Weights of Markov traces for Alexander polynomials of mixed links

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Abstract

Using the Fourier expansion of Markov traces for Ariki-Koike algebras over $\mathbb{Q}(q, u_1, \ldots, u_e)$, we give a direct definition of the Alexander polynomials for mixed links. We observe that under the corresponding specialization of a Markov parameter, the Fourier coefficients of Markov traces take quite simple form.

As a consequence, we show that the Alexander polynomial of a mixed link is essentially equal to the Alexander polynomial of the link obtained by resolving the twisted parts.

1 Introduction

In [L1] Lambropoulou initiated the mixed link theory which is the link theory in the solid torus. A mixed link is an embedding of a disjoint union of finitely many circles into the solid torus $Y$. Since the 3-sphere has the canonical genus 1 Heegaard decomposition, it can be considered as an embedding into the complement $S^3 \setminus \text{Int} Y$.

We say that two mixed links are equivalent if they are joined by an ambient isotopy of $S^3 \setminus \text{Int} Y$.

Lambropoulou showed that there is a variant of the usual braid theory, that is, every mixed link is equivalent to the closure of a mixed braid and one can consider analogues of the Markov Moves. More precisely, a mixed braid with $n$-strands is an $n$-tuple $(p_1, \ldots, p_n)$ of embeddings of the closed interval $[0, 1]$ into the closure $\text{cl}(I^3 \setminus Cyn)$, where
Cyn := \{ (x, y, z) \in I^3 \left| \left( x - \frac{1}{4} \right)^2 + \left( y - \frac{1}{2(n+1)} \right)^2 \leq \left( \frac{1}{4(n+1)} \right)^2 \right. \},

such that

- the curves \( p_i(I) \) and \( p_j(I) \) do not intersect if \( i \neq j \),
- there exists a permutation \( \sigma \) of \( \{1, \ldots, n\} \) such that
  \[
  p_i(0) = \left( \frac{1}{2}, \frac{i}{n+1}, 0 \right), \quad p_i(1) = \left( \frac{1}{2}, \frac{\sigma(i)}{n+1}, 1 \right)
  \]
  for all \( i \in \{1, \ldots, n\} \),
- the point \( p_i(t) \) lies in the interior of \( \text{cl}(I^3 \setminus \text{Cyn}) \) if \( t \in (0, 1) \),
- the 3rd coordinate of \( p_i(t) \) is increasing with respect to \( t \) for all \( i \in \{1, \ldots, n\} \).

Two mixed braids with \( n \)-strands are said to be equivalent if they are joined by an ambient isotopy of \( \text{cl}(I^3 \setminus \text{Cyn}) \) which fixes the boundary.

Then the set of equivalence classes is in bijection with the affine braid group \( B_{\text{aff},n} \) with generators

\[
t_0, t_1, \ldots, t_{n-1}
\]
satisfying fundamental relations

\[
 t_0t_1t_0t_1 = t_1t_0t_1t_0, \quad t_it_j = t_jt_i \quad (|i - j| \geq 2), \quad t_it_{i+1}t_i = t_{i+1}t_it_{i+1} \quad (1 \leq i \leq n-2).
\]

It is known that \( B_{\text{aff},n} \) is the semi-direct product of the braid group \( B_n = \langle t_1, \ldots, t_{n-1} \rangle \) and the free subgroup \( P_n \) generated by \( t'_0, t'_1, \ldots, t'_{n-1} \) where

\[
 t'_i = t_i \cdots t_1t_0t_1^{-1} \cdots t_i^{-1} \quad (i = 0, 1, \ldots, n-1).
\]

In particular, the braid group \( B_n \) is embedded in \( B_{\text{aff},n} \) as a subgroup.

Moreover, in this setting we have the following analogues of the Markov moves:

1. \( \alpha \leftrightarrow \beta \alpha \beta^{-1} \quad (\alpha, \beta \in B_{\text{aff},n}) \),
(2) $\alpha \leftrightarrow \alpha t_n^{\pm 1} \ (\alpha \in B_{\text{aff}, n})$.

One of the main interest in this paper is to construct an analogue of the Alexander polynomial explicitly.

In the usual link theory, Jones [J] discovered a way to construct the HOMFLYPT polynomial using the Markov traces of Iwahori-Hecke algebras of type $A$. He gave two methods to derive the Alexander polynomials. The first one is to use the Skein relation for HOMFLYPT polynomials and the second is to use the “Fourier expansion” of the Markov traces of Iwahori-Hecke algebras of type $A$, namely, the expression as the linear combination of irreducible characters. Note that the second method is more direct than the first one.

As for the mixed links, it is possible to give an analogue of the Alexander polynomial using the analogue of HOMFLYPT polynomials and their Skein relations given in [L2]. We point out that one can also define the Alexander polynomial of a mixed link directly following Jones’s second argument. For this purpose a result of Geck-Iancu-Malle [GIM] is quite helpful. In [GIM], they determined the rational polynomials appearing in the coefficients in the expression of the Markov traces as the linear combination of the irreducible characters of the Ariki-Koike algebra of type $G(e, 1, n)$.

Then our second observation is that when we consider the specialization of parameter, the Fourier coefficients take quite simple form. As a consequence, we show that the Alexander polynomial of a mixed link is essentially the same as the Alexander polynomial of the link obtained by resolving the twisting parts.

This paper is organized as follows. In Section 2, we recall the definition of the Ariki-Koike algebras and its irreducible ordinary representations. In Section 3, we recall the definition of the Markov traces for Ariki-Koike algebras and its Fourier expansion due to Geck-Iancu-Malle. In Section 4, we collect some fundamental definitions and results in Lambropoulou’s mixed link theory. In Section 5, we propose a definition of Alexander polynomial for a mixed link. In Section 6, we calculate the Fourier coefficient of Markov traces and prove a relation between the Alexander polynomials for mixed links and the one for the usual links.

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2 Representations of Ariki-Koike algebras

In this section we recall the definition of the Ariki-Koike algebra and its ordinary finite dimensional irreducible representations. For the details we refer the original paper [AK].

Let $e, n$ be two positive integers. We denote by $k$ the field of rational functions over $\mathbb{Q}$ with $(e + 1)$-indeterminates $q, u_1, \ldots, u_e$. 
Definition 2.1. The Ariki-Koike algebra of type $G(e, 1, n)$ is the associative $k$-algebra $H_{e,n}$ with generators $T_0, T_1, \cdots, T_{n-1}$ satisfying fundamental relations

\[
(T_0 - u_1)(T_0 - u_2) \cdots (T_0 - u_e) = 0, \\
(T_i - q)(T_i + 1) = 0 \quad (1 \leq i \leq n - 1), \\
T_0T_1T_0T_1 = T_1T_0T_1T_0, \\
T_iT_j = T_jT_i \quad ([i - j] \geq 2), \\
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad (1 \leq i \leq n - 2).
\]

Remark 2.2. When $e = 1$ and $u_1 = 1$, the corresponding Ariki-Koike algebra of type $G(1, 1, n)$ is just the Iwahori-Hecke algebra of type $A_{n-1}$. Similarly, when $e = 2$ and $u_1 = -1$, we have the Iwahori-Hecke algebra of type $B_n$ with unequal parameters.

For $i \in \{0, 1, \cdots, n - 1\}$ we put $L_i = T_i \cdots T_1T_0T_1 \cdots T_i^{-1}$. We also define an element $T_w (w \in S_n)$ as follows: let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression of $w$ where $s_i$ is the permutation which transposes $i$ and $i + 1$. Then we put $T_w = T_{i_\ell} \cdots T_{i_1}$. By [AK, Theorem 3.10] the set

\[
\{ L_0^eL_1^e \cdots L_{n-1}^eT_w \mid 0 \leq e_1 \leq e \leq 1 (0 \leq i \leq n - 1), \ w \in S_n \}
\]

forms a $k$-basis of $H_{e,n}$.

Note that for each positive integer $e$ we have the following inductive system:

\[ H_{e,1} \subset H_{e,2} \subset \cdots \subset H_{e,n} \subset \cdots \]

We denote by $H_e$ the inductive limit of the inductive system, i.e., the union

\[ \bigcup_{n=1}^{\infty} H_{e,n}. \]

Next, let us recall the definition of multi-Young tableaux and related notions.

Definition 2.3. An $e$-Young diagram of total size $n$ is an $e$-tuple $\lambda = (\lambda_1, \cdots, \lambda_e)$ consisting of sequences $\lambda_i = (\lambda_{i,1}, \cdots, \lambda_{i,p(i)})$ of positive integers which satisfies the following two conditions:

1. $\sum_{1 \leq i \leq n} \lambda_{i,j} = n,$
2. $\lambda_{i,1} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,p(i)}$ for all $i \in \{1, \cdots, e\}$.

Remark 2.4. In the definition of the multi-Young diagram, we allow the case that $\lambda_i = \emptyset$ for some $i$.

Obviously we can regard an $e$-Young diagram as an $e$-tuple of Young diagrams.

Example 2.5. The 4-Young diagram $\lambda = ((4, 3, 1), \emptyset, (3, 1, 1), \emptyset)$ of total size 13 is identified with the following 4-tuple of Young diagrams:
Let $a, b$ be boxes in an $e$-Young diagram $\lambda$. Then the content $c(a; \lambda)$ of $a$ is the difference

$$(\text{the column number of } a) - (\text{the row number of } a)$$

and the axial distance $r(a, b)$ from $a$ to $b$ is the difference $c(b; \lambda) - c(a; \lambda)$.

**Definition 2.6.** Let $\lambda$ be an $e$-Young diagram. A **standard $e$-tableau** $T$ of shape $\lambda$ is a pair of an $e$-Young diagram and an ordering on the boxes by $\{1, 2, \cdots, n\}$ which satisfies the following condition: in each Young diagram the written numbers are increasing from left to right and from top to bottom.

**Example 2.7.** Let $\lambda$ be the 4-Young diagram presented in Example 2.5. Then in the following two figures the left one is a standard 4-tableau of shape $\lambda$ but the right one is not.

$$\begin{pmatrix}
2 & 5 & 6 & 10 & \emptyset & \emptyset & \emptyset & \emptyset \\
8 & 11 & 12 & \emptyset & 3 & \emptyset & \emptyset & \emptyset \\
9 & 13 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset
\end{pmatrix}, \quad \begin{pmatrix}
2 & 5 & 6 & 10 & \emptyset & \emptyset & \emptyset & \emptyset \\
8 & 11 & 12 & \emptyset & 3 & \emptyset & \emptyset & \emptyset \\
9 & 13 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset
\end{pmatrix}$$

For an $e$-Young diagram $\lambda$, we denote by $\text{Std}(\lambda)$ the set of standard $e$-tableaux of shape $\lambda$. For an integer $k$ and an indeterminate $y$ we define a Laurent polynomial $\Delta(k, y)$ and a matrix $M(k, y)$ as follows:

$$\Delta(k, y) = 1 - q^k y, \quad M(k, y) = \frac{1}{\Delta(k, y)} \begin{bmatrix}
q - 1 & \Delta(k + 1, y) \\
q \Delta(k - 1, y) & -q^k y(q - 1)
\end{bmatrix}.$$

Finally, for a standard $e$-tableau $T$ we define the number $\tau(i)$ so that $i$ is written in the $\tau(i)$-th Young diagram of $T$. For an $e$-Young diagram $\lambda$ of total size $n$ we denote by $V(\lambda)$ the finite dimensional vector space

$$V(\lambda) = \bigoplus_{T \in \text{Std}(\lambda)} kt$$

where $t$ is the symbol corresponding to a standard $e$-tableau $T$.

Following [AK] we define a representation of $H_{e,n}$ on $V(\lambda)$ as follows:

1. $T_0 t = u_{\tau(1)} t$,

2. For $i \in \{1, \cdots, n - 1\}$ we define $T_i t$ as follows:
   (2-1) We define $T_i t = qt$, if $i$ and $i + 1$ are placed as
   $$\begin{pmatrix}
i & i + 1
\end{pmatrix}$$
We define $T_i t = -t$, if $i$ and $i + 1$ are placed as

$$
\begin{array}{c}
i \\
\hline
i+1
\end{array}
$$

In the other case we define $T_i t$ by the following:

$$T_i \langle t, t' \rangle = \langle t, t' \rangle M \left( r(i+1, i), \frac{u_{\tau(i)}}{u_{\tau(i+1)}} \right).$$

Here $t'$ is the symbol corresponding to the standard $e$-Young tableau $T'$ obtained by transposing $i$ and $i + 1$ in $T$.

**Remark 2.8.** To unify the above definitions (1),(2-1),(2-2) and (2-3) we introduce the following notation. For the permutation $s_i = (i, i + 1)$ and a standard $e$-tableau $T$ we define a vector $s_i t$ in $V(\lambda)$ by

$$s_i t = \begin{cases} t' & \text{(if } T' \text{ is standard)} \\ 0 & \text{(else)} \end{cases}.$$

Here, $T'$ is the $e$-Young tableau obtained by transposing $i$ and $i + 1$ in $T$.

Under this notation we have

$$T_i t = \frac{(q - 1) u_{\tau(i)}}{u_{\tau(i)} - q^{r(i+1,i)} u_{\tau(i+1)}} t + \frac{q(u_{\tau(i)} - q^{r(i+1,i)} - 1) u_{\tau(i+1)}}{u_{\tau(i)} - q^{r(i+1,i)} u_{\tau(i)}} (s_i t),$$

for all $i \in \{0, \ldots, n - 1\}$ and a standard $e$-tableau $T$.

The set

$$\{ V(\lambda) \mid \lambda : e\text{-Young diagram of total size } n \}$$

gives a complete list of finite dimensional irreducible representations over $k$ up to equivalence.

For an $e$-Young diagram of total size $n$ we denote by $\chi_\lambda$ the irreducible character corresponding to $\lambda$.

## 3 Fourier expansion of Markov traces

In this section we recall a result of Lambropoulou [L2] and of Geck-Iancu-Malle [GIM] concerning the Markov traces of the Ariki-Koike algebras.

Let $z, y_1, \cdots, y_{e-1}$ be elements of $k$. A $k$-linear map $\tau : H_e \rightarrow k$ is called the **Markov trace** associated to $z, y_1, \cdots, y_{e-1}$ if it satisfies the following properties:

1. $\tau(1) = 1$, 


(2) \( \tau(hh') = \tau(h'h) \)  \( (h, h' \in H_{e,n}) \),

(3) \( \tau(hT_i) = z\tau(h) \)  \( (h \in H_{e,n}, 1 \leq i \leq n) \),

(4) \( \tau(hL^j_1) = y_j\tau(h) \)  \( (h \in H_{e,n}, 0 \leq i \leq n-1, 1 \leq j \leq e-1) \).

By [L2] for fixed \( e, z, y_1, \cdots, y_{e-1} \) the Markov trace of \( H_e \) exists uniquely.

Since the Markov trace \( \tau \) satisfies (2), the restriction \( \tau|_{H_{e,n}} \) is written as the \( k \)-linear combination of irreducible characters. Geck-Iancu-Malle [GIM] determined the coefficients in this expression.

To state their result we prepare some notations. Let \( \lambda = (\lambda_1, \cdots, \lambda_e) \) be an \( e \)-Young diagram of total size \( n \). By adding some zeros we regard \( \lambda_1 \) and \( \lambda_p \) \( (p \in \{2, \cdots, e\}) \) as the sequences

\[
\lambda_1 = (\lambda_{1,1}, \cdots, \lambda_{1,n}, \lambda_{1,n+1}), \\
\lambda_p = (\lambda_{p,1}, \cdots, \lambda_{p,n})
\]

of length \( n+1 \) and \( n \) respectively. We define finite sets \( A_1, \cdots, A_e \) as follows:

\[
A_1 = \{ \alpha_{1,i} := \lambda_{1,i} + n - i + 1 \mid 1 \leq i \leq n + 1 \}, \\
A_p = \{ \alpha_{p,i} := \lambda_{p,i} + n - i \mid 1 \leq i \leq n \} \quad (2 \leq p \leq n).
\]

Finally, for a non-negative integer \( d \) we denote by \( \sigma_d \) the \( d \)-th fundamental symmetric polynomial with respect to \( u_1, \cdots, u_e \). Here, we understand \( \sigma_0 = 1 \). Under these notations, we define \( \tilde{D}^\lambda(q, u_1, \cdots, u_e) \) and \( R^\lambda(z, y_1, \cdots, y_{e-1}) \) as follows:

\[
\tilde{D}^\lambda(q, u_1, \cdots, u_e) = (-1)^{(\frac{1}{2})(n^2)} \prod_{1 \leq k \leq l \leq e} \prod_{(\alpha, \alpha') \in A_k \times A_l} (\alpha^\alpha u_k - q^{\alpha'} u_l) \prod_{k=1}^e u_k^{n_k},
\]

\[
R^\lambda(z, y_1, \cdots, y_{e-1}) = \prod_{k=1}^e \prod_{x \in \Lambda_k} (z(1-q^{c(x)}) \prod_{1 \leq i \leq e} (q^{c(x)} u_k - u_l))
\]

\[
+ (1 - q) \prod_{i=1}^{e-1} (q^{(e-i)c(x)} u_k^{e-i-1} \sum_{j=1}^i (-1)^{i-j} y_j \sigma_{i-j}) + (-1)^{e-1} \prod_{1 \leq i \leq e} u_i.
\]

Here, \( f(n, e) \) is given by

\[
\sum_{i=1}^{n-1} \binom{e(i+1)}{2} = \binom{e(n-1)+1}{2} + \binom{e(n-2)+1}{2} + \cdots + \binom{e+1}{2}.
\]

Finally, we put \( D^\lambda(q, u_1, u_2, \cdots, u_e) = \tilde{D}^\lambda(q, q^{-1} u_1, u_2, \cdots, u_e) \).
Lemma 3.1. ([GIM]) We have
\[ \tau|_{H_{e,n}} = \sum_{\lambda} (-1)^{\theta} (\prod_{k=1}^{e} u_{\lambda_k}^{|\lambda_k| - n}) D^\lambda(q, u_1, u_2, \ldots, u_e) R^\lambda(z, y_1, \ldots, y_{e-1}) \chi \]
where \( \lambda \) runs through all \( e \)-Young diagrams of total size \( n \) and \( |\lambda_k| \) stands for the number of boxes in \( \lambda_k \).

Remark 3.2. We see easily by induction on \( n \) that
\[ f(n, e) = \frac{1}{12} en(n-1)(2en-e+3). \]
In the rest of this paper, we set
\[ C(e, k, x) = \prod_{i=1}^{e-1} (q^{(e-i)c(x)} u_k^{e-i-1} \sum_{j=1}^{e} (-1)^{i-j} y_j \sigma_{i-j}) + (-1)^{e-1} \prod_{1 \leq l \leq e, l \neq k} u_l. \]

4 Mixed link theory

In this section we collect some fundamental notions and facts from Lambropoulou’s mixed link theory. For the details we refer [L1] and [L2].

Definition 4.1. (1) A \textbf{mixed link} is an embedding of a disjoint union of finitely many circles into the solid torus.

(2) Two mixed links are said to be \textbf{equivalent} if these are joined each other by an ambient isotropy of the solid torus.

By considering the canonical genus 1 Heegaard decomposition, we regard the solid torus as the complement of the interior of a solid torus:

\[ \text{Definition 4.2. (1) A mixed braid with } n \text{-strands is an } n \text{-tuple } (p_1, \ldots, p_n) \text{ of embeddings of the closed interval } [0, 1] \text{ into the closure } \text{cl}(I^3 \setminus Cyn), \text{ where} \]
\[ Cyn := \left\{ (x, y, z) \in I^3 \left| (x - \frac{1}{4})^2 + (y - \frac{1}{2(n+1)})^2 \leq \left( \frac{1}{4(n+1)} \right)^2 \right. \right\}, \]
such that
• the curves $p_i(I)$ and $p_j(I)$ do not intersect if $i \neq j$,

• there exists a permutation $\sigma$ of $\{1, \ldots, n\}$ such that

$$
p_i(0) = \left(\frac{1}{2} \cdot \frac{i}{n+1}, 0\right), \quad p_i(1) = \left(\frac{1}{2} \cdot \frac{\sigma(i)}{n+1}, 1\right)
$$

for all $i \in \{1, \ldots, n\}$,

• the point $p_i(t)$ lies in the interior of $\text{cl}(I^3 \setminus Cyn)$ if $t \in (0, 1)$,

• the 3rd coordinate of $p_i(t)$ is increasing with respect to $t$ for all $i \in \{1, \ldots, n\}$.

(2) Two mixed braids with $n$-strands are said to be **equivalent** if they are joined by an ambient isotopy of $\text{cl}(I^3 \setminus Cyn)$ which fixes the boundary.

As in the usual braid theory the set of equivalence classes of mixed links with $n$-strands is equipped with the natural group structure.

Let $B_{\text{aff}, n}$ be the $n$-th affine braid group, namely, the group with generators $t_0, t_1, \cdots, t_{n-1}$ satisfying fundamental relations

$$
t_{0}t_{1}t_{0}=t_{1}t_{0}t_{1},
\begin{align*}
t_{i}t_{j} &= t_{j}t_{i} \quad (|i - j| \geq 2), \\
t_{i}t_{i+1}t_{i} &= t_{i+1}t_{i}t_{i+1} \quad (1 \leq i \leq n-2).
\end{align*}
$$

**Lemma 4.3.** The group of equivalence classes of mixed braids with $n$-strands is isomorphic to $B_{\text{aff}, n}$ as groups. Here, $t_0$ corresponds to

and $t_i$ ($1 \leq i \leq n-1$) corresponds to

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (0,2);
\draw (1,0) -- (1,2);
\draw (2,0) -- (2,2);
\draw (3,0) -- (3,2);
\draw (4,0) -- (4,2);
\end{tikzpicture}
\end{center}
As in the usual braid theory, we can consider the closure of a mixed braid.

**Lemma 4.4.** Every mixed link is equivalent to the closure of a mixed braid.

For a mixed braid $\alpha$ we denote by $\hat{\alpha}$ the closure of $\alpha$. The analogue of the Markov Moves and Alexander’s theorem are given by the following lemma.

**Lemma 4.5.** The closures of two mixed braids are equivalent as mixed links if and only if the corresponding mixed braids are joined by a sequence of the following two transformations:

1. $\alpha \leftrightarrow \beta \alpha \beta^{-1}$ ($\alpha, \beta \in B_{\text{aff}, n}$)
2. $\alpha \leftrightarrow \alpha t_n^{\pm 1}$ ($\alpha \in B_{\text{aff}, n}$).

Let us recall the construction of the analogue of the HOMFLYPT polynomial in mixed link theory. We first introduce new variable $t$ as

$$t = \frac{1 - q + z}{q z}.$$

Let $\pi_n : B_{\text{aff}, n} \to H_{e,n}^*$ be the group homomorphism defined by $\pi(t_i) = T_i$ ($0 \leq i \leq n - 1$).

**Definition 4.6.** Let $\hat{\alpha}$ be a mixed link obtained as the closure of a mixed braid $\alpha$ with $n$-strands. Then we define the **HOMFLYPT polynomial of type $G(e, 1)$** of $\hat{\alpha}$ by

$$X_{\hat{\alpha}}(q, t) = \left[ - \frac{1 - tq}{t^2 (1 - q)} \right]^{n-1} (t^\frac{1}{2})^{\text{wr}(\hat{\alpha})} T(\pi_n(\alpha)).$$

Here $\text{wr}(\hat{\alpha})$ stands for the writhe number of the mixed link $\hat{\alpha}$.

**Lemma 4.7.** HOMFLYPT polynomials of type $G(e, 1)$ satisfy the following Skein relations:

$$(qt)^{-\frac{1}{2}} X_{L^+} - (qt)^{\frac{1}{2}} X_{L^-} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) X_{L_0},$$

$$X_{M_\pm} = a_{\pm 1} X_{M_{\pm 1}} + \cdots + a_1 X_{M_1} + a_0 X_{M_0}.$$
Here, $a_i$ is defined by

$$(T_0 - u_1) \cdots (T_0 - u_e) = T_0^e - a_{e-1}T_0^{e-1} - \cdots - a_1T_0 - a_0$$

and $L_+, L_-, L_0, M_e, \ldots, M_1, M_0$ are given by the following local mixed link diagrams:

5 Alexander polynomials for mixed links

In this section we define the Alexander polynomials for mixed links.

If we want to define the Alexander polynomial for a mixed link, we must consider the specialization $t \to q^{-1}$. However, a priori, the specialization does not make sense since the term $1 - qt$ appears in $X(q, t)$. To solve this problem we can use the Skein relations for $X(q, t)$. As explained in the previous section the link polynomials $X(q, t)$ satisfy the Skein relation. Conversely by giving some initial conditions, we can define $X(q, t)$ by using the Skein relations. In particular, we can define the Alexander polynomial for a mixed link. However, this definition is indirect and is not explicit.

In fact, thanks to the result of Geck-Iancu-Malle explained in Section 3, we can define the Alexander polynomial for a mixed link explicitly.

To see this let us focus on the term $R^\lambda$.

After the change of variables

$$z = -\frac{1 - q}{1 - tq},$$

$R^\lambda$ can be written as $\tilde{g}^\lambda / (1 - tq)^n$ where $\tilde{g}^\lambda$ is given by

$$\prod_{k=1}^e \prod_{x \in \lambda_k} \left( - (1 - q)(1 - q^{c(x)}) \prod_{1 \leq l \leq e \atop l \neq k} (q^{c(x)}u_k - u_l) + (1 - tq)(1 - q)C(e, k, x) \right).$$

Since at least one component of $\lambda$, say $\lambda_k$, is not empty, when $x$ is the (1,1)-component of $\lambda_k$, the corresponding term is given by $(1 - tq)(1 - q)C(e, k, x)$. This allows us to evaluate at $t = q^{-1}$. 

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Definition 5.1. Let \( \hat{\alpha} \) be a mixed link obtained as the closure of a mixed braid \( \alpha \). We define \textbf{Alexander polynomial of type} \( G(e, 1) \) for \( \hat{\alpha} \) by

\[
\Delta_{G(e,1)}(\hat{\alpha}) = X_{\hat{\alpha}}(q, q^{-1}).
\]

The following is a direct consequence of Lemma 4.7.

Corollary 5.2. Alexander polynomials of type \( G(e, 1) \) satisfy the following Skein relations:

\[
\Delta_{G(e,1)}(L_+) - \Delta_{G(e,1)}(L_-) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \Delta_{G(e,1)}(L_0),
\]

\[
\Delta_{G(e,1)}(M_e) = a_{e-1} \Delta_{G(e,1)}(M_{e-1}) + \cdots + a_1 \Delta_{G(e,1)}(M_1) + a_0 \Delta_{G(e,1)}(M_0).
\]

Here, the local mixed link diagrams \( L_+, L_-, L_0, M_e, M_{e-1}, \ldots, M_1, M_0 \) are as in Lemma 4.7.

Finally we discuss simplification of the Alexander polynomials of type \( G(e, 1) \).

Assume that \( \lambda \) has at least two non-empty components. Then the same consideration shows that after the specialization \( t = q^{-1} \), the corresponding \( R^\lambda \) is zero. Similarly, if \( \lambda \) has a component which has at least two diagonal boxes, the corresponding \( R^\lambda \) is also zero. This shows that when we consider the Alexander polynomial of type \( G(e, 1) \), we only have to consider the \( e \)-Young diagrams which have the form \((\emptyset, \ldots, \emptyset, \lambda^{(a)}, \emptyset, \ldots, \emptyset)\), where \( \lambda^{(a)} \) is the Young diagram \((a + 1, 1, \cdots, 1)\) of size \( n \).

Summarizing the above argument we have the following lemma.

Lemma 5.3. Let \( \alpha \) be a mixed braid. Then we have

\[
\Delta_{G(e,1)}(\alpha) = (q - 1)^{(n-1)} q^{\frac{n - u_c(\alpha) - 1}{2}} \sum_{0 \leq a \leq n-1} (-1)^en \left( \prod_{1 \leq i \leq e} u_i^{-n} \right) D_{p}^{\lambda_p^{(a)}} g_{\lambda_p^{(a)}}^{(a)}.
\]

Here,

\[
\lambda_p^{(a)} = (\emptyset, \ldots, \emptyset, \lambda^{(a)}(p), \emptyset, \ldots, \emptyset),
\]

\[
g_{\lambda_p^{(a)}}^{(a)} = (-1)^{n-1}(1 - q) C(e, p) \prod_{x \in \lambda^{(a)}, x \neq (1, 1)} (1 - q)(1 - q^{c(x)}) \prod_{1 \leq l \leq e} (q^{c(x)}u_k - u_l),
\]

\[
C(e, p) = \prod_{i=1}^{e-1} \left( u_k^{e-i-1} \sum_{j=1}^{i} (-1)^{i-j} y_j s_{i-j} \right) + (-1)^{e-1} \prod_{1 \leq l \leq e} u_l.
\]
6 Quantum calculus

In this section we calculate $D_p^{(a)} q^{N_p^{(a)}}$.

We first consider the case of $p \in \{2, \cdots, e\}$. In this case $\alpha_{i,j}$ are given by

\[ \alpha_{1,i} = n - i + 1 \quad (1 \leq i \leq n + 1), \quad \alpha_{l,i} = n - i \quad (l \neq 1, p, \ 1 \leq i \leq n), \]

\[ \alpha_{p,i} = \begin{cases} 
  a + n & (i = 1) \\
  n - i + 1 & (2 \leq j \leq b + 1) \\
  n - i & (b + 2 \leq n). 
\end{cases} \]

Here $b = n - a - 1$.

Lemma 6.1. For $l \neq 1, p$ and $i, j \geq 2$, we have the following.

1. \( \prod_{j'=1}^{j-1} (q^{\alpha_{1,j'}} - q^{\alpha_{l,j}}) = q^{(j-1)(n-j+1)} \prod_{h=1}^{j-1} (q^h - 1). \)

2. \( \prod_{j'=1}^{j-1} (q^{\alpha_{i,j'}} - q^{\alpha_{l,i}}) = q^{(j-1)(n-j)} \prod_{h=1}^{j-1} (q^h - 1). \)

3. \[ \prod_{i'=1}^{i-1} (q^{\alpha_{p,i'}} - q^{\alpha_{p,i}}) \]

   \[ = \begin{cases} 
   q^{(i-1)(n-i)} \frac{q^{a+i-1} - 1}{q^{i-1} - 1} \prod_{h=1}^{i-1} (q^h - 1) & (b + 3 \leq i \leq n), \\
   q^{(i-1)(n-i)} \frac{q^{a+i-1} - 1}{q - 1} \prod_{h=1}^{i-1} (q^h - 1) & (i = b + 2), \\
   q^{(i-1)(n-i) + i - 1} \frac{q^{a+i-1} - 1}{q^{i-1} - 1} \prod_{h=1}^{i-1} (q^h - 1) & (2 \leq i \leq b + 1). 
\end{cases} \]

Proof. We only prove the 1st identity in (3).

\[ (\text{LHS}) = \prod_{i'=1}^{i-1} (q^{\alpha_{p,i'}} - q^{n-i}) \]

\[ = q^{(i-1)(n-i)} \prod_{i'=1}^{i-1} (q^{\alpha_{p,i'}} - (n-i) - 1) \]

\[ = q^{(i-1)(n-i)} (q^{a+i-1} - 1) \prod_{i'=2}^{b+1} (q^{(n-i'+1) - (n-i) - 1}) \prod_{i'=b+2}^{i-1} (q^{(n-i') - (n-i) - 1}) \]

\[ = q^{(i-1)(n-i)} (q^{a+i-1} - 1) \prod_{i'=2}^{b+1} (q^{i'+1} - 1) \prod_{i'=b+2}^{i-1} (q^{i'-1} - 1) \]

\[ = q^{(i-1)(n-i)} \frac{q^{a+i-1} - 1}{q^{i-1} - 1} \prod_{h=1}^{i-1} (q^h - 1). \]
Now we can decompose $D^{\lambda_1^\alpha_1}_p$ into the following three factors:

$$
(1) \frac{(-1)^{\binom{n}{2} + n(e-1)}}{q^\frac{1}{2} \prod_{2 \leq l \leq e} (q^{-1}u_l - u_l)^n \prod_{2 \leq k < l \leq e} (u_k - u_l)^n}.
$$

$$
(2) \prod_{\alpha, \alpha' \in \Lambda_1} \prod_{\alpha > \alpha'} (q^\alpha - q^{\alpha'}) \times \prod_{2 \leq l \leq e} (q^\alpha - q^{\alpha'}) \times \prod_{2 \leq l \leq e} u_l^n \quad (k = l = 1),
$$

$$
\prod_{2 \leq l \leq e} (q^\alpha - q^{\alpha'}) \times \prod_{2 \leq l \leq e} u_l^n \quad (k = l \geq 2, l \neq p),
$$

$$
\prod_{2 \leq k < l \leq e} (q^\alpha - q^{\alpha'}) u_k - q^{\alpha'} u_l \quad (2 \leq k < l \leq e; k, l \neq p),
$$

$$
\prod_{1 \leq k \leq e} (q^\alpha - q^{\alpha'}) \times \prod_{1 \leq l \leq e} u_l^n \quad (k = l = p),
$$

$$
\prod_{2 \leq k < l \leq e} (q^\alpha - q^{\alpha'}) u_k - q^{\alpha'} u_l \quad (2 \leq k < l \leq e; k = p \text{ or } l = p).
$$

$$
(3) \prod_{\alpha \in \Lambda_1} \prod_{h=1}^\alpha (q^h - 1) \times q^{-\sum_{\alpha \in \Lambda_1} \sum_{\alpha' > \alpha} \sum_{u_1} u_1} \quad (k = l = p),
$$

$$
\prod_{2 \leq l \leq e} (q^{-1}u_l - u_l) \quad (k = 1, l \geq 2),
$$

$$
\prod_{2 \leq l \leq e} (q^h - 1) \times \prod_{2 \leq l \leq e; l \neq p} u_l \quad (k = l \geq 2, l \neq p),
$$

$$
\prod_{2 \leq k \leq e} \prod_{k \neq p} (q^h u_k - u_1) \quad (k \geq 2, k \neq p, l = 1),
$$

$$
\prod_{k,l \geq 2} \prod_{k \neq p; l \neq p} (q^h u_k - u_l) \quad (k \neq l, p; k, l \geq 2),
$$

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\[
\prod_{\alpha \in \mathcal{A}_p \ h=1} \prod_{\alpha} (q^h u_p - u_1) \quad (k = p, l = 1),
\]
\[
\prod_{\alpha \in \mathcal{A}_p \ h=1} \prod_{\alpha} (q^h - 1) \times u_p \sum_{\alpha \in \mathcal{A}_p} \alpha \quad (k = l = p),
\]
\[
\prod_{2 \leq l \leq e \ \alpha \in \mathcal{A}_p \ h=1} \prod_{\alpha} (q^h u_p - u_l) \quad (k = p, l \neq p, l \geq 2).
\]

We combine the terms which relate \( \mathcal{A}_p \), that is, we put

\[
D_0^{\lambda(p)} = \frac{\prod_{\alpha, \alpha' \in \mathcal{A}_p} (q^\alpha - q^{\alpha'})}{\prod_{\alpha} (q^h - 1)},
\]
\[
D_1^{\lambda(p)} = \frac{\prod_{(\alpha, \alpha') \in \mathcal{A}_1 \times \mathcal{A}_p} (q^{\alpha'} u_1 - q^{\alpha} u_p) \prod_{2 \leq k < l \leq e \ (\alpha, \alpha') \in \mathcal{A}_k \times \mathcal{A}_l} (q^h u_k - q^{\alpha'} u_l)}{\prod_{\alpha} (q^h u_p - q^{-1} u_1) \prod_{l \neq p} (q^h u_p - u_l)}.
\]

\[
g_0^{\lambda(p)} = (-1)^{n-1} (1 - q)^n C(e, p) \prod_{i=1}^{a} (1 - q^i) \prod_{j=1}^{b} (1 - q^{-j}),
\]
\[
g_1^{\lambda(p)} = \prod_{i=1}^{a} \prod_{l \neq p} (q^i u_p - u_l) \prod_{j=1}^{b} \prod_{l \neq p} (q^{-j} u_p - u_l).
\]

We note that \( g^{\lambda(p)} = g_0^{\lambda(p)} \ g_1^{\lambda(p)} \).

Lemma 6.2. We have

\[
D_0^{\lambda(p)} g_0^{\lambda(p)} = (-1)^{a+n} C(e, p) q^{\frac{n(n-1)(n-2)}{6}} \frac{(1 - q)^n}{1 - q^n}.
\]

Proof. By Lemma 6.1 (3) we have

\[
\prod_{\alpha, \alpha' \in \mathcal{A}_p} (q^\alpha - q^{\alpha'})
\]
\[
= \prod_{i=1}^{n} \prod_{i' = 1}^{i} (q^{a_i, a_{i'}} - q^{a_{i'}})
\]
\[
= q^{\sum_{i=2}^{n} (n-i) (i-1)} + \sum_{i=2}^{n} (i-1) \prod_{i=1}^{n} (q^h - 1)
\]
\[
= q^{\sum_{i=2}^{n} (n-i) (i-1)} + \sum_{i=2}^{n} (i-1) \prod_{i=1}^{n} (q^h - 1)
\]

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\frac{\prod_{i=2}^{b+1} \frac{q^{a+i-1} - 1}{q^{i-1} - 1} \times \frac{q^{n+1} - 1}{q - 1} \times \prod_{i=b+3}^{n} \frac{q^{a+i} - 1}{q^{i-b-1} - 1}}{\prod_{i=1}^{a} (1 - q^{i})}.

On the other hand, since
\prod_{j=1}^{b} (1 - q^{-j}) = q^{-\sum_{j=1}^{b} j} \prod_{j=1}^{b} (q^{j} - 1), \quad \prod_{i=1}^{a} (1 - q^{i}) = (-1)^{a} \prod_{i=1}^{a} (q^{i} - 1),
we have
\prod_{i=2}^{b+1} \frac{q^{a+i-1} - 1}{q^{i-1} - 1} \prod_{j=1}^{b} (1 - q^{-j}) = q^{-\sum_{j=1}^{b} j} \prod_{i=a+1}^{n-1} (q^{i} - 1),
\prod_{i=b+3}^{n} \frac{q^{a+i} - 1}{q^{i-b-1} - 1} \prod_{i=1}^{a} (1 - q^{i}) = (-1)^{a}(q - 1) \prod_{i=n+2}^{a+n} (q^{i} - 1).

By combining these calculations, we also have
\prod_{\alpha \in A} \prod_{h=1}^{\alpha} (q^{h} - 1).

Since
\prod_{\alpha \in A} \prod_{h=1}^{\alpha} (q^{h} - 1) = \frac{\prod_{i=1}^{n-1} \prod_{h=1}^{i} (q^{h} - 1) \prod_{i=a+1}^{a+n} (q^{i} - 1)}{\prod_{h=1}^{n-b-1} (q^{h} - 1)}
= \prod_{i=2}^{n} \prod_{h=1}^{i} (q^{h} - 1) \prod_{i=a+1}^{a+n} (q^{i} - 1),
we obtain the desired formula. \( \square \)

**Lemma 6.3.** We have
\frac{D_{1}^{\lambda^{(a)}} g_{1}^{\lambda^{(a)}}}{D_{0}^{\lambda^{(a)}} g_{0}^{\lambda^{(a)}}}
= (-1)^{n_{p}} \frac{1}{u_{p} - u_{1}} \prod_{j=1}^{n} \prod_{h=j-n}^{1} (q^{h}u_{p} - u_{1}) \prod_{2 \leq t \leq e} \frac{1}{u_{p} - u_{1}} \prod_{i=1}^{n} \prod_{h=i-n}^{0} (q^{h}u_{p} - u_{1}).

**Proof.** We first focus on the factors of \( D_{1}^{\lambda^{(a)}} \) which have the form \( q^{s}u_{1} - q^{s}u_{p} \).
It is given by
Thus the factor is given by

\[
\frac{\prod_{(a,a') \in A_1 \times A_p} (q^{a-1} u_1 - q^{a'} u_p)}{\prod_{\alpha \in A_p} \prod_{h=1}^{n} (q^h u_p - q^{-1} u_1)}.
\]

Now the numerator is equal to

\[
\prod_{i=1}^{n+1} \prod_{j=1}^{n} (q^{-1+i-1} u_1 - q^{0_p} u_p) = q^n \sum_{i=1}^{n+1} \prod_{i=1}^{n} \prod_{j=1}^{n} (q^{0_p} u_p - u_1)
\]

\[
= q^n \prod_{i=1}^{n+1} \prod_{j=1}^{n} (q^{j-1} u_p - u_1) \prod_{i=1}^{n+1} (q^{(a+n)-(i-2)} u_p - u_1)
\]

\[
= q^{\frac{n(n+1)(n-2)}{2}} \prod_{i=1}^{n+1} (q^{(n-b-1)-(i-2)} u_p - u_1)
\]

\[
= q^{\frac{n(n+1)(n-2)}{2}} \prod_{i=1}^{n+1} (q^{n-i-b+1} u_p - u_1)
\]

and the denominator is equal to

\[
\prod_{\alpha \in A_p} \prod_{h=1}^{n} (q^h u_p - q^{-1} u_1) = \prod_{i=1}^{n} \prod_{h=1}^{n} (q^h u_p - q^{-1} u_1) \prod_{i=1}^{n} (q^h u_p - q^{-1} u_1)
\]

\[
= q^{\frac{n(n+1)(n-2)}{2}} \prod_{i=1}^{n-1} (q^{h+1} u_p - u_1)
\]

Thus the factor is given by

\[
q^{\frac{n(n+1)(n-1)}{2}} \prod_{i=1}^{n+1} \prod_{j=1}^{n} (q^{-i+1} u_p - u_1) \prod_{i=1}^{n+1} (q^{a+n-i+2} u_p - u_1) \prod_{i=1}^{n-1} (q^{h+1} u_p - u_1)
\]

\[
= q^{\frac{n(n+1)(n-1)}{2}} \prod_{i=1}^{n} (q^{n-i-b+1} u_p - u_1) \prod_{i=1}^{n-1} (q^{h+1} u_p - u_1)
\]

\[
= q^{\frac{n(n+1)(n-1)}{2}} \prod_{i=1}^{n} (q^{n-i-b+1} u_p - u_1) \prod_{i=1}^{n-1} (q^{h+1} u_p - u_1)
\]
Now the following three identities hold:

\[
\prod_{i=1}^{n+1} \prod_{j=1}^{n} (q^{j-i+1} u_p - u_1) = \prod_{j=1}^{n} \prod_{h=j-n}^{j} (q^h u_p - u_1),
\]

\[
\prod_{i=1}^{n+1} \prod_{j=1}^{n} (q^{n+i-1-j} u_p - u_1) = (\prod_{j=1}^{n} (q^h u_p - u_1))^{-1},
\]

\[
\prod_{i=1}^{n+1} \prod_{j=1}^{n} (q^{n-i-1+j} u_p - u_1) = \prod_{j=1}^{n} (q^j u_p - u_1). \]

Thus by combining with the corresponding factors in \( g_{1}^{\lambda}(a) \), we have

\[
q^{n(n+1)(\alpha-1)} \prod_{i=1}^{n} \prod_{j=1}^{n} (q^h u_p - u_1).
\]

Next we focus on the factors in \( D_{1}^{\lambda}(a) \) which have the form \( q^* u_p - q^* u_l \) \((l \neq 1, p)\). It is given by

\[
\prod_{2 \leq k \leq l \leq e} (q^a u_k - q^{a'} u_l) \prod_{\alpha \in A_p} (q^h u_p - u_l) \prod_{\alpha \in A_l} (q^h u_p - u_l).
\]

We note that the numerator is equal to

\[
(-1)^{np} \prod_{2 \leq l \leq e} (q^a u_p - q^{a'} u_l).
\]

Thus a similar calculation shows that the corresponding factor in \( D_{1}^{\lambda}(a) g_{1}^{\lambda}(a) \) is given by

\[
(-1)^{np} q^{2(n+1)(\alpha-2)} \prod_{2 \leq l \leq e} \prod_{h \neq p} (q^h u_p - u_l) \prod_{i=1}^{n} \prod_{h=j-n}^{j} (q^h u_p - u_1).
\]

This completes the proof. \( \square \)
Lemma 6.4. We have
\[ D_{\lambda}^{\alpha} g_{\lambda}^{\alpha} = (-1)^{a+c_n} C(e, p) \frac{(1-q)^n}{1-q^n} \prod_{1 \leq j \leq e, \ell \neq p} u_{\ell}^{n} \prod_{1 \leq j \leq e, \ell \neq p} \frac{1}{u_{\ell} - u_{j}}. \]

Proof. The proof is divided into five parts. We use Lemma 6.2 and Lemma 6.3.

(1) The factors which have the form \( q^{a} - q^{a'} \). It is given by
\[ (1-q)^n \prod_{\alpha, \alpha' \in A_1} (q^{\alpha} - q^{\alpha'}) \prod_{2 \leq \ell \leq e, \alpha, \alpha' \in A_1} \frac{1}{(q^{\alpha} - q^{\alpha'})}. \]
By Lemma 6.1 (1) and (2) we have the following two identities:
\[ \prod_{\alpha, \alpha' \in A_1} (q^{\alpha} - q^{\alpha'}) = q^{n(n+1)(n-1)} \prod_{\alpha \in A_1} (q^{r} - 1), \]
\[ \prod_{\alpha, \alpha' \in A_1} (q^{\alpha} - q^{\alpha'}) = q^{n(n-1)(n-2)} \prod_{\alpha \in A_1} (q^{r} - 1). \]
Thus this part is given by
\[ q^{n(n-1)(e-n-2c+5)} \frac{(1-q)^n}{1-q^n}. \]

(2) The factors which have the form \( q^{a} u_k - q^{a} u_1 \). We divide into four cases.

(2-1) The factors which have the form \( q^{a} u_p - q^{a} u_1 \).
This is given by
\[ \frac{1}{u_p - u_1} \prod_{i=1}^{n} \prod_{h=i-n}^{1} (q^{r} u_p - u_{1}). \]
Now the denominator can be calculated as follows:
\[ (q^{r} u_1 - u_{p})^{n} \prod_{i=1}^{n} \prod_{h=1}^{n-i+1} (q^{r} u_1 - u_{p}) \]
\[ = \prod_{i=1}^{n} \prod_{h=0}^{n-i+1} (q^{r} u_1 - u_{p}). \]
The factors which have the form

\[ q^\alpha \in A \]

Now we can calculate as follows:

\[ \prod_{i=1}^n \prod_{h=1}^{n-i+1} (q^{1-h}u_p - u_1) \]

Thus this part is given by

\[ (-1)^{n(n+1)/2} q^{n^2-n/2} \prod_{i=1}^n \prod_{h=1}^{n-i} (q^h u_p - u_1). \]

(2-2) The factors which have the form \( q^*u_p - q^*u_l \) (\( l \geq 2, l \neq p \)).

This is given by

\[
\prod_{2 \leq i \leq e \atop l \neq p} \frac{1}{u_p - u_i} \prod_{i=1}^n \prod_{h=1}^{n-h} (q^h u_p - u_i)
\]

\[
\prod_{p \leq i \leq e} (u_p - u_i)^n \prod_{2 \leq k < p} (u_k - u_p)^n \prod_{2 \leq l \leq e \atop l \neq p} \prod_{2 \leq i \leq e} \prod_{h=1}^{n-h} (q^h u_l - u_i)
\]

Now we can calculate as follows:

\[
\prod_{\alpha \in A} \prod_{h=1}^\alpha (q^h u_l - u_p)
\]

\[
= \prod_{i=1}^n \prod_{h=1}^{n-i} (q^h u_l - u_p)
\]

\[
= (-1)^{n(1+2)(n-1)} q^{n(n+1)(n+1)/2} \prod_{i=1}^n \prod_{h=1}^{n-h} (q^h u_p - u_l)
\]

\[
= (-1)^{n(n-1)/2} q^{n^2+n} (u_p - u_l)^n \prod_{i=1}^n \prod_{h=1}^{n-h} (q^h u_p - u_l).
\]

From this calculation one can easily show that this part is given by

\[
(\prod_{2 \leq l \leq e \atop l \neq p} \prod_{\alpha, \alpha' \in A \times A} (q^{\alpha-1} u_1 - q^{\alpha'} u_l))
\]

(2-3) The factors which have the form \( q^*u_1 - q^*u_l \) (\( l \neq 1, p \)).

This is given by

\[
\prod_{2 \leq l \leq e \atop l \neq p} \prod_{\alpha \in A} \prod_{h=1}^\alpha (q^{h-1} u_1 - u_l)
\]

\[
\prod_{2 \leq l \leq e} (u_p - u_l)^n \prod_{2 \leq l \leq e \atop l \neq p} \prod_{\alpha \in A} \prod_{h=1}^\alpha (q^{h-1} u_1 - u_l)
\]

\[
\prod_{2 \leq l \leq e} \prod_{\alpha \in A \times A} (q^{\alpha-1} u_1 - q^{\alpha'} u_l)
\]

\[
\prod_{2 \leq l \leq e \atop l \neq p} \prod_{\alpha \in A} \prod_{h=1}^\alpha (q^{h-1} u_1 - q^{1-h} u_l)
\]

\[
= (-1)^{n(n-i+2)} q^{n^2+n/2} \prod_{i=1}^n \prod_{h=1}^{n-i+1} (q^{1-h} u_p - u_1)
\]

\[
= (-1)^{n(n+1)/2} q^{n^2-n/2} \prod_{i=1}^n \prod_{h=1}^{n-i} (q^h u_p - u_1).
\]
Now one can check easily the following three identities:

\[
\prod_{(\alpha, \alpha') \in A_1 \times A_l} (q^{\alpha-1}u_1 - q^{\alpha'}u_l) = q^{\frac{n(n+1)(n-2)}{2}} \prod_{i=1}^{n+1} (u_1 - q^{-j}u_l),
\]

\[
\prod_{\alpha \in A_1} (q^{h-1}u_1 - u_l) = q^{\frac{n(n+1)(n-1)}{2}} \prod_{i=1}^{n+1} (u_1 - q^{1-h}u_l),
\]

\[
\prod_{\alpha \in A_1} (q^{h}u_1 - q^{-1}u_l) = \frac{(-1)^{\frac{n(n-1)(e-2)}{2}} q^{\frac{n(n+1)(n-1)(e-2)}{6}} \prod_{i=1}^{n} (u_1 - q^{h+1}u_l)}{\prod_{h=1}^{n} (u_1 - q^{h+1}u_l)}.
\]

From these calculations we find that this part is given by

\[
(-1)^{\frac{n(n-1)(e-2)}{2}} q^{\frac{n(n+1)(n-1)(e-2)}{6}}.
\]

(2-4) The factors which have the form \(q^*u_k - q^*u_l\) (\(k, l \geq 2, k \neq l, p\)).

This is given by

\[
\prod_{2 \leq k < l \leq e, k, l \neq p, k \neq l} (q^*u_k - q^*u_l) \frac{\prod_{\alpha \in A_1} \prod_{h=1}^{n} (q^h u_k - u_l)}{\prod_{2 \leq k < l \leq e, k, l \neq p, k \neq l} (u_k - u_l)^n \prod_{\alpha \in A_k, h=1} \prod_{\alpha, l} (q^h u_k - u_l)}.
\]

Let \(k, l\) be two integers such that \(2 \leq k < l \leq e\). We first note the following identity:

\[
\prod_{(\alpha, \alpha') \in A_1 \times A_l} (q^{\alpha}u_k - q^{\alpha'}u_l) = q^{\frac{n(n-1)(e-2)}{2}} \prod_{i=1}^{n+1} (u_k - q^{-j}u_l).
\]

On the other hand, we have:

\[
\prod_{k, l \geq 2} \prod_{\alpha \in A_k, h=1} (q^h u_k - u_l)
\]

\[
= \prod_{2 \leq k < l \leq e, k, l \neq p, k \neq l} \left( \prod_{\alpha \in A_k, h=1} \prod_{\alpha \in A_l, h=1} (q^h u_k - u_l) \right)
\]

\[
= \prod_{2 \leq k < l \leq e, k, l \neq p} \left( (-1)^{\frac{n(n-1)(n+1)(n-1)}{2}} q^{\frac{n(n+1)(n-1)(n-2)}{6}} \prod_{i=1}^{n} (u_k - q^{-h}u_l) \prod_{i=1}^{n} (u_k - q^{h}u_l) \right).
\]
From these computations we find that this part is given by
\[
(-1)^{\frac{n(n-1)}{2}}q^{\frac{n(n-1)(2n-1)}{6}}(e-\frac{1}{2}).
\]

By combining (2-1), \ldots, (2-4), we can conclude that the factors which have the form \(q^*u_k - q^*u_l\) is given by
\[
(-1)\binom{n}{2} + \binom{e-\frac{1}{2}}{2} + mpq^n\left(\frac{n(n-1)}{2}\right)(e-\frac{1}{2}) + \sum_{\alpha \in A_1} \alpha + \sum_{2 \leq j \leq e, l \neq p} \frac{u_k - u_l}{u_k - u_l}.
\]

(3) The power of \(q\).

The exponent is given by
\[
-n - f(n, e) - \binom{n+1}{2} + \sum_{\alpha \in A_1} \alpha + \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)(en-n+1)}{2}.
\]
A direct computation shows that this is equal to
\[
\frac{n(n-1)(3en-2n+1)}{6} - f(n, e) - n.
\]

(4) The sign.

This is given by
\[
(-1)^{n+mp+\binom{e-\frac{1}{2}}{2} + n(e-1)}.
\]

(5) The other factor.

This is given by
\[
C(e, p) \times \prod_{k=1}^{e} u_k^n \times u_{\frac{n+1}{2}}^n \times \prod_{2 \leq j \leq e, l \neq p} u_k^n \times u_{\frac{j}{2}}^{n} \times u_l^n \times \sum_{\alpha \in A_1} \times \prod_{2 \leq j \leq e, l \neq p} u_l^n \times \sum_{\alpha \in A_1},
\]
and is equal to
\[
C(e, p) \prod_{1 \leq j \leq e, l \neq p} u_l^n.
\]

Now our desired formula follows from (1), \ldots, (5) and Remark 3.2 directly. This completes the proof. \(\square\)

Next, we must compute \(D^{\lambda^{(a)}}g^{\lambda^{(a)}}\). However, the computation of \(D^{\lambda^{(a)}}g^{\lambda^{(a)}}\) is quite similar to (and relatively easier than) that of \(D^{\lambda^{(a)}}g^{\lambda^{(a)}} (p \in \{2, \ldots, e\})\). So we omit the details. Instead, we write down the corresponding results for reader’s convenience.
In this case, $\alpha_{i,j}$ are given by

$$
\alpha_{1,i} = \begin{cases} 
    a + n + 1 & (i = 1) \\
    n - i + 2 & (2 \leq i \leq b + 1) \\
    n - i + 1 & (b + 2 \leq i \leq n + 1),
\end{cases}
$$

$$
\alpha_{l,i} = n - i & (2 \leq l \leq e, 1 \leq i \leq n).
$$

The following four lemmas correspond to Lemma 6.1, 6.2, 6.3 and 6.4 respectively.

**Lemma 6.5.** For $i, j, l \geq 2$ we have the following.

$$
(1) \prod_{j'=1}^{j-1} (q^{\alpha_{i,j'}} - q^{\alpha_{i,j}}) = q^{(j-1)(n-j)} \prod_{h=1}^{j-1} (q^h - 1)
$$

$$
(2) \prod_{i'=1}^{i-1} (q^{\alpha_{i',1}} - q^{\alpha_{i,1}})
\begin{cases} 
    q^{(i-1)(n-i+1)} \left( q^{a+i} - \frac{1}{q^{i-b-1}} \prod_{h=1}^{i-1} (q^h - 1) \right) & (b + 3 \leq i \leq n + 1) \\
    q^{(i-1)(n-i+1)} \left( q^{a+i} - \frac{1}{q} \prod_{h=1}^{i-1} (q^h - 1) \right) & (i = b + 2) \\
    q^{(i-1)(n-i+1)+1} \left( q^{a+i-1} - \frac{1}{q^{i-1} - 1} \prod_{h=1}^{i-1} (q^h - 1) \right) & (2 \leq i \leq b + 1).
\end{cases}
$$

**Lemma 6.6.** We have

$$
D_{0}^{(a)} g_{0}^{(a)} = (-1)^{a+n} C(e, 1) q^{\frac{n(n+1)(n-1)}{2}} (1 - q)^{n}.
$$

**Lemma 6.7.** We have

$$
D_{1}^{(a)} g_{1}^{(a)} = q^{\frac{n(n+1)(n-1)(e-1)}{2}} \prod_{2 \leq l \leq e} \left( \frac{1}{u_{1}-u_{l}} \prod_{i=1}^{n} u_{i}^{-1} (q^{l} u_{1} - u_{l}) \right).
$$

**Lemma 6.8.** We have

$$
D_{1}^{(a)} g_{1}^{(a)} = (-1)^{a+en} C(e, 1) \left( \frac{1 - q^{n}}{1 - q^{n}} \prod_{2 \leq l \leq e} \frac{1}{u_{1}-u_{l}} \right).
$$

The following is a direct consequence of Lemma 5.3, 6.4 and 6.8.

**Theorem 6.9.** For any mixed braid $\alpha$ with $n$-strands we have

$$
\Delta^{G(e,1)}(\hat{\alpha}) = (-1)^{n-1} q^{\frac{n-w(\hat{\alpha})-1}{2}} \sum_{0 \leq a \leq n-1} \sum_{1 \leq p \leq e} (-1)^{a} C(e, p) \left( \prod_{1 \leq l \leq e, l \neq p} \frac{1}{u_{p} - u_{l}} \right) \chi_{\lambda_{a,i}}(\pi_{n}(\alpha)).
$$
From this formula we find the relationship between the Alexander polynomial of a mixed link and of the link which is obtained by resolving the twisted parts.

**Theorem 6.10.** Let \( \alpha \) be a mixed braid, \( \alpha_0 \) be the braid obtained by avoiding the powers of \( t_0 \) appearing in \( \alpha \), and \( \Delta(\alpha_0) \) be the usual Alexander polynomial of \( \alpha_0 \). Then we have

\[
\Delta^{G(1)}(\hat{\alpha}) = \Delta(\hat{\alpha}_0) \sum_{1 \leq p \leq e} C(e, p) u_p^{w_{r_0}(\alpha)} \left( \prod_{1 \leq i \leq e, \ l \neq p} \frac{1}{u_p - u_l} \right).
\]

Here, \( w_{r_0}(\alpha) \) is the sum of exponents of \( t_0 \) appearing in \( \alpha \).

**Proof.** We first remark that by taking \( e = 1 \) we can recover the following formula

\[
\Delta(\hat{\alpha}_0) = (-1)^{n-1} q^{-\frac{n - w_{r_0}(\hat{\alpha}) - 1}{2}} \frac{1 - q^n}{1 - q^n} \sum_{0 \leq a \leq n-1} (-1)^a \chi_{\lambda_p^{(e)}}(\pi_n(\alpha_0))
\]

of Jones. Since \( \chi_{\lambda_p^{(e)}}(\alpha) = u_p^{w_{r_0}(\alpha)} \chi_{\lambda_p^{(1)}}(\alpha_0) \), we have

\[
\Delta^{G(1)}(\hat{\alpha}) = (-1)^{n-1} q^{-\frac{n - w_{r_0}(\hat{\alpha}) - 1}{2}} \frac{1 - q^n}{1 - q^n} \times \sum_{0 \leq a \leq n-1} (-1)^a C(e, p) \left( \prod_{1 \leq i \leq e, \ l \neq p} \frac{1}{u_p - u_l} \right) \chi_{\lambda_p^{(e)}}(\pi_n(\alpha))
\]

\[
= (-1)^{n-1} q^{-\frac{n - w_{r_0}(\hat{\alpha}) - 1}{2}} \frac{1 - q^n}{1 - q^n} \times \sum_{1 \leq p \leq e} C(e, p) u_p^{w_{r_0}(\alpha)} \left( \prod_{1 \leq i \leq e, \ l \neq p} \frac{1}{u_p - u_l} \right) \sum_{0 \leq a \leq n-1} (-1)^a \chi_{\lambda_p^{(1)}}(\pi_n(\alpha_0))
\]

\[
= \Delta(\alpha_0) \sum_{1 \leq p \leq e} C(e, p) u_p^{w_{r_0}(\alpha)} \left( \prod_{1 \leq i \leq e, \ l \neq p} \frac{1}{u_p - u_l} \right).
\]

as desired. \( \square \)

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