A NOTE ON SCALE FUNCTIONS FOR LÉVY PROCESSES WITH NEGATIVE PHASE-TYPE JUMPS

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Abstract. We provide a novel expression of the scale function for a Lévy processes with negative phase-type jumps. It is in terms of a certain transition rate matrix which is explicit up to a single positive number. A monotone iterative scheme for the calculation of the latter is presented and analyzed. Our numerical examples suggest that this algorithm allows to employ phase-type distributions with a hundred of phases, which is problematic when using the known formula for the scale function in terms of roots. Extensions to other distributions, such as infinite-dimensional phase-type and those with rational transform, can be anticipated.

1. Introduction

The theory of fluctuations of Lévy processes with one-sided jumps is abundant in various identities and expressions, see the review papers Avram et al. (2020); Kuznetsov et al. (2012b) and references therein for a long list of formulas and applications. These concern first passage times, extremes, reflection and refraction, limiting distributions and distributions at exponential times, Poissonian observation, and a great variety of other derived models and objectives. Most of these expressions are in terms of a so-called scale function \( W_q : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) identified by its transform, where \( q \geq 0 \) is the killing rate of the underlying Lévy process. We note that Google Scholar finds about 2070 articles containing ‘scale function’ in the context of Lévy processes as of today.

Calculation of scale functions by inversion is a feasible but nontrivial task (Kuznetsov et al., 2012b), and it may be prohibitive in a common scenario when a family of scale functions is needed for a large number of killing rates \( q \). Even though a generous number of (semi-) explicit examples of scale functions can be engineered (Hubalek and Kyprianou, 2011; Kuznetsov et al., 2012b), arguably the most important explicit examples are given by processes with jumps of rational transform and so-called meromorphic processes (Kuznetsov et al., 2012a). Our focus is on the first class of processes.

We consider a spectrally-negative Lévy process \((X_t)_{t \geq 0}\) with finite jump activity, which is often called perturbed Cramér-Lundberg risk process in actuarial science literature. That is,

\[
X_t = dt + \sigma B_t - \sum_{i=1}^{N_t} C_i, \quad t \geq 0,
\]

where \( B_t \) is a standard Brownian motion, \( N_t \) is a Poisson process of rate \( \lambda > 0 \), \( C_i \) is an iid sequence of positive random variables, and all components are independent. It is assumed that \( \sigma \geq 0, d \in \mathbb{R} \) with \( d > 0 \) when \( \sigma = 0 \) to avoid monotone paths. Certain series expansions of scale functions for such processes have been recently obtained in Landriault and Willmot (2020). We assume, however, that \( C_i \) has a phase-type (PH) distribution, which can be seen as the life-time of a transient continuous time Markov chain with finitely many states,
often called phases. These form a dense class of distributions on \( \mathbb{R}_+ \) (a proper subclass of distributions with rational transform) and lead to a profusion of tractable models in applied probability, see (Asmussen, 2003, Ch. 3) or Bladt and Nielsen (2017).

The scale function for a spectrally-negative Lévy process with PH jumps can be expressed in terms of the zeros of a certain rational function, see (Egami and Yamazaki, 2014, Prop. 2.1) and Kuznetsov et al. (2012b). The number of these (possibly complex) roots is normally close to the number of phases. Hence finding the roots may become problematic when the number of phases is substantial, which is a common scenario in applications. In this regard we note that Asmussen et al. (2019) use 20 to 100 phases in their life insurance application. Furthermore, numerical stability is of serious concern when using this expression.

An alternative approach is to use fluid embedding of PH jumps to arrive at a second order fluid flow model (Markov modulated Brownian motion) and to treat the given problem in that context, see Asmussen et al. (2004); Mordecki (2002). Indeed, there is a well-developed theory for such models (Asmussen, 1995; Ivanovs, 2010), as well as for much more general one-sided Markov additive processes (Ivanovs and Palmowski, 2012; Ivanovs, 2014). This, however, requires dealing with matrix calculus, partitioning of phases, and various further complications, not to mention a certain necessary experience. Moreover, such approach largely ignores the extensive literature on one-sided Lévy processes and related models.

In this note we use some basic insights from the analysis of fluid flow models to establish an alternative expression of the scale function, see Theorem 1 and Theorem 2 corresponding to \( \sigma > 0 \) and \( \sigma = 0 \). These simple expressions are in terms of a certain transition rate matrix \( G \), which is explicit up to a single positive number. In fact, it is fully explicit in the special case of \( q = 0 \) and \( \mathbb{E}X_1 > 0 \). Furthermore, we provide an iterative scheme yielding a monotone sequence of approximations of the unknown number, see Proposition 4 and Proposition 5, and demonstrate numerically its fast convergence. This allows to employ PH distributions with hundreds of phases, while still having access to a plethora of results and expressions in terms of scale functions. It is noted that the eigenvalues of the transition rate matrix are the above mentioned roots (not counting the non-negative one) and hence our iterative scheme can also be used to efficiently compute the roots. Extensions to distributions with rational transforms and to (heavy-tailed) infinite-dimensional PH distributions (Bladt et al., 2015) can be anticipated, but they require further investigation.

2. Preliminaries

2.1. Spectrally-negative Lévy processes. Let \( (X_t)_{t \geq 0} \) be a general Lévy process with no positive jumps, but not a process with a.s. decreasing paths. The Laplace exponent of \( X \) is denoted by \( \psi(\theta) = \log \mathbb{E}e^{\theta X_1}, \theta \geq 0 \), and the first passage times (above \( x \) and below \( -x \)) are given by

\[
\tau^+_x = \inf\{t \geq 0 : X_t > x\}, \quad \tau^-_x = \inf\{t \geq 0 : X_t < -x\}, \quad x \geq 0.
\]

For \( q \geq 0 \) let \( \Phi_q \geq 0 \) be the right-most non-negative root of \( \psi(\theta) = q \), which is known to satisfy the basic identity

\[
\mathbb{E}e^{-q\tau^+_x} = \mathbb{P}(\tau^+_x < \epsilon_q) = e^{-\Phi_q x}, \quad x \geq 0.
\]

We write \( \epsilon_q \) for an independent exponentially distributed random variable of rate \( q \geq 0 \) which is \( \infty \) for \( \epsilon = 0 \). This \( \epsilon_q \) can be seen as the killing time of \( X \), and thus \( q \geq 0 \) is just another parameter - the killing rate.
For every \( q \geq 0 \) there is a so-called scale function \( W_q : [0, \infty) \mapsto [0, \infty) \), which is a continuous, non-decreasing function identified by its transform

\[
\int_0^\infty e^{-\theta x} W_q(x) \, dx = 1/\left(\psi(\theta) - q\right), \quad \theta > \Phi(q),
\]

see (Bertoin, 1996, Thm. VII.8). The scale function is strictly positive for \( x > 0 \) and it solves the basic two-sided exit problem:

\[
\mathbb{E}(e^{-\theta \tau^+_x}; \tau^+_x < \tau^-_y) = W_q(y)/W_q(x + y), \quad x, y \geq 0, x + y > 0.
\]

We refer to Avram et al. (2020); Kuznetsov et al. (2012b) for a long list of useful formulas based on \( W_q \).

Throughout the rest of this paper we assume that

\[
(A1) \quad q > 0 \quad \text{or} \quad \psi'(0) \neq 0,
\]

which merely excludes the case of a non-killed process \( X \) with zero expectation. This case normally can be treated by taking the limit as \( q \downarrow 0 \).

### 2.2. A convenient representation of the scale function.

Define the first hitting time of \( \tau_{\{x\}} = \inf\{t > 0 : X_t = x\} \),

which a.s. coincides with \( \tau^+_x \) for \( x > 0 \). The following representation of the scale function is useful in various context, but is not widely known:

\[
W_q(x) = \frac{1}{\psi'(\Phi_q)} \left( e^{\Phi_q x} - \mathbb{P}(\tau_{\{x\}} < e_q) \right), \quad x \geq 0,
\]

see (Ivanovs and Palmowski, 2012, Eq. (12)) and also (Pistorius, 2005, Thm. 1), (Kuznetsov et al., 2012b, Eq. (95)) for some related formulas. Due to assumption (A1) the denominator \( \psi'(\Phi_q) \neq 0 \). This formula will serve as a basis for deriving our result.

For completeness let us mention that (3) can be phrased in terms of the local time \( L_t \) at zero, see (Bertoin, 1996, Ch. V) for the definition. Firstly, \( \mathbb{E}L_{e_q} = 1/\psi'(\Phi_q) \) is the expected local time of the killed process and, secondly, by the additive property of \( L_t \) we have

\[
e^{-\theta q x} W_q(x) = \mathbb{E}L_{e_q} - \mathbb{P}(\tau^+_x < e_q)\mathbb{P}(\tau_{\{x\}} < e_q)\mathbb{E}L_{e_q} = \mathbb{E}L_{\tau^+_x \wedge e_q}, \quad x \geq 0.
\]

In words, this is the expected local time at 0 of the killed process collected up to \( \tau^+_x \).

### 2.3. Phase-type jumps.

Consider a continuous time Markov chain on \( n \) transient states with an initial distribution \( \alpha \) and \( n \times n \) transition rate matrix \( T \). Here \( \alpha \) is a row vector with \( n \) non-negative elements summing up to 1, and \( t = -T1 \geq 0 \) is a column vector with \( n \) elements giving the killing rates; we write \( 1 \) and \( 0 \) for the column vectors of ones and zeros, respectively. The distribution of the life-time of this Markov chain is denoted by \( \text{PH}(\alpha, T) \).

Without loss of generality we assume that \( \alpha, T \) are such that the Markov chain has a positive probability to visit any state.

The density of \( \text{PH}(\alpha, T) \) has a matrix exponential form \( f(x) = \alpha e^{tx}, x > 0 \) and the transform is a rational function

\[
\int_0^\infty e^{-\theta x} f(x) \, dx = \alpha(\theta I - T)^{-1}t = \frac{P(\theta)}{Q(\theta)}, \quad \theta \geq 0.
\]

Here \( P \) and \( Q \) are polynomials of degree \( p - 1 \) and \( p \), respectively, with no common zeros in \( \mathbb{C} \). It must be that \( 1 \leq p \leq n \), and in the case \( p = n \) we say that the representation \( (\alpha, T) \) is minimal. It is noted that one can always provide a minimal matrix exponential representation,
whereas a minimal PH representation need not exist (Bladt and Nielsen, 2017, Sec. 4.2). Let us also point out that the zeros of $Q$ are the eigenvalues of $T$ and the latter have negative real parts.

In the following we assume that our Lévy process $X$ has the form given in (1) with $C_i \sim \text{PH}(\alpha, T)$. Equivalently,

$$\psi_q(\theta) := \psi(\theta) - q = \frac{1}{2} \sigma^2 \theta^2 + d \theta + \lambda(\theta I - T)^{-1} t - 1 - q = \frac{\hat{P}(\theta)}{Q(\theta)} , \quad \theta \geq 0. \quad (A2)$$

Note that $\hat{P}$ and $Q$ have no common zeros and the degree of $\hat{P}$ is $p + 2 - 1_{(\sigma > 0)}$. Finally, $\mathbb{E}C_i = \alpha(-T)^{-1}1$ and so $\psi'(0) = \mathbb{E}X_1 = d + \lambda \alpha T^{-1}1$.

2.4. The scale function in terms of roots. Recall that $\psi_q(\theta)$ has $p + 2 - 1_{(\sigma > 0)}$ zeros in $\mathbb{C}$ counting multiplicities for any $q \geq 0$. Let $Z_q$ be the set of these zeros excluding the single zero at $\Phi_q$:

$$Z_q = \{ z \in \mathbb{C} : \psi_q(z) = 0, z \neq \Phi_q \} . \quad (4)$$

For any $z \in Z_q$ it must be that $\Re(z) \leq 0$, which follows from the standard properties of $\psi$. Assuming that the zeros in $Z_q$ are simple, there is the identity

$$W_q(x) = \frac{e^{\Phi_q x}}{\psi'(\Phi_q)} + \sum_{z \in Z_q} \frac{e^{x z}}{\psi'(z)} , \quad x \geq 0 , \quad (5)$$

and the number of elements of $Z_q$ is $p + 1_{(\sigma > 0)}$. This result and also its less neat version in the case of multiple zeros can be found in (Egami and Yamazaki, 2014, Prop. 2.1) and in (Kuznetsov et al., 2012b, Sec. 5.4). In the case $q = 0$, $\psi'(0) < 0$ the set $Z_q$ contains 0, which is somewhat unclear in the above cited works.

Recall that for a minimal PH representation $(\alpha, T)$ we have $p = n$, and so we need to find $n + 1_{(\sigma > 0)}$ zeros of the rational function in (A2) in the left half of the complex plane. As discussed before, in some applications this number may exceed 100, and thus general root finding methods may fail to find all the zeros or may require certain adaptations. One may also use efficient numerical procedures (Bini, 1996) for locating zeros of polynomials, but that requires to pick out the numerator or to multiply by $\det(\theta I - T)$. Importantly, such methods require careful use of multiprecision due to numerical instability.

3. The scale function in terms of a transition rate matrix

In view of (3) we aim to find a simple expression of the hitting probability $\mathbb{P}(\tau_{\geq x} < e_q)$ for $x \geq 0$. This task can be achieved by fluid embedding: Consider a continuous time Markov chain $J_t$ on $n + 1$ states (phases) with a transition rate matrix

$$\begin{pmatrix} -\lambda - q & \lambda \alpha \\ t & T \end{pmatrix} ,$$

and assume that the level process $Y_t$ starts at 0 and evolves as an independent linear Brownian motion $\sigma B_t - dt$ when $J_t = 1$ and as a linear drift $t$ with unit slope when $J_t \neq 1$. This can be compactly stated as

$$Y_0 = 0, \quad dY_t = 1_{\{J_t = 1\}}(\sigma dB_t - dt) + 1_{\{J_t \neq 1\}}dt ,$$

where the process $Y$ is killed upon termination of $J$. The bivariate process $(Y_t, J_t)$ is a special case of a well-studied Markov modulated Brownian motion (MMBM), see Asmussen (1995); Ivanovs (2010). Note that the process $Y$ with time intervals $J_t \neq 1$ deleted has the law of
Theorem 1. Assuming \( \mathbb{P}(\tau_{-x}) < e_q \) coincides with the probability of \((Y, J)\) hitting \((x, 1)\) given \(J_0 = 1\).

Letting \( c_x = \inf\{t \geq 0 : Y_t > x\} \) we see that \((J_{c_x})_{x \geq 0}\) is a Markov chain, which may or may not enter state 1 according to \(\sigma > 0\) and \(\sigma = 0\). We treat these two cases separately.

3.1. Brownian component is present. Assume that \(\sigma > 0\) and let \(G\) be \((n + 1) \times (n + 1)\) transition rate matrix of the first passage Markov chain \((J_{c_x})_{x \geq 0}\). Now

\[
\mathbb{P}(\tau_{-x}) < e_q = \sum_j \mathbb{P}(J_{c_x} = j | J_{c_x} = 1) \nu_j = e_1 e^{Gx} \nu,
\]

where \(e_1 = (1, 0, \ldots, 0)\) and \(\nu\) is an \((n + 1)\) column vector with \(\nu_j\) being the probability that given \(J_0 = j\) the process \(Y\) ever hits level 0, which must be in phase 1. Clearly, \(\nu_1 = 1\) and the other elements are given by

\[
\int_0^\infty e^{Tx} t \mathbb{P}(\tau_x^+ < e_q) dx = \int_0^\infty e^{Tx} e^{-\Phi_q x} dt = (\Phi_q I - T)^{-1} t.
\]

Furthermore, a linear Brownian motion hits \((0, \infty)\) immediately a.s. and we see that

\[
G = \begin{pmatrix} -a & b \\ t & T \end{pmatrix}, \quad \nu = \begin{pmatrix} 1 \\ (\Phi_q I - T)^{-1} t \end{pmatrix}
\]

for some number \(a > 0\) and \(n\)-vector \(b > 0\) such that \(b1 \leq a\). It is thus left to characterize \(a\) and \(b\). Before doing so we note that \(a = b1\) when \(q = 0, \Sigma X_1 \leq 0\) and \(a > b1\) otherwise, which corresponds to the first passage chain \(J_{c_x}\) being recurrent and transient, respectively. Note that \(\nu, a, b\) depend on the killing rate \(q \geq 0\).

Theorem 1. Assuming (A1) and (A2) with \(\sigma, \lambda > 0, d \in \mathbb{R}, q \geq 0\) we have

\[
W_q(x) = \frac{1}{\psi'(\Phi_q)} (e^{\Phi_q x} - e_1 e^{Gx} \nu), \quad x \geq 0,
\]

where the transition rate matrix \(G\) and a non-negative vector \(\nu\) are given in (6). For \(q = 0, \Sigma X_1 > 0\) we have \(a = \frac{2d}{\sigma^2} > 0\) and \(b = \frac{2a}{\sigma^2} \alpha(-T)^{-1} > 0\), and otherwise \(b \geq 0\) and \(a \geq b1\) are characterized by

\[
\frac{1}{2} \sigma^2 (a^2 + bt) - da = \lambda + q, \quad \frac{1}{2} \sigma^2 (bt - ab) + db = -\lambda \alpha.
\]

Moreover, the latter implies

\[
\frac{2d \vee 0}{\sigma^2} < a < \frac{d + \sqrt{d^2 + 2\sigma^2(\lambda + q)}}{\sigma^2}, \quad 0 < b < \frac{2\lambda}{\sigma^2} \alpha(-T)^{-1}.
\]

Proof. In view of (3) and the above considerations, it is only required to characterize \(a > 0\) and \(b\). Note that the transition rate matrix of \(J\) is irreducible, since we have excluded certain degenerate PH representations. From the basic MMBM theory, see e.g. Asmussen (1995), (D’Auria et al., 2010, Thm. 2), (Ivanovs, 2011, Cor. 4.15) (the latter allows for phase-dependent killing), we have the matrix equation

\[
\frac{1}{2} \sigma^2 \Delta_{(1,0,\ldots,0)} G^2 - \Delta_{(-d,1,\ldots,1)} G + \begin{pmatrix} -\lambda - q \\ t \end{pmatrix} \begin{pmatrix} \lambda \alpha \\ T \end{pmatrix} = O,
\]

where \(\Delta_d\) stands for a diagonal matrix with \(d\) on the diagonal. Observe that the first row readily provides the stated identities in (7), whereas the other rows yield the above observed form of \(G\). Furthermore, the above equation uniquely characterizes \(G\) given that \(G\) is a
recurrent transition rate matrix \((a = b1)\) when \(q = 0, E X_1 \leq 0\) and transient \((a > b1)\) otherwise.

The first equation in (7) can be replaced by
\[
\frac{1}{2} \sigma^2 (a - d) (a - b1) = q, 
\]
which follows from \(\frac{1}{2} \sigma^2 (bt + a b1) - db1 = \lambda\) obtained by right-multiplying the second equation by \(1\). For \(q = 0, E X_1 > 0\) we must have \(a > b1\) and so \(a = 2d/\sigma^2\) which readily yields the above stated \(b\). For \(q > 0\) we see from (9) that \(a > b1\), as required. Finally, assume \(q = 0, E X_1 = d - \lambda x (T)^{-1} 1 < 0\). If \(a = 2d/\sigma^2 \geq 0\) then from the second equation in (7) we find \(b1 > 2d/\sigma^2 = a\). This contradiction means that \(a = b1\) as required, and the characterization is now complete. In the latter case \(a < 2d/\sigma^2\) is impossible which can be shown directly with some effort, or by appealing to continuity of \(a\) in \(q \downarrow 0\). Now the bounds in (8) easily follow, where we note that \(\left(\frac{1}{2} \sigma^2 T - (\frac{1}{2} \sigma^2 a - d) I\right)\) is a transition rate matrix with an additional killing rate \(\frac{1}{2} \sigma^2 a - d > 0\), and the proof is complete.

It may be interesting to note that in the explicit case \(q = 0, E X_1 > 0\) the rate \(a\) does not depend on \(\lambda\) and the PH distribution of jumps. In this case \((a, b)\) solve (7), but we do not claim that they are thus characterized without additionally assuming that \(a > b1\). In fact, one can provide simple examples with an additional solution satisfying \(a = b1\). In the other cases the vector \(b\) can be easily computed if the value of \(a\) is provided.

3.2. No Brownian component. Here we assume that \(\sigma = 0\) and \(d > 0\). Now the first passage chain \((Jx)_{x \geq 0}\) lives on the states \(2, \ldots, n + 1\), and we let \(G\) be its \(n \times n\) transition rate matrix. Moreover, let \(\pi\) be a row vector with \(n\) elements denoting the distribution of \(Jx\) given \(J_0 = 1\).

**Theorem 2.** Assuming (A1) and (A2) with \(\sigma = 0, d, \lambda > 0, q \geq 0\) we have
\[
W_q(x) = \frac{1}{\psi(\Phi_q)}\left(e^{\Phi_q x} - \pi e^{Gx} \nu\right), \quad x \geq 0, \quad G = T + t \pi, \quad \nu = (\Phi_q I - T)^{-1} t.
\]
For \(q = 0, E X_1 > 0\) we have \(\pi = \frac{\lambda}{\alpha} (T)^{-1} > 0\), and otherwise \(\pi \geq 0\) with \(\pi 1 \leq 1\) is characterized by
\[
\pi \left((\lambda + q - d \pi t) I - d T\right) = \lambda \alpha,
\]
which implies \(\pi t \in (0, \lambda/d)\) and \(0 < \pi < \frac{\lambda}{\alpha} (T)^{-1}\).

**Proof.** Note that \(P(\pi 1 < e_q) = \pi e^{Gx} \nu\) with \(\nu\) of the stated form. Furthermore, the form of \(G\) follows from the very simple structure of the process \((Y, J)\). More precisely, the rates are given by \(T\) plus the rates of transition via the state 1. In view of (3) it is left to identify \(\pi\). By a direct analysis or from the MMBM theory (Ivanovs, 2011, Cor. 4.15) we have
\[
d \pi G - (\lambda + q) \pi + \lambda \alpha = 0,
\]
which identifies \(G = T + t \pi\) and thus \(\pi\) together with the transience/recurrence requirement, which is \(\pi 1 < 1\) and \(\pi 1 = 1\), respectively.

Again we may right-multiply (10) by \(1\) to derive a further useful equation:
\[
(\lambda - d \pi t)(1 - \pi 1) = \pi 1 q.
\]
For \(q = 0, E X_1 > 0\) we must have \(\pi 1 < 1\) and so \(\pi t = \lambda/d\) and the expression for \(\pi\) follows. For \(q > 0\) we have \(\pi 1 < 1\) as required. If \(q = 0, \pi t = \lambda/d\) then \(\pi = \frac{\lambda}{\alpha} (T)^{-1}\) showing
that $d(1 - \pi 1) = \mathbb{E}X_1 \geq 0$. Thus for $\mathbb{E}X_1 < 0$ we must have $\pi 1 = 1$, as required. The characterization result is now complete. \hfill \Box

Similarly to the Brownian case, for $q = 0, \mathbb{E}X_1 > 0$ there may exist another solution $\pi$ of (10) with $\pi 1 = 1$. In the other cases the vector $\pi$ can be easily computed if the value of the number $\pi t$ is provided.

3.3. Relation to the formula in terms of roots. Consider the expressions of $W_q$ in Theorem 1 and in Theorem 2. Given that the corresponding transition rate matrix $G$ is diagonalizable these expressions can be rewritten as $\sum_i c_i e^{\zeta_i x}$, where $\zeta_i$ runs through $\Phi_q$ and the eigenvalues of $G$. This gives an expression analogous to (5), but even more can be said.

**Lemma 3.** Assume (A1) and minimality of PH representation $(\alpha, T)$. Then the roots in (4) are simple iff the $n+1_{\{\sigma > 0\}}$ eigenvalues of $G$ are distinct, in which case the two sets coincide.

**Proof.** Take the transform of the expression in Theorem 1 and in Theorem 2 and recall that $\pi t$ is well-defined, satisfies the bounds in (8), and converges monotonically to (9). Apply analytic continuation and use the fact that $\psi(\theta) - q$ has $n+1 + 1_{\{\sigma > 0\}}$ zeros. \hfill \Box

Finally, we remark that all the eigenvalues of $G$ have negative real parts apart from a single eigenvalue at 0 in the case $q = 0, \mathbb{E}X_1 \leq 0$. Yet deeper connections can be established using the theory of generalized Jordan chains in D’Auria et al. (2010).

4. Monotone iterative schemes

In this section we propose an efficient algorithm to compute the unknowns $a, b$ characterized by (7), as well as an algorithm for $\pi$ characterized by (10). As mentioned above, a known $a$ yields $b$ and a known $\pi t$ yields $\pi$, and so there is a single unknown number underlying the new representations of the scale function. In the MMBM theory there are various iterative schemes for calculation of $G$ and the associated initial distributions, see Asmussen (1995). Our present problem, however, has a very simple structure suggesting some particular schemes, which we now discuss and analyze.

4.1. Brownian component is present. Reconsider Theorem 1 in the non-explicit case and recall that $a > 2d/\sigma^2$ which then implies $\sigma^2 bt < 2\lambda$. For any $a_0 > 2d/\sigma^2$ we define $(b_n, a_n)_{n \geq 1}$ recursively from (7) as follows:

$$b_n = \frac{2\lambda}{\sigma^2} \alpha((a_{n-1} - 2d/\sigma^2) I - T)^{-1}, \quad a_n = \frac{d + \sqrt{d^2 + \sigma^2(2\lambda + 2q - \sigma^2 b_n t)}}{\sigma^2}.$$  

We now show that it is well-defined and converges monotonically to $(a, b)$.

**Proposition 4.** Assume the conditions of Theorem 1 and that $q > 0$ or $\mathbb{E}X_1 < 0$. For any $a_0 > 2d/\sigma^2$ the sequence $(a_n, b_n)_{n \geq 1}$ is well-defined, satisfies the bounds in (8), and exhibits monotone convergence to $(a, b)$:

$$a_0 \uparrow a, \quad b_n \downarrow b \quad \text{for } a_0 \leq a, \quad a_n \downarrow a, \quad b_n \uparrow b \quad \text{for } a_0 \geq a.$$

**Proof.** Assume that $a_{n-1} > 2d/\sigma^2$ which is true for $n = 1$. Then $T - (a_{n-1} - 2d/\sigma^2)I$ is a transient transition rate matrix implying that $b_n$ is well defined and has strictly positive elements. Moreover, we observe that $\sigma^2 b_n t < 2\lambda \alpha(-T)^{-1} t = 2\lambda$ and hence $a_n > 2d/\sigma^2 \vee 0$. Finally, interpreting $a_{n-1} - 2d/\sigma^2$ as a killing rate we see that $b_m < b_n$ iff $a_{m-1} > a_{n-1}$, in which case $a_m > a_n$. Thus $a_0 > 0$ is monotone and so is $b_n > 0$ but in the opposite way. Hence they must have a finite limit $a^* \geq 0, b^* \geq 0$ which solves (7).
For $q > 0$ we also find from (9) that $a^* \geq b^* 1$ and hence characterization result in Theorem 1 can be applied. In the case $q = 0, \mathbb{E}X_1 < 0$ we only need to show that $a^* \neq 2d/\sigma^2$, and so we may assume that $d \geq 0$. Recall that $a_{n-1} > 2d/\sigma^2$ and hence it is sufficient to show that $a_{n-1} = 2d/\sigma^2 + \epsilon$ implies $a_n > a_{n-1}$ for all small enough $\epsilon > 0$. That is, we need to show that
\[
\sqrt{d^2 + \epsilon^2 (2\lambda - 2\lambda \alpha (\epsilon I - T)^{-1} t)} > \epsilon^2 + d,
\]
which is equivalent to
\[
2\lambda(1 - \alpha (\epsilon I - T)^{-1} t)/\epsilon > \epsilon^2 + 2d.
\]
The left hand side converges to the derivative $2\lambda \alpha (-T)^{-1} 1$ as $\epsilon \downarrow 0$ which exceeds $2d$ by assumption of the negative drift, completing the proof. \(\Box\)

It is important to note that the starting value $a_0 = 2d/\sigma^2$ can be used for $q > 0$, whereas for $q = 0$ it will result in a constant sequence $a_n = 2d/\sigma^2$ irrespective of the drift and the true solution. Observe that an alternative iterative scheme can be obtained by expressing $a$ from (9). This, however, does not yield a monotone sequence and it also exhibit a much slower convergence in our numerical examples below.

The number of iterations depends on the starting position $a_0$. For large $q > 0$ we expect $b$ to be close to 0 and so a good starting position is given by the upper bound in (8) corresponding to $b = 0$. If the scale matrix is computed for a number of different $q$ then we may start from the largest value and use the last solution as initial $a_0$ for the following $q$. In general, as a rule of thumb one may use the midpoint of the interval in (8).

4.2. No Brownian component. Reconsider Theorem 2. In this case an obvious recursion is
\[
\pi_n = \lambda \alpha \left((\lambda + q - d\pi_{n-1} t) I - dT\right)^{-1}.
\]
Here we only need to specify $\pi_0$ up to the value of $\pi_0 t$.

Proposition 5. Under the assumptions of Theorem 2 for any $\pi_0 t < \lambda/d$ the sequence $(\pi_n)_{n\geq 1}$ is well-defined, satisfies $0 < \pi_n < \frac{\lambda}{2} \alpha (-T)^{-1}$, and converges monotonically to $\pi$.

Proof. The matrix under inverse is a transition rate matrix when $\pi_{n-1} t < \lambda/d$, and then it is easy to see that this condition is preserved in the sequence. Moreover, $\pi_{n+1} > \pi_n$ iff $\pi_n t > \pi_{n-1} t$. Hence the sequence $\pi_n$ is monotone and thus has a limit $\pi^*$ solving (10).

For $q > 0$ from (11) we find that $\pi^* 1 < 1$ and the characterization result completes the proof. For $q = 0, \mathbb{E}X_1 < 0$ we only need to show that $\pi^* t \neq \lambda/d$. Similar to the Brownian case it is sufficient to assume that $\pi_{n-1} t = \lambda/d - \epsilon$ and to show that $\pi_n t < \pi_{n-1} t$ for all small enough $\epsilon > 0$. That is we need to show
\[
\lambda \alpha (\epsilon I - T)^{-1} t < \lambda - \epsilon d,
\]
which readily follows from $\alpha (-T)^{-1} 1 > d$. \(\Box\)

As a rule of thumb one may use the starting value $\pi_0 t = \lambda/(2d)$.

5. Numerical illustrations

Numerical experiments in this section are performed using WOLFRAM MATHEMATICA 11 and the most straightforward implementation of required procedures. We choose $n = 50$ and a PH distribution of Coxian type which is popular in applications, see Asmussen et al. (2019). That is, the respective PH chain starts in phase 1 and it may only jump to the following phase or to terminate. The parameters are sampled randomly: inverse standard uniforms for the rates out of each phase and (0, 0.9) uniforms for the killing probabilities. On average this
construction results in expected value of about 1. Our particular sample has expectation 1.77 and its density is depicted in Figure 1. Furthermore, we choose \( \sigma = d = \lambda = 1 \) resulting in \( \mathbb{E}X_1 = -0.77 \) and consider \( q \in \{0, 0.1, 1\} \).

Let us recall the two methods of computing the scale function \( W_q(x) \):

(Old): Find the zeros of \( \theta \mapsto \psi(\theta) - q \) in \( \mathbb{C} \) and use formula (5).

(New): Form the transition rate matrix \( G \) and the vector \( \nu \) and combine them into \( W_q(x) \) as specified in Theorem 1. This requires computing \((a, b)\) which is done via the monotone iterative scheme in Proposition 4. We stop when \( |a_n - a_{n-1}| < 10^{-5} \) and return \((a_n, b_n)\) as an approximation of \((a, b)\). Convergence of this scheme is investigated below, and by default we start at the midpoint of the interval in (8).

Note that the second method requires the non-negative root \( \Phi_q \), which is trivial to compute thanks to convexity of \( \psi(\theta), \theta \geq 0 \). Furthermore, this root is needed by various fluctuation identities anyway.

![Figure 1](image1.png)

**Figure 1.** Left: PH density. Right: The complex plane with the detected zeros of \( \psi(\theta) - 1 \) in blue and the eigenvalues of \( G \) in red.

### 5.1. Comparison of the two methods.

Our first illustration concerns the correspondence between the zeros of \( \psi(\theta) - q \) and the eigenvalues of \( G \) in Theorem 1 for \( q = 1 \), see also Lemma 3. We call a generic `Solve` function and find 47 zeros, whereas the maximal number is 52. Additionally, we have tried various other `Mathematica` procedures including symbolic manipulation routines and `NRoots`, but could not make them work for \( n = 50 \). Figure 1 depicts these 47 zeros in blue over the eigenvalues in red, and we note that 2 eigenvalues are outside of the plot range, both being large negative numbers.

It seems reasonable to assume that our randomly generated PH is likely to be minimal (it is such for smaller \( n \)), and so various zeros are not detected. It is not possible to directly verify this claim by plugging the eigenvalues into \( \psi \), because the latter is often extremely sensitive at the negative values. The missing zeros may not influence \( W_q \) much due to their magnitudes and the magnitudes of the corresponding derivatives, and yet numerical stability is always a concern for this method. In this example the two methods produce almost identical scale functions, see Figure 2. Importantly, computing \( G \) and finding its eigenvalues is much faster (0.003 vs 13 seconds in this example) and unlike the current implementation of the root finding method our algorithm can easily handle \( n = 100 \) and beyond.

### 5.2. Convergence.

Here we illustrate convergence of the iterative scheme in Proposition 4 for \( q = 1 \) and \( q = 0 \). We find \( a = 3.1 \) and \( a = 2.33 \) in the two cases, respectively. For the initial \( a_0 \) we examine three choices as suggested by the interval in (8): (i) 2.001 which is almost at
the lower boundary, (ii) the upper boundary and (iii) the midpoint. As mentioned above, the algorithm stops when $|a_n - a_{n-1}| < 10^{-5}$. Recall that in the case $q = 0$ our algorithm must not be started at 2, since then it stays there. When starting at 2.001 our algorithm will take some time to escape that fixed point as can be seen in Figure 3 (left blue).

![Figure 2](image_url)

**Figure 2.** The scale function $W_1(x)$ (left) and the absolute error between such functions obtained by the two methods (right).

![Figure 3](image_url)

**Figure 3.** Convergence of $a_n$ for $q = 0$ (left) and for $q = 1$ (right) until the difference is below $10^{-5}$.

We have also tried the recursion based on (9), but that resulted in non-monotone sequences (alternating in the sign of increments) exhibiting much slower convergence.

5.3. **On the number of iterations.** Finally, we randomly generate 1000 PH distributions as specified above and compute the corresponding $a$ by starting at the midpoint (case (iii), green in Figure 3). The counts for the number of iterations for $q = 0, q = 0.1, q = 1$ are presented in Figure 4, where in the case $q = 0$ we have excluded the realizations with $E X_1 > 0$ (470 of such) corresponding to an explicit $a$. The execution times are 4.6 sec. for $q = 0$ (470 problems), 2.7 sec. for $q = 0.1$ and 1.7 sec. for $q = 1$ (1000 problems in both). It seems that the problem of finding $a, b$ is simpler for larger values of the killing rate $q \geq 0$. For $q = 2$ there are 5–6 iterations, and for $q = 1, n = 100$ the histogram is almost the same as for $q = 1, n = 50$.

In conclusion, the monotone iterative scheme in Proposition 4 performs very well for large $n$ in our setting of randomly generated PH distributions of Coxian type. Importantly, it is trivial to implement, and one can easily provide error guarantees by taking advantage of monotone convergence. Finally, the resulting exponential form of $W_q$ in Theorem 1 is convenient for various further calculations.
Figure 4. Counts for the number of iterations among 1000 replications. Left: $q = 0$ with 470 cases corresponding to $E X_1 > 0$ excluded. Right: $q = 0.1$ and $q = 1$.

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