On Higher Dimensional Point Sets in General Position

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Abstract

A finite point set in $\mathbb{R}^d$ is in general position if no $d+1$ points lie on a common hyperplane. Let $\alpha_d(N)$ be the largest integer such that any set of $N$ points in $\mathbb{R}^d$ with no $d+2$ members on a common hyperplane, contains a subset of size $\alpha_d(N)$ in general position. Using the method of hypergraph containers, Balogh and Solymosi showed that $\alpha_2(N) < \frac{5}{6}N + o(1)$. In this paper, we also use the container method to obtain new upper bounds for $\alpha_d(N)$ when $d \geq 3$. More precisely, we show that if $d$ is odd, then $\alpha_d(N) < \frac{1}{2}N^{\frac{2}{d}} + o(1)$, and if $d$ is even, we have $\alpha_d(N) < \frac{1}{2}N^{\frac{2}{d}} + o(1)$.

We also study the classical problem of determining the maximum number $a(d, k, n)$ of points selected from the grid $[n]^d$ such that no $k+2$ members lie on a $k$-flat. For fixed $d$ and $k$, we show that

$$a(d, k, n) \leq O\left(n^{\frac{d}{k+2}} \left(1 - \frac{1}{\frac{k+2}{2} \lceil \frac{k+2}{2} \rceil + 1}\right)^d\right),$$

which improves the previously best known bound of $O\left(n^{\frac{d}{k+2}}\right)$ due to Lefmann when $k+2$ is congruent to 0 or 1 mod 4.

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1 Introduction

A finite point set in $\mathbb{R}^d$ is said to be in general position if no $d+1$ members lie on a common hyperplane. Let $\alpha_d(N)$ be the largest integer such that any set of $N$ points in $\mathbb{R}^d$ with no $d+2$ members on a hyperplane, contains $\alpha_d(N)$ points in general position.

In 1986, Erdős [8] proposed the problem of determining $\alpha_2(N)$ and observed that a simple greedy algorithm shows $\alpha_2(N) \geq \Omega(\sqrt{N})$. A few years later, Füredi [10] showed that

$$\Omega(\sqrt{N \log N}) < \alpha_2(N) < o(N),$$

where the lower bound uses a result of Phelps and Rödl [20] on partial Steiner systems, and the upper bound relies on the density Hales-Jewett theorem [11, 12]. In 2018, a breakthrough was made by Balogh and Solymosi [3], who showed that $\alpha_2(N) < N^{5/6+o(1)}$. Their proof was based on the method of hypergraph containers, a powerful technique introduced independently by Balogh, Morris, and Samotij [1] and by Saxton and Thomason [24], that reveals an underlying structure of the independent sets in a hypergraph. We refer interested readers to [2] for a survey of results based on this method.
In higher dimensions, the best lower bound for $\alpha_d(N)$ is due to Cardinal, Tóth, and Wood [5], who showed that $\alpha_d(N) \geq (N \log N)^{1/4}$, for every fixed $d \geq 2$. For upper bounds, Milicic [18] used the density Hales-Jewett theorem to show that $\alpha_d(N) = O(N)$ for every fixed $d \geq 2$. However, these upper bounds in [18], just like that in [10], are still almost linear in $N$. Our main result is the following.

**Theorem 1.** Let $d \geq 3$ be a fixed integer. If $d$ is odd, then $\alpha_d(N) \leq N^{\frac{1}{4} + \frac{1}{d} + o(1)}$. If $d$ is even, then $\alpha_d(N) \leq N^{\frac{1}{2} + \frac{1}{d} + o(1)}$.

Our proof of Theorem 1 is also based on the hypergraph container method. A key ingredient in the proof is a new supersaturation lemma for $(k+2)$-tuples of the grid $[n]^d$ that lie on a $k$-flat, which we shall discuss in the next section. Here, by a $k$-flat we mean a $k$-dimensional affine subspace of $\mathbb{R}^d$.

We also study the classical problem of determining the maximum number of points selected from the grid $[n]^d$ such that no $k+2$ members lie on a $k$-flat. The key ingredient of Theorem 1 mentioned above can be seen as a supersaturation version of this Turán-type problem. When $k = 1$, this is the famous no-three-in-line problem raised by Dudeney [7] in 1917: Is it true that one can select $\alpha_d(N) = \Theta(N)$ points in $[n]^2$ such that no three are collinear? Clearly, $2n$ is an upper bound as any vertical line must contain at most 2 points. For small values of $n$, many authors have published solutions to this problem obtaining the bound of $2n$ (e.g. see [9]), but for large $n$, the best known general construction is due to Hall et al. [13] with slightly fewer than $3n/2$ points.

More generally, we let $a(d, k, r, n)$ denote the maximum number of points from $[n]^d$ such that no $r$ points lie on a $k$-flat. Since $[n]^d$ can be covered by $n^{d-k}$ many $k$-flats, we have the trivial upper bound $a(d, k, r, n) \leq (r-1)n^{d-k}$. For certain values of $d, k$, and $r$, and fixed and $n$ tends to infinity, this bound is known to be asymptotically best possible: Many authors [22, 4, 17] noticed that $a(d, d-1, d+1, n) = \Theta(n)$ by looking at the modular moment curve over a finite field $\mathbb{Z}_p$; In [21], Pór and Wood proved that $a(3, 1, 3, n) = \Theta(n^2)$; Very recently, Sudakov and Tomon [25] showed that $a(d, k, r, n) = \Theta(n^{d-k})$ when $r > d^k$.

We shall focus on the case when $r = k + 2$ and write $a(d, k, n) := a(d, k, k + 2, n)$. Surprisingly, Lefmann [17] (see also [16]) showed that $a(d, k, n)$ behaves much differently than $\Theta(n^{d-k})$. In particular, he showed that

$$a(d, k, n) \leq O\left(n^{\frac{d}{\lfloor (d+2)/2 \rfloor}}\right).$$

Our next result improves this upper bound when $k + 2$ is congruent to 0 or 1 mod 4.

**Theorem 2.** For fixed $d$ and $k$, as $n \to \infty$, we have

$$a(d, k, n) \leq O\left(n^{\frac{d}{\lfloor (d+2)/2 \rfloor} \cdot \left(1 - \frac{1}{\lfloor (d+2)/2 \rfloor} \cdot \frac{1}{d(k+1)}\right)}\right).$$

For example, we have $a(4, 2, n) \leq O(n^{\frac{16}{15}})$ while Lefmann’s bound in [17] gives us $a(4, 2, n) \leq O(n^2)$, which coincides with the trivial upper bound. In particular, Theorem 2 tells us that, if 4 divides $k + 2$, then $a(d, k, n)$ only behaves like $\Theta(n^{d-k})$ when $d = k + 1$. This is quite interesting compared to the fact that $a(3, 1, n) = \Theta(n^2)$ proved in [21]. Lastly, let us note that the current best lower bound for $a(d, k, n)$ is also due to Lefmann [17], who showed that $a(d, k, n) \geq \Omega\left(n^{\frac{d-k}{d+1}}\right)$.

For integer $n > 0$, we let $[n] = \{1, \ldots, n\}$, and $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$. We systemically omit floors and ceilings whenever they are not crucial for the sake of clarity in our presentation. All logarithms are in base two.
2 \( (k + 2) \)-tuples of \([n]^d\) on a \(k\)-flat

In this section, we establish two lemmas that will be used in the proof of Theorem 1.

Given a set \( T \) of \( k + 2 \) points in \( \mathbb{R}^d \) that lie on a \( k \)-flat, we say that \( T \) is degenerate if there is a subset \( S \subset T \) of size \( j \), where \( 3 \leq j \leq k + 1 \), such that \( S \) lies on a \((j - 2)\)-flat. Otherwise, we say that \( T \) is non-degenerate. We establish a supersaturation lemma for non-degenerate \((k + 2)\)-tuples of \([n]^d\).

▶ Lemma 3. For real number \( \gamma > 0 \) and fixed positive integers \( d, k \), such that \( k \) is even and \( d - 2\gamma > (k - 1)(k + 2) \), any subset \( V \subset [n]^d \) of size \( n^{d-\gamma} \) spans at least \( \Omega(n^{(k+1)d-(k+2)\gamma}) \) non-degenerate \((k + 2)\)-tuples that lie on a \( k \)-flat.

Proof. Let \( V \subset [n]^d \) such that \(|V| = n^{d-\gamma}\). Set \( r = \frac{k}{2} + 1 \) and \( E_r = \binom{V}{r} \) to be the collection of \( r \)-tuples of \( V \). Notice that the sum of a \( r \)-tuple from \( V \) belongs to \([rn]^d\). For each \( v \in [rn]^d \), we define

\[
E_r(v) = \{ (v_1, \ldots, v_r) \in E_r : v_1 + \cdots + v_r = v \}.
\]

Then for \( T_1, T_2 \in E_r(v) \), where \( T_1 = \{ v_1, \ldots, v_r \} \) and \( T_2 = \{ u_1, \ldots, u_r \} \), we have

\[
v_1 + \cdots + v_r = v = u_1 + \cdots + u_r,
\]

which implies that \( T_1 \cup T_2 \) lies on a common \( k \)-flat. Let

\[
E_{2r} = \bigcup_{v \in [rn]^d} \bigcup_{T_1, T_2 \in E_r(v)} \{ T_1, T_2 \}.
\]

Hence, for each \( \{T_1, T_2\} \in E_{2r} \), \( T_1 \cup T_2 \) lies on a \( k \)-flat. Moreover, by Jensen’s inequality, we have

\[
|E_{2r}| = \sum_{v \in [rn]^d} \binom{|E_r(v)|}{2} \geq (rn)^d \left( \frac{\sum_{v \in [rn]^d} |E_r(v)|}{2} \right) = (rn)^d \left( \frac{|E_r|}{(rn)^d} \right)^2 \geq \frac{|E_r|^2}{4(rn)^d}.
\]

Since \( k \) and \( d \) are fixed and \( r = \frac{k}{2} + 1 \) and \(|V| = n^{d-\gamma}\),

\[
|E_r|^2 = \left( \frac{|V|}{r} \right)^2 = \left( \frac{|V|}{(k/2) + 1} \right)^2 \geq \Omega(n^{(k+2)(d-\gamma)}).
\]

Combining the two inequalities above gives

\[
|E_{2r}| \geq \Omega(n^{(k+1)d-(k+2)\gamma}).
\]

We say that \( \{T_1, T_2\} \in E_{2r} \) is good if \( T_1 \cap T_2 = \emptyset \), and the \((k + 2)\)-tuple \((T_1 \cup T_2)\) is non-degenerate. Otherwise, we say that \( \{T_1, T_2\} \) is bad. In what follows, we will show that at least half of the pairs (i.e. elements) in \( E_{2r} \) are good. To this end, we will need the following claim.

▶ Claim 4. If \( \{T_1, T_2\} \in E_{2r} \) is bad, then \( T_1 \cup T_2 \) lies on a \((k - 1)\)-flat.

Proof. Write \( T_1 = \{ v_1, \ldots, v_r \} \) and \( T_2 = \{ u_1, \ldots, u_r \} \). Let us consider the following cases.

Case 1. Suppose \( T_1 \cap T_2 \neq \emptyset \). Then, without loss of generality, there is an integer \( j < r \) such that

\[
v_1 + \cdots + v_j = u_1 + \cdots + u_j,
\]
where \( v_1, \ldots, v_j, u_1, \ldots, u_j \) are all distinct elements, and \( v_t = u_t \) for \( t > j \). Thus \(|T_1 \cup T_2| = 2j + (r - j)\). The \( 2j \) elements above lie on a \((2j - 2)\)-flat. Adding the remaining \( r - j \) points implies that \( T_1 \cup T_2 \) lies on a \((j - 2 + r)\)-flat. Since \( r = \frac{k}{2} + 1 \) and \( j \leq \frac{k}{2} \), \( T_1 \cup T_2 \) lies on a \((k - 1)\)-flat.

**Case 2.** Suppose \( T_1 \cap T_2 = \emptyset \). Then \( T_1 \cup T_2 \) must be degenerate, which means there is a subset \( S \subset T_1 \cup T_2 \) of \( j \) elements such that \( S \) lies on a \((j - 2)\)-flat, for some \( 3 \leq j \leq k + 1 \). Without loss of generality, we can assume that \( v_1 \notin S \). Hence, \( (T_1 \cup T_2) \setminus \{v_1\} \) lies on a \((k - 1)\)-flat. On the other hand, we have

\[
v_1 = u_1 + \cdots + u_r - v_2 - \cdots - v_r.
\]

Hence, \( v_1 \) is in the affine hull of \((T_1 \cup T_2) \setminus \{v_1\}\) which implies that \( T_1 \cup T_2 \) lies on a \((k - 1)\)-flat. ▶

We are now ready to prove the following claim.

**Claim 5.** At least half of the pairs in \( E_{2r} \) are good.

**Proof.** For the sake of contradiction, suppose at least half of the pairs in \( E_{2r} \) are bad. Let \( H \) be the collection of all the \( j \)-flats spanned by subsets of \( V \) for all \( j \leq k - 1 \). Notice that if \( S \subset V \) spans a \( j \)-flat \( h \), then \( h \) is also spanned by only \( j + 1 \) elements from \( S \). So we have

\[
|H| \leq \sum_{j=0}^{k-1} |V|^{j+1} \leq kn^{k(d-\gamma)}.
\]

For each bad pair \( \{T_1, T_2\} \in E_{2r}, T_1 \cup T_2 \) lies on a \( j \)-flat from \( H \) by Claim 4. By the pigeonhole principle, there is a \( j \)-flat \( h \) with \( j \leq k - 1 \) such that at least

\[
\frac{|E_{2r}|/2}{|H|} \geq \frac{\Omega(n^{(k+1)d-(k+2)\gamma})}{2kn^{k(d-\gamma)}} = \Omega(n^{d-2\gamma})
\]

bad pairs from \( E_{2r} \) have the property that their union lies in \( h \). On the other hand, since \( h \) contains at most \( n^{k-1} \) points from \([n]^d\), \( h \) can correspond to at most \( O(n^{(k-1)(k+2)}) \) bad pairs from \( E_{2r} \). Since we assumed \( d - 2\gamma > (k - 1)(k + 2) \), we have a contradiction for \( n \) sufficiently large. ▶

Each good pair \( \{T_1, T_2\} \in E_{2r} \) gives rise to a non-degenerate \((k + 2)\)-tuple \( T_1 \cup T_2 \) that lies on a \( k \)-flat. On the other hand, any such \((k + 2)\)-tuple in \( V \) will correspond to at most \( \binom{k+2}{r} \) good pairs in \( E_{2r} \). Hence, by Claim 5, there are at least

\[
\frac{|E_{2r}|}{2} \binom{k+2}{r} = \Omega(n^{(k+1)d-(k+2)\gamma})
\]

non-degenerate \((k + 2)\)-tuples that lie on a \( k \)-flat, concluding the proof. ▶

In the other direction, we will use the following upper bound.

**Lemma 6.** For real number \( \gamma > 0 \) and fixed positive integers \( d, k, \ell \), such that \( \ell < k + 2 \), suppose \( U, V \subset [n]^d \) satisfy \(|U| = \ell \) and \(|V| = n^{d-\gamma} \), then \( V \) contains at most \( n^{(k+1-\ell)(d-\gamma)+k} \) non-degenerate \((k + 2)\)-tuples that lie on a \( k \)-flat and contain \( U \).
Proof. If $U$ spans a $j$-flat for some $j < \ell - 1$, then by definition no non-degenerate $(k+2)$-tuple contains $U$. Hence we can assume $U$ spans a $(\ell-1)$-flat. Observe that a non-degenerate $(k+2)$-tuple $T$, which lies on a $k$-flat and contains $U$, must contain a $(k+1)$-tuple $T' \subset T$ such that $T'$ spans a $k$-flat and $U \subset T'$. Then there are at most $n^{(k+1)-\ell)(d-s)}$ ways to add $k + 1 - \ell$ points to $U$ from $V$ to obtain such $T'$. After $T'$ is determined, there are at most $n^k$ ways to add a final point from the affine hull of $T'$ to obtain $T$. So we conclude the proof by multiplication.

3 The container method: Proof of Theorem 1

In this section, we use the hypergraph container method to prove Theorem 1. We follow the method outlined in [3]. Let $H = (V(H),E(H))$ denote a $(k+2)$-uniform hypergraph. For any $U \subset V(H)$, its degree $\delta(U)$ is the number of edges containing $U$. For each $\ell \in [k+2]$, we use $\Delta(\ell)$ to denote the maximum $\delta(U)$ among all $U$ of size $\ell$. For parameter $\tau > 0$, we define the following quantity

$$
\Delta(H,\tau) = \frac{2^{((k+2)/2)}-1}{(k+2)|E(H)|} \sum_{\ell=0}^{k+2} \Delta(\ell)^{\ell-1}\ell!^{\ell+2}.
$$

Then we have the following hypergraph container lemma from [3], which is a restatement of Corollary 3.6 in [24].

Lemma 7. Let $H$ be a $(k+2)$-uniform hypergraph and $0 < \epsilon, \tau < 1/2$. Suppose that $\tau < 1/(200 \cdot (k+2) \cdot ((k+2)!))$ and $\Delta(H,\tau) \leq \epsilon/(12 \cdot (k+2)!))$. Then there exists a collection $C$ of subsets (containers) of $V(H)$ such that

1. Every independent set in $H$ is a subset of some $C \in C$;
2. $\log |C| \leq 1000 \cdot (k+2) \cdot ((k+2)!)^3 \cdot |V(H)| \cdot \tau \cdot \log(1/e) \cdot \log(1/\tau)$;
3. For every $C \in C$, the induced subgraph $H[C]$ has at most $\epsilon |E(H)|$ many edges.

The main result in this section is the following theorem.

Theorem 8. Let $k,r$ be fixed integers such that $r \geq k \geq 2$ and $k$ is even. Then for any $0 < \alpha < 1$, there are constants $c = c(k,r)$ and $d = d(k,r)$ such that the following holds. For infinitely many values of $N$, there is a set $V$ of $N$ points in $\mathbb{R}^d$ such that no $r + 3$ members of $V$ lie on an $r$-flat, and every subset of $V$ of size $cN^{\frac{r+2}{2} + \frac{\alpha}{2} + \frac{1}{2(2-r)}}$ contains $k+2$ members on an $k$-flat.

Before we prove Theorem 8, let us show that it implies Theorem 1. In dimensions $d_0 \geq 3$ where $d_0$ is odd, we apply Theorem 8 with $k = r = d_0 - 1$ to obtain a point set $V$ in $\mathbb{R}^d$ with the property that no $d_0 + 2$ members lie on a $(d_0 - 1)$-flat, and every subset of size $cN^{\frac{r+2}{2} + \frac{\alpha}{2} + \frac{1}{2(2-r)}}$ contains $d_0 + 1$ members on a $(d_0 - 1)$-flat. By projecting $V$ to a generic $d_0$-dimensional subspace of $\mathbb{R}^d$, we obtain $N$ points in $\mathbb{R}^{d_0}$ with no $d_0 + 2$ members on a common hyperplane, and no $cN^{\frac{r+2}{2} + \frac{\alpha}{2} + \frac{1}{2(2-r)}}$ members in general position.

In dimensions $d_0 \geq 4$ where $d_0$ is even, we apply Theorem 8 with $k = d_0 - 2$ and $r = d_0 - 1$ to obtain a point set $V$ in $\mathbb{R}^d$ with the property that no $d_0 + 2$ members on a $(d_0 - 1)$-flat, and every subset of size $cN^{\frac{r+2}{2} + \frac{\alpha}{2} + \frac{1}{2(2-r)}}$ contains $d_0$ members on a $(d_0 - 2)$-flat. By adding another point from this subset, we obtain $d_0 + 1$ members on a $(d_0 - 1)$-flat. Hence, by projecting to $V$ a generic $d_0$-dimensional subspace of $\mathbb{R}^d$, we obtain $N$ points in $\mathbb{R}^{d_0}$ with no $d_0 + 2$ members on a common hyperplane, and no $cN^{\frac{r+2}{2} + \frac{\alpha}{2} + \frac{1}{2(2-r)}}$ members in general position. This completes the proof of Theorem 1.
Proof of Theorem 8. We set $d = d(\alpha, k, r)$ to be a sufficiently large integer depending on $\alpha$, $k$, and $r$. Let $H$ be the hypergraph with $V(H) = [n]^d$ and $E(H)$ consists of non-degenerate $(k + 2)$-tuples $T$ such that $T$ lies on a $k$-flat. Let $C^0 = [n]^d$, $C^0 = (C^0)$, and $H^0 = H$. In what follows, we will apply the hypergraph container lemma to $H^0$ to obtain a family of containers $C^1$. For each $C^1_j \in C^1$, we consider the induced hypergraph $H^1_j = H[C^1_j]$, and we apply the hypergraph container lemma to it. The collection of containers obtained from all $H^1_j$ will form another collection of containers $C^2$. We iterate this process until each container in $C^1$ is sufficiently small, and moreover, we will only produce a small number of containers.

As a final step, we apply the probabilistic method to show the existence of the desired point set. We now flesh out the details of this process.

We start by setting $C^0 = [n]^d$, $C^0 = (C^0)$, and set $H^0 = H[C^0] = H$. Having obtained a collection of containers $C^1$, for each container $C^1_j \in C^1$ with $|C^1_j| \geq n^{\frac{3k}{k+1}d+k}$, we set $H^1_j = H[C^1_j]$. Let $C^1_j = (C^1_j)$ be defined by $|V(H^1_j)| = n^{d-\gamma}$. So, $\gamma \leq \frac{d}{k+1} - k$. We set $\tau = \tau(i, j) = n^{\frac{k}{k+1}d+\gamma+\epsilon}$, and $\gamma = \epsilon(i, j) = c_1 d^{-\alpha}$, where $c_1 = c_1(d, k)$ is a sufficiently large constant depending on $d$ and $k$. Then we can verify the following condition.

\textbf{Claim 9.} $\Delta(H^1_j, \tau) \leq \epsilon/(12 \cdot (k+2)!)$.

\textbf{Proof.} Since $|V(H^1_j)| = n^{d-\gamma}$, $\gamma \leq \frac{d}{k+1} - k$, and $d$ is sufficiently large, Lemma 3 implies that $|E(H^1_j)| \geq c_2 n^{(k+1)d-(k+2)\gamma}$ for some constant $c_2 = c_2(d, k)$. Hence, we have

$$\frac{|V(H^1_j)|}{|E(H^1_j)|} \leq \frac{n^{d-\gamma}}{c_2 n^{(k+1)d-(k+2)\gamma}} = \frac{1}{c_2 n^{kd-(k+1)\gamma}}.$$ 

On the other hand, by Lemma 6, we have

$$\Delta(H^1_j, \tau) \leq n^{(d-\gamma)(k+1-\ell)+k} \quad \text{for } \ell \leq k+2,$$

and obviously $\Delta_{k+2}(H^1_j) \leq 1$.

Applying these inequalities together with the definition of $\Delta$, we obtain

$$\Delta(H^1_j, \tau) = \frac{2^{(k+2)-1}|V(H^1_j)|}{(k+2)|E(H^1_j)|} \sum_{\ell=2}^{k+2} \frac{\Delta(H^1_j, \tau)}{\tau^{\ell-1}2^{\ell-1}} \leq \frac{c_3}{n^{kd-(k+1)\gamma}} \left( \sum_{\ell=2}^{k+1} \frac{n^{(k+1-\ell)(d-\gamma)+k}}{\tau^{\ell-1}2^{\ell-1}} + \frac{1}{\tau^{k+1}} \right) \leq \frac{c_3}{n^{kd-(k+1)\gamma}} \left( \sum_{\ell=2}^{k+1} \frac{n^{(k+1-\ell)(d-\gamma)+k}}{\tau^{\ell-1}2^{\ell-1}} + \frac{c_3}{\tau^{k+1}} \right),$$

for some constant $c_3 = c_3(d, k)$. Let us remark that the summation above is where we determined our $\tau$ and $\gamma$. In order to make the last term small, we choose $\tau = n^{\frac{k}{k+1}d+\gamma+\epsilon}$. Having determined $\tau$, in order for the first term in the summation to be small, we choose $\gamma \leq \frac{d}{k+1} - k$.

By setting $\epsilon = c_1 d^{-\alpha}$ with $c_1 = c_1(d, k)$ sufficiently large, we have

$$\Delta(H^1_j, \tau) \leq c_3 \sum_{\ell=2}^{k+1} \frac{n^{(k+1-\ell)(d+\gamma)-\ell+\epsilon}}{\tau^{\ell-1}2^{\ell-1}} + c_3 n^{-(k+1)\alpha} \leq c_3 kn^{-\alpha} + c_3 n^{-(k+1)\alpha} \leq \frac{\epsilon}{12(k+2)!}.$$

This verifies the claimed condition. \hfill ▷
Given the condition above, we can apply Lemma 7 to $H^i_j$ with chosen parameters $\tau$ and $\epsilon$. Hence we obtain a family of containers $C^{i+1}_j$ such that

$$|C^{i+1}_j| \leq 2^{10^r(k+2)((k+2)!)}|\mathcal{V}(H^i_j)|\tau \log(1/\epsilon) \log(1/\tau)$$

$$\leq 2^{c_4 n^{d+\alpha} \log^2 n},$$

for some constant $c_4 = c_4(d, k)$. In the other case where $|C^i_j| < n^{\frac{d}{d+\alpha}}$, we just define $C^{i+1}_j = \{C^i_j\}$. Then, for each container $C \in C^{i+1}_j$, we have either $|C| < n^{\frac{d}{d+\alpha}}$ or $|E(H[C])| \leq \epsilon|E(H^i_j)| \leq \epsilon|E(H)|$. After applying this procedure for each container in $C^i$, we obtain a new family of containers $C^{i+1} = \bigcup C^i_j$ such that

$$|C^{i+1}| \leq |C^i|2^{c_4 n^{\frac{d}{d+\alpha}} \log^2 n} \leq 2^{(i+1)c_4 n^{\frac{d}{d+\alpha}} \log^2 n}.$$

Notice that the number of edges in $H^i_j$ shrinks by a factor of $c_1 n^{-\alpha}$ whenever $i$ increases by one, while on the other hand, Lemma 3 tells us that every large subset $C \subset [n]^d$ induces many edges in $H$. Hence, after at most $t \leq c_5/\alpha$ iterations, for some constant $c_5 = c_5(d, k)$, we obtain a collection of containers $C = C^t$ such that: each container $C \in C$ satisfies $|C| < n^{\frac{d}{d+\alpha}}$; every independent set of $H$ is a subset of some $C \in C$; and

$$|C| \leq 2^{(c_5/\alpha)c_4 n^{\frac{d}{d+\alpha}} \log^2 n}.$$

Before we construct the desired point set, we make the following crude estimate.

**Claim 10.** The grid $[n]^d$ contains at most $O(n^r d + 2r)$ many $(r + 3)$-tuples that lie on a $r$-flat.

**Proof.** Let $T$ be an arbitrary $(r + 3)$-tuple that spans a $j$-flat. There are at most $n^{(j+1)d}$ ways to choose a subset $T' \subset T$ of size $j + 1$ that spans the affine hull of $T$. After this $T'$ is determined, there are at most $n^{(r+2-j)}d$ ways to add the remaining $r + 2 - j$ points from the $j$-flat spanned by $T'$. Then the total number of $(r + 3)$-tuples that lie on a $r$-flat is at most

$$\sum_{j=1}^{r} n^{(j+1)d+(r+2-j)} \leq \sum_{j=1}^{r} n^{(j+1)d+(r+2-j)r} \leq r n^{(r+1)d + 2r},$$

since we can assume $d > r$.

Now, we randomly select a subset of $[n]^d$ by keeping each point independently with probability $p$. Let $S$ be the set of selected elements. Then for each $(r + 3)$-tuple $T$ in $S$ that lies on an $r$-flat, we delete one point from $T$. We denote the resulting set of points by $S'$. By the claim above, the number of $(r + 3)$-tuples in $[n]^d$ that lie on a $r$-flat is at most $c_6 n^{(r+1)d + 2r}$ for some constant $c_6 = c_6(r)$. Therefore,

$$\mathbb{E}[|S'|] \geq pn^{d} - c_6 p^{r+3} n^{(r+1)d + 2r}.$$  

By setting $p = (2c_6)^{-1} n^{-\frac{d}{r+2}}$, we have

$$\mathbb{E}[|S'|] \geq \frac{pm^d}{2} = \Omega(n^{2d-1}).$$

Finally, we set $m = (c_7/\alpha)n^{d/4 + 2\alpha}$ for some sufficiently large constant $c_7 = c_7(d, k, r)$. Let $X$ denote the number of independent sets of size $m$ in $S'$. Using the family of containers
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This completes the proof.

\[ C \]

such that

solution does not have a solution with \( a \)

In this section, we will give a proof of Theorem 2. Let

for some constant \( c_s = c_s(d, k, r) \). Since \( r \geq k, 0 < \alpha < 1, \) and \( d \) is large, for \( n \) sufficiently large, we have

\[ c_s \alpha n \frac{(k-r-1)d}{(k+r-1)(d-r)} + k - \frac{2r}{d} - 2\alpha < 1/2. \]

Hence, we have \( E[X] \leq o(1) \) as \( n \) tends to infinity. Notice that \( |S'| \) is exponentially concentrated around its mean by Chernoff’s inequality. Therefore, some realization of \( S' \) satisfies: \( |S'| = N = \Omega(n^{2(d-r)/(r+2)}); \) \( S' \) contains no \((r+3)\)-tuples on a \( r \)-flat; and \( H[S'] \) does not contain an independent set of size

\[ m = (c_7/\alpha)n \frac{d}{r} + 2\alpha \leq cN \frac{r+2}{r+1} + \frac{(r+2)^2}{(r+1)(d-r)} + \frac{r+2}{d} 2\alpha \leq cN \frac{r+2}{(r+1)(d-r)} + \alpha, \]

for some constant \( c = c(\alpha, d, k, r) \). Here we assume \( d \) is sufficiently large so that

\[ \frac{(r+2)r}{2(k+1)(d-r)} + \frac{r+2}{d} 2\alpha \leq \alpha. \]

This completes the proof. \( \blacklozenge \)

### 4 Avoiding non-trivial solutions: Proof of Theorem 2

In this section, we will give a proof of Theorem 2. Let \( V \subset [n]^d \) such that there are no \( k + 2 \) points that lie on a \( k \)-flat. In [17], Lefmann showed that \( |V| \leq O \left( n \frac{(k+r)^2}{r+1} \right) \). To see this, assume that \( k \) is even and consider all elements of the form \( v_1 + \cdots + v_{k+1} \), where \( v_i \neq v_j \) and \( v_i \in V \). All of these elements are distinct, since otherwise we would have \( k + 2 \) points on a \( k \)-flat. In other words, the equation

\[
\left( x_1 + \cdots + x_{\frac{k+1}{2}} \right) - \left( x_{\frac{k+2}{2}} + \cdots + x_{k+2} \right) = 0,
\]

does not have a solution with \( \{x_1, \ldots, x_{\frac{k+1}{2}}\} \) and \( \{x_{\frac{k+2}{2}}, \ldots, x_{k+2}\} \) being two different \( (\frac{k}{2} + 1) \)-tuples of \( V \). Therefore, we have \( \frac{|V|}{2^{k+2}} \leq (kn)^d \), and this implies Lefmann’s bound.

More generally, let us consider the equation

\[
c_1 x_1 + c_2 x_2 + \cdots + c_r x_r = 0,
\]

with constant coefficients \( c_i \in \mathbb{Z} \) and \( \sum_i c_i = 0 \). Here, the variables \( x_i \) takes value in \( \mathbb{Z}^d \). A solution \( (x_1, \ldots, x_r) \) to equation (1) is called **trivial** if there is a partition \( P : [r] = I_1 \cup \cdots \cup I_t, \) such that \( x_j = x_{j+} \) if and only if \( j, j+ \in I_i, \) and \( \sum_{j \in I_i} c_j = 0 \) for all \( i \in [t] \). In other words,
being trivial means that, after combining like terms, the coefficient of each $x_i$ becomes zero. Otherwise, we say that the solution $(x_1, \ldots, x_r)$ is non-trivial. A natural extremal problem is to determine the maximum size of a set $A \subset [n]^d$ with only trivial solutions to (1). When $d = 1$, this is a classical problem in additive number theory, and we refer the interested reader to [23, 19, 15, 6].

By combining the arguments of Cilleruelo and Timmons [6] and Jia [14], we establish the following theorem.

**Theorem 11.** Let $d, r$ be fixed positive integers. Suppose $V \subset [n]^d$ has only trivial solutions to each equation of the form

$$c_1 ((x_1 + \cdots + x_r) - (x_{r+1} + \cdots + x_{2r})) = c_2 ((x_{2r+1} + \cdots + x_{3r}) - (x_{3r+1} + \cdots + x_{4r})),$$

for integers $c_1, c_2$ such that $1 \leq c_1, c_2 \leq n^{\frac{d}{2}}$. Then we have

$$|V| \leq O \left( n^{\frac{d}{2}} \left( 1 - \frac{1}{2^{d/2}} \right) \right).$$

Notice that Theorem 2 follows from Theorem 11. Indeed, when $k + 2$ is divisible by 4, we set $r = (k + 2)/4$. If $V \subset [n]^d$ contains $k + 2$ points $\{v_1, \ldots, v_{k+2}\}$ that is a non-trivial solution to (2) with $x_i = v_i$, then $\{v_1, \ldots, v_{k+2}\}$ must lie on a $k$-flat. Hence, when $k + 2$ is divisible by 4, we have

$$a(d, k, n) \leq O \left( n^{\frac{d}{2}} \left( 1 - \frac{1}{2^{d/2}} \right) \right).$$

Since we have $a(d, k, n) < a(d, k - 1, n)$, this implies that for all $k \geq 2$, we have

$$a(d, k, n) \leq O \left( n^{\frac{d}{2}} \left( 1 - \frac{1}{2^{d/2}} \right) \right).$$

In the proof of Theorem 11, we need the following well-known lemma (see e.g. [6]Lemma 2.1 and [23]Theorem 4.1). For $U, T \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we define

$$\Phi_{U-T}(x) = \{(u, t) : u - t = x, u \in U, t \in T\}.$$ 

**Lemma 12.** For finite sets $U, T \subset \mathbb{Z}^d$, we have

$$\frac{|[U]|^2}{|U + T|} \leq \sum_{x \in \mathbb{Z}^d} |\Phi_{U-U}(x)| \cdot |\Phi_{T-T}(x)|.$$

**Proof of Theorem 11.** Let $d, r$, and $V$ be as given in the hypothesis. Let $m \geq 1$ be an integer that will be determined later. We define

$$S_r = \{v_1 + \cdots + v_r : v_i \in V, v_i \neq v_j\},$$

and a function

$$\sigma : \binom{V}{r} \to S_r, \{v_1, \ldots, v_r\} \mapsto v_1 + \cdots + v_r.$$

Notice that $\sigma$ is a bijection. Indeed, suppose on the contrary that $v_1 + \cdots + v_r = v'_1 + \cdots + v'_r$
for two different \( r \)-tuples in \( V \). Then by setting \((x_1, \ldots, x_r) = (v_1, \ldots, v_r), (x_{r+1}, \ldots, x_{2r}) = (v'_1, \ldots, v'_r), (x_{2r+1}, \ldots, x_{3r}) = (x_{3r+1}, \ldots, x_{4r})\) arbitrarily, and \( c_1 = c_2 = 1 \), we obtain a non-trivial solution to (2), which is a contradiction. In particular, we have \(|S_r| = \binom{|V|}{r}\).

For \( j \in [m] \) and \( w \in \mathbb{Z}_j^d \), we let
\[
U_{j,w} = \{ u \in \mathbb{Z}^d : ju + w \in S_r \}.
\]
Notice that for fixed \( j \in [m] \), we have
\[
\sum_{w \in \mathbb{Z}_j^d} |U_{j,w}| = \sum_{w \in \mathbb{Z}_j^d} |\{ v \in S_r : v \equiv w \mod j \}| = |S_r|.
\]
Applying Jensen’s inequality to above, we have
\[
\sum_{w \in \mathbb{Z}_j^d} |U_{j,w}|^2 \geq |S_r|^2 / j^d.
\] (3)

For \( i \geq 0 \), we define
\[
\Phi_{U_{j,w} - U_{j,w}}^i(x) = \{ (u_1, u_2) \in \Phi_{U_{j,w} - U_{j,w}}(x) : |\sigma^{-1}(ju_1 + w) \cap \sigma^{-1}(ju_2 + w)| = i \}.
\]
It’s obvious that these sets form a partition of \( \Phi_{U_{j,w} - U_{j,w}}(x) \). We also make the following claims.

\[ \triangleright \text{Claim 13.} \quad \text{For a fixed } x \in \mathbb{Z}_d, \text{ we have}
\]
\[
\sum_{j \in [m]} \sum_{w \in \mathbb{Z}_j^d} |\Phi_{U_{j,w} - U_{j,w}}^i(x)| \leq 1,
\]
Proof. For the sake of contradiction, suppose the summation above is at least two, then we have \((u_1, u_2) \in \Phi_{U_{j,w} - U_{j,w}}^i(x)\) and \((u_3, u_4) \in \Phi_{U_{j',w'} - U_{j',w'}}^i(x)\) such that either \((u_1, u_2) \neq (u_3, u_4)\) or \((j, w) \neq (j', w')\).

Let \( s_1, s_2, s_3, s_4 \in \mathbb{Z}_r \) such that \( s_1 = ju_1 + w, s_2 = ju_2 + w, s_3 = j'u_3 + w', s_4 = j'u_4 + w'\) and write \( \sigma^{-1}(s_1) = \{ v_{1,1}, \ldots, v_{1,r} \} \). Notice that \( u_1 - u_2 = w = u_3 - u_4 \). Putting these equations together gives us
\[
\begin{align*}
&j'((v_{1,1} + \cdots + v_{1,r}) - (v_{2,1} + \cdots + v_{2,r})) = j((v_{3,1} + \cdots + v_{3,r}) - (v_{4,1} + \cdots + v_{4,r})).
\end{align*}
\] (4)
It suffices to show that (4) can be seen as a non-trivial solution to (2). The proof now falls into the following cases.

Case 1. Suppose \( j \neq j' \). Without loss of generality we can assume \( j' > j \). Notice that \((u_1, u_2) \in \Phi_{U_{j,w} - U_{j,w}}^0(x)\) implies
\[
\{ v_{1,1}, \ldots, v_{1,r} \} \cap \{ v_{2,1}, \ldots, v_{2,r} \} = \emptyset.
\]
Then after combining like terms in (4), the coefficient of \( v_1^j \) is at least \( j' - j \), which means this is indeed a non-trivial solution to (2).

Case 2. Suppose \( j = j' \), then we must have \( s_1 \neq s_3 \). Indeed, if \( s_1 = s_3 \), we must have \( w = w' \) (as \( s_1 \) modulo \( j \) equals \( s_3 \) modulo \( j' \)) and \( s_2 = s_4 \) (as \( j'(s_1 - s_2) = j(s_3 - s_4) \)). This is a contradiction to either \((u_1, u_2) \neq (u_3, u_4)\) or \((j, w) \neq (j', w')\).

Given \( s_1 \neq s_3 \), we can assume, without loss of generality, \( v_{1,1} \not\in \{ v_{3,1}, \ldots, v_{3,r} \} \). Again, we have \( \{ v_{1,1}, \ldots, v_{1,r} \} \cap \{ v_{2,1}, \ldots, v_{2,r} \} = \emptyset \). Hence, after combining like terms in (4), the coefficient of \( v_1^j \) is positive and we have a non-trivial solution to (2).
\[\sum_{w \in \mathbb{Z}_d^j} \sum_{x \in \mathbb{Z}^d} |\Phi_{U_{j,w} - U_{j,w}}(x)| \cdot |\Phi_{T-T}(x)| \leq |V|^{2r-1}|T|.\]

**Proof.** The summation on the left-hand side counts all (ordered) quadruples \((u_1, u_2, t_1, t_2)\) such that \((u_1, u_2) \in \Phi_{U_{j,w} - U_{j,w}}(t_1 - t_2)\). For each such a quadruple, let \(s_1, s_2 \in S_v\) such that

\[s_1 = j u_1 + w \quad \text{and} \quad s_2 = j u_2 + w.\]

There are at most \(|V|^{2r-i}\) ways to choose a pair \((s_1, s_2)\) satisfying \(\sigma^{-1}(s_1) \cap \sigma^{-1}(s_2) = i\). Such a pair \((s_1, s_2)\) determines \((u_1, u_2)\) uniquely. Moreover, \((s_1, s_2)\) also determines the quantity

\[t_1 - t_2 = u_1 - u_2 = \frac{s_1 - w}{j} - \frac{s_2 - w}{j} = \frac{1}{j}(s_1 - s_2).\]

After such a pair \((s_1, s_2)\) is chosen, there are at most \(|T|\) ways to choose \(t_1\) and this will also determine \(t_2\). So we conclude the claim by multiplication. \(\blacksquare\)

Now, we set \(T = \mathbb{Z}_d^j\) for some integer \(\ell\) to be determined later. Notice that \(U_{j,w} + T \subset \{0, 1, \ldots, \lfloor rn/j \rfloor + \ell - 1\}^d\), which implies

\[|U_{j,w} + T| \leq (rn/j + \ell)^d. \quad (5)\]

By Lemma 12, we have

\[\frac{|U_{j,w}|^2|T|^2}{|U_{j,w} + T|} \leq \sum_{x \in \mathbb{Z}^d} |\Phi_{U_{j,w} - U_{j,w}}(x)| \cdot |\Phi_{T-T}(x)|.\]

Summing over all \(j \in [m]\) and \(w \in \mathbb{Z}_d^j\), and using Claims 13 and 14, we can compute

\[\sum_{j \in [m]} \sum_{w \in \mathbb{Z}_d^j} \frac{|U_{j,w}|^2|T|^2}{|U_{j,w} + T|} \leq \sum_{j \in [m]} \sum_{w \in \mathbb{Z}_d^j} \sum_{x \in \mathbb{Z}^d} |\Phi_{U_{j,w} - U_{j,w}}(x)| \cdot |\Phi_{T-T}(x)| \]

\[= \sum_{x \in \mathbb{Z}^d} \sum_{j \in [m]} \sum_{w \in \mathbb{Z}_d^j} \left(|\Phi_{U_{j,w} - U_{j,w}}(x)| + \sum_{i=1}^r |\Phi_{i_{U_{j,w} - U_{j,w}}(x)}|\right) |\Phi_{T-T}(x)| \]

\[\leq \sum_{x \in \mathbb{Z}^d} \Phi_{T-T}(x) \sum_{j \in [m]} \sum_{w \in \mathbb{Z}_d^j} |\Phi_{U_{j,w} - U_{j,w}}(x)| + \sum_{j \in [m]} \sum_{i=1}^r |V|^{2r-i-1} \ell^d \]

\[\leq \sum_{x \in \mathbb{Z}^d} \Phi_{T-T}(x) + \sum_{j \in [m]} \sum_{i=1}^{r-1} |V|^{2r-i-1} \ell^d \]

\[\leq \ell^{2d} + rm|V|^{2r-1} \ell^d.\]
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On the other hand, using (3) and (5), we can compute
\[
\sum_{j \in [m]} \sum_{w \in \mathbb{Z}^d_j} |U_{j,w}|^2 |T_j|^2 \geq \sum_{j \in [m]} \sum_{w \in \mathbb{Z}^d_j} \frac{|U_{j,w}|^2 \ell^{2d}}{(rn/j + \ell)^d}
\]
\[
\geq \sum_{j \in [m]} \frac{|S_r|^{2\ell^{2d}}}{j^d(rn/j + \ell)^d}
\]
\[
= \sum_{j \in [m]} \frac{|S_r|^{2\ell^{2d}}}{(rn + j\ell)^d}
\]
\[
\geq \frac{m|S_r|^{2\ell^{2d}}}{(rn + m\ell)^d}.
\]
Combining the two inequalities above gives us
\[
\frac{m|S_r|^{2\ell^{2d}}}{(rn + m\ell)^d} \leq \ell^{2d} + rm|V|^{2r-1} \ell^d
\]
\[
\Rightarrow |S_r|^2 \leq \frac{(rn + m\ell)^d}{m} + r|V|^{2r-1}(rn + m\ell)^d.\]

By setting \( m = n^{\frac{d-1}{2}} \) and \( \ell = n^{1-\frac{d}{2r+1}} \), we get
\[
\left(\frac{|V|}{r}\right)^2 = |S_r|^2 \leq cn^{d-\frac{d}{2r+1}} + c|V|^{2r-1}n^{\frac{d^2}{2r+1}},
\]
for some constant \( c \) depending only on \( d \) and \( r \). We can solve from this inequality that
\[
|V| = O\left(n^{\frac{d}{r}(1-\frac{d}{2r+1})}\right),
\]
completing the proof.

\section{Concluding remarks}

1. One can consider a generalization of the quantity \( \alpha_d(N) \). We let \( \alpha_{d,s}(N) \) be the largest integer such that any set of \( N \) points in \( \mathbb{R}^d \) with no \( d + s \) members on a hyperplane, contains \( \alpha_{d,s}(N) \) points in general position. Hence, \( \alpha_d(N) = \alpha_{d,2}(N) \). Following the arguments in our proof of Theorem 1 with a slight modification, we show the following.

\begin{theorem}
Let \( d, s \geq 3 \) be fixed integers. If \( d \) is odd and \( \frac{2d+s-2}{d+s-2} < \frac{d-1}{d} \), then \( \alpha_{d,s}(N) \leq N^{\frac{1}{2} + o(1)} \). If \( d \) is even and \( \frac{2d+s-2}{d+s-2} < \frac{d-2}{d} \), then \( \alpha_{d,s}(N) \leq N^{\frac{1}{2} + o(1)} \).
\end{theorem}

For example, when we fix \( d = 3 \) and \( s \geq 5 \), we have \( \alpha_{d,s}(N) \leq N^{\frac{1}{2} + o(1)} \). In the other direction, it is easy to show that \( \alpha_{d,s}(N) \geq \Omega(N^{1/d}) \) for any fixed \( d, s \geq 2 \) (see [8]).

\begin{problem}
Are there fixed integers \( d, s \geq 3 \) such that \( \alpha_{d,s}(N) \leq o(N^{\frac{1}{2}}) \)?
\end{problem}

2. We call a subset \( V \subset [n]^d \) an \( m \)-fold \( B_g \)-set if \( V \) only contains trivial solutions to the equations
\[
c_1x_1 + c_2x_2 + \cdots + c_gx_g = c_1'x'_1 + c_2'x'_2 + \cdots + c_gx'_g,
\]
with constant coefficients \( c_i \in [n] \). We call 1-fold \( B_g \)-sets simply \( B_g \)-sets. By counting distinct sums, we have an upper bound \( |V| \leq O(n^{\frac{d}{2}}) \) for any \( B_g \)-set \( V \subset [n]^d \).

Our Theorem 11 can be interpreted as the following phenomenon: by letting \( m \) grow as some proper polynomial in \( n \), we have an upper bound for \( m \)-fold \( B_g \)-sets, where \( g \) is even, which gives a polynomial-saving improvement from the trivial \( O(n^{\frac{d}{2}}) \) bound. We believe this phenomenon should also hold without the parity condition on \( g \).
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