Generalized potential for apparent forces: the Coriolis effect

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Abstract

It is well known, from Newtonian physics, that apparent forces appear when the motion of masses is described using a non-inertial frame of reference. The generalized potential of such forces is rigorously analyzed, focusing on their mathematical aspects.

Keywords: generalized potential, accelerated observers, equivalence principle

1. Introduction

In the framework of classical mechanics, the configuration $P = (P_1, ..., P_N)$ of a system with $n$ degrees of freedom can described through the parametrization $P(t, q)$, where $t$ is the time and $q = (q_1, ..., q_n)$ is the Lagrangian parameter. The well-known Eulero–Lagrange equation of motion is written as

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i, \quad (1)$$
where $T = T(t, q, \dot{q})$ is the kinetic energy, $\dot{q}_i = \frac{dq_i}{dt}$ are the generalized velocities, and $Q_i$ is the generalized components of the forces acting on the system defined as

$$Q_i = \sum_{j=1}^{N} F_j \cdot \frac{\partial P_j}{\partial q_i}, \quad i = 1, \ldots, n. \quad (2)$$

Now, for the conservative system, it is possible to introduce the potential $U = U(q)$ such that

$$Q_i = \sum_{j=1}^{N} F_j \cdot \frac{\partial P_j}{\partial q_i} = \frac{\partial U}{\partial q_i}(q), \quad i = 1, \ldots, n, \quad (3)$$

where $F_j(t, P, \dot{P})$ are the total conservative forces applied to each point $P_j$. Hence, one can introduce the Lagrangian

$$L = T + U$$

and consider the equation of motion in the equivalent form

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, \ldots, n. \quad (4)$$

Suppose now we have a generalized force that can be written in terms of a velocity-dependent potential $U(q, \dot{q}, t)$ as

$$Q_i = \frac{\partial U}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial U}{\partial \dot{q}_i} \right). \quad (5)$$

If this is the case, substituting (5) into (1), we can conclude that the Euler–Lagrange equation still holds in the form (4) for a Lagrangian function $L$ that can be defined once more as $L = T + U$. The potential $U$ may be called a generalized potential or velocity-dependent potential. It is not a potential in the conventional sense because it depends on more than just the particle position and it cannot be calculated from a line integral of the generalized force. Despite this fact, the interesting aspect of such a formalism lies in the fact that it permits once more the use of a Lagrangian and the Euler–Lagrange equation and hence to re-obtain, in a generalized context, all the consequent properties. In addition, from a velocity-dependent potential can be derived interesting forces, such as the Lorentz force of a magnetic field $B$ acting on a moving charged particle

$$F = e(E + v \times B), \quad (6)$$

where $e$ is the particle charge, $v$ is the particle velocity, and $E$ is the electric field. In particular, we will show how forces deriving from a generalized potential can be written in the form of (6), giving the suitable definition of the two vector fields $E$ and $B$. Following this line, the paper is so organized: in section 2 we recall the expression of apparent forces in non-inertial frames. The generalized potential is analyzed in section 3 and evaluated in section 4 in case of apparent forces.

Finally, we wish to point out that the level of the paper is educational, and hence we restrict ourselves to examples from classical mechanics. Let us remark that these methods could be extended to the effects of relativistic mechanics.

2. Apparent forces in non-inertial frames

Consider a free material point $(P, m)$, not subjected to effective forces, in absolute motion with respect to an inertial frame of reference $T_0 \equiv \Omega \xi_0 \eta_0 \zeta_0$ and in relative motion with respect to a non-inertial reference frame $T_\Omega \equiv \Omega \xi_\Omega \eta_\Omega \zeta_\Omega$, moving in any rigid translational motion with respect to $T_\Omega$. If so, the principle of relatives motions is valid and we have
where \( \mathbf{v}^{(a)} \), \( \mathbf{v} \), and \( \mathbf{v}_r \) are respectively absolute, relative, and translational velocity. In particular, if \( \mathbf{\omega}_r \) is the angular velocity vector, the translational velocity is given by the well-known fundamental formula of rigid kinematics:

\[
\mathbf{v}_r = \mathbf{v}_O + \mathbf{\omega}_r \times (P - O),
\]

where \( \mathbf{v}_O \) is the velocity of the origin of the non-inertial reference frame \( T_O \). In its relative motion, the material point is subjected to the apparent forces

\[
\mathbf{F} = -m\mathbf{a}_r - m\mathbf{a}_c,
\]

where

\[
\mathbf{a}_r = \mathbf{a}_O + \mathbf{\omega}_r \times (P - O) - \mathbf{\omega}_r \times (P - O) \times \mathbf{\omega}_r,
\]

is the translational acceleration, and

\[
\mathbf{a}_c = 2\mathbf{\omega}_r \times \mathbf{v}
\]

is the Coriolis acceleration. Therefore,

\[
\mathbf{F}_r = -m\mathbf{a}_r = -m[\mathbf{a}_O + \mathbf{\omega}_r \times (P - O) - \mathbf{\omega}_r \times (P - O) \times \mathbf{\omega}_r]
\]

is the translational force (sometimes known as dragging force) and

\[
\mathbf{F}_c = -m\mathbf{a}_c = -2m\mathbf{\omega}_r \times \mathbf{v}
\]

is the Coriolis force. The sum of \( \mathbf{F}_r \) and \( \mathbf{F}_c \) provides the most general inertial force acting in the non-inertial reference frame. Let us derive a generalized potential for such a force. The absolute motion of the point with respect to the inertial frame rests on Lagrange’s three scalar equations that, in this case, since we use the three Cartesian coordinates as Lagrangian coordinates, may be summarized in the following vector equation:

\[
\frac{d}{dt} \frac{\partial \mathcal{L}^{(a)}}{\partial \dot{\mathbf{v}}^{(a)}} - \frac{\partial \mathcal{L}^{(a)}}{\partial \mathbf{P}} = 0,
\]

where, by definition, we have

\[
\begin{pmatrix}
\frac{\partial}{\partial \mathbf{v}^{(a)}} \\
\frac{\partial}{\partial \mathbf{P}}
\end{pmatrix} = \begin{pmatrix}
i \frac{\partial}{\partial \xi} + j \frac{\partial}{\partial \eta} + k \frac{\partial}{\partial \zeta}, \\
i \frac{\partial}{\partial \xi} + j \frac{\partial}{\partial \eta} + k \frac{\partial}{\partial \zeta}
\end{pmatrix}
\]

Being the material point in its absolute motion, not subjected to effective forces, the absolute Lagrangian \( \mathcal{L}^{(a)} \) coincides with the absolute kinetic energy, so

\[
\mathcal{L}^{(a)} = \frac{m}{2} \mathbf{v}^{2}_{(a)} = T^{(a)}.
\]

To pass to the Lagrangian \( \mathcal{L} \) in relative motion, we only have to substitute to the absolute velocity the expression given by (7), so we obtain

\[
\mathcal{L} = \frac{m}{2}(\mathbf{v} + \mathbf{v}_r)^2 = \frac{m}{2} \mathbf{v}^2 + m \left[ \mathbf{v} \cdot \mathbf{v}_r + \frac{1}{2} \mathbf{v}_r^2 \right],
\]

where \( T = \frac{m}{2} \mathbf{v}^2 \) is the apparent kinetic energy. Afterwards, we will prove that the terms in the square brackets on the second side of (17) represent the generalized potential of apparent forces, not considering the sign and the mass \( m \).

**Remark 2.1.** Following (2), we can obtain the generalized components \( Q^D \) and \( Q^{Cor} \) of the dragging and Coriolis forces, respectively:
3. Deriving forces from a generalized potential

In the physical space, the most general force resulting from a generalized potential can be expressed by the following classical theorem [1]:

**Theorem 3.1.** If a force is of the type

\[ \mathbf{F} = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \]

where \( \mathbf{v} \) is the velocity of the material point on which the force is exerted, than the two vectorial fields (eventually depending on time) \( \mathbf{E} \) and \( \mathbf{B} \) must satisfy

\[
\begin{align*}
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0.
\end{align*}
\]

Moreover, a corresponding generalized potential is

\[ U(P, \mathbf{v}, t) = \phi(P, t) + \mathbf{A}(P, t) \cdot \mathbf{v}, \]

where \( \phi(P, t) \) is a so-called scalar potential and \( \mathbf{A}(P, t) \) is a so-called vector potential. In addition, the connection between force and generalized potential is provided by the two equations

\[
\begin{align*}
\mathbf{B} &= \nabla \times \mathbf{A}, \\
\mathbf{E} &= \nabla \phi - \frac{\partial \mathbf{A}}{\partial t}.
\end{align*}
\]

This is the case, for example, of the Lorentz electromagnetic force acting on a free point particle of charge \( e \), position \( P \), and velocity \( \mathbf{v} \) in the presence of an electric field \( \mathbf{E}(t, P) \) and magnetic induction \( \mathbf{B}(t, P) \):

\[ \mathbf{F} = e\mathbf{E} + ev \times \mathbf{B}. \]

This is a force of type (20), and the corresponding equations (21) represent the first couple of the fundamental Maxwell equations. These equations are deduced as necessary conditions for the existence of a potential, without other physical considerations.

4. Generalized potential of apparent forces

This section is devoted to showing the connection given in equation (23) between the force of type (20) and generalized potential. Precisely, starting from a force of type (20) and following theorem 3.1, it is known that it admits a generalized potential of type expressed in (22), i.e.

\[ U(P, \mathbf{v}, t) = \phi(P, t) + \mathbf{A}(P, t) \cdot \mathbf{v}. \]
Now, imposing
\[ \mathbf{A}(P, t) = mv_r = m[v_O + \omega_r \times (P - O)] \]  \hspace{1cm} (26)
as the vector potential and
\[ \phi(P, t) = \frac{A^2}{2} = m \frac{v_r^2}{2} \]  \hspace{1cm} (27)
as the ordinary scalar potential, we are interested to recover the explicit expression of the two
vector fields \( \mathbf{E} \) and \( \mathbf{B} \).

**Remark 4.1.** It could be directly done by making explicit the terms appearing in the
Lagrangian equations, but we prefer to use a different strategy based on the theorem
appearing in [2], p. 64. Precisely, we observe that the two arguments \( P \) and \( t \) of the vector
field \( \mathbf{A}(P, t) \) must be considered as independent from each other. Furthermore, once the
motion of the non-inertial reference frame \( T_O \) with respect to the fixed (inertial) one \( T_O \),
namely the translational motion, is assigned the origin \( O \), the velocity \( v_O \) and the angular
velocity \( \omega_r \) become three known functions of time, \( O = O(t), v_O = v_O(t), \omega_r = \omega_r(t) \) (see
[1], p. 70).

Let us consider (23)\(_1\), that gives back (see for example [3], equation (2.3.36) p. 293)
\[ \mathbf{B} = \nabla \times \mathbf{A} = m \nabla \times v_r = 2m\omega_r, \]  \hspace{1cm} (28)
so, by also using (11), we can write
\[ \mathbf{F} = \mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{E} - 2m\omega_r \times \mathbf{v} = \mathbf{E} - ma_c. \]  \hspace{1cm} (29)
To calculate the field \( \mathbf{E} \) we use (23)\(_2\), which gives
\[ \mathbf{E} = \nabla \phi(P, t) - \frac{\partial \mathbf{A}}{\partial t} = \frac{1}{2m} \nabla A^2 - \frac{\partial \mathbf{A}}{\partial t}. \]  \hspace{1cm} (30)
We start to calculate the first term of the right-hand side,
\[ \frac{1}{2m} \nabla ^2 A^2 = \frac{m}{2} \nabla ^2 v_r^2 = \frac{m}{2} \nabla (v_O^2 + 2v_O \cdot \omega_r \times (P - O) + [\omega_r \times (P - O)]^2). \]  \hspace{1cm} (31)
Observing that \( \nabla v_O^2 = 0 \), we can write
\[ \nabla [2v_O \cdot \omega_r \times (P - O)] = 2 \nabla [v_O \times \omega_r \times (P - O)], \]  \hspace{1cm} (32)
and so
\[ \frac{1}{2} \nabla A^2 = 2m \nabla [v_O \times \omega_r \times (P - O)] + m \nabla [\omega_r \times (P - O)]^2. \]  \hspace{1cm} (33)
Recalling that \( (P - O) \) is a potential field, hence \( \nabla \times (P - O) = \mathbf{0} \) and, using a well-known
formula from the vector analysis (see for example [4], p. 230),
\[ \nabla (\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} + (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{B} \times \nabla \times \mathbf{A} + \mathbf{A} \times \nabla \times \mathbf{B}, \]  \hspace{1cm} (34)
we obtain
\[ \nabla [v_O \times \omega_r \times (P - O)] = v_O \times \omega_r. \]  \hspace{1cm} (35)
By a substitution in (33), we can write
\[ \frac{1}{2} \nabla A^2 = \frac{m}{2} \nabla [\omega_r \times (P - O)]^2 + m v_O \times \omega_r. \]  \hspace{1cm} (36)
If $P^*$ is the projection of $P$ on the instantaneous rotation axis related to the origin $O$, that is the axis passing through $O$ and parallel to $\omega_r$, we get

$$[\omega_r \times (P - O)]^2 = \omega_r^2 (P - P^*)^2. \quad (37)$$

If, at each fixed instant $t$, we introduce a system of cylindrical coordinates $r(=|P - P^|)$, $\theta$, $z$ having as its axis $z$, the instantaneous axis of rotation related to $O$, and define $(\hat{e}_1 = \text{vers}(P - P^), \hat{e}_2, \hat{e}_3)$ the associated basis (orthonormal in this case), the following expression results:

$$\omega_r^2 r^2 = \omega_r^2 (P - P^*)^2, \quad (38)$$

and, by recalling the gradient in cylindrical coordinates

$$\nabla_{(r, \theta, z)} = \frac{\partial}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{e}_\theta + \frac{\partial}{\partial z} \hat{e}_z, \quad (39)$$

we have

$$\nabla[\omega_r \times (P - O)]^2 = \omega_r^2 \frac{\partial r^2}{\partial r} \hat{e}_r = 2\omega_r^2 r \hat{e}_r = 2\omega_r^2 (P - P^*), \quad (40)$$

that can be rewritten as

$$\frac{1}{2} \nabla[\omega_r \times (P - O)]^2 = \omega_r \times (P - O) \times \omega_r. \quad (41)$$

Thus,

$$\frac{1}{2} \nabla A^2 = m\omega_r \times (P - O) \times \omega_r + m\mathbf{v}_O \times \omega_r. \quad (42)$$

Finally, following remark 4.1,

$$\frac{\partial \mathbf{A}}{\partial t} = m \frac{\partial}{\partial t} [\mathbf{v}_O + \omega_r \times (P - O)] = m [\mathbf{a}_O + \omega_r \times (P - O) - \omega_r \times \mathbf{v}_O]. \quad (43)$$

By substituting (42) and (43) into (30) and using (10), we obtain

$$\mathbf{E} = -m[a_O + \omega_r \times (P - O) - \omega_r \times \mathbf{v}_O] = m \mathbf{a}_r. \quad (44)$$

Finally by (29) and (44), we conclude that

$$\mathbf{F} = \mathbf{E} + \mathbf{v} \times \mathbf{B} = m \mathbf{a}_r - m \mathbf{a}_r, \quad (45)$$

that is exactly the translation forces (9).

**Remark 4.2.** Collecting (8), (25), (26), and (27), we can write the explicit expression of the generalized potential. Moreover, since $\frac{1}{2} m \mathbf{v}_O^2$ is certainly independent of $P$ and $\dot{P}$, it can be neglected and hence $U(P, \mathbf{v}, t)$ can be regarded as a sum of three contributions:

$$U_1(\mathbf{v}, t) = m \mathbf{v}_O \cdot \mathbf{v};$$

$$U_2(P, \mathbf{v}, t) = m \omega_r \times (P - O) \cdot \mathbf{v};$$

$$U_3(P, \mathbf{t}) = m \mathbf{v}_O \cdot \omega_r \times (P - O) + \frac{1}{2} m [\omega_r \times (P - O)]^2. \quad (46)$$

Recalling (5), it is possible to consider

$$\frac{\partial U_1}{\partial q_j} = \frac{d}{dt} \left( \frac{\partial U_1}{\partial \dot{q}_j} \right) = -m \mathbf{a}_O \cdot \frac{\partial P}{\partial q_j} \quad (47)$$
\[
\frac{\partial U_2}{\partial q_j} - \frac{d}{dt} \left( \frac{\partial U_2}{\partial q_j} \right) = -2m \omega_r \times v \cdot \frac{\partial P}{\partial q_j} - m\omega_r \times (P - \mathbf{O}) \cdot \frac{\partial P}{\partial q_j} - m\mathbf{n}_O \times \omega_r \cdot \frac{\partial P}{\partial q_j}
\]

(48)

\[
\frac{\partial U_3}{\partial q_j} - \frac{d}{dt} \left( \frac{\partial U_3}{\partial q_j} \right) = +m \mathbf{v}_O \times \omega_r \cdot \frac{\partial P}{\partial q_j} - m\omega_r \times [\omega_r \times (P - \mathbf{O})] \cdot \frac{\partial P}{\partial q_j}
\]

(49)

from which we obviously obtain (18) and (19). From all these results we remark, as in [5], that a part of the case of uniformly rotating frames, we cannot separate the contributions to the only time-dependent Coriolis force and to the only dragging force, respectively. In other words, the generalized potential is a feature of the whole system of the general inertial forces and not separately to the generalized components of each force.

5. Conclusion remarks

In this paper, we have reviewed the analogies between the electromagnetic force and the inertial ones. The generalized potential of the Lorentz force is often studied in standard textbooks, while the analogous potential of the Coriolis field is generally overlooked. We have highlighted this aspect, describing the mathematical formalism of the inertial fields and emphasizing the role of the Coriolis and Dragging forces.

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