Abstract

We show that for $U(1)$-symmetric spacetimes on $S^3 \times R$ a constant of motion associated with the well known Geroch transformation, a functional $K[h_{ij}, \pi^{ij}]$, quadratic in gravitational momenta, is strictly positive in an open subset of the set of all $U(1)$-symmetric initial data, and therefore not weakly zero. The Mixmaster initial data appear to be on the boundary of that set. We calculate the constant of motion perturbatively for the Mixmaster spacetime and find it to be proportional to the minisuperspace Hamiltonian to the first order in the Misner anisotropy variables, i.e. weakly zero. Assuming that $K$ is exactly zero for the Mixmaster spacetime, we show that Geroch’s transformation, when applied to the Mixmaster spacetime, gives a new $U(1)$-symmetric solution of the vacuum Einstein equations, globally defined on $S^2 \times S^1 \times R$, which is non-homogeneous and presumably exhibits Mixmaster-like complicated dynamical behavior.

PACS number(s): 04.20.Cv, 04.20.Jb
I. INTRODUCTION

The homogeneous cosmological models, apart from their possible cosmological relevance, provide us with a rich theoretical laboratory in which, it is hoped, many of the problems of General Relativity can be reduced to a manageable level of complexity. One of the longstanding problems of Classical Relativity that benefited from such an approach is the problem of behavior of the solutions of Einstein’s equations close to the “singularity”, i.e., close to the boundary of the maximal globally hyperbolic development; a problem which is relevant to the issue of Strong Cosmic Censorship and possibly to the quantization of gravity program. Due to the complex nonlinear nature of the Einstein partial differential equations, an exact and full description of the asymptotic behavior seems to be unattainable, at least so in the foreseeable future. Restricted to the spatially homogeneous cosmological models, however, the Einstein field equations, in the natural symmetry-adapted foliation, reduce to a system of ordinary differential equations which is much easier to analyze, and had been extensively analyzed in the past (see [1]). Among the homogeneous models the most important special cases for our theoretical laboratory are the Kasner and Mixmaster models. The Kasner solution is explicitly known and is a very simple (generically) curvature singular solution. The Mixmaster model, despite several decades of intense effort, has not yet been explicitly solved (we do not have as detailed knowledge of its properties as we would want to), but an approximate description for the approach to the singularity has been found in terms of a discrete sequence of Kasner solutions [2]. Due to the stochastic properties of the associated discrete mapping it is hard to estimate the quality of the approximation analytically, but numerical simulations indicate that the discrete sequence approximates the exact solution the better the closer we come to the singularity [3].

The significance of the Mixmaster model for the problem of the asymptotic behavior of solutions of Einstein’s equations stems largely from the long series of papers by Belinski, Khalatnikov and Lifshitz [4], hereafter BKL. They tried to describe the approach to the singularity for non-symmetric spacetimes as a sequence of pointwise Mixmaster-like tran-
sitions between Kasner epochs. A satisfactory geometrical formulation of their method is still lacking, and it is still controversial whether their method can give any information about the global structure of the singularity \[5\]. Nevertheless, it indicates that even locally the dynamical behavior close to the singularity can be extremely complex; at least as complex as in the Mixmaster model (in fact as complex as in non-diagonal Bianchi IX and Bianchi VIII). Even though the asymptotic dynamics can be extremely complex for generic solutions, there exists a class of solutions whose asymptotic behavior is fairly simple: the asymptotic dynamics simplifies significantly for spacetimes for which, in a suitable foliation, the spatial derivative terms can be neglected asymptotically. Since the thus truncated Einstein’s equations are exactly solvable we can extract all the asymptotic properties we need from the solution of the truncated equations, the Generalized Kasner Solution; a solution which evolves pointwise like the Kasner solution. The corresponding approximation method, usually called the Velocity-Dominated Approximation (VDA) \[6,7\] or Strong Coupling Expansion has been shown to be applicable to a large class of non-homogeneous spacetimes including Gowdy spacetimes \[8\]. Although the class of spacetimes for which the VDA is valid is infinite dimensional, it does not include some of the homogeneous models. Namely, the necessary conditions for the applicability of the VDA are not satisfied for the Bianchi VIII and IX spacetimes and, as BKL showed, are not satisfied for a generic spacetime as well.

In order to test whether the Mixmaster solution (diagonal Bianchi IX), or the more general non-diagonal Bianchi VIII and IX, as claimed by BKL, are in some sense good asymptotic, pointwise approximations for the generic spacetimes, as the Kasner solution is for the velocity-dominated spacetimes, it is important to have some non-homogeneous solutions that are close to the Mixmaster in the gravitational phase space. If this is the case it is also important to have as detailed knowledge of the complex Mixmaster dynamics as possible.

To that end in this paper we use the structure associated with the well known Geroch solution generating technique for spacetimes with one Killing field \[9\]. Most importantly,
with Geroch’s transformation there is associated a new non-trivial constant of motion, a
dynamical observable which, we hope, could help us in the analysis of the asymptotic behavior
of spacetimes. We treat the Mixmaster spacetime as a special case of $U(1)$-symmetric
spacetimes by using only one of the three Mixmaster Killing fields—one that generates a
one-dimensional isometry group $U(1)$. The action of that $U(1)$ group on any Mixmaster
homogeneous spacelike slice $\Sigma$, $\Sigma \sim S^3$, induces a Hopf fibration of $\Sigma$, i.e., makes it into
the $n = 1$, $U(1)$ principal fiber bundle over $S^2$. We then calculate perturbatively the new
constant of motion for the Mixmaster spacetime which, to the first order in the Misner
anisotropy variables, turns out to be proportional to the minisuperspace Hamiltonian. This,
barring an unlikely coincidence, indicates that the constant of motion is proportional to the
minisuperspace Hamiltonian. If that is true, we can, using Geroch’s transformation, obtain
a new *globally* (in space) defined, non-homogeneous solution on $S^2 \times S^1$, with presumably
Mixmaster-like complex asymptotic behavior.

However, when calculated for the Taub-Nut special case, a spacetime that has 4 Killing
fields, the constant of motion associated with the fourth Killing field, which does *not exist*
in the general Mixmaster case, is strictly positive and therefore not proportional to the
Hamiltonian constraint (which is zero for the solutions of the Einstein equations). It is
worth emphasizing that there is no contradiction in the Mixmaster constant of motion being
zero and the Taub-NUT constant being positive, since the above mentioned Taub-NUT
constant of motion is not a restriction of the Mixmaster constant of motion to the Taub-
NUT symmetry-class because a different Killing field is used for the Taub-NUT case.

The outline of the paper is as follows: in section 2 we review the partial reduction of the
Einstein equations in the ADM hamiltonian formulation for the $U(1)$-symmetric spacetimes
as well as the conditions for global applicability of Geroch’s transformation. In section
3 we describe the Mixmaster spacetime as a $U(1)$-symmetric spacetime. In section 4 we
apply Geroch’s transformation to the Mixmaster and Taub-NUT spacetimes and find their
respective constants of motion. In section 4 we give some concluding remarks.
II. $U(1)$-SYMMETRIC SPACETIMES AND GEROCH’S TRANSFORMATION

As shown by Geroch, the vacuum Einstein equations for the class of four dimensional spacetimes with at least one Killing field can be reduced, locally, to the three dimensional Einstein equations coupled to a harmonic map, with the two-dimensional hyperbolic space as the target manifold for the harmonic map. The isometry group, $SL(2, R)$, of the target space becomes a symmetry of the reduced equations, i.e., maps solutions of the vacuum Einstein equations into new locally defined vacuum solutions with one Killing field.

It was hoped that, with some control over the global structure of the spacetime, the transformation might be applicable globally. Indeed, for the globally hyperbolic spacetimes foliated by spacelike, compact, connected and orientable 3-surfaces (Cauchy surfaces) $\Sigma$, invariant with respect to the action of the isometry group, the conditions for generation of new globally (in space) defined vacuum solutions were found by Moncrief and Cameron [10,11]. They showed that the Einstein equations for $U(1)$-symmetric spacetimes on $U(1)$ principal fiber bundles (circle bundles) $\Sigma \times R \rightarrow \tilde{\Sigma} \times R$—where the base manifold $\tilde{\Sigma} \sim \Sigma / U(1)$ is a compact, connected and orientable two-dimensional manifold—can be reduced to a 2+1 Einstein-Harmonic map system from $\tilde{\Sigma} \times R$ to the Poincaré half-plane (hyperbolic space) as the target space, provided one integrality condition is satisfied [10,11]. For each base manifold $\tilde{\Sigma}$ there is a countable infinity of isomorphism classes of principal $U(1)$-bundles $B_n$ over $\tilde{\Sigma}$, and the integrality condition depends only on the integer $n$ characterizing the isomorphism class $B_n$ of the Cauchy surface $\Sigma$. For $\tilde{\Sigma} = S^2$ all classes were explicitly constructed by Quiroz et. al. [12] by imposing suitable identifications on the $S^3$. The integer $n$ characterizing the classes is equal to both the winding number and the Chern number of the bundle; the trivial bundle $S^2 \times S^1 \sim B_0$, i.e., $n = 0$ and $S^3 \sim B_1$, i.e., $n = 1$, for example.

The restriction to the $U(1)$ isometry group, i.e., the restriction that the group orbits be closed, is not a restriction for the spacetimes with at least one dimensional isometry groups. The isometry group of a cosmological spacetime must be a compact Lie group—because
it is also the isometry group of the foliation surfaces \( \Sigma \), which are compact Riemannian manifolds—and every compact Lie group has a \( U(1) \) subgroup. The only real restriction is that there exists a \( U(1) \) subgroup of the isometry group whose action on \( \Sigma \) makes the foliation surfaces \( \Sigma \) into \( U(1) \) principal fiber bundles with compact and orientable base manifold \( \tilde{\Sigma} \), i.e., that there are no fixed points for the action and no orbits twisting around each other.

We shall now, mostly following [10], give a brief review of Geroch’s formalism. In order to express the conditions for the global applicability of Geroch’s transformation as conditions on the gravitational initial data, we shall use the usual ADM hamiltonian formulation adapted to the class of spacetimes with one spacelike Killing field. Depending on convenience, we shall use index-free or typical-component (abstract-index) notation for tensor fields. The lower case greek letter \((\mu, \nu \cdots)\) indices shall denote components in an arbitrary basis and will run from 0 to 3. When dealing with coordinates (or tetrads) specially adapted to a family of spacelike surfaces we shall use the lower case latin indices, beginning with \( i \), for the range from 1 to 3, and call such components the spacelike components. Likewise, when using further specialized coordinate systems (or tetrads), specially adapted to the group action, we shall use the lower case latin indices, beginning with \( a \), for the range from 1 to 2.

Let the \( U(1) \) isometry group be parametrized by the usual angle parameter \( \alpha \in [0, 2\pi) \) and let the Killing field \( \xi \) be just the derivative with respect to the group parameter \( \alpha \), i.e., \( \xi\{f(p)\} = \frac{d}{d\alpha} f(e^{i\alpha}p) \), where \( e^{i\alpha}p \) denotes the action of the \( U(1) \) group on the point \( p \). By definition, the spacetime can be foliated by a family of spacelike hypersurfaces \( \Sigma_t \) invariant with respect to the action of the isometry group, i.e., a family whose defining global time function \( t \) has zero Lie derivative with respect to the Killing vector field \( \xi \) generating the isometry group action. The Killing field is then necessarily tangent to the foliation.

As usual, using the unit normal vector field \( n^\mu \) to \( \Sigma \) and the spacetime metric \( g_{\mu\nu} \), one can define a new tensor field, the three-metric

\[
h_{\mu\nu} = g_{\mu\nu} + g_{\mu\lambda} g_{\nu\sigma} n^\lambda n^\sigma, \tag{1}
\]
whose restriction to the vectors tangent to \( \Sigma \) generates a Riemannian metric on \( \Sigma \). In a coordinate system generated by extending arbitrary coordinates on \( \Sigma \) using the flow of the normal vector \( n^\mu \), only spacelike components \( h_{ij} \) are non-zero, and both the three-metric \( h_{\mu\nu} \) and the foliation unit normal vector field \( n^\mu \) are group invariant.

The three-metric \( h_{\mu\nu} \) can be further decomposed as follows:

\[
h_{\mu\nu} = \lambda^{-1} \tilde{g}_{\mu\nu} + \lambda \eta_\mu \eta_\nu,
\]

using the norm \( \lambda \) of the Killing field \( \xi^\mu \),

\[
\lambda = g_{\mu\nu} \xi^\mu \xi^\nu = h_{\mu\nu} \xi^\mu \xi^\nu,
\]

and a one-form \( \eta_\mu \) defined as

\[
\eta_\mu = \lambda^{-1} g_{\mu\nu} \xi^\nu.
\]

This decomposition is \( \xi \)-invariant and induces a new Riemannian two-dimensional metric \( \tilde{g}_{\mu\nu} \) in the subspace of vectors tangent to \( \Sigma \) and orthogonal to \( \xi \). As a result one gets a \( \xi \)-invariant decomposition of the metric \( g_{\mu\nu} \to (n_\mu, \tilde{g}_{\mu\nu}, \eta_\mu, \lambda) \), and it is easy to see that

\[
\xi^\mu n_\mu = 0,
\]

\[
\xi^\mu \tilde{g}_{\mu\nu} = 0,
\]

\[
\xi^\mu \eta_\mu = 1,
\]

\[
\xi^\mu (d\eta)_{\mu\nu} = 0.
\]

The last equation being a consequence of (7) and \( L_\xi \eta = 0 \).

The bundle projection pullback \( {}^*\pi \) generates a one to one correspondence between, on the one side, the \( \xi \)-invariant covariant tensor fields on \( \Sigma \sim B_n \) all of whose contractions with \( \xi \) vanish, and on the other, the covariant tensor fields on the base \( \tilde{\Sigma} \). We can, therefore, treat the fields \( \lambda \) and \( \tilde{g}_{ab} \) as tensor fields on \( \tilde{\Sigma} \) (and \( n_\mu \) as a field on \( \tilde{\Sigma} \times R \)). The one-form \( \eta_\mu \), however, does not have vanishing contraction with \( \xi \) and cannot be treated as a field on \( \tilde{\Sigma} \), which imposes the only global restriction to the reduction of dynamics from \( \Sigma \) to \( \tilde{\Sigma} \). Even
though $\eta$ does not have vanishing contraction with $\xi$, its exterior derivative (the only form in which $\eta$ will enter the equations of motion) does, and as shown in [14], for a closed two-form $\Phi$ on $\tilde{\Sigma}$ there will exist a one-form $\eta$ on $\Sigma \sim B_n$, satisfying $\xi^\mu \eta_\mu = 1$ and $d\eta = \ast \pi(\Phi)$, if and only if

$$\int_{\tilde{\Sigma}} \Phi = 2\pi n. \quad (9)$$

To effectively use the symmetry of the problem in concrete calculations we shall be doing in this paper, we need an isometry-adapted atlas on the spacetime manifold. Using an atlas on the base manifold $\tilde{\Sigma}$, with coordinates in a representative chart labeled by $\{x^a\}$, $a = 1, 2$ one can construct, with the help of the bundle structure on $\Sigma$, a $\xi$-invariant atlas on the initial foliation surface $\Sigma$. Let the coordinates in a representative chart be labeled $\{x^i\} = \{x^a, x^3\}$, $i = 1..3$, with the third coordinate normalized so that $\partial/\partial x^3 = \xi$. Now, one can choose a $\xi$-invariant time-evolution vector field $t^\mu$ and lift the coordinates from the initial leaf $\Sigma$, along the flow generated by $t^\mu$, to all other foliation surfaces, which are labeled by the flow parameter $t$ used as the zeroth coordinate. Thus, one has created an atlas on the spacetime manifold in whose every chart, $\{x^\mu\} = \{t, x^a, x^3\}$, $\mu = 0..3$, the action of the $U(1)$ group is given by $e^{i\alpha}(t, x^a, x^3) = (t, x^a, x^3 + \alpha)$, and the bundle projection $\pi$ by $\pi: (t, x^a, x^3) \mapsto (t, x^a)$.

In any such chart we can define a one-form

$$\beta = \eta - dx^3, \quad (10)$$

which carries all important dynamical information contained in $\eta$ and whose exterior derivative, $d\beta = d\eta$, and time derivative, $\dot{\beta} \equiv \mathcal{L}_t \beta = \dot{\eta}$, are globally defined and chart independent differential forms. The two-form $d\beta$ can be identified with the (exact, globally-defined) two-form $\Phi$ on the base manifold $\tilde{\Sigma}$ and is, in an $x^a$ chart, expressible as

$$d\beta = d\eta = r \, dx^1 \wedge dx^2, \quad (11)$$

where $r$ is a scalar density whose value in the same chart,
\[ r^b = \epsilon^{ab} \beta_{a,b}, \quad (12) \]

can be calculated using the contravariant antisymmetric tensor (density) \( \epsilon^{ab} = -\epsilon^{ba}, \epsilon^{12} = 1 \),
associated with the chart.

In the new \( U(1) \)-adapted variables the isometry of the spacetime metric,
\[
\begin{align*}
  ds^2 &= g_{\mu\nu} \, dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \\
  &= \lambda^{-1} \left[ -\tilde{N} dt^2 + \tilde{g}_{ab}(dx^a + \tilde{N}^a dt)(dx^b + \tilde{N}^b dt) \right] \\
  &\quad + \lambda \left[ dx^3 + \beta_a dx^a + \tilde{N}^a \beta_a dt \right]^2, \quad (13)
\end{align*}
\]
is equivalent to all the component functions being independent of \( x^3 \), as the coordinate
one-forms \( \{ dt, dx^i \} \) are already \( \xi \)-invariant.

Using the \( U(1) \) isometry, the ADM action \( I \) on the \( \Sigma \times R \),
\[
I = \int dt \int \Sigma dx^3 \left\{ \pi^{ij} h_{ij, t} - N \mathcal{H} - N^i \mathcal{H}_i \right\}, \quad (14)
\]
with
\[
\mathcal{H} = h^{-1/2} \left[ \pi^{ij} \pi_{ij} - 1/2 (\pi^i)^2 \right] - h^{1/2} R(h),
\]
\[
\mathcal{H}_i = -2h^{1/2} \nabla_j (h^{-1/2} \pi^j_i),
\]
reduces to an equivalent action \( \tilde{I} \) on \( \tilde{\Sigma} \times R \),
\[
\tilde{I} = 2\pi \int dt \int \Sigma dx^2 \left\{ \tilde{\pi}^{ab} \tilde{g}_{ab, t} + p \lambda_{, t} + e^a \beta_{a, t} - \tilde{N} \tilde{\mathcal{H}} - \tilde{N}^a \tilde{\mathcal{H}}_a + \beta_0 e^a_a \right\}, \quad (17)
\]
where
\[
\begin{align*}
  \tilde{\mathcal{H}} &= \lambda^{-1/2} \mathcal{H} = \tilde{g}^{-1/2} \left[ \tilde{\pi}^{ab} \tilde{\pi}_{ab} - (\tilde{\pi}^a)^2 + \frac{1}{2} \lambda^2 p^2 + \frac{1}{2\lambda} \tilde{g}_{ab} e^a e^b \right] \\
  &\quad + \tilde{g}^{1/2} \left[ -R(\tilde{g}) + \frac{1}{2\lambda} \tilde{g}^{ab} \lambda, a \lambda, b + \frac{\lambda^2}{4} \tilde{g}_{ac} \tilde{g}_{bd} (d\beta)_{ab}(d\beta)_{cd} \right], \quad (18) \\
  \tilde{\mathcal{H}}_a &= \mathcal{H}_a = -2 \tilde{g}^{1/2} \nabla_b \left( \tilde{g}^{-1/2} \tilde{\pi}^a_{ab} \right) + p \lambda_{, a} + e^b (d\beta)_{ab}, \quad (19) \\
  \tilde{N} &= \lambda^{1/2} N, \quad (20) \\
  \tilde{N}^a &= N^a, \quad (21)
\end{align*}
\]
\beta_0 = \eta_i N^i, \quad (22)

\tilde{\pi}^{ab} = \lambda^{-1} \pi^{ab}, \quad (23)

e^a = 2\lambda \eta_i \pi^{ia}, \quad (24)

p = 2\pi^{ij} \eta_i \eta_j - \lambda^{-1} \pi^{ij} h_{ij}. \quad (25)

Variation of the action \tilde{I} with respect to \beta_0 gives a constraint:

\dot{e}^a_a = 0 \iff \text{div } \tilde{e} = 0 \iff d(\star \tilde{e}) = 0; \quad (26)

where we have used \tilde{e} = \tilde{g}^{-1/2} e to denote vector field associated to the vector density \tilde{e}, and \star to denote the Hodge star operator of the Riemannian metric \tilde{g} on \tilde{\Sigma}. This constraint can be easily solved since, according to the Hodge theorem, every closed form on \tilde{\Sigma}, and in our case this is the form \star \tilde{e}, can be uniquely decomposed to a sum of one exact and one harmonic form. Explicitly written in an arbitrary chart on \tilde{\Sigma}, the decomposition gives

\epsilon^b \epsilon_{ba} = \omega, + h_a, \quad (27)

with \omega the scalar field (defined up to a constant) and \tilde{h}_a the components of the unique harmonic form, both defined globally on \tilde{\Sigma}. By a slight abuse of notation, we have used \epsilon_{ab} = -\epsilon_{ba}, \epsilon_{12} = 1, to denote the covariant antisymmetric tensor (density) associated with the chart. From the equations of motion obtained by varying the action with respect to \beta_a, we can obtain

\dot{\omega}_a + \dot{h}_a = \left[ \tilde{N} \tilde{g}^{-1/2} \lambda^2 r + \tilde{N}^b (\omega, + h_b) \right], \quad (28)

which, because of the uniqueness of Hodge decomposition and the fact that the one form on the right-hand side is exact, forces the harmonic form \dot{h} to be zero, i.e., h to be time independent. Hereafter in this paper we shall be interested only in the special case of \tilde{\Sigma} \sim S^2 which has no non-zero harmonic one-forms, and shall therefore put \tilde{h} = 0.

With the decomposition of \epsilon^a included into the action \tilde{I}, the one-form \beta_a appears only in the form \epsilon^{ab} \beta_{a,b} = r and the action reduces to an equivalent 2+1 Einstein-Harmonic action
\[ J = 2\pi \int dt \int_{\Sigma} dx^2 \left\{ \tilde{\pi}^{ab} \tilde{g}_{ab,t} + p\lambda,t + r\omega,t - \tilde{N} \tilde{\mathcal{H}} - \tilde{N}^a \tilde{\mathcal{H}}_a \right\}, \tag{29} \]

where

\[ \tilde{\mathcal{H}} = \tilde{g}^{-1/2} \left[ \tilde{\pi}^{ab} \tilde{\pi}_{ab} - (\pi^a_a)^2 + \frac{1}{2} G^{AB} P_A P_B \right] \]
\[ + \tilde{g}^{1/2} \left[ -R(\tilde{g}) + \frac{1}{2} G_{AB} X_a^A X_b^B \tilde{g}^{ab} \right], \tag{30} \]
\[ \tilde{\mathcal{H}}_a = -2\tilde{g}^{1/2} \tilde{\nabla}_b \left( \tilde{g}^{-1/2} \tilde{g}^b_a \right) + p\lambda_a + r\omega_a, \tag{31} \]
\[ X^A = \{ \lambda, \omega \}, \quad P_A = \{ p, r \}, \quad A = 1, 2, \tag{32} \]

and

\[ G_{AB} = \lambda^{-2} \left[ d\lambda^2 + d\omega^2 \right]_{AB} \tag{33} \]

is the metric of constant negative curvature on the Poincaré half-plane \( \{ \lambda > 0, \omega \} \), the target space for the harmonic variables \( \lambda \) and \( \omega \). This form of the action makes evident the \( SL(2, R) \) symmetry of the equations of motion, \( SL(2, R) \) being the isometry group of the Poincaré half-plane, and enables us to apply the Geroch transformation to the harmonic variables \( \lambda \) and \( \omega \) to generate new solutions of the vacuum 3+1 Einstein equations from the known ones.

Starting with a \( U(1) \)-symmetric four-dimensional solution of the vacuum Einstein equations on \( R \times B_n \) expressed in the form (13), we can calculate the canonical variables \( (\lambda, r, p) \) locally from the metric components and their velocities. \( \lambda \) is the norm of the Killing vector field, \( r \) is given by (12) and

\[ p = 2\tilde{N}^{-1} \tilde{g}^{1/2} \lambda^{-1} \left( \lambda - \tilde{N}^a \lambda_a \right). \tag{34} \]

Only the canonical variable \( \omega \) is not a local function of the metric components and their velocities (or momenta), but it can be calculated by performing a line integral in the three-dimensional space \( R \times \tilde{\Sigma} \):

\[ \omega(t, x^1, x^2) = \int_{t_0}^t \dot{\omega}(t', x_0^1, x_0^2) \, dt + \int_{\Gamma(x_0, x)} \omega_a(t, x') \, dx'^a. \tag{35} \]
In order to do so, we need the spatial and temporal derivatives of $\omega$. Without using the equations of motion in the Einstein-Harmonic form the spatial derivatives can be calculated from (24) and (27) and the time derivative from (28) giving:

$$\omega_{,a} = -\tilde{N}^{-1}\tilde{g}^{1/2}\lambda^2 \epsilon_{ab} \tilde{g}^{bc} \dot{\beta}_c,$$

$$\omega_{,t} = \tilde{N} \tilde{g}^{-1/2}\lambda^2 r - \tilde{N}^{-1} \tilde{N}^a \tilde{g}^{1/2}\lambda^2 \epsilon_{ab} \tilde{g}^{bc} \dot{\beta}_c + c(t),$$

where $c(t)$ is an arbitrary function of time. This is to be expected, since $\omega$ is defined only up to a constant on each $\tilde{\Sigma}$, i.e., given an $\omega$ a whole class of functions $\omega + f(t)$ is associated to each four dimensional spacetime. So, it would appear that there is a gauge indeterminacy for the evolution of $\omega$. Fortunately, the new evolution equations completely eliminate that gauge freedom for $\omega$ and, given initial data for $\lambda, \omega, p$ and $r$, determine a unique solution.

Varying the action $\tilde{J}$ with respect to $r$, we find that the class representative for $\omega$ singled out by the new equations of motion is the one obtained by putting $c(t) = 0$ in (37). The class representative is not completely determined—a constant (in time and space) can be added to $\omega$—which corresponds to freedom of choice of the reference point $x_0$ in the initial $\tilde{\Sigma}$ from which the spatial line integration in (35) starts.

The action of an element,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ab - cd = 1,$$

of $SL(2, R)$ on the canonical variables, given by:

$$\lambda \rightarrow \lambda' = \frac{\lambda}{c^2(\omega^2 + \lambda^2) + 2cd\omega + d^2},$$

$$\omega \rightarrow \omega' = \frac{ac(\omega^2 + \lambda^2) + (ad + bc)\omega + bd}{c^2(\omega^2 + \lambda^2) + 2cd\omega + d^2},$$

$$p \rightarrow p' = \frac{p[c^2(\omega^2 - \lambda^2) + 2cd\omega + d^2] - r[2\lambda(cd + c^2\omega)]}{c^2(\omega^2 + \lambda^2) + 2cd\omega + d^2},$$

$$r \rightarrow r' = p(2c^2\lambda\omega + 2cd\lambda) + r\left[d^2 + c^2(\omega^2 - \lambda^2) + 2cd\omega\right],$$

then transforms our old solution of the 2+1 Einstein-Harmonic equations on $R \times \tilde{\Sigma}$ into a new solution of the same equations. To lift that solution to a new solution of the 3+1
vacuum Einstein equations on $R \times B_{\nu'}$, all we need to do is use new $\lambda'$ and integrate (11) to find the new one-form $\eta'$. As mentioned previously, this will be possible if and only if the integrality condition (43), which now becomes

$$\int_{\Sigma} r' = 2\pi n',$$

is satisfied at all times.

The above integral is a constant of motion of the Einstein-Harmonic equations and the integrality condition has to be enforced only at one time. Even better, the symmetry-action of $SL(2, R)$ results in three conserved quantities, each associated with one of the three Killing fields of the Poincaré half-plane. These three constants of motion are:

$$A = \int_{\Sigma} (2\omega r + p),$$

$$B = \int_{\Sigma} r,$$

$$C = \int_{\Sigma} \left[r(\lambda^2 - \omega^2) - p\omega\right],$$

and represent a momentum map of the $SL(2, R)$ action.

$B$ is not interesting as a constant of motion since its value does not depend on the initial data; it is just the integrality condition dictated by the topology of $\Sigma$. $A$ and $C$ are not proper constants of motion associated to the spacetime, either, because their values depend on the constant added to $\omega$, which is not observable in the spacetime. Nevertheless, the $SL(2, R)$-invariant constant of motion found by Geroch [13],

$$K[h_{ij}, \pi^{ij}] = A^2 + 4BC,$$

has the desired properties. It is easy to see that the value of $K$, not only does not depend on the unobservable constant added to $\omega$, but remains the same even if we add an arbitrary function of time to $\omega$, which means that it is a proper constant of motion of the spacetime, i.e., an observable. This also means that when calculating $\omega$ to be used for evaluating $K$, the first term in (35), which is only a function of time, can be dropped. That guarantees that the constant of motion $K$ depends only on the initial data on a single Cauchy surface,
i.e., that it is a functional of the standard gravitational variables \((h_{ij}, \pi^{ij})\) that is local in time.

In order to calculate \(K[h_{ij}, \pi^{ij}]\), given the ADM initial data \((h_{ij}, \pi^{ij})\), one has to express \(\lambda, p, r\) and \(\omega_{,a}\) in terms of the components of the ADM data in a \(U(1)\)-adapted coordinate chart:

\[
\lambda = h_{33}, \quad (48)
\]
\[
p = 2\lambda^{-2}\pi^{ij}h_{i3}h_{j3} - \lambda^{-1}\pi^{ij}h_{ij}, \quad (49)
\]
\[
r = \left(\lambda^{-1}h_{31}\right)_2 - \left(\lambda^{-1}h_{32}\right)_1, \quad (50)
\]
\[
\omega_{,a} = 2\epsilon_{ba}\pi^{bi}h_{i3}, \quad (51)
\]

and evaluate \(\omega\) by integrating \(\omega_{,a}\) along a curve in the two-sphere. The constants of motion \(A, B, \) and \(C\) are then calculated by evaluating the surface integrals. Loosely speaking, \(K\) is a quadratic functional of the ADM momenta \(\pi^{ij}\).

An important property of \(K\) is that it controls the global applicability of the Geroch transformation. As shown by Moncrief [15], any solution on a nontrivial bundle \(B_n, n \neq 0\), can be transformed to a globally defined solution on any other nontrivial bundle \(B_{n'}, n' \neq 0\); no restrictions exist. For a more interesting case of transforming a solution from a nontrivial bundle to a global solution on the trivial bundle there is a restriction, however. The transformation can generate a globally defined solution if and only if \(K \geq 0\).

In the next section we shall establish some of the properties of \(K\) and try to calculate it for the Mixmaster spacetime.

**III. MIXMASTER MODEL AS A SPECIAL CASE OF THE \(U(1)\)-SYMMETRIC SPACETIME**

In a spatially homogeneous spacetime, by definition, the orbits of the isometry group (or some subgroup if there are timelike Killing vector fields) are three-dimensional space-like hypersurfaces that foliate the spacetime. If the isometry group has a subgroup that
acts simply transitively on the orbits, the spacetime can be classified into one of the nine Bianchi classes, according to the Bianchi class of the Lie algebra of the corresponding simply transitive subgroup. If the isometry group has no simply transitive subgroup, the isometry group must be at least four-dimensional and the spacetime must be locally isometric to the interior of the Schwarzschild solution. These solutions, found by Kantowski and Sachs [16], if restricted to compact spatial sections must have the spatial sections diffeomorphic to $S^2 \times S^1$ and the four dimensional group with transitive action must be $SU(2) \times U(1)$.

The Mixmaster cosmological model, by definition, belongs to the Bianchi IX class, has $S^3$ spatial topology and $SU(2)$ as the isometry group, and the spatial metric is diagonal in the left- or right-invariant basis of the isometry group.

Let us begin with the brief analysis of the structure of the spatial sections. The three-sphere, the group manifold of $SU(2)$, can be represented as a hypersurface in $C^2 \sim R^4$ $S^3 = \{(z_1, z_2), |z_1|^2 + |z_2|^2 = 1\}$. Writing $z_1 = r_1 e^{\alpha_1}$ and $z_2 = r_1 e^{\alpha_2}$, with $r_1 \geq 0$ and $\alpha_i$ from any interval of width $2\pi$, we see that the only condition the points lying on $S^3$ must satisfy is $(r_1)^2 + (r_2)^2 = 1$, which guarantees that there exist unique $\theta \in [0, \pi]$ such that $r_1 = \cos \theta/2$ and $r_2 = \sin \theta/2$. Thus, one can introduce $\{\theta, \alpha_1, \alpha_2\}$ as coordinates on $S^3$, and can think of $S^3$ as a singular one-parameter family of two-tori $(\alpha_1, \alpha_2)$, $\theta$ being the parameter of the family. The tori for $\theta = 0, \pi$ collapse to $S^1$, making the coordinate system singular there.

Recall that we need a coordinate system in which one coordinate vector field would have closed orbits, i.e., generate a $U(1)$ group action, and that group action would have to be such that it gives $S^3$ a principal fiber bundle structure over some two-dimensional $\tilde{\Sigma}$. The most useful coordinates on $S^3$, for that purpose, are the standard Euler angle coordinates $\{\theta, \phi, \psi\}$, which can be obtained from $\{\theta, \alpha_1, \alpha_2\}$ coordinates by a simple coordinate transformation:

\[
\phi = \alpha_1 + \alpha_2, \quad \psi = \alpha_1 - \alpha_2.
\]

The Euler coordinates, just like the $\{\theta, \alpha_1, \alpha_2\}$ have a singularity at $\theta = 0, \pi$ and when we
let \{\theta, \phi, \psi\} \in \{(0, \pi), [0, 2\pi), [0, 4\pi)\} or \{\theta, \phi, \psi\} \in \{(0, \pi), [0, 4\pi), [0, 2\pi)\} they cover all of the three-sphere (except the two singular circles) exactly once. The period of both \phi and \psi coordinates is 4\pi, and if we let both \phi and \psi extend all the way to 4\pi, the three-sphere would be covered twice.

The coordinate vector fields \partial_\phi and \partial_\psi are globally defined on \(S^3\) despite the coordinate singularities in \(\theta = 0, \pi\) and they generate two \(U(1)\) group actions, which each yield a Hopf fibration of \(S^3\), i.e., they make \(S^3\) into a principal fiber bundle over \(S^2\) with bundle projection for the \(\psi\)-bundle, for example, given by:

\[
\pi_\psi : S^3 \to S^2, \quad (\theta, \phi, \psi) \mapsto (\theta, \phi).
\]

Since these two bundle structures are equivalent, we shall use \(\psi\)-bundle for the Mixmaster spacetime and, taking \(\psi/2 = x^3\), define

\[
\xi = 2 \partial_\psi,
\]

(54)
to be the Killing field we shall use to apply Geroch’s transformation to the Mixmaster solution. The coordinate system \(\{x^1, x^2, x^3\} = \{\theta, \phi, \psi/2\}\) is appropriately adapted to the bundle structure, as described in the previous section. In a spacetime with several Killing fields it is important to remember which Killing field was employed for Geroch’s transformation, so we shall denote the constant of motion (47) associated with the Killing vector \(2 \partial_\psi\) by \(K_\psi\).

The set of standard, globally-defined (and analytic) left-invariant vector fields \(\hat{X}_i\), which generate a right-action on \(SU(2)\) \([17]\), written here together with their dual one-forms \(\hat{\omega}^i\):

\[
\hat{X}_1 = \cos \psi \partial_\theta + \csc \theta \sin \psi \partial_\phi - \cot \theta \sin \psi \partial_\psi \\
\hat{X}_2 = -\sin \psi \partial_\theta + \csc \theta \cos \psi \partial_\phi - \cot \theta \cos \psi \partial_\psi \\
\hat{X}_3 = \partial_\psi \\
\hat{\omega}^1 = \cos \psi \, d\theta + \sin \theta \sin \psi \, d\phi \\
\hat{\omega}^2 = -\sin \psi \, d\theta + \sin \theta \cos \psi \, d\phi \\
\hat{\omega}^3 = d\psi + \cos \theta \, d\phi
\]

(55)
\[
\begin{aligned}
[\hat{X}_i, \hat{X}_j] &= C_{ij}^k \hat{X}_k \\
\hat{\omega}_k &= -1/2 C_{ij}^k \hat{\omega}^i \wedge \hat{\omega}^j \\
C_{ij}^k &= \epsilon_{ijk}, \quad \epsilon_{123} = 1
\end{aligned}
\]

contains \(\partial_\psi\), so, we shall chose them as the complete set of Killing fields for the Mixmaster spacetime. We shall also need a right-invariant one-form basis \(\hat{\sigma}^i\) to explicitly express the isometry of the metric. For completeness we give them with their dual vector fields \(\hat{Y}_i\) and some of their properties:

\[
\begin{aligned}
\hat{Y}_1 &= \cos \phi \partial_\theta + \csc \theta \sin \phi \partial_\psi - \cot \theta \sin \phi \partial_\phi \\
\hat{Y}_2 &= \sin \phi \partial_\theta - \csc \theta \cos \phi \partial_\psi + \cot \theta \cos \phi \partial_\phi \\
\hat{Y}_3 &= \partial_\phi \\
\hat{\sigma}^1 &= \cos \phi \ d\theta + \sin \theta \sin \phi \ d\psi \\
\hat{\sigma}^2 &= \sin \phi \ d\theta + \sin \theta \cos \phi \ d\psi \\
\hat{\sigma}^3 &= d\phi + \cos \theta \ d\psi
\end{aligned}
\]

The Mixmaster metric, invariant with respect to the right action of \(SU(2)\), can now be written as:

\[
ds^2 = -N^2 \ dt^2 + A^2(\hat{\sigma}^1)^2 + B^2(\hat{\sigma}^2)^2 + C^2(\hat{\sigma}^3)^2,
\]

where \(N, A, B\) and \(C\) are functions of time only, and when expressed in the usual Misner anisotropy variables \(\Omega, \beta_+\) and \(\beta_-\),

\[
\begin{aligned}
A &= e^{-\Omega+\beta_++\sqrt{3}\beta_-} \\
B &= e^{-\Omega+\beta_+-\sqrt{3}\beta_-} \\
C &= e^{-\Omega-2\beta_+}
\end{aligned}
\]
IV. THE CONSTANTS OF MOTION AND TRANSFORMED SPACETIMES

As the first step in calculating the constants of motion for the Mixmaster spacetime, we need to calculate the harmonic variables. The function $\lambda$ and one-form $\beta$ can be found simply by expanding the metric (59). The functions $r$ and $p$ can be straightforwardly calculated using (12) and (34), which requires only differentiation. The function $\omega$, on the other hand, is hard to calculate since that involves a line integration in the $(\theta, \phi)$ surface with

$$\lambda = A^2 \sin^2 \phi \sin^2 \theta + B^2 \cos^2 \phi \sin^2 \theta + C^2 \cos^2 \theta$$

(61)

in the denominator. We have not been able to evaluate $\omega$ in closed form, which prevented us from calculating the constant of motion $K_\psi$ exactly. But, even if one succeeded in putting $\omega$ into a complicated closed form it would probably be of little practical importance, since we would still have to evaluate the surface integrals, which seem extremely difficult.

However, the integrals are simple enough when $A = B = C$, i.e., when the anisotropy variables $\beta_+ = \beta_- = 0$, which allowed us to calculate $K_\psi$ perturbatively around the isotropic solution, to the first order in the anisotropy variables. This calculation is tedious and rather unilluminating, so we shall state only the main result:

To the first order in the anisotropy variables $(\beta_+, \beta_-)$, the constant of motion $K_\psi$ for the Mixmaster spacetime is proportional to the minisuperspace Hamiltonian,

$$H = 1/12 \, Ne^{3\Omega} \left[ -p_\Omega^2 + p_+^2 + p_-^2 + e^{-4\Omega}V(\beta_+, \beta_-) \right],$$

$$V(\beta_+, \beta_-) = 3 \left\{ e^{-8\beta_+} - 4e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+} \left[ \cosh(4\sqrt{3}\beta_-) - 1 \right] \right\}.$$  

(62, 63)

In particular, for all Mixmaster solutions whose trajectories in the anisotropy plane at least once pass through the origin, $\beta_+ = \beta_- = 0$, the constant of motion $K_\psi = 0$ exactly.

Is $K_\psi$ exactly proportional to the minisuperspace Hamiltonian? We do not know, but it seems to us that, considering the complexity of the calculations involved, it would be an extraordinary coincidence if the first two terms in the expansion of $K$ coincided with those of $H$, without $K$ and $H$ being essentially equal (proportional). In addition, if all Mixmaster
trajectories passed through the origin the value of $K_\phi$ would have to be exactly equal to zero for all Mixmaster initial data. Regrettably, it is not known how large is the set of Mixmaster trajectories passing through the origin.

Does this indicate that the constant of motion $K$ is proportional to the hamiltonian constraint, i.e., uninteresting and trivial, for all $U(1)$-symmetric spacetimes on $S^3$? Is it weakly zero? Not at all! To see that, we shall calculate $K$ for the Taub-NUT spacetime, using the additional Killing field the general Mixmaster spacetime does not have.

The Taub-NUT spacetime is a special case of the Mixmaster spacetime, obtained by putting $A = B$ in (59). Its equations of motion can be explicitly solved, and give the solution (written here in the right-invariant form, slightly different form [18], where it is given in the left-invariant form):

$$ds^2 = -U^{-1}dt^2 + (t^2 + l^2)[(\dot{\sigma}^1)^2 + (\dot{\sigma}^2)^2] + (2l)2U(\dot{\sigma}^3)^2,$$

$$U = -t^2 + 2mt + l^2 \over t^2 + l^2.$$  \hspace{1cm} (64)

Due to the additional restriction ($A = B$), the Taub-NUT spacetime has four Killing fields; the three left-invariant vector fields inherited from the Mixmaster plus one of the right-invariant vector fields, namely the $\partial_\phi$. Note that $\partial_\phi$, being a right-invariant vector field, commutes with other three Killing fields. This additional Killing field, as mentioned in the previous section, also generates a $U(1)$ group action which induces a Hopf fibration of $S^3$. We can use it (actually 2 $\partial_\phi$ because of the normalization convention we use) to calculate a new constant of motion, which we shall call $K_\phi$. Owing to the simplicity of the Taub-NUT solution all angular integrals can now be easily calculated and the constant of motion turns out to be:

$$K_\phi[Taub] = 16(8\pi)^2(2l)^2(l^2 + m^2).$$ \hspace{1cm} (65)

As we can see, $K_\phi[Taub]$ is a strictly positive, non-trivial function of the initial data ($l$ and $m$) and as such cannot be proportional to the Hamiltonian constraint, which is zero for all solutions of the Einstein equations. More generally, it guarantees that $K_\phi$ is not weakly

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zero. This implies that there exists a neighbourhood $\mathcal{O}$ of the Taub-NUT initial data, open in the set of all $U(1)$-symmetric initial data on $S^3$ with $\partial_\phi$ as a Killing field, in which the constant of motion $K_\phi$ is strictly positive. That constant of motion is local in time as a functional of the usual gravitational initial data—the three-metric and its momentum—and can be written down essentially explicitly. It seems to be the only explicitly known observable for such a large family of Einstein spacetimes.

For the left-invariant Mixmaster metric—which, as opposed to its isometric right-invariant counterpart (59), has $\partial_\phi$ but not $\partial_\psi$ as Killing field—the $K_\phi$ constant of motion can be calculated. (The left-invariant Mixmaster metric can be obtained from (59) by making $A$, $B$ and $C$ appropriate functions of the angles or exchanging $\sigma$-s for $\omega$-s.) Due to isometry of the left- and right-invariant spacetimes, it must be that $K_\phi[\text{Mixmaster}] = K_\psi[\text{Mixmaster}]$. Assuming that $K_\psi[\text{Mixmaster}]$ is indeed proportional to the Hamiltonian constraint, it follows that the Mixmaster spacetime is the special case for which the $U(1)$-generated constant of motion is (weakly) identically zero.

The positivity of $K_\phi[\text{Taub}]$ also implies that Geroch’s transformation can yield a new globally defined spacetime on $S^2 \times S^1$ when applied to Taub-NUT. Since the harmonic variables can be easily calculated in this case, the transformation can be carried out explicitly and we find that the new solution on the trivial bundle turns out to be the Kantowski-Sachs solution. This had already been established by Geroch himself [13], but only locally.

Now a word about the possibility of generating new globally defined solutions of the vacuum Einstein equations by applying Geroch’s transformation to the Mixmaster spacetime. Moncrief’s analysis of the global applicability of Geroch’s transformation [15] guarantees that it is possible to transform a solution from $S^3$, as the bundle with winding number $n = 1$, to a new solution on the trivial bundle $S^2 \times S^1$ if $K \geq 0$. We showed that for all the Mixmaster solutions that pass through the $\beta_+ = \beta_- = 0$ point in the anisotropy plane $K$ is exactly zero; the condition for global applicability of Geroch’s transformation is therefore satisfied and we can obtain a new solution on the trivial bundle. Likewise, if $K$ is exactly zero for all Mixmaster spacetimes, as suggested by our perturbative result, a new solution
globally defined on the trivial bundle can be obtained from any Mixmaster solution.

What properties does that new solution have? It is not a homogeneous solution, for the following reason. If it were homogeneous it would have to be the Kantowski-Sachs solution, which is the only homogeneous solution on $S^2 \times S^1$. But the Kantowski-Sachs constant of motion $K$ associated with the Killing vector field tangent to the $S^1$ factor is strictly positive. It is positive because it must be equal to the Taub-NUT constant $K$, since $K$ is invariant under the action of Geroch’s transformation and the Kantowski-Sachs solution is just the transformed Taub-NUT solution. The same invariance requires that the Kantowski-Sachs $K$, a strictly positive quantity, be equal to the Mixmaster constant $K$, which is identically zero, leading to a contradiction. So, the new solution on the trivial bundle is not homogeneous.

The asymptotic behavior of the new solution requires additional analysis; for now let us just say that it is probably oscillatory in the BKL sense. The BKL condition for the existence of the oscillatory behavior is an “open” condition, i.e., some quantity has to be different from zero. It is satisfied by the Mixmaster solution, hence it has to be satisfied by the transformed solution obtained by using an element of $SL(2, \mathbb{R})$ sufficiently close to unity. Whether the element of $SL(2, \mathbb{R})$ that gives the new globally defined solution on the trivial bundle violates the BKL condition or not, will be the subject of future study.

V. CONCLUSIONS

The constant of motion $K$ we calculated in the previous section is a second degree polynomial in the minisuperspace momenta $(p_0, p_+, p_-)$ and, to the first order in the Misner anisotropy variables $(\beta_+, \beta_-)$, it is proportional to the minisuperspace Hamiltonian. We believe that it is reasonable to conjecture that $K$ is exactly proportional to the minisuperspace Hamiltonian. The coincidental agreement of the first two terms in the expansion of $K$ and $H$, considering the complexity of the expression for the constant of motion $K$, is highly unlikely unless $K$ and $H$ are exactly proportional. It seems that the Mixmaster dynamical system has once more defied the attempt to find a non-trivial constant of motion
that would help us to gain more detailed knowledge of its behavior. Of course, this might just be a consequence of the simple, but not yet established, fact that there are no non-trivial constants of motion and that a generic Mixmaster trajectory fills densely an open subset of the constraint surface in the minisuperspace. Indeed, Kuchař [19] showed that for non-homogeneous spacetimes there are no (weakly) non-zero constants of motion that are linear functionals of the gravitational momenta. Presently K. Schleich and D. Witt [20] are trying to prove that any constant of motion for the Mixmaster dynamical system that is quadratic in minisuperspace momenta—which includes our $K$—has to be proportional to the minisuperspace Hamiltonian, i.e., has to be trivial. Our perturbative result for $K$ supports that.

More importantly, the probable weak vanishing of $K$ for the Mixmaster spacetime does not mean that $K$ is weakly zero for all $U(1)$-symmetric spacetimes. By calculating explicitly the constant of motion $K_{\phi}$ for the Taub-NUT spacetime, we showed that around the Taub-NUT initial data, there exists an open set in the set of all $U(1)$-symmetric initial data for which the constant of motion $K$ is a non-trivial spatially non-local, but temporally local, strictly positive functional of ADM canonical data. That constant of motion could be used as an observable in a $U(1)$-symmetric toy model for the future quantum theory of gravity. Classically, the value of this constant of motion for a particular $U(1)$-symmetric spacetime might signal whether the spacetime is asymptotically velocity-dominated or not. The value of $K$ for the Taub-NUT spacetime is positive and the spacetime is velocity-dominated, on the other hand the value of $K$ for the Mixmaster spacetime, which is not a velocity-dominated spacetime, is presumably zero, which might signal that the $K = 0$ surface in the set of $U(1)$-symmetric initial data is the boundary between the simple velocity dominated behavior and the much more complicated oscillatory BKL-like behavior. The Mixmaster spacetime would then be exactly at the boundary of the set of velocity-dominated spacetimes. Likewise, an interesting and closely related issue we would like to resolve is: Does Geroch’s transformation, which does not change the value of $K$, “conserve” the velocity-dominated behavior. i.e., are new solutions obtained from velocity-dominated solutions also velocity-
dominated?

ACKNOWLEDGMENTS

Both authors are grateful to the Institute for Theoretical Physics in Santa Barbara, where part of this research was carried out, for its support and hospitality. This work was supported in part by NSF Grants PHY-9201196 to Yale University and PHY-890435 to Institute for Theoretical Physics.
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