Virtually free groups are $p$-Schatten stable

Maria Gerasimova    Konstantin Shchepin

Abstract

In [8] was proven that finitely generated virtually free groups are stable in permutations. In this note we show that a similar strategy can be used to prove that finitely generated virtually free groups are stable with respect to a normalized $p$-Schatten norm for $1 \leq p < \infty$. In particular, this implies that virtually free groups are Hilbert-Schmidt stable.

1 Introduction

Given a system of equations in noncommutative variables one can ask if it is “stable”, meaning that each of its “almost” solutions is “close” to an actual solution. For variables $s_i$ and equations of the form $\prod_{k} s_{i_k} = 1$ this “stability” property is a property of the group generated by $s_i$ with relations given by these equations – it does not depend on the sets of generators and relations. To make this into a concrete problem one needs to specify what exactly is considered as a solution and what “almost” and “close” mean. For example, as a solution one can consider a set of permutations or a set of matrices satisfying some additional conditions. In this paper we will focus on the set of unitary matrices of arbitrary size equipped with a normalized $p$-Schatten norm for $1 \leq p < \infty$. When $p = 2$ this norm is called the Hilbert-Schmidt norm.

The systematic study of stability with respect to the Hilbert-Schmidt norm was taken in [6], where in particular was given a characterization of stability of amenable groups in terms of the approximation property for characters. More precisely, it was proven that an amenable group is stable with respect to the Hilbert-Schmidt norm if and only if each character is a pointwise limit of traces of finite-dimensional representations. Using this characterization it was shown that all finitely generated virtually abelian groups as well as the discrete Heisenberg group $H^3$ are Hilbert-Schmidt stable. For non-amenable groups this approximation property was shown to be only a necessary condition. In particular, $SL_3(\mathbb{Z})$ satisfies this property, but is not Hilbert-Schmidt stable (see [6],[2]).

Let us observe that Hilbert-Schmidt stability is preserved under taking free products (obviously) and direct products with Hilbert-Schmidt stable amenable groups (see [7]). Other examples of Hilbert-Schmidt stable groups include all one-relator groups with non-trivial center (see [6]).

In [8] was proven that finitely generated virtually free groups are stable in permutations. Although the normalized Hamming distance can be expressed using the Hilbert-Schmidt distance, this cannot be used directly to deduce Hilbert-Schmidt stability from stability in permutations. Despite this, we show that the similar strategy can be used to prove that finitely generated virtually free groups are stable with respect to a $p$-Schatten norm for $1 \leq p < \infty$. In particular, they are Hilbert-Schmidt stable.

Let us mention that in [4] was shown that for an amenable group operator norm stability implies Hilbert-Schmidt stability. Virtually free groups are known to be operator norm stable, but since they are not amenable it does not give us Hilbert-Schmidt stability.
2 Preliminaries

2.1 $p$-Schatten stability of finite groups

Let $W$ be some finite-dimensional Hilbert space and let us fix some $1 \leq p < \infty$.

**Definition 1.** For any operator $A : W \to W$ let us define the $p$-Schatten norm by

$$||A||_p := (\text{tr}(|A|^p))^{\frac{1}{p}},$$

where $|A| = \sqrt{A^*A}$.

**Definition 2.** For any operator $A : W \to W$ let us define the normalized $p$-Schatten norm by

$$||A||'_p := \left(\text{tr} \left(\frac{|A|^p}{\dim W}\right)\right)^\frac{1}{p}.$$

Let us note that $|| \cdot ||_2$ is called the Frobenius norm and $|| \cdot ||'_2$ is called the Hilbert-Schmidt norm.

Recall that any operator $A : W \to W$ can be decomposed as

$$A = U^* \Lambda V,$$

where $U, V \in U(W)$ and $\Lambda = \text{diag}(\lambda_i)$, $\lambda_i \in \mathbb{R}_{\geq 0}$. $\lambda_i$ are called singular values of $A$ and they are uniquely defined up to a permutation.

One can compute the $p$-Schatten norm as

$$||A||_p = \left(\sum_i \lambda_i^p\right)^\frac{1}{p}.$$  

From now on let $G = \langle S, R \rangle$ be a finitely presented group, $F_S$ be the free group generated by $S$. We recall some basic definitions.

**Definition 3.** A sequence of homomorphisms $\varphi_n : F_S \to U(W_n)$, where $W_n$ is a sequence of finite-dimensional Hilbert spaces, is called an asymptotic homomorphism of $G$ with respect to the normalized $p$-Schatten norm if for all $r \in R$

$$\lim_n ||\varphi_n(r) - I_n||'_p = 0,$$

where $I_n$ is an identity operator on $W_n$.

**Definition 4.** We will say that a group $G$ is stable with respect to the normalized $p$-Schatten norm if for any asymptotic homomorphism $\rho_n : F_S \to U(W_n)$ there exists a sequence of representations $\rho'_n : G \to U(W'_n)$ such that for all $s \in S$

$$\lim_n ||\rho_n(s) - \rho'_n(s)||'_p = 0.$$

**Definition 5.** We will say that a group $G$ is flexibly stable with respect to the normalized $p$-Schatten norm if for any asymptotic homomorphism $\rho_n : F_S \to U(W_n)$ there exists a sequence of representations $\rho'_n : G \to U(W'_n)$, $W_n \subset W'_n$ such that for all $s \in S$

$$\lim_n ||\rho_n(s) - P\rho'_n(s)P||'_p = 0,$$

where $P$ is an orthogonal projection $P : W'_n \to W_n$ and $\frac{\text{dim}(W'_n)}{\text{dim}(W_n)} \to 1$ as $n \to \infty$. 

2
Flexible stability of finite groups with respect to a normalized $p$-Schatten norm for $1 \leq p \leq 2$ follows from Theorem 6.9 in [5]. Later the stability result of [5] was generalized in [3] to the class of amenable groups with respect to unitary groups of von Neumann algebras equipped with any unitarily invariant, ultraweakly lower semi-continuous semi-norm. In particular, this gives flexible stability of finite groups with respect to a normalized $p$-Schatten norm for $1 \leq p < \infty$.

One can show that in the case of finite groups flexible stability with respect to a $p$-Schatten norm implies stability. For $p = 2$ it follows from Theorem 3.2 in [1]. This theorem can be generalized for all $1 \leq p < \infty$ using the basic inequality $||BAC^*||_p \leq ||A||_p ||B||_{op} ||C||_{op}$, where $A \in M_{n \times m}(C)$ and $B, C \in M_{n \times m}(C)$ (see Lemma 6.1 in [5]).

**Corollary 1.** Any finite group is stable with respect to a normalized $p$-Schatten norm for $1 \leq p < \infty$.

### 2.2 Graph of groups

We will briefly recall the main definitions and notations which we will use further. Using Serre’s notation for graphs, we will say that a graph $\Gamma$ consists of a set vertices $V(\Gamma)$ and a set of oriented edges $E(\Gamma)$, moreover, each edge has an origin $o(e) \in V(\Gamma)$ and a terminus $t(e) \in V(\Gamma)$ and admits a distinct opposite edge $\vec{e} \in E(\Gamma)$ that satisfies $\vec{e} = e$, $t(\vec{e}) = o(e)$ and $o(\vec{e}) = t(e)$. In this notation an orientation of the graph $\Gamma$ is just a subset $\vec{E}(\Gamma) \subset E(\Gamma)$ containing exactly one edge from each pair of opposite edges $\{e, \vec{e}\}$.

**Definition 6.** A graph of groups $\mathcal{G}$ is

$$\mathcal{G} = \{\Gamma, \{G_e\}_{e \in V(\Gamma)}, \{G_{\vec{e}}\}_{e \in E(\Gamma)}, \{i_e : G_e \to G_{t(e)}\}_{e \in E(\Gamma)}\}$$

where $\Gamma$ is a connected graph, $G_e$ and $G_{\vec{e}}$ - vertex and edge groups correspondingly with $G_e = G_{\vec{e}}$ and $i_e : G_e \to G_{t(e)}$ are injective homomorphisms.

Let $\mathcal{G}$ be a graph of groups. Fix a subtree $T \subset \Gamma$ and an orientation $\vec{E}(\Gamma)$, this gives the induced orientation $\vec{E}(T)$. Consider the group $\pi_1(\mathcal{G}, T)$ defined as the free product

$$\pi_1(\mathcal{G}, T) = \ast_{e \in V(\Gamma)} G_e \ast F\left(\{s_e\}_{e \in \vec{E}(\Gamma)}\right)$$

We will also fix the generating set of $\pi_1(\mathcal{G}, T)$

$$S_\mathcal{G} = \bigcup_{e \in V(\Gamma)} G_e \cup \{s_e\}_{e \in \vec{E}(\Gamma)}.$$ 

**Definition 7.** The fundamental group $\pi_1(\mathcal{G}, T)$ of the graph of groups $\mathcal{G}$ with respect to the subtree $T$ is the quotient of the free product $\pi_1(\mathcal{G}, T)$ by the normal subgroup generated by the relations

$$R_\mathcal{G} = \left\{ s_e, s_e^{-1} i_e(g_e)s_e (i_{\vec{e}}(g_e))^{-1} \quad \forall e \in \vec{E}(T), \quad \forall e \in \vec{E}(\Gamma), \quad g_e \in G_e. \right\}$$

In the definition above one can use any subtree $T$ of $\Gamma$ but for us $T$ will always be a maximal spanning tree because of the following results.

**Remark 1.** The fundamental group $\pi_1(\mathcal{G}, T)$ as well as the group $\pi_1(\mathcal{G}, T)$ are independent of the choice of a maximal spanning tree $T$ up to isomorphism.

**Theorem 1** (Stallings, [9]). The fundamental group $\pi_1(\mathcal{G}, T)$ of a finite graph of groups $\mathcal{G}$ with finite vertex groups with respect to any maximal spanning tree $T$ is virtually free. Any finitely generated virtually free group can be constructed this way.
2.3 Stable epimorphisms

One of the main definitions of \([8]\) was one of a \(P\)-stable epimorphism. We will need a natural analogue of this definition. Let \(\overline{G}\) be a group generated by a finite set \(S\), \(N \trianglelefteq \overline{G}\) be a normal subgroup normally generated by some finite set \(R \subset \overline{G}\). Denote \(G = \overline{G}/N\). We say that a representation \(\rho: \overline{G} \to U(W)\) is a \(\delta\)-almost \(G\)-representation if for all \(r \in R\)

\[||\rho(r) - I||_p < \delta.\]

For two maps \(\rho_1, \rho_2: \overline{G} \to U(W)\) and a subset \(A \subset \overline{G}\) we will denote by

\[d_A(\rho_1, \rho_2) = \max_{g \in A} ||\rho_1(g) - \rho_2(g)||_p.\]

**Definition 8.** We will say that the epimorphism \(\phi: \overline{G} \to G\) is stable with respect to the normalized \(p\)-Schatten norm if for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that for every \(\delta\)-almost \(G\)-representation \(\rho: \overline{G} \to U(W)\) there is a \(G\)-representation \(\rho': G \to U(W)\) with \(d_S(\rho, \phi^*(\rho')) < \varepsilon\).

For \(\pi_1(G, T)\) we will always use the generating set \(S_G\) and the set of relations \(R_G\). Similarly to \([8]\) one can get the following lemma.

**Lemma 1.** The stability of the epimorphism is a well-defined notion (i.e. it is independent of the choices of the finite sets \(S\) and \(R\)).

This definition is motivated by the following fact.

**Remark 2.** If \(\overline{G}\) and an epimorphism \(\phi: \overline{G} \to G\) are stable with respect to some norm then \(G\) is stable with respect to the same norm.

In our case all vertex groups of a graph of groups are finite and hence are stable with respect to the normalized \(p\)-Schatten norm. \(\pi_1(G, T)\) is stable as a free product of stable groups. So to prove stability of virtually free groups we need only the following theorem.

**Theorem 2.** The epimorphism \(\pi_1(G, T) \to \pi_1(G, T)\) is stable with respect to a \(p\)-Schatten norm for all \(1 \leq p < \infty\).

3 Stability of virtually free groups

As we mentioned already, the strategy of proving that virtually free groups are stable with respect to the normalized \(p\)-Schatten norm is very similar to the strategy used in \([8]\) to prove that virtually free groups are stable in permutations. Although the statements will look similar, the proofs will be vastly different.

3.1 On “close” representations of finite groups

In \([5]\) was proven that two representations of a finite group \(G\) which are close to each other with respect to the normalized \(p\)-Schatten norm are almost isomorphic and there is an intertwining operator close to identity. We will use a little bit more data hidden in the proof of this theorem and will change its statement accordingly (see Theorem 7.3 in \([5]\)).

**Theorem 3** (Gowers, Hatami). Let \(\rho_1, \rho_2: G \to U(W)\) be two representations of a finite group \(G\) such that \(d_G(\rho_1, \rho_2) \leq \delta\). Then for \(i = 1, 2\) there exist \(\rho_i\)-invariant subspaces \(V_i\) of dimension at least \((1 - (2\delta)^p) \cdot \dim W\) and an operator \(T\) such that \(||T - I||_p \leq 3\delta\) and \(\rho_2(x) \cdot T = T \cdot \rho_1(x)\) for every \(x \in G\). Moreover, \(T\) maps an orthonormal basis of \(V_1\) to an orthonormal basis of \(V_2\) and \(V_1^\perp\) to 0.
We will prove the following corollary which we will use further.

**Corollary 2.** If moreover $\rho_1$ and $\rho_2$ are isomorphic, then there exists non-degenerate intertwining operator $T' \in U(W)$ such that $||T' - I||_p \leq 5\delta$.

**Proof.** $T$ establishes an isomorphism between representations $\rho_1|_{V_1}$ and $\rho_2|_{V_2}$. Since $\rho_1$ and $\rho_2$ are also isomorphic, $\rho_1|_{V_1^\perp} : G \to U(V_1^\perp)$ is isomorphic to $\rho_2|_{V_2^\perp} : G \to U(V_2^\perp)$. Let $S : V_1^\perp \to V_2^\perp$ be any intertwining operator between these representations which maps an orthonormal basis to an orthonormal basis. We can extend $S$ to $S' : W \to W$ by $S'(V_1) = 0$. An operator $T' := T + S'$ maps an orthonormal basis of $W$ to an orthonormal basis of $W'$, hence $T'$ is a unitary intertwining operator. Singular values of $S'$ are either 0 or 1 and at most $\dim(V_1^\perp)$ of them are equal to 1. But $\dim(V_1^\perp) \leq (2\delta)^p \cdot \dim W$ and hence we get:

$$||T' - I||_p' \leq ||T - I||_p' + ||S'||_p' \leq 3\delta + ((2\delta)^p)^\frac{1}{p} = 5\delta.$$ 

\[\square\]

We will also need the following lemma that might be considered as a converse statement to Theorem 3.

**Lemma 2.** Let $G$ be a finite group, $\rho : G \to U(W_1)$ and $\sigma_1, \sigma_2 : G \to U(W_2)$ be some representations. If for some $0 < \delta < 1$ we have $\dim(W_2) \leq \delta^p \cdot \dim(W_1 \oplus W_2)$, then $d_G(\rho \oplus \sigma_1, \rho \oplus \sigma_2) \leq 2\delta$.

**Proof.** For any $g \in G$, $(\rho \oplus \sigma_1)(g) - (\rho \oplus \sigma_2)(g)$ has at most $\dim(W_2)$ nonzero singular values and all of them are not greater than 2. Hence,

$$||[(\rho \oplus \sigma_1)(g) - (\rho \oplus \sigma_2)(g)]||_p \leq \left(\frac{2^p \cdot \dim(W_2)}{\dim(W_1) + \dim(W_2)}\right)^\frac{1}{p} \leq 2\delta.$$ 

\[\square\]

### 3.2 Cones and representations

Since the representation theory of finite groups is nice, we can introduce cones of isomorphism classes of finite-dimensional representations, which are similar to the cones of isomorphism classes of actions on finite sets introduced in [8].

**Definition 9.** Let $\text{Repr}(G)$ be the set of isomorphism classes of finite-dimensional unitary representations of a group $G$ and $\text{Irr}(G)$ be the set of isomorphism classes of irreducible finite-dimensional unitary representations of $G$.

**Definition 10.** Let $\Theta_G := \bigoplus_{\pi \in \text{Irr}(G)} \mathbb{Z} \pi$ be the free $\mathbb{Z}$-module with basis $\text{Irr}(G)$ and let $\Theta^+ \subset \Theta_G$ be its non-negative cone.

Since every unitary representation is just a sum of irreducible representations, $\text{Repr}(G)$ can be identified with $\Theta^+_G$. For a representation $\rho : G \to U(W)$ let us denote by $\rho^\#$ the corresponding element of $\Theta^+_G$.

We will consider a norm on $\Theta_G$, defined by

$$||\lambda||_G = \sum_{\pi \in \text{Irr}(G)} |\lambda_{\pi}| \cdot \dim \pi.$$ 

For a representation $\rho : G \to U(W)$, $||\rho^\#||_G = \dim(W)$. For any $\lambda \in \Theta^+_G$ and any Hilbert space $W$ with $\dim(W) = ||\lambda||_G$ there is a representation $\rho : G \to U(W)$ with $\rho^\# = \lambda$. 

5
Given a homomorphism \( i: H \to G \) and a representation \( \rho: G \to U(W) \) one can consider the pullback representation \( i^* (\rho): H \to U(W) \). This gives a linear map \( i^# : \Theta_G \to \Theta_H \) which maps \( \rho^\# \) to \( (i^*(\rho))^\# \).

For a graph of groups
\[
\mathcal{G} = (\Gamma, \{ G_e \}_{e \in V(\Gamma)}, \{ G_e \}_{e \in E(\Gamma)}, \{ i_e : G_e \to G_{\ell(e)} \}_{e \in E(\Gamma)})
\]
we define \( \mathbb{Z} \)-modules
\[
\Theta_V = \bigoplus_{e \in V(\Gamma)} \Theta_{G_e} \quad \text{and} \quad \Theta_E = \bigoplus_{e \in E(\Gamma)} \Theta_{G_e}
\]
and the corresponding positive cones
\[
\Theta_V^+ = \bigoplus_{e \in V(\Gamma)} \Theta_{G_e}^+ \quad \text{and} \quad \Theta_E^+ = \bigoplus_{e \in E(\Gamma)} \Theta_{G_e}^+.
\]

Let us note that, since \( G_e = G_{e\sigma} \) module \( \Theta_E \) contains each \( \Theta_{G_e} \) twice. It is not essential, but will allow us to simplify notations. It will be convenient to consider these \( \mathbb{Z} \)-modules with the norms
\[
\| \cdot \|_V = \frac{1}{|V(\Gamma)|} \sum_{e \in V(\Gamma)} \| \cdot \|_{G_e} \quad \text{and} \quad \| \cdot \|_E = \frac{1}{|E(\Gamma)|} \sum_{e \in E(\Gamma)} \| \cdot \|_{G_e}.
\]

For a representation \( \rho: \pi_1(\mathcal{G}, T) \to U(W) \) (or \( \rho: \pi_1(\mathcal{G}, T) \to U(W) \)) we can define \( \rho^\# \in \Theta_V^+ \) by
\[
(\rho^\#)_v = (\rho|_{G_v})^\#
\]
for \( v \in V \).

Let
\[
d_{\mathcal{G}}|_{\Theta_{G_e}} = \sum_{e \in \ell(e) = v} i_e^\# - \sum_{e \in \varrho(e) = v} i_e^\#
\]

**Proposition 1.** For any representation \( \rho: \pi_1(\mathcal{G}, T) \to U(W) \) we have \( \rho^\# \in \Theta_V^+ \cap \ker d_{\mathcal{G}} \). For any \( \lambda \in \Theta_V^+ \cap \ker d_{\mathcal{G}} \) there exists some representation \( \rho: \pi_1(\mathcal{G}, T) \to U(W) \) with \( \rho^\# = \lambda \).

**Proof.** For any representation \( \rho: \pi_1(\mathcal{G}, T) \to U(W) \) and any \( e \in E(\Gamma) \) representations \( i^*_e (\rho|_{G_{\ell(e)}}) \) and \( i^*_e (\rho|_{G_{\varrho(e)}}) \) of \( G_e \) are conjugated by \( \rho(s_e) \), hence
\[
(\mathcal{D}_{\mathcal{G}}(\rho^\#))_{e} = i^*_e (\rho^\#_{\ell(e)}) - i^*_e (\rho^\#_{\varrho(e)}) = (i^*_e (\rho|_{G_{\ell(e)}}))^\# - (i^*_e (\rho|_{G_{\varrho(e)}}))^\# = 0
\]
and \( \rho^\# \in \ker d_{\mathcal{G}} \cap \Theta_V^+ \).

Proof of the second part has the same flavor as the proof of Remark 1. Let us note that for any \( \lambda \in \Theta_V^+ \cap \ker d_{\mathcal{G}} \), \( ||\lambda||_V = ||\lambda_v||_{G_v} \) for any \( v \in V \), since \( \Gamma \) is connected. Let us fix a Hilbert space \( W \) with \( \dim(W) = ||\lambda|| \). We will construct a representation \( \rho: \pi_1(\mathcal{G}, T) \to U(W) \) and show that relations \( R_{\chi} \) hold. At first we will construct representations \( \rho_v \) of \( G_v \) inductively using the spanning tree \( T \).

**Basis.** Let us fix some \( v \in V(\Gamma) \) and some representation \( \rho_v \) of \( G_v \) with \( \rho_v^\# = \lambda_v \).

**Induction step.** Assume that for some \( e \in E(\Gamma) \) we already have \( \rho_{\ell(e)}^\# = \lambda_{\ell(e)} \). Let us fix a representation \( \rho': G_{\ell(e)} \to U(W) \) with \( \rho'^\# = \lambda_{\ell(e)} \). Since
\[
(i^*_e (\rho_{\ell(e)}))^\# = i^*_e (\lambda_{\ell(e)}) = i^*_e (\lambda_{\varrho(e)}) = (i^*_e (\rho'))^\#,
\]
these representations are isomorphic and there is a unitary intertwining operator \( S \) such that \( i^*_e (\rho_{\ell(e)})(g_v) \cdot S = S \cdot i^*_e (\rho')(g_v) \) for all \( g_v \in G_v \). Now we can define \( \rho_{\theta(e)} := \text{ad}(S^{-1})(\rho') \), where

6
(ad(L)(ρ))(g) = L^{-1} · ρ(g) · L. Let us note that \( i^*_c(ρ_{t(c)}) = i^*_c(ρ_{o(c)}) \).

Now we will construct our representation on elements \( s_e, e ∈ E(Γ) \). For \( e ∈ E(Γ) \) we define \( ρ_e = I \). For \( e ∈ E(Γ) \setminus E(Γ) \) similar to the induction step we acquire unitary operators \( ρ_e \) such that \( i^*_c(ρ_{t(c)})(g_e) · ρ_e = ρ_e · i^*_c(ρ_{o(c)})(g_e) \) for all \( g_e ∈ G_e \). Collections of \( ρ_e \) and \( ρ_e \) define the representation \( ρ: π_1(G, T) → U(W) \) with \( ρ(s_e) = ρ_e \). The relations \( R_G \) hold by the construction. This finishes the proof.

We will write \( A ∼_X B \) if \( A < C · B \) for some constant \( C \), which depends only on \( X \).

Let us state a general fact about cones (see Lemma 5.3 in [8]) only for \( Θ^+ \).

**Lemma 3** (Lazarovich, Levitt). For any \( λ ∈ Θ^+_V \) there exists \( λ'' ∈ Θ^+_V \cap ker d_G \) satisfying \( ||λ − λ''||_V ∼_G ||d_Gλ||_E \) and \( ||λ''||_V ≤ ||λ||_V \).

### 3.3 From representations to cones and back to representations

**Proposition 2.** Let \( ρ : π_1(G, T) → U(W) \) be a unitary representation. If \( ρ \) is a \( δ \)-almost \( π_1(G, T) \)-representation for some \( δ < 1 \), then

\[
||d_G(ρ^#)||_E ∼ δ^p · ||ρ^#||_V.
\]

**Proof.** Let us fix an oriented edge \( e ∈ E(Γ) \). By the definition of a \( δ \)-almost \( π_1(G, T) \)-representation and by the unitary invariance of the normalized \( p \)-Schatten norm we have

\[
||ρ(s_e^{-1}i_e(g)s_e) − ρ(i_e(g))||_p < δ
\]

for any group element \( g ∈ G_e \). For convenience we will write \( ρ_e \) instead of \( ρ|_{G_e} \). We can apply Theorem 3 to the representations \( ad(ρ(s_e))(i^*_c(ρ_{t(c)})) \) and \( i^*_c(ρ_{o(c)}) \). It gives us isomorphic subrepresentations of these representations of dimension at least \( (1 − (2δ)^p) · dim W \). Since \( ad(ρ(s_e))(i^*_c(ρ_{t(c)})) \) and \( i^*_c(ρ_{t(c)}) \) are conjugated, \( (ad(ρ(s_e))(i^*_c(ρ_{t(c)})))^# = (i^*_c(ρ_{t(c)}))^# \) and

\[
||d_G(ρ^#)||_G_e = ||i^*_c(ρ^#) − i^*_c(ρ^#)||_G_e = ||(i^*_c(ρ_{o(c)}))^# - (i^*_c(ρ_{o(c)}))^#||_G_e =
\]

\[
= ||(ad(ρ(s_e))(i^*_c(ρ_{t(c)})))^# - (i^*_c(ρ_{o(c)}))^#||_G_e ≤ (2δ)^p · dim W ∼ δ^p · ||ρ^#||_V.
\]

This finishes the proof.

**Lemma 4.** Let \( i: H → G \) be a homomorphism of finite groups. Let \( τ: H → U(W) \) and \( ρ: G → U(W) \) be a pair of representations. Denote \( λ = ρ^# \). If \( λ' ∈ Θ^+_G \) and \( δ > 0 \) are such that

1. \( d_H(i^*(ρ); τ) < δ \),
2. \( i^*(λ') = τ^# \),
3. \( ||λ − λ'||_G ≤ δ^p · ||λ||_G \),

then there exists a unitary representation \( ρ': G → U(W) \) satisfying

1. \( i^*(ρ') = τ \),
2. \( (ρ')^# = λ' \),
3. \( d_G(ρ', ρ) < δ \).
Proof. We will denote by $\lambda_1 \in \Theta^+_G$ the common part of $\lambda$ and $\lambda'$, that is the vector $\lambda_1$ with 
$$(\lambda_1)_\pi = \min(\lambda_\pi, \lambda'_\pi)$$ 
for $\pi \in \text{Irr}(G)$. Then
$$||\lambda - \lambda'||_G = 2||\lambda - \lambda_1||_G = 2||\lambda' - \lambda_1||_G.$$  
Let us take a $\rho(G)$-invariant subspace $V_1$ such that $(\rho|_{V_1})^# = \lambda_1$. Let us define a new representation $\sigma: G \to U(V^{+}_1)$ such that $\sigma^# = \lambda' - \lambda_1 \in \Theta^+_G$. Now we can construct $\rho_1: G \to U(W)$ as a sum $\rho_1 := \rho|_{V_1} \oplus \sigma$. Let us note that $\rho_1^# = \lambda'$ and, since $2 \dim(V^{+}_1) \leq \delta^p \cdot \dim(W)$, by Lemma 2
$$d_G(\rho, \rho_1) \leq 2\delta.$$  
Since $\rho_1|_H$ and $\tau$ are two isomorphic representations of $H$ and
$$d_H(i^*(\rho_1), \tau) \leq d_H(i^*(\rho_1), i^*(\rho)) + d_H(i^*(\rho), \tau) \leq d_G(\rho_1, \rho) + \delta \leq 3\delta,$$
we can apply Corollary 2. We get an operator $T \in U(W)$ such that $||T - I||_p^* < \delta$ and $ad(T)(\rho_1|_H) = \tau$. Now we can define $\rho'$ by $\rho' := ad(T)(\rho_1)$. We have $i^*(\rho') = \tau$, $(\rho')^# = \lambda'$ and
$$d_G(\rho_1, \rho') \leq 2||T - I||_p^*,$$
hence
$$d_G(\rho', \rho) \leq d_G(\rho', \rho_1) + d_G(\rho_1, \rho) < \delta.$$  
This finishes the proof. \qed

**Proposition 3.** Let $\rho: \pi_1(G, T) \to U(W)$ be a $\delta$-almost $\pi_1(G, T)$-representation with $\lambda = \rho^#$. Let $\lambda' \in \Theta^+_G$ be any vector with $||\lambda'||_V = ||\lambda||_V$. If
1. $\lambda' \in \ker d_G$,
2. $||\lambda - \lambda'||_V \leq \delta^p \cdot ||\lambda||_V$,
then there is a representation $\rho': \pi_1(G, T) \to U(W)$ satisfying
1. $(\rho')^# = \lambda'$,
2. $d_{S_2}(\rho, \rho') <_\delta \delta$.

**Proof.** We will construct a representation $\rho'$ of $\pi_1(G, T)$ such that $\rho'(R_G) = I$, so $\rho'$ can be considered as a representation of $\pi_1(G, T)$. Let us note that for each $v \in V(\Gamma)$ we have $||\lambda_\pi - \lambda'_\pi||_{G_v} \leq \delta^p \cdot ||V(\Gamma)|| \cdot ||\lambda_\pi||_G$, and for each $e \in E(\Gamma)$ we have $i^e_\#(\lambda_\pi) = i^e_\#(\lambda'_\pi)$. We will start by constructing representations $\rho'_e$ of $G_v$ with $d_{G_v}(\rho_v, \rho'_e) < \delta$ and $(\rho'_e)^# = \lambda'_e$ inductively using the spanning tree $T$.

**Basis.** Let us fix a vertex $v \in V(\Gamma)$. We can apply Lemma 4 to the trivial inclusion of a one-element group into $G_v$, to the trivial representation of a one-element group, to $\rho_v$, and to $\lambda'_e \in \Theta^+_G$. We get a representation $\rho'_e: G_v \to U(W)$ such that $d_{G_v}(\rho_v, \rho'_e) < \delta$ and $(\rho'_e)^# = \lambda'_e$.

**Induction step.** Assume that $\rho'_t(e): G_t(e) \to U(W)$ is already defined, but $\rho'_{o(e)}$ is not yet defined for some $e \in E(\Gamma)$. By the induction assumption we know that $d_{G_{t(e)}}(\rho_{t(e)}, \rho'_{t(e)}) < \delta$ and $(\rho'_{t(e)})^# = \lambda'_{t(e)}$. To construct $\rho'_{o(e)}$ we will apply Lemma 4 to the inclusion $i^e_{t(e)}: G_e \to G_{t(e)}$, to the representation $i^*_e(\rho'_{t(e)})$ of $G_e$, to $\rho_{o(e)}$, and to $\lambda'_{o(e)}$. As we mentioned already in the beginning of the proof, the second and the third conditions of Lemma 4 are satisfied. Moreover,
$$d_G(i^*_e(\rho'_{t(e)}), i^*_e(\rho_{o(e)})) \leq d_G(i^*_e(\rho'_{t(e)}), i^*_e(\rho_{t(e)})) + d_G(i^*_e(\rho_{t(e)}), i^*_e(\rho_{o(e)})) < \delta$$
8
and hence the first condition is also satisfied. The representation given by Lemma 4 satisfies
\( d_G(\pi_e, \rho_e') < \delta \) and \( (\rho_e')^# = \rho_e^# \), thus we proved the induction step. Also, by the construction we have \( i_e(\rho_e') = i_e(\rho_e^#) \).

To finish the proof we need to define operators \( \rho'_e \) for \( e \in \tilde{E}(\Gamma) \), i.e. values of our new representation on \( s_e \). For \( e \in \tilde{E}(T) \) we can define \( \rho'_e := 1 \). Then
\[
\|\rho(s_e) - \rho'_e\|_p < \delta.
\]

For \( e \in \tilde{E}(\Gamma \setminus \tilde{E}(T)) \) we already know that
\[
d_G(\tilde{i}_e(\rho_{t(e)}), \tilde{i}_e(\rho'_{t(e)})) < \delta,
\]
\[
d_G(\tilde{i}_e(\rho_{o(e)}), \tilde{i}_e(\rho'_{o(e)})) < \delta,
\]
\[
d_G(\tilde{ad}(\rho_{s(e)})(\tilde{i}_e(\rho_{t(e)})), \tilde{i}_e(\rho_{o(e)})) < \delta,
\]
hence
\[
d_G(\tilde{ad}(\rho_{s(e)})(\tilde{i}_e(\rho_{t(e)})), \tilde{i}_e(\rho'_{o(e)})) < \delta.
\]

We also know that
\[
(ad(\rho_{s(e)})(\tilde{i}_e(\rho_{t(e)})))^# = (\tilde{i}_e(\rho_{t(e)}))^# = (\tilde{i}_e(\rho'_{o(e)}))^#,
\]
since \( (d_G \lambda)^e = (\tilde{i}_e(\rho_{t(e)}))^# - (\tilde{i}_e(\rho'_{o(e)}))^# = 0 \). So we can apply Corollary 2 and get \( T \) such that
\[
ad(T)(ad(\rho_{s(e)})(\tilde{i}_e(\rho_{t(e)}))) = ad(T\rho_{s(e)})(\tilde{i}_e(\rho'_{t(e)})) = i_e(\rho'_{o(e)})
\]
and \( ||T - I||_p < \delta \). Now we can define \( \rho'_e := T\rho(s_e) \). We have
\[
||\rho(s_e) - \rho'_e||_p = ||T - I||_p < \delta.
\]

Collections of \( \rho'_e \) and \( \rho_e' \) define the representation \( \rho' : \pi_1(\tilde{G}, T) \to U(W) \) such that \( d_{\pi_1}(\rho, \rho') < \delta \). Moreover, by the construction \( \rho'(R\tilde{G}) = I \), hence \( \rho' \) can be considered as a representation of \( \pi_1(\tilde{G}, T) \).

Now we are ready to prove Theorem 2.

**Proof of Theorem 2.** Given a \( \delta \)-almost \( \pi_1(\tilde{G}, T) \)-representation \( \rho : \pi_1(\tilde{G}, T) \to U(W) \) we can consider \( \lambda = \rho^# \). By Proposition 2
\[
||d_G(\lambda)||_E \prec \delta^p \cdot ||\lambda||_V.
\]

By Lemma 3 there exists \( \lambda'' \in \Theta^+ \cap \ker d_G \) such that
\[
||\lambda'' - \lambda||_V \prec \delta^p \cdot ||\lambda||_V \text{ and } ||\lambda''||_V \leq ||\lambda||_V.
\]

Let us note that \( ||\lambda||_V = \dim W \) and \( ||\lambda''||_V = ||\lambda''||_{\Theta^+} \in \mathbb{N} \) for all \( v \in V \), since \( \lambda'' \in \ker d_G \). We will construct \( \lambda' \in \Theta^+ \cap d_G \) by
\[
\lambda' = \lambda'' + (||\lambda||_V - ||\lambda''||_V)\pi^#,
\]
where $\pi$ is a trivial one-dimensional representations of $\pi_1(G,T)$. We have $\lambda' \in \ker d_G$, since $\pi^\# \in \ker d_G$. Moreover,

$$||\lambda' - \lambda||_V \leq ||\lambda'' - \lambda||_V + (||\lambda||_V - ||\lambda''||_V) \leq 2||\lambda'' - \lambda||_V \prec \delta^p \cdot ||\lambda||_V.$$ 

So we can apply Proposition 3 to $\lambda'$ and get the desired representation

$$\rho': \pi_1(G,T) \to U(W),$$

which satisfies $(\rho')^# = \lambda'$ and $d_G(\rho, \rho') \prec G \delta$.

\[\square\]

**Corollary 3.** Finitely generated virtually free groups are stable with respect to a normalized $p$-Schatten norm for $1 \leq p < \infty$.

Recall that a character of a group $G$ is a positive definite function on $G$ which is constant on conjugacy classes and takes value 1 at the unit. A character $\tau$ is called embeddable if it factorizes through a homomorphism to a tracial ultraproduct of matrices, that is if there is a non-trivial ultrafilter $\alpha$ on $\mathbb{N}$ and a homomorphism $f: G \to U\left(\prod_{n \in \mathbb{N}} (M_n(\mathbb{C}), \text{tr}_n)\right)$ such that $\tau_\alpha \circ f = \tau$, where $\tau_\alpha$ is a canonical trace on $\prod_{n \in \mathbb{N}} (M_n(\mathbb{C}), \text{tr}_n)$. Using the necessary condition for the Hilbert-Schmidt stability (see Theorem 3 in [6]) we get the following corollary.

**Corollary 4.** Each embeddable character of a finitely generated virtually free group is a pointwise limit of traces of finite-dimensional representations.

**Acknowledgements**

We thank Tatiana Shulman, Vadim Alekseev and Andreas Thom for helpful comments and remarks. The first author was supported by the Israel Science Foundation grants #575/16 and #957/20.

**References**

[1] Danil Akhtiamov and Alon Dogon, *On uniform Hilbert Schmidt stability of groups*, Proc. Amer. Math. Soc. 150 (2022), 1799–1809.

[2] Oren Becker and Alexander Lubotzky, *Group stability and Property (T)*, Journal of Functional Analysis 278 (2020), no. 1, 108–298.

[3] Marcus De Chiffre, Narutaka Ozawa, and Andreas Thom, *Operator algebraic approach to inverse and stability theorems for amenable groups*, Mathematika 65 (2019), no. 1, 98–118.

[4] Søren Eilers, Tatiana Shulman, and Adam Sørensen, *C*-stability of discrete groups, Advances in Mathematics 373 (2020), 107324.

[5] William Timothy Gowers and Omid Hatami, *Inverse and stability theorems for approximate representations of finite groups*, Sbornik: Mathematics 208 (2017), no. 12, 1784–1809.

[6] Don Hadwin and Tatiana Shulman, *Stability of group relations under small Hilbert–Schmidt perturbations*, Journal of Functional Analysis 275 (2018), no. 4, 761–792.

[7] Adrian Ioana and Pieter Spaas, *II1 factors with exotic central sequence algebras*, Journal of the Institute of Mathematics of Jussieu (2019), 1–26.

[8] Nir Lazarovich and Arie Levit, *Virtually free groups are stable in permutations*, arXiv preprint arXiv:2103.05583 (2021).
[9] John Robert Stallings, *On torsion-free groups with infinitely many ends*, Annals of Mathematics (1968), 312–334.

Maria Gerasimova, WWU Münster
*E-mail address*: mari9gerasimova@mail.ru

Konstantin Shchepin
*E-mail address*: shchepin.konstantin@gmail.com