Koopman Performance Analysis of Nonlinear Consensus Networks

Hossein K. Mousavi, Christoforos Somarakis, and Nader Motee

Abstract Spectral decomposition of dynamical systems is a popular methodology to investigate fundamental qualitative and quantitative properties of dynamical systems and their solutions. In this article, we consider a class of nonlinear cooperative protocols that consist of multiple agents that are coupled together via an undirected state-dependent graph. We develop a representation of system solution by decomposing the nonlinear system utilizing ideas from the Koopman theory and its spectral analysis. We use recent results on the well-known Hartman theorem for hyperbolic systems to establish a connection between the non-linear and the linearized dynamics in terms of Koopman spectral properties. The expected value of the output energy of the non-linear protocol, which relates to notions of coherence and robustness in dynamical networks, is evaluated and characterized in terms of Koopman eigenvalues, eigenfunctions, and modes. Spectral representation of the performance measure enables us to develop algorithmic methods to assess performance of this class of nonlinear dynamical networks as a function of their graph topology. Finally, we propose a scalable computational method for approximation of the components of the Koopman mode decomposition, that is necessary to evaluate the systemic performance of the nonlinear dynamic network.

Key words: Koopman Mode Decomposition, Consensus Systems, Spatially Decaying Couplings, Nonlinear Control, Polynomial Approximation

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1 Introduction

The central objective in the theory of networked control systems is to address and analyze practical challenges in implementations of real-world dynamical networks, in order to develop design algorithms with convincing proof certificates \([9, 19, 24, 39, 53, 55, 56]\). The application areas, nowadays, range from multirobot systems \([3]\) to social networks \([25]\), power systems \([20]\), metabolic pathways \([11, 45, 47]\), and brain networks \([8]\). One of the inherent unappealing features of these real-world networks is nonlinear interactions among the subsystems that stem from how subsystems affect each other’s dynamics \([2, 4, 13, 16, 26, 39]\). For example, in natural networks, physical interactions, such as fluid field coupling \([46]\), coupled biochemical reactions \([47]\), or visual coordination \([16]\) result in nonlinear coupling among the subsystems.

The main focus of the existing body of literature is on stability analysis of nonlinear dynamical networks, where some of them investigate effects of coupling topologies \([20, 26]\), time-delay \([41, 42, 52]\) and exogenous noise \([15]\). The common approach to deal with existing nonlinearities is to study linearized forms of network dynamics. There is a rich number of works devoted to performance and robustness analysis and optimal design of linear dynamical network \([5, 6, 18, 27, 31, 35, 36, 38, 40, 43, 48, 49, 50, 51, 54]\). Despite growing need to analyze and synthesize nonlinear dynamical network in non-equilibrium modes of operation, consistent and systematic methods to tackle these problems are sorely missing in the literature. The main reason is that linear network techniques, which are mainly based on eigen-decomposition, cannot be applied to nonlinear maps. Recent advances in Koopman methods to analyze dynamical systems \([10, 28, 29, 33, 34]\) has opened up a new venue to study properties of nonlinear systems in a systematic manner.

In this article, we build upon concepts and tools from Koopman methodology to assess the performance of a class of nonlinear consensus networks. These networks are defined over an undirected state-dependent interconnection graph topology, where control input of each agent is equal to a weighted combination of between its own state and its neighbors. The expected value of the output energy of the network is adopted as the performance measure. We obtain a closed-form series representation for this quadratic performance measure and show that the value of performance measure depends on spectra of the Koopman operator. The idea of spectral characterization of performance measure can be potentially utilized to analyze and design nonlinear networks; we refer to \([38, 50, 51, 54]\) for successfulness of this approach in linear dynamical network. An efficient numerical algorithm is developed to compute value of the performance measure for a given dynamical network. Several analytical and numerical examples have been provided to highlight the usefulness of our theoretical findings.
2 Problem Formulation

Consider an autonomous dynamical system

$$\dot{x} = F(x),$$  (1)

with $F(x): \mathbb{R}^n \to \mathbb{R}^n$ representing a $C^2$ vector field on $\mathbb{R}^n$. For the initial condition $x_0 \in \mathbb{R}^n$, $S(t, x_0): \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the generated flow of (1), assumed to be defined for all $t \geq 0$. We assume that $F$ attains a hyperbolic stable fixed point at the origin; i.e., $F(0) = 0$ and the eigenvalues of the Jacobian at the fixed point

$$A := \frac{\partial}{\partial x} F|_{x=0}$$  (2)

have strictly negative real part. The basin of attraction of the origin is an open neighborhood of 0, denoted as $\Omega \subset \mathbb{R}^n$. By definition, $S(t, x_0) \in \Omega$ for any $x_0 \in \Omega$ and $t \geq 0$, such that $S(t, x_0) \to 0$ as $t \to +\infty$. Define the functional space

$$\Psi = \{ f \in C^1(\Omega, \mathbb{C}) : \sup_{x \in \Omega} |f| < \infty \}.$$  

This will be the space of observables of the flow $S(\cdot, x_0)$. For fixed $t \geq 0$, the Koopman operator $U_t: \Psi \to \Psi$ associated with (1) is

$$(U_t f)(x_0) = f \circ S(t, x_0),$$  (3)

Among the basic properties of $U_t$ is that it is a linear and bounded operator in $\Psi$. In this work, we will discuss the role of Koopman operator theory in evaluating quadratic performance measures for a class of nonlinear consensus protocols that enjoy a great interest in the field of networked control systems.

2.1 On the Spectrum of the Koopman Operator

Our explorations rely on the spectral features of $U_t$ and their role in representing useful system observables. It can be easily established that $U_t$ is continuous in the operator topology and that the collection $\{U_t\}_{t \in \mathbb{R}_+}$ satisfies the semi-group property \cite{10}. The associated infinitesimal generator $L: \Omega \to \Omega$ of $\{U_t\}_{t \in \mathbb{R}_+}$ is defined as

$$(L\psi)(x_0) := \lim_{t \downarrow 0} \frac{(U_t \psi)(x_0) - \psi(x_0)}{t}.$$  (4)

The generator $L$ acts on $\Omega$ and it is, in turn both linear and bounded. The purpose of this section is to comment on the spectra of $U_t$ and $L$. The spectrum of $U_t$ will con-
sist the underlying framework to assist us in representing the performance metrics of interest.

We begin with the spectrum of $L$ that we label as $\sigma(L)$. For $L$ linear and bounded, $\sigma(L)$ is a non-empty and compact subset of $\mathbb{C}$. Moreover, $\sigma(L)$ is partitioned into discrete, continuous, and residual parts [21]. The discrete part is also known as the point spectrum defined as

$$\sigma_p(L) = \{ \lambda \in \mathbb{C} | L \phi = \lambda \phi \text{ for } \phi = \phi_\lambda \in \Psi \}.$$  \hspace{1cm} (5)

By construction of $\Psi$ and $F$, $U_t$ is continuous in the operator topology and this implies the relation between $L$ and $U_t$

$$U_t = e^{Lt}, \quad t \geq 0,$$  \hspace{1cm} (6)

where $e^{Lt} = \sum_{k=0}^{\infty} \frac{L^{(k)}t^k}{k!}$ and $L^{(k)}$ is the $k$th composition of $L$. The spectra of $\sigma(L)$ and $\sigma(U_t)$ are also closely related. In view of (6), it is easily shown that $(e^{\lambda t}, \phi_\lambda)$ belongs to $\sigma_p(U_t)$.

Throughout this paper, either $(\lambda, \phi_\lambda) \in \sigma_p(L)$ or $(e^{\lambda t}, \phi_\lambda)$ will be called Koopman pairs (of eigenvalues and eigenfunctions). Now, let $\{ (e^{\lambda t}, \phi_\lambda) \}_{\lambda \in \sigma_p(U_t)}$ be a countable subset of $\sigma_p(U_t)$. For an observable $f \in \Psi$ that can be represented as

$$f = \sum_{\lambda} v_\lambda \phi_\lambda,$$

we have the following identity by linearity

$$\left( U_t f \right)(x_0) = \sum_{\lambda} v_\lambda \left( U_t \phi_\lambda \right)(x_0) = \sum_{\lambda} v_\lambda e^{\lambda t} \phi_\lambda(x_0).$$  \hspace{1cm} (7)

Clearly, one should not expect any member of $\Psi$ to be representable in terms of the Koopman eigenfunctions. What is more, it is by no means given that $\sigma_p(U_t)$ is countable$^2$.

The rest of the manuscript is organized as follows. In §3 we will apply the extension of the Hartman’s theorem in order to investigate the conditions under which one is able to express the flow $S(\cdot, x_0)$ of (1) in terms of the Koopman pairs, i.e., to write the $i$th element of $S(\cdot, x_0)$ as

$$\left[ S(t, x_0) \right]_i = \sum_{\lambda} C_\lambda e^{\lambda t} \phi_\lambda(x_0), \quad t \geq 0$$

for some constant coefficients $C_\lambda = C_\lambda^{(i)}$. Any collection of triplets $(\lambda, \phi_\lambda, C_\lambda)$ (or $(e^{\lambda t}, \phi_\lambda, C_\lambda)$) will constitute a Koopman Mode Decomposition (KMD). We use an

$^1$ See for example Theorem VIII.2 in [21]

$^2$ Unless $U_t$ is at least a self-adjoint operator, it may be the case that $\sigma_p(U_t)$ is uncountable. See for example in §II.5.1 in [14].
interesting fact about the map created by stacking the eigenfunctions of the dynamical system with hyperbolic stable fixed points and its inverse map: any polynomial approximation of the inverse map results in Koopman Mode Decompositions.

Based on the results of 3, we proceed in 4 with the calculation of the performance measures for nonlinear consensus networks. The measures are expressed in infinite series form as function of KMD’s. In addition, we discuss a number of special cases where KMD’s can be explicitly calculated.

In Section 5 we describe a method to come-up with sparse approximation to the eigenfunctions of Koopman operator. The method strongly depends on a nearly-optimal fitting technique called Smolyak-Collocation projection. We use the same method to compute the approximate Koopman modes for the Koopman Mode decompositions. Using the above developments, we may derive quantitative information about the stability and performance of nonlinear dynamical systems. In fact, inspired by our previous work [37], we look at the performance measure of a class of nonlinear dynamical systems and illustrate how their performance can be assessed using the spectra of the Koopman operator.

3 Koopman-Based Representation of Flows

The celebrated theorem of Hartman (stated below for convenience) establishes an crucial connection between and the dynamics of the linearized system around the origin. A moment of reflection, initially mentioned in [29], can lay the groundwork of bridging the gap between spectral properties of the nonlinear and the linearized system around the fixed point. The aim of the present section is to conduct a rigorous discussion of these exact steps. We begin our analysis with parts adapted from the literature to keep the manuscript self-contained.

**Theorem 1.** [44] Consider the dynamical system (1) with the smoothness assumptions on \( F \) to hold and the origin to be a hyperbolic fixed point. Then there exists a \( C^1 \)-diffeomorphism \( H \) of a neighborhood \( U \) of the origin on an open set \( \Omega' \subset \Omega \) containing the origin such that for each \( x_0 \in \Omega' \), there is an open interval \( I(x_0) \subset \mathbb{R}_+ \) containing zero such that for all \( x_0 \in U \) and \( t \in I(x_0) \)

\[
H \circ S(t, x_0) = e^{\Lambda t} H(x_0),
\]

where \( \Lambda = \frac{\partial}{\partial x} F|_{x=0} \).

**Remark 1.** The set \( I(x_0) \) stands for the maximal interval of existence of the solution (1), that defines the flow \( S(t, x_0) \), for any \( t \geq 0 \). In this work we shall always have \( I(x_0) = \mathbb{R}_+ \) for all \( x_0 \in \Omega \).

The next result extends the theorem of Hartman to the basin of attraction of the origin.

**Theorem 2.** [29] If \( F \) is \( C^2 \) and \( \Lambda = \frac{\partial}{\partial x} F|_{x=0} \) is Hurwitz, then there exists a diffeomorphism \( \alpha : \Omega \to \mathbb{R}^n \) such that
\[ \alpha \circ S(t, x_0) = e^{At} \alpha(x_0), \]  
\( \text{for all } x_0 \in \Omega \text{ and } t \geq 0. \)

Using latter theorem and assuming that \( A \) is diagonalizable, we have the eigen-decomposition \( A = R \Lambda R^{-1} \) for \( \Lambda \) a diagonal matrix, with eigenvalues of strictly negative real part. Define
\[ H(x) := R^{-1} \alpha(x). \]  
Then
\[ H(S(t, x_0)) = R^{-1} e^{At} RH(x_0) = e^{At} H(x_0). \]  
Clearly, \( H: \Omega \to \Omega \) is a diffeomorphism and the flow \( S(\cdot, x_0) \) can be represented as
\[ S(t, x_0) = H^{-1} \left( e^{At} H(x_0) \right), \quad t \geq 0. \]

This suggests that the knowledge of the maps \( H \) and \( H^{-1} \) for this class of dynamical systems helps identify the flow of the system. The next results outlines the necessary bridge between \( U^t \) and \( H \).

**Lemma 1.** Let \( H \) as in (9) have the components \( H = [H_1, H_2, \ldots, H_n]^T \) where \( H_i: \Omega \to \Omega \) for all \( i = 1, \ldots, n \). If \( \lambda_i \) is the \( i \)-th left eigenvalue of \( A \), then \((e^{\lambda_i t}, H_i)\) constitutes a pair of Koopman eigenvalues and eigenfunctions.

**Proof.** The result immediately follows after comparing the definition of the Koopman eigenfunction (7) to (10).

We take advantage of this connection to provide a Koopman Mode Decomposition (KMD) for dynamical systems with a stable hyperbolic fixed point. One may find the general aspects of this decomposition in [10]. In this context, Extended Dynamic Mode Decomposition (EDMD) [57] is a framework with focus on derivation of numerical estimations to Koopman operator and its decomposition. At first, we make two crucial remarks before coming up with the advertised decomposition.

**Polynomial Expansion of \( H^{-1} \).** Clearly, all elements of \( H^{-1}(x) = \alpha^{-1}(Rx) \) are continuous in \( \Omega \), by virtue of the Stone-Weierstrass Theorem [17] every \( H_i \) can be approximated by multivariate polynomials. Thus we can write in concise form
\[ H_i^{-1}(x) = \sum_{\gamma \in \mathbb{Z}_+^n} c_\gamma x_1^{j_1} \cdots x_n^{j_n}, \]  
where \( i = 1, \ldots, n, \gamma = (j_1, \ldots, j_n)^T \in \mathbb{Z}_+^n \), and \( c_\gamma = c_{j_1, \ldots, j_n} \) is represented in the spirit of multi-index notation. There are various rigorous methods for calculation of the coefficients \( \{c_\gamma\}_\gamma \). One method is to verify that \( H^{-1} \) is analytic, hence it admits a Maclaurin expansion with a positive radius of convergence. Another method is via identifying \( c_\gamma \) as the Fourier-type projections with respect to orthogonal families of polynomials with several variables. These latter approaches have been systematically investigated in [22]. In either case we can assume without loss of generality
that a representation of the form (12) is always feasible for any \( x_0 \in \Omega \). We remark however that in numerical explorations in subsequent sections, we restrict to a neighborhood of the fixed point, for computational convenience.

**Superposition of Koopman Eigenpairs.** The second concept is the closedness of the set of Koopman eigenvalues and eigenfunction with respect to multiplication as stated in the next lemma.

**Lemma 2 ([10]).** Let \( \phi_1, \phi_2 \in \Psi \) with associated eigenvalues \( e^{\lambda_1 t} \) and \( e^{\lambda_2 t} \), respectively. Then \( \phi_3 := \phi_1 \cdot \phi_2 \in \Psi \) with associated eigenvalue \( e^{\lambda_3 t} = e^{(\lambda_1 + \lambda_2) t} \).

We are ready now to formulate a KMD-based expression for the flow \( S(\cdot, x_0) \). For its exposition we consider an arbitrary but fixed ordering of the elements of \( \mathbb{Z}^n_+ \), \( \mathbb{Z}^n_+ = \{ \gamma_1, \gamma_2, \ldots, \gamma_i, \ldots \} \) where \( \gamma_i = (j_1, j_2, \ldots, j_n)^T \).

**Proposition 1.** Let \( A = \frac{\partial}{\partial x} F(x)|_{x=0} \) be diagonalizable and Hurwitz. Consider \( H^{-1} \) with elements as in (9) for any \( x_0 \in \Omega \). The flow \( S(\cdot, x_0) \) of (1) attains the representation

\[
S(t, x_0) = \sum_{i \geq 1} c_i e^{\lambda_i t} \phi_i(x_0), \quad \text{for all } t \geq 0
\]

where for the ordered vector \( \gamma_i = (j_1, \ldots, j_n) \in \mathbb{Z}^n_+ \), we have

\[
\lambda_i := \sum_{k=1}^{n} j_k \lambda_k \quad \text{and} \quad \phi_i(x_0) := \prod_{k=1}^{n} H_{j_k}^k(x_0).
\]

**Proof.** Recall the expression (11) that is true for any \( x_0 \in \Omega \). Substituting \( e^{\lambda_i t} H(x_0) \) into the series (12), we may write the flow of (1) as

\[
S(t, x_0) = \sum_{\gamma \in \mathbb{Z}^n_+} c_{\gamma} \left( e^{\lambda_{j_1} t} H_1(x_0) \right)^{j_1} \cdots \left( e^{\lambda_{j_n} t} H_n(x_0) \right)^{j_n},
\]

which can be reorganized to obtain

\[
S(t, x_0) = \sum_{\gamma \in \mathbb{Z}^n_+} c_{\gamma} e^{(j_1 \lambda_1 + j_2 \lambda_2 + \cdots + j_n \lambda_n) t} H_1^{j_1}(x_0) \cdots H_n^{j_n}(x_0).
\]

Set \( \lambda_i = \sum_{k=1}^{n} j_k \lambda_k \) and \( \phi_i(x) = \prod_{k=1}^{n} H_{j_k}^k(x) \) and use Lemma 2 to deduce that \( \phi_i(x) \) is an eigenfunction of Koopman operator with eigenvalue \( e^{\lambda_i t} \). Rewriting the flow and using the introduced notation yields the desired representation.

In fact, we are deriving the explicit decomposition introduced in Proposition 1 by an extension of the material presented in [34] or [29]. We will see that this decomposition is a necessary tool for the subsequent analysis in the upcoming section. Before that, we recall a useful lemma from the literature, which identifies a partial differential equation (PDE) for the Koopman pairs.
Lemma 3. \([34]\) Let \((\lambda, \phi_\lambda(x))\) be a Koopman pair with respect to (1). The eigenfunction satisfies
\[
\nabla^T \phi_\lambda(x) F(x) = \lambda \phi_\lambda(x).
\]

4 Performance of Nonlinear Consensus Networks

The standard multi-agent setting regards a finite collection of agents labeled as \(i = 1, 2, \ldots, n\), for \(n < \infty\). The \(i^{th}\) agent is characterized by a real-valued state \(x_i\). In a consensus network with first order dynamics, agents update their states by communicating with their adjacent (neighboring) agents. Our focus in this work is on the class of dynamic protocols of the form
\[
\dot{x}_i = \sum_{i,j} w_{ij} (x_j - x_i),
\]
(14)
where \(\mathcal{E}\) is the set of edges of the undirected graph of the network whose weights are symmetric and state-dependent in the form of
\[
w_{ij} = w_{ji} = \tilde{w}_{ij} g(|x_i - x_j|),
\]
(15)
for \(g : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) a positive coupling function of the graph, and constant \(\tilde{w}_{ij} > 0\). We note that such a state-dependence of the couplings is motivated by a natural assumption: the remote or dissimilar agents less likely interact with each other. For instance, this is the case in the context of social networks, oscillatory networks [26] or biological networks. For this reason function \(g\) is usually considered to be monotonically decreasing [16, 53]. By defining the state of the network as \(x := [x_1, \ldots, x_n]^T \in \mathbb{R}^n\), we may express the collective dynamics of the agents as
\[
\dot{x} = -\mathcal{L}_x x
\]
where \(\mathcal{L}_x\) is the state-dependent graph Laplacian matrix with coupling weights that vary according to (15). For subsequent analysis we rely on two conditions, stated right below.

**Assumption 1** The function \(g\) is \(C^2\) and it satisfies \(g(0) = 1\).

**Assumption 2** The graph with coupling weights \(\{\tilde{w}_{ij}\}_{i,j} \in \mathcal{E}\) is connected.

Connectedness implies that there exists a path between any two distinct nodes \(i\) and \(j\) in the graph of the network. A consequence of the latter assumption is that the graph corresponding to \(\mathcal{L}_x\) remains connected and undirected for all \(x \in \mathbb{R}^n\), since \(w_{ij} > 0\) for every \(\{i, j\} \in \mathcal{E}\). The next result provides a standard sufficient condition for convergence of (14) to the average consensus.
Theorem 3. Let Assumptions 1 and 2 hold true. For any initial vector \( x_0 \in \mathbb{R}^n \) the long term dynamics satisfy

$$\lim_{t \to \infty} S(t, x_0) = \bar{x} 1_n,$$

where the average vector of the network is \( \bar{x} := \frac{1}{n} \sum_{i=1}^{n} x_i(0) \). The convergence to consensus occurs exponentially fast, with a rate that depends on the initial vector \( x_0 \).

Proof. At first, observe that

$$\max_{i,j=1,...,n} \left| x_i(t) - x_j(t) \right| \leq \max_{i,j=1,...,n} \left| x_i(0) - x_j(0) \right| \text{ for all } t \geq 0.$$

This is easily verified since for the node \( i \) with the maximum initial condition \( \max_i \dot{x}_i(0) < 0 \). Similarly for the node \( i \) with the minimum starting value \( \min_i \dot{x}_i(0) > 0 \). All in all, the solution \( x(t), t \geq 0 \) of (14), we have

$$\frac{d}{dt} \Lambda(x(t)) = \sum_{i \neq j} \left( x_i(t) - x_j(t) \right) \left( \dot{x}_i(t) - \dot{x}_j(t) \right) \leq -\beta(t) \Lambda(x(t))$$

where the value of \( \beta(t) \) is given by

$$\beta(t) := \min_{s \in [0,t]} \{ i, j \} \in E \ w_{ij} \left( x(s) \right) \geq w \cdot g > 0$$

for \( w = \min_{i,j=1,...,n} \tilde{w}_{ij} \) and \( g = \min_{s_1,s_2 \in \Omega_0} g(|s_1 - s_2|) > 0 \). By virtue of graph connectivity the convergence to the agreement space \( x_1 = x_2 = \cdots = x_n \), occurs exponentially fast. Finally, observe that \( \frac{1}{n} \sum_{i=1}^{n} x_i \) is a first integral of motion to conclude about the consensus point.

The average of \( x_0 \) is called the consensus equilibrium of the network over the state of interest [39]. The central objective of this work is to evaluate systemic measures of performance that quantify the necessary effort the dynamical system takes to converge to consensus. We aim at leveraging the Koopman framework, developed in the previous section. The requirement for the implementation of that machinery is to have a hyperbolic and asymptotically stable fixed point. One may notice that

$$\Lambda := -L_0,$$

with a smallest eigenvalue in magnitude is \( \lambda_1(\Lambda) = 0 \). Hence, the fixed point at the origin is not hyperbolic. In order to overcome this difficulty we introduce output dynamics vector \( y \) with elements \( y_i := x_i - \bar{x} \), or in the matrix form, \( y = M_n x \), where \( M_n \) is the the centering matrix given by.
\[ M_n := I_n - J_n / n \in \mathbb{R}^{n \times n}, \]

where \( J_n \) is the square matrix of all ones. The dynamics of \( y \) constitute the disagreement network that is associated with (14) is defined to pass this obstacle [39, 49]; that is

\[ \dot{x} = -L_d(x)x, \quad (16) \]

where the disagreement Laplacian matrix is

\[ L_d(x) := L_x + \delta J_n \]

for some \( \delta > 0 \). The next stability result is a straightforward corollary of Theorem 3 and it is stated without proof.

**Corollary 1.** The output dynamics of \( y = M_n x \) of (14) satisfy

\[ \dot{y} = -L_d(y)y, \quad (N_d) \]

with \( y = 0 \) is the a globally exponentially stable hyperbolic fixed point.

The dynamics of (N_d) satisfy \( y(t, y_0) = M_n S(t, x_0), \ t \geq 0 \). The energy of the output once weighted with a positive-definite and symmetric matrix \( Q \) is

\[ \int_0^{\infty} y^T(t, y_0)Q y(t, y_0) dt. \]

We choose the performance measure as the mean energy of this eventually vanishing signal \( y \), when the state of the consensus system starts from a random initial condition \( x_0 \). The long term energy of the output signal \( y \) that converges to zero is equivalent to the energy of the state vector \( x \) to converge to consensus. We take this mean for uncertain initial conditions, by assuming that the initial state is a random variable \( x_0 : \Omega_s \rightarrow \Omega \) from the sample space \( \Omega_s \), with some probability measure (e.g. a probability density function or a probability mass function). In either case, we define the performance measure as

\[ \rho(L) := \mathbb{E}_{x_0} \left\{ \int_0^{\infty} S^T(t, x_0) M_n^T Q M_n S(t, x_0) dt \right\}. \quad (17) \]

The next result establishes an analytical expression for the performance measure of (N_d) that reflects the contributions of the spectra of the linearized graph Laplacian and eigenfunctions of the Koopman operator.

**Theorem 4.** (Performance Measure) Consider the disagreement dynamics \( N_d \) and the associated flow \( S(\cdot, y_0) \) for all initial disagreements \( y_0 \). Then, the performance measure (17) can be expressed as
\[ \rho(\mathcal{L}) = \sum_{i,j \geq 1} \phi_{ij} c_{ij} = \frac{1}{\lambda_i + \lambda_j}, \]  

(18)

where \( \{ \lambda_i \}_{i=1,2,\ldots} \) is the sequence of Koopman eigenvalues in the KMD of \( \mathcal{L}_d \), enumerated by an arbitrary numbering of \( \gamma_i = (j_2, \ldots, j_n) \in \mathbb{Z}_+^n \) as

\[ \lambda_i := \sum_{k=2}^n j_k \lambda_k \quad \text{and} \quad \phi_i(x_0) := \prod_{k=2}^n H_k^{j_k}(x_0). \]

with \( \lambda_2, \ldots, \lambda_n \) the nonzero eigenvalues of the linearized Laplacian \( \frac{\partial}{\partial y} \mathcal{L}(x)|_{y=0} = \mathcal{L}_0 \). Moreover, \( \phi_j := \mathbb{E}_{x_0} \{ \phi_i(y) \phi_j(y) \} \) and \( c_{ij} := \mathbf{c}_i^T \mathbf{Q} \mathbf{c}_j \), are computed in terms of Koopman eigenfunctions and modes, respectively.

**Proof.** The disagreement dynamics \( \mathcal{L}_d \) attain a globally exponentially stable hyperbolic origin. In view of Assumptions 1 and 2, one can sort the eigenvalues of \( -A = \frac{\partial}{\partial y} \mathcal{L}_d(y)|_{y=0} = \mathcal{L}_0 \) as \( \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \) such that \( \lambda_1 = -\delta \). We claim that the restriction of \( \phi_1(x) = H_1(x) \) to \( 1^\perp \) is zero, since \( \phi_1(x) = 1_n^T x \). We substitute \( \phi_1(x) \), \( F(x) = -\mathcal{L}_d(x) \), and \( \lambda_1 = -\delta \) into the left hand side of (13) to obtain

\[ \nabla^T \phi(x) F(x) = 1_n^T (\mathcal{L}(x) - \delta \mathbf{1}_n/n) x, \]

which implies

\[ \nabla^T \phi(x) F(x) = 0 - \delta \sum_{i=1}^n x_i = -\delta \times 1_n^T x = -\lambda_1 \phi_1. \]

Thus \( \phi_1(x) = 1_n x \) is in fact a Koopman eigenfunction with eigenvalue \( -\delta \). We observe that for any \( y \in 1^\perp \), it holds that \( \phi_1(y) = 0 \). Considering the restricted dynamics, \( H_1(y) = \phi_1(y) = 0 \). Hence, any Koopman eigenfunction parametrized with \( \gamma_i = (j_1, j_2, \ldots, j_n) \) with \( j_1 \geq 1 \) is zero, because the corresponding eigenfunction is

\[ \phi_i(x) = \prod_{k=1}^n H_k^{j_k}(x). \]

Now let a \( \mathbf{H}^{-1} \) have the form (12) for \( y \). We consider a KMD based on Proposition 1. This implies all the term related to \( \lambda_1 \) are canceled out of the decomposition. Thus, we can restrict the numbering of summation indices to \( \mathbb{Z}_+^{n-1} \) and then write the KMD for \( y(t, y_0) = M_0 S(\cdot, x_0) \) as

\[ y(t, y_0) = \sum_{i \geq 1} \mathbf{c}_i e^{-\lambda_i t} \phi_i(y_0) \]  

(19)

where for any multi-index \( \gamma_i = (j_2, \ldots, j_n) \in \mathbb{Z}_+^{n-1} \) inducing
\[ x_i := \sum_{k=2}^n j_k \lambda_k \quad \text{and} \quad \phi_i(x_0) := \prod_{k=2}^n H_k(x_0). \]

The integrand of the integral in the performance measure is

\[ y^T(t, y_0) Q y(t, y_0) = \left( \sum_{i=1}^n e^{-\lambda_i t} \phi_i(y_0) c_i^T \right) Q \left( \sum_{j=1}^n c_j e^{-\lambda_j t} \phi_j(y_0) \right). \]

We reorganize this quadratic term as

\[ y^T(t, y_0) Q y(t, y_0) = \sum_{i,j \geq 1} e^{- (\lambda_i + \lambda_j) t} \phi_i(y_0) \phi_j(y_0) c_i^T Q c_j. \]

The induced eigenvalues satisfy \( \lambda_i = \sum_{k=2}^n j_k \lambda_k > 0 \), hence, \( \lambda_i + \lambda_j > 0 \) for all \( i \geq 1 \).

Integrating over all times yields

\[ \int_0^\infty y^T(t, y_0) Q y(t, y_0) \, dt = \sum_{i,j \geq 1} \phi_i(y_0) \phi_j(y_0) \frac{c_i^T Q c_j}{\lambda_i + \lambda_j}. \]

The result follows by virtue of the linearity of the expected value.

### 4.1 Analytic Examples

The Koopman representation of flows in consensus networks can be derived analytically, for some special cases. In this section, we discuss a few such types of networks in the form of (14) where the associated Koopman modes (subsequently \( \rho(L) \)) can be calculated in a closed form.

**Example 1 (Linear Consensus Network).** We evaluate the performance measure of a first-order LTI consensus network of order \( n \), which has the dynamics

\[ \dot{x} = -L x, \]

for a graph Laplacian \( L \) that is not state-independent but satisfies Assumption 2.

To use (18), we let \( Q = I_n \) and choose the initial conditions such that

\[ E_{x_0} \{ y_0 y_0^T \} = I_n. \]

We denote the eigenvalues of \( L \) as \( \lambda_i \) for \( i = 1, \ldots, n \). Based on Lemma 1, \( \lambda_i \) has a Koopman eigenfunction \( \phi_i(x) = H_i(x) \), that is

\[ \phi_i(x) = v_i^T x, \]
where $v_i$ is the unit eigenvector of $L$ corresponding to $\lambda_i$ (see [10, 34]). Let $V = [v_1 | v_2 | \ldots | v_n]$ be the orthonormal matrix of eigenvectors of $L$, then for the disagreement dynamics we have $H(y) = V^T y$. Since $(V^T)^{-1} = (V^{-1})^{-1} = V$ the inverse of this map is $H^{-1}(y) = Vy$. This lets us compute the components of the performance measure as follows.

$$
\phi_{ij} = \mathbb{E}_{x_0} \left\{ \phi_i(y) \phi_j(y) \right\} = \mathbb{E}_{x_0} \left\{ v_j^T y \cdot v_i^T y \right\},
$$

for all $i, j = 2, \ldots, n$. We rearrange to obtain

$$
\phi_{ij} = \mathbb{E}_{x_0} \left\{ v_j^T y y^T v_i \right\} = v_j^T \mathbb{E}_{x_0} \left\{ y y^T \right\} v_i = v_j^T v_i = \delta_{ij},
$$

since $\mathbb{E}_{x_0} \{ y y^T \} = I_n$ and $V$ is orthonormal. Obviously, $H^{-1}(y) = Vy$ is an exact polynomial representation, thus

$$
e_i = \begin{cases} v_i & \text{if } i = 2, \ldots, n, \\ 0 & \text{if } i = 1 \end{cases},
$$

which allows to compute the coefficients used in the performance measure as

$$
ce_{ij} = c_i^T c_j = \begin{cases} \delta_{ij} & \text{if } i, j = 2, \ldots, n, \\ 0 & \text{if } i \text{ or } j = 1 \end{cases}.
$$

We substitute these terms into the result in (18) to find

$$
\rho(L) = \sum_{i=2}^{n} \frac{1}{2\lambda_i},
$$

that is the $\mathcal{H}_2$-norm squared of a first order linear consensus network [49].

It turns out that for the case when we have only two agents, we may be able to compute the eigenfunctions analytically. The next two examples highlight this fact.

Example 2. Suppose that the network consists of two agents with dynamics dictated by (14) and weight functions

$$
w_{ij} = \frac{1}{(1 + (x_i - x_j)^2)^\alpha},
$$

for some constant $\alpha \in \mathbb{R}_+$. Parameter $\alpha$ in (21) defines how localized the interactions are within the network. As $\alpha$ increases, the agents update their states mainly with respect to their closest neighbors. In fact, the particular type of link implies that magnitude of interaction between two subsystems becomes weaker as their state becomes more different. For such a consensus network, in the case of two nodes, let $p$ and $q$ denote the states of the agents. Consequently, we can explain the interaction of these two nodes through the dynamics
\[
\begin{bmatrix}
\dot{p} \\
\dot{q}
\end{bmatrix} = \frac{-1}{(1+(p-q)^2)^\alpha} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
p \\
q
\end{bmatrix}.
\]

Now, we turn into the disagreement dynamics \( N_d \) with \( \delta = 1 \), whose Jacobian at the origin attains the eigenvalues \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \). Based on Lemma 1, each eigenvalue corresponds to a Koopman eigenfunction, say \( \phi_1(x) \) and \( \phi_2(x) \). We restrict the dynamics to \( \text{I}_n \), however, for the sake of simplicity, we denote the restricted variables with the same notation (i.e., \( p \) and \( q \)). Hence, \( [p, q]^T \in \text{I}_1 \); i.e., \( p + q = 0 \).

We already know that
\[\phi_1(x) = 1_n^T x,\]
whose restriction to \( \text{I}_1 \) is indeed zero. Once \( \phi_2(x) \) is restricted to \( \text{I}_1 \), we may compute it in an explicit fashion using (13), that is
\[\frac{\partial \phi_2}{\partial p} \frac{\partial \phi_2}{\partial q} = \frac{-1}{(1+(p-q)^2)^\alpha} \begin{bmatrix} q-p \\ p-q \end{bmatrix} = -2 \phi_2.\]

We let \( z := p - q \) and use the chain rule to obtain
\[\frac{\partial \phi_2}{\partial p} = \frac{\partial \phi_2}{\partial z} \text{ and } \frac{\partial \phi_2}{\partial q} = -\frac{\partial \phi_2}{\partial z}.\]

Noting that the only free variable is now \( z \), we can change the partial derivatives with respect to \( z \). The resulting scalar ordinary differential equation is
\[\frac{2z}{(1+z^2)^\alpha} \frac{d\phi_2}{dz} = 2\phi_2,\]
which is equivalent to the following indefinite integral equation
\[
\int \frac{d\phi_2}{\phi_2} = \int \frac{(1+z^2)^\alpha}{z} dz.
\]

We add \(+1\) and \(-1\) into the numerator of the integrand in the right hand side and integrate to get
\[\ln|\phi_2| = \ln|z| + \int \frac{(1+z^2)^\alpha - 1}{z} \, dz;\]
that is equivalent to
\[\phi_2 = \pm z \exp\left(\int \frac{(1+z^2)^\alpha - 1}{z} \, dz\right).\]

Without loss of generality, we choose to work with the plus sign. Using the binomial series we write the numerator of the integrand as
\[h(z) := (1+z^2)^\alpha - 1 = \alpha z^2 + \frac{(\alpha)(\alpha - 1)z^4}{2!} + \ldots,\]
for all $|z| < 1$. We integrate the series to get

$$\int \frac{h(z)}{z} \, dz = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1) \cdots (\alpha-n+1) z^{2n}}{2n \times n!}$$

which completes the evaluation of $\phi_2(z)$ as

$$\phi_2(z) = z \exp \left( \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1) \cdots (\alpha-n+1) z^{2n}}{2n \times n!} \right),$$

that is a convergent series for $|z| < 1$, since $\exp(.)$ is analytic everywhere. The identity $p + q = 0$ implies $z = 2p$, thus

$$\phi_2(p) = 2p \exp \left( \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1) \cdots (\alpha-n+1) 2^{2n-1} p^{2n}}{n \times n!} \right).$$

Thus, the component-wise description of $H(p, q)$ is

$$H(p, q) = H(p) = [\phi_2(p) \ 0]^T.$$

The definition of inverse of a map implies

$$H^{-1}(H(p, q)) = [p \ q]^T$$

so $H^{-1}_1(p, q)$ and $H^{-1}_2(p, q)$ are simply the inverse functions of $\phi_2(p)$ and $-\phi_2(p)$, respectively. These functions are locally analytic around the origin based on Lagrange inversion theorem [1]. Furthermore, one can calculate their coefficients in terms of Bell polynomials [2]. We assume that the initial value of $p$ to come from a discrete random variable that takes the values from $\{0, 0.1, 0.2, 0.3, 0.4\}$ with uniform probability distribution, where because the mean of each initial condition must be zero, the initial value of $q$ will be $-p$. We compute the value of $\rho(\mathcal{L})$ for $Q = I_n$ using the 17 order Maclaurin series of both eigenfunctions and the inverse map $H^{-1}(p)$. Because the value of the performance measure may be computed from the numerical integration of the trajectory as well, we may compare the results of the analytic approximations to the KMD and the real value of performance measure. These two value for a range of $\alpha \in [0, 0.4]$ have been illustrated in Fig. (1), where they are in good numerical agreement. The relative error is observed to increase from zero in the case that $\alpha = 0$ to less than 0.13% for $\alpha = 0.4$.

**Remark 2.** There is a limitation on the magnitude of the admissible initial conditions for the analysis conducted in the previous example. However, notice that this does not imply that it is a linear analysis, since for any value of $\alpha$, the linearization matrix of $\mathcal{M}_d$ (with $k = 1$) in Example 2 is the following matrix.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + J_n.$$
Example 3. The dynamics of oscillators have been observed to be closely related to the consensus dynamics. Kuramoto suggested a model of biological oscillation, in which each oscillator was connected to the other one; i.e., the topology was a complete graph. Instead the interactions can be limited over a certain graph [26], so the dynamics of agent $i$ can be represented as

\[
\dot{x}_i = \omega_i + \sum_{\{i,j\} \in E} w_{ij} (x_j - x_i),
\]

(22)

where $\omega_i$ is the natural frequency and the coupling weight is

\[
w_{ij} = K \frac{\sin(x_i - x_j)}{x_i - x_j}.
\]

(23)

When the agents are identical, the change of variable $x_i \to x_i - \omega t$ induces a nonlinear consensus network that is

\[
\dot{x}_i = \sum_{\{i,j\} \in E} w_{ij} (x_j - x_i).
\]

(24)

We proceed with a procedure for computation of performance measure similar to one introduced in Example 2. For two identical oscillators, with phases $\theta$ and $\gamma$, the dynamics are

\[
\frac{d}{dt} \begin{bmatrix} \theta \\ \gamma \end{bmatrix} = K \begin{bmatrix} \sin(\gamma - \theta) \\ \sin(\theta - \gamma) \end{bmatrix},
\]

Fig. 1 The performance measure of the network of two agents in Example 2 with the decaying parameter $\alpha$. This means that the value of the performance measure computed using the linearized system for each value of decaying parameter $\alpha$ is only one value.
Setting \( k = 1 \), the disagreement Jacobian has eigenvalues \( \lambda_1 = -1 \) and \( \lambda_2 = -2K \), with eigenfunctions \( \phi_1 \) and \( \phi_2 \), respectively. Again we only need the restriction of \( \phi_2 \) to \( 1^\perp \). The equation (13) for these dynamics becomes

\[
\begin{bmatrix}
\frac{\partial \phi_2}{\partial \theta} \\
\frac{\partial \phi_2}{\partial \gamma}
\end{bmatrix}^T K \sin(\theta - \gamma) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2K \phi_2.
\]

The new variable \( z := \theta - \gamma \) creates a single ordinary differential equation that is

\[
\frac{2 \sin(z)}{d\phi_2}dz = 2\phi_2 \Rightarrow \frac{d\phi_2}{\phi_2} = \frac{1}{\sin(z)}dz,
\]

which is integrated and manipulated to get

\[
\phi_2 = \pm \exp(-\ln(\cot(z/2))) = \pm \tan(z/2),
\]

where we arbitrarily choose \( + \). The eigenfunction \( \phi_2 \) is locally analytic around the origin for \( |z| < \pi \). Restricting to \( 1^\perp \), \( \theta + \gamma = 0 \), so \( z = 2\theta \), hence

\[
\phi_2(\theta, \gamma) = \phi_2(\theta) = \tan(\theta),
\]

which helps write the components of \( H(\theta, \gamma) = H(\theta, 0) \) as \( H^{-1}(\theta) = [\theta, 0]^T \). First component of \( H^{-1}(\theta) \) satisfies \( H^{-1}_1(\phi_2(\theta), 0) = \theta \). Hence, \( H^{-1}(\theta, \gamma) = \arctan(\theta) \), which is again locally analytic for \( |\theta| < 1 \) around zero.

We sample initial conditions from a uniform discrete random variable of 63 equally distributed initial conditions in \( \theta \in [0, \pi/5] \) (and \( \gamma = -\theta \)) with equal distance of 0.01. We set \( Q = I \) and use 17th order Maclaurin series of eigenfunctions and \( H^{-1}(\theta) \) to assess \( \rho(\mathcal{L}) \). As shown in Fig. 2 we alter \( K \in [0.2, 4] \) and take a look at the performance measure, once compared with the exact value of \( \rho(\mathcal{L}) \) (computed with the numerical integration of the trajectories). The numerical agreement in this experiment can be measured by the relative error of the performance using the KMD approximation, which was about \( 3 \times 10^{-4} \% \) for the case that had the worst error.

### 5 Sparse Polynomial Approximations

We have observed that any polynomial approximation to the inverse map of eigenfunctions \( H^{-1}(x) \) will result in the Koopman Mode Decompositions. In this section, first we detail a general sparse approximation technique for multivariate interpolation. Then, we demonstrate how we can use this technique for the map of Koopman eigenfunctions \( H(x) \) as well as its inverse \( H^{-1}(x) \).
5.1 Smolyak-Collocation Method

To introduce the notion of sparsity for the approximation of both Koopman eigenfunctions and Koopman Mode Decomposition, we may use sparse functional approximation methods. The idea is that instead of searching for the approximant in the whole space of polynomials, the search is carried out over a nearly optimal sparse basis, called Smolyak basis. The output of the method would be a polynomial: weighted sum of the tensor product of Chebyshev polynomials that are in the basis. Naturally, we choose the coefficients of the polynomial with respect to some error criterion. One way of doing so is named collocation, where we enforce the the approximant to perhaps approximately satisfy the governing equation of the problem at the given points of a grid, called Smolyak Sparse Grid. To describe this method, we need few basic tools (consult [7] and [32] for more details).

**Definition 1 (Chebyshev Polynomials).** The sequence of the Chebyshev polynomials of first kind \( \{T_i(x)\}_{i=1,2,...} \) are initialized with \( T_1(x) = 1 \) and \( T_2(x) = x \) and recursively defined as follows.

\[
T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x), \text{ for all } i = 2,3,\ldots
\]

Similarly, the Chebyshev polynomials of second kind \( \{U_i(x)\}_{i=1,2,...} \) start with \( U_1(x) = 1 \) and \( U_2(x) = 2x \), and then iteratively

\[
U_{i+1}(x) = 2xU_i(x) - U_{i-1}(x), \text{ for all } i = 2,3,\ldots
\]

We define the integer function \( m(i) : \mathbb{N} \to \mathbb{N} \) with \( m(1) = 1 \) and for \( i = 2,3,\ldots \), it is evaluated according to

\[
m(i) := 2^{i-1} + 1.
\]
We also define the sequence of sets \( \{ G^i \}_{i=1}^{\infty} \), wherein, \( G^1 = \{ 0 \} \), and for \( i = 2, 3, \ldots \) it holds that \( G^i = \{ \xi_1, \ldots, \xi_{i} \} \subset [-1, 1] \), that is the set of the extrema of the Chebyshev polynomials with the components given by

\[
\xi_j := -\cos \left( \frac{\pi(j-1)}{i-1} \right), \text{ for all } j \in \{1, \ldots, i\}.
\]

In the next definition, we use the multi-index notation \( i = (i_1, \ldots, i_n) \).

**Definition 2 (Smolyak Sparse Grid).** The Smolyak Sparse grid of \([-1, 1]^n\) is a union of the Cartesian products of the form

\[
H^{n, \mu} := \bigcup_{|i|=n+\mu} \left( G^{m(i_1)} \times \cdots \times G^{m(i_n)} \right),
\]

where \( n \) is the dimension of the variable and \( \mu \in \mathbb{N} \) is called the order of the grid.

**Definition 3 (Smolyak Approximant Polynomial).** The Smolyak approximant to a function \( f : [-1, 1]^n \rightarrow \mathbb{R} \) is given by

\[
\hat{f}^{n, \mu}(x) := \sum_{q \leq |i| \leq n+\mu} (-1)^{n+\mu-|i|} \binom{n-1}{n+\mu-|i|} p^i,
\]

with the tensor product polynomials for each multi-index \( i = (i_1, \ldots, i_n) \) defined as

\[
p^i(x) := \sum_{l_1=1}^{m(i_1)} \cdots \sum_{l_n=1}^{m(i_n)} \theta_{l_1, \ldots, l_n} T_{l_1}(x_1) \cdots T_{l_n}(x_n),
\]

where \( \theta_{l_1, \ldots, l_n} \) are the coefficients of the approximation.

To find the optimal vector of coefficients \( \Theta = \text{vec}(\theta_{l_1, \ldots, l_n}) \in \mathbb{R}^m \) for approximation of some function \( f \) that is \( C^k([-1, 1]^n) \), an error objective should be defined and minimized. One way is to consider the error function \( E(f, \Theta) : C^k([-1, 1]^n) \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \) to be

\[
E(f, \Theta) := \int_{[-1,1]^n} (f(x) - \hat{f}^{n, \mu}(x))^2 \prod_{y \in H^{n, \mu}} \delta(x - y) \, dx,
\]

that is certainly minimized (i.e., \( E(f, \Theta) = 0 \)) if \( \Theta \) is chosen so that

\[
f(x) = \hat{f}^{n, \mu}(x) \text{ for all } x \in H^{n, \mu}.
\]

This procedure is called collocation. A pivotal property of the overall method is that size of the vector of coefficients \( \Theta \) and the number of interpolation points is equal; i.e.,

\[
|H^{n, \mu}| = |\Theta| = M.
\]
Therefore, enforcing the equalities \((27)\) constitutes of searching for the solution to \(M\) equations involving \(M\) unknown entries of \(\Theta\). Once we evaluate the coefficients with the described scheme, the following error bound would hold, wherein the used functional norm \(\|\cdot\|: C^k([-1,1]^n) \to \mathbb{R}_+\) is defined as
\[
\|f\| = \max \{ \|D^i f\|_{\infty} : i = 1, \ldots, k \} .
\] (28)

**Theorem 5.** Suppose that the function \(f : [-1,1]^n \to \mathbb{R}\) is \(C^k([-1,1]^n)\), together with an Smolyak approximant \(\hat{f}^{n,\mu}(x)\) that interpolates \(f\) on the points of \(\mathcal{H}^{n,\mu}\). Then, the error of the approximation is bounded according to
\[
\|f - \hat{f}^{n,\mu}\| \leq c_{n,k} M^{-k} (\log M)^{(k+2)(n+1)+1}.
\] (29)

Because for a fixed \(q\), the number of grid points satisfies
\[
M \leq c_q n^q,
\] the following corollary can be obtained about the error.

**Corollary 2.** For the error bound (29), it holds that
\[
\|f - \hat{f}^{n,\mu}\| \leq c_{n,k} c_q n^{-qk} (\log c_q + q \log n)^{(k+2)(n+1)+1}.
\] (30)

Each multi-index \(i\) in \((25)\) induces a number of tensor product polynomials that are summed together as in \((26)\). We gather the indices of all these tensor product polynomials in a set \(L\); i.e.,
\[
L := \bigcup_{q \leq |i| \leq n+\mu} \{ (l_1, \ldots, l_n) : l_j \leq m(i_j) \}
\]

One can show that at the end of the day, the approximant constructed in \((25)\) using the polynomials \((26)\) boils down to the following simple representation
\[
\hat{f}^{n,\mu}(x) = \sum_{I_1 = (l'_1, \ldots, l'_n) \in L} \sum_{i=1}^M \Theta_i T_i(x) = \sum_{i=1}^M \Theta_i T_i(x),
\] (31)

where \(M = |\mathcal{H}^{n,\mu}| = |L|\), and for \(I_1 = (l'_1, \ldots, l'_n) \in L\), the coefficients \(\Theta_i\) and polynomial terms are given by
\[
\Theta_i := \theta_{l'_1, \ldots, l'_n},
\] (32)
\[
T_i(x) := T_{l'_1}(x_1) \ldots T_{l'_n}(x_n).
\] (33)

To compute the partial derivatives of approximation, we define
\[
T_j^i(x) := \begin{cases} 
T_j \frac{U_j(x_j)}{T_j(x_j)} & \text{for } l_j^i = 2, \ldots, n, \\
0 & \text{for } l_j^i = 1
\end{cases}
\]

that helps write the partial derivatives of the approximation as
\[
\frac{\partial \hat{f}^{n,\mu}(x)}{\partial x_j} = \sum_{i=1}^{M} l_j^i \Theta_i \cdot T_j^i(x).
\]

### 5.2 Sparse Approximation to Eigenfunctions

We denote the approximation to eigenfunction \(\phi(x)\) by \(\hat{\phi}(x)\). Substituting (31) and (34) into (13), we get the equality
\[
\sum_{j=1}^{n} \sum_{i=1}^{M} l_j^i \Theta_i \cdot T_j^i(x) F_j(x) = \lambda \sum_{i=1}^{n} M \Theta_i \cdot T_i(x).
\]

We change the order of summations to further obtain
\[
\sum_{i=1}^{M} \left( \sum_{j=1}^{n} \left( l_j^i \cdot T_j^i(x) F_j(x) - \lambda T_i(x) \right) \right) \Theta_i = 0.
\]

We define and denote the vector of coefficients \(\Theta \in \mathbb{R}^M\) by
\[
\Theta := [\Theta_1, \ldots, \Theta_M]^T.
\]

For a point in the grid \(x^k \in \mathcal{H}^{n,\mu}\), we may write the left hand side of (35) as
\[
\mathcal{A}_k \Theta = [\mathcal{A}_k]_{i=1, \ldots, n} \Theta,
\]
where the entries of the row vector \(\mathcal{A}_k \in \mathbb{R}^{1 \times M}\) can be computed from
\[
\mathcal{A}_k := \sum_{j=1}^{n} \left( l_j^i \cdot T_j^i(x^k) F_j(x^k) - \lambda T_i(x^k) \right), \quad \text{for all } i = 1, \ldots, M.
\]

Repeating this for \(M\) points in the Smolyak grid, the stacked left hand side of all equations becomes \(\mathcal{A} \Theta\), where \(\mathcal{A} \in \mathbb{R}^{M \times M}\) consists of rows equal to \(\mathcal{A}_k\) as
\[
\mathcal{A} := [\mathcal{A}_1^T, \ldots, \mathcal{A}_M^T]^T.
\]

Ideally, \(\mathcal{A} \Theta\) should be zero for an eigenfunction, however, if it is not possible, we would like to minimize an error function. In this paper, we choose the objective function
Consider the Koopman eigenfunction $\phi(x)$ with Koopman eigenvalue $\lambda$, which is the eigenvalue of the linearization matrix with a left eigenvector $\mathbf{R} = [r_1, \ldots, r_n] \in \mathbb{R}^{n \times n}$. We omit the index of the eigenvalues and eigenvectors in the following developments for simplicity and consider $\lambda$ to be associated with the eigenvector $\mathbf{r} \in \mathbb{R}^n$. We can show that with the fixed point at the origin,

$$\nabla \phi(x)|_{x=0} = \mathbf{r}.$$ 

Translating this for the approximant, for each $j = 1, \ldots, n$ we have

$$\frac{\partial \hat{\phi}(x)}{\partial x_j}|_{x=0} = \sum_{i=1}^{M} l_i^j \Theta T_i^j(0) = \mathcal{B}_j \Theta = r_j,$$

where the row vector $\mathcal{B}_j \in \mathbb{R}^{1 \times M}$ has the components

$$\mathcal{B}_{ji} = l_i^j T_i^j(0), \text{ for all } i = 1, \ldots, M.$$ 

The matrix form of this equality becomes

$$\mathcal{B} \Theta = \mathbf{r},$$

where the matrix $\mathcal{B} \in \mathbb{R}^{n \times M}$ is the result of stacking row vectors as

$$\mathcal{B} = [\mathcal{B}_1^T, \ldots, \mathcal{B}_n^T]^T.$$ 

A similar, but simpler, condition $\phi(0) = 0$ can be expressed as

$$\mathcal{C} \Theta = 0,$$

where $\mathcal{C} \in \mathbb{R}^{1 \times M}$ is the row vector with elements

$$\mathcal{C}_i := T_i^j(0) \text{ for all } i = 1, \ldots, M.$$ 

Now, would like to minimize the error defined by (37), while the constraints (38) and (40) are satisfied. Therefore, we define the optimization problem

$$\min_{\Theta \in \mathbb{R}^m} \|\mathcal{A} \Theta\|_2^2,$$

subject to

$$\begin{bmatrix} \mathcal{B} \\ \mathcal{C} \end{bmatrix} \Theta = \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}.$$

This program is equivalent to a Semi-Definite Program (SDP) and can be solved using the conventional convex programming such as CVX [23].
Sparse Approximation to Koopman Mode Decomposition

In the previous subsection we illustrated a way to find approximations to the Koopman eigenfunctions. Thus, the components of the map $\hat{H}(x)$ can be approximated. Here, following a similar approach, we seek approximations to the components of its inverse map $H^{-1}(x)$. The very natural equation for component $H^{-1}_j(x)$ is

$$H^{-1}_j(H(x)) = x_j, \quad \text{for all } j = 1, \ldots, n. \quad (42)$$

We only have an approximation to $H(x)$, namely $\hat{H}(x)$, and we need approximations to $H^{-1}(x)$, namely $\hat{H}^{-1}(x)$, we consider the equation

$$\hat{H}^{-1}_j(\hat{H}(x)) = x_j, \quad \text{for all } j = 1, \ldots, n. \quad (43)$$

Again, following the spirit of the collocation method, for each point of the grid $x^k \in \mathcal{X}^{\mu}t$, we enforce this equation to hold. Suppose that we have found the series approximation to each components of $H(x)$, denoted by $\hat{H}(x)$. Moreover, we define

$$z^k := \hat{H}(x^k). \quad (44)$$

Then, we consider a Smolyak series representation for this function as

$$\hat{g}^{\mu}_{j}(z^k) = \sum_{i=1}^{M} \Phi_{j}^i T_i(z^k), \quad (45)$$

Similar to essence of the method that we discussed in the previous subsection, we define the vector of coefficients $\Phi^i_j \in \mathbb{R}^M$ to be

$$\Phi^i_j := [\Phi_{1}^j, \ldots, \Phi_{M}^j].$$

Inserting (45) into (42), we get

$$D_k \Phi^i_j = [D_{ki}]_{i=1, \ldots, M} \Phi^i_j = x^k_j, \quad (46)$$

where the components of the row vector $D_k \in \mathbb{R}^{1 \times M}$ are

$$D_{ki} := T_i(z^k).$$

Concatenating these vectors and the right hand side scalars for each point in the grid (i.e., $M$ points), we may write these equations as

$$D \Phi^i_j = X_j, \quad (47)$$

where matrix $D \in \mathbb{R}^{M \times M}$ and vector $X_j \in \mathbb{R}^M$ are given by
\( D := [D_1^T, \ldots, D_M^T]^T \), \hspace{1cm} (48)

\( X_j := [x_1^j, \ldots, x_M^j]^T \). \hspace{1cm} (49)

Again one hopes that (47) holds with a minimal error for each \( j = 1, \ldots, n \). Therefore, we define the optimization problem

\[
\min_{\Phi^j \in \mathbb{R}^M} \| D\Phi^j - X_j \|_2.
\] \hspace{1cm} (50)

Note that the matrix \( D \) is deliberately denoted without index \( j \), because it is the same matrix for the optimization problem for each component \( H_j^{-1}(x) \). The solution to this least-squares optimization problem is given by

\[
\Phi^j = D^\dagger X_j \quad \text{for all} \quad j = 1, \ldots, n.
\]

We should repeat this for each component of \( H^{-1}(x) \). Putting the results in a matrix gives us

\[
\Phi := [\Phi^1, \ldots, \Phi^n].
\]

Because the value of matrix \( D \) is shared between \( M \) optimization problems defined by (50), by inspection we find that

\[
\Phi = D^\dagger X^T, \hspace{1cm} (51)
\]

where \( X \in \mathbb{R}^{n \times M} \) is the matrix containing the vector of all grid points.

### 5.4 Numerical Representations

In this section, we show that we may be able to effectively estimate the performance measure of nonlinear consensus networks with more than two subsystems. Note that in all cases, the real performance measure is calculated from the numerical solution of the network output followed by numerical integration.

**Example 4 (Complete Graphs).** Using the described numerical approximation method, we estimate the performance measure for the nonlinear consensus network with exponentially decaying weights. The corresponding linearized graph Laplacian corresponds to the undirected unweighted complete graph; i.e.

\[
A = -L_{\mathcal{G}_n} = J_{n} / n - I_n.
\]

We evaluate the performance measure from the KMD approximation based on the numerical integration of the solutions. The initial conditions are uniformly sampled random initial conditions from \([-1, 1]^n\]. The numerical values for data using Koopman approach have been obtained by implementation of the suggested numerical
Fig. 3 The performance measure of nonlinear consensus network with $\alpha = 0.25$ and the graph at the linearized Laplacian of complete graph.

Fig. 4 The performance measure of nonlinear consensus network with $n = 8$, $\alpha = 0.25$ and random graphs with different number of edges

method with and the results are shown in Fig. 3. The error among these data points in any case was less than 2%.

Example 5 (Random Graphs). We fix $n = 8$ and create Erdős-Rényi graphs with different edges probabilities (and consequently, different edge numbers). Then, we consider again the exponentially decaying weights given by (21). The performance measure from the simulations as well as the formula are also evaluated and illustrated in Fig. 4. The error of approximation in this case is less than 1.4%.

5.5 Connections to Extended Dynamic Mode Decomposition

The numerical method explained in this section is related to the notion of Extended Dynamic Mode Decomposition (EDMD) [57], that has been a promising procedure
for extracting information about the Koopman spectrum of the dynamical system. In EDMD, to find a KMD for the flow of the dynamical system, one should first assume a rich enough dictionary of basis functions such that hopefully, the Koopman eigenfunctions lie in their span. Then, using the snapshots from the trajectory, one may find a truncated approximation to the Koopman operator and finite number of approximations to the Koopman eigenfunctions and their corresponding eigenvalues. Then, the state of the dynamical system is approximated as a truncated KMD using those eigenvalues and eigenfunctions.

In the current settings we know what are the eigenvalues and eigenfunctions that are required for representation of the flow of the nonlinear system. Hence, we do not need the first step of the EDMD for the computation of the (approximate) eigenvalues and eigenfunctions. In fact, we build the dictionary that one needs for EDMD based on the principal eigenfunctions in the map $H(x)$.

On the other hand, the second step in both methods is identical in the spirit: approximating the flow of the nonlinear system by a (truncated) Koopman Mode Decomposition by using a list of eigenfunctions. More precisely, finding an approximation to map $H^{-1}(x)$ from

$$H^{-1}(H(x)) = x,$$

is the equivalent to step of representing the state of the system $x$ in terms of the basis functions in the dictionary, which is the case in the EDMD (see the reference for more details).

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**References**

1. Abramowitz, M., Stegun, I.A., et al.: Handbook of mathematical functions. Applied mathematics series 55, 62 (1966)
2. Ajorlou, A., Momeni, A., Aghdam, A.G.: Sufficient conditions for the convergence of a class of nonlinear distributed consensus algorithms. Automatica 47(3), 625–629 (2011)
3. Ali, Q., Gageik, N., Montenegro, S.: A review on distributed control of cooperating mini uavs. International Journal of Artificial Intelligence & Applications 5(4), 1 (2014)
4. Arcak, M.: Passivity as a design tool for group coordination. IEEE Transactions on Automatic Control 52(8), 1380–1390 (2007)
5. de Badyn, M.H., Mesbahi, M.: Growing controllable networks via whiskering and submodular optimization. In: Decision and Control (CDC), 2016 IEEE 55th Conference on, pp. 867–872. IEEE (2016)
6. Bamieh, B., Jovanovic, M.R., Mitra, P., Patterson, S.: Coherence in large-scale networks: Dimension-dependent limitations of local feedback. IEEE Transactions on Automatic Control 57(9), 2235–2249 (2012)

7. Barthelmann, V., Novak, E., Ritter, K.: High dimensional polynomial interpolation on sparse grids. Advances in Computational Mathematics 12(4), 273–288 (2000)

8. Bassett, D.S., Bullmore, E.: Small-world brain networks. The neuroscientist 12(6), 512–523 (2006)

9. Beard, R.W., Lawton, J., Hadaegh, F.Y.: A coordination architecture for spacecraft formation control. IEEE Transactions on control systems technology 9(6), 777–790 (2001)

10. Budišić, M., Mohr, R., Mezić, I.: Applied koopmanism a. Chaos: An Interdisciplinary Journal of Nonlinear Science 22(4), 047,510 (2012)

11. Buzi, G., Topcu, U., Doyle, J.C.: Quantitative nonlinear analysis of autocatalytic pathways with applications to glycolysis. In: American Control Conference (ACC), 2010, pp. 3592–3597. IEEE (2010)

12. Charalambides, C.A.: Enumerative combinatorics. CRC Press (2002)

13. Chiang, H.D., Wu, F.F., Varaiya, P.P.: A bcu method for direct analysis of power system transient stability. IEEE Transactions on Power Systems 9(3), 1194–1208 (1994)

14. Conway, J.B.: A Course In Functional Analysis. Springer (1994)

15. Cucker, F., Mordecki, E.: Flocking in noisy environments. arXiv preprint arXiv:0706.3343 (2007)

16. Cucker, F., Smale, S.: Emergent behavior in flocks. IEEE Transactions on Automatic Control 52(5), 852–862 (2007)

17. Cullen, H.F.: Introduction to General Topology. D C Heath & Co (1968)

18. Dai, R., Mesbahi, M.: Optimal topology design for dynamic networks. In: 2011 50th IEEE Conf. Decision Control / Euro. Control Conf., pp. 1280–1285. IEEE (2011)

19. Dorfler, F., Bullo, F.: Synchronization and transient stability in power networks and nonuniform kuramoto oscillators. SIAM J. Control Optimization 50(3), 1616–1642 (2012)

20. Dörfler, F., Chertkov, M., Bullo, F.: Synchronization in complex oscillator networks and smart grids. Proceedings of the National Academy of Sciences 110(6), 2005–2010 (2013)

21. Dunford, N., Schwartz, J.: Linear Operators. Part I: General Theory. Interscience Publishers (1964)

22. Dunkl, C., Xu, Y.: Orthogonal Polynomials of Several Variables. Encyclopedia of Mathematics and its Applications. Cambridge University Press (2014)

23. Grant, M., Boyd, S.: Cvx: Matlab software for disciplined convex programming, version 1.21 (2011). Available: cvxr.com/cvx (2010)

24. Jadababaie, A., Lin, J., Morse, A.S.: Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Trans. Autom. Control 48(6), 988–1001 (2003)

25. Jadababaie, A., Molavi, P., Sandroni, A., Tahbaz-Salehi, A.: Non-bayesian social learning. Games and Economic Behavior 76(1), 210–225 (2012)

26. Kim, Y., Mesbahi, M.: On maximizing the second smallest eigenvalue of a state-dependent graph laplacian. In: American Control Conference, 2005. Proceedings of the 2005, pp. 99–103. IEEE (2005)

27. Kust, J., Brunton, S.L., Brunton, B.W., L., P.J.: Dynamic Mode Decomposition. Data-Driven Modeling of Complex Systems. OT149. SIAM (2016)

28. Lan, Y., Mezić, I.: Linearization in the large of nonlinear systems and koopman operator spectrum. Physica D: Nonlinear Phenomena 242(1), 42–53 (2013)

29. Leonard, N.E., Fiorelli, E.: Virtual leaders, artificial potentials and coordinated control of groups. In: Decision and Control, 2001. Proceedings of the 40th IEEE Conference on, vol. 3, pp. 2968–2973. IEEE (2001)

30. Lin, F., Fardad, M., Jovanović, M.R.: Design of optimal sparse feedback gains via the alternating direction method of multipliers. IEEE Transactions on Automatic Control 58(9), 2426–2431 (2013)
32. Malin, B.A., Krueger, D., Kubler, F.: Solving the multi-country real business cycle model using a smolyak-collocation method. Journal of Economic Dynamics and Control 35(2), 229–239 (2011)
33. Mauroy, A., Goncalves, J.: Koopman-based lifting techniques for nonlinear systems identification. arXiv preprint arXiv:1709.02003 (2017)
34. Mauroy, A., Mezić, I.: Global stability analysis using the eigenfunctions of the koopman operator. IEEE Transactions on Automatic Control 61(11), 3356–3369 (2016)
35. Moghaddam, S.H., Jovanović, M.R.: An interior point method for growing connected resistive networks. In: 2015 Amer. Control Conf., pp. 1223–1228 (2015)
36. Mousavi, H.K., Somarakis, C., Bahavarnia, M., Motee, N.: Performance bounds and optimal design of randomly switching linear consensus networks. In: American Control Conference (ACC), 2017, pp. 4347–4352. IEEE (2017)
37. Mousavi, H.K., Somarakis, C., Motee, N.: Koopman performance analysis of a class of nonlinear dynamical networks. In: Decision and Control (CDC), 2016 IEEE 55th Conference on, pp. 117–122. IEEE (2016)
38. Mousavi, H.K., Somarakis, C., Motee, N.: Spectral performance analysis and design for distributed control of multi-agent systems. In: Decision and Control (CDC), 2017 IEEE 56th Annual Conference on, pp. 2918–2923. IEEE (2017)
39. Olfati-Saber, R., Murray, R.M.: Consensus problems in networks of agents with switching topology and time-delays. IEEE Transactions on automatic control 49(9), 1520–1533 (2004)
40. Olshhevsky, A., Tsitsiklis, J.N.: Convergence speed in distributed consensus and averaging. SIAM Journal on Control and Optimization 48(1), 33–55 (2009)
41. Papachristodoulou, A., Jadbabaie, A.: Synchronization in oscillator networks with heterogeneous delays, switching topologies and nonlinear dynamics. In: Decision and Control, 2006 45th IEEE Conference on, pp. 4307–4312. IEEE (2006)
42. Papachristodoulou, A., Jadbabaie, A., Munz, U.: Effects of delay in multi-agent consensus and oscillator synchronization. IEEE transactions on automatic control 55(6), 1471–1477 (2010)
43. Patterson, S., Bamieh, B.: Leader selection for optimal network coherence. In: Decision and Control (CDC), 2010 49th IEEE Conference on, pp. 2692–2697. IEEE (2010)
44. Perko, L.: Differential equations and dynamical systems, vol. 7. Springer Science & Business Media (2013)
45. van Riel, N.A., Sontag, E.D.: Parameter estimation in models combining signal transduction and metabolic pathways: the dependent input approach. IEE Proceedings-Systems Biology 153(4), 263–274 (2006)
46. Shilong, L., Mao, S.: Aerodynamic force and flow structures of two airfoils in flapping motions. Acta Mechanica Sinica 17(4), 310–331 (2001)
47. Siami, M., Motee, N.: On existence of hard limits in autocatalytic networks and their fundamental tradeoffs. IFAC Proceedings Volumes 45(26), 294–298 (2012)
48. Siami, M., Motee, N.: Performance analysis of linear consensus networks with structured stochastic disturbance inputs. In: 2015 Amer. Control Conf., pp. 4080–4085 (2015)
49. Siami, M., Motee, N.: Fundamental limits and tradeoffs on disturbance propagation in linear dynamical networks. IEEE Transactions on Automatic Control 61(12), 4055–4062 (2016)
50. Siami, M., Motee, N.: Growing linear dynamical networks endowed by spectral systemic performance measures. IEEE Transactions on Automatic Control 63(8) (2018)
51. Siami, M., Motee, N.: Network abstraction with guaranteed performance bounds. IEEE Transactions on Automatic Control (2018)
52. Somarakis, C., Baras, J.S.: Delay-independent convergence for linear consensus networks with applications to non-linear flocking systems. In: Submitted to the 12th IFAC Workshop on Time Delay Systems. Ann Arbor,ML,USA (2015)
53. Somarakis, C., Baras, J.S.: A simple proof of the continuous time linear consensus problem with applications in non-linear flocking networks. In: 14th European Control Conference. Linz,Austria (2015)
54. Somarakis, C., Haedehsarf, Y., Motee, N.: Time-delay origins of fundamental tradeoffs between risk of large fluctuations and network connectivity. arXiv preprint arXiv:1801.06856 (2018)
55. Susuki, Y., Mezic, I.: Nonlinear koopman modes and a precursor to power system swing instabilities. IEEE Transactions on Power Systems 27(3), 1182–1191 (2012)
56. Wang, P.K.: Navigation strategies for multiple autonomous mobile robots moving in formation. Journal of Field Robotics 8(2), 177–195 (1991)
57. Williams, M.O., Kevrekidis, I.G., Rowley, C.W.: A data-driven approximation of the koopman operator: Extending dynamic mode decomposition. Journal of Nonlinear Science 25(6), 1307–1346 (2015)