Super D-string Action on $AdS_5 \times S^5$

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Abstract

We present a supersymmetric and $\kappa$-symmetric D-string action on $AdS_5 \times S^5$ in supercoset construction. As in the previous work of the super D-string action in the flat background, the super D-string action on $AdS_5 \times S^5$ can be transformed to a form of the IIB Green-Schwarz superstring action with the $SL(2,\mathbb{Z})$ covariant tension on $AdS_5 \times S^5$ through a duality transformation. In order to understand a part of the duality transformation as $SO(2)$ rotation of $N = 2$ spinor coordinates, it seems to be necessary to fix the $\kappa$-symmetry in a gauge condition which simplifies the classical action. This is the article showing for the first time that there exists S-duality in type IIB superstring theory in a curved background whose validity has been conjectured in the past but not shown so far in an explicit way.

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1 Introduction

Recently an action of type IIB Green-Schwarz superstring [1] was constructed in the $AdS_5 \times S^5$ background in terms of supercoset formalism [2]. (See also [3] for super D3-brane action on this background.) This action has $\kappa$-symmetry as well as two dimensional reparametrization invariance as local symmetries, and reduces to the conventional type IIB Green-Schwarz superstring action in the flat background limit. More recently, the gauge-fixing of $\kappa$-symmetry was developed in two different approaches, the supersolvable algebra approach [4, 5] and the Killing gauge approach [6, 7], and afterwards it was shown that the gauge-fixed actions obtained in the two approaches in fact agree through appropriate rearrangement of fields [8].

In such a situation it seems to be timely to construct a super D-string action in the $AdS_5 \times S^5$ background and then ask ourselves if the duality relations such as $SL(2, \mathbb{Z})$ S-duality [9] between the type IIB Green-Schwarz superstring theory and the super D-string theory also exist in this curved background where supersymmetric and $\kappa$-symmetric D-brane actions [10, 11, 12] provided a good starting point for studying various properties of D-branes and the web of string dualities. We will show later that this is indeed the case.

The contents of this article are as follows. First of all, we construct a super D-string action on $AdS_5 \times S^5$ explicitly on the basis of the recently developed supercoset formalism. Next we verify that this super D-string action is invariant under the $\kappa$-transformation. Moreover, we construct the Nambu-Goto form of the type IIB Green-Schwarz superstring action and its $\kappa$-transformation in a similar form to the super D-string. Based on these studies it is shown that the super D-string action on $AdS_5 \times S^5$ is transformed to the type IIB Green-Schwarz superstring action with the modified tension on $AdS_5 \times S^5$ by performing the duality transformation. In the process, we need to achieve $SO(2)$ rotation with respect to $N = 2$ spinor coordinates, but to this aim it seems to be necessary to fix $\kappa$-symmetry to simplify the classical action.

2 Super D-string action

In this section, we construct the super D-string action in the $AdS_5 \times S^5$ background in terms of supercoset formalism and examine $\kappa$-symmetry of this action.

The $\kappa$-symmetric and reparametrization invariant super D-string action in the $AdS_5 \times S^5$ background is given by

$$S = S_{DBI} + S_{WZ},$$

with

$$S_{DBI} = - \int_{M_3} d^2 \sigma \sqrt{- \det(G_{ij} + F_{ij})},$$

$$S_{WZ} = \int_{M_3} H_3 (\mathcal{I}) = \int_{\partial M_3} \Omega_2 (\mathcal{I}),$$

(1)
where \( H_3(I) = d\Omega_2(I) = iL \wedge \hat{L} \wedge ILL, \) \( i \) and \( j \) run over the world-sheet indices 0 and 1, and we have defined

\[
\mathcal{E} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{I} = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{K} = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(3)

Here \( \sigma_i \) are the Pauli matrices, which operate on \( N = 2 \) spinor indices \( I, J = 1, 2 \). Throughout this article we follow the conventions and notations of references [2, 3, 7].

In the explicit parametrization \( G(X, \Theta) = g(X)e^{i\Theta}Q \) where \( X^\hat{m}, \Theta^I, \) and \( Q_I \) are respectively the bosonic and fermionic space-time coordinates and 32-component supercharges [2], the action (1) takes the form

\[
S = -\int_{M_2} d^2\sigma \left[ \sqrt{-\det(G_{ij} + F_{ij})} - 2i\epsilon^{ij} \int_0^1 ds L^\hat{a}_{is} \bar{\Theta}\Gamma^\hat{a} I L_{js} \right].
\]

(4)

The Cartan 1-form superfields \( L^I = L^I_{s=1} \) (32-component spinor) and \( L^\hat{a}_s = 1 \) (10 dimensional vector) are given by [13]

\[
L^I_s = \left( \sinh(sM) \right)^I_M D\Theta,
\]

\[
L^\hat{a}_s = e^\hat{a}_{\hat{m}}(X) dX^\hat{m} - 4i\bar{\Theta} \Gamma^\hat{a} \left( \frac{\sinh^2(sM)}{M^2} D\Theta \right)^I_M.
\]

(5)

with

\[
(M^2)^{IL} = \epsilon^{IJ}(-\gamma^a \Theta^I \Theta^L \gamma^a + \gamma^a \Theta^I \bar{\Theta}^L \gamma^a) + \frac{1}{2} \epsilon^{KL} (\gamma^{ab} \Theta^I \bar{\Theta}^K \gamma^{ab} - \gamma^{ab} \Theta^I \bar{\Theta}^K \gamma^{ab}),
\]

\[
(D\Theta)^I = \left[ d + \frac{1}{4} (\omega^{ab} \gamma_{ab} + \omega^{a'b'} \gamma_{a'b'}) \right] \Theta^I - \frac{1}{2} i\epsilon^{IJ} (\gamma_a \Theta^I + i\epsilon^{a'a'} \gamma_{a'}) \Theta^J,
\]

(6)

where for later convenience we decompose ten dimensional flat index \( \hat{a} \) in '5 + 5' and '4 + 6' ways, \( \hat{a} = (a, a') = (0, \ldots, 4, 5, \ldots, 9) = (p, t) = (0, \ldots, 3, 4, \ldots, 9) \) [6, 7]. Finally, \( G_{ij} \) and \( F_{01} \) are defined as

\[
G_{ij} = L^\hat{a}_i L^\hat{a}_j, \quad F_{01} = F_{01} + \epsilon^{ij} \Omega_{ij}(\mathcal{K}),
\]

(7)

where

\[
F_{ij} = \partial_i A_j - \partial_j A_i, \quad \Omega_{ij}(\mathcal{K}) = i \int_0^1 ds L^\hat{a}_{is} \bar{\Theta}\Gamma^\hat{a} KL_{js} - (i \leftrightarrow j).
\]

(8)

Actually it is easy to show that the action (4) precisely reduces to the super D-string action on the flat background [10, 11, 12] in the flat background limit. For instance, in that limit the Wess-Zumino term reduces to

\[
S_{WZ} = -i \int_{M_2} d^2\sigma \epsilon^{ij} \bar{\Theta}\Gamma^\hat{a} I \partial_i \Theta \cdot (\Pi^\hat{a}_j + \frac{1}{2} i\bar{\Theta}\Gamma^\hat{a} \partial_j \Theta),
\]

(9)
where $\Pi^a \equiv \partial_i X^a - i \Theta \Gamma^a \partial_i \Theta$, which is exactly the same form as in the flat background case. This follows from the fact that in the flat background limit the superfields are given by [4]

$$L^a_{is} = \partial_i X^a - is^2 \Theta^j \Gamma^a \partial_i \Theta^j, \quad L^I_{is} = s \partial_i \Theta^I.$$  \hfill (10)

Now we are ready to present the $\kappa$-transformation of the super D-string action [4] whose concrete expressions are given by

$$\delta_{\kappa} \Theta^I = \kappa^I,$$  \hfill (11)

and the projection $\Gamma$ is

$$\Gamma_{\kappa} = \kappa, \quad \Gamma^2 = 1, \quad Tr \Gamma = 0,$$

$$\Gamma = \frac{1}{2} \frac{\epsilon^{ij}}{-\det(G_{ij} + F_{ij})} (\Gamma_{ij} I + F_{ij} E),$$  \hfill (12)

with the definition of $\Gamma_{ij} \equiv \frac{1}{2}(\hat{L}_i \hat{L}_j - \hat{L}_j \hat{L}_i)$. Moreover, various superfields must transform under the $\kappa$-transformation as follows [2, 3]

$$\delta_{\kappa} L^a = 2i\tilde{L} \Gamma^a \delta_{\kappa} \Theta,$$

$$\delta_{\kappa} L = d\delta_{\kappa} \Theta - \frac{i}{2} \sigma_+ \hat{L} \delta_{\kappa} \Theta + \frac{1}{4} \tilde{L} \Gamma^{a \hat{b}} \delta_{\kappa} \Theta,$$

$$\delta_{\kappa} \tilde{L} = d\delta_{\kappa} \Theta + \frac{i}{2} \delta_{\kappa} \Theta \hat{E} \sigma_+ - \frac{1}{4} \delta_{\kappa} \Theta \Gamma^{a \hat{b}} \tilde{L}^{\hat{b}},$$

$$\delta_{\kappa} G_{ij} = 2i(\hat{L}_i \hat{L}_j + \hat{L}_j \hat{L}_i) \delta_{\kappa} \Theta,$$

$$\delta_{\kappa} F_{ij} = 2i(\tilde{L}_i K \hat{L}_j - \tilde{L}_j K \hat{L}_i) \delta_{\kappa} \Theta.$$  \hfill (13)

Then it is straightforward to show that

$$\delta_{\kappa} H_3(I) = d\Lambda_2(I),$$  \hfill (14)

with $\Lambda_2(I) = 2i\tilde{L} \wedge \hat{L} I \delta_{\kappa} \Theta$. Thus the $\kappa$-transformation of the Wess-Zumino term becomes

$$\delta_{\kappa} S_{WZ} = \int_{M_3} d\Lambda_2(I) = 2i \int_{M_2} \tilde{L} \wedge \hat{L} \mathcal{I} \delta_{\kappa} \Theta.$$  \hfill (15)

In order to prove the equation [14], we have made use of the following Fierz identity for Grassmann odd functions $A, B, C, D$ [11]

$$(\hat{A} \Gamma^a B) \cdot (\hat{C} \Gamma^a D) = -\frac{1}{2} \left[ (\hat{A} \Gamma^a e_I D) \cdot (\hat{B} \Gamma^a e_I C) + (\hat{A} \Gamma^a e_I C) \cdot (\hat{B} \Gamma^a e_I D) \right],$$  \hfill (16)

where $e_I = \{1, \mathcal{I}, \mathcal{K}, \mathcal{L} \}$, and the Maurer-Cartan equations for $su(2,2|4)$ superalgebra [2, 3]

$$dL^a = -L_{\hat{a} \hat{b}} \wedge L^{\hat{b}} - i\tilde{L} \Gamma^a \wedge L,$$

$$d\hat{L} = \frac{i}{2} \sigma_+ \hat{L} \wedge \hat{E} L - \frac{1}{4} L_{\hat{a} \hat{b}} \Gamma^{\hat{a} \hat{b}} \wedge L,$$

$$dL = \frac{i}{2} \hat{L} \hat{E} \wedge \hat{L} \sigma_+ - \frac{1}{4} \tilde{L} \Gamma^{\hat{a} \hat{b}} \wedge L^{\hat{a} \hat{b}}.$$  \hfill (17)
Then it is easy to prove the $\kappa$-invariance of the super D-string action on $AdS_5 \times S^5$ by showing

$$\delta_\kappa S_{DBI} + \delta_{\Gamma_\kappa} S_{WZ} = 0,$$

(18)

where we have made use of a useful identity

$$\epsilon^{ij} \epsilon^{kl} = \det G(G^{ik}G^{jl} - G^{il}G^{jk}).$$

(19)

3 The Nambu-Goto form of type IIB superstring

In the pioneering paper [2], the Polyakov form of type IIB Green-Schwarz superstring action was constructed in the $AdS_5 \times S^5$ background in terms of supercoset formalism. As a trivial extension of it, we shall present the Nambu-Goto form and construct its local $\kappa$-transformation which is of the form similar to that of super D-string made in the previous section.

The type IIB Green-Schwarz superstring action on $AdS_5 \times S^5$ is given by in the Polyakov form [2]

$$S = S_{Poly} + S_{WZ},$$

(20)

with

$$S_{Poly} = -\frac{1}{2} \int_{M_2} d^2\sigma \sqrt{-g} g^{ij} \tilde{L}_i \tilde{L}_j,$$

$$S_{WZ} = \int_{M_3} H_3(-\mathcal{K}) = \int_{M_2 = \partial M_4} \Omega_2(-\mathcal{K}),$$

(21)

where $H_3(-\mathcal{K})$ and $\Omega_2(-\mathcal{K})$ are defined as in the previous section but with the different argument $-\mathcal{K}$. Usually the local $\kappa$-transformation is defined by [2]

$$\delta_\kappa \Theta^I = 2 \hat{L}_i \kappa^i I,$$

$$\delta_\kappa(\sqrt{-g} g^{ij}) = -16i \sqrt{-g}(P^-L^1_k \kappa^{i1} + P^+_L L^2_k \kappa^{i2}),$$

(22)

where

$$P^-_{ij} \kappa^1_j = \kappa^{i1},$$

$$P^+_{ij} \kappa^2_j = \kappa^{i2},$$

(23)

with the definition of the projection operator $P_{ij} \equiv \frac{1}{2}(g^{ij} \pm \frac{1}{\sqrt{-g}} \epsilon^{ij})$. Of course, the $\kappa$-transformation for the superfields is defined as in (13).

Solving the field equation with respect to the auxiliary world-sheet metric $g_{ij}$ leads to the Nambu-Goto form of the type IIB Green-Schwarz superstring action

$$S = S_{NG} + S_{WZ}$$

$$= -\int_{M_2} d^2\sigma \sqrt{-\det G_{ij}} + S_{WZ},$$

(24)
where $G_{ij} = L^a_i L^a_j$ and the Wess-Zumino term remains unchanged.

It is interesting to notice that this action (24) is invariant under a similar $\kappa$-transformation to as in the super D-string action \[11\], \[12\]. Namely, the local $\kappa$-transformation can now be described by

$$\delta_\kappa \Theta^I = \kappa^I,$$  \hspace{1cm} (25)$$
and the projection $\Gamma$ is defined as

$$\Gamma \kappa = \kappa, \quad \Gamma^2 = 1, \quad Tr \Gamma = 0, \quad \Gamma = -\frac{1}{2} \epsilon^{ij} \Gamma_{ij} \kappa.$$  \hspace{1cm} (26)$$

It is straightforward to show that $\delta_\kappa L_{WZ} = -2i \epsilon^{ij} \hat{L}_i \hat{L}_j \kappa \delta_\kappa \Theta$ which is precisely canceled against a contribution from the $\kappa$ variation of the Nambu-Goto action.

4 Duality transformation between super D-string and type IIB Green-Schwarz superstring actions

In the previous sections, we have investigated super D-string and type IIB Green-Schwarz superstring actions in the $AdS_5 \times S^5$ background. In this section, we wish to clarify the relationship between the two actions, in other words, the duality transformation. For most of works done so far, analysis of the duality transformation properties of super Dp-branes has been classical and limited to the flat background although the results are expected not to depend on these restrictions \[11\]. Our purpose in this section is to remove these restrictions and carry out the analysis not only in a quantum mechanical way but also in a curved background. Of course our analysis is still unsatisfactory in that we deal with only the specific background $AdS_5 \times S^5$ and super D1-brane, but it would be the first important step towards the full analysis of the duality transformation properties of super Dp-branes in general background.

Now we are in a position to show how the super D-string action \[11\] becomes a fundamental superstring action \[24\] with the $SL(2, Z)$ covariant tension by using the path integral. We shall use the first-order Hamiltonian formalism of the path integral which was found by de Alwis and Sato \[14\] in the bosonic case and later applied to the supersymmetric case by the present author \[15\]. Note that this formalism does not rely on any approximation at least in case of string even if it is necessary to use a saddle point approximation when we want to apply this method to super Dp-branes with $p > 1$ because of the nonlinear feature of the p-brane actions.

Here it is worthwhile to comment on why the first-order Hamiltonian formalism on the $U(1)$ gauge sector plays an important role in the analysis of the duality transformation of D-branes. One reason comes from the fact that the duality is a 'symmetry' not in the Lagrangian
but in the Hamiltonian. In other words, the duality is a 'symmetry' holding only at the level of classical field equations. The other reason is that the major difference between super D-branes and super F-branes exists in the presence of $U(1)$ gauge field in the former. Hence in order to understand the duality transformation between the two actions it is enough to use the first-order Hamiltonian formalism only on the $U(1)$ gauge sector in the super D-branes.

According to the Hamiltonian formalism, let us start by introducing the canonical conjugate momenta $\pi^i$ corresponding to the gauge field $A_i$ defined as

$$\pi^i \equiv \frac{\partial S_{\text{D-string}}}{\partial \dot{A}_i} = \frac{\partial S_{\text{DBI}}}{\partial \dot{A}_i},$$

(27)

where we used the fact that the Wess-Zumino term is independent of the gauge potential. Note that compared to the flat background the position of index $i$ is important in the curved one. Then the canonical conjugate momenta $\pi^i$ are calculated to be

$$\pi^0 = 0, \quad \pi^1 = \frac{F_{01}}{\sqrt{-\det(G_{ij} + F_{ij})}},$$

(28)

from which the Hamiltonian has the form

$$\mathcal{H} = \sqrt{1 + (\pi^1)^2} \sqrt{-\det G_{ij} - \epsilon^{ij} \Omega_{ij}(\pi^1 \mathcal{K} + \mathcal{I}) - A_0 \partial_1 \pi^1 + \partial_1 (A_0 \pi^1)},$$

(29)

where we have chosen the positive sign in front of the first term without loss of generality since this sign ambiguity is related to overall normalization of the action.

At this stage, the partition function is defined by the first-order Hamiltonian form with respect to only the gauge field as follows:

$$Z = \int D\pi^0 D\pi^1 DA_0 DA_1 \exp i \int d^2 \sigma (\pi^1 \partial_0 A_1 - \mathcal{H})$$

$$= \int D\pi^1 DA_0 DA_1 \exp i \int d^2 \sigma \left[ -A_1 \partial_0 \pi^1 + A_0 \partial_1 \pi^1 - \sqrt{1 + (\pi^1)^2} \sqrt{-\det G_{ij} + \epsilon^{ij} \Omega_{ij}(\pi^1 \mathcal{K} + \mathcal{I}) - \partial_1 (A_0 \pi^1)} \right].$$

(30)

where we have canceled the trivial gauge group volume. Note that if we take the boundary conditions for $A_0$ such that the last surface term in the exponential identically vanishes, then we can carry out the integrations over $A_i$ explicitly, which gives rise to $\delta$ functions

$$Z = \int D\pi^1 \delta(\partial_0 \pi^1) \delta(\partial_1 \pi^1) \exp i \int d^2 \sigma \left[ -\sqrt{1 + (\pi^1)^2} \sqrt{-\det G_{ij} + \epsilon^{ij} \Omega_{ij}(\pi^1 \mathcal{K} + \mathcal{I})} \right].$$

(31)

Note that the existence of the $\delta$ functions reduces the integral over $\pi^1$ to the one over only its zero-modes. If we require that one space component is compactified on a circle, these zero-modes are quantized to be integers $[10]$. Consequently, the partition function becomes

$$Z = \sum_{m \in \mathbb{Z}} \exp i \int d^2 \sigma \left[ -\sqrt{1 + m^2} \sqrt{-\det G_{ij} + \epsilon^{ij} \Omega_{ij}(m \mathcal{K} + \mathcal{I})} \right].$$

(32)
Since the eigenvalues of $mK + I$ are $\pm \sqrt{1 + m^2}K$, we can redefine
\[ mK + I \equiv -\sqrt{1 + m^2}K. \] (33)

Then we finally arrive at the partition function
\[ Z = \sum_{m \in \mathbb{Z}} \exp i \int d^2\sigma \sqrt{1 + m^2} \left( -\sqrt{-\det G_{ij} - \epsilon^{ij}\Omega_{ij}(K)} \right). \] (34)

From this expression of the partition function, we can read off the effective action
\[ S = -\sqrt{1 + m^2} \int d^2\sigma \left( \sqrt{-\det G_{ij} + \epsilon^{ij}\Omega_{ij}(K)} \right). \] (35)

This is nothing but type IIB Green-Schwarz superstring action (24) with the modified tension $\sqrt{1 + m^2}$. This agrees with the tension formula for the $SL(2, \mathbb{Z})$ S-duality spectrum of strings in the flat background [4] provided that we identify the integer value $\pi^1 = m$ as corresponding to the $(m, 1)$ string. To show clearly that the tension obtained at hand is the $SL(2, \mathbb{Z})$ covariant tension, it would be more convenient to start with the following classical action instead of the action (4)
\[ S = -n \int_{M_2} d^2\sigma \left[ e^{-\phi} \left( \sqrt{-\det(G_{ij} + F_{ij})} - 2i\epsilon^{ij} \int_0^1 ds L^a_{is} \bar{\Theta}^a L^s_{jis} \right) + \frac{1}{2} \epsilon^{ij} \chi F_{ij} \right], \] (36)

where $n$ is an integer, and we confined ourselves to be only a constant dilaton $\phi$ and a constant axion $\chi$ such that the action (36) is still invariant under the same $\kappa$-transformation as before. Then following the same path of thoughts as above, we can obtain the manifestly $SL(2, \mathbb{Z})$ covariant tension
\[ T = \sqrt{(m + n\chi)^2 + n^2 e^{-2\phi}}. \] (37)

Here we would like to emphasize two important points. One point is that we have shown that there exists $SL(2, \mathbb{Z})$ S-duality in type IIB superstring even in the $AdS_5 \times S^5$ background. We think that this statement is quite nontrivial and important for future development of string theory in curved background geometry. Another important point is that we have obtained this result without appealing to any approximation so that the effective action (35) is quantum mechanically equivalent to the super D-string action (4).

It may appear that we have succeeded in deriving the $SL(2, \mathbb{Z})$ S duality of type IIB superstring theory at least within the present context. However, a reality is not simple. That is, in the above procedure of derivation we have assumed one dubious step tacitly, which amounts to the step of the redefinition (33). Note that to achieve this redefinition successfully we have to change the basis of spinor variables by an appropriate orthogonal matrix. In case of the flat background this redefinition is easily carried out in terms of an $SO(2)$ rotation of the spinor coordinates $\Theta^I$ because in this case the Wess-Zumino term consists of a function of the simple forms like $\bar{\Theta}^I \Theta^I$ and $\bar{\Theta}^I \Gamma^I \Theta^J$ as in Eq.(9) [15]. In contrast, in case of $AdS_5 \times S^5$
background, as seen in Eqs.(5),(6), the spinor and vector superfields involved in the Wess-Zumino term have quite complicated dependence on the spinor variables \( \Theta^I \) so that although it may not be impossible it seems to be difficult to perform the \( SO(2) \) rotation in the classical action.

Then how do we improve this situation? One idea is to fix the \( \kappa \)-symmetry to make the classical action of the super D-string simpler, and then carry out the \( SO(2) \) rotation. Fortunately, a consistent quantization procedure has recently appeared to the type IIB Green-Schwarz superstring action on \( AdS_5 \times S^5 \) [6, 7]. This quantization method can also be taken over the case of the super D-string on \( AdS_5 \times S^5 \) in a direct manner.

Let us explain how to carry out the \( SO(2) \) rotation in the process of the gauge-fixing procedure of the \( \kappa \)-symmetry. Before doing it, let us consider what orthogonal matrix makes \( m\mathcal{K} + \mathcal{I} \) an orthogonal form. By solving the eigenvalue equation, it is easy to show that \( U = \frac{1}{\sqrt{1 + (m - \sqrt{1 + m^2})^2}}[(m - \sqrt{1 + m^2})1 - \mathcal{E}] \) works well, that is,

\[
U^T (m\mathcal{K} + \mathcal{I}) U = -\sqrt{1 + m^2} \mathcal{K}.
\]  

(38)

Next let us recall how the \( \kappa \)-symmetry is fixed using the 'parallel to D3-brane' \( \Gamma \)-matrix projector \( \mathcal{P}_{\pm}^{IJ} \) [6, 7]. The gauge condition of the \( \kappa \)-symmetry is chosen to be

\[
\Theta^I_+ = 0,
\]

(39)

where

\[
\Theta^I_\pm \equiv \mathcal{P}_{\pm}^{IJ} \Theta^J, \quad \mathcal{P}_{\pm}^{IJ} \equiv \frac{1}{2}(\delta^{IJ} \pm \Gamma_{0123} \epsilon^{IJ}).
\]  

(40)

Then, it can be shown that the Wess-Zumino term becomes to be a quadratic form with respect to \( \theta^I_\pm \) by the change of the fermionic variables from \( \Theta^I \) to \( \theta^I \) where \( \Theta^I_+ = \frac{1}{y} \theta^I_+ \) and at the same time using the relations

\[
L^I_{is} = \frac{1}{y} \partial_i y^I, \quad L^I_{js} = sy^I \partial_i \theta^I_+,
\]

(41)

which hold in the gauge choice [34]. (See the original references [6, 7] for more detail.) The important observation here is that the projector \( \mathcal{P}_{\pm}^{IJ} \) is invariant under the orthogonal transformation \( U \), i.e., \( U^T \mathcal{P}_{\pm} U = \mathcal{P}_{\pm} \). This fact implies that if we change the spinor coordinates by \( \Theta^I = U^I J \tilde{\Theta}^J \), we have the relation \( \Theta^I_\pm = U^I J \tilde{\Theta}^J_\pm \). As a result, in the gauge condition (39), we reach the important relation \( \theta^I_\pm = U^I J \tilde{\theta}^J_\pm \). By using this relation as well as other ones, we can prove that the replacement (33) is indeed legitimated in terms of the \( SO(2) \) rotation of the spinor coordinates. (Incidentally, the Nambu-Goto action is invariant under this \( SO(2) \) rotation as can be checked easily.) Since the above arguments are a little formal, let us expose the related equations in order according to the above arguments in what follows:

\[
I \equiv \epsilon^{ij} \Omega_{ij}(m\mathcal{K} + \mathcal{I})
\]
\[
\begin{align*}
&= 2i \int_0^1 dse^{ij} \hat{L}_s \bar{\Theta} \hat{\Gamma}^{a}(mK + \mathcal{I})L_{js} \\
&= ie^{ij} \partial_i y^j \tilde{\theta}_+ \Gamma^t(mK + \mathcal{I}) \partial_j \theta_+ \\
&= ie^{ij} \partial_i y^j \tilde{\theta}_+ \Gamma^t(-\sqrt{1 + m^2}) \mathcal{K} \partial_j \tilde{\theta}_+. \quad (42)
\end{align*}
\]

Finally, we wish to close this section by commenting on one problem. In the previous work \cite{[15]}, it was shown that in the flat background geometry the super D-string action is exactly equivalent to the type IIB Green-Schwarz superstring action with some ”theta term” in terms of the same path integral method. It is then natural to ask what happens to the case of \(AdS_5 \times S^5\) background. We can easily show that this is not always the case. In fact, to demonstrate the equivalence one needs to perform a scale transformation, but the nonlinear sigma action (or the Nambu-Goto action) in the \(AdS_5 \times S^5\) background is scale invariant so that it is impossible to make the super D-string action coincide with the type IIB Green-Schwarz superstring action with ”theta term” in the case of the \(AdS_5 \times S^5\) background.

5 Discussions

In this paper, we have constructed a supersymmetric and \(\kappa\)-symmetric D-string action in the \(AdS_5 \times S^5\) background in supercoset construction. Starting with the super D-string action it has been shown that one can obtain the \(SL(2, \mathbb{Z})\) multiplet of type IIB strings with the correct tensions by performing the duality transformation. One of the most appealing points in this paper is that we have shown the existence of the \(SL(2, \mathbb{Z})\) multiplet of type IIB strings even in the \(AdS_5 \times S^5\) background, which was already expected in the past but not verified explicitly \cite{[11]}. We believe that we have shed some light on the S-duality transformation between the super D-string and fundamental Green-Schwarz superstring in a nontrivial curved background.

Finally we would like to make comments on future works. One interesting direction is to construct general super D-brane actions on the \(AdS_5 \times S^5\) and investigate various duality transformations. In particular, it is expected that the super D3-brane action \cite{[8]} transforms in the same way as in the super D-string. Moreover, the super D2-brane and D4-brane actions would transform in a manner that is expected from the relation between type IIA superstring theory and 11 dimensional M theory. This work is now under active investigation and will be reported in a separate publication \cite{[17]}.

Another valuable future work is to perform the quantization of general super D-brane actions like the super D-string and type IIB Green-Schwarz superstring on \(AdS_5 \times S^5\). We usually think that these p-brane actions with \(p > 1\) are unrenormalizable so that new dynamical degrees of freedom appear in the short distance region, and consequently problem of quantization is physically uninteresting. But it is worthwhile to point out that these arguments are entirely based on discussions of the field theory in the flat background, so the quantization of higher (super)p-branes on a curved space-time manifolds deserves further studies in future. For instance, it seems to be difficult to quantize the M5-brane action \cite{[18]} in a Lorentz covariant manner, but it may be possible to perform the quantization if one couples
supergravity background to the theory. Actually, in the quantization method adopted in [7] the Killing symmetry of the background metric field plays an essential role.

In addition to these future works, we wish to utilize the results obtained in curved background in order to understand background independent matrix models [19] in a more complete way. The progress is still under way but without fruits at present.

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