Orbits in lattices

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Abstract

Let $O(L)$ be the orthogonal group of a lattice $L$. We exhibit algorithms for calculating Tits’ buildings and orbits of vectors in $L$ for certain subgroups of $O(L)$. We discuss how these algorithms can be applied to understand the configuration of boundary components in the Baily-Borel compactification of orthogonal modular varieties and to improve the performance of computer arithmetic of orthogonal modular forms.

1 Introduction

We exhibit algorithms for calculating Tits’ buildings and orbits of vectors in a lattice $L$ for certain subgroups of the orthogonal group $O(L)$. Our motivations are twofold. In one direction, we show how our results can be applied to understand the configuration of boundary components in the Baily-Borel compactification of orthogonal modular varieties. Such questions are natural problems in algebraic geometry [Daw, HKW93, Sca87, Ste91], but are typically addressed via ad-hoc methods. We believe our results are the first broadly applicable results for orthogonal modular varieties. In another direction, we show how our results can be applied to improve the performance of computer arithmetic of orthogonal modular forms, which can be useful in a variety of situations (e.g. calculating Fourier expansions of Borcherds products [GKR13]). While theoretical frameworks for fast multiplication have been discussed in the literature (e.g. [Rau11]), practical implementations are only known for certain special cases (e.g. for Hermitian modular modular forms using ideas based on Minkowski reduction [Der04]) which do not generalise to the orthogonal group.

Our main results are Algorithms 2.1, 2.2 and 2.3 for finding orbits of vectors in $L$ for certain subgroups of $O(L)$ and Algorithms 3.2 and 3.4 for calculating Tits’ building of subgroups of $O(L)$ when $L$ is of signature $(2, n)$. We have produced software [Daw22] for calculating certain Tits’ buildings using these algorithms: example calculations can be found in §4. While our results are mostly intended for use with computer algebra packages such as [GAP21, S+22, BCP97], we have interspersed a number of examples to show the algorithms can sometimes be used for manual calculation.

1.1 Lattices

Unless otherwise stated, a lattice $L$ is an even, non-degenerate integral quadratic form on a free abelian group of finite rank [Nik80, CS99]. We denote the bilinear form of $L$ by $(-, -)$ and, for a given $\mathbb{Z}$-basis $\{x_i\}$ of $L$, we let $G(L) := ((x_i, x_j))_{i,j}$ denote the associated Gram matrix. Examples of lattices include root lattices such as $A_n$ (we will assume all roots lattices are negative definite); the rank 1 lattice $\langle d \rangle$ generated by a single element $x$ of squared length $x^2 := (x, x) = d$; and the hyperbolic plane $U$, whose Gram matrix is given by

$$G(U) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ (1)
for a suitable basis (a canonical basis for $U$). We use $L_1 \oplus L_2$ to denote the orthogonal direct sum of lattices $L_1$ and $L_2$, $mL$ to denote the orthogonal direct sum of $m$ copies of $L$ and $L(m)$ to denote the lattice obtained by multiplying the quadratic form of $L$ by $m$. The signature of $L$ is the pair $(t_+, t_-)$ consisting of the number of positive $t_+$ and negative $t_-$ squares in a diagonalisation of $G(L)$. A lattice $L$ is (positive) definite if $t_+ = 0$, (negative) definite if $t_- = 0$ and indefinite otherwise; $L$ is isotropic if there exists $0 \neq x \in L$ such that $x^2 = 0$; a sublattice $S \subset L$ is totally isotropic if the quadratic form of $L$ restricts to zero on $S$; and a vector $0 \neq x \in L$ is isotropic if $x^2 = 0$. A sublattice $S \subset L$ is primitive if $L/S$ is torsion-free and a vector $0 \neq x \in L$ is primitive if $\langle x \rangle \subset L$ is primitive. The dual lattice $L^\vee$ of $L$ is the lattice defined on the group $\text{Hom}(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}$ by the quadratic form of $L \otimes \mathbb{Q}$. For $x \in L$, we let $\hat{x} := \frac{1}{2}xG(L)$. If $0 \neq x \in L$, we let $\text{div}(x)$ (the divisor of $x$) denote the positive generator of the ideal $(x, L)$ and define $x^* := x/\text{div}(x) \in L^\vee$. We will use $\text{Iso}(L_1, L_2)$ to denote the set of isomorphisms between lattices $L_1$ and $L_2$ compatible with the quadratic forms of $L_1$ and $L_2$. A lattice $L'$ is said to belong to the same genus as $L$ if the quadratic forms $L' \otimes \mathbb{Z}_p \cong L \otimes \mathbb{Z}_p$ for all primes $p$ and $L' \otimes \mathbb{R} \cong L \otimes \mathbb{R}$. If $\text{Iso}(L, L')$ is non-trivial then $L$ and $L'$ are said to belong to the same class. We let $\text{gen}(L)$ denote the set of classes in the same genus as $L$. The (real) Witt index of $L$ is defined as the maximal dimension of a totally isotropic subspace of $L \otimes \mathbb{R}$ and, for prime $p$, the $p$-rank $\text{rank}_p(L)$ of $L$ is the maximal rank of all sublattices $S \subset L$ such that $\text{det}(S)$ is coprime to $p$.

1.2 Discriminant forms

The discriminant group of a lattice $L$ is defined as the abelian group $D(L) := L^\vee/L$ [Nik80]. There is a natural $\mathbb{Q}/2\mathbb{Z}$-valued finite quadratic form $q_L$ on $D(L)$ inherited from $L$ (the discriminant form of $L$) [Nik80]. It is known that the genus of $L$ is uniquely determined by the data $(t_+, t_-, q_L)$ [Nik80]. For lattices $L_1$ and $L_2$, we let $\text{Iso}(D(L_1), D(L_2))$ denote the set of group isomorphisms from $D(L_1)$ to $D(L_2)$, we define

$$\text{Iso}(q_{L_1}, q_{L_2}) := \{g \in \text{Iso}(D(L_1), D(L_2)) \mid q_{L_2} \circ g \equiv q_{L_1} \mod 2\mathbb{Z}\}$$

and use $O(D(L))$ to denote $\text{Iso}(q_{L_1}, q_{L_2})$. We use $[d_1, \ldots, d_{\min(m,n)}]_{m,n}$ to denote the $m \times n$ matrix whose $(i, i)$-th entry is $d_i$ and all other entries are zero. We use $(y_1 | \ldots | y_m)$ to denote the matrix whose $i$-th column is given by $y_i$. For $A \in M_{m,n}(\mathbb{Z})$, there exist $P(A) \in M_{m,m}(\mathbb{Z})$ and $Q(A) \in M_{n,n}(\mathbb{Z})$ such that

$$P(A)AQ(A) = [d_1, \ldots, d_{\min(m,n)}]_{m,n}$$

(2)

(the Smith normal form of $A$) where, for some $r \in \mathbb{N}$, $d_i \neq 0$ for $i \leq r$, $d_i = 0$ for $i > r$ and $d_i/d_{i+1}$ for $i = 1, \ldots, r - 1$ [New72]. For a fixed basis of $L$, the Gram matrix $G(L)$ defines an inclusion $G(L) : L \hookrightarrow L^\vee$, allowing one to calculate the structure of $D(L)$ and a set of generators, using (2).

1.3 The orthogonal group

Let $O(L)$ and $O(L \otimes \mathbb{F})$ denote the orthogonal groups of $L$ and $L \otimes \mathbb{F}$, where $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Every $g \in O(L \otimes \mathbb{R})$ can be written as a product of reflections

$$g = \sigma_{w_1} \ldots \sigma_{w_m}$$

(3)

where $\sigma_w \in O(L \otimes \mathbb{R})$ is defined by

$$\sigma_w : x \mapsto x - \frac{2(x,w)}{(w,w)}x \in O(L \otimes \mathbb{R})$$

for $w \in L \otimes \mathbb{R}$ [Cas78]. The spinor norm $\text{sn}_\mathbb{R}(g)$ of $g$ as in (3) is defined by [Kne02]

$$\text{sn}_\mathbb{R}(g) = \frac{-(w_1, w_1)}{2} \ldots \frac{-(w_m, w_m)}{2} \in \mathbb{R}/(\mathbb{R}^*)^2.$$  

(4)
We use $O^+(L \otimes \mathbb{R})$ to denote the kernel of $\text{sgn}(g)$ in $O(L \otimes \mathbb{R})$ and, for $\Gamma \subset O(L \otimes \mathbb{R})$, we let $\Gamma^+ := \Gamma \cap O^+(L \otimes \mathbb{R})$. There is a natural map

$$O(L) \to O(D(L)).$$

We use $\mathcal{O}$ to denote the image of $g \in O(L)$ under (5) and $\widetilde{O}(L)$ to denote the kernel of $\mathcal{O}$ (the stable orthogonal group). More generally, for $\Gamma \subset O(L)$ and $\mathcal{A} \subset O(D(L))$, we let $\widetilde{\Gamma} := \Gamma \cap \widetilde{O}(L)$ and $\Gamma_A := \{g \in \Gamma \mid \mathcal{O} \in \mathcal{A}\}$.

**Lemma 1.1** ([GHS13], Lemma 7.1). If $S \subset L$ is an inclusion of lattices then

$$\widetilde{O}(S) \subset \widetilde{O}(L)$$

where the extension of $g \in \widetilde{O}(S)$ to $\widetilde{O}(L)$ is defined by allowing $g$ to act as the identity on $S^\perp \subset L$.

### 1.4 Modular forms

For a lattice $L$ of signature $(2, n)$ we let $\Omega_L$ denote the Hermitian symmetric space

$$\Omega_L = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0, (x, \overline{x}) > 0\},$$

where $\overline{x}$ denotes the complex conjugate of $x$. We use $\mathcal{D}_L$ to denote the component of $\Omega_L$ fixed by $O^+(L)$ and $\mathcal{D}_L^*$ to denote the affine cone of $\mathcal{D}_L$.

**Definition 1.2** ([GHS13]). For a subgroup $\Gamma \subset O^+(L)$ of finite index, a modular form of weight $k$ and character $\chi : \Gamma \to \mathbb{C}^*$ for $\Gamma$ is a holomorphic function $F : \mathcal{D}_L^* \to \mathbb{C}$ such that $F(tZ) = t^{-k}F(Z)$ and $F(gZ) = \chi(g)F(Z)$ for all $Z \in \mathcal{D}_L^*$ and all $t \in \mathbb{C}^*$. We use $M_k(\Gamma, \chi)$ to denote the space of weight $k$ modular forms with character $\chi$ and let $M_k(\Gamma) := \bigoplus M_k(\Gamma, \chi)$.

### 1.5 Fourier expansions

To define Fourier expansions of modular forms, we begin by exhibiting $\mathcal{D}_L$ as a tube domain, following [GHS13]. A primitive isotropic vector $c \in L$ identifies $\mathcal{D}_L$ with the affine quadric

$$\mathcal{D}_{L,c} = \{Z \in \mathcal{D}_L^* \mid (Z, c) = 1\} \cong \mathcal{D}_L.$$

If $b \in L^\perp$ satisfies $(c, b) = 1$ then the signature $(1, n - 1)$ lattice $L_c := c^\perp/(c)$ can be realised as a sublattice of $L$ by

$$L_c \cong L_{c,b} = L \cap c^\perp \cap b^\perp$$

and we define the cone

$$C(L_c) := \{x \in L_c \otimes \mathbb{R} \mid (x, x) > 0\}.$$ 

The two components of $C(L_c)$ are interchanged by elements of negative spinor norm in $O(L_c \otimes \mathbb{R})$: we let $C^+(L_c)$ denote the component preserved by $O^+(L_c \otimes \mathbb{R})$. The tube domain $\mathcal{H}_{L,c}$ is given by

$$\mathcal{H}_{L,c} = L_c \otimes \mathbb{R} + iC^+(L_c)$$

and we define the isomorphism $\mathcal{H}_{L,c} \sim \mathcal{D}_{L,c} \cong \mathcal{D}_L$ by

$$z \mapsto [z] = z \oplus \left( b - \frac{(z, z) + (b, b)}{2}c \right).$$

If $\overline{C}^+(L_c^\perp)$ denotes the closure of $C^+(L_c^\perp)$ then each $F_j \in M_k(\Gamma, \chi)$ admits a Fourier expansion

$$F_j(Z) = \sum_{l \in \overline{C}^+(L_c^\perp)} f_j(l)e^{2\pi i(Z, l)}.$$
where $Z \in \mathcal{H}_{L,c}$ [GHS13].

One often wishes to calculate the Fourier expansion of $F_1F_2$ to precision $O(e^{2\pi i(p,Z)})$ for some $p \in L_c^\vee$. As the cones $\mathbb{C}^\dagger(L_c^\vee)$ can be large, term-by-term multiplication can be very expensive. However, as

$$F_1F_2 = \sum_{l_3 \in \mathbb{C}^\dagger(L_c^\vee)} \left( \sum_{l_1,l_2 \in \mathbb{C}^\dagger(L_c^\vee)} f_1(l_1)f_2(l_2) \right) e^{2\pi i(l_3,Z)} \in M_4(\Gamma, \chi_1 \otimes \chi_2),$$

then, as explained in [Rau11], one only needs to perform the inner multiplication for representatives of the orbits of $\mathbb{C}^\dagger(L_c^\vee)$ for a subgroup

$$\Theta \subset \text{stab}_\Gamma(C^\dagger(L_c^\vee)) \cap \text{Ker}(\chi_1 \otimes \chi_2). \quad (6)$$

One can often take $\Theta$ to be a group of the form $O^\dagger(L)$ or $SO^\dagger(L_c)$. Indeed, under the mild conditions of Theorem [L3] and Lemma [L4] this is always the case.

**Theorem 1.3** (GHS09). Let $L$ be a lattice.  

1. Suppose $L$ represents $-2$, has Witt index $\geq 2$ over $\mathbb{R}$, $\text{rank}_3(L) \geq 5$ and $\text{rank}_2(L) \geq 6$ (the Kneser conditions). If $L$ contains a single orbit of $-2$ vectors under $SO^\dagger(L)$ then $\text{Hom}(SO^\dagger(L), \mathbb{C}^*)$ is trivial.

2. If $L$ contains at least two copies of the hyperbolic plane and both $\text{rank}_2(L) \geq 6$, $\text{rank}_3(L) \geq 5$ then $\text{Hom}(SO^\dagger(L), \mathbb{C}^*)$ is trivial.

**Lemma 1.4.** Let $\Gamma \subset O(L)$ where $L$ is a lattice of signature $(2,n)$. If $\widetilde{SO}^\dagger(L) \subset \Gamma$ and $\text{Hom}(SO^\dagger(L), \mathbb{C}^*)$ is trivial then one can take $\Theta = \widetilde{SO}^\dagger(L_c)$ in (6).

**Proof.** By Lemma [L1], $\widetilde{SO}^\dagger(L_c) \subset \widetilde{SO}^\dagger(L)$ and the result follows as $\widetilde{SO}^\dagger(L_c) \subset \text{Stab}_\Gamma(\mathbb{C}^\dagger(L_c))$. \hfill \Box

### 1.6 The Baily-Borel compactification

If $L$ is a lattice of signature $(2,n)$ and $\Gamma \subset O^\dagger(L)$ is a subgroup of finite index, the quotient

$$\mathcal{F}_L(\Gamma) = \mathcal{D}_L/\Gamma$$

is known as an **orthogonal modular variety**. Orthogonal modular varieties are quasi-projective [BB66] but, in general, are non-compact. The simplest compactification of $\mathcal{F}_L(\Gamma)$ is the **Baily-Borel compactification** $\mathcal{F}_L(\Gamma)^*$, which can be defined as $\text{Proj} M_\ast(\Gamma, \mathcal{I})$ [BB66].

**Theorem 1.5** (GHS13). The Baily-Borel compactification $\mathcal{F}_L(\Gamma)^*$ decomposes as

$$\mathcal{F}_L(\Gamma)^* = \mathcal{F}_L(\Gamma) \sqcup \bigcup_{\Pi} C_{\Pi} \sqcup \bigcup_{\ell} Q_{\ell}$$

where each $C_{\Pi}$ is a modular curve, each $Q_{\ell}$ is a point and the indices $\Pi$ and $\ell$ are taken over representatives of $\Gamma$-orbits of totally isotropic planes and isotropic lines in $L \otimes \mathbb{Q}$, respectively. A point $Q_{\ell}$ is contained in the closure of $C_{\Pi}$ if and only if representatives can be chosen such that $\ell \subset \Pi$.

In §3.4 we explain how to calculate the boundary configuration of $\mathcal{F}_L(\Gamma)^*$ using Algorithms 3.1, 3.2 and 3.4.
2 Orbits of vectors

For a lattice $L$, let $\Gamma \subset O(L)$ be a subgroup and let $v_1, v_2 \in L^\vee \subset L \otimes \mathbb{Q}$. If there exists $g \in \Gamma \subset O(L)$ such that $gv_1 = v_2$ we say that $v_1$ and $v_2$ are equivalent under $\Gamma$, which we denote by $v_1 \sim_\Gamma v_2$; otherwise, $v_1$ and $v_2$ are said to be inequivalent under $\Gamma$, which we denote by $v_1 \not\sim_\Gamma v_2$. If $v_1$ and $v_2$ are non-isotropic, these relations can be calculated using Algorithms 2.1, 2.2 and 2.3. We will treat the isotropic case separately in §3. In order to prove Algorithm 2.1, 2.2 and 2.3 we shall need Lemma 2.1, which is contained but not proved in [Nik80].

**Lemma 2.1.** For $i = 1, 2$, let $S_i \subset L$ be a sublattice and let $K_i := S_i^\perp \subset L$. The inclusion

$$S_i \oplus K_i \subset L \subset L^\vee \subset S_i^\vee \oplus K_i^\vee$$

(7)

defines subgroups

$$H_i = L/(S_i \oplus K_i) \subset D(S_i) \oplus D(K_i)$$

and natural homomorphisms $p_{S_i}$ : $H_i \to D(S_i)$ and $p_{K_i}$ : $H_i \to D(K_i)$. We let $H_{S_i}$ and $H_{K_i}$ denote the respective images of $p_{S_i}$ and $p_{K_i}$, and define $\gamma_i = p_{K_i} \circ p_{S_i}^{-1}$ : $H_{S_i} \to H_{K_i}$. Then,

1. the homomorphisms $p_{S_i}$ and $p_{K_i}$ are monomorphisms if and only if both $S_i \subset L$ and $K_i \subset L$ are primitive.

2. If $\varphi : S_1 \cong S_2$ then $\varphi$ extends to an element of $g \in O(L)$ if and only if there exists $\psi : K_1 \to K_2$ such that $\psi \circ \gamma_1 = \gamma_2 \circ \varphi$. The map $g$ is given by the natural extension of $\varphi \circ \psi$ from $S \oplus K$ to $L$.

3. Let $\iota_i : D(L) \hookrightarrow (S_i^\vee \oplus K_i^\vee)/L$ be the natural map defined by (7). If $g \in O(L)$ is the extension of $\varphi \oplus \psi$, then $g \in O_{\mathcal{A}}(L)$ if and only if $\iota_1^{-1} \circ (\varphi \circ \psi) \circ \iota_1 \in \mathcal{A}$, where $\mathcal{A} \subset O(D(L))$.

**Proof.** 1. Let $S := S_i$ and $K := K_i$. For $x \in L$, let $s \in S^\vee$ and $k \in K^\vee$ be such that $x = s + k$. Suppose $S \subset L$ is primitive and $0 \not\equiv x \mod S \oplus K$. As $S$ is primitive, if $p_S(x) = 0$ then $s \in S$ and $k \in L \cap K^\vee = K$ (as $K = S^\perp$ is primitive). Hence, from the contradiction $0 \equiv x \mod S \oplus K$, $p_S$ is a monomorphism. Now suppose $p_S$ and $p_K$ are monomorphisms. If $x \mod S$ is torsion then $k = 0$ and as $p_S$ is a monomorphism, the contradiction $0 \equiv p_S(x) \equiv x \mod S$ implies $S$ is primitive. The argument for $K$ is identical.

2. The map $\varphi \oplus \psi$ extends to $g \in O(L)$ if and only if it preserves $H$ which, by definition of $\gamma_1$ and $\gamma_2$, is equivalent to $\psi \circ \gamma_1 = \gamma_2 \circ \varphi$.

3. Immediate from definition.

\[\square\]

2.1 Orbits of non-isotropic vectors

**Algorithm 2.1.** For a lattice $L$ of rank $n$, let $\Gamma \subset O(L)$ be a subgroup and let $v_1, v_2 \in L \otimes \mathbb{Q}$. If $v_1$ is non-isotropic and $v_1^\perp$ is definite then one can determine if $v_1 \sim_\Gamma v_2$ by proceeding as follows.

1. For $i \in \{1, 2\}$, let $c_i \in \mathbb{Q}_{>0}$ be minimal such that $c_i v_i \in L$.

2. If $\nu_1^2 \neq \nu_2^2$ or $c_1 \neq c_2$ return $v_1 \not\sim_\Gamma v_2$.

3. For $i \in \{1, 2\}$

   (a) Let $w_i := c_i v_i$.

   (b) Let $(q_1|\ldots|q_n) := Q(w_i)$.
(c) Let $K_i = \langle k_{ij} \mid j = 1, \ldots, n - 1 \rangle$ where $k_{ij} = q_{j+1}$.
(d) Let $\iota_i := (w_i|k_{i1}| \ldots |k_{i(n-1)})$.

4. Let $\varphi$ be the map $w_1 \mapsto w_2$.

5. For $\psi \in \text{Iso}(K_1, K_2)$
   (a) Let $\theta := \iota_2 \circ (\varphi \oplus \psi) \circ \iota_1^{-1}$
   (b) If $\theta \in \Gamma$ return $v_1 \sim_{\Gamma} v_2$.
   (c) Let $\iota_i := (w_i|k_{i1}| \ldots |k_{i(n-1)})$.

6. Return $v_1 \not\sim_{\Gamma} v_2$.

Proof. As $v_i^2$ and $c_i$ are invariant under $O(L)$, \textbf{1} and \textbf{2} serve as preliminary tests. In \textbf{3} we calculate $K_i := w_i^1 \subset L$. By the Smith normal form (as $d_2 = \ldots = d_n = 0$), $K_i \subset L$ is primitive and as $K_i \perp w_i$ then $K_i = w_i^1 \subset L$. In \textbf{4} we define an embedding $\iota_i$ of $\langle w_i \rangle \oplus K_i \subset L$. By Lemma 2.1 $w_1 \sim_{\Gamma} w_2$ if and only if there exists $\varphi := g|_{\langle w_1 \rangle} : w_1 \mapsto w_2$ and $\psi := g|_{K_1} : K_1 \rightarrow K_2$ extending to $\Gamma$. We search for suitable $\psi$ in \textbf{5}.

As $K_i$ is definite, $\text{Iso}(K_1, K_2)$ in \textbf{5} can be calculated efficiently using the fast isomorphism testing of \cite{PP85,PS97}. In \textbf{5(b)}, one has to verify whether $\theta \in \Gamma$. If $\Gamma = \text{SO}_A(L)$ or $\text{O}_A(L)$ one can simply check if $\theta \in \text{GL}(N, \mathbb{Z})$ and $\overline{\theta} \in \mathcal{A}$. However, if $\Gamma = \text{SO}_A^+(L)$ or $\text{O}_A^+(L)$ one also has to verify a spinor norm condition. Effective methods exist (e.g. \cite{Cas78} p.18-20) for decomposing $\psi$ into a product of reflections, from which one can calculate $\text{sn}_\mathbb{R}(\theta)$ using \cite{4}; however, in the case of Lorenzian $L$, we demonstrate an alternative approach in Example 2.2.

Example 2.2. Let $L = U \oplus A_3$ where

$$G(U) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad G(A_3) = -\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

and suppose $v_1, v_2 \in L \otimes \mathbb{Q}$ are given by $v_1 = (4, 4, 1, 2, -1)$ and $v_2 = (36, 144, 5, -30, 83)$. If $\Gamma = \overline{O}^+(L)$ then $v_1 \sim_{\Gamma} v_2$.

Proof. We apply Algorithm 2.1. In \textbf{1} - \textbf{3} we have $c_1 = c_2 = 1$, $v_1^2 = v_2^2 = 20$, $w_1 = v_1$, $w_2 = v_2$, $\overline{\theta}$ is given by

$$Q(\bar{w}_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q(\bar{w}_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -5 & 180 & 45 & 25 & 34 \\ 1 & -36 & -9 & -5 & -7 \end{pmatrix},$$

$$\iota_1 = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \iota_2 = \begin{pmatrix} 36 & 1 & 0 & 0 & 0 \\ 144 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 \\ -30 & 180 & 45 & 25 & 34 \\ 83 & -36 & -9 & -5 & -7 \end{pmatrix}.$$

The Gram matrices of $K_1$ and $K_2$ are given by

$$G(K_1) = \begin{pmatrix} -2 & -1 & 1 & -1 \\ -1 & 2 & 1 & -1 \\ 1 & 1 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{pmatrix} \quad \text{and} \quad G(K_2) = \begin{pmatrix} -54432 & -13607 & -7740 & -10260 \\ -13607 & -3402 & -1935 & -2565 \\ -7740 & -1935 & -1102 & -1459 \\ -10260 & -2565 & -1459 & -1934 \end{pmatrix}.$$
In \cite{Friedlander83}, we search over $\text{Iso}(K_1, K_2)$ (e.g. using \cite{Friedlander85, Piatetski-Shapiro85, Piatetski-Shapiro97}) for $\theta = \nu_2 \circ (\varphi \oplus \psi) \circ \iota_1^{-1} \in \Gamma$. We find $\theta \in O(L)$, given by

$$\theta = \begin{pmatrix} 11 & 5 & -11 & -13 & -9 \\ 43 & 21 & -46 & -51 & -36 \\ 1 & 1 & -1 & -2 & -2 \\ -9 & -5 & 10 & 12 & 8 \\ 25 & 12 & -26 & -30 & -21 \end{pmatrix} \quad \text{for} \quad \psi = \begin{pmatrix} -2 & -8 & 2 & -9 \\ -8 & -30 & 5 & -36 \\ -1 & -1 & 1 & -2 \\ 22 & 83 & -18 & 97 \end{pmatrix}.$$

The vector $w = \frac{1}{3}(0, 0, 3, -2, 1) \in L^\vee \subset L \otimes \mathbb{Q}$ represents a generator for the cyclic group $D(L)$. Therefore, to determine $\theta \in \hat{O}_1^+(L)$, we verify $w \equiv \theta w \mod L$. As $L$ is Lorentzian, $g \in O^+(L)$ if and only if $g(C(L)^+) = C(L)^+$. The quadratic form of $L$ is diagonalised to $\text{diag}(-2, -1, 1, -\sqrt{2} - 2, \sqrt{2} - 2)$ by

$$P = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \sqrt{2} & -\sqrt{2} \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and so, in new coordinates given by $P$,

$$C(L)^+ = \{(x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 \mid x_0 > 0 \text{ and } \omega_0 x_0^2 > \sum_{i=1}^4 \omega_i x_i^2\}.$$

As $x = (1, 0, 0, 0, 0) \in C(L)^+$ and $(P^{-1} \theta P)x = x = (3, 4, 6, -\frac{1}{2} \sqrt{2} + 1, \frac{1}{2} \sqrt{2} + 1)$ then $\theta$ preserves $C(L)^+$, implying $v_1 \sim_\Gamma v_2$. \hfill \Box

Algorithm 2.2. If $L$ is a lattice of rank $n$, let $v_1, v_2 \in L \otimes \mathbb{Q}$ and $\Gamma := O_A(L)$ where $A \subset O(D(L))$ is subgroup. If, simultaneously, $v_1$ is non-isotropic, $v_1^\perp$ is indefinite, the natural map $O(L) \to O(D(L))$ is surjective, then one can determine if $v_1 \sim_\Gamma v_2$ as follows.

1. For $i \in \{1, 2\}$
   (a) Let $c_i \in \mathbb{Q}_{>0}$ be minimal such that $w_i := c_i v_i \in L$.
   (b) If $c_1 \neq c_2$ or $v_1^2 \neq v_2^2$ then return $v_1 \not\sim_\Gamma v_2$.
   (c) Let $K_i := w_i^\perp \subset L$.
   (d) For the natural inclusion
   \[
   \langle w_i \rangle \oplus K_i \subset L \subset L^\vee \subset \langle w_i \rangle^\vee \oplus K_i^\vee,
   \]
   let $H_i := L / \langle w_i \rangle \oplus K_i \subset D(\langle w_i \rangle) \oplus D(K_i)$.
   (e) Let $\iota_i$ be the map $D(L) \xrightarrow{\sim} D(\langle w_i \rangle) \oplus D(K_i) \mod H_i$.

2. If $K_1 \not\cong K_2$ then return $v_1 \not\sim_\Gamma v_2$.

3. For $\varphi \oplus \psi \in \{\pm 1\} \oplus \text{Iso}(q_{K_1}, q_{K_2})$
   \[
   \text{If } (\varphi \oplus \psi)(H_1) = H_2 \text{ and } \iota_2^{-1} \circ (\varphi \oplus \psi) \circ \iota_1 \in A \text{ then return } v_1 \sim_\Gamma v_2.
   \]

4. Return $v_1 \not\sim_\Gamma v_2$.

Proof. Lemma 2.1 with (1a) and (1b) serving as preliminary tests. \hfill \Box
Algorithm 2.2 can be rephrased in coordinate form as follows.

**Algorithm 2.3.** If $L$ is a lattice of rank $n$, let $v_1, v_2 \in L \otimes \mathbb{Q}$ and $\Gamma := O_\mathcal{A}(L)$ where $\mathcal{A} \subset O(D(L))$ is a subgroup. If, simultaneously, $v_1$ and $v_2$ are non-isotropic, $v_1^2$ is indefinite, the natural map $O(L) \to O(D(L))$ is surjective, then one can determine if $v_1 \sim \Gamma v_2$ as follows.

1. **For** $i \in \{1, 2\}$
   
   (a) Let $c_i \in \mathbb{Q}_{>0}$ be minimal such that $w_i := c_i v_i \in L$.
   
   (b) Let $\alpha_i := w_i^2 / |w_i^2|$. 

2. **If** $v_1^2 \neq v_2^2$ **or** $c_1 \neq c_2$ **return** $v_1 \not\sim \Gamma v_2$.

3. **For** $i \in \{1, 2\}$
   
   (a) Let $(q_1 \ldots |q_n) := Q(\hat{w}_i)$.
   
   (b) Let $K_i := \langle k_{ij} | i, j = 1, \ldots, n-1 \rangle$ where $k_{ij} = q_{j+1}$.
   
   (c) Let $[d_1, \ldots, d_n]_{n,n} := P(G(K_i)) G(K_i) Q(G(K_i))$ and identify
      
      $D(K_i) \cong \bigoplus_j C_{d_j}$. 
      
      (8)
   
   (d) Let
      
      $f_{il} := \frac{1}{d_k} \sum_{j=1}^n q_{jl} k_{il}$
      
      for $l = 1, \ldots, n-1$.

4. **If** $K_1 \not\cong K_2$ **then** return $v_1 \not\sim \Gamma v_2$ **else**
   
   (a) Let $\xi_i := \tau(1, 0, \ldots, 0), \theta_{i1} := (w_{i1} | k_{i1} \ldots | k_{im}), \theta_{i2} := (w_i^2) \oplus G(K_i), \theta_{i3} := (\alpha_{i1} | f_{i1} \ldots | f_{i(n-1)})$ and $\lambda_i := \theta_{i3} \circ \theta_{i2} \circ \theta_{i1}^{-1}$.
   
   (b) Let $H_i$ be the subgroup of $D(\langle w_i \rangle) \oplus D(K_i)$ generated by the columns of $\lambda_i$ taken modulo $\tau(|w_i^2|, d_1, \ldots, d_n)$.
   
   (c) **If** $H_1 \not\cong H_2$ **then** return $v_1 \not\sim \Gamma v_2$.
   
   (d) Let $\xi_i := \lambda_i \circ G(L)^{-1}$.

5. **For** $\varphi \oplus \psi \in \{\pm 1\} \oplus \text{Iso}(q_{K_1}, q_{K_2})$
   
   If $(\varphi \oplus \psi)(H_1) = H_2$ and $\theta := \xi_{i2}^{-1} \circ (\varphi \oplus \psi) \circ \xi_1$ mod $L \in \mathcal{A}$ then
      
      return $v_1 \sim \Gamma v_2$.

6. **Return** $v_1 \not\sim \Gamma v_2$.

**Proof.** The algorithm is essentially a rephrasing of Algorithm 2.2. The term $\alpha_i$ in 1 corresponds to the Smith normal form of $\langle w_i \rangle$, which we will use in 4. In 3(a), we calculate bases for the lattices $K_i$ as in Algorithm 2.1 and in 3(b), we calculate the groups $D(K_i)$. Step 3(c) calculates representatives $f_{il}$ in $K_i^\vee \subset K_i \otimes \mathbb{Q}$ for the canonical basis of $D(K_i)$ in (8). In 4, we calculate maps

$$\iota_i : D(L) \to D(\langle w_i \rangle) \oplus D(K_i)$$

and

$$\lambda_i : L / \langle w_i \rangle \oplus K_i \to D(w_i) \oplus D(K_i)$$

defined on generators in $L^\vee$, $L$ and $D(\langle w_i \rangle) \oplus D(K_i)$ with $\lambda_i$ and $\iota_i$ as in 4(a) and 4(d), respectively. We conclude in 5 by verifying the existence of $\varphi \oplus \psi : \langle w_1 \rangle \oplus K_1 \to \langle w_2 \rangle \oplus K_2$ extending to $\Gamma$. \qed
Unlike in Algorithm 2.1 the lattices $K_i$ of Algorithm 2.2 and 2.3 are indefinite, precluding an application of the isomorphism tests of [PP85,PS97]. However, one can typically determine if $K_1 \cong K_2$ by using Theorem 2.3 (which is originally due to [Kne56] and rephrased in the language of discriminant forms in [Nik80]).

**Theorem 2.3** ([Kne56] [Nik80, Theorem 1.13.2/1.14.2]). Let $q$ be the discriminant form of a lattice. If both

1. $t_+ \geq 1$, $t_- \geq 1$ and $t_+ + t_- \geq 3$;
2. $t_+ + t_- \geq 2 + l(q)$ where $l(q)$ is the minimum number of generators for the underlying group of $q$,

then there exists a lattice $L$ of signature $(t_+, t_-)$ with $q_L = q$. Furthermore, the natural map $O(L) \to O(D(L))$ is surjective and $\text{gen}(L)$ contains a single class.

Under the conditions of Lemma 2.4, Algorithm 2.3 can be used to determine the equivalence of vectors under $SO^+_2(L)$.

**Lemma 2.4.** In the notation of Algorithm 2.3, if $K_1$ represents both $\pm 2$ then $v_1 \sim_{O_L} v_2$ if and only if $v_1 \sim_{SO^+_2(L)} v_2$.

**Proof.** Suppose $v \in K_1$ satisfies $v^2 = \pm 2$. By definition, $\text{sn}_q(\sigma_v) = \mp 1$ and, by [GHS07, Proposition 3.1], $\sigma_v \in \tilde{O}(K_1)$. If necessary, as $\text{det}(\sigma_v) = -1$, one can replace $\psi$ in Algorithm 2.3 by $\psi \circ \sigma_v$ so that $\varphi \oplus \psi$ in Algorithm 2.3 has the required determinant and spinor norm.

Theorem 2.3 and Lemma 2.5 can often be used to determine if a lattice represents $\pm 2$.

**Lemma 2.5.** Let $K$ be an indefinite lattice with discriminant form $q_K$ and signature $(t_+, t_-)$. If $S := \langle \pm 2 \rangle$, suppose $\delta$ is given by one of

1. $\delta = q_S \oplus (-q_K)$;
2. $\delta = ((q_S \oplus (-q_K)) \mid \Gamma^+_\gamma) / \Gamma_\gamma$, where $\Gamma_\gamma$ is the pushout for an inclusion of subgroups $\gamma : q_S \to q_L$ compatible with the forms $q_S$ and $q_L$.

If $K$ is unique in its genus and there exists a lattice of signature $(t_+, t_-)$ with discriminant form $-\delta$ then $S \subset K$.

**Proof.** Immediate from Proposition 1.15.1 of [Nik80] and Theorem 2.3.

**Example 2.6.** Let $L = U \oplus A_3$ and suppose $v_1, v_2 \in L \otimes \mathbb{Q}$ are given by $v_1 = (1, -1, 0, 0, 0)$ and $v_2 = (1, 0, 1, 0, 0)$ where $G(U)$ and $G(A_3)$ are as in Example 2.2. If $\Gamma = \tilde{SO}^+_2(L)$ then $v_1 \sim_{\Gamma} v_2$.

**Proof.** We apply Algorithm 2.3. As $L$ is of signature $(2, 5)$ and $v_1^2 > 0$ then $v_1^2$ is indefinite. By Theorem 2.3 $L$ is unique in its genus and $O(L) \to O(D(L))$ is surjective. For $i = 1, 2$, we have $w_i^2 = -2$, $w_i = v_i$, $c_i = 1$ and $\alpha_i = 1$. Bases for $K_i$ are given by

$$K_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
We calculate $SO_L$ if $x \in E$ and edge set $g$. We now explain how to calculate Tits’ buildings and orbits of isotropic vectors for a group $G$.

**3 Algorithms for Tits’ buildings and isotropic vectors**

We find that if $a, b \in \mathbb{Q}$ are defined by $x_1 := \frac{1}{4}(3, 3, 2, -2, 4)$, $x_2 := \frac{1}{4}(2, 2, 1, -2, 3)$, $y_1 := \frac{1}{2}(1, 0, -1, 2, -2)$ and $y_2 := \frac{1}{2}(0, 0, -1, 2, -3)$ then $D(K_1) \cong \langle x_1, x_2 \rangle$ mod $L$ and $D(K_2) \cong \langle y_1, y_2 \rangle$ mod $L$. We verify condition 5 by considering $P(G(L))\theta P(G(L))^{-1}$ where

$$P(G(L)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 \end{pmatrix}.$$  

We find that if $\varphi = (1)$ and $\psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$, then $\theta$ acts trivially on $D(L)$. Therefore, $v_1 \sim v_2$ and, by Lemma 2.2.1, $v_1 \sim v_2$.  

**3 Algorithms for Tits’ buildings and isotropic vectors**

We explain how to calculate Tits’ buildings and orbits of isotropic vectors for a group $G_1 \subset O^+(L \otimes \mathbb{Q})$ where $L$ is a lattice of signature $(2, n)$. We will assume throughout that $SO^+(L) \subset G_1$.

**Definition 3.1.** Let $L$ be a lattice of signature $(2, n)$. For a group $G \subset O^+(L \otimes \mathbb{Q})$, let $\mathcal{P}$ and $\mathcal{C}$ denote the $G$-orbits of totally isotropic subspaces of dimension 1 and 2 in $L \otimes \mathbb{Q}$, respectively. Then the **Tits’ building** $B(G) = (\mathcal{N}, \mathcal{E})$ of $G$ is the bipartite graph with node set $\mathcal{N} := \mathcal{P} \sqcup \mathcal{C}$ and edge set $\mathcal{E}$, where an edge is drawn between $[\Pi] \in \mathcal{C}$ and $[\ell] \in \mathcal{P}$ if and only if $g\ell \in \Pi$ for some $g \in G$.  

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When drawing Tits’ buildings, we adopt the convention that nodes in \( \mathcal{P} \) will be coloured black and nodes in \( \mathcal{C} \) will be coloured white. Nodes may or may not be numbered. We will typically use \( \ell \) to denote isotropic lines and \( \Pi \) to denote isotropic planes. We note that Definition 3.1 has an equivalent formulation with ‘primitive totally isotropic sublattice of \( L \)’ replacing ‘totally isotropic subspace of \( L \otimes \mathbb{Q} \)’.

**Algorithm 3.1.** Let \( G_1 \subset G_2 \subset O^+(L \otimes \mathbb{Q}) \) be an inclusion of groups such that \( |G_2 : G_1| < \infty \). Suppose \( \mathcal{B}(G_2) = (\mathcal{N}_2, \mathcal{E}_2) \) where \( \mathcal{N}_2 = \mathcal{P}_2 \sqcup \mathcal{C}_2, \mathcal{P}_2 = \{[\ell_i]\} \) and \( \mathcal{C}_2 = \{[\Pi_i]\} \). Then the Tits’ building \( \mathcal{B}(\mathcal{N}_1, \mathcal{E}_1) \) (where \( \mathcal{N}_1 = \mathcal{P}_1 \sqcup \mathcal{C}_1 \)) can be calculated from \( \mathcal{B}(G_2) \) as follows.

1. Let \( \mathcal{E}_1 := \emptyset, \mathcal{P}_1 := \emptyset, \mathcal{C}_1 := \emptyset \).
2. Let \( \mathcal{G} := \{g_i\} \) be a transversal for \( G_2/G_1 \).
3. For \( [E] \in \mathcal{N}_2 \)
   
   (a) Let \( \mathcal{H} \) be a set of representatives for \( \mathcal{G}/\sim \) where, if \( \mathcal{L}_j \) is a transversal for \( \text{Stab}_{G_2}(q_jE) \setminus \text{Stab}_{G_2}(q_jE) \), then \( g_i \sim g_j \) if and only if \( l_k g_j g_i^{-1} \in G_1 \) for \( l_k \in \mathcal{L}_j \).
   
   (b) If \( E \in \mathcal{P}_2 \) let \( \mathcal{P}_1 := \mathcal{P}_1 \cup \mathcal{H}.E \) else let \( \mathcal{C}_1 := \mathcal{C}_1 \cup \mathcal{H}.E \).
4. For \( ([\ell], [\Pi]) \in \mathcal{P}_1 \times \mathcal{C}_1 \)
   
   (a) Let \( X \) be a finite set of representatives for all lines in \( \Pi \) up to equivalence in \( \text{Stab}_{G_2}(\Pi) \).
   
   (b) Let \( \rho : \text{Stab}_{G_2}(\Pi) \to \text{GL}(\Pi) \) be the restriction homomorphism; let \( \Gamma \subset \text{Stab}_{G_1}(\Pi) \) be such that \( |\rho(\Gamma) : \text{Im}(\rho)| < \infty \); and let \( \mathcal{J} \subset \text{Im}(\rho) \) be a finite set containing representatives for all cosets \( \rho(\Gamma)\setminus\text{Im}(\rho) \).
   
   (c) Let \( Y := \mathcal{J}.X \).
   
   (d) Let \( \mathcal{U} \) be a transversal for \( \text{Stab}_{G_1}(\ell) \setminus \text{Stab}_{G_2}(\ell) \).
   
   (e) If, for some \( y \in Y \), both
      
      i. there exists \( \tau(y, \ell) \in \mathcal{G}_1 \) such that \( \tau(y, \ell)y = \ell \)
      
      ii. \( u\tau(y, \ell) \in \mathcal{G}_1 \) for some \( u \in \mathcal{U} \)
      
      then add the edge \([\ell] - [\Pi] \) to \( \mathcal{E}_1 \).
5. Return \( (\mathcal{N}_1, \mathcal{E}_1) \).

**Proof.** In \( \mathcal{B}(G_2) \) we calculate \( \mathcal{N}_1 \). The set \( \mathcal{G}.\mathcal{P}_2 \) contains representatives for all lines in \( L \otimes \mathbb{Q} \) up to \( G_1 \)-equivalence, which we refine by identifying equivalent classes. If \( [\ell_i], [\ell_j] \in \mathcal{P}_2 \) and \( \ell_i \not\sim_{G_2} \ell_j \) then \( \ell_i \not\sim_{G_1} \ell_j \) and so it suffices to establish conditions for \( gg_i \ell_i = g_j \ell_j \) where \( g_i, g_j \in \mathcal{G} \) and \( g \in G_1 \). By an elementary calculation, the condition \( g^{-1}g g_i \ell_i = \ell_i \) for some \( g \in G_1 \) is equivalent to \( G_1 \cap \text{Stab}_{G_2}(g_i \ell_i)g g_i^{-1} = \emptyset \), which is equivalent to \( l_k g_j g_i^{-1} \in G_1 \) for some \( l_k \in \text{Stab}_{G_2}(g_j \ell_i) \). The case of \( \mathcal{C}_1 \) follows by an identical argument. We calculate \( \mathcal{E}_1 \) in \( \mathcal{B}(G_1) \) containing representatives for all \( \text{Stab}_{G_1}(\Pi) \)-orbits of lines in \( \Pi \), we add the edge \([\ell] - [\Pi] \) to \( \mathcal{E}_1 \) if and only if \( \ell = gy_i \) for some \( g \in G_1 \) and \( y_i \in Y \) or, equivalently, if \( \ell = g \tau(y_i, \ell)^{-1} \ell \) for some \( \tau(y_i, \ell) \in G_2 \) where \( \tau(y_i, \ell) : y_i \mapsto \ell \). As this condition is equivalent to \( G_1 \cap \text{Stab}_{G_2}(\ell)\tau(y_i, \ell) = \emptyset \), the result follows.

**Remark**

1. One can calculate \( G_1 \)-orbits of isotropic vectors in \( L \) by performing steps 1-4 in Algorithm 3.1 with \( \mathcal{P}_2 \) replacing \( \mathcal{N}_2 \) in 3.

2. Given a series of groups \( G_1 \subset \ldots \subset G_m \subset O^+(L \otimes \mathbb{Q}) \), it is typically faster to calculate \( \mathcal{B}(G_1) \) by calculating each of the intermediate buildings \( \mathcal{B}(G_i) \) than working directly with \( G_1 \subset G_m \).
3.1 Maximal lattices

We now show that if \( G_1 \subset O_+^+(L) \) then one can always take \( G_2 = O_+^+(L') \) in Algorithm 3.3, where \( L \subset L' \) is a maximal overlattice of \( L \). In such a case, \( \mathcal{B}(G_2) \) can be described using the results of Attwell-Duval \[AD15\] [AD15] [AD17].

**Definition 3.2.** A lattice \( L' \) is said to be maximal if \( D(L') \) contains no non-trivial totally isotropic subgroup (where a subgroup \( H \) is said to be totally isotropic if \( q_L|H \equiv 0 \mod 2\mathbb{Z} \)). An overlattice \( L \subset L' \) with \( L' \) maximal is said to be a maximal overlattice.

**Lemma 3.3.** Every lattice \( L \) admits a maximal overlattice \( L \subset L' \).

**Proof.** As explained in §4 of [Nik80], overlattices \( L \subset L' \) are in bijection with isotropic subgroups \( H \subset D(L) \). We enlarge the initial set \( H := \{0\} \) by adding successive isotropic elements in \( H^\perp \) until \( H^\perp / H \) contains no non-trivial isotropic subgroup. As \( q_L \cong H^\perp / H \) [Nik80], \( L' \) is maximal.

**Definition 3.4.** A maximal lattice \( L' \) of signature \((2, n)\) is said to be split if

\[
L' \cong 2U \oplus L'_0
\]

for a lattice \( L'_0 \). We let \( \mathcal{S}(L'_0) = \{x_i\}^{n+2}_{i=1} \) denote a \( Z \)-basis of \( L' \) such that \( \{x_1, x_2\}, \{x_3, x_4\} \) are canonical bases for the two copies of \( U \) in \((9)\) and \( \{x_i\}^{n+2}_{i=3} \) is a basis for \( L'_0 \). We write \( \mathcal{S}(L) \) to denote \( \mathcal{S}(L'_0) \) for an arbitrary choice of \( L'_0 \) satisfying \((9)\).

It is known that if \( L' \) is maximal of signature \((2, n)\) then \( L' \) splits whenever \( n \geq 5 \) [AD15]. Split maximal lattices are also very common when \( n < 5 \), as can be established from [Nik80] Corollary 1.13.5.

**Assumption** From now on, we will assume that all maximal lattices are of signature \((2, n)\) and split.

As explained in [AD15] §5, a totally isotropic sublattice \( E \) of a maximal lattice \( L' \) defines a \( Z \)-basis \( \mathcal{S}(E) = \{x_i\} \) of \( L' \) where, if \( E = \langle e_1, e_2 \rangle \) then \( \{x_1, x_2\}, \{x_3, x_4\} \) are canonical bases for orthogonal copies of \( U \) and \( x_1 = e_1 \) and \( x_2 = e_2 \); if \( E = \langle e_1 \rangle \) then \( x_1 = e_1 \) [AD15]. Furthermore, the totally isotropic sublattices \( E_1, E_2 \subset L' \) belong to the same \( O_+^+(L') \)-orbit if and only if \( E_1^\perp / E_1 \cong E_2^\perp / E_2 \) [AD15]. Rephrasing the results of [AD15], we obtain Algorithm 3.2 and Figure 1.

**Algorithm 3.2.** If \( L' \) is a maximal lattice of signature \((2, n)\) with \( n \geq 2 \) then \( \mathcal{B}(O_+^+(L')) \) (illustrated in Figure 1) can be calculated as follows.

1. Let \( \mathcal{C} := \emptyset \) and let \( \mathcal{P} := \{e\} \) where \( e \in L' \) is a primitive isotropic vector.

2. Calculate \( \text{gen}(0, n-2, D(L)) \) (e.g. by [SH98]).

3. For each class \( L_0 \) in \( \text{gen}(0, n-2, D(L)) \)

   (a) Let \( \{x_i\}^{2+n}_{i=1} \) := \( \mathcal{S}(L_0) \) and let \( \Pi := \langle x_1, x_3 \rangle \).

   (b) Let \( \mathcal{C} := \mathcal{C} \cup [\Pi] \).

4. For \([\Pi] \in \mathcal{C} \) let \( \mathcal{E} := \mathcal{E} \cup \{[e] - [\Pi]\} \).

5. Return \( \mathcal{B}(O_+^+(L')) := (\mathcal{P} \cup \mathcal{C}, \mathcal{E}) \).
3.2 Generators and cosets

We now exhibit generators for $G_2$, $\text{Stab}_{G_2}(\ell)$ and $\text{Stab}_{G_2}(\Pi)$ where $G_2 = O^+(L')$ and $L'$ is a maximal lattice. These generators can be used to calculate representatives for each of the cosets in Algorithm 3.1.

**Definition 3.5 ([Eic52, GHS09]).** Let $L = 2U \oplus L_0$ be a lattice of signature $(2, n)$ and, for primitive isotropic $e \in L$, let $a \in e^\perp$. Then there exists an element $t(e,a) \in \tilde{\text{SO}}^+(L)$, known as an Eichler transvection, defined by

$$t(e,a) : v \mapsto v - (e,v)e + (e,v)a - \frac{1}{2}(a,a)(e,v),$$

where $v \in L$.

**Lemma 3.6 ([GHS09, Proposition 3.3]).** Suppose $L = U \oplus L_1$ is a lattice where $L_1 = U \oplus L_0$. If \( \{x_i\}_{i=1}^{n+2} \) is a $\mathbb{Z}$-basis for $L$ such that \( \{x_1, x_2\} \) is a canonical basis for $U$ then,

$$O^+(L) = \langle t(x_1,v), t(x_2,v), O^+(L_1) \mid v \in L_1 \rangle.$$

**Lemma 3.7.** Let $L'$ be a maximal lattice. If $\ell \subset L'$ is an isotropic line, let $\{x_i\}_{i=1}^{n+2} := S(\ell)$ and $L'_1 := \langle x_i \mid i = 3, \ldots, n \rangle$. Then

$$\text{Stab}_{O^+(L')}(\ell) = \langle t(x_1,v), O^+(L'_1) \mid v \in L'_1 \rangle.$$

**Proof.** By [AD17, p.30],

$$\text{Stab}_{O^+(L)}(\ell) \cong O^+(L_1) \ltimes U(\ell),$$

where the unipotent radical $U(\ell) \subset \text{Stab}_{O^+(L)}(\ell)$ is generated by matrices of the form

$$g(\underline{z}) = \begin{pmatrix}
1 & y_{1,2} & y_{1,3} & \cdots & y_{1,n+2} \\
0 & 1 & 0 & \cdots & 0 \\
0 & z_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & z_n & 0 & \cdots & 1
\end{pmatrix} I.$$

The terms $y_{1,i}$ in (11) satisfy

$$y_{1,2} = -\langle \underline{z}, \underline{z} \rangle \quad \text{and} \quad y_{1,i+2} = -\langle \underline{z}, x_i \rangle$$

where $\underline{z} = \sum_{i=1}^{n} z_i x_{i+2}$ and $(z_1, \ldots, z_n) \in \mathbb{Z}^n$ [AD17]. By (10), $t(x_1,a+b) = t(x_1,a)t(x_1,b)$ for all $a, b \in L_1$, implying $g(\underline{z}) = t(x_1,\underline{z})$, from which the result follows. \(\square\)
Lemma 3.8. If $\Pi \subset L'$ is a totally isotropic plane and $S(\Pi) = \{x_i\}_{i=1}^{n+2}$, let

$$L' : = \langle x_i \mid i = 5, \ldots, n \rangle.$$ 

Then

$$\text{Stab}_{O^+(L')}(\Pi) = \langle t(x_1, x_4), t(x_2, x_3), t(x_1, x_3), t(x_1, v), t(x_3, v), O^+(L') \mid v \in L'_0 \rangle.$$ 

Proof. As the Jacobi group $\Gamma^j(L')$ is the subgroup of $\text{Stab}_{O^+(L')}(\Pi)$ acting trivially on $L_0$ [GHS09, p.470], the result follows by noting that [GHS09] that

$$\Gamma^j(L') = \langle t(x_1, x_4), t(x_2, x_3), t(x_1, x_3), t(x_1, v), t(x_3, v) \mid v \in L_0 \rangle.$$ 

Remark One can often calculate generators for the groups $O^+(L_1)$ and $O^+(L_0)$ in Lemmas 3.6, 3.7 and 3.8 by using [Mer14, Vin75, FP85]. Furthermore, if $\Pi \subset L$ is a totally isotropic sublattice, one can also use Lemma 3.8 to understand the boundary curves $C_\Pi$ in Theorem 1.5. By [Bri83], there exists a homomorphism

$$\pi : \text{Stab}_I(E) \to \text{SL}(2, \mathbb{Z})$$

and, by [Scag87] [2], [Kon93] [2], $C_\Pi \cong \mathbb{H}^+ / \text{Im} \pi$.

Lemma 3.9. The inclusions

$$\begin{cases}
\tilde{O}^+(L) \subset O^+(L') \\
\text{Stab}_{O^+(L)}(\ell) \subset \text{Stab}_{O^+(L')}(\ell) \\
\text{Stab}_{O^+(L)}(\Pi) \subset \text{Stab}_{O^+(L')}(\Pi)
\end{cases}$$

are of finite index.

Proof. The following argument is used in [Kon93] for moduli spaces of K3 surfaces, where it is attributed to O'Grady (see also [Daw]). If $M$ is the exponent of the group $L'/L$ then

$$L'(M) \subset L \subset L'$$

and we let $Q : = L'/L'(M)$. If $I : = L/L'(M) \subset Q$ then $\text{Stab}_{O^+(L')}(I) \subset O^+(L)$ and, by the Orbit-Stabiliser theorem, $\text{Stab}_{O^+(L')}(I) \subset O^+(L')$, implying $|\tilde{O}^+(L) : O^+(L')| < \infty$. The other cases follow by an identical argument.

Remark By Schreier’s Lemma [Cam99, p.18], Lemma 3.6 can be used to obtain a set of generators for the subgroups occurring in Lemma 3.9.

3.3 The element $\tau(x, y)$

If $L$ is a split maximal lattice and $O^+(L) \subset G_2 \subset O^+(L \otimes \mathbb{Q})$, then the element $\tau(x, y)$ in Algorithm 3.1 can be calculated using Algorithm 3.3.

Algorithm 3.3. Let $L$ be a maximal lattice such that $L = U \oplus L_1$ where $L_1 = U \oplus L_0$. If $x, y \in L$ are primitive and isotropic then there exists $\tau(x, y) \in O^+(L)$ such that $\tau(x, y)x = y$. The element $\tau(x, y)$ can be calculated as follows.

1. Let

$$\theta : \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \to O^+(L)$$

be defined by

$$\boxed{12}$$
\[ \theta(Z, I) = \begin{pmatrix} d & 0 & c & 0 \\ 0 & a & 0 & -b \\ b & 0 & a & 0 \\ 0 & -c & 0 & d \end{pmatrix} \oplus I \quad \text{and} \quad \theta(I, Z) = \begin{pmatrix} d & 0 & 0 & -c \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ -b & 0 & 0 & a \end{pmatrix} \oplus I \]

for

\[ Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}). \]

2. Let \( \iota : L \to M_2(\mathbb{Z}) \) be defined by

\[ \iota : (x_1, x_2, \ldots) \mapsto \begin{pmatrix} x_3 & -x_2 \\ x_1 & x_4 \end{pmatrix}. \]

3. If \( w \in L_1 \) then

let \( g(w) := I \in O^+(L) \)

else, by calculating the Smith normal form, let \( (A, B) \in \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \) be such that \( \iota(w) \) is diagonal and let \( g(w) := \theta(A, B) \).

4. Let \( x' := g(x)x \) and \( y' := g(y)y \).

5. Let \( u', v' \in L_1 \) be such that \( (x', u') = (y', v') = 1 \) (which can be calculated using the Euclidean algorithm).

6. Return \( \tau(x, y) \) where

\[ \tau(x, y) := g(x)t(e, u')t(f, x' - f')t(e, -v')g(y)^{-1}. \]

Proof. The algorithm essentially follows the proof of the Eichler criterion given in [GHS09, Proposition 3.3]; we have simply made a few details explicit. The homomorphism (12) in \( \mathbb{Z} \) is well known (e.g. [Ste91]) and is obtained by taking coordinates \( (w, x, y, z) \) for \( 2U \) and identifying

\[ 2U \ni (x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_3 & -x_2 \\ x_1 & x_4 \end{pmatrix} \in M_2(\mathbb{Z}), \]

where the quadratic form on \( M_2(\mathbb{Z}) \) is given by \( 2 \det \). The action of \( (A, B) \in \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \) is then given by

\[ \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \ni (A, b) \mapsto AXB^{-1} \]

for \( X \in M_2(\mathbb{Z}) \). The elements \( u' \) and \( v' \) in \([5]\) both exist as \( x' \) and \( y' \) are primitive and \( L' \) is maximal, implying \( u'^* \) and \( v'^* \) modulo \( L \) are of order \( 1 \), hence \( \text{div}(x') = \text{div}(y') = 1. \]

\[ \square \]

3.4 The sets \( X, Y \) and the group \( \Gamma \)

We now show how to calculate the set \( X, Y \) and the group \( \Gamma \) in Algorithm 3.1.

Lemma 3.10. Let \( L \subset L' \) be a maximal overlattice and let \( \Pi = \langle e_1, e_2 \rangle \) be a totally isotropic plane. Then,

1. \( \text{SL}(2, \mathbb{Z}) \subset \text{Stab}_{O^+(L')}(\Pi) \).

2. If \( L'(M) \subset L \subset L' \) for some \( M > 0 \) and \( N \) is the exponent of the group \( L'/L'(M) \) then \( \Gamma(N) \subset \text{Stab}_{O^+(L)}(\Pi) \).
3. If \( \mathcal{H} \) is a transversal for \( \Gamma(N) \backslash \text{SL}(2, \mathbb{Z}) \) then the set \( \mathcal{H}.e_1 \) contains representatives for all \( \text{Stab}_{O^+(L)}(\Pi) \)-orbits of isotropic lines in \( \Pi \).

**Proof.** We fix a basis \( S(\Pi) \) for \( L' \). By \((12)\), the group \( \text{Stab}_{O^+(L')}(\Pi) \) contains a copy of \( \text{SL}(2, \mathbb{Z}) \), acting as

\[
\begin{pmatrix}
  d & 0 & c & 0 \\
  0 & a & 0 & -b \\
  b & 0 & a & 0 \\
  0 & -c & 0 & d \\
 0 & I
\end{pmatrix}
\]

for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \)

and, by Lemma 3.9, there exists \( M \) such that

\[ L'(M) \subset L \subset L' \subset L' \).

Therefore, \( \Gamma(N) \) acts trivially on \( L'/L'(M) \) and \( \Gamma(N) \subset \text{O}^+(L) \). The result follows by noting that \( \text{SL}(2, \mathbb{Z}) \) acts transitively on isotropic vectors in \( \Pi \).

\[ \square \]

### 3.5 Tits’ buildings of enclosing groups

If \( G_1 \subset G_2 \) then (as noted in [HKW93]) in connection with the moduli of abelian surfaces \( G_2 \) acts naturally on \( B(G_1) \). If \( B(G_1) \) is known, one can calculate \( B(G_2) \) using Algorithm 3.4. This can be helpful when working with groups of the form \( \text{O}^+_A(L) \). In such a case, one may be unable to apply Algorithm 3.1 directly with \( G_2 = \text{O}^+(L') \) for a split maximal lattice \( L' \), however as \( \text{O}^+_A(L) \subset \text{O}^+_A(L) \) and \( \text{O}^+_A(L) \subset \text{O}^+(L') \), one can calculate \( B(\text{O}^+_A(L)) \) using Algorithm 3.2 and 3.3.

**Algorithm 3.4.** Let \( G_1 \subset G_2 \subset \text{O}^+(L) \) and suppose \( G_1 \subset \text{O}^+(L') \) where \( L \subset L' \) is an overlattice and both \( |\text{O}^+(L) : G_1| < \infty \) and \( |\text{O}^+(L') : G_1| < \infty \). Let \( B(G_1) = (N, E) \) where \( N = \mathcal{P} \cup \mathcal{C} \) and let \([E]\) denote the \( G_1 \)-equivalence class of an isotropic subspace \( E \subset L \otimes \mathbb{Q} = L' \otimes \mathbb{Q} \). Then \( B(G_2) \) can be calculated from \( B(G_1) \) by identifying nodes as follows.

1. **Let** \( \mathcal{H} \) **be a transversal of** \( G_1 \backslash G_2 \).

2. **For** \( \mathcal{I} \in \{ \mathcal{P}, \mathcal{C} \} \)

   *For** \( h \in \mathcal{H} \) **and** \([E_1], [E_2] \) \( \in \mathcal{I} \times \mathcal{I} \)

   **If** there exists \( \hat{g} \in \text{O}^+(L') \) **such that** \( \hat{g}(hE_1) = E_2 \) **then**

   **Let** \( \mathcal{J} \) **be a transversal for** \( \text{Stab}_{\text{O}^+(L')}(hE_1) \backslash \text{Stab}_{G_1}(hE_1) \).

   **If** \( \hat{g}\mathcal{J} \cap G_1 \neq \emptyset \) **then** identify \([E_1]\) **and** \([E_2]\).

**Proof.** The classes \([E_1], [E_2] \) \( \in \mathcal{P} \) or \( \mathcal{C} \) are equivalent under \( G_2 \) if and only if \( ghE_1 = E_2 \) for some \( g \in G_1 \) and \( h \in \mathcal{H} \). In such a case, there exists \( x \in \text{O}^+(L') \) such that \( x : hE_1 \rightarrow E_2 \). The set

\[ X = \{ x \in \text{O}^+(L') \mid x(hE_1) = E_2 \} = \hat{g} \text{Stab}_{\text{O}^+(L')}(hE_1) \]

and \( X \cap G_1 \neq \emptyset \) if and only if \( \hat{g}\mathcal{J} \cap G_1 \neq \emptyset \), from which the result follows. \( \square \)

**Remark** We note that if \( L' \) is a split maximal lattice, the map \( \hat{g} \) in Algorithm 3.4 can be calculated using \( \mathfrak{g} \) and Algorithm 3.3.
4 Examples

We now calculate $\mathcal{B}(\tilde{O}^+(L))$ for $L = 2U \oplus A_2$, $L = 2U \oplus \langle -6 \rangle \oplus \langle -2 \rangle$ and $L = 2U(2) \oplus A_2$. We will use a computer for lengthier parts of the calculations: our code (written for the Sage computer algebra system) can be found at [Daw22]. Throughout, we will use $\{e_i, f_i\}$ for $i = 1, 2$ to denote canonical bases for the two copies of $U \subset L$.

4.1 $L = 2U \oplus A_2$

Let $L = 2U \oplus A_2$. We begin by calculating generators for $O^+(L_1)$, where $L_1 = U \oplus A_2$.

![Figure 2: A fundamental domain for $W(U \oplus A_2)$.](image)

**Example 4.1.** If $\{e_i\}_{i=1}^4$ is a $\mathbb{Z}$-basis for $L_1$ with Gram matrix

$$
((e_i, e_j)) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -2 & -1 \\
0 & 0 & -1 & -2
\end{pmatrix}
$$

then $O^+(L_1)$ is generated by the reflections $\{\sigma_{v_i}\}_{i=1}^4$ and the map

$$
U \oplus A_2 \ni (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_4, -x_3) \in U \oplus A_2,
$$

(13)

where $v_1 = (1, -1, 0, 0)$, $v_2 = (0, 0, -1, 0)$, $v_3 = (0, 0, 0, 1)$ and $v_4 = (0, 1, 1, -1)$.

**Proof.** We begin by applying Vinberg’s algorithm [Vin75] to obtain generators $\{\sigma_{v_i}\}$ for the reflection subgroup $W(L_1) \subset O^+(L_1)$. Let $\mathbb{H}(L_1)$ be a component of $\{[x] \in \mathbb{P}(L_1 \otimes \mathbb{C}) \mid (x, x) < 0\}$ and, for fixed $x_0 \in \mathbb{H}(L_1)$, let $H(v_i)^-$ denote the half-space of $\mathbb{H}(L_1)$ orthogonal to $v_i \in L_1$ containing $x_0$. Vinberg’s algorithm proceeds by selecting a point $x_0 \in \mathbb{H}(L_1)$ whose stabiliser is generated by reflections $\{\sigma_{v_i}\}_{i=1}^k$ and proceeds by adding additional reflections $\sigma_{v_j}$, one-by-one for $j > k$, such that

$$
\frac{|(x_0, v_j)|^2}{|(v_j, v_j)|}
$$

is minimal and $H(v_i)$ and $H(v_j)$ are opposite for all $i < j$ [Vin75, p.327]. The algorithm terminates at $j = m$ if the polyhedron $P^{(m)}$, defined by

$$
P^{(m)} = \bigcap_{i=1}^m H(v_i)^-,$$

has finite volume (for which sufficient criteria are given in [Vin75]).
If $x_0 := e_1 + e_2$ then $x_0^+ = \langle e_1 - e_2, e_3, e_4 \rangle \subset U \oplus A_2(-1)$ and $x_0^- \cong (-2) \oplus A_2(-1)$. Therefore, $\text{Stab}_{W(L_1)}(x_0) = \langle \sigma_{v_1}, \sigma_{v_2}, \sigma_{v_3} \rangle$ where $v_1 = e_1 - e_2$, $v_2 = -e_3$ and $v_3 = e_4$. If $v \in U \oplus A_2(-1)$ and $\sigma_v \in O(L_1)$ then $v^2 \mid 2 \text{div}(v)$. As $\text{div}(v)$ is the order of $v^*$ in $D(L_1)$ then $\text{div}(v) = 1$ or $3$ and (14) is minimised for $v^2 = -2$ or $-6$, respectively. If $v_4 := e_2 + e_3 - e_4$ then $H(v_4)^-$ is opposite to $H(v_i)^-$ for all $i < 4$ and $P^{(4)}$ is non-degenerate with finite volume. The Coxeter diagram of $P^{(4)}$ (which was also obtained in [Vin07]) is given in Figure 2. The result follows by noting that the group of diagram automorphisms $D$ of $P$ is generated by $\{L_3\}$ and $O(L) = W(O(L)) \rtimes D$ [Vin07].

**Example 4.2.** The building $B(O^+(L))$ is given by Figure 3.

**Proof.** We apply Algorithm 3.1 with $G_1 = O^+(L)$ and $G_2 = O^+(L)$. The lattice $L$ is maximal and $A_2$ is unique in its genus [CS99 Table 15.1 p.360]. Therefore, $B(G_1)$ is as in Figure 1 with $m = 0$ and nodes $P = \{[\ell]\}$ and $C = \{[\Pi]\}$, where $\ell = \langle e_1 \rangle$ and $\Pi = \langle e_1, e_2 \rangle$. Using the generators $I$ of Lemma 3.6 and Example 4.1, we calculate $G = G_2/G_1$. We begin with an initial set $G := \{e\}$, which we enlarge by adding $x \in \mathcal{G} \mathcal{I}$ to $\mathcal{G}$ if there exists no $y \in \mathcal{G}$ such that $y^{-1}x \in G_1$, and stop when $\mathcal{G}$ can be enlarged no further. We find $\mathcal{G} = \{g_1, g_2\}$, where

$$g_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$g_2 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$ 

Similarly, using the generators of Lemma 3.7 and 3.8, we find that $\{g_1, g_2\}$ are also representatives for $\text{Stab}_{G_1}(g_1\ell) \setminus \text{Stab}_{G_2}(g_1\ell)$ and $\text{Stab}_{G_1}(g_2\Pi) \setminus \text{Stab}_{G_2}(g_2\Pi)$, implying $\mathcal{N}_1 = (\{[\ell]\}, \{[\Pi]\})$. By Lemma 3.10, we let $X = \{\ell\}$ and $\mathcal{F} = SL(2,\mathbb{Z})/\Gamma(3)$, and as $\ell \in Y := G.X$ we add the edge $l \rightharpoonup \Pi$ to $\mathcal{E}_1$.

The calculation of $B(O^+(L))$ using [Daw22] is fast, taking less than a second on a desktop computer.

**4.2** $L = 2U \oplus \langle -6 \rangle \oplus \langle -2 \rangle$

We now consider $L = 2U \oplus \langle -6 \rangle \oplus \langle -2 \rangle$, which arises in connection with period spaces of deformation generalised Kummer varieties [Daw].

**Example 4.3.** There exists a maximal overlattice $L \subset L'$.

**Proof.** The discriminant form of $L$ is given by

$$q_L(a, b, c) = -\frac{a^2}{2} - \frac{3b^2}{2} - \frac{2c^2}{3} \mod 2\mathbb{Z}$$

where $(a, b, c) \in C_2^\mathbb{Z} \oplus C_3 \cong D(L)$. The isotropic elements of $D(L)$ are given by $(1, 1, 0)$ and $(0, 0, 0)$. By Lemma 3.3 if $H := \langle (1, 1, 0) \rangle$ then $H^\perp = \langle (1, 1, 0), (0, 0, 1) \rangle$ and $H^\perp/H \cong q_{A_2}$. The overlattice can be realised by the map

$$2U \oplus \langle -6 \rangle \oplus \langle -2 \rangle \ni (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1, x_2, x_3, x_4, -2x_5, x_6 + x_5) \in 2U \oplus A_2.$$
Example 4.4. The buildings $B(\tilde{O}^+(L))$ and $B(O^+(L))$ are given by Figure 4.

Proof. We calculate $B(\tilde{O}^+(L))$ using [Daw22], which applies Algorithm 3.1 and Example 4.3. If $\ell = \langle e_1 \rangle$ and $\Pi = \langle e_1, e_2 \rangle$ then $P = \{g_1\ell, g_2\ell\}$ and $C = \{g_1\Pi, g_2\Pi\}$, where

$$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As shown in [Daw, Lemma 2.4], $O^+(L) = \tilde{O}^+(L), \sigma_v$ where $v$ generates the $\langle -6 \rangle$ factor of $L$. We therefore apply Algorithm 3.4 with $H = \{I, \sigma_v\}$. For $I = P$, if $h = I$ we calculate a transversal $\{r_k\}$ for $\text{Stab}_{\tilde{O}^+(L)}(g_2\ell) \setminus \text{Stab}_{O^+(L)}(g_2\ell)$. The element $\hat{g} = g_2$ and we find no $r_k\hat{g}^{-1} \in G_2$. Similarly, for $h = \sigma_v$, we have $\hat{g} = g_2$ and the same conclusion follows. Therefore, $[g_1\ell]$ and $[g_2\ell]$ are not identified under $O^+(L)$. One proceeds identically for $I = C$.

Remark Figure 4 was also obtained using different methods in [Daw].

4.3 $2U(2) \oplus A_2$

For a more intricate example, we consider $L = 2U(2) \oplus A_2$.

Example 4.5. The building $B(\tilde{O}^+(L))$ is given in Figure 5.
We calculated $B(\tilde{O}^+(L))$ using [Daw22], applying Algorithm 3.1 for $G_2 = O^+(2U \oplus A_2)$ and $G_1 = \tilde{O}^+(L)$. As the index $|G_2 : G_1| = 1080$ is quite large, the calculation was rather lengthy, taking a few hours on a desktop computer. A significant improvement in performance should be possible by making use of the subgroups

$$\tilde{O}^+(2U(2) \oplus A_2) \subset \tilde{O}^+(U \oplus U(2) \oplus A_2) \subset \tilde{O}^+(2U \oplus A_2) \subset O^+(2U \oplus A_2),$$

which are of index 20, 27 and 2, respectively.

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