Robust Output Feedback Consensus for Networked Heterogeneous Nonlinear Negative-Imaginary Systems

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Abstract—This paper provides a control protocol for the robust output feedback consensus of networked heterogeneous nonlinear negative-imaginary (NI) systems. Heterogeneous nonlinear output strictly negative-imaginary (OSNI) controllers are applied in positive feedback according to the network topology to achieve output feedback consensus. The main contribution of this paper is extending the previous studies of the robust output feedback consensus problem for networked heterogeneous linear NI systems to nonlinear NI systems. Output feedback consensus is proved by investigating the internal stability of the closed-loop interconnection of the network of heterogeneous nonlinear NI plants and the network of heterogeneous nonlinear OSNI controllers through the network topology. The network of heterogeneous nonlinear NI systems is proved to be also a nonlinear NI system, and the network of heterogeneous nonlinear OSNI systems is proved to be a nonlinear NI system. Under suitable conditions, the nonlinear OSNI controllers lead to the convergence of the outputs of all nonlinear NI plants to a common limit trajectory, regardless of the system model of each plant. Hence, the protocol is robust with respect to uncertainty in the system models of the heterogeneous nonlinear NI plants in the network. This paper also describes some typical first-order and second-order nonlinear OSNI systems that can be used as controllers for the robust output feedback consensus of heterogeneous nonlinear NI plants.

Index Terms—nonlinear negative-imaginary systems, nonlinear output strictly negative-imaginary systems, heterogeneous systems, consensus, robust control.

I. INTRODUCTION

Negative-imaginary (NI) systems theory was introduced by Lanzon and Petersen in 2008 [1]. NI systems theory has attracted a lot of interest among control theory researchers (see [2]–[7], etc.). NI systems theory complements positive-real (PR) systems theory because it can be applied to systems with a relative degree from zero to two, while PR systems theory can only deal with systems with relative degree of zero or one. Typical NI systems are mechanical systems with co-located force actuators and position sensors. Positive-position feedback control is often used for NI systems, which can be applied to flexible structures with highly resonant dynamics due to the robustness of NI systems with respect to uncertainty in system models and external disturbances. NI systems theory has already achieved success in some fields, such as nano-positioning (see [8]–[10], etc.).

NI systems theory was recently extended to nonlinear systems [11]. A system is said to be nonlinear NI if it is dissipative with the supply rate $w = u^T \dot{y}$, where $u$ and $y$ are the input and output of the system, respectively, and $y$ only depends on the system state $x$. While the positive-feedback interconnection of a linear NI system and a linear strictly negative-imaginary (SNI) system is asymptotically stable if their cascaded DC gain is less than unity, the positive-feedback interconnection of a nonlinear NI system and a so-called weak strictly nonlinear NI system is proved in [11] to be also asymptotically stable under reasonable assumptions.

A class of NI systems called output strictly negative-imaginary (OSNI) systems was introduced in [12] and [13] for linear systems and was recently extended to nonlinear systems in [14]. A nonlinear system is said to be nonlinear OSNI if it is dissipative with the supply rate $w(u, \dot{y}) = u^T \dot{y} - \epsilon |y|^2$, where $u$, $x$ and $y$ are the input, state and output of the system, respectively. Also, $y$ is only dependent on $x$. Here, $\epsilon > 0$ is an index that quantifies the level of output strictness of the system. It is proved in [14] that the closed-loop interconnection of a nonlinear NI system and a nonlinear OSNI system is asymptotically stable under certain conditions.

A robust cooperative control problem for networked heterogeneous NI systems is investigated in [15] for linear systems. A network of systems is said to have output feedback consensus if the outputs of all subsystems converge to a common limit trajectory under the effect of the network communication between subsystems. With certain conditions satisfied, the outputs of heterogeneous linear NI systems connected according to an undirected connected graph can converge to the same limit trajectory if edge-based linear SNI controllers are connected to the plants in positive feedback according to the network topology.

This paper extends the investigation of the output feedback consensus problem for networked heterogeneous linear NI systems in [15] to nonlinear NI systems by using the results in [11] and [14]. Output feedback consensus of networked heterogeneous nonlinear NI systems is proved by analysing the stability of the closed-loop interconnection of a network of heterogeneous nonlinear NI plants and a network of heterogeneous nonlinear OSNI controllers. The main contribution of this paper is providing a control framework to synchronise multiple heterogeneous nonlinear NI systems under certain conditions. The protocol is robust with respect to uncertainty in the system models of the heterogeneous nonlinear NI systems. This paper is also applicable to real-world control systems considering that differences in system
models are inevitable in a network of real-world plants or controllers due to manufacturing uncertainties, even if they are designed to be identical. This paper provides theoretical support for the feasibility of real-world synchronisation problems of nonlinear NI systems when uncertainties are taken into consideration.

Notation: The notation in this paper is standard. \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers, respectively. \( \mathbb{R}^{m \times n} \) and \( \mathbb{C}^{m \times n} \) denote the spaces of real and complex matrices of dimension \( m \times n \), respectively. \( A^T \) denotes the transpose of matrix \( A \). \( \mathbb{E} \) denotes a constant vector or scalar. \( I_n \) is the \( n \times n \) identity matrix. \( A \otimes B \) denotes the Kronecker product of matrices \( A \) and \( B \). For a nonlinear system \( H \) with input \( u \) and output \( y \), \( y = H(u) \) describes its input-output relationship.

Graph theory preliminaries: \( G = (V, E) \), where \( V = \{v_1, v_2, \ldots, v_n\} \) and \( E = \{e_1, e_2, \ldots, e_l\} \subseteq V \times V \), describes an undirected graph with \( n \) nodes and \( l \) edges. The symmetric adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) is defined so that \( a_{ii} = 0 \), and \( \forall i \neq j \), \( a_{ij} = 1 \) if \( (v_i, v_j) \in E \) and \( a_{ij} = 0 \) otherwise. A sequence of unrepeated edges in \( E \) that joins a sequence of nodes in \( V \) defines a path. An undirected graph is connected if there is a path between every pair of nodes. Given an undirected graph \( G \), a corresponding directed graph can be obtained by defining a direction for each edge of \( G \). The incidence matrix \( Q = [q_{ev}] \in \mathbb{R}^{l \times n} \) of a directed graph is defined so that the elements in \( Q \) are given by

\[
q_{ev} := \begin{cases} 
1 & \text{if } v \text{ is the initial vertex of edge } e, \\
-1 & \text{if } v \text{ is the terminal vertex of edge } e, \\
0 & \text{if } v \text{ does not belong to edge } e.
\end{cases}
\]

In this paper, the initial and terminal vertices of an edge in a directed graph can both send information to each other. For an undirected graph \( G \), the choice of a corresponding directed graph is not unique. However, the Laplacian matrix \( L_n \) of \( G \) has the following relationship with the incidence matrix \( Q \) of any directed graph corresponding to \( G \): \( L_n = Q^T Q \).

II. Preliminaries

Here, we recall the definitions of nonlinear negative-imaginary systems and nonlinear output strictly negative-imaginary systems.

Consider the following general nonlinear system:

\[
\dot{x}(t) = f(x(t), u(t)); \\
y(t) = h(x(t))
\]

where \( f: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p \) is a Lipschitz continuous function and \( h: \mathbb{R}^p \to \mathbb{R}^m \) is a class \( C^1 \) function.

**Definition 1:** [14] The system \( \{1, 2\} \) is said to be a nonlinear negative-imaginary system if there exists a positive definite storage function \( V : \mathbb{R}^p \to \mathbb{R} \) of class \( C^1 \) such that

\[
\dot{V}(x(t)) \leq u(t)^T \dot{y}(t)
\]

for all \( t \geq 0 \).

**Definition 2:** [14] The system \( \{1, 2\} \) is said to be a nonlinear output strictly negative-imaginary system if there exists a positive definite storage function \( V : \mathbb{R}^p \to \mathbb{R} \) of class \( C^1 \) and a constant \( \epsilon > 0 \) such that

\[
\dot{V}(x(t)) \leq u(t)^T \dot{y}(t) - \epsilon |\dot{y}(t)|^2
\]

for all \( t \geq 0 \). The index \( \epsilon \) quantifies the level of output strictness of the system.

III. Robust Output Feedback Consensus

Consider \( n \) heterogeneous nonlinear systems \( H_{pi} (i = 1, 2, \ldots, n) \) described as

\[
\dot{x}_{pi}(t) = f_{pi}(x_{pi}(t), u_{pi}(t)); \\
y_{pi}(t) = h_{pi}(x_{pi}(t))
\]

where \( f_{pi}: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p \) is a Lipschitz continuous function and \( h_{pi}: \mathbb{R}^p \to \mathbb{R}^m \) is a class \( C^1 \) function. They operate independently in parallel and each of them has its own input \( u_{pi} \in \mathbb{R}^m \) and output \( y_{pi} \in \mathbb{R}^m \), \( (i = 1, 2, \ldots, n) \), which is shown in Fig. 1. The subscript “p” indicates that this system will play the role of a plant in what follows. We combine the inputs and outputs respectively as the vectors

\[
U_p = \begin{bmatrix} u_{p1} \\
u_{p2} \\
\vdots \\
u_{pn} \end{bmatrix} \in \mathbb{R}^{nm \times 1}, \quad \text{and} \quad Y_p = \begin{bmatrix} y_{p1} \\
y_{p2} \\
\vdots \\
y_{pn} \end{bmatrix} \in \mathbb{R}^{nm \times 1}.
\]

**Lemma 1:** If the \( H_{pi} \) are nonlinear NI systems for all \( i = 1, 2, \ldots, n \), then \( H_p \) is also a nonlinear NI system.

**Proof:** According to Definition 1, each nonlinear NI system \( H_{pi} (i = 1, 2, \ldots, n) \) must have a corresponding positive definite storage function \( V_{pi}(x_{pi}) \) such that

\[
\dot{V}_{pi}(x_{pi}) \leq u_{pi}^T \dot{y}_{pi}, \quad \text{where } x_{pi} \text{ is the state of the system } H_{pi}.
\]

We define the storage function for the system \( H_p \) as

\[
\dot{V}_p = \sum_{i=1}^{n} \dot{V}_{pi}(x_{pi}), \quad \text{which is positive definite. Then}
\]

\[
\dot{V}_p \leq \sum_{i=1}^{n} u_{pi}^T \dot{y}_{pi} = U_p^T \dot{Y}_p,
\]

which implies the NI inequality (3). Therefore, \( H_p \) is a nonlinear NI system.

Now we give a definition of output feedback consensus for a network of systems as shown in Fig. 1.
**Definition 3:** A distributed output feedback control law achieves output feedback consensus for a network of systems if \( y_{pi}(t) - y_{ss}(t) \to 0 \) as \( t \to +\infty \), \( \forall i \in \{1, 2, \ldots, n\} \). Here, \( y_{ss}(t) \) is the limit trajectory.

Consider a series of heterogeneous nonlinear OSNI systems \( H_{ck} \) \((k = 1, 2, \ldots, l)\) applied as controllers corresponding to the edges in the network. The OSNI controllers have the following state-space models:

\[
\dot{x}_{ck}(t) = f_{ck}(x_{ck}(t), u_{ck}(t)); \quad (8) \\
y_{ck}(t) = h_{ck}(x_{ck}(t)), \quad (9)
\]

where \( f_{ck} : \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^q \) is a Lipschitz continuous function and \( h_{ck} : \mathbb{R}^q \to \mathbb{R}^m \) is a class \( C^1 \) function. They operate independently in parallel and each of them has its own input \( u_{ck} \in \mathbb{R}^m \) and output \( y_{ck} \in \mathbb{R}^m \), \((k = 1, 2, \ldots, l)\), which is shown in Fig. 2. We combine the inputs and outputs respectively as the vectors

\[
U_c = \begin{bmatrix} u_{c1} \\ u_{c2} \\ \vdots \\ u_{cl} \end{bmatrix} \in \mathbb{R}^{lm \times 1}, \quad \text{and} \quad Y_c = \begin{bmatrix} y_{c1} \\ y_{c2} \\ \vdots \\ y_{cl} \end{bmatrix} \in \mathbb{R}^{lm \times 1}.
\]

![Fig. 2. System \( \mathcal{H}_c \): a nonlinear system consisting of \( l \) independent and heterogeneous nonlinear systems \( H_{ck} \) \((k = 1, 2, \ldots, l)\), with independent inputs and outputs combined as the input and output of the networked system \( \mathcal{H}_c \).](image)

Let us consider the networked controllers connected according to the graph network topology \( \mathcal{H}_c \) as shown in Fig. 3 where \( Q \) is the incidence matrix of a directed graph that represents the communication links between the heterogeneous nonlinear NI plants.

\[
\dot{U}_c = Q \otimes I_m U_c, \quad \text{and} \quad \dot{Y}_c = Q^T \otimes I_m \dot{Y}_c
\]

![Fig. 3. Heterogeneous nonlinear OSNI controllers connected according to the directed graph network topology.](image)

For the system \( \mathcal{H}_c \), we have the following lemma.

**Lemma 2:** If the \( H_{ck} \) are nonlinear OSNI systems for all \( k = 1, 2, \ldots, l \), then the system \( \mathcal{H}_c \) is a nonlinear NI system.

\[
\dot{c}_k = 1_k, \quad \dot{h}_c = \frac{\partial f_{ck}}{\partial x_{ck}} c_k + \frac{\partial f_{ck}}{\partial u_{ck}} u_c
\]

**Proof:** Let \( \hat{u}_{ci} \) denote the \( i \)-th \((i = 1, 2, \ldots, n)\) vector in the input \( \hat{U}_c \) and let \( \hat{y}_{ci} \) denote the \( i \)-th \((i = 1, 2, \ldots, l)\) vector in the output \( \hat{Y}_c \) of the system \( \mathcal{H}_c \). Let \( u_{ck} \) denote the \( k \)-th \((k = 1, 2, \ldots, l)\) \((m \times 1)\) vector in the input \( U_c \) and let \( y_{ck} \) denote the \( k \)-th \((m \times 1)\) vector in the output \( Y_c \) of the system \( \mathcal{H}_c \). Using the properties of the incidence matrix \( Q \), the following equations are obtained:

\[
u_{ck} = \sum_{i=1}^{n} q_{ki} \hat{u}_{ci}, \quad \hat{y}_{ci} = \sum_{k=1}^{l} q_{ki} \hat{y}_{ck}.
\]

For every nonlinear OSNI system \( H_{ck} \), we have a positive definite storage function \( V_{ck}(x_{ck}) \) and a constant index \( \epsilon_k > 0 \) such that

\[
\dot{V}_{ck}(x_{ck}) \leq \sum_{i=1}^{n} q_{ki} \hat{u}_{ci}^T \hat{y}_{ci} - \epsilon_k |\hat{y}_{ck}|^2,
\]

where \( \epsilon_k \) is the level of output strictness of the system \( H_{ck} \). From (10) and (11), we obtain

\[
\dot{V}_{ck}(x_{ck}) \leq \sum_{i=1}^{n} q_{ki} \hat{u}_{ci}^T \hat{y}_{ci} - \epsilon_k |\hat{y}_{ck}|^2.
\]

For the system \( \mathcal{H}_c \), we define its storage function \( \hat{V}_c \) as the sum of the storage functions of all the networked controllers:

\[
\hat{V}_c := \sum_{k=1}^{l} V_{ck}(x_{ck}) > 0.
\]

The time derivative of \( \hat{V}_c \) satisfies the following inequality due to (10) and (12):

\[
\dot{\hat{V}}_c \leq \sum_{k=1}^{l} \dot{V}_{ck}(x_{ck}) \leq \sum_{k=1}^{l} \sum_{i=1}^{n} q_{ki} \hat{u}_{ci}^T \hat{y}_{ci} - \sum_{k=1}^{l} \epsilon_k |\hat{y}_{ck}|^2
\]

\[
= \sum_{i=1}^{n} \sum_{k=1}^{l} q_{ki} \hat{y}_{ck} - \sum_{k=1}^{l} \epsilon_k |\hat{y}_{ck}|^2
\]

\[
= \sum_{i=1}^{n} \hat{u}_{ci}^T \hat{y}_{ci} - \sum_{k=1}^{l} \epsilon_k |\hat{y}_{ck}|^2.
\]

Hence, the system \( \mathcal{H}_c \) satisfies the definition of a nonlinear NI system. In addition, the term \( \sum_{k=1}^{l} \epsilon_k |\hat{y}_{ck}|^2 \) represents a non-negative output dissipation that comes from all of the controllers. This completes the proof.

We assume that the following conditions are satisfied.

\[\text{Assumption I:} \quad \text{Over any time interval } [t_a, t_b] \text{ where } t_b > t_a, \ y_{ck}(t) \text{ remains constant if and only if } x_{ck}(t) \text{ remains constant; i.e., } \ y_{ck}(t) \equiv 0 \iff \ x_{ck}(t) \equiv 0.\]

\[\text{Assumption II:} \quad \text{Over any time interval } [t_a, t_b] \text{ where } t_b > t_a, \ x_{ck}(t) \text{ remains constant only if } u_{ck}(t) \text{ remains constant; i.e., } \ x_{ck}(t) \equiv 0 \iff u_{ck}(t) \equiv 0.\]
Assumption III: In the single-input single-output (SISO) case, if the system $H_{ck}$ is in steady state; i.e., $u_{ck}(t) \equiv \bar{u}_{ck}$, $x_{ck}(t) \equiv \bar{x}_{ck}$ and $y_{ck}(t) \equiv \bar{y}_{ck}$, then $\bar{y}_{ck} > 0 \iff \bar{u}_{ck} > 0$, $\bar{y}_{ck} = 0 \iff \bar{u}_{ck} = 0$ and $\bar{y}_{ck} < 0 \iff \bar{u}_{ck} < 0$.

![Fig. 4](image-url) Open-loop interconnection of the networked nonlinear NI plants $\mathcal{H}_p$ and the networked nonlinear OSNI controllers $\mathcal{H}_c$.

b) For the open-loop interconnection of the systems $\mathcal{H}_p$ and $\mathcal{H}_c$ shown in Fig. 4, suppose:

Assumption IV: Given any constant input $U_p(t) = \bar{U}_p$ for the system $\mathcal{H}_p$, we obtain a corresponding output $Y_p(t)$, which is not necessarily constant. Given $Y_p(t)$ as input $\bar{U}_c(t)$ to the system $\mathcal{H}_c$, if the corresponding output of the system $\mathcal{H}_c$ is a constant $\bar{Y}_c(t) = \bar{Y}_5$, then there exists a constant $\gamma \in (0, 1)$ such that $\bar{U}_p$ and $\bar{Y}_c$ satisfy

$$\bar{U}_p^T \bar{Y}_c \leq \gamma |\bar{U}_p|^2. \quad (14)$$

Now consider the closed-loop interconnection of the networked plants shown in Fig. 1 and the networked controllers shown in Fig. 3 in positive output feedback, which is depicted in Fig. 5. In this paper, the robust output consensus of heterogeneous nonlinear NI plants is achieved by constructing a control system with the block diagram shown in Fig. 5 and choosing suitable controllers that satisfy certain conditions. The connections between the plants and controllers can be better visualised from the undirected graph, as shown in the example in Fig. 6.

![Fig. 5](image-url) Positive feedback interconnection of heterogeneous nonlinear NI plants and nonlinear OSNI controllers according to the directed graph network topology.

The nodes $p_i$ ($i = 1, \cdots, 5$ in this example) represent the heterogeneous nonlinear NI plants, while the heterogeneous nonlinear OSNI controllers $c_k$ ($k = 1, \cdots, 5$ in this example) correspond to the edges. Given any directed graph corresponding to the graph in Fig. 6 with the incidence matrix $Q$, each edge will have a direction. Then the corresponding connection between the plants and the controller is as shown in Fig. 7. The controller takes the difference between the outputs of the plants as its input and feeds back its output to the plants with a positive or negative sign corresponding to the edge direction. Each plant takes the sum of all the outputs of the controllers connected to it with correct signs as its input.

![Fig. 6](image-url) An example of the networked connection of plants and controllers.

![Fig. 7](image-url) Detailed block diagram corresponding to a pair of nodes connected by an edge.

For simplicity, we consider SISO systems (with $m = 1$) in the following theorem.

**Theorem 1:** Consider an undirected connected graph $G$ that models the communication links for a network of heterogeneous nonlinear NI systems $H_{pi}$ ($i = 1, 2, \cdots, n$) as shown in Fig. 1 and any directed graph corresponding to $G$ with the incidence matrix $Q$. Also, consider the heterogeneous nonlinear OSNI control laws $H_{ck}$ ($k = 1, 2, \cdots, l$) for all the edges. Suppose Assumptions I-IV are satisfied and the storage function, defined as

$$W := V_p + \dot{V}_c - Y_p^T \dot{Y}_c,$$

is positive definite, where $V_p$ and $\dot{V}_c$ are positive definite storage functions that satisfy (1) for the system $\mathcal{H}_p$ and (13) for the system $\mathcal{H}_c$, respectively. Here, $Y_p$ and $\dot{Y}_c$ are outputs of the systems $\mathcal{H}_p$ and $\mathcal{H}_c$, respectively. Then the robust output feedback consensus can be achieved via the protocol

$$U_p = (Q^T \otimes I_m) \mathcal{H}_c ((Q \otimes I_m) Y_p),$$

or equivalently,

$$u_{pi} = \sum_{k=1}^{l} q_{ki} H_{ck} \left( \sum_{j=1}^{n} q_{kj} y_{pj} \right) ,$$

for each plant $i$, as shown in Fig. 3, where $\sum_{j=1}^{n} q_{kj} y_{pj}$ represents the difference between the outputs of the plants connected by the edge $e_{kj}$.

**Proof:** We apply the Lyapunov’s direct method and take the time derivative of the storage function $W$. According to
(7) and (13), we have
\[
W = V_p + \dot{\hat{y}}_c - T_p \dot{\hat{y}}_c - y_p \dot{\hat{y}}_c
\]
\[
= V_p + \dot{\hat{y}}_c - U_p \dot{\hat{y}}_c - \dot{U}_p \dot{\hat{y}}_c
\]
\[
\leq - \sum_{k=1}^l \epsilon_k |\dot{y}_{ck}|^2
\]
\[
\leq 0. \quad (15)
\]
Hence, the closed-loop system is at least Lyapunov stable. Now we apply LaSalle’s invariance principle. According to (15), \( \dot{W} \) can remain zero only if \( \sum_{k=1}^l \epsilon_k |\dot{y}_{ck}|^2 \) remains zero, which means \( \dot{y}_{ck}(t) \) remains zero for all \( k = 1, 2, \ldots, l \).

According to Assumptions I and II, for the system \( H_{ck} \), \( \dot{y}_{ck}(t) = 0 \) implies \( \dot{x}_{ck}(t) = 0 \), which holds only if \( \dot{u}_{ck}(t) = 0 \). In other words, for all \( k = 1, 2, \ldots, l \), the controllers \( H_{ck} \) are in steady-state; i.e., \( \dot{u}_{ck}(t) = \bar{u}_{ck} \), \( x_{ck}(t) = \bar{x}_{ck} \) and \( \dot{y}_{ck}(t) = \bar{y}_{ck} \). Consider the setting in Fig. 5 in which \( U_c(t) \), \( Y_c(t) \) and \( \dot{Y}_c(t) \) are all constant vectors; i.e., \( U_c(t) \equiv \dot{U}_c \), \( Y_c(t) \equiv \dot{Y}_c \) and \( \dot{Y}_c(t) \equiv \dot{\dot{Y}}_c \). Therefore, \( U_p(t) \equiv \dot{U}_p \) also remains constant. \( U_c(t) \) and \( \dot{Y}_c(t) \) are not necessarily constant. According to the closed-loop setting that \( U_p \equiv \dot{\dot{Y}}_c \), the inequality (14) implies
\[
\dot{U}_p^T \dot{\dot{Y}}_c = |\dot{U}_p|^2 \leq \gamma |\dot{U}_p|^2 .
\]
This condition can only hold if \( \dot{U}_p = 0 \), which implies \( \dot{\dot{Y}}_c = 0 \).

We will now show that since the controllers \( H_{ck} \) are in steady-state for all \( k = 1, 2, \ldots, n \), \( \dot{Y}_c = 0 \) implies \( \dot{y}_{ck} = 0 \) for all \( k = 1, 2, \ldots, l \), according to Assumption III. We will show this by contradiction. Indeed, suppose \( \exists k \in \{1, 2, \ldots, n\} \) such that \( \dot{y}_{ck} \neq 0 \), then we have \( \dot{u}_{ck} \neq 0 \), according to Assumption III. This implies \( \forall i, j \in \{1, 2, \ldots, n\} \) such that \( \dot{y}_{pi} \neq \dot{y}_{pj} \). Consider the plants whose output is equal to the maximum of all the plant outputs. Since the graph is connected and the outputs of all the plants are not the same, there must be at least one plant \( p_e \) with the maximum output \( \dot{y}_{pr} = \max\{\dot{y}_{p1}, \dot{y}_{p2}, \ldots, \dot{y}_{pn}\} \) connected with a plant \( p_s \) with the output \( \dot{y}_{ps} < \dot{y}_{pr} \) by an edge \( e_w \). Then, we have \( \dot{u}_{cw} = \dot{y}_{pr} - \dot{y}_{ps} > 0 \) if \( v_r \) is the initial vertex of \( e_w \) and \( \dot{u}_{cw} = \dot{y}_{ps} - \dot{y}_{pr} < 0 \) if \( v_r \) is the terminal vertex of \( e_w \). According to Assumption III, \( \dot{u}_{cw} > 0 \iff \dot{y}_{cw} > 0 \) and \( \dot{u}_{cw} < 0 \iff \dot{y}_{cw} < 0 \). According to the distributed control protocol, \( \dot{u}_{pr} = \dot{y}_{cr} \) is the sum of all \( \dot{y}_{cm} > 0 \) if \( \nu_r \) is the initial vertex of \( e_m \) minus the sum of all \( \dot{y}_{cm} < 0 \) if \( \nu_r \) is the terminal vertex of \( e_m \). Therefore, \( \dot{u}_{pr} = \dot{y}_{cr} \) is positive and this contradicts the condition \( \dot{Y}_c = 0 \). Thus, we can conclude that \( \dot{y}_{ck} = 0 \) for \( k = 1, 2, \ldots, l \).

According to Assumption III, \( \dot{y}_{ck} = 0 \) implies \( \dot{u}_{ck} = 0 \), which implies \( \dot{y}_{ps}(t) = \dot{y}_{ps}(t) \) for all \( (v_i, v_j) \in E \). This means the output consensus is achieved for all the heterogeneous nonlinear NI plants. Otherwise, \( \dot{W} \) cannot remain at zero and \( \dot{W} \) will keep decreasing until \( \dot{W} = 0 \) or the states of all the plants \( p_i \ (i = 1, 2, \ldots, n) \) converge to zero, which also implies the output consensus. This completes the proof.

Remark 1: The protocol in Theorem 1 is robust with respect to uncertainty in system models for heterogeneous nonlinear NI plants connected in a network. Indeed, the output consensus can always be achieved with the protocol in Theorem 1, regardless of the system models for the heterogeneous nonlinear NI systems connected in the network.

We now provide some typical first-order and second-order dynamical systems as possible choices for nonlinear OSNI controllers.

Lemma 3: Consider a first-order system with the state-space model:
\[
\dot{x}(t) = \rho(x(t)) + \alpha u(t);
\]
\[
y(t) = x(t)
\]
where \( x(t), u(t) \) and \( y(t) \) are scalar functions of time, \( \rho: \mathbb{R} \rightarrow \mathbb{R} \) is a Lipschitz continuous function and \( \alpha > 0 \) is a constant. If the function \( V \), given by \( V(x) = -\frac{1}{\alpha} \int_0^x \rho(z)dz \) is positive definite, then the system is nonlinear OSNI with level of strictness \( \epsilon \in (0, \frac{1}{2\alpha}] \) and with \( V \) being a storage function.

Proof: Let us define \( D(x) = \dot{V}(x) - (u\dot{y} - \epsilon y^2) \). We prove in the following that \( D(x) \leq 0 \) for \( \epsilon \in (0, \frac{1}{2\alpha}] \).

\[
D(x) = \dot{V}(x) - (u\dot{y} - \epsilon y^2)
\]
\[
= \frac{\partial V(x)}{\partial x} \dot{x} - u\dot{x} + \epsilon \dot{x}^2
\]
\[
= \dot{x} \left[ \frac{\partial V(x)}{\partial x} - u + \epsilon \dot{x} \right]
\]
\[
= (\rho(x) + \alpha u) \left[ -\frac{1}{\alpha} \rho(x) - u + \epsilon (\rho(x) + \alpha u) \right]
\]
\[
= \left( \epsilon - \frac{1}{\alpha} \right) (\rho(x) + \alpha u)^2
\]
\[
\leq 0
\]
when \( \epsilon \in (0, \frac{1}{2\alpha}] \). Therefore, this system is a nonlinear OSNI system according to Definition 2.

Lemma 4: Consider a second order system with the following state-space model:
\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix}
x_2(t) \\
\eta(x_1(t)) - \beta x_2(t) + \alpha u(t)
\end{bmatrix};
\]
\[
y(t) = x_1(t)
\]
where \( x_1(t), x_2(t), u(t) \) and \( y(t) \) are scalar functions of time, \( \eta: \mathbb{R} \rightarrow \mathbb{R} \) is a Lipschitz continuous function and \( \alpha > 0 \) and \( \beta > 0 \) are constants. If the function \( V \), given by \( V(x_1, x_2) = -\frac{1}{\alpha} \int_0^{x_2} \eta(z)dz + \frac{1}{\alpha} \beta^2 x_2^2 \) is positive definite, then the system is nonlinear OSNI with level of strictness \( \epsilon \in (0, \frac{1}{2\alpha}] \) and with \( V \) being a storage function.

Proof: Define \( D(x_1, x_2) = \dot{V}(x_1, x_2) - (u\dot{y} - \epsilon y^2) \).
We prove in the following that $D(x_1, x_2) \leq 0$ for $\epsilon \in (0, \frac{\beta}{\alpha}]$.

\[ D(x_1, x_2) = \dot{V}(x_1, x_2) - (u_2 - u_1)^2 = \frac{\partial V(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial V(x_1, x_2)}{\partial x_2} \dot{x}_2 - u_1 \dot{x}_1 + \epsilon x_1^2 \]

\[ = -\frac{1}{\alpha} \eta(x_1)x_2 + \frac{1}{\alpha} x_2 \left[ \eta(x_1) - \beta x_2 + \alpha u \right] - w x_2 + \epsilon x_2^2 \]

\[ = \left( \epsilon - \frac{\beta}{\alpha} \right) x_2^2 \leq 0 \]

when $\epsilon \in \left(0, \frac{\beta}{\alpha}\right)$. Hence, this system is a nonlinear OSNI system according to Definition 2.

\section*{IV. Example}

This section illustrates the robust output feedback consensus protocol described in Theorem II with an example of networked heterogeneous pendulum systems.

Consider three pendulum systems connected by an undirected connected graph $\mathcal{G}$ as shown in Fig. 8. The Laplacian matrix of graph $\mathcal{G}$ is $L_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. To obtain a directed graph corresponding to $\mathcal{G}$, we can arbitrarily decide the direction of each edge. If we decide the directions of the edges as $e_1 = (v_1, v_2)$ and $e_2 = (v_2, v_3)$, then the incidence matrix of the directed graph corresponding to $\mathcal{G}$ is $Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

These pendulum systems have the following state-space model:

\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{m} \left( -\kappa x_1 - mg \sin x_1 + u_1 \right) \\ 0 \end{bmatrix};
\]

\[
y_1 = x_1
\]

where $m$ is the mass of each bob, $l$ is the length of each rod, $\kappa$ is the spring constant of a torsional spring installed in each pivot and $g \approx 9.8 m/s^2$ is the gravitational acceleration. $m$, $l$ and $\kappa$ are different for the three heterogeneous networked pendulums. For each pendulum system, the system state $x_1$ is the counterclockwise angular displacement from the vertically downward position and $x_2$ is the system angular velocity. The system input $u$ is an external torsional force in the counterclockwise direction, and $y$ is the system output.

The system is a nonlinear NI system with the storage function $V_1(x_1, x_2) = \frac{1}{2} \kappa x_1^2 + \frac{1}{2} ml^2 x_2^2 + ml(1 - \cos x_1)$.

According to Lemma 3, we choose the following nonlinear OSNI system as the control law corresponding to each edge:

\[
\dot{x}_c = -\beta x_c - \phi x_c^3 + \alpha u_c;
\]

\[
y_c = x_c
\]

where $\beta > 0$, $\phi > 0$ and $\alpha > 0$ are constants. The nonlinear OSNI property of this system can be proved with the storage function $V_c(x_c) = \frac{\beta}{\alpha} x_c^2 + \frac{\phi}{\alpha} x_c^4$.

Suppose the pendulums have the following parameters:

- pendulum 1: $m_1 = 1 kg$, $l_1 = 0.5 m$, $\kappa_1 = 3 Nm/\text{rad}$;
- pendulum 2: $m_2 = 1.5 kg$, $l_2 = 0.3 m$, $\kappa_2 = 5 Nm/\text{rad}$;
- pendulum 3: $m_3 = 0.5 kg$, $l_3 = 0.8 m$, $\kappa_3 = 6 Nm/\text{rad}$.

The parameters for the controllers are chosen to be:

- controller 1: $\beta_1 = 10$, $\phi_1 = 15$ and $\alpha_1 = 20$;
- controller 2: $\beta_2 = 20$, $\phi_2 = 5$ and $\alpha_2 = 30$.

According to Theorem II we control the pendulums with the distributed control law: $u_{p1} = H_{c1}(y_{p1} - y_{p2})$, $u_{p2} = -H_{c1}(y_{p1} - y_{p2}) + H_{c2}(y_{p2} - y_{p3})$ and $u_{p3} = -H_{c2}(y_{p2} - y_{p3})$, respectively. Here, $H_{ck}(\cdot)$ represents the output of controller $c_k$. It can be verified that Assumptions I-IV are satisfied and the storage function of the entire networked system is positive definite. As shown in Fig. 9, the pendulum systems approach the same limit trajectory under the effect of the heterogeneous nonlinear OSNI controllers.

\section*{V. Conclusion}

This paper provides a protocol for the output feedback consensus problem of heterogeneous nonlinear NI systems. For a network of heterogeneous nonlinear NI systems connected by an undirected and connected graph, heterogeneous edge-based nonlinear OSNI controllers can be applied in positive feedback through a network topology leading to convergence of the outputs of the nonlinear NI plants to a common limit trajectory if certain conditions are satisfied. This protocol is robust with respect to uncertainty in the system models of the nonlinear NI plants and the nonlinear OSNI controllers so that any network of heterogeneous nonlinear NI systems can be synchronised with suitable nonlinear OSNI controllers that are not necessarily identical. Some typical first-order and second-order nonlinear systems are also provided as possible choices for nonlinear OSNI controllers.
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