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A conjecture on $B$-groups

Serge Bouc

Abstract: In this note, I propose the following conjecture: a finite group $G$ is nilpotent if and only if its largest quotient $B$-group $\beta(G)$ is nilpotent. I give a proof of this conjecture under the additional assumption that $G$ be solvable. I also show that this conjecture is equivalent to the following: the kernel of restrictions to nilpotent subgroups is a biset-subfunctor of the Burnside functor.

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1. Introduction

In the study of the lattice of biset-subfunctors of the Burnside functor $KB$ over a field $\mathbb{K}$ of characteristic 0 (cf. Section 7.2 of [2], or Chapter 5 of [4]), a special class of finite groups, called $B$-groups, plays an important role. It was shown in particular in [2] Proposition 9 (see also [4] Theorem 5.4.11) that any finite group $G$ admits a largest quotient in this class, well-defined up to isomorphism, and denoted by $\beta(G)$. A few properties of $B$-groups were proved in [2], some of which will be recalled in this paper, but almost no progress was made since, until the following theorem proved recently by Mélanie Baumann ([1]): when $p$ is a prime number, recall that a finite group $G$ is called cyclic modulo $p$ (or $p$-hypo-elementary) if the group $G/O_p(G)$ is cyclic.

1.1. Theorem: [M. Baumann] Let $p$ be a prime number, and $G$ be a finite group. Then $G$ is cyclic modulo $p$ if and only if $\beta(G)$ is cyclic modulo $p$.

In this note I propose the following similar looking conjecture:

Conjecture A: Let $G$ be a finite group. Then $\beta(G)$ is nilpotent if and only if $G$ is nilpotent.

1.2. Remark: It was shown in [3] (Proposition 14) that the nilpotent $B$-groups are the groups of the form $C_n \times C_n$, where $C_n$ is a cyclic group of square free order $n$.

After recalling the basic definitions and properties of $B$-groups, I will give a proof of Conjecture A under the additional assumption that $G$ be solvable.
2. \textit{B}-groups

Let $K$ be a field of characteristic 0. Let $G$ be a finite group, let $s_G$ denote the set of subgroups of $G$, and let $[s_G]$ be a set of representatives of $G$-conjugacy classes on $s_G$.

Denote by $KB(G)$ the Burnside algebra of $G$ over $K$. It is a split semisimple commutative $K$-algebra, with two natural $K$-bases: the first one consists of the isomorphism classes of transitive $G$-sets, i.e. the set $\{[G/H]|H \in [s_G]\}$. The second one consists of the primitive idempotents of $KB(G)$, i.e. the set $\{e^G_H| H \in [s_G]\}$. The transition matrix from the first basis to the second one has been described explicitly by Gluck (5) and Yoshida (7), as follows

$$e^G_H = \frac{1}{N_G(H)} \sum_{X \leq H} |X| \mu(X, H) [G/X] ,$$

where $\mu$ is the Möbius function of the poset of subgroups of $G$.

The correspondence $G \mapsto KB(G)$ is a biset functor: when $G$ and $H$ are finite groups, and $U$ is a finite $(H, G)$-biset, the functor $S \mapsto U \times_G S$ from the category of finite $G$-sets to the category of finite $H$-sets induces a map $KB(U) : KB(G) \rightarrow KB(H)$, which is well behaved with respect to disjoint union and composition of bisets.

This involves in a single formalism the usual operations of restriction, induction, inflation, and transport by isomorphism between the corresponding Burnside groups. It also involves the less usual operation of \textit{deflation}: when $N \trianglelefteq G$, the deflation homomorphism $Def^G_{G/N} : B(G) \rightarrow B(G/N)$ corresponds to the $(G/N, G)$-biset $G/N$, and it induced by the functor $S \mapsto N\setminus S$ from $G$-sets to $(G/N)$-sets.

These elementary operations can be expressed explicitly in each of the above bases. In particular ([4] Theorem 5.2.4):

\begin{enumerate}
\item \textbf{1. Theorem :} Let $G$ be a finite group.
\begin{enumerate}
\item If $H$ is a proper subgroup of $G$, then $\text{Res}^G_H e^G_G = 0$.
\item When $N \trianglelefteq G$, set

$$m_{G,N} = \frac{1}{|G|} \sum_{X \leq G, XN= G} |X| \mu(X, G) .$$

Then $\text{Def}^G_{G/N} e^G_G = m_{G,N} e^G_{G/N}$.
\end{enumerate}
\end{enumerate}
This leads to the notion of $B$-group: the group $G$ is a $B$-group if any proper deflation of $e_G^G$ is equal to 0. In other words:

2.2. Definition: The finite group $G$ is called a $B$-group if $m_{G,N} = 0$ for any non-trivial normal subgroup $N$ of $G$.

2.3. Notation: When $G$ is a finite group, and $N \leq G$ is maximal such that $m_{G,N} \neq 0$, set $\beta(G) = G/N$.

There may be several normal subgroups $N$ with the required properties, but the group $G/N$ does not depend on the choice of $N$, up to isomorphism. More precisely ([4] Theorem 5.4.11):

2.4. Theorem: Let $G$ be a finite group.
   1. The group $\beta(G)$ is a $B$-group.
   2. If a $B$-group $H$ is isomorphic to a quotient of $G$, then $H$ is isomorphic to a quotient of $\beta(G)$.
   3. Let $M \leq G$. The following conditions are equivalent:
      (a) $m_{G,M} \neq 0$.
      (b) The group $\beta(G)$ is isomorphic to a quotient of $G/M$.
      (c) $\beta(G) \cong \beta(G/M)$.

2.5. Proposition: Let $G$ be a finite group.
   1. The group $G$ is a $B$-group if and only if $m_{G,N} = 0$ for any minimal (non-trivial) normal subgroup of $G$.
   2. Let $N$ be a minimal (non-trivial) normal subgroup of $G$. If $N$ is abelian, then
      $$m_{G,N} = 1 - \frac{|K_G(N)|}{|N|},$$
      where $K_G(N)$ is the set of complements of $N$ in $G$.

Proof: Assertion 1 follows from the transitivity of deflations. Assertion 2 is Proposition 5.6.4 of [4].
3. Proof of Conjecture A in the solvable case

3.1. Theorem: Let $G$ be a solvable finite group. Then $\beta(G)$ is nilpotent if and only if $G$ is nilpotent.

Proof: If $G$ is nilpotent, then $\beta(G)$ is nilpotent, for it is a quotient of $G$. The converse follows from an induction argument on the order of $G$: assume that if $G'$ is a finite solvable group of order $|G'| < |G|$, and if $\beta(G')$ is nilpotent, then $G'$ is nilpotent. Assume that $\beta(G)$ is nilpotent, and let $N$ be a non-trivial normal subgroup of $G$. Since $\beta(G/N)$ is a quotient of $\beta(G)$, it is nilpotent. Hence $G/N$ is nilpotent. In particular, if $Z(G) \neq 1$, then $G/Z(G)$ is nilpotent, hence $G$ is nilpotent. So we can assume that $Z(G) = 1$.

Now suppose that $M$ and $N$ are non trivial normal subgroups of $G$, such that $M \cap N = 1$. Then $G$ is nilpotent: indeed, the group $G/(M \cap N)$, isomorphic to $G$, maps injectively into $(G/M) \times (G/N)$, which is nilpotent.

It follows that we can assume that $G$ has a unique (non trivial) minimal normal subgroup $N$. Since $G$ is solvable, the group $N$ is elementary abelian, isomorphic to $(C_p)^k$, for some prime number $p$ and some integer $k \geq 1$. Let $Q$ be a Sylow $p$-subgroup of $G$. Then $Q \geq N$. Since the group $G/N$ is nilpotent, and since $Q/N$ is a Sylow $p$-subgroup of $G/N$, it follows that $Q/N \leq G/N$, i.e. $Q \leq G$.

Now $N$ is a non trivial normal subgroup of $Q$, thus $N \cap Z(Q) \neq 1$. But $N \cap Z(Q)$ is a normal subgroup of $G$, and by minimality of $N$, it follows that $N \leq Z(Q)$.

The group $G$ splits as a semidirect product $G = Q \rtimes H$, where $H$ is a (nilpotent) $p'$-subgroup of $G$. The group $H$ acts on the $p$-group $Q$, thus $Q = C_Q(H)[H, Q]$ (by [6] Theorem 3.5 Chapter 5).

Since $G/N$ is nilpotent, it follows that $G/N \cong (Q/N) \times H$. It follows that $[H, Q] \leq N$. Thus $Q = C_Q(H)N$. Now $N \cap C_Q(H)$ is centralized by $H$, and by $Q$, since $N \leq Z(Q)$. Thus $N \cap C_Q(H) \leq Z(G) = 1$, and it follows that $Q = N \times C_Q(H)$. Then $C_Q(H)$ is normalized by $Q$, and centralized by $H$. Thus $C_Q(H) \leq G$, and as $N \cap C_Q(H) = 1$, it follows that $C_Q(H) = 1$, thus $N = Q$.

But now $G = N \rtimes H$, where $N \cong (C_p)^k$, and $H$ is a $p'$-group. Since $N$ is minimal normal in $G$, it follows that $H$ acts irreducibly on $N$, and that $H$ is a maximal subgroup of $G$. Since $H$ is not normal in $G$ (as $N$ is the only minimal normal subgroup of $G$, and $N \nleq H$), it follows that $N_G(H) = H$. Finally, since $H$ is a $p'$-group, all the complements of $N$ in $G$ are conjugate, hence $|K_G(N)| = |G : N_G(H)| = |N|$. Thus $m_{G,N} = 1 - \frac{|K_G(N)|}{|N|} = 0$. Hence $G$ is a $B$-group, thus $G \cong \beta(G)$ is nilpotent. \(\blacksquare\)
3.2. **Remark** : Actually, in the situation of the end of the proof, the group $G$ is trivial: indeed, it is a nilpotent $B$-group, hence isomorphic to $C_n \times C_n$, where $n$ is a square free integer. As $G$ has a unique minimal normal subgroup by assumption, the only possibility is $n = 1$.

4. **Comments**

The following conjecture doesn’t mention $B$-groups:

**Conjecture B** : For any group $G$, let $\nu_G$ denote the restriction map

$$
\nu_G = \prod_{H \in \mathcal{N}(G)} \text{Res}_H^G : B(G) \to \prod_{H \in \mathcal{N}(G)} B(H),
$$

where $\mathcal{N}(G)$ is the set of nilpotent subgroups of $G$.

Then the correspondence $G \mapsto L(G) = \text{Ker} \nu_G$ is a biset subfunctor of $B$.

Still:

**4.1. Theorem** : Conjecture B is equivalent to Conjecture A.

**Proof** : Since $B(G)$ is a free $\mathbb{Z}$-module, it maps injectively in $\mathbb{K}B(G)$. Let $u \in B(G)$. Then $u$ can be written

$$
u_G = \left[ \sum_{H \in \mathcal{N}(G)} |u^H| e^G_H \right] \in \mathbb{K}B(G).$$

in $\mathbb{K}B(G)$. Thus $u \in L(G)$ if and only if $|u^H| = 0$ for any $H \in \mathcal{N}(G)$.

Suppose that Conjecture A holds. Proving Conjecture B amounts to proving that $L$ is invariant under the elementary biset operations of induction, restriction, inflation, deflation, and transport by isomorphism.

The latter case is clear: if $\varphi : G \to G'$ is a group isomorphism, then $\text{Iso}(\varphi)(L(G)) \subseteq L(G')$.

Now let $G$ be a group, and let $K$ be a subgroup of $G$. As nilpotent subgroups of $K$ are nilpotent subgroups of $G$, the transitivity of restrictions implies that $\text{Res}_K^G L(G) \subseteq L(K)$. Conversely, if $u \in L(K)$ and $H \in \mathcal{N}(G)$, then by the Mackey formula

$$
\text{Res}_H^G \text{Ind}_K^G u = \sum_{g \in [H \cap G/K]} \text{Ind}_H^G c_g \text{Res}_K^{H \cap G} u = 0,
$$

5
(where $c_g$ denote conjugation by $G$), since $H^g \cap K \in \mathcal{N}(K)$. It follows that $\text{Ind}^G_K L(K) \subseteq L(G)$.

Suppose now that $N \trianglelefteq G$, and that $u \in L(G/N)$. Then for any $H \in \mathcal{N}(G)$

$$\text{Res}^G_H \text{Inf}^G_G L(K) \subseteq L(G).$$

Finally, in the same situation, let $u \in L(G)$, and $K/N \in \mathcal{N}(G/N)$. Then

$$\text{Res}^G_{K/N} \text{Def}^G_{G/N} v = \text{Def}^K_{K/N} \text{Res}^G_K v .$$

Moreover

$$\text{Res}^G_K v = \sum_{X \in [s_K]} |v^X| e^K_X ,$$

and $|v^X| = 0$ for $X \in \mathcal{N}(X)$. Since moreover

$$e^K_X = \frac{1}{|N_K(X) : X|} \text{Ind}^K_X e^X_X ,$$

it follows that

$$\text{Res}^G_{K/N} \text{Def}^G_{G/N} v = \sum_{X \in [s_K]} |v^X| \frac{|N_K(X) : X|}{|N_K(X) : X|} \text{Def}^K_{K/N} \text{Ind}^K_X e^X_X$$

$$= \sum_{X \in [s_K]} |v^X| \text{Ind}^K_{X/N} \text{Ind}^X_{X/N} \text{Def}^X_{X/N} e^X_X$$

$$= \sum_{X \in [s_K]} |v^X| m_{X,X \cap N} \text{Ind}^K_{X/N} \text{Ind}^X_{X/N} e^X_{X/X \cap N}$$

$$= \sum_{X \in [s_K]} |v^X| m_{X,X \cap N} \text{Ind}^K_{X/N} e^X_{X/N} .$$

If $X \leq K$ is such that $m_{X,X \cap N} \neq 0$, then $\beta(X) \cong \beta(X/X \cap N) \cong \beta(XN/N)$. The group $XN/N$ is a subgroup of $K/N$, hence it is nilpotent, hence $\beta(X) \cong \beta(XN/N)$ is nilpotent. If Conjecture A is true, then $X$ is nilpotent, hence $|v^X| = 0$. It follows that $\text{Res}^G_{K/N} \text{Def}^G_{G/N} v = 0$, hence $\text{Def}^G_{G/N} L(G) \leq L(G/N)$.

Observe that one can still conclude that $\text{Res}^G_{K/N} \text{Def}^G_{G/N} v = 0$ without assuming Conjecture A, in the case where $N$ is solvable: indeed in this case, the group $K$ is solvable (as $N$ is solvable and $K/N$ is nilpotent), so $X$ is solvable, and one can conclude by Theorem 3.1.

Conversely, assume that Conjecture B holds, and let $G$ be a finite group. Then $|G| e^G_G$ is an element of $B(G)$, whose restrictions to all proper subgroups
of $G$ are 0. If $G$ is not nilpotent, then $|G|e_G^G \in L(G)$. Hence for any normal subgroup $N$ of $G$, the element

$$\text{Def}_{G/N}^G|G|e_G^G = m_{G,N}|G|e_{G/N}^{G/N}$$

is in $L(G/N)$. If $G/N \cong \beta(G)$, then $m_{G,N} \neq 0$, hence $|G|e_{G/N}^{G/N} \in L(G/N)$.

Since it is a non-zero element, it follows that $G/N \notin N(G/N)$, i.e. that $G/N \cong \beta(G)$ is not nilpotent. Hence Conjecture A holds.

4.2. Remark : The above proof shows that the correspondence $G \mapsto L(G)$ is a biset functor on the full subcategory of the biset category consisting of solvable groups. Actually, it proves a little more : the correspondence $G \mapsto L(G)$ is a biset functor on the category of all finite groups, if we only allow as morphisms those bisets for which left stabilizers are solvable (or equivalently, if we only allow deflations by solvable normal subgroups).

4.3. Remark : Let $G$ be a minimal counterexample to Conjecture A. Then $G$ is non-solvable, and as above $G$ has a unique minimal normal subgroup $N$, non-central in $G$. Thus $N \cong S^k$, where $S$ is a non-abelian simple group, and $C_G(N) = 1$. The group $H = G/N$ is nilpotent, and it is the largest solvable quotient of $G$. In particular $\beta(G)$ is a quotient of $H$, hence of $\beta(H)$. Since $\beta(G)$ cannot be a $p$-group (for otherwise $G$ would be cyclic modulo $p$, by Theorem 1.1, hence solvable, hence nilpotent by Theorem 3.1), it follows that $H$ is not $p$-elementary for any prime $p$ (that is, there are at least two different primes $p$ such that the Sylow $p$-subgroups of $H$ are non cyclic) : indeed, if $P$ is a $p$-group, then $\beta(P) = 1$ if $P$ is cyclic, and $\beta(P) = C_p \times C_p$ otherwise (cf. Remark 1.2).

Since $\Phi(G)$ is nilpotent, it follows that $\Phi(G) = 1$. Let $X$ be a minimal subgroup of $G$ such that $XN = G$. Then $X \cap N \leq \Phi(X)$, thus $m_{X,X \cap N} = 1$. Hence $\beta(X) \cong \beta(X/X \cap N) \cong \beta(G)$ is nilpotent. Moreover $X < G$ (as $N \notin \Phi(G) = 1$), so $X$ is nilpotent.

4.4. Remark : In view of Conjecture A and Theorem 3.1, Jacques Thévenaz has proposed the following :

**Conjecture :** Let $G$ be a finite group. Then $\beta(G)$ is solvable if and only if $G$ is solvable.

This conjecture implies Conjecture A, by Theorem 3.1. A straightforward modification of the proof of Theorem 4.1 shows that this conjecture is equivalent to saying that the correspondence sending a finite group $G$ to the
kernel of the restriction map

\[ \rho_G = \prod_{H \in \mathcal{R}(G)} \text{Res}^G_H : B(G) \to \prod_{H \in \mathcal{R}(G)} B(H), \]

where \( \mathcal{R}(G) \) is the set of solvable subgroups of \( G \), is a biset functor.

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