THREE MANIFOLDS THAT ADMIT INFINITELY MANY ANOSOV FLOWS

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Abstract. We construct an example of a graph manifold that supports infinitely many Anosov flows that are not orbit equivalent. Our construction is reminiscent of the Thurston-Handel construction, consisting of cutting open a geodesic flow on a surface of constant negative curvature, modifying the flow in each piece by taking finite covers, and gluing back along the boundary tori to get a flow on a new manifold. We study the boundary behaviour to ensure that we can perform the above modification in infinitely many different ways, always arriving at the same manifold when gluing, and we prove the flows are all topologically distinct.

1. Introduction

An Anosov flow, also called a hyperbolic flow, is a flow for which some directions are expanded and others are contracted. Such flows are fundamental examples for (idealized) chaotic dynamical systems. The study of Anosov flows in dimension three is well connected to the study of the topology of the three-manifold carrying such flows. For instance, a three-manifold carrying an Anosov flow is always irreducible [16], has a fundamental group of exponential growth [18], and carries a tight contact structure [15].

In some cases, one can classify the Anosov flows supported by a three-manifold if the manifold has a simple geometry. For instance [9] and [2] proved that a flow on a Seifert fibered three-manifold is equivalent to a geodesic flow up to finite covers, while [17] proved that an Anosov flow on a Solv manifold is a suspension of an Anosov diffeomorphism of the torus.

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Although any three-manifold carrying an Anosov flow is irreducible, it may be toroidal, i.e. it can contain essential embedded tori. When this is the case, a fundamental tool of three-dimensional topology allows one to cut the manifold along a collection of the essential tori, obtaining pieces with boundary that have better understood topological and geometric properties.

Therefore, it makes sense to cut the manifold along essential tori and analyse the resulting flows on the geometric pieces. This is the subject of a number of seminal papers by Barbot and Fenley. In particular, they prove in [4] that every essential torus is homotopic to one which is either transverse to the flow, or is quasi-transverse: It is transverse except along a finite number of periodic orbits. The manifold can then be cut along these transverse or quasi-transverse essential tori, from which they have developed a variety of classification results for the Anosov flows on the resulting pieces.

Essential tori have also been used in the converse direction, namely to glue pieces of Anosov flows together to produce new Anosov flows with surprising qualities. See for instance the examples of Fried-Williams, who construct a non-transitive Anosov flow, and Handel-Thurston who construct an Anosov flows which is transitive, but is neither a geodesic flow nor a suspension.

A recurrent problem in the field has been to determine the number of Anosov flows that can be supported by a single manifold. This appears as Problem 3.53 of Kirby’s problem list, where he asks: Given an integer $N$, does there exist a hyperbolic 3-manifold with at least $N$ Anosov flows which are topologically inequivalent?

The first explicit examples to appear in the literature were constructed by [2]. In his work, Barbot constructs a family of graph manifolds that each support two distinct Anosov flows. The surgery techniques of Goodman [10] may also produce two distinct Anosov flows on a manifold if the periodic orbit used for the Dehn surgery admits two distinct purely cosmetic surgery slopes. More recently, Bonatti, Beguin and Yu [7] proved a general theorem allowing them to glue many pieces with transverse toral boundaries. They use this new technique to construct, for each $n \in \mathbb{N}_{>0}$ a 3-manifold $M_n$ supporting at least $n$ distinct Anosov flows. A similar result for hyperbolic manifolds was recently obtained by Bowden and Mann using
an analysis of the rigidity properties of certain fundamental group actions \[8\]. This prompts the obvious question, appearing in \[6\] and \[8, \text{Question 7.4} \]:

**Question 1.1.** *Does there exist a 3-manifold \(M\) supporting infinitely many non orbit equivalent Anosov flows?*

The main result of this work is an affirmative answer to this question. We construct, via modifications to a geodesic flow, graph manifolds which support infinitely many Anosov flows, no two of which are orbit equivalent. This is surprising for two reasons. First, in all of the examples listed above the manifolds become increasingly complicated in order to support more flows. Second, our manifolds are graph manifolds—and it was generally believed to be impossible for such a manifold to support infinitely many Anosov flows. With a finite number of Seifert fiber pieces, and with the classification results of Barbot-Fenley greatly restricting the flows on each piece, it seemed unlikely that infinitely many different lifts to finite covers (and their blow ups) would ever be consistent with fixed gluing maps between pieces.

Nevertheless, we use the simplest possible graph manifold, having only two pieces. The simplicity gives us a much more precise picture of the behaviour of the possible flows when restricted to the boundary tori, allowing us to find infinitely many pairs of flows compatible with a fixed gluing map.

Our technique is inspired by the examples of Handel-Thurston. They begin by fixing a closed hyperbolic surface \(S\), and equipping the unit tangent bundle \(T^1S\) with the geodesic flow \(\psi\). Next, they choose a simple closed geodesic \(g(t)\) in \(S\), and cut \(S\) along \(g(t)\) to produce two pieces \(P_1\) and \(P_2\), and equip each of \(N_1 = T^1P\) and \(N_2 = T^1P_2\) with \(\psi|N_i\). Last, they build a gluing map \(F : \partial T^1N_1 \to \partial T^1N_2\) which is distinct from the identity, but which behaves very well with respect to the flow on each piece, so that the pieces can be assembled into a new manifold \(N_1 \cup_F N_2\) equipped with an Anosov flow built from the flows \(\psi|N_i\). For any of these gluings, the resulting flow is not automatically Anosov. The difficulty is that an Anosov flow carries two invariant foliations, one uniformly (exponentially) attracting and one uniformly repelling. The gluing usually cannot be made to preserve both these foliations, the smoothness of the flow, and the uniformity of the attraction. Therefore, one must
typically appeal to cone-field arguments to prove that the resulting flow is indeed Anosov.

Our method of proof is as follows: We consider the geodesic flow on the modular surface, which is a flow on the trefoil complement. We then observe that the trefoil complement admits an \( n \)-sheeted self-covering \( p_n : M \to M \) for certain values of \( n \). Fixing a carefully chosen gluing map \( F \), and closely following the Handel-Thurston analysis of the invariant directions for the geodesic flow, we are able to find infinitely many pairs of integers \( (n, m) \) with corresponding coverings such that the pullback of the geodesic flow from \( M \) along \( p_n \) and \( p_m \) are “compatible” with the map \( F \). With such a choice, the gluing arguments of Handel–Thurston show that the lifted geodesic flows may be glued to produce an Anosov flow on \( N = M \cup_F M \). For distinct pairs of integers \( (n, m) \) corresponding to compatible pullbacks, the resulting flows on \( N \) are never orbit equivalent. It follows that:

**Theorem 1.2.** There exists a graph manifold \( M \) supporting infinitely many non orbit equivalent Anosov flows.

Note that it is impossible to obtain a similar result by fixing a geodesic flow on a closed surface and varying the covering maps, as these coverings will have different domains. In fact, in a recent result of Barbot and Fenley [5], they prove that any cover, along the fiber direction, of a unit tangent bundle can carry at most two distinct Anosov flows up to topological equivalence. Thus, it is cardinal here to consider Anosov flows on manifolds with boundary.

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2. **BACKGROUND**

2.1. **The geodesic flow.** Let \( S \) be a hyperbolic surface \( \mathbb{H}^2 / \Gamma \) for a discrete subgroup \( \Gamma \) of \( \text{Isom}^+(\mathbb{H}^2) \). The surface \( S \) inherits a Riemannian metric from \( \mathbb{H}^2 \), which allows one to define the geodesic flow on \( S \), defined on the unit tangent bundle \( T^1(S) \cong \text{PSL}_2(\mathbb{R}) / \Gamma \).
A flow \( \phi \) on a three manifold \( M \) is called Anosov if there is a continuous \( D\phi \)-invariant decomposition of the tangent bundle \( TM = E^s \oplus E^u \oplus E^t \) and constants \( A > 0, \lambda > 1 \), so that:

- \( E^t \) is tangent to \( \phi \) at any point \( x \in M \),
- for any \( x \in M \) and any \( v \in E^u_x \), \( ||D\phi^t(v)_{\phi^t(x)}|| \geq A\lambda^t ||v_x|| \),
- for any \( x \in M \) and any \( v \in E^s_x \), \( ||D\phi^t(v)_{\phi^t(x)}|| \leq A\lambda^{-t} ||v_x|| \).

In this decomposition, \( E^u \) is called the strong unstable direction, and \( E^s \) the strong stable direction.

Anosov [1] proved that the geodesic flow on \( H^2 \) is Anosov, and that this property descends to any hyperbolic surface \( S \). We can describe the geodesic flow, together with its stable and unstable directions, as follows:

Each point in \( T^1 H^2 \) consists of a point \( x \in H^2 \) and a unit vector \( v \in T_x H^2 \). There exists a unique geodesic \( g \) in \( H^2 \) passing through \( x \) and tangent to \( v \). Denote the emanating point of \( g \) on \( S^1 = \partial H^2 \) by \( v^- \), and its terminating point by \( v^+ \). We denote the point \( x \) with direction \( v \) by \( (x, v) \). The strong stable direction at \( (x, v) \) is the direction of the horocycle passing through \( x \) based at \( v^+ \) with a direction perpendicular to the horocycle itself at any point determined by the direction of \( v \). The strong unstable direction is likewise given by the horocycle based at \( v^- \) with a similarly determined perpendicular direction, see Figure [1].

The geodesic flow moves \( (x, v) \) forward along \( g \) by a time \( t \), while mapping the stable horocycle to another stable horocycle based at \( v^+ \) (exponentially contracting in \( t \) the distance along the horocycle), and the unstable horocycle to an unstable horocycle based at \( v^- \).

In order to perform the gluings in Section 4, we need to be able to identify the directions of the Anosov decomposition \( TM = E^s \oplus E^u \oplus E^t \) for \( M = T^1 H^2 \), and we later take the quotient to find them in \( T^1 S \) for a particular choice of surface \( S \). To this end, we need to consider the action of the flow on \( T(T^1 H^2) \). Thus, we use the following coordinate system introduced in [11].

Fix a point in \( H^2 \) and \( v \in T^1_x H^2 \). Following [11], we identify \( T_{(x,v)}(T^1 H^2) \) with \( T_x H^2 \times \mathbb{R} \) as follows: At any point \( y \in H^2 \) we define the angle \( \angle(w,u) \) between any two vectors \( w, u \in T^1_y H^2 \) to be the angle measured counterclockwise from \( w \) to \( u \). Given a vector \( w \in T_x H^2 \) and a real number \( \rho \in \mathbb{R} \), define \( c_{w,\rho}(t) = \)
Figure 1. A point \((x, v) \in T^1 \mathbb{H}^2\) together with its geodesic and the stable and unstable manifolds through it.

\[
\exp(tw), u(\alpha + t \rho) \quad \text{where} \quad \alpha = \langle w, v \rangle \quad \text{and} \quad u(\alpha + t \rho) \in T^1_{\exp(tw)} \mathbb{H}^2 \quad \text{is the vector satisfying} \quad \langle T \exp(tw), u(\alpha + t \rho) \rangle = \alpha + t \rho \quad \text{(see Figure 2)}.
\]

The path \(c(t) := c_{w, \rho}(t)\) in \(T^1 \mathbb{H}^2\) satisfies \(c(0) = (x, v)\), and we identify the point \((w, \rho) \in T_x \mathbb{H}^2 \times \mathbb{R}\) with \(\frac{dc}{dt} \bigg|_{t=0}\) in \(T(x,v)(T^1 \mathbb{H}^2)\). We call these coordinates Handel-Thurston notation.

Figure 2. The path \(c(t)\) in \(T^1 \mathbb{H}^2\).
For any point \((y, w) \in T^1 \mathbb{H}^2\), define the vector \(w^\perp\) to be the vector in \(T^1_y \mathbb{H}^2\) satisfying \(\langle w^\perp, w \rangle = \pi/2\). Consider again Figure 1 showing the Anosov directions. The strong stable and unstable manifolds are both tangent to \(v^\perp\) (that is pointing downwards from \(x\) in the figure). For the unstable manifold that is the horocycle on the left, the direction of the vector of a point in it is rotating clockwise, while for the stable manifold the rotation is counterclockwise. Thus in the Handel-Thurston notation:

- \(E^s_{(x,v)}\) is generated by \((v, 0)\),
- \(E^u_{(x,v)}\) is generated by \((v^\perp, -1)\),
- \(E^u_{(x,v)}\) is generated by \((v^\perp, 1)\).

In particular both directions of the strong unstable direction at each point are contained in the \((v^\perp, \rho)\) plane, in the second and fourth quadrants.

**Remark 2.1.** Fixing a geodesic \(g\) in \(\mathbb{H}^2\), we can consider the unit tangent bundle to \(g\). At any point \((x, v) \in T^1 g\), parallel transporting along \(g\) preserves the angle \(\alpha = \angle(Tg, v)\). One may identify \(T^1 g\) with \(\mathbb{R} \times \mathbb{R} / 2\pi\) by \(t, \theta \mapsto (g(t), v)\) where \(v\) is the vector in \(T^1_{g(t)} \mathbb{H}^2\) satisfying \(\angle(Tg, v) = \theta\). In these coordinates the angles of the Anosov directions above do not depend on the point \(t\), but only on the angle \(\alpha\).

### 2.2. The modular surface.

Let \(\Gamma\) denote the group \(\text{PSL}_2(\mathbb{Z})\), generated by

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}.
\]

In the standard representation of \(\text{Isom}^+(\mathbb{H}^2)\) as \(\text{PSL}_2(\mathbb{R})\) acting by Möbius transformations, the generators of \(\Gamma \subset \text{PSL}_2(\mathbb{R})\) act as rotation by \(\pi\) about the point \(p = i\), and a rotation by \(2\pi/3\) about the point \(q = \frac{1}{2} + \frac{\sqrt{3}}{2}i\). The modular surface \(S_{\text{Mod}}\) is the hyperbolic manifold \(S_{\text{Mod}} = \mathbb{H}^2 / \Gamma\), a surface with one cusp at infinity and two cone points, one of order 2 and one of order 3. This is easy to check, e.g. using that \(\Gamma\) is the symmetry group of the Farey tessellation of \(\mathbb{H}^2 / \Gamma\).

Increasing the distance between the centers of rotations \(p\) and \(q\) in \(\mathbb{H}^2\) as in [9], turns the cusp into a funnel. There is then a unique closed geodesic around the funnel, that we denote by \(g\). The greater the distance between \(p\) and \(q\), the longer the geodesic \(g\). We cut the surface along \(g\) to obtain a compactification of \(S_{\text{Mod}}\) which is a surface with a boundary. The resulting surface \(S_{\text{Mod}}\) is depicted in Figure 3.
The unit tangent bundle $T^1\overline{S}_{Mod}$ is the bundle of all $(x,v)$ where $x \in \overline{S}_{Mod}$ and $v \in T_x\overline{S}_{Mod}$ satisfies $||v|| = 1$. Consider the boundary torus $\partial T^1\overline{S}_{Mod} = T^1g$. We may trivialize the fiber direction along this torus using the tangent to the geodesic $g$ at each point as a section, as in Remark 2.1. If $g$ has length $L$, this results in the coordinates $[0,L] \times [0,2\pi]$ on the boundary torus, $(s, \theta) \mapsto (g(s), v)$, where $g$ is parametrized by arc-length, and $v \in T^1_{g(s)}\overline{S}_{Mod}$ satisfies $(Tg, v) = \theta$. We call these the orbit-fiber coordinates. As the geodesic flow is structurally stable, it follows that it is independent of the length $L$ of the boundary geodesic. Hence, we may choose the length to be equal to $2\pi$ and $0 \leq s \leq 2\pi$.

It is uncommon to define Anosov flows on a manifolds with boundary, and to avoid this technicality we can alternatively introduce our objects of study as pieces of an Anosov flow on a closed 3-manifold cut along essential tori. Consider the surface which is a sphere with four cone points, two of order 2 and two of order 3, denoted by $S_{2,2,3,3}$. It is the double of the modular surface $\overline{S}_{Mod}$ and is also a hyperbolic surface. As in Section 2.1, we have the geodesic flow over $S_{2,2,3,3}$, which is an Anosov flow defined on $T^1S_{2,2,3,3}$. Cutting the three manifold $T^1S_{2,2,3,3}$ along the two dimensional torus $T^1g$ as in Figure 4 one obtains two copies of $T^1\overline{S}_{Mod}$, and the geodesic flow on $\overline{S}_{Mod}$ is thus a piece of an Anosov flow on a closed manifold as in [3].

Denote one half of $S_{2,2,3,3}$ by $\overline{S}_{Mod1}$, and the other by $\overline{S}_{Mod2}$, each taken with the hyperbolic metric induced from $S_{2,2,3,3}$. The surfaces $\overline{S}_{Mod1}$ and $\overline{S}_{Mod2}$ are homeomorphic and also isometric, as the boundary geodesic $g$ is of the same length in each of them. The hyperbolic metric defines the geodesic flow on each of them, denote by $\psi^1_1$
the geodesic flow on $\mathcal{S}_{\text{Mod}1}$ and by $\psi_1^2$ the geodesic flow on $\mathcal{S}_{\text{Mod}2}$. Although they are conjugate, we want to fix these specific representatives. The gluing of $\mathcal{S}_{\text{Mod}1}$ to $\mathcal{S}_{\text{Mod}2}$ induced by the embedding into $S_{2,2,3,3}$ induces a gluing of the unit tangent bundle $T^1\mathcal{S}_{\text{Mod}1}$ to $T^1\mathcal{S}_{\text{Mod}2}$, which is the identity on the boundary $\partial\mathcal{S}_{\text{Mod}1} = T^1g$. i.e., a point and direction on the boundary is glued to the same point in the boundary of the other surface with the same direction. Naturally this gluing glues the flowlines to each other smoothly, so the resulting flow is the geodesic flow on $S_{2,2,3,3}$.

We may use the coordinate system defined in Section 2.1 for $T(T^1\mathcal{S}_{\text{Mod}})$ to identify the Anosov directions over the boundary torus $\partial T^1\mathcal{S}_{\text{Mod}}$. The result is depicted in Figure 5. To obtain the figure, note that in our coordinate system $(s, \theta)$ corresponds to the point $(g(s), v)$ where $v$ is the rotation $R_\theta$ by $\theta$ of $Tg$ along the fiber at the point $g(s)$. In particular, The flow direction at $(g(s), v)$ is generated by $(v, 0) = (R_\theta(Tg), 0)$, the strong unstable direction $E^u_{(g(s), v)}$ is generated by $(v^\perp, -1)$, which is the direction in the plane perpendicular to $v$ with slope $-1$, and the unstable direction $E^s_{(x, v)}$ is generated by the vector in the same plane and slope $+1$, $(v^\perp, 1)$. In particular, as in Remark 2.1, the Anosov directions depend solely on $\theta$ and not on $s$. They are simply the rotation of the directions along $g$, all by the same angle $\theta$ in a horizontal (tangent to the surface) direction.

Note that these directions are the same for any hyperbolic surface $\mathbb{H}^2/\Gamma$, and $g$ a closed geodesic in $S$. In particular, the above torus can be seen as the boundary $T^1g = \partial T^1\mathcal{S}_{\text{Mod}1} = \partial T^1\mathcal{S}_{\text{Mod}2}$ and with $T^1\mathcal{S}_{\text{Mod}1}$ on the front of the figure, and $T^1\mathcal{S}_{\text{Mod}2}$ on the back. The gluing via the identity taking the orbit to orbit (or equivalently meridian to meridian) and fiber to fiber, takes the geodesic $g$ to itself.
the geodesic $-g$ to itself, and matches the Birkhoff annulus with the flow directed outwards from $T^1 \mathbb{S}_{\text{Mod}}^1$ with that directed inwards into $T^1 \mathbb{S}_{\text{Mod}}^2$.

2.3. The trefoil complement. In this section we will show how to identify $T^1 \mathbb{S}_{\text{Mod}}$ with the trefoil complement, and will identify the closed orbit on the boundary of $T^1 \mathbb{S}_{\text{Mod}}$ with a curve on the boundary of the trefoil complement expressed in meridan/longitude coordinates.

The trefoil complement and its Seifert fibration can be described as follows. There is an action of $S^1$ on $S^3$ given by $\lambda \cdot (z_1, z_2) = (z_1 \lambda^2, z_2 \lambda^3)$ for all $\lambda \in S^1$. Each orbit of this action is a trefoil knot lying on one of the two dimensional tori $\{ (z_1, z_2) \mid |z_1|^2 = r \}$ for $0 < r < 1$, except for the two orbits $S^1 \times \{ 0 \} = \{ (z_1, z_2) \mid |z_1| = 1, z_2 = 0 \}$ and $\{ 0 \} \times S^1 = \{ (z_1, z_2) \mid z_1 = 0, |z_2| = 1 \}$. Fixing a choice of orbit $T_{2,3} \subset T = \{ (z_1, z_2) \mid |z_1|^2 = |z_2|^2 \}$, we may remove an open neighbourhood of $T_{2,3}$ from $S^3$ consisting of a union of fibers (orbits) to get $M = S^3 \setminus \mathcal{N}(T_{2,3})$, a compact manifold.
The complement of the trefoil knot.

with boundary, as in Figure 6. The decomposition of $M$ into orbits under the $S^1$ action gives $M$ the structure of a Seifert fiber space, where the orbit $S^1 \times \{0\}$ admits a fibered torus neighbourhood with invariants $(2, 1)$ and $\{0\} \times S^1$ admits a fibered torus neighbourhood with invariants $(3, 1)$. The orbit surface for this fibration, i.e. the space obtained by collapsing each fiber to a point, is a copy of $S^2$ minus a disk, with two cone points of orders 2 and 3, i.e. it is the surface $\mathcal{S}_{\text{Mod}}$.

The trefoil complement also has a fibration over $S^1$: Consider the punctured torus $F$ composed of two disks in the solid torus $\{(z_1, z_2) \mid |z_1|^2 \leq |z_2|^2\}$ connected by three half twisted bands in the solid torus $\{(z_1, z_2) \mid |z_1|^2 \geq |z_2|^2\}$. (See [19] for more details, including an explicit parameterization of this surface). On the boundary torus $\partial M$, we can choose a basis composed of the meridian $\mu$ of the torus $\mathcal{N}(T_{2,3})$, and a regular fiber $h$ of the Seifert fibration (see Figure 6). The longitude $\lambda$ is the boundary of the Seifert surface $\partial F \subset \partial M$.

The Seifert fibers intersect the surface $F$ transversely, with each regular fiber intersecting it six times. We assume that these intersections happen at regular intervals along the fiber (that is, if $\lambda, \lambda' \in S^1$ are such that $\lambda \cdot (z_1, z_2) = \lambda' \cdot (z'_1, z'_2)$ for $(z_1, z_2), (z'_1, z'_2) \in F$, then $\lambda^{-1} \lambda' = e^{2k\pi i/6}$ for some $k \in \{0, \ldots, 5\}$). As such,
every point in $M$ can be written uniquely as $\lambda \cdot (z_1, z_2)$ for $(z_2, z_2) \in F$ and $\lambda \in S^1$ with $0 \leq \arg(\lambda) < \pi/3$. The singular fibers intersect the surface 2 and 3 times respectively. Sliding $F$ along the fibers rotates $F$ through $S^3$ until it returns to its initial position as a set—in other words, $e^{2\pi i/6} \cdot F = F$. Note this will shift each intersection point of a regular Seifert fiber with $F$ to the next intersection point along the fiber, permuting cyclically the two disks on the outside, and permuting the three bands on the inside. This yields a map $\varphi : F \to F$ given by $\varphi(x) = e^{2\pi i/6} \cdot x$ for all $x \in F$.

As each Seifert fiber has 2, 3 or 6 intersection points with the punctured torus, $\varphi^6 = \text{id}$. Moreover, $\varphi$ serves as the generator of the group of deck transformations for the natural branched covering map $p : F \to \mathbb{S}_{\text{Mod}}$ as in Figure 7.

\[ \pi \quad 2\pi/3 \]

**Figure 7.** The cover $p : F \to \mathbb{S}_{\text{Mod}}$, here $F$ is identified with the surface $T^2 \setminus D^2$ for ease of illustration.

The unit tangent bundle to $T^1 \mathbb{S}_{\text{Mod}}$ is composed of two unit tangent bundles, each one a unit tangent bundle to a closed neighbourhood of a cone point, glued along the unit tangent bundle to a segment. It is thus homeomorphic to the (compactified) trefoil complement $M$ [14].

**Proposition 2.2.** The identification of $T^1 \mathbb{S}_{\text{Mod}}$ with the trefoil complement $M$ carries the closed orbit of the geodesic flow on $\partial T^1 \mathbb{S}_{\text{Mod}}$ to the curve $\mu \subset \partial M$.

**Proof.** The covering map $p$ induces a cover $P : T^1 F \to M \cong T^1(\mathbb{S}_{\text{Mod}})$ and $\varphi$ induces $\Phi : T^1 F \to T^1 F$. As the unit tangent bundle $T^1 F$ is trivial, it has a section, say $s$. Set $\lambda = s(\partial F)$, then we can assume $\lambda$ appears as in Figure 8, where the black arrows indicate the direction of the nonsingular vector field defined by $s$. The images $\Phi^k \circ s$
are disjoint for \( k \in \{0, \ldots, 5\} \). Thus, the image under \( P \) of any such section is a copy of \( F \) embedded in the trefoil complement \( M \) with \( \partial F \subset \partial M \). Hence, we may identify the image of \( P \circ s \) with the Seifert surface for the trefoil, and in particular, choosing the orientation for \( \lambda \) accordingly, \( P(\bar{\lambda}) = \lambda \).

We can also identify a curve \( \bar{h} \) in \( \partial(T^1 F) \) that covers a regular fiber \( h \) of the Seifert fibration of \( M \). Fixing \( x \in \partial F \), set \( \bar{h} = \{(x,v) \mid ||v|| = 1\} \), oriented so that counterclockwise rotation of the vector \( v \) is the positive direction along \( \bar{h} \). Choosing an appropriate orientation for \( h \), we get \( P(\bar{h}) = h \).

Next, we define a curve \( \beta \) representing the class \([\bar{h}] + [\bar{\lambda}] \in H^1(\partial T^1 F)\) by first supposing that \( \partial F \) is identified with a curve \( \alpha : [0,L] \to F \), and choosing a point \( t_0 \in [0,L] \) for which the section is a direction tangent to the curve, i.e. \( s(\alpha(t_0)) = (\alpha(t_0), \frac{T\alpha(t_0)}{||T\alpha(t_0)||}) \). Supposing the longitude is travelled counterclockwise as well, as in Figure 8, such a point corresponds to the rightmost point of the curve \( \lambda \) in Figure 8 and reparameterizing if necessary, we assume \( t_0 = 0 \). Our curve \( \beta : [0,L] \to T^1 F \) is \( \alpha \) with its tangent direction, \( \beta(t) = (\alpha(t), \frac{T\alpha(t)}{||T\alpha(t)||}) \). To see that \([\beta] = [\bar{h}] + [\bar{\lambda}] \), note that if \( \tilde{\lambda}(t) = s \circ \alpha(t) = (\alpha(t), v_t) \) for some unit vector \( v_t \in T\alpha(t)F \), then \( \langle v_t, T\alpha(t) \rangle = \frac{2\pi t}{L} \).

Thus the vector \( v_t \) makes one complete rotation relative to \( T\alpha(t) \) as \( t \) ranges over \([0,L]\), so that \( \beta(t) \sim \tilde{\lambda} \circ \bar{h} \).

![Figure 8](image-url)  

**Figure 8.** The curve \( \tilde{\lambda} \) on \( \partial T^1 F \) in red, as a the curve \( \partial F \) with a direction at each point.
Considering Figure 7, we see that the cover \( p : F \to \mathbb{S}_\text{Mod} \) wraps \( \partial F \) six times around the boundary curve \( g \) of \( \mathbb{S}_\text{Mod} \), and thus \( P(\beta) = (g, Tg_{\|Tg\|}) \), where \( \beta \) wraps six times around its image under \( P \). On the other hand, from Figure 6 it follows that the fiber \( h \) has linking number 6 with the trefoil, hence \( [h] = \pm [\lambda] \pm 6[\mu] \). However our choices of orientations for \( \lambda \) and \( h \) yield \( [\beta] = [\bar{\lambda}] + [\bar{h}] \), so \( [P(\beta)] = [\lambda] + [h] = \pm 6[\mu] \).

Therefore, the closed orbit \( (g, Tg_{\|Tg\|}) \subset \partial T^1 \mathbb{S}_\text{Mod} \) is exactly the curve you Dehn fill to get \( S^3 \) (c.f. [9]).

**Remark 2.3.** Suppose one sees the once punctured torus on the left of Figure 7 as a half of a genus two surface. The boundary curve if oriented counterclockwise on one punctured torus, will be oriented clockwise on the other punctured torus, i.e. opposite to Figure 8, in the second copy of the punctured torus.

Taking the cover \( P \) to act on both sides at the same time, this fact is true also for the two copies of \( \mathbb{S}_\text{Mod} \) composing \( S_{2,2,3,3} \) in Figure 4. Thus, in the unit tangent bundle to one of them we have \( [\lambda] + [h] = \pm 6[\mu] \), and in the other \( [\lambda] - [h] = \pm 6[\mu] \).

In particular, the orientation of the (orbit, longitude) pair is different in each of the two copies. So that in one copy the longitude points from the orbit into the same Birkhoff annulus as the fiber, and in the second copy it points into the Birkhoff annulus in the other side of the orbit.

### 3. Lifting

With \( M \) as in the previous section, recall that the \( S^1 \) action which determined the Seifert fibration allowed every point in \( M \) to be written uniquely as \( \lambda \cdot (z_1, z_2) \) for \( (z_2, z_2) \in F \) and \( \lambda \in S^1 \) with \( 0 \leq \text{arg}(\lambda) < \pi/3 \). Then for each \( d = \pm 1 + 6k \) with \( k \in \mathbb{Z} \) we can construct a \( d \)-fold covering map \( p_d : M \to M \) as \( p_d(\lambda \cdot (z_1, z_2)) = (z_1 \lambda^{2d}, z_2 \lambda^{3d}) \) if \( d = 1 + 6k \) and \( p_d(\lambda \cdot (z_1, z_2)) = (z_1 \lambda^{-2d}, z_2 \lambda^{-3d}) \) if \( d = -1 + 6k \). From this description, it is clear that under the cover \( p_d \) each regular fiber of the Seifert fibration upstairs wraps \( d \) times around each regular fiber downstairs (either preserving or reversing orientation depending on the degree of the cover).

This same covering map can also be described in terms of mapping cylinders. Recall that \( M \cong (F \times [0, 1])/ \sim \) where \( (x, 0) \sim (\varphi(x), 1) \) for all \( x \in F \) and \( \varphi \) is of order 6. When \( d = \pm 1 + 6k \), consider the \( d \)-fold cyclic cover constructed from \( d \) “puzzle pieces”, each homeomorphic to \( M \) cut open along a Seifert surface as in [19].
The resulting manifold is homeomorphic to the mapping cylinder with respect to $\varphi^d$. As $\varphi^6 = \text{id}$, $\varphi^d = \varphi^1$ and the cover manifold is again homeomorphic to $M$. It wraps, via the covering map $p_d$, $d$ times around itself. It follows from this description that the preimage of $\lambda = \partial F$ downstairs is $d$ disjoint copies of $\lambda$ upstairs.

These descriptions and the observations about the action of $p_d$ on the fiber $h$ and the longitude $\lambda$ are sufficient to completely describe the behaviour of the maps $p_d$ upon restriction to the boundary torus $\partial M$.

Lemma 3.1. Let $M$ be the complement of the trefoil knot in $S^3$, and $p_d : M \rightarrow M$ denote a covering map of order $d = 1 \pm 6k$, where $k \in \mathbb{Z}$. Then the induced map $p_d^* : \pi_1(\partial M) \rightarrow \pi_1(\partial M)$ satisfies $p_d^{-1*}(\lambda) = d[\lambda]$, $p_d^{-1*}(\mu) = [\mu] + k[\lambda]$ if $d > 0$, and $p_d^{-1*}(\mu) = [\mu] - k[\lambda]$ if $d < 0$.

Proof. From the above descriptions of the cover we have $p_d^{-1*}(\lambda) = [h]$ when $d$ is positive, $p_d^{-1*}(\lambda) = -[h]$ when $d$ is negative, and $p_d^{-1*}(\lambda) = d[\lambda]$. For a counterclockwise longitude $\lambda$ (on $S_{\text{Mod}}$ as well as on the punctured torus cover as in Figure 8), the fact that $[h] = 6[\mu] - [\lambda]$ yields for $d$ positive:

$$p_d^{-1*}(6[\mu]) = p_d^{-1*}(6[\lambda]) + p_d^{-1*}([h]) = d[\lambda] + [h] = 6[\mu] + (d - 1)[\lambda]$$

so that $p_d^{-1*}(6[\mu]) = 6[\mu] + 6k[\lambda]$ and

$$p_d^{-1*}(\mu) = [\mu] + k[\lambda].$$

If $d$ is negative, $d = 1 - 6k$, this means that we added $6k$ copies of our trefoil complement, with the fiber going in the opposite direction than before. Thus, as we start with a counterclockwise longitude and the fiber now rotates clockwise, while the direction of the meridian did not change, we have in the cover $[\lambda] - [h] = 6[\mu]$, while in the original manifold we still have $[h] = 6[\mu] - [\lambda]$. Thus we have:

$$p_d^{-1*}(6[\mu]) = p_d^{-1*}(6[\lambda]) + p_d^{-1*}([h]) = d[\lambda] - [h] = d[\lambda] - [\lambda] + 6[\mu]$$

and thus in this case,

$$p_d^{-1*}(\mu) = [\mu] - k[\lambda].$$
When the longitude is oriented clockwise, we have that \([h] = -6[\mu] + [\lambda]\). In this case, if \(d > 0\),

\[
p^{-1}_d(6[\mu]) = p^{-1}_d([\lambda]) - p^{-1}_d([h]) = d[\lambda] - [h] = 6[\mu] + (d - 1)[\lambda]
\]
and therefore \(p^*_d([\mu]) = [\mu] + k[\lambda]\) as before.

In case \(d < 0\), \([h] = -6[\mu] + [\lambda]\) for the base manifold, but \([\lambda] + [h] = 6[\mu]\) in the cover. Therefore,

\[
p^{-1}_d(6[\mu]) = p^{-1}_d([\lambda]) - p^{-1}_d([h]) = d[\lambda] + [h] = d[\lambda] + 6[\mu] - [\lambda]
\]
and thus

\[
p^{-1}_d([\mu]) = [\mu] - k[\lambda]
\]
in this case as well. \(\square\)

**Lemma 3.2.** Let the degree of the cover be \(d = 1 \pm 6k\), the preimage of the curve \(\mu\) under the map \(p_d\) is a curve homotopic to \(\mu \pm k\lambda\).

**Proof.** That any curve in \(p^{-1}_d(\mu)\) lies in the homotopy class of \(\mu + k\lambda\) follows from the description of the map \(p^*_d\) in Lemma 3.1. What remains to be shown is that \(p^{-1}_d(\mu)\) consists of single curve.

This follows from an analysis of the deck transformations group of the cover \(p_d : \partial M \to \partial M\), which is \(\mathbb{Z}/d\mathbb{Z}\). Let \(\beta : \pi_1(\partial M) \to \mathbb{Z}/d\mathbb{Z}\) denote the map sending \([\gamma] \in \pi_1(\partial M)\) to its corresponding deck transformation, which is surjective. Note that the image of \([\gamma]\) under this map completely determines the number of connected components of \(p^{-1}_d(\gamma)\), it is exactly the index \(|\mathbb{Z}/d\mathbb{Z} : \langle \beta([\gamma]) \rangle|\).

Since we know that \(p^{-1}_d(\lambda)\) consists of \(d\) disjoint curves, it follows that \(\beta([\lambda]) = 0\). Consequently if \(\beta\) is to be surjective, then \(\beta([\mu])\) must be a generator of \(\mathbb{Z}/d\mathbb{Z}\), so that \(|\mathbb{Z}/d\mathbb{Z} : \langle \beta([\mu]) \rangle| = 1\). \(\square\)

**Proposition 3.3.** Suppose \(d = 1 \pm 6k\) and let \(\psi_d\) denote the pullback of the geodesic flow along the covering map \(p_d : M \to M\). Then \(\psi_d\) has exactly two periodic orbits of slope \(\mu \pm k\lambda\). Moreover, if \(m \neq n\) are integers with \(m, n \equiv \pm 1 \mod 6\) then \(\psi_n\) and \(\psi_m\) are orbit equivalent if and only if \(n = m\).
Proof. That $\psi_d$ has exactly two periodic orbits of the given slope follows from Lemmas 3.2 and 3.1, and the fact that the geodesic flow on $M$ has only two periodic orbits each of slope $\mu$ (however note the two orbits have opposite orientations).

Now suppose that $H : M \to M$ is a diffeomorphism carrying the orbits of $\psi_m$ to the orbits of $\psi_n$. As the symmetry group of the pair $(S^3, T_{p,q})$ is $\mathbb{Z}_2$, if $H$ is not isotopic to the identity then it is isotopic to a map that generates the symmetry group. Such a map induces, up to conjugation, a homomorphism $\phi : \langle a, b \mid a^3 = b^2 \rangle \to \langle a, b \mid a^3 = b^2 \rangle$ given by $\phi(a) = a^{-1}, \phi(b) = b^{-1}$.

The homomorphism above is conjugate to a homomorphism whose restriction to $\pi_1(\partial M) \to \pi_1(\partial M)$ has action $[\mu] \mapsto -[\mu]$ and $[\lambda] \mapsto -[\lambda]$. As such, a closed orbit $\pm \mu + k\lambda$ will be carried by $H$ to a closed orbit of the form $\mp \mu - k\lambda$, meaning that the closed orbits of $\psi_n$ and $\psi_m$ are never identified by $H$ unless $m = n$. □

Remark 3.4. In particular, the longitude $\lambda$ intersects the orbit at a single point for any cover, and thus we can always choose the longitude $[\lambda]$ and the orbit $[g_d]$ as a basis for $H_1(\partial M)$. In this basis

- If the longitude is oriented counterclockwise for the base as in Figure 8 and $d < 0$, then in the cover $[h] = -p_d^{-1*}[h] = -d[\lambda] + 6[g_d]$, and as the fiber is covered by a single curve, $h = d\lambda - 6g_d$.
- If the longitude is oriented clockwise, and $d > 0$, the fiber is again covered by itself, and so is $h = d\lambda - 6g_d$ in the orbit longitude basis of the cover.

4. GLUING PIECES

In this section we complete the proof of Theorem 1.2. Begin by denoting the modular surface on the right of Figure 4 by $\overline{S}_{\text{Mod1}}$, and assume that the longitude is oriented counterclockwise on $\overline{S}_{\text{Mod1}}$. Denote its unit tangent bundle by $T^1(\overline{S}_{\text{Mod1}}) = M_1$, and denote the fiber in $\partial M_1$ by $h_1$, the meridian in $\partial M_1$ by $\mu_1$ and the longitude by $\lambda_1$. Denote the modular surface on the left of that figure by $\overline{S}_{\text{Mod2}}$, its unit tangent bundle by $T^1(\overline{S}_{\text{Mod2}}) = M_2$ and the meridian, longitude, fiber on its boundary by $\mu_2, \lambda_2$ and $h_2$. 
Next, define the map $F: \partial M_2 \to \partial M_1$ by

\[ F: \begin{align*}
\lambda_2 &\mapsto -\lambda_1 \\
\mu_2 &\mapsto \mu_1.
\end{align*} \]

Note that the map $F$ is chosen so it takes the periodic orbits of the flow $\psi_{1+6k}$ to the periodic orbits of the flow $\psi_{1-6k}$. At the same time, the map on the longitudes means it takes the Birkhoff annulus above the orbit in one copy to the annulus above the orbit in the other copy.

**Theorem 4.1.** The manifold $N = M \cup_F M$ supports infinitely many non orbit equivalent Anosov flows $\Psi_k$ such that the restriction of $\Psi_k$ to each factor $M$ is either $\psi_{1+6k}$ or $\psi_{1-6k}$.

**Proof.** Consider the boundary torus $\partial M$, as given in Figure 5. We take $M_2 = T^1 S_{\text{Mod}2}$ (with the clockwise longitude) to be on the back side of the figure. The map $F$ glues it to $M_1 = T^1 S_{\text{Mod}1}$ which is the copy of $M$ on the front side of the figure. Note that, abusing notation, we denote by $\psi^d$ the $d$ fold cover of the geodesic flow on either $S_{\text{Mod}1}$ or $S_{\text{Mod}2}$. We also take a perpendicular normal direction on one side to a perpendicular normal direction in the other. Equivalently, the $(v, v^\perp)$ planed is matched with the $(v, v^\perp)$ plane exactly via the gluing.

For the geodesic flow $\psi_1$ itself, we may switch from the orbit-fiber coordinates $(t, \theta)$ to orbit-longitude coordinates $(t, \tau)$ where $0 \leq \tau \leq 2\pi$ is a normalized to $2\pi$ arc-length parametrisation of the longitude $\lambda$. Recall that the Anosov directions are independent of $t$, and they are a horizontal rotation by $\theta$ of the directions at $\theta = 0$. Thus, they are also independent of $t$ in the $(t, \tau)$ coordinates, and consist of a rotation by $\tau$ of the directions at points with $\tau = 0$.

Next consider the flow $\psi_d$ corresponding to a self cover of the trefoil complement of degree $d = 1 \pm 6k$. By the previous section, the tangent orbit $g_d$ is a curve of homology $[\mu] \pm k[\lambda] = [g_1] \pm k[\lambda]$. Thus, $\{[g_d], [\lambda]\}$ is also a basis for the homology. The curve $g_d$ is $d$ times the length of the boundary geodesic $g_1$ of $S_{\text{Mod}}$. Since the geodesic flow on $M$ is independent of the length of $g_1$ (see Section 2.2), we may choose its length to be $L = 2\pi/d$, so that an arc-length parameter for $g_d$ ranges from 0 to $2\pi$. 
The map $F$ takes the periodic orbit $g_{1+6k}$ of $\psi_{1+6k}$, acting on the back cover $M_2$, to the periodic orbit $g_{1-6k}$ of $\psi_{1-6k}$ acting on the front copy $M_1$.

We define the flow $\Psi_k$ to be the smooth flow resulting from gluing $\psi_{1+6k}$ to $\psi_{1-6k}$ via $F$. The flows are smooth as the gluing map thus defined matches exactly the flow directions.

We start by considering the flow $\Psi_1$. By Remark 3.4, the fiber has the form $h_2 = 7\lambda_2 - 6g_7$ on the back copy $M_2$. It is mapped to $F(h_2) = -7\lambda_1 + 6g_5$. The fiber $h_1$ of $M_1$ is $h_1 = -5\lambda_1 - 6g_5$ in the longitude-orbit coordinates for $M_1$. Thus a fiber is not glued to a fiber and the manifold is not globally Seifert fibered but is a graph manifold.

Although the orbit-fiber coordinates are now not a basis for the homology of the boundary torus, as they intersect multiple times, the coordinates $(v, v^\perp, \rho)$ are still a local basis for the tangent bundle at each point.

In $M_1$, as $5\lambda_1 = -h_1 - 6g_5$, the slope of $\lambda_1$ is $(-\frac{1}{5}, -\frac{6}{5})$ in the local orbit-fiber directions on the boundary torus. Thus, $F(h_2) = -7\lambda_1 + 6g_5$ is of slope $(\frac{7}{5}, 6 + \frac{42}{5})$.

Therefore, $DF(\rho)$ is always tangent to the boundary torus, and is in the first and third quadrant in the (local) orbit-fiber basis. This direction falls in the second and fourth quadrant in the $(v^\perp, \rho)$ plane along the core curve of the top Birkhoff annulus in Figure 9, which is the annulus one flows through to get from the back copy of $M$ to the front one. In the bottom Birkhoff annulus, one flows from the front copy to the back copy, and thus the gluing is performed via $F^{-1}$. This takes the fiber slope of the front copy to $DF^{-1}(\rho)$ that falls in the second and fourth quadrant in the back orbit fiber $(t_1, \theta_1)$ coordinates.

**Claim 4.2.** The flow $\Psi_1$ is Anosov.

This follows immediately from [11, Propositions 3 and 4], we give the idea of the proof here for sake of completeness. We first show there exist two continuous plane fields, $F^u$ and $F^s$, that intersect along the direction $F^t$ tangent to the flow $\Psi_0$, and are invariant under $D\Psi_0$.

Consider at each point $x$ of $N$ the union of the second and fourth quadrants in the $(v^\perp, h)$ plane, product with the $v$ direction. That is, the portion of the tangent
Figure 9. The orbit fiber coordinates \((t, \theta)\), and the \((v, v^\perp, \rho)\) directions to the unit tangent bundle on \(\partial \mathbb{S}_{\text{Mod}}\). The shaded regions are the second and fourth quadrants in the \((v^\perp, \rho)\) plane, and in the top annulus, where the flow direction \(v\) points from the back \(\mathbb{S}_{\text{Mod}2}\) to the front \(\mathbb{S}_{\text{Mod}1}\), the image \(DF(\rho)\) falls in these quadrants.

space \(T_xN\) consisting of the two infinite wedges

\[
\mathcal{W}^u = \{(av + bv^\perp, c) \mid \text{s.t. } bc \leq 0\}.
\]

Within each piece, the action of \(D\Psi_0\) on the tangent directions is given by the action of \(\psi_0\) and \(\psi_1\). Under these actions, both \(v^\perp\) and \(h\) are moved towards the unstable direction, and thus into the interiors of the second and fourth quadrants. When passing through the gluing, one uses the action of \(DF\) when passing from the back copy to the front one, and the action of \(DF^{-1}\) when passing from the front copy to the back one. In both of these cases, by the computation above the claim, \(v^\perp\) is invariant, while \(h\) is mapped into the interior of the second quadrant (see Figure 9). It follows that

\[
D\Psi_0^t(\mathcal{W}^u_x) \subset \mathcal{W}^u_{\psi_0^t(x)}.
\]

By fixing any point \(x \in N\) and considering \(D\Psi_0^n(\mathcal{W}^u_{\psi_0^{-n}(x)})\), these are a sequence of closed sets (each corresponding to a closed set of possible slopes in the \((v^\perp, h)\) plane), that are each contained in all its prequels. Thus, there is an invariant plane field
$W^u_x = (av + bv^\perp, -\eta(x)b)_x \subset T_xN$ at any point $x$ with some finite slope function $\eta(x) > 0$.

The continuity of $\eta$ and $W^u$ follows from the continuity of the $D\Psi_t^0$ action. By the same argument applied to the action of $\Psi_0^{-t}$ on the first and third quadrants a continuous invariant plane field $W^s$ exists as well.

Next we show there exist continuous $D\Psi_0$ invariant line fields $E^u \subset W^u$ and $E^s \subset W^s$ which together with $E^t$ yield the Anosov directions for $\Psi_t^t$. We again use a cone field argument.

Any vector in the plane field $W^u$ can be expressed as $(av + bv^\perp, -\eta(x)b)_x$. A vector $(bv^\perp, -\eta(x)b)_x$ in the intersection of the plane field with the $(v^\perp, \rho)$ plane is shifted along the $v$ direction some bounded distance, as $\rho$ is shifted. As one increases the $v$ component, so $|a|$ becomes larger, the amount of the shift decreases, as both $v$ and $v^\perp$ are invariant under the gluing.

On the other hand, within each piece, the flow takes a vector $(av + bv^\perp, -\eta(x)b)_x$ exponentially fast towards the unstable direction $(v^\perp, -1)$ in the intersection of the $(v^\perp, \rho)$ plane and the plane field (as we already know the plane field is invariant). It follows that for such a vector with $|a|$ large enough, its image under $D\Psi_0$ is a vector with $|a|$ exponentially decreasing. Therefore, we can define the cone field within $W^u$:

$$C^u = \{(a(x)v + bv^\perp, -\eta(x)b)_x | b > 0\}$$

And $a(x)$ is a function that is a large enough constant $a_0$ on most of $N$, decreasing on each piece when approaching the exiting annulus in the boundary torus, and we can fix this decrease to be slower than the decrease induced by $D\Psi_0$, but still small enough at the annulus, so that the image under the gluing has $|a| < a_0$.

It follows that the cone field is invariant under the flow

$$D\Psi_0^t(C^u_x) \subset C^u_{\Psi_0^t(x)},$$

and there exist invariant line fields $E^u$ and $E^s$. Thus, the resulting flow is indeed Anosov, as the stable and unstable directions are globally defined.

**Claim 4.3.** The flow $\Psi_k$ is Anosov for any $k$. 

The only ingredient needed in the argument is the fact the fiber is mapped by $DF$, in the top Birkhoff annulus, to a tangent vector in the second quadrant in the $(v^\perp, \rho)$ plane.

This follows as for any $d = 1 + 6k$ on $M_2$ we have $h_2 = d\lambda - 6g_d$, and thus $F(h_2) = -d\lambda_1 - 6g_d$. On $M_1$, $h_1 = d'\lambda_1 + 6g_{d'}$ where $d' = 1 - 6k$ and hence $\lambda_1$ is of slope $(\frac{1}{d'}, -\frac{6}{d'})$. Thus

$$F(h_2) = -d\lambda_1 - 6g_d = -\frac{d}{d'}h_1 + (-6 - \frac{6d}{d'})g_{7}.$$ 

Thus, as $d' < 0$, $F(\rho)$ is in the first and third quadrants in the orbit fiber coordinates for any $k \geq 1$, and in the second and fourth quadrants in the $(v^\perp, \rho)$ coordinates for any $k \in \mathbb{N}$ as is true for $k = 1$.

**Claim 4.4.** The flows $\Psi_k$ and $\Psi_m$ are conjugate only if $m = k$.

Consider the topology of the manifold $N = M \cup_F M$. As mentioned above, since $F$ does not map a Seifert fiber in one copy of $M$ to the unique fiber in the second copy, $N$ is a graph manifold (and is not Seifert fibered). Thus, its JSJ torus is unique up to isotopy [12, 13].

Let $H$ be a self homeomorphism of $N$ realizing a topological equivalence between $\Psi_k$ and $\Psi_n$. Let $T \subset N$ be the embedding of $T^1\partial S_{\text{Mod}}$ into $N$. By construction, $T$ is a Birkhoff torus for $\Psi_k$. The homeomorphism $H$ takes $T$ to a Birkhoff torus $H(T)$ for $\Psi_n$. At the same time, $T$ is also a Birkhoff torus for $\Psi_n$ and they are isotopic by the uniqueness of the JSJ torus. As the only tangent orbits in the geodesic flow that can be isotoped into the boundary $\partial M$ are $g$ and $g^{-1}$, this is also true for any of its covers. Thus, $T$ and $H(T)$ share the same tangent orbits, and $H(T)$ can be isotoped to $T$ along the flowlines transverse to $T$. This implies that the two pieces of $N \setminus H(T)$ are orbit equivalent to $(M, \psi_n) \cup (M, \psi_{-n})$. Thus, $H$ conjugates $\{\psi_k, \psi_{-k}\}$ to $\{\psi_n, \psi_{-n}\}$ and it follows from Proposition [3.3] that $\{k, -k\} = \{n, -n\}$ and therefore $k = n$. □

**Remark 4.5.** Handel–Thurston point out in [11] that one can take covers of the geodesic flows in the pieces $N_i$: “The family we have described can be enlarged by taking the $n$-fold cover of $N_i$ ($n$ independent of $i$) corresponding to the $S^1$ fiber, before gluing the $N_i$ together.” One might ask if this could lead to a generalization of our
constructions presented here. However it is not clear which basis to use in the general setting, and how to control the manifold obtained by gluing (the orbit-fiber coordinates do not work for this setting, as the orbit and the fiber necessarily intersect multiple times in the cover).

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