Abstract:

The universal field equations introduced by the author and his collaborators, which admit infinitely many inequivalent Lagrangian formulations are shown to arise as consistency conditions for the existence of non-trivial solutions to the quasi-linear equations, called equations of hydrodynamic type by Novikov, Dubrovin and others. The solutions in closed form are only implicit. A method due to Stokes, which is in essence just Fourier Analysis is resurrected for application to those equations. With the benefit of algebraic computation facilities, this method, allows the general structure of power series solutions to be conjectured.
1. Introduction.

The purpose of this article is to comment upon some features of nonlinear equations related to the Euler equations of motion for fluid flow, which have attracted considerable attention among theoretical physicists in recent months after the seminal paper of Polyakov\cite{1}. Two diverse lines of inquiry regarding the properties of these equations are followed; The first is to demonstrate how the equations which we have called Universal Field Equations\cite{2−5} may be derived as consistency conditions on the integrability of a simple set of first order equations which have been extensively investigated by Dubrovin and Novikov\cite{6,7} and by Tsarev\cite{8−10} under the name of ‘systems of hydrodynamic type’. The Universal Field Equations enjoy remarkable properties. They may be derived from an infinite number of inequivalent Lagrangians, are associated with an infinite number of conservation laws and are either reparametrization invariant\cite{3}, or else the solutions are completely covariant. They may be linearised by a Legendre Transform and thus integrated to give an implicit solution\cite{5}.

The second is concerned with an iterative method of solving such nonlinear equations which goes back to Stokes\cite{11}, as referenced in Whitham’s book\cite{12}. Stokes did not have the advantage of a sophisticated algebraic manipulation package, such as MAPLE\cite{13} which permits the rapid evaluation of approximations to a sufficient order to admit their structure to be guessed, otherwise the development of methods of dealing with integrable equations might have taken a different course. Consider first the case of (1+1) dimensions; take the simple nonlinear wave equation, \( u_t = uu_x \), or perhaps better the KdV equation \( u_t = 6uu_x + u_{xxx} \), or the Burgers equation, \( u_t = uu_x + \nu u_{xx} \), which is the Navier Stokes equation in the absence of a pressure gradient in one space dimension. All of these equations can be solved in principle; the first by the method of characteristics to give the implicit solution \( u = F(x + ut) \), the second by the Inverse Scattering method and the third by the Cole-Hopf transformation. In practice an explicit approximate solution might be more useful, as frequently the problem posed is that of the time evolution of an initial wave-form. Look for a solution of the form of a formal Fourier expansion

\[
    u(x,t) = u_0(t) + u_1(t)e^{ax} + u_2(t)e^{2ax} + u_3(t)e^{3ax} + \ldots
\]  

The initial data is coded into the values of the coefficients \( u_j(t) \) at \( t = 0 \). Then the key point is that these coefficients are determined recursively by a set of inhomogeneous linear equations, where the driving term for the equation for \( u_n(t) \) is determined by terms of the coefficients \( u_j(t), \ j < n \). \( u_0 \) is a constant and \( u_1 \) is determined by a homogeneous linear equation; \( u_2 \) is then determined by an inhomogeneous linear equation with driving term determined by \( u_1 \) and so on. The general structure of the solution takes a remarkably simple form, and allows a Taylor like expansion for solutions to the non-linear wave equation to be formally deduced. If one looks for periodic solutions of the KdV equation, then this method gives a sequence of approximations to a given initial value problem. This is just what the inverse scattering method was set up to achieve! Admittedly this construction is formal; but then the inverse scattering method permits a exact solution in closed form to the initial value problem, only when this corresponds to a collection of of solitons! Indeed the multisoliton solution of the KdV equation\cite{12} can be recovered by formal summations
using the above method\cite{14}. Another possible application of this approach is to the quantisation of such nonlinear equations, by considering the coefficients in the expansion of (1.1) as creation operators.

2. The Universal Field Equations as Consistency Conditions.

Consider the following set of equations

\[ \sum_{j=1}^{N} u_j \frac{\partial u_i(x_k)}{\partial x_j} = 0, \quad i = 1, \ldots, N. \tag{2.1} \]

A special case, in which \( u(1) = -1, \ x_1 = t \) constitutes a multidimensional version of the equations of hydrodynamic type\cite{7}, or equivalently, (2.1) may be regarded as a static limit of those multidimensional equations. For the purposes of this section it is more convenient to treat all the variables on the same footing, as in (2.1). Now suppose all \( u_j, \ j = 2, \ldots, N \) are all functions of a single function \( u_1 = v \). Then the result of differentiating (2.1) for \( i = 1 \) with respect to \( x_1, x_2, \ldots, x_N \) in turn and using the easily proved result that

\[ \frac{\partial u_i}{\partial x_j} \frac{\partial v}{\partial x_k} = \frac{\partial u_i}{\partial x_k} \frac{\partial v}{\partial x_j} \tag{2.2} \]

yields the \( N \) equations

\begin{align*}
& u_1 v_{x_1} + u_2 v_{x_2} + \ldots + u_N v_{x_N} + \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \ldots + \frac{\partial u_N}{\partial x_N} \right) v_1 = 0 \\
& u_1 v_{x_2} + u_2 v_{x_2} + \ldots + u_N v_{x_N} + \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \ldots + \frac{\partial u_N}{\partial x_N} \right) v_2 = 0 \\
& \quad \ldots \quad \ldots \quad \ldots = 0 \\
& u_1 v_{x_N} + u_2 v_{x_N} + \ldots + u_N v_{x_N} + \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \ldots + \frac{\partial u_N}{\partial x_N} \right) v_N = 0 \tag{2.3}
\end{align*}

Elimination of the \( N + 1 \) variables \( u_i, \ i = 1, \ldots, N \) and \( \sum \frac{\partial u_i}{\partial x_j} \) between the \( N + 1 \) equations (2.1) and (2.3) yields the determinantal condition

\[ \det \left( \begin{array}{cc} 0 & \frac{\partial v}{\partial x_k} \\ \frac{\partial v}{\partial x_j} & \frac{\partial^2 v}{\partial x_j \partial x_k} \end{array} \right) = 0 \tag{2.4} \]

This is just the Universal Field equation of reference [1]. Furthermore, the extension to incorporate many fields arises in a similar manner. Suppose the only the first \( n \) among the \( N \) functions \( u_j \) are functionally independent; i.e. the functions \( u_j, \ j = n + 1, \ldots, N \) are functions of \( u_i, \ i = 1, \ldots n \). Then consider the equation

\[ \sum_{j=0}^{N} u_j \sum_{k=0}^{n} a_k \frac{\partial u_k}{\partial x_j} = 0. \tag{2.5} \]
Here $\lambda_k, k = 0, \ldots, n$ are a set of arbitrary constants. Differentiation with respect to $x_i$ gives

$$
\sum_{j=0}^{N} u_j \sum_{k=0}^{n} \lambda_k \frac{\partial^2 u_k}{\partial x_i \partial x_j} + \sum_{i=0}^{n} \frac{\partial u_j}{\partial x_i} \left[ \sum_{k=0}^{n} \lambda_k \frac{\partial u_k}{\partial x_j} + \sum_{r=n+1}^{N} \frac{\partial u_r}{\partial x_j} \sum_{k=0}^{n} \lambda_k \frac{\partial u_k}{\partial x_r} \right] = 0. \tag{2.6}
$$

Elimination of the $N+n$ variables $u_j$ and

$$
(\sum_{k=0}^{n} \lambda_k \frac{\partial u_k}{\partial x_j} + \sum_{r=n+1}^{N} \frac{\partial u_r}{\partial x_j} \sum_{k=0}^{n} \lambda_k \frac{\partial u_k}{\partial x_r})
$$

from the $N$ equations (2.6) regarded as linear equations in those variables and the $n$ independent relations among the equations

$$
\sum_{j=0}^{N} u_j \frac{\partial u_k}{\partial x_j} = 0, \tag{2.7}
$$

reproduces the multifield form of the Universal equation. In compact form, these generalised Bateman equations are given by

$$
\det \left( \begin{array}{c|c} 0 & \frac{\partial u_a}{\partial x_j} \\ \hline \frac{\partial u_a}{\partial x_i} & \lambda^c \frac{\partial^2 u_c}{\partial x_i \partial x_j} \end{array} \right) = 0. \tag{2.8}
$$

Here, the determinant is that of a $(N+n) \times (N+n)$ matrix, with $n$ being the number of fields $u_a$ and $N$ the number of coordinates $x_i$ ($n < N$), and the indices $(a, i)$ and $(b, j)$ refer to rows and columns respectively (in particular, the entry “0” stands for the $n \times n$ null matrix). The quantities $\lambda^c$ are arbitrary coefficients, in terms of which the determinant is to be expanded. It is understood that (2.8) has to hold for all values of these coefficients. Hence, one obtains $(N-1) \choose (n-1)$ equations – clearly generalising the Bateman equation (2.4) for one field to many – whose general covariance under arbitrary field redefinitions in $u^a$ is easily established. Even though these equations form an over determined set, except for ($n = 1$) or ($n = N-1$) (the equations in the latter case were shown\cite{1} to be also universal), their space of solutions is non empty\cite{1}, as any arbitrary choice of functions $u_i(x_j)$, homogeneous of degree zero in the variables $x_i$ will satisfy (2.8).

3. Introduction of explicit time.

Consider the equations,

$$
u_t + uu_x + vv_y = 0; \tag{3.1}$$
$$v_t + uv_x + vv_y = 0.$$

These equations are related to the one Polyakov and others have studied in the theory of conformal turbulence through setting $u = \psi_y; v = -\psi_x$. This constraint ensures that the two dimensional velocity vector $\{u, v\}$ describes incompressible flow; i.e. $u_x + u_y = 0$. Of course the above equations simply say that the total derivative $\frac{d}{dt} u(x, y, t) = 0$, and are the Euler equations for the free motion of an inviscid fluid. The equation for the vorticity $\omega$, which is given by $\omega = u_y - v_x = \nabla^2 \psi$ which Polyakov studies is

$$
\dot{\omega} = \psi_x \omega_y - \psi_y \omega_x \tag{3.2}
$$
There is a further consistency condition on $\psi$, arising from (3.1), namely

$$\psi_{xx}\psi_{yy} - \psi_{xy}^2 = 0. \tag{3.3}$$

if in addition to (3.2) equations (3.1) are imposed. These considerations have been extended to the Navier Stokes equation\textsuperscript{15} by the introduction of a viscous force in (3.2) Somewhat curiously, if an irrotational flow with $\omega = 0$ is sought, then $\{u, v\}$ is derivable from a potential $\phi$ as $\nabla\phi$, and the equations (3.1) reduce to the following single equation for $\phi$,  

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 = 0. \tag{3.4}$$

Of course, the requirement of incompressibility imposes an additional constraint upon $\phi$, namely that it should satisfy the equation $\nabla^2\phi = 0$. It is easy to see, by transforming to complex variables $z = x + iy, \bar{z} = x - iy$, that the only solution in this case is $\{u, v\} =$ constant. There are infinitely many Lagrangians;

$$\frac{f(uv)}{u} u_t + f(uv)u_x - f(uv)v_y. \tag{3.5}$$

An implicit solution;

$$u = F(x - ut, y - vt)$$

$$v = G(x - ut, y - vt). \tag{3.6}$$

where $F$ and $G$ are arbitrary functions of two variables can easily be verified, and this result extends in an obvious manner to the solution of the equations

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{N} u_j \frac{\partial u_i}{\partial x_j} = 0. \tag{3.7}$$

That these equations admit solutions which develop singularities can be seen by looking for solutions which satisfy the ansatz

$$u_j(t, x_k) = \frac{z_j(x_k)}{(t - a)}. \tag{3.8}$$

With the choice $z_i(x_k) = \sum A_{ik}x_k$, then (3.8) is a a solution provided the matrix $A$ satisfies $A = A^T A$.  

Introduce another set of fields $v_i, \ i = 1 \ldots N$.Then

$$\mathcal{L} = \sum_{j=1}^{N} v_j \frac{\partial u_j}{\partial t} + \sum_{j=1}^{N} \sum_{k=1}^{N} u_k v_j \frac{\partial u_j}{\partial x_k} \tag{3.9}$$

is a Lagrangian for the equations (2.4). The companion set of equations, obtained by varying $\mathcal{L}$ with respect to $u_i$ is

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{N} (u_j \frac{\partial v_i}{\partial x_j} + v_i \frac{\partial u_j}{\partial x_j} - v_j \frac{\partial u_j}{\partial x_i}) = 0. \tag{3.10}$$
If \( u \) is eliminated from equations (3.1) the resulting equation for \( v \) can be written in a nice determinantal form:

\[
\det \begin{pmatrix}
0 & 0 & 0 & 1 & v \\
0 & 0 & v_x & v_y & v_t \\
0 & v_x & v_{xx} & v_{xy} & v_{xt} \\
1 & v_y & v_{xy} & v_{yy} & v_{yt} \\
v & v_t & v_{xt} & v_{yt} & v_{tt}
\end{pmatrix} = 0. \tag{3.11}
\]

Note that if \( v \) does not depend upon \( y \) this equation is the Bateman equation, various aspects of which are considered in references [2-5]. In that case the complete solution is given by the implicit solution for \( v \) of the equation

\[
t f(v) + x g(v) = c \tag{3.12}
\]

where \( f, g \) are arbitrary functions and \( c \) is a constant which may be zero. In the general situation the solution is given by the elimination of \( u \) from (3.6)

4. Explicit Formal Solution of the Nonlinear Wave Equation.

It is well known that the general solution of the nonlinear wave equation

\[
\frac{\partial u}{\partial t} = uu_x \tag{4.1}
\]

is solved implicitly as

\[
u = G(x + ut). \tag{4.2}
\]

It may however be solved for an arbitrary differentiable initial value \( F \) in terms of a Laplace, or Fourier expansion of the initial value of the form

\[
F(x) = c_0 + c_1 e^{ax} + c_2 e^{2ax} + c_3 e^{3ax} + \ldots, \tag{4.3}
\]

by the method outlined in the introduction. \( c_0 \) may be taken as 1, by rescaling \( t \). A computer calculation using MAPLE\([13]\) reveals the conjecture that the power series expansion of \( u(x,t) \) is given by

\[
u(x,t) = \sum_{n_1, n_2, \ldots} \prod_{j=1}^{j=k} c_j^{n_j} (Nt)^{n-1} \exp N(ax + t) n_j! n_j^{-1} \tag{4.4}
\]

Here \( N = n_1 + 2n_2 + \ldots kn_k, n = n_1 + n_2 + \ldots n_k \). These coefficients may displayed as a formal contour integral, and the formal sum constructed to give

\[
u(x,t) = 1 + \sum_{n=1}^{\infty} \frac{1}{2\pi i} \oint (c_1 \exp a(x + z) + c_2 \exp 2a(x + z) + \ldots)^n \left( \frac{t}{(z - t)} \right)^n \frac{dz}{nt}
\]

\[
= F(0) + \sum_{n=1}^{\infty} t^{n-1} \frac{\partial^{n-1}}{\partial x^{n-1}} \frac{(F(x + F(0)t) - F(0))^n}{n!}, \tag{4.5}
\]

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after rescaling so that $F(O) = c_0$ is arbitrary. The contour is taken sufficiently large so as to encircle the multiple poles at $z = t$. The sum may itself be formally performed, to give the result

$$u(x, t) = 1 - \frac{1}{2\pi i} \oint \log \left(1 - \frac{t \sum_j c_j \exp aj(x + z)}{z - t} \right) \frac{dz}{t}$$

(4.6)

$$= F(0) - \frac{1}{2\pi i} \oint \log \left(1 - \frac{t}{z - t} \left(F(x + F(0)z) - F(0)\right) \right) \frac{dz}{t}$$

There does not appear to be a simple proof of the result (4.5), but it may be used to obtain the explicit solution

$$u(x, t) = \frac{a + x}{1 - t},$$

(4.7)

which also follows from (4.2). It would be interesting to extend this to (3.1). We may note that if $v(t)$ is any prescribed function of $u(t)$ the two equations (3.1) reduce to a single equation,

$$u_t + uu_x + f(u)u_y = 0,$$

(4.8)

where $f$ is arbitrary. This case at least, is not tractable by the above method. However, the physically interesting case is when $u, v$ are derivable from a stream function, and $u_x + v_y = 0$. An explicit, possibly general, easily verifiable class of solutions dependent upon an arbitrary function of a linear combination of $x, y, t$ can be found by the Fourier method as

$$u(x, y, t) = u_0 + bG(ax + by + (au_0 + bv_0)t),$$

$$v(x, y, t) = v_0 - aG(ax + by + (au_0 + bv_0)t).$$

(4.9)

5. Conclusion

Some properties of the quasi linear system of equations of hydrodynamic type have been explored, and their general solution given in terms of arbitrary functions, but only in implicit form. This is a feature of some nonlinear completely integrable systems in higher dimension, and results from the fact that one of the methods of solution is to interchange the role of dependent and independent variables, whereupon the equation linearizes. However, if a Fourier series solution, with time dependent coefficients is sought, then instead of the complete decoupling of the ordinary differential equations describing the time evolution of the modes, these are determined successively in terms of the fundamental mode. It is a remarkable feature that the structure of the power series solution to the simplest nonlinear wave equation is easily elucidated with the aid of an algebraic computation package, and a formal power series solution obtained for an arbitrary initial configuration for the simplest nonlinear wave equation.
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