Transfer of high-dimensional quantum state through an $XXZ$-Heisenberg quantum spin chain

Zhe Yang

1 State Key Laboratory of Low-Dimensional Quantum Physics and Department of Physics, Tsinghua University, Beijing 100084, China

Ming Gao

1 State Key Laboratory of Low-Dimensional Quantum Physics and Department of Physics, Tsinghua University, Beijing 100084, China

Wei Qin

2 School of Physics, Beijing Institute of Technology, Beijing 100081, China

qinwei09@tsinghua.org.cn

We propose and analyze an efficient high-dimensional quantum state transfer scheme through an $XXZ$-Heisenberg spin chain in an inhomogeneous magnetic field. By the use of a combination of coherent quantum coupling and free spin-wave approximation, pure unitary evolution results in a perfect high-dimensional swap operation between two remote quantum registers mediated by a uniform quantum data bus, and the feasibility is confirmed by numerical simulations. Also, we observe that either the strong $z$-directional coupling or high quantum spin number can partly suppress the thermal excitations and protect quantum information from the thermal noises when the quantum data bus is in the thermal equilibrium state.

Keywords: spin chain, quantum state transfer, high fidelity, thermal field

1. introduction

The transfer of quantum state between two distant quantum registers is an essential task of quantum information processing (QIP).

While long-range quantum communication can be realized by the use of photons, coupled solid-state systems can act as quantum data buses to connect two separated registers for short-range communication, e.g., within a computer. Such data buses have been explored in the context of various quantum systems ranging from trapped ion, superconducting flux qubits, to cavity arrays and nanoelectromechanical oscillators. Due to the ability to provide an alternative to either direct register interactions or an interface between stationary and flying qubits, quantum spin chains have attracted much attention in recent years.

In the original scheme, S. Bose studied a uniform spin chain of Heisenberg coupling, and quantum information can be efficiently transferred between two ends of
the spin channel via natural evolution. Moreover, many strategies aiming to achieve
the perfect quantum state transfer (QST) over arbitrary distance have emerged, such
as engineering the coupling strength in a way dependent of the chain length, implement-
ing local measurements of individual spins, and designing some special configurations
of spin chains. Alternatively, coherent quantum coupling has been widely used to
achieve high-fidelity QST by tuning the registers to interact weakly with the channel.

Compared to two-dimensional systems working as qubits, high-dimensional sys-
tems as qudits also deserve to explore because they can carry large capacity and
lead to a further insight into our understanding of quantum physics. Until now,
many proposals of quantum computation and quantum communication, e.g. quan-
tum cloning, quantum teleportation, quantum key distribution, and quantum corre-
lation, have been extended to high-dimensional versions. Indeed, with some notable exceptions, where perfect high-dimensional state trans-
fer over long distance has been implemented by utilizing a repeated measurement
procedure or a free spin wave approximation, prior work on perfect QST in coupled-
spin systems has primarily focused upon qubits.

In this paper, we devote our attention to a perfect transfer of high-dimensional
quantum state through an XXZ-Heisenberg coupling spin chain of arbitrary length
in an inhomogeneous magnetic field. On employing the Holstein-Primakoff trans-
formation and the free spin wave approximation, the Hamiltonian takes the form of
free bosons and can be diagonalized through an orthogonal transformation. Tuning
the register-bus coupling in the xy plane to be much smaller than that within the
data bus enables a special data bus collective eigenmode resonating with the two
end registers. As a consequence, unitary evolution results in a perfect swap opera-
tion between the two registers in the optimal time, and numerical simulations are
performed to confirm it. Moreover, we observe that increasing either the strong
z-directional coupling or high quantum spin number is capable of protecting quantum
information from the thermal noises.

The structure of the paper is as follows. In section 2, we introduce the analysis
of the model and give the Hamiltonian. In section 3, we show that a high fidelity
QST and the thermal effects. Finally, we summarize the whole mechanism and draw
our conclusions in the section 4.

2. Model and Analysis

As shown in Fig. 1(a), an XXZ-Heisenberg model governs an (N + 2)-site spin-S
chain in an inhomogeneous magnetic field. Only the nearest-neighbor interaction is
considered and the system is described by

\[ H = H_B + H_I + H_M. \]
Transfer of high-dimensional quantum state through an XXZ-Heisenberg quantum spin chain

Fig. 1. (Color online) (a) Shown is a quantum data bus mediating two quantum registers, with an XXZ-Heisenberg coupling. We demonstrate that a perfect high-dimensional swap operation between the registers via purely unitary evolution over arbitrary distance by applying an inhomogeneous field. (b) We employ a $d$-dimensional space spanned by the low-lying level states ranging from $|0\rangle$ to $|d-1\rangle$ to encode quantum information as a qudit. The condition $2S \gg d$ predicts that the spin-wave interaction can be neglected to yield a tight-binding Hamiltonian, which can be diagonalized through an orthogonal transformation. (c) On maintaining $\frac{\omega_0}{\Omega_0} \ll 1$, there is a special data bus collective mode being resonantly coupled to the two registers, and off-resonant coupling can be neglected. Therefore, we achieve a high dimensional quantum state transfer protocol through this eigenmode-mediated quantum channel.

The Hamiltonian of the quantum data bus is

$$H_B = -\Omega_0 \sum_{i=1}^{N-1} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) - \Omega_z \sum_{i=1}^{N-1} S_i^z S_{i+1}^z.$$
where $\Omega_0 > 0$ is the coupling strength in the $xy$-plane and $\Omega_z > 0$ is that along the $z$-direction. $S_i^\nu$ is the $\nu$ ($\nu = x, y, z$) component of the spin operator $\mathbf{S}_i$ at the $i$-th site with $S_i^z = S_i^x \pm i S_i^y$. $H_I$ describes the interaction between the two end registers and the intermediate quantum data bus,

$$H_I = -\omega_0(S_i^+ S_i^- + S_r^+ S_r^- + H.c.) - \omega_z(S_i^z S_i^z + S_r^z S_r^z),$$

where $\omega_0 > 0$ is the interaction between the sender (receiver) and the quantum data bus in the $xy$-plane and $\omega_z > 0$ is that along the $z$-direction. The Zeeman term reads

$$H_M = -(B_s S_s^z + B_r S_r^z + \sum_{i=1}^{N} B_i S_i^z),$$

with $B_i$ being the local magnetic field on the $i$th-site in the $z$-direction. By implementing the Holstein-Primakoff (HP) transformation $S_i^+ = \sqrt{2S} a_i^\dagger a_i$ and $S_i^r = S_i^z - a_i^\dagger a_i$, the Hamiltonian can be rewritten in terms of boson operators, and the state of each spin is described by a Fock state instead. In general, the low-lying $d$-dimensional space of the sender is harnessed to encode quantum information, and the input state is $\ket{\varphi_s} = \sum_{a=0}^{d-1} \alpha_a \ket{u_a}_s$, while the data bus and the receiver align in a parallel way being a ferromagnetic order [14], as sketched in Fig. 1(b).

For a spin chain of $N + 2$ spins-$S$, the Hilbert space $\mathcal{H}$ is of dimension $(2S + 1)^N+2$. The Hamiltonian $\mathcal{H}$ preserves the total bosonic number $N = a_s^\dagger a_s + a_r^\dagger a_r + \sum_{i=1}^{N} a_i^\dagger a_i$ due to $[\mathcal{H}, N] = 0$. Therefore, $\mathcal{H}$ can be decomposed into an invariant subspace $\mathcal{S}_G$ spanned by $|n_s, n_1, \cdots, n_N, n_r\rangle$ for $n_s, n_i, n_r = 0, \cdots, d-1$, and the dynamics of the system is completely restricted in the the $d^{(N+2)}$-dimensional subspace $\mathcal{S}_G$. Suppose that the dimension of the transferred state is much smaller than quantum spin number, i.e., $d << 2S$, the average boson number of each site could be much smaller than $2S$, $\left\langle a_i^\dagger a_i \right\rangle << 2S$. Subsequently, the spin-wave interaction is negligible, such that the HP transformation is simplified to $S_i^+ = \sqrt{2S} a_i^\dagger a_i$ leading to a bosonized tight-binding Hamiltonian

$$H_B = -2\Omega_0 S \sum_{i=1}^{N-1} (a_i^\dagger a_{i+1} + H.c.) - \Omega_z \sum_{i=1}^{N-1} [S_i^2 - S(a_i^\dagger a_i + a_{i+1}^\dagger a_{i+1})],$$

$$H_I = -2\omega_0 S (a_s^\dagger a_s + a_r^\dagger a_r + H.c.) - \omega_z [2S^2 - S(a_s^\dagger a_s + a_r^\dagger a_r + a_s^\dagger a_r + a_r^\dagger a_s)],$$

$$H_M = - [B_s (S - a_s^\dagger a_s) + B_r (S - a_r^\dagger a_r) + \sum_{i=1}^{N} B_i (S - a_i^\dagger a_i)] .$$

In order to achieve an efficient high-dimensional state transfer, we choose

$$B_s = B_r = 2\Omega_z S,$$

$$B_1 = B_N = \Omega_z S,$$

$$B_2 = \cdots = B_{N-1} = \omega_z S,$$

(6)
and apply the following orthogonal transformation:

$$a_i^\dagger = \sqrt{\frac{2}{N+1}} \sum_{k=1}^{N} \sin\frac{k\pi}{N+1} c_k^i, \quad i = 1, ..., N, \quad (7)$$

the Hamiltonian $H$ is transformed to

$$H = \sum_{k=1}^{N} \left[ (\varepsilon_k + \Gamma) c_k^i c_k + \Gamma (a_k^i a_s + a_k^i a_r) + \sum_{k=1}^{N} t_k [a_k^i c_k + (-1)^{k-1} a_k^i c_k + H.c.] \right]. \quad (8)$$

where $\varepsilon_k = -4\Omega_0 S \cos(\frac{k\pi}{N+1})$, $t_k = -2\omega_0 S \sqrt{\frac{2N}{N+1}} \sin(\frac{k\pi}{N+1})$ and $\Gamma = (2\Omega_z + \omega_z) S$. Note that the choice of the nonuniform field is applicable for only $N \geq 3$, and in the special case of $N = 1$, the field can be chosen as $B_s = B_r = \omega_z S + h$ and $B_1 = h$ with $h \geq 0$.

3. Quantum state transfer and the thermal effects

By restricting our discussion to a case of odd $N$ chains, there exists a zero-energy data bus collective mode, corresponding to $\kappa = (N + 1)/2$, being resonantly coupled to the two end registers with strength $t_\kappa = -2\omega_0 S/A$. Under the assumption that $\omega_0/\Omega_0 << 1$, off-resonant coupling can be neglected as a result of $t_\kappa \ll |\lambda_\kappa - \lambda_{\kappa \pm 1}|$, such that evolution dynamics behaves as an effective model in which only the two end registers and the $\kappa$-th collective mode are involved, as illustrated schematically in Fig. 1(c). In this case, the effective Hamiltonian

$$H_{\text{eff}} = \Gamma (c_\kappa^i c_\kappa + a_\kappa^i a_s + a_\kappa^i a_r) + t_\kappa [a_\kappa^i c_\kappa + (-1)^{k-1} a_\kappa^i c_\kappa + H.c.] \quad (9)$$

governs the evolution of the system. By choosing evolution time $\tau \equiv \pi/\sqrt{2} t_\kappa$, it yields

$$a_\kappa^i(\tau) = (-1)^{k} e^{-it_\kappa \tau} a_\kappa^i, \quad a_k^i(\tau) = (-1)^{k} e^{-it_\kappa \tau} a_k^i, \quad (10)$$

in Heisenberg picture. Eq. (10) reveals that the quantum state of the sender can be perfectly transferred to the receiver in the optimal time $\tau$. A swap gate has been established between the sender and the receiver, up to an additional phase independent of the sent state.

To confirm the efficiency of our method, numerical simulations are performed. Initially, the whole system, including the two end registers and the intermediate data bus, is in a product state

$$|\psi(0)\rangle = |\varphi\rangle_s |0\rangle_{\text{bus}}^{\otimes N} |0\rangle_r, \quad (11)$$

and $|0\rangle_{\text{bus}}^{\otimes N} = |0\rangle_1 \otimes \cdots \otimes |0\rangle_N$. In general, the state of the receiver at time $t$ is a mixed state $\rho_r(t)$, which can be obtained by tracing off the other sites $\rho_r(\tau) = \text{Tr}_r\left( e^{-iHt} |\psi(0)\rangle \langle \psi(0)| e^{iHt} \right)$. The fidelity between the sent state of the sender and the received state of the receiver at time $\tau$ is given by $F(\tau) = \langle \varphi | \rho_r(\tau) | \varphi \rangle_s$. Correspondingly, the average fidelity over all possible input pure states is

$$\langle F(\tau) \rangle = \frac{1}{V} \int F(\tau) dV. \quad (12)$$
Here, $V = \pi^{d-1}/(d-1)$ is the total volume of the manifold of pure states, and $dV = \prod_{p=1}^{d-1} \cos \vartheta_p (\sin \vartheta)^{2p-1} \, d\vartheta d\chi_p$ is the volume element \cite{10}, where $0 \leq \vartheta_p \leq \pi/2$ and $0 \leq \chi_p < 2\pi$ are the Hurwitz parameters for $p = 1, 2, \cdots, d-1$.

In Fig. 2 the average fidelity varies as a function of quantum spin number $S$ when $N = 3$ and $d = 3$ with $\omega_0/\Omega_0 = 0.1$. Three different $z$-directional coupling strengths are chosen to demonstrate the feasibility of the method. We observe that the average fidelity increases with $S$, and when $\omega_0/\Omega_0 << 1$ and $d << 2S$, the average fidelity nearly tends to one, e.g., in a case of $S = 10$, $\langle F(\tau) \rangle$ is 0.9974 (black line), 0.9984 (red line), and 0.9986 (blue line). The leakage of quantum information results from either the off-resonant coupling or the spin-wave interaction.

In the following, an investigation on the effects of the thermal environment will be numerically given when the quantum data bus is in a thermal equilibrium state described by

$$\rho_B = \frac{1}{Z} e^{-H_B/T},$$

where $Z = \text{tr}(e^{-H_B/T})$ characterizes a partition function and $T$ represents the temperature. The density matrix of the whole system is in a product state,

$$\rho(0) = \sum_{\mu', \mu = 0}^{d-1} \alpha_\mu \alpha_{\mu'}^* \langle \mu \rangle_s |0\rangle_s^r \langle \mu' |^r_s |0\rangle_r \otimes \rho_B.$$ 

In Fig. 3 we plot the average fidelity as a function of the temperature for a bus of length $N = 1$ initially in its thermal equilibrium state: $F(\tau)$ decreases with $T/\omega_0$ owing to the validity of free spin wave approximation only in the low boson excitation regime, however, increasing either $\omega_0$ or $S$ can depress the thermal noises to prevent quantum information from leaking into the thermal environment. In
Fig. 3. (Color online) The average fidelity varies as a function of temperature with $N = 1$ and $d = 3$ for either (a) three $z$-directional coupling strengths in a case of $S = 3$, or (b) three quantum spin numbers in a case of $\omega_0 = \omega_z$. Here, $h = \omega_z S$ and the evolution time is the optimal time $\tau$.

Fig. 3(a), it should be noted that the $z$-directional coupling contains the spin-wave interaction found in nonlinear terms of the HP transformation, and such coupling can lead to the leakage of quantum information, specially at very low temperature range. However, with the temperature increasing, the $z$-directional coupling can effectively cope with the thermal effects, and provide the protection for quantum information instead. Moreover, from the model featured by $H$ of Eq. (8), both the $z$-directional coupling and the magnetic field result in $\Gamma$ being capable of protecting quantum information, in similar to the magnetic field applied on an $XX$ coupling spin chain.

4. Summary

In this paper a quantum state transfer protocol has been studied through an $XXZ$ coupling spin chain in the presence of an inhomogeneous magnetic field. Upon harnessing coherent quantum coupling and free spin-wave approximation, off-resonant
couplings and spin-wave interactions can be ignored, and consequently, an arbitrary unknown high-dimensional quantum state can be transferred between two remote registers with high fidelity via purely dynamical evolution. When the quantum data bus is in the thermal equilibrium state, the effects of the temperature on the state transfer protocol have also been numerically studied. In contrast to previous work on $XX$ coupling spin chains, an additional $z$-directional coupling can depress the thermal excitations and partly counteract the thermal effects to ensure the feasibility of the present method. With its scalability and robustness, this protocol may be applicable in a high-dimensional solid device for quantum information processing.

Acknowledgments

We gratefully thank Chao Lian, Shuzhe Shi and Hui Li for helpful discussions.

References

1. M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (UK: Cambridge University Press, 2000).
2. K. Mattle, H. Weinfurter, P. G. Kwiat, and A. Zeilinger, Phys. Rev. Lett. 76, 4656 (1996).
3. J. I. Cirac, P. Zoller, H. J. Kimble, and H. Mabuchi, Phys. Rev. Lett. 78, 3221 (1997).
4. T. Jennewein, C. Simon, G. Weihs, H. Weinfurter and A. Zeilinger, Phys. Rev. Lett. 84, 4729 (2000).
5. D. Kielpinski, C. Monroe, and D. J. Wineland, Nature (London) 417, 709 (2002).
6. F. Schmidt-Kaler, H. Hauffner, M. Riebe, S. Gulde, G. P. T. Lancaster, T. Deuschle, C. Becher, C. F. Roos, J. Eschner, and R. Blatt, Nature (London) 422, 408 (2003).
7. M. A. Sillanpää, J. I. Park, and R. W. Simmonds, Nature (London) 449, 438 (2007).
8. J. Q. You and F. Nori Nature (London) 474, 589 (2011).
9. J. Majer, J. M. Chow, J. M. Gambetta, Jens Koch, B. R. Johnson, J. A. Schreier, L. Frunzio, D. I. Schuster, A. A. Houck, A. Wallraff, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Nature (London) 449, 443 (2007).
10. C. D. Ogden, E. K. Irish, and M. S. Kim, Phys. Rev. A 78, 063805 (2008).
11. G. D. de Moraes Neto, M. A. de Pente, and M. H. Y. Mousa, Phys. Rev. A 84, 032339 (2011).
12. J. Eisert, M. B. Plenio, S. Bose, and J. Hartley, Phys. Rev. Lett. 93, 190402 (2004).
13. S. Bose, Phys. Rev. Lett. 91, 207901, (2003).
14. M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, Phys. Rev. Lett. 92, 187902 (2004).
15. M. Christandl, N. Datta, T. C. Dorlas, A. Ekert, A. Kay, and A. J. Landahl, Phys. Rev. A 71, 032312 (2005).
16. F. Verstraete, M. A. Martín-Delgado, and J. I. Cirac, Phys. Rev. Lett. 92, 087201 (2004).
17. D. Burgarth and S. Bose, Phys. Rev. A 71, 052315 (2005).
18. Y. Li, T. Shi, B. Chen, Z. Song, and C. P. Sun, Phys. Rev. A 71, 022301 (2005).
19. T. J. Osborne and N. Linden, Phys. Rev. A 69, 052315 (2004).
20. A. Wójcik, T. Łuczak, P. Kurzyński, A. Grudka, T. Gdala, and M. Bednarska, Phys. Rev. A 72, 034303 (2005).
21. A. Wójcik, T. Łuczak, P. Kurzyński, A. Grudka, T. Gdala, and M. Bednarska, Phys. Rev. A 75, 022330 (2007).
Transfer of high-dimensional quantum state through an XXZ-Hessenberg quantum spin chain

22. L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia, Phys. Rev. Lett. 99, 060401 (2007).
23. L. Campos Venuti, S. M. Giampaolo, F. Illuminati, and P. Zanardi, Phys. Rev. A 76, 052328 (2007).
24. N. Y. Yao, L. Jiang, A. V. Gorshkov, Z.-X. Gong, A. Zhai, L.-M. Duan, and M. D. Lukin, Phys. Rev. Lett. 106, 040505 (2011).
25. N. Y. Yao, Z.-X. Gong, C. R. Laumann, S. D. Bennett, and L.-M. Duan, and M. D. Lukin, and L. Jiang, and A. V. Gorshkov, Phys. Rev. A 87, 022306 (2013).
26. W. Qin, C. Wang, Y. Cao, G. L. Long, Phys. Rev. A 89, 062314 (2014).
27. L. Campos Venuti, S. M. Giampaolo, F. Illuminati, and P. Zanardi, Phys. Rev. A. 76, 052328 (2007).
28. R. H. Crooks and D. V. Khveshchenko, Phys. Rev. A 77, 062305 (2008).
29. T. J. G. Apollaro, S. Lorenzo, and F. Plastina, Int. J. Mod. Phys. B 27, 1345035 (2013).
30. J. Liu, G. F. Zhang, and Z. Y. Chen, Int. J. Mod. Phys. B 24, 1279 (2010).
31. Y. Cao, S. G. Peng, C. Zheng, and G. L. Long, Commun. Theor. Phys. 55, 790 (2011).
32. R. F. Werner, Phys. Rev. A 58, 1827 (1998).
33. M. Keyl and R. F. Werner, J. Math. Phys. 40, 3283 (1999).
34. A. Acín, N. Gisin and V. Scarani, Quantum Inf. Comput. 3, 563 (2003).
35. G. Rigolin, Phys. Rev. A 71, 032303 (2005).
36. X. Ge and Y. Shen, Phys. Lett. B 606, 184 (2005).
37. M. Jiang, X. Huang, L. L. Zhou, Y. M. Zhou, and J. Zeng, Chin. Sci. Bull. 57, 2247 (2012).
38. V. Karimipour, A. Bahraminasab, and S. Bagherinezhad, Phys. Rev. A 65, 052331 (2002).
39. H. Li, Y. S. Li, S. H. Wang, and G. L. Long, Commun. Theor. Phys. 61, 273 (2014).
40. A. Bayat, Phys. Rev. A 89, 062302 (2014).
41. W. Qin, C. Wang, and G. L. Long, Phys. Rev. A 87, 012339 (2013).
42. W. Qin, J. L. Li, and G. L. Long, Chin. Phys. B 24, 040305 (2015).
43. T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940).
44. J. M. Ziman Principles of the theory of solids, second edition (Cambridge University Press, Cambridge, UK, 1972).
45. E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (NY) 16, 407 (1961).
46. K. Życzkowski and H. Sommers, J. Phys. A 34, 7111 (2001).