Viscosity, Reversibility, Chaotic Hypothesis, Fluctuation Theorem and Lyapunov Pairing

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Abstract: Incompressible fluid equations are studied with UV cut-off and in periodic boundary conditions. Properties of the resulting ODEs holding uniformly in the cut-off are considered and, in particular, are conjectured to be equivalent to properties of other time reversible equations. Reversible equations with the same regularization and describing equivalently the fluid, and the fluctuations of large classes of observables, are examined in the context of the “Chaotic Hypothesis”, “Axiom C” and the “Fluctuation Theorem”.

I. ON THE EQUATIONS

The incompressible Euler equation, denoted E, in a periodic container $\mathcal{T}^d = [0, 2\pi]^d$, $d = 2, 3$, for a smooth velocity field $u(x), x \in \mathcal{T}^d$ is:

$$\dot{u}(x) = -(u(x) \cdot \partial_x)u(x) - \partial_x P(x),$$

$$\partial_x \cdot u(x) = 0, \int_{\mathcal{T}^d} dx \ u(x) = 0$$

(1.1)

where $P = -\sum_{i=1}^{d} \Delta^{-1}(\partial_i u_j \partial_j u_i)$ is the ’pressure’ and $\Delta$ -Laplace operator.

It is also useful to consider the $E$ equations from the “Lagrangian viewpoint”: a configuration of the fluid is described by assigning the displacement $x = q_\xi$ of a fluid element, from the reference position $\xi \in \mathcal{T}^d$, and the velocity $\dot{q}_\xi$ of the same fluid element. So the state of the fluid is $(q, \dot{q})$ where $q$ is a smooth map of $\mathcal{T}^d$ to itself and $\dot{q}$ is a smooth vector field on $\mathcal{T}^d$ with $\int_{\mathcal{T}^d} d\xi \ |\dot{q}| = 0$. Denote $\mathcal{F}$ the space of the dynamical configurations $(q, \dot{q}) \in Diff(\mathcal{T}^d) \times Lin(\mathcal{T}^d) = \mathcal{F}$ where $Diff(\mathcal{T}^d)$ is the set of diffeomorphisms of $\mathcal{T}^d$ and $Lin(\mathcal{T}^d)$ the space of the vector fields on $\mathcal{T}^d$.

Actually we concentrate on the subspace of $(q, \dot{q}) \in (SDiff(\mathcal{T}^d) \times SLin(\mathcal{T}^d)) \cong SF \subset \mathcal{F}$ where the evolution of an incompressible fluid takes place: $SDiff(\mathcal{T}^d)$ being the volume preserving diffeomorphisms and $SLin(\mathcal{T}^d)$ the 0-divergence vector fields.

A $(q, \dot{q}) = \{q_\xi, \dot{q}_\xi\}_{\xi \in \mathcal{T}^d} \in \mathcal{F}$ should be regarded as a set of Lagrangian coordinates labeled by $\xi \in \mathcal{T}^d$. And the equations Eq.(1.1) can be derived from a Hamiltonian in canonical coordinates $(q, p) \in \mathcal{F}$ which is quadratic in $p$ and which generates motions in $\mathcal{F}$ evolving leaving $SF$ invariant. Therefore the motion in $\mathcal{F}$ is a “geodesic motion” (i.e. a motion generated by a Hamiltonian quadratic in the momenta).

A key remark is that the motions that follow initial data in $SF$ remain, as long as the evolution is defined and smooth\(^1\) in $SF$, \(\mathbb{R}^{5d}\), i.e. $SF$ is an invariant surface in $\mathcal{F}$. And the equations of motion that $H$ generates can be written (using incompressibility of $(q, p) \in SF$) as:

$$\dot{q}_\xi = -\partial_{q_\xi} Q(q, p) \xi, \quad \dot{p}_\xi = p_\xi$$

(1.2)

where $Q_\xi$ is $q_\xi$-dependent and quadratic in $p$, see Appendix A.

Since $\dot{q}_\xi = \partial_\xi p_\xi + (p_\xi \cdot \partial_\xi p_\xi)$, setting $x = q_\xi, u(x) = p_\xi$ and $P(x) = \partial_\xi Q(q, p) \xi$, the equations become:

$$\dot{q}_\xi = p_\xi, \quad \partial_\xi u(x) + ((u \cdot \partial) u)(x) = -\partial P(x)$$

(1.3)

with $\partial \cdot u = 0$, and $P$ as above. The Lagrangian form of Euler’s equations, Eq.(1.2) or (1.3), will be called $E^t$. See Appendix A.

The above “geodesic” formulation of $E, E^t$ will be used to exhibit symmetry properties of Euler’s equation which may be relevant also for the IN (irreversible Navier-Stokes) equations:

$$\partial_t u(x) + ((u \cdot \partial) u)(x) = \nu \Delta u(x) - \partial P(x) + f(x)$$

(1.4)

with the conditions $\partial \cdot u = 0, \int_{\mathcal{T}^d} u(x) = 0$.

II. ULTRAVIOLET REGULARIZATION

Here we study the regularized version, see below, of $E$ or IN, Eq.(1.1), (1.4), obtained by requiring that the Fourier’s transform $u_k$ of $u$ does not vanish only for modes $k$ with components $\leq N$.

We shall focus on properties of the solutions which hold uniformly in the cut-off $N$: the space of such $u_k$‘s with 0 divergence ($\partial \cdot u = 0$) and 0 average ($\int_{\mathcal{T}^d} u(x) = 0$) will be denoted $C_N$.

Therefore the equation in dimension $d = 2, 3$ is expressed in terms of complex scalars $u_{\beta, k} = \overline{u}_{\beta, -k}, \beta = 1, \ldots, d, k \in \mathbb{Z}^d, |k| \leq N$: thus the number of real coordinates is $N = 4N(N + 1)$ in 2D and $N = 2(4N^3 + 6N^2 + 3N)$ in 3D and $\mathbb{N}$ will be the dimension of the phase space $C_N$.

For instance in 3D choose, for each $k \neq 0$, two unit vectors $e_\beta(k) = -e_\beta(-k), \beta = 1, 2, 3$, mutually orthogonal and orthogonal to $k$; data are combined to form a velocity field:

$$u(x) = \sum_{0 < |k| \leq N} u_k e^{-ik \cdot x}, \ k = (k_\beta)_{\beta=1,2,3}$$

$$u_k = \sum_{\beta=1,2} i u_{\beta, k} e_\beta(k), \ k \cdot e_\beta(k) = 0$$

(2.1)

with $|k| = \max_j |k_j|, u_{-k,j} = \overline{u}_{k,j}$.

\(^1\)This might be a very short time.
Define \( D_{k_i,k_2,k_3}^{\beta_1,\beta_2,\beta_3} = - (e_{\beta_1}(k_1) \cdot e_{\beta_2}(k_2)) (\partial_{x_{\beta_3}} u) \). Introduce also forcing \( f = \sum_{k,\beta} i f_{k,\beta} e_{\beta}(k) e^{-i k \cdot x} \) and viscosity in the form \( -\nu k^2 u_{k} \).

The IN equations Eq. (1.4) become, if \( k^2 = k_1^2 + k_2^2 + k_3^2 \) and the sum is restricted to \( |k|, |k_1|, |k_2| \leq N \):

\[
\dot{u}_{\beta,k} = \sum_{k_1,k_2 \leq N} D_{k_1,k_2,k_3}^{\beta_1,\beta_2,\beta_3} u_{\beta_1,k_1} u_{\beta_2,k_2} - \nu k^2 u_{\beta,k} + f_{\beta,k} \tag{2.2}
\]

which will define the regularized IN equation.

The 2D case is similar but simpler: no need for the labels \( \beta \) and \( e(k) \) can be taken \( k \leq N \).

The coefficients \( D_{k_1,k_2,k_3}^{\beta_1,\beta_2,\beta_3} \) can be used to check that if \( \nu = 0, f = 0 \) then for all \( u \in \mathcal{C}_N \)

\[
dt \int_{\mathcal{T}^d} u(x)^2 dx = 0, \quad dt \int_{\mathcal{T}^d} u(x) \cdot (\nabla \wedge u(x)) dx = 0 \tag{2.3}
\]

As is well known, the first of Eq. (2.3) leads to the a priori, \( N \)-independent bounds for the solutions of the E and IN equations:

\[
\|u^{X,N}(t)\|_2^2 \leq \max(E_0, |\frac{F_0}{\nu}|^2), \quad X = E, IN \tag{2.4}
\]

satisfied (for all \( X \)) by solutions \( t \rightarrow u^{X,N}(t) \) of \( S_1 X,N u \), in terms of \( E_0 = \|u(0)\|^2 = \sum |e_{\beta,k}|^2 \) and \( F_0 = \|f\|_2 \).

From now on the cut-off \( N \) will be kept constant and the solution of the equations will be denoted simply \( S_t u \) dropping the \( X,N \) as superscript of the solution map \( S_t \). Only when not clear from the context a superscript \( E \) or \( IN \) or a label \( N \) will be added to clarify whether reference is made to the evolution, or to its properties, following \( E \) or \( IN \) equation with cut-off \( N \).

By scaling, the equation can be written in a fully dimensionless form in which \( \|f\|_2 = 1 \).

The Jacobian of the Euler flow \( S_t E^N u \) with UV cut-off \( N \) is more easily written, without using the Fourier’s transform representation of \( u \), directly from Eq. (1.3) and, see Appendix B is the sum of the following convolution operator on \( (\phi_j(x))_{j=1}^d = \varphi \in L_2(T^d) \times R^d \):

\[
\frac{\partial u_i(x)}{\partial y_j(y)} = -\mathcal{P}(x-y) \delta_{ij} u_i(y), \quad i,j = 1,..,d, \tag{2.5}
\]

plus an antisymmetric operator on the same space; here \( \mathcal{P} \) is the orthogonal projection, in the \( L_2(T^d) \times R^d \) metric, on the divergenceless fields \( \varphi \).

The operator acts on the fields \( \varphi \) with 0 divergence (this is used in deriving Eq. (2.6) to discard contributions

\[
\text{that vanish on the divergenceless fields } \varphi: \text{ and in the end its symmetric part is } \mathcal{P} \text{ times the multiplication operator, on 0-divergence fields } (\varphi_j(x))_{j=1}^d = \varphi \in L_2(T^d) \times R^d, \text{ by:}
\]

\[
W_{i,j}(x) = \frac{1}{2} (\partial_x u_i(x) + \partial_y u_j(x)) \tag{2.6}
\]

i.e. \( \mathcal{P} \) times the operator \( (J \varphi)(i) = \sum_{j=1}^d W_{i,j}(x) \varphi_j(x) \).

Introducing also viscosity (and forcing, which however does not contribute) Eq. (2.6) immediately leads to express the symmetric part of the Jacobian of the regularized IN, irreversible Navier-Stokes, as \( \mathcal{P} \) times the \( J_{i,j} = \nu \delta_{i,j} \Delta + W_{i,j} \).

Defining, for \( d = 2, 3 \), \( w(x)^2 = \frac{1}{\nu^2} \sum_{j=1}^d W_{i,j}(x)^2 \)

the inequality \( J^\nu \leq \nu \Delta + w(x) \), derived in \( [33, 40] \) for the nonregularized IN equation, remains valid for the regularized one and leads to the bound, \( [33, 40] \): Theorem: the sum of the time averages of the first \( p \) eigenvalues of the (Schrödinger operator) \( \nu \Delta + w(x) \) yields an upper bound to the sum of the first \( p \) Lyapunov exponents (of any invariant distribution on \( F \)) of the flow \( S_t \).

Remark: Lyapunov exponents depend on the invariant distribution used to select data: here they will be defined as the time averages of the eigenvalues of the symmetric part of the Jacobian of the evolution equation, \( [41] \), the \( u \)-dependent non averaged eigenvalues will be called local Lyapunov exponents.

## III. REVERSIBLE EQUATIONS

The theory of nonequilibrium fluctuations has led to studying phenomena via equations considered equivalent (at least for some of the purposes of interest) to the “fundamental” ones.

Thus new non-Newtonian forces have been added to systems of particles claiming that the values of important quantities would have the same values as those implied by the fundamental equations, even in cases in which the modification was drastic: with the advantage, in several cases, of greatly facilitating simulations, \( [17, 32, 37] \).

At the same time the idea that modification of the equations would not affect, at least in some important cases, most of their predictions arose in other domains: it appeared for instance, in \( [16, 10] \), to show that the Navier-Stokes (IN above) equation could be modified, into new reversible equations, still remaining consistent with selected predictions of the Obukov-Kolmogorov theory.

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2The symmetries of \( D \) arise from the identities \( \int ((u \cdot \partial) u) \cdot u = 0 \) and \( \int (u \cdot \partial u) \cdot (\partial \wedge u) = 0 \), by integration by parts.

3Other projections could be used: this is convenient to follow the analysis in \( [40] \).

4By fundamental I mean here equations based on Newton’s principles as theoretical base for studying quantities like transport coefficients e.g. in a shear flow model, \( [15, 10] \), or properties of heat conduction, \( [4, 13] \). More generally principles of Quantum Mechanics also are modified to make various problems accessible to simulations, \( [4] \).
In [20, 21] an attempt was presented to link empirical equivalence observations to the well established theory of the equivalence of ensembles in Statistical Mechanics. And a paradigmatic example was the NS incompressible fluid in the simple case of periodic boundary conditions and forcing acting at large scale (i.e. with a force with Fourier’s coefficients non zero only for modes $|k| < K_f$ for some $K_f$). In this case new equation proposed was:

$$\dot{u} = -(u \cdot \triangledown)u + \alpha(u) \Delta u + f - \partial P$$

(3.1)

with the multiplier $\alpha(u)$ so defined that a “global” quantity becomes a constant of motion: for instance the energy $E(u) = \sum_{k,i} \frac{1}{2} |u_{k,i}|^2$ or the enstrophy $D(u) = \sum_k k^2 |u_{k,i}|^2$.

In Statistical Mechanics global conserved quantities define the ensembles, which are collections of stationary probability distributions on phase space giving the statistical fluctuations of observables in the ‘equilibrium states’.

The main property being that the “local” observables have in each state properties independent on the special global quantity that defines a given state, at least in some limiting situation (like in the “thermodynamic limit”, in which the container volume $\rightarrow \infty$).

Distinction between local and global observables is essential: in particle systems global quantities can be the total energy (microcanonical ensemble) or the total kinetic energy (isokinetic ensemble) or the total potential energy (thermodynamic ensemble) or the total kinetic energy (canonical ensemble).

Local observables: in such systems, are observables $O_k(q,p)$ whose value depends on the configuration of positions and velocities of particles located, at the time of observation, in a region $V_0$ of finite size compared to the total volume $V$ of the system.

And local observables, in most systems and in stationary states, exhibit evolving statistical properties of the values of $O_k$, which have a limit as $V \rightarrow \infty$, for all $V_0$, i.e. become independent of the “volume cut-off $V$”.

Local and global observables arise often also in connection with the theory of many systems whose evolution is controlled by differential equations not arising directly from fundamental equations (like the Lorenz96 model, 29).

In the next section the example of the fluid equations, always in presence of a UV cut-off $N$, in Eq. (14) (22) will be analyzed choosing viscosity force as $\nu \Delta u$ or $\alpha(u) \Delta u$ as in Eq. (31) with:

$$\alpha(u) = \frac{\sum_k f_k \cdot \bar{u}_k}{\sum_k k^2 |u_{k,i}|^2}$$

(3.2)

where, with $D$ introduced in Eq. (22):

$$\Lambda(u) = \sum_{k_1 + k_2 + k_3 = 0} D^{k_1,k_2,k_3} k^2 u_{k_1,i} u_{k_2,j} u_{k_3,r}$$

(3.3)

With the choice (1) the equation Eq. (3.1) generates evolutions conserving exactly the energy $E(u)$, considered in [17], while with the choice (2) evolution conserves exactly the enstrophy $D(u)$, considered in [22, 20]. But remark that $\Lambda = 0$ in 2D, implied by Eq. (2.3).

IV. ENSEMBLES

In the case of the fluid equations in Eq. (1.4) and Eq. (5.1) define:

Local observables: are functions of the velocity fields $u$ which depend on the Fourier’s modes $u_k$ with $|k| < K$ with $K \ll N$, i.e. of finite size compared to the maximum value $N$ (UV cut-off) used to make the equations meaningful.

It can be said that local observables refer to measurements that can be effected looking at large scale properties of the fluid.

While in Statistical Mechanics locality refers to events in regions in position space small with respect to the volume cut-off $V$, in fluid mechanics locality refers to events measurable in regions in Fourier’s space small compared to the ultraviolet cut-off $N$. Hence locality has a physical meaning when the aim of the theory is to study properties of “large scale” observables (i.e. expressible in terms of Fourier’s components $u_k$ of the velocity fields $u$ with $|k|^{-1}$ of the order of the linear size of the container).

Hereafter consider Eq. (5.1) and Eq. (1.4) with $f$ fixed and with only few Fourier’s components non zero, say $|k| < K_f$ fixed, and $\|f\|_2 = 1$: such $f$ will be called a “large scale forcing”.

Having named “22” “irreversible” IN, consistently the Eq. (5.2) will be named “reversible” RE in case (1), or “reversible” RN in case (2). Properties of RE, RN are

(1) they generate reversible evolutions $u \rightarrow S_t u$: i.e. if $\Pi u = -u$ is the “time reversal” then $S_{-t} = S_t$.

The latter value $N$, “ultraviolet cut-off”, is certainly necessary in 3D E, [10], just to make sense of the equations, and “might” be necessary in 3D IN, [13].
RE evolutions conserve exactly energy \( E = \mathcal{E}(\mathbf{u}) \) and RN conserve exactly enstrophy \( D = D(\mathbf{u}) \).

Stationary distributions are usually associated with chaotic evolutions: therefore the multipliers \( \alpha(\mathbf{u}(t)) \) in Eq.\( \text{(4.1)} \) should show, at large \( E \) or \( D \), chaotic fluctuations and behave effectively as constants: this leads to several “equivalence conjectures”.

For clarity we reintroduce a label \( N \) as a reminder that all quantities considered so far were defined in presence of a UV cut-off \( N \) and to discuss variations of \( N \).

Collect the invariant (i.e. stationary) distributions for IN,RE,RN and denote the collections \( \mathcal{E}^IN \), \( \mathcal{E}^RE \), \( \mathcal{E}^RN \) respectively: we call each such collection an ensemble.

The stationary distributions are parameterized by the viscosity \( \nu \) in \( \mathcal{E}^IN \) or by the energy \( E \) in the \( \mathcal{E}^RE \) or by the enstrophy \( D \) in \( \mathcal{E}^RN \).

Denoting as \( \mu^IN,\nu \), \( \mu^RE,\nu \), \( \mu^RN,D \) the stationary distributions, respectively, in the ensembles \( \mathcal{E}^IN \), \( \mathcal{E}^RE \), \( \mathcal{E}^RN \), we shall try to establish a correspondence between the elements \( \mu^IN,\nu \), \( \mu^RE,\nu \), \( \mu^RN,D \) so that corresponding distributions can be called “equivalent” in the sense discussed below.

To fix the ideas we focus first on the correspondence between the distributions in \( \mathcal{E}^IN \) and \( \mathcal{E}^RN \): the simplest situation arises when above equations, for each \( \nu \) small or \( D \) large, admit a unique stable invariant distribution, i.e. a unique “natural stationary distribution” in the sense of [39].

For an observable \( O(\mathbf{u}) \) define \( \langle O \rangle^IN,\nu \equiv \mu^IN,\nu(O) \), \( \langle O \rangle^RN,D \equiv \mu^RN,D(O) \) the respective time averages of \( O(\mathbf{u}(t)) \) observed under the (\( N \)-regularized) \( IN \) and \( RN \) evolutions.

Define also the work per unit time done by the forcing:

\[
L(\mathbf{u}) = \int_{T^d} f(x) \cdot \mathbf{u}(x) \frac{dx}{(2\pi)^d} = \sum_{k} \mathbf{k} \cdot \mathbf{m}_k. \tag{4.1}
\]

So the average work per unit time in the stationary states with parameters \( \nu \) or \( D \) of the ensembles \( \mathcal{E}^IN \), \( \mathcal{E}^RN \) is, respectively, \( \langle L \rangle^IN,\nu \equiv \mu^IN,\nu(L) \) or \( \langle L \rangle^RN,D \equiv \mu^RN,D(L) \).

Given \( \mu^RN,D, \mu^IN,\nu \): define \( \mu^IN,\nu \) to be correspondent to \( \mu^RN,D \), denote this by \( \mu^IN,\nu \sim \mu^RN,D \), if the time average of the enstrophy is equal in the two distributions:

\[
\langle D \rangle^IN,\nu = D. \tag{4.2}
\]

The natural distributions, see footnote [11] are associated with chaotic evolutions: therefore the multipliers \( \alpha(\mathbf{u}(t)) \) should show, at large \( D \), chaotic fluctuations and behave effectively as constants equal to their average.

Hence the proposal, [20, 21]: for an observable \( O(\mathbf{u}) \) define \( \langle O \rangle^N_{\nu} \equiv \mu^IN,\nu(O) \), \( \langle O \rangle^D_{\nu} \equiv \mu^RN,D(O) \) the respective time averages of \( O(\mathbf{u}(t)) \) observed under the IN and RN evolutions; then:

Conjecture 1: Under the equivalence condition Eq.\( \text{(4.2)} \), equal average enstrophy, if \( O(\mathbf{u}) \) is an observable, then:

\[
\lim_{\nu \to 0} \langle O \rangle^N_{\nu} = \lim_{\nu \to 0} \langle O \rangle^D_{\nu} \tag{4.3}
\]

The collection of stationary distributions \( \mu \in \mathcal{E}^IN \) can be assimilated to the distributions of Statistical Mechanics canonical ensemble and the distributions \( \mu \in \mathcal{E}^RN \) can be assimilated to the distributions of the microcanonical ensemble. The regularization \( N \) plays the role of the volume and the friction \( \nu \) that of temperature, the enstrophy that of energy.

So there is ’some’ similarity between the equilibrium states equivalence in Statistical Mechanics and the equivalence proposed by the conjecture 1 about averages observed following the two different evolutions \( IN \) and \( RN \), under the condition of equal average enstrophy.

V. ENSEMBLES IN FLUIDS AND STATISTICAL MECHANICS

However the need to consider the limit as \( \nu \to 0 \) in Eq.\( \text{(4.2)} \) limits strongly the analogy: the Statistical Mechanics theory of equivalence of the ensembles requires considering the thermodynamic limit \( V \to \infty \) of the volume of the system container and restricting the observables \( O \) to be local.

In the conjecture in Sec.\( \text{IV} \) instead, the observables are unrestricted and the role of the volume \( V \) is played by the cut-off \( N \). Clearly for a full analogy equivalence should hold for \( \nu \) fixed as \( N \to \infty \), provided the observables are suitably restricted [11].

To see what has to be understood to try to establish a closer connection between the theory of the ensembles in Statistical Mechanics and the proposed fluid equations equivalence the key remark is that the conjectured equivalence is based on the chaoticity of the evolution, which is ensured by the \( \nu \to 0 \) condition in Eq.\( \text{(4.2)} \).

So the same argument can simply be extended to many other equations in which the size of a parameter controls the increasingly “chaotic” motion of a system. Examples of this phenomenon have been explicitly considered adding new examples to a wide literature of homogenization phenomena: see [23, 24, 31, 33] for fluid equations or [2, 21, 29]. Thus the conjecture in Sec.\( \text{IV} \) although...
quite unsatisfactory, as pointed out, seems to hold in its
generality.\textsuperscript{29, 31}

Far more interesting would be to dispose of the condition \( \nu \to 0 \) and to realize a stronger analogy with Statistical Mechanics. The idea is that some, by far not all, equations describing macroscopic phenomena arise as scaling limits of fundamental equations governing evolutions of systems of particles interacting via forces verifying all principles and symmetries of Physics: staying within classical systems among these are Newton’s laws, time reversal and parity and charge symmetry...

The evolutions, at so fundamental a level, are certainly chaotic and the ergodic hypothesis epitomizes this property: from them, via approximations and/or heuristic arguments, arise simplified equations (\textit{models}) that generate motions apt to describe many of the features found in the observations. One of the first examples is in the derivation of the (compressible) Navier-Stokes equations in \textsuperscript{36}.

A model can even fail to respect one or more of the fundamental laws or symmetries: like the time reversal symmetry breaking which accounts phenomenologically for dissipation. This has never been considered a violation of the basic principles: it has been always clear that it was simply due to the procedure followed in the derivations.

Then the idea arises that there could (should?) exist models representing the same phenomena at the same level of accuracy and preserving some of the properties that other models do not respect, but which are properties on which there is a minor interest in the context on which one is working.\textsuperscript{23, 24}

The case of the Navier-Stokes equation has been proposed as an example of the possibility of describing an incompressible fluid via a reversible equation, without the need (as in conjecture 1 above, see also \textsuperscript{22}) of taking the limit \( \nu \to 0 \) but paying the price of restricting attention to a suitable (large) family of observables.

In the NS case the equations of motion are irreversible but they arise from a fundamental microscopic representation which is reversible and chaotic. If, as in most experimental studies, interest is on properties of “large scale” then it is natural to extend the conjecture 1 to the NS evolution without cut-off restricting attention to the case in which the macroscopic forces act at large scale and whose results have to be observed also on large scale, \textsuperscript{25, 26}

This can be formalized, \textsuperscript{24}, into the:

\textbf{Conjecture 2: Under the equal average enstrophy Eq. (3.2) and if \( O \) is a local observable, as defined in Sec. IV, then}

\[
\lim_{N \to \infty} \langle O \rangle_{\nu}^N = \lim_{N \to \infty} \langle O \rangle_D^N
\]

\textbf{for all} \( \nu > 0 \).

The conjecture 2 therefore adds to conjecture 1 the restriction that the observables \( O \) must be local and replaces the equivalence condition \( \nu \to 0 \) with the \textit{condition} \( N \to \infty \) (keeping equal average enstrophy).

The ensemble \( \mathcal{E}_N^{IN,\nu} \) is analogous to the canonical ensemble with \( \nu \) as temperature while \( \mathcal{E}_N^{IN,D} \) is analogous to the microcanonical ensemble with the enstrophy \( D \) as the energy and \( N \to \infty \) corresponds to \( V \to \infty \), \textit{i.e.} to the thermodynamic limit necessary for all local observables to show the same statistics.

The analogy with Statistical Mechanics is now ‘essentially’ complete (however see Sec. VI below) and provides an example of use of the ‘thermodynamic limit’ among the ideas emerging in nonequilibrium theory, \textsuperscript{1, 43}.

\section{VI. Equivalence and Phase Transitions}

Conjecture 2 of Sec. \textsuperscript{IV} leaves a gap in the strict analogy between Fluid Mechanics and Statistical Mechanics ensembles. Is there an analogue of the phase transitions?

So far we have considered the ensembles \( \mathcal{E}_N^{IN,\nu}, \mathcal{E}_N^{IN,D} \) assuming that for each pair of \( \nu, D \) the equations \( IN \) and \( RN \) admit just one “natural” stationary distribution controlling the fluctuations of the (local) observables.

However it is possible that initial data chosen with a distribution density \( \rho(u) > 0 \) generate a statistics which still depends on the initial \( u \) with positive probability: this case would be met if the evolution admitted several attracting sets in the phase space \( \mathcal{C}_N \).

If so, label the “indecomposable” invariant distributions by \( \mu_0 \in \mathcal{E}_N^{IN,\nu}, \theta = 1, 2, \ldots, q_{\nu,N} \)
Likewise label the “indecomposable” invariant distributions by \( \mu_0 \in \mathcal{E}_N^{RN,D}, \theta = 1, 2, \ldots, p_{D,N} \).
Each \( \mu_0 \) will be called a “pure phase”\textsuperscript{12}.

For simplicity we assume that \( q_{\nu}, p_D < \infty \) and say that at the values \( \nu \) or \( D \) there are \( q_{\nu} \) or \( p_D \) “pure phases”.

Then, keeping in mind the theory of phase transitions in Statistical Mechanics, conjecture 2 should be modified as:

\textbf{If under the equivalence condition between \( \nu \) and \( D \), Eq. (3.2), there are \( q_{\nu,N} \) respectively \( p_{D,N} \) pure phases, then \( q_{\nu,N}, p_{D,N} \) have the same limit \( q \geq 1 \) as \( N \to \infty \), and it is possible to establish a \( 1 \leftrightarrow 1 \) correspondence between the \( \mu_j \in \mathcal{E}_N^{RN,\nu} \) and the \( \mu_j \in \mathcal{E}_N^{RN,D} \) such that the distribution of the local observables become, in corresponding \( \mu_i \)’s and in the limit \( N \to \infty \), the same.}

If one thinks to the ferromagnetic Ising model in vonne V at low temperature then there are two indecomposable pure phases in which the total magnetization or just its average is fixed to some \( m = \pm m^* \neq 0 \), whether the boundary conditions are periodic or free or whether the dynamics is of Glauber type or other. Make correspondent the phases with the same \( m \) then the local observables (\textit{i.e.} the observables \( O \) which depend only on the

\textsuperscript{12}Indecomposable means that for each \( \theta \) with probability 1 with respect to \( \mu_0 \) initial data generate precisely \( \mu_0 \) itself: synonymous of ergodic.
spins located in a fixed region) have fluctuation with the same statistics in the thermodynamic limit, \( V \to \infty \).

See comments following Eq. \( (\text{11}) \) for other analogies with phase transitions arising in \( RN \) and developed in \([17]\).

### VII. Chaotic Hypothesis and Reversibility

In a general evolution equation \( \dot{x} = g(x), x \in b\mathbb{R}^n \) generating motions \( t \to S_t x \) which lead to an attracting set \( \mathcal{A} \) on which they are chaotic (i.e. have positive Lyapunov exponents) the “chaotic hypothesis” is:

**Chaotic hypothesis (CH):** The attracting sets can be considered smooth surfaces on which the motion is an Anosov flow, \([14, 29, 43]\).

The assumption implies the existence of a unique stationary probability distribution \( \mu \) on \( \mathcal{A} \) which is a natural distribution in the sense that it gives the statistical properties of the motions of almost all initial data chosen in the vicinity of \( \mathcal{A} \) with a probability with density \( \rho(x) > 0 \).

This assumption should be viewed as an extension of the analysis leading to the ergodic hypothesis in equilibrium problems, although of course examples which do not satisfy it are easy to find.

Still it is an assumption that has been proposed to be applicable to most systems undergoing chaotic motions, following a path that led to the modern ergodic hypothesis in equilibrium thermodynamics. In the case of non equilibrium the chaotic hypothesis only strengthens the key ideas of Ruelle, \([33, 41, 42]\), developed to provide a fundamental reason for the selection of the probability distributions to be used to evaluate the time averages of observables in systems out of equilibrium, remaining compatible with leading to the selection of the microcanonical ensemble in the equilibrium cases.

The real problem is to show that it not only has the merit of providing a conceptual extension of ideas at the basis of equilibrium Statistical Mechanics to nonequilibrium and Fluid Mechanics but it has also predictive power on new observations.

The simplest applications of the CH deal with reversible evolutions; hence the equations \( RN \) or \( RE \) might offer insights.

Imagine to fix the UV cut-off \( N \) and that for some \( \nu \) the evolution appears to generate trajectories of \( IN \) that visit densely the entire phase space. We expect that to be the case at small \( \nu \), at fixed \( N \); and for \( \nu = 0 \) ergodicity is expected to hold. As \( \nu \) increases the system develops an attracting set which, if the CH holds, should still be the full phase space (a consequence of the structural stability of Anosov systems\([15]\)).

For such value of \( \nu \) let \( D \) be the average endstrophy: we consider the \( RN \) evolution of initial data with endstrophy \( D(u) = D \). The phase space “contracts” at a rate \( \sigma(u) \), i.e. if \( u_{\beta,k} = u_{r,\beta,k} + iu_{i,\beta,k}, \beta = 1, 2 \), see \([21]\), at a rate equal (by Liouville’s theorem) to:

\[
-\sigma(u) = -\sum_{k,\beta} \left( \frac{\partial u_{r,k,\beta}}{\partial u_{r,k,\beta}} + \frac{\partial u_{i,k,\beta}}{\partial u_{i,k,\beta}} \right)
\]

where \( \sum_k \) denotes summation over the \( k \) so that only one \( k \) between \( \pm k \) contributes (the contribution is independent on which one is selected).

Let \( F_4 = \sum_k k^4\sigma_k, E_0 = \sum_k k^6|\mathbf{u}_k|^2, E_4 = \sum_k k^4|\mathbf{u}_k|^2, K_2 = \sum_k k^2 \), then:

\[
-\sigma(u) = 2\left( K_2 - \frac{E_6(u)}{E_4(u)} \right) \alpha(u) + \frac{F_4(u)}{E_4(u)}
\]

which has the same expression in dimension 2, 3 (but the expression of \( \alpha \) is of course different).

If CH holds the “Fluctuation theorem”, FT, can be applied and the result is that it implies a simple prediction on the non local observable

\[
p = \frac{1}{t} \int_0^t \frac{\sigma(u(t'))}{\sigma_+} dt'
\]

where \( \sigma_+ \) is the average value of \( \sigma(u(t)) \). The fluctuations of \( p \) in the stationary distribution \( \mu_{N,D}^{RN} \) have the probability that \( p \in [a, b] \) is \( \exp(t \max_{p \in [a, b]} s(p) + o(t)) \) and the “large deviations rate” \( s(p) \) has the symmetry property, \([19, 27, 28]\):

\[
s(p) - s(-p) = pt \sigma_+
\]

which follows combining CH and the time reversibility.

The observable \( \sigma(u) \) can be considered also as an observable for the IN evolution. Although it is non local it has been tested in a few cases to see whether it nevertheless obeys the same fluctuation relation Eq. \( (7.3) \) in corresponding distributions, see \([23]\) for a positive result, but no other attempt exist, so far, to check possible equivalence between corresponding fluctuation relations (in any event the fluctuation relation is not a local observable and is not covered by the conjectures).

\[\text{13}\] This is a compact set \( \mathcal{A} \) such that all initial data \( x \) close enough to \( \mathcal{A} \) are such that the distance \( d(S_t x, \mathcal{A}) \to 0 \).

\[\text{14}\] Anosov evolutions are smooth flows on bounded smooth surfaces \( \mathcal{A} \) such that at every point \( x \) the evolution is hyperbolic (i.e. in a system of coordinates following \( S_t x \) as \( t \) varies the \( S_t x \) is a hyperbolic fixed point); furthermore any open set \( U \subseteq \mathcal{A} \) is such \( S_t U \) covers any prefixed point \( x \in \mathcal{A} \) for infinitely many \( t > t_0 \) and for all \( t_0 \) (“motion of most points covers densely \( \mathcal{A} \), “recurrence”), \([4, 12]\).

\[\text{15}\] Structural stability means here that small pertubations of Anosov systems are still Anosov systems.\([2, 45, 46]\).
VIII. ATTRACTORS AND SMALL SCALES

However the assumption that at an enstrophy value $D$ the stationary distribution $\mu^{RN,D}_N$ arises from an evolution which leads to an attracting set invariant under time reversal is too strong.

Certainly it does not cover the cases in which the UV cut-off $N$ is large enough and the $u_k$ components are strongly damped for $|k|$ large (as implied by the equivalence conjecture).

Hence if $N$ is large the attracting set $A$ will shrink and its time reversal image $IA$ will become different from $A$: a spontaneous breaking of time reversal.

The consequence is that the FT cannot be applied to the observable $\sigma(u)$, not even if the CH is assumed in the reversible RN equation.

Nevertheless FT could be applied, under the CH, to the motion on $A$ if the time reversal $I$ could be replaced by another map $\tilde{I}$ which leaves $A$ invariant and on $A$ the $\tilde{I}S_t = S_{-t}\tilde{I}$ holds. Because by CH $A$ is a surface on which the evolution is of Anosov type.

In this case the fluctuation relation will be applied no longer to $\sigma(u)$, but to the sum $\sigma_A$ of the local Lyapunov exponents relative to the motion on $A$: clearly the negative exponents pertaining to the attraction to $A$ should not be counted.

Hence the question under which conditions a time reversal for the motions on $A$ exists is preliminary to the second question of how to identify the Lyapunov exponents of the motions on $A$.

Considering the RN equations with UV cut-off $N$ and fixed enstrophy $D$. Suppose that for small $N$ (i.e. at strong regularization) the motions invade densely the phase space $C_N$: i.e. the attracting set $A$ coincides with $C_N$. Increasing $N$ arrives at $N*$ beyond which the (average) viscosity affects the components $u_k$ with large $k$ so that $A$ becomes smaller than $C_N$.

So the evolution is reversible for all $N$, but for $N$ large its restriction to the attracting set $A$ is not.

In [3] the question of existence, as a “remnant” of the global symmetry $I$, of a time reversal $\tilde{I}$ operating on $A$ has been examined and a geometric property, named Axiom C property, leading to the existence of $\tilde{I}$ was identified and shown to have a “structural stability” property (as demanded to properties of physical relevance) The definition and main properties of Axiom C are described in Appendix C.

A scenario for the application to IN,RN (and more general) equations in which time reversal is a symmetry but $A$ does not coincide with the full phase space can be the following.

Assume that Axiom C holds for RN, hence there is a map $\tilde{I}: A \rightarrow A$ such that $\tilde{I}S_t = S_{-t}\tilde{I}$: to apply FT the problem still remains of identifying the phase space contraction $\sigma_A$, i.e. the local Lyapunov exponents which contribute to the phase space contraction on the surface $A$.

In studying the Lyapunov spectrum for $IN,RN$ the following “pairing symmetry” has been tested and approximately verified in a few 2D simulations and for a few values of the ensembles parameters $\nu, D$.

If the $N$ local Lyapunov exponents are arranged in decreasing order and their time averages are $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{N-1}$, then

$$ (\lambda_k + \lambda_{N-1-k}) = n + O(k^{-1}), \quad k = 0, \ldots, N/2 $$

and the constant $n < 0$ and the $\lambda_k$ turned out to have, for each $k$, in IN and RN very different fluctuations but remarkably the same average in corresponding distributions $\mu^{IN,\nu}$ and $\mu^{RN,D}$: quite unexpected a result because the $\lambda_k$ are not local observables. See figs.7,8 and, respectively, figs.5,6 in [23, 24]. Relations like Eq. (8.1) are called “pairing rules”.

So among the $N/2$ averages $\lambda_k, \lambda_{N-1-k}$ there may be, depending on the values of $\nu, D$, pairs in which both elements are < 0: i.e. there may be $n^* \leq N/2$ pairs of opposite sign and $N/2 - n^*$ negative pairs.

A natural interpretation of the above pairing rule is that the pairs of exponents < 0 represent the exponents controlling the approach to $A$ while the other $n^*$ pairs are associated with the chaotic motion on the attracting set: the phase space contraction on $A$ would then be

$$ \sigma_A(u) = \sum_{k=0}^{n^*} (\lambda_k(u) + \lambda_{N-1-k}(u)) $$

The interest of the above remarks is that if CH, axiom C and pairing are satisfied and if $O(k^{-1})$ in Eq. (8.1) can be neglected the consequent relation:

$$ \sigma_A(u) = \frac{2n^*}{N} \sigma(u) $$

can be used to define the phase space contraction on $A$. The advantage is that $\sigma_A$ is measurable simply by measuring $\sigma(u)$ from the equations of motion using Eq. (8.2).

Then, applying FT, the relation Eq. (7.4) is simply changed into:

$$ s(p) - s(-p) = p \frac{2n^*}{N} \sigma_+ $$

in the case of the RN evolution.

---

16 The local exponents are defined as the eigenvalues of the symmetric part of the Jacobian of the motion on $A$: their sum defines the contraction (or expansion) of the surface elements of $A$.

17 i.e. small perturbations of systems with the axiom C property still have the property. [4]. Persistence under perturbations is clearly essential in most Physics theories, [5].

18 Remark that this evaluation of the attracting set dimension is different from that obtained by the general Kaplan-Yorke dimension: an upper bound on the latter is, for IN, in [21].
Furthermore if the equivalence conjecture can be extended to the non local observables \( \sigma, \sigma_A \) then the fluctuation relation gives a prediction on fluctuations of both IN and RN and, if \( n^* < N/2 \), a test of the Axiom C.

The above scenario, proposed first in [7, p.445] and leading to the formulation of Axiom C, does not seem to have been tested, not even for simple test examples and it is certainly interesting if it can be confirmed in some instances: the only attempt to check Eq. (3.1) dealt, \([23]\), with cases in which \( n^* = N/2 \). Hence it does not deal with the most interesting part of the above scenario and in particular it does not test the Axiom C: however it did yield the result that the fluctuation relation holds in equivalent distributions, i.e. the observable \( \sigma(u) \), Eq. (7.2), satisfies the same Eq. (7.3) even in the irreversible evolution IN.

There are cases in which the phase space contraction can be identified with entropy creation: this is important as the entropy production is accessible, in a laboratory experiment, to measurements of heat and work exchanges with the surroundings, \([11]\); however it is very difficult to perform complete analysis of such energy exchanges and among the many experimental works very few convincingly discuss the problem.

### IX. OTHER ENSEMBLES

In Statistical Mechanics there are several equivalent ensembles. The same should hold for the fluids considered above. For instance we could compare IN with the equation that will be called RE given by Eq. (3.1) with \( a(u) \) given by the first of Eq. (3.2).

The RE is reversible and conserves the global quantity \( \mathcal{E}(u) \), energy, instead of enstrophy. The ensemble is now the collection of the stationary states \( \mu^R,E \in \mathcal{E}^{R,E} \).

The equivalence condition is equality of the average energy, hence Eq. (4.2) is modified as

\[
\langle \mathcal{E} \rangle^N_{E} = E
\]

and the analysis of the previous sections can be repeated.

Care has to be exercised because the condition Eq. (4.2) is not the same as \( E = \langle \mathcal{E} \rangle^N_{E} \) (unlike the corresponding case of the RN equations \( [19,26] \)).

This implies that a first test of the conjecture in the case of RN is obtained by fixing \( \nu \) and computing the average enstrophy \( D \) and checking that if \( \nu, D \) correspond in the sense of Eq. (4.2):

\[
\langle \alpha \rangle^R,D_N D = \nu(\mathcal{D})^N_{E}, \quad \text{i.e.} \quad \langle \alpha \rangle^R,N \cdot \nu \quad (9.2)
\]

where \( \mathcal{D}(u) \) denotes, as above, the enstrophy. While for RE, if \( \nu \) and \( E \) correspond in the sense of Eq. (4.1), the analogous test is to check

\[
\langle \alpha \rangle^R,E \cdot E \quad (9.3)
\]

The above relations have been tested in several cases, with particular care and a few positive results for the equivalence between IN and RN in 2D: only in very few cases for the IN and RE equivalence.

### Appendix A: Euler Flow is Geodesic

Here some details on the Hamiltonian representation Eq. (1.2) for the Euler flow are presented, listing again for the reader's convenience, the conventions set in Sec. II.

It has to be kept in mind that in analytic mechanics the canonical coordinates for \( n \)-degrees of freedom systems are given as strings of \( 2n \) variables \( \{p_i, q_i, i = 1 \ldots n\} \): particle \( i \) is located at position \( q_i \) and has momentum \( p_i \).

In a Lagrangian description of a fluid, coordinates will be \( (q, \dot{q}_i) = \{(\dot{q}_i, q_i) \in T_q\} \) with \( q, \dot{q}_i \) consisting in a diffeomorphism \( q : \xi \rightarrow q \in \text{space} \mathcal{D}(T^d) \) of \( C^\infty \) diffeomorphisms of \( T^d \) and \( q \in \text{Lin}(T^d) \) where \( \text{Lin}(T^d) \) is the space of the \( C^\infty \) vector fields 'tangent' to \( q^\circ \) in a pair \( (\dot{q}, \dot{q}_i) \) the vector \( \dot{q}_i \in R^d \) is considered a vector applied at the point \( q_i \).

Hence, given \( q, \dot{q}_i \), the derivative \( \partial_{q_i} \dot{q}_i \) is defined as well as the divergence \( (\text{div}\dot{q}) q_{\xi} \triangleq \sum_{i=1} d \partial_{q_i} \dot{q}_i \) of \( \dot{q}_i \).

The space of the pairs \( (q, \dot{q}_i) \) will be called \( F \) and the points of \( T^d \) become labels of a fluid element located at the point \( q \) with velocity \( \dot{q}_i \).

More formally \( (q, \dot{q}_i) \in \text{Difi}(T^d) \times \text{Lin}(T^d) \) where \( \text{Difi}(T^d) \) is the space of the \( C^\infty \) diffeomorphisms of \( T^d \) and \( \text{Lin}(T^d) \) the space of the \( C^\infty \) vector fields with 0 average: for each \( (q, \dot{q}_i) \) the vector \( \dot{q}_i \in R^d \) is considered applied to the point \( q_i \). and \( (\text{div}\dot{q}) q_{\xi} \equiv \sum_{i=1} d \partial_{q_i} \dot{q}_i \).

Actually we concentrate on the subspace of \( (q, \dot{q}_i) \in SD\text{Difi}(T^d) \times (\text{Lin}(T^d))^{def} F \) where the evolution of an incompressible fluid takes place: \( SD\text{Difi}(T^d) \), being

\[
\text{as in the analysis of the transition in [7], it is likely that the difference between imposing the condition } E = \langle \mathcal{E} \rangle^N_{E} \text{ instead of the equal average enstrophy, as in Eq. (4.1), is not appreciable.}
\]
the volume preserving diffeomorphisms and $SLin(T^d)$ the 0-divergence vector fields, i.e. for each such pair $(\mathbf{q}, \dot{\mathbf{q}})$ it is $(\text{div} \dot{\mathbf{q}})_\xi = 0$.

If the positions $\dot{q}_\xi$ are moved the variation of $\dot{q}_\xi$ is proportional to $\frac{\partial H}{\partial p_{q_\xi}}$, the Lagrangian is:

$$L(\dot{\mathbf{q}}, \mathbf{q}) = \int_{T^d} \frac{1}{2} (\dot{q}_\xi^2 - Q(\mathbf{q}, \dot{\mathbf{q}})_\xi) d\xi \quad (A.1)$$

where $Q$ is the quadratic form on $\mathcal{F}$:

$$-\frac{1}{4\pi} \int_{T^d} d\gamma \int_{T^d} d\gamma' \int_{T^d} d\gamma'' \frac{1}{|q_\gamma - q_{\gamma'}|} \left\{ \frac{1}{2} \delta(q_{\gamma''} - q_{\gamma'}) \partial_{\dot{q}_{\gamma''},s} \partial_{\dot{q}_{\gamma,r}} - \partial_{\dot{q}_{\gamma,r},s} \partial_{\dot{q}_{\gamma'',s}} \right\} \quad (A.2)$$

and $-(4\pi)^{-1} \frac{1}{|x-y|''}$ symbolizes the Green’s function for the Laplacian on $T^d$:

$$\delta_{\gamma''}(\mathbf{x}) \partial_{\gamma''}$$

Therefore the Eq. (A.7) coincide with the Navier Stokes equations and their solutions will remain in $S F$ s long as they remain smooth: for data not in $S F$ only solutions local in time can be envisaged and the equations would be more involved.

**Appendix B: Euler’s equation Jacobian**

The Jacobian is obtained by taking suitable functional derivatives of the transport term and the pressure term, before applying the projection operator $\mathcal{P}$ in Eq. (A.4). The contribution of the transport term is (before applying $\mathcal{P}$):

$$\frac{\partial \mathcal{H}}{\partial u_j(y)} = -\delta(x-y)\partial_{z_x}u_j(x) - u_k(x)\partial_{x_z}(\delta(x-y)) = 2(\partial_{z_x}\partial_{y_k}\Delta^{-1}(x-y))\partial_{y_k}u_k(z) \quad (B.1)$$

where the second term is an antisymmetric operator in $L_2(T^d) \times \mathbb{R}^d$. The contribution from the pressure term is

$$\frac{\partial \mathcal{P}}{\partial u_j(y)} = -2\partial_{z_x}\mathcal{H} = 2\partial_{z_x}\partial_{y_k}\Delta^{-1}(x-y)\partial_{y_k}u_k(z) \quad (B.2)$$

**Footnotes**

21 i.e. formally the summation over images $y + 2\pi n, n \in \mathbb{Z}^d$; which makes sense if the kernel is applied to a smooth function with 0 average.

22 Representing a constrained motion as a special case of unconstrained motion subject suitable extra forces follows a familiar prototype. A point mass constrained on a circle of radius $R$, centered at the origin $O \in \mathbb{R}^2$, can be seen as a point subject to a centripetal force evolving under the Lagrangian $L = \frac{1}{2} \dot{q}_\xi^2 - \frac{1}{2} \dot{q}_\xi^2 \cdot \dot{q}_\xi$. This leads to $\mathbf{p} = \dot{\mathbf{q}} \theta$ with $\theta = 1 - 2\Delta^{-1}(x)$ and, for the Hamiltonian, $\mathcal{H} = \frac{1}{2} \mathbf{p}^2$, to the equations $\dot{\mathbf{q}} = \frac{\mathbf{p}}{\theta R}$, $\dot{\mathbf{p}} = -\frac{\mathbf{p}}{\theta R} \cdot \dot{\mathbf{q}}$. Thus it appears that the phase space $R^6$ is analogous to $\mathcal{F}$, the maps of the circle $\xi \rightarrow s$, mapping the arc $\xi$ to the arc $s$, correspond with $S D f$ and the vectors tangent to the circle are analogous to $S L i n$. The motion is general a geodesic motion, as long as it is defined (i.e. as long as $|\mathbf{q}| \neq 0, \infty$, which for data initially on the circle and initial velocity tangent to it is a uniform rotation. On such motions the Hamiltonian value is $\frac{1}{2} \mathbf{p}^2$ as $\theta = 1$, alike the corresponding vanishing of $Q$ on $S F$ in Eq. (1.3), (A.3).
and in both Eq. (B.1), (B.2) summation over \( k \) is intended. The latter operator does not contribute to the Jacobian because acting on a divergenceless field yields \( 0 \); therefore the symmetric part of the Jacobian is the multiplication operator, in \( L_2(T^d) \times R^d \), by:

\[
W_{i,j}(x) = -\frac{1}{2}(\partial_{x,i} u_j(x) + \partial_{x,j} u_i(x)) \tag{B.3}
\]

followed by the orthogonal projection \( \mathcal{P} \) on the subspace of the divergenceless fields \( SLin(T^d) \subset L_2(T^d) \times R^d \), Sec[I].

Appendix C: The Axiom C.

To describe the main features of the Axiom C, consider first the simpler case of a reversible diffeomorphism \( S \), i.e. such that there is a diffeomorphism \( I \) such that \( IS = S^{-1}I \). Imagine that the attracting set \( A \) differs from its time reversal image \( IA = R \) and that CH holds.

The tangent space at a generic point \( z \) is supposed to be smoothly decomposed as \( T_z(A) \oplus T_z(S(A)) \oplus T_z(R) \). If \( z \in A \) or \( z \in R \) then \( T_z(A), T_z(S(A)) \) coincide with the tangent, at \( z \), to the stable manifold of \( S \) on \( A \) or \( R \) respectively; furthermore for each ball \( U_\delta(x) \subset A \), of radius \( \delta \), consider the manifolds \( W_i(x) \cap U_\delta(x), i = u, s \), and assume that they can be continued into smooth manifolds \( W^+u, W^- \) everywhere tangent \( T_u \oplus T_m \) and \( T_u \oplus T_m \) which intersect \( R \) in a single point \( \tilde{x} = Pr \delta \) if \( \delta \) is small enough: thus defining \( P \) as a map between \( A, R \).

Finally, as the labels \( s, u \) suggest, the vectors in \( T_u, T_s \) uniformly contract exponentially as time tends to \( +\infty \) or \( -\infty \) respectively, while vectors in \( T_m \) contract exponentially as \( t \to \pm \infty \) (i.e. in both directions, being ‘squeezed’ on \( A \) and \( R \)).

The case of a flow \( S_t \) can be described similarly by imagining that \( T_m \) contains also the neutral direction \( d \) intersected by the manifolds emerging from \( A \).

The latter property permits to establish the map \( P \), thus allowing to define the composition \( \tilde{I} = PI \), acting as a time reversal on \( A \) and \( R \), because the invariance of the manifolds implies that on \( A \cup R \) it is \( PS_t = S_tP \); so that \( \tilde{I}S_t = S_t\tilde{I} \) on \( A \cup R \) (note that \( \tilde{I} \) is not defined outside \( A \cup R \)). See \( [4] \) for more details.

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Erratum: This is version 3: the former Appendix C has been suppressed (together with the sentence referring to it following Eq. (A.7)) because of errors in the attempt to understand heuristically the apparent approximate symmetry of the Lyapunov exponents. The former Appendix D is now Appendix C.

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