DESCENT FOR $n$-BUNDLES

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Abstract. Given a Lie group $G$, one constructs a principal $G$-bundle on a manifold $X$ by taking a cover $U \to X$, specifying a transition cocycle on the cover, and descending the trivialized bundle $U \times G$ along the cover. We demonstrate the existence of an analogous construction for local $n$-bundles for general $n$. We establish analogues for simplicial Lie groups of Moore’s results on simplicial groups; these imply that bundles for strict Lie $n$-groups arise from local $n$-bundles. Our construction leads to simple finite dimensional models of Lie 2-groups such as $\text{String}(n)$.

1. Introduction

The nerve of a group is a simplicial set satisfying Kan’s horn-filling conditions. Grothendieck observed that the nerve provides an equivalence between the category of groups and the category of reduced Kan simplicial sets whose horns have unique fillers above dimension one. More generally, he showed that the nerve extends to an equivalence between the category of groupoids and the category of Kan simplicial sets whose horns have unique fillers above dimension one. Inspired by this, Duskin [6] defined an $n$-groupoid to be a Kan simplicial set whose horns have unique fillers above dimension $n$.

In the last decade, Henriques [12], Pridham [17] and others have begun the study of Lie $n$-groupoids: simplicial manifolds whose horn-filling maps are surjective submersions in all dimensions, and isomorphisms above dimension $n$. Examples are common. (See also Getzler [10].) Lie 0-groupoids are precisely smooth manifolds. Lie 1-groups are nerves of Lie groups. Abelian Lie $n$-groups are equivalent to chain complexes of abelian Lie groups supported between degrees 0 and $n - 1$.

Simplicial Lie groups whose underlying simplicial set is an $(n - 1)$-groupoid give rise to Lie $n$-groups, by the $\tilde{W}$-construction (Section 6). We call this special class of Lie $n$-groups strict Lie $n$-groups.

Much of the theory of principal bundles for Lie groups generalizes naturally to principal bundles for strict Lie $n$-groups. In Theorems 5.6 and 6.7, we show that the construction of a $G$-bundle from a cocycle on a cover has a close analogue for cocycles for strict Lie $n$-groups. As an application, we show how this allows for the construction of finite dimensional Lie 2-groups, such as $\text{String}(n)$, from cohomological data. The method works equally well for $n > 2$.

Outline. We develop our results within a category $C$ which has a terminal object $\ast$ and a subcategory of covers. We require the subcategory of covers to be stable under pullback, to contain the maps $X \longrightarrow \ast$ for every object $X$, to satisfy an axiom

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of right-cancelation, and to be contained within the class of effective epimorphisms. Our motivating example is the category of finite dimensional smooth manifolds, with surjective submersions as covers. Other examples include Banach manifolds, analytic manifolds over a complete normed field, and of course sets.

In Section 2, we recall the definitions and basic properties of higher stacks. This section parallels the discussion in Behrend and Getzler [2], the main difference being that in that paper, the category $\mathcal{C}$ is assumed to possess finite limits. Here, we work with categories of manifolds, so this assumption does not hold. (Conversely, they do not impose the assumption that the maps $X \rightarrow \ast$ are covers, which fails in the setting of not-necessarily smooth analytic spaces.)

In Section 3, we show that the collection of $k$-morphisms in a Lie $n$-groupoid forms a Lie $(n-k)$-groupoid (Theorem 3.6). In Section 4 we apply this to study Duskin's $n$-strictification functor $\tau_n$. (For a discussion in the absolute case, see [11, Section 3.1, Example 5], [12] Definition 3.5 or [10] Section 2.) In the Lie setting, the functor $\tau_n$ does not always exist. However, when it does, it provides a partial left adjoint to the inclusion of $n$-stacks into the category of $\infty$-stacks. We recall the relevant properties and give a necessary and sufficient criterion for existence. In Section 5, we impose the additional axiom that quotients of regular equivalence relations in $\mathcal{C}$ exist. (This was first established by Godemont for smooth manifolds, or analytic manifolds over a complete normed field.) Under this assumption, we introduce local $n$-bundles and prove our main result on descent (Theorem 5.6).

In Section 6, we extend results of Moore on simplicial groups in $\text{Set}$ to simplicial Lie groups. These results provide a ready supply of examples satisfying the hypotheses of Theorem 5.6. We conclude in Section 7 by applying our results to construct finite dimensional Lie $2$-groups. We describe the resulting model of $\text{String}(n)$ and compare it to the model constructed by Schommer-Pries [18].

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2. Higher Stacks

We work in a category $\mathcal{C}$ with a subcategory of “covers”.

Axiom 1. The category $\mathcal{C}$ has a terminal object $\ast$, and the map $X \rightarrow \ast$ is a cover for every object $X \in \mathcal{C}$.

Axiom 2. Pullbacks of covers along arbitrary maps exist and are covers.

Axiom 3. If $g$ and $f$ are composable maps such that $fg$ and $g$ are covers, then $f$ is also a cover.

If it exists, the kernel pair of a map $f : X \rightarrow Y$ in a category $\mathcal{C}$ is the pair of parallel arrows

$$X \times_Y X \rightarrow X$$

The map $f : X \rightarrow Y$ is an effective epimorphism if $f$ is the coequalizer of this pair:

$$X \times_Y X \rightarrow X \xrightarrow{f} Y$$

Axiom 4. Covers are effective epimorphisms.
Axioms 1 and 2 ensure that isomorphisms are covers, that \( C \) has finite products, that projections along factors of products are covers, and that covers form a pre-topology on \( C \). Axiom 3, which we borrow from Behrend and Getzler [2], ensures that being a cover is a local property: it is preserved and reflected under pullback along covers. Likewise, Axiom 4 ensures that being an isomorphism is a local property.

We will be interested in the following examples of categories with covers satisfying these assumptions:

1. \( \textbf{Set} \), the category of sets, with surjections as covers;
2. \( \textbf{Smooth} \), the category of finite-dimensional smooth manifolds, with surjective submersions as covers;
3. the category of Banach manifolds, with surjective submersions as covers (see [12]);
4. the category of analytic manifolds over a complete normed field, with surjective submersions as covers (see [19, Chapter III]).

Denote by \( sC \) the category of simplicial objects in \( C \). In particular, we have the category of simplicial sets \( s\textbf{Set} \). The category \( C \) embeds fully faithfully in \( sC \) as the category of constant simplicial diagrams. We do not distinguish between the category \( C \) and its essential image under this embedding.

Let \( \Delta^k \) denote the standard \( k \)-simplex, that is, the simplicial set \( \Delta^k = \Delta(\ast, [k]) \).

A simplicial set \( S \) is the colimit of its simplices

\[
\colim_{\Delta^k \to S} \Delta^k \cong S
\]

**Definition 2.1.** Let \( X_\bullet \) be a simplicial object in \( C \). Let \( S \) be a simplicial set. Denote by \( \text{hom}(S, X) \) the limit

\[
\text{hom}(S, X) := \lim_{\Delta^k \to S} X_k
\]

Note that such limits do not exist in general.

By a Lie group, we will mean a group internal to the category \( C \), that is, an object \( G \) with product \( m : G \times G \to G \), inverse \( i : G \to G \), and identity \( e : \ast \to G \), satisfying the usual axioms. We may associate to a Lie group its nerve \( N_\bullet G \in sC \), which is the simplicial object

\[
N_k G = G^k.
\]

The face maps \( d_i : G^k \to G^{k-1} \) are defined for \( i = 0 \) and \( i = k \) by projection along the first and last factor respectively, and for \( 0 < i < k \) by

\[
d_i = G^{i-1} \times m \times G^{k-1}.
\]

The degeneracy maps \( s_i : G^k \to G^{k+1} \) are defined by

\[
s_i = G^i \times e \times G^{k-i}.
\]

In fact, the above construction does not use the existence of an inverse, and works if \( G \) is only a monoid in \( C \).

We will use the following simplicial subsets of \( \Delta^k \):

1. the boundary \( \partial \Delta^k \) of \( \Delta^k \);
2. the \( i^{th} \) horn \( \Lambda^k_i \subset \partial \Delta^k \), obtained from \( \partial \Delta^k \) by omitting its \( i^{th} \) face.
Definition 2.2. Let \( f : X \to Y \) be a map in \( sC \). The \textit{matching object} \( M_k(f) \) is the limit
\[
\hom(\partial \Delta^k, X) \times_{\hom(\partial \Delta^k, Y)} Y_n.
\]
Denote by \( \mu_k(f) \) the induced map from \( X_k \to M_k(f) \).

The object of \textit{relative} \( \Lambda^k \)-\textit{horns} \( \Lambda^k(f) \) is the limit
\[
\hom(\Lambda^k, X) \times_{\hom(\Lambda^k, Y)} Y_k.
\]
Denote by \( \lambda^k(f) \) the induced map from \( X_k \to \Lambda^k(f) \).

A section of \( M_k(f) \) is a \( k \)-simplex of \( Y \) together with a lift of its boundary to \( X \), and \( \lambda^k(f) \) measures the extent to which these relative spheres are filled by \( k \)-simplices of \( X \). Similarly, a section of \( \Lambda^k(f) \) is a \( k \)-simplex of \( Y \) together with a lift of the \( \Lambda^k \)-horn to \( X \), and \( \lambda^k(f) \) measures the extent to which these relative horns are filled by \( k \)-simplices of \( X \).

In the absolute case, where the target \( Y \) of the simplicial map \( f \) is the terminal object \( * \), we write \( M_k(X) \) and \( \Lambda^k(X) \) instead of \( M_k(f) \) and \( \Lambda^k(f) \), and similarly for the induced maps \( \mu_k(X) \) and \( \lambda^k(X) \).

As an example, we have
\[
\Lambda^k(N\bullet G) \cong \begin{cases} *	ext{,} & k = 0, 1, \\ G^k, & k > 1. \end{cases}
\]
This is easily seen if \( 0 < i < k \); for \( i = 0 \) or \( i = k \), the proof requires the existence of the inverse for \( G \). In fact, the isomorphisms \( \Lambda^k(N\bullet G) \cong N_k G \), \( k > 1 \), together with the condition \( N_0 G \cong * \), characterize the nerves of groups, and indeed give an alternative axiomatization of the theory of groups.

Grothendieck extended this observation, omitting the condition \( N_0 G \cong * \).

Definition 2.3. A \textit{Lie groupoid} \( G \) in \( C \) is an internal groupoid in \( C \), with morphisms \( G_1 \) and objects \( G_0 \), source and target maps \( s, t : G_1 \to G_0 \), multiplication
\[
m : G_1 \times^{l.s}_{G_0} G_1 \to G_1,
\]
unit \( e : G_0 \to G_1 \), and inverse \( i : G_1 \to G_1 \), such that \( s \) and \( t \) are covers.

The \textit{nerve} \( N\bullet G \) of a groupoid is the simplicial object \( N\bullet G \in sC \),
\[
N_k G = \begin{cases} G_0, & k = 0, \\ G_1, & k = 1, \\ G_1 \times_{G_0}^{t.s} \cdots \times_{G_0}^{t.s} G_1, & k > 1. \end{cases}
\]
On 1-simplices, the face maps \( d_0 \) and \( d_1 \) correspond to the target \( t \) and source \( s \). The \textit{degeneracy} \( s_0 : G_0 \to G_1 \) corresponds to the unit. On \( k \)-simplices for \( k > 1 \), the face maps
\[
d_i : G_1 \times_{G_0}^{t.s} \cdots \times_{G_0}^{t.s} G_1 \to G_1 \times_{G_0}^{t.s} \cdots \times_{G_0}^{t.s} G_1
\]
are defined for \( i = 0 \) and \( i = k \) by projection along the first and last factor respectively, and for \( 0 < i < k \) by
\[
d_i = G_1 \times_{G_0}^{t.s} \cdots \times_{G_0}^{t.s} G_1 \times m \times G_1 \times_{G_0}^{t.s} \cdots \times_{G_0}^{t.s} G_1.
\]
The degeneracy maps
\[ s_i : G_1 \times_{G_0} \cdots \times_{G_0} G_1 \to G_1 \times_{G_0} \cdots \times_{G_0} G_1 \]
are defined by
\[ s_i = G_1 \times_{G_0} \cdots \times_{G_0} G_1 \times e \times G_1 \times_{G_0} \cdots \times_{G_0} G_1. \]

Grothendieck’s observation, generalized to Lie groupoids from his setting of discrete groupoids to, is as follows.

**Proposition 2.4** (Grothendieck). A simplicial object \( X_\bullet \in sC \) is isomorphic to the nerve of a Lie groupoid if and only if the horn-filler maps
\[ \lambda^k_i(X) : X_k \to \Lambda^k_i(X) \]
are covers for \( k = 1 \), and isomorphisms for \( k > 1 \).

In particular, a simplicial object \( X_\bullet \in sC \) is isomorphic to the nerve of a Lie group if and only if the above conditions are fulfilled and \( X_0 \equiv \ast \).

Maps between Lie groupoids are in bijection with simplicial maps between their nerves. As a result, the full subcategory of \( sC \) consisting of those simplicial objects satisfying the above conditions is equivalent to the category of Lie groupoids.

Motivated by Grothendieck’s observation, Duskin [6] introduced a notion of \( n \)-groupoid valid in any topos. Duskin’s notion was adapted by Henriques [12] (see also [10]) to cover higher Lie groupoids.

**Definition 2.5.** Let \( n \in \mathbb{N} \cup \{\infty\} \). A **Lie \( n \)-groupoid** is a simplicial object \( X_\bullet \in sC \) such that for all \( k > 0 \) and \( 0 \leq i \leq k \), the limit \( \Lambda^k_i(X) \) exists in \( C \), the map
\[ \lambda^k_i(X) : X_k \to \Lambda^k_i(X) \]
is a cover, and it is an isomorphism for \( k > n \).

A **Lie \( n \)-group** \( X_\bullet \) is a Lie \( n \)-groupoid such that \( X_0 = \ast \).

As an example, a Lie 0-groupoid is the same as an object of \( C \) (viewed as a constant simplicial diagram).

**Definition 2.6** (Verdier). Let \( n \in \mathbb{N} \cup \{\infty\} \). A map \( f : X_\bullet \to Y_\bullet \) of Lie \( \infty \)-groupoids is an **\( n \)-hypercover** if, for all \( k \geq 0 \), the limit \( M_k(f) \) exists in \( C \), the map
\[ \mu_k(f) : X_k \to M_k(f) \]
is a cover for all \( k \), and it is an isomorphism for \( k \geq n \).

Hypercovers in \( sSet \) are the same as trivial fibrations, that is, Kan fibrations which are also weak homotopy equivalences. Hypercovers play much the same role in the theory of Lie \( n \)-groupoids. We refer to an \( \infty \)-hypercover simply as a “hypercover.”

A 0-hypercover is an isomorphism, while a 1-hypercover of a Lie 0-groupoid is isomorphic to the nerve of the cover \( f_0 : X_0 \to Y_0 \). In other words,
\[ X_k \cong X_0 \times_{Y_0} \cdots \times_{Y_0} X_0 \]
Definition 2.7. An *augmentation* of a simplicial object $X_\bullet \in s\mathcal{C}$ is a simplicial map to an object $Y \in \mathcal{C} \subset s\mathcal{C}$.

This amounts to the same thing as a map $\varepsilon : X_0 \rightarrow Y$ that renders the diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{d_0} & X_0 \\
\downarrow{d_1} & & \downarrow{\varepsilon} \\
X_0 & \xrightarrow{\varepsilon} & Y
\end{array}
$$

commutative.

Definition 2.8. The *orbit space* $\pi_0(X)$ of a Lie $\infty$-groupoid $X_\bullet$ is a cover

$$X_0 \rightarrow \pi_0(X)$$

which coequalizes the fork

$$
\begin{array}{ccc}
X_1 & \xrightarrow{d_0} & X_0 \\
\downarrow{d_1} & & \downarrow{\pi_0(X)} \\
X_0 & \xrightarrow{\pi_0(X)} & Y
\end{array}
$$

In other words, for any augmentation $\varepsilon : X_0 \rightarrow Y$ of $X_\bullet$, there is an induced map

$$
\begin{array}{ccc}
X_1 & \xrightarrow{d_0} & X_0 \\
\downarrow{d_1} & & \downarrow{\pi_0(X)} \\
X_0 & \xrightarrow{\pi_0(X)} & Y
\end{array}
$$

Furthermore, if $\varepsilon$ is a cover, then so is the induced map from $\pi_0(X)$ to $Y$, by Axiom [3]

It is characteristic of the theory of Lie $\infty$-groupoids that the orbit space $\pi_0(X)$ need not exist. One case in which the orbit space of $X_\bullet$ exists, however, is when $X_\bullet$ admits an augmentation $\varepsilon : X_\bullet \rightarrow Y$ which is a hypercover.

Proposition 2.9.

1. An augmentation $\varepsilon : X_\bullet \rightarrow Y$ is an $n$-hypercover if and only if the maps $\varepsilon : X_0 \rightarrow Y$ and $\mu_1(\varepsilon) : X_1 \rightarrow X_0 \times Y X_0$ are covers, and the maps $\mu_k(X) : X_k \rightarrow M_k(X)$ are covers for $k > 1$ and isomorphisms for $k \geq n$.

2. If the augmentation $\varepsilon : X_\bullet \rightarrow Y$ is a hypercover, then $\pi_0(X) \cong Y$.

Proof. If $\varepsilon : X_\bullet \rightarrow Y$ is an augmentation, we have

$$M_k(\varepsilon) = \begin{cases} Y, & k = 0; \\ X_0 \times Y X_0, & k = 1; \\ M_k(X), & k > 1. \end{cases}$$

The first part follows by inspection.

The second part is a restatement of Axiom [4].

Let $\Delta_{\leq n} \subseteq \Delta$ be the full subcategory with objects $\{[m] \mid m \leq n\}$. An $n$-truncated simplicial object is a functor

$$X_{\leq n} : \Delta_{\leq n} \rightarrow \mathcal{C}.$$
Denote by $s_{\leq n}C$ the category of $n$-truncated simplicial objects in $C$. Restriction along $\Delta_{\leq n} \hookrightarrow \Delta$ induces the functor of $n$-truncation:

$$\text{tr}_n : sC \longrightarrow s_{\leq n}C.$$ 

When $S$ is a simplicial set of dimension less than or equal to $n$, we abuse notation and write $S$ for $\text{tr}_n S$.

When $C$ has finite limits, $n$-truncation $\text{tr}_n : sC \longrightarrow s_{\leq n}C$ admits a right-adjoint $\text{cosk}_n : s_{\leq n}C \longrightarrow sC$, called the $n$-coskeleton. The composition $\text{cosk}_n \circ \text{tr}_n : sC \longrightarrow sC$ is denoted $\text{Cosk}_n$.

In the category of simplicial sets, there is also a left-adjoint $\text{sk}_n : s_{\leq n}\text{Set} \longrightarrow \text{Set}$ to $n$-truncation, called the $n$-skeleton.

Lemma 2.10. Suppose that $S$ is $n$-dimensional. Let $Y_\bullet \in sC$ be a simplicial object such that the limit $\text{hom}(T, Y)$ exists. Let $f : \text{tr}_n X_\bullet \longrightarrow \text{tr}_n Y_\bullet$ be a map in $s_{\leq n}C$ such that the matching object $M_k(f)$ exists for all $k \leq n$, and the map $\mu_k(f) : X_k \longrightarrow M_k(f)$ is a cover for all $k \leq n$. Then the limit

$$\text{hom}(S, X) \times_{\text{hom}(S, Y)} \text{hom}(T, Y)$$

exists.

Proof. Filter the simplicial set $S$

$$\emptyset = S_0 \hookrightarrow \ldots \hookrightarrow S_N = S$$

where $S_\ell \cong S_{\ell-1} \cup_{\partial \Delta^n} \Delta^n$. Here, $n_\ell \leq n$ for all $\ell$.

Suppose that the limit $Z_j = \text{hom}(S_j, X) \times_{\text{hom}(S_j, Y)} \text{hom}(T, Y)$ exists for $j < \ell$. This is true for $\ell = 1$, since $Z_0 \cong \text{hom}(T, Y)$. The limit $Z_\ell$ is the pullback

$$
\begin{array}{ccc}
Z_\ell & \longrightarrow & X_{n_\ell} \\
\downarrow & & \mu_{n_\ell}(f) \\
Z_{\ell-1} & \longrightarrow & M_{n_\ell}(f)
\end{array}
$$

This pullback exists because $\mu_{n_\ell}(f)$ is a cover. \qed
Lemma 2.11. Let \( f : X_\bullet \longrightarrow Y_\bullet \) be a hypercover such that the limit
\[
\text{hom}(S, X) \times_{\text{hom}(S, Y)} \text{hom}(T, Y)
\]
exists. Then the limit \( \text{hom}(T, X) \) exists and the map
\[
\text{hom}(T, X) \longrightarrow \text{hom}(S, X) \times_{\text{hom}(S, Y)} \text{hom}(T, Y)
\]
is a cover.

Proof. Filter the simplicial set \( T \)
\[
S = S_0 \hookrightarrow \ldots \hookrightarrow S_N = T
\]
where
\[
S_\ell \cong S_{\ell-1} \cup_{\partial \Delta^n} \Delta^n \ell.
\]
Suppose that the limit
\[
Z_j = \text{hom}(S_j, X) \times_{\text{hom}(S_j, Y)} \text{hom}(T, Y)
\]
exists for \( j < \ell \), and that the map \( Z_j \longrightarrow Z_0 \) is a cover. The limit
\[
Z_0 = \text{hom}(S, X) \times_{\text{hom}(S, Y)} \text{hom}(T, Y)
\]
extists by hypothesis. The limit \( Z_\ell \) is the pullback
\[
\begin{array}{ccc}
Z_\ell & \longrightarrow & X_{n_\ell} \\
\downarrow & & \downarrow \mu_{n_\ell}(f) \\
Z_{\ell-1} & \longrightarrow & M_{n_\ell}(f)
\end{array}
\]
This pullback exists because \( \mu_{n_\ell}(f) \) is a cover.

We conclude that the limit \( Z_N = \text{hom}(T, X) \) exists, and that the morphism \( Z_N \longrightarrow Z_0 \) is a cover. \( \Box \)

Theorem 2.12.

(1) The composition of two \( n \)-hypercovers is an \( n \)-hypercover.

(2) The pullback of an \( n \)-hypercover along a map of Lie \( \infty \)-groupoids exists in \( sC \), and is an \( n \)-hypercover.

Proof. Consider a composable pair of \( n \)-hypercovers
\[
X_\bullet \xrightarrow{g} Y_\bullet \xrightarrow{f} Z_\bullet
\]
Suppose that the matching object \( M_j(fg) \) exists and that the map \( \mu_j(fg) \) is a cover for \( j < k \). This is certainly the case for \( k = 1 \), since \( M_0(fg) \cong Z_0 \), and \( \mu_0(fg) = f_0g_0 \) is the composition of the two covers \( f_0 \) and \( g_0 \). Lemma 2.11 now shows that the matching object \( M_k(fg) \) exists. The square in the commuting diagram
\[
\begin{array}{ccc}
X_k & \xrightarrow{\mu_k(g)} & M_k(g) \\
\downarrow \mu_k(fg) & & \downarrow \mu_k(f) \\
M_k(fg) & \xrightarrow{M_k(fg)} & M_k(f)
\end{array}
\]
is a pullback. Since \( f \) and \( g \) are \( n \)-hypercovers, we see that \( \mu_k(fg) \) is a (composition of) cover(s) for all \( k \), and an isomorphism if \( k > n \).
We turn to the second statement. Consider an \( n \)-hypercover \( f : X_\bullet \rightarrow Z_\bullet \) and a map \( g : Y_\bullet \rightarrow Z_\bullet \) of Lie \( \infty \)-groupoids. Suppose that the limits \( g^*X_j \) and \( M_j(g^*f) \) exist and that the maps \( \mu_j(g^*f) \) are covers for \( j < k \). Lemma 2.10 shows that the matching object \( M_k(g^*f) \) exists. The limit \( g^*X_k \) is the pullback

\[
\begin{array}{ccc}
g^*X_k & \rightarrow & X_k \\
\mu_k(g^*f) & \downarrow & \mu_k(f) \\
M_k(g^*f) & \rightarrow & M_k(f)
\end{array}
\]

The map \( \mu_k(f) \) is a cover for all \( k \) because \( f \) is an \( n \)-hypercover. This shows that the pullback \( g^*X_k \) exists, that the map \( \mu_k(g^*f) \) is a cover for all \( k \), and that it is an isomorphism for \( k \geq n \). \( \square \)

There is also a relative version of the notion of a Lie \( n \)-groupoid, modeled on the definition of a Kan fibration in the theory of simplicial sets.

**Definition 2.13.** Let \( n \in \mathbb{N} \cup \{ \infty \} \). A map \( f : X_\bullet \rightarrow Y_\bullet \) of Lie \( \infty \)-groupoids is an \( n \)-stack if for all \( k > 0 \) and \( 0 \leq i \leq k \), the limit \( \Lambda^k_\bullet(f) \) exists, the map

\[
\lambda^k_i(f) : X_k \rightarrow \Lambda^k_i(f)
\]

is a cover, and it is an isomorphism if \( k > n \).

**Remark 2.14.** If we wanted to emphasize the origin in simplicial homotopy theory, we might well have called \( \infty \)-stacks “Kan fibrations”, as in [12]. The present terminology emphasizes their relation with geometry.

There are analogues of Lemmas 2.10 and 2.11 for \( n \)-stacks, due to Henriques [12], but only under certain additional conditions on the simplicial sets \( S \) and \( T \).

**Definition 2.15.** An inclusion of finite simplicial sets \( S \hookrightarrow T \) is an expansion if it can be written as a composition

\[
S = S_0 \hookrightarrow \ldots \hookrightarrow S_N = T
\]

where

\[
S_\ell \cong S_{\ell-1} \cup_{N_\ell} \Delta^n_{\ell}.
\]

A finite simplicial set \( S \) is collapsible if the inclusion of some, and hence any, vertex is an expansion.

Let \( S \hookrightarrow T \) be a monomorphism of finite simplicial sets.

**Lemma 2.16.** Suppose that \( S \) is \( n \)-dimensional and collapsible. Let \( Y_\bullet \in sC \) be a simplicial object such that the limit \( \hom(T, Y) \) exists and the restriction to any vertex

\[
\hom(T, Y) \rightarrow Y_0
\]

is a cover. Let \( f : \text{tr}_nX_\bullet \rightarrow \text{tr}_nY_\bullet \) be a map in \( s_{\leq n}C \) such that the limit \( \Lambda^k_\bullet(f) \) exists for all \( 0 < k \leq n \) and \( 0 \leq i \leq n \), and the map

\[
\lambda^k_i(f) : X_k \rightarrow \Lambda^k_i(f)
\]

is a cover for all \( 0 < k \leq n \) and \( 0 \leq i \leq n \). Then the limit

\[
\hom(S, X) \times_{\hom(S, Y)} \hom(T, Y)
\]

exists.
Proof. Filter the simplicial set $S$

$$\Delta^0 = S_0 \hookrightarrow \ldots \hookrightarrow S_N = S$$

where

$$S_\ell \cong S_{\ell-1} \cup_{\Lambda^{n_\ell}_\ell} \Delta^{n_\ell}.$$  

Here, $n_\ell \leq n$ for all $\ell$.

Suppose that the limit

$$Z_j = \text{hom}(S_j, X) \times_{\text{hom}(S_j, Y)} \text{hom}(T, Y).$$

does not exist for $j < \ell$. This is true for $\ell = 1$, by the hypotheses on $\text{hom}(T, Y)$. We have the pullback diagram

$$\begin{array}{ccc}
Z_\ell & \to & X_{n_\ell} \\
\downarrow & & \downarrow \\
Z_{\ell-1} & \to & \Lambda^{n_\ell}_\ell(f)
\end{array}$$

This pullback exists because $\lambda^{n_\ell}_\ell(f)$ is a cover. \hfill \Box

Lemma 2.17. Let $f : X_\bullet \longrightarrow Y_\bullet$ be an $\infty$-stack such that the limit

$$\text{hom}(S, X) \times_{\text{hom}(S, Y)} \text{hom}(T, Y)$$

does not exist. Suppose that the inclusion $S \hookrightarrow T$ is an expansion. Then the limit $\text{hom}(T, X)$ exists, and the map

$$\text{hom}(T, X) \longrightarrow \text{hom}(S, X) \times_{\text{hom}(S, Y)} \text{hom}(T, Y)$$

does not exist.

Proof. Filter the simplicial set $T$

$$S = S_0 \hookrightarrow \ldots \hookrightarrow S_N = T$$

where

$$S_\ell \cong S_{\ell-1} \cup_{\Lambda^{n_\ell}_\ell} \Delta^{n_\ell}.$$  

Suppose that the limit

$$Z_j = \text{hom}(S_j, X) \times_{\text{hom}(S_j, Y)} \text{hom}(T, Y).$$

does not exist for $j < \ell$, and the map $Z_j \longrightarrow Z_0$ is a cover. The limit $Z_0 = \text{hom}(S, X) \times_{\text{hom}(S, Y)} \text{hom}(T, Y)$ exists by hypothesis. We have the pullback diagram

$$\begin{array}{ccc}
Z_\ell & \to & X_{n_\ell} \\
\downarrow & & \downarrow \\
Z_{\ell-1} & \to & \Lambda^{n_\ell}_\ell(f)
\end{array}$$

This pullback exists because $\lambda^{n_\ell}_\ell(f)$ is a cover.

We conclude that the limit $Z_N = \text{hom}(T, X)$ exists, and that the morphism $Z_N \longrightarrow Z_0$ is a cover. \hfill \Box

Theorem 2.18.
Lemma 2.10 shows that the limit $\Lambda^k_{\bullet}$ and, by Axiom 4, an epimorphism. The diagram (2.1) now implies that $\lambda^k_{\bullet}$ is a pullback. Since $f$ is an $\infty$-hypercover, we see that $f$ is a pullback. If $k > n$, then the pullback of $f_0$ along $g_0$ exists in $\mathcal{C}$, then the pullback of $f$ along $g$ exists in $\mathcal{C}$ and this pullback is an $n$-stack.

**Proof.** Let $f : X_{\bullet} \rightarrow Y_{\bullet}$ be an $\infty$-hypercover. We see that $f$ is an $\infty$-stack by considering the finite inclusions $\Lambda^k_{\bullet} \hookrightarrow \Delta^k$ and applying Lemma 2.11. It remains to show that if $f$ is an $n$-hypercover, then $\lambda^k_{\bullet}(f)$ is an isomorphism when $k > n$ (and $0 \leq i \leq k$).

The square in the commuting diagram

$$
\begin{array}{ccc}
X_k & \xrightarrow{\mu_k(f)} & M_k(f) \\
\downarrow{\lambda^k_{\bullet}(f)} & & \downarrow{\mu_{k-1}(f)} \\
\Lambda^k_{\bullet}(f) & \rightarrow & M_{k-1}(f)
\end{array}
$$

(2.1)

is a pullback. If $f$ is an $n$-hypercover and $k > n$, then the maps $\mu_k(f)$ and $\mu_{k-1}(f)$ are isomorphisms, and we see that $\lambda^k_{\bullet}(f)$ is an isomorphism.

To prove the second part, we consider (2.1) in the case where $f$ is a hypercover and an $n$-stack. If $k > n$, then $\lambda^k_{\bullet}(f)$ is an isomorphism. The map $\mu_k(f)$ is a cover, and, by Axiom 4, an epimorphism. The diagram (2.1) now implies that $\mu_k(f)$ is an isomorphism. Similarly, the map $M_k(f) \rightarrow \Lambda^k_{\bullet}(f)$ is an isomorphism.

The map $\Lambda^k_{\bullet}(f) \rightarrow M_{k-1}(f)$ is induced by the inclusion $\partial\Delta^{k-1} \hookrightarrow \Delta^k$. Lemma 2.10 shows that it is a cover. The pullback of $\mu_{k-1}(f)$ along this cover is the map $M_k(f) \rightarrow \Lambda^k_{\bullet}(f)$. This map is an isomorphism for $k > n$. Axiom 4 therefore implies that $\mu_{k-1}(f)$ is an isomorphism for $k > n$. Hence $f$ is an $n$-hypercover.

Turning to the third part of the theorem, consider a composable pair of $n$-stacks

$$
X_{\bullet} \xrightarrow{g} Y_{\bullet} \xrightarrow{f} Z_{\bullet}
$$

Suppose that the limit $\Lambda^j_{\bullet}(fg)$ exists and that the map $\lambda^j_{\bullet}(fg)$ is a cover, for all $0 < j < k$ (and $0 \leq i \leq j$). Lemma 2.11 now shows that the limit $\Lambda^k_{\bullet}(fg)$ exists. The square in the commuting diagram

$$
\begin{array}{ccc}
X_k & \xrightarrow{\lambda^k_{\bullet}(g)} & \Lambda^k_{\bullet}(g) \\
\downarrow{\lambda^k_{\bullet}(fg)} & & \downarrow{\lambda^k_{\bullet}(f)} \\
\Lambda^k_{\bullet}(fg) & \rightarrow & \Lambda^k_{\bullet}(f)
\end{array}
$$

is a pullback. Since $f$ and $g$ are $n$-stacks, we see that $\lambda^k_{\bullet}(fg)$ is a (composition of) cover(s) for all $k > 0$ and $0 \leq i \leq k$, and an isomorphism if $k > n$.

We turn to the third statement. Consider an $n$-stack $f : X_{\bullet} \rightarrow Z_{\bullet}$ and a map $g : Y_{\bullet} \rightarrow Z_{\bullet}$ of Lie $\infty$-groupoids. Suppose that, for $j < k$ (and $0 \leq i \leq j$), the pullback $g^*X_j$ exists, the limit $\Lambda^j_{\bullet}(g^*f)$ exists, and the map $\lambda^j_{\bullet}(g^*f)$ is a cover. Lemma 2.10 shows that the limit $\Lambda^k_{\bullet}(g^*f)$ exists for $0 \leq i \leq k$. The limit $g^*X_k$ is
the pullback

\[
g^*X_k \xrightarrow{\lambda^k(g^*f)} X_k \xrightarrow{\lambda^k(f)} \Lambda^k_i(g^*f) \xrightarrow{\lambda^k_i(f)} \Lambda^k_i(f)
\]

The map \(\lambda^k_i(f)\) is a cover for all \(k > 0\) because \(f\) is an \(n\)-hypercover. This shows that the pull-back \(g^*X_k\) exists, that the map \(\lambda^k_i(g^*f)\) is a cover for all \(k > 0\) and \(0 \leq i \leq k\), and that this map is an isomorphism for \(k > n\). □

3. Higher Morphism Spaces in Higher Stacks

Order preserving maps of all finite ordinals, including \(0\), form a category \(\Delta_+\) extending \(\Delta\). An augmented simplicial set is a functor

\[
\Delta_+ \longrightarrow \mathbf{Set}
\]

Such a functor consists of a simplicial set \(S\) equipped with a map to a constant simplicial set \(S_{-1}\).

The ordinal sum

\[
[n] + [m] := \{0 \leq \ldots \leq n \leq 0' \leq \ldots \leq m'\} = [n + m + 1]
\]

endows the category \(\Delta_+\) with a monoidal structure. This structure extends along the Yoneda embedding

\[
\Delta_+ \longrightarrow \mathbf{sSet}_+
\]

to give a closed monoidal structure on \(\mathbf{sSet}_+\) called the join, and denoted \(\ast\). Given an augmented simplicial set \(K\), denote the right adjoint to \(K \ast (-)\) by

\[
\mathbf{sSet}_+ \overset{(-)K\ast}{\longrightarrow} \mathbf{sSet}_+
\]

**Example 3.1.** The functor \((-)^{\Delta_n-\ast}\) is Illusie’s \(\text{Dec}_n(-)\) (c.f. [13, Chapter VI]).

The inclusion \(i : \Delta \hookrightarrow \Delta_+\) provides a forgetful functor

\[
\mathbf{sSet}_+ \overset{i^\ast}{\longrightarrow} \mathbf{sSet}
\]

Its right adjoint

\[
\mathbf{sSet} \overset{i_*}{\longrightarrow} \mathbf{sSet}_+
\]

augments a simplicial set by a point.

**Definition 3.2.**

1. Let \(S\) and \(T\) be simplicial sets. Denote by \(S \ast T\) the simplicial set

\[
S \ast T = i^*((i_*S) \ast (i_*T)).
\]

2. Let \(X\) be in \(\mathbf{sC}\) and let \(S\) be a finite simplicial set. Denote by \(X^S\ast\) the putative simplicial object with \(k\)-simplices

\[
X^S_k := \text{hom}(S \ast \Delta^k, X)
\]

Face and degeneracy maps are given by \(1 \ast d_i\) and \(1 \ast s_i\).
In [7], Duskin gave a construction of the collection of morphisms in a higher category.\footnote{Duskin called this the “path-homotopy complex”. His construction is hinted at in the earlier treatment of nerves in [13], and of weak Kan complexes in [4].}

**Definition 3.3.** Let $X_\bullet$ be an $\infty$-groupoid in $\text{Set}$. Define $P^{\geq 1}X_\bullet$ to be the pullback

$$
\begin{array}{c}
P^{\geq 1}X_\bullet \\
\downarrow \\
X_0 \\
\downarrow \\
X_\bullet
\end{array}
$$

The simplicial set $P^{\geq 1}X$ is an $\infty$-groupoid which models the “space” of $1$-morphisms in $X_\bullet$.\footnote{Lurie [16] denotes this construction ‘$\text{hom}_L$’. Given $x, y \in X_0$, Lurie’s $\text{hom}_L(x, y)$ is the fiber at $(x, y)$ of a canonical map $P^{\geq 1}X \to \text{hom}(\partial \Delta^1, X)$.}

We will generalize $P^{\geq 1}X_\bullet$ to higher morphism spaces $P^{\geq k}X_\bullet$, for $k > 0$, and at the same time, we will define a relative version of the construction. The discussion follows the lines of Joyal’s proof of [15, Theorem 3.8], except that we restrict attention to the case

$$(S \hookrightarrow T) = (\partial \Delta^{k-1} \hookrightarrow \Delta^{k-1}),$$

and work in the Lie setting.

Given a proper, non-empty subset $J$ of $[n]$, let

$$\Lambda^n_j = \bigcup_{i \in J} \partial_i \Delta^n \subset \partial \Delta^n.$$  

An induction shows that $\Lambda^n_j$ is collapsible.

Let $f : X_\bullet \to Y_\bullet$ be an $\infty$-stack. The $\ell$-simplices of the simplicial object

$$(3.1) \quad X_\bullet \partial \Delta^{k-1} \times_{Y_\bullet \partial \Delta^{k-1}} Y_\bullet \Delta^{k-1}$$

are given by the limit

$$\text{hom}(\partial \Delta^{k-1} \ast \Delta^\ell, X) \times_{\text{hom}(\partial \Delta^{k-1} \ast \Delta^\ell, Y)} \text{hom}(\Lambda_0^{k+\ell}, X) \times_{\text{hom}(\Lambda_0^{k+\ell}, Y)} Y_{k+\ell}.$$

Lemma [2.10] implies that this limit exists. In the special case of vertices $\ell = 0$, we obtain a natural identification with the space of relative horns $\Lambda_k^\ell(f)$.

Denote by $f^\partial \Delta^{k-1}$ the induced map

$$f^\partial \Delta^{k-1} : X_{\Delta^{k-1}} \times_{Y_\bullet \partial \Delta^{k-1}} Y_\bullet \Delta^{k-1} \times_{Y_\bullet \Delta^{k-1}} Y_\bullet \Delta^{k-1}.$$

**Lemma 3.4.** Let $f : X_\bullet \to Y_\bullet$ be an $\infty$-stack. For $\ell > 0$, the map $\Lambda_0^\ell(f^\partial \Delta^{k-1})$ is canonically isomorphic to $\Lambda_0^{k+1}(f)$.

**Proof.** An exercise in the combinatorics of joins shows that

$$(\Delta^{k-1} \ast \Lambda_0^\ell \hookrightarrow \Delta^{k-1} \ast \Delta^\ell) \cong (\Lambda_0^{k+\ell} \hookrightarrow \Delta^{k+\ell}),$$

and

$$(\partial \Delta^{k-1} \hookrightarrow \Delta^{k-1} \ast \Delta^\ell) \cong (\Lambda_0^{k+\ell} \hookrightarrow \Delta^{k+\ell}).$$
In this way, we obtain a pushout square

$$\partial \Delta^{k-1} \ast \Lambda_i \longrightarrow \partial \Delta^{k-1} \ast \Delta^f$$

which gives rise to the pullback square

$$\Lambda^k(f) \longrightarrow (X \circ \partial \Delta^{k-1} \ast Y_{\partial \Delta^{k-1} \ast})_f$$

But this is also the pullback square defining $\Lambda^f(f_{\partial \Delta^{k-1} \ast})$.

It follows that if $f$ is an $\infty$-stack, then so is $\partial \Delta^{k-1} \ast$, and if $f$ is an $n$-stack, $\partial \Delta^{k-1} \ast$ is an $(n - k)$-stack.

**Definition 3.5.** For $k > 0$, the relative higher morphism space $P^{\geq k}(f)_\bullet$ is the pullback

$$\xymatrix{P^{\geq k}(f)_\bullet \ar[r] \ar[d] & X_{\ast}^{\Delta^{k-1} \ast} \ar[d] \\
\Lambda^k(f) \ar[r] & X_{\ast}^{\partial \Delta^{k-1} \ast} \times Y_{\partial \Delta^{k-1} \ast} \times Y_{\partial \Delta^{k-1} \ast}^\ast}$$

The following result is an immediate consequence of Lemma 3.4 and Theorem 2.18.

**Theorem 3.6.** If $f : X \longrightarrow Y$ is an $n$-stack, the relative higher morphism space $P^{\geq k}(f)_\bullet$ is a Lie $(n - k)$-groupoid.

The vertices of $P^{\geq k}(f)_\bullet$ are the $k$-simplices of $X$. Its 1-simplices correspond to $(k + 1)$-simplices $x \in X_{k+1}$ such that

$$f(x) = s_k d_k f(x)$$

and, for $i < k$,

$$d_i x = s_{k-1} d_{k-1} d_i x.$$  

We interpret $x$ as a path rel boundary in the fiber of $f$, which begins at $d_{k+1} x$ and ends at $d_k x$.

When the matching object $M_k(f)$ exists, it provides a natural augmentation

(3.2) \hspace{1cm} \pi : \xymatrix{P^{\geq k}(f)_\bullet \ar[r] & M_k(f)}

The map underlying $\pi : P^{\geq k}(f)_0 \cong X_k \longrightarrow M_k(f)$ is $\mu_k(f)$. This determines an augmentation because the diagram

$$\xymatrix{P^{\geq k}(f)_1 \ar[r]^{d_0} \ar[d]_{d_1} & P^{\geq k}(f)_0 \ar[r]^\pi & M_k(f)}$$
commutes.

The augmentation (3.2) encodes the idea that vertices in the relative higher morphism space are lifts of a \( k \)-morphism in \( Y_* \), that edges are paths rel boundary between these lifts such that the paths live in the fiber of \( f \), etc.

**Theorem 3.7.** If \( f \) is an \( n \)-hypercover, then the augmentation (3.2) exists and is an \( (n - k) \)-hypercover.

**Proof.** We prove that \( \pi \) is an \( (n - k) \)-hypercover by showing that the square

\[
\begin{array}{ccc}
P^{\geq k}(f)_{\ell} & \longrightarrow & X_{k+\ell} \\
\downarrow^{\mu_\ell(\pi)} & & \downarrow^{\mu_{k+\ell}(f)} \\
M_\ell(\pi) & \longrightarrow & M_{k+\ell}(f) \\
\end{array}
\]

is a pullback. For \( \ell = 0 \), this is evident: both horizontal maps are isomorphisms in this case.

For \( \ell > 0 \), the above square is the top half of a commutative diagram

\[
\begin{array}{ccc}
P^{\geq k}(f)_{\ell} & \longrightarrow & X_{k+\ell} \\
\downarrow^{\mu_\ell(\pi)} & & \downarrow^{\mu_{k+\ell}(f)} \\
M_\ell(\pi) & \longrightarrow & M_{k+\ell}(f) \\
\downarrow & & \downarrow \\
\Lambda^k(f) & \longrightarrow & (X^{\partial \Delta^{k-1}_*} \times_{Y^{\partial \Delta^{k-1}_*}} Y^\Delta^{k-1}_*)_{\ell}
\end{array}
\]

The outer rectangle is a pullback by definition. Thus, it suffices to prove that the bottom square is a pullback.

For \( \ell = 1 \), this may be checked directly. For \( \ell > 1 \), an exercise in the combinatorics of joins shows that we have a pushout square

\[
\begin{array}{ccc}
\partial \Delta^{k-1} \times \partial \Delta \longrightarrow & \partial \Delta^{k-1} \times \Delta \\
\downarrow & & \downarrow \\
\Delta^{k-1} \times \partial \Delta \longrightarrow & \partial \Delta^{k+\ell}
\end{array}
\]

which gives rise to a pullback square

\[
\begin{array}{ccc}
M_{k+\ell}(f) & \longrightarrow & M_\ell(X^{\Delta^{k-1}_*}) \\
\downarrow & & \downarrow \\
(X^{\partial \Delta^{k-1}_*} \times_{Y^{\partial \Delta^{k-1}_*}} Y^\Delta^{k-1}_*)_{\ell} & \longrightarrow & M_\ell(X^{\partial \Delta^{k-1}_*} \times_{Y^{\partial \Delta^{k-1}_*}} Y^\Delta^{k-1}_*)
\end{array}
\]
This square embeds in a commutative diagram

\[
\begin{array}{ccc}
M_t(P\geq k(f)) & \longrightarrow & M_{k+t}(f) \\
\downarrow & & \downarrow \\
\Lambda^k(f) & \longrightarrow & (X^{\partial X^{k-1}} \times Y_{\partial X^{k-1}}. Y^{\Delta^{k-1}})_{t} \longrightarrow M_t(X^{\partial X^{k-1}} \times Y_{\partial X^{k-1}}. Y^{\Delta^{k-1}})
\end{array}
\]

The outer rectangle is a pullback because \(M_t(\cdot)\) commutes with limits. We observed above that the right square is a pullback. We conclude that the left square is a pullback, completing the proof that \(\mu_t(\pi)\) is a cover.

4. Strictification

In this section we recall Duskin’s “\(n\)-strictification” functor \(\tau_n\) for \(n \geq 0\). This is a partially defined left-adjoint to the inclusion of the category of \(n\)-stacks into the category of \(\infty\)-stacks. We establish its main properties in Propositions 4.5 and 4.6.

Let \(f: X_\bullet \longrightarrow Y_\bullet\) be an \(\infty\)-stack such that the orbit space \(\pi_0(P\geq n(f))\) exists. We define a map

\[
(4.1) \quad \text{tr}_n \tau_n(f): \text{tr}_n \tau_n(X, f)_\bullet \longrightarrow \text{tr}_n Y_\bullet
\]

On \(k\)-simplices, for \(k < n\), \(\text{tr}_n \tau_n(f)\) is the map

\[
f_k: X_k \longrightarrow Y_k
\]

On \(n\)-simplices, \(\text{tr}_n \tau_n(f)\) is the canonical map

\[
\pi_0(P\geq n(f)) \longrightarrow Y_n
\]

**Lemma 4.1.** Let \(f: X_\bullet \longrightarrow Y_\bullet\) be an \(\infty\)-stack such that the orbit space \(\pi_0(P\geq n(f))\) exists. The maps \(\lambda^k_i(\tau_n(f))\) are covers for \(k \leq n\). For all \(i\), the limit \(\Lambda^{n+1}_i(\tau_n(f))\) exists and the map \(\Lambda^{n+1}_i(\tau_n(f)) \longrightarrow \Lambda^{n+1}_i(\tau_n(f))\) is a cover.

**Proof.** For \(k < n\), the natural map \(\text{tr}_n X_\bullet \longrightarrow \text{tr}_n \tau_n(X, f)_\bullet\) induces an isomorphism between the maps \(\lambda^k_i(f)\) and \(\lambda^k_i(\text{tr}_n \tau_n(f))\). For \(k = n\), we have a commuting square

\[
\begin{array}{ccc}
X_n & \longrightarrow & \tau_n(X, f)_n \\
\downarrow \lambda^n_i(f) & & \downarrow \lambda^n_i(\tau_n(f)) \\
\Lambda^n_i(f) & \longrightarrow & \Lambda^n_i(\tau_n(f))
\end{array}
\]

This square guarantees that the map \(\lambda^n_i(\tau_n(f))\) is a cover for all \(i\). Indeed, the top horizontal map is the cover \(X_n \longrightarrow \pi_0(P\geq n(f))\), the map \(\lambda^n_i(f)\) is a cover by assumption, and the bottom horizontal map is an isomorphism. Axiom 3 implies that \(\lambda^n_i(\tau_n(f))\) is a cover.

Lemma 2.10 guarantees that, for all \(i\), the limit \(\Lambda^{n+1}_i(\tau_n(f))\) exists. It remains to show that the map \(\Lambda^{n+1}_i(\tau_n(f)) \longrightarrow \Lambda^{n+1}_i(\tau_n(f))\) is a cover for all \(i\).

Recall that given a proper, non-empty subset \(J\) of \([n + 1]\),

\[
\Lambda^{n+1}_J = \bigcup_{i \in J} \partial_i \Delta^{n+1} \subset \partial \Delta^{n+1}.
\]
For each $J$, there is a map
\[
\Lambda_{J}^{n+1}(f) = \text{hom}(\Lambda_{J}^{n+1}, X) \times_{\text{hom}(\Lambda_{J}^{n+1}, Y)} Y_{n+1}
\]
\[
\to \Lambda_{J}^{n+1}(\tau_n(f)) = \text{hom}(\Lambda_{J}^{n+1}, \tau_n(X, f)) \times_{\text{hom}(\Lambda_{J}^{n+1}, Y)} Y_{n+1}.
\]
We will show that it is a cover, by induction on $|J|$.

Let $J_+ = J \cup \{j\}$, where $j \notin J$, and let
\[
(4.2) \quad \Lambda_{J_+}^{n+1}(f) = \text{hom}(\Lambda_{J_+}^{n+1} \cap \partial_J \Delta^{n+1}, X) \times_{\text{hom}(\Lambda_{J_+}^{n+1} \cap \partial_J \Delta^{n+1}, Y)} Y_{n+1}.
\]
We have a pair of pullback diagrams in which the vertical maps are covers:
\[
\Lambda_{J_+}^{n+1}(f) \times_{\Lambda_{J_+}^{n+1}(f)} \pi_0(P^{\leq n}(f)) \to \Lambda_{J_+}^{n+1}(f)
\]
\[
\Lambda_{J_+}^{n+1}(\tau_n(f)) \to \Lambda_{J_+}^{n+1}(\tau_n(f))
\]
and
\[
\Lambda_{J_+}^{n+1}(f) \to X_n
\]
\[
\Lambda_{J_+}^{n+1}(f) \times_{\Lambda_{J_+}^{n+1}(f)} \pi_0(P^{\leq n}(f)) \to \pi_0(P^{\leq n}(f))
\]

Composing the left vertical arrows, we see that the map
\[
\Lambda_{J_+}^{n+1}(f) \to \Lambda_{J_+}^{n+1}(\tau_n(f))
\]
is a cover; this completes the induction step. \hfill \Box

Our goal is now to construct, for any $i$, a “missing face map”
\[
d_i: \Lambda_{i}^{n+1}(\tau_n(f)) \to \pi_0(P^{\geq n}(f)) = \tau_n(X, f)_n.
\]

Compose the covers $\Lambda_{i}^{n+1}(f) \to \Lambda_{i}^{n+1}(\tau_n(f))$ and $\Lambda_{i}^{n+1}(f)$ to obtain a cover $X_{n+1} \to \Lambda_{i}^{n+1}(\tau_n(f))$. Denote by $qd_i$ the composite
\[
X_{n+1} \xrightarrow{d_i} X_n \to \pi_0(P^{\geq n}(f))
\]

**Lemma 4.2.** The diagram
\[
(4.3) \quad X_{n+1} \times_{\Lambda_{i}^{n+1}(\tau_n(f))} X_{n+1} \xrightarrow{qd_i} \pi_0(P^{\geq n}(f))
\]
commutes.

**Remark 4.3.** Recall that a point of $C$ is a functor $p: C \to \text{Set}$ which preserves finite limits, which preserves arbitrary colimits, and which takes covers to surjections. We say $C$ has enough points if for every pair of maps $f \neq g$ in $C$, there exists a point $p$ such that $p(f) \neq p(g)$.

We prove the lemma under the assumption that $C$ has enough points. This assumption is satisfied in many examples of interest, and has the benefit of allowing for an elementary proof.

One could proceed without this assumption by using Ehresman’s theory of sketches in combination with Barr’s theorem on the existence of a Boolean cover of
the topos of sheaves on C. The latter approach is discussed in [3] Section 2 – “For Logical Reasons” or in more depth in [14, Chapter 7, especially 7.5].

Proof. To show that 4.3 commutes, we extend it to a diagram

\[
\begin{tikzcd}
K \arrow[r, g] & X_{n+1} \times_{\Lambda_i^{n+1}(\tau_n(f))} X_{n+1} \arrow[r, q_d] \arrow[r] & \pi_0(P^{\geq n}(f))
\end{tikzcd}
\]

where the fork

\[
\begin{tikzcd}
K \arrow[r] & X_{n+1}
\end{tikzcd}
\]

is a “fattened version” of the kernel pair of the cover \(X_{n+1} \to \Lambda_i^{n+1}(\tau_n(f))\). We show that the diagram

\[
\begin{tikzcd}
K \arrow[r, g] & X_{n+1} \arrow[r, q_d] \arrow[r] & \pi_0(P^{\geq n}(f))
\end{tikzcd}
\]

commutes and that the map \(g\) is an epi. This implies that 4.3 commutes.

We begin by defining the map \(g\): \(K \to X_{n+1} \times_{\Lambda_i^{n+1}(f)} X_{n+1}\). Define \(H^{n+1}_i(f)\) to be the pullback

\[
\begin{tikzcd}
H^{n+1}_i(f) \arrow[r] \arrow[d, (d_0)^{\times(n+1)}] & (P^{\geq n}(f))^{\times(n+1)} \arrow[d, (d_0)^{\times(n+1)}] \arrow[d, (X_n)^{\times(n+1)}] \\
\Lambda_i^{n+1}(f) \arrow[r] & (X_n)^{\times(n+1)} \arrow[r] & (\pi_0(P^{\geq n}(f)))^{\times(n+1)}
\end{tikzcd}
\]

Observe that the pullback exists because the right vertical maps are both covers.

In addition to the map \((d_0)^{\times(n+1)}: H^{n+1}_i(f) \to \Lambda_i^{n+1}(f)\), there is another map \((d_1)^{\times(n+1)}: H^{n+1}_i(f) \to \Lambda_i^{n+1}(f)\). Define \(K\) to be the iterated pullback

\[
\begin{tikzcd}
K \arrow[r] \arrow[r] & X_{n+1} \arrow[d, \Lambda_i^{n+1}(f)] \\
X_{n+1} \times_{\Lambda_i^{n+1}(f)} H^{n+1}_i(f) \arrow[r, (d_1)^{\times(n+1)}] \arrow[r] & H^{n+1}_i(f) \arrow[r, (d_0)^{\times(n+1)}] \arrow[r] & \Lambda_i^{n+1}(f) \\
X_{n+1} \arrow[r, \Lambda_i^{n+1}(f)] & \Lambda_i^{n+1}(f)
\end{tikzcd}
\]

The limit \(K\) exists because \(\Lambda_i^{n+1}(f)\) is a cover (\(f\) is an \(\infty\)-stack). The projections along the left and right \(X_{n+1}\) factors induce a map

\[
\begin{tikzcd}
K \arrow[r, g] & X_{n+1} \times_{\Lambda_i^{n+1}(\tau_n(f))} X_{n+1}
\end{tikzcd}
\]

We show that 4.4 commutes by constructing a sequence of covers

\[
\begin{align*}
K_{n+3} & \to \cdots \to K_{n+3-i} \to K_{n+1-i} \to \cdots \to K_0 = K
\end{align*}
\]
fitting into a commuting square

\[
\begin{array}{c}
K_{n+3} \xrightarrow{h} P^{2n}(f) \\
\downarrow{c} \quad \downarrow{q d_0, d_1} \\
\mathbb{K} \xrightarrow{(d_i, d_j, g)} X_n \times X_n
\end{array}
\]

(4.6)

Recall that \( q \) denotes the map \( X_n \rightarrow \pi_0(P^{2n}(f)) \). By definition,

\[
q d_0 = q d_1 : P^{2n}(f) \rightarrow \pi_0(P^{2n}(f))
\]

The square 4.6 implies that

\[
qd_i \mathsf{pr}_1 gc = qd_0 h = qd_1 h = qd_i \mathsf{pr}_2 gc
\]

The map \( c : K_{n+3} \rightarrow \mathbb{K} \) is an epimorphism, because it is cover (Axiom 4). We conclude that

\[
qd_i \mathsf{pr}_1 g = qd_i \mathsf{pr}_2 g,
\]

or equivalently, that 4.4 commutes.

The construction we have just described arises from the observation that \( \mathbb{K} \) encodes the data of pairs \((x_0, x_1)\) of \((n+1)\)-simplices and explicit homotopies rel \((n-1)\)-skeleta \((p_j)_{0\leq j \neq i}^{n+1}\), between all but their \( i \)th faces. For \( \ell < n + 3 \), the sequence of covers \( K_\ell \rightarrow K_{\ell-1} \) amounts to an explicit sequence of combinatorial moves, by which we use the homotopies \((p_j)_{0\leq j \neq i}^{n+1}\) to replace \( x_1 \) by a simplex whose \( i \)th-horn equals that of \( x_0 \). The final stage \( K_{n+3} \) amounts to an explicit homotopy rel boundary between the \( i \)th faces of two simplices all of whose other faces agree.

We now construct 4.5. In detail, a section of \( \mathbb{K} \) is a tuple \((x_0, x_1, (p_j)_{0\leq j \neq i}^{n+1})\) where

\[
x_j \in X_{n+1}, \text{ and } \\
p_j \in P^{2n}(f),
\]

such that

\[
f(x_0) = f(x_1),
\]

and, for all \( j, k = 0, 1 \),

\[
d_{n+k} p_j = d_j x_k.
\]

For concreteness, we describe the induction under the assumption \( i < n - 1 \). The inductions when \( i = n - 1, n, \text{ and } n + 1 \) are simpler, because we can omit one of the first three steps below. In all cases, we continue the induction until we have constructed \( K_\ell \) for \( n + 2 - i \neq \ell \leq n + 2 \). We then construct \( K_{n+3} \) as a final step.

The assignment

\[
(x_0, x_1, (p_j)_{0\leq j \neq i}^{n+1}) \mapsto ((d_0 s_n x_1, \ldots, d_{n-1} s_n x_1, \ldots, x_1, p_{n+1}), s_n f(x_1))
\]

defines a map \( \mathbb{K} \rightarrow \Lambda_n^{n+2}(f) \). Denote by \( K_1 \) the pullback

\[
K_1 := \mathbb{K} \times_{\Lambda_n^{n+2}(f)} X_{n+2}
\]
The projection $K_1 \to K$ is a cover, because it is the pullback of the cover $\lambda_{n+2}^n(f)$.

Denote a section of $K_1$ by a tuple $(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, y_{n+1})$ where

\[(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}) \in K, \quad \text{and} \quad y_{n+1} \in X_{n+2}.
\]

The assignment

\[(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, y_{n+1}) \mapsto ((d_0 s_{n+1} d_n y_{n+1}, \ldots, d_{n-1} s_{n+1} d_n y_{n+1}, p_n, -d_n y_{n+1}), s_{n+1} d_n f(y_{n+1}))
\]

defines a map $K_1 \to \Lambda_{n+2}^n(f)$. Denote by $K_2$ the pullback

\[K_2 := K_1 \times_{\Lambda_{n+2}^n(f)} X_{n+2}\]

The projection $K_2 \to K_1$ is a cover, because it is the pullback of the cover $\lambda_{n+2}^n(f)$.

Denote a section of $K_2$ by a tuple $(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, y_{n+1}, y_n)$ where

\[(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, y_{n+1}) \in K_1, \quad \text{and} \quad y_n \in X_{n+2}.
\]

The assignment

\[(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, y_{n+1}, y_n) \mapsto ((d_0 s_{n+1} d_n y_{n+1} y_n, \ldots, p_n-1, d_n s_n d_{n+1} y_n, -d_{n+1} y_{n+1}), s_{n+1} d_{n+1} f(y_{n+1}))
\]

defines a map $K_2 \to \Lambda_{n+2}^n(f)$. Denote by $K_3$ the pullback

\[K_3 := K_2 \times_{\Lambda_{n+2}^n(f)} X_{n+2}\]

If $i = n - 2$, we have constructed $K_3 = K_{n+1-i}$. If $i < n - 2$, suppose that, for $3 \leq \ell < n + 1 - i$, we have constructed a cover $K_{\ell} \to K_{\ell-1}$, such that sections of $K_{\ell}$ are tuples $(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (y_j)_{j=\ell+1}^{n+1-\ell})$ with

\[(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (y_j)_{j=\ell+1}^{n+1-\ell}) \in K_{\ell-1}, \quad \text{and} \quad y_{n+2-\ell} \in X_{n+2},
\]

such that

\[f(y_{n+2-\ell}) = s_{n+1} d_{n+2} f(y_{n+2-\ell})
\]

\[d_j y_{n+2-\ell} = \begin{cases} d_{n+1} y_{n+3-\ell} & j = n + 2 \\ d_{j+1} s_{n+1} d_{n+1} y_{n+3-\ell} & n + 1 - \ell < j < n + 1 \\ p_{n+1-\ell} & j = n + 1 - \ell \\ d_j s_n d_{n+1} y_{n+3-\ell} & j < n + 1 - \ell \end{cases}
\]

The assignment

\[(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (y_j)_{j=\ell+1}^{n+1-\ell}) \mapsto ((d_0 s_{n+1} d_{n+1} y_{n+2-\ell} \ldots, d_{n-1-\ell} s_{n+1} d_{n+1} y_{n+2-\ell}, p_{n-\ell}, d_{n+1-\ell} s_{n+1} d_{n+1} y_{n+2-\ell}, \ldots, d_n s_{n+1} d_{n+1} y_{n+2-\ell}, -d_{n+1} y_{n+2-\ell}), s_{n+1} d_{n+1} f(y_{n+2-\ell}))
\]
defines a map $K_\ell \to \Lambda_{n+1}^{n+2}(f)$. Denote by $K_{\ell+1}$ the pullback

$$K_{\ell+1} := K_\ell \times_{\Lambda_{n+1}^{n+2}(f)} X_{n+2}$$

This completes the induction step for $\ell < n + 1 - i$.

Let $\ell = n + 1 - i$. If $i = 0$, we have completed the induction. If $i > 0$, the assignment

$$(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (y_j)_{j=1}^{n+1})$$

defines a map $K_{n+1-i} \to \Lambda_{n+1}^{n+2}(f)$. Denote by $K_{n+3-i}$ the pullback

$$K_{n+1-i} \times_{\Lambda_{n+1}^{n+2}(f)} X_{n+2}$$

If $i = 1$, we have completed the induction. If $i > 1$ the assignment

$$(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (y_j)_{j=1}^{n+1})$$

defines a map $K_{n+3-i} \to \Lambda_{n+1}^{n+2}(f)$. Denote by $K_{n+4-i}$ the pullback

$$K_{n+3-i} \times_{\Lambda_{n+1}^{n+2}(f)} X_{n+2}$$

If $i = 2$, we have completed the induction. If $i > 2$, suppose that, for $\ell$ at least $n + 4 - i$ but less than $n + 2$, we have constructed a cover $K_\ell \to K_{\ell-1}$, where sections of $K_\ell$ are tuples $(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (y_j)_{n+2-\ell \leq j \neq i}^{n+1})$ with

$$(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (y_j)_{n+3-\ell \leq j \neq i}^{n+1}) \in K_{\ell-1}, \quad y_{n+2-\ell} \in X_{n+2},$$

such that

$$f(y_{n+2-\ell}) = s_{n+1}d_{n+2}f(y_{n+2-\ell})$$

$$d_jy_{n+2-\ell} = \begin{cases} 
  d_{n+1}y_{n+3-\ell} & j = n + 2 \\
  d_js_{n+1}d_{n+1}y_{n+3-\ell} & n + 1 - \ell < j < n + 1 \\
  p_{n+1-\ell} & j = n + 1 - \ell \\
  d_js_{n+1}d_{n+1}y_{n+3-\ell} & j < n + 1 - \ell
\end{cases}$$

The assignment

$$(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (y_j)_{n+2-\ell \leq j \neq i}^{n+1})$$

defines a map

$$(d_0s_{n+1}d_{n+1}y_{n+2-\ell}, \ldots, d_{n-1-\ell}s_{n+1}d_{n+1}y_{n+2-\ell}, p_{n-\ell}, d_{n+1-\ell}s_{n+1}d_{n+1}y_{n+2-\ell}, \ldots, d_ns_{n+1}d_{n+1}y_{n+2-\ell}, s_{n+1}d_{n+1}f(y_{n+2-\ell}))$$
defines a map \( K_{\ell} \rightarrow \Lambda_{n+2}^{n+2}(f) \). Denote by \( K_{\ell+1} \) the pullback
\[
K_{\ell+1} := K_{\ell} \times_{\Lambda_{n+2}^{n+2}(f)} X_{n+2}
\]
This completes the induction step, in all cases.
For \( i = 0 \), we conclude the existence of the sequence of covers
\[
K_{n+1} \rightarrow \cdots \rightarrow K_0 = K
\]
The construction allows us to denote a section of \( K_{n+1} \) by
\[(x_0, x_1, (p_j)_{j=1}^{n+1}, (y_j)_{j=1}^{n+1})\]
where
\[s_{n+1} f(x_0) = f(y_0)\]
and, for \( j > 0 \),
\[d_j x_0 = d_j d_{n+1} y_0.\]
The assignment
\[
(x_0, x_1, (p_j)_{j=1}^{n+1}, (y_j)_{j=1}^{n+1}) \mapsto
((-, d_1 s_{n+1} x_0, \ldots, d_n s_{n+1} x_0, x_0, d_{n+1} y_1), s_{n+1} f(x_0))
\]
defines a map \( K_{n+1} \rightarrow \Lambda_{n+2}^{n+2}(f) \). Denote by \( K_{n+3} \) the pullback
\[
K_{n+3} := K_{n+1} \times_{\Lambda_{n+2}^{n+2}(f)} X_{n+2}
\]
Note that the map \( K_{n+3} \rightarrow K_{n+1} \) is a cover, because it is a pullback of the cover \( \Lambda_{n+2}^{n+2}(f) \). Denote a section of \( K_{n+3} \) by \((x, z)\) where \( x \in K_{n+1} \) and \( z \in X_{n+2} \).
For \( i > 0 \), the induction above demonstrates the existence of the sequence of covers
\[
K_{n+2} \rightarrow \cdots \rightarrow K_{n+3-i} \rightarrow K_{n+1-i} \rightarrow \cdots \rightarrow K_0 = K
\]
The construction allows us to denote a section of \( K_{n+2} \) by
\[(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (y_j)_{0 \leq j \neq i}^{n+1})\]
where
\[s_{n+1} f(x_0) = f(y_0),\]
and, for \( j \neq i \),
\[d_j x_0 = d_j d_{n+1} y_0.\]
For \( i \neq n + 1 \), the assignment
\[
(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (y_j)_{0 \leq j \neq i}^{n+1}) \mapsto
((d_0 s_{n+1} x_0, \ldots, d_{i-1} s_{n+1} x_0, -, d_{i+1} s_{n+1} x_0, \ldots, d_n s_{n+1} x_0, x_0, d_{n+1} y_0), s_{n+1} f(x_0))
\]
defines a map \( K_{n+2} \rightarrow \Lambda_{i+2}^{n+2}(f) \).
For \( i = n + 1 \), the assignment
\[
(x_0, x_1, (p_j)_{j=0}^{n+1}, (y_j)_{j=0}^{n+1}) \mapsto
((d_0 s_{n+1} x_0, \ldots, d_{n-1} s_{n+1} x_0, x_0, d_{n+1} y_0, -), s_n f(x_0))
\]
defines a map \( K_{n+2} \rightarrow \Lambda_{n+2}^{n+2}(f) \). Denote by \( K_{n+3} \) the pullback
\[
K_{n+3} := K_{n+2} \times_{\Lambda_{i+2}^{n+2}(f)} X_{n+2}
\]
Note that the map $K_{n+3} \to K_{n+2}$ is a cover, because it is a pullback of the cover $\lambda_i^{n+2}(f)$. Denote a section of $K_{n+3}$ by $(x, z)$ where $x \in K_{n+2}$ and $z \in X_{n+2}$. The construction, in all cases, guarantees that the assignment

$$(x, z) \mapsto d_i z$$

defines a map $h: K_{n+3} \to P^{\geq n}(f)_1$ which, along with $c: K_{n+3} \to \mathbb{K}$, gives to \ref{4.4}.

It remains to show that the map $g$ is an epi. Because we assume that $C$ has enough points, it suffices to check that the map $p(g)$ is a surjection for any point $p: C \to \text{Set}$. A point $p$ preserves finite limits and arbitrary colimits, and takes covers to surjections. As a result, $p$ takes each construction we have been considering to its analogue in the category $\text{Set}$.

It suffices to show that if $f: X_{\bullet} \to Y_{\bullet}$ is a Kan fibration of simplicial sets, then the map

$$\mathbb{K} = X_{n+1} \times \Lambda_{n+1}^i(f) \coprod_{\Lambda_{n+1}^i(f)} X_{n+1} \to X_{n+1} \times \Lambda_{n+1}^i(\tau_n(f)) X_{n+1}$$

is surjective. This map fits into a pullback square

$$\begin{array}{ccc}
\mathbb{K} & \to & (P^{\geq n}(f)_1)^{(n+1)} \\
\downarrow & & \downarrow \\
X_{n+1} \times \Lambda_{n+1}^i(\tau_n(f)) X_{n+1} & \to & (X_n \times \tau_0(P^{\geq n}(f)) X_n)^{(n+1)}
\end{array}$$

The map

$$P^{\geq n}(f)_1 \to X_n \times \tau_0(P^{\geq n}(f)) X_n$$

is surjective because the simplicial set $P^{\geq n}(f)_\bullet$ is Kan. As a result, the map $g$ is surjective, because surjections of sets are preserved under products and pullbacks.

By Axiom \ref{4.3} determines a map

$$\Lambda_{n+1}^i(f) \to \tau_n(X, f)_n,$$

We now extend \ref{4.1} to a map

$$\text{tr}_{n+1} \tau_n(f): \text{tr}_{n+1} \tau_n(X, f)_\bullet \to \text{tr}_{n+1} Y_{\bullet}$$

On $k$-simplices, for $k \leq n$, the map $\text{tr}_{n+1} \tau_n(f)$ equals the map $\text{tr}_n \tau_n(f)$. On $(n+1)$-simplices, $\text{tr}_{n+1} \tau_n(f)$ in the canonical map

$$\Lambda_{n+1}^i(\tau_n(f)) \to Y_{n+1}$$

The missing face map

$$\text{tr}_{n+1} \tau_n(X, f)_{n+1} = \Lambda_{n+1}^i(\tau_n(f)) \to \tau_0(P^{\geq n}(f)) = \text{tr}_{n+1} \tau_n(X, f)_n$$

makes $\text{tr}_{n+1} \tau_n(X, f)_\bullet$ into an $(n+1)$-truncated simplicial object.

**Definition 4.4.** Let $f: X_{\bullet} \to Y_{\bullet}$ be an $\infty$-stack such that $\tau_0(P^{\geq n}(f))$ exists. Define $\tau_n(X, f)_\bullet$ to be the limit

$$\tau_n(X, f)_\bullet := \cosk_{n+1} \text{tr}_{n+1} \tau_n(X, f)_\bullet \times \cosk_{n+1} Y_{\bullet} Y_{\bullet}$$
The $n$-strictification of $f$ is the map
\[
\tau_n(f) : \tau_n(X, f) \to Y.
\]

**Proposition 4.5.** Let $f : X \to Y$ be an $\infty$-stack such that $\pi_0(P^{\geq n}(f))$ exists. The $n$-strictification $\tau_n(f) : \tau_n(X, f) \to Y$ is an $n$-stack. The maps $f$ and $\tau_n(f)$ are isomorphic if and only if $f$ is an $n$-stack.

**Proof.** We show that $\tau_n(f)$ is an $n$-stack. Lemma 4.1 establishes that $\lambda^k_i(\tau_n(f))$ is a cover for $k \leq n$ and all $i$. We now show that the map $\lambda^{n+1}_i(\tau_n(f))$ is an isomorphism for all $i$. The inclusion $\Lambda^{n+1}_i \hookrightarrow \partial \Delta^{n+1}$ induces a map
\[
d_i : M_{n+1}(\tau_n(f)) \to \Lambda^{n+1}_i(\tau_n(f)).
\]

Observe that, in the notation of 4.2, if $J = [n + 1] \setminus \{i\}$, then
\[
M_{n+1}(\tau_n(f)) \cong \Lambda^{n+1}_i(\tau_n(f)) \times_{\Lambda^{n+1}_{\partial \Delta^{n+1}}(\tau_n(f))} \pi_0(P^{\geq n}(f))
\]
The missing face map $d_i : \Lambda^{n+1}_i(\tau_n(f)) \to \pi_0(P^{\geq n}(f))$ induces a map
\[
(1, d_i) : \Lambda^{n+1}_i(\tau_n(f)) \to M_{n+1}(\tau_n(f)).
\]

Note that the map $(1, d_i)$ is a right inverse for the map $d_i$.

For all $i$, the map $\lambda^{n+1}_i(\tau_n(f))$ factors as the composite
\[
\tau_n(X, f)_{n+1} = \Lambda^{n+1}_i(\tau_n(f)) \xrightarrow{(1, d_i)} M_{n+1}(\tau_n(f)) \xrightarrow{d_i} \Lambda^{n+1}_i(\tau_n(f)).
\]

These maps fit into a commuting diagram
\[
\begin{array}{ccc}
X_{n+1} & \xrightarrow{(1, d_i)} & \Lambda^{n+1}_i(\tau_n(f)) \\
\downarrow d_i & & \downarrow \Lambda^{n+1}_i(\tau_n(f)) \\
M_{n+1}(\tau_n(f)) & \xrightarrow{(1, d_i)} & \Lambda^{n+1}_i(\tau_n(f)) \\
\end{array}
\]

The proof of Lemma 4.2 shows that
\[
(1, d_i) \circ d_i \circ (1, d_i) = (1, d_i)
\]
\[
(1, d_i) \circ d_i \circ (1, d_i) = (1, d_i)
\]

We conclude that
\[
(d_i(1, d_i)) \circ \lambda^{n+1}_i(\tau_n(f)) = (d_i(1, d_i)) \circ (d_i(1, d_i)) = d_i \circ (1, d_i) = \lambda^{n+1}_i(\tau_n(f)).
\]

Similarly,
\[
\lambda^{n+1}_i(\tau_n(f)) \circ (d_i(1, d_i)) = (d_i(1, d_i)) \circ (d_i(1, d_i)) = \lambda^{n+1}_i(\tau_n(f)).
\]

We have shown that $\lambda^{n+1}_i(\tau_n(f))$ is an isomorphism for all $i$.

Lemma 2.16 guarantees that the limit $\Lambda^{n+2}_i(\tau_n(f))$ exists. Because $\lambda^{n+1}_i(\tau_n(f))$ is an isomorphism for all $i$, the map
\[
\Lambda^{n+2}_i(\tau_n(f)) \hookrightarrow M_{n+2}(\tau_n(f)) = \tau_n(X, f)_{n+2}
\]
is an isomorphism, with inverse given by $\lambda^{n+2}_i(\tau_n(f))$.  

\[
\begin{array}{ccc}
\end{array}
\]
For $k > n + 2$, the map $\lambda^k_!(\tau_n(f))$ is an isomorphism because the map $\tau_n(f)$ is $(n + 1)$-coskeletal, and the inclusion $\Delta^k \hookrightarrow \Delta^n$ is the identity on $(n + 1)$-skeleta. By inductively applying Lemma 2.16, we conclude that $\tau_n(X, f)_k$ is an object of $\mathcal{C}$ for all $k$ and that $\tau_n(f)$ is an $n$-stack.

We have shown that if $f$ is isomorphic to $\tau_n(f)$, then $f$ is an $n$-stack. Conversely, suppose that $f$ is an $n$-stack. For any $k$ and any map, the $(k - 1)$-skeleton of the map determines its $\Lambda^k_+$-horns. The horn-filling maps for $\tau_n(f)$ are isomorphisms above dimension $n$. If $f$ is an $n$-stack, then the horn-filling maps for $f$ are also isomorphisms above dimension $n$. We conclude that the canonical map from $f$ to $\tau_n(f)$ is an isomorphism if $f$ is an $n$-stack and if the map from $f$ to $\tau_n(f)$ induces an isomorphism on $n$-skeleta.

When $f$ is an $n$-stack, the map from $f$ to $\tau_n(f)$ is automatically an isomorphism on $n$-skeleta. Indeed, if $f$ is an $n$-stack, then $P^{\geq n}(f)_*$ is a Lie 0-groupoid (Theorem 3.1). As a result, the map from $X_n$ to $\pi_0(P^{\geq n}(f))$ is an isomorphism. □

If $f : X_* \rightarrow Y_*$ is a hypercover, then Proposition 2.30 and Theorem 3.7 imply that

$$\pi_0(P^{\geq n}(f)) \cong M_n(f)$$

In particular, the $n$-strictification $\tau_n(f)$ is an $n$-stack.

**Proposition 4.6.** If $f : X_* \rightarrow Y_*$ is a hypercover, then

1. the map $\tau_n(f)$ is an $n$-hypercover, and
2. the map $X_* \rightarrow \tau_n(X, f)_*$ is a hypercover.

**Proof.** We show that $\tau_n(X, f)_{n+1} = \Lambda_{n+1}^{-1}(\tau_n(f)) \cong M_{n+1}(\tau_n(f))$. Consider the commuting triangle

$$\begin{align*}
    X_{n+1} & \rightarrow \Lambda_{n+1}^{-1}(\tau_n(f)) \\
    \Lambda_{n+1}^{-1}(\tau_n(f)) & \leftarrow M_{n+1}(\tau_n(f))
\end{align*}$$

We showed in Lemma 4.13 that the map $X_{n+1} \rightarrow \Lambda_{n+1}^{-1}(\tau_n(f))$ is a cover. A similar argument shows that, because $f$ is a hypercover, the map $X_{n+1} \rightarrow M_{n+1}(\tau_n(f))$ is a cover. Axiom 3 implies that both $d_1$ and $(1, d_1)$ are covers. Axiom 4 implies that they are both epimorphisms. By construction

$$d_1(1, d_1) = 1_{\Lambda_{n+1}^{-1}(\tau_n(f))}$$

As a result,

$$(1, d_1)d_1(1, d_1) = (1, d_1)
= 1_{M_{n+1}(\tau_n(f))}(1, d_1)$$

Because $(1, d_1)$ is an epimorphism, we conclude that $(1, d_1)d_1 = 1_{M_{n+1}(\tau_n(f))}$.

This isomorphism, combined with the isomorphism

$$\tau_n(X, f)_n \cong M_n(f)$$

$$= M_n(\tau_n(f))$$
to show that
\[ \tau_n(X, f) \cong \text{Cosk}_n(X) \times_{\text{Cosk}_n(Y)} Y. \]

We have shown that \( \tau_n(f) \) is an \( n \)-hypercover.

For \( k < n \), \( M_k(q) \cong X_k \) by inspection. A similar check shows that the map from \( M_n(q) \) to \( \tau_n(X, f)_n \) is an isomorphism. We observed above that \( \tau_n(X, f)_n \cong M_n(f) \) when \( f \) is a hypercover. This implies that the map \( X_n \to M_n(q) \) is isomorphic to the cover \( X_n \to M_n(f) \). For \( k > n \), an exercise in combinatorics shows that the map \( X_k \to M_k(q) \) is isomorphic to the cover \( X_k \to M_k(f) \). We conclude the proof. \( \square \)

5. \textit{N-Bundles and Descent}

A classical construction produces a principal bundle for a Lie group from local data on the base. This local data is frequently presented in the form of a cocycle \( \varphi \) on a cover \( U \to X \). If we pass to the nerve of the cover \( f : U \to X \), we can encode the cocycle as a 0-stack

\[ E^\varphi \to U. \]

The principal bundle corresponding to the cocycle is the 0-strictification

\[ \tau_0(E^\varphi, fp) \to \tau_0(fp) \to X. \]

From this perspective, the principal bundle is representable because the 0-stack \( p \) has the structure of a twisted Cartesian product. Twisted Cartesian products were studied by Barratt, Guggenheim and Moore in their work on principal and associated bundles for simplicial groups \( \Pi \).

\textbf{Definition 5.1.} A map \( p : E \to X \) is a twisted Cartesian product, if there exists \( Y \in \mathcal{S} \), and there exist isomorphisms

\[ \begin{array}{ccc}
E_k & \xrightarrow{\varphi_k} & X_k \times Y_k \\
\downarrow{p} & & \downarrow{\pi_{X_k}} \\
X_k & & \\
\end{array} \]

for each \( k \in \mathbb{N} \), such that, for \( i < k \),

\[ \varphi_{k-1}d^E_i = (d^X_i \times d^Y_i)\varphi_k, \]

and, for all \( i \),

\[ \varphi_k s^E_i = (s^X_i \times s^Y_i)\varphi_k. \]

\textbf{Definition 5.2.} A local \( n \)-bundle is a twisted Cartesian product which is also an \( n \)-stack.

In analogy with 5.1, we consider local \( n \)-bundles on the total spaces of \( (n + 1) \)-hypercovers

\[ \begin{array}{ccc}
E & \xrightarrow{p} & U \\
\downarrow{f} & & \downarrow{\tau_n(fp)} \\
X & & \\
\end{array} \]

Our goal in this section is to show that the \( n \)-strictification

\[ \tau_n(E, fp) \to \tau_n(fp) \to X. \]

exists.
From the definition, it suffices to show that the orbit space \( \pi_0 P^{\geq n}(fp) \) exists. Because the map \( fp \) is an \((n+1)\)-stack, the relative higher morphism space \( P^{\geq n}(fp) \) is a Lie 1-groupoid (Theorem 5.3). We see that the existence of the \( n \)-strictification is equivalent to the existence of the orbit space of a Lie groupoid.

**Theorem 5.3** (Godemont). Let \( G \) be a Lie groupoid in the category of analytic manifolds over a complete normed field. If the map \((s,t) : G_1 \to G_0 \times G_0\) is a closed embedding, then \( \pi_0(G) \) is an analytic manifold and the map \( G_0 \to \pi_0(G) \) is a surjective submersion.

Serre gives a proof in [19, Theorem III.12.2] which applies mutatis mutandis to the category of smooth manifolds. Inspired by this, we formulate an analogue of Godemont’s Theorem for categories with covers. We begin by defining an analogue of closed embeddings.

**Definition 5.4.** Let \( f : X \to Y \) be a morphism in \( C \). The graph \( \Gamma_f \) of \( f \) is the inclusion

\[
X \times_Y Y \subseteq \Gamma_f \subseteq X \times Y
\]

The subcategory of *regular embeddings* is the smallest sub-category of \( C \) which is closed under pullback along covers and which contains all graphs.

An isomorphism is a regular embedding, because it is a pullback of the graph

\[
\begin{array}{ccc}
* & \xrightarrow{\Delta} & * \\
& \downarrow & \\
* & \rightarrow & * \\
\end{array}
\]

along a cover \( X \to \ast \). A regular embedding in the category of smooth manifolds is a certain type of closed embedding.

**Definition 5.5.** A regular Lie \( n \)-groupoid is a Lie \( n \)-groupoid \( X_\bullet \) such that the map \( \mu_1(X) : X_1 \to M_1(X) \cong X_0 \times X_0 \) is a regular embedding.

**Axiom 5.** (Godemont’s Theorem) Let \( X_\bullet \) be a regular Lie \( n \)-groupoid. The orbit space \( \pi_0(X_\bullet) \) exists.

**Theorem 5.6** (Descent for \( n \)-bundles). Suppose that Godemont’s Theorem holds in \( C \). If \( p : E_\bullet \to U_\bullet \) is a local \( n \)-bundle and \( f : U_\bullet \to X_\bullet \) is an \((n+1)\)-hypercover, then the \( n \)-strictification

\[
\tau_n(E,f) \to \tau_n(fp) \to X_\bullet
\]

exists.

**Remark 5.7.** To study bundles for simplicial Lie groupoids, one should use twisted fiber products rather than twisted Cartesian products in the definition of local \( n \)-bundles. The theorem also holds for this more general notion.

The following lemma is the crux of the proof.

**Lemma 5.8.** There exists an isomorphism

\[
P^{\geq n}(fp)_1 \xrightarrow{\psi} P^{\geq n}(f)_1 \times Y_n
\]

such that the maps \( \psi d_n \) and \((d_n \times 1_{Y_n}) \psi \) are equal.
Proof. The definition of \( P^{\geq n}(fp)_1 \) allows us to view it as a sub-object of \( E_{n+1} \).
Since \( p \) is a twisted Cartesian product, there exists \( Y_\bullet \in C \) and isomorphisms
\[
E_k \xrightarrow{\varphi_k} U_k \times Y_k
\]
such that, for \( i < k \),
\[
\varphi_k^{-1}d^E_1 = (d^X_1 \times d^Y_1)\varphi_k,
\]
and, for all \( i \),
\[
\varphi_k s^E_1 = (s^X_1 \times s^Y_1)\varphi_k.
\]
Using this, we see that sections of \( P^{\geq n}(fp)_1 \) consist of pairs
\[
(u, y) \in U_{n+1} \times Y_{n+1}
\]
such that, for \( i < n \),
\[
d_i y = s_{n-1} d_{n-1} d_i y,
\]
\[
d_i u = s_{n-1} d_{n-1} d_i u,
\]
and
\[
f(u) = s_n d_n f(u).
\]
Similarly, sections of \( P^{\geq n}(p)_1 \) consist of pairs
\[
(u, y) \in U_{n+1} \times Y_{n+1}
\]
such that, for \( i < n \),
\[
d_i y = s_{n-1} d_{n-1} d_i y,
\]
and
\[
u = s_n d_n u.
\]
These equations say that if \((u, y)\) is a section of \( P^{\geq n}(fp)_1 \), then \( u \) is a section of
\( P^{\geq n}(f)_1 \) and the natural map
\[
P^{\geq n}(fp)_1 \longrightarrow P^{\geq n}(f)_1 \times U_n \quad P^{\geq n}(p)_1 \]
\[
(u, y) \longrightarrow (u, (s_n d_n u, y))
\]
is an isomorphism.
Because \( p \) is an \( n \)-stack, \( P^{\geq n}(p)_\bullet \) is a Lie 0-groupoid (Theorem 3.6). The target map
\[
P^{\geq n}(p)_1 \longrightarrow P^{\geq n}(p)_0
\]
is therefore an isomorphism. Using this isomorphism and the map (5.2) we obtain the desired isomorphism
\[
P^{\geq n}(fp)_1 \xrightarrow{\psi} P^{\geq n}(f)_1 \times Y_n
\]
Proof of Theorem 5.6. Axiom 5 reduces the proof to showing that

\[ P^{\geq n}(fp)_1 \longrightarrow P^{\geq n}(fp)_0 \times P^{\geq n}(fp)_0 \]

is a regular embedding. The map

\[ P^{\geq n}(f)_{\bullet} \longrightarrow M_n(f) \]

is a 1-hypercover, because \( f \) is an \((n+1)\)-hypercover (Theorem 3.7). In particular, the map \( \partial \Delta^1(\pi) \) gives an isomorphism

\[ P^{\geq n}(f)_1 \mathop{\cong}^{\partial \Delta^1(\pi)} U_n \times M_n(f) U_n \]

The canonical map

\[ U_n \times M_n(f) U_n \longrightarrow U_n \times U_n \]

is a regular embedding. Composing this with the map above, we obtain a regular embedding

\[ P^{\geq n}(f)_1 \mathop{\longrightarrow}^{\partial \Delta^1(\pi)} U_n \times U_n \]

The isomorphisms

\[ P^{\geq n}(fp)_0 \mathop{\cong} \longrightarrow E_n \mathop{\longrightarrow}^{\cong} U_n \times Y_n \]

and

\[ P^{\geq n}(fp)_1 \mathop{\longrightarrow}^{\psi} P^{\geq n}(f)_1 \times Y_n \]

allow us to factor the map

\[ P^{\geq n}(fp)_1 \longrightarrow P^{\geq n}(fp)_0 \times P^{\geq n}(fp)_0 \]

as the composite

\[ P^{\geq n}(fp)_1 \mathop{\longrightarrow}^{\Gamma_n} P^{\geq n}(fp)_1 \times Y_n \mathop{\longrightarrow}^{\psi \times 1_Y} Y_n \times P^{\geq n}(f)_1 \times Y_n \mathop{\longrightarrow}^{1_Y \times (\partial \Delta^n(\pi)) \times 1_Y} Y_n \times U_n \times U_n \times Y_n \]

Each map in this sequence is a regular embedding. \( \square \)

6. Strict Lie \(n\)-Groups and their Actions

Principal and associated bundles for discrete simplicial groups provide examples of local \(n\)-bundles in \(s\text{Set}\). This theory was developed by Barratt, Gugenheim and Moore \([1]\). In this section, we develop analogous results for simplicial Lie groups. While we restrict to simplicial groups for ease of exposition, the results and proofs carry over to simplicial Lie groupoids.

Definition 6.1. A simplicial Lie group \(G_{\bullet}\) in \(C\) is a simplicial diagram in the category of group objects in \(C\). Denote by \(s\text{Group}(C)\) the category of simplicial Lie groups.

Eilenberg and Mac Lane \([8]\) introduced a pair of functors \(W\) and \(\overline{W}\) from simplicial groups to simplicial sets which generalize the universal bundle and nerve of a group.
**Definition 6.2.** Let $G_\bullet$ be a simplicial group.

1. The **total space** $W_\bullet G$ of the universal $G_\bullet$-bundle is the simplicial set with
   \[
   W_n G := G_0 \times \cdots \times G_n
   \]
   \[
d_i(g_0, \ldots, g_n) := (g_0, \ldots, g_{i-2}, g_{i-1}d_i g_i, d_{i+1}g_{i+1}, \ldots, d_ng_n)
   \]
   \[
s_i(g_0, \ldots, g_n) := (g_0, \ldots, g_{i-1}, e, s_i g_i, s_{i+1}g_{i+1}, \ldots, s_ng_n)
   \]
2. The **nerve** $\overline{W}_\bullet G$ of $G_\bullet$ is the simplicial set with
   \[
   \overline{W}_0 G := *
   \]
   and, for $n > 0$,
   \[
   \overline{W}_n G := G_0 \times \cdots \times G_{n-1}
   \]
   \[
d_i(g_0, \ldots, g_{n-1}) := (g_0, \ldots, g_{i-2}, g_{i-1}d_i g_i, d_{i+1}g_{i+1}, \ldots, d_ng_{n-1})
   \]
   \[
s_i(g_0, \ldots, g_{n-1}) := (g_0, \ldots, g_{i-1}, e, s_i g_i, s_{i+1}g_{i+1}, \ldots, s_ng_{n-1})
   \]
3. The assignment which sends $(g_0, \ldots, g_n) \in W_n G$ to $(g_0, \ldots, g_{n-1}) \in \overline{W}_n G$ defines a twisted Cartesian product $W_\bullet G \to \overline{W}_\bullet G$. This is the **universal $G_\bullet$-bundle**.

Because $C$ has finite products, the same formulas give functors $W$ and $\overline{W}$ from the category of simplicial Lie groups to the category $sC$.

**Definition 6.3.** If $G_\bullet$ is a simplicial Lie group and $X_\bullet$ is a simplicial object in $C$, then a left **action** $G_\bullet \times X_\bullet$ consists of maps
\[
G_k \times X_k \to X_k
\]
\[
(g, x) \mapsto gx
\]
for each $k$, such that, in all dimensions and for all $i$:
\[
g_1(g_2 x) = (g_1g_2)x,
\]
\[
ex = x,
\]
\[
d_i(gx) = (d_i g)(d_i x), \text{ and }
\]
\[
s_i(gx) = (s_i g)(s_i x).
\]
Right actions are defined analogously.

When $G_\bullet$ and $X_\bullet$ are constant simplicial diagrams, this is the usual notion of a left action of a Lie group. The right action of $G_\bullet$ on itself induces a right $G_\bullet$-action on $W_\bullet G$.

**Definition 6.4.** Suppose we have a left action $G_\bullet \times X_\bullet$. The **homotopy quotient** $(WG \times_G X)_\bullet$ is defined by
\[
(WG \times_G X)_n := \overline{W}_n G \times X_n
\]
\[
d_i(g_0, \ldots, g_{n-1}, x) := (g_0, \ldots, g_{i-2}, g_{i-1}d_i g_i, d_{i+1}g_{i+1}, \ldots, d_ng_{n-1}, d_ix)
\]
\[
s_i(g_0, \ldots, g_{n-1}, x) := (g_0, \ldots, g_{i-1}, e, s_i g_i, s_{i+1}g_{i+1}, \ldots, s_ng_{n-1}, s_ix)
\]
If $G_\bullet$ acts on $X_\bullet$ and $Y_\bullet$ and $f : X_\bullet \to Y_\bullet$ is $G_\bullet$-equivariant, then $f$ induces a map of homotopy quotients
\[
(WG \times_G X)_\bullet \xrightarrow{1 \times_G f} (WG \times_G Y)_\bullet
\]
given on \( n \)-simplices by \( \pi_1 G \times f_n \).

**Definition 6.5.** A strict Lie \( n \)-group is a simplicial Lie group \( G_\bullet \) such that the horn-filling maps \( \lambda^k_i(G) \) are isomorphisms for \( k \geq n \).

We might have defined a strict Lie \( n \)-group as a simplicial Lie group such that the maps \( \lambda^k_i(G) \) were also covers for all \( k < n \). An argument due to Moore shows that this follows from our definition.

**Proposition 6.6.** Let \( G_\bullet \) be a strict Lie \( n \)-group. The simplicial object underlying \( G_\bullet \) is a Lie \((n - 1)\)-groupoid.

**Proof.** We perform an induction on the dimension of the horns to show that the maps \( \lambda^k_i(G) \) are covers for \( k < n \).

Suppose that for \( l < k \) and all \( i \), the limit \( \Lambda^l_i(G) \) exists and the map \( \lambda^l_i(G) \) is a cover. Lemma 2.16 shows that the limit \( \Lambda^k_i(G) \) exists for all \( i \).

Sections of \( \Lambda^k_i(G) \times G_k \) are tuples \((g_0, \ldots, -i, -i, \ldots, g_k, g) \in \Lambda^k_i(G) \times G_k\) such that, for \( j < m \neq i \),

\[
d_{m-1}g_j = d_jg_m.
\]

We perform an induction on \( 0 \leq \ell \leq k + 1 \) to construct sections \( g^\ell \in G_k \) such that for \( j < \ell \) we have

\[
d_jg^\ell = g_j.
\]

Fix \((g_0, \ldots, -i, -i, \ldots, g_k, g) \in \Lambda^k_i(G) \times G_k\), and set

\[
g^0 := g.
\]

Now suppose that for \( 0 \leq \ell \), we have \( g^\ell \in G_k \) such that, for \( j < \ell \),

\[
d_jg^\ell = g_j.
\]

We define

\[
a^\ell_j := g_j(d_jg^\ell)^{-1} \in G_{k-1}.
\]

The horn relations on the \( g_j \) ensure that \((a^0_j, \ldots, -i, -i, \ldots, a^\ell_k)\) defines a section of \( \Lambda^k_i(G) \), so we have

\[
((a^0_j, \ldots, -i, -i, \ldots, a^\ell_k), g^\ell) \in \Lambda^k_i(G) \times G.
\]

We define

\[
g^{\ell+1} := (st a^\ell_j)g^\ell.
\]

A short exercise shows that, for \( j < \ell + 1 \),

\[
d_jg^{\ell+1} = g_j.
\]

This completes the induction step. We now define

\[
\Lambda^k_i(G) \times G_k \xrightarrow{\varphi} \Lambda^k_i(G) \times G_k
\]

\[
((g_0, \ldots, -i, -i, \ldots, g_k, g) \xrightarrow{\varphi} ((g_0(d_0g^{k+1})^{-1}, \ldots, -i, -i, \ldots, g_k(d_kg^{k+1})^{-1}, g^{k+1})
\]

By construction, \( \varphi \) is an isomorphism. It factors the projection

\[
\Lambda^k_i(G) \times G_k \xrightarrow{\varphi} \Lambda^k_i(G)
\]
as

\[
\Lambda_i^k(G) \times G_k \xrightarrow{\varphi} \Lambda_i^k(G) \times G_k
\]

\[
\Lambda_i^k(G) \xrightarrow{\pi_{G_k}} G_k
\]

Axiom 3 guarantees that \( \lambda_i^k(G) \) is a cover. Lemma 2.16 shows that the limit \( \Lambda_i^{k+1}(G) \) exists for all \( i \). This concludes the induction step. \( \square \)

Observe that a strict Lie 1-group is a Lie group viewed as a constant simplicial group, or let \( X \to Y \) be a \( G \)-equivariant \( n \)-stack. We show that, in the first case, \( \Lambda_i^k(G) \) are isomorphisms. For \( k \), the maps \( \Lambda_i^k(WG) \to \Lambda_i^{k-1}(G) \) given by

\[
(g^0, \ldots, \hat{g^i}, \ldots, g^k) \mapsto \begin{cases} 
(g^k, (g^0_{k-2}, \ldots, -), \ldots, (g^k_{k-2}), (g^k_{k-1})), & i < k - 1, \\
(g^k, (g^0_{k-2}, \ldots, g^k_{k-2}, -)), & i = k - 1, \\
(g^k_{k-1}, (g^0_{k-2}, \ldots, g^k_{k-2}, -)), & i = k,
\end{cases}
\]

are isomorphisms. For \( k > 0 \) and \( i < k \), the maps \( \Lambda_i^k(WG \times_G \varphi) \to \Lambda_i^k(\varphi) \) given by

\[
((g^0, x_0), \ldots, \hat{(g^i, x_i)}, \ldots, (g^k, x_k)), (h, y)) \mapsto \begin{cases} 
(h_i, ((x_0, \ldots, x_i, \ldots, x_{k-1}, (h_{k-1})^{-1} x_k), y)), & i < k, \\
(h_i, ((x_0, \ldots, x_{k-1}, -), y)), & i = k,
\end{cases}
\]

\textbf{Theorem 6.7.}

1. The nerve of a strict Lie \( n \)-group is a Lie \( n \)-group.
2. The homotopy quotient of a \( G \)-equivariant \( n \)-stack is an \( n \)-stack.

\textbf{Proof.} Let \( G \) be a strict \( n \)-group, or let \( \varphi : X \to Y \) be a \( G \)-equivariant \( n \)-stack. We show that, in the first case, \( W_G \) is a Lie \( n \)-group, and, in the second, that \( 1 \times_G \varphi \) is an \( n \)-stack.

Denote sections of \( W_kG \) by

\[
g^j = (g_0^j, \ldots, g_{k-1}^j) \in G_0 \times \cdots \times G_{k-1} = W_kG.
\]

Denote sections of \( \Lambda_i^k(WG) \) by

\[
(g^0, \ldots, -, \ldots, g^k) \in \Lambda_i^k(WG).
\]

We now give a series of isomorphisms which relate horns in the nerve or homotopy quotient to horns in the strict Lie \( n \)-group or equivariant \( n \)-stack. The formulas proceed from the observation that the highest face in these horns determines all but the last coordinates of the lower ones; these last coordinates themselves determine a horn in the original simplicial Lie group or \( G \)-map.

For \( k > 0 \), the maps \( \Lambda_i^k(WG) \to W_{k-1}G \times \Lambda_i^{k-1}(G) \) given by

\[
((g^0, x_0), \ldots, \hat{(g^i, x_i)}, \ldots, (g^k, x_k)), (h, y)) \mapsto \begin{cases} 
(h_i, ((x_0, \ldots, x_i, \ldots, x_{k-1}, (h_{k-1})^{-1} x_k), y)), & i < k, \\
(h_i, ((x_0, \ldots, x_{k-1}, -), y)), & i = k,
\end{cases}
\]
are isomorphisms. For $k > 1$, the isomorphisms for horns in the nerve fit into the following commuting squares:

For $k = 1$, we have

Similarly, for $k \geq 1$, the isomorphisms for horns in the homotopy quotients fit into the commuting squares

These squares show that the relevant horn-filling maps for $\mathbb{W}_\bullet G$ and $1 \times_G \varphi$ are covers for all $k$ and isomorphisms for $k > n$. □

Remark 6.8. While we do not need it for this paper, one could define a strict $n$-stack as a homomorphism of simplicial Lie groups such that the relative horn-filling maps are isomorphisms in dimensions at least $n$. The analogues of the results above hold in the relative case, with minimal changes to the proofs. One could also make analogous definitions for simplicial Lie groupoids. The analogues of the results above hold, with minimal changes to the proofs.

7. A Finite Dimensional String 2-Group

In this section, we specialize to the category of smooth manifolds and apply our results to construct finite dimensional Lie 2-groups. Let $A$ be an abelian group. For each natural number $n$, Eilenberg and MacLane introduced a simplicial abelian group $K(A, n)_\bullet$, whose geometric realization represents the cohomology functor
$H^n(\_; A)$. They further observed that $\mathcal{W}_\bullet K(A, n)$ is isomorphic to $K(A, n+1)_\bullet$. This construction and identification also exist for Abelian Lie groups.

**Definition 7.1.** Let $G$ be a Lie group. Let $A$ an Abelian Lie group.

1. An $A$-valued $n$-cocycle on $\mathcal{W}_\bullet G$ is a span

   \[
   \begin{array}{c}
   \mathcal{W}_\bullet G \\
   \downarrow \downarrow \\
   U_\bullet \\
   \end{array} \quad \begin{array}{c}
   \rightarrow \rightarrow \\
   \leftarrow \leftarrow \\
   K(A, n)_\bullet \\
   \end{array}
   \]

   such that $U_\bullet \to \mathcal{W}_\bullet G$ is a hypercover.

2. An equivalence of cocycles is a commuting diagram of cocycles

   \[
   \begin{array}{c}
   \mathcal{W}_\bullet G \\
   \downarrow \downarrow \\
   U_\bullet \\
   \end{array} \quad \begin{array}{c}
   \rightarrow \rightarrow \\
   \leftarrow \leftarrow \\
   \end{array} \quad \begin{array}{c}
   \begin{array}{c}
   \begin{array}{c}
   \mathcal{W}_\bullet G \\
   \downarrow \downarrow \\
   U_\bullet \\
   \end{array} \end{array} \quad \begin{array}{c}
   \rightarrow \rightarrow \\
   \leftarrow \leftarrow \\
   \end{array} \end{array} \quad \begin{array}{c}
   \begin{array}{c}
   \begin{array}{c}
   K(A, n)_\bullet \\
   \downarrow \downarrow \\
   V_\bullet \\
   \end{array} \end{array} \end{array}
   \]

   such that the maps $V_\bullet \to U_\bullet^i$ are hypercovers.

The connected 2-types $G_\bullet \in s\text{Smooth}$ which have arisen in the literature are determined by

1. a Lie group $G$,
2. an Abelian Lie group $A$, and
3. an equivalence class of 3-cocycles

   \[
   \begin{array}{c}
   \mathcal{W}_\bullet G \\
   \downarrow \downarrow \\
   U_\bullet \\
   \end{array} \quad \begin{array}{c}
   \rightarrow \rightarrow \\
   \leftarrow \leftarrow \\
   K(A, 3)_\bullet \\
   \end{array}
   \]

Much work has gone into finding geometric models for smooth 2-types. By pulling back the universal twisted $K(A, 2)$-bundle along a cocycle, we obtain a local 2-bundle

\[
\begin{array}{c}
E_\bullet \\
\downarrow \downarrow \\
U_\bullet \\
\end{array} \quad \begin{array}{c}
\rightarrow \rightarrow \\
\leftarrow \leftarrow \\
\mathcal{W}_\bullet G \\
\end{array}
\]

The composite

\[
\begin{array}{c}
E_\bullet \\
\downarrow \downarrow \\
U_\bullet \\
\end{array} \quad \begin{array}{c}
\rightarrow \rightarrow \\
\leftarrow \leftarrow \\
\mathcal{W}_\bullet G \\
\end{array}
\]

is an $\infty$-stack. This shows that connected smooth 2-types can be realized as finite dimensional Lie $\infty$-groups.

Over the last decade there have been many attempts to do better. The most relevant of these is provided by Schommer-Pries [18] who showed that connected smooth 2-types can be realized as weak group objects in the bicategory of finite dimensional Lie groupoids. Zhu [20], drawing on ideas of Duskin, constructed a nerve for such weak group objects and showed that the nerve is a Lie 2-group.

The tools in this article allow us to construct a Lie 2-group $X_\bullet$ directly from the data above. The object produced is equivalent to the one obtained by Zhu from Schommer-Pries. Our methods extend to $n > 2$.

**Proposition 7.2.** Any $A$-valued $n$-cocycle

\[
\begin{array}{c}
\mathcal{W}_\bullet G \\
\downarrow \downarrow f \\
U_\bullet \\
\end{array} \quad \begin{array}{c}
\rightarrow \rightarrow \\
\leftarrow \leftarrow \\
K(A, n)_\bullet \\
\end{array}
\]
factors uniquely through $\tau_n(f)$ as in the diagram

\[
\begin{array}{ccc}
W \cdot G & \xrightarrow{f} & U \\
\downarrow \tau_n(f) & & \downarrow \tau_n(U, f) \\
K(A, n) & \xrightarrow{\pi} & K(A, n)
\end{array}
\]

This factorization is an equivalence of cocycles.

**Proof.** The data of a cocycle is equivalent to a commuting triangle

\[
\begin{array}{ccc}
U & \xrightarrow{\tau_n(U, f)} & W \cdot G \\
\downarrow & & \downarrow \\
K(A, n) & \xrightarrow{\pi} & K(A, n)
\end{array}
\]

Theorem 2.18 shows that both diagonal maps are $\infty$-stacks over $W \cdot G$.

We apply $\tau_n$ to obtain the commuting diagram

\[
\begin{array}{ccc}
\tau_n(U, f) & \xrightarrow{\tau_n(WG \times K(A, n), \pi_{WG})} & \tau_n(W \cdot G \\
\downarrow & & \downarrow \\
U & \xrightarrow{\tau_n(WG \times K(A, n), \pi_{WG})} & K(A, n)
\end{array}
\]

Proposition 4.5 and Theorem 6.7 together show that the map

\[
W \cdot G \times K(A, n) \xrightarrow{\tau_n(WG \times K(A, n), \pi_{WG})}
\]

is an isomorphism. Proposition 4.6 shows that $\tau_n(f)$ is an $n$-hypercover and that the map

\[
U \xrightarrow{\tau_n(U, f)}
\]

is a hypercover. □

We can now use Theorem 5.6 to produce a Lie 2-group from an $A$-valued 3-cocycle on $W \cdot G$. We can assume, without loss of generality, that the 3-cocycle (7.1)

\[
W \cdot G \xrightarrow{f} U \xrightarrow{\varphi} K(A, 3)
\]

has $U_0 = *$ and $f$ a 3-hypercover. We pull back the universal $K(A, 2)$-bundle along $\varphi$ to obtain a local 2-bundle

\[
\varphi^* WK(A, 2) \xrightarrow{p} U
\]

We descend this local 2-bundle along the 3-hypercover $f$, as in Theorem 5.6. We obtain a 2-stack

\[
X : = \tau_2(\varphi^* WK(A, 2), fp) \xrightarrow{} W \cdot G
\]

The object $X$ is the desired Lie 2-group.

We now examine $X$ in more detail. For simplicity, we ignore degeneracies. The hypercover in the cocycle (7.1) is determined by its 2-skeleton (Proposition 4.6). The 2-skeleton consists of

1. a cover $f_1 : U \to G$, which we view as a 1-truncated hypercover, and
2. a cover $f_2 : V \to (\cosk_1 U \times_{\cosk_1 WG} WG)_2$. 
The Lie 2-group $X_\bullet$ has the same 1-skeleton as this hypercover. The 2-simplices of $X_\bullet$ are determined by the Lie 1-groupoid $P^{\geq 2}(fp)_\bullet$. Its vertex manifold, $P^{\geq 2}(fp)_{1,0}$, is isomorphic to $V \times A$. We showed that

$$P^{\geq 2}(fp)_1 \cong (V \times (\coskel_1 U \times \coskel_1 W G) \times V) \times K(A,2)$$

at the end of the proof of Theorem 5.6. The target map of $P^{\geq 2}(fp)_\bullet$ is given by

$$(v_2, v_3, a) \mapsto (v_2, a)$$

We abuse notation and denote by $\varphi$ the restriction of the map $U \xrightarrow{\varphi} K(A,3)$ to $P^{\geq 2}(f)_1 \subset U_3$. The source in the Lie groupoid $P^{\geq 2}(f)_\bullet$ is given by

$$(v_2, v_3, a) \mapsto (v_3, \varphi(v_2, v_3) + a)$$

The Lie 1-groupoid structure on $P^{\geq 2}(fp)_\bullet$ ensures that $\varphi$ is an $A$-valued 1-cocycle on the cover

$$V \longrightarrow (\coskel_1 U \times \coskel_1 W G) \times V$$

The orbit space

$$X_2 \longrightarrow (\coskel_1 U \times \coskel_1 W G) \times V$$

is the principal $A$-bundle determined by this data.

We now describe the higher simplices. The map $\Lambda_k(\tau_2(fp))$ is an isomorphism for $k > 2$ and all $i$ (Proposition 4.5). For $k = 3$, the data of these isomorphisms can be reduced to a trivialization $\zeta$ of the bundle

$$d^*_3 P^\vee \otimes d^*_2 P \otimes d^*_1 P^\vee \otimes d^*_0 P \longrightarrow (\coskel_1 U \times \coskel_1 W G)$$

Here $d_i$ denotes the face map from 3-simplices to 2-simplices in the simplicial object $(\coskel_1 U \times \coskel_1 W G)_\bullet$, and $P^\vee$ denotes the dual bundle of $P$.

Proposition 4.5 implies that $X_\bullet$ is determined by its 3-skeleton. In the present context, this is equivalent to requiring that $\zeta$ satisfy a pentagonal coherence condition coming from the 1-skeleton of the 4-simplex.

Summing up, we see that the Lie 2-group $X_\bullet$ is reducible to the following data:

1. a cover $f: U \longrightarrow G$,
2. a principal $A$-bundle $P \longrightarrow (\coskel_1 U \times \coskel_1 W G) \times V$,
3. a trivialization $\zeta$ of

$$d^*_3 P^\vee \otimes d^*_2 P \otimes d^*_1 P^\vee \otimes d^*_0 P \longrightarrow (\coskel_1 U \times \coskel_1 W G)$$

which satisfies a pentagonal coherence condition coming from the 1-skeleton of the 4-simplex.

If we take $\text{Spin}(n)$ for $G$, $U(1)$ for $A$, and a cocycle representing the fractional first Pontrjagin class $\frac{p}{2}$, then the resulting Lie 2-group is the nerve, à la Duskin–Zhu, of Schommer-Pries’s model for $\text{String}(n)$.

The discussion above gives a construction of higher central extensions of Lie groups. More generally, we might consider higher abelian extensions. We consider an Abelian Lie group $A$ on which the Lie group $G$ acts by automorphisms. The data which specifies a higher abelian extension of a Lie group, and the construction
which produces a Lie 2-group from this data, are analogous to the case of higher central extensions above. We sketch the necessary changes.

For each $n > 0$, the $G$-action on $A$ induces $G$-actions on $W_k(A; n-1)$ and $K(A; n)_*$ such that the universal $K(A; n-1)$-bundle

$$W_k(A; n-1) \longrightarrow K(A; n)_*$$

is $G$-equivariant with respect to these actions. We define the twisted universal bundle

$$W^G_k(A; n-1) \longrightarrow K^G(A; n)_*$$

by taking the homotopy quotient of the universal $K(A; n-1)$-bundle with respect to the $G$-action. An $A$-valued $n$-cocycle on $W^G_k$ is now a commuting triangle

$$\begin{array}{ccc}
W^G_k & \longrightarrow & K^G(A; n)_* \\
\downarrow & & \downarrow \\
W_k & \longrightarrow & K(A; n)_*
\end{array}$$

such that $U_* \to W^G_k$ is a hypercover. Equivalences of cocycles are defined analogously. The analogue of Proposition 7.2 holds, and the construction proceeds just as above. The only difference is that, if we unpack the construction of the Lie 2-group $X_*$ produced from this more general notion of cocycle, we observe that instead of the 2-simplices $X_2$ being the total space of a principal $A$-bundle, $X_2$ is now the total space of a $G$-twisted principal $A$-bundle. We leave the remaining details to the interested reader.

The methods above additionally allow for the construction of Lie $n$-groups for $n > 2$. In particular, there exists a finite dimensional model of Fivebrane($n$) as a Lie 7-group extending String($n$) by $K(Z,7)$. We expect that a model of Fivebrane($n$) also exists as a Lie 6-group extending String($n$) by $K(U(1),6)$.

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