On a Raychaudhuri equation for hot gravitating fluids

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Abstract. We generalize the Raychaudhuri equation for the evolution of a self gravitating fluid to include an Abelian and non-Abelian hybrid magnetofluid at a finite temperature. The aim is to utilize this equation for investigating the dynamics of astrophysical high temperature Abelian and non-Abelian plasmas.

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1. Introduction

It is perhaps one of the finest tributes to the simplicity and elegance of Einstein’s theory of Gravitation and to the human spirit, that a college teacher sitting in Kolkata was able to conceive the equation for the evolution of a gravitating fluid now known as Raychaudhuri’s equation [1]. The equation served as a lemma for the Penrose-Hawking singularity theorems and for the study of exact solutions in general relativity [2,3]. It provides a simple validation of our expectation that gravitation should be a universal attractive force between any two particles in general relativity [4]. This equation has stood the test of time and has been generalized in many ways. It has found applications in modern theories of strings and membranes [5,6]. We attempt a more modest generalization to tackle the statistical properties of a hot astrophysical plasma. These may be electromagnetic or chromomagnetic (quark gluon) plasmas. We are guided here by a recent formalism that has been used to investigate the dynamics of a hot charged fluid in terms of a
hybrid magneto-fluid and the changes brought about to the Raychaudhuri equation by the introduction of a statistical attributes associated with finite temperature [7] are many and interesting. We give also give a brief outline of how to generalize this to the evolution of gravitating non-Abelian plasmas in the early universe [8].

2. Raychaudhuri’s equation for charged gravitating fluids.

In this section, for the sake of completeness, we first review the derivation of the standard Raychaudhuri equation for the evolution of a self gravitating fluid. We follow the derivation from Raychaudhuri’s original work for geodesic flow [9]. We then incorporate the Lorentz force into the Raychaudhuri equation for non geodesic flow in the presence of an electromagnetic field.

Consider a self gravitating fluid in which the fluid particles have a velocity \( U^\mu \). In a spacetime with metric \( g_{\mu\nu} \), these velocity vectors allow us to define an orthogonal projection vector

\[
h_{\mu\nu} = g_{\mu\nu} + U^\mu U_\nu.
\]

The following properties hold for \( h_{\mu\nu} \):

1. \( h^\mu_\mu = g^\mu_\mu + U^\mu U_\mu = 3 \), assuming \( U^\mu U_\mu = -1 \). This implies that \( h_{\mu\nu} \) is three dimensional.
2. \( U^\mu h_{\mu\nu} = U^\mu g_{\mu\nu} + U^\mu U_\mu U_\nu = 0 \) \( (2) \)

and

\[
h_{\mu\nu} U^\nu = g_{\mu\nu} U^\nu + U^\mu U_\nu U^\nu = U_\mu - U_\mu = 0
\]

This implies \( U^\mu \) is orthogonal to \( h_{\mu\nu} \). Since \( U^\mu \) is tangent to timelike curves, \( h_{\mu\nu} \) is purely spacelike. It is the metric of the three dimensional hypersurface orthogonal to \( U^\mu \).

For any time-like vector \( U^\mu \) in a spacetime, the geodesic deviation equation can be written as

\[
U^{\mu}_{;\alpha\beta} - U^{\mu}_{;\beta\alpha} = R^{\mu}_{\alpha\beta\gamma} U^\gamma
\]

Although Eqn.4 is the geodesic deviation equation, it holds for any timelike vector \( U^\mu \) and not just the tangent vector to the geodesic. This is an important observation since Electromagnetic and Yang-Mills fields cause the fluid motion to be non-geodesic. Let us consider the parallel transport of \( U^\alpha \), \( B_{\alpha\beta} = U_{\alpha;\beta} \). It has the following properties:

1. \( U^\alpha B_{\alpha\beta} = U^\alpha U_{\alpha;\beta} = \frac{1}{2} (U^\alpha U_{\alpha})_{;\beta} = 0 \).
2. \( B_{\alpha\beta} U^\beta = U_{\alpha;\beta} U^\beta \neq 0 \) unless geodesic.
Thus, $B_{\alpha\beta}$ is a purely spatial tensor for geodesic motion, but fails to be fully spatial for non geodesic motion. In particular, with an EM field present

$$U_{\alpha;\beta}U^\beta = \frac{q}{m}F_{\alpha\beta}U^\beta$$

(5)

Following Raychaudhuri, let us decompose $B_{\alpha\beta}$ into its irreducible parts:

- i. The antisymmetric part : $\Omega_{\alpha\beta}$,
- ii. The symmetric, trace free part : $\Sigma_{\alpha\beta}$,
- iii. The trace : $\frac{1}{3}\Theta h_{\alpha\beta}$.

We can write

$$B_{\alpha\beta} = -U_{\alpha;\gamma}U^\gamma U_\beta + \Sigma_{\alpha\beta} + \Omega_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta},$$

(6)

with $\Theta$, giving the rate of dilation of a three space element locally orthogonal to $U^\alpha$, emerging as the trace

$$B^\alpha_{\quad \alpha} = U^\alpha_{;\alpha} = \Theta.$$  

(7)

We can define

$$\Omega_{\alpha\beta} = \frac{1}{2}(U_{\alpha;\beta} - U_{\beta;\alpha}) + \frac{1}{2}(\dot{U}_\alpha U^\beta - \dot{U}_\beta U^\alpha),$$

(8)

where, $\dot{U}_\alpha = U_{\alpha;\beta}U^\beta$ is the acceleration vector, which gives the departure of the velocity field from geodesity. Similarly

$$\Sigma_{\alpha\beta} = \frac{1}{2}(U_{\alpha;\beta} + U_{\beta;\alpha}) + \frac{1}{2}(\dot{U}_\alpha U^\beta + \dot{U}_\beta U^\alpha) - \frac{1}{3}\Theta h_{\alpha\beta}.$$  

(9)

Consider

$$U^\alpha \Omega_{\alpha\beta} = \frac{1}{2}(U^\alpha U_{\alpha;\beta} - U^\alpha U_{\beta;\alpha}) + \frac{1}{2}(U^\alpha \dot{U}_\beta - U^\beta \dot{U}_\alpha)$$

$$= -\frac{1}{2}(U_{\alpha;\beta}U^\alpha + \frac{1}{2}(U_{\beta;\alpha}) + \frac{1}{2}(U^\alpha U_{\alpha;\gamma} U^\gamma)U^\beta + \frac{1}{2}U_{\beta;\alpha} U^\alpha = 0,$$  

(10)

and

$$\Omega_{\alpha\beta} U^\beta = \frac{1}{2}(U_{\alpha;\beta} U^\beta - U_{\beta;\alpha} U^\beta) + \frac{1}{2}(\dot{U}_\alpha U^\beta - \dot{U}_\beta U^\alpha) = 0.$$  

(11)

These relations imply that $\Omega_{\alpha\beta}$ is spatially antisymmetric. Similarly,

$$U^\alpha \Sigma_{\alpha\beta} = \frac{1}{2}(U^\alpha U_{\alpha;\beta} + U^\alpha U_{\beta;\alpha}) + \frac{1}{2}(U^\alpha \dot{U}_\beta + U^\beta \dot{U}_\alpha) - \frac{1}{3}\Theta U^\alpha h_{\alpha\beta}$$

$$= \frac{1}{2}(\dot{U}^\beta - \dot{U}_\beta) = 0.$$  

(12)
\[
\Sigma_{\alpha\beta} U^\beta = \frac{1}{2} (U_{\alpha;\beta} U^\beta + U_{\beta,\alpha} U^\beta) + \frac{1}{2} (\dot{U}_{\alpha} U^\beta U^\beta + \dot{U}_{\beta} U^\beta U^\beta) - \frac{1}{3} \Theta h_{\alpha\beta} U^\beta
\]
\[
= \frac{1}{2} (\dot{U}^\alpha - U^\alpha) = 0. \quad (13)
\]
So we have taken care of the non geodesic nature of \( U^\alpha \) to define \( \Sigma_{\alpha\beta} \) and \( \Omega_{\alpha\beta} \), by including the acceleration due to the EM force and made them spatial.

Let us now consider a parallely transported \( B_{\alpha\beta;\gamma} \), \( B_{\alpha\beta;\gamma} = U_{\alpha;\beta\gamma} \). Eq. 4 allows us to write
\[
B_{\alpha\beta;\gamma} = (U_{\alpha;\gamma\beta} - R_{\alpha\mu\gamma\beta} U^\mu) \quad (14)
\]
Thus,
\[
B_{\alpha\beta;\gamma} U^\gamma = (U_{\alpha;\gamma\beta} U^\gamma - R_{\alpha\mu;\gamma\beta} U^\mu U^\gamma)
\]
\[
= (U_{\alpha;\gamma\beta} U^\gamma;\beta - U_{\alpha;\gamma} U^{\gamma;\beta}) - R_{\alpha\mu;\gamma\beta} U^\mu U^\gamma
\]
\[
= (\dot{U}_{\alpha})_{\beta} - B_{\alpha\gamma} B^\gamma_{\beta} - R_{\alpha\mu;\gamma\beta} U^\mu U^\gamma. \quad (15)
\]
Taking the trace over \( \alpha \) and \( \beta \) we get
\[
B^{\alpha}_{\alpha;\gamma} U^\gamma = (\dot{U}^\alpha)_{\alpha} - B_{\alpha\gamma} B^{\gamma\alpha} - R_{\alpha\mu;\gamma\alpha} U^\mu U^\gamma. \quad (16)
\]
But
\[
B^{\alpha}_{\alpha} = \Theta \quad (17)
\]
since \( \Sigma_{\alpha\beta} \) is traceless and \( \Omega_{\alpha\beta} \) is antisymmetric. Therefore Eq.16 becomes
\[
\Theta_{\gamma} U^\gamma = (\dot{U}^\alpha)_{\alpha} - B_{\alpha\gamma} B^{\gamma\alpha} - R_{\mu;\gamma\alpha} U^\mu U^\gamma \quad (18)
\]
where \( R_{\mu;\gamma} \) is the Ricci tensor. Using the Lorentz force law equation \( \dot{U}^\alpha = \frac{q}{m} F^{\alpha}_{\gamma} U^\gamma \), we have
\[
\dot{\Theta} = (\frac{q}{m} F^{\alpha}_{\gamma} U^\gamma)_{\alpha} - B_{\alpha\gamma} B^{\gamma\alpha} - R_{\mu;\gamma\alpha} U^\mu U^\nu
\]
\[
= \frac{q}{m} F^{\alpha}_{\gamma;\alpha} U^\gamma + \frac{q}{m} F^{\alpha}_{\gamma} B^{\gamma\alpha} + B_{\alpha\gamma} B^{\gamma\alpha} - R_{\mu;\gamma\alpha} U^\mu U^\nu
\]
\[
= \frac{q}{m} F^{\alpha}_{\gamma;\alpha} U^\gamma + \frac{q}{m} F^{\alpha}_{\gamma} \Omega^{\gamma\alpha} - B_{\alpha\gamma} B^{\gamma\alpha} - R_{\mu;\gamma\alpha} U^\mu U^\nu \quad (19)
\]
Where we have the antisymmetric nature of \( F_{\alpha\beta} \), so that \( F_{\alpha\beta} B^{\beta\alpha} = F_{\alpha\beta} \Omega^{\alpha\beta} \). We also have
\[
B_{\alpha\gamma} B^{\gamma\alpha} = \Sigma_{\alpha\gamma} \Sigma^{\gamma\alpha} + \Omega_{\alpha\gamma} \Omega^{\gamma\alpha} + \frac{1}{9} \Theta^2 h_{\alpha\gamma} h^{\gamma\alpha} \quad (20)
\]
Thus
\[
\dot{\Theta} = \frac{q}{m} F^{\alpha}_{\gamma;\alpha} U^\gamma + \frac{q}{m} F^{\alpha}_{\gamma} \Omega^{\gamma\alpha} - \Sigma_{\alpha\gamma} \Sigma^{\gamma\alpha} - \Omega_{\alpha\gamma} \Omega^{\gamma\alpha} - \frac{1}{3} \Theta^2 = R_{\mu;\gamma\alpha} U^\mu U^\nu. \quad (21)
\]
The Ricci tensor is related to the energy momentum tensor by the Einstein’s equation

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi t_{\mu\nu} \]  

(22)

or, equivalently, by

\[ R_{\mu\nu} = 8\pi (t_{\mu\nu} - \frac{1}{2} t g_{\mu\nu}) \]  

(23)

Substituting for \( R_{\mu\nu} \) we find

\[ \dot{\Theta} = f U^\rho \left( \frac{q}{m} F^\alpha_{\gamma;\mu} U^\gamma + \frac{q}{m} F^\alpha_{\gamma;\alpha} \Omega^\gamma - \Sigma_{\alpha\beta} \Sigma^\gamma_{\alpha\gamma} - \Omega_{\alpha\gamma} \Omega^\gamma_{\alpha} \right) - \frac{1}{3} \Theta^2 - 8\pi (t_{\mu\nu} - \frac{1}{2} t g_{\mu\nu}) U^\mu U^\nu. \]  

(24)

With our notation, \( (U^\mu U^\mu = -1) \), the limit \( F_{\mu\nu} \rightarrow 0 \) is the standard (geodesic) Raychaudhuri equation. \( \Sigma_{\alpha\beta} \) is called the shear tensor, \( \Omega_{\alpha\beta} \) is called the vorticity tensor and \( \theta \) is called the expansion scalar. These names arise because one can define an orthogonal triad of vectors in the three dimensional hypersurface orthogonal to \( U^\mu \). Under Fermi-Walker transport, the volume defined by the triad expands, gets deformed and also rotated.

3. Raychaudhuri’s equation for a ”unified” charged gravitating fluid

For the use of the Raychaudhuri equation to describe astrophysical plasmas in which gravitational effects are compatible with the fluid attributes, we have to account for the temperature and pressure of the fluid. For this, we incorporate into a small volume element of the fluid a statistical factor \( f \) which represents a temperature dependent statistical attribute of the fluid, and is related to the enthalpy \( h \), the scalar density in the rest frame \( n \) and the mass \( m \) of the fluid particles by the relation \( h = mn f(T) \). When one does the kinetic theory of high temperature plasmas \( f(T) \) seems to emerge as the most useful variable to represent temperature effects. For relativistic plasmas \( h = mn \frac{K^3}{(ms/T)^2} \) and \( f(T) \) is purely a function of temperature. The velocity vector of the fluid is obtained as the average velocity of this small volume of the fluid and is written as \( V^\mu = f U^\mu \). This drastically alters the character of the terms in the evolution equation. For example, now \( V^\mu V_\mu = -f^2 \) in contrast with \( U^\mu U_\mu = -1 \) and unlike \( U^\alpha U_{\alpha;\beta} = \frac{1}{2} (U^\alpha U_\alpha)_{,\beta} = 0 \), now \( V^\alpha V_{\alpha;\beta} = -f \partial_\beta f \). These terms significantly change the spatial terms of the Raychaudhuri equation and necessitating a generalization to account for these statistical factors. Indeed, this statistical limit for the fluid velocity \( f(T) \), unlike the particle limit, provides a natural factor for producing acceleration forces from within the fluid due to pressure and temperature gradients. In the unified magnetofluid picture one can write the equation of motion of a magnetofluid with entropy \( \sigma \) as

\[ T \partial^\rho \sigma = g M^{\mu\nu} U_\mu \]  

(25)
where
\[ M^{\mu \nu} = F^{\mu \nu} + \frac{m}{g} S^{\mu \nu}, \quad (26) \]
and
\[ S_{\mu \nu} = \partial_\mu (fU_\nu) - \partial_\nu (fU_\mu), \quad (27) \]
are antisymmetric second rank "flow" tensors defined in [7]. In this sense, \( M_{\mu \nu} \) represents an anti-symmetric "unified" field-flow tensor constructed from the kinematic \( (U^\mu) \), statistical \( (f(T)) \) and electromagnetic \( (F_{\mu \nu}) \) attributes of the magneto-fluid.

While the statistical factors provide us with additional acceleration terms which alter the purely spatial character of the shear and vorticity tensors, there are some relations that remain unaltered. A trivial example is that \( V^\mu \) remains orthogonal to the hypersurface with metric \( h_{\alpha \beta} \) defined as the projector \( h_{\alpha \beta} = g_{\alpha \beta} + U_\alpha U_\beta \).

To derive the geodesic deviation for \( V^\mu \) notice that
\[ V^\mu;_{\alpha \beta} = (fU^\mu;_\alpha + U^\mu f;_\alpha)_\beta = f;_\beta U^\mu;_\alpha + U^\mu;_\beta f;_\alpha + U^\mu f;_\alpha + fU^\mu;_{\alpha \beta}. \quad (28) \]

It is easy to see that anti symmetrization in indices \( \alpha, \beta \) reduces the geodesic deviation to
\[ V^\mu;_{\alpha \beta} - V^\mu;_{\beta \alpha} = f(U^\mu;_{\alpha \beta} + U^\mu;_{\beta \alpha}) = f R^\mu_{\sigma \alpha \beta} U^\sigma = R^\mu_{\sigma \alpha \beta} V^\sigma, \quad (29) \]
and its character is unchanged.

The question that arises now is if it is possible to define the generalizations of the standard definitions of the shear and vorticity tensors in a similar manner, and can they be constructed to be purely spatial tensors.

Let us, following the standard procedure, decompose \( \tilde{B}_{\alpha \beta} \) into its irreducible parts and define:
\[ \tilde{B}_{\alpha \beta} = \tilde{\Sigma}_{\alpha \beta} + \tilde{\Omega}_{\alpha \beta} + \frac{1}{3} \tilde{\Theta} h_{\alpha \beta} + \frac{1}{f} V^\alpha \partial_\alpha f. \quad (30) \]

Let us examine the trace of \( \tilde{B}_{\alpha \beta} \) as this defines the expansion scalar \( \tilde{\Theta} \)
\[ \tilde{B}^\alpha = \tilde{\Theta} + \frac{1}{f} V^\alpha \partial_\alpha f = V^\alpha;_\alpha \quad (31) \]
where we have assumed that the generalized shear tensor \( \tilde{\Sigma}_{\mu \nu} \) is symmetric and traceless. If we now define
\[ \tilde{\Sigma}_{\alpha \beta} = \frac{1}{2} (V_{\alpha ; \beta} + V_{\beta ; \alpha}) - \frac{1}{3} \tilde{\Theta} h_{\alpha \beta} - \frac{1}{2 f} (V_\alpha \partial_\beta f + V_\beta \partial_\alpha f), \quad (32) \]
then its trace

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\[
\tilde{\Sigma}_\mu = V^\mu - \hat{\Theta} - \frac{1}{f} V^\mu \partial_\mu f
\]

because of equation (31), goes to zero; the trace free condition also reproduces the required definition of \( \hat{\Theta} \).

The generalized vorticity tensor \( \tilde{\Omega}_{\alpha\beta} \) may also be written as

\[
\tilde{\Omega}_{\alpha\beta} = \frac{1}{2} (V_{\alpha;\beta} - V_{\beta;\alpha}) - \frac{1}{2f} (V_\alpha \partial_\beta f - V_\beta \partial_\alpha f).
\]

(34)

We see that the tensor \( S_{\mu\nu} \) defined in Eqn.27 allows us the following identification:

\[
\hat{\Omega}_{\alpha\beta} = \frac{1}{2} S_{\beta\alpha} - \frac{1}{2f} (V_\alpha \partial_\beta f - V_\beta \partial_\alpha f).
\]

(35)

Then, a straightforward analysis shows that

\[
V^\alpha (\tilde{\Sigma}_{\alpha\beta} + \tilde{\Omega}_{\alpha\beta}) = 0.
\]

(36)

An easy consequence of this result is that

\[
V^\alpha \tilde{B}_{\alpha\beta} = 0.
\]

(37)

From earlier work, we know that the gradient of \( f \), for a perfect fluid, is related to the pressure gradient and to Lorentz forces from the presence of an electromagnetic field [7]. So, unlike the standard particle picture of the shear and vorticity, the unified picture of shear and vorticity automatically includes accelerations coming from the internal forces of the fluids. This in turn ensures that the shear is now not a purely spatial tensor; the vorticity tensor behaves in a similar manner. Indeed, we see from the above equation that it is the sum of the shear and vorticity tensors that allows simplification.

Let us examine \( \tilde{B}_{\alpha\beta} V^\beta \): Again, it is easy to see that

\[
(\tilde{\Sigma}_{\alpha\beta} + \tilde{\Omega}_{\alpha\beta} + \frac{1}{3} \tilde{\Theta} h_{\alpha\beta} + \frac{1}{f} V_\alpha \partial_\beta f) V^\beta = V_{\alpha;\beta} V^\beta = \dot{V}_\alpha,
\]

(38)

implying \( \tilde{B}_{\alpha\beta} \)

\[
V^\alpha \tilde{B}_{\alpha\beta} = 0
\]

(39)

and

\[
\tilde{B}_{\alpha\beta} V^\beta = V_{\alpha;\beta} V^\beta = \dot{V}_\alpha.
\]

(40)

The decomposition of \( V_{\alpha;\beta} \) into its irreducible components can therefore be written as:

\[
V_{\alpha;\beta} = \tilde{B}_{\alpha\beta} = \tilde{\Sigma}_{\alpha\beta} + \tilde{\Omega}_{\alpha\beta} + \frac{1}{3} \tilde{\Theta} h_{\alpha\beta} + \frac{1}{f} V_\alpha \partial_\beta f.
\]

(41)

So, while the generalized expansion scalar, shear and vorticity tensors show evidence of internal fluid forces, the tensor \( \tilde{B}_{\alpha\beta} = V_{\alpha;\beta} \) itself shows similarity with the
original tensor, $B_{\alpha\beta} = U_{\alpha;\beta}$ defined earlier in the particle picture of the gravitating fluid. I.e;

$$V^\alpha B_{\alpha\beta} = 0,$$

(42)

and

$$\tilde{B}_{\alpha\beta}V^\beta = V_{\alpha;\beta}V^\beta = \dot{V}_\alpha.$$  

(43)

A careful analysis shows why this is so. From the definitions of $\tilde{\Theta}$, $\tilde{\Sigma}_{\mu\nu}$ and $\tilde{\Omega}_{\mu\nu}$, it is easy to see that the following equations relate the scalar of expansion, shear and vorticity in the particle view to those in the unified view of gravitating fluids:

$$\tilde{\Theta} = f\Theta$$

$$\tilde{\Sigma}_{\mu\nu} = f\Sigma_{\mu\nu}$$

$$\tilde{\Omega}_{\mu\nu} = f\Omega_{\mu\nu},$$

(44)

which then imply,

$$\tilde{B}_{\mu\nu} = fB_{\mu\nu} + V_\mu \partial_\nu \ln f.$$  

(45)

In analogy with the derivation of the standard Raychaudhuri equation, if we define $\tilde{B}_{\alpha\beta} = V_{\alpha;\beta}$, then from the deviation equation (29),

$$\tilde{B}_{\alpha\beta;\gamma}V^\gamma = (V_{\alpha;\gamma\beta}V^\gamma - R_{\alpha\sigma\gamma\beta}V^\sigma V^\gamma)$$

$$= (V_{\alpha;\gamma\beta}V^\gamma - V_{\alpha;\gamma\beta}V^\gamma - R_{\alpha\sigma\gamma\beta}V^\sigma V^\gamma)$$

$$= (V_{\alpha;\gamma\beta}V^\gamma - \tilde{B}_{\alpha\gamma}V^\gamma - R_{\alpha\sigma\gamma\beta}V^\sigma V^\gamma)$$

(46)

Taking the trace over the indices $\alpha\beta$ and using $\tilde{B}_{\alpha\alpha} = V_{\alpha;\alpha} = \tilde{\Theta} + \frac{1}{f} V^\alpha \partial_\alpha \ln f$, the new expansion scalar, we have

$$\left(\tilde{\Theta} + \frac{1}{f} V^\mu \partial_\mu f\right)^{\gamma\gamma} = (V_{\alpha;\alpha} - \tilde{B}_{\alpha\gamma} \tilde{B}^\gamma - R_{\alpha\sigma\gamma\beta} V^\sigma V^\gamma).$$

(47)

As before, we use Einstein’s equation to get

$$\left(\tilde{\Theta} + \frac{1}{f} V^\mu \partial_\mu f\right)^{\gamma\gamma} = (V_{\alpha;\alpha} - \tilde{B}_{\alpha\beta} \tilde{B}^{\beta\alpha} - 8\pi (t_{\mu\nu} - \frac{1}{2} t g_{\mu\nu}) V^\mu V^\nu).$$

(48)

The acceleration forces are twofold, one coming from the em forces within the fluid and the second from the fluid forces themselves (pressures, etc). To examine what these are, we return to the equation of motion for a fluid with entropy $\sigma$ in the unified picture Eqn.26 and use the fact that

$$S_{\mu\nu} = \partial_\mu (f(T)U_\nu) - \partial_\nu (f(T)U_\mu) = V_{\nu\mu} - V_{\mu\nu}$$

(49)

For an isentropic fluid,
\[ U^\mu \partial_\mu \sigma = \sigma \gamma V^\gamma = 0 \]  

and therefore, we can write

\[
T \partial_\nu \sigma = -T U^\mu \{ U_\mu \partial_\nu \sigma - U_\nu \partial_\mu \sigma \} \\
= -\frac{T}{f^2} V^\mu \{ V_\mu \partial_\nu \sigma - V_\nu \partial_\mu \sigma \}
\]  

Putting this together with (26), we find

\[-\frac{T}{f^2} V^\mu \{ V_\mu \partial_\nu \sigma - V_\nu \partial_\mu \sigma \} = q U^\mu (F_{\nu\mu} + \frac{m}{q} S_{\nu\mu}) \]  

or,

\[m V^\mu S_{\mu\nu} = q V^\mu F_{\nu\mu} + \frac{T}{f} V^\mu \{ V_\mu \partial_\nu \sigma - V_\nu \partial_\mu \sigma \} \]

Therefore,

\[V^\mu V_{\nu;\mu} - V^\mu V_{\mu;\nu} = \frac{q}{m} (V^\mu F_{\nu\mu} + \frac{mT}{qf} V^\mu \{ V_\mu \partial_\nu \sigma - V_\nu \partial_\mu \sigma \}) \]

and,

\[\dot{V}_\nu = -f \partial_\nu f + \frac{q}{m} (V^\mu F_{\nu\mu} + \frac{mT}{qf} V^\mu \{ V_\mu \partial_\nu \sigma - V_\nu \partial_\mu \sigma \}) \]

Defining a pure fluid factor, \( N_{\mu\nu} \) as

\[ N_{\mu\nu} = \frac{T}{f} (\sigma_{\mu\nu} V_\nu - \sigma_{\nu\nu} V_\mu), \]

the acceleration vector simplifies to

\[\dot{V}_\nu = -f \partial_\nu f + \frac{q}{m} V^\mu (F_{\nu\mu} + \frac{m}{q} N_{\mu\nu}). \]

Defining

\[ G_{\mu\nu} = F_{\mu\nu} + \frac{m}{q} N_{\mu\nu}, \]

substitution into (48) yields

\[
(\dot{\Theta} + \frac{1}{f} V^\mu \partial_\mu f)\gamma V^\gamma = \left( -f \partial^\alpha f + \frac{q}{m} V_\alpha \tilde{G}^{\alpha\beta} \right)_{\beta} - \tilde{B}_{\alpha\beta} \tilde{B}^{\beta\alpha} \\
- 8\pi t_{\mu\nu} V^\mu V^\nu - 4\pi f^2 t
\]

which simplifies to
\[
\dot{\Theta} + \dot{\zeta} = (-f \partial^{\alpha} f)_{,\alpha} + \frac{q}{m} V_{\beta,\alpha} G^{\alpha \beta} + \frac{q}{m} V_{\beta} G^{\alpha \beta} :_{,\alpha} \\
- \tilde{B}_{\alpha \beta} \tilde{B}^{\beta \alpha} - 8\pi t_{\mu \nu} V^\mu V^\nu - 4\pi f^2 t
\] (60)

where we have defined \( V^\mu \partial_\mu \ln f = \zeta \). Using the relationship between \( B_{\mu \nu} \) and \( \tilde{B}_{\mu \nu} \), we find

\[
\dot{\Theta} + \dot{\zeta} = (-f \partial^{\alpha} f)_{,\alpha} + \frac{q}{m} V_{\beta,\alpha} G^{\alpha \beta} + \frac{q}{m} V_{\beta} G^{\alpha \beta} :_{,\alpha} \\
- f B_{\alpha \beta} f B^{\beta \alpha} - 2B^{\alpha \gamma} V_\gamma \partial_\alpha \ln f + \dot{\zeta}^2 - 8\pi t_{\mu \nu} V^\mu V^\nu - 4\pi f^2 t \\
= \frac{q}{m} V_{\beta,\alpha} G^{\alpha \beta} + \frac{q}{m} V_{\beta} G^{\alpha \beta} :_{,\alpha} - f B_{\alpha \beta} f B^{\beta \alpha} \\
+ \dot{\zeta}^2 - f^2 (\partial^2 \ln f) - 8\pi t_{\mu \nu} V^\mu V^\nu - 4\pi f^2 t
\] (61)

In the limit \( f \to 1 \) as \( T \to 0 \) this reduces to the standard Raychaudhuri equation. Simplifying the left hand side,

\[
\dot{\Theta} = \frac{q}{m} S_{\alpha \beta} G^{\alpha \beta} + \frac{q}{m} V_{\beta} G^{\alpha \beta} :_{,\alpha} - f B_{\alpha \beta} f B^{\beta \alpha} + \dot{\zeta}^2 \\
- \dot{\zeta} - f^2 (\partial^2 \ln f) - 8\pi t_{\mu \nu} V^\mu V^\nu - 4\pi f^2 t
\] (62)

Simplifying the \( B_{\mu \nu} B^{\nu \mu} \) terms, we have

\[
\dot{\Theta} = \frac{q}{2m} S_{\alpha \beta} G^{\alpha \beta} + \frac{q}{m} V_{\beta} G^{\alpha \beta} :_{,\alpha} - (\dot{\Sigma}_{\alpha \beta} \dot{\Sigma}^{\beta \alpha} \\
+ \ddot{\Omega}_{\alpha \beta} \dot{\Omega}^{\beta \alpha} + \frac{1}{3} \dot{\Theta}^2) + \dot{\zeta}^2 - \dot{\zeta} - f^2 (\partial^2 \ln f) - 8\pi t_{\mu \nu} V^\mu V^\nu - 4\pi f^2 t
\] (63)

This final form of the generalized Raychaudhuri equation shows all the terms which reduce to the standard equation in the limit \( f(T) \to 1 \).

4. A Raychaudhuri equation for the unified non-Abelian magneto fluid.

In a recent paper [8], we had generalized the proposed unification of the Abelian magneto fluid [7] to a non-Abelian magneto fluid. The equations of motion for the magneto fluid clearly indicated the possibility of solitonic solutions which are normally absent in the Abelian case. The inherent non-linearities present in the Yang Mills magneto fluid allow such possibilities and a natural question arises as to what will self gravity do to the dynamics of a non-Abelian magneto fluid.

This is particularly relevant in view of the compelling experimental evidence from the relativistic heavy ion collider at Brookhaven National Laboratory (BNL) that the universe in its first few moments may have existed as a quark-gluon fluid. Since, large gravitational fields are also present in this epoch, it provides us with a motivation to give a generalization of the Raychaudhuri equation for the non-Abelian magneto fluid.

The suggestion in [8] was that each worldline would now carry an internal index labelling the non-Abelian species. Generalizing the Lorentz force law to the non-Abelian case, we had derived the equation of motion for the magneto fluid in
terms of a unified antisymmetric tensor, $M^{i\mu\nu} = F^{i\mu\nu} + \frac{2i}{g} S^{i\mu\nu}$, where $F^{i\mu\nu}$ is the standard Yang Mills field strength tensor while $S^{i\mu\nu}$ is given by:

$$S^{i\mu\nu} = \partial_\mu (f U^{i\nu}) - \partial_\nu (f U^{i\mu}) - ig f [A_\mu, U_i] + ig f [A_\nu, U_i] - im f^2 (U_\mu, U_\nu)$$

(64)

with the gauge covariant derivative being defined by $D_\mu = \partial_\mu - ig [A_\mu, \cdot]$. The non-Abelian fluid equations of motion corresponding to Eqn.26 for a "Yang-Mills Magneto-fluid", with entropy $\sigma$ are given by

$$T \partial_\nu \sigma = g M^{\mu\nu}_a U^{a\mu}_\nu.$$  

(65)

For a non-Abelian magneto fluid velocity vector $U^{i\mu}$, the deviation equation can easily be written as

$$U^{i\mu}_{\nu\sigma} - U^{i\mu}_{\sigma\nu} = R^{\mu\rho\sigma\nu} U^{i\rho}.$$  

(66)

Examining $B^{i\mu\nu} = U^{i\mu}_{\nu\alpha} U^{i\alpha}$,

$$B^{i\mu\nu}_{\alpha\beta} U^{i\alpha}_\beta = (U^{i\mu}_{\alpha\beta} U^{i\alpha})_{\nu} - R^{\mu\rho\sigma\nu} U^{i\rho\mu}_\beta - U^{i\mu}_{\alpha\beta} U^{i\alpha}_{\nu}.$$  

(67)

From our earlier work [8], we write

$$tr \dot{U}^i = U^{i\mu}_{\nu\rho} U^{i\nu}_\rho = \frac{g}{m} F^{i\mu\rho} U^{i\nu}_\rho$$

for the generalization of the Lorentz force law in the particle picture of the non-Abelian magneto fluid; the trace is over the internal, gauge group indices $(i = 1 \ldots N = \text{dim}(\text{gauge group}))$. Clearly, since now, $U^{i\mu} U^{i\mu} = -N$, we shall henceforth, assume the $U^{i\mu}$ are normalized so that $U^{i\mu} U^{i\mu} = -1$, i.e; we assume $U^{i\mu} \rightarrow U^{i\mu}/\sqrt{N}$. With this proviso, we can define an orthogonal projection tensor as before

$$h^{i\mu\nu} = g^{i\mu\nu} + U^{i\mu}_{\nu\rho} U^{i\rho}.$$  

(69)

It follows that

$$U^{j\mu} h^{i\mu\nu} = U^{j\mu} g^{i\mu\nu} + U^{j\mu}_{\mu\nu} U^{i\nu} = U^{j\nu} - \delta^{ij} U^{i\nu} = 0$$

(70)

and

$$h^{i\mu\nu} U^{j\nu} = g^{i\mu\nu} U^{j\nu} + U^{i\mu}_{\mu\nu} U^{j\nu} = U^{j\mu} - \delta^{ij} U^{i\mu} = 0$$

(71)

where we have used

$$U^{i\mu}_\mu U^{j\nu} = -\delta^{ij}$$

(72)

from which it follows that $U^{i\mu}_\mu U^{i\mu} = -1$. Equation(67) can be reduced to

$$B^{i\mu\nu}_{\alpha\beta} U^{i\alpha}_\beta = (\frac{g}{m} F^{i\mu\alpha} U^{i\alpha})_{\nu} - R^{\mu\rho\sigma\nu} U^{i\rho\mu}_\beta - U^{i\mu}_{\alpha\beta} U^{i\alpha}_{\nu}.$$  

(73)

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Tracing over indices $\mu$ and $\nu$, and defining

$$\Theta_i = B_i^{i\mu\mu},$$

we find

$$\dot{\Theta} = \left(\frac{g}{m} F^{i\mu\alpha}_{\mu\alpha} U_i^\alpha\right)_{\mu\alpha} - R_{\mu\rho\alpha} U^{i\alpha\rho} - B_i^{i\mu\alpha} B_i^{i\alpha\mu}.$$

Once again, we can decompose the tensor $B_i^{i\mu\nu}$ into its irreducible parts and write

$$B_i^{i\mu\nu} = U_i^{i\mu\nu} = \Sigma_i^{i\mu\nu} + \Omega_i^{i\mu\nu} + \frac{1}{3} \Theta_i h_{\mu\nu}.$$  

It is to be noted here that we are considering a special sector of the magneto fluid dynamics, that of a non-interacting sector of the full non-Abelian fluid as is seen from our definition of the acceleration vector $a_{\mu} = U_{\mu\nu} U^\nu_i$. From a fluid point of view, we are dealing with a multi-species model of the non-Abelian fluid. The full non-Abelian interactions of the fluid will be explored in a future study. Returning to the decomposition of $B_i^{i\mu\nu}$, we write

$$\Sigma_i^{i\mu\nu} = \frac{1}{2} (U_i^{i\mu\nu} + U_i^{i\nu\mu}) - \frac{1}{3} \Theta_i h_{\mu\nu}$$

$$\Omega_i^{i\mu\nu} = \frac{1}{2} (U_i^{i\mu\nu} - U_i^{i\nu\mu})$$

Clearly, these definitions give a decomposition of $B_i^{i\mu\nu}$ into its irreducible components. The resulting Raychaudhuri equation, upon substitution of $B_i^{i\mu\nu} B_i^{i\nu\mu}$ is, as expected, a multi-species generalization of the standard Raychaudhuri equation. This Non-Abelian generalization of Raychaudhuri’s equation gives us a means of studying the shear and vorticity of quark-gluon astrophysical plasmas.

5. Conclusion

High temperature plasmas have an important role to play in the early universe. In what is known as the "classical" or the "radiation" epoch of the universe, both gravitational and high temperature effects are of equal importance. Thus for the evolutionary dynamics of these hot gravitating plasmas, a generalization of the Raychaudhuri equation to include finite temperature fluid forces as well as electromagnetic effects was called for. Using a unified magneto-fluid approach to construct a generalized Raychaudhuri equation, we have attempted to respond to this call. Non-Linear plasmas also find an application in cosmology through the (now compelling) evidence that a non-Abelian, non-linear quark-gluon fluid existed in the early epochs of the universe. To deal with the evolution of this fluid when the gravitational effects are strong, we have laid the foundation of a generalized Raychaudhuri equation for the evolution of non-Abelian gravitating plasmas. Most interesting plasmas involve collective effects which are non-linear even in the special relativistic situation and the inclusion of gravity can only lead to more intriguing
highly non linear phenomena. The investigation of further implications of the ideas germinated in this paper is a promising avenue for future research. Amalgamation and unification of ideas that cut across barriers always enriches physics and we feel that any work of this kind is a fitting tribute to a man whose physics cut across international boundaries.

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