n-PROJECTIVE MODEL STRUCTURES ON CHAIN COMPLEXES

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Abstract
We study completeness of the cotorsion pairs 
\((\widetilde{P}_n, dg\widetilde{P}_n)\) and 
\((dg\widetilde{P}_n, \widetilde{P}_n)\) in 
\(\text{Ch}(R)\) induced by the cotorsion pair \((P_n, P_n)\) in \(R\)-Mod, where 
\(\widetilde{P}_n\) is the class of left \(R\)-modules with projective dimension at most \(n\). The completeness of the pair \((dg\widetilde{P}_n, \widetilde{P}_n)\) is a consequence of a result proved by J. Gillespie in \[6\]. We use a generalization of the zig-zag argument to show that \((\widetilde{P}_n, dg\widetilde{P}_n)\) is also complete. This gives rise to a new model structure on \(\text{Ch}(R)\).

1 Introduction

The process of constructing model structures on an arbitrary bicomplete category is not trivial at all. However, if we are working on an bicomplete abelian category, we can use several tools to construct a certain type of model structures, called abelian model structures. One of those tools is the theory of cotorsion pairs. In \[9\], M. Hovey describes the connection between complete cotorsion pairs and abelian model categories. Namely, one can always obtain two complete cotorsion pairs from a given abelian model category. The converse is also true, that means that one can construct an abelian model structure from two complete cotorsion pairs satisfying certain conditions. Such cotorsion pairs are called compatible by J. Gillespie in \[8\].

In the theory of cotorsion pairs, there have been several influential contributors, such as P. Eklof, J. Trlifaj. They proved an important theorem which gives us a wide range of complete cotorsion pairs. This result is known as the Eklof and Trlifaj Theorem,
which states that every cotorsion pair cogenerated by a set is complete. Among all the complete cotorsion pairs obtained by using this tool, we shall focus on the pair \((\mathcal{P}_n, \mathcal{P}_n^\perp)\), where \(\mathcal{P}_n\) is the class of all left \(R\)-modules whose projective dimension is at most \(n\). We shall refer to the modules in \(\mathcal{P}_n\) as \(n\)-projective modules. The completeness of \((\mathcal{P}_n, \mathcal{P}_n^\perp)\) was proven by S. T. Aldrich et al. (see [4]), using, along with the Eklof and Trlifaj Theorem, a technique known as the \textbf{zig-zag procedure}.

In [8], Gillespie provides some results which help us to induce compatible cotorsion pairs \((\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})\) and \((dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})\) in the category of chain complexes \(Ch(R)\) from a complete cotorsion pair \((\mathcal{A}, \mathcal{B})\) in \(R\text{-Mod}\). As far as the author knows, these induced cotorsion pairs are not necessarily complete. In this sense, our goal is to prove that \((\tilde{\mathcal{P}}_n, dg\tilde{\mathcal{P}}_n^\perp)\) is complete.

This paper is organized as follows. First, we recall some definitions and introduce the notation we shall use. Then, we shall present the \textit{zig-zag} procedure. This technique allows us to construct a small \(n\)-projective module \(X'\) from a given \(n\)-projective module \(X\) such that the quotient \(X/X'\) is also \(n\)-projective. Using a generalization of this technique, we shall prove that \((\tilde{\mathcal{P}}_n, dg\tilde{\mathcal{P}}_n^\perp)\) is a complete cotorsion pair. In order to do that, we characterize \(\tilde{\mathcal{P}}_n\) as the class of chain complexes having a projective resolution of length \(n\). As we mentioned in the abstract, the completeness of \((dg\tilde{\mathcal{P}}_n, \tilde{\mathcal{P}}_n^\perp)\) follows by a result proved by Gillespie. At this point, we shall have two compatible cotorsion pairs \((\tilde{\mathcal{P}}_n, dg\tilde{\mathcal{P}}_n^\perp)\) and \((dg\tilde{\mathcal{P}}_n, \tilde{\mathcal{P}}_n^\perp)\) from which we shall obtain, using a theorem by M. Hovey, a new abelian model structure in \(Ch(R)\), which we shall name the \(n\)-\textbf{projective model structure}. We shall comment some properties about this model structure, such as the fact that it is cofibrantly generated, but it is not monoidal with respect to two well known tensor products on \(Ch(R)\). Finally, we conclude this paper by commenting another method to obtain the \(n\)-projective model structure in the particular case when \(R\) is a two-sided Artinian ring.
2 Preliminaries

This section is devoted to recall some notions and to introduce some notation. Denote by \( \text{Ch}(\mathcal{C}) \) the category of chain complexes and chain maps over an abelian category \( \mathcal{C} \). Given a chain complex \( A = (A_m)_{m \in \mathbb{Z}} \) with boundary maps \( \partial^A_m : A_m \to A_{m-1} \), the object \( Z_m(A) := \ker(\partial^A_m) \) is called the \( m \)-cycle object of \( A \), and \( B_m(A) := \text{im}(\partial^A_{m+1}) \) the \( m \)-boundary object of \( A \). A chain complex \( A \) is said to be exact if \( Z_m(A) = B_m(A) \), for every \( m \in \mathbb{Z} \).

Now consider the category \( \text{Ch}(\mathcal{R}) \) of chain complexes over \( \mathcal{R} \)-Mod, the category of left \( \mathcal{R} \)-modules. A chain complex \( B \) is said to be a subcomplex of \( A \) if there exists a monomorphism \( i : B \to A \). Sometimes, we shall denote monomorphisms (resp. epimorphisms) in \( \mathcal{R} \)-Mod and \( \text{Ch}(\mathcal{R}) \) by \( \hookrightarrow \) (resp. \( \twoheadrightarrow \)). If \( B \) is a subcomplex of \( A \), we define the quotient complex \( A/B \) as the complex whose components are given by \( (A/B)_m = A_m/B_m \) and whose boundary maps \( \partial^{A/B}_m : A_m/B_m \to A_{m-1}/B_{m-1} \) are given by \( a + B_m \mapsto \partial^A_m(a) + B_{m-1} \). Given a chain map \( f : A \to B \), the image complex \( \text{Im}(f) \) is the chain complex given by \( (\text{Im}(f))_m = \text{im}(f_m) = f_m(A_m) \), whose boundary maps \( \partial^{\text{Im}(f)}_m : f_m(D_m) \to f_{m-1}(D_{m-1}) \) are defined by

\[
\partial^{\text{Im}(f)}_m(x) := \partial^B_m \circ f_m(x) = f_{m-1} \circ \partial^A_m(x).
\]

The kernel complex \( \text{Ker}(f) \) is the chain complex given by \( (\text{Ker}(f))_m = \ker(f_m) \), whose boundary maps \( \partial^{\text{Ker}(f)}_m : \ker(f_m) \to \ker(f_{m-1}) \) are defined by

\[
\partial^{\text{Ker}(f)}_m(x) := \partial^A_m \circ i_m(x), \text{ where } i_m \text{ denotes the inclusion } \ker(f_m) \hookrightarrow X_m.
\]

Let \( \mathcal{C} \) be an abelian category. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two classes of objects in \( \mathcal{C} \). The pair \( (\mathcal{A}, \mathcal{B}) \) is called a cotorsion pair in \( \mathcal{C} \) if the following conditions are satisfied:
(1) $A = \perp B := \{ X \in \text{Ob}(\mathcal{C}) / \text{Ext}^1(X, B) = 0 \text{ for every } B \in \mathcal{B} \}$.

(2) $B = A^\perp := \{ X \in \text{Ob}(\mathcal{C}) / \text{Ext}^1(A, X) = 0 \text{ for every } A \in \mathcal{A} \}$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$ is said to be **complete** if:

(a) $(\mathcal{A}, \mathcal{B})$ has **enough projectives**: for every object $X$ there exist objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and a short exact sequence

$$0 \to B \to A \to X \to 0.$$ 

(b) $(\mathcal{A}, \mathcal{B})$ has **enough injectives**: for every object $X$ there exist objects $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$, and a short exact sequence

$$0 \to X \to B' \to A' \to 0.$$ 

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be **cogenerated** by a set $\mathcal{S} \subseteq \mathcal{A}$ if $B = \mathcal{S}^\perp$. There is a wide range of complete cotorsion pairs, thanks to the following result, known as the Eklof and Trlifaj Theorem:

**Theorem 2.1.** [3, Theorem 10] Every cotorsion pair in $R$-Mod cogenerated by a set is complete.

**Remark 2.1** (Salce's Lemma). If $\mathcal{C}$ is an abelian category with enough projectives and injectives, then a cotorsion pair $(\mathcal{A}, \mathcal{B})$ has enough projectives if and only if it has enough injectives.

A class of objects $\mathcal{A}$ is said to be:
(a) **Closed under extensions** if given a short exact sequence

\[ 0 \rightarrow A'' \rightarrow A \rightarrow A' \rightarrow 0 \]

with \( A', A'' \in \mathcal{A} \), then \( A \in \mathcal{A} \).

(b) **Resolving** if \( \mathcal{A} \) is closed under extensions, contains the projective objects, and is closed under taking kernels of epimorphisms, that means that if

\[ 0 \rightarrow A'' \rightarrow A \rightarrow A' \rightarrow 0 \]

is a short exact sequence with \( A, A' \in \mathcal{A} \), then \( A'' \in \mathcal{A} \).

(c) **Coresolving** if \( \mathcal{B} \) is closed under extensions, contains the injective objects, and is closed under taking cokernels of monomorphisms, that means that if

\[ 0 \rightarrow A'' \rightarrow A \rightarrow A' \rightarrow 0 \]

is a short exact sequence with \( A'', A \in \mathcal{A} \) then \( A' \in \mathcal{A} \).

**Remark 2.2.** Given a cotorsion pair \((\mathcal{A}, \mathcal{B})\) in \( \mathcal{C} \), it is not hard to show that the class \( \mathcal{A} \) is closed under extensions and contains the projective objects. Dually, \( \mathcal{B} \) is closed under extensions and contains the injective objects. If \( \mathcal{C} \) is an abelian category with enough projectives and injectives, \( \mathcal{A} \) is resolving if and only if \( \mathcal{B} \) is coresolving.

**Example 2.1.** If \( \text{Proj} \) denotes the class of projective modules, then \((\text{Proj}, \text{R-Mod})\) is a cotorsion pair. We shall refer to \((\text{Proj}, \text{R-Mod})\) as the **projective cotorsion pair**. Similarly, if \( \text{Inj} \) denotes the class of injective modules, then \((\text{R-Mod}, \text{Inj})\) is a cotorsion pair. Using the Baer’s Criterion, one can show that \((\text{R-Mod}, \text{Inj})\) is cogenerated by the set of modules of the form \( \text{R}/I \), where \( I \) is a left ideal. So by the Eklof
and Trlifaj Theorem, \((R\text{-Mod}, \text{Inj})\) is complete. A less trivial example of a complete cotorsion pair is given by the flat cotorsion pair \((\mathcal{F}, \mathcal{C})\), where \(\mathcal{F}\) is the class of flat modules and \(\mathcal{C} = \mathcal{F}^\perp\). This result, known as the flat cover conjecture, was proven by Enochs by using the Eklof and Trlifaj Theorem, and independently and at about the same time, by Bican and El Bashir by a different method (see \[2\]).

We recall the notion of a model category. Given a category \(\mathcal{C}\), a map \(f\) in \(\mathcal{C}\) is a retract of a map \(g\) in \(\mathcal{C}\) if there is a commutative diagram of the form

\[
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow f & & \downarrow g \\
B & \longrightarrow & D \\
\end{array}
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow f & & \downarrow g \\
B & \longrightarrow & D \\
\end{array}
\]

where the horizontal composites are identities. Let \(f : A \longrightarrow B\) and \(g : C \longrightarrow D\) be two maps in \(\mathcal{C}\). We shall say that \(f\) has the left lifting property with respect to \(g\) (or that \(g\) has the right lifting property with respect to \(f\)) if for every pair of maps \(u : A \longrightarrow C\) and \(v : B \longrightarrow D\) with \(g \circ u = v \circ f\), there exists a map \(d : B \longrightarrow C\) such that \(g \circ d = v\) and \(d \circ f = u\).

\[
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow f & & \downarrow g \\
B & \longrightarrow & D \\
\end{array}
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow f & & \downarrow g \\
B & \longrightarrow & D \\
\end{array}
\]

A model category is a bicomplete category \(\mathcal{C}\) equipped with three classes of maps named cofibrations, fibrations and weak equivalences, satisfying the following properties:

(a) \textbf{3 for 2}: If \(f\) and \(g\) are maps of \(\mathcal{C}\) such that \(g \circ f\) is defined and two of \(f\), \(g\) and \(g \circ f\) are weak equivalences, then so is the third.
(b) If \( f \) and \( g \) are maps of \( C \) such that \( f \) is a retract of \( g \) and \( g \) is a weak equivalence, cofibration, or fibration, then so is \( f \).

Define a map to be a **trivial cofibration** if it is both a weak equivalence and a cofibration. Similarly, define a map to be a **trivial fibration** if it is both a weak equivalence and a fibration.

(c) Trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.

(d) Every map \( f \) can be factored as \( f = \alpha \circ \beta = \gamma \circ \delta \), where \( \alpha \) (resp. \( \delta \)) is a cofibration (resp. fibration), and \( \gamma \) (resp. \( \beta \)) is a trivial cofibration (resp. trivial fibration).

An object \( X \) in \( C \) is called **cofibrant** if the map \( 0 \to X \) is a cofibration, **fibrant** if the map \( X \to 0 \) is a fibration, and **trivial** if the map \( 0 \to X \) is a weak equivalence.

Given a bicomplete abelian category \( C \), a model structure on it is said to be **abelian** if the following conditions are satisfied:

(a) A map is a cofibration if and only if it is a monomorphism with cofibrant cokernel.

(b) A map if a fibration if and only if it is an epimorphism with fibrant kernel.

**Example 2.2.** A chain complex \( X \) is called **dg-projective** if each \( X_m \) is a projective module and every chain map \( Y \to X \) is chain homotopic to \( 0 \) (or nullhomotopic) whenever \( Y \) is an exact complex. The category of chain complexes \( \text{Ch}(R) \) has an abelian model structure, where a map \( f \) is:

(a) a weak equivalence if the induced map \( H_m(f) \) on homology is an isomorphism for all \( m \),
(b) a fibration if \( f_m \) is surjective for all \( m \),

(c) a cofibration if \( f_m \) is a dimensionwise split injection and \( \text{CoKer}(f) \) is a DG-projective complex.

We shall refer to this model structure as the **projective model structure** on \( \text{Ch}(R) \).

A subcategory \( D \) of an abelian category \( C \) is said to be **thick** if the following two conditions are satisfied:

(a) \( D \) is **closed under retracts**, i.e., given a sequence

\[
D' \xrightarrow{f} D \xrightarrow{g} D'
\]

with \( g \circ f = \text{id}_{D'} \) and \( D \in D \), then \( D' \in B \).

(b) If two out of three of the terms in a short exact sequence

\[
0 \rightarrow D'' \rightarrow D \rightarrow D' \rightarrow 0
\]

are in \( D \), then so is the third.

**Example 2.3.** The class \( E \) of exact complexes is thick.

### 3 Completeness of the cotorsion pair \((P_n, P_n^\perp)\)

Every left \( R \)-module \( X \) has a projective resolution, i.e. an exact sequence

\[
\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0
\]

where each \( P_i \) is a projective module. The **projective dimension** of \( X \) is defined as the integer
\[ \text{pd}(X) := \min \{ n \geq 0 : \text{Ext}^j(X, -) = 0, \forall j > n \}, \]

provided such an integer exists. Otherwise, set \( \text{pd}(X) = \infty \). It is known that \( \text{pd}(X) \leq n \) if and only if there exists a finite projective resolution of length \( n \), i.e. an exact sequence

\[ 0 \to P_n \to \cdots \to P_1 \to P_0 \to X \to 0 \]

where \( P_i \) is projective for every \( 0 \leq i \leq n \). A module \( X \) is called \( n \)-projective if \( \text{pd}(X) \leq n \). Recall that \( \mathcal{P}_n \) denotes the class of \( n \)-projective modules.

Given a left \( R \)-module \( X \), a disk complex is a chain complex of the form

\[ D^n(X) := \cdots \to 0 \to X \to 0 \to \cdots \]

where \( X \) appears at the \( n \)-th and \( n-1 \)-th entries. By the Eilenberg Trick, for each \( i \) there exists a free module \( F_i \) such that \( P_i \oplus F_i \cong F_i \). Consider the disks complexes \( D^{i+1}(F_i) \) where \( 0 \leq i \leq n-1 \), and \( D^n(F_n) \). Taking the direct sum of these disks and the given projective resolution of \( X \), we get an exact sequence

\[ 0 \to P_n \oplus F_n \oplus F_{n-1} \to P_{n-1} \oplus F_{n-1} \oplus F_n \to \cdots \to P_1 \oplus F_1 \oplus F_0 \to P_0 \oplus F_0 \to X \to 0 \]

which turns out to be a free resolution of \( X \).

From now on, let \( \kappa \) be a fixed infinite cardinal such that \( \kappa \geq \text{Card}(R) \). We shall say that a set \( S \) is small if \( \text{Card}(S) \leq \kappa \).

The following result is due to Aldrich et al. (see [1, Proposition 4.1]). It is a tool used to prove that \( (\mathcal{P}_n, \mathcal{P}_n^+) \) is a complete cotorsion pair. For the reader convenience, we give its proof to introduce a technique called the zig-zag procedure.
Lemma 3.1 (Aldrich, Enochs, Jenda & Oyonarte). If $X$ is an $n$-projective module with $x \in X$, then there exists a small $n$-projective submodule $X' \subseteq X$ containing $x$ such that $X/X'$ is also $n$-projective.

Proof. By the comments above, we can find a free resolution

$$0 \rightarrow F_n \xrightarrow{\partial_n} F_{n-1} \rightarrow \cdots \rightarrow F_0 \xrightarrow{\partial_0} X \rightarrow 0.$$ 

Let $B_i$ be a basis of $F_i$. We shall find small linearly independent sets $B_i \subseteq B_i$ and construct an exact sequence

$$0 \rightarrow \langle B_n \rangle \rightarrow \cdots \rightarrow \langle B_1 \rangle \rightarrow \langle B_0 \rangle,$$

and then take $X' = \text{Im} \left( \partial_0 \langle B_0 \rangle \right)$. For this, we use a technique called the zig-zag procedure. Since $\partial_0$ is surjective, we can find a finite subset $Z_0 \subseteq B_0$ such that $x \in \partial_0 \langle Z_0 \rangle$. Now we choose a small subset $Z_1 \subseteq B_1$ such that $\partial_1 \langle Z_1 \rangle \supseteq \text{Ker} \left( \partial_0 \langle Z_0 \rangle \right)$. Then we choose a small subset $Z_2 \subseteq B_2$ such that $\partial_2 \langle Z_2 \rangle \supseteq \text{Ker} \left( \partial_1 \langle Z_1 \rangle \right)$. We continue this procedure until we get a set a small subset $Z_n \subseteq X_n$ satisfying

$$\partial_n \langle Z_n \rangle \supseteq \text{Ker} \left( \partial_{n-1} \langle Z_{n-1} \rangle \right).$$

Now choose a small subset $Z_{n-1} \subseteq B_{n-1}$ containing $Z_{n-1}$ such that $\partial_n \langle Z_n \rangle \subseteq \langle Z_{n-1} \rangle$. Then choose a small subset $B_{n-2} \supseteq Z_{n-2} \supseteq B_{n-2}$ such that $\partial_{n-1} \langle Z_{n-1} \rangle \subseteq \langle Z_{n-2} \rangle$. Continue this procedure to construct small sets $Z_{n-3}, \ldots, Z_0$ satisfying $X_{n-i} \supseteq Z_{n-i} \supseteq Z_{n-i-1}$, and $\partial_{n-i} \langle Z_{n-i+1} \rangle \subseteq \langle Z_{n-i} \rangle$, where $3 \leq i \leq n$. Now choose a small set $Z_1(1) \subseteq Z_1(2) \subseteq B_1$ such that $\partial_1 \langle Z_1(2) \rangle \supseteq \text{Ker} \left( \partial_0 \langle Z_0^{(1)} \rangle \right)$. Then enlarge $Z_2(1)$ and so on. Continue this zig-zag procedure indefinitely, and set $B_i = \bigcup_{k=0}^{\infty} Z_i^{(k)}$, where $Z_i^{(0)} = Z_i$. Note that each $B_i$ is small. Let $\overline{\partial}_i = \partial_i \langle B_i \rangle$. By construction, we get an exact sequence

$$0 \rightarrow \langle B_n \rangle \xrightarrow{\overline{\partial}_n} \cdots \xrightarrow{\overline{\partial}_2} \langle B_1 \rangle \xrightarrow{\overline{\partial}_1} \langle B_0 \rangle.$$
Let \( X' = \partial_0(\langle B_0 \rangle) \). We have a free (and hence projective) resolution of \( X' \) given by the sequence

\[
0 \longrightarrow \langle B_n \rangle \overset{\bar{\partial}_n}{\longrightarrow} \cdots \overset{\bar{\partial}_2}{\longrightarrow} \langle B_1 \rangle \overset{\bar{\partial}_1}{\longrightarrow} \langle B_0 \rangle \overset{\bar{\partial}_0}{\longrightarrow} X' \longrightarrow 0.
\]

Hence \( X' \) is \( n \)-projective. Note also that \( X' \) is small and that \( x \in X' \). The quotient of the resolutions of \( X \) and \( X' \) gives rise to a projective resolution of \( X/X' \) of length \( n \), and so \( X/X' \) is also \( n \)-projective.

**Example 3.1** (Aldrich, Enochs, Jenda & Oyonarte). \((\mathcal{P}_n, \mathcal{P}_n^\perp)\) is a hereditary complete cotorsion pair. The previous lemma implies that \( S = \{ L \in \mathcal{P}_n \mid L \text{ is small} \} \) is a cogenerating set for \((\mathcal{P}_n, \mathcal{P}_n^\perp)\), and hence by the Eklof and Trlifaj Theorem, this pair is complete. You can check the details in [4, Theorem 7.4.6] or in [1, Theorem 4.2].

## 4 \( n \)-Projective Model Structures

Let \((\mathcal{A}, \mathcal{B})\) be a cotorsion pair in an abelian category \( \mathcal{C} \). Define:

1. \( \tilde{\mathcal{A}} \) (resp. \( \tilde{\mathcal{B}} \)) as the class of all exact complexes \( X \) such that \( Z_m(X) \in \mathcal{A} \) (resp. \( Z_m(X) \in \mathcal{B} \)) for every \( m \in \mathbb{Z} \).

2. \( \text{dg} \tilde{\mathcal{A}} \) (resp. \( \text{dg} \tilde{\mathcal{B}} \)) as the class of all complexes \( X \) with \( X_m \in \mathcal{A} \) (resp. \( X_m \in \mathcal{B} \)) for every \( m \in \mathbb{Z} \), such that every chain map \( Y \longrightarrow X \) (resp. \( X \longrightarrow Y \)) is null-homotopic whenever \( Y \in \tilde{\mathcal{B}} \) (resp. \( Y \in \tilde{\mathcal{A}} \)).

**Proposition 4.1** (Gillespie, [5, Proposition 3.6]). If \((\mathcal{A}, \mathcal{B})\) is complete in \( \mathcal{C} \), then \((\tilde{\mathcal{A}}, \text{dg} \tilde{\mathcal{B}})\) and \((\text{dg} \tilde{\mathcal{A}}, \tilde{\mathcal{B}})\) are cotorsion pairs in \( \text{Ch}(\mathcal{C}) \).
The following definitions are due to J. Gillespie [8, Definition 3.7]. Let \((A, B)\) be a cotorsion pair in an abelian category \(C\). Whenever \((\tilde{A}, dg\tilde{B})\) and \((dg\tilde{A}, \tilde{B})\) are indeed cotorsion pairs, we shall call them the induced \textbf{cotorsion pairs} (of chain complexes over \(C\)). We say that the induced cotorsion pairs are \textbf{compatible} if \(\tilde{A} = dg\tilde{B} \cap E\) and \(\tilde{B} = dg\tilde{B} \cap \mathcal{E}\), where \(\mathcal{E}\) is the class of exact complexes.

\textbf{Proposition 4.2} (Gillespie, [8, Theorem 3.12]). \textit{If \((A, B)\) is a complete and hereditary cotorsion pair in an abelian category \(C\), then the induced cotorsion pairs \((\tilde{A}, dg\tilde{B})\) and \((dg\tilde{A}, \tilde{B})\) are compatible.}

We know that \((P_n, P_n^\perp)\) is a hereditary complete cotorsion pair. So the induced cotorsion pairs \((\tilde{P}_n, dg\tilde{P}_n^\perp)\) and \((dg\tilde{P}_n, \tilde{P}_n^\perp)\) in \(\text{Ch}(R)\) are compatible. If these two cotorsion pairs turn out to be complete, then we shall obtain a new model structure on \(\text{Ch}(R)\).

Given an abelian category \(C\), an object \(X\) is called a \textbf{generator} of \(C\) if for any pair of distinct morphisms \(f, g : A \rightarrow B\) there exists a morphism \(h : X \rightarrow A\) such that \(f \circ h \neq g \circ h\). For example, any Grothendieck category has a generator. Another example is given by \(R\)-Mod, where \(R\), as a left module over itself, is a generator.

The completeness of the pair \((dg\tilde{P}_n, \tilde{P}_n^\perp)\) is a consequence of the following result:

\textbf{Proposition 4.3} (Gillespie, [7, Proposition 3.8]). \textit{Let \((A, B)\) be a cotorsion pair in a Grothendieck category \(C\) which has a generator \(G \in A\). If \((A, B)\) is cogenerated by a set \(S = \{A_i\}_{i \in I_0}\), then the induced cotorsion pair \((\perp B, B)\) is cogenerated by the set}

\[ S = \{S^n(G) / n \in \mathbb{Z}\} \cup \{S^n(A_i) / n \in \mathbb{Z} \text{ and } i \in I_0\}. \]
Furthermore, suppose \((A, B)\) is small with generating monomorphisms the map \(0 \to G\) together with monomorphisms \(k_i\), with \(i \in I_0\), as below:

\[
0 \to Y_i \xrightarrow{k_i} Z_i \to A_i \to 0.
\]

Then \((\perp B, B)\) is small with generating monomorphisms the set

\[
I = \{0 \to D^n(G)\} \cup \{S_i^{n-1}(G) \to D^n(G)\} \cup \{S_i^n(Y_i) \xrightarrow{S^n(k_i)} S_i^n(Z_i) / i \in I_0\}.
\]

We shall recall later the notions of a small cotorsion pair and generating sets of monomorphisms. We know that the pair \((P_n, P_\perp_n)\) is cogenerated by the set of all small \(n\)-projective modules. Also, \(R \in P_n\) as a left \(R\)-module and it is a generator in \(R\)-Mod. It follows that \((\tilde{P}_n, \tilde{P}_\perp_n)\) is a complete cotorsion pair in \(\text{Ch}(R)\). Hence our problem reduces to show that \((\tilde{P}_n, d\tilde{P}_\perp_n)\) is also complete.

We first study the case \(n = 0\). Note that \(P_0 = \text{Proj}\), the class of projective modules, and that \(P_\perp_0 = R\)-Mod. In this case we get the projective cotorsion pair \((P_0, R\text{-Mod})\). Then we have that \(\tilde{P}_0 = \mathcal{E}\) and that \(\tilde{P}_0\) is the class of exact complexes such that \(Z_m(X)\) is a projective module, for every \(m \in \mathbb{Z}\).

**Lemma 4.1.** Every chain complex in \(\tilde{P}_0\) is contractible.

**Proof.** Let \(X \in \tilde{P}_0\). Then \(X\) is exact and so for each \(n \in \mathbb{Z}\) we have a short exact sequence

\[
0 \to Z_n(X) \to X_n \to Z_{n-1}(X) \to 0
\]

where \(Z_{n-1}(X)\) is projective. Then this sequence splits, i.e. \(X_n \cong Z_n(X) \oplus Z_{n-1}(X)\). It follows \(X \cong \bigoplus_{n \in \mathbb{Z}} D^{n+1}(Z_n(X))\), where each \(D^{n+1}(Z_n(X))\) is contractible. Hence so is \(X\).

\(\square\)
It follows that every map $Y \rightarrow X$ with $Y \in \widetilde{P}_0$ is null homotopic, since $Y$ is contractible. Hence $dg\widetilde{P}_0^\perp = Ch(R)$ and so we have the cotorsion pair $\left(\widetilde{P}_0, Ch(R)\right)$. In other words, $\widetilde{P}_0$ is the class of projective chain complexes. Note also that $dg\widetilde{P}_0$ is the class of DG-projective chain complexes. Then, the projective objects of $Ch(R)$ are the complexes that are DG-projective and exact. The projective model structure on $Ch(R)$ given by Hovey in [9] has $dg\widetilde{P}_0$ as the class of cofibrant objects, $Ch(R)$ as the class of fibrant objects, and $E$ as the class of trivial objects. Hovey also proved the following result concerning the relation between cotorsion pairs and model categories (see [10, Theorem 2.2]).

**Theorem 4.1** (M. Hovey). Let $C$ be a bicomplete abelian category with a model structure on it. Let $A$ denote the full subcategory of cofibrant objects, $B$ denote the full subcategory of fibrant objects, and $E$ denote the full subcategory of trivial objects. Then:

(a) $E$ is a thick subcategory; and

(b) $(A, B \cap E)$ and $(A \cap E, B)$ are complete cotorsion pairs.

Conversely, given classes $A$, $B$, and $E$ satisfying the two conditions above, there is a unique abelian model structure on $C$ such that $A$ is the class of cofibrant objects, $B$ is the class of fibrant objects, and $E$ is the class of trivial objects.

Using this theorem and the fact that the projective model structure is abelian, we conclude that the induced cotorsion pair $\left(\widetilde{P}_0, dg\widetilde{P}_0^\perp\right)$ is complete.

Now we study the case when $n > 0$. Before proving the completeness of $\left(\widetilde{P}_n, dg\widetilde{P}_n^\perp\right)$, we need some lemmas.
Lemma 4.2. Let \( 0 \rightarrow A^n \xrightarrow{f^n} A^{n-1} \rightarrow \cdots \rightarrow A^1 \xrightarrow{f^1} A^0 \xrightarrow{f^0} X \rightarrow 0 \) be an exact sequence of chain complexes such that \( A^i \) is exact for every \( 0 \leq i \leq n \). Then \( X \) is also an exact complex.

Proof. We use induction on \( n \). The case \( n = 0 \) is trivial. For the case \( n = 1 \), we have a short exact sequence

\[
0 \rightarrow A^1 \rightarrow A^0 \rightarrow X \rightarrow 0,
\]

where \( A^0 \) and \( A^1 \) are exact complexes. Since \( E \) is thick, we have that \( X \) is also exact.

Now suppose the statement is true for \( n - 1 \). Consider the following exact sequence

\[
0 \rightarrow A^n \rightarrow A^{n-1} \rightarrow \cdots \rightarrow A^1 \rightarrow \text{Im}(f^1) \rightarrow 0.
\]

Then \( \text{Im}(f^1) \) is an exact complex by the induction hypothesis. On the other hand, we have a short exact sequence

\[
0 \rightarrow \text{Ker}(f^0) \rightarrow A^0 \rightarrow X \rightarrow 0,
\]

where \( \text{Ker}(f^0) = \text{Im}(f^1) \) and \( A^0 \) are exact complexes. Hence \( X \) is exact since \( E \) is thick.

\[\square\]

Lemma 4.3. Let \( 0 \rightarrow A^n \xrightarrow{f^n} A^{n-1} \rightarrow \cdots \rightarrow A^1 \xrightarrow{f^1} A^0 \rightarrow 0 \) be an exact sequence of exact chain complexes. Then, for every \( m \in \mathbb{Z} \), there exists an exact sequence of modules

\[
0 \rightarrow Z_m(A^n) \rightarrow Z_m(A^{n-1}) \rightarrow \cdots \rightarrow Z_m(A^1) \rightarrow Z_m(A^0) \rightarrow 0.
\]

Proof. The proof follows by induction on \( n \). We prove first the case \( n = 2 \). We are given a short exact sequence \( 0 \rightarrow A^2 \xrightarrow{f^2} A^1 \xrightarrow{f^1} X \rightarrow 0 \). For each \( m \in \mathbb{Z} \), we have a commutative diagram
Let \( x \in \text{Im}(\partial_{A}^{2}) \), then \( x = \partial_{A}^{2}(a) \) for some \( a \in A_{m}^{2} \). We have

\[
f_{m-1}^{2}(x) = f_{m-1}^{2} \circ \partial_{m}^{2}(a) = \partial_{m}^{1} \circ f_{m}^{2}(a) \in \text{Im}(\partial_{m}^{1}).
\]

Then we obtain a map \( \tilde{f}_{m-1}^{2} : \text{Im}(\partial_{m}^{2}) \to \text{Im}(\partial_{m}^{1}) \) given by \( x \mapsto f_{m-1}^{2}(x) \). Similarly, we get a map \( \tilde{f}_{m-1}^{1} : \text{Im}(\partial_{m}^{1}) \to \text{Im}(\partial_{m}^{0}) \) given by \( x \mapsto f_{m-1}^{1}(x) \). Consider the restrictions \( \tilde{\partial}_{m}^{2} : A_{m}^{2} \to \text{Im}(\partial_{m}^{2}), \tilde{\partial}_{m}^{1} : A_{m}^{1} \to \text{Im}(\partial_{m}^{1}) \) and \( \tilde{\partial}_{m}^{0} : A_{m}^{0} \to \text{Im}(\partial_{m}^{0}) \).

We prove that the sequence \( 0 \to \text{Im}(\partial_{m}^{2}) \to \text{Im}(\partial_{m}^{1}) \to \text{Im}(\partial_{m}^{0}) \) is exact.

(i) It is clear that \( \tilde{f}_{m-1}^{2} \) is injective.

(ii) It is easy to see that \( \text{Im}(\tilde{f}_{m-1}^{2}) \subseteq \text{Ker}(\tilde{f}_{m-1}^{1}) \). Now let \( y \in \text{Ker}(\tilde{f}_{m-1}^{1}) \). Then \( f_{m-1}^{1}(y) = 0 \) and so there exists \( x \in A_{m-1}^{2} \) such that \( y = f_{m-1}^{2}(x) \). We show that \( x \in \text{Im}(\partial_{m}^{2}) = \text{Ker}(\partial_{m}^{1}) \). We know that there exists \( b \in A_{m}^{1} \) such that \( y = \partial_{m}^{1}(b) \). Then we have

\[
0 = f_{m-1}^{1}(y) = f_{m-1}^{1} \circ \partial_{m}^{1}(b) = \partial_{m}^{0} \circ f_{m}^{1}(b)
\]

and so \( f_{m}^{1}(b) \in \text{Ker}(\partial_{m}^{0}) = \text{Im}(\partial_{m+1}^{0}) \), i.e. there exists \( c \in A_{m+1}^{0} \) such that \( f_{m}^{1}(b) = \partial_{m+1}^{0}(c) \). Since \( f_{m+1}^{1} \) is surjective, there exists \( b' \in A_{m+1}^{1} \) such that \( c = f_{m+1}^{1}(b') \). We have

\[
f_{m}^{1}(b) = \partial_{m+1}^{0} \circ f_{m+1}^{1}(b') = f_{m}^{1} \circ \partial_{m+1}^{1}(b').
\]
So \( b - \partial_{m+1}^{A_1}(b') \in \text{Ker}(f_m) = \text{Im}(f_m^2) \). It follows there exists \( a \in A_m^2 \) such that \( b = f_m^2(a) + \partial_{m+1}^{A_1}(b') \). Then

\[
f_{m-1}^2(x) = y = \partial_m^{A_1}(b) = \partial_m^{A_1} \circ f_m^2(a) + \partial_m^{A_1} \circ \partial_{m+1}^{A_1}(b') = f_{m-1}^2 \circ \partial_m^{A_2}(a).
\]

Since \( f_{m-1}^2 \) is injective, we get \( x = \partial_m^{A_2}(a) \). Therefore, \( \text{Ker}(\tilde{f}_{m-1}^2) = \text{Im}(\tilde{f}_{m-1}^1) \).

Then we have a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & A_m^2 & \xrightarrow{f_m^2} & A_m^1 & \xrightarrow{f_m^1} & A_m^0 & \rightarrow & 0 \\
& & \downarrow{\partial_m^{A_2}} & & \downarrow{\partial_m^{A_1}} & & \downarrow{\partial_m^{A_0}} & & \\
0 & \rightarrow & \text{Im}(\partial_m^{A_2}) & \xrightarrow{\tilde{f}_{m-1}^2} & \text{Im}(\partial_m^{A_1}) & \xrightarrow{\tilde{f}_{m-1}^1} & \text{Im}(\partial_m^{A_0}) & \rightarrow & 0
\end{array}
\]

By the Snake Lemma, we get an exact sequence

\[
0 \rightarrow \text{Ker}(\tilde{\partial}_m^{A_2}) \rightarrow \text{Ker}(\tilde{\partial}_m^{A_1}) \rightarrow \text{Ker}(\tilde{\partial}_m^{A_0}) \rightarrow \\
\rightarrow \text{CoKer}(\tilde{\partial}_m^{A_2}) \rightarrow \text{CoKer}(\tilde{\partial}_m^{A_1}) \rightarrow \text{CoKer}(\tilde{\partial}_m^{A_0}).
\]

On the other hand, we have \( \text{Ker}(\tilde{\partial}_m^{A_2}) = Z_m(A^2) \), \( \text{Ker}(\tilde{\partial}_m^{A_1}) = Z_m(A^1) \), \( \text{Ker}(\tilde{\partial}_m^{A_0}) = Z_m(A^0) \) and \( \text{CoKer}(\tilde{\partial}_m^{A_2}) = 0 \). Hence we obtain a short exact sequence

\[
0 \rightarrow Z_m(A^2) \rightarrow Z_m(A^1) \rightarrow Z_m(A^0) \rightarrow 0.
\]

Now suppose the statement is true for \( n-1 \). Consider the exact sequence

\[
0 \rightarrow A^n \rightarrow A^{n-1} \rightarrow \cdots \rightarrow A^2 \rightarrow \text{Im}(f^2) \rightarrow 0,
\]

where \( \text{Im}(f^2) \) is an exact complex by the previous lemma. By the induction hypothesis, we have an exact sequence
We get the following commutative diagram with exact rows and columns:

\[
\begin{array}{cccccccc}
0 & & & & & & & \\
\downarrow & & & & & & & \\
0 & \rightarrow & Z_m(A^n) & \rightarrow & Z_m(A^{n-1}) & \rightarrow & \cdots & \rightarrow & Z_m(A^2) & \rightarrow & Z_m(\text{Im}(f^2)) & \rightarrow & 0 \\
\downarrow & & & & & & & \\
0 & \rightarrow & Z_m(A^n) & \rightarrow & Z_m(A^{n-1}) & \rightarrow & \cdots & \rightarrow & Z_m(A^2) & \rightarrow & Z_m(\text{Im}(f^2)) & \rightarrow & 0 \\
\downarrow & & & & & & & \\
& & & & & & & \downarrow & & & & & & & \\
& & & & & & & h & & & & & & & \\
& & & & & & & Z_m(A^1) & \rightarrow & Z_m(A^0) & \rightarrow & 0 \\
\end{array}
\]

where \( Z_m(A^2) \xrightarrow{h} Z_m(A^1) \) is given by the composition of \( Z_m(A^2) \rightarrow Z_m(\text{Im}(f^2)) \) and \( \text{Im}(f^2) \rightarrow Z_m(A^1) \). It is not hard to show that

\[
0 \rightarrow Z_m(A^n) \rightarrow Z_m(A^{n-1}) \rightarrow \cdots \rightarrow Z_m(A^2) \xrightarrow{h} Z_m(A^1) \rightarrow Z_m(A^0) \rightarrow 0
\]

is an exact sequence of modules, for every \( m \in \mathbb{Z} \).

The \textbf{projective dimension of a chain complex} \( X \in \text{Ch}(R) \), denoted \( \text{pd}(X) \), is defined as the smallest nonnegative integer \( n \) such that there exists an exact sequence

\[
0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0,
\]

where \( P_i \) is a projective chain complex for every \( 0 \leq i \leq n \). If such an integer does not exists, then set \( \text{pd}(X) = \infty \). We shall say that \( X \) is an \textbf{\( n \)-projective complex} if \( \text{pd}(X) \leq n \).
Proposition 4.4. $\mathcal{P}_n$ is the class of $n$-projective complexes.

Proof. We use induction on $n$. The case $n = 0$ is clear. We prove the case $n = 1$:

$(\subseteq)$ Let $X \in \mathcal{P}_1$. Consider a short exact sequence

$$0 \to K \to P \to X \to 0$$

where $P \in \mathcal{P}_0$. We shall see that $K \in \mathcal{P}_0$. Since $P$ and $X$ are exact complexes and $\mathcal{E}$ is thick, we have $K$ is also exact. By the previous lemma, we have a short exact sequence

$$0 \to Z_m(K) \to Z_m(P) \to Z_m(X) \to 0$$

where $Z_m(P) \in \mathcal{P}_0$ and $Z_m(X) \in \mathcal{P}_1$. Then there exists a short exact sequence

$$0 \to Q_1 \to Q_0 \to Z_m(X) \to 0$$

where $Q_0, Q_1 \in \mathcal{P}_0$. Taking the pullback of the morphisms $Z_m(P) \to Z_m(X)$ and $Q_0 \to Z_m(X)$, we have a commutative diagram

$$
\begin{array}{c}
\begin{tikzcd}
0 & 0 \\
Q_1 & Q_1 \\
0 & Z_m(K) & E & Q_0 & 0 \\
0 & Z_m(K) & Z_m(P) & Z_m(X) & 0 \\
0 & 0
\end{tikzcd}
\end{array}
$$
Since $Z_m P_1, Q_1 \in \mathcal{P}_0$ and $\mathcal{P}_0$ is closed under extensions, we have $E \in \mathcal{P}_0$. Now we have $Q_0, E \in \mathcal{P}_0$. It follows $Z_m K \in \mathcal{P}_0$ since $\mathcal{P}_0$ is resolving. Therefore, $K \in \mathcal{P}_0$ and so $\text{pd}(X) \leq 1$.

(≥) Let $X$ be a complex with $\text{pd}(X) \leq 1$. Then there exists a short exact sequence in $\text{Ch}(R)$

$$0 \rightarrow P^1 \rightarrow P^0 \rightarrow X \rightarrow 0$$

such that $P^0, P^1 \in \mathcal{P}_0$. Then $P^0, P^1 \in \mathcal{E}$ and hence $X \in \mathcal{E}$, since $\mathcal{E}$ is thick. It is only left to show that each $Z_m(X) \in \mathcal{P}_1$. By the previous lemma we have a short exact sequence

$$0 \rightarrow Z_m(P^1) \rightarrow Z_m(P^0) \rightarrow Z_m(X) \rightarrow 0,$$

for each $m \in \mathbb{Z}$ where $Z_m(P^0), Z_m(P^1) \in \mathcal{P}_0$. Hence $Z_m(X) \in \mathcal{P}_1$.

Now suppose the equality holds for $n - 1$. Let $X \in \text{Ch}(R)$ be an $n$-projective complex. Then there is an exact sequence

$$0 \rightarrow P^n \rightarrow P^{n-1} \rightarrow \cdots \rightarrow P^1 \rightarrow P^0 \rightarrow X \rightarrow 0$$

where each $P^i \in \mathcal{P}_0$. By Lemma 4.2 we have that $X$ is an exact complex. Moreover, by Lemma 4.3 there is, for every $m \in \mathbb{Z}$, an exact sequence of modules

$$0 \rightarrow Z_m(P^n) \rightarrow Z_m(P^{n-1}) \rightarrow \cdots \rightarrow Z_m(P^1) \rightarrow Z_m(P^0) \rightarrow Z_m(X) \rightarrow 0$$

such that each $Z_m(P^i)$ is a projective module. Then $Z_m(X) \in \mathcal{P}_n$, and hence $X \in \mathcal{P}_n$.

Now let $X \in \mathcal{P}_n$. Then $X$ is an exact complex and each $Z_m X \in \mathcal{P}_n$, for every $m \in \mathbb{Z}$. We know there is a short exact sequence

$$0 \rightarrow K \rightarrow P \rightarrow X \rightarrow 0$$
where \( P \) is a projective chain complex and \( K \) is the kernel of the epimorphism \( P \to X \).

Since \( \mathcal{E} \) is thick, we have that \( K \) is exact. By Lemma 4.3 we obtain a short exact sequence

\[
0 \to Z_m(K) \to Z_m(P) \to Z_m(X) \to 0
\]

for every \( m \in \mathbb{Z} \). Consider an epimorphism \( q_0 : Q_0 \to Z_m(X) \) with \( Q_0 \) a projective module. As we did in the case \( n = 1 \), we have the following commutative diagram with exact rows and columns:

\[
\begin{array}{cccccc}
  & 0 & 0 \\
  & \downarrow & \downarrow \\
\text{Ker}(q_0) & \longrightarrow & \text{Ker}(q_0) \\
  & \downarrow & \downarrow \\
0 & \longrightarrow & Z_m(K) & \longrightarrow & E & \longrightarrow & Q_0 & \longrightarrow & 0 \\
  & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & Z_m(K) & \longrightarrow & Z_m(P) & \longrightarrow & Z_m(X) & \longrightarrow & 0 \\
  & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Since the two columns are exact, we have

\[
\text{pd}(\text{Ker}(q_0)) \leq \max\{\text{pd}(Q_0), \text{pd}(Z_mX) - 1\} \leq \max\{0, n - 1\} = n - 1,
\]

\[
\text{pd}(E) \leq \max\{\text{pd}(\text{Ker}(q_0)), \text{pd}(Z_mP)\} \leq \max\{n - 1, 0\} = n - 1.
\]

So \( \text{Ker}(q_0) \in \mathcal{P}_{n-1} \) and \( E \in \mathcal{P}_{n-1} \). Hence \( Z_mK \in \mathcal{P}_{n-1} \) since \( \mathcal{P}_n \) is resolving. It follows \( K \in \widehat{\mathcal{P}}_{n-1} \). By the induction hypothesis, there exists a projective resolution of \( K \) of length \( n - 1 \):
$0 \rightarrow L^{n-1} \rightarrow L^{n-2} \rightarrow \cdots \rightarrow L^1 \rightarrow L^0 \rightarrow K \rightarrow 0$

As we did in the proof of Lemma 4.3, we get an exact sequence

$0 \rightarrow L^{n-1} \rightarrow L^{n-2} \rightarrow \cdots \rightarrow L^1 \rightarrow L^0 \rightarrow P \rightarrow X \rightarrow 0$

by composing the maps $K \rightarrow P$ and $L^0 \rightarrow K$. Therefore, $X$ is $n$-projective.

Given a chain complex $F \in \text{Ch}(R)$, we shall say that $F$ is a free complex if $F$ is exact and each $Z_n(F)$ is a free left $R$-module. Note that every free complex is projective.

**Lemma 4.4.** Every free complex can be decomposed into a direct sum of free disks. Conversely, any direct sum of free disks is free.

**Proof.** Let $F$ be a free complex. As we did in the proof of Lemma 4.1, we have $F \cong \bigoplus_{m \in \mathbb{Z}} D^{m+1}(Z_m(F))$, where each $D^{m+1}(Z_m(F))$ is a free disk.

Now let $F = \bigoplus_{n \in \mathbb{Z}} D^{n+1}(F_n)$, where each $F_n$ is a free module. It is clear that $F$ is exact. Note that $F$ has the form

$$\cdots \rightarrow F_n \oplus F_{n+1} \overset{\partial_n^F}{\rightarrow} F_{n-1} \oplus F_n \overset{\partial_n^F}{\rightarrow} F_{n-2} \oplus F_{n-1} \rightarrow \cdots$$

where each boundary map $\partial_n^F : F_{n-1} \oplus F_n \rightarrow F_{n-2} \oplus F_{n-1}$ is given by $(x, y) \mapsto (0, x)$. We have that $Z_n(F) \cong F_n$ is a free module.

**Proposition 4.5** (Eilenberg Trick in Ch(R)). If $P$ is a projective complex, then there exists a free complex $F$ such that $P \oplus F \cong F$. 

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Proof. Let $P$ be a projective module. Then $P$ is isomorphic to a direct sum of projective disks $P \cong \bigoplus_{m \in \mathbb{Z}} D^{m+1}(Z_m(P))$. By the Eilenberg Trick in $R$-Mod there exists a free module $F_m$ such that $Z_m(P) \oplus F_m \cong F_m$. It follows

$$D^{m+1}(Z_m(P)) \oplus D^{m+1}(F_m) \cong D^{m+1}(F_m).$$

Setting $F = \bigoplus_{m \in \mathbb{Z}} D^{m+1}(F_m)$, which is a free complex by the previous lemma, we obtain

$$P \oplus F \cong \left( \bigoplus_{m \in \mathbb{Z}} D^{m+1}(Z_m(P)) \right) \oplus \left( \bigoplus_{m \in \mathbb{Z}} D^{m+1}(F_m) \right) \cong \bigoplus_{m \in \mathbb{Z}} (D^{m+1}(Z_m(P)) \oplus D^{m+1}(F_m)) \cong \bigoplus_{m \in \mathbb{Z}} D^{m+1}(F_m) = F.$$

Lemma 4.5. Every $X \in \widehat{\mathcal{P}}_n$ has a free resolution of length $n$.

Proof. We only prove the case $n = 1$. The general case can be proven similarly. Let $X \in \widehat{\mathcal{P}}_1$ and let $0 \rightarrow P^1 \rightarrow P^0 \rightarrow X \rightarrow 0$ be a projective resolution of $X$. By the Eilenberg Trick in $\text{Ch}(R)$, there exist free complexes $F^0$ and $F^1$ such that $P^0 \oplus F^0 \cong F^0$ and $P^1 \oplus F^1 \cong F^1$. Consider the short exact sequences

$$0 \rightarrow F^1 \rightarrow F^1 \rightarrow 0 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow F^0 \rightarrow F^0 \rightarrow 0 \rightarrow 0.$$

Adding the sequences above, we get

$$0 \rightarrow P^1 \oplus F^1 \oplus F^0 \rightarrow P^0 \oplus F^0 \oplus F^1 \rightarrow X \rightarrow 0 \cong 0 \rightarrow F^1 \oplus F^0 \rightarrow F^0 \oplus F^1 \rightarrow X \rightarrow 0.$$


**Lemma 4.6.** Given the following commutative diagram with exact rows and columns:

\[
\begin{array}{cccc}
0 & F^n_m & F^n_{m+1} & F^n_{m+2} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & F^n_{m-1} & F^n_{m-2} & F^n_m \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & F^n_{m-2} & F^n_{m-3} & F^n_m \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\end{array}
\]

For every \( m \in \mathbb{Z} \) and every \( i \in \{0, 1, \ldots, n\} \), one has \( f^i_m(F^n_m) \subseteq F^{i-1}_m \). Moreover, the sequence

\[
0 \rightarrow F^n_{m-1} \oplus F^n_m \rightarrow \cdots \rightarrow F^1_{m-1} \oplus F^1_m \rightarrow F^1_{m-1} \oplus F^1_m \rightarrow F^0_{m-1} \oplus F^0_m \rightarrow \cdots \rightarrow X_{m+1} \rightarrow X_m \rightarrow X_{m-1} \rightarrow \cdots
\]

is exact.

**Proof.** Let \((0, b) \in F^i_m\). We have \( f^i_m(0, b) = f^i_m \circ \partial^i_{m+1}(b, 0) = \partial^i_{m+1} \circ f^i_{m+1}(b, 0) \in F^{i-1}_m \).

It follows that \( \text{Im}(f^i_m|_{F^n_m}) \subseteq \text{Ker}(f^{i-1}_m|_{F^n_{m-1}}) \), for every \( m \in \mathbb{Z} \). To prove the other inclusion, we start with \( i = n \). Let \((0, b) \in F^n_m\) such that \( f^n_m(0, b) = (0, 0) \). Then there exists \((\alpha, \beta) \in F^n_{m-1} \oplus F^n_m\) such that \((0, b) = f^n_m(\alpha, \beta)\), since the \( m \)-th column is exact.
On the other hand,

\[
    f_{n-1}^m(0, \alpha) = f_{n-1}^m \circ \partial_m^n(\alpha, \beta) = \partial_{m-1}^n \circ f_m^n(\alpha, \beta) = \partial_{m-1}^n(0, b) = (0, 0).
\]

Since \( f_{n-1}^m \) is injective, we have \( (0, \alpha) = (0, 0) \) and so \( (0, b) = f_m^n(0, \beta) \), i.e. \( \text{Im}(f_m^n|_{F_{n-1}}) \) contains \( \text{Ker}(f_m^{n-1}|_{F_{n-1}}) \), for every \( m \in \mathbb{Z} \). Now we show that \( \text{Im}(f_m^{n-1}|_{F_{n-1}}) \) contains \( \text{Ker}(f_m^{n-2}|_{F_{n-2}}) \). Let \( (0, b) \in \text{Ker}(f_m^{n-2}) \). Since the central column is exact, there exists \( (\alpha, \beta) \in F_{n-1}^m \oplus F_{n-1}^m \) such that \( (0, b) = f_m^{n-1}(\alpha, \beta) \). On the other hand,

\[
    f_{n-1}^m(0, \alpha) = f_{n-1}^m \circ \partial_{m-1}^n(\alpha, \beta) = \partial_{m-2}^n \circ f_m^{n-1}(\alpha, \beta) = \partial_{m-2}^n(0, b) = (0, 0).
\]

Then \( (0, \alpha) \in \text{Ker}(f_m^{n-1}|_{F_{n-1}}) = \text{Im}(f_m^{n-1}|_{F_{n-1}}) \) and so there exists an element \( (0, \gamma) \) in \( F_{m-2}^n \oplus F_{m-1}^n \) such that \( (0, \alpha) = f_m^{n-1}(0, \gamma) \). Since \( (\alpha, \beta) - f_m^n(\gamma, 0) \in F_{m-1}^n \oplus F_{m-1}^n \), we have

\[
    \partial_{m-1}^n((\alpha, \beta) - f_m^n(\gamma, 0)) = (0, \alpha) - \partial_{m-1}^n \circ f_m^n(\gamma, 0)
    = (0, \alpha) - f_{m-1}^n \circ \partial_{m-2}^n(\gamma, 0)
    = (0, \alpha) - f_{m-1}^n(0, \gamma) = (0, 0),
\]

i.e. \( (\alpha, \beta) - f_m^n(\gamma, 0) \in \text{Ker}(\partial_{m-1}^n) = F_{m-1}^n \). Also,

\[
    f_m^{n-1}((\alpha, \beta) - f_m^n(\gamma, 0)) = f_m^{n-1}(\alpha, \beta) = (0, b).
\]

Hence \( \text{Im}(f_m^{n-1}|_{F_{m-1}}) \supset \text{Ker}(f_m^{n-2}|_{F_{m-2}}) \). Repeating the same argument several times, we get the exact sequence

\[
    0 \rightarrow F_{m-1}^n \oplus F_m^n \rightarrow \cdots \rightarrow F_{m-1}^1 \oplus F_m^1 \xrightarrow{f_m^n|_{F_{m-1}}} F_{m-1}^0 \oplus F_m^0 \xrightarrow{f_m^n|_{F_{m-1}}} X_m.
\]
Given a chain complex $X = (X_m, \partial^X_m)_{m \in \mathbb{Z}}$, its cardinal number is defined as $\text{Card}(X) = \sum_{m \in \mathbb{Z}} \text{Card}(X_m)$. We shall say that $X$ is small if $\text{Card}(X) \leq \kappa$. We shall denote $x \in X$ whenever there exists $m \in \mathbb{Z}$ such that $x \in X_m$.

Lemma 4.7. If $X$ is an $n$-projective complex and $x \in X$, then there exists a small $n$-projective subcomplex $X' \subseteq X$ with $x \in X'$, such that $X/X'$ is also $n$-projective.

Proof. We only prove the case when $n = 1$. The general case follows similarly. Consider a free resolution of $X$ of length 1 in $\text{Ch}(R)$:

$$0 \longrightarrow L^1 \xrightarrow{f^1} L^0 \xrightarrow{f^0} X \longrightarrow 0.$$  

The idea of the proof is to apply a generalization of the zig-zag argument to produce small free subcomplexes of $L^0$ and $L^1$, say $T^0$ and $T^1$, and a short exact sequence of the form

$$0 \longrightarrow T^1 \xrightarrow{f^1|_{T^1}} L^0 \xrightarrow{f^0|_{T^0}} X$$

where $X' = \text{CoKer}(f^1|_{T^1})$ is a subcomplex of $X$ with $x \in X'$.

Since $L^0$ and $L^1$ are free complexes,

$$L^0 \cong \bigoplus_{i \in \mathbb{Z}} D^i(F^0_{i-1}) \quad \text{and} \quad L^1 \cong \bigoplus_{i \in \mathbb{Z}} D^i(F^1_{i-1}),$$

where $F^0_i$ and $F^1_i$ are free modules, for every $i \in \mathbb{Z}$. Let $B^0_i$ and $B^1_i$ be bases of $F^0_i$ and $F^1_i$, respectively. Then $B^0_{i-1} \sqcup B^0_i$ and $B^1_{i-1} \sqcup B^1_i$ are bases of $F^0_{i-1} \oplus F^0_i$ and $F^1_{i-1} \oplus F^1_i$, respectively. Suppose $x \in X_m$. At the $m$-th level, we have the following exact sequence

$$0 \longrightarrow F^1_{m-1} \oplus F^1_m \xrightarrow{f^1_m} F^0_{m-1} \oplus F^0_m \xrightarrow{f^0_m} X_m \longrightarrow 0.$$
Consider the free resolution above as the following commutative diagram with exact rows and columns:

\[
\begin{array}{cccccc}
0 & \rightarrow & F_1^m \oplus F_1^{m+1} & \oplus & \rightarrow & F_1^{m-1} \oplus F_1^m & \oplus & \rightarrow & F_1^{m-2} \oplus F_1^{m-1} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & F_0^m \oplus F_0^{m+1} & \oplus & \rightarrow & F_0^{m-1} \oplus F_0^m & \oplus & \rightarrow & F_0^{m-2} \oplus F_0^{m-1} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & X_{m+1} & \rightarrow & X_m & \rightarrow & X_{m-1} & \rightarrow & \cdots \\
0 & & 0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

Let \( W_{m-1}^0 \cup W_m^0 \) be a finite subset of \( B_{m-1}^0 \cup B_m^0 \) (we mean \( W_{m-1}^0 \subseteq B_{m-1}^0 \) and \( W_m^0 \subseteq B_m^0 \)) such that \( x \in f_m^1 (\langle W_{m-1}^0 \cup W_m^0 \rangle) \). Now let \( W_{m-1}^1 \cup W_m^1 \subseteq B_{m-1}^1 \cup B_m^1 \) be a small set such that

\[
f_m^1 (\langle W_{m-1}^1 \cup W_m^1 \rangle) \supseteq \operatorname{Ker} (f_{m-1}^0 |_{\langle W_{m-1}^0 \cup W_m^0 \rangle}).
\]

It is not necessarily true that \( f_m^1 (\langle W_{m-1}^1 \cup W_m^1 \rangle) \supseteq \operatorname{Ker} (f_{m-1}^0 |_{\langle W_{m-1}^0 \rangle}) \). Since the sequence

\[
0 \rightarrow F_{m-1}^1 \xrightarrow{f_{m-1}^1} F_{m-1}^0 \xrightarrow{f_{m-1}^0} X_{m-1}
\]

is exact, there exists a small set \( \widetilde{W}_{m-1}^1 \subseteq B_{m-1}^1 \) such that \( f_{m-1}^1 (\langle \widetilde{W}_{m-1}^1 \rangle) \) does contain \( \operatorname{Ker} (f_{m-1}^0 |_{\langle W_{m-1}^0 \rangle}) \). Adding to \( \widetilde{W}_{m-1}^1 \) the elements of \( W_{m-1}^1 \) which are not in \( \widetilde{W}_{m-1}^1 \), we may assume that \( \widetilde{W}_{m-1}^1 \supseteq W_{m-1}^1 \). So we have

- \( f_m^1 (\langle \widetilde{W}_{m-1}^1 \cup W_m^1 \rangle) \supseteq f_m^1 (\langle W_{m-1}^1 \cup W_m^1 \rangle) \supseteq \operatorname{Ker} (f_{m-1}^0 |_{\langle W_{m-1}^0 \cup W_m^0 \rangle}), \)

- \( f_{m-1}^1 (\langle \widetilde{W}_{m-1}^1 \rangle) \supseteq \operatorname{Ker} (f_{m-1}^0 |_{\langle W_{m-1}^0 \rangle}). \)
Summarizing, we can choose a small set \( W_{m-1}^1 \cup W_m^1 \subseteq B_{m-1}^1 \cup B_m^1 \) such that

- \( f_m^1 \left( \langle W_{m-1}^1 \cup W_m^1 \rangle \right) \supseteq \text{Ker} \left( f_m^0 \mid_{\langle W_{m-1}^0 \cup W_m^0 \rangle} \right) \),
- \( f_{m-1}^1 \left( \langle W_{m-1}^1 \rangle \right) \supseteq \text{Ker} \left( f_{m-1}^0 \mid_{\langle W_{m-1}^0 \rangle} \right) \).

Notice that \( \langle W_{m-1}^j \cup W_m^j \rangle = \langle W_{m-1}^j \rangle \oplus \langle W_m^j \rangle \), for \( j = 0, 1 \).

We go back to \( F_{m-1}^0 \oplus F_m^0 \). Choose a small set \( W_{m-1}^{0,1} \cup W_m^{0,1} \subseteq B_{m-1}^1 \cup B_m^1 \) containing \( W_{m-1}^0 \cup W_m^0 \) (and so \( W_{m-1}^{0,1} \supseteq W_{m-1}^0 \) and \( W_m^{0,1} \supseteq W_m^0 \)), such that

\[
\begin{align*}
f_m^1 \left( \langle W_{m-1}^1 \rangle \oplus \langle W_m^1 \rangle \right) &\subseteq \langle W_{m-1}^{0,1} \rangle \oplus \langle W_m^{0,1} \rangle.
\end{align*}
\]

Note that \( f_{m-1}^1 \left( \langle W_{m-1}^1 \rangle \right) \subseteq \langle W_{m-1}^{0,1} \rangle \).

Now we enlarge \( W_{m-1}^1 \cup W_m^1 \), i.e. we choose a small set \( W_{m-1}^{1,1} \cup W_m^{1,1} \subseteq B_{m-1}^1 \cup B_m^1 \) containing \( W_{m-1}^1 \cup W_m^1 \) such that

- \( f_m^1 \left( \langle W_{m-1}^{1,1} \rangle \oplus \langle W_m^{1,1} \rangle \right) \supseteq \text{Ker} \left( f_m^0 \mid_{\langle W_{m-1}^{0,1} \rangle \oplus \langle W_m^{0,1} \rangle} \right) \),
- \( f_{m-1}^1 \left( \langle W_{m-1}^{1,1} \rangle \right) \supseteq \text{Ker} \left( f_{m-1}^0 \mid_{\langle W_{m-1}^{0,1} \rangle} \right) \).

Then enlarge \( W_{m-1}^{0,1} \cup W_m^{0,1} \) to a small subset \( W_{m-1}^{0,2} \cup W_m^{0,2} \subseteq B_{m-1}^0 \cup B_m^0 \) and so on. For \( k = 0, 1, \) set

- \( B_{m-1}^{k,(0)} = \bigcup_{j=0}^{\infty} W_{m-1}^{k,(j)} \), where \( W_{m-1}^{k,(0)} = W_{m-1}^k \) and \( W_{m-1}^{k,(1)} \subseteq W_{m-1}^{k,(0)} \subseteq \cdots \),
- \( B_m^{k,(0)} = \bigcup_{j=0}^{\infty} W_m^{k,(j)} \), where \( W_m^{k,(0)} = W_m^k \) and \( W_m^{k,(1)} \subseteq W_m^{k,(0)} \subseteq \cdots \).

Note that the sets above are linearly independent and small. By construction, we have

\[
\begin{align*}
f_m^1 \left( \langle B_{m-1}^{1,(0)} \rangle \oplus \langle B_m^{1,(0)} \rangle \right) &\subseteq \langle B_{m-1}^{0,(0)} \rangle \oplus \langle B_m^{0,(0)} \rangle \quad \text{and} \quad f_{m-1}^1 \left( \langle B_{m-1}^{1,(0)} \rangle \right) \subseteq \langle B_{m-1}^{0,(0)} \rangle.
\end{align*}
\]
Moreover, the following diagram commutes and has exact rows and columns:

\[
\begin{array}{ccc}
\langle B_m^{1,(0)} \rangle \oplus \langle 0 \rangle & \xrightarrow{\partial_{m+1}} & \langle B_m^{0,(0)} \rangle \oplus \langle 0 \rangle \\
\downarrow f_1^m & & \downarrow \partial_0^{m+1} \\
\langle B_{m-1}^{1,(0)} \rangle \oplus \langle B_m^{1,(0)} \rangle & \xrightarrow{f_1^m} & \langle B_{m-1}^{0,(0)} \rangle \oplus \langle B_m^{0,(0)} \rangle \\
\downarrow \partial_1^m & & \downarrow \partial_0^m \\
\langle 0 \rangle \oplus \langle B_{m-1}^{1,(0)} \rangle & \xrightarrow{f_1^{m-1}} & \langle 0 \rangle \oplus \langle B_{m-1}^{0,(0)} \rangle \\
\downarrow 0 & & \downarrow 0 \\
\vdots & & \vdots
\end{array}
\]

where the morphisms appearing in it are their corresponding restrictions. At this point, the problem is that we do not know if the \(m+1\)-th row is exact. Actually, we do not even know if \(f_1^{m+1} \left( \langle B_m^{1,(0)} \rangle \right) \subseteq \langle B_m^{0,(0)} \rangle \). In order to fix this problem, we are going to refine the sets of generators just obtained applying the zig-zag argument again, without destroying exactness in the other rows.

Choose a small set \(Y_0^0 \sqcup Y_0^{m+1} \subseteq B_0^0 \sqcup B_0^{m+1}\) containing \(B_0^{m,(0)}\) (and so \(B_0^{m,(0)} \subseteq Y_0^0\)) such that

\[
f_1^{m+1} \left( \langle Y_0^{m+1} \rangle \right) \subseteq \langle Y_0^0 \rangle \oplus \langle Y_0^{m+1} \rangle.
\]

Note that

\[
f_1^m \left( \langle B_{m-1}^{1,(0)} \rangle \oplus \langle B_m^{1,(0)} \rangle \right) \subseteq \langle B_{m-1}^{0,(0)} \rangle \oplus \langle Y_0^0 \rangle.
\]

Now choose a small set \(Y_1^0 \sqcup Y_1^{m+1} \subseteq B_1^0 \sqcup B_1^{m+1}\) containing \(B_1^{1,(0)}\) (i.e. \(B_1^{1,(0)} \subseteq Y_1^0\)) such that

\[
f_1^{m+1} \left( \langle Y_1^0 \rangle \oplus \langle Y_1^{m+1} \rangle \right) \supseteq \text{Ker} \left( f_1^m |_{Y_1^0 \oplus \langle Y_1^{m+1} \rangle} \right).
\]
It is not necessarily true that $f_m^1 \left( \langle B_{m-1}^{1,(0)} \rangle \oplus \langle Y_{m}^1 \rangle \right) \supseteq \text{Ker} \left( f_m^0 \mid _{B_{m-1}^{0,(0)} \oplus \langle Y_{m}^0 \rangle} \right)$. But there exists a small set $\bar{Y}_{m-1}^1 \cup \bar{Y}_m^1 \subseteq B_{m-1}^1 \cup B_m^1$ containing $B_{m-1}^{1,(0)} \sqcup B_m^{1,(0)}$ such that

$$f_m^1 \left( \langle \bar{Y}_{m-1}^1 \rangle \oplus \langle \bar{Y}_m^1 \rangle \right) \supseteq \text{Ker} \left( f_m^0 \mid _{B_{m-1}^{0,(0)} \oplus \langle Y_{m}^0 \rangle} \right).$$

We may assume that $\bar{Y}_m^1 \supseteq Y_m^1$. So

$$f_{m+1}^1 \left( \langle \bar{Y}_m^1 \rangle \oplus \langle Y_{m+1}^1 \rangle \right) \supseteq \text{Ker} \left( f_{m+1}^0 \mid _{Y_m^0 \oplus \langle Y_{m+1}^0 \rangle} \right).$$

Note that

$$f_{m-1}^1 \left( \langle \bar{Y}_{m-1}^1 \rangle \right) \supseteq \text{Ker} \left( f_{m-1}^0 \mid _{B_{m-1}^{0,(0)} \oplus \langle Y_{m}^0 \rangle} \right).$$

Summarizing, we may choose small sets $Y_{m-1}^1 \subseteq B_{m-1}^1$, $Y_m^1 \subseteq B_m^1$, $Y_{m+1}^1 \subseteq B_{m+1}^1$, containing $B_{m-1}^{1,(0)}$, $B_m^{1,(0)}$ and $B_{m+1}^{1,(0)}$, respectively, such that

- $f_{m+1}^1 \left( \langle Y_m^1 \rangle \oplus \langle Y_{m+1}^1 \rangle \right) \supseteq \text{Ker} \left( f_{m+1}^0 \mid _{Y_m^0 \oplus \langle Y_{m+1}^0 \rangle} \right)$;
- $f_m^1 \left( \langle Y_{m-1}^1 \rangle \oplus \langle Y_m^1 \rangle \right) \supseteq \text{Ker} \left( f_m^0 \mid _{B_{m-1}^{0,(0)} \oplus \langle Y_{m}^0 \rangle} \right)$, and
- $f_{m-1}^1 \left( \langle Y_{m-1}^1 \rangle \right) \supseteq \text{Ker} \left( f_{m-1}^0 \mid _{B_{m-1}^{0,(0)} \oplus \langle Y_{m}^0 \rangle} \right)$.

Now choose a small set $Y_{m}^{0,(1)} \sqcup Y_{m+1}^{0,(1)} \subseteq B_{m}^{0} \sqcup B_{m+1}^{0}$ containing $Y_{m}^{0} \sqcup Y_{m+1}^{0}$ such that

$$f_{m+1}^1 \left( \langle Y_{m}^1 \rangle \oplus \langle Y_{m+1}^1 \rangle \right) \subseteq \langle Y_{m}^{0,(1)} \rangle \oplus \langle Y_{m+1}^{0,(1)} \rangle.$$

On the other hand, choose a small set $B_{m-1}^{0,(0)} \sqcup Y_m^{0,(1)} \subseteq \bar{Y}_{m-1}^{0,(1)} \sqcup \bar{Y}_m^{0,(1)} \subseteq B_{m-1}^{0} \sqcup B_{m}^{0}$ such that

$$f_m^1 \left( \langle Y_{m}^1 \rangle \oplus \langle Y_{m+1}^1 \rangle \right) \subseteq \langle \bar{Y}_{m-1}^{0,(1)} \rangle \oplus \langle \bar{Y}_m^{0,(1)} \rangle.$$

Note that $f_{m-1}^1 \left( Y_{m-1}^1 \right) \subseteq \langle \bar{Y}_{m-1}^{0,(1)} \rangle$ and that we may choose $\bar{Y}_m^{0,(1)}$ containing $Y_{m}^{0,(1)}$. Summarizing, there exist small sets $Y_{m-1}^{0,(1)} \subseteq B_{m-1}^{0}$, $Y_m^{0,(1)} \subseteq B_{m}^{0}$ and $Y_{m+1}^{0,(1)} \subseteq B_{m+1}^{0}$ containing $B_{m-1}^{0,(0)}$, $Y_{m}^{0}$ and $Y_{m+1}^{0}$, respectively, such that

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• $f_{m+1}^1 \left( \langle Y_1^m \rangle \oplus \langle Y_{m+1}^1 \rangle \right) \subseteq \langle Y_{m+1}^0 \rangle \oplus \langle Y_{m+1}^0 \rangle$,
• $f_m^1 \left( \langle Y_{m-1}^1 \rangle \oplus \langle Y_{m-1}^1 \rangle \right) \subseteq \langle Y_{m-1}^0 \rangle \oplus \langle Y_{m-1}^0 \rangle$, and
• $f_{m-1}^1 \left( \langle Y_{m-1}^1 \rangle \right) \subseteq \langle Y_{m-1}^0 \rangle$.

Now choose small sets $Y_{m-1}^{1(1)} \subseteq B_{m-1}^1$, $Y_{m}^{1(1)} \subseteq B_{m}^1$ and $Y_{m+1}^{1(1)} \subseteq B_{m+1}^1$ containing $Y_{m-1}^1$, $Y_{m}^1$ and $Y_{m+1}^1$, respectively, such that

• $f_{m+1}^1 \left( \langle Y_{m}^1 \rangle \oplus \langle Y_{m+1}^1 \rangle \right) \supseteq \text{Ker} \left( f_{m+1}^0 | \langle Y_{m+1}^0 \rangle \oplus \langle Y_{m+1}^0 \rangle \right)$, and
• $f_m^1 \left( \langle Y_{m-1}^1 \rangle \oplus \langle Y_{m}^1 \rangle \right) \supseteq \text{Ker} \left( f_m^0 | \langle Y_{m}^0 \rangle \oplus \langle Y_{m}^0 \rangle \right)$.

It is not necessarily true that $f_{m-1}^1 \left( Y_{m-1}^{1(1)} \right) \supseteq \text{Ker} \left( f_{m-1}^0 | \langle Y_{m-1}^0 \rangle \right)$. But using similar arguments as above, we can enlarge $Y_{m-1}^{1(1)}$ in such a way that the previous inclusion is satisfied.

Continue using the zig-zag procedure to enlarge $Y_{m-1}^{0(1)}$, $Y_{m}^{0(1)}$, $Y_{m+1}^{0(1)}$ and so on. Then set

• $B_{m-1}^{0(1)} = Y_{m-1}^{0(1)} \cup Y_{m-1}^{0(2)} \cup \cdots$,
• $B_{m-1}^{1(1)} = Y_{m-1}^{1(1)} \cup Y_{m-1}^{1(1)} \cup \cdots$,
• $B_m^{0(1)} = Y_{m}^{0(1)} \cup Y_{m}^{0(1)} \cup \cdots$,
• $B_m^{1(1)} = Y_{m}^{1(1)} \cup Y_{m}^{1(1)} \cup \cdots$,
• $B_{m+1}^{0(1)} = Y_{m+1}^{0(1)} \cup Y_{m+1}^{0(1)} \cup \cdots$,
• $B_{m+1}^{1(1)} = Y_{m+1}^{1(1)} \cup Y_{m+1}^{1(1)} \cup \cdots$.
Hence we have the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
\langle B^{1,(1)}_{m+1} \rangle \oplus \langle 0 \rangle & \rightarrow & \langle B^{0,(1)}_{m+1} \rangle \oplus \langle 0 \rangle \\
\downarrow \partial^1_{m+2} & & \downarrow \partial^0_{m+2} \\
\langle B^{1,(1)}_m \rangle \oplus \langle B^{1,(1)}_{m+1} \rangle & \rightarrow & \langle B^{0,(1)}_m \rangle \oplus \langle B^{0,(1)}_{m+1} \rangle \\
\downarrow \partial^1_{m+1} & & \downarrow \partial^0_{m+1} \\
\langle B^{1,(1)}_{m-1} \rangle \oplus \langle B^{1,(1)}_m \rangle & \rightarrow & \langle B^{0,(1)}_{m-1} \rangle \oplus \langle B^{0,(1)}_m \rangle \\
\downarrow \partial^1_{m} & & \downarrow \partial^0_{m} \\
\langle 0 \rangle \oplus \langle B^{1,(1)}_{m-1} \rangle & \rightarrow & \langle 0 \rangle \oplus \langle B^{0,(1)}_{m-1} \rangle \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\end{array}
\]

As in the previous iteration, we do not know if \(f^1_{m+2} \left( \langle B^{1,(1)}_{m+1} \rangle \oplus \langle 0 \rangle \right) \subseteq \langle B^{0,(1)}_{m+1} \rangle \oplus \langle 0 \rangle\).

Then repeat the same process and so on. In the \(i\)-th iteration, we get, for \(j = 0, 1\), small sets \(B^{j,k}_{m+k}\) such that:

- \(B^{j,(i)}_{m+i} \subseteq B^j_{m+i}\),
- \(B^{j,(i-1)}_{m+i-1} \subseteq B^{j,(i-1)}_{m+i-1} \subseteq B^j_{m+i-1}\),
- \(B^{j,(i-2)}_{m+i-2} \subseteq B^{j,(i-1)}_{m+i-2} \subseteq B^{j,(i)}_{m+i-2} \subseteq B^j_{m+i-2}\),
  \vdots
- \(B^{j,(0)}_{m} \subseteq \cdots \subseteq B^{j,(i-1)}_{m} \subseteq B^{j,(i)}_{m} \subseteq B^j_{m}\),
- \(B^{j,(0)}_{m-1} \subseteq \cdots \subseteq B^{j,(i-1)}_{m-1} \subseteq B^{j,(i)}_{m-1} \subseteq B^j_{m-1}\).
and the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
\langle B_1^{1,(i)} \rangle \oplus \langle 0 \rangle & \rightarrow & \langle B_0^{0,(i)} \rangle \oplus \langle 0 \rangle \\
\downarrow \partial_1^{m+i+1} & & \downarrow \partial_0^{m+i+1} \\
\langle B_0^{1,(i)} \rangle \oplus \langle B_1^{1,(i)} \rangle & \rightarrow & \langle B_0^{0,(i)} \rangle \oplus \langle B_1^{0,(i)} \rangle \\
\downarrow \partial_1^{m+i} & & \downarrow \partial_0^{m+i} \\
\vdots & & \vdots \\
\langle B_0^{1,(i)} \rangle \oplus \langle B_1^{1,(i)} \rangle & \rightarrow & \langle B_0^{0,(i)} \rangle \oplus \langle B_1^{0,(i)} \rangle \\
\downarrow \partial_1^{m+1} & & \downarrow \partial_0^{m+1} \\
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\end{array}
\]

Finally, for \( j = 0, 1 \), set

- \( B_{m-1}^{j} = B_{m-1}^{j,(0)} \cup B_{m-1}^{j,(1)} \cup \ldots \),
- \( B_{m}^{j} = B_{m}^{j,(0)} \cup B_{m}^{j,(1)} \cup \ldots \),
- \( B_{m+1}^{j} = B_{m+1}^{j,(1)} \cup B_{m+1}^{j,(2)} \cup \ldots \),
  \vdots
- \( B_{m+i}^{j} = B_{m+i}^{j,(i)} \cup B_{m+i}^{j,(i+1)} \cup \ldots \),
  \vdots
All of these sets are small. We have the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
0 & \longrightarrow & \langle B_{m+i}^1 \rangle \oplus \langle B_{m+i+1}^1 \rangle \\
\downarrow & & \downarrow f_{m+i+1}^1 \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \partial_{m+i+1}^1 \\
0 & \longrightarrow & \langle B_{m-i}^1 \rangle \\
\downarrow & & \downarrow f_m^1 \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \partial_m^1 \\
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \partial_{m-1}^1 \\
\cdots & \longrightarrow & \cdots
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & \langle B_{m+i}^0 \rangle \oplus \langle B_{m+i+1}^0 \rangle \\
\downarrow & & \downarrow f_{m+i+1}^0 \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \partial_{m+i+1}^0 \\
0 & \longrightarrow & \langle B_{m-i}^0 \rangle \\
\downarrow & & \downarrow f_m^0 \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \partial_m^0 \\
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \partial_{m-1}^0 \\
\cdots & \longrightarrow & \cdots
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & X_{m+i+1} \\
\downarrow & & \downarrow \partial_{m+i+1}^1 \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \partial_{m+i+1}^0 \\
0 & \longrightarrow & X_m \\
\downarrow & & \downarrow \partial_m^1 \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \partial_m^0 \\
0 & \longrightarrow & X_{m-1} \\
\downarrow & & \downarrow \partial_{m-1}^1 \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \partial_{m-1}^0 \\
0 & \longrightarrow & X_{m-2} \\
\end{array}
\]

We have obtained the following exact sequence in \( \text{Ch}(R) \):

\[
\bigoplus_{i \geq m} D^i \left( \langle B_{i-1}^1 \rangle \right) \xrightarrow{f_1^1 \bigoplus_{i \geq m} D^i \left( \langle B_{i-1}^1 \rangle \right)} \bigoplus_{i \geq m} D^i \left( \langle B_{i-1}^0 \rangle \right) \xrightarrow{f_0^0 \bigoplus_{i \geq m} D^i \left( \langle B_{i-1}^0 \rangle \right)} X'
\]

where \( X' = \text{CoKer} \left( f_1^1 \bigoplus_{i \geq m} D^i \left( \langle B_{i-1}^1 \rangle \right) \right) \). Note that \( x \in X' \), that \( X' \) is a subcomplex of \( X \), and that \( \bigoplus_{i \geq m} D^i \left( \langle B_{i-1}^1 \rangle \right) \) and \( \bigoplus_{i \geq m} D^i \left( \langle B_{i-1}^0 \rangle \right) \) are small subcomplexes of \( L^1 \) and \( L^0 \), respectively. Since \( f_0^0 \bigoplus_{i \geq m} D^i \left( \langle B_{i-1}^0 \rangle \right) \) is surjective, we have that \( X' \) is also small. The exact sequence above is a projective resolution of \( X' \) of length 1, hence \( X' \) is 1-projective.
Now consider the following commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{i \geq m} D^i \left( \langle B^1_{i-1} \rangle \right) & \xrightarrow{f^1} & \bigoplus_{i \geq m} D^i \left( \langle B^0_{i-1} \rangle \right) \\
\downarrow f^1 & & \downarrow f^0 \\
L^1 & \xrightarrow{f^1} & L^0 & \xrightarrow{f^0} & X' \xrightarrow{f_0} X
\end{array}
\]

Taking the quotient of the resolution of \(X\) by the resolution of \(X'\), we get a free resolution of \(X/X'\), and so \(X/X'\) is also 1-projective.

\[\square\]

Given a chain complex \(X \in \text{Ch}(R)\) and an ordinal number \(\lambda\), a **filtration of \(X\)** indexed by \(\lambda\) is a family \((X_\alpha)_{\alpha \leq \lambda}\) of subcomplexes of \(X\) such that

(a) \(X_\lambda = X\),

(b) \(X_0 = 0\),

(c) \(X_\alpha\) is a subcomplex of \(X_{\alpha'}\) whenever \(\alpha \leq \alpha'\), and

(d) \(X_\beta = \bigcup_{\alpha < \beta} X_\alpha\) whenever \(\beta\) is a limit ordinal.

By the **union of chain complexes** \(X_\beta = \bigcup_{\alpha < \beta} X_\alpha\) we mean the chain complex whose objects are given by

\[
X_{\beta,n} = \bigcup_{\alpha < \beta} X_{\alpha,n} = \{(\alpha, x) : \alpha < \beta \text{ and } x \in X_{\alpha,n}\},
\]

and whose morphisms are given by

\[
\partial_{n}^{X_\beta}: X_{\beta,n} \rightarrow X_{\beta,n-1}
\]

\[
(\alpha, x) \mapsto (\alpha, \partial_{n}^{X_\alpha}(x)).
\]
If $S$ is some class of complexes in $\text{Ch}(R)$, we shall say that a filtration $(X_\alpha)_{\alpha \leq \lambda}$ of $X$ is an $S$-filtration if for each $\alpha + 1 < \lambda$ we have that $X_0$ and $X_{\alpha+1}/X_\alpha$ are isomorphic to elements of $S$.

**Lemma 4.8** (Eklof). Let $X$ and $Y$ be chain complexes and let $(X_\alpha)_{\alpha \leq \lambda}$ be a $\perp \{Y\}$-filtration of $X$. Then $\text{Ext}^1(X,Y) = 0$.

The proof of the previous result is given in [Eklof] Theorem 7.3.4 in the category $R$-Mod carries over directly to the category $\text{Ch}(R)$. In the same way, the Eklof and Trlifaj Theorem also applies to $\text{Ch}(R)$.

**Theorem 4.2.** The cotorsion pair $\left(\widetilde{P}_n, \widetilde{P}_n^\perp\right)$ is complete.

*Proof.* Applying Lemma 4.7 we have that any complex $X \in \widetilde{P}_n$ can be written as a union $X = \bigcup_{\alpha < \lambda} X_\alpha$, where $(X_\alpha)_{\alpha \leq \lambda}$ is a $\widetilde{P}_n$-filtration, and $X_0$ and $X_{\alpha+1}/X_\alpha$ are small complexes whenever $\alpha + 1 < \lambda$. We construct such a filtration by using transfinite induction. Set $X_0 = 0$. Now choose any $x \neq 0$ in $X$ and let $X_1$ be the complex given by Lemma 4.7. We have that $X_0, X_1 \in \widetilde{P}_n$ and they are small. Also, $X/X_1 \in \widetilde{P}_n$. If $X_1 \subseteq X$ then choose $x' + X_1 \neq 0 + X_1$ in $X/X_1$. Using Lemma 4.7 again, we can construct a chain complex $X_2/X_1 \subseteq X/X_1$ such that $x' + X_1 \in X_2/X_1$, $X_2/X_1$ is small and $X_2/X_1 \in \widetilde{P}_n$. Now suppose that $\beta$ is an ordinal and that for any $\alpha < \beta$ one has constructed chain complexes $X_\alpha \subseteq X$ such that:

(a) $X_\alpha \subseteq X_{\alpha'}$ whenever $\alpha \leq \alpha'$,

(b) $X_{\alpha+1}/X_\alpha \in \widetilde{P}_n$ and $X_{\alpha+1}/X_\alpha$ is small whenever $\alpha + 1 < \beta$,

(c) $X_\gamma = \bigcup_{\alpha < \gamma} X_\alpha$ for every limit ordinal $\gamma < \beta$.

If $\beta$ is a limit ordinal, set $X_\beta = \bigcup_{\alpha < \beta} X_\alpha$. Otherwise there exists an ordinal $\alpha < \beta$ such that $\beta = \alpha + 1$. Then construct $X_{\alpha+1}$ from $X_\alpha$ as we just constructed $X_2$ from
By transfinite induction, we have a \( \tilde{\mathcal{P}}_n \)-filtration \( (X_\alpha : \alpha < \lambda) \) of \( X \) such that \( X_{\alpha+1}/X_\alpha \) is small whenever \( \alpha + 1 \leq \lambda \). Now consider the set

\[
\mathcal{S} = \left\{ L \in \tilde{\mathcal{P}}_n / L \text{ is small complex} \right\}.
\]

Note that \( \tilde{\mathcal{P}}_n^\perp \subseteq \mathcal{S} \subseteq \tilde{\mathcal{P}}_n \). Now let \( Y \in \mathcal{S} \) and \( X \in \tilde{\mathcal{P}}_n \). Write \( X = \bigcup_{\alpha < \lambda} X_\alpha \) with \( (X_\alpha)_{\alpha < \lambda} \) as above. Then \( X_0, X_{\alpha+1}/X_\alpha \in \mathcal{S} \). Hence \( \text{Ext}^1(X_0, Y) = 0 \) and \( \text{Ext}^1(X_{\alpha+1}/X_\alpha, Y) = 0 \), i.e. \( (X_\alpha)_{\alpha < \lambda} \) is a \( \perp \{ Y \} \)-filtration of \( X \). By the Eklof Lemma, we have \( \text{Ext}^1(X, Y) = 0 \) and so \( \mathcal{S}^\perp \subseteq \tilde{\mathcal{P}}_n^\perp \). We have \( \tilde{\mathcal{P}}_n^\perp = \mathcal{S}^\perp \). Therefore, by the Eklof and Trlifaj Theorem we conclude that \( (\tilde{\mathcal{P}}_n, \tilde{\mathcal{P}}_n^\perp) \) is a complete cotorsion pair.

\[ \blacksquare \]

We have obtained two complete cotorsion pairs

\[
(\tilde{\mathcal{P}}_n, \text{dg}\tilde{\mathcal{P}}_n^\perp) = (\text{dg}\tilde{\mathcal{P}}_n \cap \mathcal{E}, \text{dg}\tilde{\mathcal{P}}_n^\perp) \quad \text{and} \quad (\text{dg}\tilde{\mathcal{P}}_n, \text{dg}\tilde{\mathcal{P}}_n^\perp) = (\text{dg}\tilde{\mathcal{P}}_n, \text{dg}\tilde{\mathcal{P}}_n^\perp \cap \mathcal{E})
\]

in \( \text{Ch}(R) \), where the class of exact complexes \( \mathcal{E} \) is thick. By Theorem 4.1 we have:

**Corollary 4.1.** There exists a unique abelian model structure on \( \text{Ch}(R) \) such that:

(a) \( \text{dg}\tilde{\mathcal{P}}_n \) is the class of cofibrant objects,

(b) \( \text{dg}\tilde{\mathcal{P}}_n^\perp \) is the class of fibrant objects,

(c) \( \mathcal{E} \) is the class of trivial objects.

We shall call this this model structure the \( n \)-**Projective Model Structure on** \( \text{Ch}(R) \).
5 Some properties of the \( n \)-projective model structures

From the definition of an abelian model structure, we have that the \( n \)-projective model structure has the following classes of morphisms:

1. A chain map \( f : X \rightarrow Y \) is a (trivial) cofibration if and only if it is a monomorphism and \( \text{CoKer}(f) \in \text{dg}(\mathcal{P}_n) \) (resp. \( \text{CoKer}(f) \in \text{dg}(\mathcal{P}_n) \cap \mathcal{E} \)).

2. A chain map \( f : X \rightarrow Y \) is a (trivial) fibration if and only if it is an epimorphism and \( \text{Ker}(f) \in \text{dg}(\mathcal{P}_n^\perp) \) (resp. \( \text{Ker}(f) \in \text{dg}(\mathcal{P}_n^\perp) \cap \mathcal{E} \)).

In [10], Hovey proved that the set of weak equivalences in an abelian model structure are given by the composition of trivial cofibrations followed by trivial fibrations. We show that weak equivalences in the \( n \)-projective model structures are given by the class of quasi-isomorphisms. Let \( f : X \rightarrow Y \) be a chain map such that \( f = p \circ i \), where \( p \) is a trivial fibration and \( i \) is a trivial cofibration. Consider the chain map \( i \), we have a short exact sequence

\[
0 \rightarrow X \xrightarrow{i} W \rightarrow \text{CoKer}(i) \rightarrow 0
\]

where \( C \in \text{dg}(\mathcal{P}_n) \cap \mathcal{E} \). It follows there is the following long exact homology sequence

\[
\cdots \rightarrow H_{n+1}(\text{CoKer}(i)) \rightarrow H_n(X) \xrightarrow{H_n(i)} H_n(W) \rightarrow H_n(\text{CoKer}(i)) \rightarrow \cdots,
\]

where \( H_n(\text{CoKer}(i)) = 0 \) for every \( n \in \mathbb{Z} \) since \( \text{CoKer}(i) \) is exact. Then \( i \) is a quasi-isomorphism. Similarly, \( p \) is also a quasi-isomorphism, and then so is \( f = p \circ i \). Now suppose that \( f : X \rightarrow Y \) is a quasi-isomorphism. We can write \( f = p \circ i \), where \( p \) is a fibration and \( i \) is a trivial cofibration. We show that \( p \) is a trivial fibration, i.e. that \( \text{Ker}(p) \in \mathcal{E} \). Since \( f \) and \( i \) are quasi-isomorphism, then so is \( p \). From the short exact sequence
\[ 0 \to \text{Ker}(p) \to W \xrightarrow{p} Y \to 0, \]

we obtain the following long exact homology sequence

\[ \cdots \to H_{n+1}(Y) \to H_n(\text{Ker}(p)) \to H_n(W) \xrightarrow{H_n(p)} H_n(Y) \to \cdots. \]

Since \( H_n(p) \) is an isomorphism and the previous sequence is exact, we have \( H_n(\text{Ker}(p)) = 0 \) for every \( n \in \mathbb{Z} \). Hence \( \text{Ker}(p) \in \mathcal{E} \). Therefore, we have proven:

**Proposition 5.1.** The class of weak equivalences in the \( n \)-projective model structures is the class of quasi-isomorphisms.

Now we show that the \( n \)-projective model structure is cofibrantly generated. An abelian model structure is **cofibrantly generated** when there is a set \( I \) of cofibrations and a set \( J \) of trivial cofibrations such that:

1. A map \( f \) is a trivial fibration if and only if it has the right lifting property with respect to \( I \).
2. A map \( f \) is a fibration if and only if it has the right lifting property with respect to \( J \).

Recall that the pair \( (\text{dg}^n \cap \mathcal{E}, \text{dg}^n) \) is cogenerated by the set of small \( n \)-projective complexes

\[ \mathcal{S}_1 = \{ X \in \text{Ch}(R) / X \in \mathcal{P}_n \text{ and Card}(X) \leq \kappa \}, \]

where \( \kappa \geq \text{Card}(R) \). By Proposition 4.3 the pair \( (\text{dg}^n, \text{dg}^n \cap \mathcal{E}) \) is cogenerated by the set

\[ \mathcal{S}_2 = \{ S^n(R) / n \in \mathbb{Z} \} \cup \{ S^n(A) / n \in \mathbb{Z} \text{ and } A \in \mathcal{P}_n \text{ with Card}(A) \leq \kappa \}. \]
So the $n$-projective model structure is cofibrantly generated due to the following result:

**Proposition 5.2.** [10, Lemma 6.7] Given two compatible cotorsion pairs in $\mathcal{G}$ is a Grothendieck Category with enough projectives, each cogenerated by a set, the corresponding abelian model structure is cofibrantly generated.

Now we find the sets $I$ and $J$ of cofibrations and trivial cofibrations satisfying the conditions (1) and (2) above. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in a Grothendieck Category $\mathcal{G}$ such that $\mathcal{A}$ contains a generator $G$ for $\mathcal{G}$. The pair $(\mathcal{A}, \mathcal{B})$ is said to be **small** if there is a set $S$ which cogenerated $(\mathcal{A}, \mathcal{B})$ and for each $S \in S$ there is a monomorphism $i_S$, with $\text{CoKer}(i_S) = S$, satisfying the following condition: for all $X \in \text{Ob}(\mathcal{G})$, if $\text{Hom}_\mathcal{G}(i_S, X)$ is surjective for all $S \in S$, then $X \in \mathcal{B}$. The set of all morphisms $i_S$ satisfying the previous condition is called a set of **generating monomorphisms** for $(\mathcal{A}, \mathcal{B})$. For the reader’s convenience, we state and prove the following result.

**Proposition 5.3.** [10, Corollary 6.8] If $\mathcal{G}$ is a Grothendieck category with enough projectives, then every cotorsion pair cogenerated by a set is small.

**Proof.** If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $\mathcal{G}$ cogenerated by a set $S$, then for every $S \in S$ we choose an epimorphism $P_S \rightarrow S$, where $P_S$ is projective, with kernel $i_S : K_S \rightarrow P_S$. Let $G$ be a generator for $\mathcal{G}$. Then $I = \{0 \rightarrow G\} \cup \{i_S : K_S \rightarrow P_S / S \in S\}$ is a set of generating monomorphisms for $(\mathcal{A}, \mathcal{B})$.

Recall that $\text{Ch}(R)$ is a Grothendieck category with generator $D^n(R)$, for any $n \in \mathbb{Z}$. Then we have that the cotorsion pair $\left(\text{dg}P_n \cap \mathcal{E}, \text{dg}P_n^\perp\right)$ is small. Since this pair is generated by $S_1$, for every $S \in S_1$ an epimorphism $P_S \rightarrow S$. Then we consider the cokernel $i_S : K_S \rightarrow P_S$ of the previous epimorphism. The generating set of
monomorphisms is given by

\[ J = \{ 0 \rightarrow D^n(R) \} \cup \{ K_{S} \xrightarrow{i_S} P_S / S \in S_1 \}. \]

Applying the same result in the Grothendieck category \( \text{R-Mod} \) with generator \( \text{R} \), we have that \( (\mathcal{P}_n, \mathcal{P}^1_n) \) is a small cotorsion pair with generating monomorphisms the set

\[ \{ k_S : Y_S \rightarrow Z_S / S \in \mathcal{P}_n \text{ and Card}(S) \leq \kappa \}, \]

where each \( k_S \) appears in an exact sequence

\[ 0 \rightarrow Y_S \xrightarrow{k_S} Z_S \rightarrow S \rightarrow 0. \]

It follows the cotorsion pair \( (\text{dgP}_n, \text{dgP}^1_n \cap \mathcal{E}) \) is small, with generating monomorphisms the set

\[ I = \{ 0 \rightarrow D^n(R) \} \cup \{ S^{n-1}(R) \rightarrow D^n(R) \} \cup \{ S^n(Y_S) \xrightarrow{S^n(k_S)} S^n(Z_S) / S \in \mathcal{P}_n \text{ and Card}(S) \leq \kappa \}. \]

From the proof of Proposition 5.2, we have that \( I \) and \( J \) are the sets of cofibrations and trivial cofibrations satisfying the conditions (1) and (2) above.

Another interesting question about a new model structure is whether it is monoidal or not. It is known that the projective model structure is monoidal with respect to the usual tensor product on \( \text{Ch}(\text{R}) \) (see [9]). We shall consider two tensor products on \( \text{Ch}(\text{R}) \) and prove that the \( n \)-projective model structure is not monoidal with respect to these two tensor products whenever \( n > 0 \). First, we recall the notion of a monoidal model structure. Given two model categories \( \mathcal{C} \) and \( \mathcal{D} \), a functor \( F : \mathcal{C} \rightarrow \mathcal{D} \) is said to be a left Quillen functor if \( F \) is a left adjoint and preserves cofibrations and trivial cofibrations. We call a functor \( G : \mathcal{D} \rightarrow \mathcal{C} \) a right Quillen functor if \( G \) is a right adjoint and preserves fibrations and trivial fibrations. An adjunction \( (F, G, \varphi) \) from \( \mathcal{C} \) to \( \mathcal{D} \) is called a Quillen adjunction if \( F \) is a Quillen left functor. Now given
three categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, an **adjunction of two variables** from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ is a quintuple $(\otimes, \text{Hom}_r, \text{Hom}_l, \varphi_r, \varphi_l)$, where $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, $\text{Hom}_r : \mathcal{D}^{\text{op}} \times \mathcal{E} \to \mathcal{C}$, and $\text{Hom}_l : \mathcal{C}^{\text{op}} \times \mathcal{E} \to \mathcal{D}$, are functors, $\varphi_r$ and $\varphi_l$ are natural isomorphisms

\[
\begin{array}{ccc}
\text{Hom}_C(C, \text{Hom}_r(D, E)) & \xrightarrow{\varphi_l} & \text{Hom}_E(C \otimes D, E) \\
\downarrow{\cong} \quad & & \downarrow{\cong} \\
\text{Hom}_D(D, \text{Hom}_l(C, E)) & & 
\end{array}
\]

Given a map $f : U \to V$ in $\mathcal{C}$ and a map $g : W \to X$ in $\mathcal{D}$, we have two maps

\[
f \otimes \text{id}_W : U \otimes W \to V \otimes W \quad \text{and} \quad \text{id}_U \otimes g : U \otimes W \to U \otimes X
\]
in $\mathcal{E}$. If $\mathcal{E}$ has pushouts, we consider the pushout of these two maps. Now consider the maps

\[
\text{id}_V \otimes g : V \otimes W \to V \otimes X \quad \text{and} \quad f \otimes \text{id}_X : U \otimes X \to V \otimes X.
\]

Since $\otimes$ is a functor of two variables, $(\text{id}_V \otimes g) \circ (f \otimes \text{id}_W) = (f \otimes \text{id}_X) \circ (\text{id}_U \otimes g)$. By the universal property of pushouts, there exists a unique map

\[
f \Box g : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \to V \otimes X
\]
such that the following diagram commutes:

\[
\begin{array}{cccccc}
U \otimes W & \xrightarrow{f \otimes \text{id}_W} & V \otimes W & \xrightarrow{\text{id}_V \otimes g} & (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \\
\downarrow{\text{id}_U \otimes g} & & \downarrow{f \Box g} & & \\
U \otimes X & \xrightarrow{(V \otimes W) \coprod_{U \otimes W} (U \otimes X)} & V \otimes X
\end{array}
\]
If in addition $C$, $D$ and $E$ are model categories, an adjunction of two variables

$$(\otimes, \Hom_r, \Hom_l, \varphi_r, \varphi_l) : C \times D \to E$$

is called a Quillen adjunction of two variables if given a cofibration $f : U \to V$ in $C$ and a cofibration $g : W \to X$ in $D$, the induced map $f \Box g$ is a cofibration in $E$ which is trivial if either $f$ or $g$ is.

A closed monoidal category $C$ is a category equipped with a monoidal structure $(C, \otimes, S)$ such that there is an adjunction of two variables $(\otimes, \Hom_r, \Hom_l, \varphi_r, \varphi_l) : C \times C \to C$ satisfying the following condition: Let $q : QS \to S$ be a cofibrant replacement for the unit $S$, obtained by using functorial factorizations to factor $0 \to S$ into a cofibration followed by a trivial fibration. Then the natural map

$$QS \otimes X \xrightarrow{q \otimes \text{id}_X} S \otimes X$$

is a weak equivalence for all cofibrant $X$. Similarly, the natural map

$$X \otimes QS \xrightarrow{\text{id}_X \otimes q} X \otimes S$$

is a weak equivalence for all cofibrant $X$.

On Ch($R$) there are two closed monoidal structures given by two tensor products $\otimes$ and $\boxtimes$. By $\otimes$ we shall mean the usual tensor product on Ch($R$), and by $\boxtimes$ the one introduced by Enochs and García Rozas (see [5]). We shall see that the $n$-projective model structure is not necessarily closed monoidal with respect either to $\otimes$ or $\boxtimes$, whenever $n \geq 1$. Recall that given two chain complexes $X$ and $Y$, the usual tensor product of $X$ and $Y$ is the chain complex $X \otimes Y$ given by $(X \otimes Y)_n = \bigoplus_{k \in \mathbb{Z}} X_k \otimes Y_{n-k}$, whose boundary maps $\partial_n^{X \otimes Y} : (X \otimes Y)_n \to (X \otimes Y)_{n-1}$ are defined as
Consider the case \( R = \mathbb{Z} \). Note that \( \mathbb{Z}_2 \) is a \( \mathbb{Z} \)-module in \( \mathcal{P}_1 \), since it is not projective and there exists a short exact sequence

\[ 0 \rightarrow \mathbb{Z} \xrightarrow{2x} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \rightarrow 0 \]

where \( 2x \) is the map \( x \mapsto 2 \times x \) and \( \pi \) is the canonical projection \( x \mapsto \pi \in \{0, 1\} \).

Now let \( X \) be the complex given by the previous sequence, where \( X_1 = \mathbb{Z}, \ X_0 = \mathbb{Z} \) and \( X_{-1} = \mathbb{Z}_2 \). We have \( X \in \mathcal{P}_1 \subseteq \mathcal{P}_n \). Consider also the complex \( S^0(\mathbb{Z}_2) \). We check \( S^0(\mathbb{Z}_2) \in \text{dgP}_n \). Let \( Y \in \mathcal{P}_n \), we have

\[ \text{Ext}^1(S^0(\mathbb{Z}_2), Y) \cong \text{Ext}^1(\mathbb{Z}_2, Z_0 Y) = 0 \] (see [8, Lemma 3.1]).

Then \( S^0(\mathbb{Z}_2) \in \text{dgP}_n \). Now we compute \( S^0(\mathbb{Z}_2) \otimes X \):

\[ (S^0(\mathbb{Z}_2) \otimes X)_m = \bigoplus_{k \in \mathbb{Z}} S^0(\mathbb{Z}_2)_k \otimes X_{m-k} = \mathbb{Z}_2 \otimes X_m = \begin{cases} 
\mathbb{Z}_2 \otimes \mathbb{Z} & \text{if } m = 1, \\
\mathbb{Z}_2 \otimes \mathbb{Z} & \text{if } m = 0, \\
\mathbb{Z}_2 \otimes \mathbb{Z}_2 & \text{if } m = -1, \\
0 & \text{otherwise.}
\end{cases} \]

It is not hard to see that \( \partial_1^{S^0(\mathbb{Z}_2) \otimes X} \) is the zero map, so the sequence

\[ \cdots \rightarrow \mathbb{Z}_2 \otimes \mathbb{Z} \xrightarrow{\partial_1^{S^0(\mathbb{Z}_2) \otimes X}} \mathbb{Z}_2 \otimes \mathbb{Z} \xrightarrow{\partial_0^{S^0(\mathbb{Z}_2) \otimes X}} \mathbb{Z}_2 \otimes \mathbb{Z}_2 \rightarrow 0 \]

is not exact. We have that \( 0 \rightarrow S^0(\mathbb{Z}_2) \) is a cofibration, \( 0 \rightarrow X \) is a trivial cofibration, but \( (0 \rightarrow S^0(\mathbb{Z}_2)) \square (0 \rightarrow X) = 0 \rightarrow S^0(\mathbb{Z}_2) \otimes X \) is cofibration but not a weak equivalence. Therefore, the \( n \)-projective model structure on \( \text{Ch}(\mathbb{Z}) \) is not monoidal with respect to the tensor product \( \otimes \).
To conclude this section we show that the $n$-projective model structure, with $n \geq 1$, is not monoidal with respect to $\boxtimes$. Given two chain complexes $X$ and $Y$, $X \boxtimes Y$ is the chain complex given by $(X \boxtimes Y)_n = (X \otimes Y)_n/B_n(X \otimes Y)$ whose boundary maps are defined as

$$\partial_n : (X \boxtimes Y)_n \rightarrow (X \boxtimes Y)_{n-1}$$

$$x \otimes y \mapsto \partial_k(x) \otimes y$$

whenever $x \in X_k$ and $y \in Y_{n-k}$.

Consider the complex $X$ of the previous counterexample. We have

$$(S^0(\mathbb{Z}_2) \boxtimes X)_m = \begin{cases} (S^0(\mathbb{Z}_2) \otimes X)_1/B_1(S^0(\mathbb{Z}_2) \otimes X) & \text{if } m = 1, \\ (S^0(\mathbb{Z}_2) \otimes X)_0/B_0(S^0(\mathbb{Z}_2) \otimes X) & \text{if } m = 0, \\ (S^0(\mathbb{Z}_2) \otimes X)_{-1}/B_{-1}(S^0(\mathbb{Z}_2) \otimes X) & \text{if } m = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $S^0(\mathbb{Z}_2) \boxtimes X$ is the complex

$$\cdots \rightarrow \mathbb{Z}_2 \otimes \mathbb{Z} \xrightarrow{\partial^0_{S^0(\mathbb{Z}_2) \boxtimes X}} \mathbb{Z}_2 \otimes \mathbb{Z} \rightarrow 0 \rightarrow \cdots$$

where $\partial^0_{S^0(\mathbb{Z}_2) \boxtimes X}$ is the zero map. Hence $S^0(\mathbb{Z}_2) \boxtimes X$ is not an exact complex, and so the induced map $(0 \rightarrow S^0(\mathbb{Z}_2)) \boxtimes (0 \rightarrow X) = 0 \rightarrow S^0(\mathbb{Z}_2) \boxtimes X$ is not a trivial cofibration.
6  $n$-projective model structures on chain complexes over Artinian rings

We conclude this paper commenting the particular case when $R$ is a two-sided artinian ring. In this case, there is another way to construct the $n$-projective model structure. This way is based on some results from homological algebra and a theorem proved by Gillespie in [7]. Before stating that theorem, we need to recall some definitions [7, Definitions 4.9, A.1 and A.6].

An infinite cardinal $\kappa$ is regular if it is not the sum of a smaller number of smaller cardinals. Let $\mathcal{C}$ be a category and let $\kappa$ be a regular cardinal.

- A $\kappa$-filtered category is a category $\mathcal{K}$, for which every subcategory with less than $\kappa$ morphisms has a cocone. A $\kappa$-filtered diagram is simply a functor $F : \mathcal{K} \to \mathcal{C}$ in which $\mathcal{K}$ is a small $\kappa$-filtered category. By a $\kappa$-filtered colimit, we mean the colimit of a $\kappa$-filtered diagram.

- An object $X$ in $\mathcal{C}$ is called $\kappa$-presentable if $\text{Hom}_\mathcal{C}(X, -)$ preserves $\kappa$-filtered colimits.

- An object $X$ in $\mathcal{C}$ is called $\kappa$-generated if $\text{Hom}_\mathcal{C}(X, -)$ preserves $\kappa$-filtered colimits of monomorphisms, i.e. preserves the colimits of diagrams $F : \mathcal{K} \to \mathcal{C}$ for which $F(d) : F(c) \to F(c')$ is a monomorphism for each $d : c \to c'$ in $\mathcal{K}$.

- Given a class of objects $\mathcal{F}$ in an abelian category $\mathcal{C}$. We say $\mathcal{F}$ is a $\kappa$-Kaplansky class if the following property holds: Given $X \subseteq F \neq 0$ where $F \in \mathcal{F}$ and $X$ is a $\kappa$-generated, there exists a $\kappa$-presentable object $S \neq 0$ such that $X \subseteq S \subseteq F$ and $S, F/S \in \mathcal{F}$.

- Any cocomplete category $\mathcal{C}$ is called locally $\kappa$-presentable if each object is a $\kappa$-filtered colimit of $\kappa$-presentable objects and the class of $\kappa$-presentable objects
is essentially small, meaning there is only a set worth of $\kappa$-presentable objects up to isomorphism.

**Theorem 6.1** (Gillespie). Let $\mathcal{G}$ be a locally $\kappa$-presentable Grothendieck category. Suppose $\mathcal{F}$ is a class of objects which satisfies the following:

1. $\mathcal{F}$ is a $\kappa$-Kaplansky class.
2. $\mathcal{F}$ contains a $\kappa$-presentable generator $G$ for $\mathcal{G}$.
3. $\mathcal{F}$ is closed under direct limits, extensions and retracts.
4. $\text{dg} \mathcal{F} \cap \mathcal{E} = \mathcal{F}$.

Then we have an induced model category structure on $\text{Ch}(\mathcal{G})$ where the weak equivalences are the homology isomorphisms. The cofibrations (resp. trivial cofibrations) are the monomorphisms whose cokernels are in $\text{dg} \mathcal{F}$ (resp. $\mathcal{F}$). The fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are in $\text{dg} \mathcal{F}^\perp$ (resp. $\mathcal{F}^\perp$).

At this point, the question is: does the class $\mathcal{P}_n$ satisfy the four conditions above? The class $\mathcal{P}_n$ is a $\kappa$-Kaplansky class (see [1]). The ring $R$, considered as a left module over itself, is a $\kappa$-presentable generator for $R$-$\text{Mod}$, and also $R \in \mathcal{P}_n$. We already know the fourth condition is true for $\mathcal{P}_n$. This class is also closed under extensions since it is the left class of a cotorsion pair. It is straightforward to check that $\mathcal{P}_n$ is closed under retracts. However, it is not true in general that $\mathcal{P}_n$ is closed under direct limits. This condition seems to be very related to the ring $R$. H. Krause proved in [11, Lemma 5] that if $R$ is a two-sided artinian ring, then $\mathcal{P}_n$ is closed under direct limits. Hence, in this case we can use the previous theorem and get a model structure which coincides with the $n$-projective model structure.
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