Deterministic Discrepancy Minimization via the Multiplicative Weight Update Method

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Abstract

A well-known theorem of Spencer shows that any set system with \( n \) sets over \( n \) elements admits a coloring of discrepancy \( O(\sqrt{n}) \). While the original proof was non-constructive, recent progress brought polynomial time algorithms by Bansal, Lovett and Meka, and Rothvoss. All those algorithms are randomized, even though Bansal’s algorithm admitted a complicated derandomization.

We propose an elegant deterministic polynomial time algorithm that is inspired by Lovett-Meka as well as the Multiplicative Weight Update method. The algorithm iteratively updates a fractional coloring while controlling the exponential weights that are assigned to the set constraints.

A conjecture by Meka suggests that Spencer’s bound can be generalized to symmetric matrices. We prove that \( n \times n \) matrices that are block diagonal with block size \( q \) admit a coloring of discrepancy \( O(\sqrt{n} \cdot \sqrt{\log(q)}) \).

Bansal, Dadush and Garg recently gave a randomized algorithm to find a vector \( x \) with entries in \( \{-1,1\} \) with \( \|Ax\|_\infty \leq O(\sqrt{\log n}) \) in polynomial time, where \( A \) is any matrix whose columns have length at most 1. We show that our method can be used to deterministically obtain such a vector.

1 Introduction

The classical setting in (combinatorial) discrepancy theory is that a set system \( S_1, \ldots, S_m \subseteq \{1, \ldots, n\} \) over a ground set of \( n \) elements is given and the goal is to find bi-coloring \( \chi : \{1, \ldots, n\} \to \{\pm 1\} \) so that the worst imbalance \( \max_{i=1,\ldots,m} |\chi(S_i)| \) of a set is minimized. Here we abbreviate \( \chi(S_i) := \sum_{j \in S_i} \chi(j) \). A seminal result of Spencer [Spe85] says that there is always a coloring \( \chi \) where the imbalance is at most \( O(\sqrt{n} \cdot \log(2m/n)) \) for \( m \geq n \). The proof of Spencer is based on the partial coloring method that was first used by Beck in 1981 [Beck81]. The argument applies the pigeonhole principle to obtain that many of the \( 2^n \) many colorings \( \chi, \chi' \) must satisfy \( |\chi(S_i) - \chi'(S_i)| \leq O(\sqrt{n} \cdot \log(2m/n)) \) for all sets \( S_i \). Then one can take the difference between such a pair of colorings with \( |\{j \mid \chi(j) \neq \chi'(j)\}| \geq \frac{n}{2} \) to obtain a partial coloring of low discrepancy. This partial coloring can be used to color half of the elements. Then one iterates the argument and again finds a partial coloring. As the remaining set system has only half the elements, the bound in the second iteration becomes better by a constant factor. This process is repeated until all elements are colored; the total discrepancy is then given by a convergent series with value \( O(\sqrt{n} \cdot \log(2m/n)) \). More general arguments based on convex geometry were given by Gluskin [Glu90] and by Giannopoulos [Gia97], but their arguments still relied on a pigeonhole principle with exponentially many pigeons and pigeonholes and did not lead to polynomial time algorithms.

In fact, Alon and Spencer [AS08] even conjectured that finding a coloring satisfying Spencer’s theorem would be intractable. In a breakthrough, Bansal [Ban10] showed that one could set up a semi-definite
program (SDP) to find at least a vector coloring, using Spencer’s Theorem to argue that the SDP has to be feasible. He then argued that a random walk guided by updated solutions to that SDP would find a coloring of discrepancy $O(\sqrt{n})$ in the balanced case $m = n$. However, his approach needed a very careful choice of parameters.

A simpler and truly constructive approach that does not rely on Spencer’s argument was provided by Lovett and Meka [LM12], who showed that for $x^{(0)} \in [-1, 1]^n$, any polytope of the form $P = \{x \in [-1, 1]^n : |\langle v_i, x - x^{(0)} \rangle| \leq \lambda, \forall i \in [m] \}$ contains a point that has at least half of the coordinates in $\{-1, 1\}$. Here it is important that the polytope $P$ is large enough; if the normal vectors $v_i$ are scaled to unit length, then the argument requires that $\sum_{i=1}^m e^{-\lambda_i^2/16} \leq \frac{n}{16}$ holds. Their algorithm surprisingly simple: start a Brownian motion at $x^{(0)}$ and stay inside any face that is hit at any time. They showed that this random walk eventually reaches a point with the desired properties.

More recently, the third author provided another algorithm which simply consists of taking a random Gaussian vector $x$ and then computing the nearest point to $x$ in $P$. In contrast to both of the previous algorithms, this argument extends to the case that $P = Q \cap [-1, 1]^n$ where $Q$ is any symmetric convex set with a large enough Gaussian measure.

However, all three algorithms described above are randomized, although Bansal and Spencer [BS13] could derandomize the original arguments by Bansal. They showed that the random walk already works if the directions are chosen from a 4-wise independent distribution, which then allows a polynomial time derandomization.

In our algorithm, we think of the process more as a multiplicative weight update procedure, where each constraint has a weight that increases if the current point moves in the direction of its normal vector. The potential function we consider is the sum of those weights. Then in each step we simply need to select an update direction in which the potential function does not increase.

The multiplicative weight update method is a meta-algorithm that originated in game theory but has found numerous recent applications in theoretical computer science and machine learning. In the general setting one imagines having a set of experts (in our case the set constraints) that are assigned an exponential by their exponential weights.

1.1 Related work

If we have a set system $S_1, \ldots, S_m$ where each element lies in at most $t$ sets, then the partial coloring technique described above can be used to find a coloring of discrepancy $O(\sqrt{t} \cdot \log n)$ [Str97]. A linear programming approach of Beck and Fiala [BF81] showed that the discrepancy is bounded by $2t - 1$, independent of the size of the set system. On the other hand, there is a non-constructive approach of Banaszczyk [Ban98] that provides a bound of $O(\sqrt{t} \log n)$ using convex geometry arguments. Only very recently, a corresponding algorithmic bound was found by Bansal, Dadush and Garg [BDG14]. A conjecture of Beck and Fiala says that the correct bound should be $O(\sqrt{t})$. This bound can be achieved for the vector coloring version, see Nikolov [Nik13].

More generally, the theorem of Banaszczuk [Ban98] shows that for any convex set $K$ with Gaussian measure at least $\frac{1}{2}$ and any set of vectors $v_1, \ldots, v_m$ of length $\|v_i\|_2 \leq \frac{1}{\sqrt{m}}$, there exist signs $\varepsilon_i \in \{\pm 1\}$ so that $\sum_{i=1}^m \varepsilon_i v_i \in K$.

A set of $k$ permutations on $n$ symbols induces a set system with $kn$ sets given by the prefix intervals. One can use the partial coloring method to find a $O(\sqrt{k} \log n)$ discrepancy coloring [SST], while a linear programming approach gives a $O(k \log n)$ discrepancy [Boh90]. In fact, for any $k$ one can always color half of the elements with a discrepancy of $O(\sqrt{k})$ — this even holds for each induced sub-system [SST]. Still, [NNN12] constructed 3 permutations requiring a discrepancy of $\Theta(\log n)$ to color all elements.

\footnote{We should mention for the sake of completeness that our update choice is \textit{not} a convex combination of the experts weighted by their exponential weights.}
Also the recent proof of the Kadison-Singer conjecture by Marcus, Spielman and Srivastava [MSS13] can be seen as a discrepancy result. They show that a set of vectors \( v_1, \ldots, v_m \in \mathbb{R}^n \) with \( \sum_{i=1}^m v_i v_i^T = I \) can be partitioned into two halves \( S_1, S_2 \) so that \( \sum_{i \in S_j} v_i v_i^T \preceq \left( \frac{1}{2} + O(\sqrt{\varepsilon}) \right) I \) for \( j \in \{1, 2 \} \). Here \( \varepsilon = \max_{i=1, \ldots, m} \| v_i \|_2^2 \) and \( I \) is the \( n \times n \) identity matrix. Their method is based on interlacing polynomials; no polynomial time algorithm is known to find the desired partition.

For a symmetric matrix \( A \in \mathbb{R}^{m \times m} \), let \( \| A \|_{\text{op}} \) denote the largest singular value; in other words, the largest absolute value of any eigenvalue. The discrepancy question can be generalized from sets to symmetric matrices \( A \). The matrix discrepancy can be seen as a discrepancy result. They show that a set of vectors \( \{ v_i \}_{i=1}^m \) and \( A \) can be partitioned into two halves \( A_i \) corresponding to the incidence vector of element \( i \) would exactly encode the set coloring setting. Again the interesting case is \( m = n \); in contrast to the diagonal case it is only known that the discrepancy is bounded by \( O(\sqrt{n \log(n)}) \), which is already attained by a random coloring. Meka conjectured that the discrepancy of \( n \) matrices can be bounded by \( O(\sqrt{n}) \).

For a very readable introduction into discrepancy theory, we recommend Chapter 4 in the book of Matoušek [Mat90] or the book of Chazelle [Cha01].

### 1.2 Our contribution

Our main result is a deterministic version of the theorem of Lovett and Meka:

\[ \text{Theorem 1.} \quad \text{Let } v_1, \ldots, v_m \in \mathbb{R}^n \text{ unit vectors, } x(0) \in [-1, 1]^n \text{ be a starting point and let } \lambda_1 \geq \ldots \geq \lambda_m \geq 0 \text{ be parameters so that } \sum_{i=1}^m \exp(-\lambda_i^2/16) \leq \frac{1}{m}. \text{ Then there is a deterministic algorithm that computes a vector } x \in [-1, 1]^n \text{ with } \langle v_i, x - x(0) \rangle \leq 8 \lambda_i \text{ for all } i \in [m] \text{ and } |\{ i : x_i = \pm 1 \}| \geq \frac{n}{2}, \text{ in time } O(\min\{n^4 m, n^3 m \lambda_1^2\}). \]

By setting \( \lambda_i = O(1) \) this yields a deterministic version of Spencer’s theorem in the balanced case \( m = n \):

**Corollary 2.** Given \( n \) sets over \( n \) elements, there is a deterministic algorithm that finds a \( O(\sqrt{n}) \)-discrepancy coloring in time \( O(n^3) \).

Furthermore, Spencer’s hyperbolic cosine algorithm [Spe77] can also be interpreted as a multiplicative weight update argument. However, the techniques of [Spe77] are only enough for a \( O(\sqrt{n \log(n)}) \) discrepancy bound for the balanced case. Our hope is that similar arguments can be applied to solve open problems such as whether there is an extension of Spencer’s result to balance matrices [Zou12] and to better discrepancy minimization techniques in the Beck-Fiala setting. To demonstrate the versatility of our arguments, we show an extension to the matrix discrepancy case.

We say that a symmetric matrix \( A \in \mathbb{R}^{m \times m} \) is \( q \)-block diagonal if it can be written as \( A = \text{diag}(B_1, \ldots, B_m/q) \), where each \( B_i \) is a symmetric \( q \times q \) matrix.

\[ \text{Theorem 3.} \quad \text{For given } q \text{-block diagonal matrices } A_1, \ldots, A_n \in \mathbb{R}^{m \times m} \text{ with } \| A_i \|_{\text{op}} \leq 1 \text{ for } i = 1, \ldots, n \text{ one can compute a coloring } x \in \{-1, 1\}^n \text{ with } \| \sum_{i=1}^n x_i A_i \|_{\text{op}} \leq O(\sqrt{n \log(2m n)}) \text{ deterministically in time } O(n^5 + n^4 m^3). \]

Finally, we can also give the first deterministic algorithm for the result of Bansal, Dadush and Garg [BDG16].

**Theorem 4.** Let \( A \in \mathbb{R}^{m \times n} \) be a matrix with \( \| A \|_2 \leq 1 \) for all columns \( j = 1, \ldots, n \). Then there is a deterministic algorithm to find a coloring \( x \in \{-1, 1\}^n \) with \( \| Ax \|_{\infty} \leq O(\sqrt{\log n}) \) in time \( O(n^3 \log(n) \cdot (m + n)) \).

While [BDG16] need to solve a semidefinite program in each step of their random walk, our algorithm does not require solving any SDPs. Note that we do not optimize running times such as by using fast matrix

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2See the blog post https://windowsontheory.org/2014/02/07/discrepancy-and-beating-the-union-bound/
algorithm is as follows:

\[ \text{For space reasons, we defer the proof of Theorem 3 to Appendix B.} \]

2 The algorithm for partial coloring

We will now describe the algorithm proving Theorem 1. First note that for any \( \lambda > 2\sqrt{n} \) we can remove the constraint \( \langle v_i, x - x_0 \rangle \leq \lambda \), as it does not cut off any point in \([-1, 1]^n\). Thus we assume without loss of generality that \( 2\sqrt{n} \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0 \). Let \( \delta = \frac{1}{\sqrt{n}} \) denote the step size of our algorithm. The algorithm will run for \( O(n/\delta^2) \) iterations, each of computational cost \( O(n^2m) \). Note that \( \delta = O(1/\sqrt{n}) \) so the algorithm terminates in \( O(n^2) \) iterations. The total runtime is hence \( O(n^2m \cdot n/\delta^2) = O(n^3m^2) \leq O(n^4m) \).

For a symmetric matrix \( M \in \mathbb{R}^{n \times n} \) we know that an eigendecomposition \( M = \sum_{j=1}^{n} \mu_j u_j u_j^T \) can be computed in time \( O(n^3) \). Here \( \mu_j := \mu_j(M) \) is the \( j \)th eigenvalue of \( M \) and \( u_j := u_j(M) \) is the corresponding eigenvector with \( \|u_j\|_2 = 1 \). We make the convention that the eigenvalues are sorted as \( \mu_1 \geq \cdots \geq \mu_n \). The algorithm is as follows:

1. Set weights \( w_i^{(0)} = \exp(-\lambda_i^2) \) for all \( i = 1, \ldots, m \).
2. For \( t = 0 \) to \( \infty \) do
   3. Define the following subspaces
      - \( U_1^{(t)} := \text{span}\{e_j : -1 < x_j^{(t)} < 1\} \)
      - \( U_2^{(t)} := \{x \in \mathbb{R}^n | \langle x, x^{(t)} \rangle = 0\} \)
      - \( U_3^{(t)} := \{x \in \mathbb{R}^n | \langle v_i, x \rangle = 0 \ \forall i \in I(t)\} \). Here \( I(t) \subseteq [m] \) are the \( |I(t)| = \frac{n}{\log n} \) indices with maximum weight \( w_i^{(t)} \).
      - \( U_4^{(t)} := \{x \in \mathbb{R}^n | \langle v_i, x \rangle = 0 \ \forall i \text{ with } \lambda_i \leq 1\} \)
      - \( U_5^{(t)} := \{x \in \mathbb{R}^n | x, \sum_{i=1}^{m} \lambda_i w_i^{(t)} \cdot \exp\left(-\frac{4\delta^2 \lambda_i^2}{n}\right) v_i = 0\} \)
      - \( U_6^{(t)} := \text{span}\{u_j(M^{(t)}) : \frac{1}{15} n \leq j \leq n\} \), for \( M^{(t)} := \sum_{i=1}^{m} w_i^{(t)} \lambda_i^2 v_i v_i^T \).
      - \( U^{(t)} := U_1^{(t)} \cap \ldots \cap U_6^{(t)} \).
4. Let \( z^{(t)} \) be any unit vector in \( U^{(t)} \).
5. Choose a maximal \( \alpha^{(t)} \in (0, 1] \) so that \( x^{(t+1)} := x^{(t)} + \delta \cdot y^{(t)} \in [-1, 1]^n \), with \( y^{(t)} = \alpha^{(t)} z^{(t)} \).
6. Update \( w_i^{(t+1)} := w_i^{(t)} \cdot \exp(\lambda_i \cdot \delta \cdot \langle v_i, y^{(t)} \rangle) \cdot \exp\left(-\frac{4\delta^2 \lambda_i^2}{n}\right) \).
7. Let \( A^{(t)} := \{i \in [n] : -1 < x_i^{(t)} < 1\} \). If \( |A^{(t)}| < \frac{n}{2} \), then set \( T := t \) and stop.

The intuition is that we maintain weights \( w_i^{(t)} \) for each constraint \( i \) that increase exponentially with the one-sided discrepancy \( \langle v_i, x^{(t)} - x^{(0)} \rangle \). Those weights are discounted in each iteration by a factor that is slightly less than 1 — with a bigger discount for constraints with a larger parameter \( \lambda_i \). The subspaces \( U_1^{(t)} \) and \( U_2^{(t)} \) ensure that the length of \( x^{(t)} \) is monotonically increasing and fully colored elements remain fully colored.

2.1 Bounding the number of iterations

First, note that if the algorithm terminates, then at least half of the variables in \( x^{(T)} \) will be either \(-1\) or \(+1\). In particular, once a variable is set to \( \pm 1 \), it is removed from the set \( A^{(t)} \) of active variables and the subsequent updates will leave those coordinates invariant.
First we bound the number of iterations. Here we use that the algorithm always makes a step of length $\delta$ orthogonal to the current position — except for the steps where it hits the boundary.

**Lemma 5.** The algorithm terminates after $T = O(\frac{n}{\delta^2})$ iterations.

**Proof.** First, we can analyze the length increase
\[ \|x^{(t+1)}\|_2^2 = \|x^{(t)} + \delta \cdot y^{(t)}\|_2^2 = \|x^{(t)}\|_2^2 + 2\delta \langle x^{(t)}, y^{(t)} \rangle + \delta^2 \|y^{(t)}\|_2^2, \]
using that $y^{(t)} \in U_2^{(t)}$. Whenever $\alpha^{(t)} = 1$, we have $\|x^{(t+1)}\|_2^2 \geq \|x^{(t)}\|_2^2 + \delta^2$. It happens that $\alpha^{(t)} < 1$ at most $n$ times, simply because in each such iteration $|A^{(t)}|$ must decrease by at least one. We know that $x^{(T)} \in [-1, 1]^n$. Suppose for the sake of contradiction that $T > \frac{2n}{\delta^2}$, then $\|x^{(T)}\|_2^2 \geq (T - n) \cdot \delta^2 > n$, which is impossible. We can hence conclude that the algorithm will terminate in step (7) after at most $\frac{2n}{\delta^2}$ iterations. □

### 2.2 Properties of the subspace $U^{(t)}$

One obvious condition to make the algorithm work is to guarantee that the subspace $U^{(t)}$ satisfies $\dim(U^{(t)}) \geq 1$. In fact, its dimension will even be linear in $n$.

**Lemma 6.** In any iteration $t$, one has $\dim(U^{(t)}) \geq \frac{n}{8}$.

**Proof.** We simply need to account for all linear constraints that define $U^{(t)}$ and we get
\[ \dim(U^{(t)}) \geq |A^{(t)}| - |I^{(t)}| - |\{i : \lambda_i \leq 1\}| - \frac{n}{16} - 2 \geq \frac{n}{2} - \frac{n}{16} - \frac{n}{8} - \frac{n}{16} - 2 \geq \frac{n}{8} \]
assuming that $n \geq 16$. □

Another crucial property will be that every vector in $U^{(t)}$ has a bounded quadratic error term:

**Lemma 7.** For each unit vector $y \in U^{(t)}$ one has $y^TM^{(t)}y \leq \frac{16}{n} \sum_{i=1}^{m} w^{(t)}_i \lambda_i^2$.

**Proof.** We have $\text{Tr}[v_i v_i^T] = 1$ since each $v_i$ is a unit vector, hence $\text{Tr}[M^{(t)}] = \sum_{i=1}^{m} w^{(t)}_i \lambda_i^2 \text{Tr}[v_i v_i^T] = \sum_{i=1}^{m} w^{(t)}_i \lambda_i^2$. Because $M^{(t)}$ is positive semidefinite, we know that $\mu_1, \ldots, \mu_n \geq 0$, where $\mu_j := \mu_j(M^{(t)})$ is the $j$th eigenvalue. Then by **Markov’s inequality** at most a $\frac{16}{n}$ fraction of eigenvalues can be larger than $\frac{16}{n} \text{Tr}[M^{(t)}]$. The claim follows as $U^{(t)}_0$ is spanned by the $\frac{16}{n}n$ eigenvectors $v_j(M^{(t)})$ belonging to the smallest eigenvalues, which means $\mu_j \leq \frac{16}{n} \text{Tr}[M^{(t)}]$ for $j = \frac{1}{16}n, \ldots, n$. □

### 2.3 The potential function

So far, we have defined the weights by iterative update steps, but it is not hard to verify that in each iteration $t$ one has the explicit expression
\[
  w^{(t)}_i = \exp \left( \lambda_i \langle v_i, x^{(t)} - x^{(0)} \rangle - \lambda_i^2 \cdot \left( 1 + t \cdot \frac{4\delta^2}{n} \right) \right).
\]

Inspired by the multiplicative weight update method, we consider the potential function $\Phi^{(t)} := \sum_{i=1}^{m} w^{(t)}_i$ that is simply the sum of the individual weights. At the beginning of the algorithm we have $\Phi^{(0)} = \sum_{i=1}^{m} w^{(0)}_i = \sum_{i=1}^{m} \exp(-\lambda_i^2/16) \leq \frac{1}{16}$ using the assumption in Theorem 1. Next, we want to show that the potential function does not increase. Here the choice of the subspaces $U^{(t)}_5$ and $U^{(t)}_6$ will be crucial to control the error.
Lemma 8. In each iteration \(t\) one has \(\Phi^{(t+1)} \leq \Phi^{(t)}\).

Proof. Let us abbreviate \(\rho_t := \exp\left(-\frac{4\delta^2 \lambda_t^2}{n}\right)\) as the discount factor for the \(i\)th constant. Note that in particular \(0 < \rho_t \leq 1\) and \(\rho_t \leq 1 - \frac{2\delta^2 \lambda_t^2}{n}\). The change in one step can be analyzed as follows:

\[
\Phi^{(t+1)} = \sum_{i=1}^{m} w_i^{(t+1)} = \sum_{i=1}^{m} w_i^{(t)} \cdot \exp\left(\lambda_i \delta \langle v_i, y^{(t)} \rangle\right) \cdot \rho_t
\]

\[
\leq \sum_{i=1}^{m} w_i^{(t)} \cdot \left(1 + \lambda_i \delta \langle v_i, y^{(t)} \rangle + \lambda_i^2 \delta^2 \langle v_i, y^{(t)} \rangle^2\right) \cdot \rho_t
\]

\[
= \sum_{i=1}^{m} w_i^{(t)} \cdot \rho_t + \delta \sum_{i=1}^{m} \lambda_i w_i^{(t)} \rho_t v_i, y^{(t)}\rangle + \delta^2 \sum_{i=1}^{m} w_i^{(t)} \lambda_i^2 \rho_t \langle v_i, y^{(t)} \rangle^2
\]

\[
\leq \sum_{i=1}^{m} w_i^{(t)} \cdot \rho_t + \delta^2 \cdot (y^{(t)})^T M^{(t)} y^{(t)} \leq \sum_{i=1}^{m} w_i^{(t)} \cdot \rho_t + \delta^2 \frac{16}{n} \sum_{i=1}^{m} w_i^{(t)} \lambda_i^2
\]

\[
\leq \sum_{i=1}^{m} w_i^{(t)} = \Phi^{(t)}.
\]

In (\(\ast\)), we use the inequality \(e^x \leq 1 + x + x^2\) for \(|x| \leq 1\) together with the fact that \(\lambda_i \delta \langle v_i, y^{(t)} \rangle| \leq \lambda_i \delta \leq 1\). In (\(\ast\ast\)) we bound \((y^{(t)})^T M^{(t)} y^{(t)}\) using Lemma 7. In (\(\ast\ast\ast\)) we finally use the fact that \(\rho_t \leq 1 - \frac{2\delta^2 \lambda_t^2}{n} \leq 1\). \(\square\)

Typically in the multiplicative weight update method one can only use the fact that \(\max_{i \in [m]} w_i^{(t)} \leq \Phi^{(t)}\) which would lead to the loss of an additional \(\sqrt{\log n}\) factor. The trick in our approach is that there is always a linear number of weights of order \(\max_{i \in [m]} w_i^{(t)}\) since the updates are always chosen orthogonal to the \(\frac{n}{16}\) constraints with highest weight.

Lemma 9. At the end of the algorithm, \(\max\{w_i^{(T)} : i \in [m]\} \leq 2\).

Proof. Suppose, for contradiction, that \(w_i^{(T)} > 2\) for some \(i\). Let \(t^*\) be the last iteration when \(i\) was not among the \(\frac{n}{16}\) constraints with highest weight. After iteration \(t^* + 1\), \(w_i^{(t)}\) only decreases in each iteration, due to the factor \(\exp\left(-\frac{4\delta^2 \lambda_t^2}{n}\right)\). Then

\[
2 < w_i^{(T)} = w_i^{(t^* + 1)} = w_i^{(t^*)} \cdot \exp(\lambda_i \cdot \delta \cdot \langle v_i, y^{(t^*)} \rangle) \cdot \rho_t \leq w_i^{(t^*)} \cdot e,
\]

and hence, \(w_i^{(t^*)} > \frac{2}{e}\). This would imply that \(\Phi^{(t^*)} \geq \frac{n}{16} \cdot \frac{2}{e} > \frac{n}{32}\), contradicting Lemma 8. \(\square\)

Lemma 10. If \(w_i^{(T)} \leq 2\), then \(\langle v_i, x^{(T)} - x^{(0)} \rangle \leq 11\lambda_i\).

Proof. First note that the algorithm always walks orthogonal to all constraint vectors \(v_i\) if \(\lambda_i \leq 1\) and in this case \(\langle v_i, x^{(T)} - x^{(0)} \rangle = 0\). Now suppose that \(\lambda_i > 1\). We know that \(w_i^{(T)} \leq \exp\left(\lambda_i \cdot \langle v_i, x^{(T)} - x^{(0)} \rangle - \lambda_i^2 \cdot \left(1 + 4 \cdot T \cdot \frac{\delta^2}{n}\right)\right) \leq 2\). Taking logarithms on both sides and dividing by \(\lambda_i\) then gives

\[
\langle v_i, x^{(T)} - x^{(0)} \rangle \leq \frac{\log(2)}{\lambda_i} + \lambda_i \left(1 + 4 \cdot T \cdot \frac{\delta^2}{n}\right) \leq 11\lambda_i.
\]

This lemma concludes the proof of Theorem 11. \(\square\)
2.4 Application to set coloring

Now we come to the main application of the partial coloring argument from Theorem 1 which is to color set systems:

**Lemma 11.** Given a set system $S_1, \ldots, S_m \subseteq [n]$, we can find a coloring $x \in \{-1, 1\}^n$ with $|\sum_{j \in S_i} x_j| \leq O(\sqrt{n \log \frac{2m}{n}})$ for every $i$ deterministically in time $O(n^3 m \log(\frac{2m}{n}))$.

**Proof.** For a fractional vector $x$, let us abbreviate $\text{disc}(S, x) := |\sum_{j \in S} x_j|$ as the discrepancy with respect to set $S$. Set $x^{(0)} := \mathbf{0}$. For $s = 1, \ldots, \log_2(n)$ many phases we do the following. Let $A^{(s)} := \{i \in [n] : -1 < x^{(s-1)}_i < 1\}$ be the not yet fully colored elements. Define a vector $v_i := \frac{1}{\sqrt{|A^{(s)}|}} 1_{S_i \cap A^{(s)}}$ of length $\|v_i\|_2 \leq 1$ with parameters $\lambda_i := C\sqrt{\log(\frac{2m}{|A^{(s)}|})}$. Then apply Theorem 1 to find $x^{(s)} \in [-1, 1]^n$ with $\text{disc}(S, x^{(s)}) - x^{(s-1)} \leq O(\sqrt{|A^{(s)}| \log(\frac{2m}{|A^{(s)}|})})$ such that $x_i^{(s)} = x_i^{(s-1)}$ for $i \notin A^{(s)}$. Since each time at least half of the elements get fully colored we have $|A^{(s)}| \leq 2^{-(s-1)} n$ for all $s$. Then $x := x^{(\log_2 n)} \in \{-1, 1\}^n$ and

$$\text{disc}(S_i, x) \leq \sum_{s \geq 1} O\left(\sqrt{2^{-(s-1)} n \log \left(\frac{2m}{2^{-(s-1)} n}\right)}\right) \leq O\left(\sqrt{n \log \left(\frac{2m}{n}\right)}\right)$$

using that this convergent sequence is dominated by the first term.

In each application of Theorem 1 one has $\delta \geq \Omega(1/\sqrt{\log(\frac{2m}{n})})$. Thus phase $s$ runs for $O(2^{-(s-1)} n / \delta^2) = O(2^{-(s-1)} n \log(\frac{2m}{n}))$ iterations, each of which takes $O((2^{-(s-1)} n)^2 m)$ time. This gives a total runtime of $O((2^{-(s-1)} n)^3 m \log(\frac{2m}{n}))$ in phase $s$. Summing the geometric series for $s = 1, \ldots, \log_2 n$ results in a total running time of $O(n^3 m \log(\frac{2m}{n}))$. $lacksquare$

By setting $m = n$ in Lemma 11 we obtain Corollary 2.

3 Matrix balancing

In this section we prove Theorem 4. We begin with some preliminaries. For matrices $A, B \in \mathbb{R}^{n \times n}$, let $A \cdot B := \sum_{i=1}^n \sum_{j=1}^n A_{ij} \cdot B_{ij}$ be the Frobenius inner product. Recall that any symmetric matrix $A \in \mathbb{R}^{n \times n}$ can be written as $A = \sum_{j=1}^n \mu_j u_j u_j^T$, where $\mu_j$ is the eigenvalue corresponding to eigenvector $u_j$. The trace of $A$ is $\text{Tr}[A] = \sum_{j=1}^n A_{jj} = \sum_{j=1}^n \mu_j$ and for symmetric matrices $A, B$ one has $\text{Tr}[AB] = A \cdot B$. If $A$ has only nonnegative eigenvalues, we say that $A$ is positive semidefinite and write $A \succeq 0$. Recall that $A \succeq 0$ if and only if $y^T A y \geq 0$ for all $y \in \mathbb{R}^n$. For a symmetric matrix $A$, we denote $\mu_{\text{max}} := \max \{\mu_j : j = 1, \ldots, n\}$ as the largest Eigenvalue and $\|A\|_{\text{op}} := \max \{|\mu_j| : j = 1, \ldots, n\}$ as the largest singular value. Note that if $A \succeq 0$, then $|A \cdot B| \leq \text{Tr}[A] \cdot \|B\|_{\text{op}}$. If $A, B \succeq 0$, then $A \cdot B \succeq 0$. Finally, note that for any symmetric matrix $A$ one has $A^2 := AA \succeq 0$.

From the eigendecomposition $A = \sum_{j=1}^n \mu_j u_j u_j^T$, one can easily show that the maximum singular value also satisfies $\|A\|_{\text{op}} = \max \{|\mu_j| \geq 1\}$ and $\|A\|_{\text{op}} = \max \{|\mu_j| \geq 1\}$. For any function $f : \mathbb{R} \rightarrow \mathbb{R}$ we define $f(A) := \sum_{j=1}^n f(\mu_j) u_j u_j^T$ to be the symmetric matrix that is obtained by applying $f$ to all Eigenvalues. In particular we will be interested in the matrix exponential $\exp(A) := \sum_{j=1}^n e^{\mu_j} u_j u_j^T$. For any symmetric matrices $A, B \in \mathbb{R}^n$, the Golden-Thompson inequality says that $\text{Tr}[\exp(A + B)] \leq \text{Tr}[\exp(A) \exp(B)]$. (It is not hard to see that for diagonal matrices one has equality.) We refer to the textbook of Bhattacharya [Bha97] for more details.

**Theorem 12.** Let $A_1, \ldots, A_m \in \mathbb{R}^{n \times n}$ be $q$-block diagonal matrices with $\|A_i\|_{\text{op}} \leq 1$ for $i = 1, \ldots, m$ and let $x^{(0)} \in \{-1, 1\}^n$ be a starting point. Then there is a deterministic algorithm that finds an $x \in \{-1, 1\}^n$
that will bound the quadratic error term.

The algorithm is as follows: suppose that the potential function is simply the sum of the potential function applied to each individual block. The proof.

Proof. By exactly the same arguments as in Lemma 5 we know that the algorithm terminates after \( \frac{3}{2} \) coordinates of \( x \) will be in \( \{-1, 1\} \).

Our algorithm computes a sequence of iterates \( x^{(0)}, \ldots, x^{(T)} \) such that \( x^{(T)} \) is the desired vector \( x \) with half of the coordinates being integral. In our algorithm the step size is \( \delta = \frac{1}{\sqrt{n}} \) and we use a parameter \( \varepsilon = \frac{1}{\sqrt{n}} \) to control the scaling of the following potential function:

\[
\Phi^{(t)} := \text{Tr} \left[ \exp \left( \varepsilon \sum_{i=1}^{n} (x_i^{(t)} - x_i^{(0)}) \cdot A_i \right) \right].
\]

Suppose \( B_{i,k} \in \mathbb{R}^{n \times q} \) are symmetric matrices so that \( A_i = \text{diag}(B_{i,1}, \ldots, B_{i,m/q}) \). Then we can decompose the weight function as \( \Phi^{(t)} = \sum_{k=1}^{m/q} \Phi_k^{(t)} \) with \( \Phi_k^{(t)} := \text{Tr} \left[ \exp \left( \varepsilon \sum_{i=1}^{n} (x_i^{(t)} - x_i^{(0)})B_{i,k} \right) \right] \). In other words, the potential function is simply the sum of the potential function applied to each individual block. The algorithm is as follows:

1. \( FOR \ t = 0 \ TO \ \infty \ DO \)
2. Define weight matrix \( W^{(t)} := \exp(\varepsilon \sum_{i=1}^{n} (x_i^{(t)} - x_i^{(0)})A_i) \)
3. Define the following subspaces
   - \( U_1^{(t)} := \text{span}\{e_i : -1 < x_i^{(t)} < 1\} \)
   - \( U_2^{(t)} := \{x \in \mathbb{R}^n \mid \langle x, x^{(t)} \rangle = 0\} \)
   - \( U_3^{(t)} := \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i B_{i,k} = 0 \ \forall k \in I^{(t)}\} \). Here \( I^{(t)} \subseteq [m] \) are the \( |I^{(t)}| = \frac{1}{16} \cdot \frac{m}{q} \) indices \( k \) with maximum weight \( \Phi_k^{(t)} \).
   - \( U_4^{(t)} := \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \cdot (W^{(t)} \cdot A_i) = 0\} \)
   - \( U_5^{(t)} \) is the subspace defined in Lemma 14 with \( k = 16 \).
   - \( U^{(t)} := U_1^{(t)} \cap \ldots \cap U_5^{(t)} \)
4. Let \( z^{(t)} \) be any unit vector in \( U^{(t)} \).
5. Choose a maximal \( \alpha^{(t)} \in (0, 1] \) so that \( x^{(t+1)} := x^{(t)} + \delta \cdot y^{(t)} \in [-1, 1]^n \), where \( y^{(t)} = \alpha^{(t)} z^{(t)} \).
6. Let \( A^{(t)} := \{j \in [n] : -1 < x_j^{(t)} < 1\} \). If \( |A^{(t)}| < \frac{n}{q} \), then set \( T := t \) and stop.

The analysis of our algorithm follows a sequence of lemmas, the proofs of most of which we defer to Appendix A. By exactly the same arguments as in Lemma 5 we know that the algorithm terminates after \( T \leq \frac{2m}{q} \) iterations. Each iteration can be done in time \( O(n^2 m^3 + n^3) \) (c.f. Lemma 14).

**Lemma 13.** In each iteration \( t \) one has \( \dim(U^{(t)}) \geq \frac{n}{4} \).

**Proof.** We simply need to account for all linear constraints that define \( U^{(t)} \) and we get

\[
\dim(U^{(t)}) \geq |A^{(t)}| - |I^{(t)}| - \frac{n}{16} \overset{\varepsilon^{(t)}_1}{\geq} - \frac{2}{U_5^{(t)}} \overset{\varepsilon^{(t)}_2}{\geq} \frac{n}{2} - \frac{n}{16q^2} \cdot q^2 - \frac{n}{16} - 2 \geq \frac{n}{4}
\]

assuming that \( n \geq 16 \).

To analyze the behavior of the potential function, we first prove the existence of a suitable subspace \( U_5^{(t)} \) that will bound the quadratic error term.
Lemma 14. Let $W \in \mathbb{R}^{m \times m}$ be a symmetric positive semidefinite matrix, let $A_1, \ldots, A_n \in \mathbb{R}^{m \times m}$ be symmetric matrices with $\|A_i\|_{op} \leq 1$ and let $k > 0$ be a parameter. Then in time $O(n^2 m^3 + n^3)$ one can compute a subspace $U \subseteq \mathbb{R}^n$ of dimension $\text{dim}(U) \geq (1 - \frac{1}{k})n$ so that

$$W \cdot \left(\sum_{i=1}^{n} y_i A_i\right)^2 \leq k \cdot \text{Tr}[W] \quad \forall y \in U \text{ with } \|y\|_2 = 1. \quad (2)$$

Proof. See Appendix A.

Again, we bound the increase in the potential function:

Lemma 15. In each iteration $t$, one has $\Phi^{(t+1)} \leq (1 + 16\varepsilon^2 \delta^2) \cdot \Phi^{(t)}$.

Proof. See the Appendix A.

This gives us a bound on the potential function at the end of the algorithm.

Lemma 16. At the end of the algorithm, $\Phi^{(T)} \leq m \cdot \exp(32\varepsilon^2 n)$.

Proof. Since $\Phi^{(0)} = \text{Tr}[\exp(0)] = \text{Tr}[I] = m$, we get that $\Phi^{(T)} \leq m \cdot (1 + 16\varepsilon^2 \delta^2)^T \leq m \cdot \exp(32\varepsilon^2 n)$, using the fact that $T \leq \frac{2n}{\delta^2}$.

Lemma 17. We have $\mu_{\text{max}}(\sum_{i=1}^{n} (x_i^{(T)} - x_i^{(0)}) \cdot A_i) = O(\sqrt{n \log(\frac{2mn}{n})})$.

Proof. See Appendix A.

These lemmas put together give us Theorem 12: an algorithm that yields a partial coloring with the claimed properties. We run the algorithm in phases to obtain Theorem 3 by boosting the partial coloring to a full coloring using a similar technique as in Lemma 11. The interested reader may refer to Appendix A for details.

References

[AHK12] S. Arora, E. Hazan, and S. Kale. The multiplicative weights update method: a meta-algorithm and applications. Theory of Computing, 8(6):121–164, 2012.

[AS08] N. Alon and J. H. Spencer. The probabilistic method. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2008. With an appendix on the life and work of Paul Erdős.

[Ban98] W. Banaszczyk. Balancing vectors and Gaussian measures of $n$-dimensional convex bodies. Random Structures Algorithms, 12(4):351–360, 1998.

[Ban10] N. Bansal. Constructive algorithms for discrepancy minimization. In FOCS, pages 3–10, 2010.

[BDG16] Nikhil Bansal, Daniel Dadush, and Shashwat Garg. An algorithm for komlós conjecture matching banaszczyk’s bound. CoRR, abs/1605.02882, 2016.

[Bec81] J. Beck. Roth’s estimate of the discrepancy of integer sequences is nearly sharp. Combinatorica, 1(4):319–325, 1981.

[BF81] J. Beck and T. Fiala. “Integer-making” theorems. Discrete Appl. Math., 3(1):1–8, 1981.

[Bha97] Rajendra Bhatia. Matrix analysis, volume 169 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1997.
[Boh90] G. Bohus. On the discrepancy of 3 permutations. *Random Structures Algorithms*, 1(2):215–220, 1990.

[BS13] Nikhil Bansal and Joel Spencer. Deterministic discrepancy minimization. *Algorithmica*, 67(4):451–471, 2013.

[Cha01] B. Chazelle. *The discrepancy method - randomness and complexity*. Cambridge University Press, 2001.

[Gia97] A. Giannopoulos. On some vector balancing problems. *Studia Mathematica*, 122(3):225–234, 1997.

[Glu89] E. D. Gluskin. Extremal properties of orthogonal parallelepipeds and their applications to the geometry of banach spaces. *Mathematics of the USSR-Sbornik*, 64(1):85, 1989.

[LM12] S. Lovett and R. Meka. Constructive discrepancy minimization by walking on the edges. In *FOCS*, pages 61–67, 2012.

[Mat99] J. Matoušek. *Geometric discrepancy*, volume 18 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 1999. An illustrated guide.

[MSS13] A. Marcus, D. A Spielman, and N. Srivastava. Interlacing Families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem. *ArXiv e-prints*, June 2013.

[Nik13] A. Nikolov. The Komlos Conjecture Holds for Vector Colorings. *ArXiv e-prints*, January 2013.

[NNN12] A. Newman, O. Neiman, and A. Nikolov. Beck’s three permutations conjecture: A counterexample and some consequences. In *FOCS*, pages 253–262, 2012.

[Spe77] Joel Spencer. Balancing games. *J. Comb. Theory, Ser. B*, 23(1):68–74, 1977.

[Spe85] J. Spencer. Six standard deviations suffice. *Transactions of the American Mathematical Society*, 289(2):679–706, 1985.

[Sri97] A. Srinivasan. Improving the discrepancy bound for sparse matrices: Better approximations for sparse lattice approximation problems. In *SODA’97*, pages 692–701, Philadelphia, PA, 1997. ACM SIGACT, SIAM.

[SST] J. H. Spencer, A. Srinivasan, and P. Tetali. The discrepancy of permutation families. Unpublished manuscript.

[Zou12] Anastasios Zouzias. A matrix hyperbolic cosine algorithm and applications. In *Automata, Languages, and Programming - 39th International Colloquium, ICALP 2012, Warwick, UK, July 9-13, 2012, Proceedings, Part I*, pages 846–858, 2012.
A Proofs from Section 3

of Lemma 14 Let $M \in \mathbb{R}^{n \times n}$ be the matrix with entries $M_{ij} := W \cdot A_i A_j$ for all $i, j \in [n]$. First note that the matrix $M$ is symmetric, because $M_{ij} = W \cdot A_i A_j = (A_j A_i)^T = W \cdot A_j A_i = M_{ji}$. Now, for any $y \in \mathbb{R}^n$, $(\sum_{i=1}^n y_i A_i)^2$ is symmetric and positive semidefinite and hence $y^T M y = \sum_{i,j=1}^n y_i y_j (W \cdot A_i A_j) = W \cdot (\sum_{i=1}^n y_i A_i)^2 \geq 0$, proving that $M$ is positive semidefinite.

Consider the eigendecomposition $M = \sum_{i=1}^n \mu_i u_i u_i^T$ where $\mu_i \geq 0$. Define the subspace $U := \text{span}\{u_i : \mu_i < k \text{Tr}[W] \}$. The desired inequality (2) follows immediately from the definition of $U$. All that remains is to verify that $\dim(U) \geq (1 - \frac{1}{n}) n$. Since $\mu_i \geq 0$ we may apply Markov’s inequality to deduce that $\#\{i : \mu_i \geq k \cdot \text{Tr}[W] \} \leq \frac{\text{Tr}[M]}{k \cdot \text{Tr}[W]} \leq \frac{n}{k}$, where in the second inequality we have used the bound $\text{Tr}[M] = \sum_{i=1}^n M_{ii} = \sum_{i=1}^n W \cdot A_i^2 \leq n \cdot \text{Tr}[W]$. Hence $\dim(U) \geq n - \#\{i : \mu_i \geq k \text{Tr}[W] \} \geq (1 - \frac{1}{n}) n$, as desired.

Finally, to bound the running time, we observe that computing $M$ takes time $O(n^2 m^3)$ and the eigendecomposition of $M$ can be computed in time $O(n^3)$. \hfill \Box

Proof of Lemma 16. We estimate that

$$
\Phi(t+1) = \text{Tr}[\exp\left(\varepsilon \sum_{i=1}^n (x_i^{(t+1)} - x_i^{(0)}) A_i\right)]
$$

$$
= \text{Tr}\left[\exp\left(\varepsilon \sum_{i=1}^n (x_i^{(t)} - x_i^{(0)}) A_i + \varepsilon \delta \sum_{i=1}^n y_i^{(t)}\right)\right]
$$

$$
(\ast) \leq \text{Tr}\left[\exp\left(\varepsilon \sum_{i=1}^n (x_i^{(t)} - x_i^{(0)}) \cdot A_i\right) \exp\left(\varepsilon \delta \sum_{i=1}^n y_i^{(t)} A_i\right)\right]
$$

$$
= W(t) \cdot \exp\left(\varepsilon \delta \sum_{i=1}^n y_i^{(t)} A_i\right)
$$

$$
(\ast\ast) \leq W(t) \cdot \left(I + \varepsilon \delta \sum_{i=1}^n y_i^{(t)} A_i + \varepsilon^2 \delta^2 \left(\sum_{i=1}^n y_i^{(t)} A_i\right)^2\right)
$$

$$
= W(t) \cdot \left(I + \varepsilon \delta \left(\sum_{i=1}^n y_i^{(t)} A_i\right) + \varepsilon^2 \delta^2 \left(\sum_{i=1}^n y_i^{(t)} A_i\right)^2\right)
$$

$$
(\ast\ast\ast) \leq \Phi(t) \cdot (1 + 16\varepsilon^2 \delta^2).
$$

In $(\ast)$ we use the Golden-Thompson inequality. In $(\ast\ast)$ we use that $\exp(X) \preceq I + X + X^2$ for any symmetric matrix $X$ with $\|X\|_{\text{op}} \leq 1$ together with the triangle inequality

$$
\left\|\varepsilon \delta \sum_{i=1}^n y_i^{(t)} A_i\right\|_{\text{op}} \leq \varepsilon \delta \sum_{i=1}^n \|y_i^{(t)} \cdot A_i\|_{\text{op}} \leq \varepsilon \delta n = 1.
$$

In $(\ast\ast\ast)$ we use Lemma 14 and the fact that $y(t) \in U_5(t)$. \hfill \Box

Proof of Lemma 17. Let $\mu_{\text{max}} := \mu_{\text{max}}(\sum_{i=1}^n (x_i^{(T)} - x_i^{(0)}) \cdot A_i)$. Suppose the eigenspace corresponding to $\mu_{\text{max}}$ lies in block $k$, for some $k \in \{1, \ldots, m/q\}$. Let $t^*$ be the last iteration when $k$ was not among the

\footnote{One has to be careful as the product $A_i A_j$ is in general not symmetric, even if $A_i$ and $A_j$ are symmetric.}
\[ n/(16q^2) \] indices with maximum weight. We then have
\[
\Phi_k^{(T)} = \Phi_k^{(t^* + 1)} = \text{Tr} \left[ \exp \left( \varepsilon \sum_{i=1}^{n} (x_i^{(t^*)} + \delta y_i^{(t^*)} - x_i^{(0)}) B_i, k \right) \right]
\]
\[
\leq \text{Tr} \left[ \exp \left( \varepsilon \sum_{i=1}^{n} (x_i^{(t^*)} - x_i^{(0)}) B_i, k \right) \exp \left( -\varepsilon \delta \sum_{i=1}^{n} y_i^{(t^*)} B_i, k \right) \right] \leq e \cdot \Phi_k^{(t^*)},
\]
where in (\(\ast\)) we use the Golden-Thompson inequality. In (\(\ast\ast\)) we have used the bounds \(\|B_i, k\|_{op} \leq \|A_i\|_{op} \leq 1\) and \(|y_i^{(t^*)}| \leq 1\) together with the triangle inequality to deduce that \(\|\varepsilon \sum_{i=1}^{n} y_i^{(t^*)} B_i, k\|_{op} \leq 1\). Hence
\[
e^{\mu_{max}} \leq \Phi_k^{(T)} \leq e \cdot \Phi_k^{(t^*)} \leq e \cdot \frac{16q^2}{n} \Phi^{(T)} \leq \frac{16q^2}{n} \cdot m \exp(32\varepsilon^2 n).
\]
Then taking logarithms and dividing by \(\varepsilon\) gives
\[
\mu_{max} \leq \frac{1}{\varepsilon} \cdot \log \left( \frac{16q^2 m}{n} \right) + 32\varepsilon n = O \left( \sqrt{n \log \frac{qm}{n}} \right),
\]
where in the final inequality we have used that \(\varepsilon = \sqrt{\log(qm/n)}\).

Proof of Theorem 3

Set \(x^{(0)} := 0\). For \(s = 1, \ldots, \log_2(n)\) many phases we do the following. Let \(J^{(s)} := \{ i \in [n] : -1 < x_i^{(s-1)} < 1 \}\) be the not yet fully colored elements. Apply Theorem 12 to find \(x^{(s)} \in [-1, 1]^n\) with
\[
\left\| \sum_{i \in J^{(s)}} (x_i^{(s)} - x_i^{(s-1)}) \cdot A_i \right\|_{op} = O \left( \sqrt{|J^{(s)}| \log \frac{2qm}{|J^{(s)}|}} \right),
\]
and such that \(x_i^{(s)} = x_i^{(s-1)}\) for all \(i \notin J^{(s)}\). Since each time at least half of the elements get fully colored we have \(|J^{(s)}| \leq 2^{-s+1} n\) for all \(s\). Then \(x := x^{(\log_2 n)} \in \{-1, 1\}^n\) and
\[
\left\| \sum_{i=1}^{n} x_i \cdot A_i \right\|_{op} = \sum_{s \geq 1} O \left( \sqrt{2^{-s+1} n \log \frac{2qm}{2^{-s+1} n}} \right) = O \left( \sqrt{n \log \frac{2qm}{n}} \right),
\]
using that the sum of a subgeometric sequence is dominated by its first term. Phase \(s\) has a running time of \(O((2^{-s+1} n)^5 + (2^{-s+1} n)^4 m^3)\) and summing this geometric series over \(s = 1, \ldots, \log_2 n\) yields a total runtime of \(O(n^5 + n^4 m^3)\).

B Discrepancy minimization for matrices with bounded column length

In this section we prove Theorem 3. Fix a matrix \(A \in \mathbb{R}^{m \times n}\) with \(\|A^j\|_2 \leq 1\) for each column \(j = 1, \ldots, n\). Recently Bansal, Dadush and Garg [BDG16] gave the first polynomial time algorithm to find a coloring \(x \in \{-1, 1\}^n\) with \(\|Ax\|_{\infty} \leq O(\sqrt{\log n})\). Their method is based on a random walk, where the random updates in each iteration are chosen using a semidefinite program that has to be re-solved each time. We show that instead a deterministic walk can be used, guided by a suitable exponential potential function. The update directions will be chosen from the intersection of subspaces satisfying certain constraints; no SDP has to be solved in our method. We should also mention that the more general non-constructive result of
Banasczyk [Ban98] even guarantees signs $x$ so that $Ax \in 5 \cdot K$, where $K$ is any convex body with Gaussian measure at least $1/2$.

In this section, let $C > 0$ be a sufficiently large constant. For a row $i$ with $\|A_i\|_2^2 \leq \frac{1}{n}$, any coloring $x$ will satisfy $|\langle A_i, x \rangle| \leq \|A_i\|_2 \|x\|_2 \leq 1$ and we can safely remove such a row. From now on we can assume that $\|A_i\|_2^2 \geq \frac{1}{n}$ and hence $m \leq n^2$. Note that it also suffices to find an $x \in \{-1, 1\}^n$ satisfying the one-sided error $\langle A_i, x \rangle \leq O(\sqrt{\log n})$ as one can simply stack $\sqrt{2} A$ and $-\sqrt{2} A$ together. Next, replace each row $A_i$ with two rows: one row is the light row containing all entries of size $|A_i| \leq \frac{1}{C^{2} \sqrt{\log n}}$ and the other row is the heavy row whose only nonzero entries have size $|A_i| > \frac{1}{C^{2} \sqrt{\log n}}$. After this modification, we abbreviate the indices as $I_{\text{light}} := \{ i \in [m] : \|A_i\|_\infty \leq \frac{1}{C^{2} \sqrt{\log n}} \}$ and $I_{\text{heavy}} := \{ i \in [m] : \|A_i\|_\infty > \frac{1}{C^{2} \sqrt{\log n}} \}$. As in the previous settings, our algorithm will compute a sequence $x^{(0)}, \ldots, x^{(T)} \in [−1, 1]^n$, starting at $x^{(0)} = 0$ so that the final point $x^{(T)}$ has coordinates only in $\{-1, 1\}$. For the point $x^{(t)} \in [−1, 1]^n$ and some parameters $\alpha, \beta > 0$ that we specify later, we define a potential function $\Phi^{(t)} := \sum_{i \in I_{\text{light}}} w_i^{(t)}$ with

$$w_i^{(t)} := \exp \left( \alpha \left( A_i, x^{(t)} \right) \right) + \beta \min \left\{ C, \sum_{j=1}^{n} \left( 1 - \langle x^{(t)} \rangle^2 \cdot A_{ij}^2 \right) \right\}.$$

Here the quantity $L(i, x) := \sum_{j=1}^{n} (1 - x_j^2) \cdot A_{ij}^2$ can be interpreted as the effective length of row $i$ with $L(i, 0) = \|A_i\|_2^2$ and $L(i, x) = 0$, if $x \in \{-1, 1\}^n$.

The intuition behind the algorithm is as follows: at the beginning one has $x^{(0)} = 0$ and the whole weight of the potential function comes from the $\beta$-term. Then in the course of the algorithm the weight is transferred from the $\beta$-term to the $\alpha$-term until all elements are colored and the effective length of all constraints is 0. In fact, if $\beta \geq \Omega(\alpha^2)$, we show that the potential function is nonincreasing.

To keep the notation readable, for vectors $x, y \in \mathbb{R}^n$ we write $(x \circ y)_i := x_i \cdot y_i$ and $x^{\otimes 2} := x \circ x$. Moreover, $x^{\otimes 2} := xx^T$ is the tensor product. As before, we find an update vector in each iteration so that the potential function does not increase, by choosing it from the intersection of certain subspaces. We postpone some linear algebra arguments till Section B.2. We use the following algorithm:

1. Set $x^{(0)} := 0$ and $A^{(0)} := [n]$.
2. FOR $t = 0$ TO $T$ DO
3. Let $I^{(t)} := \{ i \in I_{\text{light}} | L(i, x^{(t)}) < C \} \cup \{ i \in I_{\text{heavy}} | \sum_{j \in A^{(t)}} A_{ij}^2 < C \}$. Define the subspaces
   - $U^{(t)}_0 := \text{span}\{e_j \mid j \in A^{(t)}\} \subseteq \mathbb{R}^n$
   - $U^{(t)}_1 := \{ x \in U^{(t)}_0 | \langle x, A_i \rangle = 0 \}$
   - $U^{(t)}_2 := \{ x \in U^{(t)}_0 | \langle x, A_i \rangle = 0 \forall i \notin I^{(t)} \}$
   - $U^{(t)}_3 := \{ x \in U^{(t)}_0 | \sum_{j \in I^{(t)}} w_j^{(t)} A_{ij}, x \} = 0 \}$
   - $U^{(t)}_4 := \{ x \in U^{(t)}_0 | \sum_{j \in I^{(t)}} w_j^{(t)} \cdot (A_{i}^{\otimes 2} \circ x^{(t)}, x) = 0 \}$
   - $U^{(t)}_5 \subseteq U^{(t)}_0$ with $\sum_{j \in I^{(t)}} w_j^{(t)} \langle A_i, x \rangle^2 \leq \frac{\beta}{16 \sigma^2} \sum_{i \in I^{(t)}} w_i^{(t)} \sum_{j=1}^{n} x_j^2 A_{ij}^2$ for all $x \in U^{(t)}_5$ and $\text{dim}(U^{(t)}_5) \geq \frac{|I^{(t)}|}{\beta}$ (see Sec. B.2)
   - $U^{(t)}_6 \subseteq U^{(t)}_0$ with $\text{dim}(U^{(t)}_6) \geq \frac{15}{16} |A|^{(t)}$ and $\sum_{j \in I^{(t)}} w_j^{(t)} \langle A_{i}^{\otimes 2} \circ x^{(t)}, x \rangle^2 \leq \frac{1}{8 \sigma^2} \sum_{i \in I^{(t)}} w_i^{(t)} \sum_{j=1}^{n} x_j^2 A_{ij}^2$ for all $x \in U^{(t)}_6$ (see Sec. B.2)
   - $U^{(t)} := U^{(t)}_1 \cap \ldots \cap U^{(t)}_6$
4. Let $z^{(t)}$ be any unit vector in $U^{(t)}$.
5. Choose a maximal $\alpha^{(t)} \in (0, 1]$ so that $x^{(t+1)} := x^{(t)} + \delta \cdot z^{(t)} \in [−1, 1]^n$ with $y^{(t)} = \alpha^{(t)} z^{(t)}$.
6. Let $A^{(t)} := \{ j \in [n] : -1 < x_j^{(t)} < 1 \}$. If $|A^{(t)}| \leq C$, then set $T := t$ and stop.

Technically speaking, the final point $x^{(T)}$ still has a constant number of entries not in $[−1, 1]$ — these entries can be rounded arbitrarily. The first step is to guarantee that the subspace $U^{(t)}$ is indeed non-empty in each iteration.
Lemma 18. In each iteration \( t \), we have \( \dim(U^{(t)}) \geq \frac{1}{2} |A^{(t)}| \), if \( C \) is chosen large enough.

Proof. Observe that for \( i \in I_{\text{light}} \setminus I^{(t)} \),

\[
\sum_{j \in A^{(t)}} A_{ij}^2 \geq \sum_{j \in A^{(t)}} (1 - (x^{(t)})^2) A_{ij}^2 = \sum_{j=1}^{n} (1 - (x^{(t)})^2) A_{ij}^2 = L(i, x^{(t)}) \geq C,
\]

and hence \( \sum_{j \in A^{(t)}} A_{ij}^2 \geq C \) holds for all \( i \notin I^{(t)} \). Now, since the \( t \)-norm of each column \( A^t \) is at most \( 1 \), we have

\[
|A^{(t)}| \geq \sum_{j \in A^{(t)}} \sum_{i \in [m]} A_{ij}^2 \geq \sum_{i \notin I^{(t)}} \sum_{j \in A^{(t)}} A_{ij}^2 \geq \sum_{i \notin I^{(t)}} \sum_{j \in A^{(t)}} A_{ij}^2 \geq C(m - |I^{(t)}|).
\]

Hence, \( \text{codim}(U^{(t)}) \leq |A^{(t)}|/C \). We can hence bound

\[
\dim(U^{(t)}) \geq \frac{|A^{(t)}|}{C} - \frac{|A^{(t)}|}{16} - \frac{|A^{(t)}|}{16} - \frac{(1 + 1 + 1)}{16} \geq \frac{|A^{(t)}|}{2},
\]

if \( C \) is chosen large enough. \( \square \)

As before, one always has \( \|x^{(t+1)}\|_2^2 \geq \|x^{(t)}\|_2^2 \) and in each but at most \( n \) iterations one has \( \|x^{(t+1)}\|_2^2 = \|x^{(t)}\|_2^2 + \delta^2 \). Then the algorithm terminates after \( T \leq n + \frac{n}{C} \leq \frac{2n}{\beta} \) iterations, given that \( 0 < \delta \leq 1 \).

The main part of the analysis lies in guaranteeing that the potential function is nonincreasing.

Lemma 19. Suppose that \( \beta \geq C \cdot \alpha^2 \) where \( C > 0 \) is a large enough constant with \( 0 < \delta \leq \frac{1}{4\sqrt{\beta}} \) and \( \|A_i\|_\infty \leq \frac{1}{C\sqrt{\beta}} \) for \( i \in I_{\text{light}} \). Then in each iteration \( t \) we have \( \Phi^{(t+1)} \leq \Phi^{(t)} \).

Proof. Note that \( u_i^{(t+1)} \leq u_i^{(t)} \) for any light index with \( L(i, x^{(t)}) \geq C \). In fact, one can only have strict inequality if \( L(i, x^{(t)}) > C \geq L(i, x^{(t+1)}) \). Hence we only need to prove that \( \sum_{i \in I^{(t)} \cap I_{\text{light}}} u_i^{(t+1)} \leq \sum_{i \in I^{(t)}} \sum_{\text{light}} u_i^{(t)} \). For ease of notation, we drop the index \( t \) and also write \( x' = x + \delta y \) instead of \( x^{(t+1)} = x^{(t)} + \delta y^{(t)} \), and \( I \) instead of \( I^{(t)} \cap I_{\text{light}} \). We estimate that

\[
\sum_{i \in I} u_i^{(t+1)} = \sum_{i \in I} \exp \left( \alpha \langle A_i, x + \delta y \rangle + \beta \sum_{j=1}^{n} (1 - (x_j + \delta y_j)^2) \cdot A_{ij}^2 \right)
\]

\[
= \sum_{i \in I} \exp \left( \alpha \langle A_i, x \rangle + \beta \sum_{j=1}^{n} (1 - x_j^2) \cdot A_{ij}^2 \right)
\]

\[
\cdot \exp \left( \alpha \delta \langle A_i, y \rangle - 2\beta \delta \langle A_i \cdot x, y \rangle - \beta \delta^2 \sum_{j=1}^{n} \sum_{j=1}^{n} y_j^2 A_{ij}^2 \right)
\]

Now we bound the second exponential term using the inequality \( e^{x_1 + x_2 + x_3} \leq 1 + x_1 + x_2 + x_3 + 9x_1^2 + 9x_2^2 + 9x_3^2 \) for \( \max\{x_1, x_2, x_3\} \leq 1 \). We obtain
\[ \sum_{i \in I} w_i + \left[ \alpha \delta \sum_{i \in I} w_i \cdot \langle A_i, y \rangle + 9\alpha^2 \delta^2 \sum_{i \in I} w_i \cdot \langle A_i, y \rangle^2 \right] \]

\[ = 0 \text{ as } y \in U_i^{(t)} \]

\[ + \left[ -2\beta \sum_{i \in I} w_i \cdot \langle A_i^{(2)} \circ x, y \rangle + 9 \cdot 4\beta^2 \delta^2 \sum_{i \in I} w_i \cdot \langle A_i^{(2)} \circ x, y \rangle^2 \right] \]

\[ = 0 \text{ as } y \in U_i^{(t)} \]

\[ + \left[ -\beta \sum_{i \in I} w_i \sum_{j=1}^{n} y_j^2 A_{ij}^2 + 9\beta \delta^4 \sum_{i \in I} w_i \cdot \left( \sum_{j=1}^{n} y_j^2 A_{ij}^2 \right)^2 \right] \]

\[ \leq -\frac{3\beta\delta^2}{2} \sum_{i \in I} w_i \sum_{j=1}^{n} y_j^2 A_{ij}^2 \]

Now, we use the fact that \( 9\beta^2 \sum_{j=1}^{n} y_j^2 A_{ij}^2 \leq 9\beta^2 \leq \frac{1}{2} \) to get

\[ \leq \sum_{i \in I} w_i + \delta^2 \sum_{i \in I} w_i \cdot \left( 9\alpha^2 \langle A_i, y \rangle^2 - \frac{\beta}{4} \sum_{j=1}^{n} y_j^2 A_{ij}^2 \right) \]

\[ \leq 0 \text{ since } y \in U_i^{(t)} \]

\[ + \beta \delta^2 \sum_{i \in I} w_i \cdot \left( 36\beta \langle A_i^{(2)} \circ x, y \rangle^2 - \frac{1}{4} \sum_{j=1}^{n} y_j^2 A_{ij}^2 \right) \leq \sum_{i \in I} w_i. \]

\[ \leq 0 \text{ since } y \in U_i^{(t)} \]

This proves the claim. \( \square \)

### B.1 The discrepancy guarantee

We can now prove that the algorithm indeed finds a vector satisfying the desired discrepancy bound:

**Lemma 20.** For a proper choice of \( \alpha := \Theta(\sqrt{\log n}) \) and \( \beta := \Theta(\log n) \), the algorithm returns a vector \( x := x^{(T)} \) with \( \langle A_i, x \rangle \leq O(\sqrt{\log n}) \) for each row \( i \in [m] \).

**Proof.** First consider a light index \( i \in I_{\text{light}} \). The potential function never increases, hence

\[ e^{\alpha \langle A_i, x \rangle} \leq \Phi^{(T)}(x) \leq \Phi^{(0)}(x) \leq m \cdot e^{\beta \cdot C} \leq n^2 \cdot e^{\beta \cdot C} \]

Taking logarithms and dividing by \( \alpha \) gives

\[ \langle A_i, x \rangle \leq \frac{2 \log(n)}{\alpha} + \frac{C \beta}{\alpha} \leq (C^2 + 2) \cdot \sqrt{\log(n)}. \]

Here the last inequality follows for choices of if \( \alpha := \sqrt{\log(n)} \) and \( \beta := C \log(n) = C \alpha^2 \). Now consider a heavy index \( i \). Let \( t \) be the last iteration when \( \sum_{j \in A^{(t)}} A_{ij}^2 > C. \) Until this point one has \( \langle A_i, x^{(t)} \rangle = 0. \) Since \( |A_{ij}| \geq 1/(C^2 \sqrt{\log n}) \) for every non-zero entry, one has \( |\{ j \in A^{(t+1)} : A_{ij} \neq 0 \}| \leq C^3 \log(n). \) Hence, regardless how those elements are colored, one has \( \langle A_i, x^{(T)} \rangle = \langle A_i, x^{(T)} - x^{(t)} \rangle \leq 2 \sum_{j \in A^{(t+1)}} |A_{ij}| \leq O(\sqrt{\log n}). \) \( \square \)
B.2 Quadratic error in subspaces

It remains to prove that the subspaces \( U_5^{(t)} \) and \( U_6^{(t)} \) used in the algorithm exist with high enough dimensions. We will prove two lemmas that we keep general:

**Lemma 21.** Let \( A, B \in \mathbb{R}^{m \times n} \) be any matrices with \( |B_{ij}| \leq |A_{ij}| \) for all \((i, j) \in [m] \times [n]\) and let \( w_1, \ldots, w_m \geq 0 \) be any weights. Then for any \( k \in \mathbb{N} \), one can compute a subspace \( U \) of dimension at least \( \dim(U) \geq (1 - \frac{1}{k})n \) in time \( O(n^2(m + n)) \) so that

\[
\sum_{i=1}^{m} w_i \cdot (B_i B_i^T \cdot yy^T) \leq k \sum_{i=1}^{m} w_i \cdot (\text{diag}(A_i^2) \cdot yy^T) \quad \forall y \in U.
\]

**Proof.** Consider the matrix \( L := \sum_{i=1}^{m} w_i B_i B_i^T \) and \( R := k \cdot \sum_{i=1}^{m} w_i \cdot \text{diag}(A_i^2) \). Then the goal is to find a subspace \( U \) so that \((L \cdot yy^T) \leq (R \cdot yy^T)\) for all \( y \in U \). First, if we replace \( A_i' := \sqrt{w_i} A_i \) and \( B_i' := \sqrt{w_i} B_i \), then the assumption \(|B_{ij}| \leq |A_{ij}|\) is preserved and the claim is not changed. Hence we may assume that \( w_i = 1 \) for all \( i \in [m] \). If \( A_j = 0 \), then also \( B_j = 0 \) and \((L \cdot e_j e_j^T) = 0 = (R \cdot e_j e_j^T)\) which means that \( e_j \) can be added to the subspace. So let us assume that \( A_j \neq 0 \) for all \( j \). Next, if we scale a columns \( A_j \) and \( B_j \) by some scalar \( s \) and we scale \( y_j \) by \( \frac{1}{s} \), then the claim remains invariant. Hence we assume that \( \|A_j\|_2 = 1 \) for all \( j \in [n] \). Then

\[
\text{Tr}[L] = \sum_{i=1}^{m} \|B_i\|_2^2 = \sum_{j=1}^{n} \|B_j\|_2^2 \leq k \sum_{j=1}^{n} \|A_j\|_2^2 = n
\]

On the other hand,

\[
R = k \sum_{i=1}^{m} \text{diag}(A_i^2) = k \cdot \text{diag}\left(\left(\sum_{j=1}^{n} A_{ij}^2\right)_{j \in [n]}\right) = k \cdot I
\]

Then \( L \) must have less than \( \frac{n}{k} \) eigenvalues of value more than \( k \). Then we can define \( U \) as the span of the eigenvectors of \( L \) that have eigenvalue at most \( k \). Computing the matrices \( L, R \) takes time \( O(mn^2) \) and the eigendecomposition can be done in time \( O(n^3) \).

The existence of the subspace \( U_5^{(t)} \) follows from choosing \( B := A \) with \( k := 16 \) and \( \frac{\beta}{16 \alpha^2} \geq 16 \). The second lemma that we need is the following

**Lemma 22.** Let \( A \in [-\gamma, \gamma]^{m \times n} \) and \( x \in [-1, 1]^n \). Then for any \( k \in \mathbb{N} \) one can compute a subspace \( U \subseteq \mathbb{R}^n \) with \( \dim(U) \geq n \cdot (1 - \frac{1}{k}) \) in time \( O(n^2(m + n)) \) so that

\[
\sum_{i=1}^{m} w_i \cdot ((A_i^2 \circ x) \otimes 2 \cdot yy^T) \leq k \cdot \gamma^2 \cdot \sum_{i=1}^{m} w_i \cdot (\text{diag}(A_i^2) \cdot yy^T) \quad \forall y \in U
\]

**Proof.** We define a matrix \( B \in \mathbb{R}^{m \times n} \) by letting \( B_i := \frac{A_{ij} \circ x}{\gamma} \). Then \( |B_{ij}| \leq |A_{ij}| \) and applying Lemma 21 gives the claim.

Then applying Lemma 22 with \( \gamma := \frac{1}{\sqrt{\log n}} \) and \( k = 16 \) guarantees the subspace \( U_6^{(t)} \). For the running time analysis of Theorem 3 one can set \( \delta := \Theta\left(\frac{1}{\sqrt{\log n}}\right) \) and the algorithm only takes \( O(n \log(n)) \) iterations, each taking time \( O(n^2(m + n)) \).