Combinatorial $t$-designs from special polynomials

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Abstract

Combinatorial $t$-designs have nice applications in coding theory, finite geometries and several engineering areas. There are two major methods of constructing $t$-designs. One of them is via group actions of certain permutation groups which are $t$-transitive or $t$-homogeneous on some point set. The other is a coding-theoretical one. The objectives of this paper are to introduce two constructions of $t$-designs with special polynomials over finite fields $\text{GF}(q)$, and obtain 2-designs and 3-designs with interesting parameters. A type of $d$-polynomials is defined and used to construct 2-designs. Under the framework of the first construction, it is shown that every $o$-polynomial over $\text{GF}(2^m)$ gives a 2-design, and every $o$-monomial over $\text{GF}(2^m)$ yields a 3-design. Under the second construction, every $o$-polynomial gives a 3-design. Some open problems and conjectures are also presented in this paper.

Keywords: Hyperoval, $o$-polynomial, polynomial, projective plane, $t$-design.

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1. Introduction

Let $\mathcal{P}$ be a set of $v \geq 1$ elements, and let $\mathcal{B}$ be a set of $k$-subsets of $\mathcal{P}$, where $k$ is a positive integer with $1 \leq k \leq v$. Let $t$ be a positive integer with $t \leq k$. The pair $\mathbb{D} = (\mathcal{P}, \mathcal{B})$ is called a $t$-$(v, k, \lambda)$ design, or simply $t$-design, if every $t$-subset of $\mathcal{P}$ is contained in exactly $\lambda$ elements of $\mathcal{B}$. The elements of $\mathcal{P}$ are called points, and those of $\mathcal{B}$ are referred to as blocks. We usually use $b$ to denote the number of blocks in $\mathcal{B}$. A $t$-design is called simple if $\mathcal{B}$ does not contain repeated blocks. In this paper, we consider only simple $t$-designs. A $t$-design is called symmetric if $v = b$. It is clear that $t$-designs with $k = t$ or $k = v$ always exist. Such $t$-designs are trivial. In this paper, we consider only $t$-designs with $v > k > t$. A $t$-$(v, k, \lambda)$ design is referred to as a Steiner system if $t \geq 2$ and $\lambda = 1$, and is denoted by $S(t, k, v)$.

By a special polynomial over a finite field we mean a polynomial either of special form or with special property. For instance, monomials and permutation polynomials are special polynomials. Special polynomials have interesting applications in combinatorial designs. For instance,
the Dickson polynomials $x^5 + ax^3 + a^2x$ over $\text{GF}(3^m)$ led to a 70-year breakthrough in searching for new skew Hadamard difference sets [6].

A hyperoval in the projective space $\text{PG}(2, \text{GF}(2^m))$ is a set of $2^m + 2$ points such that no three of them are collinear. O-polynomials are a special type of polynomials over $\text{GF}(2^m)$ and correspond to hyperovals in the projective space $\text{PG}(2, \text{GF}(2^m))$ (see Theorems 9 and 10). This means that o-polynomials can be used to construct 2-designs indirectly (via their corresponding hyperovals). Motivated by this fact, in this paper we present two constructions of t-designs using o-polynomials directly. Specifically, we obtain 3-designs from o-monomials and 2-designs from general o-polynomials using the first construction. We obtain 3-designs from o-polynomials using the second construction. We also introduce new types of polynomials over finite fields, which give also 2-designs.

The rest of this paper is arranged as follows. Section 2 presents the general construction of t-designs from polynomials over general finite fields, and introduces a special type of polynomials, called design polynomial (in short, d-polynomials). Section 3 investigates the designs of d-polynomials over $\text{GF}(2^m)$. Section 4 studies 3-designs from o-monomials and 2-designs from general o-polynomials using o-polynomials. Section 5 deals with designs from o-polynomials over $\text{GF}(q)$ for odd $q$. Section 6 introduces an extended construction of the one documented in Section 2, and deals with 2-designs and 3-designs via this extended construction. Section 7 concludes this paper and makes concluding remarks.

2. A construction for t-designs with polynomials over $\text{GF}(q)$

Let $q$ be a prime power, and let $f$ be a polynomial over $\text{GF}(q)$, which is always viewed as a function from $\text{GF}(q)$ to $\text{GF}(q)$ throughout this paper. For each $(b, c) \in \text{GF}(q)^2$, define

$$B_{(f,b,c)} = \{f(x) + bx + c : x \in \text{GF}(q)\}. \quad (1)$$

Let $k$ be an integer with $2 \leq k \leq q$. Define

$$\mathcal{B}_{(f,k)} = \{B_{(f,b,c)} : |B_{(f,b,c)}| = k, b, c \in \text{GF}(q)\}. \quad (2)$$

The incidence structure $\mathcal{D}(f, k) := (\text{GF}(q), \mathcal{B}_{(f,k)})$ may be a $t$-$\{q,k,\lambda\}$ design for some $\lambda$, where $\text{GF}(q)$ is the point set, and the incidence relation is the set membership. In this case, we say that the polynomial $f$ supports a $t$-$\{q,k,\lambda\}$ design.

The following is a general result about monomials. It shows an interesting application of monomials in the theory of combinatorial designs.

**Theorem 1.** Let $f(x) = x^e$ be a permutation polynomial of $\text{GF}(q)$, and let $k \geq 2$ be a positive integer such that $|\mathcal{B}_{(f,k)}| \geq 1$. Then the incidence structure $\mathcal{D}(f, k) := (\text{GF}(q), \mathcal{B}_{(f,k)})$ is a $2$-$(q,k,\lambda)$ for some $\lambda$.

**Proof.** The general affine group $\text{GA}_1(\text{GF}(q))$ is defined by

$$\text{GA}_1(\text{GF}(q)) := \{ux + v : (u,v) \in \text{GF}(q)^* \times \text{GF}(q)\}.$$

Let $\sigma(x) = ux + v \in \text{GA}_1(\text{GF}(q))$, where $u \in \text{GF}(q)^*$ and $v \in \text{GF}(q)$. Note that $\gcd(e,q-1) = 1$. Let $1/e$ denote the multiplicative inverse of $e$ modulo $q-1$. We have then

$$u(f(x) + bx + c) + v = u(x^e + ubx + cu + v) = (u^{1/e}x)^e + u^{1-1/e}b(u^{1/e}x) + cu + v.$$
We then deduce that \( \sigma(B_{(f,b,c)}) = B_{(f,ad^{-1}(rb,rc+bv))} \). This means that the general affine group \( \text{GA}_1(\mathbb{F}_q) \) fixes \( B_{(f,k)} \). It is well known that \( \text{GA}_1(\mathbb{F}_q) \) acts on \( \mathbb{F}_q \) doubly transitively. The desired conclusion then follows.

Two designs \( \mathbb{D}(P, B) \) and \( \mathbb{D}(P', B') \) are said to be isomorphic if there is a 1-to-1 mapping \( \sigma \) from \( P \) to \( P' \) such that \( \sigma \) sends each block in \( B \) to a block in \( B' \). Such a \( \sigma \) is called an isomorphism from \( \mathbb{D}(P, B) \) to \( \mathbb{D}(P', B') \). An isomorphism from \( \mathbb{D}(P, B) \) to \( \mathbb{D}(P, B) \) is called an automorphism of \( \mathbb{D}(P, B) \). All automorphisms of \( \mathbb{D}(P, B) \) form a group under the function composition, and is called the automorphism group of \( \mathbb{D}(P, B) \). It is straightforward to prove the following theorem.

**Theorem 2.** Let \( f \) and \( g \) be two polynomials over \( \mathbb{F}_q \) such that \( \mathbb{D}(f,k) \) and \( \mathbb{D}(g,k) \) are \( t \)-designs. If there are \( h \in \mathbb{F}_q^* \), \( u \in \mathbb{F}_q^* \) and \( v \in \mathbb{F}_q \) such that \( g(x) = hf(ux+v) \) for all \( x \in \mathbb{F}_q \), then \( \mathbb{D}(f,k) \) and \( \mathbb{D}(g,k) \) are isomorphic.

We define the value spectrum of a polynomial over \( \mathbb{F}_q \) to be the multiset

\[
\text{VS}(f) = \{ |B_{(f,b,c)}| : (b,c) \in \mathbb{F}_q^2 \}.
\]

To determine the parameters of \( t \)-designs supported by a polynomial \( f \), we need to know the value spectrum of a polynomial \( f \). The value spectrum of a polynomial is hard to determine in general, but can be done in special cases.

We call a permutation polynomial of \( \mathbb{F}_q \) a design polynomial (in short, d-polynomial) if the size \( |\{f(x) + bx\}| \) is a constant for all \( b \in \mathbb{F}_q^* \). As will be seen later, some d-polynomials supports 2-designs or 3-designs with interesting parameters under this construction framework.

### 3. Designs from d-monomials over \( \mathbb{F}_{2^m} \)

Throughout this section, let \( q = 2^m \) for some positive integer \( m \). Our objective in this section is to search for d-monomials and consider the parameters of their 2-designs. As will be seen soon, determining the block size and the number of blocks in the 2-design supported by a d-monomial could be extremely hard. There are a number of such d-monomials. Some of them are treated in this section, and some will be investigated in Section 4.

**Lemma 3.** Let \( f(x) = x^e \) be a polynomial over \( \mathbb{F}_q \) such \( \gcd(e-1, q-1) = 1 \). Then \( f(x) \) is a d-polynomial over \( \mathbb{F}_q \).

**Proof.** Since \( x^{e-1} \) is a permutation of \( \mathbb{F}_q \), for each \( b \in \mathbb{F}_q^* \) there is a unique \( u \in \mathbb{F}_q^* \) such that \( u^{e-1} = b \). We have then

\[
|\{x^e + bx : x \in \mathbb{F}_q\}| = |\{(ay)^e + buy : y \in \mathbb{F}_q\}| = |\{a^e(y^e+y) : y \in \mathbb{F}_q\}| = |\{y^e + y : y \in \mathbb{F}_q\}|.
\]

By definition, \( f(x) = x^e \) is a d-polynomial. \( \square \)

There are a number of monomials \( \overline{f}(x) = x^e \) satisfying \( \gcd(e-1, q-1) = 1 \). Such d-monomials over \( \mathbb{F}_q \) support 2-designs by Theorem 1. It will be shown later that some of them support 3-designs.

The following theorem presents a large number of 2-designs supported by a class of d-monomials.
Theorem 4. Let $e$ be a positive integer with $\gcd(e-1, q-1) = 1$. Define
\[ J_e = \{ x^e + x : x \in \text{GF}(q) \} \]  
and
\[ \text{Stab}_{AG_1(\text{GF}(q))}(J_e) = \{ ux + v : (u, v) \in \text{GF}(q)^* \times \text{GF}(q), \ uJ_e + v = J_e \} . \]

Then the incidence structure $D(x^e, k) := (\text{GF}(q), B(x^e, k))$ is a $2$-$(q, k, (k-1)k/\mu)$ design, where $k = |J_e|$ and $\mu = |\text{Stab}_{AG_1(\text{GF}(q))}(J_e)|$.

Proof. It follows from Lemma 3 and its proof that $f(x) = x^e$ is a $d$-monomial and that
\[ B(x^e, k) = \{ bJ_e + c : (b, c) \in \text{GF}(q)^* \times \text{GF}(q) \} . \]
Note that the general affine group acts on $\text{GF}(q)$ doubly transitively and fixes $B(x^e, k)$. The desired conclusion then follows from Proposition 4.6 in [2, p. 175].

Theorem 5. Let $m \geq 3$ be odd and $q = 2^m$. The following is a list of $d$-monomials $x^e$ over $\text{GF}(q)$ satisfying the condition of Theorem 4.

- $e = 2^h + 1$, where $\gcd(h, m) = 1$.
- $e = 2^{(m-1)/2} + 3$.
- $e = 2^h - 2^h + 1$, where $\gcd(h, m) = 1$.
- $e = 2^{(m-1)/2} + 2^{(m-1)/4} - 1$, where $m \equiv 1 \pmod{4} \geq 5$.
- $e = 2^{(m-1)/2} + 2^{(3m-1)/4} - 1$, where $m \equiv 3 \pmod{4}$.

Proof. In all the cases above, it can be verified that $\gcd(e-1, q-1) = 1$. It then follows from Lemma 3 that $x^e$ is a $d$-polynomial.

Theorem 6. Let $m$ be even and $q = 2^m$. The following is a list of $d$-monomials $x^e$ over $\text{GF}(q)$ satisfying the condition of Theorem 4.

- $e = 2^h + 1$, where $m/\gcd(h, m)$ is odd.
- $e = 2^{m/2} + 2^{(m+2)/4} + 1$, where $m \equiv 2 \pmod{8}$.
- $e = 2^{(m-2)/2} - 3$, where $m \equiv 4 \pmod{24}$ or $m \equiv 20 \pmod{24}$.
- $e = 2^{(m+2)/2} - 3$, where $m \equiv 0 \pmod{24}$ or $m \equiv 8 \pmod{24}$ or $m \equiv 16 \pmod{24}$.

Proof. In all the cases above, it can be verified that $\gcd(e-1, q-1) = 1$. It then follows from Lemma 3 that $x^e$ is a $d$-polynomial.

All the d-monomials in Theorems 5 and 6 can be plugged into Theorem 4 for obtaining $2$-$(q, k, \lambda)$ designs. But determining the parameters $k$ and $\lambda$ in the designs seems quite difficult. The reader is warmly invited to attack this problem.
4. Designs from o-polynomials over GF(2^m)

Throughout this section \( q = 2^m \) for some positive integer \( m \). The objective of this section is to construct 2-designs and 3-designs from o-polynomials over GF(q). Since o-polynomials and hyperovals can be viewed as the same and hyperovals were used to construct two types of 2-designs in the literature, we have to introduce hyperovals and their designs, so that we will be able to compare our newly constructed designs with hyperoval designs in the literature.

4.1. Hyperovals and their designs

An arc in the projective plane PG(2, GF(q)) is a set of at least three points in PG(2, GF(q)) such that no three of them are collinear (i.e., on the same line). For any arc \( A \) of PG(2, GF(q)), it is well known that \( |A| \leq q + 2 \).

A hyperoval \( H \) in PG(2, GF(q)) is a set of \( q + 2 \) points such that no three of them are collinear, i.e., an arc with \( q + 2 \) points. Hyperovals are maximal arcs, as they have the maximal number of points as arcs. Two hyperovals are said to be equivalent if there is a collineation (i.e., an automorphism) of PG(2, GF(q)) that sends one to the other. Note that the automorphism group of PG(2, GF(q)) is the projective general linear group PGL(3, GF(q)). The automorphism group of a hyperoval is the set of all collineations of PG(2, GF(q)) that leave the hyperoval invariant.

The next theorem shows that all hyperovals in PG(2, GF(q)) can be constructed with a special type of permutation polynomials of GF(q) [12, p. 504].

**Theorem 7** (Segre). Let \( m \geq 2 \). Any hyperoval in PG(2, GF(q)) can be written in the form

\[
H(f) = \{(f(c), c, 1) : c \in GF(q)\} \cup \{(1, 0, 0)\} \cup \{(0, 1, 0)\},
\]

where \( f \in GF(q)[x] \) is such that

1. \( f \) is a permutation polynomial of GF(q) with \( \deg(f) < q \) and \( f(0) = 0 \), \( f(1) = 1 \);
2. for each \( a \in GF(q) \), \( g_a(x) = (f(x + a) + f(a))x^{q-2} \) is also a permutation polynomial of GF(q).

Conversely, every such set \( H(f) \) is a hyperoval.

Polynomials satisfying the two conditions of Theorem 7 are called o-polynomials, i.e., oval-polynomials. For example, \( f(x) = x^2 \) is an o-polynomial over GF(q) for all \( m \geq 2 \). In the next section, we will summarize known o-polynomials over GF(q).

Two o-monomials \( f \) and \( g \) are said to be equivalent if the two hyperovals \( H(f) \) and \( H(g) \) are equivalent. The following result was presented in [20].

**Lemma 8.** Let \( q \geq 4 \). Two monomial hyperovals \( H(x^i) \) and \( H(x^e) \) in PG(2, GF(q)) are equivalent if and only if \( i \equiv e, 1/e, 1-e, 1/(1-e), e/(e-1) \) or \( (e-1)/e \pmod{q-1} \).

Any hyperoval \( H \) in PG(2, GF(q)) meets each line either in 0 or 2 points. A line is called an interior line (also called secant) of \( H \) if it meets the hyperoval in two points, and an exterior line otherwise. Hence, a hyperoval partitions the lines of PG(2, GF(q)) into two classes, i.e., interior and exterior lines. This property allows us to define the so-called hyperoval designs as follows.

Let \( H \) be a hyperoval in the Desarguesian projective plane PG(2, GF(q)). The hyperoval design \( W(q, H) \) is the incidence structure with points the lines of PG(2, GF(q)) exterior to \( H \) and blocks the points of PG(2, GF(q)) not on the hyperoval; incidence is given by the incidence in PG(2, GF(q)). We have then the following conclusion on the incidence structure \( W(q, H) \).
Theorem 9 (10). The incidence structure \( W(q, \mathcal{H}) \) defined by a hyperoval \( \mathcal{H} \) in \( \text{PG}(2, \text{GF}(q)) \) is a 2-((q - 1)q/2, q/2, 1) design, i.e., a Steiner system.

The second type of 2-designs from hyperovals are constructed as follows. Let \( \mathcal{H} \) be a hyperoval in \( \text{PG}(2, \text{GF}(q)) \). Let \( \mathcal{P} \) be the set of \( q^2 - 1 \) exterior points to \( \mathcal{H} \), i.e., the set of points in \( \text{PG}(2, \text{GF}(q)) \setminus \mathcal{H} \). For each point \( x \in \mathcal{P} \), define a block

\[ B_x = \{ y \in \mathcal{P} \setminus \{ x \} : xy \text{ is a secant to } \mathcal{H} \} \cup \{ x \}. \]

Define further \( \mathcal{B} = \{ B_x : x \in \mathcal{P} \} \). We have then the following conclusion.

Theorem 10 (10, 11, 12). The incidence structure \( S(q, \mathcal{H}) := (\mathcal{P}, \mathcal{B}) \) is a symmetric 2-(q^2 - 1, \frac{1}{2}q^2 - 1, \frac{1}{2}q^2 - 1) design.

It is known that the Hadamard design \( S(q, \mathcal{H}) \) can be extended into a 3-(q^2, \frac{1}{2}q^2, \frac{1}{3}q^2 - 1) design, denoted by \( S(q, \mathcal{H})^{*} \).

4.2. Known o-polynomials over GF(2^m)

Recall that \( q = 2^m \). To construct 2-designs and 3-designs subsequently, we need o-polynomials over GF(q). The objective of this section is to summarise known constructions of o-polynomials over GF(q) and consequently hyperovals in \( \text{PG}(2, \text{GF}(q)) \).

In the definition of o-polynomials, it is required that \( f(1) = 1 \). However, this is not essential, as one can always normalise \( f(x) \) by using \( f(1)^{-1}f(x) \) due to the fact that \( f(1) \neq 0 \). In this section, we do not require that \( f(1) = 1 \) for o-polynomials.

For any permutation polynomial \( f(x) \) over GF(q), we define \( \overline{f}(x) = xf(x^{q-2}) \), and use \( f^{-1} \) to denote the compositional inverse of \( f \), i.e., \( f^{-1}(f(x)) = x \) for all \( x \in \text{GF}(q) \).

The following two theorems introduce basic properties of o-polynomials whose proofs can be found in references about hyperovals.

Theorem 11. Let \( f \) be an o-polynomial over GF(q). Then the following statements hold:

- \( f^{-1} \) is also an o-polynomial;
- \( f(x^{2^j}) \) is an o-polynomial for any \( 1 \leq j \leq m - 1 \);
- \( \overline{f} \) is also an o-polynomial; and
- \( f(x + 1) + f(1) \) is also an o-polynomial.

Theorem 12. Let \( x^e \) be an o-polynomial over GF(q). Then every polynomial in

\[ \{ x^{\frac{1}{2}}, x^{1-e}, x^{rac{1}{1-2e}}, x^{rac{1}{e+1}}, x^{rac{1}{1-e}} \} \]

is also an o-polynomial, where \( 1/e \) denotes the multiplicative inverse of \( e \) modulo \( q - 1 \).

Theorem 13 (11). A polynomial \( f \) over GF(q) with \( f(0) = 0 \) is an o-polynomial if and only if \( f_u := f(x) + ux \) is 2-to-1 for every \( u \in \text{GF}(q)^* \).

Below we summarise some classes of o-polynomials over GF(q). The translation o-polynomials are described in the following theorem (13).
Theorem 14. Trans\((x) = x^{2^h}\) is an o-polynomial over \(\text{GF}(q)\), where \(\gcd(h, m) = 1\).

The following is a list of known properties of translation o-polynomials.

- \(\text{Trans}^{-1}(x) = x^{2^{m-h}}\) and
- \(\text{Trans}(x) = xf(x^{q-2}) = x^{q-2^{m-h}}\).

The following theorem describes a class of o-polynomials, which are called Segre o-polynomials \([16, 17]\).

Theorem 15. Let \(m\) be odd. Then \(\text{Segre}(x) = x^6\) is an o-polynomial over \(\text{GF}(q)\).

For this o-monomial, we have the following.

1. \(\text{Segre}(x) = x^{q-6}\).
2. \(\text{Segre}^{-1}(x) = x^{\frac{6 \cdot 2^{m-1}-1}{2}}\).

Glynn discovered two families of o-polynomials \([8]\). The first is described as follows.

Theorem 16. Let \(m\) be odd. Then \(\text{Glynni}(x) = x^{3 \times 2^{(m+1)/2} + 4}\) is an o-polynomial.

The second family of o-polynomials discovered by Glynn is documented in the following theorem.

Theorem 17. Let \(m\) be odd. Then
\[
\text{Glynnii}(x) = \begin{cases} 
  x^{2^{(m+1)/2} + 2^{(3m+1)/4}} & \text{if } m \equiv 1 \pmod{4}, \\
  x^{2^{(m+1)/2} + 2^{(m+1)/4}} & \text{if } m \equiv 3 \pmod{4}
\end{cases}
\]
is an o-polynomial over \(\text{GF}(q)\).

The following describes another class of o-polynomials discovered by Cherowitzo \([3, 4]\).

Theorem 18 \([7]\). Let \(m\) be odd and \(e = (m + 1)/2\). Then \(\text{Cherowitzo}(x) = x^{2^e} + x^{2^e+2} + x^{3\times 2^e+4}\) is an o-polynomial over \(\text{GF}(q)\).

For this o-trinomial, we have the following conclusions.

1. \(\text{Cherowitzo}(x) = x^{q-2^e} + x^{q-2^e-2} + x^{q-3\times 2^e-4}\).
2. \(\text{Cherowitzo}^{-1}(x) = x(x^{2^e+1} + x^3 + x)^{2^e-1} \cdot 1\).

The following documents a family of o-trinomials due to Payne.

Theorem 19 \([13]\). Let \(m\) be odd. Then \(\text{Payne}(x) = x^{\frac{e}{2}} + x^{\frac{3}{2}} + x^{\frac{5}{2}}\) is an o-polynomial over \(\text{GF}(q)\).

We have the following statements regarding the Payne o-trinomial.

1. \(\text{Payne}(x) = xD_5(x^{\frac{e}{2}}, 1)\), where \(D_5(a, x)\) is the Dickson polynomial of order 5.
2. \(\text{Payn}(x) = \text{Payne}(x)\).
3. Note that
\[
\frac{1}{6} = \frac{5 \times 2^{m-1} - 2}{3}.
\]
We have then
\[
Payne(x) = x^{2^{m-1}+3} + x^{2^{m-1}} + x^{5 \times 2^{m-1} - 2}.
\]

**Theorem 20 ([7]).** Let \( m \) be odd. Then
\[
Payne^{-1}(x) = \left(D_{1+2^{m-2}, 2^m}(x, 1)\right)^6. \tag{4}
\]

The Subiaco o-polynomials are given in the following theorem [5].

**Theorem 21.** Define
\[
Subiaco_a(x) = ((a^2(x^4 + x) + a^2(1 + a + a^2)(x^3 + x^2))(x^4 + a^2x^2 + 1)^{q-2} + x^{2m-1},
\]
where \( \text{Tr}(1/a) = 1 \) and \( d \notin GF(4) \) if \( m \equiv 2 \mod 4 \). Then \( Subiaco_a(x) \) is an o-polynomial over \( GF(q) \).

As a corollary of Theorem [21] we have the following.

**Corollary 22.** Let \( m \) be odd. Then
\[
Subiaco_1(x) = (x + x^2 + x^3)(x^4 + x^2 + 1)^{q-2} + x^{2m-1} \tag{5}
\]
is an o-polynomial over \( GF(q) \).

### 4.3. Combinatorial \( t \)-designs from o-polynomials

In this section, we plug o-polynomials into the construction of Section 2 to construct 2-designs and 3-designs. By Theorem [13] o-polynomials are d-polynomials. This fact will play an important role.

#### 4.3.1. Families of 2-designs and 3-designs from o-polynomials

We start with a few auxiliary results. Let \( g(x) \) be a polynomial over \( GF(q) \). The value set of \( g(x) \) is the image of the induced map \( g : GF(q) \to GF(q) \). Thus the value set is
\[
V(g) = \{g(x) : x \in GF(q)\}.
\]

We denote the cardinality of \( V(g) \) by \( v(g) \).

**Lemma 23.** Let \( f(x) \in GF(q)[x] \) be an o-polynomial. For any \( u_1, u_2, u_3 \in GF(q) \) with \( (u_1 + u_2)(u_2 + u_3)(u_3 + u_1) \neq 0 \), define
\[
I(u_1, u_2, u_3) = \{(a, b, c) \in GF(q)^3 : ab \neq 0, \{u_1, u_2, u_3\} \subseteq V(af(x) + bx + c)\}.
\]
Then
\[
|I(u_1, u_2, u_3)| = \frac{q(q-1)(q-4)}{8}.
\]
Proof. Put
\[ T = \left\{ (a, b, c, x_1, x_2, x_3) \in \text{GF}(q)^6 : af(u_i) + bx_i + c = u_i \ (i = 1, 2, 3) \right\}. \]

Then
\[ |T| = \sum_{(a,b,c)\in\text{GF}(q)^3} J(a,b,c) = \sum_{(x_1,x_2,x_3)\in\text{GF}(q)^3} K(x_1,x_2,x_3), \]

where
\[ J(a,b,c) = |\{(x_1,x_2,x_3)\in\text{GF}(q)^3 : af(x_i) + bx_i + c = u_i \ (i = 1, 2, 3)\}|, \]

and
\[ K(x_1,x_2,x_3) = |\{(a,b,c)\in\text{GF}(q)^3 : af(x_i) + bx_i + c = u_i \ (i = 1, 2, 3)\}|. \]

Notice that \( g(x) = af(x) + bx + c \) is 2-to-1 when \( ab \neq 0 \). We have \( v(g) = q/2 \) if \( ab \neq 0 \). If \( ab = 0 \) and \( a \neq b \), then \( g(x) \) is a permutation. We deduce then
\[ v(af(x) + bx + c) = \begin{cases} 1, & \text{if and only if } a = b = 0, \\ q, & \text{if and only if } ab = 0 \text{ and } a \neq b, \\ q/2, & \text{if and only if } ab \neq 0. \end{cases} \]

Since \( g(x) = af(x) + bx + c \) is 2-to-1 when \( v(g) = q/2 \) and is a permutation when \( v(g) = q \), we have
\[ J(a,b,c) = \begin{cases} 0, & \text{if } \{u_1,u_2,u_3\} \not\subseteq V(g), \\ 1, & \text{if } \{u_1,u_2,u_3\} \subseteq V(g) \text{ and } v(g) = q, \\ 8, & \text{if } \{u_1,u_2,u_3\} \subseteq V(g) \text{ and } v(g) = q/2. \end{cases} \]

It then follows that
\[ |T| = \sum_{(a,b,c)\in\text{GF}(q)^3} J(a,b,c) \]
\[ = |\{(a,b,c)\in\text{GF}(q)^3 : v(af(x) + bx + c) = q\}| + 8|\{u_1,u_2,u_3\}| \]
\[ = 2(q - 1)q + 8|\{u_1,u_2,u_3\}|. \]

Let \( x_1, x_2 \) and \( x_3 \) be three pairwise distinct elements in \( \text{GF}(q) \). Then \( (f(x_1), x_1, 1), (f(x_2), x_2, 1), \) and \( (f(x_3), x_3, 1) \) are three points in the hyperoval defined by the \( a \)-polynomial \( f(x) \), and thus are linearly independent over \( \text{GF}(q) \). We then deduce that
\[ K(x_1,x_2,x_3) = \begin{cases} 0, & |\{x_1,x_2,x_3\}| < 3, \\ 1, & |\{x_1,x_2,x_3\}| = 3. \end{cases} \]

Thus,
\[ |T| = \sum_{(x_1,x_2,x_3)\in\text{GF}(q)^3} K(x_1,x_2,x_3) = q(q - 1)(q - 2). \]
Consequently,
\[
I(u_1, u_2, u_3) = \frac{1}{8}(q(q-1)(q-2) - 2(q-1)q) = \frac{q(q-1)(q-4)}{8}.
\]

This completes the proof. □

**Lemma 24.** Let \(a \in \text{GF}(q)^*\) and \(f(x) = x^d \in \text{GF}(q)[x]\) be an o-monomial. For any \(u_1, u_2, u_3 \in \text{GF}(q)\) with \((u_1 + u_2)(u_2 + u_3)(u_3 + u_1) \neq 0\), define
\[
I_a(u_1, u_2, u_3) = \{(b, c) \in \text{GF}(q)^2 : b \neq 0, \{u_1, u_2, u_3\} \subseteq V(af(x) + bx + c) \}.
\]
Then, \(|I_a(u_1, u_2, u_3)| = \frac{q(q-4)}{8}\).

**Proof.** Recall that \(f(x) = x^d\) is a permutation of \(\text{GF}(q)\). We have then
\[
V(af(x) + bx + c) = \{ax^d + bx + c : x \in \text{GF}(q)\} = \{((a^{d-1})x)^d + ba^{-d-1}(a^{d-1})x + c : x \in \text{GF}(q)\} = V(x^d + ba^{-d-1}x + c),
\]
where \(d^{-1}\) is a positive integer such that \(dd^{-1} \equiv 1 \pmod{q-1}\). Thus, \((b, c) \rightarrow (ba^{-d-1}, c)\) induces a bijective mapping from \(I_a(u_1, u_2, u_3)\) to \(I_1(u_1, u_2, u_3)\). Then, \(|I_a(u_1, u_2, u_3)| = |I_1(u_1, u_2, u_3)|\). We then deduce by Lemma 23 that
\[
|I_a(u_1, u_2, u_3)| = \frac{1}{q-1}|I(u_1, u_2, u_3)| = \frac{q(q-4)}{8}.
\]
This completes the proof. □

We are now ready to prove the following result, which is one of the main results of this paper.

**Theorem 25.** Let \(f(x) = x^e\) be an o-monomial over \(\text{GF}(q)\). Then \(\mathbb{D}(f, q/2) := (\text{GF}(q), B_{(f, q/2)})\) is a \(3(q, q/2, q(q-4)/8 \mu)\) design, where
\[
\mu = |\text{Stab}_{\text{AG}(\mu, \text{GF}(q))}(J_c)| = |\{(u, v) \in \text{GF}(q)^* \times \text{GF}(q) : uJ_c + v = J_c\}|
\]
and
\[
J_c = \{y^c : y \in \text{GF}(q)\}.
\] (6)

**Proof.** We follow the notation of Lemmas 23 and 24 and their proofs. By the definition of o-polynomials, we have \(\gcd(e(e-1), q-1) = 1\). Define the following multiset:
\[
\tilde{B}_{(f, q/2)} = \{\{x^e + bx + c : x \in \text{GF}(q)\} : b \in \text{GF}(q)^*, c \in \text{GF}(q)\}\}
\]
By the proof of Lemma 24,
\[
\tilde{B}_{(f, q/2)} = \{\{bJ_c + c : b \in \text{GF}(q)^*, c \in \text{GF}(q)\}\}
\]
and
\[
B_{(f, q/2)} = \{bJ_c + c : b \in \text{GF}(q)^*, c \in \text{GF}(q)\}.
\]
Clearly, the general affine group \( \text{AG}_1(\text{GF}(q)) \) fixes both \( \tilde{B}_{(f,q/2)} \) and \( B_{(f,q/2)} \). The stabilizer of \( J_e \) under \( \text{AG}_1(\text{GF}(q)) \) is defined by

\[
\text{Stab}_{\text{AG}_1(\text{GF}(q))}(J_e) = \{ux+v : (u,v) \in \text{GF}(q)^* \times \text{GF}(q), uJ_e + v = J_e\}.
\]

We then deduce that

\[
|B_{(f,q/2)}| = \frac{(q-1)q}{|\text{Stab}_{\text{AG}_1(\text{GF}(q))}(J_e)|}.
\]

Note that

\[
V(x^e + bx + c) = V(bx/(e-1)(x^e + x) + c).
\]

Consequently, the multiset

\[
\{\{V(x^e + bx + c) : (b,c) \in I_1(u_1,u_2,u_3)\}\}
\]

is the same as the multiset

\[
|\text{Stab}_{\text{AG}_1(\text{GF}(q))}(J_e)|\{\{B_{(f,b,c)} : \{u_1,u_2,u_3\} \subset B_{(f,b,c)}\}\},
\]

where \( \{u_1,u_2,u_3\} \) is a set of three distinct elements in \( \text{GF}(q) \), and \( I_1(u_1,u_2,u_3) \) was defined in Lemma 24. It then follows that \( (\text{GF}(q),\tilde{B}_{(f,q/2)}) \) is a \( t-(q,q/2,3) \) design if and only if \( (\text{GF}(q),B_{(f,q/2)}) \) is a \( t-(q,q/2,\lambda/\mu) \) design, where \( \mu \) was defined earlier.

By Lemma 24, \( (\text{GF}(q),\tilde{B}_{(f,q/2)}) \) is a \( 3-(q,q/2,q(q-4)/8) \) design, which may contain repeated blocks. As a result, \( (\text{GF}(q),B_{(f,q/2)}) \) is a \( 3-(q,q/2,q(q-4)/8\mu) \) simple design.

Theorem 25 says that every o-monomial \( x^e \) supports a 3-design \( \mathbb{D}(x^e,q/2) \). The determination of the parameters of the 3-design boils down to that of the size \( \mu \) of the stabiliser of the block \( J_e \) under the action of \( \text{GA}_1(\text{GF}(q)) \).

The following is a corollary of Theorem 25. We give a direct proof of it below.

**Corollary 26.** Let \( f(x) = x^e \) be an o-monomial over \( \text{GF}(q) \) such that \( |B_{(f,q/2)}| = (q-1)q \). Then \( \mathbb{D}(f,q/2) := (\text{GF}(q),\tilde{B}_{(f,q/2)}) \) is a \( 3-(q,q/2,q(q-4)/8) \) design.

**Proof.** It follows from Theorem 13 that \( |B_{(f,b,c)}| = q/2 \) for all \( (b,c) \in \text{GF}(q)^* \times \text{GF}(q) \). By assumption, all blocks \( B_{(f,b,c)} \) with \( (b,c) \in \text{GF}(q)^* \times \text{GF}(q) \) are pairwise distinct. The design property then follows from Lemma 24.

Only o-monomials support 3-designs with respect to this construction. O-polynomials do not support 3-designs in general, but do support 2-designs with respect to this construction. Below we prove this general result. To this end, we need prove the next two auxiliary results.

**Lemma 27.** Let \( f(x) \in \text{GF}(q)[x] \) be an o-polynomial. For any \( u_1,u_2 \in \text{GF}(q) \) with \( u_1 \neq u_2 \), define

\[
I(u_1,u_2) = \{(b,c) \in \text{GF}(q)^2 : b \neq 0, \{u_1,u_2\} \subseteq V(f(x) + bx + c)\}.
\]

Then, \( |I(u_1,u_2)| = \frac{q(q-2)}{2} \).
Proof. Set
\[ T = \{ (b,c,x_1,x_2) \in \text{GF}(q)^4 : f(x_i) + bx_i + c = u_i \ (i = 1,2) \} \).

Then
\[ |T| = \sum_{(b,c) \in \text{GF}(q)^2} J(b,c) = \sum_{(x_1,x_2) \in \text{GF}(q)^2} K(x_1,x_2), \]

where
\[ J(b,c) = | \{ (x_1,x_2) \in \text{GF}(q)^2 : f(x_i) + bx_i + c = u_i \ (i = 1,2) \} |, \]

and
\[ K(x_1,x_2) = | \{ (b,c) \in \text{GF}(q)^2 : f(x_i) + bx_i + c = u_i \ (i = 1,2) \} |. \]

For \( J(b,c) \), we have
\[
J(b,c) = \begin{cases} 
0, & \{u_1,u_2\} \not\subseteq V(g), \\
1, & \{u_1,u_2\} \subseteq V(g) \text{ and } v(g) = q, \\
4, & \{u_1,u_2\} \subseteq V(g) \text{ and } v(g) = q/2,
\end{cases}
\]

where \( g = f(x) + bx + c \).

Note that
\[
v(f(x) + bx + c) = \begin{cases} q, & b = 0, \\
q/2, & b \neq 0.
\end{cases}
\]

We have
\[
|T| = \sum_{(b,c) \in \text{GF}(q)^2} J(b,c)
= |\{ (b,c) \in \text{GF}(q)^2 : v(f(x) + bx + c) = q \}| + 4|I(u_1,u_2)|
= q + 4|I(u_1,u_2)|.
\]

For \( K(x_1,x_2) \), we have
\[
K(x_1,x_2) = \begin{cases} 
0, & x_1 = x_2, \\
1, & x_1 \neq x_2.
\end{cases}
\]

Thus,
\[
|T| = \sum_{(x_1,x_2) \in \text{GF}(q)^2} K(x_1,x_2) = q(q - 1).
\]

Finally,
\[
I(u_1,u_2) = \frac{1}{4} (q(q - 1) - q) = \frac{q(q - 2)}{4}.
\]

This completes the proof. \( \square \)
Another major result of this paper is the following.

**Theorem 28.** Let \( f(x) \) be an o-polynomial over \( GF(q) \) such that \( |B_{(f,q/2)}| = (q - 1)q \). Then \( B(f,q/2) := (GF(q),B_{(f,q/2)}) \) is a 2-\((q,q/2,q(q - 2)/4)\) design.

**Proof.** It follows from Theorem 13 that \( |B_{(f,q/2)}| = q/2 \) for all \((b,c) \in GF(q)^+ \times GF(q)\). By assumption, all blocks \( B_{(f,b,c)} \) with \((b,c) \in GF(q)^+ \times GF(q)\) are pairwise distinct. The design property then follows from Lemma 27.

Regarding Theorem 28, one basic question is which of the known o-polynomials satisfy \( |B_{(f,q/2)}| = q(q - 1)/2 \). It will be shown later that \( |B_{(f,q/2)}| = 2(q - 1) \) for translation o-monomials \( x^{2^j} \) and their variants \( (ax)^{2^j} \). For other o-polynomials, we have the following conjecture, which is strongly supported by experimental data.

**Conjecture 1.** Let \( f(x) \) be any o-polynomial over \( GF(q) \) such that \( f(x) \neq (ax)^{2^j} \) for all \( a \in GF(q)^+ \) and all \( h \) with \( 1 \leq h < m \) and \( \gcd(h,m) = 1 \). Then \( |B_{(f,q/2)}| = q(q - 1) \).

As pointed out earlier, o-polynomials do not support 3-designs in general with respect to the construction of Section 2. However, if an o-polynomial \( g(x) \) can be expressed as \( (ux + v)^{2^j} + c \), where \( x^c \) is an o-monomial, then \( g(x) \) does support a 3-design. For example, \( g(x) = x^6 + x^4 + x^2 = (x + 1)^5 + 1 \). Since \( x^6 \) is an o-monomial over \( GF(2^m) \), where \( m \) is odd, \( g(x) \) supports a 3-design.

We would make the following comments on 2-designs \( D(f,q/2) \) supported by o-polynomials \( f(x) \) such that \( f(x) \neq (ax + b)^{2^j} + b' \) for all o-polynomials \( y^c \).

1. They are not 3-designs in general. For example, when \( m = 5 \) and \( m = 7 \), the Cherowitzo o-polynomial, Payne o-polynomial, and Subiaco o-polynomial support only 2-designs.
2. The 2-designs \( D(f,q/2) \) from these o-polynomials are not affine-invariant, as their automorphism groups are smaller than the general affine group \( AG_1(GF(q)) \).
3. For example, when \( m = 5 \), the sizes of the automorphism groups of the 2-designs supported by the Cherowitzo o-polynomial, Payne o-polynomial and Subiaco o-polynomial are 160, while \( |AG_1(GF(q))| = 993 \).
4. These 2-designs \( D(f,q/2) \) cannot be isomorphic to the hyperoval 2-designs documented in Theorems 9 and 10 as their parameters do not match.

For the 3-designs \( D(f,q/2) \) supported by o-monomials, we have the following remarks.

1. They are not 4-designs according to Magma experiments.
2. They are affine-invariant, i.e., \( AG_1(GF(q)) \) is a subgroup of their automorphism groups.

Experimental data indicates that their automorphism groups are larger than \( AG_1(GF(q)) \).

For example, when \( m = 5 \) and \( m = 7 \), the automorphism groups of the 3-designs supported by the first Glynn o-monomial, second Glynn o-monomial, and the Segre o-monomial have size \( q(q - 1)m \), while \( |AG_1(GF(q))| = q(q - 1) \). In these two cases, the automorphism groups of these designs are

\[
\Gamma A_1(GF(q)) = \left\{ ax^{2^j} + v : (u,v) \in GF(q)^+ \times GF(q), \ 0 \leq i \leq m - 1 \right\}.
\]

The degree of transitivity of the group \( \Gamma A_1(GF(q)) \) acting on \( GF(q) \) is only 2, and cannot be used to prove the 3-design property of these designs.

When \( m = 5 \), the automorphism group of the design supported by the translation o-monomial \( x^2 \) has size 319979520, while \( |AG_1(GF(q))| = 992 \). This is a special and degenerated case, and will be treated shortly.
3. They are not symmetric designs, as only trivial 3-designs exist. Only the designs supported by the translation o-monomials are quasi-symmetric. Other 3-designs have many block intersection numbers according to experimental data.

**Open Problem 1.** Find the automorphism groups of the designs \( \mathbb{D}(f, q/2) \) supported by the known o-polynomials \( f(x) \).

4.3.2. The parameters of the 3-designs from the translation o-monomial \( x^{2h} \)

Let \( \gcd(h, m) = 1 \). Recall that

\[ J_{2h} = \{ y^{2h} + y : y \in GF(q) \} . \]

Obviously, \( J_{2h} \) is an additive subgroup of \( (GF(q), +) \) with order \( q/2 \).

Let \( (u, v) \in GF(q)^* \times GF(q) \) with \( uJ_{2h} + v = J_{2h} \). Note that \( uJ_{2h} \) is also an additive subgroup of \( (GF(q), +) \) with order \( q/2 \). It then follows that \( J_{2h} + v \) is also an additive subgroup of order \( q/2 \), which forces \( v \in J_{2h} \). Consequently,

\[ uJ_{2h} = J_{2h} . \quad (7) \]

Let \( J_{2h}^* = J_{2h} \setminus \{ 0 \} \). It is known that \( J_{2h}^* \) is a Singer difference set with parameters \((q - 1, (q - 2)/2, (q - 4)/4)\) in the group \((GF(q)^*, \times)\) (see Theorem 31). It then follows from (7) that \( u = 1 \).

Consequently,

\[ \text{Stab}_{AG_1(GF(q))}(J_{2h}) = \{ x + v : v \in J_{2h} \} \]

and

\[ \mu = |\text{Stab}_{AG_1(GF(q))}(J_{2h})| = q/2 . \]

The following then follows from Theorem 23.

**Corollary 29.** Let \( \gcd(h, m) = 1 \) and \( f(x) = x^{2h} \). Then \( \mathbb{D}(f, q/2) := (GF(q), B_{(f,q/2)}) \) is a 3-(\(q, q/2, (q - 4)/4\)) design.

Note that the number of blocks in the design of Corollary 29 is \( 2(q - 1) \). Therefore, it is not a symmetric design. It is also well known that nontrivial symmetric 3-designs do not exist. Below we prove that the 3-design in Corollary 29 is quasi-symmetric.

**Theorem 30.** The 3-design of Corollary 29 has two block intersection numbers 0 and \( q/4 \), and is thus quasi-symmetric.

**Proof.** We prove the conclusion only for odd \( m \), as the proof for the other case is similar. Let notation be the same as before. Since \( m \) is odd, \( J_{2h} \) does not contain 1. In this case the block set becomes

\[ B_{(f,q/2)} = \{ uJ_{2h} + uv : u \in GF(q)^*, v \in GF(2) \} . \]

Let \((u_1, v_1)\) and \((u_2, v_2)\) be two elements in \( GF(q) \times GF(2) \). Define

\[ I = (u_1J_{2h} + u_1v_1) \cap (u_2J_{2h} + u_2v_2) . \]

We now consider the value \(|I|\) by distinguishing among the following cases.

Assume that \((v_1, v_2) = (0, 0)\). Then

\[ I = u_1J_{2h} \cap u_2J_{2h} = \left( u_1J_{2h}^* \cap u_2J_{2h}^* \right) \cup \{0\} . \]
Since $J_{2^h}$ is a $(q-1,(q-2)/2,(q-4)/4)$ Singer difference set in $(\text{GF}(q)^*, \times)$ (see Theorem 31), we have then

$$|I| = \left| \left\{ u_1 J_{2^h} \cap u_2 J_{2^h} \right\} \right| + 1 = \begin{cases} q/4 & \text{if } u_1 \neq u_2, \\ q/2 & \text{if } u_1 = u_2. \end{cases}$$

Assume that $(v_1,v_2) = (0,1)$ or $(v_1,v_2) = (1,0)$. Note that $1 \not\in J_{2^h}$ and $uJ_{2^h}$ is an additive subgroup of $(\text{GF}(q), +)$. It is easily seen that $I = \emptyset$.

Finally, assume that $(v_1,v_2) = (1,1)$. We have then

$$I = u_1 (J_{2^h} + 1) \cap u_2 (J_{2^h} + 1).$$

It is known that $J_{2^h} + 1$ is a $(q-1,q/2,q/4)$ Singer difference set in $(\text{GF}(q)^*, \times)$. We then deduce that $|I| = q/4$ if $u_1 \neq u_2$, and $|I| = q/2$ otherwise. This completes the proof.

The foregoing discussions in this section showed that the $3$-$(q,q/2,\frac{q(q-4)}{8})$ designs from the translation o-monomials $x^{2^h}$ are related to the Singer difference sets with parameters $(q-1,\frac{q-2}{2},\frac{q-4}{4})$. It is very likely that they are isomorphic to the extended designs of the Singer difference sets with parameters $(q-1,\frac{q-2}{2},\frac{q-4}{4})$. This is because every quasi-symmetric $3$-design with the block intersection number $0$ is the extension of a symmetric $2$-design [18]. Anyway, our construction of the quasi-symmetric $3$-designs uses the direct approach $\mathbb{D}(f,q/2)$, and relates the designs to translation hyperovals.

### 4.3.3. Parameters of the $3$-designs from other o-monomials

To determine the $\lambda$ value of the $3$-$(q,q/2,\frac{q(q-4)}{8})$ design $\mathbb{D}(x^e,q/2)$ from an o-monomial other than the translation o-monomials $x^e$, we need determine the size of the stabilizer $\text{Stab}_{\text{AG}_1(\text{GF}(q))}(J_e)$ of $J_e$, both of which were defined in Theorem 25. Experimental data strongly supports the next conjecture.

**Conjecture 2.** Let $x^e$ be an o-monomial, where $e$ is not a power of $2$. Then

$$\text{Stab}_{\text{AG}_1(\text{GF}(q))}(J_e) = \{x\}.$$\[\text{Consequently, the design } \mathbb{D}(x^e,q/2) \text{ has parameters } 3$-$(q,q/2,q(q-4)/8)$.

To settle this conjecture, one may need the following result proved by Maschietti [11].

**Theorem 31.** Let $e$ be a positive integer with $\text{gcd}(e(e-1),q-1) = 1$. Then $x^e$ is an o-monomial if and only if $J_e^* = J_e \setminus \{0\}$ is a $(q-1,\frac{q-2}{2},\frac{q-4}{4})$ difference set in $(\text{GF}(q)^*, \times)$.

Below we prove Conjecture 2 for several o-monomials. Let $J_e$ be defined in (6). Define the following Boolean function $h(x)$ from $\text{GF}(q)$ to $\text{GF}(2)$:

$$h(x) = \begin{cases} 1, & \text{if } x \in J_e, \\ 0, & \text{otherwise}. \end{cases}$$

(8)

To prove Conjecture 2 for several o-monomials, we need the following lemma.
Lemma 32 ([19]). Let \( m \) be odd and \( e = 2^i + 2^j \) with \( 1 \leq i < j \leq m - 1 \). If \( f(x) = x^e \) is an o-polynomial over \( \text{GF}(2^m) \), then

\[
\hat{h}(\beta) = \begin{cases} 
0, & \text{if } \text{Tr}(\beta^i) = 0, \\
\pm 2^\frac{m+i}{2}, & \text{if } \text{Tr}(\beta^i) = 1,
\end{cases}
\]

where \( \hat{h} \) denotes the Walsh transform of \( h \) and

\[
\ell \equiv \frac{e - 1}{e} \quad (\text{mod } (2^m - 1)) .
\]

By Lemma [5] or [12], \( x^i \) is also an o-monomial over \( \text{GF}(q) \). We will make use of this fact shortly below. We now prove the following lemma, which settles Conjecture [2] for several o-monomials over \( \text{GF}(q) \).

Lemma 33. Let \( m \) be odd and \( e = 2^i + 2^j \) with \( 1 \leq i < j \leq m - 1 \). Let \( (b, c) \in \text{GF}(q)^* \times \text{GF}(q) \). If \( f(x) = x^e \) is an o-polynomial over \( \text{GF}(q) \), then

\[
\text{Stab}_{\text{AGL}(\text{GF}(q))}(J_e) = \{ x \},
\]

where \( J_e \) was defined in [2].

Proof. Let \( h(x) \) be defined in [8], which is the characteristic function of the set \( J_e \). Let \( (b, c) \in \text{GF}(q)^* \times \text{GF}(q) \) such that \( h(bx + c) = h(x) \). The desired conclusion is the same as that \( (b, c) = (1, 0) \).

Let \( A = \sum_{y \in \text{GF}(q)} (-1)^{h(x) + h(bx + c)} \). Since \( h(bx + c) = h(x) \), we have \( A = q \). We now compute \( A \) in a different way. Note that

\[
\sum_{\beta \in \text{GF}(q)} (-1)^{\text{Tr}(\beta x + y)} = \begin{cases} 
q & \text{if } x = y, \\
0 & \text{if } x \neq y.
\end{cases}
\]

We have then

\[
qA = \sum_{x, y \in \text{GF}(q)} (-1)^{h(x) + h(by + c)} \sum_{\beta \in \text{GF}(q)} (-1)^{\text{Tr}(\beta x + y)}
= \sum_{\beta \in \text{GF}(q)} \sum_{x, y \in \text{GF}(q)} (-1)^{h(x) + \text{Tr}(\beta x)} \sum_{y \in \text{GF}(q)} (-1)^{h(by + c) + \text{Tr}(\beta y)}
= \sum_{\beta \in \text{GF}(q)} \hat{h}(\beta) \sum_{y \in \text{GF}(q)} (-1)^{h(by + c) + \text{Tr}(\beta(by + c) + \frac{\beta}{y})}
= \sum_{\beta \in \text{GF}(q)} \hat{h}(\beta) \hat{h}(\frac{\beta}{y}) (-1)^{\text{Tr}(\frac{\beta}{y})}.
\]

Since \( A = q \), we then deduce that

\[
q^2 = \sum_{\beta \in \text{GF}(q)} \hat{h}(\beta) \hat{h}(\frac{\beta}{y}) (-1)^{\text{Tr}(\frac{\beta}{y})}.
\]

Using this equation and Lemma [12] below we prove that \( (b, c) = (1, 0) \).
Recall that $x^\ell$ is a permutation of $\mathrm{GF}(q)$, where $\ell$ was defined in \(\text{(3)}\). Suppose that $b \neq 1$. Then $b^\ell \neq 1$. Consequently, the total number of $\bar{\beta}$ in $\mathrm{GF}(q)$ such that $\mathrm{Tr}(\bar{\beta}^\ell) = 1$ and $\mathrm{Tr}((\bar{\beta}/b)^\ell) = 1$ is $2^{m-2}$. It then follows from Lemma\(\text{(5)}\) that
\[
\sum_{\beta \in \mathrm{GF}(q)} h(\bar{\beta}) \hat{h}(\frac{\bar{\beta}}{b}) (-1)^{\mathrm{Tr}(\bar{\beta})} = \sum_{\mathrm{Tr}(\bar{\beta}) = 1, \mathrm{Tr}((\bar{\beta}/b)^\ell) = 1} \hat{h}(\bar{\beta}) \hat{h}(\frac{\bar{\beta}}{b}) (-1)^{\mathrm{Tr}(\bar{\beta})} \\
\leq \sum_{\mathrm{Tr}(\bar{\beta}) = 1, \mathrm{Tr}((\bar{\beta}/b)^\ell) = 1} 2^{\frac{m+1}{2}} 2^\frac{m+1}{2} \times 1 \\
= 2^{m-2} 2^{\frac{m+1}{2}} 2^\frac{m+1}{2} \\
= 2^{2m-1} \\
< q^2,
\]
which is contrary to \(\text{(10)}\). Consequently, we must have $b = 1$. Since $b = 1$, by Lemma\(\text{(5)}\) Equation \(\text{(10)}\) becomes
\[
q^2 = 2^{m+1} \sum_{\mathrm{Tr}(\bar{\beta}) = 1} (-1)^{\mathrm{Tr}(\bar{\beta})_1}, \tag{11}
\]
This equation forces $\mathrm{Tr}(c\bar{\beta}) = 0$ for all the $2^{m-1}$ nonzero elements $\bar{\beta} \in \mathrm{GF}(q)$ such that $\mathrm{Tr}(\bar{\beta}^\ell) = 1$. Note that $\mathrm{Tr}(c x) = 0$. Thus, $\mathrm{Tr}(c x) = 0$ has at least $2^{m-1} + 1$ solutions, which is possible only if $c = 0$. This completes the proof. \hfill \square

The next result follows directly from Theorem \(\text{(25)}\) and Lemma \(\text{(33)}\).

**Corollary 34.** The incidence structure $\mathbb{D}(f,q/2) := (\mathrm{GF}(q), B_{f,q/2})$ is a 3-$(q,q/2,q(q-4)/8)$ design if $f(x) = \text{Segre}(x)$ or $f(x) = \text{Glynnii}(x)$.

It can be easily proved that $\mathbb{D}(f,q/2)$ is isomorphic to $\mathbb{D}(f^{-1},q/2)$ if $f$ is an o-monomial over $\mathrm{GF}(q)$. The conclusion of Corollary \(\text{(34)}\) is also true for the two designs $\mathbb{D}(\text{Segre}^{-1}(x),q/2)$ and $\mathbb{D}(\text{Glynnii}^{-1}(x),q/2)$. Note that Conjecture \(\text{(2)}\) is still open for the o-monomials $\text{Segre}(x)$ and $\text{Glynnii}(x)$.

It is well known that the development of the difference set $J^*_e$ can be extended into a 3-$(q,q/2,q(q-4)/4)$ design. For any o-monomial $x^e$, where $e$ is not a power of 2, the 3-design $\mathbb{D}(x^e,q/2)$ has parameters 3-$(q,q/2,q(q-4)/4)$. Therefore, our 3-designs $\mathbb{D}(x^e,q/2)$ supported by such o-monomials $x^e$ cannot be isomorphic to the extended 3-design of the development of the difference set $J^*_e$. Recall that the translation o-monomials are exceptions.

### 4.3.4. The isomorphism of designs $\mathbb{D}(f,q/2)$ from o-polynomials $f$

First of all, we point out that two equivalent o-polynomials $f$ and $g$ may give two non-isomorphic designs $\mathbb{D}(f,q/2)$ and $\mathbb{D}(g,q/2)$. For example, by Lemma \(\text{(8)}\) the two o-monomials $x^2$ and $x^{e+2}$ are equivalent, but $\mathbb{D}(x^2,q/2)$ and $\mathbb{D}(x^{e+2},q/2)$ are not isomorphic, as $\mathbb{D}(x^2,q/2)$ is a 3-$(q,q/2,(q-4)/4)$ design and $\mathbb{D}(x^{e+2},q/2)$ is a 3-$(q,q/2,q(q-4)/8)$ design. By Lemma \(\text{(8)}\) the two hyperovals $H(x^2)$ and $H(x^{e+2})$ are equivalent, while it can be proved that the two designs $\mathbb{D}(x^2,q/2)$ and $\mathbb{D}(x^{e+2},q/2)$ are isomorphic. Hence, the equivalence of o-polynomials is different from the isomorphism of designs $\mathbb{D}(f,q/2)$ from o-polynomials.

If $f(x) = x^e$ is an o-polynomial, then it is easily seen that $\mathbb{D}(f,q/2)$ and $\mathbb{D}(f^{-1},q/2)$ are isomorphic. But $\mathbb{D}(f,q/2)$ and $\mathbb{D}(f^{-1},q/2)$ may not be isomorphic if $f$ is not a monomial. For example, $\mathbb{D}(\text{Cherowitzo}(x),q/2)$ and $\mathbb{D}(\text{Cherowitzo}^{-1}(x),q/2)$ are not isomorphic when $m = 5$. 


Since it is hard to do a theoretical isomorphy classification of designs \( D(f, q/2) \) from \( \alpha \)-polynomials \( f \), we have done an isomorphy classification for the following set of \( \alpha \)-polynomials for the case \( m = 5 \) with Magma:

\[
\{ \text{Segre}(x), \text{Segre}(x), \text{Glynni}(x), \text{Glynnii}(x), \text{Cherowitzo}(x), \\
\text{Cherowitzo}(x), \text{Cherowitzo}^{-1}(x), \text{Payne}(x), \text{Subiaco}_1(x) \}.
\]

Their designs \( D(f, q/2) \) for \( m = 5 \) are pairwise not isomorphic, except that \( D(\text{Segre}(x), q/2) \) and \( D(\text{Glynni}(x), q/2) \) are isomorphic. But \( D(\text{Segre}(x), q/2) \) and \( D(\text{Glynni}(x), q/2) \) are not isomorphic when \( m = 7 \). Hence, the 3-designs of these \( \alpha \)-monomials are pairwise not isomorphic in general.

5. Designs from special polynomials over \( GF(q) \) for odd \( q \)

Let \( q \) be odd throughout this section. Theorem \([\text{I}]\) says that any permutation monomial \( x^e \) over \( GF(q) \) supports 2-designs. Since \( x^e \) is a permutation, \( e \) must be odd. Let \( d = \gcd(e - 1, q - 1) \). Then \( d \geq 2 \).

An interesting case is that \( d = \gcd(e - 1, q - 1) = 2 \). In this case, it can be shown that there are at most two block sizes \( |\{x^e + y : y \in GF(q)\}| \) and \( |\{x^e + \alpha y : y \in GF(q)\}| \), where \( \alpha \) is a generator of \( GF(q)^* \). In this case, \( x^e \) supports at most two nontrivial 2-designs with different block sizes.

Motivated by the foregoing discussions, we call a monomial \( x^e \) over \( GF(q) \) for odd \( q \) a \( t \)-monomial (i.e., twin design monomial) if \( \gcd(e, q - 1) = 1 \) and \( |\{x^e + bx : x \in GF(q)\}| \) takes only two distinct values for all \( b \) in \( GF(q)^* \).

**Theorem 35.** Let \( p \) be odd and \( m \geq 2 \). Below is a list of monomials \( x^e \) over \( GF(p^m) \) such that \( \gcd(e, p^m - 1) = 1 \) and \( \gcd(e - 1, p^m - 1) = 2 \).

- \( e = 3 \) and \( p = 3 \).
- \( e = 3 \), \( p \equiv 5 \pmod{6} \) and \( m \) is odd.
- \( e = 5 \), \( p \in \{3, 7\} \) and \( m \) is odd.
- \( e = p^m - 2 \).
- \( e = (p^m - 3)/2 \), \( p \equiv 1 \pmod{4} \) and \( m \) is even, or \( p \equiv 3 \pmod{4} \).
- \( e = p^m - p - 1 \) and \( m \) is odd.

**Proof.** It is straightforward to prove the desired conclusions for the values \( e \). The details are left to the reader. \( \blacksquare \)

These are candidates of \( t \)-monomials. But it may be technical to prove that they are \( t \)-monomials. In fact, the Dickson permutation monomial \( x^5 \) over \( GF(3^5) \) is in fact a \( d \)-monomial, as the block size \( |B_{(x^5, b, c)}| \) is 153 for all \( (b, c) \in GF(3^5)^* \times GF(3^5) \).

Monomials \( x^e \) with \( \gcd(e, q - 1) \neq 1 \) may also support 2-designs. For example, \( x^2 \) over \( GF(3^m) \) supports a 2-\((3^m, (3^m + 1)/2, (3^m + 1)/4)\) symmetric design, which is the development of the difference set defined by all the squares in \( GF(3^m) \). We have also the following conjecture.
Conjecture 3. Let \( m \geq 3 \) be odd. Define \( k_m \) by the recurrence relation

\[
k_m = \frac{3^m + 1}{2} + 3^{m-1} - 3k_{m-2}
\]

with initial value \( k_1 = 2 \). Then

\[
|B_f(k_1, b, c)| = \left\{ \begin{array}{ll}
\frac{3^m+1}{2k_m} & \text{with } 3^m \text{ times,} \\
\frac{3^{m-1}}{2k_m} & \text{with } (3^m - 1)3^m \text{ times}
\end{array} \right.
\]

and

\[
|B_f((3^m+1)/2)| = 3^m; \quad |B_f(k_1, k_m)| = \frac{3^m(3^m - 1)}{2}.
\]

Further,

- \( (GF(3^m), B_f((3^m+1)/2)) \) is a \( 2 \cdot (3^m, (3^m + 1)/2, (3^m + 1)/4) \) symmetric design, which is the development of the difference set consisting of all the squares in \( GF(3^m) \); and
- \( (GF(3^m), B_f(k_1, k_m)) \) is a \( 2 \cdot (3^m, k_m, k_m(k_m - 1)/2) \) design.

If Conjecture 3 is true, the design \( (GF(3^m), B_f(k_1, k_m)) \) would be interesting. The following is a fundamental open problem.

Open Problem 2. Is there a polynomial \( f(x) \) over \( GF(q) \) with odd \( q \) such that \( D(f, k) \) is a 3-design for some \( k \)?

6. An extended construction of \( t \)-designs from polynomials

In the construction of designs introduced in Section 2, not every polynomial \( f \) supports a 2-design \( D(f, k) \). Only special polynomials over \( GF(q) \) can support a 2-design. In this section, we outline an extended construction of 2-designs from polynomials over finite fields \( GF(q) \).

Let \( f(x) \) be a polynomial over \( GF(q) \). For each \( (a, b, c) \in GF(q)^3 \), we define

\[
\hat{B}_{(f, a, b, c)} = \{ af(x) + bx + c : x \in GF(q) \}.
\]

Let \( k \) be any integer with \( 2 \leq k \leq q \). Define

\[
\hat{B}_{(f, k)} = \{ \hat{B}_{(f, a, b, c)} : |\hat{B}_{(f, a, b, c)}| = k, \ (a, b, c) \in GF(q)^3 \}. \tag{12}
\]

We have then the following result.

Theorem 36. Let notation be the same as before. If \( |\hat{B}_{(f, k)}| > 1 \), then the incidence structure \( \hat{D}(f, k) = (GF(q), \hat{B}_{(f, k)}) \) is a \( 2 \cdot (q^m, k, \lambda) \) design for some \( \lambda \).

Proof. The desired conclusion follows from the facts that the general affine group \( GA_1(GF(q)) \) is a subgroup of the automorphism group of the incidence structure \( \hat{D}(f, k) \), \( GA_1(GF(q)) \) fixes \( \hat{B}_{(f, k)} \), and \( GA_1(GF(q)) \) acts on \( GF(q) \) doubly transitively.
Theorem 36 tells us that almost every polynomial over GF(q) gives 2-designs under this extended construction \( \mathcal{D}(f, k) \). This fact makes the extended construction \( \mathcal{D}(f, k) \) less interesting than the previous one \( \mathcal{D}(f, k) \), though many 2-designs with nice parameters may be obtained by choosing special types of polynomials. However, it would be very nice if this extended construction \( \mathcal{D}(f, k) \) can produce t-designs with \( t \geq 3 \).

It is easily seen that for any o-monomial \( x^r \) over GF(2^m), we have \( \mathcal{D}(x^r, 2^{m-1}) = \mathcal{D}(x^r, 2^{m-1}) \). Hence, it is indeed a 3-design, but was already covered by the construction \( \mathcal{D}(x^r, 2^{m-1}) \).

Recall that \( \mathcal{D}(f, 2^{m-1}) \) is only a 2-design if \( f \) is the Cherowitzo or Payne trinomial. What will happen if we plug the Cherowitzo and Payne trinomials into this extended construction? Regarding this question, we have the following.

**Theorem 37.** Let \( m \geq 4 \) and \( q = 2^m \). Then the incidence structure \( \mathcal{D}(f, q/2) = (\text{GF}(q), \mathcal{B}_{(f,q/2)}) \) is a \( 3(q,q/2, (q-4)(q-1)/8 \) design if \( f \) is an o-polynomial over GF(q) with \( |\mathcal{B}_{(f,q/2)}| = q(q-1)^2 \).

**Proof.** Lemmas 23 and 24 can be modified into a proof of the desired result. The details are omitted. \( \square \)

Theorem 37 is valuable only when there is an o-polynomial over GF(q) with \( |\mathcal{B}_{(f,k)}| = q(q-1)^2 \). In fact, we have the following conjecture.

**Conjecture 4.** Let \( m \geq 5 \) be odd and \( q = 2^m \). Let \( f(x) \) be an o-polynomial over GF(q) such that \( f(x) \neq (ax + b)^2 + b' \) for all o-monomials \( y^r \) and all \( (a, b) \in \text{GF}(q)^2 \). Then \( |\mathcal{B}_{(f,q/2)}| = q(q-1)^2 \).

It might be hard to settle Conjecture 4 in general. But it is possible to prove the conjecture for the Cherowitzo, Payne and Subiaco o-polynomials. The reader is invited to attack this conjecture.

We inform the reader that Conjecture 4 is indeed true for the Cherowitzo trinomial, Payne trinomial and Subiaco polynomials for \( m \in \{5, 7, 9\} \) according to Magma experimental data. Hence, 3-designs have been indeed obtained from this extended construction \( \mathcal{D}(f, q/2) \) with o-polynomials introduced in this section. Recall that \( \mathcal{D}(f, q/2) \) is always a 2-design for any o-polynomial \( f \) over GF(q) by Theorem 37 and a 3-design for any o-monomial over GF(q), where \( q = 2^m \).

### 7. Summary and concluding remarks

The main contributions of this paper are the following:

1. The first one is the two general constructions of t-designs with polynomials over finite fields documented in Sections 2 and 3. Many types of polynomials may be plugged into the two constructions for obtaining t-designs with different parameters.
2. The second is the application of o-polynomials in t-designs under the frameworks of the two general constructions. The first construction has produced infinite families of 3-designs from o-monomials over GF(2^m), and infinite families of 2-designs from o-polynomials over GF(2^m). The second construction has given 2-designs and also 3-designs from o-polynomials over GF(2^m).

Some of the 2-designs obtained in this paper are affine-invariant, while other 2-designs are not affine-invariant and thus interesting. Some 3-designs presented in this paper are indeed affine-invariant, but their automorphism groups are only doubly transitive on their point sets. So the
3-design property of these designs had to be proved with direct approaches. This makes these 3-designs very special.

Since the two constructions of \(t\)-designs are quite general, a lot of work can be done in this direction. Several open problems and conjectures were presented in this paper. The reader is cordially invited to join the venture into the topic of this paper.

References

[1] E. F. Assmus Jr., J. D. Key, Designs and Their Codes, Cambridge University Press, Cambridge, 1992.
[2] T. Beth, D. Jungnickel, H. Lenz, Design Theory, Cambridge University Press, Cambridge, 1999.
[3] W. Cherowitzo, Hyperovals in Desarguesian planes of even order, Annals of Discrete Math. 37 (1988) 87–94.
[4] W. Cherowitzo, Hyperovals in Desarguesian planes: an update, Disc. Math. 155 (1996) 31–38.
[5] W. Cherowitzo, T. Penttila, I. Pinneri, G. F. Royle, Flocks and ovals, Geometriae Dedicata 60 (1996) 17–37.
[6] C. Ding, J. Yuan, A family of skew Hadamard difference sets, J. Combinatorial Theory Ser. A, 113 (2006) 1526–1535.
[7] C. Ding, P. Yuan, Five constructions of permutation polynomials over \(\text{GF}(q^2)\), unpublished manuscript, 2015, http://arxiv.org/abs/1511.00322.
[8] D. G. Glynn, Two new sequences of ovals in finite Desarguesian planes of even order, in: L. R. A. Casse (Ed.), Combinatorial Mathematics X, Lecture Notes in Mathematics 1036, Heidelberg, Springer Verlag, 1983, pp. 217–229.
[9] D. G. Glynn, A condition for the existence of ovals in \(\text{PG}(2,q)\), \(q\) even, Geometriae Dedicata 32 (1989), pp. 247–252.
[10] W. A. Jackson, A characterisation of Hadamard designs with \(\text{SL}(2,q)\) acting transitively, Geom. Dedicata 46 (1993) 197–206.
[11] A. Maschietti, Difference set and hyperovals, Des. Codes Cryptogr. 14 (1998) 89–98.
[12] R. Lidl, H. Niederreiter, Finite Fields, 2nd Edition, Cambridge University Press, Cambridge, 1997.
[13] S. E. Payne, A new infinite family of generalized quadrangles, Congressus Numerantium 49 (1985) 115–128.
[14] A. Pott, Finite Geometry and Character Theory, Lecture Notes in Computer Science 1601, Springer, Berlin, 1995.
[15] B. Segre, Sui k-archi nei piani finiti di caratteristica 2, Revue de Math. Pure Appl. 2 (1957) 289–300.
[16] B. Segre, Ovali e curve nei piani di Galois di caratteristica due, Atti Accad. Naz. Lincei Rend. 32 (1962), pp. 785–790.
[17] B. Segre, U. Bartocci, Ovali ed alte curve nei piani di Galois di caratteristica due, Acta Arith. 18 (1971) 423–449.
[18] M. S. Shrikhande, Quasi-symmetric designs, in: C. J. Colbourn, and J. H. Dinitz, (Eds.), Handbook of Combinatorial Designs, 2nd Edition, CRC Press, New York, 2007, pp. 578–582.
[19] Q. Xiang, On balanced binary sequences with two-level autocorrelation functions, IEEE Trans. Inf. Theory 44 (1998) 3153–3156.
[20] Q. Xiang, Recent results on difference sets with classical parameters, in: A. Pott et al. (Eds.), Difference Sets, Sequences and their Correlation Properties, Kluwer, 1999, pp. 419–437.