ON A MATHEMATICAL MODEL WITH NON-COMPACT BOUNDARY CONDITIONS DESCRIBING BACTERIAL POPULATION (II)

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Abstract. This work is a natural continuation of an earlier one [1] in which a mathematical model has been studied. This model is based on maturation-velocity structured bacterial population. The bacterial mitosis is mathematically described by a non-compact boundary condition. We investigate the spectral properties of the generated semigroup and we give an explicit estimation of the bound of its infinitesimal generator.

1. Introduction. In this work, we continue our investigation already started in [1]. This study concerns a mathematical model describing a maturation-velocity structured bacterial population. Each bacterium is distinguished by its degree of maturity $\mu$ and its maturation velocity $v$. The degree of maturity of a daughter bacterium is $\mu = 0$ while that of a mother bacterium is $\mu = 1$. Between birth and division, the degree of maturity of each bacterium is $0 < \mu < 1$. As each bacterium may not become less mature, its maturation velocity $v$ must be positive (i.e., $0 < v < \infty$). The bacterial density $f = f(t, \mu, v)$, with respect to the degree of maturity $\mu$ and the maturation velocity $v$, fulfills

$$\frac{\partial f}{\partial t} = -v \frac{\partial f}{\partial \mu} - \left[ \int_0^\infty s(\mu, v', v)dv' \right] f + \int_0^\infty r(\mu, v, v')f(t, \mu, v')dv',$$

where, $r(\mu, v, v')$ (resp. $s(\mu, v', v)$) stands for the transition rate at which bacteria change their velocities from $v'$ to $v$ (resp. from $v$ to $v'$).

During mitosis, there may be a correlation $\tau := \tau(v, v')$ between the maturation velocity $v'$ of a mother bacterium and that of its daughter $v$. If $\alpha \geq 0$ denotes the average number of bacteria daughter viable per mitotic, then this correlation (called Transition Rule) is mathematically described by

$$f(t, 0, v) = \frac{\alpha}{v} \int_0^\infty \tau(v, v')f(t, 1, v')dv'.$$

The model (1)–(2) was proposed in [9] and numerically treated in [5]. In our knowledge, the first theoretical approach of the model (1)–(2) is due to [6]. One have proved that this model is governed by a contractive strongly continuous semigroup provided that the average number of daughter bacteria is less than 1 (i.e., $\alpha < 1$).

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The most interesting case, \( \alpha \geq 1 \) (corresponding to an increasing number of daughter bacteria), was studied in [2]. We have then proved that the model (1)–(2) is governed by a strongly continuous semigroup.

During mitosis, there also may be a total inheritance of the maturation velocity between a mother bacterium and its daughters (i.e., \( \tau(v, v') = \delta_{v}(v') \)). If \( \beta \geq 0 \) denotes the average number of bacteria daughter viable per mitotic, then this inheritance (called Perfect Memory Rule) is mathematically described by

\[
f(t, 0, v) = \beta f(t, 1, v).
\]

We have proved (see [3]) that the model (1)–(3) is governed by a contractive strongly continuous semigroup if and only if \( \beta < 1 \). In other words, there are no solutions for the most interesting case \( \beta \geq 1 \) corresponding to an increasing number of daughter bacteria.

In most observed mitosis, the bacterial population is divided into two subpopulations. The first one obeys to Transition Rule described by (2) while the second one obeys to Perfect Memory Rule described by (3). We are then facing a third biological rule mathematically described by

\[
f(t, 0, v) = \alpha \int_{0}^{\infty} \tau(v, v') f(t, 1, v') dv' + \beta f(t, 1, v).
\]

The full model (1)–(4) has recently been the subject of a mathematical study (see [1]). We have then proved that this model is governed by a strongly continuous semigroup for all \( t \geq 0 \); this was our first challenge. For more explanations, we refer to [1] and the references therein. (See also [7] for another mathematical background slightly different than that used in [1]).

Now, our new challenge is to prove that the bacterial population, described by the full model (1)–(4), possesses "Asynchronous Growth Property". This property describes the bacterial profile whose privileged direction is mathematically interpreted by an eigenvector corresponding to the leading eigenvalue. This is what the biologist observes in his laboratory.

Mathematically, Asynchronous Growth Property is closely related to the inequality

\[
\omega_{0}(U_{\alpha, \beta}) > \omega_{\text{ess}}(U_{\alpha, \beta}),
\]

where, \( \omega_{0}(U_{\alpha, \beta}) \) and \( \omega_{\text{ess}}(U_{\alpha, \beta}) \) denote respectively the type and the essential type of the full semigroup \( U_{\alpha, \beta} = (U_{\alpha, \beta}(t))_{t \geq 0} \).

It is well known that strict inequalities, such as (5), are too hard to prove and therefore adequate strategies are needed.

Henceforth, we are looking for \( \delta \) such that

\[
\omega_{0}(U_{\alpha, \beta}) > \delta \quad \text{and} \quad \delta \geq \omega_{\text{ess}}(U_{\alpha, \beta}).
\]

Our computations have shown that the most suitable \( \delta \) is

\[
\delta := -s \quad \text{where} \quad s := \inf_{(\mu, v') \in \Omega} \int_{0}^{\infty} |s(\mu, v', v)| dv',
\]

and for which (6) can be considered instead of (5); that is our strategy.

In this work, we focus our attention only on the first inequality of (6), that is to say that, \( \omega_{0}(U_{\alpha, \beta}) > -s \). We then start by preparing the necessary matter and putting all the relevant hypotheses on the kernel of correlation \( \tau := \tau(\cdot, \cdot) \) and on both transition rates \( s \) and \( r \). Next, we characterize the type \( \omega_{0}(T_{\alpha, \beta}) \) of the unperturbed semigroup \( T_{\alpha, \beta} = (T_{\alpha, \beta}(t))_{t \geq 0} \) (i.e., \( s = r = 0 \)) as the unique
solution of a certain characteristic equation. We also prove that both semigroups $T_{\alpha,\beta} = (T_{\alpha,\beta}(t))_{t\geq0}$ and $U_{\alpha,\beta} = (U_{\alpha,\beta}(t))_{t\geq0}$ are ordered. At the end of this work, we prove the desired strict inequality (see Theorem 4). We end this work by some useful Remarks.

Regarding to the second inequality of (6) (i.e., $-g \geq \omega_{\text{ess}}(U_{\alpha,\beta})$), this one is clearly hardest than the first one and needs to be proved separately. Its proof needs different technics than those used in this works and it will be the aim of [4]. Finally, note the novelty of this work.

2. Full model (1)-(4). This section deals with useful results for the sequel. Let then $Y_1$ be the following trace space

$$Y_1 := L^1\left((0, \infty), \, \nu dv\right)$$

whose norm is $\|\psi\|_{Y_1} = \int_0^\infty |\psi(v)| \, v dv$.

Until the end of this work, both average numbers $\alpha \geq 0$ and $\beta \geq 0$, of daughter bacteria viable per mitosis, are assumed to be fixed unless stated otherwise. The correlation kernel $\tau := \tau(\cdot, \cdot)$, between the maturation velocity $v' \geq 0$ of a mother bacterium and that of its daughter bacteria $v \geq 0$, is also assumed to be fixed and likely to fulfill the following hypotheses

$$(H^1_{\alpha,\beta}) : \quad \tau_o := \text{ess sup}_{v' \geq 0} \int_0^\infty |\tau(v,v')| \, dv < \infty,$$

$$(H^2_{\alpha,\beta}) : \quad \exists \omega_\alpha \geq 0 \text{ such that } \alpha \text{ ess sup}_{v' \geq \omega_\alpha} \int_0^\infty |\tau(v,v')| \, dv \geq \beta < 1.$$

Let $L_{\alpha,\beta}$ be the following linear operator

$$L_{\alpha,\beta} \psi(v) := \frac{\alpha}{v} \int_0^\infty \tau(v,v') \psi(v') v' dv' + \beta \psi(v)$$

$$:= L_{\alpha,\beta}^0 \psi(v) + \tilde{L}_{\alpha,\beta} \psi(v)$$

whose useful properties are given by

**Lemma 1.** Let $\alpha \geq 0$ and $\beta \geq 0$. If $(H^1_{\alpha,\beta})$ holds, then $L_{\alpha,\beta}$ is a bounded linear operator from $Y_1$ into itself satisfying,

$$\|L_{\alpha,\beta}\|_{L^1(Y_1)} \leq \alpha \tau_o + \beta.$$  \hspace{1cm} (9)

Furthermore, if $(H^2_{\alpha,\beta})$ holds, then

$$\|L_{\alpha,\beta} \ll \omega\|_{L^1(Y_1)} < 1 \quad \text{for all } \omega \geq \omega_\alpha$$ \hspace{1cm} (10)

where, $\ll \omega$ denotes the characteristic operator of the subset $(\omega, \infty)$.

**Proof.** The boundedness of $L_{\alpha,\beta}$ and (9) follow from [1, Lemma 3.2].

Let $\psi \in Y_1$ and let $\omega$ be such that $\omega \geq \omega_\alpha$ ($\omega_\alpha$ is given in $(H^2_{\alpha,\beta})$). Using (8), we can write that

$$\|L_{\alpha,\beta} \ll \omega \psi\|_{Y_1} \leq \|L_{\alpha,\beta} \ll \omega \psi\|_{Y_1} + \|L_{\alpha,\beta} \ll \omega \psi\|_{Y_1}.$$ \hspace{1cm} (11)

Using once more (8), we get that

$$\|L_{\alpha,\beta} \ll \omega \psi\|_{Y_1} = \int_0^\infty \frac{\alpha}{v} \int_0^\infty \tau(v,v') \ll \omega(v') \psi(v') v' dv' \, v dv \leq \alpha \int_0^\infty \tau(v,v') \psi(v') v' dv'$$
\begin{align*}
\leq \alpha \int_{\omega_0}^{\infty} \left[ \int_{0}^{\infty} |\tau(v, v')| \, dv \right] |\psi(v')| \, v' \, dv' \\
\leq \alpha \left[ \text{ess sup}_{v' \geq \omega_0} \int_{0}^{\infty} |\tau(v, v')| \, dv \right] \int_{\omega_0}^{\infty} |\psi(v')| \, v' \, dv'
\end{align*}

and therefore

\[ \|L_{\alpha,0} \|_{Y_1} \leq \alpha \left[ \text{ess sup}_{v' \geq \omega_0} \int_{0}^{\infty} |\tau(v, v')| \, dv \right] \|\psi\|_{Y_1}. \] \tag{12}

Similarly,

\[ \|L_{\alpha,0} \|_{Y_1} = \beta \int_{\omega}^{\infty} |\psi(v)| \, dv \leq \beta \|\psi\|_{Y_1}. \] \tag{13}

Putting now (12) and (13) into (11), we obtain

\[ \|L_{\alpha,\beta} \|_{Y_1} \leq \alpha \left[ \text{ess sup}_{v' \geq \omega_0} \int_{0}^{\infty} |\tau(v, v')| \, dv \right] + \beta \|\psi\|_{Y_1}. \]

which readily leads to the desired (10) because of (H'\_1).

Let \( \Omega := (0,1) \times (0,\infty) \) and let \( L_1 \) and \( W_1 \) be the following Banach spaces

\[ L_1 := L^1(\Omega) \quad \text{whose norm is} \quad \|\varphi\|_1 = \int_{\Omega} |\varphi(\mu, v)| \, d\mu \, dv \]

\[ W_1 := \left\{ \varphi \in L_1 : \ v \frac{\partial \varphi}{\partial \mu} \in L_1 \quad \text{and} \quad v\varphi \in L_1 \right\}. \] \tag{14}

Let \( T_{\alpha,\beta} \) be the following unbounded linear operator

\[ T_{\alpha,\beta} \varphi := -v \frac{\partial \varphi}{\partial \mu} \quad \text{on} \quad D(T_{\alpha,\beta}) := \left\{ \varphi \in W_1 : \varphi(0, \cdot) = L_{\alpha,\beta}(\varphi(1, \cdot)) \right\} \] \tag{15}

and let \( V_{\alpha,\beta} \) and \( U_{\alpha,\beta} \) be two perturbations of \( T_{\alpha,\beta} \) given by

\[ V_{\alpha,\beta} := T_{\alpha,\beta} + S \quad \text{on the domain} \quad D(V_{\alpha,\beta}) := D(T_{\alpha,\beta}) \]

and

\[ U_{\alpha,\beta} := V_{\alpha,\beta} + R \quad \text{on the domain} \quad D(U_{\alpha,\beta}) := D(T_{\alpha,\beta}). \]

The linear operators \( S \) and \( R \) are defined by

\[ S\varphi(\mu, v) := -\left[ \int_{0}^{\infty} s(\mu, v', v) \, dv' \right] \varphi(\mu, v), \] \tag{16}

and

\[ R\varphi(\mu, v) := \int_{0}^{\infty} r(\mu, v, v') \varphi(\mu, v') \, dv'. \]

whose kernels \( s \) and \( r \) are subject to the following hypotheses

\((H'\_1) : \quad \bar{s} := \text{ess sup}_{(\mu, v) \in \Omega} \int_{0}^{\infty} |s(\mu, v', v)| \, dv' < \infty \) \tag{17}

\((H'\_2) : \quad \bar{r} := \text{ess sup}_{(\mu, v) \in \Omega} \int_{0}^{\infty} |r(\mu, v', v)| \, dv' < \infty. \)
Lemma 2. Suppose that \((H^1_1)\) holds and let \(\alpha \geq 0\) and \(0 \leq \beta < 1\). If \((H^1_1)\) holds, then \(T_{\alpha,\beta}\) generates, on \(L_1\), a strongly continuous semigroup \(T_{\alpha,\beta} = (T_{\alpha,\beta}(t))_{t \geq 0}\) given by

\[
T_{\alpha,\beta}(t) := T_{0,0}(t) + T_{\alpha,\beta}(t), \quad t \geq 0,
\]

where,

\[
T_{0,0}(t)\varphi(\mu, v) := \chi(t, \mu, v)\varphi(\mu - tv, v), \quad (\mu, v) \in \Omega, \quad (18)
\]

and \(T_{\alpha,\beta}(0) = 0\) and if \(t > 0\),

\[
T_{\alpha,\beta}(t)\varphi(\mu, v) := \frac{\xi(t, \mu, v)}{v} \int_0^\infty \tau(v, v')T_{\alpha,\beta} \left( t - \frac{\mu}{v} \right) \varphi(1, v')dv' + \beta \xi(t, \mu, v)T_{\alpha,\beta} \left( t - \frac{\mu}{v} \right) \varphi(1, v) \tag{20}
\]

with

\[
\chi(t, \mu, v) := \begin{cases} 1 & \text{if } \mu \geq tv \\ 0 & \text{if } \mu < tv \end{cases} \text{ and } \xi := 1 - \chi. \tag{21}
\]

Furthermore,

1. If \((H^1_1)\) holds, then \(V_{\alpha,\beta}\) generates, on \(L_1\), a strongly continuous semigroup \(V_{\alpha,\beta} = (V_{\alpha,\beta}(t))_{t \geq 0}\).

2. If \((H^1_1)\) and \((H^1_1)\) hold, then \(U_{\alpha,\beta}\) generates, on \(L_1\), a strongly continuous semigroup \(U_{\alpha,\beta} = (U_{\alpha,\beta}(t))_{t \geq 0}\).

Proof. The boundedness of \(L_{\alpha,\beta}\) from \(Y_1\) into itself (Lemma 1) together with [1, Lemma 4.1] give a sense to \(D(T_{\alpha,\beta})\) and therefore, \(T_{\alpha,\beta}\) is a generator because of [1, Theorem 4.3].

Next, (10) says that \(L_{\alpha,\beta}\) is an admissible operator (in the sense of [2, Remark 3.4]) which leads, by virtue of [2, Theorem 4.1], to the desired (18) with

\[
T_{\alpha,\beta}(t)\varphi(\mu, v) := \xi(t, \mu, v)L_{\alpha,\beta} \left( T_{\alpha,\beta} \left( t - \frac{\mu}{v} \right) \varphi(1, \cdot) \right)(v).
\]

This together with (8) obviously yield the explicit form (20).

Finally, both points (1) and (2) follow from [1, Theorems 5.2 and 6.2].

3. Stability properties. The aim of this section is to prove two stability results of the unperturbed semigroup \(T_{\alpha,\beta} = (T_{\alpha,\beta}(t))_{t \geq 0}\). The first one concerns the case \(\alpha = 0\) while the second one concerns the case \(\alpha > 0\). The case \(\alpha = 0\) can be formulated as follows.

Theorem 1. Let \(0 \leq \beta < 1\). Then, for all \(\varphi \in L_1\) we have

\[
\int_0^\infty \|T_{0,\beta}(t)\varphi(1, \cdot)\|_{Y_1} dt \leq \frac{1}{1 - \beta} \|\varphi\|_1. \tag{22}
\]

Proof. Suppose that \(\tau = 0\) and let \(\alpha = 0\) and let \(0 \leq \beta < 1\). Both hypotheses \((H^1_0)\) and \((H^1_1)\) (i.e., \((H^1_1)\) and \((H^1_2)\) with \(\tau = 0\)) hold true and therefore the semigroup \(T_{0,\beta} = (T_{0,\beta}(t))_{t \geq 0}\) exists because of Lemma 2 (with \(\alpha = 0\)).

So let \(\varphi \in L_1\). Using (18) (with \(\alpha = 0\)) we can write that

\[
T_{0,\beta}(t)\varphi = T_{0,0}(t)\varphi + T_{0,\beta}(t)\varphi
\]
and therefore
\[
\int_0^\infty \left\| T_{0,\beta}(t) \varphi(1, \cdot) \right\|_{Y_1} \, dt \leq \int_0^\infty \left\| T_{0,0}(t) \varphi(1, \cdot) \right\|_{Y_1} \, dt + \int_0^\infty \left\| T_{0,\beta}(t) \varphi(1, \cdot) \right\|_{Y_1} \, dt.
\] (23)

On one hand, using (19) we get that
\[
\int_0^\infty \left\| T_{0,0}(t) \varphi(1, \cdot) \right\|_{Y_1} \, dt = \int_0^\infty \left[ \int_0^\infty |T_{0,0}(t, v)| \varphi(1, v) \right] \, dt
\]
\[
= \int_0^\infty \int_0^\infty \chi(t, v) |\varphi(1 - tv, v)| \, dt \, dv
\]
\[
= \int_0^\infty \int_0^1 \left(1 - \frac{\mu}{v}, 1, v\right) |\varphi(\mu, v)| \, d\mu \, dv
\]
which leads, by virtue of (21) (i.e., \( \chi(1 - \mu, 1, v) = 1 \) iff \( \mu > 0 \)), to
\[
\int_0^\infty \left\| T_{0,0}(t) \varphi(1, \cdot) \right\|_{Y_1} \, dt = \int_0^\infty \int_0^1 |\varphi(\mu, v)| \, d\mu \, dv = \| \varphi \|_1.
\]

On the other hand, using (20) we get that
\[
\int_0^\infty \left\| T_{0,\beta}(t) \varphi(1, \cdot) \right\|_{Y_1} \, dt = \int_0^\infty \left[ \int_0^\infty |T_{0,\beta}(t, v)| \varphi(1, v) \right] \, dt
\]
\[
= \beta \int_0^\infty \int_0^\infty \xi(t, v) \left| T_{0,\beta} \left(1 - \frac{1}{v}, \varphi(1, v)\right) \right| \, dt \, dv
\]
\[
= \beta \int_0^\infty \int_0^1 \xi \left(1 + \frac{1}{v}, 1, v\right) |\varphi(1, v)| \, d\mu \, dv
\]
which leads, by virtue of (21) (i.e., \( \xi(1 + \frac{1}{v}, 1, v) = 1 \) iff \( x > 0 \)), to
\[
\int_0^\infty \left\| T_{0,\beta}(t) \varphi(1, \cdot) \right\|_{Y_1} \, dt = \beta \int_0^\infty \left[ \int_0^\infty |T_{0,\beta}(x) \varphi(1, v)| \, dv \right] \, dx
\]
\[
= \beta \int_0^\infty \| T_{0,\beta}(x) \varphi(1, \cdot) \|_{Y_1} \, dx.
\]

Now, (23) becomes
\[
\int_0^\infty \left\| T_{0,\beta}(t) \varphi(1, \cdot) \right\|_{Y_1} \, dt \leq \| \varphi \|_1 + \beta \int_0^\infty \left\| T_{0,\beta}(t) \varphi(1, \cdot) \right\|_{Y_1} \, dt
\]
which obviously proves the desired (22) because of \( 0 \leq \beta < 1 \).

\[ \square \]

**Corollary 1.** Let \( 0 \leq \beta < 1 \). Then, for all \( \varphi \in L_1 \) we have
\[
\int_0^t \| T_{0,\beta}(x) \varphi(1, \cdot) \|_{Y_1} \, dx \leq \frac{1}{1 - \beta} \| \varphi \|_1 \quad \text{for all} \quad t \geq 0.
\] (24)

**Proof.** This follows from Theorem 1. \[ \square \]

Theorem 1 holds only for \( \alpha = 0 \). In order to prove a similar result for the case \( \alpha > 0 \), we put

\[(H_{\alpha}^\beta) : \quad \exists \omega_i > 0 \quad \text{such that} \quad \alpha \left[ \operatorname{ess sup} \int_{\omega_i}^\infty |\tau(v, v')| \, dv \right] + \beta < 1.\]
Remark 1. The hypothesis \( (H''_1) \) can also be formulated as follows

\( (H''_1) \) : \[ \exists \omega_i > 0 \quad \text{such that} \quad \tau_{\alpha,\omega_i} < 1 \]

where,

\[ \tau_{\alpha,\omega} := \alpha \left[ \text{ess sup}_{v' \geq 0} \int_{\omega} |\tau(v, v')| \, dv \right] + \beta. \]  

(25)

Hence, if \( (H'_1) \) holds and \( \alpha \eta_n + \beta < 1 \), then

\[ \tau_{\alpha,\omega} \leq \alpha \left[ \text{ess sup}_{v' > 0} \int_{0}^{\infty} |\tau(v, v')| \, dv \right] + \beta = \alpha \eta_n + \beta < 1 \quad \text{for all} \omega \geq 0 \]

and therefore \( (H''_1) \) holds true (for any \( \omega_i > 0 \)).

Theorem 2. Suppose that \( (H'_1) \) holds and let \( \alpha > 0 \) and \( 0 \leq \beta < 1 \) be such that \( (H''_1) \) holds. If \( (H''_1) \) holds, then there exists a finite constant \( \lambda_0 \) such that, for all \( \varphi \in L_1 \) we have

\[ \int_{0}^{\infty} e^{-\lambda_0 t} \| T_{\alpha,\beta}(t) \varphi(1, \cdot) \|_{Y_1} \, dt \leq \frac{2}{(1 - \tau_{\alpha,\omega_1})} \| \varphi \|_1 \]  

(26)

where, \( \tau_{\alpha,\omega_1} \) is defined by \( (25) \).

Proof. Firstly, suppose that \( \tau = 0 \). Then, \( L_{\alpha,\beta} = L_{0,\beta} \) (because of \( (8) \)) which leads, by virtue of \( (15) \), to \( (T_{0,\beta}, D(T_{0,\beta})) = (T_{0,\beta}, D(T_{0,\beta})) \). By the uniqueness of the generated semigroup we then infer that \( T_{\alpha,\beta}(t) = T_{0,\beta}(t) \) for all \( t \geq 0 \).

On the other hand, \( (25) \) yields that \( \tau_{\alpha,\omega_1} = \beta \). Lastly, by choosing \( \lambda_0 = 0 \), we infer that the desired \( (26) \) is nothing else but \( (22) \).

Next, suppose that \( \tau \neq 0 \). Let \( \varphi \in L_1 \) and let \( \lambda \geq 0 \). Since \( (18) \), we can write that

\[ \int_{0}^{\infty} e^{-\lambda t} \| T_{0,\beta}(t) \varphi(1, \cdot) \|_{Y_1} \, dt \leq A_\lambda + B_\lambda \]  

(27)

where,

\[ A_\lambda := \int_{0}^{\infty} e^{-\lambda t} \| T_{0,\beta}(t) \varphi(1, \cdot) \|_{Y_1} \, dt, \]

and

\[ B_\lambda := \int_{0}^{\infty} e^{-\lambda t} \| T_{\alpha,\beta}(t) \varphi(1, \cdot) \|_{Y_1} \, dt. \]

For convenience, we divide the rest of the proof into several steps.

Step I. (Computation of \( A_\lambda \)).

Using the explicit form \( (19) \), it yields that

\[ A_\lambda = \int_{0}^{\infty} e^{-\lambda t} \left[ \int_{0}^{\infty} |T_{0,\beta}(t)\varphi(1, v)| \, dv \right] \, dt \]

\[ = \int_{0}^{\infty} \int_{0}^{\infty} e^{-\lambda t} \chi(t, 1, v) |\varphi(1 - tv, v)| \, v \, dt \, dv \]

\[ = \int_{0}^{1} \int_{-\infty}^{\infty} e^{-\lambda \left( \frac{1}{2} - \frac{v}{2} \right)} \chi \left( \frac{1 - \mu}{v}, 1, v \right) |\varphi(\mu, v)| \, d\mu \, dv \]

which leads, by virtue of \( (21) \) (i.e., \( \chi \left( \frac{1 - \mu}{v}, 1, v \right) = 1 \) iff \( \mu > 0 \)), to

\[ A_\lambda = \int_{0}^{1} \int_{0}^{\infty} e^{-\lambda \chi(1 - v)} |\varphi(\mu, v)| \, d\mu \, dv \leq \| \varphi \|_1. \]  

(28)
Step II. (Estimation of $B_\lambda$).

Using the explicit form (20), it follows that
\[ B_\lambda = \int_0^\infty \frac{e^{-\lambda t}}{v} \int_0^\infty |\mathbb{T}_{\alpha,\beta}(t)(1, v)| vdv dt \]
\[ = \int_0^\infty \int_0^\infty \frac{e^{-\lambda t}}{v} \xi(t, 1, v) |\mathcal{L}_{\alpha,\beta} \left( \mathbb{T}_{\alpha,\beta} \left( \frac{t - 1}{v} \right) \phi(1, \cdot) \right)(v)| vdvdt \]
\[ = \int_0^\infty \int_{-\frac{1}{v}}^\infty e^{-\lambda(x + \frac{1}{2})} \xi \left( x + \frac{1}{v}, 1, v \right) \left| \mathcal{L}_{\alpha,\beta} \left( \mathbb{T}_{\alpha,\beta}(x)\phi(1, \cdot) \right)(v) \right| vdv dx \]

which leads, by virtue of (21) \text{i.e.,} $\xi \left( x + \frac{1}{v}, 1, v \right) = 1$ iff $x > 0$, to
\[ B_\lambda = \int_0^\infty \int_0^\infty e^{-\lambda x} \left| \mathcal{L}_{\alpha,\beta} \left( \mathbb{T}_{\alpha,\beta}(x)\phi(1, \cdot) \right)(v) \right| vdv dx \]
\[ = \int_0^\infty e^{-\lambda x} \left[ \int_0^\infty e^{-\hat{\lambda}x} \mathcal{L}_{\alpha,\beta} \left( \mathbb{T}_{\alpha,\beta}(x)\phi(1, \cdot) \right)(v) vdv \right] dx \]
\[ = \int_0^\infty e^{-\lambda x} \left\| e^{-\hat{\lambda}x} \mathcal{L}_{\alpha,\beta} \left( \mathbb{T}_{\alpha,\beta}(x)\phi(1, \cdot) \right) \right\|_{Y_1} dx. \]

To simplify $B_\lambda$, we put
\[ \mathcal{L}_{\alpha,\beta,\lambda}(v) := e^{-\hat{\lambda}x} \mathcal{L}_{\alpha,\beta} \psi(v). \] (29)

Obviously, $\mathcal{L}_{\alpha,\beta,\lambda}$ is a bounded linear operator, from $Y_1$ into itself, because of the boundedness of the linear operator $\mathcal{L}_{\alpha,\beta}$ (Lemma 1). Hence, $B_\lambda$ becomes
\[ B_\lambda = \int_0^\infty e^{-\lambda x} \left\| \mathcal{L}_{\alpha,\beta,\lambda} \left( \mathbb{T}_{\alpha,\beta}(x)\phi(1, \cdot) \right) \right\|_{Y_1} dx. \]

and therefore
\[ B_\lambda \leq \left\| \mathcal{L}_{\alpha,\beta,\lambda} \right\|_{\mathcal{L}(Y_1)} \int_0^\infty e^{-\lambda x} \left\| \mathbb{T}_{\alpha,\beta}(x)\phi(1, \cdot) \right\|_{Y_1} dx. \] (30)

In order to improve (30), the norm $\left\| \mathcal{L}_{\alpha,\beta,\lambda} \right\|_{\mathcal{L}(Y_1)}$ needs to be estimated.

Step III. (Estimation of $\left\| \mathcal{L}_{\alpha,\beta,\lambda} \right\|_{\mathcal{L}(Y_1)}$).

Let $\psi \in Y_1$. Observe that (29) can, by virtue of (8), be written as
\[ \mathcal{L}_{\alpha,\beta,\lambda}(v) := \frac{e^{-\frac{x}{v}}}{v} \int_0^\infty \tau(v, v')(\psi(v')v')dv' + \beta e^{-\frac{x}{v}}\psi(v) \]
\[ := \mathcal{L}_{\alpha,\beta,\lambda}(v) + \mathcal{L}_{\psi,\beta,\lambda}(v). \] (31)

As the norm of $\mathcal{L}_{\psi,\beta,\lambda}$ is obviously less than $\beta$, then
\[ \left\| \mathcal{L}_{\alpha,\beta,\lambda} \right\|_{\mathcal{L}(Y_1)} \leq \left\| \mathcal{L}_{\alpha,\beta,\lambda} \right\|_{\mathcal{L}(Y_1)} + \beta. \] (32)

On the other hand, using (31) we can write
\[ \left\| \mathcal{L}_{\alpha,\beta,\lambda}(\psi) \right\|_{Y_1} = \int_0^\infty \left| \frac{e^{-\frac{x}{v}}}{v} \int_0^\infty \tau(v, v')(\psi(v')v')dv' \right| vdv \]
\[ \leq \alpha \int_0^\infty \left[ \int_0^\infty e^{-\frac{x}{v}}|\tau(v, v')|dv \right] |\psi(v')|v'dv' \]
\[ \leq \alpha \left[ \text{ess sup}_{v' \geq 0} \int_0^\infty e^{-\frac{x}{v}}|\tau(v, v')|dv \right] \int_0^\infty |\psi(v')|v'dv'. \]
That is to say that,
\[
\| \mathcal{Z}_{\alpha,0,\lambda} \|_{\mathcal{L}(Y_1)} \leq \alpha \left[ \operatorname{ess} \sup_{v' > 0} \int_0^\infty e^{-\frac{\lambda}{T}} |\tau(v,v')| \, dv \right].
\] (33)

However, for almost all \( v' \in (0, \infty) \), we have
\[
\int_0^\infty e^{-\frac{\lambda}{T}} |\tau(v,v')| \, dv = \int_0^{\omega_1} e^{-\frac{\lambda}{T}} |\tau(v,v')| \, dv + \int_{\omega_1}^\infty e^{-\frac{\lambda}{T}} |\tau(v,v')| \, dv
\leq e^{-\frac{\lambda}{T}} \int_0^{\omega_1} |\tau(v,v')| \, dv + \int_{\omega_1}^\infty |\tau(v,v')| \, dv
\leq e^{-\frac{\lambda}{T}} \operatorname{ess} \sup_{v' > 0} \int_0^{\omega_1} |\tau(v,v')| \, dv + \operatorname{ess} \sup_{v' > 0} \int_{\omega_1}^\infty |\tau(v,v')| \, dv
= e^{-\frac{\lambda}{T}} \tau_0 + \operatorname{ess} \sup_{v' > 0} \int_{\omega_1}^\infty |\tau(v,v')| \, dv
\]
where, \( \omega_1 \) is given in (H\(_1^{\alpha,1} \)). Accordingly, (33) becomes
\[
\| \mathcal{Z}_{\alpha,0,\lambda} \|_{\mathcal{L}(Y_1)} \leq \alpha \tau_0 e^{-\frac{\lambda}{T}} + \alpha \left[ \operatorname{ess} \sup_{v' > 0} \int_0^{\omega_1} |\tau(v,v')| \, dv \right]
\]
which we put into (32), to finally get the desired estimation
\[
\| \mathcal{Z}_{\alpha,\beta,\lambda} \|_{\mathcal{L}(Y_1)} \leq \alpha \tau_0 e^{-\frac{\lambda}{T}} + \mathcal{T}_{\alpha,\beta,\omega_1}.
\] (34)

Step IV. (Conclusion).
Combining (34) and (30), we obtain that
\[
B_\lambda \leq \left( \alpha \tau_0 e^{-\frac{\lambda}{T}} + \mathcal{T}_{\alpha,\beta,\omega_1} \right) \int_0^\infty e^{-\lambda x} \| T_{\alpha,\beta}(x) \varphi(1, \cdot) \|_{Y_1} \, dx.
\]
Putting it together with (28) into (27) we finally get, for all \( \lambda \geq 0 \), that
\[
\left( 1 - \alpha \tau_0 e^{-\frac{\lambda}{T}} - \mathcal{T}_{\alpha,\beta,\omega_1} \right) \int_0^\infty e^{-\lambda t} \| T_{\alpha,\beta}(t) \varphi(1, \cdot) \|_{Y_1} \, dt \leq \| \varphi \|_1.
\] (35)
Now, let \( \lambda_0 \) be the following finite constant
\[
\lambda_0 := \omega_1 \max \left\{ \ln \left( \frac{2 \alpha \tau_0}{1 - \mathcal{T}_{\alpha,\beta,\omega_1}} \right), \ 0 \right\}
\]
which is well defined because of (H\(_1^{\alpha,1} \)) (i.e., \( 0 \leq \mathcal{T}_{\alpha,\beta,\omega_1} < 1 \)) and \( \alpha > 0 \). Replacing \( \lambda = \lambda_0 \) into (35), we readily get that
\[
\frac{1 - \mathcal{T}_{\alpha,\beta,\omega_1}}{2} \int_0^\infty e^{-\lambda_0 t} \| T_{\alpha,\beta}(t) \varphi(1, \cdot) \|_{Y_1} \, dt \leq \| \varphi \|_1
\]
which proves the desired (26) and completes the proof.

\[
\Box
\]

Corollary 2. Suppose that (H\(_1^1 \)) holds and let \( \alpha > 0 \) and \( 0 \leq \beta < 1 \) be such that (H\(_1^{\beta,1} \)) holds. If (H\(_1^{\alpha,1} \)) holds, then for all \( \varphi \in L_1 \) we have
\[
\int_0^t \| T_{\alpha,\beta}(x) \varphi(1, \cdot) \|_{Y_1} \, dx \leq C_1 \| \varphi \|_1 \quad \text{for all} \quad t \geq 0
\] (36)

where, \( C_1 \) is a finite constant.
Proof. Let \( t \geq 0 \). For all \( \varphi \in L_1 \), we can write that
\[
\int_0^t \| T_{\alpha,\beta}(x) \varphi(1, \cdot) \|_{Y_1} \, dx \leq e^{\lambda_0 t} \int_0^t e^{-\lambda_0 s} \| T_{\alpha,\beta}(x) \varphi(1, \cdot) \|_{Y_1} \, dx \\
\leq e^{\lambda_0 t} \int_0^\infty e^{-\lambda_0 x} \| T_{\alpha,\beta}(x) \varphi(1, \cdot) \|_{Y_1} \, dx
\]
which leads, by virtue of (26), to
\[
\int_0^t \| T_{\alpha,\beta}(x) \varphi(1, \cdot) \|_{Y_1} \, dx \leq e^{\lambda_0 t} \frac{2}{(1 - T_{\alpha,\beta,\omega_1})} \| \varphi \|_1
\]
which proves the desired (36) with \( C_t := 2e^{\lambda_0 t}(1 - T_{\alpha,\beta,\omega_1})^{-1} \).

4. Lattice property. This section is devoted to some lattice properties of the semigroups \( T_{\alpha,\beta} = (T_{\alpha,\beta}(t))_{t \geq 0} \) and \( V_{\alpha,\beta} = (V_{\alpha,\beta}(t))_{t \geq 0} \) and \( U_{\alpha,\beta} = (U_{\alpha,\beta}(t))_{t \geq 0} \) whose existence is already proved in Lemma 2. So, let us put
\[(H^2) : \quad \tau \geq 0\]
and let us start with the following result.

Lemma 3. Suppose that \((H^1)\) holds and let \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \) be such that \((H^2)\) holds. If \((H^2)\) holds, then \( T_{\alpha,\beta} = (T_{\alpha,\beta}(t))_{t \geq 0} \) is a positive semigroup and
\[
0 \leq T_0,\beta(t) \leq T_{\alpha,\beta}(t) \quad t \geq 0.
\]
(37)

Proof. The positivity of \( T_{\alpha,\beta} = (T_{\alpha,\beta}(t))_{t \geq 0} \) follows from [1, Theorem 4.3(1)]. The positivity of \( T_0,\beta = (T_0,\beta(t))_{t \geq 0} \) follows from that of \( T_{\alpha,\beta} = (T_{\alpha,\beta}(t))_{t \geq 0} \) (with \( \alpha = 0 \)). It remains therefore to prove (37).

Step I. (Preparative Step).

Let \( \lambda \geq 0 \) and let \( L_{\alpha,\beta,\lambda} \) be the following linear operator
\[
L_{\alpha,\beta,\lambda} \psi(v) := \frac{\alpha}{\nu} \int_0^\infty e^{-\frac{\lambda}{\nu} \tau(v,v')} \psi(v') \nu' \, dv' + \beta e^{-\frac{\lambda}{\nu} \psi(v)}
\]
\[
:= L_{\alpha,0,\lambda} \psi(v) + L_{0,\beta,\lambda} \psi(v)
\]
which is positive and bounded from \( Y_1 \) into itself ([1, Lemma 3.4]) and satisfying,
\[
\| L_{\alpha,\beta,\lambda} \|_{L(Y_1)} < 1 \quad \text{for all large } \lambda
\]
(39)
because of [1, Proposition 3.5]. From (38), we get that \( L_{\alpha,\beta,\lambda} \geq L_{0,\beta,\lambda} \geq 0 \) which leads, by induction on the integer \( n \geq 1 \), to
\[
L_{\alpha,\beta,\lambda} \geq L_{0,\beta,\lambda} \geq 0 \quad n = 1, 2, 3, \ldots
\]
Similarly, \( L_{\alpha,\beta} \geq L_{0,\beta} \geq 0 \) because of (8) and therefore
\[
L_{\alpha,\beta,\lambda} \geq L_{0,\beta,\lambda} \geq 0 \quad n = 1, 2, 3, \ldots
\]
and by summing both sides,
\[
\left( \sum_{n \geq 1} L_{\alpha,\beta,\lambda}^{n,\alpha,\beta} \right) + L_{\alpha,\beta} \geq \left( \sum_{n \geq 1} L_{0,\beta,\lambda}^{n} \right) + L_{0,\beta} \geq 0.
\]
That is to say
\[ \left( \sum_{n \geq 0} c_{n} n^{\alpha,\beta} \right) \mathcal{L}_{\alpha,\beta} \geq \left( \sum_{n \geq 0} c_{n} n^{\alpha,\beta} \right) \mathcal{L}_{0,\beta} \geq 0. \]

Now, (39) obviously yields that
\[ (I_{Y_1} - \mathcal{L}_{\alpha,\beta})^{-1} \mathcal{L}_{\alpha,\beta} \geq (I_{Y_1} - \mathcal{L}_{0,\beta})^{-1} \mathcal{L}_{0,\beta} \geq 0 \quad \text{for all large} \quad \lambda \quad (40) \]
where, \( I_{Y_1} \) denotes the identity operator into \( Y_1 \).

**Step II (Conclusion).**
Let \( \lambda \) be large and let \( g \in L_1 \) be such that \( g \geq 0 \). Rewriting [1, (4.11)], i.e.,
\[ (\lambda - T_{\alpha,\beta})^{-1} g = (\lambda - T_{0,0})^{-1} g + \varepsilon \left( I_{Y_1} - \mathcal{L}_{\alpha,\beta} \right) - (\lambda - T_{0,0})^{-1} g(1, \cdot) \]
and by the way (with \( \alpha = 0 \)),
\[ (\lambda - T_{0,\beta})^{-1} g = (\lambda - T_{0,0})^{-1} g + \varepsilon \left( I_{Y_1} - \mathcal{L}_{\alpha,\beta} \right) - (\lambda - T_{0,0})^{-1} g(1, \cdot) \]
where, \( \varepsilon(\mu, v) = e^{-\lambda \alpha} \) and \( (\lambda - T_{0,0})^{-1} g \) is defined by [1, (4.3)], i.e.,
\[ (\lambda - T_{0,0})^{-1} g(\mu, v) = \int_{0}^{\xi} e^{-\lambda s} g(\mu - s v, v) d s \geq 0. \quad (41) \]

It then follows, by virtue of (40) together with \( (\lambda - T_{0,0})^{-1} g(1, \cdot) \geq 0 \), that
\[ (\lambda - T_{\alpha,\beta})^{-1} g \geq (\lambda - T_{0,\beta})^{-1} g \geq 0 \]
and by induction on the integer \( n \geq 1 \),
\[ \left[ \lambda (\lambda - T_{\alpha,\beta})^{-1} \right]^{n} g \geq \left[ \lambda (\lambda - T_{0,\beta})^{-1} \right]^{n} g \geq 0 \quad n = 1, 2, 3, \ldots \quad (42) \]
Let \( t > 0 \). Putting \( \lambda = \frac{n}{t} \) \((n \text{ big enough})\) into (42), we finally get that
\[ \left[ \frac{n}{t} \left( \frac{n}{t} - T_{\alpha,\beta} \right)^{-1} \right]^{n} g \geq \left[ \frac{n}{t} \left( \frac{n}{t} - T_{0,\beta} \right)^{-1} \right]^{n} g \geq 0 \]
and at the limit \( n \to \infty \), the desired (37) follows. \( \square \)

In the sequel, the following hypotheses will be needed
\[ (H^2_s) \quad s \geq 0 \]
\[ (H^2_r) \quad r \geq 0. \]

**Theorem 3.** Suppose that \((H^1_s), (H^2)\) and \((H^1_r)\) hold and let \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \) be such that \((H^1_s)\) holds. Then \( V_{\alpha,\beta} = (V_{\alpha,\beta}(t))_{t \geq 0} \) is a positive semigroup satisfying
\[ 0 \leq e^{-\pi t} T_{\alpha,\beta}(t) \leq V_{\alpha,\beta}(t), \quad t \geq 0, \quad (43) \]
and
\[ 0 \leq V_{\alpha,\beta}(t) \leq e^{-\pi t} T_{\alpha,\beta}(t), \quad t \geq 0, \quad (44) \]
where, \( s \) and \( \pi \) are defined by (7) and (17). Furthermore
1. If \((H^2_s)\) holds, then
\[ 0 \leq V_{\alpha,\beta}(t) \leq T_{\alpha,\beta}(t), \quad t \geq 0 \quad (45) \]
and
\[ 0 \leq V_{\alpha,\beta}(t) - V_{0,\beta}(t) \leq T_{\alpha,\beta}(t) - T_{0,\beta}(t), \quad t \geq 0. \quad (46) \]
2. If \((H^1_2)\) and \((H^2)\) hold, then \(U_{\alpha,\beta} = (V_{\alpha,\beta}(t))_{t \geq 0}\) is a positive semigroup satisfying
\[
0 \leq V_{\alpha,\beta}(t) \leq U_{\alpha,\beta}(t) \quad t \geq 0.
\]

Proof. The positivity of \(V_{\alpha,\beta} = (V_{\alpha,\beta}(t))_{t \geq 0}\) follows from [1, Theorem 5.2(1)]. Furthermore,
\[
V_{\alpha,\beta}(t) = \lim_{n \to \infty} \left[ e^{\frac{t}{n} S T_{\alpha,\beta} \left( \frac{t}{n} \right)} \right]^n \varphi \quad t \geq 0
\]
for all \(\varphi \in L_1\), where, \(S\) is defined by (16).

Let \(\varphi \in L_1\) be such that \(\varphi \geq 0\). Due to
\[
e^{-\tau x} \varphi \leq e^{xS} \varphi \leq e^{-\tau x} \varphi \quad \text{for all } x \geq 0
\]
it follows, by the positivity of \(T_{\alpha,\beta} = (T_{\alpha,\beta}(t))_{t \geq 0}\) (Lemma 3), that
\[
0 \leq e^{-\tau x} T_{\alpha,\beta}(x) \varphi \leq e^{xS} T_{\alpha,\beta}(x) \varphi \leq e^{-\tau x} T_{\alpha,\beta}(x) \varphi \quad \text{for all } x \geq 0.
\]

Let \(t \geq 0\). Using (49) we get, for all integers \(n \geq 1\), that
\[
\left[ e^{\frac{t}{n} S T_{\alpha,\beta} \left( \frac{t}{n} \right)} \right]^n \varphi \leq e^{-\frac{t}{n} T_{\alpha,\beta} \left( \frac{t}{n} \right)} \varphi
\]
and similarly,
\[
\left[ e^{\frac{t}{n} S T_{\alpha,\beta} \left( \frac{t}{n} \right)} \right]^n \varphi \geq e^{-\frac{t}{n} T_{\alpha,\beta} \left( \frac{t}{n} \right)} \varphi
\]
and therefore
\[
0 \leq e^{-\tau x} T_{\alpha,\beta}(t) \varphi \leq \left[ e^{\frac{t}{n} S T_{\alpha,\beta} \left( \frac{t}{n} \right)} \right]^n \varphi \leq e^{-\tau x} T_{\alpha,\beta}(t) \varphi \quad n = 1, 2, 3, \ldots
\]

Passing now to the limit \(n \to \infty\) and using (48), both (43) and (44) obviously follow.

(1). Firstly, (45) obviously follows from (44) because of \((H^2)\) \(i.e., s \geq 0\).

Next, let \(t \geq 0\) and let \(\varphi \in L_1\) be such that \(\varphi \geq 0\). So, (37) readily yields, for all integers \(n \geq 1\), that
\[
0 \leq \left[ e^{\frac{t}{n} S \varphi} \right]^n \varphi \leq e^{-\tau x} T_{\alpha,\beta}(t) \varphi
\]
and therefore
\[
0 \leq V_{\alpha,\beta}(t) \leq U_{\alpha,\beta}(t)
\]
(50)
and therefore
\[
0 \leq V_{\alpha,\beta}(t) - V_{0,\beta}(t).
\]

On the other hand, \(T_{\alpha,\beta} = (T_{\alpha,\beta}(t))_{t \geq 0}\) and \(V_{\alpha,\beta} = (V_{\alpha,\beta}(t))_{t \geq 0}\) (generated respectively by \(T_{\alpha,\beta}\) and \(V_{\alpha,\beta} = T_{\alpha,\beta} + S\)) are related by
\[
V_{\alpha,\beta}(t) = T_{\alpha,\beta}(t) + \int_0^t V_{\alpha,\beta}(t-s) S T_{\alpha,\beta}(s) \, ds
\]
and similarly (for $\alpha = 0$),

$$V_{0,\beta}(t) = T_{0,\beta}(t) + \int_0^t V_{0,\beta}(t-s)ST_{0,\beta}(s)ds.$$ 

Now, the positivity of $(-S)$ (because of $(H^2)$) together with (37) and (50) allow us to write that

$$V_{\alpha,\beta}(t) - T_{\alpha,\beta}(t) = \int_0^t V_{\alpha,\beta}(t-s)ST_{\alpha,\beta}(s)ds$$

$$\leq -\int_0^t V_{\alpha,\beta}(t-s)(-S)T_{0,\beta}(s)ds$$

$$\leq -\int_0^t V_{0,\beta}(t-s)(-S)T_{0,\beta}(s)ds$$

$$= \int_0^t V_{0,\beta}(t-s)ST_{0,\beta}(s)ds$$

$$= V_{0,\beta}(t) - T_{0,\beta}(t)$$

which leads to

$$V_{\alpha,\beta}(t) - V_{0,\beta}(t) \leq T_{\alpha,\beta}(t) - T_{0,\beta}(t).$$

Finally, this together (51) obviously prove the desired (46).

(2) The positivity of $U_{\alpha,\beta} = (U_{\alpha,\beta}(t))_{t \geq 0}$ follows from [1, Theorem 6.2(1)] while (47) follows from [1, (6.3)].

5. Spectral properties. The aim of this section is to estimate the type $\omega_0(U_{\alpha,\beta})$, of the full semigroup $U_{\alpha,\beta} = (U_{\alpha,\beta}(t))_{t \geq 0}$, defined by

$$\omega_0(U_{\alpha,\beta}) := \lim_{t \to \infty} \frac{1}{t} \ln \|U_{\alpha,\beta}(t)\|_{L(L_1)}. \quad (52)$$

Obviously, (52) needs the explicit form of $U_{\alpha,\beta} = (U_{\alpha,\beta}(t))_{t \geq 0}$ which is unfortunately unavailable. In order to overcome this difficulty, we firstly prove that the spectral bound $s(T_{\alpha,\beta})$, of the generator $T_{\alpha,\beta}$ defined by

$$s(T_{\alpha,\beta}) := \begin{cases} \sup \{ \text{Re}(\lambda), \; \lambda \in \sigma(T_{\alpha,\beta}) \} & \text{if } \sigma(T_{\alpha,\beta}) \neq \emptyset \\ -\infty & \text{if } \sigma(T_{\alpha,\beta}) = \emptyset \end{cases} \quad (53)$$

is the unique solution of a certain characteristic equation (see the proof of Theorem 4). We then conclude the desired estimation of the type $\omega_0(U_{\alpha,\beta})$ through both (43) and (47). Notice, by the way, that we have

$$s(T_{\alpha,\beta}) \leq \omega_0(T_{\alpha,\beta}). \quad (54)$$

Until now, our functional framework $L_1$ was always assumed to be a Banach space over the real field. However, in this section, $L_1$ will sometimes be assumed to be over the complex field. The transition between $L_1$ over the real field and its complexification is well known. Therefore, in the sequel we will not distinguish between $L_1$ over the real field and its complexification.

Let $L_{\alpha,\beta,\lambda}$ be the following linear operator
As we are going to see, the operator \( L_{\alpha,\beta,\lambda} \) will play a crucial role in this spectral study and therefore, it is natural to prove all its useful properties. We let, in the sequel, \( \mathbb{C}_+ := \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) \geq 0 \} \). The first useful property of \( L_{\alpha,\beta,\lambda} \) is given as follows.

**Lemma 4.** Let \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \). If \((H^1)\) holds, then \( L_{\alpha,\beta,\lambda} (\lambda \in \mathbb{C}_+) \) is a bounded linear operator from \( Y_1 \) into itself satisfying,

\[
\| L_{\alpha,\beta,\lambda} \|_{L(Y_1)} \leq \frac{\alpha}{(1-\beta)} \tau_0. \tag{56}
\]

Furthermore, if \((H^2)\) holds, then

\[
| L_{\alpha,\beta,\lambda} \psi | \leq | L_{\alpha,\beta,\lambda,\text{Re}\lambda} | | \psi | \quad \text{for all } \psi \in Y_1. \tag{57}
\]

**Proof.** Let \( \lambda \in \mathbb{C}_+ \) and let \( \psi \in Y_1 \). According to \( 0 \leq \beta < 1 \), we can write that

\[
\| L_{\alpha,\beta,\lambda} \psi \|_{Y_1} = \int_0^{\infty} \left| \frac{1}{(e^{\frac{\lambda}{v}} - \beta)} \right| | L_{\alpha,\beta,\lambda,\text{Re}\lambda} | \psi(v) | vdv
\]

\[
\leq \int_0^{\infty} \left| \frac{1}{(e^{\frac{\lambda}{v}} - \beta)} \right| | L_{\alpha,\beta,\lambda} | \psi(v) | vdv
\]

\[
\leq \frac{1}{(1-\beta)} \int_0^{\infty} | L_{\alpha,\beta,\lambda} | \psi(v) | vdv
\]

and therefore

\[
\| L_{\alpha,\beta,\lambda} \|_{L(Y_1)} \leq \frac{1}{(1-\beta)} \| L_{\alpha,\beta,\lambda,\text{Re}\lambda} \|_{L(Y_1)}.
\]

Now (9) (with \( \beta = 0 \)) obviously leads to the desired (56).

Next, for almost all \( v \in (0,\infty) \), we have

\[
| L_{\alpha,\beta,\lambda} \psi(v) | = \left| \frac{1}{(e^{\frac{\lambda}{v}} - \beta)} \right| \left| \frac{\alpha}{v} \int_0^{\infty} \tau(v, v') | \psi(v') | v' dv' \right|
\]

\[
\leq \frac{1}{(e^{\frac{\lambda}{v}} - \beta)} \left( \frac{\alpha}{v} \int_0^{\infty} \tau(v, v') | \psi(v') | v' dv' \right)
\]

\[
= L_{\alpha,\beta,\lambda,\text{Re}\lambda} | \psi | (v)
\]

which proves the desired (57). \(\square\)

Before we continue, notice that if \( \alpha = \beta = 0 \), then (15) becomes

\[
T_{0,0} \varphi = -v \frac{\partial \varphi}{\partial \mu} \quad \text{on} \quad D(T_{0,0}) = \{ \varphi \in W_1 : \varphi(0, \cdot) = 0 \}
\]

where, \( W_1 \) is defined by (14).

**Lemma 5.** Suppose that \((H^1)\) holds and let \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \). Suppose, for given \( \lambda \in \mathbb{C}_+ \) and \( \phi \in D(T_{0,0}) \), the following equation

\[
(E_{\varphi}) : \quad \psi = L_{\alpha,\beta,\lambda} \psi + \frac{1}{1-\beta e^{-\frac{\lambda}{v}}} \phi(1, \cdot)
\]

\[
(58)
\]
admits a solution \( \psi \in Y_1 \), where \( \mathcal{L}_{\alpha, \beta, \lambda} \) is defined by (55). Then, 
\[
\phi := \theta_{\lambda} \mathcal{L}_{\alpha, \beta} \psi + \phi
\]  
(59)
fulfills: \( \phi \in D(T_{\alpha, \beta}) \) and \( (\lambda - T_{\alpha, \beta}) \phi = (\lambda - T_{0,0}) \phi \) 
(60)
where, \( \theta_{\lambda}(\mu, v) := e^{-\lambda \frac{v}{2}} \) and \( \mathcal{L}_{\alpha, \beta} \) and \( T_{\alpha, \beta} \) are defined by (8) and (15).

Proof. Let \( \lambda \in \mathbb{C}_+ \) and let \( \phi \in D(T_{0,0}) \) be given. As \( \phi \in D(T_{0,0}) \subset W_1 \), then \( \phi(1, \cdot) \in Y_1 \) because of [1, Lemma 4.1] and therefore 
\[
\| \frac{1}{1 - \beta e^{-\frac{v}{2}}} \phi(1, \cdot) \|_{Y_1} = \int_0^\infty \left| \frac{1}{1 - \beta e^{-\frac{v}{2}}} \phi(1, v) \right| \, \mathrm{d}v < \infty.
\]
This together with the fact that \( \mathcal{L}_{\alpha, \beta, \lambda} \) acts into \( Y_1 \) (Lemma 4) obviously show that the equation \((E_\phi)\) is well posed into \( Y_1 \). Now, let us prove (60).

**Step I.** (Proof of \( \phi \in W_1 \)). Firstly, integrating \( \phi \) and \( v \phi \) into \( L_1 \), we get that 
\[
\| \phi \|_1 \leq \| \theta_{\lambda} \mathcal{L}_{\alpha, \beta} \psi \|_1 + \| \phi \|_1 = \int_0^\infty \int_0^1 \left| e^{-\lambda \frac{v}{2}} \right| \| \mathcal{L}_{\alpha, \beta} \psi(v) \| \, \mathrm{d}\mu + \| \phi \|_1
\]
\[
= \int_0^\infty \left[ \int_0^1 e^{-\frac{(\mathrm{Re}\lambda)v}{2}} \, \mathrm{d}\mu \right] \| \mathcal{L}_{\alpha, \beta} \psi(v) \| \, \mathrm{d}v + \| \phi \|_1
\]
\[
= \int_0^\infty \frac{v}{(\mathrm{Re}\lambda)} \left[ 1 - e^{-\frac{(\mathrm{Re}\lambda)v}{2}} \right] \| \mathcal{L}_{\alpha, \beta} \psi(v) \| \, \mathrm{d}v + \| \phi \|_1
\]
\[
\leq \frac{1}{(\mathrm{Re}\lambda)} \| \mathcal{L}_{\alpha, \beta} \psi \|_{Y_1} + \| \phi \|_1
\]
and 
\[
\| v \phi \|_1 \leq \| v \theta_{\lambda} \mathcal{L}_{\alpha, \beta} \psi \|_1 + \| v \phi \|_1
\]
\[
= \int_0^\infty \int_0^1 v \left| e^{-\lambda \frac{v}{2}} \right| \| \mathcal{L}_{\alpha, \beta} \psi(v) \| \, \mathrm{d}\mu + \| v \phi \|_1
\]
\[
= \int_0^1 \int_0^\infty ve^{-\frac{(\mathrm{Re}\lambda)v}{2}} \| \mathcal{L}_{\alpha, \beta} \psi(v) \| \, \mathrm{d}v + \| v \phi \|_1
\]
\[
\leq \int_0^\infty \| \mathcal{L}_{\alpha, \beta} \psi(v) \| \, \mathrm{d}v + \| v \phi \|_1
\]
\[
= \| \mathcal{L}_{\alpha, \beta} \psi \|_{Y_1} + \| v \phi \|_1
\]
which leads, by virtue of (9), to 
\[
\| \phi \|_1 \leq \frac{1}{(\mathrm{Re}\lambda)} (\alpha \tau_0 + \beta) \| \psi \|_{Y_1} + \| \phi \|_1 < \infty
\]
(61)
and 
\[
\| v \phi \|_1 \leq (\alpha \tau_0 + \beta) \| \psi \|_{Y_1} + \| v \phi \|_1 < \infty.
\]
(62)
Next, deriving $v \varphi$ with respect to $\mu$, we get that
\[ v \frac{\partial \varphi}{\partial \mu} = v \frac{\partial}{\partial \mu} \left( \theta \mathcal{A}_{\alpha,\beta} \psi + \phi \right) \]
\[ = v \frac{\partial}{\partial \mu} \left( \theta \mathcal{A}_{\alpha,\beta} \psi \right) - \left( -v \frac{\partial \phi}{\partial \mu} \right) \]
\[ = -\lambda \theta \mathcal{A}_{\alpha,\beta} \psi - T_{0,0} \phi \]
which leads to
\[ \left\| v \frac{\partial \varphi}{\partial \mu} \right\| \leq \frac{\lambda}{|\mathcal{A}|} \left\| \theta \mathcal{A}_{\alpha,\beta} \psi \right\|_1 + \left\| T_{0,0} \phi \right\|_1 \]
\[ = \lambda \int_0^\infty \int_0^1 \left| e^{-\frac{\lambda}{v} \frac{\mu}{v}} \right| \left| \mathcal{A}_{\alpha,\beta} \psi(v) \right| \, d\mu \, dv + \left\| T_{0,0} \phi \right\|_1. \]
Same previous calculations show that
\[ \left\| v \frac{\partial \varphi}{\partial \mu} \right\| \leq \frac{\lambda}{|\mathcal{A}|} \left\| \mathcal{A}_{\alpha,\beta} \psi \right\|_y + \left\| T_{0,0} \phi \right\|_1 \]
and by (9),
\[ \left\| v \frac{\partial \varphi}{\partial \mu} \right\| \leq \frac{\lambda}{(\text{Re}\lambda)} \left( \alpha \sigma + \beta \right) \left\| \psi \right\|_y + \left\| T_{0,0} \phi \right\|_1 < \infty. \] (63)
Now, (61), (62) and (63) obviously prove that $\varphi \in W_1$.

**Step II.** (Proof of $\varphi(0, \cdot) = \mathcal{A}_{\alpha,\beta} (\varphi(1, \cdot))$).

Firstly, using (8) we can write that
\[ \psi - e^{-\frac{\lambda}{v} \frac{\mu}{v}} \mathcal{A}_{\alpha,\beta} \psi = \psi - e^{-\frac{\lambda}{v} \frac{\mu}{v}} \left( \mathcal{A}_{\alpha,\beta} \psi + \beta \psi \right) \]
\[ = \left( 1 - \beta \right) \psi - e^{-\frac{\lambda}{v} \frac{\mu}{v}} \mathcal{A}_{\alpha,\beta} \psi \]
\[ = \left( 1 - \beta \right) \left( \psi - \frac{1}{(e^{-\frac{\lambda}{v} \frac{\mu}{v}} - \beta)} \mathcal{A}_{\alpha,\beta} \psi \right) \]
which is, by virtue of (55), nothing else but
\[ \psi - e^{-\frac{\lambda}{v} \frac{\mu}{v}} \mathcal{A}_{\alpha,\beta} \psi = \left( 1 - \beta \right) \left( \psi - \mathcal{A}_{\alpha,\beta,\lambda} \psi \right). \]
That is to say, by (58),
\[ \psi - e^{-\frac{\lambda}{v} \frac{\mu}{v}} \mathcal{A}_{\alpha,\beta} \psi = \phi(1, \cdot). \] (64)

Next, using (59) we can write that
\[ \varphi(0, \cdot) - \mathcal{A}_{\alpha,\beta} (\varphi(1, \cdot)) = \left( \mathcal{A}_{\alpha,\beta} \psi + \phi(0, \cdot) \right) - \mathcal{A}_{\alpha,\beta} \left( \left( \theta \mathcal{A}_{\alpha,\beta} \psi + \phi(1, \cdot) \right) \right) \]
\[ = \mathcal{A}_{\alpha,\beta} \psi + \phi(0, \cdot) - \mathcal{A}_{\alpha,\beta} \left( e^{-\frac{\lambda}{v} \frac{\mu}{v}} \mathcal{A}_{\alpha,\beta} \psi + \phi(1, \cdot) \right) \]
\[ = \mathcal{A}_{\alpha,\beta} \left( \psi - e^{-\frac{\lambda}{v} \frac{\mu}{v}} \mathcal{A}_{\alpha,\beta} \psi - \phi(1, \cdot) \right) + \phi(0, \cdot) \]
which leads, by virtue of $\phi(0, \cdot) = 0$ (because of $\phi \in D(T_{0,0})$), to
\[ \varphi(0, \cdot) - \mathcal{A}_{\alpha,\beta} (\varphi(1, \cdot)) = \mathcal{A}_{\alpha,\beta} \left( \psi - e^{-\frac{\lambda}{v} \frac{\mu}{v}} \mathcal{A}_{\alpha,\beta} \psi - \phi(1, \cdot) \right). \] (65)
Combining now (64) and (65), the desired $\varphi(0, \cdot) = \mathcal{A}_{\alpha,\beta} (\varphi(1, \cdot))$ follows.

**Step III.** (Proof of (60)).
Firstly, both Steps I and II obviously yield that \( \varphi \in D(T_{\alpha,\beta}) \). Next, from (59), we can write that
\[
(\lambda - T_{\alpha,\beta}) \varphi = \lambda \left( \theta_\lambda L_{\alpha,\beta} \psi + \phi \right) + v \frac{\partial}{\partial \mu} \left( \theta_\lambda L_{\alpha,\beta} \psi + \phi \right)
= \lambda \theta_\lambda L_{\alpha,\beta} \psi + \lambda \phi - \lambda \theta_\lambda L_{\alpha,\beta} \psi + v \frac{\partial \phi}{\partial \mu}
= \lambda \phi - \left( -v \frac{\partial \phi}{\partial \mu} \right)
= (\lambda - T_{0,0}) \phi
\]
which completes the proof of the desired (60).

The spectrum \( \sigma(T_{\alpha,\beta}) \) of the generator \( T_{\alpha,\beta} \) can be characterized as follows

**Proposition 1.** Suppose that \( (H_1^\beta) \) holds and let \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \). Let \( \lambda \in \mathbb{C}_+ \). Then,
\[
1 \in \sigma_p \left( L_{\alpha,\beta,\lambda} \right) \implies \lambda \in \sigma(T_{\alpha,\beta}) \quad (66)
\]
and
\[
\lambda \in \sigma \left( T_{\alpha,\beta} \right) \implies 1 \in \sigma \left( L_{\alpha,\beta,\lambda} \right) \quad (67)
\]
where, \( \sigma(\cdot) \) and \( \sigma_p(\cdot) \) denote the spectrum and the point spectrum.

**Proof.** Let \( \lambda \in \mathbb{C}_+ \) be given.

Proof of (66). Suppose firstly that \( 1 \in \sigma_p \left( L_{\alpha,\beta,\lambda} \right) \). There exists then \( \psi \in Y_1 \) such that \( \psi = L_{\alpha,\beta,\lambda} \psi \). That is to say that \( (E_\lambda) \) (i.e., \( (58) \) with \( \phi = 0 \in D(T_{0,0}) \)) admits the solution \( \psi \). Accordingly, Lemma 5 yields that \( \varphi := \theta_\lambda L_{\alpha,\beta} \psi \) (i.e., \( (59) \) with \( \phi = 0 \)) fulfils (60) i.e.,
\[
\varphi \in D \left( T_{\alpha,\beta} \right) \quad \text{and} \quad (\lambda - T_{\alpha,\beta}) \varphi = (\lambda - T_{0,0}) \varphi = 0
\]
and therefore \( \lambda \in \sigma_p \left( T_{\alpha,\beta} \right) \subset \sigma \left( T_{\alpha,\beta} \right) \). The desired (66) is now proved.

Proof of (67). Let us prove the converse of (67), i.e.,
\[
1 \in \rho \left( L_{\alpha,\beta,\lambda} \right) \implies \lambda \in \rho(T_{\alpha,\beta}). \quad (68)
\]
Suppose that \( 1 \in \rho \left( L_{\alpha,\beta,\lambda} \right) \) and let \( g \in L_1 \). Let \( \phi := (\lambda - T_{0,0})^{-1} g \in D(T_{0,0}) \). Due to (41), we get that
\[
\phi(1,v) = (\lambda - T_{0,0})^{-1} g(1,v) = \int_0^{\frac{1}{2}} e^{-\lambda s} g(1 - sv,v) ds \quad (69)
\]
and therefore
\[
\left\| \frac{1}{1 - \beta e^{-\frac{1}{2}}} \phi(1,\cdot) \right\|_{Y_1} \leq \int_0^\infty \left| \int_0^{\frac{1}{2}} e^{-\lambda s} g(1 - sv,v) ds \right| dv
\leq \frac{1}{1 - \beta} \int_0^\infty \int_0^{\frac{1}{2}} |g(1 - sv,v)| dv ds dv
= \frac{1}{1 - \beta} \int_0^\infty \int_0^1 |g(\mu,v)| d\mu dv
< \infty.
\]
Lemma 6. Let
\[
\frac{1}{1 - \beta e^{-\frac{v}{\beta}}} \phi(1, \cdot) \in Y_1.
\]

That is to say that \(264\) MOHAMED BOULANOUAR \(\hat{\tau}_\alpha\) where,
\[\int_0^\infty \text{ess sup}_{v' > 0} |\tau(v, v')| dv < \infty.\]

Remark 2. The hypothesis \((\hat{H}_1^1)\) can also be formulated as follows
\[
\hat{\tau}_\alpha := \int_0^\infty \tau_\infty(v) dv < \infty \quad \text{with} \quad \tau_\infty(\cdot) := \text{ess sup}_{v' > 0} |\tau(\cdot, v')|.
\]

So, if \((\hat{H}_1^1)\) holds then
\[\tau_\alpha = \text{ess sup}_{v' > 0} \int_0^\infty |\tau(v, v')| dv \leq \int_0^\infty \tau_\infty(v) dv = \hat{\tau}_\alpha < \infty\]
and therefore \((\hat{H}_1^1)\) is stronger than \((H_1^1)\). Hence, all results of \([1]\), related to \((H_1^1)\), still true whenever \((\hat{H}_1^1)\) holds true.

The first consequence of \((\hat{H}_1^1)\) is given by the following result.

Lemma 6. Let \(\alpha \geq 0\) and \(0 \leq \beta < 1\). If \((\hat{H}_1^1)\) holds, then
\[
\lim_{\lambda \to \infty} \|\mathcal{L}_{\alpha, \beta, \lambda}\|_{C(Y_1)} = 0
\]
and
\[
\lim_{\lambda \to \eta} \|\mathcal{L}_{\alpha, \beta, \lambda} - \mathcal{L}_{\alpha, \beta, \eta}\|_{C(Y_1)} = 0.
\]

Proof. Let \(\lambda \geq 0\) and let \(\psi \in Y_1\). Using \((55)\), we can write that
\[
\|\mathcal{L}_{\alpha, \beta, \lambda} \psi\|_{Y_1} = \int_0^\infty \left| \frac{1}{e^{\frac{v}{\beta}} - \beta} \int_0^\infty \tau(v, v') \psi(v') v' dv' \right| v dv
\]
\[
\leq \int_0^\infty \frac{\alpha}{e^{\frac{v}{\beta}} - \beta} \left[ \int_0^\infty |\tau(v, v')| \psi(v') v' dv' \right] dv
\]
\[
\leq \left[ \int_0^\infty \frac{\alpha}{e^{\frac{v}{\beta}} - \beta} \text{ess sup}_{v' > 0} |\tau(v, v')| dv \right] \left[ \int_0^\infty |\psi(v')| v' dv' \right]
\]
which leads to
\[
\|\mathcal{L}_{\alpha, \beta, \lambda}\|_{C(Y_1)} \leq \int_0^\infty \frac{\alpha}{e^{\frac{v}{\beta}} - \beta} \tau_\infty(v) dv
\]
where, \(\tau_\infty\) is defined by \((70)\). However,
\[
0 \leq \frac{\alpha}{e^{\frac{v}{\beta}} - \beta} \tau_\infty \leq \frac{\alpha}{1 - \beta} \tau_\infty \in L^1(0, \infty)
\]
and
\[
\lim_{\lambda \to \infty} \frac{\alpha}{e^{\frac{v}{\beta}} - \beta} \tau_\infty(v) = 0 \quad \text{a.e.} \quad v \in (0, \infty).
\]
Now, Dominated Convergence Theorem applied to (73) leads to (71).
Similarly, if \( \eta \geq 0 \) then,
\[
\| \mathcal{L}_{\alpha,\beta,\lambda} \psi - \mathcal{L}_{\alpha,\beta,\eta} \psi \|_{Y_1} \leq \left[ \int_0^\infty \left| \frac{\alpha}{(e^{\frac{\lambda}{\eta}} - \beta)} - \frac{\alpha}{(e^{\frac{\lambda}{\eta}} - \beta)} \right| \tau_\infty(v) dv \right] \| \psi \|_{Y_1}
\]
and therefore
\[
\| \mathcal{L}_{\alpha,\beta,\lambda} - \mathcal{L}_{\alpha,\beta,\eta} \|_{L_1(Y_1)} \leq \int_0^\infty \left| \frac{\alpha}{(e^{\frac{\lambda}{\eta}} - \beta)} - \frac{\alpha}{(e^{\frac{\lambda}{\eta}} - \beta)} \right| \tau_\infty(v) dv. \tag{74}
\]
However,
\[
0 \leq \left| \frac{\alpha}{(e^{\frac{\lambda}{\eta}} - \beta)} - \frac{\alpha}{(e^{\frac{\lambda}{\eta}} - \beta)} \right| \tau_\infty \leq \frac{2\alpha}{1 - \beta} \tau_\infty \in L^1(0,\infty)
\]
and
\[
\lim_{\lambda \to \eta} \left| \frac{\alpha}{(e^{\frac{\lambda}{\eta}} - \beta)} - \frac{\alpha}{(e^{\frac{\lambda}{\eta}} - \beta)} \right| \tau_\infty(v) = 0 \quad \text{a.e.} \quad v \in (0,\infty).
\]
Finally, Dominated Convergence Theorem applied to (74) leads to (72). This completes the proof.

The second consequence of \((\hat{H}_1^1)\) is given by the following result.

**Lemma 7.** Suppose that \((H^2_2)\) holds and let \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \). If \((\hat{H}_1^1)\) holds, then \( \mathcal{L}_{\alpha,\beta,\lambda} (\lambda \in C_+) \) is a compact operator into \( Y_1 \).

**Proof.** The boundedness of \( \mathcal{L}_{\alpha,\beta,\lambda} (\lambda \in C_+) \) follows from Lemma 4.

**Step I.** Let \( \eta \geq 0 \) and let \( \psi \in B_0 \) (\( B_0 \) denotes the unit closed ball of \( Y_1 \)). Obviously, (55) shows that \( \mathcal{L}_{\alpha,\beta,\eta} \) is a positive operator because of \((H^2_2)\). Accordingly, for almost all \( v \in (0,\infty) \), we can write that
\[
\mathcal{L}_{\alpha,\beta,\eta} \psi (v) = \frac{1}{(e^{\frac{\lambda}{\eta}} - \beta)} \frac{\alpha}{v} \int_0^\infty \tau(v,v') |\psi(v')| v' dv' \leq \frac{1}{(1 - \beta)} \frac{\alpha}{v} \int_0^\infty \tau(v,v') |\psi(v')| v' dv' \leq \frac{1}{(1 - \beta)} \frac{\alpha}{v} \left[ \sup_{v' \geq 0} \tau(v,v') \right] \int_0^\infty |\psi(v')| v' dv'
\]
which can be written as
\[
\mathcal{L}_{\alpha,\beta,\eta} \psi (v) \leq \frac{\alpha}{(1 - \beta)} \frac{\tau_\infty(v)}{v}
\]
where, \( \tau_\infty \) is defined by (70). Therefore,
\[
\sup_{\psi \in B_0} \int_a^b \mathcal{L}_{\alpha,\beta,\eta} \psi (v) dv \leq \frac{\alpha}{(1 - \beta)} \int_a^b \tau_\infty(v) dv \quad \text{for all } 0 < a < b < \infty
\]
\[
\sup_{\psi \in B_0} \int_T^\infty \mathcal{L}_{\alpha,\beta,\eta} \psi (v) dv \leq \frac{\alpha}{(1 - \beta)} \int_T^\infty \tau_\infty(v) dv \quad \text{for all } T > 0.
\]
At the limit,
\[
\lim_{|a-b| \to 0} \left[ \sup_{\psi \in B_0} \int_a^b \mathcal{L}_{\alpha,\beta,\eta} |\psi| \, dv \right] \leq \frac{\alpha}{(1-\beta)} \lim_{|a-b| \to 0} \int_a^b \tau_\infty(v) \, dv \quad (75)
\]
\[
\lim_{T \to \infty} \left[ \sup_{\psi \in B_0} \int_T^\infty \mathcal{L}_{\alpha,\beta,\eta} |\psi| \, dv \right] \leq \frac{\alpha}{(1-\beta)} \lim_{T \to \infty} \int_T^\infty \tau_\infty(v) \, dv. \quad (76)
\]
As \(\tau_\infty \in L^1(0, \infty)\), it follows that
\[
\lim_{|a-b| \to 0} \int_a^b \tau_\infty(v) \, dv = 0 \quad \text{and} \quad \lim_{T \to \infty} \int_T^\infty \tau_\infty(v) \, dv = 0
\]
and therefore, both (75) and (76) become
\[
\lim_{|a-b| \to 0} \left[ \sup_{\psi \in B_0} \int_a^b \mathcal{L}_{\alpha,\beta,\eta} |\psi| \, dv \right] = 0 \quad (77)
\]
\[
\lim_{T \to \infty} \left[ \sup_{\psi \in B_0} \int_T^\infty \mathcal{L}_{\alpha,\beta,\eta} |\psi| \, dv \right] = 0. \quad (78)
\]
**Step II.** Let \(\lambda \in \mathbb{C}_+\). Then,
\[
\mathcal{L}_{\alpha,\beta,\lambda} \psi = \text{Re} \left( \mathcal{L}_{\alpha,\beta,\lambda} \psi \right) + i \text{Im} \left( \mathcal{L}_{\alpha,\beta,\lambda} \psi \right) = A\psi + i B\psi
\]
which leads, by virtue of (57), to
\[
|A\psi| \leq |\mathcal{L}_{\alpha,\beta,\lambda} \psi| \leq |\mathcal{L}_{\alpha,\beta,\eta \Re \lambda} |\psi|
\]
\[
|B\psi| \leq |\mathcal{L}_{\alpha,\beta,\lambda} \psi| \leq |\mathcal{L}_{\alpha,\beta,\eta \Re \lambda} |\psi|.
\]
This together with (77) and (78) (with \(\eta = \Re \lambda \geq 0\)) yield that
\[
\lim_{|a-b| \to 0} \left[ \sup_{\psi \in B_0} \int_a^b |A\psi| \, dv \right] \leq \lim_{|a-b| \to 0} \left[ \sup_{\psi \in B_0} \int_a^b \mathcal{L}_{\alpha,\beta,\eta \Re \lambda} |\psi| \, dv \right] = 0
\]
\[
\lim_{T \to \infty} \left[ \sup_{\psi \in B_0} \int_T^\infty |A\psi| \, dv \right] \leq \lim_{T \to \infty} \left[ \sup_{\psi \in B_0} \int_T^\infty \mathcal{L}_{\alpha,\beta,\eta \Re \lambda} |\psi| \, dv \right] = 0
\]
and similarly,
\[
\lim_{|a-b| \to 0} \left[ \sup_{\psi \in B_0} \int_a^b |B\psi| \, dv \right] \leq \lim_{|a-b| \to 0} \left[ \sup_{\psi \in B_0} \int_a^b \mathcal{L}_{\alpha,\beta,\eta \Re \lambda} |\psi| \, dv \right] = 0
\]
\[
\lim_{T \to \infty} \left[ \sup_{\psi \in B_0} \int_T^\infty |B\psi| \, dv \right] \leq \lim_{T \to \infty} \left[ \sup_{\psi \in B_0} \int_T^\infty \mathcal{L}_{\alpha,\beta,\eta \Re \lambda} |\psi| \, dv \right] = 0.
\]
Hence, \(A(B_0)\) and \(B(B_0)\) are weakly compact subsets of \(Y_1\) and therefore, \(A\) and \(B\) are weakly compact operators into \(Y_1\). Now, Dunford-Pettis Property yields that \(A^2\) and \(B^2\) and \(AB\) and \(BA\) are compact operators into \(Y_1\) and therefore,
\[
\mathcal{L}_{\alpha,\beta,\lambda}^2 = (A^2 - B^2) + i(AB + BA)
\]
is a compact operator into \(Y_1\). The proof is now completed. \(\square\)
Let \( (H^2) \) be the following hypothesis
\[
(H^2) : \quad \tau(v, v') > 0 \quad \text{for almost all } v \geq 0 \text{ and } v' \geq 0
\]
which is obviously stronger than \( (H^2) \). Now, another interesting property of the operator \( \mathcal{L}_{\alpha, \beta, \lambda} \) is given by

**Proposition 2.** Suppose that \( (H^2) \) holds and let \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \). If \( \tilde{H} \) holds, then the spectral radius \( r_\sigma \left( \mathcal{L}_{\alpha, \beta, \lambda} \right) \) of \( \mathcal{L}_{\alpha, \beta, \lambda} \) fulfills,

\[
r_\sigma \left( \mathcal{L}_{\alpha, \beta, \lambda} \right) > 0 \quad \text{and} \quad r_\sigma \left( \mathcal{L}_{\alpha, \beta, \lambda} \right) \in \sigma_p \left( \mathcal{L}_{\alpha, \beta, \lambda} \right) \quad \text{for all } \lambda \geq 0. \tag{79}
\]
Furthermore,

\[
\lambda \geq 0 \quad \longrightarrow \quad r_\sigma \left( \mathcal{L}_{\alpha, \beta, \lambda} \right) \tag{80}
\]
is a continuous and strictly decreasing mapping.

**Proof.** Let \( \lambda \geq 0 \) and \( \eta \geq 0 \) be such that \( \lambda > \eta \). Firstly, \( \mathcal{L}_{\alpha, \beta, \lambda} \) is a strongly positive operator (because of \( (H^2) \)) with a compact square (Lemma 7). Now, since [8], it yields that

\[
(\mathbf{P}_\lambda) \quad \left\{ \begin{array}{l}
r_\sigma \left( \mathcal{L}_{\alpha, \beta, \lambda} \right) > 0 \quad \text{and} \quad \exists \text{ a quasi-interior vector } \psi_\lambda \\
\text{of } (Y_1)_+ \text{ such that } \|\psi_\lambda\|_{Y_1} = 1 \text{ and } \mathcal{L}_{\alpha, \beta, \lambda} \psi_\lambda = r_\sigma \left( \mathcal{L}_{\alpha, \beta, \lambda} \right) \psi_\lambda
\end{array} \right. \tag{81}
\]
which proves the desired (79). By the way, for almost all \( v \geq 0 \),

\[
\mathcal{L}_{\alpha, \beta, \lambda} \psi_\lambda(v) = \frac{\alpha}{(e^\frac{\alpha}{\rho} - \beta)} \frac{1}{v} \int_0^\infty \tau(v, v') \psi_\lambda(v') v' dv' \\
< \frac{\alpha}{(e^\frac{\alpha}{\rho} - \beta)} \frac{1}{v} \int_0^\infty \tau(v, v') \psi_\lambda(v') v' dv'
\]
and therefore

\[
\left( \mathcal{L}_{\alpha, \beta, \eta} \psi_\lambda - \mathcal{L}_{\alpha, \beta, \lambda} \psi_\lambda \right)(v) > 0 \quad \text{for almost all } v \geq 0. \tag{82}
\]
Since [8], it also yields that

\[
(\mathbf{P}'_\lambda) \quad \left\{ \begin{array}{l}
\exists \text{ a strictly positive functional } \psi^*_\lambda \in (Y_1')_+ \\
\text{such that } \|\psi^*_\lambda\|_{Y_1'} = 1 \text{ and } \mathcal{L}^*_{\alpha, \beta, \lambda} \psi^*_\lambda = r_\sigma \left( \mathcal{L}^*_{\alpha, \beta, \lambda} \right) \psi^*_\lambda
\end{array} \right. \tag{83}
\]
where, \( \mathcal{L}^*_{\alpha, \beta, \lambda} \) is the adjoint operator of \( \mathcal{L}_{\alpha, \beta, \lambda} \).

Next, let \( (\mathbf{P}_\eta) \) and \( (\mathbf{P}'_\eta) \) be (81) and (83) with \( \eta \) instead of \( \lambda \).

Applying then the strictly positive functional \( \psi^*_\eta \) to both quasi-interior vectors \( \psi_\lambda \) (see (81)) and \( (\mathcal{L}_{\alpha, \beta, \eta} \psi_\lambda - \mathcal{L}_{\alpha, \beta, \lambda} \psi_\lambda) \) (see (82)), we can write that

\[
\langle \psi^*_\eta, \psi_\lambda \rangle > 0 \quad \text{and} \quad \langle \psi^*_\eta, \mathcal{L}_{\alpha, \beta, \eta} \psi_\lambda - \mathcal{L}_{\alpha, \beta, \lambda} \psi_\lambda \rangle > 0. \tag{84}
\]
Using now \( (\mathbf{P}_\lambda) \) and \( (\mathbf{P}'_\lambda) \) and \( (\mathbf{P}_\eta) \) and \( (\mathbf{P}'_\eta) \), then

\[
\langle \psi^*_\eta, \mathcal{L}_{\alpha, \beta, \eta} \psi_\lambda - \mathcal{L}_{\alpha, \beta, \lambda} \psi_\lambda \rangle = \langle \psi^*_\eta, \mathcal{L}_{\alpha, \beta, \eta} \psi_\lambda \rangle - \langle \psi^*_\eta, \mathcal{L}_{\alpha, \beta, \lambda} \psi_\lambda \rangle \\
= \left( \mathcal{L}^*_{\alpha, \beta, \eta} \psi^*_\eta, \psi_\lambda \right) - \left( \mathcal{L}^*_{\alpha, \beta, \lambda} \psi^*_\eta, \psi_\lambda \right) \\
= \left( r_\sigma \left( \mathcal{L}_{\alpha, \beta, \eta} \right) \psi^*_\eta, \psi_\lambda \right) - \left( \psi^*_\eta, r_\sigma \left( \mathcal{L}_{\alpha, \beta, \lambda} \right) \psi_\lambda \right) \\
= r_\sigma \left( \mathcal{L}_{\alpha, \beta, \eta} \psi^*_\eta, \psi_\lambda \right) - r_\sigma \left( \mathcal{L}_{\alpha, \beta, \lambda} \psi^*_\eta, \psi_\lambda \right) \\
= \left( r_\sigma \left( \mathcal{L}_{\alpha, \beta, \eta} \right) - r_\sigma \left( \mathcal{L}_{\alpha, \beta, \lambda} \right) \right) \langle \psi^*_\eta, \psi_\lambda \rangle.
\]
which leads, by virtue of (84), to
\[
  r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\sigma}) - r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\lambda}) = \frac{\langle \psi^*_\eta, \mathfrak{L}_{\alpha,\beta,\eta} \psi_\lambda - \mathfrak{L}_{\alpha,\beta,\lambda} \psi_\lambda \rangle}{\langle \psi^*_\eta, \psi_\lambda \rangle} > 0 \tag{85}
\]
and therefore (80) is a strictly decreasing mapping because of \( \lambda > \eta \).

Now, let us prove that (80) is a continuous mapping. Let \( \lambda \geq 0 \) and \( \eta \geq 0 \). If \( \lambda > \eta \), then (85) leads to
\[
  |r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\lambda}) - r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\eta})| \leq \frac{1}{\langle \psi^*_\eta, \psi_\lambda \rangle} \| \mathfrak{L}_{\alpha,\beta,\lambda} \|_{L(Y_1)} \| \mathfrak{L}_{\alpha,\beta,\eta} \|_{L(Y_1)}.
\]
and therefore,
\[
  |r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\lambda}) - r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\eta})| \leq \frac{1}{\langle \psi^*_\eta, \psi_\lambda \rangle} \| \mathfrak{L}_{\alpha,\beta,\lambda} - \mathfrak{L}_{\alpha,\beta,\eta} \|_{L(Y_1)}.
\]
However, if \( \lambda < \eta \), by permuting the roles of \( \lambda \) and \( \eta \), we similarly get that
\[
  |r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\lambda}) - r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\eta})| \leq \frac{1}{\langle \psi^*_\eta, \psi_\lambda \rangle} \| \mathfrak{L}_{\alpha,\beta,\lambda} - \mathfrak{L}_{\alpha,\beta,\eta} \|_{L(Y_1)}.
\]
Finally, (72) leads, in both cases, to
\[
  \lim_{\lambda \rightarrow \eta} |r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\lambda}) - r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\eta})| = 0 \tag{86}
\]
which proves the desired continuity of (80) and completes the proof. \( \square \)

As we have pointed out in the beginning of this section, our aim is to estimate the type \( \omega_0 (U_{\alpha,\beta}) \) of the full semigroup \( U_{\alpha,\beta} = (U_{\alpha,\beta}(t))_{t \geq 0} \). So, this one can now be announced as follows

**Theorem 4.** Suppose that (H10), (H2), (H10), (H11) and (H12) hold and let \( \alpha > 0 \) and \( 0 \leq \beta < 1 \) be such that (H11) holds. If the spectral radius
\[
  r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\pi}) > 1 \tag{87}
\]
then
\[
  \omega_0 (U_{\alpha,\beta}) > -\bar{s} \tag{88}
\]
where, \( \mathfrak{L}_{\alpha,\beta,\pi-s} \) and \( \pi \) and \( s \) are respectively defined by (55), (17) and (7).

**Proof.** Firstly, let us put \( \lambda_s := \pi - \bar{s} \). So, due to (H11) we get that \( 0 \leq \lambda_s < \infty \). For convenience, we divide the rest of the proof into several steps.

**Step I.** Proposition 2 yields that \( \lambda \geq 0 \longrightarrow r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\lambda}) \) is a continuous and strictly decreasing mapping. This together with (86) and
\[
  \lim_{\lambda \rightarrow \infty} r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\lambda}) \leq \lim_{\lambda \rightarrow \infty} \| \mathfrak{L}_{\alpha,\beta,\lambda} \|_{L(Y_1)} = 0
\]
because of (71), yield that there exists a unique \( \lambda_0 \) such that
\[
  \lambda_s < \lambda_0 < \infty \quad \text{and} \quad r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\lambda_0}) = 1. \tag{88}
\]
Due to (79), we then infer that
\[
  1 = r_{\sigma} (\mathfrak{L}_{\alpha,\beta,\lambda_0}) \in \sigma_p (\mathfrak{L}_{\alpha,\beta,\lambda_0})
\]
and therefore \( \lambda_0 \in \sigma (T_{\alpha,\beta}) \) because of (66). Finally,
\[
  \lambda_0 \leq s (T_{\alpha,\beta}) \tag{89}
\]
where, \( s(T_{\alpha,\beta}) \) is the spectral bound of the generator \( T_{\alpha,\beta} \), defined by (53).

**Step II.** Let \( \lambda \in \sigma(T_{\alpha,\beta}) \) be such that \( \text{Re}(\lambda) \geq 0 \). Using (67) we get that \( 1 \in \sigma\left( L_{\alpha,\beta,\lambda} \right) \) and by the compactness of \( \mathcal{L}^{2}_{\alpha,\beta,\lambda} \) (Lemma 7) together with the spectral mapping, it then follows that

\[
1 = 1^2 \in \left( \sigma\left( L_{\alpha,\beta,\lambda} \right) \right)^2 = \sigma\left( \mathcal{L}^{2}_{\alpha,\beta,\lambda} \right) = \sigma_{p}\left( \mathcal{L}^{2}_{\alpha,\beta,\lambda} \right)
\]

and therefore, there exists \( \psi_{\lambda} \neq 0 \) such that \( \mathcal{L}^{2}_{\alpha,\beta,\lambda} \psi_{\lambda} = \lambda \psi_{\lambda} \). Using now (57) (with \( \psi = \psi_{\lambda} \)), it yields that

\[
|\psi_{\lambda}| = \left| \mathcal{L}^{2n}_{\alpha,\beta,\lambda} \psi_{\lambda} \right| = \left| \mathcal{L}_{\alpha,\beta,\lambda} \left( \mathcal{L}_{\alpha,\beta,\lambda} \psi_{\lambda} \right) \right| \leq \mathcal{L}^{2}_{\alpha,\beta,\lambda} |\psi_{\lambda}|
\]

and by induction on the integer \( n \geq 1 \),

\[
|\psi_{\lambda}| \leq \mathcal{L}^{2n}_{\alpha,\beta,\lambda} |\psi_{\lambda}| \quad n = 1, 2, 3, \ldots
\]

and therefore

\[
1 \leq \left\| \mathcal{L}^{2n}_{\alpha,\beta,\lambda} \right\|_{L(Y_{1})}^{\frac{1}{n}} \quad n = 1, 2, 3, \ldots
\]

Passing now to the limit \( n \to \infty \) and using (88), we get that

\[
\text{Re} \lambda \leq \lambda_{0}
\]

because (80) is a strictly decreasing (Proposition 2). Hence,

\[
s(T_{\alpha,\beta}) \leq \lambda_{0}.
\]  

(90)

**Step III.** Firstly, (89) together with (90) and (54) yield that

\[
\lambda_{s} < \lambda_{0} = s(T_{\alpha,\beta}) \leq \omega_{0}(T_{\alpha,\beta}).
\]  

(91)

Next, due to (43) and (47) we can write that

\[
0 \leq e^{-\beta t} T_{\alpha,\beta}(t) \leq U_{\alpha,\beta}(t) \quad t \geq 0
\]

which leads to

\[
-\beta + \frac{\ln \| T_{\alpha,\beta}(t) \|}{t} \leq \frac{\ln \| U_{\alpha,\beta}(t) \|}{t} \quad t \geq 0
\]

and by passing to the limit \( t \to \infty \), it follows that

\[
-\beta + \omega_{0}(T_{\alpha,\beta}) \leq \omega_{0}(U_{\alpha,\beta}).
\]  

(92)

Finally, combining (92) and (91) we get that

\[
-\beta + \lambda_{s} < -\beta + \omega_{0}(T_{\alpha,\beta}) \leq \omega_{0}(U_{\alpha,\beta})
\]

which in nothing else but the desired (87) because of \( \lambda_{s} := \bar{\sigma} - \beta \).

The type \( \omega_{0}(T_{\alpha,\beta}) \) of the unperturbed semigroup \( T_{\alpha,\beta} = (T_{\alpha,\beta}(t))_{t \geq 0} \) can be readily estimated as follows

**Corollary 3.** Suppose that (\( \tilde{H}^{1}_{1} \)) and (\( H^{2}_{1} \)) hold and let \( \alpha > 0 \) and \( 0 \leq \beta < 1 \) be such that (\( H^{2}_{1} \)) holds. If the spectral radius

\[
r_{s}(\mathcal{L}_{1,0}) > \frac{1 - \beta}{\alpha}
\]

(93)

where, \( \mathcal{L}_{1,0} \) is defined by (8), then

\[
\omega_{0}(T_{\alpha,\beta}) > 0.
\]

(94)
Proof. Suppose that \( s = r = 0 \). Then \((H_1^s), (H_1^r)\) and \((H_2^r)\) obviously hold true. Furthermore, \( s = s = 0 \) and by (55) and (93) we get that
\[
\alpha \left( \frac{1}{1 - \beta} \right) L_{1, \alpha, 0} = \alpha \left( \frac{1}{1 - \beta} \right) r \in (L_{1, \alpha, 0}) > \frac{1 - \beta}{\alpha} = 1
\]
and therefore (86) is fulfilled. Now, Theorem 4 completes the proof.

6. Remarks and comments. Let us end this work with some Remarks. The first one is

Remark 3. The average number \( \alpha \) was always assumed to be positive \((i.e., \alpha \geq 0)\) except in Theorem 4 which requires \( \alpha > 0 \). This follows from (86) together with (56) because of
\[
1 < r \sigma \left( L_{1, \alpha, \beta, \lambda} \right) \leq \left\| L_{1, \alpha, \beta, \lambda} \right\|_{C(L_{1, \alpha, \beta, \lambda})} \leq \frac{\alpha}{(1 - \beta)} \tau_0.
\]

Remark 4. The unique role of the hypothesis \((H_2^r)\) is to insure the irreducibility of the integral operator \( L_{1, \alpha, \beta, \lambda} \) defined by (55). Accordingly, this hypothesis can be weakened without losing any results of this work.

Remark 5. The hypothesis \((H_2^s)\) can easily be replaced by
\[
(H_2^s) : \quad \text{ess inf} \left( \int_0^\infty s(\mu, v', v)dv' \right) \geq 0.
\]
which is obviously less strong than \((H_2^s)\).

Remark 6. Suppose that the kernel \( \tau \) can be decomposed as follows
\[
\tau(v, v') = (f_\tau \otimes g_\tau)(v, v') = f_\tau(v)g_\tau(v').
\]
Then, the hypotheses \((H_1^s)\) and \((\hat{H}_1^s)\) becomes
\[
(H_1^s) : \quad \tau_0 := \left[ \text{ess sup}_{v' \geq 0} \left| g_\tau(v') \right| \right] \left[ \int_0^\infty \left| f_\tau(v) \right| dv \right] < \infty
\]
\[
(\hat{H}_1^s) : \quad \hat{\tau}_0 := \left[ \text{ess sup}_{v' \geq 0} \left| g_\tau(v') \right| \right] \left[ \int_0^\infty \left| f_\tau(v) \right| dv \right] < \infty.
\]
That is to say that, both hypotheses \((H_1^s)\) and \((\hat{H}_1^s)\) are the same. Furthermore, (55) becomes
\[
\cal{L}_{\alpha, \beta, \lambda} \psi(v) := \frac{\alpha}{v^\beta} f_\tau(v) \int_0^\infty g_\tau(v') \psi(v')dv'
\]
which is obviously a rank one operator into \( L_1 \).

Remark 7. We point out that an addendum was attached to [2].

Remark 8. The choice of \( L_1 \) were natural. Indeed, if \( f(t, \mu, v) \) denotes the bacterial density, at time \( t \geq 0 \), with respect to the degree of maturity \( \mu \) and the maturation velocity \( v \), then
\[
\| f(t) \|_1 = \int_0^1 \int_0^\infty |f(t, \mu, v)| dv d\mu
\]
denotes the number of all bacteria at time \( t \geq 0 \). Even so, we claim that all results of this work can easily be extended to \( L^p(\Omega) \) \((p > 1)\). It then suffices to update all hypotheses to the desired context \( L^p(\Omega) \) \((p > 1)\).
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