ASYMPTOTICS AND PREASYMPTOTICS AT SMALL $x^*$

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This talk discusses the relative impact of running-coupling and other higher-order corrections on the small-$x$ gluon-gluon splitting function. Comments are made on similarities with some aspects of the Balitsky-Kovchegov equation, which arise because of the presence of an effective infrared cutoff in both cases. It is emphasised that, at least in the splitting-function case, the asymptotic small-$x$ behaviour has little relevance to the phenomenologically interesting preasymptotic region. This is illustrated with the aid of a convolution of the resummed splitting function with a toy gluon distribution.

1 Introduction

Detailed introductions to the more theoretical aspects of small-$x$ physics have been given in the contributions to these proceedings by Ciafaloni [1] and by Mueller [2]. The former concentrated on our understanding of the all-orders perturbative structure of the linear problem of small-$x$ parton multiplication, while the latter discussed the new phenomena that occur when the gluon density becomes so high that the small-$x$ growth saturates.

In the linear regime there have been extensive studies of the higher-order corrections. These are essential, insofar as the leading-logarithmic (LL$^x$) BFKL equation [3] for small-$x$ growth is clearly inconsistent with data (see for example [4,5]). However, the pure next-to-leading logarithmic (NLL$^x$) contributions to the evolution, [6, 7], are so large that the problem appears perturbatively unstable. Techniques have been developed over the past few years to help understand the origin of the poor perturbative convergence, in the hope that one may then use that understanding to help reorganise the perturbative series into a more stable hierarchy. As discussed in [1], methods based on the combination of collinear and small-$x$ resummations [9–19] seem to be particularly successful in this respect.

The situation in the context of saturation studies is less developed. Firstly, there is no definitive understanding of how to extend linear LL$x$ BFKL evolution to the saturation regime. One of the most widely studied models is the Balitsky-Kovchegov (BK) equation [20,21], which can be understood as resumming pomeron fan diagrams [22], and for which an additional mean-field approximation is nearly always made (formally valid only for a thick nucleus, and over a limited energy range). Other more sophisticated approaches to saturation are currently being investigated (e.g. [23–26]), a number of which aim to account for pomeron loops, first shown to be important in some early numerical calculations [27, 28] within the dipole approach [29]. Secondly, even within the simplest, BK, approach to saturation, studies of higher-order corrections have been less extensive than for the linear BFKL equation.

One purpose of this talk is to examine some general lessons that have been learnt about the effects of higher-order corrections in the case of linear evolution and to discuss how they might be relevant also in the BK saturation case.

A second part of this talk will examine briefly the outlook for phenomenological applications of the higher-order linear BFKL framework.

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aThough there are certain observables for which specific implementations of the pure NLL$x$ corrections may have reduced instabilities [8].
2 General aspects of higher-order BFKL corrections

In order to discuss higher-order corrections in linear and saturating BFKL, it is important to understand what precisely to compare. A critical feature of the BK equation in this context is that its non-linear term provides an effective transverse infrared cutoff on the evolution, usually known as the saturation scale $Q_s$. This cutoff scale varies as a function of $y = \ln 1/x$. A very elegant formalisation of these properties has recently been given in [30].

Infrared cutoffs have long been investigated in linear BFKL. They arise (a) when one imposes them ad-hoc to eliminate the non-perturbative regime, or (b) implicitly, in the study of the gluon-gluon splitting function $P_{gg}(z)$, which through factorisation contains just ultraviolet evolution, while infrared evolution (that below the factorisation scale) is entirely in the gluon distribution function, $g(x, Q^2)$, as illustrated in figure 1 (see e.g. [12, 15, 31]).

In all BFKL-type problems the question of higher-order corrections is doubly complicated, because in addition to the usual NLL$_x$ corrections (relative order $\alpha_s$ compared to LL$_x$), the iteration of the kernel means that it is not possible to identify a unique scale at which to evaluate the kernel. In problems without cutoffs (and with two probes at similar hard scales), it turns out that it is nevertheless a reasonable approximation to use the hard scale of the problem as the effective scale, at least over a moderately large range of $Y$, because fluctuations in scale due to BFKL diffusion are, to a first approximation, symmetric around the hard scale [32–34]. In contrast, in each of the three cutoff-contexts mentioned above, it was discovered, independently [11, 35, 36], that if one uses the cutoff scale (the only physically unambiguously identifiable scale) as the renormalisation scale, then there are large negative corrections to the BFKL power, of relative order $\alpha_s^{2/3}$, which come about because the cutoff introduces an asymmetry: the BFKL evolution can only take place at scales larger than the cutoff, where the coupling is reduced by its running.

That the correction should go as $\alpha_s^{2/3}$ can be seen as follows. Recall that in the usual fixed-coupling saddle-point approximation the gluon Green function between transverse scales $k$ and $k_0$ at rapidity $Y$ goes as

$$G(Y; k, k_0) \sim \frac{e^{\omega Y - (\ln^2 k^2/k_0^2)/(2\bar{\alpha}_s \chi'') + 1/2Y}}{\sqrt{2\pi \bar{\alpha}_s \chi'' Y}},$$

where $\omega = \bar{\alpha}_s \chi(1/2)$, $\bar{\alpha}_s = \alpha_s N_c/\pi$, and $\chi$, the BFKL characteristic function, and $\chi''$, its second derivative, are assumed to be evaluated at $\gamma = 1/2$, unless otherwise stated. By examining $\partial_Y \ln G(Y, k, k) = \omega - 1/2Y$, one sees that the effective BFKL power receives a correction, $\delta \omega$, of order $1/2$; the corresponding width in $\ln k^2$ of the solution is of order $\sqrt{\bar{\alpha}_s \chi'' Y}$. Equivalently if, for ‘external’ reasons, the width in $\ln k^2$ of the solution is limited to be $\Delta t$, then the BFKL evolution power will be suppressed by an amount $\delta \omega \sim (\bar{\alpha}_s \chi'')/\Delta t^2$ (cf. [37,38] for more precise calculations). When the evolution takes place with a running coupling, the width

![Figure 1. Evolution paths between two transverse scales, $Q_1$ and $Q_2$, and their separation into the parton distribution (paths that go below $Q_1$, blue if viewed in colour) and the perturbative DGLAP evolution (remaining paths, red).](image-url)
is naturally limited by the cutoff in the infrared and by the low value of the coupling in the ultraviolet. The actual width of the solution is such that the two sources of suppression, i.e. the finite width of the solution and the running of the coupling, are of similar importance: $\Delta t^3 \sim \chi''/(b\bar{\alpha}_s\chi)$, or equivalently $\delta \omega \sim \bar{\alpha}^{5/3}(b\chi)^{2/3}/\chi''^{1/3}$, i.e. a correction of relative order $\alpha_s^{2/3}$.

This simple argument actually reproduces the whole of the suppression’s leading functional dependence on $\alpha_s$, $\chi$, $\chi''$ and $b$.

Supplementing the above result with the relevant extra numerical coefficients, and additionally the NLL corrections, the small-$x$ power growth, $\omega_c$, of the $P_{gg}$ splitting function at scale $Q^2$ becomes

$$\omega_c \simeq 4 \ln 2 \bar{\alpha}_s(Q^2) \cdot \left(1 - 4.0\bar{\alpha}_s^{2/3} - 6.5\bar{\alpha}_s + O(\bar{\alpha}_s^{4/3})\right)$$

(2)

One sees that, numerically, the running coupling and NLL$_B$ contributions are both large, negative, and of the same order of magnitude. In order to make a phenomenological prediction it is necessary to take into account the running of the coupling at all orders and to supplement the NLL$_x$ corrections with the yet higher-order collinear-enhanced terms (which we refer to as NLL$_B$). The results for the power are shown in figure 2. One sees that despite their different parametric dependence on $\alpha_s$, in practice if one takes individually either the running coupling or the NLL$_B$ contributions, they lead to almost identical suppressions. Interestingly though, when taking both running and NLL$_B$ contributions, there is only limited extra suppression compared to either one individually.$^b$

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$^b$An alternative way of viewing the results, [39], is to consider not the absolute change in $\omega_c$, but rather the fractional change as one includes various higher-order contributions — one then notices (with, say, $\alpha_s = 0.2$) that the inclusion of a first higher-order contribution leads to about a 50% reduction in $\omega_c$, while the second has almost as large an effect, being a further 35% reduction.

![Figure 2. The small-x power growth $\omega_c$ for the $P_{gg}$ splitting function in various approximations: LL$x$ and NLL$_B$ (NLL$x$ with additional enhanced higher-order corrections), each with fixed and running coupling [13].](image-url)

In the context of studies of the evolution of the saturation scale with both higher-order corrections and running coupling [40], a similar phenomenon has been observed, though there, the sequence of results that was shown corresponded to fixed-coupling LL$x$, running coupling LL$x$ and running coupling NLL$x$ — if one considers just this combination of results then it is tempting (as was done in [2, 40]) to make the statement that running coupling effects are dominant and the NLL$x$ corrections are rather small. However, recently, the fixed-coupling (approximate collinearly improved) NLL$x$ results were presented [41] for the evolution of the
saturation scale, and together with [40], those results suggest that the picture is actually very similar to the splitting-function case: each of NLL\textsubscript{x} and running coupling contributions are individually large and negative, but combining them leads to only a small amount of further suppression.

Actually, such a result is quite natural: while at the lowest orders, e.g. Eq.(2), different sources of higher-order effects combine linearly, at higher orders there are strong non-linear effects. In the case of running coupling and NLL\textsubscript{x} effects there are actually three physical mechanisms at play: (a) since the cutoff causes the solution of the BFKL equation to be dominated by higher scales, where \( \alpha_s \) is smaller due to its running, NLL\textsubscript{x} effects are reduced; (b) NLL\textsubscript{x} effects themselves reduce the dependence of \( \omega_c \) on \( \alpha_s \), (suppressing it more at large \( \alpha_s \) than at small \( \alpha_s \)), slowing the running of \( \omega_c \) with transverse scale, as if there were a reduced ‘effective’ \( \beta \)-function, and this leads to a smaller running coupling correction; (c) the NLL\textsubscript{x} corrections cause a very strong suppression of the diffusion coefficient, \( \chi'' \), which means that limiting the width of diffusion, as happens due to the running of the coupling, has a smaller effect on the asymptotic power.

The discussion so far has concentrated just on the power-growth of splitting functions and saturation scales. In the case of the splitting function, with the aid of recent technical developments, it has become possible to study the whole \( x \)-dependence of the splitting function, even at preasymptotic values of \( x \) [13, 18]. Again, one can examine what happens when switching on, separately, running coupling and NLL\textsubscript{x} effects, as shown in figure 3. As was the case when studying just the asymptotic power, \( \omega_c \), one sees that, individually, running coupling and NLL\textsubscript{x} (or rather NLL\textsubscript{B}) effects are of similar magnitude. What should be noted here though, is that when considering the size of the (phenomenologically relevant) preasymptotic region of \( x \) without growth, there is a rather large additional effect from the combination of running-coupling and NLL\textsubscript{B} contributions — for example the point at which the resummed splitting function starts to become larger than the LO DGLAP splitting function is \( x \sim 10^{-1} \) for fixed-coupling LL\textsubscript{x}, \( x \sim 10^{-3} \) for running-coupling LL\textsubscript{x} or fixed-coupling NLL\textsubscript{B}, and \( x \sim 10^{-5} \) for running-coupling NLL\textsubscript{B}.

Figure 3 conveys what is perhaps one of the main general lessons to be retained from studies of resummed splitting functions: asymptotic properties of small-\( x \) resummation have little relevance at today’s energies. This statement holds in two senses: the behaviour of the splitting function at moderately small values of \( x \) is definitely not power-like; and general properties that one may deduce from studies of the asymptotic region (e.g. that combining NLL\textsubscript{B} and running-coupling effects provides only a modest extra suppression relative to each one individually) do not hold in the preasymptotic region. In the case of the splitting functions, the
specificity of the preasymptotic region can be traced to the appearance of new hierarchies in the perturbative structure, finite towers of terms \( \alpha_s^p(\alpha_s, \ln^2 1/x)^n \), discussed in [1,42], leading to the characteristic dip structure at \( \ln 1/x \sim 1/\sqrt{\alpha_s} + O(1) \).

In the case of the BK equation, a full study of preasymptotic effects including higher orders has yet to be carried out. It would presumably require that one know the structure of the NLLx terms not only for the linear part of the evolution, but also for the non-linear term.\(^5\) Currently however, the higher-order corrections to the non-linear term are not known. In the meantime it would nevertheless be of interest to have even just a full study of the \( x \) and \( Q^2 \) structure of the BK-equation in which only the linear term was supplemented by higher-order corrections. It is to be noted though that some general information on the impact of higher-order corrections on preasymptotics in the BK-equation can already be obtained from studies [43–45] which solve the BK-equation in \( x, Q^2 \) space with additional terms that partially mimic the linear NLLx corrections.

3 Phenomenological impact of resummed splitting functions

We have seen, Fig.3, that preasymptotic effects are large in the resummation of the gluon-gluon splitting function, so much so that the BFKL growth only sets in for \( z \sim 10^{-5} \). This suggests that resummation may have only a modest impact on DGLAP fits. To determine robustly whether or not this is the case would however require that one carry out a complete DGLAP fit, with not only the \( P_{gg} \) splitting function, but also the whole matrix of splitting functions and the coefficient functions, preferably in the \( \overline{\text{MS}} \) scheme, so as to aid comparison with existing fixed-order DGLAP fits, \( e.g. \) [46–48]. This represents a major programme of work, some aspects of which are currently being investigated.

Nevertheless, some degree of insight into the possible phenomenological impact can be obtained simply by taking a fixed gluon distribution (here CTEQ6M [48], which has the advantage of being smooth at small \( x \)) and examining the convolution \( P_{gg} \otimes g(x, Q^2) \), shown in Fig.4 normalised to \( g(x, Q^2) \). As well as the convolution with the resummed (NLL\(^B\)) splitting function, the plot shows the convolution with the fixed-order splitting function up to NNLO [49]. The comparison is to be used only for illustrative purposes since the fixed-order splitting functions are in the \( \overline{\text{MS}} \) scheme (though actually, at small \( x \), the scheme is usually important only \( \)

\(^5\)Whereas the universality features demonstrated in [30] ensure that the asymptotic properties of the solutions [40,41] are independent of the details of the non-linear term.
starting from $\text{N}^3\text{LO}$), while the resummed splitting function is in the $Q_0$ scheme [50]. Furthermore at large $x$ the NLL$_B$ resummation has been matched only to the LO DGLAP splitting function.

In Fig. 4, because the gluon distribution itself rises at small-$x$, a feature of the splitting function at some given $x$ value manifests itself in the convolution at somewhat smaller $x$. Thus, though the NLL$_B$ splitting function drops below the LO splitting function for $x \sim 10^{-1}$ (cf. Fig. 3, though the $Q^2$ value there is different), this crossover in the convolution takes place at $x \sim 10^{-2}$. For the crossover in the opposite direction the effect is much stronger, the NLL$_B$ splitting function overtaking the LO splitting function at $x \sim 5 \cdot 10^{-5}$, whereas in the convolution this occurs below $10^{-8}$.

Looking at the comparison with higher orders, one notices that at small $x$, the resummed convolution coincides quite closely with the NNLO convolution — this is perhaps not unsurprising, since down to $x \sim 10^{-3}$ there is a good deal of similarity between the NNLO and resummed splitting functions [1, 42]. Only for $x \lesssim 10^{-4}$ does one start to see a difference between the NNLO and NLL$_B$ convolutions and, over the remaining phenomenologically accessible region, the NLL$_B$ convolution is intermediate between the NLO and NNLO results. If one is courageous (i.e. one believes that the main characteristics will remain the same after scheme changes and inclusion of the full matrix of splitting functions and the coefficient functions), one may take this to suggest that current NNLO fits [47] should be adequate down to $x \sim 10^{-4}$ and that only beyond does the fixed-order truncation truly start to break down.\[^d\]

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\[^d\]If one takes higher-order fixed order calculations, they may break down earlier, because large LL ($\alpha^0_s \ln^{n-1} x/x$ terms, absent in NLO and NNLO, appear starting N$^3$LO.
