INTEGRAL WITH RESPECT TO THE G-BROWNIAN LOCAL TIME

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Abstract. Let $L$ be the local time of $G$-Brownian motion $B$. In this paper, we prove the existence of the quadratic covariation $\langle f(B), B \rangle_t$ and the integral $\int_{\mathbb{R}} f(x) L(dx, t)$. Moreover, a sublinear version of the Bouleau-Yor identity

$$\int_{\mathbb{R}} f(x) L(dx, t) = -\langle f(B), B \rangle_t$$

is showed to hold under some suitable conditions. These allow us to write the Itô’s formula for $C^1$-functions.

- Key words and phrases: Nonlinear expectation, $G$-Brownian motion, Local times, Young integration, Quadratic covariation;
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1. Introduction

Motivated by various types of uncertainty and financial problems, Peng \cite{15} has introduced a new notion of nonlinear expectation, the so-called $G$-expectation (see also Peng \cite{16, 17, 18}), which is associated with the following nonlinear heat equation

$$\begin{cases}
\frac{\partial}{\partial t} u(t, x) = G(\Delta u), & (t, x) \in [0, +\infty) \times \mathbb{R}, \\
u(0, x) = \varphi(x),
\end{cases}$$

where $\Delta$ is Laplacian, the sublinear function $G$ is defined as

$$G(\alpha) = \frac{1}{2} \left( \sigma^2 \alpha^+ - \sigma^2 \alpha^- \right), \quad \alpha \in \mathbb{R}$$

with two given constants $0 < \underline{\sigma} < \overline{\sigma}$. Together with the notion of $G$-expectations Peng also introduced the related $G$-normal distribution, the $G$-Brownian motion and related stochastic calculus under $G$-expectation, and moreover an Itô’s formula for the $G$-Brownian motion was established. $G$-Brownian motion has a very rich and interesting new structure which non-trivially generalizes the classical one. Briefly speaking, a $G$-Brownian motion $B$ is a continuous process with independent stationary increments $B_{t+s} - B_t$ being $G$-normally distributed under a given sublinear expectation $\hat{E}$. A very interesting new phenomenon of $G$-Brownian motion $B$ is that its quadratic process $\langle B \rangle$ is a continuous process with independent and stationary increments, but not a deterministic process.

On the other hand, in the theory and applications of classical stochastic calculus, the Itô’s formula plays a central role. But, the restriction of Itô’s formula to functions with twice differentiability often encounter difficulties in applications. For the classical Itô formula, many authors have given some extension to the Itô formula. Some approach extending Itô’s formula are to use the classical quadratic covariation and the local time-space calculus. More works for the problem can be found in Bouleau–Yor \cite{1}, Eisenbaum \cite{4}.

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Recently, Li-Peng \cite{11} gave a more general Itô integral with respect to $G$-Brownian motion and Itô’s formula. Moreover, Lin \cite{12} studied local time of $G$-Brownian motion and established a Tanaka formula under $G$-expectation. These motivate us to consider the extensions of the classical local time-space calculus under $G$-expectation. Let us first recall some known results concerning the quadratic variation and Itô’s formula. Let $F$ be an absolutely continuous function with locally square integrable derivative $f$, that is,

\[
F(x) = F(0) + \int_0^x f(y)dy
\]

with $f$ being locally square integrable. Bouleau-Yor \cite{1} and Föllmer et al \cite{9} introduced the following formulas:

\begin{align}
[f(W), W]_t &= -\int_{\mathbb{R}} f(x)\mathcal{L}^W(dx, t) \quad \text{(1.1)}
\end{align}

and

\begin{align}
F(W_t) &= F(0) + \int_0^t f(W_s)dW_s + \frac{1}{2} [f(W), W]_t, \quad \text{(1.2)}
\end{align}

where $W$ is the classical Brownian motion, $\mathcal{L}$ is local time of Brownian motion $W$ and $[f(W), W]$ is the classical quadratic covariation of $f(W)$ and $W$. If $f \in C^1(\mathbb{R})$, (1.2) is the classical Itô formula. This result has been extended to some classical semimartingales, smooth nondegenerate martingales and fractional Brownian motion by Russo–Vallois \cite{20}, Moret–Nualart \cite{13} and Yan et al. \cite{21,22}.

In this paper we consider the identity (1.1) and Itô’s formula (1.2) under sublinear expectation. Our start point is to define the integral

\begin{align}
\int_{\mathbb{R}} f(x)\mathcal{L}(dx, t)
\end{align}

by using Young integration, where $\mathcal{L}$ is local time of $G$-Brownian motion. If $f$ is of bounded $p$-variation with $1 \leq p < 2$ we show that the integral (1.3) and quadratic covariation $\langle f(B), B \rangle$ exist, where the quadratic covariation $\langle f(B), B \rangle$ is defined as

\[
\langle f(B), B \rangle_t := \lim_{n \to \infty} \sum_{k=0}^{n-1} \{f(B_{t_{k+1}}) - f(B_{t_k})\}(B_{t_{k+1}} - B_{t_k})
\]

in $L^1(\Omega)$, where $\{t_k\}$ is a partitions of $[0, T]$ such that $\max_k \{t_k - t_{k-1}\} \to 0$ as $n \to \infty$.

This paper is organized as follows. In Section 2 we present some preliminaries for $G$-Brownian motion. In Section 3 we show that the one parameter integral (1.3) exists and establish the generalized Itô formula

\begin{align}
F(B_t) &= F(0) + \int_0^t f(B_s)dB_s - \frac{1}{2} \int_{\mathbb{R}} f(x)\mathcal{L}(dx, t), \quad \text{(1.4)}
\end{align}

where $F \in C^1(\mathbb{R})$ and $F' = f$ is of bounded $p$-variation with $1 \leq p < 2$. In Section 4 we prove the existence of the quadratic covariation $\langle f(B), B \rangle$. As a result, we get a sublinear version of the Bouleau-Yor identity

\begin{align}
\langle f(B), B \rangle_t &= -\int_{\mathbb{R}} f(x)\mathcal{L}(dx, t), \quad \text{(1.5)}
\end{align}

under some suitable conditions. In Section 5 we extend these results to the time-dependent case. In Appendix we give the other representation of quadratic covariation:

$$\frac{1}{\varepsilon} \int_0^t (B_{s+\varepsilon} - B_s)^2 ds \rightarrow \langle B \rangle_t$$

in \(L^1\), as \(\varepsilon \downarrow 0\).

2. Preliminaries

In this section, we briefly recall some basic notations and results for \(G\)-Brownian motion under \(G\)-framework. For more aspects on these material we refer to Li-Peng [11], Lin [12] and Peng [13, 16, 17, 18]. For simplicity throughout this paper we let \(C\) stand for a positive constant depending only on the subscripts and its value may be different in different appearance.

2.1. Sublinear expectation space. Let \(\Omega \neq \emptyset\) be a given set and let \(\mathcal{H}\) be a linear space of real valued functions defined on \(\Omega\) such that \(1 \in \mathcal{H}\) and \(|X| \in \mathcal{H}\) for all \(X \in \mathcal{H}\).

Definition 2.1. A sublinear expectation \(\hat{\mathbb{E}}\) on \(\mathcal{H}\) is a functional with the following properties: for all \(X, Y \in \mathcal{H}\), we have

- Monotonicity : if \(X \geq Y\), then \(\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]\);
- Constant preserving : \(\hat{\mathbb{E}}[c] = c\), for all \(c \in \mathbb{R}\);
- Sub-additivity : \(\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y]\);
- Positive homogeneity : \(\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]\), for all \(\lambda \geq 0\).

The triple \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\) is called a sublinear expectation space, and \(\mathcal{H}\) is considered as the space of random variables on \(\Omega\).

It is important to note that we can suppose that

\(\varphi(X_1, \ldots, X_d) \in \mathcal{H}\)

if \(X_i \in \mathcal{H}, i = 1, \ldots, d\), for all \(\varphi \in C_{b,Lip}(\mathbb{R}^d)\), where \(C_{b,Lip}(\mathbb{R}^d)\) denotes the space of all bounded and Lipschitz functions on \(\mathbb{R}^d\). In a sublinear expectation space \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\), a random vector \(Y = (Y_1, \ldots, Y_n), Y_i \in \mathcal{H}\) is said to be independent under \(\hat{\mathbb{E}}\) from another random vector \(X = (X_1, \ldots, X_m), X_i \in \mathcal{H}\), if for each test function \(\varphi \in C_{b,Lip}(\mathbb{R}^{m+n})\) we have

\(\hat{\mathbb{E}}[\varphi(X,Y)] = \hat{\mathbb{E}}_1 [\hat{\mathbb{E}}[\varphi(x,Y)]_{x=x}]\).

Two \(n\)-dimensional random vectors \(X\) and \(Y\) defined respectively in the sublinear expectation spaces \((\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)\) and \((\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)\) are called identically distributed, denoted by \(X \sim Y\), if

\(\hat{\mathbb{E}}_1[\varphi(X)] = \hat{\mathbb{E}}_2[\varphi(Y)]\)

for all \(\varphi \in C_{b,Lip}(\mathbb{R}^n)\).

Let \(\underline{\sigma}, \bar{\sigma}\) be two real numbers with \(0 < \underline{\sigma} < \bar{\sigma}\). A random variable \(X\) in a sublinear expectation space \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\) is called \(G\)-normal distributed, denoted by \(\xi \sim N(0, [\underline{\sigma}^2, \bar{\sigma}^2])\), if for each \(\varphi \in C_{b,Lip}(\mathbb{R})\), the function defined by

\(u(t, x) := \hat{\mathbb{E}} \left[ \varphi(x + \sqrt{t} \xi) \right], \quad (t, x) \in [0, \infty) \times \mathbb{R}\)

is the unique viscosity solution of the following nonlinear heat equation:

\[
\begin{cases}
\frac{\partial}{\partial t} u(t, x) = G(\Delta u), & (t, x) \in [0, +\infty) \times \mathbb{R}, \\
u(0, x) = \varphi(x), & \end{cases}
\]
where $\Delta$ is Laplacian and the sublinear function $G$ is defined as
\[
G(\alpha) = \frac{1}{2} \left( \sigma^2 \alpha^+ - \sigma^2 \alpha^- \right), \quad \alpha \in \mathbb{R}.
\]

**Example 2.1** (Peng [15]). Let $\xi \sim N(0, [\sigma^2, \mathcal{I}^2])$. We then have
\[
\hat{E} [\varphi(\xi)] = \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} \varphi(x) e^{-\frac{1}{2\sigma^2} x^2} dx
\]
for all convex functions $\varphi$, and
\[
\hat{E} [\psi(\xi)] = \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} \varphi(x) e^{-\frac{1}{2\sigma^2} x^2} dx
\]
for all concave functions $\psi$.

### 2.2. Some spaces.

In this paper we throughout let $\Omega = C_0(\mathbb{R}^+) \subseteq \mathbb{R}$ be the space of all real valued continuous functions on $[0, \infty)$ with initial value 0, equipped with the distance
\[
\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i} \left[ \max_{t \in [0,i]} |\omega^1_t - \omega^2_t| \right] \wedge 1,
\]
where $\omega^1, \omega^2 \in \Omega$.

We denote by $\mathcal{B}(\Omega)$ the Borel-algebra on $\Omega$. We also denote, for each $t \in [0, \infty)$,
\[
\Omega_t = \{ \omega \cdot \wedge t, \omega \in \Omega \},
\]
and $\mathcal{F}_t = \mathcal{B}(\Omega_t)$, where $x \wedge y = \min\{x, y\}$. We also denote

- $L^0(\Omega)$ : the space of all $\mathcal{B}(\Omega)$-measurable real valued functions on $\Omega$;
- $L^0(\Omega_t)$ : the space of all $\mathcal{B}(\Omega_t)$-measurable real valued functions on $\Omega_t$;
- $L_b(\Omega)$ : the space of all bounded elements in $L^0(\Omega)$;
- $L_b(\Omega_t)$ : the space of all bounded elements in $L^0(\Omega_t)$.

Let $L^p_G(\Omega)$ be the closure of $\mathcal{H}$ with respect to the norm
\[
\|X\|_p = \hat{E} [\|X\|^p]^{1/p}
\]
with $p \in [1, \infty)$. Clearly, the space $L^p_G(\Omega)$ is a Banach space and the space $C_b(\Omega)$ of bounded continuous functions on $\Omega$ is a subset of $L^1_G(\Omega)$, and moreover, for the sublinear expectation space $(\Omega, L^p_G(\Omega), \hat{E})$ there exists a weakly compact family $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that
\[
\hat{E} = \sup_{P \in \mathcal{P}} E_P.
\]
So we can introduce the Choquet capacity $\hat{C}$ by taking
\[
\hat{C}(A) = \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).
\]

**Definition 2.2.** A set $A \subset \Omega$ is called polar if $\hat{C}(A) = 0$. A property is said to hold "quasi surely" (q.s.) if it holds outside a polar set.

By using the above family of probability measures $P$ we can characterize the space $L^p_G(\Omega)$ as
\[
L^p_G(\Omega) = \left\{ L^0(\Omega) \ni X \text{ is continuous, q.s., and } \sup_{P \in \mathcal{P}} E_P[\|X\|^p] < \infty \right\}
\equiv \left\{ L^0(\Omega) \ni X \text{ is continuous, q.s., and } \lim_{n \to \infty} \sup_{P \in \mathcal{P}} E_P[|X|1_{\{|X|>n\}}] = 0 \right\}.
\]

The following three results can be consulted in Denis et al [2] and Hu-Peng [10].
Lemma 2.1 (Denis et al [2] and Hu-Peng [10]). Let \( \{X_n, n = 1, 2, \ldots\} \) be a monotonically decreasing sequence of nonnegative random variances in \( C_b(\Omega) \). If \( X_n \) converges to \( G \), then we have

\[
\lim_{n \to 0} \hat{E}[X_n] = 0.
\]

Moreover, if \( X_n \uparrow X \) and \( \hat{E}[X], \hat{E}[X_n] \) are finite for all \( n = 1, 2, \ldots \), we then have

\[
\lim_{n \to 0} \hat{E}[X_n] = \hat{E}[X].
\]

Lemma 2.2 (Denis et al [2] and Hu-Peng [10]). Let \( 1 \leq p < \infty \). Consider the sets \( \mathbb{L}_p^p(\Omega) \) and \( \mathbb{L}_p = \mathcal{L}^p/N \), where

\[
\mathcal{L}^p = \left\{ X \in L^0(\Omega) : \hat{E}(|X|^p) = \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty \right\},
\]

\[
\mathcal{N} = \left\{ X \in L^0(\Omega) : X = 0 \text{ q.s.} \right\}.
\]

Then

- \( \mathbb{L}_p \) is a Banach space with respect to the norm \( \| \cdot \|_p \);
- \( \mathbb{L}_p^p \) is the completion of \( C_b(\Omega) \) with respect to the norm \( \| \cdot \|_p \).

Lemma 2.3 (Denis et al [2] and Hu-Peng [10]). For a given \( p \in (0, +\infty) \), if the sequence \( \mathbb{L}_p \ni \{X_n\} \) converges to \( X \) in \( \mathbb{L}_p \), then there exists a subsequence \( \{X_{n_k}\} \) such that \( X_{n_k} \) converges to \( X \) quasi-surely.

We denote by \( \mathbb{L}^p_0(\Omega) \) the completion of \( L_0(\Omega) \) with respect to the norm \( \| \cdot \|_p \).

2.3. G-Brownian motion. Now, let us recall the definition of G-Brownian motion and related Itô's integral.

Definition 2.3. A process \( B = \{B_t, t \geq 0\} \subset \mathcal{H} \) in a sublinear expectation space \( (\Omega, \mathcal{H}, \hat{E}) \) is called a G-Brownian motion if the following properties are satisfied:

- \( B_0 = 0 \);
- For each \( t, s \geq 0 \), the increment \( B_{t+s} - B_t \) is \( N(0, [\sigma^2 s, \sigma^2 s]) \)-distributed and is independent from \( (B_{t_1}, \ldots, B_{t_n}) \), for all \( n = 0, 1, 2, \ldots \) and \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t \).

The G-Brownian motion \( B \) has following properties:

1. For all \( \xi \in \mathbb{L}_2^2(\Omega_t) \), we have \( \hat{E}[\xi(B_T - B_t)] = 0 \) with \( 0 \leq t \leq T \);
2. For all \( \mathcal{B}(\Omega_t) \)-measurable real valued, bounded functions \( \xi \), we have

\[
\hat{E}[\xi^2(B_T - B_t)^2] \leq \sigma^2(T - t)\hat{E}[\xi^2], \quad 0 \leq t \leq T;
\]

3. For all \( t \geq 0 \), we have \( \hat{E}[B_t] = \hat{E}[-B_t] = 0 \);
4. \( t \mapsto B_t \) is Hölder continuous of order \( \delta < \frac{1}{2} \), quasi-surely.

In Li-Peng [11], a generalized Itô integral and a generalized Itô formula with respect to the G-Brownian motion are introduced. For arbitrarily fixed \( p \geq 1 \) and \( T \in \mathbb{R}_+ \), we denote by \( M_{b,p}^{p,0}([0, T]) \) the set of step processes:

\[
(2.1) \quad \eta(t) = \sum_{j=1}^{N} \xi_j(\omega) 1_{[t_{j-1}, t_j)}(t), \quad \xi_j \in L_b(\Omega_{t_j})
\]
with \(0 = t_0 < \cdots < t_N = T\). For the process of the form (2.1) we define the related Bochner integral as follows
\[
\int_0^T \eta_t \, dt = \sum_{j=1}^N \xi_j(\omega)(t_{j-1}, t_j).
\]

For every \(\eta \in M^p_b([0,T])\) we set
\[
\hat{E}_T(\eta) := \frac{1}{T} \hat{E} \int_0^T \eta_t \, dt.
\]

Then \(\hat{E}_T\) forms a sublinear expectation. Moreover, we denote by \(M^p_*([0,T])\) the completion of \(M^p_b([0,T])\) under the norm
\[
\|\eta\|_{M^p_*([0,T])} = \left( \hat{E} \left[ \int_0^T |\eta_s|^p \, ds \right] \right)^{\frac{1}{p}}.
\]

**Definition 2.4.** For every \(\eta \in M^p_b([0,T])\) of the form (2.1), we define the Itô integral of \(\eta\) with respect to \(G\)-Brownian motion \(B\) as
\[
I(\eta) := \int_0^T \eta_s d\mathbf{B}_s = \sum_{j=1}^N \xi_j(B_{s_j} - B_{s_{j-1}}).
\]

The mapping \(I : M^p_b([0,T]) \to L^2_*\Omega)\) is a linear continuous mapping and thus can be continuously extended to \(I : M^2_*([0,T]) \to L^2_\Omega\), which is called the Itô integral of \(\eta \in M^2_*([0,T])\) with respect to \(G\)-Brownian motion \(B\), and define
\[
\int_0^t \eta_s d\mathbf{B}_s = \int_0^T 1_{\{0 \leq s \leq t\}} \eta_s d\mathbf{B}_s
\]
for all \(\eta \in M^2_*([0,T])\) and \(t \in [0,T]\). We have
\[
\hat{E} \left( \int_0^T \eta_s d\mathbf{B}_s \right) = 0
\]
and
\[
\hat{E} \left[ \left( \int_0^T \eta_s d\mathbf{B}_s \right)^2 \right] \leq \sigma^2 \hat{E} \left[ \int_0^T \eta_s^2 \, ds \right]
\]
for all \(\eta \in M^2_*([0,T])\). Moreover, the process \(\{\int_0^t \eta_s d\mathbf{B}_s, t \in [0,T]\}\) is continuous in \(t\) quasi surely, and
\[
\int_0^T \eta_s d\mathbf{B}_s \in M^2_*([0,T])
\]
for all \(\eta \in M^2_*([0,T])\).

**Definition 2.5 (Quadratic Variation).** Let \(\pi^N_t = \{0 = t^N_0 < t^N_1 < \cdots < t^N_{N-1} = t\}\) be a partition of \([0,t]\) for \(t > 0\), such that \(\mu(\pi^N_t) := \max\{t_j - t_{j-1}\} \to 0\) as \(N \to \infty\). The quadratic variation of \(G\)-Brownian motion \(B\) is defined as
\[
\langle B \rangle_t = \lim_{\mu(\pi^N_t) \to 0} \sum_{k=0}^{N-1} (B_{t^N_{k+1}} - B_{t^N_k})^2 = B^2_t - 2 \int_0^t B_s d\mathbf{B}_s
\]
in \(L^2_\Omega\).
The function $t \mapsto \langle B \rangle_t$ is continuous and increasing outside a polar set. We can define the integral
\[
\int_0^T \eta_t d\langle B \rangle_t := \sum_{j=1}^N \xi_j (\langle B \rangle_{t_j} - \langle B \rangle_{t_{j-1}})
\]
as a map from $M^{1,0}_b([0, T])$ into $L^1_\ast(\Omega_T)$, and the map is linear and continuous, and it can be extended continuously to $M^1_\ast([0, T])$.

**Theorem 2.1** (Itô’s formula). Let $F \in C^2(\mathbb{R} \times \mathbb{R}_+)$. We then have
\[
F(B_t, t) = F(B_0, 0) + \int_0^t \frac{\partial}{\partial x} F(B_s, s) dB_s + \int_0^t \frac{\partial}{\partial t} F(B_s, s) ds + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} F(B_s, s) d\langle B \rangle_s
\]
for all $t \geq 0$.

Finally, recall that the $G$-Brownian motion $B$ has a jointly continuous local time $\mathcal{L}(x, t)$ which satisfies (see Lin [12])
\[
\mathcal{L}(x, t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(x-\varepsilon, x+\varepsilon)}(B_s) d\langle B \rangle_s
\]
in $L^2$, and $(x, t) \mapsto \mathcal{L}(x, t)$ is Hålder continuous of order $0 < \gamma < \frac{1}{2}$. Moreover, the following Tanaka formula holds:
\[
|B_t - x| = |x| + \int_0^t \text{sign}(B_s - x) dB_s + \mathcal{L}(x, t).
\]
for all $x \in \mathbb{R}$ and $t \geq 0$. As a result of Tanaka formula we have
\[
\mathcal{L}(x, t) = 0
\]
if $|x| \geq \sup_{0 \leq s \leq t} |B_s|$, i.e., the function $x \mapsto \mathcal{L}(x, t)$ has a compact support for all $0 \leq t < \infty$.

**Theorem 2.2** (Theorem 5.4 in Lin [12]). Let $\sigma > 0$ and let $p \geq 1, a \leq b$. Then the convergence in $L^p$
\[
\lim_{n \to \infty} \sum_{i=0}^{2^n-1} (\mathcal{L}(a_{i+1}^n, t) - \mathcal{L}(a_i^n, t))^2 = 4 \int_a^b \mathcal{L}(x, t) dx
\]
holds uniformly in $t \in [0, T]$ for all the sequence of partitions $\pi_n = \{a_i^n = a + 2^{-n}i(b-a), i = 0, 1, 2, \ldots, 2^n\}$, $n \geq 1$, of the interval $[a, b]$.

### 3. One Parameter Integrals of Local Time

In this section, we study the integral
\[
(3.1) \quad \int_{\mathbb{R}} f(x) \mathcal{L}(dx, t)
\]
where $f$ is a real function, $B$ is a $G$-Brownian motion and $\mathcal{L}$ is the local time of $B$. To use Young integration to establish the integral, we first investigate $p$-variation of the mapping $x \mapsto \mathcal{L}(x, t)$ for every $t \geq 0$. Throughout this paper we let $\underline{a} > 0$.

**Lemma 3.1** (Occupation times formula). For all $t \geq 0$ and every bounded function $f$, we have
\[
\int_0^t f(B_s) d\langle B \rangle_s = \int_{\mathbb{R}} f(x) \mathcal{L}(x, t) dx.
\]
This formula is first introduced by Lin [12] for \( f(x) = 1_{[a,b]}(x) \) (see Theorem 5.2 in Lin [12]). By approximation we can give the above formula. In fact, for any simple function \( f_{\Delta}(x) = \sum_{i} a_{i} 1_{[x_{i-1}, x_{i})}(x) \), where \( \{x_{0}, x_{1}, \cdots \} \) is a partition of \( \mathbb{R} \), we have

\[
\int_{0}^{t} f_{\Delta}(B_{s})d\langle B \rangle_{s} = \int_{\mathbb{R}} f_{\Delta}(x)\mathcal{L}(x,t)\,dx, \quad q.s.
\]

by the linearity of the integral.

Now, for every bounded function \( f \), one can show that there is a sequence of bounded simple functions \( f_{n} \), \( n \geq 1 \) such that \( f_{n}(x) \uparrow f(x) \) for all \( x \in \mathbb{R} \) (of course, there also exists a sequence of simple functions converging to \( f \) uniformly)

\[
\int_{0}^{t} f_{n}(B_{s})d\langle B \rangle_{s} \to \int_{0}^{t} f(B_{s})d\langle B \rangle_{s}
\]

and

\[
\int_{\mathbb{R}} f_{n}(x)\mathcal{L}(x,t)\,dx \to \int_{\mathbb{R}} f(x)\mathcal{L}(x,t)\,dx,
\]

in \( L^{1} \), by Lemma 2.1 as \( n \to \infty \), which gives

\[
\int_{0}^{t} f(B_{s})d\langle B \rangle_{s} = \int_{\mathbb{R}} f(x)\mathcal{L}(x,t)\,dx, \quad q.s.
\]

and the lemma follows.

**Definition 3.1.** Let \( p \geq 1 \) be a fixed real number. A function \( f : [a,b] \mapsto \mathbb{R} \) is said to be of bounded \( p \)-variation if

\[
v_{p}(f) = \sup_{\Delta_{n}} \sum_{i=0}^{n} |f(x_{i+1}) - f(x_{i})|^{p} < \infty,
\]

where the supremum is taken over all partition \( \Delta_{n} = \{a = x_{0} < x_{1} < \cdots < x_{n} = b\} \) of \([a,b]\).

**Corollary 3.1.** Let \( p > 2 \). Then the mapping \( x \mapsto \mathcal{L}(x,t) \) is of bounded \( p \)-variation q.s. for any \( 0 \leq t \leq T \).

We now can establish one parameter integral (3.1). Denote by \( \mathcal{W}_{p}([a,b]) \) \((p \geq 1)\) the set of all measurable functions \( f \) on \([a,b]\) such that \( v_{p}(f) < \infty \). For \( 1 \leq p < \infty \) define

\[
\|f\|_{(p)} := v_{p}(f)^{1/p}.
\]

Then \( \| \cdot \|_{(p)} \) is a seminorm on \( \mathcal{W}_{p}([a,b]) \), which is called the \( p \)-variation seminorm. For \( 1 \leq p \leq \infty \) define

\[
\|f\|_{[p]} := \|f\|_{(p)} + \|f\|_{\infty},
\]

where \( \|f\|_{\infty} = \sup_{x \in [a,b]} |f(x)| \). Then \( \| \cdot \|_{[p]} \) is a norm on \( \mathcal{W}_{p}([a,b]) \), which is called the \( p \)-variation norm. The space \( (\mathcal{W}_{p}([a,b]), \| \cdot \|_{[p]} \) \) is a Banach space for \( p \geq 1 \), and \( f \in \mathcal{W}_{p} \) means that \( f \in \mathcal{W}_{p}([a,b]) \) for any \( a, b \in \mathbb{R} \), and moreover \( f \in \mathcal{W}_{p} \) is locally bounded. For these, see Dudley-Norvaiša [3] and Young [24].

**Lemma 3.2** (Dudley-Norvaiša [3]). Let \( f \in \mathcal{W}_{p}([a,b]) \) and \( g \in \mathcal{W}_{q}([a,b]) \), where \( p, q \geq 1 \) and \( \frac{1}{p} + \frac{1}{q} > 1 \). If \( f \) and \( g \) have no common discontinuities, then the Young integral (Bochner integral)

\[
\int_{a}^{b} f(x)dg(x) := \lim_{|\Delta_{n}| \to 0} \sum_{j=1}^{n} f(\xi_{j})(g(x_{j}) - g(x_{j-1}))
\]
exists, where \( \xi_j \in [x_{j-1}, x_j] \) \( (j = 1, 2, \ldots) \), \( |\Delta_n| = \max_{1 \leq j \leq n} |x_j - x_{j-1}| \) and the Love-Young inequality

\[
\left| \int_a^b f(x) dg(x) \right| \leq C_{p,q} \|f\|_{[p]} \|g\|_{(p)}
\]

holds.

**Definition 3.2.** Let \( f \) be a Borel function. We define Bochner integral

\[
\int_{\mathbb{R}} f(x) L(dx, t) = \lim_{|\Delta_n| \to 0} \sum_{j=1}^n f(x_j) (L(x_j, t) - L(x_{j-1}, t))
\]

exists in \( L^1 \) for any \( 0 \leq t \leq T \), where \( \{a = x_0 < x_1 < \cdots < x_n = b\} \) is any partition of \( [a, b] \) and \( |\Delta_n| = \max \{x_i - x_{i-1}\} \).

**Proposition 3.1.** Let \( f \) be of bounded \( p \)-variation with \( 1 \leq p < 2 \). Then

\[
\int_{\mathbb{R}} L(x, t) df(x)
\]

exists in \( L^1 \) and we have

\[
\int_{\mathbb{R}} f(x) L(dx, t) = -\int_{\mathbb{R}} L(x, t) df(x)
\]

for all \( t \geq 0 \). In particular, for \( f \in C^1(\mathbb{R}) \) we have

\[
\int_{\mathbb{R}} f(x) L(dx, t) = -\int_{\mathbb{R}} L(x, t) f'(x) dx.
\]

**Proof.** Clearly, the Bochner integral

\[
\int_{\mathbb{R}} L(x, t) df(x)
\]

exists q.s. for all \( t \geq 0 \), and

\[
\int_{\mathbb{R}} L(x, t) df(x) \in L^1,
\]

because \( x \mapsto L(x, t) \) is continuous and has a compact support for each \( 0 \leq t < \infty \). It follows that

\[
\sum_{j=1}^n f(x_{j-1}) (L(x_j, t) - L(x_{j-1}, t))
\]

\[=
\sum_{j=1}^n f(x_{j-1}) L(x_j, t) - \sum_{j=0}^{n-1} f(x_j) L(x_j, t)
\]

\[=
-\sum_{j=1}^n (f(x_j) - f(x_{j-1})) L(x_j, t)
\]

by adding some points in the partition \( \{x_i\} \) to make \( L(x_n, t) = 0 \) and \( L(x_1, t) = 0 \). This completes the proof. \( \square \)

Define the mollifier \( \theta \) by

\[
\theta(x) = \begin{cases} 
  c \exp \left( \frac{1}{(x-1)^2-1} \right), & x \in (0, 2), \\
  0, & x \notin (0, 2),
\end{cases}
\]

(3.2)
where \( c \) is a normalizing constant such that \( \int_{\mathbb{R}} \theta(x)dx = 1 \). Set \( \theta_n(x) = n\theta(nx) \). For a locally integrable function \( g(x) \) we define
\[
g_n(x) = \int_{\mathbb{R}} \theta_n(x-y)g(y)dy = \int_0^2 \theta(z)g(x - \frac{z}{n})dz, \quad n \geq 1.
\]

**Lemma 3.3.** Let \( g \) be of bounded \( p \)-variation with \( 1 \leq p < 2 \). Suppose that \( g_n \) is defined as above, then \( g_n \) is of bounded \( p \)-variation with \( 1 \leq p < 2 \) and \( g_n \in C^\infty(\mathbb{R}) \) for every \( n \geq 1 \), and moreover, the convergence
\[
\lim_{n \to \infty} \int_{\mathbb{R}} g_n(x)L(dx,t) = \int_{\mathbb{R}} g(x)L(dx,t)
\]
holds in \( L^1 \).

**Proof.** We have
\[
\left| \int_{\mathbb{R}} L(x,t)dg_n(x) - \int_{\mathbb{R}} L(x,t)dg(x) \right| = \left| \int_{\mathbb{R}} L(x,t)d\left(g_n(x) - g(x)\right) \right| \\
\leq C_{p,q}\|L(\cdot,t)\|_{[p]}\|g_n(x) - g(x)\|_{(p)}
\]
for all \( 0 \leq t < \infty \) because \( x \mapsto L(x,t) \) is continuous and has a compact support for each \( 0 \leq t < \infty \), and the lemma follows. \( \Box \)

By using the above lemmas, we immediately get an extension of Itô formula stated as follows, which is an analogue of the Bouleau-Yor type formula.

**Theorem 3.1.** Let the function \( f \) be of bounded \( p \)-variation with \( 1 \leq p < 2 \) and let \( F \) be an absolutely continuous function with derivative \( F' = f \). Then
\[
F(B_t) = F(0) + \int_0^t f(B_s)dB_s - \frac{1}{2} \int_{\mathbb{R}} f(x)L(dx,t).
\]

Recall that if \( F \) is the difference of two convex functions, then \( F \) is an absolutely continuous function with derivative of bounded variation. Thus, the Itô-Tanaka formula
\[
F(B_t) = F(0) + \int_0^t F'(B_s)dB_s + \frac{1}{2} \int_{\mathbb{R}} L(x,t)F''(dx)
\equiv F(0) + \int_0^t F'(B_s)dB_s - \frac{1}{2} \int_{\mathbb{R}} F'(x)L(dx,t)
\]
holds.

**Proof of Theorem 3.1.** For \( n \geq 1 \), we set
\[
F_n(x) := \int_{\mathbb{R}} \theta_n(y)f(x-y)dy = \int_0^2 f(x - \frac{y}{n})\theta(y)dy,
\]
where \( \theta_n, n \geq 1 \) are the mollifiers defined as above. Then \( F_n \in C^\infty(\mathbb{R}) \),
\[
F_n(B_t) = F_n(B_0) + \int_0^t F'_n(B_s)dB_s + \frac{1}{2} \int_0^t F''_n(B_s)d(B)_s
\]
and \( F'_n \) is of bounded \( p \)-variation with \( 1 \leq p < 2 \) for all \( n = 1, 2, \ldots \). Now, we prove the theorem in two steps.
**Step I.** Let $f$ be bounded uniformly in $\mathbb{R}$. Then $F_n \to F$ uniformly, as $n$ tends to infinity, because

$$|F_n(x) - F(x)| \leq \int_0^2 |F(x - \frac{y}{n}) - F(x)|\theta(y)dy \leq \sup_z |f(z)| \left( \int_0^2 y\theta(y)dy \right) \frac{1}{n}$$

for a constant $C > 0$. Moreover, we have $F_n', f \in W_p$ for all $n = 1, 2, \ldots$, and $F_n' \to f$ in $W_p$, as $n$ tends to infinity. Noting that

$$|F_n'(x) - f(x)| \leq \|F_n' - f\|_p, \quad n = 1, 2, \ldots$$

for all $x \in \mathbb{R}$, we get $F_n' \to f$ uniformly, as $n$ tends to infinity. It follows that

$$F_n(B_t) \to F(B_t), \quad \int_0^t F_n'(B_s)dB_s \to \int_0^t f(B_s)dB_s,$$

in $L^1$, as $n \to \infty$. On the other hand, we have

$$\int_0^t F_n''(B_s)d\langle B \rangle_s = \int_\mathbb{R} \mathcal{L}(x,t)F_n''(x)dx = -\int_\mathbb{R} F_n'(x)\mathcal{L}(dx,t)$$

$$\to -\int_\mathbb{R} f(x)\mathcal{L}(dx,t)$$

in $L^1$, by Lemma 3.3 and Proposition 3.1, which deduces, as $n \to \infty$

$$\int_0^t F_n'(B_s)dB_s = F_n(B_t) - F_n(0) - \frac{1}{2} \int_0^t F_n''(B_s)d\langle B \rangle_s$$

$$\to F(B_t) - F(0) + \frac{1}{2} \int_\mathbb{R} f(x)\mathcal{L}(dx,t)$$

in $L^1$, as $n \to \infty$, and the theorem follows if $f$ is bounded uniformly in $\mathbb{R}$.

**Step II.** Let $f$ be of bounded $p$-variation with $1 \leq p < 2$ and let $F$ be an absolutely continuous function with derivative $F' = f$. Then $f$ is locally bounded on $\mathbb{R}$. Consider the sets

$$\Omega_k = \left\{ \sup_{0 \leq s \leq t} |B_s| < k \right\}, \quad k = 1, 2, \ldots.$$

Let $f^{[k]} \in W_p$ be an uniformly bounded function on $\mathbb{R}$ such that $f^{[k]} = f$ on $[-k,k]$ and vanishes outside, and let $F^{[k]}$ be an absolutely continuous function with derivative $\frac{d}{dx}F^{[k]} = f^{[k]}$. Then the **Step I** implies that the formula

$$F^{[k]}(B_t) = F^{[k]}(0) + \int_0^t f^{[k]}(B_s)dB_s - \frac{1}{2} \int_\mathbb{R} f^{[k]}(x)\mathcal{L}(dx,t)$$

holds q.s. on the set $\Omega_k$ for all $k = 1, 2, \ldots$. Letting $k$ tend to infinity we deduce the desired Itô formula (3.3) if $f$ is of bounded $p$-variation with $1 \leq p < 2$. □

### 4. The quadratic covariation

In this section, we consider the quadratic covariation of $f(B)$ and $B$, where $B$ is a $G$-Brownian motion and $f$ is a Borel function on $\mathbb{R}$.

**Definition 4.1.** For a Borel function $f : \mathbb{R} \mapsto \mathbb{R}$ we define the quadratic covariation $(f(B), B)$ of $f(B)$ and $B$ as follows

$$\langle f(B), B \rangle_t := \lim_{n \to \infty} \sum_{k=0}^{n-1} (f(B_{t_{k+1}}) - f(B_{t_k}))(B_{t_{k+1}} - B_{t_k})$$
in $L^1$, where $\{t_k\}$ is a partitions of $[0,T]$ such that $\max_k\{t_k - t_{k-1}\} \to 0$ as $n \to \infty$.

**Lemma 4.1.** For $f \in C^1(\mathbb{R})$, we have

$$\langle f(B), B \rangle_t = \int_0^t f'(B_u)d(B_u).$$

**Proof.** By the Taylor’s expansion, we have

$$f(B_{t_{k+1}}) - f(B_{t_k}) = f'(B_{t_k})(B_{t_{k+1}} - B_{t_k}) + o(B_{t_{k+1}} - B_{t_k})$$

for all $k = 1, 2, \ldots$. We claim that the convergence

$$(4.1) \quad \sum_{k=0}^{n-1} \left[ o(B_{t_{k+1}} - B_{t_k}) \right] (B_{t_{k+1}} - B_{t_k}) \to 0$$

and

$$(4.2) \quad \sum_{k=0}^{n-1} f'(B_{t_k})(B_{t_{k+1}} - B_{t_k})^2 \to \int_0^t f'(B_u)d\langle B \rangle_u$$

hold in $L^1$, as $n \to \infty$. The convergence (4.1) follows from the Hölder continuity of $G$-Brownian motion. In order to prove (4.2), noting that $f'(B_t) \in L^1_G$, we get

$$(4.3) \quad \mathbb{E} \left[ f'(B_s)f'(B_{s'}) \int_s^t (B_u - B_s)dB_u \int_{s'}^{t'} (B_u - B_{s'})dB_{s'} \right] = 0$$

for $0 < s' < t' < s < t$. In fact by

$$f'(B_{s'}) \int_{s'}^{t'} (B_u - B_{s'})dB_u \in L^1_G$$

for $0 < s' < t' < s < t$, we have

$$(4.3) \quad \mathbb{E} \left[ f'(B_s)f'(B_{s'}) \int_s^t (B_u - B_s)dB_u \int_{s'}^{t'} (B_u - B_{s'})dB_u \right]$$

$$= \mathbb{E} \left[ \left( f'(B_{s'}) \int_{s'}^{t'} (B_u - B_{s'})dB_u \right)^+ \mathbb{E} \left[ \int_s^t (B_u - B_s)dB_u | \mathcal{F}_s \right] \right]$$

$$+ \left( f'(B_{s'}) \int_{s'}^{t'} (B_u - B_{s'})dB_u \right)^- \mathbb{E} \left[ - \int_s^t (B_u - B_s)dB_u | \mathcal{F}_s \right] = 0$$

for $0 < s' < t' < s < t$. On the other hand, we have

$$(B_{t_{k+1}} - B_{t_k})^2 = \langle B \rangle_{t_{k+1}} - \langle B \rangle_{t_k} + 2 \int_{t_k}^{t_{k+1}} (B_u - B_{t_k})dB_u$$
by using the identity $B_t^2 = \langle B \rangle_t + 2 \int_0^t B_s dB_s$. Combining this with (4.3), we get
\[
\hat{E} \left[ \sum_{k=0}^{n-1} f'(B_{t_k})(B_{t_{k+1}} - B_{t_k})^2 - \sum_{k=0}^{n-1} f'(B_{t_k})(\langle B \rangle_{t_{k+1}} - \langle B \rangle_{t_k})^2 \right] \\
= \hat{E} \left[ \sum_{k=0}^{n-1} 2f'(B_{t_k}) \int_{t_k}^{t_{k+1}} (B_u - B_{t_k}) dB_u \right]^2 \\
\leq 4 \sum_{k=0}^{n-1} \hat{E} \left[ f'(B_{t_k}) \int_{t_k}^{t_{k+1}} (B_u - B_{t_k}) dB_u \right]^2 \\
= 4 \sum_{k=0}^{n-1} \hat{E} \left[ f'(B_{t_k}) \right]^2 \hat{E} \left[ \left( \int_{t_k}^{t_{k+1}} (B_u - B_{t_k}) dB_u \right)^2 \right] \\
\leq 4M^2 \sum_{k=0}^{n-1} \hat{E} \left[ \left( \int_{t_k}^{t_{k+1}} (B_u - B_{t_k}) dB_u \right)^2 \right] \\
\leq 4(M\sigma)^2 \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (u - t_k) du = (M\sigma)^2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \rightarrow 0,
\]
as $n \rightarrow \infty$, where $M = \sup_{0 \leq s \leq t} \hat{E} \left[ |f'(B_s)|^2 \right]$, which deduces the convergence (4.2), and the lemma follows. \hfill \Box

As a direct consequence of the above lemma, we can rewrite the Itô’s formula as follows.

**Corollary 4.1.** If $F \in C^2(\mathbb{R})$ and $F' = f$, we have
\[
F(B_t) = F(B_0) + \int_0^t f(B_u) dB_u + \frac{1}{2} \langle f(B), B \rangle_t,
\]
and the Bouleau-Yor identity
\[
\langle f(B), B \rangle_t = -\int_\mathbb{R} f(x) \mathcal{L}(dx, t), \quad t \geq 0,
\]
holds.

**Theorem 4.1.** Let the function $x \mapsto f(x)$ be of bounded $p$-variation with $1 \leq p < 2$ and let
\[
\left| \hat{E}[f_z(B_t)f_z(B_{t'}) (B_{t'} - B_t)(B_{t'} - B_s)] \right| \\
\leq C \frac{(t - s)|t' - s'|}{t'(t - t')} \left| \hat{E}[f_z(B_t)f_z(B_{t'}) B_{t'}(B_t - B_{t'})] \right|
\]
for all $t > s > t', s' > 0$ and $z \in \mathbb{R}$, where $f_z(\cdot) = f(\cdot - z) - f(\cdot)$ for all $z \in \mathbb{R}$. Then the quadratic covariation $\langle f(B), B \rangle$ exits in $\mathbb{L}^1$, and the Bouleau-Yor identity
\[
\langle f(B), B \rangle_t = -\int_\mathbb{R} f(x) \mathcal{L}(dx, t)
\]
holds q.s., for all $t \in [0, T]$.

It is important to note that the condition (4.4) is clear and it is an identity with $C = 1$ if $B = W$ is a classical standard Brownian motion. In order to see that the condition (4.4) holds for a classical standard Brownian motion $W$, by approximating we may assume that $f$ is an infinitely differentiable function with compact support and $t > s \geq t' > s' > 0$. It
follows from the formula of integration by parts (between divergence integral and Malliavin derivative operator $D$)
\begin{align*}
E \left[ F \int_0^T u_s dW_s \right] = E \left[ \int_0^T D_s F u_s ds \right]
\end{align*}
with $F$ being a smooth random variable that
\begin{align*}
E \left[ f_z(W_t)f_z(W_{t'}) (W_t - W_s)(W_{t'} - W_{s'}) \right]
= E \left[ f_z(W_t)f_z(W_{t'}) (W_t - W_s) \int_s^{t'} dW_t \right] \\
= E \int_0^{t'} 1_{[s', t']} (u) D_u [f_z(W_t)f_z(W_{t'}) (W_t - W_s)] du \\
= (t' - s') E \left[ f_z(W_t)f_z(W_{t'}) (W_t - W_s) \right] \\
+ (t' - s') E \left[ f_z(W_t)f_z(W_{t'}) (W_{t'} - W_s) \right] \\
= (t' - s')(t - s) E \left[ f_z''(W_t)f_z(W_{t'}) \right] + (t' - s')(t - s) E \left[ f_z'(W_t)f_z'(W_{t'}) \right].
\end{align*}
For Malliavin calculus, see Nualart [14]. Let now
\begin{align*}
\varphi_{t,s}(x, y) = \frac{1}{2\pi \rho_{t,s}} \exp \left( -\frac{1}{\rho_{t,s}^2} (sx^2 - 2\mu_{t,s} xy + ty^2) \right)
\end{align*}
be the density function of $(W_t, W_s)$ with $\mu_{t,s} = E(W_t W_s) = s \wedge t = s$ and $\rho_{t,s}^2 = st - \mu_{t,s}^2 = s(t - s)$. We then have, by integration by parts,
\begin{align*}
E \left[ f_z''(W_t)f_z(W_{t'}) \right] &= \int_{\mathbb{R}^2} f_z(x)f_z(y) \frac{\partial^2}{\partial x^2} \varphi_{t,t'}(x, y) dx dy \\
&= \int_{\mathbb{R}^2} f_z(x)f_z(y) \left\{ \frac{1}{\rho_{t,t'}^2} (t'x - t'y)^2 - \frac{t'}{\rho_{t,t'}^2} \right\} \varphi_{t,t'}(x, y) dx dy
\end{align*}
\begin{align*}
E \left[ f_z'(W_t)f_z'(W_{t'}) \right] &= \int_{\mathbb{R}^2} f_z(x)f_z(y) \left\{ \frac{1}{\rho_{t,t'}^2} (ty - t'x)(t'x - t'y) + \frac{t'}{\rho_{t,t'}^2} \right\} \varphi_{t,t'}(x, y) dx dy.
\end{align*}
Combining these with the fact
\begin{align*}
(ty - t'x)(t'x - t'y) = \rho_{t,t'}^2 y(x - y) - (t')^2 (x - y)^2,
\end{align*}
we get
\begin{align*}
E \left[ f_z''(W_t)f_z(W_{t'}) + f_z'(W_t)f_z'(W_{t'}) \right]
&= \frac{1}{\rho_{t,t'}^2} \int_{\mathbb{R}^2} f_z(x)f_z(y)(x - y) y \varphi_{t,t'}(x, y) dx dy \\
&= \frac{1}{\rho_{t,t'}^2} E \left[ f_z(W_t)f_z(W_{t'}) (W_t - W_s) \right],
\end{align*}
which gives
\begin{align*}
E \left[ f_z(W_t)f_z(W_{t'}) (W_t - W_s)(W_{t'} - W_{s'}) \right]
&= (t' - s')(t - s) \frac{1}{\rho_{t,t'}^2} E \left[ f_z(W_t)f_z(W_{t'}) (W_t - W_s) \right].
\end{align*}
Thus, the assumption (4.4) is a natural condition under the sublinear expectation $\hat{E}$. 
Lemma 4.2 (Lemma 4.3 in Dudley-Norvaisa [3]). For $1 \leq p < \infty$, $f$ is of bounded $p$-variation on $[a,b]$ if and only if

$$f = g \circ h$$

for a bounded nondecreasing nonnegative function $h$ on $[a,b]$ and a function on $[h(a), h(b)]$ satisfying a Hölder condition with exponent $1/p$.

Proof of Theorem 4.1. If $f \in C^1(\mathbb{R})$, we then have by Corollary 4.1

$$\langle f(B), B \rangle_t = \int_0^t f'(B_u)d\langle B \rangle_u = -\int_{\mathbb{R}} f(x)\mathcal{L}(dx,t).$$

Let now $f \notin C^1(\mathbb{R})$. By the localization argument similar to proof of Theorem 3.1 we may assume that $f$ is uniformly bounded in the next discussion. For $n \geq 1$, we set

$$f_n(x) := \int_{\mathbb{R}} \theta_n(x - y)f(y)dy,$$

where $\theta_n$, $n \geq 1$ are the mollifiers defined by (3.2). Then $f_n \in C^\infty(\mathbb{R})$ is bounded and

(4.6) $$\langle f_n(B), B \rangle_t = -\int_{\mathbb{R}} f_n(x)\mathcal{L}(dx,t), \quad n \geq 1$$

for all $t \in [0,T]$. Consider now the double sequence

$$\alpha_{m,t}(f_n) = \sum_{k=0}^{m-1} \{f_n(B_{tk+1}) - f_n(B_{tk})\} (B_{tk+1} - B_k), \quad m, n \geq 1.$$ 

In order to prove (4.3), we claim that

$$\hat{\mathbb{E}} \left[ \left| \alpha_{m,t}(f) + \int_{\mathbb{R}} f(x)\mathcal{L}(dx,t) \right| \right] \rightarrow 0,$$

as $m \rightarrow \infty$. We have

(4.7) $$\hat{\mathbb{E}} \left[ \left| \alpha_{m,t}(f) + \int_{\mathbb{R}} f(x)\mathcal{L}(dx,t) \right| \right] \leq \hat{\mathbb{E}} \left[ |\alpha_{m,t}(f) - \alpha_{m,t}(f_n)| \right]$$

$$+ \hat{\mathbb{E}} \left[ \left| \alpha_{m,t}(f_n) + \int_{\mathbb{R}} f_n(x)\mathcal{L}(dx,t) \right| \right]$$

$$+ \hat{\mathbb{E}} \left[ \left| \int_{\mathbb{R}} f(x)\mathcal{L}(dx,t) - \int_{\mathbb{R}} f_n(x)\mathcal{L}(dx,t) \right| \right].$$

Let us estimate the three terms above. Denote $\Delta_jB = B_{t_j} - B_{t_{j-1}}$ for $j = 1, 2, \ldots, m$ and

$$g_n(y) = \int_{\mathbb{R}} [f(x - y) - f(y)] \theta_n(x)dx$$

$$= \int_0^2 \left[ f(y - \frac{x}{n}) - f(y) \right] \theta(x)dx, \quad n = 1, 2, \ldots.$$
It follows that

\[
\mathbb{E} [\alpha_{m,t}(f) - \alpha_{m,t}(f_n)]^2 = \mathbb{E} \left( \sum_{j=1}^{m} (g_n(B_{t_j}) - g_n(B_{t_{j-1}})) \Delta_j B \right)^2 \\
\leq \sum_{i<j} \mathbb{E} (g_n(B_{t_i}) - g_n(B_{t_{i-1}}))(g_n(B_{t_j}) - g_n(B_{t_{j-1}})) \Delta_j B \Delta_i B \\
+ \sum_{j=1}^{m} \mathbb{E} (g_n(B_{t_j}) - g_n(B_{t_{j-1}}))^2 (\Delta_j B)^2 \\
\leq \sum_{i<j} \mathbb{E} (g_n(B_{t_i}) - g_n(B_{t_{i-1}}))(g_n(B_{t_j}) - g_n(B_{t_{j-1}})) \Delta_j B \Delta_i B \\
+ \sum_{j=1}^{m} \mathbb{E} \left[ (g_n^2(B_{t_j}) + g_n^2(B_{t_{j-1}})) (\Delta_j B)^2 \right] \\
\equiv I + II
\]

for all \( t \geq 0 \). Denote

\[
\Delta_n(s,r,x,y) := \mathbb{E} \left[ (f(B_x - \frac{x}{n}) - f(B_s))^2 (f(B_y - \frac{y}{n}) - f(B_r))^2 \right]
\]

for any \( s,r > 0, x,y \in \mathbb{R} \) and \( n = 1, 2, \ldots \). Notice that

\[
|f(B_x - \frac{x}{n}) - f(B_s)| \leq 2 \sup_{x} |f(x)|
\]

for all \( s \geq 0 \) and \( n = 1, 2, \ldots \). We have, by the condition (4.4)

\[
|I| \leq \sum_{1 \leq i < j \leq m} \left| \mathbb{E} \left[ g_n(B_{t_i})g_n(B_{t_j})\Delta_j B \Delta_i B \right] + \mathbb{E} \left[ g_n(B_{t_i})g_n(B_{t_{i-1}})\Delta_j B \Delta_i B \right] \right|
\]

\[
\leq C \sum_{1 \leq i < j \leq m} \frac{(t_j - t_{j-1})(t_i - t_{i-1})}{\sqrt{t_i(t_j - t_i)}} \int_{0}^{2} \int_{0}^{2} \Delta_n(t_j,t_i,x,y)\theta(x)\theta(y) dx dy \\
+ C \sum_{1 \leq i < j \leq m} \frac{(t_j - t_{j-1})(t_i - t_{i-1})}{\sqrt{t_i(t_j - t_i)}} \int_{0}^{2} \int_{0}^{2} \Delta_n(t_j,t_{i-1},x,y)\theta(x)\theta(y) dx dy \\
\rightarrow C \int_{0}^{t} ds \int_{0}^{s} \frac{dr}{\sqrt{r(s-r)}} \int_{0}^{2} \int_{0}^{2} \Delta_n(s,r,x,y)\theta(x)\theta(y) dx dy
\]

and

\[
|II| \leq C \sigma^2 \sum_{j=1}^{m} (t_j - t_{j-1}) \int_{0}^{2} \Delta_n(t_j,t_j,x,x)\theta(x) dx \\
+ C \sigma^2 \sum_{j=1}^{m} (t_j - t_{j-1}) \int_{0}^{2} \mathbb{E} \left[ (f(B_{t_{j-1}} + \frac{x}{n}) - f(B_{t_{j-1}}))^2 \right] \theta(x) dx \\
\rightarrow C \sigma^2 \int_{0}^{t} ds \int_{0}^{2} \Delta_n(s,s,x,x)\theta(x) dx \\
+ C \sigma^2 \int_{0}^{t} ds \int_{0}^{2} \mathbb{E} \left[ (f(B_s + \frac{x}{n}) - f(B_s))^2 \right] \theta(x) dx,
\]

as \( m \to \infty \) for all \( n \geq 1 \). Thanks to Lemma [4.2], we see that there exists a bounded nondecreasing nonnegative function \( h \) such that

\[
|f(B_s - \frac{x}{n}) - f(B_s)| \leq C_p \left( h(B_s) - h(B_s - \frac{x}{n}) \right)^{1/p} \downarrow 0
\]
q.s., as \( n \to \infty \), which deduces
\[
\lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E} [ |\alpha_{m,t}(f) - \alpha_{m,t}(f_n)|] = 0
\]
for all \( t \geq 0 \) by Lemma 2.1 and Lebesgue’s dominated convergence theorem. Noting that
\[
\langle f_n(B), B \rangle_t = - \int_{\mathbb{R}} f_n(x) \mathcal{L}(dx,t)
\]
leads to
\[
\mathbb{E} \left[ |\alpha_{m,t}(f_n) + \int_{\mathbb{R}} f_n(x) \mathcal{L}(dx,t)\right] = \mathbb{E} [ |\alpha_{m,t}(f_n) - \langle f_n(B), B \rangle_t|] \to 0,
\]
as \( m \to \infty \), for all \( n \geq 1 \). It follows from (4.7) and Lemma 3.3 that
\[
\lim_{m \to \infty} \mathbb{E} \left[ |\alpha_{m,t}(f_n) + \int_{\mathbb{R}} f_n(x) \mathcal{L}(dx,t)\right] \leq \lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E} [ |\alpha_{m,t}(f) - \alpha_{m,t}(f_n)|] + \lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E} \left[ \left| \int_{\mathbb{R}} f(x) \mathcal{L}(dx,t) - \int_{\mathbb{R}} f_n(x) \mathcal{L}(dx,t) \right| \right] = 0,
\]
and the theorem follows. \( \square \)

According to Theorem 3.1 and Theorem 4.1, we get an extension of Itô’s formula (Föllmer-Protter-Shiryayev’s formula).

**Corollary 4.2.** Let the function \( f : \mathbb{R} \to \mathbb{R} \) be of bounded \( p \)-variation with \( 1 \leq p < 2 \) and let \( F \) be an absolutely continuous function with derivative \( F' = f \). If the condition (4.4) holds, we then have
\[
F(B_t) = F(0) + \int_0^t f(B_s)dB_s + \frac{1}{2} \langle f(B), B \rangle_t
\]
for all \( t \in [0,T] \).

## 5. Two parameter integrals of local time

In this section we turn to consider two parameter integrals
\[
\int_0^t \int_{\mathbb{R}} g(x,s) \mathcal{L}(dx,ds).
\]

We first give an extension to the Lemma 3.1.

**Lemma 5.1** (Occupation times formula). For all \( t \geq 0 \) and every bounded continuous Borel function \( \Phi \), we have
\[
\int_0^t \Phi(B_s)d\langle B \rangle_s = \int_{\mathbb{R}} dx \int_0^t \Phi(x,s) \mathcal{L}(x,ds) \quad q.s.
\]

**Proof.** Let \( \Phi(x,s) = 1_{[a,b]}(x)1_{[t_1,t_2]}(s) \) with \( a, b \in \mathbb{R} \) and \( t_1, t_2 \in [0,t] \). We have
\[
\int_0^t \Phi(B_s,s)d\langle B \rangle_s = \int_{t_1}^{t_2} 1_{[a,b]}(B_s)d\langle B \rangle_s
\]
and
\[
\int_{\mathbb{R}} dx \int_{0}^{t} \Phi(x, s) \mathcal{L}(x, ds) = \int_{a}^{b} dx \int_{t_{1}}^{t_{2}} \mathcal{L}(x, ds)
\]
\[
= \int_{a}^{b} dx [\mathcal{L}(x, t_{2}) - \mathcal{L}(x, t_{1})]
\]
\[
= \int_{a}^{b} \mathcal{L}(x, t_{2}) dx - \int_{a}^{b} \mathcal{L}(x, t_{1}) dx
\]
\[
= \int_{t_{1}}^{t_{2}} 1_{[a, b]}(B_{s}) d(B_{s}) \quad \text{q.s.}
\]
which imply that (5.2) holds. Consequently, (5.2) deduces for the simple functions
\[
\Phi_{\Delta}(x, s) = \sum_{i,j} a_{ij} 1_{[x_{i}, x_{i+1})}(x) 1_{[t_{j}, t_{j+1})}(s),
\]
where \(\{a = x_{0} < x_{1} < \cdots < x_{n} = b, 0 = t_{0} < x_{1} < \cdots < t_{n} = T\}\) is an arbitrary partition of \([a, b] \times [0, T]\).

On the other hand, for every bounded continuous Borel function \(\Phi(x, s)\) there is a sequence of bounded simple functions \(\Phi_{n}(x, s); n \geq 1\) such that \(\Phi_{n} \to \Phi\) uniformly, in \(\mathbb{R} \times \mathbb{R}_{+}\), and hence
\[
\int_{0}^{t} \Phi_{n}(B_{s}, s) d(B_{s}) \to \int_{0}^{t} \Phi(B_{s}, s) d(B_{s})
\]
in \(L^{1}\). It follows that there exists a subsequence \(\Phi_{n_{k}}, k \geq 1\) such that
\[
\int_{0}^{t} \Phi_{n_{k}}(B_{s}, s) d(B_{s}) \to \int_{0}^{t} \Phi(B_{s}, s) d(B_{s}), \quad \text{q.s.}
\]
and
\[
\int_{\mathbb{R}} dx \int_{0}^{t} \Phi_{n_{k}}(x, s) \mathcal{L}(x, ds) \to \int_{\mathbb{R}} dx \int_{0}^{t} \Phi(x, s) \mathcal{L}(x, ds), \quad \text{q.s.}
\]
which deduce
\[
\int_{0}^{t} \Phi(B_{s}, s) d(B_{s}) = \int_{\mathbb{R}} dx \int_{0}^{t} \Phi(x, s) \mathcal{L}(x, ds) \quad \text{q.s.}
\]
and the lemma follows. \(\square\)

In order to define the two parameter integrals (5.1) we use the idea from Feng-Zhao [7]. Recall that a function \((x, y) \mapsto F(x, y)\), defined on \([a, b] \times [c, d]\) is of bounded \(p\)-variation in \(x\) uniformly in \(y\), if
\[
\sup_{y \in [c, d]} \sum_{i=1}^{m} |F(x_{i}, y) - F(x_{i-1}, y)|^{p} < \infty,
\]
where \(\{a = x_{0} < x_{1} < \cdots < x_{m} = b\}\) is an arbitrary partition of \([a, b]\), and furthermore, it is of bounded \(p, q\)-variation in \((x, y)\), if
\[
\sup_{[a, b] \times [c, d]} \sum_{j=1}^{n} \sum_{i=1}^{m} |\Delta F(x_{i}, y_{j})|^{p} q^{q} < \infty,
\]
where
\[
\Delta F(x_{i}, y_{j}) = F(x_{i}, y_{j}) - F(x_{i-1}, y_{j}) - F(x_{i}, y_{j-1}) + F(x_{i-1}, y_{j-1}),
\]
and \(\{a = x_{0} < x_{1} < \cdots < x_{m} = b; c = y_{0} < y_{1} < \cdots < y_{n} = d\}\) is arbitrary partition of \([a, b] \times [c, d]\).
Let now $G: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function of bounded $q_1$-variation in $x$ uniformly in $y$, and be of bounded $q_2$-variation in $y$ uniformly in $x$; the continuous function $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be of bounded $p,q$-variation in $(x,y)$, where $p, q, q_1, q_2 \geq 1$. Then the Young integral (see Theorem 3.1 in Feng-Zhao [7])

$$\int_a^b \int_c^d G(x,s)F(dx,dy) := \lim_{\Delta_{m,n} \to 0} \sum_{i,j=1}^n \sum_{i=1}^m G(x_{i-1},y_{j-1})\Delta F(x_i,y_j)$$

is well defined, where $\Delta_{m,n} = \max_{i,j} \{|(x_i,y_j) - (x_{i-1},y_{j-1})|\}$, if there exits two monotone increasing functions $\rho: \mathbb{R} \to \mathbb{R}_+$ and $\sigma: \mathbb{R} \to \mathbb{R}_+$ such that

$$\sum_{n,m} o\left(\frac{1}{n^{1/q_1}}\right)\sigma\left(\frac{1}{n^{1/q_2}}\right)\frac{1}{n^{1/p}m^{1/pq}} < \infty.$$ 

More works for two-parameter $p,q$-variation path integrals can be fund in Feng-Zhao [7]. By taking $\rho(u) = u^\alpha$ and $\sigma(u) = u^{1-\alpha}$ with $\alpha \in (0,1)$, one can prove the following.

**Proposition 5.1.** Let the continuous function $F: \mathbb{R} \times [0,t] \to \mathbb{R}$ be of bounded $p,q$-variation in $(x,t)$, and of bounded $\gamma$-variation in $x$ uniformly in $t$. Assume that the conditions

$$1 \leq \gamma < 2, \quad pq < q + \frac{1}{2}, \quad p, q \geq 1 \tag{5.5}$$

holds, then the Young integral (Bochner integral) of two parameters

$$\int_0^t \int_0^t \mathcal{L}(x,s)F(dx,ds)$$

exists q.s., and

$$\int_0^t \int_0^t F(x,s)\mathcal{L}(dx,ds) = \int_0^t \int_0^t \mathcal{L}(x,s)F(dx,ds) - \int_0^t \mathcal{L}(x,t)F(dx,t)$$

Consider the smooth approximation of the function $F$

$$F_n(x,s) := \int_0^2 \int_0^2 \theta(r)\theta(z)F(x - \frac{r}{n}, s - \frac{z}{n})drdz, \quad n \geq 1, \tag{5.6}$$

where $\theta$ are the mollifiers defined in (3.2). Then under the conditions of Proposition 5.1 we have

$$\int_0^t \int_0^t F_n(x,s)\mathcal{L}(dx,ds) \to \int_0^t \int_0^t F(x,s)\mathcal{L}(dx,ds) \quad q.s.$$ 

as $n \to \infty$.

**Corollary 5.1.** Under the conditions of Proposition 5.1 we have

$$\int_0^t \int_0^t F(x,s)\mathcal{L}(dx,ds) = -\int_0^t dx \int_0^t \frac{\partial}{\partial x} F(x,s)\mathcal{L}(dx,ds)$$

provided $F \in C^{1,1}(\mathbb{R} \times [0,T])$.

**Theorem 5.1.** $F \in C^{1,1}(\mathbb{R} \times \mathbb{R}_+)$. Suppose that the function $(x,t) \mapsto \frac{\partial}{\partial x} F(x,t)$ is of bounded $p,q$-variation in $(x,t)$, and of bounded $\gamma$-variation in $x$ uniformly in $t$, where these parameters $p,q,\gamma$ satisfy the condition (5.5). Assume that

$$(x,t) \mapsto F(x,t), \quad (x,t) \mapsto \frac{\partial}{\partial x} F(x,t), \quad (x,t) \mapsto \frac{\partial}{\partial t} F(x,t)$$
are uniformly continuous in $\mathbb{R}^2$. Then the following Itô formula holds:

$$F(B_t, t) = F(0, 0) + \int_0^t \frac{\partial}{\partial t} F(B_s, s) ds + \int_0^t \frac{\partial}{\partial x} F(B_s, s) dB_s - \frac{1}{2} \int_0^t \int_\mathbb{R} \frac{\partial}{\partial x} F(x, s) \mathcal{L}(dx, ds).$$

Proof. Let $F_n$ be defined in (5.6) for $n \geq 1$. Then $F_n \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+)$ and $\frac{\partial}{\partial x} F_n, n \geq 1$ are of bounded $p, q$-variation in $(x, t)$, and of bounded $\gamma$-variation in $x$ uniformly in $t$, and we have

$$F_n(B_t, t) = F_n(0, 0) + \int_0^t \frac{\partial}{\partial t} F_n(B_s, s) ds + \int_0^t \frac{\partial}{\partial x} F_n(B_s, s) dB_s - \frac{1}{2} \int_0^t \int_\mathbb{R} \frac{\partial}{\partial x} F_n(x, s) \mathcal{L}(dx, ds),$$

for all $n \geq 1$. On the other hand, we have

$$\mathbb{E} \left| \int_0^t \frac{\partial}{\partial t} F_n(B_s, s) ds - \int_0^t \frac{\partial}{\partial t} F_n(B_s, s) ds \right| \leq \int_0^t \int_0^2 \theta(r) \theta(z) \mathbb{E} \left| \frac{\partial}{\partial t} F(B_s - \frac{r}{n}, s - \frac{z}{n}) - \frac{\partial}{\partial t} F(B_s, s) \right| drdz \rightarrow 0,$$

and

$$\mathbb{E} \left| \int_0^t \frac{\partial}{\partial x} F_n(B_s, s) dB_s - \int_0^t \frac{\partial}{\partial x} F_n(B_s, s) dB_s \right|^2 \leq C\sigma^2 \int_0^t \left( \int_0^2 \theta(r) \theta(z) \mathbb{E} \left| \frac{\partial}{\partial x} F(B_s - \frac{r}{n}, s - \frac{z}{n}) - \frac{\partial}{\partial x} F(B_s, s) \right| drdz \right)^2 ds \rightarrow 0,$$

as $n \rightarrow \infty$. Moreover, the occupation times formula (5.2) and Corollary 5.1 imply that

$$\int_0^t \frac{\partial^2}{\partial x^2} F_n(B_s, s) dB(s)_s = \int_\mathbb{R} \int_0^t \frac{\partial^2}{\partial x^2} F_n(x, s) \mathcal{L}(x, ds)$$

$$= - \int_\mathbb{R} \int_0^t \frac{\partial}{\partial x} F_n(x, s) \mathcal{L}(dx, ds)$$

$$\rightarrow - \int_\mathbb{R} \int_0^t \frac{\partial}{\partial x} F(x, s) \mathcal{L}(dx, ds)$$

q.s., as $n \rightarrow \infty$. Combining this with $F_n \rightarrow F$ uniformly, as $n \rightarrow \infty$, we obtain the theorem. \qed

As the end of this paper, we consider the quadratic covariation $\langle f(B, \cdot), B \rangle$ of $f(B, \cdot)$ and $B$ defined by

$$\langle f(B, \cdot), B \rangle := \lim_{n \rightarrow \infty} \sum_{j=1}^n \{f(B_{t_j}, t_j) - f(B_{t_{j-1}}, t_{j-1})\}(B_{t_j} - B_{t_{j-1}})$$

in $\mathbb{L}^1$, where $f : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ is a Borel function and $\{t_j\}$ is a partitions of $[0, T]$ such that $|\Delta_n| := \max\{t_j - t_{j-1}\} \rightarrow 0$, as $n \rightarrow \infty$. 
Theorem 5.2. Let \( f \in C(\mathbb{R} \times \mathbb{R}_+) \) be of bounded \( p,q \)-variation in \((x,t)\), and of bounded \( \gamma \)-variation in \( x \) uniformly in \( t \), and let the parameters \( p,q,\gamma \) satisfy the condition (5.5). Assume that

\[
\left| E[f_z(B_t,t)f_z(B_{t'},t')(B_t-B_s)(B_{t'}-B_{s'})] \right| \leq C \frac{(t-s)|t'-s'|}{t'(t'-t)} \left| E[f_z(B_t,t)f_z(B_{t'},t')B_t(B_{t'}-B_{t'})] \right|
\]

for all \( t > s \geq t', s' > 0 \) and \( z \in \mathbb{R} \), where \( f_z(\cdot) = f(\cdot + z) - f(\cdot) \) for all \( z \in \mathbb{R} \), if the function \( f \) is uniformly continuous in \( \mathbb{R}^2 \), then the quadratic covariation \( \langle f(B,\cdot), B \rangle \) exits in \( L^1 \), and the Bouleau-Yor identity

\[
\langle f(B,\cdot), B \rangle_t = - \int_0^t \int_{\mathbb{R}} f(x,s) \mathcal{L}(dx,ds)
\]

holds q.s. for all \( t \in [0,T] \).

Proof. Let \( f \in C^{1,1}(\mathbb{R} \times \mathbb{R}_+) \). Similar to the proof of Lemma 4.1 we can get

\[
\langle f(B,\cdot), B \rangle_t = \int_0^t \frac{\partial}{\partial x} f(B_s,s) dB_s
\]

for all \( t \in [0,T] \). It follows from the occupation times formula (5.2) that

\[
\langle f(B,\cdot), B \rangle_t = \int_{\mathbb{R}} dx \int_0^t \frac{\partial}{\partial x} f(x,s) \mathcal{L}(x,ds)
\]

\[
= - \int_0^t \int_{\mathbb{R}} f(x,s) \mathcal{L}'(dx,ds)
\]

for all \( t \in [0,T] \).

Let now \( f \notin C^{2,1}(\mathbb{R} \times \mathbb{R}_+) \). For \( n \geq 1 \) we define \( f_n \) as follows

\[
f_n(x,s) := \int_0^2 \int_0^2 \theta(r)\theta(z) f\left(x - \frac{r}{n}, s - \frac{z}{n}\right) dr dz, \quad n \geq 1,
\]

where \( \theta \) is the mollifier defined in (3.2). Then \( f_n \in C^{1,1}(\mathbb{R} \times [0,T]) \) and we have

\[
\langle f_n(B,\cdot), B \rangle_t = - \int_0^t \int_{\mathbb{R}} f_n(x,s) \mathcal{L}(dx,ds)
\]

for all \( t \in [0,T] \). Thus, similar to the proof of Theorem 4.1 we can obtain the identity (5.8). \( \square \)

According to the above theorems, we get an analogue of Föllmer-Protter-Shiryayev’s formula (see [9]).

Corollary 5.2. Let \( F \in C^{1,1}(\mathbb{R} \times \mathbb{R}_+) \) and let the conditions in Theorem 5.1 and Theorem 5.2 hold. Then

\[
F(B_t,t) = F(0,0) + \int_0^t \frac{\partial}{\partial t} F(B_s,s) ds + \int_0^t f(B_s,s) dB_s + \frac{1}{2} \langle f(B,\cdot), B \rangle_t
\]

for all \( t \in [0,T] \).
6. Appendix: Quadratic Variation

In this appendix, we give the other representation of quadratic covariation \( \langle B, B \rangle \). From the properties of quadratic variation of \( G \)-Brownian motion we have

\[
\frac{1}{\varepsilon} \mathbb{E} \int_0^\varepsilon \langle B \rangle_r \, dr \longrightarrow 0,
\]
as \( \varepsilon \) tends to 0.

**Lemma 6.1.** For all \( t \geq 0 \), we have

\[
\frac{1}{\varepsilon} \int_0^t (\langle B \rangle_{s+\varepsilon} - \langle B \rangle_s) \, ds \longrightarrow \langle B \rangle_t
\]
in \( L^1 \), as \( \varepsilon \) tends to 0.

**Proof.** Notice that

\[
\frac{1}{\varepsilon} \int_0^t (\langle B \rangle_{s+\varepsilon} - \langle B \rangle_s) \, ds = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \langle B \rangle_r \, dr - \frac{1}{\varepsilon} \int_0^\varepsilon \langle B \rangle_s \, ds
\]

\[
= \frac{1}{\varepsilon} \int_0^\varepsilon \langle B \rangle_{t+r} \, dr - \frac{1}{\varepsilon} \int_0^\varepsilon \langle B \rangle_r \, dr.
\]

We get

\[
\mathbb{E} \left| \frac{1}{\varepsilon} \int_0^t (\langle B \rangle_{s+\varepsilon} - \langle B \rangle_s) \, ds - \langle B \rangle_t \right| \leq \mathbb{E} \left| \frac{1}{\varepsilon} \int_0^\varepsilon (\langle B \rangle_{t+r} - \langle B \rangle_t) \, dr \right| + \mathbb{E} \left| \frac{1}{\varepsilon} \int_0^\varepsilon \langle B \rangle_r \, dr \right|
\]

\[
\longrightarrow 0
\]
for all \( t \geq 0 \), as \( \varepsilon \) tends to 0. \( \square \)

From the proof and the definition of the integral with respect to \( \langle B \rangle \) we get

\[
(6.1) \quad \frac{1}{\varepsilon} \int_0^t g(B_s)(\langle B \rangle_{s+\varepsilon} - \langle B \rangle_s) \, ds \longrightarrow \int_0^t g(B_s) \, d\langle B \rangle_s
\]
in \( L^1 \), as \( \varepsilon \) tends to 0, for all \( g \in C(\mathbb{R}) \).

**Proposition 6.1.** Let \( g \in C(\mathbb{R}) \). Then, for all \( t \geq 0 \), we have

\[
\frac{1}{\varepsilon} \int_0^t g(B_s)(B_{s+\varepsilon} - B_s)^2 \, ds \longrightarrow \int_0^t g(B_s) \, d\langle B \rangle_s
\]
in \( L^1 \), as \( \varepsilon \) tends to 0.

**Proof.** It is enough to show that the following convergence holds:

\[
\frac{1}{\varepsilon} \int_0^t (B_{s+\varepsilon} - B_s)^2 \, ds \longrightarrow \int_0^t d\langle B \rangle_s = \langle B \rangle_t
\]
in \( L^1 \), as \( \varepsilon \) tends to 0. Clearly, we have

\[
(B_{s+\varepsilon} - B_s)^2 = \langle B \rangle_{s+\varepsilon} - \langle B \rangle_s + 2 \int_s^{s+\varepsilon} (B_r - B_s) \, dB_r
\]
for all \( s \geq 0 \). Thus, it is enough to show

\[
(6.2) \quad \mathbb{E} \left| \frac{1}{\varepsilon} \int_0^t ds \int_s^{s+\varepsilon} (B_r - B_s) \, dB_r \right| \longrightarrow 0
\]
for all $t \geq 0$, as $\varepsilon$ tends to 0, by (6.1). To end this we have, by (4.3),

$$\frac{1}{\varepsilon^2} \mathbb{E} \left[ \int_0^t ds \int_s^{s+\varepsilon} (B_t - B_s) dB_t \right]^2$$

$$= \frac{1}{\varepsilon^2} \mathbb{E} \left[ \int_0^t \int_0^{l+\varepsilon} (B_u - B_l) dB_u \right] \int_l^{l+\varepsilon} (B_v - B_l) dB_v ds dl$$

$$\leq \frac{1}{\varepsilon^2} \left[ \int_0^t ds \int_s^{s+\varepsilon} \mathbb{E} \left[ \int_s^{s+\varepsilon} (B_u - B_l) dB_u \right] \right] \int_l^{l+\varepsilon} (B_v - B_l) dB_v dl$$

$$+ \frac{1}{\varepsilon^2} \left[ \int_0^t ds \int_s^{s+\varepsilon} \mathbb{E} \int_s^{s+\varepsilon} (B_u - B_l) dB_u \int_l^{l+\varepsilon} (B_v - B_l) dB_v dl \right].$$

Notice that

$$\mathbb{E} \left[ \int_s^{s+\varepsilon} (B_u - B_l) dB_u \right] \int_l^{l+\varepsilon} (B_v - B_l) dB_v$$

$$\leq \mathbb{E} \left( \int_l^{l+\varepsilon} (B_u - B_l) dB_u \right)^2 + \mathbb{E} \left( \int_s^{l+\varepsilon} (B_u - B_s) dB_u \right) \mathbb{E} \left( B_s - B_l \right) \mathbb{E} \left( B_{l+\varepsilon} - B_s \right)$$

$$\leq C\sigma^2 \int_s^{l+\varepsilon} (u-s) du = C\sigma^2 (l + \varepsilon - s)^2 + C\sigma^2 (l + \varepsilon - s)^2 \frac{s}{2} (s - l)^{\frac{3}{2}}$$

for $l < s < l + \varepsilon < s + \varepsilon$, and

$$\mathbb{E} \left( \int_s^{s+\varepsilon} (B_u - B_l) dB_u \right) \int_l^{l+\varepsilon} (B_v - B_l) dB_v$$

$$\leq C\sigma^2 \int_l^{s+\varepsilon} (u-s) du = C\sigma^2 (s + \varepsilon - l)^2 + C\sigma^2 (s + \varepsilon - l)^2 \frac{l}{2} (l - s)^{\frac{3}{2}}$$

for $s < l < s + \varepsilon < l + \varepsilon$. We get

$$\frac{1}{\varepsilon^2} \mathbb{E} \left| \int_0^t ds \int_s^{s+\varepsilon} (B_t - B_s) dB_t \right|^2 \leq C\sigma^2 t \varepsilon \to 0,$$

as $\varepsilon$ tends to 0, and the lemma follows. \qed

**Corollary 6.1.** We have

$$\frac{1}{\varepsilon} \int_0^t (B_{s+\varepsilon} - B_s)^2 ds \to \langle B \rangle_t$$

in $L^1$, as $\varepsilon \to 0$.

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