A CONSTRUCTION OF KNOT FLOER HOMOTOPY

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Abstract. Given a knot presented in a grid diagram, we associate to it a partially ordered set with certain properties, and then construct a CW complex whose cells correspond to the elements of the partially ordered set, and whose attaching maps correspond to the covering relations. This space is well-defined, its homology is the grid homology, and its stable homotopy type is a knot invariant. Thus to each knot, we can associate an invariant spectrum, whose $\mathbb{F}_2$ homology is the knot Floer homology over $\mathbb{F}_2$.

1. Introduction

Heegaard Floer homology is a powerful invariant for closed oriented 3-manifolds, introduced by Peter Ozsváth and Zoltán Szabó [OSz04b, OSz04c]. This invariant was later generalized by them [OSz04a] and independently by Jacob Rasmussen [Ras03] to an invariant for knots in 3-manifolds called knot Floer homology, which was later even further generalized to include the case of links [O Sz08a].

We will mostly be concerned with the case of links inside $S^3$, and indeed for the most part, we will be dealing with knots. There are several variants of the knot Floer homology, but we will be only working with the hat version and the minus version denoted by $\widehat{\text{HF}}^\text{K}$ and $\text{HF}^\text{K}^-$ respectively. They are bigraded modules over $\mathbb{Z}$ and $\mathbb{Z}[U]$ respectively, although, we will often ignore the $U$ action on $\text{HF}^\text{K}^-$ and treat them simply as bigraded abelian groups. The two gradings $M$ and $A$ are called the Maslov grading and the Alexander grading, and they both assume integer values for knots in $S^3$. These groups are obtained as the homology of certain chain complexes, and the Maslov grading $M$ is in fact the homological grading.

If $\widehat{\text{HF}}^\text{K}_{i,j}(L)$ is knot Floer homology of a link $L$ in $(M, A)$ bigrading $(i, j)$, then the Euler characteristic $\sum_j \sum_i (-1)^i \text{rk}(\widehat{\text{HF}}^\text{K}_{i,j}(L)) t^j$ is the symmetric Alexander polynomial $\Delta(t)$ for the link $L$, and hence the name of the second grading.

The strength of knot Floer homology can be demonstrated by the following few theorems. Peter Ozsváth and Zoltán Szabó proved that the absolute value of the highest or the lowest Alexander grading $j$ for which $\bigoplus_i \widehat{\text{HF}}^\text{K}_{i,j}(K)$ is non-trivial, is equal to the genus of the knot $K$ [OSz04a]. This was later generalized by them to show that $\widehat{\text{HF}}^\text{K}(L)$ detects the Thurston norm of a link $L$ [OSz08b]. Yi Ni showed that a knot $K$ is fibered if and only if $\bigoplus_i \widehat{\text{HF}}^\text{K}_{i,g(K)}(K) = \mathbb{Z}$, where $g(K)$ is the genus of the knot [Ni07]. Peter Ozsváth and Zoltán Szabó also constructed an invariant $\tau$ coming from knot Floer homology which gives a lower bound on the 4-ball genus of the knot [O Sz03].

In [MOS09], based on a grid presentation of the knot, chain complexes over $\mathbb{F}_2$ are constructed, whose homologies agree with knot Floer homologies with coefficients in $\mathbb{F}_2$. A sign refined version of the grid chain complexes was constructed by Ciprian Manolescu, Peter Ozsváth, Zoltán Szabó and Dylan Thurston in [MOSzT07], where they also gave a combinatorial proof of the invariance of the homology of the chain complex. In this paper, we associate partially ordered sets to the grid chain complexes, and prove that the partially ordered sets satisfy certain conditions. Such partially ordered sets are called GSS posets, and we
prove the following central theorem about GSS posets. It is stated more precisely as Theorem 7.2 and is proved in Section 7.

**Theorem 1.1.** Given a GSS poset $P$ and a sufficiently large integer $n$, there is a well-defined CW complex $X_P(n)$, whose cells correspond to the elements of $P$, and whose attaching maps correspond to the covering relations in $P$.

Using GSS posets coming from grid chain complexes, we can associate a CW complex to a grid chain complex, such that the cells of the CW complex correspond to the generators of the grid chain complex, and the attaching maps correspond to the boundary maps. This implies that the homology of the CW complex is isomorphic to the grid homology. We mimic the proof of invariance from [MOSzT07] to show that the stable homotopy type of these CW complexes is also a knot invariant. The following theorem summarizes Theorem 7.4 and Theorem 7.5 of Section 7.

**Theorem 1.2.** Given a knot $K$ presented in a grid diagram $G$, an Alexander grading $m$, and a sufficiently large integer $n$, there is a well-defined CW complex $X_{G-m}(n)$, whose homology is the minus version of the grid homology and whose stable homotopy type depends only on $K$, $m$ and $n$, such that its cells and attaching maps correspond to the generators and the boundary maps respectively, in the minus version of the grid chain complex.

It is unclear whether the stable homotopy types of these spaces is a stronger invariant than knot Floer homology, or in other words, whether there exists knots $K_1$ and $K_2$ with the same knot Floer homology, such that the corresponding spaces have different stable homotopy types.

On an unrelated note, it is possible that GSS posets arise in many other contexts, and therefore Theorem 1.1 might have applications outside of knot Floer homology. Even restricted to the world of homology theories of knots and 3-manifolds, it is an interesting problem to investigate whether Khovanov homology and Heegaard Floer invariants for 3-manifolds and 4-manifolds admit similar constructions.

In Section 2 we state and prove certain properties of partially ordered sets, and we introduce the notion of a GSS poset. All the results are well-known, but we still reprove the easy ones to keep the text readable. In Section 3 we associate certain posets to a grid diagram of a knot such that the associated chain complexes are the grid chain complexes. Almost all the results presented in this section are available elsewhere, most notably in [MOSzT07], but once more we reprove some of the easier results. In Section 4 we prove that the posets associated to a grid diagram are GSS posets, and in Section 5 we construct some CW complexes of a special type from a GSS poset based on some of its properties. In Section 6 we give an explicit and a very detailed construction of the dual of CW complexes of that special type, and finally in Section 7 using the duals, we associate a CW complex to a GSS poset. We also prove that, for a CW complex associated to a poset coming from a grid diagram, its homology is the grid homology and its stable homotopy type is a knot invariant. In Section 8 we give examples of some other GSS posets and conclude with the computation for the index 5 grid diagram for the trefoil.

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2. Partially ordered sets

A set $P$ with a binary relation $\preceq$ is a partially ordered set if $a \preceq b, b \preceq c \Rightarrow a \preceq c$ and $a \preceq b, b \preceq a \Rightarrow a = b$. If $a \preceq b, a \not= b$, then we often say that $a$ is less than $b$ and write $a \prec b$. If $\not\exists z, b \prec z$, then we say that $b$ is a maximal element. Minimal elements are defined similarly. We also often abbreviate partially ordered sets as posets.

We say that $b$ covers $a$, and write $a \leftarrow b$ if $a \prec b$ and $\not\exists z, a \prec z \prec b$. Any subset of a poset has an induced partial order. A subset $C \subseteq P$ is called a chain if the induced order is a total order. Chains themselves are partially ordered by inclusion. Maximal chains are the maximal elements under this order. Submaximal chains are chains which are covered by maximal chains under this order. The length of a chain is the cardinality of the chain considered just as a set.

The Cartesian product of two posets $P$ and $Q$ is defined as the poset $P \times Q$, whose elements are pairs $(p, q)$ with $p \in P$ and $q \in Q$, and we declare $(p', q') \preceq (p, q)$ if and only if $p' \preceq p$ in $P$ and $q' \preceq q$ in $Q$.

The order complex of a poset is a simplicial complex, whose $k$-simplexes are chains of length $(k + 1)$. The boundary maps are defined naturally.

We define a closed interval $[a, b]$ as $\{ z \in P | a \preceq z \preceq b \}$. Open intervals and half-closed intervals are defined similarly. We also define $(-\infty, b]$ as $\{ z \in P | z \preceq b \}$ and $[a, \infty)$ as $\{ z \in P | a \preceq z \}$.

A poset is said to be graded if in every interval, all the maximal chains have the same length, in which case the common length is known as the length of the interval. A graded poset is said to be thin, if every submaximal chain is covered by exactly 2 maximal chains. A graded poset is subthin if it is not thin, and every submaximal chain is covered by at most 2 maximal chains.

A graded poset is said to be shellable if the maximal chains have a total ordering $\preceq$, such that $m_i < m_j \Rightarrow \exists m_k < m_j$ and $\exists x \in m_j$ such that $m_i \cap m_j \subseteq m_k \cap m_j = m_j \setminus \{ x \}$.

**Lemma 2.1.** Any interval (closed, half-closed, open) of a shellable poset is itself shellable.

*Proof.* We just prove this for the case of an interval of the form $(a, b)$. The other cases follow similarly. Choose a maximal chain $c_1$ in $(-\infty, a]$, and choose a maximal chain $c_2$ in $[b, \infty)$. The maximal chains in $(a, b]$ can be put in an one-one correspondence with the maximal chains of the original poset which start with $c_1$ and end with $c_2$. However, such maximal chains have a total ordering induced from the shellable structure, and it is routine to check that such an ordering suffices. 

**Lemma 2.2.** Let $P$ be a shellable poset with a unique minimum $z$. If we construct a new poset $P'$ by adjoining a single element $z'$ which covers nothing and is itself covered by precisely the elements that cover $z$, then $P'$ is shellable.

*Proof.* Note that the maximal chains in $[z', \infty)$ correspond to the maximal chains in $[z, \infty)$, and thus a shellable total ordering of the maximal chains in $[z, \infty)$ gives us a shellable total ordering of the maximal chains in $[z', \infty)$. We extend this to a total ordering on the maximal chains in $P'$ by declaring any maximal chain in $[z, \infty)$ to be smaller than any maximal chain in $[z', \infty)$. It is again easy to check that this ordering satisfies all the required properties.

**Lemma 2.3.** Let $P$ be a shellable poset with two minimums $z$ and $z'$, which are covered by the same elements. If we construct a new poset $P'$ by adjoining a single element $w$ which is covered by $z$ and $z'$, then $P'$ is shellable.

*Proof.* Note that the maximal chains of $P'$ correspond to the maximal chains of $P$. Thus a shellable total ordering of the maximal chains in $P$ induces a total ordering of the maximal chains in $P'$, which is easily checked to be shellable.
A graded poset is said to be edge-lexicographically shellable or \textit{EL-shellable} if there is a map $f$ from the set of all covering relations (alternatively, closed intervals of length 2) to a totally ordered set, such that for any closed interval $[x_1, x_n]$ of length $n$, if we associate the $(n-1)$-tuple $(f([x_1, x_2]), \ldots, f([x_{n-1}, x_n]))$ to a maximal chain $x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n$, then there is a unique maximal chain for which the $(n-1)$-tuple is increasing, and under the lexicographic ordering, the corresponding $(n-1)$-tuple is smaller than any $(n-1)$-tuple coming from any other maximal chain between $x_1$ and $x_n$.

For the rest of the paper, we shall mainly use the following theorems.

\textbf{Theorem 2.4.} \textit{BJ70} If a poset is EL-shellable, then every closed interval of the poset is shellable.

\textit{Proof.} Let $[x_1, x_n]$ be a closed interval of length $n$. There is a map from the set of all covering relations to a totally ordered set, and the lexicographic ordering induces a partial ordering of the maximal chains. The reason that this is a partial order, and not a total order, is because two different maximal chains might have the same labeling. To fix this, totally order the maximal chains that have the same labeling in some arbitrary way, and extend the partial ordering to a total ordering of all the maximal chains in $[x_1, x_n]$. We now prove that this total ordering satisfies all the requirements of shellability.

Let $m_1$ and $m_2$ be two maximal chains with $m_1 < m_2$. Each maximal chain is a sequence of $n$ elements from the poset, starting at $x_1$ and ending at $x_n$. Thus $m_1$ and $m_2$ agree up to some $x_k$, and start being different, and then agree again at $x_1$ (and maybe disagree again later). In other words, $m_1$ starts as $x_1 \prec \cdots \prec x_k \prec y_{k+1} \prec \cdots \prec y_l \prec x_1 \prec \cdots$, and $m_2$ starts as $x_1 \prec \cdots \prec x_k \prec z_{k+1} \prec \cdots \prec z_l \prec x_1 \prec \cdots$, and the set $\{y_{k+1}, \ldots, y_l\}$ is disjoint from the set $\{z_{k+1}, \ldots, z_l\}$. Look at the interval $[x_k, x_l]$, and let $n_i = m_i \cap [x_k, x_l]$. Since the interval $[x_k, x_l]$ has a unique maximal chain whose labeling is increasing, which in addition happens to the minimum one, the labeling in $n_2$ cannot be increasing. Hence there is a first place $z_{l-1} \prec z_l \prec z_{l+1}$, where the labeling decreases. However there must be an increasing chain $z_{l-1} \prec z' \prec z_{l+1}$ in the interval $[z_{l-1}, z_{l+1}]$. Therefore, if $m_3 = m_2 \cup \{z'\} \setminus \{z_l\}$, then $m_3 < m_2$, and $m_1 \cap m_2 \subseteq m_3 \cap m_2 = m_2 \setminus \{z_l\}$. This shows that $[x_1, x_n]$ is shellable. $\square$

\textbf{Theorem 2.5.} \textit{DK74} The order complex of a finite, shellable and thin poset is PL-homeomorphic to a sphere. The order complex of a finite, shellable and subthin poset is PL-homeomorphic to ball, and the boundary of the ball corresponds to those submaximal chains, which are covered by exactly one maximal chain.

\textit{Proof.} Let $P$ be a finite, shellable poset which is either thin or subthin. Choose some shellable total ordering on the maximal chains, and under that ordering, let the maximal chains be $m_1 < m_2 < \cdots < m_k$. Let $n$ be the length of each maximal chain. The order complex of $P$ is the union of the order complexes of the maximal chains $m_i$, each of which is an $(n-1)$-simplex $\Delta^{n-1}$.

Let us construct the order complex of $P$ in the following manner. Let $X_i$ be the order complex of the union of the elements in $m_1, \ldots, m_i$. We glue to it the order complex of $m_{i+1}$ to get $X_{i+1}$.

We start with $X_1$ which is an $(n-1)$-simplex $\Delta^{n-1}$ (and hence PL-homeomorphic to a ball). By induction, each of the $X_i$'s (for $1 \leq i < k$) is PL-homeomorphic to an $(n-1)$-dimensional ball. A careful consideration reveals that while gluing $\Delta^{n-1}$, the order complex of $m_{i+1}$, to $X_i$ (which by induction is an $(n-1)$-ball), thinness or subthinness along with shellability implies that the gluing is done along a union of $(n-2)$-simplices on the boundary. The proof finishes after the (slightly non-trivial) observation that the union of a non-empty collection of $(n-2)$-simplices on $\partial \Delta^{n-1}$ is either an $(n-1)$-ball or an $(n-1)$-sphere. $\square$

We will often encounter posets with the following properties. A \textit{sign assignment} is a map from the set of all covering relations to $\{\pm 1\}$, such that every closed interval of length 3 has exactly two maximal chains and the product of the signs for all the four covering relations is $(-1)$. Two such sign assignments are said to be equivalent if one can be obtained from another by a sequence of moves, where at each move we choose
an element of the poset and change the signs of all the covering relations involving that element. A *grading assignment* is a map \( g \) from the elements of the poset to \( \mathbb{Z} \), such that whenever \( a \leftarrow b \), \( g(b) = g(a) + 1 \). Having a grading assignment is weaker than being graded, but is stronger than each closed interval being graded.

**Definition 2.6.** A poset equipped with a sign assignment and a grading assignment, whose every closed interval of the form \([a, b]\) is shellable, is called a graded signed shellable poset, or in other words, a GSS poset.

For most of the time, we will be working with GSS posets. Given a GSS poset, it is very easy to associate a chain complex to it. The generators of the chain complex are the elements of the poset with gradings determined by the grading assignment, and the boundary map is given by

\[
\partial x = \sum_{y, y \leftarrow x} s(y, x)y
\]

where \( s(y, x) \) is the sign assigned to the covering relation \( y \leftarrow x \). It is easy to see that this is indeed a chain complex, and the chain homotopy type of the chain complex remains unchanged if the sign assignment is replaced by an equivalent one. We call this complex to be the chain complex associated to the GSS poset.

3. **Grid diagrams**

In this section we will introduce three types of diagrams, grid diagrams, commutation diagrams and stabilization diagrams. All of them are pictures on the standard torus, and we will associate certain posets to each one of them. We often think of diagrams on the torus as diagrams on the unit square in the plane. This viewpoint allows us to work with certain transformations. We can rotate the diagrams by an angle of \( \theta \), where \( \theta \in \{ \frac{\pi}{2}, \pi, \frac{3\pi}{2} \} \), and we call it the rotation \( R(\theta) \). We can reflect the whole diagram along a horizontal line or a vertical line, and we call them the reflections \( R(h) \) and \( R(v) \) respectively. The transformations \( R(\frac{\pi}{2}), R(\frac{3\pi}{2}), R(h) \) and \( R(v) \) keep the elements of the posets unchanged but reverse the partial order. However, if a poset is a GSS poset, it stays a GSS poset even after reversing its partial order, so as far as being GSS is concerned, it does not matter.

A grid diagram with index \( N \), is a picture on the standard torus \( T \). There are \( N \alpha \) circles, which are pairwise disjoint and parallel to the meridian, such that they cut up the torus into \( N \) horizontal annuli, and there are \( N \beta \) circles, which are pairwise disjoint and parallel to the longitude, such that they cut up the torus into \( N \) vertical annuli. Furthermore, each \( \alpha \) circle intersects with each \( \beta \) circle exactly once, so clearly \( T \setminus (\alpha \cup \beta) \) has \( N^2 \) components. There are \( 2N \) markings on \( T \setminus (\alpha \cup \beta) \), \( N \) of them marked \( X \), \( N \) of them marked \( O \), such that each component contains at most one marking, each horizontal annulus contains one \( X \) and one \( O \), and each vertical annulus contains one \( X \) and one \( O \). At this point, a reader familiar with knot Floer homology, will observe that \((T, \alpha, \beta, X, O)\) is a genus one Heegaard diagram for a link inside \( S^3 \).

Given a grid diagram, we can construct a link inside \( \mathbb{R}^3 \) in the following way. If \( T \) is embedded in \( \mathbb{R}^3 \) in the standard way, with the meridian bounding a disk inside the torus, and the longitude bounding a disk outside, then the link is obtained by joining \( O \) to \( X \) in the same horizontal annulus, inside the torus \( T \), and by joining \( X \) to \( O \) in the same vertical annulus, outside the torus \( T \). Therefore at every crossing, the vertical strand is the overpasses. In the other direction, given a link \( L \subset \mathbb{R}^3 \), it is not difficult to get a grid diagram for \( L \).

**Lemma 3.1.** Given a link \( L \subset \mathbb{R}^3 \), there is a grid diagram that represents \( L \).

**Proof.** Let \( L \) be represented by a PL-link diagram in the \( xy \)-plane. That means that there are a bunch of vertices and a bunch of straight edges joining some of the vertices, such that each vertex has exactly two
edges coming into it. By moving the vertices slightly, we can ensure that no two vertices lie in the same horizontal line or the same vertical line. We then replace each edge by a pair of horizontal and vertical edges, in one of two possible ways, as shown in Figure 3.1. Thus $L$ is now represented by horizontal edges (with no two on the same horizontal line) and vertical edges (with no two on the same vertical line).

![Figure 3.1. Converting all edges to horizontal and vertical ones](image)

If in any crossing, the horizontal edge is the overpass, then we change the local picture as shown in Figure 3.2 to ensure that the vertical edge is the overpass. Such a diagram then easily corresponds to a grid diagram.

![Figure 3.2. Changing the horizontal overpasses to vertical ones](image)

There are two processes on the grid diagram, namely commutation and stabilization, which do not change the isotopy class of the underlying link. Commutations can be either horizontal or vertical, but one can be obtained from the other by a rotation $R(\frac{\pi}{2})$, so we only describe a horizontal commutation.

We view markings in a particular horizontal annulus as an embedded 0-sphere in one of the bounding $\alpha$ circles. In a horizontal commutation, we choose two adjacent horizontal annuli, such that the markings in one of them is unlinked with the markings in the other. Then we interchange the markings for the two annuli. This process can also be viewed as changing the $\alpha$ circle that lies between the two adjacent horizontal annuli. Note that a commutation does not change the index of the grid diagram, and it also keeps the isotopy class of the link unchanged.

We can represent the process of commutation by a single grid like diagram on the torus. Let $G$ be a grid diagrams with index $N$, drawn on the torus $T$. Let us do a horizontal commutation on $G$, by changing a circle $\alpha_c$ to a circle $\alpha'_c$, to get a grid diagram $G'$. Therefore, $G'$ looks exactly like $G$, except it has a circle $\alpha'_c$ instead of $\alpha_c$. We can represent the whole commutation by a single diagram $G_{\alpha_c}$, which is basically the grid diagram $G$, along with an extra circle $\alpha'_c$. The circles $\alpha_c$ and $\alpha'_c$ intersect in exactly two points, and we
ensure that none of the $\beta$ circles pass through either of those two points. Thus the diagram has $(N^2 + N + 2)$ regions, of which 4 are triangles, 4 are pentagons, and the rest are squares. There are two triangles and two pentagons around each point of $\alpha_c \cap \alpha'_c$, and we can ensure that for each of those two points, either the triangle to the right or the triangle to the left has an $X$ marking. Of the two points of intersection between $\alpha$ and $\alpha'$, let $\rho$ be the one with $\alpha$ on its top-left. We call the pair $(G_c, \rho)$ a commutation diagram. Note that due to presence of the point $\rho$, the definition is not symmetric regarding the roles of $G$ and $G'$.

In a stabilization, we choose a marking $X$, and change the vertical annulus through the marking into two parallel vertical annuli by adding a $\beta$ circle, and change the horizontal annulus through the marking into two parallel horizontal annuli by adding an $\alpha$ circle. The component containing the original $X$ marking has now become 4 components, and we put two $X$ markings in two diagonally opposite components, and put one $O$ marking in one of the two other components. The original horizontal and vertical annuli through our $X$ marking contained two $O$ markings, and their position in the new diagram gets fixed by the condition that each horizontal and each vertical annulus must contain exactly one $X$ and exactly one $O$ marking. Again note that a stabilization keeps the isotopy class of the link unchanged, but increases the index of the grid diagram by 1. The roles of $X$ and $O$ seem asymmetric in this definition, but the other type of stabilization, where the roles of $X$ and $O$ are reversed, can be obtained as a composition of stabilization of this type and a few commutations.

Note that after stabilization, in the new grid diagram, a neighborhood of the original $X$ marking looks like Figure 3.5. The new $\alpha$ and $\beta$ circles are denoted by thick lines. The cases (c) and (d) can be obtained from cases (a) and (b) respectively after the rotation $R(\pi)$. Thus we will only be concentrating on the cases (a) and (b). (Indeed the case (b) can be obtained from the case (a) by a rotation $R(\pi)$, but the two cases behave quite differently, and hence they deserve separate attention). Let $\alpha_s$ and $\beta_s$ be the new $\alpha$ circle and $\beta$ circle, and let their intersection be $\rho$. If the new grid diagram is $G$, we call the pair $(G, \rho)$ a stabilization diagram. Therefore a stabilization diagram is basically just a grid diagram with a distinguished $\alpha$ and a distinguished $\beta$ circle such that the neighborhood of their intersection looks like Figure 3.5.

The following theorem shows that commutations and stabilizations provide a full set of Reidemeister moves for grid diagrams. That means that if we want to show that a grid diagram invariant is a link invariant, then we only have to investigate what happens during a commutation and a stabilization.

**Theorem 3.2.** [Cro95] If two grid diagrams represent the same link, then we can apply sequences of commutations and stabilizations on each of them, such that the final two grid diagrams are the same.
homologous in the torus $T$.

Observing that the torus $T$ construct a periodic domain $T$.

All periodic domains are generated by vertical and horizontal annuli.

Lemma 3.3. Let $\mu$.

Proof. Let $D$ be a periodic domain and let $\partial D = \sum n_i \alpha_i + \sum m_i \beta_i$. Since each $\alpha_i$ is homologous to the meridian, and each $\beta_i$ is homologous to the longitude, this implies that $(\sum n_i \alpha_i) + (\sum m_i \beta_i)$ is null-homologous in the torus $T$. Therefore, $\sum n_i = \sum m_i = 0$. Then, it is pretty easy to see that we can construct a periodic domain $D_\alpha$ out of only vertical annuli such that $\partial D_\alpha = \sum m_i \beta_i$, and we can construct a periodic domain $D_\beta$ out of only horizontal annuli such that $D_\beta = \sum n_i \alpha_i$. Thus $D - D_\alpha - D_\beta$ is a periodic domain without any boundary, and therefore has to be equal to $kT$ for some $k$. We finish the proof by observing that the torus $T$ is also generated by vertical annuli.

Lemma 3.3. All periodic domains are generated by vertical and horizontal annuli.

Proof. Let $D$ be a periodic domain and let $\partial D = \sum n_i \alpha_i + \sum m_i \beta_i$. Since each $\alpha_i$ is homologous to the meridian, and each $\beta_i$ is homologous to the longitude, this implies that $(\sum n_i \alpha_i) + (\sum m_i \beta_i)$ is null-homologous in the torus $T$. Therefore, $\sum n_i = \sum m_i = 0$. Then, it is pretty easy to see that we can construct a periodic domain $D_\alpha$ out of only vertical annuli such that $\partial D_\alpha = \sum m_i \beta_i$, and we can construct a periodic domain $D_\beta$ out of only horizontal annuli such that $D_\beta = \sum n_i \alpha_i$. Thus $D - D_\alpha - D_\beta$ is a periodic domain without any boundary, and therefore has to be equal to $kT$ for some $k$. We finish the proof by observing that the torus $T$ is also generated by vertical annuli.

Figure 3.5. Different types of stabilization

3.1. Grid diagram. Given a grid diagram of index $N$ representing a link $L$, we can define two GSS posets $\mathcal{G}$ and $\mathcal{G}^-$, such that the homology of the associated chain complex in the first case depends only on $L$ and $N$, and in the second case depends only on $L$. The homologies turn out to be closely related to the hat version and the minus version of the knot Floer homologies. The elements of the poset $\mathcal{G}$ are indexed by formal sums $\hat{x} = x_1 + x_2 + \cdots + x_N$ of $N$ points, such that each $\alpha$ circle contains one point and each $\beta$ circle contains one point. The elements of $\mathcal{G}^-$ are indexed elements of the form $x = \hat{x} \prod_{\alpha \in L} U^\alpha$ where $\hat{x} \in \mathcal{G}$ and $k_i \in \mathbb{N} \cup \{0\}$. We need the following few definitions to understand the partial order in the poset.

First number the $O$ markings as $O_1, O_2, \ldots, O_N$ and let $\mathcal{O}$ denote the formal sums $\sum_i O_i$. Similarly, number the $X$ markings as $X_1, X_2, \ldots, X_N$ and let $\mathcal{X}$ denote the formal sum $\sum_i X_i$. A domain $D$ connecting a generator $\hat{x}$ to another generator $\hat{y}$, is a 2-chain generated by the components of $T \setminus (\alpha \cup \beta)$ such that $\partial(\partial D) = \hat{y} - \hat{x}$. The set of all domains connecting $\hat{x}$ to $\hat{y}$ is denoted by $\mathcal{D}(\hat{x}, \hat{y})$. For a point $p \in T \setminus (\alpha \cup \beta)$ and a domain $D \in \mathcal{D}(\hat{x}, \hat{y})$, we define $n_p(D)$ to be the coefficient of the 2-chain $D$ at the point $p$. We define $\mathcal{D}^0(\hat{x}, \hat{y})$ as a subset of $\mathcal{D}(\hat{x}, \hat{y})$ consisting of all the domains $D$ with $n_p(D) = 0$ whenever $p$ is any of the $N$ $X$ markings, and we define $\mathcal{D}^0(\hat{x}, \hat{y})$ as the subset of $\mathcal{D}(\hat{x}, \hat{y})$ consisting of all the domains $D$ with $n_p(D) = 0$ whenever $p$ is one of the $2N$ $X$ or $O$ markings. For $x = \hat{x} \prod_i U^\alpha_i$ and $y = \hat{y} \prod_i U^\beta_i$ in $\mathcal{G}$, we define $\mathcal{D}^0(x, y)$ as the subset of $\mathcal{D}(\hat{x}, \hat{y})$ consisting of all the domains with $n_{O_i} = i - k_i$. A domain $D$ is positive if $n_p(D) \geq 0$ for all points $p \in T \setminus (\alpha \cup \beta)$. For $v$, a point of intersection between an $\alpha$ curve and a $\beta$ curve, and $D \in \mathcal{D}(\hat{x}, \hat{y})$, we define $n_v(D)$ as the average of the coefficients of $D$ in the four components of $T \setminus (\alpha \cup \beta)$ around $v$. Domains in $\mathcal{D}(\hat{x}, \hat{y})$ are said to be periodic domains.

Lemma 3.3. All periodic domains are generated by vertical and horizontal annuli.

Proof. Let $D$ be a periodic domain and let $\partial D = \sum n_i \alpha_i + \sum m_i \beta_i$. Since each $\alpha_i$ is homologous to the meridian, and each $\beta_i$ is homologous to the longitude, this implies that $(\sum n_i \alpha_i) + (\sum m_i \beta_i)$ is null-homologous in the torus $T$. Therefore, $\sum n_i = \sum m_i = 0$. Then, it is pretty easy to see that we can construct a periodic domain $D_\alpha$ out of only vertical annuli such that $\partial D_\alpha = \sum m_i \beta_i$, and we can construct a periodic domain $D_\beta$ out of only horizontal annuli such that $D_\beta = \sum n_i \alpha_i$. Thus $D - D_\alpha - D_\beta$ is a periodic domain without any boundary, and therefore has to be equal to $kT$ for some $k$. We finish the proof by observing that the torus $T$ is also generated by vertical annuli.

For two generators $\hat{x} = \sum_i x_i$ and $\hat{y} = \sum_i y_i$, and a domain $D \in \mathcal{D}(\hat{x}, \hat{y})$, the Maslov index is defined to be $\mu(D) = \sum (n_{x_i}(D) + n_{y_i}(D))$. We choose an $\alpha$ circle and a $\beta$ circle on the grid diagram $G$ and cut open the torus $T$ along those circles to obtain a diagram in $[0, N] \times [0, N] \subset \mathbb{R}^2$. In this planar diagram, the $\alpha$ circles become the lines $y = i$ and the $\beta$ circles become the lines $x = i$ for $0 \leq i < N$. Let the $X$ marking and the $O$ markings occupy half-integral lattice points. Now for two points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{R}^2$, we define $J(a, b) = \frac{1}{2}$ if $(a_1 - b_1)(a_2 - b_2) > 0$ and $0$ otherwise. We extend $J$ bilinearly for formal sums.
and differences of points. For \( \hat{x} \in \hat{G} \), we define the Maslov grading \( M(\hat{x}) = J(\hat{x} - \emptyset, \hat{x} - \emptyset) + 1 \) and the Alexander grading \( A(\hat{x}) = J(\hat{x} - \frac{X+\hat{X}}{2}, X - \emptyset) - \frac{N^+1}{2} \). The following is mere verification.

**Lemma 3.4.** [MOSz107] A(\( \hat{x} \)) and M(\( \hat{x} \)) are independent of the choice of \( \alpha \) and \( \beta \) circles along which the torus is cut open. The Maslov grading M(\( \hat{x} \)) always takes integral values and the Alexander grading A(\( \hat{x} \)) takes integral values for a knot. Furthermore, for a domain \( D \in \mathcal{D}(\hat{x}, \hat{y}) \), \( M(\hat{x}) - M(\hat{y}) = \mu(D) - 2 \sum_n n_{iO}(D) \) and \( A(\hat{x}) - A(\hat{y}) = \sum_{i} (n_{iX}(D) - n_{iO}(D)) \).

We extend the assignment of Maslov and Alexander gradings from \( \hat{G} \) to \( \mathcal{G}^- \). We define \( M(\hat{x} \prod U_i^k) = M(\hat{x}) - 2 \sum_i k_i \) and \( A(\hat{x} \prod U_i^k) = A(\hat{x}) - \sum_i k_i \). (In other words, we assign an \((M,A)\) bigrading of \((-2,-1)\) to each \( U_i \)). We define \( \mathcal{G}_m \) to be the the subset of \( \hat{G} \) which has Alexander grading \( m \), and we define \( \mathcal{G}^-_m \) to be the subset of \( \mathcal{G}^- \) which has Alexander grading \( m \). Note that even though \( \mathcal{G}^- \) is an infinite set, for each \( m \), \( \mathcal{G}^-_m \) is a finite set. For each of the objects \( \hat{G}, \mathcal{G}_m, \mathcal{G}^- \) and \( \mathcal{G}^-_m \), we define \( M_c = M + c \), and call it the Maslov grading shifted by \( c \).

If the reader is following the analogies from the Floer homology picture, it should be pretty clear by this point that the positive domains of index one are of special importance to us. The following lemma characterizes them. The lemma is in fact a consequence of Lemma [5.4] whose proof does not really require this result, but we give an independent proof here, so as to not disrupt the flow of the text.

**Lemma 3.5.** [MOS09] Let \( D \in \mathcal{D}(\hat{x}, \hat{y}) \) be a positive domain with \( \mu(D) = 1 \). Then \( \hat{x} \) and \( \hat{y} \) differ in exactly two coordinates. Furthermore, \( D \) has coefficients 0 and 1 everywhere, and the closure of the regions where \( D \) has coefficients 1 form a rectangle which does not contain any x-coordinate or any y-coordinate in its interior.

**Proof.** The domain \( D \) cannot just be copies of the torus, since each copy of the torus has index \( 2N \). Therefore \( D \) must have boundary, and without loss of generality, let \( \partial D \) be non-zero on some \( \alpha \) circle, say \( \alpha_1 \). Then, it is easy to see that \( \partial D \) must also be non-zero on some other \( \beta \) circle, say \( \beta_2 \). Let \( x_i \) and \( y_i \) be the \( x \) and \( y \) coordinates on \( \alpha_1 \). Then \( n_{p}(D) \neq 0 \) for \( p \in \{ x_1, x_2, y_1, y_2 \} \), and since each of them is at least \( \frac{1}{2} \), each of them is exactly \( \frac{1}{2} \). Thus \( \partial D \) must look like the boundary of a rectangle, and \( D \) itself must be a rectangle. Furthermore, it is also clear that \( D \) cannot contain any \( x \)-coordinate or any \( y \)-coordinate in its interior. \( \Box \)

Positive index one domains in \( \mathcal{D}(\hat{x}, \hat{y}) \) are called empty rectangles and the set of empty rectangles joining \( \hat{x} \) to \( \hat{y} \) is denoted by \( \mathcal{R}(\hat{x}, \hat{y}) \). Note that \( \mathcal{R}(\hat{x}, \hat{y}) = \emptyset \) unless \( \hat{x} \) and \( \hat{y} \) differ in exactly two coordinates, and even then \( \# \mathcal{R}(\hat{x}, \hat{y}) \leq 2 \). We define \( \mathcal{R}^0(\hat{x}, \hat{y}) = \mathcal{R}(\hat{x}, \hat{y}) \cap \mathcal{D}^0(\hat{x}, \hat{y}) \) and \( \mathcal{R}^{0,0}(\hat{x}, \hat{y}) = \mathcal{R}(\hat{x}, \hat{y}) \cap \mathcal{D}^{0,0}(\hat{x}, \hat{y}) \). For \( x = \hat{x} \prod U_i^{k_i} \) and \( y = \hat{y} \prod U_i^{l_i} \) in \( \mathcal{G}^- \), we define \( \mathcal{R}^0(x,y) = \mathcal{R}(\hat{x}, \hat{y}) \cap \mathcal{D}^0(x,y) \). The following lemma characterizes positive index \( k \) domains.

**Lemma 3.6.** Let \( D \in \mathcal{D}(\hat{x}, \hat{y}) \) be a positive domain. Then there exist generators \( \hat{u}_0, \hat{u}_1, \ldots, \hat{u}_k \in \hat{G} \) with \( \hat{u}_0 = \hat{x} \) and \( \hat{u}_k = \hat{y} \), and domains \( D_i \in \mathcal{R}(\hat{u}_{i-1}, \hat{u}_i) \) such that \( D = \sum D_i \).

**Proof.** We can assume that \( D \) is not a trivial domain, and thereby assume without loss of generality that \( n_{x_1}(D) \neq 0 \). Furthermore since \( \partial(\partial D_{o\alpha}) = \hat{y} - \hat{x} \), the coefficient of \( D \) at either the top-right square or the bottom-left square of \( x_1 \) must be non-zero. Assume after a rotation \( R(\pi) \) if necessary, that it is the top-right one.

Consider all rectangles \( R \), such that \( R \) is contained in \( D \) as 2-chains, and \( R \) has \( x_1 \) as its bottom-left corner. Partially order such rectangles by inclusion. Let \( R_0 \) be a maximal element under such an order, and let \( p_0 \) be the top-right corner of \( R_0 \). We want to show that \( R_0 \) contains an \( x \)-coordinate other than \( x_1 \).

Assume that \( D \) has non-zero coefficient at the square to the top-left of \( p_0 \). Since \( R_0 \) is a maximal element, either \( p_0 \) must lie on the \( \alpha \) circle immediately below the \( \alpha \) passing through \( x_1 \), or \( D \) must have zero
coefficient at some square above the top edge of $R$. In the first case, $R_0$ contains the $x$-coordinate lying on the $\beta$ circle passing through $p_1$ and so we are done. For the second case, let us start at $p_0$ and proceed left along the top edge of $R_0$ until we reach the first point $p_1$, such $D$ has non-zero coefficient at the top-right square of $p_1$, but has zero coefficient at the top-left square of $p_1$. Then it is easy to see that $p_1$ must be an $x$-coordinate, and once more, we are done. A similar analysis shows that if $D$ has non-zero coefficient at the bottom-right square of $p_0$, then too $R_0$ contains an $x$-coordinate other than $x_1$. Finally, if the coefficient of $D$ is zero at both the top-left and the bottom-right square of $p_0$, then $p_0$ itself is an $x$-coordinate.

Thus $D$ contains a rectangle $R_1$, with two $x$-coordinates, say $x_1$ and $x_2$, being the bottom-left corner and the top-right corner respectively. Now consider the partial order on rectangles that we have defined earlier, but restrict only to the ones whose top-right corner is an $x$-coordinate. Once again the poset is non-empty, since it contains $R_2$. Let $R_3$ be a minimal element. Then the rectangle $R_3$ is an index 1 domain connecting $\hat{x}$ to some generator $\hat{u}_1$. The positive domain $D \setminus R_3$ has index 1 less (alternatively, has a smaller sum of coefficients as 2-chains), and hence an induction finishes the proof. $\square$

From now on, until the rest of the section, we only consider the case for knots. There is a combinatorial sign assignment $s : \{(\hat{x}, \hat{y}, D)|\hat{x}, \hat{y} \in \hat{G}, D \in R(\hat{x}, \hat{y})\} \rightarrow \{-1, 1\}$, satisfying the following properties: for $D_1 \in R(\hat{x}, \hat{y})$ and $D_2 \in R(\hat{y}, \hat{x})$, if $D_1 + D_2$ is a horizontal annulus then $s(\hat{x}, \hat{y}, D_1)s(\hat{y}, \hat{x}, D_2) = 1$ and if $D_1 + D_2$ is a vertical annulus then $s(\hat{x}, \hat{y}, D_1)s(\hat{y}, \hat{x}, D_2) = -1$; for distinct $\hat{x}, \hat{y}, \hat{z}, \hat{w}$, if $D_1 \in R(\hat{x}, \hat{y})$, $D_2 \in R(\hat{y}, \hat{z})$, $D_3 \in R(\hat{z}, \hat{w})$ and $D_4 \in R(\hat{w}, \hat{z})$, then $s(\hat{x}, \hat{y}, D_1)s(\hat{y}, \hat{z}, D_2) = -s(\hat{x}, \hat{w}, D_3)s(\hat{w}, \hat{z}, D_4)$.

Two such sign assignments are said to be equivalent if one can be obtained from another by a sequence of moves, such that at each move, we fix a generator $\hat{x}$ and we switch the sign of every triple of the form $(\hat{x}, \hat{y}, D)$ or $(\hat{y}, \hat{x}, D)$.

The partial order in $\hat{G}$ is defined by declaring $\hat{y} \preceq \hat{x}$ if and only if there exists a positive domain in $D^{0,0}(\hat{x}, \hat{y})$. The partial order in $\hat{G}^-$ is defined by declaring $y \preceq x$ if and only if there exists a positive domain in $D^0(x, y)$. It is clear that in both the cases the elements in different Alexander gradings are not comparable. The covering relations are indexed by elements of $R^{0,0}(\hat{x}, \hat{y})$ and $R^0(x, y)$. It is routine to prove the following.

**Lemma 3.7.** [MOSzT07] For knots, with sign assignment as defined above, and the grading assignment being the Maslov grading, for each $m$, $\hat{G}_m$ and $\hat{G}^-_m$ are well-defined, finite, graded and signed posets.

In Section 4 we will see that the closed intervals in each of these posets are also shellable, and hence they will be GSS posets. However, just being graded and signed is enough for us to associate a chain complex to each of them. Let $C^-$ and $\hat{C}$ be the associated chain complexes. Their homology is bigraded, with the Maslov grading being the homological grading, and the Alexander grading being an extra grading.

**Theorem 3.8.** [MOSzT07] There is a bigraded abelian group $HF^-K(L)$ which depends only on the knot $L$, which is isomorphic (as bigraded abelian groups) to the homology of $C^-$. 

**Theorem 3.9.** [MOSzT07] There is a bigraded abelian group $\hat{HF}^K(L)$ which depends only on the knot $L$, such that the homology of $\hat{C}$ is isomorphic (as bigraded abelian groups) to $\hat{HF}^K(L) \otimes^{N-1} \mathbb{Z}^2$, where the $(M,A)$ bigrading of the two generators in $\mathbb{Z}^2$ are $(0,0)$ and $(-1, -1)$.

If everything is computed with coefficients in $\mathbb{F}_2$, then these two groups have to the hat version and the minus version of the knot Floer homology respectively. However with coefficients in $\mathbb{Z}$, the groups $\hat{HF}^K(L)$ and $HF^-K(L)$ do not have to be the hat and the minus version of the link Floer homology. This is because there could be a different sign convention on the grid poset whose homology is the knot Floer homology. However, by a slight abuse of notation, we still denote these groups as $\hat{HF}^K$ and $HF^-K$.

The following is a crucial piece of observation.
Lemma 3.10. If the grid diagram represents a knot, then $D^{0,0}(\hat{x}, \hat{y})$ consists of only the trivial domain. In particular, for any pair $\hat{x}, \hat{y} \in \hat{G}$, $\# |D^{0,0}(\hat{x}, \hat{y})| \leq 1$, and for any pair $x, y \in G^-$, $\# |D^0(x, y)| \leq 1$.

Proof. Number the $O$ points and the $X$ points modulo $N$ such that $O_i$ and $X_i$ lie in the same horizontal annulus and $O_{i+1}$ and $X_i$ lie in the same vertical annulus. Since the grid diagram represents a knot, such a numbering can be done.

Now let $A_i$ be the horizontal annulus through $O_i$, and let $B_i$ be the vertical annulus through $O_i$. Let $D \in D^{0,0}(\hat{x}, \hat{y})$ be a periodic domain with $D = \sum_i n_i A_i + \sum_i m_i B_i$. Since $n_{O_i}(D) = n_{X_i}(D) = 0$, we have $m_i = -n_i = m_{i+1}$. This implies that all the $m_i$’s are equal, and all the $n_i$’s are equal, and they are opposite of one another. Thus $D$ is the trivial domain. □

3.2. Commutation diagram. Many of the above results are true if we work with a commutation diagram instead of a grid diagram. Let $(G_c, \rho)$ be the commutation diagram, where we commute from $G$ to $G'$ by replacing $\alpha_c$ by $\alpha'_c$, and $\rho$ is one of the two intersection points between $\alpha_c$ and $\alpha'_c$. We define new posets $\hat{G_c}$ and $\hat{G^-}$ corresponding to the commutation diagram as follows. If $\hat{G}$ and $\hat{G'}$ are the generators of $G$ and $G'$, then $\hat{G_c} = \hat{G} \cup \hat{G'}$ and $\hat{G^-} = \hat{G}^c \cup (\hat{G'})^c$. For $\hat{x}, \hat{y} \in \hat{G}$, let $x_c$ and $y_c$ be the coordinates of $\hat{x}$ and $\hat{y}$ on $\alpha_c$ or $\alpha'_c$. If $\hat{x}, \hat{y} \in \hat{G}$, a domain joining $\hat{x}$ to $\hat{y}$ is a 2-chain $D$, such that $\partial(\partial D_{\{\alpha_c, \alpha'_c\}}) = \hat{y} - \hat{x}$ and $\partial D_{\{\alpha_c, \alpha'_c\}} = 0$; if $\hat{x}, \hat{y} \in \hat{G}$, a domain joining $\hat{x}$ to $\hat{y}$ is a 2-chain $D$ with $\partial(\partial D_{\{\alpha_c, \alpha'_c\}}) = \hat{y} - \hat{x}$ and $\partial D_{\{\alpha_c, \alpha'_c\}} = \rho - x_i$ and $\partial(\partial D_{\{\alpha'_c\}}) = y_i - \rho$. (We are not interested in domains that join points in $\hat{G'}$ to points in $\hat{G}$). The set of all such domains is denoted by $D(\hat{x}, \hat{y})$. The subset of $D(\hat{x}, \hat{y})$ consisting of domains which have coefficient zero at every $X$ marking is denoted by $D^0(\hat{x}, \hat{y})$, and the subset of $D(\hat{x}, \hat{y})$ consisting of domains which have coefficient zero at every $X$ marking and every $O$ marking is denoted by $D^{0,0}(\hat{x}, \hat{y})$. For $x = \hat{x} \prod_i U_i$ and $y = \hat{y} \prod_i U_i$ in $\hat{G^-}$, we define $D^0(x, y)$ as the subset of $D^0(\hat{x}, \hat{y})$ consisting of domains with $n_{O_i} = l_i - k_i$. We call a domain to be positive if it has non-negative coefficients everywhere. A 2-chain $D$ is said to be periodic if $\partial D$ is a collection of whole copies of $\alpha$ and $\beta$ circles. Note that there are more periodic domains than the ones in $D(\hat{x}, \hat{y})$ for any given generator $\hat{x}$.

The Alexander grading in $\hat{G}$, is the one induced from the Alexander grading in $\hat{G}$ and the Alexander grading in $\hat{G'}$. The Maslov grading in $\hat{G}$, is the one induced from the Maslov grading in $\hat{G}$ and the Maslov grading in $\hat{G'}$ shifted by $-1$. The $(M, A)$ bigrading of each $U_i$ is still $(−2, −1)$. For each of the objects $\hat{G_c}$ and $\hat{G^-}$, the Maslov grading shifted by $c$ is defined as $M_c = M + c$. Given a domain $D \in D(\hat{x}, \hat{y})$, we define the Maslov index as $\mu(D) = M(\hat{x}) - M(\hat{y}) + 2 \sum_i n_{O_i}(D)$. Note that this is different from the standard way of defining the Maslov index. We will soon encounter objects called empty pentagons, and according to our definition they have index 1, but according to the standard definition they have index 0. However, this should not be a cause for concern. This is an extremely minor issue, and we will be consistent and stick to our definition for the rest of the paper.

The partial orders are defined similarly. In $\hat{G}$, we define $\hat{y} \preceq \hat{x}$ if and only if there is a positive domain in $D^{0,0}(\hat{x}, \hat{y})$, and in $(\hat{G'})^c$, we define $y \preceq x$ if and only if there is a positive domain in $D^0(x, y)$. There exists a sign assignment for covering relations with properties analogous to the case for the grid diagrams.

We define $(\hat{G_c})_m$ to be the subset of $\hat{G}$ consisting of the elements with Alexander grading $m$, and we define $(\hat{G^-})_m$ to be the subset of $\hat{G^-}$ consisting of elements with Alexander grading $m$.

The following is a list of lemmas, very similar to the case for the grid diagrams. Most of them are mere verifications. We provide details of the proofs for some of the trickier cases.
Lemma 3.11. Periodic domains are generated by annuli. For horizontal annuli, we consider both the annuli coming from $G$ and the annuli coming from $G'$. Thus, periodic domains are generated by the annuli in $G$ and the special domain $D_c$ as shown in Figure 3.6.

\[
\begin{array}{cccc}
\alpha_c & 0 & \rho & 0 \\
\alpha'_c & 0 & 1 & 0 \\
0 & -1 & 0
\end{array}
\]

**Figure 3.6. Coefficients of the special domain $D_c$**

Lemma 3.12. For any positive domain $D$ in $D^0(\hat{x}, \hat{y})$, each of the coefficients of $D$ in the four regions around $\rho$ is at most 1.

Proof. Recall that one of the 4 regions around $\rho$ is an $X$ marking, and hence the coefficient of $D$ at that region is 0. After the rotation $R(\pi)$ if necessary, we can assume that the region is to the right of $\rho$. If $D$ is a domain in either $G$ or $G'$, then it is easy to see that $n_\rho(D)$ is either 0 or $\frac{1}{2}$. So let us assume that $\hat{x} \in \hat{G}$ and $\hat{y} \in \hat{G}'$. If $x_c$ is the $\hat{x}$-coordinate on $\alpha_c$ and $y_c$ is $\hat{y}_c$-coordinate on $\alpha'_c$, then $\partial(\partial D_{\alpha_c}) = \rho - x_c$ and $\partial(\partial D_{\alpha'_c}) = y_c - \rho$. Therefore, there is a path which goes from $x_c$ to $\rho$ along $\alpha_c$ and then from $\rho$ to $y_c$ along $\alpha'_c$, such that we never have to make an $180^\circ$ turn along the path, and the path coincides with $\partial D_{(\alpha_c, \alpha'_c)}$ as 1-chains. Now we will prove that such a path hits $\rho$ exactly once. Note that the lemma will follow from such an observation.

Assume if possible that the path hits $\rho$ at least twice. Then look at the part of the path between the first hit and the second hit. This part has to one copy of either $\alpha_c$ or $\alpha'_c$, and neither is allowed since either has an $X$ marking immediately on its left. \[\square\]

Lemma 3.13. For a knot, a periodic domain $D$ with $n_{X_i}(D) = n_{\alpha_i}(D) = 0$ for all $i$, is generated by the special domain $D_c$. That implies that, given $x, y \in G_c^0$, there can be at most 2 positive domains in $D^0(x, y)$.

Lemma 3.14. $\text{MOSZT07}$ $A(x)$ and $M(x)$ are well-defined and they take integral values for a knot.

A domain $R$ in $D(\hat{x}, \hat{y})$ is said to be an empty rectangle if either $\hat{x}, \hat{y} \in \hat{G}$ and $R$ is an empty rectangle in $G$, or if $\hat{x}, \hat{y} \in \hat{G}'$ and $R$ is an empty rectangle in $G'$. A domain $P$ in $D(\hat{x}, \hat{y})$ is defined to be an empty pentagon if $P$ has coefficients 0 or 1 everywhere and the closure of the regions where $P$ has coefficients 1 forms a pentagon which does not contain any $x$-coordinate or any $y$-coordinate in its interior. The set of all empty rectangles and empty pentagons joining $\hat{x}$ to $\hat{y}$ is denoted by $R(\hat{x}, \hat{y})$. If $\hat{x}, \hat{y} \in \hat{G}_c$, we define $R^0(\hat{x}, \hat{y}) = R(\hat{x}, \hat{y}) \cap D^0(\hat{x}, \hat{y})$, and $R^{0,0}(\hat{x}, \hat{y}) = R(\hat{x}, \hat{y}) \cap D^{0,0}(\hat{x}, \hat{y})$; if $x, y \in G_c$, we define $R^0(x, y) = R(x, y) \cap D^0(x, y)$.

Lemma 3.15. Empty rectangles and empty pentagons have index 1.

Lemma 3.16. Let $D \in D(\hat{x}, \hat{y})$ be a positive domain. Then there exists generators $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_k \in \hat{G}_c$ with $\hat{u}_0 = \hat{x}$ and $\hat{u}_k = \hat{y}$, and domains $D_i \in R(\hat{u}_{i-1}, \hat{u}_i)$ such that $D = \sum_i D_i$. This implies that the positive domains have non-negative index.
Proof. We only prove the first part of the lemma. The second part is a trivial implication.

If both \( \tilde{x} \) and \( \tilde{y} \) belong to \( \tilde{\mathcal{G}} \), or if both of them belong to \( \tilde{\mathcal{G}}' \), then the lemma follows directly from Lemma 3.6. So let us assume that \( \tilde{x} \in \tilde{\mathcal{G}} \) and \( \tilde{y} \in \tilde{\mathcal{G}}' \). If \( x_c \) is the \( x \)-coordinate on \( \alpha_c \), then \( n_{x_c}(D) \neq 0 \), and therefore either the region to the top-right, or the region to the top-left of \( x_c \) has non-zero coefficient. If both the regions have non-zero coefficients, choose the region which lies between \( x_c \) and \( \rho \) and not the region that lies between \( x_c \) and the other intersection point between \( \alpha_c \) and \( \alpha'_c \). After a rotation by \( R(\pi) \) if necessary, assume that this region is to the top-right of \( x_c \).

Now consider all the 2-chains \( R \), such that \( R \) either looks like a rectangle whose top edge is not on \( \alpha'_c \), or looks like a pentagon with \( \rho \) as a vertex, \( R \) lies inside \( D \) as 2-chains and \( R \) contains \( x_c \) as its bottom-left corner. Order such 2-chains by inclusion. A careful observation reveals that the set of such 2-chains is non-empty, and we let \( R_0 \) be a maximum. An analysis similar to the one in Lemma 3.6 shows that \( R_0 \) must contain some \( x \)-coordinate other than \( x_c \). We then restrict our partial order to only those rectangles and pentagons, whose top-right corners are \( x \)-coordinates. If \( R_1 \) is a minimum, then it is easy to see that \( R_1 \) is either an empty rectangle or an empty pentagon, and as 2-chains \( D \setminus R_1 \) has a smaller sum of coefficients than \( D \), thus completing the induction. \( \square \)

Lemma 3.17. Let \( D \in \mathcal{D}(\tilde{x}, \tilde{y}) \) be a positive domain with \( \mu(D) = 1 \). Then \( D \) is either an empty rectangle or an empty pentagon.

Proof. This follows directly from Lemma 3.16. \( \square \)

Lemma 3.18. [MOSZ107] For knots, with sign assignment as referred to in the beginning of this subsection, and with the grading being the Maslov grading, each of the posets \( (\mathcal{G}_c)_m \) and \( (\mathcal{G}_c)_m^- \) is well-defined, finite, signed and graded.

In Section 4 we will see that the closed intervals in commutation posets are also shellable, and so like grid posets, they will also be GSS posets.

3.3. Stabilization diagram. Now we repeat the whole process for the stabilization diagram. We only consider the cases (a) and (b) of Figure 3.5. The other two cases are obtained by the rotation \( R(\pi) \).

Let \( H \) be the grid diagram before the stabilization and let \( G \) be the diagram after. Let \( G_s = (G, \rho) \) be the stabilization diagram, where \( \alpha_s \) and \( \beta_s \) are the extra circles, and \( \rho \) is their intersection point. Define \( \tilde{\mathcal{L}} \) to be the set of all the intersection points in \( \tilde{\mathcal{G}} \) which contain \( \rho \) as one of its coordinates, and define \( \tilde{\mathcal{I}}^- \) to be set of all the generators in \( \tilde{\mathcal{G}}^- \) which come from \( \tilde{\mathcal{L}} \). Let \( \tilde{\mathcal{N}}^- = \tilde{\mathcal{G}}^- \setminus \tilde{\mathcal{I}}^- \) and let \( \mathcal{N}^- = \mathcal{G}^- \setminus \mathcal{I}^- \).

Let us number the \( X \) and \( O \) marking in \( G \) as \( X_0, X_1, \ldots, X_N \) and \( O_0, O_1, \ldots, O_N \) such that the neighborhood of \( \rho \) contains the points \( O_0, X_0, X_1 \) with \( O_0 \) directly above \( X_0 \), and \( O_1 \) in the same horizontal annulus as \( X_0 \). Thus \( H \) is obtained from \( G \) by deleting \( \alpha_s, \beta_s, O_0 \) and \( X_0 \), and the rest of the points being numbered the same.

Note that there is a natural bijection \( \tilde{f} \) from \( \tilde{\mathcal{L}} \) to \( \tilde{\mathcal{H}} \), and we will always identify them in this subsection using this bijection. This bijection induces a map \( f^- \) from \( \tilde{\mathcal{I}}^- \) to \( \mathcal{H}^- \) given by \( f^-(\tilde{x} \prod_{i=0}^N U_i^{n_i}) = f(\tilde{x})U_0^{n_0+n_1} \prod_{i=2}^N U_i^{n_i} \).

We define \( \tilde{\mathcal{G}}_s \) to be a disjoint union of \( \tilde{\mathcal{G}} \) and two copies of \( \tilde{\mathcal{H}} \), denoted by \( \tilde{\mathcal{H}} \) of \( \tilde{\mathcal{H}}' \). In case (a), the \((M, A)\) bigrading in \( \tilde{\mathcal{G}}_s \) is obtained from the one induced from \( \tilde{\mathcal{G}} \), the one induced from \( \tilde{\mathcal{H}} \) shifted by \((-1, 0)\), and the one induced from \( \tilde{\mathcal{H}}' \) shifted by \((-2, -1)\). In case (b), the \((M, A)\) bigrading in \( \tilde{\mathcal{G}}_s \) is obtained from the one induced from \( \tilde{\mathcal{G}} \), the one induced from \( \tilde{\mathcal{H}} \) shifted by \((1, 0)\), and the one induced from \( \tilde{\mathcal{H}}' \) shifted by \((0, -1)\).
For $\hat{x}, \hat{y}$ both in $\hat{G}$ or $\hat{H}$ or $\hat{H}'$, $D(\hat{x}, \hat{y})$ is the set of all domains joining $\hat{x}$ to $\hat{y}$ in the corresponding grid diagram. In case (a), for $\hat{x} \in \hat{G}$ and $\hat{y} \in \hat{H}$ or $\hat{y} \in \hat{H}'$, we define $D(\hat{x}, \hat{y}) = D(\hat{x}, \hat{f}^{-1}(\hat{y}))$. In case (b), for $\hat{x} \in \hat{H}$ or $\hat{x} \in \hat{H}'$ and $\hat{y} \in \hat{G}$, we define $D(\hat{x}, \hat{y}) = D(\hat{f}^{-1}(\hat{x}), \hat{y})$. We are not interested in domains joining other possible pairs of generators. The Maslov index $\mu$ and the set of all empty rectangles $R$ are defined in a way similar to the case for the grid diagrams. Unless one of the generators $\hat{x}, \hat{y}$ belongs to $\hat{G}$ and the other one belongs to $\hat{H}$, the sets $D^0, D^{0,0}, R^0$ and $R^{0,0}$ are also defined analogously. However, for $\hat{x} \in \hat{G}$ and $\hat{y} \in \hat{H}$ in case (a), or for $\hat{x} \in \hat{H}$ and $\hat{y} \in \hat{G}$ in case (b), while defining $D^0(\hat{x}, \hat{y}), D^{0,0}(\hat{x}, \hat{y}), R^0(\hat{x}, \hat{y})$ and $R^{0,0}(\hat{x}, \hat{y})$, we require all the domains to have $n_{x_0} = 1$.

In $G_s$, the partial order is given by $\hat{y} \preceq \hat{x}$ if and only if there is a positive domain in $D^{0,0}(\hat{x}, \hat{y})$. Note that for $\hat{x} \in \hat{G}$ and $\hat{f}(\hat{x}) \in \hat{H}'$, in case (a), the trivial domain is a positive domain in $D^{0,0}(\hat{x}, \hat{f}(\hat{x}))$ and hence $\hat{f}(\hat{x}) \prec \hat{x}$, and in case (b), the trivial domain is a positive domain in $D^{0,0}(\hat{f}(\hat{x}), \hat{x})$ and hence $\hat{x} \prec \hat{f}(\hat{x})$. However for $\hat{x} \in \hat{G}$ and $\hat{y} \in \hat{H}$ in case (a), or for $\hat{x} \in \hat{H}$ and $\hat{y} \in \hat{G}$ in case (b), the partial order does not come from trivial domains, since the domains are required to have $n_{x_0} = 1$.

We define $G_s^-$ to be a disjoint union of $G^-$ and $H^-$. The Alexander grading in $G_s^-$ is the one induced from $G^-$ and $H^-$, and the Maslov grading in $G_s^-$ is induced from the one in $G^-$ and the one in $H^-$ shifted by $-1$ in case (a) and $1$ in case (b).

For $G_s^-$, if $x, y$ both lie in either $G^-$ or $H^-$, the partial order is the usual one given by the existence of $\bar{D}^0(\hat{x}, \hat{y})$ for the relevant grid diagram. In case (a), for $\hat{x} \in \hat{G}$ and $\hat{y} \in \hat{H}$, we declare the partial order to be given by $\hat{y}U_{i=1}^{n_{x_0}+n_{y_0}+k_0+i_0} \preceq \hat{x} \Pi_{i \geq 0} U_{i}^{n_i}$ if and only if there is a positive domain in $D(\hat{x}, \hat{f}(\hat{y}))$ which has $n_{O_i} = k_i$ for all $i$, $n_{X_i} = 0$ for all $i \geq 1$ and $n_{x_0} = 1$. In case (b), for $\hat{x} \in \hat{H}$ and $\hat{y} \in \hat{G}$, we declare the partial order to be given by $\hat{y}U_{i=1}^{k_0} \Pi_{i \geq 0} U_{i}^{n_i} \preceq \hat{x} \Pi_{i \geq 0} U_{i}^{n_i}$ if and only if there is a positive domain in $D(\hat{f}(\hat{x}), \hat{y})$ which has $n_{O_i} = k_i$ and $n_{X_i} = 0$ for all $i$. The set of all such domains is called $\bar{D}^0(\hat{x}, \hat{y})$. Once again note that, in case (a), the trivial domains do not contribute to the partial order.

For both $G_s$ and $G_s^-$, in both the cases (a) and (b), we choose a sign assignment $s$ on the grid diagram $G$, and then declare the signs in $G$ and $G^-$ to be the signs induced from $s$, the signs in $H$ and $H^-$ to be the signs induced from $-s$, and the signs of covering relations corresponding to the trivial domain to be 1. Most of the results proved in the subsection for the grid diagrams are true here with some minor modifications. We just mention the few results that are slightly different.

**Lemma 3.19.** If $\hat{x}, \hat{y}$ belong to $G_s$ and $D$ is a positive domain in $D^0(\hat{x}, \hat{y})$, or if $x, y$ belong to $G_s^-$, and $D$ is a positive domain in $D^0(x, y)$, at most two regions around $\rho$ have non-zero coefficients, and each of the coefficient is at most 1.

**Lemma 3.20.** If $D$ is a periodic domain in the grid diagram $G$, such that $n_{X_i} = 0$ for all $i$, $n_{O_i} = 0$ for all $i > 1$, and $n_{O_0} + n_{O_1} = 0$, then $D$ is generated by the special domain $D_s$, which is the vertical annulus through $X_0$ minus the horizontal annulus through $X_0$.

**Lemma 3.21.** In case (a), for $x \in G^-$ and $y \in H^-$, there are at most two positive domains in $D^0(x, y)$. For any other combination of $x, y$ in $G_s^-$ or $\hat{x}, \hat{y}$ in $G_s$ in either case (a) or case (b), there is at most one such positive domain.

Like before, in each Alexander grading $m$, the stabilization posets turn out to be well-defined, finite, graded and signed. In the next section, we will prove that the closed intervals in these posets are also shellable.
4. GSS shellability

As promised earlier, in this section we will show that the different posets that we encountered in the previous section are indeed GSS posets.

Let $G$ be a grid diagram index $N$ drawn on a torus $T$, which represents a knot $K$. Recall that if $\hat{x}, \hat{y} \in \hat{G}$, then $\hat{y} \preceq \hat{x}$ if and only if there is a positive domain in $D_{0,n}(\hat{x}, \hat{y})$, and if $x, y \in G^{-}$, then $y \preceq x$ if and only if there is a positive domain in $D_{0}(x, y)$. We now show that each closed interval in either of these posets is EL-shellable. First note that it is enough to prove it in the case of $G^{-}$. Draw a circle $l$ which is disjoint from all the $\beta$ circles and intersects each $\alpha$ circle exactly once.

Let $R \in \mathcal{R}(x, y)$ be an empty rectangle not containing any $X$ marking. To such an empty rectangle $R$, we associate a triple $(s(R), i(R), t(R))$, where $s(R)$ is 0 if $R$ intersects $l$ and is 1 otherwise. If $s(R) = 0$, $i(R)$ is the minimum number of $\beta$ circles we have to intersect to reach the leftmost arc of $R$, starting at $l$ and going left throughout. If $s(R) = 1$, $i(R)$ is the minimum number of $\beta$ circles we have to intersect to reach the leftmost arc of $R$, starting at $l$ and going right throughout. The number $t(R)$ always denotes the thickness of a rectangle $R$. The set of such triples is ordered lexicographically, and thus we have a map from the set of all covering relations to a totally ordered set.

**Theorem 4.1.** Let $x, y \in G^{-}$. The map which sends a covering relation represented by an empty rectangle $R$ to $(s(R), i(R), t(R))$ induces an EL-shelling on the closed interval $[y, x]$.

Note that the interval $[y, x]$ is non-empty if and only if $y \preceq x$. From now on, we only consider that case. Recall from Section 2 that given a maximal chain $m = y \leftarrow z_1 \leftarrow \cdots \leftarrow z_n \leftarrow x$ in $[y, x]$, we associate to it the labeling $((s, i, t)(y \leftarrow z_1), \ldots, (s, i, t)(z_n \leftarrow x))$, where $(s, i, t)(p \leftarrow q)$ is the $(s, i, t)$-triple associated to the empty rectangle corresponding to the covering relation $p \leftarrow q$. Also note that given a generator $z \in G^{-}$, and a triple $(s, i, t)$, there is at most one generator $z'$ covering $z$, such that the covering relation corresponds to that triple. Thus each maximal chain in $[y, x]$ has a unique labeling. Therefore there is a unique maximal chain $m_0$ for which the labeling is lexicographically the minimum. The following two lemmas prove the above theorem.

**Lemma 4.2.** The lexicographically minimum labeling is an increasing labeling.

**Proof.** Assume not. Let $m_0$ be the unique maximal chain whose labeling is lexicographically the minimum. Let $m \leftarrow n \leftarrow p$ be the first place in $m_0$ where the labeling decreases. Let $R_1$ and $R_2$ be the two empty rectangles corresponding to the two covering relations. Since each vertical annulus and each horizontal annulus has at least one $X$ marking, $\partial(R_1 + R_2)$ must be non-zero on at least three $\beta$ circles (and clearly on at most four $\beta$ circles).

If it is non-zero on exactly four $\beta$ circles, then switch $R_1$ and $R_2$, thereby producing a new maximal chain whose labeling is smaller than the labeling for $m_0$ and thus contradicting the assumption that the labeling for $m_0$ was the minimum. If on the other hand, $\partial(R_1 + R_2)$ is non-zero on exactly three $\beta$ circles, then $R_1 + R_2$ looks like a hexagon. Depending on the shape of the hexagon and the position of the line $l$, only the cases as shown in Figure 4.1 can occur. In each of the cases, the lexicographically best way to divide the hexagon is shown, and in each case, that happens to correspond to a chain where the labeling is increasing. This proves that the labeling for the maximal chain $m_0$ is increasing.

**Lemma 4.3.** If the labeling corresponding to a maximal chain is an increasing labeling, then that maximal chain is the one whose labeling is lexicographically the minimum.

**Proof.** Let $m$ be the maximal chain whose labeling is increasing, and let $m_0$ be the unique maximal chain whose labeling is lexicographically the minimum. We want to show that $m = m_0$. 
Starting at $y$, let us assume that they agree up to a generator $z$. Let $D$ be the unique positive domain in $D^0(x,z)$. Let $m_1 = m_0 \cap [z,x]$ and let $m_2 = m \cap [z,x]$. Let $R$ and $R'$ be the rectangles corresponding to the two covering relations on $z$ coming from the two chains $m_1$ and $m_2$. We will show that $(s(R), i(R), t(R)) = (s(R'), i(R'), t(R'))$ which would imply that $r = r'$; that, in turn would imply that $m$ and $m_0$ agree for at least one more generator, thus concluding the proof.

Now, if $D$ does not intersect $l$, then $s$ is forced to be 1. On the other hand, if $D$ does intersect $l$, then eventually in both $m_1$ and $m_2$, some covering relation will have $s = 0$, and since the labelings in both $m_1$ and $m_2$ are increasing, they both must start with $s = 0$. Therefore, we see that $s$ is fixed.

First we analyze the case when $s = 1$. So assume that the whole domain $D$ lies to the right of $l$, and let $i_0$ be the minimum number of $\beta$ circles we have to cross to reach $D$ from $l$ going right throughout. Clearly $i$, the second coordinate in the triple $(s, i, t)$, can never be smaller than $i_0$. Furthermore, since the whole domain $D$ has to be used up in both the chains $m_1$ and $m_2$, so at some point, $i$ will be equal to $i_0$. Since the labelings in both $m_1$ and $m_2$ are increasing, this fixes $i = i_0$.

To see that $t$ is also fixed, we need an induction statement. Look at all $p$ of the form $z \leftarrow p \leq x$, such that the covering relation $z \leftarrow p$ is by a rectangle with $i = i_0$. Let $R_0$ be the thinnest rectangle among them and let $t_0$ be the thickness of $R_0$. Our induction claim is that, at some point in the chain, we have to use a rectangle with $i = i_0$ and $t \leq t_0$. The induction is done on the length of the interval $[z,x]$. Clearly when this length is 2, the statement is true. Let us assume that we do not start with the thinnest rectangle, but rather start with a rectangle $\overline{R_0}$. Since both $R_0$ and $\overline{R_0}$ are index 1 domains, they do not contain any coordinate of $z$ in their interior, and hence the local diagram must look like Figure 4.2.
Figure 4.2. Fixing the thickness of the starting rectangle when $s = 1$

Since the Maslov index of $D \setminus \tilde{R}_0$ is 1 lower than that of $D$, and since it has a starting rectangle with $(i, t) = (i_0, t_0)$, so induction applies finishing the proof. Thus in both the chains $m_1$ and $m_2$, at some point we have to use a rectangle with $i = i_0$ and $t \leq t_0$. But since the labelings in both $m_1$ and $m_2$ are increasing, and $(i_0, t_0)$ is the smallest value of $(i, t)$ that we can start with, we have to start with $t = t_0$. Thus, this fixes $t$.

Now, let us assume that $s = 0$. We need an induction statement to show that $i$ is fixed. For each coordinate $z_i$ of $z$, consider the horizontal line segment $h_i$ lying on some $\alpha$-curve, which starts at $z_i$ and ends at $l$ and goes right throughout. We call $z_i$ to be admissible if every point just below the line segment $h_i$ belongs to $D$. Since the starting rectangles in the chains $m_1$ and $m_2$ have $s = 0$, so there is at least one admissible coordinate. Among all the admissible coordinates, let $z_1$ be the one with $h_i$ having the smallest length. Let $i_0$ be the smallest length, measured by number of intersections with $\beta$ curves. Our induction claim is that, at some point in the chain, we have to use a rectangle with $(s, i) = (0, i_0)$ and $t \leq t_0$, and the induction is done on the length of $[z, x]$. Clearly when the length is 2, the claim is true. Let us assume that we start with a rectangle $R_0$ with $s = 0$ and $i > i_0$. Since both have increasing labelings, we must start with a rectangle with $(s, i) = (0, i_0)$. Next, we want to show that $t$ is also fixed. This is also done by an induction very similar to the ones above. Consider all $p$ with $z \leftarrow p \leq x$, such that the covering relation $x \leftarrow p$ has $(s, i) = (0, i_0)$. Let $R_0$ be the thinnest rectangle among all such covering relations, and let $t_0$ be the thickness of $R_0$. The induction claim is that at some point in the chain, we have to use a rectangle with $(s, i) = (0, i_0)$ and $t \leq t_0$, and the induction is done on the length of $[z, x]$. Once again, it is trivial when the length is 2. Assume that we start with a rectangle $\tilde{R}_0$ with $(s, i) = (0, i_0)$ and $t > t_0$. Since both $R_0$ and $\tilde{R}_0$ have index one, they must look like Figure 4.3.

Note that $D \setminus R_0$ has index one lower than $D$ and it still intersects $l$, and it still has an admissible coordinate with $h = i_0$. Thus induction applies. Since the labelings for $m_1$ and $m_2$ are both increasing, this implies that they both must start with a rectangle with $(s, i, t) = (0, i_0, t_0)$. Thus we see that the thickness is fixed.
As explained earlier, this finishes the proof.

Using the theorems from Section 2, this implies the following.

**Theorem 4.4.** Each subinterval of a closed interval in the grid poset is shellable. For intervals of the form \((y, x)\), the order complex is a sphere, and for intervals of the form \([y, x]\), \([y, x)\) or \((y, x]\), the order complex is a ball.

Therefore, using the results from Section 3 we see that \(\hat{G}, \hat{G}_m\), and \(G_m\) (in each Alexander grading \(m\)) are GSS posets.

Now, we concentrate the commutation posets \(\hat{G}_c\) and \(G_m^\cdot\). Let \((G_c, \rho)\) be a commutation diagram. We are trying to prove that the closed intervals in these posets are shellable. Once more, it is enough to restrict our attention to the closed intervals in \(G_m^\cdot\).

**Theorem 4.5.** Closed intervals in the commutation poset are shellable.

**Proof.** We do not know if the closed intervals are always EL-shellable; the theorem only guarantees the shellability of closed intervals. For \(\hat{x}, \hat{y} \in \hat{G}_c\), let \(D \in D^0(\hat{x}, \hat{y})\) be a positive domain with \(n_{\alpha}(D) = k_i\). If \(x = \hat{x}\) and \(y = \hat{y} \prod U_i^{k_i}\), we will prove that the closed interval \([y, x]\) is shellable. Note that \(n_{\rho}(D) < 1\). So we prove this by taking cases.

**Case 1:** \(D\) is the unique positive domain joining \(x\) to \(y\) and \(n_{\rho}(D) \neq \frac{1}{4}\).

We can choose any vertical line \(l\) disjoint from all the \(\beta\) circles (indeed we can choose a vertical line through \(\rho\)) and define \((s, i, t)\) as in the proof of the previous theorem. Essentially the same proof shows that this is an EL-shelling. It is important to note that we can also apply the rotation \(R(\pi/2)\) (such that the horizontal commutation becomes a vertical commutation), and then take a vertical line \(l\) (this time disjoint from all the \(\alpha\) circles), and then define \((s, i, t)\) which still induces an EL-shelling. The line \(l\) has to be disjoint from \(\alpha_c\) and \(\alpha_c'\) (which are now vertical circles), and we stipulate (for defining \(i\) and \(t\)) that both of them are equidistant from \(l\).

**Case 2:** \(n_{\rho} = \frac{1}{4}\).
In this case, using Lemma 3.12, \( D \) is the unique positive domain joining \( x \) to \( y \). Choose a vertical line \( l \) passing through \( \rho \), the chosen intersection point between \( \alpha_c \) and \( \alpha'_c \). To each covering relation, associate a 4-tuple \((s, i, t, p)\), where \( s, i \) and \( t \) are defined similarly and \( p = 1 \) if the covering relation corresponds to a pentagon, and is 0 otherwise. Thus given \( y \), and a 4-tuple \((s, i, t, p)\), there is at most one \( x \) with \( y \leftarrow x \) corresponding to that 4-tuple. The tuples are ordered lexicographically, and thus all maximal chains in \([y, x]\) have their edges labeled by a totally ordered set, and hence themselves get an induced total ordering. We claim that this ordering gives the required shelling.

We follow the general outline of the proof of Theorem 2.4. Let \( m_1 \) and \( m_2 \) be two maximal chains, with \( m_1 < m_2 \). Let \( m_1 \) and \( m_2 \) agree from \( y \) to \( y_1 \) and then start to disagree, and then agree once more at \( x_1 \) (and then maybe disagree again). Thus we can restrict our attention on the interval \([y_1, x_1]\), which has a smaller length. Hence by induction, we will be done. Therefore, we can assume that \( y_1 = y \) and \( x_1 = x \), i.e. \( m_1 \) and \( m_2 \) never agree. The domain \( D \) corresponding to \([y, x]\) might now have \( n_\rho(D) \neq \frac{3}{4} \). But note that \( D \) is still the unique positive domain joining \( x \) to \( y \), and therefore if \( n_\rho(D) \neq \frac{3}{4} \), then we have reduced this case to the previous case. Hence assume that \( D \) still has \( n_\rho = \frac{3}{4} \).

If \( m_2 \) has a subchain \( y_{k-1} \leftarrow y_k \leftarrow y_{k+1} \), where the 4-tuples corresponding to the two covering relations decrease, and the domain corresponding to \([y_{k-1}, y_{k+1}]\) does not look like any of the two domains in Figure 4.4(a), then we can change \( m_2 \) by replacing \( y_k \) with \( y'_k \) with \( y_{k-1} \leftarrow y'_k \leftarrow y_{k+1} \). Call such an operation a switching operation. A case by case analysis shows that the new maximal chain obtained after a switching operation is smaller than the original. Call the operation of changing one element of a maximal chain to get a smaller maximal chain, a generalized switching operation. Thus a switching operation is a generalized switching operation.

**Figure 4.4.** The special index 2 domain

Since we are trying to prove shellability, hence we can assume that \( m_2 \) does not admit any generalized switching operation. In that case there is an element \( z \) in \( m_2 \), such that \( m_2 \cap [y, z] \) is a chain in \((\mathcal{G}')^-\) with increasing labeling, and \( m_2 \cap [z, x] \) is a chain with increasing labeling which starts with an empty pentagon but there is no \( x' \in [z, x] \) such that the domain corresponding to \([z, x']\) looks any of the two domains in Figure 4.4(b). We call such a maximal chain to be quasi-increasing. Thus in a quasi-increasing chain, there
exists \( y' \in [y, z] \) and \( x' \in [z, x] \) such that \( y' \leftarrow z \leftarrow x' \) and the index 2 domain corresponding to \([y', x']\) is one of domains shown in Figure 4.4. In all the cases, the \( z \)-coordinates are marked.

Now, we want to show that \( m_2 \) is the smallest chain. This will rule out the possibility of having a chain \( m_1 \) with \( m_1 < m_2 \), thereby finishing the proof. Thus, if possible, let \( m_1 < m_2 \). We can do the generalized switching operations as described above, on \( m_1 \), such that \( m_1 \) also becomes quasi-increasing. Now if we show \( m_1 = m_2 \), we will have the required contradiction.

Thus it boils down to showing that there is a unique quasi-increasing chain. The proof is essentially the same as the proof of uniqueness of a maximal chain with increasing labeling, in the previous theorem. Therefore in this case, we are done.

**Case 3:** There are exactly two positive domains \( D \) and \( D' \) joining \( x \) to \( y \).

By assumption, note that both \( D \) and \( D' \) have \( n_{O_i} = k_i \). Also both \( D \) and \( D' \) must have \( n_p = \frac{1}{4} \). For simplicity, we apply the rotation \( R(\hat{y}) \). After the rotation, all the \( \alpha \) circles (incl. \( \alpha_c \) and \( \alpha'_c \)) become vertical circles. Let \( D \) be the domain which has non-zero coefficient in the region immediately to the left of \( \rho \). We choose the vertical circle \( l \) to be line immediately to the left of \( \alpha_c \) and \( \alpha'_c \). We define \( (s, i, t) \) as in the proof of the previous theorem. Note that we assume both \( \alpha_c \) and \( \alpha'_c \) be to distance 1 to the right of \( l \). Furthermore, note that given \( y \) and a triple \((s, i, t)\), there is at most one \( x \) with \( y \leftarrow x \) corresponding to that triple. Thus each maximal chain gets a unique labeling. We use this labeling to totally order all the maximal chains that come from \( D \), and also all the maximal chains that come from \( D' \). We then declare all the maximal chains that come from \( D' \) to be smaller than all the maximal chains that come from \( D \). We claim that this ordering is a shelling.

Again following the general outline of the proof of Theorem 2.3, let \( m_1 \) and \( m_2 \) be two maximal chains with \( m_1 < m_2 \). By restricting to smaller chains if necessary, we can assume that the two maximal chains are disjoint. After restricting to smaller chains, we can still assume that \( D \) and \( D' \) are two distinct domains joining \( x \) to \( y \), or else we have reduced this to an earlier case.

Now we can assume that the labeling on \( m_2 \) is non-decreasing, since otherwise we can do a switching operation to make it smaller. But each of the domains \( D \) and \( D' \) has a unique chain for which the labeling is non-decreasing, which in addition is the smallest maximal chain among all maximal chains coming from that domain. Since \( m_1 \) is a maximal chain which is smaller than \( m_2 \), hence \( m_2 \) must be the unique maximal chain with non-decreasing labeling, coming from \( D \).

By assumption, the line \( l \) lies entirely inside \( D \), and hence the first two covering relations in \( m_2 \) starting at \( y \) must have \((s, i, t)\) as \((0, 1, 1)\). Thus we can do a switch, where this index 2 domain can be replaced another index 2 domain, which is this domain minus \( D_s \), where \( D_s \) is the special domain from Figure 3.6. After the switch, the new maximal chain comes from \( D' \), and hence is smaller than \( m_2 \). This completes the proof of shellability.

Since the commutation poset was already a graded and signed poset, this completes the proof that it is a GSS poset. We now prove that the closed intervals in the stabilization poset in either case \((a)\) or case \((b)\) from Figure 3.5 are also shellable.

**Theorem 4.6.** Closed intervals in the stabilization poset are shellable.

**Proof.** In both \( \hat{G}_x \) and \( \hat{G}^-_x \), even if we are allowed to pass through \( X_0 \), in most cases, the proof of shellability (indeed EL-shellability) follows directly from the proof of EL-shellability of intervals in the grid poset. There are only three cases which are slightly different.

The first case that is slightly different is when \( \hat{x} \in \hat{G} \) and \( \hat{y} \in \hat{H}' \) in case \((a)\), or when \( \hat{x} \in \hat{H}' \) and \( \hat{y} \in \hat{G} \) in case \((b)\). Here in any maximal chain, there will be exactly one covering relation corresponding to the trivial domain. Let us assign the \((s, i, t)\)-label to each of those covering relations as \((-1, 0, 0)\). It is easy to see that this labeling still induces an EL-shelling of the interval \([\hat{y}, \hat{x}]\). In fact this interval is the Cartesian product.
**Order Complex.** We can give a CW complex structure on the order complex where the $k$-cells correspond to the closed intervals of length $(k + 1)$. The boundary map maps to the cells corresponding to all the closed subintervals of length $k$. The boundary map is well defined because the union of such cells forms a sphere of the right dimension.

**Theorem 5.1.** The above defined CW complex is well defined and is homeomorphic to the order complex.

**Proof.** Recall that the order complex of an interval $[y, x]$ of length $(k + 1)$ is a ball of dimension $k$. The order complex of the whole poset is just the union of all such balls; thus we only need to understand the boundary map. The boundary of the order complex of $[y, x]$ consists of the order complexes of all submaximal chains that are covered by exactly one chain, or in other words, maximal chains in either $[y, x]$ or $(y, x)$. But the order complex of each of $[y, x]$ and $(y, x)$ is a ball of dimension $(k - 1)$, with common boundary being the order complex of $(y, x)$ which is a sphere of dimension $(k - 2)$. Thus the order complexes of $[y, x]$ and $(y, x)$ glue to form a sphere of dimension $(k - 1)$, and is the boundary of the order complex of $[y, x]$. Therefore, the boundary map in the order complex is the same as the boundary map in our CW complex. This shows that the CW complex is well defined and is same as the order complex.

**5.2. Fake moduli space.** Given a positive domain $D \in \mathcal{D}^0(x, y)$ with $\mu(D) = k$, we construct a CW complex which has many properties of what the actual moduli space should have, although it is not clear whether the real moduli space will always be homeomorphic to this space. The 0-cells correspond to the maximal chains in $[y, x]$, the 1-cells correspond to the submaximal chains in $[y, x]$ containing both the...
endpoints, and in general an \( r \)-cell corresponds to a chain in \([y, x]\) containing \((k - r + 1)\) points including both the endpoints, and the unique \((k - 1)\)-cell corresponds to the 2 element chain \(\{y, x\}\). The boundary map is injective and corresponds to co-inclusion.

**Theorem 5.2.** The above defined CW complex is well defined. It is homeomorphic to a ball, and its boundary is homeomorphic to the order complex of \((y, x)\).

*Proof.* Let us prove this by induction on \(k\), so assume that the theorem holds for \(\mu(D) \leq k - 1\). By boundary of our CW complex, we mean everything except the top dimensional \((k - 1)\)-cell. So by induction, the boundary of our CW complex is a \((k - 2)\)-dimensional manifold \(M\). All we need to show is that \(M\) is PL-homeomorphic to the order complex of \((y, x)\). Once we have proved that, both being spheres of dimension \((k - 2)\), the attaching map of the \((k - 1)\)-cell is forced, thus completing the induction.

Consider the order complex of \((y, x)\). Its \(r\)-simplices correspond to chains of length \(r\) in \((y, x)\). On the other hand, \(M\) is a CW complex whose \(r\)-cells correspond to chains of length \((k - 1 - r)\) in \((y, x)\). The boundary map of \((y, x)\) is same as the coboundary map of \(M\), which is given by inclusion. Since the order complex of \((y, x)\) is a manifold (in fact a sphere) of dimension \((k - 1)\), hence \(M\) is just its dual triangulation. This completes the proof. \(\square\)

### 5.3. Grid spectral sequences

We try to construct CW complexes whose attaching maps correspond to the grid chain complex boundary map. This will ensure that the homology of the CW complex is the grid homology. We start with a very simple example. Consider the order complex of \((y, \infty)\). It has a CW complex structure where the \(r\)-cells correspond to the elements \(z \in G\) with \(y \prec z\) and \(M(z, y) = r + 1\), and the boundary maps correspond to the covering relations.

**Theorem 5.3.** The above defined CW complex is well defined and is homeomorphic to the order complex of \((y, \infty)\).

*Proof.* For any \(z\) with \(y \prec z\) and \(M(z, y) = r + 1\), the order complex of \((y, z)\) is a ball of dimension \(r\), or in other words an \(r\)-cell. The union of such cells make the order complex, thus we only need to show that the boundary maps are the same for the order complex and the CW complex. The boundary of the \(r\)-cell in the order complex corresponds precisely to the maximal chains in \((y, z)\), or in other words maximal chains of \((y, p)\) where \(p\) is covered by \(z\). Since \(p\) being covered by \(z\) in the grid poset is equivalent to saying that \(p\) appears in \(\partial z\) in the grid chain complex map, we conclude that the boundary maps for the order complex are same as the ones for the CW complex. \(\square\)

In the later sections, we will constantly be dealing with pointed CW complexes, so now is as good a time as any, to introduce them. In a pointed CW complex \(X\), the \((-1)\)-skeleton \(X^{-1}\) is a point, which is the basepoint, but itself is not considered as a cell. If there are \(k\) 0-cells, then the 0-skeleton is a discrete union of \((k + 1)\) points. There are no attaching maps for the 0-cells. The construction of the rest of the CW complex is standard. We define a CW complex to be finite if it has finite number of cells. A finite CW complex is clearly finite dimensional.

We define a pointed CW complex to be nice if the following properties hold.

- There is a unique 0-cell (such that the 0-skeleton is a discrete union of 2 points)
- The attaching maps for all the other cells are injective.
- We define a partial order on the cells of the CW complex and the basepoint, by declaring \(a \prec b\) if \(a \subseteq \partial b\). This poset is a GSS poset, with the grading being the dimension of the cell and the sign being the homological sign of the boundary map.
We can extend the above theorem and construct a pointed CW complex whose \((k + r)\)-cells correspond to the elements \(z \in [y, \infty)\) with \(M(z, y) = r\) and whose attaching maps correspond to the covering relations in \([y, \infty)\).

**Theorem 5.4.** For every \(k \geq 0\), there is a well-defined pointed CW complex \(S_y(k)\), such that the cells correspond to the elements of \([y, \infty)\), the boundary maps correspond to the boundary maps of the chain complex induced from \([y, \infty)\) and agree with any given sign convention on it, the cell corresponding to \(y\) has dimension \(k\), and the boundary map of every other cell is injective (which implies that \(S_y(0)\) is nice). Furthermore, we have \(S_y(k) = S_y(0) \wedge^k S^3\), where \(\wedge\) denotes the smash product.

**Proof.** We extend the shellable poset \([y, \infty)\) by attaching elements \(x_0, x_1, y_1, \ldots, x_k, y_k, x_{k+1}\), such that \(x_0\) is covered by precisely the elements that cover \(y\) and with the same sign for each covering relation, and each of \(x_i\) and \(y_i\) is covered \(x_{i-1}\) and \(y_{i-1}\) with positive and negative signs respectively. Using Lemmas 2.2 and 2.3, we see that this new poset is also shellable. Let \(P_0\) be the poset defined as \(P_0 = (x_{k+1}, y) \cup \{x_{k+1}, x_0\}\).

Now consider the order complex of \((x_{k+1}, \infty)\). It has a CW complex structure whose cells correspond to the elements of \((x_{k+1}, \infty)\), and the boundary maps represent the covering relations. But \(P_0\) is a thin shellable poset, and hence the order complex of \(P_0\) is a sphere of dimension \(k\). Thus we can treat the order complex of \(P_0\) as the cell corresponding to \(y\) in our pointed CW complex. The order complex of \((x_{k+1}, \infty)\) then has a pointed CW complex structure, whose cells correspond to elements of \([y, \infty)\) and whose boundary maps correspond to the chain complex boundary maps.

Recall that a sign convention \(s\) assigns 1 or \(-1\) to each covering relation in the poset \(G\). Two sign conventions are said to be equivalent if one can be obtained from another by reversing the signs of all the covering relations \(z \leftarrow x\), where exactly one of \(z\) and \(x\) belongs to a fixed subset of generators. A property that sign conventions must satisfy is that the grid chain complex boundary map must actually be a boundary map. This means that, if \(z \leftarrow \{p, q\} \leftarrow x\) is an interval of length three, then the product of the signs of the four covering relations is \(-1\).

Note that the boundary maps in the CW complex \([y, \infty)\) also has this property and this equivalence. The equivalence is obtained by reversing the orientations of the cells corresponding to the fixed subset of generators. To see that it also has the above mentioned property, let \(z \leftarrow \{p, q\} \leftarrow x\) be an interval of length three. The generator \(x\) corresponds to an \(r\)-cell, whose boundary contains two \((r - 1)\)-cells corresponding to \(p\) and \(q\). These two cells have a common \((r - 2)\)-cell on their boundaries, coming from \(z\). Thus it is easy to see that the product of the signs of the four boundary maps has to be negative.

Now we will show that this equivalence and this property is enough to determine the sign in \([y, \infty)\). View the Hasse diagram of the poset \(G\) as an unoriented graph. The sign convention assigns signs to the edges of the graph. This induces a group homomorphism from the first homology of the graph to the multiplicative group \([-1, 1]\), given by first representing an element of the first homology by a closed circuit, and then taking the product of the signs on the edges along the circuit. An interval of length three gives rise to a 4-cycle in the graph, and the property of sign convention states that such 4-cycles should map to \(-1\).

Fix a maximal tree in the graph. Using the equivalence, we can ensure that all the edges in this maximal tree have positive sign. We need to show that the property will fix the sign of every other edge. Whenever we add an edge, we get a cycle in the graph, consisting of that edge and a few edges from the maximal tree. We need to show that the value of the homomorphism on such a cycle is determined by the property. If we can show that any cycle is generated (in the first homology of the graph) by the 4-cycles coming from intervals of length three, then we are done.

Consider two maximal chains in \([y, x]\). They combine to form a cycle. Call such cycles to be simple cycles. It is easy to see that any cycle in \([y, \infty)\) is a sum (in the first homology of the graph) of simple cycles. So we only need to show that any simple cycle is a sum of 4-cycles coming from length three intervals. Let
Let \( m_1 \) and \( m_2 \) be two maximal chains in \( [y, x] \). Since \( [y, x] \) is shellable, it follows from the definition that there is some total ordering on the maximal chains, such that we can replace the bigger maximal chain by a smaller one \( m_3 \) by adding a 4-cycle coming from a length three interval. This completes the proof that there is a unique sign assignment on \([y, \infty)\) and hence we can choose the orientations of the cells properly to ensure that the CW complex boundary maps respect the sign conventions.

Also note that during the construction of the CW complex, when we were trying to attach an \( n \)-dimensional cell, its boundary had to map injectively to an \((n-1)\)-sphere respecting some sign. Thus throughout there was only one option, and hence there is only one such CW complex that can be constructed with the above properties. This shows that the CW complex is well-defined. Since with the obvious CW complex structure, \( S_y(0) \land^k S^1 \) is another CW complex with the same properties, we have \( S_y(k) = S_y(0) \land^k S^1 \).

Indeed, the above proof shows that if \( y \prec x \) with \( M(x, y) = r \), then the \((k + r)\)-ball has a pointed CW complex structure, whose cells are the generators in \([y, x]\) and the boundary maps are the grid chain complex boundary maps. This has the following corollaries.

**Lemma 5.5.** For any interval \([y, x]\) with \( y \neq x \), the homology of the chain complex induced from the poset, is trivial.

**Proof.** The homology of the chain complex induced from \([y, x]\) is the reduced homology of the pointed CW complex whose cells correspond to the generators of \([y, x]\) and whose boundary maps correspond to the chain complex boundary maps. However that CW complex is the ball, and hence the reduced homology is trivial.

**Lemma 5.6.** Given \( x, y \in G \), there are even number of generators \( z \) with \( y \prec z \prec x \).

**Proof.** We can assume that \( y \prec x \). Consider the chain complex induced from the poset \([y, x]\). Since it has trivial homology, there must be an even number of generators in \([y, x]\), and hence in \((y, x)\).

We digress for a bit to explore some consequences of the previous lemma. For the rest of this subsection, we work with coefficients in \( F_2 = \mathbb{Z}/2\mathbb{Z} \). We define a filtered chain complex, whose generators are the elements in \( G \) with the filtration grading being the Maslov grading. We define the boundary map as

\[
\delta x = \sum_{y \prec x} y
\]

We know that \( \delta \) decreases the grading, and the previous lemma implies that \( \delta^2 = 0 \). Therefore, we can look at the Leray spectral sequence on the filtered chain complex \((G, \delta)\). The \( E_0 \) page and the \( E_1 \) page is \( G \), and the \( E_2 \) page is the grid homology.

### 6. CW Complexes

In the previous section, we defined a nice CW complex to be a pointed CW complex with a unique 0-cell, such that the attaching maps for all the other cells are injective, and the poset, whose elements are the cells and the basepoint, is a GSS poset. Hence the order complex of any closed interval of the poset is a ball, and the order complex of any interval of the form \((-\infty, a]\) is also a ball.

In fact, using Theorem 5.4, given a suitable GSS poset, there is only one nice pointed CW complex, satisfying these properties. Furthermore, we can fix the orientation of the 0-cell arbitrarily, but once that orientation is fixed, the orientation of every other cell and the basepoint is fixed by the sign convention of the GSS poset.
Let $X$ be a nice finite CW complex. Let the dimension $k$ cells of $X$ be $e_1^k, e_2^k, \ldots, e_n^k$. We define its dual in the following way. We first fix a map $P$ from the discrete union of all the cells to $\mathbb{R}^2$, such that each cell maps to a single point in $\mathbb{R}^2$, and different cells map to different points in $\mathbb{R}^2$. Let the image of the cell $e_i^k$ be $p_i^k$; let $\mu^k_i : [0, 1] \to \mathbb{R}^2$ be the straight line path from the origin to $p_i^k$ at constant speed, and let $g^k_i \subset \mathbb{R}^2 \times \mathbb{R}$ be the graph of the function $\mu_i^k$.

Given such a map $P$, we will construct a PL-embedding $f_P$ of $X$ in $\mathbb{R}^n$, with $n \geq 3d + 1$, where $d$ is the dimension of $X$. We will embed the $(k-1)$-skeleton $X^{k-1}$ in $\mathbb{R}^{3k-2}$, and then view $\mathbb{R}^{3k-2}$ as the subspace $\mathbb{R}^{3k-2} \times \{0\}^3$ in $\mathbb{R}^{3k+1} = \mathbb{R}^{3k-2} \times \mathbb{R}^3$ and extend this embedding to the $k$-skeleton. Thus we will be able to embed $X$ in $\mathbb{R}^{3d+1}$ which we view as the subspace $\mathbb{R}^{3d+1} \times \{0\}^{n-3d-1}$ in $\mathbb{R}^n = \mathbb{R}^{3d+1} \times \mathbb{R}^{n-3d-1}$.

**Theorem 6.1.** Given a map $P$, there is a PL-embedding $f_P$ of the type as described in the previous paragraph.

**Proof.** For clarity, we explicitly write down the embedding of $X^k$ for a few small values of $k$. We embed the 0-skeleton in $\mathbb{R}$ by mapping the basepoint to the origin and the 0-cell to 1.

The 1-cells are $e_1^1, e_2^1, \ldots, e_{k_1}^1$. We view $\mathbb{R}^4$ as $\mathbb{R} \times \mathbb{R}^3$, and embed $e_i^1$ as an union of $\partial e_i^1 \times \{\mu_i^1(t)\} \times \{t\}$ for $t \in [0, 1]$, and $[0, 1] \times \{p_i^1\} \times \{1\}$. Note that, since $p_i^1$’s are distinct, this is indeed an embedding.

There is a different way of viewing the above process. For each 1-cell $e_i^1$, its boundary is a 0-sphere $s_i^0$ in $\mathbb{R}$, and bounds a disk $d_i^1$ in $\mathbb{R}$ (which in our case always happens to be the unit interval $I$). We then embed the 1-cell $e_i^1$ as an union of an annulus $S^0 \times I$ embedded in $\mathbb{R} \times \mathbb{R}^3$ as $s_i^0 \times g_i^1$, and a disk $D^1$ embedded in $\mathbb{R} \times \mathbb{R}^3$ as $d_i^1 \times \{p_i^1\} \times \{1\}$.

Now, to embed $X^k$ in $\mathbb{R}^{3k+1}$, we proceed inductively. We assume that $X^{k-1}$ is already embedded in $\mathbb{R}^{3k-2}$, and we view $\mathbb{R}^{3k+1} = \mathbb{R}^{3k-2} \times \mathbb{R}^3$. For each $k$-cell $e_i^k$, its boundary is a $(k-1)$-sphere $s_i^{k-1}$ embedded in $\mathbb{R}^{3k-2}$. If that sphere $s_i^{k-1}$ bounds a disk $d_i^{k-1}$ in $\mathbb{R}^{3k-2}$, then we can embed the $k$-cell $e_i^k$ as an union of an annulus $S^{k-1} \times I$ embedded in $\mathbb{R}^{3k+1}$ as $s_i^{k-1} \times g_i^k$, and a disk $D^k$ embedded in $\mathbb{R}^{3k+1}$ as $d_i^k \times \{p_i^k\} \times \{1\}$. Note that, since $p_i^k$’s are distinct points in $\mathbb{R}^2$, this is still an embedding.

Thus to show that there is a well-defined embedding depending only on the choice of the map $P$, we need to produce a disk $d_i^k$ bounding $s_i^{k-1}$, which does not depend on anything other than the choice of the map $P$. Without loss of generality, let $i = 1$. Let $s_1^{k-1}$ be the boundary of a $k$-cell $e_1^k$. Note that, the order complex of $[e_0^1, e_1^k]$ is a disk of the same dimension as $d_1^k$. So we will produce an embedding of this order complex with the proper boundary.

To present a clearer picture, let us explicitly describe how we define the embeddings of the vertices and edges of this order complex. We embed $e_1^0$ and $e_i^k$ as $(1) \times \{0\}^{3k-3}$ and $\{0\}^{3k-2}$ respectively. For $1 \leq l \leq k-1$, we embed $e_l^1$ as $(1) \times \{0\}^{3l-3} \times \{p_l^1\} \times \{1\} \times \{0\}^{3k-3l-3}$. The edge joining $e_l^k$ to $e_0^1$ is $I \times \{0\}^{3k-3}$, the edge
Now let us describe in general, how a simplex of this order complex coming from a chain $e_1^0 \prec e_1^1 \prec \cdots \prec e_{l_m}^0$ is embedded in $\mathbb{R}^{3k-2}$ with $l_m < k$. We embed this as the disk $\{1\} \times \{0\}^{3l_1-3} \times g_1^0 \times \{0\}^{3l_2-3l_1-3} \times g_2^0 \times \cdots \times g_{l_m}^0 \times \{0\}^{3k-3l_m-3}$. For the rest of this paragraph, let us call this subspace $\{1\} \times A$, where $A$ is
a subspace of \( \mathbb{R}^{3k-3} \). The simplex of the order complex coming from the chain \( e_1^{i_1} \prec \cdots \prec e_n^{i_m} \) is a suitable part of the boundary of the above order complex, and again for the rest of this paragraph, let us denote that subspace to be \( \{1\} \times B \), where \( B \) is a subspace of \( \partial A \). Then the simplex of the order complex coming from a chain \( e_1^{i_1} \prec \cdots \prec e_n^{i_m} \prec e^k_1 \) is embedded as the union of \( \{0\} \times A \) followed by \( I \times B \), and the simplex of the order complex coming from a chain \( e_1^{i_1} \prec \cdots \prec e_n^{i_m} \prec e^k_1 \) is embedded as \( I \times A \).

Thus we have embedded the order complex of \([e_0^1, e_1^k]\) in \( \mathbb{R}^{3k-2} \), and this is the required disk \( d_k^i \) bounding \( s_1^k \). Using such disks \( d_k^i \)'s, we can then embed \( X^k \) in \( \mathbb{R}^{3k+1} \), thus completing the proof.  

There are a few observations that we should make now. The only choice that we made in defining the embedding, is the choice of the map \( P \). But we can connect any two such maps \( P \) and \( P' \) by an isotopy of \( \mathbb{R}^2 \), and this induces an isotopy in \( \mathbb{R}^n \) connecting the embeddings \( f_P \) and \( f_{P'} \).

Furthermore, this embedding is also an embedding of the order complex of the whole poset coming from the CW complex. The basepoint is embedded as the origin, the 0-cell is embedded as \( \{1\} \times \{0\}^{n-1} \), and the vertex corresponding to the cell \( e_1^0 \) is embedded as \( \{1\} \times \{0\}^{3k-3} \times \{1\} \times \{2\} \times \{0\}^{n-3k-1} \). A simplex of this order complex coming from a chain \( e_1^0 \prec e_2^1 \prec \cdots \prec e_n^m \prec e^k_1 \) is embedded in \( \mathbb{R}^n \), as the disk \( \{1\} \times \{0\}^{3k-3} \times \{1\} \times \{0\}^{3k-3} \times \{2\} \times \{0\}^{n-3k-1} \). Once more for the rest of this paragraph, let us call this subspace \( \{1\} \times A \), where \( A \) is a subspace of \( \mathbb{R}^{n-1} \). The simplex of the order complex coming from the chain \( e_1^0 \prec \cdots \prec e_n^m \) is a suitable part of the boundary of the above order complex, and again for the rest of this paragraph, let us denote that subspace to be \( \{1\} \times B \), where \( B \) is a subspace of \( \partial A \). Then the simplex of the order complex coming from a chain \( b \prec e_1^0 \prec \cdots \prec e_n^m \), where \( b \) is the basepoint, is embedded as the union of the closure of \( (\{1\} \times 2A) \setminus (\{1\} \times A) \), followed by \( I \times 2B \) followed by \( \{0\} \times 2A \). Note that this embedding of the order complex is slightly different from the one that we used in the previous proof.

Thus the closure of a regular neighborhood of \( X \) in \( \mathbb{R}^n \) will give an \( n \)-dimensional manifold \( N_X \) (with boundary) with same homotopy type as that of \( X \). We construct \( N_X \) in the following way. Let \( N_k \) be the set of all points in \( \mathbb{R}^n \) with \( L^2 \) distance less than or equal to \( \epsilon_k \) from \( X^k \). We assume that \( \epsilon_k \)'s are decreasing in \( k \) and we choose positive \( \epsilon_0 \) to be small enough such that the interior of \( \cup N_k \) is a regular neighborhood of \( X \). We choose positive \( \epsilon_1 \) to be small enough such that \( N_1 \cap \partial N_0 \) has exactly two components for each 1-cell \( e_1^1 \), and for each \( k > 1 \), after we have already chosen \( \epsilon_0, \ldots, \epsilon_{k-1} \), we choose positive \( \epsilon_k \) to be sufficiently small such that \( N_k \cap \partial(\cup_{j=0}^{k-1} N_j) \) has exactly one component for each \( k \)-cell \( e_k^k \). We define \( N_X = \cup_i N_i \). Note that \( \partial N_X \) is not a smooth manifold.

Let \( b \) be the image of the basepoint \( X^{-1} \) in the embedding, and let \( B \) be the small neighborhood of \( b \), lying in the interior of \( N_0 \). Let us view \( W = N_X \setminus B \) as a cobordism from \( \partial B \) to \( \partial N_X \). Note that this cobordism is obtained by starting with \( \partial B \), adding disks corresponding to the embeddings of the order complexes of \( (-\infty, e^k_1] \), and then taking a regular neighborhood. Let us assume that there is a Morse function and a corresponding gradient-like flow for this cobordism, such that the flow is transverse to \( \partial N_X \) and \( \partial B \), the only index 2 critical points are the images of the vertices in the order complex corresponding to \( e^k_1 \) and the unstable disks are the embeddings of the order complexes corresponding to \( (-\infty, e^k_1] \). Then the original pointed CW complex \( X \) can be recovered from this gradient-like flow in the following way. Quotient out \( \partial B \) to the basepoint, and the cells for the CW complex are the unstable disks with the attaching maps given by the flow. We construct the dual of \( X \) in a very similar way. We look at the stable disks, and regard the cobordism as one obtained from \( \partial N_X \) by adding those disks and then taking a regular neighborhood. Thus to construct the CW complex dual to \( X \), we should quotient out \( \partial N_X \) to the basepoint, and have cells correspond to stable disks with attaching maps given by the flow. However, to define the dual in this way, we first need to find a Morse function and a corresponding gradient-like flow satisfying the above conditions. The dual then might depend on the choice of the Morse function and the gradient-like flow and...
also on the map $P$. We will bypass the construction of the Morse function and the gradient-like flow, and define the stable disks directly, depending only on the choice of the map $P$.

We will define the stable disk $r^k_i$ corresponding to the critical point coming from the vertex $e^k_i$ of the order complex in several stages. Recall that the regular neighborhood $N_X$ is constructed as a union $\cup_j N_j$. Let $r^k_{i,j} = N_j \cap r^k_i$. We will define $r^k_{i,j}$ starting at $j = 0$, then gradually extending the definition to $j = 1, 2, \ldots$, and finally define $r^k_i = \cup_j r^k_{i,j}$.

Furthermore, note that $r^k_{i,j} = \emptyset$ for $j < k$. So for $j = 0$, we only need to define $r^0_{1,0}$. We define $r^0_{1,0}$ as the connected component of $N_0$ not containing $\partial B$. For $j = 1$, define $r^1_{i,1}$ as the intersection of $N_1$ with the hyperplane $\mathbb{R}^3 \times \{ \frac{1}{2} \} \times \mathbb{R}^{n-4}$ and extend $r^0_{1,0}$ to $r^0_{1,1}$ as the set of all points in $N_1$ whose $L^\infty$ distance from $e^0_1$ (embedded as $\{ 1 \} \times \{ 0 \}^{n-1}$) is at most $\frac{1}{2}$. It is easy to see that $\partial r^0_{1,1}$ lies in the union of $\partial N_1$ and $r^1_{1,1}$ and each stable disk is still a ball of the correct dimension. The way we extended the definition of $r^0_{1,0}$ to that of $r^0_{1,1}$ can also be described as follows. Since $r^0_{1,0}$ is one of the components of $N_0$, $N_1 \cap \partial r^0_{1,0}$ is a disjoint union of $(n-1)$-dimensional balls, one for each $e^1_1$. We then take the ball corresponding to $e^1_1$ and extend it like a horn in the direction of $e^1_1$ until we reach the vertex corresponding to $e^1_1$. Since different balls on $\partial r^0_{1,0}$ corresponding to different $e^1_1$’s are disjoint, after extending the horns, $r^0_{1,1}$ is still a ball of dimension $n$. Suitable parts of $\partial r^0_{1,0}$ are defined as $r^1_{1,1}$.

Now to define $r^k_{i,j}$, by induction, let us assume, we have defined $r^j_{i,j'}$ for all $j' < j$. We define $r^j_{i,j}$ as the intersection of $N_j$ with the plane $\mathbb{R} \times (\mathbb{R}^2 \times \{ 0 \})^{j-1} \times \mathbb{R}^2 \times \{ \frac{1}{2} \} \times \mathbb{R}^{n-3j-1}$. For $k < j$, by induction, $r^k_{i,j-1}$ is already defined. $N_j \cap \partial r^j_{i,j-1}$ is a disjoint union of $(n-k-1)$-dimensional balls, one for each $e^j_l$, with $e^k_i \prec e^j_l$. We extend the ball corresponding to $e^j_l$ in the direction given by embedding of the order complex of $[e^k_i, e^j_l]$ until we reach the boundary of the order complex. We define $r^k_{i,j}$ as $r^k_{i,j-1}$ after these extensions. Since $r^k_{i,j-1}$ was a $(n-k)$-dimensional ball, and we extended along disks starting at different portions of $\partial r^k_{i,j-1}$, $r^k_{i,j}$ is still a ball of the correct dimension. Note that $r^0_{i,j}$ is still the set of all points in $N_j$ whose $L^\infty$ distance from $e^1_1$ is at most $\frac{1}{2}$, and thus it is particularly easy to see that $r^0_{1,j}$ is an $n$-dimensional ball, since $N_j$ is an $n$-dimensional manifold. Finally, we define $r^k_i = \cup_j r^k_{i,j}$. Note that $r^k_i$ lies in the boundary of $\partial r^k_i$ if and only if $e^k_i \prec e^k_i$, and in that case, it is actually embedded.

![Figure 6.3](image-url)  

**Figure 6.3.** Constructing the stable disks for the CW complex of Figure 6.1 and Figure 6.2

We then quotient out $\partial N_X$ to the basepoint, and define the dual CW complex using the stable disks as described above. Note that the union of all the stable disks is simply $r^0_i$, and thus if $\overline{X}_n = (r^0_i, \partial N_X \cap r^0_i)$, we can also construct the dual by starting with $\overline{X}_n$ and then quotienting out $\partial N_X \cap r^0_i$ to the basepoint. This
construction might a priori depend on the map $P$, but we can connect any two such maps $P$ and $P'$ by an isotopy of $\mathbb{R}^2$. During the isotopy, for each $k > 1$ (resp. $k = 1$) we can make the $\epsilon_k$'s used in the definition of $N_k$'s sufficiently small such that the condition about $N_k \cap \partial(\cup_{j=0}^{k-1} N_j)$ having exactly one component (resp. exactly two components) for each $k$-cell $e^k_l$, holds. Then this induces an isotopy joining the two $\widetilde{X}_n$'s, and hence induces a homeomorphism between the two duals. Thus the dual of a nice pointed CW complex $X$ does not depend on the map $P$ and depends only on the ambient dimension $n$. Let us denote this dual by $\overline{X}_n$.

Before we prove any other properties of the dual, we need to understand the dependence of $\overline{X}_n$ on $n$. The following result makes this precise.

**Theorem 6.2.** For a nice pointed CW complex $X$, we have $\overline{X}_{n+1} = \overline{X}_n \wedge S^1$, where $\wedge$ denotes the smash product.

**Proof.** After fixing a map $P$, we can construct an embedding of $X$ in $\mathbb{R}^n$ in a well-defined way, and we extend this embedding to an embedding into $\mathbb{R}^{n+1}$ by embedding $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \{0\}$. After fixing an embedding to $\mathbb{R}^m$, we define $\overline{X}_n$ as a pair $(A_n, B_n)$ with $B_n$ lying in $\partial A_n$, and we define $\overline{X}_m$ as a quotient of $\overline{X}_n$ obtained by quotienting out $B_n$ to the basepoint.

However $A_{n+1}$ is homeomorphic to $A_n \times [-\epsilon, \epsilon]$ and $B_{n+1}$ is homeomorphic to $(A_n \times \{\pm \epsilon\}) \cup (B_n \times [-\epsilon, \epsilon])$. Since $[-\epsilon, \epsilon]/\{\pm \epsilon\}$ is the circle $S^1$, hence $\overline{X}_{n+1} = A_{n+1}/B_{n+1} = (A_n/B_n) \wedge S^1 = \overline{X}_n \wedge S^1$. \hfill $\square$

Now we are in a position to state and prove the following important properties of duals. Let $X$ be a nice pointed CW complex, and let $Y$ be a subcomplex. $Y$ is also clearly nice and pointed. We can thus define the duals $\overline{X}_n$ and $\overline{Y}_n$ for $n$ sufficiently large (in fact $n$ simply has to be larger than $3d$ where $d$ is the dimension of $X$). Then the following holds,

**Theorem 6.3.** For $Y$, a subcomplex of a nice pointed CW complex $X$, the dual $\overline{Y}_n$ can be obtained from $\overline{X}_n$ by quotienting out the cells corresponding to the cells in $Y$ that are not in $X$.

**Proof.** Note that it is enough to prove the case when there is exactly one cell $e^k_l$ that is in $Y$ but not in $X$. Thus to embed $Y$ in $\mathbb{R}^n$, we embed $X$ in $\mathbb{R}^n$ and then delete the cell $e^k_l$ (which was embedded as an embedding of the order complex of $(-\infty, e^k_l]$). Another way to see this is the following. Take the embedding of $X$, view it as an embedding of the order complex, and delete the vertex corresponding to $e^k_l$. Then the new space deform retracts to the embedding of $Y$. Let $N_X$ and $N_Y$ be regular neighborhoods of $X$ and $Y$ respectively, as defined earlier in this section. Let $R$ be a small neighborhood of the stable disk $r^k_l$ of $e^k_l$ in the embedding of $X$. Then $N_X \setminus R$ deform retracts to $N_Y$.

The stable disks required for defining the dual $\overline{Y}_n$ come from the manifold $N_Y$. The stable disks required for defining the dual $\overline{X}_n$ come from the manifold $N_X$, and when these stable disks are restricted to $N_X \setminus \hat{R}$, they define the quotient complex of $\overline{X}_n$ obtained by quotienting out $r^k_l$, the cell corresponding to $e^k_l$. A properly chosen deformation retract of $N_X \setminus \hat{R}$ to $N_Y$ gives the required homeomorphism between this quotient complex and $\overline{Y}_n$. \hfill $\square$

A very similar property holds for quotient complexes. However if $X$ is a nice pointed CW complex, quotient complexes of $X$ in general will not be nice. Let $e^1_l$ be an 1-cell of $X$, and consider the quotient complex $Z$ of $X$ obtained by keeping only the cells $e^k_l$ with $e^1_l \preceq e^k_l$ and quotienting out everything else. Let us assume that there is a nice pointed CW complex $Y$, such that $Y \wedge S^1$ with the natural CW complex structure is the same CW complex as $Z$ (in fact using Theorem 6.3, we can always assume this). If $d$ is the dimension of $X$, then for $n > 3d$, we can define the duals $\overline{X}_n$ and $\overline{Y}_n$. Then the following is true.
Theorem 6.4. The dual $Y_{n-1}$ is homeomorphic to the subcomplex of $X_n$ obtained by considering only the cells corresponding to the ones present in $Z$.

Proof. First observe that the order complex of the poset coming from $X$ restricted to the cells of $Z$, can also be obtained from the order complex of $Y$ by removing the element corresponding to the basepoint. Now choose an embedding of $X$ (which is also an embedding of the order complex of the poset coming from $X$) to $\mathbb{R}^n$. Let us restrict to the order complex of $Z \cup \{b\}$, where $b$ is the basepoint in $X$, and delete all the simplices which use the edge coming from $b \leftarrow e_1$. This is same as the order complex coming from $Y$. Thus an embedding of $X$ in $\mathbb{R}^n$ gives an embedding of this order complex in $\mathbb{R}^n$. We will now modify this embedding such that it agrees with a standard embedding of $Y$ in $\mathbb{R}^{n-1}$. Observing how the stable disks change under this modification will complete the proof.

At time $t$ for $t \in [0, 1]$, $e_1$ is embedded as $\{1\} \times \{\frac{1}{2}p_1\} \times \{\frac{1}{2}\} \times \{0\}^{n-4}$, the basepoint $b$ is embedded as $\{0\} \times \{\frac{1}{2}p_1\} \times \{\frac{1}{2}\} \times \{0\}^{n-4}$ and a vertex $e_i$ for $k > 1$ is embedded as $\{1\} \times \{\frac{1}{2}p_1\} \times \{\frac{1}{2}\} \times \{0\}^{3k-6} \times \frac{1}{2}g_k \times \{0\}^{n-3k-1}$. The simplex coming from a chain that does not involve $e_1$, is a shifted version of the original, with the second, third and fourth coordinate being changed from $0$ to $\frac{1}{2}p_1$ and the stable disk of $e_1$ for $k > 1$ is a shifted version of the original, where we delete the part that intersects with $\mathbb{R} \times \{\frac{1}{2}p_1\} \times \{\frac{1}{2}\} \times \mathbb{R}^{n-4}$. Note that at $t = 0$, this is an embedding of the order complex of $Y$ as induced from an embedding of $X$. At $t = 1$, this is the standard embedding of the order complex of $Y$ in $\mathbb{R}^3 \times \{\frac{1}{2}\} \times \mathbb{R}^{n-4} = \mathbb{R}^{n-1}$. To complete the proof, we should observe how the stable disks change during this isotopy. At time $t$, we can define the stable disk of $e_1$ as a truncated version of the original stable disk by deleting the part that intersects with $\mathbb{R} \times \{\frac{1}{2}p_1\} \times \{\frac{1}{2}\} \times \mathbb{R}^{n-4}$ and the stable disk of $e_i$ for $k > 1$ as a shifted version of the original stable disk with the second, third and fourth coordinate shifted from $0$ to $\frac{1}{2}p_1$. This gives an explicit isotopy connecting the subcomplex of $X_n$, coming from the cells corresponding to those in $Z$, to $Y_{n-1}$.

Thus given a GSS poset with one minimum, by Theorem \ref{thm:iso} we can construct a nice CW complex corresponding to the poset, and then construct its dual. We can assign an orientation to the top dimensional cell (the one corresponding to the unique minimum in the poset) arbitrarily, but once that is fixed, the orientations of the remaining cells are determined by the sign convention on the GSS poset. This extra information coming from the orientation of the top-dimensional cell allows us to strengthen Theorem \ref{thm:iso}. In that theorem, we showed that there is an isomorphism between $Y_{n-1}$ and a subcomplex of $X_n$, but there might be more than one such isomorphism. However after we orient the top-dimensional cells in both $X_n$ and $Y_{n-1}$ (and hence using the sign convention on the poset of $X$, orient every cell in each of these two CW complexes), we choose the isomorphism that matches the orientations. Thus for oriented CW complexes, there is a well-defined isomorphism between $Y_{n-1}$ and a subcomplex of $X_n$. This will be of use to us in Section \ref{sec:iso}.

Before concluding this section, we should note that our explicit construction of a dual actually agrees with the Alexander dual, which is obtained by embedding the space $X$ in the sphere $S^n$, and then taking the homotopy type of the complement. Thus the Alexander dual is homotopic to $A(X) = S^n \setminus N_X$. The way to see this is as follows. Let us embed $X$ as described above into $\mathbb{R}^n$ and let $S^n$ be viewed as the one point compactification of that $\mathbb{R}^n$ with that extra point being denoted by $\ast$. Let $\bar{b}$ be the basepoint in the dual $\overline{X}_n$ and let $A(X) = S^n \setminus N_X$ be the Alexander dual. If $\sim$ denotes the homotopy equivalence of pairs of spaces, we have

$$\overline{X}_n, \bar{b}) \sim (N_X \setminus B, \partial N_X) \sim (S^n \setminus B, (S^n \setminus N_X)) \sim (S^n \setminus \{b\}, A(X))$$
However we have an exact sequence of spaces
\[(A(X), *) \xrightarrow{\epsilon} (S^n \setminus \{b\}, *) \xrightarrow{\iota} (S^n \setminus \{b\}, A(X)) \sim (X_n, b)\]

This induces the Puppe map from \(X_n\) to \(A(X) \wedge S^1\), and since \(H_\ast(S^n \setminus \{b\}, *) = 0\), the map induces isomorphism in \(H_\ast\) and hence induces a homotopy equivalence.

7. Grid homotopy

Let \(P\) be a GSS poset. In this section, we construct a pointed CW complex whose cells correspond to the elements of \(P\), and whose attaching maps correspond to the covering relations in \(P\). For most of the time, \(P\) will be a grid poset, a commutation poset or a stabilization poset.

If we take the poset \(P\), and reverse the partial order, each closed interval in the new poset still remains shellable. This follows from the definition of shellability. Thus the applications of Section \(5\) can all be constructed. In particular, we can construct a pointed CW complex corresponding to the interval \((-\infty, x]\) in the old poset, whose cells correspond to the elements of \((-\infty, x]\) and the attaching maps correspond to the coboundary maps in the chain complex induced from the poset.

**Theorem 7.1.** There is a well-defined nice CW complex \(P_x\), such that the cells correspond to the elements of \((-\infty, x]\), the attaching maps correspond to the coboundary map of the chain complex induced from \((-\infty, x]\) and agree with any given sign convention on it, the cell corresponding to \(x\) has dimension 0, and the boundary map every other cell is injective.

**Proof.** We reverse the partial order of the poset \((-\infty, x]\) and construct the pointed CW complex \(S_x(0)\) as described in Theorem \(5.4\). This is the required pointed CW complex \(P_x\).

We now state and prove the main result of this section.

**Theorem 7.2.** Given a GSS poset \(P\), for sufficiently large \(n\), there is a well-defined CW complex \(X_P(n)\) whose \(k\)-cells correspond to the elements of \(P\) of grading \((k + n)\) and whose attaching maps correspond to the covering relations in \(P\), even up to sign.

**Proof.** If \(M_1\) and \(M_2\) are the maximum and the minimum gradings in the poset, then we choose \(n > 2M_1 - 3M_2\). For each \(x \in P\), we construct \(P_x\) as in Theorem \(7.1\). Each of these CW complexes is a nice pointed CW complex, and hence we can construct their duals \((P_x)_{g(x) + n}\), where \(g(x)\) is the grading of \(x\). In each of these CW complexes, we orient the top-dimensional cell arbitrarily, and that fixes an orientation of every cell. For \(y \prec x\), \((-\infty, y]\) is a subcomplex of \((-\infty, x]\). A repeated application of Theorem \(6.4\) allows us to construct a well-defined injection of \((P_y)_{g(y) + n}\) to \((P_x)_{g(x) + n}\) which matches the orientations. Thus we have a space for each \(x \in P\) and a map for each pair \(x, y \in P\) with \(y \prec x\). We take the discrete union of all these spaces and glue them together using these maps, and call it \(X_P(n)\). It is easy to see that \(X_P(n)\) is well-defined and satisfies the conditions of the theorem.

However note that the same poset \(P\) can carry two different non-equivalent sign conventions. Figure \(7.3\) demonstrates that such posets can indeed give rise to different spaces. In the diagram we have significantly reduced the dimensions of the spaces.

Now we want to state and prove certain properties of this space \(X_P(n)\).

**Theorem 7.3.** If \(P\) is a GSS poset, \(Q\) is a subposet and \(R\) is a quotient poset, then for \(n\) sufficiently large, the following are true.
\[
X_P(n + 1) = X_P(n) \wedge S^1.
\]
\[
X_Q(n)\] is a subcomplex of \(X_P(n)\) containing only the cells corresponding to the elements in \(Q\).
\[
X_R(n)\] is a quotient complex of \(X_P(n)\) containing only the cells corresponding to the elements in \(R\).
Proof. The space $X_P(n)$ is constructed as a union of spaces of the form $(P_x)_{g(x)+n}$, and the proof follows after observing that each of these spaces has the three properties mentioned above, as proved in Theorems 6.2, 6.3 and 6.4.

Thus by taking $P$ to be $\hat{G}, \hat{G}_m$ or $G^{-}_m$ (for any Alexander grading $m$), and for $n$ sufficiently large, we can construct CW complexes $X_P(n)$. In fact for $n$ sufficiently large, $X_{\hat{G}}(n) = \vee^\infty_{m=-\infty} X_{\hat{G}_m}(n)$, where $\vee$ is the wedge sum.

Since $X_P(n+1) = X_P(n) \wedge S^1$, we can associate finite spectra $S(P)$ to each GSS poset $P$, whose $n$-th space is $X_P(n)$. The previous note implies that $S(\hat{G}) = \vee^\infty_{m=-\infty} S(\hat{G}_m)$. We can also define a spectrum $S(G^-)$ corresponding to $G^-$ by declaring it to be $\vee^\infty_{m=-\infty} G^-_m$.

Now, we want to show that some of these objects that we associate to grid diagrams of knots are actually knot invariants. First note that, given any two grid diagrams for the same knot, we can apply sequences of commutations and stabilizations to each one of them, such that the final two grid diagrams are the same. Therefore, we only need to prove invariance when the two diagrams can be related by either a single commutation or a single stabilization. We consider each of the cases in great detail.

7.1. Commutation.

Theorem 7.4. If two grid diagrams $G$ and $G'$ differ by a commutation, then for any Alexander grading $m$, and with $n$ sufficiently large, $X_{\hat{G}_m}(n)$ is homotopic to $X_{\hat{G'}_m}(n)$, and $X_{G^-_m}(n)$ is homotopic to $X_{(G'^-)_m}(n)$.
3.5. Recalling how the stabilization posets are defined, we see that, in the hat version of case (a) of Figure 7.2, the annulus as a manifold is homotopic to $X$ while in the minus version of case (a) and case (b) of Figure 7.2, the annulus as a manifold is homotopic to $X$.

Proof. For the rest of the proof, let $G$, $G'$, $G_\alpha$ denote $G_m$, $G_m'$, $(G_\epsilon)_m$ respectively if we are working in the hat version, or $G_m^\perp$, $(G_m')^\perp$, $(G_\epsilon)_m$ respectively if we are working in the minus version.

In either case, $G'$ is a subcomplex of $G_\epsilon$ and $G$ is the corresponding quotient complex; therefore, we have a long exact sequence of spaces

$$X_{G'}(n-1) \longrightarrow X_{G_\epsilon}(n) \longrightarrow X_G(n)$$

This induces the Puppe map from $X_G(n)$ to $X_G'(n-1) \wedge S^1 = X_G(n)$. As proved in [MOSZ17], this map induces an isomorphism in homology, and since we can choose $n$ large enough to ensure that both the sides are simply connected, the map is a homotopy equivalence.

7.2. Stabilization. The situation for stabilization is slightly different. In the hat version, we can no longer hope for any sort of homotopy equivalence.

**Theorem 7.5.** If $H$ and $G$ are the grid diagrams before and after stabilization, then for any Alexander grading $m$ and with $n$ sufficiently large, $X_{G_m}(n)$ is homotopic to $X_{G_m'}(n) \vee X_{G_{m+1}^\perp}(n-1)$, and $X_{G_m}(n)$ is homotopic to $X_{G_m'}(n)$.

Proof. There are two types of stabilization that we have to work with, namely case (a) and case (b) of Figure 7.2. Recalling how the stabilization posets are defined, we see that, in the hat version of case (a), there is an exact sequence of spaces

$$X_{G_m}(n-1) \vee X_{G_{m+1}^\perp}(n-2) \longrightarrow X_{(G_\epsilon)_m}(n) \longrightarrow X_{G_m}(n)$$

while in the minus version of case (a), there is an exact sequence of spaces

$$X_{G_m}(n-1) \longrightarrow X_{(G_\epsilon)_m}(n) \longrightarrow X_{G_m}(n)$$

Similarly, in the hat version of case (b), there is an exact sequence of spaces

$$X_{G_m}(n-1) \vee X_{(G_\epsilon)_m}(n-1) \longrightarrow X_{H_m}(n) \vee X_{H_{m+1}^\perp}(n-1)$$

and in the minus version of case (b), there is an exact sequence of spaces

$$X_{G_m}(n-1) \longrightarrow X_{(G_\epsilon)_m}(n-1) \longrightarrow X_{H_m}(n)$$

In all the four cases, the Puppe map gives a map between the relevant spaces. If $n$ is large enough, so as to ensure that all the spaces are simply connected, and if the Puppe map induces isomorphism in homology, then the Puppe map would be a homotopy equivalence. Therefore, we only need to show that, in each of the four cases, the map induced in homology is an isomorphism. Following the lines of the proof in [MOSZ17], we prove that the map induced on the chain complexes is a quasi-isomorphism. Since that implies that the quasi-isomorphism is induced from a homotopy equivalence of spaces, we see that the map on the chain complexes is actually a homotopy equivalence.

Recall that, if $G_\alpha = (G, \rho)$ is the stabilization diagram, then $\rho$ is the intersection of the two new circles $\alpha_s$ and $\beta_s$. Furthermore, a small neighborhood of $\rho$ contains the points $X_0$, $X_1$ and $O_0$, such that $O_0$ lies immediately above $X_0$, $X_1$ lies immediately to the right or left of $O_0$, and $O_1$ lies in the same horizontal annulus as $X_0$. Let $C$ be the union of the vertical annulus and the horizontal annulus through $X_0$.

For the rest of the proof, we will be working with empty rectangles that are supported in $C$; therefore, let us familiarize ourselves with the picture, as illustrated in Figure 7.2. The two $\alpha$ circles and the two $\beta$ circles that lie on $\partial C$, are numbered as shown. Note that $\beta_1 = \beta_s$ in case (a) and $\beta_2 = \beta_s$ in case (b), while $\alpha_2 = \alpha_s$ in both the cases. Let $p$ be the intersection $\alpha_2 \cap \beta_1$ and let $q$ be the intersection $\alpha_2 \cap \beta_2$. Observe...
that in case \((a)\), \(p = \rho\), while in case \((b)\), \(q = \rho\). Based on the placement of the coordinates of the generators inside \(C\), we have divided the generators coming from the grid diagram \(G\) into 11 groups \(X\), \(Y\), \(A\), \(A'\), \(B\), \(B'\), \(Q\), \(R\), \(S\), \(T\) and \(M\); ten generators, one from each of the first ten groups, are shown in the figure. The last group \(M\) consists of all the generators whose none of the coordinates is one of the four intersection points in \((\alpha_1 \cup \alpha_2) \cap (\beta_1 \cup \beta_2)\). Recall that \(\hat{I} \subset \hat{G}\) is the subset consisting of all the generators whose one of the coordinates is \(\rho\); therefore, in case \((a)\), \(\hat{I} = X \cup A \cup A'\), while in case \((b)\), \(\hat{I} = Y \cup B \cup B'\).

**Figure 7.2.** Some of the different types of generators

We fix some Alexander grading \(m\), and only work with generators of that grading. We put special marked points on all the squares in the grid diagram \(G\), except the ones that lie on \(C\). This allows us to introduce a filtration on the grid chain complexes, given by the following relative grading. Given two generators \(\hat{x}, \hat{y} \in \hat{G}\), choose a domain \(D \in D^0(\hat{x}, \hat{y})\), and define the relative grading between \(\hat{x}\) and \(\hat{y}\) as the sum of the coefficients of \(D\) at all the special marked points. This definition is easily seen to be independent of the choice of the domain \(D \in D(\hat{x}, \hat{y})\). For the minus version, define the relative grading between \(\hat{x}\) and \(\hat{y}\) as \(\sum \alpha x_i\) to be zero. Via the bijection \(\hat{f} : \hat{I} \to \hat{H}\), this gives a filtration on the chain complexes coming from \(H\) too.

Let us first work with the hat version. The boundary maps in the associated graded object correspond to rectangles that avoid all the \(X\) markings and all the \(O\) markings, and are supported in \(C\). Therefore, the empty rectangles are all supported in the cross-shaped area, as shown in Figure 7.2. The following are direct summands of the chain complex on the associated graded object: \(X\), \(Y\), \(A \cup Q \cup R\), \(A'\), \(Y\), \(J \cup S \cup T\), \(J'\) and \(M\). The homology of \(M\) is zero; there are no boundary maps in \(X\), \(Y\), \(A'\) and \(B'\); the boundary maps in \(A \cup Q \cup R\) are of the following form: each element \(q \in Q\) is covered by some element \(a \in A\) and some element \(r \in R\); the boundary maps in \(B \cup S \cup T\) are of the following form: each element \(s \in S\) covers some element \(b \in B\) and some element \(t \in T\). Therefore, the homology of the associated graded object for \(\hat{G}\) is freely generated by elements of \(X\), \(Y\), \(A'\), \(B'\), elements of the form \(a \pm r\) where \(a \in A\) and \(r \in R\) cover the same element in \(Q\), and elements of the form \(b \pm t\) where \(b \in B\) and \(t \in T\) are covered by the same element in \(S\).
We want to show that the map between the chain complex induced from $\hat{G}$ and the chain complex induced from $\hat{H} \cup \hat{H}'$ is a quasi-isomorphism. For this, it is enough to prove that the induced map on the homology of the associated graded objects is an isomorphism. It is not difficult to see that the induced map is, in fact, a bijection between the free generators of the respective homologies. This shows that the required map is a quasi-isomorphism, and as argued earlier, completes the proof of the statement for the hat version.

The minus version deserves a bit more work. The empty rectangles corresponding to the boundary maps in the associated graded object in $G^{-}$, are now allowed to pass through $O_{0}$ and $O_{1}$. The chain complex has four direct summands, $X \cup Y$, $A' \cup B \cup S \cup T$, $A \cup B' \cup Q \cup R$ and $M$. The homology of $M$ is trivial, while the boundary maps in the other three direct summands are arranged in staircases, as shown in Figure 7.3 for small $U$ powers. Therefore, the homology of the associated graded object in $G^{-}$, is freely generated by elements of the form $U_{1}^{l}z$, for $z \in (Y \cup B \cup B')$.

**Figure 7.3.** Some of the boundary maps in the associated graded object in $G^{-}$

We want to show that the map between the chain complex induced from $G^{-}$ and the chain complex induced from $H^{-}$ is a quasi-isomorphism. However, it is easy to see that, in both case (a) and case (b), the map on the homology of the associated graded objects is a bijection between the free generators of the two homologies. As explained earlier, this finishes the proof for the minus version. □

Thus to every knot $K$ and in every Alexander grading $m$, we can associate an invariant spectrum $S(G^{-}_{m})$, and hence after taking an infinite wedge, the spectrum $S(G^{-})$. We call these spectra $S_{m}^{-}$ and $S^{-}$ to stress the fact that they only depend on the knot $K$, and not on the grid diagram representing $K$. Therefore, any invariant of the spectrum is also a knot invariant. The homology of the spectrum $S^{-}$ is the well-known invariant $HF^{-}(K)$. Stable homotopy groups can constitute an interesting collection of invariants. Another invariant to consider would be the Steenrod operations, acting on the cohomology of $S^{-}$. A very natural question is whether $S^{-}$ computes anything new. It will interesting to find two knots $K_{1}$ and $K_{2}$, such that $S^{-}(K_{1})$ and $S^{-}(K_{2})$ have the same homology, but are not homotopic to one another.

For the hat version, unfortunately we do not have a knot invariant. Given an index $N$ grid diagram $G$ for a knot, we can construct finite spectra $S(\hat{G}_{m})$ and their wedge $S(\hat{G})$.

It is not clear whether the homotopy type of these spectra depend only on $K$ and $N$. However, there is some partial answer to this question. Let $g$ be the highest Alexander grading $m$, such that the homology of $\hat{G}_{m}$ is non-trivial. It is easy to see that $g$ depends only on the knot $K$. Then we claim that the homotopy type of the spectrum $S(\hat{G}_{g})$ depends only on the knot $K$, and henceforth we will denote it by $\hat{S}_{g}$. The way to see this is as follows. For sufficiently large $k$ and for $m > g$, the spaces $X_{\hat{G}^{-}_{m}}(k)$ are acyclic because they are simply connected and have trivial homology. Commutation does not change the homotopy type of $S(\hat{G}_{g})$.
and when we stabilize to go from a grid diagram $H$ to a grid diagram $G$, for sufficiently large $k$, we have $X_{\hat{G}_m}(k) = X_{\hat{H}_m}(k) \lor X_{\hat{H}_{m+1}}(k-1) \sim X_{\hat{G}_n}(k)$, since $X_{\hat{H}_{m+1}}(k-1)$ is acyclic.

In fact, this proof shows a possible way to answer the above question positively. We are trying to show that the homotopy type of $S(\hat{G}_m)$ depends only on $K$, the Alexander grading $m$, and the grid number $N$. First note that it is enough to prove the following fact. If the stabilizations of $G$ and $G'$ have spectra that are homotopy equivalent, then the spectra for $G$ and $G'$ are homotopy equivalent. We have already proved this for $m \geq g$. So by induction, assume that it is true for all Alexander gradings bigger than $m$. Thus for $k$ sufficiently large, we have $X_{\hat{G}_m}(k) \lor X_{\hat{G}_{m+1}}(k-1) \sim X_{\hat{G}_n}(k) \lor X_{\hat{G}_{n+1}}(k-1)$. But by induction, we already know $X_{\hat{G}_{m+1}}(k-1) \sim X_{\hat{G}_{m+1}}(k-1)$. Thus our proof would be complete if, for finite CW complexes $X$, $Y$ and $A$, $X \lor A$ being homotopic to $Y \lor A$ would imply that $X$ is stably homotopic to $Y$.

However, irrespective of that, we can still construct certain stable homotopy invariants from the spectra $S(\hat{G})$ which depend only on $K$ and $N$. One such example is the stable homotopy group.

**Theorem 7.6.** The stable homotopy groups of $S(\hat{G}_m)$ depend only on the knot $K$, the Alexander grading $m$ and the grid index $N$.

**Proof.** We just mimic our attempted proof for showing that the homotopy type of $S(\hat{G}_m)$ depends only on $K$, $m$ and $N$. Call two grid diagrams $G$ and $G'$ to be $r$-equivalent if after stabilizing both of them $r$ times, the two diagrams can be related by commutations. We are trying to prove that $\pi^*_i(S(\hat{G}_m)) = \pi^*_i(S(\hat{G}'_m))$ for two $r$-equivalent diagrams $\hat{G}_m$ and $\hat{G}'_m$. This is true if either $r = 0$ or the Alexander grading $m'$ is sufficiently large. We prove this by the induction on the pairs $(r, -m)$, ordered lexicographically.

If two diagrams $G$ and $G'$ are $r$-equivalent, then their stabilizations are $(r-1)$-equivalent, and hence from the induction on $(r, -m)$, we have $\pi^*_i((S^1 \land S(\hat{G}_m)) \lor S(\hat{G}'_{m+1})) = \pi^*_i((S^1 \land S(\hat{G}'_m)) \lor S(\hat{G}'_{m+1}))$. However, for spectra coming from finite CW complexes, the stable homotopy groups are finitely generated and abelian; for wedges of spaces, the stable homotopy groups are products; therefore, using the classification of finitely generated abelian groups, we get $\pi^*_i(S(\hat{G}_m)) = \pi^*_i(S(\hat{G}'_m))$. \(\square\)

8. **Examples**

In this section, we give examples of some GSS posets $P$, and construct the spaces $X_P(n)$ corresponding to them. We conclude the section by computing the homotopy type of $X_{\hat{G}}(n)$ for the grid diagram $G$ of the trefoil, as shown in Figure 8.1.

![Grid diagram for the trefoil](image)

**Figure 8.1.** Grid diagram for the trefoil
Let $I$ be the poset consisting of two elements 0 and 1, with $0 \preceq 1$. Let $I^n$ be the $n$-fold Cartesian product of $I$ with itself. For very natural reasons, let us call this the $n$-cube poset.

The elements of $I^n$ look like $n$-tuples $a = (a_1, a_2, \ldots, a_n)$, where each $a_i$ is 0 or 1. We put a grading on this poset by declaring the grading of $a$ to be the number of 1’s in the $n$-tuple. We can also put a sign assignment on this poset in the following way. Observe that if $a \leftarrow b$, then there is a unique $k$ for which $a_k = 0$ and $b_k = 1$, and for every $i \neq k$, $a_i = b_i$. We assign a sign of $(-1)^{\sum_{i=1}^{n} a_i}$ to this covering relation, and it is easy to check that this is indeed a sign assignment. Since the $n$-cube poset has a unique minimum, this is the only sign assignment up to equivalence.

This poset is also EL-shellable. In a covering relation $a \leftarrow b$, if $k$ is the unique place where $a_k < b_k$, we label the covering relation by the integer $k$. It is easy to see that this map from the set of all covering relations to the integers, totally ordered in the standard way, is indeed an EL-shelling. However, since the homology of the chain complex associated to $I^n$ is trivial, the CW complex $X_{I^n}(m)$ is contractible for sufficiently large $m$.

The $n$-cube poset is naturally isomorphic to the subset poset, whose elements are the subsets of $\{x_1, \ldots, x_n\}$, partially ordered by inclusion. An element $a$ of $I^n$ corresponds to a subset $S$, such that $x_i \in S$ if and only if $a_i = 1$.

Now consider the $(n + 1)$-cube poset, restricted to the elements of positive grading. Let us reduce the grading of each element by 1, and then rename it as the simplex poset $\Delta_n$, since the grading $k$ elements of this poset correspond to the $k$-simplices lying inside an $n$-simplex $\Delta^n$, with partial order given by inclusion. Thus $\Delta_n$ is graded with $k$-simplices having grading $k$. It has a sign assignment obtained by restricting the sign assignment of the subset poset, and this is the unique sign assignment, since $\Delta_n$ has a unique maximum. It is also shellable, since it is isomorphic to the interval $(\emptyset, \infty)$ of the subset poset.

We can also construct the reduced simplex poset $\tilde{\Delta}_n$, where we label one of the vertices of the $n$-simplex $\Delta^n$ to be the basepoint $b$, and define $\tilde{\Delta}_n = \Delta_n \setminus \{b\}$ with the same partial order. This poset also has grading and sign assignments, and each closed interval in this poset is still shellable, since closed intervals in the poset $\Delta_n$ are shellable. Thus $\tilde{\Delta}_n$ is also a GSS poset.

**Theorem 8.1.** Consider the space $\Delta^n \cup \{b\}$, the one-point compactification of the $n$-simplex $\Delta^n$, and let the point $b$ be declared its basepoint. Then for $m$ large enough, there is a well-defined homeomorphism $h_{\Delta_n,m}$ between $X_{\Delta_n}(m)$ and $(\Delta^n \cup \{b\}) \setminus S^m$.

**Proof.** Let $d_n$ be the maximum element in $\Delta_n$. Since the poset $(-\infty, d_n)$ is shellable and thin, its order complex is $S^{n-1}$. After reversing the partial order, if we recall the construction from Theorem 5.3, then we see that this partially ordered set comes from a CW complex structure, whose $k$-cells correspond to elements of grading $n - 1 - k$. However $S^{n-1}$ can also be thought of as the boundary of the $n$-simplex with the inherited simplicial structure, where the $k$-cells correspond to elements of grading $k$. It is relatively easy to check that this is the dual triangulation of the CW complex structure.

Now recall how $X_{\Delta_n}(m)$ is defined. We embed the order complex of the reverse of $\Delta_n$ into $\mathbb{R}^{n+m}$ in some standard way. We take the image of the point corresponding to $d_n$ (denoted in Section 6 as $e_1^0$), and construct the first step of stable disk $r_{1,0}^0$ which is simply an $(n + m)$-dimensional ball $B^{n+m}$. We extend $r_{1,j}^0$ to $r_{1,j+1}^0$ by marking some thickened $j$-cells lying on $\partial B^{n+m} = S^{n+m-1}$, one for each element of grading $(n - j - 1)$ in $\Delta_n$. Finally, we quotient out everything in $B^{n+m}$ that is not marked, to a point, to obtain $X_{\Delta_n}(m)$.

Thus we are embedding $S^{n-1}$, with the CW complex structure as described in the first paragraph, into $S^{n+m-1} = \partial B^{n+m}$, taking its regular neighborhood in $S^{n+m-1}$, and then quotienting out its complement in $S^{n+m-1}$ to a point, to obtain $X_{\Delta_n}(m)$. But the dual triangulation of the $S^{n-1}$ is the simplicial complex $\partial \Delta^n$, and since $m$ is sufficiently large, this embedding of $\partial \Delta^n$ in $\partial B^{n+m}$ can be extended in a standard
way to a proper embedding of $\Delta^n$ in $B^{n+m}$. For $m$ large enough, we then can view $B^{n+m}$ as $\Delta^n \times D^m$, where $D^m$ is the $m$-dimensional disk, such that the closure of the regular neighborhood of $\partial \Delta^n$ in $S^{n+m-1}$ is $\partial \Delta^n \times D^m$. The space $X_{\Delta_n}$ is obtained by quotienting $\Delta^n \times \partial D^m$ to a point. This is illustrated in Figure S.2 for $n = 2$ and $m = 1$.

\[
\begin{array}{c}
\text{Figure 8.2. Construction of } X_{\Delta_n}(1)
\end{array}
\]

We end the proof by noting that

\[
(\Delta^n \times D^m)/(\Delta^n \times \partial D^m) = (\Delta^n \times S^m)/(\Delta^n \times \{pt\}) = (\Delta^n \cup \{b\}) \wedge S^m
d\square
\]

**Theorem 8.2.** Let $b \in \Delta^n$ be declared the basepoint of the $n$-simplex $\Delta^n$, and consider the reduced simplex poset $\tilde{\Delta}_n = \Delta_n \setminus \{b\}$. Then for $m$ large enough, there is a well-defined homeomorphism $h_{\tilde{\Delta}_n,m}$ between $X_{\tilde{\Delta}_n}(m)$ and $\Delta^n \wedge S^m$.

**Proof.** We construct $X_{\tilde{\Delta}_n}(m)$ in a similar way. View $\tilde{\Delta}_n$ as a quotient poset of $\Delta_n$. Let $d_n$ be the maximum element of $\Delta_n$, and let $(-\infty, d_n) = \Delta_n \setminus \{d_n\}$. We embed $S^{n-1}$, the order complex of the reverse of $(-\infty, d_n)$, into $S^{n+m-1} = \partial B^{n+m}$. We know that the dual triangulation of $S^{n-1}$ is the simplicial structure on $\partial \Delta^n$, and for $m$ large enough we can view $B^{n+m}$ as $\Delta^n \times D^m$, such that $S^{n-1}$ is embedded as $\partial \Delta^n \times \{pt\}$, and the closure of its regular neighborhood is $\partial \Delta^n \times D^m$.

However, since we are working with $\tilde{\Delta}_n = \Delta_n \setminus \{b\}$, we actually embed the order complex of the reverse of $(-\infty, d_n) \cap \tilde{\Delta}_n$. This order complex is $S^{n-1}$ minus the $(n-1)$-dimensional cell corresponding to the vertex $b$ in $\Delta^n$. Thinking in terms of the dual triangulation, it equals $\partial \Delta^n \setminus N(b)$, where $N(b)$ is a small neighborhood of the basepoint $b \in \Delta^n$. The complement of a regular neighborhood of this order complex in $\partial(\Delta^n \times D^m)$ can be thought of as $(\Delta^n \times \partial D^m) \cup (N(b) \times D^m)$. We obtain $X_{\tilde{\Delta}_n}(m)$ by starting with $\Delta^n \times D^m$, and then quotienting out $(\Delta^n \times \partial D^m) \cup (N(b) \times D^m)$ to a point.

We once more end the proof by noting that

\[
(\Delta^n \times D^m)/(\Delta^n \times \partial D^m) \cup (N(b) \times D^m)) = (\Delta^n \times D^m)/(\Delta^n \times \partial D^m) \cup (\{b\} \times D^m)) = \Delta^n \wedge S^m
d\square
\]

For the next theorem, let $h_{\tilde{\Delta}_n}$ denote either $h_{\tilde{\Delta}_n}$ or $h_{\Delta_n}$ depending on whether or not $\Delta^n$ contains a special marked vertex $b$. Similarly, let $\Delta_n$ denote either $\tilde{\Delta}_n$ or $\Delta_n$, and correspondingly let $S^m(\Delta_n)$ denote either $\Delta_n \wedge S^m$ or $(\Delta_n \cup \{b\}) \wedge S^m$.

**Theorem 8.3.** Let $\Delta^{n-1}$ be a codimension-1 face in $\Delta^n$. There can be three cases regarding the role of the basepoint $b$, namely, $b \in \Delta^{n-1}$, $b \in (\Delta^n \setminus \Delta^{n-1})$ or $b \notin \Delta^n$. In either case, for sufficiently large $m$, the following diagram commutes

\[
\begin{array}{ccc}
X_{\tilde{\Delta}_n}(m) & \xrightarrow{h_{\tilde{\Delta}_n}} & S^m(\Delta^{n-1}) \\
\downarrow & & \downarrow \\
X_{\Delta_n}(m) & \xrightarrow{h_{\Delta_n}} & S^m(\Delta_n)
\end{array}
\]
where the inclusion on the left is given by Theorem 7.3 and the inclusion on the right is induced from the inclusion of $\Delta^{n-1}$ into $\Delta^n$.

Proof. Let us just do the case when $b \notin \Delta^n$. Recall that $X_{\Delta_n}(m)$ is obtained from $\Delta^n \times D^m$ by quotienting out $\Delta^n \times \partial D^m$. However, the inclusion of $\Delta^{n-1}$ into $\Delta^n$ induces both the inclusion on the left and the one on the right, and hence the diagram commutes. \hfill \□

The above theorems have a very interesting corollary which shows that the CW complexes $X_P(m)$ can be quite complicated.

**Theorem 8.4.** Let $K$ be a simplicial complex with a special vertex marked as the basepoint $b$. Then there exists a GSS poset $P$, such that for sufficiently large $m$, $X_P(m) = K \land S^m$.

Proof. Consider the poset whose elements in grading $k$ are the $k$-simplices of $K$, partially ordered by inclusion. Let $P$ be the quotient poset consisting of all the elements except the element corresponding to the basepoint $b$. Let us fix an orientation on every simplex of $K$, and then assign a sign of $\pm 1$ to each covering relation based on whether the attaching map preserves the orientation or reverses it. The closed intervals in this poset are isomorphic to the subset poset, and hence are shellable. Thus $P$ is a GSS poset. For $m$ large enough, let us consider the pointed CW complex $X_P(m)$.

The $(m + k)$-cells of $X_P(m)$ correspond to the $k$-cells in $K$ except for the 0-cell $b$, and the boundary maps of $X_P(m)$ correspond to the boundary maps in $K$. Observe that $K \land S^m$ with its natural pointed CW complex structure also has this property. Now recall how we construct $X_P(m)$. For each element $x \in P$, we construct a CW complex corresponding to the poset $(-\infty, x ]$, and whenever $y \leq x$, there is an embedding of the CW complex corresponding to $y$ into the CW complex corresponding to $x$. Since such an inclusion can be viewed as a composition of inclusions coming from covering relations like $y \leftarrow x$, we can just restrict our attention to those maps.

If $x$ corresponds to an $n$-simplex, then the poset $(-\infty, x ]$ is either $\tilde{\Delta}_n$ or $\Delta_n$, depending on whether or not $b$ is in $\Delta_n$. From the previous theorems, we know that the CW complex corresponding to $x$ is either $\Delta^n \land S^m$ or $(\Delta^n \cup \{ b \}) \land S^m$, and the inclusion maps coming from $y \leftarrow x$ are induced from inclusions of simplices in $K$. Therefore, $X_P(n)$ and $K \land S^m$ have the same CW complex structure, and hence are homeomorphic. \hfill \□

**Theorem 8.5.** There exist GSS posets $P_1$ and $P_2$ with the same homology, but with different homotopy types of their associated spectra.

Proof. We want to find GSS posets $P_1$ and $P_2$ with same homology, such that $X_{P_1}(m)$ is not homotopic to $X_{P_2}(m)$ for all $m$. We choose $P_1$ to be a poset consisting of only two elements, which are non-comparable and have gradings 2 and 4. We choose $P_2$ to be a poset coming from a simplicial complex structure on $\mathbb{C}P^2$, as described in Theorem 7.3. Clearly, both have homology $\mathbb{Z}^2$ supported in gradings 2 and 4.

Furthermore, $X_{P_1}(m) = S^{m+2} \lor S^{m+4}$ and $X_{P_2}(m) = \mathbb{C}P^2 \land S^m$. We want to show that these two spaces are not homotopic for any $m$, or in other words, we want to show that $S^2 \lor S^4$ is not stably homotopic to $\mathbb{C}P^2$. This can be seen in several ways. If $a_2$ and $a_4$ (resp. $b_2$ and $b_4$) are the generators in $H^2$ and $H^4$ of $S^2 \lor S^4$ (resp. $\mathbb{C}P^2$) with coefficients in $F_2$, then $Sq^2(a_2) = 0$ but $Sq^2(b_2) = b_4$, where $Sq^2$ is the second Steenrod square operation. Also, $\pi_3(S^2 \lor S^4) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_3(\mathbb{C}P^2) = 0$, where $\pi_3$ is the third stable homotopy group. \hfill \□

Now, as promised at the beginning of the section, we do the computation for the hat version of the trefoil, presented in the grid as shown in Figure 8.1. If $a \in \mathcal{G}$ is the generator whose coordinate on the vertical line marked $i$, lies on the horizontal line marked $a_i$, then we denote $a$ by the sloppy notation $a_0a_1a_2a_3a_4$. A component of $\mathcal{G}$ is a path connected component of the Hasse diagram of the partial order on $\mathcal{G}$, viewed as

an unoriented graph. A simple computation shows that there are 25 components in \( \hat{G} \), of which 22 of them contain only one element. There are two components \( C_1 \) and \( C_2 \) with 26 elements each, and homology \( \mathbb{Z}^6 \), and there is one component \( D \) with 46 elements and homology \( \mathbb{Z}^{14} \).

The CW complex \( X_\hat{G} \) is a wedge of the CW complexes coming from the different components. The spaces coming from the components with only one element, are simply spheres of the right dimension; therefore, we can restrict our attention to \( C_1, C_2 \) and \( D \). Let us first consider the case of \( C_1 \).

Each of \( C_1 \) and \( C_2 \) has a unique element of maximum Maslov grading (however, neither of them has a unique maximum), which happens to be 12340 and 23401 respectively. However, these two generators swap when we apply a rotation of \( R(\pi) \) and reverse the roles of the \( X \) markings and the \( O \) markings. This shows that \( C_1 \) and \( C_2 \) are isomorphic posets, and hence we can work with \( C_1 \). The following are the 26 elements of \( C_1 \).

- Maslov grading 2: 12340
- Maslov grading 1: 12304, 02341, 21340, 13240, 12430
- Maslov grading 0: 20134, 12034, 03124, 21304, 41203, 13204, 01423, 01342, 40231, 31240, 03241, 14230, 02431, 21430
- Maslov grading \(-1\): 21034, 31204, 03214, 04231, 01432

The homology \( \mathbb{Z}^6 \) lies entirely in grading 0. There are six maxima in \( C_1 \), which are 20134, 03124, 41203, 01423, 40231, 31240 in grading 0 and 12340 in grading 2. Let \( C \) be the poset \((-\infty, 12340]\), which turns out to be \( C_1 \) \(-\{20134, 03124, 41203, 01423, 40231\}\). Since \( C \) is a subposet of \( C_1 \), \( X_{C_1}(m) \) is obtained by adding five \( m \)-cells to \( X_C(m) \). However, the homology of \( C \) is \( \mathbb{Z} \) supported in grading 0, hence \( H_i(X_C(m)) = 0 \) for all \( i < m \). Since we can assume all the spaces to be simply connected, Hurewicz theorem implies that \( \pi_m(X_C(m)) = \mathbb{Z} \).

If a map \( S^m \to X_C(m) \) represents the generator of \( \pi_m(X_C(m)) \), then the map induces isomorphism in all the homology groups. Therefore, \( X_C(m) \sim S^m \). Furthermore, since \( \pi_m \sim S^m \), there is a unique way, up to homotopy, to add the five \( m \)-cells to \( X_C(m) \). Thus we get \( X_{C_1}(m) \sim X_C(m) \lor (\lor_{i=1}^{6} S^m) \sim \lor_{i=1}^{6} S^m \).

Observe that our argument is entirely independent of the choice of a sign assignment. In fact, \( C_1 \) has only one sign assignment up to equivalence. This is because \( C \), being a GSS poset with a unique maximum, has only one sign assignment; and that sign assignment extends uniquely to \( C_1 \), since every element of \( C_1 \setminus C \) covers exactly one element in \( C \).

Let us now concentrate on the poset \( D \). Its homology is \( \mathbb{Z}^{10} \), supported in grading \(-1 \). The following are the 46 elements of \( D \).

- Maslov grading 0: 40123, 12403, 42301, 20341, 23140, 13420
- Maslov grading \(-1\): 30124, 41023, 13024, 04123, 20314, 40213, 23104, 42103, 14203, 01423, 20413, 21403, 41302, 43201, 20143, 40132, 12043, 03142, 42031, 10342, 30241, 32140, 24130, 34120, 31420, 30421, 14320
- Maslov grading \(-2\): 31024, 30214, 32104, 04213, 21043, 41032, 04132, 10432, 04321, 43210

Consider the subposet \( D_1 \) of \( D \), consisting of the elements \{42103, 10423, 20143, 43120, 40321, 13024, 20314, 14203, 41320, 03142\} in grading \(-1 \) and all the ten elements in grading \(-2 \). The Hasse diagram of the poset \( D_1 \), viewed as an unoriented graph, has ten components. Hence \( X_{D_1}(m) \sim \{pt\} \). Let \( D_2 \) be the subposet of \( D \), consisting of all the elements in gradings \(-1 \) and \(-2 \). Since \( D_1 \) is a subposet of \( D_2 \), \( X_{D_2}(m) \) is obtained by adding twenty \((m - 1)\)-cells to \( X_{D_1}(m) \), and there is only one way of doing that, leading to \( X_{D_2}(m) \sim \lor_{i=1}^{20} S^{m-1} \). The space \( X_D(m) \) is obtained from \( X_{D_2}(m) \) by attaching six \( m \)-cells to it, and the choice depends on \( \pi_m(X_{D_1}(m)) = \mathbb{Z}^{20} \). However in \( D \), the six elements of grading 0 cover disjoint elements; therefore, after attaching those six \( m \)-cells, we get \( X_D(m) \sim \lor_{i=1}^{14} S^{m-1} \). Notice once more that this is entirely independent of the sign assignment.
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