Analytical Composition of Differential Privacy via the Edgeworth Accountant

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Abstract

Many modern machine learning algorithms are composed of simple private algorithms; thus, an increasingly important problem is to efficiently compute the overall privacy loss under composition. In this study, we introduce the Edgeworth Accountant, an analytical approach to composing differential privacy guarantees of private algorithms. The Edgeworth Accountant starts by losslessly tracking the privacy loss under composition using the f-differential privacy framework [9], which allows us to express the privacy guarantees using privacy-loss log-likelihood ratios (PLLRs). As the name suggests, this accountant next uses the Edgeworth expansion [14] to the upper and lower bounds the probability distribution of the sum of the PLLRs. Moreover, by relying on a technique for approximating complex distributions using simple ones, we demonstrate that the Edgeworth Accountant can be applied to the composition of any noise-addition mechanism. Owing to certain appealing features of the Edgeworth expansion, the (ε, δ)-differential privacy bounds offered by this accountant are non-asymptotic, with essentially no extra computational cost, as opposed to the prior approaches in [16, 13], wherein the running times increase with the number of compositions. Finally, we demonstrate that our upper and lower (ε, δ)-differential privacy bounds are tight in federated analytics and certain regimes of training private deep learning models.

1 Introduction

Differential privacy (DP) provides a mathematically rigorous framework to analyze and develop private algorithms working on datasets containing sensitive information about individuals [10]. However, this framework often faces with challenges when it comes to analyzing the privacy loss of complex algorithms such as privacy-preserving deep learning and federated analytics [19] [22], which are composed of simple private building blocks. Therefore, the central question in this increasingly
active area is understanding how the overall privacy guarantees degrade from the repetition of simple algorithms that are applied to the same dataset.

Continued efforts to address this question have led to the development of relaxations of differential privacy and privacy analysis techniques [12, 11, 5, 6]. A recent flurry of activity in this line of research was triggered by [1], who proposed a technique called “moments accountant” to provide upper bounds on the overall privacy loss from the iterative training of private deep learning models. A shortcoming of the moments accountant technique is that the privacy bounds are generally not tight despite being computationally efficient. This is because this technique was enabled by Rényi DP [17], whose privacy loss profile is lossy for certain mechanisms, although some improvements exist [2, 23]. Alternatively, other lines of works directly compose $(\varepsilon, \delta)$-DP guarantees via numerical methods such as the fast Fourier transform [16, 13]. However, this approach can be computationally expensive because the number of algorithms under composition is large; this unfortunately is often the case when training deep neural networks.

Instead of these techniques, this study aims to develop computationally efficient lower and upper privacy bounds for composing private algorithms with finite-sample guarantees. This does so by relying on a new privacy definition called $f$-differential privacy ($f$-DP) [9]. $f$-DP offers a complete characterization of differential privacy guarantees using a hypothesis testing interpretation, which was first introduced in [15], and enables a precise tracking of the privacy loss under composition using a certain operation between the functional privacy parameters. Moreover, [9] developed an approximation tool to evaluate the overall privacy guarantees using a central limit theorem (CLT), which can lead to approximate $(\varepsilon, \delta)$-DP guarantees using the duality between $(\varepsilon, \delta)$-DP and Gaussian Differential Privacy (GDP, a special type of $f$-DP) [9]. Although the $(\varepsilon, \delta)$-DP guarantees are asymptotically accurate, a usable finite-sample guarantee is still missing in the $f$-DP framework.

In this study, we introduce the Edgeworth Accountant as an analytically efficient approach to obtaining finite-sample $(\varepsilon, \delta)$-DP guarantees by leveraging the $f$-DP framework. In short, the Edgeworth Accountant makes use of the Edgeworth approximation [14], which is a refinement of the CLT with a better convergence rate, to estimate the distribution of the sum of certain random variables that we refer to as “privacy-loss log-likelihood ratios” (PLLRs). By leveraging a Berry–Esseen type bound derived for the Edgeworth approximation, we obtain non-asymptotic upper and lower privacy bounds that are applicable to privacy-preserving deep learning and federated analytics. At a higher level, we compare the approach of our Edgeworth Accountant to the Gaussian Differential Privacy approximation in Figure 1. Additionally, we note that while the rate of the Edgeworth approximation is well conceived, the explicit finite-sample error bounds are highly non-trivial. To the best of our knowledge, this is the first time such a bound has been established in the statistical and differential privacy communities and it is also of interest on its own.

We have made two versions of our Edgeworth Accountant available to better fulfill practical needs: the approximate Edgeworth Accountant (AEA), and the exact Edgeworth Accountant (EEAI). The AEA can provide an estimate with an asymptotically accurate bound for any composition number $m$. By using a higher-order Edgeworth expansion, such an estimate can be arbitrarily accurate, provided that the Edgeworth series converges; therefore, it is useful in practice to quickly estimate privacy parameters. The EEAI provides an accurate finite-sample bound for any $m$ thus, efficiently providing a rigorous bound on the privacy parameters.

Our proposal is important as an efficiently computable DP-accountant. The runtime of our

\[ \text{Here, “sample” refers to the number of compositions of DP algorithms. Hereafter, we use the term “finite-sample” to mean that the bound is non-asymptotic in the number of compositions.} \]
algorithm becomes $O(1)$ to compute the privacy loss for composition of $m$ identical mechanisms. For the general case, (where we need to compose $m$ heterogeneous algorithms) the runtime becomes $O(m)$, which is information-theoretically optimal. In contrast, most existing privacy accountant algorithms can only achieve polynomial runtime for the general composition of private algorithms \cite{Liu2020, Wang2022}. Our proposal relies on the Edgeworth approximation and its corresponding finite-sample bounds. Another generality of the Edgeworth Accountant is that it can be applied to the composition of any (subsampled) noise-addition mechanisms, including the well known NoisySGD algorithm in privacy-preserving deep learning \cite{Kamath2020} and federated analytics \cite{Mcmahan2017, Chi2019}.

The remainder of the study is organized as follows. We briefly summarize the related work on the privacy accounting of differentially private algorithms as well as our contributions in Section 1. In Section 2 we introduce the concept of $f$-DP and its important properties. We then introduce the notion of privacy-loss log-likelihood ratios in Section 3 and establish how to use them for privacy accountants based on the distribution function approximation. In Section 4 we provide a new method, the Edgeworth Accountant, that can efficiently and almost accurately evaluate the privacy guarantees, while providing finite-sample error bounds. The simulation results and conclusions can be found in Sections 5 and 6, respectively. The proofs and technical details are provided in the appendices.

1.1 Motivating applications

We now discuss two motivating applications — NoisySGD \cite{Kamath2020, Li2020, Kamath2020, Li2020} and the Federated Analytics and Federated Learning \cite{Mcmahan2017, Chi2019}. The analysis of DP guarantees of these applications is important and especially challenging owing to the large number of compositions involved. Our primary goal is to devise a general tool to analyze the DP guarantees for these applications.

**NoisySGD.** NoisySGD is one of the most popular algorithms for training differentially private neural networks. In contrast to the standard SGD, the NoisySGD has two additional steps in each iteration — clipping (to bound the sensitivity of the gradients) and noise addition (to guarantee the privacy of the models). Details of the NoisySGD algorithm are provided in Algorithm 1 in Appendix A.

**Federated Analytics.** Federated analytics is a distributed analytical model \cite{Mcmahan2017, Chi2019}, which performs statistical tasks through interactions between a central server and local devices. To complete a global analytical task, the central server randomly selects a subset of devices in each iteration to carry out local analytics and then aggregates the results for the statistical analysis. The total number of
iterations is usually very large in federated analytics, requiring a tight analysis of its DP guarantee.

1.2 Related work

In this section, we present a survey of several related works on DP accountants.

Moments Accountant and Rényi DP. [1] proposed “moments accountant”, which uses Rényi DP [17] to provide an upper bound for the DP guarantee of the composition of DP algorithms. With the help of the moments accountant technique, [1] proposed the differentially private stochastic gradient descent (DP-SGD) algorithm, whose privacy loss can be effectively bounded. However, as mentioned before, the Rényi DP only yields lossy conversion to $(\varepsilon, \delta)$-DP, often making the upper bound impractical to use. The runtime of the accountant is independent of $m$, the number of compositions, for DP-SGD, and is $O(m)$ for the composition of general algorithms.

Numerical Composition obtained via FFT. In another work, [16, 13] approximated the privacy loss of compositions using fast Fourier transform on the convolutions of the privacy-loss random variables (PRVs). This notion is closely related to our definition of the PLLRs. Although both definitions allow for computing compositions by understanding the convolutions of random variables, we note that the two concepts stem from different analysis frameworks. Specifically, PRV amounts to finding a pair of random variables that reparametrizes the privacy curve, which is dual to the trade-off function. On the other hand, PLLRs are defined naturally from the hypothesis-testing perspective of $f$-DP; hence, the random variables directly decompose into the sum of their log-likelihood ratios. Consequently, Proposition 3.2 in our study is a strict generalization that encompasses their Theorem 3.2 as a special case when $m = 1$. Note that their FFT accountant is the first algorithm that can approximate the privacy loss up to arbitrary levels of precision; the runtime of their algorithm is $O(\sqrt{m})$ for DP-SGD and $O(m^{2.5})$ for general compositions.

Analytical Composition via Characteristic Functions. Recently, [26] proposed using a characteristic function to analytically compute composition of privacy algorithms. Their algorithm, the Analytical Fourier Accountant, yields tight privacy accounting but fails to perform efficient computations for the sub-sampled mechanisms. Their time complexity is $O(1)$ if the characteristic function of their dominating Privacy Loss Distribution (PLD) of $m$-fold is simple enough for closed-form composition, and is at least $\Omega(m^2)$ when no closed-form solution is available.

$f$-DP accountant via Edgeworth expansion. It is worth mentioning that [24] also uses the Edgeworth expansion for DP guarantees. Specifically, they use the Edgeworth approximation as a refinement to the CLT to better approximate the $f$-DP trade-off curve. The most important difference between the two approaches is that we provide a finite-sample error bound that allows for an exact DP accountant, while they focus solely on an asymptotic approximation to the trade-off curves. In addition, we use the Edgeworth approximation on PLLRs to obtain an estimate of the exact characterization of $(\varepsilon, \delta)$-DP (the lower path in Figure [1]), while they directly approximate the trade-off function $f$ (the same as GDP, using the upper path in Figure [1]). Therefore, we focus more on the practical aspect (finite-sample guarantee), and interpretability (directly dealing with $(\varepsilon, \delta)$-DP).

Table [1] presents a comparison of the existing works on DP accountant. Specifically, we focus on their theoretical guarantees and the runtime complexities when the number of compositions is $m$.

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2 The number of iterations can be small for a single analytical task. However, in most practical cases, many statistical tasks are performed on the same base of users which leads to a large number of total iterations.
| Method     | Finite-sample guarantee | Tightness of guarantee | Computational complexity |
|------------|-------------------------|------------------------|-------------------------|
| GDP/GDP-E  | No                      | N/A                    | $O(1), O(m)$            |
| MA         | Only upper bound        | Loose conversion to $(\varepsilon, \delta)$-DP | $O(1), O(m)$ |
| FFT        | Yes                     | Yes                    | $O(\sqrt{m}), O(m^{2.5})$ |
| EA         | Yes                     | Yes*                   | $O(1), O(m)$            |

Table 1: Comparison of different DP accountants. Each entry in the computation complexity contains two columns — (Left) the runtime for the composition of $m$ identical algorithms; and (Right) the runtime for the composition of $m$ general algorithms. GDP: the Gaussian differential privacy accountant [9]; GDP-E: the Edgeworth refinement to the GDP accountant [24]; MA: the moments accountant using Rényi-DP [1]; FFT: the fast Fourier transform accountant for privacy random variables [13]; EA: the Edgeworth Accountant we propose, including both the AEA (Definition 4.1), and the EEAI (Definition 4.2). *The guarantee of EA is tight when the order of the underlying Edgeworth expansion $k$ is high, or when $m$ is large for $k = 1$.

1.3 Our contributions

We now briefly summarize our three main contributions.

Improved time-complexity and estimation accuracy. We propose a new DP accountant method, termed the Edgeworth Accountant, which provides a finite-sample error bound in constant/-linear time complexity for the composition of identical/general mechanisms. In practice, our method outperforms the GDP and moments accountant, with almost the same runtime.

Unified framework for efficient and computable evaluation of $f$-DP guarantee. Although the evaluation of $f$-DP guarantee is #P-hard, we provide a general framework to efficiently approximate it. Leveraging this framework, any approximation scheme applied to the CDFs of the sum of the privacy-loss log-likelihood ratios (PLLRs) can be directly transformed to a new DP accountant.

Exact finite-sample Edgeworth bound analysis. To the best of our knowledge, we are the first to use Edgeworth expansion with finite-sample bounds in the statistics and machine learning communities. The analysis of these finite-sample bounds of Edgeworth expansion is of its own interest, and has many potential applications. We further derive an explicit adaptive exponential decaying bound for the Edgeworth expansion of the PLLRs, which is the first such result for the Edgeworth expansion.

2 Preliminaries and Problem Setup

In this section, we first mathematically define the notion of differential privacy and $f$-DP. Then, we set up the problem by revisiting our motivating applications.

A differentially private algorithm promises that an adversary with perfect information about the entire private dataset in use, except for a single individual, will find it difficult to distinguish between its presence or absence based on the output of the algorithm [10]. Formally, for $\varepsilon > 0$, and $0 \leq \delta < 1$, we consider a (randomized) algorithm $M$ that takes a dataset as its input.

**Definition 2.1.** A randomized algorithm $M$ is $(\varepsilon, \delta)$-DP if for any neighboring dataset $S, S'$ differs by an arbitrary sample, and for any event $E$, $\Pr[M(S) \in E] \leq e^{\varepsilon} \cdot \Pr[M(S') \in E] + \delta$.

In [9], the authors have proposed the use of the trade-off between type-I and type-II errors instead of a few privacy parameters in $(\varepsilon, \delta)$-DP. To formally define this new privacy notion, we denote the distribution of $M(S)$ and $M(S')$ by $P$ and $Q$; let $\phi$ be a (possibly randomized) rejection
We aim to compute the explicit DP guarantees for the general composition of trade-off functions of the form \( f = \bigotimes_{i=1}^{m} f_i \). For the \( i \)-th composition, the trade-off function \( f_i = T(P_i, Q_i) \) is realized by the following two hypotheses:

\[ H_{0,i} : w_i \sim P_i \text{ vs. } H_{1,i} : w_i \sim Q_i, \]

The following facts regarding \( f \)-DP have been established in previous studies:\footnote{4} \footnote{9}.

**Fact 1** (Duality to \((\varepsilon, \delta)\)-DP). A mechanism is \( f \)-DP if and only if it is \((\varepsilon, \delta(\varepsilon))\)-DP for all \( \varepsilon > 0 \), where \( \delta(\varepsilon) = 1 + f^*(-e^{\varepsilon}) \), and \( g^*(y) = \sup_{-\infty < x < \infty} yx - g(x) \) is the convex conjugate of \( g \).

**Fact 2** (Composition). Letting \( M_1 \) and \( M_2 \) be two mechanisms, whose composition algorithm \( M \) is defined as \( M(S) = (M_1(S), M_2(S, M_1(S))) \). In general, compositions of more than two algorithms follow recursively. Given the trade-off functions \( f = T(P, Q) \) and \( g = T(P', Q') \), let \( f \otimes g = T(P \times P', Q \times Q') \). We assume \( M_t \) is \( f_t \)-DP for \( t = 1, \ldots, m \). The composition theorem states that their \( m \)-fold composition algorithm is \( f_1 \otimes \cdots \otimes f_m \)-DP, which is tight in general.

**Fact 3** (Subsampling). Consider the following two most common subsampling schemes — (1) (Poisson subsampling) for each individual in dataset \( S \), includes its datum in the subsample independently with probability \( p \); (2) (Uniform subsampling) draws a subsample of \( S \) that is chosen uniformly at random among all \( s \) \( |S| \)-sized subsets of \( S \). Denote \( \text{Id}(\alpha) = 1 - \alpha \), and suppose an algorithm \( M \) is \( f \)-DP. The subsampling for \( f \)-DP states that the Poisson and uniformly subsampled algorithms are both \( \min \{ f_p, f_p^{-1}\} \)-\( \ast \)-DP, where \( f_p = pf + (1 - p) \text{Id} \).

**Fact 4** (Gaussian Differential Privacy (GDP)). To address the composition of \( f \)-DP guarantees, \footnote{4} \footnote{9} introduce the concept of \( \mu \)-GDP, which is a special case of \( f \)-DP with \( f = G_\mu = T(N(0,1),N(\mu,1)) \). They prove that when all the \( f \)-DP guarantees are close to the identity, their composition is asymptotically a \( \mu \)-GDP with some computable \( \mu \), which can then be converted to \((\varepsilon, \delta)\)-DP via duality. However, it does not include an usable finite-sample bound to the approximation error.

Based on these facts, we characterize the \( f \)-DP guarantee for motivating applications in Section\footnote{1.1}.

**NoisySGD.** For a NoisySGD with \( m \) iterations, subsampling ratio of \( p \), and noise multiplier \( \sigma \), it is known that it is \( \min \{ f, f^{-1}\} \)-\( \ast \)-DP \footnote{4} \footnote{9}, with \( f = (pG_1/\sigma + (1 - p)\text{Id})^{\otimes m} \).

**Federated Analytics.** Suppose there are \( m \) tasks, and each task is \( f_t \)-DP, with \( f_t = T(P_t, Q_t) \). Then, the overall DP guarantee is characterized by \( \bigotimes_{i=1}^{m} f_i \)-DP.

It is easy to see that the \( f \)-DP guarantee of NoisySGD is a special case of the \( f \)-DP guarantee of federated analytics with each trade-off function being \( f_i = \min \{ f_p, f_p^{-1}\} \)-\( \ast \) — with an identical composition of subsampled Gaussian mechanisms. Therefore, our goal is to efficiently and accurately evaluate the privacy guarantee of the general \( \bigotimes_{i=1}^{m} f_i \)-DP with an explicit finite-sample error bound.

### 3 Privacy-Loss Log-likelihood Ratios (PLLs)

We aim to compute the explicit DP guarantees for the general composition of trade-off functions of the form \( f = \bigotimes_{i=1}^{m} f_i \). For the \( i \)-th composition, the trade-off function \( f_i = T(P_i, Q_i) \) is realized by the following two hypotheses:

\[ H_{0,i} : w_i \sim P_i \text{ vs. } H_{1,i} : w_i \sim Q_i, \]
where $P_i, Q_i$ are two distributions. To evaluate the trade-off function $f = \bigotimes_{i=1}^m f_i$, we essentially distinguish between the two composite hypotheses

$$H_0 : w \sim P_1 \times P_2 \times \cdots \times P_m \text{ vs. } H_1 : w \sim Q_1 \times Q_2 \times \cdots \times Q_m,$$

where $w = (w_1, \ldots, w_m)$ is the concatenation of all $w_i$ values. Motivated by the optimal test asserted by the Neyman-Pearson Lemma, we provide the following definition.

**Definition 3.1.** The associated pair of privacy-loss log-likelihood ratios (PLLRs) is defined as the logarithm of the Radon-Nikodym derivatives of the two hypotheses under null and alternative hypotheses, respectively. Specifically, PLLRs can be expressed with respect to $H_{0,i}$ and $H_{1,i}$ as

$$X_i \equiv \log \left( \frac{dQ_i}{dP_i}(\xi_i) \right), \quad Y_i \equiv \log \left( \frac{dQ_i}{dP_i}(\zeta_i) \right), \quad \text{where } \xi_i \sim P_i, \zeta_i \sim Q_i.$$

Note that the definition of PLLRs depends only on two hypotheses. This allows us to convert $(\varepsilon, \delta)$-DP guarantees into a collection of $(\varepsilon, \delta)$-DP guarantees losslessly. The following proposition characterizes the relationship between $\varepsilon$ and $\delta$ in terms of the distribution functions of PLLRs:

**Proposition 3.2.** Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$ be the PLLRs defined above. Let $F_{X,m}, F_{Y,m}$ be the CDFs of $X_1 + \cdots + X_m$ and $Y_1 + \cdots + Y_m$, respectively. Then, the composed mechanism is $f$-DP (with $f = \bigotimes_{i=1}^m f_i$) if and only if it is $(\varepsilon, \delta)$-DP for all $\varepsilon > 0$ with $\delta$ defined by

$$\delta = 1 - F_{Y,m}(\varepsilon) - e^\varepsilon (1 - F_{X,m}(\varepsilon)). \quad (3.1)$$

Proposition 3.2 establishes a relationship between $f$-DP and a collection of $(\varepsilon, \delta(\varepsilon))$-DP, which reflects the primal-dual relationship between them. Note that some special forms of this general proposition have been previously proven in terms of privacy loss random variables. See [33, 26, 24] for details. The key contribution of Proposition 3.2 is that we express the $(\varepsilon, \delta)$-DP characterization of the $\#P$-complete $f$-DP in terms of (3.1), which can be approximated directly. Of note, the above relationship is general in the sense that we make no assumptions about the private mechanisms.

Definition 3.1 can be applied directly when $\frac{dQ_i}{dP_i}(\xi_i)$ is easy to compute, which corresponds to the case without subsampling. To deal with the case with subsampling, one must consider that the subsampled DP guarantee is the double conjugate of the minimum of two asymmetric trade-off functions (for example, recall that the trade-off function of a single sub-sampled Gaussian mechanism is $\min\{f_p, f_p^{-1}\}$, where $f_p = (pG_{1/\sigma} + (1-p)\text{Id})$). In general, the composition of multiple subsampled mechanisms satisfies $f$-DP for $f = \min\{\bigotimes_{i=1}^m f_{p_i}, \bigotimes_{i=1}^m f_{p_i}^{-1}\}$

**Lemma 3.3.** Suppose that for each $\alpha$, the functions $f^{(\alpha)}$ and $\delta^{(\alpha)}$ satisfy the condition that a mechanism is $f^{(\alpha)}$-DP if and only if it is $(\varepsilon, \delta^{(\alpha)}(\varepsilon))$-DP for all $\varepsilon > 0$. Then, a mechanism is $f = \left(\min_{\alpha \in \mathcal{I}} \{f^{(\alpha)}\}\right)^*$-DP if and only if it is $(\varepsilon, \sup_{\alpha} \{\delta^{(\alpha)}(\varepsilon)\})$-DP for all $\varepsilon > 0$.

\[\text{For completeness, we explicitly require that all } \xi_i \text{ and } \zeta_i \text{ be independent.}\]
We provide the proof of this lemma in the appendices. The intuition is that both \((\inf_{\alpha \in \mathbb{Z}} \{f^{(\alpha)}\})^{**}\)-DP and \((\varepsilon, \sup_{\alpha} \{\delta^{(\alpha)}(\varepsilon)\})\)-DP correspond to the tightest possible DP-guarantee for the entire collection.

Lemma 3.3 allows us to characterize the subsampled Gaussian mechanism using two sequences of PLLRs. As mentioned above, it is \(f\)-DP with \(f = \min\{\otimes_{i=1}^{m} f_{i,p}, \otimes_{i=1}^{m} f_{i,p}^{-1}\}^{**}\), where each \(f_{i,p} = (pG_{1/\sigma} + (1-p)\text{Id})\). For the first part, the PLLRs corresponding to \(\otimes_{i=1}^{m} f_{i,p} = (pG_{1/\sigma} + (1-p)\text{Id})^{\otimes m}\) are given by

\[
X_i^{(1)} = \log(1 - p + pe^{\mu \xi_i - \frac{1}{2} \mu^2}), \quad \text{and} \quad Y_i^{(1)} = \log(1 - p + pe^{\mu \xi_i - \frac{1}{2} \mu^2}),
\]

for \(1 \leq i \leq m\), with \(\xi_i \sim \mathcal{N}(0,1), \xi_i \sim p\mathcal{N}(0,1) + (1-p)\mathcal{N}(\mu,1)\). And for the second part, the PLLRs corresponding to \(\otimes_{i=1}^{m} f_{i,p}^{-1} = ((pG_{1/\sigma} + (1-p)\text{Id})^{-1})^{\otimes m}\) are given by

\[
X_i^{(2)} = -\log(1 - p + pe^{\mu \xi_i - \frac{1}{2} \mu^2}), \quad \text{and} \quad Y_i^{(2)} = -\log(1 - p + pe^{\mu \xi_i - \frac{1}{2} \mu^2}),
\]

for \(1 \leq i \leq m\), with \(\xi_i \sim \mathcal{N}(0,1), \xi_i \sim p\mathcal{N}(0,1) + (1-p)\mathcal{N}(\mu,1)\). By substituting \(F_{X^{(1)},m}\) and \(F_{Y^{(1)},m}\) by any approximation (for example, using CLT or Edgeworth), we obtain a computable relationship in terms of the \((\varepsilon, \delta^{(1)}(\varepsilon))\)-DP. Similarly, we can obtain a relationship in terms of the \((\varepsilon, \delta^{(2)}(\varepsilon))\)-DP. We conclude that the subsampled Gaussian mechanism is \((\varepsilon, \max\{\delta^{(1)}(\varepsilon), \delta^{(2)}(\varepsilon)\})\)-DP.

### 3.1 Transferred error bound based on CDF approximations

As discussed above, Lemma 3.3 allows us to characterize the double conjugate of the infimum of a collection of \(f^{(\alpha)}\)-DPs by analyzing each sequence of PLLRs separately. As a result, our aim is to compute the bounds of \(\delta^{(\alpha)}\) for each single trade-off function \(f^{(\alpha)}\). To fulfill this purpose, we seek an efficient algorithm to approximate distribution functions of the sum of PLLRs, namely, \(F_{X^{(\alpha)},m}, F_{Y^{(\alpha)},m}\). This perspective provides a general framework that naturally encompasses many existing methods, including the fast Fourier transform [13] and the characteristic function method [26]. These can be viewed as different methods for determining the upper and lower bounds of \(F_{X^{(\alpha)},m}, F_{Y^{(\alpha)},m}\). Specifically, we denote the upper and lower bounds of \(F_{X^{(\alpha)},m}\) by \(F_{X^{(\alpha)},m}^{+}\) and \(F_{X^{(\alpha)},m}^{-}\), and similarly for \(F_{Y^{(\alpha)},m}\). These bounds can be easily converted to the error bounds on the privacy parameters of the form \(g^{(\alpha)-}(\varepsilon) \leq \delta^{(\alpha)}(\varepsilon) \leq g^{(\alpha)+}(\varepsilon)\), for all \(\varepsilon > 0\), where

\[
g^{(\alpha)+}(\varepsilon) = 1 - F_{Y^{(\alpha)},m}^{-}(\varepsilon) - e^{\varepsilon} (1 - F_{X^{(\alpha)},m}^{+}(\varepsilon)), \quad g^{(\alpha)-}(\varepsilon) = 1 - F_{Y^{(\alpha)},m}^{+}(\varepsilon) - e^{-\varepsilon} (1 - F_{X^{(\alpha)},m}^{-}(\varepsilon)).
\]

(3.2)

Thus, the DP guarantee of \((\inf_{\alpha} f^{(\alpha)})^{**}\)-DP in the form of \((\varepsilon, \delta(\varepsilon))\) satisfies \(\sup_{\alpha} \{g^{(\alpha)-}(\varepsilon)\} \leq \delta(\varepsilon) \leq \sup_{\alpha} \{g^{(\alpha)+}(\varepsilon)\}\), for all \(\varepsilon > 0\). To convert the guarantee of the form \((\varepsilon, \delta(\varepsilon))\) for all \(\varepsilon > 0\) to the guarantee of the form \((\varepsilon(\delta), \delta)\) for all \(\delta \in [0,1]\), we can invert the above bounds on \(\delta(\varepsilon)\) and obtain the bounds of the form \(\varepsilon^{-}(\delta) \leq \varepsilon(\delta) \leq \varepsilon^{+}(\delta)\). Here, \(\varepsilon^{+}(\delta)\) is the largest root of the equation \(\delta = \sup_{\alpha} \{g^{(\alpha)+}(\cdot)\}\), and \(\varepsilon^{-}(\delta)\) is the smallest non-negative root of the equation \(\delta = \sup_{\alpha} \{g^{(\alpha)-}(\cdot)\}\).

**Remark 3.4.** In practice, we often need to solve for these roots numerically and specify a finite range in which we can find all roots. In Appendix A, we exemplify how to find such a range for NoisySGD (see Remark A.1 in the appendices for details).
4 Edgeworth Accountant with Finite-Sample Guarantee

In this section, we present a new approach — the Edgeworth Accountant. This is based on the Edgeworth expansion to approximate the distribution functions of the sum of PLLRs. For simplicity, we demonstrate how to obtain the Edgeworth Accountant for any trade-off function \( f^{(a)} \) based on a single sequence of PLLRs \( \{X_i^{(a)}\}_{i=1}^m, \{Y_i^{(a)}\}_{i=1}^m \). Henceforth, we drop the superscript \( \alpha \) when it is clear from the context. Specifically, we derive an approximate Edgeworth Accountant (AEA) and the associated exact Edgeworth Accountant interval (EEAI) for \( f \) with PLLRs \( \{X_i\}_{i=1}^m, \{Y_i\}_{i=1}^m \). We define AEA and EEAI for the general trade-off function of the form \((\inf_{\alpha} f^{(a)})^*\) in Appendix B.

4.1 Edgeworth Accountant

To approximate the CDF of a random variable \( X = \sum_{i=1}^m X_i \), we introduce the Edgeworth expansion in its most general form, where \( X_i \)'s are independent but not necessarily identical. This generality allows us to account for the composition of heterogeneous DP algorithms. Suppose \( \mathbb{E}[X_i] = \mu_i \) and \( \gamma_{p,i} := \mathbb{E}[(X_i - \mu_i)^p] < +\infty \) for some \( p \geq 4 \). We define \( B_m := \sqrt{\sum_{i=1}^m \mathbb{E}[(X_i - \mu_i)^2]} \), and \( \sum_{i=1}^m \mu_i = M_m \). Thus, the standardized sum can be written as \( S_m := (X - M_m)/B_m \). We denote \( E_{m,k,X}(x) \) as the \( k \)-th order Edgeworth approximation of \( S_m \). Note that the central limit theorem (CLT) can be viewed as the 0-th order Edgeworth approximation. The first-order Edgeworth approximation is obtained by adding one extra order \( O(1/\sqrt{m}) \) term to the CLT, that is, \( E_{m,1,X}(x) = \Phi(x) - \frac{\lambda_{3,m}}{\sqrt{m}} (x^2 - 1) \phi(x) \). Here, \( \Phi \) and \( \phi \) are the CDF and PDF of a standard normal distribution, respectively, and \( \lambda_{3,m} \) is a constant defined in Lemma 4.3. It is known that (see for example, [14]) the Edgeworth approximation of order \( p \) has an error rate of \( O(m^{-(p+1)/2}) \). This desirable property motivates us to use the rescaled Edgeworth approximation,

\[
G_{m,k,X}(x) = E_{m,k,X} ((x - M_m)/B_m) \quad \text{and} \quad G_{m,k,Y}(x) = E_{m,k,Y} ((x - M_m)/B_m),
\]

to approximate \( F_{X,m}(x) \) and \( F_{Y,m}(x) \), respectively, in (3.1). This is what we term the approximate Edgeworth Accountant (AEA).

**Definition 4.1** (AEA). The \( k \)-th order AEA that defines \( \delta(\varepsilon) \) for \( \varepsilon > 0 \) is given by \( \delta(\varepsilon) = 1 - G_{m,k,Y}(\varepsilon) - \varepsilon \gamma_{\varepsilon}(1 - G_{m,k,X}(\varepsilon)) \), for all \( \varepsilon > 0 \).

Asymptotically, the AEA is an exact accountant, because of the rate of convergence that the Edgeworth approximation admits. However, in practice, the finite-sample guarantee is still missing because the exact constant of such a rate is unknown. To obtain a computable \((\varepsilon, \delta(\varepsilon))\)-DP bound via (3.1), we require finite-sample bounds on the approximation error of the CDF for any finite number of iterations \( m \). Suppose that we can provide a finite-sample bound using the Edgeworth approximation of the form \( |F_{X,m}(x) - G_{m,k,X}(x)| \leq \Delta_{m,k,X}(x) \), where \( \Delta_{m,k,X}(x) \) can be computed. Then we have

\[
F^+_{X,m}(x) = G_{m,k,X}(x) + \Delta_{m,k,X}(x) \quad \text{and} \quad F^-_{X,m}(x) = G_{m,k,X}(x) - \Delta_{m,k,X}(x), \quad (4.1)
\]

and similarly, for \( F_{Y,m} \). We now define the exact Edgeworth Accountant interval (EEAI).

**Definition 4.2** (EEAI). The \( k \)-th order EEAI associated with the privacy parameter \( \delta(\varepsilon) \) for \( \varepsilon > 0 \) is given by \([\delta^-, \delta^+], \) where for all \( \varepsilon > 0 \)

\[
\delta^-(\varepsilon) \equiv 1 - G_{m,k,Y}(\varepsilon) - \Delta_{m,k,Y}(\varepsilon) - \varepsilon \gamma_{\varepsilon}(1 - G_{m,k,X}(\varepsilon) + \Delta_{m,k,X}(\varepsilon)), \\
\delta^+(\varepsilon) \equiv 1 - G_{m,k,Y}(\varepsilon) + \Delta_{m,k,Y}(\varepsilon) - \varepsilon \gamma_{\varepsilon}(1 - G_{m,k,X}(\varepsilon) - \Delta_{m,k,X}(\varepsilon)). \quad (4.2)
\]
To bound the EEAI, it suffices to have a finite-sample bound on $\Delta_{m,k,X}(\varepsilon)$ and $\Delta_{m,k,Y}(\varepsilon)$.

4.2 Uniform bound on PLLRs

We now address the bounds of the Edgeworth approximation on the PLLRs in (4.1). Our starting point is a uniform bound of the form $\Delta_{m,k,X}(x) \leq c_{m,k,X}$, for all $x$. The bound for $\Delta_{m,k,Y}(x)$ follows identically. To achieve this goal, we follow the analysis of the finite-sample bound in [3]. We state the bound of the first-order Edgeworth expansion.

**Lemma 4.3.** We define the average individual standard deviation $\tilde{B}_m := B_m / \sqrt{m}$ and the average standardized $r$-th cumulant as $\lambda_{k,m} := \frac{1}{m} \sum_{j=1}^{m} k_{r,j} / \tilde{B}_m^3$, where $k_{r,j}$ is the $r$-th centralized cumulant of the $j$-th sample. With bounded moments of order four, that is, $\gamma_{4,i} < +\infty$ for $1 \leq i \leq m$, we calculate the (uniform) bound on the Edgeworth expansion as

$$
\Delta_{m,1,X} \leq \frac{0.1995 \tilde{K}_{3,m}}{\sqrt{m}} + \frac{0.031 \tilde{K}_{3,m}^2 + 0.195 K_{4,m} + 0.054 |\lambda_{3,m}| \tilde{K}_{3,m} + 0.038 \lambda_{3,m}^2}{m} + r_{1,m},
$$

where $K_{p,m} := m^{-1} \sum_{i=1}^{m} E[|X_i - \mu_i|^p] / (\tilde{B}_m)^p$, which is the average standardized $p$-th absolute moment, and $\tilde{K}_{3,m} := K_{3,m} + \frac{1}{m} \sum_{i=1}^{m} E[|X_i - \mu_i| \gamma_{2,i} / \tilde{B}_m^3]$. Here $r_{1,m}$ is a remainder term of order $O(1/m^{5/4})$ that depends only on $K_{3,m}, K_{4,m}$ and $\lambda_{3,m}$, and is defined in Equation (F.1).

This lemma deals with the first-order Edgeworth approximation, which can be generalized to higher-order Edgeworth approximations. We present an analysis of the second- and third-order approximations in the appendices. The expression for $r_{1,m}$ only involves real integration with known constants, which can be numerically evaluated in a constant time.

**Remark 4.4.** The precision of the EEAI highly depends on the rate of the finite-sample bound of the Edgeworth expansion. Any better bounds for higher-order Edgeworth expansions can be directly applied to our EEAI by substituting $\Delta_{m,k,X}(\varepsilon)$; here we simply demonstrate when $k = 1$ leveraging the first-order expansion. Observe that Lemma 4.3 provides a bound of order $O(1/\sqrt{m})$ because we want to deal with general independent but not necessarily identical random variables. We demonstrate how one can obtain an $O(1/m)$ rate in the i.i.d. case in Appendix F. Our current first-order bound is primarily useful when $m$ is sufficiently large, but a bound for higher-order Edgeworth expansions can further improve the precision for all values of $m$.

4.3 Adaptive exponential decaying bound for NoisySGD

One specific concern regarding the bound derived in the previous section is that it is uniform in $\varepsilon$. Note that in (3.1), there is an amplification factor of the error by $e^\varepsilon$ in front of $F_{X,m}$. Therefore, as long as $\varepsilon$ grows in $m$ with an order of at least $\varepsilon \sim \Omega(\log m)$, the error term in (3.1) scales with order $e^{\Omega(\log m)} / O(m) = \Omega(1)$.

In this section, we study the composition of the subsampled Gaussian mechanism (including NoisySGD and many federated learning algorithms), where the previous bound can be improved when $\varepsilon$ is large. Informally, by omitting the dependence on $m$, we want to have a bound of the form $|F_{X,m}(\varepsilon) - G_{m,k,X}(\varepsilon)| = O(e^{-\varepsilon^2})$ to offset the effect of $e^\varepsilon$ in front of $F_{X,m}$. To this end, we first prove that the tail bound of $F_{X,m}(\varepsilon)$ is of the order $O(e^{-\varepsilon^2})$ with an exact constant. Combining this with the tail behavior of the Edgeworth expansion, we then conclude that the difference has
the desired convergence rate. Following the discussion in Section 3 we must calculate the bounds for the two PLLR sequences separately. Here we focus on the sequence of PLLRs corresponding to \((pG_{1/\sigma} + (1-p)\text{Id})^\otimes m\). These PLLRs are given by \(X_i = \log(1 - p + pe^{\mu \xi_i - \frac{1}{2} \mu^2})\), where \(\xi_i \sim N(0,1)\). The following theorem characterizes the tail behavior of \(F_{X,m}\). The tail bound of the sum of the other PLLR sequence corresponding to \((pG_{1/\sigma} + (1-p)^{-1})^\otimes m\) has the same rate, and can be proved similarly.

**Theorem 1.** There exists a positive constant \(a\) and some associated constant \(\eta(a) > 0\), such that the tail of \(F_{X,m}\) can be bounded as

\[
1 - F_{X,m}(\varepsilon) = \mathbb{P}\left( \sum_{i=1}^{m} X_i \geq \varepsilon \right) \leq 2 \exp\left( \frac{-(\varepsilon + m\eta)^2}{8m\tau^2} \right),
\]

where \(\tau^2 = \max\left\{ \frac{\log(1-p+pe^{\mu a - \frac{1}{2} \mu^2}) + \mu(a^+ - \mu) - \log(1-p)}{4}, \mu^2, \frac{(a^+ - \mu)^2 \mu^2}{2\log(\Phi(a^+) - \Phi(a))} \right\}\) and \(a^+ = \frac{\phi(a)}{1 - \Phi(a)}\).

The proof of Theorem 1 is provided in Appendix E along with its dependent technical lemmas. From the above theorem, we know that the tail of \(F_{X,m}(\varepsilon)\) is \(O(e^{-\max\{\varepsilon^2/m, m\}}) = o(e^{-\varepsilon})\), as long as \(\varepsilon = o(m)\). Note that in this case, the tail of the rescaled Edgeworth expansion is of the same order \(O(e^{-\max\{\varepsilon^2/m, m\}}) = o(e^{-\varepsilon})\). Therefore, we can give a finite-sample bound of the same rate for the difference between \(F_{X,m}(\varepsilon)\) and its approximation \(G_{m,k,X}\) for a large \(\varepsilon\). Note that this finite-sample bound scales better than the uniform bound in Lemma 4.3 when \(m\) and \(\varepsilon\) are both large.

### 4.4 Extension to other mechanisms

Note that our analysis framework is applicable to a wide range of common noise-adding mechanisms. Specifically, Lemma 4.3 only requires the distribution of PLLRs to have bounded fourth moments. For many common mechanisms, the counterpart of Theorem 1 can be similarly proved. We now demonstrate the generalization of our analysis to the Laplace Mechanism.

**The Laplace Mechanism.** It is straightforward to verify that the trade-off function for subsampled Laplace Mechanisms is given by \(\min\{(pL_{\mu} + (1-p)\text{Id})^\otimes m, (pL_{\mu} + (1-p)^{-1}\text{Id})^\otimes m \}^\star\), where \(L_{\mu} = T(\text{Lap}(0,1), \text{Lap}(\mu,1))\). The two associated sequences of PLLRs, \(X_i\) and \(Y_i\) can be expressed as \(X_i^{(1)} = \log(1 - p + pe^{\xi - |\xi - \mu|}), Y_i^{(1)} = \log(1 - p + pe^{|\xi - \mu|})\), and \(X_i^{(2)} = -\log(1 - p + pe^{\xi - |\xi - \mu|}), Y_i^{(2)} = -\log(1 - p + pe^{\xi - |\xi - \mu|})\), where \(\xi \sim \text{Lap}(0,1), \zeta \sim p\text{Lap}(\mu,1) + (1-p)\text{Lap}(0,1)\). Note that all PLLRs are bounded and thus sub-Gaussian. This implies that we can apply Lemma 4.3 directly and also bound the tail similar to Theorem 1.

**Proposition 4.5.** We denote \(\eta = -\max\{\mathbb{E}(X_i^{(1)}), \mathbb{E}(X_i^{(2)})\} > 0\). The tail of the sum of both sequences of PLLRs under the Laplace Mechanism exhibits the following inverse exponential behavior:

\[
\max\left\{ \mathbb{P}\left( \sum_{i=1}^{m} X_i^{(1)} \geq \varepsilon \right), \mathbb{P}\left( \sum_{i=1}^{m} X_i^{(2)} \geq \varepsilon \right) \right\} \leq \exp\left( \frac{-2(\varepsilon + m\eta)^2}{m\tau^2} \right),
\]

where \(\tau^2 = (\log(1 - p + pe^{\mu}) - \log(1 - p + pe^{-\mu}))^2\).
5 Numerical Experiments

In this section, we illustrate the advantages of the Edgeworth Accountant by presenting the plots of DP accountant curves under different settings. Specifically, we plot the privacy curve of $\varepsilon$ against the number of compositions and compare our methods (AEA and EEAI) with those of existing DP accountants. The implementation details of our Edgeworth Accountant is provided in Appendix C, and Python code of it can be found at https://github.com/HuaWang-wharton/EdgeworthAccountant.

The AEA. We first demonstrate that our proposed approximate Edgeworth Accountant (AEA) is indeed very accurate, outperforming the existing Rényi DP and the CLT approximations in experiments. The first experiment had the same setting as in Figure 1(b) in [13], where the authors reported that both RDP and GDP were inaccurate, whereas the second setting corresponded to a real federated learning task. The results are shown in Figure 2, in which we describe the specific settings of the caption. For each sub-figure, the dotted lines “FFT_LOW” and “FFT_UPP” denote the lower and upper bounds computed by FFT [13] which are used as the underlying ground truth. The “GDP” curve is computed by the CLT approximation [4], the “RDP” curve is computed by moments accountant using Rényi DP with subsampling amplification [23], and the “EW_EST” curve is computed by our (second-order) AEA. As is evident from the figures, our AEA outperforms the GDP and RDP.

![Figure 2: The privacy curve computed via several different accountants. Left: The setting in Figure 1(b) in [13], where $p = 0.01$, $\sigma = 0.8$, and $\delta = 0.015$. Middle and Right: The setting of a real application task in federated learning for 10 epochs, with $p = 0.05$, $\sigma = 1$, and $\delta = 10^{-5}$. Here, “EW_EST” is the estimate given by our approximate Edgeworth Accountant. We omit the RDP curve in the middle subfigure for better comparisons with other approaches.]

The EEAI. We now present the empirical performance of the EEAI obtained in Section 4.1. We still experiment with the NoisySGD. Details of the experiments are provided in the caption of Figure 3. The two error bounds of EEAI are represented by “EW_UPP” and “EW_LOW,” and all other curves are defined in the same way as in the previous setting.

6 Conclusion

Currently, there is a growing number of applications for differential privacy in large-scale deep learning and federated analytics. To enable real-world deployment, it is critical to accurately depict the privacy loss introduced by DP mechanisms when practitioners face a common utility-privacy trade-off. In this study, we provide a novel way to efficiently evaluate the composition of $f$-DP, which
Figure 3: We demonstrate the comparisons between our Edgeworth Accountant (both AEA and EEAI), the RDP accountant, and the FFT accountant (whose precision of $\varepsilon$ is set to be 0.1). The three settings are set so that the privacy guarantees do not change dramatically as $m$ increases. Specifically, in all three settings, we set $\delta = 0.1$, $\sigma = 0.8$, and $p = 0.4/\sqrt{m}$ (left), $p = 1/\sqrt{m}\log{m}$ (middle), and $p = 0.1\sqrt{\log{m}/m}$ (right). We omit the GDP curve here, because the performance is fairly close to the AEA (“EW_EST” curve) when $m$ is large.

serves as a general framework for constructing DP accountants based on approximations to PLLRs. Specifically, we introduce the Edgeworth Accountant, an efficient approach to compose DP algorithms via the Edgeworth approximation. By contrast, existing privacy accountant algorithms either fail to provide a finite-sample bound, or only achieve polynomial runtime for general compositions. Importantly, our approach complements the existing literature when the number of compositions is large, which is typically the case for applications such as large-scale deep learning and federated analytics.

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Appendix

A Analysis of NoisySGD

We present the algorithms we considered in Section 1.1. To start with, suppose we have a neural network $h$ that is governed by weights $w$, with samples $x_i$ and labels $y_i$ ($i = 1, ..., n$). The prediction for each example is $h(x_i, w)$, and the per-sample loss is given by $\ell(h(x_i, w), y_i)$ for some loss function $\ell$. We define the objective function $L$ to be the average of per-sample losses

$$L(w) = \frac{1}{n} \sum_{i=1}^{n} \ell(h(x_i, w), y_i).$$

Stochastic Gradient Descent (SGD) algorithm uses a mini-batch as a proxy to this objective function. To control the potential privacy leak in each step of SGD, we need to clip the gradients to control the sensitivity, after which a Gaussian noise is added to it. The details of the algorithm is shown below.

**Algorithm 1 NoisySGD (with local or global flat per-sample clipping)**

**Parameters:** initial weights $w_0$, learning rate $\eta$, subsampling probability $p$, number of iterations $m$, noise scale $\sigma$, gradient norm bound $R$.

for $t = 0, \ldots, m - 1$ do

- Subsample a batch $I_t \subseteq \{1, \ldots, n\}$ from training set with probability $p$
- for $i \in I_t$ do
  - $v_t^{(i)} \leftarrow \nabla_w \ell(f(x_i, w_t), y_i)$
  - $\bar{v}_t^{(i)} \leftarrow \min \{1, R/\|v_t^{(i)}\|_2\} \cdot v_t^{(i)}$  \(\triangleright\) Clip the gradient
  - $\bar{V}_t \leftarrow \sum_{i \in I_t} \bar{v}_t^{(i)}$  \(\triangleright\) Sum over batch
  - $w_{t+1} \leftarrow w_t - \frac{\eta}{|I_t|} (\bar{V}_t + \sigma R \cdot \mathcal{N}(0, I))$  \(\triangleright\) Apply Gaussian mechanism and descend

Output $w_{r,m}$

Recall that in Section 3.1, in order to transfer the bounds from CDF approximations to privacy parameters, we need to find a range that contains all possible roots of $\delta = g^+(\varepsilon), \delta = g^-(\varepsilon)$. Here we showcase how to find such bound in the case of NoisySGD.

**Remark A.1.** For NoisySGD, we can express such range analytically. Specifically, for any $\alpha \in \{1, 2\}$ (the index of the sequence of PLLRs), we focus on finding roots in the range $[0, C]$ for $\varepsilon^+(\alpha)$ and $\varepsilon^-(\alpha)$, where $C$ is the smallest value of $\varepsilon$ such that

$$C \geq \sup_{S \subseteq \mathbb{R}, \mathbb{P}(Y^{(\alpha)} \in S) \geq \delta} \log \left( \frac{\mathbb{P}(Y^{(\alpha)} \in S)}{\mathbb{P}(X^{(\alpha)} \in S)} \right).$$

This is clearly a (sub-optimal) upper bound. A loose bound of the range can be easily proved to be

$$C = \min \left\{ m \log \left( \frac{p \delta}{1 - \Phi(z_\delta + \mu)} \right), \log \left( \frac{\delta}{1 - \Phi(z_\delta/\sqrt{m + \mu})} \right) \right\},$$

where $z_\delta$ is the upper $\delta$ quantile of a standard normal distribution. We can find $0 < \varepsilon^-(\alpha) \leq \varepsilon^+(\alpha)$ in the range defined above.
B Full definition of AEA and EEAI

Recall that in Section 4.1, the AEA and EEAI are defined for a specific trade-off function \(f^{(\alpha)}\). This is only for the simplicity of notations. We now demonstrate how to generalize the definitions to the general trade-off function of the form \((\inf_\alpha f^{(\alpha)})^*\).

**Definition B.1** (AEA for general trade-off function). The k-th order AEA of \((\inf_\alpha f^{(\alpha)})^*\)-DP that defines \(\delta(\varepsilon)\) for \(\varepsilon > 0\) is given by \(\delta(\cdot) = \sup_\alpha \delta^{(\alpha)}(\cdot)\), where

\[
\delta^{(\alpha)}(\varepsilon) = 1 - G_{m,k,Y^{(\alpha)}(\varepsilon)}(\varepsilon) - e^\varepsilon (1 - G_{m,k,X^{(\alpha)}(\varepsilon)}(\varepsilon)),
\]

for any \(\alpha\).

**Definition B.2** (EEAI for general trade-off function). The k-th order EEAI of \((\inf_\alpha f^{(\alpha)})^*\)-DP associated with privacy parameter \(\delta(\varepsilon)\) for \(\varepsilon > 0\) is given by \([\delta^-, \delta^+],\) where \(\delta^-(\cdot) = \sup_\alpha \delta^{(\alpha)-}(\cdot),\)

\[
\delta^{(\alpha)-}(\varepsilon) \equiv 1 - G_{m,k,Y^{(\alpha)}(\varepsilon)}(\varepsilon) - \Delta_{m,k,Y^{(\alpha)}(\varepsilon)} - e^\varepsilon (1 - G_{m,k,X^{(\alpha)}(\varepsilon)}(\varepsilon) + \Delta_{m,k,X^{(\alpha)}(\varepsilon)}),
\]

\[
\delta^{(\alpha)+}(\varepsilon) \equiv 1 - G_{m,k,Y^{(\alpha)}(\varepsilon)}(\varepsilon) + \Delta_{m,k,Y^{(\alpha)}(\varepsilon)} - e^\varepsilon (1 - G_{m,k,X^{(\alpha)}(\varepsilon)}(\varepsilon) - \Delta_{m,k,X^{(\alpha)}(\varepsilon)}).
\]

C Implementation of Edgeworth Accountant

We now present the detailed implementation of AEA and EEAI.

**Algorithm 2 AEA**

**Parameters:** \(m\) general mechanisms \(M_1,\ldots,M_m\). An epsilon \(\varepsilon \geq 0\), and an order \(k \geq 1\).

for \(i = 1,\ldots,m\) do

Analytically encode all the corresponding PLLRs for \(M_i\), \(\{(X_i^{(\alpha)}, Y_i^{(\alpha)})\}_{\alpha}\) for all \(\alpha\).

Numerically calculate the cumulants up to order \(k + 2\) for \(X_i^{(\alpha)}\) and \(Y_i^{(\alpha)}\) for all \(\alpha\).

Calculate \(G_{m,k,X^{(\alpha)}(\varepsilon)}(\varepsilon)\) and \(G_{m,k,Y^{(\alpha)}(\varepsilon)}(\varepsilon)\) for each \(\alpha\) using k-th order Edgeworth expansion.

Calculate \(\delta^{(\alpha)}(\varepsilon)\) for each \(\alpha\) by \((B.1)\).

**Output** \(\sup_\alpha \delta^{(\alpha)}(\varepsilon)\).

And similarly, we present the algorithm for the general EEAI.

**Algorithm 3 EEAI**

**Parameters:** \(m\) general mechanisms \(M_1,\ldots,M_m\). An epsilon \(\varepsilon \geq 0\), and fix the order \(k = 1\).

for \(i = 1,\ldots,m\) do

Analytically encode all the corresponding PLLRs for \(M_i\), \(\{(X_i^{(\alpha)}, Y_i^{(\alpha)})\}_{\alpha}\) for all \(\alpha\).

Numerically calculate the cumulants up to order \(4\) for \(X_i^{(\alpha)}\), and \(Y_i^{(\alpha)}\) for all \(\alpha\).

Calculate \(G_{m,1,X^{(\alpha)}(\varepsilon)}(\varepsilon)\) and \(G_{m,1,Y^{(\alpha)}(\varepsilon)}(\varepsilon)\) for each \(\alpha\) using first order Edgeworth expansion.

Calculate \(\Delta_{m,1,X^{(\alpha)}(\varepsilon)}(\varepsilon)\) and \(\Delta_{m,1,Y^{(\alpha)}(\varepsilon)}(\varepsilon)\) for each \(\alpha\) using Lemma 4.3 or Theorem 1.

Calculate \(\delta^{(\alpha)+}(\varepsilon)\) and \(\delta^{(\alpha)-}(\varepsilon)\) for each \(\alpha\) by \((B.2)\).

**Output** \([\sup_\alpha \delta^{(\alpha)-}(\varepsilon), \sup_\alpha \delta^{(\alpha)+}(\varepsilon)]\).
Note that Algorithm 2 is an algorithm to find an estimate (bounds) of \( \delta \) given an \( \varepsilon \). And both algorithms run in constant/linear time for \( m \) identical/general compositions. In practice, people often would like to find an estimate or bounds on \( \varepsilon \) given an \( \delta \). To get such an estimate of \( \varepsilon \) given \( \delta \), we can directly inverse the Algorithm 2. And to get upper and lower bounds of \( \varepsilon \) given \( \delta \), we can use the inversion method discussed in Section 3.1, and specifically, the equations in (3.2).

D Proofs in Section 3

D.1 Proof of Proposition 3.2

We present the proof of Proposition 3.2 in this section. The proof relies on two Lemmas that are of self-interest and we first present the lemmas. The proof of Proposition 3.2 is straightforward from results of Lemmas. Recall that the trade-off functions \( f_i = T(P_i, Q_i) \) we consider are realized by the two following hypotheses:

\[
H_{0,i} : w_i \sim P_i \text{ vs. } H_{1,i} : w_i \sim Q_i,
\]

where \( P_i, Q_i \) are two distributions. To evaluate the trade-off function \( f = \bigotimes_{i=1}^m f_i \), we are essentially distinguishing between the two composite hypotheses

\[
H_0 : \mathbf{w} \sim P_1 \times P_2 \times \ldots \times P_m \text{ vs. } H_1 : \mathbf{w} \sim Q_1 \times Q_2 \times \ldots \times Q_m,
\]

where \( \mathbf{w} = (w_1, \ldots, w_m) \) is the concatenation of all the \( w_i \)'s. The following lemma shows how to connect PLLRs of each \( f_i \) to the trade-off function \( f \).

Lemma D.1. Let \( X_1, \ldots, X_m \) be the PLLR under the null hypothesis and, likewise, \( Y_1, \ldots, Y_m \) be the PLLR under the alternative. Let \( F_{X,m}, F_{Y,m} \) be the CDFs of \( x \equiv X_1 + \ldots + X_m \) and \( Y \equiv Y_1 + \ldots + Y_m \), respectively. Then we have the following relationship between privacy parameters and privacy-loss log-likelihood ratios

\[
\varepsilon = \log \frac{F_{Y,m}'(c)}{F_{X,m}'(c)},
\]

\[
\delta = \frac{F_{X,m}(c) (1 - F_{Y,m}(c)) - F_{Y,m}(c) (1 - F_{X,m}(c))}{F_{X,m}'(c)},
\]

where \( c \) is some constant.

Proof of Lemma D.1 To distinguish between \( H_0 : P_1 \times P_2 \times \ldots \times P_m \text{ vs. } H_1 : Q_1 \times Q_2 \times \ldots \times Q_m \), By the Neyman-Pearson lemma, we know that each point of the trade-off function \( f \) is realized by a likelihood ratio test (cut-off at some threshold \( c \)). So, the trade-off function takes a parametric form \( f(\alpha) = \beta \), where \( \alpha \) is the type-I error of the test, and \( \beta \) is type-II error of the test:

\[
\alpha = \mathbb{P}_{H_0} \left( \log \frac{dP_1 \times P_2 \times \ldots \times P_m}{dQ_1 \times Q_2 \times \ldots \times Q_m}(\mathbf{w}) > c \right)
\]

\[
\beta = \mathbb{P}_{H_1} \left( \log \frac{dP_1 \times P_2 \times \ldots \times P_m}{dQ_1 \times Q_2 \times \ldots \times Q_m}(\mathbf{w}) \leq c \right)
\]

\(^4\text{Note that if we substitute Edgeworth approximation with the true CDF of PLLRs, it is direct to show (by taking derivative) that } \delta^{(\alpha)}(\varepsilon) \text{ is always a decreasing function of } \varepsilon, \text{ and the supremum of decreasing functions is still a decreasing function. Therefore, we can always take inversion.}\)
Note that under $H_0$, we have
\[
\log \left( \frac{dP_1 \times P_2 \times \cdots \times P_m}{dQ_1 \times Q_2 \times \cdots \times Q_m} (w) \right) = \log \left( \frac{dP_1}{dQ_1} (w_1) \times \cdots \times \frac{dP_m}{dQ_m} (w_m) \right) = \log \left( \frac{dP_1}{dQ_1} (w_1) \right) + \cdots + \log \left( \frac{dP_m}{dQ_m} (w_m) \right) = X_1 + \cdots + X_m = X.
\]
and similarly under $H_1$,
\[
\log \left( \frac{dP_1 \times P_2 \times \cdots \times P_m}{dQ_1 \times Q_2 \times \cdots \times Q_m} (w) \right) = Y_1 + \cdots + Y_m = Y.
\]
So, we can simplify the parametric form of $f$ by $f(\alpha) = \beta$, where
\[
\alpha = \mathbb{P}(X_1 + \cdots + X_m > c) = 1 - F_{X,m}(c)
\]
\[
\beta = \mathbb{P}(Y_1 + \cdots + Y_m \leq c) = F_{Y,m}(c).
\]
This allows us to simply write
\[
f(\alpha) = F_{Y,m} \circ F_{X,m}^{-1}(1 - \alpha).
\]
For a point $(\alpha, \beta)$ on the trade-off function $f$, where
\[
\beta = F_{Y,m}(c) = F_{Y,m} \left( F_{X,m}^{-1}(1 - \alpha) \right),
\]
and $\alpha$ is small. By the equivalence given in Proposition 2.5 in [9], we know that the slope of the tangent line passing through $(\alpha, \beta)$ (for small $\alpha$) is given by
\[
-e^\varepsilon = \frac{df}{d\alpha}(\alpha) = F_{Y,m}' \left( F_{X,m}^{-1}(1 - \alpha) \right) \cdot \frac{1}{F_{X,m}' \left( F_{X,m}^{-1}(1 - \alpha) \right)} \cdot (-1) = \frac{F_{Y,m}' \left( F_{X,m}^{-1}(1 - \alpha) \right)}{F_{X,m}' \left( F_{X,m}^{-1}(1 - \alpha) \right)},
\]
which gives
\[
\varepsilon = \log \frac{F_{Y,m}' \left( F_{X,m}^{-1}(1 - \alpha) \right)}{F_{X,m}' \left( F_{X,m}^{-1}(1 - \alpha) \right)} = \log \frac{F_{Y,m}'(c)}{F_{X,m}'(c)}.
\]
The equation of the tangent line takes the form of
\[
y = -\frac{F_{Y,m}'(c)}{F_{X,m}'(c)} \left( x - 1 + F_{X,m}(c) \right) + F_{Y,m}(c).
\]
Using the same proposition, we know the intercept of the line is $1 - \delta$, so we should have
\[
1 - \delta = \frac{F_{Y,m}(c)(1 - F_{X,m}(c))}{F_{X,m}(c)} + F_{Y,m}(c) = \frac{F_{Y,m}'(c)(1 - F_{X,m}(c)) + F_{X,m}'(c)F_{Y,m}(c)}{F_{X,m}(c)}.
\]
which gives
\[ \delta = 1 - \frac{\frac{F'_Y(m)(c)}{F_X(m)(c)}(1 - F_X(m)(c)) + F'_X(m)(c)F_Y(m)(c)}{F'_X(m)(c)} \]
\[ = \frac{F'_X(m)(c)(1 - F_Y(m)(c)) - F'_Y(m)(c)(1 - F_X(m)(c))}{F'_X(m)(c)}. \]

Therefore, \( \varepsilon \) and \( \delta \) takes the following parametric form as in the statement of the lemma,
\[ \varepsilon = \log \frac{F'_Y(m)(c)}{F_X(m)(c)}, \]
\[ \delta = \frac{F'_X(m)(c)(1 - F_Y(m)(c)) - F'_Y(m)(c)(1 - F_X(m)(c))}{F'_X(m)(c)}. \]

To simplify the relation in (D.2), we observe the following interesting lemma about PLLRs.

**Lemma D.2.** Let \( X_1, X_2, \ldots, X_m \) and \( Y_1, Y_2, \ldots, Y_m \) and \( F_{X,m}, F_{Y,m} \) be defined as in Lemma D.1. Let \( f_{X,m}, f_{Y,m} \) be the PDFs of \( \sum_{i=1}^m X_i \) and \( \sum_{i=1}^m Y_i \). Then we have for any \( c \in \mathbb{R}, \)
\[ c = \log \frac{f_{Y,m}(c)}{f_{X,m}(c)}. \] (D.3)

**Proof of Lemma D.2.** We use induction on \( m \), the number of compositions, to prove this Lemma.

**Base Case:** \( m = 1 \). When \( m = 1 \), we write out the forms of \( X \) and \( Y \) explicitly as
\[ X = \log \frac{Q_1(w_1)}{P_1(w_1)} \quad \text{where } w_1 \sim P_1, \]
\[ Y = \log \frac{Q_1(w_1)}{P_1(w_1)} \quad \text{where } w_1 \sim Q_1. \]

As a result, for any measurable function \( g : \mathbb{R} \rightarrow \mathbb{R} \), we have
\[ \mathbb{E}_Y[g(Y)] = \mathbb{E}_{w_1 \sim Q_1} \left[ g \left( \log \frac{Q_1(w_1)}{P_1(w_1)} \right) \right] \]
\[ = \mathbb{E}_{w_1 \sim P_1} \left[ g \left( \log \frac{Q_1(w_1)}{P_1(w_1)} \right) \frac{Q_1(w_1)}{P_1(w_1)} \right] \]
\[ = \mathbb{E}_{w_1 \sim P_1} \left[ g \left( \log \frac{Q_1(w_1)}{P_1(w_1)} \right) e^X \right] \]
\[ = \mathbb{E}_X[g(X)e^X]. \]

Since the above equality holds for all \( g \), we must have that their exists a version of both PDFs such that \( f_{Y,1}(t) = f_{X,1}(t)e^t \). This shows that for \( m = 1, \)
\[ c = \log \frac{f_{Y,1}(c)}{f_{X,1}(c)}. \]

**Induction Step:** Suppose the result is true for \( m \), we now show that it is also true for \( m + 1 \) compositions. We now claim the following Lemma.
Lemma D.3. Let $A_1, A_2, B_1, B_2$ be four random variables. Denote the PDFs of $A_1, A_2, B_1, B_2$ by $f_{A_1}, f_{A_2}, f_{B_1}, f_{B_2}$ respectively. Suppose further that

$$f_{B_1}(t) = g(t)f_{A_1}(t), \quad f_{B_2}(t) = g(t)f_{A_2}(t) \quad \text{for all } t,$$

for some function $g$ satisfying $g(x + y) = g(x)g(y)$. Let $f_{A_2}, f_{B_2}$ denote the density function for $A_1 + A_2, B_1 + B_2$. Then

$$f_{B,2}(t) = g(t)f_{A,2}(t) \quad \text{for all } t.$$

Proof of Lemma D.3 will be given at the end of the proof. Applying Lemma D.3 on random variables $A_1 = \sum_{i=1}^m X_i, A_2 = X_{m+1}$ and $B_1 = \sum_{i=1}^m Y_i, B_2 = Y_{m+1}$, we will show that we get the desired relationship for $m+1$ compositions. By induction hypothesis we know that $f_{B_1}(t) = g(t)f_{A_1}(t)$ for $g(t) = e^t$. Since $f_{Y_{m+1}}(t) = g(t)f_{X_{m+1}}(t)$ by the base case in induction, and $g(x + y) = g(x)g(y)$, we have $f_{Y_{m+1}}(t) = g(t)f_{X_{m+1}}(t)$ for all $t$. This indicates that we have

$$c = \log \frac{f_{Y_{m+1}}(c)}{f_{X_{m+1}}(c)}$$

for any $c$. Hence we have completed the induction step and concluded the proof.

Proof of Lemma D.3. We use the convolution formula on $B_1, B_2$ and obtain

$$f_{B,2}(t) = \int_{-\infty}^{\infty} f_{B_1}(t-u)f_{B_2}(u)du$$

$$= \int_{-\infty}^{\infty} g(t-u)g(u)f_{A_1}(t-u)f_{A_2}(u)du$$

$$= \int_{-\infty}^{\infty} g(t)f_{A_1}(t-u)f_{A_2}(u)du$$

$$= g(t)f_{A_2}(t)$$

by convolution formula on $A_1, A_2$.

Proof of Lemma 3.3. Define

$$h(x) = \left(\inf_{\alpha \in I} f^{(\alpha)}\right)^*(x),$$

which is convex and lower semi-continuous by definition of convex conjugate. By Fenchel–Moreau theorem, we have $h^{**} = h$. Denote $(\varepsilon, \delta(\varepsilon))$ to be the equivalent dual relationship to $f = \inf_{\alpha} \{ f^{(\alpha)} \}^{**}$-DP. From [9] we know that

$$\delta(\varepsilon) = 1 + \left(\inf_{\alpha \in I} f^{(\alpha)}\right)^*(-e\varepsilon) = 1 + h^{**}(-e\varepsilon) = 1 + h(-e\varepsilon).$$

By order reversing property of convex conjugate, we have

$$\delta(\varepsilon) = 1 + h(-e\varepsilon)$$

$$= 1 + \left(\inf_{\alpha \in I} f^{(\alpha)}\right)^*(-e\varepsilon)$$
\[
= 1 + \sup_{\alpha \in \mathcal{I}} f^{(\alpha)}(-\varepsilon)
= \sup_{\alpha \in \mathcal{I}} \left( 1 + f^{(\alpha)}(-\varepsilon) \right)
= \sup_{\alpha \in \mathcal{I}} \delta^{(\alpha)}(\varepsilon)
\]

where we used dual relationship for each \(\alpha \in \mathcal{I}\) again in the last step. And the other direction follows directly from the duality of \(f\text{-DP}\) and \((\varepsilon, \delta(\varepsilon))\text{-DP}\), meaning that if the mechanism satisfies \((\varepsilon, \sup_{\alpha \in \mathcal{I}} \delta^{(\alpha)})\text{-DP}\), then it also satisfies \(f = \left( \inf_{\alpha} \{ f^{(\alpha)} \} \right)^{**}\text{-DP}\).

\[\square\]

E Proofs in Section 4

E.1 Proof of Theorem 1

Proof of Theorem 1. We first briefly introduce the idea of the proof. The main idea is to construct a random variable \(\tilde{X}_i\) by choosing an \(a \geq 0\), such that it stochastically dominates \(X_i\) (that is, \(X_i \leq \tilde{X}_i \) a.s.), and satisfies \(\mathbb{E}(\tilde{X}_i) < 0\). We then choose \(\eta(a) = -\mathbb{E}(\tilde{X}_i)\) which is a positive number. In what follows, we will explicitly construct \(\tilde{X}_i\) so that \(\tilde{X}_i\) can be decomposed into the sum of two sub-Gaussian random variables with parameters \(\sigma_A, \sigma_B\). Then since \(X_i \leq \tilde{X}_i \) a.s., we deduce that

\[
\mathbb{P}\left( \sum_{i=1}^{m} X_i \geq \varepsilon \right) \leq \mathbb{P}\left( \sum_{i=1}^{m} \tilde{X}_i \geq \varepsilon \right) = \mathbb{P}\left( \sum_{i=1}^{m} \tilde{X}_i - \sum_{i=1}^{m} \mathbb{E}(\tilde{X}_i) \geq \varepsilon + m\eta \right),
\]

The final conclusion, which will be proved at the end, follows from the sub-Gaussian bounds.

For notation-wise convenience, we first define a quantity depending on the value of \(\xi_i\), where

\[
\Delta(\xi_i) := X_i - (\xi_i \mu - \frac{1}{2} \mu^2) = \log \left( p + \frac{1 - p}{e^{\mu \xi_i - \frac{\mu^2}{2}}} \right).
\]

It is obvious that \(\Delta(\xi_i)\) is a strictly decreasing function of the value of \(\xi_i\). Now, we construct the random variable \(\tilde{X}_i\) as follows. Define

\[
\tilde{X}_i = A_i + B_i
\]

where

\[
A_i = \begin{cases} X_i & \text{if } \xi_i < a, \\ a^+ \mu - \frac{1}{2} \mu^2 + \Delta(a) & \text{if } \xi_i \geq a. \end{cases}
\]

and

\[
B_i = \begin{cases} 0 & \text{if } \xi_i < a, \\ \xi_i \mu - a^+ \mu & \text{if } \xi_i \geq a. \end{cases}
\]

Here, we define \(a^+ = \frac{\phi(a)}{1 - \Phi(a)}\). Note that \(a^+ > a\) for any \(a > 0\) by bounds on Mills ratio. To shed light on this decomposition, we first show that \(\tilde{X}_i\) stochastically dominates \(X_i\). Since \(\Delta(\xi_i)\) is a decreasing function in \(\xi_i\), hence when \(\xi_i \geq a\), we have \(\Delta(\xi_i) \leq \Delta(a)\). As a result, when \(\xi_i > a\),

\[
\tilde{X}_i = \xi_i \mu - \frac{1}{2} \mu^2 + \Delta(a) \geq \xi_i \mu - \frac{1}{2} \mu^2 + \Delta(\xi_i) = X_i.
\]

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This guarantees that $\tilde{X}_i \geq X_i$ a.s.. Another good property of this decomposition, we observe that $A_i$ is sub-Gaussian due to Lemma E.1, and $B_i$ is a mean-zero sub-Gaussian random variable from Lemma E.2. Proof of the two Lemmas is postponed to the next section.

Note that the above construction is valid for any $a$. Now we show that there exists some $a > 0$ such that $\mathbb{E}(\tilde{X}_i) = -\eta(a) < 0$ where $\eta(a)$ only depends on $a$. Note that

$$\mathbb{E}(\tilde{X}_i) - \mathbb{E}(X_i) = \int_a^\infty (\Delta(a) - \Delta(\xi)) \phi(\xi) d\xi,$$

where $\phi(x)$ is the density for standard Normal random variable. Also recall that $\mathbb{E}(X_i) < 0$. Then

$$\mathbb{E}(\tilde{X}_i) = \mathbb{E}(X_i) + \int_a^\infty (\Delta(a) - \Delta(\xi)) \phi(\xi) d\xi$$

$$= \mathbb{E}(X_i) + e(a),$$

where $e(a)$ satisfies that $\lim_{a \to \infty} e(a) = 0$. This is because by construction $\int_a^\infty (\Delta(a) - \Delta(\xi)) \phi(\xi) d\xi \geq 0$ and that

$$e(a) = \int_a^\infty \Delta(a) \phi(\xi) d\xi - \int_a^\infty \Delta(\xi) \phi(\xi) d\xi.$$

The second term vanishes as $\xi \to \infty$ since $\Delta(\xi)$ is integrable. For the first term, if $\Delta(a) < 0$ the integral is already negative. If $\Delta(a) > 0$ we have $\int_a^\infty \Delta(a) \phi(\xi) d\xi < \int_a^\infty \Delta(0) \phi(\xi) d\xi$ which also vanishes. As a result, we have shown that $\lim_{a \to \infty} e(a) \leq 0$. Combined with what we have above, we deduce that $\lim_{a \to \infty} e(a) = 0$ as required. Then we can pick $a$ large enough such that $e(a) = -\frac{1}{2} \mathbb{E}(X_i)$ and we have $\mathbb{E}(\tilde{X}_i) = \frac{1}{2} \mathbb{E}(X_i) < 0$.

Now we can combine the previous results and prove the tail bound of $\sum_{i=1}^m X_i$. Recall we have constructed random variable $\tilde{X}_i = A_i + B_i$ such that $\tilde{X}_i \geq X_i$ a.s. with $\mathbb{E}(\tilde{X}_i) = -\eta(a) < 0$. Moreover, $A_i, B_i$ are both sub-Gaussian with parameters $\sigma_A^2$ and $\sigma_B^2$. Then we have

$$\mathbb{P}\left(\sum_{i=1}^m X_i \geq \varepsilon\right) \leq \mathbb{P}\left(\sum_{i=1}^m \tilde{X}_i \geq \varepsilon\right) = \mathbb{P}\left(\sum_{i=1}^m \tilde{X}_i - \sum_{i=1}^m \mathbb{E}(\tilde{X}_i) \geq \varepsilon + m\eta\right)$$

$$\leq \mathbb{P}\left(\sum_{i=1}^m A_i - \sum_{i=1}^m \mathbb{E}(A_i) + \sum_{i=1}^m B_i - \sum_{i=1}^m \mathbb{E}(B_i) \geq \varepsilon + m\eta\right)$$

$$\leq \mathbb{P}\left(\sum_{i=1}^m A_i - \sum_{i=1}^m \mathbb{E}(A_i) \geq \frac{\varepsilon + m\eta}{2}\right) + \mathbb{P}\left(\sum_{i=1}^m B_i - \sum_{i=1}^m \mathbb{E}(B_i) \geq \frac{\varepsilon + m\eta}{2}\right),$$

where the last inequality follows from the union bound. Finally, since $A_i, B_i$ are both sub-Gaussian, we know that their sum $\sum_{i=1}^m A_i, \sum_{i=1}^m B_i$ are still sub-Gaussian. Hence

$$\mathbb{P}\left(\sum_{i=1}^m A_i - \sum_{i=1}^m \mathbb{E}(A_i) \geq \frac{\varepsilon + m\eta}{2}\right) \leq \exp\left(-\frac{(\varepsilon + m\eta)^2}{8m\sigma_A^2}\right),$$

$$\mathbb{P}\left(\sum_{i=1}^m B_i - \sum_{i=1}^m \mathbb{E}(B_i) \geq \frac{\varepsilon + m\eta}{2}\right) \leq \exp\left(-\frac{(\varepsilon + m\eta)^2}{8m\sigma_B^2}\right).$$
As a result,
\[
P\left( \sum_{i=1}^{m} X_i \geq \varepsilon \right) \leq \exp\left( -\frac{(\varepsilon + m\eta)^2}{8m\sigma_A^2} \right) + \exp\left( -\frac{(\varepsilon + m\eta)^2}{8m\sigma_B^2} \right) \leq 2 \exp\left( -\frac{(\varepsilon + m\eta)^2}{8m\tau^2} \right),
\]
where
\[
\tau^2 = \max\{\sigma_A^2, \sigma_B^2\}
= \max \left\{ \frac{(\log(1 - p + pe^{\mu a - \frac{1}{2}\mu^2}) + \mu(a^+ - a) - \log(1 - p))^2}{4}, \mu^2, \frac{(a^+ - a)^2 \mu^2}{2\log(\Phi(a^+) - \Phi(a))} \right\}.
\]

\section*{E.2 Technical Lemmas}

\textbf{Lemma E.1.} The random variable \(A_i\), defined in (E.1), is sub-Gaussian random variable with parameter \(\sigma_A^2\) where
\[
\sigma_A^2 = \frac{(\log(1 - p + pe^{\mu a - \frac{1}{2}\mu^2}) + \mu(a^+ - a) - \log(1 - p))^2}{4}.
\]

\textbf{Proof of Lemma [E.1]} The proof of Lemma [E.1] is straightforward, we show that \(A_i\) is bounded and thus sub-Gaussian by Hoeffding’s inequality. Note that when \(\xi_i < a\), we have \(A_i = X_i = \log(1 - p + pe^{\mu a - \frac{1}{2}\mu^2}) < \log(1 - p + pe^{\mu a - \frac{1}{2}\mu^2})\). Moreover, since \(X_i\) is bounded below by \(\log(1 - p)\), we deduce that when \(\xi_i < a\), we have
\[
\log(1 - p) < A_i < \log(1 - p + pe^{\mu a - \frac{1}{2}\mu^2}),
\]
which is bounded as desired.

On the other hand, when \(\xi_i > a\), by definition of \(\Delta(\xi_i)\),
\[
A_i = a^+ \mu - \frac{1}{2} \mu^2 + \log \left( p + \frac{1 - p}{e^{\mu a - \frac{1}{2}\mu^2}} \right) = \log(1 - p + pe^{\mu a - \frac{1}{2}\mu^2}) + \mu(a^+ - a),
\]
which is a constant. Since \(a^+ > a\), in this case the above constant is greater than \(\log(1 - p + pe^{\mu a - \frac{1}{2}\mu^2})\). Combine the above two settings, we deduce that \(A_i \in (\log(1 - p), \log(1 - p + pe^{\mu a - \frac{1}{2}\mu^2}) + \mu(a^+ - a))\) is a bounded random variable. By Hoeffding’s inequality, it is sub-Gaussian with parameter defined in the Lemma.

\textbf{Lemma E.2.} The random variable \(B_i\), defined in Equation (E.2), is a mean-zero sub-Gaussian random variable with parameter \(\sigma_B^2 = \mu^2 \max \left\{ 1, \frac{(a^+ - a)^2}{2\log(\Phi(a^+) - \Phi(a))} \right\}\).

\textbf{Remark E.3.} We note that as a function of \(a\), \(\frac{(a^+ - a)^2}{2\log(\Phi(a^+) - \Phi(a))}\) is in fact a decreasing function, and is always less than 1 as \(a > 0\). Its plot can be found in Figure [4]. Therefore, by truncating normal at \(a\), we essentially loss nothing, since \(B_i\) is still a sub-Gaussian random variable with parameter \(\mu\).
Proof of Lemma E.2. Recall the definition in Equation (E.2), we can re-write $B_i$ as a mixture random variable

$$B_i = \begin{cases} 0 & \text{w.p. } \Pr(\xi_i < a), \\ \tilde{\xi}_i \mu - a^+ \mu & \text{w.p. } \Pr(\xi_i \geq a) \end{cases}$$

where $\tilde{\xi}_i = \xi_i | \xi_i > a > 0$ is the normal $\mathcal{N}(0, 1)$ truncated at $a > 0$, whose probability density function is given by

$$f(t) = \frac{\phi(t)}{1 - \Phi(a)}, \text{ for } t > a.$$ 

From Lemma E.4, we know the expectation of $B_i$ is

$$\mathbb{E}[B_i] = 0 + (1 - \Phi(a)) (\mathbb{E}[\tilde{\xi}_i] - a^+) \mu = 0.$$ 

Therefore, to prove that the mean-zero variable $B_i$ is sub-Gaussian, we only need to bound the probability of $\Pr(B_i > t)$ and $\Pr(B_i < -t)$ for any $t > 0$ with the form of $\exp(-\frac{t^2}{2\sigma^2})$ for some $\sigma > 0$.

We will first prove the part for $\Pr(B_i > t)$ Note that

$$\Pr(B_i > t) = (1 - \Phi(a)) \Pr(\tilde{\xi}_i \mu - a^+ \mu > t)$$

$$= (1 - \Phi(a)) \Pr(\tilde{\xi}_i \mu - a^+ \mu > t)$$

$$= (1 - \Phi(a)) \Pr(\tilde{\xi}_i > a^+ + t/\mu)$$

$$= (1 - \Phi(a)) \left(1 - \frac{\Phi(a^+ + t/\mu) - \Phi(a))}{1 - \Phi(a)}\right)$$

$$= 1 - \Phi(a^+ + t/\mu)$$

$$\leq 1 - \Phi(t/\mu) \leq \exp\left(-\frac{t^2}{2\mu^2}\right),$$

where the fourth equality is due to (2) in Lemma E.4, the first inequality is due to the fact that $a^+ \geq a > 0$, and the last inequality is due to the fact that $\mathcal{N}(0, \mu^2)$ is sub-Gaussian with parameter $\mu^2$. 

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We now prove the other side. Observe that $B_i > \mu(a^+ - a)$, so for $t > \mu(a^+ - a)$ we have $\mathbb{P}(B_i < -t) = 0$. Therefore, we only need to bound $\mathbb{P}(B_i < -t)$ for $0 < t \leq \mu(a^+ - a)$. Note that in this range,

$$\mathbb{P}(B_i < -t) \leq \mathbb{P}(B_i < 0) = (1 - \Phi(a))\mathbb{P}(\xi_i - a^+ < \mu)$$

$$= (1 - \Phi(a))\mathbb{P}(\xi_i < a^+)$$

$$= \Phi(a^+) - \Phi(a)$$

where the last equality is again due to (2) of Lemma E.4. On the other hand, we have for any $\sigma > 0$,

$$\exp\left(-\frac{t^2}{2\sigma^2}\right) \geq \exp\left(-\frac{(\mu(a^+ - a))^2}{2\sigma^2}\right), \text{ for any } 0 < t \leq \mu(a^+ - a).$$

And with the choice of $\sigma^2 = \frac{(a^+ - a)^2}{2\log(\Phi(a) - \Phi(a))}$, we have

$$\exp\left(-\frac{t^2}{2\sigma^2}\right) \geq \exp\left(-\frac{(\mu(a^+ - a))^2}{2\sigma^2}\right) = \Phi(a^+) - \Phi(a) \geq \mathbb{P}(B_i < -t)$$

holds for all $0 < t \leq \mu(a^+ - a)$.

Therefore, combining the two sides, we know that $B_i$ is sub-Gaussian with parameter $\max\{\mu^2, \frac{\mu^2(a^+ - a)^2}{2\log(\Phi(a) - \Phi(a))}\}$, which concludes the proof.

\[\square\]

**Lemma E.4.** For a truncated normal distribution $\tilde{\xi}_i$ with density $f(t) = \frac{\phi(t)}{1 - \Phi(a)}$, for $t > a$ we have

1. $\mathbb{E}(\tilde{\xi}_i) = \frac{\phi(a)}{1 - \Phi(a)}$.
2. $\mathbb{P}(\tilde{\xi}_i \leq t) = \frac{\Phi(t) - \Phi(a)}{1 - \Phi(a)}$.

**Proof of Lemma E.4.** This is based on several well-known truncated normal properties, and is easy to prove from the density function. Therefore we omit the proof here. \[\square\]

### F Details of Edgeworth approximation error

The following discussion is largely adapted from [8] to be self-contained. For a distribution $P$, let $f_P$ denote its characteristic function; similarly, for a random variable $X$, we denote by $f_X$ its characteristic function. We recall that $f_{N(0, 1)}(t) = e^{-t^2/2}$. Some constants are used in the definition.

- Denote by $\chi_1$ the constant $\chi_1 := \sup_{x > 0} x^{-3} |\cos(x) - 1 + x^2/2| \approx 0.099162$ [20].
- Denote by $\theta_1^*$ the unique root in $(0, 2\pi)$ of the equation $\theta^2 + 2\theta \sin(\theta) + 6(\cos(\theta) - 1) = 0$.
- Denote by $t_1^* := \theta_1^*/(2\pi) \approx 0.635967$ [20].
F.1 Details of first-order Edgeworth expansion

We now provide details on the first-order Edgeworth expansion in Lemma 4.3. The main narrative is adapted from [8].

We first define the reminder term $r_{1,m}$. To this end, we define

$$
\Psi(t) := \frac{1}{2} \left( 1 - |t| + i \left( (1 - |t|) \cot(\pi t) + \frac{\text{sign}(t)}{\pi} \right) \right) 1 \{ |t| \leq 1 \}
$$

where $i$ is the imaginary number. Note that from [18] we have the following bound for function $\Psi$:

$$
|\Psi(t)| \leq \frac{1.0253}{2\pi|t|} \quad \text{and} \quad |\Psi(t) - \frac{i}{2\pi t}| \leq \frac{1}{2} \left( 1 - |t| + \frac{\pi^2}{18} t^2 \right).
$$

We further define

$$
I_{3,1}(T) := \frac{2}{T} \int_{0}^{\sqrt{\pi}(m/K_{4,m})^{1/4}} |\Psi(u/T)| \left| f_{S_m}(u) - e^{-u^2/2} \left( 1 - \frac{iu^3\lambda_{3,m}}{6\sqrt{m}} \right) \right| du
$$

$$
I_{3,2}(T) := \frac{2}{T} \int_{0}^{t_{u}T} |\Psi(u/T)| \left| f_{S_m}(u) - e^{-u^2/2} \right| du
$$

$$
I_{3,3}(T) := \frac{2}{T} \frac{\lambda_{3,m}}{6\sqrt{m}} \int_{0}^{t_{u}T} |\Psi(u/T)| e^{-u^2/2} |u|^3 du.
$$

and $r_{1,m}$ is defined to be

$$
r_{1,m} := \frac{(14.1961 + 67.0415)K_{3,m}^4}{16\pi^4m^2} + \frac{\lambda_{3,m} \exp \left( -\frac{2m^2}{K_{3,m}^4} \right)}{3\pi\sqrt{m}} + I_{3,2}(T) + I_{3,3}(T)
$$

$$
+ \frac{1.0253}{\pi} \int_{0}^{\sqrt{\pi}(m/K_{4,m})^{1/4}} u e^{-u^2/2} R_{1,m}(u, \varepsilon) du.
$$

(F.1)

For $\varepsilon \in (0, 1/3)$ and $t \geq 0$, we further define

$$
R_{1,m}(t, \varepsilon) := \frac{U_{1,1,m}(t) + U_{1,2,m}(t)}{2(1 - 3\varepsilon)^2} + e_1(\varepsilon) \left( \frac{t^8 K_{4,m}^2}{2m^2} \left( \frac{1}{24} + \frac{P_{1,m}(\varepsilon)}{2(1 - 3\varepsilon)^2} \right)^2 + \frac{|t|^7 |\lambda_{3,m}| K_{4,m}}{6m^{3/2}} \left( \frac{1}{24} + \frac{P_{1,m}(\varepsilon)}{2(1 - 3\varepsilon)^2} \right) \right),
$$

$$
P_{1,m}(\varepsilon) := \frac{144 + 48\varepsilon + 4\varepsilon^2 + \left( 96\sqrt{2}\varepsilon + 32\varepsilon + 16\sqrt{2}\varepsilon^{3/2} \right) \{ \exists i \in \{1, \ldots, m\} : \mathbb{E} \left[ (X_i - \mu_i)^3 \right] \neq 0 \}}{576},
$$

$$
e_1(\varepsilon) := \exp \left( \varepsilon^2 \left( \frac{1}{6} + \frac{2P_{1,m}(\varepsilon)}{(1 - 3\varepsilon)^2} \right) \right),
$$

$$
U_{1,1,m}(t) := \frac{t^6}{24} \left( \frac{K_{4,m}}{m} \right)^{3/2} + \frac{t^8}{24} \left( \frac{K_{4,m}}{m} \right)^2,
$$

$$
U_{1,2,m}(t) := \left( \frac{|t|^5}{6} \left( \frac{K_{4,m}}{m} \right)^{5/4} + \frac{t^6}{36} \left( \frac{K_{4,m}}{m} \right)^{3/2} + \frac{|t|^7}{72} \left( \frac{K_{4,m}}{m} \right)^{7/4} \right) \{ \exists i \in \{1, \ldots, m\} : \mathbb{E} \left[ (X_i - \mu_i)^3 \right] \neq 0 \}.
$$

Observe the bound from Lemma 4.3 is a bound of leading order $O(1/\sqrt{m})$, which is due to the fact that the variables in the sequence may not be identical since we may encounter non-identical
where \( a_m := 2t^*_m \pi \sqrt{m}/K_{3,m}, b_n := 16\pi^4 m^2 / \tilde{K}_{3,m}^4, \) and \( r_{2,m} \) is a remainder term that depends only on \( K_{3,m}, K_{4,m} \) and \( \lambda_{3,m} \). Specifically, the term \( r_{2,m} \) is defined by

\[
r_{2,m} := \frac{1.2533 \tilde{K}_{3,m}^4}{16\pi^4 m^2} + \frac{0.3334 |\lambda_{3,m}| \tilde{K}_{3,m}^4}{16\pi^4 m^{5/2}} + \frac{14.1961 \tilde{K}_{3,m}^{16}}{16^4 \pi^{16} m^8} + \frac{|\lambda_{3,m}| \exp \left( -128\pi^6 m^4 / \tilde{K}_{3,m}^8 \right)}{3\pi\sqrt{m}}
\]

+ \( I_{5,2}(T) + I_{5,3}(T) + I_{5,4}(T) + J_3(T) + J_5(T) \)

Here,

\[
I_{5,2}(T) := E_{1,m} \frac{|\lambda_{3,m}|}{3T\sqrt{m}} \int_{\sqrt{2\varepsilon}(m/K_{4,m})^{1/4}}^{T^{1/4}/\pi} |\Psi(u/T)|u^3 e^{-u^2/2} du,
\]

\[
I_{5,3}(T) := E_{1,m} \frac{2}{T} \int_{\sqrt{2\varepsilon}(m/K_{4,m})^{1/4}}^{T^{1/4}/\pi} |\Psi(u/T)| \left| f_{S_n}(u) - e^{-u^2/2} \right| du,
\]

\[
I_{5,4}(T) := E_{2,m} \frac{|\lambda_{3,m}|}{3T\sqrt{m}} \int_{T^{1/4}/\pi}^{T^{1/4}/\pi} |\Psi(u/T)|u^3 e^{-u^2/2} du,
\]

where \( E_{1,m} := \frac{1}{\{\sqrt{2\varepsilon}(m/K_{4,m})^{1/4} < T^{1/4}/\pi \}} \) and \( E_{2,m} := \frac{1}{\{T^{1/4} \}} \). Further, \( T = 16\pi^4 m^2 / \tilde{K}_{3,m}^4 \). Note that if \( T^{1/4} > T \) or \( \sqrt{2\varepsilon}(m/K_{4,m})^{1/4} > T^{1/4}/\pi \), our bounds are still valid and can even be improved in the sense that the corresponding integrals can be removed. Further, we have the following bound for the terms \( I_{5,2}, I_{5,3} \) and \( I_{5,4} \).

\[
I_{5,2}(T) \leq \frac{|\lambda_{3,m}|}{3\sqrt{m}} \int_{\sqrt{2\varepsilon}(m/K_{4,m})^{1/4}}^{T^{1/4}/\pi} \frac{1.0253}{2\pi} u^2 e^{-u^2/2} du
\]

\[
= \frac{1.0253 |\lambda_{3,m}|}{3\pi\sqrt{2\varepsilon}} \left( \Gamma \left( 3/2, \varepsilon (m/K_{4,m})^{1/2} \right) - \Gamma \left( 3/2, T^{1/2}/2\pi^2 \right) \right),
\]

\[
I_{5,3}(T) \leq \frac{2}{T} \int_{\sqrt{2\varepsilon}(m/K_{4,m})^{1/4}}^{T^{1/4}/\pi} |\Psi(u/T)| K_{3,m} \frac{1}{6\sqrt{m}} |t|^3 \exp \left( -\frac{t^2}{2} + \frac{\chi_{1,t}^3 \tilde{K}_{3,m}}{\sqrt{m}} + \frac{t^2 \sqrt{\tilde{K}_{3,m}}}{2\sqrt{m}} \right) du
\]

\[
= \frac{K_{3,m}}{3\sqrt{m}} J_2 \left( 3, \sqrt{2\varepsilon} (mK_{4,m})^{1/4}, T^{1/4}/\pi, \tilde{K}_{3,m}, K_{4,m}, T, m \right)
\]

and

\[
I_{5,4}(T) = \frac{1.0253 |\lambda_{3,m}|}{3\pi\sqrt{2\varepsilon}} \left( \Gamma \left( 3/2, T^{1/2}/2\pi^2 \right) - \Gamma \left( 3/2, T^{1/2}/2\pi^2 \right) \right),
\]

and all the terms converge exponentially fast to zero. Here \( \Gamma(a, x) \) is the incomplete Gamma function and can be numerically evaluated.
For the other terms, we have

\[ J_3(T) := \frac{2}{T} \int_{T^{1/4}/\pi}^{T^{1/4}} |\Psi(u/T)| |f_{S_m}(u)| du = \frac{2}{T^{3/4}} \int_{1/\pi}^{T^{1/4}} |\Psi\left(\frac{v}{T^{3/4}}\right)| |f_{S_m}\left(T^{1/4}v\right)| dv, \]

\[ J_4(T) := \frac{1}{T} \int_{T^{1/4}/\pi}^{T/\pi} |\Psi(u/T)| |f_{S_m}(u)| du, \]

\[ J_5(T) := \frac{2}{T} \int_{T^{1/4}/\pi}^{T/\pi} |\Psi(u/T)| e^{-u^2/2} du. \]

Obviously, now all the above bounds are real integrations, and can be calculated numerically.

### F.2 Extension to higher-order Edgeworth expansion

We now briefly state that how we can extend the current first-order Edgeworth bound to higher-orders. We essentially need to upper bound the approximation error by a careful decomposition. For example, when extending to the second-order Edgeworth expansion, we have the following new smoothing Lemma.

**Lemma F.1.** For every \( t_0 \in (0,1] \) and every \( T > 0 \), we have

\[ \Delta_{m,2} \leq \Omega_1 (t_0, T) + \Omega_2 (t_0, T) + \Omega_3 (t_0, T), \]

where

\[ \Omega_1 (t_0, T) := 2 \int_{t_0}^{T} \left| \Psi(t) - \frac{i}{2\pi t} e^{-(Tt)^2/2} \left( 1 + \frac{1}{24m} \frac{|Tt|^4}{\lambda_{4,m}} - \frac{\lambda_{3,m} |Tt|^6}{72m} \right) \right| dt \]

\[ + \frac{1}{\pi} \int_{t_0}^{\infty} \frac{e^{-(Tt)^2/2}}{t} \left( 1 + \frac{1}{24m} \frac{|Tt|^4}{\lambda_{4,m}} - \frac{\lambda_{3,m} |Tt|^6}{72m} \right) \left| \lambda_{3,m} |Tt|^3 \right| \frac{1}{6\sqrt{m}} dt, \]

\[ \Omega_2 (t_0, T) := 2 \int_{t_0}^{1} |\Psi(t)| |f_{S_m}(Tt)| dt, \]

\[ \Omega_3 (t_0, T) := 2 \int_{t_0}^{T} |\Psi(t)| \left| f_{S_m}(Tt) - e^{-(Tt)^2/2} \left( 1 - \frac{\lambda_{3,m} |Tt|^3}{6\sqrt{m}} + \frac{\lambda_{4,m} |Tt|^4}{24m} - \frac{\lambda_{3,m} |Tt|^6}{72m} \right) \right| dt. \]

Using such bound we can numerically compute \( \Omega_1 (t_0, T) \), \( \Omega_2 (t_0, T) \), \( \Omega_3 (t_0, T) \) for suitably chosen \( t_0, T \) and get the uniform bound on Edgeworth expansion of different orders. It is expected that the order of approximation error decays as we increase the order of Edgeworth expansion. In practice however, we notice that first-order Edgeworth expansion already yields accurate results.