Optimal Three-Dimensional Antenna Array for Direction Finding With Geometric Constraint

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Abstract This paper studies the optimal three-dimensional (3-D) antenna array for direction finding when the antenna array is subject to geometric constraint. Two-dimensional (2-D) planar array has been widely used in existing direction finding systems, and the optimal geometry for planar array has also been investigated in existing works. However, the optimal antenna array in the 3-D space, especially when the positions of the antennas are subject to geometric constraint, has not been studied yet. In this paper, a novel approach based on coordinate transformation is proposed for optimal antenna array design in the 3-D space with geometric constraint. Due to geometric constraint, the position of each antenna has only two degrees-of-freedom even thought it has three coordinate values, so that coordinate transformation can be used to convert 3-D coordinates of the array elements to 2-D ones. Then, existing optimization approach for 2-D planar array can be applied to address the issue of optimal 3-D antenna array. Simulation results show that the proposed antenna array in this paper can outperform the defacto uniform circular array (UCA) even in the presence of geometric constraint. As a potential application, the result in this paper can be applied to design the high-frequency (HF) direction finding antenna arrays that are placed in mountain area.

Index Terms Direction finding, optimal antenna array, three-dimensional antenna array, geometric constraint.

I. INTRODUCTION

Direction finding using antenna array has been widely used for direction-of-arrival (DOA) estimation of impinging signal. It can determine the directions (or even the locations when multiple direction finding receivers are available) of the unknown signal sources, and thus has been widely applied in wireless communications, radars, and radio monitoring systems [1], [2]. When narrowband signals are impinging from far-filed onto the antenna array, the phases of the received signals at different array elements are determined by the relative positions of the array elements. Therefore, the accuracy of a direction finding system depends heavily on the geometry of the antenna array. Although direction finding have been widely studied in the past decades [3]–[7], the optimal array geometry, especially for the 3-D antenna array, is not investigated sufficiently, and as a consequence, uniform circular array (UCA) has been adopted as the defacto standard array in many existing direction finding systems [8]–[10].

Due to the simple geometric structure, the optimality of uniform linear array (ULA) and its two-dimensional (2-D) extensions (such as L-shaped and rectangular arrays) are studied in [11], [12]. As the defacto reference, UCA is also investigated in [13]–[15]. For a general 2-D antenna array where the array elements are located in $\mathbb{R}^2$, an exhaustive searching approach is proposed in [8] to find out the optimal geometry for the omni-directional planar array. For the omni-directional antenna array, the key property lies in that it can achieve the same estimation performance for different
azimuth angles. As in [14], [15], the antenna array has to be specially designed to achieve the omni-directional property. Due to the omni-directional property, the derived Cramer-Rao Bound (CRB) is independent on the azimuth angle, and as a result, the problem of optimal array design can be converted to a constrained optimization problem. Exhaustive searching is adopted in [8] to solve such optimization problem. It is shown in [8] that the optimal antenna array, whose geometry is similar to a “V” shape, can outperform the traditional UCA. As a improvement to [8], a simplified approach is proposed in [16] by observing that the optimal omni-directional array has an axis symmetry property. Compared to the approach in [8], the simplified approach in [16] can reduce the computational complexity substantially, and it is therefore can be used for the optimal design of large-scale antenna array.

For three-dimensional (3-D) antenna array, array elements are located in $\mathbb{R}^3$. Therefore, 3-D antenna array can be located in 3-D space rather than only in 2-D planar space. The flexibility of placement makes 3-D antenna array much more important than traditional 2-D planar arrays. For example, traditional high-frequency (HF) direction finding system has to occupy a large planar area to place 2-D antenna array. Due to the size limit, the required large planar area may be unavailable, which limits the deployment of the HF direction finding system. This issue can be addressed by placing the 3-D antenna array on the mountain area, and thus the large planar area is not required any more. By doing this, we can not only save the physical space but also expand the monitoring range of HF direction finding systems because it is impossible to place 2-D planar array in mountain area. The flexibility of placement makes the 3-D antenna array widely investigated in existing works [15], [17]. In [17], potential performance improvement using 3-D array is investigated. Omni-directional 3-D antenna array is studied in [15], and the geometry to achieve omni-directional 3-D array is also shown. However, the optimal geometry for 3-D antenna array is not well investigated yet to the best knowledge of the authors.

In this paper, we propose a coordinate transformation approach to address the design of optimal 3-D antenna array. For a practical 3-D antenna array, placement of the antenna array is always subject to some geometric constraint. For example, if a 3-D HF antenna array is placed in mountain area, the positions of the antennas will be constrained by the surface of the mountain. As a result, the location of each array element has only two degrees-of-freedoms, even though it has three coordinate values (i.e., $x$, $y$, $z$) in Cartesian coordinate system). This observation inspires a coordinate transformation approach, which can convert the positions of the antennas in 3-D space into the positions in 2-D space through coordinate transformation. On this basis, we will show that the optimal design of 3-D array in the presence of geometric constraint can be converted into a problem of 2-D array design. Therefore, the optimal 3-D array design in the presence of geometric constraint is mathematically equivalent to the optimal design of the 2-D planar array, so that exhaustive searching approach in [8], which is dedicated for 2-D planar array, can be used for the design of the optimal 3-D antenna array. The result in this paper provides theoretical insights on the structure of 3-D array with geometric constraint, and as a potential application, it can be applied in HF direction finding systems that are placed on mountain area.

The rest of this paper is organized as follows. In Section II, signal model is presented and the problem for optimal 3-D antenna array design is formulated. In Section III, the optimal planar array design is reviewed briefly. In Section IV, the proposed coordinate transformation approach is discussed. Simulation results are presented in Section V and the conclusions are drawn in Section VI.

II. SYSTEM MODEL

As in Fig. 1, consider a narrowband signal impinging from far field on an $M$ antenna 3-D array. The position of the $m$-th antenna is given by a position vector

$$r_m = (x_m, y_m, z_m)^T.$$  \hspace{1cm} (1)

Accordingly, the centroid of the array can be expressed as

$$r_c = \frac{1}{M} \sum_{m=1}^{M} r_m.$$ \hspace{1cm} (2)

Theoretically, the result in this paper can be applied for an arbitrary number of antennas. In practice, however, as the optimal array is obtained using exhaustive searching, the computation burden will be substantially increased when the number of antennas is large. In this case, low complexity algorithm can be adopted as in [16].

![FIGURE 1. A signal from far-filed source imping onto the array.](image)

In reality, placement of a practical 3-D antenna array is always subject to some geometric constraint, that is, we can always have $z_m = f(x_m, y_m)$ where the function $z = f(x, y)$ describes the geometric constraint as in Fig. 1. In other words,
\[ z_m = ax_m + by_m. \]  

(3)

The geometric constraint plane in (3) is not only easy for analysis but also practical for real mountain area, because the local of any mountain area can be approximated by a plane.

Denote \( s(t) \) to be the signal from the far-field source with power \( E(\{s(t)\})^2 = \sigma_s^2 \), and the signal arrives at the array with azimuth angle \( \phi \) and elevation angle \( \theta \). Denote \( x(t) = [x_1(t), x_2(t), \ldots, x_M(t)]^T \) to be the \( M \times 1 \) output vector of the antenna array, where \( x_m(t) \) indicates the received signal on the \( m \)-th antenna, then

\[ x(t) = a(\phi, \theta)s(t) + n(t), \]  

(4)

where \( n(t) \) denotes the additive Gaussian white noise (AWGN) vector with power \( E[\{n(t)n^H(t)\}] = \sigma_n^2 \mathbf{I} \). and \( a(\phi, \theta) = \begin{bmatrix} a_1(\phi, \theta), a_2(\phi, \theta), \ldots, a_M(\phi, \theta) \end{bmatrix}^T \) indicates the steering vector. The steering vector is usually affected by mutual coupling among the antennas, especially when the antennas are close to each other. Accuracy of a direction finding system will be substantially deteriorated if the mutual coupling effect is not well addressed [18]–[24]. As the mutual coupling effect is inversely proportional to the distance between antennas [25], an effective approach for mutual coupling reduction is to increase the distance between adjacent antennas so that the mutual coupling effect can be ignored. When the distance between adjacent antennas is large enough (say half wave length), the \( m \)-th entry of \( a(\phi, \theta) \) is given by

\[ a_m(\phi, \theta) = e^{j\lambda \left( x_m \sin \phi \cos \theta + y_m \sin \phi \sin \theta + z_m \cos \theta \right)}, \]  

(5)

where \( \lambda \) is the wave length of the signal.

For optimal design of the antenna array, CRB is often used as the performance measure. It presents the lower bound on the mean-square-error (MSE) of the estimated parameters \( \phi \) and \( \theta \). For the single source case, the \( 2 \times 2 \) CRB matrix can be given by

\[ C = \begin{pmatrix} C_{\phi\phi} & C_{\phi\theta} \\ C_{\theta\phi} & C_{\theta\theta} \end{pmatrix}, \]  

(6)

where the diagonal entries, \( C_{\phi\phi} \) and \( C_{\theta\theta} \), refer to the CRBs for the azimuth and the elevation angles, respectively. The CRB matrix is usually given as the inverse of the Fisher information matrix \( F \), that is,

\[ C = F^{-1}. \]  

(7)

Denote \( F_{kl} \) \( (k, l = 1, 2) \) to be the entry of \( F \) on the \( k \)-th row and the \( l \)-th column, then the result in [26] shows that

\[ F_{kl} = C_{\text{SNR}} \cdot \left[ a_k^H(\phi, \theta)a_l(\phi, \theta) - \frac{1}{M} a_k^H(\phi, \theta)a(\phi, \theta)a^H(\phi, \theta)a_l(\phi, \theta) \right], \]  

(8)

where

\[ C_{\text{SNR}} = \frac{2N\sigma_s^4}{\sigma_n^4(\sigma_n^2 + \sigma_s^2)}; \]  

(9)

is a constant that only depends on the SNR \( = \sigma_s^2/\sigma_n^2 \) and the number of snapshots. In (8), \( a_1(\phi, \theta) \) and \( a_2(\phi, \theta) \) are the partial derivatives of the steering vector with respect to the azimuth angle and the elevation angle, that is, \( a_1(\phi, \theta) = \partial a(\phi, \theta)/\partial \phi \) and \( a_2(\phi, \theta) = \partial a(\phi, \theta)/\partial \theta \). Compared to the elevation angle, the azimuth angle is of more practical interest because it can be used to determine the location of the unknown signal source. As a result, the design of optimal geometry of the antenna array is to minimize the CRB of the azimuth angle [26].

In general, \( C_{\phi\phi} \) is not only a function of the array geometry but also a function of the azimuth angle. It is therefore difficult to derive a unique array geometry that achieves the minimum CRB for different azimuth angles. Omni-directional antenna array can be adopted to address the above issue. The key property of omni-directional antenna array lies in that it can achieve the same estimation accuracy regardless of the azimuth angle of the impinged signal. In this case, \( C_{\phi\phi} \) is a function of the array geometry only. Therefore, optimal array geometry is unique and thus can be investigated. For a 3-D antenna array, the omni-directional property can be ensured if the positions of the antennas satisfy [15]

\[ \frac{1}{M} \sum_{m=1}^{M} (r_m - r_c)(r_m - r_c)^H = 0, \]  

(10)

where 0 indicates an \( M \times M \) all-zero matrix.

From the above discussion, the optimal design of the 3-D antenna array can be formulated as a constrained optimization problem, that is

\[ \begin{align*}
\min_{\{r_m\}_{m=1}^{M}} & \quad C_{\phi\phi} \\
\text{s.t.} & \quad \frac{1}{M} \sum_{m=1}^{M} (r_m - r_c)(r_m - r_c)^H = kI, \\
& \quad ax_m + by_m - z_m = 0,
\end{align*} \]  

(11)

where \( k \) is a constant, and the first and the second constraints are caused by the requirement of the omni-directional array and the geometric constraint, respectively. In general, it is difficult to obtain an analytical result for the the optimization problem in (11) because further analysis on \( C_{\phi\phi} \) is intractable.

III. REVIEW ON OPTIMAL PLANAR ARRAY

Optimal design of planar array has been investigated in [8], and the approach there can be exploited to solve the optimization in (11). In this section, we will review the optimal planar array design in [8] under the framework of (11).

For the planar array, the constraint in (11c) is reduced to

\[ z_m = 0, \]  

(12)

and thus the position of the antenna is reduced to

\[ r_m = (x_m, y_m, 0)^T. \]  

(13)
Apparently, the constraint in (12) can be always satisfied for a planar array. Meanwhile, it is shown in (13) that the position of the antenna in planar array is determined by only two coordinate values. Therefore, complex number can be used to describe the positions of the antennas, which can greatly simplify the analysis. In this situation, the position of the \( m \)-th antenna is given by

\[
\rho_m = x_m + jy_m. \tag{14}
\]

Accordingly, the centroid of the array is given by

\[
\rho_c = \frac{1}{M} \sum_{m=1}^{M} \rho_m. \tag{15}
\]

With the complex position in (14), the optimization target \( C_{\phi \phi} \) in (11) can be rewritten, for the planar array, as [8]

\[
C_{\phi \phi} = \frac{1}{2\pi^2 C_{\text{SNR}}} \cdot \frac{1}{\sin^2 \theta} \cdot \frac{S_0 + \text{Re}(S_1 e^{-j2\phi})}{S_0^2 - |S_1|^2}, \tag{16}
\]

where \( S_0 \) and \( S_1 \) are given by

\[
S_0 = \frac{1}{\lambda^2} \left( \sum_{m=1}^{M} |\rho_m|^2 - M|\rho_c|^2 \right), \tag{17}
\]

\[
S_1 = \frac{1}{\lambda^2} \left( \sum_{m=1}^{M} \rho_m^2 - M\rho_c^2 \right). \tag{18}
\]

On the other hand, by substituting (13) into (11b), we have

\[
\sum_{m=1}^{M} (x_m^2 - y_m^2) = M \cdot (x_c^2 - y_c^2), \tag{19}
\]

\[
\sum_{m=1}^{M} x_m y_m = M \cdot x_c y_c, \tag{20}
\]

from which, we can obtain

\[
\sum_{m=1}^{M} \frac{(x_m^2 - y_m^2 + j2x_m y_m) - M(x_c^2 - y_c^2 + j2x_c y_c)}{\rho_m^2 - \rho_c^2} = 0, \tag{21}
\]

By noting that the first term in the summarization of (21) is \( \rho_m^2 \) and the second term is \( \rho_c^2 \), the left hand side of (21) is exactly \( S_1 \) in (18) except for a constant \( 1/\lambda^2 \). Therefore, the constraint in (11b) is equivalent to

\[
S_1 = 0, \tag{22}
\]

for the planar array.

From (16) and (22), the optimization problem in (11) can be rewritten as minimizing \( C_{\phi \phi} \) in (16) subject to the constraint \( S_1 = 0 \). In fact, if \( S_1 = 0 \), \( C_{\phi \phi} \) in (16) can be further simplified to be \( 1/S_0 \) except for a constant that is independent on the array geometry. Therefore, for the planar array, the optimization problem in (11) can be rewritten as

\[
\min_{(x_m, y_m)_{m=1}^{M}} \frac{1}{S_0}, \quad \text{s.t.} \quad S_1 = 0, \tag{23}
\]

As in [8], the optimization problem in (23) can be solved directly using the exhaustive searching approach. It is shown in [8] that the optimal geometry of the planar antenna array is similar to a “V” shape, and the optimal antenna array can outperform the traditional UCA.

Although efficient, the proposed approach in [8] is dedicated for planar array, and therefore it cannot be directly used for 3-D antenna arrays because the geometry of the 3-D array is much more complex than the 2-D planar array.

IV. OPTIMAL 3-D ANTENNA ARRAY DESIGN

To solve the optimization problem in (11), the coordinate transformation is adopted so that the optimal 3-D array design can be converted to a 2-D array design in the transformed coordinate system. Then, we will prove that the optimal 3-D array design is mathematically equivalent to the optimal design of the 2-D planar array in the transformed coordinate system, and therefore the approach in [8] can be used for 3-D array design.

A. COORDINATE TRANSFORMATION

As in Section II, due to the geometric constraint, the position of each antenna has only two degrees-of-freedom, even thought it has three coordinate values. It indicates that the
position of each antenna can be uniquely determined by two coordinate values. In view of this, the purpose of coordinate transformation is to convert the original position vector $\mathbf{r}_m$ into a transformed position vector $\tilde{\mathbf{r}}_m$.

$$\mathbf{r}_m = \mathbf{T}\tilde{\mathbf{r}}_m,$$

(24a)

$$\tilde{\mathbf{r}}_m = T^H\mathbf{r}_m,$$

(24b)

via a $3 \times 3$ orthogonal matrix $\mathbf{T}$ with $\mathbf{T}^H\mathbf{T} = \mathbf{I}$, so that the third coordinate value can be zero, that is

$$\tilde{\mathbf{r}}_m = (\tilde{x}_m, \tilde{y}_m, 0)^T,$$

(25)

where $\tilde{x}_m, \tilde{y}_m$ are the transformed coordinates. When there are only two coordinate values in the transformed coordinate system, similar to (13) and (14), complex number can be used to describe the position of the antennas in the transformed coordinate system, that is

$$\tilde{\rho}_m = \tilde{x}_m + j\tilde{y}_m.$$  

(26)

Accordingly, the centroid of can be given by

$$\tilde{\rho}_c = \frac{1}{M} \sum_{m=1}^{M} \tilde{\rho}_m.$$  

(27)

To determine the transformation matrix $\mathbf{T}$, consider an example as in Fig. 2 (a). In the figure, a group of points are distributed on a line crossing the origin, and the coordinate of each point is $(x_m, y_m)$. If the original coordinate system is rotated by $\Delta \psi$, a new coordinate system can be obtained and the positions of those points in the new coordinate system will be $(\tilde{x}_m, 0)$. As a result, the second coordinate is removed.

Inspired by the example above, $\mathbf{r}_m$ can be converted to $\tilde{\mathbf{r}}_m$ if the original coordinate system can be rotated properly such that the geometric constraint plane can become the $\tilde{x}\tilde{y}$ plane in the transformed coordinate system, as in Fig. 2 (b).

In this case, the transformation matrix $\mathbf{T}$ denotes exactly the transformation from the original coordinate system $(x, y, z)$ to the new coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$. Denote $\mathbf{e}_x, \mathbf{e}_y$ and $\mathbf{e}_z$ to be the basis vectors in the transformed coordinate system, which are mutually orthogonal

$$\mathbf{e}_x^H\mathbf{e}_x = \mathbf{e}_y^H\mathbf{e}_y = \mathbf{e}_z^H\mathbf{e}_z = 0,$$

(28)

then $\mathbf{T}$ can be expressed as

$$\mathbf{T} = \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{pmatrix}.$$

(29)

In Appendix, it is shown that the basis vectors are given by

$$\mathbf{e}_x = (b, -a, 0)^T,$$

(30a)

$$\mathbf{e}_y = (a, b, a^2 + b^2)^T,$$

(30b)

$$\mathbf{e}_z = (a, b, -1)^T.$$

(30c)

For more insights, note that $\mathbf{e}_z$ in (30a) is on the $xoy$ plane in the original coordinate system, then if denote $\Delta \theta$ to be the angle between axis $\alpha \tilde{z}$ and axis $\alpha \tilde{x}$, and $\Delta \phi$ to be the angle between axis $\alpha \tilde{z}$ and axis $\alpha \tilde{y}$, $\mathbf{T}$ can be rewritten as

$$\mathbf{T} = \begin{pmatrix} \cos \Delta \phi & -\sin \Delta \phi & \sin \Delta \phi \cos \Delta \theta & \sin \Delta \phi \sin \Delta \theta \\ \sin \Delta \phi & \cos \Delta \phi & \cos \Delta \phi \cos \Delta \theta & -\cos \Delta \phi \sin \Delta \theta \\ 0 & 0 & \sin \Delta \theta & \cos \Delta \theta \end{pmatrix},$$

(31)

which can be further decomposed into

$$\mathbf{T} = \mathbf{T}_1 \mathbf{T}_2,$$

(32)

where $\mathbf{T}_1, \mathbf{T}_2$ are given by

$$\mathbf{T}_1 = \begin{pmatrix} \cos \Delta \phi & -\sin \Delta \phi & 0 \\ \sin \Delta \phi & \cos \Delta \phi & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

(33)

and

$$\mathbf{T}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Delta \theta & -\sin \Delta \theta \\ 0 & \sin \Delta \theta & \cos \Delta \theta \end{pmatrix}.$$  

(34)

It is shown that the transformation matrix in (32) can be obtained by first rotating the original coordinate system around axis $\alpha \tilde{z}$ (denoted by $\mathbf{T}_1$), and then rotating around axis $\alpha \tilde{x}$ (denoted by $\mathbf{T}_2$), c.f. Fig. 2 (b).

B. OPTIMAL 3-D ARRAY DESIGN

The transformation in (32) not only changes the position vectors of the array, but also alters the azimuth and the elevation angles. Denote $\hat{\phi}$ and $\hat{\theta}$ to be the azimuth and the elevation angles of the received signal after the first rotation around axis $\alpha \tilde{z}$, and $\tilde{\phi}$ and $\tilde{\theta}$ to be the azimuth and the elevation angles after the second rotation around axis $\alpha \tilde{x}$, respectively, then we can obtain, from Fig. 2 (b), that

$$\hat{\phi} = \phi - \Delta \phi,$$

(35)

$$\hat{\theta} = \theta,$$

(36)

and

$$\tilde{\phi} = \arccos(\cos \Delta \theta \cdot \cos \hat{\phi}),$$

(37)

$$\tilde{\theta} = \hat{\theta} - \Delta \theta.$$  

(38)

Note that although the coordinate transformation changes the azimuth and the elevation angles, the optimization problem in (11) should be conducted with respect to the original azimuth angle $\phi$. In the following, we will prove that minimizing $C_{\hat{\phi} \hat{\theta}}$ is equivalent to minimizing $C_{\tilde{\phi} \tilde{\theta}}$.

Denote $\tilde{\mathbf{C}}$ to be the CRB matrix with respect to $\tilde{\phi}$ and $\tilde{\theta}$,

$$\tilde{\mathbf{C}} = \begin{pmatrix} C_{\tilde{\phi} \tilde{\phi}} & C_{\tilde{\phi} \tilde{\theta}} \\ C_{\tilde{\theta} \tilde{\phi}} & C_{\tilde{\theta} \tilde{\theta}} \end{pmatrix}.$$  

(39)

then from [27], $\tilde{\mathbf{C}}$ can be expressed as

$$\tilde{\mathbf{C}} = \begin{pmatrix} \frac{\partial \tilde{\phi}}{\partial \phi} \frac{\partial \tilde{\phi}}{\partial \theta} & \frac{\partial \tilde{\phi}}{\partial \phi} \frac{\partial \tilde{\theta}}{\partial \theta} \\ \frac{\partial \tilde{\theta}}{\partial \phi} \frac{\partial \tilde{\phi}}{\partial \theta} & \frac{\partial \tilde{\theta}}{\partial \phi} \frac{\partial \tilde{\theta}}{\partial \theta} \end{pmatrix} \mathbf{C} \begin{pmatrix} \frac{\partial \tilde{\phi}}{\partial \phi} & \frac{\partial \tilde{\phi}}{\partial \theta} \\ \frac{\partial \tilde{\theta}}{\partial \phi} & \frac{\partial \tilde{\theta}}{\partial \theta} \end{pmatrix}.$$  

(40)

By substituting (35) and (36) into (40), we can obtain $\tilde{\mathbf{C}} = \mathbf{C}$ and thus $C_{\tilde{\phi} \tilde{\theta}} = C_{\hat{\phi} \hat{\theta}}$. It means that minimizing $C_{\hat{\phi} \hat{\theta}}$ in (11a)
FIGURE 3. Optimal array geometry projected onto the geometry constraint plane (a), and optimal array geometry in the 3-D space (b).

can be converted to minimizing $C_{\tilde{\phi} \tilde{\phi}}$. Denote $\hat{C}$ to be the CRB matrix with respect to $\tilde{\phi}$ and $\tilde{\theta}$, then similar to (40), we can obtain

$$C_{\hat{\phi} \hat{\phi}} = \frac{\cos^2 \Delta \theta \cdot \sin^2 \tilde{\phi}}{1 - \cos^2 \Delta \theta \cdot \cos^2 \tilde{\phi}} C_{\tilde{\phi} \tilde{\phi}}.$$  \hfill (41)

It is shown in (41) that $C_{\hat{\phi} \hat{\phi}}$ is equal to $C_{\tilde{\phi} \tilde{\phi}}$ except for a scaling constant which is independent on the array geometry. Therefore, minimizing $C_{\hat{\phi} \hat{\phi}}$ is also equivalent to minimizing $C_{\tilde{\phi} \tilde{\phi}}$. As a result, we can conclude that minimizing $C_{\phi \phi}$ in (11a) can be converted to minimizing $C_{\tilde{\phi} \tilde{\phi}}$.

On the other hand, if multiplying $T^H$ on the left and $T$ on the right of (11b), we can obtain

$$\frac{1}{M} \sum_{m=1}^{M} (\tilde{r}_m - \tilde{r}_c)(\tilde{r}_m - \tilde{r}_c)^H = kI,$$  \hfill (42)

where $\tilde{r}_c$ is the centroid of the transformed position vectors,

$$\tilde{r}_c = \frac{1}{M} \sum_{m=1}^{M} \tilde{r}_m.$$  \hfill (43)

By substituting (25) into (42), the constraint in (11b) can be rewritten, following a similar procedure from (19) to (22), as

$$\tilde{S}_1 = 0,$$  \hfill (44)

where the definition of the $\tilde{S}_1$ is similar to $S_1$ in (18) except replacing $\rho_m$ and $\rho_0$ with $\tilde{\rho}_m$ and $\tilde{\rho}_c$. Meanwhile, it is easy to verify that the third constraint in (11c) can be always satisfied if the antennas are placed on the geometric constraint plane.

From the discussion above, the optimization problem in (11) can be rewritten as

$$\min\limits_{\{\tilde{\theta}_m, \tilde{\phi}_m\}_{m=1}^{M}} C_{\tilde{\phi} \tilde{\phi}}, \text{ s.t. } \tilde{S}_1 = 0.$$  \hfill (45)
Different from the original problem in (11) which is formulated in a 3-D space, the optimization problem in (45) is formulated in a 2-D plane. To proceed, similar to (16), we note that \( C_{\hat{\phi} \hat{\theta}} \) can be obtained with respect to \( \hat{\phi} \) and \( \hat{\theta} \) as

\[
C_{\hat{\phi} \hat{\theta}} = \frac{1}{2\pi^2 C_{SNR}} \cdot \frac{1}{\sin^2 \hat{\theta}} \cdot \frac{\hat{S}_0 + \text{Re}(\hat{S}_1 e^{-j2\hat{\phi}})}{\hat{S}_0^2 - |\hat{S}_1|^2},
\]

where \( \hat{S}_0 \) is similar to \( S_0 \) in (17) except replacing \( \rho_m \) and \( \rho_c \) with \( \rho_{\hat{\phi}} \) and \( \rho_{\hat{\theta}} \). When \( \hat{S}_1 = 0 \), \( C_{\hat{\phi} \hat{\theta}} \) in (46) can be further simplified to be \( 1/\hat{S}_0 \) except for a constant that is independent on the array geometry. As a result, the optimization problem in (45) can be further rewritten as

\[
\min_{\{C_{\hat{\phi} \hat{\theta}}(\hat{S}_0, \hat{S}_1)\}_{\hat{\phi}, \hat{\theta} = 1}} 1/\hat{S}_0, \quad \text{s.t. } \hat{S}_1 = 0.
\]

Apparently, the optimization problem in (47) has the same form with the one in (23). Therefore, exhaustive searching in [8] can be adopted to solve (47). Note that the optimization in (47) is conducted with respect to the transformed coordinate system, and solution to (47) only gives the transformed position vectors \( \hat{r}_m \)'s. Therefore, \( r_m = T \hat{r}_m \) should be used to obtain the practical position vectors in the 3-D space.

V. SIMULATION RESULTS

Computer simulation is used in this section to demonstrate the effectiveness of the optimal array geometry. In the simulation, exhaustive searching in [8] is first adopted to generate the optimal 2-D antenna array. Then, equation (24a) is used to convert the optimal 2-D antenna array into the optimal 3-D antenna array. For the geometry constraint plane, \( \Delta \phi = \pi/4 \) and \( \Delta \theta = \pi/12 \), which corresponds to a plane \( z = -\sqrt{2}/2 \cdot x + \sqrt{2}/2 \cdot y \). For comparison, a UCA array is considered to be placed on the geometric constraint plane, that is, the projection of the array onto the geometry constraint plane forms a circle. Since the UCA is not placed on a 2-D plane, it is termed as sloped UCA in the simulation. The distance between antennas is normalized by the wavelength, the result is valid for any frequency bands.

Fig. 3 shows the geometry of the optimal antenna array. In Fig. 3 (a), the projection of the optimal antenna array on the geometric constraint plane is plotted. Since the projection of the 3-D array onto the geometric constraint plane is equivalent to the traditional 2-D plane, the obtained optimal array geometry is therefore the same with the traditional 2-D optimal array design [8]. Fig. 3 (b) shows the optimal array geometry in the 3-D space, and the 3-D array geometry in Fig. 3 (b) is obtained by applying (24a) to the 2-D array geometry in Fig. 3 (b). As expected, the projection on the geometric constraint plane has a “V” shape especially when the antenna number is large [8], while the optimal array in the 3-D space is thus shown to be a rotated “V” shape.

Fig. 4 shows the performance of the optimal antenna array, which is evaluated using the root-mean-square (RMS). Instead of numerical simulation, the CRB is used to derive the RMS. In Fig. 4 (a), it shows that the performance of the optimal array can be improved as the number of antennas increases. This is not surprising because the aperture of the array will increase as the antenna number increases, as demonstrated in Fig. 3. Fig. 4 (b) shows the comparison between the optimal 3-D array geometry and the sloped UCA. When \( M = 4 \), the geometric structure of the proposed array is the same with the sloped UCA, and thus they have the same performance. When \( M > 4 \), the geometric structure of the proposed array is different from the sloped UCA, and thus as shown in the figure, the proposed 3-D array geometry can outperform the sloped UCA.

VI. CONCLUSION

This paper has studied the optimal 3-D antenna array design for direction finding when the antenna array is subject to geometric constraint. A novel approach based on coordinate transformation had been proposed for optimal antenna array design in the 3-D space with geometric constraint. In this approach, the three-dimensional coordinate of the array element is converted into a two-dimensional one, then the two-dimensional optimization approach can be used with respect to the azimuth and elevation angles in the transformed coordinate system. Simulation results have also been presented and it is shown that the proposed optimal antenna array can outperform the sloped UCA when the antenna number is more than four.

APPENDIX

To derive the expression of \( e_\gamma \) in (30a), denote

\[
e_\gamma = (\alpha_1, \beta_1, \gamma_1)^T.
\]

From Fig. 2 (b), \( e_\gamma \) is the intersection of \( xy \) plane and the geometry constraint plane \( z = ax + by \). It means that \( e_\gamma \) is not only on the \( xy \) plane but also on the geometry constraint plane. Therefore,

\[
\gamma_1 = a\alpha_1 + b\beta_1,
\]

\[
\gamma_1 = 0,
\]

Using (A.2), one can obtain \( \beta_1 = -a\alpha_1/b \), and thus \( e_\gamma \) in (A.1) can be rewritten as

\[
e_\gamma = \frac{\alpha_1}{b} \cdot (b, -a, 0)^T,
\]

which is exactly the same with \( e_\gamma \) in (30a) except for a scaling factor \( \alpha_1/b \). The scaling factor will not change the direction of \( e_\gamma \), and it can be eliminated by normalizing the length of the basis vector to be unit, as in (29).

To derive \( e_\gamma \), denote

\[
e_\gamma = (\alpha_2, \beta_2, \gamma_2)^T.
\]

Two conditions can help to determine the basis vector \( e_\gamma \). First, \( e_\gamma \) is on the geometry constraint plane \( z = ax + by \), as shown in Fig. 2 (b). Second, \( e_\gamma \) should be orthogonal to \( e_\beta \), c.f. (28). Therefore,

\[
\gamma_2 = a\alpha_2 + b\beta_2,
\]

\[
0 = b\alpha_2 + (-a)\beta_2,
\]
from which, one can obtain $\beta_2 = ba_3/a$ and $\gamma_2 = (a^2 + b^2)a_2/a$. Therefore, $e_7$ in (A.4) can be rewritten as

$$e_7 = \frac{a_2}{a} \cdot \frac{a_3}{a} \cdot (a, b, a^2 + b^2)^T,$$

which is also similar to $e_7$ in (30b) except for a scaling factor $a_2/a$.

For the derivation of $e_7$, denote

$$e_7 = (\alpha_3, \beta_3, \gamma_3)^T.$$

The basis vector $e_7$ can be determined because it is orthogonal to $e_7$ and $e_7$. Therefore,

$$ba_3 + (-a)\beta_3 = 0,$$

$$a\alpha_3 + b\beta_3 + (a^2 + b^2)\gamma_3 = 0,$$

from which, one can obtain $\beta_3 = ba_3/a$ and $\gamma_3 = -a_3/a$. Therefore, $e_7$ in (A.8) can be rewritten as

$$e_7 = \frac{a_3}{a} \cdot (a, b, -1)^T.$$

Similarly, except for a scaling factor $a_3/a$, $e_7$ in (A.11) is the same with that in (30c).

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