Light Spanner and Monotone Tree

Hao-Hsiang Hung

Dept. of Math & CS, Emory University, {hhung2}@mathcs.emory.edu

Abstract. In approximation algorithm design, light spanners has applications in graph-metric problems such as metric TSP (the traveling salesman problem) [8] and others [3]. We have developed an efficient algorithm in [9] for light spanners in bounded pathwidth graphs, based on an intermediate data structure called monotone tree. In this paper, we extended the results to include bounded catwidth graphs.

1 Introduction

Light Spanners. Suppose $G$ is a connected undirected graph where each edge $e$ has length (or weight) $w(e) \geq 0$. Let $d_G(u, v)$ denote the length of the shortest path between vertices $u$ and $v$. Suppose $G'$ is a spanning subgraph of $G$, where each edge of $G'$ inherits its weight from $G$; evidently $d_G(u, v) \leq d_{G'}(u, v)$. Fix $\epsilon > 0$. If $d_{G'}(u, v) \leq (1 + \epsilon) \cdot d_G(u, v)$ (for all $u, v$), then we say that $G'$ is a $(1 + \epsilon)$-spanner of $G$. In other words, the metric $d_{G'}$ closely approximates the metric $d_G$.

Let $w(G')$ denote the total edge weight of $G'$, and let MST($G$) denote the minimum weight of a spanning tree in $G$. We are interested in conditions on $G$ that guarantee the existence of a $(1 + \epsilon)$-spanner $G'$ with bounded $w(G')$/MST($G$). More formally, suppose $G$ is a family of undirected graphs. We say $G$ has light spanners if the following holds: for every $\epsilon > 0$ there is a bound $f(\epsilon)$, so that for any edge-weighted $G$ from $G$, $G$ has a $(1 + \epsilon)$-spanner $G'$ with $w(G') \leq f(\epsilon) \cdot$ MST($G$). Less formally, we say such a $G'$ is a light spanner for $G$. Note $f(\epsilon)$ depends on $\epsilon$ and $G$, but not on $G$ or $w$.

We know that if a graph family has unbounded clique minors, then it does not have light spanners; just consider a clique with uniform edge weights. We conjecture the converse [3]:

Conjecture 1. Any graph family with a forbidden minor has light spanners.

Our motivation comes from metrical optimization problems such as metric TSP and any other problems [4] need light spanner as an algorithmic tool. We are given an edge-weighted graph $G$, and we seek a cyclic order of its vertices with minimum total distance as measured by $d_G$. Equivalently, we want a minimum weight cyclic tour in $G$ visiting each vertex at least once. Let OPT($G$) denote the minimum tour weight; it is well known that MST($G$) $\leq$ OPT($G$) $\leq 2 \cdot$ MST($G$). We seek an approximation scheme: an algorithm which takes as inputs the weighted graph $G$ and $\epsilon > 0$, and which outputs a tour with weight at most $(1 + \epsilon) \cdot$ OPT($G$).
The problem is MAX SNP-hard \cite{14}, so we consider approximation schemes where the input graph $G$ is restricted to some graph family $\mathcal{G}$ (e.g., planar graphs). We would like a PTAS (an approximation scheme running in time $O(n^\omega(\epsilon))$, for some function $g$), or better yet an EPTAS (an approximation scheme running in time $O(g(\epsilon) \cdot n^c)$, where the constant $c$ is independent of $\epsilon$).

Suppose $\mathcal{G}$ is a graph family, and that for any $G \in \mathcal{G}$ we can compute a $(1 + \epsilon)$-spanner $G'$ with $w(G') \leq f(\epsilon) \cdot \text{MST}(G)$. Then we may attempt to design a PTAS (or an EPTAS) for the metric TSP on $\mathcal{G}$, as follows:

1. On input $G$ and $\epsilon$, first compute $G'$, a $(1 + \epsilon/2)$-spanner of $G$, with weight at most $f(\epsilon/2) \cdot \text{MST}(G)$.
2. Choose $\delta = (\epsilon/2)/f(\epsilon/2)$. Apply some algorithm finding a tour in $G'$ with cost at most $\text{OPT}(G') + \delta \cdot w(G')$.
3. Return the tour, with cost at most $(1 + \epsilon/2) \cdot \text{OPT}(G) + \delta \cdot f(\epsilon/2) \cdot \text{MST}(G) \leq (1 + \epsilon) \cdot \text{OPT}(G)$. (For other metric optimization problems, it may be less trivial to lift a solution from $G'$ back to $G$.)

Step 2 looks like the original problem, except now we allow an error term proportional to $w(G')$ instead of $\text{OPT}(G')$. This approach has already succeeded for planar graphs \cite{2,12} and bounded genus graphs \cite{7,8}.

A recent result of Demaine et al. \cite{6, Thm. 2} implies a PTAS for metric TSP when $\mathcal{G}$ is any graph class with a fixed forbidden minor. Since we do not know that $\mathcal{G}$ has light spanners, for step 1 they substitute a looser result \cite{10}, finding a $(1 + \epsilon)$-spanner $G'$ with weight $O((\log n)/\epsilon) \cdot \text{MST}(G)$ (the hidden constant depending on $\mathcal{G}$). In step 2 their algorithm runs in time $2^{O(1/\delta + \log n)}$. Their $1/\delta$ is $O(w(G')/(\text{MST}(G) \cdot \epsilon)) = O((\log n)/\epsilon^2)$, so their running time is $n^{O(1/\epsilon^2)}$.

If we could compute light spanners for $\mathcal{G}$, then $\delta$ would improve to something independent of $n$, and this would yield an EPTAS for metric TSP on $\mathcal{G}$. (Or alternatively, it would yield an approximation scheme allowing $\epsilon$ to slowly approach zero, as long as $1/\delta$ stays $O(\log n)$.)

**Our Work.** A series of previous work of Grigni et al. has discovered light spanners in bounded genus graphs \cite{8}, apex graphs \cite{10}, and bounded pathwidth graphs \cite{9}. Obviously, we know a list of graph families with bounded treewidth constraints can also been handled: see \cite{13, Ch. 7} for reference.

We know a intermediate data structure called **monotone tree** could handle bounded pathwidth graphs well; however, there are simple bounded treewidth graphs which do not have any monotone tree \cite{9} but has simple charging schemes. What we are interested in is the strength of the monotone tree.

In this paper, we study the monotone tree in bounded catwidth graphs. Given a graph $G$ with pathwidth $pw(G)$, treewidth $tw(G)$, and catwidth $catw(G)$, we know that $tw(G) \leq catw(G) \leq pw(G)$ \cite{15}. We handle bounded catwidth graphs based on the fact that for any graph $H$ of catwidth $k$, there is a graph of pathwidth $k$ with $(p,q)$-flaps (to be defined later) stitched and it contains $H$ as a subgraph.

We prove the following theorem in Section 4.
Theorem 1. Bounded catwidth graphs have light spanners, computable by a greedy algorithm.

The study of catwidth of graphs origins from the memory allocation problems in dynamic programming, simply just because the size of the table required does not grow as fast as bounded treewidth graphs. Habib et al. [11] shows that there is a linear time algorithm for recognizing the catwidth of the graph. Additionally, we know metric TSP in bounded catwidth graphs could be solved in polynomial time [5], based on the fact that bounded catwidth graphs are subclasses of bounded treewidth graphs, and metric TSP is fixed-parameter-tractable in bounded treewidth graphs. See Section 5 for some further remarks.

2 Preliminaries

2.1 Charging Schemes

In order to exhibit light spanners in a weight-independent way, we use charging schemes [10]. (We use the notion called “0-schemes” in [10], not the more general “ε-schemes” required for apex graphs.) Suppose each edge of graph G can hold some quantity of charge, initially zero. A detour is an edge e ∈ E and a path P such that e + P is a simple cycle in G. For each detour (e, P) we introduce a variable \( x_{(e,P)} \geq 0 \). Each \( x_{(e,P)} \) describes a charging move: it subtracts \( x_{(e,P)} \) units of charge from edge e, and adds \( x_{(e,P)} \) units of charge to each edge of P. When \( x_{(e,P)} > 0 \), we say “e charges P”.

Given graph G, a spanning tree T, and a number v, a charging scheme from G to T of value v is an assignment of nonnegative values to the \( x_{(e,P)} \) variables (i.e., a fractional sum of detours) meeting the three conditions listed below. Here out(e) denotes the total charge subtracted from edge e, in(e) denotes the total charge added to e (as part of various detour paths), and net(e) = in(e) − out(e) is the total charge on e after all the moves are done:

1. \( \text{out}(e) \geq 1 \) for all \( e \in G - T \),
2. \( \text{net}(e) \leq 0 \) for all \( e \in G - T \),
3. \( \text{net}(e) \leq v \) for all \( e \in T \).

Note “\( e \in G - T \)” means e is an edge of G but not T. As we’ll see in Theorem 2, charging schemes imply light spanners.

Definition 1. An acyclic scheme is a charging scheme with two additional properties:

4. If edge e charges some path, then \( e \in G - T \).
5. There is an ordering of the edges such that whenever edge \( e_1 \) charges a path containing edge \( e_2 \), \( e_1 \) precedes \( e_2 \).

For example, planar graphs have integral acyclic schemes of value \( v = 2 \) [11].

Definition 2. Suppose we have detours \((e_1, P_1)\) and \((e_2, P_2)\), with \( e_2 \in P_1 \) and \( e_1 \notin P_2 \). Their shortcut is the detour \((e_1, P')\), where \( P' \) is the path derived from \( P_1 \) by replacing \( e_2 \) with \( P_2 \), and then reducing that walk to a simple path.
Lemma 1. Suppose we have an acyclic scheme of value \( v \) from \( G \) to \( T \), and an edge \( e \) in \( G - T \). Then there is an acyclic scheme of value \( v \) from \( G - e \) to \( T \).

Proof. Let \( e_2 = e \). While \( \text{in}(e_2) \) is positive, we find some \( e_1 \) charging a path \( P_1 \) containing \( e_2 \). Since \( \text{net}(e_2) \leq 0 \), \( e_2 \) also charges some path \( P_2 \). \( P_2 \) cannot contain \( e_1 \), since the scheme is acyclic. Let \( \alpha = \min(x(e_1, P_1), x(e_2, P_2)) \). Now reduce both \( x(e_1, P_1) \) and \( x(e_2, P_2) \) by \( \alpha \), and increase \( x(e_1, P'_1) \) (their shortcut) by \( \alpha \). After this change all the conditions are still satisfied, except possibly for condition (1) at \( e_2 \). Repeat until \( \text{in}(e_2) \) reaches zero. Finally remove \( e_2 \) and any remaining charges out of \( e_2 \). \( \square \)

Theorem 2. Suppose \( G \) is a graph with spanning tree \( T \), and we have an acyclic scheme from \( G \) to \( T \) of value \( v \). Then for any \( \epsilon > 0 \), and for any non-negative edge-weighting \( w \) on \( G \), a simple greedy algorithm finds a \((1 + \epsilon)\)-spanner \( G' \) in \( G \) containing \( T \), with total weight \( w(G') \leq (1 + \frac{v}{\epsilon}) \cdot w(T) \).

We use the following greedy algorithm of Althöfer et al. [1], modified to force the edges of \( T \) into \( G' \):

\[
\text{Spanner}(G, T, 1 + \epsilon):
\begin{align*}
G' &= T \\
&\text{for each edge } e \in G - T, \text{ in non-decreasing } w(e) \text{ order} \\
&\quad \text{if } (1 + \epsilon) \cdot w(e) < d_{G'}(e) \text{ then} \\
&\quad \quad \text{add edge } e \text{ to } G' \\
&\text{return } G'
\end{align*}
\]

The proof of Theorem 2 is a variant of previous arguments by LP duality [8, 10], based on Lemma 1 that we have an acyclic scheme from \( G' \) to \( T \) of value \( v \). We omit the detail here.

2.2 Treewidth, Pathwidth, Catwidth and Monotone Trees

Suppose \( G = (V,E) \) is a graph, \( T \) is a tree, and \( B = (B_i)_{i \in T} \) is a collection of subsets of \( V \) (bags) indexed by vertices \( i \) (identical copies) in \( T \). We call the pair \((T, B)\) a tree decomposition of \( G \) if the following conditions hold: (1) \( \bigcup_{i \in T} B_i = V \); (2) for every \( e = \{u,v\} \in E \), there is at least one bag \( B_i \) where \( u \in B_i \) and \( v \in B_i \); (3) for every \( v \in V \), the collection of \( B_i \) containing \( v \) is connected (an interval) in \( T \). The treewidth of the decomposition is the maximum bag size minus one, and the treewidth of \( G \) is the minimum treewidth of any tree decomposition of \( G \).

If \( T \) is a path (that is, \((P, B)\) instead of \((T, B)\) where \( P \) stands for a path), then we call it a path decomposition of \( G \) (and of course, the pathwidth of \( G \) is the minimum width of any path decomposition of \( G \)). Additionally, if \( T \) is a caterpillar (a path with added leaves, \((C, B)\) where \( C \) stands for a caterpillar) then we call it a caterpillar decomposition of \( G \), with ‘catwidth’ defined in a similar way.

How to represent a bounded pathwidth graph? Given \((P, B)\), we may lay out \( P \) on the line, and regard \( G \) as a subgraph of an interval graph. That is, for each
we have a line interval $I_v$ (corresponding to an interval in $P$), and we have $I_u \cap I_v \neq \emptyset$ whenever $\{u, v\} \in E$, and at most $k + 1$ intervals overlap at any point of the line. For convenience we may eliminate ties, so that all the interval endpoints are distinct. In particular, let $\text{left}(v)$ denote the leftmost point of $I_v$.

Next, we define the immediate data structure for helping the analysis later. Suppose $T$ is a rooted tree in $G$ (here we abuse the notation of $T$ a little bit without confusion). We say $T$ is monotone if for every vertex $v$ in $T$ with parent $p$, we have $\text{left}(p) < \text{left}(v)$. When $T$ is a path rooted at an endpoint, we say it is a monotone path. In particular if $T$ is a monotone spanning tree in $G$, then from any vertex $v$, we can find a monotone path in $T$ from $v$ to the root of $T$ (the vertex with the leftmost interval). For this process, it is convenient to imagine that edges connect intervals at their leftmost intersection point.

2.3 $(p, q)$-flap

We first introduce parameters to the graph structure as follows. A $k$-clique is a clique of size $k$. A $k$-leaf is the union of one external vertex $v$ attached to a $k$-clique, with additional edges connecting $v$ to all the vertices of this clique. A $k$-tree is a graph which can be constructed as follows: starting from a $k$-clique, repeatedly adding $k$-leaves to the existent graph. A $k$-path is a graph which can be constructed by repeatedly adding $k$-leaves to a proper $k$-path such that the neighbor of each $k$-leaf is a separator of the original proper $k$-path.

A $(p, q)$-flap is a partition $(P, Q)$ of vertices from a clique (in a graph $G$) of size $p + q$ such that (1) $|P| = p$; (2) $|Q| = q$; and (3) $P$ separates $Q$ from the rest of $G$. We also define the attachment of a $(p, q)$-flap to a graph $G$ as follows: first identify a clique $P$ of size $p$ in $G$, then add an external clique $Q$ of size $q$ with additional edges $(u, v)$ between all $u \in P$ and all $v \in Q$. Note that a $k$-leaf is actually a $(k, 1)$-flap.

A $k$-caterpillar is a $k$-path with attached $(p, q)$-flaps such that $p + q = k + 1$, in fact, it is also a $k$-tree. It is known that a graph $G$ has catwidth at most $k$ if and only if $G$ is a partial $k$-caterpillar [15].

3 Surgery for $k$-caterpillars

We are given $\epsilon > 0$, a connected edge-weighted graph $G$ with $n$ vertices, and an interval representation $\{I_v\}$ of $G$ with pathwidth $k$, with $(p, q)$-flaps attached (that is, a $k$-caterpillar). We want to find a $(1 + \epsilon)$-spanner $G'$ in $G$ of low weight. First we apply some reductions to simplify $G$ in the following subsections. In each of the following subsections, the first paragraph discusses the $k$-path (of bounded pathwidth) alone, and the second paragraph discusses the attachment of the flaps.

3.1 Nice Path Decomposition
We first assume that each pair of consecutive bags (as vertex sets) differ by only one vertex. This can be enforced by an argument similar to the construction of nice tree-decompositions \[13\]: if two consecutive bags differ on \(m \geq 2\) vertices, we introduce \(m - 1\) intermediate bags, in such a way that each pair differs on only one vertex, and we do not increase the maximum bag size. This does not modify \(G\) at all.

Because a \((p,q)\)-flap is attached to a \(q\)-clique of the \(k\)-path, and by nice path decomposition there should be one bag containing all vertices of a \(q\)-clique (if the last vertex added in one bag but some vertices of the clique are removed then it is impossible to form a \(q\)-clique), therefore we simply attach each one of them to a corresponding bag (one bag may have many flaps attached).

### 3.2 Bounded Degree Assumption

We may assume each vertex appears in \(O(k)\) bags, and so the maximum degree of \(G\) is \(O(k)\). To enforce this, we copy the bags of \(G\) from left to right. After each group of \(k\) original bags, ending with a bag \(B\), we insert \(|B|\) “replacer” bags, each of which replaces one vertex \(v \in B\) with a copy \(v'\), connected to \(v\) by an edge of length zero. This ends with a bag \(B'\), where every vertex \(v \in B\) has been replaced by a copy \(v' \in B'\). We continue in this way (using the copies in place of the originals) across the entire path decomposition. If we aren’t careful the pathwidth may increase by one, but this does not matter for our asymptotic results. The original graph is obtained by contracting a set \(S\) of weight-zero edges in the modified graph. So given a spanner \(G'\) in this modified graph, we may contract \(S\) in \(G' \cup S\) to recover a spanner (of no greater weight) in the original.

The \(O(k)\) maximum degree assumption does not introduce any difficulty to the representation of the flaps at all because the degree of each attached vertex is also bounded by \(k\). Therefore even if a vertex appears in multiple \(q\)-cliques and has to be separated into different identical copies by this assumption, we can attach these \((p,q)\)-leaves individually according to which \(q\)-clique they attach to.

### 3.3 Completion Assumption

We may assume that \(G\) is completed; that is, it contains all edges allowed by its overlapping intervals (in other words: we have a clique in each bag, \(G\) is an interval graph). For each absent edge \(e = \{u,v\}\), we simply add it with weight \(w(e)\) equal to the shortest path length \(d_G(u,v)\). This does not change \(d_G\) at all. Given a spanner \(G'\) in the completed graph, we recover a spanner in the original graph by replacing each completion edge by the corresponding shortest path, and apparently this assumption provides an upper bound of the charges towards the edges in \(T\).

Since we attach each flap to a really ‘complete’ subgraph of the original graph, we can limit the charging moves introduced by the new additional \(p \cdot q\) edges within the subgraph induced by these \(k + 1\) vertices of the clique.
4 Our Approach

We start by proving the main theorem as follows (assume all the reductions in Section 3 are applied).

Proof (of Theorem 1). We assume all the above reductions have been applied: the input graph \( G \) is completed (an interval graph), each bag introduces at most one vertex, and each vertex has degree \( O(k) \).

By Lemma 2 (below), we compute a monotone spanning tree \( T \) with \( w(T) = O(k^2) \cdot \text{MST}(G) \). We handle the charges from the \( k \)-paths and the flaps individually. By Lemma 3 (below), we exhibit an acyclic charging scheme from \( G \) to \( T \) of value \( v = O(k) \). By Lemma 4 (below), we deal with charges from the flaps to \( T \) of value \( v = O(k) \). Finally we apply the greedy algorithm, which computes a \((1+\epsilon)\)-spanner \( G' \). By Theorem 2 \( w(G') \leq (1+\epsilon) \cdot w(T) = O(k^3/\epsilon) \cdot \text{MST}(G) \).

\[ \square \]

Lemma 2. Given a weighted, bounded catwidth graph \( G \) is completed under shortest path metric. There is a monotone spanning tree \( T \) inside of the \( k \)-path part with \( w(T) \leq O(k^2) \cdot \text{MST}(G) \).

Proof. Choose a minimum spanning tree \( T^* \), so \( w(T^*) = \text{MST}(G) \). Let \( I_l \) and \( I_r \) be the leftmost and rightmost intervals. Let \( P_1 \) be a shortest path from \( I_l \) to \( I_r \); since \( G \) is completed, we may assume \( P_1 \) is monotone, as in Figure 1. Note \( w(P_1) \leq w(T^*) \).

Consider the components \( T_1^*, T_2^*, ..., T_m^* \) of \( T^* - V(P_1) \). Let \( e_i \) be an edge connecting the leftmost point of \( T_i^* \) to a vertex of \( P_1 \) (it exists by completion). For each \( T_i^* \), we recursively compute a monotone spanning tree \( T_i \) of \( G[V(T_i^*)] \).

Finally, \( T = P_1 \cup \bigcup_i (T_i \cup e_i) \).

We also have to consider the vertices from the \((p, q)\)-flaps. Given a \((p, q)\)-flap, we consider the edges of MST of \( G \) in the flap. The edges might be divided into \( j \) components \((1 \leq j \leq n, \text{and the connection paths are at } G - P - Q)\), and we denote them as \( c_1, \ldots, c_j \). Within each component if the removal of its \( Q \) side will partition it into sub-components of MST then we connect ends of the detour paths of the MST induced in its \( P \) side, with the weights of these completion edges setup as the weights of the shortest path distances inside of the MST.

It is clear that \( T \) is monotone (although we allow some twist paths inside of the same bag by flaps), but we must account for the total weight of \( w(T) \). For each component \( T_i^* \), let \( f_i \) be an edge of \( T^* \) connecting \( T_i^* \) to \( P_1 \) (there must be at least one). By triangle inequality, we see \( w(e_i) \) is at most \( w(T_i^*) + w(f_i) + w(P_{1,i}) \), where \( P_{1,i} \) is a sub-path of \( P_1 \) from the endpoint of \( e_i \) to the endpoint of \( f_i \). Note the \( f_i \)'s and \( T_i^* \)'s are disjoint parts of \( T^* \), but the sub-paths may overlap inside \( P_1 \).

An edge \( e \in P_1 \) appears in at most \( k-1 \) of the \( P_{1,i} \) sub-paths, since each sub-path witnesses another vertex (from \( T_i^* \)) that must appear in the bag with \( e \). So \( \sum_i w(e_i) \leq \sum_i[w(f_i) + w(T_i^*) + w(P_{1,i})] \leq w(T^*) + (k-1)w(P_1) \leq k \cdot w(T^*) \). Since \( w(T) \leq O(k \cdot w(T^*)) \), \( \sum_i w(T_i^*) \leq w(T^*) \), a simple depth-\( k \) recursion finishes our bound.

\[ \square \]
Remark: we do not have to compute $T$ as in Lemma 2; it suffices to use any light enough monotone spanning tree. A natural choice is to let $T$ be the lightest monotone spanning tree, which we compute as follows. Start with just the root (in the leftmost bag), and grow the tree in a left-to-right scan of the bags: each time a bag $B$ introduces a new vertex $v$, add an edge connecting $v$ to its nearest neighbor in $B$ (which is already in $T$).

In the completed $G$ (please refer to 3), a triangle move is a charging move where a non-tree edge $e$ charges a path $P$ of length two, where at most one edge of $P$ is not in $T$. We now define $T^{(2)}$, a graph whose edges represent triangle moves. Each vertex $jk$ of $T^{(2)}$ corresponds to an edge $\{j,k\}$ in $G$. We also represent the vertex $jk$ by the interval $I_{jk} = I_j \cap I_k$. To define the edges of $T^{(2)}$, we first define a parent for each vertex $jk$. If $\{j,k\}$ is an edge of $T$, then $jk$ has no parent. Otherwise, suppose left($j$) < left($k$) (else swap them), and let $i$ be the parent of $k$ in $T$; $i$ must exist since $k$ is not the root. Note $\{i,j,k\}$ is a triangle in $G$. Now we say the parent of $jk$ is $ij$, and we add the edge $\{ij,jk\}$ in $T^{(2)}$. Note left($ij$) < left($jk$), so these parent links are acyclic. Thus $T^{(2)}$ is a forest, with each component rooted at a vertex corresponding to an edge of $T$.

**Lemma 3.** Suppose a weighted, bounded catwidth graph $G$ is completed under shortest path metric. There is an acyclic charging scheme from the $k$-path of $G$ to $T$ of value $O(k)$.

**Proof.** Recall $T^{(2)}$ is a forest. Fix a component $C$ of $T^{(2)}$; it is a tree, rooted at a vertex $r$ corresponding to an edge of $T$, and that is the only such vertex in $C$. Consider a directed Euler tour of $C$, traversing each edge twice. Delete each tour edge out of $r$, so we get a list of directed paths, each of the form

$$e_1 \rightarrow e_2 \rightarrow \cdots e_m \rightarrow r$$

where each vertex $e_i$ corresponds to some edge of $G - T$. Since $C$ is a tree, these paths are vertex disjoint (except at $r$). However, a vertex may appear more than once on the same path; call an appearance $e_i$ a repeat if the same vertex appeared earlier on the path. Let $\mathcal{P}$ be the collection of all these paths, from all components of $T^{(2)}$. 
We now propose a charging scheme (which fails to be acyclic). Recall how we constructed edges in $T^{(2)}$: we connect each vertex $jk$ (corresponding to an edge of $G - T$) to its parent $ij$. If a path in $P$ traverses this edge in the direction $jk \rightarrow ij$, we add the triangle move where edge $\{j, k\}$ charges one unit to path $j - i - k$. If a path traverses this edge in the other direction $ij \rightarrow jk$ (so $ij$ is not a tree edge), we add the triangle move where edge $\{i, j\}$ charges one unit to path $i - k - j$. In either direction, the tree edge $\{i, k\}$ is charged.

We now verify the proposed charging scheme with the properties in Section 2.1. For an edge $e \in G - T$, the corresponding vertex appears at least once on a path, and it has at least as many out-edges as in-edges, so our proposed scheme has properties (1) and (2). For an edge $e \in T$, we must bound the number of times it is charged. Since $G$ has maximum degree $O(k)$, $e$ appears in $O(k)$ distinct triangles, and it is charged at most twice per triangle (this includes the charges it receives in its role as $r$). So if we choose $v = O(k)$, we have property (3). Also there are no charges out of tree edges, so we have property (4).

However, this charging scheme does not have property (5); if a vertex (corresponding to an edge $e \in G - T$) has a repeat appearance on its path, then there is no consistent way to order the edges. To fix this, we eliminate all “repeat” appearances using shortcuts. That is, whenever we have a sequence $e_1 \rightarrow e_2 \rightarrow e_3$ where $e_2$ is a repeat, we shortcut out $e_2$. Note that this can be repeated. For example if we have a sequence $e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_4 \rightarrow e_5$, corresponding to four triangle moves, it is possible to shortcut out $e_2, e_3, e_4$ (in any order), and the result is a single charge from $e_1$ to a path containing $e_5$ (the rest of the charged path is all tree edges). After eliminating all repeats by shortcuts, we get the desired acyclic scheme.

Lemma 4. Suppose a weighted, bounded catwidth graph $G$ is completed under shortest path metric. There is an acyclic charging scheme from $(p, q)$-flaps to $T$ of value $O(k)$.

Proof. Inside of each MST component in a flap (in particular the $Q$ part), the charges are localized with value $O(k)$ because the detour paths are in the flap, too. In addition, we have to consider influx charges from the $k$-path, and by Lemma 3 we know it introduced charges of value $O(k)$.

For edges crossing different MST components in a flap, the outflux charges towards the $k$-path are also of value $O(k)$. Consider two components $c_i$ and $c_j$ ($i \neq j$, $|c_i| \leq k$, and $|c_j| \leq k$). For each $u \in c_i$, we group up all $|c_j|$ edges (with one end $u$) with an arbitrary acyclic charging ordering, and let the last edge charge toward the $k$-path. Since $|c_i| \leq k$, the value is $O(k)$.

5 Conclusion and Further Work

In this paper, we extended the technique that we used for finding light spanners in bounded pathwidth graphs towards bounded catwidth graphs, with additional charges from ‘local’ structures called $(p, q)$-flaps. However, a different approach might require in the progress towards Conjecture 1.
We have discussed the major difficulty in handling bounded treewidth graphs in [9]: we have no control over the MST topology inside of the bounded treewidth graphs, and a trimming might introduce too heavy overload (relative to the total weight of the MST).

We are investigating known subclasses of bounded treewidth graphs, and considering introduce some constraints such as bounded degree of the original graphs, the diameter of the decomposition tree, etc.

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