On differences between DP-coloring and list coloring

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Abstract

DP-coloring (also known as correspondence coloring) is a generalization of list coloring introduced recently by Dvořák and Postle [12]. Many known upper bounds for the list-chromatic number extend to the DP-chromatic number, but not all of them do. In this note we describe some properties of DP-coloring that set it aside from list coloring. In particular, we give an example of a planar bipartite graph with DP-chromatic number 4 and prove that the edge-DP-chromatic number of a $d$-regular graph with $d \geq 2$ is always at least $d + 1$.

1 Introduction

1.1 Basic notation and conventions

We use $\mathbb{N}$ to denote the set of all nonnegative integers. For a set $S$, Pow($S$) denotes the power set of $S$, i.e., the set of all subsets of $S$. All graphs considered here are finite, undirected, and simple, except in Section 4, which mentions (loopless) multigraphs. For a graph $G$, $V(G)$ and $E(G)$ denote the vertex and the edge sets of $G$ respectively. For a subset $U \subseteq V(G)$, $G[U]$ is the subgraph of $G$ induced by $U$. For two subsets $U_1, U_2 \subseteq V(G)$, $E_G(U_1, U_2) \subseteq E(G)$ is the set of all edges of $G$ with one endpoint in $U_1$ and the other one in $U_2$. The maximum degree of $G$ is denoted by $\Delta(G)$.

1.2 Graph coloring, list coloring, and DP-coloring

Recall that a proper coloring of a graph $G$ is a function $f: V(G) \to C$, where $C$ is a set of colors, such that $f(u) \neq f(v)$ for each edge $uv \in E(G)$. The chromatic number $\chi(G)$ of $G$ is the smallest $k \in \mathbb{N}$ such that there exists a proper coloring $f: V(G) \to C$ with $|C| = k$.

List coloring is a generalization of ordinary graph coloring that was introduced independently by Vizing [23] and Erdős, Rubin, and Taylor [13]. As in the case of ordinary graph coloring, let $C$ be a set of colors. A list assignment for a graph $G$ is a function $L: V(G) \to \text{Pow}(C)$; if $|L(u)| = k$ for all $u \in V(G)$, then $L$ is called a $k$-list assignment. A proper coloring $f: V(G) \to C$ is called an $L$-coloring if $f(u) \in L(u)$ for each $u \in V(G)$. The list-chromatic number $\chi_L(G)$ of $G$ is the smallest $k \in \mathbb{N}$ such that $G$ admits an $L$-coloring for every $k$-list assignment $L$ for $G$. An immediate consequence of this definition is that $\chi_L(G) \geq \chi(G)$ for all graphs $G$, since ordinary coloring is the same as $L$-coloring with $L(u) = C$ for all $u \in V(G)$. On the other hand, it is well-known

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that the gap between $\chi(G)$ and $\chi_\ell(G)$ can be arbitrarily large; for instance, $\chi(K_{n,n}) = 2$, while $\chi_\ell(K_{n,n}) = (1 + o(1)) \log_2(n) \to \infty$ as $n \to \infty$, where $K_{n,n}$ denotes the complete bipartite graph with both parts having size $n$.

In this paper we study a further generalization of list coloring that was recently introduced by Dvořák and Postle [12]; they called it correspondence coloring, and we call it DP-coloring for short. In the setting of DP-coloring, not only does each vertex get its own list of available colors, but also the identifications between the colors in the lists can vary from edge to edge.

**Definition 1.1.** Let $G$ be a graph. A *cover* of $G$ is a pair $\mathcal{H} = (L, H)$, consisting of a graph $H$ and a function $L : V(G) \to \text{Pow}(V(H))$, satisfying the following requirements:

- (C1) the sets $\{L(u) : u \in V(G)\}$ form a partition of $V(H)$;
- (C2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;
- (C3) if $E_H(L(u), L(v)) \neq \emptyset$, then either $u = v$ or $uv \in E(G)$;
- (C4) if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching.

A cover $\mathcal{H} = (L, H)$ of $G$ is $k$-fold if $|L(u)| = k$ for all $u \in V(G)$.

**Remark 1.2.** The matching $E_H(L(u), L(v))$ in Definition 1.1(C4) does not have to be perfect and, in particular, is allowed to be empty.

**Definition 1.3.** Let $G$ be a graph and let $\mathcal{H} = (L, H)$ be a cover of $G$. An $\mathcal{H}$-coloring of $G$ is an independent set in $H$ of size $|V(G)|$.

**Remark 1.4.** By definition, if $\mathcal{H} = (L, H)$ is a cover of $G$, then $\{L(u) : u \in V(G)\}$ is a partition of $H$ into $|V(G)|$ cliques. Therefore, an independent set $I \subseteq V(H)$ is an $\mathcal{H}$-coloring of $G$ if and only if $|I \cap L(u)| = 1$ for all $u \in V(G)$.

**Definition 1.5.** Let $G$ be a graph. The *DP-chromatic number* $\chi_{DP}(G)$ of $G$ is the smallest $k \in \mathbb{N}$ such that $G$ admits an $\mathcal{H}$-coloring for every $k$-fold cover $\mathcal{H}$ of $G$.

**Example 1.6.** Figure 1 shows two distinct 2-fold covers of the 4-cycle $C_4$. Note that $C_4$ admits an $\mathcal{H}_1$-coloring but not an $\mathcal{H}_2$-coloring. In particular, $\chi_{DP}(C_4) \geq 3$; on the other hand, it can be easily seen that $\chi_{DP}(G) \leq \Delta(G) + 1$ for any graph $G$, and so we have $\chi_{DP}(C_4) = 3$. A similar argument demonstrates that $\chi_{DP}(C_n) = 3$ for any cycle $C_n$ of length $n \geq 3$.

![Figure 1: Two distinct 2-fold covers of a 4-cycle.](image)

One can construct a cover of a graph $G$ based on a list assignment for $G$, thus showing that list coloring is a special case of DP-coloring and, in particular, $\chi_{DP}(G) \geq \chi_\ell(G)$ for all graphs $G$. 

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More precisely, let $G$ be a graph and suppose that $L : V(G) \to \text{Pow}(C)$ is a list assignment for $G$, where $C$ is a set of colors. Let $H$ be the graph with vertex set

$$V(H) := \{(u, c) : u \in V(G) \text{ and } c \in L(u)\},$$

in which two distinct vertices $(u, c)$ and $(v, d)$ are adjacent if and only if

- either $u = v$,
- or else, $uv \in E(G)$ and $c = d$.

For each $u \in V(G)$, set

$$L'(u) := \{(u, c) : c \in L(u)\}.$$

Then $\mathcal{H} := (L', H)$ is a cover of $G$, and there is a natural bijective correspondence between the $L$-colorings and the $\mathcal{H}$-colorings of $G$. Indeed, if $f : V(G) \to C$ is an $L$-coloring of $G$, then the set

$$I_f := \{(u, f(u)) : u \in V(G)\}$$

is an $\mathcal{H}$-coloring of $G$. Conversely, given an $\mathcal{H}$-coloring $I \subseteq V(H)$ of $G$, $|I \cap L'(u)| = 1$ for all $u \in V(G)$, so one can define an $L$-coloring $f_I : V(G) \to C$ by the property

$$(u, f_I(u)) \in I \cap L'(u)$$

for all $u \in V(G)$.

### 1.3 DP-coloring vs. list coloring and the results of this note

Some upper bounds on list-chromatic number hold for DP-chromatic number as well. For instance, it is easy to see that $\chi_{DP}(G) \leq d + 1$ for any $d$-degenerate graph $G$. Dvořák and Postle [12] observed that for any planar graph $G$, $\chi_{DP}(G) \leq 5$ and, moreover, $\chi_{DP}(G) \leq 3$ if $G$ is a planar graph of girth at least 5 (these statements are extensions of classical results of Thomassen [20, 21] on list colorings).

Furthermore, there are statements about list coloring whose only known proofs involve DP-coloring in essential ways. For example, the reason why Dvořák and Postle originally introduced DP-coloring was to prove that every planar graph without cycles of lengths 4 to 8 is 3-list-colorable [12, Theorem 1], thus answering a long-standing question of Borodin [7, Problem 8.1]. Another example can be found in [5], where Dirac's theorem on the minimum number of edges in critical graphs [10, 11] is extended to the framework of DP-colorings, yielding a solution to the problem, posed by Kostochka and Stiebitz [17], of classifying list-critical graphs that satisfy Dirac's bound with equality.
On the other hand, DP-coloring and list coloring are also strikingly different in some respects. For instance, Bernshteyn [3, Theorem 1.6] showed that the DP-chromatic number of every graph with average degree $d$ is $\Omega(d / \log d)$, i.e., close to linear in $d$. Recall that due to a celebrated result of Alon [2], the list-chromatic number of such graphs is $\Omega(\log d)$, and this bound is sharp for “small” bipartite graphs. In spite of this, known upper bounds on list-chromatic numbers often have the same order of magnitude as in the DP-coloring setting. For example, by Johansson’s theorem [16], triangle-free graphs $G$ of maximum degree $\Delta$ satisfy $\chi'_\ell(G) = O(\Delta / \log \Delta)$. The same asymptotic upper bound holds for $\chi_{DP}(G)$ [3, Theorem 1.7]. Recently, Molloy [18] refined Johansson’s result to $\chi'_\ell(G) \leq (1 + o(1))\Delta / \ln \Delta$, and this improved bound, including the constant factor, also generalizes to DP-colorings [4].

Important tools in the study of list coloring that do not generalize to the framework of DP-coloring are the orientation theorems of Alon and Tarsi [1] and the closely related Bondy–Boppana–Siegel lemma (see [1]). Indeed, they can be used to prove that even cycles are 2-list-colorable, while the DP-chromatic number of any cycle is 3, regardless of its length (see Example 1.6). In this note we demonstrate the failure in the context of DP-coloring of two other list-coloring results whose proofs rely on either the Alon–Tarsi method or the Bondy–Boppana–Siegel lemma.

A well-known application of the orientation method is the following result:

**Theorem 1.7** (Alon–Tarsi [1, Corollary 3.4]). Every planar bipartite graph is 3-list-colorable.

We show that Theorem 1.7 does not hold for DP-colorings (note that every planar triangle-free graph is 3-degenerate, hence 4-DP-colorable):

**Theorem 1.8.** There exists a planar bipartite graph $G$ with $\chi_{DP}(G) = 4$.

This answers a question of Grytczuk (personal communication, 2016). We prove Theorem 1.8 in Section 2.

Our second result concerns edge colorings. Recall that the line graph $\text{Line}(G)$ of a graph $G$ is the graph with vertex set $E(G)$ such that two vertices of $\text{Line}(G)$ are adjacent if and only if the corresponding edges of $G$ share an endpoint. The chromatic number, the list-chromatic number, and the DP-chromatic number of $\text{Line}(G)$ are called the chromatic index, the list-chromatic index, and the DP-chromatic index of $G$ and are denoted by $\chi'(G)$, $\chi'_\ell(G)$, and $\chi'_{DP}(G)$ respectively. The following hypothesis is known as the Edge List Coloring Conjecture and is a major open problem in graph theory:

**Conjecture 1.9** (Edge List Coloring Conjecture, see [15]). For every graph $G$, $\chi'_\ell(G) = \chi'(G)$.

In an elegant application of the orientation method, Galvin [14] verified the Edge List Coloring Conjecture for bipartite graphs:

**Theorem 1.10** (Galvin [14]). For every bipartite graph $G$, $\chi'_\ell(G) = \chi'(G) = \Delta(G)$.

We show that this famous result fails for DP-coloring; in fact, it is impossible for a $d$-regular graph $G$ with $d \geq 2$ to have DP-chromatic index $d$:

**Theorem 1.11.** If $d \geq 2$, then every $d$-regular graph $G$ satisfies $\chi'_{DP}(G) \geq d + 1$.

We prove Theorem 1.11 in Section 3.

Vizing [22] proved that the inequality $\chi'(G) \leq \Delta(G) + 1$ holds for all graphs $G$. He also conjectured the following weakening of the Edge List Coloring Conjecture:

**Conjecture 1.12** (Vizing). For every graph $G$, $\chi'_\ell(G) \leq \Delta(G) + 1$. 


We do not know if Conjecture 1.12 can be extended to DP-colorings:

**Problem 1.13.** Do there exist graphs $G$ with $\chi'_{DP}(G) \geq \Delta(G) + 2$?

In Section 4 we discuss two natural ways to define edge-DP-colorings for multigraphs. According to one of them, the DP-chromatic index of the multigraph $K^d_2$ with two vertices joined by $d$ parallel edges is $2d$.

## 2 Proof of Theorem 1.8

In this section we construct a planar bipartite graph $G$ with DP-chromatic number 4. The main building block of our construction is the graph $Q$ shown in Figure 3 on the left, i.e., the skeleton of the 3-dimensional cube. Let $\mathcal{F} = (L, F)$ denote the cover of $Q$ shown in Figure 3 on the right.

![Figure 3: The graph $Q$ (left) and its cover $\mathcal{F}$ (right).](image)

**Lemma 2.1.** The graph $Q$ is not $\mathcal{F}$-colorable.

**Proof.** Suppose, towards a contradiction, that $I$ is an $\mathcal{F}$-coloring of $Q$. Since $L(a) = \{x\}$, we have $x \in I$, and, similarly, $y \in I$. Since $z_1$ is the only vertex in $L(c_1)$ that is not adjacent to $x$ or $y$, we also have $z_1 \in I$, and, similarly, $z_2 \in I$. This leaves only 2 vertices available in each of $L(d_1)$, $L(d_2)$, $L(d_3)$, and $L(d_4)$, and it is easy to see that these 8 vertices do not contain an independent set of size 4 (cf. the cover $\mathcal{H}_2$ of the 4-cycle shown in Figure 1 on the right). \[\Box\]

Consider 9 pairwise disjoint copies of $Q$, labeled $Q_{ij}$ for $1 \leq i, j \leq 3$. For each vertex $u \in V(Q)$, its copy in $Q_{ij}$ is denoted by $u_{ij}$. Let $\mathcal{F}_{ij} = (L_{ij}, F_{ij})$ be a cover of $Q_{ij}$ isomorphic to $\mathcal{F}$. Again, we assume that the graphs $F_{ij}$ are pairwise disjoint and use $u_{ij}$ to denote the copy of a vertex $u \in V(F)$ in $F_{ij}$. Let $G$ be the graph obtained from the (disjoint) union of the graphs $Q_{ij}$ by identifying the vertices $a_{11}$, ..., $a_{33}$ to a new vertex $a^*$ and the vertices $b_{11}$, ..., $b_{33}$ to a new vertex $b^*$. Let $H$ be the graph obtained from the union of the graphs $F_{ij}$ by identifying, for each $1 \leq i, j \leq 3$, the...
vertices $x_{i1}$, $x_{i2}$, $x_{i3}$ to a new vertex $x_i$ and the vertices $y_{ij}$, $y_{2j}$, $y_{3j}$ to a new vertex $y_j$. Define the map $L^*: V(G) \to \text{Pow}(V(H))$ as follows:

$$L^*(u) := \begin{cases} 
L_{ij}(u) & \text{if } u \in V(Q_{ij}); \\
\{x_1, x_2, x_3\} & \text{if } u = a^*; \\
\{y_1, y_2, y_3\} & \text{if } u = b^*.
\end{cases}$$

Then $\mathcal{H} := (L^*, H)$ is a 3-fold cover of $G$. We claim that $G$ is not $\mathcal{H}$-colorable. Indeed, suppose that $I$ is an $\mathcal{H}$-coloring of $G$ and let $i$ and $j$ be the indices such that $\{x_i, y_j\} \subset I$. Then $I$ induces an $\mathcal{F}_{ij}$-coloring of $Q_{ij}$, which cannot exist by Lemma 2.1. Since $G$ is evidently planar and bipartite, the proof of Theorem 1.8 is complete.

## 3 Proof of Theorem 1.11

Let $d \geq 2$ and let $G$ be an $n$-vertex $d$-regular graph. If $\chi'(G) = d + 1$, then $\chi'_{DP}(G) \geq d + 1$ as well, so from now on we will assume that $\chi'(G) = d$. In particular, $n$ is even. Indeed, a proper coloring of $\text{Line}(G)$ is the same as a partition of $E(G)$ into matchings, and if $n$ is odd, then $d$ matchings can cover at most $d(n - 1)/2 < dn/2 = |E(G)|$ edges of $G$.

Let $uv \in E(G)$ and let $G^u := G - uv$. Our argument hinges on the following simple observation:

**Lemma 3.1.** Let $C$ be a set of size $d$ and let $f : E(G') \to C$ be a proper coloring of $\text{Line}(G')$. For each $w \in \{u, v\}$, let $f_w$ denote the unique color in $C$ not used in coloring the edges incident to $w$. Then $f_w = f_{\bar{w}}$.

**Proof.** For each $c \in C$, let $M_c \subseteq E(G')$ denote the matching formed by the edges $e$ with $f(e) = c$. Then $|M_c| \leq n/2$ for all $c \in C$. Moreover, by definition, $\max\{|M_w|, |M_{\bar{w}}|\} \leq n/2 - 1$. Thus, if $f_w \neq f_{\bar{w}}$, then

$$\frac{dn}{2} - 1 = |E(G')| = \sum_{c \in C} |M_c| \leq \frac{dn}{2} - 2;$$

a contradiction. \hfill \blacksquare

Let $\mathbb{Z}_d$ denote the additive group of integers modulo $d$ and let $H$ be the graph with vertex set

$$V(H) := E(G) \times \mathbb{Z}_d,$$

in which the following pairs of vertices are adjacent:

- $(e, i)$ and $(e, j)$ for $e \in E(G)$ and $i, j \in \mathbb{Z}_d$ with $i \neq j$,
- $(e, i)$ and $(h, i)$ for $eh \in E(\text{Line}(G'))$ and $i \in \mathbb{Z}_d$,
- $(uv, i)$ and $(u', i)$ for $uv \in E(G')$ and $i \in \mathbb{Z}_d$,
- $(uv, i)$ and $(u'v, i + 1)$ for $u'v \in E(G')$ and $i \in \mathbb{Z}_d$.

For each $e \in E(G)$, let $L(e) := \{e\} \times \mathbb{Z}_d$. Then $\mathcal{H} := (L, H)$ is a $d$-fold cover of $\text{Line}(G)$. We claim that $\text{Line}(G)$ is not $\mathcal{H}$-colorable (which proves Theorem 1.11). Indeed, suppose that $I$ is an $\mathcal{H}$-coloring of $\text{Line}(G)$. For each $e \in E(G')$, let $f(e)$ denote the unique element of $\mathbb{Z}_d$ such that $(e, f(e)) \in I$. Then $f$ is a proper coloring of $\text{Line}(G')$ with $\mathbb{Z}_d$ as its set of colors. Let $f_w$ be the
unique element of $Z_d$ that is not used in coloring the edges incident to $u$. Then the only element of $L(uv)$ that can, and therefore must, belong to $I$ is $(uv, i)$. On the other hand, Lemma 3.1 implies that $i$ is also the unique element of $Z_d$ that is not used in coloring the edges incident to $v$, and, in particular, for some $u'v \in E(G')$, $f(u'v) = i + 1$. Since $(uv, i)$ and $(u'v, i + 1)$ are adjacent vertices of $H$, $I$ is not an independent set, which is a contradiction.

4 Edge-DP-colorings of multigraphs

One can extend the notion of DP-coloring to loopless multigraphs, see [6]. The definitions are almost identical; the only difference is that in Definition 1.1, (C4) is replaced by the following:

(C4′) If $u$ and $v$ are connected by $t \geq 1$ edges in $G$, then $E_H(L(u), L(v))$ is a union of $t$ matchings.

An interesting property of DP-coloring of multigraphs is that the DP-chromatic number of a multigraph may be larger than its number of vertices. For example, the multigraph $K^t_k$ obtained from the complete graph $K_k$ by replacing each edge with $t$ parallel edges satisfies $\chi_{DP}(K^t_k) = \Delta(K^t_k) + 1 = tk - t + 1.$ (See [6, Lemma 7].)

Similarly to the case of simple graphs, the line graph $\text{Line}(G)$ of a multigraph $G$ is the graph with vertex set $E(G)$ such that two vertices of $\text{Line}(G)$ are adjacent if and only if the corresponding edges of $G$ share at least one endpoint. Notice that, in particular, $\text{Line}(G)$ is always a simple graph. Sometimes, instead of $\text{Line}(G)$, it is more natural to consider the line multigraph $\text{MLine}(G)$, where if two edges of $G$ share both endpoints, then the corresponding vertices of $\text{MLine}(G)$ are joined by a pair edges. Line multigraphs were used, e.g., in the seminal paper by Galvin [14] and also in [8, 9].

Somewhat surprisingly, Shannon’s bound $\chi'(G) \leq 3\Delta(G)/2$ [19] on the chromatic index of a multigraph $G$ does not extend to $\chi_{DP}(\text{MLine}(G))$. Indeed, if $G \cong K^d_2$, i.e., if $G$ is the 2-vertex multigraph with $d$ parallel edges, then $\text{MLine}(G) \cong K^d_2$, so

$$\chi_{DP}(\text{MLine}(G)) = \chi_{DP}(K^d_2) = 2d - 1 = 2\Delta(G) - 1.$$ 

This is in contrast with the result in [8] that $\chi'_G(G) \leq 3\Delta(G)/2$ for every multigraph $G$. However, we conjecture that the analog of Shannon’s theorem holds for line graphs:

**Conjecture 4.1.** For every multigraph $G$, $\chi_{DP}(\text{Line}(G)) \leq 3\Delta(G)/2$. 

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