SINC APPROXIMATION OF ALGEBRAICALLY DECAYING FUNCTIONS

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Abstract. An extension of sinc interpolation on \( \mathbb{R} \) to the class of algebraically decaying functions is developed in the paper. Similarly to the classical sinc interpolation we establish two types of error estimates. First covers a wider class of functions with the algebraic order of decay on \( \mathbb{R} \). The second type of error estimates governs the case when the order of function’s decay can be estimated everywhere in the horizontal strip of complex plane around \( \mathbb{R} \). The numerical examples are provided.

Introduction

We begin by introducing some necessary notation. Let

\[
sinc(x) = \frac{\sin \pi x}{\pi x},
\]

\[
S\{k, h\}(x) = sinc \left( \frac{x}{h} - k \right), \quad h > 0, \ k \in \mathbb{Z}.
\]

(0.1)

By \( H^1(D_d) \) in the paper we denote the class of functions \( f(x) \) analytic in the horizontal strip \( D_d \)

\[
D_d = \{z = x + iy \mid x \in (-\infty, \infty), \ |y| \leq d\},
\]

(0.2)

and such, that the quantity

\[
N_1(f, D_d) \equiv \int_{\partial D_d} \left| f(z) \right| dz,
\]

is bounded. Next, for some given \( h > 0 \) and integer \( N > 0 \) we define a sinc interpolation polynomial as

\[
C_N\{f, h\}(x) = \sum_{k=-N}^{N} f(kh)S\{k, h\}(x).
\]

(0.3)

The following classical result characterize the accuracy of interpolation of \( f \in H^1(D_d) \) by \( C_N\{f, h\}(x) \) for the case, when \( f(s) \) is exponentially decaying.

**Theorem** (Stenger [6, p. 137]) Assume that the function \( f \in H^1(D_d) \) is bounded by

\[
|f(x)| \leq Le^{-\alpha|x|}, \quad \forall x \in \mathbb{R},
\]

(0.4)

with some \( \alpha, L > 0 \). Then the error of \( 2N + 1 \) term sinc interpolation of \( f(x) \) by \( C_N\{f, h\}(x) \), satisfies the following estimate

\[
\sup_{x \in \mathbb{R}} |f(x) - C_N\{f, h\}(x)| \leq cE_N,
\]

\[
E_N = N^{1/2}e^{-\sqrt{\pi d\alpha}N},
\]

(0.5)

\[\text{†} \] Key words: sinc methods, sinc interpolation, algebraically decaying functions, Lambert-W function, polynomial order of convergence, approximation on real-line.
provided that
\[ h = \sqrt{\frac{\pi d}{\alpha N}}. \]  
(0.6)

Here \( c > 0 \) is some constant dependent on \( f, d, \alpha \) and independent on \( N \). In this paper we extend the results of the above theorem to a class of algebraically decaying functions on \( \mathbb{R} \). All theoretical considerations are given in sections 1, 2. Section 3 is devoted to numerical examples and discussion.

1. Interpolation of functions with algebraic decay on real line

In this section we study the convergence of sinc interpolation for the class of algebraically decaying functions. Specifically, we consider the situation when the function \( f(x) \) satisfies
\[ |f(x)| \leq L + |x|^\alpha, \quad \forall x \in \mathbb{R} \]  
(1.1)

instead of inequality (0.4), convenient for the classical sinc methods [6].

\textbf{Theorem 1.1} Assume that the function \( f \in H^1(D_d) \) has an algebraic decay defined by (1.1) with some \( \alpha > 1, L > 0 \). Then the error of 2N + 1-term sinc interpolation (0.3) satisfies the following estimate
\[ \sup_{x \in \mathbb{R}} |f(x) - C_N\{f, h\}(x)| \leq c\mathcal{E}_N, \quad \forall x \in \mathbb{R}, \]  
(1.2)

provided that \( h \) in (0.3) is chosen as
\[ h = \frac{\pi d}{\alpha} \left( \mathcal{W} \left( \frac{\pi d}{\alpha} \left( \frac{\alpha - 1}{\pi d} \right)^\frac{1}{\alpha} (N + 1)^\frac{\alpha - 1}{\alpha} \right) \right)^{-1}. \]  
(1.3)

Here \( \mathcal{W}[\cdot] \) denotes a positive branch of the Lambert-W function, \( c = c_1N_1(f, D_d) + 2L \) and \( c_1 > 1 \) is the constant independent of \( N \):
\[ c_1 = \frac{(\pi d)^2(\alpha - 1)^2}{(\pi d)^2(\alpha - 1)^2 - \alpha^2 \mathcal{W}^2(\frac{\pi d}{\alpha} \sqrt{\frac{\alpha - 1}{\pi d}})} \]  
(1.4)

\textbf{Proof.} For any fixed \( h \) the error of sinc interpolation can be represented as follows [6, equation (3.1.29)]
\[ |f(x) - C_N\{f, h\}(x)| \leq |f(x) - C_{\infty}\{f, h\}(x)| + \sum_{|k|>N} |f(kh)| \]

Bound of the first term on the right-hand side of this formula was obtained in Theorem 3.1.3 from [6]. For \( x \in \mathbb{R} \) this term satisfies
\[ |f(x) - C_{\infty}\{f, h\}(x)| \leq \frac{N_1(f, D_d)}{2\pi d \sinh \frac{\pi d}{h}} \leq \frac{c_1N_1(f, D_d)}{\pi d} \exp \left( -\frac{\pi d}{h} \right), \]  
(1.5)
where \( c_1 > 1 \) is some constant to be determined later. For the second term we get

\[
\sum_{|k|>N} |f(kh)| \leq 2L \sum_{k=N+1}^{\infty} (kh)^{-\alpha} \leq 2L \int_{N+1}^{\infty} (th)^{-\alpha} dt
\]

\[
\leq \frac{2L(N+1)^{1-\alpha}}{(\alpha - 1)h^\alpha}.
\]  

(1.6)

The above sequence of inequalities is justified as long as \( f(x) \) satisfy (1.1) with some \( \alpha > 1 \). For such \( f(x) \), truncation error (1.6) decays algebraically as \( N \to \infty \). In order to balance it with exponentially decaying discretization error (1.5) one needs to solve for \( h \) the equation

\[
e^{-\frac{\pi d}{c_2}} = \frac{(N+1)^{1-\alpha}}{(\alpha - 1)h^\alpha}.
\]  

(1.7)

Let \( s = \frac{\pi d}{c_2}h^{-1} \) and assume that \( c_2 > 0 \) is some fixed parameter. Then, equation (1.7) takes the form

\[
\frac{\pi d}{\alpha} \left( \frac{\alpha - 1}{c_2} (N+1)^{\alpha-1} \right)^{\frac{1}{\alpha}} = se^s,
\]

which has a unique solution

\[
s = \text{W}\left( \frac{\pi d}{\alpha} \left( \frac{\alpha - 1}{c_2} (N+1)^{\alpha-1} \right)^{\frac{1}{\alpha}} \right).
\]

Next, we set \( c_2 = \pi d \) and substitute back the expression for \( s \) in terms of \( h \) to obtain (1.3). The proof of (1.2) is straightforward

\[
|f(x) - C_N\{f, h\}(x)| \leq (c_1N_1(f, D_d) + 2L) \frac{(N+1)^{1-\alpha}}{(\alpha - 1)h^\alpha} \leq \frac{c^\alpha(N+1)^{1-\alpha}}{(\alpha - 1)(\pi d)^\alpha} \left( \text{W}\left( \frac{\pi d}{\alpha} \left( \frac{\alpha - 1}{\pi d} \right)^{\frac{2}{\alpha}} (N+1)^{\frac{1}{\alpha}} \right) \right)^{\alpha}.
\]

Now, let us come back to the determination of \( c_1 \). The smallest \( c_1 \) suitable for (1.5) can be defined as follows

\[
c_1 = \sup_{N \in \mathbb{Z}^+} \left\{ \frac{e^{\pi d}}{2 \sinh \frac{\pi d}{N}} \right\} = \max_{N \in \mathbb{Z}^+} \left( 1 - e^{-\frac{2\pi d}{N}} \right)^{-1}.
\]

Its not hard to see that the maximum is attained at \( N = 0 \). Therefore, the value of \( c_1 \):

\[
c_1 = \left( 1 - \exp \left( -\frac{\pi d}{\alpha} \left( \frac{\alpha - 1}{\pi d} \right)^{\alpha} \right) \right)^{-1}
\]

is clearly greater than one, for any \( \alpha > 1, d > 0 \). To get (1.4) we apply the identity \( \exp(-\text{W}(x)) = \text{W}(x)/x \) to the above formula for \( c_1 \) and rearrange the result accordingly

\[
c_1 = \left( 1 - \frac{\alpha^2}{(\pi d)^2(\alpha - 1)^2} \left( \text{W}\left( \frac{\pi d}{\alpha} \sqrt{\frac{\alpha - 1}{\pi d}} \right) \right)^{2\alpha} \right)^{-1}
\]

\[
= \frac{(\pi d)^2(\alpha - 1)^2}{(\pi d)^2(\alpha - 1)^2 - \alpha^2 \text{W}^{2\alpha} \left( \frac{\pi d}{\alpha} \sqrt{\frac{\alpha - 1}{\pi d}} \right)}
\]

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The presence of $W(x)$ in estimate (1.2) makes it harder to perceive the asymptotic behavior of the interpolation error intuitively. To fix that we recall a well-established result \[5\] on the asymptotic properties of $W(x)$, valid for any $x > 0$:

$$\ln x - \ln (\ln x) + \frac{\ln (\ln x)}{2 \ln x} \leq W(x) \leq \ln x - \ln (\ln x) + \frac{e \ln (\ln x)}{(e - 1) \ln x}.$$  

By using the above inequality along with the definition of $W(x)$ and (1.7) we transform (1.2) in the following way

$$|f(x) - C_N \{f, h\}(x)| \leq \frac{c}{e^{\alpha s}} \leq c \left( \frac{\ln \left( \frac{\pi d}{\alpha} \left( \frac{\alpha - 1}{\alpha c} \right)^{\frac{1}{1}} (N + 1) \frac{\alpha - 1}{\alpha} \right)}{(\pi d)^{1-\frac{1}{\alpha}}} \right)^{1-\alpha} \ln^{\alpha} \left( \frac{\alpha - 1}{\alpha} \right) \ln^{\frac{1}{\alpha}} (N + 1),$$

whence it is clear that the error of sinc interpolation provided by Theorem 1.1 is asymptotically equal to $(N + 1)^{1-\alpha} \ln^{\alpha}(N + 1)$ as $N \to \infty$. To analyze the error for small $N$ we note that, in the view of (1.7), $E_N$ is bounded by the exponent with a strictly decreasing negative argument. Consequently, for any $\alpha > 1$, $x \in \mathbb{R}$, the error $\sup_{x \in \mathbb{R}} |f(x) - C_N \{f, h\}(x)|$ lies within the interval $[0, c]$ and decreases as $N \to \infty$.

One might conclude from the foregoing analysis that a simple asymptotic formula $W(x) \approx \ln(x)$ can be used to redefine $h$ (1.3) in terms of logarithms, which are computationally more favorable than the Lambert-W function. To explore this possibility we set

$$h = \frac{\pi d}{\alpha} \left( \ln \left( \frac{\pi d}{\alpha} \left( \frac{\alpha - 1}{\alpha c} \right)^{\frac{1}{1}} (N + 1) \frac{\alpha - 1}{\alpha} \right) \ln^{\frac{1}{\alpha}} (N + 1) \right)^{-1},$$

and study the corresponding error terms of the approximation. Discretization error (1.5) is positive and monotonically decreasing in $N$ for any $c_2 > 0$, since $h$ is monotonic. The principal part $(N + 1)^{1-\alpha}$ of truncation error (1.6) has one global maximum at $N = N_0$:

$$N_0 = \left( \frac{\alpha}{\pi d} \right) \frac{\alpha}{\alpha - 1} \exp \left( \frac{\alpha}{\alpha - 1} \left( \frac{\alpha - 1}{c_2} \right)^{\alpha - 1} \right) - 1.$$

To guarantee a monotonous decrease of the truncation error for all $N \geq 0$ we must require $N_0 = 0$, which yields $c_2 = (\alpha - 1) \left( \frac{\pi d}{\alpha} \right)^{\alpha}$. The aforementioned formula for $h$ is thereby reduced to

$$h = \frac{\pi d}{\alpha + (\alpha - 1) \ln(N + 1)}.$$  

(1.8)

For such $h$, the error of sinc interpolation will be bounded by (1.2) with

$$E_N = \frac{(N + 1)^{1-\alpha}}{(\alpha - 1)(\pi d)^{\alpha}} \left( \alpha + (\alpha - 1) \ln(N + 1) \right)^{\alpha}.$$

(1.9)

and $c = (\alpha - 1) \left( \frac{\pi d}{\alpha} \right)^{\alpha} N_1(f, D_d) + 2L$. The main concern with (1.9), is the presence of additional summand $c$ when compared to (1.2).

**Remark 1.2** The definition of $h$ from Theorem 1.1 can not be simplified by adopting $W(x) \approx \ln(x)$, since such simplification, as described by (1.8), (1.9), would make the approximation method ineffective for large $\alpha$. 

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With an additional a priori knowledge about $f(x)$ we should be able to improve the convergence properties of $C_N\{f, h\}(x)$ described by Theorem 1.1. The following improvement of (1.2) offers a more realistic balance of discretization and truncation errors, presuming that both $N_1(f, D_d)$ and $L$ are known.

**Corollary 1.3** Assume that the function $f(x)$ satisfies the conditions of Theorem 1.1. If

$$h = \pi d \left( W \left( \frac{\pi d}{\alpha} \left( \frac{N_1(f, D_d)(\alpha - 1)}{\pi d L} \right)^{\frac{1}{\alpha}} \right) \right)^{-1},$$

(1.10)

then the error of sinc interpolation fulfills estimate (1.2), with $c = (c_1 + 2)L$ and $E_N$ given by

$$E_N = \frac{(N + 1)^{1-\alpha}}{(\alpha - 1)} h^{-\alpha}.$$  

Formula (1.10) was obtained in the same way as (1.3), except this time we set $c_2 = \pi d L / N_1(f, D_d)$.

### 2. Interpolation of functions with algebraic decay in the strip

Corollary 1.3 is difficult to apply as it is, because the evaluation of $N_1(f, D_d)$ requires computation of the contour integral over $\partial D_d$. In order to make this result more applicable we note, that if $f \in H^1(D_r)$, for some $r > 0$, then $\lim_{x \to \pm \infty} f(x + iy) = 0$ uniformly with respect to $y \in [d, d]$, for all $d \in (0, r)$ [2] Proposition 6. Hence, for any $r > 0$ there exist a nonempty subspace of $H^1(D_r)$, such that its elements $f$ satisfy

$$|f(z)| \leq \frac{L}{1 + |z|^\alpha}, \quad \forall z \in D_d,$$

(2.1)

with some $d \in (0, r)$.

**Theorem 2.1** Assume that the function $f(z)$ is analytic in the horizontal strip $D_d$, $d > 0$. If $f(z)$ is bounded by (2.1) with some $\alpha > 1$, $L > 0$, then the error of sinc interpolation (0.3) satisfies the following estimate

$$\sup_{x \in \mathbb{R}} |f(x) - C_N\{f, h\}(x)| \leq cE_N,$$

(2.2)

provided that

$$h = \pi d \left( W \left( \frac{\pi d}{\alpha} \left( \frac{2(\alpha - 1)}{\pi d} \right)^{\frac{1}{\alpha}} \right) \right)^{-1},$$

(2.3)

with $\beta = \min \left\{ \frac{2}{\sin(\alpha)}, \left( \frac{\alpha}{2} \right)^{\alpha-1} B \left( \frac{\alpha}{2}, 1, \frac{\alpha}{2} + \frac{1}{2} \right) \right\}$. Here $B(\cdot, \cdot)$ is the beta function, $c = c_1 L$ and $c_1$ is the constant dependent on $\alpha, d$. 

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Proof. For small values of $d$ we proceed as follows

\begin{equation}
\int_{-\infty}^{+\infty} |f(x+id)| \, dx \leq \int_{-\infty}^{+\infty} \frac{Ldx}{1+|x+id|^\alpha} = 2L \int_{0}^{+\infty} \frac{dx}{1+(x^2+d^2)^{\alpha/2}} \tag{2.4}
\end{equation}

\begin{equation}
\int_{0}^{+\infty} \frac{dx}{1+(x^2+d^2)^{\alpha/2}} \leq \lim_{x \to +\infty} \frac{x \Phi(-x^n,1,\alpha^{-1})}{\alpha}.
\end{equation}

Here $\Re z$ and $\Im z$ is real and imaginary part of $z$ correspondingly. To evaluate the last limit we employ Corollary 1 from [3]. It offers a convergent expansion of the Hurwitz-Lerch zeta function $\Phi(z, s, a)$, when its second parameter $s$ is an integer number.

\begin{equation}
z \Phi \left( z^{\alpha}, 1, \frac{1}{\alpha} \right) = \pi \left( \text{sgn} \{ \arg(\alpha \ln(z)) \} i + \cot \frac{\pi}{\alpha} \right) - \sum_{k=1}^{\infty} \frac{z^{1-\alpha k}}{1/\alpha - k}. \tag{2.5}
\end{equation}

The expression on the right of (2.5) is bounded and uniformly convergent to the left-hand side for any $\alpha > 1$, $|z| > 1$, such that $z^{\alpha} \notin (-\infty, -1) \cup (1, \infty)$. Therefore

\begin{equation}
\lim_{\Re z \to +\infty} \left| \frac{z \Phi(-z^n,1,\alpha^{-1})}{\alpha} \right| = \frac{\pi}{\alpha} \sqrt{1 + \cot^2 \frac{\pi}{\alpha}} - \frac{1}{\alpha} \sum_{k=1}^{\Re z \to +\infty} \frac{z^{1-\alpha k}}{1/\alpha - k},
\end{equation}

which leads us to the bound

\begin{equation}
\int_{-\infty}^{+\infty} |f(x+id)| \, dx \leq \frac{2\pi L}{\alpha} \sqrt{1 + \cot^2 \frac{\pi}{\alpha}} = 2L \sin^{-1} \left( \frac{1}{\alpha} \right). \tag{2.6}
\end{equation}

For large $d$, the integral from (2.4) can be estimated as follows

\begin{equation}
\int_{0}^{+\infty} \frac{1}{1+(x^2+d^2)^{\alpha/2}} \, dx \leq \int_{0}^{+\infty} \frac{1}{(x^2+d^2)^{\alpha/2}} \, dx = \sqrt{\pi} d^{1-\alpha} \Gamma \left( \frac{(\alpha-1)/2}{2} \right) 2\Gamma \left( \frac{\alpha}{2} \right) = \frac{d^{1-\alpha} \Gamma \left( \frac{(\alpha-1)/2}{2} \right) \Gamma \left( \frac{(\alpha+1)/2}{2} \right)}{2^{2-\alpha} \Gamma(\alpha)} \leq \frac{1}{2} B \left( \frac{\alpha}{2} - \frac{1}{2} \frac{\alpha}{2} + \frac{1}{2} \right) \left( \frac{2}{d} \right)^{\alpha-1}.
\end{equation}

To obtain the above estimate we used a well-known multiplication theorem [1, p. 4] for Gamma function $\Gamma(\cdot)$. The next bound is a direct consequence of the above formula and (2.6)

\begin{equation}
\int_{-\infty}^{+\infty} |f(x+id)| \, dx \leq LB \left( \frac{\alpha}{2} - \frac{1}{2} \frac{\alpha}{2} + \frac{1}{2} \right) \left( \frac{2}{d} \right)^{\alpha-1} \tag{2.7}
\end{equation}
By combining bounds (2.6), (2.7) and taking into account the fact that the expression on the right of (2.1) is invariant with respect to \( z \rightarrow \overline{z} \) we arrive at the following estimate

\[
N_1(f, D_d) \leq 2L \min \left\{ \frac{2}{\text{sinc} \left( \frac{\alpha}{2} \right)}, \left( \frac{2}{d} \right)^{\alpha-1} B \left( \frac{\alpha}{2}, \frac{\alpha}{2} + \frac{1}{2} \right) \right\}.
\]

To finalize the proof, we evaluate (1.10) assuming that the value of \( N_1(f, D_d) \) is equal to its estimate provided by the previous formula. This will get us (2.3).

3. Examples and discussion

In this section we consider several examples of the developed approximation method. As a measure of experimental error we use a discrete norm

\[
\text{err} = \max_{x \in X} |f(x) - C_N\{f, h\}(x)|,
\]

defined on a uniform grid \( X = \{jh/2 \mid j = -2N, 2N\} \). With such choice of \( X \) the specified discrete norm ought to capture the contribution from both the discretization and truncation parts of the error. To experimentally check the convergence of \( C_N\{f, h\}(x) \) we repeat the approximation procedure on a sequence of grids determined by

\[ N_i \in \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\}, \]

and the corresponding \( h_i \) evaluated by one of the formulas (1.3), (1.10) or (2.3).

Example 3.1 Let

\[
f(x) = \frac{4}{2 + x^{2a}}.
\]

where \( a \geq 2 \) is integer. Then, the largest possible value of \( d \) such that \( f(x) \) remains analytic in \( D_d \), is equal to \( \frac{\sqrt{2}}{2} \sin \frac{\pi}{2a} \). For the purpose of the illustration we set \( d = 0.9 \sqrt{2} \sin \frac{\pi}{2a}, \ a = 2 \), then \( N_1(f, D_d) \approx 17.05467564, L \approx 4, \alpha = 4 \). The behaviour of an error \( \text{err}(x) = f(x) - C_{32}\{f, h\}(x) \) for the values of \( h \), calculated by three different formulas (1.3), (1.10), (2.3), is depicted in Fig. 1. Predictably, the maximum of \( \text{err}(x) \)

\[
\begin{align*}
\text{Fig. 1. Graphs of } \text{err}(x) = f(x) - C_{32}\{f, h\}(x) \text{ from Example 3.1 for } h \text{ calculated by (1.3) – left graph, (1.10) – central graph and (2.3) – graph on the right.}
\end{align*}
\]

for \( h = 0.3149022805 \) calculated by (1.10) (see central plot from Fig. 1) is superior to the error with \( h = 0.3589479879 \) calculated by (1.3) (left plot from Fig. 1). The value
of \( h \) calculated by (2.3) is close to the one obtained from (1.10), that is why the error function \( \text{err}(x) \) (see plot on the right from Fig. 1) is close to \( \text{err}(x) \) obtained with help of (1.10). One can see a discernible spike in the error function from central plot of Fig. 1 at \( x_0 = N_0 h \approx 10.0769 \). The values of \( \text{err}(x) \) on the left of \( x_0 \) corresponds to the discretization error, whilst the values on the right of \( x_0 \) corresponds to the truncation error. The magnitude of those errors almost match. This highlight the fact that the chosen \( h \) is quite close to the theoretically optimal value.

**Example 3.2** In this example we set \( f(x) \in H^1(D_d) \) as

\[
f(x) = \frac{6 \cos 2x}{(5 + \cos^2 x)(1 + x^4)},
\]

and choose formula (1.3) for the evaluation of \( h \). The function \( f(x) \) is meromorphic and bounded in \( D_d \) for any \( d \) smaller than the imaginary part of zeros of \( (5 + \cos^2 x)(1 + x^4) \). The zeros of the polynomial part of this expression lie closer to the real line than any zero of \( 5 + \cos^2 x \), so \( d \leq \sqrt{-1} = \frac{\sqrt{2}}{2} \approx .707106781186550 \). Therefore it is safe to set \( d = 0.7 \). For given \( f(x) \) we can also explicitly find the parameters of algebraic decay bound (1.1): \( L = f(0) = 1 \), \( \alpha = 4 \).

Note, that for a more general function \( f(x) \) the corresponding \( L, \alpha \) can be calculated numerically from a sequence of its values. For explicitly given \( f(x) \) the possible values of \( d \) can be calculated numerically as well, for example using Analytic routine from Maple [4].

The graphs of the approximated function \( f(x) \) and the error of its interpolation by \( C_{32}\{f,h\}(x) \) are given in Fig. 2. The precise values of \( \text{err}_i \) for \( i = 1, \ldots, 11 \) are presented in Table 1. Here we additionally supply the theoretical estimate \( E_{N_i} \) defined in Theorem 1.1 and the value of \( c_i = \text{err}_i/E_{N_i} \).

The data from in Table 1 demonstrates that the approximation method presented by Theorem 1.1 converges to \( f(x) \). The magnitude of the observed approximation errors are consistent with the estimate provided by (1.2). Moreover the estimated value of \( c \) from (1.2) remains bounded by 2.1 for all \( i = 1,6 \). All this prove the effectiveness of the developed method.
Table 1. Result of the numerical experiments for $f(x)$ from Example 3.2. The step size $h$ is calculated by (1.3), the quantities $E_N$ and $c$ are evaluated with help of (1.2).

| $i$ | $N_i$ | $\text{err}_i$ | $E_N$ | $c_i$ |
|-----|------|----------------|-------|-------|
| 1   | 2    | 6.373770E-02  | 3.641222E-02 | 1.750448 |
| 2   | 4    | 4.011175E-02  | 1.904281E-02 | 2.106399 |
| 3   | 8    | 1.019463E-02  | 8.186076E-03 | 1.245362 |
| 4   | 16   | 3.765622E-03  | 2.948999E-03 | 1.279151 |
| 5   | 32   | 1.368552E-03  | 9.160491E-04 | 1.493972 |
| 6   | 64   | 1.777309E-04  | 2.523604E-04 | 0.704274 |
| 7   | 128  | 7.216260E-05  | 6.312895E-05 | 1.143098 |
| 8   | 256  | 7.698800E-06  | 1.460731E-05 | 0.527051 |
| 9   | 512  | 2.505400E-06  | 3.171023E-06 | 0.790092 |
| 10  | 1024 | 3.281000E-07  | 6.528835E-07 | 0.502540 |

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