ON THE CLASSIFICATION OF TYPE II CODES OF LENGTH 24

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Abstract. We give a new, purely coding-theoretic proof of Koch’s criterion on the tetrad systems of Type II codes of length 24 using the theory of harmonic weight enumerators. This approach is inspired by Venkov’s approach to the classification of the root systems of Type II lattices in \( \mathbb{R}^{24} \), and gives a new instance of the analogy between lattices and codes.

1. Background

We denote by \( \mathbb{F}_2 \) the two-element field \( \mathbb{Z}/2\mathbb{Z} \). By a “code” we mean a binary linear code of length \( n \), that is, a linear subspace of \( \mathbb{F}_2^n \). For such a code \( C \), and any integer \( w \), we define

\[
C_w := \{ c \in C : \text{wt}(c) = w \},
\]

where \( \text{wt}(c) := |\{ i : c_i = 1 \}| \) is the Hamming weight. Recall that the dual code of \( C \), denoted \( C^\perp \), is defined by

\[
C^\perp := \{ c' \in \mathbb{F}_2^n : \langle c, c' \rangle = 0 \text{ for all } c \in C \},
\]

where \( \langle c, c' \rangle \) is the usual bilinear pairing \( \langle x, y \rangle = \sum_{i=1}^n x_i y_i \) on \( \mathbb{F}_2^n \). We have \( \dim(C) + \dim(C^\perp) = n \). A code \( C \) is said to be self-dual if \( C = C^\perp \). Such a code must have \( \dim(C) = n/2 \); in particular \( 2 \mid n \). Because \( \langle c, c \rangle \equiv \text{wt}(c) \pmod{2} \), it follows that a self-dual code \( C \) is even: \( 2 \mid \text{wt}(c) \) for every word \( c \in C \); equivalently, \( C_w = \emptyset \) unless \( 2 \mid w \). A code \( C \) is said to be doubly even if \( 4 \mid \text{wt}(c) \) for all \( c \in C \); equivalently, if \( C_w = \emptyset \) unless \( 4 \mid w \).

A self-dual code is said to be of Type II if it is doubly even, and of Type I otherwise. Type II codes are especially rare. A Type II code must have length \( n = 8n' \) for some integer \( n' \), and for small values of \( n' \) the Type II codes have been entirely classified. Indeed, Pless [15] classified all the self-dual codes of length \( n \), both of Type I and of Type II, for \( n \leq 20 \); Pless and Sloane [16] extended this classification to lengths \( n = 22 \) and 24, citing unpublished work of Conway for the Type II case; and Conway and Pless [4, 5] classified the Type II codes of length 32. In particular, there is a unique Type II code of length 8 (the \([8, 4, 4] \) extended Hamming code), two Type II codes of length 16, and nine of length 24.

For sufficiently large \( n \), a complete classification of self-dual codes is likely out of reach. Indeed, Rains and Sloane [17] remarked that “length 32 is probably a good place to stop [seeking such classification results],” using the mass formula of MacWilliams, Sloane, and Thompson [12] to compute that there are at least 17493 Type II codes of length 40. King [9] further showed that at least 12579 of these have \( C_4 = \emptyset \), suggesting that it may even be unreasonable to ask for a classification of extremal Type II codes of length 40.

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1 Recall that an \([n, k, d] \) code is a code of length \( n \), dimension \( k \), and minimal (nonzero) distance \( d \).

2 Mallows and Sloane [13] showed that a Type II code \( C \) of length \( n \) must contain nonzero words of weight at most \( 4\lfloor n/24 \rfloor + 4 \) (see also [6, p. 194]). If \( C_w = \emptyset \) for all positive \( w < 4\lfloor n/24 \rfloor + 4 \), then \( C \) is said to be “extremal”: it has the largest minimal distance among all Type II codes of its length.
Binary codes $C$ are related with certain lattices $L_C$ in $\mathbb{R}^n$, via the following construction originally due to Leech and Sloane [11]:

**Construction A** ([6] pp. 182–183). For a code $C \subseteq \mathbb{F}_2^n$, the lattice $L_C \subseteq \mathbb{R}^n$ consists of all $x \in \mathbb{R}^n$ such that $2^{1/2}x \in \mathbb{Z}^n$ and $(2^{1/2}x) \mod 2 \in C$.

The lattice $L_C^\perp$ associated to $C^\perp$ is the dual of $L_C$; that is, it consists of all $x' \in \mathbb{R}^n$ such that $(x, x') \in \mathbb{Z}$ for all $x \in L_C$. In particular, $L_C$ is self-dual if and only if $C$ is. A lattice $L$ for which $(x, x) \in 2\mathbb{Z}$ for all $x \in L$ is said to be even; a self-dual lattice is said to be of Type II if it is even and of Type I if not. Thus a Construction A lattice $L_C$ is even if and only if $C$ is doubly even, and if $C$ is a Type I (resp. Type II) code then $L_C$ is a Type I (resp. Type II) lattice. As with codes in $\mathbb{F}_2^n$, Type II lattices in $\mathbb{R}^n$ exist if and only if $8 \mid n$ (see [18] p. 53 (Cor. 2) and [18] p. 109). For $n = 8$ and $n = 16$, Witt [20] proved that the only Type II lattices are those of the form $L_C$ for one of the Type II codes of the same length (see also [6] p. 48).

Niemeier [14] was the first to classify the Type II lattices of rank 24. There are 24, including the 9 lattices $L_C$ where $C$ is one of the Type II codes of length 24. His technique was later greatly simplified by Venkov [19] (also in [6] Ch. 18), who used weighted theta functions to constrain the possible root systems of Type II lattices. Specifically, Venkov showed that any rank-24 Type II lattice has one of exactly twenty-four root systems; the work of Niemeier [14] furthermore shows that each of these root systems corresponds to exactly one rank-24 Type II lattice.

Koch [10] developed a theory of tetrad systems for codes analogous to the theory of root systems for lattices. He then obtained a condition on the tetrad systems of Type II codes of length 24 through an appeal to Venkov’s results [19]. In particular, Koch [10] showed that any Type II code of length 24 has one of nine tetrad systems; Conway’s classification [16] of such codes furthermore implies that each of these nine arises for a unique code.

In this paper, we give a new, purely coding-theoretic proof of the Koch condition [10] on tetrad systems of Type II codes of length 24. Our method uses Bachoc’s theory [1, 2] of harmonic weight enumerators, a coding-theoretic analogue of weighted theta functions, which had not been developed at the time of Koch’s work. This approach gives a new instance of the analogy between lattices and codes: our method is directly analogous to that of Venkov [19] for the classification of the root systems of Type II lattices in $\mathbb{R}^{24}$.

The remainder of this paper is organized as follows. Sections 1.1 and 1.2 respectively introduce relevant results from the theories of tetrad systems and harmonic weight enumerators. Section 2 states and proves the Koch condition.

### 1.1. Tetrad systems.

For a doubly even code $C \subseteq \mathbb{F}_2^n$, the set $C_4$ is called the tetrad system of $C$. In analogy with the theory of root systems for lattices, the code $\tau(C)$ generated by $C_4$ is called the tetrad subcode of $C$, and if $\tau(C) = C$ then $C$ is called a tetrad code. The irreducible tetrad codes are exactly

- the codes $d_{2k}$ ($k \geq 2$), consisting of all words $c \in \mathbb{F}_2^{2k}$ of doubly even weight such that $c_{2j-1} = c_{2j}$ for each $j = 1, 2, \ldots, k$;
- the $[7, 3, 4]$ dual Hamming code, called $e_7$ in this context; and
- the $[8, 4, 4]$ extended Hamming code, here called $e_8$ (see [10]). We use the names $d_{2k}$, $e_7$, $e_8$ because the Construction A lattices $L_{d_{2k}}$, $L_{e_7}$, and $L_{e_8}$ are isomorphic with the root lattices $D_{2k}$, $E_7$, and $E_8$ respectively.

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3Here $(\cdot, \cdot)$ denotes the standard inner product on $\mathbb{R}^n$.

4See Ebeling [5] for further discussion.
Thus either \( \tau \in \{ \} \) or each irreducible component of (3) has tetrad number \( \eta(C) \) of an irreducible tetrad code \( C \) of length \( m \) to be \( |C_4|/m \). A quick computation shows that each of the \( m \) coordinates of \( C \) takes the value 1 on exactly \( 4\eta(C) \) words in \( C_4 \), and that \( \eta(d_{2k}) = (k - 1)/4 \) for each \( k \), while \( \eta(e_8) = 7/4 \).

1.2. Harmonic weight enumerators. Delsarte \cite{Delsarte} introduced the theory of discrete harmonic polynomials. For any code \( C \subseteq F_2^n \) and any discrete harmonic polynomial \( P : F_2^n \to C \), the harmonic weight enumerator \( W_{C,P}(x, y) \) is defined by

\[
W_{C,P}(x, y) = \sum_{c \in C} P(c)x^{|c|}y^{|c|} = \sum_{w=0}^n \left( \sum_{c \in C_w} P(c) \right) x^n y^w.
\]

These generalized weight enumerators are analogous to the weighted theta functions of lattice theory; they encode the distributions of codewords in \( C_w \) on the Hamming sphere of radius \( w \).

Bachoc \cite{Bachoc1} showed that harmonic weight enumerators satisfy the following identity generalizing the MacWilliams identity for Hamming weight enumerators:

\[
W_{C,P}(x, y) = (-xy)^{\deg P} \cdot \frac{2^{n/2}}{|C_4|} \cdot W_{C^\perp, P} \left( \frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}} \right).
\]

Furthermore, by Bachoc \cite[Cor. 2.1]{Bachoc1}, if \( C \) is Type II and if \( P \) is homogeneous of degree 1, then \( W_{C,P}(x, y) \) is in a space of covariant homogeneous polynomials, called \( \mathcal{I}_{C_1, C_2} \) in \cite[Lem. 3.1]{Bachoc1}. By the same lemma \cite[Lem. 3.1]{Bachoc1}, \( \mathcal{I}_{C_1, C_2} \) contains no nonzero polynomials of degree less than 30.

2. The Koch criterion

**Theorem 1.** If \( C \) is a Type II code of length 24, then \( C \) has one of the following nine tetrad systems:

\[
0, \ 6d_4, \ 4d_6, \ 3d_8, \ 2d_{12}, \ d_{24}, \ 2e_7 + d_{10}, \ 3e_8, \ e_8 + d_{10}.
\]

This result follows from the classification of Type II codes of length 24, and also appears in \cite{Gopal}. In the proof of Theorem 1, we will use the following proposition.

**Proposition 2.** Let \( C \) be a Type II code of length 24. Then,

- either \( C_4 = \{ \} \) or for each \( i \in \{ 1, 2, \ldots, 24 \} \) there exists \( c \in C_4 \) such that \( c_i = 1 \).
- each irreducible component of \( \tau(C) \) has tetrad number equal to \( |C_4|/24 \).

**Proof.** For \( i = 1, 2, \ldots, 24 \) let \( P_i \) be the discrete harmonic polynomial of degree 1 given by \( P_i(c) = 24c_i - |c| \), where “24c” is the real number 0 or 24 according as the \( F_2 \) element \( c_i \) is 0 or 1. As we saw at the end of Section \cite{Gopal} the harmonic weight enumerator \( W_{C,P_i}(x, y) \) must vanish for each \( i \). Extracting the \( x^{24-w}y^w \) coefficient from \( (1) \), we deduce that

\[
\sum_{c \in C_w} (24c_i - w) = 0
\]

for each \( i \) and \( w \). Taking \( w = 4 \) and reorganizing (2) gives

\[
\sum_{c \in C_w} (24c_i - w) = 0
\]

Thus either \( C_4 \) is empty or each \( i \) is contained in the support of some \( c \in C_4 \), and in the latter case each irreducible component of \( \tau(C) \) has tetrad number \( |C_4|/6 = |C_4|/24 \). \( \square \)
For any irreducible tetrad code $C$, the set $C^4$ is a “1-design”. Our Proposition 2 shows that $C^4$ is also a 1-design whenever $C$ is a Type II code of length 24. Theorem 1 now follows:

Proof of Theorem 1. For each $\eta \notin \{1, 7/4\}$, there is at most one tetrad system with tetrad number $\eta$. For each $\eta \in \{1, 7/4\}$, there are two tetrad systems with tetrad number $\eta$, namely $d_{10}$ and $e_7$ for $\eta = 1$, and $d_{16}$ and $e_8$ for $\eta = 7/4$.

By Lemma 2 we see that if $C^4$ is nonempty then either it consists of $m$ tetrad systems of type $d_{2k}$ for some $m$ and $k$ such that $m \cdot 2k = 24$, or it has one of the following forms:

- $\delta_{10}d_{10} + \varepsilon_7e_7$, with $\varepsilon_7 > 0$ and $10\delta_{10} + 7\varepsilon_7 = 24$, or
- $\delta_{16}d_{16} + \varepsilon_8e_8$, with $\varepsilon_8 > 0$ and $16\delta_{16} + 8\varepsilon_8 = 24$.

The list in Theorem 1 then follows almost immediately. □

As we mentioned in Section 1, our approach is directly analogous to that of Venkov 19. We may now explain this analogy explicitly: in our proof of Theorem 1, we use the combinatorial 1-design property of the tetrad system of $C$ in the same way that Venkov 19 uses the spherical 2-design property of the root system of a rank-24 Type II lattice.

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