Structure Constants in $\mathcal{N} = 4$ SYM and Separation of Variables

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We propose a new framework for computing three-point functions in planar $\mathcal{N} = 4$ super Yang-Mills where these correlators take the form of multiple integrals of Separation of Variables type. We test this formalism at weak coupling at leading and next-to-leading orders in a non-compact SL(2) sector of the theory and all the way to next-to-next-to-leading orders for a compact SU(2) sector. We find evidence that wrapping effects can also be incorporated.

I. INTRODUCTION

Solving planar $\mathcal{N} = 4$ SYM in a satisfactory way would mean efficiently computing both the spectrum as well as higher point correlation functions – starting with three points – at any value of the ’t Hooft coupling.

A formalism for computing three point functions by means of integrability exists. It is the so called hexagon approach [1]. It is conjectured to hold for any coupling indeed but it is not easy to use, at least not by the remarkable standards of the spectrum quantum spectral curve approach [2]. With hexagons one needs to go over infinitely many sums and integrals to produce such correlators. At weak coupling perturbation theory these sums and integrals truncate [3–5] and we can use hexagons to produce a wealth of data to test any putative new framework. We will do it all the time below.

Here we suggest a new approach for correlation functions in $\mathcal{N} = 4$ SYM based on the Baxter Q-functions. The final representations are of so-called separation of variables (SoV) type where these Baxter functions are integrated against simple universal measures to produce the structure constants.

Q-functions play the central role in the quantum spectral curve, the top of the line tech for computing the dimension of any single trace operator in this conformal gauge theory so it is only natural to look for a similar central role for these objects in the context of other physical observables such as the OPE structure constants.

In the most conventional integrable spin chains, Q-functions are polynomials whose roots are the so called Bethe roots $v_k$. In SYM these polynomials are present at leading order at weak coupling but at higher coupling they get dressed by quantum non-polynomial factors. This is expected; as we crank up the coupling we are no longer dealing with a spin chain or with a classical string but something in between and so these Baxter polynomials get naturally deformed. For the so-called SL(2) sector of the theory which includes all operators of the schematic form $\text{Tr}(D^J Z^L) + \text{permutations}$ we have

$$Q(u) = \left( \prod_{k=1}^{J} \frac{u - v_k}{v_k^+ v_k^-} \right) e^{\frac{2}{\beta} Q_1^+ H_1^+(u) + \frac{2}{\beta} Q_1^- H_1^-(u) + O(\beta^4)}$$

where the charges $Q_1^\pm$ are simple functions of the Zhukowsky variables $x_k^\pm = x^\pm(v_k)$ and $H_k^\pm$ are harmonic functions, see appendix A.1

An operator is given by a Q-function [1]. What we are after is thus a functional eating these functions and spitting out a number, the structure constant.

FIG. 1. Three point functions of operators with sizes $L_1$, $L_2$ and $L_3$. At tree level there are $\ell_{ij} = (L_i + L_j - L_k)/2$ propagators between operators $i$ and $j$; these integers $\ell_{ij}$ are called the bridge lengths. We will usually use $L \equiv L_1$ for the length of the non-BPS operator and $\ell \equiv \ell_{12}$ for its so-called left adjacent bridge (the right adjacent bridge will have length $L - \ell$); the bottom bridge we will often denote as $\ell_{23} \equiv \ell_{23}$. 
For the most part we will consider a single non-BPS operator and two BPS operators. The geometry of the three point function depends on the size of all these operators or — equivalently — on the so-called bridges that connect them as reviewed in figure [\text{I}].

The proposal is that (the square of) the structure constant is given by a ratio of SoV like scalar products

\[ C_{200}^2 = \frac{\langle \mathcal{J}_1 \rangle^2}{2!} \langle Q_1, Q_2 \rangle_L \]  

(2)

Here \( \mathcal{J} = J + g^2 Q_1^+ \) and the scalar product

\[ \langle Q_1, Q_2 \rangle_\ell \equiv \left( \frac{J_1 + J_2 + \ell - 1}{\ell - 1} \right) \int d\mu_{\ell-1} \prod_{i=1}^{\ell-1} Q_1(u_i) Q_2(u_i) \]  

(3)

with a nice factorized measure

\[ d\mu_\ell = \prod_{i=1}^{\ell} du_i \mu_1(u_i) \prod_{i=1}^{\ell-1} \prod_{j=i+1}^{\ell} \mu_2(u_i, u_j) \]  

(4)

which is constructed out of the building blocks (using, \( s_a, c_a, t_a \) for \( \sinh(\pi a), \cosh(\pi a), \tanh(\pi a) \))

\[ \mu_1(u) = \frac{2\pi}{2\pi^2} \left( 1 + g^2 \pi^2 (3u^2_a - 1) \right) + \ldots \]  

(5)

\[ \mu_2(u, v) = \frac{\pi(u-v) s_a v - (1+g^2 \pi^2 ((t_a + v)^2 - \frac{3}{4}) + \ldots)}{c_a c_v} \]  

(6)

valid to leading order (LO) and next-to-leading order (NLO); the dots in the above expressions are NNLO corrections which start at \( O(g^4) \). The structure of the result might be corrected at subleading orders as explained below. The contours in (2) are the real axis for all \( u_i \).

We can prove (3) by exhaustion by comparing it with hexagon produced data for numerous \( L \)'s and \( \ell \)'s and for different operators corresponding to different \( \mathcal{Q} \)-functions. We did it; (2) is correct. We can also establish it more honestly as discussed in the next section.

Representations like (2) are the main results of this letter. In section \[ \text{III} \] we present an SU(2) counterpart of this representation in \[ \text{[19]} \]; we managed to fully test it to LO, NLO and NNLO. We compare these two rank one sectors in section \[ \text{IV} \]. In the discussion section \[ \text{V} \] we discuss further generalizations such as multiple spinning operators and speculate on all loop structures we expect to find. Many appendices complement the main text.

II. SL(2)

At leading order, that is at tree level, correlation functions are given by Wick contractions. Each operator can be through of as a spin chain and these Wick contractions are thus given by spin chain scalar products \[ \text{[6,8]} \]. For the SL(2) sector such scalar products can be cast as SoV integrals \[ \text{[9]} \] — see also \[ \text{[10]} \] in the \( N = 4 \) context.

Once we properly normalize all these scalar products as in \[ \text{[7]} \] to extract the structure constant we precisely end up with \[ \text{[2]} \] for \( g = 0 \).

What we find remarkable is that this classical \( g = 0 \) expression seems to have a nice quantum lift as anticipated in the introduction.

Let us first discuss the simplest possible case where a twist two operator \( (L = 2) \) splits evenly into two BPS operators \( (\ell = 1) \) so that the proposal \[ \text{[2]} \] simply reads

\[ C_{200}^2 = \frac{\langle \mathcal{J}_1 \rangle^2}{2!} \left( \int d\mu_1(u) Q(u)^2 \right)^{-1}. \]  

(7)

We derived the single-particle measure \( \mu_1 \) as well as the quantum deformed Baxter functions \( Q \) in two ways. Both are based on looking for a measure such that, for two different twist–two states (i.e. with different spins \( J \) and \( J' \)), we have an orthogonality relation

\[ \int \mu_1(u) Q_J(u) Q_{J'}(u) \propto \delta_{J,J'}. \]  

(8)

a powerful relation which has been extensively exploited by \[ \text{[11,13]} \] in numerous SoV studies, most of which for rational spin chains or for the fishnet reduction \[ \text{[14]} \] of SYM, see most notably \[ \text{[15]} \].

The first uses the fact that the Bethe roots \( v_k \) of twist two operators are given by the zeros of Hahn polynomials, a fact that persists at NLO. We have that \( \prod_k (u - v_k) \) is proportional to the Hahn polynomial \( \text{[16,17]} \) \( p_J(u|a,b,b,a) \) where \( a - b = 2i\sqrt{2g} \) and \( a + b = 1 + 4g^2 H_1(J) \). Hahn polynomials are orthogonal with a simple measure as reviewed in appendix \[ \text{[11]} \] which allows us to derive \[ \text{[3]} \] up to some simple tuning due to the mild \( J \) dependence in the polynomial parameters \( a,b \), see appendix for details. It is the \( J \) dependence that renders the derivation non-trivial and which is responsible for the needed modification of the Baxter functions in \[ \text{[4]} \].

The second derivation follows \[ \text{[12]} \] closely (the novelty being the extension to \( g^2 \) corrections) and makes use of the Baxter equation \[ \text{[18]} \]

\[ B \circ Q = T(u) Q(u) \]  

(9)

where \( Q(u) \) are the Baxter polynomials (i.e. just the parentheses in \[ \text{[1]} \]) and the Baxter operator

\[ B \equiv (x^+) L \left( 1 - \frac{g^2}{x^- (u)} (Q_1^+ - iQ_1^-) \right) e^{i\partial_u} + c.c. \]  

(10)

Note that at \( g = 0 \) we have \( x^+ = u + i/2 \) and most importantly, \( B \) becomes a simple linear operator but as we turn on \( g \) corrections this is no longer true since the charges \( Q_1^\pm \) depend on \( Q \). Consider first \( g = 0 \) and \( L = 2 \) so that the transfer matrix is \( T(u) = 2u^2 + c_J \). Multiplying \[ \text{[9]} \] by the sought after measure \( \mu_1 \) and by another Baxter polynomial with a different spin, subtracting that to the same thing with the spin swapped and integrating yields

\[ \int \mu_1 Q_J Q_{J'} = \int \mu_1 (Q_J(B \circ Q_J) - (B \circ Q_{J'}))Q_J. \]  

(11)
If we manage to make $B$ self-adjoint we will thus have the required orthogonality. An $i$-periodic $\mu_1$ would do the job since under shifts of contour the two terms of Baxter would swap and cancel in the right hand side. In detail, $Q_j(u)(u \pm \frac{i}{2})^2 Q_j(u \pm i)\mu_1(u)$ becomes $Q_j(u \mp i)(u \mp \frac{i}{2})^2 Q_j(u)\mu_1(u \mp i)$ under a shift of contour by $\mp i$ leading to an interchange (and thus cancellation) of the two terms in (11) once we use that the measure is periodic. To make this manipulation kosher we need to make sure no singularities are picked when deforming the contour and to make the measure acceptable we need to make sure it decays fast enough at infinity so that it can be integrated against polynomials of arbitrary degree. Both this problems are solved at once with

$$\mu_1(u) = \frac{\pi/2}{\cosh^2(\pi u)} + O(g^2).$$  

The function decays faster than any polynomial and the double poles at $\pm i/2$ precisely cancel the double zeroes in the potential terms $(u \mp i/2)^L$ when $L = 2$ so that they lead to no extra contribution when deforming the contours. A periodic function without these double poles would not decay fast enough and a function with more than double poles would lead to extra contributions when deforming the contours; (12) is the sweet spot.

Turning on $g$ corrections is not complicated. The redefinition (1) brings the Baxter operator to a linear operator again but introduces some further poles at $\pm i/2$ in the (no longer polynomial) Baxter $Q$ functions so that the measure now needs some extra poles to cancel the contribution of these when deforming the contour. This explains (1) as well as (3); more details in appendix B 2.

Having derived the measure $\mu_1$ and the Baxter polynomial dressing it remains to fix the overall normalization of the structure constant to be sure everything is in order. Evaluating the SoV integral using the loop corrected Bethe roots for various spins immediately leads to the cute result

$$\left(\int du \mu_1(u)Q(u)^2\right)^{-1} = 1 + g^2(4H_2(J) + \frac{8H_1(J)}{2J+1}) + \ldots \frac{2J+1}{4J+3}$$

which is indeed almost the correct loop level structure constant computed in (19). The prefactor in (7) with $J$ deformed into $\bar{J}$ in a reciprocity reminiscent fashion (20) neatly combines with this expression to give the full NLO structure constants for twist-two operators.

At this point it is straightforward to guess the general structure constant for any $\text{SL}(2)$ operators of any twists by simply taking the tree level $g = 0$ result and deforming the new ingredient for higher twists – the two particle measure $\mu_2$ – by a bunch of hyperbolic tangents following what was derived for twist two. The coefficients of these hyperbolic tangents are then fixed by requiring orthogonality between any two different Baxter solutions. The last line deformation of the Baxter functions in (1) – which was invisible for twist two operators for which odd charges vanish – is also fixed by imposing orthogonality.

In the end, we just need to check that with a minimal reciprocity friendly prefactor as in (2) we precisely agree with perturbative data produced by hexagons. We do.

Even without matching with data, there is a nice self-consistency check of the full construction including the deformed pre-factor: The structure should be invariant under swapping left and right bridges $\ell \leftrightarrow L - \ell$; we checked that this is indeed realized by our expressions once the prefactors are deformed as in (2).

We made some progress at higher loops, in particular at NNLO (two loops) and for the smallest possible sizes and bridge lengths. For twist $L = 2$ operators for example we found a Baxter function dressing as well as an orthogonal one-particle measure realizing (8) as

$$Q(u) = \prod_{k=1}^{J} \frac{u - v_k}{\sqrt{x_k^+ x_k^-}} \times e^{\frac{5}{2} s^2 Q_1^+ H_1^+ + \frac{3}{2} s^4 Q_2^+ Q_2^+ H_2^+ - \frac{5}{2} s^4 Q_2^+ H_2^+}$$

and

$$\mu_1(u) = \frac{\pi/2}{2s^2} \left(1 + \pi^2 g^2 (4t_u^2 - 1) + \pi^4 g^4 \left(\frac{5}{2} - 7t_u^4 + \frac{11}{2} t_u^4\right) - \frac{g^4}{8} H_1^+(Q_1^+(v_1)Q_2^+(v_2) + Q_1^+(v_2)Q_2^+(v_1))\right),$$

where we use the fact that $Q_k^+ = 0$ for twist-2 operators.

The last line is exotic as it depends now on the charges of the two operators in (8) with Bethe roots $v_1$ and $v_2$ respectively. Note, however, that when considering the pairings $\langle Q, 1 \rangle$ and $\langle Q, Q \rangle$ a single Baxter function shows up and thus this mixing term can be absorbed as new factor dressing $Q$. One would then have different dressings in each pairing, a phenomenon we observe in the next section as well (in the $\text{SU}(2)$ sector).

The first line in (14) is also not any random combination of trigonometric functions. Take the tree level measure $1/\cosh^2(\pi u)$ – which is periodic with period $i$ and has poles at all the imaginary half integers – and promote it to a periodic function where all these poles are opened up into small cuts following (15), see figure (2).

$$\hat{\mu}_1 \equiv \int dv \frac{\pi/2}{2\pi i \cosh^2(\pi(u - v))} \frac{1}{x(v)}.$$

The integration contour encircles the Zhukovsky cut $v \in [-2g, 2g]$. Evaluating this integral in perturbation theory precisely reproduces the first line in (14) up to an overall normalization constant! It is tempting to conjecture that the finite coupling expression may play an important role in an all loop SoV formulation. We make further comments on NNLO structure constants in the discussion.

Other sectors such as the $\text{SU}(2)$ sector might also hint at other important structures in a putative all loop formulation. This is what we turn to now.
with \( \partial_i = \partial_\theta \) and \( \partial_{i,i+1} = \partial_i - \partial_{i+1} \), see appendix [C.1] for \( \delta M_{\text{NNLO- b}}, \) and where \( \Lambda_2 \) is a simple norm factor, (A.15). To NLO this is just the construction of [21]. We fix the NNLO part through comparison with the hexagons prediction, see appendix [C.1] for details.

The second observation is that given the known integral representation for the scalar products in the inhomogeneous XXX\( _{1/2} \) spin-chain, derived from Sklyanin’s [22], Baxter’s [11] and hexagon methods [10, 23], one can straightforwardly derive the functional SOV representation for the NNLO structure constants simply by acting with the \( \theta \)-morphism operator on this classical measure.

The result is once again (the square of) a \( \ell - 1 \) dimensional integral over an \( L - 1 \) dimensional integral,

\[
C^2_{\omega \phi} = \Lambda_{\ell}(Q) \times \frac{(J)!}{(2J)!} \times \frac{\langle Q, 1 \rangle_{\ell,L}^2}{\langle Q, Q \rangle_{\ell,L}^2} + O(g^6),
\]

with \( \langle f, g \rangle_{\ell,L} \equiv \langle f, g \rangle_{\ell,L}/\langle 1, 1 \rangle_{\ell,L} \) and

\[
\langle Q_1, Q_2 \rangle_{\ell,L} = \left( \begin{array}{c} J_1 + J_2 \\ \ell \end{array} \right) \int d\mu_{\ell,L} \prod_{i=1}^{\ell-1} Q_1(u_i) Q_2(u_i).
\]

The contour of integration is a circle wrapping the singularities of the measure (i.e. both Zhukowsky cuts), which once again is factorized:

\[
d\mu_{\ell,L} = \prod_{i=1}^{\ell-1} du_i \mu^1_{\ell,L}(u_i) \prod_{j \neq i}^{\ell-1} \mu^2_{\ell,L}(u_i, u_j),
\]

with

\[
\mu^1_{\ell,L}(u) = \frac{\sinh(2\pi u)}{(x_u x^{-}_u)^2} e^{\delta_{\ell \neq L} k_L k_L(u)}
\]

\[
\mu^2_{\ell,L}(u, v) = \frac{\sinh(2\pi(u-v))((u-v))}{2x_u x_v x^{-}_u x^{-}_v} e^{\delta_{\ell \neq L} k_L k_L(u,v)}
\]

\[
Q(u) = \prod_{k=1}^J \frac{u - v_k}{\sqrt{x_k x^{-}_k}} e^{\delta_{\ell \neq L} k_L k_L(u)}
\]

The scalar product in the denominator of \([19]\) enters with \( \ell = L \), and therefore the exponential in all these expressions should be dropped in that case. For the numerator we need these extra quantum dressing factors. They read

\[
\Lambda_{\ell,L}(u) = g^2(q^+_1 - 2) + g^4 \left( \frac{1}{2} q^+_2 - 2 \right) + 4 \alpha q^+_1 - 6 \pi^2 q^+_2 \delta_{\ell=2},
\]

\[
\Lambda_{\ell,L}(u, v) = g^2 q^+_1 q^-_1 + g^4 \left( \frac{1}{2} q^+_2 q^-_2 + 2 \alpha q^+_1 q^-_1 \right),
\]

\[
B(u) = g^2 q^+_1 Q^-_1 + g^4 \left( \frac{1}{2} q^+_2 Q^-_2 + \alpha Q^-_2 + 4 \alpha q^+_1 Q^-_1 \right) - \pi^2 q^+_1 Q^-_1 \delta_{\ell=2} + \pi^2 q^+_1 Q^-_1 \delta_{L=L-1},
\]

where the charges \( q^+_2 = q^+_2(u) \) and \( q^-_k = q^-_k(v) \) are defined in appendix [A] and \( \alpha = \frac{3}{2} \pi^2 - 1 \). Finally \( \Lambda_{\ell}(Q) \) is a simple function of the higher conserved charges also given in appendix [A].
Having expressed the results, a number of remarks are in order. The first is that \( \text{(19)} \) is an exact result capturing finite-volume effects around the seams adjacent to the NBPS operator \( [24] \). These effects start at \( N^3 \text{LO} \) in the SL(2) sector and therefore were not considered in the previous section, but for SU(2) they are present already at NNLO. Their effect in the SOV representation is encoded in the dressing of the Q-function through the \( \delta_{\ell,L-1} \) factor, see appendix C.3 for a lengthy discussion on these finite volume effects. Here we emphasize that the structure constant geometry dress the Q-function: \( \mathcal{Q} \) depends both on the state and \( \ell \).

The second remark is the presence of the \( \delta_{\ell=2} \) “anomaly” in \( \text{(25)} \). Its origin is not clear to us. Should it be put on foot with the \( \ell = 1 \) result \( \text{(27)} \) in the SL(2) sector at NNLO in which a new integral appears? Is it an indication that we are integrating out a simpler higher-dimensional integral for this short bridge overlap?

To a more basic point, in contrast to the SL(2) sector result \( \text{(2)} \), different measures and Q-dressings enter the numerator and denominator of \( \text{(19)} \) already at NLO. From the \( \theta \)-morphism point of view, the mismatch is due to the boundary terms in the second line of \( \text{(18)} \) which cancel when acting on the denominator, a symmetric function of the \( \theta_i \), but do not on the numerator. It turns out that the denominator’s Gaudin measure follows as in the SL(2) case from an orthogonality principle – see appendix C.2. The numerator measure is more complicated and we could not identify an orthogonality principle that fixes it. Is there an alternative principle that generalizes to higher loops and allow us to move forward? We hope so. As usual, one can always rely on hexagons to compare any new proposal with data, as we did to confirm the correctness of \( \text{(19)} \).

**IV. SL(2) VS SU(2)**

It is amusing to compare our results so far in the following summary table:

|                      | SL(2)       | SU(2)       |
|----------------------|-------------|-------------|
| Main result          | \( \text{(2)} \) and \( \text{(26)} \) | \( \text{(19)} \) |
| Factorized measure   | yes         | yes         |
| Same \( \mu \) and \( \mathcal{Q} \) for \( A \) and \( B \) | yes         | no          |
| Contour              | Real axis   | Encircling \( x^2(\mu) \) cuts |
| Manifest transcendality | yes         | no          |
| Wrapping effects incorporated | Hahn polynomials | \( \theta \)-morphism |
|                      | Baxter orthogonality | Baxter |
|                      | Zhukowskization | Hexagon data |
|                      | Hexagon data | Adj. Wrapping |
|                      | Bottom Wrapping | Selection rules |
|                      | Reciprocity | |
| Next step            | Finish NNLO | Do NNLO |
|                      | \( \& \) guess all loops | \( \& \) guess all loops |

Many things are common to both SU(2) and SL(2) correlators most notably both are given by a SoV like scalar products involving factorized quantum corrected measures and quantum dressed Baxter functions. As far as we checked, both are capable of capturing wrapping corrections.

There are also differences. Some seem minor: for example, the counting of transcendality in the SU(2) expressions is a bit weird specially due to the mixed transcendality factor \( \alpha = \tfrac{2}{3} x^2 - 1 \) \[25\] \[26\].

Some differences seem deeper: The contour of the non-compact SL(2) sector is non-compact while the contour for the SU(2) compact sector is a closed contour, see figure 3. (This was already observed before in rational SoV explorations, see e.g. [10] [12]).

For states with two particles of opposite momenta \( \{ v, -v \} \), SL(2) three-point functions of length \( L \) and adjacent bride \( \ell \) should match SU(2) correlators of length \( L+2 \) and bridge \( \ell + 1 \). We checked this by explicit evaluation but it is amusing that this type of relation – which follows from some SU(2) and SL(2) states being in the same supermultiplet – is manifest in the spectrum problem \[25\] but is totally obscure here. Would be desirable to make it more manifest; this might hint at an even more unified description of both sectors.

Let us conclude this section highlighting the beautiful appendix G of [10] by Jiang, Komatsu, Kostov and Serban. There the starting point are the hexagon sums over partitions for \( A \). These sums are cast as contour integrals and after several clever manipulations these contours end up being recast as new SoV like integrals. They do this for both SU(2) and SL(2). (The SL(2) derivation is more involved with some ”straightforward (but complicated and tedious)” steps.) In other words, they derive SoV from hexagons. The drawback if that this derivation was done at tree level and there are a few steps (see previous quote) that don’t seem to be obvious to lift to all loops. Revisiting this appendix in light of what we learned seems promising.
V. DISCUSSION

We have initiated here a project for recasting correlators in \( \mathcal{N} = 4 \) SYM in a language closer to that of the spectrum quantum spectral curve. With all recent SoV related advances [11, 13, 15, 29, 30, 55, 57] we believe the time is ripe for seeking out for such a new approach. We are probably scratching the tip of an iceberg as far as the full finite coupling structure goes but some elements are clearly emerging.

One is the central importance of SoV quantum corrected measures \( \mu(u_1, \ldots, u_p) \) which can be found by combining a myriad of different approaches from orthogonality of quantum corrected Baxter equations, matching SoV integrals with hexagon combinatorics, converting hexagon sums into contour integrals which one then massages into SoV like integrals, \( \theta \)-morphism operations on in-homogeneous SoV integrals, re-summing wrapping corrections, extending orthogonality relations of known classical polynomials, Zhukowsky upgrades of trigonometric functions etcetera. (This etcetera often includes a good deal of inspired guesswork.)

These measures are then used to build scalar product like integrals which couple Baxter functions \( \mathcal{Q}(u) \) for the various involved operators. The Baxter functions also are corrected away from their tree level polynomial forms.

The most obvious question is then how to fix the measures and the Baxter Q-functions. Do these measures obey any sort of all loop bootstrap set of axioms? They must. It is crucial to work it out. Are these Q-functions some of the finite coupling solutions to the Quantum Spectral Curve? We hope so.

For the most part we considered a single non-trivial non-BPS operator but we could have considered correlation functions involving more than one non-BPS operators using the very same universal measures and Baxter functions. Take for the example the case of two spinning operators as studied in [31] and consider the so-called abelian polarization there, see figure [1]. We found a compact representation of all such correlators at NLO as

\[
\left( \sum_{\ell} \tilde{C}_{\ell}^{\bullet \bullet} \right)^2 = \frac{(J_1 + J_2)^2}{(2J_1)!(2J_2)!} \frac{(\mathcal{Q}_1, \mathcal{Q}_2)^2}{(\mathcal{Q}_1, \mathcal{Q}_2)_L (\mathcal{Q}_2, \mathcal{Q}_2)_L}. \tag{26}
\]

Would be very interesting to find an SoV representation for the other spinning correlator tensor structures studied in [31]: a strategy could be to embed the external operator SL(2)'s into higher rank SL(N) where we could borrow recent SoV technology from [12].

Is the number of SoV integrals going to remain constant or at least grow in a controllable way? And if they grow, can we re-sum the full result into some kind of exotic infinite dimensional sort of scalar products? We have little to say about the last speculative question but as far as the growth of the number of integrals goes, we do seem to find some interesting structure. Take for example the SL(2) correlation function involving a single non-BPS operator with twist \( L = 2 \) and adjacent bridges \( \ell = 1 \).

\[
\langle \mathcal{Q}, \mathcal{1} \rangle_1 \rightarrow 1 + \eta g^4 \int du \nu(u) \frac{\mathcal{Q}(u)}{\mathcal{Q}(i/2)} \tag{27}
\]

+ simple contact terms + \( O(g^6) \),

with the various ingredients spelled out in appendix [33] and where \( \eta = 1 \) for \( \ell_B > 1 \) (no wrapping effects) and \( \eta = 2 \) for \( \ell_B = 1 \) (wrapping effects included). It is very encouraging to see wrapping and asymptotic effects cast in such unified way. We see it as evidence that the number of integrals in this SoV approach should increase as we go to higher and higher loops but that increase is not directly related to wrapping effects which can be automatically incorporated (as seen in this bottom wrapping example and also in the SU(2) adjacent wrapping example, see appendix [33]). This growth of the number of SoV integrals is consistent with the picture that at finite coupling we are dealing with three quantum strings with infinitely many degrees of freedom and in the SoV approach the number of integrals is related to the number of such degrees of freedom [15].

As we go to higher loops, multiple wrapping effects come in at once and we should make contact with [34] where \( PSU(2, 2|4) \) transfer matrices are shown to appear. Again, our hope is that even in those cases, wrapping and

![FIG. 4. The NBPS operators have twists \( L_i \) and spins \( J_i \). Their polarizations are orthogonal. In the conventions of [31] it is equivalent to \( \sum_\ell C^{\bullet \bullet}(J_1, J_2, \ell) \).

With \( \ell = 1 \) the numerator \( \langle \mathcal{Q}, \mathcal{1} \rangle_1 = 1 \) to LO and NLO since the scalar product in (27) is an \( \ell - 1 \) dimensional integral. This is of course consistent with the hexagon approach where this numerator is given by the sum over partitions of magnons \( A_1 \) weighted by the hexagon form factors – see (A13) – and this sum over partitions normalized as in this paper indeed evaluated to \( 1 + O(g^4) \).

At NNLO \( O(g^6) \), however, it is no longer unity. Instead it is given by a cute expression involving Bernouli numbers which we summarize in appendix [33]. Also, at this loop order, if the bottom bridge \( \ell_{23} = \ell_B = 1 \) there is a new wrapping correction to the structure constant [1]. Both these new effects can be compactly incorporated by simply modifying the trivial scalar product

\[
\langle \mathcal{Q}, \mathcal{1} \rangle_1 \rightarrow 1 + \eta g^4 \int du \nu(u) \frac{\mathcal{Q}(u)}{\mathcal{Q}(i/2)} \tag{27}
\]

+ simple contact terms + \( O(g^6) \),
no-wrapping are all cast in the same unified way.

We should fit strong coupling in this framework. For large operators there could be interesting semi-classical limits [23, 35, 37] where one might be able to make progress; for small operators we know very little about what to expect for the Baxter functions and we have nothing intelligent to say. A good starting point there might be [25] where important sets of wrapping corrections were beautifully resummed for structure constants involving one small operator and two large operators.

Another limit worth exploring is large spin. This is a particular limit where SoV should be quite powerful since the number of integrals does not grow with spin. Very interesting structures seem to emerge [39] which might also shed light on WL/correlator dualities [40, 41] along the lines of the recent works [33, 42].

It would also be very interesting to think about how all this fits into a higher point function picture [43]. What is the SoV description of a four point function of BPS operators? Does it involve integrating over intermediate Baxter functions? Will the measures derived here play a role in this integration? A starting point for these explorations could be recasting the Coronado’s all loop octagon correlator [44, 45] in this language, perhaps using its determinant representations [46]. The octagon is the SoV description of a four point function of BPS operators. (Another motivation for studying this limit.)

Another limit worth exploring is large spin. This is a particular limit where SoV should be quite powerful since the number of integrals does not grow with spin. Very interesting structures seem to emerge [39] which might also shed light on WL/correlator dualities [40, 41] along the lines of the recent works [33, 42].

We explored the SU(2) and SL(2) sectors of $N = 4$ SYM. There is life beyond it. We should look for it.

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Appendix A: Notation

1. Roots and Charges

Up to at least 4 loops, the Bethe roots $v_k$ are the solutions to the asymptotic Bethe equations [27, 28]

\[ 1 = \left( \frac{x_k^+}{x_k^-} \right)^L \prod_{j \neq k} \frac{x_j^- - x_k^+}{x_j^+ - x_k^-} \eta \left( 1 - \frac{g^2}{x_j^- x_k^+} \right) \sigma(v_k, v_j)^2 \]

for the simplest SU(2) and SL(2) rank one sectors where $\eta = \pm 1$ respectively. Here the Zhuikovsky variables

\[ x_k^\pm = x(v_k \pm \theta) \quad \text{with} \quad x(u) = \frac{u + \sqrt{u^2 - 4g^2}}{2} \approx u - \frac{g^2}{u} + \ldots \]

where $g^2 = \lambda/(4\pi)^2$ is the coupling. (For three point functions $g^{2k}$ effects are $N^k LO$ effects while for the quantum anomalous dimensions they are $N^{k-1} LO$ effects.) We sometimes also use $x[i\pi] \equiv x(u \pm i\pi/2)$. Finally the dressing phase starts only at very high loop order, $\sigma = 1 + O(g^k)$ and we will not use its explicit expression in this paper. Generating Bethe roots for SL(2) is simple since they are real; we can use an extension of the code in [17] for instance; for SU(2) we used [35] to produce tree level solutions and then we found the loop corrections by linearizing Bethe equations around these seed values.

We also define the auxiliary real functions

\[ q_k^\pm(u) = i^{k+0}(x^+(u))^{-k} + (-i)^{k+0}(x^-(u))^{-k}, \]

\[ q_k^\pm(u) = i^{k-1}(x^+(u))^{-k} + (-i)^{k-1}(x^-(u))^{-k}, \]

which can be used to construct the conserved charges

\[ Q_k^\pm = \sum_{i=1}^J q_k^\pm(v_j). \]

The $Q_k^\pm$ are even and $Q_k^\mp$ are odd functions of the Bethe roots. The anomalous dimension of an operator is nothing by $\gamma = 2g^2 Q_1^+$ for instance.
Each closed spin chain operator is split into two open chain operators. We sum over all the ways its excitations can end up in either one of the chains. Gluing the hexagons together amounts to integrating over all possible mirror states.

We can also package the Bethe roots into Baxter polynomials \( Q(u) \equiv \prod_{k=1}^{J} (u-v_k) \) or dressed Baxter functions as in \([1]\) or \([25]\). Then these charges can be extracted by simple contour integrals as well. For instance:

\[
Q_1^+ = \int \frac{u \, du}{\pi g^2 (u^2 - 4g^2) \log \frac{Q(u + i/2)}{Q(u - i/2)}}
\]  

(A4)

where the contour encircles the Zhukowsky cut \( u \in [-2g, 2g] \). This is an interesting definition since it now applies to any function \( Q \), be it a polynomial or not. We can use such definitions to pair up arbitrary functions in scalar products such as \((3)\).

The concept of transcendentality is often useful. At loop order \( n \) we expect functions of uniform transcendentality \( n \) to show up only properly counted. It is tempting to assign transcendentality \( k \) to the charges \( q_k \). Some identities like \((q_1^+)^2 - (q_1^-)^2 = 2q_1^+ \) nicely preserve this counting but others such as \((q_1^+)^2 + (q_1^-)^2 = 4q_1^+ + O(g^2) \) can lead to ambiguities in perturbation theory. In the \( SL(2) \) sector Harmonic numbers \( H_k \) of transcendentality \( k \) often show up. We use the mathematica notation \( H_k(x) = \text{HarmonicNumber}(x, k) \) and

\[
H_n^+(u) = H_n(-1/2 + iu) + H_n(-1/2 - iu),
\]  

(A5)

\[
i H_n^-(u) = H_n(-1/2 + iu) - H_n(-1/2 - iu).
\]  

(A6)

2. Structure Constants and Hexagons

The hexagon formalism \([1]\) is a non-perturbative integrability framework developed to evaluate structure constants in planar \( N = 4 \) SYM. It was used throughout this letter to generate a myriad of perturbative data. This formalism entails two main components: asymptotic sums and mirror corrections, as depicted in figure [5].

When we cut the three point function pair of pants into two hexagons the excitations on each operator can end up on either hexagon so we must sum over the ways of partitioning such excitations. In the end when we glue back the hexagons into a pair of pants we must sum over all possible quantum states along each edge where we glue. In this appendix we ignore these second effect, focusing only in the asymptotic contributions.

The three point function depicted in figure \([1]\) in the hexagon formalism is given by the ratio of two quantities

\[
(C_{\text{hex}}^2)_{\ell} = \frac{\mathcal{A}_\ell^2}{B}. 
\]  

(A7)
Appendix B: SL(2) material

1. Hahn Polynomials and Measures

A Hahn polynomial \( p_J(x|a, b, c, d) \) is given by
\[
\frac{\mathcal{F}_2}{a + ix, a + b + c + d - 1 + J, -J \mid a + c, a + d}
\]
and admits a simple orthogonality relation
\[
\int dx \mu_{\text{Hahn}}(x) p_J(x|a, b, c, d) p_J(x|a, b, c, d) \propto \delta_{JJ'}
\]
with
\[
\mu_{\text{Hahn}}(x) = \Gamma(a + ix) \Gamma(b + ix) \Gamma(c - ix) \Gamma(d - ix).
\]
At LO the twist-2 Baxter functions are given by these polynomials with \( a, b, c, d = \frac{1}{2} \) so that the measure becomes \( \Gamma(\frac{1}{2} + ix)^2 \Gamma(\frac{1}{2} - ix)^2 \propto \text{sech}^2(\pi x) \) as quoted in [11].

At loop level the corrections to 1/2 for the coefficients \( a, b, c, d \) depend on \( J \) albeit mildly, through the Harmonic number \( H_J(J) \), so that \( \mu_{\text{Hahn}} \) reads
\[
\frac{3\mathcal{F}_2}{\frac{1}{2} + i\sqrt{2}g + 2g^2H_1 + i\gamma, J + 1 + 8g^2H_1, -J \mid 1 + 4g^2H_1, 1 + 2i\sqrt{2}g + 2g^2H_1}
\]
where we highlighted the NLO corrections in magenta. If we plug the \( J \) dependent one loop values for \( a, b, c, d \) in [13] we would obtain
\[
\mu_{\text{Hahn}}(x) \propto \frac{1}{\cosh^2(\pi x)} \left( 1 + 2\pi^2g^2(\alpha \tanh^2(\pi x)) + 4g^2H_1(J)H_1^+(x) + O(g^4) \right).
\]
where \( \alpha = 1 \). This is unsatisfactory as it is \( J \) dependent. To fix this we tried to absorb the second line into the definition of \( Q_J \); recalling that these Harmonic numbers are nothing but the energy \( Q_0 \) of these twist-two operators we end up with the correction in the first line in [11]. (We can not derive this second line with the simple derivation since the odd charge \( Q_1^- \) vanishes for the operators in this leading Regge trajectory.) What we did is then absorb this second line of [11] into the Baxter functions and look for an orthogonal measure for these no longer polynomial Baxter functions of the form of the first line of (B4). It exists with \( \alpha = 3/2 \). In [11] we fixed the overall normalization of the measure so that the vacuum \( (Q_0 = 1) \) is unit normalized.

There are still interesting open problems to pursue along these lines. NNLO corrections to twist-two Baxter polynomials are also known [17]. Can we use them to infer the next correction to the measure? Some twist three families are also known [17] [49]. Can we use them to shed light on the multi-particle measure \( \mu_2 \)?

2. SL(2) Baxter and Measure

It is easy [50] to check that for on-shell solutions of the one-loop twist 2 Baxter equation [9] the dressed polynomials [11] solve the simplified difference equation
\[
\tilde{B} \circ Q(u) = T(u)Q(u)
\]
with \( \tilde{B} \equiv (x^+)\gamma^\mu + c.c. \) without any charge dependence in contradistinction with the original \( B \).

From this is easy to build a one loop orthogonal inner product for the twist 2 solutions. We look for a measure so that \( B \) is self adjoint in the inner product
\[
\langle Q_i, Q_j \rangle \equiv \int dx \mu_1(x) Q_i(x) Q_j(x)
\]
on the space of dressed polynomials [11]. Deforming the contours in \( \tilde{B} \circ Q_1, Q_2 \) as done in section II shows that it is enough to consider a periodic measure with fast enough decay at infinity provided all poles of the integrand – which now receive contributions from \( \mu_1, Q \) and \( B \) have vanishing combined residue. See [12] for similar ideas applied to SL(N) spin chains. Writing an ansatz
\[
\mu_1(x) = \frac{\pi/2}{\cosh^2(\pi x)} \left( a_1 + g^2 \left[ a_2 \tanh^2(\pi u) + a_3 \right] \right)
\]
and requiring the cancelation of poles combined with the requirement that the vacuum \( (Q_0 = 1) \) is normalized to one in (B6) fix \( a_1 = 1, a_2 = 3\pi^2, a_3 = -1 \). This is [5].

3. Hexagons \( \mathcal{A} \) at \( \ell = 1 \)

For minimal left bridge length \( \ell = 1 \) the hexagon sum over partitions simplifies dramatically. We thank Frank Coronado for highlighting this and for help in establishing the asymptotic formulae [B8] a few years ago.

The asymptotic result [A8] for \( \ell = 1 \) simplifies to
\[
(A_{\ell=1}^{\text{asymptotic}})^2 = 1 + g^4 \sum_{k=3}^{S} \sum_{i < k < l} \frac{i^k(4-k)B_{3-k}}{\prod_{i=1}^{k} (v_i + \frac{1}{2})} + c.c.
\]
Plus \( O(g^6) \). Here \( B_n \) are the Bernoulli numbers.

Since the roots \( v_i \) that appear in the denominator product never appear repeated we can cast the sum as
\[
1 + g^4 \frac{P(v_j)}{Q(-i/2)} + c.c.
\]
where \( P(v_k) \) is a polynomial linear in each of the Bethe roots \( v_k \), just like the Baxter polynomial \( Q(u) \equiv \prod (u - v_k) = \sum c_n u^n \) is. We can thus look for a linear integral operator acting on \( Q \) and producing such \( P(v_j) \),
\[
\langle Q(u) \rangle = \sum_{n=0}^{S} c_n \langle v_j \rangle \langle u^n \rangle = P(v_j).
\]
It is not granted that such operation exists for all \( S \). We imposed it spin by spin and observed that we can indeed satisfy (B9) as long as we fix the linear map moments as 
\[
\langle 1 \rangle = \langle u^2 \rangle = 0, \langle u^3 \rangle = \frac{1}{2}, \langle u^4 \rangle = 1, \langle u^5 \rangle = -\frac{7i}{6}, \ldots
\]
We computed the first few dozens such moments from which we guessed that they can be generated from
\[
\langle F \rangle = \frac{i}{2\pi} F \left( \frac{-1}{\pi} \right) + 3i\zeta_3 F' \left( \frac{-1}{\pi} \right) - \frac{i}{2\pi} F'' \left( \frac{-1}{\pi} \right) + \int du \nu(u) F(u)
\]
where the measure
\[
\nu(u) = \frac{i\pi - i(u + \frac{1}{2})^2\tan(\pi u)}{2(u + \frac{1}{2})^3 \cosh(\pi u)^2}.
\]  
(B10)

This came as a surprise as this very same measure arose before in a very different context. We use the fact that \( Q(\frac{1}{2}) = Q(-\frac{1}{2}) \) for physical states to combine both terms in (58); we see that (the real part of) this measure is nothing but the measure which arose when computing the first wrapping correction when the bottom bridge \( \ell_B = 1 \), see equations (44),(55) in [1]! We conclude that we cannot cast the two loop asymptotic contribution (58) as a simple SoV looking integral but we can also trivially incorporate the bottom wrapping effects:
\[
\left( A_{\ell=1} \right)^2 = 1 + g^4 \left( \eta_{12} Q_1^{\frac{1}{2}} - \eta_{1\ell} Q_{1\ell}^\dagger - 6(2 - \eta) \zeta(3) Q_1^{\frac{1}{2}} + \eta_{1\ell} Q_{1\ell}^\dagger + 16\pi\eta \int \frac{1 - 12u^2 + 2\pi u(1 + 4u^2)\tan(\pi u) - (\pi u)^2}{(1 + 4u^2)^3 \cosh(\pi u)^2} Q(u) \right) + \ldots
\]
where \( \eta = 1 \) for \( \ell_B > 1 \) and \( \eta = 2 \) for \( \ell_B = 1 \) [51].

Do we continue to have identical formulas with and without wrapping at higher loops? The NNNLO effective wrapping measure correction was nicely worked out in appendix B of [5] (in appendix E of [3] an equivalent representation -- a sort of Fourier transform -- was derived). Can the next loop order asymptotic result still be neatly combined with the wrapping correction? At higher loops we will get more mirror corrections which we might be able to cast as SoV like integrals. We should probably expect a similar growth in the number of integrals for the asymptotic part of the result as well. Ultimately, at finite coupling, the distinction between the two should fade away as in the spectrum problem.

### Appendix C: SU(2) material

#### 1. \( \theta \)-morphism at NNLO

The morphism operator \( \mathcal{M} \) performs a “Zhukowskization” of the rational propagation of magnons in the XXX\(_{1/2}\) through the action on background inhomogeneities. From our perspective \( \mathcal{M} \) is defined by equation [17]. We write an ansatz and fix it by requiring the match of the RHS of (17) with \( C_{10}^2 \), which can be computed from hexagons. It can be divided into a closed part and a boundary part,
\[
\mathcal{M} = \mathcal{M}_c \cdot \mathcal{M}_b + O(g^6).
\]  
(C1)

At NLO we recover the result in [21]; at NNLO we obtain
\[
\mathcal{M}_c = \exp \left( \sum_{i=1}^L \left( g^2 (\partial_i \partial_i+)^2 - \frac{1}{4} g^4 (\partial_i \partial_i+)^2 (\partial_i \partial_i+)^2 \right) \right),
\]
and
\[
\mathcal{M}_b = \exp \left( -ig^2 \Phi_1^\dagger (\partial_1 - \partial_L) + g^4 \delta \mathcal{M}_{\text{NNLO-b}} \right).
\]  
(C2)

\( \mathcal{M}_{\text{NNLO-b}} = \frac{1}{2} (2(Q_1^\dagger - i Q_2^- - Q_1^- Q_1^\dagger) \partial_1^2 + i Q_1^\dagger \partial_i^2 + \frac{1}{2} (Q_1^\dagger Q_1^\dagger - (Q_1^\dagger)^2) \partial_i^2 - \frac{1}{2} Q_1^\dagger \delta_1 \partial^2_1 + \frac{1}{2} (i Q_2^- - (Q_1^\dagger)^2) \partial_1 \partial_2 - (\partial_1 \leftrightarrow \partial_L, \partial_2 \leftrightarrow \partial_{L-1}) \right).\]
  
(C3)

It would be interesting to re-derive \( \mathcal{M}_c \) and \( \mathcal{M}_b \) along the lines of [21].

The closed action satisfies the morphism property when acting on symmetric functions \( f_{\text{sym}}(\theta) \) of the inhomogeneities \( \theta \):
\[
\mathcal{M}_c \circ f_{\text{sym}} = (\mathcal{M}_c \circ f_{\text{sym}}) (\mathcal{M}_c \circ g)
\]  
(C4)

for a generic function \( g \). It also satisfies the “Zhukowskization” property
\[
\mathcal{M}_c \circ \prod_{i=1}^L (u - \theta_i \pm i/2)^{k} = (x^+)^{kL}.
\]  
(C5)

The inhomogeneous Gaudin norm [16] is given in the SOV representation, see [22] for details, by
\[
\mathcal{B}_\theta = f_{\text{sym}}(\theta) \times \oint d\bar{\mu} \mu \prod_{j=1}^{L-1} Q(u_j)^2
\]  
(C6)

with the \( \theta \) dependence entering only through the symmetric normalization function \( f_{\text{sym}}(\theta) \) and the measure
\[
\mu_\theta = \prod_{i=1}^{L-1} \prod_{j=1}^L ((u_i - \theta_j)^2 + 1/4)^{-1}.
\]  
(C7)

Combining (C4), (C5) we get the NNLO Gaudin measure
\[
d\mu_{L,L} = d\mu \times (\mathcal{M}_c \circ \mu_\theta = (x^+)^{-L})(x^{-})
\]  
(C8)

as in [21][22][23]. The same measure is derived from a orthogonality principle in section (C2). Comparison with the NNLO gaudin determinant fixes the conversion factor \( \Lambda_{\theta} \) to (A15), so that in the end we are left with
\[
|\Lambda_{\theta} \mathcal{M} \circ \mathcal{B}_\theta| = \mathcal{B},
\]
with \( \mathcal{B} \) given in (A14) [52]. Note that \( \mathcal{M}_b \), (C2), acts trivially in \( \mathcal{B}_\theta \) since it is a symmetric function of \( \theta \).

Having fixed the morphism operator through hexagons, the SOV result [10] then follows [52] from the action of the morphism operator (C1) on the inhomogeneous XXX\(_{1/2}\) spin chain SOV overlaps [10][11][22].
2. SU(2) orthogonality

The Gaudin measure defining \(\langle Q, Q\rangle_{L,L}\) was derived in sections 11–1 [1] from the \(\theta\)-morphism action. Here we show it also defines an orthogonal scalar product for NNLO Baxter polynomials \(Q\), meaning

\[
\langle Q_1, Q_2\rangle_{L,L} = N_1 \delta_{12} \tag{C9}
\]

if \(Q_1, Q_2\) are solutions to the Baxter equation \(\mathcal{B} \circ Q = T(u)Q(u)\) with

\[
\mathcal{B} = (x^+)^L e^{-i\partial u} + (x^-)^L e^{i\partial u}. \tag{C10}
\]

What follows is a simple loop generalization of the XXX case [11]. Consider the pairing

\[
(f, g)_\mu = \oint \gamma \frac{d\mu(u)f(u)g(u)}{2\pi i} \tag{C11}
\]

with \(\gamma\) contour being the boundary of the \([-3g, 3g] \times [-2i, 2i]\) square. Inserting \(\mathcal{B}\),

\[
\langle (x^+)^L e^{-i\partial u} + (x^-)^L e^{i\partial u} \circ f, g\rangle_\mu, \tag{C12}
\]

and shifting contours down by \(i\) in the first term and up by \(i\) in the second term, as done below equation (11), we find that \(\mathcal{B}\) is self-adjoint with respect to (C11) provided

\[
\mu^{[2]} / \mu = (x^{-1}/x)\mu^L, \tag{C13}
\]

so that \(\mu = (x^+ x^-)^{-L} \mu_0\) with \(\mu_0\) an \(i\)-periodic factor.

Note that \(T(u)\) is polynomial for physical zero-momentum states at NNLO

\[
T(u) = 2u^L + \sum_{i=0}^{L-2} c_i(Q)u^i \tag{C14}
\]

with \(c_i(Q)\) being the state-dependent integrals of motion. Consider the family of measures

\[
\mu_j = \sinh(2\pi u) \times \exp(2\pi u(2j - L)) \tag{C15}
\]

with \(j = 1, \ldots, L - 1\). Let \(Q_1, Q_2\) be solutions of the Baxter equation. We then have, from self-adjointness,

\[
\sum_{i=0}^{L-2} (c_i(Q_1) - c_i(Q_2))(Q_1, u^i Q_2)\mu_j = 0 \tag{C16}
\]

since the LHS is simply \(\langle \mathcal{B} \circ Q_1, Q_2\rangle_{\mu_j} - \langle Q_1, \mathcal{B} \circ Q_2\rangle_{\mu_j} = 0\). The integrals of motion \(c_i\) are generic and therefore the linear system (C16) should be non degenerate, implying

\[
\det \left[ (Q_1, u^{i-1} Q_2)\right]_{\mu_j} = N_1 \delta_{12}. \tag{C17}
\]

with \(i, j = 1, \ldots, L - 1\). Our claim is that expanding the Vandermonde-like determinant (C17) into a \(L - 1\) dimensional integral reproduces the main text result (21) for \(\ell = L\) up to a combinatorial normalization factor,

\[
\det \left[ (Q_1, u^{i-1} Q_2)\right] \propto \oint_{\gamma} d\mu_{L,L} \prod_{i=1}^{L-1} Q_1(u_i)Q_2(u_i). \tag{C18}
\]

We conclude that the Gaudin measure defines and orthogonal scalar product and, more over, the Gaudin norm takes determinant form in the SOV representation.

For \(A\) things are more involved as explained in the main text with the additional exponential dressings in (22,23) kicking in.

3. SU(2) structure constants at finite volume

At leading order mirror particles have infinite energy and vanishing phase space, and therefore can be ignored. Their contribution, which starts at NNLO, is however crucial. It is only when they are taken into account that selection rules are realized for instance. In this appendix, we review how the first mirror contributions are computed in the hexagon formalism, adapting the SL(2) computations of [3,5] to the SU(2) case. We then discuss how these virtual effects are taken into account in the SOV approach by appropriately dressing the \(Q\)-functions.

Part I: mirror particles on the hexagon

When gluing hexagons together to reconstruct the closed string geometry one must insert a complete basis of states along the seams \((\ell_{12}, \ell_{31}, \ell_{23})\). This complete set of states is given by the Fock space of mirror particles labeled by their rapidities \(v\) and bound state index \(a\). The contribution from terms with \(n_{ij}\) particles on edge \(\ell_{ij}\) is

\[
A(n_{ij}) = \int_{\{a\}} d\mu(v) e^{-E(v)\ell_{ij}} \mathcal{H}^2(v,a) \tag{A1}
\]

where \(\{a\} = \{a_{ij}^k\}\) is a collective index for the bound state label \(a_{ij}^k\) of particle \(k\) on edge \((ij)\) and similar for the mirror rapidities \(v_{ij}^k\). Above \(\mu(v)\) and \(E\) are respectively the phase space measure and the energy of the mirror particle and \(\mathcal{H}^2(v,a)\) are the glued hexagon form factors with mirror particles labeled by \((v,a)\) inserted along the seams, the dependence on the external operators being left implicit. Below we provide explicit expressions for the case of interest, see [3,5] for general expressions.

At weak coupling multiparticle mirror states are suppressed, both the energies and the measure being of \(O(g^2)\). Mirror contributions are also suppressed for large geometries so that at NNLO only edges with bridge length \(\ell_{ij} = 1\) can support mirror excitations. Moreover, at this order only one edge can be excited at a time: we may have an excitation in the adjacent or in the opposite edge to the non BPS operator.

Adjacent virtual corrections in the SU(2) sector provide an example of the selection rules restoration aforementioned. We now delve into this in detail to understand what is expected from the SOV formulas at NNLO. Structure constants in the SU(2) sector for states with R-charge \(M\) when an adjacent bridge length \(\ell < M\) vanish. At the classical level \((g = 0)\) this is simply the statement that we cannot contract \(J\) scalars through a bridge of
length $\ell \equiv \ell_2$ [21, 33]. The asymptotic contribution $n_{ij} = 0$ correctly reproduce this selection rule at LO and NLO, but at NNLO a non-zero contribution is obtained when $\ell_2 = 1$. The claim is that the mirror factor cancels this contribution and restore the symmetry [23]. This reads

$$\mathcal{A}_f^{\text{asymptotic}} = -\mathcal{A}(n_{12} = 1), \quad (C18)$$

with

$$\mathcal{A}(n_{12} = 1) = \text{N}\,\text{I},\ell \int \frac{dv}{2\pi} \frac{ag^4}{v^2 + a^2/4} \tau_{SU(2)}(v^2, \mathcal{O}_1) \times \sum_{\alpha,\bar{\alpha}} (-1)^{\bar{\alpha}} \frac{e^{-ips} h_n(v^2, \alpha)}{h_{SU(2)}^{SU(2)}(v^2, \alpha, v^2)} \quad (C19)$$

where $h_n$ are the fused hexagon dynamical factors, $v^2$ denotes analytic continuation to mirror kinematics across the $x^+_{[\alpha]}$ cuts, $h_{SU(2),\alpha}$ are the SU(2) dynamical factors [A10] and $\tau_{SU(2)}(v^2, \mathcal{O}_1)$ is the forward SU(2) transfer-matrix eigenvalue for the Bethe state describing operator $\mathcal{O}_1$. We sum over partitions of the bethe state $\{u = \alpha \cup \bar{\alpha}, \text{ and use the short hand notation } h_n(\bar{\alpha}, v^2) = \prod_{\alpha \in \bar{\alpha}} h_n(u_{\alpha}, v^2) \}$ and similar for $h_n(v^2, \alpha)$. To leading order the objects in (C19) read

$$h_n(v^2, u_j) = \frac{u_j + i/2}{u_j + i/2 - v + a i/2}$$

$$h_n(\ell, v^2) = \frac{-g^2}{u_j + i/2 (v - u_j)^2 + (a - 1)^2/4}$$

and $\tau_{SU(2)}(v^2, \mathcal{O}_1)$ is given by

$$(a + 1) \sum_{j=1} f_j^+ + a \sum_{j=1} f_j^- + a \sum_{j=1} g_j f_j^+ + (a - 1) \sum_{j=1} g_j f_j^- \quad (C19)$$

where $g_j = (u_j - v - i/2(a + 1))(u_j + i/2)/((u_j - v - i/2(a - 1))(u_j - i/2))$ and $f_j^+ = (u_j - v + i/2(a + 1))/(u_j - i/2(a - 2))$. Equation (C18) holds off-shell. The LHS is a rational function of the rapidities $u_j$. The RHS integral can be evaluated by residues. Poles at $v = u_j \pm i/2(a - 1)$ cancel after summing over the bound-state index while those at $v = \pm i/2(a$ sum to a rational function matching the LHS.

Performing an R-symmetry transformation permuting the polarizations of operators $\mathcal{O}_2 \leftrightarrow \mathcal{O}_3$ should leave the structure constant invariant. In the integrability description this amounts to the replacement $\ell \leftrightarrow L - \ell$. The structure constant must therefore also vanish when $\ell = L - 1$. The story in this case is more interesting. First, the LO and NLO asymptotic contributions only vanish on-shell, since after all the expression [A8] only knows that the right bridge $\ell_3 = 1$ through the Bethe roots which solve Bethe equations on a chain of size $L = \ell_1 + \ell_2$. At the NNLO the rational result is non-zero on-shell. The mirror contribution $\mathcal{A}(n_{11} = 1)$ is now given by an expression identical to (C19) with the last line replaced by

$$\sum_{\alpha,\bar{\alpha}} (-1)^{\bar{\alpha}} \frac{e^{-ips} h_n(v^2, \alpha)}{h_{SU(2)}^{SU(2)}(v^2, \alpha, v^2)} \quad (C20)$$

After summing over the bound state index in the wrapping corrections we now obtain a complicated expression full of $\zeta$ and Polygamma functions whose coefficients vanish on-shell so that in the end the we are left with simple algebraic numbers that cancel the asymptotic result, reproducing the selection rule.

**Part II: dressing the Q-functions**

Wrapping effects at NNLO are due to virtual particles propagating over bridges of size one. Since nontrivial SU(2) operators have R-charge $M \geq 2$, NNLO wrapping effects are only present when restoring the selection rules discussed in Part I of [C3] There are two cases to consider in the SOV proposal: $\ell_2 \equiv 1$ and $\ell_31 = 1 \text{ i.e. } \ell = L - 1$. The selection rules are realized trivially for $\ell = 1$ simply due to the binomials in (C20). This holds off-shell, as in the hexagon method. We henceforth focus on the nontrivial case. In other words, we seek to restore the important $\ell \leftrightarrow L - \ell$ symmetry for any possible lengths.

The NNLO SOV expression for structure constants in the SU(2) sector [19] was derived from the morphism action, $\mathcal{S}^{SU(2)}$ [17]. Note that $\mathcal{A}_0$ depends only on $\theta_1$ with $i \leq \ell$. One might therefore naively think that the method is unaware of $\ell_31$ being short or long and therefore cannot distinguish when mirror corrections are relevant. The solution is provided by the right boundary terms, i.e. terms with explicit $\partial_\ell$ and $\partial_{\ell-1}$ dependence in the second line of [18, 34]. These can be ignored whenever $\ell < L - 1$. Equation [17] then reproduces the asymptotic structure constants. However, when $\ell = L - 1$ the right boundary acts non-trivially. Its action generates the extra terms proportional to $\delta_{\ell,L-1}$ in [25]. Once these terms are properly taken into account, the selection rules are correctly reproduced when (19) is evaluated on-shell. Note that this corresponds to an infinite number of constrains on the on-shell action of the right boundary terms, and therefore the match is quite non-trivial.

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[25] It would be desirable to restore naive transcendentality counting, see also discussion at the end of appendix A.
[26] There is a fun fact about all the $\pi$'s floating around in the SU(2) result – see the anomaly $\delta_{1,2}$ as well as $\alpha$: they are there simply to cancel $\pi$'s generated from the integrations so that the final result is a rational function of the Bethe roots. From a practical point of view we could drop these $\pi$'s provided we also drop any $\pi$'s generated when picking up residues. At higher loops the structure constants are no longer rational and such games would be dangerous.
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[50] One simply use the Baxter equation (9) to solve for $Q^{+2}$ in terms of $Q$ and $Q^{-2}$, plug the result in (B5), use recurrence identities to cancel the Harmonic functions and note that that what is left is zero provided $Q$ is an even function. This is the case for twist 2 Bethe states.
[51] For $\ell_B > 1$ the right hand side of (B11) is a rational number; the $\pi$’s and the $\zeta(3)$ in the second line are simply cancelling the $\pi$’s and the $\zeta(3)$ generated by the integral. For $\ell_B = 1$ the $\pi$’s still cancel but there is no $\zeta(3)$ in the second line. (In that sense the expression is even simpler in this case when wrapping is present!) A $\zeta(3)$ term is generated by the integral so that the rhs of (B11) is of the form $\text{rational} + \text{rational} \zeta(3)$ as it should.
[52] In writing (19), (A14) we introduced the vacuum integrals $\langle 1,1 \rangle_{\ell,L}$ relative to the naive $\theta$-morphism action in order to simplify the resulting measure and normalizations factors. We also massaged the expression into a manifestly real form.
[53] Note that primary SU(2) operators have $M \geq 2$ and therefore all structure constants for $\ell_1 = 1$ vanish.
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