Self-adjointness of semi-relativistic Pauli-Fierz Hamiltonian

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Abstract

The spinless semi-relativistic Pauli-Fierz Hamiltonian

\[ H = \sqrt{(p \otimes 1 - A)^2 + M^2} + V \otimes 1 + 1 \otimes H_f, \quad M \geq 0, \]

in quantum electrodynamics is considered. Here \( p \) denotes a momentum operator, \( A \) a quantized radiation field, \( M > 0 \), \( H_f \) the free hamiltonian of a Boson Fock space and \( V \) an external potential. The self-adjointness and essential self-adjointness of \( H \) are shown. It is emphasized that it includes the case of \( M = 0 \). Furthermore, the self-adjointness and the essential self-adjointness of the semi-relativistic Pauli-Fierz model with a fixed total momentum \( P \in \mathbb{R}^d \):

\[ H(P) = \sqrt{(P - P_f - A(0))^2 + M^2} + H_f, \quad M \geq 0, \]

is also proven for arbitrary \( P \).

1 Introduction

1.1 Fundamental facts

In this paper we are concerned with the self-adjointness of the so-called semi-relativistic Pauli-Fierz (SRPF) Hamiltonian \( H \) in quantum electrodynamics. Essential self-adjointness
of $H$ is shown in [Hir13, Theorem 4.5] by a path measure approach under some conditions. We furthermore show its self-adjointness under weaker conditions in this paper. Our result is independent of coupling constants. In this sense the result is non-perturbative.

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and $h$ be a symmetric operator with the domain $D_0$. In general $h$ has the infinite number of self-adjoint extensions. Let $h_0$ be one self-adjoint extension, which defines the Schrödinger equation

$$i \frac{\partial}{\partial t} \Phi_t = h_0 \Phi_t$$

(1.1)

with the initial condition $\Phi_0 = \Phi \in \mathcal{H}$. Then the self-adjointness of $h_0$ ensures the uniqueness of the solution to (1.1) and it is given by $\Phi_t = e^{-ith_0} \Phi$. The time-evolution of a physical system governed by the Schrödinger equation (1.1) is different according to which self-adjoint extension is chosen. Hence it is important to find a core of $h$ or a domain on which $h$ is self-adjoint in order to determine the unique time-evolution of the physical system.

A semi-relativistic Schrödinger operator with nonnegative rest mass $M \geq 0$ is defined as a self-adjoint operator in $L^2(\mathbb{R}^d)$, which is given by

$$H_p = \sqrt{p^2 + M^2} + V.$$  

(1.2)

Here $p = (-i\partial_{x_1}, \ldots, -i\partial_{x_d})$ denotes the momentum operator and $V : \mathbb{R}^d \to \mathbb{R}$ is an external potential. The SRPF model is defined by $H_p$ coupled to a quantized radiation field $A$. Let $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(W) = \bigoplus_{n=0}^{\infty} \otimes^n W$ be the Boson Fock space over Hilbert space $W = \bigoplus_{n=0}^{d-1} L^2(\mathbb{R}^d)$, $d \geq 3$. Although the physically reasonable choice of the spatial dimension is $d = 3$, we generalize it. Let $\omega : \mathbb{R}^d \to \mathbb{R}$ be a dispersion relation. We introduce assumptions on the dispersion relation.

**Assumption 1.1** $\omega(k) \geq 0$ a.e. $k \in \mathbb{R}^d$.

Physically reasonable choice of dispersion relation is $\omega(k) = |k|$ or $\omega(k) = \sqrt{|k|^2 + \nu^2}$ with some $\nu > 0$. In [Hir13] the dispersion relation such that $\omega \in C^1(\mathbb{R}^d; \mathbb{R})$, $\nabla \omega \in L^\infty(\mathbb{R}^d)$, $\inf_{k \in \mathbb{R}^d} \omega(k) \geq m$ with some $m > 0$ and $\lim_{|k| \to \infty} \omega(k) = \infty$ is treated. The free field Hamiltonian $H_f$ of the Boson Fock space is given by the second quantization of the multiplication operator by $\omega$ on $W$, i.e., $H_f = d\Gamma(\omega)$. SRPF Hamiltonian is defined by the minimal coupling of a quantized radiation field $A$ to

$$H_0 = H_p \otimes 1 + 1 \otimes H_f.$$  

(1.3)

$H_0$ is self-adjoint on $D(H_p \otimes 1) \cap D(1 \otimes H_f)$. The creation operator and the annihilation operator in $\mathcal{F}$ are denoted by $a^\dagger(f)$ and $a(f)$, $f \in W$, respectively. They
are linear in \( f \) and satisfy canonical commutation relations: \([a(f), a^\dagger(g)] = (\tilde{f}, g)_W\) and \([a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)]\). Here and in what follows the scalar product \((f, g)_\mathcal{H}\) on a Hilbert space \(\mathcal{H}\) is linear in \( g \) and anti-linear in \( f \). We formally write as \(a^{\#r}(f) = \int a^{\#r}(k)f(k)dk\) for \(a^\dagger(F)\) with \(F = \bigoplus_{s=1}^{d-1} \delta_{sr} f\) and

\[
H_f = \sum_{r=1}^{d-1} \int \omega(k) a^{\dagger r}(k) a^r(k) dk.
\]

Let \(e^r(k) = (e_1^r(k), \ldots, e_d^r(k))\) be \(d\)-dimensional polarization vectors, i.e., \(e^r(k) \cdot e^s(k) = \delta_{rs}\) and \(k \cdot e^r(k) = 0\) for \(k \in \mathbb{R}^d \setminus \{0\}\) and \(r = 1, \ldots, d - 1\). For each \(x \in \mathbb{R}^d\) a quantized radiation field \(A(x) = (A_1(x), \ldots, A_d(x))\) is defined by

\[
A_\mu(x) = \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \int e_\mu^r(k) \left( \frac{\hat{\varphi}(k) e^{-ik \cdot x}}{\sqrt{\omega(k)}} a^{\dagger r}(k) + \frac{\hat{\varphi}(-k) e^{ik \cdot x}}{\sqrt{\omega(k)}} a^r(k) \right) dk. \tag{1.4}
\]

Here \(\hat{\varphi}\) is an ultraviolet cutoff function, for which we introduce assumptions below.

**Assumption 1.2** \(\hat{\varphi}/\sqrt{\omega}, \omega \sqrt{\omega} \hat{\varphi} \in L^2(\mathbb{R}^d)\) and \(\hat{\varphi}(k) = \overline{\hat{\varphi}(-k)}\).

Note that \(\sqrt{\omega} \hat{\varphi} \in L^2(\mathbb{R}^d)\) follows from Assumption 1.2. We fix \(\hat{\varphi}\) and \(\omega\) satisfying Assumptions 1.1 and 1.2 throughout this paper. Then \(\hat{\varphi}(k) = \overline{\hat{\varphi}(-k)}\) implies that \(A_\mu(x)\) is essentially self-adjoint for each \(x\). We denote the self-adjoint extension by the same symbol \(A_\mu(x)\). We identify \(\mathcal{H}\) with \(\int_{\mathbb{R}^d}^\oplus \mathcal{F} dx\), and under this identification we define the self-adjoint operator \(A_\mu\) in \(\mathcal{H}\) by

\[
A_\mu = \int_{\mathbb{R}^d}^\oplus A_\mu(x) dx.
\]

Set \(A = (A_1, \ldots, A_d)\). Let \(N = d\Gamma(\mathbb{1})\) be the number operator on \(\mathcal{F}\) and \(C^\infty(\mathbb{1} \otimes N) = \cap_{n=1}^{\infty} D(\mathbb{1} \otimes N^n)\). Let

\[
\sum_{\mu=1}^{d} (p_\mu \otimes \mathbb{1} - A_\mu)^2 = (p \otimes \mathbb{1} - A)^2. \tag{1.5}
\]

**Lemma 1.3** \(D(p^2 \otimes \mathbb{1}) \cap C^\infty(\mathbb{1} \otimes N) \cap D(\mathbb{1} \otimes H_f)\) is a core of \((p \otimes \mathbb{1} - A)^2\).

**Proof:** See Appendix 13. \(\blacksquare\)

The closure of \((p \otimes \mathbb{1} - A)^2[D(p^2 \otimes \mathbb{1}) \cap C^\infty(\mathbb{1} \otimes N) \cap D(\mathbb{1} \otimes H_f)]\) is denoted by \((p \otimes \mathbb{1} - A)^2\) for simplicity. Thus \(\sqrt{(p \otimes \mathbb{1} - A)^2 + M^2}\) is defined through the spectral measure of \((p \otimes \mathbb{1} - A)^2\). Set

\[
T_M = \sqrt{(p \otimes \mathbb{1} - A)^2 + M^2}. \tag{1.6}
\]
Proposition 1.4 [Hir13, Lemma 3.10, Theorem 4.5] Let $M > 0$. Then (1) and (2) follow.

(1) Let $V = 0$. Then $H$ is essentially self-adjoint on $\mathcal{D}$.

(2) Suppose that $V$ is relatively bounded (resp. form bounded) with respect to $\sqrt{p^2 + M^2}$ with a relative bound $a$. Then $V$ is also relatively bounded (resp. form bounded) with respect to $T_M + H$ with a relative bound smaller than $a$.

1.2 Potential classes and definition of SRPF Hamiltonian

We introduce two classes, $V_{\text{qf}}$ and $V_{\text{rel}}$, of potentials.

**Definition 1.5** ($V_{\text{qf}}$) $V = V_+ - V_- \in V_{\text{qf}}$ if and only if $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $V_-$ is relatively form bounded with respect to $\sqrt{p^2 + M^2}$ with a relative bound strictly smaller than one, i.e., $D((p^2 + M^2)^{1/4}) \subset D(V_{-1/2}^2)$ and there exist $0 \leq a < 1$ and $b \geq 0$ such that

$$\|V_{-1/2}f\| \leq a\|p^2 + M^2\|^{1/4}f\| + b\|f\|$$

for all $f \in D((p^2 + M^2)^{1/4})$.

($V_{\text{rel}}$) $V \in V_{\text{rel}}$ if and only if $V$ is relatively bounded with respect to $\sqrt{p^2 + M^2}$ with a relative bound strictly smaller than one, i.e., $D(\sqrt{p^2 + M^2}) \subset D(V)$ and there exist $0 \leq a < 1$ and $b \geq 0$ such that

$$\|Vf\| \leq a\|\sqrt{p^2 + M^2}f\| + b\|f\|$$

for all $f \in D(\sqrt{p^2 + M^2})$.

It can be shown that $V_{\text{rel}} \subset V_{\text{qf}}$. By Proposition 1.4 we can define the SRPF Hamiltonian as a self-adjoint operator through quadratic form sums. Let $V \in V_{\text{rel}} \cap V_{\text{conf}}$. We define the quadratic form by

$$q : (F, G) \mapsto (T_{M}^{1/2}F, T_{M}^{1/2}G) + (H_{t}^{1/2}F, H_{t}^{1/2}G) + (V_{+}^{1/2}F, V_{+}^{1/2}G) - (V_{-}^{1/2}F, V_{-}^{1/2}G)$$

with the form domain

$$Q(q) = D(T_{M}^{1/2}) \cap D(H_{t}^{1/2}) \cap D(V_{+}^{1/2}).$$

By Proposition 1.4, we note that $Q(q) = D(T_{M}^{1/2}) \cap D(H_{t}^{1/2}) \cap D(V_{+}^{1/2}) \cap D(V_{-}^{1/2})$. It can be checked that $Q(q)$ is densely defined semi-bounded closed form. Then there exists the unique self-adjoint operator $\hat{H}$ associated with the quadratic form $q$, i.e., $D([\hat{H}]^{1/2}) = Q(q)$ and $q(F, G) = \int_{\sigma(\hat{H})} \lambda d(E_{\lambda}F, G)$. Here $E_{\lambda}$ denotes the spectral measure associated with $H$. We write $\hat{H}$ as

$$H = T_{M} + V_{+} \otimes 1 - V_{-} \otimes 1 + 1 \otimes H_{t}.$$
Definition 1.6 Let $V \in V_{qf}$. Then the SRPF Hamiltonian is defined by (1.9).

We do not write tensor notation $\otimes$ for notational convenience in what follows. Thus $H$ can be simply written as $H = T_M + H_f + V_+ - V_-.$

1.3 Essential self-adjointness of $H$

Let

$$D = D(|p|) \cap D(V) \cap D(H_f).$$

When $V \in V_{rel}$, $D(V) \subset D(|p|) \cap D(H_f)$ and it follows that $D = D(|p|) \cap D(H_f)$. We introduce a subclass $V_{conf} \subset V_{qf}$, which include confining potentials.

Definition 1.7 $(V_{conf}) V = V_+ - V_- \in V_{conf}$ if and only if $V_- = 0$ and $V_+$ is twice differentiable, and

$$\partial_\mu V_+, \partial^2_\mu V_+ \in L^\infty(\mathbb{R}^d) \text{ for } \mu = 1, \ldots, d,$$

and $D(V) \subset D(|x|)$.

When $V \in V_{conf}$, $V \in L^2_{loc}(\mathbb{R}^d)$ and nonnegative. Then $p^2 + V$ is essentially self-adjoint on $C^\infty_c(\mathbb{R}^d)$ by Kato's inequality. It is established in [Hir13, Theorem 4.5] that $H$ with $M > 0$ is essentially self-adjoint on $D$ for $V \in V_{rel}$. We extend this to $V \in V_{rel} \cup V_{conf}$.

Proposition 1.8 Let $V \in V_{rel} \cup V_{conf}$ and $M > 0$. Then $H$ is essentially self-adjoint on $D$.

Proof: When $V \in V_{rel}$, the proposition follows from (2) of Proposition 1.4 and the Kato-Rellich theorem. The proof of the proposition for $V \in V_{conf}$ is a minor modification of [Hir13, Theorem 4.5]. Then we give it in Appendix C.

1.4 Literatures and organization

In [GS12, HH13, Hir13, KMS11, MS09] the spectral properties of the SRPF Hamiltonian are studied. In particular the case of $M = 0$ is studied in [HH13, Hir13], and the self-adjointness of the Hamiltonian is discussed in [MS09], where the SRPF Hamiltonian with spin $1/2$ is treated. We are not sure however how we can show the self-adjointness of the spinless SRPF Hamiltonian. Furthermore in [HH13] it is shown that the SRPF Hamiltonian with a confining potential has the unique ground state. Then the main purpose of this paper is to show the self-adjointness and the essential self-adjointness of $H$ not only for $V \in V_{rel}$ but also for $V \in V_{conf}$.

This paper is organized as follows.

In Section 2 we show that $H$ is self-adjoint on $D(|p|) \cap D(V) \cap D(H_f)$ and essentially self-adjoint on $\mathcal{H}_f$ which is defined in (2.2).

In Section 3 we discuss the translation invariant SRPF Hamiltonian which is defined by $H$ with $V = 0$. Then $H \cong \int_{\mathbb{R}^d} H(P) dP$ is obtained and $H(P)$ is called the SRPF Hamiltonian with total momentum $P \in \mathbb{R}^d$. The self-adjointness of $H(P)$ on $D(|P|) \cap D(H_f)$, and essential self-adjointness on $\mathcal{H}_f$ defined in (2.1).
2 Self-adjointness

We define the dense subset $\mathcal{H}_{\text{fin}}$. Let

$$\mathcal{F}_{\text{fin}} = L.H.\{\Omega, a^\dagger(h_1)\cdots a^\dagger(h_n)\Omega | h_j \in C_\infty^\infty(\mathbb{R}^d), j = 1, \cdots, n, n \geq 1\} \quad (2.1)$$

and

$$\mathcal{H}_{\text{fin}} = C_\infty^\infty(\mathbb{R}^d) \hat{\otimes} \mathcal{F}_{\text{fin}}, \quad (2.2)$$

where $\hat{\otimes}$ denotes the algebraic tensor product. The main theorem in this paper is to extend Proposition 1.8 as follows.

**Theorem 2.1** Let $V \in V_{\text{rel}} \cup V_{\text{conf}}$ and $M \geq 0$. Then $H$ is self-adjoint on $\mathcal{D}$, and essentially self-adjoint on $\mathcal{H}_{\text{fin}}$.

Note that Theorem 2.1 includes the case of $M = 0$. In order to prove Theorem 2.1 we need several lemmas.

**Lemma 2.2** Let $M \geq 0$. It follows that $D(|p|) \cap D(H^{1/2}_t) \subset D(T_M)$, and for all $\Psi \in D(|p|) \cap D(H^{1/2}_t)$,

$$\|T_M \Psi\| \leq C(\||p|\Psi\| + \|H^{1/2}_t \Psi\| + \|\Psi\|) \quad (2.3)$$

with some constant $C > 0$. In particular

$$\|H \Psi\| \leq C(\||p|\Psi\| + \|H_t \Psi\| + \|V \Psi\| + \|\Psi\|) \quad (2.4)$$

follows for $\Psi \in \mathcal{D}$ with some constant $C > 0$.

**Proof:** It follows that $\|T_M \Psi\|^2 = \sum_{\mu=1}^d \|(p_\mu - A_\mu)\Psi\|^2 + M^2 \|\Psi\|^2$ for $\Psi \in \mathcal{H}_{\text{fin}}$. Then (2.3) follows from the well-known bound

$$\|A_\mu \Psi\| \leq C\|(H_t + 1)\Psi\|^2$$

with some constant $C > 0$ for $\Psi \in \mathcal{H}_{\text{fin}}$. Furthermore since $|p| + H^{1/2}_t$ is essentially self-adjoint on $\mathcal{H}_{\text{fin}}$, the lemma follows from a limiting argument.

Let $\mathcal{H}_0 = \{\{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{H} | \Psi^{(n)} = 0 \text{ for all } n \geq n_0 \text{ with some } n_0 \geq 1\}$ and

$$\mathcal{D}_1 = \mathcal{D} \cap \mathcal{H}_0. \quad (2.5)$$

**Lemma 2.3** Let $V \in V_{\text{conf}}$ and $M > 0$. Then $\mathcal{D}_1$ is a core of $H$. 


Proof: Let \( P_n = 1_{[0,n]}(N) \) for \( n \in \mathbb{N} \). Take an arbitrary \( \Psi \in \mathcal{D} \). Hence \( P_n \Psi \in \mathcal{D}_1 \). We see that \( P_n \Psi \rightarrow \Psi \) as \( n \rightarrow \infty \). Since

\[
\|H(P_n - P_{n'})\Psi\| \leq C\|(P_n - P_{n'})||p|\Psi|| + \|(P_n - P_{n'})V\Psi\| + \|(P_n - P_{n'})H_1\Psi\|,
\]

we also see that \( \{HP_n\Psi\}_n \) is a Cauchy sequence in \( \mathcal{H} \). By the closedness of \( H \), \( \Psi \in D(H) \) and \( HP_n\Psi \rightarrow H\Psi \). Thus \( \mathcal{D}_1 \) is a core of \( H \). \[\square\]

Let

\[
\mathcal{D}_2 = \{\{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{D}_1 | \Psi^{(n)}(\cdot, k) \in C_c^\infty(\mathbb{R}^d) \text{ a.e. } k \in \mathbb{R}^d, n \geq 1\}. \tag{2.7}
\]

**Lemma 2.4** Let \( V \in \mathcal{V}_{\text{conf}} \) and \( M > 0 \). Then \( \mathcal{D}_2 \) is a core of \( H \).

Proof: Take an arbitrary \( \Phi \in \mathcal{D}_1 \). Let \( j \in C_c(\mathbb{R}^d) \) and \( g \in C_c(\mathbb{R}^d; [0, 1]) \) such that \( \int_{\mathbb{R}^d} j(x)dx = 1 \) and \( g(x) = 1 \) for \( |x| \leq 1 \). For each \( \epsilon > 0 \) we set \( j_\epsilon(x) = \epsilon^{-d}j(x/\epsilon) \),

\[
\Phi^{(n)}_{\epsilon,L}(x, k) = g(x/L) \int_{\mathbb{R}^d} j_\epsilon(x - y)\Phi^{(n)}(y, k)dy,
\]

and \( \Phi_{\epsilon,L} = \{\Phi_{\epsilon,L}^{(n)}\}_{n=0}^{\infty} \). We see that \( \Phi_{\epsilon,L} \rightarrow \Phi \), \( p_\mu\Phi_{\epsilon,L} \rightarrow p_\mu\Phi \), \( V\Phi_{\epsilon,L} \rightarrow V\Phi \) and \( H_1\Phi_{\epsilon,L} \rightarrow H_1\Phi \) strongly as \( \epsilon \downarrow 0 \) and \( L \rightarrow \infty \). Then by inequality (2.4) and the closedness of \( H \), we see that \( \Phi \in D(H) \) and \( \lim_{L \rightarrow \infty} \lim_{\epsilon \downarrow 0} H\Phi_{\epsilon,L} = H\Phi \) in \( \mathcal{H} \). Thus the lemma follows. \[\square\]

**Lemma 2.5** Let \( V \in \mathcal{V}_{\text{conf}} \) and \( M > 0 \). Let \( \Phi \in \mathcal{D}_2 \). Then it follows that

\[
\|p^2\Phi\| + \|V\Phi\| + \|H_1\Phi\| \leq C\|(p^2 + V + H_1 + 1)\Phi\|
\]

with some constant \( C > 0 \).

Proof: Note that \( \|(p^2 + V)\Phi\|^2 = \|p^2\Phi\|^2 + 2\text{Re}(p^2\Phi, V\Phi) + \|V\Phi\|^2 \). Let \( V_\mu = \partial_\mu V \).

Since

\[
2\text{Re}(p^2\Phi, V\Phi) \geq 2\sum_{\mu} \text{Re}(p_\mu\Phi, [p_\mu, V]\Phi) \geq -2\sum_{\mu} \|p_\mu\Phi\|\|V_\mu\|\|\Phi\|,
\]

for an arbitrary \( \epsilon > 0 \), we have \( \|(p^2 + V)\Phi\|^2 \geq (1 - \epsilon)\|p^2\Phi\|^2 + \|V\Phi\|^2 - \epsilon\|\Phi\|^2 \) and

\[
\|(p^2 + V + H_1)\Phi\|^2 \geq \|(p^2 + V)\Phi\|^2 + \|H_1\Phi\|^2 \geq (1 - \epsilon)\|p^2\Phi\|^2 + \|V\Phi\|^2 - \epsilon\|\Phi\|^2 + \|H_1\Phi\|^2.
\]

Then (2.9) follows. \[\square\]
Lemma 2.6 Let $V \in \mathcal{V}_{\text{rel}} \cup \mathcal{V}_{\text{conf}}$ and $M > 0$. Then $\mathcal{H}_{\text{fin}}$ is a core of $H$.

Proof: Let $\Phi \in \mathcal{D}_2$. Let $V \in \mathcal{V}_{\text{conf}}$. Note that $p^2 + V + H_f$ is essentially self-adjoint on $\mathcal{H}_{\text{fin}}$. We see that there exists a sequence $\{\Phi_n\}$, $\Phi_n \in \mathcal{H}_{\text{fin}}$, such that $\Phi_n \to \Phi$, and $(p^2 + V + H_f)\Phi_n \to (p^2 + V + H_f)\Phi$ as $n \to \infty$. From (2.4) it follows that $p^2\Phi_n \to p^2\Phi$, $V\Phi_n \to V\Phi$ and $H_f\Phi_n \to H_f\Phi$ as $n \to \infty$. Then we can also see that $\{H\Phi_n\}$ is a Cauchy sequence by (2.4), and $\lim_n H\Phi_n = H\Phi$ follows. Thus $\mathcal{H}_{\text{fin}}$ is a core of $H$.

Next we suppose that $V \in \mathcal{V}_{\text{rel}}$. By the argument above it is seen that operator $T_M + H_f$ is essentially self-adjoint on $\mathcal{H}_{\text{fin}}$. By Proposition 1.4 we also see that $\|V\Phi\| \leq a\|(T_M + H_f)\Phi\| + b\|\Phi\|$ with some constant $0 \leq a < 1$ and $b \geq 0$. The Kato-Rellich theorem yields that $H$ is essentially self-adjoint on $\mathcal{H}_{\text{fin}}$.

The key inequality to show the self-adjointness of $H$ on $\mathcal{D}$ is the following inequality.

Lemma 2.7 Let $V \in \mathcal{V}_{\text{conf}}$. Let $M_0 > 0$ be fixed and $0 \leq M \leq M_0$. Then for all $\Psi \in D(H)$,

\[
\|\|p|\Psi\|^2 + \|V\Psi\|^2 + \|H_f\Psi\|^2 \leq C\|(H + 1)\Psi\|^2 \tag{2.10}
\]

with some constant $C$ independent of $M$.

Proof: Suppose that $M = 0$. In the case of $M > 0$, the proof is parallel with that of $M = 0$, but rather easier.

(Step 0) Let $\Psi \in \mathcal{H}_{\text{fin}}$. Let $H_0 = |p - A| + H_f$. We have

\[
\|H\Psi\|^2 = \|H_0\Psi\|^2 + \|V\Psi\|^2 + 2\text{Re}(H_0\Psi, V\Psi), \\
\|H_0\Psi\|^2 = \|p - A|\Psi\|^2 + \|H_f\Psi\|^2 + 2\text{Re}(p - A|\Psi, H_f\Psi).
\]

Then

\[
\|H\Psi\|^2 = \|\|p - A|\Psi\|^2 + \|H_f\Psi\|^2 + 2\text{Re}(p - A|\Psi, H_f\Psi) + \|V\Psi\|^2 + 2\text{Re}(H_0\Psi, V\Psi) \tag{2.11}
\]

We estimate three terms $\|\|p - A|\Psi\|^2$, $\text{Re}(p - A|\Psi, H_f\Psi)$ and $\text{Re}(H_0\Psi, V\Psi)$ on the right-hand side of (2.11) from below.

(Step 1) We estimate $\text{Re}(p - A|\Psi, H_f\Psi)$. Since the operator $|p - A|$ is singular, we introduce an artificial positive mass $m > 0$ and

\[
T_m = \sqrt{(p - A)^2 + m^2}. \tag{2.12}
\]

We fix $m$ throughout. Note that $|p - A| - T_m$ is bounded. Thus

\[
|p - A| = T_m + (|p - A| - T_m) \tag{2.13}
\]

can be regarded as a perturbation of $T_m$, and the perturbation $|p - A| - T_m$ is bounded. We have $\text{Re}(p - A|\Psi, H_f\Psi) = (T_m\Psi, H_f\Psi) + ((|p - A| - T_m)\Psi, H_f\Psi)$. Since $\Psi \in \mathcal{H}_{\text{fin}}$
\[ \mathcal{H}_m, H_t \Psi \in D(p^2) \cap D(H_t). \] In particular \( H_t \Psi \in D(T_m) \) and then \( H_t \Psi \in D(T_m^{1/2}) \).

Furthermore we show that
\[ T_m^{1/2} \Psi \in D(H_t) \] (2.14)
in Appendix D. So we can see that

\[ \text{Re}(|p - A| \Psi, H_t \Psi) = (T_m^{1/2} \Psi, H_t T_m^{1/2} \Psi) + \text{Re}(T_m^{1/2} \Psi, [T_m^{1/2}, H_t] \Psi) + ((|p - A| - T_m) \Psi, H_t \Psi) \]
\[ \geq \text{Re}(T_m^{1/2} \Psi, [T_m^{1/2}, H_t] \Psi) + ((|p - A| - T_m) \Psi, H_t \Psi). \]

We estimate \((|p - A| - T_m) \Psi, H_t \Psi \). Since \( \|(p - A| - T_m) \Psi \| \leq m \| \Psi \| \), we see that for each \( \epsilon > 0 \) there exists \( C_1 > 0 \) such that
\[ \|(p - A| - T_m) \Psi, H_t \Psi \| \geq -\epsilon \| H_t \Psi \|^2 - C_1 \| \Psi \|^2. \] (2.15)

On the other hand we estimate \( \text{Re}(T_m^{1/2} \Psi, [T_m^{1/2}, H_t] \Psi) \). Let \( \epsilon > 0 \) be given. Then there exists \( C_2 > 0 \) such that
\[ \text{Re}(T_m^{1/2} \Psi, [T_m^{1/2}, H_t] \Psi) \geq -c \| T_m^{1/2} \Psi \| \| (H_t + 1) \Psi \| \]
\[ \geq -\epsilon \| p - A \| \Psi \|^2 - \epsilon \| H_t \Psi \|^2 - C_2 \| \Psi \|^2. \] (2.16)

The first inequality of (2.16) is derived from
\[ \| [T_m^{1/2}, H_t] \Psi \| \leq c \| (H_t + 1)^{1/2} \Psi \| \]
with some constant \( c > 0 \). This is shown in Appendix E. Hence we have
\[ \text{Re}(|p - A| \Psi, H_t \Psi) \geq -\epsilon \| p - A \| \Psi \|^2 - 2\epsilon \| H_t \Psi \|^2 - (C_1 + C_2) \| \Psi \|. \] (2.18)

(Step 2) We estimate \( \text{Re}(H_0 \Psi, V \Psi) \). For each \( \epsilon > 0 \) there exists \( C_3 > 0 \) such that
\[ \text{Re}(H_0 \Psi, V \Psi) = \text{Re}((H_0 - T_m - H_t) \Psi, V \Psi) + \text{Re}(T_m \Psi, V \Psi) + (H_t \Psi, V \Psi) \]
\[ \geq -\epsilon \| V \Psi \|^2 - C_3 \| \Psi \|^2 + \text{Re}(T_m \Psi, V \Psi). \]

We also see that
\[ \text{Re}(T_m \Psi, V \Psi) = (T_m^{1/2} \Psi, VT_m^{1/2} \Psi) + \text{Re}(T_m^{1/2} \Psi, [T_m^{1/2}, V] \Psi) \geq \text{Re}([T_m^{1/2}, V] \Psi, V \Psi). \]

Note that integral representation \( T_m^{1/2} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{w^{3/4}} (T_m^2 + w)^{-1} T_m^2 dw \), commutation relations
\[ [(T_m^2 + w)^{-1} T_m^2, V] = (T_m^2 + w)^{-1} [T_m^2, V] + (T_m^2 + w)^{-1} [T_m^2, V] (T_m^2 + w)^{-1} T_m^2, \]
and facts \([T^2_m, V] = 2 \sum_{\mu=1}^d (p_\mu - A_\mu) V_\mu + \sum_{\mu=1}^d V_{\mu\mu}\), where \(V_\mu = \partial_\mu V\) and \(V_{\mu\mu} = \partial^2_\mu V\). Then we have

\[
|([T^2_m, V] \Psi, \Phi)| = \left| \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{w^{3/4}} \left| (T^2_m + w)^{-1} T^2_m, V \right| \Psi, \Phi \right| \, dw \leq \frac{\sqrt{2}}{\pi} \|\Psi\| \|\Phi\| \int_0^\infty \frac{dw}{w^{3/4}} \sum_{\mu=1}^d \left( \frac{2\|V_\mu\|_\infty}{\sqrt{w+m^2}} + \frac{\|V_{\mu\mu}\|_\infty}{w+m^2} \right).
\]

(2.19)

Thus for each \(\epsilon > 0\) there exists \(C_4 > 0\) such that

\[
\text{Re}(H_0 \Psi, V \Psi) \geq -\epsilon \|V \Psi\|^2 - C_4 \|\Psi\|^2.
\]

(2.20)

(Step 3) We estimate \(\|p - A|\Psi\|\). Note that

\[
\|p_\mu \Psi\|^2 = \|(p_\mu - A_\mu) \Psi\|^2 + 2\text{Re}(A_\mu \Psi, (p_\mu - A_\mu) \Psi) - \|A_\mu \Psi\|^2.
\]

For each \(\epsilon > 0\), there exist \(C_5 > 0\) and \(C_6 > 0\) such that

\[
|\text{Re}(A \Psi, (p - A) \Psi)| \leq \epsilon(\|p - A|\Psi\|^2 + \|H_f \Psi\|^2) + C_5 \|\Psi\|^2
\]

\[
\|p|\Psi\|^2 \leq (1 + \epsilon)\|p - A|\Psi\|^2 + \epsilon\|H_f \Psi\|^2 + C_6 \|\Psi\|^2.
\]

Hence we have

\[
\|p - A|\Psi\|^2 \geq \frac{1}{1+\epsilon} \|p|\Psi\|^2 - \frac{\epsilon}{1+\epsilon} \|H_f \Psi\|^2 - \frac{C_6}{1+\epsilon} \|\Psi\|^2.
\]

(2.21)

(Step 4) By (2.18), (2.20), (2.21) and (2.11), we can see (2.10) for \(\Psi \in \mathcal{H}_{\text{fin}}\). Let \(\Psi \in D(H)\). Since \(H\) is essentially self-adjoint on \(\mathcal{H}_{\text{fin}}\), by a limiting argument we can see (2.10) for \(\Psi \in D(H)\).

Proof of Theorem 2.1.

We emphasis the dependence on \(M\) by writing \(H_M\) for \(H\). Let \(M > 0\). Suppose that \(V \in V_{\text{conf}}\). By Lemma 2.7, \(H_M\) is closed on \(\mathcal{D}\). Then it implies that \(H_M\) is self-adjoint on \(\mathcal{D}\) since it is essentially self-adjoint on \(\mathcal{D}\). Next suppose that \(V \in V_{\text{rel}}\). Then \(T_M + H_f\) is self-adjoint on \(\mathcal{D}\). Since \(V\) is also relatively bounded with respect to \(T_M + H_f\) with a relative bound strictly smaller than one. Thus \(H\) is self-adjoint on \(\mathcal{D}\).

Let \(M = 0\). Then \(H_0 = H_M + (H_0 - H_M)\) and \(\|(H_0 - H_M) \Psi\| \leq M \|\Psi\|\). Thus \(H_0\) is self-adjoint on \(\mathcal{D}\) and essentially self-adjoint on \(\mathcal{H}_{\text{fin}}\) by the Kato-Rellich theorem.
3 Translation invariant case

The momentum operator in $\mathcal{F}$ is defined by the second quantization of the multiplication by $k_\mu$. I.e., $P f_\mu = \sum_{r=1}^{d-1} k_\mu a^{\dagger}(k)a(r(k)dk, \mu = 1, \ldots, d$. Let $P_{\text{tot}} = p_\mu \otimes 1 + 1 \otimes P_{f_\mu}$, $\mu = 1, \ldots, d$, be the total momentum operator, and we set $P_{\text{tot}} = (P_{\text{tot}1}, \ldots, P_{\text{tot}d}$). Let $V = 0$. Then we can see that $[H, P_{\text{tot}1}] = 0$ and hence $H$ can be decomposed with respect to the spectrum of $P_{\text{tot}1}$. Thus $H \cong \int_{R^d} H_{pd}dP$, where $H_{pd}$ is called the fiber Hamiltonian with the total momentum $P \in R^d$.

We can see the explicit form of the fiber Hamiltonian. Let $L(P) = (P - P_f - A(0))^2 + M^2$. (3.1)

Proposition 3.1 [Hir07, Theorem 2.3 (2), Lemma 3.11] Let $P \in R^d$. Then $L(P)$ is essentially self-adjoint on $C_0 = D(P^2_f) \cap D(H_f)$.

Set

$\bar{L}(P) = \overline{L(P)}_{C_0}$. (3.2)

Definition 3.2 Let $P \in R^d$. We define $H(P)$ by

$H(P) = \sqrt{\bar{L}(P)} + H_f$. (3.3)

Lemma 3.3 It follows that

$T_M + H_f \cong \int_{R^d} H(P)dP$. (3.4)

Proof: We define the unitary operator $U$ on $H$ by $(UF)(\cdot) \in H$ for $F(\cdot) \in H$ by

$(UF)(P) = (2\pi)^{-d/2} \int_{R^d} e^{ixP} e^{-iP_f^x} F(x)dx$. (3.5)

It is shown that

$U^{-1} \left( \int_{R^d} \bar{L}(P)dP \right) U = (p - A)^2$ (3.6)

in [Hir07 Theorem 2.3]. Actually it is shown that

$(F, T_M^2 G) = \int_{R^d} dP \langle (UF)(P), \bar{L}(P)(UG)(P) \rangle_\mathcal{F}$ (3.7)
for $F, G \in \mathcal{H}_{\text{fin}}$. From (3.6) we see that $U^{-1} \left( \int_{\mathbb{R}^d} e^{-tL(P)} dP \right) U = e^{-tT^2}$ for all $t \geq 0$ by [RS78] Theorem XIII 85 (c)]. Let $F \in \mathcal{H}_{\text{fin}}$. By the formula

$$T^\alpha_M = C_\alpha \int_0^\infty (1 - e^{-\lambda T^2_M}) \frac{d\lambda}{\lambda^{1+\alpha/2}},$$

we can see that

$$(F, T_M F) = C_1 \int_0^\infty \int_{\mathbb{R}^d} \frac{d\lambda}{\lambda^{3/2}} dP \left( (UF)(P), (1 - e^{-\lambda L(P)})(UF)(P) \right).$$

By Fubini’s theorem we have

$$(F, T_M F) = C_1 \int_{\mathbb{R}^d} dP \int \frac{d\lambda}{\lambda^{3/2}} ((UF)(P), (1 - e^{-\lambda L(P)})(UF)(P)).$$

Note that $(UF)(P) \in \mathcal{F}_{\text{fin}}$ for each $P \in \mathbb{R}^d$. Hence $(UF)(P) \in D(\tilde{L}(P)) \subset D(\sqrt{L}(P))$, which implies that

$$(F, T_M F) = \int_{\mathbb{R}^d} dP \left( (UF)(P), \sqrt{\tilde{L}(P)}(UF)(P) \right).$$

By the polarization identity and (3.10) we have

$$(F, T_M G) = \int_{\mathbb{R}^d} dP \left( (UF)(P), \sqrt{\tilde{L}(P)}(UG)(P) \right).$$

Furthermore we see that

$$(F, (T_M + H_f) G) = \int_{\mathbb{R}^d} dP \left( (UF)(P), H(P)(UG)(P) \right),$$

which implies that

$$T_M + H_f = U^{-1} \left( \int_{\mathbb{R}^d} H(P) dP \right) U$$

on $\mathcal{H}_{\text{fin}}$. Since $\mathcal{H}_{\text{fin}}$ is a core of the left hand side of (3.11),

$$T_M + H_f \cong \int_{\mathbb{R}^d} H(P) dP$$

holds true as self-adjoint operators. Note that $T_M + H_f = T_M + H_f$ on $D(|\rho|) \cap D(H_{\text{fin}})$ and $T_M + H_f$ is self-adjoint on $D(|\rho|) \cap D(H_{\text{fin}})$. Then the lemma follows.

Let $\mathcal{C} = D(|P_{\text{fin}}|) \cap D(H_{\text{fin}})$. Note that $D(|P - P_{\text{fin}}|) = D(|P_{\text{fin}}|)$ for all $P \in \mathbb{R}^d$. The essential self-adjointness of $H(P)$ is established in [Hir13].

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Proposition 3.4  \cite{Hir13, Corollary 7.2} Let $M > 0$. Then $H(P)$ is essentially self-adjoint on $\mathcal{C}$.

The main result in this section is as follows.

Theorem 3.5 Let $M \geq 0$. Then $H(P)$ is self-adjoint on $\mathcal{C}$ and essentially self-adjoint on $\mathcal{F}_{\text{fin}}$.

Proof: The proof is parallel with that of $H$. We show the outline of the proof. It can be seen that there exists a constant $C > 0$ such that for arbitrary $\Psi \in \mathcal{F}_{\text{fin}}$,

$$
\| \sqrt{(P - P_f - A(0))^2 + M^2}\Psi \| \leq C (\| P - P_f \| \Psi \| + \| H_1^{1/2}\Psi \| + \| \Psi \|).
$$

Then we can derive that

$$
\| H(P)\Psi \| \leq C (\| P - P_f \| \Psi \| + \| H_f \Psi \| + \| \Psi \|)
$$

for $\Psi \in \mathcal{F}_{\text{fin}}$. In a similar manner to Lemma 2.3 from (3.13) we can see that $\mathcal{C}_1 = \mathcal{C} \cap \mathcal{F}_{\text{fin}}$ is a core of $H(P)$ for $M > 0$. Since $|P - P_f|^2$ and $H_f$ are strongly commutative and positive, it is trivial to see that

$$
\| (|P - P_f|^2 + H_f)\Psi \| \geq \| P - P_f \| \Psi \| + \| H_f \Psi \|^2.
$$

Since $\mathcal{F}_{\text{fin}}$ is a core of $|P - P_f|^2 + H_f$, in a similar manner to Lemma 2.6 we can see that $\mathcal{F}_{\text{fin}}$ is also a core of $H(P)$ by (3.14). The key inequality to show the self-adjointness of $H(P)$ is

$$
\| P - P_f \| \Psi \|^2 + \| H_f \Psi \|^2 \leq C \| (H(P) + \mathbb{1}) \Psi \|^2
$$

with some $C > 0$ for $\Psi \in \mathcal{F}_{\text{fin}}$. This is shown by using the inequality

$$
\| [T_m(P)^{1/2}, H_f] \Psi \| \leq c \| (H_f + \mathbb{1})^{1/2} \Psi \|,
$$

where $T_m(P) = \sqrt{(P - P_f - A(0))^2 + m^2}$. (3.16) is proven in Appendix E. Thus by (3.15) in a similar manner to the proof of Theorem 2.1 we can see that $H(P)|_{\mathcal{C}}$ is closed. Then $H(P)$ is self-adjoint on $\mathcal{C}$ for $M \geq 0$.

A  Stochastic preliminary

In this appendix we review functional integral representations of the semigroup generated by semi-relativistic Pauli-Fierz model. This is established in \cite[Theorem 3.13]{Hir13}. These representations play an important roles to estimate some commutation relations in this paper.
A.1 Semi-relativistic Pauli-Fierz model

Let $(B_t)_{t \geq 0}$ be the $d$-dimensional Brownian motion defined on a Wiener space with Wiener measure $P^x$ starting from $x$. Let $(T_t)$ be the subordinator on a probability space with a probability measure $\nu$ such that $\mathbb{E}_\nu[e^{-t^2}] = e^{-t^2}$. We denote the expectation with respect to the measure $P^x \otimes \nu$ by $\mathbb{E}_{P^x \otimes \nu}[\cdots]$. Let $a = (a_1(x), \cdots, a_d(x))$ be electromagnetic fields. Then the semi-relativistic Schrödinger operator is defined by

$$H = T_M + V + H_f.$$ 

We can give the functional integral representation of $e^{-tH}$ in [Hir13, Theorem 3.13]. Let

$$q(F, G) = \frac{1}{2} \sum_{\mu, \nu=1}^d (\hat{F}_\mu, \delta^\perp_{\mu \nu} \hat{G}_\nu)$$

be the quadratic form on $\oplus^d L^2(\mathbb{R}^d)$, where $\delta^\perp_{\mu \nu}(k) = \delta_{\mu \nu} - k_\mu k_\nu/|k|^2$ denotes the transversal delta function. Let $\mathcal{A}(F)$ be a Gaussian random variables on a probability space $(Q, \Sigma, \mu)$, which is indexed by $F = (F_1, \cdots, F_d) \in \oplus^d L^2(\mathbb{R}^d)$. The mean of $\mathcal{A}(F)$ is zero and the covariance is given by $\mathbb{E}[\mathcal{A}(F)\mathcal{A}(G)] = q(F, G)$. Furthermore we introduce the Euclidean version of $\mathcal{A}$. Let

$$q_E(F, G) = \frac{1}{2} \sum_{\mu, \nu=1}^d (\hat{F}_\mu, \delta_{\mu \nu} \hat{G}_\nu)$$

be the quadratic form on $\oplus^d L^2(\mathbb{R}^{d+1})$. On the right-hand side of (A.2), we note that $(\hat{F}_\mu, \delta_{\mu \nu} \hat{G}_\nu) = \int_{\mathbb{R} \times \mathbb{R}^d} F_\mu(k_0, k) \delta_{\mu \nu} \hat{G}_\nu(k_0, k) dk_0 dk$ and $\delta_{\mu \nu} \hat{G}_\nu$ is independent of $k_0$. Let $\mathcal{A}_E(F)$ be a Gaussian random variables on a probability space $(Q_E, \Sigma_E, \mu_E)$, which is indexed by $F \in \oplus^d L^2(\mathbb{R}^{d+1})$. The mean of $\mathcal{A}_E(F)$ is zero and the covariance is given by $\mathbb{E}[\mathcal{A}_E(F)\mathcal{A}_E(G)] = q_E(F, G)$. Let us identify $H$ with $L^2(\mathbb{R}^d; \mathcal{F})$. Thus $\Phi \in \mathcal{H}$ can be an $\mathcal{F}$-valued $L^2$-function on $\mathbb{R}^d$, $\mathbb{R}^d \ni x \mapsto \Phi(x) \in \mathcal{F}$. It is well known that there exists the family of isometries $J_t : L^2(Q) \to L^2(Q_E)$ ($t \in \mathbb{R}$) and $j_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{d+1})$ ($t \in \mathbb{R}$) such that $J_t^* J_s = e^{-|t-s|^2|H|}$ and $j_t^* j_s = e^{-|t-s|^2|\mathcal{H}E|}$. 

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Proposition A.1 Let $F, G \in \mathcal{H}$. Then
\[
(F, e^{-tH}G) = e^{-tM} \int_{\mathbb{R}^d} d^2x \mathbb{E}_P^x \left[ e^{-\int_0^t V(B_t)dt} (J_0 F(B_{T_0}), e^{-i\mathcal{D}_E(I[0,t])} J_t G(B_{T_t}))_{L^2(Q_E)} \right]. 
\] (A.3)

Here $I[0,t] = \oplus_{i=1}^d \int_0^{T_t} \partial_s \tilde{\varphi}(-B_s) dB^i_s$ is defined by the limit of $\oplus^d L^2(\mathbb{R}^{d+1})$-valued stochastic integrals of $\tilde{\varphi} = (\tilde{\varphi}/\sqrt{2})$, and $T^*_s = \inf\{t; T_t = s\}$.

Proof: See [Hir13] Theorem 3.13 and Remark 3.8.

Furthermore let
\[
K = \frac{1}{2} (p - A)^2 
\] (A.4)
be the kinetic term of the Pauli-Fierz model $K + V + H_t$. The Feynman-Kac formula of $e^{-tK}$ is also established as follows.

Proposition A.2 Let $F, G \in \mathcal{H}$. Then it follows that
\[
(F, e^{-tK}G) = \int_{\mathbb{R}^d} d^2x \mathbb{E}_P^x \left[ e^{-\int_0^t V(B_t)dt} (F(B_0), e^{-i\mathcal{D}_E(K[0,t])} G(B_t))_{L^2(Q)} \right], 
\] (A.5)
where $K[0,t] = \oplus_{i=1}^d \int_0^t \partial_s \tilde{\varphi}(-B_s) dB^i_s$ is a $\oplus^d L^2(\mathbb{R}^d)$-valued stochastic integral.

Proof: See [Hir00] (4.20), Theorem 4.8 and [LHB11] (7.3.18).

A.2 Semi-relativistic Pauli-Fierz model with a fixed total momentum

Let $H(P) = \sqrt{(P - P_t - A(0))^2 + M^2} + H_t$ be the semi-relativistic Pauli-Fierz model with total momentum $P \in \mathbb{R}^d$. The rigorous definition of $H(P)$ is given by (3.3). The Feynman-Kac formula of $e^{-tH(P)}$ is also established.

Proposition A.3 Let $F, G \in L^2(Q)$. Then
\[
(F, e^{-tH(P)}G) = e^{-tM} \mathbb{E}_P^0 \left[ (J_0 F(B_{T_0}), e^{-i\mathcal{D}_E(I[0,t])} e^{i(P-P_t)\cdot B_{T_t}} J_t G(B_{T_t}))_{L^2(Q_E)} \right]. 
\] (A.6)

Proof: This is proven by a minor modification of [Hir07] Theorem 3.3.

Furthermore the kinetic term of the Pauli-Fierz model with total momentum $P \in \mathbb{R}^d$ is given by
\[
K(P) = \frac{1}{2} (P - P_t - A(0))^2, \quad P \in \mathbb{R}^d.
\]

The Feynman-Kac formula of $e^{-tK(P)}$ is also established as follows.

Proposition A.4 Let $F, G \in L^2(Q)$. Then
\[
(F, e^{-tK(P)}G) = \mathbb{E}_P^0 \left[ (F(B_0), e^{-i\mathcal{D}_E(K[0,t])} e^{i(P-P_t)\cdot B_t} G(B_t))_{L^2(Q)} \right]. 
\] (A.7)

Proof: This is also proven by a minor modification of [Hir07] Theorem 3.3.
B Proof of Lemma 1.3

Proof of Lemma 1.3
It is shown that $e^{-tK}$ leaves $D(p^2) \cap C^\infty(N)$ invariant in [LHB11, Lemma 7.53]. See also [Hir00, Theorem 2.6]. It is enough to show that $e^{-tK}D(H_f) \subset D(H_f)$. By the Feynman-Kac formula we have

\[
(H_f F, e^{-tK}G) = \int_{\mathbb{R}^d} dx \mathbb{E}_P^x[(H_f F(B_0), e^{-i\phi(K[0,t])}G(B_t))]
\]

\[
= (F, e^{-tK}H_f G) + \int_{\mathbb{R}^d} dx \mathbb{E}_P^x[(F(B_0), [H_f, e^{-i\phi(K[0,t])}]G(B_t))].
\]

We can estimate as $[H_f, e^{-i\phi(K[0,t])}] = e^{-i\phi(K[0,t])}(\Pi(K[0,t]) + \xi)$, where $\Pi(K[0,t]) = [H_f, \mathcal{A}(K[0,t])]$ and $\xi = q(K[0,t], K[0,t])$. Thus we see that

\[
\left| \int_{\mathbb{R}^d} dx \mathbb{E}_P^x[(F(B_0), [H_f, e^{-i\phi(K[0,t])}]G(B_t))] \right| \leq C(t + \sqrt{t})\|F\|\|H_f + \mathbb{1}\|^{1/2}G\|. \quad (B.1)
\]

Here we used that $\|\Pi(K[0,t])\Psi \| \leq C(\|K[0,t]\| + \|K[0,t]/\sqrt{\omega}\|)\|H_f + \mathbb{1}\|^{1/2}\Psi\|$ and BDG-type inequality ([Hir00, Theorem 4.6] and [LHB11, Lemma 7.21]):

\[
\mathbb{E}_P^0[\xi^2] \leq t^2 C, \quad (B.2)
\]

\[
\mathbb{E}_P^0[(\|K[0,t]\| + \|K[0,t]/\sqrt{\omega}\|)^2] \leq Ct. \quad (B.3)
\]

Then we have

\[
\|(H_f F, e^{-tK}G)\| \leq C(t + \sqrt{t})\|F\|\|(H_f + \mathbb{1})^{1/2}G\| + \|F\|\|H_f G\|,
\]

and the desired results follow.

C Proof of Proposition 1.8

Lemma C.1 Let $V \in V_{\text{conf}}$. Then $e^{-tH}$ leaves $D(V)$ invariant, i.e., $e^{-tH}D(V) \subset D(V)$.

Proof: Let $F, G \in D(V)$. We define $Q_{[0,t]}$ by $Q_{[0,t]} = e^{-tM}e^{-\int_0^t V(B_s)ds}J_t^*e^{-i\phi_E(I[0,t])}J_t : \mathcal{H} \rightarrow \mathcal{H}$. Then we have

\[
(VF, e^{-tH}G) = \int_{\mathbb{R}^d} dx \mathbb{E}_{P \times \nu}^x[(V(B_{T_0})F(B_{T_0}), Q_{[0,t]}G(B_{T_1}))].
\]

[16]
Hence we see that
\[
(VF, e^{-tH}G) = (F, e^{-tH}VG) + \int_{\mathbb{R}^d} dx E_{\mu}(F(B_{T_0}), Q[0,t])
\]
and, by the Taylor expansion \(V(x) - V(B_{T_t} + x) = \sum \mu(\partial_\mu V(\xi))B_{T_t}^\mu\) with some \(\xi \in \mathbb{R}^d\), we can estimate as
\[
\left| \int_{\mathbb{R}^d} dx E_{\mu}(F(B_{T_0}), Q[0,t]) G(B_{T_t}) \right| \leq \|F\| \|x\| \sup_x \sqrt{\sum \mu |\partial_\mu V(x)|^2}.
\]
Here we used the fact that \(G \in D(|x|)\). Then we have
\[
|(VF, e^{-tH}G)| \leq C \|F\| (\|x\| G + \|VG\|)
\]
with some constant \(C > 0\). Then \(e^{-tH}G \in D(V)\) follows.

Proof of Proposition 1.8
Suppose that \(V\) satisfies (2) of Assumption 1.7. It is shown in [Hir13, Lemmas 4.3 and 4.4] that \(D(H) \subset \bigcap \mu D(p_\mu) \bigcap\bigcap D(H_f)\) and \(e^{-tH}\) leaves \(\bigcap \mu D(p_\mu) \bigcap D(H_f)\) invariant, which implies that \(e^{-tH}\) leaves \(\bigcap p \bigcap D(H_f)\) invariant. Combining this with Lemma C.1 we see that \(D(H) \subset \mathcal{D}\) and \(e^{-tH}\) leaves \(\mathcal{D}\) invariant. Then \(\mathcal{D}\) is a core of \(H\) by [RS75, Theorem X.49].

D Proof of (2.14)
Note that \(T_m^{1/2} = (2K + m^2)^{1/4}\), where \(K\) is given by (A.4). We have
\[
(2K + m^2)^{\alpha/2} = C_\alpha \int_0^\infty (1 - e^{-\lambda(2K + m^2)}) \frac{d\lambda}{\lambda^{1+\alpha/2}}
\]
for \(0 \leq \alpha < 2\) with some constant \(C_\alpha\). From this formula we have the lemma below:

Lemma D.1 There exists \(C > 0\) such that
\[
(F, T_m^{1/2}G) = C \int_0^\infty \left\{ (F, G) - e^{-\lambda m^2/2} \int_{\mathbb{R}^d} dx E_{\mu}(F(B_0), e^{-iA(0,\lambda)B_{T\lambda})} G(B_{T\lambda})) \right\} \frac{d\lambda}{\lambda^{5/4}}.
\]
Proof: This can be derived from Proposition A.2 (D.1) and changing the variable. ■

Proof of (2.14):
Let \(F \in D(H_f)\) and \(G \in \mathcal{H}_{\text{fin}}\). Thus \(H_f G \in \mathcal{H}_{\text{fin}}\). By (D.1) we have
\[
(H_f, T_m^{1/2}G) = C \int_0^\infty \left\{ (H_f F, G) - e^{-\lambda m^2/2} \int_{\mathbb{R}^d} dx E_{\mu}(H_f F(B_0), e^{-iA(0,\lambda)B_{T\lambda})} G(B_{T\lambda})) \right\} \frac{d\lambda}{\lambda^{5/4}}.
\]
Then we have
\[
(H_f F, T_{m}^{1/2}G) - (F, T_{m}^{1/2}H_f G) = -C \int_{0}^{\infty} \frac{e^{-\lambda m^2/2}}{\lambda^{5/4}} \, d\lambda \int_{\mathbb{R}^d} dx E_P^x \left[ (F(B_0), [H_f, e^{-i\mathcal{A}(K[0,\lambda])}] G(B_\lambda)) \right]. \tag{D.2}
\]

We have
\[
[H_f, e^{-i\mathcal{A}(K[0,\lambda])}] = e^{-i\mathcal{A}(K[0,\lambda])} (\Pi(K[0,\lambda]) + \xi),
\]
where \(\Pi(K[0,\lambda]) = [H_f, A(K[0,\lambda])]\) and \(\xi = q(K[0,\lambda], K[0,\lambda])\). Thus we see that in a similar manner to (B.1), (B.2) and (B.3),
\[
| \int_{0}^{\infty} \left\{ \int_{\mathbb{R}^d} dx E_P^x \left[ (F(B_0), [H_f, e^{-i\mathcal{A}(K[0,\lambda])}] G(B_\lambda)) \right] \right\} e^{-\lambda m^2/2} d\lambda \alpha^{5/4} |
\]
\[
\leq C \int_{0}^{\infty} \frac{\sqrt{\lambda + \lambda}}{\lambda^{5/4}} e^{-\lambda m^2/2} d\lambda \parallel F \parallel \| (H_f + 1) G \|. \tag{D.3}
\]

Then we see that \(|(H_f H_f, T_{m}^{1/2}G)| \leq C \| F \| \| (H_f + 1) G \|\) with some constant \(C > 0\). Hence \(T_{m}^{1/2} G \in D(H_f)\) follows.

\section*{E Proof of (2.17)}

**Proof of (2.17):**
The proof of (2.17) is similar to that of (2.14). Let \(G \in \mathcal{H}_{\text{fin}}\). By (D.2) and (D.3), it follows that \(|(F, H_f, T_{m}^{1/2}G)| \leq C \| F \| \| (H_f + 1) G \|\). This implies (2.17).

\section*{F Proof of (3.16)}

**Proof of (3.16):**
The idea of the proof of (3.16) is similar to (2.14) and (2.17). We have
\[
(\Phi, [H_f, T_{m}(P)^{1/2}] \Psi) = (H_f \Phi, T_{m}(P)^{1/2} \Psi) - (\Phi, T_{m}(P)^{1/2} H_f \Psi)
\]
The Feynman-Kac formula yields that
\[
(\Phi, [H_f, T_{m}(P)^{1/2}] \Psi) = \int_{0}^{\infty} \frac{e^{-m^2/2} \lambda^{m/5}}{\lambda^{5/4}} \, d\lambda \int_{\mathbb{R}^d} dx E_P^x \left[ e^{iP \cdot B_\lambda} (\Phi(B_0), [H_f, e^{-i\mathcal{A}(K[0,\lambda])}] e^{-iP \cdot B_\lambda} \Psi(B_\lambda)) \right].
\]
Since \([H_f, e^{-i\mathcal{A}(K[0,\lambda])}] = e^{-i\mathcal{A}(K[0,\lambda])} (\Pi(K[0,\lambda]) + \xi)\). Then in a similar manner to (D.3) we can derive the desired results.
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