Further Spectral Properties of the Weighted Finite Fourier Transform Operator and Related Applications.

NourElHouda Bourguiba\textsuperscript{a} and Ahmed Souabni\textsuperscript{a,1}

\textsuperscript{a} University of Carthage, Department of Mathematics, Faculty of Sciences of Bizerte, Tunisia.

Abstract— In this work, we first give some mathematical preliminaries concerning the generalized prolate spheroidal wave function (GPSWFs). These set of special functions have been introduced in [16] and [7] and they are defined as the infinite and countable set of the eigenfunctions of a weighted finite Fourier transform operator. Then, we show that the set of the singular values of this operator decay at a super-exponential decay rate. We also give some local estimates and bounds of these GPSWFs. As a first application of the spectral properties of the GPSWFs and their associated eigenvalues, we give their quality of approximation in a periodic Sobolev space. Our second application is related to a generalization in the context of the GPSWFs, of the Landau-Pollak approximate dimension of the space of band-limited and almost time-limited functions, given in the context of the classical PSWFs, the eigenfunctions of the finite Fourier transform operator. Finally, we provide the reader with some numerical examples that illustrate the different results of this work.

1 Introduction

We first recall that for $c > 0$, the classical prolate spheroidal wave functions (PSWFs) were first discovered and studied by D. Slepian and his co-authors, see [13, 14]. These PSWFs are defined as the solutions of the following energy maximization problem:

\[
\text{Find } g = \arg \max_{f \in B_c} \frac{\int_{-1}^{1} |f(t)|^2 dt}{\int_\mathbb{R} |f(t)|^2 dt}
\]

Where $B_c$ is the classical Paley-Winer space, defined by

\[
B_c = \{ f \in L^2(\mathbb{R}), \text{ Support } \hat{f} \subseteq [-c, c] \}.
\]

Here, $\hat{f}$ is the Fourier transform of $f \in L^2(\mathbb{R})$, defined by $\hat{f}(\xi) = \lim_{A \to +\infty} \int_{[-A,A]} e^{-ix\xi} f(x) \, dx$. Moreover, it has been shown in [13], that the PSWFs are also the eigenfunctions of the integral operator $Q_c$ defined on $L^2(-1,1)$ by

\[
Q_c f(x) = \int_{-1}^{1} \frac{\sin(c(x-y))}{\pi(x-y)} f(y) \, dy
\]

as well as the eigenfunctions of a commuting differential operator $\mathcal{L}_c$ with $Q_c$ and given by

\[
\mathcal{L}_c f(x) = (1 - x^2) f''(x) - 2x f'(x) - c^2 x^2 f(x)
\]

In this paper we are interested in a weighted family of PSWFs called the generalized prolate spheroidal wave functions (GPSWFs), recently given in [16] and [7]. They are defined as the eigenfunctions of the Gegenbauer perturbed differential operator,

\[
\mathcal{L}_c^{(\alpha)} \varphi(x) = (1 - x^2) \varphi''(x) - 2 (\alpha + 1) x \varphi'(x) - c^2 x^2 \varphi(x) = \mathcal{L}_0 \varphi(x) - c^2 x^2 \varphi(x), \text{ where } \alpha > -1; c > 0,
\]

1 Corresponding author: Ahmed Souabni, email: souabniahmed@yahoo.fr
This work was supported in part by the Tunisian DGRST research grants UR 13ES47.
as well as the eigenfunctions of the following commuting weighted finite Fourier transform integral operator,

$$\mathcal{F}_c^{(\alpha)} f(x) = \int_{-1}^{1} e^{icxy} f(y) (1 - y^2)^{\alpha} dy, \alpha > -1.$$  

The importance of PSWFs is due to their wide range of applications in different scientific research areas such as signal processing, physics, and applied mathematics. The question of quality of approximation by PSWFs has attracted some interests. In fact, the authors in [12] have given an estimate of the decay of the PSWFs expansion of $f \in H^s(I)$. Here $H^s(I)$ is the Sobolev space over $I = [-1, 1]$ and of exponent $s > 0$. Later in [16] authors studied the convergence of the expansion of functions $f \in H^s(I)$ in a basis of PSWFs. We should mention that the problem of the best choice of the value of the bandwidth $c > 0$ arises here. Numerical answers were given in [16]. Recently in [3], the authors have given a precise answer to this question. Note that the convergence rate of the projection $S_N \cdot f = \sum_{k=0}^{N} < f, \psi_{n,c}^{(\alpha)} > \psi_{n,c}^{(\alpha)}$ to $f$ where $f \in B^{(\alpha)}_c$ or $f \in H^s_{\text{per}}(I)$, a periodic Sobolev space. This is done by using an estimate for the decay of the eigenvalues $(\lambda_n^{(\alpha)}(c))$ of the self-adjoint operator $\mathcal{Q}_c^{(\alpha)} = \frac{c}{2\pi} \mathcal{F}_c^{(\alpha)_*} \mathcal{F}_c^{(\alpha)}$, that is $\lambda_n^{(\alpha)}(c) = \frac{c}{2\pi} |\mu_n^{(\alpha)}(c)|^2$, where $\mu_n^{(\alpha)}$ is the $n$-th eigenvalues of the integral operator (2). That's why the first part of this work will be devoted to the study of the spectral decay of the eigenvalues of the integral operator $(\lambda_n^{(\alpha)}(c))$.

On the other hand, in [9], the authors have shown that the PSWFs are well adapted for the approximation of functions that are band-limited and also almost time-limited. That is for an $\varepsilon > 0$, this space is defined by

$$E_c(\varepsilon) = \{ f \in B_c : \| f \|_{L^2(\mathbb{R})} = 1, \| f \|_{L^2(I)}^2 \geq 1 - \varepsilon \}.$$

Here $I = [-1, 1]$ and $B_c$ is the Paley-Wiener space given by (1). In particular, the approximate dimension of the previous space of functions has been given in [9]. Moreover, in [10], the author has extended this result in the context of the circular prolate spheroidal wave functions (CPSWFs). These later are defined as the different eigenfunctions of the finite Hankel transform operator, based on the properties of the CPSWFs, we give a procedure for recovering the orthogonality over $\mathbb{R}$, of these later. In section 3, we give some further estimates of the CPSWFs and some estimates of the eigenvalues of both integral and differential operators that define CPSWFs which will be useful later on in the study of the quality of approximations by CPSWFs. In section 4, we first extend to the case of the CPSWFs, the result given in [5], concerning the quality of approximation by the CPSWFs of functions from the periodic Sobolev space $H^s_{\text{per}}(I)$. Then, we show that the CPSWFs best approximate almost time-limited functions and belonging to the restricted Paley-Wiener space $B^{(\alpha)}_c$, given by

$$B^{(\alpha)}_c = \{ f \in L^2(\mathbb{R}), \text{ Support } \hat{f} \subseteq [-c, c], \hat{f} \in L^2((-c, c), \omega_{-\alpha} \hat{\omega}(\cdot)) \}.$$  

Finally, in section 5 we give some numerical examples that illustrate the different results of this work.
2 Mathematical Preliminaries

In this section, we give some mathematical preliminaries concerning some properties as well as the computation of the GPSWFs. These mathematical preliminaries are used to describe the different results of this work.

We first mention that some content of this paragraph has been borrowed from [7]. Since, the GPSWFs are defined as the eigenfunctions of the weighted finite Fourier transform operator $F_c^{(\alpha)}$, defined by

$$F_c^{(\alpha)} f(x) = \int_{-1}^{1} e^{icxy} f(y) \omega_\alpha(y) \, dy, \quad \omega_\alpha(y) = (1 - y^2)^\alpha, \quad \alpha > 0, \quad (3)$$

then, they are also the eigenfunctions of the operator $Q_c^{(\alpha)} = \frac{c}{2\pi} F_c^{(\alpha)} \circ F_c^{(\alpha)}$, defined on $L^2(I, \omega_\alpha)$ by

$$Q_c^{(\alpha)} g(x) = \int_{-1}^{1} \frac{c}{2\pi} K_\alpha(c(x - y)) g(y) \omega_\alpha(y) \, dy, \quad K_\alpha(x) = \sqrt{\pi} \frac{2^{\alpha+1/2}}{\Gamma(\alpha + 1)} \frac{J_{\alpha+1/2}(x)}{x^{\alpha+1/2}}. \quad (4)$$

Here $J_\alpha(\cdot)$ is the Bessel function of the first kind and order $\alpha > -1$. Moreover, the eigenvalues $\mu_n^{\alpha}(c)$ and $\lambda_n^{\alpha}(c)$ of $F_c^{\alpha}$ and $Q_c^{\alpha}$ are related to each others by the identity $\lambda_n^{\alpha}(c) = \frac{c}{2\pi} |\mu_n^{\alpha}(c)|^2$. Note that the previous two integral operators commute with the following Gegenbauer-type Sturm-Liouville operator $\mathcal{L}_c^{(\alpha)}$, defined by

$$\mathcal{L}_c^{(\alpha)}(f)(x) = -\frac{1}{\omega_\alpha(x)} \frac{d}{dx} \left[ \omega_\alpha(x)(1 - x^2)f'(x) \right] + c^2 x^2 f(x), \quad \omega_\alpha(x) = (1 - x^2)^\alpha.$$  

Also, note that the $(n + 1)$–th eigenvalue $\chi_n^{\alpha}(c)$ of $\mathcal{L}_c^{(\alpha)}$ satisfies the following classical inequalities,

$$n(n + 2\alpha + 1) \leq \chi_n^{\alpha}(c) \leq n(n + 2\alpha + 1) + c^2, \quad \forall n \geq 0. \quad (5)$$

For more details, see [7]. We will denote by $(\psi_{n,c}^{(\alpha)})_{n \geq 0}$, the set of the eigenfunctions of $F_c^{(\alpha)}$, $Q_c^{(\alpha)}$ and $\mathcal{L}_c^{(\alpha)}$. They are called generalized prolate spheroidal wave functions (GPSWFs). It has been shown that $\{\psi_{n,c}^{(\alpha)}, n \geq 0\}$ is an orthogonal basis of $L^2(I, \omega_\alpha), I = [-1, 1]$. We recall that the restricted Paley-Wiener space of weighted $c$–band-limited functions has been defined in [7] by

$$B_c^{(\alpha)} = \{ f \in L^2(\mathbb{R}), \text{ Support } \tilde{f} \subseteq [-c, c], \tilde{f} \in L^2((-c, c), \omega_\alpha(\frac{t}{c})) \}. \quad (6)$$

Here, $L^2((-c, c), \omega_\alpha(\frac{t}{c}))$ is the weighted $L^2(-c, c)$–space with norm given by

$$\| f \|_{L^2((-c,c),\omega_\alpha(\frac{c}{t}))}^2 = \int_{-c}^{c} |f(t)|^2 \omega_\alpha \left( \frac{t}{c} \right) \, dt.$$

Note that when $\alpha = 0$, the restricted Paley-Wiener space $B_c^{(\alpha)}$ is reduced to the usual space $B_c$.

Also, it has been shown that

$$B_c^{(\alpha)} \subseteq B_c^{(\alpha')}, \quad \forall \alpha \geq \alpha' \geq 0. \quad (7)$$

As a consequence, it has been shown in [7] that the eigenvalues $\lambda_n^{\alpha}(c)$ decay with respect to the parameter $\alpha$, that is for $c > 0$, and any $n \in \mathbb{N}$,

$$0 < \lambda_n^{\alpha}(c) \leq \lambda_n^{\alpha'}(c) < 1, \quad \forall \alpha \geq \alpha' \geq 0. \quad (8)$$

Hence, by using the precise behaviour as well as the sharp decay rate of the $\lambda_n^{\alpha}(c)$, given in [4], one gets a first result concerning the decay rate of the $(\lambda_n^{\alpha}(c))_n$, for any $\alpha \geq 0$. Moreover, since in [4], the authors have shown that for any $c \geq 1$ and any $0 < b < 4/e$, there exists $N_b \in \mathbb{N}$ such that

$$\lambda_n^{0}(c) < e^{-2n \log(\frac{b}{c})}, \quad \forall n \geq N_b, \quad (9)$$

3
then by combining (8) and (9), one gets
\[
0 < \lambda_n^\alpha(c) \leq \lambda_n^0(c) < e^{-2n\log(\frac{4c}{n})} \quad \forall n \geq N_0, \quad \alpha \geq 0.
\] (10)

Note that a second decay rate of the $\lambda_n^\alpha(c)$ and valid for $0 < \alpha < \frac{3}{2}$, has been recently given in [8]. More precisely, it has been shown in [8], that if $c > 0$ and $0 < \alpha < \frac{3}{2}$, then there exist $N_\alpha(c) \in \mathbb{N}$ and a constant $C_\alpha > 0$ such that
\[
\lambda_n^\alpha(c) \leq C_\alpha \exp\left( -(2n+1) \left[ \log \left( \frac{4n + 4\alpha + 2}{2n + 1} \right) + C_\alpha \frac{c^2}{2n+1} \right] \right), \quad \forall n \geq N_\alpha(c).
\] (11)

Also, by using the fact that
\[
\psi_{n,c}^{(\alpha)}(x) = \frac{1}{\mu_n^\alpha} \int_{-1}^{1} \psi_{n,c}^{(\alpha)}(t)(1 - t^2)\alpha e^{itc} dt = \frac{1}{c\mu_n^\alpha} \int_{\mathbb{R}} \psi_{n,c}^{(\alpha)}(t/c)(1 - (t/c)^2)\alpha e^{itx} dt,
\]
then, we have
\[
\hat{\psi}_{n,c}^{(\alpha)}(x) = \frac{\sqrt{2\pi}}{\sqrt{c\lambda_n}} \psi_{n,c}^{(\alpha)} \left( \frac{x}{c} \right) \left( 1 - \left( \frac{x}{c} \right)^2 \right) \alpha 1_{[-c,c]}(x).
\] (12)

Note that the GPSWFs are normalized by the following rule,
\[
\|\psi_{n,c}^{(\alpha)}\|^2_{L^2(I,\omega_\alpha)} = \int_{-1}^{1} \left( \psi_{n,c}^{(\alpha)}(t) \right)^2 \omega_\alpha(t) dt = \lambda_n^{\alpha}(c).
\] (13)

The extra weight function generates new complications concerning the orthogonality of $\psi_{n,c}^{(\alpha)}$ over $B_{e}^{(\alpha)}$. To solve this problem, we propose the following procedure. We define on $B_{e}^{(\alpha)}$, the following inner product
\[
< f, g >_\alpha = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(x)\hat{g}(x)\omega_{-\alpha} \left( \frac{x}{c} \right) dx
\] (14)

By using (12) and (13), we have:
\[
< \psi_{n,c}^{(\alpha)}, \psi_{m,c}^{(\alpha)} >_\alpha = \frac{1}{c\lambda_n^{\alpha}(c)} \int_{-c}^{c} \psi_{n,c}^{(\alpha)} \left( \frac{x}{c} \right) \psi_{m,c}^{(\alpha)} \left( \frac{x}{c} \right) \omega_\alpha \left( \frac{x}{c} \right) dx = \delta_{nm}
\] (15)

Here $\delta_{nm}$ is the Kronecker’s symbol. For more details, see [7]. Now for $f \in B_{e}^{(\alpha)}$, we have $\hat{f}(x) = g(x)\omega_\alpha(x/c)$ for some $g \in L^2((-c,c),\omega_\alpha(\cdot))$. By using the Fourier inversion formula and the fact that $\{\psi_{n,c}^{(\alpha)}, n \in \mathbb{N}\}$ is an orthogonal basis of $L^2(I,\omega_\alpha)$, one gets
\[
f(x) = \frac{1}{2\pi} \int_{-c}^{c} e^{ixy} \hat{f}(y)dy = \frac{1}{2\pi} \int_{-c}^{c} e^{ixy} g(y)\omega_\alpha \left( \frac{y}{c} \right)dy
\]
\[
= \frac{c}{2\pi} \int_{-1}^{1} e^{ixy} g(cy)\omega_\alpha(y)dy = \frac{c}{2\pi} \int_{-1}^{1} e^{ixy} \left( \sum_{n=0}^{\infty} \alpha_n \psi_{n,c}^{(\alpha)}(y) \right)\omega_\alpha(y)dy
\]
\[
= \frac{c}{2\pi} \sum_{n=0}^{\infty} \alpha_n \int_{-1}^{1} \psi_{n,c}^{(\alpha)}(y)e^{ixy}\omega_\alpha(y)dy = \frac{c}{2\pi} \sum_{n=0}^{\infty} \alpha_n \mu_n^{\alpha}(c)\psi_{n,c}^{(\alpha)}(x)
\] (16)
where $\alpha_n = \int_{-1}^{1} g(cy)\psi_{n,c}^{(\alpha)}(y)\omega_\alpha(y) dy$. Consequently, the set $\{\psi_{n,c}^{(\alpha)}, n \in \mathbb{N}\}$ is an orthogonal basis of $L^2(I,\omega_\alpha)$ and an orthonormal basis of $B_{e}^{(\alpha)}$, when this later is equipped with the inner product $< \cdot, \cdot >_\alpha$.  

4
Also, in [7], the authors have proposed the following scheme for the computation of the GPSWFs.

In fact, since \( \psi^{(a)}_{n,c} \in L^2(I, \omega_a) \), then its series expansion with respect to the Jacobi polynomials basis is given by

\[ \psi^{(a)}_{n,c}(x) = \sum_{k \geq 0} \beta_k^{n} \tilde{P}_k^{(a,a)}(x), \quad x \in [-1,1]. \]  

(17)

Here, \( \tilde{P}_k^{(a,a)} \) denotes the normalized Jacobi polynomial of degree \( k \), given by

\[ \tilde{P}_k^{(a,a)}(x) = \frac{1}{h_k} P_k^{(a,a)}(x), \quad h_k = \frac{2^{2a+1} \Gamma^2(k + a + 1)}{k!(2k + 2a + 1) \Gamma(k + 2a + 1)}. \]  

(18)

The expansion coefficients \( \beta_k^n \) as well as the eigenvalues \( \chi_n^{(a)}(c) \) this system is given by

\[
\begin{align*}
\chi_n^{(a)}(c) & = \sum_{k \geq 0} \beta_k^n \chi_{2k}^{(a,a)}(c) \\
& \sim \sqrt{(k+1)(k+2)(k+2a+1)(k+2a+2)} \frac{c^2 \beta_k^n}{(2k+3) \Gamma(2k+2a+3)} \frac{c^2 \beta_k^n}{(2k+2a+2) \Gamma(2k+2a+3)} \frac{c^2 \beta_k^n}{(2k+2a+1) \Gamma(2k+2a+3)} \\
& \quad + \frac{\sqrt{k(k-1)(k+2a+1)(k+2a-1)}}{(2k+2a-1) \Gamma(2k+2a+3)} \frac{c^2 \beta_k^n}{(2k+2a+1) \Gamma(2k+2a+3)} \frac{c^2 \beta_k^n}{(2k+2a+2) \Gamma(2k+2a+3)} \frac{c^2 \beta_k^n}{(2k+2a+1) \Gamma(2k+2a+3)} \\
& = \chi_n^{(a)}(c) \beta_k^n, \quad k \geq 0.
\end{align*}
\]

(19)

3 Further Estimates of the GPSWFs and their associated eigenvalues.

In this section, we first give some new estimates of the eigenvalues of the eigenvalues \( \mu_n^{(a)}(c) \) associated with the GPSWFs. Then, we study a local estimate of these laters.

3.1 Estimates of the eigenvalues.

We first prove in a fairly simple way, the super-exponential decay rate of the eigenvalues of the weighted Fourier transform operator. We should the techniques used in this proof is inspired from the recent paper [6], where a similar result has been given in the special case \( \alpha = 0 \).

**Proposition 1.** For given real numbers \( c > 0, \ \alpha > -1 \) and for any integer \( n > \frac{ec+1}{2} \), we have

\[ |\mu_n^{(a)}(c)| \leq k_n \frac{k_n}{c 2^{2n-1} \log \left( \frac{2n-1}{2} \right)} \left( \frac{ec+1}{c 2^{2n-1}} \right)^{n+\frac{\alpha+1}{2}}, \quad k_n = \left( \frac{2}{c} \right)^{\frac{3+\alpha}{2}} \pi^{5/4} \Gamma(\alpha+1)^{1/2}. \]

(20)

**Proof:** We first recall the Courant-Fischer-Weyl-Min-Max variational principle concerning the eigenvalues of a compact self-adjoint operator \( T \) on a Hilbert space \( \mathcal{G} \), with positive eigenvalues arranged in the decreasing order \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq \cdots \), then we have

\[ \lambda_n = \min_{f \in S_n} \max_{f \in S_n^\perp, \|f\|_2=1} \langle Tf, f \rangle \geq 0, \]

where \( S_n \) is a subspace of \( \mathcal{G} \) of dimension \( n \). In our case, we have \( T = F_c^{\alpha} F_c^{\alpha}, \mathcal{G} = L^2(I, \omega_a) \). We consider the special case of

\[ S_n = \text{Span} \left\{ P_0^{(a,a)}, P_1^{(a,a)}, \ldots, P_{n-1}^{(a,a)} \right\} \]

and

\[ f = \sum_{k \geq n} a_k P_k^{(a,a)} \in S_n^\perp, \quad \|f\|_{L^2(\omega_a)}^2 = \sum_{k \geq n} |a_k|^2 = 1. \]
By using the well-known bound of the Bessel function given by the variational principle.

From [11] page 456, we have
\[ \| x^k \|_{L^2(I,\omega_n)}^2 = \beta(k+1/2, \alpha +1) \leq \frac{\sqrt{\pi} \Gamma(\alpha+1)}{e^{-k\alpha+1}}. \] (22)

By using the well-known bound of the Bessel function given by
\[ |J_\alpha(x)| \leq \frac{|x|^\alpha}{2^\alpha \Gamma(\alpha+1)} \quad \forall \alpha > -1/2, \quad \forall x \in \mathbb{R}, \] (23)

one gets,
\[ \| F_c^\alpha P_k^{(\alpha,\alpha)} \|_{L^2(I,\omega_n)} \leq \frac{\sqrt{\pi} \Gamma(k+ \alpha +1)}{\Gamma(k+1)\Gamma(k+ \alpha+3/2)} \frac{c^k}{2} \| x^k \|_{L^2(I,\omega_n)}. \] (24)

Next, we use the following useful inequalities for the Gamma function, see [17]
\[ \sqrt{2e} \left( \frac{x+1/2}{e} \right)^{x+1/2} \leq \Gamma(x+1) \leq \sqrt{2\pi} \left( \frac{x+1/2}{e} \right)^{x+1/2}, \quad x > 0. \] (26)

Then, we have
\[ \| F_c^\alpha P_k^{(\alpha,\alpha)} \|_{L^2(I,\omega_n)} \leq \frac{\pi^{5/4} \Gamma(\alpha+1)}{\sqrt{ce^{3/4}}} \frac{1}{k^{\alpha+2}} \left( \frac{ce}{2k+1} \right)^{k+1/2}. \] (27)

Hence, for the previous \( f \in S_n^+ \), and by using Hölder’s inequality, combined with the Minkowski’s inequality for an infinite sums, and taking into account that \( \| f \|_{L^2(I,\omega_n)} = 1 \), so that \( |a_k| \leq 1 \), for \( k \geq n \), one gets:
\[ |< F_c^\alpha F_c^\alpha f, f >_{L^2(I,\omega_n)}| = |< F_c^\alpha f, F_c^\alpha f >_{L^2(I,\omega_n)}| \leq \sum_{k \geq n} |a_k|^2 \| F_c^\alpha P_k^{(\alpha,\alpha)} \|_{L^2(I,\omega_n)}^2 \leq \left( \frac{\pi^{5/4} \Gamma(\alpha+1)}{\sqrt{ce^{3/4}}} \sum_{k \geq n} \left( \frac{ce}{2k+1} \right)^{k+1/2} \right)^2. \] (28)

The decay of the sequence appearing in the previous sum, allows us to compare this later with its integral counterpart, that is
\[ \sum_{k \geq n} \left( \frac{ce}{2k+1} \right)^{k+1/2} \leq \int_{n-1}^{\infty} e^{-(x+1/2) \log(2(n-1)/ce)} \frac{x^{\alpha+2}}{(n-1)^{\alpha+2}} dx \leq \int_{n-1}^{\infty} e^{-(x+1/2) \log(2(n-1)/ce)} \frac{1}{(n-1)^{\alpha+2}} dx. \] (30)

Hence, by using (29) and (30), one concludes that
\[ \max_{f \in S_n^+, \| f \|_{L^2(I,\omega_n)} = 1} |< F_c^\alpha f, F_c^\alpha f >_{L^2(I,\omega_n)}|^{1/2} \leq \frac{\pi^{5/4} \Gamma(\alpha+1)}{\sqrt{ce^{3/4}}} \frac{1}{\log(2n-1/ce)} \left( \frac{ce}{2n-1} \right)^{n-1/2} \frac{1}{(n-1)^{\alpha+2}}. \] (32)

To conclude the proof of the theorem, it suffices to use the Courant-Fischer-Weyl Min-Max variational principle. \( \square \)
Remark 1. By the fact that \( \lambda_n^\alpha(c) = \frac{c}{\pi^2} |\mu_n^\alpha(c)|^2 \) and straightforward computations one gets for \( c, \alpha > 0 \) and for all \( n > \frac{c+1}{2} \) :

\[
\lambda_n^\alpha(c) \leq \frac{K_\alpha}{c^{n+2} \log^2 \left( \frac{2n-1}{c} \right)^{2n+\alpha+1}} \quad \text{with} \quad K_\alpha = \frac{\pi^{3/2}}{2} (2/e)^{\alpha+3} \Gamma(\alpha+1)
\]

Remark 2. The decay rate given by the last proposition improve the two results mentioned above \([10]\) and \([11]\). Indeed, we have a more precise result furthermore the proof given in \([8]\) contains two serious drawbacks. It is only valid in the particular case \( 0 < \alpha < 3/2 \) and clearly more sophisticated than the previous one.

### 3.2 Local estimates of the GPSWFs

In this paragraph, we give a precise local estimate of the GPSWFs, which is valid for \( 0 \leq \alpha \leq 1/4 \). Then, this local estimate will be used to provide us with a new lower bound for the eigenvalues \( \lambda_n^\alpha(c) \) of the differential operator \( \mathcal{L}_c^\alpha \), for \( 0 \leq \alpha \leq 1/4 \). For this purpose, we first recall that \( \psi_{n,c}^{(\alpha)} \) are the bounded solutions of the following ODE:

\[
\omega_n(x) \mathcal{L}_c^\alpha \psi(x) + w_n(x) \chi_n \alpha \psi(x) = (w_n(x) \psi(x)(1-x^2))' + w_n(x)(\chi_n^\alpha - c^2 x^2) \psi(x) = 0, \quad x \in [-1, 1].
\]  

(33)

As it is done in \([8]\), we consider the incomplete elliptic integral

\[
S(x) = \int_x^1 \frac{1 - q t^2}{1 - t^2} dt, \quad q = \frac{c^2}{\lambda_n^\alpha(c)} < 1.
\]  

(34)

Then, we write \( \psi_{n,c}^{(\alpha)} \) into the form

\[
\psi(x) = \phi_n(x) V(S(x)), \quad \phi_n(x) = (1 - x^2)^{(-1/2 + \alpha/2)} (1 - qx^2)^{-1/2}.
\]  

(35)

By combining \([33]\), \([55]\) and using straightforward computations, it can be easily checked that \( V(\cdot) \) satisfies the following second order differential equation

\[
V''(s) + (\chi_n \alpha + \theta_n(s)) V(s) = 0, \quad s \in [0, S(0)]
\]  

(36)

with

\[
\theta_n(S(x)) = \frac{1}{\phi_n(x) w_n(x)} \frac{d}{dx} \left( \frac{Q_n'(x)}{Q_n(x)} \right)
\]

(37)

Define \( Q_n(x) = w_n(x)^2 (1 - x^2)(1 - qx^2) \), then we have

\[
\frac{\phi_n(x)}{w_n(x)} = -1/4 \frac{Q_n'(x)}{Q_n(x)}.
\]

It follows that \( \theta_n(s) \) can be written as

\[
\theta_n(S(x)) = \frac{1}{16(1 - qx^2)} \left[ \frac{Q_n'(x)}{Q_n(x)} \right]^2 (1 - x^2) - 4 \frac{d}{dx} \left( \frac{Q_n'(x)}{Q_n(x)} \right) - 4(1 - x^2) \frac{Q_n'(x)}{Q_n(x)} \frac{w_n'(x)}{w_n(x)}.
\]

Since \( Q_n(x) = w_n^2(x) Q_0(x) \), then we have

\[
\frac{Q_n'(x)}{Q_n(x)} = 2 \frac{w_n'(x)}{w_n(x)} + \frac{Q_0'(x)}{Q_0(x)} \text{ and } \frac{w_n'(x)}{w_n(x)} = -\frac{2\alpha x}{1 - x^2}.
\]

Hence, we have

\[
\theta_n(S(x)) = \theta_0(S(x)) + \frac{1}{4(1 - qx^2)} \left[ \frac{w_n'(x)}{w_n(x)} \right]^2 (1 - x^2) + 2 \frac{d}{dx} \left[ (1 - x^2) \frac{w_n'(x)}{w_n(x)} \right]
\]

(38)

The previous equality allows us to prove the following lemma.
Lemma 1. For any $0 \leq \alpha \leq \frac{1}{4}$ and $0 < q = \frac{\epsilon^2}{\chi_n^\alpha(c)} < \frac{3}{17}$, we have $\theta_\alpha(S(x))$ is increasing on $[0,1]$.

Proof: We use the notation $u = 1 - x^2$ and with Straightforward computations, we have

$$(\theta_\alpha(S(x)))' = (\theta_0(S(x)))' + \frac{2x}{(1 - qx^2)^2(1 - x^2)^2} [q\alpha(\alpha + 1)(1 - x^2)^2 - q\alpha^2(1 - x^2) - \alpha^2(1 - qx^2)]$$

$$= \frac{2xH(u)}{4u^2(1 - q + qu)^2}$$

where

$$H(u) = G(u) + 4(1 - q + qu^2)[q(\alpha^2 + \alpha)u^2 - \alpha^2(1 - q + 2qu)]$$

$$\geq (1 - q + qu^2)(G(u) + 4[q(\alpha^2 + \alpha)u^2 - \alpha^2(1 - q + 2qu)]$$

with $G(u) \geq \frac{(1 - q)^2}{4}(4 - 4q + u(17q - 3)) \geq 0, \forall q > 0, \alpha > 0$ given by [??] then

$$H(u) \geq (1 - q + qu^2)^2 \left[ \frac{(1 - q)^2}{4}(4 - 4q + u(17q - 3)) + 4[-q\frac{\alpha^3}{\alpha + 1} - (1 - q)^2] \right]$$

$$\geq (1 - q + qu^2)^2 \left[ (1 - q)^3 - 4\alpha^2(1 - q)\frac{1}{\alpha + 1} + \frac{(1 - q)^2}{4}(17q - 3) \right]$$

Then if $0 \leq q \leq \frac{3}{17}$ and $\alpha \leq \frac{1}{4}$ we have

$$H(u) \geq (1 - q + qu^2)^2 \left[ (1 - q)^3 - 4\alpha^2(1 - q)\frac{1}{\alpha + 1} + \frac{(1 - q)^2}{4}(17q - 3) \right]$$

$$\geq (1 - q + qu^2)^2 \left[ \frac{(1 - q)^2}{4}(1 + 13q) - 4\alpha^2 \right] \geq 0$$

□

As a consequence of the previous lemma, one gets.

Lemma 2. For any $0 \leq \alpha \leq \frac{1}{4}$ and for any integer $n \in \mathbb{N}$, with $0 < q = \frac{\epsilon^2}{\chi_n^\alpha(c)} < \frac{3}{17}$, we have

$$\sup_{x \in [0,1]} \sqrt{(1 - x^2)(1 - qx^2)}\omega_\alpha(x)|\psi_n^{(\alpha)}(x)|^2 \leq |\psi_n^{(\alpha)}(0)|^2 + \frac{|\psi_n^{(\alpha)'}(0)|^2}{\chi_n^\alpha(c)} = A^2 \leq 2\alpha + 1 \quad (39)$$

Proof. For $0 \leq \alpha \leq \frac{1}{4}$ and $0 < q < \frac{3}{17}$, we have $\theta_\alpha \circ S$ is increasing. Since $S$ is decreasing then $\theta$ is also decreasing. We define $K(s) = |V(s)|^2 + \frac{1}{\chi_n^\alpha(c)+\theta(s)}|V'(s)|^2$. By the fact that $K(s)$ and $\frac{1}{\chi_n^\alpha(c)+\theta(s)}$ has the same monotonicity and $\theta$ is decreasing, we conclude that $K$ is increasing. Then

$$|V(S(x))|^2 \leq |K(S(x))| \leq |K(S(0))|$$

We remark easily that $V(S(0)) = \psi_n^{(\alpha)}(0)$ and $V'(S(0)) = \psi_n^{(\alpha)'}(0)$ then we conclude for the first inequality of (39). For the second inequality see remark.[3]

□

Next, we give some lower and upper bounds for the eigenvalues $\chi_n^\alpha(c)$, the $n + 1$th eigenvalues of the differential operator. This is given by the following proposition.

Proposition 2. For $c$ and $n$ such that $0 \leq \alpha \leq \frac{1}{4}$ and $0 < q < \frac{3}{17}$, we have :

$$n(n + 2\alpha + 1) + C_n\epsilon^2 \leq \chi_n^\alpha(c) \leq n(n + 2\alpha + 1) + \epsilon^2,$$

where $C_n = 2(2\alpha + 1)^2 + 1 - 2(2\alpha + 1)\sqrt{1 + (2\alpha + 1)^2}$. 

8
Proof. By differentiating with respect to $c$ the differential equation satisfied by GPSWFs, one gets

$$(1 - x^2)\partial_c \psi_{n,c}^{(\alpha)}(x) - 2(\alpha + 1)x\partial_c \psi_{n,c}^{(\alpha)}(x) + (\chi_n^\alpha(c) - c^2x^2)\partial_c \psi_{n,c}^{(\alpha)}(x) + \left(\partial_c \chi_n^\alpha(c) - 2cx^2\right)\psi_{n,c}^{(\alpha)}(x) = 0$$

Hence, we have

$$\left(L_c^{(\alpha)} + \chi_n^\alpha(c)Id\right)\partial_c \psi_{n,c}^{(\alpha)} + \left(\partial_c \chi_n^\alpha(c) - 2cx^2\right)\psi_{n,c}^{(\alpha)} = 0$$

It’s well known that $L_c^{(\alpha)}$ is a self-adjoint operator and have $\psi_{n,c}^{(\alpha)}$ as eigenfunctions. Hence, we have

$$< \left(L_c^{(\alpha)} + \chi_n^\alpha(c)Id\right)\partial_c \psi_{n,c}^{(\alpha)}; \psi_{n,c}^{(\alpha)} >_{\omega_n} = 0$$

It follows that

$$\int_{-1}^{1} \left(\partial_c \chi_n^\alpha(c) - 2cx^2\right)\psi_{n,c}^{(\alpha)}(x)dx = 0.$$ 

From the fact that $\left\|\psi_{n,c}^{(\alpha)}\right\|_{L^2(I,\omega_n)} = 1$, one gets

$$\partial_c \chi_n^\alpha(c) = 2c \int_{-1}^{1} x^2\psi_{n,c}^{(\alpha)}(x)\omega_n(x)dx \quad (40)$$

As it is done in [2], we denote by $A = \left[\psi_{n,c}^{(\alpha)}(0) + \frac{\psi_{n,c}^{(\alpha)}(0)(1)}{\chi_n^\alpha(c)}\right]^{1/2}$ and consider the auxiliary function

$$K_n(t) = -(1 - t^2)^{2\alpha + 1}\psi_{n,c}^{(\alpha)} - \frac{(1 - t^2)2\alpha + 2}{\chi_n^\alpha(c)(1 - qt^2)}(\psi_{n,c}^{(\alpha)})^2$$

Straight forward computations give us

$$K_n'(t) = 2(2\alpha + 1)t(1 - t^2)2\alpha\psi_{n,c}^{(\alpha)}(t) - H(t)\psi_{n,c}^{(\alpha)}(t)^2$$

with $H(t) \geq 0$ for $t \in [0, 1]$. Hence

$$K_n'(t) \leq 2(2\alpha + 1)t\omega_n(t)^2\psi_{n,c}^{(\alpha)}(t) \leq 2(\alpha + 1)t\omega_n(t)\psi_{n,c}^{(\alpha)}(t)^2.$$ 

So one has the inequality

$$K_n(1) - K_n(0) = A^2 \leq 2(\alpha + 1) \int_{0}^{1} t|\psi_{n,c}^{(\alpha)}(t)|^2\omega_n(t)dt \leq (\alpha + 1) \int_{-1}^{1} t^2|\psi_{n,c}^{(\alpha)}(t)|^2\omega_n(t)dt \leq (\alpha + 1)^{1/2}. \quad (41)$$

That is

$$A^2 \leq (2\alpha + 1)B^{1/2}. \quad (42)$$

Remark that (39) implies that

$$1 - B \leq 2A^2 \quad (43)$$

By combining (42) and (43) we conclude that $B^{1/2}$ is bounded below by the largest solution of the equation $X^2 + 2(\alpha + 1)X - 1 = 0$

Remark 3. By using (42) and since $B \leq 1$, one concludes that the constant $A$, given in (39) satisfies $A^2 \leq 2\alpha + 1.$
4 Qualities of approximation by GPSWFs

4.1 Approximation by the GPSWFs in Weighted Sobolev spaces

In this section, we study the issue of the quality of spectral approximation of a function \( f \in H^s_{\alpha}([-1,1]) \) by its truncated GPSWFs series expansion. Note that a different spectral approximation result by GPSWFs has already given in [16]. It is important to mention here that this approximations are given in a different approach. More precisely, by considering the weighted Sobolev space associated with the differential operator defined by

\[
\tilde{H}^{\alpha}_\omega(I) = \{ f \in L^2(I,\omega) : \|f\|_{\tilde{H}^{\alpha}_\omega(I)} = \sum_{k=0}^{\infty} (\chi_{n_k}^\alpha)^r |f_k|^2 < \infty \}
\]

where \( f_k \) are expansions coefficients of \( f \) in the GPSWFs's basis. Then it has been shown that:

For any \( f \in \tilde{H}^{\alpha}_\omega(I) \) with \( r \geq 0 \)

\[
\|S_{N,c} f - f\|_{L^2(I,\omega)} \leq (\chi_{N+1}^\alpha)^{-r/2} \|f\|_{\tilde{H}^{\alpha}_\omega(I)}
\]

For more details, we refer the reader to [16].

In the following, we need a theorem of [7] Let \( c > 0 \), be a fixed positive real number. Then, for all positive integers \( n, k \) such that \( q < 1 \) and \( k(k + 2\alpha + 1) + C'_\alpha c^2 \leq \chi_{n}^\alpha(c) \), we have

\[
|\beta_n^k| \leq C_\alpha \left( \frac{2\sqrt{\chi_{n}^\alpha(c)}}{c} \right)^k |\mu_n^\alpha(c)|. \tag{44}
\]

With \( C_\alpha \) a constant depends only on \( \alpha \), and \( C_\alpha = \frac{2^\alpha (3/2)^{3/4} (3/2 + 2\alpha)^{3/4 + \alpha}}{c^{2\alpha + 3/2}} \)

**Remark 4.** The condition \( k(k + 2\alpha + 1) + C_\alpha c^2 \leq \chi_{n}^\alpha(c) \) of previous theorem can be replaced with the following more explicit condition. \( n \geq cA \) and \( k \leq n/B \), for any real \( A, B \) such that

If \( 0 \leq \alpha \leq 1/4 \), just take \( A^2 = B^2 = 2.18 \)

If \( -1 < \alpha < 0 \) or \( 1/4 \leq \alpha \), just take \( A^2 = B^2 = 2.8 \)

**Lemma 3.** Let \( c > 0 \) and \( \alpha > -1 \), then for all positive integers \( n, k \) such that, \( k \leq n/1.7 \) and \( n \geq (75 + 40\alpha)^{0.7} c \) we have

\[
|\beta_n^k| \leq k_{\alpha,c} e^{-an} \tag{45}
\]

with \( k_{\alpha,c} \) and a positive constants .

**Proof.** By [44], [20], the inequality \( \chi_{n}^\alpha(c) \leq n(n + 2\alpha + 1) + c^2 \) and the previous remark. One concludes that for \( c > 0 \) and \( \alpha > -1 \), and for all positive integers \( n, k \) such that, \( k \leq n/A \) and
\[ n \geq Ac \text{ with } A \geq 1.7, \text{ we have,} \]
\[
|\beta^n_k| \leq C_\alpha \left( \frac{2\sqrt{\chi^0_n(c)}}{c} \right)^k |\mu^n_{\alpha}(c)| \quad (46)
\]
\[
\leq C_\alpha \left( \frac{1}{c \log(\frac{2n-1}{ec})} \right) \left( \frac{ec}{2n-1} \right)^{2A} \left( \frac{2\sqrt{n(n+2\alpha+1)+c^2}}{c} \right)^\frac{n+2+1}{2} \left( \frac{n+2\alpha+1+2A}{2n-1} \right) \quad (47)
\]

For the appropriate value of value of \( A = 1.7 \), we have
\[
|\beta^n_k| \leq C_\alpha \left( \frac{1}{c \log(\frac{2n-1}{ec})} \right) \left( \frac{c(1.35 + \frac{2\alpha+1}{1.7c})^{0.7}}{2n(0.36 - \frac{0.1}{c})^{2A}} \right) \quad (48)
\]
\[
\leq C_\alpha \left( \frac{1}{c \log(\frac{2n-1}{ec})} \right) \left( \frac{c(1.94 + 1.18\alpha)^{0.7}}{0.078n} \right) \quad (49)
\]
\[
\leq C_\alpha \left( \frac{1}{c \log(\frac{2n-1}{ec})} \right) \left( \frac{c(74.6 + 45.4\alpha)^{0.7}}{n} \right) \quad (50)
\]

Then for all \( n \geq (75 + 46\alpha)^{0.7}c \). We have \( (45) \) \( \square \)

**Lemma 4.** for \( c \geq 0, \alpha > -1, \) for all positif real \( n,k \) such that \( k \leq 0.14n \) and \( n \geq (75 + 46\alpha)^{0.7}c \). There exist \( C_{\alpha,c}, \delta > 0 \) such that
\[
|\langle e^{i k \pi x}, \psi_{n,c}^\alpha \rangle|_{L^2(I,\omega_n)}| \leq C_{\alpha,c}e^{-\delta n} \quad (51)
\]

**Proof.** We have
\[
|\langle e^{i k \pi x}, \psi_{n,c}^\alpha \rangle|_{L^2(I,\omega_n)}| \leq \sum_{m=0}^{[n/4]} |\beta^n_m||\langle e^{i k \pi x}, \tilde{\phi}_m(\alpha,\alpha) \rangle|_{L^2(I,\omega_n)}| \leq \sum_{m=0}^{[n/4]} |\beta^n_m||\langle e^{i k \pi x}, \tilde{\phi}_m(\alpha,\alpha) \rangle|_{L^2(I,\omega_n)}| + \sum_{m=[n/4]+1}^{[n/4]} |\beta^n_m||\langle e^{i k \pi x}, \tilde{\phi}_m(\alpha,\alpha) \rangle|_{L^2(I,\omega_n)}| = I_1 + I_2 \quad (52)
\]
For $I_2$ since $|\beta_m^m| \leq 1$ and by using (21), (23), one gets

$$I_2 \leq \sum_{m \geq [n/A]+1} \left| \langle e^{ik\pi x}, F_m^{(\alpha,\alpha)} \rangle_{L^2(I,\omega_{\alpha})} \right| \leq \sum_{m \geq [n/A]+1} \sqrt{\pi} \frac{2}{k \pi} \frac{(2m+1)}{(m+1)} \frac{(k\pi)^{m}}{\Gamma(m+1)} |J_{m+\alpha+1/2}(k\pi)|$$

$$\leq \sum_{m \geq [n/A]+1} \sqrt{\pi} \frac{2}{k \pi} \frac{2^m}{2m+1} \frac{(k\pi)^m}{\Gamma(m+1)} \leq \sum_{m \geq [n/A]+1} \sqrt{\pi} \frac{2}{2m+1} \frac{(k\pi)^m}{\Gamma(m+1)} \leq K \sqrt{2c+1} \frac{(k\pi)^{n}}{2m+1}$$

(53)

Where $K$ is a positive constant.

It is clear that the appropriate value of $A$ is 1.7 then for $k \leq \frac{12n}{\pi} \simeq 0.14n$ there is $b > 0$ such as

$$I_2 \leq K \sqrt{2c+1} \frac{(k\pi)^{n}}{2m+1} \leq Ke^{-bn}$$

(54)

For $I_1$, we have $|\langle e^{ik\pi x}, F_m^{(\alpha,\alpha)} \rangle_{L^2(I,\omega_{\alpha})}| \leq 1$, so we use (45). For $n \geq (75 + 46\alpha)^{0.7}c$ we have

$$I_1 \leq \sum_{m=0}^{[n/1.7]} |\beta_m^m| \leq K\alpha e^{-an}$$

(55)

We first recall that if $f \in H^s_{\alpha, per}$ then $f(x) = \sum_{k \geq 0} \langle f, \tilde{P}_{n}^{(\alpha,\alpha)} \rangle_{L^{2}_{\alpha}(I)} \tilde{P}_{n}^{(\alpha,\alpha)}$ and

$$\|f\|_{H^s_{\alpha, per}}^2 = \sum_{k \geq 0} \langle f, \tilde{P}_{n}^{(\alpha,\alpha)} \rangle_{L^{2}_{\alpha}(I)}^2 (1 + k^2)^s.$$

**Proposition 3.** Let $c > 0$ and $\alpha > -1$, then there exist constants $K > 0$ and $a > 0$ such that, when $N > (75 + 46\alpha)^{0.7}c$ and $f \in H^s_{\alpha, per}; s > 0$, we have the inequality

$$\|f - S_N(f)\|_{L^2_{\alpha}(I)} \leq \left( 1 + \left( \frac{N}{2} \right)^2 \right)^{1/2} \|f\|_{H^s_{\alpha, per}} + Ke^{-aN} \|f\|_{L^2_{\alpha}(I)}$$

(56)

Where $S_N(f)(t) = \sum_{n < N} \langle f, \psi_{n,c}^{\alpha} \rangle_{L^2_{\alpha}(I)} \psi_{n,c}^{\alpha}$

**Proof.** Assume that $f \in H^s_{\alpha, per}$ such that $\|f\|_{L^2_{\alpha}(I)} = 1$ and

$$f(x) = g(x) + h(x) = \sum_{k \geq [\frac{N}{2}]} \langle f, \tilde{P}_{n}^{(\alpha,\alpha)} \rangle_{L^{2}_{\alpha}(I)} \tilde{P}_{n}^{(\alpha,\alpha)} + \sum_{k < [\frac{N}{2}]} \langle f, \tilde{P}_{n}^{(\alpha,\alpha)} \rangle_{L^{2}_{\alpha}(I)} \tilde{P}_{n}^{(\alpha,\alpha)}$$

then

$$\|g\|_{L^2_{\alpha}(I)}^2 = \sum_{k \geq [\frac{N}{2}]} \langle f, \tilde{P}_{n}^{(\alpha,\alpha)} \rangle_{L^{2}_{\alpha}(I)} (1 + k^2)^s \leq \frac{1}{(1 + (\frac{N}{2})^2)^s} \|f\|_{H^s_{\alpha, per}}^2$$

(57)
On the other hand by (15) there exist $a > 0$ and $K > 0$ such that, for all $N \geq (74.6 + 45.4a)^{0.7c}$

$$
\|h - S_N(h)\|_{L^2(I)}^2 = \sum_{n \geq N} |<f, \psi_{\alpha,c} >_{L^2(I)}|^2 = \sum_{n \geq N} \sum_{k \leq (\frac{a}{2})} |\beta_n|^{2} | <f, \tilde{\chi}_{\alpha,c} >_{L^2(I)}|^2 \\
\leq Ke^{-aN} \|f\|_{L^2(I)}^2
$$

finally by combining (57) and (58), on gets (56) \( \square \)

**Remark 5.** The previous proposition is considered as the generalization of a similar result given in [12], in the special case $\alpha = 0$

### 4.2 Approximate dimension of the space of almost time-limited functions from the restricted Paley-Wiener space.

The aim of this paragraph is to show that the Landau-Pollak approximate dimension of the space of band-limited and almost time-limited functions [19] initially proved in the context of the classical PSWFs, can be generalized in the context of the GPSWFs. That is for functions from $B_{c,1}^{(\alpha)}$, that are almost time-limited. In this case, we have some additional difficulties, compared to the classical case. In fact and unlike the PSWFs, the GPSWFs do not have the the double orthogonality property over $[-1, 1]$ and over $\mathbb{R}$. To overcome this problem, we will use the new inner product $<\cdot, \cdot>_{\alpha}$, given by [14].

In the sequel, for $\varepsilon > 0$, we let $E^{(\alpha)}_{c,\varepsilon}$ denote the space of almost time-limited functions from the restricted Paley-Wiener space $B_{c,1}^{(\alpha)}$, given by

$$
E^{(\alpha)}_{c,\varepsilon} = \{ f \in B_{c,1}^{(\alpha)} : \|f\|_{\alpha} = 1, \|f\|_{L^2(I, \omega_\alpha)} = 1 - \varepsilon^2 \}.
$$

**Definition 1.** Let $\phi_0, \phi_1, \ldots, \phi_{N-1} \in B_{c,1}^{(\alpha)}$ and let us denote by $S_N^{\phi}$ the subspace spanned by the $\phi_i$'s, $0 \leq i \leq N - 1$. The deflection $\delta_N(E^{(\alpha)}_{c,\varepsilon}, S_N^{\phi})$ of $E^{(\alpha)}_{c,\varepsilon}(\varepsilon)$ from $S_N^{\phi}$, is defined as follows:

$$
\delta_N(E^{(\alpha)}_{c,\varepsilon}, S_N^{\phi}) = \sup_{f \in E^{(\alpha)}_{c,\varepsilon}(\varepsilon)} \| f - P_N(\phi) \cdot f \|_{\alpha}.
$$

Here $P_N(\phi)$ is the projection operator over $S_N^{\phi}$.

We first recall the following result from [12] that gives us a first result of approximation of band-limited functions by GPSWFs over $[-1, 1]$.

**Proposition 4.** Let $c > 0$, $\alpha \geq 0$ be two real numbers and let $f \in B_{c,1}^{(\alpha)}$. For any positive integer $N > \frac{2e}{\varepsilon}$, let

$$
S_N(f)(x) = \sum_{k=0}^{N} <f, \Psi_{k,c}^{(\alpha)} >_{L^2(I, \omega_\alpha)} \Psi_{k,c}^{(\alpha)}(x).
$$

Here, $\Psi_{k,c}^{(\alpha)}(x) = \frac{1}{\sqrt{\lambda_k^{(\alpha)}(c)}} \psi_{k,c}^{(\alpha)}(x)$. Then, we have

$$
\left( \int_{-1}^{1} |f(t) - S_N f(t)|^2 \omega_\alpha(t) dt \right)^{1/2} \leq C_1 \sqrt{\lambda_k^{(\alpha)}(c)} (\chi_N(c))^{(1+\alpha)/2} \|f\|_{L^2(\mathbb{R})},
$$

and

$$
\sup_{x \in [-1, 1]} |f(x) - S_N f(x)| \leq C_1 \sqrt{\lambda_k^{(\alpha)}(c)} (\chi_N(c))^{1+\alpha/2} \|f\|_{L^2(\mathbb{R})},
$$

for some uniform constant $C_1$ depending only on $\alpha$.
Recall that in the previous proposition $\chi^\alpha_n(c)$ is the $n$-th eigenvalue of the differential operator $L_c$ and satisfying the bounds given by (5). The following proposition improves the result of the previous one, in the case where the function $f$ belongs to $B^{(\alpha)}_c$.

**Proposition 5.** Let $c > 0$, $\alpha \geq 0$ be two real numbers and let $f \in B^{(\alpha)}_c$. For any positive integer $N \geq \frac{2c\pi}{\alpha}$, let

$$S_N(f)(x) = \sum_{k=0}^{N} < f, \Psi^{(\alpha)}_{k,c} >_{L^2(I,\omega_\alpha)} \Psi^{(\alpha)}_{k,c}(x).$$

Then, we have

$$\left( \int_{-1}^{1} |f(t) - S_N f(t)|^2 \omega_\alpha(t) dt \right)^{1/2} \leq C_1 \sqrt{\lambda^{(\alpha)}_N(c)} \| f \|_\alpha. \quad (62)$$

Moreover, for any $N \geq \frac{2c\pi}{\alpha}$, we have

$$\sup_{x \in [-1,1]} |f(x) - S_N f(x)| \leq C_1 \sqrt{\lambda^{(\alpha)}_N(c)} (\chi_N(c))^{1/2+\alpha/2} \| f \|_\alpha. \quad (63)$$

for some uniform constant $C_1$ depending only on $\alpha$.

**Proof:** We first note that since the $\psi^{(\alpha)}_{k,c}$ form an orthogonal basis of $L^2(I,\omega_\alpha)$ with $\| \psi^{(\alpha)}_{k,c} \|_{L^2(I,\omega_\alpha)} = \sqrt{\lambda^{(\alpha)}_k(c)}$, then the $\Psi^{(\alpha)}_{k,c}$ form an orthonormal basis of $L^2(I,\omega_\alpha)$. Hence, for any $f \in B^{(\alpha)}_c$, we have

$$f(x) = \sum_{n=0}^{\infty} \beta_n \Psi^{(\alpha)}_{n,c}(x) = \sum_{n=0}^{\infty} \beta_n \sqrt{\lambda^{(\alpha)}_n(c)} \psi^{(\alpha)}_{n,c}(x). \quad (64)$$

Moreover, from Parseval’s equality, one gets

$$\int_{-1}^{1} |f(t) - S_N f(t)|^2 \omega_\alpha(t) dt = \sum_{n=N+1}^{\infty} |\beta_n|^2. \quad (65)$$

On the other hand, we have already shown that the $\psi^{(\alpha)}_{n,c}(x)$ form an orthonormal basis of $B^{(\alpha)}_c$, hence, we have

$$f(x) = \sum_{n=0}^{\infty} \gamma_n \psi^{(\alpha)}_{n,c}(x), \quad \forall x \in \mathbb{R}. \quad (66)$$

Since both equalities (64) and (66) agree on $[-1,1]$, then one concludes that

$$\beta_n = \sqrt{\lambda^{(\alpha)}_n(c)} \gamma_n, \quad \forall n \geq 0.$$

Also, Parseval’s equality applied to (66) gives us

$$\sum_{n=0}^{\infty} |\gamma_n|^2 = \| f \|_{\alpha}^2.$$

By combining the previous two equalities, one concludes that

$$\sum_{n=N+1}^{\infty} |\beta_n|^2 \leq \lambda^{(\alpha)}_N(c) \| f \|_{\alpha}^2.$$
Finally, by using the fact that the $\lambda_n^\alpha(c)$ have a fast decay starting from $n = 2c/\pi$, one gets the error bound (62). Finally, to get the uniform error given by (63), it suffices to combine the previous analysis together with the following bound the $\Psi_n(c)$, given in [7]

$$
\sup_{x \in [-1,1]} |\Psi_n(c)(x)| \leq C_\alpha (\lambda_n^\alpha(c))^{1/2 + \alpha/2}, \quad n \geq 2c/\pi. \quad \square
$$

Next, we define the projection operator over $B_c^{(\alpha)}$, denoted by $\Pi_c^{(\alpha)}$ and given by

$$
\Pi_c^{(\alpha)} f(x) = \int_{-c}^c \tilde{f}(t) e^{i\alpha x} \left( \frac{t}{c} \right) dt. \quad (67)
$$

It is clear that $\Pi_c^{(\alpha)} f \in B_c^{(\alpha)}$. By using the finite Fourier transform of the weight function

$$
\int_{-1}^1 e^{ixy} \omega_\alpha(y) dy = \sqrt{\pi} 2^{\alpha+1/2} \Gamma(\alpha+1) \frac{J_{\alpha+1/2}(x)}{x^{\alpha+1/2}} = K_\alpha(x), \quad x \in \mathbb{R}, \quad (68)
$$

one gets the expression of $\Pi_c^{(\alpha)} f$ in terms of $f$, that is

$$
\Pi_c^{(\alpha)} f(x) = c \int_{\mathbb{R}} K_\alpha(c(x-u)) f(u) du. \quad (69)
$$

Note that

$$
\Pi_c^{(\alpha)} 1_{[-1,1]} = \frac{1}{2\pi} Q_c^{(\alpha)}, \quad (70)
$$

where $Q_c^{(\alpha)}$ is as given by [4]. The following theorem tells us that the $\psi_n^{(\alpha)}$ are actually the best basis for the approximation of almost-time limited functions from the space $B_c^{(\alpha)}$.

**Remark 6.** By using [7] and straightforward computations one gets

$$
\left( \int_{-1}^1 |f(t) - S_N f(t)|^2 \omega_\alpha(t) dt \right)^{1/2} \leq C_\alpha \frac{1}{\log\left( \frac{2N-1}{ec} \right)} \left( \frac{ec}{2N-1} \right)^{N+\alpha+1} \|f\|_\alpha \quad (71)
$$

**Theorem 1.** For every positive integer $N$, $S_N^{\psi}$ best approximate $E_c^{(\alpha)}(\varepsilon)$ in the sense that the deflexion of $E_c^{(\alpha)}(\varepsilon)$ for $\alpha \geq 0$ from that subspace is smaller than from any other subspace of dimension $N$. Moreover, for any $N \geq \left[ \frac{2\varepsilon}{\alpha} \right] + 1$, we have

$$
\delta_N(E_c^{(\alpha)}(\varepsilon), S_N^{\psi}) \leq \frac{\varepsilon^2}{1 - \lambda_N^{(\alpha)}}, \quad (72)
$$

**Proof.** We first compute the deflexion of $E_c^{(\alpha)}(\varepsilon)$ from $S_N^{\psi}$. As it is done in [9], we have to consider two cases:

**Case 1:** If $1 - \varepsilon^2 \leq \lambda_N^{(\alpha)}(c)$. Since $\|\psi_n^{(\alpha)}\|_\alpha = 1$ and $\|\psi_n^{(\alpha)}\|_{L^2(\mathbb{R}, \omega_\alpha)} = \lambda_n^{\alpha}(c)$, then $\delta_N(E_c^{(\alpha)}(\varepsilon), S_N^{\psi}) = 1$. To prove this it suffices to consider the function

$$
f = a_N \psi_n^{(\alpha)} + a_{N+1} \psi_{n+1}^{(\alpha)},
$$

where

$$
a_N^2 + a_{N+1}^2 = 1 \quad \text{and} \quad \lambda_n^{\alpha} a_N^2 + \lambda_{n+1}^{\alpha} a_{N+1}^2 = 1 - \varepsilon^2.
$$
Case 2: If \( 1 - \varepsilon^2 \geq \lambda_N^{(\alpha)}(c) \), then \( \delta_N(E_c^{(\alpha)}(\varepsilon), S^N_\psi) = \frac{\lambda_0^{(\alpha)}(c) - (1 - \varepsilon^2)}{\lambda_0^{(\alpha)}(c) - \lambda_N^{(\alpha)}(c)} \). In fact, let \( f \in E_c^{(\alpha)}(\varepsilon) \), then by the fact that \( \{\psi_{n,c}, n \in \mathbb{N}\} \) is an orthonormal basis of \( B_c^{(\alpha)} \), one gets
\[
    f = \sum_{n=0}^{\infty} a_n \psi_{n,c}^{(\alpha)} \quad a_n = \langle f, \psi_{n,c}^{(\alpha)} \rangle^\alpha
\]
with
\[
    \sum_{n=0}^{\infty} a_n^2 = 1 \quad \text{and} \quad \sum_{n=0}^{\infty} \lambda_n^\alpha a_n^2 = 1 - \varepsilon^2. \tag{73}
\]
Note that
\[
    \left\| f - \sum_{n=0}^{N-1} a_n \psi_{n,c}^{(\alpha)} \right\|_\alpha^2 = \sum_{k=N}^{\infty} |a_k|^2
\]
From \[73\], one can easily see that
\[
    \lambda_0^{(\alpha)}(c) - (1 - \varepsilon^2) = \sum_{k=0}^{N-1} (\lambda_0^{(\alpha)}(c) - \lambda_k^{(\alpha)}(c)) a_k^2 + \sum_{k=N}^{\infty} (\lambda_0^{(\alpha)}(c) - \lambda_k^{(\alpha)}(c)) a_k^2.
\]
Hence, by using the monotonicity of the \( \lambda_n^\alpha \), one gets
\[
    \left\| f - \sum_{n=0}^{N-1} a_n \psi_{n,c}^{(\alpha)} \right\|_\alpha^2 \leq \frac{\lambda_0^{(\alpha)}(c) - (1 - \varepsilon^2)}{\lambda_0^{(\alpha)}(c) - \lambda_N^{(\alpha)}(c)}. \tag{74}
\]
To conclude for the proof, it suffices to consider \( f = a_0 \psi_{0,c}^{(\alpha)} + a_N \psi_{N,c}^{(\alpha)} \), where
\[
    a_0^2 + a_N^2 = 1 \quad \text{and} \quad a_0^2 \lambda_0^{(\alpha)}(c) + a_N^2 \lambda_N^{(\alpha)}(c) = 1 - \varepsilon^2.
\]
Consequently, we have
\[
    \delta_N(E_c^{(\alpha)}(\varepsilon), S^N_\psi) = \begin{cases} 1 & \text{if } 1 - \varepsilon^2 \leq \lambda_N^{(\alpha)}(c) \\ \frac{\lambda_0^{(\alpha)}(c) - (1 - \varepsilon^2)}{\lambda_0^{(\alpha)}(c) - \lambda_N^{(\alpha)}(c)} & \text{if } 1 - \varepsilon^2 \geq \lambda_N^{(\alpha)}(c) \end{cases}.
\]
Now, we consider the following map:
\[
    \Phi : B_c^{(\alpha)} \rightarrow \mathbb{R}^2 \quad f \rightarrow \left( x = \frac{\|1_{[-1,1]} f\|}{\|f\|_\alpha^2}, y = \frac{\|f\|_\alpha^2 - \|P_N f\|_\alpha^2}{\|f\|_\alpha^2} \right)
\]
Note that the \( x \)-coordinate of points in \( R(\alpha) \) satisfy \( 0 \leq x \leq \lambda_0^{(\alpha)}(c) \), while the \( y \)-coordinates satisfy \( 0 \leq y \leq 1 \). Also, \( x = \lambda_0^{(\alpha)}(c) \) is achieved by \( \psi_{0,c}^{(\alpha)} \) and \( y = 1 \) is achieved by functions orthogonal to \( S^N_\psi \). Therefore by Weyl-Courant lemma, we have
\[
    \sup_{y = 1} x \geq \lambda_N^{(\alpha)}(c).
\]
To prove that \( R(\alpha) = \Phi(B_c^{(\alpha)}) \) is convex, we use the same techniques developed in \[9\]. We should only mention that for \( a, b \in \mathbb{R} \), the operator \( a P_c^{(\alpha)} 1_{[-1,1]} + b P_N \) is completely continuous. Here, \( P_N \) is the orthogonal projection over \( S^N_\psi \). In fact, \( P_c^{(\alpha)} 1_{[-1,1]} \) is a Hilbert-Schmidt operator and \( P_N^{(\alpha)} \) is a bounded operator. On the other hand, \( P_N \) is a projection over a subspace of finite dimension.
so it is completely continuous. The rest of the proof in [9] is independent of the operators $P_c^{(\alpha)}$ and $I_{[-1,1]}$. By combining the convexity of $R(\alpha)$ with the fact that $(\lambda_N, 1), (0, \lambda_0) \in R(\alpha)$ one gets

$$\delta_N(E_N^{(\alpha)}(\varepsilon), S_N^N) = \sup_{x=1-\varepsilon^2} y \begin{cases} = 1 \text{ if } 1 - \varepsilon^2 \leq \lambda_N^{(\alpha)}(c) \\ \geq \frac{\lambda_N^{(\alpha)}(c) - (1-\varepsilon^2)}{\lambda_0^{(\alpha)}(c) - \lambda_N^{(\alpha)}(c)} \text{ if } 1 - \varepsilon^2 \geq \lambda_N^{(\alpha)}(c) \end{cases}.$$ 

Hence, for $\lambda_N^{(\alpha)}(c) < 1 - \varepsilon^2$ we have $\frac{\lambda_N^{(\alpha)}(c) - (1-\varepsilon^2)}{\lambda_0^{(\alpha)}(c) - \lambda_N^{(\alpha)}(c)} < \frac{\varepsilon^2}{1-\lambda_N^{(\alpha)}(c)}$ and for $\lambda_N^{(\alpha)}(c) \geq 1 - \varepsilon^2$, that is $1 \leq \frac{\varepsilon^2}{1-\lambda_N^{(\alpha)}(c)}$, we have $\delta_N \leq \frac{\varepsilon}{\sqrt{1-\lambda_N^{(\alpha)}(c)}}$. 

5 Numerical results

In this section, we give some numerical examples that illustrate the different results of this work.

Example 1: In this second example, we illustrate the quality of approximation of a band-limited functions by the GPSWFs. To do this, we have considered $\alpha = 1, c = 50$ and the band-limited function $f(x) = \sin(40x)/(40x)$. Then, we have computed the projection $P_N f$ given by:

$$S_N(f)(x) = \sum_{k=0}^{N} < f, \psi_{k,c}^{(\alpha)} >_{L^2(I, \omega_{\alpha})} \psi_{k,c}^{(\alpha)}(x), \quad x \in I = [-1,1].$$

In Figure 2, we have plotted the graphs of $f$, together with the approximation errors $E_N = f - S_N f$, with $N = 20, 30$. Note that as predicted by Proposition 1, Figure 2 indicate that our GPSWFs based approximation method provides us with high accurate approximation of the band-limited functions.

Example 2: In this example, we illustrate the quality of approximation of a band-limited functions by GPSWFs, given by Proposition 2. For this purpose, we have considered $\alpha = 1, c = 50$ and the function $g(x) = c\sqrt{\pi} \left( \frac{2}{cx} \right)^{\alpha+1/2} \Gamma(\alpha+1) J_{\alpha+1/2}(cx)$. Using (68), we have $\tilde{g}(x) = \left( 1-\left( \frac{1}{c} \right)^2 \right)^{\alpha} \chi_{[-c, c]}(x), \chi_{[-c, c]}(x)$. 

Figure 1: (a) Graph of $f(x) = \sin(40x)/(40x)$, (b) graph of the error $E_N = f(x) - P_N f(x)$, with $N = 20$. (c) same as (b) with $N = 30$. 

In Figure 2, we have plotted the graphs of $f$, together with the approximation errors $E_N = f - S_N f$, with $N = 20, 30$. Note that as predicted by Proposition 1, Figure 2 indicate that our GPSWFs based approximation method provides us with high accurate approximation of the band-limited functions.
so that \( g \in B^{(\alpha)}_c \). Then, we have computed the projection \( P_N g \) with \( N = 32, 40 \). In Figure 3, we have plotted the graphs of \( g \), together with the approximation errors \( E_N(x) = f(x) - S_N f(x) \), with \( N = 30, 40 \) and for \( x \in [0, 1] \). By symmetry, the graphs of \( g \) and the associated approximation errors inside \([-1, 0]\) are similar to those obtained in \([0, 1]\). It is interesting to note the high precision with which the GPSWFs approximate functions from the space \( B^{(\alpha)}_c \).

Example 3: In this last example, for any positive real number \( s \), we consider the random function

\[
B_s(x) = \sum_{k=1}^{\infty} \frac{X_k}{k^s} \cos(k\pi x) \quad -1 \leq s \leq 1
\]

Here \( X_k \) is a sequence of independent standard Gaussian random variable.

We compute for \( c = 5\pi \), the truncated series expansion of \( B_s \) in the GPSWFs basis to the order \( N \). The graph of \( B_s \) and the graph of its approximation \( B_{s} \) are given by the figure.

References

[1] G. E. Andrews, R. Askey and R. Roy, Special Functions, Cambridge University Press, Cambridge, New York, 1999.

[2] A. Bonami and A. Karoui, Uniform bounds of prolate spheroidal wave functions and eigenvalues decay, *C. R. Math. Acad. Sci. Paris. Ser. I*, 352 (2014), 229–234.

[3] A. Bonami and A. Karoui, Uniform approximation and explicit estimates of the Prolate Spheroidal Wave Functions, *Constr. Approx.*, (2015) Doi:10.1007/s00365-015-9295-1

[4] A. Bonami and A. Karoui, Spectral decay of time and frequency limiting operator, *Appl. Comput. Harmon. Anal*, (2015), doi: 10.1016/j.acha.2015.05.003.

[5] A. Bonami and A. Karoui, Approximation in Sobolev spaces by Prolate Spheroidal Wave Functions, *Appl. Comput. Harmon. Anal* 42, 361-377 (2017)
Figure 3: (a) Graph of $B_s(x)$ (b) graph of $B_s^{(N)}(x)$ (c) graph of the error $E_N = B_s(x) - B_s^{(N)}$, with $N = 90$.

[6] A. Bonami, P. Jamming and A. Karoui, Non asymptotic behavior of the spectrum of the sinc kernel operator and related applications, arXiv:1804.01257 [math.CA]

[7] A. Karoui and A. Souabni, Generalized Prolate Spheroidal Wave Functions: Spectral Analysis and Approximation of Almost Band-limited Functions, J. Four. Anal. Appl., 22 (2), (2016), 383–412.

[8] A. Karoui & A. Souabni Weighted Finite Fourier Transform Operator: Uniform Approximations of the Eigenfunctions, Eigenvalues Decay and Behaviour. J. Sci. Comp. 71 (2017), 547–570.

[9] H. J. Landau and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty-III. The dimension of space of essentially time-and band-limited signals, Bell System Tech. J. 41, (1962), 1295–1336.

[10] T. Moumni, On essentially time and Hankel band-limited function Integral transforms special functions, 23 (2), (2012), 83–95.

[11] Frank W. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, NIST Handbook of Mathematical Functions, Cambridge University Press, New York, NY, USA, 1st edition, 2010.

[12] Q. Chen, D. Gottlieb, J. S. Hesthaven, Spectral methods based on prolate spheroidal wave functions for hyperbolic PDEs, SIAM J Numer. Anal 43(5) (2005) 1912-1933

[13] D. Slepian and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty I, Bell System Tech. J. 40 (1961), 43–64.

[14] D. Slepian, Prolate spheroidal wave functions, Fourier analysis and uncertainty–IV: Extensions to many dimensions; generalized prolate spheroidal functions, Bell System Tech. J. 43 (1964), 3009–3057.

[15] G. Szegő, Orthogonal polynomials, Fourth edition, American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975.
[16] L. L. Wang and J. Zhang, A new generalization of the PSWFs with applications to spectral approximations on quasi-uniform grids, *Appl. Comput. Harmon. Anal.* 29, (2010), 303–329.

[17] Batir, N. *Inequalities for the gamma function*. Arch. Math. 91, 554563 (2008).