F-threshold functions: syzygy gap fractals and the two-variable homogeneous case

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Abstract

In this article we study F-pure thresholds (and, more generally, F-thresholds) of homogeneous polynomials in two variables over a field of characteristic $p > 0$. Passing to a field extension, we factor such a polynomial into a product of powers of pairwise prime linear forms, and to this collection of linear forms we associate a special type of function called a syzygy gap fractal. We use this syzygy gap fractal to study, at once, the collection of all F-pure thresholds of all polynomials constructed with the same fixed linear forms. This allows us to describe the structure of the denominator of such an F-pure threshold, showing in particular that whenever the F-pure threshold differs from its expected value its denominator is a multiple of $p$. This answers a question of Schwede in the two-variable homogeneous case. In addition, our methods give an algorithm to compute F-pure thresholds of homogenous polynomials in two variables.

1 Introduction

Fix an arbitrary field $k$ of characteristic $p > 0$, and consider a polynomial $g$ in $k[x_1, \ldots, x_r]$ with $g(0) = 0$. By utilizing properties of the Frobenius endomorphism on the ambient polynomial ring, one may show that

$$fpt(g) := \inf \left\{ \frac{a}{p^e} : g^a \in \langle x_1^{p^e}, \ldots, x_r^{p^e} \rangle \right\}$$

is a well-defined nonzero real number contained in the unit interval. This invariant, called the F-pure threshold of $g$ (at the origin), was originally introduced in [TW03], though the definition we give here follows [MTW05]. Though it is not obvious from this definition, it turns out that the F-pure threshold of a polynomial is always a rational number [BMS08 Corollary 2.30, Theorem 3.1].
Note that if, in the description of \( \text{fpt}(g) \) given above, one replaces the Frobenius power \( m^p = \langle x_1^p, \ldots, x_r^p \rangle \) of the maximal ideal \( m = \langle x_1, \ldots, x_r \rangle \) with the ordinary power \( m^p \), one would instead obtain the reciprocal of the multiplicity of \( g \) at the origin (that is, the largest \( N \) such that \( g \in m^N \)). Thus, the \( F \)-pure threshold may be thought of as a sort of “Frobenius multiplicity”, with smaller values corresponding to “worse” singularities at the origin.

In this article we are motivated by the relationship between \( F \)-pure thresholds and another important invariant, traditionally defined for polynomials over fields of characteristic zero. Consider a polynomial \( g \) over a field of characteristic zero that vanishes at the origin. By referring to a (log) resolution of singularities, one may assign to \( g \) the numerical invariant \( \text{lct}(g) \), called the log canonical threshold of \( g \) (at the origin). Like the \( F \)-pure threshold, the log canonical threshold of a polynomial is always a nonzero rational number contained in the unit interval and may be thought of as a measure of the singularity of \( g \) (at the origin), with smaller values corresponding to “worse” singularities. For more on this invariant, we refer the reader to the survey [BL04] and the references cited therein. Throughout the rest of this article, we shall always consider polynomials vanishing at the origin and shall omit the phrase “at the origin” when referring to \( F \)-pure and log canonical thresholds.

Remarkably, \( F \)-pure and log canonical thresholds are intimately related. Consider a polynomial \( g_0 \) over \( \mathbb{Q} \) and, for \( p \gg 0 \), let \( g_p \) denote the polynomial over \( \mathbb{F}_p \), the field with \( p \) elements, obtained by reducing the coefficients of \( g_0 \) modulo \( p \). It follows from work of Hara and Yoshida [HY03] that \( \text{fpt}(g_p) \leq \text{lct}(g_0) \) and \( \lim_{p \to \infty} \text{fpt}(g_p) = \text{lct}(g_0) \) (see [MTW05, Theorem 3.4]). In general, little is known about how \( \text{fpt}(g_p) \) varies with \( p \), and an important (open) conjecture predicts that \( \text{fpt}(g_p) = \text{lct}(g_0) \) for infinitely many primes. Motivated by understanding the situation when \( \text{fpt}(g_p) \neq \text{lct}(g_0) \), the following question was asked by Karl Schwede.

**Question 1.1 (Schwede).** Fix \( g_0 \in \mathbb{Z}[x_1, \ldots, x_r] \) vanishing at the origin. Assume \( \text{fpt}(g_p) \neq \text{lct}(g_0) \), for a prime \( p \gg 0 \). Write \( \text{fpt}(g_p) = a/b \) in lowest terms. Does \( p \) divide \( b \)?

In recent work, Bhatt and Singh (and, in a subsequent generalization, Núñez-Betancourt, Witt, Zhang, and the first author) have shown the following: Suppose \( g_0 \) is a polynomial over \( \mathbb{Q} \) that is homogeneous under some \( N \)-grading and such that the ideal generated by the partial derivatives of \( g_0 \) is primary to the ideal generated by the variables. If \( p \gg 0 \) and \( \text{fpt}(g_p) \neq \text{lct}(g_0) \), then the denominator of \( \text{fpt}(g_p) \) is a not just a multiple of \( p \) but in fact a power of \( p \) \([BS, HNWZ14]\). In particular, a “stronger” form of Schwede’s question has a positive answer for such polynomials. As far as the authors are aware, there are no such

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1Schwede’s question was asked during the Computational Workshop on Frobenius Singularities and Invariants, held in Ann Arbor, MI, in 2012. His question, and others, can be found at [https://sites.google.com/site/computingfinvariantsworkshop/open-questions](https://sites.google.com/site/computingfinvariantsworkshop/open-questions).

2The question of under what circumstances the denominator of \( \text{fpt}(g_p) \) must be a power of \( p \) was asked by the first author during the aforementioned workshop.
general descriptions of $F$-pure thresholds of polynomials whenever one relaxes
the hypothesis on the ideal generated by the corresponding partial derivatives.

In this article we shed some further light on Question 1.1 answering it in
what is perhaps the simplest nontrivial case.

**Theorem 1** (see Theorem 7.7). If $G_0 \in \mathbb{Q}[x, y]$ is a non-constant homogeneous
polynomial$^3$ then Question 1.1 has a positive answer for $G_0$. More precisely, if
$p \geq 0$ and $fpt(G_p) \neq fpt(G_0)$, then the minimal denominator of $fpt(G_p)$ is of
the form $kp^{e}$, where $e \geq 1$ and $k$ divides the multiplicity of some linear factor
(over $\mathbb{C}$) of $G_0$.

We now briefly describe the main ideas in this article. For the remainder of
this introduction, $G$ will denote a homogeneous polynomial in $k[x, y]$, where $k$
is a field of characteristic $p > 0$. Many of our results deal with a generalization
of $F$-pure thresholds called, simply, $F$-thresholds; this generality pays off later,
allowing us to extend our main result to polynomials that are homogeneous
under non-standard $\mathbb{N}$-gradings—see Theorem 8.2. Given an ideal $b \subseteq k[x, y]$,
the $F$-threshold of $G$ with respect to $b$, denoted $ft^b(G)$, is a numerical invariant
describing the complexity of the hypersurface defined by $G$. $F$-thresholds gen-
erealize $F$-pure thresholds, in the sense that $fpt(G) = ft^{(x, y)}(G)$ (see [MTW05]).

Rather than considering $F$-thresholds with respect to arbitrary ideals $b$, we focus
instead on the case when $b$ is generated by two non-constant, relatively prime
forms. The motivation for this restriction is that it allows us to apply the theory
of syzygy gap fractals, introduced by Han in her thesis [Han91] and generalized
and studied by the second author in [Tei02, Tei12].

Over $k$ there exists a collection of pairwise prime linear forms $\ell = (\ell_1, \ldots, \ell_n)$
such that $G = \ell_1^{a_1} \cdots \ell^n_{a_n}$, for some $a_1, \ldots, a_n \in \mathbb{N}_{>0}$. In Section 4 we define a
continuous function $ft^b(\ell^n) : \mathbb{R}_{>0}^n \to \mathbb{R}$ with the property that $k \mapsto ft^b(\ell_1^{a_1} \cdots \ell^n_{a_n})$
whenever $k = (k_1, \ldots, k_n) \in \mathbb{Z}_{>0}^n$. This function, called an $F$-threshold function
(see Definition 4.2), encodes the $F$-thresholds (with respect to the fixed ideal $b$)
of all homogeneous polynomials with the same linear factors as $G$, and will play
a key role in this article. The $F$-threshold function is described in terms of a
syzygy gap fractal associated to the ideal $b$ and the linear forms $\ell_1, \ldots, \ell_n$, and
properties of syzygy gap fractals worked out in [Tei12] allow us to understand $F$-
threshold functions well enough to prove our main results. More precisely, we see
that the $F$-threshold function (and, hence, the $F$-thresholds of all homogeneous
polynomials with the same linear factors as $G$) is completely determined by a
family of distinguished points, called critical points. It turns out that every
coordinate of a critical point is a rational number whose denominator is a power
of $p$, and it is precisely this fact that allows us to say something about the
denominators of $F$-pure thresholds of homogeneous polynomials.

From this, something surprising will emerge: that even if we are solely inter-
ested in computing the $F$-pure threshold of a single polynomial $G = \ell_1^{a_1} \cdots \ell^n_{a_n}$,
or equivalently, describing the $F$-threshold function $t \mapsto ft^{(x, y)}(\ell^n_t)$ along a

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3Ongoing work by the first author and Emily Witt suggests that this theorem may be false
for certain non-homogeneous polynomials in two variables.
single line, we often find that what happens along that line is determined by something—a critical point—that occurs elsewhere. The authors believe that this phenomenon replicates in much more general situations where polynomials (or elements of more general rings) are constructed with simple “building blocks” that mimic the linear forms, and that the approach introduced here can help us understand $F$-pure thresholds in those more general situations.

Finally, we point out that our methods are effective, and provide us with an algorithm to compute $F$-pure thresholds of homogeneous polynomials in two variables, which has been implemented by the second author in the Macaulay2 \cite{GS} package PosChar \cite{HKM}. This implementation is remarkably efficient when the polynomial factors over a relatively small field. For instance, if $a$ is algebraic over $\mathbb{F}_5$, satisfying $a^3 + a + 1 = 0$, and

$$G = x^{420}y^{419}(x + y)^{417}(x + ay)^{390}(x + a^2y)^{402}(x + a^3y)^{438} \in \mathbb{F}_5[a][x,y],$$

then our current implementation takes only about 0.3 seconds to report that

$$\text{fpt}(G) = \frac{46636216675560574859117627837996765605705641779512143}{2 \cdot 3 \cdot 5^{76} \cdot 73}.$$

### 1.1 Outline

This paper is organized as follows. In Section 2 we recall the basics of base $p$ expansions of real numbers as well as the properties of syzygy gap fractals needed in this article. In Section 3 we introduce and study a pair of functions $\Delta$ and $\Phi$, the former being a special instance of a syzygy gap fractal, and in Section 4 we use these functions to define $F$-threshold functions. In Section 5 we define the notion of critical points and precisely describe in which ways these points determine the values of $F$-threshold functions. In Section 6 we consider $F$-threshold functions attached to three linear forms. In Sections 7 and 8 we apply our methods to the study and computations of $F$-thresholds of polynomials in two variables that are homogeneous under either standard or non-standard $\mathbb{N}$-gradings.

### 1.2 Notations and conventions

The following are some notations and conventions used throughout:

- $p$ denotes a prime number and $q$ always denotes a (variable) power of $p$.
- $\mathbb{k}$ is a field of characteristic $p$.
- If $\mathfrak{a}$ is an ideal of $\mathbb{k}[x,y]$, then $\mathfrak{a}^{[q]}$ denotes the $q$th Frobenius power of $\mathfrak{a}$, that is, $\mathfrak{a}^{[q]} := \langle f^q : f \in \mathfrak{a} \rangle$. Also, $\deg \mathfrak{a}$ denotes the degree (or colength) of $\mathfrak{a}$, that is, $\deg \mathfrak{a} := \dim_{\mathbb{k}}(\mathbb{k}[x,y]/\mathfrak{a})$.
- The term form is used as a synonym for nonzero homogenous polynomial. In this context, $\deg H$ denotes the typical degree of a form $H$ under some fixed $\mathbb{N}$-grading (usually the standard $\mathbb{N}$-grading) on the ambient polynomial ring.


• If $S \subseteq \mathbb{R}$, then $S_q$ denotes the set consisting of rational elements of $S$ with denominator $q$, and $S_p^\infty$ denotes the union of all $S_q$.

• Vectors are denoted by bold face letters, and their components are denoted by the same letter in regular font (e.g., $u = (u_1, \ldots, u_n)$). The canonical basis vectors of $\mathbb{R}^n$ are denoted by $e_1, \ldots, e_n$.

• Unary operations on real numbers are extended to vectors in a componentwise fashion (e.g., $\lfloor u \rfloor = (\lfloor u_1 \rfloor, \ldots, \lfloor u_n \rfloor)$).

2 Background

2.1 Base $p$ expansions

We review here some terminology and notation concerning base $p$ expansions.

Definition 2.1. Consider $\alpha \in (0, 1]$. An expression of the form

$$\alpha = \sum_{s \geq 1} \alpha_s \cdot p^{-s}$$

with $\alpha_s \in \mathbb{N} \cap [0, p)$ is called a base $p$ expansion of $\alpha$. If $\alpha_s = 0$ for $s \gg 0$, the base $p$ expansion is terminating; otherwise it is non-terminating.

Since every number in $(0, 1]$ has a unique non-terminating base $p$ expansion, we shall work almost exclusively with those (exceptions will be clearly flagged).

Definition 2.2. Consider a real number $\alpha$ in $(0, 1]$ with non-terminating base $p$ expansion $\alpha = \sum_{s \geq 1} \alpha_s \cdot p^{-s}$. Given $e \geq 0$, we call the number

$$\langle \alpha \rangle_e := \sum_{s=1}^{e} \alpha_s \cdot p^{-s} \in \mathbb{Q}_p^e$$

the $e$th truncation of $\alpha$ (base $p$). Given an arbitrary $\alpha \in \mathbb{R}_{>0}$, there exist unique $N \in \mathbb{N}$ and $\beta \in (0, 1]$ such that $\alpha = N + \beta$, and we define the $e$th truncation of $\alpha$ as $\langle \alpha \rangle_e := N + \langle \beta \rangle_e$.

Note that if $\alpha \in (0, 1]$ then $\langle \alpha \rangle_0 = 0$, and thus for arbitrary $\alpha \in \mathbb{R}_{>0}$ we have $\langle \alpha \rangle_0 = \lfloor \alpha \rfloor$, if $\alpha \notin \mathbb{N}$, and $\langle \alpha \rangle_0 = \alpha - 1$, otherwise.

Remark 2.3. Truncations can be characterized as follows: $\langle \alpha \rangle_e$ is the unique $\lambda \in \mathbb{Q}_p^e$ such that $\lambda < \alpha \leq \lambda + 1/p^e$. (For later use, note that if our choice had been to use terminating base $p$ expansions whenever available, then strict and weak inequalities would be interchanged.)
2.2 Syzygy gap fractals

We gather here some definitions and results concerning syzygy gaps and syzygy gap fractals from [Tei12] and adapt them to suit our needs. In what follows, $F, G, H \in R := k[x, y]$ are forms with no common factor.

**Definition 2.4.** Let $M = R(-\deg F) \oplus R(-\deg G) \oplus R(-\deg H)$, so that, by the Hilbert Syzygy Theorem, there exists an exact sequence of graded $R$-modules

$$0 \to R(-m) \oplus R(-n) \to M \to R \to R/\langle F, G, H \rangle \to 0.$$ 

The syzygy gap of $F, G,$ and $H$ is the nonnegative integer

$$\delta(F, G, H) = |m - n|.$$ 

Perhaps one of the most important aspects of syzygy gaps is their relation with the degrees of certain ideals, which we recall below.

**SG 1 ([Tei12 Proposition 2.2]).** The syzygy gap $\delta(F, G, H)$ and the degree of the ideal $\langle F, G, H \rangle$ are related as follows:

$$4 \deg \langle F, G, H \rangle = Q(\deg F, \deg G, \deg H) + \delta(F, G, H)^2,$$

where $Q(a, b, c) = 2ab + 2ac + 2bc - a^2 - b^2 - c^2$.

The proof of SG 1 relies on the fact that the Hilbert series of $R/\langle F, G, H \rangle$ (and, consequently, $\deg \langle F, G, H \rangle$) can be calculated from the free resolution of $R/\langle F, G, H \rangle$ appearing in Definition 2.4. Though we omit the proof of SG 1, we use a similar idea to establish the following identity.

**Lemma 2.5.** Let $U, V \in k[x, y]$ be relatively prime forms. Then

$$\deg \langle U, V \rangle = \deg U \deg V.$$ 

**Proof.** Set $u = \deg U$ and $v = \deg V$. As $U$ and $V$ are relatively prime, the sequence

$$0 \to R(-u - v) \to R(-u) \oplus R(-v) \to R \to R/\langle U, V \rangle \to 0,$$

in which the second map is given by $1 \mapsto (V, -U)$ and the third map by $(A, B) \mapsto AU + BV$, is exact. If $\text{Hilb}(t)$ denotes the Hilbert series of $R/\langle U, V \rangle$, then the above exact sequence shows that

$$\text{Hilb}(t) = \frac{1 - t^u - t^v + t^{u+v}}{(t - 1)^2}.$$ 

Applying l’Hôpital’s rule twice, we find that $\text{Hilb}(1) = uv = \deg U \deg V$, and the lemma follows, as $\deg \langle U, V \rangle = \dim_k R/\langle U, V \rangle = \text{Hilb}(1)$.

**Corollary 2.6.** If $F$ and $G$ are relatively prime, then

$$\delta(F, G, H)^2 = 4(\deg \langle F, G, H \rangle - \deg \langle F, G \rangle) + (\deg H - \deg FG)^2.$$ 


Proof. Standard algebraic manipulations of SG 1 produce the identity
\[
\delta(F, G, H)^2 = 4(\deg\langle F, G, H \rangle - \deg F \deg G) + (\deg H - \deg FG)^2,
\]
and the claim then follows from Lemma 2.5.

SG 2 ([Tei12, Remark 2.6]). If \( \deg H \geq \deg F + \deg G \) and \( F \) and \( G \) are relatively prime, then \( \delta(F, G, H) = \deg H - \deg F - \deg G \).

SG 3 ([Tei12, Proposition 2.7(2)]). If a form \( P \in k[x, y] \) is prime to \( H \), then \( \delta(PF, PG, H) = \delta(F, G, H) \).

SG 4 ([Tei12, Equation (2)]). If \( \ell \in k[x, y] \) is a linear form, then \( \delta(F, G, H\ell) = \delta(F, G, H) \pm 1 \).

While the above results hold in arbitrary characteristic, from this point on the assumption that \( k \) is a field of positive characteristic \( p \) will become essential. Due to the flatness of the Frobenius map over \( k[x, y] \), we have the following:

SG 5 ([Tei12, Equation (3)]). \( \delta(F^q, G^q, H^q) = q \cdot \delta(F, G, H) \).

Definition 2.7. Let \( \ell_1, \ldots, \ell_n \in k[x, y] \) be pairwise prime linear forms. A cell (with respect to \( \ell_1, \ldots, \ell_n \)) is a triple of forms \( C = (F, G, H) \) such that \( F, G, \) and \( H\ell_1 \cdots \ell_n \) have no common factor. If \( C = (F, G, 1) \), we shall dispense with the third component and simply write \( C = (F, G) \).

In the remainder of this section, \( \ell_1, \ldots, \ell_n \in k[x, y] \) are fixed pairwise prime linear forms, \( \ell = (\ell_1, \ldots, \ell_n) \), and \( C \) is a cell \( (F, G, H) \) with respect to the \( \ell_i \).

Notation. For each \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \), \( \ell^a \) denotes the product \( \ell_1^{a_1} \cdots \ell_n^{a_n} \).

Definition 2.8. The syzygy gap fractal \( \delta_C : (\mathbb{Q}_{\geq 0})^n_{p^\infty} \to \mathbb{Q} \) is defined as follows:
\[
\delta_C(a^q) = \frac{1}{q} \cdot \delta(F^q, G^q, H^q \ell^a),
\]
for each \( q \) and each \( a \in \mathbb{N}^n \). (SG 5 shows that this is well defined.)

In [Tei12] these functions were defined on \( [0, 1]^n_{p^\infty} \), but it will be convenient in this paper to extend them to \( (\mathbb{Q}_{\geq 0})^n_{p^\infty} \).

Definition 2.9. \( d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) and \( \|\cdot\| : \mathbb{R}^n \to \mathbb{R} \) are the taxicab metric and norm on \( \mathbb{R}^n \). That is, \( d(t, u) = \sum_{i=1}^n |t_i - u_i| \) and \( \|t\| = \sum_{i=1}^n |t_i| \), for each \( t, u \in \mathbb{R}^n \).

SG 4 gives us the following result:

SG 6 ([Tei12, Proposition 4.2]). \( |\delta_C(t) - \delta_C(u)| \leq d(t, u) \), for all \( t, u \in (\mathbb{Q}_{\geq 0})^n_{p^\infty} \).
This shows that \( \delta_C \) is uniformly continuous, so it extends (uniquely) to a continuous function \( \mathbb{R}^n_{\geq 0} \to \mathbb{R} \). \textbf{Henceforth, \( \delta_C \) will denote this extension.}

The next three results were stated in [Tei12] for the original \( \delta_C \), defined on \([0, 1]_p^n \), but also hold for the extension to \((\mathbb{Q}_{\geq 0})_p^n \) (with identical proofs, with one exception noted below) and extend to \( \delta_C : \mathbb{R}^n_{\geq 0} \to \mathbb{R} \), via density and continuity.

\textbf{SG 7 ([Tei12] Proposition 3.4).} For each cell \((F, G, H)\), there exists a cell \((U, V)\) such that \( \delta_{(F, G, H)} = \delta_{(U, V)} \).

\textbf{SG 8 ([Tei12] Theorem II).} Suppose the restriction of \( \delta_C \) to \((\mathbb{Q}_{\geq 0})_q^n \) attains a local maximum at \( u_0 \), in the sense that the values of \( \delta_C \) at all points of \((\mathbb{Q}_{\geq 0})_q^n \) adjacent to \( u_0 \) are smaller than \( \delta_C(u_0) \). Then

\[ \delta_C(t) = \delta_C(u_0) - d(t, u_0), \]

for all \( t \in \mathbb{R}^n_{\geq 0} \) with \( d(t, u_0) \leq \delta_C(u_0) \). In particular, \( \delta_C \) is piecewise linear on that region and has a local maximum at \( u_0 \) in the usual sense.

\textbf{SG 9 ([Tei12] Theorem III).} Suppose \( \delta_C \) has a local maximum at \( a/q \), where \( q > 1 \) and \( a \in \mathbb{N}_0 \) has some coordinate not divisible by \( p \). Then

\[ \delta_C\left(\frac{a}{q}\right) \leq \frac{n - 2}{q}. \]

As the proof of [Tei12] Theorem III uses the fact that the \( \delta_C \) are defined on a unit hypercube in an essential way, relying on symmetry and reflections, this requires explanation. Let \( a/q \) be as in the above statement. Set \( b = \lfloor a/q \rfloor \) and \( C_b = (F, G, H b) \). Then \( \delta_C(t) = \delta_{C_b}(t - b) \) for each \( t \) with \( t - b \in \mathbb{R}^n_{\geq 0} \), so \( \delta_{C_b} \) has a local maximum at \( a/q - b \). Since some \( a_i \) is prime to \( p \), so is the corresponding numerator \( a_i - b_i q \) of \( a/q - b \). Thus, the hypotheses of [Tei12] Theorem III hold for \( \delta_{C_b} \) at the point \( a/q - b \in [0, 1]_q^n \), and that result shows that \( \delta_C(a/q) = \delta_{C_b}(a/q - b) \leq (n - 2)/q \).

\textbf{Remark 2.10.} The function \( \delta_C \) is linear outside a bounded subset of \( \mathbb{R}^n_{\geq 0} \). To see that, we may assume that \( C = (U, V) \), by SG 7 and SG 8 allows us to assume that \( U \) and \( V \) are relatively prime; SG 7 then shows that \( \delta_C(a/q) = \|a/q\| - \deg UV \) whenever \( \|a/q\| \geq \deg UV \). By continuity, \( \delta_C(t) = \|t\| - \deg UV \), for each \( t \in \mathbb{R}^n_{\geq 0} \) with \( \|t\| \geq \deg UV \).

\textbf{Definition 2.11.} The largest unbounded closed set on which \( \delta_C \) is linear is the \textbf{trivial region of \( \delta_C \)}, and its complement in \( \mathbb{R}^n_{\geq 0} \) is the \textbf{nontrivial region}.

We close this section with some consequences of SG 8.

\textbf{Proposition 2.12.} Suppose \( t_0 \) lies in the nontrivial region of \( \delta_C \) and \( \delta_C(t_0) > 0 \). Then \( \delta_C \) attains a local maximum at a point \( u_0 \in \mathbb{Q}_{\geq 0}^n \) with \( d(t_0, u_0) < \delta_C(u_0) \). In particular, \( \delta_C(t) = \delta_C(u_0) - d(t, u_0) \) on a neighborhood of \( t_0 \).
Proof. Using the continuity of the function $t \mapsto \delta_C(t) - d(t, t_0)$ and the density of $\mathbb{Q}_p^n$ in $\mathbb{R}^n$, choose $u$ in some $\mathbb{Q}_p^n$ such that $\delta_C(u) - d(u, t_0) > 0$. Then $\delta_C(u) > d(u, t_0) \geq 0$, and SG [6] shows that $\delta_C > 0$ on the line segment joining $u$ and $t_0$. Since $\delta_C$ vanishes on the boundary of its trivial region, this shows that $u$ is also in the nontrivial region. Consider the equivalence relation on the set $S = \{v \in (\mathbb{Q}_{\geq 0})^n : \delta_C(v) > 0\}$ generated by adjacency. We shall show that the equivalence class of $u$ is contained in the nontrivial region, and is therefore finite. For that, it suffices to show that points $z, w \in S$, one in the trivial region and the other in the nontrivial region, cannot be adjacent. And indeed, if they were adjacent, then $|\delta_C(z) - \delta_C(w)| = 1/q = d(z, w)$, by SG [4] and SG [6] would imply that $\delta_C$ is linear on the line segment joining $z$ and $w$. This is impossible, since $\delta_C(z)$ and $\delta_C(w)$ are both positive, and $\delta_C$ vanishes on the boundary of the trivial region.

Since the equivalence class of $u$ is finite, we can choose a point $u_0$ in this class where $\delta_C$ is maximum. Then SG [3] shows that $\delta_C$ attains a local maximum at $u_0$ and that $\delta_C(u) = \delta_C(u_0) - d(u, u_0)$. To complete the proof, note that $d(t_0, u_0) \leq d(t_0, u) + d(u, u_0) < \delta_C(u) + (\delta_C(u_0) - \delta_C(u)) = \delta_C(u_0). \quad \square$

Remark 2.13. The above proof shows that if $u \in \mathbb{Q}_q^n$ lies in the nontrivial region of $\delta_C$ and $\delta_C(u) > 0$, then the local maximum $u_0$ that determines the behavior of $\delta_C$ near $u$ is also in $\mathbb{Q}_q^n$.

Two corollaries follow immediately:

Corollary 2.14. The function $\delta_C$ is piecewise linear with coefficients in $\mathbb{Q}_{p^\infty}$ on each connected component of its positive locus. \quad \square

Corollary 2.15. The local maxima of $\delta_C$ are attained at points in $\mathbb{Q}_q^n$. \quad \square

3 The functions $\Delta$ and $\Phi$; the upper and lower regions

Throughout this and the next two sections, $\ell$ is an $n$-tuple $(\ell_1, \ldots, \ell_n)$ of pairwise prime linear forms in $k[x, y]$, and $\mathfrak{b}$ is an ideal of $k[x, y]$ generated by relatively prime non-constant forms $U$ and $V$.

Definition 3.1. We use $\Delta$ to denote $\delta_{(U, V)}$, the syzygy gap fractal associated with the cell $(U, V)$ with respect to the linear forms $\ell_1, \ldots, \ell_n$. That is, $\Delta$ is the unique continuous function $\mathbb{R}_{\geq 0}^n \to \mathbb{R}$ such that

$$\Delta \left( \frac{a}{q} \right) = \frac{1}{q} \cdot \delta(U^q, V^q, \ell^a),$$

for each $q$ and for each $a \in \mathbb{N}^n$. We use $\Phi$ to denote the unique continuous function $\mathbb{R}_{\geq 0}^n \to \mathbb{R}$ such that

$$\Phi \left( \frac{a}{q} \right) = \frac{1}{q^2} \cdot \deg(U^q, V^q, \ell^a), \quad (3.1)$$
for each \( q \) and for each \( a \in \mathbb{N}^n \).

Because \( \mathbb{k}[x, y] \) is regular of dimension 2, for each ideal \( a \) of \( \mathbb{k}[x, y] \) we have \( \deg a[p] = p^2 \cdot \deg a \). Thus, \( \Phi(a/q) \) is well defined, and (3.1) gives us a function \((\mathbb{Q}_{\geq 0})^n_p \to \mathbb{Q}\). Below we shall justify the existence of the continuous extension of this function to \( \mathbb{R}^n_{\geq 0} \), tacitly assumed above. Furthermore, we shall see that \( \Delta \) and \( \Phi \) are both independent of the choice of the two homogeneous generators of the ideal \( b \), so they can be thought of as functions attached to \( \ell \) and \( b \).

**Lemma 3.2.** There exists a unique continuous extension to \( \mathbb{R}^n_{\geq 0} \) of the function \( \Phi : (\mathbb{Q}_{\geq 0})^n_p \to \mathbb{Q} \) defined by (3.1). Furthermore, both \( \Delta \) and \( \Phi \) depend only on the ideal \( b \), and not on the particular choice of generators \( U \) and \( V \). 

**Proof.** Both claims can be shown directly (the former, in much greater generality). Instead, we argue by relating \( \Delta \) and \( \Phi \). Corollary 2.6 shows that \( \Delta \) and \( \Phi \) are related as follows on \((\mathbb{Q}_{\geq 0})^n_p \):

\[
\Delta(t)^2 = 4(\Phi(t) - \deg b) + (\|t\| - \deg UV)^2. \tag{3.2}
\]

Since \( \Delta \) is defined and continuous on \( \mathbb{R}^n_{\geq 0} \), (3.2) can be used to (uniquely) extend \( \Phi \) to a continuous function \( \mathbb{R}^n_{\geq 0} \to \mathbb{R} \), establishing the first claim. To prove the second claim, first note that \( \deg UV \) depends only on \( b \), and not on the particular choice of generators \( U \) and \( V \): as \( \sqrt{b} = \langle x, y \rangle \), any two forms generating \( b \) must be a minimal set of generators; consequently, their degrees are univocally determined by \( b \). Next, observe that, as the restriction of \( \Phi \) to \((\mathbb{Q}_{\geq 0})^n_p \) is clearly independent of the choice of \( U \) and \( V \), so is \( \Phi \) itself, by continuity, and \textit{a fortiori} \( \Delta \), by (3.2) and the above observation on \( \deg UV \).

**Example 3.3.** The function \( \Phi \) attached to \( \ell = (x, y) \) and \( m = \langle x, y \rangle \) is as follows:

\[
\Phi\left(\frac{a}{q}\right) = \frac{1}{q^2} \cdot \deg\{x^q, y^q, x^{a_1}y^{a_2}\} = \begin{cases} \frac{a_1}{q} + \frac{a_2}{q} - \frac{a_1a_2}{q^2} & \text{if } a_1, a_2 < q \\ 1 & \text{otherwise} \end{cases}
\]

for each \( q \) and each \( a \in \mathbb{N}^2 \); so

\[
\Phi(t) = \begin{cases} t_1 + t_2 - t_1t_2 & \text{if } t_1, t_2 < 1 \\ 1 & \text{otherwise} \end{cases}
\]

for each \( t \in \mathbb{R}^2_{\geq 0} \), by continuity.

**Definition 3.4.** For \( u, v \in \mathbb{R}^n \) we write \( u \leq v \) if \( u_i \leq v_i \), for each \( i \). The relations \( \geq, <, \) and \( > \) on \( \mathbb{R}^n \) are defined likewise.

**Definition 3.5.** Let \( u, v \in \mathbb{R}^n \). Then

\[
[u, v] := \{t \in \mathbb{R}^n : u \leq t \leq v\} = [u_1, v_1] \times \cdots \times [u_n, v_n].
\]

The “intervals” \( (u, v) \), \( [u, v) \), \( [u, \infty) \), etc., are defined analogously.
Definition 3.6. The set \( \mathcal{T} = \{ t \in \mathbb{R}^n_{\geq 0} : \|t\| \geq \deg UV \} \) is the trivial region attached to \( b \), and its complement in \( \mathbb{R}^n_{\geq 0} \) is the nontrivial region.

Proposition 3.7 (Basic properties of \( \Phi \)).

1. \( \Phi \) is (weakly) increasing: \( t \leq u \Rightarrow \Phi(t) \leq \Phi(u) \).
2. \( 0 \leq \Phi(t) \leq \deg b \), for each \( t \).
3. \( \Phi(t) = \deg b \iff \Delta(t) = \|t\| - \deg UV \). In particular, \( \Phi \equiv \deg b \) on the trivial region \( \mathcal{T} = \{ t \in \mathbb{R}^n_{\geq 0} : \|t\| \geq \deg UV \} \).

Proof. (1) and (2) are clear for \( t, u \in (\mathbb{Q}^n_{\geq 0})_{p=\infty} \), and extend to all \( t, u \in \mathbb{R}^n_{\geq 0} \) by continuity. The first statement in (3) follows immediately from (3.2), while the second statement follows from the first and Remark 2.10, which states that \( \Delta(t) = \|t\| - \deg UV \) whenever \( \|t\| \geq \deg UV \).

Convention. All topological notions used will refer to the subspace topology of \( \mathbb{R}^n_{\geq 0} \) induced by the standard topology of \( \mathbb{R}^n \).

Notation. If \( X \subseteq \mathbb{R}^n_{\geq 0} \), then \( \overline{X} \) and \( \partial X \) denote the closure and the boundary of \( X \) in \( \mathbb{R}^n_{\geq 0} \).

Definition 3.8. The upper and lower regions attached to \( \ell \) and \( b \) are the sets

\[ \mathcal{U} = \{ t \in \mathbb{R}^n_{\geq 0} : \Phi(t) = \deg b \} = \Phi^{-1}(\deg b) \]

and

\[ \mathcal{L} = \{ t \in \mathbb{R}^n_{\geq 0} : \Phi(t) < \deg b \} = \Phi^{-1}([0, \deg b)). \]

The set \( \mathcal{B} \) is the common boundary \( \partial \mathcal{U} = \partial \mathcal{L} \) of those regions in \( \mathbb{R}^n_{\geq 0} \).

Very often the choice of \( \ell \) and \( b \) will be clear from the context (or fixed in advance, as in this section), so we shall omit the phrase “attached to \( \ell \) and \( b \)” and ask that the reader rely on the context to determine the exact setup.

Remark 3.9. If \( a \in \mathbb{N}^n \) and \( q \) is a power of \( p \), then

\[ a/q \in \mathcal{U} \iff \ell^a \in b^{[q]} \quad \text{and} \quad a/q \in \mathcal{L} \iff \ell^a \not\in b^{[q]} . \]

Example 3.10. In the setting of Example 3.3, the lower region \( \mathcal{L} \) is the square \([0,1)^2\). Less trivial instances can be seen in Examples 4.7, 6.1, and 7.13.

Some properties of the regions \( \mathcal{U} \), \( \mathcal{L} \), and \( \mathcal{B} \) follow immediately from Proposition 3.7 and the continuity of \( \Phi \):

Corollary 3.11 (Basic properties of the regions \( \mathcal{U} \), \( \mathcal{L} \), and \( \mathcal{B} \)).

1. \( \mathcal{L} \) is open and \( \mathcal{U} \) and \( \mathcal{B} \) are closed in \( \mathbb{R}^n_{\geq 0} \).
2. \( \mathcal{U} \) contains the trivial region \( \mathcal{T} = \{ t \in \mathbb{R}^n_{\geq 0} : \|t\| \geq \deg UV \} \).
3. \( \mathcal{L} \) is contained in the nontrivial region, and is therefore bounded.
(4) If $u \in \mathcal{U}$, then $[u, \infty) \subseteq \mathcal{U}$.

(5) If $u \in \mathcal{L}$, then $[0, u] \subseteq \mathcal{L}$.

(6) If $u \in \mathcal{B}$, then $[0, u] \subseteq \mathcal{L}$ and $[u, \infty) \subseteq \mathcal{U}$.

The lower region (in the case where $b = \langle x, y \rangle$) was studied by Pérez [Per13] under a different guise—as the first constancy region of the mixed test ideals $\tau(\ell^t)$. The upper and lower regions (and a function that measures how far from the origin the upper region is in each direction; see Definition 4.2) are the central concepts of this paper, and can be defined in much greater generality and independently of our functions $\Phi$ and $\Delta$ (as Pérez’s work shows). But not only do their definitions arise naturally in the context of those functions, but structural properties of syzygy gap fractals yield some nice properties of those regions, with interesting consequences—hence the approach chosen here.

4 The $F$-threshold function

Let $\ell = (\ell_1, \ldots, \ell_n)$ and $b = \langle U, V \rangle \subseteq k[x, y]$ be as in the previous section. As before, we use $\Delta$ and $\Phi$ to denote the unique continuous functions $\mathbb{R}_{\geq 0}^n \to \mathbb{R}$ such that $\Delta(a/q) = q^{-1} \cdot \delta(U^q, V^q, \ell^a)$ and $\Phi(a/q) = q^{-2} \cdot \deg(U^q, V^q, \ell^a)$, for every $a \in \mathbb{N}^n$ and for every $q = p^r$.

**Discussion 4.1.** The $F$-threshold of a polynomial $G \in \mathfrak{m} = \langle x, y \rangle$ with respect to $b$ can be defined as follows:

$$ft_b(G) = \inf \{k/q \in (\mathbb{Q}_{> 0})_{p^{\infty}} : G^k \in b^{[q]} \}.$$  

Though it is not at all obvious from this definition, it turns out that $ft_b(G)$ is a rational number (see [BMS08, Corollary 2.30, Theorem 3.1]). When $b = \mathfrak{m}$, $ft_b(G)$ is the $F$-pure threshold of $G$, denoted by $\text{fpt}(G)$.

The condition $G^k \in b^{[q]}$ is equivalent to $\deg(U^q, V^q, G^k) = \deg(U^q, V^q)$ (where $\deg(U^q, V^q) = q^2 \cdot \deg b$), so in the special case where $G = \ell^a$, for some $a \in \mathbb{N}^n_{> 0}$, we find that

$$ft_b(\ell^a) = \inf \{k/q \in (\mathbb{Q}_{> 0})_{p^{\infty}} : \Phi(k/q \cdot a) = \deg b \}.$$  

The continuity of $\Phi$ and the density of $(\mathbb{Q}_{> 0})_{p^{\infty}}$ in $\mathbb{R}_{> 0}$ then allow us to write

$$ft_b(\ell^a) = \min \{\lambda \in \mathbb{R}_{> 0} : \Phi(\lambda a) = \deg b \} = \min \{\lambda \in \mathbb{R}_{> 0} : \lambda a \in \mathcal{U} \},$$  

which motivates the following definition.

**Definition 4.2.** For each $t \in \mathbb{R}_{\geq 0}^n$ we define

$$ft_b(t^t) = \min \{\lambda \in \mathbb{R}_{> 0} : \Phi(\lambda t) = \deg b \} = \min \{\lambda \in \mathbb{R}_{> 0} : \lambda t \in \mathcal{U} \}.$$

These descriptions agree, by definition of the upper region $\mathcal{U}$, and the latter description shows that $ft_b(t^t)$ is well defined: as $\mathcal{U}$ is closed and its complement
\( \mathcal{L} \) is bounded (see Corollary 3.11), the minima appearing in these definitions exist. The function \( t \mapsto f^b(\ell t) \) is the \( F \)-threshold function attached to \( \ell \) and \( b \).

The following are alternate characterizations of \( f^b(\ell t) \):

\[
\begin{align*}
 f^b(\ell t) &= \min \left\{ \lambda \in \mathbb{R}_> 0 : \Delta(\lambda t) = |\lambda||t| - \deg UV \right\} \\
 &= \max \left\{ \lambda \in \mathbb{R}_> 0 : \lambda t \in \mathcal{L} \right\} \\
 &= \text{the unique } \lambda \in \mathbb{R}_> 0 \text{ such that } \lambda t \in \mathcal{B}.
\end{align*}
\]

The last characterization, which depends on Corollary 3.11(6), is perhaps the best way to think of the \( F \)-threshold function: \( f^b(\ell t) \) is the exact factor by which \( t \) needs to be scaled to obtain a point of the boundary \( \mathcal{B} \).

Remark 4.3. Fix \( t \in \mathbb{R}_n > 0 \). The observation that \( \deg UV \|t\| \cdot t \) has norm \( \deg UV \) (and hence lies in the trivial region \( \mathcal{T} \), and therefore in \( \mathcal{W} \)) shows that the factor by which \( t \) needs to be scaled to obtain a point of the boundary \( \mathcal{B} \) is at most \( \deg UV/\|t\| \). Restated, we see that \( f^b(\ell t) \leq \deg UV/\|t\| \).

In future sections we shall try to understand in what ways the value of \( f^b(\ell t) \) differs from this natural upper bound.

Example 4.4. In the situation of Examples 3.3 and 3.10 (\( m = \langle x, y \rangle \) and \( \ell = \langle x, y \rangle \)), the closure of the lower region is the unit square \([0, 1]^2\), so \( f^m(\ell t) = \max \left\{ \lambda \in \mathbb{R}_> 0 : \lambda t \in [0, 1]^2 \right\} = \min \left\{ \frac{1}{t_1}, \frac{1}{t_2} \right\} \).

Directly from the definition we have:

Proposition 4.5. \( f^b(\ell \lambda t) = \frac{1}{\lambda} \cdot f^b(\ell t) \), for each \( \lambda \in \mathbb{R}_> 0 \) and \( t \in \mathbb{R}_n > 0 \).

The above result and the rationality of \( F \)-thresholds of polynomials imply that \( f^b(\ell t) \) is rational whenever \( t \in \mathbb{Q}_n > 0 \).

Proposition 4.6. The \( F \)-threshold function is continuous.

Proof. To show continuity at \( c \in \mathbb{R}_n > 0 \), Proposition 4.5 allows us to scale \( c \) and assume that \( c \in \mathcal{B} \) or, equivalently, \( f^b(\ell c) = 1 \). If \( t \in \mathbb{R}_n > 0 \), then the factors by which \( t \) needs to be scaled to obtain points of \( \partial(0, c) \) and \( \partial(c, \infty) \) are, respectively, \( \min\{c_i/t_i\} \) and \( \max\{c_i/t_i\} \). By Corollary 3.11(6) the scaling factor needed to obtain a point of \( \mathcal{B} \) lies somewhere in between:

\[
\min\left\{ \frac{c_1}{t_1}, \ldots, \frac{c_n}{t_n} \right\} \leq f^b(\ell t) \leq \max\left\{ \frac{c_1}{t_1}, \ldots, \frac{c_n}{t_n} \right\}. \tag{4.1}
\]

(Figure 1 illustrates this for \( n = 2 \).) As \( t \to c \), both \( \min\{c_i/t_i\} \) and \( \max\{c_i/t_i\} \) tend to 1, so \( f^b(\ell t) \to 1 = f^b(\ell c) \), showing continuity at \( c \). \( \square \)

\(4\)In convex geometry this kind of function is called a radial function; specifically, this is the radial function attached to the closure of \( \mathcal{L} \).
Figure 1: An illustration of inequalities (4.1) in the case $n = 2$

Figure 2: The syzygy gap fractal of Example 4.7 and its upper and lower regions

Notation. Let $x \in \mathbb{R}^n_{\geq 0}$ and $r \in \mathbb{R}_{>0}$. Then $B_r(x)$ and $\overline{B}_r(x)$ denote the open and closed taxicab balls (in $\mathbb{R}^n_{\geq 0}$) of radius $r$ centered at $x$.

Example 4.7. Let $b = \langle x^3 + y^3 + xy^2, x^2y^3 \rangle \subseteq \mathbb{F}_5[x, y]$ and $\ell = (x, y)$. Figure 2(a) shows a density plot of the function $\Delta$ attached to $\ell$ and $b$ on the square $[0, 8]^2$, and Figure 2(b) shows the corresponding upper and lower regions. As the plot suggests, $\Delta$ attains a local maximum at $c = (2, 3)$. SG 8 confirms that, and also shows that $\Delta(t) = 3 - |t_1 - 2| - |t_2 - 3|$ on $\overline{B}_3(c)$. In particular, on $\overline{B}_3(c)$ we have $\Delta(t) = 8 - ||t||$ if and only if $t \geq c$. Proposition 3.7(3) then shows that $\mathcal{W} \cap \overline{B}_3(c) = [c, \infty) \cap \overline{B}_3(c)$, and thus $\mathcal{B} \cap \overline{B}_3(c) = \partial(c, \infty) \cap \overline{B}_3(c)$. Consequently, $\mathcal{B}^b(t^*) = \max\{2/t_1, 3/t_2\}$ on the cone over $\partial(c, \infty) \cap \overline{B}_3(c)$, which appears shaded in Figure 2(b).

The point $c$ plays a special role because it is a local maximum “adjacent to the trivial region”. There are five such points in this example: $c_1 = c = (2, 3)$, $c_2 = (0, 7)$, $c_3 = (1, 6)$, $c_4 = (5, 1)$, and $c_5 = (7, 0)$, all shown in Figure 2(b). Each of those points determines the $F$-threshold function locally, and together they determine the $F$-threshold function globally: $\mathcal{W} = \bigcup_{i=1}^5 [c_i, \infty)$, and therefore
Figure 3: A critical point \( c \)

\[ \text{ft}^b(\ell^t) \] equals the minimum value of the set

\[ \left\{ \max\left\{ \frac{2}{t_1}, \frac{3}{t_2} \right\}, \max\left\{ \frac{0}{t_1}, \frac{2}{t_2} \right\}, \max\left\{ \frac{1}{t_1}, \frac{6}{t_2} \right\}, \max\left\{ \frac{5}{t_1}, \frac{1}{t_2} \right\}, \max\left\{ \frac{7}{t_1}, \frac{0}{t_2} \right\} \right\}. \]

Special points like those in the above example will be examined in detail in the next section. As we shall see, Example 4.7 is illustrative of the general situation in that the lower region has a “staircase” aspect determined by such special points, and that the \( F \)-threshold function is determined by those points. But Example 4.7 is also misleadingly simple. In more general situations there are typically infinitely many of those special points; very often those points do not have integer coordinates, and \( \mathcal{W} \) is not the union of boxes \([c, \infty)\) determined by them.

5 Critical points

Throughout this section we fix \( \ell = (\ell_1, \ldots, \ell_n) \) and \( b = (U, V) \) as in Section 4 and again use \( \Delta \) and \( \Phi \) to denote the functions \( \mathbb{R}^n_\geq \to \mathbb{R} \) defined in terms of these choices, as in Definition 3.1.

**Definition 5.1.** A point \( c \in \mathbb{R}^n_\geq \) is a critical point associated with \( \ell \) and \( b \) if \( \Delta \) attains a local maximum at \( c \) and that local maximum is adjacent to the trivial region \( \mathcal{T} \), in the sense that \( \Delta(c) = \deg UV - \|c\| \) (so the ball \( B_{\Delta(c)}(c) \) on which \( \Delta \) is determined by \( c \) touches the trivial region; see Figure 3).

If \( \ell \) and \( b \) have been fixed (as they were in this section) or are clear from the context, we shall call those points simply critical points. By Corollary 2.15 all critical points lie in \( \mathbb{Q}^n_{\geq} \); this observation will be fundamental in the proofs of our main results, in Sections 7 and 8. Critical points forge the (often extremely complex) boundary \( \mathcal{B} \) locally—if \( c \) is a critical point, then \( \mathcal{B} \) agrees with the (very simple) boundary of \([c, \infty)\) near \( c \), as shown in the next
proposition. Consequently, the $F$-threshold function agrees with the function $t \mapsto \max\{c_1/t_1, \ldots, c_n/t_n\}$ near $c$. (Proposition 5.3 will make precise what “near” means.)

**Proposition 5.2.** Let $c$ be a critical point, and set $r = \Delta(c)$.

1. For $t \in \overline{B}_r(c)$ we have $\Delta(t) = \deg UV - \|t\|$ if and only if $t \geq c$.

Consequently,

2. $\mathcal{W} \cap \overline{B}_r(c) = [c, \infty) \cap \overline{B}_r(c)$,

3. $\mathcal{B} \cap \overline{B}_r(c) = \partial[c, \infty) \cap \overline{B}_r(c)$ (so, in particular, $c \in \mathcal{B}$), and

4. $[c, \infty) \subseteq \mathcal{W}$.

**Proof.** If $t \in \overline{B}_r(c)$, then $\Delta(t) = \Delta(c) - d(t, c) = \deg UV - \|c\| - \|t - c\|$, by SG and the definition of critical point, so $\Delta(t) = \deg UV - \|t\|$ if and only if $\|c\| + \|t - c\| = \|t\|$. As $t$ and $c$ have nonnegative coordinates, the latter condition is equivalent to $t \geq c$, giving (1). Each $t \in \overline{B}_r(c)$ has norm $\leq \deg UV$, so $t \in \mathcal{W}$ if and only if $\Delta(t) = \deg UV - \|t\|$, by Proposition 3.7(3) parts (2)–(4) now follow easily from (1).

**Proposition 5.3.** If $t \in \mathbb{R}^n_{>0}$ and $c$ is a critical point, then

$$ft^b(\ell^t) = \max\left\{\frac{c_1}{t_1}, \ldots, \frac{c_n}{t_n}\right\} \iff c \leq \frac{\deg UV}{\|t\|} \cdot t.$$

**Proof.** In what follows, we shall show that

$$ft^b(\ell^t) = \max\left\{\frac{c_1}{t_1}, \ldots, \frac{c_n}{t_n}\right\} \iff \max\left\{\frac{c_1}{t_1}, \ldots, \frac{c_n}{t_n}\right\} \leq \frac{\deg UV}{\|t\|}, \quad (5.1)$$

which is clearly equivalent to the asserted statement. Set $\gamma = \max\{c_i/t_i\}$. Assuming $ft^b(\ell^t) = \gamma$, Remark 4.3 implies that $\gamma = ft^b(\ell^t) \leq \deg UV/\|t\|$. To establish the remaining implication, suppose $\gamma \leq \deg UV/\|t\|$. By definition of $\gamma$, we have $\gamma t \in \partial[c, \infty)$. In particular, $\gamma t \geq c$, and combining this with our assumption that $\gamma \leq \deg UV/\|t\|$ we see that

$$\|\gamma t - c\| = \gamma \|t\| - \|c\| \leq \deg UV - \|c\| = \Delta(c),$$

where the last equality follows from the fact that $c$ is a critical point. In summary, we have just seen that $\gamma t \in \partial[c, \infty) \cap \overline{B}_{\Delta(c)}(c)$, and Proposition 5.3 then shows that $\gamma t \in \mathcal{B}$ as well. The fact that $ft^b(\ell^t) = \gamma$ then follows, as $ft^b(\ell^t)$ is the unique factor by which $t$ is scaled to obtain an element of $\mathcal{B}$.

We now move towards the proof of some simple characterizations of critical points. For that, we need the following elementary result.

**Lemma 5.4.** Let $F, G \in R := \mathbb{k}[x, y]$ be relatively prime forms of degrees $a$ and $b$, respectively. Then $\langle x, y \rangle^{a+b-1} \subseteq \langle F, G \rangle$. 

16
Proof. As \( F \) and \( G \) are relatively prime, the \( k \)-linear map \( R_{b-1} \times R_{a-1} \rightarrow R_{a+b-1} \), \((A, B) \mapsto AF + BG\), is injective. But the \( k \)-dimensions of the domain and the codomain of the map are the same, so the map is surjective as well. \(\Box\)

**Proposition 5.5.** Let \( c = a/q \in (\mathbb{Q}_{\geq 0})^n \). The following are equivalent:

1. \( c \) is a critical point;
2. \( c \in \mathcal{U} \) and \( c - t e_i \in \mathcal{L} \), for each \( i \) such that \( c_i > 0 \) and \( 0 < t \leq c_i \);
3. \( c \in \mathcal{U} \) and \( c - e_i/q \in \mathcal{L} \), for each \( i \) such that \( c_i > 0 \);
4. \( \ell^a \in b[q] \) and \( \ell^{a-e_i} \not\in b[q] \), for each \( i \) such that \( a_i > 0 \).

Proof. Proposition 5.2(2) and Corollary 3.11(5) show that (1) \(\Rightarrow\) (2); (2) \(\Rightarrow\) (3) is immediate; Remark 3.9 shows that (3) \(\iff\) (4). We now show that (3) \(\Rightarrow\) (1).

Rewrite equation (3.2) as follows:

\[
A (\deg UV - \|t\| + \Delta(t)) = B (\deg UV - \|t\| - \Delta(t)) = C (\deg b - \Phi(t)).
\]

(5.2)

Substituting \( t = c \), the first clause of (3) shows that \( C = 0 \). Moreover, as \( \langle x, y \rangle^q \deg UV - 1 \subseteq b[q] \), by Lemma 5.4, the second clause of (3) implies that \( \|c\| < \deg UV \), so \( B > 0 \). It follows that \( A = 0 \), so \( \Delta(c) = \deg UV - \|c\| > 0 \).

It remains to prove that \( \Delta \) attains a local maximum at \( c \). Note that SG 4 tells us that each \( \Delta(c \pm e_i/q) \) is either \( \Delta(c) - 1/q \) or \( \Delta(c) + 1/q \). We claim that the latter cannot happen; the result will then follow from SG 8.

Substituting \( t = c + e_i/q \) in (5.2) and supposing that \( \Delta(c + e_i/q) = \Delta(c) + 1/q \) we reach an impossibility: \( A = -2/q \) and \( B = 2\Delta(c) > 0 \), while \( C = 0 \) (since \( c + e_i/q \in \mathcal{U} \)). If \( c_i > 0 \), then substituting \( t = c - e_i/q \) in (5.2) and supposing that \( \Delta(c - e_i/q) = \Delta(c) + 1/q \) we again reach an impossibility: \( A = 0 \), while \( C > 0 \), by the second clause of (3.) This establishes the claim. \(\Box\)

The above proposition shows that a critical point is a minimal point of \( \mathcal{U} \) with respect to \( \leq \). It also shows a partial converse: a minimal point of \( \mathcal{U} \cap Q^n_{\geq 0} \) is a critical point. The actual converse, however, does not hold—there are minimal points of \( \mathcal{U} \) that are not critical points.

A consequence of that minimality is that if \( b \) and \( c \) are critical points with \( b \leq c \), then \( b = c \). We shall use this fact often, without further comment.

**Corollary 5.6.** Let \( u \in (\mathbb{Q}_{\geq 0})^n_\mathcal{U} \). Then \( u \in \mathcal{U} \) if and only if there exists a critical point \( c \in (\mathbb{Q}_{\geq 0})^n_\mathcal{U} \) such that \( c \leq u \).

Proof. If \( u \in \mathcal{U} \), choose \( c \in \mathcal{U} \cap \mathcal{U} \cap Q^n_{\geq 0} \) of minimum norm; then \( c \) satisfies condition (3) of Proposition 5.5, and is therefore the desired critical point. The converse follows directly from Proposition 5.4(1). \(\Box\)
Remark 5.7. While a point in $\mathcal{U}$ is typically (strictly) greater than several critical points, that is not so for points in $\partial \mathcal{F} = \{ t \in \mathbb{R}^n_{>0} : \| t \| = \deg UV \}$.

Indeed, suppose $b \not= c$ are critical points with $b, c \leq u \in \partial \mathcal{F}$. Consider the sum $d(u, b) + d(u, c)$. Since $u \in \partial \mathcal{F}$, this sum equals $\Delta(b) + \Delta(c)$, which is $\leq d(b, c)$, by SG\[1] The triangle inequality then implies that $d(u, b) + d(u, c) = d(b, c)$.

This fact, the assumption that $\max b$ lies in $\mathcal{U}$, and the identity $|b - c| + b + c = 2\max\{b, c\}$ show that $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} \max\{b_i, c_i\}$. But $u_i \geq \max\{b_i, c_i\}$, for each $i$, and thus $u_i = \max\{b_i, c_i\}$, for each $i$. In particular, $b, c \not\leq u$.

The facts proven above lead up to the following “structure theorem” for the $F$-threshold function.

Theorem 5.8. Let $\ell = (\ell_1, \ldots, \ell_n)$, where the $\ell_i$ are pairwise prime linear forms in $k[x, y]$, and $b = \langle U, V \rangle$, where $U$ and $V$ are relatively prime non-constant forms in $k[x, y]$. Fix $t \in \mathbb{R}^n_{>0}$ and set $\lambda = \deg UV / \| t \|$. If a truncation $(\lambda t)_e$ lies in $\mathcal{U}$, then there exists a (unique) critical point $c \leq (\lambda t)_e$ associated with $\ell$ and $b$ with coordinates in $\mathbb{Q}p^\ell$, and $\ft^b(t)$ is determined by $c$: 

$$\ft^b(t) = \max\left\{ \frac{c_1}{t_1}, \ldots, \frac{c_n}{t_n} \right\} < \lambda.$$ 

Otherwise, $\ft^b(t)$ is determined by the trivial region attached to $b$:

$$\ft^b(t) = \lambda = \frac{\deg UV}{\| t \|}.$$ 

Proof. If $(\lambda t)_e \in \mathcal{L}$, for each $e$, then $\lambda t \in \partial \mathcal{F} \subseteq \mathcal{U}$, $\lambda t \in \mathcal{F}$, whence $\ft^b(t) = \lambda$. If $(\lambda t)_e \in \mathcal{W}$, for some $e$, then there exists a critical point $c = a/p^e \leq (\lambda t)_e < \lambda t$, by Corollary 5.6 and Proposition 5.3 shows that $\ft^b(t) = \max\{c_i/t_i\}$.

The way the $F$-threshold function is determined by critical points (and the trivial region) can be compactly stated as follows.

Corollary 5.9. For each $t \in \mathbb{R}^n_{>0}$,

$$\ft^b(t) = \min\left\{ \left\{ \max\left\{ \frac{c_1}{t_1}, \ldots, \frac{c_n}{t_n} \right\} : c \text{ is a critical point} \right\} \cup \left\{ \frac{\deg UV}{\| t \|} \right\} \right\}.$$ 

Remark 5.10. Suppose that $c$ is a critical point, so that, in particular, $\Delta(c) = \deg UV - \| c \|$. It follows from the inequality $\| c \| / \| t \| \leq \max\{c_i/t_i\}$ that

$$\frac{\Delta(c)}{\| t \|} = \frac{\deg UV}{\| t \|} - \| c \| / \| t \| \geq \frac{\deg UV}{\| t \|} - \max\left\{ \frac{c_1}{t_1}, \ldots, \frac{c_n}{t_n} \right\}. \quad (5.3)$$

If, in addition, we suppose that $\max\{c_i/t_i\} \leq \deg UV / \| t \|$, then \[5.1\] and \[5.3\] give us the following:

$$\frac{\deg UV}{\| t \|} - \ft^b(t) \leq \frac{\Delta(c)}{\| t \|}.$$ 

18
This will be particularly relevant to us in the case where $c$ is not in $\mathbb{N}^n$; then we can write $c = a/q$, where $q > 1$ and some $a_i$ is prime to $p$, and SG 9 gives
\[
\frac{\deg UV}{\|t\|} - \tilde{t}^b(\ell^t) \leq \frac{n - 2}{q\|t\|}.
\]

6 The case $n = 3$: the “Sierpiński staircases”

In this section we examine the case $n = 3$ in detail. Let $\ell_1, \ell_2, \ell_3 \in k[x, y]$ be pairwise prime linear forms, $\ell = (\ell_1, \ell_2, \ell_3)$, $m = (x, y)$, and $b = (U, V)$, where $U, V \in k[x, y]$ are relatively prime non-constant forms.

Example 6.1. Figure 4 shows the lower regions attached to $\ell = (\ell_1, \ell_2, \ell_3)$ and $m$ in various characteristics. Figure 5 shows the lower regions attached to $\ell = (x, y, x + y)$ and various ideals $b$ in characteristic 5. Because of the conspicuous presence of Sierpiński gaskets and analogues, we call those regions “Sierpiński staircases”.

Remark 6.2. SG 9 tells us that if the function $\Delta$ attached to $\ell$ and $b$ attains a local maximum at $a/q$, where $q > 1$ and some $a_i$ is prime to $p$, then $\Delta(a/q) = 1/q$. This is reflected in some unique features of critical points in the case $n = 3$. Suppose $c = a/q$, with $a$ and $q$ as above, is a critical point.

(1) As $\Delta(c) = 1/q$, we have $\|c\| + 1/q = \deg UV$, by definition of critical point. That is, $c$ is at distance $1/q$ from the trivial region.

(2) If $u \in (c, \infty) \cap \overline{B}_{1/q}(c)$, then $c$ is a truncation of $u$. This follows from Remark 2.3, since $c < u \leq c + (1, 1, 1)/q$ and $c \in \mathbb{Q}^3_q$.

Since $m$ is invariant under linear changes of variables, those regions do not depend on the choice of $\ell$; in fact, through a change of variables we may assume that $\ell = (x, y, x + y)$. 
Theorem 6.3. Let \( \ell = (x, y, x + y) \) and the ideals \((x^3, y^3), (x^5 + y^5), (x^3 + x^2y + y^3)\)

(3) Likewise, if \( u \in [c, \infty) \cap B_{1/q}(c) \), then \( c \) is a truncation of the “possibly terminating” base \( p \) expansion of \( u \).

The following theorem describes the \( F \)-threshold function attached to \( \ell \) and \( b \). The notation \( \{ \lambda \} \) is used for the fractional part of a real number \( \lambda \); that is, \( \{ \lambda \} = \lambda - |\lambda| \). Furthermore, we adopt notation analogous to standard decimal notation when expressing real numbers base \( p \). For example, \((1.234)_{5}\) denotes the base 5 representation of \( 1 + 2/5 + 3/25 + 4/125 = 194/125 \).

**Theorem 6.3.** Let \( t \in \mathbb{R}^3_{>0} \), and set \( u = \deg UV \cdot t/\|t\| \).

1. If \( |u| \in \mathcal{L} \), then there exists a critical point \( c \leq u \) with integer coordinates that determines \( ft^b(\ell^t) \), that is, \( ft^b(\ell^t) = \max\{c_1/t_1, c_2/t_2, c_3/t_3\} \).

2. If \( |u| \in \mathcal{L} \), write \( \{u_1\} = (0, \alpha_1\alpha_2\alpha_3 \cdots)_p \), \( \{u_2\} = (0, \beta_1\beta_2\beta_3 \cdots)_p \), and \( \{u_3\} = (0, \gamma_1\gamma_2\gamma_3 \cdots)_p \), and let \( e = \inf\{s \geq 1 : \alpha_s + \beta_s + \gamma_s \neq 2p - 2\} \).

Then:

(a) If \( e = \infty \), then \( u \in \mathcal{B} \) and \( ft^b(\ell^t) = \deg UV/\|t\| \).

(b) If \( e < \infty \), then \( \alpha_e + \beta_e + \gamma_e \) equals either \( 2p - 1 \) or \( 2p - 3 \).

(i) If \( \alpha_e + \beta_e + \gamma_e = 2p - 3 \), then \( u \in \mathcal{B} \) and \( ft^b(\ell^t) = \deg UV/\|t\| \).

(ii) If \( \alpha_e + \beta_e + \gamma_e = 2p - 1 \), then the truncation \( \langle u \rangle_e \) is a critical point, and \( ft^b(\ell^t) = \max\{\langle u_1 \rangle_{t_1}, \langle u_2 \rangle_{t_2}, \langle u_3 \rangle_{t_3}\} \).

**Remark 6.4.** Fix \( t \in \mathbb{N}^3_{>0} \), and adopt notation as above. It can be shown that the infimum \( e \) depends only on the congruence class of \( p \) modulo \( \|t\| \), and so does the sum \( \alpha_e + \beta_e + \gamma_e \), if \( e < \infty \). Thus, if \( |u| \in \mathcal{L} \), then whether \( ft^b(\ell^t) \) falls under case (2a), (2b-i), or (2b-ii) depends only on the class of \( p \) modulo \( \|t\| \).

It follows that for each prime \( p \) and each \( \ell \) and \( b \) in characteristic \( p \) such that \( |u| \in \mathcal{L} \), the \( F \)-threshold \( ft^b(\ell^t) \) is an expression in \( p \) that depends only on the congruence class of \( p \) modulo \( \|t\| \) and the degrees of the generators of \( b \).
Remark 6.5. It is a general fact (see, e.g., [MTW05, Proposition 1.9] or [Her12, Key Lemma 3.1]) that the numerical invariant \( ft^b(t^\ell) \) determines the value of \( \nu(p^\ell) = \max\{a : (t^\ell)^a \notin b[p^\ell]\} \) for every nonnegative integer \( \ell \). Specializing to the context of Theorem 6.3 with \( t \in \mathbb{N}_{>0} \), the relation between \( ft^b(t^\ell) \) and \( \nu(p^\ell) \) allows us to restate the conclusion of Remark 6.4 in the following way: for each prime \( p \) and each \( \ell \) in characteristic \( p \) such that \( |u| \in \mathcal{L} \), if \( \ell \geq 1 \), then \( \nu(p^\ell) \) is a polynomial in \( p \) (of degree \( \ell \)) whose coefficients depend only on the class of \( p \) modulo \( |t| \) and the degrees of the generators of \( b \). Thus, in this setting, we are able to give a positive answer to [MTW05, Problem 3.10].

Remark 6.6. When \( b = m \), part (1) of Theorem 6.3 may be replaced by the following more precise statement:

(1') If \( u_i \geq 1 \), for some \( i \), then \( ft^m(t^\ell) = 1/t_i \).

Indeed, \( |u| \in \mathcal{W} \) if and only if there exists a critical point \( c \leq u \) with integer coordinates, and in this case the only critical points with integer coordinates are the canonical basis vectors \( e_1, e_2, \) and \( e_3 \) (a consequence of Proposition 5.5(4)). So \( |u| \in \mathcal{W} \) if and only if \( u_i \geq 1 \), for some (unique) \( i \), in which case \( ft^m(t^\ell) \) is determined by the critical point \( e_i \): \( ft^m(t^\ell) = 1/t_i \).

Proof of Theorem 6.3. Part (1) follows from Corollary 5.6 and Proposition 5.3. Suppose \( |u| \in \mathcal{L} \) and write each \( \{u_i\} \) as in the statement of the theorem. If \( \alpha_s + \beta_s + \gamma_s = 2p - 2 \), for all \( s \), then no truncation of \( u \) lies in \( \mathcal{W} \). This is indeed the case for the 0th truncation, as \( \langle u \rangle_0 \leq |u| \in \mathcal{L} \), and if any other truncation were in \( \mathcal{W} \), it would lie above a non-integral critical point, by Corollary 5.6. That critical point would be a truncation of \( u \), by Remark 6.4, but each truncation of \( u \) violates Remark 6.2(1). This establishes our claim, and Theorem 5.8 gives (2a).

Suppose now that \( e = \inf\{s \geq 1 : \alpha_s + \beta_s + \gamma_s \neq 2p - 2\} < \infty \). As \( \langle x, y \rangle^{\deg UV - 1} \leq b \), by Lemma 5.4, the assumption that \( |u| \in \mathcal{L} \) implies that \( \{u_1\} + \{u_2\} + \{u_3\} \leq \deg UV - 2 \). Since \( |u| = \deg UV \), we conclude that \( \{u_1\} + \{u_2\} + \{u_3\} = \deg UV - 2 \) and \( \{u_1\} + \{u_2\} + \{u_3\} = 2 \), and the last equation and the fact that \( \alpha_s + \beta_s + \gamma_s = 2p - 2 \) for each \( s < e \) imply that \( \alpha_s + \beta_s + \gamma_s \) is either \( 2p - 1 \) or \( 2p - 3 \).

If \( \alpha_s + \beta_s + \gamma_s = 2p - 3 \), then the subsequent digits of each \( \{u_i\} \) must all be \( p - 1 \) (otherwise the \( \{u_i\} \) would not add up to 2). The argument used in the first paragraph then shows that no truncation of \( u \) lies in \( \mathcal{W} \), and thus \( u \in \mathcal{B} \).

If \( \alpha_s + \beta_s + \gamma_s = 2p - 1 \), we must show that \( c := \langle u \rangle \) is a critical point. As \( \|c\| + 1/p^e = \deg UV \), the point \( c \) is at distance 1/p from the boundary of the trivial region, where the function \( \Delta \) attached to \( \ell \) and \( b \) vanishes. SG 3 shows that \( \Delta(c) = 1/p^e \), so \( \Delta(c) = \deg UV - \|c\| \), and Proposition 5.7(3) shows that \( c \in \mathcal{W} \). By Corollary 5.6, there exists a critical point \( c' \leq c \). Either \( c' \in \mathbb{N}^3 \) or \( c' \) is a truncation of the terminating base \( p \) expansion of \( c \), by Remark 6.3. But the assumption that \( |u| \in \mathcal{L} \) rules out the first possibility, and each truncation of \( c \) other than \( c \) itself violates Remark 6.2(1) so \( c' = c \). \( \square \)
The two situations in Theorem 6.3(2) that lead to the conclusion that \( u \in \mathcal{B} \) are characterized by the following property:

\[
\{u_1\}, \{u_2\}, \text{ and } \{u_3\} \text{ have base } p \text{ expansions (terminating or not)}
\]

whose digits add up to \( 2p - 2 \) in every spot.

Indeed, in (2b-i) we may obtain such base \( p \) expansions by replacing the non-terminating expansion of one of those numbers with the terminating expansion, while in (2b-ii) that would not do, as replacing non-terminating expansions with terminating ones can only increase the digit sums in the first \( e \) spots.

When \( b = m \), this characterization of boundary points can be extended to include points with \( \lfloor u \rfloor \in \mathcal{U} \) (which lie in \( \mathcal{B} \) if and only if some \( u_i = 1 \)):

\[
\text{A point } u \in [0, 1]^3 \text{ with } \|u\| = 2 \text{ lies in } \mathcal{B} \text{ if and only if } u_1, u_2, \text{ and } u_3 \text{ have base } p \text{ expansions (terminating or not)} \text{ whose digits add up to } 2p - 2 \text{ in every spot.}
\]

Alternatively, let \( L = (0, 1, 1), M = (1, 0, 1), \text{ and } N = (1, 1, 0) \). Any \( u \in [0, 1]^3 \) with \( \|u\| = 2 \) can be written uniquely as \( u = \lambda L + \mu M + \nu N \), where \( \lambda, \mu, \nu \in [0, 1] \) and \( \lambda + \mu + \nu = 1 \), and the above characterization can be rephrased as follows:

\[
u \text{ have base } p \text{ expansions (terminating or not)} \text{ whose digits add up to } p - 1 \text{ in every spot.}
\]

This is a well-known description of the Sierpiński gasket (if \( p = 2 \)) or analogues, with vertices \( L, M, \) and \( N \), so we have the following corollary:

**Corollary 6.7.** If \( b = m \), then the intersection of the boundary \( \mathcal{B} \) and the plane \( \|t\| = 2 \) is a Sierpiński gasket or a higher characteristic analogue.

These fractals have area 0, so \( \mathfrak{F}^m(\ell^3) \) is “almost never” determined by the trivial region. Moreover, because the union of these countably many fractals still has area 0, for “most” \( t \in \mathbb{R}_3^0 \), \( \mathfrak{F}^m(\ell^3) \) is not determined by the trivial region in any characteristic. The following result then comes as a surprise:

**Corollary 6.8.** Let \( a = (a_1, a_2, a_3) \in \mathbb{N}^3 \) with \( a_i < a_j + a_k \), for each \( \{i, j, k\} = \{1, 2, 3\} \). Then \( \mathfrak{F}^m(\ell^3) \) is determined by the trivial region (i.e., \( \mathfrak{F}^m(\ell^3) = 2/\|a\| \)) in infinitely many characteristics.

**Proof.** By Dirichlet’s theorem on primes in arithmetic progressions, there are infinitely many primes \( p \) such that \( p \equiv 1 \pmod{\|a\|} \). For such \( p \), the non-terminating base \( p \) expansions of the components of \( u = 2a/\|a\| \) are “constant”: \( u_1 = (0.\alpha\alpha\alpha\cdots)_p, u_2 = (0.\beta\beta\beta\cdots)_p, \) and \( u_3 = (0.\gamma\gamma\gamma\cdots)_p \). Since \( \|u\| = 2 \), we must have \( \alpha + \beta + \gamma = 2p - 2 \), and Theorem 6.3(2a) gives the result.

\[\text{So the “normalized” point } u = 2(a_1, a_2, a_3)/(a_1 + a_2 + a_3) \text{ has components < 1. This is to avoid the trivial case (1) of Theorem 6.3.} \]
7 \textit{F}-pure thresholds of homogeneous polynomials in two variables

Let $G$ be a non-constant form in $k[x,y]$ and $b=\langle U, V \rangle$, where $U, V \in k[x,y]$ are non-constant relatively prime forms. Extending $k$, if necessary, we write $G = \ell^a$, where $\ell = (\ell_1, \ldots, \ell_n)$ is a collection of pairwise prime linear forms and $a \in \mathbb{N}^{\geq 0}$. Let $\lambda = \deg UV/\deg G$. We shall say that “$f^b(G)$ is determined by a critical point $c$” if $f^b(G) = \max\{c_1/a_1, \ldots, c_n/a_n\} < \lambda$ or, equivalently, $c < \lambda a$ (see Proposition 5.3 and Theorem 5.8). According to Theorem 5.8, we have three (mutually exclusive) possibilities for $f^b(G) = f^b(a)$:

(A) $f^b(G)$ is determined by a critical point in $\mathbb{N}^n$. This is the case if and only if $\langle \lambda a \rangle_0 \in \mathcal{W}$.

(B) $f^b(G)$ is determined by a critical point not in $\mathbb{N}^n$. This is the case if and only if $\langle \lambda a \rangle_0 \notin \mathcal{W}$ but $\langle \lambda a \rangle_e \in \mathcal{W}$ for some $e \geq 1$.

(C) $f^b(G)$ is determined by the trivial region: $f^b(G) = \lambda$.

\textbf{Remark 7.1.} In case (B), taking $q = p^e > 1$ to be the least power of $p$ such that $qa \in \mathbb{N}^n$, Remark 5.10 shows that \[0 < \lambda - f^b(G) \leq \frac{n-2}{q \deg G} < \frac{1}{q}.\]

If $f^b(G) \in \mathbb{Q}_q$ (which will be the case, e.g., when $G$ is square free), then the above inequalities and Remark 2.3 show that $f^b(G) = \langle \lambda \rangle_e$. This reproduces a result obtained by Núñez-Betancourt, Witt, Zhang, and the first author, through completely different methods [HNWZ14]. (For more on this, see Theorem 8.13.)

\textbf{Example 7.2.} In the above remark, the conclusion that $f^b(G)$ is a truncation of $\lambda$ does not hold under the looser assumption that $f^b(G) \in \mathbb{Q}_{p^n}$. If $G = (xy)^{49}(x+y)(x+2y)(x+4y)^{13} \in \mathbb{P}_7[x,y]$ and $m = \langle x, y \rangle$, for instance, then $f^m(G) = 4/343 \in \mathbb{Q}_{7^3}$ (determined by the critical point $(4, 4, 1, 1)/7$ associated with $\ell = (x, y, x+y, x+2y, x+4y)$ and $m$), while $\langle \lambda \rangle_3 = \langle 2/137 \rangle_3 = 5/343$.

We now specialize to the case where $b = m = \langle x, y \rangle$. Recall that $f^m(G)$ is the $\textit{F-pure threshold}$ of $G$ (at the origin), denoted by $\text{fpt}(G)$.

\textbf{Remark 7.3.} As noted in Remark 6.6 [Proposition 5.3.4] shows that the only critical points with integer coordinates associated with $\ell$ and $m$ are $e_1, \ldots, e_n$. Moreover, those are the only critical points that have some positive integer coordinate. Indeed, if $c$ is a critical point and $c_i \in \mathbb{N}_{>0}$, then $c \geq e_i$, and therefore $c = e_i$.

The scarcity of positive integer coordinates in critical points observed above has interesting consequences. Case (A) becomes relegated to a “degenerate case” where the multiplicity of some $\ell_i$ in $G$ is too large—$\text{fpt}(G)$ is determined by $e_i$ if and only if $\lambda a = (2/ \deg G)a > e_i$ if and only if $a_i > (\deg G)/2$, in which case
fpt(G) = \max\{0, 1/a_i\} = 1/a_i. Also, in case (B) no nonzero coordinate of the critical point \(c\) is integral, so the minimal denominator of fpt(G) = \max\{c_i/a_i\} is of the form \(kp^e\), with \(e \geq 1\) and \(k\) a factor of one of the multiplicities \(a_i\). To put this observation in context, we must digress momentarily with some characteristic 0 considerations.

**Definition 7.4.** Let \(G_0 \in \mathbb{Q}[x, y]\) be a non-constant form. We say that a prime \(p\) is a good prime associated with \(G_0\) if there exists a reduction modulo \(p\) of \(G_0\) in \(\mathbb{F}_p[x, y]\), which we denote by \(G_p\), and the factorization of \(G_p\) over \(\mathbb{F}_p\) is similar to the factorization of \(G_0\) over \(\mathbb{C}\), in the sense that those factorizations have the same number of pairwise prime linear factors and the same multiplicities. If that is not the case, we say that \(p\) is a bad prime.

**Lemma 7.5.** There are at most finitely many bad primes associated with a fixed non-constant form \(G_0 \in \mathbb{Q}[x, y]\).

**Proof.** The proof relies on the following facts. Given a finitely generated \(\mathbb{Z}\)-algebra \(A\) containing \(\mathbb{Z}\), for all but finitely many primes \(p\) there exists a maximal ideal of \(A\) containing \(p\). Moreover, if \(\mathfrak{M}\) is a maximal ideal of \(A\) containing a prime number \(p\), then \(A/\mathfrak{M}\) is a finite field of characteristic \(p\). For a justification of these facts, see, e.g., [Her11, Corollary 3.2].

We now proceed with the proof. Suppose \(G_0\) factors over \(\mathbb{C}\) as

\[G_0 = \gamma \cdot x^i y^m (x - \alpha_1 y)^{k_1} \cdots (x - \alpha_r y)^{k_r},\]

where \(\gamma, \alpha_1, \ldots, \alpha_r \in \mathbb{C}^\times\) and the \(\alpha_i\) are distinct. Let \(A\) be the \(\mathbb{Z}\)-algebra generated by \(\gamma\) and the \(\alpha_i\), together with \(\prod_{i<j} (\alpha_i - \alpha_j)^{-1}, (\gamma \cdot \alpha_1 \cdots \alpha_r)^{-1}\).

According to the facts cited above, given a prime \(p \gg 0\), there exists a maximal ideal \(\mathfrak{M}\) of \(A\) containing \(p\). Furthermore, for such \(p\) and \(\mathfrak{M}\), the field \(k = A/\mathfrak{M}\) is a finite extension of \(\mathbb{F}_p\) over which the image of \(G_0\) has a factorization similar to that of \(G_0\), because, by design, the images of \(\gamma\) and the \(\alpha_i\) and \(\alpha_i - \alpha_j\) in \(k\) are nonzero, since they are invertible. So \(p\) is a good prime. \(\square\)

**Definition 7.6.** A form \(F\) in two variables over some field \(K\) is degenerate, of degeneracy type \(m\), if it has a linear factor (over \(\overline{K}\)) with multiplicity \(m > (\deg F)/2\). (Note that each degenerate form has a unique degeneracy type.)

Let \(G_0 \in \mathbb{Q}[x, y]\) be a non-constant form. As discussed in the introduction, the log canonical threshold of \(G_0\), denoted by \(\text{lct}(G_0)\), is an invariant measuring the singularity of \(G_0\) at the origin, and is defined via a (log) resolution of singularities. In the context of this paper, the most important property of \(\text{lct}(G_0)\) is that \(\lim_{p \to \infty} \text{fpt}(G_p) = \text{lct}(G_0)\).

If \(G_0\) is degenerate, of degeneracy type \(m\), then for each good prime \(p\) the computation of \(\text{fpt}(G_p)\) falls under case (A), and \(\text{fpt}(G_p) = 1/m\). Consequently, \(\text{lct}(G_0) = \lim_{p \to \infty} \text{fpt}(G_p) = 1/m\). If \(G_0\) is non-degenerate, then for each good prime \(p\) the computation of \(\text{fpt}(G_p)\) falls under cases (B) or (C), and the inequalities in Remark 7.1 show that \(\text{lct}(G_0) = \lim_{p \to \infty} \text{fpt}(G_p) = \lambda = 2/\deg(G_0)\). Thus, in the paragraph before Definition 7.4 we have shown the
following result, which provides an affirmative answer to Question 1.1 in the two-variable homogeneous setting.

**Theorem 7.7.** Let $G_0 \in \mathbb{Q}[x,y]$ be a non-constant form. Let $p$ be a good prime associated with $G_0$, and let $G_p$ be the image of $G_0$ in $\mathbb{F}_p[x,y]$. If $\text{fpt}(G_p) \neq \text{lct}(G_0)$, then the minimal denominator of $\text{fpt}(G_p)$ is of the form $kp^e$, where $e \geq 1$ and $k$ divides the multiplicity of some linear factor (over $\mathbb{C}$) of $G_0$. \[\square\]

**Remark 7.8.** If we weaken the notion of good prime, requiring only that $G_0$ and $G_p$ be both non-degenerate or both degenerate, of the same degeneracy type, then an alternate version of the above theorem still holds, where the conclusion states that $k$ divides the multiplicity of some linear factor (over $\mathbb{F}_p$) of $G_p$.

**Example 7.9.** Let $G_0 = x(x+y)(x+6y)$. Then $G_5 = x(x+y)^2$, so $\text{fpt}(G_5) = 1/2 \neq 2/3 = \text{lct}(G_0)$, and yet $\text{fpt}(G_5)$ has a denominator prime to 5. This shows the need for requiring $p$ to be a good prime in Theorem 7.7. That result may otherwise not hold when the factorizations of $G_0$ and $G_p$ are “too different”.

Going back to the three cases discussed earlier in this section, while case (A) is clearly delimited, distinguishing between cases (B) and (C) is delicate. For that, it is useful to know an upper bound for the denominator of the critical point that determines $\text{fpt}(G)$ in case (B). When $\deg G$ is prime to $p$ we have such a bound—a consequence of the “forbidden intervals” theorem of Blickle, Mustaţă, and Smith [BMS09, Proposition 4.3], generalized by the first author [Her12, Proposition 4.8], which states that for any $\beta \in (0,1)$ there are no $F$-pure thresholds (of hypersurfaces in characteristic $p$) in the interval $(\beta, \beta q/(q-1))$.

**Lemma 7.10.** Let $\lambda = a/b \in (0,1] \cap \mathbb{Q}$. Suppose $b$ is prime to $p$, and let $\mu$ be the multiplicative order of $p$ modulo $a$. Then no $F$-pure threshold of a polynomial over a field of characteristic $p$ lies in the interval $(\langle \lambda \rangle_\mu, \lambda)$.

**Proof.** Let $q = p^k$ and $k = (q-1)\lambda$; then $k \in \mathbb{N}$ and

$$\lambda = \frac{k}{q-1} = \frac{k}{q} \left(1 + \frac{1}{q} + \frac{1}{q^2} + \cdots \right).$$

Since $k < q$, the above equation shows that $\langle \lambda \rangle_\mu = k/q$, so $\lambda = \langle \lambda \rangle_\mu q/(q-1)$, and the “forbidden intervals” theorem gives the result. \[\square\]

**Proposition 7.11.** Let $G \in \mathbb{k}[x,y]$ be a form of degree $d > 0$. Suppose $\text{fpt}(G) < \lambda = 2/d$ and the minimal denominator of $\lambda$ is prime to $p$. Let $\mu$ be the multiplicative order of $p$ modulo that denominator. Then $\text{fpt}(G)$ is determined by a critical point with coordinates in $\mathbb{Q}_p^\mu$.

**Proof.** Write $G = \ell^n$, as in the beginning of this section, and let $c$ be the critical point that determines $\text{fpt}(G)$. Lemma 7.10 gives us the following inequalities:

$$\max \left\{ \frac{c_1}{a_1}, \ldots, \frac{c_n}{a_n} \right\} = \text{fpt}(G) \leq \langle \lambda \rangle_\mu < \lambda = \frac{2}{d}.$$
As \( \max\{c_i/a_i\} \cdot a \in \partial(c, \infty) \) and \( \frac{3}{2} \cdot a = \frac{2}{\|a\|} \cdot a \in \partial \mathcal{T} \), multiplying each of the terms in the above inequalities by \( a \) shows that \( \langle \lambda \rangle \mu a \) lies in \( B_{\Delta(c)}(c) \), the region where the function \( \Delta \) attached to \( \ell \) and \( m \) is determined by \( c \). Because \( \langle \lambda \rangle \mu a \in \mathbb{Q}_p^n \), Remark 2.13 tells us that \( c \in \mathbb{Q}_p^n \) as well.

We summarize the above observations in the following theorem.

**Theorem 7.12.** Let \( G \in \mathbb{k}[x, y] \) be a form of degree \( d > 0 \). Write \( G = \ell_1^{a_1} \cdots \ell_m^{a_m} \), where the \( \ell_i \) are pairwise prime linear forms in \( \mathbb{k}[x, y] \) and \( a_i > 0 \), for each \( i \).

1. If \( a_i > d/2 \), for some \( i \), then \( \text{fpt}(G) = 1/a_i \).
2. Suppose \( a_i \leq d/2 \), for each \( i \). Set \( \ell = (\ell_1, \ldots, \ell_n) \) and \( a = (a_1, \ldots, a_n) \). Then exactly one of the following holds:
   - \( \langle \frac{3}{2} \cdot a \rangle \) lies in the upper region attached to \( \ell \) and \( m = \langle x, y \rangle \), for some \( e \geq 1 \). Thus, there exists a (unique) critical point \( c \in \mathbb{Q}_p^n \) associated with \( \ell \) and \( m \) such that \( c \leq \langle \frac{3}{2} \cdot a \rangle \), and
     \[
     \text{fpt}(G) = \max \left\{ \frac{c_1}{a_1}, \ldots, \frac{c_n}{a_n} \right\}.
     \]

The critical point \( c \) has no nonzero integer coordinates, and thus the minimal denominator of \( \text{fpt}(G) \) is of the form \( kp^n \), where \( 1 \leq m \leq e \) and \( k \) is a factor of some multiplicity \( a_i \). Moreover, if the minimal denominator of \( 2/d \) is prime to \( p \), then the above holds for some \( e \) no greater than the multiplicative order of \( p \) modulo that denominator. Finally, if \( \text{fpt}(G) \in \mathbb{Q}_p^n \) (e.g., if \( G \) is square free), then \( \text{fpt}(G) = (2/d)_e \).

- \( \text{fpt}(G) = 2/d \). 

**Example 7.13.** Consider the forms \( G_1 = x^2 y^2 (x^2 + 2xy + 3y^2)^7 \) and \( G_2 = x^2 y^2 (x^2 + 2xy + 3y^2)^3 \) in \( \mathbb{F}_{25}[x, y] \). Let \( \ell_1 = x \), \( \ell_2 = y \), and \( \ell_3 \ell_4 = x^2 + 2xy + 3y^2 \), and set \( \ell = (\ell_1, \ell_2, \ell_3, \ell_4) \). Finally, let \( a_1 = (2, 2, 7, 7) \), \( a_2 = (2, 2, 1, 1) \), \( \lambda_1 = 2/\deg G_1 = 1/9 \), and \( \lambda_2 = 2/\deg G_2 = 1/3 \). Since the multiplicative order of \( 5 \) (mod 9) is 6, to find \( \text{fpt}(G_1) \) we look for a truncation \( \langle \lambda_1 a_1 \rangle \) with \( e \leq 6 \) that lies in the upper region \( \mathcal{U} \) attached to \( \ell \) and \( m \). We find that \( \langle \lambda_1 a_1 \rangle = (27, 27, 97, 97)/125 \) lies in \( \mathcal{U} \), and is itself a critical point. Thus \( \text{fpt}(G_1) = \max \left\{ \frac{27}{125}, \frac{97}{125} \right\} = \frac{97}{125} \). As for \( G_2 \), the multiplicative order of \( 5 \) (mod 3) is 2, but \( \langle \lambda_2 a_2 \rangle \) does not lie in \( \mathcal{U} \), so \( \text{fpt}(G_2) \) is determined by the trivial region: \( \text{fpt}(G_2) = \lambda_2 = 1/3 \). Figure 7 shows a density plot of a section of the function \( \Delta \) attached to \( \ell \) and \( m \), together with the lines spanned by \( a_1 \) and \( a_2 \).

**Remark 7.14.** Let \( G = \ell_1^{a_1} \cdots \ell_m^{a_m} = \ell^n \) and \( \lambda = 2/\deg G \). Directly checking if truncations \( \langle \lambda a \rangle \) are in the upper region \( \mathcal{U} \) attached to \( \ell \) and \( m \) (i.e., checking if \( \ell^n(\lambda a) \in m^{\langle \beta \rangle} \) can quickly lead to impractical computations involving
polynomials of extremely large degrees. We can, however, get around this problem. Write the non-terminating base $p$ expansion of $\lambda a$ as follows:

$$\lambda a = \frac{d_1}{p} + \frac{d_2}{p^2} + \frac{d_3}{p^3} + \cdots.$$  

(Here each $d_e$ consists of the $e$th digits of the components of $\lambda a$.) Set $b_0 := m$, and successively compute $b_e := (b_e^{[p]} : \ell^{d_e})$. Then

$$b_e = \langle m^{[p^e]} : p^{e-1}d_1 + p^{e-2}d_2 + \cdots + d_e \rangle = \langle m^{[p^e]} : p^{e-\langle \lambda a \rangle_e} \rangle,$$

so $b_e = \langle 1 \rangle$ if and only if $\langle \lambda a \rangle_e \in \mathcal{U}$. When computing the $b_e$ we never raise polynomials to powers greater than $p$; moreover, it can be shown that each $b_e$ can be generated by two forms whose degrees add up to at most $n$. So we can check whether $\langle \lambda a \rangle_e \in \mathcal{U}$ for arbitrarily large $e$ without ever having to deal with large degree polynomials. If $p$ does not divide the minimal denominator of $\lambda = \frac{2}{\deg G}$, this check only needs to be done for $e \leq \ell^\lambda(p)$ modulo that denominator. If $\langle \lambda a \rangle_e \notin \mathcal{U}$ for each $e$ in this range, then $\text{fpt}(G) = \lambda$. If $\langle \lambda a \rangle_e \in \mathcal{U}$, for some $e$, we search for a critical point $c \in \mathbb{Q}_p^n$ that lies under $\langle \lambda a \rangle_e$ (using the colon ideals $b_e$ to avoid degree explosion).

In cases where $p$ divides the minimal denominator of $\lambda = \frac{2}{\deg G}$, we must test if $\text{fpt}(G) = \lambda$ first and only if that is not the case we set off on a search for a truncation $\langle \lambda a \rangle_e$ in $\mathcal{U}$ and then for a critical point $c \leq \langle \lambda a \rangle_e$. Testing whether $\text{fpt}(G) = \lambda$ (or whether $\text{fpt}(G)$ equals any other rational number) can be done rather efficiently by a method of Schwede, which compares the “non-$F$-pure ideals” of FSTII and test ideals. This was implemented by Schwede in the Macaulay2 package PosChar [HKM+] and used by the second author in the implementation of computations of $F$-pure thresholds of binary forms in that same package, using the above ideas.

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In our Macaulay2 implementation, this test is also performed first whenever $G$ is not already factored into linear forms. This is because factoring $G$ is the weakness of this method, as this factorization often happens in very large field extensions, and cannot be handled by Macaulay2.

27
We close this section highlighting a key difference between $F$-pure thresholds and $F$-thresholds with respect to ideals $b \neq m$, which is that in the latter setting the analogue of Remark 7.3 does not hold. Integral critical points tend to abound (see, for instance, Example 4.7) and, as the following example will show, critical points may have both (nonzero) integral coordinates and non-integral coordinates—so $F$-thresholds with a denominator prime to $p$ may arise from non-integral critical points.

Example 7.15. Let $\ell = (x, y, x+y, x+2y) \in \mathbb{F}_5[x, y]^4$ and $b = \langle x, y^2 \rangle \subseteq \mathbb{F}_5[x, y]$. Then $c = (2/5, 3/5, 4/5, 1)$ is a critical point associated with $\ell$ and $b$, and it is easy to produce forms whose $F$-thresholds with respect to $b$ are determined by the last coordinate of $c$ and have a denominator prime to 5. For instance, if $G = x^7y^{10}(x+y)^{13}(x+2y)^{16}$, then $ft^b(G) = \max\left\{ \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{16} \right\} = \frac{1}{16}$.

8 $F$-pure thresholds of quasi-homogeneous polynomials in two variables

If $K$ is an arbitrary field, we endow $K[X, Y]$ with a non-standard $\mathbb{N}$-grading where $\deg X = u$, $\deg Y = v$, and $uv \neq 0$. We shall refer to the homogeneous elements under this grading as quasi-homogeneous polynomials and reserve the terms form and homogeneous polynomial for polynomials that are homogeneous under the standard grading. We extend the notion of degeneracy (see Definition 7.3) to this setting by saying that a quasi-homogeneous polynomial $f \in K[X, Y]$ is degenerate, of degeneracy type $m$, if it has an irreducible factor (over $K$) with multiplicity $m > \deg(f)/(u+v)$.

Notation. Given $f \in K[X, Y]$ and an irreducible polynomial $h \in \overline{K}[X, Y]$, $\text{mult}_h(f)$ denotes the multiplicity of $h$ in $f$, that is, the largest $m \in \mathbb{N}$ (possibly 0) such that $h^m$ divides $f$ (in $\overline{K}[X, Y]$).

Definition 8.1. Let $g_0 \in \mathbb{Q}[X, Y]$ be a non-constant quasi-homogeneous polynomial. A prime $p$ is a good prime associated with $g_0$ if the following hold:

1. there exists a reduction modulo $p$ of $g_0$ in $\mathbb{F}_p[X, Y]$, denoted by $g_p$;
2. the factorization of $g_p$ over $\mathbb{F}_p$ is similar to the factorization of $g_0$ over $\mathbb{C}$, in the sense that those factorizations have the same number of non-associated irreducible factors and the same multiplicities;
3. $\text{mult}_X(g_p) = \text{mult}_X(g_0)$ and $\text{mult}_Y(g_p) = \text{mult}_Y(g_0)$.

If that is not the case, then $p$ is a bad prime. (Remark 8.7 will show that there exist at most finitely many bad primes associated with a fixed $g_0$.)

Our first and main goal in this section will be to extend Theorem 7.7 to the quasi-homogeneous setting, proving:
Theorem 8.2. Let $g_0 \in \mathbb{Q}[X,Y]$ be a non-constant quasi-homogeneous polynomial. Let $p$ be a good prime associated with $g_0$, and let $g_p$ be the image of $g_0$ in $\mathbb{F}_p[X,Y]$. If $\text{fpt}(g_p) \neq \text{fpt}(g_0)$, then the minimal denominator of $\text{fpt}(g_p)$ is of the form $kp^e$, where $e \geq 1$ and $k$ is a factor of one of the following: $\text{mult}_X(g_0) \cdot \deg X$, $\text{mult}_Y(g_0) \cdot \deg Y$, or $\text{mult}_h(g_0)$, where $h$ is some irreducible factor (over $\mathbb{C}$) of $g_0$ other than $X$ or $Y$.

Remark 8.3. As it is the case with Theorem 7.7 (see Remark 7.8), an alternate version of the above result, where, in the conclusion, multiplicities in $g_0$ are replaced with multiplicities in $g_p$, may be obtained under a looser notion of good prime, that only requires that $g_0$ and $g_p$ be both non-degenerate or both degenerate, of the same degeneracy type.

Fix $g_0$ as in the statement of Theorem 8.2, and a good prime $p$; let $k = \mathbb{F}_p$, and $g = g_p$, the image of $g_0$ in $k[X,Y]$. Since Theorem 8.2 is already known for (standard) homogenous polynomials, we may assume that $g_0$ is not homogeneous—so $g_0$ is not a monomial (and, thus, neither is $g$) and $u \neq v$. In fact we assume (by possibly changing the grading) that $u$ and $v$ are coprime.

Definition 8.4. $\psi : k[X,Y] \to k[x,y]$ is the map $f(X,Y) \mapsto f(x^u, y^v)$.

Let $G = \psi(g)$ and $b = (x^u, y^v)$. The above graded, injective $k$-algebra homomorphism will allow us to extend our methods to the quasi-homogeneous setting, via the following proposition.

Proposition 8.5. $\text{fpt}(g) = \text{ft}^b(G)$.

Proof. Because of the description of the $F$-threshold of a polynomial given in Discussion 4.1, it suffices to show that, for each $a \in \mathbb{N}$ and $q = p^e$, we have that $a^q \in (X,Y)^{[q]}$ if and only if $G^a \in b^{[q]}$. Let $A = k[x^u, y^v]$ and $B = k[x,y]$. Let $\mathfrak{a}$ be the ideal of $A$ generated by $x^u$ and $y^v$. Then $\psi$ induces an isomorphism from $k[X,Y]$ to $A$, mapping $g$ to $G$ and $m = (X,Y)$ to $\mathfrak{a}$, and hence, $a^q \in m^{[q]}$ if and only if $G^a \in \mathfrak{a}^{[q]}$. But $A$ is a direct summand of $B$ as an $A$-module, and therefore $b^{[q]} \cap A = (a^{[q]}B) \cap A = a^{[q]}$. As $G \in A$, we see that $G^a \in \mathfrak{a}^{[q]}$ if and only if $G^a \in b^{[q]}$, which allows us to conclude the proof.

Lemma 8.6. The polynomial $g$ can be factored as

$$g = \xi \cdot X^{j_1}Y^{j_2} \cdot \prod_{i=1}^{m} (X^u - \mu_i Y^v)^{k_i},$$

for some $j_1, j_2 \in \mathbb{N}$, $m, k_1, \ldots, k_m \in \mathbb{N}_{>0}$, $\xi \in k^\times$, and distinct $\mu_1, \ldots, \mu_m \in k^\times$.

Remark 8.7. The polynomial $g_0$, of course, has a similar factorization over $\mathbb{C}$, and the argument used in the proof of Lemma 7.5 can be adapted to show that there exist at most finitely many bad primes associated with $g_0$.

Proof. Write $g$ as $X^{j_1}Y^{j_2}h$, where $h$ is a quasi-homogeneous polynomial prime to $XY$. As $g$ is not a monomial, $h$ has at least terms $X^a$ and $Y^b$, with $au = bv$. 

29
Then, as \( k \) and \( c \) are coprime, \( cu + dv = au \), so \( v \mid c - a \); but \( v \mid a \) as well, so we conclude that \( v \mid c \). Similarly we find that \( u \mid d \), so in each term of \( h \) the exponents of \( X \) and \( Y \) are divisible by \( v \) and \( u \), respectively, and therefore \( h = H(X^v, Y^u) \) for some form \( H \). The result is then obtained by factoring \( H \) into linear forms.

In our proof of Theorem 8.2, we assume that \( j_1j_2 \neq 0 \); our argument can be adapted to handle the other cases (which are simpler). As \( u = \deg X \) and \( v = \deg Y \) are coprime, \( p \) can only divide one of them; we assume that \( u \) is prime to \( p \) and write \( v = \bar{q} \), where \( w \) is prime to \( p \) and \( \bar{q} \) is a power of \( p \) (possibly 1).

Then, as \( k = \overline{k} \), there exist (unique) \( \mu_i^{1/\bar{q}} \in k \) with \( (\mu_i^{1/\bar{q}}) \bar{q} = \mu_i \), so that

\[
G = \psi(g) = \xi \cdot x^{n_{j_1}}y^{n_{j_2}} \cdot \prod_{i=1}^{m} (x^{uw} - \mu_i^{1/\bar{q}}y^{uw})^{\bar{q}k_i},
\]

where the factors \( x^{uw} - \mu_i^{1/\bar{q}}y^{uw} \) are square free and pairwise prime. As our methods will depend on factoring \( G \) into a product of linear forms, it will be necessary to factor each \( x^{uw} - \mu_i^{1/\bar{q}}y^{uw} \) into a product of linear forms. Let \( \zeta \) be a primitive \((uw)\)th root of unity in \( k \), and let \( \nu_i \) be a \((uw)\)th root of \( \mu_i^{1/\bar{q}} \) in \( k \), for each \( i \); then \( x^{uw} - \mu_i^{1/\bar{q}}y^{uw} = \prod_{j=1}^{w}\left(x - \nu_i\zeta^jy\right) \). Substituting this into (8.1) produces the following factorization of \( G \) into a product of linear forms:

\[
G = \xi \cdot x^{n_{j_1}}y^{n_{j_2}} \cdot \prod_{i=1}^{m} \prod_{j=1}^{w} (x - \nu_i\zeta^jy)^{\bar{q}k_i}.
\]

Let \( n = 2 + muw \) (the number of pairwise prime linear factors of \( G \)). It will be convenient to label the \( n \) linear factors of \( G \) and the canonical basis vectors of \( \mathbb{R}^n \) in a non-standard way. Let \( \ell_1 = x, \ell_2 = y, \) and for each \( i \in \{1, \ldots, m\} \) and \( j \in \{1, \ldots, uw\} \) let \( \ell_{i,j} = x - \nu_i\zeta^jy \); set

\[
\ell = (\ell_1, \ell_2, \ell_{1,1}, \ldots, \ell_{1,uw}, \ldots, \ell_{m,1}, \ldots, \ell_{m,uw}).
\]

For each \( \ell_{i,j} \), let \( f_{i,j} \) be the corresponding canonical basis vector of \( \mathbb{R}^n \), and set \( f_i = \sum_{j=1}^{uw} f_{i,j} \) (so \( \ell f_i = \prod_{j=1}^{uw}\left(x - \nu_i\zeta^jy\right) = x^{uw} - \mu_i^{1/\bar{q}}y^{uw} \), by definition).

We now explore some symmetries in the critical points associated with \( \ell \) and \( b = (x^n, y^n) \) coming from the special shape of \( G \).

**Lemma 8.8.** Suppose \( c = c_1 e_1 + c_2 e_2 + \sum_{i,j} c_{i,j} f_{i,j} \) is a critical point. Let \( c' \) be the point obtained from \( c \) by replacing each set of coordinates \( c_{i,1}, c_{i,2}, \ldots, c_{i,uw} \) with \( c_{i,uw}, c_{i,1}, \ldots, c_{i,uw-1} \). Then \( c' \) is also a critical point.

**Proof.** Suppose \( c \in \mathbb{Q}^n \). Then the \( k \)-automorphism of \( k[x, y] \) that maps \( x \rightarrow x \) and \( y \rightarrow \zeta y \) transforms \( c^e \) into a constant multiple of \( \ell^e c' \), while fixing \( b \).

**Lemma 8.9.** Let \( t = t_1 e_1 + t_2 e_2 + \sum_{i=1}^{m} t_i f_i \in \partial \mathcal{T} \). Suppose \( c \) is a critical point such that \( c < t \). Then \( c = c_1 e_1 + c_2 e_2 + \sum_{i=1}^{m} c_i f_i \), for some \( c_i, c_i^* \in \mathbb{Q}_{p\infty} \).
Proof. Write \( c = c_1e_1 + c_2e_2 + \sum_{i,j} c_{i,j}f_{i,j} \), and construct \( c' \) as in Lemma 8.9. Then \( c' \) is a critical point and \( c' < t \) as well. Since there can be no more than one critical point lying under a point in \( \mathcal{P} \) (see Remark 5.7), we conclude that \( c = c' \). Iterating this process, we see that for each \( i \in \{1, \ldots, m\} \) the coordinates \( c_{i,1}, \ldots, c_{i,u_i} \) of \( c \) are all equal, and the result follows. \( \square \)

Lemma 8.10. Suppose \( c = \alpha e_1 + \beta e_2 + \sum_{i=1}^m \gamma_i f_i \in \mathbb{Q}^n \) is a critical point. Then \( \alpha/u \) and \( \beta/w \) both lie in \( \mathbb{Q} \).

Proof. Suppose \( \alpha \neq 0 \). Write \( \alpha = a/q, \beta = b/q, \) and \( \gamma_i = c_i/q \). As \( c \in \mathcal{W} \),

\[
elev^c = x^a y^b \prod_{i=1}^m \left(x^{uw} - \mu_{i}^{1/q} y^{uw}\right)^{c_i} \in b[\xi] = \langle x^{uw}, y^{vw} \rangle.
\]

Since \( c \) is a critical point, there is a monomial \( M = x^{a+iuw} y^{b+juw} \) in the support of \( \ell^c \) such that \( M/x \notin b[\xi] \). Clearly \( a + iuw \leq uq \), and if the inequality were strict, then \( \frac{a + iuw}{q} \) would imply that \( b + juw \geq vq \), and \( M/x \) would be in \( b[\xi] \), a contradiction. So \( a + iuw = uq \), whence \( \alpha/u = a/(uq) = 1 - iw/q \in \mathbb{Q} \). The argument showing that \( \beta/w \in \mathbb{Q} \) is analogous. \( \square \)

Corollary 8.11. Suppose \( c = \alpha e_1 + \beta e_2 + \sum_{i=1}^m \gamma_i f_i \) is a critical point.

1. If \( \alpha \) is a positive integer, then \( c = u e_1 \).

2. If \( \beta \) is a positive integer, then either \( c = v e_2 \) or \( \tilde{q} \mid \beta \).

Proof. If \( \alpha \in \mathbb{N}_{>0} \), then, bearing in mind that \( u \) is prime to \( p \), Lemma 8.10 shows that \( u \mid \alpha \). Thus, \( c \geq u e_1 \), and since \( u e_1 \) is a critical point we must have \( c = u e_1 \). Likewise, if \( \beta \in \mathbb{N}_{>0} \), then \( v \mid \beta \). If \( \tilde{q} \mid \beta \) as well, then \( v = w \tilde{q} \mid \beta \), and arguing as before we conclude that \( c = v e_2 \). \( \square \)

We are now ready to conclude the proof of Theorem 8.2.

Proof of Theorem 8.2. Recall that \( \text{fpt}(g) = \text{ft}^b(G) \), by Proposition 8.5. Set \( \lambda = (u + v)/\deg G \) and write \( G = \xi \ell^a \), where \( a = u j_1 e_1 + v j_2 e_2 + \sum_{i=1}^m q_i f_i \). Consider the following cases:

- **\( \lambda k_i > 1 \) for some \( i \)**. This situation is not common—\( \lambda k_i > 1 \) is equivalent to \( \deg G < (u + v)k_i \) and, as \( \deg G \geq uvk_i \), this requires either \( u \) or \( v \) to be 1. In this case, \( \tilde{q} f_i \) is a critical point lying under \( \lambda a \), so \( \text{fpt}(g) = \max \left\{ 0, \frac{q}{\tilde{q} f_i} \right\} = \frac{1}{f_i} \).

- **\( \lambda j_i > 1 \) for some \( i \)**. Suppose \( \lambda j_1 > 1 \) (the other case is analogous). Then \( \lambda a \) lies above the critical point \( u e_1 \), so \( \text{fpt}(g) = \max \left\{ \frac{u}{u j_1}, 0 \right\} = \frac{1}{j_1} \).

- **\( \lambda j_i \leq 1 \) and \( \lambda k_i \leq 1 \), for each \( i \)**. If \( \text{ft}^b(G) \) is determined by a critical point \( c \), then \( c < \lambda a \), so \( c = \alpha e_1 + \beta e_2 + \sum_{i=1}^m \gamma_i f_i \), for some \( \alpha, \beta, \gamma_i \in \mathbb{Q}_{p^\infty} \), by Lemma 8.9. The inequalities \( \lambda j_i \leq 1 \) ensure that \( c \) is neither \( u e_1 \) nor \( v e_2 \), and thus Corollary 8.11 shows that \( \alpha \) is not a positive integer, and that if \( \beta \) is a positive integer, then \( \tilde{q} \mid \beta \). The inequalities \( \lambda k_i \leq 1 \) on the
other hand, ensure that $\gamma_i < \lambda qk_i \leq \bar{q}$. Being determined by $c$, $\text{fpt}(g) = \text{ft}^b(G)$ equals the maximum among $\alpha/(uj_1), \beta/(\bar{q}w_1j_2)$, and $\gamma_i/(qk_i)$ ($i = 1, \ldots, m$), hence its minimal denominator has the desired form. Alternatively, $\text{ft}^b(G)$ may be determined by the trivial region, and $\text{fpt}(g) = \text{ft}^b(G) = \lambda$.

We now allow $p$ to vary, to find $\text{lct}(g_0)$ in each of the above cases. Note that the conditions “$\lambda j_1 > 1$” and “$\lambda k_1 > 1$” are equivalent to degeneracy conditions on $g_0$, which are inherited from $g_0$, and thus independent of the choice of the good prime $p$. It follows that $\text{lct}(g_0) = \text{fpt}(g_p)$ in the first two cases. As for the last case, note that if $p \nmid v$ then $\alpha$ and $\beta$ cannot be nonzero integers, by Corollary [8.11] and $\gamma_i < \bar{q} = 1$, so $c \notin \mathbb{N}^n$. Thus, $\text{fpt}(g_p)$ is either $\lambda$ or is determined by a non-integral critical point, for all $p \gg 0$, and the inequalities in Remark [7.4] show that $\text{lct}(g_0) = \lim_{p \to \infty} \text{fpt}(g_p) = \lambda$. So we have shown that the minimal denominator of $\text{fpt}(g_p)$ has the required form whenever $\text{fpt}(g_p) \neq \text{lct}(g_0)$. 

A byproduct of our proof is the following:

**Corollary 8.12.** If $g_0$ is non-degenerate, then $\text{lct}(g_0) = (u + v)/\deg(g_0)$. If $g_0$ is degenerate, of degeneracy type $m$, then $\text{lct}(g_0) = 1/m$.

The following theorem was recently proved by Núñez-Betancourt, Witt, Zhang, and the first author.

**Theorem 8.13 (\cite[Theorem 4.4]{HNWZ14}).** Let $g \in \mathbb{k}[X,Y]$ be a non-constant quasi-homogeneous polynomial that is square free over $\mathbb{k}$. Set $\lambda = (u + v)/\deg g$. Then either $\text{fpt}(g) = \min\{1, \lambda\}$ or $\text{fpt}(g) = (\lambda)_{e}$, for some $e \geq 1$.

We conclude this section and the paper with a simple proof of this theorem under the additional assumption that $u$ and $v$ are prime to $p$.

**Proof.** Suppose $g$ is not homogeneous (or else simply use Theorem [7.12]—so $g$ is not a monomial and $u \neq v$). As before, we assume that $u$ and $v$ are coprime and $k = \mathbb{k}$. Using Lemma [8.6] and the assumption that $g$ is square free, we write

$$G = \psi(g) = \xi \cdot x^{ju}y^{kv} \cdot \prod_{i=1}^{m}(x^{u_i} - \mu_i y^{v_i}),$$

where $j, k \in \{0, 1\}$ and the factors $x^{u_i} - \mu_i y^{v_i}$ are square free and pairwise prime. We consider the case $j = k = 1$ (other cases are analogous). Set $\lambda = (u + v)/\deg G$ and $b = (x^w, y^v)$ and, adopting the setup used earlier (mind that here $v = w$ and $\bar{q} = 1$), write $G = \xi \ell^a$, where $a = u e_1 + v e_2 + \sum_{i=1}^{m} f_i$.

If $\text{ft}^b(G) \neq \lambda$, then $\text{ft}^b(G)$ is determined by a critical point $c < \lambda a$. Since $\lambda < 1$, we find that $c \notin \mathbb{N}^n$ and Lemmata [8.9] and [8.10] show that $c = \alpha e_1 + \beta e_2 + \sum_{i=1}^{m} \gamma_i f_i$ for some $\alpha, \beta, \gamma_i \in \mathbb{Q}_q$, where $q = p^e > 1$, and $\text{ft}^b(G) = \max\{\alpha/u, \beta/v, \gamma_i\} \in \mathbb{Q}_q$. Remark [7.4] then shows that $\text{ft}^b(G) = (\lambda)_{e}$. As $\text{fpt}(g) = \text{ft}^b(G)$, by Proposition [8.5] we have shown that either $\text{fpt}(g) = \lambda = \min\{1, \lambda\}$ or $\text{fpt}(g) = (\lambda)_{e}$, for some $e \geq 1$. 

\footnote{Note that if $x$ or $y$ are not factors of $g$ (i.e., $jk = 0$), then $\lambda$ may—in some rare instances—be $\geq 1$, in which case $\text{ft}^b(G)$ is determined by an integral critical point and $\text{ft}^b(G) = 1$.}

32
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