On Gorenstein Graphs with Independence Number at Most Three

Mohammad Reza Oboudi, Ashkan Nikseresht*

Department of Mathematics, Shiraz University, 71457-13565, Shiraz, Iran
E-mail: mr_oboudi@yahoo.com
E-mail: ashkan_nikseresht@yahoo.com

Abstract

Suppose that $G$ is a simple graph on $n$ vertices and $\alpha = \alpha(G)$ is the independence number of $G$. Let $I(G)$ be the edge ideal of $G$ in $S = K[x_1, \ldots, x_n]$. We say that $G$ is Gorenstein when $S/I(G)$ is so. Here, first we present a condition on $G$ equivalent to $G$ being Gorenstein and use this to get a full characterization of Gorenstein graphs with $\alpha = 2$. Then we focus on Gorenstein graphs with $\alpha = 3$ and find the number of edges and the independence polynomial of such graphs. Finally, we present a full characterization of triangle-free Gorenstein graphs with $\alpha = 3$.

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1 Introduction

Throughout this paper, $K$ is a field, $S = K[x_1, \ldots, x_n]$ and $G$ denotes a simple undirected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. Recall that the edge ideal $I(G)$ of $G$ is the ideal of $S$ generated by $\{x_ix_j | v_iv_j \in E(G)\}$. Many researchers have studied how algebraic properties
S/I(G) relates to the combinatorial properties of G (see [2–4, 11] and references therein). An important algebraic property that recently has gained attention is being Gorenstein. Recall that Gorenstein rings and Cohen-Macaulay rings (CM rings for short) are central concepts in commutative algebra. We refer the reader to [1] for their definitions and basic properties. We say that G is a Gorenstein (resp, Cohen-Macaulay) graph over K, if S/I(G) is a Gorenstein (resp. CM) ring. When G is Gorenstein (resp. CM) over every field, we say that G is Gorenstein (resp. CM). In [3] and [4] characterizations of planar Gorenstein graphs of girth at least four and triangle-free Gorenstein graphs are presented, respectively. (Recall that G is said to be triangle-free when no subgraph of G is a triangle). Also in [11] a condition on a planar graph equivalent to being Gorenstein is stated. An importance of characterizing Gorenstein graphs comes from the Charney-Davis conjecture which states that “flag simplicial complexes” which are Gorenstein over \( \mathbb{Q} \) satisfy a certain condition (see [9, Problem 4]). In [9], Richard P. Stanley mentioned this conjecture as one of the “outstanding open problems in algebraic combinatorics” at the start of the 21st century. An approach to solve this conjecture is trying to give a characterization of Gorenstein “flag simplicial complexes” which is equivalent to a characterization of Gorenstein graphs.

Denote by \( \alpha(G) \) the independence number of G, that is, the maximum size of an independent set of G. It is well-known that if \( \alpha(G) = 1 \) (that is, G is complete) then G is Gorenstein if and only if \( G = K_1 \) or \( G = K_2 \) (where \( K_n \) denotes the complete graph on n vertices). Moreover, it is easy to see that if \( \alpha(G) = 2 \), then G is Gorenstein if and only if G is the complement of a cycle of length at least four (see Corollary 2.4 below). Therefore, it is natural to ask “if \( \alpha(G) = 3 \), then when is G Gorenstein?” In this paper, we present some properties of Gorenstein graphs with \( \alpha = 3 \) and find all Gorenstein graphs with \( \alpha = 3 \) which are also triangle-free. Before stating our main results and since we need some results on Gorenstein simplicial complexes, first we briefly review simplicial complexes and their Stanley-Reisner ideal.

Recall that a simplicial complex \( \Delta \) on the vertex set \( V = V(\Delta) = \{v_1, \ldots, v_n\} \) is a family of subsets of V (called faces) with the property that \( \{v_i\} \in \Delta \) for each \( i \in [n] = \{1, \ldots, n\} \) and if \( A \subseteq B \in \Delta \), then \( A \in \Delta \). In the sequel, \( \Delta \) always denotes a simplicial complex. Thus the family \( \Delta(G) \) of all cliques of a graph G is a simplicial complex called the clique complex of G. Also \( \Delta(\overline{G}) \) is called the independence complex of G, where \( \overline{G} \) denotes the complement of G. Note that the elements of \( \Delta(\overline{G}) \) are independent sets of
The ideal of $S$ generated by $\{\prod_{x_i \in F} x_i | F \subseteq V \}$ is called the Stanley-Reisner ideal of $\Delta$ and is denoted by $I_{\Delta}$ and $S/I_{\Delta}$ is called the Stanley-Reisner algebra of $\Delta$ over $K$. Therefore we have $I_{\Delta(\overline{G})} = I(G)$. If the Stanley-Reisner algebra of $\Delta$ over $K$ is Gorenstein (resp. CM), then $\Delta$ is called Gorenstein (resp. CM) over $K$. The relation between combinatorial properties of $\Delta$ and algebraic properties of $S/I_{\Delta}$ is well-studied, see for example [2, 6–8] and the references therein.

The dimension of a face $F$ of $\Delta$ and the simplicial complex $\Delta$ are defined to be $|F| - 1$ and $\max\{\dim(F) | F \in \Delta\}$, respectively. Denote by $\alpha(G)$ the independence number of $G$, that is, the maximum size of an independent set of $G$. Note that $\alpha(G) = \dim \Delta(\overline{G}) + 1$. A graph $G$ is called well-covered, if all maximal independent sets of $G$ have size $\alpha(G)$ and we say that $G$ is a $W_2$ graph, if $|V(G)| \geq 2$ and every pair of disjoint independent sets of $G$ are contained in two disjoint maximum independent sets. In some texts, $W_2$ graphs are called 1-well-covered graphs. The reader can see [10] for more on $W_2$ graphs. In the following lemma, we have collected two known results on $W_2$ graphs and their relation to Gorenstein graphs.

**Lemma 1.1.**  
(i) ([3, Lemma 3.1] or [4, Lemma 3.5]) If $G$ is a graph without isolated vertices and $G$ is Gorenstein over some field $K$, then $G$ is a $W_2$ graph.

(ii) ([4, Proposition 3.7]) If $G$ is triangle-free and without isolated vertices, then $G$ is Gorenstein if and only if $G$ is $W_2$.

Let $f_i$ be the number of $i$-dimensional faces of $\Delta$ (if $\Delta \neq \emptyset$, we set $f_{-1} = 1$), then $(f_{-1}, \ldots, f_{d-1})$ is called the $f$-vector of $\Delta$, where $d - 1 = \dim(\Delta)$. Now define $h_i$'s such that $h(t) = \sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-1}$. Then $h(t)$ is called the $h$-polynomial of $\Delta$. Recall that the polynomial $I(G, x) = \sum_{i=0}^{\alpha(G)} a_i x^i$, where $a_i$ is the number of independent sets of size $i$ in $G$ and $a_0 = 1$, is called the independence polynomial of $G$. Note that $(a_0, a_1, \ldots, a_{\alpha(G)})$ is the $f$-vector of $\Delta(\overline{G})$. There are many papers related to this polynomial in the literature, see for example [5].

For $F \in \Delta$, let $\text{link}_{\Delta}(F) = \{A \setminus F | F \subseteq A \in \Delta\}$. If all maximal faces of $\Delta$ have the same dimension and for each $F \in \Delta$, we have $\sum_{i=-1}^{d_F} (-1)^i f_i(F) = (-1)^{d_F} \quad (\ast)$, where $d_F = \dim \text{link}_{\Delta}(F)$ and $(f_i(F))_{i=-1}^{d_F}$ is the $f$-vector of $\text{link}_{\Delta}(F)$, then $\Delta$ is said to be an Euler complex. Another complex constructed from $\Delta$ is $\text{core}(\Delta)$. The set of vertices of $\text{core}(\Delta)$ is $V_{\text{core}} = \{v \in V(\Delta) | \exists G \in \Delta \quad G \cup \{v\} \notin \Delta\}$ and $\text{core}(\Delta) = \{F \in \Delta | F \subseteq V_{\text{core}}\}$.
2 Main results

First we establish a theorem which presents a condition on graphs equivalent to being Gorenstein. For this, we need the following lemma.

**Lemma 2.1.** If $G$ is a graph without any isolated vertex and Gorenstein over some field $K$, then $I(G, -1) = (-1)^{\alpha(G)}$.

**Proof.** Let $\Delta = \Delta(G)$ and $d = \text{dim} \Delta = \alpha(G) - 1$. If $v \in V(G)$, then as $v$ is not isolated, there is a $u \in V(G)$ such that $uv \in E(G)$. Hence $G = \{u\} \in \Delta$ and $G \cup \{v\} \notin \Delta$. This means that $v \in \text{core}(\Delta)$ and hence $\text{core}(\Delta) = \Delta$. So according to [1, Theorem 5.5.2], $\Delta$ is an Euler complex. In particular, noting that $\text{link}_\Delta(\emptyset) = \Delta$, if we write (\ref{eqn:link}) with $F = \emptyset$, we get $\sum_{i=-1}^{d} (-1)^i f_i = (-1)^d$ where $(f_i)_{i=-1}^{d}$ is the $f$-vector of $\Delta$. Thus

$$I(G, -1) = \sum_{i=0}^{\alpha(G)} a_i (-1)^i = \sum_{i=0}^{\alpha(G)} (-1)^i f_{i-1} = - \sum_{j=-1}^{d} (-1)^j f_j$$

$$= (-1)^{d+1} = (-1)^{\alpha(G)}.$$ 

\qed

**Remark 2.2.** The above argument shows that if $G$ has no isolated vertex, then $\Delta(G) = \text{core}(\Delta(G))$ and if moreover $G$ is Gorenstein over some field, then $\Delta(G)$ is an Euler complex.

Suppose that $F \subseteq V(G)$. Here by $N[F]$ we mean $F \cup \{v \in V(G) | uv \in E(G) \text{ for some } u \in F\}$ and we set $G_F = G \setminus N[F]$. Our next theorem presents a condition on graphs with $\alpha \geq 2$ equivalent to being Gorenstein. Recall that if $\alpha(G) = 1$, then $G$ is Gorenstein if and only if $G$ is isomorphic to $K_1$ or $K_2$. Note that if $\Delta$ is a simplicial complex with dimension at most one, then we can view $\Delta$ as a graph on vertex set $V(\Delta)$ by considering 1-dimensional faces of $\Delta$ as edges.

**Theorem 2.3.** Suppose that $G$ is a graph without any isolated vertex and $\alpha(G) \geq 2$. Then $G$ is Gorenstein over $K$ if and only if all of the following conditions hold:

(i) $G$ is CM over $K$;

(ii) $I(G, -1) = (-1)^{\alpha(G)}$.
(iii) For each independent set $F$ of $G$ with size $\alpha(G) - 2$, the graph $\overline{G_F}$ is a cycle of length at least 4.

Proof. ($\Rightarrow$): By Lemma 2.1, $I(G, -1) = (-1)^{\alpha(G)}$ and $G$ is CM over $K$ by definition. If $G = K_2$, then it has no independent set of size 2 and hence (iii) holds trivially. So we assume that $G \neq K_2$.

Suppose that $F$ is an independent set of $G$ with size $\alpha(G) - 2$. First we claim that $\overline{G_F}$ is not a triangle or a path of length 1 or 2. Suppose that the claim is not true. First, assume that $\overline{G_F}$ consists of just an edge $ab$. Let $A = F \cup \{a\}$ and $B = \{b\}$ which are two disjoint independent sets of $G$. By Lemma 1.1(i), $G$ is $W_2$ and there exist independent sets $A_0$ and $B_0$ of size $\alpha(G)$ such that $A \subseteq A_0$, $B \subseteq B_0$ and $A_0 \cap B_0 = \emptyset$. But the only vertex of $G$ which is not adjacent to any vertex of $A$ in $G$ is $b$, so $A_0 = A \cup \{b\}$. But $b \in B \subseteq B_0$ which contradicts $A_0 \cap B_0 = \emptyset$. From this contradiction it follows that $\overline{G_F}$ is not a path of length 1.

Now assume that $\overline{G_F}$ is a path of length 2 with edges $ab$ and $bc$. Let $A = \{a, b\}$ and $B = F \cup \{c\}$. Again $A$ and $B$ are disjoint independent sets of $G$ and since $G$ is $W_2$, there exist independent sets $A_0$ and $B_0$ of size $\alpha(G)$ such that $A \subseteq A_0$, $B \subseteq B_0$ and $A_0 \cap B_0 = \emptyset$. But $b$ is the only vertex of $G$ not adjacent to any vertex of $B$. Therefore $b \in A_0 \cap B_0$, a contradiction, thus $\overline{G_F}$ is not a path of length 2. Also we have $\alpha(G_F) = 2$ and hence $\overline{G_F}$ can not be a triangle. This concludes the proof of the claim.

Note that $\Delta = \Delta(\overline{G})$ is not empty and since $G \neq K_1, K_2$, its independence complex $\Delta$ does not consist of only one or two isolated vertices. Also by Remark 2.2, $\text{core}(\Delta) = \Delta$. Hence by equivalence of parts (a) and (e) of [8, Chapter II, Theorem 5.1], $\text{link}_\Delta(F)$ is either a cycle or a path of length 1 or 2 (when viewed as a graph, as mentioned in the notes before this theorem). Let $A \in \text{link}_\Delta(F)$, then there is a $B \in \Delta$ containing $F$, such that $A = B \setminus F$. So $B = A \cup F$ is an independent set of $G$. Hence $A$ is an independent set of $G_F$, that is, $A \in \Delta(\overline{G_F})$. Conversely, if $A \in \Delta(\overline{G_F})$, then $A \cup F$ is an independent set of $G$ and hence $A \in \text{link}_\Delta(F)$. Therefore, $\text{link}_\Delta(F) = \Delta(\overline{G_F})$. If we view this 1-dimensional simplicial complex as a graph, then $\Delta(\overline{G_F}) = \overline{G_F}$. Hence $\overline{G_F}$ is either a cycle or a path of length 1 or 2. But as we showed above, $\overline{G_F}$ can not be a triangle or a path of length 1 or 2. Consequently, $\overline{G_F}$ is a cycle of length at least 4, as required.

($\Leftarrow$): Let $F$ be an independent set of size $\alpha(G) - 2$ and $\Delta = \Delta(\overline{G})$. According to the argument in the above paragraph, $\text{link}_\Delta(F) = \Delta(\overline{G_F}) = \overline{G_F}$. Therefore, $\text{link}_\Delta(F)$ is a cycle for each independent set of size $\alpha(G) - 2$
of $G$. Also core($\Delta$) = $\Delta$ by Remark 2.2. Hence the result follows from the equivalence of (a) and (e) of [8, Chapter II, Theorem 5.1].

As an instant corollary of the previous theorem, we get the following characterization of Gorenstein graphs with $\alpha(G) = 2$.

**Corollary 2.4.** Suppose that $G$ is a graph without any isolated vertex and $\alpha(G) = 2$. Then the following are equivalent.

(i) $G$ is Gorenstein over some field.

(ii) $G$ is Gorenstein over every field.

(iii) $G$ is the complement of a cycle of length at least four.

**Proof.** (ii) $\Rightarrow$ (i) is trivial and (i) $\Rightarrow$ (iii) follows from Theorem 2.3(iii) with $F = \emptyset$.

(iii) $\Rightarrow$ (ii): Suppose that $G$ is the complement of an $n$-cycle with $n \geq 4$. If we view $\Delta(G)$ as a graph, then $\Delta(G)$ is a cycle and hence is connected. Therefore, by [1, Exercise 5.1.26(c)], $\Delta(G)$ or equivalently $G$ is CM over every field. Also $I(G, x) = 1 + nx + nx^2$ and $I(G, -1) = 1 = (-1)^\alpha(G)$. Thus the result follows from Theorem 2.3.

Next we find some properties of Gorenstein graphs with $\alpha(G) = 3$. For this, we need a lemma.

**Lemma 2.5.** Assume that $G$ is a graph without any isolated vertex and Gorenstein over a field $K$ and suppose that $\alpha(G)$ is odd, then $I(G, -\frac{1}{2}) = 0$.

**Proof.** Let $\Delta = \Delta(G)$ and $\alpha = \alpha(G)$. By Remark 2.2, $\Delta$ is an Euler complex of dimension $\alpha - 1$. Thus by [1, Theorem 5.4.2], we have $h_i = h_{\alpha-i}$, where $h(t) = \sum_{i=0}^{\alpha} h_i t^i = \sum_{i=0}^{\alpha} f_{i-1} t^i (1-t)^{\alpha-i}$ is the $h$-polynomial of $\Delta$. Since $\alpha$ is odd, this means that $h(-1) = 0$. Now

$$I \left(G, -\frac{1}{2}\right) = \sum_{i=0}^{\alpha} a_i \left(-\frac{1}{2}\right)^i = \sum_{i=0}^{\alpha} f_{i-1} \left(-\frac{1}{2}\right)^i = \frac{1}{2^\alpha} \sum_{i=0}^{\alpha} f_{i-1} (-1)^i (1-(-1))^{\alpha-i} = \frac{1}{2^\alpha} h(-1) = 0.$$ 

$\square$
Now we can prove the following which determines the number of edges and the independence polynomial of Gorenstein graphs with independence number three.

Proposition 2.6. Suppose that $G$ is a graph with $n$ vertices and $m$ edges and without any isolated vertex. Also assume that $G$ is Gorenstein over some field and $\alpha(G) = 3$. Then the following hold.

(i) $m = \frac{n^2 - 7n + 12}{2}$.

(ii) $G$ has exactly $2n - 4$ independent sets of size three and exactly $3n - 6$ independent sets of size two.

(iii) $I(G, x) = 1 + nx + (3n - 6)x^2 + (2n - 4)x^3$.

Proof. Suppose that $I(G, x) = \sum_{i=0}^{\alpha(G)} a_i x^i$ is the independence polynomial of $G$. Then $a_0 = 1$, $a_1 = n$ and $a_2$ and $a_3$ are the number of independent sets of size two and three of $G$, respectively. According to Lemmas 2.1 and 2.5, $I(G, -1) = (-1)^3 = -1$ and $I(G, -\frac{1}{2}) = 0$. It follows that $a_2 - a_3 = n - 2$ and $2a_2 - a_3 = 4n - 8$. Therefore $a_2 = 3n - 6$ and $a_3 = 2n - 4$, which proves (ii) and (iii). On the other hand, independent sets of size two are exactly those 2-subsets $\{u, v\}$ of $V(G)$ such that $uv \notin E(G)$. Thus $3n - 6 = a_2 = \binom{n}{2} - m$ and $m = \frac{n^2 - 7n + 12}{2}$.

Next we characterize triangle-free Gorenstein graphs with independence number three. In what follows, by $G_v$ we mean $G_{\{v\}} = G \setminus N[v]$, where $v \in V(G)$. Also $C_n$ denotes the $n$-vertex cycle.

Theorem 2.7. Suppose that $G$ is triangle-free, $\alpha(G) = 3$ and $G$ has no isolated vertices. The following are equivalent:

(i) $G$ is Gorenstein over every field;

(ii) $G$ is Gorenstein over some field;

(iii) $G$ is $W_2$.

(iv) $G$ is isomorphic to one of the graphs depicted in Figure 1.
Figure 1: Triangle-free Gorenstein graphs without isolated vertex and with \( \alpha = 3 \)

Proof. (i) \( \iff \) (iii) is an especial case of [4, Proposition 3.7] and (i) \( \Rightarrow \) (ii) is trivial.

(iv) \( \Rightarrow \) (iii): Let \( G \) be one of the graphs in Figure 1. For each \( v \in V(G) \), one can check that \( G \setminus v \) is well-covered and \( \alpha(G \setminus v) = \alpha(G) \). According to [10, Theorem 1], this means that \( G \) is \( W_2 \).

(ii) \( \Rightarrow \) (iv): Suppose that \( n = |V(G)| \) and \( m = |E(G)| \) and let \( v \in V(G) \). According to Theorem 2.3(iii), \( G_v \) is the complement of a cycle on at least 4 vertices. Since complement of any cycle with at least 6 vertices has a triangle, thus \( G_v \) is isomorphic to \( \bar{C}_4 \) or \( \bar{C}_5 \). In particular, \( n - 1 - \deg(v) = |V(G_v)| \in \{4, 5\} \), that is, either \( \deg(v) = n - 5 \) or \( \deg(v) = n - 6 \). Assume that \( a \) and \( b \) are the number of vertices of \( G \) with degree \( n - 5 \) and \( n - 6 \), respectively. Therefore \( a + b = n \) and \( (n - 5)a + (n - 6)b = 2m = n^2 - 7n + 12 \) by Proposition 2.6. Solving these two equations for \( a \) and \( b \), we get \( a = 12 - n \) and \( b = 2n - 12 \). In particular, it follows that \( 6 \leq n \leq 12 \).

Suppose that \( n \geq 7 \). Then as \( b > 0 \), there exists a vertex of \( G \), say \( v \), with degree \( n - 6 \). Let \( A = N(v) \) and \( B = V(G) \setminus (N[v]) \). By Theorem 2.3, the subgraph of \( G \) induced by \( B \) is \( \bar{C}_5 \cong C_5 \) and has five edges. Also there is no edge of \( G \) with both end-vertices in \( N(v) \), because \( G \) is triangle free. Consequently, the subgraph of \( G \) induced by \( N[v] \) has exactly \( n - 6 \) edges and all the remaining edges of \( G \) has one end-vertex in \( A \) and one end-vertex in \( B \). Assume that a vertex \( x \in A \) is adjacent to at least three vertices \( u_1, u_2, u_3 \) in \( B \). Since \( u_i \)'s \( (i = 1, 2, 3) \) lie on a 5-cycle, at least two of them, say \( u_1 \) and \( u_2 \), are adjacent and \( xu_1u_2 \) is a triangle in \( G \), a contradiction. Therefore, each vertex of \( A \) can be adjacent to at most two of the vertices in \( B \) and

\[ \text{(c)} \]
the number of edges between $A$ and $B$ is at most $2|A| = 2(n - 6)$. Hence $m \leq 2(n - 6) + (n - 6) + 5$ and by applying Proposition 2.6, we deduce that $n^2 - 13n + 38 \leq 0$ which only holds for $n \leq 8$. We conclude that $6 \leq n \leq 8$.

If $n = 6$, then $a = 6$ and $b = 0$, that is, $G$ consists of 6 vertices of degree one and is isomorphic to the graph (a) of Figure 1. If $n = 7$, then $G$ has $a = 5$ vertices of degree 2 and $b = 2$ vertices of degree 1. This means that $G$ is the disjoint union of a 5-cycle and an edge, that is, $G$ is isomorphic to Figure 1(b).

Now assume that $n = 8$ and hence $m = 10$ and $G$ has $a = 4$ vertices of degree 3 and $b = 4$ vertices of degree 2. Let $v$ be a vertex of degree 3. Then $G_v$ is $C_4$ and has two edges, say $xy$ and $uw$. Assume that $N(v) = \{v_1, v_2, v_3\}$. Since there is no edge between vertices of $N(v)$ (because $G$ is triangle-free), the number of edges with one end-vertex in $N(v)$ and one end-vertex in $V(G_v)$ is $m - \deg(v) - |E(G_v)| = 5$. Each of the four vertices of $G_v$ has degree at least two in $G$ and exactly one in $G_v$. Thus each of them is adjacent to at least one of the $v_i$’s, and exactly one of the vertices of $G_v$, say $x$, is adjacent to two of the $v_i$’s, say $v_1$ and $v_3$ (see Figure 1(c)). So $v_1$ and $v_3$ are not adjacent to $y$ (else $v_i xy$ is a triangle for $i = 1$ or 3) and $y$ must be adjacent to $v_2$. According to Theorem 2.3, $G_x$ is also isomorphic to $C_4$. But $G_x$ is the subgraph of $G$ induced by $\{u, w, v, v_2\}$ and we know that $uw, vv_2 \in E(G_x)$. Thus there is no edge in $G$ with one end-vertex in $\{u, w\}$ and one end-vertex in $\{v, v_2\}$. In particular, $u$ and $w$ are not adjacent to $v_2$ and one of them say $u$ must be adjacent to $v_1$ and the other must be adjacent to $v_3$. Consequently, $G$ is the graph (c) in Figure 1.

It is well-known that a graph is Gorenstein (over $K$) if and only if each of its connected components are Gorenstein (over $K$). The above theorem shows that there is just one connected triangle-free Gorenstein graph with independence number 3. This poses the following question.

**Question 2.8.** Is there an infinite family of connected Gorenstein graphs with $\alpha = 3$?

Note that if we remove connectedness from the above question, then the family $\{C_n \cup K_2 | 4 \leq n \in \mathbb{N}\}$, is an easy answer to the question.
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