Generators of Picard modular groups

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Abstract
In this paper, we extend the method in [FFLP] to obtain the generators of the Picard modular groups $\text{PU}(2, 1; \mathcal{O}_d)$ with $d = 3, 7, 11$.

2000 Mathematics Subject Classification. 30M05, 22E40, 32M15.

Keywords: Picard modular groups, Complex hyperbolic space

1. Introduction
Picard modular group $\text{PU}(2, 1; \mathcal{O}_d)$ is the subgroup of $\text{PU}(2, 1)$ with entries in $\mathcal{O}_d$, where $\mathcal{O}_d$ is the ring of algebraic integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$ for any positive square-free integer $d$. If $d \equiv 1, 2 \pmod{4}$, then $\mathcal{O}_d = \mathbb{Z}[\sqrt{d}]$, and if $d \equiv 3 \pmod{4}$ then $\mathcal{O}_d = \mathbb{Z}[\frac{1+\sqrt{1+d}}{2}]$. The Picard modular groups $\text{PU}(2, 1; \mathcal{O}_d)$ are the simplest arithmetic lattices in $\text{PU}(2, 1)$.

In [FP], Falbel and Parker studied the group $\text{PU}(2, 1; \mathbb{Z}[\omega])$, where $\omega$ is a cube root of unity. They constructed a fundamental domain for the action of $\text{PU}(2, 1; \mathbb{Z}[\omega])$ on complex hyperbolic space $H^2$. Moreover, they gave a generator system and the corresponding presentation.

It is well known that the modular group $\text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\{\pm I\}$ is generated by the transformations $z \mapsto z + 1$ and $z \mapsto -\frac{1}{z}$. Motivated by the statement of $\text{PSL}(2, 1; \mathbb{Z})$, in [KPS], A. Kleinschmidt and D. Persson asked if there is a simple description of $\text{SU}(2, 1; \mathbb{Z}[i])$ in terms of generators. In [FFLP], they proved that the Gauss Picard modular group $\text{SU}(2, 1; \mathbb{Z}[i])$ can be generated by four transformations, two Heisenberg translations, a rotation and an involution. It means that they gave a positive answer for the question raised by A. Kleinschmidt and D. Persson in [KPS].

In this paper, we extended the method in [FFLP] to the Picard modular groups $\text{PU}(2, 1; \mathcal{O}_d)$ with $d = 3, 7, 11$. As $d \equiv 3 \pmod{4}$ when $d = 3, 7, 11$, the elements of the ring $\mathcal{O}_d$ can be described as $\mathcal{O}_d = \mathbb{Z}[-\frac{1+i\sqrt{d}}{2}]$, where the ring $\mathbb{Z}[-\frac{1+i\sqrt{2}}{2}]$ equals to the ring $\mathbb{Z}[\frac{1+\sqrt{2}}{2}]$. Let $\omega_d = \frac{-1+i\sqrt{d}}{2}$, then the Picard modular groups can be denoted as $\text{PU}(2, 1; \mathbb{Z}[\omega_d])$. We get the following results.

Theorem 1. The Picard modular group $\text{PU}(2, 1; \mathbb{Z}[\omega_3])$ with $\omega_3 = \frac{-1+i\sqrt{3}}{2}$ is generated by the Heisenberg translations

$$N_{(\omega_3, \sqrt{3})} = \begin{pmatrix} 1 & -\bar{\omega}_3 & \omega_3 \\ 0 & 1 & \omega_3 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_{(1, \sqrt{3})} = \begin{pmatrix} 1 & -1 & \omega_3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

the rotation

$$M_{-\omega_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\omega_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the involution

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$
Theorem 2. The Picard modular group $\text{PU}(2, 1; \mathbb{Z}[\omega_7])$ with $\omega_7 = \frac{-1+i\sqrt{7}}{2}$ is generated by the Heisenberg translations

$$N_{(\omega_7, 0)} = \begin{pmatrix} 1 & -\overline{\omega_7} & -1 \\ 0 & 1 & \omega_7 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_{(1, \sqrt{7})} = \begin{pmatrix} 1 & -1 & \omega_7 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

the rotation

$$M_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the involution

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Theorem 3. The Picard modular group $\text{PU}(2, 1; \mathbb{Z}[\omega_{11}])$ with $\omega_{11} = \frac{-1+i\sqrt{11}}{2}$ is generated by the Heisenberg translations

$$N_{(\omega_{11}, \sqrt{11})} = \begin{pmatrix} 1 & -\overline{\omega_{11}} & -1 + \omega_{11} \\ 0 & 1 & \omega_{11} \\ 0 & 0 & 1 \end{pmatrix}, \quad N_{(1, \sqrt{11})} = \begin{pmatrix} 1 & -1 & \omega_{11} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

the rotation

$$M_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the involution

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Similar to the modular group $\text{PSL}(2, \mathbb{Z})$, by using the generators of Picard modular group, for which we can construct a fundamental domain. Such as the Gauss Picard modular group $\text{PU}(2, 1; \mathbb{Z}[i])$, in [FFP], they gave a construction of a fundamental domain.

2. Preliminaries

Let $\mathbb{C}^{2,1}$ denote the 3 dimension complex vector space $\mathbb{C}^3$ equipped with the Hermitian form

$$\langle z, w \rangle = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1,$$

where $z = (z_1, z_2, z_3)^t$ and $w = (w_1, w_2, w_3)^t$. The vector $x^t$ stands for the transposition of vector $x$. Consider the subspaces of $\mathbb{C}^{2,1}$:

$$V_- = \{ z \in \mathbb{C}^{2,1} | \langle z, z \rangle < 0 \};$$

$$V_0 = \{ z \in \mathbb{C}^{2,1} - \{0\} | \langle z, z \rangle = 0 \}.$$

Complex hyperbolic space $\mathbb{H}_x^2$ is defined to be the complex projective subspace $\mathbb{P}(V_-)$ equipped with the Bergman metric, where $\mathbb{P} : \mathbb{C}^{2,1} - \{0\} \to \mathbb{C}P^2$ is the canonical projection onto the complex projective space. We consider the complex hyperbolic space $\mathbb{H}_x^2$ as the Siegel domain $\{ z = (z_1, z_2) \in \mathbb{C}^2 : 2\mathcal{R}(z_1) + |z_2|^2 < 0 \}$. The boundary of complex hyperbolic space is $\partial \mathbb{H}_x^2 = \mathbb{P}(V_0)$, which can be identified with the one point compactification $\hat{\mathbb{R}}$ of Heisenberg group $\mathbb{R}$ by stereographic projection. The Heisenberg group $\mathbb{R}$ is the set $\mathbb{C} \times \mathbb{R}$ with the group law

$$(z_1, t_1)(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\Im(z_1 \bar{z}_2)).$$
The point at infinity is \( q_\infty = (1, 0, 0)^t \).

The group of biholomorphic transformations of complex hyperbolic space \( H^2 \) is \( \text{PU}(2, 1) \), which is the projectivisation of the unitary group \( \text{U}(2, 1) \) preserving the Hermitian form. It is well known that if \( A \in \text{PU}(2, 1) \) fixes \( q_\infty \) then \( A \) is upper triangular. There are three important classes of transformations fixing \( q_\infty \), Heisenberg translation, rotation and dilation. The group generated by all Heisenberg translations, rotations and dilations is the stabilizer of \( q_\infty \) in \( \text{PU}(2, 1) \).

Heisenberg translation by \( (z, t) \in \partial H^2 \) is given by

\[
N_{z,t} \equiv \begin{pmatrix}
1 & -\bar{z} & (-|z|^2 + it)/2 \\
0 & 1 & z \\
0 & 0 & 1
\end{pmatrix}.
\]

The product of two Heisenberg translations \( N_{(z_1,t_1)} \) and \( N_{(z_2,t_2)} \) is the Heisenberg translation

\[
N_{(z_1,t_1)} \circ N_{(z_2,t_2)} = N_{(z_1+z_2,t_1+t_2+2\Im(z_1\bar{z}_2))}
\]

corresponding to the product of two points in the Heisenberg group \( \mathfrak{H} \).

The Heisenberg rotation by \( \beta \in S^1 \) is given by

\[
M_\beta \equiv \begin{pmatrix}
1 & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

The Heisenberg dilation by \( \lambda \in \mathbb{R}_+ \) is given by

\[
A_\lambda \equiv \begin{pmatrix}
\lambda & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \lambda^{-1}
\end{pmatrix}.
\]

The holomorphic involution \( R \), which swaps the point \( q_0 = (0, 0) \) and the point at infinity \( q_\infty \), is given by

\[
R \equiv \begin{pmatrix}
0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

Let \( \Gamma_\infty \) be the stabilizer subgroup of \( q_\infty \) in \( \text{PU}(2, 1) \). Using Langlands decomposition, any element \( P \in \Gamma_\infty \) can be decomposed as a product of a Heisenberg translation, dilation, and a rotation:

\[
P = \begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
0 & p_{22} & p_{23} \\
0 & 0 & p_{33}
\end{pmatrix} = NAM = \begin{pmatrix}
\lambda & -\beta \bar{z} & -|z|^2 + it \\
0 & \beta & \lambda^{-1} \bar{z} \\
0 & 0 & \lambda^{-1}
\end{pmatrix}.
\]

The parameters satisfy the corresponding conditions.

Through we have known that an element belonging to the subgroup \( \Gamma_\infty \) of \( \text{PU}(2, 1) \) is upper triangular, the following lemma gives a neccessary and sufficient condition to determine that an element of \( \text{PU}(2, 1) \) lies in \( \Gamma_\infty \).

**Lemma 2.1** Let \( G = (g_{jk}) \in \text{PU}(2, 1) \). Then \( G \in \Gamma_\infty \) if and only if \( g_{31} = 0 \).

In [FL1] and [FL2], they have shown that the Langlands decomposition can also be used to descripr a holomorphic transformation \( G \in \text{PU}(2, 1) \) which is not in the stabilizer subgroup of the infinity point \( q_\infty \). Let \( N_{G(q_\infty)} \) be the Heisenberg translation which maps \( q_0 \) to \( G(q_\infty) \). It is quite easy to see that the transformation \( P = RN^{-1}_{G(q_\infty)}G \) belongs to \( \Gamma_\infty \), so

\[
G = N_{G(q_\infty)}RP = N_{G(q_\infty)}RNAM.
\]
The transformations $N$ and $P$ in the decomposition of $G$ are not necessarily in the Picard modular group $\Gamma \equiv PU(2, 1; \mathcal{O}_{d})$, even if $G \in \Gamma$. It is clear that the entries of $N$ and $P$ are not necessarily integers in the ring $\mathcal{O}_{d}$.

3. The Picard modular groups $PU(2, 1; \mathcal{O}_{d})$

3.1 The case $d = 3$

In this section, we consider the Picard modular group $PU(2, 1; \mathcal{O}_{d})$ when $d = 3$. Let $\omega_3 = \frac{-1 + i\sqrt{3}}{2}$, then the ring $\mathcal{O}_{d}$ can be written as $\mathbb{Z}[\omega_3]$. We first consider the stabilizer subgroup $\Gamma_{\infty}$ of the Picard modular group $PU(2, 1; \mathbb{Z}[\omega_3])$.

Lemma 3.1. Let $\Gamma_{\infty}(2, 1; \mathbb{Z}[\omega_3])$ denote the subgroup $\Gamma_{\infty}$ of Picard modular group $PU(2, 1; \mathbb{Z}[\omega_3])$. Then any element $P \in \Gamma_{\infty}(2, 1; \mathbb{Z}[\omega_3])$ if and only if the parameters in the Langlands decomposition of $P$ satisfy the conditions

$$
\lambda = 1, \; t \in \sqrt{3}\mathbb{Z}, \; z \in \mathbb{Z}[\omega_3], \; \beta = \pm 1, \pm \omega_3, \pm \omega_3^2
$$

and the integers $\frac{t}{\sqrt{3}}$ and $|z|^2$ have the same parity.

Proof. It is quite easy to see that $\lambda = 1$. Considering the Langlands decomposition when $P \in \Gamma_{\infty}(2, 1; \mathbb{Z}[\omega_3])$, we can get that $|\beta| = 1, \; z \in \mathbb{Z}[\omega_3]$ and $t \in \sqrt{3}\mathbb{Z}$. Since the entries $\frac{|z|^2 + it}{2} \in \mathbb{Z}[\omega_3]$, $\frac{t}{\sqrt{3}} \in \mathbb{Z}$ and $|z|^2 \in \mathbb{Z}$, the integers $\frac{t}{\sqrt{3}}$ and $|z|^2$ have the same parity. As $\omega_3$ is a cube root of unit, if $|\beta| = 1$, then $\beta = \pm 1, \pm \omega_3, \pm \omega_3^2$. q.e.d.

Proposition 3.2. The stabilizer subgroup $\Gamma_{\infty}$ of the infinity point $q_{\infty}$ in the Picard modular group $PU(2, 1; \mathbb{Z}[\omega_3])$ is generated by the Heisenberg translations $N_{(\omega_3, \sqrt{3})}$, $N_{(1, \sqrt{3})}$, and the rotation $M_{-\omega_3}$.

Proof. For any $P \in \Gamma_{\infty}(2, 1; \mathbb{Z}[\omega_3])$, we know that $P$ is upper triangular. According to Lemma 1, there is no dilation component in its Langland decomposition, that is

$$
P = NM = \begin{pmatrix}
1 & -\bar{z} & (-|z|^2 + it)/2 \\
0 & 1 & z \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Since $\beta^6 = 1$, the rotation in $P$ is $M_{-\omega_3}$, $M_{\omega_3^2} = M_{\omega_3}$, $M_{1} = M_{3\omega_3}$, $M_{\omega_3} = M_{4\omega_3}$, $M_{-\omega_3} = M_{5\omega_3}$, or $I = M_{6\omega_3}^5$. Therefore the rotation component of $P$ in the Langland decomposition is generated by $M_{-\omega_3}$.

We now consider the Heisenberg translation part of $P$, $N_{(z, t)}$. Let $z = a + b\omega_3$, where $a, b \in \mathbb{Z}$, since $z \in \mathbb{Z}[\omega_3]$. Then $N_{(z, t)}$ splits as

$$
N_{(z, t)} = N_{(a + b\omega_3, t)} = N_{(b\omega_3, \sqrt{3}b)} \circ N_{(a, \sqrt{3}a)} \circ N_{(0, t - \sqrt{3}ab - \sqrt{3}a - \sqrt{3}b)}.
$$

Here $N_{(b\omega_3, \sqrt{3}b)}$ can be written as

$$
N_{(b\omega_3, \sqrt{3}b)} = N_{(\omega_3, \sqrt{3})}^b,
$$

since $b \in \mathbb{Z}$. Obviously, the Heisenberg translation $N_{(a, \sqrt{3}a)}$ can be written as

$$
N_{(a, \sqrt{3}a)} = N_{(1, \sqrt{3})}^a,
$$

since $a \in \mathbb{Z}$.

To obtain

$$
N_{(0, t - \sqrt{3}ab - \sqrt{3}a - \sqrt{3}b)} = N_{(0, 2\sqrt{3})}^{t - \sqrt{3}ab - \sqrt{3}a - \sqrt{3}b},
$$

we have completed the proof.
it suffice to show that the number \( \frac{t - \sqrt{3}(ab + a + b)}{2\sqrt{3}} \) is an integer, namely,

\[
\frac{t}{\sqrt{3}} - (ab + a + b) \in 2\mathbb{Z}.
\]

According to Lemma 3.1, the integers \( \frac{t}{\sqrt{3}} \) and \( |z|^2 = |a + b\omega_3|^2 = a^2 - ab + b^2 \) have the same parity. It can be easily seen that

\[
a^2 - ab + b^2 + (ab + a + b) = a(a + 1) + b(b + 1) \in 2\mathbb{Z}.
\]

Hence \( \frac{t}{\sqrt{3}} \) and \( ab + a + b \) have the same parity. This prove that

\[
\frac{t}{\sqrt{3}} - (ab + a + b) \in 2\mathbb{Z}.
\]

The Heisenberg translation \( N_{(0, 2\sqrt{3})} \) can be generated by \( N_{(1, \sqrt{3})} \) and \( M_{-1} \), i.e.

\[
N_{(0, 2\sqrt{3})} = (N_{(1, \sqrt{3})} \circ M_{-1})^2.
\]

This proposition is proved. q.e.d.

**Proof of Theorem 1.** Let \( G = (g_{jk})_{j,k=1}^3 \) be an element of the group \( \text{PU}(2, 1; \mathbb{Z}[\omega_3]) \). Since the result is obviously when \( G \in \Gamma_{\infty} \), which is the stabilizer subgroup of infinity \( q_{\infty} \), we may assume that \( G \) does not belong to the subgroup \( \Gamma_{\infty} \). Then \( g_{31} \neq 0 \) according to Lemma 2.1 and \( G \) maps \( q_{\infty} \) to \( (g_{11}/g_{31}, g_{21}/g_{31}) \). Since \( G(q_{\infty}) \) is in \( \partial \mathbb{H}_2^3 \), then

\[
2\Re\left(\frac{g_{11}}{g_{31}}\right) = -\left|\frac{g_{21}}{g_{31}}\right|^2.
\]  

Consider the Heisenberg translation \( N_{G(q_{\infty})} \) that maps \( q_0 \) to \( G(q_{\infty}) \). Note that the translation \( N_{G(q_{\infty})} \) is not necessarily in the Picard modular group \( \text{PU}(2, 1; \mathbb{Z}[\omega_3]) \) except \( |g_{31}| = 1 \). However, we known that

\[
RN_{G(q_{\infty})}^{-1} G = P.
\]

It is well known that the ring \( \mathcal{O}_3 = \mathbb{Z}[\omega_3] \) is Euclidean. Then we will successively approximate \( N_{G(q_{\infty})}^{-1} \) by Heisenberg translations in the Picard modular group to decrease the value \( |g_{31}|^2 \in \mathbb{Z} \) until it becomes 0. Therefore, \( G \) belongs to the subgroup \( \Gamma_{\infty} \) according to Lemma 2.1 and can be expressed as a product of the generators according to Proposition 3.2.

We calculate the entry in the lower left corner of the product

\[
G_1 \equiv RN_{(z, t)} G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -z \\ 1 & -\bar{z} & -|z|^2 + it \end{pmatrix} G.
\]

It follows that the entry \( g_{31}^{(1)} \) lying in the lower left corner of \( G_1 = (g_{jk}^{(1)}) \) is equal to

\[
g_{31}^{(1)} = g_{11} - g_{21} \bar{z} + g_{31} - \frac{|z|^2 + it}{2} = g_{31}\left(\Re\left(\frac{g_{11}}{g_{31}}\right) - \Re\left(\frac{g_{21}}{g_{31}}\frac{1}{\bar{z}}\right) - \frac{|z|^2}{2}\right) + i\Im\left(\frac{g_{11}}{g_{31}} - \Im\left(\frac{g_{21}}{g_{31}}\frac{1}{\bar{z}}\right) + \frac{t}{2}\right) = g_{31}(J_1 + iJ_2).
\]
We can simplify $I_1$ to

$$ I_1 = \frac{1}{2} \left( \frac{g_{21}}{g_{31}} + z \right)^2, $$

according to equation (1).

Let $\frac{2n}{g_{31}} = x + iy$. Since $z = a + b\omega_3 = (a - \frac{b}{2}) + \frac{b\sqrt{3}}{2}$, we can select two appropriate integers $a$ and $b$ satisfying $|x + (a - \frac{b}{2})| \leq \frac{1}{2}$ and $|y + \frac{b\sqrt{3}}{2}| \leq \frac{\sqrt{3}}{4}$. Hence we obtain the upper bound

$$ |I_1| \leq \frac{1}{4} \left( \left( \frac{1}{2} \right)^2 + \left( \frac{\sqrt{3}}{4} \right)^2 \right) = \frac{7}{32}. $$

Selecting some $t$ in $I_2$, we can get the inequality $|I_2| = |\Im(\frac{2n}{g_{31}}) - \Im(\frac{2n}{g_{31}}z) + \frac{1}{2}| \leq \frac{\sqrt{3}}{4}$ since $t \in \sqrt{3}Z$. Therefore, we have the estimation of $g^{(1)}_{31}$

$$ |g^{(1)}_{31}|^2 = |g_{31}|^2 |I_1 + iI_2|^2 = |g_{31}|^2 (I_1^2 + I_2^2) \leq |g_{31}|^2 [(\frac{7}{32})^2 + (\frac{\sqrt{3}}{4})^2] < \frac{1}{4}|g_{31}|^2. $$

The preceding inequality tell us that we can reduce the matrix of the transformation $G$ to the matrix of a transformation $G_n$ with $g^{(n)}_{31} = 0$ by repeating this approximation procedure finitely many times. However, according to Lemma 2.1, this condition implies that the $G_n$ belongs to the subgroups $\Gamma_\infty$. As we shown in Proposition 3.2, the subgroup $\Gamma_\infty$ can be generated by the Heisenberg translation $N_{(\omega_3, \sqrt{3})}$, $N_{(1, \sqrt{3})}$ and the Heisenberg rotation $M_{\omega_3}$. Since the approximation procedure just uses the transformations in $\Gamma_\infty$ and the transformation $R$. Hence the proof of Theorem 1 is completed. q.e.d.

3.2 The case $d = 7$

In this section, we consider the Picard modular group $\text{PU}(2, 1; O_d)$ when $d = 7$. Let $\omega_7 = \frac{-1+i\sqrt{7}}{2}$, then the ring $O_d$ can be written as $\mathbb{Z}[\omega_7]$. In order to prove theorem 2, we start by considering the stabilizer subgroup $\Gamma_\infty$.

**Lemma 3.3** Let $\Gamma_\infty(2, 1; \mathbb{Z}[\omega_7])$ denote the subgroup $\Gamma_\infty$ of Picard modular group $\text{PU}(2, 1; \mathbb{Z}[\omega_7])$. Then any element $P \in \Gamma_\infty(2, 1; \mathbb{Z}[\omega_7])$ if and only if the parameters in the Langlands decomposition of $P$ satisfy the conditions

$$ \lambda = 1, \ t \in \sqrt{3}Z, \ z \in \mathbb{Z}[\omega_7], \ \beta = \pm 1, $$

and the integers $\frac{t}{\sqrt{7}}$ and $|z|^2$ have the same parity.

**Proof.** It is quite easy to see that $\lambda = 1$. Considering the Langlands decomposition when $P \in \Gamma_\infty(2, 1; \mathbb{Z}[\omega_7])$, we can get that $|\beta| = 1$, $z \in \mathbb{Z}[\omega_7]$ and $t \in \sqrt{3}Z$. Since the entries $\frac{-|z|^2+it}{2} \in \mathbb{Z}[\omega_7]$, $\frac{t}{\sqrt{7}} \in \mathbb{Z}$ and $|z|^2 \in \mathbb{Z}$, the integers $\frac{t}{\sqrt{7}}$ and $|z|^2$ have the same parity. As $|\beta| = 1$ and there is not an element in $\mathbb{Z}[\omega_7]$ except $-1, 1$ satisfying the preceding condition. This prove the lemma. q.e.d.

**Proposition 3.4** The stabilizer subgroup $\Gamma_\infty$ of the infinity point $q_\infty$ in the Picard modular group $\text{PU}(2, 1; \mathbb{Z}[\omega_7])$ is generated by the Heisenberg translations $N_{(\omega_7, 0)}$, $N_{(1, \sqrt{7})}$ and the rotation $M_{-1}$.

**Proof.** It is similar to the proof of Proposition 1. Let $P \in \Gamma_\infty(2, 1; \mathbb{Z}[\omega_7])$. According to Lemma 1, it can be decomposed as the product of a Heisenberg translation $N_{(z, t)}$ and a rotation $M_{\beta}$. The Heisenberg rotation $M_{\beta}$ is $M_{-1}$ and $I = M_{-1}^2$ since $\beta^2 = 1$. Therefore, the rotation component of $P$ in the Langland decomposition is generated by $M_{-1}$.

Let $z = a + b\omega_7$ with $a, b \in \mathbb{Z}$, the Heisenberg translation $N_{(z, t)}$ can be decomposed as

$$ N_{(z, t)} = N_{(a+b\omega_7, t)} = N_{(b\omega_7, 0)} \circ N_{(a, \sqrt{7}a)} \circ N_{(0, t-\sqrt{7}ab-\sqrt{7}a)}. $$
Here \( N_{(b\omega,0)} \) and \( N_{(a,\sqrt{\tau}a)} \) can be written as
\[
N_{(b\omega,0)} = N_{(0,0b)}, \quad b \in \mathbb{Z},
\]
and
\[
N_{(a,\sqrt{\tau}a)} = N_{(1,\sqrt{\tau})}, \quad a \in \mathbb{Z}.
\]
According to Lemma 3.3, integers \( \frac{a}{\sqrt{7}} \) and \( a^2 - ab + 2b^2 \) have the same parity. It can be easily seen that
\[
a^2 - ab + 2b^2 + (a + ab) = a(a + 1) + 2b^2 \in 2\mathbb{Z}.
\]
Therefore, \( \frac{a}{\sqrt{7}} \) and \( a + ab \) have the same parity, it means that
\[
\frac{t}{\sqrt{7}} = a + ab \in 2\mathbb{Z}.
\]
Hence the Heisenberg translation \( N_{(0,t - \sqrt{7}a - \sqrt{7}a)} \) can be written as
\[
N_{(0,t - \sqrt{7}a - \sqrt{7}a)} = N_{(0,2\sqrt{7})}^{t-\sqrt{7}(a+b)}.
\]
As \( N_{(0,2\sqrt{7})} \) can be decomposed as
\[
N_{(0,2\sqrt{7})} = (N_{(1,\sqrt{7})} \circ M_{-1})^2,
\]
this prove the proposition. \( \square \)

**Proof of Theorem 2** Let \( G = (g_{jk})^3_{j,k=1} \) be an element of the group \( \text{PU}(2,1;\mathbb{Z}[^{\omega_7}]) \). We may assume that \( G \) does not belong to the subgroup \( \Gamma_{\infty} \), which is the stabilizer subgroup of infinity \( q_{\infty} \). Then \( g_{31} \neq 0 \) and \( G \) maps \( q_{\infty} \) to \( (g_{11}/g_{31}, g_{21}/g_{31}) \). Since \( G(q_{\infty}) \) is in \( \partial \mathbb{H}^2 \), then
\[
2\Re\left(\frac{g_{11}}{g_{31}}\right) = -\left|\frac{g_{21}}{g_{31}}\right|^2. \tag{3}
\]
Consider the Heisenberg translation \( N_{G(q_{\infty})} \), that maps \((0,0)\) to \((G(q_{\infty}))\). Note that the translation \( N_{G(q_{\infty})} \) is not necessarily in the Picard modular group \( \text{PU}(2,1;\mathbb{Z}[^{\omega_7}]) \) except \( |g_{31}| = 1 \). However, we know that
\[
RN_{G(q_{\infty})}^{-1} = P.
\]
We will successively approximate \( N_{G(q_{\infty})}^{-1} \) by Heisenberg translations in the Picard modular group to decrease the value \( |g_{31}|^2 \in \mathbb{Z} \) until it becomes 0. Then \( G \) belongs to the subgroup \( \Gamma_{\infty} \) according to Lemma 1 and can be expressed as a product of the generators according to Proposition 3.4. The approximation step uses the fact that the ring \( \mathcal{O}_3 = \mathbb{Z}[^{\omega_7}] \) is Euclidean.

We calculate the entry in the lower left corner of the product
\[
G_1 \equiv RN_{(z,t)}G = \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & -1 & -z \\ 1 & -\bar{z} & -|z|^2 + it \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right)
\]
so \( g_{31}^{(1)} \), the entry in the lower left corner of \( G_1 \) is equal to
\[
g_{31}^{(1)} = g_{11} - g_{21}\bar{z} + g_{31} \frac{-|z|^2 + it}{2} \\
= g_{31}\left( \frac{g_{11}}{g_{31}} - \frac{g_{21}}{g_{31}}\bar{z} + \frac{-|z|^2 + it}{2} \right)
\]
\[
= g_{31}\left( \Re(\frac{g_{11}}{g_{31}}) - \Re(\frac{g_{21}}{g_{31}}\bar{z}) - \frac{|z|^2}{2} \right) + i(\Im(\frac{g_{11}}{g_{31}}) - \Im(\frac{g_{21}}{g_{31}}\bar{z}) + \frac{t}{2}) \\
= g_{31}(I_1 + iI_2). \tag{4}
\]
We can simplify $I_1$ to

$$I_1 = -\frac{1}{2} \left( \frac{g_{21}}{g_{31}} + z \right)^2,$$

according to equation (3). Let $\frac{g_{21}}{g_{31}} = x + iy$. Since $z = a + b\omega = (a - \frac{b}{\sqrt{4}}) + \frac{b\sqrt{7}}{2}$, we can select two appropriate integers $a$ and $b$ satisfying $|x + (a - \frac{b}{\sqrt{4}})| \leq \frac{1}{2}$ and $|y + \frac{b\sqrt{7}}{2}| \leq \frac{\sqrt{7}}{4}$. Hence we get the upper bound

$$|I_1| \leq \frac{1}{2} \left( \left( \frac{1}{2} \right)^2 + \left( \frac{\sqrt{7}}{4} \right)^2 \right) = \frac{11}{32}.$$

Selectting some $t$ in $I_2$, we can get the inequality $|I_2| \leq \frac{\sqrt{7}}{4}$. Therefore, we have the estimation of $g^{(1)}_{31}$

$$|g^{(1)}_{31}| = |g_{31}|^2 |I_1 + iI_2|^2 = |g_{31}|^2 (I_1^2 + I_2^2) \leq |g_{31}|^2 \left( \left( \frac{11}{32} \right)^2 + \left( \frac{\sqrt{7}}{4} \right)^2 \right) < \frac{37}{64} |g_{31}|^2.$$

Repeating this approximation procedure finitely many times we reduce the matrix of the transformation $G$ to the matrix of a transformation $G_n$ with $g^{(n)}_{31} = 0$. However, according to Lemma 2.1, this condition implies that the $G_n$ belongs to the stabilizer subgroups of $q_\infty, \Gamma_\infty$. According to Proposition 3.4, the transformation $G_n$ is generated by the Heisenberg translations $N_{(\omega_7, 0), n_1, (1, \sqrt{7})}$ and the rotation $M_{-1}$. Since the approximation procedure uses the transformation $R$ and transformations in $\Gamma_\infty$, this completes the proof of Theorem 2. q.e.d.

### 3.3 The case $d = 11$

In this section, we consider the Picard modular group $\textbf{PU}(2, 1; O_d)$ when $d = 11$. Let $\omega_{11} = \frac{-1 + \sqrt{11}}{2}$, then the ring $O_d$ can be written as $\mathbb{Z}[\omega_{11}]$. We use the same method to prove theorem 3.

**Lemma 3.5** Let $\Gamma_\infty(2, 1; \mathbb{Z}[\omega_{11}])$ denote the subgroup $\Gamma_\infty$ of Picard modular group $\textbf{PU}(2, 1; \mathbb{Z}[\omega_{11}])$. Then any element $P \in \Gamma_\infty(2, 1; \mathbb{Z}[\omega_{11}])$ if and only if the parameters in the Langlands decomposition of $P$ satisfy the conditions

$$\lambda = 1, \ t \in \sqrt{3}\mathbb{Z}, \ z \in \mathbb{Z}[\omega_{11}], \ \beta = \pm 1,$$

and the integers $\frac{1}{\sqrt{11}}$ and $|z|^2$ have the same parity.

**Proof.** It is quite easy to see that $\lambda = 1$. Considering the Langlands decomposition when $P \in \Gamma_\infty(2, 1; \mathbb{Z}[\omega_{11}])$, we can get that $|\beta| = 1, \ z \in \mathbb{Z}[\omega_7]$ and $t \in \sqrt{11}\mathbb{Z}$. Since the entries $-\frac{|z|^2 + d}{2} \in \mathbb{Z}[\omega_{11}], \ \frac{1}{\sqrt{11}} \in \mathbb{Z}$ and $|z|^2 \in \mathbb{Z}$, the integers $\frac{1}{\sqrt{d}}$ and $|z|^2$ have the same parity. As $|\beta| = 1$ and there is not an element in $\mathbb{Z}[\omega_{11}]$ except $-1, 1$ satisfying the preceding condition. This prove the lemma. q.e.d.

**Proposition 3.6** The stabilizer subgroup $\Gamma_\infty$ of the infinity point $q_\infty$ in the Picard modular group $\textbf{PU}(2, 1; \mathbb{Z}[\omega_{11}])$ is generated by the Heisenberg translations $N_{(\omega_{11}, \sqrt{11}), n_1, (1, \sqrt{11})}$ and the rotation $M_{-1}$. q.e.d.

**Proof.** It can be arised from the identical arguments in the proofs of the proposition 3.4. Therefore we can obtain the following decomposition for the Heisenberg translation $N_{(z, t)}$, where $z = a + b\omega_7$ with $a, b \in \mathbb{Z}$

$$N_{(a + b\omega_{11}, t)} = N_{(\omega_{11}, \sqrt{11})}^b \circ N_{(1, \sqrt{11})}^a \circ N_{(0, \sqrt{11})}^{\frac{-\sqrt{7}a + \sqrt{7}b + \sqrt{11}t}{2\sqrt{11}}}.$$

And we just want to notice that the Heisenberg translation $N_{(0, \sqrt{11})}$ can be decomposed as

$$N_{(0, \sqrt{11})} = (N_{(1, \sqrt{11})} \circ M_{-1})^2.$$

q.e.d.
Proof of Theorem 3  The theorem can be proved by the same arguments in the proof of theorem 2. We only need to mention that the upper bound of $I_1$ and $I_2$ are

$$|I_1| \leq \frac{1}{2} \left( \frac{1}{2} \right)^2 + \left( \frac{\sqrt{11}}{4} \right)^2 = \frac{15}{32},$$

and $|I_2| \leq \frac{\sqrt{11}}{4}$. Hence the entry $g^{(1)}_{31}$ has the following estimation

$$|g^{(1)}_{31}|^2 = |g_{31}|^2 |I_1 + iI_2|^2 = |g_{31}|^2 (I_1^2 + I_2^2) \leq |g_{31}|^2 \left( \frac{15}{32} \right)^2 + \left( \frac{\sqrt{11}}{4} \right)^2 \leq \frac{15}{16} |g_{31}|^2.

q.e.d.

4. Remarks

In [FP], Falbel and Parker gave a presentation for the Picard modular group $\text{PU}(2, 1; \mathbb{Z}[\omega_3])$

$$\langle P, Q, R \rangle R^2 = (QP^{-1})^6 = P^{-1}RQP^{-1}R = P^3Q^{-2} = (RP)^3 = 1.$$ 

Moreover, the stabilizer subgroup of infinity $q_{\infty}$, $\Gamma_{\infty} = \langle P, Q \rangle$. The elements $P$, $Q$ and $R$ are

According to Proposition 3.2, it is clear that $PQ^{-1} = M_{-\omega_3}$, $Q = N_{(1, \sqrt{3})} \circ M_{-\omega_3}$ and $P = M_{-\omega_3} \circ Q = M_{-\omega_3} \circ N_{(1, \sqrt{3})} \circ M_{-\omega_3}^3$. It means that the subgroup $\Gamma_{\infty}$ of $\text{PU}(2, 1; \mathbb{Z}[\omega_3])$ can be generated by a Heisenberg translation $N_{(1, \sqrt{3})}$ and a rotation $M_{-\omega_3}$. Hence the Picard modular group $\text{PU}(2, 1; \mathbb{Z}[\omega_3])$ is generated by $N_{(1, \sqrt{3})}$, $M_{-\omega_3}$ and $R$.

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