Abstract

In this short note we will revisit the large $N$ solution of $\mathbb{C}P^N$ sigma model on a finite interval of length $L$. We will find a family of boundary conditions for which the large $N$ saddle point can be found analytically. For a certain choice of the boundary conditions the theory has only one phase for all values of $L$. Also, we will provide an example when there are two phases: for large $L$ there is a standard phase with an unbroken $U(1)$ gauge symmetry and for small $L$ there is Higgs phase with a broken gauge symmetry.

1 Introduction

Two dimensional $\mathbb{C}P^N$ sigma model in the large $N$ limit was first solved in [1] and [2]. The theory exhibits a plethora of non-trivial properties: asymptotic freedom, confinement and dynamical scale $\Lambda$ generation via the dimensional transmutation:

$$\Lambda^2 = \Lambda_{\text{av}}^2 \exp \left( - \frac{4\pi}{g^2} \right)$$  \hspace{1cm} (1)

where $g$ is the coupling constant.

Physically, 2D $\mathbb{C}P^N$ model naturally arises as a low-energy effective action of non-Abelian strings in QCD-like models, see [3] for a review. Therefore, a finite interval geometry corresponds to a string stretched between two branes or a monopole–anti-monopole pair. Such configuration was studied in [4].

Recently $\mathbb{C}P^N$ sigma model on a finite interval of length $L$ with Dirichlet boundary conditions(BC) was investigated in [5] and [6] using large $N$ expansion. In the earlier work [5] the large $N$ saddle point equations were solved only approximately and two distinct phases were found. In [6] saddle-point equations were solved numerically and it was argued that there is only one phase. In this paper we will find a set boundary conditions...
conditions for which the saddle point equations can be solved analytically. Strictly speaking, we will study $\mathbb{C}P^{2N+1}$ sigma model. We will consider two different boundary conditions:

- Mixed Dirichlet-Neumann (D-N) boundary conditions which will break global $SU(2N+1)$ to $SU(N) \times SU(N)$. We will show that the system has at least two phases: for $L > \pi/4\Lambda$ there is a standard “Coulomb” phase with an unbroken $U(1)$ gauge symmetry. This phase takes place for the $\mathbb{C}P^N$ model on usual $\mathbb{R}^2$. For $L < \pi/4\Lambda$ there is “Higgs” phase with broken $U(1)$. Global $SU(N) \times SU(N)$ stays unbroken in both phases.

- Dirichlet-Dirichlet and Neumann-Neumann (D-D and N-N) boundary conditions which will break $SU(2N+1)$ to $SU(N) \times SU(N+1)$. In this case, for all values of $L$ there is a standard phase with an unbroken $U(1)$ gauge symmetry. Higgs phase is prohibited in this case, because it will break global $SU(N) \times SU(N+1)$ to $SU(N) \times SU(N)$.

In case of simple Dirichlet boundary conditions studied in [3, 6], Higgs phase does not break any global symmetries, so we expect that the system will have two phases as was predicted in [5]. Let us note that the large $N$ $\mathbb{C}P^N$ model on a cylinder also possesses multiple phases [7].

2 Generalized saddle point equations

Let us study $\mathbb{C}P^{2N+1}$ model in the large $N$ limit. The field content consists of $2N + 1$ fields $n^i$, $i = 0, \ldots, 2N$, real vector field $A_\mu$ and real scalar $\lambda$. In the Euclidian space the Lagrangian reads as:

$$\mathcal{L} = (D_\mu n^i)^* (D^\mu n^i) + \lambda (n^* n^i - r)$$

(2)

where $D_\mu = \partial_\mu - iA_\mu$, $\mu = t, x$ and $r = 2N/g^2$. Time coordinate $t$ takes values from $-\infty$ to $+\infty$ and $x \in [0, L]$.

Non-dynamical Lagrangian multipliers $A_\mu$ and $\lambda$ forces $n^i$ to lie on $\mathbb{C}P^{2N+1}$ space: integration over $\lambda$ yields $\sum_i n^* n^i = r$ and $A_\mu$ is responsible for $U(1)$ invariance $n^i \sim e^{i\phi} n^i$.

We will proceed in a standard fashion: we will integrate out $2N$ fields $n^i$, $i = 1, \ldots, 2N$ fields and then find the large $N$ saddle point values of $\lambda$, $A_\mu$ and the remaining $n^0$ which we will denote by $\sigma = n^0$. After integrating out $2N$ $n^i$, fields we have:

$$S_{eff} = \text{tr} \log(-D_x^2 - D_t^2 + \lambda) + \int d^2x \left((D\sigma)^2 + \lambda(|\sigma|^2 - r)\right)$$

(3)

So far we do not have a factor of $2N$ in front of the determinant because we will impose different boundary conditions for these $2N$ fields.
We will study this model on a finite interval of length $L$ with various boundary conditions. Note that the translational symmetry in $x$ direction is explicitly broken. However, we still have the time translations so we will consider only time translation invariant saddle points. By the choice of gauge we can always set $A_t = 0$. This allows us to rewrite eq. (3) as:

$$S_{\text{eff}} = \sum_n E_n + \int d^2x \left( (D_x \sigma)^2 + \lambda(|\sigma|^2 - r) \right)$$

(4)

Note that we have already integrated out time frequencies, so we have energies $E_n$ instead of the usual log det. The sum over $n$ is the sum over the eigenvalues $E_n^2$ of the following equation:

$$(-D_x^2 + \lambda(x))\psi_n = E_n^2 \psi_n(x)$$

(5)

$\psi_n$ are required to be normalized.

Varying effective action (4) with respect to $\lambda$ we get the first saddle point equation:

$$\frac{1}{2} \sum_n \frac{|\psi_n(x)|^2}{E_n} + |\sigma(x)|^2 - r = 0$$

(6)

To obtain this equation we have used the standard quantum mechanical first order perturbation theory for (5).

The second saddle-point equation coincides with the $\sigma$ equation of motion:

$$D_x^2 \sigma - \lambda(x)\sigma = 0$$

(7)

Finally, we have to vary with respect to $A_x$:

$$i \frac{1}{2} \sum_n \frac{\psi_n(D_x \psi_n)^* - \psi_n^* D_x \psi_n}{E_n} = i\sigma(D_x \sigma)^* - i\sigma_n^* D_x \sigma_n$$

(8)

Below we will study the case $A_x = 0$ with real $\psi_n$ and $\sigma$ and so this equation will be trivially satisfied.

3  **D-N boundary conditions: two phases**

Now it is time to choose boundary conditions. Let us consider the following: For $N$ fields $n^i, i = 1, \ldots, N$ we will use Dirichlet-Neumann (D-N):

$$n^i(0) = 0, \ D_x n^i(L) = 0$$

(9)

And for $N$ fields $n^i, i = N + 1, \ldots, 2N$ we will use Neumann-Dirichlet (N-D):

$$D_x n^i(0) = 0, \ n^i(L) = 0$$

(10)
And for $\sigma$ we will impose Neumann-Neumann (N-N):

$$D_x \sigma(0) = D_x \sigma(L) = 0$$ (11)

This choice breaks global $SU(2N + 1)$ to $SU(N) \times SU(N)$.

Then in the D-N sector we have:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{\pi x (n - 1/2)}{L} \right), \quad E_n^2 = \left( \frac{\pi (n - 1/2)}{L} \right)^2 + \lambda, \quad n = 1, \ldots$$ (12)

In the N-D sector:

$$\tilde{\psi}_n(x) = \sqrt{\frac{2}{L}} \cos \left( \frac{\pi x (n - 1/2)}{L} \right), \quad E_n^2 = \left( \frac{\pi (n - 1/2)}{L} \right)^2 + \lambda, \quad n = 1, \ldots$$ (13)

If we plug this into the first saddle point equation (6) we will notice that $\sin^2$ and $\cos^2$ will sum up to 1 and the $x$-dependence will disappear! So we can consider $\sigma$ to be constant. Let us first study the phase with non-zero $\lambda$. From the second saddle-point equation (7) we see that we have to put $\sigma = 0$. We will call this phase "Coulomb" phase because $n^i$ has zero VEV which leaves the $U(1)$ unbroken.

The first saddle-point equation now reads as:

$$\frac{N}{\pi} \sum_{n=1}^{\infty} \frac{1}{\sqrt{(n - 1/2)^2 + (\lambda L/\pi)^2}} - r = 0$$ (14)

We need to separate the divergent part:

$$\frac{N}{\pi} \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{(n - 1/2)^2 + (\lambda L/\pi)^2}} - \frac{1}{n} \right) + \frac{N}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} - r = 0$$ (15)

Introducing the cut-off:

$$\sum_{n=1}^{\infty} \frac{\exp(-n\pi/L\Lambda_{uv})}{n} = -\log(1 - \exp(-\pi/L\Lambda_{uv})) \approx -\log(\pi/L\Lambda_{uv})$$ (16)

Renormalizing $r$ using eq. (11) we will have:

$$\sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{(n - 1/2)^2 + (\lambda L/\pi)^2}} - \frac{1}{n} \right) = \log(\pi/L)$$ (17)

Now it is easy to see the presence of two phases: the maximum of the LHS is reached when $\lambda = 0$, the corresponding value is

$$\sum_{n=1}^{\infty} \left( \frac{1}{n - 1/2} - \frac{1}{n} \right) = \log(4)$$ (18)
It means that if \( \log(\pi/\Lambda L) > \log(4) \) the saddle-point equations do not have a solution with non-zero \( \lambda \).

Let consider the limit \( L \to 0 \). We can expand the LHS in power series in \( \lambda L \):

\[
\frac{1}{\sqrt{(n - 1/2)^2 + (\lambda L/\pi)^2}} = \frac{1}{n - 1/2} - 4 \left( \frac{\lambda L}{\pi} \right)^2 \frac{1}{(2n - 1)^3} + \ldots \tag{19}
\]

Using the following identity:

\[
\sum_{n=1}^{\infty} \frac{4}{(2n - 1)^3} = \frac{7}{2} \zeta(3) \tag{20}
\]

we have:

\[
\frac{7\zeta(3)}{2} \left( \frac{\lambda L}{\pi} \right)^2 = \log(4\Lambda L/\pi) \tag{21}
\]

We see that the Coulomb phase does not exist for \( L < \pi/4\Lambda \).

Let us now show that the "Higgs" phase \( \sigma = \text{const}, \lambda = 0 \) exists only for \( L < \pi/4\Lambda \). We call this phase "Higgs" because non-zero \( \sigma \) breaks \( U(1) \) gauge symmetry. In this case the second saddle-point equation is satisfied. The first one reads as:

\[
\frac{N}{\pi} \sum_{n=1}^{\infty} \left( \frac{1}{n - 1/2} - \frac{1}{n} \right) + \sigma^2 = \frac{N}{\pi} \log(\pi/\Lambda L) \tag{22}
\]

Again using eq. (20) we have:

\[
\sigma^2 = \frac{N}{\pi} \log(\pi/4\Lambda L) \tag{23}
\]

4 D-D and N-N boundary conditions: one phase

Instead of the D-N and N-D boundary conditions let us investigate the case with Dirichlet-Dirichlet(D-D) and Neumann-Neumann(N-N) boundary conditions. As we will see shortly Coulomb phase is possible for all values of \( L \). For the D-D case we have the following set of eigenfunctions:

\[
\psi_n(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{\pi x n}{L} \right), \quad E_n^2 = \left( \frac{\pi n}{L} \right)^2 + \lambda, \quad n = 1, \ldots \tag{24}
\]

And for N-N:

\[
\psi_n(x) = \sqrt{\frac{2}{L}} \cos \left( \frac{\pi x n}{L} \right), \quad E_n^2 = \left( \frac{\pi n}{L} \right)^2 + \lambda, \quad n = 0, \ldots \tag{25}
\]
Note that now we can have $n = 0$ which corresponds to a constant mode. Note that if $\lambda = 0$ we have a genuine zero mode. It means that the phase with $\lambda = 0$ can not exist for this choice of boundary conditions. In the saddle-point equations $\cos^2$ and $\sin^2$ again sum to 1, so we can have a saddle-point with constant $\sigma$ and $\lambda$. From now on, we will assume that $\lambda = \text{const} \neq 0$. Then from the second saddle-point equation it follows that $\sigma = 0$. The first saddle-point equation now reads as:

$$\frac{N}{\pi} \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{n^2 + (\lambda L/\pi)^2}} - \frac{1}{n} \right) + \frac{N}{\lambda L} + \frac{N}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} - r = 0$$

(26)

After $r$ renormalization we have:

$$\frac{N}{\pi} \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{n^2 + (\lambda L/\pi)^2}} - \frac{1}{n} \right) + \frac{N}{\lambda L} = \frac{N}{\pi} \log(\pi/\Lambda L)$$

(27)

Unlike the D-N and N-D case now the LHS is not bounded from above because of the $\frac{N}{\lambda L}$ term, which is essentially the contribution from the N-N constant mode. It easy to show that for a fixed $\Lambda$ and $L$ we can always find the corresponding value of $\lambda$ (for example one can plot the LHS as a function of $\lambda$ and see that it takes values from $-\infty$ to $+\infty$).

5 Conclusion

In this paper we studied the large $N$ $\mathbb{C}P^N$ model on a finite interval. We have shown that for a specific choice of boundary conditions the saddle-point equations admit a simple analytical solution. Under the Dirichlet-Dirichlet and Neumann-Neumann boundary condition the system possesses a Coulomb phase with the uniform $\lambda$ VEV, usual for the $\mathbb{C}P^N$ in the infinite space. This phase exists for all values of the interval length $L$. However, under the mixed Dirichlet-Neumann boundary conditions the system has two phases: Coulomb phase which exists for $L > \pi/4\Lambda$ and unusual Higgs phase for $L < \pi/4\Lambda$ with the uniform $n^0$ VEV. Strictly speaking, it is possible to have additional phases with non-constant VEVs, similar to the FFLO phase in superconductivity. It is even possible that the Coulomb and Higgs phases in the N-D case are not adjacent on the phase diagram because of the presence of additional phases. We will postpone this analysis for future work.

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