From Petrov-Einstein to Navier-Stokes in Spatially Curved Spacetime

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Abstract

We generalize the framework in arXiv:1104.5502 to the case that an embedding may have a non-vanishing intrinsic curvature. Directly employing the Brown-York stress tensor as the fundamental variables, we study the effect of finite perturbations of the extrinsic curvature while keeping the intrinsic metric fixed. We show that imposing a Petrov type I condition on the hypersurface geometry may reduce to the incompressible Navier-Stokes equation for a fluid moving in spatially curved spacetime in the near-horizon limit.
I. INTRODUCTION

It has been more than three decades since Damour firstly found that the excitations of a black hole horizon behave very much like those of a fluid\cite{1,2}. Later the analogy of gravity dynamics with hydrodynamics has been further disclosed in\cite{3} and then in the context of AdS/CFT correspondence\cite{4-9}. In particular it is shown in\cite{10} that in a suitably defined near horizon limit, the dynamics of gravity on an arbitrary cutoff surface can be governed by the incompressible Navier-Stokes equation if one identifies the Brown-York tensor of the bulk gravity with the stress-energy tensor of a fluid moving on the surface (and other relevant work can be found, for instance, in\cite{11-15}). More specifically, in this approach one embeds a $p + 1$-dimensional timelike hypersurface into a $p + 2$-dimensional spacetime, with a small distance from the horizon. Then imposing a Dirichlet-like boundary condition on the hypersurface while demanding regularity on the future horizon and no incoming flux across the past horizon, one can solve the bulk Einstein equations explicitly under the non-relativistical limit with a long-wavelength expansion, and the incompressible Navier-Stokes equation can be obtained as a constraint on the hypersurface.

Very recently it has been remarkably noted in\cite{16} that imposing a Petrov type I condition on the hypersurface geometry exactly reduces the degrees of freedom in the extrinsic curvature to those of a fluid such that the leading-order Einstein constraint leads to the Navier-Stokes equation provided that the mean curvature of the embedding is large enough. This observation strongly implies that regularity on the future horizon and the Petrov type I condition are equivalent at least in the near horizon limit. However, mathematically imposing the Petrov condition is much simpler and elegant than imposing regularity. It is very worthy to further understand its role in exploring the deep relations between the Einstein equations and Navier-Stokes equations.

Since in\cite{16} only an intrinsically flat $p + 1$-dimensional embedding is taken into account, in this paper we intend to generalize this framework to the case that the embedded hypersurface has a non-vanishing intrinsic curvature. Through explicit construction we will show that for a spatially curved embedding, it is still possible to obtain an incompressible Navier-Stokes equation for a fluid moving in this background. We organize the paper as follows. In next section we present the generalized framework in which the embedded hypersurface is intrinsically curved. In section three and four we explicitly construct two models in which
the background metric is spatially curved, and then study the fluctuations of the extrinsic curvature on the hypersurface. Imposing the Petrov type I condition as well as the Hamiltonian constraint, we obtain the Navier-Stokes equations with incompressible condition in the near horizon limit with a large mean curvature. Moreover, in contrast to the scheme used in [16] where a new traceless stress tensor is introduced, we insist to expand the effect of fluctuations directly in terms of the Brown-York stress tensor. To demonstrate that our scheme derives the same results at least at the leading orders of the expansion as in [16], we present two simple examples which has a Minkowski limit in the appendix.

II. THE FRAMEWORK FOR AN INTRINSICALLY CURVED EMBEDDING

Given a \( p + 2 \)-dimensional spacetime with a bulk metric \( g_{\mu\nu} \) which satisfies the vacuum Einstein’s equations

\[
G_{\mu\nu} = 0, \quad \mu, \nu = 0, ..., p + 1. \tag{1}
\]

We consider an embedding \( \Sigma_c \) whose \( p + 1 \)-dimensional spacetime with an induced metric \( \gamma_{ab} \) may be intrinsically curved. Suppose that the extrinsic curvature of the hypersurface is \( K_{ab} \), then the \( p + 1 \) “momentum constraints” on \( \Sigma_c \) reads as

\[
\nabla^a (K_{ab} - \gamma_{ab} K) = 0, \quad a, b = 0, ..., p \tag{2}
\]

where \( \nabla_a \) is compatible with the induced metric on \( \Sigma_c \), namely \( \nabla_a \gamma_{bc} = 0 \). While the “Hamiltonian constraint” is

\[
p + 1 R + K_{ab} K^{ab} - K^2 = 0. \tag{3}
\]

When the bulk metric satisfies the vacuum Einstein equation, the Riemann curvature tensor and the Weyl tensor are equal, thus we can decompose the Weyl tensor in \( p + 2 \) dimensions in terms of the intrinsic curvature of the \( p + 1 \) hypersurface and its extrinsic curvature. It turns out that the result is

\[
C_{abcd} = p + 1 R_{abcd} + K_{ad} K_{bc} - K_{ac} K_{bd}
\]

\[
C_{abc(n)} = \nabla_a K_{bc} - \nabla_b K_{ac}
\]

\[
C_{a(n)b(n)} = - p + 1 R_{ab} + K K_{ab} - K_{ac} K^c_b, \tag{4}
\]
where $C_{abc(\mu)} \equiv C_{abc\mu}$ and $n^\mu$ is the unit normal to $\Sigma_c$. The Petrov type I condition is defined as

$$C(\ell)_{ij} = \ell^\mu m_i^\nu \ell^\alpha m_j^\beta C_{\mu\nu\alpha\beta} = 0,$$

where $p + 2$ Newman-Penrose-like vector fields should satisfy the relations

$$\ell^2 = k^2 = 0, \quad (k, \ell) = 1, \quad (k, m_i) = (\ell, m_i) = 0, \quad (m^i, m_j) = \delta^i_j.$$

III. THE PETROV TYPE I CONDITIONS FOR SPATIALLY CURVED EMBEDDING

In this section we consider a $p + 2$ dimensional space time with a metric as

$$ds^2_{p+2} = -r dt^2 + 2 dt dr + h_{ij}(x^i) dx^i dx^j,$$

where $h_{ij}(x^i)$ is a general spatial metric but independent of the coordinates $t$ and $r$.

We consider an embedding $\Sigma_c$ by setting $r = r_c$ such that the induced metric $\gamma_{ab}$ on $\Sigma_c$ is

$$ds^2_{p+1} = -r_c dt^2 + h_{ij}(x^i) dx^i dx^j \equiv -(dx^0)^2 + h_{ij}(x^i) dx^i dx^j.$$

We also require that the induced metric on $\Sigma_c$ is fixed and then only consider the effects of the fluctuations of the extrinsic curvature. Now it is straightforward to obtain the components of $K_{ab}$ as

$$K_{tt} = -\frac{1}{2\sqrt{r_c}}, \quad K_{ti} = 0,$$

$$K_{ij} = 0, \quad K = \frac{1}{2\sqrt{r_c}},$$

where $K$ is the trace of the extrinsic curvature. Equivalently we may define the Brown-York stress tensor on $\Sigma_c$ as

$$t_{ab} \equiv K_{\gamma_{ab}} - K_{ab}.$$

Next we introduce a parameter $\lambda$ by rescaling the time coordinate with $\tau = \lambda x^0$ in order to discuss the dynamical behavior of the geometry in the non-relativistic limit, i.e.

$$ds^2_{p+1} = -\frac{1}{\lambda^2} d\tau^2 + h_{ij}(x^i) dx^i dx^j.$$

Moreover, we identify the parameter $\lambda$ with the location of the hypersurface by setting $r_c = \lambda^2$ such that the limit $\lambda \to 0$ means a large mean curvature and can be thought of as
a kind of near-horizon limit. In this coordinate system we obtain the relations between the Brown-York stress tensor and the extrinsic curvature as follows

\[ K^\tau_\tau = K - t^\tau_\tau = \frac{t}{p} - t^\tau_\tau, \quad K^\tau_i = -t^\tau_i, \]
\[ K^i_j = -t^i_j + \delta^i_j \frac{t}{p}, \quad K = \frac{t}{p}. \quad (12) \]

It is easy to show that except \( \Gamma^i_{jk} \), all the other components of the connection with the induced metric \((11)\) on \( \Sigma_c \) vanish.

Furthermore, the requirement that the background \((7)\) should satisfy the Einstein vacuum equations in \( p + 2 \)-dimensional spacetime leads to

\[ p+1 R = \frac{p}{R} = 0, \quad p+1 R_{ij} = \frac{p}{R_{ij}} = 0. \quad (13) \]

Next in contrast to defining a new traceless stress tensor as in \([16]\), we insist to take the Brown-York stress tensor as the fundamental variables and consider its fluctuations over the background (To demonstrate that our scheme derives the same results at the leading order of perturbations as in \([16]\), we present two examples in the appendix where the \( p+1 \)-dimensional hypersurface has a Minkowski limit). We expand the components of Brown-York tensor in powers of \( \lambda \)

\[ t^\tau_i = 0 + \lambda t^\tau_i^{(1)} + \ldots \]
\[ t^\tau_\tau = 0 + \lambda t^\tau_\tau^{(1)} + \ldots \]
\[ t^i_j = \frac{1}{2\sqrt{r_c}} \delta^i_j + \lambda t^i_j^{(1)} + \ldots \]
\[ t = \frac{p}{2\sqrt{r_c}} + \lambda t^{(1)} + \ldots. \quad (14) \]

By definition in our formalism the relation \( t = pK \) holds for arbitrary order of the expansion and \( t^{(n)} = t^\tau_\tau^{(n)} + t^i_j^{(n)} \). Substituting it into the “Hamiltonian constraint” in Eq.\((3)\) we find

\[ (t^\tau_\tau)^2 - \frac{2}{\lambda^2} (t^\tau_i)^2 + t^i_j t^j_i - \frac{t^2}{p} = 0. \quad (15) \]

Note that all the indices here are lowered or raised with \( \gamma_{ab} \) or \( \gamma^{ab} \). The leading order of the constraint is \( \lambda^{-2} \), which is automatically satisfied by the background, while the sub-leading order of the expansion gives rise to

\[ t^\tau_\tau^{(1)} = -2(t^\tau_i^{(1)})(t^\tau_j^{(1)})\gamma^{ij}. \quad (16) \]
Next we turn to the Petrov type I condition. Firstly we choose the vector fields as
\[ \sqrt{2}\ell = \partial_0 - n, \quad \sqrt{2}k = -\partial_0 - n. \] (17)
Then the condition becomes
\[ 2C = C_{\partial_0 n} + C_{\partial_0 (n)} + C_{0j(n)} + C_{i(n)j(n)} = 0. \] (18)
With the use of Eq.(4), the Petrov type I condition can be rewritten in terms of the Brown-York tensors as follows
\[ \begin{equation}
\tau^i_{(1)} = 2\gamma^{ik}t^\tau_k t^\tau_j - 2\lambda t^i_{j\tau} - t^i_k t^j_k - \frac{2}{\lambda} \gamma^{ik} t^\tau_{(k,j)} + \\
\delta^i_j \left[ t^\tau_p \frac{t}{p} - t^\tau_{(p)} \right] + \frac{2}{\lambda} \gamma^{ik} \Gamma^m_{kj} t^\tau_m = 0.
\end{equation} \] (19)
First of all, after expanding in powers of \( \lambda \), we find the background satisfies this condition automatically at the order of \( \lambda^2 \):
\[ \begin{equation}
-\frac{1}{4\lambda^2} \delta^i_j + \frac{1}{4\lambda^2} \delta^i_j = 0.
\end{equation} \] (20)
The next non-vanishing order is \( \lambda^0 \), which gives rise to the following equation
\[ \begin{equation}
t^i_{(1)} = 2\gamma^{ik}t^\tau_k t^\tau_j - 2\gamma^{ik} t^\tau_{(k,j)} + \delta^i_j \frac{t^\tau_{(1)}}{p} + 2\gamma^{ik} \Gamma^m_{kj} t^\tau_m^{(1)}.
\end{equation} \] (21)
Finally we come to the momentum constraints which is
\[ \nabla_a t^a_b = 0. \] (22)
The time component gives at leading order
\[ \begin{equation}
D_i t^\tau_{(1)i} = \partial_i t^\tau_{(1)i} + \Gamma^i_{ik} t^\tau_{(1)k} = 0,
\end{equation} \] (23)
where \( D_i \gamma_{jk} = 0 \). The space components at leading order can be written as
\[ \begin{equation}
\partial_\tau t^\tau_{(1)i} + D_k t^\tau_{(1)k_i} = 0.
\end{equation} \] (24)
Then plugging the solution to the Petrov type I condition in equation (21) and identifying
\[ \begin{equation}
t^\tau_{(1)i} = \frac{v_i}{2}, \quad t^{(1)} = \frac{p}{2} P,
\end{equation} \] (25)
we finally have the incompressible condition and the Navier-Stokes equation in spatially curved spacetime as
\[ \begin{equation}
D_k v^k = 0,
\end{equation} \] (26)
\[ \begin{equation}
\partial_\tau v_i + v^k D_k v_i + D_i P - (D^2 v_i + R^k_{ij} v_k) = 0.
\end{equation} \] (27)
Since in this simple case \( p R_{ij} = 0 \), the last term in above equation vanishes.
IV. NAVIER-STOKES EQUATIONS IN CURVED SPACETIME WITH NON-VANISHING $^{p}R_{ij}$

In this section we explicitly construct a model with a non-vanishing Ricci tensor $^{p}R_{ij}$. We assume that the metric of $p+2$ dimensional spacetime has the following form

$$ds_{p+2}^2 = -f(r)dt^2 + 2dtdr + e^{p(r,x^i)}\delta_{ij}dx^i dx^j.$$  \hfill (28)

Now the spatial components of the metric is conformally flat, but both $f$ and $\rho$ are functions of radial coordinate $r$. In particular, we specify the function $f(r)$ has the following form

$$f(r) = r(1 + a_1 r + a_2 r^2 + \ldots)$$  \hfill (29)

such that the Rindler horizon is fixed at $r = 0$.

The hypersurface is located at $r = r_c$, then the induced metric $\gamma_{ab}$ is

$$ds^2 = -f(r_c)dt^2 + e^{\rho(r_c,x^i)}\delta_{ij}dx^i dx^j$$

$$= -dx^0^2 + e^{\rho(r_c,x^i)}dx^i dx^j \equiv -\frac{1}{\lambda^2}d\tau^2 + e^{\rho} \delta_{ij}dx^i dx^j.$$  \hfill (30)

Now it is straightforward to compute the components of the extrinsic curvature, which are

$$K = \frac{1}{2\sqrt{f}} \partial_r f + \frac{1}{2} p \sqrt{f} \partial_r \rho$$

$$K^\tau_\tau = \frac{1}{2\sqrt{f}} \partial_r f$$

$$K^\tau_i = 0$$

$$K^i_j = \frac{1}{2} \sqrt{f} \partial_r \rho \delta^i_j.$$  \hfill (31)

On the other hand, the intrinsic quantities of the hypersurface can be obtained with the following components of connection:

$$\Gamma^\tau_\tau = \Gamma^\tau_i = \Gamma^i_\tau = \Gamma^i_j = 0$$

$$\Gamma^i_j = \frac{1}{2}(\delta^i_k \partial_j \rho + \delta^i_j \partial_k \rho - \delta^{im} \delta_{kj} \partial_m \rho).$$  \hfill (32)

Specifically, the components of Ricci tensor and the Ricci scalar have the following form

$$^{p+1}R_{\tau\tau} = ^{p+1}R_{\tau i} = 0$$

$$^{p+1}R_{ij} = \frac{2-p}{2} \partial_i \partial_j \rho - \frac{1}{2} \delta_{ij} \delta^{km} \partial_k \partial_m \rho + \frac{p-2}{4} (\partial_i \rho)(\partial_j \rho) - \frac{p-2}{4} \delta_{ij} \delta^{km} (\partial_k \rho)(\partial_m \rho)$$

$$^{p+1}R = (1-p)\gamma^{ij} \partial_i \partial_j \rho + \frac{(1-p)(p-2)}{4} \gamma^{ij} (\partial_i \rho)(\partial_j \rho).$$  \hfill (33)
Now we require that the metric in equation (28) be a solution to the Einstein vacuum equations in \( p + 2 \) dimensions. After a direct calculation, we find these equations to be

\[
\begin{align*}
\partial_i \partial_r \rho &= 0 \\
\partial_r^2 \rho + \frac{1}{2} (\partial_r \rho)^2 &= 0 \\
\partial_r^2 f + \frac{1}{2} p (\partial_r f)(\partial_r \rho) &= 0
\end{align*}
\]

(34)

and

\[
\begin{align*}
p+1 R_{ij} &= \gamma_{ij} \left( \frac{1}{2} (\partial_r f)(\partial_r \rho) + \frac{p f}{4} (\partial_r \rho)^2 + \frac{1}{2} f \partial_r^2 \rho \right).
\end{align*}
\]

(35)

Observing the first equation in (34), we notice that its solution is the linear combination of arbitrary functions \( F(x^i) \) and \( G(r) \). This immediately leads to the fact that \( p+1 R_{ij} \) is \( r \)-independent which can be seen from its definition in Eq.(33)\(^1\). As a matter of fact, employing the equations in (34) we can show that the right hand side of Equation (35) is also \( r \)-independent by taking the partial derivative with respect to \( r \). Besides the trivial solution corresponding to the flat spacetime, we can find the general solutions for \( f(r) \) and \( \rho(r, x^i) \) to be

\[
\begin{align*}
f(r) &= \frac{(r + c_1)^{1-p}}{1-p} c_2 + c_3 \\
\rho(r, x^i) &= F(x^i) + 2 \ln (r + c_1),
\end{align*}
\]

(36)

where \( c_1, c_2 \) and \( c_3 \) are arbitrary constants, and the function \( F(x^i) \) is subject to the equation

\[
\begin{align*}
p+1 R_{ij} &= c_3 (p - 1) e^{F(x^i)} \delta_{ij}.
\end{align*}
\]

(37)

Obviously, non-trivial solutions to this equation exist, but we need not to know their specific forms in our paper. Now requiring the function \( f(r) \) has the form as in (29), we can arrive at the following solution after an appropriate choice of the constants (here we set \( c_1 = c_2 = 1 \) and \( c_3 = \frac{1}{p-1} \)).

\[
\begin{align*}
f(r) &= \frac{(r + 1)^{1-p} - 1}{1-p} = r - \frac{p}{2} r^2 + \frac{1}{6} (p + p^2) r^3 + \ldots \\
\rho(r, x^i) &= F(x^i) + 2 \ln (r + 1) = F(x^i) + 2r - r^2 + \frac{2}{3} r^3 + \ldots
\end{align*}
\]

(38)

\(^1\) However \( p+1 R \) is \( r \)-dependent, but only through the induced metric \( \gamma^{ij} \).
Since we are mainly concerned with the behavior of the fluctuations in the near horizon limit, we have expanded both functions $f(r)$ and $\rho(r, x^i)$ in powers of $r$. As a consequence, the induced metric $\gamma_{ij}$ can also be expanded as

$$
\gamma_{ij} = e^{F(x^i)} \delta_{ij} (1 + r)^2 \equiv \gamma_{ij}^{(0)} + r \gamma_{ij}^{(1)} + r^2 \gamma_{ij}^{(2)},
$$

(39)

where $\gamma_{ij}^{(0)} \equiv e^{F(x^i)} \delta_{ij} = p + 1 R_{ij}$. Moreover, we point out that the “spatially covariant derivative" $D_i$ compatible with $\gamma_{ij}$ is also compatible with $\gamma_{ij}^{(n)}$ since the connection is $r$-independent, which can be seen by substituting the general form of $\rho = F(x^i) + G(r)$ into Eq. (32).

Next we consider the effects of fluctuations of the extrinsic curvature in a similar manner. The components of the Brown-York stress tensor on $\Sigma_c$ are expanded as

$$
t_{\tau i} = 0 + \lambda t_{\tau i}^{(1)} + \ldots \\
t_{\tau \tau} = \frac{1}{2} p \sqrt{f} \partial_r \rho + \lambda t_{\tau \tau}^{(1)} + \ldots \\
t_{ij} = (\frac{1}{2 \sqrt{f}} \partial_r f + \frac{p - 1}{2} \sqrt{f} \partial_r \rho) \delta_{ij} + \lambda t_{ij}^{(1)} + \ldots \\
t = (\frac{p}{2 \sqrt{f}} \partial_r f + \frac{1}{2} p^2 \sqrt{f} \partial_r \rho) + \lambda t^{(1)} + \ldots.
$$

(40)

When substituting these quantities into the Hamiltonian constraint as well as the Petrov type I condition and expanding these equations in powers of $\lambda$, we point out that the following quantities should also be expanded since the location of the hypersurface $r_c$ is identified with $\lambda^2$:

$$
\partial_r \rho|_{r_c} = 2 - 2 r_c + 2 r_c^2 + \ldots \\
\partial_r f|_{r_c} = 1 - pr_c + \frac{1}{2} (p + p^2) r_c^2 + \ldots \\
f^{1/2}|_{r_c} = r_c^{1/2} - \frac{1}{4} pr_c^{3/2} + \frac{1}{96} (8p + 5p^2) r_c^{5/2} + \ldots.
$$

(41)

The “Hamiltonian” constraint is

$$
(t_{\tau \tau})^2 - \frac{2}{\lambda^2} (t_{\tau i})^2 + t_{ij} t^{ij} - \frac{t^2}{p} + p^{p+1} R = 0.
$$

(42)

The leading order of the expansion automatically vanishes with

$$
\frac{p}{4} - \frac{p}{4} = 0.
$$

(43)
while the non-trivial sub-leading order gives rise to

$$t^{\tau}_{i}^{(1)} = -2\gamma^{ijk}(0)t^{\tau}_{k}^{(1)}t^{\tau}_{j}^{(1)}. \quad (44)$$

Next we turn to Petrov type I condition,

$$t^{\tau}_{i}t^{j}_{j} + \frac{2}{\lambda^2}\gamma^{ik}\Gamma^{l}_{kj}t^{\tau}_{i}t^{\tau}_{j} - 2\lambda t^{i}_{i,\tau} - t^{i}_{k}t^{k}_{j} - \frac{2}{\lambda}\gamma^{ij}t^{(k,j)} + \delta^{i}_{j}\left[\frac{t}{p}(t - t^{\tau}_{\tau}) + 2\lambda\partial_{\tau}t^{i}_{i}\right] + \frac{2}{\lambda}\gamma^{ik}\Gamma^{m}_{kj}t^{\tau}_{m} - \gamma^{ik}R_{ki,j} = 0. \quad (45)$$

Similarly, taking the expansion we find the leading order is automatically satisfied by the background quantities with

$$-\frac{1}{4}\delta^{i}_{j} + \frac{1}{4}\delta^{j}_{i} = 0, \quad (46)$$

and the non-trivial sub-leading order is \(\lambda^0\):

$$t^{i}_{j}^{(1)} = 2\gamma^{ik}(0)t^{\tau}_{k}^{(1)}t^{\tau}_{j}^{(1)} - 2\gamma^{ik}(0)t^{\tau}_{(k,j)}^{(1)} + 2\gamma^{ik}(0)\Gamma^{m}_{kj}t^{\tau}_{m}^{(1)} + \frac{t^{(1)}}{p}\delta^{i}_{j}. \quad (47)$$

Using the momentum constraint on the hypersurface and identifying

$$t^{\tau}_{i}^{(1)} = \frac{1}{2}v_{i}, \quad P = \frac{2}{p}t^{(1)}, \quad (48)$$

we finally obtain the incompressible condition and the Navier-Stokes equation in spatially curved background as

$$D_{i}v^{i} = \partial_{i}v^{i} + \Gamma^{i}_{ik}v^{k} = 0 \quad (49)$$

$$\partial_{\tau}v_{i} + D_{i}P + v^{k}D_{k}v_{i} - (D^{k}D_{k}v_{i} + R^{m}_{i}v_{m}) = 0, \quad (50)$$

where we have used the fact that \(D_{i}R^{i}_{j} = 0\) since \(R_{ij} \propto \gamma_{ij}\).

**V. SUMMARY AND DISCUSSIONS**

As a summary, we have generalized the framework in [16] by considering an embedding which may be intrinsically curved. Directly employing the Brown-York stress tensor as the fundamental variables of fluctuations, we explicitly construct models with spatially curved embedding and demonstrate that the incompressible Navier-Stokes equations can be derived for a fluid moving on \(\Sigma_{c}\) provided that the fluctuations are subject to the Petrov type I condition as well as the “Hamiltonian constraint”. The fact that the Petrov type I condition
can be applied to a class of more general spacetime strongly implies that this boundary condition would play a more important (and perhaps fundamental) role in linking the Einstein equation and Navier-Stokes equation, and this importance might be further disclosed by manifestly proving the equivalence of imposing Petrov type I condition with imposing the horizon regularity at least in the near horizon limit.

Through the paper we need assume that the background is fixed without time dependence such that the dynamics can be described by a Navier-Stokes equation in a sense of a non-relativistic limit. In the appendix the discussed models may have a dynamics for the background, but constrained by the condition that it has a Minkowski limit. It is an open question whether this framework could be generalized to a general background which might be dynamical and intrinsically curved.

In the end of this paper we propose that our current framework is applicable to the Schwarzschild black holes and the spacetime in the presence of matter fields with a cosmological constant. The investigation is under progress and will be presented elsewhere. We also expect that the higher order expansions of the momentum constraint can be investigated in the future.

Appendix: Two examples with a Minkowski limit

A. \( ds^2_{\mu+2} = -r dt^2 + 2dtdr + e^\rho \delta_{ij} dx^i dx^j \)

Now we require that the hypersurface goes back to the flat spacetime as \( \lambda \to 0 \), then the function \( \rho \) can be expanded as

\[
\rho = 0 + \rho^{(1)} \lambda + \rho^{(2)} \lambda^2 + \ldots \tag{51}
\]

We stress that in this case the background need not to be fixed and \( \rho \) can be a general function of \((\tau, r, x^i)\). The hypersurface is located at \( r = r_c \) and the components of the connection corresponding to the induced metric in \((\tau, x^i)\) coordinate system read as

\[
\Gamma^\tau_{\tau\tau} = \Gamma^\tau_{\tau i} = \Gamma^i_{\tau\tau} = 0 \tag{52}
\]

\[
\Gamma^\tau_{ij} = \frac{1}{2} \lambda^2 e^\rho (\partial_\tau \rho) \delta_{ij} \tag{53}
\]

\[
\Gamma^i_{\tau j} = \frac{1}{2} \delta^i_j \partial_\tau \rho \tag{54}
\]

\[
\Gamma^i_{jk} = \frac{1}{2} (\delta^i_k \partial_j \rho + \delta^i_j \partial_k \rho - \delta^i_m \delta_{kj} \partial_m \rho). \tag{55}
\]
Now it is straightforward to write down the “Hamiltonian constraint” in terms of the Brown-York tress tensor and the intrinsic curvature as

\[
(t^\tau_\tau)^2 - \frac{2}{\lambda^2} (t^\tau_i)^2 + t^i_j t^j_i - \frac{t^2}{p} + p\lambda^2 \partial^2 \rho + \frac{p(p+1)}{4} \lambda^2 (\partial_\tau \rho)^2 \\
+ \frac{(1-p)(p-2)}{4} e^{-\rho} \delta^{ij}(\partial_i \rho)(\partial_j \rho) + (1-p) e^{-\rho} \delta^{ij} \partial_i \partial_j \rho = 0. \tag{56}
\]

Now consider the fluctuation effects of both the extrinsic curvature and the background, we expand the variables in powers of \(\lambda\)

\[
t^\tau_i = 0 + \lambda t^\tau_i^{(1)} + \ldots
\]

\[
t^\tau_\tau = \frac{\lambda}{2} p \partial_\tau \rho + \frac{p}{2} \sqrt{r_c} \partial_\tau \rho + \lambda t^{\tau_\tau}^{(1)} + \ldots
\]

\[
t^i_j = \left( \frac{1}{2\sqrt{r_c}} + \frac{p-1}{2} \lambda \partial_\tau \rho + \frac{p-1}{2} \sqrt{r_c} \partial_\tau \rho \right) \delta^{ij} + \lambda t^i_j^{(1)} + \ldots
\]

\[
t = \frac{p}{2\sqrt{r_c}} + \frac{\lambda}{2} p^2 \partial_\tau \rho + \frac{p^2}{2} \sqrt{r_c} \partial_\tau \rho + \lambda t^{(1)} + \ldots
\]

\[
R = 0 + \lambda R^{(1)} + \ldots, \tag{57}
\]

where \(R^{(1)}\) can be found using Eq. (51). At the sub-leading order of the hamiltonian constraint we have

\[
t^\tau_\tau^{(1)} = -2t^\tau_i^{(1)} t^\tau_j^{(1)} \delta^{ij}. \tag{58}
\]

In a parallel way the Petrov type I condition leads to the following form

\[
t^\tau_\tau t^i_j + \frac{2}{\lambda^2} \gamma^{ik} t^\tau_k t^\tau_j - 2\lambda t^i_j,\tau - t^i_k t^\tau_j + \frac{2}{\lambda} \gamma^{ik} t^\tau_{(k,j)} + \delta^{ij} \left[ \frac{t}{p} \left( \frac{t}{p} - t^\tau_\tau \right) + 2\lambda \partial^2 \rho + \lambda [-t^i_j \partial_\tau \rho + t^\tau_k \partial_\tau \rho - \frac{1}{\lambda^2} \gamma^{ik} t^\tau_{(k,j)} + \delta^{ij} \left[ \frac{t}{p} \left( \frac{t}{p} - t^\tau_\tau \right) + 2\lambda \partial^2 \rho + \lambda [-t^i_j \partial_\tau \rho + t^\tau_k \partial_\tau \rho - \frac{1}{\lambda^2} \gamma^{ik} t^\tau_{(k,j)} + \delta^{ij} \right] \right]ight] - \frac{1}{4} \lambda^2 \delta^{ij} (\partial_\tau \rho)^2 + \lambda^2 \delta^{ij} \partial^2 \rho
\]

\[
+ \frac{p-2}{4} \gamma^{ik} (\partial_\rho) (\partial_j \rho) + \frac{2-p}{2} \gamma^{ik} \partial_\rho \partial_j \rho + \frac{2-p}{4} \delta^{ij} \gamma^{km} (\partial_\rho) (\partial_m \rho) - \frac{1}{2} \delta^{ij} \gamma^{km} \partial_\rho \partial_m \rho = 0. \tag{59}
\]

The sub-leading term of \(t^i_j\) is

\[
t^i_j^{(1)} = 2\delta^{ik} t^\tau_k^{(1)} t^\tau_j^{(1)} - 2\delta^{ik} t^\tau_{(k,j)}^{(1)} + \delta^{ij} \frac{t^{(1)}}{p}. \tag{60}
\]

Finally, from the momentum constraint

\[
\nabla_a t^a_b = 0, \tag{61}
\]

we have the time component as

\[
\partial_\tau t^\tau - \frac{1}{\lambda^2} \partial_i t^\tau i + \frac{p+1}{2} t^\tau \partial_\tau \rho - \frac{p}{2\lambda^2} t^\tau_k \partial_k \rho - \frac{t}{2} \partial_\tau \rho = 0. \tag{62}
\]
The leading order of the expansion gives
\[ \partial_t t^{\tau(1)} = 0. \]  (63)

The space components of the constraint is
\[ (\partial_{\tau} t_{i}^{\tau} - \frac{1}{2} t^{\tau} \partial_{\tau} \rho) + \partial_k t^k_i + \frac{1 + p}{2} t^{\tau} \partial_{\tau} \rho + \frac{p}{2} t^{k} \partial_k \rho - \frac{1}{2} (t - t^{\tau}) \partial_{\tau} \rho = 0. \]  (64)

We find the leading order of the expansion is automatically satisfied with a form as
\[ \frac{p}{4} \partial_{\tau} \rho^{(1)} - \frac{p}{4} \partial_{\tau} \rho^{(1)} = 0, \]  (65)

while the sub-leading order leads to
\[ \partial_{\tau} t^{\tau(1)} + 2 t^{k(1)} \partial_k t^{\tau(1)} - \partial^2 t^{\tau(1)} + \frac{\partial_{\tau} t^{(1)}}{p} = 0. \]  (66)

Identifying
\[ t^{\tau(1)} = \frac{\nu_i}{2}, \quad t^{(1)} = \frac{p}{2} P, \]  (67)

we obtain a Navier-Stokes equation with incompressible condition for a fluid in p-dimensional flat space.
\[ \partial_i v^i = 0, \]  (68)
\[ \partial_{\tau} v_i + v^k \partial_k v_i - \partial^2 v_i + \partial_{\tau} P = 0. \]  (69)

From above expansion, we notice that the contribution from the fluctuations of the background are higher order such that the leading order of the solutions to the Hamiltonian constraint and the Petrov I condition are the same as those in flat embedding with a fixed background, which of course is not surprising since the spacetime is subject to the condition of having a Minkowski limit. However, we would like to point out that the fluctuations of the background certainly will provide corrections to the higher order expansions, which should be different from the results with a fixed background and we leave this issue for study in future.

B. \[ ds^2_{p+2} = -re^\rho dt^2 + 2drdt + e^\rho \delta_{ij} dx^i dx^j \]

The induced metric on the hypersurface \( r = r_c \) is conformally flat, even the effects of fluctuations are taken into account. Therefore, we may set the trace of the Brown-York
tensor to be a constant without fluctuations, as suggested in [16]. The components of the connection in $p + 1$ dimensional spacetime are

$$
\Gamma^\tau_\tau = \frac{1}{2} \partial_\tau \rho, \quad \Gamma^\tau_i = \frac{1}{2} \partial_t \rho, \quad \Gamma^\tau_i = \frac{1}{2} \lambda^2 \delta_{ij} \partial_\tau \rho, \quad \Gamma^i_\tau = \frac{1}{2} \lambda^2 \delta_{ij} \partial_j \rho,
$$

$$
\Gamma^i_j = \frac{1}{2} \delta^i_j \partial_\tau \rho, \quad \Gamma^i_j = \frac{1}{2} (\delta^i_k \partial_j \rho + \delta^i_j \partial_k \rho - \delta^{jm} \delta_k j \partial_m \rho).
$$

(70)

The “Hamiltonian constraint” has the form

$$
(t^\tau_\tau)^2 - \frac{2}{\lambda^2} t^i t^j \delta^{ij} + t^i j - \frac{t^2}{p} + e^{-\rho} (p \lambda^2 \partial_\tau^2 \rho - p \delta^{ij} \partial_i \rho \partial_j \rho) + \frac{p - p^2}{4} e^{-\rho} [-\lambda^2 (\partial_\tau \rho)^2 + \delta^{ij} (\partial_i \rho) (\partial_j \rho)] = 0.
$$

(71)

It is easy to check that the leading order are satisfied by the quantities of the background, and the sub-leading order vanishes with

$$
\frac{\rho (1)}{4} p - \frac{\rho (1)}{4} p = 0.
$$

(72)

The subsequent order is $\lambda^0$ which gives

$$
t^\tau_\tau (1) = -2 t^i (1) t^j (1) \delta^{ij}.
$$

(73)

While for Petrov type I condition we introduce the following vector fields

$$
m_i = e^{-\frac{\rho}{2}} \partial_i, \quad \sqrt{2} \ell = e^{-\frac{\rho}{2}} \partial_0 - n, \quad \sqrt{2} k = -e^{-\frac{\rho}{2}} \partial_0 - n,
$$

(74)

then the condition is written as

$$
2C = e^{-\rho} C_{0i0j} + e^{-\frac{\rho}{2}} C_{0ij(n)} + e^{-\frac{\rho}{2}} C_{0ji(n)} + C_{i(n)j(n)} = 0.
$$

(75)

In terms of the extrinsic curvature and the components of the induced metric, it can be rewritten as

$$
\begin{align*}
&t^i t^j + \frac{2}{\lambda^2} \delta^{ik} t^k t^j - 2 \lambda e^{-\frac{\rho}{2}} t^i j - t^i j + \frac{t}{p} \left[ \frac{(t^\tau_\tau) - \frac{t^2}{p}}{p} + 2 \lambda e^{-\frac{\rho}{2}} \partial_\tau \frac{t^\tau_\tau}{p} \right] \\
&\quad - \frac{2}{\lambda} e^{-\frac{\rho}{2}} \delta^{ik} t^i j k - [\lambda^2 e^{-\rho} \delta^{ij} \partial_\tau \rho + \frac{p}{2} \gamma^{ik} \partial_k \partial_j \rho + \frac{p}{4} \delta^{ij} \gamma^{km} (\partial_k \rho) (\partial_m \rho)] \\
&\quad - \frac{p - 1}{4} \lambda^2 e^{-\rho} \delta^{ij} (\partial_j \rho)^2 + \frac{1}{2} \delta^{ij} \gamma^{km} \partial_k \partial_m \rho] \\
&\quad + e^{-\frac{\rho}{2}} \lambda [(-t^i j + \delta^i j t^\tau_\tau) \partial_\tau \rho + \frac{1}{\lambda^2} \left( \frac{1}{2} t^i j \delta^{ik} \partial_k \rho - e^\rho \delta^i j t^\tau_\tau \partial_k \rho + \frac{1}{2} e^\rho t^i j \partial_\tau \rho \right)]
= 0.
\end{align*}
$$

(76)
Although in general this equation is rather complicated, its leading expansions in power of $\lambda$ become very simple when the metric of the background has a Minkowski limit as described in (51). It turns out that the leading order of $t^i_j$ still has the form

$$t^i_j = \frac{t^{(1)}}{p} \delta^i_j - 2\delta^{ik}t^{(1)}_{(k,j)} + 2t^{\tau(1)}_\tau t^{(1)}_\tau. \quad (77)$$

The time component of the momentum constraint is

$$\partial_\tau t^\tau - \frac{1}{\lambda^2} \partial_i(t^{\tau i} e^\rho) - \frac{1}{2\lambda^2} \delta^{ik} t^\tau_k \partial_i \rho + \frac{1}{2} p t^\tau \partial_\tau \rho - \frac{1}{2} l \partial_\tau \rho - \frac{p}{2\lambda^2} e^\rho t^\tau_k \partial_k \rho = 0. \quad (78)$$

Its leading order gives rise to

$$\partial_\tau t^{\tau(1)} = 0. \quad (79)$$

While the space components of the constraint is

$$\partial_\tau t^\tau_i + \partial_k t^k_i + p + \frac{1}{2} t^\tau_k \partial_k \rho + \frac{1}{2} p t^\tau_i \partial_\tau \rho - \frac{t}{2} \partial_i \rho = 0. \quad (80)$$

Its leading and sub-leading orders of the expansion respectively lead to

$$\partial_i \rho^{(1)} = 0, \quad (81)$$

and

$$\partial_\tau t^{\tau(1)} + \frac{1}{p} \partial_\tau t^{(1)} + 2t^{\tau k(1)} \partial_k t^\tau_i^{(1)} + \frac{1}{2} \partial_i \rho^{(2)} - \partial^2 t^\tau_i^{(1)} = 0. \quad (82)$$

Identifying

$$t^\tau_i^{(1)} = \frac{\upsilon_i}{2}, \quad \rho^{(2)} = P \quad (83)$$

and taking $t^{(1)} = 0$, we arrive at the Navier-Stokes equation with incompressible condition.

$$\partial_\tau v^i = 0, \quad (84)$$

$$\partial_\tau v_i + \upsilon^k \partial_k v_i - \partial^2 v_i + \partial_i P = 0. \quad (85)$$

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