Periodic Response Analysis of A Jeffcott-Rotor System By The Integral Equation Method

Wenxin Zhang
Zhengzhou University

Yueli Chen (chenyueli@zzu.edu.cn)
Zhengzhou University  https://orcid.org/0000-0002-4025-6268

Research Article

Keywords: Nonlinear saturation controller, The integral equation method, Quadratic velocity coupling term, a four-degree-of-freedom Jeffcott-rotor system

Posted Date: May 13th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-439004/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License. Read Full License
Periodic response analysis of a Jeffcott-rotor system by the integral equation method

Wenxin Zhang\textsuperscript{a}, Yueli Chen\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a}Henan Academy of Big Data/School of Mathematics and Statistics, Zhengzhou University, 450001 Zhengzhou, Henan, China
\textsuperscript{b}Zhengzhou Key Laboratory of Big Data Analysis and Application, Henan Academy of Big Data, Zhengzhou University, 450001 Zhengzhou, Henan, China

Abstract

In this paper, two modified nonlinear saturation-based controllers and negative velocity feedback controllers are integrated to suppress the horizontal and vertical vibrations of a horizontally supported Jeffcott-rotor system at primary resonance excitation and the presence of 1:1 and 1:2 internal resonances. The second order approximations and the amplitude equations are obtained by applying the integral equation method to analyze the nonlinear behavior of this model. The stability of the steady-state solutions is ascertained based on the Floquet theory. The necessity of adding a negative velocity feedback to the main system is stated. The effects of different control parameters on the frequency-response curves and the force-response curves are investigated. Time histories of the whole system are included to show the response with and without control. It is shown that the saturation-based controller can reduce the system response to almost zero and the negative velocity feedback can suppress the transient vibrations and prevent the main system having the large amplitude vibration. The analyses show that analytical solutions are in excellent agreement with the numerical simulations. Finally, a comparison with previously published works is included.

Keywords: Nonlinear saturation controller, The integral equation method,\textsuperscript{*}Corresponding author

Email address: chenyueli@zzu.edu.cn (Yueli Chen)

Preprint submitted to Elsevier

April 12, 2021
Quadratic velocity coupling term, a four-degree-of-freedom Jeffcott-rotor system

1. Introduction

Mechanical vibrations are undesired phenomena in dynamical structures because they always cause damages, disturbances, and dangerous accidents. Thus, many research papers focused on how to suppress the nonlinear vibrations in those machines, especially the periodic vibration. To study the periodic motions of a nonlinear vibration system, many methods have been proposed, such as the method of multiple scales, the averaging method, the Lindstedt-Poincare method [1, 2], the asymptotic perturbation method [3–6] and so on. Macca-ri [3–6] applied the asymptotic perturbation method to study the parametric resonance and the primary resonance of the van der Pol oscillator with a time delay state feedback. By means of the averaging method and the multiple scales method, El-Bassiouny analyzed the qualitative behavior of the response of a ship rolling in longitudinal waves in [7]. In [8], the HB-AFT method was applied to obtain the periodic solutions of a four-degree-of-freedom Misaligned rotor model. In [9], M.Eissa employed the multiple scales perturbation technique to discuss the vibration suppression of a nonlinear magnetic levitation system via a nonlinear saturation-based controller. In [10–12], authors used the method of multiple scales to obtain the evolution equations of the amplitude and the phase of the whole controlled system. Previous published works showed that the multiple scales perturbation method had remarkable accuracy in solving the resonance solution of multi degree of freedom vibration systems. At present, the multiple scales perturbation method is almost the only way to solve the approximate solution of the multi-degree-of-freedom system, especially the four-degree-of-freedom system. However, the calculation process of this method is a little complicated, and the first step using the multiple scales perturbation method is to re-scale the parameters of the system, which has no formula to
| Nomenclature                  | Description                                                                 |
|------------------------------|-----------------------------------------------------------------------------|
| $u$, $\dot{u}$, $\ddot{u}$  | Dimensionless displacement, velocity and acceleration of the horizontal oscillation mode |
| $v$, $\dot{v}$, $\ddot{v}$  | Dimensionless displacement, velocity and acceleration of the vertical oscillation mode |
| $x$, $\dot{x}$, $\ddot{x}$  | Dimensionless displacement, velocity and acceleration of the controller of the horizontal oscillation mode |
| $y$, $\dot{y}$, $\ddot{y}$  | Dimensionless displacement, velocity and acceleration of the controller of the vertical oscillation mode |
| $f$                          | The rotor eccentricity                                                      |
| $\alpha$                     | Quadratic nonlinearity coefficient                                          |
| $\beta_1$, $\beta_3$        | Dimensionless cubic nonlinearity coefficient                                |
| $\gamma_1$, $\gamma_3$      | Dimensionless control signal gains of both the horizontal and vertical oscillation modes |
| $\gamma_2$, $\gamma_4$      | Dimensionless feedback signal gains of the controllers of both the horizontal and vertical oscillation modes |
| $\mu_1$, $\mu_3$            | Dimensionless linear damping coefficients of Jeffcott-rotor system          |
| $\mu_2$, $\mu_4$            | Dimensionless linear damping coefficients of saturation-based controller    |
| $\Omega$                    | Normalized rotor-spinning speed                                            |
| $\omega_1$, $\omega_3$      | Linear natural frequencies of Jeffcott-rotor system                         |
| $\omega_2$, $\omega_4$      | Linear natural frequencies of saturation-based controller                  |
| $\lambda_1$, $\lambda_2$    | Dimensionless negative velocity feedback signal gains                      |
follow. Therefore, it is still a very meaningful subject to develop an efficient quantitative analysis method for multi degree of freedom vibration system.

Another research hot spot is to design various controllers to suppress the mechanical vibration. The most commonly used controllers are NSC (nonlinear saturation controller), PPF controller (positive position feedback controller), PD controller (proportional-derivative controller), P controller and $X^3$ controller, etc. In [13], the authors utilized four different types controllers (saturation controller, PPF controller, P controller and $X^3$ controller) to suppress the nonlinear vibrations of a nonlinear beam-like structure and found that PPF and NSC controllers were the most effective. Compared to the PPF controller, using the saturation controller, there would be no double peaks on the frequency-response curve of the controlled system. That is to say, under the saturation control, the controlled system will not have large amplitude vibration in the whole frequency band. In [14], a saturation-based active controller had been proposed to reduce the vibration of a four-degree-of-freedom rotor-AMB at the primary resonance excitation and the presence of 1:1 and 1:2 internal resonances. They concluded that NSC controller could effectively eliminate the nonlinear phenomena when the natural frequencies of the controllers were properly tuned to half of the rotor-AMB natural frequency. Li et al. [15] employed a nonlinear saturation-based controller to suppress the free vibration of a self-excited plant. Kamel et al. [16] studied a nonlinear saturation controller (NSC), which was proposed to reduce the horizontal vibration of a magnetically levitated body. It illustrated that there was a region where the main system amplitude was notched down to have a minimum value after control. In [17], Hamed et al. proposed a nonlinear saturation controller that used to suppress the vibration amplitude of a nonlinear composite beam at simultaneous sub-harmonic and internal resonance excitation. In [18], Sayed et al. utilized the nonlinear time delay saturation-controller to mitigate the steady-state vibrations of a Model of helicopter rotor blade flapping motion in the presence of 1:4 internal resonances. Tusset et al. [19] presented two control strategies (NSC, MR) for a parametrically excited pendulum. It showed that saturation control method could suppress
the chaotic behavior. Kandil et al. [20] considered the vibration suppression of a compressor blade via a nonlinear saturation controller. It showed that the jump phenomena, saddle-node and Hopf bifurcations could be eliminated by the saturation-based controller. In [22], two modified saturation-based controllers were proposed to reduce the whirling motions of a vertically supported nonlinear Jeffcott-rotor system. The influences of time-delays on the controller performance and system stability were discussed. In order to make up for the narrow frequency band of transitional saturation controller, Xu et al.[24] improved the saturation controller by replacing quadratic position coupling term with the quadratic velocity coupling term.

In this paper, two modified saturation-based controllers and negative velocity feedbacks are applied to suppress the nonlinear vibrations of a horizontally supported Jeffcott-rotor system subjected to harmonic excitation force. The integral equation method [23–26], which was introduced by G.Schmidt, is utilized to obtain the second order approximations and the amplitude equations. The stability of the system is investigated by applying a combination of the Floquet theory and Hill’s determinant. The stable and unstable solutions are determined depending on the real parts of all eigenvalues of the Hill’s determinant. The frequency response curves, the force response curves and time histories are presented to illustrate the performance of the control law. In addition, the effects of the controller parameters on the main systems and controllers are explored and optimal working conditions of the system are extracted from the force-response curves and the frequency-response curves. Numerical simulations are presented to validate the analytical predictions.

2. Mathematical modeling

After integrating a negative velocity feedback and a nonlinear saturation-based controller to each oscillation mode of the uncontrolled main system, the modified dimensionless equations of motion of a horizontally supported Jeffcott-rotor system([11, 27, 28]) subjected to harmonic excitation force can be written
as follows:

\[ \ddot{u} + \mu_1 \dot{u} + \omega_1^2 u + 2\alpha uv + \beta_1(u^3 + v^2 u) = f\Omega^2 \cos(\Omega t) + \gamma_1 x^2 - 2\lambda_1 \omega_1 \dot{u} \]

\[ \ddot{x} + \mu_2 \dot{x} + \omega_2^2 x = \gamma_2 \dot{u} \dot{x} \]

\[ \ddot{v} + \mu_3 \dot{v} + \omega_3^2 v + \alpha(u^2 + 3v^2) + \beta_3(v^3 + u^2 v) = f\Omega^2 \sin(\Omega t) + \gamma_3 y^2 - 2\lambda_2 \omega_3 \dot{v} \]

\[ \ddot{y} + \mu_4 \dot{y} + \omega_4^2 y = \gamma_4 \dot{v} \dot{y} \]

(1)

Where \( \gamma_1 x^2 \) and \( \gamma_3 y^2 \) are the control forces, \( \gamma_2 \dot{u} \dot{x} \) and \( \gamma_4 \dot{v} \dot{y} \) represent the feedback signals, \(-2\lambda_1 \omega_1 \dot{u} \) and \(-2\lambda_2 \omega_3 \dot{v} \) represent negative velocity feedback signals. The schematic diagram of the whole controlled system is shown in Fig.1.

3. Amplitude equations

In this section, we examine the case of the simultaneous resonance (\( \Omega \cong \omega_1 \cong 2\omega_2 \cong \omega_3 \cong 2\omega_4 \)). To quantitatively describe the closeness of the considered resonance case, we introduce four detuning parameters \( \sigma_1, \sigma_2, \sigma_3, \sigma_4 \) defined by

\[ \Omega = \omega_1 + \sigma_1, \quad \omega_2 = \frac{\omega_1 + \sigma_3}{2}, \]

\[ \omega_3 = \omega_1 + \sigma_2, \quad \omega_4 = \frac{\omega_3 + \sigma_4}{2}. \]

(2)

In this section, the integral equation method (see Appendix)[23] is applied to seek an approximate solutions to Eq.(1). Then, a dimensionless time is introduced by

\[ t_1 = \frac{\Omega t}{2}. \]

(3)
Then Eq.(1) are transformed into the following form:

\[
\ddot{u} + 4u = \frac{4}{\Omega^2} (-\frac{\Omega}{2} \mu_1 \dot{u} + (2\Omega\sigma_1 - \sigma_1^2)u - 2\alpha uv - \beta_1(u^3 + v^2u) + f\Omega^2 \cos(2t) + \gamma_1 x^2
- \Omega \lambda_1(\Omega - \sigma_1) \dot{u})]
\]

\[
\ddot{x} + x = \frac{4}{\Omega^2} (-\frac{\Omega}{2} \mu_2 \dot{x} - (\frac{\sigma_3 - \sigma_1}{4} + \frac{\Omega(\sigma_3 - \sigma_1)}{2})x + \frac{\Omega^2}{4} \gamma_2 \dot{x})
\]

\[
\ddot{v} + 4v = \frac{4}{\Omega^2} (-\frac{\Omega}{2} \mu_3 \dot{v} + (2\Omega(\sigma_2 - \sigma_1) + (\sigma_2 - \sigma_1)^2)v - \alpha(u^2 + 3v^2) - \beta_3(v^3 + u^2v
+ f\Omega^2 \sin(2t) + \gamma_3 y^2 - \Omega \lambda_2(\Omega + \sigma_2 - \sigma_1) \dot{v})
\]

\[
\ddot{y} + y = \frac{4}{\Omega^2} (-\frac{\Omega}{2} \mu_4 \dot{y} - (\frac{\sigma_4 + \sigma_2 - \sigma_1}{4} + \frac{\Omega(\sigma_4 + \sigma_2 - \sigma_1)}{2})y + \frac{\Omega^2}{4} \gamma_4 \dot{y})
\]

For Eq.(4), the corresponding generalized Green’s functions are

\[
\lambda_{1,3} = 4,
\]

\[
G_{1,3}(t, \sigma) = \frac{1}{\pi} \left[ \frac{1}{8} + \frac{1}{4} \cos(t - \sigma) - \frac{1}{4} \cos(2(t - \sigma)) + \sum_{j=3}^{\infty} \frac{\cos(j(t - \sigma))}{4 - j^2} \right],
\]

\[
\lambda_{2,4} = 1,
\]

\[
G_{2,4}(t, \sigma) = \frac{1}{\pi} \left[ \frac{1}{12} - \cos(t - \sigma) + \sum_{j=2}^{\infty} \frac{\cos(j(t - \sigma))}{4 - j^2} \right].
\]

From Eq.(A.5) and Eq.(A.6) in Appendix, we can obtain a first-order approximate solution to Eqs.(4) as follows:

\[
u_1(t) = r_1 \cos(2t) + s_1 \sin(2t)
\]

\[
x_1(t) = r_2 \cos t + s_2 \sin t
\]

\[
v_1(t) = r_3 \cos(2t) + s_3 \sin(2t)
\]

\[
y_1(t) = r_4 \cos t + s_4 \sin t
\]

Substituting Eqs.(6)-(9) into Eqs.(A.5), the second order approximate solution
can be obtained as

\[ u_2(t) = \frac{2\pi}{0} \left( \frac{1}{8} + \frac{3}{4} \cos(t - \sigma) - \frac{1}{4} \cos(2(t - \sigma)) \right) + \sum_{j=3}^{\infty} \cos(j(t-\sigma)) \]

\[ \times \frac{4\pi}{f^2} \left( -\frac{\Omega}{2} \mu_1 u_1 + (2\Omega \sigma_1 - \sigma_1^2) u_1 - 2\sigma_1 u_1 - \beta_1 (u_1^3 + v_1^2 u_1) + f \Omega^2 \cos(2t) \right) 
+ \gamma_3 x_1^2 - \Omega \lambda_1 (\Omega - \sigma_1) u_1 \right) d\sigma + r_1 \cos(2t) + s_1 \sin(2t) - r_1 \beta_1 - s_1 \beta_1 \]

\[ + \frac{r_1^2 \beta_1}{12} - \frac{r_1^2 \gamma_1}{12} + \frac{2r_1^2 \gamma_1}{12} + \frac{3r_1^2 \beta_1}{12} + \frac{3r_1^2 \gamma_1}{12} + \frac{r_1 \beta_1 \gamma_1}{12} + \frac{r_1 \beta_1 \lambda_1}{12} + \frac{r_1 \gamma_1 \lambda_1}{12} \cos(2t) + \left( \frac{r_1 \gamma_1}{12} \right) \cos(6t) \]

\[ + \left( s_1 - 2r_1 \lambda_1 + \frac{3s_1 \beta_1}{12} + \frac{s_1 \beta_1 \lambda_1}{12} + \frac{3s_1 \gamma_1 \lambda_1}{12} + \frac{3s_1 \beta_1 \gamma_1}{12} + \frac{r_1 \beta_1 \lambda_1}{12} + \frac{r_1 \lambda_1 \gamma_1}{12} \right) \sin(2t) + \left( \frac{s_1 \gamma_1}{12} + \frac{r_1 \gamma_1}{12} \right) \sin(4t) \]

\[ + \left( -\frac{s_1 \gamma_1}{12} + \frac{s_1 \beta_1 \gamma_1}{12} + \frac{s_1 \beta_1 \lambda_1}{12} + \frac{r_1 \beta_1 \gamma_1}{12} \right) \sin(6t) \]

\[ (10) \]

\[ x_2(t) = \frac{2\pi}{0} \left( \frac{1}{8} - \frac{3}{4} \cos(t - \sigma) - \frac{1}{4} \cos(2(t - \sigma)) \right) + \sum_{j=2}^{\infty} \cos(j(t-\sigma)) \]

\[ \times \frac{4\pi}{f^2} \left( -\frac{\Omega}{2} \mu_2 x_1 - \left( \frac{(\sigma_2 - \sigma_1)^2}{4} \right) \right) \]

\[ + \left( \frac{\Omega (\sigma_2 - \sigma_1)}{2} \right) x_1 + \frac{\Omega^2}{12} \gamma_2 x_1 u_1 \right) d\sigma + r_2 \cos(t) + s_2 \sin(t) = (r_2 - r_1 r_2 \gamma_2 - s_1 r_2 \gamma_2 \]

\[ + \frac{3r_2 \beta_1}{12} - \frac{2r_2 \gamma_1}{12} + \frac{2r_2 \gamma_1}{12} + \frac{2r_2 \sigma_1}{12} + \frac{2r_2 \sigma_1}{12} \cos(2t) + \left( \frac{r_2 \gamma_1}{8} - \frac{s_2 \gamma_2}{2} \right) \cos(3t) + \left( s_2 - s_1 r_2 \gamma_2 + r_1 s_2 \gamma_2 + \frac{s_2 \gamma_2}{2} \right) - \frac{2s_2 \gamma_2}{12} + \frac{2s_2 \gamma_2}{12} \sin(t) \]

\[ + \left( \frac{2s_2 \gamma_2}{12} \right) \sin(3t) \]

\[ (11) \]

\[ v_2(t) = \frac{2\pi}{0} \left( \frac{1}{8} + \frac{3}{4} \cos(t - \sigma) - \frac{1}{4} \cos(2(t - \sigma)) \right) + \sum_{j=3}^{\infty} \cos(j(t-\sigma)) \]

\[ \times \frac{4\pi}{f^2} \left( -\frac{\Omega}{2} \mu_3 v_1 - \left( \frac{(\sigma_2 - \sigma_1)^2}{4} \right) \right) \]

\[ + \left( \frac{\Omega (\sigma_2 - \sigma_1)}{2} \right) v_1 + f \Omega^2 \sin(2t) + \frac{\gamma_3 y_1 (-\Omega \lambda_2 (\Omega - \sigma_2 - \sigma_1) v_1 - \alpha (u_1^3 + 3v_1^2) - \beta_3 (v_1^3) + u_1^2 v_1) + f \Omega^2 \sin(2t) + \gamma_3 y_1 \right) \right) d\sigma + r_3 \cos(2t) \]

\[ + \frac{s_3 \sin(2t)}{\Omega} = \frac{r_3 \beta_1}{12} - \frac{3r_3 \gamma_1}{12} + \frac{3r_3 \lambda_1}{12} + \frac{r_3 \beta_1 \gamma_1}{12} - \frac{r_3 \beta_1 \lambda_1}{12} + \frac{r_3 \gamma_1 \lambda_1}{12} \right) \cos(2t) + \left( \frac{r_3 \gamma_1}{12} \right) \cos(6t) \]

\[ \frac{r_3 \beta_1}{12} \cos(3t) + \frac{3r_3 \beta_1}{12} + \frac{r_3 \beta_1 \gamma_1}{12} - \frac{r_3 \beta_1 \lambda_1}{12} + \frac{r_3 \gamma_1 \lambda_1}{12} \right) \sin(2t) + \left( \frac{r_3 \gamma_1}{12} \right) \sin(4t) \]

\[ \frac{r_3 \beta_1}{12} \sin(3t) + \frac{3r_3 \beta_1}{12} + \frac{r_3 \beta_1 \gamma_1}{12} - \frac{r_3 \beta_1 \lambda_1}{12} + \frac{r_3 \gamma_1 \lambda_1}{12} \right) \sin(6t) \]

\[ (12) \]
Substituting Eqs. (10)-(13) into Eqs. (A.3) (see Appendix) yields a set of nonlinear algebraic equations on \( r_i, s_i, \ i = 1, 2, 3, 4 \)

\[
y_2(t) = \int_0^{2\pi} \left( \frac{1}{r} - \cos(t - \sigma) + \sum_{j=2}^{\infty} \frac{\cos(j(t-\sigma))}{j^2} \right) \times \frac{4}{1\pi^2} \left( -\frac{\Omega}{2} \mu_1 y_1 - \frac{(\sigma_4 + \sigma_2 - \sigma_1)^2}{4} \right) + \frac{\Omega (\sigma_4 + \sigma_2 - \sigma_1)}{2} y_1 + \frac{\Omega^2}{4} \gamma_4 y_1 \right) d\sigma + r_4 \cos(t) + s_4 \sin(t)
\]

\[
= (r_4 - r_3 r_4 \gamma_4 - s_3 s_4 \gamma_4 + \frac{r_4 \sigma_2^2}{12} - \frac{r_4 \sigma_4^2}{12} - \frac{r_4 \sigma_6^2}{12} + \frac{2r_4 \sigma_4 \sigma_2}{12} + \frac{2r_4 \sigma_2 \sigma_6}{12}) \cos(t) + \left( \frac{r_4 \gamma_4 \tau_4}{8} - \frac{s_4 s_4 \tau_4}{8} \right) \cos(3t)
\]

\[
+ (s_4 - s_3 r_4 \gamma_4 + r_3 s_4 \gamma_4 + \frac{s_4 \sigma_2^2}{12} - \frac{2s_4 \sigma_1 \sigma_2}{12} + \frac{s_4 \sigma_2^2}{12} - \frac{2s_4 \sigma_4 \sigma_2}{12} + \frac{2s_4 \sigma_4 \sigma_6}{12} + \frac{s_4 \sigma_6^2}{12}) \sin(t) + \left( \frac{s_4 s_4 \gamma_4}{8} + \frac{r_4 s_4 \gamma_4}{8} \right) \sin(3t)
\]

\[
(13)
\]

Substituting Eqs. (10)-(13) into Eqs. (A.3) yields a set of nonlinear algebraic equations on \( r_i, s_i, \ i = 1, 2, 3, 4 \)

\[
4f \Omega^2 - 8s_1 \lambda_1 \Omega^2 - 3r_1^3 \beta_1 - 3r_1 r_3^2 \beta_1 - 3r_1 s_3^2 \beta_1 - 2r_3 s_3 s_3 \beta_1 - r_1 s_3^2 \beta_1 + 2r_2^2 \gamma_1 - 2s_2^2 \gamma_1 - 4r_1 \sigma_1^2 - 4s_1 \mu_1 \Omega + 8r_1 \sigma_1 \Omega + 8s_1 \lambda_1 \sigma_1 \Omega = 0
\]

\[
(14)
\]

\[
8 \Omega^2 r_1 \lambda_1 - 3s_1^3 \beta_1 - s_1 r_3^2 \beta_1 - 3s_1 r_3^2 \beta_1 - 2r_3 s_3 \beta_1 - 3s_1 s_3^2 \beta_1 + 4r_2 s_2 \gamma_1 - 4s_1 \sigma_1^2 + 4r_1 \mu_1 \Omega + 8s_1 \sigma_1 \Omega - 8r_1 \lambda_1 \sigma_1 \Omega = 0
\]

\[
(15)
\]

\[
- r_1 r_2 \gamma_2 \Omega^2 - s_1 s_2 \gamma_2 \Omega^2 + r_2 \sigma_1^2 - 2r_2 \sigma_1 \sigma_3 + r_2 \sigma_3^2 + 2s_2 \mu_2 \Omega - 2r_2 \sigma_1 \Omega + 2r_2 \sigma_3 \Omega = 0
\]

\[
(16)
\]

\[
- s_1 r_2 \gamma_2 \Omega^2 + r_1 s_2 \gamma_2 \Omega^2 + s_2 \sigma_1^2 - 2s_2 \sigma_1 \sigma_3 + s_2 \sigma_3^2 - 2r_2 \mu_2 \Omega - 2s_2 \sigma_1 \Omega + 2s_2 \sigma_3 \Omega = 0
\]

\[
(17)
\]

\[
8s_3 \lambda_2 \Omega^2 + 3r_3^3 \beta_3 + 3r_3 r_3^2 \beta_3 + 3r_3 s_3^2 \beta_3 + 2r_3 s_3 s_3 \beta_3 + r_3 s_3^2 \beta_3 - r_3^2 \gamma_3 + 2s_3^2 \gamma_3 - 4r_3 s_3 \sigma_1^2 + 4r_3 \sigma_3^2 + 4s_3 \mu_3 \Omega - 8r_3 \sigma_3 \Omega - 8s_3 \lambda_2 \sigma_3 \Omega + 8r_3 \sigma_2 \Omega + 8s_3 \lambda_2 \sigma_2 \Omega = 0
\]

\[
(18)
\]

\[
- 4f \Omega^2 - 8r_3 \lambda_2 \Omega^2 + 2r_3 s_3 \sigma_1^2 + 3s_3^3 \beta_3 + s_3 r_3^2 \beta_3 + s_3 s_3^2 \beta_3 + 3s_3 s_3^2 \beta_3 - 3s_3 s_3^2 \beta_3 - 3s_3^2 s_3^2 \beta_3 + 3s_3^2 s_3^2 \beta_3 - 3s_3^2 s_3^2 \beta_3 - 3s_3^2 s_3^2 \beta_3 - 3s_3^2 s_3^2 \beta_3 - 3s_3^2 s_3^2 \beta_3 = 0
\]

\[
(19)
\]

\[
+ 8s_3 \sigma_2 \Omega - 8r_3 \lambda_2 \sigma_2 \Omega = 0
\]

\[
- r_3 r_4 \gamma_4 \Omega^2 - s_3 s_4 \gamma_4 \Omega^2 + r_4 \sigma_1^2 - 2r_4 \sigma_1 \sigma_3 + r_4 \sigma_3^2 + 2r_4 s_4 \sigma_2 + 2r_4 \mu_4 \Omega + 2r_4 \sigma_4 \Omega + 2r_4 \sigma_2 \Omega = 0
\]

\[
(20)
\]

\[
s_4 r_3 \gamma_4 \Omega^2 - r_4 s_4 \gamma_4 \Omega^2 + s_4 \sigma_1^2 - 2s_4 \sigma_1 \sigma_3 + s_4 \sigma_3^2 - 2s_4 \sigma_4 \sigma_2 + 2s_4 \sigma_4 \Omega + 2s_4 \sigma_2 \Omega + 2s_4 \sigma_4 \Omega = 0
\]

\[
(21)
\]
For convenience to investigate the dynamics of the whole controlled system in the following sections, we denote the amplitude of the horizontal oscillation mode as \( A_1 = \sqrt{r_1^2 + s_1^2} \), the amplitude of the vertical oscillation mode as \( A_3 = \sqrt{r_3^2 + s_3^2} \), the amplitude of the controller of the horizontal oscillation mode as \( A_2 = \sqrt{r_2^2 + s_2^2} \), and the amplitude of the controller of the vertical oscillation mode as \( A_4 = \sqrt{r_4^2 + s_4^2} \).

4. Stability of periodic solutions

To investigate the stability of the periodic solutions of Eq.(1), we suppose \( z_1(t) \), \( z_2(t) \), \( z_3(t) \), \( z_4(t) \) are small perturbations about the approximate solutions \( u_2(t) \), \( x_2(t) \), \( v_2(t) \), \( y_2(t) \). And then substituting \( u_2(t) + z_1(t) \), \( x_2(t) + z_2(t) \), \( v_2(t) + z_3(t) \), \( y_2(t) + z_4(t) \) into Eq.(1), expanding for the small \( z_1(t) \), \( z_2(t) \), \( z_3(t) \) and \( z_4(t) \) with keeping the linear terms only, yields the following linearized system:

\[
\ddot{z}_1 + 4z_1 = \frac{4}{\Omega^2} (\mu_1 \dot{z}_1 + (2\Omega \sigma_1 - \sigma_1^2)z_1 - 2\alpha z_1 (-r_1^2 \sigma_1 - 3r_2 \sigma_1 - s_1^2 \sigma_1 - \frac{3r_2^2 \sigma_1}{2r_1^2} + r_1 \gamma_1 + r_1 \gamma_2 + r_3 \cos(2t) + s_3 \sin(2t)) - 2\alpha z_3 (-r_1^2 \sigma_1 - 3r_2 \sigma_1 - s_1^2 \sigma_1 - \frac{3r_2^2 \sigma_1}{2r_1^2} + r_1 \gamma_1 + r_1 \gamma_2 + r_3 \cos(2t) + s_3 \sin(2t)) - \beta_1 (3(-r_1^2 \sigma_1 - 3r_2 \sigma_1 - s_1^2 \sigma_1 - \frac{3r_2^2 \sigma_1}{2r_1^2} + r_1 \gamma_1 + r_1 \gamma_2 + r_3 \cos(2t) + s_3 \sin(2t)) + r_3 \cos(2t) + s_3 \sin(2t)) z_3 + 2r_2 \cos(t) + s_2 \sin(t) \dot{z}_2 - \Omega \lambda_1 (\Omega - \sigma_1) \dot{z}_1) 
\]

\[
\dot{z}_2 + z_2 = \frac{4}{\Omega^2} (\mu_2 \dot{z}_2 - \left( \frac{\sigma_3 - \sigma_1}{4} \right) \dot{z}_2 + \frac{\Omega^2}{4} \gamma_2 \left( z_2 - 2s_2 \cos(t) \right) - 2r_1 \sin(2t) \dot{z}_2 + (s_2 \cos t - r_2 \sin t) \dot{z}_1) 
\]

(22) (23)
\[
\ddot{z}_3 + 4z_3 = \frac{1}{17}\left(-\Omega_\mu_3\dot{z}_3 - ((\sigma_2 - \sigma_1)^2 + 2\Omega(\sigma_2 - \sigma_1))z_3 - \alpha(6z_3(-\frac{r^2\sigma}{21\Omega})
- \frac{3r^2\alpha}{21\Omega} - \frac{s^2\alpha}{21\Omega} + \frac{r^3\gamma_1}{21\Omega} + \frac{s^3\gamma_1}{21\Omega} + r_3\cos(2t) + s_3\sin(2t)) + 2z_1(-\frac{r^2\sigma}{21\Omega} - \frac{r^3\gamma_1}{21\Omega} + \frac{s^3\gamma_1}{21\Omega} + s_3\cos(2t) + r_3\sin(2t)) - \beta_3(3(-\frac{r^2\gamma_1}{21\Omega} - \frac{r^3\gamma_1}{21\Omega} + \frac{s^3\gamma_1}{21\Omega} + s_3\cos(2t) + r_3\sin(2t))z_3 + 2(-\frac{r^2\gamma_1}{21\Omega} - \frac{s^3\gamma_1}{21\Omega} + \frac{r^3\gamma_1}{21\Omega} + \frac{s^3\gamma_1}{21\Omega} + r_3\cos(2t) + s_3\sin(2t))z_1(-\frac{r^2\gamma_1}{21\Omega} - \frac{r^3\gamma_1}{21\Omega} + \frac{s^3\gamma_1}{21\Omega} + \frac{r^3\gamma_1}{21\Omega} + r_3\cos(2t) + s_3\sin(2t)))
+ r_3(\cos(2t) + s_3\sin(2t)) + (-\frac{s^3\alpha}{17\Omega} - \frac{r^3\gamma_1}{21\Omega} + \frac{s^3\gamma_1}{21\Omega} + s_3\cos(2t) + r_3\sin(2t))z_1(-\frac{r^2\gamma_1}{21\Omega} - \frac{r^3\gamma_1}{21\Omega} + \frac{s^3\gamma_1}{21\Omega} + \frac{r^3\gamma_1}{21\Omega} + r_3\cos(2t) + s_3\sin(2t))
+ r_3(\cos(2t) + s_3\sin(2t)) + (-\frac{r^3\gamma_1}{21\Omega} - \frac{s^3\gamma_1}{21\Omega} + \frac{s^3\gamma_1}{21\Omega} + r_3\cos(2t) + s_3\sin(2t))
+ s_3(\cos(2t) + s_3\sin(2t))z_3 - \Omega\lambda_2(\Omega + \sigma_2 - \sigma_1)\dot{z}_3)
\]
\[\text{(24)}\]

\[
\ddot{z}_4 + z_4 = \frac{1}{17}\left(-\Omega\mu_4\dot{z}_4 - ((\sigma_1 + \sigma_2 - \sigma_1)^2 + \frac{\Omega(\sigma_1 + \sigma_2 - \sigma_1)}{2})z_4 + \frac{\Omega^2}{4}\gamma_4((2s_3\cos(2t)
- 2r_3\sin(2t)))\dot{z}_4 + (s_4\cos(t - r_4\sin(t))\dot{z}_3)\right)\]
\[\text{(25)}\]

According to Floquet theory, the solutions of Eq.(22)-Eq.(25) can be expressed in the form
\[z_i(t) = \exp(\rho t)Z_i(t), \quad i = 1, 2, 3, 4\]
\[\text{(26)}\]

Inserting Eq.(26) into Eqs.(22)-(25), we get

\[
\ddot{Z}_1 + 4Z_1 = (-\rho^2 - \frac{4\sigma_1(\sigma_1 - 2\Omega)}{17\Omega} - \frac{2\mu\rho}{17} - \frac{4\lambda_1(\sigma_1 + \Omega)}{17\Omega} - \frac{3\beta_1}{17})
- 2r_1\sigma_3 - 2s_3\sigma_3 + r_2^2\gamma_1 + s_2^2\gamma_1 + 2r_1\Omega^2\cos(2t) + 2s_1\Omega^2\sin(2t))^2
+ \frac{4\alpha}{17}(r_1^2\sigma + 3r_3^2\sigma + s_1^2\sigma + s_3^2\sigma - r_1^2\gamma_3 - s_1^2\gamma_3 - 2r_3\Omega^2\cos(2t)
- 2s_3\Omega^2\sin(2t)) - \frac{\beta_1}{17}(r_1^2\sigma + 3r_3^2\sigma + s_1^2\sigma + s_3^2\sigma - r_1^2\gamma_3 - s_1^2\gamma_3
- 2r_3\Omega^2\cos(2t) - 2s_3\Omega^2\sin(2t))^2)Z_1 + \frac{s^2\gamma_1}{17}(r_2\cos(t) + s_2\sin(t))Z_2 + \frac{4\alpha}{17}
\[\text{(27)}\]

\[
(2r_1\sigma_3 + 2s_3\sigma_3 - r_2^2\gamma_1 - s_2^2\gamma_1 - 2r_1\Omega^2\cos(2t) - 2s_1\Omega^2\sin(2t))
+ \frac{2\beta_1}{17}(2r_1\sigma_3 + 2s_3\sigma_3 - r_2^2\gamma_1 - s_2^2\gamma_1 - 2r_1\Omega^2\cos(2t) - 2s_1\Omega^2\sin(2t))
- 2s_3\Omega^2\sin(2t))Z_3 + (-2\rho - \frac{2\mu}{17} - \frac{4\lambda_1(\sigma_1 + \Omega)}{17})Z_1
\[\text{(28)}\]
The first approximations of Eqs.(27)-(30) can be expressed in the following form

\[ \ddot{Z}_3 + 4Z_3 = \left( \frac{22}{15} \right)(2r_1\sigma_4 - 2s_1s_3\alpha - r_2^2\gamma_1 - 2r_1\Omega^2 \cos(2t) - s_2^2\gamma_1 - 2s_1\Omega^2 \sin(2t)) - \frac{4\rho}{15}(\sigma_1 - \sigma_2 - \Omega) - 2\rho^2 - \frac{2\rho^3}{15} - \frac{\rho}{15}(2r_1\sigma_4 - 2s_1s_3\alpha + r_2^2\gamma_1 + 2r_1\Omega^2 \cos(2t) + 2s_1\Omega^2 \sin(2t) + s_2^2\gamma_1) + \frac{4\rho}{15}(\sigma_1 + \sigma_2 + \Omega) + \frac{12\rho}{15}(r_1^2\alpha + 3r_3^2\alpha + s_1^2\alpha + 3s_3^2\alpha - r_2^2\gamma_3 - s_2^2\gamma_3 - 2r_3\Omega^2 \cos(2t) - 2s_3\Omega^2 \sin(2t)) - \frac{3\rho}{15}(r_1^2\alpha + 3r_3^2\alpha + s_1^2\alpha + 3s_3^2\alpha - r_2^2\gamma_3 - s_2^2\gamma_3 - 2r_3\Omega^2 \cos(2t) - 2s_3\Omega^2 \sin(2t)))Z_3 + \frac{8\rho}{15}(r_4 \cos t + s_4 \sin t)Z_4 + \left( \frac{4\rho}{15}(\sigma_1 + \sigma_2 + \Omega) - 2\rho - \frac{2\rho^3}{15} \right) \dot{Z}_3 \]

\[ \ddot{Z}_4 + Z_4 = \frac{1}{15}(\gamma_4^2 \rho^2 (s_4 \cos t - r_4 \sin t))Z_3 + (\sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2 - 2\sigma_1\sigma_4 + 2\sigma_2\sigma_4 + 2\sigma_4^2\rho - 2\sigma_1\sigma_2\Omega + 2\sigma_4\Omega + 2s_4\Omega + \rho^2\Omega^2 - 2s_3\gamma_4 \Omega^2 \cos(2t) + \frac{3\rho}{15}(\sigma_1^2 + 2s_4\gamma_4 \Omega^2 \sin(2t)))Z_4 + (\gamma_4^2 (s_4 \cos t - r_4 \sin t)Z_3 - 2(\mu_4 + \rho \Omega - s_3\gamma_4 \Omega \cos(2t) + r_3 \gamma_4 \Omega \sin(2t))) \dot{Z}_4) \]
least one eigenvalue is positive, the approximate periodic solution is unstable. The Routh-Hurwitz criterion can be employed to investigate the stability of the periodic solution.

In the following sections, all bifurcation diagrams are constructed by solving the nonlinear algebraic equations (14)-(21) in terms of one of the control parameter as a bifurcation parameter. Meanwhile, the stability of the obtained approximate solutions are checked according to Eq.(35).

5. Effects of the control parameters

In this section, the steady state vibration amplitudes at both the horizontal and vertical directions are investigated analytically and numerically. The adopted values of the dimensionless system parameters are fixed at: $f = 0.015$, $\alpha = 0.05$, $\omega_1 = 1.0247$, $\beta_1 = 0.05$, $\beta_3 = 0.05$, $\gamma_1 = 0.3$, $\gamma_2 = 2.5$, $\gamma_3 = 0.3$, $\gamma_4 = 2.5$, $\lambda_1 = 0.01$, $\lambda_2 = 0.01$, $\mu_1 = 0.015$, $\mu_2 = 0.001$, $\mu_3 = 0.025$, $\sigma_2 = 0.05$, $\mu_4 = 0.001$, unless otherwise is stated. Effects of various control parameters $f$, $\lambda_1$, $\lambda_2$, $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$, $\mu_2$, $\mu_4$ on both response curves and the stability regions are investigated. The main results are presented in graphical forms as steady-state amplitudes versus the detuning parameters or the rotor eccentricity. In all bifurcation diagrams, the solid lines represent stable solutions, while the dashed ones represent unstable solutions.

(1) The necessity of negative velocity feedbacks

Fig 2 shows the frequency-response curves under three states: without any control, controlled only by the negative velocity feedback and controlled simultaneously by the negative velocity feedback and the saturation-based controller (i.e. the combined control). It can be seen that, when only the negative velocity feedback control is utilized, the amplitudes of the main systems of the horizontal oscillation mode and the vertical oscillation mode are dramatically reduced, and the nonlinear behaviors are simultaneously disappeared. After integrating the saturation-based controllers to main system, the vibration reduction effect is better than that controlled only by the negative velocity feedback control,
Fig. 1: Schematic diagram of the whole controlled system

Fig. 2: Frequency-response curves of two main systems in three states: before control (blue, $\lambda_1 = \lambda_2 = 0$), controlled only by the negative velocity feedback control (red, $\lambda_1 = \lambda_2 = 0.1$) and the combined control (green, $\lambda_1 = \lambda_2 = 0.1$).
even at the resonance point, it can be reduced to almost zero.

Fig. 3 shows the frequency-response curves under the combined control. In Fig. 3, as $\sigma_1$ increase gradually from a negative value (point A), the amplitudes of the main systems increase gradually until they reach point B. In this process, the saturation controllers does not activate. At point B, the saturation controller starts to work automatically, the main system amplitudes (A1 and A3) decrease with the increase of $\sigma_1$, until it reaches point C (optimum operating point) at $\sigma_1 = 0$. Beyond C, the oscillation continues to rise again with an observable rate until $\sigma_1$ reaches point E. At point E, saddle-node bifurcations occur. Subsequent to this bifurcation point, the vibration amplitudes (A1 and A3) exhibit a jump from point E to point F, then the vibration amplitudes along the path $F \rightarrow G$ and undergo small amplitude vibrations until $\sigma_1$ reach point G. As $\sigma_1$ decreases gradually from point G, the vibration amplitudes will increase gradually following the $G \rightarrow F \rightarrow D$ path until $\sigma_1$ reaches D. At point D, the saturation controllers react and the main system amplitudes gradually decrease along the path $D \rightarrow C$. Beyond C, the vibration amplitudes increases until $\sigma_1$ reaches point H. At point H, there are saddle-node bifurcations. Subsequent to this point, the two modes jump down from point H to I. Then the oscillation amplitudes decrease along the $I \rightarrow A$ path.

![Fig. 3: Frequency-response curves under the combined control. (a): the horizontal oscillation mode; (b): the vertical oscillation mode](image-url)
Fig. 4: The frequency-response curves at different values of the rotor eccentricity $f$ before control.

Fig. 5: Frequency $\sigma_1$-response curves at different values of the rotor-eccentricity $f$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode.
(2) Effects of the rotor eccentricity $f$ on the frequency-response curves with and without control

Fig. 4 shows the effect of varying the rotor eccentricity $f$ on the frequency-response curve of the considered systems without any control. As can be seen from this figure, with the increase of $f$, the frequency-response curves at both the horizontal and vertical oscillation modes change from a single-valued linear curve to a nonlinear curve with multiple steady-state solutions, and the jump phenomenon appears clearly due to the domination of the nonlinearity. It also illustrates that the steady state amplitude is a monotonic increasing function of the rotor eccentricity $f$.

Fig. 5 shows the effect of varying the rotor eccentricity on the frequency-response curve of the main system ($A_1, A_3$) and the controller ($A_2, A_4$) under the combined control (the negative velocity feedback control and the saturation-based control). It can be seen that each amplitude of the whole controlled system is a monotonic increasing function with respect to the rotor eccentricity $f$. We also observe that increasing $f$ can broaden the effective frequency bandwidth of the saturation controller. Fig. 5(a) and (c) illustrate that, under three different values of $f$, the frequency $\sigma_1$-response ($A_1, A_3$) curves of two main systems coincide when the saturation-based controllers activate. This is because that the saturation phenomenon occurs and all the extra energy added to the main systems by increasing the excitation amplitude is transferred to the saturation controllers.

(3) Effects of the detuning parameter $\sigma_1$

Fig. 6 shows the effect of varying the detuning parameter $\sigma_1$ on the force-response curve in three different control states (i.e. without control, controlled only by the negative velocity feedback, controlled by the combination of the negative velocity feedback and the saturation-based controller). The figure shows that, when there is no any control, a slight increase of the rotor-eccentricity $f$ produces a large vibration amplitudes, and the increase of $\sigma_1$ results in multivalued solutions and jump phenomenons. We also observe that increasing $\sigma_1$ leads to a significant broadening of the multiple-solution region. That is to say, the
Fig. 6: Varying $\sigma_1$ on force-response curves in three different control states. Blue: before control, red: controlled only by the negative velocity feedback, black: under the combined control.
increase of $\sigma_1$ would enhance the nonlinear characteristics of the uncontrolled systems. After integrating a negative velocity feedback to the main system of each mode, the amplitudes of two oscillation modes ($A_1, A_3$) become linear functions of the rotor eccentricity $f$, the jump phenomena have been eliminated even at larger rotor eccentricity and the vibration of the considered systems have been suppressed drastically. In addition, we add a nonlinear saturation-based controller to each oscillation mode of the considered system. In Fig.6, We also observe that, after the combined control of the negative velocity feedback and the saturation-based controller, both the amplitudes of the horizontal oscillation and the vertical oscillation are almost reduced to zero and keep constant when $f$ increases to a certain value. This value is the threshold for the system to enter the saturation control state. Fig.6(a), (c) and (e) show that increasing $\sigma_1$ will increase both the threshold value of $f$ and the amplitude of the horizontal oscillation. However, it can be seen from Fig.6(b),(d) and (f) that, increasing $\sigma_1$ will decrease both the threshold value of $f$ and the amplitude of the vertical oscillation.

(4) Effects of the internal detuning parameter $\sigma_3$

Fig.8 shows the frequency-response curves of the whole system under the combined control for three different values of the internal detuning parameter $\sigma_3$. Fig.8 (a) shows that for $\sigma_3 = -0.15$ the minimum steady-state amplitude of the horizontal oscillation mode occurs at $\sigma_1 = -0.15$, for $\sigma_3 = 0$ the minimum main system steady-state amplitude occurs at $\sigma_1 = 0$, and for $\sigma_3 = 0.15$ the minimum steady-state amplitude occurs at $\sigma_1 = 0.15$. From this figure, it can be deduced that the minimum steady-state amplitude of the horizontal oscillation mode occurs at $\sigma_1 = \sigma_3$ (i.e. $\omega_2 = \frac{\Omega}{2}$ from Eq.(2)). That is to say, it is necessary to keep the natural frequency of the controller of the horizontal oscillation mode ($\omega_2$) equal to half of the rotor-spinning speed($\Omega$). From Fig.8 (b), we observe that, when $\sigma_3$ is negative, the frequency-response curve of the controller of the horizontal oscillation mode ($\sigma_1-A_2$) moves to the left. And for the positive value of $\sigma_3$, the frequency-response curve moves to the right. From Fig.8 (c), (d), it can be seen that, the change of $\sigma_3$ has little effect on the frequency-response
Fig. 7: Frequency-response curve at different values of $\lambda_1$, $\lambda_2$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode
Fig. 8: Frequency-response curves at different values of $\sigma_3$ under the combined control: $(a), (b)$ the horizontal oscillation mode; $(c), (d)$ the vertical oscillation mode
curves of the main system and the controller of the vertical oscillation mode.

Fig. 9: Frequency-response curves at different values of $\sigma_4$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode

(5) Effects of the internal detuning parameter $\sigma_4$

Fig. 9 shows the frequency-response curves of the whole system under the combined control for three different values of the internal detuning parameter $\sigma_4$. Fig. 9(a) and (b) illustrate that $\sigma_4$ has little effect on the frequency-response curves of the main system and the controller of the horizontal oscillation mode. Fig. 9(c) shows that for $\sigma_4 = -0.15$ the minimum steady-state amplitude of the vertical oscillation mode occurs at $\sigma_1 = -0.1$, for $\sigma_4 = 0$ the minimum main system steady-state amplitude occurs at $\sigma_1 = 0.05$, and for $\sigma_4 = 0.15$ the minimum steady-state amplitude occurs at $\sigma_1 = 0.2$. From this figure, it can
be deduced that the minimum steady-state amplitude of the vertical oscillation mode occurs at $\sigma_1 = \sigma_4 + 0.05 = \sigma_4 + \sigma_2$. From Eq.(2), we can conclude that the natural frequency of the controller of the vertical oscillation mode ($\omega_4$) should always be equal to half of the rotor-spinning speed ($\Omega$). Similar to Fig.8(b), Fig.9(d) illustrates that for the negative value of $\sigma_4$ the frequency response curve of the controller of the vertical oscillation mode ($\sigma_1-A_4$) moves to the left, and for the positive value of $\sigma_4$ the frequency response curve of the controller ($\sigma_1-A_4$) moves to the right.

Fig. 10: Frequency-response curves at different values of $\gamma_2$, $\gamma_4$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode

(6) Effects of the feedback signal gain $\gamma_2$, $\gamma_4$

Fig.10 shows the frequency-response curves of the whole system under the
combined control at three different values of the feedback signal gain $\gamma_2$, $\gamma_4$. Fig. 8 (a) and (c) indicate that increasing $\gamma_2$, $\gamma_4$ can widen the effective frequency bandwidth of the saturation-based controller. We also observe that the larger $\gamma_2$, $\gamma_4$, the smaller the amplitudes of the main systems of two oscillation modes under the saturation-based control. It means that increasing $\gamma_2$, $\gamma_4$ can enhance the vibration reduction effect of the main systems at both the horizontal and vertical oscillation modes. Therefore, the feedback gains $\gamma_2$, $\gamma_4$ are the key factors to ensure the damping effect of the saturation-based controllers. However, the increase of $\gamma_2$, $\gamma_4$ will result in large amplitude vibration of the saturation-based controllers, especially when $\sigma_1$ deviates from the origin.

![Frequency-response curves at different values of $\gamma_1$, $\gamma_3$ under the combined control](image)

Fig. 11: Frequency-response curves at different values of $\gamma_1$, $\gamma_3$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode.
(6) Effects of the control signal gain $\gamma_1$, $\gamma_3$

Effects of varying the control signal gains $\gamma_1$, $\gamma_3$ on the system frequency-response curves are illustrated in Fig.11. It can be seen from Fig.11(a) and (c) that varying the control gain $\gamma_1$, $\gamma_3$ has little effect on the amplitudes of the main systems of two oscillation modes and the effective frequency bandwidth of the saturation-based controllers. However, Fig.11 (b) and (d) show that the amplitudes of the two saturation-based controllers decrease as $\gamma_1$, $\gamma_3$ increase. Therefore, the control signal gains $\gamma_1$, $\gamma_3$ can be used as important parameters to prevent the controllers overload risk.

(7) Effects of the damping coefficients $\mu_2$, $\mu_4$

Fig. 12: Frequency-response curves at different values of $\mu_2$, $\mu_4$ under the combined control : (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode
Fig. 12 displays the effects of the linear damping coefficients $\mu_2$, $\mu_4$ of the saturation-based controllers on the frequency-response curves. From Fig. 12(a) and (c), it is noted that decreasing $\mu_2$, $\mu_4$ can reduce the main system amplitudes of the two oscillation modes better. Meanwhile, Fig. 12(b) and (d) show that decreasing $\mu_2$, $\mu_4$ can increase the amplitudes of the saturation-based controllers due to enhance energy transfer between the main systems and the controllers. That is to say, the smaller the $\mu_2$, the bigger the overload risk of the saturation controller.

6. Numerical simulations

![Fig. 13: Comparison of frequency-response curves between numerical solutions and approximate solutions when $f = 0.015$, $\sigma_3 = 0$, $\sigma_4 = 0$ before control. The red dots for numeric solutions. (a) horizontal oscillations; (b) vertical oscillations](image)

To validate the results of the integral equation method, the analytical results are verified by integration of the original Eqs.(4) numerically, and the numerical results for steady-state solutions are marked as red points on Figs.13, 15-18. Fig.13 show comparison of frequency-response curves between numerical solutions and approximate solutions when $f = 0.015$, $\sigma_3 = 0$, $\sigma_4 = 0$ without control. From the figure, we can see that the main systems without any control both have narrow multiple-solution regions and jump phenomenons around
Fig. 14: Time histories for the main systems of two oscillation modes when $\sigma_1 = 0.018$, (a),(c): $u(0) = 0.1$, $v(0) = 0.1$; (b),(d): $u(0) = 0.8$, $v(0) = 0.8$. 
$\sigma_1 = 0.02$. Fig.14 confirm the coexistence of two stable solutions at $\sigma_1 = 0.018$ with two different initial conditions.

Fig. 15: Comparison of frequency-response curves between numerical solutions and approximate solutions when $\sigma_3 = 0$, $\sigma_4 = -0.05$. (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode.

Fig.15-Fig.17 show comparisons of frequency $\sigma_1, \sigma_3, \sigma_4$-response curves between numerical solutions and approximate solutions obtained by the integral equation method. Fig.15 shows that the system exhibits a large oscillation amplitudes before control, but after control, the maximum oscillation amplitude becomes extremely small. From Fig.16, we can see that both amplitudes of the main system $A_3$ and the controller $A_4$ at the vertical oscillation mode are independent of the detuning parameter $\sigma_3 = 0$. Similarly, Fig.17 shows that the detuning parameter $\sigma_4$ has no effect on both amplitudes of the main system.
and the controller at the horizontal oscillation mode. These figures also confirm a good agreement between the numerical solutions and the approximate analytical solution.

![Frequency-response curves comparison](image)

**Fig. 16**: Comparison of frequency-response curves between numerical solutions and approximate solutions when $\sigma_1 = 0$, $\sigma_4 = 0$. (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode

Fig. 18(a) and (c) show comparisons of force $f$-response curves between numerical solutions and approximate solutions obtained by the integral equation method when $\sigma_1 = 0$, $\sigma_3 = 0$, $\sigma_4 = 0$. In these two figures, $A_1$ and $A_3$ are plotted versus $f$ for three different cases. Blue corresponds to the case without any control, green corresponds to the case of only negative velocity feedback control, and black represents the case of the combined control. In Fig. 18(b) and (d), $A_2$ and $A_4$ under the combined control are plotted as a function of
Fig. 17: Comparison of frequency-response curves between numerical solutions and approximate solutions when $\sigma_1 = 0$, $\sigma_3 = 0$. (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode.
Fig. 18: Comparison of excitation force-response curves between numerical solutions and approximate solutions when $\sigma_1 = 0$, $\sigma_3 = 0$, $\sigma_4 = 0$. (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode
Based on Figs. 13, 15-18, we found that theoretical results obtained by the integral equation method are in good agreement with numerical simulation.

Fig. 19: Time histories before control when $\sigma_1 = 0, \sigma_3 = 0, \sigma_4 = -0.05$. (a) horizontal oscillations; (b) vertical oscillations

Fig. 19 show time-histories of the uncontrolled main systems when $\sigma_1 = 0, \sigma_3 = 0, \sigma_4 = -0.05$. It can be seen that the amplitudes of the two main systems are about 0.7 and 0.14, respectively. Fig. 20 show time histories of the whole system under the combined control when $\sigma_1 = 0, \sigma_3 = 0, \sigma_4 = -0.05$, $\lambda_1 = 0.01, \lambda_2 = 0.01$ with the initial conditions $u(0) = 0, v(0) = 0, x(0) = 0.1, y(0) = 0.1, \dot{u} = 0, \dot{v} = 0$. By comparing Fig. 19 and Fig. 20, it is clear that the two main systems’ steady-state amplitude under the combined control is reduced by about 99.9% and 99.7%, respectively. Fig. 21 show time histories of the whole system under the combined control when $\sigma_1 = 0, \sigma_3 = 0, \sigma_4 = -0.05$, $\lambda_1 = 0.05, \lambda_2 = 0.05$ with the same initial conditions in Fig. 20. Compared with Fig. 20 and Fig. 21, we observe that the increase of $\lambda_1$ and $\lambda_2$ can effectively reduce the transient vibration time.

7. Conclusion

A saturation controller and a negative velocity feedback are integrated to control the transient and steady-state vibrations of a horizontally supported
Fig. 20: Time histories of the whole system under the combined control for 
\( \sigma_1 = 0, \sigma_3 = 0, \sigma_4 = -0.05, \gamma_2 = \gamma_4 = 5, \gamma_1 = \gamma_3 = 0.3, \lambda_1 = 0.01, \lambda_2 = 0.01. \)
Fig. 21: Time histories of the whole system under the combined control for 
\( \sigma_1 = 0, \sigma_3 = 0, \sigma_4 = -0.05, \gamma_2 = \gamma_4 = 5, \gamma_1 = \gamma_3 = 0.3, \lambda_1 = 0.05, \lambda_2 = 0.05. \)
Jeffcott-rotor system at primary resonance excitation and the presence of 1:1 and 1:2 internal resonances. An uncommon method called the integral equation method is used to find the second-order approximate solutions and the amplitude equations. Then, the Floquet theory is utilized to analyze the stability of the approximate periodic solutions. The analyses revealed that:

1. Both the steady state approximate solutions and the stability analysis based on the integral equation method are in excellent agreement with numerical simulations.

2. It can be seen from the solving process of integral equation method that, once the Generalized Green’s functions are determined according to the resonance condition, the amplitude equations and the concrete form of steady state approximate solutions can be obtained in two step. It can be said that compared with the multiple scales method, the integral equation method does not need to rescale parameters and get the amplitude equations more quickly. The mechanism of this method is simple and clear, and it is easier to program than other perturbation methods. Therefore, the integral equation method is worth popularizing to solve the steady-state approximate solutions of multiple degree of freedom vibration system.

3. Using the saturation controller can avoid the phenomenon of double peaks on the frequency-response curve of the controlled system. The negative velocity feedback can suppress the transient vibrations, thus shorten the time to reach a steady state. The above studies illustrate that the negative velocity feedback can also enhance the efficiency of the vibration reduction and suppress the nonlinear behavior of the system. Therefore, the combination of negative velocity feedback control and saturation control can achieve better results.

4. Under the combined control, the minimum steady-state amplitudes occur when the natural frequencies of the controllers ($\omega_2, \omega_4$) are equal to half the rotor-spinning speed($\Omega$). So we can select the natural frequencies of the controllers based on the value of the rotor-spinning speed, so that the amplitude of the system can always be reduced to a minimum.

5. Under the combined control, as the excitations force increases and the
controller linear damping coefficient ($\mu_2, \mu_4$) decreases, the steady-state amplitudes of the main system saturate to a very small value.

6. Increasing the feedback signal gain ($\gamma_2, \gamma_4$) can broaden the effective frequency bandwidth of the controller but increase the overload risk of the controller. The control signal gain ($\gamma_1, \gamma_3$) has no effect on the amplitude of the main system, but it can suppress the vibration of the controller. So we can use control signal gain as an important parameter to prevent the controller overload.

8. Comparison with previously published work

In recent years, various types of active controllers [11, 21, 22, 29, 30] were designed to suppress the nonlinear vibrations of Jeffcott-rotor system. In [11], two PPF controllers are integrated to mitigate the horizontal and vertical vibrations of a horizontally supported Jeffcott-rotor system. It concluded that PPF controllers could reduce the amplitudes of the considered system to almost zero. However, in the region outside the resonance region, there would be double peaks with large amplitude on the frequency response curves. In [21], a nonlinear PD-controller was proposed to suppress the nonlinear vibrations of a horizontally supported Jeffcott-rotor system. They concluded that the PD controller could effectively eliminate the nonlinear phenomena of the Jeffcott-rotor system even in extremely large eccentricity, and a positive position feedback and a negative velocity feedback controller were considered to be optimal method. In [22], two modified saturation-based controllers are coupled to reduce the nonlinear vibration of a vertically supported Jeffcott-rotor system via the combination of quadratic nonlinear velocity coupling term and quadratic nonlinear displacement coupling term. Time delay in the control loop were considered. It illustrated that the modified saturation-based controller could effectively suppress the whirling motions. In [29], a nonlinear time-delayed position-velocity feedback controller was utilized to control the lateral vibrations of a horizontally suspended Jeffcott-rotor system. This paper showed that under the positive position feedback control, the appropriate time delay could
improve the damping effect, but under the negative position feedback control, it had the opposite effect. In [30], the same authors proposed a time-delayed states feedback controller to suppress the nonlinear vibrations of a horizontally supported Jeffcott-rotor system. In addition, the approximate solutions of the considered system in the above works were obtained by utilizing the multiple time scales perturbation technique and the stability of the steady-state solution is ascertained by examining the eigenvalues of Jacobian matrix.

Within this novel work, we integrate a modified saturation-based controller via a quadratic velocity coupling term and add a negative velocity feedback to each oscillation mode of a horizontally supported Jeffcott-rotor system, which form a four-degree-of-freedom vibration system, to suppress the horizontal and vertical vibrations of the main systems. In this paper, it is the first time to utilize the integral equation method to construct the second order approximate solutions and amplitude equations for a four-degree-of-freedom vibration system. It illustrates that the modified saturation-based controller can effectively reduce the vibration of the system to almost zero, and the negative velocity feedback could effectively suppress transient vibration and avoid large amplitude vibration of the main system.

Acknowledgements This work was supported by the National Natural Science Foundation of China under the Grant Nos. 11702250, 52071298 and 11972327.

Declarations
Conflict of Interest The authors declare that they have no conflict of interest.

Appendix

The integral equation method
Vibration of various kinds can be modelled by a differential equation
\[ \ddot{x}_i + \lambda_i x_i = \phi_i(x_i, \dot{x}_i, x_j, \dot{x}_j; \varepsilon_p_i, t) = \phi_i[t], \quad i, j = 1, 2, \ldots \] (A.1)
Theorem 1. Every periodic solution of the differential equation (A.1) is a solution of the integro-differential equation (see [20]-[24])

\[
x_i(t) = \int_0^{2\pi} G_i[t, \sigma] \phi_i[\sigma] \, d\sigma + \delta_{\lambda_i}^{n_i^2} (r_i \cos n_i t + s_i \sin n_i t),
\]

(A.2)

where \(G_i[t, \sigma] = \frac{1}{\pi} \left( \frac{1}{\lambda_i} + \sum_{j=1}^{\infty} \cos \frac{j(t-\sigma)}{\lambda_i} \right)\) is the corresponding Generalized Green’s function. And if \(\lambda_i = n_i^2\) (\(n_i\) being an integer for the resonance case), the parameters \(r_i, s_i\) can be determined by the periodicity equations

\[
\begin{align*}
r_i &= \frac{1}{\pi} \int_0^{2\pi} x_i(t) \cos n_i t \, dt, \\
s_i &= \frac{1}{\pi} \int_0^{2\pi} x_i(t) \sin n_i t \, dt,
\end{align*}
\]

(A.3)

which are equivalent to

\[
\int_0^{2\pi} \phi_i[t] \cos n_i t \, dt = \int_0^{2\pi} \phi_i[t] \sin n_i t \, dt = 0.
\]

(A.4)

The solutions of (A.2) can be found by successive approximations of \(x_{ik}(t)\), \(k = 1, 2, 3, \ldots\), which are given by

\[
x_{ik}(t) = \int_0^{2\pi} G_i[t, \sigma] \phi_{i,k-1}[\sigma] \, d\sigma + \delta_{\lambda_i}^{n_i^2} (r_i \cos n_i t + s_i \sin n_i t),
\]

(A.5)

where

\[
\begin{align*}
\phi_{i0}[t] &= \phi_i[0, 0, 0; \varepsilon, t], \\
\phi_{ik}[t] &= \phi_i[x_{ik}(t), \dot{x}_{ik}(t), x_{jk}(t), \dot{x}_{jk}(t); \varepsilon, t] \\
i, j, k &= 1, 2, 3, \ldots
\end{align*}
\]

(A.6)

References

[1] Nayfeh, A.H., Mook, D.T: Non-linear Oscillations. Wiley, New York (1979)

[2] Nayfeh, A.H: Perturbation Techniques. Wiley, New York (1979)

[3] Attilio Maccari: The response of a parametrically excited van der pol oscillator to a time delay state feedback. Nonlinear Dyn. 26, 105-119 (2001)
[4] Attilio Maccari: Vibration control for an externally excited nonlinear system. Phys. Scr. 70, 79-85 (2004)

[5] Attilio Maccari: Delayed feedback control for a parametrically excited van der pol oscillator. Phys. Scr. 76, 526-532 (2007)

[6] Attilio Maccari: Vibration amplitude control for a van der Pol-Duffing oscillator with time delay. J. Sound Vib. 317, 20-29 (2008)

[7] El-Bassiouny A.F: Nonlinear analysis for a ship with a general roll-damping model. Phys. Scr. 75, 691-701 (2007)

[8] H. L. Li, Y. S. Chen, L. Hou, Z. Y. Zhang: Periodic response analysis of a misaligned rotor system by harmonic balance method with alternating frequency/time domain technique. Sci. China Technol. Sc. 59, 1717-1729 (2016)

[9] M. Eissa, Kandil A: Vibration suppression of a nonlinear magnetic levitation system via time delayed nonlinear saturation controller. Int. J. Nonlin. Mech. 72, 23-41 (2015)

[10] N. A. Saeed, M. Kamel: Active magnetic bearing-based tuned controller to suppress lateral vibrations of a nonlinear Jeffcott rotor system. Nonlinear Dyn. 90, 457-478(2017)

[11] M. Eissa, N.A. Saeed: Nonlinear vibration control of a horizontally supported Jeffcott-rotor system. J. Vib. Control. 87, 109-124 (2018)

[12] N.A. Saeed, Ali Kandil: Lateral vibration control and stabilization of the quasiperiodic oscillations for rotor-active magnetic bearings system. Nonlinear Dyn. 98, 1191-1218 (2019)

[13] Jerzy Warminski: Active suppression of nonlinear composite beam vibrations by selected control algorithms. Commun. Nonlinear Sci. Numer. Simul. 16, 2237-2248 (2011)
[14] M. Eissa, N.A. Saeed, W.A. El-Ganaini: Saturation-based active controller for vibration suppression of a four-degree-of-freedom rotor-AMB system. Nonlinear Dyn. 76, 743-764 (2014)

[15] J. Li, X.B. Li, H. X. Hua: Active nonlinear saturation-based control for suppressing the free vibration of a self-excited plant. Commun. Nonlinear Sci. Numer. Simul. 15, 1071-1079 (2010)

[16] M. Kamel, Kandil A, W.A. El-Ganaini, M. Eissa: Active vibration control of a nonlinear magnetic levitation system via Nonlinear Saturation Controller (NSC). Nonlinear Dyn. 77, 605-619 (2014)

[17] Y. S. Hamed, Amer Y.A: Nonlinear saturation controller for vibration supersession of a nonlinear composite beam. J. Mech. Sci. Technol. 28, 2987-3002 (2014)

[18] M. Sayed, Mousa A.A, Alzaharani D.Y: Non-linear time delay saturation controller for reduction of a non-linear vibrating system via 1:4 internal resonance. J. Vibroeng. 18, 2515-2536 (2016)

[19] Tusset A.M, Janzen F.C, Piccirillo V, Rocha R.T, Balthazar J.M, Latka G: On nonlinear dynamics of a parametrically excited pendulum using both active control and passive rotational (MR) damper. J. Vib. Control. 24, 1587-1599 (2018)

[20] Ali Kandil, Hany A El-Gohary: Suppressing the nonlinear vibrations of a compressor blade via a nonlinear saturation controller. J. Vib. Control. 24, 1488-1504 (2018)

[21] N.A. Saeed, M. Kamel: Nonlinear PD-controller to suppress the nonlinear oscillations of horizontally supported Jeffcott-rotor system. Int. J. Non-Linear Mech. 87, 109-124 (2016)

[22] N.A. Saeed, H.A. El-Gohary: Influences of time-delays on the performance of a controller based on the saturation phenomenon. Eur. J. Mech. A-Solid. 66, 125-142 (2017)
[23] G. Schmidt, A. Tondl: Nonlinear vibrations. Cambridge University Press, Cambridge, UK, 1986.

[24] J. Xu, Y. L. Chen, K. W. Chung: An improved time-delay saturation controller for suppression of nonlinear beam vibration. Nonlinear Dyn. 82, 1691-1707 (2015)

[25] Y. L. Chen, J. Xu: Applications of the integral equation method to delay differential equations. Nonlinear Dyn. 73, 2241-2260 (2013)

[26] Y. L. Chen, K. W. Chung, J. Xu: Analysis of vibration suppression of master structure in nonlinear system using nonlinear delayed absorber. Int. J. Dyn. Control. 2, 55-67 (2014)

[27] Hiroshi Yabuno, Takuma Kashimura, Tsuyoshi Inoue, Yukio Ishida.: Nonlinear normal modes and primary resonance of horizontally supported Jeffcott rotor. Nonlinear Dyn. 66, 377-387 (2011)

[28] N.A. Saeed, H.A.EI-Gohary: On the nonlinear oscillations of a horizontally supported Jeffcott rotor with a nonlinear restoring force. Nonlinear Dyn. 88, 293-314 (2017)

[29] N.A. Saeed, W.A.El-Ganaini: Time-delayed control to suppress the nonlinear vibrations of a horizontally suspended Jeffcott-rotor system. Appl. Math. Model. 44, 523-539 (2017)

[30] N.A. Saeed, W.A.El-Ganaini: Utilizing time-delays to quench the nonlinear vibrations of a two-degree-of-freedom system. Meccanica. 52, 2969-2990 (2017)
Figures

Figure 1

Schematic diagram of the whole controlled system

Figure 2
Frequency-response curves of two main systems in three states: before control (blue, \( \lambda_1 = \lambda_2 = 0 \)), controlled only by the negative velocity feedback control (red, \( \lambda_1 = \lambda_2 = 0.1 \)) and the combined control (green, \( \lambda_1 = \lambda_2 = 0.1 \)).

**Figure 3**

Frequency-response curves under the combined control. (a): the horizontal oscillation mode, (b): the vertical oscillation mode

**Figure 4**

The frequency-response curves at different values of the rotor eccentricity \( f \) before control.
Figure 5

Frequency $\sigma_1$-response curves at different values of the rotor-eccentricity $f$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode
Figure 6

Varying $\sigma_1$ on force-response curves in three different control states. Blue: before control, red: controlled only by the negative velocity feedback, black: under the combined control.
Figure 7

Frequency-response curve at different values of $\lambda_1, \lambda_2$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode
Figure 8

Frequency-response curves at different values of $\sigma_3$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode.
Figure 9

Frequency-response curves at different values of $\sigma_4$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode
Figure 10

Frequency-response curves at different values of $\gamma_2$, $\gamma_4$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode
Figure 11

Frequency-response curves at different values of $\gamma_1$, $\gamma_3$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode
Figure 12

Frequency-response curves at different values of $\mu_2$, $\mu_4$ under the combined control: (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode.
Comparison of frequency-response curves between numerical solutions and approximate solutions when $f = 0.015$, $\sigma_3 = 0$, $\sigma_4 = 0$ before control. The red dots for numeric solutions. (a) horizontal oscillations; (b) vertical oscillations.

**Figure 14**

Time histories for the main systems of two oscillation modes when $\sigma_1 = 0.018$, (a),(c): $u(0) = 0.1$, $v(0) = 0.1$, (b),(d): $u(0) = 0.8$, $v(0) = 0.8$. 
Figure 15

Comparison of frequency-response curves between numerical solutions and approximate solutions when $\sigma_3 = 0$, $\sigma_4 = -0.05$. (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode.
Figure 16

Comparison of frequency-response curves between numerical solutions and approximate solutions when \( \sigma_1 = 0, \sigma_4 = 0 \). (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode.
Comparison of frequency-response curves between numerical solutions and approximate solutions when $\sigma_1 = 0, \sigma_3 = 0$. (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode.
Figure 18

Comparison of excitation force-response curves between numerical solutions and approximate solutions when $\sigma_1 = 0$, $\sigma_3 = 0$, $\sigma_4 = 0$. (a), (b) the horizontal oscillation mode; (c), (d) the vertical oscillation mode.

Figure 19
Time histories before control when $\sigma_1 = 0, \sigma_3 = 0, \sigma_4 = -0.05$. (a) horizontal oscillations; (b) vertical oscillations

Figure 20

Time histories of the whole system under the combined control for $\sigma_1 = 0, \sigma_3 = 0, \sigma_4 = -0.05, \gamma_2 = \gamma_4 = 5, \gamma_1 = \gamma_3 = 0.3, \lambda_1 = 0.01, \lambda_2 = 0.01$. 
Figure 21

Time histories of the whole system under the combined control for $\sigma_1 = 0$, $\sigma_3 = 0$, $\sigma_4 = -0.05$, $\gamma_2 = \gamma_4 = 5$, $\gamma_1 = \gamma_3 = 0.3$, $\lambda_1 = 0.05$, $\lambda_2 = 0.05$. 