Quantum enhancement in time-delays estimation through spectrally resolved two-photon interference

Danilo Triggiani*1, Giorgos Psaroudis1 and Vincenzo Tamma†1,2

1School of Mathematics and Physics, University of Portsmouth, Portsmouth PO1 3QL, UK
2Institute of Cosmology and Gravitation, University of Portsmouth, Portsmouth PO1 3FX, UK

December 23, 2021

Abstract

The quantum interference occurring between two indistinguishable photons impinging on the two input faces of a beam-splitter can be exploited for a range of applications, from quantum optical coherence tomography, to quantum metrology including time intervals measurements. In the latter, recent advances managed to reach a resolution in the estimation of the delay between the two photons of the order of attoseconds, i.e. in the nanometer scale. Unfortunately, these techniques are highly affected in the estimation precision by any experimental distinguishability between the photons at the detectors. Here, we perform an analysis of the precision achievable in the estimation of the delay between two independent photons interfering at a beam-splitter when frequency-resolved measurements are employed. Remarkably, we show that the observation of the spectra of the photons at the output ports when coincidence and bunching events are recorded, largely enhance the precision of the estimation for any degree of distinguishability between the photons at the detectors. In particular, we find that such scheme is effective also for temporal delays much larger than the coherence time of each photons, a regime in which standard two-photon interferometers or spectral analyses at the single-photon level, do not provide any information. Furthermore, we show that by increasing the bandwidth of the photons it is possible to further increase quadratically the precision in the estimation, differently from non-resolved two-photon interference where the precision degrades for large bandwidth values. Therefore, such estimation scheme with frequency-resolving detectors allows to substantially enhance the precision of measurements of time delays, including the ones induced by one of the two photons going through a transparent materials or biological tissue with unknown thickness. Relevant applications can range from the characterization of two-dimensional nanomaterials to the analysis of biological samples, including DNA and cell membranes.

1 Introduction

Photons manifest unique quantum properties which might appear odd and counter intuitive from a classical standpoint. A paradigmatic example of this unusual behaviour is given by the interference pattern of two photons when two identical photons simultaneously impinge on the two faces of a beam-splitter, and on-off detectors are employed at each one of its output arms, no coincidence event is ever observed.
Instead the two photons always end up in the same output arm of the beam-splitter [1–8]. This tendency of the photons to ‘bunch’ together is a prerogative of all quanta of bosonic fields (e.g. the electromagnetic field), and arises from the lack of information on which path the two photons undertake when interfering at the beam splitter. Such interference phenomenon tends to disappear at the increasing of the distinguishability between the two photons, for example in their colours, polarizations or injection times at the beam splitter. This allows us to conceive estimation schemes for high-precision measurements of the two-photon time delay [1, 9–11], or the state of polarization of the photons [12]. Remarkably, the estimation of time intervals has achieved in this way precisions ranging from sub-picoseconds [1], even up, recently, to the attoseconds regime [9].

The employment of notions borrowed from estimation theory and the subsequent study of the Fisher Information [13,14] – a way to quantify the ultimate amount of information that can be possibly obtained about an unknown parameter with a given estimation scheme – have recently allowed for a further boost in the levels of precision achievable [9,11,15]. Finding the optimal value of the time interval to be estimated [9], or engineering an efficient spectral distribution of the two-photon quantum probe state employed [11], are two paths made possible by the analysis of the Fisher Information. However, an analysis of the Fisher information in the estimation of the delay between two interfering photons has been limited to scenarios where the detectors do not resolve any photonic degrees of freedom [9]. Unfortunately, the sensitivity of such scenarios depends on the overlap between the photonic wave-packets and therefore on the values of time delays to be estimated, becoming completely insensitive to values beyond the photonic coherence time [9]. Furthermore any distinguishability between the photons at the detectors in any other degrees of freedom, such as polarization or spatial inner modes, also affects substantially the precision in the estimation [9].

A different approach with respect to standard two-photon interference takes advantage of current detectors capable to resolve the time of arrival of the photons [16–20], or their frequencies [21,22], in order to restore quantum interference even for photonic wave-packets not overlapping in the frequency or time domain, respectively. Indeed, one can observe quantum beating [16,17,23,24], i.e. the oscillations in the count of coincidence and bunching events of two photons with different frequencies or with a certain delay as a function of the detection times or the detected frequencies, respectively. The period of the measured quantum beats can then be exploited to infer the value of the delay or the frequency shift between the two photons independently of the overlap between the photonic wave-packets. However, although a quantum advantage for inner-mode interference of independent photons has been already demonstrated from a computational point of view [25–27], the ultimate precision of such technique from a metrological perspective is yet not known. This motivates the following important questions pertinent to the development of high-precision quantum sensing technologies. Is there a quantum metrological advantage arising from the inner-mode quantum interference of independent photons? If yes, is it possible to quantify such an advantage in terms of the associated Fisher information? Indeed, an enhancement in the precision of estimation schemes, particularly for time delays, would largely increase the feasibility of protocols for the observation of nanomaterials or nanostructures in biological samples, such as cell membranes and DNA structures.

In this work, we analyse the ultimate precision achievable in the estimation of time delays between independent photons with same frequency distribution, interfering at a balanced beam-splitter and detected at the output with frequency-resolving detectors, as described in Figure 1. Differently from the photon pair generated with spontaneous parametric down-conversion, these probes can be generated by two independent single-photon sources, such as heralded spontaneous parametric down conversion sources [21,28] or quantum dots [29]. We show that the measurement scheme proposed is the optimal one for identical photons with an arbitrary time delay, namely it maximises the Fisher information. Remarkably, no entanglement between the photons is required overcoming therefore the associated detrimental effects of decoherence. Indeed, the only metrolog-
ical resource in our scheme is the quantum interference of two photons emitted at different times by measuring their frequencies.

We show that even when a non-vanishing distinguishability between the two photons in any photonic parameter different from the one we want to estimate (e.g. polarizations) is present, and such distinguishability is not ‘erased’ at the detectors by suitable measurements, the quantum advantage of this technique is still retained. In particular, we demonstrate that this scheme is effective also in the regime of photons coherence times much smaller than their temporal delay, a regime in which standard two-photon interferometry, without resolving the photonic frequencies, fades away becoming ‘blind’ blind to the delay value. On the other hand, we show that in our scheme, for regular photonic spectra, such as Gaussian spectra, the Fisher information only depends on the bandwidth of the photons, while their spectral shape and temporal overlap do not affect the precision. Therefore, the precision in the estimation of the photonic time-delay is remarkably independent of its value. Furthermore, such precision can be increased quadratically with the increasing of the photonic bandwidth for large bandwidth values. This is not the case if the detectors do not resolve the photonic frequencies resulting in a ‘classical’ average of the measurement outcomes over all the possible frequencies which hinders the sensitivity in the estimation \[9\].

2 Experimental setup

We describe now in more detail the setup in Figure 1, where two independent photons are injected separately at the two input ports of a balanced beam splitter. The quantum state of the two independent photons before impinging on the beam-splitter is

\[
|\psi\rangle = \int d\omega_1 \xi_1(\omega_1) \left( \eta a_{1,\omega_1}^\dagger + \sqrt{1 - \eta^2} b_{1,\omega_1}^\dagger \right) |\text{vac}\rangle \\
\quad \quad \otimes \int d\omega_2 \xi_2(\omega_2) \bar{a}_{2,\omega_2}^\dagger |\text{vac}\rangle ,
\]

where \(\xi_i(\omega) = \bar{\xi}(\omega)e^{-i\omega t_i}\) is the frequency probability amplitude of the photon injected in the \(i\)-th channel, with \(i = 1, 2\), with \(|\xi_1(\omega)| = |\xi_2(\omega)| = \bar{\xi}(\omega)\), (photons with the same frequency probability distribution) and \(t_i\) is the emission time of the \(i\)-th photon.

Figure 1: Scheme of the interferometric setup. Two independent photons in the state in Eq. (1) with identical frequency distributions are produced by independent sources \(S_1\) and \(S_2\) with a given degree of indistinguishability \(\eta\) deriving from different properties of the two photons other than their temporal properties (e.g. different polarizations, or inner spacial modes). The two photons then impinge onto the two faces of a balanced beam splitter with an unknown time delay \(\Delta t\). The photons are eventually observed by the frequency-resolving detectors \(D_1\) and \(D_2\). The probability distributions in Eqs. (3)-(4) for coincidence (a) and bunching (b-c) events, respectively, are plotted in Figure 2 as functions of all the possible detected frequencies \(\omega\) and \(\omega'\) within the photonic spectra.
The quantity $\Delta t = t_2 - t_1$ is thus the unknown delay to be measured. For a given frequency $\omega$ and spatial channel $i$, the commuting creation operators $\hat{a}^\dagger_{i,\omega}$ and $\hat{b}^\dagger_{i,\omega}$ denote orthogonal modes in any given additional degree of freedom, such as polarisation or inner spatial modes: for $\eta = 1$, the two photons differ only in their injection times (no contribution from the mode operator $\hat{b}^\dagger_{i,\omega}$) while for $0 \leq \eta < 1$ a further distinction between the states of the two photons arises (e.g. they can be in different polarisations, where $\eta = 0$ is the limit case of orthogonal polarizations).

In our model, for each repetition of the experiment, other than detecting the photons in the output ports of the balanced beam-splitter, their frequencies are measured with a resolution $\delta\omega \ll \sigma$, (2)

where $\sigma$ is the width of the frequency spectrum $\bar{\xi}(\omega)^2$ appearing in the quantum state in Eq. (1). This condition restores the indistinguishability between the photons at the detectors [23]. Indeed, in Appendix B we show that, for imperfect detectors able to detect an incoming photon with a finite probability $\gamma < 1$, the probability that both photons are detected in the same channel with frequencies $\omega, \omega'$ is

$$P^B_\eta(\omega, \omega') = \gamma^2 \bar{\xi}(\omega)^2 \bar{\xi}(\omega')^2 \times (1 + \eta^2 \cos((\omega - \omega')\Delta t)),$$  

while the probability that the photons are detected in different channels with frequencies $\omega, \omega'$ is given by

$$P^C_\eta(\omega, \omega') = \gamma^2 \bar{\xi}(\omega)^2 \bar{\xi}(\omega')^2 \times (1 - \eta^2 \cos((\omega - \omega')\Delta t)),$$  

up to a factor $\delta\omega^2$ due to the resolution of the detectors.

As evident in the corresponding plots in Figure 2, such probabilities manifest quantum beats as a function of the difference between the detected frequencies with period proportional to the photonic time delay to be estimated. We point out that any distinguishability ($0 \leq \eta < 1$) between the photonic states in parameters other than emission times and frequencies affect the visibility of the measured probabilities and therefore the estimation precision, as it will be clear in the analysis of the Fisher information.

### Figure 2: Plots of the probability distributions $P^B_\eta(\omega, \omega')$ (blue solid line) for bunching events and $P^C_\eta(\omega, \omega')$ (green solid line) for coincidences (Eqs. (3) and (4), respectively), integrated in $\omega + \omega'$, as functions of the difference $\omega - \omega'$ of the measured frequencies of two Gaussian wave-packets (bandwidth $\sigma = 2.5\Delta t^{-1}$), with degree of indistinguishability $\eta = 0.75$ in any photonic parameters other than the emission times. The frequency of the oscillations (quantum beating) is proportional to the time delay $\Delta t$ between the two photons, which can be in this way inferred. The partial distinguishability ($\eta < 1$) between the two photons reduces the visibility of the quantum beats with respect to the case of completely indistinguishable ($\eta = 1$) photons (plots with finely dashed lines). The sum of the coincidence and bunching probabilities yields the distribution of the difference in frequencies of the two photon at the input (dashed red line).

### 3 Bounds on the precision

To analyse the optimal precision achievable when estimating the time delay $\Delta t$ between two interfering...
photons when frequency-resolving detectors are employed in the two-photon interference setup in Figure 1, we evaluate the smallest possible variance of any unbiased estimator $\tilde{\Delta t}$, given by the Cramér–Rao bound \[13,14\]
\[
\text{Var}[\tilde{\Delta t}] \geq \frac{1}{NF(\Delta t)}
\]
when $N$ repetitions of the measurement are performed. Here, $F(\Delta t)$ is the Fisher Information \[13,14\]
\[
F(\Delta t) = \mathbb{E}\left[\left(\frac{d}{d\Delta t} \log p(X|\Delta t)\right)^2\right],
\]
associated with the probability distribution $p(X|\Delta t)$ of the possible outcomes $X$ of the experiment (i.e. the channels where the photons are detected and their frequencies), where $\mathbb{E}[Z]$ denotes the expected value of the variable $Z$. Moreover, the maximum likelihood estimator \[13,14\] can be employed to asymptotically saturate the bound in Eq. (5), in the limit of large $N$ \[13,14\].

Before determining the expression of the Fisher information in (6) for frequency-resolved measurements, it is useful to assess the maximum precision achievable with the probe state given in Eq. (1), assuming that there is an optimal measurement scheme which can be chosen. Such benchmark is given by the Quantum Cramér–Rao bound \[30,31\], which is obtained from the Cramér–Rao bound in Eq. (5) by maximising the Fisher Information (6) over all possible measurement schemes, namely

\[
\text{Var}[\tilde{\Delta t}] \geq \frac{1}{NF(\Delta t)} \geq \frac{1}{NH(\Delta t)},
\]

where $H(\Delta t)$, called Quantum Fisher Information, is such maximum, and it can be evaluated by \[30,31\]
\[
H(\Delta t) = 4 \left(\langle \partial_{\Delta t} \psi | \partial_{\Delta t} \psi \rangle + (\partial_{\Delta t} \psi | \psi \rangle)^2\right),
\]
where $\partial_{\Delta t}$ is a shorthand for the derivative with respect to $\Delta t$. We show in appendix A that this expression for the probe state $|\psi\rangle$ in Eq. (1) is independent of the value of $\Delta t$ to estimate:

\[
H(\Delta t) \equiv H = 2\sigma^2,
\]

where $\sigma^2$ is the squared bandwidth of each photon frequency probability distribution, i.e. the variance of $\xi(\omega)^2$. Compared to the Quantum Fisher information in the case of entangled photons generated with spontaneous parametric down-conversion with spectral bandwidth $\sigma$, the one in Eq. (9) is halved \[10,15\]. This means that, by only relying on independent photons, it is possible to achieve an ultimate precision which differs only by a constant factor $1/\sqrt{2}$ from the one achievable with entangled sources.

We notice how the indistinguishability factor $\eta$ in any other photonic parameters which we are not estimating does not affect the quantum Fisher information for the estimation of the time delay. This result is to be expected. In fact, in principle, it is possible to apply an unitary rotation between the modes $\hat{a}_{1,\omega 1}$ and $\hat{b}_{1,\omega 1}$ such that $(\eta \hat{a}_{1,\omega 1} + \sqrt{1-\eta^2} \hat{b}_{1,\omega 1}) \rightarrow \hat{a}_{1,\omega 1}$ (in the case of distinguishability originated by different polarization, this would be equivalent to rotate the polarization of one of the two photons), restoring the indistinguishability between them. However, in this work, we will use the expression of the Quantum Fisher information in Eq. (9) as a comparative benchmark to discuss the efficiency of the estimation through frequency resolving measurements in the worst case scenario where any further distinguishability ($\eta < 1$) between the photons cannot be ‘erased’ in the measurement.

4 Fisher information for frequency-resolved measurements

We now determine the expression of the Fisher information for frequency-resolved measurements in the setup in Figure 1. This includes the contribution form all the possible frequency-resolved events of photon bunching and coincidences occurring with the probabilities in Eqs. (3) and (4), respectively. Therefore, as shown in Appendix C, the Fisher informa-
tion (6) for such scheme becomes

\[
F_\eta(\Delta t) = \int_{R^2} d\omega d\omega' \frac{1}{P^{B}_\eta(\omega, \omega')} \left( \frac{d}{d\Delta t} P^{B}_\eta(\omega, \omega') \right)^2 
+ \int_{R^2} d\omega d\omega' \frac{1}{P^{C}_\eta(\omega, \omega')} \left( \frac{d}{d\Delta t} P^{C}_\eta(\omega, \omega') \right)^2 
= \eta^4 \gamma^2 \int_{R^2} d\omega d\omega' \xi(\omega)^2 \xi(\omega')^2 \times (\omega - \omega')^2 \zeta_\eta((\omega - \omega')\Delta t),
\]

(10)

where

\[
\zeta_\eta(x) = \frac{\sin^2 x}{1 - \eta \cos^2 x}
\]

(11)
is for $\eta \neq 1$ a periodic function of period $\pi$ oscillating between 0, for $x = k\pi$, and 1, for $x = (1/2 + k)\pi$, $k \in \mathbb{Z}$, whilst for $\eta = 1$ the function $\zeta_{\eta=1}(x) = 1$ identically and $F_{\eta=1}(\Delta t) \equiv F_{\eta=1}$ becomes independent of $\Delta t$:

\[
F_{\eta=1} = \gamma^2 \int_{R^2} d\omega d\omega' \xi(\omega)^2 \xi(\omega')^2(\omega - \omega')^2 
= 2\gamma^2 \sigma^2 \equiv \gamma^2 H,
\]

(12)

where we introduced the variance $\sigma^2$ of the frequency distribution $\xi(\omega)^2$. From Eq. (10) we see that the events with equal observed frequencies $\omega = \omega'$ do not contribute in general to the overall precision due to the presence of the term $(\omega - \omega')^2$ in the integral. Furthermore, the detector efficiency only affects the Fisher information and therefore the estimation precision through a constant factor $\gamma^2$.

A typical experimental example is the case of Gaussian wave packets for which the Fisher information in Eq. (10) reduces to (see Appendix C for derivation)

\[
F_{\eta}^G(\Delta t) = \eta^4 \gamma^2 I_\eta(\Delta t),
\]

(13)

with the integral

\[
I_\eta(\Delta t) = \sqrt{\frac{1}{4\pi \sigma^2}} \int_{-\infty}^{\infty} d\delta \ e^{-\frac{\delta^2}{4\pi \sigma^2}} \delta^2 \zeta_\eta(\delta \Delta t),
\]

(14)
in $\delta \equiv \omega - \omega'$, dependent on the periodic function $\zeta_\eta(\delta \Delta t)$ in Eq. (11). Such expression manifests a clear metrological advantage with respect to the Fisher information associated with non-resolved (NR) mea-
measurements (see Appendix C and Ref. 9)

\[ F_{\eta}(\Delta t) = \frac{\eta^4}{\exp[2\Delta t^2\sigma^2] - \eta^4 \Delta t^2 \sigma^2} \] ...

(15)

as evident from the corresponding plots in Figure 3.

In the next two sections we will describe such a quantum metrological advantage in the two cases of completely indistinguishable \((\eta = 1)\) and partially distinguishable \((\eta < 1)\) photons at the two detectors.

### 4.1 Indistinguishable photons at the detectors

We analyse first the Fisher information \(F_{\eta=1}\) in Eq. (12) in the case of indistinguishable photons at the detectors \((\eta = 1)\). We notice that, for lossless detectors, i.e. for \(\gamma = 1\), the Fisher information equals the Quantum Fisher information \(H\) found in Eq. (9), meaning that this estimation scheme is optimal, and thus for a large number of iterations of the measurement it saturates the Quantum Cramér-Rao bound through the maximum-likelihood estimator [13, 14].

This implies that frequency resolved measurements after the two photons interfere at a beam splitter are optimal measurements in the estimation of the photonic delay \(\Delta t\).

Furthermore, \(F_{\eta=1}\), as well as \(H\), is independent of the structure of the photonic wave-packets, including their injection times, and it is proportional to the spectral variance \(\sigma^2 = 1/\tau^2\), where \(\tau\) is the coherence time. Therefore, it is possible to increase the sensitivity quadratically by increasing the photonic bandwidth \(\sigma\) for any value of the time delay \(\Delta t\). Such remarkable metrological advantage of the frequency-resolving approach with respect to the standard two-photon interferometry based on correlation measurements insensitive to the photonic inner modes [19], is also evident in the solid-line plots in Figure 3. Indeed, in the latter, the time delay must remain smaller or comparable with the photons coherence time, so that the two wave-packets overlap in the time domain in order to observe their interference. Here, instead, the wave-packet overlap can be arbitrarily small without affecting the precision, as long as the interference is observed in the frequency domain, and, in principle, there is no limit to the achievable precision by increasing the photonic bandwidth.

In summary, employing frequency-resolving detectors gives us access to the information on the time delay that statistically would be ‘averaged out’ otherwise, and such information increases with the number of possible detectable frequencies at the increasing of the bandwidth. To retrieve such information, quantified by the Fisher information in Eq. (15), it is necessary to erase any distinguishability at the detectors between the photons in their emission times through frequency-resolved measurement. In fact, the detectors at each single event cannot distinguish each photonic injection time, although the correlation measurement allows to ‘statistically’ infer their difference \(\Delta t\). We emphasize that it is important that the two photons have a similar (in our case identical) frequency distribution \(\bar{\xi}(\omega)^2\): measuring the frequencies of two photons with no overlap in their frequency distributions would in fact automatically reveal which photon is measured, hence preventing the quantum interference to take place. It is also important to notice that the action of the beam-splitter is in this setup essential: no information can be distilled about the delay through a spectral analysis of the two separate photons before they impinge on the beam-splitter. Indeed, the sensitivity to the time delay of this scheme arises entirely from the frequency-resolved quantum interference between the two photons.

### 4.2 Partially distinguishable photons at the detectors

The quantum advantage of frequency-resolved measurements in the estimation of the photonic time delays is also retained for photons partially distinguishable at the detectors in any degrees of freedom other than frequency and time \((\eta < 1)\), as evident from Figure 3 in the example of Gaussian wave-packets. Indeed, in the latter, the time delay must remain smaller or comparable with the photons coherence time, so that the two wave-packets overlap in the time domain in order to observe their interference. Here, instead, the wave-packet overlap can be arbitrarily small without affecting the precision, as long as the interference is observed in the frequency domain, and, in principle, for non-resolving measurements.
We also notice in the left plot in Figure 3 that, for particular values of $\Delta t$, the Fisher information $F_\eta^G(\Delta t)$ presents higher peaks with respect to $F_\eta^\text{NR}(\Delta t)$ for any value of $\eta < 1$. Therefore, if prior knowledge about $\Delta t$ is available, the interferometer can be tuned to operate optimally within the neighbourhood of the maximum of $F_\eta^G(\Delta t)$ for given values of $\eta$ and $\sigma$. Furthermore, in the right plot in Figure 3 it is evident that, for any value $\eta < 1$, the Fisher information $F_\eta^G(\Delta t)$ increases substantially at the increasing of the frequency variance $\sigma^2$ with respect to $F_\eta^\text{NR}(\Delta t)$ which eventually starts to fade away. In the equivalent regimes of large delays $\Delta t$ for fixed spectral bandwidth $\sigma$, or large variances $\sigma^2$ for fixed $\Delta t$, where $F_\eta^G(\Delta t)$ tends to $\Delta t$-independent values proportional to $\sigma^2$, as opposed to the vanishing $F_\eta^\text{NR}(\Delta t)$. In fact, both these regimes correspond to the interesting case of large time delays $\Delta t$ with respect to the coherence time $\tau = 1/\sigma$. Indeed, it is always possible to reduce the estimation problem to this case even for small unknown delays by adding a known large delay to the one we wish to estimate, or, alternatively, by reducing the photonic coherence time, depending on the particular experimental scenario.

We now analyse in more detail the Fisher information $F_\eta(\Delta t)$ in Eq. (10) in the aforementioned scenario $\Delta t \gg \tau$. We first notice that the function $\zeta_\eta(\omega - \omega')\Delta t)$ in Eq. (11) oscillates in $\omega - \omega'$, with period $\pi/\Delta t$. If the frequency spectrum $\bar{\xi}(\omega)^2$ of the two photons is regular enough (as for Gaussian wave-packets) so that it does not present intrinsic fast fluctuations, we can assume that the rest of the integrand in the Fisher information (10) is essentially constant within the periodicity interval $\pi/\Delta t$ much smaller than the photonic bandwidth $\sigma = 1/\tau$ in the regime $\Delta t \gg \tau$. It is then possible to substitute $\zeta_\eta(\omega - \omega')\Delta t)$ with its average over its period in Eq. (10), which reads

$$F_\eta(\Delta t \gg \tau) \approx 2(1 - \sqrt{1 - \eta^2})\gamma^2 \sigma^2 = (1 - \sqrt{1 - \eta^2})F_{\eta=1}, \quad (17)$$

which, for any value of $\eta < 1$, is finite, independent of $\Delta t$, proportional to the spectral variance $\sigma^2$ and independent of any other parameter in the wave-packet distribution, as per the Fisher information $F_{\eta=1}$ in Eq. (10) for completely indistinguishable photons at the detectors. Indeed, the effect of $\eta$ in this regime is only a constant factor that is vanishing only when the two photons are completely distinguishable at the detectors, i.e. for $\eta = 0$. As discussed, this is the case, for example, of Gaussian wave-packets, as depicted in Figure 3.

5 Conclusions

We have demonstrated a remarkable quantum metrological advantage in the estimation of the time delay between two independent (non-entangled) photons with same frequency distribution in a two-photon interference setup with frequency-resolving detectors. We have shown that the observation of both the coincidence and bunching events, together with the photons frequencies, is an optimal estimation strategy – in the sense that it saturates the ultimate precision given by the Quantum Cramer-Rao Bound – when the two photons are completely indistinguishable beside their time delay. Furthermore, such ultimate precision is only reduced by a factor $1/\sqrt{2}$ comparing with the potential maximum precision (Quantum Fisher Information) which may be achievable through entangled photons, and it has the advantage of avoiding any associated detrimental effect due to decoherence. Additionally, imperfect detectors only affects such a precision by a constant factor.

For pairs of photons with some degree of distinguishability in the measurement arising from inner parameters other than the injection times (e.g. polarization), this frequency-resolved technique still outperforms measurements non resolving the frequencies, for any degree of photonic distinguishability and for any value of the bandwidth and of the time delay to be estimated. This enhancement is especially
significant in the regime of time delays much larger than the coherence time of the two photons, condition which can be always attainable introducing, if needed, an additional known time delay between the photons, or by increasing the photonic bandwidth. The degree of distinguishability of the photons at the detectors only affects the precision through a multiplicative factor, in standard experimental scenarios where, for example, Gaussian photonic wave packets are employed.

These are remarkable results since, without the need of entanglement, our scheme outclasses the traditional techniques of two-photon interferometry not resolving the photonic spectra, which are completely insensitive to time delays much larger than the photonic coherence time. The combined action of the spectral analysis and the counting of bunching and coincidence events unveils information about temporal properties of the two-photon system that would otherwise be hidden with measurements which do not resolve the frequencies. In particular, such information emerges from the quantum interference arising from the impossibility to distinguish the photons, in their interferometric paths and injection times, through detecting their frequencies.

We emphasize that, since time and frequency are conjugate parameters, it is possible to obtain the same results, inverting the role of time and frequency in this estimation scheme. In other words it is possible to enhance the precision of two-photon interference protocols for the estimation of the frequency shift between two independent photons through time-resolving measurements. These results are therefore, in principle, applicable to general estimation schemes based on any pair of conjugate parameters, such as position and momentum or angular position and orbital angular momentum. Furthermore, an extension of such results to more than two photons could lead in future works to the demonstration of a quantum metrological advantage of inner-mode multi-photon interference in more general linear optical networks with a larger number $N$ of non-entangled input photons [23].

**Acknowledgement**

We thank Paolo Facchi and Frank A. Narducci for the helpful discussions. This work was supported by the Office of Naval Research Global (N62909-18-1-2153).
Appendix

A Quantum Fisher information

To evaluate the Quantum Fisher information through the expression given in Eq. (9), we first need to rewrite the probe state $|\psi\rangle$ in Eq. (1) in terms of the delay $\Delta t$ between the two photons. In the following treatment, we will suppose that all the moments of the frequency distribution of the two photons are known, as well as the factor $\eta$, while the time delay $\Delta t = t_1 - t_2$ and the sum of the emission times $t_{\text{tot}} = t_1 + t_2$ are unknown, as it is customary in experimental scenarios. Since in this setup two unknown parameter are present, the formal approach to this estimation problem requires the employment of the $2 \times 2$ Quantum Fisher information matrix (QFIM) $\mathcal{H}$, whose elements are given by

$$
\mathcal{H}_{ij} = 4 \text{Re}[\langle \partial_i \psi | \partial_j \psi \rangle - \langle \psi | \partial_i \psi \rangle^* \langle \psi | \partial_j \psi \rangle], \quad i, j = 1, 2,
$$

where we denoted with $\partial_1 \equiv \partial_-, \quad \text{and} \quad \partial_2 \equiv \partial_+$ the derivatives with respect to $\Delta t$ and $t_{\text{tot}}$ respectively, while the Quantum Cramér-Rao bound is given in matrix form

$$
\text{Cov}([\Delta t, t_{\text{tot}}]) \geq \mathcal{H}^{-1}.
$$

We will show that the QFIM is diagonal, so that the Cramér-Rao bound associated with the time delay can be written only in terms of $\mathcal{H}_{11} \equiv H(\Delta t)$ with $H(\Delta t)$ shown in Eq. (9).

First, we rewrite the state $|\psi\rangle$ from Eq. (1)

$$
|\psi\rangle = \int_{\mathbb{R}^2} d\omega_1 d\omega_2 \bar{\xi}(\omega_1)\bar{\xi}(\omega_2)e^{-i\omega_1 t_1-i\omega_2 t_2} \left(\eta \hat{a}^\dagger_{1,\omega_1} + \sqrt{1 - \eta^2} \hat{b}^\dagger_{1,\omega_1}\right) \hat{a}^\dagger_{2,\omega_2} |\text{vac}\rangle,
$$

$$
= \int_{\mathbb{R}^2} d\omega_1 d\omega_2 \bar{\xi}(\omega_1)\bar{\xi}(\omega_2)e^{-i t_{\text{tot}}(\omega_1+\omega_2)/2-i\Delta t(\omega_1-\omega_2)/2} \left(\eta \hat{a}^\dagger_{1,\omega_1} + \sqrt{1 - \eta^2} \hat{b}^\dagger_{1,\omega_1}\right) \hat{a}^\dagger_{2,\omega_2} |\text{vac}\rangle.
$$

It is straightforward to evaluate the derivatives $|\partial_1 \psi\rangle \equiv |\partial_- \psi\rangle$ and $|\partial_2 \psi\rangle \equiv |\partial_+ \psi\rangle$ as

$$
|\partial_\pm \psi\rangle = -\frac{i}{2} \int_{\mathbb{R}^2} d\omega_1 d\omega_2 (\omega_1 \pm \omega_2)\bar{\xi}(\omega_1)\bar{\xi}(\omega_2)e^{-i t_{\text{tot}}(\omega_1+\omega_2)/2-i\Delta t(\omega_1-\omega_2)/2} \left(\eta \hat{a}^\dagger_{1,\omega_1} + \sqrt{1 - \eta^2} \hat{b}^\dagger_{1,\omega_1}\right) \hat{a}^\dagger_{2,\omega_2} |\text{vac}\rangle
$$

To evaluate the scalar products in Eq. (18), it is useful to first notice that

$$
\langle \text{vac} | (\eta \hat{a}^\dagger_{1,\omega_1} + \sqrt{1 - \eta^2} \hat{b}^\dagger_{1,\omega_1}) \hat{a}_{2,\omega_4} | \text{vac} \rangle = \delta(\omega_1 - \omega_3) \delta(\omega_2 - \omega_4),
$$

where $\delta(\cdot)$ is the Dirac delta distribution. Now we can easily evaluate the scalar products

$$
\langle \partial_- \psi | \partial_- \psi \rangle = \frac{1}{4} \int_{\mathbb{R}^2} d\omega_1 d\omega_2 (\omega_1 - \omega_2)^2 \bar{\xi}(\omega_1)^2 \bar{\xi}(\omega_2)^2 = \frac{1}{2} \sigma^2
$$

$$
\langle \partial_+ \psi | \partial_+ \psi \rangle = \frac{1}{4} \int_{\mathbb{R}^2} d\omega_1 d\omega_2 (\omega_1 + \omega_2)^2 \bar{\xi}(\omega_1)^2 \bar{\xi}(\omega_2)^2 = \frac{1}{2} (\sigma^2 + 2\omega_0^2)
$$

$$
\langle \partial_- \psi | \partial_+ \psi \rangle = \frac{1}{4} \int_{\mathbb{R}^2} d\omega_1 d\omega_2 (\omega_1^2 - \omega_2^2) \bar{\xi}(\omega_1) \bar{\xi}(\omega_2)^2 = 0
$$
\[
\langle \psi | \partial_\psi \rangle = -\frac{i}{2} \int_{\mathbb{R}^2} d\omega_1 d\omega_2 \ (\omega_1 - \omega_2) \vec{\xi}(\omega_1)^2 \vec{\xi}(\omega_2)^2 = 0
\]
\[
\langle \psi | \partial_+ \psi \rangle = -\frac{i}{2} \int_{\mathbb{R}^2} d\omega_1 d\omega_2 \ (\omega_1 + \omega_2) \vec{\xi}(\omega_1)^2 \vec{\xi}(\omega_2)^2 = -i\omega_0,
\]
(23)
where \(\omega_0\) and \(\sigma^2\) are the central frequency and variance of the frequency distribution \(\vec{\xi}(\omega)^2\), and finally, by substituting Eq. (23) in Eq. (18), we obtain the QFIM
\[
\mathcal{H} = \begin{pmatrix}
2\sigma^2 & 0 \\
0 & 2\sigma^2
\end{pmatrix},
\]
(24)
which is diagonal, and the element \(H(\Delta t)\) associated with the delay \(\Delta t\) is the value in Eq. (9) appearing in the Quantum Cramér-Rao bound shown in Eq. (7).

B Bunching and coincidence probabilities

Here, we evaluate the probabilities in Eqs. (3)-(4) that the two photons are observed with frequencies \(\omega_1, \omega_2\) in the same and in different output channels, respectively.

The balanced beam splitter, on which the two photons in the state \(\psi'\) in Eq. (1) impinge, can be described with a \(2 \times 2\) unitary matrix \(U_{BS}\) of transition amplitudes
\[
U_{BS} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix},
\]
(25)
and it acts on the injected probe through the map \(\hat{U}_{BS} \hat{a}_1^\dagger \hat{U}_{BS}^\dagger = \sum_{j=1,2} (U_{BS})_{ij} \hat{a}_j^\dagger\), and equivalently for \(\hat{b}_i^\dagger\).

With reference to Eq. (1), the two-photon state \(|\psi'\rangle\) at the output of the beam splitter thus reads
\[
|\psi'\rangle = \hat{U}_{BS} |\psi\rangle
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} d\omega_1 d\omega_2 \ xi(\omega_1) x_2(\omega_2) \left( \eta (\hat{a}_{1,\omega_1}^\dagger - \hat{a}_{2,\omega_2}^\dagger) + \sqrt{1 - \eta^2} (\hat{b}_{1,\omega_1}^\dagger - \hat{b}_{2,\omega_2}^\dagger) \right) (\hat{a}_{1,\omega_2}^\dagger + \hat{a}_{2,\omega_2}^\dagger) |\text{vac}\rangle
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} d\omega_1 d\omega_2 \ xi(\omega_1) x_2(\omega_2) \left( \eta (\hat{a}_{1,\omega_1}^\dagger \hat{a}_{1,\omega_2}^\dagger - \hat{a}_{2,\omega_2}^\dagger \hat{a}_{2,\omega_2}^\dagger) + \hat{a}_{1,\omega_1}^\dagger \hat{a}_{2,\omega_2}^\dagger - \hat{a}_{2,\omega_1}^\dagger \hat{a}_{1,\omega_2}^\dagger + \sqrt{1 - \eta^2} (\hat{b}_{1,\omega_1}^\dagger \hat{b}_{1,\omega_2}^\dagger - \hat{b}_{2,\omega_1}^\dagger \hat{b}_{2,\omega_2}^\dagger + \hat{b}_{1,\omega_1}^\dagger \hat{b}_{2,\omega_2}^\dagger - \hat{b}_{2,\omega_1}^\dagger \hat{b}_{1,\omega_2}^\dagger) \right)
\]
(26)
In order to evaluate the probabilities that two photons with frequencies \(\omega\) and \(\omega'\) are observed in each configuration (bunching or coincidence), it is convenient to further manipulate (26), to more easily take into account the indistinguishability of identical photons, so that
\[
|\psi'\rangle = \frac{\eta}{2} \int_{\omega_1 < \omega_2} d\omega_1 d\omega_2 \ (\xi(\omega_1) \xi_2(\omega_2) + \xi(\omega_2) \xi_2(\omega_1)) \left( \hat{a}_{1,\omega_1}^\dagger \hat{a}_{1,\omega_2}^\dagger \hat{a}_{2,\omega_2} + \hat{a}_{2,\omega_1}^\dagger \hat{a}_{1,\omega_2}^\dagger \right) |\text{vac}\rangle
\]
\[
+ \frac{\eta}{2} \int_{\omega_1 < \omega_2} d\omega_1 d\omega_2 \ (\xi(\omega_1) \xi_2(\omega_2) - \xi(\omega_2) \xi_2(\omega_1)) \left( \hat{a}_{1,\omega_1}^\dagger \hat{a}_{2,\omega_2}^\dagger - \hat{a}_{2,\omega_1}^\dagger \hat{a}_{1,\omega_2}^\dagger \right) |\text{vac}\rangle
\]
\[
+ \sqrt{1 - \eta^2} \int_{\mathbb{R}^2} d\omega_1 d\omega_2 \ \xi(\omega_1) \xi_2(\omega_2) \left( \hat{b}_{1,\omega_1}^\dagger \hat{a}_{1,\omega_2}^\dagger - \hat{b}_{2,\omega_1}^\dagger \hat{a}_{2,\omega_2}^\dagger + \hat{b}_{1,\omega_1}^\dagger \hat{a}_{2,\omega_2}^\dagger - \hat{b}_{2,\omega_1}^\dagger \hat{a}_{1,\omega_2}^\dagger \right) |\text{vac}\rangle,
\]
(27)
can be written as a sum of three contributions: the first associated with photons ending up in the same channel and in the mode described by the operator $\hat{a}^\dagger_{i,\omega}$, the second with the photons ending up in different channels in the mode described by the operator $\hat{a}_{i,\omega}$, and the third is the contribution given when the distinguishability in the inner properties other than time and frequency (e.g., polarization) is observed at the detectors. Hence, the probability $p_B(\omega, \omega')$ to observe the two photons together in any of the output channels with frequencies $\omega, \omega'$ within the resolution $\delta \omega$ satisfying Eq. (2) is given by

$$p_B(\omega, \omega') = \sum_{i=1,2} \left| \langle \text{vac} | \hat{a}_{i,\omega} \hat{a}_{i,\omega'} | \psi' \rangle \right|^2 + \left| \langle \text{vac} | \hat{a}_{i,\omega} \hat{b}_{i,\omega'} | \psi' \rangle \right|^2 + \left| \langle \text{vac} | \hat{b}_{i,\omega} \hat{a}_{i,\omega'} | \psi' \rangle \right|^2$$

$$= \frac{\eta^2}{2} \left| \xi_1(\omega) \xi_2(\omega') + \xi_1(\omega') \xi_2(\omega) \right|^2 +$$

$$+ \frac{1 - \eta^2}{2} \left( \left| \xi_1(\omega) \xi_2(\omega') \right|^2 + \left| \xi_1(\omega') \xi_2(\omega) \right|^2 \right)$$

$$= \frac{1}{2} \left( \left| \xi_1(\omega) \xi_2(\omega') \right|^2 + \left| \xi_1(\omega') \xi_2(\omega) \right|^2 + 2\eta^2 \text{Re}[\xi_1(\omega) \xi_2(\omega') \xi_1^*(\omega') \xi_2^*(\omega)] \right), \quad (28)$$

where we omitted the multiplicative factor $\delta \omega^2$ for simplicity, while the probability $p_C(\omega, \omega')$ for the photons ending up in different channels reads

$$p_C(\omega, \omega') = \left| \langle \text{vac} | \hat{a}_{1,\omega} \hat{a}_{2,\omega'} | \psi' \rangle \right|^2 + \left| \langle \text{vac} | \hat{a}_{1,\omega} \hat{b}_{2,\omega'} | \psi' \rangle \right|^2 + \left| \langle \text{vac} | \hat{b}_{1,\omega} \hat{a}_{2,\omega'} | \psi' \rangle \right|^2$$

$$+ \left| \langle \text{vac} | \hat{a}_{1,\omega} \hat{a}_{2,\omega} | \psi' \rangle \right|^2 + \left| \langle \text{vac} | \hat{a}_{1,\omega} \hat{b}_{2,\omega} | \psi' \rangle \right|^2 + \left| \langle \text{vac} | \hat{b}_{1,\omega} \hat{a}_{2,\omega} | \psi' \rangle \right|^2$$

$$= \frac{\eta^2}{2} \left| \xi_1(\omega) \xi_2(\omega') - \xi_1(\omega') \xi_2(\omega) \right|^2 +$$

$$+ \frac{1 - \eta^2}{2} \left( \left| \xi_1(\omega) \xi_2(\omega') \right|^2 + \left| \xi_1(\omega') \xi_2(\omega) \right|^2 \right)$$

$$= \frac{1}{2} \left( \left| \xi_1(\omega) \xi_2(\omega') \right|^2 + \left| \xi_1(\omega') \xi_2(\omega) \right|^2 - 2\eta^2 \text{Re}[\xi_1(\omega) \xi_2(\omega') \xi_1^*(\omega') \xi_2^*(\omega)] \right), \quad (29)$$

where as well we omitted the factor $\delta \omega^2$. It is straightforward to check that the previous probabilities are correctly normalized, since all the probabilities sum to unity

$$\int_{\omega \leq \omega'} d\omega d\omega' \left( p_B(\omega, \omega') + p_C(\omega, \omega') \right) = 1 \quad (30)$$

Let us now suppose that the frequency distributions are of the form $\xi_i(\omega) = \bar{\xi}(\omega) e^{-i\omega t_i}$, with $\bar{\xi}(\omega)$ real and independent on $t_i$, for $i = 1, 2$, so that

$$p_B(\omega, \omega') = \bar{\xi}(\omega)^2 \bar{\xi}(\omega')^2 \left( 1 + \eta^2 \cos((\omega - \omega') \Delta t) \right),$$

$$p_C(\omega, \omega') = \bar{\xi}(\omega)^2 \bar{\xi}(\omega')^2 \left( 1 - \eta^2 \cos((\omega - \omega') \Delta t) \right).$$

(31)

In the case of imperfect detectors detecting a single incoming photon with probability $\gamma^2$, the probabilities associated with the events of two-photon bunching and two-photon coincidences at the detectors are respectively

$$P_B^\gamma(\omega, \omega') = \gamma^2 p_B(\omega, \omega') = \gamma^2 \bar{\xi}(\omega)^2 \bar{\xi}(\omega')^2 \left( 1 + \eta^2 \cos((\omega - \omega') \Delta t) \right)$$

$$P_C^\gamma(\omega, \omega') = \gamma^2 p_C(\omega, \omega') = \gamma^2 \bar{\xi}(\omega)^2 \bar{\xi}(\omega')^2 \left( 1 - \eta^2 \cos((\omega - \omega') \Delta t) \right).$$
\[ P^C_\eta(\omega, \omega') = \gamma^2 p_C(\omega, \omega') = \gamma^2 \bar{\xi}(\omega)^2 \bar{\xi}(\omega')^2 (1 - \eta^2 \cos((\omega - \omega') \Delta t)) , \]  

(32)

from which we find the probabilities [3] and [4], respectively. On the other hand, as one may expect, the probabilities

\[ P_0 = \gamma^2 \]

\[ P_1(\omega) = \gamma(1 - \gamma) \left( \int_\mathbb{R} d\omega' \left( p^B_\eta(\omega, \omega') + p^C_\eta(\omega, \omega') \right) \right) = 2\gamma(1 - \gamma) |\bar{\xi}(\omega)|^2 \]

(33)

do not contribute to the Fisher information.

### C Fisher information

In order to evaluate the Fisher Information in Eq. (6) associated with the estimation of \( \Delta t \), we evaluate from the relevant expressions of the probabilities of bunching and coincidence in Eq. (32)

\[
\frac{d}{d\Delta t} P^B_\eta(\omega, \omega') = -\frac{d}{d\Delta t} P^C_\eta(\omega, \omega') = -\gamma^2 \bar{\xi}(\omega)^2 \bar{\xi}(\omega')^2 \eta^2 (\omega - \omega') \sin((\omega - \omega') \Delta t) \]

(34)

and thus Eq. (6) becomes

\[
F_\eta(\Delta t) = \int_{\omega < \omega'} d\omega d\omega' \left( \frac{1}{P^B_\eta(\omega, \omega')} \left( \frac{d}{d\Delta t} P^B_\eta(\omega, \omega') \right)^2 + \frac{1}{P^C_\eta(\omega, \omega')} \left( \frac{d}{d\Delta t} P^C_\eta(\omega, \omega') \right)^2 \right) 
\]

\[
= \eta^4 \gamma^2 \int_{\mathbb{R}^2} d\omega d\omega' \bar{\xi}(\omega)^2 \bar{\xi}(\omega')^2 (\omega - \omega')^2 \frac{\sin^2((\omega - \omega') \Delta t)}{1 - \eta^4 \cos^2((\omega - \omega') \Delta t)} , \]

(35)

which corresponds to Eq. [10].

Now we obtain from Eq. (35) the specialised expressions for the Fisher information shown in Eq. (13) associated with Gaussian photons, with spectra

\[
\bar{\xi}(\omega)^2 = \sqrt{\frac{1}{2\pi \sigma^2}} \exp \left[ -\frac{(\omega - \Omega_0)^2}{2\sigma^2} \right] , \]

(36)

where \( \Omega_0 \) is the central frequency and \( \sigma^2 \) the variance. By substituting Eq. (36) in Eq. (35), we obtain

\[
F_\eta(\Delta t) = \eta^4 \gamma^2 \frac{1}{2\pi \sigma^2} \int d\omega d\omega' e^{\frac{1}{4\pi}((\omega - \Omega_0)^2+(\omega' - \Omega_0)^2)} (\omega - \omega')^2 \zeta((\omega - \omega') \Delta t) . \]

(37)

With a change of variables \( \delta = \omega - \omega' \), \( s = \omega + \omega' - 2\Omega_0 \), we get

\[
F_\eta(\Delta t) = \eta^4 \gamma^2 \frac{1}{4\pi \sigma^2} \int ds d\delta \exp\left(\frac{1}{4\pi}(s^2+\delta^2)\right) \delta^2 \zeta(\delta \Delta t) 
\]

\[
= \eta^4 \gamma^2 \frac{1}{4\pi \sigma^2} \int d\delta \exp\left(-\frac{1}{4\pi \sigma^2} \delta^2\right) \delta^2 \zeta(\delta \Delta t) , \]

(38)

corresponding to the expression for the Fisher information in Eq. [13].
If instead the frequencies of the two photons are not observed, the probabilities of bunching and coincidence events can be found by summing the probabilities in Eqs. (3) and (4) over all the possible frequencies $\omega, \omega'$ obtaining

$$P^B_\eta = \int_{\omega < \omega'} d\omega d\omega' P^B_\eta(\omega, \omega'),$$  \hspace{1cm} (39a)$$

$$P^C_\eta = \int_{\omega < \omega'} d\omega d\omega' P^C_\eta(\omega, \omega').$$  \hspace{1cm} (39b)$$

Notice how the effect of non-resolving the frequencies anticipates the integration over all the frequencies of the photons, which is here performed on the probabilities instead of while evaluating the expectation value of the Fisher information in Eq. (10). If we assume that the photons are Gaussian, with spectra given in Eq. (36), the probabilities in (39) become

$$P^B_\eta = \frac{\gamma}{2} \left( 1 + \eta^2 e^{-\Delta t^2 \sigma^2} \right),$$  \hspace{1cm} (40a)$$

$$P^C_\eta = \frac{\gamma}{2} \left( 1 - \eta^2 e^{-\Delta t^2 \sigma^2} \right).$$  \hspace{1cm} (40b)$$

It is straightforward to see, by applying the definition in Eq. (6), that the expression of the Fisher information associated with non-resolving measurements is

$$F^C(\Delta t) = 4\gamma^2 \frac{\eta^4}{\exp[2\Delta t^2 \sigma^2]} - \eta^4 \Delta t^2 \sigma^4 \equiv 2F_{\eta=1} \frac{\eta^4}{\exp[2\Delta t^2 \sigma^2]} - \eta^4 \Delta t^2 \sigma^2,$$  \hspace{1cm} (41)$$

as in Eq. (15) with $F_{\eta=1}$ given in Eq. (12).

References

[1] C. K. Hong, Z. Y. Ou, and L. Mandel, “Measurement of subpicosecond time intervals between two photons by interference,” *Phys. Rev. Lett.*, vol. 59, pp. 2044–2046, Nov 1987.

[2] Y. H. Shih and C. O. Alley, “New type of einstein-podolsky-rosen-bohm experiment using pairs of light quanta produced by optical parametric down conversion,” *Phys. Rev. Lett.*, vol. 61, pp. 2921–2924, Dec 1988.

[3] I. Abram, R. K. Raj, J. L. Oudar, and G. Dolique, “Direct observation of the second-order coherence of parametrically generated light,” *Phys. Rev. Lett.*, vol. 57, pp. 2516–2519, Nov 1986.

[4] S. Prasad, M. O. Scully, and W. Martienssen, “A quantum description of the beam splitter,” *Optics Communications*, vol. 62, no. 3, pp. 139–145, 1987.

[5] Z. Ou, C. Hong, and L. Mandel, “Relation between input and output states for a beam splitter,” *Optics Communications*, vol. 63, no. 2, pp. 118–122, 1987.

[6] H. Fearn and R. Loudon, “Quantum theory of the lossless beam splitter,” *Optics Communications*, vol. 64, no. 6, pp. 485–490, 1987.
[7] J. G. Rarity and P. R. Tapster, “Fourth-order interference in parametric downconversion,” *J. Opt. Soc. Am. B*, vol. 6, pp. 1221–1226, Jun 1989.

[8] F. Bouchard, A. Sit, Y. Zhang, R. Fickler, F. M. Miatto, Y. Yao, F. Sciarrino, and E. Karimi, “Two-photon interference: the hong-ou-mandel effect,” *Reports on Progress in Physics*, vol. 84, p. 012402, Jan 2021.

[9] A. Lyons, G. C. Knee, E. Bolduc, T. Roger, J. Leach, E. M. Gauger, and D. Faccio, “Attosecond-resolution hong-ou-mandel interferometry,” *Science Advances*, vol. 4, no. 5, 2018.

[10] Y. Chen, M. Fink, F. Steinlechner, J. P. Torres, and R. Ursin, “Hong-ou-mandel interferometry on a biphoton beat note,” *npj Quantum Information*, vol. 5, p. 43, May 2019.

[11] N. Fabre and S. Felicetti, “Parameter estimation of time and frequency shifts with generalized hong-ou-mandel interferometry,” *Phys. Rev. A*, vol. 104, p. 022208, Aug 2021.

[12] N. Harnchaiwat, F. Zhu, N. Westerberg, E. Gauger, and J. Leach, “Tracking the polarisation state of light via hong-ou-mandel interferometry,” *Opt. Express*, vol. 28, pp. 2210–2220, Jan 2020.

[13] H. Cramér, *Mathematical methods of statistics*, vol. 9. Princeton university press, 1999.

[14] V. K. Rohatgi and A. M. E. Saleh, *An introduction to probability and statistics*. John Wiley & Sons, 2000.

[15] H. Scott, D. Branford, N. Westerberg, J. Leach, and E. M. Gauger, “Beyond coincidence in hong-ou-mandel interferometry,” *Phys. Rev. A*, vol. 102, p. 033714, Sep 2020.

[16] T. Legero, T. Wilk, A. Kuhn, and G. Rempe, “Time-resolved two-photon quantum interference,” *Applied Physics B*, vol. 77, pp. 797–802, Dec 2003.

[17] T. Legero, T. Wilk, M. Hennrich, G. Rempe, and A. Kuhn, “Quantum beat of two single photons,” *Phys. Rev. Lett.*, vol. 93, p. 070503, Aug 2004.

[18] V. Tamma and S. Laibacher, “Multiboson correlation interferometry with arbitrary single-photon pure states,” *Phys. Rev. Lett.*, vol. 114, p. 243601, Jun 2015.

[19] X.-J. Wang, B. Jing, P.-F. Sun, C.-W. Yang, Y. Yu, V. Tamma, X.-H. Bao, and J.-W. Pan, “Experimental time-resolved interference with multiple photons of different colors,” *Phys. Rev. Lett.*, vol. 121, p. 080501, Aug 2018.

[20] V. Prakash, A. Sierant, and M. W. Mitchell, “Autoheterodyne characterization of narrow-band photon pairs,” *Phys. Rev. Lett.*, vol. 127, p. 043601, Jul 2021.

[21] R.-B. Jin, T. Gerrits, M. Fujiwara, R. Wakabayashi, T. Yamashita, S. Miki, H. Terai, R. Shimizu, M. Takeoka, and M. Sasaki, “Spectrally resolved hong-ou-mandel interference between independent photon sources,” *Opt. Express*, vol. 23, pp. 28836–28848, Nov 2015.

[22] P. Yepiz-Graciano, A. M. A. Martinez, D. Lopez-Mago, H. Cruz-Ramirez, and A. B. U’Ren, “Spectrally resolved hong-ou-mandel interferometry for quantum-optical coherence tomography,” *Photon. Res.*, vol. 8, pp. 1023–1034, Jun 2020.
[23] S. Laibacher and V. Tamma, “Symmetries and entanglement features of inner-mode-resolved correlations of interfering nonidentical photons,” Phys. Rev. A, vol. 98, p. 053829, Nov 2018.

[24] V. V. Orre, E. A. Goldschmidt, A. Deshpande, A. V. Gorshkov, V. Tamma, M. Hafezi, and S. Mittal, “Interference of temporally distinguishable photons using frequency-resolved detection,” Phys. Rev. Lett., vol. 123, p. 123603, Sep 2019.

[25] S. Laibacher and V. Tamma, “From the physics to the computational complexity of multiboson correlation interference,” Phys. Rev. Lett., vol. 115, p. 243605, Dec 2015.

[26] V. Tamma and S. Laibacher, “Multi-boson correlation sampling,” Quantum Information Processing, vol. 15, pp. 1241–1262, Mar 2016.

[27] V. Tamma and S. Laibacher, “Scattershot multiboson correlation sampling with random photonic inner-mode multiplexing,” 2021.

[28] M. Tanida, R. Okamoto, and S. Takeuchi, “Highly indistinguishable heralded single-photon sources using parametric down conversion,” Opt. Express, vol. 20, pp. 15275–15285, Jul 2012.

[29] M. D. Eisaman, J. Fan, A. Migdall, and S. V. Polyakov, “Invited review article: Single-photon sources and detectors,” Review of Scientific Instruments, vol. 82, no. 7, p. 071101, 2011.

[30] C. W. Helstrom, “Quantum detection and estimation theory,” Journal of Statistical Physics, vol. 1, pp. 231–252, Jun 1969.

[31] A. Holevo, Probabilistic and Statistical Aspects of Quantum Theory. Publications of the Scuola Normale Superiore, Scuola Normale Superiore, 2011.