Testing isomorphism of circular-arc graphs in polynomial time

Roman Nedela *  Ilia Ponomarenko †  Peter Zeman ‡

Abstract
A graph is said to be circular-arc if the vertices can be associated with arcs of a circle so that two vertices are adjacent if and only if the corresponding arcs overlap. It is proved that the isomorphism of circular-arc graphs can be tested by the Weisfeiler-Leman algorithm after individualization of two vertices.

1 Introduction
A finite graph is said to be circular-arc if the vertices can be associated with arcs of a circle so that two vertices are adjacent if and only if the corresponding arcs overlap. The circular-arc graphs arise in many applications, see [9], and form a natural border between isomorphism complete classes of graphs and the classes admitting polynomial-time isomorphism testing [2].

An efficient algorithm recognizing circular-arc graphs was constructed in [9]. There are also polynomial-time algorithms testing isomorphism in various subclasses of circular-arc graphs, e.g., interval graphs, co-bipartite circular-arc graphs, proper circular-arc graphs, and Helly circular-arc graphs, see [4, 8]. Some partial results were also obtained in [1]. There were several attempts to solve the problem of testing isomorphism for the whole class of circular-arc graphs in polynomial time, but each time the proof contained a gap (for more details, we refer the reader to a discussion in [4]). In the present paper, we solve the isomorphism problem for circular-arc graphs, which has remained open for more than 40 years.

Theorem 1.1. Given an $n$-vertex circular-arc graph $X$ and any graph $Y$, one can test whether $X$ is isomorphic to $Y$ in time polynomial in $n$.

Our approach is different from all the previous ones and it is based on the theory of coherent configurations, see [3] and Section 3. Such a configuration can be viewed as an arc-colored complete directed graph with a regularity condition. Namely, the number of triangles with fixed base and fixed colors of the sides depends on the color of the base (and does not depend on the chosen base). These numbers form the intersection number array of the coherent configuration.

To every graph $X$, one can associate a unique coherent configuration $\mathcal{X} := \text{WL}(X)$ such that the edge set of $X$ is a union of color classes of $\mathcal{X}$. The crucial part in the proof of Theorem 1.1 is that if $X$ is a circular-arc graph with two distinguished non-adjacent vertices, then the intersection number array of $\mathcal{X}$ form a full set of invariants of $X$ with respect to isomorphism (Theorem 6.1). Thus to test whether the graph $X$ is isomorphic to a graph $Y$, it suffices to verify whether there exist two non-adjacent vertices of $Y$ such that the intersection arrays of the corresponding coherent configurations are equal.

Given an $n$-vertex graph $X$, the coherent configuration $\text{WL}(X)$ (and the corresponding intersection number array) can be constructed by the Weisfeiler-Leman algorithm in time...
Figure 1: Replacement an edge \((\alpha, \beta)\) in the construction of the graph \(Y_n\).

\[ O(n^3 \log n) \] [7]. Since there are at most \(\binom{n}{2}\) pairs of non-adjacent vertices in \(X\), the running time of our algorithm is \(O(n^5 \log n)\). Although the algorithm does not construct an isomorphism of two isomorphic graphs, the induction used in the proof of Theorem 6.1 can be modified to find the isomorphism (and even the set of all of them) explicitly.

We complete the introduction by establishing a relation between the technique we used to prove Theorem 1.1 and the \(m\)-dimensional Weisfeiler-Leman algorithm [6, Section 3.5.2]. Following Grohe [6, Definition 18.4.2], we define the WL-dimension \(\dim_{\text{WL}}(K)\) of a class \(K\) of graphs to be the minimal integer \(m\) such that the \(m\)-dimensional WL-algorithm distinguishes each graph \(X \in K\) from all graphs not isomorphic to \(X\). Now if \(K\) is the class of circular-arc graphs, then

\[ 2 \leq \dim_{\text{WL}}(K) \leq 5. \] (1.1)

Indeed, the upper bound immediately follows from Theorem 6.1. To prove the lower bound, we construct non-isomorphic circular-arc graphs of the same order and degree.

Let \(n = 5k\), where \(k \geq 3\). Set \(X_n\) to be a circulant graph with vertex set \(\{0, \ldots, n-1\}\), in which two vertices are adjacent if and only if \(|i - j| \in \{1, 2\}\). On the other hand, denote by \(Y_n\) the graph obtained from a cycle on \(k\) vertices by replacing each edge with a 4-clique as depicted in Fig. 1 One can see that \(X_n\) and \(Y_n\) are 4-regular circular-arc graphs. Moreover, they are not isomorphic, because \(X_n\) is vertex-transitive and \(Y_n\) is not.

2 Preliminaries

2.1 Relations

Throughout the paper, \(\Omega\) denotes a finite set of cardinality \(n\). The diagonal of the Cartesian product \(\Omega \times \Omega\) is denoted by \(1_\Omega\); given \(\alpha \in \Omega\), we write \(1_\alpha\) instead of \(1_{\{\alpha\}}\). For a relation \(r\) on \(\Omega\) and \(\Delta, \Gamma \subseteq \Omega\), we set

- \(r^* = \{(\beta, \alpha) : (\alpha, \beta) \in r\}\),
- \(\alpha r = \{\beta \in \Omega : (\alpha, \beta) \in r\}\) for all \(\alpha \in \Omega\), and
- \(r_{\Delta,\Gamma} = r \cap (\Delta \times \Gamma)\).

For a set \(S\) of relations on \(\Omega\) and two sets \(\Delta, \Gamma \subseteq \Omega\), we define \(S_{\Delta,\Gamma}\) to be the collection of all nonempty relations \(s_{\Delta,\Gamma}\), \(s \in S\), and put \(S_{\Delta} := S_{\Delta,\Delta}\). Moreover, we denote by \(S^\cup\) the set of all unions of the elements of \(S\), and put \(S^* := \{s^* : s \in S\}\).

2.2 Graphs

By a graph we mean a pair \(X = (\Omega, E)\), where \(\Omega\) is a finite set and \(E \subseteq \Omega^2\) is an irreflexive symmetric relation; the elements of the sets \(\Omega(X) := \Omega\) and \(E(X) := E\) are called vertices and edges of \(X\), respectively. According to our notation, the neighborhood of a vertex \(\gamma\) is denoted by \(\gamma E\). For a set \(\Delta \subseteq \Omega\), the subgraph induced by \(\Delta\) is denoted by \(X_{\Delta}\).
Let $X = (\Omega, E)$ and $X' = (\Omega', E')$ be two graphs. A bijection $\Omega \to \Omega'$ is called an isomorphism from $X$ to $X'$ if it maps edges to edges and non-edges to non-edges. The set of all isomorphisms from $X$ to $X'$ is denoted by $\text{Iso}(X, X')$.

Let $X = (\Omega, E)$ be a graph. Two vertices $\alpha$ and $\beta$ are called twins if for any other vertex $\gamma$, the set $\gamma E$ contains either both $\alpha$ and $\beta$, or none of them. If the vertices $\alpha$ and $\beta$ are adjacent (respectively, non-adjacent), we say that they are 1-twins (respectively, 0-twins). One can see that the relations “to be 1-twins” and “to be 0-twins” are equivalence relations on $\Omega$.

A complete graph with $n$ vertices (respectively, complete bipartite graph with parts of cardinalities $m$ and $n$) is denoted by $K_n$ (respectively, $K_{m,n}$).

2.3 Circular-arc graphs

Let $C$ be an undirected cycle with at least 3 vertices. Under an arc of $C$ we mean a subset of its vertices, that induces a path. The set of all arcs of $C$ is denoted by $A(C)$. The arc $A \in A(C)$ starting at the point $c_1 \in C$ and ending at the point $c_2 \in C$ (in the clockwise orientation of $C$) is denoted by $[c_1, c_2]$. In this case, we set $A_h := c_1$ and $A_t := c_2$.

Let $X = (\Omega, E)$ be a graph and $C$ as above. A function $R: \Omega \to A(C)$ is called a circular-arc representation, or briefly a CA-representation of $X$, if

$$\{\alpha, \beta\} \in E \iff R(\alpha) \cap R(\beta) \neq \emptyset.$$ 

A graph $X$ is a circular-arc graph, or briefly a CA-graph, if it has a CA-representation. Clearly, the class of circular-arc graphs is closed with respect to taking induced subgraphs. In what follows the vertices of $C$ are called points, to distinguish them from the vertices of $X$.

A set $\Delta \subseteq \Omega$ is said to be $R$-interval if the union of all arcs $R(\alpha)$, $\alpha \in \Delta$, is a proper subset of the points of $C$. In this case, $X_\Delta$ is an interval graph. The set $\Delta$ is said to be $R$-covering if it is not interval.

A CA-representation $R$ of a graph $X$ can always be chosen to be normal, i.e., such that $R(\alpha) = R(\beta)$ if and only if the vertices $\alpha$ and $\beta$ are 1-twins.

For further information on CA-graphs, we refer the reader to [4].

3 Coherent configurations

In our presentation of coherent configurations, we follow [3].

3.1 Definitions

A pair $\mathcal{X} = (\Omega, S)$, where $\Omega$ is a finite set and $S$ is a partition of $\Omega \times \Omega$, is called a coherent configuration on $\Omega$ if $1_\Omega \in S^{\cup}$, $S^* = S$, and for all $r, s, t \in S$, the number

$$c^t_{rs} := |\alpha r \cap \beta s^*|$$

does not depend on a choice of $(\alpha, \beta) \in t$. The elements of $\Omega$, $S$, $S^{\cup}$, and the numbers $c^t_{rs}$ are called the points, basis relations, relations and intersection numbers of $\mathcal{X}$, respectively. A unique basis relation containing a pair $(\alpha, \beta) \in \Omega \times \Omega$ is denoted by $r(\alpha, \beta)$.

3.2 Fibers and valency

A set $\Delta \subseteq \Omega$ is called a fiber of the coherent configuration $\mathcal{X}$ if $1_\Delta \in S$; the set of all fibers is denoted by $F = F(\mathcal{X})$. The point set $\Omega$ is a disjoint union of fibers. If $\Delta$ is a union of fibers, then the pair

$$\mathcal{X}_\Delta = (\Delta, S_\Delta)$$

is a coherent configuration, called the restriction of $\mathcal{X}$ to $\Delta$. 
For any basis relation \( s \in S \), there exist uniquely determined fibers \( \Delta, \Gamma \) such that \( s \subseteq \Delta \times \Gamma \); in particular, the union
\[
S = \bigcup_{\Delta, \Gamma \in F} S_{\Delta, \Gamma}
\]
is disjoint. The number \(|\delta s|\) with \( \delta \in \Delta \) equals the intersection number \( c^1_{ss'} \), and hence does not depend on the choice of the point \( \delta \). It is called the \textit{valency} of \( s \) and denoted by \( n_s \).

### 3.3 Isomorphisms

Let \( \mathcal{X} = (\Omega, S) \) and \( \mathcal{X}' = (\Omega', S') \) be coherent configurations. A bijection from \( \Omega \) to \( \Omega' \) is called an \textit{isomorphism} from \( \mathcal{X} \) to \( \mathcal{X}' \) if the induced bijection from \( \Omega \times \Omega \) to \( \Omega' \times \Omega' \) takes \( S \) to \( S' \).

Coherent configurations \( \mathcal{X} \) and \( \mathcal{X}' \) are called isomorphic if there exists an isomorphism from \( \mathcal{X} \) to \( \mathcal{X}' \).

A bijection \( \varphi : S \to S' \), \( s \mapsto s' \) is called an \textit{algebraic isomorphism} from \( \mathcal{X} \) to \( \mathcal{X}' \) if
\[
c^1_{rs} = c^1_{r's'} \quad \text{for all} \quad r, s, t \in S.
\]
The set of all such \( \varphi \) is denoted by \( \text{Iso}_{\text{alg}}(\mathcal{X}, \mathcal{X}') \); when this set is nonempty, the coherent configurations \( \mathcal{X} \) and \( \mathcal{X}' \) are said to be \textit{algebraically isomorphic}.

An algebraic isomorphism \( \varphi : S \to S' \) is said to be induced by a bijection \( f : \Omega \to \Omega' \) if
\[
\varphi(s) = s^f \quad \text{for all} \quad s \in S.
\]
In this case, \( f \) takes \( S \) to \( S' \) and hence is an isomorphism from \( \mathcal{X} \) to \( \mathcal{X}' \). Conversely, each isomorphism \( f \) from \( \mathcal{X} \) to \( \mathcal{X}' \) obviously preserves the intersection numbers and hence induces an algebraic isomorphism
\[
\varphi_f : S \to S', \ s \mapsto s^f.
\]
It should be emphasized that not every algebraic isomorphism is induced by a bijection.

An algebraic isomorphism \( \varphi \in \text{Iso}_{\text{alg}}(\mathcal{X}, \mathcal{X}') \) can naturally be extended to a bijection between the relations of the coherent configurations \( \mathcal{X} \) and \( \mathcal{X}' \). Namely, given \( r \in S^\cup \), \( r = s \cup t \cup \cdots \), we set \( \varphi(r) := \varphi(s) \cup \varphi(t) \cup \cdots \). Similarly, given a fiber \( \Delta \in F \), we set \( \Delta^\varphi \) to be the fiber of \( \mathcal{X}' \) such that \( 1_{\Delta^\varphi} = \varphi(1_\Delta) \).

### 3.4 Separable coherent configurations

A coherent configuration \( \mathcal{X} \) is said to be \textit{separable} if every algebraic isomorphism from \( \mathcal{X} \) to any coherent configuration is induced by an isomorphism. The following statement is a special case of [5, Lemma 9.4], giving a sufficient condition for a coherent configuration to be separable. In what follows a fiber \( \Delta \in F \) is said to be \textit{unessential} if there exist a fiber \( \Gamma \neq \Delta \) and a relation \( s \in S_{\Gamma, \Delta} \) such that \( n_s = 1 \).

**Lemma 3.1.** Let \( \mathcal{X} \) be a coherent configuration on \( \Omega \). Assume that \( \Delta \) is an unessential fiber of \( \mathcal{X} \). Then \( \mathcal{X} \) is separable if so is \( \mathcal{X}_{\Omega \setminus \Delta} \).

### 3.5 Coherent closure

There is a natural partial order \( \leq \) on the set of all coherent configurations on the same set. Namely, given two coherent configurations \( \mathcal{X} = (\Omega, S) \) and \( \mathcal{X}' = (\Omega, S') \), we set
\[
\mathcal{X} \leq \mathcal{X}' \iff S^\cup \subseteq (S')^\cup.
\]
The minimal and maximal elements with respect to this ordering are the \textit{trivial} and \textit{discrete} coherent configurations: the basis relations of the former one are the reflexive relation \( 1_\Omega \) and (if \( n > 1 \)) its complement in \( \Omega \times \Omega \), whereas the basis relations of the latter one are singletons.
Given a set \( S \) of binary relations on a set \( \Omega \), there exists the smallest coherent configuration \((\Omega, S)\) such that \( S \subseteq S^u \). It is called the coherent closure of \( S \) and denoted by \( WL(S) \). The well-known Weisfeiler-Leman algorithm described in detail in [10, Section B] finds the coherent closure in polynomial time in \(|S|\) and \(|\Omega|\). The canonical version of the Weisfeiler-Leman algorithm has been studied in [10, Section M] (under the name “simultaneous stabilization”), where, in fact, the following statement was proved.

**Theorem 3.2.** Let \( S_i \) be a set of \( m \) binary relations on a set of size \( n \), \( i = 1, 2 \). Then given a bijection \( \psi: S_1 \to S_2 \) one can check in time \( mn^{O(1)} \) whether or not there exists an algebraic isomorphism \( \phi: WL(S_1) \to WL(S_2) \) such that \( \phi|_{S_1} = \psi \). Moreover, if \( \phi \) does exist, it can be found within the same time.

### 3.6 Coherent configuration of a graph

Let \( X \) be a graph. The coherent configuration of \( X \) is defined to be the coherent closure \( WL(X) := WL(\{E\}) \) with \( E = E(X) \). In this case, \( E \) is a relation of \( WL(X) \).

**Lemma 3.3.** Let \( X \) be a graph, \( X \geq WL(X) \) a coherent configuration, and let \( \Delta \) and \( \Gamma \) be fibers of \( X \). Then the number \( |\alpha E \cap \Gamma| \) does not depend on the choice of \( \alpha \in \Delta \).

**Proof.** One can see that \( E_{\Delta, \Gamma} \) belongs to the set \((S_{\Delta, \Gamma})^{\cup}\). Therefore,

\[
|\alpha E \cap \Gamma| = \sum_{s \in S_{\Delta, \Gamma}, s \subseteq E} n_s.
\]

Since the right-hand side does not depend on \( \alpha \), we are done. \( \square \)

**Corollary 3.4.** Let \( X \) be a graph and \( X \geq WL(X) \) a coherent configuration. Assume that \( \alpha \) is a vertex of \( X \) such that \( \{\alpha\} \) is a fiber of \( X \). Then \( \alpha \) is adjacent with all or none vertex of any fiber of \( X \).

Under the hypothesis of Lemma 3.3 denote by \( X_{\Delta, \Gamma} \) the graph on \( \Delta \cup \Gamma \), in which a vertex of \( \Delta \) is adjacent with a vertex of \( \Gamma \) if and only if they are adjacent in the graph \( X \). Thus if \( \Delta \neq \Gamma \), this graph is bipartite with parts \( \Delta \) and \( \Gamma \), whereas if \( \Delta = \Gamma \), then \( X_{\Delta, \Gamma} = X_{\Delta} \). Now the conclusion of the lemma implies that the graph \( X_{\Delta, \Gamma} \) is biregular, i.e., any two vertices of \( \Delta \) (respectively, \( \Gamma \)) have the same number of neighbors. In particular, the induced graph \( X_{\Delta} \) is regular.

### 4 Twins in a coherent configuration

Let \( \mathcal{X} \) be a coherent configuration on \( \Omega \). Two points \( \alpha, \beta \in \Omega \) are called **twins** of \( \mathcal{X} \) if

\[
\tau(\alpha, \gamma) = \tau(\beta, \gamma) \quad \text{for all } \gamma \in \Omega, \ \alpha \neq \beta.
\]

Assume that \( |\Omega| \geq 3 \). Then the twins \( \alpha \) and \( \beta \) belong to the same fiber \( \Delta \in F \) and the relation “to be twins” is an equivalence relation on \( \Delta \); this relation is called the **twin parabolic** (of \( \mathcal{X} \)) associated with \( \Delta \). The twin parabolic is said to be nontrivial if it does not coincide with \( 1_{\Delta} \).

**Lemma 4.1.** Let \( e \) be a nontrivial twin parabolic associated with a fiber \( \Delta \). Then

1. \( e \setminus 1_{\Delta} \) is a basis relation of \( \mathcal{X} \),
2. given \( \varphi \in \text{Iso}_{\text{alg}}(\mathcal{X}, \mathcal{X}') \), \( \varphi(e) \) is twin parabolic associated with \( \Delta^\varphi \).
Proof. To prove statement (1), we have to verify that given pairwise distinct twins $\alpha, \beta, \gamma \in \Delta$ the relations $r(\alpha, \beta)$ and $r(\alpha, \gamma)$ coincide. However, this immediately follows, because $\beta$ and $\gamma$ are twins.

To prove statement (2), it suffices to note that by statement (1), an equivalence relation $e \in S^\Delta$ is a twin parabolic associated with $\Delta$ if and only if $e = 1_{\Delta} \cup s$ for some $s \in S$ such that

$$c'_{sr} = \begin{cases} k - 2 & \text{if } (r, t) = (s, s) \text{ and } t \neq 1_{\Delta}, \\ k - 1 & \text{if } (r, t) \neq (s, s) \text{ and } c'_{sr} \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $k$ is the cardinality of a class of $e$ (here we made use of the fact that $k$ does not depend on the class, see [3, Corollary 2.1.23]). \hfill \Box

Let $e \in S^\Delta$ be a twin parabolic associated with a fiber $\Delta$. Take an arbitrary full system $\bar{X}$ of distinct representatives for the classes of $e$, and set

$$\bar{\Omega} = (\Omega \setminus \Delta) \cup \bar{\Delta} \quad \text{and} \quad \bar{S} = S_{\Omega}.$$}

The pair $(\bar{\Omega}, \bar{S})$ is called an $e$-residue of $X$. It is easily seen that this pair coincides with $X$ if and only if the twin parabolic associated with $\Delta$ is trivial.

There is a natural bijection taking a point $\alpha \in \bar{\Omega}$ to the class of the equivalence relation $e \cup 1_{\Omega \setminus \Delta}$, that contains $\alpha$. This bijection sends the relations of $\bar{S}$ to the basis relations of the quotient $X_{\Omega/e}$ of the coherent configuration $X$ modulo $e$ [3, Section 3.1.2]. This shows that an $e$-residue of $X$ is a coherent configuration (isomorphic to that quotient).

Lemma 4.2. Let $X$ be a coherent configuration, $e$ a twin parabolic of $X$, and $\bar{X}$ an $e$-residue of $X$. Then $X$ is separable if so is $\bar{X}$.

Proof. Assume that the coherent configuration $\bar{X}$ is separable. We need to verify that given a coherent configuration $X' = (\Omega', S')$, any algebraic isomorphism

$$\varphi \in \text{Iso}_{\text{alg}}(X, X')$$

is induced by a bijection. To this end, we note that $e' := \varphi(e)$ is a twin parabolic (of $X'$) associated with the fiber $\Delta' := \Delta \varphi$ (statement (2) of Lemma 4.1). Choose a full system $\bar{X'}$ of distinct representatives for the classes of $e'$, and let $\bar{X'} = (\bar{\Omega'}, \bar{S'})$ be the corresponding $e'$-residue of $X'$.

The identification of $X$ (respectively, $X'$) with $X_{\Omega/e}$ (respectively, $X'_{\Omega'/e'}$) shows (see [3, Section 3.1.2]) that the mapping

$$\bar{\varphi} : \bar{S} \rightarrow \bar{S'}, \ s_{\bar{\Omega}} \mapsto \varphi(s)_{\bar{\Omega'}}$$

belongs to $\text{Iso}_{\text{alg}}(\bar{X}, \bar{X'})$. By the assumption on $\bar{X}$, this implies that $\bar{\varphi}$ is induced by a certain bijection $\bar{f} : \bar{\Omega} \rightarrow \bar{\Omega}'$.

Let

$$\bar{\Delta} = \{\delta_1, \ldots, \delta_k\} \quad \text{and} \quad \bar{\Delta'} = \{\delta'_1, \ldots, \delta'_k\},$$

where $k \geq 1$; without loss of generality we assume that the indices are chosen so that $\delta_i^\bar{f} = \delta'_i$ for each $i \in \{1, \ldots, k\}$. Denote by $\Delta_i$ (respectively, $\Delta'_i$) the class of $e$ (respectively, $e'$), that contains $\delta_i$ (respectively, $\delta'_i$), and choose arbitrary bijection $f_i : \Delta_i \rightarrow \Delta'_i$ taking $\delta_i$ to $\delta'_i$.

Let us extend the bijection $f$ to a bijection $f : \Omega \rightarrow \Omega'$ defined as follows:

$$\alpha' = \begin{cases} \alpha^\bar{f} & \text{if } \alpha \in \Omega \setminus \Delta, \\ \alpha^{f_i} & \text{if } \alpha \in \Delta_i \text{ with } i \in \{1, \ldots, k\}. \end{cases}$$
From the definition, it immediately follows that \( s^f = \varphi(s) \) for all \( s \in S_{\Omega \setminus \Delta} \). Let \( s = r(\delta, \alpha) \) with \( \delta \in \Delta \) and \( \alpha \in \Omega \setminus \Delta \). Then \( \delta \in \Delta_i \) for some \( i \), and

\[
(\delta_i, \alpha) \in s \iff (\delta_i, \alpha^f) \in \varphi(s) \iff (\delta_i^f, \alpha^f) \in \varphi(s).
\]

Similarly, one can verify that \( s^f = \varphi(s) \) for \( s = r(\delta_i, \delta_j) \) with \( i \neq j \). Finally, if \( i = j \), then the corresponding statement follows from statement (1) of Lemma 4.1. Thus the algebraic isomorphism \( \varphi \) is induced by the bijection \( f \), and we are done. □

Let \( X \) be a graph. A fiber \( \Delta \) of a coherent configuration \( X \geq WL(X) \) is said to be \( X \)-twinless if there are no distinct twins of \( X \) belonging to \( \Delta \). The following lemma is obvious.

**Lemma 4.3.** Let \( X \) be a graph, and let \( \Delta \) be a fiber of a coherent configuration \( X \geq WL(X) \). Assume that \( \Delta \) is not \( X \)-twinless. Then the twin parabolic associated with \( \Delta \) is not trivial.

5 Circular-arc graphs and coherent configurations

A key point in the proof of Theorem 1.1 is the following statement which almost immediately implies (see Theorem 6.1) that the coherent configuration of a CA-graph with two distinguished vertices is separable.

**Theorem 5.1.** Let \( X \) be a CA-graph with at least three vertices, and let \( X \geq WL(X) \) be a coherent configuration. Assume that

(i) each fiber of \( X \) is \( X \)-twinless,

(ii) \( X \) has distinct nonadjacent vertices \( \alpha \) and \( \beta \) such that \( \{\alpha\}, \{\beta\} \in F(X) \).

Then \( X \) has an unessential fiber other than \( \{\alpha\} \) and \( \{\beta\} \).

The proof of Theorem 5.1 is given later. First, we prove some auxiliary lemmas. In what follows, we fix a normal CA-representation of a graph \( X = (\Omega, E) \),

\[
R : \Omega \rightarrow A(C),
\]

a coherent configuration \( X \geq WL(X) \), and set \( F := F(X) \). We also assume that \( X \) and \( X \) satisfy the hypothesis of Theorem 5.1. In particular, \( X \) is not a complete graph.

**Lemma 5.2.** Let a set \( \Delta \in F \) be \( R \)-interval and \( X \)-twinless. Then the graph \( X_\Delta \) is the union of cliques of the same size.

**Proof.** The statement is straightforward if \( X_\Delta \) is a regular interval graph: in such a graph the neighborhood of each vertex induces a clique. It remains to note that the graph \( X_\Delta \) is interval, because \( \Delta \) is \( R \)-interval, and regular by the remark after Corollary 3.4. □

In the lemma below, we use the following a consequence of the condition (ii) in Theorem 5.1 and Corollary 3.4. Namely, there are three possibilities for any fiber \( \Gamma \in F \):

- \( \Gamma \subseteq \alpha E \cap \beta E \),
- \( \Gamma \subseteq \alpha E \setminus \beta E \) or \( \Gamma \subseteq \beta E \setminus \alpha E \),
- \( \Gamma \cap (\alpha E \cup \beta E) = \emptyset \).

**Lemma 5.3.** Let \( \Delta \in F \). Assume that the following conditions are satisfied for a fiber \( \Gamma \in F \) other than \( \{\alpha\} \) and \( \{\beta\} \):

(i) the set \( \Delta \cup \Gamma \) is \( R \)-covering,
(ii) the graph $X_{\Delta,E}$ contains three pairwise non-adjacent vertices,

(iii) the graph $X_{\Delta}$ is disconnected.

Then $\Delta \cap (\alpha E \cup \beta E) = \emptyset$ and $\Gamma \subseteq (\alpha E \cap \beta E)$.

**Proof.** From the assumption (i), it follows that each of the sets $\alpha E$ and $\beta E$ contains $\Delta$ or $\Gamma$ or both. Furthermore,

$$\Delta \cup \Gamma \not\subseteq \alpha E \cap \beta E.$$  

Indeed, otherwise the subgraph of $X$ induced by the three vertices from assumption (ii) and the vertices $\alpha$ and $\beta$ is isomorphic to $K_{3,2}$. However, this is impossible, because the graph $X$ is circular-arc.

Thus each $\lambda \in \{\alpha, \beta\}$ is adjacent with all the vertices of exactly one of the sets $\Delta$, $\Gamma$. It remains to show that $\Delta \cap \lambda E = \emptyset$. Assume on the contrary that $\Delta \subseteq \lambda E$, i.e., $\lambda$ is adjacent with each vertex of $\Delta$. Then by the above, $R(\lambda) \cap R(\Gamma) = \emptyset$. By the assumption (i), this implies that

$$R(\lambda) \subsetneq R(\Delta'),$$

where $\Delta'$ is the vertex set of a connectivity component of the graph $X_{\Delta}$. Therefore, by assumption (iii), $\lambda$ is adjacent with at least one vertex of $\Delta'$ and with no vertex of another component, a contradiction. $\square$

Let $\Delta$ be an $R$-covering fiber of $X$. Then for any vertex $\delta \in \Delta$, the arc $R(\delta)$ contains at least one of the ends of each of the arcs $R(\alpha)$ and $R(\beta)$ (here we use the fact that $\alpha$ and $\beta$ are non-adjacent). In what follows, the vertex $\delta$ is said to be special if the arc $R(\delta)$ contains at least three ends of the arcs $R(\alpha)$ and $R(\beta)$. Obviously, in this case

$$\Delta \subseteq \alpha E \cap \beta E$$

and the graph $X_{\Delta}$ is a clique.

**Lemma 5.4.** Assume that $\Delta \in F$ contains a special vertex, and each non-singleton fiber of $X$ is $R$-covering. Then $\Delta$ is $X$-twinless only if $\Delta$ is a singleton.

**Proof.** Denote by $\Omega_0$ the union of all singleton fibers of $X$. The definition of special vertex together with the assumption imply that

$$\Omega \setminus \Omega_0 \subseteq \delta E$$

for each special and hence for all vertex $\delta \in \Delta$. Moreover, from Corollary 3.4 it follows that for any two vertices $\delta, \varepsilon \in \Delta$,

$$\delta E \cap \Omega_0 = \varepsilon E \cap \Omega_0.$$  

Thus $\delta$ and $\varepsilon$ are twins of $X$, and we are done. $\square$

**Proof of Theorem 5.1.** Every singleton fiber of $X$ is obviously unessential. Let $\Delta \in F$ be a non-singleton fiber. It is $X$-twinless by the hypothesis. Therefore any two distinct vertices $\delta, \varepsilon \in \Delta$ are separated by a certain fiber $\Gamma$, i.e.,

$$(\delta E \cap \Gamma) \Delta (\varepsilon E \cap \Gamma) \neq \emptyset,$$

where $\Delta$ denotes the symmetric difference. Certainly, the fiber $\Gamma$ is not necessarily unique and depends on $\delta$ and $\varepsilon$.

Now if $|\Delta| = 2$, then $\Gamma \neq \Delta$ and each relation of the set $S_{\Gamma, \Delta}$ is of valency 1. In particular, the fiber $\Delta$ is unessential. Thus without loss of generality, we may assume that given $\Delta \in F$,

$$\Delta = \{\alpha\} \quad \text{or} \quad \Delta = \{\beta\} \quad \text{or} \quad |\Delta| \geq 3. \quad (5.1)$$

8
In the following lemmas, we fix an $R$-interval non-singleton fiber $\Delta \in F$ (recall that $\Delta$ is also $X$-twinless). By Lemma 5.2, there exist positive integers $m = m(\Delta)$ and $n = n(\Delta)$ such that

$$X_\Delta = nK_m.$$  

**Lemma 5.5.** Assume that $m \geq 3$. Then the fiber $\Delta$ is unessential.

**Proof.** Choose an arbitrary $m$-clique $K = \{ \delta_1, \ldots, \delta_m \}$ in the graph $X_\Delta$. Since the set $\Delta$ is $R$-interval, the indices $i = 1, \ldots, m$ can be chosen so that in the clockwise orientation, the ends of the arcs $R(\delta_i)$ are located in the cycle $C$ in the following order:

$$R(\delta_1)_h, \ldots, R(\delta_m)_h, R(\delta_1)_t, \ldots, R(\delta_m)_t.$$  

(5.2)

Indeed, if this is not true, then $R(\delta_i) \subseteq R(\delta_j)$ for some distinct $i$ and $j$. Since the vertices $\delta_i$ and $\delta_j$ are not 1-twins, the normality of $R$ implies that $\delta_i E \subseteq \delta_j E$. It follows that the vertices $\delta_i$ and $\delta_j$ have different valencies in the graph $X$, which is impossible by Lemma 3.3.

Let $\Gamma$ be a fiber separating the vertices $\delta_1$ and $\delta_2$. Then $\Gamma \neq \Delta$ and there exists a vertex $\gamma \in \Gamma$ adjacent with $\delta_2$ and not adjacent with $\delta_1$. In view of (5.2), this implies that

$$R(\delta_2)_t, \ldots, R(\delta_m)_t \in R(\gamma),$$

i.e., $\gamma$ is adjacent with each vertex of $K$ other than $\delta_1$, see Fig. 2 (a). Take arbitrary $\gamma' \in \delta_2 s^*$, where $s = r(\gamma, \delta_1)$. Then it is easily seen that $\gamma'$ is adjacent with each vertex of $K$ other than $\delta_2$. In view of (5.2), this implies that

$$[R(\delta_2)_t, R(\delta_1)_h] \subseteq R(\gamma').$$

It follows that $\gamma'$ is adjacent with each vertex of $\Delta$ other than $\delta_2$. Thus the relation $s$ is of valency 1 and the fiber $\Delta$ is unessential.

**Lemma 5.6.** Assume that $m = 2$. Then the fiber $\Delta$ is unessential.

**Proof.** By the assumption, Lemma 5.2 and relation (5.1), we have $X_\Delta = nK_2$ with $n \geq 2$. In particular, $E_\Delta$ is a basis relation of $X$. Denote the vertices of $\Delta$ by $\delta_0, \ldots, \delta_{k-1}$, where $k = 2n$. Then the indices can be chosen so that in the clockwise orientation, the ends of the arcs $R(\delta_i)$ are located in $C$ as follows:

$$R(\delta_0)_h, R(\delta_1)_h, R(\delta_0)_t, R(\delta_1)_t, \ldots, R(\delta_{k-2})_h, R(\delta_{k-1})_h, R(\delta_{k-2})_t, R(\delta_{k-1})_t.$$  

(5.3)

Let $\Gamma$ be a fiber separating two vertices of $\Delta$, that are adjacent in $X$. Then $\Gamma \neq \Delta$ and $\Gamma$ separates every two adjacent vertices of $\Delta$ (recall that $E_\Delta$ is a basis relation of $X$). For each $i = 0, \ldots, k-1$, denote by $\gamma_i$ a vertex of $\Gamma$ such that

$$\{\gamma_i, \delta_i\} \in E \quad \text{and} \quad \{\gamma_i, \delta_{i+e}\} \notin E,$$
where the addition is taken modulo \( k \) and \( \epsilon = 1 \) or \(-1\) depending on whether or not the index \( i \) is even. Without loss of generality, we may assume that \( \gamma_i \) is also adjacent with \( \delta_{i-\epsilon} \) and is not adjacent with \( \delta_{i+2\epsilon} \), for otherwise the relations \( r(\gamma_i, \delta_{i}) \) and \( r(\gamma_i, \delta_{i+\epsilon}) \), respectively, have valency 1, and the fiber \( \Delta \) is unessential.

The fibers \( \Delta \) and \( \Gamma \) satisfy the hypothesis of Lemma 5.3, the conditions (i) and (iii) are satisfied trivially, whereas the vertices in the condition (ii) can be chosen as \( \delta_0, \delta_2, \) and \( \delta_4 \) if \( n \geq 3 \), and \( \gamma_0, \delta_1, \) and \( \delta_2 \) if \( n = 2 \), see Fig. 2(b). Thus by this lemma,

\[
\Delta \cap \alpha E = \emptyset \quad \text{and} \quad \Gamma \subseteq \alpha E. 
\]

Then there exists an odd \( i \in \{0, \ldots, k-1\} \) such that the arc \( R(\alpha) \) is properly contained in the arc \( [R(\delta_i), R(\delta_{i+1})] \). But in this case, \( R(\alpha) \cap R(\gamma_{i-1}) = \emptyset \), i.e., \( \Gamma \not\subseteq \alpha E \), a contradiction. \( \square \)

In view of Lemmas 5.5 and 5.6, the following statement reduces the rest of proof to the case, where each non-singleton fiber is \( R \)-covering. Below, we denote by \( F_0 \) the set of all \( R \)-interval non-singleton fibers of \( X \); in particular, \( \Delta \in F_0 \).

**Lemma 5.7.** Assume that \( m(\Gamma) = 1 \) for each \( \Gamma \in F \). Then the set \( F_0 \) contains an unessential fiber.

**Proof.** First, assume that there exists a fiber \( \Gamma \in F_0 \) other than \( \Delta \). We claim that in this case the graph \( X_{\Delta \cup \Gamma} \) is interval. Indeed, suppose on the contrary that \( \Delta \cup \Gamma \) is an \( R \)-covering set. Then the hypothesis of Lemma 5.3 is satisfied for the pairs \((\Delta, \Gamma)\) and \((\Gamma, \Delta)\). Thus by this lemma applied to each of these pairs,

\[
(\Delta \cup \Gamma) \cap (\alpha E \cup \beta E) = \emptyset,
\]

which is impossible by the supposition.

From the above claim it follows that the bipartite graph \( X_{\Delta \cup \Gamma} \) defined in Subsection 3.6 is interval. This is possible only if this graph has no cycles. But then it contains a vertex of valency one. Since \( X_{\Delta \cup \Gamma} \) is also biregular, it follows that it is a union of stars. Now if the roots of these stars form \( \Delta \) (respectively, \( \Gamma \)), then the relation \( r(\gamma, \delta) \) (respectively, \( r(\delta, \gamma) \)) with \( \gamma \in \Gamma \) and \( \delta \in \Delta \), is of valency one, and hence the fiber \( \Delta \) (respectively, \( \Gamma \)) is unessential.

Thus at this point we may assume that \( F_0 = \{\Delta\} \). In particular, each fiber \( \Gamma \) separating two vertices of \( \Delta \) is an \( R \)-covering set. Therefore the hypothesis of Lemma 5.3 is satisfied. Consequently,

\[
\Delta \cap (\alpha E \cup \beta E) = \emptyset \quad \text{and} \quad \Gamma \subseteq (\alpha E \cap \beta E). \quad (5.4)
\]

It follows that each vertex of \( \Delta \) is not adjacent with \( \alpha \). Denote by \( \delta \) and \( \epsilon \) the vertices of \( \Delta \) such that the arc \( [R(\epsilon)\alpha, R(\alpha)\delta] \) (respectively, \( [R(\delta)\epsilon, R(\epsilon)\delta] \)) contains no end of the arc \( R(\delta') \), \( \delta' \in \Delta \), except for \( \delta' = \epsilon \) (respectively, \( \delta' = \epsilon \) or \( \delta \)), see Fig. 2(c).

Without loss of generality, we may assume that the fiber \( \Gamma \) separates the vertices \( \delta \) and \( \epsilon \), i.e., there exists a vertex \( \gamma \in \Gamma \) adjacent with \( \delta \) and non-adjacent with \( \epsilon \). Since \( \gamma \) is adjacent with both \( \alpha \) and \( \beta \) (see (5.4)), it follows that

\[
[R(\alpha)\delta, R(\delta)\epsilon] \subseteq R(\gamma).
\]

Consequently, \( \gamma \) is adjacent with all the vertices of \( \Delta \) other than \( \epsilon \). Thus the relation \( r(\gamma, \epsilon) \) is of valency 1 and the fiber \( \Delta \) is unessential. \( \square \)

By Lemmas 5.5, 5.6, and 5.7, to complete the proof of Theorem 5.1, we may assume that each non-singleton fiber \( \Delta \in F \) is an \( R \)-covering set. It remains to prove that \( \Delta \) is unessential. By Lemma 5.4, such a fiber contains no special vertex. Therefore, \( \Delta \) is the disjoint union of the sets

\[
\Delta_1 = \{ \delta \in \Delta : R(\alpha)\delta, R(\beta)\epsilon \in R(\delta) \quad \text{and} \quad R(\alpha)\epsilon, R(\beta)\delta \not\subseteq R(\delta) \}.
\]
\[ \Delta_2 = \{ \delta \in \Delta : R(\alpha)_h, R(\beta)_t \not\in R(\delta) \text{ and } R(\alpha)_t, R(\beta)_h \in R(\delta) \}. \]

Obviously, the sets \( \Delta_1 \) and \( \Delta_2 \) are \( R \)-interval. Furthermore since the graph \( X_\Delta \) is a clique, so are the graphs \( X_{\Delta_1} \) and \( X_{\Delta_2} \).

Let \( \Delta_1 = \{ \delta_1, \ldots, \delta_m \} \). By the above, the indices can be chosen so that the ends of the arcs \( R(\delta_i) \) are located in the cycle \( C \) as in (5.2). Let \( \Gamma \) be a fiber separating the vertices \( \delta := \delta_{m-1} \) and \( \varepsilon := \delta_m \), and a vertex \( \gamma \in \Gamma \) is adjacent with \( \delta \) and is not adjacent with \( \varepsilon \). Since \( \gamma \) is not special,
\[
R(\alpha)_h, R(\beta)_t \not\in R(\gamma) \text{ and } R(\alpha)_t, R(\beta)_h \in R(\gamma).
\]
Therefore, \( \gamma \) is adjacent with each vertex of \( \Delta_1 \setminus \{ \varepsilon \} \) and each vertex of \( \Delta_2 \), and hence with all vertices of \( \Delta \setminus \{ \varepsilon \} \). This implies that the relation \( r(\gamma, \varepsilon) \) is of valency 1 and the fiber \( \Delta \) is unessential.

## 6 Proof of Theorem 1.1

The key point of the proof is the following statement which is obtained as a combination of results established in Section 4 and Theorem 5.1.

**Theorem 6.1.** Let \( X \) be a CA-graph and \( \mathcal{X} \geq \text{WL}(X) \) a coherent configuration. Assume that \( X \) contains at least two non-adjacent vertices each of which forms a fiber of \( \mathcal{X} \). Then the coherent configuration \( \mathcal{X} \) is separable.

**Proof.** Denote the vertex set of \( X \) by \( \Omega \), and let \( \alpha \) and \( \beta \) be non-adjacent vertices each of which forms a fiber of \( \mathcal{X} \). If \( \Omega = \{ \alpha, \beta \} \), then the statement holds trivially. We assume that \( |\Omega| \geq 3 \). Then by the theorem hypothesis, \( X \) and \( \mathcal{X} \) satisfy the condition (ii) of Theorem 5.1.

Assume first that each fiber of \( \mathcal{X} \) is \( X \)-twinless. By Theorem 5.1 this implies that \( \mathcal{X} \) contains an unessential fiber \( \Delta \) other than \( \{ \alpha \} \) and \( \{ \beta \} \). Thus the required statement follows from Lemma 3.1 by induction on \( |\Omega| \) applied to the graph \( X_{\Omega \setminus \Delta} \) and coherent configuration \( \mathcal{X}_{\Omega \setminus \Delta} \).

Now we may assume that \( \mathcal{X} \) has a nontrivial twin parabolic \( e \) (Lemma 4.3). Then the underlying set \( \Omega \) of an \( e \)-residue \( \mathcal{X} \) of the coherent configuration \( \mathcal{X} \) is a proper subset of \( \Omega \). Thus the required statement follows from Lemma 4.2 by induction on \( |\Omega| \) applied to the graph \( X_{\Omega} \) and coherent configuration \( \mathcal{X} \).

To complete the proof of Theorem 1.1 let \( X = (\Omega, E) \) and \( X' = (\Omega', E') \) be two \( n \)-vertex graphs. Without loss of generality we may assume that each of them is a non-complete graph with at least three vertices. Fix two non-adjacent vertices \( \alpha, \beta \in \Omega \). Given two (non-adjacent) vertices \( \alpha', \beta' \in \Omega' \), denote by \( \psi_{\alpha', \beta'} \) the bijection between the 3-element sets \( \{1_{\alpha}, 1_{\beta}, E\} \) and \( \{1_{\alpha'}, 1_{\beta'}, E'\} \),
\[
1_{\alpha} \mapsto 1_{\alpha'}, \quad 1_{\beta} \mapsto 1_{\beta'}, \quad \{E\} \mapsto \{E'\}.
\]

**Lemma 6.2.** In the above notation, assume that \( X \) is a CA-graph. Then \( X \) and \( X' \) are isomorphic if and only if there exist (non-adjacent) vertices \( \alpha', \beta' \) of \( X' \) and algebraic isomorphism
\[
\varphi_{\alpha', \beta'} : \text{WL}(\{1_{\alpha}, 1_{\beta}, E\}) \to \text{WL}(\{1_{\alpha'}, 1_{\beta'}, E'\})
\]
(6.1) extending \( \psi_{\alpha', \beta'} \).

**Proof.** Given an isomorphism \( f \in \text{Iso}(X, X') \), set \( \alpha' = \alpha f \) and \( \beta' = \beta f \), and denote by \( \varphi_{\alpha', \beta'} \) the algebraic isomorphism induced by \( f \). Then, obviously, \( \varphi_{\alpha', \beta'} \) extends \( \psi_{\alpha', \beta'} \). This proves the “only if” part.

To prove the “if” part, assume that there exist (non-adjacent) vertices \( \alpha', \beta' \in \Omega \) and algebraic isomorphism (6.1) extending \( \psi_{\alpha', \beta'} \). From Theorem 6.1 it follows that this algebraic
isomorphism is induced by a bijection \( f : \Omega \to \Omega' \). This implies that \( E' = E' \). Consequently, \( f \in \text{Iso}(X, X') \) and hence the graphs \( X \) and \( X' \) are isomorphic. \( \square \)

By Theorem 3.2, given \( \alpha', \beta' \in \Omega' \) one can test in time \( \text{poly}(n) \) whether there exists the algebraic isomorphism (6.1). However, there are at most \( n(n - 1) \) possible choices for the pair of non-adjacent vertices \( \alpha' \) and \( \beta' \). Thus using Lemma 6.2 one can test whether \( X \) and \( X' \) are isomorphic also in time \( \text{poly}(n) \). Since the coherent closure can be computed in \( O(n^3 \log n) \) time, the overall complexity of the algorithm deciding isomorphism of circular-arc graphs is \( O(n^5 \log n) \).

References

[1] M. Chandoo, Deciding circular-arc graph isomorphism in parameterized logspace, 33rd Symposium on Theoretical Aspects of Computer Science, Art. No. 26, LIPIcs. Leibniz Int. Proc. Inform., 47, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern (2016).

[2] S. Chaplick and P. Zeman, Combinatorial problems on \( H \)-graphs, Electronic Notes in Discrete Mathematics, no. 61, 223–229, (2017).

[3] G. Chen and I. Ponomarenko, Lectures on Coherent Configurations (2019), http://www.pdmi.ras.ru/~inp/ccNOTES.pdf.

[4] A. R. Curtis, M. C. Lin, R. M. McConnell, Y. Nussbaum, F. J. Soulignac, J. P. Spinrad, and J. I. Szwarcfiter, Isomorphism of graph classes related to the circular-ones property, Discrete Math. Theor. Comput. Sci., 15, no. 1, 157–182 (2013).

[5] S. Evdokimov and I. Ponomarenko, Characterization of cyclotomic schemes and normal Schur rings over a cyclic group, St. Petersburg Math J. 14, no. 2, 189–221 (2002).

[6] M. Grohe, Descriptive complexity, canonisation, and definable graph structure theory, Cambridge University Press, Cambridge (2017).

[7] N. Immerman and E. Lande, Describing graphs: a first-order approach to graph canonization, Complexity Theory Retrospective, Springer-Verlag (1990), pp. 59–81.

[8] J. Köbler, S. Kuhnert, and O. Verbitsky, On the isomorphism problem for Helly circular-arc graphs, Inform. and Comput., 247, 266–277 (2016).

[9] A. Tucker, An efficient test for circular-arc graphs, SIAM J. Comput., 9, no. 1, 1–24 (1980).

[10] B. Weisfeiler (editor), On construction and identification of graphs, Lecture Notes in Math., 558 (1976).