Poisson deformations of symplectic quotient singularities

Victor Ginzburg and Dmitry Kaledin

To the memory of Andrey Tyurin

Abstract

We establish a connection between smooth symplectic resolutions and symplectic deformations of a (possibly singular) affine Poisson variety.

In particular, let $V$ be a finite-dimensional complex symplectic vector space and $G \subset \text{Sp}(V)$ a finite subgroup. Our main result says that the so-called Calogero-Moser deformation of the orbifold $V/G$ is, in an appropriate sense, a versal Poisson deformation. That enables us to determine the algebra structure on the cohomology $H^*(X, \mathbb{C})$ of any smooth symplectic resolution $X \rightarrow V/G$ (multiplicative McKay correspondence). We prove further that if $G \subset GL(\mathfrak{h})$ is an irreducible Weyl group and $V = \mathfrak{h} \oplus \mathfrak{h}^*$, then no smooth symplectic resolution of $V/G$ exists unless $G$ is of types $A, B, C$.

Contents

1 Main results. 2
  1.1 Introduction. 2
  1.2 Orbifold cohomology and Quantization. 4
  1.3 Poisson deformations. 6
  1.4 Deforming symplectic resolutions. 8
  1.5 Comparison. 10
  1.6 Calogero-Moser deformation. 11
  1.7 Proofs of Theorem 1.1 and Theorem 1.2. 13

2 Generalities on Poisson deformations. 13
  2.1 Poisson cohomology. 14
  2.2 Globalization. 14

3 The case of $\dim V = 2$. 17

4 The computation of $HP^2(V/G)$. 20

5 Resolutions. 24
  5.1 Geometry of the resolution. 24
  5.2 Globalizing the deformation. 26
  5.3 Comparison with the Calogero-Moser deformation. 29
  5.4 Restriction to strata and the end of the proof. 30

6 Applications of Hochschild cohomology. 33
1 Main results.

1.1 Introduction. Let $V$ be a finite-dimensional symplectic vector space over $\mathbb{C}$, and $G \subset \text{Sp}(V)$ a finite subgroup. The quotient $V/G$ has a natural structure of an irreducible affine algebraic variety with coordinate ring $\mathbb{C}[V/G] := \mathbb{C}[V]^G$, the subalgebra of $G$-invariant polynomials on $V$. The algebraic variety $V/G$ is normal and Gorenstein. However, it is always singular whenever $G \neq \{1\}$.

The symplectic structure on $V$ gives rise to a Poisson algebra structure on $\mathbb{C}[V]$ which induces, by restriction, a Poisson algebra structure on $\mathbb{C}[V]^G$. Thus, the space $V/G$ becomes a Poisson variety.

Recall that a resolution of singularities $X \to Y$ of an irreducible Gorenstein variety $Y$ is called crepant provided the smooth manifold $X$ has trivial canonical class. Assume now $Y$ has a Poisson structure which is generically non-degenerate (i.e., symplectic). Then a resolution of singularities $X \to Y$ is called symplectic provided the pull-back of the symplectic 2-form on the generic locus of $Y$ extends to a (non-degenerate) symplectic 2-form on the whole of $X$. Any symplectic resolution is crepant. Conversely, it was shown in [K1] that any crepant resolution $X \to Y$ is necessarily symplectic, i.e., the pull-back of symplectic 2-form on the generic locus of $Y$ automatically extends to a non-degenerate 2-form on $X$.

In this paper we study crepant (equivalently, symplectic) resolutions $X \to V/G$ of a symplectic quotient singularity $V/G$. Our first result concerns the existence of such resolutions. It has been shown by M. Verbitsky [Ve] that if the quotient $V/G$ admits a crepant resolution, then the group $G$ must be generated by the so-called symplectic reflections (see [Ve] or Definition 1.15 below).

Example. If $\dim V = 2$ then any finite subgroup $G \subset SL(V)$ is generated by symplectic reflections, and $V/G$ is the so-called du Val surface singularity. It is well-known that there exists a canonical minimal resolution of singularities $X \to V/G$, which is at the same time a symplectic resolution.

An important series of groups generated by symplectic reflections are provided by Coxeter groups. Specifically, let $\mathfrak{h}$ be the complexification of a euclidean real vector space with root system of some Dynkin type, and let $G \subset GL(\mathfrak{h})$ be the corresponding Weyl group. Further let $\mathbb{C}^2$ be the standard space with a fixed volume 2-form, and set $V := \mathfrak{h} \otimes \mathbb{C}^2$. The tensor product of the symmetric bilinear form (coming from the euclidean inner product) on $\mathfrak{h}$ and the 2-form
on \( \mathbb{C}^2 \) gives a nondegenerate 2-form on \( V \), hence, makes \( V \) a symplectic vector space. The group \( G \) acts on \( V = \mathfrak{h} \otimes \mathbb{C}^2 \), via the action on the first factor, by symplectic automorphisms.

For types \( A_n, B_n, C_n \) the quotient \( V/G \) is known to admit a crepant resolution of singularities (of Hilbert scheme type, see e.g. [K1]). Using the results of I. Gordon [Go], we resolve the existence question for other Dynkin graphs to the negative:

**Theorem 1.1.** For a root system of type \( D_n, E_n, F_4 \) or \( G_2 \), the quotient \( V/G \) where \( V = \mathfrak{h} \otimes \mathbb{C}^2 \), does not admit a resolution of singularities \( X \to V/G \) with trivial canonical bundle.

Following [Al], define an increasing filtration \( F_i(\mathbb{C}[G]) \) on the group algebra \( \mathbb{C}[G] \) by letting \( F_k(\mathbb{C}[G]), k \geq 0 \), be the \( \mathbb{C} \)-linear span of the elements \( g \in G \) such that \( \text{rk}(\text{id}_V - g) \leq k \). (Thus, \( 1 \in F_0(\mathbb{C}[G]) \) and symplectic reflections belong to \( F_2(\mathbb{C}[G]) \)). This filtration is clearly compatible with the algebra structure on \( \mathbb{C}[G] \). Let \( F_*(ZG) \) denote the induced filtration on \( ZG \), the center of \( \mathbb{C}[G] \). Write \( \text{gr}^F_*(ZG) \) for the corresponding graded algebra.

Let \( X \to V/G \) be a symplectic resolution. Our second result describes the algebra structure in the cohomology \( H^*(X, \mathbb{C}) \) of the manifold \( X \).

**Theorem 1.2 (Multiplicative McKay correspondence).** Let \( X \to V/G \) be a resolution of singularities with trivial canonical bundle. Then there is a canonical graded algebra isomorphism: \( H^*(X, \mathbb{C}) \cong \text{gr}^F_*(ZG) \). In particular, \( X \) has no odd (rational) cohomology.

We note that the dimension equality: \( \dim H^i(X, \mathbb{C}) = \dim \text{gr}_i(ZG) \) has been known for some time, see [BD]. Later on, an explicit basis in the Borel-Moore homology \( H^*_{BM}(X, \mathbb{C}) \) parametrized by conjugacy classes in \( G \) was constructed in [K2]. This constitutes the so-called general McKay correspondence and amounts, in our notation, to a linear bijection \( \gamma_* : H^*_{BM}(X, \mathbb{C}) \to \text{gr}_*(ZG) \), where \( d = \dim_{\mathbb{C}} X \).

The multiplication structure in the cohomology was first computed independently by Vasserot [Vas] and Lehn-Sorger [LS] (cf. also [LQW]) in the case when \( X \) is the Hilbert scheme of \( n \) points on \( \mathbb{C}^2 \). At the same time, Vasserot (and the first author) conjectured the existence of a natural algebra isomorphism: \( \gamma^* : \text{gr}^F_*(ZG) \to H^*(X, \mathbb{C}) \) for arbitrary symplectic quotient singularities; our result proves this conjecture.

The natural \( C^* \)-action on \( V \) by dilations induces a \( C^* \)-action on \( V/G \). The latter may be canonically lifted to the symplectic resolution \( X \), cf. [K1] and Proposition 5.2 below. Recall further that \( C^* \)-equivariant cohomology of a \( C^* \)-variety is an algebra over \( \mathbb{C}[u] \), the cohomology of the classifying space. Further, using the filtration \( F_*(ZG) \) one forms the corresponding graded Rees algebra \( \text{Rees}_*(ZG) := \sum_i F_i(ZG) \cdot u^i \subset ZG[u] \).

**Conjecture 1.3 (\( C^* \)-equivariant cohomology).** There is a canonical graded \( \mathbb{C}[u] \)-algebra isomorphism: \( H^*_{C^*}(X, \mathbb{C}) \cong \text{Rees}_*(ZG) \).
In the special case of the Hilbert scheme of \(n\) points in \(\mathbb{C}^2\) a proof of this
conjecture is implicitly contained in [Vas].

Theorem 1.2 leads further to an interesting question of computing the Poincaré
duality isomorphism for \(X\) in terms of the group \(G\); in other words we propose

**Problem 1.4.** Compute the composite map

\[
\text{gr}^d(ZG) \xrightarrow{\gamma^*} \overset{\text{Poincaré duality}}{H^d(X, \mathbb{C})} \xrightarrow{\gamma_*} \text{gr}(ZG).
\]

This map seems to be closely related to the character table of the group \(G\).

### 1.2 Orbifold cohomology and Quantization

Let \(M\) be an arbitrary smooth symplectic algebraic variety, and \(G\) a finite group of symplectic automorphisms of \(M\). There are many examples, cf. [Ba], [BKR], in which, given a crepant resolution \(X \to M/G\) one has a canonical equivalence of triangulated categories:

\[
D^b(\text{Coh}(X)) \cong D^b(\text{Coh}_G(M)).
\]

In such a case, taking the Grothendieck groups of both categories, one obtains an isomorphism: 

\[
K(X) \cong K_G(M)
\]

of the algebraic K-groups.

In the special case where \(V = M\) is a symplectic vector space and \(G \subset Sp(V)\), the Borel-Moore homology group \(H^\text{BM}_q(X, \mathbb{C})\) is known, by [K2], to be spanned by the algebraic cycles. Hence the Chern character map gives a ring isomorphism \(ch: \mathbb{C} \otimes K(X) \xrightarrow{\sim} H^*(X, \mathbb{C})\). One also has the Thom isomorphism:

\[
K_G(V) \cong R(G),
\]

where \(R(G)\) stands for the representation ring of \(G\). Thus, in addition to Problem 1.4 we arrive at the following

**Problem 1.5.** Compute the composite map (cf. also [EG, Problem 17.11]):

\[
ZG \xrightarrow{\sim} \mathbb{C} \otimes R(G) \xrightarrow{\sim} \mathbb{C} \otimes K_G(V) \xrightarrow{\sim} \mathbb{C} \otimes K(X) \xrightarrow{ch} H^*(X, \mathbb{C}) \xrightarrow{\sim} \text{gr}(ZG).
\]

Note that the group \(ZG\) on the left is viewed as the algebra of class-functions on \(G\) with pointwise multiplication. Note further, that the isomorphism \(K(X) \cong K_G(V)\) is *not* compatible with the ring structures.

Associated with a finite group action on a Calabi-Yau manifold \(M\), one can introduce an orbifold (= ‘stringy’) cohomology \(H^*_\text{orb}(M; G)\), see [Ba]. We consider the special case where \(M\) is a holomorphic symplectic manifold and the group \(G\) acts by symplectic automorphisms. In such a case the definition reads:

(1.1) \[
H^*_\text{orb}(M; G) := \left( \bigoplus_{g \in G} H^{*-\dim M^g}(M^g) \right)^G,
\]

where \(M^g\) denotes the fixed point set of \(g \in G\), and where \(H^{*-\dim M^g}(M^g)\) is a shorthand notation for \(\bigoplus_{\alpha} H^{*-\dim M_{\alpha}}(M_{\alpha})\), a direct sum ranging over the set of connected components, \(M_{\alpha}\), of the manifold \(M^g\). Further, for any \(g, h \in G\),
one has a cup-product pairing $\cup: H^\ast(M^g) \times H^\ast(M^h) \to H^\ast(M^g \cap M^h)$, and the Gysin map $i_*: H^\ast(M^g \cap M^h) \to H^\ast(M^g)$, induced by the imbedding $i: M^g \cap M^h \hookrightarrow M^g$. Following Ruan, cf. [R], [R2], Fantechi and Göttsche [FG] have introduced a certain cohomology class $c(g,h) \in H^\ast(M^g \cap M^h)$, and showed that the assignment

$$H^\ast(M^g) \times H^\ast(M^h) \longrightarrow H^\ast(M^g^h), \quad a, b \longmapsto i_*(a \cup b \cup c(g,h))$$

gives rise to an associative product on the direct sum in the RHS of (1.1). This product puts a graded algebra structure on orbifold cohomology.

It is known further, see [Ba], [DL] and also [Ba], that, given an arbitrary Calabi-Yau orbifold $M/G$ and a smooth crepant resolution $X \to M/G$, there is a graded space isomorphism $H^\ast_{orb}(M; G) \cong H^\ast(X)$. Moreover, it has been conjectured in [CR], [R1] and [FG], that there is a graded algebra isomorphism $H^\ast_{orb}(M; G) \cong H^\ast(X)$, provided $M$ is a holomorphic symplectic manifold with symplectic $G$-action and $X \to M/G$ is a symplectic resolution.

The above conjecture is supported by the special case where $M = V$ is a symplectic vector space and $G \subset Sp(V)$. Then each fixed point set $V^g$ is contractible, and formula (1.1) reduces to $H^\ast_{orb}(V; G) = gr_\ast(ZG)$ (lemma A.16 from §6 below insures compatibility of the gradings on both sides). Thus, the conjectured algebra isomorphism $H^\ast_{orb}(M; G) \cong H^\ast(X)$ becomes nothing but our Theorem 1.2 above.

For a general algebraic symplectic manifold $M$, the above conjecture may be approached as follows. The symplectic form on $M$ makes the structure sheaf into a sheaf, $(\mathcal{O}_M, \{-,-\}_M)$, of Poisson algebras. Consider a deformation-quantization of $\mathcal{O}_M$, i.e., a sheaf $\text{Quant}_M$ of locally free complete $\mathbb{C}[[\varepsilon]]$-algebras with a $G$-equivariant star-product $a, b \mapsto a \star b$, such that $a \star b - b \star a \equiv \varepsilon \cdot \{a, b\}_M \mod \varepsilon^2$. Let $\text{Quant}_M[\frac{1}{\varepsilon}]$ be the localization of $\text{Quant}_M$ with respect to $\varepsilon$, a sheaf of $\mathbb{C}[[\varepsilon]]$-algebras on $M$. We form the cross-product algebra $\text{Quant}_M[\frac{1}{\varepsilon}] \# G$. Let $(\text{Quant}_M[\frac{1}{\varepsilon}] \# G) \text{-mod}$ be the abelian category of coherent $\text{Quant}_M[\frac{1}{\varepsilon}] \# G$-modules, that is, the category of $G$-equivariant coherent $\text{Quant}_M[\frac{1}{\varepsilon}]$-modules.

On the other hand, given a smooth symplectic resolution $X \to M/G$, we consider similarly a sheaf $\text{Quant}_X$ of $\mathbb{C}[[\varepsilon]]$-algebras, which is a deformation-quantization of the structure sheaf $(\mathcal{O}_X, \{-,-\}_X)$, viewed as a sheaf of Poisson algebras on $X$. Let $\text{Quant}_X[\frac{1}{\varepsilon}]$ be its localization with respect to $\varepsilon$, a sheaf of $\mathbb{C}[[\varepsilon]]$-algebras on $X$. Write $\text{Quant}_X[\frac{1}{\varepsilon}] \text{-mod}$ for the abelian category of sheaves of coherent $\text{Quant}_X[\frac{1}{\varepsilon}]$-modules.

We propose the following conjecture that may be thought of as a ‘quantum analogue’ of Bridgeland-King-Reid theorem [BKR].

**Conjecture 1.6.** For appropriate choices of deformation-quantizations of symplectic manifolds $X$ and $M$, respectively, there is a category equivalence

$$\text{Quant}_X[\frac{1}{\varepsilon}] \text{-mod} \simeq (\text{Quant}_M[\frac{1}{\varepsilon}] \# G) \text{-mod}.$$
Remark 1.7. Notice that, unlike the situation considered in \[BKR\], the conjecture above involves no derived categories. This is somewhat reminiscent of the ‘$D$-affineness’ property, proved by Beilinson-Bernstein \[BB\] for flag manifolds.

Now, given a space $U$ and a sheaf of algebras $\mathcal{A}_U$ on $U$, define the Hochschild cohomology of $\mathcal{A}_U$ by the formula $HH^q(\mathcal{A}_U) := \text{Ext}^q_{\mathcal{A}_U \otimes \mathcal{A}_U^{op}}(\mathcal{A}_U, \mathcal{A}_U)$, where $\mathcal{A}_U$ is viewed as a sheaf on the diagonal $U \subset U \times U$. It was shown in \[EG, \S15\] that Kontsevich’s Formality theorem \[Kon\] yields a graded $\mathbb{C}((\varepsilon))$-algebra isomorphism:

\[
\mathbb{C}((\varepsilon)) \otimes H^*(X) \cong HH^*(\text{Quant}_X[\frac{1}{\varepsilon}]).
\]

Further, we expect that there is a graded $\mathbb{C}((\varepsilon))$-algebra isomorphism

\[
\mathbb{C}((\varepsilon)) \otimes H^\text{arb}_*(M; G) \cong HH^*((\text{Quant}_M[\frac{1}{\varepsilon}]) \# G),
\]

that may be viewed as a ‘quantization’ of the isomorphism of Proposition \[6.2\] (see \[6\] below). Assuming this, the equivalence of Conjecture \[1.6\] would yield an isomorphism between the Hochschild cohomology algebras in the right-hand sides of \[1.2\] and \[1.3\]. Hence, the corresponding left-hand sides should also be isomorphic, and we would get the desired algebra isomorphism $H^\text{arb}_*(M; G) \cong H^*(X)$.

1.3 Poisson deformations. The main idea of this paper, used in particular in the proofs of Theorem \[1.1\] and Theorem \[1.2\] is to relate smooth symplectic resolutions of $V/G$ to smooth symplectic deformations of $V/G$. Specifically, a certain canonical Poisson deformation of the Poisson algebra $\mathbb{C}[V]^G$, which we propose to call the Calogero-Moser deformation, has been introduced in \[EG\]. Our main Theorem \[1.18\] claims that – under some assumptions, and in an appropriate sense – the Calogero-Moser deformation is a versal deformation of $V/G$ in the class of Poisson algebras. In the course of proving the Theorem, we establish some general basic results on Poisson deformations, which may be of independent interest.

We will now introduce the necessary definitions and state, one by one, the technical results leading up to and including Theorem \[1.18\]. At the end of this section, we will show how the stated results imply Theorem \[1.1\] and Theorem \[1.2\].

Let $A$ be a commutative unital $\mathbb{C}$-algebra with product $(a, b) \mapsto a \cdot b$, and $R \subset A$ a (unital) subalgebra. Recall that $A$ is said to be a Poisson algebra over $R$, or a Poisson $R$-algebra, if $A$ is equipped with an $R$-linear skew-symmetric bracket $\{\cdot, \cdot\}$ that satisfies the Leibniz rule

\[
\{a, (b \cdot c)\} = \{a, b\} \cdot c + \{a, c\} \cdot b \quad \text{for all } a, b, c \in A,
\]

and the Jacobi identity

\[
\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0 \quad \text{for all } a, b, c \in A.
\]
Geometrically, the embedding $R \hookrightarrow A$ corresponds to a scheme morphism $f : \text{Spec} A \rightarrow \text{Spec} R$, and the Poisson $R$-algebra structure on $A$ makes each fiber of $f$ a Poisson scheme. In particular, if $R$ is a local Artin algebra with maximal ideal $m$, then the fiber $A/m$ over the special point $o \in S = \text{Spec} R$ is a Poisson algebra over $\mathbb{C}$.

**Definition 1.8.** A Poisson deformation of a Poisson algebra $A$ over the spectrum $S = \text{Spec} R$ of a local Artin algebra $R$ with maximal ideal $m \subset R$ is a pair of a flat Poisson $R$-algebra $A_R$ and a Poisson algebra isomorphism $A_R/m \cong A$.

It is useful to introduce gradings into the picture. Say that a unital associative algebra $A = \bigoplus_{n \geq 0} A_n$ is positively graded if for all $n, m \geq 0$ we have $A_n \cdot A_m \subset A_{n+m}$, and $A_0 = \mathbb{C} \cdot 1$. Let $A$ be a commutative positively graded $\mathbb{C}$-algebra, and let $R \subset A$ be a graded subalgebra. Fix $l > 0$, a positive integer.

A Poisson $R$-algebra $A_R$ over a complete local $\mathbb{C}$-algebra $(\langle R, m_R \rangle)$ equipped with a Poisson isomorphism $A_R/m_R \cong A$ is called a universal formal Poisson deformation of the Poisson algebra $A$ if for every Poisson deformation $A_S$ over a local Artin base $S$ there exists a unique map $\tau : S \rightarrow \text{Spec} R$ such that the isomorphism $A \cong A_R/m_R$ extends to a Poisson isomorphism

\[(1.6) \quad A_S \cong \tau^* A_R.\]

Analogously, say that a graded Poisson $R$-algebra $A_R$ over a positively graded $\mathbb{C}$-algebra $R$ equipped with a graded Poisson isomorphism $A_R/R^{>0} \cong A$ is a universal graded Poisson deformation of the Poisson algebra $A$ if for every graded Poisson deformation $A_S$ over a local Artin base $S$ with an action of $\mathbb{C}^*$ there exists a unique $\mathbb{C}^*$-equivariant map $\tau : S \rightarrow \text{Spec} R$ such that the isomorphism $A \cong A_R/R^{>0}$ extends to a graded Poisson isomorphism (1.6).

To study Poisson deformations of a given Poisson $\mathbb{C}$-algebra $A$, we develop a cohomology theory, $HP^*(A)$, the Poisson cohomology of $A$. Very much like Hochshcild cohomology of an associative algebra, first order deformations of a Poisson algebra $A$ are controlled by the group $HP^2(A)$, while obstructions to deformations are controlled by the group $HP^3(A)$. The Poisson cohomology groups are sufficiently functorial; in particular, for a graded Poisson algebra $A$, each space $HP^k(A)$ carries a natural additional grading.

**Remark 1.9.** For Poisson algebras such that $\text{Spec} A$ is smooth, the theory of Poisson cohomology goes back to J.-L. Koszul [Ko], and J.-L. Brylinski [Br]. In the singular case, the approach must be quite different; it is based on the general operadic formalism (see [Fr], and Appendix below).
Assume now that $HP^1(A) = 0$ and $\dim HP^2(A) < \infty$. Then, the result below says that there exists a formal universal Poisson deformation of $A$; moreover, under additional assumptions, in the graded setting the formal universal deformation comes from a deformation over a base of finite type over $\mathbb{C}$.

To make a precise statement, let $\hat{HP}^2(A)$ denote the completion of the affine space $HP^2(A)$.

**Theorem 1.10 (Universal Poisson deformations).** Let $A$ be a Poisson algebra. Assume that $HP^1(A) = 0$ and that $HP^2(A)$ is a finite-dimensional vector space over $\mathbb{C}$.

(i) There exists a closed subscheme $S \subset \hat{HP}^2(A)$ and a Poisson $\mathbb{C}[S]$-algebra $A_S$ which is a universal formal Poisson deformation of the algebra $A$.

(ii) Assume in addition that the Poisson algebra $A$ is positively graded of some positive degree $l$, and that the induced grading on $HP^2(A)$ is also positive. View $HP^2(A)$ as a $\mathbb{C}^*$-variety via the grading.

Then there exists a $\mathbb{C}^*$-stable closed subvariety $S \subset HP^2(A)$ and a positively graded Poisson $\mathbb{C}[S]$-algebra $A_S$ of degree $l$ which is a universal graded Poisson deformation of the algebra $A$.

The proof of Theorem 1.10 is contained in the Appendix, in Subsection A.8.

**Remark 1.11.** We would like to note that the isomorphism (1.6) is not canonical, nor indeed is it unique. Thus what we obtain is more than a “coarse deformation space” and less than a “fine moduli space”. In the language of the stack formalism, our deformation problems are classified by the quotient stack of some variety $S$ by a trivial action of some group $G$. The group in question is the connected component of unity in the group of Poisson automorphisms of the algebra $A$.

1.4 Deforming symplectic resolutions. Let $X\text{aff} := \text{Spec} H^0(X, \mathcal{O}_X)$ denote the affinization of an algebraic variety $X$. This is an affine algebraic variety.

We first introduce

**Definition 1.12.** An irreducible smooth algebraic variety $X$ will be called convex if the canonical morphism $X \to X\text{aff}$ is projective and birational. The affine algebraic variety $X\text{aff}$ is then a normal variety.

For a convex symplectic manifold $X$, the symplectic structure on $X$ induces a Poisson bracket on the algebra $H^0(X, \mathcal{O}_X)$, hence, a Poisson structure on $X\text{aff}$.

Our next result concerns deformations of symplectic convex manifolds. Related results on infinitesimal and formal deformations have been studied in the paper [KV]. In order to generalize the main theorem of [KV] to a global setting, we need to assume that the affinization $Y = X\text{aff}$ is equipped with an expanding action of the multiplicative group $\mathbb{C}^*$. Equivalently, we assume that
the algebra $\mathbb{C}[Y] = H^0(X, \mathcal{O}_X)$ carries a grading by non-negative integers such that the Poisson bracket on $\mathbb{C}[Y]$ is of some fixed degree $l > 0$, that is, for any homogeneous functions $a_1, a_2 \in \mathbb{C}[Y]$, we have

$$\deg\{a_1, a_2\} = \deg a_1 + \deg a_2 - l.$$ 

With these assumptions, the result below says that any convex symplectic manifold can be deformed nicely to a smooth affine symplectic manifold without changing the rational cohomology algebra.

**Theorem 1.13.** Let $X$ be a symplectic convex variety, and $X \to Y = X^{\text{aff}}$ the corresponding resolution. Assume that the algebra $\mathbb{C}[Y]$ carries a grading by non-negative integers such that the Poisson bracket has degree $l > 0$. Put $B := H^2(X, \mathbb{C})$ and equip the affine space $B$ with a $\mathbb{C}^*$-action by $z \cdot b = z^{-l}b$, $z \in \mathbb{C}^*$, $b \in B = H^2(X, \mathbb{C})$.

Then there exists a smooth $\mathbb{C}^*$-variety $X_B$ and a smooth $\mathbb{C}^*$-equivariant morphism $\pi : X_B \to B$ such that

(i) $X_B$ is a relative symplectic manifold over $B$, i.e., we have a relative 2-form $\omega \in H^0(X_B, \Omega^2(X_B/B))$ which induces a symplectic structure on each fiber $X_b$, $b \in B = H^2(X, \mathbb{C})$.

(ii) The relative cohomology sheaves $R^k\pi_*\mathbb{Q}$ are constant sheaves on $B$ for all $k \geq 0$, and the canonical base change morphism

$$H^k(X_b, \mathbb{Q}) \to (R^k\pi_*\mathbb{Q})_b$$

is an isomorphism for every point $b \in B$.

(iii) The affinization, $(X_B)^{\text{aff}}$, is flat over $B$. The canonical map $X_B \to (X_B)^{\text{aff}}$ is projective, birational, and it is an isomorphism over the generic point $b \in B$.

(iv) The special fiber $X_0$ over $0 \in B = H^2(X, \mathbb{C})$ is isomorphic to $X$ as a symplectic algebraic variety. The special fiber $(X_0)^{\text{aff}}$ of the affinization $(X_B)^{\text{aff}}$ is isomorphic to $X^{\text{aff}} = Y$ as a Poisson algebraic variety with $\mathbb{C}^*$-action.

---

\[1\text{this is similar to the deformation of the Springer resolution } \mathcal{N} \to \mathcal{N} \text{ (of the nilpotent variety } \mathcal{N} \text{ in a semisimple Lie algebra } g) \text{ provided by Grothendieck’s simultaneous resolution } \hat{\mathfrak{g}} \to \mathfrak{g}. \text{ Here } X = \mathcal{N}, X^{\text{aff}} = \mathcal{N}, \text{ and } X_B \text{ in the Theorem below plays the role of } \hat{\mathfrak{g}}.\]
In other words, there exists a Zariski open, dense subset $B^{\text{generic}} \subset B$, such that one has a commutative diagram

\[
\begin{array}{ccccccccc}
X_{B^{\text{generic}}} & \xrightarrow{\varphi} & X_B & \xleftarrow{\pi} & X_B^{\text{aff}} & \xrightarrow{\varphi} & X \\
\downarrow{\pi} & & \downarrow{\pi} & & \downarrow{\pi} & & \downarrow{\pi} \\
(X_{B^{\text{generic}}})^{\text{aff}} & \xrightarrow{\varphi} & (X_B)^{\text{aff}} & \xleftarrow{\pi} & (X_B^{\text{aff}})^{\text{aff}} & \xrightarrow{\varphi} & Y \\
B^{\text{generic}} & \xrightarrow{\varphi} & B & \xleftarrow{\pi} & \{o\} & \xrightarrow{\pi} & \{o\} \\
\end{array}
\]

In this diagram, the subscript '$B^{\text{generic}}$' indicates restriction to $B^{\text{generic}}$ of a scheme over $B$. Note, in particular, that according to Theorem 1.13, the map $X_{B^{\text{generic}}} \to B^{\text{generic}}$ is affine, so the arrow $(X_{B^{\text{generic}}})^{\text{aff}} \to (X_B^{\text{aff}})^{\text{aff}}$, on the left of the diagram above is an isomorphism.

The proof of Theorem 1.13 is contained in Subsection 5.2.

1.5 Comparison. Let $X$ be a convex symplectic manifold whose affinization $Y = X^{\text{aff}}$ is a positively-graded Poisson algebra of degree $l$, so that the assumptions of Theorem 1.13 are satisfied. Assume in addition that $H^1(C[Y]) = 0$, $\dim H^2(C[Y]) < \infty$, and the natural grading on $H^2(C[Y])$ is positive, so that we can apply Theorem 1.10 to get the universal graded Poisson deformation $(Y)$ over a base $S$. On the other hand, let $X/B$ be the deformation provided by Theorem 1.13. Then the affinization $(X_B)^{\text{aff}}$ is by construction a positively-graded flat Poisson $\mathcal{O}_B$-algebra. Hence, by Theorem 1.10 (ii), we get the classifying map

\[\tau : B \to S := \text{base of universal deformation}.\]

In Subsection 5.2 we will prove

**Proposition 1.14.** The classifying map $\tau : B \to S$ is a finite map onto an irreducible component of the variety $S$.

Now, fix a symplectic vector space $V$ and a finite group $G \subset \text{Sp}(V)$.

**Definition 1.15** ([EG]). An element $g \in G$ is called a symplectic reflection if $\text{rk}(\text{id}_V - g) = 2$.

Let $\Sigma$ denote the set of symplectic reflections in $G$. The group $G$ acts on $\Sigma$ by conjugation, and we put

\[n := \text{number of } G\text{-conjugacy classes in } \Sigma.\]

Introduce a grading on the polynomial algebra $\mathbb{C}[V]$ by assigning degree 1 to all linear functions. This turns $\mathbb{C}[V]$ into a positively graded Poisson algebra of degree $l = 2$. Set $Y = V/G$. Both the grading and the Poisson bracket descend to the algebra $\mathbb{C}[Y] = \mathbb{C}[V]^G$. In Section 4 we will prove
Proposition 1.16. We have $H^1_p(C[V]^G) = 0$ and $\dim H^2_{p}(C[V]^G) = n$. Moreover, the natural grading on $H^2_{p}(C[V]^G)$ is positive.

Let $X \to V/G$ be a symplectic resolution. The Betti numbers of $X$ are known by [BD] (see [K2] for an alternative proof), in particular, we have $\dim H^2(X, \mathbb{C}) = n$. Applying Proposition 1.14 we obtain a sequence of maps

$$H^2(X, \mathbb{C}) \to S \hookrightarrow H^2_{p}(C[V/G]),$$

where the first map is a finite surjection onto an irreducible component, and the second map is a closed embedding. Since $\dim H^2_{p}(C[V/G]) = n = \dim H^2(X, \mathbb{C})$, we conclude that $S$ is the whole $H^2_{p}(C[V/G])$ and $\tau : H^2(X, \mathbb{C}) \to S$ is a finite dominant map.

Remark 1.17. The composite map $H^2(X, \mathbb{C}) \to H^2_{p}(C[V/G])$ is not a bijection. Although both the source and the target of this map are affine spaces, the map is not linear; it is a ramified covering, and its differential at $0 \in H^2(X, \mathbb{C})$ usually vanishes.

1.6 Calogero-Moser deformation. Let $V$ and $G$ be as above. Let $\Sigma \subset G$ be the set of all symplectic reflections in $G$, and let $C$ be the vector space of all $G$-invariant functions $c : \Sigma \to \mathbb{C}$. Clearly, $\dim C = n$. We regard $\mathbb{C}[C]$, the polynomial algebra, as a positively graded algebra by assigning degree 2 to linear polynomials.

In [EG] P. Etingof and the first author have constructed a certain flat graded Poisson $\mathbb{C}[C]$-algebra $B$ of degree 2, which gives a graded $n$-parameter deformation of $\mathbb{C}[V]^G$. Let $B_c$ denote the specialization of $B$ at a point $c \in C$, and put $\mathcal{M}_c = \text{Spec} B_c$. Thus, $\{\mathcal{M}_c\}_{c \in C}$ is a flat family of Poisson affine algebraic varieties and, by construction, we have $\mathcal{M}_0 = V/G$. We will refer to $\mathcal{M}_c$ as a Calogero-Moser variety with parameter $c$, and we call the projection $\mathcal{M} := \text{Spec} B \to C$ the Calogero-Moser deformation of $V/G$.

According to Proposition 1.16 there exists a canonical classifying map $\kappa : C \to S \subset H^2_{p}(C[V]^G)$ such that the Calogero-Moser deformation is obtained by pull-back from the universal deformation. We will see that the map $\kappa$ is never bijective. The best we can prove is provided by the following

Theorem 1.18. If the Calogero-Moser space $\mathcal{M}_c$ is smooth for generic values of the parameter $c \in C$ then:

(i) The base $S$ of the universal Poisson deformation of $V/G$ coincides with the whole vector space $H^2_{p}(C[V]^G)$.

(ii) The classifying map $\kappa : C \to H^2_{p}(C[V]^G)$ is surjective and generically étale.

Our terminology is motivated by the special case where $G = S_n$ is the symmetric group, acting diagonally on $V = \mathbb{C}^n \oplus \mathbb{C}^n$ by permutation of coordinates. In that case, the deformation $\mathcal{M}_c$ of $V/S_n$ is known (see [EG] and references therein) to be the usual Calogero-Moser space.
The proof of this theorem is contained in Section 6. Part (ii) of the Theorem says, roughly speaking, that generically, every fiber of the universal deformation is isomorphic to \( B_c \) for a suitable value of the parameter \( c \in C \). Moreover, for general \( c \), there exists only a finite number of other values \( c' \in C \) such that \( B_c \cong B_{c'} \). This is what we mean by saying that the Calogero-Moser deformation is “versal”.

It is not unreasonable to conjecture that the claim of Theorem 1.18 holds in the general situation. Even stronger, we propose the following.

**Conjecture 1.19.** For an arbitrary symplectic quotient singularity \( V/G \), the classifying map \( C \to S \) of the Calogero-Moser deformation \( \mathcal{M}/C \) is finite.

The conjecture is motivated by analogy with the case of a symplectic resolution \( X \to V/G \) and the corresponding deformation \( (\mathfrak{X}_B)^\text{aff} \). The parameter spaces of these two deformations are the same vector spaces with the same grading. Moreover, we will see in Section 3 that they actually coincide when \( \dim V/G = \dim V = 2 \).

In general, view \( V/G \) as a \( \mathbb{C}^* \)-variety, with \( \mathbb{C}^* \)-action being induced from the natural one on the vector space \( V \). Then, the relation between symplectic resolutions of \( V/G \) and the Calogero-Moser deformation is provided by the following theorem.

**Theorem 1.20 (Symplectic resolutions and Calogero-Moser).** Let \( X \to Y = V/G \) be a symplectic resolution of a symplectic quotient singularity \( V/G \).

- Let \( \mathcal{Y}/S \) be the universal Poisson deformation of the Poisson variety \( Y \),
- Let \( \mathfrak{X}_B \) be the deformation over \( B = H^2(X, \mathbb{C}) \) provided by Theorem 1.13,
- Let \( \mathcal{M}/C \) be the Calogero-Moser deformation, and denote by \( \kappa : C \to S \) its classifying map.

Then for \( c \in C \) general enough, the image \( \kappa(c) \in S \) is generic in the sense of Theorem 1.13(iii). More precisely, for every point \( b \in B \) lying over \( \kappa(c) \in S \), the canonical map \( \bar{\mathfrak{X}}_b \to \mathcal{Y}_{\kappa(c)} \) is an isomorphism.

Put \( \bar{C} = C \times_S B \) and let \( \psi : \bar{C} \to B \) and \( \varphi : \bar{C} \to C \) be the natural projections. Then the statement of Theorem 1.20 can be summarized by the following commutative diagram.
The proof of Theorem 1.20 is contained in Subsection 5.3.

**Corollary 1.21.** The existence of a symplectic resolution \( X \to V/G \) implies that the generic fiber \( M_c \) of the Calogero-Moser deformation \( M/C \) is smooth.

**Proof.** By definition we have \( M_c \cong \mathcal{Y}_{\kappa(c)} \), where \( \mathcal{Y}_B = (\mathfrak{X}_B)^{\text{aff}} \). The latter is isomorphic, for \( b \in B \) such that \( \kappa(c) = \tau(b) \in S \) is general enough, to \( \mathfrak{X}_b \), which is smooth. \( \square \)

### 1.7 Proofs of Theorem 1.1 and Theorem 1.2.

Let \( V \) be a symplectic vector space, let \( G \subset \text{Sp}(V) \) be a finite group, and let \( X \to Y = V/G \) be a crepant, hence symplectic, resolution.

To prove Theorem 1.1 note that in the assumptions of the Theorem, the existence of \( X \) implies by Corollary 1.21 that the generic fiber \( M_c \) of the Calogero-Moser deformation \( M/C \) is smooth. This contradicts [Go, Proposition 7.3], see also [EG, Proposition 16.4(ii)].

To prove Theorem 1.2 consider the universal deformation \( Y/S \) and the deformation \( \mathfrak{X}/B \) provided by Theorem 1.13. Consider a general fiber \( M_c \) of the Calogero-Moser deformation \( M/C \). Choose a point \( b \in B \) lying over \( \kappa(c) \in S \).

By definition, we have isomorphisms of fibers \( M_c \cong \mathcal{Y}_{\kappa(c)} \cong \mathfrak{X}_b \), which induce algebra isomorphisms of the rational cohomology:

\[
H^*(M_c, \mathbb{Q}) \cong H^*(\mathcal{Y}_{\kappa(c)}, \mathbb{Q}) \cong H^*(\mathfrak{X}_b, \mathbb{Q}) \cong H^*(X, \mathbb{Q}),
\]

where the last isomorphism is due to Theorem 1.13(ii). By Corollary 1.21 the generic fiber \( B_c \) is smooth. By [EG, Theorem 1.8(ii)], this implies that the left-hand-side is isomorphic to \( \text{gr}(ZG) \). \( \square \)

**Acknowledgments.** We are grateful to V. Baranovsky, R. Bezrukavnikov, P. Etingof, and V. Ostrik for many useful discussions. We also thank Y. Ruan for bringing the question of the validity of Lemma A.16 to our attention. The second author was partially supported by CRDF Award RM1-2354-MO02.

### 2 Generalities on Poisson deformations.

In this section, we review basic results on the deformation theory of Poisson algebras that will be used later. We have been unable to find an adequate reference in the literature, so in the Appendix to this paper the reader may find some details of proofs and precise definitions.

**Convention.** Given a commutative \( \mathbb{C} \)-algebra \( A \), we write \( \text{Hom}_A, \otimes_A \), and \( \Lambda^k_A \) for Hom, tensor product and \( k \)-th wedge product over \( A \), respectively. If no subscript \( A \) is indicated, then the corresponding functors are understood to be taken over \( \mathbb{C} \), e.g. \( \otimes = \otimes_{\mathbb{C}}. \)
We let $\mathcal{T}_M$ denote the tangent sheaf (or tangent bundle) of a smooth algebraic variety $M$. Further, given a finitely generated commutative algebra $A$ we write $\Omega^1 A$ and $\mathcal{T}(A) = \mathcal{T}(\text{Spec } A)$ for the $A$-modules of (global) Kähler differentials and vector fields on the scheme $\text{Spec } A$, respectively.

2.1 Poisson cohomology. For smooth Poisson manifolds, the notion of Poisson cohomology (in the differential geometric setting) is due to J.-L. Koszul [Ko] and J.-L. Brylinski [Br]. In the algebraic setting, the Poisson cohomology has been introduced by B. Fresse [Fr]. We review and extend it below (relations with deformation theory were not discussed in [Fr]). For a more general formalism of deformation theory of an algebra over an arbitrary operad the reader may consult [KS].

Let $A$ be a finitely generated commutative algebra. The standard Lie bracket of vector fields extends to the so-called Schouten bracket $\{−, −\}$ on $\Lambda^*_A \mathcal{T}(A)$, the space of polyvector fields. It is well-known that if $\text{Spec } A$ is smooth then any Poisson structure on $A$ defines (and is defined by) a bivector field $\Theta \in \Lambda^2_A \mathcal{T}(A)$ which satisfies the integrability condition $\{\Theta, \Theta\} = 0$.

Given a smooth Poisson algebra $A$ one defines a map

$$d : \Lambda^*_A \mathcal{T}(A) \to \Lambda^*_A \mathcal{T}(A), \quad a \mapsto da := \{\Theta, a\}.$$ 

We have $d \circ d = 0$. Taking $d$ as the differential makes $\Lambda^*_A \mathcal{T}(A)$ into a complex. The cohomology groups of this complex are called the Poisson cohomology groups of the algebra $A$ and denoted by $HP^*(A)$.

It turns out that this formalism can be extended to arbitrary, not necessarily smooth Poisson algebras $A$. The precise constructions are a little bit technical; we give them in full in the Appendix. Here we only describe the end result.

Recall that for an arbitrary finite-type commutative algebra $A$, one can define the so-called Harrison complex $\mathsf{Har}(A)$, a certain canonical complex of free $A$-modules representing the cotangent complex of the algebra $A$. We recall the precise construction of $\mathsf{Har}(A)$ in Subsection A.1. Here we only note that when the algebra $A$ is smooth, the complex $\mathsf{Har}(A)$ has non-trivial cohomology only in degree 0, and this non-trivial cohomology module is isomorphic to the module $\Omega^1 A$ of Kähler differentials.

Let now $A$ be an arbitrary Poisson algebra, not necessarily smooth. Heuristically, to extend the Brylinski construction to $A$, one replaces everywhere the module $\Omega^1 A$ of Kähler differentials with the Harrison complex $\mathsf{Har}(A)$. More precisely, one considers the exterior powers $\Lambda^k_A \mathsf{Har}(A)$ of the complex $\mathsf{Har}(A)$ of flat $A$-modules and sets

$$DP^{*, k}(A) \cong \text{Hom}_A(\Lambda^k_A \mathsf{Har}(A), A), \quad k \geq 0.$$ 

This defines a canonical bigraded vector space $DP^{*, *}(A)$ and a differential $d : DP^{*, *}(A) \to DP^{*, 1, *}(A)$. In particular, we have

$$DP^{0, 2}(A) \cong \text{Hom}_A(\Lambda^2 A \otimes A, A) = \text{Hom}_C(\Lambda^2 A, A).$$
There is a natural Gerstenhaber bracket on $D^{p,\cdot}(A)$, see (A.3):

$$\{-,-\} : DP^{p,q}(A) \otimes DP^{q',q''}(A) \to DP^{p+p'+q+q'-1}(A),$$

that makes $D^{p,\cdot}(A)$ a DG Lie algebra (with shifted grading). The Poisson structure on $A$ defines (and is defined by) an element $\Theta \in DP^{0,2}(A) = \text{Hom}(\Lambda^2 A, A)$ satisfying $d\Theta = 0$ and $\{\Theta, \Theta\} = 0$. We call this element the Poisson cochain. Given such an element, one defines the differential $\delta : D^{p,\cdot}(A) \to D^{p,\cdot+1}(A)$ by setting

$$\delta : D^{p,\cdot}(A) \to D^{p,\cdot+1}(A), \quad a \mapsto \{\Theta, a\}.$$

Thus we have a bicomplex $D^{p,\cdot}(A)$ with differentials $d, \delta$. We define Poisson cohomology of $A$ to be the cohomology groups $HP^{\cdot}(A)$ of the total complex associated with this bicomplex, with respect to the total differential $d + \delta$.

If the algebra $A$ is smooth, then we have a quasiisomorphism $\text{Har}_A \cong \Omega^1 A$, so that

$$D^{p,\cdot,k}(A) \cong \text{Hom}(\Lambda^k \Omega^1 A, A) \cong \Lambda^k_A T(A),$$

and the general Poisson cohomology complex $D^{p,\cdot}(A)$ is quasiisomorphic to the Brylinski complex $(\Lambda^k_A T(A), d)$.

If the algebra $A$ is a Poisson graded algebra in the sense of Subsection 1.3, then the Poisson cohomology bicomplex $D^{p,\cdot}(A)$ acquires an additional grading, called the $A$-grading. However, since the Poisson cochain $\Theta$ is of degree $l$ with respect to the grading, the differential $\delta : D^{p,\cdot}(A) \to D^{p,\cdot+1}(A)$ does not preserve the $A$-grading but rather shifts it by $l$. To cure this, we redefine the $A$-grading by shifting it by $(k-1)l$ on $D^{p,k}(A)$.

### 2.2 Globalization

The Poisson cohomology complex $D^{p,\cdot}(A)$ becomes very simple if the scheme $A$ is not only smooth but also symplectic. In that case the canonical isomorphism $\Omega^1 A \cong T(A)$ extends to an isomorphism $\Omega^* A \cong \Lambda^* T(A)$ between the Brylinski complex $\Lambda^* T(A)$ and the de Rham complex $\Omega^* A$. Thus the Poisson cohomology $HP^{\cdot}(A)$ coincides with the de Rham cohomology $H^*(\text{Spec} A)$ of the scheme $\text{Spec} A$. It turns out that several features of the de Rham cohomology formalism extends to the general case.

Firstly, one can define Poisson cohomology with coefficients, which is analogous to the de Rham cohomology with coefficients in a local system (or, more generally, in a $D$-module). For this one defines a Poisson module $M$ over an arbitrary Poisson algebra $A$ in an natural way. Then to every Poisson module $M$ one associates a canonical bicomplex $D^{p,\cdot}(A, M)$ called the cohomology complex with coefficients in $M$. The algebra $A$ is a Poisson module over itself, and we have $D^{p,\cdot}(A, A) = D^{p,\cdot}(A)$.

Secondly, one generalizes the notion of Poisson cohomology to the scheme case. One defines a Poisson scheme $X$ and a Poisson sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules in the natural way. To a Poisson scheme $X$ with a Poisson sheaf $\mathcal{F}$ (or, more generally, to a complex $\mathcal{F}$ of Poisson sheaves) one associates a canonical complex $\mathcal{H}P^*(X, \mathcal{F})$ of Zariski sheaves on $X$ called the local Poisson cohomology
complex with coefficients in $\mathcal{F}^*$. In the particular case $\mathcal{F} = \mathcal{O}_X$, one obtains the local Poisson cohomology complex $\mathcal{H}P^*(X) = \mathcal{H}(\mathcal{P}^*(X, \mathcal{O}_X))$. When the Poisson scheme $X$ is smooth and symplectic, a Poisson sheaf on $X$ is the same as a $D$-module, the Poisson cohomology complex $\mathcal{H}P^*(X)$ is the de Rham complex of the scheme $X$, and the Poisson cohomology complex with coefficients $\mathcal{H}P^*(X, \mathcal{F})$ is the de Rham complex of the $D$-module $\mathcal{F}$.

Taking the hyperhomology, one obtains the groups

$$HP^*(X) = \mathbb{H}^*(X, \mathcal{H}P^*(X))$$

of global Poisson cohomology of the scheme $X$ and the groups

$$HP^*(X, \mathcal{F}) = \mathbb{H}^*(X, \mathcal{H}P^*(X, \mathcal{F}))$$

of global Poisson cohomology of $X$ with coefficients in the complex $\mathcal{F}^*$. When the scheme $X = \text{Spec} A$ is affine and the complex $\mathcal{F}^*$ comes from a complex $\mathcal{M}^*$ of Poisson $A$-modules, we have $HP^*(X, \mathcal{F}) \cong HP^*(A, \mathcal{M}^*)$. When the scheme $X$ is smooth and symplectic, $HP^*(X, \mathcal{F}^*)$ is the singular cohomology $H^*(X, \mathcal{F}^*)$ with coefficients in the $D$-module $\mathcal{F}^*$.

Taking the hyperhomology, one obtains the groups

$$HP^k(X) = \bigoplus_{p+q=k} HP^p(Y) \otimes HP^q(Z), \quad k \geq 0.$$

The notion of a Poisson sheaf is sufficiently functorial; in particular, for any morphism $f : X \to Y$ between Poisson schemes, the direct image $f_* \mathcal{F}$ of a Poisson sheaf $\mathcal{F}$ on $X$ is a Poisson sheaf on $Y$. Moreover, one can represent the direct image $R^q f_* \mathcal{F}$ in the derived category by a complex of Poisson sheaves. In the particular case of an open embedding $j : U \hookrightarrow X$, we obtain a Poisson structure on the direct image $R^q j_* \mathcal{O}_U$. Moreover, in this particular case we have

$$\mathcal{H}P^*(X, R^q j_* \mathcal{O}_U) \cong R^q j_* (\mathcal{H}P^*(U)).$$

One can also obtain a Poisson structure on the third term $i_* \mathcal{O}_Z$ in the exact triangle

$$i_* \mathcal{O}_Z \longrightarrow \mathcal{O}_X \longrightarrow R^q j_* \mathcal{O}_U \longrightarrow$$

where $Z \subset X$ is the closed complement to $U \subset X$ and $i : Z \hookrightarrow X$ is its embedding. We define the Poisson cohomology $HP^*_Z(X)$ of the scheme $X$ with supports in $Z \subset X$ by setting $HP^*_Z(X) \cong HP^*(X, i_* \mathcal{O}_Z)$. When $X$ is smooth and symplectic, this is the ordinary singular cohomology $H^*_Z(X)$ with supports in $Z$. By definition, for a general Poisson $X$ we have the canonical exact triangle

$$HP^*_Z(X) \longrightarrow HP^*(X) \longrightarrow HP^*(U) \longrightarrow$$

Note that the scheme $Z$ actually enters into this construction only through its open complement $U \subset X$. In particular, there is no need to assume that $Z \subset X$ is a Poisson subscheme in any sense.
The reader will find the precise definitions and statements on Poisson cohomology in the Appendix, with all the proofs. The only statements that we will actually use in the main body of the paper are contained in Proposition A.11, Corollary A.12, and in Lemma A.9.

The reason we are interested in Poisson cohomology is its role in the study of Poisson deformations of a Poisson scheme \( X \). This role is completely analogous to the role of the standard cotangent complex \( \Omega_{\cdot}(X) \) and the groups \( \text{Ext}^\cdot(\Omega_{\cdot}(X), \mathcal{O}_X) \) in the standard deformation theory of a scheme \( X \). The analogy can be pushed quite far. We will use Poisson deformation theory in Subsection A.8 to prove Theorem 1.10.

3 The case of \( \dim V = 2 \).

Before we proceed to the study of general symplectic quotient singularities \( V/G \), we need to consider the particular case \( \dim V = 2 \). This is the case of the so-called McKay correspondence. Starting with the paper [McK], it has been studied excessively by many authors. We recall here some of the results.

Let \( V \) be a complex vector space of dimension \( \dim V = 2 \). To every finite subgroup \( G \subset \text{Sp}(V) \), one canonically associates a simply-laced root system whose rank is equal to the number of non-trivial conjugacy classes in \( G \). Let \( g \subset \mathfrak{g} \) be the simple Lie algebra associated to this root system. The Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) is naturally dual, \( \mathfrak{h} \cong \mathbb{C}^* \), to the base \( \mathbb{C} \) of the Calogero-Moser deformation \( \mathcal{M}/\mathbb{C} \) of the quotient variety \( V/G \). The conjugacy classes of elements \( g \in G \) define a basis in the vector space \( \mathbb{C} \cong \mathfrak{h}^* \), which is in fact a basis of simple roots. The dual space \( \mathfrak{g}^* \) is naturally a Poisson scheme. The nilpotent cone \( \mathcal{N} \subset \mathfrak{g}^* \) is a Poisson subscheme consisting of a finite number of coadjoint orbits, all of which are symplectic. There exists a unique orbit \( \mathcal{N}_{\text{subreg}} \subset \mathcal{N} \) of dimension \( \dim \mathcal{N}_{\text{subreg}} = \dim \mathcal{N} - 2 \), called the subregular nilpotent orbit. Take an arbitrary element \( n \in \mathcal{N}_{\text{subreg}} \) and let \( \mathfrak{p} \subset \mathfrak{g}^* \) be an affine space passing through \( n \subset \mathfrak{g}^* \) and transversal to \( \mathcal{N}_{\text{subreg}} \subset \mathfrak{g}^* \). The Poisson structure on \( \mathfrak{g}^* \) induces a Poisson structure on the affine space \( \mathfrak{p} \). The intersection \( \mathfrak{p} \cap \mathcal{N} \subset \mathfrak{p} \) is a Poisson subscheme. It turns out that the Poisson scheme \( \mathfrak{p} \cap \mathcal{N} \) is naturally isomorphic to the quotient \( V/G \). Moreover, the Poisson algebra of functions on \( \mathfrak{p} \) has a large center, so that in fact we have a Poisson deformation \( \mathfrak{p}/S \) over a base \( S \) of dimension \( \dim S = \text{rk} \mathfrak{g} \). The base \( S \) of the deformation \( \mathfrak{p}/S \) is canonically isomorphic to the quotient \( S = \mathfrak{h}^*/W \) of the dual Cartan algebra \( \mathfrak{h}^* \) by the Weyl group \( W \). In particular, \( S \) is smooth. The subscheme \( Y \equiv \mathfrak{p} \cap \mathcal{N} \subset \mathfrak{p} \) is the fiber of \( \mathfrak{p}/S \) over the point \( 0 \in S \equiv \mathfrak{h}^*/W \). In other words, we obtain a Poisson deformation \( \mathfrak{p}/S \) of the Poisson scheme \( V/G \) over a smooth base \( S \) of dimension \( \dim S = \text{rk} \mathfrak{g} \).

The natural \( \mathbb{C}^* \)-action on the vector space \( \mathfrak{g}^* \) by dilatations induces the natural grading on the Poisson algebra \( \mathbb{C}[V/G] \). The deformation \( \mathfrak{p}/S \) is a graded deformation. The grading on \( S \equiv \mathfrak{h}^*/W \) is induced by the \( \mathbb{C}^* \)-action by dilatations on \( \mathfrak{h}^* \).

From now on, set \( Y = V/G \). It is known that the deformation \( \mathfrak{p}/S \) is a
miniversal deformation of $Y$ in the category of affine schemes, with the Poisson structure forgotten. In particular, the canonical map

$$T_o S \to \text{Ext}^1(\Omega_*(Y), \mathcal{O}_Y)$$

between the Zariski tangent space $T_o S$ and the group which classifies infinitesimal deformations of the scheme $Y$ is an isomorphism. Moreover, the total space $p$ and the base $S$ of the deformation $p/S$ are smooth; therefore the scheme $Y$ is a complete intersection, and the cotangent complex $\Omega_*(Y)$ only has non-trivial cohomology in degree 0.

Using these facts, we can now compute the Poisson cohomology groups $HP^1(Y)$ and $HP^2(Y)$.

**Lemma 3.1.** We have $HP^1(Y) = 0$, and

$$HP^2(Y) \cong \text{Ext}^1(\Omega(Y), \mathcal{O}_Y) \cong T_o S.$$ 

Moreover, the degrees of the natural grading on the group $HP^2(Y)$ coincide with the exponents of the Weyl group $W$.

**Proof.** Denote $A = \mathbb{C}[Y]$. By definition of the Poisson cohomology bicomplex $DP^{*,*}(A)$ we have $DP^{0,0}(A) \cong A$, $DP^{0,k}(A) = 0$ for $k \geq 1$ and

$$DP^{1,*}(A) \cong \mathcal{R}\text{Hom}^*(\Omega^1 A, A).$$

In particular, there exists a canonical map $\kappa : HP^2(A) \to \text{Ext}^1(\Omega^1 A, A)$ induced by the natural projection $DP^{*,k}(A) \to DP^{1,k}(A)$, $k \geq 1$. On the level of deformation theory, the map $\kappa$ corresponds to forgetting the Poisson structure. Since the universal deformation $p/S$ of the scheme $Y$ does admit a Poisson structure, the map $\kappa : HP^2(A) \to \text{Ext}^1(\Omega^1 A, A)$ is surjective. Thus to prove the Lemma, it suffices to prove that $HP^1(A) = 0$ and that $\kappa$ is an injective map.

Consider the spectral sequence associated to the bicomplex $HP^{*,*}(A)$. We have

$$E_1^{p,q} = \text{Ext}^q(\Lambda^p \Omega_*(A), A), \quad p, q \geq 0.$$ 

The only term which contributes to $HP^1(A)$ is the term $E^{1,0}_\infty$. The only non-trivial terms which contribute to $HP^2(Y)$ are $E^{2,0}_1$, $E^{1,1}_1$ and $E^{0,2}_1$. The term $E^{0,2}_1 = \text{Ext}^2(A, A)$ vanishes. The term $E^{1,1}_1$ is precisely $\text{Ext}^1(\Omega^1 A, A)$. Moreover, the claim about the gradings on this term follows from the identification $S \cong \mathfrak{h}^*/W$. To prove that the map $HP^2(Y) \to E^{1,1}_1$ is injective, it suffices to prove that the term $E^{2,0}_\infty$ vanishes. Thus it suffices to prove that $E^{2,0}_\infty = E^{1,0}_\infty = 0$. We will prove that already $E^{2,0}_p = 0$ for every $p \geq 1$.

Indeed, denote by $j : U \hookrightarrow Y$ the embedding of the open complement $U = Y \setminus \{0\} \subset Y$ to the origin $o \in Y$. By definition we have

$$E_1^{p,0} \cong \text{Hom}(\Lambda^p \Omega_Y, \mathcal{O}_Y) \cong \text{Hom}(\Lambda^p \Omega_Y, j_* \mathcal{O}_Y)$$

$$\cong \text{Hom}_U(\Lambda^p \Omega_U, \mathcal{O}_U) \cong H^0(U, \Lambda^p \mathfrak{T}_U).$$

18
The quotient map \( \pi : V \to Y = V/G \) is \( \acute{e} \)tale over \( U \), so that
\[
H^0(U, \Lambda^p T_U) \cong H^0(\pi^{-1}(U), \Lambda^p T_{\pi^{-1}(U)})^G.
\]
Moreover, since \( \pi^{-1}(U) \subset V \) is the complement to a point in a smooth scheme, the right-hand side is isomorphic to \( H^0(V, \Lambda^p T_V) \). The differential \( d_1 : E_r^{p, 0} \to E_{r+1}^{p, 0} \) in the spectral sequence is induced by the Poisson differential on the space \( H^0(V, \Lambda^p T_V) \). Hence, \( E_r^{p, 0} \cong H^p(V, \Lambda^p T_V)^G \). Since \( V \) is smooth and symplectic, and \( H^p(V, \mathbb{C}) = 0 \) for \( p \geq 1 \), this implies that \( E_r^{p, 0} = 0 \) for \( p \geq 1 \). This finishes the proof. □

In particular we see that \( \mathfrak{p}/S \) is the universal Poisson deformation of the Poisson scheme \( Y = V/G \).

Recall the Calogero-Moser deformation \( M/C \) introduced in [EG]. The deformation \( M/C \) does not coincide with the universal deformation \( \mathfrak{p}/S \) but there is a Cartesian square
\[
\begin{array}{ccc}
\mathcal{M} & \longrightarrow & \mathfrak{p} \\
\downarrow & & \downarrow \\
C & \longrightarrow & S
\end{array}
\]
Here \( C \cong \mathfrak{h}, S \cong \mathfrak{h}/W \), and the bottom row in the square may be identified with the canonical projection \( C = \mathfrak{h} \to \mathfrak{h}/W = S \) (which is not an isomorphism). Thus, Lemma 3.1 provides a canonical identification \( HP^2(Y) \cong C/W \) (this is an identification of algebraic varieties, not of vector spaces).

The identification \( HP^2(Y) \cong C/W \) yields: \( \dim HP^2(Y) = \dim C = n \), the equality claimed in Proposition 1.16. To study the higher-dimensional case we will need the following twisted version of this equality.

**Lemma 3.2.** Let \( G \subset G' \subset \text{Sp}(V) \) be two finite groups acting on the two-dimensional symplectic space \( V \). Assume that \( G \subset G' \) is normal, and let \( H = G'/G \) be the quotient group, which act naturally on the quotient variety \( V/G \) and on the vector space \( C \).

Then we have: \( \dim HP^2(Y)^H = \dim C^H \).

**Proof.** The group \( H \) preserves the root system \( \Delta \subset C \) corresponding to \( G \subset \text{Sp}(V) \) and the base of simple roots defined by the conjugacy classes in \( G \). Moreover, \( H \) commutes with the action of the Weyl group \( w \). We have \( HP^2(Y) \cong C/W \). The space \( HP^2(Y)^H \) of \( H \)-invariant vectors is the subvariety \( (C/W)_H \subset C/W \) of \( H \)-fixed points in \( C/W \). Therefore it suffices to use the following standard result.

**Lemma 3.3.** Let \( \Delta \subset C_{\mathbb{R}} \) be a root system with Weyl group \( W \), and let \( H \) be a finite group of automorphisms of the root system \( \Delta \) which preserves a Weyl chamber \( C^+_\mathbb{R} \subset C_{\mathbb{R}} \). Let \( C = C_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \) be the complexification of the real vector space \( C_{\mathbb{R}} \). Then the quotient map \( C \to C/W \) induces a surjective finite map
\[
C^H \to (C/W)_H
\]
onto the set of \( H \)-fixed points in the quotient variety \( C/W \).
Proof. It is well-known that the set \( C^+ = C^+_R + \sqrt{-1}C_R \) is a fundamental domain for the \( W \)-action on the vector space \( C \), so that we have an isomorphism \( C^+ \cong C/W \). Since \( H \) preserves \( C^+ \), it induces an isomorphism

\[ C^+_H \cong C^+ \cap C^H \cong (C/W)_H, \]

which gives a section of the map \( C^H \to (C/W)_H \).

To describe the Calogero-Moser deformation \( M/C \) more functorially, we recall that the quotient \( V/G \) admits a canonical smooth symplectic resolution \( X \to Y \) (coming from the Springer resolution \( \tilde{N} \to N \) of the nilpotent cone in \( g^* \)). Moreover, the scheme \( X \) has a universal deformation \( X/C \), and the total space \( X \) is symplectic over \( C \). Therefore the algebra \( H^0(X, \mathcal{O}_X) \) of global functions on \( X \) has a natural Poisson structure. This algebra coincides with the algebra of functions on the Calogero-Moser deformation \( M/C \). Geometrically, we have a natural birational projective map \( \pi : X \to M \) and an isomorphism \( \pi_* \mathcal{O}_X \cong \mathcal{O}_M \). The cohomology group \( H^2(X, \mathbb{C}) \) of the resolution \( X \) is naturally identified with the base \( C \) of the deformation \( X/C \). The map \( \pi \) is compatible with the projections to the base \( C \). Moreover, it is an isomorphism over a generic point \( c \in C \).

These facts taken together immediately imply Proposition 1.16, Theorem 1.18 and Theorem 1.20 in the case \( \dim V = 2 \).

4 The computation of \( HP^2(V/G) \).

Let \( V \) be an arbitrary finite-dimensional symplectic vector space, and let \( G \subset Sp(V) \) be a finite subgroup. In this section we will compute the Poisson cohomology groups \( HP^1(V/G) \) and \( HP^2(V/G) \) of the quotient \( Y = V/G \). In particular, we will prove Proposition 1.16.

First, we introduce some notation. Notice that the vector space \( V \) is naturally stratified by subspaces \( V^H \subset V \) of \( H \)-invariant vectors for various subgroups \( H \subset G \). All these subspaces are symplectic, hence even-dimensional. This induces a stratification of the quotient variety \( Y = V/G \). The strata of the stratification are known to be smooth symplectic locally-closed subvarieties in \( V/G \), in particular have even dimension. Moreover, these strata turn out to be exactly the symplectic leaves of the standard Poisson structure on \( V/G \), cf. e.g. [BG].

In more detail, let \( \Gamma \subset G \) be a subgroup which is the isotropy group of an element of \( V \). Then \( V^\Gamma \subset V \) is a nonzero vector subspace such that the symplectic form on \( V \) restricts to a nondegenerate 2-form on \( V^\Gamma \). Further, let \( U^\Gamma \subset V \) be the set of points of \( V \) whose stabilizer (in \( G \)) is equal to \( \Gamma \). It is clear that \( U^\Gamma \subset V^\Gamma \), moreover, it is known from the theory of finite group actions that \( U^\Gamma \) is a non-empty Zariski open, hence dense, subset in \( V^\Gamma \). The strata of the stratification of \( V/G \) that has been mentioned in the previous paragraph are defined to be the images under the projection \( V \to V/G \) of the sets of the form \( U^\Gamma \), as \( \Gamma \) varies inside \( G \).
It is straightforward to verify that \( N(\Gamma) \), the normalizer of \( \Gamma \) in \( G \), preserves the set \( U^T \). Furthermore, the resulting \( N(\Gamma)/\Gamma \)-action on \( U^T \) is free, and the projection \( V \to V/G \) induces an isomorphism of \( U^T/(N(\Gamma)/\Gamma) \) with its image in \( V/G \), that is, with the corresponding stratum of the stratification.

Now, recall the notation \( Y = V/G \) and write \( U \subset Y \) for the complement to the union of all the strata in \( Y \) of codimension \( \geq 4 \). Thus we have \( U = U_0 \coprod \bigcup_{i \geq 1} U_i \), where \( U_0 \) is the unique open stratum in \( Y \), and \( U_i \), \( i = 1, 2, \ldots \), are all the codimension two strata of the stratification. By the earlier discussion, each stratum is the image under the projection \( V \to V/G \) of the set \( U^{G_i} \subset V \) where \( G_i \subset G \) is a subgroup such that the fixed point set \( V_i := V^{G_i} \) is a codimension 2 vector subspace in \( V \) and, moreover, such that the set \( U^{G_i} = \{ v \in V \mid \text{isotropy group of } v = G_i \} \) is a Zariski dense subset in \( V^{G_i} \). Two subgroups as above give rise to the same stratum \( U \) if and only if they are conjugate within \( G \).

Let \( G'_i := N(G_i) \subset G \) denote the normalizer of the subgroup \( G_i \), and \( H_i := G'_i/G_i \) the quotient group. As we have explained above, the group \( G'_i \) preserves the set \( U^{G_i} \), the induced \( H_i \)-action on \( U^{G_i} \) is free; furthermore, we have an isomorphism \( U^{G_i}/H_i \simto U_i \).

For each \( i \geq 1 \), let \( W_i \) denote the annihilator of \( V_i \) with respect to the symplectic form. Thus, \( \dim W_i = 2 \), and there is a canonical \( G'_i \)-stable direct sum decomposition \( V = V_i \oplus W_i \). Let \( Y_i = W_i/G_i \) be the quotient variety. The group \( H_i \) acts on the variety \( Y_i \) and preserves the origin \( o \in Y_i \). The direct sum decomposition \( V \cong V_i \oplus W_i \) induces a map \( \eta_i : (V_i \times Y_i)/H_i \to Y_i \). We have a Zariski open subset \( U'_i \subset V_i \) such that \( \eta_i \) maps \( (U'_i \times \{o\})/H_i \) isomorphically onto \( U_i \) and, moreover, is étale in a Zariski open neighborhood of \( U_i = (U'_i/H_i) \times \{o\} \subset (V_i \times Y_i)/H_i \). We note for further record that \( V_i \setminus U'_i \) is a finite union of vector subspaces in \( V_i \) of (complex) codimension \( \geq 2 \). It follows that \( H^1(U'_i, \mathbb{C}) = 0 \), for \( i = 1, 2 \).

As in Subsection 1.5, let \( C \) be the space of all \( G \)-invariant functions on the set \( \Sigma \) of symplectic reflections in \( G \). By definition we have \( \dim C = n \). For every \( i \), denote by \( C_i \) the vector space of \( G_i \)-invariant \( C \)-valued functions on the group \( G_i \), and let \( B_i \subset C_i \) be the subspace of \( G'_i \)-invariant functions. The group \( H_i \) acts on the space \( C_i \) and we have \( B_i = C_i^{H_i} \subset C_i \). Every element \( g \in G_i \setminus \{1\} \) pointwise fixes the codimension two subspace \( V_i \), hence is a symplectic reflection. Therefore we have a natural restriction map \( C \to C_i \). This map factors through a map \( C \to B_i = C_i^{H_i} \subset C_i \). The map \( C \to B_i \) is surjective. Indeed, it suffices to check that two elements \( g_1, g_2 \in G_i \) conjugate in \( G \) are already conjugate in \( G'_i \); and if \( g \cdot g_1 \cdot g^{-1} = g_2 \), then \( g \) preserves the subspace \( V_i = V^{g_1} \subset V \), hence lies in \( G'_i \). Moreover, every symplectic reflection \( g \in G \) lies in one of the subgroups \( G_i \subset G \), – namely, the stabilizer of the invariant subspace \( V^g \subset V \). Therefore the projections \( C \to B_i \) give a natural splitting

\[
C \cong \bigoplus_i B_i = \bigoplus_i C_i^{H_i}.
\]

We recall that the quotient singularity \( Y = V/G \) carries a natural Poisson structure, so that we have the Poisson cohomology groups \( H^{P^*}(Y) \). Moreover,
$\mathbb{C}[Y]$ is a positively-graded Poisson algebra of degree 2, and this grading induces a grading on $HP^k(Y)$.

**Proof of Proposition 1.10** The scheme $Y = V/G$ is normal, and the smooth locus $U_0 \subset Y$ carries a non-degenerate symplectic form. Moreover, by [W] the scheme $Y$ is Gorenstein, hence Cohen-Macaulay. By definition, the complement to the open subset $U \subset Y$ is of codimension $\geq 4$. Therefore by Lemma [A.9] the natural map $HP^k(Y) \to HP^k(U)$ is an isomorphism for $k = 1, 2$.

Since the open subscheme $U_0 \subset U$ is smooth and symplectic, we have $HP^k(U_0) \cong H^k(U_0, \mathbb{C})$. But $U_0 = (V \setminus Z)/G$ is the quotient of the complement in the vector space $V$ to some closed subscheme $Z \subset V$ of codim $\geq 2$. Therefore the singular cohomology $H^k(U, \mathbb{C}) = H^k(V \setminus Z)^G$ vanishes in low degrees,

$$HP^k(U_0) \cong H^k(U_0, \mathbb{C}) \cong H^k(V \setminus Z, \mathbb{C})^G \cong H^k(V, \mathbb{C})^G = 0, \quad k = 1, 2, 3.$$  

We conclude that for $k = 1, 2$ the natural map $HP^k(U \setminus U_0) (U) \to HP^k(U)$ is an isomorphism, and the isomorphism $HP^k(Y) \cong HP^k(U)$ factors through a canonical isomorphism

$$HP^k(Y) \cong HP^k(U \setminus U_0)(U).$$

The complement $U \setminus U_0$ is the disjoint union of the closed strata $U_i \subset U$, $i \geq 1$, and for every $i \geq 1$, we have a map $\eta_i : (V_i \times Y_i)/H_i \to Y$. The map $\eta_i$ identifies $(U_i'/H_i) \subset (V_i \times Y_i)/H_i$ with the stratum $U_i \subset Y$, and it is étale in an open neighborhood of $U_i \cong (U_i'/H_i) \subset (V_i \times Y_i)/H_i$. By Corollary [A.12]i), we have

$$HP^*_{U_i}(U) \cong HP^*_{U_i}(U_i' \times Y_i)/H_i).$$

By Proposition [A.11]iii), the right-hand hand side is naturally isomorphic to

$$HP^*_{U_i}(U_i' \times Y_i)/H_i) \cong \left(HP^*_{U_i}(U_i' \times Y_i)\right)^{H_i}.$$ 

Hence, we obtain a canonical direct sum decomposition

$$HP^k(Y) \cong HP^k(U \setminus U_0) (U) \cong \bigoplus_i \left(HP^k_{U_i}(U_i' \times Y_i)\right)^{H_i}.$$ 

The Kunneth formula yields

$$HP^k_{U_i}(U_i' \times Y_i) \cong \bigoplus_{0 \leq l \leq k} HP^l(U_i') \otimes HP^{k-l}(Y_i).$$ 

Further, recall that by construction $U_i'$ is an open subset in $V_i$ whose complement has complex codimension $\geq 2$. In particular, $U_i'$ is smooth symplectic and connected. Therefore, $HP^l(U_i') = H^l(U_i', \mathbb{C})$; moreover, this group is 1-dimensional (and has trivial action of the group $H_i$) if $l = 0$, and is equal to zero if $l = 1, 2$. We conclude that the product decomposition [A.10] reduces to an $H_i$-equivariant isomorphism

$$HP^k_{U_i}(U_i' \times Y_i) \cong HP^k_0(Y_i) \otimes HP^0(U_i') \cong HP^k_0(Y_i), \quad k \leq 2.$$
Therefore for \( k = 1, 2 \) we have

\[
HP^k(Y) \cong \bigoplus_i HP^k(Y_i)^{H_i}.
\]

But for every \( i \geq 0 \), the complement \( Y_i \setminus \{o\} \cong (W_i \setminus \{0\})/G_i \) is smooth, symplectic, and satisfies \( H^k(Y_i \setminus \{0\}) \cong H^k(W_i \setminus \{0\})^{G_i} = 0 \), \( k = 1, 2, 3 \). Therefore \( HP^k(Y_i \setminus \{0\}) = 0 \) for \( k = 1, 2, 3 \), and we have an isomorphism

\[
HP^k(Y_i) \cong HP^k(Y_i).
\]

Collecting together all of the above, we conclude that for \( k = 1, 2 \) there exists a natural isomorphism

\[
HP^k(Y) \cong \bigoplus_i (HP^k(Y_i))^{H_i}.
\]

But we have already computed \( HP^k(Y_i), k = 1, 2 \) in Section 3. Lemma 3.1 immediately implies that \( HP^1(Y) = 0 \), and Lemma 3.2 shows that

\[
\dim HP^2(Y) = \sum_i \dim HP^2(Y_i)^{H_i} = \sum_i \dim C^H_i.
\]

By (4.1) the right-hand side is equal to \( \dim C \). Moreover, the natural grading on each of the \( HP^2(Y_i) \) has positive degrees. \( \square \)

This Proposition of course applies to any pair \( \langle V, G \rangle \). In particular, one can take the trivial group \( G = \{e\} \), and obtain, as expected, the equality \( HP^2(V) = 0 \). Taking the quotient by some non-trivial \( G \subset Sp(V) \) increases the second Poisson cohomology group and creates new deformations. These deformations are indeed new: they do not lift to \( G \)-equivariant deformations of the symplectic vector space \( V \). We do not formulate this fact precisely because we will not need it. However, in Section 5 we will need the following claim.

**Lemma 4.1.** Let \( \mathcal{Y}/S \) be a Poisson deformation of a symplectic quotient singularity \( V/G \) over a local Artin base \( S \). Assume that there exists a Poisson deformation \( V/S \) of the vector space \( V \) and a map \( \eta : V/S \to \mathcal{Y}/S \) which extends the quotient map \( \eta : V \to Y \). Then the Poisson cocycle \( \Theta_{\mathcal{Y}} \in HP^2(\mathcal{Y}/S) \) is trivial.

**Proof.** Assume that \( \Theta_{\mathcal{Y}} \neq 0 \). Let \( m \subset \mathbb{C}[S] \) be the maximal ideal, and let \( k \) be the largest integer such that \( \Theta_{\mathcal{Y}} = 0 \mod m^k \). Then \( \Theta_{\mathcal{Y}} \mod m^{k+1} \) is a non-trivial element in the Poisson cohomology group

\[
HP^2(\mathcal{Y}/S, m^k/m^{k+1}) \cong H^2(Y) \otimes_{\mathbb{C}} (m^k/m^{k+1})
\]

But the group \( HP^2(Y) \) admits a natural map \( \eta : H^2(Y) \to H^2(Y, \eta_\ast\mathcal{O}_V) \) which identifies it with the direct summand \( H^3(Y, \eta_\ast\mathcal{O}_V)^G \subset H^3(Y, \eta_\ast\mathcal{O}_V) \). Therefore \( \eta(\Theta_{\mathcal{Y}}) \) must be a non-trivial class in the group

\[
H^2(Y, \eta_\ast\mathcal{O}_V) \otimes_{\mathbb{C}} (m^k/m^{k+1})
\]

23
This is impossible. Indeed, by the functoriality of the Poisson cohomology, the class \( \eta(\Theta_Y) \) comes from the class \( \Theta_V \in HP^2(V) \) of the Poisson cocycle on the vector space \( V \), and the group \( HP^2(V) \) is a trivial group. □

In truth, under the assumptions of the Lemma the whole deformation \( Y/S \) must be trivial, not only its Poisson cocycle \( \Theta_Y \). But the proof of this fact is slightly harder. We settle for the weaker version to save space.

One final remark is the following: since the Poisson algebra \( \mathbb{C}[Y] \) is positively graded, Lemma A.15 immediately shows that all the results of this Section are valid not only for \( Y \), but also for its completion \( \hat{Y} \) at the origin \( o \in Y \).

5 Resolutions.

In this section we will prove Theorem 1.13 and Theorem 1.20.

Below, we use the notation \( H^i(M) = H^i(M, \mathbb{C}) \) for the singular cohomology of a variety \( M \) with complex coefficients. The homology of \( M \) is usually taken with rational coefficients, i.e., \( H_i(M) = H_i(M, \mathbb{Q}) \).

5.1 Geometry of the resolution. We will need several general facts on the geometry of symplectic resolutions. Most of them are well-known. The reader can find the proofs, for instance, in the papers [K1], [K3].

Let \( X \) be an irreducible smooth variety over \( \mathbb{C} \) equipped with a closed nowhere-degenerate 2-form \( \omega \), and let \( Y = X^{\text{aff}} \) be its affinization. Assume that the natural map \( \pi: X \to Y \) is projective and birational. By definition \( Y \) is normal, and we have \( \pi_*\mathcal{O}_X \cong \mathcal{O}_Y \). The canonical bundle \( K_X \) of the manifold \( X \) is trivial (it is trivialized by the top power of the symplectic form \( \omega \)). Therefore map \( \pi: X \to Y \) is one-to-one over the smooth locus \( Y_0 \subset Y \). By the Grauert-Riemenschneider Vanishing Theorem, we have

\[
H^i(X, \mathcal{O}_X) = H^0(Y, R^i\pi_*\mathcal{O}_X) = H^0(Y, R^i\pi_*K_X) = 0 \quad \text{for} \quad i \geq 1.
\]

for \( i \geq 1 \). Considering the exponential exact sequence on \( X \), one easily deduces that \( R^1\pi_*\mathcal{O}_X = 0 \), where \( \mathcal{O}_X \) is the constant sheaf on \( X \).

The symplectic form \( \omega \) on the smooth variety \( X \) induces a Poisson structure on the algebra \( \mathbb{C}[Y] \cong H^0(X, \mathcal{O}_X) \). Further, the resolution \( X \to Y \) is known, see e.g. [K2], to be semismall (that is, \( \dim X \times_Y X = \dim X \)). This implies that the Leray spectral sequence which computes the group \( H^2(X, \mathbb{Q}) \) degenerates, and we have a canonical short exact sequence of rational cohomology groups

\[
0 \longrightarrow H^2(Y, \mathbb{Q}) \longrightarrow H^2(X, \mathbb{Q}) \longrightarrow H^0(Y, R^2\pi_*\mathbb{Q}) \longrightarrow 0
\]

Set

\[
H_2(X/Y) := \ker(H_2(X, \mathbb{Q}) \to H_2(Y, \mathbb{Q}))
\]

the subgroup in the homology group \( H_2(X, \mathbb{Q}) \) dual to \( H^0(Y, R^2\pi_*\mathbb{Q}) \). The following fact will be very important.
Lemma 5.1. Let \([\omega] \in H^2(X, \mathbb{C})\) be the cohomology class of the symplectic form \(\omega\). Then for every homology class \(\alpha \in H_2(X/Y)\), we have \(\langle \alpha, [\omega] \rangle = 0\), and

\[
(5.3) \quad [\omega] = \pi^*[\omega]|_Y,
\]

for some cohomology class \([\omega]|_Y \in H^2(Y, \mathbb{C})\). Moreover, the class \([\omega]|_Y\) depends only on the Poisson structure on the variety \(Y\), not on the resolution \(X\).

Proof. This is not new, see e.g. \([CF]\), \([K3\, Lemma\, 2.2]\), \([WW]\). We only give a sketch of the proof. By (5.2), to establish (5.3) it suffices to prove that \([\omega]|_Y\) vanishes on all the fibers of the map \(Y \to X\). Taking a resolution of singularities, it suffices to prove that \(f^*[\omega] = 0\) for every map \(f : Z \to X\) from a smooth projective manifold \(Z\). By Hodge theory, this is equivalent to proving that \(f^*[\omega] = 0 \in H^0,2(Z) = H^2(Z, \mathcal{O}_Z)\), where \([\omega]\) is the complex-conjugate cohomology class. But already \([\omega]\) = 0, since \(H^2(X, \mathcal{O}_X) = 0\) by (5.1). To prove that the class \([\omega]|_Y\) is canonical, note that its restriction to the non-singular part \(U \subset Y\) is represented by an explicit 2-form \(\omega_U\) which is inverse to the non-degenerate Poisson bivector \(\Theta\). Thus \([\omega]|_Y|_U\) depends only on \(\Theta\). Moreover, we know that the form \(\omega_U\) extends to some, hence to an arbitrary smooth resolution \(X \to Y\) and represents a cohomology class \([\omega]|_Y \subset H^2(X, \mathbb{C})\) coming from \(H^2(Y, \mathbb{C})\). The class \([\omega]|_Y\) is completely determined by the form \(\omega_U\) since the map \(H^2(Y, \mathbb{C}) \to H^2(X, \mathbb{C})\) is injective. \(\square\)

In order to study deformations of the pair \(\langle X, \omega \rangle\) and introduce the period map. To do this, note that for every smooth deformation \(X/B\) over a local Artin base \(B\), the Gauss-Manin connection trivializes the relative de Rham cohomology \(H^2_{DR}(X/B)\),

\[
(5.4) \quad H^2_{DR}(X/B) \cong H^2(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[B].
\]

Therefore the class \([\omega]\) can be canonically considered as a class in the group \(H^2(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[B]\). This induces a map \(P : B \to H^2(X, \mathbb{C})\) called the period map. The fundamental theorem of \([KV]\) claims that the period map completely determines the deformation. More precisely, for every local Artin scheme \(B\) and a map \(P : B \to H^2(X, \mathbb{C})\) which sends the closed point \(o \in B\) to the class \([\omega] \in H^2(X, \mathbb{C})\), there exists a deformation \(X/B\) of the pair \(\langle X, \omega \rangle\) with the period map \(P\). Moreover, such a deformation \(X/B\) is unique up to a non-canonical isomorphism.

Set \(B := H^2(X, \mathbb{C})\), and let \(\hat{B}\) denote the formal neighborhood of \([\omega] \in H^2(X, \mathbb{C})\). This way, one obtains a universal deformation \(\hat{X}/\hat{B}\) of the pair \(\langle X, \omega \rangle\). Since one has to pass to the limit, the universal deformation \(\hat{X}/\hat{B}\) is only a formal scheme.
5.2 Globalizing the deformation. We can now start proving Theorem 1.13. Our method will be to extend the formal deformation from [KV] to an actual deformation defined over a global base.

Throughout the subsection, let \( X \) be a convex symplectic manifold with symplectic form \( \omega \). We write \( B := H^2(X, \mathbb{C}) \), and equip the vector space \( B \) with \( \mathbb{C}^* \)-action so that \( z \in \mathbb{C}^* \) acts via multiplication by \( z^{-l} \). Further, let \( \hat{B} \) denote the formal neighborhood of \( [\omega] \in B \), and let \( m \subset \mathbb{C}[\hat{B}] \) be the maximal ideal of the complete algebra \( \mathbb{C}[\hat{B}] \). Write \( \hat{X} / \hat{B} \) for the universal formal deformation of the pair \( (X, \omega) \).

Put \( Y := X^{\text{aff}} \). Assume that the Poisson algebra \( \mathbb{C}[Y] = H^0(X, \mathcal{O}_X) \) is a finitely-generated positively graded Poisson algebra of degree \( l > 0 \), and consider the corresponding \( \mathbb{C}^* \)-action on the scheme \( Y \). We split the proof of Theorem 1.13 into three Propositions below.

**Proposition 5.2.** The \( \mathbb{C}^* \)-action on \( Y \) lifts uniquely to a \( \mathbb{C}^* \)-action on \( X/Y \), and the resulting action on \( X \) extends to a \( \mathbb{C}^* \)-action on \( \hat{X}/\hat{B} \). Moreover, we have \( [\omega] = 0 \) in \( H^2(X) \).

This result is known, and we refer to [Fu], [K1], and [Ve] for the proofs. We also have the following infinitesimal version of the above proposition.

**Lemma 5.3.** Let \( Y \) be an irreducible algebraic variety, and let \( X \to Y \) be smooth projective semismall resolution of \( Y \). If the canonical bundle is trivial then every vector field \( \xi \) (a derivation of \( \mathcal{O}_Y \)) on \( Y \) lifts canonically to a vector field on \( X \).

**Proof.** (Compare [CF], where a similar statement is proved for isolated singularities, but without the semismallness assumption.) The vector field \( \xi \) canonically lifts to a vector field defined outside of the exceptional locus of the map \( \pi : X \to Y \). Since the variety \( X \) is smooth, the sheaf \( \mathcal{T}(X) \) of vector fields is reflexive. Therefore it suffices to prove that \( \xi \) to the generic point of an arbitrary exceptional Weil divisor \( E \subset X \). Since \( \pi : X \to Y \) is semismall, the image \( \pi(E) \subset Y \) of the divisor \( E \) is a subvariety of codimension 2. Therefore near the generic point of the subvariety \( \pi(E) \subset Y \), the variety \( Y \) is the product of a smooth variety and an isolated surface singularity, and it suffices to prove the Lemma in the case \( \dim Y = 2 \). In this case, the triviality of the canonical bundle implies that \( Y \) is a Du Val point. Then Lemma easily follows from an explicit construction of the minimal resolution as a transversal slice to a subregular nilpotent orbit, see Section [3].

**Proposition 5.4.** The formal scheme \( \hat{X} \) extends to an actual scheme \( \hat{X}/B \) defined over the whole affine space \( B = H^2(X) \). The map \( \sigma : \hat{X} \to H^2(X) \) is smooth and \( \mathbb{C}^* \)-equivariant, and the scheme \( \hat{X} \) is symplectic over \( B \). Moreover, the relative cohomology sheaves \( R^i \sigma_* \mathbb{Q} \) are constant sheaves on \( H^2(X) \), and the canonical base change morphism

\[
H^k(X_b, \mathbb{Q}) \to (R^k \pi_* \mathbb{Q})_b
\]
is an isomorphism for every point \( b \in B \).

**Proof.** To extend the formal scheme \( \hat{X}/\hat{B} \) to a scheme over the whole \( B \), we repeat the argument of [K3] Lemma 4.2. Take an ample line bundle \( L \) on \( X \) and note that, since by (5.1) \( H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0 \), the bundle \( L \) canonically extends to the formal scheme \( \hat{X}/\hat{B} \). Moreover, using the same cohomology vanishing it has been shown in [K3] that \( H^0(\hat{X}, \mathcal{O}_\hat{X}) \), the algebra of global functions on \( \hat{X} \), is a filtered algebra whose associated graded algebra is isomorphic to \( H^0(X, \mathcal{O}_X) \otimes \mathbb{C}[t] \), a Noetherian algebra. It follows that \( H^0(\hat{X}, \mathcal{O}_\hat{X}) \) is a Noetherian algebra, which is also complete with respect to the \( m \)-adic topology. Thus,

\[
\hat{Y} = \text{Spf}(H^0(\hat{X}, \mathcal{O}_\hat{X})),
\]

the formal spectrum, is an affine Noetherian formal scheme flat over \( \hat{B} \), and \( \hat{X} \) is projective over \( \hat{Y} \), with an ample line bundle \( L \). Therefore we can apply the Grothendieck algebraization theorem [EGA] III, Théorème 5.4.5] to \( \hat{X}/\hat{Y} \) and conclude that the formal scheme \( \hat{X} \) extends to an actual scheme \( X \) over \( \hat{Y} \) (and a posteriori, over \( B \)). Since the algebra \( \mathbb{C}[B] \) of functions on the vector space \( B = H^2(X) \) is positively graded, we are done by Lemma A.15. Moreover, by construction the scheme \( X/B \) is symplectic and smooth, and the map \( X \to B \) is \( \mathbb{C}^* \)-equivariant.

Let us now analyze the relative de Rham cohomology sheaves \( H^n_{DR}(\hat{X}/\hat{B}) \). For every \( k \geq 0 \), denote by \( B_k = \text{Spec} \mathcal{O}_\hat{B}/m^k \subset \hat{B} \) the \( k \)-th infinitesimal neighborhood of the special point in the local scheme \( \hat{B} \), and let \( x_k = \hat{X} \times \hat{B} B_k \). By [EGA] III, Théorème 4.1.5], for every \( p, q \geq 0 \) we have canonical isomorphisms

\[
H^q(\hat{X}, \Omega^p(\hat{X}/\hat{B})) \cong \lim_{\leftarrow} H^q(\hat{x}_k, \Omega^p(\hat{x}_k/B_k)),
\]

and the projective system on the right-hand side satisfies the Mittag-Leffler condition [EGA] 0, 13.1]. Moreover, for every \( k, n \) the de Rham cohomology module \( H^n_{DR}(\hat{x}_k/B_k) \) is a finitely-generated module over \( \mathbb{C}[B_k] \), an Artinian algebra. Therefore for every \( n \geq 0 \), the projective system \( H^n_{DR}(\hat{x}_k/B_k) \) also satisfies the Mittag-Leffler condition, and by [EGA] 0, Proposition 13.2.3] the maps (5.5) induce canonical isomorphisms \( H^n_{DR}(\hat{X}/\hat{B}) \cong \lim_{\leftarrow} H^n_{DR}(\hat{x}_k/B_k) \).

We conclude that for every \( n \geq 0 \), the \( \mathbb{C}[\hat{B}] \)-module \( H^n_{DR}(\hat{X}/\hat{B}) \) is finitely generated, complete and separated with respect to the \( m \)-adic topology. Since it carries a flat connection – namely, the Gauss-Manin connection – it must be a free \( \mathbb{C}[\hat{B}] \)-module.

By Lemma A.15 this implies that the relative de Rham cohomology sheaves \( H^n_{DR}(X/B) \) are also free sheaves of \( \mathbb{C}[B] \)-modules. Therefore the \( D \)-modules \( R^s\sigma_*\mathcal{O}_X \) are constant \( D \)-modules on \( B \). This means that the sheaves \( R^s\sigma_*\mathcal{O}_X \) are constant sheaves.

Finally, let \( b \in B \) be an arbitrary point with embedding \( i_b : b \to B \), and let \( \hat{x}_b \subset \hat{X} \) be the fiber of \( \hat{X} \) over \( b \), with embedding \( \hat{x}_b \hookrightarrow \hat{X} \) denoted by the same letter \( i_b \). Since the scheme \( X/B \) is smooth, we can apply Poincare
duality together with the Proper Base Change Theorem and obtain a canonical isomorphism

\[ H^q(X_b, \mathbb{Q}) \cong H^q(X, i_\ast^! \mathbb{Q})[\dim B] \cong i_\ast^! R^s \sigma_\ast \mathbb{Q}[\dim B]. \]

Since the sheaves \( R^s \sigma_\ast \mathbb{Q} \) are locally constant on \( B \), the right-hand side is isomorphic to \( (R^s \sigma_\ast \mathbb{Q})_b \).

\[ \square \]

Below, by ‘general enough’ or ‘generic’ point of \( B \) we mean a point from an appropriately chosen Zariski open dense subset of \( B \).

To finish the proof of Theorem 1.13, it remains to prove the following.

**Proposition 5.5.** The map \( \pi: X \to Y \) is semismall. Moreover, for a general enough point \( b \in B \), the induced map \( X_b \to Y_b \) of fibers over \( b \) in the varieties \( X/B \), resp., \( Y/B \), is an isomorphism.

**Proof.** (Compare [K3, Proposition 4.6].) To prove that \( \pi: X \to Y \) is semismall, it suffices to prove that for every closed subvariety \( Z \subset X \), the dimension of the generic fiber of the map \( Z \to \pi(Z) \) does not exceed the codimension \( \text{codim}(Z, X) \). Let \( Z = X \cap Z \). Since \( X \subset X \) is the fiber of a smooth map \( X \to H^2(X) \), we have \( \text{codim}(Z, X) \leq \text{codim}(Z, X) \). Since the map \( \pi: X \to Y \) is semismall, the dimension of the generic fiber of the map \( \pi: Z \to \pi(Z) \) does not exceed the codimension \( \text{codim}(Z, X) \). Therefore the same is true for the generic fiber of the map \( \pi: Z \to \pi(Z) \).

Let \( b \in B \). Then, since \( Y \) is normal, the variety \( Y_b \) is also normal. By construction, the map \( X_b \to Y_b \) is projective and generically one-to-one. The set of points \( b \in B \) such that the map \( X_b \to Y_b \) has fibers of dimension \( >1 \) is clearly a closed subset \( B_{BAD} \subset B \). Hence, to complete the proof of the Proposition it suffices to prove that \( B_{BAD} \) is a proper subset in \( B \).

To this end, we choose \( b \in B \) such that the cohomology class \( [\omega] = b \in H^2(X) = H^2(X, \mathbb{C}) \) does not annihilate any rational homology class \( Z \in H_2(X/Y) \subset H_2(X, \mathbb{Q}) \) (the latter obviously form a countable subset \( H_2(X, \mathbb{Q}) \subset H_2(X, \mathbb{C}) \)). We claim that \( b \notin B_{BAD} \), hence, \( B_{BAD} \neq B \). Indeed, if not, then there is a fiber of the map \( X_b \to Y_b \) that contains a projective curve \( Z \subset X_b \). The corresponding homology class \( [Z] \in H_2(X/Y) \subset H_2(X, \mathbb{Q}) \) annihilates \( \omega \), i.e., we have \( \langle [Z], [\omega] \rangle = 0 \), by Lemma 5.1. But that would contradict the choice of \( b \) which was made so that \( [\omega] \) does not annihilate any rational homology class \( [Z] \in H_2(X/Y) \). This completes the proof. \[ \square \]

**Remark 5.6.** We included the claim that \( X \to Y \) semismall because it was used in the proof of Lemma 5.3 (if applied to the map \( X \to Y \)); this case of the Lemma will be used at one point of the argument later, in Subsection 5.4.

We have completed the proof of Theorem 1.13.

**Proof of Proposition 1.14.** Assume that the graded Poisson algebra \( \mathbb{C}[Y] \) satisfies the assumptions of Theorem 1.10 so that we have the universal graded Poisson
deformation $\mathcal{Y}_S/S$. Let $\tau: B \to S$ be the classifying map of the deformation $\mathcal{Y}/B = \mathcal{X}^{\text{aff}}/B$. We have to prove that the map $\tau$ is a finite map onto an irreducible component of the variety $S$.

Let $b \in B = H^2(X)$ be a point sufficiently generic so that the map $\mathcal{X}_b \to \mathcal{Y}_b$ is an isomorphism. Let $S' \subset S$ be the irreducible component which contains the image $\tau(c)$ of the point $b \in B$.

Since $\mathcal{X}_b \to \mathcal{Y}_b$ is one-to-one, the Poisson scheme $\mathcal{Y}_b$ is in fact symplectic, and the Poisson deformation theory for $\mathcal{Y}_b$ reduces to the deformation theory of the pair $(\mathcal{X}_b = \mathcal{Y}_b, \omega)$. Since the deformations $\mathcal{X}/B$, $\mathcal{Y}/S$ are universal, this implies that the map $\tau: B \to S'$ is étale at the point $b$.

We see that the map $\tau: B \to S'$ is generically étale. To prove that it is in fact finite, it suffices to prove that all of its geometric fibers are finite. Moreover, since the map $\tau$ is $\mathbb{C}^*$-equivariant, it suffices to consider the fiber over the origin point $o \in S'$. We claim that this fiber in fact consists of the point $0$.

This implies $b = 0$ by the definition of the period map. \hfill \qed

5.3 Comparison with the Calogero-Moser deformation. We now turn to the proof of Theorem 1.20. Let $Y = V/G$ be a symplectic quotient singularity with a resolution $X$, let $\mathcal{X}/B$ be the deformation provided by Theorem 1.13 and let $\mathcal{M}/C$ be the Calogero-Moser deformation. Consider also the universal deformation $\mathcal{Y}/S$ of the Poisson scheme $Y$.

By construction, we have a multi-valued map $\kappa: C \to B$. As we have noted after stating Conjecture 1.19 the domain and the range of this map are in fact canonically isomorphic as vector spaces. If $\dim V = 2$, then the map $\kappa$ is single-valued and provides an isomorphism between $C$ and $B$.

It is natural to conjecture that the same is true in the general case: $\kappa: C \to B$ is single-valued and an isomorphism. It would immediately imply Theorem 1.20 Unfortunately, we were not able to prove it, and we have to settle for less. Namely, we consider the direct sum decomposition $C = \bigoplus_i B_i$, see (1.11), and the associated decomposition of the vector space $B = H^2(X, \mathbb{C}) \cong C$. As the main technical step in the proof of Theorem 1.20 we show that the map $\kappa: C \to B$ is compatible with these decompositions.

**Proposition 5.7.** Let $c \in B_i$ be a point in the subspace $B_i \subset C$ of the base of the Calogero-Moser deformation $\mathcal{M}/C$. Let $\kappa: C \to S$ be the classifying map of the deformation $\mathcal{M}/C$, and let $b \in B$ be an arbitrary point lying over $\kappa(c) \in S$.

(i) The cohomology class $[\omega]_b \subset B \cong H^2(X, \mathbb{C})$ of the symplectic form $\omega \in \Omega^2(\mathcal{X}_b)$ lies in the subspace $B_i \subset B$.

(ii) Assume that the point $c \in B_i$ is generic. Then a rational homology class $\alpha \in H_2(X, \mathbb{Q})$ satisfies $\langle \alpha, [\omega]_b \rangle = 0$ if and only if $\alpha$ is orthogonal to the whole subspace $B_i \subset B$. 

29
By definition of the period map, Proposition 5.7(i) means that the point $b \in B$ lies in $B_i$. Proposition 5.7(ii) means that for a generic $c \in B_i$, the point $b \in B_i$ is generic.

The proof of Proposition 5.7 is rather technical. We will give it in the next subsection. But first, we will deduce Theorem 1.20 from the proposition.

**Proof of Theorem 1.20.** Let $C^{\text{good}} \subset C$ be the set of points such that the claim of the Theorem holds for $c$. This is clearly a Zariski open subset in $C$. Hence, we must only show that $C^{\text{good}}$ is nonempty.

To this end, for every rational homology class $0 \neq \varphi \in H_2(X, \mathbb{Q})$, denote by $C_{\varphi} \subset C$ the subset of elements $c \in C$ such that

- For some point $b \in B$ lying over $\kappa(c) \in S$, the class $\varphi \in H_2(X, \mathbb{Q}) \cong H_2(\mathfrak{X}_b, \mathbb{Q}) = H_2(\mathfrak{X}_b)$ lies in the subgroup $H_2(\mathfrak{X}_b/\mathcal{Y}_{\kappa(c)}) \subset H_2(\mathfrak{X}_b, \mathbb{Q})$.

All the subsets $C_{\varphi} \subset C$ are closed algebraic subvarieties.

Assume first that there exists $\varphi \in H_2(X, \mathbb{Q}) \setminus \{0\}$, such that $C_{\varphi} = C$. Then by definition, for any $c \in C$, the class $\varphi$ belongs to the subgroup $H_2(\mathfrak{X}_b/\mathcal{Y}_{\kappa(c)}) \subset H_2(\mathfrak{X}_b, \mathbb{Q})$. It follows, by the first claim of Lemma 5.1, that $\varphi$ is orthogonal to $[\omega]_b$, for all elements $c \in C$. In particular, this applies to generic elements in $B_i \subset C$ for each $i \geq 1$. By Proposition 5.7, we deduce that $\varphi \in H_2(X, \mathbb{Q})$ is orthogonal to all the subspaces $B_i \subset C$. Thus, $\varphi = 0$, contradicting the assumption $\varphi \neq 0$.

The contradiction implies that $C_{\varphi} \neq C$, for any nonzero class $\varphi \in H_2(X, \mathbb{Q})$. Since there is only a countable set of homology classes $\varphi \in H_2(X, \mathbb{Q})$, it follows that there exists $c \in C$ which is not contained in $C_{\varphi}$, for any $\varphi$. It follows that, for such a $c$, no nonzero element in $H_2(X, \mathbb{Q})$ belongs to $H_2(\mathfrak{X}_b/\mathcal{Y}_{\kappa(c)})$ for any $b$ lying over $\kappa(c)$. This means $H_2(\mathfrak{X}_b/\mathcal{Y}_{\kappa(c)}) = 0$. We deduce, similarly to the argument in the proof of Proposition 5.7, that the fibers of $\mathfrak{X}_b \to \mathcal{Y}_{\kappa(c)}$ do not contain projective curves. As in the proof of Theorem 1.13, this implies that the map $\mathfrak{X}_b \to \mathcal{Y}_{\kappa(c)}$ is an isomorphism. Thus, $c \in C^{\text{good}}$, and we are done. □

### 5.4 Restriction to strata and the end of the proof

It remains to prove Proposition 5.7. To do this, we need some information on the Calogero-Moser deformation $\mathcal{M}/C$ and on the geometry of the resolution $X/Y$.

Somewhat surprisingly, we need to know very little about the Calogero-Moser deformation – it suffices to know how it behaves with respect to the change of the subgroup $G \subset \text{Sp}(V)$. The precise statement is as follows. Let $G_1 \subset G$ be a subgroup, let $Y_1 = V/G_1$ be the associated symplectic quotient singularity, and let $\mathcal{M}_1/C_1$ be its Calogero-Moser deformation with its base $C_1$. We have a canonical restriction map $C \to C_1$. Denote by $B_1 \subset C_1$ its image, and consider the splitting $\iota : B_1 \to C$ of the projection $C \to B_1$ defined by

$$\iota(c)(g) = 0 \text{ unless } g \in G \text{ is conjugate to an element } g_1 \in G_1.$$
Proposition 5.8. The canonical projection $\eta : Y' \to Y$ extends to a commutative diagram

\[
\begin{array}{ccc}
M_1 \times_{C_1} B_1 & \xrightarrow{\eta} & M \\
\downarrow & & \downarrow \\
B_1 & \xrightarrow{\iota} & C
\end{array}
\]

(5.6)

where $M_1 \times_{C_1} B_1$ is the restriction of the Calogero-Moser deformation $M_1/C_1$ to $B_1 \subset C_1$. □

Proposition 5.8 immediately follows from the definition of the Calogero-Moser deformation, see [EG]. We will apply it to the terms in the direct sum decomposition (4.1).

Remark 5.9. (Added on Feb. 16, 2010.) As G. Bellamy kindly indicated to us, the above Proposition is not contained in [EG], nor does it follow directly from the definition, and in fact it is not clear if the fact is true or not. Fortunately, in his beautiful recent paper [arXiv:1001.0239] I. Losev has proved that if one chooses a point $v \in V$, and lets $G_1 \subset G$ be the stabilizer of this point, then a slightly stronger statement becomes true after completing both sides near $v$. This is Theorem 1.2.1 of [arXiv:1001.0239]. Losev’s Theorem is sufficient to save our proof of Proposition 5.7, the only place where Proposition 5.8 is used.

Recall (see Section 4) that the subspaces $B_i \subset C$ correspond to codimension-2 strata $U_i$ in the natural stratification of the quotient variety $Y = V/G$. Every stratum $U_i$ is of the form $U_i = V_0^i / G'_i$, where $V_i \subset V$ is a symplectic vector subspace of codimension 2, $G'_i \subset G$ is the subgroup of elements which preserve $V_i \subset V$, and $V_0^i \subset V_i$ is the open subset of elements with minimal possible stabilizer. This stabilizer is a subgroup $G_i \subset G'_i \subset G$. All elements of the subgroup $G_i$ are symplectic reflections. They act trivially on the subspace $V_i \subset V$. This induces a natural action of the group $G_i$ on the 2-dimensional quotient $W_i = V/V_i$. The Calogero-Moser deformation of the quotient $V/G_i$ is simply the product of the Calogero-Moser deformation of the quotient $Y_i = W_i / G_i$ and the symplectic vector space $V_i$. Its base $C_i$ is the space of $G_i$-invariant $\mathbb{C}$-valued functions on the set of all non-trivial elements $g \in G_i$. The image $B_i \subset C_i$ of the restriction map $C \to C_i$ is the subspace

$$B_i = C_i^{H_i} \subset C_i$$

of functions invariant with respect to the natural action of the group $H_i = G'_i / G_i$. By Proposition 5.8, for every stratum $U_i$ we have a commutative diagram (5.6), where the scheme in the left top corner is the product $M_i \times V_i$ of the Calogero-Moser deformation of the quotient $Y_i$ and the symplectic vector space $V_i$. The Calogero-Moser deformation $M_i$ is the standard deformation described in Section 3.
These are all the properties of the Calogero-Moser deformation that we will need. Proposition \ref{prop:5.7} hence also Theorem \ref{thm:1.20} will hold for every Poisson deformation \( \mathcal{M}/C \) of the quotient singularity \( Y = V/G \) which admits a diagram \ref{diag:5.10} for every \( G_i \subset G \), with \( \mathcal{M}' \cong \mathcal{M}_i \times V_i 
olimits \).

Aside from Proposition \ref{prop:5.8}, the decomposition \ref{diag:4.1} has a very clear geometric interpretation in terms of the isomorphism \( H^2(X, \mathbb{C}) \cong B \) provided by the generalized McKay correspondence \cite{K2}. Namely, let \( y \in U_i \) be an arbitrary point in the stratum \( U_i \subset Y \). The map \( V_i \times U_i \rightarrow Y \) induced by the projection \( V_i \rightarrow Y \) is étale at \( y \). Therefore the formal neighborhood \( \widehat{Y}_y \) of the point \( y \in Y \) naturally decomposes

\[(5.7) \quad \widehat{Y}_y = \widehat{Y}_i \times \widehat{V}_i, \quad \text{product in the sense of formal schemes,} \]

where \( \widehat{Y}_i \) is the completion of the quotient \( Y_i = W_i/G_i \) at the origin \( o \in Y_i \), and \( \widehat{V}_i \) is the completion of the vector space \( V_i \) at \( 0 \in V_i \). Let \( \widehat{X}_i \) be the completion of the variety \( X \) at the fiber \( \pi^{-1}(y) \subset X \). Then by \cite{K1} Proposition 5.2 the decomposition \ref{diag:5.7} lifts to the formal scheme \( \widehat{X}_y \). Namely, we have a canonical decomposition

\[ \widehat{X}_y = \widehat{X}_i \times \widehat{V}_i, \]

where \( \widehat{X}_i \) is the completion of a smooth symplectic resolution \( X_i/Y_i \) of the 2-dimensional quotient \( Y_i \) at the exceptional divisor \( E \subset X_i \) of the map \( X_i \rightarrow Y_i \). The cohomology space \( H^2(\widehat{X}_i) \cong H^2(X_i) \cong H^2(E) \) is naturally identified with the space \( C_i \), and the natural map \( B \rightarrow C_i \) is given by the restriction map \( H^2(X, \mathbb{C}) \rightarrow H^2(E, \mathbb{C}) \).

**Proof of Proposition \ref{prop:5.7}**. We begin with (i). Consider the restriction \( \mathcal{M}/B_i \) of the Calogero-Moser deformation \( \mathcal{M}/C \) to the subspace \( B_i \subset C \). Let \( \eta : \mathcal{M}_i \times V_i \rightarrow \mathcal{M}/B_i \) be the projection provided by Proposition \ref{prop:5.8}. It suffices to prove that \( [\omega_i] \) maps to 0 under every projection \( B \rightarrow B_j \) with \( j \neq i \).

Choose an arbitrary point \( y \in U_j \) and consider the completion \( \widehat{Y}_y \) together with the induced deformation \( \widehat{\mathcal{M}}_{y}/\widehat{B}_i \) (here \( \widehat{B}_i \) is the completion of the space \( B_i \) at \( 0 \in B_i \)). Since \( j \neq i \), there exists a point \( y' \in Y_i \times V_i \) such that \( \eta(y') = y \), the map \( \eta \) is étale at \( y' \), and the scheme \( Y_i \times V_i \) is smooth at \( y' \in Y_i \times V_i \). Then the completion \( \widehat{(Y_i \times V_i)}_{y'} \) of \( Y_i \times V_i \) at \( y' \) is isomorphic to the completion \( \widehat{V} \) of the vector space \( V \) at 0, and the projection

\[ \widehat{V} \cong (Y_i \times V_i)_{y'} \rightarrow \widehat{Y}_y \cong \widehat{Y}_i \times \widehat{V}_i \]

is induced by the quotient map \( V \rightarrow Y_j \times V_j \cong V/G_j \). By Lemma \ref{lem:4.1} this implies that the Poisson cocycle \( \Theta \in HP^2(\widehat{\mathcal{M}}_{y}/\widehat{B}_i) \) of the Poisson scheme \( \widehat{\mathcal{M}}_{y}/\widehat{B}_i \) is a coboundary. In other words, we have \( \Theta = d\xi \) for some vertical vector field \( \xi \) on \( \widehat{\mathcal{M}}_{y}/\widehat{B}_i \).

Now replace, if necessary, the base \( B_i \) of the deformation \( \mathcal{M}/B_i \) with a finite cover \( B'_i \), and consider the resolution \( \pi : \mathcal{X} \rightarrow \mathcal{M}/B'_i \) provided by Theorem \ref{thm:1.13}. Let \( \mathcal{X}_y \) be its completion at the fiber \( \pi^{-1}(y) \subset \mathcal{X} \). The map \( B \rightarrow B_j \subset C_j \) is given by the restriction from \( B \cong H^2(X, \mathbb{C}) \) to \( H^2(X_j, \mathbb{C}) \cong B_j \) and factors
through the restriction to $H^2(\mathfrak{X}_b, \mathbb{C})$. By Lemma 5.3, the vector field \( \xi \) extends to a vertical vector field on \( \mathfrak{X}_b / \mathfrak{X}_b' \). Since \( \Theta = d\xi \), the Cartan homotopy formula gives \( \omega = d(\omega \cdot \theta) \), where \( \omega \in \Omega^2(\mathfrak{X}_b / \mathfrak{X}_b') \) is the relative symplectic form. This implies that \( [\omega] = 0 \) tautologically on \( \mathfrak{X}_b \). Therefore the same is true on the whole variety \( \mathfrak{X} / \mathfrak{X}_b \) and for every fiber \( \mathfrak{X}_b \).

This proves (i). To prove (ii), denote by \( C^i_\perp \subset H_2(X, \mathbb{Q}) \) the orthogonal to the subspace \( B_i \subset B \cong H_2(X, \mathbb{C}) \), and denote by \( \omega^\perp \subset H_2(X, \mathbb{Q}) \) the orthogonal to the cohomology class \( [\omega]_b \in H^2(X, \mathbb{C}) \). We have to prove that the embedding \( C^i_\perp \subset \omega^\perp \) is in fact an equality. To prove this, it suffices to show that

\[
\dim \omega^\perp = \dim C^i_\perp = \dim B - \dim B_i.
\]

To prove this inequality, it suffices to exhibit a \( \mathbb{Q} \)-vector space subspace \( P \) of dimension \( \dim P = \dim B_i \) and a map \( f : P \to H_2(X, \mathbb{Q}) \) such that the pairing with \( [\omega]_b \) induces an embedding \( P \to \mathbb{C} \). We claim that we can take

\[
P = (H_2(X_i, \mathbb{Q}))^{H_i} = (H_2(E_i, \mathbb{Q}))^{H_i},
\]

and \( f = \eta_* : H_2(E_i, \mathbb{Q}) \to H_2(X, \mathbb{Q}) \).

Indeed, by Lemma 5.3, the class \( [\omega]_b \in H^2(\mathfrak{X}_b, \mathbb{C}) \) comes from a class \( [\omega] \in H^2(\mathcal{M}_c, \mathbb{C}) \). Moreover, the class \( [\omega] \in H^2(\mathcal{M}_c, \mathbb{C}) \) is canonical. In particular, the restriction \( \eta^* [\omega] \in H^2((\mathcal{M}_c)_c \times V_i, \mathbb{C}) \) coincides with the class \( [\omega]_c \otimes 1 \), where \( [\omega]_c \in H^2((\mathcal{M}_c)_c, \mathbb{C}) \) is the class corresponding to the Calogero-Moser deformation \( \mathcal{M}_c \). But the variety \( Y_i \) is 2-dimensional. Therefore the Calogero-Moser deformation \( \mathcal{M}_c \) comes from the universal symplectic deformation of the resolution \( X_i \), and the class \( [\omega]_c \in C_i \cong H^2(X_i, \mathbb{C}) \) coincides with \( c \in B_i = C^i_{\perp} \subset C_i \). Thus we have to show that for a generic element \( c \in C^i_{\perp} \cong P^* \otimes \mathbb{C} \), the pairing with \( c \) induces an embedding \( P \to \mathbb{C} \). This is clear.

\[\square\]

6 Applications of Hochschild cohomology.

The primary goal of this section is to prove Theorem 1.18. We begin however with some general results that relate Hochschild cohomology to orbifold cohomology.

Let \( A \) be an associative algebra. Recall the Gerstenhaber bracket \( [-,-] : HH^p(A) \times HH^q(A) \to HH^{p+q-1}(A) \) on Hochschild cohomology (see [Lo] and §7 below). Given a Hochschild cocycle \( \Theta \in HH^2(A) \) such that \( [\Theta, \Theta] = 0 \), we introduce a twisted Hochschild cohomology algebra of \( A \) as follows.

**Definition 6.1.** For \( \Theta \in HH^2(A) \) such that \( [\Theta, \Theta] = 0 \), define \( HH^*_{\Theta}(A) \) to be the cohomology of the complex \( (HH^*(A), d_{\Theta}) \) where the differential \( d_{\Theta} : HH^*(A) \to HH^{*+1}(A) \) is given by \( d_{\Theta}(a) = [\Theta, a] \).

Let \( A = \mathbb{C}[M] \) be the coordinate ring of a smooth affine algebraic variety \( M \). Then we have \( HH^*(A) = \Gamma(M, \Lambda^* \mathcal{O}_M) \), by Hochschild-Kostant-Rosenberg
theorem, cf. [Lo]. Furthermore, the Gerstenhaber bracket on $HH'(A)$ reduces, in this case, to the Schouten bracket on polyvector fields, cf. [A, 17]. Thus, the cocycle $\Theta \in HH^2(A)$ may be viewed as a bivector. The equation $[\Theta, \Theta] = 0$ says that this bivector gives a Poisson structure on $A$. Moreover, according to (A.17) of the Appendix, there is a natural isomorphism $HH_\Theta^2(A) \cong HP'(A)$. Note that if $M$ is not smooth, both sides in the isomorphism are still well-defined, but they are not necessarily isomorphic any more.

Assume that the smooth affine variety $M$ is a symplectic manifold. We let the 2-cocycle $\Theta$ be the bivector on $M$ corresponding to (the inverse of) the symplectic 2-form. In such a case, the Poisson cohomology reduces to De Rham cohomology, see [Br] and section A.7 below, hence we obtain graded algebra isomorphisms

$$HH_\Theta^*(\mathbb{C}[M]) \cong HP^*(\mathbb{C}[M]) \cong H^*(M, \mathbb{C}).$$

Assume next that a finite group $G$ acts on $M$, a smooth affine symplectic variety, by symplectic automorphisms. The action on $M$ induces one on $\mathbb{C}[M]$, and we form the cross-product algebra

$$A := \mathbb{C}[M] \# G.$$

The bivector corresponding to (the inverse of) the symplectic form on $M$ is $G$-invariant, and the class $\Theta \in HH^2(\mathbb{C}[M]) = \Gamma(M, \Lambda^2 T_M)$ gives rise, by the standard deformation theory, to a $G$-equivariant first-order infinitesimal deformation of the associative algebra $\mathbb{C}[M]$. Moreover, the equation $[\Theta, \Theta] = 0$ insures that this first-order deformation may be extended (not uniquely) to a $G$-equivariant second order deformation. The latter gives rise, via the cross-product construction, to a second order deformation of $\mathbb{C}[M] \# G$, hence, to a class $\widehat{\Theta} \in HH^2(A) = HH^2(\mathbb{C}[M] \# G)$, such that $[\widehat{\Theta}, \Theta] = 0$. We form the corresponding twisted Hochschild cohomology algebra $HH^*_\Theta(\mathbb{C}[M] \# G)$.

On the other hand, associated with the $G$-action on $M$ one has the orbifold cohomology algebra $H^*_{\text{orb}}(M; G)$, see (1.1). We have

**Proposition 6.2.** Given a symplectic $G$-action on an affine symplectic manifold, there is a natural graded algebra isomorphism

$$HH^*_\Theta(\mathbb{C}[M] \# G) \cong H^*_{\text{orb}}(M; G) \left( = \left( \bigoplus_{g \in G} H^* - \dim M^g (M^g) \right)^G \right).$$

**Proof.** The action of the finite group $G$ on $\mathbb{C}[M]$ being semisimple, the Hochschild cohomology of the cross-product may be expressed in terms of Ext-groups of $\mathbb{C}[M]$-bimodules as follows, cf. [AFLS]:

$$HH^*(\mathbb{C}[M] \times G) = \text{Ext}_{\text{\mathbb{C}[M]-bimod}}^* \left( \mathbb{C}[M], \mathbb{C}[M] \# G \right)^G$$

$$= \left( \bigoplus_{g \in G} \text{Ext}_{\mathbb{C}[M] \times M^g \text{-mod}}^* \left( \mathbb{C}[M], \mathbb{C}[M] \cdot g \right) \right)^G.$$

(6.2)
Here $G$-invariants are taken with respect to the adjoint $G$-action, and in the rightmost term of the formula we identify $\mathbb{C}[M]$-bimodules with $\mathbb{C}[M \times M]$-modules. Thus $\mathbb{C}[M] \cdot g$ stands for the $\mathbb{C}[M \times M]$-module arising from the coordinate ring of the graph-subvariety $\text{Graph}(g : M \to M) \subset M \times M$.

In general, let $W$ be a smooth affine variety containing two smooth (closed) subvarieties $E, F \subset W$, such that the intersection $E \cap F$ is clean. The latter means that $E \cap F$ is smooth and that the equation $T_{E \cap F} = T_{E|_{E \cap F}} \cap T_{F|_{E \cap F}}$ holds for the corresponding tangent bundles. In such a case, we put

$$d := \dim W - \dim E - \dim F + \dim(E \cap F).$$

A standard argument based on Koszul complexes shows that the group $\text{Ext}^i_{\mathbb{C}[W]}(\mathbb{C}[E], \mathbb{C}[F])$ vanishes for all $i < d$, and for $i \geq d$ we have

$$\text{Ext}^i_{\mathbb{C}[W]}(\mathbb{C}[E], \mathbb{C}[F]) = \Gamma \left( E \cap F, \Lambda^{i-d} \left( \frac{T_W|_{E \cap F}}{T_{E|_{E \cap F}} + T_{F|_{E \cap F}}} \right) \right) \otimes \det \left( \frac{T_E|_{E \cap F} + T_{F|_{E \cap F}}} {T_{E|_{E \cap F}}} \right),$$

where $\det(\ldots)$ denotes the top wedge power.

For any $g \in G$, there is a natural $g$-action on $T_M|_{M^g}$ by vector bundle endomorphisms, and we have a canonical direct sum decomposition

$$T_M|_{M^g} = \text{Image}(\text{id} - g) \bigoplus T_{M^g},$$

where $\text{Image}(\text{id} - g)$ denotes the subbundle (of locally constant rank) formed by the images of the fiberwise action of the operator $\text{id} - g$.

Writing $E \subset M \times M$ for the diagonal, we get $E \cap \text{Graph}(g) \cong M^g$. We deduce canonical isomorphisms

$$\frac{T_E|_{M^g} + T_{\text{Graph}(g)}|_{M^g}} {T_E|_{M^g}} \cong \text{Image}(\text{id} - g), \quad \text{and,}$$

$$\frac{T_{M \times M}|_{M^g} + T_{\text{Graph}(g)}|_{M^g}} {T_E|_{M^g} + T_{\text{Graph}(g)}|_{M^g}} \cong (T_M|_{M^g})/\text{Image}(\text{id} - g) \cong T_{M^g}.$$

Thus, $M^g$ is a symplectic submanifold and $d = \dim M^g$. Further, the symplectic form restricts to a non-degenerate 2-form on the fibers of the vector bundle $\text{Image}(\text{id} - g)$, and this gives canonical trivializations $\det(\text{Image}(\text{id} - g)) \cong \mathbb{C}$. Being canonical, the trivializations are compatible with $G$-action (that permutes the fixed point sets $M^g$ for various elements $g$). Thus, from (6.2) and (6.3) we obtain

$$\text{HH}^i(\mathbb{C}[M] \times G) \cong \left( \bigoplus_{g \in G} \Gamma(M^g, \Lambda^{i-\dim M^g} T_{M^g}) \right)^G.$$

Further, the symplectic form on $M^g$ induces an vector bundle isomorphism $T_{M^g} \cong (T_{M^g})^*$, hence a graded algebra isomorphism $\Gamma(M^g, \Lambda^* T_{M^g}) \cong \Omega^*(M^g)$. Combining the formulas above, we obtain

$$\text{HH}^i(\mathbb{C}[M] \times G) \cong \left( \bigoplus_{g \in G} \Omega^{i-\dim M^g}(M^g) \right)^G.$$
Observe next that the cochain $\tilde{\Theta} \in HH^2(C[M] \times G)$ is given, essentially, by the bivector corresponding to the symplectic 2-form. Therefore, it is easy to see (as e.g. in [Br]), that the differential $[\Theta, -]$ on Hochschild cohomology gets transported under the isomorphism (6.4) to the direct sum of the de Rham differentials $d: \Omega^{-\dim M^g}(M^g) \to \Omega^{1+\dim M^g}(M^g)$. The cohomology of the latter is nothing but $H^{-\dim M^g}(M^g, \mathbb{C})$, the singular cohomology of $M^g$ (up to shift).

This establishes the isomorphism of the Proposition. Compatibility of the isomorphism with the algebra structures is more difficult (it involves Kontsevich’s theorem, see [Ko] §8.4, on the cup-product on tangent cohomology), and it will not be given here. Below, see (6.5), we will only use a very special case of Proposition 6.2 where such a compatibility is immediate from definitions. □

For the rest of this section, assume $M = V$ a symplectic vector space, and $G \subset Sp(V)$, a finite group, so that the de Rham cohomology of each fixed point set $V^g$ is trivial in all degrees but zero. Thus, writing $C(k)$ for a 1-dimensional graded vector space concentrated in degree $k$, the orbifold cohomology algebra in the RHS of the isomorphism of Proposition 6.2 reads

$$H^*_\text{orb}(V; G) \cong \left( \bigoplus_{g \in G} C(\dim V^g) \right)^G \cong (\text{gr}_F^* C[G])^G \cong \text{gr}_F^* (ZG),$$

where the associated graded algebra is taken with respect to the filtration $F_i(C[G])$ considered in §1.

In [EG], the authors construct a certain deformation $H_{t,c}$ of the cross-product algebra $H_{0,0} := C[V] \# G$, which is parametrized by an affine line with coordinate $t$ and the space $C \cong \text{gr}_F^* (ZG)$ of $G$-invariant functions on the set on symplectic reflections in $G$. When $t = 0$, the algebra $H_{0,c}$ has a large center $\mathbb{Z}$c. The Calogero-Moser space $\mathcal{M}_c$ is obtained by taking its spectrum, $\mathcal{M}_c = \text{Spec} \mathbb{Z}_c$. The deformation in the $t$-direction induces a Poisson structure on the algebra $\mathbb{Z}_c$ and on the variety $\mathcal{M}_c$. When, on the other hand, $t$ is generic, the center of the algebra $H_{t,c}$ is the one-dimensional $\mathbb{C}$-vector space spanned by the unit element.

In [EG], the authors consider the Hochschild cohomology $HH^*(H_{t,c})$. They construct a canonical map $\chi: \text{gr}_F^* (ZG) \to HH^*(H_{t,c})$, which is shown to be an isomorphism for generic $t$. Moreover, the map $\chi$ is compatible with the deformation $H_{t,c}$, i.e., for every pair $(t_0, c_0)$, the composite map:

$$C \xrightarrow{\sim} \text{gr}_2^* (ZG) \xrightarrow{\chi} HH^2(H_{t_0, c_0})$$

is the Kodaira- Spencer map for the family $H_{t,c}$ near the point $(t_0, c_0)$. In particular, if one fixes a generic enough $t = t_0$, then the family $H_{t_0,c}/C$ is the universal deformation of the associative algebra $H_{t_0,c}$.

When $t = 0$, the map $\chi$ is still defined, but its image no longer generates the Hochschild cohomology groups $HH^*(H_{0,c})$. In fact, these groups become infinite-dimensional as $\mathbb{C}$-vector spaces.

Now, fix a $c \in C$, and consider $H_{t,c}$ as a family depending on $t$. Applying the Kodaira-Spencer map at the point $t = 0$, we obtain a cohomology class $\Theta_c \in
Lemma 6.3. The canonical map \( \chi : \text{gr}^F(ZG) \rightarrow H^\ast(H_{0,c}) \) descends to a map \( \chi : \text{gr}^F(ZG) \rightarrow H^\ast_{\Theta_c}(H_{0,c}) \), and the latter map is an isomorphism.

Proof. The map \( \chi : \text{gr}^F(ZG) \rightarrow H^\ast(H_{t,c}) \) is defined for all \((t,c)\). It follows that, for \(t = 0\), the image of this map commutes with \(\Theta_0\), hence, the map \(\chi\) induces a well-defined map \(\text{gr}^F(ZG) \rightarrow H^\ast_{\Theta_0}(H_{0,c})\).

First, consider the case \(c = 0\) where \(H_{0,0} = \mathbb{C}[V]\#G\). Applying Proposition 6.2 and using (6.5), we obtain a graded algebra isomorphism \(H^\ast_{\Theta_0}(H_{0,0}) \cong \text{gr}(ZG)\), in particular, all odd twisted Hochschild cohomology groups of \(H_{0,0}\) vanish. Moreover, a calculation carried out in [Al] for a Weyl algebra instead of the polynomial algebra \(\mathbb{C}[V]\) shows that the isomorphisms above is the inverse of the map \(\chi : \text{gr}^F(ZG) \rightarrow H^\ast_{\Theta_0}(H_{0,0})\) considered in [EG]. This completes the proof in the special case: \(c = 0\).

To complete the proof in the general case recall from [EG] that, for any \(c\), the algebra \(H_{0,c}\) comes equipped with a canonical increasing filtration such that the associated graded algebra \(\text{gr}H_{0,c}\) is isomorphic to the algebra \(H_{0,0}\), see [EG]. This filtration induces a filtration on the twisted Hochschild complex which is compatible with the differential \([\Theta, -]\), and therefore gives rise to a spectral sequence

\[
    H^\ast_{\Theta_0}(H_{0,0}) = E_1^{\ast\ast} \quad \Rightarrow \quad E_\infty^{\ast\ast} = \text{gr}^r(H^\ast_{\Theta_0}(H_{0,c})).
\]

Since \(H_{0,0}\) has no odd twisted Hochschild cohomology, all the differentials in this spectral sequence vanish, and we have: \(E_1^{\ast\ast} = E_\infty^{\ast\ast}\). Thus, it follows from the case \(c = 0\) of the Lemma that the map \(\chi : \text{gr}^F(ZG) \rightarrow \text{gr}(H^\ast_{\Theta_0}(H_{0,c}))\) is a bijection. Therefore, for any \(c\), the map \(\chi : \text{gr}^F(ZG) \rightarrow H^\ast_{\Theta_c}(H_{0,c})\) is also a bijection, and the Lemma is proved.

We now apply the above result assuming in addition that \(c \in C\) is such that the Calogero-Moser space \(\mathcal{M}_c\) is smooth.

Lemma 6.4. Let \(c \in C\) be such that the Calogero-Moser space \(\mathcal{M}_c = \text{Spec} \mathcal{B}_c\) is smooth. Then the Calogero-Moser family \(\mathcal{M}/C\) considered over the formal neighborhood of the point \(c\) gives a universal Poisson deformation of the Poisson variety \(\mathcal{M}_c\).

Proof. It is proved in [EG] that whenever \(\mathcal{M}_c = \text{Spec} \mathcal{B}_c\) is smooth, the algebra \(H_{0,c}\) is Morita-equivalent to \(\mathcal{B}_c\). Therefore we have \(H^\ast(H_{0,c}) \cong H^\ast(\mathcal{B}_c)\) and \(H^\ast_{\Theta_0}(H_{0,c}) \cong HH^\ast_{\Theta_0}(\mathcal{B}_c)\). Using the smoothness of \(\mathcal{M}_c\) again, we conclude, see (A.17), that the right-hand side is isomorphic to \(HP^\ast(\mathcal{B}_c)\). Combining this with Lemma 6.3, we see that the Kodaira-Spencer map for the family \(\mathcal{M}/C\) computed at the point \(c \in C\) induces an isomorphism \(C \cong HP^2(\mathcal{B}_c)\).

We can now prove Theorem 1.18 using the gradings and the dimension estimate obtained in Section 4.
Proof of Theorem 1.18. Denote $S' = HP^2(\mathbb{C}[Y])$, and let $S \subset S'$ be the base of the universal graded Poisson deformation of the graded Poisson algebra $\mathbb{C}[Y] = \mathbb{C}[V]^G$. By Lemma 6.2, the classifying map $\kappa : C \to S$ of the Calogero-Moser family $\mathcal{M}/C$ is étale at a generic point $c \in C$. In particular, its differential

$$\text{d} \kappa : T_cC \to T_{\kappa(c)}S'$$

is injective. Since, $\dim S' \leq n = \dim C$, this differential is also surjective, and $S = S'$.

It remains to prove that the map $\kappa : C \to S = S'$ is surjective. The Calogero-Moser deformation $\mathcal{M}/C$ is graded, with grading given by assigning degree 2 to every element in the space $C$. Therefore the map $\kappa : C \to S \subset S'$ is compatible with the gradings. Taking quotients with respect to $\mathbb{C}^*$, we obtain a rational map $\overline{\kappa} : P(C) \dashrightarrow P(S')$, where $P(C) \cong \mathbb{P}^{n-1}$ is the projectivization of $C$, and $P(S')$ is the projectivization of the vector space $S'$ (with its grading, whatever it may be). Its image $\overline{\kappa}(P(C)) \subset P(S')$ is a closed subvariety. Since $\kappa : C \to S = S'$ is generically étale, the image $\overline{\kappa}(P(C)) \subset P(S)$ coincides with the whole $P(S)$. \qed

7 Appendix.

A.1 Harrison cohomology. Consider an arbitrary vector space $A$ over $\mathbb{C}$ (or, more generally, over an arbitrary field of characteristic 0). Let $L_*A$ be the free graded Lie coalgebra generated by the vector space $A$ placed in degree $-1$, so that $L_1A = A$ and $L_2A = S^2A$, the symmetric square of $A$. Denote by $DL^*(A)$ the graded Lie algebra of coderivations of the Lie coalgebra $L_*A$. Since $L_*$ is free, we have $DL^k(A) \cong \text{Hom}(L_kA, A)$. In particular, every map $m : S^2A \to A$ extends to a coderivation $d \in DL^1$. The following is easily checked by a direct computation.

Lemma A.1. A commutative product $m : S^2A \to A$ is associative if and only if the corresponding coderivation $m \in DL^1$ satisfies $\{m, m\} = 0 : L_2A \to A$. \qed

Assume that $A$ is a commutative associative algebra. Then we can apply the Lemma and obtain a canonical coderivation $m \in DL^1(A)$ satisfying $\{m, m\} = 0$. Setting $a \mapsto \{m, a\}$ defines a differential $d : DL^*(A) \to DL^{*+1}(A)$ and turns $DL^*(A)$ into a DG Lie algebra. The differential $d : DL^0(A) \to DL^1(A) \cong \text{Hom}(S^2A, A)$ can be explicitly described in the following way:

(A.1) \hspace{1cm} (df)(a \otimes b) = f(ab) − af(b) − bf(a) , \hspace{0.5cm} a \otimes b \in S^2A.

The spaces $DL^*(A) = \text{Hom}(L_*A, A)$ carry natural $A$-module structure — $A$ acts on the target space $A$. It is easy to check that $d : DL^*(A) \to DL^{*+1}(A)$ is an $A$-module map. Moreover, we have

$$DL^k(A) \cong \text{Hom}(L_kA, A) \cong \text{Hom}_A(L_kA \otimes A, A),$$

38
and the the differential $d$ is dual to an $A$-module map $d : L_{*+1} \otimes A \to L_* A \otimes A$. The complex $(L_* \otimes A, d)$ is denoted by $\text{Har}_*(A)$ and called the Harrison complex of the commutative associative algebra $A$. Its first two terms are

\[
\begin{array}{cccc}
\text{S}^2 A & \otimes A & \xrightarrow{d} & A \otimes A & \xrightarrow{d} & 0 \\
\end{array}
\]

with the differential given by $d : ab \otimes c \mapsto a \otimes bc + b \otimes ac - ab \otimes c$. It is well-known that the Harrison homology complex is quasiisomorphic to the so-called cotangent complex $\Omega_*(A)$ of the algebra $A$.\footnote{In degree 0, this is essentially the standard representation of the Kähler differentials module $\Omega^1 A$ as the quotient of the diagonal ideal $I \subset A \otimes A$ by its square.}

### A.2 Poisson cohomology.

Let $A$ be a unital commutative $\mathbb{C}$-algebra with a Poisson bracket $\{-, -\} : \Lambda^2_*(A) \to A$ that satisfies the Leibniz formula (1.4), the Jacobi identity (1.5), and such that $\{1, a\} = 0$, $\forall a$.

Let $P_*(A)$ be the free graded Poisson coalgebra generated by the vector space $L_*(A)$ placed in degree $-1$ (here ‘free’ means that our coalgebra is a universal object in the category of graded Poisson coalgebras). It is easy to see, cf. e.g. [FT], that the coalgebra $P_*(A)$ may be constructed as the free super-symmetric (co)algebra on the vector space $L_*(A)$ (= free graded Lie coalgebra generated by $A$). In more details, for each $m \geq 0$, one has a canonical decomposition

\[
(A.2) \quad P_m(A) = \bigoplus_{p+q=m} P_{p,q} \quad \text{where} \quad P_{*,k} = \Lambda^k(L_*(A)), \quad k = 0, 1, \ldots.
\]

This gives a bigrading $P(A) = \bigoplus P_{p,q}$, $P_{p,q} = P_{p,q}(A)$ such that $P_{*,1}(A) = L_*(A)$. The comultiplication in $P_*(A)$ preserves the bigrading, while the co-bracket is of bidegree $(0, 1)$.

Denote by $DP^{*,*}(A)$ the graded Lie algebra of coderivations of the Poisson coalgebra $P_*$. For reasons of convenience, we will shift the bigrading on $DP^{*,*}(A)$ by $(1, 0)$, so that the first non-trivial term is $DP^{0,0}(A)$. Since $P_*$ is free, we have

\[
DP^k(A) \cong \text{Hom}(P_k(A), A), \quad k \geq 0.
\]

Moreover, (A.2) gives an identification

\[
(A.3) \quad DP^{k,*}(A) \cong \text{Hom}(P_{k,*}(A), A) \cong \text{Hom}(\Lambda^k(DL^*)(A)), A).
\]

In particular, we have $DP^{0,2}(A) = \text{Hom}(\Lambda^2(A), A)$, the space of all skew-commutative binary operations on $A$.

Assume now that $A$ is an associative commutative algebra. Then by definition, we have the Harrison complex $\text{Har}_*(A)$ of flat (in fact, free) $A$-modules. The identification (A.3) can be re-written as

\[
(A.4) \quad DP^{*,k}(A) \cong \text{Hom}_A(\Lambda^k_*(\text{Har}_*(A)), A), \quad k \geq 0,
\]

and the differential in the Harrison complex $\text{Har}_*(A)$ induces a differential $d : DP^{*,*}(A) \to DP^{*,*+1}(A)$. This turns the graded Lie algebra $DP^*(A)$ into a DG Lie algebra.
Lemma A.2.

(i) A skew-linear operation \(\{−, −\} : \Lambda^2(A) \to A\) satisfies the Leibnitz rule (1.4) if and only if for the corresponding element \(\Theta \in DP^{0,2}(A) \cong \text{Hom}(\Lambda^2(A), A)\) we have \(d\Theta = 0\).

(ii) The operation \(\{−, −\} : \Lambda^2(A) \to A\) satisfies the Jacobi identity (1.5) if and only if \([\Theta, \Theta] = 0\).

(iii) Let \(1 \in A \cong DP^{0,0}(A)\) be the unit element. We have \(\{−, 1\} = 0\) if and only if \([\Theta, 1] = 0\).

Proof. The first claim immediately follows from (A.1). To prove (ii), note that \(DP^{q,0}(A) = \text{Hom}(\Lambda^q(A), A)\) is in fact the graded Lie algebra of coderivations of the free skew-commutative coalgebra \(\Lambda^k(A)\) generated by \(A\). Therefore \([\Theta, \Theta] = 0\) if and only if the associated map \(\{-, -\} : \Lambda^2(A) \to A\) extends to a coderivation \(\delta : \Lambda^q+1(A) \to \Lambda^q(A)\) satisfying \(\delta \circ \delta = 0\). It is well-known (and easily checked) that the latter is equivalent to the Jacobi identity (1.5) on the operation \(\{-, -\} : \Lambda^2(A) \to A\). Finally, (iii) is clear. □

By Lemma A.2, giving a Poisson structure on a commutative associative algebra \(A\) is equivalent to giving an element \(\Theta \in DP^{0,2}(A)\) such that \(d\Theta = 0\), \([\Theta, \Theta] = 0\), \([\Theta, 1] = 0\).

We will call the element \(\Theta \in DP^2(A)\) the Poisson cochain corresponding to the Poisson structure on \(A\). For every Poisson algebra \(A\), the map

\[
\delta : DP^{*,*}(A) \to DP^{*,*+1}(A) \quad , \quad a \mapsto \delta(a) := [\Theta, a]
\]

satisfies \(\delta \circ \delta = 0\).

Definition A.3. The Poisson cohomology \(HP^*(A)\) of a Poisson algebra \(A\) is the cohomology of the total complex \((DP^*, d + \delta)\) associated to the Poisson bi-complex \(DP^{*,*}(A)\) with differentials \(d : DP^{*,*}(A) \to DP^{*,*+1}(A)\) and \(\delta : DP^{*,*}(A) \to DP^{*,*+1}(A)\).

Here are the first few terms in the bicomplex \(DP^{*,*}(A)\).

\[
\begin{array}{ccc}
A & \xrightarrow{\delta} & \text{Hom}(A, A) \\
\downarrow & & \downarrow \\
\text{Hom}(S^2A, A) & \xrightarrow{\delta} & \text{Hom}(S^2A \otimes A, A) \\
\downarrow & & \downarrow \\
\text{Hom}(S^2A \otimes A, A) & \xrightarrow{\delta} & \text{Hom}(A, A)
\end{array}
\]

40
Here the leftmost vertical column is the Harrison complex $\text{Hom}(L,A,A)$, the 2-d vertical column is formed by graded pieces of $\Lambda^2(\text{Hom}(L,A,A))$, etc., and the bottom row is the standard cochain complex for $A$ viewed as a Lie algebra with respect to the Poisson bracket. In particular we get

1. $HP^0(A) = \{ z \in A \mid \{z,a\} = 0, \forall a \in A \}$ = Poisson center of $A$.
2. $HP^1(A) = ZP^1/BP^1$ where $ZP^1, BP^1 \subset \text{Hom}(A, A)$ are defined by
   
   \[ ZP^1 := \left\{ f \mid f(ab) = af(b) + f(a) b, f(\{a,b\}) = \{f(a), b\} + \{a, f(b)\}, \forall a, b \right\} \]
   and
   
   \[ BP^1 := \left\{ f = f_c : a \mapsto \{c,a\} \mid c \in A \right\}. \]

   Next, let $\varphi \in \text{Hom}(S^2A, A)$ and $\psi \in \text{Hom}(A^2A, A)$. Define two $C$-bilinear maps $A \otimes A \to A[\varepsilon]/(\varepsilon^2)$ by the formulas

   \[ (A.7) \quad a, b \mapsto a \cdot \varepsilon b = a \cdot b + \varepsilon \cdot \varphi(a, b), \quad a, b \mapsto \{a, b\}_\varepsilon = \{a, b\} + \varepsilon \cdot \psi(a, b) \]

   It is straightforward to verify that

   (iii) $\varphi \oplus \psi$ is a Poisson 2-cocycle if and only if formulas (A.7) give a 1-st order infinitesimal deformation of $A$ as a Poisson algebra. Furthermore, the group $HP^2(A)$ classifies 1-st order infinitesimal deformations up to equivalence.

**Remark A.4.** The Poisson bracket gives a canonical cocycle $\Theta \in DP^2(A)$, which may or may not be trivial in Poisson cohomology, depending on the algebra $A$.

### A.3 Relation to Hochschild cohomology and the Küneth formula.

It is well-known that the Poisson operad may be viewed as a ‘degeneration’ of the Associative operad. In this way, the Poisson cohomology bicomplex of a Poisson algebra may be viewed as a ‘degeneration’ of the Hochschild cochain complex.

To explain the analogy, recall that, given a vector space $A$, one defines three graded Lie algebras $DL^*(A)$, $DP^*(A)$, and $DT^*(A)$ as the Lie algebras of coderivations of the free Lie coalgebra $L_*(A)$, free Poisson coalgebra $P_*(A)$ or free associative coalgebra (with counit) $T_*(A)$, respectively. As in the case of $DP^*(A)$, we shift the grading on $DT^*(A)$ by one, so that

\[ DT^k(A) \cong \text{Hom}(T_k(A), A) = \text{Hom}(A^\otimes k, A), \quad k \geq 0. \]

Under this identification, the Lie bracket in $DT^*(A)$ becomes the Gerstenhaber bracket given, for every $f \in DT^k(A)$, $g \in DT^l(A)$, by the standard formula

\[ [f, g](a_1 \otimes \cdots \otimes a_{k+l-1}) = \]

\[ \sum_{1 \leq i \leq l} (-1)^i g(a_1 \otimes \cdots \otimes f(a_i \otimes \cdots \otimes a_{i+k-1}) \otimes \cdots \otimes a_{k+l-1}) - \sum_{1 \leq i \leq k} (-1)^i f(a_1 \otimes \cdots \otimes g(a_i \otimes \cdots \otimes a_{i+l-1}) \otimes \cdots \otimes a_{k+l-1}). \]
Recall that the free associative coalgebra $T_*(A)$ is known to be isomorphic to the universal enveloping coalgebra of the free Lie coalgebra $L_*(A)$. This means that there is a canonical decreasing filtration on the associative coalgebra $T_*(A)$, such that the corresponding associated graded $gr_1 T_*(A)$ is isomorphic, by Poincaré-Birkhoff-Witt theorem, to the free super-symmetric coalgebra on the vector space $L_*(A)$. As we have mentioned earlier, this free super-symmetric coalgebra is nothing but the free Poisson coalgebra $P_* (A)$ generated by $A$. In other words, there is a canonical Poisson coalgebra (bigraded) isomorphism

$$\text{gr}_1 T_*(A) \cong P_*(A).$$

Further, the decreasing filtration on $T_*(A)$ induces an increasing filtration on the graded Lie algebra $DT^*(A)$ which we call the PBW filtration. The associated graded $gr_1 DT^*(A)$ acts on the associated graded $gr_1 T_*(A) \cong P_*(A)$ by Poisson coderivations; therefore we have a Lie algebra map $gr_1 DT^*(A) \rightarrow DP^{***}(A)$. It is easy to check that this map is an isomorphism. A similar relation between $DT^*(A)$ and $DP^{***}(A)$, formulated in terms of eulerian idempotents, was considered in [Lo].

For every vector space $A$ we have $DT^1(A) \cong \text{Hom}(A \otimes A, A)$. An element $m \in \text{Hom}(A \otimes A, A)$ satisfies $[m, m] = 0$ if and only if the corresponding binary operation $A \otimes A \rightarrow A$ is associative. In this case, one defines a differential $DT^*(A) \rightarrow DT^{*+1}(A)$ and obtains the Hochschild cochain complex $DT^*(A)$. Its cohomology groups are called the Hochschild cohomology and denoted by $HH^*(A)$. If the associative algebra $A$ is commutative, then the Hochschild differential $DT^*(A) \rightarrow DT^{*+1}(A)$ preserves the PBW filtration. The complex $gr_1 DT^*(A)$ with induced differential is then isomorphic to the Poisson cohomology bicomplex $HP^*(A)$ of the Poisson algebra $A$, which is the commutative algebra $A$ equipped with trivial Poisson bracket $\{ -, - \} = 0 : \Lambda^2(A) \rightarrow A$. Note that the PBW filtration on the complex $DT^*(A)$ gives rise to a PBW filtration $F^\text{PBW} HH^*(A)$ on the Hochschild cohomology.

We will use the relationship between Hochschild and Poisson complexes to prove the Künneth formula for Poisson cohomology. To do this, we recall that for every two associative unitary algebras $A, B$, there exists a canonical quasi-isomorphism

$$\text{sh} : DT^*(A \otimes B) \cong DT^*(A) \otimes DT^*(B),$$

given by the shuffle product, see [Lo] §4.2. In more detail, consider the category $\Delta$ of finite linearly ordered sets, and denote by $[n] \in \Delta$, $n \geq 1$ the set of cardinality $n$. By an $(l, n)$-shuffle we will understand an order-preserving embedding $\Phi : [l] \rightarrow [n]$. Given an $(l, k + l)$-shuffle $\Phi$, one defines the complementary $(k, k + l)$-shuffle $\overline{\Phi} : [k] \rightarrow [k + l]$. Taken together, $\Phi$ and $\overline{\Phi}$ define a permutation $\sigma_{\Phi} : [l] \cup [k] \rightarrow [k + l]$ of the set of $k + l$ elements. For every shuffle
We define a map $\Phi : [l] \to [k + l]$, we define a map $sh_\Phi : A \otimes l \otimes B \otimes k \to (A \otimes B)^{\otimes k+l}$ by

$$sh_\Phi(a_1 \otimes \cdots \otimes a_l \otimes b_1 \otimes \cdots \otimes b_k) = c_1 \otimes \cdots \otimes c_{k+l},$$

where

$$c_i = \begin{cases} a_i \otimes 1, & i = \Phi(p), p \in [l], \\ 1 \otimes b_i, & i = \Phi(q), p \in [k]. \end{cases}$$

The map $sh : DT^{k+l}(A \otimes B) \to DT^k(A) \otimes DT^k(B) \cong \text{Hom}(A^{\otimes l} \otimes B^{\otimes k}, A \otimes B)$ is given by

$$sh(f)(a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_l) = \sum_{\Phi : [l] \to [k+l]} \text{sign}(\sigma_\Phi) f(sh_\Phi(a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_l))$$

for every $f \in DT^{k+l}(A \otimes B) = \text{Hom}(A^{\otimes l} \otimes B^{\otimes k}, A \otimes B)$.

In addition to the quasiisomorphism $sh$, we also have an embedding $\kappa_A : DT^*(A) \to DT^*(A \otimes B)$ given by

$$\kappa_A(f)(a_1 \otimes b_1 \otimes \cdots \otimes a_k \otimes b_k) = f(a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_k), \quad f \in DT^k(A),$$

and an analogously defined embedding $\kappa_B : DT^*(B) \to DT^*(A \otimes B)$. Looking at the formula (A.8) for the Gerstenhaber bracket, we immediately see that both $\kappa_A$ and $\kappa_B$ are Lie algebra maps. Moreover, say that a cochain $\omega \in DT^k(A)$ is reduced if $f(a_1 \otimes \cdots \otimes a_k) = 0$ whenever at least one of the elements $a_1, \ldots, a_k \in \mathfrak{g}$ is equal to 1. Then one can easily derive from (A.8) and (A.10) that for every reduced cochain $\omega \in DT^*(A)$ and an arbitrary cochain $f \in DT^*(A \otimes B)$ we have

$$sh([\kappa_A \omega, f]) = [\omega, sh(f)]$$

(here the bracket on the right-hand side acts on the first factor in $DP^*(A) \otimes DP^*(B)$). The same statement holds for $A$ replaced by $B$.

If the algebras $A$ and $B$ are commutative, then the quasiisomorphism (A.10) is compatible with the PBW filtrations. The associated graded map

$$sh : DP^*(A) \otimes DP^*(B) \to DP^*(A \otimes B)$$

is also a quasiisomorphism; it is induced by the standard quasiisomorphism

$$(\Omega^*(A \otimes B) \oplus (A \otimes \Omega^*(B)) \cong \Omega^*(A \otimes B)$$

between cotangent complexes.

**Lemma A.5.** For every two Poisson algebras $A$, $B$, the shuffle map (A.10) induces a natural quasiisomorphism

$$sh : DP^*(A \otimes B) \cong DP^*(A) \otimes DP^*(B).$$
Proof. It suffices to prove that the map $\text{sh}$ is compatible with the differentials: since the associated graded map (A.12) is a quasiisomorphism, $\text{sh} : DP^*(A \otimes B) \cong DP^*(A) \otimes DP^*(B)$ will be a quasiisomorphism as well. By definition, the differential in the Poisson cohomology complex is the sum of two differentials, which we denoted by $d$ and $\delta$. Compatibility with $d$ is contained in (A.12). Thus we have to prove that
\[
\text{sh}(\delta_{A \otimes B} f) = (\delta_A \otimes \text{id} + \text{id} \otimes \delta_B) \text{sh}(f)
\]
for every cochain $f \in DP^*(A \otimes B)$. But by definition, the Poisson cochain $\Theta_{A \otimes B}$ is of the form
\[
\Theta_{A \otimes B} = \kappa_A \Theta_A + \kappa_B \Theta_B.
\]
Since both Poisson cochains $\Theta_A, \Theta_B$ are clearly reduced, we can apply (A.11) and obtain
\[
\text{sh}([\Theta_{A \otimes B}, f]) = [\Theta_A, \text{sh}(f)] + [\Theta_B, \text{sh}(f)].
\]
By (A.6), this is precisely what we had to prove. \hfill \square

A.4 Graded algebras. As in Subsection 1.3 by a graded Poisson algebra of degree $l$ we will understand a Poisson algebra $A$ equipped with a grading such that the multiplication in $A$ is compatible with the grading, and the Poisson bracket is of degree $-l$. For a graded Poisson algebra $A$ of degree $l \neq 0$, the Poisson cocycle $\Theta \in DP^2(A)$ is canonically a coundary. Indeed, the grading $A = \bigoplus A^*$ induces a canonical derivation $\xi : A \to A$ by setting $\xi = k \text{id}$ on $A^k$. The formula $c \mapsto [\xi, c]$ extends this derivation to a derivation of the Lie algebra $DP^{*,*}(A)$. Since the Poisson bracket is of degree $-l$ with respect to the grading, we have $[\xi, \Theta] = -l \cdot \Theta$, which can be rewritten as $\Theta = -\frac{1}{l} d\xi$.

The grading on $A$ induces an additional grading on the Poisson cohomology bicomplex $DP^{*,*}(A)$, called the $A$-grading. The differential $d : DP^{*,*}(A) \to DP^{*,*+1}(A)$ preserves the $A$-grading, and the differential $\delta : DP^{*,*}(A) \to DP^{*,*+1}(A)$ shifts it by $l$. It will be convenient to redefine the $A$-grading by shifting it by $(k-1)l$ on $DP^{k,*,*}(A)$, so that it is preserved by both differentials $d, \delta$. We obtain a Poisson cohomology graded bicomplex $DP^{*,*}(A)$. Note that after the shift, the Poisson cochain $\Theta \in DP^{2,0}(A)$ has $A$-degree 0, and the $A$-grading becomes compatible with the Lie bracket on $DP^{*,*}(A)$.

A.5 Modules and étale descent. Let $A$ be a Poisson algebra. By a Poisson module $M$ over $A$ we will understand an $A$-module $M$ equipped with an additional operation $\{\cdot, \cdot\} : A \otimes M \to M$ such that
\[
\{ab, m\} = a\{b, m\} + b\{a, m\}, \quad \{a, bm\} = \{a, b\}m + b\{a, m\},
\]
\[
\{a, b, m\} = \{a, \{b, m\}\} - \{b, \{a, m\}\}.
\]
Given a Poisson module $M$ one defines, following [Fr], its Poisson cohomology bicomplex $DP^{*,*}(A, M)$ (with coefficients in $M$) by
\[
DP^{p,q}(A, M) = \text{Hom}(P_{p,q}(A), M), \quad p, q \geq 0,
\]

44
To get the differentials, it is convenient to treat $\hat{A} := A \oplus M$ as a Poisson algebra by letting both operations to be zero on $M \subset \hat{A}$ (the trivial square-zero extension). We grade this algebra by assigning $\deg A = 0$, $\deg M = 1$, so that $\hat{A}$ becomes a graded Poisson algebra of degree $l = 0$. The differentials on $DP^{\ast,\ast}(\hat{A})$ preserve the induced grading, and $DP^{\ast,\ast}(A, M)$ is nothing but the graded component in $DP^{\ast,\ast}(\hat{A})$ of minimal degree.

In the special case $M = A$, we get back the original Poisson cohomology bicomplex $DP^{\ast,\ast}(A)$. Indeed, the algebra $\hat{A} \cong A[\varepsilon]/(\varepsilon^2)$ is simply the truncated polynomial algebra over $A$. The bicomplex $DP^{\ast,\ast}(A, M)$ naturally lies in the degree-0 part of $DP^{\ast,\ast}(\hat{A})$, and it is easy to check that, for $M = A$, multiplication by $\varepsilon$ provides an isomorphism $DP^{\ast,\ast}(A, M) \cong DP^{\ast,\ast}(A).

Cohomology with coefficients appear naturally if one wants to consider the functoriality properties of Poisson cohomology groups. Let $f : A \to B$ be a Poisson map between Poisson algebras. Then there is no natural map between the groups $HP^\ast(A)$, $HP^\ast(B)$ – just as there is no natural map between the spaces of derivations of the algebras $A$, $B$. However, there exist obvious natural maps $f : DP^{\ast,\ast}(A) \to DP^{\ast,\ast}(A, f_\ast B)$ and $df : DP^{\ast,\ast}(B) \to DP^{\ast,\ast}(A, f_\ast B)$. If $\Theta_A$, $\Theta_B$ are the Poisson cochains of the algebra $A$, $B$, then we have $f(\Theta_A) = df(\Theta_B)$.

More generally, for every Poisson module $M$ over $B$, there exists a natural map $f : DP^{\ast,\ast}(B, M) \to DP^{\ast,\ast}(A, f_\ast M).

The main reason we want to consider functoriality is the following compatibility result.

**Lemma A.6.** Let $A \to B$ be an étale map between Poisson algebras, and let $M$ be a Poisson module over the algebra $B$. Then the natural map

$$f : DP^{k,\ast}(B, M) \to DP^{k,\ast}(A, f_\ast M)$$

induced by the map $A \to B$ is a quasiisomorphism for every $k \geq 0$. Consequently, the induced map $DP^{\ast}(B, M) \to DP^{\ast}(A, f_\ast M)$ of total complexes is a quasiisomorphism.

**Proof.** By (A.13), it suffices to prove that the canonical map

$$\text{Har}_\ast(A) \otimes B \to \text{Har}_\ast(B)$$

of Harrison complexes is a quasiisomorphism – in other words, that the pullback $\Omega_\ast(A) \otimes B$ of the cotangent complex $\Omega_\ast(A)$ of the algebra $A$ is naturally quasiisomorphic to the cotangent complex $\Omega_\ast(B)$ of the algebra $B$. This is very well-known.

Notice that for every Poisson algebra $A$ and every multiplicative system $S \subset A$, the localization $A[S^{-1}]$ carries a natural Poisson structure. The natural map $A \to A[S^{-1}]$ is an example of a Poisson étale map. This example will be used later in constructing Poisson cohomology complexes of Poisson schemes.

---

4In degree 0, this is the definition of an étale map.
A.6 Poisson schemes. We define a Poisson scheme as a scheme $X$ equipped with a Poisson bracket on the structure sheaf $\mathcal{O}_X$. For any Poisson algebra $A$, the affine scheme $X = \text{Spec } A$ is a Poisson scheme. Poisson morphisms between Poisson schemes, sheaves of Poisson $\mathcal{O}_X$-modules and complexes of such sheaves are defined in the natural way. For every Poisson module $M$ over a Poisson algebra $A$, the affine scheme $X = \text{Spec } A$ is a Poisson scheme. Poisson morphisms between Poisson schemes, sheaves of Poisson $\mathcal{O}_X$-modules and complexes of such sheaves are defined in the natural way. For every Poisson morphism $f : X \to Y$ and a sheaf $\mathcal{F}$ of Poisson $\mathcal{O}_X$-modules on $X$, the direct image sheaf $f_* \mathcal{F}$ is a sheaf of Poisson $\mathcal{O}_Y$-modules. The same is true for the higher direct image sheaves $R^p f_* \mathcal{F}$, $p \geq 1$, and for the whole complex $R^q f_* \mathcal{F}$.

The construction of the Poisson cohomology complex $DP^q(A)$ generalizes straightforwardly to the scheme case. Namely, we notice that the functors $P_k$, $k \geq 0$ from the category of vector spaces into itself can be easily defined in an arbitrary symmetric tensor category with the unit object. The terms $DP^k(A) = \text{Hom}(P_k(A), A)$ of the Poisson cohomology complex can be defined in any symmetric monoidal category which admits internal $\text{Hom}$’s. Moreover, the differential in $DP^q(A)$ for a Poisson algebra $A$ is defined essentially by linear algebra, and the definition also works just as well in the general categorical setting. The same is true for the cohomology complex with coefficients $HP^q(A, M)$.

We use this and define the Poisson cohomology complex $HP^q(X, \mathcal{F})$ of the Poisson scheme $X$ with coefficients in an injective sheaf $\mathcal{F}$ of Poisson $\mathcal{O}_X$-modules by setting

$$HP^q(X, \mathcal{F}) = \text{Hom}(P_k(\mathcal{O}_X), \mathcal{F})$$

where all tensor products and the $\text{Hom}$ are taken in the category of Zariski sheaves of groups on $X$. This complex carries the same combinatorial structures as in the case of algebras, in particular, the PBW filtration.

This definition makes sense for an arbitrary sheaf $\mathcal{F}$ of Poisson modules – in particular, for the structure sheaf $\mathcal{O}_X$ – but it gives the wrong result: the resulting functor does not have reasonable exactness properties, and it becomes impossible to prove a crucial compatibility condition (Proposition A.10 below). To get the correct definition, we use canonical injective resolutions. Namely, let $\overline{X}$ be the union of all scheme-theoretic points of $X$, and let $j : \overline{X} \to X$ be the canonical map. Recall that for every sheaf $\mathcal{F}$ of abelian groups on $X$, the canonical map $\mathcal{F} \to j_* j^* \mathcal{F}$ is an embedding, and the sheaf $j_* j^* \mathcal{F}$ is injective. Iterating this construction, one obtains the so-called canonical Godement resolution $\mathcal{F}^*$ of the sheaf $\mathcal{F}$. If $\mathcal{F}$ is a sheaf of Poisson $\mathcal{O}_X$-modules, then all the terms in the Godement resolution carry canonical structures of Poisson $\mathcal{O}_X$-modules.

**Definition A.7.** Let $\mathcal{F}$ be a sheaf of Poisson $\mathcal{O}_X$-modules on a Poisson scheme $X$. The local Poisson cohomology complex $HP^*(X, \mathcal{F})$ is the total complex of the bicomplex

$$HP^*(X, \mathcal{F}^*) = \text{Hom}(P_k(\mathcal{O}_X), \mathcal{F}^*)$$

where $\mathcal{F}^*$ is the canonical Godement resolution of the sheaf $\mathcal{F}$. The Poisson cohomology

$$HP^*(X, \mathcal{F}) = \mathbb{H}^*(X, HP^*(X, \mathcal{F}))$$
with coefficients in $\mathcal{F}$ is the hypercohomology of the local Poisson cohomology complex $\mathcal{H}P^*(X, \mathcal{F})$.

We note that the functors $P_q^*(\mathcal{O}_X) = P_{q*}(X)$ carry an additional grading, so that in fact the local Poisson cohomology complex $\mathcal{H}P^q(X, \mathcal{F}) = \mathcal{H}P^{q*}(X, \mathcal{F})$ is a bicomplex. For every $k \geq 0$, we have a quasiisomorphism

$$\mathcal{H}P_k^q(X, \mathcal{F}) \cong \mathcal{R}Hom^q(\Lambda^k \Omega(X), \mathcal{F}),$$

where $\Omega(X)$ is the cotangent complex of the scheme $X$, and $\Lambda^k$ is understood in the derived-category sense. Moreover, the complex $\mathcal{H}P_k^q(X, \mathcal{F})$ is concentrated in degrees $\geq 0$. Therefore we obtain a canonical embedding

$$\text{Hom}(\Lambda^k \Omega(X), \mathcal{O}_X) \to \mathcal{H}P_k^q(X, \mathcal{F}),$$

where $\Omega(X)$ is the cotangent sheaf.

Since the Godement resolution is functorial, Definition A.7 immediately generalizes to the case when $\mathcal{F}$ is itself not a single sheaf of Poisson $\mathcal{O}_X$-modules, but a complex of such sheaves. Then it descends to a triangulated functor on the corresponding derived category. In the case when $\mathcal{F} = \mathcal{O}_X$ is the structure sheaf, we will denote the local Poisson cohomology complex $\mathcal{H}P^*(X, \mathcal{O}_X)$ simply by $\mathcal{H}P^*(X)$, and we will denote the Poisson cohomology $\mathcal{H}P^*(X, \mathcal{O}_X)$ by $\mathcal{H}P^*(X)$.

Another important situation is the following. Let $Z \subset X$ be a closed subscheme in a Poisson scheme $X$, and let $U \subset X$ be the open complement. Let $i : Z \hookrightarrow X$, $j : U \hookrightarrow X$ be the embeddings. Then there exists a canonical map $\mathcal{O}_X^* \to j_* \mathcal{O}_U^*$, where $\mathcal{O}_X^*$ is the Godement resolution of the sheaf $\mathcal{O}_U$, and $\mathcal{O}_U^*$ is the Godement resolution of the sheaf $\mathcal{O}_U$. This map is compatible with the natural Poisson structures on both sides. Therefore we also have a Poisson module structure on its cone, denoted by $i! \mathcal{O}_Z^* \subset \mathcal{O}_X^*$ (this is a particular model of the canonical quasicoherent complex $i_! \mathcal{O}_Z$ used in the local duality theory).

**Definition A.8.** The Poisson cohomology $\mathcal{H}P_Z^*(X) = \mathcal{H}P^*(X, i! \mathcal{O}_Z^*)$ is called the Poisson cohomology of the scheme $X$ with supports in $Z \subset X$.

Note that the closed subscheme $Z \subset X$ enters into this construction only through its open complement $U \subset X$. In particular, there is no need to assume that $Z \subset X$ is a Poisson subscheme in any sense. We will need the following easy vanishing result.

**Lemma A.9.** Assume that the Poisson scheme $X$ is Cohen-Macaulay, and the closed subscheme $Z \subset X$ is of codimension $\text{codim} Z \geq k$ for some integer $k \geq 2$. Then $\mathcal{H}P_Z^i(X) = 0$ for all $i \leq k - 2$.

**Proof.** Since $X$ is Cohen-Macaulay, the complex $i_! \mathcal{O}_Z$ is trivial in degrees $\leq k - 1$. □
Proposition A.10. Let $X = \text{Spec } A$ be a Poisson scheme with a Poisson sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. Then the canonical map $H^P(X, \mathcal{F}) \to H^P(X, \mathcal{F})$ is a quasiisomorphism.

Proof. The claim is functorial in $\mathcal{F}$. By Definition A.7, it suffices to consider sheaves of the form $\mathcal{F} = i_* M$, where $i : x \hookrightarrow X$ is the embedding of a scheme-theoretic point $x \in X$, and $M \cong \Gamma^*_x(A, \mathcal{F})$ is a Poisson module over the local ring $\mathcal{O}_{X,x}$. Let $\mathcal{F} = i_* M$ be such a sheaf. Then by definition the local Poisson cohomology complex $H^P(X, \mathcal{F})$ is quasiisomorphic $H^P(X, F) \cong i_* H^P(\mathcal{O}_{X,x}, M)$, so that $H^P(X, \mathcal{F}) \cong H^P(\mathcal{O}_{X,x}, M)$. To prove the claim, it suffices to note that the natural map $A \to \mathcal{O}_{X,x}$ is étale, and apply Lemma A.6. □

Proposition A.11.

(i) For every two Poisson schemes $X, Y$, there exists a canonical quasiisomorphism

$$H^P(X \times Y) \cong H^P(X) \otimes H^P(Y).$$

(ii) Let $f : X \to Y$ be an étale map of Poisson schemes. Then for every complex $\mathcal{F}^*$ of sheaves of Poisson modules on $X$, there exists a canonical quasiisomorphism

$$H^P(Y, f_* \mathcal{F}^*) \cong H^P(X, \mathcal{F}^*).$$

(iii) Let $f : X \to Y$ be an étale Galois cover with Galois group $G$. Then there exists a canonical quasiisomorphism $H^P(Y) \cong (H^P(X))^G$.

Proof. By taking affine covers, we reduce (i) and (ii) to the case of affine schemes $X, Y$. Then Proposition A.10 reduces both statements to their algebraic versions Lemma A.6 and Lemma A.6. Finally, (iii) is a direct corollary of (ii): we have

$$(H^P(X))^G \cong (H^P(Y, f_* \mathcal{O}_X))^G \cong H^P(Y, (f_* \mathcal{O}_X)^G) \cong H^P(Y, \mathcal{O}_Y).$$ □

Corollary A.12.

(i) Let $f : X \to Y$ be a map between Poisson schemes, and assume the restriction of $f$ to a closed subscheme $Z \subset X$ is injective. Then, there exists a canonical quasiisomorphism

$$H^P_{\otimes Z}(Y) \to H^P_{\otimes Z}(X).$$

(ii) Let $Z \subset X$ be a closed subscheme in a Poisson scheme $X$, and let $Y$ be another Poisson scheme. Then there exists a canonical isomorphism

$$H^P_{\otimes X \times Y}(X \times Y) \cong H^P_{\otimes Z}(X) \otimes H^P_{\otimes Y}(Y).$$
Proof. (ii) immediately follows from Proposition A.11(i). To prove (i), let \( i^X : Z \hookrightarrow X, i^Y = f \circ i^X : Z \hookrightarrow Y \) be the embeddings. Notice that the map \( f \) must be étale in a Zariski neighborhood of the subscheme \( Z \subset X \), and we have \( f_* i^X_* \mathcal{O}_Z \cong i^Y_* \mathcal{O}_Z \). The claim then follows from Proposition A.11(ii). □

A.7 The smooth case. Let now \( X \) be a smooth Poisson scheme. In this case, we have the original Koszul-Brylinski formalism for local Poisson cohomology. Namely, one considers the graded Lie algebra \( \Lambda^* \mathcal{T}(X) \) of polyvector fields on \( X \) with the so-called Schouten bracket. The Poisson structure is defined by a bivector field \( \Theta \in \Lambda^2 \mathcal{T}(X) \) such that \([\Theta, \Theta] = 0\). Taking commutator with \( \Theta \) defines a differential \( d : \Lambda^q \mathcal{T}(X) \to \Lambda^{q+1} \mathcal{T}(X); \) we will call the complex \( \mathcal{H}P^q(X) \cong (\Lambda^* \mathcal{T}(X), d) \) the Koszul-Brylinski complex.

For a general smooth Poisson scheme \( X \), we have canonical embeddings \( \Lambda^k \mathcal{T}(X) \to \mathcal{H}P^k(X) \). These embeddings are compatible with the Koszul-Brylinski differential and define a quasiisomorphism \( \mathcal{H}P^q(X) \to \mathcal{H}P^q(X) \). Indeed, since the embeddings are canonical, it suffices to check this fact locally, so that we can assume that the scheme \( X = \text{Spec} \ A \) is affine. Then since \( A \) is smooth, the Harrison homology \( \text{Har}_q(A) \) is quasiisomorphic to the module \( \Omega^1 A \) of Kähler differentials of the algebra \( A \), and this module is flat over \( A \). Therefore the complex \( DP^{k,*}(A) \) only has non-trivial cohomology in degree 0, and \( HP^k_0(A) \cong \Lambda^k \mathcal{T}(A) \), cf. [Fr]. We have \( DP^{k,l}(A) = 0 \) for \( l < 0 \) and a canonical embedding \( \Lambda^k \mathcal{T}(A) \cong HP^k_0(A) \to DP^{k,0}(A) \) sends the Poisson bivector \( \Theta \) into the Poisson cochain, and it also sends the Schouten bracket in \( \Lambda^* \mathcal{T}(A) \) into the Gerstenhaber bracket in \( \mathcal{D}P^*(A) \). Therefore it sends the Koszul-Brylinski differential into the differential \( \delta : \mathcal{D}P^{*,*}(A) \to \mathcal{D}P^{*,*+1}(A) \), see also [Fr].

Recall that in case of a smooth affine Poisson variety \( X = \text{Spec} \ A \) there are Hochschild-Kostant-Rosenberg isomorphisms

\[
\mathcal{H}H^*(A) \cong \Lambda^*_{\text{ar}}(\text{Har}_1(A)) \cong \Lambda^*_{\mathcal{A}}(A),
\]

and the Koszul-Brylinski complex of \( X \) may be identified via this isomorphism with the ‘twisted’ Hochschild complex of \( A \) considered in §6, cf. Definition 6.1. Thus, we have

(A.17) \[
HP^*_A(X) \cong \mathcal{H}H^*_A(A).
\]

Also, in the smooth case, for the PBW filtration on Hochschild cohomology we get

\[
F^\text{PBW}_i \mathcal{H}H^j(A) = \begin{cases} \mathcal{H}H^j(A) & \text{if } i \geq j \\ 0 & \text{if } i < j. \end{cases}
\]

Furthermore, the spectral sequence associated to the bicomplex (A.6) collapses.

If the Poisson scheme \( X \) is not only smooth but also symplectic, then the symplectic form provides an identification \( \mathcal{T}(X) \cong \Omega(X) \). This identification
extends to an isomorphism $\mathcal{H}P^*_\Theta(X) \cong \Omega^*(X)$, the standard de Rham complex of the smooth scheme $X$. Thus for symplectic schemes we have

$$\mathcal{H}P^*(X) \cong \mathcal{H}P^*_{\Theta}(X) \cong \Omega^*(X) \quad \text{and} \quad HP^*(X) \cong H^*(X, \mathbb{C}),$$

the ordinary singular cohomology of the scheme $X$. We can also obtain isomorphisms for cohomology with coefficients: Poisson modules over $X$ are the same as $D$-modules, and the Poisson cohomology $HP^*(X, \mathcal{F})$ with coefficients in a Poisson module is isomorphic to the de Rham cohomology with coefficients in the corresponding $D$-module. In particular, the Poisson cohomology with supports is isomorphic to the singular cohomology with supports.

### A.8 Deformations

The shortest route to the classification of Poisson deformations (in particular, to the proof of Theorem 1.10) is through the formalism of deformation groupoids. This consists of obtaining deformed structures from solutions of the so-called Maurer-Cartan equation $d\gamma = \{\gamma, \gamma\}$ in a certain differential-graded Lie algebra $L^\ast$.

More precisely, assume given a DG Lie algebra $L^\ast$ with differential $d : L^\ast \to L^{\ast+1}$ and commutator $\{\cdot,\cdot\}$, and a local Artin algebra $S$ with maximal ideal $m \subset S$. Consider the DG Lie algebra $L^\ast \otimes S$ over $S$. Set $g = L^0$. For any $g \in g \otimes m$, $a \in L^1 \otimes S$ set

$$g \cdot a = dg + [g, a] \in L^1 \otimes S.$$

Consider $g \cdot a$ as a tangent vector to the vector space $L^1 \otimes S$ at the point $a \in L^1 \otimes S$. Then the collection $g \cdot a$, $a \in L^1 \otimes S$ defines a vector field on $L^1 \otimes S$ for every $g \in g \otimes m$. These vector fields glue together to an action of the Lie algebra $g \otimes m$ on $L^1 \otimes S$. Since the Lie algebra $g \otimes m$ is nilpotent, this action extends to an action of the corresponding nilpotent Lie group $G_m$.

**Definition A.13.** The Maurer-Cartan groupoid $MC(L^\ast, S)$ associated to the pair $(L^\ast, S)$ is defined in the following way:

(i) Objects of $MC(L^\ast, S)$ are elements $\gamma \in L^1 \otimes m$ satisfying the Maurer-Cartan equation

$$d\gamma = \{\gamma, \gamma\}. \quad (A.18)$$

(ii) Morphisms between objects $\gamma_1, \gamma_2 \in MC(L^\ast, S)$ are elements $g \in G_m$ in the group $G_m$ such that $g \cdot \gamma_1 = \gamma_2$.

To check that this definition is consistent, one has to prove that the $G_m$-action preserves the Maurer-Cartan equation, which is an elementary computation.

The Maurer-Cartan formalism admits an obvious graded version. If the DG Lie algebra $L^\ast$ is equipped with a grading, $-$ or, equivalently, with a $\mathbb{C}^\ast$-action, $-$ then one can consider local Artin algebras $S$ equipped with a $\mathbb{C}^\ast$-action and form
the groupoid $MC_{gr}(\mathcal{L}^*, S)$ of the $\mathbb{C}^*$-invariant solutions to the Maurer-Cartain equation (A.18).

Application of the Maurer-Cartan formalism to the Poisson deformation theory is an immediate corollary of Lemma A.2 and Lemma A.1.

Lemma A.14. Let $A$ be a Poisson algebra over $\mathbb{C}$. Consider the Poisson cohomology complex $DP^*(A)$, and let

$$L^k = \begin{cases} DP^{k+1}(A), & k \geq 0, \\ 0, & k < 0. \end{cases}$$

Let $S$ be a local Artin algebra with maximal ideal $m$.

Then the category $Def(A, S)$ of flat Poisson algebras $\tilde{A}$ over $S$ equipped with an isomorphism $\tilde{A}/m \cdot \tilde{A} \cong A$ is equivalent to the Maurer-Cartan groupoid $MC(L^1, S)$.

Proof. By Lemma A.2 and Lemma A.1, defining a Poisson algebra structure on a vector space $V$ is equivalent to giving a cochain $\alpha = m + \Theta \in DP^2(V) = D_L^1(V) \oplus DP^{2,0}(V)$ which satisfies $[\alpha, \alpha] = 0$. To define a functor $MC(\text{Der}^*(A), S) \to Def(A, S)$, it suffices to notice that for every solution $\gamma \in L^1 = DP^2(A) \otimes m$ of the Maurer-Cartan equation, the element

$$\tilde{\alpha} = \alpha - \frac{1}{2} \gamma \in L^1 \otimes S$$

satisfies $\{\tilde{\alpha}, \tilde{\alpha}\} = 0$, hence defines a Poisson algebra structure on $A \otimes S$. This functor is surjective on objects – indeed, by definition for every deformation $\tilde{A}$ we have $\tilde{A} \cong A \otimes S$ as $S$-modules. To prove that it is an equivalence, it suffices to prove that it is surjective on morphisms. But the Lie algebra $L^0 = DP^1(A) \cong \text{Hom}(A, A)$ by definition consists of all linear endomorphisms of the vector space $A$. Therefore morphisms in the Maurer-Cartan groupoid are all $S$-module morphisms $A \otimes S \to A \otimes S$ which reduce to identity modulo $m \subset S$. This is the same as morphisms in $Def(A, S)$. □

For any DG Lie algebra $\mathcal{L}^*$, the Maurer-Cartan groupoids $MC(\mathcal{L}^*, S)$ form a stack over the category of local Artin algebras $S$. The general machinery shows that this stack is (pro)-representable. The representing object is a certain canonically defined subscheme in the vector space $H^1(\mathcal{L}^*)$ completed at zero, modulo the action of the (pro)-algebraic group corresponding to the Lie algebra $\mathcal{L}^0$.

The machinery works in the case of graded DG Lie algebras without any change: the (pro)-representing object carries a natural $\mathbb{C}^*$-action compatible with the induced grading on $H^1(\mathcal{L}^*)$, and it (pro)-represents the Maurer-Cartan stack $MC_{gr}(\mathcal{L}^*, S)$ on the category of local Artin schemes equipped with a $\mathbb{C}^*$-action. However, if the degrees of the induced gradings on $H^1(\mathcal{L}^*)$, $H^0(\mathcal{L}^*)$ are strictly positive, we can obtain more.

Say that a commutative algebra $A = \bigoplus A^*$ is positively graded if $\dim A^k = 0$ for $k < 0$, $\dim A^0 = 1$ and $\dim A^k < \infty$ for $k > 0$. Further, we say that a
complete local algebra \( \langle A, m \rangle \) is equipped with a good \( \mathbb{C}^* \)-action if the group \( \mathbb{C}^* \) acts on \( \widehat{A} \), and this action induces a positive grading on every Artin quotient \( A/m^k A \) and on the associated graded quotient \( \text{gr}^* A = \bigoplus_k m^k/m^{k+1} \). For every positively graded algebra \( A^* \), denote by \( \widehat{A} \) the completion of the algebra \( A^* \) with respect to the maximal ideal \( m = A^{\geq 1} \subset A^* \). We note the following standard fact (essentially, this is a toy version of the Grothendieck algebraization, \cite[EGA III, 5.4]{EGA}).

**Lemma A.15.**  
(i) The correspondence \( A^* \mapsto \widehat{A} \) is an equivalence between the category of positively graded commutative algebras \( A^* \) and the category of complete local algebras \( \langle \widehat{A}, m \subset \widehat{A} \rangle \) equipped with a good \( \mathbb{C}^* \)-action.

(ii) For any positively graded algebra \( A^* \), the completion induces an equivalence between the category of finitely generated graded \( A^* \)-modules and the category of finitely generated \( \mathbb{C}^* \)-equivariant \( \widehat{A} \)-modules.

(iii) If the algebra \( A^* \) is finitely generated, then the completion induces an equivalence between the category of projective \( \mathbb{C}^* \)-equivariant schemes of finite type over \( \text{Spec} A^* \) and the category of projective \( \mathbb{C}^* \)-equivariant schemes of finite type over \( \text{Spec} \widehat{A} \).

(iv) Let \( X \) be a projective \( \mathbb{C}^* \)-equivariant scheme of finite type over \( \text{Spec} A^* \), and let \( \mathcal{X}/\text{Spec} \widehat{A} \) be its completion. Then completion induces an equivalence between the category of coherent \( \mathbb{C}^* \)-equivariant sheaves on \( X \) and coherent \( \mathbb{C}^* \)-equivariant sheaves on \( \mathcal{X} \). A sheaf \( \mathcal{F} \) on \( X \) is flat if and only if its completion \( \widehat{\mathcal{F}} \) is flat. Moreover, for every sheaf \( \mathcal{F} \) on \( X \), the completion \( H^k(X, \mathcal{F}) \) of the cohomology group \( H^k(X, \mathcal{F}) \) with respect to the \( m \)-adic topology is canonically isomorphic to the cohomology group \( H^k(\mathcal{X}, \widehat{\mathcal{F}}) \) of the completion \( \widehat{\mathcal{F}} \).

**Sketch of proof.** To prove (i), we will just give the inverse to the equivalence of categories. It is given by

\[
(A, m) \mapsto \bigoplus_p \lim_p (A/m^k)^p.
\]

By our definition of a good \( \mathbb{C}^* \)-action, for every integer \( p \) the projective system of the corresponding graded components \( (A/m^k)^p \) actually stabilizes at some finite level \( k \geq 0 \), and the limit is a finite-dimensional vector space. Thus the right-hand side is indeed a positively graded commutative algebra. To prove (ii), we define the inverse equivalence by the same formula \((A, m) \mapsto \bigoplus_p \lim_p (A/m^k)^p\).

To prove (iii), we write \( X = \text{Proj} \widehat{B}^* \) for a scheme \( \widehat{X} \) projective over \( \text{Spec} \widehat{A} \), apply \((A, m) \mapsto \bigoplus_p \lim_p (A/m^k)^p\) and obtain a scheme \( X = \text{Proj} B \) over \( \text{Spec} A \) whose completion is \( \widehat{X} \) (here \( B \) has two independent gradings: the first one is related to the \( \mathbb{C}^* \)-action, and the second one comes from the definition of \text{Proj}). This defines the inverse equivalence on the level of objects. To lift it to morphisms, one identifies morphisms with their graphs.
there is an orthogonal direct sum decomposition $V \in u \oplus H$

Let $A$ be a linear algebra result.

Proof. It is clear that $H^k(X, \mathcal{F}) = 0$ for some $k$ if and only if $\mathcal{F}$ is a vector space, and $G$ is a space of $x \in H^1(G)$ which coincides with the Lie algebra of all Poisson derivations.

This Lie algebra is rather big; in particular, it contains all the Hamiltonian vector fields on $H^1(G)$. To prove the statement on cohomology, it suffices to identify the Lie algebra $H^0(G)$ acts trivially on the vector space $H^1(G)$.

Proof of Theorem 1.10. Assume that $H^1(A)$. We have to prove that the Lie algebra $H^0(G)$ acts trivially on the vector space $H^1(G)$. By assumption, the Lie algebra $H^0(G)$ is induced by the action of the Lie algebra $H^1(G)$. By assumption, the Lie algebra $H^0(G)$ is trivial.

A linear algebra result. Let $V$ be a finite dimensional complex symplectic vector space, and $G \subset Sp(V)$ a finite subgroup. Write $V^g$ for the fixed point space of $g \in G$.

Lemma A.16. For any elements $g, h \in G$ such that $V = V^g + V^h$, we have $V^{gh} = V^g \cap V^h$.

Proof. It is clear that $V^g \cap V^h \subseteq V^{gh}$. To prove the opposite inclusion, let $x \in V^{gh}$. The assumption $V = V^g + V^h$, implies that $x = u + v$, for some $u \in V^g$ and $v \in V^h$. Thus, $gh(u + v) = u + v$, hence, $hu + v = u + g^{-1}v$, or equivalently $(h-1)u = (g^{-1}-1)v$. But, for any $a \in Sp(V)$ of finite order, there is an orthogonal direct sum decomposition $V = \text{Image} \text{id} - a \oplus V^a$. We
deduce that \((h - 1)u\) in the LHS of the last equation is orthogonal to \(V^h\), resp. \((g^{-1} - 1)v\) in the RHS is orthogonal to \(V^g\), with respect to the symplectic form. Thus, each side is orthogonal to \(V = V^g + V^h\), hence, vanishes. It follows that \((h - 1)u = (g^{-1} - 1)v = 0\). Therefore \(u, v \in V^g \cap V^h\), hence \(x \in V^g \cap V^h\), and we obtain \(V^{g,h} \subseteq V^g \cap V^h\). □

References

[AFLS] J. Alev, M.A. Farinati, T. Lambre, and A.L. Solotar, Homologie des invariants d’une algèbre de Weyl sous l’action d’un groupe ﬁni. J. of Algebra 232 (2000), 564–577.

[Al] M.S. Alvarez, Algebra structure on the Hochschild cohomology of the ring of invariants of a Weyl algebra under a ﬁnite group. J. Algebra 248 (2002), 291-306.

[Ba] V. Baranovsky, Orbifold Cohomology as Periodic Cyclic Homology. arXiv:math.AG/0206256

[BB] A. Beilinson, J. Bernstein, Localisation de \(g\)-modules. C. R. Acad. Sci. Paris 292 (1981), 15–18.

[BD] V. Batyrev and D. Dais, Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry, Topology, 35 (1996), 901–929. Preprint alg-geom/9410001.

[BKR] T. Bridgeland, A. King, and M. Reid, The McKay correspondence as an equivalence of derived categories. J. Amer. Math. Soc. 14 (2001), 535–554.

[BG] K. Brown, I. Gordon, Poisson orders, symplectic reflection algebras and representation theory, arXiv:math.RT/020104.

[Br] J.-L. Brylinski, A differential complex for Poisson manifolds. J. Diff. Geom. 28 (1988), 93–114.

[CF] F. Campana and H. Flenner, Contact singularities, [arXiv:math.AG/010906 ].

[CR] W. Chen, Y. Ruan, A new cohomology theory for orbifold, arXiv:math.AG/0004129.

[DL] J. Denef, F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration. Invent. Math. 135 (1999), 201–232.

[EG] P. Etingof and V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism, Invent. Math. 147 (2002), 243-348, [arXiv:math.AG/0011114 ].

[FG] B. Fantechi, L. Göttsche, Orbifold cohomology for global quotients. Duke Math. J. 117 (2003), 197–227. [arXiv:math.AG/0104207].

[Fi] H. Flenner, Extendability of differential forms on non-isolated singularities, Inv. Math. 94 (1988), 317-326.

[Fr] B. Fresse, Homologie de Quillen pour les algèbres de Poisson, C.R. Acad. Sci. Paris 326 (1998), 1053–1058.

[Fu] B. Fu, Symplectic Resolutions for Nilpotent Orbits, Invent. Mathem. 151 (2003), 167–186. [arXiv:math.AG/0205048]
[Go] I. Gordon, Baby Verma modules for rational Cherednik algebras, Bull. London Math. Soc. 35 (2003), 321–336, [arXiv:math.RT/0202301].

[EGA] A. Grothendieck, EGA, 0, III, Publ. Math. IHES 11.

[K1] D. Kaledin, Dynkin diagrams and crepant resolutions of quotient singularities, arXiv:math.AG/9903157, to appear in Selecta Math.

[K2] D. Kaledin, McKay correspondence for symplectic quotient singularities, Invent. Math. 148 (2002), 151–175, [arXiv:math.AG/9907087].

[K3] D. Kaledin, Symplectic resolutions: deformations and birational maps, preprint arXiv:math.AG/0012008.

[KV] D. Kaledin and M. Verbitsky, Period map for non-compact holomorphically symplectic manifolds, Geom. Funct. Anal. 12 (2002), 1205–1295.

[KS] M. Kontsevich, Y. Soibelman, Deformations of algebras over operads and Deligne’s conjecture. [arXiv:math.QA/0001151].

[Kon] M. Kontsevich, Deformation quantization of Poisson manifolds. arXiv:q-alg/9709040.

[Ko] J.-L. Koszul, Crochet de Schouten-Nijenhuis et cohomologie. Astérisque 1985, Numero Hors Serie, 257–271.

[Ku] A. Kuznetsov, Quiver varieties and Hilbert schemes, arXiv:math.AG/0111092.

[LS] M. Lehn, Ch. Sorger, Symmetric groups and the cup product on the cohomology of Hilbert schemes, Duke Math. J. 110 (2001), 345–357, [arXiv:math.AG/0009131].

[Lo] J.-L. Loday, Cyclic homology. Grundlehren der Mathematischen Wissenschaften 301 Springer-Verlag, Berlin, 1992.

[LQW] W. Li, Z. Qin, and W. Wang, Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces, Math. Ann. 324 (2002), 105–133, [arXiv:math.AG/0009132].

[McK] J. McKay, Graphs, singularities and finite groups, in The Santa Cruz Conference on Finite Groups, Proc. Symp. Pure Math. 37 (1980), 183–186.

[Ri] D.S. Rim, Formal Deformation Theory. SGA 7 I, Lect. Notes in Mathem. 288 Springer-Verlag 1972, pp.32–132.

[R1] Y. Ruan, Stringy Geometry and Topology of Orbifolds, Symposium in Honor of C. H. Clemens, 187–233, Contemp. Math., 312, Amer. Math. Soc., Providence, RI, 2002, arXiv:math.AG/0011149.

[R2] Y. Ruan, Stringy Orbifolds. Orbifolds in mathematics and physics (Madison, WI, 2001), 259–299, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002, [arXiv:math.AG/0201123].

[Vas] E. Vasserot, Sur l’anneau de cohomologie du schéma de Hilbert de C², C.R. Acad. Sci. Paris, 332 (2001), 7–12, [ arXiv:math.AG/0009127 ].

[Ve] M. Verbitsky, Holomorphic symplectic geometry and orbifold singularities, Asian Journal of Mathematics, 4 (2000), 553-564.

[W] K. Watanabe, Certain invariant subrings are Gorenstein, I, II, Osaka J. Math. 11 (1974), 1–8; 379–388.

[WW] J. Wierzba and J. Wisniewski, Small Contractions of Symplectic 4-folds, preprint arXiv:math.AG/020102.
V.G.: Department of Mathematics, University of Chicago, Chicago, IL 60637, USA;
ginzburg@math.uchicago.edu

D.K.: Steklov Mathematical Institute, Moscow, Russia
kaledin@balthi.dntm.ru