Analysis of $F(R,T)$ Gravity Models Through Energy Conditions

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Abstract

This paper is devoted to study the energy conditions in $F(R,T)$ gravity for FRW universe with perfect fluid, where $R$ is the Ricci scalar and $T$ is the torsion scalar. We construct the general energy conditions in this theory and reduce them in $F(R)$ as well as $F(T)$ theory of gravity. Further, we assume some viable models and investigate bounds on their constant parameters to satisfy the energy condition inequalities. We also plot some of the cases using present-day values of the cosmological parameters. It is interesting to mention here that the model $F(R,T) = \mu R + \nu T$ satisfies the energy conditions for different ranges of the parameters.

Keywords: $F(R,T)$ gravity; Ricci scalar; Torsion scalar; Energy conditions.

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1 Introduction

Modified theories of gravity have recently attained much attention to explore the dark energy (DE) and late accelerated expansion of the universe...
Dark energy is almost equally distributed in the universe and different models have been developed in the last decade to describe its nature. Dark energy is related with the modification of Einstein gravity in such a way that it would give the gravitational alternative to DE. The modified theories of gravity explain the unification of dark matter and DE, deceleration to acceleration epochs of the universe, description of hierarchy problem, dominance of effective DE, which help to solve the coincidence problem, viability of DE models through energy conditions and many more [1,2]. The proposed modified theories of gravity include Gauss-Bonnet gravity with $G$ invariant [3], $F(R)$ takes Ricci scalar $R$ [4], $F(T)$ inherits torsion scalar $T$ [5], $F(R, T)$ with $T$ as the trace of stress-energy tensor [6], $F(R, G)$ carries both $R$ and $G$ [7] etc. These theories use their corresponding invariant and scalars in order to meet the acceleration of the expanding universe.

In alternative theories, the $F(R)$ theory has shown significant progress in cosmology. This theory can directly be achieved by replacing the Ricci scalar $R$ by $F(R)$ as an arbitrary function in the Einstein-Hilbert action [1, 4], i.e.,

$$S_{EH} = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} R \rightarrow S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} F(R),$$  \hspace{1cm} (1)$$

where $\kappa^2 = 8\pi G$, $G$ is the gravitational constant and $g$ is the determinant of the metric coefficients. This theory uses Levi-Civita connection having only curvature for its formation and admits $1/R$ type terms in the Lagrangian to discuss the expanding universe with acceleration. Following the same strategy, the $F(T)$ theory of gravity is obtained by replacing torsion scalar $T$ with a general function $F(T)$ in the action of teleparallel equivalent of general relativity (GR) [5], i.e.,

$$S_T = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} T \rightarrow S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} F(T).$$  \hspace{1cm} (2)$$

This theory uses Weitzenböck connection which has only torsion but no curvature. The accelerated expansion of the universe in $F(T)$ theory is remarked with $T$ formed by the tetrad field.

The unification of $F(R)$ and $F(T)$ gravity theories as $F(R, T)$ is the prospective and interesting version in modified theories. The torsion as well as curvature scalar with no matter source leads to the formation of this unification. There are many extensions of GR in which torsion effects are included [8]. The $F(R, T)$ theory of gravity incorporates all the properties of
its constituent gravities. This theory is considered as an important gravitational theory which represents the evolution of the universe. Myrzakulov [9] constructed different reductions of $F(R, T)$ gravity and derived some torsion scalar solutions, in particular, the exact de Sitter solution by assuming a particular model $F(R, T) = \mu R + \nu T$. He concluded that these exact analytic solutions describe the evolving universe in phantom acceleration phase. Chattopadhyay [10] found a quintom-like behavior and transition from deceleration to acceleration phase of the universe using the same model with interacting Ricci DE in this theory. Also, the statefinder parameters indicate that the model interpolates between dust and ΛCDM phases of the universe.

The energy conditions are used to restrict some higher order terms of GR models or its extensions to make them simpler with viability. García et al. [11] formulated these energy conditions in modified Gauss-Bonnet gravity for two DE models and provided the ranges of some parameters in the model. Santos et al. [12] examined the bounds on the parameters of two families of $F(R)$ gravity. Liu and Rebouças [13] investigated the constraints on the logarithmic and exponential models in the framework of $F(T)$ gravity. Recently, Alvarenga et al. [14] explored energy conditions in $F(R, T)$ gravity with $T$ as trace of the energy-momentum tensor for two specific models. They found that for some values of the input parameters of the models, the de Sitter and power-law solutions may be stable with the validity of energy conditions.

In this paper, we construct the energy conditions on $F(R, T)$ gravity with $T$ as a torsion scalar in FRW spacetime with perfect fluid. The constraints on the parameters are investigated for the general as well as particular cases by taking viable models of DE. The paper is organized as follows. In next section, we provide preliminaries related to the energy conditions and $F(R, T)$ gravity. Section 3 is devoted to represent the general form of the energy conditions in $F(R, T)$ gravity and their reduction to $F(R)$ and $F(T)$ theories of gravity. Also, the graphical behavior of the constraints on the model parameters is shown. The last section summarizes the results.

2 Preliminaries

In this section, we briefly discuss the energy conditions and their cosmological implications in GR as well as in modified theory of gravity. The energy-momentum distribution and stress due to matter or any other non-
gravitational field is described by the energy-momentum tensor $T_{\mu\nu}$. For a realistic matter source satisfied by all states of matter and non-gravitational fields, it must satisfy some conditions on this tensor. There are several different types of these conditions, so-called energy conditions such as averaged energy conditions (depend on the average stress-energy tensor along a desirable curve) and point-wise energy conditions (count the stress-energy tensor at a given point in the space) etc. The standard point-wise energy conditions are the null (NEC), weak (WEC), strong (SEC) and dominant energy condition (DEC). Basically these conditions are formulated with the help of Raychaudhuri equation that describes the behavior of timelike, spacelike or lightlike curves of a congruence and attractiveness of the gravity [15].

The Raychaudhuri equation for timelike and spacelike curves is given by

$$ R_{\mu\nu} u^\mu u^\nu + \sigma_{\mu\nu} \sigma^{\mu\nu} - \omega_{\mu\nu} \omega^{\mu\nu} + \frac{1}{3} \theta^2 + \frac{d\theta}{d\pi} = 0, $$

$$ R_{\mu\nu} k^\mu k^\nu + \sigma_{\mu\nu} \sigma^{\mu\nu} - \omega_{\mu\nu} \omega^{\mu\nu} + \frac{1}{2} \theta^2 + \frac{d\theta}{d\lambda} = 0. $$

Here $R_{\mu\nu}$ is the Ricci tensor, $u^\mu$ and $k^\mu$ are the timelike and lightlike tangent vectors, $\sigma^{\mu\nu}$ and $\omega^{\mu\nu}$ are the shear and vorticity tensors describing the distortion of volume and rotation of the curves respectively. The expansion scalar $\theta$ shows the expansion of volume, while $\pi$ and $\lambda$ are the positive parameters which represent the curved of the congruence under spacetime manifold. The quadratic terms in Raychaudhuri equation may be neglected for small distortions of the volume without rotation, leading to

$$ \theta = -\pi R_{\mu\nu} u^\mu u^\nu = -\lambda R_{\mu\nu} u^\mu u^\nu. \quad (3) $$

The expansion scalar, $\theta < 0$, provides the attractiveness property for any hypersurface of orthogonal congruence ($\omega_{\mu\nu} = 0$) yielding $R_{\mu\nu} u^\mu u^\nu \geq 0$ and $R_{\mu\nu} k^\mu k^\nu \geq 0$. In the framework of GR, through the Einstein field equations, the Ricci tensor is replaced by the energy-momentum tensor $T_{\mu\nu}$, which leads to point-wise energy conditions. Taking into account the perfect fluid ($T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$) with energy density $\rho$ and pressure $p$ in the effective gravitational field, these energy conditions are classified as follows.

The NEC is the consequence of $T_{\mu\nu} k^\mu k^\nu \geq 0$, which leads to familiar form $\rho_{eff} + p_{eff} \geq 0$. The NEC implies that density of the universe falls with its expansion and its violation may yield the Big Rip of the universe. It ensures the validity of second law of black hole thermodynamics. The
positivity condition for timelike vector $u^\mu$ results the SEC which reduces to $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$, $\rho_{\text{eff}} + 3p_{\text{eff}} \geq 0$ in the effective field. The violation of this condition represents the accelerated expansion of the universe. The WEC requires the positivity of the energy density for any observer at any point, i.e., $\rho_{\text{eff}} \geq 0$, $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$ in addition to NEC. The Hawking-Penrose singularity theorems require the validity of SEC and WEC. The condition on energy that it must not flow faster than light ($p \leq \rho$) yields one inequality of DEC whose complete set is $\rho_{\text{eff}} \geq 0$, $\rho_{\text{eff}} \pm p_{\text{eff}} \geq 0$. The WEC and NEC are the most important of all energy conditions as their violation lead to the violation of other energy conditions.

This procedure is trivially true for $F(R)$ gravity. However, for any other theory, the replacement of the energy-momentum tensor should be taken under consideration. It may not affect the procedure if NEC is satisfied for any matter and is maintaining by the physical motivation of geodesic congruences. The Raychaudhuri equation and attractiveness property hold for any gravitational theory, and hence can be used to check the energy conditions of the recently developed modified theories of gravity.

There is an equivalent role treated by both curvature and torsion scalars in the fundamental framework. It should not be surprising that such a comparability between curvature and torsion should be found in the construction of the gravitational Lagrangian of the theory. Consider the flat, homogenous and isotropic FRW universe as

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),$$

where $a$ is the time dependent scale factor. The corresponding Lagrangian of $M_{37} - F(R, T)$ gravity model is given by [9, 10]

$$L_{37} = a^3[F - (T - v)F_T - (R - u)F_R + L_m] - 6a\ddot{a}(FR + FT) - 6a^2\dot{a}(\dot{R}FR + \dot{T}FR).$$

Here $L_m$ is the matter Lagrangian, the subscripts represent the first and second order derivatives of $F$ with respect to the corresponding scalar and dot represents the time derivative. The functions $u$ and $v$ are generally defined as $u = u(t, a, \dot{a}, \ddot{a}, ...,)$ and $v = v(t, a, \dot{a}, \ddot{a}, ...,)$, which are related with the geometry of the spacetime. The Ricci and torsion scalars are defined as

$$R = u + g^{\mu\nu}R_{\mu\nu}, \quad T = v - S_{\rho}^{\mu\nu}T_{\rho}^{\mu\nu},$$
where \( g^{\mu \nu} \) is the metric tensor and \( R_{\mu \nu}, \ S^\rho_{\mu \nu}, \ T^\lambda_{\mu \nu} \) are the Ricci, antisymmetric and torsion tensors respectively, given by

\[
R_{\mu \nu} = \partial_\lambda \Gamma^\lambda_{\mu \nu} - \partial_\mu \Gamma^\lambda_{\nu \lambda} + \Gamma^\lambda_{\mu \rho} \Gamma^\rho_{\nu \lambda} - \Gamma^\lambda_{\nu \rho} \Gamma^\rho_{\mu \lambda},
\]

\[
S^\rho_{\mu \nu} = -\frac{1}{4}(T^\mu_{\rho \nu} - T^\nu_{\rho \mu} - T^\rho_{\mu \nu}) + \frac{1}{2}(\delta^\rho_{\mu} T^{\theta \nu}_{\rho} - \delta^\nu_{\rho} T^{\theta \mu}_{\rho}),
\]

\[
T^\lambda_{\mu \nu} = \Gamma^\lambda_{\nu \mu} - \Gamma^\lambda_{\mu \nu} = h^a_\lambda (\partial_\mu h^a_\mu - \partial_\mu h^a_\mu).
\]

Here \( h^a_\mu \) are the tetrad components and \( \Gamma^\lambda_{\mu \nu} = \frac{1}{2}g^{\lambda \rho} (\partial_\mu g_{\nu \rho} + \partial_\nu g_{\mu \rho} - \partial_\rho g_{\mu \nu}) \) are the Christoffel symbols with Greek alphabets \( (\mu, \nu, \rho, ... = 0, 1, 2, 3) \) denote spacetime components while the Latin alphabets \( (a, b, c, ... = 0, 1, 2, 3) \) are used to describe components of tangent space..

Using Eq. (11) and tetrad \( h^a_\mu = \text{diag}(1, a, a, a) \), the Ricci and torsion scalars become \( R = u + 6(\dot{H} + 2H^2) \) and \( T = v - 6H^2 \) with Hubble parameter \( H = \dot{a}/a \). Taking \( a, R \) and \( T \) as the generalized coordinates of the configuration space, the total energy (Hamiltonian) corresponding to the Lagrangian (5) yields the following field equations (assuming \( \kappa^2 = 1 \))

\[
6a^2 \dot{a} \dot{R} F_{RR} - (6a^2 \dot{a} + a^3 \dot{a} \frac{\partial u}{\partial \dot{a}}) F_R + 6a^2 \dot{a} \dot{T} F_{RT} + (12a \dot{a}^2 - a^3 \frac{\partial v}{\partial \dot{a}}) F_T \\
+ a^3 F = 2a^3 p_m,
\]

\[
-6a^2 \dot{R}^2 F_{RRR} - (12a \dot{a} \dot{R} + 6a^2 \ddot{R} - a^3 \dot{R} \frac{\partial u}{\partial \dot{a}}) F_{RR} + (12 \dot{a}^2 + 6a \ddot{a}) \\
+ 3a^2 \dot{a} \frac{\partial u}{\partial \dot{a}} + a^3 \frac{\partial u}{\partial t} \left( \frac{\partial u}{\partial \dot{a}} \right) - a^3 \frac{\partial u}{\partial \dot{a}} F_R - (12a \dot{a} \dot{T} - a^3 \dot{T} \frac{\partial v}{\partial \dot{a}}) F_{TT} \\
- (24 \dot{a}^2 + 12 a \ddot{a} - 3a^2 \frac{\partial v}{\partial \dot{a}} - a^3 \frac{\partial v}{\partial t} \left( \frac{\partial v}{\partial \dot{a}} \right) + a^3 \frac{\partial v}{\partial \dot{a}}) F_T - 12a^2 \dot{R} \dot{F}_{RR} \\
- 6a^2 \dot{T}^2 F_{RTT} - (12a \dot{a} \dot{T} + 12a \dot{a} \dot{R} + 6a^2 \ddot{T} - a^3 \dot{R} \frac{\partial v}{\partial \dot{a}} - a^3 \dot{T} \frac{\partial v}{\partial \dot{a}}) F_{RT} \\
- 3a^2 F = 6a^2 p_m.
\]

### 3 Energy Conditions of \( F(R, T) \) Gravity

In this section, we construct the energy conditions of \( M_{37} \) - \( F(R, T) \) gravity model. The \( M_{37} \) - model admits some important features from the physically and geometrically viewpoints. We assume the functions \( u \) and \( v \) in power-law form [10] as \( u = \alpha a^n \) and \( v = \beta a^m \) where \( \alpha, \beta \) are non-zero real constants and \( m, n \) are positive integers. The Ricci and torsion scalars with their
derivatives can be expressed in terms of deceleration \((q)\), jerk \((j)\) and snap \((s)\) parameters \([16]\) as

\[
R = \alpha a^n + 6H^2(1 - q), \quad \dot{R} = \alpha a^n H + 6H^3(j - q - 2), \quad \ddot{R} = \alpha a^n(n - q - 1)H^2 + 6H^4(4q^2 + 15q + 2j + s + 9), \quad (12)
\]

\[
T = \beta a^m - 6H^2, \quad \dot{T} = \beta ma^m H + 12H^2(1 + q), \quad \ddot{T} = \beta ma^m(m - q - 1) - 12H^4(q^2 + 5q + j + 3), \quad (13)
\]

where \(q = -\frac{\ddot{a}}{H^2}\), \(j = \frac{\dot{\ddot{a}}}{H^3}\), \(s = \frac{\dot{\dot{\ddot{a}}}}{H^4}\). The present-day values of \(H\) and negative \(q\) describe the expansion rate of the accelerating universe, while \(j\) classifies different DE models representing acceleration of the universe. Equations \((10)\) and \((11)\) can be written as the effective gravitational field equations in the following form

\[
3H^2 = \rho_{\text{eff}}, \quad -(2\dot{H} + 3H^2) = p_{\text{eff}}, \quad (14)
\]

where

\[
\rho_{\text{eff}} = \rho_m - 3H^2(\alpha a^n + 6H^2(j - q - 2))F_{RR} - (3H^2 q + \frac{\alpha a^n}{2q})F_R - 3H^2(\beta ma^m + 12H^2(1 + q))F_{RT} - (6H^2 + \frac{\beta ma^m}{2q})F_T + 3H^2 - \frac{F}{2}, \quad (15)
\]

\[
p_{\text{eff}} = \rho_m + (2q - 1)H^2 + H^2(\alpha a^n + 6H^2(j - q - 2))^2F_{RRR} + [\alpha a^n H^2(1 - q + n) + 6H^4(4q^2 + 4j + 13q + s + 5)] + \frac{\alpha a^n}{6q}(\alpha a^n + 6H^2(j - q - 2))\]  

\[
\times (\frac{n + 2}{q} + 1)]F_R + (2H^2 + \frac{\beta ma^m}{6q})(\beta ma^m + 12H^2(1 + q))F_{TT} + [2(2 - q)H^2 + \frac{\beta ma^m}{6}(\frac{m + 2}{q} + 1)]F_T + 2H^2(\alpha a^n)
\]

\[
+ 6H^2(j - q - 2)(\beta ma^m + 12H^2(1 + q))F_{RRT} + H^2(\beta ma^m + H^2(1 + q))^2F_{RRT} + \frac{H^2(1 + q)^2}{2}F_{RRT} + [\beta ma^m(m - q + 1)H^2 + 2\alpha a^n H^2]
\]

\[
- 12H^4(q^2 + 4q + 3) + \frac{\beta ma^m}{6q}(\alpha a^n + 6H^2(j - q - 2)) + \frac{\alpha a^n}{6q}(\beta ma^m + 12H^2(1 + q))F_{RT} + \frac{F}{2}. \quad (16)
\]

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Now we discuss energy conditions for a particular model [9, 10]
\[ F(R, T) = \mu R + \nu T, \]
where \(\mu\) and \(\nu\) are non-zero real constants. This model describes the accelerated expansion of the universe in phantom era by taking power-law form for the functions \(u\) and \(v\). Inserting model (17) in Eqs.(15) and (16), we obtain the effective field equations as
\[
\rho_{\text{eff}} = \rho_m + 3H^2 - \mu(3H^2_q + \frac{\alpha n a^n}{2q}) - v(6H^2 + \frac{\beta m a^m}{2q}) - \frac{\mu R + \nu T}{2},
\]
\[
p_{\text{eff}} = p_m + (2q - 1)H^2 - \mu[(2 - q)H^2 - \frac{\alpha n a^n}{6}(\frac{n + 2}{q} + 1)] + \nu(2(2 - q)H^2 + \frac{\beta m a^m}{6}(\frac{m + 2}{q} + 1) + \frac{\mu R + \nu T}{2}).
\]
The expressions for NEC, WEC, SEC and DEC using Eqs.(18) and (19) take the form
\[
\text{NEC: } \rho_{m0} + p_{m0} + 2(1 + q_0)H_0^2 - \mu[2(1 + q_0)H_0^2 + \frac{\alpha n a^n}{2}(\frac{1 - n - q_0}{3q_0}) - \nu[2(1 + q_0)H_0^2 + \frac{\beta m a^m}{2}(\frac{1 - m - q_0}{q_0})] \geq 0,
\]
\[
\text{WEC: } \rho_{m0} + 3H_0^2 - \mu(3H_0^2 q_0 + \frac{\alpha n a^n}{2q_0}) - \nu(6H_0^2 + \frac{\beta m a^m}{2q_0}) - \frac{\mu}{2}(\alpha a^n + 6H_0^2(1 - q_0)) - \frac{\nu}{2}(\beta a^m - 6H_0^2) \geq 0,
\]
\[
\text{SEC: } \rho_{m0} + 3p_{m0} + 6H_0^2 q_0 - \mu(6H_0^2 - \frac{\alpha n a^n}{2q_0}(\frac{1 + n + q_0}{q_0}) + \nu(6(1 - q_0)H_0^2 + \frac{\beta m a^m}{2}(\frac{1 + m + q_0}{q_0})) + \frac{\mu}{2}(\alpha a^n + 6H_0^2(1 - q_0)) + \frac{\nu}{2}(\beta a^m - 6H_0^2) \geq 0,
\]
\[
\text{DEC: } \rho_{m0} - p_{m0} + 2(2 - q_0)H_0^2 - \mu(2(2q_0 - 1)H_0^2 + \frac{\alpha n a^n}{2}(\frac{n + q_0 + 5}{3q_0})) - \nu(2(5 - q_0)H_0^2 + \frac{\beta m a^m}{2}(\frac{m + q_0 + 5}{3q_0}) - \frac{\mu}{2}(\alpha a^n + 6H_0^2(1 - q_0)) - \frac{\nu}{2}(\beta a^m - 6H_0^2) \geq 0,
\]
where \( \rho_{\text{eff}} + p_{\text{eff}} \geq 0 \) for NEC, \( \rho_{\text{eff}} \geq 0 \) for WEC, \( \rho_{\text{eff}} + 3p_{\text{eff}} \geq 0 \) for SEC, \( \rho_{\text{eff}} - p_{\text{eff}} \geq 0 \) for DEC are formulated and the subscript 0 denotes the present-day values of the corresponding parameters. In particular, we consider the following present-day values [16]: \( H_0 = 0.718, q_0 = -0.64, j_0 = 1.02, s_0 = -0.39 \) and \( a_0 = 1 \) to discuss the energy conditions. Also, we assume \( \rho_{m0} = 0 = p_{m0} \) so that if the vacuum \( F(R, T) \) model satisfies the WEC then one can always add a positive energy density or pressure from matter and radiation to any model satisfying the WEC.

The inequalities (20) and (21) contain six constants \( \alpha, \beta, \mu, \nu, m \) and \( n \). We impose constraints on \( \mu \) and \( \nu \) with the signs of \( \alpha \) and \( \beta \) and draw the graph against \( m \) and \( n \). The WEC is satisfied for \((-0.1 \leq (\mu, \nu) \leq 0.1 \) with \( \alpha < 0 \) and \( \beta > 0 \) or \( \alpha > 0 \) and \( \beta < 0 \). Figure 1 shows the corresponding plot for some specific values from these ranges. The general model (17) satisfies the WEC inequalities for the particular values of the involved parameters. Such types of constraints on the model parameters are widely constructed in the literature [11]-[14, 17].

In the following, we present some special reductions of \( M_{37} \) - model and discuss their energy conditions (particularly WEC and NEC) using some viable models.
3.1 The M44 - Model

Let the function \( F(R, T) \) be independent of the torsion scalar, i.e., \( F(R, T) = F(R) \). It is the M44 - model whose Lagrangian (5) takes the form

\[
L_{44} = a^3[F - (R - u)F_R + L_m] - 6a^2a^2F_R - 6a^2a^2\dot{R}F_{RR}.
\] (24)

The effective field equations (15) and (16) reduce to

\[
\rho_{\text{eff}} = \rho_m - 3H^2(\alpha_\alpha n^2 + 6H^2(j - q - 2))F_{RR} - (3H^2q + \frac{\alpha_\alpha n^2}{2q})F_R
\]
\[+ 3H^2 - \frac{F}{2}, \] (25)

\[
p_{\text{eff}} = p_m + (2q - 1)H^2 + H^2(\alpha_\alpha n^2 + 6H^2(j - q - 2))^2F_{RRR}
\]
\[+ \frac{\alpha_\alpha n}{6q}(\alpha_\alpha n^2 + 6H^2(j - q - 2))F_{RR} - [(2 - q)H^2
\]
\[- \frac{\alpha_\alpha n}{6}(\frac{n + 2}{q} + 1)]F_R + \frac{F}{2}. \] (26)

The energy conditions will become

\text{NEC} : \quad \rho_m + p_m + 2(q + 1)H^2 + H^2(\alpha_\alpha n^2 + 6H^2(j - q - 2))^2F_{RRR}
\]
\[- [\alpha_\alpha n^2(q + n + 2)H^2 - 6H^4(4q^2 + j + 16q + s + 11) - \frac{\alpha_\alpha n}{6q}(\alpha_\alpha n^2
\]
\[+ 6H^2(j - q - 2)]F_{RR} - [2(1 + q)H^2 + \frac{\alpha_\alpha n}{2}(\frac{1 - n - q}{3q})]F_R \geq 0, \] (27)

\text{WEC} : \quad \rho_m - 3H^2(\alpha_\alpha n^2 + 6H^2(j - q - 2))F_{RR} - (3H^2q + \frac{\alpha_\alpha n^2}{2q})F_R
\]
\[+ 3H^2 - \frac{F}{2} \geq 0, \] (28)

\text{SEC} : \quad \rho_m + 3p_m + 6H^2 + 3H^2(\alpha_\alpha n^2 + 6H^2(j - q - 2))^2F_{RRR}
\]
\[-3[\alpha_\alpha n^2(2 - q - n)H^2 + 6H^4(4q^2 + 5j + 12q + s + 3) + \frac{\alpha_\alpha n}{6q}(\alpha_\alpha n^2
\]
\[+ 6H^2(j - q - 2)]F_{RR} - [6H^2 - \frac{\alpha_\alpha n^2}{2}(\frac{1 + n + q}{q})]F_R + F \geq 0, \] (29)
\[ \text{DEC:} \quad \rho_m - p_m + 2(2 - q)H^2 - H^2(\alpha na^n + 6H^2(j - q - 2))^2F_{RRR} \\
\quad - [\alpha na^n(4 - q + n)H^2 + 6H^4(4q^2 + 7j + 10q + s - 1) + \frac{\alpha na^n}{6q} \\
\quad \times (\alpha na^n + 6H^2(j - q - 2))]F_{RR} - [2(2q - 1)H^2 + \frac{\alpha na^n}{2}(\frac{5 + n + q}{3q})]F_R \\
\quad - F \geq 0. \] (30)

These are the general conditions of the \( M_{44} \) - model which can be used to evaluate the energy conditions for any viable model. It is noted that for \( u = 0 \) the inequalities (27)-(30) become equivalent to the standard energy conditions in \( F(R) \) gravity [12].

We assume here an arbitrary viable \( F(R) \) model [4, 18] given by

\[ F(R) = R - \frac{b^2}{3R}, \] (31)

where \( b \) is a positive constant and factor 3 is used only to simplify the equations. This model belongs to the family \( R - \frac{1}{R} \) which is referred to the cosmic expansion of the universe. Also, it is consistent with the X-rays galaxy cluster distance data [19], which leads to the deceleration and snap parameters consistent with three data sets [4, 20]. Inserting Eq. (31) in (27) and (28), the expressions for WEC (contains NEC) become

\[ \rho_{m0} + p_{m0} + 2(q_0 + 1)H_0^2 + \frac{2b^2H_0^2(\alpha na_0^n + 6H_0^2(j_0 - q_0 - 2))^2}{(\alpha a_0^n + 6H_0^2(1 - q_0))^4} \\
\quad + \frac{2b^2}{3(\alpha a_0^n + 6H_0^2(1 - q_0))^3}[\alpha na_0^n(q_0 + n + 2)H_0^2 - 6H_0^4(4q_0^2 + j_0 + 16q_0 \\
\quad + s_0 + 11) - \frac{\alpha a_0^n}{6q_0}(\alpha a_0^n + 6H_0^2(j_0 - q_0 - 2))] - [2(1 + q_0)H_0^2 \\
\quad + \frac{\alpha a_0^n}{2}(\frac{1 - n - q_0}{3q_0})](1 + \frac{b^2}{3(\alpha a_0^n + 6H_0^2(1 - q_0))^2}) \geq 0, \] (32)

\[ \rho_{m0} + \frac{2b^2H_0^2}{(\alpha a_0^n + 6H_0^2(1 - q_0))^3}(\alpha a_0^n + 6H_0^2(j_0 - q_0 - 2)) \\
\quad - (3H_0^2q_0 + \frac{\alpha a_0^n}{2q_0})(1 + \frac{b^2}{3(\alpha a_0^n + 6H_0^2(1 - q_0))^2}) + 3H_0^2 \\
\quad - \frac{1}{2}(\alpha a_0^n + 6H_0^2(1 - q_0) - \frac{b^2}{3(\alpha a_0^n + 6H_0^2(1 - q_0))}) \geq 0. \] (33)
Figure 2: Plot of WEC versus $n$ and $b$. The left graph corresponds to $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$ and the right graph represents $\rho_{\text{eff}} \geq 0$ for $\alpha = -5.08$.

We take here the same assumptions on cosmological parameters as for the $M_{37}$ - $F(R, T)$ model. Inequalities (32) and (33) contain three constants $n$, $b$ and $\alpha$. We plot these conditions against $n$ and $b$ by fixing the range of $\alpha$ as shown in Figure 2. For the scales of $n$ and $b$, the value of $\alpha$ must lie in the range $[-6.1, -5.08]$ to satisfy these inequalities. Beyond this range, the WEC does not satisfy for $\alpha > -5.08$ whereas for $\alpha < -6.1$, the inequalities are supported by higher values of $n$ and $b$ upto $n, b >> 70$. Such type of constraints on the parameters for the validity of the inequalities have also been constructed in modified theories of gravity [11]-[14].

3.2 The $M_{45}$ - Model

If the function $F(R, T)$ is independent of $R$, then the model becomes $M_{45}$ keeping only the dynamics of torsion scalar. In this case, the Lagrangian (5) becomes

$$L_{45} = a^3 [F - (T - v)F_T + L_m] - 6a \dot{a}^2 F_T,$$

yielding

$$\rho_{\text{eff}} = \rho_m - (6H^2 + \frac{\beta m a^m}{2q})F_T + 3H^2 - \frac{F}{2},$$  \hspace{1cm} (35)
\[ p_{\text{eff}} = p_m + (2q - 1)H^2 + (2H^2 + \frac{\beta ma^m}{6q})(\beta ma^m + 12H^2(1 + q))F_{TT} \]
\[ + [2(2 - q)H^2 + \frac{\beta ma^m}{6}(\frac{m + 2}{q} + 1)]F_T + \frac{F}{2}. \quad (36) \]

Using the above effective field equations, the general form of the energy conditions are given as follows

**NEC:** \[ \rho_m + p_m + 2(q + 1)H^2 + (2H^2 + \frac{\beta ma^m}{6q})(\beta ma^m + 12H^2(1 + q))F_{TT} \]
\[ - [2(1 + q)H^2 + \frac{\beta ma^m}{2}(\frac{1 - m - q}{3q})]F_T \geq 0, \quad (37) \]

**WEC:** \[ \rho_m - (6H^2 + \frac{\beta ma^m}{2q})F_T + 3H^2 - \frac{F}{2} \geq 0, \quad (38) \]

**SEC:** \[ \rho_m + 3p_m + 6H^2q + 3(2H^2 + \frac{\beta ma^m}{6q})(\beta ma^m + 12H^2) \]
\[ \times (1 + q))F_{TT} + [6(1 - q)H^2 + \frac{\beta ma^m}{2}(\frac{1 + m + q}{q})]F_T + F \geq 0, \quad (39) \]

**DEC:** \[ \rho_m - p_m + 2(2 - q)H^2 - (2H^2 + \frac{\beta ma^m}{6q})(\beta ma^m + 12H^2) \]
\[ \times (1 + q))F_{TT} - [2(5 - q)H^2 + \frac{\beta ma^m}{2}(\frac{m + 3q + 5}{q})]F_T - F \geq 0. \quad (40) \]

These conditions reduce to the well-known energy conditions of \( F(T) \) gravity in the limit \( \nu = 0 \) [13].

Now we evaluate these energy conditions for the following power-law model [44]
\[ F(T) = \epsilon T^\delta, \quad (41) \]
where \( \delta \) is non-zero integer and \( \epsilon \) is non-zero real constant. For \( \delta = 1 = \epsilon \), the model corresponds to teleparallel gravity. Also, this model helps to remove four types of singularities emerging in the late-time accelerated era with specific conditions \( \epsilon \neq 0 \) and \( \delta < 0 \). We check energy conditions for this model by taking some assumptions on these parameters. Applying the present-day values of cosmological parameters and inserting model (41) in
Figure 3: Plot of WEC versus $m$ and $\epsilon$. The left graph corresponds to $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$ and the right graph represents $\rho_{\text{eff}} \geq 0$ for $\beta = 4$ and $\delta = 2$.

Eqs. (37) and (38), the WEC is obtained as

\[
\begin{align*}
\rho_{m0} + p_{m0} + 2(q_0 + 1)H_0^2 + \epsilon \delta (\delta - 1)(-6H_0^2)^{\delta - 2}(2H_0^2 + \frac{\beta m a_0^m}{6q_0}) \\
\times (\beta m a_0^m + 12H_0^2(1 + q_0)) - \epsilon \delta (6H_0^2)^{\delta - 1}[2(1 + q_0)H_0^2 \\
+ \frac{\beta m a_0^m}{2}(1 - \frac{m - q_0}{3q_0})] \geq 0, \\
\rho_{m0} - \epsilon \delta (-6H_0^2)^{\delta - 1}(6H_0^2 \frac{\beta m a_0^m}{2q_0}) + 3H_0^2 - \frac{1}{2} \epsilon \delta (-6H_0^2)^{\delta} \geq 0.
\end{align*}
\]  

(42)  

(43)

Taking the present-day values of cosmological parameters and the condition of vacuum $F(T)$ model similar to the $M_{37}$ - model, we plot the WEC versus $m$ and $\epsilon$. We have to restrict here one more constant as compared to the $F(R)$ model. Thus for the model (41), the WEC is valid for $\beta \gg 0$ with the following cases:

- **When $\delta$ is positive,**
  - (i) for even $\delta$, it must take $\epsilon < 0$,
  - (ii) if $\delta$ is odd, $\epsilon > 0$ is applied.

- **When $\delta$ is negative,**
  - (i) for even $\delta$, $\epsilon < 0$ with $\delta \leq -6$ whereas $\epsilon > 0$ with $\delta \leq -10$,
  - (ii) if $\delta$ is odd, then the ranges for the parameters are $\epsilon < 0$ with $\delta \leq -11$ whereas $\epsilon > 0$ with $\delta \leq -5$.
Figure 3 represents the plot of the first case of positive $\delta$. The negative $\delta$ constraints satisfying the energy conditions make the model more reliable to discuss the accelerated expansion of the universe. In [13], the WEC is fulfilled for the specific range of single parameter of the exponential and logarithmic $F(T)$ models.

4 Summary

The energy conditions in the cosmological modeling play an important role in interpreting different aspects of the universe including the current accelerated expansion of the universe and singularity theorems. These conditions are very useful in order to constraint different constant parameters of the model to be viable in the underlying framework, in particular NEC and WEC are very important. In this paper, we use this phenomenon in a recently developed $F(R, T)$ modified theory of gravity for FRW universe model. Here $T$ is the torsion scalar involved independently in the Lagrangian and plays its role with the Ricci scalar $R$. We have constructed expressions of energy conditions for $F(R, T)$ gravity as well as some special cases with viable models. The ranges of the parameters are investigated for which the energy conditions hold. Also, we give graphical representations of some of the parameters for the WEC. The results of the paper are summarized as follows.

Firstly, we have constructed the inequalities for the energy conditions of $M_{37} - F(R, T)$ gravity for a particular model $F(R, T) = \mu R + \nu T$. This model shows consistent results for $-0.1 \leq (\mu, \nu) \leq 0.1$ with different signs of $\alpha$ and $\beta$. Also, this model represents the accelerated expansion of the universe [9] and satisfies the energy conditions with specific ranges of the parameters. Secondly, we have found two particular reductions of $M_{37}$ model by taking torsion and curvature as independent cases. These reductions form $M_{44} - F(R)$ and $M_{45} - F(T)$ theories of gravity inheriting general functions $u$ and $v$ coming from the spacetime geometry. For two viable DE models having specific properties, we have found some constraints on constant parameters of the models to satisfy the energy conditions.

One can use the effective gravitational field equations (15) and (16) to construct the expressions of energy conditions for any viable model. The role of torsion depends upon the source associated with spin matter density whereas in $F(R, T)$ gravity, torsion as well as curvature can propagate without the presence of spin matter density.
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