HOMOTOPY INVARINTS IN SMALL CATEGORIES

I. CARCACÍA-CAMPOS, E. MACÍAS-VIRGÓS AND D. MOSQUERA-LOIS

CITMAga Universidade de Santiago de Compostela, 15782-Spain

ABSTRACT. Tanaka introduced a notion of Lusternik Schnirelmann category, denoted $c\text{cat}\, C$, of a small category $C$. Among other properties, he proved an analog of Varadarajan’s theorem for fibrations, relating the LS-categories of the total space, the base and the fiber.

In this paper we recall the notion of homotopic distance $cD(F,G)$ between two functors $F,G : C \to D$, later introduced by us, which has $c\text{cat}\, C = cD(\text{id}_C, \bullet)$ as a particular case. We consider another particular case, the distance $cD(p_1,p_2)$ between the two projections $p_1, p_2 : C \times C \to C$, which we call the categorical complexity of the small category $C$. Moreover, we define the higher categorical complexity of a small category and we show that it can be characterized as a higher distance.

We prove the main properties of those invariants. As a final result we prove a Varadarajan’s theorem for the homotopic distance for Grothendieck bi-fibrations between small categories.

INTRODUCTION

In the thirties, Lusternik and Schnirelmann introduced a homotopic invariant of manifolds which bounded from the below the number of critical points of any smooth function defined on them (see [4]). For a topological space $X$, this homotopic invariant is known as the Lusternik-Schnirelmann category of $X$, denoted $\text{cat}(X)$.

Twenty years ago, M. Farber introduced the topological complexity of a topological space ([6]). It is a homotopic invariant which measures the difficulty in finding a motion planning algorithm on the space under consideration.

Both, the Lusternik-Schnirelmann category and the topological complexity are instances of a more general homotopic invariant called homotopic distance between maps, introduced by us in [13]. It provides a way to relate and extend several other homotopic invariants (see [14]).

Recently, it has grown the interest in developing topological and homotopic invariants in the setting of small categories. For example, this is the case of the Euler characteristic by Leinster ([2, 11]) or the integration with respect to the Euler-Poincaré characteristic by Tanaka ([22]). The definition of homotopy between functors by M-J. Lee ([9]) made it possible for Tanaka ([21]) to extend the Lusternik-Schnirelmann category to the setting of small categories. Later, two of the authors extended the notion of homotopic distance to the context of categories ([12]).

E-mail address: isaac.carcacia@rai.usc.es, quique.macias@usc.es, david.mosquera.lois@usc.es.
In this work, we continue the study begun in [12]. First, we introduce a novel homotopic invariant for small categories: a notion of higher homotopic distance. Second, we study fibrations between small categories culminating with a result in the spirit of Varadarajan’s theorem ([23]) for the homotopic distance for Grothendieck bi-fibrations between small categories (Theorem 5.1), which can be seen both as a generalization of [21, Theorem 4.5] and as an extension of our previous work ([13]) to the context of categories.

The paper is organized as follows. Section 1 is devoted to fixing notation while recalling some preliminaries about homotopies and homotopic distance in the setting of small categories. Moreover, the novel notion of higher categorical distance is introduced (Definition 1.15) and it is proved to be a particular case of a homotopic distance (Theorem 1.16). In Section 2 bi-fibrations are presented in this setting from scratch. The exposition is intended to be self-contained so no previous knowledge from the reader is necessary on the topic. Section 4 addresses the equivalence of fibers whose base objects are connected by an arrow (Theorem 4.1). In Section 5, we state and prove the main result (Theorem 5.1).

1. Categorical distance between functors

We will assume that all categories are small unless stated otherwise. If $E$ is a category, we also denote by $E$ its set of objects, and by $E(e_1, e_2)$ the set of arrows between the objects $e_1, e_2 \in E$. If $P: E \rightarrow B$ is a functor, we denote by $Pe$ and $P\phi$ the image of the object $e$ and the arrow $\phi$, respectively.

1.1. Homotopies between functors. We recall the notion of homotopy between functors introduced by Lee ([9, 10]).

**Definition 1.1.** The *interval category* $I_m$ of length $m \geq 0$ consists of $m+1$ objects with zigzag arrows,

$$0 \rightarrow 1 \leftarrow 2 \rightarrow \cdots \rightarrow (\leftarrow)m.$$ 

Given two small categories $\mathcal{C}$ and $\mathcal{D}$ we denote its product by $\mathcal{C} \times \mathcal{D}$. Recall that the objects of $\mathcal{C} \times \mathcal{D}$ are pairs of objects in $\mathcal{C}$ and objects in $\mathcal{D}$, and its arrows are products of arrows in $\mathcal{C}$ and arrows in $\mathcal{D}$.

**Definition 1.2.** Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two functors between small categories. We say that $F$ and $G$ are **homotopic**, denoted by $F \simeq G$, if, for some $m \geq 0$, there exists a functor $H: \mathcal{C} \times I_m \rightarrow \mathcal{D}$, called a homotopy (of length $m$), such that $H_0 = F$ and $H_m = G$.

Alternatively, the notion of homotopy between functors can be defined as follows.

**Proposition 1.3.** The functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ are homotopic if and only if there is a finite sequence of functors $F_0, \ldots, F_m: \mathcal{C} \rightarrow \mathcal{D}$, with $F_0 = F$ and $F_m = G$, such that for each $i \in \{0, \ldots, m-1\}$ there is a natural transformation either between $F_i$ and $F_{i+1}$ or between $F_{i+1}$ and $F_i$.

That both definitions are equivalent follows from the following Lemma.

**Lemma 1.4.** There is a natural transformation $\Phi: F \Rightarrow G$ if and only if there exists a homotopy $H: \mathcal{C} \times I_1 \rightarrow \mathcal{D}$ such that $H_0 = F$ and $H_1 = G$. 


Proof. For an object, we define $H(c, 0) = Fc$, $H(c, 1) = Gc$. For an arrow $f: c \to c'$ we define $H(f \times \text{id}_0) = Ff$, $H(f \times \text{id}_1) = Gf$ and, for the only arrow $s: 0 \to 1$ in $I_1$, we define

$$H(c \times s) = \Phi_c: Fc \to Gc.$$  

The homotopy relation defined above is an equivalence relation. Also, it behaves well with respect to compositions, that is, if $F \simeq F'$ and $G \simeq G'$, then $F \circ G \simeq F' \circ G'$ whenever $F \circ F'$ and $G \circ G'$ make sense.

1.2. **Categorical distance between functors.** We introduce now the categorical homotopic distance between functors (see [12]).

**Definition 1.5.** Let $C$ be a small category, a family $\{U_i\}_{i \in I}$ of subcategories of $C$ if a geometric cover of $C$ if for every chain of arrows $c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_m} c_m$ in $C$ there is some $i \in I$ such that the chain lies in $U_i$.

There are at least three reasons for adopting this definition: first, just covering objects would leave the arrows between different subcategories uncovered; second, an arbitrary covering of the arrows would leave the compositions of arrows between different subcategories uncovered; third, geometric coverings correspond to coverings of the classifying space of the category, thus making easy to compare the categorical distance with the topological distance of the classifying space.

**Definition 1.6.** Let $F, G: C \to D$ be two functors between small categories. The homotopical distance $c_D(F, G)$ between $F$ and $G$ is the least positive integer $n \geq 0$ such that there is geometric cover $\{U_0, \ldots, U_n\}$ of $C$ such that $F|_{U_i} \simeq G|_{U_i}$ for every $0 \leq i \leq n$.

If there is no such cover we define $c_D(F, G) = \infty$.

We call $\{U_0, \ldots, U_n\}$ a geometric cover by homotopy domains for $F$ and $G$.

1.3. **Properties.** The following properties are obvious:

1. $c_D(F, G) = c_D(G, F)$.
2. $c_D(G, F) = 0$ if and only if $F \simeq G$.
3. If $F \simeq \tilde{F}$ and $G \simeq \tilde{G}$ then $c_D(F, G) = c_D(\tilde{F}, \tilde{G})$.

We finish the section by introducing a notion of higher homotopic distance in the setting of small categories.

**Definition 1.7.** Let $C$ and $D$ be two small categories and let $\{F_i\}_{i=1}^n : C \to D$ be a finite set of functors between them. The higher homotopical distance $c_D(F_1, \ldots, F_n)$ is the least integer $m \geq 0$ such that there is a geometric cover $\{U_0, \ldots, U_m\}$ that satisfies $F_i|_{U_k} \simeq F_j|_{U_k}$ for every $i, j \in \{1, \ldots, n\}$ and $0 \leq k \leq m$.

If there is no such cover we define $c_D(F_1, \ldots, F_n) = \infty$.

1.4. **LS-Category.** The following notion of “categorical LS-category” is due to Tanaka [21].

A subcategory $\mathcal{U}$ of a small category $\mathcal{C}$ is called 0-categorical if the inclusion functor $i: \mathcal{U} \to \mathcal{C}$ is homotopic to a constant functor. The geometric cover $\{U_0, \ldots, U_m\}$ of the small category $\mathcal{C}$ is called categorical if every $U_i$ is 0-categorical.
Definition 1.8. Let $\mathcal{C}$ be a small category. We define the (normalized) Lusternik-Schnirelmann category of $\mathcal{C}$, denoted by $\text{ccat}(\mathcal{C})$, as the least integer $n \geq 0$ such that there is a categorical cover of $\mathcal{C}$.

If there is no such cover we define $\text{ccat}(\mathcal{C})$ as $\infty$.

We state a result from [13] and provide a proof since we will make use of it in the proof of Theorem 5.1.

Proposition 1.9. For every connected small category $\mathcal{C}$ we have that

$$
\text{ccat}(\mathcal{C}) = c\text{D}(\text{id}_{\mathcal{C}}, \bullet)
$$

where $\bullet$ is any constant functor.

Proof. Let $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ be a geometric cover satisfying that $n = \text{ccat}(\mathcal{C})$ and $\mathcal{U}_i \simeq \bullet_i$ for some constant functors $\bullet_i$, and every $i \in \{0, ..., n\}$. We have to prove that the constant functors $\bullet_i$ and $\bullet$ are homotopic. But this follows because there is some arrow which connects the objects $\bullet_i$ and $\bullet$, and this arrow defines a natural transformation between the corresponding constant functors.

It is now evident that $\text{id}|_{\mathcal{U}_i} = \mathcal{U}_i \simeq \bullet_i$ so $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ is a homotopy domain for $\text{id}_{\mathcal{C}}$ and $\bullet$. So we have proved that $\text{ccat}(\mathcal{C}) \geq c\text{D}(\text{id}_{\mathcal{C}}, \bullet)$.

Alternatively if $\mathcal{U}_0, \ldots, \mathcal{U}_n$ are homotopy domains for $\text{id}_{\mathcal{C}}$ and $\bullet$ it follows that $\text{id}|_{\mathcal{U}_i} = \mathcal{U}_i \simeq \bullet_i$ so $\text{ccat}(\mathcal{C}) = c\text{D}(\text{id}_{\mathcal{C}}, \bullet)$.

\[ \square \]

Corollary 1.10. The small category $\mathcal{C}$ is contractible if and only if $\text{ccat}(\mathcal{C}) = 0$.

Example 1.11. If the small category $\mathcal{C}$ admits finite products, then $\mathcal{C}$ is contractible.

Proof. Fix an object $c_0$ in $\mathcal{C}$ and define the functor $c_0 \times - : \mathcal{C} \to \mathcal{C}$ which sends the object $c$ into $c_0 \times c$, and the morphism $f : c_1 \to c_2$ into $\text{id}_{c_0} \times f$. Denote by $\text{id}_c : c \to \mathcal{C}$ the constant functor.

Then there are natural transformations (that is, homotopies)

$$
\text{id}_{\mathcal{C}} \Leftarrow c_0 \times - \Rightarrow c_0
$$

given by the two projections $p_1 : c_0 \times c \to c_0$ and $p_2 : c_0 \times c \to c$.

Let $\mathcal{C}$ be a category and let $c_0$ be an object in $\mathcal{C}$. There are two functors $i_1, i_2 : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ defined as follows:

- For every object $c$ in $\mathcal{C}$ we have that $i_1(c) = (c, c_0)$ and $i_2(c) = (c_0, c)$.
- Let $f$ be a morphism in $\mathcal{C}$; then $i_1(f) = (f, \text{id}_{c_0})$ and $i_2(f) = (\text{id}_{c_0}, f)$.

Proposition 1.12 ([12, Proposition 6]). Let $\mathcal{C}$ be a small category and $c$ an object in $\mathcal{C}$. We claim that:

$$
\text{ccat}(\mathcal{C}) = c\text{D}(i_1, i_2).
$$

1.5. Categorical complexity. If $\mathcal{C}$ is a small category, we define the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ as the functor that takes every object $c$ to $\Delta(c) = (c, c)$ and that takes every morphism $f$ to $\Delta(f) = (f, f)$.

Definition 1.13. (1) We say that the subcategory $\mathcal{U}$ of $\mathcal{C} \times \mathcal{C}$ is a Farber subcategory if there is a functor $F : \mathcal{U} \to \mathcal{C}$ such that $\Delta \circ F \simeq \mathcal{U}$.

(2) The (normalized) categorical complexity of $\mathcal{C}$, denoted by $c\text{TC}(\mathcal{C})$ is the least integer $n \geq 0$ such that there is a geometric cover $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ of $\mathcal{C} \times \mathcal{C}$ by Farber subcategories.

If there is no such cover we define $c\text{TC}(\mathcal{C})$ as $\infty$. 

**Proposition 1.14** ([12, Theorem 1]). Let $C$ be a small category, we have that
\[ cTC(C) = cD(p_1, p_2), \]
where $p_1, p_2 : C \times C \to C$ are the projections.

1.6. **Higher categorical complexity.** We sketch now a definition of a higher topological complexity, analogous to that existing in the topological setting ([18]).

Let $C$ be a small category. We denote by $C^n$ the product $C \times \cdots \times C$. The higher $n$-diagonal functor $\Delta_n : C \to C^n$ is the functor that takes every object $X$ in $C$ to $\Delta_n X = (X, \ldots, X)$ and that takes every morphism $f$ to $\Delta_n f = (f, \ldots, f)$.

**Definition 1.15.** (1) We say that the subcategory $\iota : \Omega \hookrightarrow C^n$ is a higher $n$-Farber subcategory if there is a functor $F : \Omega \to C$ which is a right homotopy inverse of $\Delta_n$, that is, $\Delta_n \circ F \simeq \iota_{\Omega}$.

(2) The (normalized) higher $n$-categorical complexity of $C$, denoted by $cTC_n(C)$ is the least integer number $n \geq 0$ such that there is a geometric cover $\{\Omega_0, \ldots, \Omega_n\}$ of $C^n$ by higher Farber subcategories.

The following result guarantees that higher categorical complexity can be seen as a homotopic distance.

**Theorem 1.16.** The higher topological complexity equals the higher categorical distance between the projections $p_i : C^n \to C$, $1 \leq i \leq n$, that is,
\[ cTC_n(C) = cD(p_1, \ldots, p_n). \]

**Proof.** ($\leq$) Let $cD(p_1, \ldots, p_n) = m$. Let $\{U_0, \ldots, U_m\}$ be a geometric cover of $C^n$ by homotopy domains, that is, subcategories $U_k$ such that $p_i|U_k \simeq p_i|U_0$ for all $i, j \in \{0, \ldots, n\}$. We will see that they are also higher Farber subcategories.

Indeed, for every $U_k$ we have that the first projection $p_1 : U_k \to C$ verifies $\Delta_n \circ p_1 = \iota_{U_k}$. To check that, for every $i \in \{1, \ldots, n\}$ consider the homotopy $H_i : U_k \times \mathcal{I}_m \to C$ between $p_i|U_k$ and $p_i|U_0$. We can normalize all the homotopies by taking $m = \max\{m_i\}_{i=0}^n$ and extending $H_i : U_i \times \mathcal{I}_m \to C$ by identities if $j \geq m_i$. Now we can define a new homotopy $G : U_k \times \mathcal{I}_m \to C$ as
\[ G(c_1, c_2, \ldots, c_n, j) = (H_1(c_1, \ldots, c_n), H_2(c_1, \ldots, c_n, j), \ldots, H_n(c_1, \ldots, c_n, j)). \]

Let us check that $G$ is a homotopy between $\Delta \circ p_1$ and the inclusion $\iota_{U_k}$:
\begin{align*}
G(c_1, \ldots, c_n, 0) &= (H_1(c_1, \ldots, c_n, 0), \ldots, H_n(c_1, \ldots, c_n, 0)) \\
&= (p_1(c_1, \ldots, c_n), \ldots, p_1(c_1, \ldots, c_n)) \\
&= (c_1, \ldots, c_1) \\
&= \Delta_n p_1(c_1, \ldots, c_n),
\end{align*}
and
\begin{align*}
G(c_1, c_2, \ldots, c_n, m) &= G(H_1(c_1, \ldots, c_n, m), \ldots, H_n(c_1, \ldots, c_n, m)) \\
&= (p_1(c_1, c_2, \ldots, c_n), \ldots, p_n(c_1, \ldots, c_n)) \\
&= (c_1, c_2, \ldots, c_n).
\end{align*}
This shows that $c\text{TC}_n(C) \leq m$.

$(\geq)$ Now, let $c\text{TC}_n(C) = m$. Let $\{\Omega_0, \ldots, \Omega_m\}$ be a geometric cover of $C^n$ by Farber subcategories. Let us fix some index $k \in \{0, \ldots, n\}$ and let us see that $p_i|_{U_k} \simeq p_j|_{U_k}$ for every $i, j \in \{1, \ldots, n\}$.

Since $U_k$ is a Farber subcategory, there is a functor $F : U_k \to C$ such that $\Delta \circ F \simeq \iota_{U_k}$. Hence, there is a homotopy $H : U_k \times I_m \to C^n$ such that $H_0 = \Delta_n \circ F$ and $H_m = \iota_{U_k}$. We take the functor $K : U_k \times I_{2m} \to C$ given by

$$K(c_1, \ldots, c_n, l) = \begin{cases} p_i \circ H(c_1, \ldots, c_n, m - l) & \text{if } 0 \leq l \leq m \\ p_j \circ H(c_1, \ldots, c_n, l - m) & \text{if } m \leq l \leq 2m \end{cases}$$

The functor $K$ is well-defined because, for $l = m$:

$$p_i \circ H(X_1, \ldots, X_n, 0) = p_i \circ \Delta \circ F(c_1, \ldots, c_n) = p_i(F(c_1, \ldots, c_n), \ldots, F(c_1, \ldots, X_n)) = F(c_1, \ldots, c_n)$$

Moreover $K$ is a homotopy between $p_i$ and $p_j$:

$$K(c_1, \ldots, c_n, 0) = p_i \circ H(c_1, \ldots, c_n, m) = p_i \circ \iota_{U_k}(c_1, \ldots, c_n) = p_i(c_1, \ldots, c_n),$$

and

$$K(c_1, \ldots, c_n, 2m) = p_j \circ H(c_1, \ldots, c_n, m) = p_j \circ \iota_{U_k}(c_1, \ldots, c_n) = p_j(c_1, \ldots, c_n).$$

We have proven that $cD(p_1, \ldots, p_n) \leq m$. \hfill $\square$

1.7. Properties. We recall several basic properties of the categorical distance. They serve to give short proofs of many results.

**Proposition 1.17** ([12, Propositions 7 and 8]).

1. Let $C, D$ and $B$ be three small categories and let $F, G : C \to D$ and $H : D \to B$ be three functors. We have that:

$$cD(H \circ F, H \circ G) \leq cD(F, G).$$

2. Analogously, let $C, D$ and $B$ be three small categories and let $F, G : C \to D$ and $H : B \to C$ be three functors. We have that:

$$cD(H \circ F, H \circ G) \leq cD(F, G).$$

**Corollary 1.18.** Let $C$ be a connected small category. Then we have that

$$ccat(C) \leq c\text{TC}(C).$$

**Proof.**

$$ccat(C) = cD(\text{id}_C, C_0) = cD(p_1 \circ i_1, p_2 \circ i_1) \leq cD(p_1, p_2) = c\text{TC}(C).$$

In what follows we note that category and categorical complexity are in some sense dual and extreme cases of categorical distance.
Corollary 1.19. Let $F, G: C \to D$ be two functors between small categories, then we have that
\[ cD(F, G) \leq cTC(D). \]

Proof. We define the functor $(F, G): C \to D \times D$ such that $p_1 \circ (F, G) = F$ and $p_2 \circ (F, G) = G$. Therefore we have that
\[ cD(F, G) = cD(p_1 \circ (F, G), p_2 \circ (F, G)) \leq cD(p_1, p_2) = cTC(D). \]

\[
\square
\]

Proposition 1.20 ([12, Theorem 2]). Let $F, G: C \to D$ be two functors between small categories, then
\[ cD(F, G) \leq ccat(C). \]

2. Bi-fibrations

We recall the notions of cartesian morphism and Grothendieck fibration. We follow the references [5], [8, Appendix A], [20], [24] and [25, Chapter 12], as well as [21, Section 4].

2.1. Cartesian arrows. Let $P: E \to B$ be a functor.

Definition 2.1. The morphism $\phi \in \mathcal{E}(e_1, e_2)$ is cartesian (with respect to $P$) if, for every arrow $\beta \in \mathcal{E}(e, e_2)$ and every arrow $\overline{\alpha} \in \mathcal{B}(Pe, Pe_1)$ such that $P\phi \circ \overline{\alpha} = P\beta$, there exists a unique arrow $\alpha \in \mathcal{E}(e, e_1)$ such that $\phi \circ \alpha = \beta$ and $P\alpha = \overline{\alpha}$ (see Diagram (1)).

\[ (1) \]

The name cartesian seems to have its origin in the following characterization, whose proof is left to the reader:

Proposition 2.2. The morphism $\phi$ is cartesian if and only if the following commutative square in the category of Sets is a pullback:
\[
\begin{array}{ccc}
\mathcal{E}(e, e_1) & \xrightarrow{\phi} & \mathcal{E}(e, e_2) \\
\downarrow{P(-)} & & \downarrow{P(-)} \\
\mathcal{B}(Pe, Pe_1) & \xrightarrow{P(\phi) \circ -} & \mathcal{B}(Pe, Pe_2)
\end{array}
\]

Example 2.3. Let $P: C \to C$ the identity functor. Then any arrow in $C$ is cartesian. This is obvious because $P\alpha = \alpha$.

Example 2.4. Let $P: C \to \bullet$ be the constant functor. An arrow $\phi$ in $C$ is $P$-cartesian if and only if $\phi$ is an isomorphism.

Let us check it. From the diagram
we obtain an arrow such that $\phi \alpha = \text{id}_{e_1}$. Now, $\phi(\alpha \phi) = \phi$ means that we have the diagram

\[
\begin{array}{ccc}
\alpha \phi & \rightarrow & e_1 \\
\downarrow \phi & & \downarrow \phi \\
e_1 & \rightarrow & e_2 \\
\end{array}
\]

and by unicity it must be $\alpha \phi = \text{id}_{e_1}$. So the morphism $\phi$ is an isomorphism.

The converse is immediate.

**Example 2.5.** Posets can be viewed as small categories, with an arrow existing between two objects if and only if $c_1 \leq c_2$. Functors between posets are the order preserving maps $P: \mathcal{E} \to \mathcal{B}$. For such a functor, a map $e_1 \leq e_2$ is cartesian if and only if it verifies the following condition: if $e \leq e_2$ and $Pe \leq e_1$ then $e \leq e_1$.

This condition resembles the definition of a *down beat point* [1]. In fact, down beat points are cartesian for any functor.

For a study of fibrations in the setting of posets we refer the reader to [3].

### 2.2. Fibrations

**Definition 2.6.** The functor $P: \mathcal{E} \to \mathcal{B}$ is a *fibration* if for any arrow $\phi \in \mathcal{B}(b_1, b_2)$ and any object $e_2 \in \mathcal{E}$ such that $P e_2 = b_2$, there exists a cartesian arrow $\phi \in \mathcal{E}(e_1, e_2)$ such that $P \phi = \phi$.

The arrow $\phi$ is called a *cartesian lift* of $\phi$ with codomain $e_2$.

**Definition 2.7.** If $P: \mathcal{E} \to \mathcal{B}$ is a fibration, we define the *fiber* of $P$ over $b \in \mathcal{B}$ as the subcategory of $\mathcal{E}$ with objects $e \in \mathcal{E}$ such that $Pe = b$, and with arrows $\nu \in \mathcal{E}(e_1, e_2)$ such that $P \nu = \text{id}_b$. These arrows are called *vertical* arrows.

The cartesian lift of a given $\phi: b_1 \to b_2$ with a given codomain $e_2$ is unique, up to a unique vertical arrow. This follows from the unicity in the definition of cartesian arrow.

By using the axiom of choice, we can take a particular lift, which will be denoted by $\text{Cart}(\phi, e_2): \phi^* e_2 \to e_2$.

This particular choice defines a functor $\phi^*: \mathcal{E}_{b_2} \to \mathcal{E}_{b_1}$, where the image of a vertical arrow $\nu \in \mathcal{E}_{b_2}$ is given by the unique arrow $\phi^* \nu$ making the following diagram commute:

\[
\begin{array}{ccc}
\phi^* e_2 & \rightarrow & \text{Cart}(\phi, e_2) \\
\downarrow \phi^* \nu & & \downarrow \nu \\
\phi^* e_2 & \rightarrow & \text{Cart}(\phi, e_2') \\
\end{array}
\]

It corresponds to the diagram

\[
\begin{array}{ccc}
\phi^* e_2 & \rightarrow & e_2 \\
\downarrow \phi^* \nu & & \downarrow \phi^* \nu \\
\phi^* e_2 & \rightarrow & \text{Cart} \\
\end{array}
\quad
\begin{array}{ccc}
b_1 & \rightarrow & b_2 \\
\phi & \downarrow & \phi^* \\
b_1 & \rightarrow & b_2 \\
\end{array}
\]

The functoriality of $\phi^*$ follows again from unicity. It is called the *pullback functor*. 
Example 2.8. Every group $G$ can be considered as a category with one object, where the arrows are the elements $g \in G$, and the composition is given by the operation in $G$. A group homomorphism $F : H \to G$ can be considered as a functor. An arrow in $H$ is always cartesian; hence $F$ is a fibration if and only if it is surjective. The fiber is the kernel.

Example 2.9. Let $C^I_1$ be the category whose objects are the arrows in $C$, and whose arrows are the commutative squares. The codomain functor $P : C^I_1 \to C$ is a fibration if and only if $C$ has enough pullbacks. The cartesian arrows are the pullbacks in $C$. It is called the fundamental fibration.

We introduce another example of a fibration in the setting of locally small categories.

Example 2.10. Let $U : \text{Top} \to \text{Sets}$ be the forgetful functor from topological spaces to sets. It associates to each topological space $X$ its underlying set $UX$, and to each continuous map the set map itself. It is a fibration. The $U$-cartesian maps are the continuous maps $f : e_1 \to e_2$ such that the topology on $e_1$ is the initial topology, that is, the smallest (coarsest) topology making $f$ continuous.

2.3. Op-cartesian arrows and op-fibrations. Let $P^{op} : E^{op} \to B^{op}$ be the opposite functor of $P : E \to B$.

Definition 2.11. The arrow $\varphi : e_1 \to e_2$ in $E$ is op-cartesian for the functor $P$ if the opposite arrow $\varphi^{op}$ in $E^{op}$ is cartesian for $P^{op}$. Explicitly, that means that for any given $\beta \in \mathcal{E}(e_1,e)$ and any given $\pi \in B(Pe_2,Pe)$ such that $\pi \circ P\varphi = P\beta$, there exists a unique $\alpha \in \mathcal{E}(e_2,e)$ such that $\alpha \circ \varphi = \beta$ and $P\alpha = \pi$, as in the following diagram:

\[
\begin{array}{ccc}
    e_1 & \xrightarrow{\beta} & e \\
    \varphi \downarrow & & \downarrow \alpha \\
    e_2 & \xrightarrow{\varphi} & e_1 \\
\end{array}
\quad
\begin{array}{ccc}
    Pe_1 & \xrightarrow{P\beta} & Pe \\
    P\varphi \downarrow & & \downarrow \pi \\
    Pe_2 & \xrightarrow{P\varphi} & Pe_1 \\
\end{array}
\]

Definition 2.12. A functor $P : E \to B$ is an op-fibration if for any map $\overline{\varphi} : b_1 \to b_2$ in $B$, and for any object $e_1$ in $E$ with $P e_1 = b_1$, there exists an op-cartesian arrow $\varphi : b_1 \to b_2$ such that $P \varphi = \overline{\varphi}$.

Again, this op-cartesian lifting is unique up to a unique vertical isomorphism. Then we can choose some particular lifting

$\text{opCart}(\overline{\varphi}, e_1) : e_1 \to \overline{\varphi} e_1$

so defining a functor

$\overline{\varphi}_* : \mathcal{E}_{b_1} \to \mathcal{E}_{b_2}$

between the fibers.

Example 2.13. The codomain functor $P : C^I_1 \to C$ of Example 2.9 is always an op-fibration.

Example 2.14. Analogously, the domain functor $P : C^I_1 \to C$ is an op-fibration if and only if $C$ has enough push-outs.

Definition 2.15. We say that the functor $P : \mathcal{E} \to \mathcal{B}$ is a bi-fibration if it is both a fibration and an op-fibration.
We prefer this terminology instead of “fibration” and “cofibration”, see [16] for a discussion.

**Example 2.16.** The forgetful functor $U: \text{Top} \to \text{Sets}$ from topological spaces to sets is an op-fibration. Op-cartesian maps are the continuous maps $e_1 \to e_2$ where $e_2$ has the quotient topology.

**Example 2.17.** Let $\varphi: K \to K'$ be a simplicial map of simplicial complexes. Let $\mathcal{A}(\varphi): \mathcal{A}(K) \to \mathcal{A}(K')$ be the corresponding map between the associated face posets. A poset can be considered as a small category with the obvious arrows. Then the functor $\varphi$ is a fibrations, but it is not an op-fibration ([25, Example 13.5]).

**Example 2.18.** Take a base category $B$ and another category $C$, then the first projection $P_1 : B \times C \to B$ is a bi-fibration. In fact, given an arrow $\phi: b_1 \to b_2$ in $C$, and an object $(b_2, c)$ in $B \times C$, the morphism $\phi = \varphi \times \text{id}_c: (b_1, c) \to (b_2, c)$ is a cartesian lifting. In order to prove it we only must take into account the diagram:

$$
\begin{array}{ccc}
(b_1, c) & \xrightarrow{\phi} & (b_2, c) \\
\downarrow & & \downarrow \\
(b, c) & \xrightarrow{\varphi} & (b, c')
\end{array}
$$

Analogously, $P_1$ is an op-fibration.

**2.4. Further examples.**

**Example 2.19.** Let $\mathcal{E}$ be the category with objects the integers $n \in \mathbb{Z}$ and a unique morphism $m \to n$ if and only if $m \leq n$.

$$
\cdots \to -1 \to 0 \to 1 \to \cdots
$$

and let $\mathcal{B}$ be the monoid of the natural numbers $n \geq 0$ as a category.

$$
\begin{array}{c}
\bigcup_{n \in \mathbb{N}} n
\end{array}
$$

The functor $P: \mathcal{E} \to \mathcal{B}$ with $P(n) = \bullet$ and $P(m \to n) = n - m$ is a fibration. In fact, let us consider $n: \bullet \to \bullet$ a morphism in $\mathcal{B}$ and $m$ any object in $\mathcal{E}$; then the morphism $m - n \to m$ covers $n$. Moreover it is $P$-cartesian as indicated in the following commutative diagrams:

$$
\begin{array}{ccc}
m - n & \xrightarrow{l} & m \\
\downarrow & & \downarrow \\
m & \xrightarrow{m - n} & m - l
\end{array}
$$

Furthermore, $P$ is an op-fibration by a similar argument.

**Example 2.20.** Let $\mathcal{E}$ be the category with objects the integer numbers $n \in \mathbb{Z}$, and ziz-zag arrows:

$$
\cdots \leftarrow -2 \leftarrow -1 \leftarrow 0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots
$$

and let $\mathcal{B}$ be the following category

$$
\begin{array}{ccc}
1 & \xrightarrow{1} & 2 \\
\downarrow & & \downarrow \\
3 & \xrightarrow{3} & 1
\end{array}
$$
It is easy to show that the functor $P(n) = n \mod 4$ is a fibration and an op-fibration.

3. Lifting properties

The lifting properties of Grothendieck fibrations were considered by Gray in [7].

Recall that we denote by $\mathcal{I}_n$ the $n$-chain category generated by the following diagram:

$$0 \rightarrow 1 \rightarrow \cdots \rightarrow n.$$ 

In particular, $\mathcal{I}_1$ denotes the category $0 \rightarrow 1$ consisting of two objects and one non-identity morphism $s$.

Gray proved that the functor $P : \mathcal{E} \rightarrow \mathcal{B}$ is a fibration if and only if homotopies have cartesian liftings. Since we represent a natural transformation ending at $\eta$ as a functor $\eta : \mathcal{C} \times \mathcal{I}_1 \rightarrow \mathcal{B}$ where $\eta_1 = P \circ G$, we can state the following, which improves [21, Prop.4.2]:

**Proposition 3.1.** The functor $P : \mathcal{E} \rightarrow \mathcal{B}$ is a fibration if and only for any category $\mathcal{C}$ and for any functor $H : \mathcal{C} \times \mathcal{I}_1 \rightarrow \mathcal{B}$ such that $H_1 = H \circ i_1 = P \circ G$ there exists a functor $\tilde{H} : \mathcal{C} \times \mathcal{I}_1 \rightarrow \mathcal{E}$ such that $P \circ \tilde{H} = H$ and $\tilde{H}_1 = \tilde{H} \circ i_1 = G$, as in the following diagram:

$$\begin{align*}
\mathcal{C} & \xrightarrow{G} \mathcal{E} \\
i_1 \downarrow & \quad \quad \downarrow \tilde{H} \\
\mathcal{C} \times \mathcal{I}_1 & \xrightarrow{P} \mathcal{B}
\end{align*}$$

and moreover $\tilde{H}(f \times s)$ is a cartesian arrow for $P$ for any arrow $f$ in $\mathcal{C}$.

Here, $i_1 : \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{I}_1$ is the functor sending the object $c$ into $(c, 1)$, and the arrow $f : c_1 \rightarrow c_2$ into $f \times \text{id}_1$, and $s$ is the only arrow $0 \rightarrow 1$ in $\mathcal{I}_1$.

**Proof.** Obviously we must define $\tilde{H}_1 = G$. The problem is how to define $\tilde{H}_0$ in order to have a functor. To do that we have to use the definition of fibered category. As we know, if we have an arrow $\phi$ in $\mathcal{B}$ and we have an object $e_2$ in the fiber of the codomain $b_2$, we can lift the arrow into a cartesian morphism $\text{Cart}(\phi, e_2) : \phi^* e_2 \rightarrow e_2$. In particular, we can choose $\text{Cart}(\text{id}_b, e) = \text{id}_e$.

Now, if we have the arrow $H(\text{id}_c \times s) : H(c, 0) \rightarrow H(c, 1)$ in $\mathcal{B}$ and the object $G(X) \in \mathcal{E}_{H(c, 1)}$, there is a cartesian morphism

$$\text{Cart}(\text{id}_c \times s)) : H(\text{id}_c \times s)^* G(c) \rightarrow G(c),$$

so we define

$$\tilde{H}(c, 0) = H(\text{id}_c \times s)^* G(c)$$

and

$$\tilde{H}(\text{id}_c \times s) = \text{Cart}H(\text{id}_c \times s).$$

Finally, for every morphism $f : c_1 \rightarrow c_2$ in $\mathcal{C}$ we define $\tilde{H}(f \times s)$ as the unique cartesian arrow filling the diagram

$$\begin{align*}
\tilde{H}(c_1, 0) \xrightarrow{\tilde{H}(\text{id}_c \times s)} \tilde{H}(c_1, 1) &= Gc_1 \\
\tilde{H}(c_2, 0) \xrightarrow{\tilde{H}(\text{id}_c \times s)} \tilde{H}(c_2, 1) &= Gc_2
\end{align*}$$
For the converse statement, let \( \bar{\phi} : b_1 \to b_2 \) in \( B \) be an arrow and consider the diagram

\[
\begin{array}{ccc}
\bullet & \xrightarrow{G} & \mathcal{E} \\
\downarrow{\bar{\iota}_1} & \searrow{\bar{H}} & \downarrow{P} \\
\bullet \times \mathcal{I}_1 & \xrightarrow{H} & B
\end{array}
\]

where \( g(\bullet) = e_2 \) and \( H(\text{id}_\bullet \times s) = \bar{\phi} \). Then the map \( \phi = \bar{H}(\text{id}_\bullet \times s) \) is cartesian and verifies that \( P\phi = \bar{\phi} \). This ends the proof. \( \Box \)

Analogously, we have the lifting property for op-fibrations, where the left vertical arrow is \( \iota_0 \) instead of \( \iota_1 \):

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{G} & \mathcal{E} \\
\downarrow{\iota_0} & \searrow{\bar{H}} & \downarrow{P} \\
\mathcal{C} \times \mathcal{I}_1 & \xrightarrow{H} & B
\end{array}
\]

Taking in account this and the previous proposition we have the following general lifting property.

**Corollary 3.2.** Let \( P : \mathcal{E} \to B \) be a bi-fibration. Then the lifting property holds for any chain category \( \mathcal{I}_n \):

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{G} & \mathcal{E} \\
\downarrow{\iota_0} & \searrow{\bar{H}} & \downarrow{P} \\
\mathcal{C} \times \mathcal{I}_n & \xrightarrow{H} & B
\end{array}
\]

### 4. Homotopic fiber

The next result is a crucial one. It proves that in a bi-fibration, two objects \( b_1, b_2 \in B \) in the base which are connected by an arrow \( \bar{\tau} : b_1 \to b_2 \), have fibers which are homotopically equivalent as small categories.

**Theorem 4.1** ([21, Proposition 4.4]). Let \( P : \mathcal{E} \to B \) be a bi-fibration. If \( b_1 \) and \( b_2 \) are two objects in \( B \) such that there is a morphism \( u : b_1 \to b_2 \), then there is a categorical equivalence between the fibers \( \mathcal{E}_{b_1} \) and \( \mathcal{E}_{b_2} \).

**Proof.** Remember that in a bi-fibration we have both the pushforward functor \( u_* : \mathcal{E}_{b_1} \to \mathcal{E}_{b_2} \) and the pullback functor \( u^* : \mathcal{E}_{b_2} \to \mathcal{E}_{b_1} \).

We defined \( u_* \) as follows: for every object \( e_1 \) in the fiber \( \mathcal{E}_{b_1} \) there is a unique object \( u_*e_1 \) in \( \mathcal{E}_{b_2} \) such that \( \text{opCart}(u_*e_1) : e_1 \to u_*e_1 \) is the unique op-cartesian lift of \( u \).

Alternatively, for every \( e_2 \) in \( \mathcal{E}_{b_2} \) there is a unique object \( u^*e_2 \in \mathcal{E}_{b_1} \) such that \( \text{Cart}(u, e_2) : u^*e_2 \to e_2 \) is the unique cartesian lift of \( u \).

Now in order to show that \( u_* \) and \( u^* \) induce a categorical equivalence it is enough to show that for every object \( e_2 \) in \( \mathcal{E}_{b_2} \) there are natural arrows \( \alpha \) and \( \beta \) as in the
The following diagram:

and that their compositions are the identities because Cart\( (u, e_2) \) and opCart\( (u, u^* e_2) \) are a cartesian and op-cartesian morphism (respectively). Thus, we have a natural isomorphism between the functors \( \text{id}_{E_{b_2}} \) and \( u^* \circ u^* \).

Analogously we have another natural isomorphism between \( \text{id}_{E_{b_1}} \) and \( u^* \circ u^* \) using the followings diagrams:

A consequence of the latter theorem is that we can speak about the homotopic invariants of “the fiber” of a bi-fibration.

Remark 1. In the proof of Theorem 4.1 we implicitly assumed that both fibers \( E_{b_1} \) and \( E_{b_2} \) are non empty. This might not be true. However, the Theorem is still true, because for a bi-fibration, if two objects \( b_1, b_2 \) in the base are connected by an arrow, say \( \phi : b_1 \rightarrow b_2 \), then either \( E_{b_1} = \emptyset = E_{b_2} \) or \( E_{b_1} \neq \emptyset \neq E_{b_2} \) simultaneously.

In fact, assume first that \( P \) is only a fibration. If \( E_{b_2} = \emptyset \) the pullback functor is not defined, because we have no codomain \( e_2 \) to lift the arrow \( \phi \). Since a fibration \( P : E \rightarrow B \) may not be surjective-on-objects (see Example 4.2), it may happen that \( E_{b_2} \) is empty, while \( E_{b_1} \) is not.

However, for a fibration, if \( E_{b_2} \neq \emptyset \) then \( E_{b_1} \neq \emptyset \) too, because to each codomain \( e_2 \in E_{b_2} \) we associate the domain \( \phi(e_2) \) of Cart\( (\phi, e_2) \in E_{b_1} \).

Dually, for an op-fibration, if \( E_{b_1} \neq \emptyset \) then \( E_{b_2} \neq \emptyset \), because to each domain \( e_1 \) we associate the codomain \( \phi(e_1) \) of opCart\( (\phi, e_1) \).

As a consequence, in a bi-fibration there are no arrows connecting objects in the image with objects outside the image, or conversely.

Example 4.2. A fibration which is not surjective-on-objects is for instance the constant functor \( P : \mathcal{I}_1 \rightarrow \mathcal{I}_1 \) with \( P(0) = P(1) = 0 \), and \( P(s) = \text{id}_0 \). We have \( E_0 = \mathcal{I}_1 \) while \( E_1 = \emptyset \). This shows that \( P \) is not an op-fibration. In fact, given the arrow \( s : 0 \rightarrow 1 \) the domain \( 1 \in E_0 \) there is no op-cartesian lift of \( s \).

Notice that the arrow \( s : 0 \rightarrow 1 \) can not be lifted to any cartesian arrow, but this does not contradict the definition of fibration (there are no possible codomains).

Example 4.3. The following functor \( P : E \rightarrow B = \mathcal{I}_1 \) is a bi-fibration, but the fibers over 0 and 1 are not isomorphic. However, they are homotopically equivalent, as stated in Theorem 4.1.
Let $E$ be the category generated by the following diagram:

\[
\begin{array}{ccc}
0 & \xrightarrow{s} & 1 \\
g \downarrow & & \downarrow f \\
\bar{1} & \xrightarrow{g} & \bar{1}
\end{array}
\]

where $f \circ g = \text{id}_1$ and $g \circ f = \text{id}_1$. Let $B = \mathcal{I}_1$ the category with only one arrow $0 \xrightarrow{s} 1$.

The functor $P$ is defined as $P(0) = 0$, $P(1) = P(\bar{1}) = 1$, $P(s) = s$ and $P(f) = P(g) = \text{id}_1$. It is a bi-fibration because the arrows $\text{id}_0$, $\text{id}_1$, $s$ and $g \circ s$ are cartesian and op-cartesian. The fiber over 0 is the discrete category with one object 0 and the fiber over 1 is the following category:

\[
\begin{array}{cccc}
\bar{1} & \xrightarrow{g} & \bar{1} \\
f \downarrow & & \downarrow \\
\bar{1} & & \bar{1}
\end{array}
\]

It is contractible by the natural transformation $\alpha$ between the identity and the constant functor $\bar{1}$ given by $\alpha(1) = g$, $\alpha(\bar{1}) = \text{id}_1$. In fact, we have

\[
\begin{array}{cccc}
1 & \xrightarrow{\alpha(1)=g} & \bar{1} \\
g \downarrow & & \downarrow \text{id}_1 \\
1 & \xrightarrow{\alpha(1)=\text{id}_1} & \bar{1}
\end{array}
\]

\[
\begin{array}{cccc}
\bar{1} & \xrightarrow{\alpha(1)=\text{id}_1} & \bar{1} \\
f \downarrow & & \downarrow \text{id}_1 \\
1 & \xrightarrow{\alpha(1)=g} & 1
\end{array}
\]

5. Varadarajan’s theorem

In the classical LS-category theory of topological spaces, Varadarajan [23] proved a formula relating the categories of the fiber, the base and the total space for a Hurewicz fibration. We generalized this result for the homotopic distance, in [13, Theorem 6.1].

In [21, Theorem 4.5], Tanaka proved an analogous result for a bi-fibration $P : \mathcal{E} \rightarrow \mathcal{B}$ with fiber $\mathcal{F}$ between small categories, with path-connected base, namely

(2) $\text{ccat} \mathcal{E} + 1 \leq (\text{ccat} \mathcal{B} + 1) \cdot (\text{ccat} \mathcal{F} + 1)$.

Recall that by Corollary 4.1, if the base category $\mathcal{B}$ of a bi-fibration $P : \mathcal{E} \rightarrow \mathcal{B}$ is path-connected, then every two fibers are homotopy equivalent. We will refer to any of them as (the homotopy type of) “the fiber $\mathcal{F}$ of the bi-fibration”.

We will both extend our [13, Theorem 6.1] to the context of bi-fibrations of small categories and generalize Tanaka’s result, as follows.

**Theorem 5.1.** Let $(F, \mathcal{F})$ and $(G, \mathcal{G})$ be two morphisms between the bi-fibrations $P : \mathcal{E} \rightarrow \mathcal{B}$ and $P' : \mathcal{E}' \rightarrow \mathcal{B}'$. Let $\mathcal{B}$ and $\mathcal{B}'$ be path-connected. Let $b$ be an object in $\mathcal{B}$ such that $F(b) = G(b) = b'$ and let $F_b, G_b : \mathcal{E}_b \rightarrow \mathcal{E}_b'$ be the induced functors between the fibers. Then

\[
cD(F, G) + 1 \leq (cD(F_b, G_b) + 1) \cdot (\text{ccat}(\mathcal{B}) + 1).
\]

**Proof.** We assume that $cD(F_b, G_b)$ and $\text{ccat} \mathcal{B}$ are both finite, otherwise the result is trivial.

Let $\text{ccat}(\mathcal{B}) = m$, with $\{U_0, ..., U_m\}$ a categorical cover of $\mathcal{B}$, and let $cD(F_b, G_b) = n$, with $\{V_0, ..., V_n\}$ a covering of $\mathcal{E}_b$ by homotopy domains for $F_b$ and $G_b$. 

For every $i \in \{0, \ldots, m\}$ we have a homotopy $C^i: U_i \times \mathcal{I}_{k_i} \to \mathcal{B}$ between the inclusion $U_i \hookrightarrow \mathcal{B}$ and a constant functor $\bullet_i$ for some object $*_i \in \mathcal{B}$. Since $\mathcal{B}$ is connected we can assume that all $\bullet_i$ is the same object $*$ for all $i$ (see the proof of Proposition 1.9).

Let $\mathcal{P}^{-1}(U_i)$ be the subcategory of $\mathcal{E}$ whose objects are the objects $e \in \mathcal{E}$ with $Pe \in U_i$, and whose arrows are the arrows $\alpha \in \mathcal{E}(e_1, e_2)$ such that $P\alpha$ is an arrow in $U_i$.

By the homotopy lifting property applied to the following diagram:

$$
\begin{array}{ccc}
P^{-1}(U) & \xrightarrow{\mathcal{P}} & \mathcal{E} \\
\downarrow{\mathcal{P}} & & \downarrow{P} \\
P^{-1}(U) \times \mathcal{I}_{k_i} & \xrightarrow{\mathcal{P} \times \text{id}} & U \times \mathcal{I}_{k_i} \xrightarrow{C^i} \mathcal{B}
\end{array}
$$

we have a homotopy $\tilde{C}^i: U_i \times \mathcal{I}_{k_i} \to \mathcal{E}$ such that $\tilde{C}^i_0$ is the inclusion $\mathcal{P}^{-1}(U_i) \hookrightarrow \mathcal{E}$ and $\tilde{C}^i_{k_i}$ lies inside the fiber $\mathcal{E}_b$.

For each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ we define

$$
W_{i,j} = \mathcal{P}^{-1}(U_i) \cap (\tilde{C}^i_{k_i})^{-1}(V_j).
$$

We claim that $\{W_{i,j}\}_{0 \leq i \leq m, 0 \leq j \leq n}$ is a geometric cover of $\mathcal{E}$ such that each $W_{i,j}$ is a homotopy domain for $F$ and $G$.

1. $\{W_{i,j}\}$ is a geometric cover of $\mathcal{E}$.

Let $C: C_1 \to C_2 \to \cdots \to C_l$ be a chain in $\mathcal{E}$. Then we obtain the chain

$$
P(C): P(C_1) \to P(C_2) \to \cdots \to P(C_l)
$$

in $\mathcal{B}$. Since $\{U_0, \ldots, U_m\}$ is a geometric cover of $\mathcal{B}$, there is some $i$ such that the chain $pC$ lies in $U_i$, so the chain $C$ lies in $\mathcal{P}^{-1}(U_i)$. Moreover, the functor $\tilde{C}^i_{k_i}$ is defined in $C$, hence we have a new chain $\tilde{C}^i_{k_i}(C)$ that lies in the fiber $\mathcal{E}_b$. Now, we know that $\{V_j\}$ is a geometric cover of $F_b$, so $\tilde{C}^i_{k_i}(C)$ lies in some $V_j$. We conclude that $C$ is in $W_{i,j}$.

2. Each $W_{i,j}$ is a homotopy domain for $F$ and $G$.

For the sake of simplicity we change the notations as follows: $U = U_i$, $C = \tilde{C}^i$, $k = k_i$, $V = V_j$.

Let $K: V \times \mathcal{I}_l \to \mathcal{E}_b$ be a homotopy between $F|_V$ and $G|_V$ and let $\iota: \mathcal{E}_b \hookrightarrow \mathcal{E}$ be the inclusion of the fiber into the total category $\mathcal{E}$. The homotopy that we need is the functor

$$
H: W_{i,j} \times \mathcal{I}_{k+l+k} \to \mathcal{E}'
$$

given by

$$
H(c, n) = \begin{cases} 
F\tilde{C}(c, n) & \text{if } 0 \leq n \leq k \\
\iota K(\tilde{C}_c, n-k) & \text{if } k \leq n \leq k+l \\
G\tilde{C}(c, k+l+k-n) & \text{if } k+l \leq n \leq k+l+k
\end{cases}
$$
It only remains to check that $H$ is well defined and that it is the wanted homotopy:

$$H(c,0) = F\tilde{C}(c,0) = Fc$$

since $\tilde{C}_0$ is the inclusion,

$$\iota K(\tilde{C}_k,0) = \iota F(\tilde{C}_k) = Fc \tilde{C}(c,k),$$

$$\iota K(\tilde{C}_k,l) = \iota G\tilde{C}(c,k) = Gc$$

since $\tilde{C}_0$ is the inclusion.

This proves that $cD(F,G) \leq m + n$, as stated. \(\square\)

References

[1] J. A. Barmak, *Algebraic topology of finite topological spaces and applications*. Berlin: Springer (2011).
[2] C. Berger and T. Leinster. The Euler characteristic of a category as the sum of a divergent series. *Homology Homotopy Appl.* 10, No. 1, 41–51 (2008; Zbl 1132.18007)
[3] N. Cianci and M. Ottina. Fibrations between finite topological spaces. *arXiv:1907.03972*. (2019).
[4] O. Cornea, G. Lupton, J. Oprea, and D. Tanré. *LusternikSchnirelmann category*. Volume 103 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003.
[5] E. Balzin. *Grothendieck fibrations and homotopical algebra*. General Mathematics, Université Nice Sophia Antipolis, 2016.
[6] M. Farber. *Discrete Comput. Geom.* 29, No. 2, 211–221. (2003).
[7] J. W. Gray. *Fibred and cofibred categories*. Proc. Conf. Categor. Algebra, La Jolla 1965, 21–83. (1966).
[8] W. Koen van Woerkom. *Algebraic models of type theory*. MSc Thesis Universiteit van Amsterdam. (1992).
[9] M-J. Lee. Homotopy for functors. *Proc. Am. Math. Soc.* 36, 571–577. (1973).
[10] M-J. Lee. Erratum to: Homotopy for functors. *Proc. Am. Math. Soc.* 42, 648–650. (1974).
[11] T. Leinster. The Euler characteristic of a category. *Doc. Math.* 13, 21–49 (2008).
[12] E. Macías-Virgós and D. Mosquera-Lois. *J. Homotopy Relat. Struct.* 15, No. 3–4, 537–555. (2020).
[13] E. Macías-Virgós and D. Mosquera-Lois. Homotopic distance between maps. *Proc. Camb. Philos. Soc.* 172, No. 1, 73–93. (2022).
[14] E. Macías-Virgós, D. Mosquera-Lois, M-J. Pereira-Sáez. Homotopic distance and generalized motion planning, to appear in *Medit. J. Math.* arXiv:2105.13006.
[15] Y. B. Rudyak. On higher analogs of topological complexity. *Topology Appl.* 157, No. 5. (2010).
[16] Y. B. Rudyak. Erratum to: On higher analogs of topological complexity. *Topology Appl.* 157, No. 6. (2010).
[17] T. Streicher. Fibered Categories à la Jean Bénabou, April 1999–April. *arXiv:1801.02927*. (2022).
[18] K. Tanaka. Lusternik-Schnirelmann category for categories and classifying spaces. *Topology Appl.* 239, 65–80. (2018).
[19] K. Tanaka. *Topology Appl.* 204, 185–197 (2016).
[20] K. Varadarajan. On fibrations and category. *Math. Zeitschr.* 88, 267–273. (1965).
[21] A. Vistoli. Notes on Grothendieck topologies, fibered categories and descent theory. *arXiv arXiv:math/0412512*. (2007).
[22] R. Penner. *Topology and K-theory*. Lectures by Daniel Quillen. With a contribution by Mikhail Kapranov. Springer. (2020).