v_1}(s) = \lambda_{\phi}(V(s))$$

     $$\quad = V(s)$$

     $$\quad = \|\circ_{\lambda_{\psi}}\|_0(\sigma_\phi(s))$$

   c) $\phi = \circ_{\lambda_{\psi}}$: For simplicity, we abbreviate Hom-functors $\text{Hom}(\bar{A}, -)$ and $\text{Hom}(T\bar{P}(\bar{A}), -)$ as $\bar{H}$ and $\bar{HT}\bar{P}$ respectively in the following derivation. Recall that for the functor $\bar{H}$, we have $H(f) = g \circ f$ for morphisms $g$ and $f$ of appropriate types. Then, we have

     $$\|\circ_{\lambda_{\psi}}\|_{v_1}(s) = \lambda_{\psi}(\psi)(\|\cdot\|_n(s))$$

     $$\quad = \lambda_{\psi}(\psi)(\|\cdot\|_{n-1} \circ \sigma_{n-1}(\sigma_\phi(s)))$$

     $$\quad = \lambda_{\psi}(\psi)(\|\cdot\|_{n-1} \circ \sigma_{n-1}(\sigma_\phi(s)))$$

Recall that there is a unique map $\!_X : X \to 1$ for any set $X$ because $1$ is the terminal object. Hence, if $Y$ is a set of the form $\text{Hom}(P,A) \times X$, we can define a surjective map $!_{V} : Y \rightarrow 1$ by $\!_{V} := \text{id}_{\text{Hom}(P,A)} \times !_{X}$. In particular, when $Y = T_P \overline{1}$, we write the surjection $!_{T_P\overline{1}}$ as $\gamma^\psi$ and it has a right inverse $\delta : 1 \rightarrow T_P \overline{1}$ so that $\gamma^\psi \circ \delta = \text{id}_{T_P \overline{1}}$. For any $n \geq 1$, let $\gamma^n = T^n_P \gamma^\psi : T^n_P \overline{1} \rightarrow T^n_P \overline{1}$ and $\delta^n = T^n_P \delta : T^n_P \overline{1} \rightarrow T^n_P \overline{1}$. We obtain $\gamma^n \circ \delta^n = \text{id}_{T^n_P \overline{1}}$. Note that $(T^n_P \overline{1}, \delta^n)$ is a $T_P$-coalgebra for every $n \in \omega$. Hence, by respectively substituting $S$ and $\sigma_V$ with $T^n_P \overline{1}$ and $\delta^n$ in the definition of the sequence $(\sigma_k)_{k \in \omega}$ above, we can also define $i^n_k$ for $k \in \omega$ inductively

$$i^n_k : T^n_P \overline{1} \rightarrow 1; (\nu, d) \mapsto (\nu, \bullet)$$

$$i^n_k = T^n_P \delta_{k-1} \circ \delta^n.$$

**Lemma 1:** For all $k \leq n$, $i^n_k = T^n_P \delta_{T_P \overline{1} \rightarrow T^{n+1}_P \overline{1}}$.

**Proof:** The proof is similar to that for Lemma 4.11 in [14] and we omit it here. □

**Theorem 3:** Suppose $\phi \in \mathcal{L}_n$, we have $\models \phi \iff \models \phi$.

**Proof:** For the only if direction, let us consider the coalgebra $\mathcal{C} = (T^n_P \overline{1}, \delta^n)$. Then, by the assumption, $\|\phi\|_n(t) = 1$ for all $t \in T^n_P \overline{1}$. Using Lemma 1, we have $\delta^n = \text{id}_{T^n_P \overline{1}}$. Hence, by Theorem 2, we have $\|\phi\|_{n+1}(t) = \|\phi\|_n \circ (\delta^n)(t) = \|\phi\|_n(t) + 1$ for all $t \in T^n_P \overline{1}$.

For the other direction, suppose that $\models \phi$. Then, for all $t \in T^n_P \overline{1}$, we have $\|\phi\|_{n+1}(t) = 1$. Hence, by Theorem 2, for any $T_P$-coalgebra $\mathcal{C} = (S, \sigma_V)$, we can construct the sequence $(\sigma_k)_{k \in \omega}$ such that for any $s \in S$, $\|\phi\|_n(s) = \|\phi\|_n(\sigma_\phi(s)) = 1$.

**Corollary 1:** Suppose $\Gamma \cup \{\phi\} \in \mathcal{L}_n$, we have $\Gamma \models \phi$.

**Proof:** The proof is a straightforward extension of that for Theorem 3. □

**IV. PROOF SYSTEM**

In this section, we characterize semantic consequences in $\mathcal{ML}$ and its rank-$n$ fragment with provability relations. First, we introduce the concept of basic derivation relation, called a consecution.

**Definition 9:** A consecution in $\mathcal{ML}$ is a pair $(\Gamma, \phi)$, where $\Gamma \subseteq \mathcal{ML}$ and $\phi \in \mathcal{ML}$. We usually write a consecution $(\Gamma, \phi)$ as $\Gamma \vdash \phi$ and for a set of consecutions $\mathbf{L}$, we always abbreviate $\Gamma \vdash \phi \in \mathbf{L}$ as $\Gamma \vdash_{\mathbf{L}} \phi$ and omit $\Gamma$ when it is empty.

For a set of consecutions to be a nice logical system, we need to impose some closure properties on it. As usual, an axiom is regarded as a schema. Hence, a logical system should contain all substitution instances of its axioms. More precisely, a **substitution** is a map $\rho : P \rightarrow \mathcal{ML}$. We will use the notation $(\phi/p_i : p_i \in I)$ for the substitution that maps each variable $p_i \in I$ to the formula $\phi$ and remains identical in other variables $p \in P \setminus I$, where $I$ is a (typically finite) subset of $P$. The application of a substitution $\rho = (\phi/p_i : p_i \in I)$ to a formula $\phi$ results in a new formula $\phi[\rho]$ in which the nullary modality $\circ_{\lambda_\phi}$ is uniformly replaced by $\phi$ for each $p_i \in I$ and we say that $\phi[\rho]$ is an instance of $\phi$. An instance of a consecution $(\Gamma, \phi)$ is $(\Gamma, \phi[\rho])$ where $\Gamma[\rho] := \{\phi[\rho] : \phi \in \Gamma\}$. If the range of
a substitution ρ is confined to $L_n$, i.e. ρ : $P \rightarrow L_n$, then we call ρ an n-substitution.

As we have assumed the existence of a logical axiomatization $Ax(\mathbb{A})$ for the FL algebra $\mathbb{A}$, we can regard the derivability of ϕ from Γ in $Ax(\mathbb{A})$ as a consequence and denote it by $\Gamma \vdash_{Ax(\mathbb{A})} \phi$. Let us also fix a set of consecutions $Ax(\Lambda) \subseteq \mathbb{P}(L_\Lambda) \times L_\Lambda$ for rank-1 modal axioms. Then, we can define the general consequence relation as follows.

**Definition 10:** A set of consecutions $L$ is called a general consequence relation if it is the least set satisfying the following closure properties

- If $\Gamma \vdash_{Ax(\mathbb{A})} \phi$, then $\Gamma \vdash_L \phi$.
- If $\Gamma \vdash_{Ax(\mathbb{A})} \phi$, then $\Gamma \vdash_L \phi \rho \phi$ for each substitution ρ.
- If $\Gamma \vdash_L \phi$, then $\bigcup \lambda \Gamma \vdash_L \phi \lambda \phi$ for each λ ∈ Λ where $\bigcup \lambda := \{ \phi \lambda \phi \mid \phi \lambda \phi \in \Gamma \}$. The role of modal axioms in $Ax(\Lambda)$ is to characterize predicate liftings in Λ. In the general framework, we do not restrict to any specific functor T and predicate liftings. Thus, we do not consider a concrete set of axioms in $Ax(\Lambda)$. For instance, one might include $\{ 0, 1, 1 \}$ or $\{ \bigcup \phi \land \bigcup \psi, \bigcup (\phi \land \psi) \}$ in $Ax(\Lambda)$ for a particular predicate lifting. In addition, because $L$ includes the derivation relation in $Ax(\mathbb{A})$, it must include all instances of axioms in $Ax(\mathbb{A})$ and be closed under inference rules of $Ax(\Lambda)$. For example, $\mathcal{L}$ should be closed under the following rule if Modus Ponens is an inference rule of $Ax(\mathbb{A})$.

$$
\frac{\Gamma \vdash_L \phi \quad \Gamma \vdash_L \phi \rightarrow \psi}{\Gamma \vdash_L \psi} \quad \text{(MP)}
$$

Since $\mathcal{ML}$ can be stratified into the union of $L_n$ for all $n \in \omega$, we can also define the step-n consequence relation $L_n$ for each $L_n$.

**Definition 11:**

1. $n = 0$: We define the step-0 consequence relation $L_0 \subseteq \mathbb{P}(L_0) \times L_0$ by $\Gamma \vdash_{L_0} \phi$ iff $\Gamma \vdash_{Ax(\mathbb{A})} \phi$ for any $\Gamma \vdash \phi \subseteq L_0$.

2. $n > 0$: A set of consecutions $L_n \subseteq \mathbb{P}(L_n) \times L_n$ forms the step-n consequence relation if it is the least set satisfying the following closure properties:

- If $\Gamma \vdash_{Ax(\mathbb{A})} \phi$, then $\Gamma \vdash_{L_n} \phi$ for any $\Gamma \vdash \phi \subseteq L_n$.
- If $\Gamma \vdash_{Ax(\mathbb{A})} \phi$, then $\bigcup \rho \Gamma \vdash_{L_n} \phi \rho \phi$ for any $\rho \vdash \phi \subseteq L_n$.
- If $\Gamma \vdash_{L_{n-1}} \phi$, then $\bigcup \lambda \Gamma \vdash_{L_n} \phi \lambda \phi$ for any $\lambda \in \Lambda$ and $\Gamma \vdash \phi \subseteq L_{n-1}$.

As there is an intimate connection between $\vdash$ and $\vdash_n$, we can also find an analogous relationship between general and step-n consequence relations.

**Theorem 4:** Suppose $\Gamma \subseteq L_n$ and $\phi \in L_n$. Then $\Gamma \vdash \phi$ iff $\Gamma \vdash_{L_n} \phi$.

**Proof:** It is obvious that $L_n \subseteq L$ for any $n \in \omega$. Thus, we only need to prove that $\Gamma \vdash \phi$ implies $\Gamma \vdash_{L_n} \phi$. We prove this by induction on n. For $n = 0$, the only way that $\Gamma \vdash \phi$ is $\Gamma \vdash_{Ax(\mathbb{A})} \phi$. Therefore, $\Gamma \vdash \phi$ follows by definition. For $n > 0$, let us consider all possible ways for the derivation of $\Gamma \vdash \phi$. First, if $\Gamma \vdash_{Ax(\mathbb{A})} \phi$, then $\Gamma \vdash_{L_n} \phi$ holds by definition. Second, if $\Gamma \vdash L \phi$ is a substitution instance of $Ax(\Lambda)$, then the substitution must be an $(n-1)$-substitution because of the form of modal axioms. Hence, we have $\Gamma \vdash_{L_n} \phi$. Finally, if $\Gamma \vdash L \phi$ was derived by $\Gamma \vdash L \psi$ such that $\Gamma = \bigcup \lambda \Gamma \phi$ and $\phi = \bigcup \lambda \psi$ for some $\Gamma \vdash \phi \subseteq L_{n-1}$, then by the inductive hypothesis, we have $\Gamma \vdash_{L_{n-1}} \psi$, which leads to the desired result. □

**V. ONE-STEP LOGIC: SOUNDNESS AND COMPLETELESS**

We have defined both logical systems and semantics for $\mathcal{ML}$ and its rank-n fragments. Now, we can consider the soundness and completeness of a logical system with respect to its corresponding semantics. As usual, we say that $\vdash$ is sound (resp. complete) with respect to $\vdash$ if $\Gamma \vdash \phi$ implies $\Gamma \vdash \phi$ (resp. $\Gamma \vdash \phi$ implies $\Gamma \vdash \phi$). When $\Gamma$ is restricted to finite sets, the soundness and completeness are called finitary. By the definition of step-n consequence relation, we know that $L_0$ is simply the derivation relation of the underlying many-valued logic, which is sound and complete with respect to $\vdash_0$ [25]. For $n > 0$, we define the one-step version of soundness and completeness for each $L_n$.

**Definition 12:** We say that $\vdash_{L_n}$ is one-step sound (resp. complete) if the soundness (resp. completeness) of $\vdash_{L_{n-1}}$ with respect to $\vdash_n$ implies that of $\vdash_{L_n}$ with respect to $\vdash_n$. In addition, the general consequence relation $\vdash_L$ is one-step sound (resp. complete) if $\vdash_{L_n}$ is one-step sound (resp. complete) for each $n > 0$.

Because of the full generality of the definition, to check the one-step soundness and completeness of a system will depend on the specification of the particular functor and predicate liftings, as well as their characteristic modal axioms. While it is still an open issue, there is some preliminary discussion in the full paper [22] regarding general conditions on $Ax(\Lambda)$ and Λ that can sufficiently guarantee the one-step soundness and completeness.

Next, we show that the soundness and completeness of many-valued coalgebraic modal logics can be determined at the one-step level.

**Theorem 5:** If $\vdash_L$ is one-step sound (resp. complete), then $\vdash_{L_n}$ is sound (resp. complete) with respect to $\vdash_n$ for each $n \in \omega$.

**Proof:** The proof is an easy induction on n. For $n = 0$, the claim follows from the soundness and completeness of $Ax(\Lambda)$. For $n > 0$, one-step soundness and completeness of $\vdash_L$ implies that the soundness and completeness of $\vdash_{L_{n-1}}$ can be transferred to the next level inductively. □

**Theorem 6:** If $\vdash_L$ is one-step sound (resp. complete), then $\vdash_L$ is finitary sound (resp. complete) with respect to $\vdash_L$.

**Proof:** Because $\mathcal{ML} = \bigcup_{n \in \omega} L_n$, for any finite set $\Gamma \vdash \phi \subseteq \mathcal{ML}$, we may assume that $\Gamma \vdash \phi \subseteq L_n$ for some n. Hence, we have

$$
\Gamma \vdash \phi \iff \Gamma \vdash_{L_n} \phi \iff \Gamma \vdash_n \phi \iff \Gamma \vdash \phi
$$

by Theorem 4, Theorem 5, and Corollary 1. □

**VI. FINITE MODEL PROPERTY**

In the previous section, we have seen the connection between coalgebraic model $\mathcal{C}$ and the terminal sequence
and over the $\in M L \langle = \in C \phi \in L$ be a finite functor defined above. Then, is $2$ \parallel with $\parallel EV \phi \phi is finite because $\in \parallel \in ONCLUSION \phi if \tilde{\phi} over the slice category is satisfiable, then $\phi$ by Theorem 2, we have \bigcup L is a finite set. 

\textbf{Conclusion}:

We have used the stratification method and induction on the modal rank of formulas to prove that the soundness and completeness of many-valued coalgebraic modal logics are determined at the one-step level. Besides, we also prove finite model property under the finiteness assumption of the functor. This is different from methods of canonical model construction and filtration employed in [1]. As a consequence, we no longer require that the underlying many-valued language must have the expressive power to internalize the meta-level truth valuation operations, as it is needed during the construction of canonical model in [1].

In [16], [26], the stratification method is also used to prove either cut-free completeness for coalgebraic modal logics via predicate lifting or completeness for coalgebraic modal logics via relation lifting. In the future work, we will extend these results to the many-valued case by considering an endofunctor $T$ over the slice category $\text{Set} / Hom(P, A)$ rather than the product functor $T_P = EV_P \times T$ over the $\text{Set}$ category. Furthermore, in this paper, we do not have a concrete example for one-step soundness and completeness. Hence, to prove one-step soundness and completeness of concrete many-valued modal logics is another pressing issue for further research.

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