EVOLUTION VARIATIONAL INEQUALITY AND WASSERSTEIN CONTROL IN VARIABLE CURVATURE CONTEXT

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ABSTRACT. In this note we continue the analysis of metric measure space with variable lower Ricci curvature bounds. First, we study $(\kappa, N)$-convex functions on metric spaces where $\kappa$ is a lower semi-continuous function, and gradient flow curves in the sense of a new evolution variational inequality that captures the information that is provided by $\kappa$. Then, in the spirit of previous work by Erbar, Kuwada and Sturm, we introduce an entropic curvature-dimension condition $CD^e(\kappa, N)$ for metric measure spaces and lower semi-continuous $\kappa$. This condition is stable with respect to Gromov convergence and we show that is equivalent to the reduced curvature-dimension condition $CD^*(\kappa, N)$ provided the space is essentially non-branching. Finally, we introduce a Riemannian curvature-dimension condition in terms of an evolution variational inequality on the Wasserstein space. A consequence is a new differential Wasserstein contraction estimate.

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1. INTRODUCTION

In [Ket], the author introduces a curvature-dimension condition $CD(\kappa, N)$ for a metric measure space $(X, d_X, m_X)$ in terms of displacement convexity on the $L^2$-Wasserstein space where $\kappa : X \to \mathbb{R}$ is a lower semi-continuous function and $N \in [1, \infty)$. If $\kappa$ is constant, the condition is precisely the definition that was proposed by Lott, Sturm and Villani in [Stu06a, Stu06b, LV09, Vil09]. The condition $CD(\kappa, N)$ has geometric consequences as a generalized Bishop-Gromov estimates and a generalized Bonnet-Myers theorem. But it cannot recognize Riemannian-type spaces which are characterized by linearity of the induced heat flow of their Cheeger energy. In the context of constant lower curvature bounds this obstacle was resolved by Ambrosio, Gigli and Savaré in [AGS14a, AGS14b] who showed that displacement convexity in combination with linearity of the heat flow is equivalent to the existence of gradient flow curves in the sense of an evolution variational inequality (EVI) for the Boltzmann Entropy on the $L^2$-Wasserstein.
There are two important extensions of this idea. On the one hand, Erbar, Kuwada and Sturm [EKS15] introduce a finite dimensional version of the EVI-formula to define a Riemannian curvature-dimension condition. As part of their program they also introduce a so-called entropic curvature-dimension condition that is a more PDE-friendly modification of the original CD-condition. On the other hand, Sturm [Stu] defines EVI*-gradient flow curves where κ is a lower semi-continuous function. He proves several implications and equivalences, and also stability of this concept under measured Gromov convergence.

In this note we introduce a Riemannian curvature-dimension condition for variable lower curvature bounds. For this purpose, we study (κ, N)-convex functions on metric spaces where κ : X → R is lower semi-continuous. We use a new characterization of (κ, N)-convexity in terms of an integrated inequality [Ket] that involves so-called generalized distortion coefficients ακ(1) (Definition 3.4, Definition 3.14). Provided the metric space (X, dX) admits a first variation formula, we can deduce an evolution variational inequality for gradient flow curves of (κ, N)-convex functions.

More precisely, we say that an absolutely continuous curve (x_s)_{s ∈ [0, ∞)} is an EVIκ,N gradient flow curve of f starting in x_0 ∈ X if \lim_{s→0} x_s = x_0, and if for all z ∈ D(f) there exists a constant speed geodesic γ : [0, 1] → X between x_s and z such that the evolution variational inequality

\begin{equation}
\frac{d}{dt} \sigma_{κ/N}(s)^i_j \big|_t = 1 \geq \frac{1}{2N} \frac{d}{ds} dX(x_s, z)^2 + \frac{d}{dt} \sigma_{κ/N}(s)^i_j \big|_t = \frac{U_N(z)}{U_N(x_s)}
\end{equation}

holds for a.e. s > 0 where \( U_N(x) = e^{-\frac{x^2}{N}} \). By monotonicity of the derivatives of the distortion coefficients at 0 and 1 the evolution variational inequality \( \Pi \) is consistent with the previous versions of evolution variational inequalities by Ambrosio, Gigli, and Savaré and by Erbar, Kuwada and Sturm. Furthermore, an \( EVI_{κ,N} \)-gradient flow is also an \( EVI_{κ,N} \)-gradient flow in the sense of Sturm (Lemma 3.27). In the special case of constant κ our EVI-inequality becomes the one in [EKS15] (Remark 3.23). In addition, we prove that the existence of \( EVI_{κ,N} \)-gradient curves yields strong \((κ, N)\)-convexity (Theorem 3.28).

Then, we use this idea in the context of the \( L^2 \)-Wasserstein space and the Boltzmann Entropy over some metric measure space \((X, dX, m_X)\). In the spirit of Erbar, Kuwada and Sturm we introduce an entropic curvature-dimension condition \( CD^κ(κ, N) \). More precisely, a metric measure space \((X, dX, m_X)\) satisfies the entropic curvature-dimension condition for some admissible function κ and N ≥ 0 if for any pair \( μ_0, μ_1 \in D(Ent) \) with compact support there exists a \( L^2 \)-Wasserstein geodesic Π between \( μ_0 \) and \( μ_1 \) such that for all \( t \in [0, 1] \)

\[
U_N(μ_t) \geq \sigma^{(1-t)}_{κ/N_0^2} U_N(μ_0) + \sigma^{(t)}_{κ/N_0^2} U_N(μ_1)
\]

where \( U_N(μ) = e^{-\frac{μ^2}{N}} \), (e_t)_t Π = μ_t, Θ = W_2(μ_0, μ_1) and

\[
κ_0(t(Θ)Θ^2 = ∫ κ(ε_1(γ))|γ|^2 dΠ(γ).
\]

We show that the condition \( CD^κ \) is stable under measured Gromov convergence and that is equivalent to the reduced curvature-dimension condition \( CD^κ(κ, N) \) [Ket] provided a non-branching assumption is satisfied. Moreover, the entropic curvature-dimension condition already implies local compactness and finite Hausdorff dimension of the underlying metric measure space.
Then, we introduce a Riemannian curvature-dimension that is defined via combination of the entropic curvature-dimension condition and linearity of the heat flow. We show that the Riemannian curvature-dimension condition can be characterized by the existence of Wasserstein $EV_{\kappa,N}$-gradient flow curves that is a straightforward modification of inequality (1) in Wasserstein space context (Theorem 5.7). Furthermore, the Riemannian curvature-dimension condition is stable w.r.t. measured Gromov convergence (Theorem 5.8). Hence, these spaces arise naturally as non-smooth limit spaces of Riemannian manifolds (Corollary 5.9). By monotonicity an $EV_{\kappa,N}$ gradient flow curve is also an $EV_{\kappa}$-gradient flow curve in the sense of Sturm [Stu]. The latter implies the following contraction estimate. Let $\Pi_s$ be the unique $L^2$-Wasserstein geodesic between $\mu_s$ and $\nu_s$. Then the following contraction estimate holds
\[
\frac{d}{ds} W_2(\mu_s, \nu_s)^2 \leq - \int_0^1 \kappa(\gamma(t)) |\dot{\gamma}(t)|^2 d\Pi(\gamma)^s dt.
\]
We also show differential contraction estimates for $N < \infty$ that imply previous differential control estimates (Theorem 6.4).

In section 2 we recall some important preliminaries on metric measure space and Wasserstein geometry. In section 3 introduce generalized distortion coefficients, $(\kappa, N)$-convexity on metric spaces, the evolution variational inequality, and prove several implications. In section 4 we introduce the entropic curvature-dimension condition for variable lower curvature bounds, and prove equivalence with the reduced curvature-dimension condition in the context of essentially non-branching metric measure spaces. In section 5 we define Wasserstein-$EV_{\kappa,N}$-gradient flow curves that characterize a Riemannian curvature dimension condition. In section 6 we deduce differential contraction estimates.

2. Preliminaries

Definition 2.1 (Metric measure space). Let $(X, d_X)$ be a complete and separable metric space, and let $m_X$ be a locally finite Borel measure on $(X, d_X)$. That is, for all $x \in X$ there exists $r > 0$ such that $m_X(B_r(x)) \in (0, \infty)$. Let $O_X$ and $B_X$ be the topology of open sets and the family of Borel sets, respectively. A triple $(X, d_X, m_X)$ will be called metric measure space. We assume that $m_X(X) \neq 0$.

$(X, d_X)$ is called length space if $d_X(x, y) = \inf L(\gamma)$ for all $x, y \in X$, where the infimum runs over all rectifiable curves $\gamma$ in $X$ connecting $x$ and $y$. $(X, d_X)$ is called geodesic space if every two points $x, y \in X$ are connected by a curve $\gamma$ such that $d_X(x, y) = L(\gamma)$. Distance minimizing curves of constant speed are called geodesics. A length space, which is complete and locally compact, is a geodesic space and proper ([BB10], Theorem 2.5.23]). Rectifiable curves always admit a reparametrization proportional to arc length, and therefore become Lipschitz curves. In general, we assume that a geodesic $\gamma : [0, 1] \to X$ is parametrized proportional to its length. The set of all such geodesics $\gamma : [0, 1] \to X$ is denoted with $G(X)$ and the set of all Lipschitz curves $\gamma : [0, 1] \to X$ parametrized proportional to arc-length is denoted with $LC(X)$. $G(X)$ and $LC(X)$ are equipped with the topology that is induced by uniform convergence. More precisely, we always consider the distance $d_{\infty}(\gamma, \tilde{\gamma}) = \sup_{t \in [0, 1]} |\gamma(t) - \tilde{\gamma}(t)|$. 


Let $\mathcal{P}_2(X)$ be the $L^2$-Wasserstein space over $(X,d_X)$ equipped with the $L^2$-Wasserstein distance $W_2$. The subspace of absolutely continuous probability measure with respect to $m_X$ is denoted by $\mathcal{P}_2(m_X)$. Recall that a dynamical optimal coupling between $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ is a probability measure $\Pi$ on $\mathcal{P}(\mathcal{G}(X))$ such that $(\varepsilon_0, \varepsilon_1), \Pi$ is an optimal coupling of $\mu_0$ and $\mu_1$. Then, the curve $t \in [0,1] \mapsto \mu_t = (\varepsilon_t), \Pi$ is a geodesic in $\mathcal{P}_2(X)$ with respect to $W_2$. Moreover, for each geodesic $\mu_t$ in $\mathcal{P}_2(X)$ there exists a dynamical optimal plan $\Pi_t$. In rest of the article, we will not distinguish between $\mu_t$ and the corresponding probability measure $\Pi_t$ on $\mathcal{P}(X)$.

3. $(\kappa, N)$-convex function and EVI gradient flow curves

**Theorem 3.1** (J. C. F. Sturm’s comparison theorem). Let $\kappa, \kappa' : [a, b] \to \mathbb{R}$ be continuous function such that $\kappa' \geq \kappa$ on $[a, b]$ and $s_{\kappa'} > 0$ on $(a, b]$. Then $s_\kappa \geq s_{\kappa'}$ on $[a, b]$.

A generalization of the previous theorem is the following result.

**Theorem 3.2** (Sturm-Picone oscillation theorem). Let $\kappa, \kappa' : [a, b] \to \mathbb{R}$ be continuous such that $\kappa' \geq \kappa$ on $[a, b]$. Let $u$ and $v$ be solutions of (2) with respect to $\kappa$ and $\kappa'$ respectively. If $u(a) = u(b) = 0$ and $u > 0$ on $(a, b)$, then either $u = \lambda v$ for some $\lambda > 0$ or there exists $x_1 \in (a, b)$ such that $v(x_1) = 0$.

**Definition 3.3** (generalized sin-functions). Let $\kappa : [0, L] \to \mathbb{R}$ be a continuous function. The generalized sin function $s_\kappa : [0, L] \to \mathbb{R}$ is the unique solution of

$$v'' + \kappa' v = 0.$$ 

such that $s_\kappa(0) = 0$ and $s_\kappa'(0) = 1$. The generalized cos-function is $c_\kappa = s_\kappa'$.

**Definition 3.4** (generalized distortion coefficients). Consider $\kappa : [0, L] \to \mathbb{R}$ that is continuous and $\theta \in [0, L]$. Then

$$\sigma^{(\kappa)}(\theta) = \begin{cases} s_{\kappa}(\theta) & \text{if } s_{\kappa}(t) > 0 \text{ for all } t \in (0, \theta] \\ \infty & \text{otherwise} \end{cases}.$$ 

If $\sigma^{(\kappa)}(\theta) < \infty$, $t \mapsto \sigma^{(\kappa)}(\theta)$ is a solution of

$$u''(t) + \kappa(t\theta)^2 u(t) = 0$$

satisfying $u(0) = 0$ and $u(1) = 1$. We set $\sigma^{(\kappa)}(1) = \sigma^{(\kappa)}$. Then $\sigma^{(\kappa)}(\theta) = \sigma^{(\kappa)}_{\kappa \theta^2}$.

If $\kappa : [0, L] \to \mathbb{R}$ is just lower semi-continuous, we can extend the previous definition in the following way. Define bounded continuous functions by

$$\kappa_n(x) = \min \left\{ \min_{y \in [0, L]} \{ \kappa(y) + n|x - y| \} \right\}, \quad n \in \mathbb{N}.$$ 

The sequence $(\kappa_n)$ is montone increasing and converges pointwise to $\kappa$. Then, $\sigma^{(\kappa)}_{\kappa_n}(\theta)$ is monotone increasing we define the generalized distortion coefficient with respect to $\kappa$ by

$$\sigma^{(\kappa)}(\theta) = \lim_{n \to \infty} \sigma^{(\kappa)}_{\kappa_n}(\theta) \in [0, \infty].$$

**Lemma 3.5** (Ket). Let $\kappa : [0, L] \to \mathbb{R}$ be lower semi-continuous and $\theta \in [0, L]$. If $\sigma^{(\kappa)}_{\kappa_{t_0}}(\theta) = \infty$ for some $t_0 \in (0, 1)$ then $\sigma^{(\kappa)}(\theta) = \infty$ for any $t \in (0, 1)$.

In particular, either one has $\sigma^{(\kappa)}(\theta) < \infty$ for any $t \in (0, 1)$ and

$$\sigma^{(\kappa)}(\theta) = s_{\kappa}(t\theta)/s_{\kappa}(\theta).$$
Lemma 3.9. More precisely

Proof. First, we assume that any distortion coefficient is finite. Let \( x \) for any \( x \) for each \( t \) (\[Ket\]) Proposition 3.8 \( \blacksquare \)

If we apply the theorem of dominated convergence to \( g \) \( \delta > 0 \) each \( \kappa \) sequence \( \kappa \) Lemma 3.7. If \( \sigma_\kappa^\iota(\theta) < \infty \), we have

\[
\sigma_\kappa^\iota(\theta) = \int_0^1 g(s, t)\theta^2 \kappa \circ \gamma(s)\sigma_\kappa^\iota(\theta) ds + t.
\]

with \( g(s, t) \) being the Green function of \([0, 1]\).

Proof. If \( \kappa \) is continuous, this is clear. If \( \kappa \) is lower semi-continuous, we can choose \( \kappa_n \uparrow \kappa \). Then \( \sigma_{\kappa_n}^\iota(\theta) \uparrow \sigma_\kappa^\iota(\theta) \) uniformly as \( n \to \infty \) by Dini’s theorem. Then

\[
\int_0^1 g(s, t)\theta^2 \kappa \circ \gamma(s)\sigma_\kappa^\iota(\theta) ds = \lim_{M \to \infty} \int_0^1 g(s, t)\theta^2 \kappa \circ \gamma(s)\sigma_{\kappa_n}^\iota(\theta) \wedge M ds = \lim_{n \to \infty} \lim_{M \to \infty} \int_0^1 g(s, t)\theta^2 \kappa_n \circ \gamma(s)\sigma_{\kappa_n}^\iota(\theta) \wedge M ds
\]

\[
\leq \liminf_{n \to \infty} \int_0^1 g(s, t)\theta^2 \kappa_n \circ \gamma(s)\sigma_{\kappa_n}^\iota(\theta) ds = \sigma_\kappa^\iota(\theta) + t < \infty
\]

where \( t \) is fixed and \( M > 0 \). Hence, \( g(s, t)\theta^2 \kappa \circ \gamma(s)\sigma_\kappa^\iota(\theta) \) is integrable in \( s \in [0, 1] \). If we apply the theorem of dominated convergence to \( g(s, t)\theta^2 \kappa_n \circ \gamma(s)\sigma_{\kappa_n}^\iota(\theta) \) we obtain equality in \([\mathbb{H}].\]

Proposition 3.8 (\textbf{Ker}). \( \sigma_\kappa^\iota(\theta) \) is non-decreasing with respect to \( \kappa : [0, \theta] \to \mathbb{R} \). More precisely

\( \kappa(x) \geq \kappa'(x) \forall x \in [0, \theta] \) implies \( \sigma_\kappa^\iota(\theta) \geq \sigma_{\kappa'}^\iota(\theta) \forall t \in [0, 1] \).

Lemma 3.9. Let \( \kappa, \kappa : [0, L] \to \mathbb{R} \) be lower semi-continuous, such that

\[
\liminf_{i \to \infty} \kappa_i(x) \geq \kappa(x)
\]

for any \( x \in [0, L] \). Then \( \liminf_{i \to \infty} \sigma_{\kappa_i}^\iota(\theta) \geq \sigma_\kappa^\iota(\theta) \) for any \( t \in [0, 1] \).

Proof. First, we assume that any distortion coefficient is finite. Let \( \epsilon > 0 \), and let \( i_0 \in \mathbb{N} \) such that for each \( i \geq i_0 \)

\[
\kappa_i(x) \geq \kappa(x) - \epsilon \geq \kappa_n - \epsilon
\]

for each \( x \in [0, L] \) and \( \kappa_n \) as before. By monotonicity, we have

\[
\liminf_{i \to \infty} \sigma_{\kappa_i}^\iota(\theta) \geq \sigma_{\kappa_n - \epsilon}^\iota(\theta)
\]

for each \( t \in [0, 1] \). \( \kappa_n \) is a sequence of monotone increasing continuous functions converging to \( \kappa \). Then, by stability of \( \sigma_{\kappa_i}^\iota(\theta) \) with respect to uniform change of \( \kappa_i \), for each \( \delta > 0 \) there exists \( \epsilon > 0 \) such that

\[
\sigma_{\kappa}^\iota(\theta) = \lim_{n \to \infty} \sigma_{\kappa_n}^\iota(\theta) \leq \lim_{n \to \infty} \sigma_{\kappa_n - \epsilon}^\iota(\theta) + \delta \leq \sigma_{\kappa_n - \epsilon}^\iota(\theta) + \delta \leq \liminf_{i \to \infty} \sigma_{\kappa_i}^\iota(\theta) + \delta
\]
where the first equality is the definition of \( \sigma^{(t)}_\kappa(\theta) \) and the first inequality is continuity with respect to uniform changes of \( \kappa_n \). The only case that is left is when \( \sigma^{(t)}_\kappa(\theta) = \infty \). But then by monotonicity it follows for a subsequence that \( \kappa_i \uparrow \kappa \).

From that we can deduce that \( \lim_{i \to \infty} \sigma^{(t)}_\kappa(\theta) = \infty \) for any \( t \in [0, 1] \) by comparison with the coefficients w.r.t. \( \kappa_i \) and with the coefficients w.r.t. \( \kappa_n \). \( \square \)

We define \( \kappa^- : [0, \theta] \to \mathbb{R} \) by \( \kappa^-(x) = \kappa(\theta - x) \).

**Lemma 3.10** ([Ket]). Consider \( \kappa, \kappa' : [0, \theta] \to \mathbb{R} \). Then

\[
\sigma^{(1-t)}_\kappa(\theta)^{1-\lambda} \cdot \sigma^{(t)}_\kappa(\theta)^{\lambda} \geq \sigma^{(t)}_{(1-\lambda)\kappa + \lambda \kappa'}(\theta).
\]

Especially, \( \kappa \in \text{LSC}([0,1]) \mapsto \log \sigma_\kappa \) and \( \kappa \mapsto \log \sigma_{\kappa^-} \) are convex. \text{LSC}([0,1]) denotes the space of lower semi-continuous functions.

**Corollary 3.11.** For \( t \in [0, 1] \) the function \( G : \mathbb{R}^2 \times \text{LSC}([0,1]) \to \mathbb{R} \cup \{\infty\} \) given by

\[
G(x, y, \kappa) = \log \left[ \sigma^{(1-t)}_{\kappa^-} e^x + \sigma^{(t)}_{\kappa^+} e^y \right] \text{ is convex.}
\]

**Proof.** We argue like in [EKS15, Lemma 2.11]. Note that

\[
G(x, y, \kappa) = F(\log \sigma^{(1-t)}_{\kappa^-} + x, \log \sigma^{(t)}_{\kappa^+} + y)
\]

with \( F(u, v) = \log(e^u + e^v) \). Then the assertion follows since \( F \) is convex, \( x \mapsto F(u + x, v + x) \) is monotone increasing and \( \kappa \mapsto \log \sigma_{\kappa^-} \) is convex. \( \square \)

**Remark 3.12.** If \( \Pi \in \mathcal{P}(\text{LSC}([0,1])) \), then \( \kappa_{\Pi} : t \mapsto \int \kappa(t)d\Pi(\kappa) \in \text{LSC}([0,1]) \) by Fatou’s Lemma. Hence, if \( f : \text{LSC}([0,1]) \to \mathbb{R} \) is convex, we obtain

\[
\int f(\kappa)d\Pi(\kappa) \leq f(\kappa_{\Pi}).
\]

For the rest of the article we use the following notation. Let \( (X, d_X) \) be a metric space, and let \( \kappa : X \to \mathbb{R} \) be lower semi-continuous. We set \( \kappa_\gamma = \kappa \circ \bar{\gamma} \) where \( \gamma : [0,1] \to X \) is a constant speed geodesic in \( X \) and \( \bar{\gamma} \) its unit speed reparametrization. We denote with \( \gamma^-(t) = \gamma(1-t) \) the reverse parametrization of \( \gamma \), and we also write \( \gamma = \gamma^+ \), and \( \kappa^-_\gamma := \kappa_{\gamma^-/} \).

**Proposition 3.13** ([Ket]). Let \( \kappa : [a,b] \to \mathbb{R} \) be lower semi-continuous and \( u : [a, b] \to \mathbb{R}_{\geq 0} \) be upper semi-continuous. Then the following statements are equivalent:

(i) \( u'' + \kappa u \leq 0 \) in the distributional sense, that is

\[
\int_a^b \varphi''(t)u(t)dt \leq -\int_a^b \varphi(t)\kappa(t)u(t)dt
\]

for any \( \varphi \in C^\infty_0((a, b)) \) with \( \varphi \geq 0 \).

(ii) It holds

\[
u(\gamma(t)) \geq (1-t)u(\gamma(0)) + tu(\gamma(1)) + \int_0^1 g(t,s)\kappa(\gamma(s))\theta^2 u(\gamma(s))ds
\]

for any constant speed geodesic \( \gamma : [0,1] \to [a, b] \) where \( \theta = |\gamma| = L(\gamma) \) with \( g(s,t) \) being the Green function of \([0,1] \).
S is (weakly) \Corollary{3.16}. If
\begin{proof}
Apply Proposition \Proposition{3.8} and Lemma \Lemma{3.10}.
\end{proof}
\begin{proof}
Let us assume that
\begin{align*}
\text{for all } t \in [a, b],\end{align*}
\end{proof}
\begin{remark}
\begin{align}
\text{Lemma \Lemma{3.15}}.
\end{align}
\begin{align}
\text{Let } \gamma : [0, 1] \to [a, b].
\end{align}
\end{remark}
\begin{definition}
Consider a metric space \( Y, d_y \) and a continuous function \( \kappa : Y \to \mathbb{R} \). We say a function \( u : Y \to [0, \infty) \) is \( \kappa \)-concave if for each geodesic \( \gamma : [0, 1] \to D(u) = \{ u > 0 \} \)
\begin{align}
\sigma_{\kappa, N}(t)u(\gamma(0)) + \sigma_{\kappa, N}(t)u(\gamma(1)) \end{align}
for any constant speed geodesic \( \gamma : [0, 1] \to Y \) with \( \theta = |\gamma| = L(\gamma) \leq L \).
We use the convention \( \cdot \cdot \cdot = 0 \).
\( \text{Remark} \text{ holds for any constant speed geodesic } \gamma : [0, 1] \to [a, b]. \)
\end{definition}
\begin{lemma}
Let \( Y \) be a metric space as in the previous definition.
\begin{enumerate}
\item If \( S \) is (weakly) \( (\kappa, N) \)-convex, then \( \lambda \cdot f \) is (weakly) \( (\lambda \kappa, \lambda N) \)-convex for \( \lambda > 0 \).
\item If \( S_1 \) is weakly \( (\kappa_1, N_1) \)-convex and \( S_2 \) is strongly \( (\kappa_2, N_2) \)-convex, then \( S_1 + S_2 : D(S_1) \cap D(S_2) \to [-\infty, \infty) \) is weakly \( (\kappa_1 + \kappa_2, N_1 + N_2) \)-convex.
\item If \( S \) is (weakly) \( (\kappa, N) \)-convex and \( \kappa' \leq \kappa \) and \( N' \geq N \) then \( S \) is (weakly) \( (\kappa', N') \)-convex.
\end{enumerate}
\end{lemma}
\begin{proof}
Apply Proposition \Proposition{3.8} and Lemma \Lemma{3.10}.
\end{proof}
\begin{corollary}
If \( S : Y \to (-\infty, \infty] \) is a (weakly) \( (\kappa, N) \)-convex function then \( S \) is (weakly) \( \kappa \)-convex in the sense that for each geodesic \( \gamma \) in \( D(S) \) (for each pair \( x_0, x_1 \in D(S) \) there exists a geodesic \( \gamma : [0, 1] \to Y \) with)
\begin{align}
S(\gamma_t) \leq (1 - t)S(x_0) + tS(x_1) - \int_0^1 g(s, t) d_y(x_0, x_1)^2 \kappa \circ \gamma(s) dt
\end{align}
for all \( t \in [0, 1] \).
\end{corollary}
\begin{proof}
Let us assume that \( S \) is \( (\kappa, N) \)-convex. Consider a geodesic \( \gamma \). Then \( (3) \) holds for \( u(s) = e^{-\frac{S_{\kappa, N}}{\gamma(s)}} \) for \( \kappa / N \) instead of \( \kappa \). Multiply with \( N \) and let \( N \to \infty \). We obtain a distributional inequality for \( S \) along \( \gamma \) that characterizes \( \kappa \)-convexity.
\end{proof}
\begin{remark}
\begin{align}
\text{Let } (M, g_M) \text{ be a Riemannian manifold. A smooth function } u : M \to [0, \infty) \text{ is (weakly)} \text{ \( \kappa u \)-convex if } \nabla^2 u \geq \kappa g_M \text{ and a smooth function } f : M \to \mathbb{R} \text{ is (weakly)} \text{ \( (\kappa, N) \)-convex if } \nabla^2 f \geq \kappa g_M + \frac{1}{N} (df \otimes df). \end{align}
\end{remark}
For a function \( f : [a, b] \rightarrow \mathbb{R} \) the right and left derivatives are denoted by
\[
\frac{d^+}{dt} f(t) = \limsup_{h \downarrow 0} \frac{f(t + h) - f(t)}{h} \quad \text{and} \quad \frac{d^-}{dt} f(t) = \liminf_{h \uparrow 0} \frac{f(t + h) - f(t)}{h}.
\]
We can consider the derivative of \( \sigma^{(i/(1-t))}_\kappa(\theta) \) in the following sense:
\[
\left\{ \begin{array}{ll}
\frac{d^+}{d\theta} \sigma^{(i/(1-t))}_\kappa(\theta) \in \mathbb{R} & \text{if } \sigma^{(i/(1-t))}_\kappa(\theta) < \infty \\
\infty & \text{otherwise}.
\end{array} \right.
\]

**Lemma 3.18.** Consider \( \kappa \in LSC([0, L]) \) and \( \theta \in [0, L] \) such that \( \sigma^{(i/(1-t))}_\kappa(\theta) < \infty \). Then
\[
\frac{d^+}{dt} \bigg|_{t=0} \sigma^{(i)}(\theta) \uparrow \text{ and } \frac{d^-}{dt} \bigg|_{t=1} \sigma^{(i)}(\theta) = -\frac{d^+}{dt} \bigg|_{t=0} \sigma^{(i/(1-t))}_\kappa(\theta) \downarrow
\]
if \( \kappa \) is non-decreasing.

**Proof.** The distortion coefficients \( \sigma^{(i)}(\theta) \) are monotone increasing with respect to \( \kappa \) and satisfy \( \sigma^{(i)}(0) = 0 \) and \( \sigma^{(i)}(\theta) = 1 \).

**Remark 3.19.** If \( \sigma^{(i)}_\kappa(\theta) < \infty \), \( \frac{d^+}{d\theta} \big|_{t=0} \sigma^{(i)}_\kappa(\theta) \) has the following representation.
\[
\frac{d^+}{dt} \bigg|_{t=0} \sigma^{(i)}(\theta) = \int_0^1 (1-s)\theta^2 \kappa \circ \gamma(s) \sigma^{(i)}(\theta) ds + 1.
\]
This follows from
\[
\sigma^{(i)}(\theta) - t = \int_0^1 g(s, t)\theta^2 \kappa \circ \gamma(s) \sigma^{(i)}(\theta) ds
\]
and \( \frac{d}{dt} \big|_{t=0} g(s, t) = (1-s) \). Similar for \( \frac{d^-}{d\theta} \big|_{t=1} \sigma^{(i)}(\theta) \).

**Lemma 3.20.** Let \( \kappa_n, \kappa : [0, \theta] \rightarrow \mathbb{R} \) be lower semi-continuous such that
\[
\liminf_{n \rightarrow \infty} \kappa_n(x) \geq \kappa(x).
\]
Assume that \( \frac{d^+}{dt} \big|_{t=0} \sigma^{(i)}(\kappa_n), \frac{d^+}{d\theta} \big|_{t=0} \sigma^{(i)}(\kappa) < \infty \). Then
\[
\liminf_{n \rightarrow \infty} \frac{d^+}{dt} \bigg|_{t=0} \sigma^{(i)}(\kappa_n) \geq \frac{d^+}{dt} \bigg|_{t=0} \sigma^{(i)}(\kappa).
\]

**Proof.** If \( \kappa \) is continuous, then uniform changes of \( \kappa \) imply that \( \frac{d^+}{dt} \big|_{t=0, 1} \sigma^{(i)}_\kappa(\theta) \) changes uniformly by the previous remark. The rest of the proof works as in Lemma 3.9.

**Lemma 3.21.** Let \( \kappa \) be continuous and let \( u \in C^1((a, b)) \). Then \( u \) satisfies one of the equivalent statements in Proposition 3.13 if and only if there exists a constant \( L \in (0, b - a) \) such that
\[
\frac{d}{dt} \sigma^{(i)}_\kappa(\gamma) \big|_{t=0} u \circ \gamma(1) \leq \frac{d}{dt} \sigma^{(i)}(\gamma) \big|_{t=1} u \circ \gamma(0) + (u \circ \gamma)'(0)
\]
for any constant speed geodesic \( \gamma : [0, 1] \rightarrow [a, b] \).

**Proof.** “\( \Rightarrow \)” Consider \[\Box\]. We add \( u \circ \gamma(0) \) on both sides of the inequality and devide by \( t \).
\[
\frac{1}{t} (u(\gamma(t)) - u(\gamma(0))) \geq \frac{1}{t} \left( \sigma^{(i/(1-t))}_\kappa(\gamma) - 1 \right) u(\gamma(0)) + \frac{1}{t} \sigma^{(i)}_\kappa(\gamma) u(\gamma(1))
\]
Taking the limit $t \to 0$ yields
\[
(u \circ \gamma)'(0) \geq -\frac{d}{dt} \sigma^{(t)}_{\kappa_\gamma}(|\gamma|)|_{t=1} u(\gamma(0)) + \frac{d}{dt} \sigma^{(t)}_{\kappa_\gamma}(|\gamma|)|_{t=0} u(\gamma(1)).
\]

"\(\Rightarrow\)" If $\kappa_\gamma \geq K \in \mathbb{R}$, Lemma 3.18 implies
\[
\frac{d}{dt}|_{t=0} \sigma^{(t)}_{\kappa_\gamma}(|\gamma|) \geq \frac{d}{dt}|_{t=0} \sigma^{(t)}_{\kappa}|(|\gamma|).
\]
and similar for $\sigma^{(1-t)}_{\kappa_\gamma}(|\gamma|)$. Hence
\[
\mathbf{g}_K(|\gamma|)|_{t=0} (u \circ \gamma)'(0) \geq -\kappa(|\gamma|) u(\gamma(0)) + u(\gamma(1))
\]
Now, we pick a point $r \in [a, b]$. Then, for each $\epsilon > 0$ one can pick a geodesic $\gamma : [0, 1] \to [a, b]$ such that $\gamma(1) = r$ and $|\gamma| = \epsilon$. If we set $\min \kappa_\gamma = K_\epsilon$, we can deduce exactly like in Lemma 2.2 in [EKS15] that
\[
(u \circ \gamma)'(0) \geq -\kappa(|\gamma|) u(\gamma(0)) + u(\gamma(1))
\]
pointwise on $[0, \epsilon]$. Since $K_\epsilon \to \kappa(r)$ for $\epsilon \to 0$ and $\Im(\gamma) \to r$, the result follows. □

**Lemma 3.22.** Let $f$ be a smooth $(\kappa, N)$-convex function on a Riemannian manifold $(M, g)$, and let $U_N$ be as in Definition 2.14. Then, a smooth curve $x : [0, \infty) \to M$ is a gradient flow curve of $f$ if and only if for each $z \in M$ and all $t > 0$ we have that
\[
\frac{1}{2N} \frac{d}{ds} d_M(x_s, z)^2 + \frac{d}{dt} \sigma^{(t)}_{\kappa_\gamma}(|\gamma^s|)|_{t=1} \geq \frac{d}{dt} \sigma^{(t)}_{\kappa_\gamma}(|\gamma^s|)|_{t=0} \frac{U_N(z)}{U_N(x_s)}
\]
where $\gamma^s : [0, 1] \to M$ is the constant speed geodesic between $x_s$ and $z$.

**Remark 3.23.** In the case of constant curvature $\kappa$ the inequality becomes
\[
\cos_{\kappa/N}(d_M(x_s, z)) - \frac{1}{2N} \sin_{\kappa/N}(d_M(x_s, z)) \frac{d}{ds} d_M(x_s, z) \geq \frac{U_N(z)}{U_N(x_s)}
\]
If $\kappa > 0$, we can write
\[
\kappa_{\kappa/N}(d_M(x_s, z)) + \frac{1}{2N} \frac{d}{ds} \kappa_{\kappa/N}(d_M(x_s, z)) \geq \frac{U_N(z)}{U_N(x_s)}
\]
And similar for $\kappa < 0$. Using the transformation
\[
\frac{1}{N} g_{\kappa/N}(x/2)^2 = \frac{2N}{2N}(1 - \kappa_{\kappa/N}(x))
\]
this becomes the e
ti algorithm formula from [EKS15].

**Proof.** "$\Rightarrow\)" Let $x_t$ be a gradient flow curve of $f$. Then, by the first variation formula we can compute
\[
\frac{d}{dt}|_{t=0} U_N(\gamma_t) = -\frac{1}{N} U_N(x_t) g_M(\nabla f|_{x_t}, \gamma_0) = -\frac{1}{2N} U_N(x_t) \frac{d}{dt} d_M(x_t, z)^2
\]
with Lemma 3.21 the result follows immediately.

"\(\Leftarrow\)" Since $\kappa$ is bounded from below, the backward direction follows from monotonicity of the distortion coefficients and their derivatives like in Lemma 2.4 in [EKS15]. □
Evolution variational inequality in metric spaces. Let \((X, d_x)\) be a complete and separable metric space, and let \(f : X \to (-\infty, \infty]\) be a lower semi-continuous function. We repeat some definitions from differential calculus on metric spaces.

The descending slope of \(f\) at \(x\) is

\[
|\nabla^- f|(x) := \limsup_{y \to x} \frac{|f(x) - f(y)|_+}{d_x(x, y)}.
\]

A curve \(x : [a, b] \to X\) is called absolutely continuous if

\[
d_x(x_t, x_s) \leq \int_t^s g(r)dr \quad \text{for all } t, s \in [a, b] \text{ such that } s \leq t
\]

and some \(g \in L^1([a, b])\). We say \(x\) is locally absolutely continuous if \((14)\) holds locally in \([a, b]\). For an absolutely continuous curve \(x\) the metric speed

\[
|\dot{x}|(t) := \lim_{h \to 0} \frac{d_x(x_{t+h}, x_t)}{|h|}
\]

exists for a.e. \(t \in [a, b]\) and is the minimal \(g\) in \((14)\).

**Definition 3.24.** A locally absolutely continuous curve \(x : [0, \infty) \to X\) with \(x(0) \in X\) is a gradient flow curve of \(f\) starting in \(x(0)\) if the energy dissipation equality

\[
f(x(s)) = f(c(t)) + \frac{1}{2} \int_s^t (|\dot{x}|^2(r) + |\nabla^- f|(x(r))) \, dr
\]

for all \(0 \leq s \leq t\) holds.

Lemma 3.22 motivates the following definition.

**Definition 3.25.** Let \(f\) be as before. Let \(\kappa : X \to \mathbb{R}\) be a lower semi-continuous function, \(N \geq 1\) and let \(x : (0, \infty) \to D(f)\) be a locally absolutely continuous curve. We say that \(x_s\) is an \(EVI_{\kappa, N}\) gradient flow curve of \(f\) starting in \(x_0 \in X\) if \(\lim_{s \to 0} x_s = x_0\), and if for all \(z \in D(f)\) there exists a constant speed geodesic \(\gamma^s : [0, 1] \to X\) between \(x_s\) and \(z\) such that

\[-\frac{d}{dt}\sigma^{(t)}_{\kappa, N}(|\dot{\gamma}^s|)|_{t=1} < \infty \quad \text{and} \quad \frac{d}{dt}\sigma^{(t)}_{\kappa, N}(|\dot{\gamma}^s|)|_{t=0} < \infty\]

and the evolution variational inequality

\[-\frac{1}{2N} \frac{d}{ds} d_x(x_s, z)^2 + \frac{d}{dt}\sigma^{(t)}_{\kappa, N}(|\dot{\gamma}^s|)|_{t=1} \geq \frac{d}{dt}\sigma^{(t)}_{\kappa, N}(|\dot{\gamma}^s|)|_{t=0} U_N(z) / U_N(x_s)\]

holds for a.e. \(s > 0\). If \(N = \infty\), we say \(x_s\) is an \(EVI_{\kappa, \infty}\) gradient flow curve of \(f\) if for all \(z \in D(f)\) there exists a constant speed geodesic \(\gamma^s : [0, 1] \to X\) between \(x_s\) and \(z\) such that

\[
\frac{d}{ds} \frac{d_x(x_t, z)^2}{2} + \int_0^1 \kappa(\gamma^s(t)) dt d_x(x_s, z)^2 \leq f(x_s) - f(z)
\]

holds for a.e. \(s > 0\).

**Remark 3.26.** The definition of \(EVI_{\kappa, \infty}\) for gradient flows already appears [Sim].

**Lemma 3.27.** If \((x_s)_{s \in [0, \infty)}\) is an \(evi_{\kappa, N}\) gradient flow curve of \(f\), then it is also an \(evi_{\kappa', N'}\) gradient flow curve for any \(\kappa' \leq \kappa\) and \(N' \geq N\) where \(\kappa'\) is a lower semi-continuous function and \(N' \in [N, \infty)\).
Proof. The case \( N' < \infty \) follows directly from Lemma 3.18. If \( N' = \infty \), we rewrite the \( EVI_{\kappa, N} \) formula as

\[
- \frac{1}{2N} \frac{d}{ds} d_x(x_s, z)^2 - \frac{d}{dt} \left[ \sigma^{(1-\tau)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|) + \sigma^{(\tau)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|) \right]_{t=0} \\
\geq \frac{d}{dt} \sigma^{(\tau)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|)_{t=0} \left[ \frac{U_N(z)}{U_N(x_s)} - 1 \right]
\]

If we multiply the inequality with \( N \), we see that the right hand side converges to 
\(-f(z) + f(x_s)\) for \( N \to \infty \). To see what happens on the left hand side we define for fixed \( s \in (0, \infty) \)

\[
\phi(t) := \sigma^{(1-\tau)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|) + \sigma^{(\tau)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|) - 1.
\]

\( \phi \) solves \( u''(t) + \kappa(\gamma(t))/N d_x(z, x_s) [\phi(t) + 1] = 0 \) with \( u(0) = u(1) = 0 \). Hence, we have the following representation

\[
\phi(t) = \int_0^1 g(\tau, t) \kappa(\gamma^\tau(\tau))/N d_x(z, x_s) \left[ \sigma^{(1-\tau)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|) + \sigma^{(\tau)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|) \right] d\tau
\]

and

\[
\phi'(0) = \int_0^1 (1 - \tau) \kappa(\gamma^\tau(\tau))/N d_x(z, x_s) \left[ \sigma^{(1-\tau)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|) + \sigma^{(\tau)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|) \right] d\tau.
\]

If \( N \to \infty \), we see that \( \phi \) converges uniformly to 0. Therefore, by the previous formula of \( \phi'(0) \), \( N\phi'(0) \) converges to \( \int_0^1 (1 - \tau) \kappa(\gamma^\tau(\tau)) d_x(z, x_s) d\tau \).

**Theorem 3.28.** Let \((X, d_x)\) be a locally compact metric measure space, and let \( f : D(f) \to \mathbb{R} \) be lower semi-continuous. Assume that for every \( x_0 \in D(f) \) there exists an \( EVI_{\kappa, N} \) gradient flow curve \((x_s)_{s \in (0, \infty)}\) starting in \( x_0 \). Then \( f \) is strongly \((\kappa, N)\)-convex.

**Proof.** First, we assume that \( \kappa : X \to \mathbb{R} \) is continuous. Let \( c : [0, 1] \to X \) be a constant speed geodesic, and let \( \tilde{c} : [0, \theta] \to X \) is its 1-speed reparametrization. Let \( \delta > 0 \) be arbitrary. Since \((X, d_x)\) is locally compact, we can find \( h > 0 \) and points \( r_i \in [0, \theta] \) for \( i = 1, \ldots, N \) such that

\[
\max \kappa|_{B_{2h}(\tilde{c}(r_i))} - \min \kappa|_{B_{2h}(\tilde{c}(r_i))} < \delta
\]

for each \( i = 1, \ldots, N \). Now, we pick \( \tilde{r} \in [0, \theta] \) and \( \epsilon > 0 \), and consider \( \tilde{\gamma} = \tilde{c}|_{[\tilde{r} - \epsilon, \tilde{r} + \epsilon]} \) such that \( \tilde{r} \pm \epsilon \in [r_i - h, r_i + h] \) for some \( i = 1, \ldots, N \). Its constant speed reparametrization is \( \gamma : [0, 1] \to X \). Let \( x_s \) be the \( EVI_{\kappa, N} \) gradient flow curve starting in \( \gamma(\frac{1}{2}) \). Then, we obtain

\[
- \frac{1}{2N} \frac{d}{ds} d_x(x_s, \gamma(0))^2 + \frac{d}{dt} \sigma^{(i)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|)_{t=1} \geq \frac{d}{dt} \sigma^{(i)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|)_{t=0} \frac{U_N(\gamma(0))}{U_N(x_s)}
\]

and

\[
- \frac{1}{2N} \frac{d}{ds} d_x(x_s, \gamma(1))^2 + \frac{d}{dt} \sigma^{(i)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|)_{t=1} \geq \frac{d}{dt} \sigma^{(i)}_{\kappa^*_N/\tilde{N}}(|\hat{\gamma}_s^\tau|)_{t=0} \frac{U_N(\gamma(1))}{U_N(x_s)}
\]

where \( \gamma^\tau : [0, 1] \to X \) \( i = 0, 1 \) are constant speed geodesics between \( x_s \) and \( \gamma(i) \). Since \( \gamma \) is a constant speed geodesic, it follows

\[
\frac{1}{2} d_x(\gamma(0), x_s)^2 + \frac{1}{2} d_x(\gamma(1), x_s)^2 \geq \frac{1}{2} d_x(\gamma(0), x_0)^2 + \frac{1}{2} d_x(\gamma(1), x_0)^2
\]
and therefore
\[
\frac{1}{2} \frac{d}{ds} |_{s=0} dx(\gamma(0), x_s)^2 + \frac{1}{2} \frac{d}{ds} |_{s=0} dx(\gamma(1), x_s)^2 \geq 0
\]
Together with the previous observation it follows
\[
\begin{align*}
U_N(x_s) \left[ \frac{d}{dt} \sigma^{(s)}_{\gamma/\kappa} |_{t=1} \left( |\dot{\gamma}_1| \right) \right] + \frac{d}{dt} \sigma^{(s)}_{\kappa/\gamma} |_{t=0} U_N(\gamma(0)) \geq \frac{d}{dt} \sigma^{(s)}_{\kappa/\gamma} |_{t=0} U_N(\gamma(1))
\end{align*}
\]
Now, let \( s \to 0 \). Since \((X, dx, m_X)\) is locally compact and the length of \( \gamma_i^s (i = 0, 1) \) is uniformly bounded, there exist uniformly converging subsequences of \( \gamma_0^s \) and \( \gamma_1^s \) w.r.t. \( d_{\infty} \). The limits are denoted by \( \varsigma_0 \) and \( \varsigma_1 \). By lower semi-continuity of the length function one can see that the composition of \( \varsigma_0 \) and \( \varsigma_1 \) is again a geodesic between \( \gamma(0) \) and \( \gamma(1) \). Its constant speed reparametrization is denoted with \( \varsigma : [0, 1] \to X \) and its 1-speed reparametrization is denoted with \( \xi \). By construction we have \( \varsigma(\frac{1}{2}) = \gamma(\frac{1}{2}) \). Note, that \( |\xi| = \frac{1}{2} |\varsigma_0| = \frac{1}{2} |\varsigma_1| = |\varsigma| = 2\varepsilon \) and \( \text{Im} \varsigma \subset B_{2\varepsilon}(\xi(\varepsilon_1)) \).

Recall that by Lemma 3.20 \( \frac{d}{dt} |_{0} \sigma^{(s)}_{\kappa} \) and \( \frac{d}{dt} |_{1} \sigma^{(s)}_{\kappa} \) are lower and upper semi-continuous respectively with respect to \( \kappa \) (Lemma 3.20). Hence
\[
\begin{align*}
U_N(\gamma(\frac{1}{2})) \left[ \frac{d}{dt} \sigma^{(s)}_{\kappa/\gamma} |_{t=0} |\varsigma_0| \right] + \frac{d}{dt} \sigma^{(s)}_{\kappa/\gamma} |_{t=0} U_N(\gamma(0)) \geq \frac{d}{dt} \sigma^{(s)}_{\kappa/\gamma} |_{t=0} U_N(\gamma(1))
\end{align*}
\]
Now, we use Taylor expansion of the coefficients. Recall from [3.20] that for \( t \) fixed \( f : h \mapsto \sigma^{(s)}_h(h) \) is twice differentiable at \( h = 0 \) and we have
\[
\begin{align*}
\sigma^{(s)}_h(t) = t \left[ 1 + \frac{1}{6} (1 - t^2) \kappa(0) h^2 \right] + o(h^2 t^2).
\end{align*}
\]
If \( \pi \geq \kappa \geq \kappa \), then
\[
\begin{align*}
\frac{1}{3} (1 - t^2) (\kappa - \pi) h^2 + o(h^2 t^2) \leq o(h^2 t^2) \leq \frac{1}{3} (1 - t^2) (\pi - \kappa) h^2 + o(h^2 t^2).
\end{align*}
\]
Therefore, if \( \gamma \in \mathcal{G}(X) \)
\[
\begin{align*}
\frac{d}{dt} \sigma^{(s)}_{\kappa/\gamma} |_{t=0} |\dot{\gamma}| \geq 1 + \frac{1}{6} \kappa(\gamma(0)) |\dot{\gamma}|^2 + \frac{1}{3} (\kappa - \pi) |\dot{\gamma}|^2 + \frac{d}{dt} o(|\dot{\gamma}|^2) |_{t=0},
\end{align*}
\]
and
\[
\begin{align*}
\frac{d}{dt} \sigma^{(s)}_{\kappa/\gamma} |_{t=0} |\dot{\gamma}| \geq 1 + \frac{1}{6} \kappa(\gamma(1)) |\dot{\gamma}|^2 + \frac{1}{3} (\kappa - \pi) |\dot{\gamma}|^2 + \frac{d}{dt} o(|\dot{\gamma}|^2) |_{t=0}.
\end{align*}
\]
Now, consider \( \varsigma_0 \) and \( \varsigma_1 \). Let \( \kappa := \min \kappa|_{B_{2\varepsilon}(\xi(\varepsilon_1))} \) and \( \pi := \max \kappa|_{B_{2\varepsilon}(\xi(\varepsilon_1))} \). We plug this into \( 16 \):
\[
\begin{align*}
U_N(\gamma(\frac{1}{2})) \left[ 2 - \frac{1}{3} \kappa(\gamma(0)) e^2 - \frac{d}{dt} o(e^2) |_{t=0} - \frac{1}{3} \kappa(\gamma(1)) e^2 - \frac{d}{dt} o(e^2) |_{t=0} - \frac{2}{3} (\kappa - \pi) e^2 \right] \geq \left[ 1 - \frac{1}{6} \kappa(\gamma(\frac{1}{2})) e^2 + \frac{d}{dt} o(e^2) |_{t=0} + \frac{1}{3} (\kappa - \pi) e^2 \right] U_N(\gamma(0))
\end{align*}
\]
\[
+ \left[ 1 - \frac{1}{6} \kappa(\gamma(\frac{1}{2})) e^2 + \frac{d}{dt} o(e^2) |_{t=0} + \frac{1}{3} (\kappa - \pi) e^2 \right] U_N(\gamma(1)).
\]
Rearranging the terms yields
\[
2U_N(\gamma(\frac{1}{2})) - U_N(\gamma(0)) - U_N(\gamma(1)) \\
\geq -U_N(\gamma(\frac{1}{2}))\epsilon^2 \left[ \frac{1}{3} \kappa(0) + \frac{1}{3} \kappa(1) \right] - \frac{1}{6} \kappa(\gamma(\frac{1}{2}))\epsilon^2 U_N(\gamma(0)) \\
- \frac{1}{6} \kappa(\gamma(\frac{1}{2}))\epsilon^2 U_N(\gamma(1)) + o(\epsilon^2) + \frac{1}{3}(\kappa - \bar{\kappa})\epsilon^2 \left[ U_N(\gamma(1)) + U_N(\gamma(0)) + 2U_N(\gamma(\frac{1}{2})) \right].
\]
In terms of the geodesic \(c\) this is
\[
2U_N(\check{c}(\check{r})) - U_N(\check{c}(\check{r} - \epsilon)) - U_N(\check{c}(\check{r} + \epsilon)) \\
\geq -U_N(\check{c}(\check{r})) \left[ \frac{1}{3} \kappa(\check{r} - \epsilon)\epsilon^2 + \frac{1}{3} \kappa(\check{r} + \epsilon)\epsilon^2 \right] \\
- \frac{1}{6} \kappa(\check{r})\epsilon^2 U_N(\check{c}(\check{r} - \epsilon)) - \frac{1}{6} \kappa(\check{r})\epsilon^2 U_N(\check{c}(\check{r} + \epsilon)) + o(\epsilon) \\
+ \frac{1}{3}(\kappa - \bar{\kappa})\epsilon^2 \left[ U_N(\check{c}(\check{r} - \epsilon)) + U_N(\check{c}(\check{r} + \epsilon)) + 2U_N(\check{c}(\check{r})) \right].
\]
Deviding by \(\epsilon^2 > 0\), multiplication with \(\phi \in C_0^\infty((0, \theta))\) such that \(\phi \geq 0\), integration with respect to \(\epsilon\), a change of variables and taking the limit \(\epsilon \to 0\) yields
\[
\int_0^\theta U_N(\check{c}(t))\phi''(t)dt \leq \int_0^\theta \kappa(t) + \frac{4}{3} \delta U_N(\check{c}(t))\phi(t)dt.
\]
Recall that \(U_N\) is upper semi-continuous. Since \(\delta\) was arbitrary, the theorem follows from the characterization result of Proposition \ref{prop:characterization}.

Finally, if \(\kappa : X \to \mathbb{R}\) is lower semi-continuous, we choose \(\kappa_n \uparrow \kappa\) pointwise for \(\kappa_n\) bounded and continuous. By monotonicity the assumptions are satisfied for \(\kappa_n\) instead of \(\kappa\) for each \(n \in \mathbb{R}\). Hence, we can apply the first part of the proof, and we obtain \ref{eq:inequality} for \(\kappa_n\). But by the theorem of monotone convergence, this differential inequality still holds for \(\kappa\). \(\square\)

**Corollary 3.29.** If \((M, g_M)\) is a Riemannian manifold, \(f \in C^2(M)\) and \(\kappa \in C(M)\), the following statements are equivalent

(i) For every \(x_0 \in D(f)\) there exists an EVI \(\kappa, N\) gradient flow curve \((x_s)_{s \in (0, \infty)}\) starting in \(x_0\).

(ii) \(f\) is \((\kappa, N)\)-convex.

4. REDUCED AND ENTROPIC CURVATURE DIMENSION CONDITION

In this section we introduce an entropic curvature-dimension condition for metric measure spaces and semi-continuous lower curvature bound \(\kappa\). For this purpose we will apply the results of the previous section in the following context.

For \(\mu \in \mathcal{P}_2(X)\) we define the relative entropy by
\[
\text{Ent}(\mu) := \int \rho \log \rho d m_X
\]
if \(\mu \in \mathcal{P}(m_X)\) and \((\rho \log \rho)_+\) is integrable. Otherwise, we set \(\text{Ent}(\mu) = \infty\). Moreover, for \(N \in (0, \infty)\) we introduce the functional \(U_N : \mathcal{P}_2(X) \to [0, \infty]\) by
\[
U_N(\mu) := \exp \left(-\frac{1}{N} \text{Ent}(\mu)\right).
\]
If we assume the following volume growth condition
\begin{equation}
\int e^{-c(d(p,x)^2)} \, dm_x(x) < \infty
\end{equation}
it is well known that \( \text{Ent} \) does not take the value \(-\infty\) on \( \mathcal{P}_2(X) \) and \( \text{Ent} \) is lower semi-continuous with respect to \( W_2 \).

**Definition 4.1.** A metric measure space \( (X, d_X, m_X) \) satisfies the entropic curvature-dimension condition \( CD^\epsilon(\kappa, N) \) for some lower semi-continuous function \( \kappa \) and \( N \geq 0 \) if for any pair \( \mu_0, \mu_1 \in D(\text{Ent}) \) with compact support there exists a \( L^2 \)-Wasserstein geodesic \( \Pi \) connection \( \mu_0 \) and \( \mu_1 \) such that for all \( t \in [0,1] \)
\begin{equation}
U_N(\mu_t) \geq \sigma^{(i-t)}_{\kappa_{\Pi}/N}(\Theta)U_N(\mu_0) + \sigma^{(i)}_{\kappa_{\Pi}/N}(\Theta)U_N(\mu_1)
\end{equation}
where \( (e_t)_\pi = \mu_t, \Theta = W_2(\mu_0, \mu_1) \) and
\[ \kappa_{\Pi}(t\Theta) = \frac{1}{\Theta^2} \int \kappa(e_t(\gamma)|\dot{\gamma}|^2 d\Pi(\gamma). \]

Recall that \( \kappa_{\Pi} \) is lower semi-continuous by Remark 3.12. If (20) holds for any geodesic \( \Pi \in \mathcal{P}(G(X)) \) we say that \( (X, d_X, m_X) \) is a strong \( CD^\epsilon(\kappa, N) \) space. \( \mu \in D(\text{Ent}) \) implies that \( \mu \in \mathcal{P}(m_X) \).

**Remark 4.2.** The coefficient \( \sigma^{(i)}_{\kappa_{\Pi}/N}(\Theta) \) solves \( v''(t) + \int \kappa(e_t(\gamma)|\dot{\gamma}|^2 d\Pi(\gamma)v(t) = 0 \).
Therefore, we can write \( \sigma^{(i)}_{\kappa_{\Pi}/N}(\Theta) = \frac{1}{\Theta^2} \int \kappa(e_t(\gamma)|\dot{\gamma}|^2 d\Pi(\gamma). \)

**Remark 4.3.** The entropic curvature-dimension condition is not \( (K, N) \)-convexity of \( U_N \) for some function \( K \) on \( \mathcal{P}_2(X) \). More precisely, it is \( (\kappa_{\Pi}, N) \)-convexity on Wasserstein geodesics \( \Pi \) where \( \kappa_{\Pi} \) depends on the geodesic \( \Pi \).

**Lemma 4.4.** Let \( (X, d_X, m_X) \) be a metric measure space, and let \( \kappa \) be lower semi-continuous. Then
\[ \liminf_{t \to \infty} \sigma^{(i)}_{\kappa_{\Pi_i}/N}(\Theta) \geq \sigma^{(i)}_{\kappa_{\Pi}/N}(\Theta) \text{ for each } t \in [0,1] \]
if \( \Pi_i \) converges weakly to \( \Pi \).

**Proof.** One can easily check that for each \( t \in [0,1] \) \( \Pi \mapsto \int \kappa(e_t(\gamma)|\dot{\gamma}|^2 d\Pi(\gamma) = \kappa_{\Pi}(t\Theta)\Theta^2 \) is lower semi-continuous w.r.t. weak convergence. Then, the statement follows from Lemma 3.11.

**Lemma 4.5.** Let \( (X, d_X, m_X) \) be a metric measure space satisfying \( CD^\epsilon(\kappa, N) \) for some lower semi-continuous \( \kappa \) and \( N > 0 \).
\begin{enumerate}
\item[(i)] If \( \kappa' \) is admissible with \( \kappa' \leq \kappa \) and \( N' \geq N \), then \( (X, d_X, m_X) \) satisfies \( CD^\epsilon(\kappa', N') \).
In particular, \( (X, d_X, m_X) \) satisfies \( CD(\kappa, \infty) \) in the sense of [Stu].
\item[(ii)] Let \( V : X \to \mathbb{R} \) be a measurable function that is bounded from below and that is \( (\kappa', N') \)-convex in sense of Definition 3.14 for some admissible \( \kappa' \) and \( N' > 0 \).
\end{enumerate}
Proof. (i) The first part is an immediate consequence of Proposition 3.8. The second part follows by monotonicity in $N > 0$. Consider

\[
\begin{align*}
(1 - t) \text{Ent}(\mu_0) + t \text{Ent}(\mu_1) - \text{Ent}(\mu_t) \\
&= \lim_{N \to \infty} (1 - t) N (U_N(\mu_0) - 1) - t N (U_N(\mu_1) - 1) + N (U_N(\mu_t) - 1) \\
&\leq \lim_{N \to \infty} \left[ \frac{\sigma_{1/N}(\Theta) (1 - t)}{N} + \frac{\sigma_{t/N}(\Theta) (1 - t)}{N} \right] \\
&\leq \lim_{N \to \infty} N \left[ \frac{\sigma_{1/N}(\Theta)}{N} + \frac{\sigma_{t/N}(\Theta)}{N} - 1 \right] \\
&= \infty(t).
\end{align*}
\]

For large $N$ the function $v$ solves

\[
v'' + \int \kappa(e_1(\gamma)) \gamma \, d\Pi(\gamma) \leq 0.
\]

Therefore the RHS in (21) is smaller or equal than $\int \kappa(e_1(\gamma)) \gamma \, d\Pi(\gamma)$.

(ii) We define $\nabla : \mathcal{P}_2(X) \to (-\infty, \infty]$ by $\nabla(\mu) = \int Vd\mu$. We show that $\nabla$ is $(\kappa', N')$-convex. Recall that by Lemma 3.11 $g : (x, y, \kappa) \mapsto \log \left( \frac{\kappa + \sigma_{(1-t)\kappa}e^x}{\kappa + \sigma_{t\kappa}e^y} \right)$ is convex. Hence, if $\Pi$ is a geodesic in $\mathcal{P}_2(X)$, then

\[
-\frac{1}{N'} \nabla(\mu_\mu) = -\frac{1}{N'} \int V(e_1(\gamma)) \, d\Pi(\gamma) \\
\geq \int g \left( -\frac{1}{N'} V(e_0(\gamma)), -\frac{1}{N'} V(e_1(\gamma)), \kappa' \gamma |^2 / N' \right) \, d\Pi(\gamma) \\
\geq g \left( -\frac{1}{N'} \nabla(\mu_0), -\frac{1}{N'} \nabla(\mu_1), \kappa' \Theta^2 \right).
\]

After taking the exponential this $(\kappa', N')$-convexity of $\nabla$. Finally, since by lower boundedness of $V$ we have $\mathcal{P}(e^{-V} m_x) \subset \mathcal{P}(m_x)$, and since $\text{Ent}_{e^{-V} m_x} (\mu) = \text{Ent}_{m_x} (\mu) + \nabla(\mu)$, we obtain the result by Lemma 3.11.

**Definition 4.6** (Minkowski content). Consider $x_0 \in X$ and $B_r(x_0) \subset X$. Set $v(r) = m_x(\partial B_r(x_0))$. The Minkowski content of $\partial B_r(x_0)$ (the $r$-sphere around $x_0$) is defined as

\[
s(r) := \limsup_{\delta \to 0} \frac{1}{\delta} m_x(\partial B_{r+\delta}(x_0) \setminus B_r(x_0)).
\]

**Theorem 4.7.** Assume $(X, d_x, m_x)$ satisfies CD$(\kappa, N)$ for a lower semi-continuous function $\kappa$ and $N \in [1, \infty)$. Then, $(X, d_x)$ is a proper metric space, each bounded set has finite measure and satisfies a doubling property, and either $m_x$ is supported by one point or all points and all sphere have mass 0.

In particular, if $N > 1$ then for each $x_0 \in X$, for all $0 < r < R$ and $k \in \mathbb{R}$ such that $\kappa |_{B_R(x_0)} \geq k$ and $R \leq \pi \sqrt{N/k} \sqrt{0}$, we have

\[
\frac{s(r)}{s(R)} \geq \frac{\sin^{N/2} x}{\sin^{N/2} R} \quad \text{and} \quad \frac{m_x(B_r(x_0))}{m_x(B_R(x_0))} \geq \frac{\int_0^r \sin^{N/2} x \, dt}{\int_0^R \sin^{N/2} R \, dt}.
\]

If $N = 1$ and $\kappa \leq 0$, then $s(r) / s(R) \geq 1$ and $m_x(B_r(x_0)) / m_x(B_R(x_0)) \geq \frac{r}{R}$.

**Proof.** Theorem follows as in the proof of Theorem 5.3 in [Ket].

Remark 4.8. The estimates in the previous theorem are not sharp. Though, it is enough to prove that \((X, d_X, m_X)\) satisfies a doubling property and has finite Hausdorff dimension bounded from above by \(N\). See Corollary 5.4, Corollary 5.5 in [Ket] for the proofs. The entropic curvature-dimension condition \(CD^r(\kappa, N)\) yields that \((\text{supp} \, m_X, d_X)\) is a length space. Therefore, by local compactness and completeness it is geodesic, and therefore \(P_2(\text{supp} \, m_X)\) is geodesic as well. Additionally, if \(\kappa\) is bounded from below, the volume growth estimate \(22\) implies that

\[
\int e^{-c d_X(p, x)^2} \, d m_X(x) < \infty
\]

for some point \(p \in X\) and \(c > 0\), and in particular, \(\text{Ent} > -\infty\).

Recall that a sequence of \((X_i, d_{X_i}, m_{X_i})\) with \(m_{X_i}(X_i) < \infty\) converges in Gromov sense to a metric measure space \((X, d_X, m_X)\) if there exists a metric space \((Z, d_Z)\) and isometric embeddings \(i_i : X_i \to Z\) for \(i \in \mathbb{N}\) such that \((i_i)_* m_{X_i}\) converges weakly to \((i)_* m_X\). This can be equivalently defined in terms of Sturm’s transportation distance \(D\) (see [Stu06a]).

**Definition 4.9.** Let \((X_i, d_{X_i}, m_{X_i})\) be metric measure spaces converging in Gromov sense to a metric measure space \((X, d_X, m_X)\). Let \(\kappa_i, \kappa : X_i, X \to \mathbb{R}\) be lower semi-continuous functions. We say

\[
\liminf_{i \to \infty} \kappa_i \geq \kappa
\]

if for each \(\eta > 0\) there exists \(i_\eta \in \mathbb{N}\) such that \(\kappa_i(x) \geq \kappa(f_i(x)) - \eta\) if \(i \geq i_\eta\) for each \(x \in X_i\). We say \((\kappa_i)\) converges uniformly to \(\kappa\) if for each \(\eta > 0\) there exists \(\delta\) and \(i_\eta \in \mathbb{N}\) such that \(|\kappa_i(x) - \kappa(y)| < \eta\) if \(i \geq i_\eta\) and \(d(x, y) < \delta\).

**Theorem 4.10.** Let \((X_i, d_{X_i}, m_{X_i})\) be a sequence of metric measure spaces satisfying the condition \(CD^r(\kappa_i, N_i)\) respectively for lower semi-continuous functions \(\kappa_i\) and \(N_i \in [1, \infty)\). Assume \((X_i, d_{X_i}, m_{X_i})\) converges to a metric measure space \((X, d_X, m_X)\) in Gromov sense, and consider an admissible function \(\kappa : X \to \mathbb{R}\) and \(N \in [1, \infty)\) such that

\[
(23) \quad \liminf_{i \to \infty} \kappa_i \geq \kappa \geq K \quad \& \quad \limsup_{i \to \infty} N_i \leq N
\]

Then \((X, d_X, m_X)\) satisfies \(CD^r(\kappa, N)\).

**Proof.** Since \(m_X\) and \(m_X\) are finite, the Entropy functional on \(P_2(Z)\) is lower semi-continuous. Then, the proof is the same as the proof of corresponding stability results in [CMS]. The only additional information one needs is that

\[
\liminf_{i \to \infty} \int \kappa_i(\gamma(t)) |\dot{\gamma}(t)|^2 d\Pi_i(\gamma) \geq \int \kappa(\gamma(t)) |\dot{\gamma}(t)|^2 d\Pi(\gamma)
\]

if \((\Pi_i)_{i \in \mathbb{N}}\) with \(\Pi_i \in \mathcal{P}(G(X_i))\) converges weakly in \(\mathcal{P}(G(Z))\) to \(\Pi \in \mathcal{P}(G(X))\). Then, we can apply Lemma 3.9 to obtain the result. □

**Definition 4.11** ([Ket]). Consider a lower semi-continuous function \(\kappa : X \to \mathbb{R}\), and let \(N \in \mathbb{R}\) with \(N \geq 1\). \((X, d_X, m_X)\) satisfies the reduced curvature-dimension condition \(CD^r(\kappa, N)\) if and only if for each pair \(\nu_0, \nu_1 \in \mathcal{P}_2(X, m_X)\) with bounded support there exists a dynamical optimal coupling \(\Pi\) of \(\nu_0 = \theta_0 d m_X\) and \(\nu_1 = \theta_1 d m_X\) and a geodesic \((\nu_t)_{t \in [0, 1]} \subset \mathcal{P}_2(X, m_X)\), such that

\[
(24) \quad S_N(\nu_t) \leq -\left[ \sigma^{(1-t)}_{\kappa, N/\nu}([\gamma]) \theta_0(\nu(\gamma))^{-\frac{1}{N}} + \sigma^{(t)}_{\kappa, N/\nu}([\gamma]) \theta_1(\nu(\gamma))^{-\frac{1}{N}} \right] d\Pi(\gamma)
\]
for all $t \in [0,1]$ and all $N' \geq N$.

**Definition 4.12.** We say that a metric measure space $(X,d_X,m_X)$ is essentially non-branching if for any optimal dynamical plan $\Pi \in \mathcal{P}(\mathcal{G}(X))$ between absolutely continuous probability measures is supported on set of non-branching geodesics. More precisely, there exists $A \subset \mathcal{G}(X)$ such that $\Pi(A) = 1$ and for all $\gamma, \gamma' \in A$ we have the following property

$$
\gamma(t) = \gamma'(t) \quad \text{for all } t \in [0,\varepsilon] \quad \text{for some } \varepsilon > 0 \implies \gamma = \gamma'.
$$

**Lemma 4.13 (EKS15).** Let $(X,d_X,m_X)$ be an essentially non-branching metric measure space, and let $\Pi$ be an optimal dynamical coupling. Assume $\Pi = \sum_{i=1}^n \alpha_i \Pi_i$ for optimal dynamical couplings $\Pi_i$. If $(\epsilon_0), \Pi_i$ are mutually singular, then the family $(\epsilon_i), \Pi_i$ is mutually singular as well.

**Theorem 4.14.** Let $(X,d_X,m_X)$ be an essentially non-branching metric measure space, and let $\kappa$ be a lower semi-continuous function and $N \geq 1$. Then the following statements are equivalent.

(i) $(X,d_X,m_X)$ satisfies $CD^+(\kappa, N)$.

(ii) For each pair $\mu_0, \mu_1 \in \mathcal{P}_2(m_X)$ with bounded support there exists an optimal dynamical transference plan $\Pi$ with $(\epsilon_1), \Pi \in \mathcal{P}_2(m_X)$ such that

$$
\frac{1}{N} \log \phi_t(\gamma_t) \geq \sigma_{\kappa/\lambda/(\gamma)|\pi}^{1-t} \phi_t(\gamma_0) + \sigma_{\kappa/\lambda/(\gamma)|\pi}^{t} \phi_t(\gamma_1) - \frac{\alpha}{N}.
$$

for all $t \in [0,1]$ and $\Pi$-a.e. $\gamma \in \mathcal{G}(X)$. $\phi_t$ is the density of the push-forward of $\Pi$ under the map $\gamma \mapsto \gamma_t$.

(iii) $(X,d_X,m_X)$ satisfies $CD^+(\kappa, N)$.

**Proof.** “(i) $\iff$ (ii)”: The same equivalence was shown in [Kerry13] for the curvature-dimension condition $CD(\kappa, N)$ provided $(X,d_X,m_X)$ is a non-branching metric measure space, and it is obvious that same proof also works for the condition $CD^+(\kappa, N)$. If we assume that $(X,d_X,m_X)$ is essentially non-branching, we can apply the same proof using Lemma 4.13 (Compare also with Theorem 3.12 in EKS15)

“(ii) $\implies$ (iii)”: By the same argument as in [BS10] Lemma 2.11 one can show that the statement (ii) holds for all $\mu_0, \mu_1 \in \mathcal{P}_2(m_X)$. Now, let $\Pi$ be an optimal dynamical plan between $\mu_0$ and $\mu_1$ satisfying (26). It follows

$$
\frac{1}{N} \log \phi_t(\gamma_i) \geq \log \left[ \frac{\alpha}{\kappa/\lambda/(\gamma)|\pi}^{1-t} \phi_t(\gamma_0) + \frac{\alpha}{\kappa/\lambda/(\gamma)|\pi}^{t} \phi_t(\gamma_1) \right].
$$

Again, Lemma 3.11 says that $g_t : (x,y,\kappa) \mapsto \log \left( \frac{\alpha}{\kappa/\lambda/(\gamma)|\pi} e^\kappa + \frac{\alpha}{\kappa/\lambda/(\gamma)|\pi} e^{\kappa} \right)$ is convex. Therefore, integrating (26) with respect to $\Pi$ and applying Jensen’s inequality yields

$$
-\frac{1}{N} \text{Ent}(\mu_t) \geq g_t \left( -\frac{1}{N} \text{Ent}(\mu_0), -\frac{1}{N} \text{Ent}(\mu_1), \int \kappa(\epsilon_t(\gamma)|\Pi)|\pi| \kappa^2 d\Pi \right).
$$

Hence, statement (iii) follows by taking the exponential on both sides.

“(iii) $\implies$ (ii)”: We roughly follow the argument in the proof of EKS15 Theorem 3.12]. Let $\mu_0, \mu_1 \in \mathcal{P}_2(m_X)$ be with bounded support, and let $\Pi$ be an optimal dynamical coupling between $\mu_0$ and $\mu_1$. Let $\{M_t\}_{t \in [0,1]}$ be an $\cap$-stable generator of the Borel $\sigma$-field of $(X,d_X)$. For each $n$ we define a disjoint covering of $X$ of $2^n$ sets by $L_I = \bigcap_{i \in I} M_i \cap \bigcap_{i \in I^c} M_i^c$ where $I \subset \{1, \ldots, n\}$. 
We fix \( n \), and we define \( A_{i,j} = \{ \gamma \in \mathcal{G}(X) : (\gamma(0), \gamma(1)) \in L_i \times L_j \} \) for \( i, j = 1, \ldots, 2^n \), and probability measures \( \mu^{i,j}_t = \Pi(A_{i,j})^{-1}(\alpha) \), \( \Pi(A_{i,j}) \) for \( t = 0, 1 \) provided \( \Pi(A_{i,j}) > 0 \). By (iii) there are optimal dynamical plans \( \Pi^{i,j} \) between \( \mu^{i,j}_0 \) and \( \mu^{i,j}_1 \) such that (20) holds, and \( \mu^{i,j}_t = (\epsilon_t)_\Pi^{i,j} \) are absolutely continuous w.r.t. \( \mu_X \).

Then, we define an optimal dynamical coupling between \( \mu_0 \) and \( \mu_1 \) by

\[
\Pi^n = \sum_{i,j=1}^{2^n} \Pi(A_{i,j}) \Pi^{i,j}.
\]

Since \( \mu^{i,j}_t \) are absolutely continuous, \( \mu^{i,j}_t = (\epsilon_t)_\Pi^{i,j} \), and \( \Pi^n \) is absolutely continuous as well, and since the measures \( \mu^{i,j}_0 \) are mutually singular, \( \mu^{i,j}_t \) are mutually singular as well by Lemma 4.13. Therefore, \( \rho^n_t = \sum \Pi(A_{i,j}) \rho^{i,j}_t \) with \( \rho^{i,j}_t |_{A_{i,j}} = \Pi(A_{i,j}) \rho^{i,j}_t \). Hence, (20) becomes after taking logarithms (set \( \frac{\Pi(A_{i,j})}{n} = \alpha_{i,j} \))

\[
- \alpha_{i,j} \int_{A_{i,j}} \log \rho^{i,j}_t(\epsilon_t)\gamma) d\Pi^n(\gamma)
\]

(27)

Since \( \mu_0, \mu_1 \) have bounded support, all measures under consideration are supported by a common compact subset in \( X \) and \( \mathcal{G}(X) \) respectively independent of \( n \). Therefore, up to extraction of subsequences Prohorov’s yields that \( \Pi^n \) converges weakly to a dynamical coupling \( \bar{\Pi} \in \mathcal{P}(\mathcal{G}(X)) \) that is optimal by lower semi-continuity of the Wasserstein distance under weak convergence.

Now, note that the relative Entropy \( \operatorname{Ent}_{\mu_X | \mu_i(A_{i,j})} \) is lower semi-continuous. First, this implies that we can pass to the limit in the LHS of (27) for \( n \to \infty \). Second, by Lemma 4.3

\[
\lim_{n \to \infty} \sigma_{\kappa_n}^{(t)} \geq \sigma_{\bar{\kappa}}^{(t)}
\]

where \( \bar{\kappa}(t) := \int_{A_{i,j}} \kappa((\epsilon_t)_\gamma) |\gamma|^2 d\Pi(\gamma) \). Hence, (27) also holds if we replace \( \Pi^n \) by \( \Pi \). Finally, by convexity of \( g_t \) and Jensen’s inequality (27) also holds when we replace \( A_{i,j} \) by \( A \) where \( A \) is a disjoint union of sets \( A_{i,j} \) for \( i, j \in \{1, \ldots, 2^n\} \) and \( n \in \mathbb{N} \). Therefore, (27) holds for any set in the \( \cap \)-stable generator and consequently the inequality holds for \( \Pi \)-a.e. \( \gamma \). \( \square \)

5. Riemannian curvature-dimension condition

Let \( (X, d_X, m_X) \) be a metric measure space. We will briefly repeat some concepts for calculus on metric measure spaces. For any function \( u : X \to \mathbb{R} \) in \( L^2(m_X) \) the Cheeger energy \( \operatorname{Ch}^X(u) \) can be defined by

\[
\operatorname{Ch}^X(u) = \frac{1}{2} \inf \left\{ \liminf_{h \to 0} \int_X |\nabla u_h|^2 d m_X : \|u_h - u\|_{L^2(m_X)} \to 0 \right\}.
\]

The \( L^2 \)-Sobolev space is given by \( D(\operatorname{Ch}^X) = \{ u \in L^2(m_X) : \operatorname{Ch}^X(u) < \infty \} \). An important fact is that \( \operatorname{Ch} \) is not a quadratic form in general.

**Definition 5.1.** We say that a metric measure space \( (X, d_X, m_X) \) is *infinitesimally Hilbertian* if the associated Cheeger energy is quadratic.
**Definition 5.2.** Let \((X, d_X, m_X)\) be a metric measure space, and let \(\kappa : X \to \mathbb{R}\) be lower semi-continuous and bounded from below. We say that \((X, d_X, m_X)\) satisfies the Riemannian curvature-condition \(RCD^*(\kappa, N)\) for \(N \geq 1\) if \((X, d_X, m_X)\) is infinitesimally Hilbertian and satisfies the condition \(CD^*(\kappa, N)\).

In [AGS13] the authors show that \(\text{Ch}^X\) can be represented by

\[
\text{Ch}^X(u) = \frac{1}{2} \int_X |\nabla u|^2 d m_X \quad \text{if } u \in D(\text{Ch})
\]

and \(+\infty\) otherwise where \(|\nabla u| : X \to [0, \infty)\) is Borel measurable and called the minimal weak upper gradient of \(u\).

In particular, \(\text{Ch}^X\) is convex and lower semi-continuous. This allows to define a Laplacian \(L^X\) on \(L^2(m_X)\) as the \(L^2\)-norm subdifferential of \(\text{Ch}^X\). \(L^X\) is not a linear operator in general. Still, the classical theory of gradient flows of convex functionals in Hilbert spaces yields that for any \(f \in L^2(m_X)\) there is a unique, locally absolutely continuous flow curve \((f_t)_{t>0}\) starting at \(f\) such that

\[
d^+ \frac{d}{dt} f_t = L f_t \quad \text{for all } t > 0.
\]

On the other hand, one can study the metric gradient flow of the relative entropy \(\text{Ent} \in \mathcal{P}_2(X)\) in the sense of Lott, Sturm and Villani. This also gives a semi-group \(H_t\) on \(\mathcal{P}_2(X)\). Then, the main result in [AGS14a] is the following identification between the two gradient flows.

**Theorem 5.3.** Let \((X, d_X, m_X)\) be a \(CD(K, \infty)\) space and let \(f \in L^2(m_X)\) such that \(d\mu = f d m_X \in \mathcal{P}_2(X)\). Then \(dH_t \mu = (P_t f) d m_X\).

**Definition 5.4.** We say that a metric measure space \((X, d_X, m_X)\) satisfies the evolution-variational inequality - \(EVI_{\kappa, N}\) - for some lower semi-continuous function \(\kappa : X \to \mathbb{R}\) and \(N \geq 1\) if for every \(\overline{\mu} \in \mathcal{P}_2(X)\) there exists a curve \((\mu^s)_{s \in (0, \infty)}\) in \(D(\text{Ent})\) with \(\lim_{s \to 0} \mu^s = \overline{\mu}\), and for each \(s > 0\) and each \(\nu \in \mathcal{P}_2(X)\) there exists a geodesic \(\Pi^s \in \mathcal{P}(\mathcal{G}(X))\) between \(\mu^s\) and \(\nu\) such that

\[
\frac{d}{dt} \bigg|_{t=1} \sigma_{\kappa^{\mu^s}/N}^{(t)}(\Theta^s) > -\infty \quad \text{and} \quad \frac{d}{dt} \bigg|_{t=0} \sigma_{\kappa^{\mu^s}/N}^{(t)}(\Theta^s) < \infty
\]

for each \(s > 0\) and

\[
-\frac{1}{N} \frac{d}{ds} W_2(\mu^s, \nu)^2 + \frac{d}{dt} \bigg|_{t=1} \sigma_{\kappa^{\mu^s}/N}^{(t)}(\Theta^s) \geq \frac{d}{dt} \bigg|_{t=0} \sigma_{\kappa^{\mu^s}/N}^{(t)}(\Theta^s) \frac{U_N(\nu)}{U_N(\mu^s)}
\]

where \(\Theta^s = W_2(\mu^s, \nu)\). We also say that \(\mu^s\) is a \(L^2\)-Wasserstein \(EVI_{\kappa, N}\) gradient flow curve.

**Remark 5.5.** In [Stu] Sturm makes the following definition. We say that a metric measure space \((X, d_X, m_X)\) satisfies \(EVI_{\kappa, \infty}\) if for every \(\overline{\mu} \in \mathcal{P}_2(X)\) there exists a curve \((\mu^s)_{s \in (0, \infty)}\) in \(D(\text{Ent})\) and \(\Pi^s \in \mathcal{P}(\mathcal{G}(X))\) between \(\mu^s\) and \(\nu\) as in Definition 5.4 such that

\[
\frac{d}{ds} W_2(\mu^s, \nu)^2 + \int_0^1 (1-t) \kappa^{\mu^s}(t\Theta^s) dt \leq \text{Ent}(\mu^s) - \text{Ent}(\nu)
\]
holds for a.e. \( t > 0 \) where \( \Theta^* := W_2(\mu^s, \nu) \). \( \mu^s \) is a \( L^2 \)-Wasserstein EVI\(_{K,\infty} \) gradient flow curve.

**Remark 5.6.** The implications of Lemma 3.24 hold as well on the level of Wasserstein gradient flows. In particular EVI\(_{K,N} \) implies EVI\(_{K,\infty} \).

**Theorem 5.7.** Let \( (X, d_X, m_X) \) be a metric measure space with \( \text{supp} m_X = X \), and let \( \kappa \) be a lower semi-continuous function with \( \kappa \geq K \in \mathbb{R} \) and \( N \geq 1 \). Then the following tree statements are equivalent:

1. \( (X, d_X, m_X) \) is infinitesimally Hilbertian and satisfies \( CD^s(\kappa, N) \).
2. \( (X, d_X, m_X) \) is infinitesimally Hilbertian and satisfies \( CD^c(\kappa, N) \).
3. \( (X, d_X, m_X) \) is a length space that satisfies the volume growth condition \( 19 \) and EVI\(_{K,N} \).

**Proof.** “(i) \( \Leftrightarrow \) (ii)”: Both conditions - \( CD^s(\kappa, N) \) and \( CD^c(\kappa, N) \) - imply a condition \( CD(K, \infty) \) (see also [Ket]). Therefore, from AGS14b follows that \( (X, d_X, m_X) \) satisfies EVI\(_K \). Hence, \( (X, d_X, m_X) \) is essentially non-branching by RST14, and Theorem 4.14 yields the equivalence of \( CD^s(\kappa, N) \) and \( CD^c(\kappa, N) \).

“(ii) \( \Rightarrow \) (iii)”: By Remark 4.18, \( (X, d_X) \) is a geodesic space that satisfies the volume growth condition \( 19 \). Therefore, since \( CD^c(\kappa, N) \) implies \( RCD(K, \infty) \) the main result of AGS14b yields the existence of EVI\(_K\)-gradient flow curves. Additionally, in AGMR15 the authors prove that for “good” geodesics \( \mu_t \) in \( \mathcal{P}_2(X) \), one has

\[
\frac{d}{dt} \int X d\mu_t^s = \frac{1}{2} \int d^2_{W_2} \mu_t^s \mu_t^1 \leq \frac{d}{dt} \text{Ent}(\mu_t) \bigg|_{t=0}.
\]

But \( (X, d_X, m_X) \) already satisfies \( CD(K, \infty) \) and has a quadratic Cheeger energy. Hence, it satisfies the condition \( RCD(K, \infty) \) in the sense of AGS14b. Wasserstein geodesics in \( \mathcal{P}_2(m_X) \) are unique, and therefore are good geodesic in the sense of AGMR15. Then, we can copy the proof of Lemma 3.22.

“(iii) \( \Rightarrow \) (ii)”: Since \( \kappa \) is bounded from below and by monotonicity of

\[
\left. \frac{d}{dt} \right|_{t=0} \sigma^{(\kappa)}_{\kappa} & \frac{d}{dt} \left|_{t=1} \sigma^{(\kappa)}_{\kappa} \right.
\]

\( (X, d_X, m_X) \) already satisfies EVI\(_K \), and consequently it is infinitesimally Hilbertian by AGS14b.

We will prove the entropic curvature-dimension condition \( CD^c(\kappa, N) \) following the proof of Theorem 3.28. But, recall that the entropic curvature dimension condition for variable \( \kappa \) is not just \( (\kappa, N) \)-convexity of the entropy.

First, assume \( \kappa \) is continuous on \( X \). Pick a \( L^2 \)-Wasserstein geodesic \( \Pi \) in \( \mathcal{P}(\mathcal{G}(X)) \) and let \( (e_i)_\Pi \Pi = \mu_t \). Let \( \hat{\mu} : [0, \Theta] \rightarrow X \) be its 1-speed reparametrization. Note that \( \kappa_\Pi(\Theta) \Theta^2 =: K_\Pi(t) \) is just lower semi-continuous. Therefore we replace it by functions \( K_{\Pi,n} : [0, 1] \rightarrow \mathbb{R} \) that are continuous monotone non-decreasing and converge pointwise to \( K_\Pi \). Let \( \delta > 0 \) be arbitrary. Since \( K_{\Pi,n} \) is continuous, we have that \( K_{\Pi,n}(\cdot/\Theta) \) is uniformly continous on \([0, \Theta]\). Hence, we can find \( h > 0 \) and points \( r_i \in [0, \Theta] \) for \( i = 1, \ldots, N \) such that

\[
\max_{B_{2h}(r_i)} K_{\Pi} - \min_{B_{2h}(r_i)} K_{\Pi} < \delta
\]
for each $i = 1, \ldots, N$. Now, we pick $\tilde{r} \in [0, \theta]$ and $\epsilon > 0$, and consider $\tilde{\gamma} = \tilde{\mu} |_{[\tilde{r} - \epsilon, \tilde{r} + \epsilon]}$ such that $\tilde{r} \pm \epsilon \in [r_i - h, r_i + h]$ for some $i = 1, \ldots, N$. Its constant speed reparametrization is $\gamma : [0, 1] \to P_2(X)$. Let $\nu^s$ be the $EVI_{K,N}$ gradient flow curve starting in $\mu^1_1$. Then, we obtain

$$-\frac{1}{2N} \frac{d}{ds} W_2(\mu^s, (e_0), \Pi)^2 + \frac{d}{dt} \kappa_0^{\nu^s} / N (\tilde{\Theta}_0^s)_{|t=1} \geq \frac{d}{dt} \kappa_0^{\nu^s} / N (\tilde{\Theta}_0^s)_{|t=0} \frac{U_N(\gamma(0))}{U_N(\mu^s)}$$

where $\tilde{\Pi}_0^s$ and $\tilde{\Pi}_1^s$ are geodesics between $(e_0), \Pi$ and $\nu^s$, and $(e_1), \Pi$ and $\nu^s$ respectively, and $\tilde{\Theta}_0^s = W_2((e_0), \Pi, \nu^s)$ and $\tilde{\Theta}_1^s = W_2((e_1), \Pi, \nu^s)$. Local compactness of $X$ yields weak convergence of $\tilde{\Pi}_0^s$ for $s \to 0$.

By lower semi-continuity of the $L^2$-Wasserstein distance the limits $\tilde{\Pi}_0$ and $\tilde{\Pi}_1$ are geodesic between $(e_0), \Pi$ and $\nu^s$, and $(e_1), \Pi$ and $\nu^s$ respectively. Additionally, the lower semi-continuity yields that the concatenation of the geodesics $\tilde{\Pi}_0$ and $\tilde{\Pi}_1$ as absolutely continuous curves in $P_2(X)$ w.r.t. $W_2$ is a geodesic as well. But we know that $X$ already satisfies a condition $RCD^*(K, \infty)$ for some $K$. Hence, $L^2$-Wasserstein geodesics between absolutely continuous probability measures are unique, and therefore we have $\Pi = \tilde{\Pi}$.

As in the proof of Theorem 3.28 we obtain a weak differential inequality for $U_N$ along $\Pi$, $K_{\Pi \nu}$ and $\delta$. By standard convergence results and monotonicity properties the statement follows as in the proof of Theorem 3.28.

**Theorem 5.8.** Let $(X_i, d_{X_i}, m_{X_i})_{i \in \mathbb{N}}$ be a sequence of metric measure spaces with $m_{X_i} < \infty$ converging in Gromov sense to a metric measure space $(X, d_X, m_X)$. Let $\kappa_i : X_i \to \mathbb{R}$ be lower semi-continuous functions such that $(X_i, d_{X_i}, m_{X_i})$ satisfies the condition $RCD^*(K_i, N_i)$. Additionally, consider an admissible function $\kappa : X \to \mathbb{R}$ and $N \in [1, \infty)$ such that

$$\liminf_{i \to \infty} \kappa_i \geq \kappa \geq K \in \mathbb{R} \quad \text{and} \quad \limsup_{i \to \infty} N_i \leq N$$

Then $(X, d_X, m_X)$ satisfies the condition $RCD^*(\kappa, N)$.

**Proof.** Since $\kappa_i$ and $\kappa$ are bounded from below by a constant $K$, $(X, d_X, m_X)$ already satisfies the condition $RCD^*(K, N)$. Then, by combination of Theorem 4.10 and Theorem 5.7 the result follows.

**Corollary 5.9.** Let $(M_i, g_{M_i})_{i \in \mathbb{N}}$ be a family of compact Riemannian manifolds such that $\text{ric}_{M_i} \geq \kappa_i \geq C \text{ and } \dim_{M_i} \leq N$ where $\kappa_i : M_i \to \mathbb{R}$ is a family of equi-continuous functions such that $\kappa_i \geq -C$ for some $C > 0$. There exists subsequence of $(M_i, g_{M_i}, m_{M_i})$ that converges in measured Gromov-Hausdorff sense to a metric measure space $(X, d_X, m_X)$, and there exists a subsequence of $\kappa_i$ such that $\lim \kappa_i = \kappa$. Then $X$ satisfies the condition $RCD^*(\kappa, N)$.

**Proof.** Since there is uniform lower bound for the Ricci curvature, Gromov’s compactness theorem yields a converging subsequence. Then, Gromov’s Arzela-Ascoli theorem also yields a uniformly converging subsequence of $\kappa_i$ with limit $\kappa$. Finally, if we apply the previous stability theorem, we obtain the result.
6. Wasserstein contraction

From EVI_{\kappa,\infty} one can deduce easily a Wasserstein contraction estimate.

**Theorem 6.1.** Let \((X, d_X, m_X)\) be a metric measure spaces satisfying EVI_{\kappa,N} where \(\kappa : X \to \mathbb{R}\) is lower semi-continuous. Consider Wasserstein EVI_{\kappa,N}-gradient flow curves \(\mu^s\) and \(\nu^s\) with initial measures \(\mu\) and \(\nu\). Let \(\Pi^s\) be the \(L^2\)-Wasserstein geodesic between \(\mu^s\) and \(\nu^s\). Then the following contraction estimate holds

\[
\frac{d^+}{ds} W_2(\mu_s, \nu_s)^2 \leq -2 \int_0^1 \kappa(\gamma(t)) |\dot{\gamma}|^2 d \Pi(\gamma)^* dt.
\]

**Proof.** Note that by lower semi-continuety of \(\kappa \circ d_X \circ m_X\) satisfies a condition RCD*(\(K, N\)) for some constant \(K\). In particular, Wasserstein geodesics between measures in \(\mathcal{P}^2(m_X)\) are unique. Consider \(s_0, s_1 \in [0, \infty)\). In [29] we set \(\mu^s = \mu^s\) and \(\nu = \nu^s\). Integration from \(s_0\) to \(s_1\) in \(s\) yields

\[
\frac{1}{2} W_2(\mu_{s_1}, \nu_{s_1})^2 - \frac{1}{2} W_2(\mu_{s_0}, \nu_{s_0})^2 + \int_{s_0}^{s_1} \left[ \int_0^1 (1-t) \kappa_{\mu^s}(t \Theta^s) dt \right] W_2(\mu^s, \mu^s)^2 ds
\]

\[
\leq [\text{Ent}(\mu_{s_0}) - \text{Ent}(\nu_{s_0})] (s_1 - s_0)
\]

where \(\Pi^s\) is the optimal dynamical plan between \(\mu^s\) and \(\nu\). We used that \(\text{Ent}\) is monotone decreasing along gradient flow curves. Similar, if we put \(\nu = \nu^{s_0}\) and \(\nu^s = \nu^s\) and integrate again from \(s_0\) to \(s_1\), then we obtain

\[
\frac{1}{2} W_2(\mu_{s_0}, \nu_{s_0})^2 - \frac{1}{2} W_2(\mu_{s_0}, \nu_{s_0})^2 + \int_{s_0}^{s_1} \left[ \int_0^1 (1-t) \kappa_{\nu^{s_0}}(t \Theta^{s_0}) dt \right] W_2(\mu^{s_0}, \nu^{s_0})^2 ds
\]

\[
\leq (\text{Ent}(\nu_{s_0}) - \text{Ent}(\mu_{s_0}))(s_1 - s_0)
\]

where \(\Pi^{s_0}\) is the optimal dynamical plan between \(\nu^{s_0}\) and \(\mu^{s_0}\). Adding the last two inequalities, deviding by \(s_1 - s_0\) and letting \(s_1 \to s_0\) yields

\[
\frac{d^+}{ds} \frac{1}{2} W_2(\mu_s, \nu_s)^2 \leq \left[ - \int_0^1 (1-t) \kappa_{\mu^{s_0}}(t \Theta^{s_0}) dt - \int_0^1 (1-t) \kappa_{\nu^{s_0}}(t \Theta^{s_0}) dt \right] W_2(\mu_{s_0}, \nu_{s_0})^2.
\]

Since there is a unique optimal dynamical plan between \(\nu^{s_0}\) and \(\mu^{s_0}\), we have that \(\Pi^{s_0} = \Pi^{s_0^-}\). Therefore

\[
\int_0^1 (1-t) \kappa_{\mu^{s_0}}(t \Theta^{s_0}) dt = \int_0^1 (1-t) \kappa_{\nu^{s_0}}((1-t) \Theta^{s_0}) dt = \int_0^1 t \kappa_{\mu^{s_0}}(t \Theta^{s_0}) dt
\]

Hence

\[
\frac{d^+}{ds} \frac{1}{2} W_2(\mu_s, \nu_s)^2 \leq \left[ - \int_0^1 \kappa_{\mu^s}(t \Theta^s) dt \right] W_2(\mu_s, \nu_s)^2 = - \int_0^1 \kappa(\gamma(t)) |\dot{\gamma}|^2 d \Pi(\gamma)^* dt.
\]

\(\square\)

**Remark 6.2.** Following the same lines as in the proof of previous theorem one obtains the following. If \((X, d_X)\) is a locally compact complete length space with unique geodesics and a \(\kappa\)-convex function \(f : X \to [0, \infty)\), we can deduce

\[
\frac{d^+}{ds} d_X(x_s, y_s)^2 \leq -2 \int_0^1 \kappa(\gamma^s(t)) dt d_X(x_0, y_0)^2
\]
where \( x_s \) and \( y_s \) are \( EVI_{\kappa,N} \)-gradient flow curves of \( f \). Then, an application of Gronwall’s lemma yields
\[
d_x(x_s, y_s)^2 \leq e^{-2 \int_0^t f_s^1 \kappa(\gamma^r(t)) dt} d_x(x_0, y_0)^2.
\]

The case \( N < \infty \). First, we deduce a contraction estimate for \( EVI_{\kappa,N} \)-gradient flow curves \( x_s \) and \( y_s \) for \( f \) on a metric space \((X, d_X)\) as in the previous remark. Consider
\[
\frac{1}{2N} \frac{d}{ds} d_x(x_s, z)^2 + \frac{d}{dt} \sigma^{(i)}_{\kappa_{s}^{-}/N}(|\dot{\gamma}_s|) \big|_{t=0} = \frac{1}{d_X} \frac{d}{dt} \sigma^{(i)}_{\kappa_{s}^{+}/N}(|\dot{\gamma}_s|) \big|_{t=0} \frac{U_N(z)}{U_N(x_s)}
\]
and rewrite as follows
\[
\frac{1}{2N} \frac{d}{ds} d_x(x_s, z)^2 + \frac{d}{dt} \bigg|_{t=0} \sigma^{(i)}_{\kappa_{s}^{-}/N}(|\dot{\gamma}_s|) + \sigma^{(i)}_{\kappa_{s}^{+}/N}(|\dot{\gamma}_s|) \bigg| = w
\]
rewrite as follows
\[
\frac{d}{dt} \bigg[ \sigma^{(i)}_{\kappa_{s}^{-}/N}(|\dot{\gamma}_s|) + \sigma^{(i)}_{\kappa_{s}^{+}/N}(|\dot{\gamma}_s|) \bigg] = \int_0^1 \left( \frac{1}{N} \right) \frac{\kappa(\tau)}{\kappa_{s}^{+}} \sigma^{(i)}_{\kappa_{s}^{+}/N}(|\dot{\gamma}_s|) + \sigma^{(i)}_{\kappa_{s}^{-}/N}(|\dot{\gamma}_s|) \bigg) d\tau |\dot{\gamma}|^2
\]
Hence, we can rewrite the right hand side of (31) as follows
\[
\frac{1}{2N} e^{-2N \int_0^s a_{s, r} d\tau} \frac{d}{ds} \left[ e^{2N \int_0^s a_{s, r} d\tau} d_x(x_s, z)^2 \right].
\]
Then (31) becomes
\[
\frac{1}{2N} e^{-2N \int_0^s a_{s, r} d\tau} d_x(x_s, z)^2 \leq e^{2N \int_0^s a_{s, r} d\tau} \frac{\sigma(\dot{\gamma}_s)}{\sigma(\dot{\gamma}_s)} \left[ 1 - \frac{U_N(z)}{U_N(x_s)} \right]
\]
and integration with respect to \( s \) from \( s_1 \) to \( s_2 \) yields
\[
\frac{1}{2N} e^{2N \int_{s_1}^{s_2} a_{s, r} d\tau} d_x(x_{s_2}, z)^2 - \frac{1}{2N} d_x(x_{s_1}, z)^2 \leq \int_{s_1}^{s_2} e^{2N \int_{s_1}^{s_2} a_{s, r} d\tau} \frac{\sigma(\dot{\gamma}_s)}{\sigma(\dot{\gamma}_s)} \left[ 1 - \frac{U_N(z)}{U_N(x_s)} \right] d\tau \]
Since \( s \rightarrow \frac{\sigma(\dot{\gamma}_s)}{\sigma(\dot{\gamma}_s)} \) is continuous, and since \( U_N \) is increasing w.r.t. \( x_s \), the right hand side can be estimated by
\[
\max_{\sigma \in [s_1, s_2]} \frac{\sigma(\dot{\gamma}_s)}{\sigma(\dot{\gamma}_s)} \int_{s_1}^{s_2} e^{2N \int_{s_1}^{s_2} a_{s, r} d\tau} d\tau - \min_{\sigma \in [s_1, s_2]} \frac{\sigma(\dot{\gamma}_s)}{\sigma(\dot{\gamma}_s)} \int_{s_1}^{s_2} e^{2N \int_{s_1}^{s_2} a_{s, r} d\tau} \frac{U_N(z)}{U_N(x_s)} d\tau.
\]
It follows
\[
m_{\gamma}(\sigma_1, s_2) U_N(z) \geq U_N(x_s) \leq m_{\gamma}(s_1, s_2)
\]

\[
\left[ 2N \int_{s_1}^{s_2} e^{2N \int_{s_1}^{s_2} a_{s, r} d\tau} d\tau \right]^{-1} d_x(x_{s_2}, z)^2 + \left[ 2N \int_{s_1}^{s_2} e^{2N \int_{s_1}^{s_2} a_{s, r} d\tau} d\tau \right]^{-1} d_x(x_{s_1}, z)^2
\]
Consider gradient flow curves \(x_s\) and \(y_s\) and choose \(\lambda, r > 0\). We apply the previous inequality for \(z = y_{\lambda^{-1}r}\) and \(s_1 = \lambda r\) and \(s_2 = \lambda(r + \epsilon)\) for some \(\epsilon > 0\).

\[
m_{\gamma}(\lambda r, \lambda(r + \epsilon)) \frac{U_N(y_{\lambda^{-1}r})}{U_N(x_{\lambda(r+\epsilon)})} \leq M_{\gamma}(\lambda r, \lambda(r + \epsilon))
- \left[ 2N \int_{\lambda r}^{\lambda(r+\epsilon)} e^{-2N \int_s^{\lambda(r+\epsilon)} a_{\gamma,s} ds} \right]^{-1} d_x(x_{\lambda(r+\epsilon)}, y_{\lambda^{-1}r})^2
+ \left[ 2N \int_{\lambda r}^{\lambda(r+\epsilon)} e^{2N \int_s^{\lambda(r+\epsilon)} a_{\gamma,s} ds} \right]^{-1} d_x(x_{\lambda r}, y_{\lambda^{-1}r})^2
\]

And similar if we switch the roles of \(x_s\) and \(y_s\), and if we set \(z = x_{\lambda(r+\epsilon)}\) and \(s_1 = \lambda^{-1}r, s_2 = \lambda^{-1}(r + \epsilon)\).

\[
m_{\hat{\gamma}}(\lambda^{-1}r, \lambda^{-1}(r + \epsilon)) \frac{U_N(x_{\lambda(r+\epsilon)})}{U_N(y_{\lambda^{-1}(r+\epsilon)})} \leq M_{\hat{\gamma}}(\lambda^{-1}r, \lambda^{-1}(r + \epsilon))
- \left[ 2N \int_{\lambda^{-1}r}^{\lambda^{-1}(r+\epsilon)} e^{-2N \int_s^{\lambda^{-1}(r+\epsilon)} a_{\gamma,s} ds} \right]^{-1} d_x(y_{\lambda^{-1}(r+\epsilon)}, x_{\lambda(r+\epsilon)})^2
+ \left[ 2N \int_{\lambda^{-1}r}^{\lambda^{-1}(r+\epsilon)} e^{2N \int_s^{\lambda^{-1}(r+\epsilon)} a_{\gamma,s} ds} \right]^{-1} d_x(y_{\lambda^{-1}r}, x_{\lambda(r+\epsilon)})^2
\]

where \(\hat{\gamma}\) is the geodesic between \(y^s\) and \(z\). We set
\[
\int_{\lambda r}^{\lambda(r+\epsilon)} e^{-2N \int_s^{\lambda(r+\epsilon)} a_{\gamma,s} ds} = e(\lambda, \epsilon, -a_{\gamma,s}), \quad \int_{\lambda r}^{\lambda^{-1}(r+\epsilon)} e^{2N \int_s^{\lambda^{-1}(r+\epsilon)} a_{\gamma,s} ds} = e(\lambda, \epsilon, a_{\gamma,s}).
\]

If we multiply the resulting two formulas, take square roots and use Young’s inequality \(2\sqrt{ab} \leq \lambda a + \lambda^{-1}b\) for \(\lambda\) as before, we obtain
\[
2N \sqrt{m_{\gamma}(\lambda r, \lambda(r + \epsilon)) \frac{U_N(y_{\lambda^{-1}r})}{U_N(x_{\lambda(r+\epsilon)})} m_{\hat{\gamma}}(\lambda^{-1}r, \lambda^{-1}(r + \epsilon))}
\leq 2N \left[ M_{\gamma}(\lambda r, \lambda(r + \epsilon))\lambda^{-1} + M_{\hat{\gamma}}(\lambda^{-1}r, \lambda^{-1}(r + \epsilon))\lambda \right]
+ d_x(y_{\lambda^{-1}r}, x_{\lambda(r+\epsilon)})^2 \left[ \frac{\lambda^{-1}}{e(\lambda^{-1}, \epsilon, a_{\gamma,s})} - \frac{\lambda}{e(\lambda, \epsilon, -a_{\gamma,s})} \right]
+ d_x(x_{\lambda r}, y_{\lambda^{-1}r})^2 \left[ \frac{\lambda}{e(\lambda, \epsilon, a_{\gamma,s})} - \frac{\lambda^{-1}}{e(\lambda^{-1}, \epsilon, -a_{\gamma,s})} \right]
- \frac{\epsilon}{e(\lambda^{-1}, \epsilon, -a_{\gamma,s})} \left[ d_x(y_{\lambda^{-1}(r+\epsilon)}, x_{\lambda r})^2 - d_x(y_{\lambda^{-1}r}, x_{\lambda r})^2 \right].
\]

Now, let \(\epsilon \to 0\). First, note that
\[
\frac{\lambda^{-1}\epsilon}{e(\lambda^{-1}, \epsilon, -a_{\gamma,s})} \to 1
\]
and
\[
\frac{\lambda}{e(\lambda, \epsilon, a_{\gamma,s})} \to -N(\lambda^{-1}a_{\gamma^{-1}r} + \lambda a_{\gamma^{-1}r}),
\frac{\lambda^{-1}}{e(\lambda^{-1}, \epsilon, -a_{\gamma,s})} \to -N(\lambda^{-1}a_{\gamma^{-1}r} + \lambda a_{\gamma^{-1}r}).
\]
Also note, that 

\[ m_\gamma(\lambda r, \lambda(r + \epsilon)), \quad M_\gamma(\lambda r, \lambda(r + \epsilon)) \to \frac{|\dot{\gamma}^{\lambda r}|}{\delta_{\gamma^{\lambda r}/N}(\dot{\gamma}^{\lambda r})} \]

\( \gamma^{\lambda r} \) is the unique geodesic between \( x_{\lambda r} \) and \( y_{\lambda^{-1} r} \). Therefore \( \gamma^{\lambda r} = (\dot{\gamma}^{\lambda^{-1} r})^{-1} \). And since 

\[ a_{\gamma^{\lambda r}, \gamma^{\lambda^{-1} r}} = \int_0^1 (1 - \tau)^\kappa(\gamma(\tau)) \left[ \sigma_{\kappa_{\gamma^{\lambda r}, N}^{\gamma^{\lambda^{-1} r}}/(\gamma^{\lambda r})} + \sigma_{\kappa_{\gamma^{\lambda^{-1} r}, N}^{\gamma^{\lambda r}}/(\gamma^{\lambda^{-1} r})} \right] d\tau \]

we obtain, that 

\[ N \left[ \lambda a_{\gamma^{\lambda r}, \gamma^{\lambda^{-1} r}} + \lambda^{-1} a_{\gamma^{\lambda^{-1} r}, \gamma^{\lambda r}} \right] \]

\[ = \int_0^1 \kappa(\gamma(\tau)) \left[ ((1 - \tau)\lambda + \tau \lambda^{-1}) \left[ \sigma_{\kappa_{\gamma^{\lambda r}, N}^{\gamma^{\lambda^{-1} r}}/(\gamma^{\lambda r})} + \sigma_{\kappa_{\gamma^{\lambda^{-1} r}, N}^{\gamma^{\lambda r}}/(\gamma^{\lambda^{-1} r})} \right] \right] d\tau =: b_{\gamma^{\lambda r}} \]

Therefore, if we set \( g(r) = d_x(y_{\lambda^{-1} r}, x_{\lambda r})^2 \), we obtain 

\[ \frac{d}{dr} g(r) \leq -2b_{\gamma^{\lambda r}} g(r) + 2N \left[ \frac{\lambda}{\delta_{\kappa_{\gamma^{\lambda r}, N}^{\gamma^{\lambda r}}}(\dot{\gamma}^{\lambda r})} + \frac{\lambda^{-1}}{\delta_{\kappa_{\gamma^{\lambda^{-1} r}, N}^{\gamma^{\lambda r}}}(\dot{\gamma}^{\lambda^{-1} r})} - \frac{2}{\delta_{\kappa_{\gamma^{\lambda r}, N}^{\gamma^{\lambda^{-1} r}}}(\dot{\gamma}^{\lambda r})} \right] |\dot{\gamma}^{\lambda r}|^2 \]

or equivalently 

\[ (32) \]

\[ \frac{d}{dr} g(r) \leq -2b_{\gamma^{\lambda r}} g(r) + 2N \left[ \sqrt{\lambda \frac{d}{dt} |t=0} \sigma_{\kappa_{\gamma^{\lambda r}}^{(t)}}(\dot{\gamma}^{\lambda r}) | \right] - \sqrt{\lambda^{-1} \frac{d}{dt} |t=0} \sigma_{\kappa_{\gamma^{-1} r}}^{(t)}(\dot{\gamma}^{\lambda^{-1} r}) |}^2 \]

Remark 6.3. If we set \( \lambda = 1 \), this becomes 

\[ \frac{d}{dr} d_x(y_s, x_s)^2 \leq -2 \int_0^1 \kappa(\gamma(\tau)) \left[ \sigma_{\kappa_{\gamma^{\lambda r}, N}^{\gamma^{\lambda^{-1} r}}}(\dot{\gamma}^{\lambda^{-1} r}) + \sigma_{\kappa_{\gamma^{\lambda^{-1} r}, N}^{\gamma^{\lambda r}}}(\dot{\gamma}^{\lambda r}) \right] d\tau d_x(y_s, x_s)^2 \]

\[ + 2N \left[ \sqrt{\frac{d}{dt} |t=0} \sigma_{\kappa_{\gamma^{\lambda r}}^{(t)}}(\dot{\gamma}^{\lambda r}) | - \sqrt{\frac{d}{dt} |t=0} \sigma_{\kappa_{\gamma^{-1} r}}^{(t)}(\dot{\gamma}^{\lambda^{-1} r}) |}^2 \]

If \( \kappa = K \) is constant, then \( (33) \) simplifies to 

\[ \frac{d}{dr} d_x(y_s, x_s)^2 \leq -2K \int_0^1 [\sigma_{\kappa_{\gamma^{\lambda r}, N}^{\gamma^{\lambda^{-1} r}}}(d_x(y_s, x_s)) + \sigma_{\kappa_{\gamma^{\lambda^{-1} r}, N}^{\gamma^{\lambda r}}}(d_x(y_s, x_s))] d\tau d_x(y_s, x_s)^2 \]

And if \( N \to \infty \), \( (33) \) becomes 

\[ \frac{d}{dr} d_x(y_s, x_s)^2 \leq -2 \int_0^1 \kappa(\gamma(\tau)) d\tau d_x(y_s, x_s)^2 \]

If \( \lambda \neq 1 \), the second term in the right hand side in \( (32) \) tends to \( \infty \) for \( N \to \infty \).

If we follow the same reasoning, in the context of Wasserstein EVI_{\kappa_{\gamma, N}}-gradient flow curves for a compact metric measure spaces, we obtain the next theorem.
Theorem 6.4. Let $(X, d_X, m_X)$ be a compact metric measure spaces satisfying the condition $RCD^*(\kappa, N)$ where $\kappa : X \to \mathbb{R}$ is lower semi-continuous. Consider Wasserstein $EV_{\kappa, N}$-gradient flow curves $\mu^t$ and $\nu^t$ with initial measures $\mu$ and $\nu$. Let $\lambda, r > 0$. Then the following contraction estimate holds

$$
\frac{d}{dr} W_2(\mu^r, \nu^{r-1})^2 \leq -2 \int_0^1 \kappa(\gamma(\tau)) \left[ ((1-\tau)\lambda + \tau \lambda^{-1}) \left( \sigma^{(1-\tau)}_{\kappa, \lambda, r} \Theta_{\lambda, r}/N + \sigma^{(\tau)}_{\kappa, \lambda, r} \Theta_{\lambda, r}/N \right) \right] d\tau 
+ 2N \left[ \lambda^{-1} \frac{d}{d\tau} \left| t=0 \right| \sigma^{(\tau)}_{\kappa, \lambda, r} \Theta_{\lambda, r}/N - \lambda^{-1} \frac{d}{d\tau} \left| t=0 \right| \sigma^{(\tau)}_{\kappa, \lambda, r} \Theta_{\lambda, r}/N \right]^2
$$

(34)

where $\Pi^r$ is the unique $L^2$-Wasserstein geodesic between $\mu^r$ and $\nu^{r-1}$.

Remark 6.5. If we consider $\kappa = K$ constant, in the limit $W_2(\mu_0, \nu_0) \to 0$ the right hand side of (34) is

$$
\sim -K (\lambda + \lambda^{-1}) W_2(\mu_0, \nu_0)^2 + 2N \left[ \sqrt{\lambda} - \sqrt{\lambda^{-1}} \right]^2.
$$

(35)

That is the same asymptotic behaviour as the corresponding contraction estimates in [EKS15] (Remark 2.20) or in [Kuw15]. Hence, in Wasserstein space context with constant lower curvature bound our estimate yields the corresponding Bakry-Ledoux gradient estimate [EKS15, Kuw15, BL06].

References

[AGMR15] Luigi Ambrosio, Nicola Gigli, Andrea Mondino, and Tapio Rajala, Riemannian Ricci curvature lower bounds in metric measure spaces with $\sigma$-finite measure, Trans. Amer. Math. Soc. 367 (2015), no. 7, 4661–4701. MR 3335397

[AGS13] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré, Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces, Rev. Mat. Iberoam. 29 (2013), no. 3, 969–996. MR 3090143

[AGS14a] , Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below, Invent. Math. 195 (2014), no. 2, 289–391. MR 3152751

[AGS14b] , Metric measure spaces with Riemannian Ricci curvature bounded from below, Duke Math. J. 163 (2014), no. 7, 1405–1490. MR 3205729

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov, A course in metric geometry, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR 1835418, http://arxiv.org/abs/1506.03279.

[BL06] Dominique Bakry and Michel Ledoux, A logarithmic Sobolev form of the Li-Yau parabolic inequality, Rev. Mat. Iberoam. 22 (2006), no. 2, 683–702. MR 2294794 (2008m:58051)

[BS10] Kathrin Bacher and Karl-Theodor Sturm, Localization and tensorization properties of the curvature-dimension condition for metric measure spaces, J. Funct. Anal. 259 (2010), no. 1, 28–56. MR 2610378 (2011i:53050)

[EKS15] Matthias Erbar, Kazumasa Kuwada, and Karl-Theodor Sturm, On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces, Invent. Math. 201 (2015), no. 3, 993–1071. MR 3385639

[GMS] Nicola Gigli, Andrea Mondino, and Giuseppe Savaré, Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows, http://arxiv.org/abs/1311.4907.

[Ket] Christian Ketterer, On the geometry of metric measure space with variable curvature bounded from below, http://arxiv.org/abs/1506.03279.

[Kuw15] Kazumasa Kuwada, Space-time Wasserstein controls and Bakry–Ledoux type gradient estimates, Calc. Var. Partial Differential Equations 54 (2015), no. 1, 127–161. MR 3385156
[LV09] John Lott and Cédric Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2) **169** (2009), no. 3, 903–991. MR 2480619 (2010i:53068)

[RS14] Tapio Rajala and Karl-Theodor Sturm, *Non-branching geodesics and optimal maps in strong CD(K,∞)-spaces*, Calc. Var. Partial Differential Equations **50** (2014), no. 3-4, 831–846. MR 3216835

[Stu] Karl-Theodor Sturm, *Metric Measure Spaces with Variable Ricci Bounds and Couplings of Brownian Motions*, preprint.

[Stu06a] , *On the geometry of metric measure spaces. I*, Acta Math. **196** (2006), no. 1, 65–131. MR 2237206 (2007k:53051a)

[Stu06b] , *On the geometry of metric measure spaces. II*, Acta Math. **196** (2006), no. 1, 133–177. MR 2237207 (2007k:53051b)

[Vil09] Cédric Villani, *Optimal transport, old and new*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009. MR 2459454 (2010f:49001)

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