Dynamic compressibility of dense granular shear flows

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received 29 May 2013; accepted in final form 2 August 2013
published online 29 August 2013

PACS 83.80.Fg – Rheology: Granular solids
PACS 63.50.-x – Vibrational states in disordered systems
PACS 47.40.-x – Compressible flows; shock waves

Abstract – We provide evidence that an assembly of hard non-deformable particles can behave as a compressible medium when slowly sheared. By means of discrete element simulations of frictional discs, we show the existence of transverse and sagittal waves. These waves are associated to a dynamic compressibility which originates from the fact that the average density of such a system effectively depends on the confining pressure. Analysing velocity fluctuations, we show the existence of spontaneous oscillations, whose frequency coincides with that observed at the resonance of forced oscillations. The results are independent of the contact stiffness and the restitution coefficient. They are well reproduced by a continuum model based on a local constitutive relation.

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Acoustics in granular media is a rapidly developing field that is attractive both from the fundamental physics point of view [1–4] and for its applications in material science, soil mechanics and geophysics [5]. It provides a unique tool which allows to probe the mechanical response of such materials [6], to detect heterogeneities, and to study aging and avalanche precursors [7]. Surprisingly, wave propagation in granular shear flows has never been studied in isolation. It has been revealed indirectly by instabilities, in situations where acoustic waves are spontaneously emitted during a dense granular flow (see [3] for a review). For example, when sufficiently dry, avalanches flowing at the surface of a dune produce a loud sound known as the “song of dunes” [8–12]; vibrations are also produced during the discharge of a granular flow from a silo or from any other elongated tube [13]; soft monodisperse particles flowing on an inclined plane near the angle of repose exhibit spontaneous oscillations at the resonant frequency of elastic waves [14,15]. These phenomena have mostly been associated with the elastic deformations of the grains. However, Bagnold [8] and others [3,10,12] have hypothesized the existence of waves that could propagate through a granular material composed of hard non-deformable grains.

How could such waves possibly exist? When slowly sheared at a rate \( \dot{\gamma} \), an assembly of grains of density \( \rho \) and diameter \( d \) is submitted to a dynamic pressure, controlled by the grain inertia, which diverges as \( P \propto \rho \dot{\gamma}^2 d^2 (\phi_c - \phi)^{-2} \) when the volume fraction \( \phi \) approaches its critical value \( \phi_c \) at the jamming point [16–18]. It has been related to the non-affine cooperative motion of the grains [19–22]. Despite corresponding flow heterogeneities [23,24], can waves propagate due to the existence of a dynamic compressibility?

In this letter we demonstrate theoretically that such waves indeed exist. The dispersion relation presents three branches, two of which become non-dissipative as they approach the jamming point. Our analytical predictions are confirmed by numerical simulations performed with quasi-rigid grains, which show no dependence on their elasticity and on their restitution coefficient.

Numerical simulations. – We consider a two-dimensional system constituted of \( \approx 2 \cdot 10^3 \) circular particles of mass \( m_i \) and diameter \( d_i \), with a \( \pm 50\% \) polydispersity (fig. 1(a)). The shear cell is composed of two rough walls distant by \( H \) and moving along the \( x \)-direction at a constant velocity. Simulations are performed at a constant, imposed volume fraction \( \phi \). Periodic boundary conditions are applied along \( x \). The particle (inertial) dynamics is integrated using the Verlet algorithm. The particles are submitted to contact forces modeled as viscoelastic forces (linear spring-dashpot model), and with a Coulomb friction along the tangential direction [17,25,26]. Gravity is switched off. Quantities used in the model are expressed in terms of the grain density \( \rho \), of the average pressure \( P \), and of the mean grain diameter \( d \). In this
system of units, the normal spring constant $\kappa_n$ is chosen above the crossover value $\simeq 10^3$ in order to reach the rigid asymptotic regime where the results are independent of its value [17]. Here, the Coulomb friction coefficient is chosen equal to $\mu_p = 0.4$ and the tangential spring constant $\kappa_t = 0.5\kappa_n$. The influence of viscoelastic parameters is encoded in the restitution coefficient $e$, which is varied between 0.1 and 0.9. The average shear rate $\dot{\gamma}$ is kept constant during all simulations (see caption of fig. 1).

**Spontaneous and forced oscillations.** – Looking at the time fluctuations of the velocity field (fig. 1(b)), simple shear flows are found to present spontaneous vibrations. As seen from the velocity autocorrelation function $C$ (fig. 1(c)) and the power spectral density $D$ (fig. 2(b), (c)) of the velocity signal, this vibration is not a broadband noise but oscillations at a well-defined angular frequency $\omega^*$.

Keeping the shear rate $\dot{\gamma}$ constant and approaching the jamming point $\phi_c$, $\omega^*$ is found to diverge as $(\phi_c - \phi)^{-3/2}$ (fig. 2(a)). The coherence time can be determined from the spectrum, which presents a narrow peak, whose width at half-maximum is around $\omega^*/4$ (fig. 2(b)). However, as $\phi \to \phi_c$, the peak develops a tail towards small frequencies, which is reminiscent of the spectral density of elastic modes on the solid side of jamming, for $\phi > \phi_c$ [27–30].

To analyze the vibration mode by mode, we have performed simulations in which each grain is submitted to a force along $z$ of the form $f_z \sin(\omega t) \sin(k z)$, with $k = \pi/H$. We analyze the response of the system to this forcing with a simple “lock-in amplifier”: we compute the Fourier component of the velocity field at the angular frequency $\omega$ and the wave number $k$. The amplitude of the forcing is chosen small enough to remain in the linear regime —i.e., the velocity response is proportional to the forcing $f_z$. As displayed in fig. 3, this response presents a resonance, characterized by a maximum of amplitude and by a vanishing phase. The resonant angular frequency precisely coincides with the angular frequency $\omega^*$ of spontaneous oscillations (fig. 1). A similar forcing along the $x$-direction gives very similar results, see fig. 4.

Importantly, the resonance curves (figs. 3 and 4) as well as the autocorrelation function of spontaneous vibrations (fig. 1(c)) are insensitive to the values of the restitution coefficient $e$ and of the spring constants $\kappa_n, \kappa_t$ of the grains. This shows that these dense flows do not behave like a granular gas [31–34], whose compressibility is controlled by binary collisions and thus depends on $e$. The nature of these oscillations is also fundamentally different from those exhibited in [14,15] which are specifically observed for monodisperse grains and whose frequency scales as $\sqrt{\phi}$. The vibration modes of a system of sheared hard particles are therefore different from both elastic modes in solids and sound waves in granular gases.

**Continuum model.** – To shed light on the origin of these oscillations, we consider a description of the granular material as a continuum compressible medium characterized by a volume fraction $\phi$ and a velocity field $\vec{u}$. Under
the assumption that the rheology is local \([16,21,35]\), the dimensional analysis constrains the form of the constitutive relation between the stress \(\sigma_{ij}\) and the strain rate \(\dot{\gamma}_{ij} = (\partial_i u_j + \partial_j u_i)\) to

\[
\frac{\sigma_{ij}}{\rho \dot{d}^2} = f(\phi)|\dot{\gamma}| \left[ -|\dot{\gamma}| \delta_{ij} + \mu(\phi) \dot{\gamma}_{ij} + g(\phi) \dot{\gamma}_{il} \delta_{ij} \right],
\]

where \(\mu(\phi)\) is the effective friction coefficient, \(f(\phi)\) is the rescaled dilatancy, and \(g(\phi)\) is the rescaled bulk viscosity.

Note that index summation convention is used here and all through the paper. The behaviour of the two functions \(f\) and \(\mu\) have been studied in the literature. Considering a homogeneous shear flow for which \(\dot{\gamma}_{ill} = 0\) and taking the trace of eq. (1), we see that \(f\) is related to the pressure as \(\mathcal{P} = f(\phi)\rho d^2|\dot{\gamma}|^2\). As shown in the inset of fig. 2(a), it diverges as

\[
f(\phi) = \frac{b^2}{(\phi_c - \phi)^2}
\]

at the critical volume fraction \(\phi_c\) \([18–21]\) \((b\) is a numerical constant). The friction coefficient is related to the shear stress. It is regular in the limit \(\phi \to \phi_c\) and is well approximated by a function of the form

\[
\mu(\phi) = \mu_{c} + \frac{\delta \mu}{1 + ab/(\phi_c - \phi)},
\]

where \(\mu_c\) is the critical friction at the jamming point, and \(a\) and \(\delta \mu\) are two other constants \([35,36]\). Using homogeneous shear flows, we have previously calibrated these functions \(f\) and \(\mu\) for our 2D system and obtained \(\phi_0 \approx 0.817\), \(\mu_c \approx 0.277\), \(\delta \mu \approx 0.57\), \(a \approx 0.36\) and \(b \approx 0.33\) \([21]\). Finally, we hypothesize here that the rescaled bulk viscosity, which has never been measured so far, is regular as well: \(g \to g_c\) as \(\phi \to \phi_c\). Importantly, these rheological laws are the same for 2D and 3D systems, except for the values of the numerical constants.

The compressible Navier-Stokes equations read

\[
\partial_t \phi + u_i \partial_i \phi = -\phi \partial_i u_j,
\]

\[
\rho \phi (\partial_i u_i + u_j \partial_j u_i) = \partial_j \sigma_{ij} + F_i,
\]

where \(F_i\) is a bulk force density. The base state is a homogeneous and steady shear flow, and from now on we
denote its volume fraction by $\phi$, its shear rate by $\dot{\gamma}$ and its pressure by $P$. Plane waves will be studied as linear perturbations of this base state and first order corrections of the volume fraction and the velocity components will be denoted by $\phi^1$ and $u^1_z$.

Let us consider first the case of waves propagating along the transverse $y$-direction, i.e. of the form $\exp(iwt + ik y)$, without any bulk forcing ($\vec{F} = 0$). This case is relevant for 3D systems but, although our simulations are 2D, this is the simplest situation from the theoretical point of view, and for which the dispersion relation is compact. It is then a useful first step before the more interesting case of waves along $z$ addressed afterwards and in appendix A. In order to induce compression, the wave must have a transverse velocity component $u^1_y$, which corresponds to a strain perturbation $\dot{\gamma}^{1}_{yy} = 2iku^1_y$. The continuity equation (4) and the equation of motion (5) projected along $y$ then respectively read

$$i\omega\phi^1 = -\phi ik u^1_y,$$  \hspace{0.5cm} (6)
$$\rho \phi i\omega u^1_y = ikP \left[ -\frac{f'}{\phi} \phi^1 + \frac{2(\mu + g)}{\gamma} iku^1_y \right].$$  \hspace{0.5cm} (7)

In this equation and in the following, it is understood that the functions $f$, $f'$, $g$ and $\mu$ are evaluated at the volume fraction $\phi$ of the base state. These two equations have non-zero solutions only if the following dispersion relation is verified:

$$\frac{\omega^2}{\phi^2|\gamma|^2k^2} = f' + i\frac{2(\mu + g)f}{\phi} \omega. \tag{8}$$

When the imaginary term can be neglected in front of the real one, the wave propagation is found non-dispersive, with a speed

$$c = \frac{\omega}{k} = \sqrt{\frac{1}{\rho} \frac{\partial P}{\partial \phi}} = |\dot{\gamma}|d \sqrt{\frac{f'}{\rho}} \sim \sqrt{\frac{2b|\gamma|}{(\phi_c - \phi)^{3/2}}} \omega. \tag{9}$$

At a fixed shear rate, the speed of sound is therefore found to diverge at the jamming point as $(\phi_c - \phi)^{-3/2}$. The imaginary part of the dispersion relation gives the quality factor $Q$ which, at the jamming point scales as

$$Q = \frac{\phi f'}{2(\mu + g)f} \frac{\dot{\gamma}}{\omega} \sim \left(1 - \frac{\phi_c}{\phi} \right) \frac{\dot{\gamma}}{\omega}. \tag{10}$$

This term diverges as $(\phi_c - \phi)^{-1}$, which means that the propagation of this transverse mode becomes asymptotically non-dissipative.

We now proceed similarly for modes propagating along the $z$-direction, i.e. of the form $\exp(iwt + ik z)$. In this case, the perturbations of the strain rate components are $\dot{\gamma}^{1}_{zz} = ik u^1_z$ and $\dot{\gamma}^{1}_{zz} = 2iku^1_z$. The Navier-Stokes equations then read

$$i\omega\phi^1 + \phi ik u^1_z = 0,$$  \hspace{0.5cm} (11)
$$\rho \phi i\omega u^1_z = ikP \left[ -\frac{f'}{\phi} \phi^1 + \frac{2(\mu + g)}{\gamma} iku^1_z \right].$$  \hspace{0.5cm} (12)

This system can be solved to derive the dispersion relation (see eq. (A.1) in appendix A), which presents two branches with the following properties. Approaching the jamming transition $\dot{\gamma} \rightarrow \dot{\gamma}_c$, the first of these two modes is a wave which propagates at a speed equal to $c$ (eq. (9)). It presents an elliptical polarization (see eq. (A.2)) described by

$$\frac{u^1_z}{u^1_y} = -\mu_c + i\frac{\dot{\gamma}}{\omega}, \tag{14}$$

and a quality factor diverging at the jamming point as

$$Q \sim \frac{2}{(2\mu_c + g_c)(\phi_c/\phi - 1)} \frac{\dot{\gamma}}{\omega}. \tag{15}$$

The second mode is critically damped and presents a diffusive dispersion relation ($k^2 \propto \omega$), which, close to jamming, writes:

$$kd \sim \pm 1 - i\frac{1}{\sqrt{2}} \sqrt{\frac{a\phi_c w}{b\delta(\mu)\phi_c - \phi}}. \tag{16}$$

As it is highly dissipative, this shear mode cannot be observed. Finally, a plane wave emitted along the $x$-axis would rotate due to convection by the base velocity field $u_x = \dot{\gamma} z$. This effect can be alternatively seen as a refraction by an index gradient.

We can now compare the prediction of this theory to the numerical data. Figure 2(a) shows that the spontaneous angular frequency coincides, without any adjustable parameter, to the resonant mode $2\pi c/H$ deduced from eq. (9). In order to compute the resonance curves, a sinusoidal bulk force density $\vec{F}$ must be added to the right-hand side of eq. (5). A system of equations similar to (11)–(13) but with the corresponding additional forcing term can be solved for the velocity response $u^1_z$ —see appendix B for a forcing along $z$ and appendix C for a forcing along $x$. These resonance curves depend on the second viscosity $g$, which is the single adjustable parameter. The overall fit of the data by these prediction is very good, both for $z$ and $x$ forcing (figs. 3 and 4, respectively), surprisingly even at frequencies much larger than the shear rate $\dot{\gamma}$. As shown in these figures, the resonant frequency, the phase curve and the tails of the amplitude response are in fact largely insensitive to the value of $g$. The refined fit of the response amplitude around the resonance for different volume fractions is consistent with a roughly constant bulk viscosity: $\gamma(\phi) \simeq g_c \simeq 2$.

**Conclusion.** — The dependence of the normal stress on volume fraction and shear rate does lead to a dynamic compressibility. Granular shear flows can propagate waves in transverse and sagittal modes, which are specific to the dense liquid granular regime: they neither depend on the restitution coefficient nor on the elasticity of the grains.
As these modes become asymptotically non-dissipative at the jamming point, they could in principle be observed experimentally and could be used to probe the bulk of granular flows.

The existence of a dynamic compressibility has important consequences for Navier-Stokes granodynam-ics. Compressibility effects cannot be neglected when the Mach number

$$M = \frac{\gamma H}{c} \sim \left(\frac{P}{\rho_0^2 a^2}\right)^{3/4} \frac{H}{d},$$

(18)

based on the system height, approaches or exceeds 1 (here $M \approx 0.5$ in fig. 1). As the granular fluid viscosity depends on pressure, one may even expect a coupling of the mean flow with vibrations at low Mach numbers [13].

The theoretical description derived here, based on the assumption that the granular rheology is local, is able to reproduce quantitatively the entire behavior of the resonance curves. In particular, the resonance angular frequency is well captured and found to be equal to the natural angular frequency $\omega^{\ast}$ (i.e. without forcing). The agreement between the predictions and the data is becoming progressively less good at frequencies lower than $\omega^{\ast}$ when volume fraction $\phi$ is closer to its critical value $\phi_c$. This can be an effect of non-locality, which becomes increasingly important when $\phi \to \phi_c$ [37]. It could also be due to the remissiveness elastic modes from the solid side of jamming, for $\phi > \phi_c$, which appears close to the transition.

Finally, this study suggests that the bulk viscosity $g(\phi)$ can play an interesting role in granular dense flows, but more work is clearly needed to measure and characterise this function, which requires further investigations of non-homogeneous flows.

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We thank E. Clément for discussions. This work is funded by ANR JamVibe. BA is supported by Institut Universitaire de France.

Appendix A: waves along $z$. – Equations (11)–(13) lead to the following dispersion relation for plane waves along the $z$-direction:

$$4P \frac{k^2}{\omega^2} = \frac{\gamma}{\omega} \frac{\phi}{z} - \mu \frac{\gamma}{\omega} \frac{\phi f'}{f} + 2i(\mu + g)$$

$$\times \left\{ i \frac{\gamma}{\omega} \left(2 + \frac{f'}{f}\right) - 2(2\mu + g) \right\}$$

$$\pm i \left\{ \frac{\gamma}{\omega} \left(8i\phi f + \frac{\gamma}{\omega} \left(2 + \frac{f'}{f}\right)^2 + 16i\mu \right) \right\}$$

$$+ 4i \frac{\gamma}{\omega} \left(2 + \frac{f'}{f}\right) g - 4g^2 \right\}^{1/2},$$

(A.1)

with the following polarization:

$$\frac{u_k}{u_z} = \frac{1}{4} \left\{ - i \frac{\gamma}{\omega} \left(2 + \phi\frac{f'}{f}\right) + 2g \right. \right.$$  

$$\mp i \left[ \frac{\gamma}{\omega} \left(8i\phi f + \frac{\gamma}{\omega} \left(2 + \phi\frac{f'}{f}\right)^2 + 16i\mu \right) \right. \right.$$  

$$+ 4i \frac{\gamma}{\omega} \left(2 + \phi\frac{f'}{f}\right) g - 4g^2 \right\}^{1/2} \}.$$

(A.2)

Appendix B: forcing along $z$. – With an additional forcing term $F_z$ in the RHS of (13), eqs. (11)–(13) give the following expressions for the velocity response:

$$\frac{u_k^1 k^2 P}{F_z \gamma} = \frac{1}{D} k^2 P \gamma (i\omega \rho \phi + ik^2 \phi f P),$$

(B.1)

$$\frac{u_z^1 k^2 P}{F_z \gamma} = \frac{1}{D} k^2 P \omega (-i\gamma \rho \phi - 2k^2 \mu P),$$

(B.2)

with

$$D = \gamma^2 \omega^2 \rho^2 \phi^2 - i\kappa k^2 \omega \left(2\gamma + i\gamma \phi f' + 4i\omega \mu \right) \rho \phi P$$

$$- 2k^4 \left[ i\gamma \phi f + \mu \left(-i\gamma \phi f' + \omega (g + 2\mu) \right) P \right]^2.$$  

(B.3)

Appendix C: forcing along $x$. – In the case of an additional forcing term $F_x$ in the RHS of (12), we get

$$\frac{u_k^1 k^2 P}{F_x \gamma} = - \frac{1}{D} ik^2 P \left[ i\gamma \omega^2 \rho \phi \right.$$

$$- k^2 \left( i\gamma \phi f' + i\omega (g + 2\mu) \right) \phi P \right],$$

(C.1)

$$\frac{u_z^1 k^2 P}{F_x \gamma} = \frac{1}{D} 2k^4 \omega P^2.$$  

(C.2)

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