Relativistically Invariant Description of Fluctuations Against the Background of Classical Kink Solutions

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A relativistically invariant scheme for the description of excited states in a one-kink sector is formulated. The normal oscillations of fluctuations against the background of a moving kink are determined. Zero mode of these oscillations is excluded automatically due to the properties of the integral equation describing normal oscillations. A Hamiltonian for elementary excitations is obtained reflecting the relativistic nature of the problem considered.

1 Introduction

The idea to regard stable localized solutions of nonlinear equations as dynamic objects has been under intense study in a number of papers published in the middle of seventies (for references see [1]). In the one-dimensional case a coordinate $X$ of the kink solution $u_{c}(x-X)$ of the Klein-Gordon nonlinear equation has been regarded as a dynamic variable and the initial field has been written in the form $[2, 3, 4]$

$$\Phi(x) = u_{c}(x-X) + \chi(x-X).$$

The coupling condition due to fluctuation

$$\langle \chi u'_{c} \rangle \equiv \int dx \chi(x)u'_{c}(x) = 0$$

conserved the number of independent variables in the configurational space and removed the zero mode $u'_{c}$ from the spectrum of excitations. At the same time it was noted that decomposition of the field into a solitonic and fluctuating parts (1) is relativistically non-invariant and first of all is adapted to studying the system properties in the center-of-mass frame. In this case the Ansatz $u'_{c}(x)$ describes the classic field configuration in the ground state exactly and the variable $\chi(x)$ describes small perturbations related to the presence of field fluctuations as well as to kink trembling. Thus, e.g., the quantum corrections to
the soliton mass \cite{4} have been determined and the system spectrum in the center-of-mass frame \cite{1} has been studied. Besides, as the relativistic invariance of the Hamiltonian does not depend on the concrete choice of variables, there has always existed a possibility to restore the relativistic form of the heavy particle energy in one way or other. In Ref. \cite{3} this problem has been solved via summation of tree diagrams of the perturbation theory and Ref. \cite{4} has found the corresponding special solutions of the equations of motion for the fluctuations. At the following on the ground of the expansions of type (1) the studies of meson-soliton scattering in one-dimensional model have been attempted \cite{6, 7, 8, 9}. Similar studies have also been made in Skyrme model \cite{10, 11, 12, 13}. The total momentum in these problems does not vanish and therefore it is unjustified to treat the field $\chi(x)$ as a purely fluctuational one. This treatment fails to take into account the translational kink motions and contradicts the relativistic nature of the theory.

Concerning a touched problem we note, that by virtue of a relativism, the investigation of a system in the center-of-mass reference frame allows, in principle, to restore a character of its evolution in any frame of reference. However, such passage should be adjust with used perturbation theory and its details are not quite clear. Therefore, it is desirable, from the very beginning to present the scheme in the relativistic invariant form. For this purpose let us note some properties of one kink states. First of all, note that one cannot introduce a relativistically invariant vacuum state in a topological sector in contrast to the conventional field theory. The self-similar solutions of equations of motion in the form of moving kinks realize only local minima of the Hamiltonian that can be parametrized with the total momentum of the system. From the relativistic point of view these minima are equivalent, degenerate with respect to spatial translations and absolutely stable with respect to arbitrary fluctuations of field variables due to the conservation of the total momentum of the system. Thus, when one studies the topological sector of the theory from the viewpoint of an arbitrary reference frame, one should find a local minimum relating to the respective value of the total momentum of the system instead of defining the relativistically invariant vacuum state. To study the corresponding variational problem one should introduce the cyclic coordinate $X$ as a collective variable with the help of the substitution

$$\Phi(x) = \varphi(x - X)$$

imposing on the function $\varphi(x)$ certain limitations providing for the non-degenerate nature of the transformation to new variables. Developing the canonical procedure on the ground of the variables $X$, $\varphi(x)$ and minimizing the Hamiltonian with respect to variations of the field $\varphi(x)$ and the momentum conjugate to it yield a usual root at the extremum point defining the energy of a relativistic particle. The momentum of this particle coincides with the total momentum of the system and its mass corresponds to the kink’s mass. Fluctuations of field variables against the local minimum are defined in a relativistically invariant way and the study of their spectrum permits one to present the Hamiltonian in the form of a series in powers of the amplitudes of corresponding normal oscillations. The terms of this series depend on the total momentum of the system that is not an observable quantity from the physical viewpoint. Splitting the total momentum of the system into the kink and field parts one can distinguish the individual dynamic variables of the kink and the mesons. As the field momentum is small due to small fluctuations, one should again expand the Hamiltonian of the system in powers of the meson field amplitudes. Performing these procedures, one can obtain the Hamiltonian adapted for calculating the meson-soliton scattering effects. The paper shows that the kinetic term of
this Hamiltonian describes the multitude of relativistic particles including a heavy particle - kink and a set of light particles corresponding to the states of the background scattering of mesons by the kink. Such a structure of the system spectrum in the presence of a kink has been noted in Ref. [14]. A detailed treatment of the outlined scheme is divided into three Sections and an Appendix. In the next Section a local minimum is determined and the part of the Hamiltonian bilinear in fluctuations is calculated. Section 3 deals with the normal oscillations of the system. Section 4 introduces amplitudes of the meson field and calculates the kinetic part of the Hamiltonian. In the Appendix the indefinite dot product used for the field quantization is derived.

2 Determining a local minimum and the Hamiltonian of fluctuations

Consider a nonlinear scalar system described by the Lagrangian.

\[
L = \int dx \left\{ \frac{1}{2} \left( \Phi'^2(x) - \Phi''^2(x) \right) - U(\Phi(x)) \right\}.
\]

Kink solutions to this system realize a multitude of local minima of the Hamiltonian degenerated with respect to spatial translations. Taking into account the dominant role of degeneration parameters of vacuum solutions in the formation of a low-energy spectrum, let us introduce a cyclic coordinate \(X\) as a collective variable with the help of substitution (3) limiting the admissible variations of the field \(\varphi(x)\) by the condition

\[
\langle \Phi' \delta \varphi \rangle = 0.
\]

\(\Phi(x)\) is the time-independent field Ansatz whose form will be determined later. Taking into account that velocity variations \(\delta \dot{\varphi}(x)\) also satisfy the coupling condition \(\langle \Phi' \delta \dot{\varphi} \rangle = 0\), we find the canonical momenta conjugate to new variables

\[
P = \partial L / \partial \dot{X} = \langle (\dot{X} \varphi' - \dot{\varphi}) \varphi' \rangle, \quad \pi(x) = \delta L / \delta \dot{\varphi}(x) = (1 - \mathcal{P}) \left( \dot{\varphi}(x) - \dot{X} \varphi'(x) \right),
\]

where

\[
\mathcal{P}(x, x') = E^{-1} \Phi'(x) \Phi'(x'), \quad E \equiv \langle \Phi'^2 \rangle, \quad \mathcal{P}^2 = \mathcal{P}
\]

is the projection operator. Solving relation (5) with respect to the velocities \(\dot{X}, \dot{\varphi}(x)\) we find the Hamiltonian of the system in new variables

\[
H = \dot{X} P + \langle \dot{\varphi} \pi \rangle - L = \frac{1}{2} \left\{ \langle \pi^2 + \varphi'^2 \rangle + 2U(\varphi) + \frac{(\langle \varphi' \pi \rangle + P)^2}{\langle \varphi' \mathcal{P} \varphi' \rangle} \right\}.
\]

The equations for the extremals \(\varphi(x), \pi(x)\)

\[
\delta H / \delta \pi = 0, \quad \delta H / \delta \varphi = 0
\]

determine the local minima of the Hamiltonian parametrized by the total momentum of the system \(P\). As, according to (5), the variations of the momentum satisfy the condition \(\mathcal{P} \delta \pi = 0\), then the first of equations (8) assumes the form

\[
(1 - \mathcal{P}) \left\{ \frac{\pi + \varphi'}{\langle \varphi' \mathcal{P} \varphi' \rangle} \right\} = 0.
\]
Let us now fix the gauge Ansatz \( \Phi(x) \) identifying it with the extremal of the Hamiltonian: \( \Phi(x) \equiv \varphi(x) \). Then from equation (9) it follows that \( \pi(x) = 0 \) and the second of equations (8) assumes the form

\[- \left( 1 - \frac{P^2}{E^2} \right) \Phi'' + U'(\Phi) = 0.\]

That is,

\[\Phi(x) = u_c \left( \frac{E}{\sqrt{E^2 - P^2}} x \right),\]

where \( u_c(x) \) is the static kink solution. Substituting the solutions \( \pi(x) = 0, \varphi(x) = \Phi(x) \) into the definition of the parameter \( E \) (6) and in formula (7) we prove that the local minimum of the Hamiltonian coincides with the energy of a relativistic particle

\[H_0 = E = \sqrt{P^2 + M^2}, \quad (10)\]

\( M \equiv < u_c^2 > \) is the kink’s mass. Fluctuations of canonical variables \( \varphi(x) - \varphi(x), \pi(x) \) are obviously defined in a relativistically invariant way. Keeping for them the initial notation, let us write the Hamiltonian bilinear in these fluctuations

\[H_2 = \frac{1}{2} \left( \pi^2 + \varphi'^2 + 2 \frac{P}{E} \varphi' \pi + U''(\Phi) \varphi^2 + \frac{3P^2}{E^2} \varphi' \pi \varphi \right), \quad (11)\]

According to (4), (5) Poisson’s brackets of the variables \( \varphi(x), \pi(x), P \) have the form

\[\{ \varphi(x), \pi(x') \} = (1 - P)(x, x'), \quad \{ P, \varphi \} = \{ P, \pi \} = 0, \quad \{ \varphi, \varphi \} = \{ \pi, \pi \} = 0. \quad (12)\]

### 3 Normal oscillations

The Hamiltonian \( H_2 \) describes the small oscillations of the system in the vicinity of a local minimum. Using (11), (12) we will write the corresponding equations of motion for the variables \( \varphi(x), \pi(x) \) as

\[\dot{\pi}(x) \equiv - \frac{\delta H_2}{\delta \varphi(x)} = (1 - P) \left( \frac{P}{E} \frac{\partial}{\partial x} \pi(x) - \mathcal{L}_P \varphi(x) + \frac{3P^2}{E^2} \frac{\partial}{\partial x} \varphi(x) \right),\]

\[\dot{\varphi}(x) \equiv - \frac{\delta H_2}{\delta \pi(x)} = (1 - P) \left( \pi(x) + \frac{P}{E} \frac{\partial}{\partial x} \varphi(x) \right), \quad \mathcal{L}_P \equiv - \frac{\partial^2}{\partial x^2} + U''(\Phi).\]

Rewriting the second equation in the form

\[\pi(x) = (1 - P)D\varphi(x), \quad D \equiv \frac{\partial}{\partial t} - \frac{P}{E} \frac{\partial}{\partial x}, \quad (13)\]

we find the equation for the field fluctuations \( \varphi(x) \)

\[\left( \mathcal{L}_P + D^2 - 4 \frac{P^2}{E^2} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \right) \varphi(x) = 0. \quad (14)\]

To solve this equation, we define a new variable \( f(z) \) according to the formula

\[\varphi(x, t) = e^{-i \pi(Mt - Pz)} f(z), \quad (15)\]
z \equiv (E/M)x is the scale transformation of the coordinate. Using the explicit form of the operators \(L_p, D\) and differentiating, we find

\[
(L_p + D^2) e^{-i\Phi'(Mt-Pz)} f(z) = e^{-i\Phi'(Mt-Pz)} \left((\mathcal{L} - \omega^2) f(z)\right), \quad \mathcal{L} \equiv -\frac{\partial^2}{\partial z^2} + U''(\Phi).
\]

Then, taking into account that the function \(\Phi'(x)\) is zero mode of the operator \(L\) and using the coupling condition (4) written in the form

\[
P e^{i\omega E} z f(z) = 0,
\]

we obtain from (14) the equation for the function \(f(z)\)

\[
(L - \omega^2) f(z) = -F(\omega, z) \int dz' F^*(\omega, z') f(z') \equiv -F(\omega, z) Q(\omega),
\]

where

\[
F(\omega, z) \equiv e^{-i\Phi'(z)} \left(\frac{M}{E} \omega - 2i \frac{P}{M} \frac{\partial}{\partial z}\right) \psi_0(z)
\]

and

\[
\psi_0(z) = \frac{1}{\sqrt{M}} u'_c(z) = \sqrt{\frac{M}{E}} \Phi'(x), \quad \int dz \psi_0^2(z) = 1
\]

is the zero mode of the operator normalized to unity in the variables \(z\). Formally one may regard equation (17) as a linear equation with a right-hand side. The solution of a homogeneous problem

\[
L \psi_\alpha(z) = \omega^2_\alpha \psi_\alpha(z), \quad \alpha \equiv \{i, q\},
\]

determines the frequencies of normal oscillations of the system at rest. To solve the equation (17) at the points of the spectrum \(\omega = \omega_\alpha\) of the operator \(L\) the eigenstates \(\psi_\alpha\) must be orthogonal to the right-hand side of this equation

\[
\int dz \psi_\alpha(z) F(\omega_\alpha, z) = 0.
\]

At the same time the corresponding solutions of equation (17) will be written in the form

\[
f_\alpha(z) = \psi_\alpha(z) - (L - \omega^2_\alpha)^{-1} F(\omega_\alpha, z) Q(\omega_\alpha).
\]

Inserting this solution in the definition (17) for the constants \(Q(\omega_\alpha)\) and using (19) yield the equation

\[
\left(1 + \left\langle F^*(\omega)(L - \omega^2_\alpha)^{-1} F(\omega)\right\rangle_z\right) Q(\omega_\alpha) = 0, \quad < ... >_z \equiv \int dz ... (21)
\]

For the zero mode \(\omega_\alpha = 0\) equation (21) possesses only zero solution \(Q(0) = 0\) because the operator \(L\) is positively defined. As the function \(f_0\) (20) coincides with the zero mode \(\psi_0\), the gauge condition (16) is not met for it and, consequently, one should exclude the mode \(f_0\) from the fluctuation spectrum. One should note in this connection that the form of the equation for the function \(f(z)\) (17) is not related to the gauge choice for the function \(\varphi(x)\) (3) [15]. Only the concrete values of the constants \(Q(\omega_\alpha)\) depend on this choice. The constant \(Q(0)\) vanishes with any gauge. Therefore the function \(f_0\) (20) coinciding with zero mode of the operator \(L\) (18) should be excluded from the spectrum of excitations
because a fixed function cannot satisfy an arbitrary gauge. For other modes \((\omega \neq 0)\) the parameters \(Q(\omega)\), generally speaking, must be different from zero and must ensure the fulfilment of arbitrary gauge conditions. This means that the following equality must hold

\[
1 + \langle F^*(\omega)(\mathcal{L} - \omega_2^2)^{-1}F(\omega) \rangle_z = 0, \quad \omega \neq 0.
\] (22)

Performing the integration by parts, one may show that the eigenfunctions of the operator \(\mathcal{L}\) (18) satisfy the identity

\[
2i \nu \int dz e^{-i \nu z} \psi_\alpha(z) \psi_0'(z) = (\omega_\alpha^2 - \nu^2) \int dz e^{-i \nu z} \psi_\alpha(z) \psi_0(z)
\]

that establishes the link between the integrals in both parts of this equality for arbitrary \(\nu\). Using this link and the relation \(E^2 = M^2 + P^2\), one proves easily the validity of the solubility condition (19). The same identity together with the completeness condition

\[
\sum_\alpha \psi_\alpha(z) \psi_\alpha^*(z') = \delta(z - z'), \quad \sum_\alpha \equiv \int dq + \sum_i
\]

permits one to prove formula (22) for arbitrary \(\omega \neq 0\). Similarly, one can get rid of the resolvent \((\mathcal{L} - \omega_\alpha^2)^{-1}\) in (20) and write the solution of equation (17) in the form

\[
f_\alpha(z) = \psi_\alpha(z) + \frac{E}{M \omega_\alpha} e^{-i \omega_\alpha E^2 z} \psi_0(z)Q(\omega_\alpha).
\]

Determining the constants \(Q(\omega_\alpha)\) from the gauge condition (16), one may write this solution in the form

\[
f_\alpha(z) = (1 - \mathcal{P}_\alpha) \psi_\alpha(z),
\] (23)

where

\[
\mathcal{P}_\alpha(z, z') = e^{-i \omega_\alpha E^2 z} \psi_0(z) \psi_0'(z') e^{i \omega_\alpha E^2 z'}, \quad \mathcal{P}_\alpha^2 = \mathcal{P}_\alpha
\]

is the projection operator. Obviously, the solution of equation (14) splits into components with positive and negative frequencies. If one denotes with \(\omega_i^2 \neq 0, \ i = 1, ..., N; \ \omega_q^2 = q^2 + m^2\) the eigenvalues of the operator \(\mathcal{L}\) belonging to the states of the discrete and continuous spectra, where \(q\) is the wave vector of the states from the continuous spectrum, \(m^2\) being its boundary, then, according to (15), (23), one may write all multitude of solutions of equation (14) determining the normal oscillations in the vicinity of the local minimum in the form

\[
\varphi_\alpha^{(+)}(x, t) = \frac{1}{\sqrt{2 \omega_\alpha}} \exp \left(-i \frac{\omega_\alpha}{E}(Mt - Pz)\right) (1 - \mathcal{P}_\alpha) \psi_\alpha(z), \ \varphi_\alpha^{(-)} = \varphi_\alpha^{(+)*}, \ \alpha = i, q.
\] (24)

Here and in what follows \(\omega_\alpha\) are positive quantities. Functions \(\psi_\alpha\) in (24) are normalized conventionally in \(L_2\)

\[
\int dz \psi_\alpha^*(z) \psi_q(z) = \delta(q - q'), \quad \int dz \psi_i(z) \psi_j(z) = \delta_{ij}.
\] (25)

and the orthogonality properties of the functions \(\varphi_\alpha^{(\pm)}(x, t)\) and their normalization with respect to the indefinite dot product (A4) are given in the Appendix.
4 Elementary excitations

Expanding the field $\varphi(x,t)$ over the functions orthonormalized in the sense of the dot product (A4)

$$\varphi(x,t) = \sum_\alpha \left( c_\alpha \varphi_\alpha^+(x,t) + c_\alpha^* \varphi_\alpha^-(x,t) \right)$$

(26)

and using (26), (13) let us express the amplitudes $c_\alpha$, $c_\alpha^*$ through the canonical variables $\varphi(x)$, $\pi(x)$

$$c_\alpha = (\varphi_\alpha^+, \varphi) = i \left\langle \varphi_\alpha^- \pi - (D\varphi_\alpha^-) \varphi \right\rangle, \quad c_\alpha^* = -(\varphi_\alpha^-, \varphi) = -i \left\langle \varphi_\alpha^+ \pi - (D\varphi_\alpha^+) \varphi \right\rangle.$$  

(27)

With the help of formulas (12), (A6), (A7), (27), one may calculate the Poisson’s brackets of a new set of variables $P$, $c_\alpha$, $c_\alpha^*$.

$$\{P, c_\alpha\} = \{P, c_\alpha^*\} = 0, \quad \{c_\alpha, c_\alpha'\} = \{c_\alpha^*, c_\alpha'^*\} = 0, \quad \{c_\alpha, c_\alpha'^*\} = -i \delta_{\alpha\alpha'}.$$  

(28)

To write (11) in terms of the amplitudes, let us present formally the nonlocal Hamiltonian bilinear in the field variables $\varphi(x)$, $\pi(x)$ in the form

$$H_2 = \left(1/2\right) \left\langle \varphi V \varphi + 2 \varphi K \pi + \pi S \pi \right\rangle,$$

(29)

where in the general case $V$, $K$, $S$ are some integro-differential operators. Canonical equations of motion can be written in the form

$$V \varphi = -\dot{\pi} - K \pi, \quad \pi S = \dot{\varphi} - \varphi K.$$

Inserting these expressions into the first and third terms of (29), respectively, we find

$$H_2 = \left(1/2\right) \left\langle \dot{\varphi} \pi - \varphi \dot{\pi} \right\rangle.$$  

(30)

In this Hamiltonian the role of dynamic variables is played by the integration constants of the equations of motion. Substituting expressions (26), (13) in (30) with the account of the temporal dependence (15) and using formulas (A4-A7), let us write the Hamiltonian $H_2$ in terms of the amplitudes

$$H_2 = \frac{M}{E} \left( \int dq \omega_q c_q^* c_q + \sum_i \omega_i c_i^* c_i \right), \quad \omega_q = \sqrt{q^2 + m^2}.$$  

(31)

Using formulas (24), (26) one can also write the next terms of the expansion of the Hamiltonian in terms of the amplitudes of normal oscillations $c_\alpha$, $c_\alpha^*$. The Hamiltonian thus obtained describes the excitations of the system against the local topologic minimum with the given value of the total momentum of the system. These excitations possess a hybrid nature taking into account meson field oscillations and kink’s trembling simultaneously. To study the individual movements of the kink and the mesons and to describe the interaction between them one needs to split the total momentum of the system into the purely kink part $p$ and one related to field fluctuations $P_f$: $P = p + P_f$. After that

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¹One can restore the spectrum of this Hamiltonian obtained under the condition that the amplitude are small from the spectrum of the system at rest with the help of formula $E^{(\alpha)}(P) = \sqrt{P^2 + E^{(\alpha)}(0)}$.  

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one should re-expand the obtained Hamiltonian using the small value of the field momentum $P_f$. To determine the kinetic part of the Hamiltonian restructured in this way, it is sufficient to use the momentum $P_f$ in the leading approximation. One can establish the momentum of fluctuations as follows. The wave vector $q$ parametrizing the eigenfunctions and eigenvalues of the continuous spectrum of problem (18) in the leading (zero) approximation with respect to the reflection coefficient is the momentum of the corresponding state in the center-of-mass reference frame. In the reference frame relative to which the system moves with the velocity $V = P/E$, the momentum of this state $\overline{q}$ and its energy $\omega_{\overline{q}}$ in accord with the Lorentz transformations will amount to

$$\overline{q} = \frac{q + V \omega_q}{\sqrt{1 - V^2}}, \quad \omega_{\overline{q}} = \frac{\omega_q + V q}{\sqrt{1 - V^2}}. \quad (32)$$

Therefore, multiplying the momentum $\overline{q}$ by the number of ”particles” in this state $c_q^* c_q$, and performing the integration over the parameter $q$, we find the momentum of the field in the reference frame relative to which the system moves with the velocity $V = P/E$

$$P_f = \int dq \overline{q} c_q^* c_q.$$

As $dq/d\overline{q} = \omega_q/\omega_{\overline{q}}$, then passing to the integration over $\overline{q}$ in this formula and introducing the meson amplitudes

$$a_q = \sqrt{\frac{\omega_q}{\omega_{\overline{q}}}} c_q, \quad a_q^* = \sqrt{\frac{\omega_{\overline{q}}}{\omega_q}} c_q^*,$$

we write the momentum of fluctuations in the usual form

$$P_f = \int dq a_q^* a_q, \quad (34)$$

Note that the expression for the momentum of fluctuations (34) may be obtained by separating the contribution proportional to the volume of the system from the integral $- <\pi \varphi' >$ and neglecting the reflection effects. Using (28), (33) one can prove that the Poisson’s brackets of the variables $p$, $a_q$, $a_q^*$ have the form

$$\{a_q, a_{q'}\} = \{a_q^*, a_{q'}^*\} = 0, \quad \{a_q, a_{q'}^*\} = -i\delta(q - q'), \quad \{p, a_q\} = -iqa_q, \quad \{p, a_q^*\} = iqa_q^*.$$

Taking into account formulas (10), (31), (33) and keeping the linear term in $P_f$ in Hamiltonian $H_0$, we find

$$H_0 = \sqrt{p^2 + M^2} + V \int d\overline{q} \overline{q} a_{\overline{q}}^* a_{\overline{q}} + O(a^4),$$

$$H_2 = \sqrt{1 - V^2} \int d\overline{q} \omega_q a_q^* a_q + \left(\frac{M}{\sqrt{p^2 + M^2}} \right) \sum_i \omega_i c_i^* c_i + O \left((ca)^2 \right).$$

Hence summing main contributions in amplitudes and using transformations (32) we find kinetic term of full Hamiltonian.

$$\mathcal{H}_0 = \sqrt{p^2 + M^2} + \int dq \omega_q a_q^* a_q + \left(\frac{M}{\sqrt{p^2 + M^2}} \right) \sum_i \omega_i c_i^* c_i.$$

This Hamiltonian reflects the natural pattern of the spectrum in the one-kink spectrum sector noted in work [14] and describes a free relativistic particle-kink, a set of light particles related to the states of background scattering of mesons on a kink as well as excitations of the internal degrees of freedom.
5 Conclusion

The paper formulates the relativistically invariant procedure for the description of excited states in a one-kink sector. The main part of the paper deals with studying the dynamics of the system with a fixed total momentum. It includes the determination of the local minimum of the Hamiltonian formally corresponding to a free moving kink as well as studies of fluctuations in the vicinity of this minimum. A relativistic form of the corresponding normal oscillations is established. Zero mode of these oscillations is excluded automatically due to the properties of the integral equation describing normal oscillations. In this connection it should be noted that the role of the gauge condition imposed on the function $\varphi(x)$ (3) reduces only to the non-contradictory introduction of new variables. It is also evident that the choice of the gauge affects the form of equations of motion. The gauge condition used in the work is defined by the relativistically invariant way. It leads to zero canonical momentum of fluctuations in the extremum point and essentially to simplify the equations of motion. Keeping in mind that with the fixed momentum of the system the fluctuations have a hybrid pattern describing the oscillations of the meson field as well as the kink’s trembling, the momentum of the system is split into field and kink parts in the final part of the paper. This permits one to attribute to the kink the sense of a full-scale dynamic object and to write the kinetic term of the Hamiltonian in terms of natural elementary excitations characteristic for a relativistic theory.

Appendix

Let us use the identity

$$\langle \varsigma^*(\mathcal{L}_P + D^2 - 4 \frac{P^2}{E^2} \frac{\partial}{\partial x} P \frac{\partial}{\partial x}) \varphi \rangle = 0, \quad A1$$

where $\varsigma$, $\varphi$ are the solutions of equation (14) to introduce an indefinite dot product applied for the field quantization. Performing the integration by parts, we obtain

$$\frac{\partial}{\partial t} \langle \varsigma^*(D \varphi) - (D \varsigma)^* \varphi \rangle + \frac{P}{E} \left((D \varsigma)^* \varphi - \varsigma^*(D \varphi)\right)|_{-\infty}^{+\infty} + (\varsigma^* \varphi - \varsigma^* \varphi')|_{-\infty}^{+\infty} = 0.$$

Introducing according to (15), the substitution

$$\varphi(x,t) = \exp\left(-\frac{P}{E} \frac{z}{\partial t}\right) h(z,t), \quad \varsigma(x,t) = \exp\left(-\frac{P}{E} \frac{z}{\partial t}\right) g(z,t)$$

and using it for the transformation of the terms located outside the integral sign, we get

$$\frac{\partial}{\partial t} \langle \varsigma^*(D \varphi) - (D \varsigma)^* \varphi \rangle + \frac{M}{E} \exp\left(-\frac{P}{E} \frac{z}{\partial t}\right) \left(\frac{\partial g^*}{\partial z} h - g^* \frac{\partial h}{\partial z}\right)|_{-\infty}^{+\infty} = 0. \quad A2$$

The term outside the integral here is not vanishing only for functions $g$, $h$ belonging to the continuous spectrum and possessing the same sign of the frequency. As for the concrete modes with the wave vectors $q' \simeq \pm q$

$$\omega_{q'} - \omega_q \simeq \pm \frac{q}{\omega_q} (q' \mp q),$$
where \( \omega_q \) is the positive value of the root \( \sqrt{q^2 + m^2} \), then taking into account the temporal dependence \((15)\) one can change the exponential factor in \((A2)\) with unity. Indeed, after differentiating with respect to \( t \), the exponent of this factor will be equal to

\[
\frac{P}{E} (\omega_q - \omega_{q'}) z \simeq \pm \frac{P}{E} \frac{q}{\omega_q} (q' \mp q) z.
\]

As \( |(P/E)(q/\omega_q)| < 1 \) then due to formula \( A3 \) this exponential factor will not influence the values of the limits in \((A2)\) that are determined by the phases \((q' \mp q) z\) of the second factor in this term. Thus, if one defines the indefinite dot product with the formula

\[
(\varsigma, \varphi) \equiv -i \langle (D\varsigma^*) \varphi - \varsigma^* (D\varphi) \rangle = - (\varphi, \varsigma)^*,
\]

then for the functions satisfying equation \((14)\) the identity \((A1)\) may be written in the form

\[
\frac{\partial}{\partial t} (\varsigma, \varphi) = i \frac{M}{E} \left( g_0 \frac{\partial h_0}{\partial z} - \frac{\partial g_0^*}{\partial z} h_0 \right) \bigg|_{-\infty}^{+\infty}.
\]

Here \( g_0, h_0 \) are the asymptotics of the functions \( g, h \) at \( |x| \to \infty \). Without accounting for the phase shifts they are defined by the equation \((\partial^2/\partial t^2 - \partial^2/\partial z^2 + m^2) g_0 = 0\). Hence with the account of the temporal dependence and using formula \((A3)\) there follows that

\[
(\varphi(-), \varphi(+)) = 0, \quad (\varphi_i, \varphi_q) = 0.
\]

Besides, the functions \((24)\) are obey the normalisation conditions

\[
(\varphi_q^{(\pm)}, \varphi_{q'}^{(\pm)}) = \pm \delta(q - q'), \quad (\varphi_i^{(\pm)}, \varphi_q^{(\pm)}) = \pm \delta_{ij}.
\]

The first of formulas \((A7)\) is obvious actually due to \((A5)\) and the assumption about the normalization of the functions \(\psi_q\) of the continuous spectrum in \(L_2(25)\). One can prove the validity of the second one using formula \((24)\) in the direct calculation of the integral \((A4)\) and including the normalization of the functions \(\psi_i\) of the discrete spectrum in \(L_2(25)\).

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\(^{2}\text{This definition is similar to the dot product } (\varsigma, \varphi) \equiv -i \int dx (\varsigma^* \varphi - \varsigma \varphi) \text{used while quantizing the Klein-Gordon free field.}^{10}\)
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