A DISCRETE ADAPTED HIERARCHICAL BASIS SOLVER FOR RADIAL BASIS FUNCTION INTERPOLATION

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ABSTRACT. In this paper we develop a discrete Hierarchical Basis (HB) to efficiently solve the Radial Basis Function (RBF) interpolation problem with variable polynomial degree. The HB forms an orthogonal set and is adapted to the kernel seed function and the placement of the interpolation nodes. Moreover, this basis is orthogonal to a set of polynomials up to a given degree defined on the interpolating nodes. We are thus able to decouple the RBF interpolation problem for any degree of the polynomial interpolation and solve it in two steps: (1) The polynomial orthogonal RBF interpolation problem is efficiently solved in the transformed HB basis with a GMRES iteration and a diagonal (or block SSOR) preconditioner. (2) The residual is then projected onto an orthonormal polynomial basis. We apply our approach on several test cases to study its effectiveness.

1. INTRODUCTION

The computational cost for extracting RBF representations can be prohibitively expensive for even a moderate amount of interpolation nodes. For an $N$-point interpolation problem using direct methods it requires $O(N^2)$ memory and $O(N^3)$ computational cost. Moreover, since many of the most accurate RBFs have globally supported and increasing kernels, this problem is often badly conditioned and difficult to solve with iterative methods. In this paper we develop a fast, stable and memory efficient algorithm to solve the RBF interpolation problem based on the construction of a discrete HB.

Development of RBF interpolation algorithms has been widely studied in scientific computing. In general, current fast solvers are not yet optimal. One crucial observation of the RBF interpolation problem is that it can be posed as a discrete form of an integral equation. This insight allows us to extend the techniques originally introduced for integral equations to the efficient solution of RBF interpolation problems.

RBF interpolation has been studied for several decades. In 1977 Duchon [23] introduced one of the most well known RBFs, the thin-plate spline. This RBF is popular in practice since it leads to minimal energy interpolant between the interpolation nodes in 2D. In [24] Franke studied the approximation capabilities of a large class of RBFs and concluded that the biharmonic spline and the multiquadric give the best approximation. Furthermore, error estimates for RBF interpolation have been developed by Schaback et al. [52, 46, 47] and more recently by Narcowich et al. [36].

RBFs are of much interest in the area of visualization and animation. They have found applications to point cloud reconstructions, denoising and repairing of meshes [13]. In general, they have been used for the reconstruction of 3-D objects and deformation of these objects [38]. For these areas of applications it is usually sufficient to consider zero and linear-degree polynomials in the RBF problems. However, other applications, such as Neural Networks and classification [55], boundary and finite element methods [21, 22], require consideration of higher-degree polynomials.

More recently, the connection between RBF interpolation, Generalized Least Squares (GLSQ) [45] and its extension to the Best Unbiased Linear Estimator (BLUE) problem has been established.
If the covariance matrix of a GLSQ (and BLUE) problem is described by a symmetric kernel matrix of an RBF problem among other conditions, the two problems become equivalent. Although GLSQ is of high interest to the statistics community, as shown by the high number of citations of [45], the lack of fast solvers limits its application to small to medium size problems [31, 32]. Moreover, many of these statistical problems involve higher than zero- and linear-degree polynomial regression [54, 53, 43, 33, 34, 50]. By exploiting the connection between GLSQ and RBFs, we will be able to solve GLSQ using the fast solvers developed in the RBF and integral equation communities.

For the BLUE Kriging estimator there is less need of higher order polynomials. In many cases quadratic is sufficient for high accuracy estimation. The quadratic interpolant leads to much better estimate than constant or linear. In addition, in [29] the author uses second degree polynomial BLUE for repairing surfaces.

Recently Gumerov et al. [27] developed a RBF solver with a Krylov subspace method in conjunction with a preconditioner constructed from Cardinal functions. We note that this approach, to our knowledge, is the state of the art for zero-degree interpolation in $\mathbb{R}^3$ with a biharmonic spline. This makes it very useful for interpolation problems in computer graphics. On the other hand, its application to regression problems such as GLSQ is limited.

A domain decomposition method was developed in [8] by Beatson et al. This method is a modification of the Von Neumann’s alternating algorithm, where the global solution is approximated by iterating a series of local RBF interpolation problems. This method is promising and has led to (coupled with multi-pole expansions) $O(N \log(N))$ computational cost for certain interpolation problems.

Although the method is very efficient and exhibits $O(N \log(N))$ computational complexity, this seems to be true for small to medium size problems (up to 50,000 nodes in $\mathbb{R}^3$) with smooth data. Beyond that range the computational cost increases quadratically as shown in [8]. Other results for non smooth data shows that the computational complexity is more erratic [14]. Furthermore, in many cases, it is not obvious how to pick the optimal domain decomposition scheme.

An alternative approach was developed by Beatson et al. [7], which is based on preconditioning and coupled with GMRES iterations [44]. This approach relies on the construction of a polynomial orthogonal basis, similar to the HB approach in our paper. This approach gives rise to a highly sparse representation of the RBF interpolation matrix that can be very easily preconditioned by means of a diagonal matrix. The new system of equations exhibits condition number growth of no more than $O(\log N)$. The downside is that this basis is not complete. This is ameliorated by the introduction of non decaying elements, but no guarantees on accuracy can be made.

Our approach is based on posing the RBF interpolation problem as a discretization of an integral equation and applying preconditioning techniques. This approach has many parallels with the work developed by Beatson et al. [7]. However, our approach was developed from work done for fast integral equation solvers.

Most of the work in the area of fast integral equation solvers has been restricted to the efficient computation of matrix vector products as part of an iterative scheme. For the Poisson kernel the much celebrated multi-pole spherical harmonic expansions leads to a fast summation algorithm that reduces each matrix-vector multiplication to $O(N)$ computational steps [26]. This technique has been extended to a class of polyharmonic splines and multiquadrics [5, 19]. More recently L. Ying et al. has developed multipole algorithms for a general class of kernels [56]. In contrast, the development of optimal (or good) preconditioners for integral equations has been more limited.

A unified approach for solving integral equations efficiently was introduced in [1, 2, 9]. A wavelet basis was used for sparsifying the discretized operator and only $O(N \log^3(N))$ entries of the discretization matrix are needed to achieve optimal asymptotic convergence. The downside is that it was limited to 1D problems.

In [17] a class of multiwavelets based on a generalization of Hierarchical Basis (HB) functions was introduced for sparsifying integral equations on conformal surface meshes in $\mathbb{R}^3$. These wavelets
are continuous, multi-dimensional, multi-resolution and spatially adaptive. These constructions are based on the work on Lifting by Schroder and Sweldens [49] and lead to a class of adapted HB of arbitrary polynomial degree. A similar approach was also developed in [51].

These constructions provide compression capabilities that are independent of the geometry and require only $O(N\log^3(N))$ entries to achieve optimal asymptotic convergence. This is also true for complex geometrical features with sharp edges. Moreover, this basis has a multi-resolution structure that is related to the BPX scheme [39], making them an excellent basis to precondition integral and partial differential equations. In [20] Heedene et al. demonstrate how to use this basis to build scale decoupled stiffness matrices for partial differential equations (PDEs) over non uniform irregular conformal meshes.

In this paper, we develop a discrete HB for solving isotropic RBF interpolation problems efficiently. Our HB construction is adapted to the topology of the interpolating nodes and the kernel. This new basis decouples the polynomial interpolation from the RBF part, leading to a system of equations that are easier to solve. With our sparse SSOR [25, 30] or diagonal preconditioner, the RBF interpolation problem can be solved efficiently.

Our contributions include a method with asymptotic complexity costs similar to Gumerov et al [27] for problems in $\mathbb{R}^3$. However, their approach is restricted to only constant degree RBF interpolation. Due to the decoupling of the polynomial interpolation, our approach is more flexible and works well for higher degree polynomials. We show similar results for the multiquadric RBFs in $\mathbb{R}^3$. In contrast we did not observe multiquadric results for $\mathbb{R}^3$ in [27] and to our knowledge this result is not available. Note that the idea of decoupling the RBF system of equations from the polynomial interpolation has also been proposed in [49] and [8].

In the rest of Section 1 we explicitly pose the RBF interpolation problem. In Section 2 we construct an HB that is adapted to the interpolating nodes and the kernel seed function. In Section 3 we demonstrate how the adapted HB is used to form a multi-resolution RBF matrix, which is used to solve the interpolation problem efficiently. With our sparse SSOR [25, 30] or diagonal preconditioner, the interpolating nodes are randomly placed, moreover the interpolating values themselves contain random noise. We summarize our conclusions in Section 5.

During the writing of this paper we became aware of the H-Matrix approach by Hackbusch [10] applied to stochastic capacitance extraction [57] problem. In [11] the authors apply an H-matrix approach to sparsify the kernel matrix arising from a Gaussian process regression problem to $O(N\log N)$. In our paper, we apply HB to precondition the RBF system, although we could also use them to sparsify it. Instead, we use a fast summation approach to compute the matrix-vector products.

1.1. Radial Basis Function Interpolation. In this section we pose the problem of RBF interpolation for bounded functions defined on $\mathbb{R}^3$. Although our exposition is only for $\mathbb{R}^3$, the RBF problem and our HB approach can be extended to any finite dimension.

Consider a function $f(\vec{x}) : \mathbb{R}^3 \to \mathbb{R}$ in $L_2(\mathbb{R}^3)$ and its evaluation on a set of user-specified sampling of distinct nodes $X := \{\vec{x}_1, ..., \vec{x}_N\} \subset \mathbb{R}^3$, where $\vec{x} = [x_1, x_2, x_3]^T$, unisolvent with respect to all polynomials of degree at most $m$. We are interested in constructing approximations to $f(\vec{x})$ of the form

$$s(\vec{x}) = \sum_{i=1}^{M(m)} c[i] q_i(\vec{x}) + \sum_{j=1}^{N} u[j] K(\vec{x}, \vec{x}_j),$$

where $K : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $u \in \mathbb{R}^N$, $c \in \mathbb{R}^{M(m)}$ and $P := \{q_1(\vec{x}), ..., q_{M(m)}(\vec{x})\}$ is a basis for $\mathcal{P}^m(\mathbb{R}^3)$, i.e. the set of all polynomials of total degree at most $m$ in $\mathbb{R}^3$ (Note that $M(m)$ is the number of polynomials that form a basis for $\mathcal{P}^m(\mathbb{R}^3)$ i.e. $M(m) = \binom{m+3}{3}$). This interpolant must satisfy the following condition

$$s(\vec{x}_j) = f(\vec{x}_j), \quad j = 1, ..., N,$$
for all \( \vec{x}_j \) in \( X \). Moreover, to ensure the interpolation is unique we add the following constraint

\[
\sum_{j=1}^{N} u_j q(\vec{x}_j) = 0,
\]

for all polynomials \( q(\vec{x}) \) of degree at most \( m \). Now, since \( M(m) \) is the minimum amount of nodes needed to solve the polynomial problem, we need at least \( N \geq M(m) \) RBF centers. The interpolation problem can be rewritten in matrix format as

\[
\begin{pmatrix} K & Q^T \\ Q & 0 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix},
\]

where \( K_{i,j} = K(\vec{x}_i, \vec{x}_j) \) with \( i = 1 \ldots N \) and \( j = 1 \ldots N \); \( d_j = f(\vec{x}_j) \); \( c \in \mathbb{R}^{M(m)} \); and \( Q_{i,j} = q_j(\vec{x}_i) \) with \( i = 1 \ldots N \), \( j = 1 \ldots M(m) \). Denote the columns of \( Q \) as \( [q_1, \ldots, q_{M(m)}] \). This is the general form of the RBF interpolation isotropic problem. The properties of this approximation mostly depend on the seed function \( K(\vec{x}, \vec{y}) \). An example of a well known isotropic kernel in \( \mathbb{R}^3 \) is the biharmonic spline

\[
K(\vec{x}, \vec{x}_j) := K(|\vec{x} - \vec{x}_j|) = |\vec{x} - \vec{x}_j|.
\]

This is a popular kernel due to the optimal smoothness of the interpolant [8]. This kernel has been successfully applied in point cloud reconstructions, denoising and repairing of meshes [13]. More recently, there has been interest in extensions to anisotropic kernels [15][16], i.e.

\[
K(\vec{x}, \vec{x}_j) := K(|T_j(\vec{x} - \vec{x}_j)|),
\]

where \( T_j \) is a \( 3 \times 3 \) matrix. The stabilization method introduced in this paper can be extended to solving efficiently the RBF problem with spatially varying kernels. By using the sparsification properties of the adapted HB a sparse representation of the spatially varying RBF matrix can be constructed in optimal time. However, in this paper we restrict the analysis to isotropic kernels in \( \mathbb{R}^3 \), i.e. \( T_j = \alpha I \) where \( \alpha > 0 \).

One aspect of RBF interpolation is the invertibility of the matrix in Equation (2). In [35] it is shown that the interpolation problem (2) has a unique solution if we assume that the interpolating nodes in \( X \) are unisolvent with respect to \( \mathcal{P}^m(\mathbb{R}^3) \) and the continuous kernel is strictly conditionally positive (or negative) definite. Before we give the definition, we provide some notation.

**Definition 1.** Suppose that \( X \subset \mathbb{R}^3 \) is a set of interpolating nodes and \( \{q_1(\vec{x}), q_2(\vec{x}), \ldots, q_{M(m)}(\vec{x})\} \) is a basis for \( \mathcal{P}^m(\mathbb{R}^3) \), then we use \( \mathcal{P}^m(X) \) to denote the column space of \( Q \).

We now assume the kernel matrix \( K \) satisfies the following assumption.

**Definition 2.** We say that the symmetric function \( K(\cdot, \cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) is strictly conditionally positive definite of degree \( l \) if for all sets \( X \subset \mathbb{R}^3 \) of distinct nodes

\[
v^H K v = \sum_{i,j=1}^{N} v_i v_j K(\vec{x}_i, \vec{x}_j) > 0,
\]

for all \( v \in \mathbb{R}^N \) such that \( v \perp \mathcal{P}^l(X) \) and \( v \neq 0 \). Alternatively, under the same assumptions, \( K(\cdot, \cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) is strictly conditionally negative definite if

\[
v^H K v < 0,
\]

for all \( v \in \mathbb{R}^N \) such that \( v \perp \mathcal{P}^l(X) \).

The invertibility of the RBF interpolation problem can be proven by the basis construction developed in this paper. Although this is not necessary, it does cast insights on how to construct a basis that can solve the RBF Problem (2) efficiently.
1.2. Decoupling of the RBF interpolation problem. Suppose there exists a matrix \( T : \mathbb{R}^{N-M} \rightarrow \mathbb{R}^N \), where \( M := \dim(\mathcal{P}_m(X)) \), such that \( T^H \) annihilates any vector \( v \in \mathcal{P}_m(X) \) (i.e. \( T^H v = 0 \) \( \forall v \in \mathcal{P}_m(X) \)). Furthermore, suppose there exists a second matrix \( L : \mathbb{R}^M \rightarrow \mathbb{R}^N \) such that the combined matrix \( P := [L \ T] \) is orthonormal such that \( P^H : \mathbb{R}^N \rightarrow \mathbb{R}^N \) maps \( \mathbb{R}^N \) onto 
\[ \mathcal{P}_m(X) \oplus W, \]
where \( W := (\mathcal{P}_m(X))^\perp \). Suppose that \( u \in \mathcal{P}_m(X)^\perp \), then \( u = Tw \) for some \( w \in \mathbb{R}^{N-M} \). Problem (2) can now be re-written as 
\[ T^H KT w + T^H Q c = T^H d. \]
However, since the columns of \( Q \) belong in \( \mathcal{P}_m(X) \) then 
\[ T^H KT w = T^H d. \]
From Definition (2) and the orthonormality of \( P \) we conclude that \( w \) can be solved uniquely. The second step is to solve the equation \( L^H Q c = L^H d - L^H KT w \). From the unisolvent property of the nodes \( X \) the matrix \( Q \) has rank \( \dim(\mathcal{P}_m(\mathbb{R}^1)) \), moreover, \( L \) also has rank \( \dim(\mathcal{P}_m(\mathbb{R}^3)) \), thus \( L^H Q \) has full rank and it is invertible.

Although proving the existence of \( P \) and hence the uniqueness of the RBF problem is an interesting exercise, there are more practical implications to the construction of \( P \). First, the coupling of \( Q \) and \( K \) can lead to a system of ill-conditioned equations depending on the scale of the domain [8]. The decoupling property of the transform \( P \) leads to a scale independent problem, thus correcting this source of ill-conditioning. But more importantly, we focus on the structure of \( T^H KT \) and how to exploit it to solve the RBF interpolation problem (2) efficiently. The key idea is the ability of \( T^H \) to vanish discrete polynomial moments and its effect on the matrix \( K(\cdot, \cdot) \). We shall now restrict our attention to Kernels that satisfy the following assumption.

**Assumption 1.** Let \( D_\alpha^x := \frac{\partial^{\alpha_1} \alpha_2 \alpha_3}{\partial x_1 \partial x_2 \partial x_3} \) and similarly for \( D_\beta^y \), we assume that 
\[ D_\alpha^x D_\beta^y K(\vec{x}, \vec{y}) \leq \frac{C}{|\vec{x} - \vec{y}| \alpha + |\alpha| + |\beta|}, \]
where \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3 \), \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \), and \( q \in \mathbb{Z} \). In addition, we assume that \( K(\cdot, \cdot) \) and \( K(\cdot, \vec{y}) \) are analytic everywhere except for \( \vec{x} = \vec{y} \).

This assumption is satisfied by many practical kernels, such as multiquadrics and polyharmonic splines [24] [8].

2. Adapted Discrete Hierarchical Basis Constructions

In this section we show how to construct a class of discrete HB that is adapted to the kernel function \( K(\cdot, \cdot) \) and to the local interpolating nodes (or interpolating nodes) contained in \( X \). The objective is to solve RBF interpolation Problem (2) efficiently. The HB method will be divided into the following parts:

- **Multi-resolution domain decomposition.** The first part is in essence a preprocessing step to build cubics at different levels of resolution as place holders for the interpolation nodes belonging to \( X \).
- **Adapted discrete HB construction.** From the multi-resolution domain decomposition of the interpolating nodes in \( X \), an adapted multi-resolution basis is constructed that annihilates any polynomial in \( \mathcal{P}_m(X) \), where \( p \in \mathbb{Z}^+ \) and \( p \geq m \). \( p \) will be in essence the degree of the Hierarchical Basis, which is not to be confused with \( m \).
- **GMRES iterations with fast summation method.** With the adapted HB a multi-resolution RBF interpolation matrix is implicitly obtained through a fast summation method and solved iteratively with a GMRES algorithm and an SSOR or diagonal preconditioner.
2.1. **Multi-resolution Domain Decomposition.** Without loss of generality, it is assumed that the interpolating nodes in $X$ are contained in a cube $B^0_l := [0, 1]^3$. The next step is to form a series of level dependent cubes that serve as place holders for the interpolating nodes at each level of resolution.

The basic algorithm is to subdivide the cube $B^0_l$ into eight cubes if $|B^0_l| > M(p)$, where $|B^0_l|$ denotes the total number of interpolating nodes contained in the cube $B^0_l$. Subsequently, each cube $B^0_l$ is sub-divided if $|B^0_l| > M(p)$ until there are at most $M(p)$ interpolating nodes at the finest level. The algorithm is explained more in detail in the following pseudo-code:

**Input:** $X := \{x_1, x_2, \ldots, x_N\}$, $M(p)$

**Output:** $B^j_k \forall k \in \{\mathcal{K}(0), \ldots, \mathcal{K}(n)\}, n$

**begin**

pre-processing:

$j \leftarrow 0; B^0_0 \leftarrow [0, 1]^3; \mathcal{K}(0) \leftarrow \{0\}$

main:

while $|B^j_k| > M(p)$ for any $k \in \mathcal{K}(j)$ do

$\mathcal{K}(j + 1) \leftarrow \emptyset$;

for $k \leftarrow 0$ to $|\mathcal{K}(j)|$ do

forming $B^i_{8k} + 1, \ldots, B^i_{8k + 7}; \mathcal{K}(j + 1) \leftarrow \mathcal{K}(j + 1) \cup_{w=0}^{7} 8k + w$;

end

$n \leftarrow j$

**end**

**Algorithm 1:** Multi-resolution Domain Decomposition

**Remark 1.** $\mathcal{K}(j)$ is an index set for all the cubes at level $j$. We use $|\mathcal{K}(j)|$ to denote the cardinality of $\mathcal{K}(j)$.

**Remark 2.** Finding the distance between any two boxes can be performed in $O(N(n+1))$ computational steps by applying an octree algorithm. Therefore the Multi-resolution Domain Decomposition algorithm can be performed in $O(N(n+1))$ computational steps. This can be easily seen since the maximum number of boxes at any level $j$ is bounded by $N$ and there is a total of $n+1$ levels.

Before describing the construction of the adapted discrete HB, we introduce some more notations to facilitate our discussion.

**Definition 3.** Let $\mathcal{B}_j$ be the set of all the cubes $B^j_k$ at level $j$ that contain at least one interpolating center from $X$.

**Definition 4.** Let $\mathcal{C} := \{e_1, \ldots, e_N\}$, where $e_i[1] = 1$ and $e_i[j] = 0$ if $i \neq j$. Furthermore, define the bijective mapping $F_p: \mathcal{C} \rightarrow X$ such that $F_p(e_i) = \tilde{x}_i$ for $i = 1 \ldots N$ and $F_q: \mathcal{C} \rightarrow Z^+$ s.t. $F_q(e_i) = i$. Now, for each cube $B^j_k \in \mathcal{B}_n$ at the finest level $n$, let

$$B^0_k := \{e_i \mid F_p(e_i) \in B^0_k\}.$$ 

and for all $l = 1, \ldots, n-1$

$$\tilde{B}^j_k := \{e_i \mid F_p(e_i) \in B^j_k\}.$$

**Definition 5.** Let $\mathcal{C}_n := \bigcup_{k \in \mathcal{K}(n)} B^0_k$.

**Definition 6.** For all $j = 0, \ldots, n-1$, let $children(B^j_k)$ be the collection of nonempty subdivided cubes $B^{i-1}_{j+1} \in \mathcal{B}_j$, for some $l \in \mathbb{N}$, of the cube $B^j_k$.

**Definition 7.** For every non empty $B^j_k$ let the set parent$(B^j_k) := \{B^{i-1}_{j-1} \mid B^j_k \in children(B^{i-1}_{j-1})\}$. 
2.2. Basis Construction. From the output of the multi-resolution decomposition Algorithm[1] we can now build an adapted discrete HB that annihilates any polynomial in \( \mathcal{P}(X) \). To construct such a basis, we apply the stable completion [12] procedure. This approach was followed in [17]. However, the basis is further orthogonalized by using a modified Singular Value Decomposition (SVD) orthonormalization approach introduced in [51].

Suppose \( v_1, \ldots, v_s \) are a set of orthonormal vectors in \( \mathbb{R}^N \), where \( s \in \mathbb{Z}^+ \), a new basis is constructed such that

\[
\phi_j := \sum_{i=1}^{s} c_{i,j} v_i, \quad j = 1, \ldots, a; \quad \psi_j := \sum_{i=1}^{s} d_{i,j} v_i, \quad j = a + 1, \ldots, s,
\]

where \( c_{i,j}, d_{i,j} \in \mathbb{R} \) and for some \( a \in \mathbb{Z}^+ \). We desire that the new discrete HB vector \( \psi_j \) to be orthogonal to \( \mathcal{P}(X) \), i.e.

\[
\sum_{k=1}^{N} r[k] \psi_j[k] = 0,
\]

for all \( r \in \mathcal{P}(X) \). Notice that the summation and the vectors \( r \) and \( \psi_j \) are in the same order as the entries of the set \( X \).

Due to the orthonormality of the basis \( \{v_i\}_{i=1}^{s} \) this implies that Equation (5) is satisfied if the vector \( [d_{1,1}, \ldots, d_{1,s}] \) belongs to the null space of the matrix

\[
M_{s,p} := Q^H V,
\]

where the columns of \( Q \) are a basis for \( \mathcal{P}(X) \) (i.e. all the polynomial moments) and \( V = [v_1, v_2, \ldots, v_s] \).

(Notice that the order of the summation is done with respect to the set \( X \).) Suppose that the matrix \( M_{s,p} \) is a rank \( a \) matrix and let \( U_{s,p} D_{s,p} V_{s,p} \) be the SVD decomposition. We then pick

\[
\begin{bmatrix}
  c_{0,1} & \ldots & c_{a,1} \\
  c_{0,2} & \ldots & c_{a,2} \\
  \vdots & \ddots & \vdots \\
  c_{0,s} & \ldots & c_{a,s}
\end{bmatrix}
\begin{bmatrix}
  d_{0,1} & \ldots & d_{s,1} \\
  d_{0,2} & \ldots & d_{s,2} \\
  \vdots & \ddots & \vdots \\
  d_{0,s} & \ldots & d_{s,s}
\end{bmatrix}
:= V_{s,p}^H
\]

where the columns \( a+1, \ldots, s \) form an orthonormal basis of the nullspace \( \mathcal{N}(M_{s,p}) \). Similarly, the columns \( 1, \ldots, a \) form an orthonormal basis of \( \mathbb{R}^N \setminus \mathcal{N}(M_{s,p}) \).

Remark 3. If \( \{v_1, \ldots, v_s\} \) is orthonormal, then new basis \( \{\phi_1, \ldots, \phi_a, \psi_{a+1}, \ldots, \psi_s\} \) is orthonormal, and spans the same space as \( \text{span}\{v_1, \ldots, v_s\} \). This is due to the orthonormality of the matrix \( V_{s,p} \).

Remark 4. If \( s \) is larger than the total number of vanishing moments, then \( M_{s,p} \) is guaranteed to have a nullspace of at least rank \( s - M(p) \), i.e. there exist at least \( s - M(p) \) orthonormal vectors \( \{\psi_j\} \) that satisfy Equation (5).

2.3. Finest Level. We can now build an orthonormal multi-resolution basis. First, choose a priori the degree of moments \( p \) and start at the finest level \( n \). The next step is to progressively build the adapted HB as the levels are traversed.

At the finest level \( n \), for each cube \( B_{k}^{p} \in \mathcal{C}_{n} \) let \( v_i := e_i \) for all \( e_i \in B_{k}^{p} \). As described in the previous section, the objective is to build new functions

\[
\phi_{i,l}^{n} := \sum_{i=1}^{s} c_{n,i,l,k} v_i, \quad l = 1, \ldots, a_{n,k}, \quad \psi_{i,l}^{n} := \sum_{i=1}^{s} d_{n,i,l,k} v_i, \quad l = a_{n,k} + 1, \ldots, s,
\]

such that Equation (5) is satisfied.

The first step is to form the matrix \( M_{s,p}^{k} := Q^H V \), where the columns of \( Q \) are a basis for \( \mathcal{P}(X) \). Notice that since \( e_i[w] = 0 \) for \( w \neq i \) and \( e_i \in B_{k}^{p} \), then only \( |B_{k}^{p}| \) columns of \( Q^H \) are needed to form the matrix \( M_{s,p}^{k} \) and the rest can be thrown away since they multiply with zero.
The next step is to apply the SVD procedure such that $M_{s,p}^{n,k} \rightarrow U_{s,p}^{n,k} D_{s,p}^{n,k} V_{s,p}^{n,k}$. The coefficients $c_{n,i,k}$ and $d_{n,i,k}$ are then obtained from the rows of $V_{s,p}^{n,k}$ and $a_{n,k} := \text{rank} M_{s,p}^{n,k}$.

Now, for each $B_k^i \in \mathcal{B}^n$ denote $C_k^i$ as the collection of basis vectors $\{\phi_{k,1}^n, \ldots, \phi_{k,a_{n,k}}^n\}$, and similarly denote $D_k^i$ as the collection of basis vectors $\{\psi_{k,\{a_{n,k}+1\ldots, a_n\}}^n\}$. Furthermore, we define the detail subspace
\[
W_k^n := \text{span}\{\psi_{k,a_{n,k}+1}^n, \ldots, \psi_{k,s}^n\}
\]
and the average subspace
\[
V_k^n := \text{span}\{\phi_{k,1}^n, \ldots, \phi_{k,a_{n,k}}^n\}.
\]
By collecting the transformed vectors from all the cubes in $\mathcal{G}_n$, we form the subspaces
\[
V^n := \oplus_{k \in \mathcal{X}(n)} V_k^n, \quad W^n := \oplus_{k \in \mathcal{X}(n)} W_k^n,
\]
where $\oplus$ is a direct sum and $\mathcal{X}(i) := \{k \mid B_k^i \in \mathcal{B}^i\}$.

**Remark 5.** We first observe that $\mathbb{R}^N = V^n \oplus W^n \oplus \bar{V}_n$, where $\bar{V}_n$ is the span of all the unit vectors contained in $\{\{B_k^i\}_{k \in \mathcal{X}(i)}\}_{i=1}^{n-1}$. This is true since the number of interpolating nodes is equal to $N$ and $\mathbb{R}^N = \text{span}\{e_1, e_2, \ldots, e_N\}$.

**Remark 6.** It is possible that $W_k^n = \emptyset$ for some particular cube $B_k^i$. This will be the case if the cardinality of $B_k^i$ is less or equal to $M(p)$ i.e. the dimension of the nullspace of $M_{s,p}^{n,k}$ is zero. However, this will not be a problem. As we shall see in section 2.4, the next set of HB are built from the vectors in $C_k^n$ and its siblings.

**Lemma 1.** The basis vectors of $V^n$ and $W^n$ form an orthonormal set.

**Proof.** First notice that since $B_k^i \cap B_k^j = \emptyset$ whenever $k \neq l$ then $V_{k}^{n,k} \perp V_{n,l}, W_{k}^{n,k} \perp W_{n,l}$ and $V_{n,l} \perp W_{n,l}$. The result follows from the fact that the rows $V_{s,p}^{n,k}$ form an orthonormal set. \qed

It is clear that the detail subspace $W^n \perp \mathcal{B}^i(\bar{x})$, but the average subspace $V^n$ is not. However, we can still perform the SVD procedure to further decompose $V^n$. To this end we need to accumulate the average basis vectors of $V^n$ and all the unit basis vectors in $\{B_k^{n-1}\}_{k \in \mathcal{X}(n-1)}$. For each $B_k^{n-1} \in \mathcal{B}_{n-1}$ identify the set children$(B_k^{n-1})$. Form the set $B_k^{n-1} := \{\tilde{C}_k^n \mid B_k^i \in \text{children}(B_k^{n-1})\}$. If $B_k^{n-1}$ has no children then $B_k^{n-1} = B_k^{n-1}$. We can now apply the SVD procedure on each set of average vectors in $B_k^{n-1}$.

2.4. **Intermediate Level.** Suppose we have the collection of sets $B_k^i$ for all $k \in \mathcal{X}(i)$. For each $B_k^i$ perform the matrix decomposition $M_{s,p}^{i,k} = U_{s,p}^{i,k} D_{s,p}^{i,k} V_{s,p}^{i,k}$ for all $v \in B_k^i$. From the matrix $V_{s,p}^{i,k}$ obtain the decomposition
\[
\phi_{k,l}^i := \sum_{j=1}^{s} c_{i,j,k,l} v_j, \quad l = 1, \ldots, a_{i,k}, \quad \psi_{k,l}^i := \sum_{j=1}^{s} d_{i,j,k,l} v_j, \quad l = a_{i,k} + 1, \ldots, s,
\]
where the coefficients $c_{i,j,k,l}$ and $d_{i,j,k,l}$ are obtained from the rows of $V_{i,p}^{i,k}$ in (6) and $a_{i,k} := \text{rank} M_{s,p}^{i,k}$.

Then we form the subspaces
\[
W_k^i := \text{span}\{\psi_{k,a_{n,k}+1}^i, \ldots, \psi_{k,s}^i\}, \quad V_k^i := \text{span}\{\phi_{k,1}^i, \ldots, \phi_{k,a_{n,k}}^i\},
\]
and
\[
V^i := \oplus_{k \in \mathcal{X}(i)} V_k^i, \quad W^i := \oplus_{k \in \mathcal{X}(i)} W_k^i.
\]
It is easy to see that $V^{i+1} = V^i \oplus W^i \oplus \bar{V}_i$, where $\bar{V}_i$ is the span of all the unit vectors contained in $\{\{B_k^i\}_{k \in \mathcal{X}(i)}\}_{i=1}^{n-1}$. The basis vectors are collected into two groups:

**Definition 8.** For each $B_k^i \in \mathcal{B}^i$ that have children let the sets, for $i = 0, \ldots, n-1$, $\tilde{C}_k^i := \{\phi_{k,1}^i, \ldots, \phi_{k,a_{n,k}}^i\}$, and $D_k^i := \{\psi_{k,a_{n,k}+1}^i, \ldots, \psi_{k,s}^i\}$.\}
Just as for the finest level case, we can further decompose $V_i$. To this end, for each $B_{k}^{i-1} \in \bar{B}_{i-1}$ identify the set $\text{children}(B_{k}^{i-1})$ and form the set $B_{k}^{i-1} := \{ C_{i}^{|} | B_{i}^{|} \in \text{children}(B_{k}^{i-1}) \}$. If $B_{k}^{i-1}$ has no children then $B_{k}^{i-1} = \bar{B}_{k}^{i-1}$.

2.5. **Coarse Level.** It is clear that when the iteration reaches $V_0$ the basis function no longer annihilates polynomials of degree $p$. However, a new basis can be obtained that can vanish polynomials of degree $m$.

Recall that for the RBF interpolation problem with polynomial degree $m$ it is imposed that $u \perp \mathcal{P}^m(X)$. If $p = m$ then it is clear that $u \in W^0 \oplus \ldots W^n$ and RBF problem decouples as shown in Section 1. However, if $p > m$ then $u \in (\mathcal{P}^p(X) \setminus \mathcal{P}^m(X)) \oplus W^0 \oplus \ldots W^n$ and the RBF problem does not decouple. It is then of interest to find an orthonormal basis to $\mathcal{P}^p(X) \setminus \mathcal{P}^m(X)$. This can be easily achieved. Let the columns of the matrix $Q$ be a basis for $\mathcal{P}^p(X)$, where each function $q_i(x)$ corresponds to the $i$th moment. Now, the first $M(m)$ columns correspond to a basis for $\mathcal{P}^m(X)$. Thus an orthonormal basis for $\mathcal{P}^p(X) \setminus \mathcal{P}^m(X)$ is easily achieved by applying the Gram-Schmidt process.

Alternatively the matrix $M_{0,m}$ can now be formed by applying the SVD decomposition and a basis that annihilates all polynomials of degree $m$ or lower is obtained. The matrix $C_{0}^{1}$ can now be replaced with the matrix $[C_{0}^{1},D_{0}^{1},\ldots,D_{p}^{1}]$, where the columns of $C_{0}^{1}$ form an orthonormal basis for $\mathcal{P}^m(X)$ and $D_{0}^{1}$ is an orthonormal basis for $\mathcal{P}^p(X) \setminus \mathcal{P}^m(X)$.

The complete algorithm to decompose $\mathbb{R}^N$ into a multi-resolution basis with respect to the interpolating nodes $X$ is described in Algorithms 2 and 3.

**Algorithm 2:** Adapted Discrete HB Construction

\begin{verbatim}
Input: Finest level $n$; Degree of RBF $m$; $B_{i}^{j} \forall k \in \mathcal{X}(j), j = -1 \ldots n$; $B_{i}^{0} \forall k \in \mathcal{X}(n)$;
\{ $\{ B_{i}^{j} \}_{k \in \mathcal{X}(i)} \}_{i=1}^{n}$; Degree of vanishing moments $p \geq m$; $X$.
Output: $\{ C_{0}^{1},D_{0}^{1},D_{0}^{0},D_{0}^{1},\ldots,D_{p}^{p} \}$

main;
for $j \leftarrow n$ to 1 step -1 do
  for $k \leftarrow 1$ to $|\mathcal{X}(j-1)|$ do
    $B_{k}^{j-1} \leftarrow \emptyset$
  end
  for $k \leftarrow 1$ to $|\mathcal{X}(j)|$ do
    \{ $D_{k}^{j},\tilde{C}_{k}^{j}$ \} \leftarrow PolyOrtho($B_{k}^{j}, p$); $U \leftarrow \text{parent}(B_{k}^{j})$;
    for all the $B_{k}^{j-1} \in U$ do
      $B_{k}^{j-1} \leftarrow B_{k}^{j-1} \cup \tilde{C}_{k}^{j}$;
    end
    for all $\tilde{B}_{k}^{j-1} \in \bar{B}_{j-1}$ let $\tilde{B}_{k}^{j-1} = \tilde{B}_{k}^{j-1}$;
  end
\{ $D_{0}^{1},\tilde{C}_{0}^{1}$ \} \leftarrow PolyOrtho($B_{0}^{0},m$);
end
\end{verbatim}

**Lemma 2.**

$\mathbb{R}^N = V^0 \oplus W^0 \oplus \ldots W^n = \text{span}\{\tilde{C}_{0}^{1},D_{0}^{1},D_{0}^{0},D_{1}^{0},\ldots,D_{p}^{p}\}$.

for $j = 0 \ldots n$ and for all $k \in \mathcal{X}(j)$

**Proof.** The result follows from Remark 5 and that $V_i$ is decomposed into $V^{i-1} \oplus W^{i-1} \oplus \bar{V}_{i-1}$ for all $i = 1 \ldots n$.

**Remark 7.** When Algorithm 2 terminates at level $i = 0$, there will be $M(p)$ orthonormal vectors that span $\mathcal{P}^p(X)$. 


Remark 8. At the finest level $n$, the number of vectors in each matrix $C_k^n$ corresponding to $B_k^n$ is bounded by $M(p)$. Now, for each $B_k^{n-1}$ there are at most $8M(p)$ vectors from the children of $B_k^{n-1}$. From the procedure for the basis construction in section 2.2 for each $B_k^{n-1}$ there are at most $M(p)$ vectors in $C_k^n$. Furthermore, there are no more than $8M(p)$ vectors in $D_k^n$ formed. The same conclusion follows for each $B_k^i$ for all levels $i = 0, \ldots, n$.

Input: $B_k^i$, Degree of vanishing moment $p$
Output: $\hat{D}_k^i$, $\hat{C}_k^i$

1. $\hat{C}_k^i \leftarrow \emptyset$; $\hat{D}_k^i \leftarrow 0$ s.t. $|\hat{B}_k^i| = [v_1, \ldots, v_s]$; $M_{s,p} \leftarrow QHV$;
2. $[U_{s,p},D_{s,p},V_{s,p}] \leftarrow \text{SVD}(M_{s,p})$;
3. $a_{j,k} \leftarrow \text{rank of } D_{j,p}$;
4. for $l \leftarrow 1$ to $a_{j,k}$ do
   - $\phi_{k,l} \leftarrow \sum_{i=1}^s c_{j,i,l}v_i$; $\hat{C}_k^i \leftarrow [\hat{C}_k^i, \phi_{k,l}]$;
5. end
6. for $l \leftarrow a_{j,k} + 1$ to $s$ do
   - $\psi_{k,l} \leftarrow \sum_{i=1}^s d_{j,i,l}v_i$; $\hat{D}_k^i \leftarrow [\hat{D}_k^i, \psi_{k,l}]$;
7. end

Algorithm 3: PolyOrtho($B_k^i, p$)

Definition 9. For any $B_k^i$, $i = 0, \ldots, n$, let $|B_k^i|$ be the number of vectors in $B_k^i$.

Theorem 1. The complexity cost for Algorithm 3 is bounded by $O(N)$.

Proof. Suppose we start at the finest level $n$. Now, for each box in $B_k^n$, the vectors $e_i \in B_k^n$ have at most one non-zero entry. This implies that the matrix $M_{s,p} = QHV$, $Q$ is a $M(p) \times |B_k^n|$ matrix and $V$ is at most a $|B_k^n| \times |B_k^n|$ matrix. Then the total cost to computing $M_{s,p}^{n,k}$ for all $k \in \mathcal{K}(n)$ is bounded by

$$C \sum_{k \in \mathcal{K}(n)} |B_k^n|^2 M(p)$$

for some $C > 0$. Now since $|\cup_{k \in \mathcal{K}(n)} B_k^n| = N$ and $|B_k^n|$ is at most $M(p)$ for all $k \in \mathcal{K}(n)$, then the cost for computing $\hat{C}_k^i$ and $\hat{D}_k^i$, $\forall k \in \mathcal{K}(n)$, is at most $O(N)$.

At level $n-1$, from Remark 8 we see that there are at most $8M(p)$ vectors in each $B_k^{n-1}$ $\forall k \in \mathcal{K}(n-1)$. Forming the the matrix $M_{s,p}^{n-1,k} = QHV$, $Q$ is at most $M(p) \times |B_k^{n-1}|$ and $V$ is at most $|B_k^{n-1}| \times |B_k^{n-1}|$. Now, since $|\cup_{k \in \mathcal{K}(n-1)} B_k^{n-1}| = N$ it follows that the cost for computing $M_{s,p}^{n-1,k}$, $\forall k \in \mathcal{K}(n-1)$, is at most $O(N)$. Furthermore, we have from Remark 8 that $|B_k^{n-2}| \leq 8M(p)$, $\forall k \in \mathcal{K}(n-2)$.

Since for each level $i$, $|\cup_{k \in \mathcal{K}(i)} B_k^i| \leq N$, then the total cost of computing $M_{s,p}^{i,k}$, $\forall k \in \mathcal{K}(i)$, is at most $O(N)$ and $|B_k^{i-1}| \leq 8M(p)$, $\forall k \in \mathcal{K}(i-1)$. The result follows.

2.6 Properties. The adapted HB construction has some interesting properties. In particular, the space $\mathcal{R}^N$ can be decomposed in a series of nested subspaces that are orthogonal to $\mathcal{P}^M(X)$ and the basis forms an orthonormal set. As a side benefit, this series of nested subspaces can be used to prove the uniqueness of the RBF interpolation problem. One important property of the adapted HB is presented in the following lemma.

Lemma 3. The basis of $R^N$ described by the vectors of $\{C_0^{i-1}, D_0^{i-1}, D_0^0, D_0^1, \ldots, D_0^n\}$, $j = 0 \ldots n$, $k \in \mathcal{K}(j)$ form an orthonormal set.

Proof. We prove this by a simple induction argument. Assume that for level $i$ the set of vectors $\{B_k^i\}$ are orthonormal. Since the rows of the set $V_{s,p}^{j,k}$ are orthonormal and $B_j^i \cap B_k^j = \emptyset$ whenever $l \neq k$, then
it follows that the vectors $\bigcup_{k \in X(0)} \{ \tilde{C}_k, \tilde{D}_k \}$ form an orthonormal basis. The result then follows from Lemma 1.

**Definition 10.** Given a set of unisolvent interpolating nodes $X \subset \mathbb{R}^3$ with respect to $\mathcal{P}_p(\mathbb{R}^3)$, we form the matrix $P$ from the basis vectors $\{ \tilde{C}_0^{-1}, \tilde{D}_0^{-1}, \tilde{D}_1, \ldots, \tilde{D}_k \}$.

From Lemmas 2 and 3, the matrix $P$ has the following properties

1. If $v \in \mathcal{P}_p(X)$ then $P^H v$ has $\dim(C^0_0)$ non-zero entries.
2. $P P^H = P^H P = I$.

### 3. Multi-Resolution RBF Representation

The HB we constructed above is adapted to the kernel and the location of the interpolation nodes. It also satisfies the vanishing moment property. The construction of such an HB leads to several important consequences. First, we can use the basis to prove the existence of a unique solution of the RBF problem, but more importantly, this basis can be used to solve the RBF problem efficiently.

As the reader might recall from section 1.1, the construction of the adapted HB decouples the polynomial interpolant from the RBF functions if the degree of the vanishing moments $p$ is equal to the degree of the RBF polynomial interpolant $m$. This simple result can be extended if $p \geq m$.

**Theorem 2.** Suppose $X$ is unisolvent with respect to $\mathbb{R}^3$ and $u$ solves the interpolation problem of equation (2) uniquely, where $u \perp \mathcal{P}^m(X)$ and the kernel satisfies Definition 2. If the number of vanishing moments $p \geq m$ then

$$
\begin{pmatrix}
C^H K \bar{C}_1 \bar{C}^H & C^H K T \\
T^H K \bar{C}_1 & T^H K T
\end{pmatrix}
\begin{pmatrix}
s \\
w
\end{pmatrix}
= \begin{pmatrix}
C^H d \\
T^H d
\end{pmatrix},
$$

for some $s \in \mathbb{R}^{N-O}$ and $w \in \mathbb{R}^{N-M}$, where $T := [D_0, D_1, \ldots, D_k], C_1 = D_0^{-1}$ and $O = \dim(\mathcal{P}^m(X))$. Moreover, $u = C_1 s + Tw$.

**Proof.** Since $u \in (\mathcal{P}_p(X) \setminus \mathcal{P}^m(X)) \oplus W^0 \oplus \cdots W^n$, then $u = C_1 s + Tw$ for some $s \in \mathbb{R}^{N-O}$ and $w \in \mathbb{R}^{N-M}$, where $O = \dim(\mathcal{P}^m(X))$. Replacing $u$ into (2), pre-multiplying by $[C^H T^H]^H$ and recalling that $C_1, T \perp \mathcal{P}^m(X)$ the result follows.

Once $u$ is found, $c$ is easily obtained by solving the set of equations $L^H Q c = L^H (d - Ku)$, where $L^H Q \in \mathbb{R}^{(p+1) \times (p+1)}$.

There are two ways we can solve this, since $L$ and $Q$ span the same space and have full column rank, then $L^H Q$ is invertible and

$$
c = (L^H Q)^{-1} L^H (d - Ku).
$$

Alternatively, we can define the interpolation problem in terms of the basis vectors in $L$ directly i.e.

$$
Q := L,
$$
which leads to

$$
c = L^H (d - Ku).
$$

For the rest of this section we describe the algorithms for solving the previous system of equations. The entries of the matrix

$$
K_W := \begin{pmatrix}
C^H K \bar{C}_1 & C^H K T \\
T^H K \bar{C}_1 & T^H K T
\end{pmatrix}
$$

are from all the pairwise matching of any two vectors $\psi_{k,m}$, $\psi_{l,g}$ from the set $\mathcal{O} := \{ D_0^{-1}, D_0, D_1, \ldots, D_k \}$. The entries of $K_W$ take the form

$$
\sum_{k \in X(0)} \sum_{e_a \in X(0)} \sum_{e_b \in X(0)} K(F_p(e_a), F_p(e_b)) \psi_{k,m}^l |F_q(e_a)| \psi_{l,g}^j |F_q(e_b)|,
$$

Notice that the summation is over all the vectors $e_o$ s.t. $o = 1, \ldots, N$. However, the entries of $\psi_{k,m}$ are mostly zeros, thus in practice the summation is over the non-zero terms.
Continuing with the same notation, the entries of \( d_W := Td \) have the form
\[
\sum_{k \in K} \sum_{e_a \in B_k} \psi_{i_k m_k} \left( F_q(e_a) \right) f(\overline{F_p(e_a)}),
\]
Since \( w = Pu \) and \( u \perp \mathcal{P}(X) \), then entries of \( w \) have the form
\[
\sum_{k \in K} \sum_{e_a \in B_k} \psi_{i_k m_k} \left( F_q(e_a) \right) u \left( F_q(e_a) \right), \quad \forall \psi_{i_k m_k} \in \mathcal{D}.
\]
It is clear that from the set \( \mathcal{D} \) the matrix \( K_W \) is ordered such that the entries of any row of \( K_W \) sums over the same vector \( \psi_{i_k} \in \mathcal{D} \). In Figure 3 a block decomposition of the matrix \( K_W \) is shown. One interesting observation of the matrix \( K_W \) is that most of the information of the matrix is contained in a few entries. Indeed, for integral equations it can be shown that an adapted HB discretization matrix requires only \( O(N \log(N)^{1.5}) \) entries to achieve optimal asymptotic convergence [17]. This has been the approach that was followed behind the idea of wavelet sparsification of integral equations [9, 1, 2, 17, 3, 40, 41]. However, it is not necessary to compute the entries of \( K_W \) for efficiently inverting the matrix, but instead we only have to compute matrix vector products of the submatrices \( K_{i,j}^W \) in \( O(N) \) or \( O(\log(N)) \) computational steps.

### 3.1. Preconditioner.
One key observation of the matrix \( K_W \) is that each of the blocks \( K_{i,j}^W \) is well conditioned. Our experiments indicate that this is the case even for non uniform placement of the nodes. We propose to use two kinds of preconditioners on the decoupled RBF problem: a block SSOR and a diagonal preconditioner based on the multi-resolution matrix \( K_W \). The block SSOR multi-resolution preconditioner shows better iteration counts and is a novel approach to preconditioning. However, in practice, the simplicity of the diagonal preconditioner makes it easier to code and is faster per iteration count for the size of problems in which we are interested.

The preconditioner on the decoupled RBF takes the form of the following problem:

\[
P^{-1}K_W w = P^{-1}d_W,
\]
where \( K_W \rightarrow L_W + D_W + L_W^H \) and
\[
L_W = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \ddots & 0 & 0 \\
K_W^{n,0} & \ldots & K_W^{n,n-1} & 0
\end{bmatrix} \quad \text{and} \quad D_W = \begin{bmatrix}
K_W^{\alpha,0} & 0 & 0 & 0 \\
0 & K_W^{\alpha,1} & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & K_W^{\alpha,n}
\end{bmatrix}.
\]

The block preconditioner is constructed as \( \tilde{P} = (L_W + D_W)D_W^{-1}(L_W^H + D_W) \).

We can solve this system of equations with a restarted GMRES (or MINRES since the matrices are symmetric) iteration \([44]\). To compute each iteration efficiently we need each of the matrix vector products of the blocks \( K_W^{\alpha,i} \) to be computed with a fast summation method. We have the choice of either computing each block as matrix-vector products from a fast summation directly, or a sparse preconditioner that can be built and stored.

### 3.1.1. Fast Summation.
It is not necessary to compute the matrix \( K_W \) directly, but to employ approximation methods to compute matrix-vector products \( K_W \alpha_W \) efficiently. To such end we make the following assumption.

**Assumption 2.** Let \( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{N_t} \in \mathbb{R}^3 \), \( c_1, c_2, \ldots, c_{N_t} \in \mathbb{R} \), \( R_{BF} := \text{span}(K(x, \tilde{y}_1), K(x, \tilde{y}_2), \ldots, K(x, \tilde{y}_{N_t})) \), and \( T = \text{span}(\phi_1, \phi_2, \ldots, \phi_q) \), for some set of linearly independent functions \( \phi_1, \phi_2, \ldots, \phi_q \). We are interested in the evaluation of the RBF map
\[
\phi(x; \tilde{y}_1, \ldots, \tilde{y}_{N_t}) := \sum_{i=1}^{N_t} c_i K(x, \tilde{y}_i),
\]
where \( x \in \mathbb{R}^3 \). Suppose there exists a transformation \( F(\phi(x; \tilde{y}_1, \ldots, \tilde{y}_{N_t})) : R_{BF} \rightarrow T \) with \( O(N_t) \) computational and storage cost. Moreover, any successive evaluation of \( F(\phi(x; \tilde{y}_1, \ldots, \tilde{y}_{N_t})) \) can be performed on the basis functions of \( T \) in \( O(1) \) operations and
\[
|F(\phi(\cdot)) - \phi(\cdot)| \leq C_F A \left( \frac{1}{a} \right)^{\tilde{p}+1},
\]
where \( \tilde{p} \in \mathbb{Z}^+ \) is the order of the fast summation method, \( A = \sum_{i=1}^{N_t} |c_i| \), \( C_F > 0 \) and \( a > 1 \).

There exist several methods that satisfy, or nearly satisfy, Assumption 2. In particular we refer to those based on multi-pole expansions and the Non-equidistant Fast Fourier Transform \([6, 42, 56]\).

The system of equations (9) can now be solved using an inner and outer iteration procedure. For the outer loop a GMRES algorithm is used, where the search vectors are based on the matrix \( \tilde{P}^{-1} K_W \).

The inner loop consists of computing efficiently the matrix-vector products \( \tilde{P}^{-1} K_W \alpha_W \), for some vector \( \alpha_W \in \mathbb{R}^N \). This computation is broken down into two steps:

**Step One** To compute efficiently \( K_W \alpha_W \) for each matrix vector product \( K_W^{\alpha,j} \alpha_W \), we fix \( \psi_{j,m}^{\alpha} \) from Equation (8) and then transform the map
\[
\sum_{\psi_{l,g}^{\alpha} \in \mathcal{D}^{\alpha}} \sum_{\tilde{y}_g \in X} K(\tilde{x}_a, \tilde{y}_g) \psi_{l,g}^{\alpha} [F_g(F_p^{-1}(\tilde{y}_g))] \alpha_{l,g}^{\alpha},
\]
for all the vectors \( \psi_{l,g}^{\alpha} \in \mathcal{D}^{\alpha} \), into a new basis \( \{\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_q\} \). The computational cost for this procedure is \( O(N_t) \), where \( N_t \) corresponds to the number of non-zero entries of all \( \psi_{l,g}^{\alpha} \in \mathcal{D}^{\alpha} \). Since the computational cost for evaluating the new basis on any point \( \tilde{x}_a \in X \) is \( O(1) \), then the total cost for calculating each row of \( K_W^{\alpha} \alpha_W \) is \( O(N_t + N_2) \), where \( N_2 \) is equal to all the non-zero entries of \( \psi_{l,m}^{\alpha} \).
Now, since for each $j = 0, \ldots, n$, $| \cup_{k \in \mathcal{K}_{j}} B_k^j | \leq N$, then $N_1$ is bounded by $CN$ for some $C > 0$.

For the same reason $N_2$ is also bounded by $CN$. This implies that the total cost for evaluating the matrix vector products $K_W a_W$ is $O((n+1)^2 N)$.

**Step Two:** The computation of $P^{-1} \beta_W$, where $\beta_W := K_W a_W$ is broken up into three stages. First, let $\gamma_W := (L_W + D_W)^{-1} \beta_W$, then

$$
\begin{bmatrix}
K_W^{1,1} & 0 & \cdots & 0 \\
K_W^{2,1} & K_W^{2,2} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
K_W^{n,1} & \cdots & K_W^{n,n-1} & K_W^{n,n}
\end{bmatrix}
\begin{bmatrix}
\gamma_W^{1} \\
\gamma_W^{2} \\
\vdots \\
\gamma_W^{n}
\end{bmatrix}
= 
\begin{bmatrix}
\beta_W^{1} \\
\beta_W^{2} \\
\vdots \\
\beta_W^{n}
\end{bmatrix}.
$$

Since $(L_W + D_W)$ has a block triangular form, we can solve the inverse-matrix vector product with a back substitution scheme. Suppose that we have found $\gamma_W^{1}, \ldots, \gamma_W^{i-1}$, then it is easy to see from the triangular structure of $(L_W + D_W)$ that

$$\gamma_W^{i} = (K_W^{i,i})^{-1}[\alpha_W^{i} - \sum_{k=1}^{i-1} k W_{W,k}^{i,k} \gamma_W^{k}].$$

The cost for evaluating this matrix vector product with a fast summation method is $O((n+1)^2 N + k(n+1)N)$. The last term comes from the block matrices in $D_W$, which are inverted indirectly with $k$ Conjugate Gradient (CG) iterations $[28, 25]$. In Section 4 we show numerical evidence that $k$ converges rapidly for large numbers of interpolating nodes.

The second matrix vector product, $\eta := D_W \gamma_W$ is evaluated in $O(N)$ using a fast summation method. Finally the last matrix vector product $P_W^{-1} K_W a_W$ can be solved in $O((n+1)^2 N + knN)$ by again using a back substitution scheme.

**Remark 9.** For many practical distributions of the interpolating nodes in the set $X$, the total number of refinement levels $n + 1$ is bounded by $C_1 \log N$ $[6]$. For these types of distributions the total cost for evaluating $P_W^{-1} K_W a_W$ is $O(N \log^2 N)$ assuming $k$ is bounded.

This approach is best for large scale problems where memory becomes an issue and for large vanishing moments. For small to medium size problems the blocks $K_W^{i,i}$ can be computed in sparse form and then stored for repeated use.

3.1.2. Sparse Preconditioners. In this section we show how to produce two types of sparse preconditioners by leveraging the ability of HB to produce compact representations of the discrete operator matrices.

The key idea is to produce a sparsified matrix $\tilde{P}$ of $P$ from the entries of the blocks $K_W^{i,i}$. This is done by choosing an appropriate strategy that decides which entries to keep, and which ones not to compute.

Although it is possible to construct an accurate approximation of $\tilde{P}$ and $K_W$ for all the blocks $K_W^{i,i}$ ($i = 0 \ldots n$), the computational bottleneck lies in computing the matrix vector products with $(K_W^{i,i})^{-1}$. Thus it is sufficient to compute the sparse diagonal blocks of $K_W$. The off-diagonal blocks are computed using the fast summation method described in section 3.1.1.

**Definition 11.** For every vector $\psi_{k,m}^{j} \in \mathcal{P}$ and the associated support box $B_m^{j} \in \mathcal{B}$, define the set $L_m^{j}$ to be the union of $B_m^{j}$ and all boxes in $\mathcal{B}_j$ that share a face, edge or corner with $B_m^{j}$ i.e. the set of all adjacent boxes.

To produce the sparse matrix $\tilde{P}$ we execute the following strategy: For each entry in $K_W^{i,i}$ corresponding to the adapted HB vectors $\psi_{k,m}^{j}, \psi_{l,s}^{j} \in \mathcal{P}$, we only compute this entry if

$$\text{dist}(L_k^{j}, L_l^{j}) := \inf_{\bar{x}, \bar{y}} \| L_k^{j}(\bar{x}) - L_l^{j}(\bar{y}) \|_{L_1(\mathbb{R}^n)} \leq \tau_{i,j},$$

for some $\tau_{i,j}$.
where \( \tau_i \in R^+ \) for \( i = 0 \ldots n \). For an appropriate distance criterion \( \tau_i \), we can produce a highly sparse matrix \( \tilde{K}_W^{ij} \) that is close to \( K_W^{ij} \) (and respectively \( \tilde{P} \)) in a matrix 2-norm sense.

**Definition 12.** The distance criterion \( \tau_i \) is set to
\[
\tau_i := 2^{n-i},
\]

With this distance criterion it is now possible to compute a sparse representation of the diagonal blocks of \( K_W \). It is not hard to show that for kernels that satisfy Assumption 1 the decay of the entries of the matrix \( K_W \) is dependent on the distance between the respective blocks and the number of vanishing moments. If \( p \) is chosen sufficiently large (for a biharmonic \( p = 3 \) is sufficient), the entries of \( K_W \) decay polynomially fast, which leads to a good approximation to \( K_W \).

Under this sparsification strategy, it can be shown that \( \| K_W - \tilde{K}_W \|_2 \) decays exponentially fast as a function of the degree of vanishing moments \( p \) with only \( O(Np^2) \) entries in \( \tilde{K}_W \). The accuracy results have been derived in more detail in an upcoming paper we are writing for anisotropic spatially varying RBF interpolation [18].

**Lemma 4.** Let \( N(A) : R^{N \times N} \rightarrow R^+ \), be the number of non-zero entries for the matrix \( A \), then we have
\[
N(\tilde{K}_W^{ij}) \leq 8M(p)7^3N
\]

**Proof.** First, identify the box \( L^i_{k,m} \) that embeds \( \psi_{k,m}^i \) and the distance criterion \( \tau_i \), and corresponding embedding \( L^i_{l,g} \) that intersect the boundary traced by \( \tau \) is equal to \((2^{-i+3} + 2\tau_j + 2^{-i+1})/2 \leq 2^{(i-1)}7^3 = 7^3 \) (as shown in Figure 2). From Remark 8 there are at most \( 8M(p) \) HB vectors per cube. The result follows.

To compute the block diagonal entries of \( \tilde{K}_W^{ij} \), for \( i = 0, \ldots, n \) in \( O(N \log N) \) computational steps, we employ a strategy similar to the fast summation strategy in section 3.1.1. For each row of \( \tilde{K}_W^{ij} \), locate the corresponding HB \( \psi_{k,m}^i \) from Equation (8) and transform the map
\[
\sum_{k \in \mathcal{X}(m)} \sum_{\tilde{k} \in \mathcal{X}(n)} \sum_{e \in B^i_g} K(F_p(\tilde{x}, e_u)) \psi_{k,m}^i [F_q(e_u)]
\]
into an approximation \( G(\tilde{x}, \psi_{k,m}^i) := \sum_{j=1}^q \psi_{l,m}^j \tilde{g}_j \) by applying a fast summation method that satisfies Assumption 2. Any entry of the form \( a_{\psi_{l,m}^j} \psi_{l,m}^i \) can be computed by sampling \( G(\tilde{x}) \) at locations corresponding to the non-zero entries of \( \psi_{l,m}^j \), and the sampled values can be used to multiply and sum through the non-zero values of \( \psi_{l,m}^j \).

**Theorem 3.** Each block \( \tilde{K}_W^{ij} \) is computed in at most \( O(N) \) steps.

**Proof.** The cost for computing the basis of \( G(\tilde{x}) \) corresponding to \( \psi_{k,m}^i \) is at most \( O(N_{k,m}^i) \), where \( N_{k,m}^i \) is the number of non-zeros of \( \psi_{k,m}^i \). Now, since \( \bigcup_{k \in \mathcal{X}(i)} B^i_k \) \leq N the cost of computing \( G(\tilde{x}, \psi_{k,m}^i) \) for all the vectors \( \psi_{k,m}^i \) at level \( i \) is \( \sum_{k \in \mathcal{X}(i)} N_{k,m}^i = O(N) \).

For each row in \( \tilde{K}_W^{ij} \), from Lemma 4 there is at most \( 8M(p)7^3 \) entries. This implies that for each vector \( \psi_{k,m}^i \) we need only \( O(1) \) evaluations of \( G(\tilde{x}, \psi_{k,m}^i) \) to compute a row of \( \tilde{K}_W^{ij} \). Now, if we sum up the cost of evaluating \( G(\tilde{x}, \psi_{k,m}^i) \) for all the rows then the total cost for evaluating \( \tilde{K}_W^{ij} \) is \( O(N) \).

**Remark 10.** For each entry in \( \tilde{K}_W^{ij} \), the corresponding basis vectors \( \psi_{k,m}^i, \psi_{l,m}^j \) can be found in \( O(n) \) computational steps. This is easily achieved by sorting the set of cubes \( \{B^i_k\}_{k \in \mathcal{X}, j=1 \ldots n} \) with an octree structure, i.e. a parent-child sorting.
Remark 11. Note that further improvements in computation can be done by observing that $\psi_{k,m}$ is a linear combination of the vectors $\phi_{l,o}^{-1} \in V^{-1}$. Thus equation (13) can be written as a linear combination of

$$\sum_{k \in \mathcal{X}(n) K \in \mathcal{X}(n)} \sum_{e_a \in B^i_k} K(e, F_p(e_a)) \phi_{l,o}^{-1}[F_q(e_a)].$$

If two vectors $\psi_{k,m}$ and $\psi_{k,m'}$ are in the same cube then it is sufficient to compute equation (13) once and apply the coefficients computed in the construction of the entire HB. In addition, if two vectors $\psi_{k,m}$ and $\psi_{k,m'}$ share the same vector $\phi_{l,o}^{-1} \in V^{-1}$, the same procedure can be applied. In our results in Section 4 we apply this scheme to compute the SSOR and diagonal blocks.

Remark 12. As our results show a very simple, but effective, diagonal preconditioner can be built from the blocks of $K^{1,i}_W$. In particular

$$P := \text{diag} \left( \begin{array}{cccc} K^{1,1}_W & 0 & \ldots & 0 \\ 0 & K^{2,2}_W & \ddots & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & K^{n,n}_W \end{array} \right).$$

This preconditioner is also much easier to construct in practice.
Figure 3. (a) **Test Case #1 Cube RBF interpolating set**: Interpolating set with a thousand nodes with orthographic views. The colorbar indicates the height (z-axis) of the interpolating nodes. (b) **Test Case #2 V-plane RBF interpolating set**, with one thousand nodes.
4. Numerical Results

In this section we apply the multi-resolution method developed in section 3 to RBF interpolation problems. These will be of different sizes and polynomial orders for the biharmonic, multi-quadrpic and inverse multi-quadrpic function kernel in $\mathbb{R}^3$. These kernels can be written in a common form $K(r) := (r^2 + \delta^2)^l/2$, where $r := |x|$, $\delta \in \mathbb{R}$ and $l \in \mathbb{Z}$. The distribution of the nodes in $X$ are separated into two cases.

**Test Case 1:** We test our method on several sets of randomly generated interpolating nodes in the unit cube in $\mathbb{R}^3$ as shown in Figure 3. The sets of interpolating nodes $\{X_1, \ldots, X_e\}$ vary from 1000 to 512,000 nodes. Each set of interpolating nodes is a subset of any other set with bigger cardinality, i.e., $X_i \subset X_{i+1}$. The function values on each node are also grouped into $e$ sets $\{b_1, \ldots, b_e\}$ with randomly presented values and satisfy also $b_1 \subset b_{i+1}$.

**Test Case 2:** For this second test we apply a projection of the data nodes generated in Test Case 1 onto two non-orthogonal planes $\mathbb{R}^3$, then remove any two nodes that are less than $10^{-4}$ distance from each other. The V-plane intersecting are shown in Figure 4. Due to the sharp edges, this test case is significantly harder than Test Case 1 and the test examples in [27]. Note, that only about 0.1% of the centers were eliminated and the number of nodes in the table is approximate.

**Test Setup:** The implementation of the multi-resolution discrete HB method is performed in C++ and compiled with the Intel CC compiler. The GMRES algorithm is incorporated from PETSc (Parallel, Extensible Toolkit for Scientific computation) libraries [4] into our C++ code. Inner and outer iterations are solved using a GMRES algorithm with 100-iteration restart. In the rest of this section when we refer to GMRES iterations, we imply restarted GMRES with a restart for every 100 iterations. Since the preconditioned system will introduce errors in the RBF residual of the original problem, the accuracy of the GMRES is adjusted such that the residual $\epsilon$ of the unpreconditioned RBF system is less than $10^{-3}$. In Tables 2 and 3 the GMRES accuracy residual are reported.

All the numerical tests with a fast summation method are performed with a single processor version of the Kernel-Independent Fast Multipole Method (KIFMM) 3D code (http://mrl.nyu.edu/~harper/kifmm3d/documentation/index.html). This code implements the algorithm described in [54]. The accuracy is set to relative medium accuracy ($10^{-6}$ to $10^{-8}$). In addition, all numerical timings presented in this paper are wall clock times.

The C++ code was also compiled for a single core on the Dell Precision T7500 workstation with Linux Ubuntu 11.04, 12 core Xeon X5650 at 2.67 GHZ with 12 MB Cache. All results (except for the Condition number test) where performed sequentially on a single core of the same processor. Similar results where also observed with a single thread of Core i7 1.66 GHZ processor and on a single core of an Intel(R) Core(TM)2 Quad CPU Q9450 @2.66GHZ processor (12 MB L2 Cache). At some point we will make available the code for the public with instructions for compilation.

**Test Examples:**

**Condition number $\kappa$ of underlying system of equations with respect to scaling all the domain.** One immediate advantage our method has over a direct method is the invariance of the conditioning of the system of equations with respect to the scale of the polynomial domain. This is a consequence of the construction of the HB polynomial orthogonal basis.

Removing the polynomial source of ill-conditioning makes the system easier to solve. The condition number of the full RBF interpolation matrix is sensitive to the scaling of the domain. We show this by scaling the domain by a constant $\alpha \in \mathbb{R}$.

As shown in Table 1 the condition number for the 1000 center problem with $m = 3$ and $K(r) = r$ deteriorates quite rapidly with scale $\alpha$. In particular, for a scaling of 1000 or larger (0.01 or smaller) an iterative method, such as GMRES or CG, stagnates. We note that the invariance of the condition number of the decoupled system was also observed in [8, 49].

Another important observation is that the same result will apply for a multi-quadrpic, or inverse multi-quadrpic of the form $K(r) = (r^2 + \delta^2)^{\pm l}$, $\delta \in \mathbb{R}$, due to the polynomial decoupling from the RBF.
matrix. In general, this will be true for any strictly conditionally positive (or negative) definite RBF. However, the matrix $K_W$ will still be subject to the underlying condition number of $K$. In other words, if $\kappa(K)$ deteriorates significantly with scale then $K_W$ will also be ill-conditioned.

| Scale | $\alpha = 0.01$ | $\alpha = 0.1$ | $\alpha = 1$ | $\alpha = 100$ | $\alpha = 1000$ |
|-------|-----------------|----------------|-------------|---------------|---------------|
| $\kappa$ of RBF system | $4.5 \times 10^{-8}$ | $5.7 \times 10^{-11}$ | $5.2 \times 10^{0}$ | $7.9 \times 10^{11}$ | $5.3 \times 10^{17}$ |
| $\kappa(K_W)$ | 762 | 762 | 762 | 762 | 762 |

Table 1. Condition number for RBF system matrix (equation $2$) versus scale of the problem for a thousand nodes for Test Case 1 with respect to the biharmonic $K(\mathbf{r}) := r$. As observed, increasing the scale by alpha the condition number deteriorates very rapidly. In particular, for a condition number higher than the reciprocal of machine position and the GMRES or CG algorithm stagnates.

**Biharmonic RBF, $m = 3$ (cubic) and $p = 3$** This is an example of a higher order polynomial RBF interpolation. We test both the SSOR and diagonal preconditioner on Test Case 1 & 2. For the preconditioner the accuracy of the GMRES outer iterations is set such that the residual $\varepsilon := \|K_Ww - d_W\| \leq 10^{-3}$. Due to the condition number of the blocks they quickly converge with either a CG, or a GMRES solver. Moreover, the number of iterations appear to grow slowly with size.

In Table 2(a) the iteration and timing results for the sparse SSOR preconditioner for Test Case 1 & 2 are shown. For Test Case 1, the number of restarted GMRES iterations grows as $O(N^{0.55})$. Fitting a linear regression function to the log-log plot leads to a growth of $O(N^{1.85})$ for time complexity. Test Case 2 (v-plane) is a harder problem due to the corner and the projection of the random data from Test Case 1 onto two planes at 135 degrees to each other. The total GMRES iteration grows as $O(N^{0.54})$ at time complexity $O(N^{1.85})$.

In Table 2(b) the iteration and timing results for the diagonal preconditioner with Test Case 1 & 2 are shown. We can observe that although the GMRES iteration count is higher than that of the SSOR preconditioner ($CN^{0.51}$), the simplicity of the preconditioner allows every matrix-vector product to be computed much faster. Fitting a line to the log data leads to a total time complexity increases of $O(N^{1.6})$. The memory constraints are also much lower than the SSOR since only $N$ entries are needed to be stored for the preconditioner. For Test Case 2 (v-plane), the increase of time complexity ($O(N^{1.7})$) and the GMRES iteration count $O(N^{0.73})$ reflects that it is a harder problem than Test Case 1. Another observation is that the time required to compute the diagonal preconditioner is about one half compared to Test Case 1. This is due to the adaptive way we compute the diagonal, recall Remark 1.1.

**Multiquadric and inverse multiquadric RBF, $m = 3$ (cubic), $p = 3$**. For the case of the multiquadrics with $\delta = 0.01$, the iteration count increases significantly, as shown in Table 3(a). The number of GMRES iterations increases as $CN^{0.7}$. This is a harder problem to solve due to the ill-conditioning introduced by the constant term $\delta$, as reflected by the increase in the number of GMRES iterations. Fitting a line through the log-log plot of the total time leads to a $CN^{1.8}$ time complexity.

In contrast, the inverse multiquadrics result shown in Table 3(b) is a better conditioned problem leading to around the same complexity as for the biharmonic case, but the constant is lower. We note that to achieve comparable interpolation accuracy, the value of $\delta$ for the inverse multiquadric generally needs to be larger than for the multiquadric case. And the larger the $\delta$ the more ill-conditioned the RBF interpolation problem.

5. Conclusions

In this paper we construct a class of discrete HB that are adapted both to the RBF kernel function and the location of the interpolating nodes. The adapted basis has two main advantages: First the
### Table 2. Wall clock times results for biharmonic $K(r) = r$, $m = 3$ (Cubic), $p = 3$. (a) Iteration and timing results for the sparse SSOR preconditioner for Test Case 1 (uniform cube) & 2 (v-plane). The first column is the number of interpolating points. The second column is the number of iterations such that $\epsilon$, the residual error for the unpreconditioned system, is less than $10^{-3}$. The third column is the time (in seconds) to compute the sparse inner blocks $K_{ii}^{ri}$, and the fourth is the time for GMRES iterations. The fifth column is the total time (in seconds) for solving the RBF problem. The remaining columns are for Test Case 2 and follow the same order as results for Test Case 1. (b) Iteration and timing results for diagonal preconditioner for Test case 1 & 2. The columns are in the same order as before, except that the third column and seventh columns are the time involved in computing the diagonal preconditioner.

RBF problem is decoupled, thus solving the scale dependence between the polynomial and RBF interpolation. Second with a block SSOR scheme, or a simple diagonal matrix built from the multi-resolution matrix $K_W$, an effective preconditioner is built that reduces significantly the iteration count. Our result shows a promising approach for many RBF interpolation problems.

Further areas of interest as future work:

- **Sparsification of $K_W$ matrix.** Due to orthogonality properties of the discrete HB a sparse representation $\tilde{K}_W$ of $K_W$ can be constructed where $\|K_W - \tilde{K}_W\|$ is small. The sparse representation is used at each iteration in lieu of the dense matrix, thus opening the possibility of significantly increasing the time efficiency of each matrix vector product.

| N    | GMRES | $K_W^{ri}$ (s) | Itr (s) | Total (s) | GMRES | $K_W^{ri}$ (s) | Itr (s) | Total (s) |
|------|--------|---------------|--------|-----------|--------|---------------|--------|-----------|
| 1000 | 10     | 1             | 8      | 9         | 22     | 1             | 16     | 17        |
| 2000 | 15     | 2             | 13     | 15        | 29     | 2             | 41     | 43        |
| 4000 | 21     | 6             | 39     | 45        | 43     | 5             | 178    | 183       |
| 8000 | 29     | 61            | 275    | 336       | 66     | 44            | 623    | 667       |
| 16000| 48     | 194           | 798    | 993       | 91     | 118           | 2298   | 2416      |
| 32000| 71     | 815           | 3907   | 4722      | 118    | 612           | 12288  | 12900     |
| 64000| 99     | 1754          | 13841  | 15595     | 195    | 951           | 28500  | 29451     |
| 128000| 134   | 5547          | 29165  | 34712     | 305    | 2572          | 98505  | 101078    |

| N    | GMRES | Diag. (s) | Itr (s) | Total (s) | GMRES | Diag. (s) | Itr (s) | Total (s) |
|------|--------|-----------|--------|-----------|--------|-----------|--------|-----------|
| 1000 | 33     | 1         | 5      | 6         | 90     | 1         | 10     | 11        |
| 2000 | 45     | 2         | 6      | 8         | 102    | 2         | 11     | 13        |
| 4000 | 66     | 6         | 16     | 22        | 147    | 5         | 30     | 34        |
| 8000 | 87     | 62        | 56     | 117       | 269    | 44        | 121    | 165       |
| 16000| 128    | 195       | 148    | 344       | 355    | 118       | 286    | 404       |
| 32000| 184    | 813       | 749    | 1563      | 876    | 569       | 2924   | 3493      |
| 64000| 281    | 1752      | 1817   | 3569      | 1242   | 951       | 4135   | 5087      |
| 128000| 385   | 5555      | 3949   | 9505      | 3033   | 2573      | 21917  | 24491     |
| 256000| 573   | 14350     | 12130  | 26480     | -      | -         | -      | -         |
| 512000| 769   | 47082     | 44309  | 91391     | -      | -         | -      | -         |
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(a) Test Case 1, Multiquadric
(b) Test Case 1, Inverse Multiquadric

| N    | GMRES | Diag (s) | Itr (s) | Total (s) | GMRES | Diag (s) | Itr (s) | Total (s) |
|------|-------|----------|---------|-----------|-------|----------|---------|-----------|
| 1000 | 38    | 1        | 1       | 2         | 7     | 1        | 1       | 1         |
| 2000 | 55    | 2        | 5       | 7         | 8     | 3        | 1       | 4         |
| 4000 | 86    | 6        | 13      | 18        | 14    | 8        | 4       | 11        |
| 8000 | 128   | 32       | 41      | 73        | 17    | 45       | 9       | 54        |
| 16000| 195   | 99       | 155     | 254       | 27    | 138      | 28      | 166       |
| 32000| 362   | 233      | 486     | 720       | 63    | 343      | 119     | 462       |
| 64000| 684   | 757      | 2217    | 2975      | 84    | 1131     | 414     | 1546      |
| 128000|1059  | 2357     | 7637    | 9994      | 112   | 3494     | 985     | 4480      |

Table 3. Iteration and timing results for diagonal preconditioner, multiquadric $K(r) := (r^2 + 0.01^2)^{1/2}$ and test case 1 (uniform cube), $m = 3$, $p = 3$ for (a) Multiquadric $(+1/2)$ and (b) Inverse multiquadric $(-1/2)$.

- **High Dimensional RBF Problems.** In principle the method that we have developed can be extended to high dimensional RBF problems.
- **Spatially varying anisotropic kernels.** An interesting observation is that the adapted discrete HB leads to a sparse multi-resolution RBF matrix representation for spatially varying kernels. This type of RBF interpolation has been gaining some interest lately due to the ability to better steer each local RBF function to increase accuracy. Due to the spatially varying kernel, we cannot use a fast summation method to optimally compute each matrix vector product. However, preliminary results show that we can sparsify the RBF matrix while retaining high accuracy of the solution. Full error bounds and numerical results will be described in a following paper that we are currently writing.

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