Abstract
The shadow of a rotating black hole with nonvanishing gravitomagnetic charge has been studied. It was shown that in addition to the angular momentum of black hole the gravitomagnetic charge term deforms the shape of the black hole shadow. From the numerical results we have obtained that for a given value of the rotation parameter, the presence of a gravitomagnetic charge enlarges the shadow and reduces its deformation with respect to the spacetime without gravitomagnetic charge. Finally we have studied the capture cross section for massive particles by black hole with the nonvanishing gravitomagnetic charge.

Keywords Photon motion Shadow of Black hole NUT spacetime

1 Introduction
Until now it is not found any observational proof of existence of gravitomagnetic monopole, i.e. so-called NUT (Newman et al. 1963) charge. Investigation of the massive and massless particles motion in NUT spacetime may provide tool for studying new important general relativistic effects which are associated with nondiagonal components of the metric tensor and have no Newtonian analogues. Nour-Zonoz (2004); Kagramanova et al. (2008); Morozova and Ahmedov (2009) studied the solutions for electromagnetic waves and interferometry in spacetime with NUT parameter. Aliev et al. (2008) considered Kerr-Taub-NUT spacetime with Maxwell and dilation fields. In our preceding papers (Morozova et al. 2008; Abdujabbarov et al. 2008) we have studied the plasma magnetosphere around a rotating, magnetized neutron star and charged particle motion around compact objects immersed in external magnetic field in the presence of the NUT parameter. The Penrose process in the spacetime of rotating black hole with nonvanishing gravitomagnetic charge has been considered by Abdujabbarov et al. (2011).

The geodesics of test charged (both electrically and magnetically charged) test particles in the Reissner-Nordström and Taub-NUT space-times fully analyzed by Grunau and Kagramanova (2011); Kagramanova et al. (2010); Hackmann and Lämmerzahl (2012); Kubizňák et al. (2009) considered higher-dimensional black hole spacetimes and null geodesics. In refs. (Connel et al. 2008; Frolov and Krtouš 2011; Krtouš et al. 2008) the parallel transport equations in the higher-dimensional Kerr-NUT-(A)dS spacetimes have been studied in detail. Magnetized black hole on the Taub-NUT instanton has been considered by Nedkova and Yazadjiev (2012). Virmani (2011) made detailed analyse of few geometrical properties of Taub-NUT space-time metric. Amarilla et al. (2010); Amarilla and Eiroa (2012, 2013) have studied the shadow of rotating black holes in Chern-Simons modified gravity, braneworld gravity and Kaluza-Klein model.

In this paper we consider photon motion and circular orbits around the rotating NUT black hole and its general relativistic effects which are associated with nondiagonal components of the metric tensor and have no Newtonian analogues.
shadow paying attention to the influence by gravitomagnetic charge. The paper organized as follows: first, in Sect. 2 we analyse the geodesics of massless particles in the Kerr-Taub-NUT space-time metric. Sect. 3 is devoted to study the shapes of the shadows of rotating black holes for the different values of the rotating parameter and gravitomagnetic charge. In Sect. 4 we study the capture cross section for massive particles by black hole with nonvanishing gravitomagnetic charge. We conclude and discuss all obtained results of the paper in Sect. 5. In this paper a space-like signature \((-, +, +, +)\) and a system of units in which \(G = 1 = c\) have been used. Greek indices are taken to run from 0 to 3 and Latin indices from 1 to 3; covariant derivatives are denoted with a semi-colon and partial derivatives with a comma.

2 Photon motion around rotating Taub-NUT black holes

In this section we will study massless particles motion around a rotating black hole with the total mass \(M\) in the presence of non vanishing gravitomagnetic charge. This black hole is described by the space-time metric \(\text{[Newman et al., 1963; Morozova et al., 2008; Abdujabbarov et al., 2008]}\):

\[
ds^2 = -\frac{1}{\Sigma} \left( \Delta - a^2 \sin^2 \theta \right) dt^2 + \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{1}{\Sigma} \left[ (\Sigma + a \chi)^2 \sin^2 \theta - \chi^2 \Delta \right] d\phi^2 + \frac{2}{\Sigma} (\Delta \chi - a(\Sigma + a \chi) \sin^2 \theta) dtd\phi. \tag{1}
\]

In above expression the following notations \(\Delta, \Sigma, \chi\) and \(\Sigma\) are defined as

\[
\Delta = r^2 + a^2 - l^2 - 2Mr, \quad \Sigma = r^2 + (l + a \cos \theta)^2, \quad \chi = a \sin^2 \theta - 2l \cos \theta,
\]

and \(a\) is the specific angular momentum of the black hole \((a = J/M)\) and \(l\) is the gravitomagnetic charge. One can determine the event horizon by largest root of the equation \(\Delta = 0\). The solution has the following form

\[
r_+ = M + (M^2 - a^2 + l^2)^{1/2}. \tag{2}
\]

It is obvious that light coming from the distant source will be deflected by the gravitational influence of the black hole originated between observer and the source of the light. The deflection is going to increase with the decreasing of impact parameter of the photons and eventually the photon emitted by the distant source can be captured by black hole. At the end this effect cause no-light zone or dark zone in the sky due to the existence of black hole between observer and light source. The above mentioned dark zone is called shadow of the black hole and the shape of it is totally defined geodesics of massless particles. one can use the Hamilton-Jacobi equation to obtain the equation of motion of photons in the given space-time metric, in our case in space-time metric of rotating black hole with non vanishing gravitomagnetic charge:

\[
\frac{\partial S}{\partial \tau} = -\frac{1}{2} g^{\alpha \beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta}, \tag{3}
\]

here \(\tau\) is an affine parameter along the null-geodesics.

In order to separate variables (the separable problem of Hamilton-Jacobi equation in Kerr-Taub-NUT space-time has been studied by \text{Dadhich and Turakulov (2002)}\), one can choose the action for photons in the following form:

\[
S = \frac{1}{2} m^2 \tau - \mathcal{E} t + \mathcal{L} \phi + S_t(r) + S_\theta(\theta), \tag{4}
\]

here \(m\) is the mass of a test particle. By \(\mathcal{E}\) and \(\mathcal{L}\) we designated the energy and the angular momentum of the particle, respectively. Putting the rest mass of photon as zero \(m = 0\) one may solve the Hamilton-Jacobi equation for null-geodesics

\[
\frac{\partial S}{\partial \tau} = \sqrt{\mathcal{R}}, \tag{5}
\]

\[
\frac{\partial \phi}{\partial \tau} = \frac{a}{\Delta} \left[ (r^2 + a^2 + l^2) \mathcal{E} - a \mathcal{L} \right] + \frac{1}{\sin^2 \theta} (\mathcal{L} - \mathcal{E}), \tag{6}
\]

\[
\frac{\partial r}{\partial \tau} = \sqrt{\mathcal{R}}, \tag{7}
\]

\[
\frac{\partial \theta}{\partial \tau} = \sqrt{\Theta}, \tag{8}
\]

here \(\mathcal{R}(r)\) and \(\Theta(\theta)\) are introduced notations and they have the following form:

\[
\mathcal{R} = \left[ (r^2 + a^2 + l^2) \mathcal{E} - a \mathcal{L} \right] - \Delta \left[ \mathcal{K} + (\mathcal{L} - a \mathcal{E})^2 \right] \tag{9}
\]

\[
\Theta = \mathcal{K} + \cos^2 \theta \left[ (a^2 - \frac{4l^2}{\sin^2 \theta}) \mathcal{E}^2 - \frac{\mathcal{L}^2}{\sin^2 \theta} \right] + 4l \mathcal{E} \cos \theta \left( \mathcal{E} a - \frac{\mathcal{L}}{\sin^2 \theta} \right), \tag{10}
\]
where $\mathcal{K}$ is Carter constant. Defining the effective potential for massless particles as $(dr/d\tau)^2 = V_{\text{eff}}$ one may study the radial motion of photons in the presence of gravitomagnetic charge. In the Fig. 1 the radial dependence of the effective potential of radial photon motion is shown. From the figure it is seen that with the increase of the gravitomagnetic charge the shape of the effective potential is going to shift to the observer at infinity. This corresponds to increasing the event horizon of the Kerr-Taub-NUT black hole. Moreover, one may conclude from the Fig. 1 that with the increase of the gravitomagnetic charge the circular photon orbits become unstable.

Figure 1: The radial dependence of the effective potential of radial motion of the massless particles for the different values of the gravitomagnetic charge: solid line for $l/M = 0.1$, dashed line for $l/M = 0.5$, and dot-dashed line for $l = 0.9$.

Photon motion around rotating black hole with non vanishing gravitomagnetic charge can be described using the expressions (5)–(8). One may easily introduce the following two impact parameters:

$$\xi = L/E, \quad \eta = \mathcal{K}/E^2$$

in order to clarify the photon motion in complete way. Now it useful to use the equation (7) for defining the shape of the dark zone created by rotating black hole with non vanishing gravitomagnetic charge. The condition of being boundary of shadow is the following:

$$R(r) = 0 = dR(r)/dr.$$ 

Using this condition one can obtain the following equations:

$$\alpha = \lim_{r \to \infty} \left(-r^2 \sin \theta \frac{d\phi}{dr}\right), \quad (13)$$

and

$$\beta = \lim_{r_0 \to \infty} r_0^2 \frac{d\theta}{dr}, \quad (14)$$

if the distant observer is located at long distance from the rotating black hole with NUT charge then limit $r \to \infty$ can be used (Amarilla et al. 2010; Amarilla and Eiroa 2012, 2013). The geometrical structure of celestial coordinates is schematically shown in Fig. 2.

Calculating $d\phi/dr$ and $d\theta/dr$ using the spacetime metric (1) and putting the results into (13), (14), using (9), (7), and (8) we obtain equations for $(\alpha, \beta)$ coordinates in the following form:

$$\alpha = -\xi \csc \theta_0, \quad (15)$$

and

$$\beta = \pm \left[\eta + \frac{4l^2}{\sin^2 \theta_0} \cos^2 \theta_0 - \xi^2 \cot^2 \theta_0 + 4l \cos \theta_0 (a - \frac{\xi}{\sin^2 \theta_0})\right]^{1/2}, \quad (16)$$

Figure 2: The scheme of the gravitational lens system. A reference coordinate system $(x, y, z)$ with the black hole at the origin can be set up by an observer far away from the black hole. The straight continuation of observing light ray intersects the plane $\alpha - \beta$ at the position $(\alpha_i, \beta_i)$.

3 Kerr-Taub-NUT black hole shadow

To obtain the silhouette of the rotating black hole with non vanishing NUT charge it is very convenient to use the celestial coordinates (Vázquez and Esteban (2004)):

$$\alpha = \lim_{r \to \infty} \left(-r^2 \sin \theta \frac{d\phi}{dr}\right), \quad (13)$$

and

$$\beta = \lim_{r_0 \to \infty} r_0^2 \frac{d\theta}{dr}, \quad (14)$$

if the distant observer is located at long distance from the rotating black hole with NUT charge then limit $r \to \infty$ can be used (Amarilla et al. 2010; Amarilla and Eiroa 2012, 2013). The geometrical structure of celestial coordinates is schematically shown in Fig. 2.

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$$\beta = \pm \left[\eta + \frac{4l^2}{\sin^2 \theta_0} \cos^2 \theta_0 - \xi^2 \cot^2 \theta_0 + 4l \cos \theta_0 (a - \frac{\xi}{\sin^2 \theta_0})\right]^{1/2}, \quad (16)$$
Fig. 3 The shapes of the rotating black hole shadow with non vanishing NUT charge originated at centre of coordinates when $\theta = \pi/2$. (a): $a/M = 0.5$, (b): $a/M = 0.7$, (c): $a/M = 0.8$, and (d): $a/M = 0.99$. In all figures solid lines correspond to $l/M = 0.1$, dashed lines correspond to $l/M = 0.5$, and dashed-dotted lines correspond to $l/M = 0.9$. The region bounded by each curve corresponds to the black hole shadow.
While considering the shape of the shadow of rotating black hole with non vanishing gravitomagnetic charge one may introduce the radius $R_{sh}$ and the distortion parameter $\delta_{sh}$ of the silhouette related by the expression $\delta_{sh} = D_{cs}/R_{sh}$. The schematic explanation of these parameters are shown in Fig. 4 (Hioki and Maeda 2009). 

\[ \alpha = -\xi, \quad (17) \]

and

\[ \beta = \sqrt{\eta}. \quad (18) \]

Obtained numerical results are shown in Fig. 3 where the shape of the silhouettes of rotating Taub-NUT black hole for the different values of the rotation parameter and gravitomagnetic charge are presented. From the plots one can see that the presence of the gravitomagnetic charge will increase the effective size of the shadow. In the figure the shapes of the silhouette of rotating black hole with the gravitomagnetic charge are given for the different values of black hole angular momentum $a$: $a/M = 0.5$, $a/M = 0.7$, $a/M = 0.8$, and $a/M = 0.99$. One can easily compare the effect of the NUT parameter and the black hole rotation parameter on modification of the shape of the shadow of black hole. It appears they have opposite effects on black hole shadow size. The gravitomagnetic charge increases the size of the shadow shape while black hole’s angular momentum decreases its size. The parameters $R_{sh}$ (radius) and $\delta_{sh}$ (distortion) are shown as functions of the gravitomagnetic charge $l$ in Fig. 4. From the dependence of $R_{sh}$ from the NUT parameter one can again see that the gravitomagnetic charge forces to increase the size of the black hole shadow. The dependence of $\delta_{sh}$ from NUT charge shows that gravitomagnetic charge forces to shadow to get the shape of circle than ellipse. In the case of rotation, with the increase of black hole’s angular momentum the shape of black hole shadow takes form of ellipse rather than circle.

### 4 Particle capture cross sections for black hole with gravitomagnetic charge

In this section we will study the pure effect of NUT parameter assuming that the the angular momentum of the black hole is equal to zero. It has been shown in the paper of Abdujabbarov et al. (2008) that variables in the Hamilton-Jacobi equation for the particle motion around NUT black hole can be separated in the equatorial plane. In the space-time metric we assume that the central object is non-rotating and particles are confined at the equatorial plane $(a = 0, \text{ and } \theta = \pi/2)$. It was first shown by Zimmerman and Shahir (1989) for the spherical symmetric case (NUT spacetime) and later in the paper of Bini et al. (2003) for the axial symmetric case (Kerr-Taub-NUT spacetime) that the orbits of the test particles are confined to a cone with the opening angle $\theta$ given by $\cos \theta = 2l \mathcal{E} / \mathcal{L}$. It also follows that in this case the equations of motion on the cone depend on $l$ only via $l^2$ (Bini et al. 2003; Abdujabbarov et al. 2008).

The main point is that the small value for the upper limit for gravitomagnetic moment has been obtained by comparing theoretical results with experimental data as (i) $l/M < 10^{-24}$ from the gravitational microlensing (Rahvar and Habibi 2004), (ii) $l/M \leq 1.5 \times 10^{-18}$ from the interferometry experiments on ultracold atoms (Morozova and Ahmedov 2009), (iii) and similar limit has been obtained from the experiments on Mach-Zehnder interferometer (Kagramanova et al. 2008) (here $M$ is the total mass of central gravitating object).

Due to the smallness of the gravitomagnetic charge let us consider the motion in the quasi-equatorial plane when the motion in $\theta$ direction changes as $\theta = \pi/2 + \delta \theta(t)$, where $\delta \theta(t)$ is the term of first order in $l$, then it is easy to expand the trigonometric functions as $\sin \theta = 1 - \delta \theta^2(t)/2 + O(\delta \theta^3(t))$ and $\cos \theta = \delta \theta(t) - O(\delta \theta^3(t))$. 

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**Fig. 4** Scheme of radius and distortion of the silhouette of rotating black hole with gravitomagnetic charge. The distortion parameter is defined as $\delta_{sh} = D_{cs}/R_{sh}$, where $R_{sh}$ is the average radii of shadow and $D_{cs}$ is the distance of deviation of the shape of the shadow from circle shape. 

\[ R_{sh} \]

\[ D_{cs} \]

\[ \delta_{sh} = D_{cs}/R_{sh} \]
Neglecting the small terms $O(\delta^2(t))$, one can write $\Sigma = r^2 + l^2$, $\Delta = r^2 - 2Mr - l^2$, and $\chi = 0$ and consequently the equation of motion for the radial motion takes the following form

$$r^4 \left( \frac{dr}{d\tau} \right)^2 = R(r) = \left[ E^2 - 1 - 2U_{\text{eff}}(r, l, L) \right] r^4,$$  \hspace{1cm} (19)

where $E$ and $L$ is the energy and angular momentum of the particle per unit of its mass and the quantity

$$U_{\text{eff}}(r, l, L) = -\frac{l^2 +Mr}{\Sigma} + \frac{\Delta L^2}{2\Sigma^2} \hspace{1cm} (20)$$

can be interpreted as effective potential of the radial motion of the test particle at equatorial plane. The radial dependence of the effective potential of radial motion of the massive particles for the different values of the gravitomagnetic charge is presented in Fig. 6.

Assuming that the uncharged particle is moving slowly at infinity, i.e. $E \approx 1$ one can easily rewrite the expression (19) in the following form:

$$R(\rho) = \rho^3 + \left( \tilde{l} - \tilde{L} \right) \rho^2 + \tilde{L} \rho + \frac{\tilde{l} \tilde{L}}{2},$$  \hspace{1cm} (21)

where

$$\rho = \frac{r}{M}, \quad \tilde{l} = \left( \frac{l}{M} \right)^2, \quad \tilde{L} = \left( \frac{L}{M} \right)^2.$$.

Gravitational capture of the particle occurs for $L \leq L_{\text{cr}}$. For the $L = L_{\text{cr}}$ orbit spirals into a circular orbit, the radius of which is determined by the value of the multiple root of the polynomial (21), i.e. discriminant of the later should vanish. Now it is easy to find the expression for $L_{\text{cr}}$ as

$$L_{\text{cr}}^2 = 16M^2 - 6l^2 - \frac{13l^4}{16M^2}. \hspace{1cm} (22)$$

In the Fig 7 the dependence of $L_{\text{cr}}$ from dimensionless NUT parameter is presented. The dependence shows that the presence of the gravitomagnetic charge decreases the capture cross section for particles by black hole.

5 Conclusion

In this paper, we have studied the shadow of black hole with nonvanishing gravitomagnetic charge and analyzed how the shadow of the black hole will be distorted by the presence of the NUT parameter. From the numerical results we have obtained that the NUT parameter forces to increase the size of the black hole shadow. The dependence of the distortion parameter $\delta$,
Fig. 7  The dependence of the critical angular momentum for capturing by central black hole from gravitomagnetic monopole momentum.

from the NUT charge shows that the gravitomagnetic charge forces black hole’s shadow to get the shape of circle than ellipse. We have also studied the capture cross section for massive particles by black hole with non-vanishing gravitomagnetic charge and found its strong dependence from the NUT parameter.

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