Semi-analytic pricing of double barrier options with time-dependent barriers and rebates at hit

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We continue a series of papers devoted to construction of semi-analytic solutions for barrier options. These options are written on underlying following some simple one-factor diffusion model, but all the parameters of the model as well as the barriers are time-dependent. We managed to show that these solutions are systematically more efficient for pricing and calibration than, eg., the corresponding finite-difference solvers. In this paper we extend this technique to pricing double barrier options and present two approaches to solving it: the General Integral transform method and the Heat Potential method. Our results confirm that for double barrier options these semi-analytic techniques are also more efficient than the traditional numerical methods used to solve this type of problems.

Introduction

Classical problems of financial mathematics recently got new attention due to several factors. Among them one could mention:

- Very small or even negative interest rates observed at the market, and also forced by the Federal Reserve for achieving its macroeconomic goals, see, eg., (Itkin et al., 2020a) and reference therein. Therefore, financial models that allow negative rates recently redrew much attention.
- Negative oil prices due to the COVID-19 pandemic and the following economic recession, see (Bouchouev, 2020; Farrington and Cesa, 2020).
- Another consequence of the COVID-19 was a huge shift to electronic trading since major options exchanges temporarily closed their floors, and brokers and market makers were adjusting to working from home. That raised the need for real-time tools for fast calculating the option prices and Greeks, (Brogan, 2020).
Those and some other aspects forced the financial society to critically reassess even simple classical one-factor models of mathematical finance, and reanimate some of them, for instance the Ornstein-Uhlenbeck (OU) process, that traditionally have been referred to as defective/ill-posed or problematic. In (Doff, 2020) it is advocated that risk managers could even use Black-Scholes to help drive strategy. Therefore, nowadays, for instance, fast pricing of barrier options even for those simple models could be of a increasing importance. That is what this paper is devoted to as applied to double barrier options.

In what follows we consider these options written on the underlying which temporal dynamics is driven by a simple one-factor diffusion process but with time-dependent coefficients. Also, both barriers are assumed to be time dependent. Finally, when the underlying process hits any of the barriers, the Call option holder gets a rebate-at-hit (different for the upper and lower barriers), and they are also time-dependent. It is important that in this paper we consider only the underlying dynamics whose option pricing problem by using the Feynman-Kac theorem and also some transformations could be reduced to the heat equation. Nevertheless, to the best of our knowledge, even with this simplification a closed-form solution of this problem is yet unknown.

However, we have to mention (Mijatovic, 2010), where a similar problem was solved by using a probabilistic approach to obtain a decomposition of the barrier option price into the corresponding European option price minus the barrier premium for a wide class of payoff functions, barrier functions and linear diffusions (i.e., the drift is constant and the local volatility is a function of the underlying only). For this setting it is shown in (Mijatovic, 2010) that the barrier premium can be expressed as a sum of integrals along the barriers of the option’s delta at the barriers, and that those deltas solve a system of Volterra integral equations of the second kind. This is similar to the idea of the generalized integral transform (GIT) method that we use in this paper, while our setting is more general. Indeed, we allow any diffusion model with time-dependent coefficients and time-dependent barriers and rebates at hit subject to the condition that the pricing partial differential equation (PDE) can be reduced to the heat equation (or, as shown in (Carr et al., 2020) to the Bessel equation). It can also be checked that the pricing PDE in (Mijatovic, 2010) by a simple change of the spatial variable can be transformed to the heat equation.

Our approach advocated in this paper further extends the technique we elaborated in a series of papers which dealt with a similar problem for single barrier options. In (Carr and Itkin, 2020) we developed semi-analytic solutions for the barrier (perhaps, time-dependent) and American options where the underlying stock is driven by a time-dependent OU process with a lognormal drift. This model is equivalent to the familiar Hull-White model in Fixed Income that was separately considered in (Itkin and Muravey, 2020). In all cases the solution was obtained by using the method of heat potentials (HP) and the GIT method. While the HP method is well-known in mathematical physics and engineering, (Tikhonov and Samarskii, 1963; Friedman, 1964.; Kartashov, 2001), it is less known as applied to finance. The first use of this method in finance is due to (Lipton, 2002) for pricing path-dependent options with curvilinear barriers, and more recently in (Lipton and Kaushansky, 2018; Lipton and de Prado, 2020) (also see references therein).

The GIT method is also known in physics, (Kartashov, 1999, 2001), but was unknown in finance until the first use in (Carr and Itkin, 2020). It is important, that it solves the problems where the underlying is defined at the domain $S \in [0, y(t)]$ with $S$ being the stock price, and $y(t)$ being the time-dependent barrier, however, for other domains the solution was unknown even in physics. Then in (Itkin and Muravey, 2020) the GIT solution for the first time was constructed for the domain $S \in [y(t), \infty)$.

Latter this technique was elaborated also for the CIR and CEV models, (Carr et al., 2020), and the Black-Karasinski model, (Itkin et al., 2020a). In particular, in (Carr et al., 2020) the HP method was further generalized to be capable to solving not just the heat but also the Bessel equations, and was called the Bessel potential (BP) method. In (Itkin et al., 2020a) the PDE is also of a special kind. It is a flavor of the time-dependent Schrödinger equation with the unsteady Morse potential (this can be obtained by the change of variables $x \rightarrow -x$ and $\tau \rightarrow -i\tau$, $i = \sqrt{-1}$).
To make it rigorous, in this context a semi-analytic solution means that given a model with the time-dependent drift and volatility functions, and also with the time-dependent barriers, we obtain the barrier option price in the explicit (analytic) form as an integral in the time \( t \). However, this integral contains yet unknown function \( \Psi(t) \) which solves some Volterra equation of the second kind which also obtained in our papers. Therefore, we think that "semi-analytic" is an appropriate term. Also, in some situations \( \Psi(t) \) can be found analytically, see eg., (Carr and Itkin, 2020; Itkin and Muravey, 2020).

In addition to the explicit analytic representation of the solution, another advantage of this approach is computational speed and accuracy. As this is demonstrated in the above cited papers, our method is more efficient than both the backward and forward finite difference (FD) methods while providing better accuracy and stability. To briefly explain this, let us mention that the FD method we used (and this is pretty standard) provides accuracy \( O(h^2) \) in space and \( O(\tau^2) \) in time, where \( h, \tau \) are the corresponding grid steps. Since in our method the solution is represented as a time integral, it can be computed with higher accuracy in time (eg., by using high order quadratures), while the dependence on the space coordinate \( x \) is explicit. Contrary, increasing the accuracy for the FD method is not easy (i.e., it significantly increases the complexity of the method, e.g., see (Itkin, 2017)). Then the total accuracy is determined by the accuracy of solving the Volterra equation which is also determined by the order of quadratures used to compute the integral in this equation. For instance, using Gaussian quadratures allows small number of nodes and also high accuracy, in more detail see (Itkin and Muravey, 2020; Carr et al., 2020).

Also, as mentioned in (Carr et al., 2020), another advantage of our approach is computation of option Greeks. Since the option prices in both the HP and GIT methods are represented in closed form via integrals, the explicit dependence of prices on the model parameters is available and transparent. Therefore, explicit representations of the option Greeks can be obtained by a simple differentiation under the integrals. This means that the values of Greeks can be calculated simultaneously with the prices almost with no increase in time. This is because differentiation under the integrals slightly changes the integrands, and these changes could be represented as changes in weights of the quadrature scheme used to numerically compute the integrals. Since the major computational time has to be spent for computation of densities which contain special functions, they can be saved during the calculation of the prices, and then reused for computation of Greeks.

One can be curious why we need two methods - the HP and GIT, if they are used to solve the same problem and demonstrate the same performance. The answer is kind of elegant. As shown in (Carr et al., 2020), the GIT method produces very accurate results at high strikes and maturities (i.e. where the option price is relatively small) in contrast to the HP method. This can be verified by looking at the exponents under the GIT solution integral which are proportional to the time \( \tau \). Contrary, when the price is higher (short maturities, low strikes) the GIT method is slightly less accurate than the HP method, as the exponents in the HP solution integral are inversely proportional to \( \tau \). Thus, both methods are complementary.

This situation is well investigated for the heat equation with constant coefficients. There exist two representation of the solution: one - obtained by using the method of images, and the other one - by the Fourier series. Despite both solutions are equal as the infinite series, their convergence properties are different, (Lipton, 2002).

Going back to the problem considered in this paper, we skip the explicit formulation of the model. Instead we define a wide class of models where pricing double barrier options can be translated to solving the heat equation with time-dependent boundaries (barriers) and time-dependent boundary conditions (rebates-at-hit). Note, that the problems considered in the above cited paper - pricing barrier and American options in the time-dependent OU process, pricing barrier options in the Hull-White model, etc., also fit to this class as this is shown in the corresponding papers. Then we construct the solution by using both the GIT and the HP methods. The latter was already shortly presented in (Itkin and
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Muravey, 2020), but for the homogeneous boundary conditions. Also, here we present full derivation of the explicit value of the solution spatial gradient \( u_x \) at the lower \( x = y(\tau) \) and upper \( x = z(\tau) \) boundaries. This derivation differs from that in (Lipton and Kaushansky, 2018) (and is closer in sense to (Tikhonov and Samarskii, 1963)), but provides a similar result. Also, all the results obtained in this paper are new.

The rest of the paper is organized as follows. Section 1 describes the double barrier pricing problem for the time-dependent barriers and rebates at hit and shows that it can be reduced to solving inhomogeneous PDE with homogeneous boundary conditions. Section 2 describes in detail the solution of this problem by using the GIT method. We provide two alternative integral representations of the solution - one via the Jacobi theta functions, and the other one - using the Poisson summation formula. Despite these solutions are equal in a sense of infinite series, their convergence properties are different. A system of the Volterra equations for the gradient of the solution at both boundaries is obtained for both representations. Section 2 provides the same development but using the HP method. The final section concludes.

1 Statement of the problem

Let us consider some one-factor diffusion model of the type

\[
dS_t = \mu(t, S)dt + \sigma(t, S)dW_t, \quad S_t(t = 0) = S_0.
\]  

Here \( t > 0 \) is the time, \( S_t \) is the spot price, \( \mu(t, S) \) is the drift, \( \sigma(t, S) \) is the volatility of the process, \( W_t \) is the standard Brownian motion under the risk-neutral measure. By using a standard argument, to price options written on \( S_t \) as an underlying, one can apply the Feynman-Kac theorem to obtain the following partial differential equation (PDE) for, eg., the European Call option price

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2(t, S) \frac{\partial^2 C}{\partial S^2} + \mu(t, S)S \frac{\partial C}{\partial S} = r(t)C.
\]  

Here in case of Equities we treat \( S_t \) as the stock price, then \( r(t) \) is the deterministic interest rate. If \( S_t \) is the stochastic interest rate, then \( r(t) \) in the RHS of Eq. (2) should be replaced with \( S_t \).

The Eq. (2) should be solved subject to the terminal condition at the option maturity \( t = T \)

\[
C(T, S) = (S - K)^+,
\]  

where \( K \) is the option strike, and some boundary conditions. Below in this paper we are concentrated only on double barrier options with moving barriers: the lower barrier at \( S = L(t) \) and the upper barrier at \( S = H(t) > L(t) \), so \( S \in [L(t), H(t)] \).

Our main assumption in this paper is that the PDE in Eq. (2) by a series of transformations of the dependent variable \( C(S, t) \mapsto U(x, \tau) \) and independent variables \( t \mapsto \tau(t, S) \) can be reduced to the heat equation

\[
\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2},
\]  

which should be solved at the new domain \( x \in [y(\tau), z(\tau)], \quad \tau \in [0, \tau(0, S_0)] \), subject to the terminal condition

\[
U(0, x) = U_0(x),
\]  

and the boundary conditions

\[
U(\tau, y(\tau)) = f^- (\tau), \quad U(\tau, z(\tau)) = f^+ (\tau).
\]  

Here \( f^\pm(\tau), y(\tau), z(\tau) \) are some continuous functions of time \( \tau \). From the financial point of view the problem in Eq. (4), Eq. (5), Eq. (6) (the B problem) could be viewed as a pricing problem for double
barrier options with the moving lower \( y(\tau) \) and upper \( z(\tau) \) barriers and the rebates \( f^\pm(\tau) \) paid at hit, i.e. when the underlying process hits either the lower or the upper barrier.

Note, that many well-known financial models fit this framework. For instance, the time dependent OU process used in (Carr and Itkin, 2020) to model barrier and American options is such an example. Also, the time-dependent Hull-White model considered in (Itkin and Muravey, 2020) for pricing barrier options is another example. The number of models that fit this framework could be significantly expanded if one transforms the original PDE in Eq. (2) to its multilayer version. This approach is discussed in detail in (Itkin et al., 2020b) and will be reported elsewhere.

Below we present solution of the \( \mathcal{B} \) problem by using two analytic methods - the GIT and HP methods. As mentioned in Introduction, the methods are complementary in a sense that despite both solutions are equal as the infinite series, their convergence properties are different. In particular, the GIT method is more accurate at high strikes and maturities while the HP method - at low strikes and maturities.

It is worth mentioning that the \( \mathcal{B} \) problem is with inhomogeneous boundary conditions, hence from the very beginning it is useful to transform it to a similar problem but with homogeneous boundary conditions. This could be done by the change of variables

\[
\bar{u}(\tau, p) = \int_0^{z(\tau)} u(\tau, x) \sinh(p[x-y(\tau)]) dx,
\]

which transforms the PDE in Eq. (2) to the inhomogeneous PDE but with the homogeneous boundary conditions

\[
\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + g(\tau, x),
\]

where

\[
g(\tau, x) \equiv -A'(\tau) - B'(\tau)x, \quad (\tau, x) \in \mathbb{R}_+ \times [y(\tau), z(\tau)],
\]

\[u(0, x) = U_0(x) - A(0) - B(0)x \equiv u_0(x), \quad u(\tau, y(\tau)) = u(\tau, z(\tau)) = 0.
\]

## 2 Solution by the GIT method

In this section we solve the problem in Eq. (8) by using the GIT method, see (Kartashov, 1999; Carr and Itkin, 2020; Itkin and Muravey, 2020; Itkin et al., 2020a) and references therein. However, as mentioned in (Kartashov, 2001), an analytic solution for the domain with two moving boundaries is yet unknown. Therefore, our solution presented in this Section is new, and it extends the results of (Carr and Itkin, 2020) obtained for the domain \([0, y(\tau)]\).

In (Carr and Itkin, 2020) the authors used the GIT proposed in (Kartashov, 1999) which is a map \( u(\tau, x) \mapsto \bar{u}(\tau, p) \) of the form

\[
\bar{u}(\tau, p) = \int_0^{y(\tau)} u(\tau, x) \sinh(x \sqrt{p}) dx,
\]

where \( p = a + i\omega \) is a complex number with \( \Re(p) \geq \beta > 0 \), and \( -\frac{\pi}{4} < \arg(\sqrt{p}) < \frac{\pi}{4} \). Here we proceed with a similar idea by introducing the transform

\[
\bar{u}(\tau, p) = \int_{y(\tau)}^{z(\tau)} u(\tau, x) \sinh(p[x-y(\tau)]) dx.
\]

With a special choice of the lower boundary \( y(\tau) \equiv 0 \) this transform replicates that in Eq. (9) subject to the point that here we use the spectral parameter \( p \) instead of \( \sqrt{p} \) as in (Carr and Itkin, 2020). Therefore, here \( -\frac{\pi}{4} < \arg(p) < \frac{\pi}{4} \).
Since the kernel of Eq. (10) is time-dependent it doesn’t make much sense to apply this transform directly to the inhomogeneous heat equation in Eq. (8). Therefore, we represent the image \( \tilde{u} \) as a difference of two other images

\[
\tilde{u} = \frac{1}{2}(\tilde{u}_+ - \tilde{u}_-), \quad \tilde{u}_\pm(\tau, p) = \int_{y(\tau)}^{z(\tau)} u(\tau, x)e^{\pm p|x-y(\tau)|}dx.
\] (11)

To determine \( \tilde{u}(\tau, p) \) let us multiply both parts of the first line in Eq. (8) by \( e^{\pm p|x-y(\tau)|} \) and integrate on \( x \). This yield

\[
\int_{y(\tau)}^{z(\tau)} \frac{\partial u(\tau, x)}{\partial \tau} e^{\pm p|x-y(\tau)|}dx = \frac{\partial \tilde{u}_\pm(\tau, p)}{\partial \tau} - u(\tau, z(\tau))e^{\pm p z(\tau)}z'(\tau) + u(\tau, y(\tau))e^{\pm p y(\tau)}y'(\tau)
\] (12)

\[
\pm p y'(\tau) \int_{y(\tau)}^{z(\tau)} u(\tau, x)e^{\pm p|x-y(\tau)|}dx = \frac{\partial \tilde{u}_\pm}{\partial \tau} \pm p y'(\tau)\tilde{u}_\pm,
\]

\[
\int_{y(\tau)}^{z(\tau)} \frac{\partial^2 u(\tau, x)}{\partial x^2} e^{\pm p|x-y(\tau)|}dx = \Phi(\tau)e^{\pm p[z(\tau)-y(\tau)]} + \Psi(\tau) + p^2\tilde{u}_\pm(\tau, p),
\]

\[
\bar{g}_\pm(\tau, p) \equiv \int_{y(\tau)}^{z(\tau)} g(\tau, x)e^{\pm p|x-y(\tau)|}dx = \frac{B'(\tau)}{p^2} \left(e^{\pm p[z(\tau)-y(\tau)]} - 1\right)
\]

where terms proportional to \( u(\tau, y(\tau)) \) and \( u(\tau, z(\tau)) \) vanish due to the boundary conditions in Eq. (8), and by definition

\[
\Psi(\tau) = -\frac{\partial u(\tau, x)}{\partial x} \bigg|_{x=y(\tau)} \quad \Phi(\tau) = \frac{\partial u(\tau, x)}{\partial x} \bigg|_{x=z(\tau)}.
\] (13)

Collecting terms in Eq. (12) yields two initial value problems, one for the function \( \tilde{u}_+ \) and the other one - for \( \tilde{u}_- \)

\[
\frac{\partial \tilde{u}_\pm(\tau, p)}{\partial \tau} + \tilde{u}_\pm \left[ \pm p y'(\tau) - p^2 \right] = \Psi(\tau) + \Phi(\tau)e^{\pm p[z(\tau)-y(\tau)]} + \bar{g}_\pm(\tau, p),
\] (14)

\[
\tilde{u}_\pm(0, p) = \int_{y(0)}^{z(0)} u(0, x)e^{\pm p|x-y(0)|}dx.
\]

Each problem in Eq. (14) can be solved explicitly

\[
\tilde{u}_\pm(\tau, p) = e^{p^2 \tau} \int_{y(0)}^{z(0)} u(0, x)e^{\pm p|x-y(\tau)|}dx
\] (15)

+ \int_0^\tau e^{p^2 (\tau-s)} \left[ \Phi(s)e^{\pm p[z(s)-y(\tau)]} + (\Psi(s) + \bar{g}_\pm(s, p)) e^{\pm p[y(s)-y(\tau)]} \right] ds.

Note that the last term in under the second integral in Eq. (15) can be re-written in a more convenient form

\[
ge^{\pm p[y(s)-y(\tau)]} = \frac{B'(s)}{p^2} \left(e^{\pm p[z(s)-y(s)]} - 1\right) e^{\pm p[y(s)-y(\tau)]}
\]

\[
\pm \frac{1}{p} \left[A'(s) \left(1 - e^{\pm p[z(s)-y(s)]}\right) e^{\pm p[y(s)-y(\tau)]} + B'(s) \left(y(s) - z(s) e^{\pm p[z(s)-y(s)]}\right) e^{\pm p[y(s)-y(\tau)]}\right]
\]

\[
= \frac{B'(s)}{p^2} \left(e^{\pm p[z(s)-y(\tau)]} - e^{\pm p[y(s)-y(\tau)]}\right)
\]
\[ \pm \frac{1}{p} \left[ A'(s) \left( e^{\pm p[y(s)-y(\tau)]} - e^{\pm p[z(s)-y(\tau)]} \right) + B'(s) \left( y(s)e^{\pm p[y(s)-y(\tau)]} - z(s)e^{\pm p[z(s)-y(\tau)]} \right) \right]. \]

The explicit representation for \( \tilde{u} \) then follows from its definition in Eq. (11)

\[
\tilde{u}(\tau, p) = e^{p^2 \tau} \int_{y(0)}^{z(\tau)} u(0, x) \sinh \left( p[x - y(\tau)] \right) dx \tag{16}
\]

\[ + \int_0^\tau e^{p^2(\tau-s)} \left[ \Phi(s) \sinh \left( p[z(s) - y(\tau)] \right) + \Psi(s) \sinh \left( p[y(s) - y(\tau)] \right) \right] h(s, p) ds, \]

\[ h(s, p) = \frac{B'(s)}{p^2} [\sinh(p[z(s) - y(\tau)]) - \sinh(p[y(s) - y(\tau)])]
+ \frac{1}{p} \left[ (A'(s) + B'(s)y(s)) \cosh(p[y(s) - y(\tau)]) - (A'(s) + B'(s)z(s)) \cosh(p[z(s) - y(\tau)]) \right]. \]

\section{2.1 The inverse transform}

General theory of the heat equation tells us that the solution at the space domain \( a < x < b, a, b \in \mathbb{R} - \text{const} \), can be represented as Fourier series of the form, (Polyanin, 2002)

\[ u(\tau, x) = \sum_{n=1}^{\infty} a_n e^{-\frac{\pi^2 n^2}{b-a} \tau} \sin \left( \frac{\pi n(x-a)}{b-a} \right) \]

Therefore, by analogy let us look for the inverse transform of \( \tilde{u} \) (which actually is the solution \( u(\tau, x) \) of Eq. (8)) to be a generalized Fourier transform of the form (Carr and Itkin (2020))

\[ u(\tau, x) = \sum_{n=0}^{\infty} A_n(\tau) \sin \left( \frac{\pi n}{z(\tau) - y(\tau)} \frac{x - y(\tau)}{z(\tau) - y(\tau)} \right), \tag{17} \]

where \( A_n(\tau) \) are some yet unknown Fourier coefficients (weights). Applying the direct transform in Eq. (10) to the series in Eq. (17) yields

\[ \tilde{u}(\tau, x) = \int_{y(\tau)}^{z(\tau)} \sum_{n=1}^{\infty} A_n(\tau) \sin \left( \frac{\pi n}{z(\tau) - y(\tau)} \frac{x - y(\tau)}{z(\tau) - y(\tau)} \right) \sinh \left( p[x - y(\tau)] \right) dx. \tag{18} \]

Using the identity

\[ \int_y^z \sin \left( \frac{\pi n}{z-y} \frac{x-y}{z-y} \right) \sinh \left( p[x-y] \right) dx = (-1)^{n+1} \frac{\pi n}{n^2 \pi^2 + p^2(z-y)^2} \sinh \left( p[z-y] \right), \tag{19} \]

we obtain another representation for \( \tilde{u} \)

\[ \tilde{u}(\tau, x) = \frac{1}{l(\tau)} \sum_{n=1}^{\infty} (-1)^{n+1} \pi n A_n(\tau) \sinh \left( pl(\tau) \right) \left[ \frac{p + i n \pi}{l(\tau)} \right] \left[ \frac{p - i n \pi}{l(\tau)} \right]. \tag{20} \]

Combining Eq. (20) and Eq. (16) yields an equation for \( A_n(\tau) \)

\[ \frac{1}{l(\tau)} \sum_{n=1}^{\infty} (-1)^{n+1} \pi n A_n(\tau) \left[ \frac{p + i n \pi}{l(\tau)} \right] \left[ \frac{p - i n \pi}{l(\tau)} \right] = \frac{1}{\sinh \left( pl(\tau) \right)} \left[ e^{p^2 \tau \int_{y(0)}^{z(0)} u(0, x) \sinh \left( p[x - y(\tau)] \right) dx \right]
+ \int_0^\tau e^{p^2(\tau-s)} \left[ \Phi(s) \sinh \left( p[z(s) - y(\tau)] \right) + \Psi(s) \sinh \left( p[y(s) - y(\tau)] \right) \right] h(s, p) ds \right]. \tag{21} \]
The LHS and RHS of Eq. (21) as the functions of $p$ are analytic in the whole complex plane domain except the poles

$$p_k^\pm = \pm i\pi k/l(\tau), \quad k = 1, 2, \ldots, \quad (22)$$

because $h(s, p)$ is regular and well-behaved at $p \to 0$. Also, as this is easy to check, these poles are common for the LHS and RHS of Eq. (21). For what follows we need the following residues

$$\text{Res}_{p = p_k^\pm} \frac{1}{(p + i n \pi / l(\tau))(p - i n \pi / l(\tau))} = \frac{\pm l(\tau)}{2i\pi k}, \quad \text{Res}_{p = p_k^\pm} \frac{1}{\sinh (pl(\tau))} = \frac{(\pm 1)^k}{l(\tau)}. \quad (23)$$

The Fourier coefficients $A_k(\tau)$ can now be found from Eq. (21) by applying contour integration on $p$ to both sides. We integrate using the contours $L_k^\pm$, $k = 1, 2, \ldots$, where the integration contours look like it is depicted in Fig. 1. Thus, we have

$$\frac{1}{l(\tau)} \int \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \pi n A_n(\tau)}{(p + i n \pi / l(\tau))(p - i n \pi / l(\tau))} dp = \int_{L_k^+} \frac{1}{\sinh (pl(\tau))} \left\{ e^{p^2 \tau} \int_{y(0)}^{z(0)} u(0, x) \sinh (p[x - y(\tau)]) dx \right. \left. + \int_0^\tau e^{p^2(\tau - s)} \left[ \Phi(s) \sinh (p[z(s) - y(\tau)]) + \Psi(s) \sinh (p[y(s) - y(\tau)]) + h(s, p) \right] ds \right\} dp. \quad (24)$$

Figure 1: Contours of integration of Eq. (21) in the complex plane $p \in \mathbb{C}$ with poles at $p_1^\pm, p_2^\pm, \ldots$. 
By the Cauchy’s residue theorem each integral in Eq. (24) is equal to the sum of the corresponding residues that can be computed with the help of Eq. (23). This yields the following formula for $A_k(\tau)$

$$A_k(\tau) = \frac{2}{i l(\tau)} \bar{u} \left( \tau, i \frac{\pi k}{l(\tau)} \right).$$

(25)

With allowance for Eq. (16) this can be finally represented as

$$A_k(\tau) = \frac{2}{l(\tau)} \left\{ e^{-\frac{\pi^2 k^2}{i^2 l(\tau)^2}} \int_{y(0)}^{z(0)} u(0, x) \sin \left( \frac{\pi k}{l(\tau)} [x - y(\tau)] \right) dx \right\}$$

$$+ \int_0^\tau e^{-\frac{\pi^2 k^2}{i^2 l(\tau)^2} (\tau-s)} \left[ \Phi(s) \sin \left( \frac{\pi k}{l(\tau)} [z(s) - y(\tau)] \right) + \Psi(s) \sin \left( \frac{\pi k}{l(\tau)} [y(s) - y(\tau)] \right) + h_1(k, s, \tau) \right] ds \right\},$$

with

$$h_1(k, s, \tau) = -\frac{B'(s)l^2(\tau)}{\pi^2 k^2} \left[ \sin \left( \frac{\pi k}{l(\tau)} [z(s) - y(\tau)] \right) - \sin \left( \frac{\pi k}{l(\tau)} [y(s) - y(\tau)] \right) \right]$$

$$- \frac{l(\tau)}{\pi k} \left[ (A'(s) + B'(s)y(s)) \cos \left( \frac{\pi k}{l(\tau)} [y(s) - y(\tau)] \right) - (A'(s) + B'(s)z(s)) \cos \left( \frac{\pi k}{l(\tau)} [z(s) - y(\tau)] \right) \right].$$

(27)

Keeping in mind that

$$A(\tau) + B(\tau)y(\tau) = f^-(\tau) \quad A(\tau) + B(\tau)z(\tau) = f^+(\tau)$$

we re-arrange Eq. (27) as

$$h_1(k, s, \tau) = -\frac{B'(s)l^2(\tau)}{\pi^2 k^2} \left[ \sin \left( \frac{\pi k}{l(\tau)} [z(s) - y(\tau)] \right) - \sin \left( \frac{\pi k}{l(\tau)} [y(s) - y(\tau)] \right) \right]$$

$$- \frac{l(\tau)}{\pi k} \left[ (f'_-(s) - B(s)y'(s)) \cos \left( \frac{\pi k}{l(\tau)} [y(s) - y(\tau)] \right) - (f'_+(s) - B(s)z'(s)) \cos \left( \frac{\pi k}{l(\tau)} [z(s) - y(\tau)] \right) \right].$$

(28)

Substituting this result into Eq. (17), we obtain the solution $u(\tau, x)$ of the problem Eq. (8)

$$u(\tau, x) = \frac{2}{l(\tau)} \sum_{n=1}^{\infty} \sin \left( \pi n \frac{x - y(\tau)}{l(\tau)} \right) \left\{ e^{-\frac{\pi^2 n^2}{i^2 l(\tau)^2}} \int_{y(0)}^{z(0)} u(0, \xi) \sin \left( \frac{\pi n}{l(\tau)} [\xi - y(\tau)] \right) d\xi \right\}$$

$$+ \int_0^\tau e^{-\frac{\pi^2 k^2}{i^2 l(\tau)^2} (\tau-s)} \left[ \Phi(s) \sin \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) + \Psi(s) \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) + h_1(n, s, \tau) \right] ds \right\},$$

(29)

This expression can be further simplified, see Appendix A. Returning to the original variable $U(\tau, x)$ yields the final representation

$$U(\tau, x) = \frac{2}{l(\tau)} \sum_{n=1}^{\infty} \sin \left( \pi n \frac{x - y(\tau)}{l(\tau)} \right) \left\{ e^{-\frac{\pi^2 n^2}{i^2 l(\tau)^2} \int_{y(0)}^{z(0)} U(0, \xi) \sin \left( \frac{\pi n}{l(\tau)} [\xi - y(\tau)] \right) d\xi \right\}$$

$$+ \int_0^\tau e^{-\frac{\pi^2 k^2}{i^2 l(\tau)^2} (\tau-s)} \left[ \Phi(s) \sin \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) + \Psi(s) \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) + \beta(\tau, s, n) \right] ds \right\},$$

(30)

where $\beta(\tau, s, n)$ is defined in Eq. (A.2).

It is worth mentioning that the exact same formalism can be developed by using another integral transform

$$\bar{u}(\tau, p) = \int_{y(\tau)}^{z(\tau)} \sinh (p[z(\tau) - x]) u(\tau, x) dx,$$

with the result being same as in Eq. (29).
2.2 Connection to the Jacobi theta function

As observed in (Carr and Itkin, 2020), the sums in Eq. (29) could be expressed via the Jacobi theta functions of the third kind, (Mumford et al., 1983). Using their definition

\[
\theta_3(z, \omega) = 1 + 2 \sum_{n=1}^{\infty} \omega^{n^2} \cos(2nz),
\]

and the identities

\[
\frac{\partial \theta_3(z, \omega)}{\partial z} = \theta'_3(z, \omega) = -4 \sum_{n=1}^{\infty} n \omega^{n^2} \sin(2nz).
\]

we obtain from Eq. (29)

\[
4 \sum_{n=1}^{\infty} e^{-\frac{2\pi^2}{l^4}(r-s)^2} \sin \left( \frac{n\pi(x - y(\tau))}{l(\tau)} \right) \sin \left( \frac{n\pi(x - y(\tau))}{l(\tau)} \right) = \theta_3(\phi_-(x, \xi), \omega_1) - \theta_3(\phi_+(x, \xi), \omega_1),
\]

\[
4 \sum_{n=1}^{\infty} e^{-\frac{2\pi^2}{l^4}(r-s)^2} \sin \left( \frac{n\pi(x - y(\tau))}{l(\tau)} \right) \sin \left( \frac{n\pi(y - y(\tau))}{l(\tau)} \right) = \theta_3(\phi_-(x, \xi), \omega_2) - \theta_3(\phi_+(x, \xi), \omega_2),
\]

\[
8 \sum_{n=1}^{\infty} e^{-\frac{2\pi^2}{l^4}(r-s)^2} \sin \left( \frac{n\pi(x - y(\tau))}{l(\tau)} \right) \cos \left( \frac{n\pi(x - y(\tau))}{l(\tau)} \right) = - \left( \theta'_3(\phi_-(x, \xi), \omega_2) + \theta'_3(\phi_+(x, \xi), \omega_2) \right).
\]

\[
\omega_1 = e^{-\frac{\pi^2}{l^2}(r-s)}, \quad \omega_2 = e^{-\frac{\pi^2}{l^2}(r-s)}, \quad \phi_-(x, \xi) = \frac{\pi(x - \xi)}{l(\tau)}, \quad \phi_+(x, \xi) = \frac{\pi(x + \xi - 2y(\tau))}{l(\tau)}.
\]

With the help of Eq. (33) the final formula for \( u(\tau, x) \) simplifies

\[
2l(\tau)u(\tau, x) = \int_{y(0)}^{z(0)} u(0, \xi) \left[ \theta_3(\phi_-(x, \xi), \omega_1) - \theta_3(\phi_+(x, \xi), \omega_1) \right] d\xi
\]

\[
+ \int_{0}^{\tau} \left\{ \left[ \Psi(s) - B(s) - f_- (s)y'(s) \right] \left[ \theta_3(\phi_-(x, y(s)), \omega_2) - \theta_3(\phi_+(x, y(s)), \omega_2) \right]
\]

\[
+ \left[ \Phi(s) + B(s) + f_+(s)z'(s) \right] \left[ \theta_3(\phi_-(x, z(s)), \omega_2) - \theta_3(\phi_+(x, z(s)), \omega_2) \right]
\]

\[
+ \frac{2\pi}{l(\tau)} \left[ f_+(s) \left[ \theta'_3(\phi_-(x, z(s)), \omega_2) + \theta'_3(\phi_+(x, z(s)), \omega_2) \right]
\]

\[
- f_-(s) \left[ \theta'_3(\phi_-(x, y(s)), \omega_2) + \theta'_3(\phi_+(x, y(s)), \omega_2) \right] \right\} ds.
\]

Note, that if rebates at hit are not paid, the boundary conditions become homogeneous, and the last line with some summands from third line in Eq. (34) disappear.

2.3 The Poisson summation formula and alternative representations

It is known that for the fixed spatial domain \( x \in [y(\tau), z(\tau)], \ y(\tau) = 0, \ z(\tau) = const \) there exist two representations of the solution of the heat equation: one - obtained by using the method of images, and the other one - by the Fourier series. Both solutions are equal in a sense of infinite series, but their

\[^1\text{Which is not a surprise since it is known that the Jacobi theta functions is the solution of the heat equation with periodic boundary conditions. As applied to the problem considered in this paper, an example is a double barrier option with zero rebate at hit.}\]
convergence properties are different, see eg., (Lipton, 2002). It turns out that for a curvilinear strip we can also obtain an alternative representation.

The solution $u(\tau, x)$ found in Eq. (29) already has the form of the Fourier series. However, applicability of the method of images to the problem Eq. (8) is not transparent due to time-dependency of the boundaries. Instead, we can find an alternative representation by using the following property known as the Poisson Summation formula, (van der Pol and Bremmer, 1950)

**Proposition 2.1** (Poisson Summation formula). Let $\hat{h}(\nu)$ be the Fourier transform of the appropriate function $h(x)$

$$
\hat{h}(\nu) = \int_{-\infty}^{+\infty} h(x)e^{-2\pi i\nu x} dx.
$$

The following identity holds

$$
\sum_{n=-\infty}^{+\infty} h(n) = \sum_{k=-\infty}^{+\infty} \hat{h}(k). \quad (35)
$$

**Proof.** See (van der Pol and Bremmer, 1950).

Applying Eq. (35) to the functions

$$
\begin{align*}
  h_1(x) &= e^{-\frac{x^2}{2\beta}} \cos (\pi x \alpha), \quad \hat{h}_1(\nu) = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\beta} - 2\pi i\nu x} \cos (\pi x \alpha) dx, \\
  h_2(x) &= xe^{-\frac{x^2}{2\beta}} \sin (\pi x \alpha), \quad \hat{h}_2(\nu) = \int_{-\infty}^{+\infty} xe^{-\frac{x^2}{2\beta} - 2\pi i\nu x} \sin (\pi x \alpha) dx.
\end{align*}
$$

we obtain the following identities

$$
\begin{align*}
  \sum_{n=-\infty}^{+\infty} e^{-\frac{n^2 \pi^2}{2\beta}} \cos (\pi n \alpha) &= \sqrt{\frac{2\beta}{\pi}} e^{-\frac{\alpha^2 \beta}{2}} \sum_{n=-\infty}^{+\infty} e^{-2n^2 \beta} \cosh (2n \alpha \beta) \\
  &= \sqrt{\frac{\beta}{2\pi}} \sum_{n=-\infty}^{+\infty} \left[ e^{-\frac{\beta}{4}(2n-\alpha)^2} + e^{-\frac{\beta}{4}(2n+\alpha)^2} \right] = 2 \sqrt{\frac{\beta}{2\pi}} \sum_{n=-\infty}^{+\infty} e^{-\frac{\beta}{4}(2n+\alpha)^2}, \\
  \sum_{n=-\infty}^{+\infty} \pi n e^{-\frac{n^2 \pi^2}{2\beta}} \sin (\pi n \alpha) &= \frac{\beta^{3/2}}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} e^{-\frac{\beta}{4}(2n+\alpha)^2} \left[ a + 2n + (a - 2n)e^{4\alpha \beta n} \right] \\
  &= \frac{\beta^{3/2}}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \left[ e^{-\frac{\beta}{4}(2n+\alpha)^2}(a + 2n) + e^{-\frac{\beta}{4}(2n-\alpha)^2}(a - 2n) \right] \\
  &= 2 \frac{\beta^{3/2}}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} e^{-\frac{\beta}{4}(2n+\alpha)^2}(a + 2n).
\end{align*}
$$

Since each summand in Eq. (29) can be represented in the form of the LHS of Eq. (36), by using a simple trigonometric formula for the product of sines we immediately arrive at another form of $U(\tau, x)$

$$
U(\tau, x) = \frac{1}{l(\tau)} \sum_{n=1}^{\infty} \left\{ e^{-\frac{n^2 \pi^2}{4l(\tau)^2}} \int_{y(0)}^{y(\tau)} U(0, \xi) \left[ \cos \left( \frac{\pi n}{l(\tau)} [x - \xi] \right) - \cos \left( \frac{\pi n}{l(\tau)} [x + \xi - 2y(\tau)] \right) \right] d\xi \right\} + \int_{0}^{\tau} e^{-\frac{s^2 \pi^2}{4l(\tau)^2}} \left[ \Phi(s) + B(s) + f_+(s)z'(s) \right] \left[ \cos \left( \frac{\pi n}{l(\tau)} [x - z(s)] \right) - \cos \left( \frac{\pi n}{l(\tau)} [x + z(s) - 2y(\tau)] \right) \right] ds \\
+ \int_{0}^{\tau} e^{-\frac{s^2 \pi^2}{4l(\tau)^2}} \left[ \Psi(s) - B(s) - f_-(s)y'(s) \right] \left[ \cos \left( \frac{\pi n}{l(\tau)} [x - y(s)] \right) - \cos \left( \frac{\pi n}{l(\tau)} [x + y(s) - 2y(\tau)] \right) \right] ds
$$
which can be also slightly modified by replacing
\[ \sum_{n=1}^{\infty} \mapsto \frac{1}{2} \sum_{n=-\infty}^{\infty}. \]

Finally, applying Eq. (36) to thus obtained formula yields an alternative representation of \( U(\tau, x) \)

\[
U(\tau, x) = \sum_{n=-\infty}^{\infty} \left\{ \int_{y(0)}^{(38)} \frac{U(0, \xi) \Upsilon_n(x, \tau | \xi, 0) d\xi}{2\sqrt{\pi \tau}} + \int_{0}^{\tau} \frac{[\Phi(s) + B(s) + f_+(s)z'(s)] \Upsilon_n(x, \tau | z(s), s)}{2\sqrt{\pi(\tau - s)}} ds, \right.
\]

\[
+ \int_{0}^{\tau} \frac{[\Psi(s) - B(s) - f_-(s)y'(s)] \Upsilon_n(x, \tau | y(s), s)}{2\sqrt{\pi(\tau - s)}} ds \]

\[
+ \int_{0}^{\tau} \frac{f_-(s)\Lambda_n(x, \tau | y(s), s) - f_+(s)\Lambda_n(x, \tau | z(s), s)}{4\sqrt{\pi(\tau - s)^3}} ds \right\},
\]

\[
\Upsilon_n(x, \tau | \xi, s) = e^{- \frac{(2n+l(\tau)+\xi)^2}{4(\tau-s)}} - e^{- \frac{(2n+l(\tau)+\xi-2y(\tau))^2}{4(\tau-s)}},
\]

\[
\Lambda_n(x, \tau | \xi, s) = [x - \xi + 2nl(\tau)] e^{- \frac{(2n+l(\tau)+\xi)^2}{4(\tau-s)} - [(x + \xi - 2y(\tau) + 2nl(\tau))] e^{- \frac{(2n+l(\tau)+\xi-2y(\tau))^2}{4(\tau-s)}}.\]

Note that the Fourier series in these expressions usually converge rapidly when \( n \) grows. Similarly, their differentiated forms provide a convenient way of calculating the corresponding derivatives, (DLMF).

2.4 A system of Volterra equations for \( \Psi(\tau) \) and \( \Phi(\tau) \)

Taking the derivative in Eq. (29) or Eq. (34) with respect to \( x \) and substituting \( x = y(\tau) \) and \( x = z(\tau) \), we get two equivalent systems of Volterra equations:

\[
-\Psi(\tau) = \frac{2\pi}{l^2(\tau)} \sum_{n=1}^{\infty} n \left\{ e^{- \frac{\pi^2 n^2}{l^2(\tau)}} \int_{y(0)}^{z(0)} U(0, \xi) \sin \left( \frac{\pi n}{l(\tau)} [\xi - y(\tau)] \right) d\xi \right.\]

\[
+ \int_{0}^{\tau} e^{- \frac{\pi^2 n^2}{l^2(\tau)}(\tau-s)} \left[ \Phi(s) + B(s) + f_+(s)z'(s) \right] \sin \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) \right.
\]

\[
+ \left. \left[ \Psi(s) - B(s) - f_-(s)y'(s) \right] \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) \right. \]

\[
+ \left. \frac{\pi n}{l(\tau)} \left[ f_-(s) \cos \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) - f_+(s) \cos \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) \right] ds \right\},
\]

\[
\Phi(\tau) = \frac{2\pi}{l^2(\tau)} \sum_{n=1}^{\infty} (-1)^n n \left\{ e^{- \frac{\pi^2 n^2}{l^2(\tau)}} \int_{y(0)}^{z(0)} U(0, \xi) \sin \left( \frac{\pi n}{l(\tau)} [\xi - y(\tau)] \right) d\xi \right.\]

\[
+ \int_{0}^{\tau} e^{- \frac{\pi^2 n^2}{l^2(\tau)}(\tau-s)} \left[ \Phi(s) + B(s) + f_+(s)z'(s) \right] \sin \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) \right.
\]

\[
+ \left. \left[ \Psi(s) - B(s) - f_-(s)y'(s) \right] \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) \right. \]

\[
+ \left. \frac{\pi n}{l(\tau)} \left[ f_-(s) \cos \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) - f_+(s) \cos \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) \right] ds \right\}.
\]
the terms corresponding to third line in Eq. (38) vanish.

Since the following identities holds

\[
\Phi(\tau) = \sum_{n=-\infty}^{\infty} \left\{ \int_{y(0)}^{z(0)} \frac{U(0,\xi)v_n(\tau|\xi,0)\,d\xi}{2\sqrt{\pi}} \right. \\
+ \int_{0}^{\tau} \left[ \Phi(s) + B(s) + f_+(s)z'(s) \right] v_n(\tau|z(s),s) + \frac{\left[ \Theta(s) - B(s) - f_-(s)y'(s) \right] v_n(\tau|y(s),s)}{2\sqrt{\pi(\tau-s)}} \right\} ds,
\]

and

\[
-\Psi(\tau) = \sum_{n=-\infty}^{\infty} \left\{ \int_{y(0)}^{z(0)} \frac{U(0,\xi)v_{n+1/2}(\tau|\xi,0)\,d\xi}{2\sqrt{\pi}} \right. \\
+ \int_{0}^{\tau} \left[ \Phi(s) + B(s) + f_+(s)z'(s) \right] v_{n+1/2}(\tau|z(s),s) + \frac{\left[ \Theta(s) - B(s) - f_-(s)y'(s) \right] v_{n+1/2}(\tau|y(s),s)}{2\sqrt{\pi(\tau-s)}} \right\} ds,
\]

here

\[
v_n(\tau|\xi,s) = \frac{2n!}{\pi} \frac{e^{-\frac{(2n!(\tau-y(s)+\xi)^2}{4(\tau-s)}}}{\tau-s}.
\] (41)

Since the following identities holds

\[
\sum_{n=-\infty}^{+\infty} \frac{\partial \Lambda_n(x,\tau|\xi,s)}{\partial x} \bigg|_{x=y(\tau)} = \sum_{n=-\infty}^{+\infty} \frac{\partial \Lambda_n(x,\tau|\xi,s)}{\partial x} \bigg|_{x=z(\tau)} = 0.
\]

the terms corresponding to third line in Eq. (38) vanish.

The Eq. (39) again can be re-written with no sums in terms of the theta functions \( \theta_3(z,\omega) \). For doing that, one can differentiate both parts of Eq. (34) on \( x \). Then first let \( x = y(\tau) \) to get \( 2l(\tau)\Psi(\tau) \) in the LHS, and alternatively let \( x = z(\tau) \) to get \( 2l(\tau)\Phi(\tau) \) in the LHS.

\[
-\frac{2l^2(\tau)}{\pi} \Psi(\tau) = \int_{y(0)}^{z(0)} U(0,\xi) \left[ \theta_3'(\phi_-(y(\tau),\xi),\omega_1) - \theta_3'(\phi_+(y(\tau),\xi),\omega_1) \right] d\xi
\] (42)

\[
+ \int_{0}^{\tau} \left\{ \left[ \Psi(s) - B(s) - f_-(s)y'(s) \right] \left[ \theta_3'(\phi_-(y(s),s),\omega_2) - \theta_3'(\phi_+(y(s),s),\omega_2) \right] \right. \\
+ \left[ \Phi(s) + B(s) + f_+(s)z'(s) \right] \left[ \theta_3'(\phi_-(y(\tau),z(s),s),\omega_2) - \theta_3'(\phi_+(y(\tau),z(s),s),\omega_2) \right] \\
+ \frac{2\pi}{l(\tau)} \left[ f_+(s) \left[ \theta_3''(\phi_-(y(\tau),z(s),s),\omega_2) + \theta_3''(\phi_+(y(\tau),z(s),s),\omega_2) \right] - f_-(s) \left[ \theta_3''(\phi_-(y(\tau),y(s),s),\omega_2) + \theta_3''(\phi_+(y(\tau),y(s),s),\omega_2) \right] \right. \\
\left. \right\} ds.
\]

\[
\frac{2l^2(\tau)}{\pi} \Phi(\tau) = \int_{y(0)}^{z(0)} U(0,\xi) \left[ \theta_3'(\phi_-(z(\tau),\xi),\omega_1) - \theta_3'(\phi_+(z(\tau),\xi),\omega_1) \right] d\xi
\]

\[
+ \int_{0}^{\tau} \left\{ \left[ \Psi(s) - B(s) - f_-(s)y'(s) \right] \left[ \theta_3'(\phi_-(z(s),s),\omega_2) - \theta_3'(\phi_+(z(s),s),\omega_2) \right] \right. \\
+ \left[ \Phi(s) + B(s) + f_+(s)z'(s) \right] \left[ \theta_3'(\phi_-(z(\tau),z(s),s),\omega_2) - \theta_3'(\phi_+(z(\tau),z(s),s),\omega_2) \right] \\
+ \frac{2\pi}{l(\tau)} \left[ f_+(s) \left[ \theta_3''(\phi_-(z(\tau),z(s),s),\omega_2) + \theta_3''(\phi_+(z(\tau),z(s),s),\omega_2) \right] - f_-(s) \left[ \theta_3''(\phi_-(z(\tau),y(s),s),\omega_2) + \theta_3''(\phi_+(z(\tau),y(s),s),\omega_2) \right] \right. \\
\left. \right\} ds.
\]
Semi-analytic pricing of double barrier options...

\[-f_-(s)\left[\theta_3''(\phi_-(z(\tau), y(s)), \omega_2) + \theta_3''(\phi_+(z(\tau), y(s)), \omega_2)\right]\] \, ds.

Also, since the theta function \(\theta_3(z, \omega)\) solves the heat equation

\[\frac{4i}{\pi} \frac{\partial \theta_3(z, \omega)}{\partial \omega} = \frac{\partial^2 \theta_3(z, \omega)}{\partial z^2},\]

the second derivatives with respect to the first argument could be expressed via the first derivatives with respect to the second argument.

3 Solution by the HP method

Similar to Section 2, the HP method, (Tikhonov and Samarskii, 1963; Friedman, 1964.; Kartashov, 2001), can be used to price double barrier options by solving the problem in Eq. (8). The idea was first proposed and developed in (Itkin and Muravey, 2020) and is a generalization of the standard HP method for the case of two moving boundaries. Note, that to the best of authors’ knowledge, yet the closed form (or even semi-closed form) solution of this problem was not known in physics, even not mentioning finance. Below we explain our approach paying attention to all intermediate details as the behavior of the solution at the boundaries is not trivial.

Following the main idea of the HP method, let us search for the solution of the \(B\) problem in Eq. (4) Eq. (6), Eq. (5) in the form

\[U(\tau, x) = q(\tau, x) + 1 \sqrt{\frac{1}{4\pi \tau}} \int_{y(0)}^{z(0)} U(0, x') e^{-\frac{(x-x')^2}{4\tau}} \, dx',\]  \hspace{1cm} (43)

so function \(q(\tau, x)\) solves a problem with the homogeneous initial condition

\[\frac{\partial q(\tau, x)}{\partial \tau} = \frac{\partial^2 q(\tau, x)}{\partial x^2};\] \hspace{1cm} (44)

\[q(0, x) = 0, \quad y(0) < x < z(0),\]

\[q(\tau, y(\tau)) = \phi_1(\tau) \equiv f^-(\tau) - \frac{1}{2\sqrt{\pi \tau}} \int_{y(0)}^{z(0)} u(0, x') e^{-\frac{4(u(x)-x')^2}{4\tau}} \, dx',\]

\[q(\tau, z(\tau)) = \psi_1(\tau) \equiv f^+(\tau) - \frac{1}{2\sqrt{\pi \tau}} \int_{y(0)}^{z(0)} u(0, x') e^{-\frac{4(u(x)-x')^2}{4\tau}} \, dx'.\]

In (Itkin and Muravey, 2020) it is proposed to search for the solution of Eq. (44) in the form of a generalized heat potential

\[q(x, \tau) = \frac{1}{4\sqrt{\pi}} \int_{0}^{\tau} \frac{1}{\sqrt{(\tau - k)^3}} \left((x - y(k))\Omega(k)e^{-\frac{(x-y(k))^2}{4(\tau-k)}} + (x - z(k))\Theta(k)e^{-\frac{(x-z(k))^2}{4(\tau-k)}}\right) dk,\]  \hspace{1cm} (45)

where \(\Omega(k), \Theta(k)\) are the heat potential densities. In other words, the solution is represented as a sum of two heat potentials: one corresponds to the lower barrier, and the other one - to the upper barrier. It is easy to check, that each such a potential solves the heat equation in Eq. (44), see (Tikhonov and Samarskii, 1963) as the derivative with respect to \(\tau\) of the RHS of Eq. (45) can be pulled into the integral since the value of both integrands at \(k = \tau\) vanishes.

To find the unknown functions \(\Omega(k), \Theta(k)\) one can substitute into Eq. (45) the values \(x = y(\tau)\) and \(x = z(\tau)\), and get a system of two integral equations that the functions \(\Omega(k), \Theta(k)\) solve. However, it
is well-known, (Tikhonov and Samarskii, 1963), that these integrals at $x \to y(\tau)$ and $x \to z(\tau)$ have a discontinuity, but with the finite value at $x = y(\tau) \pm 0$ and $x = z(\tau) \pm 0$. To investigate this discontinuity in more detail and derive the value of heat potential integral at the boundary $x \to y(\tau) \pm 0$, we consider a problem similar to Eq. (44)

$$
\mathcal{L}q(\tau, x) = 0, \quad (x, \tau) \in \Omega : [y(\tau), \infty) \times \mathbb{R}_+, \quad \mathcal{L}q(0, x) = 0, \quad y(0) < x < \infty, \quad q(\tau, y(\tau)) = \chi(\tau), \quad q(\tau, x)|_{x \to \infty} = 0.
$$

with the operator $\mathcal{L}$ defined as

$$
\mathcal{L} = -\frac{\partial}{\partial \tau} + \sigma^2 \frac{\partial^2}{\partial x^2}, \quad (47)
$$

where $\sigma = \text{const}$. Using the HP method, the solution of this problem can be expressed as

$$
q(\tau, x) = \int_0^\tau \Omega(k) \frac{x - y(k)}{4\sigma^3 \sqrt{\pi(\tau - k)^3}} e^{-\frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau - k)}} dk, \quad (48)
$$

where $\Omega(\tau)$ is the heat potential density, and $y(\tau)$ is a smooth curve (the moving boundary). Our aim below is to derive the value of this integral at $x \to y(\tau) \pm 0$, and the gradient $\partial q(\tau, x)/\partial x$ in the same limit, namely

$$
\varphi(\tau) = \lim_{x \to y(\tau) \pm 0} q(\tau, x), \quad \psi(\tau) = \lim_{x \to y(\tau) \pm 0} \frac{\partial q(\tau, x)}{\partial x}. \quad (49)
$$

### 3.1 The limiting value of $\varphi(t)$

This result is obtained, eg., in (Tikhonov and Samarskii, 1963). Consider a function $W(\tau, x) = 2\sigma^2 \phi(t)$

$$
W(\tau, x) = \int_0^\tau \Omega(k) \frac{x - y(k)}{2\sigma \sqrt{\pi(\tau - k)^3}} e^{-\frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau - k)}} dk. \quad (50)
$$

Also consider an auxiliary integral

$$
\tilde{V}(\tau, x) = \int_0^\tau \frac{y'(k)\Omega(k)}{\sigma \sqrt{\pi(\tau - k)}} e^{-\frac{(x - y(k))^2}{4\sigma^2(\tau - k)}} dk. \quad (51)
$$

Assume that $y(k)$ is differentiable. As shown in (Tikhonov and Samarskii, 1963), $\tilde{V}(\tau, x)$ is continuous along the curve $x = y(\tau)$ because it converges uniformly and $y'(k)$ is bounded, while $W(\tau, x)$ is discontinuous. To show this, first assume that $\Omega(\tau) = \Omega_0 = \text{const}$. Then the difference $W_0 - \tilde{V}_0$, where the sub-index 0 means that we use $\Phi_0$ instead of $\Phi(\tau)$ in the definitions Eq. (50), Eq. (51), can be calculated directly with the change of variables $k \mapsto a = (x - y(k))/(2\sigma \sqrt{\tau - k})$

$$
W_0 - \tilde{V}_0 = \frac{1}{2\sigma \sqrt{\pi}} \int_0^\tau \Omega_0 e^{-\frac{(x - y(k))^2}{4\sigma^2(\tau - k)}} \left[ \frac{x - y(k)}{(\tau - k)^{3/2}} - 2y'(k) \right] \frac{(\tau - k)^{1/2}}{(\tau - k)^{3/2}} dk = \Omega_0 \frac{2}{\sqrt{\pi}} \int_{\zeta^-}^{\zeta^+} e^{-a^2} da, \quad (52)
$$

$$
\zeta^- = \frac{x - y(0)}{2\sigma \sqrt{\tau}}, \quad \zeta^+ = \begin{cases} 
\infty, & x > y(\tau), \\
0, & x = y(\tau), \\
-\infty, & x < y(\tau).
\end{cases}
$$

Accordingly, at $x \to y(\tau) + 0$ we obtain

$$
[W_0(\tau, y(\tau) + 0) - W_0(\tau, y(\tau))] - [\tilde{V}_0(\tau, y(\tau) + 0) - \tilde{V}_0(\tau, y(\tau))] = \Omega_0 \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-a^2} da = \Omega_0. \quad (53)
$$
Since the function \( \tilde{V}_0 \) is continuous, the expression in the second square brackets in Eq. (53) vanishes, and so
\[
W_0(\tau, y(\tau) + 0) - W_0(\tau, y(\tau)) = \Omega_0. \tag{54}
\]
If \( \Omega(\tau) \) is not constant, then
\[
W(\tau, x) = W_0(\tau, x) - \int_0^\tau \frac{x - y(k)}{2\sqrt{\pi} (\tau - k)^{3/2}} e^{-\frac{(x-y(k))^2}{4\sigma^2(\tau-k)}} \Omega(\tau) - \Omega(k) \, dk. \tag{55}
\]
We assume that the boundary \( y(\tau) \) and the potential density \( \Omega(k) \) are differentiable functions of their arguments, i.e., at least \( C^1 \). Then the integral in Eq. (55) has the same singularity as the function \( \tilde{V}(\tau, x) \), converges uniformly, and thus is a continuous function on the curve \( x = y(\tau) \). This implies that
\[
W(\tau, x_0 + 0) - W(\tau, x_0) = W_0(\tau, x_0 + 0) - W_0(\tau, x_0) = \Omega(\tau), \tag{56}
\]
and, in particular, this is true for \( x_0 = y(\tau) \). In a similar way one can show that
\[
W(\tau, x_0 - 0) = W_0(\tau, x_0) - \Omega(\tau), \tag{57}
\]
Combining these results together, we obtain the final formula for \( \varphi(t) \)
\[
\varphi(\tau) = \pm \frac{\Omega(\tau)}{2\sigma^2} + \int_0^\tau \Omega(k) \frac{y(\tau) - y(k)}{4\sigma^3 \sqrt{\pi (\tau - k)^3} e^{-\frac{(y(\tau)-y(k))^2}{4\sigma^2(\tau-k)}} \, dk. \tag{58}
\]

### 3.2 The limiting value of \( \psi(t) \)

Using the definition of \( q(\tau, x) \) in Eq. (48) we need an explicit formula for
\[
\psi(\tau) = \lim_{x \to y(\tau)\pm 0} \frac{\partial q(\tau, x)}{\partial x} = \lim_{x \to y(\tau)\pm 0} \frac{\partial}{\partial x} \int_0^\tau \Omega(k) \frac{x - y(k)}{4\sigma^3 \sqrt{\pi (\tau - k)^3} e^{-\frac{(x-y(k))^2}{4\sigma^2(\tau-k)}} \, dk. \tag{59}
\]

However, as shown in Section 3.1, this integral is discontinuous at \( x \to y(\tau) \) (this is an improper Riemann integral of second kind). Hence, we cannot compute \( \psi(\tau) \) directly by taking derivative of \( q(\tau, x) \) with respect to \( x \).

Therefore, to proceed let us represent this integral as
\[
\int_0^\tau \Omega(k) \frac{x - y(k)}{4\sigma^3 \sqrt{\pi (\tau - k)^3} e^{-\frac{(x-y(k))^2}{4\sigma^2(\tau-k)}} \, dk = \Omega(\tau) \int_0^\tau \frac{x - y(k)}{4\sigma^3 \sqrt{\pi (\tau - k)^3} e^{-\frac{(x-y(k))^2}{4\sigma^2(\tau-k)}} \, dk + \int_0^\tau [\Omega(k) - \Omega(\tau)] \frac{x - y(k)}{4\sigma^3 \sqrt{\pi (\tau - k)^3} e^{-\frac{(x-y(k))^2}{4\sigma^2(\tau-k)}} \, dk = I_1 + I_2. \tag{60}
\]

We showed in Section 3.1 that the second integral in Eq. (60) has the same singularity as the function \( \tilde{V}(\tau, x) \), converges uniformly, and thus is a continuous function on the curve \( x = y(\tau) \). Then, it is a continuous function for \( x \in \mathbb{R} \). Thus, by the standard theorem of integral calculus we can differentiate this integral by parameter \( x \), and the result is continuous in \( x \), (Butuzov and Butuzova, 2016)
\[
\lim_{x \to y(\tau)\pm 0} \frac{\partial}{\partial x} \int_0^\tau [\Omega(k) - \Omega(\tau)] \frac{x - y(k)}{4\sigma^3 \sqrt{\pi (\tau - k)^3} e^{-\frac{(x-y(k))^2}{4\sigma^2(\tau-k)}} \, dk = \lim_{x \to y(\tau)\pm 0} \int_0^\tau [\Omega(k) - \Omega(\tau)] \frac{e^{-\frac{(x-y(k))^2}{4\sigma^2(\tau-k)}}}{4\sigma^3 \sqrt{\pi (\tau - k)^3} (1 - \frac{(x-y(k))^2}{2\sigma^2(\tau-k)})} \, dk \tag{61}
\]
\[ \psi(\tau) = \int_0^\tau [\Omega(k) - \Omega(\tau)] e^{\frac{(y(\tau) - y(0))^2}{4\sigma^2(\tau - k)}} \left( 1 - \frac{(y(\tau) - y(0))^2}{2\sigma^2(\tau - k)} \right) dk. \]

As far as the first integral \( I_1 \) in Eq. (60) is concerned, it was already considered in Section 3.1, and is denoted as \( W_0(\tau, x)/2\sigma^2 \) in Eq. (52). Since the integral on \( a \) in the RHS of Eq. (52) can be computed explicitly, we have

\[ W_0 - \tilde{V}_0 = \Omega_0 \frac{2}{\sqrt{\pi}} \int_{\varepsilon}^{\varepsilon^+} e^{-a^2} da = \Omega_0 \begin{cases} \text{Erfc} \left( \frac{x - y(0)}{2\sigma\sqrt{\tau}} \right), & x > y(\tau), \\
\text{Erf} \left( \frac{x - y(0)}{2\sigma\sqrt{\tau}} \right), & x = y(\tau), \\
\text{Erfc} \left( \frac{x - y(0)}{2\sigma\sqrt{\tau}} \right), & x < y(\tau). \end{cases} \]  

(62)

Also, recall that the function \( \tilde{V}_0(\tau, x) \) is the continuous function along the curve \( x = y(\tau) \) as \( y'(\tau) \) is bounded, and the integral converges uniformly. Therefore

\[ \frac{\partial W_0}{\partial x} = \frac{\partial \tilde{V}_0}{\partial x} - \Omega_0 \Lambda(\tau, x), \]  

(63)

\[ \Lambda(\tau, x) = \begin{cases} \frac{1}{\pi\sigma^2} e^{-\frac{(x-y(0))^2}{4\sigma^2}}, & x > y(\tau), \\
\frac{1}{\pi\sigma^2} e^{-\frac{(x-y(0))^2}{4\sigma^2}}, & x < y(\tau). \end{cases} \]

Thus, \( \Lambda(\tau, y(\tau) - 0) = \Lambda(\tau, y(\tau) + 0) \), hence the function \( \Lambda(\tau, x) \) is differentiable at this point. This implies

\[ \frac{\partial W_0}{\partial x} = -\Omega_0 \int_0^\tau y'(k) \frac{x - y(k)}{2\sigma^2 \sqrt{\pi(\tau - k)^3}} e^{-\frac{(x-y(0))^2}{4\sigma^2(\tau - k)}} dk - \frac{\Omega_0}{\sigma \sqrt{\pi\tau}} e^{-\frac{(x-y(0))^2}{4\sigma^2\tau}}. \]  

(64)

As it was mentioned, the function \( \tilde{V}_0(\tau, x) \) is continuous over the curve \( x = y(\tau) \). However, its derivative with respect to \( x \) at \( x = y(\tau) \) in Eq. (63) has a form of the RHS in Eq. (50). Therefore, according to the result of Section 3.1, in the limit \( x \to y(\tau) \), again using Eq. (58), we obtain

\[ \lim_{x \to y(\tau) \pm 0} \frac{\partial W_0}{\partial x} = \mp \Omega_0 \frac{y'(\tau)}{\sigma^2} - \Omega_0 \int_0^\tau y'(k) \frac{y(\tau) - y(k)}{2\sigma^2 \sqrt{\pi(\tau - k)^3}} e^{-\frac{(y(\tau) - y(0))^2}{4\sigma^2(\tau - k)}} dk - \frac{\Omega_0}{\sigma \sqrt{\pi\tau}} e^{-\frac{(y(\tau) - y(0))^2}{4\sigma^2\tau}}. \]  

(65)

Combining Eq. (61) and Eq. (65) together yields the final result

\[ \psi(\tau) = \int_0^\tau [\Omega(k) - \Omega(\tau)] e^{\frac{(y(\tau) - y(0))^2}{4\sigma^2(\tau - k)}} \left( 1 - \frac{(y(\tau) - y(0))^2}{2\sigma^2(\tau - k)} \right) dk - \Omega(\tau)f(\tau), \]  

(66)

\[ f(\tau) = \pm \frac{y'(\tau)}{2\sigma^2} + \frac{1}{2\sigma^2 \sqrt{\pi\tau}} e^{-\frac{(\tau-y(0))^2}{4\sigma^2\tau}} + \int_0^\tau e^{-\frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau - k)}} \left[ 1 + \frac{y'(k)[y(\tau) - y(k)]}{\sigma^2} - \frac{(y(\tau) - y(k))^2}{2\sigma^2(\tau - k)} \right] dk. \]

Thus, we proved that the derivative \( \partial \psi(\tau, x)/\partial x \) is also discontinuous at \( x = y(\tau) \), and obtained its explicit representation in Eq. (66). Note, that this derivative should not be confused with the normal (directional) derivative of \( u(\tau, x) \) which is continuous at \( x = y(\tau) \). Indeed, the function \( q \), as defined in Eq. (48), is the double layer heat potential. The claim that this derivative is continuous is commonly referred as the Lyapunov-Tauber theorem of classic potential theory, see (Lyapunov, 1949), and (Guenter, 1967; Quaife, 2011; Costabel, 1990; Kristensson, 2009) and references therein for the extension to the multi-dimensional case.
It is worth mentioning, that the formula for \( f(\tau) \) can be further simplified. Indeed

\[
\begin{align*}
    d \left( \frac{e^{-\left(\frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau-k)}\right)}}{\sqrt{\tau-k}} \right) &= \left[ \frac{e^{-\left(\frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau-k)}\right)}}{2\sqrt{\tau-k}} - \frac{e^{-\left(\frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau-k)}\right)}}{\sqrt{\tau-k}} \right] \left( -\frac{y'(k)(y(\tau) - y(k))}{2\sigma^2(\tau-k)} + \frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau-k)^2} \right) \right] dk \\
    &= e^{-\left(\frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau-k)}\right)} \left( 1 + \frac{y'(k)(y(\tau) - y(k))}{\sigma^2(\tau-k)} + \frac{(y(\tau) - y(k))^2}{2\sigma^2(\tau-k)^2} \right) dk.
\end{align*}
\]

Therefore,

\[
\frac{e^{-\left(\frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau-k)}\right)}}{2\sqrt{\tau-k}} \left( 1 + \frac{y'(k)(y(\tau) - y(k))}{\sigma^2(\tau-k)} + \frac{(y(\tau) - y(k))^2}{2\sigma^2(\tau-k)^2} \right) dk = d \left( \frac{e^{-\left(\frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau-k)}\right)}}{\sqrt{\tau-k}} - 1 \right) + \frac{dk}{2\sqrt{\tau-k}}.
\]

Plugging this expression into the formula for \( f(\tau) \) and integrating yields an alternative representation for \( f(\tau) \)

\[
f(\tau) = \pm \frac{y'(\tau)}{2\sigma^4} + \frac{1}{2\sigma^3\sqrt{\pi}} + \int_0^\tau \frac{dk}{4\sigma^3\sqrt{\pi(\tau-k)^3}} \tag{67}
\]

and for \( \psi(\tau) \), respectively

\[
\psi(\tau) = -\Omega(\tau) \left( \frac{1}{2\sigma^3\sqrt{\pi}} \pm \frac{y'(\tau)}{2\sigma^4} \right) + \int_0^\tau \frac{\Omega(k)e^{-\left(\frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau-k)}\right)}}{4\sigma^3\sqrt{\pi(\tau-k)^3}} - \Omega(\tau) dk \tag{68}
\]

\[
- \int_0^\tau \Omega(k) \frac{(y(\tau) - y(k))^2}{8\sigma^5\sqrt{\pi(\tau-k)^5}} \frac{e^{-\left(\frac{(y(\tau) - y(k))^2}{4\sigma^2(\tau-k)}\right)}}{4\sigma^2(\tau-k)} dk.
\]

The last formula for the particular case \( \sigma = 1/\sqrt{2} \) was also obtained in (Lipton et al., 2019) by using a different method.

### 3.3 A system of Volterra equations

With allowance for the representation obtained in Eq. (58), by substituting the limiting values \( x \to y(\tau) \) and \( x \to z(\tau) \) into Eq. (45), we obtain a system of two integral equations for functions \( \Omega(\tau), \Theta(\tau) \)

\[
2\phi_1(\tau) = \Omega(\tau) + \frac{1}{2\sqrt{\pi}} \int_0^\tau \left[ \Omega(k) \frac{y(\tau) - y(k)}{(\tau-k)^{3/2}} e^{-\left(\frac{(y(\tau) - y(k))^2}{4(\tau-k)}\right)} + \Theta(k) \frac{y(\tau) - z(k)}{(\tau-k)^{3/2}} e^{-\left(\frac{(y(\tau) - z(k))^2}{4(\tau-k)}\right)} \right] dk, \tag{69}
\]

\[
2\psi_1(\tau) = -\Theta(\tau) + \frac{1}{2\sqrt{\pi}} \int_0^\tau \left[ \Omega(k) \frac{z(\tau) - y(k)}{(\tau-k)^{3/2}} e^{-\left(\frac{(z(\tau) - y(k))^2}{4(\tau-k)}\right)} + \Theta(k) \frac{z(\tau) - z(k)}{(\tau-k)^{3/2}} e^{-\left(\frac{(z(k) - z(\tau))^2}{4(\tau-k)}\right)} \right] dk.
\]

Each equation in this system is a Volterra equation of the second kind. The system can be solved, eg., by the Variational Iteration Method (VIM), see (Wazwaz, 2011) with a linear complexity by using the Fast Gaussian Transform. Once this is done, the solution of our double barrier problem is found.

It is interesting that the expression of the solution gradient in Eq. (68) provides connection between the GIT and HP methods. Indeed, by definition in Eq. (13) and also using Eq. (7), Eq. (43)

\[
\Psi(\tau) = -\frac{\partial u(\tau, x)}{\partial x} \bigg|_{x=y(\tau)} = -\frac{\partial U(\tau, x)}{\partial x} \bigg|_{x=y(\tau)} + B(\tau) \tag{70}
\]
\begin{align*}
\Phi(\tau) &= \frac{\partial u(\tau, x)}{\partial x} \bigg|_{x = z(\tau)} = \frac{\partial U(\tau, x)}{\partial \tau} \bigg|_{x = z(\tau)} - B(\tau) \\
&= \frac{\partial q(\tau, x)}{\partial x} \bigg|_{x = z(\tau) - 0} - B(\tau) + \frac{1}{4 \sqrt{\pi \tau^3}} \int_{y(0)}^{z(0)} U(0, x')(z(\tau) - x') e^{-\frac{(z(\tau) - x')^2}{4\tau}} \, dx',
\end{align*}

Therefore, once the pair $\Omega(\tau), \Theta(\tau)$ is known, the other pair $\Psi(\tau), \Phi(\tau)$ can be obtained explicitly from Eq. (70). The opposite is also true, i.e., once the pair $\Psi(\tau), \Phi(\tau)$ is known, the heat potential densities $\Omega(\tau), \Theta(\tau)$ can be found by solving this system of Volterra equations of the second kind. Thus, both the GIT and HP methods are interchangeable. But as was mentioned in Introduction, despite both solutions are equal as the infinite series, their convergence properties are different.

## 4 Discussion

In this paper we extend the technique of semi-analytic (or semi-closed form) solutions, developed in (Carr and Itkin, 2020; Itkin and Muravey, 2020; Carr et al., 2020; Itkin et al., 2020a; Lipton and Kaushansky, 2018; Lipton and de Prado, 2020), to pricing double barrier options and present two approaches to solving it: the General Integral transform method and the Heat Potential method. By semi-analytic solution we mean that first, we need to solve a system of two linear Volterra equations of the second kind, and then the option price is represented as a one-dimensional integral.

Therefore, perhaps the main point is about efficiency and robustness of the proposed approach. As shown in (Carr and Itkin, 2020; Itkin and Muravey, 2020; Carr et al., 2020; Itkin et al., 2020a), from the computational point of view the solution proposed by using the same technique for pricing single barrier options under various models with time-dependent barriers is very efficient and, at least theoretically, of the same complexity, or even faster than the forward finite-difference (FD) method. On the other hand, our approach provides high accuracy in computing the options prices, as this is regulated by quadrature rule used to discretize the integral kernel in Eq. (42), or in Eq. (69). Therefore, the accuracy of the method in $x$ space can be easily increased by using high order quadratures. For instance, using the Simpson instead of the trapezoid rule doesn’t affect the complexity of our method but increases the accuracy, while increasing the accuracy for the FD method is not easy (i.e., it significantly increases the complexity of the method, e.g., see (Itkin, 2017)).

As applied to pricing double barrier options - the problem considered in this paper, the difference is that instead of a single Volterra equation of the second kind we now have to solve a system of two equations, either in Eq. (42), or in Eq. (69). This can be done in the same way as for the single barrier problem. The Volterra equation is solved by discretizing the kernel of the integral in time using some quadrature rule which yields a system of linear equations with respect to the discrete values of $\Psi(\tau), \Phi(\tau)$. It can be checked that the matrix of this system is of the form

$$
M = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
$$

where $A, D$ are lower triangular matrices with ones on the main diagonal, and $B, C$ are lower triangular matrices with zeros on the main diagonal. Therefore, this system can be solved by a simple Gauss elimination method (by a set of algebraic multiplications and additions) with complexity $O(2N)$ where $N$ is the number of the discretization points in $\tau$ for $\Psi(\tau), \Phi(\tau)$. Alternatively, when using Eq. (69) or
Eq. (40), since the kernel is proportional to Gaussians, the discrete sum approximating the integral can be computed with linear complexity $O(2N)$ using the Fast Gauss Transform, see e.g., (Spivak et al., 2010).

Once the vectors $\Psi(\tau), \Phi(\tau)$ (for the GIT method), or $\Omega(\tau), \Theta(\tau)$ (for the HP method) are found, they can be substituted into Eq. (34) or Eq. (38) for the GIT method, or into Eq. (45) (for the HP method). Then the final solution is obtained by computing the integral(s) numerically. Various numerical examples illustrating this technique for a single barrier pricing problem can be found in (Carr and Itkin, 2020; Itkin and Muravey, 2020; Carr et al., 2020; Itkin et al., 2020a). Also, those examples demonstrate that computationally our method is more efficient than both the backward and even forward FD methods (if one uses them to solve this kind of problems), while providing better accuracy and stability.

Somebody could be a bit confused of this terminology, since despite the solution is found explicitly as an integral, the latter depends on the unknown function of time $\Psi(\tau)$. In support of this terminology, we can mention that the solution is definitely of a closed form on variable $x$. On variable $\tau$ the integrand explicitly depends on yet unknown function $\Psi(\tau)$ which solves the Volterra integral equation of the second kind. However, this equation can be solved with no iterations. Indeed, after the function $\Psi(\tau)$ is discretized on some grid in $\tau$ (so now it is represented by a finite vector $\psi$), the integral equation reduces to the linear equation for $\psi$, with the matrix being low triangular. Thus, the solution can be immediately obtained by a simple Gauss elimination with no iterations. Therefore, this is explicit and as such, the solution is given by a series of algebraic operations (substitutions). The finer is the grid, the closer is the solution to the exact one.

Also, we can make a reference to Lipton and de Prado (2020); Carr et al. (2020) where the phrase 'semi-closed' was used verbatim. And in Lipton et al. (2019); Lipton and Kaushansky (2018) it is called as 'semi-analytical' solution. Going back in time, in Kartashov and Lyubov (1974); Kartashov (1999, 2001) both GIT and HP methods are claimed as analytical. One can also look at Tikhonov and Samarskii (1963), page 533, subsection 2, which from the very beginning says, 'Heat potentials are a convenient analytical device for solving boundary-value problems'. Therefore, we think this terminology is appropriate.

Also, as mentioned in (Carr et al., 2020), another advantage of the approach advocated in this paper is computation of option Greeks. Indeed, in both the HP and GIT methods the option prices are represented in an explicit analytic form on $x$ (via the integrals on $\tau$ and the auxiliary variable $\xi$). This means that an explicit dependence of the option prices on the model parameters is available and transparent. Thus, explicit representations of the option Greeks can be obtained by a simple differentiation under the integrals. This means that the Greek values can be computed simultaneously with the option prices with almost no additional increase in the elapsed time. This is possible because differentiation under the integrals slightly changes the integrands, while these changes could be represented as changes in weights of the quadrature scheme used to compute the integrals.

Also, the integrands in the integral representation of the solution could be treated as a product of some density function and weights. The major computational time is spent for computing the densities as they contain special functions. However, once computed the results can be saved during the calculation of prices, and then reused when computing the Greeks. Therefore, computing Greeks can be done very fast. This is also true e.g., for Vega and other Greeks that cannot be computed by the FD method together with prices and require a separate run of the FD machinery. Here we don’t have such a problem as differentiation of the integral representation with respect to the model parameters is done analytically.

Finally, as mentioned in (Itkin and Muravey, 2020), the GIT and HO methods are complementary. In more detail, this means the following. Our experiments showed that performance of both the GIT and HP methods is same. However, the GIT method produces more accurate results at high strikes and maturities (i.e. where the option price is relatively small) in contrast to the HP method which is more accurate at short maturities and low strikes. For the CIR and CEV models this behavior was explained in (Carr et al., 2020), and for the Hull-White model - in (Itkin and Muravey, 2020). Briefly, for the heat equation that we consider in this paper, the exponents in both the HP and GIT integrals are inversely
proportional to $\tau$. However, the GIT integrals contain a difference of two exponents (see the definition of $\Upsilon_n(x,\tau|\xi,s)$ in Eq. (38) which becomes small at large $\tau$. On contrary, the HP exponent in Eq. (45) tends to 1 at large $\tau$. Therefore, the convergence properties of two methods are different at large $\tau$.

This situation is well known for the heat equation with constant coefficients. There exist two representation of the solution: one - obtained by using the method of images, and the other one - by the Fourier series. Despite both solutions are equal as the infinite series, their convergence properties are different.

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**References**

I. Bouchouev. Negative oil prices put spotlight on investors. *Risk.net*, 2020.

R. Brogan. Options traders adapt to electronic markets in pandemic, 2020. URL [https://flextrade.com/options-traders-adapt-to-electronic-markets-in-pandemic/](https://flextrade.com/options-traders-adapt-to-electronic-markets-in-pandemic/).

V.F. Butuzov and M.V. Butuzova. *Integrals depending on parameters*. Moscow State University, Moscow, 2016. in Russian.

P. Carr and A. Itkin. Semi-closed form solutions for barrier and American options written on a time-dependent Ornstein Uhlenbeck process, March 2020. Arxiv:2003.08853.

P. Carr, A. Itkin, and D. Muravey. Semi-closed form prices of barrier options in the time-dependent cev and cir models. *Journal of Derivatives*, 28(1):26–50, 2020.

M. Costabel. Boundary integral operators for the heat equation. *Integral Equations and Operator Theory*, 13(4):498–552, 1990.

DLMF. *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.28 of 2020-09-15. URL [http://dlmf.nist.gov/](http://dlmf.nist.gov/). F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

R. Doff. Valuing scenarios with real option pricing. *Risk.net*, August 2020.

S. Farrington and M. Cesa. Podcast: Kaminski and ronn on negative oil and options pricing. *Risk.net*, May 2020.

A. Friedman. *Partial Differential Equations of Parabolic Type*. Prentice-Hall, New Jersey., 1964.

I.S. Gradshtein and I.M. Ryzhik. *Table of Integrals, Series, and Products*. Elsevier, 2007.

N.M. Guinter. *Potential Theory and Its Applications to Basic Problems of MathematicalPhysics*. Frederick Ungar, New York, 1967.

A. Itkin. *Pricing Derivatives Under Lévy Models. Modern Finite-Difference and Pseudo-Differential Operators Approach.*, volume 12 of *Pseudo-Differential Operators*. Birkhauser, 2017.

A. Itkin and D. Muravey. Semi-closed form prices of barrier options in the Hull-White model, April 2020. Arxiv:2004.09591.
A. Itkin, A. Lipton, and D. Muravey. From the black-karasinski to the verhulst model to accommodate the unconventional fed’s policy, June 2020a. URL https://arxiv.org/abs/2006.11976.

A. Itkin, A. Lipton, and D. Muravey. Multilayer heat equations: application to finance. in preparation, 2020b.

E. M. Kartashov. Analytical methods for solution of non-stationary heat conductance boundary problems in domains with moving boundaries. Izvestiya RAS, Energetika, (5):133–185, 1999.

E.M. Kartashov. Analytical Methods in the Theory of Heat Conduction in Solids. Vysshaya Shkola, Moscow, 2001.

E.M. Kartashov and B. Ya Lyubov. Analytical methods in the theory of heat conduction in solids. Izv. Akad. Nauk SSSR, Energ. Trans., (6):83–111, 1974.

G. Kristensson. Jump conditions for single and double layer potentials, 2009. file:///C:/AndreyItkin/MyFinance/FiPapers/BK/liter/JumpConditions.pdf.

A. Lipton. The vol smile problem. Risk, pages 61–65, February 2002.

A. Lipton and M.L. de Prado. A closed-form solution for optimal mean-reverting trading strategies, 2020. available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3534445.

A. Lipton and V. Kaushansky. On the first hitting time density of an ornstein-uhlenbeck process, October 2018. URL https://arxiv.org/pdf/1810.02390.pdf.

A. Lipton, V. Kaushansky, and C. Reisinger. Semi-analytical solution of a McKean-Vlasov equation with feedback through hitting boundary. Euro. Jnl of Applied Mathematics, pages 1–34, 2019.

A.M. Lyapunov. Works on the theory of potential. Technical and Theoretical State Publishing House, Moscow - Leningrad, 1949. in Russian.

A. Mijatovic. Local time and the pricing of time-dependent barrier options. Finance and Stochastics, 14 (1):13–48, 2010.

D. Mumford, C. Musiliand M. Nori, E. Previato, and M. Stillman. Tata Lectures on Theta. Progress in Mathematics. Birkhäuser Boston, 1983. ISBN 9780817631093.

A.D. Polyanin. Handbook of linear partial differential equations for engineers and scientists. Chapman & Hall/CRC, 2002.

B. Quaife. Fast Integral Equation Methods for the Modified Helmholtz Equation. PhD thesis, University of Calgary, 2011.

M. Spivak, S.K. Veerapaneni, and L. Greengard. The fast generalized gauss transform. SIAM Journal on Scientific Computing, 32(5):3092–3107, 2010.

A.N. Tikhonov and A.A. Samarskii. Equations of mathematical physics. Pergamon Press, Oxford, 1963.

B. van der Pol and H. Bremmer. Operational calculus based on the two-sided Laplace integral. Cambridge University Press, Cambridge, UK, 1950.

A. M. Wazwaz. Linear and Nonlinear Integral Equations. Higher Education Press, Beijing and Springer-Verlag GmbH Berlin Heidelberg, 2011.
Appendices

A Simplification of Eq. (29)

To simplify Eq. (29) we proceed with the integration by parts of the last summand in Eq. (29).

\[
\int_0^\tau e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)} h_1(n,s,\tau)} ds = -\frac{B(\tau) l^2(\tau)}{\pi^2 n^2} \left[ \sin \left( \frac{\pi n}{l(\tau)} [z(\tau) - y(\tau)] \right) - \sin \left( \frac{\pi n}{l(\tau)} [y(\tau) - y(\tau)] \right) \right] \\
+ \frac{B(0) l^2(\tau)}{\pi^2 n^2} e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)}} \left[ \sin \left( \frac{\pi n}{l(\tau)} [z(0) - y(\tau)] \right) - \sin \left( \frac{\pi n}{l(\tau)} [y(0) - y(\tau)] \right) \right] \\
- \frac{l(\tau)}{\pi n} \left[ f_-(\tau) \cos \left( \frac{\pi n}{l(\tau)} [y(\tau) - y(\tau)] \right) - f_+ (\tau) \cos \left( \frac{\pi n}{l(\tau)} [z(\tau) - y(\tau)] \right) \right] \\
+ \frac{l(\tau)}{\pi n} e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)}} \left[ f_-(0) \cos \left( \frac{\pi n}{l(\tau)} [y(0) - y(\tau)] \right) - f_+ (0) \cos \left( \frac{\pi n}{l(\tau)} [z(0) - y(\tau)] \right) \right] \\
+ \frac{l^2(\tau)}{\pi^2 n^2} \int_0^\tau B(s) e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)}} \left[ \sin \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) - \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) \right] ds \\
+ \frac{\pi n}{l(\tau)} \left[ z'(s) \cos \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) - y'(s) \cos \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) \right] ds \\
+ \frac{\pi n}{l(\tau)} \int_0^\tau f_-(s) e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)}} \left[ \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right] - \frac{\pi n}{l(\tau)} y'(s) \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) ds \\
- \frac{\pi n}{l(\tau)} \int_0^\tau f_+(s) e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)}} \left[ \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right] - \frac{\pi n}{l(\tau)} z'(s) \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) ds \\
+ \frac{\pi n}{l(\tau)} \int_0^\tau B(s) e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)}} \left[ y'(s) \cos \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) - z'(s) \cos \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) \right] ds,
\]

or

\[
\int_0^\tau e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)} h_1(n,s,\tau)} ds = -\frac{l(\tau)}{\pi n} \left[ f_-(\tau) + (-1)^{n+1} f_+(\tau) \right] \\
+ \frac{B(0) l^2(\tau)}{\pi^2 n^2} e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)}} \left[ \sin \left( \frac{\pi n}{l(\tau)} [z(0) - y(\tau)] \right) - \sin \left( \frac{\pi n}{l(\tau)} [y(0) - y(\tau)] \right) \right] \\
+ \frac{l(\tau)}{\pi n} e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)}} \left[ f_-(0) \cos \left( \frac{\pi n}{l(\tau)} [y(0) - y(\tau)] \right) - f_+ (0) \cos \left( \frac{\pi n}{l(\tau)} [z(0) - y(\tau)] \right) \right] \\
+ \frac{l(\tau)}{\pi n} \left[ \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right] - \frac{\pi n}{l(\tau)} \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) ds \\
+ \frac{l(\tau)}{\pi n} \int_0^\tau f_-(s) e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)}} \left[ \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right] - \frac{\pi n}{l(\tau)} y'(s) \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) ds \\
- \frac{l(\tau)}{\pi n} \int_0^\tau f_+(s) e^{-\frac{\pi^2 n^2}{\tau^2 (\tau-s)}} \left[ \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right] - \frac{\pi n}{l(\tau)} z'(s) \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) ds.
\]
Let us introduce the notation

\[ \alpha(\tau, n) = \frac{B(0)l^2(\tau)}{\pi^2 n^2} \left[ \sin \left( \frac{\pi n}{l(\tau)} [z(0) - y(\tau)] \right) - \sin \left( \frac{\pi n}{l(\tau)} [y(0) - y(\tau)] \right) \right] \]  
(A.2)

\[+ l(\tau) \frac{\tau - \pi n}{\pi n} \left[ f_-(0) \cos \left( \frac{\pi n}{l(\tau)} [y(0) - y(\tau)] \right) - f_+(0) \cos \left( \frac{\pi n}{l(\tau)} [z(0) - y(\tau)] \right) \right], \]

\[ \beta(\tau, s, n) = B(s) \left[ \sin \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) - \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) \right] \]

\[+ f_-(s) \left( \frac{\pi n}{l(\tau)} \cos \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) - y'(s) \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) \right) \]

\[- f_+(s) \left( \frac{\pi n}{l(\tau)} \cos \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) - z'(s) \sin \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) \right). \]

This allows re-writing Eq. (A.1) as

\[ \int_{0}^{\tau} e^{- \frac{\tau^2 z^2}{\pi^2 l^2(\tau) s}} h_1(n, s, \tau) ds = \frac{l(\tau)}{\pi n} \left[ (-1)^n f_+(\tau) - f_-(\tau) \right] + \alpha(\tau, n) e^{- \frac{\tau^2 z^2}{\pi^2 l^2(\tau) s}} + \int_{0}^{\tau} e^{- \frac{\tau^2 z^2}{\pi^2 l^2(\tau) s}} \beta(\tau, s, n) ds. \]  
(A.3)

Applying well-known identities (Gradshteyn and Ryzhik, 2007)

\[ \sum_{k=0}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2}, \quad [0 < x < 2\pi], \quad \sum_{k=0}^{\infty} (-1)^{k-1} \frac{\sin kx}{k} = \frac{x}{2}, \quad [0 < x < 2\pi] \]

yields

\[ \sum_{n=1}^{\infty} \frac{2}{\pi n} \left[ (-1)^n f_+(\tau) - f_-(\tau) \right] \sin \left( \frac{\pi n}{l(\tau)} [x - y(\tau)] \right) = - \frac{2}{\pi} \left[ \frac{f_+(\tau) \pi (x - y(\tau))}{l(\tau)} + \frac{f_-(\tau) \pi (x - y(\tau))}{l(\tau)} \right] \]

\[= - \frac{f_+(\tau) - f_-(\tau)}{l(\tau)} x + \frac{f_+(\tau) y(\tau) - f_-(\tau) z(\tau)}{l(\tau)} = - [A(\tau) + B(\tau) x]. \]

Using another identity

\[ \int_{y(0)}^{z(0)} [A(0) + B(0) \xi] \sin \left( \frac{\pi n}{l(\tau)} [\xi - y(\tau)] \right) d\xi = \frac{l(\tau)}{\pi^2 n^2}, \]

\[\left\{ \begin{array}{l}
\pi n (A(0) + B(0) y(0)) \cos \left( \frac{\pi n (y(0) - y(\tau))}{l(\tau)} \right) \\
- \pi n (A(0) + B(0) z(0)) \cos \left( \frac{\pi n (z(0) - y(\tau))}{l(\tau)} \right) \\
+ B(0) l(\tau) \left( \sin \left( \frac{\pi n (z(0) - y(\tau))}{l(\tau)} \right) - \sin \left( \frac{\pi n (y(0) - y(\tau))}{l(\tau)} \right) \right) \end{array} \right\}, \]

we get

\[ \int_{y(0)}^{z(0)} u(0, \xi) d\xi = \int_{y(0)}^{z(0)} U(0, \xi) d\xi - \alpha(\tau, n). \]  
(A.4)
Returning to the original variable $U(\tau, x)$ yields the following formula

$$
U(\tau, x) = \frac{2}{l(\tau)} \sum_{n=1}^{\infty} \sin \left( \frac{\pi n}{l(\tau)} (x - y(\tau)) \right) \left\{ e^{-\frac{\pi^2 n^2}{l^2(\tau)} \tau} \int_{y(0)}^{z(0)} U(0, \xi) \sin \left( \frac{\pi n}{l(\tau)} \xi - y(\tau) \right) d\xi \right\} + \int_{0}^{\tau} e^{-\frac{\pi^2 n^2}{l^2(\tau)} (\tau-s)} \left[ \Phi(s) \sin \left( \frac{\pi n}{l(\tau)} [z(s) - y(\tau)] \right) + \Psi(s) \sin \left( \frac{\pi n}{l(\tau)} [y(s) - y(\tau)] \right) + \beta(\tau, s, n) \right] ds.
$$

(A.5)