PARTIAL INNER PRODUCTS ON ANTIDUALS

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Abstract. We discuss extensions of an inner product from a vector space to its full antidual. None of these extensions is weakly continuous, but partial extensions recapture some familiar structure including the Hilbert space completion and the antiduality pairing.

Let \( V \) be an infinite-dimensional complex vector space on which \((\cdot,\cdot)\) is an inner product. We adopt the convention according to which \((x|y)\) is antilinear in \(x\) and linear in \(y\); we make no assumption regarding completeness of the inner product. Let \( V' \) be the full antidual of \( V \): thus, \( V' \) comprises precisely all antilinear maps \( V \to \mathbb{C} \) whether bounded or otherwise. The inner product \((\cdot,\cdot)\) engenders a canonical linear embedding of \( V \) in \( V' \): explicitly, for each \( v \in V \) we define \( v' \in V' \) by the rule that if \( z \in V \) is arbitrary then
\[
v'(z) = (z|v).
\]

Our aim is to investigate inner products on \( V' \) that are compatible with the given inner product \((\cdot,\cdot)\) on \( V \). At the very least, we should insist that compatibility requires the embedding \( V \to V' \) to be isometric. Were this our only compatibility requirement, a suitable inner product on \( V' \) could of course be defined by transporting the given inner product to \( \widetilde{V} = \{v' : v \in V\} \subset V' \) and choosing a (purely algebraic) decomposition \( V' = \widetilde{V} \oplus W \), providing \( W \) with an inner product and making the decomposition orthogonal.

We shall demand more of compatibility. The full antidual \( V' \) naturally carries the (weak) topology of pointwise convergence, according to which a net \((\zeta_\delta : \delta \in \Delta)\) converges to \( \zeta \) in \( V' \) precisely when \( \zeta_\delta(v) \to \zeta(v) \) for every \( v \in V \). We shall say that the inner product \([\cdot,\cdot]\) on \( V' \) is compatible with the original inner product \((\cdot,\cdot)\) on \( V \) precisely when:

(i) the canonical embedding \( V \to V' \) is isometric, so that if \( x, y \in V \) then
\[
[x'|y'] = (x|y);
\]

(ii) \([\cdot,\cdot]\) is weakly continuous in each slot, so that if \( \xi_\delta \to \xi \) and \( \eta_\delta \to \eta \) then
\[
[\xi_\delta|\eta_\delta] \to [\xi|\eta] \quad \text{and} \quad [\xi|\eta_\delta] \to [\xi|\eta].
\]

Our approach to this investigation will be by way of finite-dimensional approximation. Write \( \mathcal{F}(V) \) for the set comprising all finite-dimensional complex subspaces of \( V \); this set is naturally directed by inclusion. Let \( \zeta : V \to \mathbb{C} \) be an antilinear functional on \( V \). If \( M \in \mathcal{F}(V) \) is any finite-dimensional subspace of \( V \) then the restriction \( \zeta|_M : M \to \mathbb{C} \) is given by taking inner product against a vector in \( M \): there exists a unique vector \( \zeta_M \in M \) such that if \( z \in M \) then
\[
\zeta(z) = (z|\zeta_M).
\]

Theorem 1. If \( \zeta \in V' \) then the net \((\zeta'_M : M \in \mathcal{F}(V))\) converges weakly to \( \zeta \) in \( V' \).

Proof. Let \( z \in V \) be arbitrary: on the one hand, if \( M \in \mathcal{F}(M) \) then \( \zeta'_M(z) = (z|\zeta_M) \); on the other, if also \( z \in M \) then \( (z|\zeta_M) = \zeta(z) \). Thus the given net is eventually constant at each point and so pointwise convergent, with the correct limit. \( \square \)

Otherwise said, if \( \zeta \in V' \) then
\[
\zeta = \lim_{M \uparrow \mathcal{F}(V)} \zeta'_M.
\]

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We may use these elementary finite-dimensional approximating nets to analyze a compatible inner product.

First we note that a compatible inner product in $V'$ restricts to reproduce the natural ‘duality’ pairing between $V$ and its antidual.

**Theorem 2.** Let $[\cdot|\cdot]$ be a compatible inner product on $V'$. If $x, y \in V$ and $\xi, \eta \in V'$ then

$$[x'|\eta] = \eta(x), \quad [\xi|y'] = \xi(y).$$

**Proof.** We need only establish the first identity. For this, let $M \in \mathcal{F}(V)$: once $M$ contains $x$ it follows that

$$\eta(x) = \langle x|\eta_M \rangle = [x'|\eta_M] \rightarrow [x'|\eta]$$
on account of compatibility and Theorem 1. \hfill $\square$

Next we note that finite-dimensional approximants permit the reconstruction of a compatible inner product in its entirety.

**Theorem 3.** Let $[\cdot|\cdot]$ be a compatible inner product on $V'$. If $\xi, \eta \in V'$ then

$$[\xi|\eta] = \lim_{M \uparrow \mathcal{F}(V)} \langle \xi_M|\eta_M \rangle.$$

**Proof.** If $M \in \mathcal{F}(V)$ then Theorem 2 ensures that

$$\langle \xi_M|\eta_M \rangle = \eta(\xi_M) = [\xi_M|\eta]$$

whereupon compatibility and Theorem 1 complete the argument. \hfill $\square$

Accordingly, $V'$ carries at most one compatible inner product; we now address the question of existence.

It will help to have available the relationship between the approximants relative to a finite-dimensional subspace and one of its hyperplanes.

**Theorem 4.** Let $N = M \oplus Cu$ where $u \in V$ is a unit vector orthogonal to $M \in \mathcal{F}(V)$. If $\zeta \in V'$ then

$$\zeta_N = \zeta_M + \zeta(u)u.$$ 

**Proof.** Immediate: if $z \in M$ then $\zeta(z) = \langle z|\zeta_M \rangle$ and if $\lambda \in \mathbb{C}$ then $\zeta(\lambda u) = \bar{\lambda}\zeta(u) = \langle \lambda u|\zeta(u)u \rangle$. \hfill $\square$

It follows at once that if $\xi, \eta \in V'$ then

$$\langle \xi_N|\eta_N \rangle = \langle \xi_M|\eta_M \rangle + \bar{\xi}(u)\eta(u).$$

Inductively, if $N = M \oplus L$ is an orthogonal decomposition then

$$\|\zeta_N\|^2 = \|\zeta_M\|^2 + \|\zeta_L\|^2;$$

indeed, $\zeta_M$ and $\zeta_L$ are the respective orthogonal projections of $\zeta_N$ on $M$ and $L$.

The following result will also be useful.

**Theorem 5.** If $x, y \in V$ are unit vectors then the supremum of $\|\langle x|u\rangle \langle u|y \rangle\|$ as $u \in V$ runs over all unit vectors is at least $1/2$. 
Proof. Define $T : V \rightarrow V$ by $T(z) = \langle x \mid z \rangle y$; note that $T$ has unit operator norm. Quite generally, the numerical radius $w(T)$ of the operator $T$ is defined by

$$w(T) = \sup\{\|u[Tu]\| : \|u\| = 1\};$$

for example, see [1]. Here,

$$\langle u | Tu \rangle = \langle x | u \rangle \langle y | v \rangle$$

so that $w(T)$ is precisely the supremum described in the statement of the theorem. If $u, v \in V$ are unit vectors then by polarization

$$2\langle u | Tv \rangle + 2\langle v | Tu \rangle = \langle u + v | T(u + v) \rangle - \langle u - v | T(u - v) \rangle$$

whence the parallelogram law yields

$$2\langle u | T v \rangle + 2\langle v | T u \rangle \leq w(T)\{\|u + v\|^2 + \|u - v\|^2\} = 4w(T);$$

furthermore, if $v = T(u)/\|T(u)\|$ then

$$\langle u | T v \rangle + \langle v | T u \rangle = \frac{\langle u | T^2 u \rangle}{\|T(u)\|^2} + \|T(u)\|.$$

Choose the unimodular scalar $\lambda$ so that $\lambda^2 \langle u | T^2 u \rangle \geq 0$ and apply the foregoing analysis to $\lambda T$ in place of $T$ itself, to deduce that $2w(T) \geq \|Tu\|$. Finally, take the supremum as $u$ runs over all unit vectors, to conclude that $2w(T) \geq \|T\| = 1$. □

Our present purposes are adequately served by this estimate, but the identification of this supremum as a numerical radius leads to an exact formula. The cleanest formula obtains when $(\langle \overline{\cdot} \cdot \rangle)$ is a real inner product, in which case the set of all reals $(\langle x | u \rangle \langle u | y \rangle)$ as $u$ runs over all unit vectors is a closed interval of unit length, namely

$$[((\langle x | y \rangle) - 1)/2, ((\langle x | y \rangle) + 1)/2].$$

Notice that the operator norm of any (bounded) antifunctional on $V$ can be identified in terms of its finite-dimensional approximants.

Theorem 6. If $\zeta \in V'$ then

$$\|\zeta\| = \sup\{\|\zeta_M\| : M \in \mathcal{F}(V)\} \in [0, \infty].$$

Proof. In one direction, let $K$ be the indicated supremum: if $z \in V$ then let $M = \mathbb{C}z$ and calculate $|\zeta(z)| = |\langle z | M \rangle| \leq \|z\| \|\zeta_M\| \leq K \|z\|$. In the opposite direction, if $M \in \mathcal{F}(V)$ then $\zeta_M \in M$ so that $\|\zeta_M\|^2 = \langle \zeta_M | \zeta_M \rangle = \langle \zeta_M | M \zeta_M \rangle \leq \|\zeta\| \|\zeta_M\|$ and cancellation ends the argument. □

In fact, the net $(\|\zeta_M\| : M \in \mathcal{F}(V))$ is increasing, as the remark after Theorem 4 makes clear; consequently,

$$\|\zeta\| = \lim_{M \in \mathcal{F}(V)} \|\zeta_M\|.$$

We coordinate these theorems to effect a proof of the next.

Theorem 7. If the antifunctionals $\zeta, \eta \in V'$ are unbounded then the net $(\langle \zeta_M | \eta_M \rangle : M \in \mathcal{F}(V))$ does not converge.

Proof. The net $(\langle \zeta_M | \eta_M \rangle : M \in \mathcal{F}(V))$ is not Cauchy. In fact, let the finite-dimensional subspace $M \in \mathcal{F}(V)$ be arbitrary. The restrictions of $\zeta$ and $\eta$ to the orthocomplement $M^\perp$ being unbounded, Theorem 6 provides $L \in \mathcal{F}(M^\perp)$ such that $\|\zeta_L\|$ and $\|\eta_L\|$ are as large as we please; say greater than unity. Theorem 5 provides a unit vector $u \in L$ such that $\langle \zeta(u) | \eta(u) \rangle = \langle \xi_L | u \rangle \langle u | \eta_L \rangle$ has modulus greater than $1/2$. Finally, Theorem 4 shows that $N = M \oplus \mathbb{C}u \in \mathcal{F}(V)$ satisfies $|\langle \zeta_N | \eta_N \rangle - \langle \xi_M | \eta_M \rangle| > 1/2$. □

Theorem 3 and Theorem 7 together imply that compatible inner products are nonexistent. We are led to ask what can be salvaged from this negative result.
Taking a cue from Theorem 3 we define the partial inner product $[\mathbf{*}] = [\mathbf{*}]_V$ in $V'$ by the rule that if $\xi, \eta \in V'$ then

$$[\xi|\eta] := \lim_{M \uparrow \mathcal{F}(V')} \langle \xi_M|\eta_M \rangle$$

whenever this limit exists.

This rule does define a partial inner product extending $(\mathbf{*})$. It is plainly Hermitian, in the sense that if $[\xi|\eta]$ is defined then so is $[\eta|\xi]$ and

$$[\eta|\xi] = [\xi|\eta].$$

It is plainly also linear in the second slot (and therefore antilinear in the first) in the sense that if $[\xi|\eta_1]$ and $[\xi|\eta_2]$ are defined then so is $[\xi|\lambda_1 \eta_1 + \lambda_2 \eta_2]$ and

$$[\xi|\lambda_1 \eta_1 + \lambda_2 \eta_2] = [\xi|\eta_1] + [\xi|\eta_2]$$

whenever $\lambda_1, \lambda_2 \in \mathbb{C}$. Finally, Theorem 6 makes it clear that $[\zeta|\zeta]$ is defined precisely when the antifunctional $\zeta$ is bounded, in which case $[\zeta|\zeta] \geq 0$ with equality if and only if $\zeta = 0$.

This last point can be amplified a little. Let us denote by $V^* \subset V'$ the subspace comprising all bounded antifunctionals. Let $\xi, \eta \in V^*$: polarization in $V$ shows that if $M \in \mathcal{F}(V)$ then

$$4\langle \xi_M|\eta_M \rangle = \sum_{n=0}^{3} i^{-n}||\xi_M + i^n \eta_M||^2 = \sum_{n=0}^{3} i^{-n}||\xi + i^n \eta||_M^2$$

and the remark after Theorem 1 justifies passage to the limit as $M \uparrow \mathcal{F}(V)$ producing

$$4[\xi|\eta] = \sum_{n=0}^{3} i^{-n}||\xi + i^n \eta||^2$$

with operator norm on the right. Thus the partial inner product $[\mathbf{*}]$ is defined on $V^*$ where it becomes a true inner product underlying the operator norm. More is true: it may be checked (as an instructive exercise) that if $\zeta \in V^*$ then the net $(\zeta_M : M \in \mathcal{F}(V))$ converges to $\zeta$ in operator norm, improving Theorem 1 in this circumstance; so $V^*$ furnishes a canonical model for the Hilbert space completion of $V$.

This partial inner product $[\mathbf{*}]$ is also defined on $V \times V'$ and $V' \times V$ upon which it induces the natural pairing between $V$ and $V'$: an argument akin to the one for Theorem 2 shows that if $x, y \in V$ and $\xi, \eta \in V'$ then

$$[x'|\eta] = \eta(x), \quad [\xi|y'] = \xi(y).$$

We may extend this partial inner product by replacing the directed set $\mathcal{F}(V)$ with one of its cofinal subsets $S \subseteq \mathcal{F}(V)$: if the net $(\{\xi_M|\eta_M \} : M \in \mathcal{F}(V))$ converges then so does its subnet $(\{\xi_M|\eta_M \} : M \in S)$ and the limits coincide; however, the latter net may converge even though the former does not. Of course, each such extension will continue to reproduce both the Hilbert space completion of $V$ and the canonical pairing with its antidual; but such an extension may have further properties.

One example will suffice as an illustration. Let

$$V = X \oplus Y$$

be an orthogonal decomposition, with $P_X : V \rightarrow X$ and $P_Y : V \rightarrow Y$ as corresponding orthogonal projectors. Note that antilinear extension by zero on orthocomplements yields the canonical embeddings $X' \rightarrow V' : \xi \mapsto \xi \circ P_X$ and $Y' \rightarrow V' : \eta \mapsto \eta \circ P_Y$. Write $\mathcal{F}(X, Y)$ for the set comprising all finite-dimensional subspaces $M$ of $V$ that split under this decomposition as $M = M_X \oplus M_Y$ where $M_X = P_X(M)$ and $M_Y = P_Y(M)$. The subset $\mathcal{F}(X, Y) \subseteq \mathcal{F}(V)$ is certainly cofinal: indeed, each $M \in \mathcal{F}(V)$ is contained in $M_X \oplus M_Y \in \mathcal{F}(X, Y)$. Now, when $\xi, \eta \in V'$ let us agree to write

$$[\xi|\eta]_{X, Y} = \lim_{M \uparrow \mathcal{F}(X, Y)} \langle \xi_M|\eta_M \rangle.$$
This new partial inner product circumvents Theorem 7 in being defined on certain pairs of unbounded antifunctionals; for example, as in the following result.

**Theorem 8.** If \( \xi \in X' \) and \( \eta \in Y' \) then \( [\xi \circ P_X|\eta \circ P_Y]_{X,Y} = 0 \).

*Proof.* As \( X \) and \( Y \) are orthogonal, this follows at once from the fact (left as another exercise) that if \( M \in \mathcal{F}(X,Y) \) then \( (\xi \circ P_X)_M = \xi_{M_X} \in M_X \subseteq X \) and \( (\eta \circ P_Y)_M = \eta_{M_Y} \in M_Y \subseteq Y \).

It is appropriate here to issue the reminder that if \( \xi \in X' \) and \( \eta \in Y' \) are unbounded then \( [\xi \circ P_X|\eta \circ P_Y]_V \) is undefined: the net \( \{(\xi \circ P_X)_M| (\eta \circ P_Y)_M : M \in \mathcal{F}(V)\} \) does not converge, even though \( \xi \circ P_X \) vanishes on \( Y = X^\perp \) and \( \eta \circ P_Y \) vanishes on \( X = Y^\perp \).

**References**

[1] G.K. Pedersen, *Analysis Now*, Springer Graduate Texts in Mathematics 118 (1989).