Multidimensional Chebyshev Systems - just a definition

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August 15, 2008

Abstract

We provide a definition of Multidimensional Chebyshev Systems of order $N$ which is satisfied by the solutions of a wide class of elliptic equations of order $2N$. This definition generalizes a very large class of Extended Complete Chebyshev systems in the one-dimensional case.

This is the first of a series of papers in this area, which solves the long-standing problem of finding a satisfactory multidimensional generalization of the classical Chebyshev systems introduced already by A. Markov more than hundred years ago.

1 History of Chebyshev systems

1.1 Developments in the Moment problem by A. Markov and M. Krein

It was namely in the Moment problem where the Chebyshev systems appeared for the first time on the big stage, and provided a very natural and beautiful generalizations of the results of Gauss, Jacobi, Chebyshev, Stieltjes and Markov, and others.

The classical Moment problem is defined as follows: Find a non-negative measure $d\mu$ such that

$$\int_a^b t^j d\mu(t) = c_j \quad \text{for } j = 0, 1, \ldots, N.$$  

The solution of the problem includes conditions on the constants $c_j$ providing solubility. In the case of $N = 2n - 1$ the problem has been solved by the famous Gauss-Jacobi quadrature; this solutions is based on the orthogonal polynomials
$P_n$ (of degree $n$) which are orthogonal with respect to the inner product defined by
\[ \langle t^j, t^k \rangle := c_{j+k}. \]

The history is well described in the book of Krein and Nudelman "The Markov Moment Problem", actually based on the 1951 paper of M. Krein. There it is said that A. Markov has realized that one may consider the Moment problem of the type
\[ \int_a^b u_j(t) \, d\mu(t) = c_j \quad \text{for } j = 0, 1, \ldots, N, \]

where the system of continuous functions $\{u_j(t)\}_{j=0}^N$ represent a Chebyshev system in the interval $[a, b]$, i.e. any linear combination
\[ u(t) = \sum_{j=0}^N \alpha_j u_j(t) \]

has no more than $N$ zeros in $[a, b]$. The subspace of $C([a, b])$ generated by the Chebyshev system is defined by
\[ U_N := \left\{ u(t) : u(t) = \sum_{j=0}^N \alpha_j u_j(t) \right\}. \]

1.2 Further developments in Approximation theory and Spline theory

Let us remind also the famous Chebyshev alternance theorem which has been proved for Chebyshev systems, and which one would like to see in a multivariate setting:

**Theorem 1 (Chebyshev-Markov)** Let $f \in C([a, b])$. A necessary and sufficient condition for the element $u_0 \in U_N$ to solve problem
\[ \min_{u \in U_N} \| f - u \|_C = \min_{u \in U_N} \left( \max_{x \in [a, b]} |f(x) - u(x)| \right) = \delta \]

is the existence of $N + 2$ points
\[ t_1 < \cdots < t_{N+2} \]
such that
\[ \delta \varepsilon (-1)^j = (f(t_j) - u_0(t_j)) \quad \text{for } j = 1, \ldots, N + 2 \]

where $\varepsilon = 1$ or $\varepsilon = -1$.

What concerns other areas where Chebyshev systems have found numerous applications, one has to mention the book of L. Schumaker, *Spline Functions: basic theory*, 1983, which contains an exhaustive consideration of spline theory where splines are piecewise elements of Chebyshev systems.
2 Definitions

Let us provide some basic definitions.

Consider the system of functions \( \{ u_j(t) \}_{j=0}^N \) defined on some interval \([a, b]\) in \(\mathbb{R}\) and the linear space

\[
U := \left\{ \sum_{j=0}^N \lambda_j u_j(t) \right\}
\]

**Definition 2** We call the system of functions \( \{ u_j(t) \}_{j=0}^N \) *Chebyshev (of T−system)* iff for every set of constants \( \{ c_j \}_{j=0}^N \) and every choice of the points \( t_j \in [a, b] \) with

\[ t_0 < t_1 < \cdots < t_N \]

we have unique solution \( u \in U \) of the equations

\[ u(t_j) = c_j \quad \text{for } j = 0, 1, \ldots, N. \]

It is equivalent to say that

\[ u(t_j) = 0 \quad \text{for } j = 0, 1, \ldots, N \]

implies

\[ u \equiv 0. \]

**Proposition 3** Assume that the space \( U \subset C[a, b] \) is given. Then if for some set of knots \( t_j \in [a, b] \) with

\[ t_0 < t_1 < \cdots < t_N, \]

and for arbitrary constants \( \{ c_j \} \) we have unique solution \( u \in U \) of the equations

\[ u(t_j) = c_j \quad \text{for } j = 0, 1, \ldots, N \]

it follows that \( \dim U = N + 1. \)

One may formulate this in an equivalent way:

**Proposition 4** The following are equivalent
1. the system \( \{ u_j(t) \}_{j=0}^N \) is T−system iff
2. for every \( u \in U_N \) the number of zeros in the interval \([a, b]\) is \( \leq N. \)
3. the following determinants satisfy

\[
\det \begin{bmatrix}
 u_0(t_0) & u_0(t_1) & \cdots & u_0(t_N) \\
 u_1(t_0) & u_1(t_1) & \cdots & u_1(t_N) \\
 \vdots & \vdots & \ddots & \vdots \\
 u_N(t_0) & u_N(t_1) & \cdots & u_N(t_N)
\end{bmatrix} \neq 0
\]
2.1 Examples

The classical polynomials, the trigonometric polynomials in smaller intervals
$[0, 2\pi]$!

1. the system
   \[ \{ u_j(t) = t^{\alpha_j} \}_{j=0}^{N} \] on subintervals of $[0, \infty]$

2. the system
   \[ u_j(t) = \frac{1}{s_j + t} \] for $0 < s_0 < s_1 < \cdots < s_N$ on closed subint. of $(0, \infty)$.

3. the system
   \[ u_j(t) = e^{-(s_j-t)^2} \] for $0 < s_0 < s_1 < \cdots < s_N$ on $(-\infty, \infty)$.

4. if $G(s, t)$ is the Green function associated with the operator
   \[ Lf = -\frac{d}{dx} \left( p \frac{df}{dx} \right) + qf \]
   and some boundary conditions on the interval $[a, b]$, then the system
   \[ u_j(t) = G(s_j, t) \] for $0 < s_0 < s_1 < \cdots < s_N$ on closed subint. of $[a, b]$

See M. Krein and A. Nudel’man "The Markov Moment Problem and extremal problems", AMS, 1978, translation from Russian.
S. Karlin and W. Studden "Tchebysheff systems", Wiley, 1966.

3 Extended systems

We usually work with differentiable systems of functions, and we count multiplicities of the zeros.

**Definition 5** Let $\{ u_j(t) \}_{j=0}^{N} \in C^N[a, b]$ be a $T$-system. We call it Extended Chebyshev system ( ET-system ) if in $U_N$ we may uniquely solve the problem (Hermite interpolation problem)
   \[ u^{(k)}(t_j) = c_{j,k} \quad \text{for } k = 0, 1, \ldots, d_j \]
   with arbitrary data $\{ c_{j,k} \}$ where
   \[ \sum (d_j + 1) = N + 1. \]
   It is equivalent to say that if for some $u \in U$ holds
   \[ u^{(k)}(t_j) = 0 \quad \text{for } k = 0, 1, \ldots, d_k \]
   then
   \[ u \equiv 0. \]
Proposition 6 The following are equivalent:
1. the system \( \{ u_j(t) \}_{j=0}^N \in C^N[a, b] \) is and ET-system
2. for every \( u \in U \) the number of zeros counted with the multiplicities is \( \leq N \).
3. the modified (!!!) determinants
\[
\det \begin{bmatrix}
u_0(t_0) & u_0'(t_0) & \cdots & u_0(t_N) \\
u_1(t_0) & u_1'(t_0) & \cdots & u_1(t_N) \\
u_N(t_0) & u_N'(t_0) & \cdots & u_N(t_N)
\end{bmatrix} \neq 0
\]

There is a nice characterization of ET-systems. A basic example of ET-systems is the following: Let \( w_i \in C^{N-i}[a, b] \) be positive on \([a, b]\) for \( i = 0, 1, \ldots, N \). Then the functions
\[
\begin{align*}
u_0(t) &= w_0(t) \\
u_1(t) &= w_0(t) \int_a^t w_1(t_1) \, dt_1 \\
u_2(t) &= w_0(t) \int_a^t w_1(t_1) \int_a^{t_1} w_2(t_2) \, dt_2 \, dt_1 \\
&\vdots \\
u_N(t) &= w_0(t) \int_a^t w_1(t_1) \int_a^{t_1} w_2(t_2) \cdots \int_a^{t_{N-1}} w_N(t_N) \, dt_N \cdots dt_1
\end{align*}
\]
form an ET-system.

Theorem 7 If the space \( U_N \) corresponds to an ET-system of order \( N \) then it has a basis \( \{ v_j \}_{j=0}^N \) which is representable in the above form.

4 The multivariate case – attempts

The brute generalization of the Chebyshev systems fails:

4.0.1 Generalization by zero sets – theorem of Mairhuber

Apparently, the first attempt has been to generalize the Chebyshev systems by considering the set of zeros:

Definition 8 Let \( K \) be a compact topological space. The system of functions \( \{ u_j \}_{j=0}^N \) is called Chebyshev of order \( N \) iff
\[
Z(u) \leq N
\]
for every \( u \in U \).
Theorem 9 (Mairhuber, 1956?) The only spaces $K$ having a Chebyshev system satisfy $K \subset \mathbb{R}$ or $K \subset S^1$.

In general, this cannot be considered as a serious attempt to define multivariate Chebyshev systems, since it is clear that a generic function in $C(K)$ has a zero set which is a subset of $K$ of codimension 1. In particular, if $K = \mathbb{R}^2$ then the zero set is roughly speaking union of some curves, and it would be more reasonable to speculate about the number of these components then to consider Definition 8 above.

4.0.2 Generalization by Haar property

Theorem of Haar:

Theorem 10 Let the space $U_N \subset C[a, b]$ represent a Chebyshev system. Then for every $f \in C[a, b]$ the best approximation problem

$$\min_{u \in U_N} \|f - u\|_C$$

has unique solution.

Extending this definition to the multivariate case seems to be very reasonable but the work with best approximations is very heavy and until now has not lead to success.

5 Systems to be generalized

One needs a new point of view on the Chebyshev systems. We suppose the point of view of boundary value problems which may be generalized to the multivariate situation: We consider a special class of ET—systems which are "generalizable".

Definition 11 We say that the system $\{u_j\}_{j=0}^{2N-1} \in C^\infty[a, b]$ is a Dirichlet type Chebyshev system — DT—system in the interval $[a, b]$ if for every two points $\alpha$ and $\beta$ there and for every set of constants $c_j$ and $d_j$ we are able to solve uniquely the problem with $u \in U_N$ as follows

$$u^{(k)}(\alpha) = c_k \quad \text{for } k = 0, 1, ..., N - 1$$
$$u^{(k)}(\beta) = d_k \quad \text{for } k = 0, 1, ..., N - 1$$

Remark 12 Obviously, all ET—systems are DT—systems.

6 The multivariate case

We consider a space of functions $U$ and assume that $U \subset C^\infty(D)$ for some bounded domain $D \subset \mathbb{R}^n$ such that its boundary $\partial D \in C^\infty$ and assume that $D$
locally "lies on one side of the boundary". These are usual conditions for the solubility of Elliptic Boundary Value problems, see e.g. Lions-Magenes "Non-homogeneous Boundary Value Problems and Applications", Springer, 1970.

For simplicity assume that $D$ is simply connected as well.

**Definition 13** We say that $U$ is a **Multivariate Chebyshev system** of order $N$ iff the following hold:

1. The "approximate" solubility of the Dirichlet problem on subdomains: Let $D_1$ be an arbitrary subdomain of $D$ with $D_1 \subset D$, and such that $D_1$ satisfies the above conditions as $D$, and $D \setminus D_1$ has only non-compact connected components.

Let $c_j \in C^\infty(\partial D_1)$ for $j = 0, 1, \ldots, N-1$. Then for every $\varepsilon > 0$ there exists an element $u \in U$ such that

$$\left| \frac{\partial^j u(x)}{\partial n^j} - c_j(x) \right| \leq \varepsilon \quad \text{for all } x \in \partial D_1$$

and for $j = 0, 1, \ldots, N-1$.

In the case $D_1 = D$ the above solubility holds for $\varepsilon = 0$.

2. The **unique** solubility of the Dirichlet problem on subdomains: If for some $u \in U$ holds

$$\frac{\partial^j u(x)}{\partial n^j} = 0 \quad \text{for all } x \in \partial D_1$$

and for $j = 0, 1, \ldots, N-1$

then

$$u \equiv 0.$$

**Remark 14** In the case of the domain $D$ we have unique solubility of the Dirichlet problem since we may take $\varepsilon = 0$!

### 7 Examples

For some integer $N \geq 1$ let us consider the space

$$U := \{ u : \Delta^N u(x) = 0 \quad \text{in } D \}.$$

**Theorem 15** The space $U$ represents a Chebyshev system of order $N$.

**Proof.** We have property 2 of the Definition first: It is well known that we have uniqueness of the solubility for the Dirichlet problem for the elliptic operators $\Delta^N$.

Property 1, the approximate solubility of the Dirichlet problem: First we solve the Dirichlet problem in the domain $D_1$, namely we find some $u$ which satisfies

$$\Delta^N v(x) = 0 \quad \text{in } D_1$$

$$\frac{\partial^j v(x)}{\partial n^j} = c_j(x) \quad \text{for all } x \in \partial D_1$$

and for $j = 0, 1, \ldots, N-1$. 

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So far this $v$ is not in the space $U$! Now since $D \setminus D_1$ has only noncompact components, we may apply the theorem of Runge to the solutions of the equation which says that for every $\varepsilon > 0$ there exists a function $u \in U$ such that

$$\left| \frac{\partial^j u (x)}{\partial n^j} - \frac{\partial^j v (x)}{\partial n^j} \right| \leq \varepsilon$$

for all $x \in \partial D_1$ and for $j = 0, 1, \ldots, N - 1$.

This ends the proof.

**Remark 16** We may take also a large class of elliptic operators of order $N$ since they have the same properties as $\Delta^N$ for solvability of the Dirichlet problems as well as the Runge type property for the approximation on subdomains.

For the Runge type property see the paper of Felix Browder, Approximation by Solutions of Partial Differential Equations, *American Journal of Mathematics*, vol. 84, no. 1, p. 134, 1962; Functional analysis and partial differential equations. II, Mathematische Annalen, vol. 145, no. 2, pp. 81226, 1962, or the book of L. Bers, F. John and M. Schechter, *Partial Differential Equations*, New York, 1964.

### 7.1 The one-dimensional case

**Proposition 17** In the one-dimensional case the notion of Multivariate Chebyshev Systems of order $N$ coincides with the Dirichlet type Chebyshev system $DT$ of order $N$ defined above.

**Proof.** Indeed, let us take the set $D = [a, b]$ and apply the interpolation property to the case $D_1 = D$. Then we know that for all constants $c_j$ and $d_j$ we have unique solubility of the problem

$$u^{(k)} (a) = c_k \quad \text{for } k = 0, 1, \ldots, N - 1$$

$$u^{(k)} (b) = d_k \quad \text{for } k = 0, 1, \ldots, N - 1.$$ 

Hence, $U$ is $2N$ dimensional; we may take the solution $v_j$ to the problem

$$u^{(k)} (a) = \delta_{j, k} \quad \text{for } k = 0, 1, \ldots, N - 1$$

$$u^{(k)} (b) = 0 \quad \text{for } k = 0, 1, \ldots, N - 1$$

and the solution $w_j$ to the problem

$$u^{(k)} (a) = 0 \quad \text{for } k = 0, 1, \ldots, N - 1$$

$$u^{(k)} (b) = \delta_{j, k} \quad \text{for } k = 0, 1, \ldots, N - 1$$

will make a basis for $U$. The approximate solubility of the Dirichlet problem for $D_1 \subset [a, b]$ implies now the exact solubility since $U$ is finite-dimensional. Indeed, we will take a sequence of solutions $u_\varepsilon (t)$ and the limit.

**Proof.**
Remark 18 In the case of Multivariate Chebyshev Systems we need the approximate solubility of the Dirichlet problem since we have infinite-dimensional spaces, and there is no equivalence between the uniqueness and the existence (in general, we have Fredholm operators).

7.2 Conservative Chebyshev systems

We may define the conditions above only for the case of spherical $N$ concentric spheres which are different. Then we may adopt the following definition

Definition 19 We say that the space $U$ is an order $N$ Multivariate Chebyshev System iff

1. For every $\varepsilon > 0$ and for every set of data $f_j$ defined on the sphere $S_{R_j}$ we have an element $u \in U$ such that

$$|u(x) - f_j(x)| \leq \varepsilon \quad \text{for} \quad x \in S_{R_j} \quad \text{for all} \quad j = 1, 2, ..., N$$

and

2. If for some $u \in U$ happens

$$u(x) = 0 \quad \text{for all} \quad x \in \Gamma_j \quad \text{for all} \quad j = 1, 2, ..., N$$

then it follows that $u \equiv 0$.

These conditions are much simpler; so far for now we can prove them only for the polyharmonic functions. Especially difficult is the uniqueness condition. But also the existence seems to be nasty.

References

[1] Krein, M., Nudelman A., The Markov Moment problem and Extremal problems, AMS translation from the Russian edition of 1973.

[2] S. Karlin, W. Studden, Tchebycheff Systems: with applications in analysis and statistics, Intersci. Publ., 1968.