Reconstructing a Simple Polytope from its Graph

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Abstract. Blind and Mani [2] proved that the entire combinatorial structure (the vertex-facet incidences) of a simple convex polytope is determined by its abstract graph. Their proof is not constructive. Kalai [15] found a short, elegant, and algorithmic proof of that result. However, his algorithm has always exponential running time. We show that the problem to reconstruct the vertex-facet incidences of a simple polytope $P$ from its graph can be formulated as a combinatorial optimization problem that is strongly dual to the problem of finding an abstract objective function on $P$ (i.e., a shelling order of the facets of the dual polytope of $P$). Thereby, we derive polynomial certificates for both the vertex-facet incidences as well as for the abstract objective functions in terms of the graph of $P$. The paper is a variation on joint work with Michael Joswig and Friederike Körner [12].

1 Introduction

The face lattice $L_P$ of a (convex) polytope $P$ is any lattice that is isomorphic to the lattice formed by the set of all faces of $P$ (including $\emptyset$ and $P$ itself), ordered by inclusion. It is well-known to be determined by the vertex-facet incidences of $P$, i.e., by any graph that is isomorphic to the bipartite graph whose nodes are the vertices and the facets of $P$, where the edges are defined by the pairs $\{v, f\}$ of vertices $v$ and facets $f$ with $v \in f$. In lattice theoretic terms, $L_P$ is a ranked, atomic, and coatomic lattice, and thus, the sub-poset formed by its atoms and coatoms already determines the whole lattice. Actually, one can compute $L_P$ from the vertex-facet incidences of $P$ in $O(\eta \cdot \alpha \cdot \lambda)$ time, where $\eta$ is the minimum of the number of vertices and the number of facets, $\alpha$ is the number of vertex-facet incidences, and $\lambda$ is the total number of faces of $P$ [14].

The graph $G_P = (V_P, E_P)$ of a polytope $P$ is any graph that is isomorphic to the graph whose nodes are the vertices of $P$, where two nodes are adjacent if and only if the convex hull of the corresponding two vertices is a one-dimensional face of $P$. Phrased differently, $G_P$ is the graph defined on the rank one elements of $L_P$.

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where two rank one elements are adjacent if and only if they are below a common rank two element. While the vertex-facet incidences completely determine the face-lattice of any polytope, the graph of a polytope in general does not encode the entire combinatorial structure. This can be seen, e.g., from the examples of the cut polytope associated with the complete graph on \(n\) nodes and the \(\binom{n}{2}\)-dimensional cyclic polytope with \(2^n - 1\) vertices, which both have complete graphs. Another example is the four-dimensional polytope shown in Fig. 1 whose graph is isomorphic to the graph of the five-dimensional cube.

**Fig. 1.** A Schlegel-diagram (projection onto one facet) of a four-dimensional polytope with the graph of a five-dimensional cube, found by Joswig & Ziegler [13].

Actually, in all dimensions less than four such ambiguities cannot occur. For one- or two dimensional polytopes this is obvious, and for three-dimensional polytopes it follows from Whitney’s theorem [18] saying that every 3-connected planar graph has a unique (up to reflection) plane embedding.

A \(d\)-dimensional polytope \(P\) is **simple** if every vertex of \(P\) is contained in precisely \(d\) facets, which is equivalent to \(G_P\) being \(d\)-regular (the polytope is non-degenerate in terms of Linear Programming). Every face of a simple polytope is simple as well. None of the examples showing that it is, in general, impossible to reconstruct the face lattice of a polytope from its graph, is simple.

In fact, Blind and Mani [2] proved in 1987 that the face lattice of a simple polytope is determined by its graph. Their proof (which we sketch in Sect. 2) is not constructive and crucially relies on the topological concept of homology. In 1988, Kalai [15] found a short and elegant proof (reviewed in Sect. 3) that does only use elementary geometric and combinatorial reasoning with the main advantage of being algorithmic. However, the running time of the method that can be devised from it is exponential in the size of the graph.

Perles conjectured in the 1970's (see [18]) that for a \(d\)-dimensional simple polytope \(P\) every subset \(F \subseteq V_P\) that induces a \((d - 1)\)-regular, connected, and non-separating subgraph of \(G_P\) corresponds to the vertex set of a facet of \(P\). A proof of this conjecture would have lead immediately to a polynomial time
algorithm that, given the graph \( G_P = (V_P, E_P) \) of a simple polytope \( P \), decides for a set of subsets of \( V_P \) if it corresponds to the set of vertex sets of facets of \( P \). However, Haase and Ziegler [10] recently disproved Perles’ conjecture. They found a four-dimensional simple polytope whose graph has a 3-regular, non-separating, and even 3-connected induced subgraph that does not correspond to any facet.

Refining ideas from Kalai’s proof (Sect. 4), we show that the problem of reconstructing the vertex-facet incidences of a simple polytope \( P \) from its graph \( G_P \) can be formulated as a combinatorial optimization problem that has a well-stated strongly dual problem (Sect. 5). The optimal solutions to this dual problem are certain orientations of \( G_P \) (induced by “abstract objective functions”) that are important also in different contexts. In particular, we provide short certificates for both the vertex-facet incidences of a simple polytope and for the abstract objective functions in terms of \( G_P \). We conclude with some remarks on the complexity status of the problem to decide whether a claimed solution to the reconstruction problem indeed are the vertex-facet incidences of the respective polytope in Sect. 6.

The material presented here has evolved from joint work with Michael Joswig and Friederike Körner [12]. The basic ideas and results are the same in both papers. However, the concept of a “facoidal system of walks” is newly introduced here. It differs from the corresponding notion of a “2-system” (introduced in [12]) with the main effect that one knows how to compute efficiently some facoidal system of walks from the graph \( G_P \) of a simple polytope \( P \) (see Proposition 3), while it is unclear how to find a 2-system from \( G_P \) efficiently. Furthermore, the proof of Theorem 3 we give here is different from the corresponding proof in [12]. Finally, the complexity theoretic statement of Corollary 5 does not appear in [12].

For all notions from the theory of polytopes that we use without (sufficient) explanations, we refer to Ziegler’s book [19]. We use the terms \( d \)-polytope and \( k \)-face for \( d \)-dimensional polytopes and \( k \)-dimensional faces, respectively. Often we will identify a face \( F \) of a simple polytope \( P \) with the subset of nodes of \( G_P \) that corresponds to the vertex set of \( F \). Whenever we talk about “polynomial time” or “efficient” this refers to the size of the graph \( G_P \) of the respective (simple) polytope \( P \).

2 The Theorem of Blind and Mani

Blind and Mani [2] proved their theorem in the dual setting, i.e., for simplicial rather than for simple polytopes. Nevertheless, we sketch parts of their proof in terms of simple polytopes here. The starting point is the observation that, while a priori it is by no means clear if the graph \( G_P = (V_P, E_P) \) of a simple polytope \( P \) determines the face lattice of \( P \), it is easy to see that the 2-faces of \( P \) (as subsets of \( V_P \)) carry the entire information on the combinatorial structure of \( P \).

Let \( P \) be a simple polytope. For a node \( v \in V_P \) of \( G_P \) denote by \( \Gamma(v) \subseteq V_P \) the subset of nodes that are adjacent to \( v \) (the neighbors of \( v \)). For any \( k \)-element
subset $S \subset \Gamma (v)$ there is a $k$-face of $P$ that contains the vertices that correspond to $S \cup \{v\}$ (and no vertex that corresponds to a node in $\Gamma (v) \setminus S$). We call the subset $F (S \cup \{v\}) \subseteq V_P$ of nodes corresponding to the vertices of that face the $k$-face spanned by $S \cup \{v\}$.

For an edge $e = \{v, w\} \in E_P$ let $\Psi_{(v,w)} : \Gamma (v) \setminus \{w\} \rightarrow \Gamma (w) \setminus \{v\}$ be the map that assigns to each subset $S \subset \Gamma (v) \setminus \{w\}$ the subset $T \subset \Gamma (w) \setminus \{v\}$ with $F (S \cup \{v, w\}) = F (T \cup \{w, v\})$. The maps $\Psi_{(v,w)}$ are cardinality preserving bijections, where $\Psi_{(v,w)}$ is the inverse of $\Psi_{(v,w)}$.

**Proposition 1.** Let $P$ be a simple polytope. For each $e = \{v, w\} \in E_P$ and $S \subseteq \Gamma (v) \setminus \{w\}$ we have

$$\Psi_{(v,w)}(S) = \Psi_{(v,w)}(\overline{S})$$

(where $\overline{U}$ is the respective complement of the set $U$).

**Proof.** This follows from the fact that we have $\Psi_{(v,w)}(S_1 \cap S_2) = \Psi_{(v,w)}(S_1) \cap \Psi_{(v,w)}(S_2)$ for all $S_1, S_2 \subseteq \Gamma (v) \setminus \{w\}$.

With the notations of Proposition 1 denote by $\psi^k_{(v,w)}$ the restriction of the map $\Psi^k_{(v,w)}$ to the $(k-1)$-element subsets of $\Gamma (v) \setminus \{w\}$. There are quite obvious algorithms that compute from the maps $\psi^k_{(v,w)}$, $\{v, w\} \in E_P$, the $k$-faces of $P$ (as subsets of $V_P$), and vice versa, in polynomial time in the number $f_k(P)$ of $k$-faces of the simple $d$-polytope $P$. Since both $f_2(P)$ as well as $f_{d-1}(P)$ are bounded polynomially in the size of $G_P$, the following result follows.

**Corollary 1.** There are polynomial time algorithms that, given the graph $G_P$ of a simple $d$-polytope $P$, compute the set of facets of $P$ from the set of 2-faces of $P$ (both viewed as sets of subsets of $V_P$), and vice versa.

For the rest of this section let $P_1$ and $P_2$ be two simple polytopes, and let $g : V_{P_1} \rightarrow V_{P_2}$ be an isomorphism of the graphs $G_{P_1} = (V_{P_1}, E_{P_1})$ and $G_{P_2} = (V_{P_2}, E_{P_2})$ of $P_1$ and $P_2$, respectively (i.e., $g$ is an in both directions edge preserving bijection).

The core of Blind and Mani’s paper [2] is the following result.

**Proposition 2.** The graph isomorphism $g$ maps every cycle in $G_{P_1}$ that corresponds to a 2-face of $P_1$ to a cycle in $G_{P_2}$ that corresponds to a 2-face of $P_2$.

Blind and Mani’s proof proceeds in the dual setting, i.e., in terms of the boundary complexes $\partial P_1^*$ and $\partial P_2^*$ of the simplicial dual polytopes $P_1^*$ and $P_2^*$ of $P_1$ and $P_2$, respectively. The strategy is to show that, if some cycle in $G_{P_1}$ corresponding to a 2-face of $P_1$ was mapped to some cycle in $G_{P_2}$ that does not correspond to any 2-face of $P_2$, then a certain sub-complex of $\partial P_2^*$ would have a certain non-vanishing (reduced) homology group. They complete their proof of Proposition 2 by showing that the respective homology group, however, is zero. The key ingredient they use to prove this is the following. For each face $F$ of $P_1^*$
there is a *shelling order* of the facets of $P_1^*$ (i.e., an ordering satisfying certain convenient topological properties, which, however, can be expressed completely combinatorially, see Sect. 3) in which the facets containing $F$ come first.

From Proposition 2 one can deduce that the graph isomorphism $g$ actually induces a bijection between the cycles in $G_{P_1}$ that correspond to 2-faces of $P_1$ and the cycles in $G_{P_2}$ that correspond to 2-faces of $P_2$. Once this is established, Proposition 1 yields the following result.

**Theorem 1 (Blind & Mani [2]).** Every isomorphism between the graphs $G_{P_1}$ and $G_{P_2}$ of two simple polytopes $P_1$ and $P_2$, respectively, induces an isomorphism between the vertex-facet incidences of $P_1$ and $P_2$. In particular, the graph of a simple polytope determines its entire face lattice.

### 3 Kalai’s Constructive Proof

Kalai realized that the existence of shelling orders as exploited by Blind and Mani can be used directly in order to devise a simple proof which does not rely on any topological notions like homology [15]. He formulated his proof in the original setting, i.e., for simple polytopes, where the notion corresponding to “shelling” is called “abstract objective function.”

From now on, let $P$ be a simple $d$-polytope with $n$ vertices. For simplicity of notation, we will identify each face of $P$ not only with the corresponding subset of $V_P$, but also with the corresponding induced subgraph of $G_P$. Furthermore, by saying that $w \in W \subset V_P$ is a sink of $W$ we mean that $w$ is a sink of the orientation induced on the subgraph of $G_P$ that is induced by $W$.

**Definition 1.** Every bijection $\varphi : V_P \rightarrow \{1, \ldots, n\}$ induces an acyclic orientation $O_\varphi$ of the graph $G_P$ of $P$, where an edge is directed from its larger end-node to its smaller end-node (with respect to $\varphi$). The map $\varphi$ is called an abstract objective function (AOF) if $O_\varphi$ has a unique sink in every non-empty face of $P$ (including $P$ itself). Such an orientation of $G_P$ is called an AOF-orientation.

The inverse orientation of an AOF-orientation is an AOF-orientation as well (this follows, e.g., from Theorem 3). Thus, every AOF-orientation also has a unique source in every non-empty face.

From the fact that the simplex algorithm works correctly (on every face) one easily derives that every linear function that assigns pairwise different values to the vertices of $P$ induces an AOF-orientation (this is a consequence of the convexity of the faces). From this observation, the following fact follows (which is dual to the existence of the shelling orders required in Blind and Mani’s proof).

**Lemma 1.** Let $W \subset V_P$ be any face of $P$. There is an AOF-orientation of $G_P$ for which $W$ is terminal, i.e., no edge in the cut defined by $W$ is directed from $W$ to $V_P \setminus W$.

In a sense, this statement can be reversed.
Lemma 2. Let $W \subset V_P$ be a set of nodes inducing a $k$-regular connected subgraph of $G_P$, and let $\mathcal{O}$ be an AOF-orientation for which $W$ is terminal. Then $W$ is a $k$-face of $P$.

Proof. Since $\mathcal{O}$ is acyclic, it has a source $s$ in $W$. Let $w_1, \ldots, w_k \in W$ be the neighbors of $s$ in $W$, and let $F := F(\{t, w_1, \ldots, w_k\}) \subset V_P$ be the $k$-face of $P$ that is spanned by $t, w_1, \ldots, w_k$. Since $\mathcal{O}$ has unique sources on non-empty faces, $s \in W \cap F$ must be the unique source of $F$. By the acyclicity of $\mathcal{O}$ there hence is a monotone path from $s$ to every node in $F$. Since $W$ is terminal this implies $F \subseteq W$. Because both $F$ and $W$ induce $k$-regular connected subgraphs of $G_P$, $F = W$ follows.

Lemma 1 and Lemma 2 imply that one can compute the vertex-facet incidences of $P$, provided that one knows all AOF-orientations of $G_P$. Kalai’s crucial discovery is that one can compute the AOF-orientations just from $G_P$ (i.e., without explicitly knowing the faces of $P$).

Definition 2. For an orientation $\mathcal{O}$ of $G_P$ let $h_k(\mathcal{O})$ be the number of nodes with in-degree $k$. The number

$$H(\mathcal{O}) := \sum_{k=0}^{d} h_k(\mathcal{O}) \cdot 2^k$$

is called the $H$-sum of $\mathcal{O}$.

Since every subset of neighbors of a vertex $v$ of $P$ together with $v$ span a face of $P$ containing no other neighbors of $v$, one finds (by double-counting) that

$$H(\mathcal{O}) = \sum_{F \text{ face of } P} \left( \# \text{ sinks of } \mathcal{O} \text{ in } F \right)$$

is the total number of sinks induced by $\mathcal{O}$ on faces of $P$. Consequently, since every acyclic orientation has at least one sink in every non-empty face, we have the following characterization.

Lemma 3. An orientation $\mathcal{O}$ of $G_P$ is an AOF-orientation if and only if it is acyclic and has minimal $H$-sum among all acyclic orientations of $G_P$ (which then equals the number of non-empty faces of $P$).

Thus, by enumerating all $2^\frac{d+1}{2} = \sqrt{2}^d$ orientations of $G_P$ one can find all AOF-orientations of $G_P$.

Theorem 2 (Kalai [15]). There is an algorithm that computes the vertex-facet incidences of a simple $d$-polytope with $n$ vertices from its graph in $O(\sqrt{2}^d n)$ steps.
4 Walks and Orientations

In this section, we refine the ideas of Kalai’s proof and combine them with the observation (exploited by Blind and Mani) that it suffices to identify the 2-faces from the graph of a simple polytope, even with respect to the question for polynomial time reconstruction algorithms (see Corollary 1). Let us start with a result that emphasizes the importance of the 2-faces even more. The result was known for cubes [11]; for three-dimensional simple polytopes it was independently proved by Develin [4]). For general simple polytopes it seems that it was assumed to be false (see [19, Ex. 8.12 (iv)]).

Theorem 3. An acyclic orientation $O$ of the graph $G_P$ of a simple polytope $P$ is an AOF-orientation if and only if it has a unique sink on every 2-face of $P$.

Proof. The “only if” part is clear by definition. For the “if” part, let $\varphi : V_P \rightarrow \{1, \ldots, n\}$ be a bijection inducing an acyclic orientation $O = O_{\varphi}$ that has a unique sink in every 2-face of $P$. Suppose there is a face $F$ of $P$ in which $O$ has two sinks $t_1, t_2 \in F$ ($t_1 \neq t_2$). We might assume $F = P$ (because $F$ itself is a simple polytope with every 2-face of $F$ being a 2-face of $P$).

Since $G_P$ is connected, there is a path in $G_P$ connecting $t_1$ and $t_2$. Let $\Pi \neq \emptyset$ be the set of all these paths. For every $\pi \in \Pi$ we denote by $\mu(\pi)$ the maximal $\varphi$-value of any node in $\pi$. Let $\pi_{\text{min}} \in \Pi$ be a path with minimal $\mu$-value among all paths in $\Pi$, where $v \in V_P$ is the node in $\pi_{\text{min}}$ with $\varphi(v) = \mu(\pi_{\text{min}})$ (see Fig. 2).

![Fig. 2. Illustration of the proof of Theorem 3. The fat grey path is the one yielding the contradiction.](image)

Obviously, $v$ is a source in the path $\pi_{\text{min}}$ (in particular, $v \notin \{t_1, t_2\}$). Let $C$ be the 2-face spanned by $v$ and its two neighbors $v_1$ and $v_2$ in $\pi_{\text{min}}$. Since $v$ is the unique source of $O$ in $C$, $v$ has the largest $\varphi$-value among all nodes in the union $U$ of $C$ and $\pi_{\text{min}}$. But $U \setminus \{v\}$ induces a connected subgraph of $G_P$ containing both $t_1$ and $t_2$, which contradicts the minimality of $\mu(\pi_{\text{min}})$.

From now on let, again, $P$ be a simple polytope. The ultimate goal is to find the system of cycles in the graph $G_P$ that corresponds to the set of 2-faces
of $P$. However, we even do not know how to prove or disprove efficiently that a given system of cycles actually is the one we are searching for. We now define more general systems having the property that one can at least generate one of them in polynomial time (which in general, of course, will not be the desired one), and among which the one corresponding to the set of 2-faces of $P$ can be characterized using AOF-orientations.

**Definition 3.**

1. A sequence $W = (w_0, \ldots, w_{l-1})$ (with $l \geq 3$) of nodes in $G_P$ is called a closed smooth walk in $G_P$, if $\{w_i, w_{i+1}\}$ is an edge of $G_P$ and $w_{i-1} \neq w_{i+1}$ for all $i$ (where, as in the following, all indices are taken modulo $l$). Note that the $w_i$ need not be pairwise disjoint. We will identify two closed smooth walks if they differ only by a cyclic shift and/or a “reflection” of their node sequences.

2. A set $W$ of closed smooth walks in $G_P$ is a facoidal system of walks if for every triple $v, v_1, v_2 \in V_P (v_1 \neq v_2)$ such that both $v_1$ and $v_2$ are neighbors of $v$ there is a unique closed smooth walk $(w_0, \ldots, w_{l-1}) \in W$ with $(w_{i-1}, w_i, w_{i+1}) \in \{(v_1, v, v_2), (v_2, v, v_1)\}$ for some $i$, which is also required to be unique.

The system of 2-faces of $P$ yields a uniquely determined (recall the identifications mentioned in part 1 of the definition) facoidal system of walks in $G_P$, which is denoted by $C_P$. In general, there are many other facoidal systems of walks (see Fig. 3).

![Fig. 3. A facoidal system of four walks in the graph of the three-dimensional cube.](image)

For each path $\lambda$ in $G_P$ of length two denote by $v(\lambda)$ the inner node of $\lambda$. Let $G_P'$ be the graph defined on the paths of length two in $G_P$, where two paths $\lambda_1$ and $\lambda_2$ are adjacent if and only if they share a common edge and $v(\lambda_1) \neq v(\lambda_2)$ holds (see Fig. 4).

A 2-factor in a graph $G$ is a set of (not self-intersecting) cycles in $G$ such that every node is contained in a unique cycle. Checking whether a graph has a 2-factor and finding one (if it exists) can be reduced (by a procedure due to
Tutte \cite{17} to searching for a perfect matching in a related graph (which can be performed in polynomial time by Edmonds’ algorithm \cite{6}).

**Proposition 3.** For simple polytopes $P$,

1. there is a (polynomial time computable) bijection between the facoidal systems of walks in $G_P$ and the 2-factors of $G_P^*$,
2. checking whether a given set of node-sequences in $G_P$ is a facoidal system of walks can be done in polynomial time, and
3. one can find a facoidal system of walks in $G_P$ in polynomial time.

**Proof.** Part 2 is obvious, part 3 follows from part 1 by Tutte’s reduction \cite{17} and Edmonds algorithm \cite{6}, and part 1 is readily obtained from the definitions.

Proposition 3 shows that facoidal systems of walks have quite convenient algorithmic properties. However, they become useful only due to the fact that the system $C_P$ corresponding to the 2-faces of $P$ can be well-characterized among them, as we will demonstrate next.

**Definition 4.** Let $O$ be any orientation of $G_P$.

1. The $\mathcal{H}_2$-sum of $O$ is defined as

$$\mathcal{H}_2(O) := \sum_{k=0}^{d} h_k(O) \cdot \binom{k}{2}.$$

2. A closed smooth walk $(w_0, \ldots, w_{l-1})$ in $G_P$ has a sink (source, respectively) at position $i$ (with respect to the orientation $O$), if the edges $\{w_i, w_{i-1}\}$ and $\{w_i, w_{i+1}\}$ both are directed towards (away from, respectively) $w_i$.

The following follows immediately from the definitions.

**Lemma 4.** For every orientation $O$ of $G_P$ the sum $\mathcal{H}_2(O)$ equals the total number of sinks (with respect to $O$) in every facoidal system of walks in $G_P$.

Now we can formulate and prove the main result of this section (where $f_2(P)$ denotes the number of 2-faces of $P$).
Theorem 4. Let $P$ be a simple polytope, $W$ a facoidal system of walks in $G_P$, and $O$ an acyclic orientation of $G_P$. Then

$$\#W \leq f_2(P) \leq \mathcal{H}_2(O)$$

holds.

1. The first inequality holds with equality if and only if $W = \mathcal{C}_P$ (i.e., $W$ “is” the set of 2-faces of $P$).
2. The second inequality holds with equality if and only if $O$ is an AOF-orientation of $G_P$.

Proof. Since $O$ is acyclic, every closed smooth walk in $G_P$ must have at least one sink with respect to $O$. Thus, Lemma 4 implies

$$\#W \leq \mathcal{H}_2(O),$$

yielding

$$f_2(P) = \#\mathcal{C}_P \leq \mathcal{H}_2(O),$$

where by Theorem 3 equality holds in (4) if and only if $O$ is an AOF-orientation. Because $G_P$ has an AOF-orientation $O_0$ (see Lemma 3), inequality (3) gives

$$\#W \leq \mathcal{H}_2(O_0) = f_2(P).$$

Hence, it remains to prove that $\#W = \#\mathcal{C}_P$ implies $W = \mathcal{C}_P$. Suppose, that $\#W = \#\mathcal{C}_P$ holds. It thus suffices to show $\mathcal{C}_P \subseteq W$ (since we know already $\#\mathcal{C}_P \geq \#W$). Let $C \in \mathcal{C}_P$ be any closed smooth walk corresponding to a 2-face of $P$. By Lemma 2 there is an AOF-orientation $O_C$ of $G_P$ such that $C$ is terminal with respect to $O_C$. Let $w_1 \in V_P$ be the unique source in $C$ (with respect to $O_C$), and let $w_0$ and $w_2$ be the two neighbors of $w_1$ in $C$. By definition, there is a (unique) $W = (w_0, w_1, w_2, \ldots, w_{l-1}) \in W$. Because of $\#W = \#\mathcal{C}_P = \mathcal{H}_2(O_C)$ the closed smooth walk $W$ has a unique sink at some position $j$ and its unique source at position 1. Thus, the two paths $(w_1, w_2, \ldots, w_j)$ and $(w_1, w_0, \ldots, w_j)$ both are monotone. Since $C$ is terminal, this implies that these two paths are contained in $C$. Therefore we have $C = W \in W$.

5 Good Characterizations

Theorem 4 immediately yields characterizations of sets of 2-faces and of AOF-orientations that are similar to Kalai’s characterization of AOF-orientations (see Lemma 3).

Corollary 2. Let $P$ be a simple polytope.

1. A facoidal system of walks in $G_P$ is the system $\mathcal{C}_P$ of 2-faces of $P$ if and only if it has maximal cardinality among all facoidal systems of walks in $G_P$.
2. An acyclic orientation of $G_P$ is an AOF-orientation if and only if it has minimal $\mathcal{H}_2$-sum among all acyclic orientations of $G_P$. 
Unfortunately, for arbitrary graphs the problem of finding a 2-factor with as many cycles as possible is \textsf{NP}-hard. This follows from the fact that the question whether a graph can be partitioned into triangles is \textsf{NP}-complete [7, Prob. GT11].

With respect to algorithmic questions, the following good characterizations (in the sense of Edmonds [5,6]) of the set of 2-faces (and thus, by Proposition 1 of the vertex-facet incidences) as well as of AOF-orientations may be more valuable than those in Corollary 2.

**Corollary 3.** Let \( P \) be a simple polytope.

1. Let \( \mathcal{W} \) be a facoidal system of walks in \( G_P \). Either there is an acyclic orientation of \( G_P \) having a unique sink in every walk of \( \mathcal{W} \), or there is a facoidal system of walks in \( G_P \) of larger cardinality than \( \# \mathcal{W} \). In the first case, \( \mathcal{W} = C_P \) “is” the set of 2-faces of \( P \), in the second, it is not.
2. Let \( \mathcal{O} \) be an acyclic orientation of \( G_P \). Either there is a facoidal system \( \mathcal{W} \) of walks in \( G_P \) such that \( \mathcal{O} \) has a unique sink in every walk in \( \mathcal{W} \), or there is an acyclic orientation of \( G_P \) with smaller \( H_2 \)-sum than \( H_2 (\mathcal{O}) \). In the first case, \( \mathcal{O} \) is an AOF-orientation, in the second, it is not.

For graphs \( G \) of simple polytopes let us define Problem (A) as

\[
\max \# \mathcal{W} \quad \text{subject to} \quad \mathcal{W} \text{ facoidal system of walks in } G
\]

and Problem (B) as

\[
\min H_2 (\mathcal{O}) \quad \text{subject to} \quad \mathcal{O} \text{ acyclic orientation of } G.
\]

A third consequence of Theorem 4 is the following result.

**Corollary 4.** The Problems (A) and (B) form a pair of strongly dual combinatorial optimization problems. The optimal solution of Problem (A) yields the 2-faces of the respective polytope (and thus its vertex-facet incidences, see Proposition 1). Every optimal solution to Problem (B) is an AOF-orientation of the graph.
Thus, the answer to Perles’ original question whether the vertex-facet incidences of a simple polytope are at all determined by its graph, is not only “yes” (as proved by Blind and Mani), or “yes, and they can be computed” (as shown by Kalai), but at least “yes, and they can be computed by solving a combinatorial optimization problem that has a well-stated strongly dual problem.”

6 Remarks

Corollary 4 suggests to design a primal-dual algorithm for the problem of reconstructing (the vertex-facet incidences of) a simple polytope from its graph. Such an algorithm would start by computing an arbitrary facoidal system $\mathcal{W}$ of walks in the given graph (see Proposition 3) and any acyclic orientation $\mathcal{O}$. Then it would check for $\# \mathcal{W} = \mathcal{H}_2(\mathcal{O})$. If equality holds then by Theorem 4 one is done. Otherwise, the algorithm would try to improve either $\mathcal{W}$ or $\mathcal{O}$ by exploiting the reasons for $\# \mathcal{W} \neq \mathcal{H}_2(\mathcal{O})$. For a concise treatment of different classical and recent applications of the primal-dual method in Combinatorial Optimization see [8].

Such a (polynomial time) primal-dual algorithm would in particular yield polynomial time algorithms for the problem to determine an (arbitrary) AOF-orientation from the graph $G_P$ of a simple polytope $P$ and the set of 2-faces of $P$, as well as for the problem to determine the 2-faces of $P$ from $G_P$ and an AOF-orientation. As for the first of these two problems it is worth to mention that no polynomial time method is known that would find any AOF-orientation even if the input is the entire face lattice of $P$. For the second problem no polynomial time algorithm is known as well.

Let $(C)$ be the problem to decide for the graph $G_P$ of a simple polytope $P$ and a set $\mathcal{C}$ of subsets of nodes of $G_P$ if $\mathcal{C}$ is the set of the subsets of nodes of $G_P$ that correspond to the 2-faces of $P$. Let $(D)$ be the problem to decide for the graph $G_P$ of a simple polytope $P$ and an orientation $\mathcal{O}$ of $G_P$ if $\mathcal{O}$ is an AOF-orientation. The good characterizations in Corollary 3 may tempt one to conjecture that these two problems can be solved in polynomial time.

Unfortunately, from the complexity theoretic point of view, Corollary 3 does not provide us with any evidence for that. In particular, it does not imply that problems $(C)$ and $(D)$ are contained in $NP \cap coNP$. The reason is that the problem $(G)$ to decide for a given graph if there is any simple polytope $P$ such that $G$ is isomorphic to the graph of $P$ is neither known to be in $NP$ nor in $coNP$. Corollary 3 only shows that both problems $(C)$ and $(D)$ are in $NP \cap coNP$, if one restricts them to any class of graphs for which problem $(G)$ is in $NP \cap coNP$.

The problem $(S)$ to decide for a given lattice $L$ if there is a simple polytope $P$ such that the face lattice $L_P$ of $P$ is isomorphic to $L$ is known as the Steinitz problem for simple polytopes.

**Corollary 5.** If problem $(G)$ is contained in $NP$ or $coNP$, then problem $(S)$ is contained in $NP$ or $coNP$, respectively.
**Proof.** The face lattice $L_P$ of a polytope $P$ is ranked. The graph $G$ having the rank one elements of $L_P$ as its nodes, where two rank one elements are adjacent if and only if they are below a common rank two element, is isomorphic to the graph of $P$. It can be computed from $L_P$ in polynomial time. Corollary 1 shows that one can compute the vertex-facet incidences (and thus the entire face lattice $L_P$, see the first paragraph of the introduction) of a simple polytope $P$ in polynomial time (in the size of $L_P$) from the poset that is induced by the elements of rank one (corresponding to the vertices), rank two (corresponding to the 1-faces), and rank three (corresponding to the 2-faces). Together with the first part of Corollary 3 this proves the claim.

Extending results of Mnëv by using techniques described in one finds (see [1, Cor. 9.5.11]) that there is a polynomial (Karp-)reduction of the problem to decide whether a system of linear inequalities has an integral solution to problem (S). Thus, problem (S) (and therefore, by Corollary 3, problem (G)) is not contained in $\text{coNP}$, unless $NP = \text{coNP}$. Furthermore, there are rational simple polytopes $P$ with the property that every rational simple polytope $Q$ whose graph is isomorphic to the graph of $P$ has vertices with super-polynomial coding lengths in the size of the graphs (this follows from Theorem B in [9]). Thus, it seems also unlikely that problem (G) is contained in $NP$.

The results presented in Sect. 4 hence do neither lead to efficient algorithms nor to new examples of problems in $NP \cap \text{coNP}$ not (yet) known to be solvable in polynomial time. Nevertheless, they show that the problem to reconstruct a simple polytope from its graph can be modeled as a combinatorial optimization problem with a strongly dual problem. We hope that this is an appearance of Combinatorial Optimization Jack Edmonds is pleased to see in this volume dedicated to him.

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