Field of moduli and field of definition of Galois covers

Stefan Wewers

Abstract. In this paper we investigate the cohomological obstruction for the field of moduli of a $G$-cover to be a field of definition, in the case of local fields and covers with tame admissible reduction. This applies in particular to $p$-adic fields where $p$ does not divide the order of the group $G$. We give examples of $G$-covers with field of moduli $\mathbb{Q}_p$ that cannot be defined over $\mathbb{Q}_p$, for all primes $p > 5$.

Introduction

In the context of the regular inverse Galois problem, the method of rigidity and its generalizations using the braid action and Hurwitz spaces have been very successful. One drawback of these methods is that they apply a priori only to groups with trivial center. To give an example, let $G$ be a finite group and $f : Y \to \mathbb{P}^1$ a $G$-Galois cover of the projective line defined over $\overline{\mathbb{Q}}$. Assume that $f$ has rational branch points $z_1, \ldots, z_r \in \mathbb{P}^1(\mathbb{Q})$ and that the associated tuple of conjugacy classes $(C_1, \ldots, C_r)$ is rational and rigid. Then $\mathbb{Q}$ is the field of moduli of the $G$-cover, i.e. for each $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the conjugate cover $\sigma f$ is isomorphic to $f$. If, moreover, $G$ has trivial center, then $\mathbb{Q}$ is a field of definition of $f$, i.e. there exists a model $f_\mathbb{Q} : Y_\mathbb{Q} \to \mathbb{P}^1_\mathbb{Q}$ of $f$ over $\mathbb{Q}$. This yields the basic Rigidity Criterion, see e.g. \cite{17}, \cite{22} or \cite{24}.

However, if the center $C$ of $G$ is not trivial, the field of moduli $k_m$ of a $G$-cover need not be a field of definition. There is an obstruction $\omega \in H^2(k_m, C)$ such that $f$ can be defined over an extension $k/k_m$ if and only if the restriction of $\omega$ to $k$ vanishes. This cohomological approach has been developed in several papers, in the context of Galois covers and in more general situations, see e.g. \cite{4}, \cite{5}. In \cite{6} it is used to prove a local-to-global principle: a $G$-cover with field of moduli $\mathbb{Q}$ is defined over $\mathbb{Q}$ if and only if it can be defined over $\mathbb{Q}_p$, for each prime number $p$. In \cite{8} it is shown that $\omega|_{\mathbb{Q}_p}$ vanishes for each prime number $p$ at which the cover $f$ has good reduction; this includes all “good” primes in the sense of \cite{2}. In the present paper we describe a method that allows, in many cases, to explicitly compute the local obstruction $\omega|_{\mathbb{Q}_p}$, at a prime $p$ at which the cover $f$ has bad reduction (but which does not divide the order of $|G|$). We also apply this method in a nontrivial example.
To explain our results in more detail, let $k$ be a field which is henselian with respect to a discrete valuation $v$, and let $f : Y \to \mathbb{P}^1$ be a $G$-cover, defined over the algebraic closure of $k$, with field of moduli $k$. Our main assumption is that $f$ has \textit{tame admissible} reduction. For instance, this condition holds if the order of $G$ is prime to the residue characteristic of $k$. In this case, bad reduction is caused by degeneration of the branch locus of the cover modulo $v$. In particular, the results we present here are interesting only for covers with at least 4 branch points. By standard lifting results, the special fiber of a tame admissible cover, together with certain degeneration data, ‘knows everything’ about the generic fiber, see e.g. [20]. Therefore, one can hope to compute the obstruction for $k$ to be a field of definition purely in terms of the special fiber of $f$.

Consider the following short exact sequence of cohomology groups:

\begin{equation}
0 \to H^2(k_0, C) \to H^2(k, C) \to H^1(k_0, C(-1)) \to 0,
\end{equation}

where $k_0$ denotes the residue field of $k$, see [23], Section II.A.2. Suppose $f : Y \to \mathbb{P}^1$ is a $G$-cover with field of moduli $k$ and $\omega \in H^2(k, C)$ is the associated obstruction for $k$ to be a field of definition. The class $r_v(\omega) \in H^1(k_0, C(-1))$ is called the \textit{residue} of $\omega$ at $v$. If $r_v(\omega) = 0$, then $\omega$, regarded as a class in $H^2(k_0, C)$, represents the obstruction for the special fiber $\tilde{f}$ of $f$ to be defined over $k_0$ (as an admissible $G$-cover). This happens, for instance, if $f$ has good reduction. On the other hand, if $H^2(k_0, C) = 0$ (for instance, if $k_0$ is a finite field) then the special fiber $\tilde{f}$ has a $k_0$-model $\tilde{f}_{k_0}$, and then $r_v(\omega)$ represents the obstruction for $\tilde{f}_{k_0}$ to lift to a $k$-model of $f$. These considerations immediately give a new proof of the main result of [8].

If the residue field $k_0$ has characteristic 0, one can describe the degeneration behavior of $f$ purely in terms of the Hurwitz data attached to $f$. A substantial part of the present paper is devoted to showing that, in many cases, this description is sufficient for computing $r_v(\omega)$.

To compute $\omega$ over a $p$-adic field, we use a specialization technique. Given a $p$-adic field $k$ and a $G$-cover $f_t$ with field of moduli $K_t := k((t))$, such that $f = f_a$ is a ‘specialization’ of $f_t$, for some value $a \in k$ with $v(a) > 0$. Since $k$ has characteristic 0, one can apply the more geometric methods referred to in the preceding paragraph to compute the obstruction $\omega_t$ for $K_t$ to be a field of definition of $f_t$. In this situation, we prove that the obstruction $\omega$ corresponding to $f$ can be computed by ‘specializing’ $\omega_t$ to $t = a$. The proof uses the theory of Hurwitz spaces in mixed characteristic, see e.g. [23].

Let $A_5$ be the unique nonsplit central extension of $A_5$ by $\pm 1$. We construct, for each prime number $p > 5$, an $A_5$ cover with field of moduli $\mathbb{Q}_p$, which is not defined over $\mathbb{Q}_p$. To my knowledge, this kind of example is essentially new ([14], Example 2.6, for instance, uses only the real numbers).

We should compare this paper to several approaches in the literature to understand the absolute Galois group of $\mathbb{Q}$ acting on the fundamental groups of moduli spaces. [16] provides a Hurwitz space context for the Drinfeld-Ihara-Grothendieck relations (that apply to elements of the absolute Galois group; call these DIG relations). The approach was through \textit{tangential base points}. We follow that tradition, though we are not taking the exact same tangential base points. For example, we often use complex conjugate pairs of branch points, while they always used sets of real branch points. Ihara’s use of the DIG relations has been primarily to describe the Lie algebra of the absolute Galois group acting through various pronilpotent...
braid groups, especially on the 3 punctured $\lambda$-line. Fried in \cite{Fri95a, App. C} proposed Modular Towers, a profinite construction, as suitably like finite representations of the fundamental group to see the DIG relations at a finite level. There is an analogy with Ihara in that modular curves are close to considerations about the $\lambda$-line, though there is no phenomenon related to them that suggests seeing the DIG relations. Still, modular curves are just one case of Modular Towers.

Fried specifically proposed the Modular Tower attached to $A_5$ and four 3-cycles as a candidate where one would see a system of Serre obstruction situations from covers of $A_5$. \cite[Prop. 8.12]{1} uses a specific case of Modular Tower levels having a tower of these Serre obstructions. Over the reals, there are points, Harbater-Mumford (HM) reps. at every level of this Modular Tower where (in our notation) the Serre obstruction vanishes and has value 0, and other points (near HM reps.) where it vanishes, but does not have value 0. At levels 1 and beyond, these are the only real points and the real components through them end at cusps of reduced Hurwitz spaces (covering the $j$-line). These computations are slightly differently than ours, and they give their own approach to Serre’s obstruction through direct use of rational points on covers \cite[Prop. 6.8]{1}. Still, restricting our Prop. 2.13 to the reals is essentially equivalent to an example of theirs.

This suggests there are analogs of $p$-adic points on these Modular Towers levels that have a similar tangential base point (cusp geometry) analysis to the near HM and HM reps. Further, this geometry should reveal the DIG relations on actual covers, instead of as a Lie algebra relation. While there may be technical difficulties with carrying this out at all levels, the computations of \cite[Prop. 9.8]{1} at level 1 of this $A_5$ Modular Tower should be feasible.

The paper is organized in two sections. In Section 1, we recall general results about the field of moduli and the field of definition of Galois covers, from our point of view. Section 2 is concerned with the case of a henselian ground field and contains the main results.

The author would like to thank the organizers of the special semester Galois Groups and Fundamental Groups for inviting him and for providing financial support. He would also like to thank the MSRI for its hospitality, and the referee for useful comments. The final version of this paper was written while the author received a grant from the Deutsche Forschungsgemeinschaft.

1. General results

In Section 1.1 we recall the necessary definitions. In Section 1.2 we introduce some notation concerning the Hurwitz description of Galois covers and compute the f.o.d.-obstruction in an example (the group is $\text{SL}_2(\ell)$ and the ramification type is $(4, \ell, \ell)$). In Section 1.3 we study in some detail Galois covers over the real numbers.

1.1. The field of moduli condition and the f.o.d.-obstruction. Let $G$ be a finite group and $k$ a field. We denote by $k^s$ a fixed separable closure of $k$ and by $\Gamma_k := \text{Gal}(k^s/k)$ the absolute Galois group of $k$. Suppose we are given a $G$-cover

$$f : Y \rightarrow \mathbb{P}^1_{k^s}$$

of the projective line over $k^s$: by this we mean that $f$ is a finite Galois cover of smooth projective curves over $k^s$, with Galois group $G$. We will always assume that $f$ is tamely ramified, and we let $S = \{z_1, \ldots, z_r\} \subset \mathbb{P}^1_{k^s}$ be the set of branch points
of \( f \). Thus, the \( G \)-cover \( f \) corresponds to a surjective morphism \( \Phi : \pi_1^s(U) \to G \), where \( \pi_1^s(U) \) denotes the tame fundamental group of \( U := \mathbb{P}^1_k - S \).

The question we are concerned with is the following: is \( k \) a field of definition of \( f \), i.e. does there exist a \( G \)-cover \( f_k : Y_k \to \mathbb{P}^1_k \) defined over \( k \) such that \( f = f_k \otimes_k k^s \)?

A necessary condition for this to hold is that the branch locus \( S \) is \( \Gamma_k \)-invariant. We will assume this from now on, and let \( S_k \subset \mathbb{P}^1_k \) be the closed subset of \( \mathbb{P}^1_k \) corresponding to \( S \) and \( U_k := \mathbb{P}^1_k - S_k \) its complement. By definition, \( k \) is a field of definition of \( f \) if and only if \( \Phi \) can be extended to a morphism \( \Phi_k : \pi_1^s(U_k) \to G \):

\[
\begin{array}{cccccc}
1 & \to & \pi_1^s(U) & \to & \pi_1^s(U_k) & \to & \Gamma_k & \to & 1 \\
& & \downarrow \Phi & & \downarrow \Phi_k & & \downarrow G & & \\
& & 1 & & 1 & & 1 & & 
\end{array}
\]

Let us fix a section \( s : \Gamma_k \to \pi_1^s(U_k) \) of the natural projection \( \pi_1^s(U_k) \to \Gamma_k \), and let \( \Gamma_k \) act on \( \pi_1^s(U) \) as follows: \( s^\gamma := s(\gamma) s(\sigma)^{-1} \).

**Definition 1.1.** We say that \( k \) is a field of moduli of the \( G \)-cover \( f : Y \to \mathbb{P}^1_k \), if for all \( \sigma \in \Gamma_k \) there exists an element \( h_\sigma \in G \) such that

\[
\Phi(s^\gamma) = h_\sigma \Phi(\gamma) h_\sigma^{-1}, \quad \text{for all } \gamma \in \pi_1^s(U).
\]

Note that the field of moduli condition does not depend on the choice of the section \( s \). Obviously, in order for \( k \) to be a field of definition of \( f \), it is necessary that \( k \) be a field of moduli. However, this is not a sufficient condition, in general. Let \( C \) be the center of \( G \) and \( G := G/C \) the quotient. Assume that \( k \) is a field of moduli of \( f \), let \( h_\sigma \), for each \( \sigma \in \Gamma_k \), be as in Definition 1.1 and set \( \varphi(\gamma) := h_\sigma(h_\tau^{-1}) \) (the class of \( h_\sigma \) in \( G \)). Clearly, \( \varphi : \Gamma_k \to G \) is well defined and a group homomorphism. It follows that for all \( \sigma, \tau \in \Gamma_k \), the element

\[
c_{\sigma, \tau} := h_\sigma h_\tau h_\tau^{-1}
\]

lies in the center \( C \), and that \( (\sigma, \tau) \mapsto c_{\sigma, \tau} \) is a 2-cocycle (where \( C \) is regarded as constant \( \Gamma_k \)-module). Let \( \omega \in H^2(k, C) \) be the cohomology class of this cocycle. By definition, \( \omega \) is the obstruction for the existence of a (weak) solution \( \varphi \) of the central embedding problem \( (q, \varphi) \), see e.g. [17] IV.6.1 (here \( q : G \to \overline{G} \) is the natural map).

\[
\begin{array}{cccccc}
\Gamma_k & \xrightarrow{\varphi} & G & \xrightarrow{\varphi} & \overline{G} & \to & 1. \\
& & \uparrow c & & \uparrow \overline{\varphi} & & \\
1 & \to & C & \to & G & \to & 1.
\end{array}
\]

A formal verification shows (see e.g. [5]):

**Proposition 1.2.** The class \( \omega \) is independent of the choice of the section \( s : \Gamma_k \to \pi_1^s(U_k) \). Moreover, \( \omega = 0 \) if and only if \( k \) is a field of definition of \( f \).

We call \( \omega \) the f.o.d.-obstruction for the \( G \)-cover \( f \), relative to \( k \) (‘f.o.d.’ stands for ‘field of definition’). The rest of the paper is concerned with studying and, if possible, computing \( \omega \), in various situations. First we make some general remarks.

**Remark 1.3.** (i) The formation of \( \omega \) is functorial in \( k \). More precisely, let \( K/k \) be a field extension, \( f_{K^s} : Y \otimes_k K^s \to \mathbb{P}^1_{K^s} \), the base change of \( f \) to \( K^s \) (we embed \( k^s \) in \( K^s \)) and \( \omega|_K \) the image of \( \omega \) under the restriction
homomorphism $H^2(k, C) \to H^2(K, C)$; then $\omega|_K$ is the f.o.d.-obstruction for $f|_K$.

(ii) A $G$-cover $f: Y \to P^1_k$ with $r$ branch points corresponds to a $k^*$-point $[f]: \text{Spec} k^* \to \mathcal{H}_r(G)$ on a certain moduli scheme $\mathcal{H}_r(G)$, called the Hurwitz space. The field $k$ is a field of moduli for $f$ if and only if $[f]$ is $k$-rational, i.e., factors through a morphism $\text{Spec} k \to \mathcal{H}_r(G)$. See [10] and 25.

(iii) There exists a cohomology class $\tilde{\omega} \in H^2_{\text{et}}(\mathcal{H}_r(G), C)$ which specializes to $\omega$, i.e., $\omega = [f]^*\tilde{\omega}$. The class $\tilde{\omega}$ represents the obstruction for the existence of a global versal $G$-cover over $\mathcal{H}_r(G)$, see 25 or 7.

For a study of f.o.d.-obstructions for more general (e.g., non-Galois) covers, see 3 and 7.

1.2. The Hurwitz description. We let $k, S = \{z_1, \ldots, z_r\}$, $U := P^1_k - S$ and $U_k$ be as before. We assume that $k$ has characteristic 0; we may therefore write $\pi_1(U)$ instead of $\pi_1^t(U)$. Let us choose a $k$-rational base point $z_0$ on $U$. The point $z_0$ will either be a $k$-rational point $\text{Spec} k \to U_k$ or a “tangential base point” $\text{Spec} k((z)) \to U_k$, see 15. In both cases, we obtain a section $s: \Gamma_k \to \pi_1(U_k)$ (unique up to an inner automorphism of $\pi_1(U)$) and hence an action of $\Gamma_k$ on $\pi_1(U)$. We will write $\pi_1(U, z_0)$ to denote the profinite group $\pi_1(U)$ together with this $\Gamma_k$-action.

We denote by $\hat{\mathbb{Z}}(1) := \varprojlim_n \mathbb{Z}_p$ the Tate module of $\mathbb{G}_m$. Let

$$\Pi := \langle \gamma_1, \ldots, \gamma_r | \prod_i \gamma_i = 1 \rangle$$

be the free profinite group with generators $\gamma_1, \ldots, \gamma_r$, subject to the usual product one relation.

Definition 1.4. An isomorphism $\rho: \Pi \cong \pi_1(U, z_0)$ of profinite groups is a presentation of $\pi_1(U, z_0)$ if it has these properties:

(i) $\rho(\gamma_i)$ generates an inertia subgroup $I_{\gamma_i} \subset \pi_1(U, z_0)$ corresponding to a point $z_i \in S$, and

(ii) under the natural identification $I_{\gamma_i} \cong \hat{\mathbb{Z}}(1)$, all the $\rho(\gamma_i)$ correspond to the same element in $\hat{\mathbb{Z}}(1)$.

It is a well known fact that such a presentation $\rho$ always exists; let us choose one. To simplify the notation, we will usually identify $\gamma_i$ with $\rho(\gamma_i)$. The branch cycle argument states that for all $\sigma \in \Gamma_k$ there exist elements $\beta_1, \ldots, \beta_r \in \pi_1(U, z_0)$ such that

$$\sigma_{\gamma_i} = \beta_i^{\chi(\sigma)} \beta_i^{-1}.$$  

(1.3)

Here $\chi: \Gamma_k \to \hat{\mathbb{Z}}^\times$ is the cyclotomic character and $\sigma(i)$ is defined by $\sigma(z_i) = z_{\sigma(i)}$.

Let $f: Y \to P^1_k$ be a $G$-cover with branch locus $S$, corresponding to a homomorphism $\Phi: \pi_1(U, z_0) \to G$. Setting $g_i := \Phi(\gamma_i)$, we obtain an $r$-tuple $g = (g_1, \ldots, g_r)$ of generators of $G$ such that $\prod_i g_i = 1$. We write $N_i(G)$ for the set of all such $g$ and $N_i^*(G) := N_i(G)/\text{Inn}(G)$ for the quotient of this set under the action of inner automorphisms of $G$, compare 10. We obtain a bijection between $N_i^*(G)$ (the set of Nielsen classes of length $r$) and the set of isomorphism classes of $G$-covers with branch locus $S$. We say that the $G$-cover $f$ has Hurwitz description $[g] \in N_i^*(G)$ (with respect to the presentation $\rho$).
The \( \Gamma_k \)-action on \( \pi_1(U, z_0) \) induces a \( \Gamma_k \)-action on \( \text{Hom}(\pi_1(U, z_0), G) \), and therefore on \( \text{N}_1(G) \). Our convention is to let \( \Gamma_k \) act on \( \text{N}_1(G) \) from the right, i.e. we define
\[
\ell^\sigma := (\tilde{g}_1, \ldots, \tilde{g}_r), \quad \text{with} \quad \tilde{g}_i := \Phi(\sigma \gamma_i \sigma^{-1}).
\]
We remark that this is not a standard convention in the literature. The branch cycle argument (1.3) becomes
\[
\ell^\sigma = b_i \cdot \gamma^\ell(\sigma) b_i^{-1},
\]
with \( b_i := \Phi(\beta_i) \). The \( \Gamma_k \)-action on \( \text{N}_1(G) \) induces a \( \Gamma_k \)-action on \( \text{N}_1(G) \). The following Proposition merely rephrases the definitions of Section 1.1.

**Proposition 1.5.** Let \( f: Y \to \mathbb{P}^1_k \), be a \( G \)-cover with branch locus \( S \) and Hurwitz description \( g \) (with respect to a presentation \( \rho : \Pi \to \pi_1(U, z_0) \)). Then

(i) \( k \) is a field of moduli of \( f \) if and only if \( [g]^\sigma = [g] \) for all \( \sigma \in \Gamma_k \).

(ii) If (i) holds then there is a unique homomorphism \( \varphi : \Gamma_k \to G \) such that
\[
[g]^\sigma = h_\sigma \cdot g \cdot h_\sigma^{-1},
\]
where \( h_\sigma \in G \) is a lift of \( \varphi(\sigma) \in \tilde{G} \), for all \( \sigma \in \Gamma_k \).

(iii) \( k \) is a field of definition of \( f \) if and only if it is a field of moduli and the homomorphism \( \varphi \) in (ii) lifts to a homomorphism \( \tilde{\varphi} : \Gamma_k \to \tilde{G} \).

The formula \( g^\sigma = h_\sigma \cdot g \cdot h_\sigma^{-1} \) may seem counterintuitive at first. But we have to keep in mind that the homomorphism \( \varphi \) depends on \( g \), and not only on its Nielsen class \([g]\). In more geometric terms, \( g \) corresponds to a *pointed* \( G \)-cover \( f : (Y, y_0) \to (\mathbb{P}^1, z_0) \). If \( k \) is a field of definition of \( f \) and \( \varphi : \Gamma_k \to \tilde{G} \) is as in Proposition 1.5 (iii), then there exists a unique model \( \tilde{f}_k : Y \to \mathbb{P}^1_k \) of \( f \) such that the fiber \( f^{-1}(z_0) \), as a \( G \)-torsor over \( k \), corresponds to \( \varphi \in H^1(k, G) \).

**Example 1.6.** Let \( r := 3, k := \mathbb{Q} \), and let \( \ell \) be an odd prime. We set \( \ell^* := (-1)^{(\ell-1)/2} \ell \), \( S := \{0, \sqrt{\ell}, -\sqrt{\ell}\} \subset \mathbb{P}^1_k \) and \( U := \mathbb{P}^1 \setminus S \). We identify \( \mathbb{Q}(z) \) with the function field of \( U_Q := \mathbb{P}^1 - S \) and let \( z_0 : \text{Spec} \mathbb{Q}(z) \to U \) be the tangential base point at 0, with parameter \( z \).

It is clear from 13 that there exists a presentation \( \rho : \Pi \to \pi_1(U, z_0) \) such that \( \gamma_1 \in \pi_1(U, z_0) \) corresponds to the closed path \( t \mapsto e^{2\pi i t} \) (for \( 0 \leq t \leq 1 \)). Then
\[
\gamma_1 = \gamma_1^\chi(\sigma), \quad \gamma_2 \sim \gamma_2(\sigma), \quad \gamma_3 \sim \gamma_3(\sigma),
\]
for all \( \sigma \in \Gamma_Q \). Here \( \sigma \) fixes the indices 2 and 3 if \( \sigma \sqrt{\ell} = \sqrt{\ell} \) and permutes them if \( \sigma \sqrt{\ell} = -\sqrt{\ell} \).

From now on we assume that \( \ell \neq \pm 1 \) (mod 8), and we set \( G := \text{SL}_2(\ell) \). Let \( \ell A, \ell B \) be the two conjugacy classes of \( G \) containing elements of order \( \ell \) and 4\( A \) the unique conjugacy class containing elements of order 4. The classes \( \ell A \) and \( \ell B \) are conjugate over the quadratic extension \( \mathbb{Q}(\sqrt{\ell})/\mathbb{Q} \). By 24 1.3.3.6, the triple \((4A, \ell A, \ell B)\) is *rigid*, i.e. there exists exactly one class \([g_1, g_2, g_3] \in \text{N}_i^r(G) \) such that \( g_1 \in 4A, g_2 \in \ell A \) and \( g_3 \in \ell B \). Let \( f : Y \to \mathbb{P}^1_Q \) be the \( G \)-cover with Hurwitz description \([g] = [g_1, g_2, g_3] \) (with respect to \( \rho \)). By Rigidity and 1.4, \([g]^\sigma = [g] \) for all \( \sigma \in \Gamma_Q \). Therefore, \( \mathbb{Q} \) is a field of moduli of \( f \).

The center \( C \) of \( G \) consists of the two diagonal matrices \( \pm I \); we identify it with \( \{ \pm 1 \} \). Let \( \omega \in H^2(\mathbb{Q}, \pm 1) \) be the f.o.d.-obstruction of \( f \). We may identify \( H^2(\mathbb{Q}, \pm 1) \) with \( \text{Br}_2(\mathbb{Q}) \), the 2-torsion of the Brauer group of \( \mathbb{Q} \).
Proposition 1.7. We have \( \omega = (-1, -1) \), i.e. \( \omega \) is represented (as an element of \( \text{Br}_2(\mathbb{Q}) \)) by the quaternion algebra \( \mathbb{Q}[i, j \mid i^2 = -1, j^2 = -1, ij = -ji] \). In particular, \( \mathbb{Q} \) is not a field of definition of \( f \).

It follows that the regular \( \text{PSL}_2(\ell) \)-extensions of \( \mathbb{Q}(z) \) with ramification of type \((2, \ell, \ell)\) (see e.g. [22], Section 8.3.3) do not lift to an \( \text{SL}_2(\ell) \)-extension. We will see in Section 1.3 that in fact no \( \text{PSL}_2(\ell) \)-extension with three branch points lifts to an \( \text{SL}_2(\ell) \)-extension.

Proof. The class \( \omega \in H^2(\mathbb{Q}, \pm 1) \) is the obstruction for lifting the homomorphism \( \varphi : \Gamma_\mathbb{Q} \to \bar{G} = \text{PSL}_2(\ell) \) to a homomorphism \( \varphi : \Gamma_\mathbb{Q} \to G = \text{SL}_2(\ell) \). By the definition of \( \varphi \) and by Equation (1.4), we have (by abuse of notation)

\[
\varphi(\sigma)g_1\varphi(\sigma)^{-1} = g_1^\chi(\sigma), \quad \text{for all } \sigma \in \Gamma_\mathbb{Q}.
\]

In particular, the image of \( \varphi \) is contained in \( \bar{N} := N/C \), where \( N \) is the normalizer in \( G \) of the cyclic subgroup generated by \( g_1 \). It is shown e.g. in [14], Abschnitt II.8, that \( \bar{N} \) is a dihedral group of order \( 4n \), where \( n := (\ell - 1)/4 \) if \( \ell \equiv 1 \) (mod 4) and \( n := (\ell + 1)/4 \) otherwise. Let \( \bar{H} < \bar{N} \) be the unique cyclic normal subgroup of order \( n \). By our assumption on \( \ell \), \( n \) is odd. Therefore, there exists a unique cyclic normal subgroup \( H \subset N \) of order \( n \) mapping onto \( \bar{H} \). Since \( N/H \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \) and every involution in \( G \) lifts to an element of order 4 in \( G \), \( N/H \) is isomorphic to the quaternion group \( Q_8 \). It is easy to check that in the following diagram

\[
\begin{array}{ccc}
\{1\} & \to & N \\
\downarrow & & \downarrow \\
\{1\} & \to & N/H \cong Q_8 \\
\end{array}
\]

the group extension in the upper row is the pullback of the lower row with respect to the projection \( \bar{N} \to N/C \). Let \( \tilde{\psi} : \Gamma_\mathbb{Q} \to \bar{N}/CH \) be the composition of \( \tilde{\psi} \) with the projection \( \bar{N} \to N/C \). It follows from [3], Exercise IV.3.1, that \( \omega = \tilde{\psi}\eta \), where \( \eta \in H^2(\bar{N}/CH, C) \) is the cohomology class corresponding to the lower row of (1.4). In other words, \( \omega \) is the obstruction for lifting \( \tilde{\psi} \) to a homomorphism \( \psi : \Gamma_\mathbb{Q} \to N/H \cong Q_8 \). Therefore, by a result of Witt (see [1], (7.7) (ii)),

\[
\omega = (-1, a) + (-1, b) + (a, b) \in \text{Br}_2(\mathbb{Q}),
\]

where \( E = \mathbb{Q}[x, y \mid x^2 = a, y^2 = b] \), \( a, b \in \mathbb{Q} \), is the étale \( \mathbb{Q} \)-algebra corresponding to \( \tilde{\psi} \). Let \( \bar{H}' < \bar{N} \) be the maximal cyclic normal subgroup, of order \( 2n \). By (1.3), the composition of \( \bar{\psi} \) with \( N/C \bar{H} \to \bar{N}/\bar{H}' \cong \mathbb{Z}/2 \cong (\mathbb{Z}/4)^\times \) equals the cyclotomic character modulo 4. Hence we may choose \( a = -1 \), and \( \omega = (-1, -1) \) follows.

1.3. G-covers over the real numbers. In this section we discuss the case \( k = \mathbb{R} \). Since the action of \( \Gamma_\mathbb{R} \) on \( \pi_1(U, z_0) \) is known, we get some easy results and examples which illustrate several key concepts of this paper. For more general results on related questions, see [1], in particular §6.

Let \( f : Y \to \mathbb{P}^1_\mathbb{C} \) be a G-cover defined over the complex numbers, and assume that the branch locus \( S \) of \( f \) is defined over \( \mathbb{R} \). For simplicity, we will also assume that \( \infty \notin S \). We can write \( S = \{z_1, \ldots, z_s\} \), with \( z_i = z_{2s-i+1} \) for \( i = 1, \ldots, s \), and \( z_{2s+j} \in \mathbb{R} \) for \( j = 1, \ldots, r - 2s \). For \( z, w \in \mathbb{C} \), we write \( z > w \) if either \( \text{Im} \, z > \text{Im} \, w \) or \( \text{Im} \, z = \text{Im} \, w \) and \( \text{Re} \, z > \text{Re} \, w \). In this notation, we may assume that \( z_1 > \ldots > z_s \) and \( z_{2s+1} < \ldots < z_r \). Let \( U := \mathbb{P}^1_\mathbb{C} - S \) and choose a real base point...
\[ z_0 < 2z_{s+1}. \] By [17], Section I.1.1, there exists a presentation \( \rho : \Pi \xrightarrow{\sim} \pi_1(U, z_0) \) with the following properties: (i) \( \gamma_i \) is represented by a simple loop based at \( z_0 \) winding counterclockwise around \( z_i \), (ii) these loops are pairwise disjoint and (iii) the action of complex conjugation \( \kappa \in \Gamma_R \) is given by
\[ \kappa \gamma_i = \gamma_{2s-i+1}^{-1}, \quad \text{for } i = 1, \ldots, 2s, \]
and
\[ \kappa \gamma_{2s+j} = \beta_j \gamma_{2s+j}^{-1} \beta_j^{-1}, \quad \text{for } j = 1, \ldots, r - 2s, \]
where \( \beta_j := \gamma_{2s+1} \cdots \gamma_{2s+j-1} \) (see [17, Fig. 1.2]). We say that \( \rho \) is a normalized presentation of \( \pi_1(U, z_0) \).

**Proposition 1.8.** Let \( g = (g_1, \ldots, g_r) \) be the Hurwitz description for the G-cover \( f : Y \rightarrow \mathbb{P}_\mathbb{C}^1 \), with respect to a normalized presentation \( \rho \), as above. Then
(i) \( \mathbb{R} \) is a field of moduli for \( f \) if and only if there exists \( b \in G \) such that
\[ b g_i b^{-1} = g_{2s-i+1}^{-1} \]
for \( i = 1, \ldots, 2s \), and
\[ b g_{2s+j} b^{-1} = h_j g_{2s+j}^{-1} h_j^{-1} \]
for \( j = 1, \ldots, r - 2s \), with \( h_j := g_{2s+1} \cdots g_{2s+j-1} \).
(ii) \( \mathbb{R} \) is a field of definition of \( f \) if and only there exists \( b \) as in (i) such that \( b^2 = 1 \).

**Proof.** Follows from Proposition 1.5 and equations (1.7) and (1.8).

For certain groups \( G \), the condition in Proposition 1.8 (ii) is rather restrictive, as shown by the next theorem.

**Theorem 1.9.** Let \( G \) be a finite group. Assume that all elements of \( G \) of order 2 lie in the center of \( G \). Let \( f : Y \rightarrow \mathbb{P}_\mathbb{C}^1 \) be a G-cover, with branch locus \( S \) defined over \( \mathbb{R} \), and let \( g \) be a Hurwitz description of \( f \) with respect to a normalized presentation \( \rho : \Pi \rightarrow \pi_1(U, z_0) \) (see above). Then \( \mathbb{R} \) is a field of definition of \( f \) if and only if
\[ g = (g_1, \ldots, g_s, g_s^{-1}, \ldots, g_1^{-1}, c_1, \ldots, c_{r-2s}), \]
with \( c_j^2 = 1 \) and \( \prod c_j = 1 \).

The quaternion group \( Q_8 \) and \( \text{SL}_2(\mathbb{F}_\ell) \), for \( \ell \neq 2 \) verify the conditions of the theorem. Elements of order 2 in \( A_n \) are products of 2s disjoint 2-cycles. They lift to have order 4 in the 2-fold coverings \( A_n \) of \( A_n \) exactly when \( s \) is odd [1, Prop. 5.8]. So this also holds for \( A_n \) if \( 4 \leq n \leq 7 \).

**Proof.** If \( g \) is as in (1.9), then \( ^*g = g \), so \( f \) can be defined over \( \mathbb{R} \). Conversely, if \( f \) can be defined over \( \mathbb{R} \) then there exists \( b \in G \) of order 2 such that \( b g_i b^{-1} = g_{2s-i+1}^{-1} \) and \( b g_{2s+j} b^{-1} = h_j g_{2s+j}^{-1} h_j^{-1} \), by Proposition 1.8 (ii). By the hypothesis of the theorem, \( b \) is a central element of \( G \). Therefore, \( g_i = g_{2s-i+1}^{-1} \), for \( i = 1, \ldots, s \). Moreover, \( g_{2s+1} = g_{2s+1}^{-1} \), so \( c_1 := g_{2s+1} \) has order 2 and hence lies in the center of \( G \). Continuing by induction, we find that \( c_j := g_{2s+j} \) is central of order 2, for \( j = 1, \ldots, r-2s \). This proves the theorem. \( \square \)
the free profinite group $\Pi$, with respect to an
Let $\rho$ compatible automorphisms cover has good reduction. In Section 2.2 we define that says that the residue of the f.o.d.-obstruction of a $G$ that the presentation $\rho$ of complex conjugation on $\pi$ corresponding f.o.d.-obstruction $t$ infinitesimal neighborhood of $t$. On the other hand, any element $\tilde{A}_5$ is a lift of $(1 2)(4 5) \in \tilde{A}_5$. Therefore, $R$ is a field of moduli of $f_{t_0}$. On the other hand, any element $b \in \tilde{A}_5$ such that (1.11) holds has order 4 (there are exactly two of them), so $R$ is not a field of definition of $f_{t_0}$. In other words: for $t \in (-1, 1) - \{0\}$, let $\omega_t \in H^2(R, \{\pm 1\})$ be the f.o.d.-obstruction for $f_t$. Then $\omega = 0$ for $t > 0$ and $\omega = (-1, -1) \neq 0$ for $t < 0$.

Example 1.10 illustrates a key point of this paper: the obstruction $\omega_t$, as a function of $t \in (-1, 1) \subset R$, has a 'jump' at $t = 0$. In the next section we will give the following algebraic interpretation for this phenomenon. Let $\bar{f} : \tilde{Y} \to \mathbb{P}^1_{k^s}$ be the $G$-cover over $k^s$, with $k = \mathbb{R}((t))$, corresponding to the family $f_t$ in an infinitesimal neighborhood of $t = 0$. Then $k$ is a field of moduli for $\bar{f}$, and the corresponding f.o.d.-obstruction $\omega \in H^2(k, \{\pm 1\})$ has a pole at $t = 0$, i.e. the residue $r_v(\omega) \in H^1(R, \{\pm 1\})$ of $\omega$ at the valuation $v$ of $k$ does not vanish.

2. Galois covers over henselian fields

In Section 2.1 we recall the definition of the residue map and we state a theorem that says that the residue of the f.o.d.-obstruction of a $G$-cover vanishes if the cover has good reduction. In Section 2.2 we define compatible automorphisms of the free profinite group $\Pi$, with respect to an ordered tree. This is preparatory.
work for Section 2.3, where we state some results on the Galois action on \( \pi_1(U) \). Here we restrict our attention to henselian fields with residue characteristic 0. In Section 2.4 we explicitly compute the f.o.d.-obstruction for some \( \tilde{A}_5 \)-covers over finite extensions of \( \mathbb{Q}(t) \). In Section 2.5 we explain how to “specialize” our previous results to \( p \)-adic fields.

2.1. The residue map. Throughout this section, we assume that \( k \) has characteristic 0 and is henselian with respect to a discrete valuation \( v \). We denote the residue field of \( v \) by \( k_0 \) and assume that \( k_0 \) is perfect. We write \( \Gamma_k \) for the maximal tame quotient of \( \Gamma_k \) and \( I' \triangleleft \Gamma_k \) for its tame inertia subgroup. We choose a section for the natural map \( \Gamma_k' \to \Gamma_{k_0} \) and identify \( \Gamma_k' \) with \( I' \times \Gamma_{k_0} \). We remark that, as a \( \Gamma_{k_0} \)-module, \( I' \) is canonically isomorphic to \( \hat{\mathbb{Z}}'(1) := \varprojlim_n \mu_n(k_0^\times) \), where \( n \) runs over the integers prime to the characteristic of \( k_0 \).

We are given a \( G \)-cover \( f : Y \to \mathbb{P}^1_{k_0} \) with branch locus \( S \) and field of moduli \( k \). Let \( \omega \in H^2(k, C) \) be the f.o.d.-obstruction for \( f \). We choose a \( k \)-rational branch point \( z_0 \) on \( U := \mathbb{P}^1_{k_0} - S \). As explained in Sections 1.1 and 1.2, we obtain a certain homomorphism \( \bar{\varphi} : \Gamma_k \to \hat{G} \), and \( \omega \) is the cohomological obstruction for lifting \( \bar{\varphi} \) to a homomorphism \( \varphi : \Gamma_k \to G \). We assume that the following tameness condition holds.

**Condition 2.1.**

(i) The order of \( C \) is prime to the characteristic of \( k_0 \).

(ii) The homomorphism \( \bar{\varphi} : \Gamma_k \to \hat{G} \) is at most tamely ramified at \( v \).

We may therefore regard \( \bar{\varphi} \) as a homomorphism \( \Gamma_k' \to \hat{G} \) and \( \omega \) as the obstruction for lifting \( \bar{\varphi} \) to a homomorphism \( \varphi : \Gamma_k' \to G \) (Condition 2.1 (i) implies that there is a natural isomorphism \( H^2(k, C) \cong H^2(\Gamma_k', C) \)).

According to 2.3, Section II.A.2, and Condition 2.1 (i), we have a short exact sequence

\[
\begin{align*}
0 & \rightarrow H^2(k_0, C) \rightarrow H^2(k, C) \xrightarrow{r_v} H^1(k_0, C(-1)) \rightarrow 0
\end{align*}
\]

(with \( C(-1) := \text{Hom}(I', C) \)). This sequence can be deduced from the Hochschild–Serre spectral sequence for the group extension determined by the inclusion \( I \to \Gamma_k \).

The map \( r_v \) is called the **residue map**, and \( r_v(\omega) \) is called the residue of \( \omega \) at \( v \). If \( r_v(\omega) = 0 \) we say that \( \omega \) is regular at \( v \); otherwise, \( \omega \) is said to have a pole at \( v \). If \( \omega \) is regular at \( v \), we can identify \( \omega \) with a class in \( H^2(k_0, C) \); we denote this class by \( \omega_0 \) and refer to it as the value of \( \omega \) at \( v \).

Proposition 2.2 below gives a description of \( r_v(\omega) \) in terms of \( \bar{\varphi} \). To state it, we need some more notation. Let us choose a topological generator \( q_0 \) of the inertia group \( I' \). To simplify the notation, we will identify \( C(-1) = \text{Hom}(I', C) \) with \( C \) (as abelian groups), via \( \eta \mapsto \eta(q_0) \). We let \( \Gamma_{k_0} \) act on \( G \) as follows:

\[
\sigma g := \bar{\varphi}(\sigma) g \bar{\varphi}(\sigma)^{-1}, \quad \sigma \in \Gamma_{k_0} \subset \Gamma_k.
\]

Finally, let us choose an element \( a_0 \in G \) lifting \( \bar{\varphi}(q_0) \in \hat{G} \). For all \( \sigma \in \Gamma_{k_0} \), we find that

\[
c_\sigma := a_0 (\sigma a_0)^{-\chi(\sigma)} = a_0 ((\sigma a_0)^{\chi(\sigma)} - 1)
\]

is an element of \( C \).

**Proposition 2.2.**

(i) The map \( \sigma \mapsto c_\sigma \) is a 1-cocycle with values in \( C \cong C(-1) \). Its cohomology class equals the residue \( r_v(\omega) \).
If } r_v(\omega) = 0 \text{, then } \omega_0 \in H^2(k_0, C) \text{ is the obstruction for lifting the homomorphism } \phi_0 := \phi|_{k_0} \text{ to } G.

**Proof.** For each } \sigma \in \Gamma_{k_0}, \text{ let us choose a lift } b_\sigma \in G \text{ lifting } \phi_0(\sigma). \text{ We may assume that } \sigma \mapsto b_\sigma \text{ is continuous. Every element } \tau \in \Gamma_k \text{ can be written in a unique way as } \tau = \sigma q^i, \text{ with } \sigma \in \Gamma_{k_0} \text{ and } i \in \hat{\mathbb{Z}}^\times. \text{ Therefore,}

\begin{equation}
\phi'(\sigma q^i) := b_\sigma a_i^0
\end{equation}

defines a continuous, set-theoretic lift } \phi' : \Gamma_k \to G \text{ of } \phi. \text{ By definition, the class } \omega \in H^2(k, C) \text{ is represented by the 2-cocycle}

\begin{equation}
\omega_{\tau_1, \tau_2} := \phi'(\tau_1) \phi'(\tau_2) \phi'(\tau_1 \tau_2)^{-1}, \quad \tau_1, \tau_2 \in \Gamma_k.
\end{equation}

Using (2.4), one checks that

\begin{equation}
\phi'(\sigma q^i) \equiv \phi'(\sigma) a_i \equiv \phi'(\sigma) \phi'(\sigma a_i^{q_i})^{-1} \text{ (mod ord } C)
\end{equation}

for which (2.6) becomes

\begin{equation}
\sigma \mapsto \omega_{q_0, \sigma} = \phi'(q_0) \phi'(\sigma) \phi'(q_0 \sigma)^{-1} = \phi'(q_0) \phi'(\sigma) \phi'(\sigma q_0^{\text{ord } \sigma})^{-1} = a_0 \sigma a_0 = c_\sigma.
\end{equation}

This proves the first part of the proposition. The second part is obvious from the definition of the sequence (2.7).

**Corollary 2.3.** Assume that the order of } \phi(q_0) \in \bar{G} \text{ is prime to the order of } C. \text{ Then } r_v(\omega) = 0.

**Proof.** It follows from our assumption that there exists a lift } a_0 \in G \text{ of } \phi(q_0) \text{ such that ord } a_0 = \text{ord } \phi(q_0) \text{ is prime to the order of } C. \text{ By the definition of } c_\sigma, \text{ we have}

\begin{equation}
\sigma a_0 = (c_\sigma^{-1} a_0) \chi(\sigma),
\end{equation}

for } \sigma \in \Gamma_{k_0}. \text{ Choose an integer } n \text{ such that } n \equiv 1 \text{ (mod ord } a_0) \text{ and } n \equiv 0 \text{ (mod ord } C). \text{ Then}

\begin{equation}
\sigma a_0 = (c_\sigma^{-n} a_0^n) \chi(\sigma) = a_0 \chi(\sigma).
\end{equation}

It follows that } c_\sigma = 1, \text{ for all } \sigma \in \Gamma_{k_0}.

Let us denote by } R \text{ the ring of integers of } k^\times. \text{ Let } S_R \subset \mathbb{P}_R^1 \text{ be the closure of } S \text{ in } \mathbb{P}_R^1. \text{ We say that the } G\text{-cover } f : Y \to \mathbb{P}_k^1 \text{ has good reduction if (i) } S_R \text{ is } \text{étale over Spec } R \text{ and (ii) } f \text{ extends to a finite morphism } f_R : Y_R \to \mathbb{P}_R^1 \text{ which is tamely ramified along } S_R \text{ and } \text{étale everywhere else. If } f \text{ has good reduction, then }

\begin{equation}
f : \bar{Y} := Y_R \otimes k_0^\times \to \mathbb{P}_k^1 \text{ is a } G\text{-cover, with branch locus } \bar{S} := S_R \otimes k_0^\times.
\end{equation}

**Theorem 2.4.** Assume that } f \text{ has good reduction at } v. \text{ Then } r_v(\omega) = 0. \text{ Moreover, the value } \omega_0 \in H^2(k_0, C) \text{ of } \omega \text{ at } v \text{ is the f.o.d.-obstruction for the } G\text{-cover } \bar{f} \text{ (for which } k_0 \text{ is a field of moduli).}

This theorem can be seen as a consequence of Remark 1.3 (iii). Let us give a proof which does not rely on the results of [25].
PROOF. Let \( \bar{U} := \mathbb{P}^1_{k_0} - \bar{S} \) and let \( \lambda : \pi_1(U, z_0) \to \pi_1(\bar{U}, \bar{z}_0) \) the specialization morphism, defined in \([13]\). Since \( f \) is assumed to have good reduction, \( \Phi : \pi_1(U, z_0) \to G \) factors through \( \lambda \) and the resulting morphism \( \Phi : \pi_1(\bar{U}, \bar{z}_0) \to G \) corresponds to the reduction \( \bar{f} \). Because \( \lambda \) is \( \Gamma_k \)-equivariant and the inertia group \( I < \Gamma_k \) acts trivially on \( \pi_1(U, \bar{z}) \), the homomorphism \( \bar{\varphi} : \Gamma_k \to G \) is unramified, i.e. corresponds to a homomorphism \( \varphi_{k_0} : \Gamma_{k_0} \to \bar{G} \). Thus, Proposition 2.2 implies that \( r_v(\omega) = 0 \) and that \( \omega_0 \) is the obstruction for lifting \( \bar{\varphi}_{k_0} \) to \( G \), proving the theorem.

Theorem 2.2 above implies Theorem 3.1 of \([8]\). Namely, assume that the characteristic of \( k_0 \) does not divide the order of \( G \) and that \( S_R \) is étale over Spec \( R \). Then \( f \) has good reduction, by \([2]\), and hence \( r_v(\omega) = 0 \). If we assume in addition that \( \text{cd} k_0 \leq 1 \) (e.g. if \( k_0 \) is finite) then \( \omega = 0 \), because the sequence \([2.1]\) is exact.

2.2. Ordered trees and compatible automorphisms. Let \( T = (V, E) \) be a tree, i.e. a finite and simply connected graph. We choose a distinguished vertex \( v_0 \in V(T) \) and call it the root of \( T \). The choice of \( v_0 \) induces a natural orientation on \( T \); we represent an edge \( e \in E(T) \) as an ordered pair \( e = (v_1, v_2) \) of vertices such that \( v_1 \) is closer to \( v_0 \) than \( v_2 \). For \( v \in V(T) \), let \( p_v = (v_0, \ldots, v) \) be the shortest path leading from \( v_0 \) to \( v \). We define \( T_v \) as the subtree of \( T \) which contains all the vertices \( v' \) such that \( p_{v'} \) passes through \( v \). We set

\[
A_v := \{ v' \mid (v, v') \in E(T) \}.
\]

For each vertex \( v' \) there exists a unique vertex \( v = \text{pre}(v') \) such that \( v' \in A_v \), called the predecessor of \( v' \). A vertex \( v \in V(T) \) - \( \{v_0\} \) will be called a leaf if \( A_v = \emptyset \). We write \( B(T) \) for the set of leaves of \( T \).

Definition 2.5. An order \( \psi : \{1, \ldots, r\} \to B(T) \) such that for each \( v \in V(T) \) the set

\[
I_v := \{ i \mid \psi(i) \in V(T_v) \} \subset \{1, \ldots, r\}
\]

is an interval, i.e. \( I_v = \{v', \ldots, v''\} \). The triple \((T, v_0, \psi)\) is called an ordered tree.

An order \( \psi \) on \( T \) induces a strict ordering of the sets \( A_v \) for \( v', v'' \in A_v \), we write \( v' < v'' \) if \( i \in I_v \) and \( j \in I_{v''} \) implies \( i < j \). Conversely, if we choose a strict ordering on each set \( A_v \) then there exists a unique order \( \psi \) on \( T \) inducing these orderings. In particular, an order \( \psi \) always exists. An automorphism of the ordered tree \((T, v_0, \psi)\) is an automorphism \( \kappa : T \to T \) such that \( \kappa(v_0) = v_0 \). Via \( \psi \), \( \kappa \) induces a permutation of \( \{1, \ldots, r\} \). Note that \( \kappa \) is uniquely determined by this permutation.

Let \( \Pi = \langle \gamma_1, \ldots, \gamma_r \mid \prod \gamma_i = 1 \rangle \) be the free profinite group defined earlier, and let \((T, v_0, \psi)\) be an ordered tree. For each \( v \in V(T) \) we define an element

\[
\gamma_v := \prod_{i \in I_v} \gamma_i
\]

and a subgroup

\[
\Pi_v := \langle \gamma_v \mid v' \in A_v \rangle \subset \Pi
\]

of \( \Pi \). Note that \( \gamma_{v_0} = 1 \), \( \gamma_{\psi(i)} = \gamma_i \) and that for all \( v \in V(T) \) we have

\[
(2.7) \quad \gamma_v = \prod_{v' \in A_v} \gamma_{v'},
\]

where the product is taken with respect to the ordering on \( A_v \) induced by \( \psi \).
DEFINITION 2.6. An automorphism \( \tau : \Pi \to \Pi \) is compatible with \((T, v_0, \psi)\) if there exists an automorphism \( \kappa \) of \((T, v_0, \psi)\), an element \( \chi \in \mathbb{Z}^s \) and elements \( \alpha_v \in \Pi_{\text{pre}(v)} \), for all \( v \in V(T) \), such that the following holds. For each \( v \in V(T) \) \( - \{ v_0 \} \) let \( \beta_v := \alpha_{v_1} \cdots \alpha_v \), where \( p_v = (v_0, v_1, \ldots, v) \) is the shortest path from \( v_0 \) to \( v \). Then

\[
\tau(\gamma_v) = \beta_{\kappa(v)} \gamma_{\kappa(v)}^{-1} \beta_{\kappa(v)}^{-1}.
\]

It is easy to see that \( \chi = \chi(\tau) \) and \( \kappa = \kappa(\tau) \) are uniquely determined by \( \tau \).

EXAMPLE 2.7. Let \( \rho : \Pi \to \pi_1(U, z_0) \) be a presentation, as in Definition 1.4. An element \( \sigma \in \Gamma_k \) induces an automorphism of \( \pi_1(U) \), and hence, via \( \rho \), an automorphism of \( \Pi \). Let \( V := \{0, 1, \ldots, r\}, E := \{(0, 1), \ldots, (0, r)\} \) and \( T := (V, E) \). We declare \( 0 \in V \) to be the root of \( T \) and let \( \psi : \{1, \ldots, r\} \to B(T) \) be the identity. Then \((T, 0, \psi)\) is an ordered tree, and the automorphism \( \Pi \to \Pi \) induced by \( \sigma \in \Gamma_k \) is compatible with \((T, 0, \psi)\), by (1.3).

DEFINITION 2.8. Let \( T = (T, v_0, \psi) \) be an ordered tree. A topological realization of \( T \) consists of

- pairwise distinct points \( z_0, \ldots, z_r \in \mathbb{P}^1_C \) (we set \( U := \mathbb{P}^1_C - \{z_1, \ldots, z_r\} \)),
- pairwise disjoint, connected, open subsets \( U_v \subset U \), for \( v \in V(T) \), and
- pairwise disjoint simple closed arcs \( \gamma_v : [0, 1] \to U_v \), for \( e \in E(T) \),

such that the following holds. The open subsets \( U_v \) are the connected components of \( U - (\cup_e c_e) \). For each edge \( e = (v_1, v_2) \), the arc \( c_e \) lies on the boundary of \( U_{v_1} \) and \( U_{v_2} \), encircling \( U_{v_1} \) in clockwise and \( U_{v_2} \) in counterclockwise direction. For \( i = 1, \ldots, r \), the set \( U_{\psi(i)} \cup \{z_i\} \) is homeomorphic to an open disk. Finally, \( z_0 \in U_{v_0} \).

Fix \( v \in V(T) - v_0 \), let \( p_v = (v_0, v_1, \ldots, v_{k-1}, v) \) be the shortest path from \( v_0 \) to \( v \), and set \( e := (v_{k-1}, v) \). Set \( z_v := c_e(0) \) and \( z_{v_0} := z_0 \). Choose a simple path \( a_v : [0, 1] \to U_v \) leading from \( z_{v_{k-1}} \) to \( z_v \). Define

\[
b_v := a_{v_1} \cdots a_v, \quad \text{and} \quad \gamma_v := b_v c_e b_v^{-1}.
\]

We regard \( \gamma_v \) as an element of \( \pi_1(U, z_0) \). To simplify the notation, set \( \gamma_i := \gamma_{\psi(i)} \), for \( i = 1, \ldots, r \), and \( \gamma_{v_0} := 1 \).

LEMMA 2.9. One can choose the paths \( a_v \) such that

\[
\gamma_v = \prod_{v' \in A_v} \gamma_{v'}
\]

holds for all \( v \in V(T) \) (in particular, \( \gamma_1 \cdots \gamma_r = 1 \)) and that the elements \( \gamma_1, \ldots, \gamma_r \) induce a presentation \( \rho : \Pi \to \pi_1(U, z_0) \).

A collection \( (a_v) \) of paths as in Lemma 2.9 will be called a skeleton of the topological realization \((z_i, U_v, c_e)\). The presentation \( \rho \) in Lemma 2.9 will be said to be induced by \((a_v)\).

PROPOSITION 2.10. Let \((z_i, U_v, c_e), (a_v)\) and \( \rho \) be as before. Let \( \tau : U \to U \) be an orientation preserving diffeomorphism of \( U \). Assume that \( \tau(z_0) = z_0 \) and that for all \( v \in V(T) \) the restriction of \( \tau \) to \( U_v \) is a diffeomorphism \( U_v \to U_v \), for some \( v' \in V(T) \). Then the induced automorphism of \( \pi_1(U, z_0) \cong \Pi \) is compatible with \( T \).
It is clear that \( \tau \) induces an automorphism \( \kappa \) of \( T \). We will use the notation \( v' := \kappa(v) \). Replacing \( \tau \) by a homotopic diffeomorphism, we may assume that \( \tau(c_e) = c_e' \), for all \( e \in E(T) \) (in particular, \( \tau(z_v) = z_v' \)). Fix \( v \in V(T) - \nu_0 \), let \( p_0 = (v_0, \ldots, v_{k-1}, v) \) be the shortest path from \( v_0 \) to \( v \) and set \( e := (v_{k-1}, v) \). Define

\[
\alpha_{v'} := b_{v'_{k-1}} \tau(a_v) a_{v'}^{-1} b_{v'_{k-1}}^{-1}
\]

and

\[
\beta_v := \tau(b_v) b_{v'}^{-1} = \alpha_{v'} \ldots \alpha_{v'}
\]

Since the closed path \( \tau(a_v) a_{v'} \) lies entirely in \( \tilde{U}_{v'_{k-1}} \), \( \alpha_{v'} \) is an element of the subgroup \( \Pi_{v'_{k-1}} \). Moreover, we have

\[
\tau(\gamma_v) = \tau(b_v) c_{v'} \tau(b_v)^{-1} = \beta_v \gamma_v \beta_v^{-1}.
\]

Therefore, \( \tau \) is compatible with \( T \) (note that \( \chi = 1 \)). \( \square \)

2.3. The fundamental group of a degenerating curve. Let \( k \) be as in Section 2.1 and \( S \subset \mathbb{P}^1_k \) a finite, \( \Gamma_k \)-invariant set and \( z_0 \) a \( k \)-rational base point for \( U := \mathbb{P}^1_k - S \). In this section, we associate to \( (\gamma, z_0) \) an ordered tree \( T \). The tree \( T \) encodes the way in which the points in \( S \cup \{z_0\} \) coalesce on the special fiber. If \( k_0 \) has characteristic 0 then there exists a presentation \( \rho : \Pi \xrightarrow{\sim} \pi_1(U, z_0) \) such that the \( \Gamma_k \)-action on \( \pi_1(U, z_0) \equiv \Pi \) is compatible with the tree \( T \), in the sense of Definition 2.6 above. If \( k_0 \) is a subfield of \( \mathbb{C} \), we can actually write down such a presentation. Moreover, we can compute the action of the inertia group on \( \pi_1(U, z_0) \) explicitly, in terms of the braid action.

The valuation \( v \) of \( k \) extends uniquely to a valuation on \( k^s \), which we also denote by \( v \). Let \( R \) be the ring of integers of \( k^s \). We identify the residue field of \( R \) with the algebraic closure \( \bar{k}_0 \) of \( k_0 \). Let \( (X_R: z_R, \ldots, z_r) \) be the (unique) model over \( R \) of \( (\mathbb{P}^1_k: z_0, \ldots, z_r) \) as a stable \( (r + 1) \)-pointed tree of projective lines, in the sense of 2.4. We denote by \( \hat{X} := X_R \otimes_R k_0^s \) the special fiber. By definition, \( \hat{X} \) is a tree of projective lines, i.e. each irreducible component of \( \hat{X} \) is non-canonically isomorphic to \( \mathbb{P}^1 \). For a point \( z \in S \cup \{z_0\} \), we denote by \( \hat{z} \) its specialization to \( \hat{X} \). The points \( z_0, \ldots, z_r \) are pairwise distinct, smooth points of \( \hat{X} \). Since the model \( X_R \) is unique, the natural \( \Gamma_k \)-action on \( \mathbb{P}^1_k \) extends to an action on \( X_R \). This yields an action of \( \Gamma_k \) on \( \hat{X} \). In particular, the inertia group \( I \subset \Gamma_k \) acts \( k_0^s \)-linearly on \( \hat{X} \).

We define the tree \( T \) as follows. The set of vertices \( V(T) \) is the union of \( S \) with the set of irreducible components of \( \hat{X} \). For every singular point \( x \) of \( \hat{X} \), we draw an edge between the two vertices corresponding to the components meeting in \( x \). Moreover, for each \( z \in S \), we draw an edge between the vertex corresponding to \( z \) and the vertex corresponding to the component containing \( \hat{z} \). We declare the component \( \hat{X}_0 \) which contains \( \hat{z}_0 \) as the root of \( T \). It is clear that the elements of \( S \) are the leaves of \( T \). If \( v \in V(T) \) is not a leaf, we write \( \hat{X}_v \) for the corresponding component of \( \hat{X} \). The numbering \( \{z_1, \ldots, z_r\} \) of the set \( S \) corresponds to a bijection \( \psi : \{1, \ldots, r\} \xrightarrow{\sim} S = B(T) \). By changing this numbering, we may assume that \( \psi \) is an order. From now on, we fix \( \psi \) and regard \( T = (T, \hat{X}_0, \psi) \) as an ordered tree. We remark that the \( \Gamma_k \)-action on \( \hat{X} \) induces an action on \( T \). It is clear that this action is determined by the action of \( \Gamma_k \) on \( S = B(T) \).
Example 2.11. Let $k := \mathbb{Q}((t))$, $d \in \mathbb{Z}$, $S := \{ \pm \sqrt{d} \pm i\sqrt{7} \}$ and $z_0 := 0$. The curve $\tilde{X}$ consists of three components $\tilde{X}_0$, $\tilde{X}_5$ and $\tilde{X}_6$. Each of these components corresponds to the choice of a coordinate $w$ which identifies the function field of $\mathbb{P}^1_k$ with $k^*(w)$, modulo the action of $\text{PGL}_2(R)$. For the component $\tilde{X}_0$, we can choose the standard coordinate $w_0 := z$. For $\tilde{X}_5$, we choose $w_5 := (z + \sqrt{d})/\sqrt{7}$, and for $\tilde{X}_6$ we choose $w_6 := (z - \sqrt{d})/\sqrt{7}$. We let $z_1 := -\sqrt{d} + i\sqrt{7}$, $z_2 := -\sqrt{d} - i\sqrt{7}$, $z_3 := \sqrt{d} - i\sqrt{7}$ and $z_4 := \sqrt{d} + i\sqrt{7}$. Then $z_1, z_2$ reduce to $\tilde{z}_1, \tilde{z}_2 \in \tilde{X}_5$ and $z_3, z_4$ reduce to $\tilde{z}_3, \tilde{z}_4 \in \tilde{X}_6$. See Figure 2. We remark that the standard generator $q_0$ of the inertia group $I$ (which sends $t^{1/n}$ to $e^{2\pi i/n}t^{1/n}$) acts as an involution on $\tilde{X}_5$ and $\tilde{X}_6$.

The following theorem can be thought of as a rigid-analytic analogue of Proposition 2.10. Many similar results can be found in the literature, see e.g. [18] or [19].

Theorem 2.12. Assume that $\text{char}(k_0) = 0$. There exists a presentation $\rho : \Pi \xrightarrow{\sim} \pi_1(U, z_0)$ such that the $\Gamma_k$-action on $\Pi$ induced by $\rho$ is compatible with $T$.

Sketch of proof. We follow [20], with some modifications. The tree $T$ can be equipped with a structure $\mathcal{G}$ of a graph of groups. For instance, to each vertex $v \in V(T)$ corresponding to a component $\tilde{X}_v$ of $\tilde{X}$ we associate the fundamental group $\pi_1(\tilde{U}_v)$, where $\tilde{U}_v$ is the open subset of $\tilde{X}$ with all the points $\tilde{z}_i$, $i > 0$, and all singular points removed. To each leaf $v(i) \in V(T)$, we associate the group $\tilde{Z}(1)$. We obtain a “canonical” isomorphism

$$\pi_1(U, z_0) \xrightarrow{\sim} \pi_1(T, \mathcal{G}). \tag{2.8}$$

In analogy to Lemma 2.13, one can choose a “skeleton” $(a_0)$ of $(T, \mathcal{G})$ which induces a presentation $\Pi \xrightarrow{\sim} \pi_1(T, \mathcal{G})$ (actually, to make this precise, one has to consider $(T, \mathcal{G})$ as a graphs of groupoids). Let $\rho : \Pi \xrightarrow{\sim} \pi_1(U, z_0)$ be the composition of this presentation with the isomorphism (2.8). The rest of the proof of Theorem 2.12 is formally the same as the proof of Proposition 2.10.

Remark 2.13. Assume that $\text{char}(k_0) = p > 0$, and let $\pi_1^{\text{adm}}(U, z_0)$ be the admissible fundamental group, i.e. the inverse limit over the finite quotients of $\pi_1(U, z_0)$ corresponding to Galois covers with admissible reduction (see [20]). There exists a surjective homomorphism $\rho : \Pi \rightarrow \pi_1^{\text{adm}}(U, z_0)$ which induces an isomorphism on the maximal prime-to-$p$ quotients such that the action of $\Gamma_k$ on $\pi_1^{\text{adm}}(U, z_0)$ is

![Figure 2. The curve $\tilde{X}$ and the associated tree $T$](image-url)
compatible with \( \rho \) and \( T \), in an obvious sense. In particular, Theorem 2.12 remains true if we replace \( \Pi \) and \( \pi_1(U, z_0) \) by their maximal prime-to-\( p \) quotients.

For a more detailed proof of Theorem 2.12 in a special case, see [27]. Under some extra assumptions, we can improve on Theorem 2.12 and define a concrete presentation \( \rho \) with the claimed properties. This presentation has the advantage that the inertia action on \( \pi_1(U, z_0) \) is known explicitly.

We assume that \( k_0 \) is a subfield of the complex numbers, and that \( k := k_0(t)^h \) is the henselization of \( k_0(t) \) at the place \( t = 0 \). For practical applications, this is not a serious restriction. We will regard \( k \) as a subfield of \( \mathbb{C}(t) \) and its algebraic closure \( k^a \) as a subfield of \( \mathbb{C}\{\{t\}\} \). We identify \( \Gamma_{k_0} \) as a subgroup of \( \Gamma_k \), via its action on the coefficients of the Puiseux-expansion of elements of \( k \). We let \( q_0 \) be the “canonical” generator of the inertia group \( I \triangleleft \Gamma_k \), i.e. we have \( q_0(t^{1/n}) = e^{2\pi i/n}t^{1/n} \).

For simplicity, we assume moreover that \( n \) be the smallest positive integer such that all points \( z_0, \ldots, z_r \), regarded as elements of \( k^a \), lie in \( \mathbb{C}(t^{1/n}) \). We let \( \tilde{t} := t^{1/n} \) and regard the \( z_i \) as germs of analytic functions of \( \tilde{t} \). Choose \( \epsilon > 0 \) such that \( z_0, \ldots, z_r \) are meromorphic on the disk

\[
\tilde{D} := \{ \tilde{t} \in \mathbb{C} \mid |\tilde{t}| < \epsilon \}
\]

and holomorphic on \( \tilde{D}^* := \tilde{D} - \{0\} \). If \( \epsilon \) is sufficiently small, the values \( z_i(\tilde{t}) \) are pairwise distinct, for all \( \tilde{t} \in \tilde{D}^* \). The R-curve \( X_R \) gives rise to an analytic space \( X_{\tilde{D}} \), together with a map \( p : X_{\tilde{D}} \to \tilde{D} \) such that \( p^{-1}(\tilde{D}^*) = \mathbb{P}^1_C \times \tilde{D}^* \) and \( p^{-1}(0) = X \otimes \mathbb{C} \). Let \( e = (v_1, v_2) \) be an edge corresponding to a singular point \( x_e \in X \). The complete local ring of \( x_e \) on \( X_R \) has the form

\[
\mathcal{O}_{X_R, x_e} = R[[u_e, v_e \mid u_ev_e = \tilde{t}^{n_e}]],
\]

(for some well determined positive integer \( n_e \)) such that \( u_e = 0 \) (resp. \( v_e = 0 \)) defines the component \( X_{v_1} \) (resp. \( X_{v_2} \)) in a neighborhood of \( x_e \). We may assume that \( u_e, v_e \) are analytic functions in a neighborhood of \( x_e \) on \( X_{\tilde{D}} \). Choosing \( \epsilon \) sufficiently small, we may identify the analytic set

\[
V_e := \{ (u_e, v_e, \tilde{t}) \in \mathbb{C}^3 \mid u_ev_e = \tilde{t}^{n_e}, |u_e|, |v_e| < \epsilon^{n_e/2n} \}
\]

with an open neighborhood of \( x_e \) on \( X_{\tilde{D}} \). We may also assume that the sets \( V_e \) are pairwise disjoint.

Fix a positive real number \( t_0 \) such that \( 0 < t_0 < \epsilon \) and let \( \bar{z}_i := z_i(t_0^{1/n}) \), for \( i = 0, \ldots, r \), and

\[
U_{t_0} := \mathbb{P}^1_C - \{\bar{z}_1, \ldots, \bar{z}_r\} \subset p^{-1}(t_0^{1/n}).
\]

For each edge \( e \) corresponding to a singular point \( x_e \), define the closed arc

\[
c_e := \left\{ \begin{array}{ll}
0,1 & \rightarrow \\
\{0,1\} & \\
s & \rightarrow \\
& \left( t_0^{n_e/2n} e^{2\pi i s}, t_0^{n_e/2n} e^{-2\pi i s}, t_0^{1/n} \right) \quad \mbox{.}
\end{array} \right.
\]

For the edge \( e \) adjacent to the leaf \( \psi(i) \), let \( c_e : [0,1] \to U_{t_0} \) be a small closed arc encircling \( \bar{z}_i \) in counterclockwise direction. We may assume that all the arcs \( c_e \) are pairwise disjoint. For \( v \in V(T) - \{t_0\} \), set \( e := (\pre(v), v) \in E(T) \) and let \( U_e \) be the connected component of \( U_{t_0} - (U_{e,c_e}) \) containing the annulus \( \{ (u_e, v_e, \tilde{t}) \in V_e \mid |u_e| > |v_e| \} \).
Lemma 2.14. The data \((\tilde{z}, U_\gamma, c_\gamma)\) is a topological realization of the ordered tree \(T\).

Let us choose a skeleton \((a_\gamma)\) for \((\tilde{z}, U_\gamma, c_\gamma)\) and let \(\rho_0 : \Pi \rightarrow \pi_1(U_{t_0}, \tilde{z}_0)\) be the induced presentation (see Lemma 2.9). We define the presentation \(\rho : \Pi \rightarrow \pi_1(U, z_0)\) as the composition of \(\rho_0\) with the canonical isomorphism
\[
(2.9) \quad \pi_1(U_{t_0}, \tilde{z}_0) \cong \pi_1(U, z_0).
\]

A careful modification of the proof of Theorem 2.12 yields (compare with \([18]\)):

Proposition 2.15. Theorem 2.12 holds with the presentation \(\rho\) constructed above.

Let \(0 < |t| < \epsilon\) and choose a root \(t^{1/n}\). The set
\[
S_t := \{ z_1(t^{1/n}), \ldots, z_r(t^{1/n}) \} \subset \mathbb{P}^1_\mathbb{C}
\]
depends only on \(t\), since changing the root \(t^{1/n}\) only permutes the points of \(S_t\). Let \(\theta : [0, 1) \rightarrow \mathbb{C}\) be the closed arc \(\theta(s) := t_0 e^{2\pi is}\). The map \(s \mapsto S_{\theta(s)}\) corresponds to an element \(Q \in \mathcal{B}_r\) of the Artin braid group on \(r\) strings. Here we identify \(\mathcal{B}_r\) with the fundamental group of the space of \(r\) unordered points in \(U_{t_0}\), with base point \(S_{t_0} = \{ z_1, \ldots, \tilde{z}_r \}\). There is a well known action of \(\mathcal{B}_r\) on \(\pi_1(U_{t_0}, \tilde{z}_0)\); see e.g. \([24]\), Section II.10. By the definition of this action, the braid \(Q\) defined above corresponds, via the canonical isomorphism \((2.9)\), to the inertia generator \(q_0\) acting on \(\pi_1(U, z_0)\). This gives us a practical way to compute this action explicitly, see \([3]\).

Example 2.16. Let \(k, S, z_0\) be as in Example 2.11. See Figure \(1\) for a picture of the presentation \(\rho_0\), for \(t_0 > 0\) sufficiently small and \(d > 0\).

It is easily seen that the inertia generator \(q_0\) corresponds to the braid \(Q = Q_1Q_3\) (where \(Q_1, Q_2, Q_4\) are the standard generators of \(\mathcal{B}_r\), see \([24]\), Section II.10). Therefore, \(q_0\) acts on \(\Pi\) via
\[
(2.10) \quad q_0^6 = \gamma_1 \gamma_2^{-1}, \quad q_0^2 = \gamma_2, \quad q_0^3 = \gamma_3 \gamma_4 \gamma_5^{-1}, \quad q_0^4 = \gamma_3.
\]
Set \(\gamma_5 := \gamma_1 \gamma_2, \quad \gamma_6 := \gamma_3 \gamma_4, \quad \Pi_5 := \langle \gamma_1, \gamma_2 \rangle \subset \Pi\) and \(\Pi_6 := \langle \gamma_3, \gamma_4 \rangle \subset \Pi\). Let \(\sigma \in \Gamma_Q\). By Theorem 2.12 and Proposition 2.15, the action of \(\sigma\) on \(\Pi \cong \pi_1(U, z_0)\) is compatible with the tree \(T\). We conclude that
\[
(2.11) \quad \sigma \gamma_i = \beta_{\sigma(i)} \gamma_{\sigma(i)} \beta_{\sigma(i)}^{-1}, \quad \text{with} \quad \beta_i \in \begin{cases} \Pi_5, & \text{for } i = 1, 2 \\ \Pi_6, & \text{for } i = 3, 4 \end{cases}
\]
Moreover, we have
\[
(2.12) \quad \sigma \gamma_5 = \gamma_5^{\epsilon(\sigma)}, \quad \sigma \gamma_6 = \gamma_6^{\epsilon(\sigma)}
\]
where \(\epsilon : \Gamma_Q \rightarrow \{ \pm 1 \}\) is the Kummer character \(\epsilon(\sigma) := \sigma \sqrt{d}/\sqrt{d}\).

2.4. Computation of the residue. In this section we compute the residue \(r_\gamma(\omega)\) (and the value \(\omega_0\), if \(r_\gamma(\omega) = 0\)) in a nontrivial example. Let us fix a square free integer \(d\) and let \(k := \mathbb{Q}(t)\), \(S := \{ \pm \sqrt{d} \pm i \sqrt{d} \}\) and \(z_0 := 0\), as in Example 2.11 and Example 2.16. Let \(\rho : \Pi \rightarrow \pi_1(U, z_0)\) be the presentation constructed in the last subsection (see Figure \(8\)). Then the action of \(\Gamma_k\) on \(\pi_1(U, z_0)\) is subject to \((2.10), (2.11)\) and \((2.12)\).
Let $G := \tilde{A}_5$ be the nonsplit central extension of $A_5$ by $C := \{\pm 1\}$. We denote by $3A$ the (unique) conjugacy class of elements of order 3 in $G$ and by $\text{Ni}^n(3A^5)$ the set of Nielsen classes $[g] = [g_1, g_2, g_3, g_4]$ with $g_i \in 3A$, for $i = 1, \ldots, 4$.

**Proposition 2.17.** Given $g = (g_1, g_2, g_3, g_4)$ such that $[g] \in \text{Ni}^n(3A^5)$, let $f : Y \to \mathbb{P}_k^1$ be the $G$-cover with branch locus $S$ and Hurwitz description $g$, with respect to the presentation $\rho$. We denote by $k'$ the smallest field containing $k$ which is a field of moduli for $f$ and by $\omega \in H^2(k', C)$ the f.o.d.-obstruction. Set $g_5 := g_1g_2 = (g_3g_4)^{-1}$. Then one of the following cases occurs:

1. **Case 1:** $g_5 = 1$. We have $k' = k = \mathbb{Q}(t)$ and $r_\omega(\omega) = (-d) \in \mathbb{Q}^*/\mathbb{Q}^2$. If $d = -1$ then $r_\omega(\omega) = 0$ and $\omega_0 = (-1, -1) \in Br_2(\mathbb{Q})$.
2. **Case 2:** $g_5$ has order 10. Then $k' = \mathbb{Q}(\sqrt{5})((t))$, with $5v(\tilde{t}) = v(t)$, $r_\omega(\omega) = 0$ and $\omega_0 = (-1, -d) \in Br_2(\mathbb{Q}(\sqrt{5}))$.
3. **Case 3:** $g_5$ has order 6. Then $k' = \mathbb{Q}(\sqrt{-3})((t))$, with $3v(\tilde{t}) = v(t)$, $r_\omega(\omega) = 0$ and $\omega_0 = (-1, d) = (3, d) \in Br_2(\mathbb{Q}(\sqrt{-3}))$.

**Proof.** There are exactly 18 classes $[g_1, g_2, g_3, g_4]$ in $\text{Ni}^n(3A^5)$. Among them, there are two classes with $g_1g_2 = 1$, 10 classes with $g_5 := g_1g_2$ of order 10 and 6 classes with $g_5$ of order 6. See e.g. [1, Prop. 5.8] for the use of a Theorem of Serre [21] showing how to compute the orders of products of odd order elements in $\tilde{A}_n$.

**Case 1:** $g_5 = 1$. The two Nielsen classes $[g] \in \text{Ni}^n(3A^5)$ such that $g_1g_2 = 1$ are permuted by an outer automorphism of $G$. Therefore, it suffices to consider one of them. So we assume that $g = (g_1, g_1^{-1}, g_3, g_3^{-1})$ and that $g_1$ (resp. $g_3$) is the (unique) lift of $(123) \in A_5$ (resp. $(345) \in A_5$) to an element of order 3. By (2.10), we have

\[(2.13) \quad g^{g_0} = (g_1^{-1}, g_1, g_3^{-1}, g_3) = a_0 g a_0^{-1},\]

where $a_0 \in G$ is a lift of $(12)(45) \in A_5$. Note that $a_0$ is of order 4. Equation (2.11) implies that

\[(2.14) \quad g^\sigma = \left\{ \begin{array}{ll}
(g_1^{\chi_\sigma}, g_1^{-\chi_\sigma}, g_3^{\chi_\sigma}, g_3^{-\chi_\sigma}), & \text{if } \epsilon = 1 \\
(g_3^{\chi_\sigma}, g_3^{-\chi_\sigma}, g_1^{\chi_\sigma}, g_1^{-\chi_\sigma}), & \text{if } \epsilon = -1,
\end{array} \right.
\]

for all $\sigma \in \Gamma_Q$ (we have used that the groups $G_5 := \langle g_1, g_1^{-1} \rangle$ and $G_6 := \langle g_3, g_3^{-1} \rangle$ are cyclic). It follows that $g^\sigma = b_\sigma g b_\sigma^{-1}$, where

\[(2.15) \quad b_\sigma := a_0^{(1-\chi_\sigma)/2} b^{(1-\epsilon)/2},\]

and $b \in G$ is a lift of $(14)(25) \in A_5$. Together with (2.13), we get that $[g]^\sigma = [g]$, for all $\tau \in \Gamma_k$. This shows that $k = \mathbb{Q}(t)$ is the field of moduli of the $G$-cover $f$. 

---

**Figure 3.**

\[
\begin{array}{c}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\gamma_5 \\
\gamma_6 \\
\end{array}
\]
In $\tilde{A}_5$, we have the equality $b a_0 b^{-1} = a_0^{-1} = -a_0$. Using this and (2.15), we can compute the $\Gamma_Q$-action on $a_0$ (defined by (2.2)):

$$ (2.16) \quad \sigma a_0 := b_\sigma a_0 b_\sigma^{-1} = a_0^{\sigma}. $$

By Proposition 2.2 (i), the residue $r_v(\omega) \in H^1(Q, \{\pm 1\})$ is represented by the cocycle

$$ (2.17) \quad c_\omega = a_0 (\sigma a_0)^{-1} = a_0^{1-\epsilon_\omega \chi_\omega}, $$

where $\eta_\sigma := \sigma i/i$. Therefore, $r_v(\omega) = (-d) \in Q^*/Q^{*2} = H^1(Q, \{\pm 1\})$. Since we assumed $d$ to be square free, we have $r_v(\omega) = 0$ if and only if $d = -1$.

If $d = -1$, then $r_v(\omega) = 0$ and the value $\omega_0 = H^2(Q, \{\pm 1\}) = Br_2(Q)$ of $\omega$ at $v$ is well defined. By Proposition 2.2 (ii), $\omega_0$ is the obstruction for lifting the homomorphism $\bar{\varphi} : \Gamma_Q \to G = A_5$ given by

$$ (2.18) \quad \bar{\varphi}(\sigma) = \bar{b}_\sigma = (1 2)(4 5) (1-\chi_\omega)/2 (2 5) (1-\epsilon_\sigma)/2 $$

to a homomorphism $\varphi : \Gamma_Q \to \tilde{A}_5$. In other words, it is the obstruction for lifting the $\mathbb{Z}/2 \times \mathbb{Z}/2$-extension $\mathbb{Q}(\sqrt{5}, i)/\mathbb{Q}$ to a quaternion extension. As in the proof of Proposition 1.7, we conclude that

$$ (2.19) \quad \omega_0 = (-1, 3) + (-1, -1) + (3, -1) = (-1, -1) \neq 0. $$

**Case 2: $g_5$ has order 10.** Let $O$ be the set of Nielsen classes $[g] \in N_i^\sim(3A^4)$ such that $g_5 := g_1 g_2$ has order 10. The inertia generator $q_0$, which acts on $N_i^\sim(3A^4)$, stabilizes $O$ and has two orbits $O_1$ and $O_2$ of length 5, where a Nielsen class $[g] \in O$ belongs to $O_1$ (resp. $O_2$) if $g_5 \in 10A$ (resp. $10B$). Here $10A$ and $10B$ are the two conjugacy classes in $A_5$ of elements of order 10. Since the classes $10A$ and $10B$ are conjugate over $\mathbb{Q}(\sqrt{5})$, it follows from (2.12) that $O_1^\sigma = O_2$ for $\sigma \in \Gamma_Q$ with $\sigma \sqrt{5} = -\sqrt{5}$. Therefore, $\Gamma_k$ acts transitively on $O$, and the fixed field of any class $[g] \in O$ is of the form $k' := \mathbb{Q}(\sqrt{5})(i \bar{t})$, with $5i \bar{t} = \bar{t}$. Hence, all classes $[g]$ with $g_5$ of order 10 are conjugate under the action of $\Gamma_k$. We may therefore assume that $g := (g_1, g_2, g_1^{-1}, g_1^{-1})$, where $g_1$ is a lift of $(123) \in A_5$ and $g_2$ is a lift of $(145)$. The $G$-cover $f$ with Hurwitz description $[g]$ has field of moduli $k'$ as above. Therefore, there exists an element $a_0 \in G$ such that $g_5^{a_0} = a_0 g_5 a_0^{-1}$. In particular, $a_0 g_5 a_0^{-1} = g_5$. Since any subgroup of order 5 in $A_5$ is self-centralizing, the image of $a_0$ in $A_5$ has odd order. From Corollary 2.4 we conclude that $r_\varphi(\omega) = 0$.

For all $\sigma \in \Gamma_Q(\sqrt{5})$, there exists $b_\sigma \in G$ such that $g_5^{\sigma} = b_\sigma g_5 b_\sigma^{-1}$. By (2.13), we have

$$ (2.20) \quad g_5^\sigma = g_5^{\chi_\sigma \epsilon_\sigma} = b_\sigma g_5 b_\sigma^{-1}. $$

In particular, $b_\sigma$ lies in $N$, the normalizer of $<g_5> \subset \tilde{A}_5$. Let $\tilde{N}$ be the image of $N$ in $\tilde{A}_5$. The group $\tilde{N}$ is dihedral of order 10. Let $\tilde{\varphi}(\sigma) := b_\sigma$ and $\tilde{\psi} : \Gamma_Q(\sqrt{5}) \to <\pm 1>$ the composition of $\bar{\varphi}$ with the sign character $\tilde{N} \to \{\pm 1\}$. By (2.21), $\tilde{\psi}(\sigma) \equiv \chi_\sigma \epsilon_\sigma$ (mod 5). A computation shows that $\tilde{\psi}$ corresponds to the quadratic extension

$$ \mathbb{Q}(\sqrt{5}, \sqrt{-10 + 2\sqrt{5}}) / \mathbb{Q}(\sqrt{5}). $$

to lift $\bar{\varphi}$ to $\tilde{A}_5$ it suffices to lift $\tilde{\psi}$ to a homomorphism $\psi : \Gamma_Q(\sqrt{5}) \to \mathbb{Z}/4$ (compare with the proof of Proposition 1.5). Using $10 - 2\sqrt{5} = (1 - \sqrt{5})^2 + 2^2$, we conclude
that
\[ \omega_0 = (-1, d(-10 + 2\sqrt{5})) = (-1, -d). \]

**Case 3:** \(g_5\) has order 6. Let \(O \in \text{Nil}(3A^2)\) be the set of classes \([g]\) with \(g_5\) of order 6. As in Case 2, \(q_0\) acts on \(O\) and has two orbits, of length 3. Therefore, the fixed field of \([g]\) in \(O\) is of the form \(k' = k_0((\ell))\), with \(3v(\ell) = v(t)\) and \(k_0/Q\) at most a quadratic extension. We claim that \(k_0 = \mathbb{Q}(\sqrt{-3})\), for all \([g]\) in \(O\). It suffices to show the following. Let \(v \in \Gamma_0\) such that \(g_v^\sigma = b_\sigma g b_\sigma^{-1}\), for some \(b_\sigma \in G\). Then \(\chi(\sigma) \equiv 1 \pmod{6}\).

The subgroups \(G_5 := <g_1, g_2>\) and \(G_6 := <g_3, g_4>\) of \(\tilde{A}_5\) are of order 24, isomorphic to \(A_4\) (two 3-cycles in \(A_5\) generate either a cyclic subgroup of order 3 or a subgroup isomorphic to \(A_4\)). Moreover, \(G_5 \cap G_6 = \text{cyclic of order 6}, \) generated by \(g_5\). Now assume that \(g_5^\sigma = b_\sigma g b_\sigma^{-1}\) and \(\epsilon(\sigma) = 1\). It follows from (2.10) that \(b_\sigma G_5 b_\sigma^{-1} = G_5\) and \(b_\sigma G_6 b_\sigma^{-1} = G_6\). It is easy to see that this implies \(b_\sigma \in G_5 \cap G_6 = <g_5>\). Using (2.12), we get \(g_5^{\chi(\sigma)} = b_\sigma g_5 b_\sigma^{-1} = g_5\), hence \(\chi(\sigma) \equiv 1 \pmod{6}\).

Let \(f\) be the \(G\)-cover with Hurwitz description \([g]\) in \(O\). We have shown that the minimal field of moduli of \(f\) containing \(k\) is of the form \(k' = \mathbb{Q}(\sqrt{-3})((\ell))\), with \(3v(\ell) = v(t)\). Therefore, \(g_5^\sigma = a_0 g a_0\), for some \(a_0 \in G\). As in Case 2, we conclude that \(a_0 \in <g_5>\) and can therefore be assumed to be of odd order. By Corollary 2.3, we have \(r_v(\omega) = 0\).

Recall that \(g_\sigma^\sigma = b_\sigma g b_\sigma\) implies \(b_\sigma g_5 b_\sigma^{-1} = g_5^{\epsilon(\sigma)}\). Therefore, the image of \(\hat{\varphi}(\sigma) := \hat{b}_\sigma\) is contained in \(N \subset A_5\), the image of the normalizer of \(g_5\) in \(A_5\). The group \(N\) is dihedral of order 6, and the composition of \(\hat{\varphi}\) with the character \(N \to \{\pm 1\}\) equals \(\epsilon\). As in Case 2, we conclude that \(\omega_0 = (-1, d) = (3, d) \in \text{Br}_2(\mathbb{Q}(\sqrt{-3}))\).

### 2.5. Specialization to \(p\)-adic fields.

In this section, we let \(k\) be a field which is complete with respect to a discrete valuation \(v\), and assume that the residue field \(k_0\) is perfect of characteristic \(p > 0\). We denote by \(\mathfrak{O}_k\) the ring of integers of \(k\) and by \(\mathfrak{p}\) the maximal ideal of \(\mathfrak{O}_k\). We set \(K_t := k((t))\) and denote by \(v_t\) the valuation of \(K_t\) which has \(t\) as a uniformizer. We let \(f_t : Y_t \to \mathbb{P}^1\) be a \(G\)-cover over \(K_t^* = k^*\{\{t\}\}\) such that \(K_t\) is a field of moduli for \(f_t\).

Our first goal is to define, for any element \(a \in \mathfrak{p} - \{0\}\), the **specialization** \(f_a : Y_a \to \mathbb{P}^1\) of \(f_t\) at \(a = t\), which should be a \(G\)-cover over \(k^a\) with field of moduli \(k\). Second, we would like to compute the residue \(r_{v_t}(\omega_a)\) of the f.o.d.-obstruction \(\omega_a\) of \(f_a\) in terms of \(a\) and the residue \(r_{v_t}(\omega_t)\) of the f.o.d.-obstruction \(\omega_t\) of \(f_t\). Both these goals are problematic, in general, and we need some extra assumptions to succeed.

Let \(A := \mathfrak{O}_k[[t]]\) and \(K := \text{Frac}(A)\). The ring \(A\) is a complete local domain, regular of dimension 2 and factorial. The field \(K_t = k((t))\) is the completion of \(K\) at the valuation corresponding to the ideal \((t) \subset A\). We denote this valuation also by \(v_t\). Moreover, we identify \(K^a\) with the algebraic closure of \(K\) inside \(K_t^* = k^*\{\{t\}\}\).

**Condition 2.18.**
(a) The branch locus $S_t \subset \mathbb{P}^1_{A}$ of the $G$-cover $f_t$ is defined over $K$, i.e.,
$$S_t = \{ z \mid F(z) = 0 \}, \quad \text{with} \quad F(Z) \in K(Z).$$
We may choose $F(Z) \in A(Z)$ such that the gcd of the coefficients is 1. We let $\delta(S) \subset A$ be the ideal generated by the discriminant of $F$.

(b) We have $\delta(S) = (t^n)$, with $n \geq 0$.

(c) The order of the group $G$ is prime to $p = \text{char}(k_0)$.

We assume from now on that Condition 2.18 is in force. By Condition 2.18 (a), the branch locus $S_t \subset \mathbb{P}^1_{A}$ of the $G$-cover $f_t$ descends to a closed subscheme $S \subset \mathbb{P}^1_{K}$. Let $V := \text{Spec} \ A[1/t]$ and $S_V \subset \mathbb{P}^1_V$ the closure of $S$ inside $\mathbb{P}^1_V$. By Condition 2.18 (b), $S_V \to V$ is étale. Therefore, the embedding $S_V \subset \mathbb{P}^1_V$ corresponds to a morphism $V \to \mathcal{U}_r$, where $\mathcal{U}_r$ is the fine moduli space for the moduli problem “$r$ distinct unordered points in $\mathbb{P}^1$”. In more concrete terms, we have $\mathcal{U}_r = \mathbb{P}^r - \delta_r$ ($\delta_r$ denotes the discriminant hypersurface), and the coefficients of the polynomial $F(X) \in A[1/t][X]$ which defines $S_V$ are the projective coordinates for the morphism $V \to \mathcal{U}_r$. Let $\mathcal{H}_r(G)$ be the Hurwitz space classifying $G$-Galois covers over $\mathbb{P}^1$ with exactly $r$ branch points. To simplify the notation, we consider $\mathcal{U}_r$ and $\mathcal{H}_r(G)$ as schemes over $A$. Since the order of $G$ is prime to the residue characteristic of $\mathfrak{o}$ (by Condition 2.18 (c)), the natural morphism $\mathcal{H}_r(G) \to \mathcal{U}_r$ (which associates to a $G$-cover its branch locus) is finite étale, see [25]. The $G$-cover $f_t$ corresponds to a morphism $[f_t] : \text{Spec} \ K_t \to \mathcal{H}_r(G)$, see Remark 1.3 (ii).

**Lemma 2.19.** The morphism $V \to \mathcal{U}_r$ lifts to a morphism $\varphi : V \to \mathcal{H}_r(G)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\text{Spec} \ K_t & \xrightarrow{[f_t]} & \mathcal{H}_r(G) \\
\downarrow & \nearrow & \downarrow \\
V & \longrightarrow & \mathcal{U}_r.
\end{array}
$$

**Proof.** Since $\mathcal{H}_r(G) \to \mathcal{U}_r$ is a finite and étale morphism, the fiber product $V \times_{\mathcal{U}_r} \mathcal{H}_r(G)$ will be finite and étale over $V$. Let $V' \subset V \times_{\mathcal{U}_r} \mathcal{H}_r(G)$ be the irreducible component which is the image of the morphism $\text{Spec} \ K_t \to V \times_{\mathcal{U}_r} \mathcal{H}_r(G)$. Then $V'$ is integral, its fraction field $K'$ is a finite extension of $K$ and a subfield of $K_t$, and $V'$ is the integral closure of $V$ in $K'$. We have to show that $K' = K$.

Let $A'$ be the integral closure of $A$ in $K'$. Clearly, $A'$ is a finite $A$-algebra, with fraction field $K'$, and $A'[1/t] = \text{étale over } A[1/t]$. Using Purity of Branch Locus and Abhyankar’s Lemma, one shows that there exists an embedding $A' \hookrightarrow \mathfrak{o}'[t^{1/n}]$ of $A$-algebras, for some $n \geq 1$ and some finite unramified extension $\mathfrak{o}'/\mathfrak{o}$. In particular, the valuation $v_t$ of $K$ is totally ramified in $K'$. But $K'$ was defined as a subfield of $K_t = k((t))$. Therefore, $v_t$ is actually unramified in $K'$. Thus, by Purity of Branch locus, $A'/A$ is étale. Since $A$ is henselian, a finite étale extension of $A$ is uniquely determined by its residue field extension. It follows that $A' = \mathfrak{o}'[[t]]$, for some finite unramified extension $\mathfrak{o}'/\mathfrak{o}$. Using again the embedding $A' \subset K_t = k((t))$, we conclude that $A' = A$. This proves the lemma.

Lemma 2.19 shows in particular that $f_t$ descends to a $G$-cover $f : Y \to \mathbb{P}^1_{K^s}$ over $K^s$ such that $K$ is a field of moduli for $f$, see Remark 1.3 (ii).

Let us fix an element $a$ in $p - \{0\}$. We denote by $v_a$ the discrete valuation of $K$ corresponding to the ideal $(t - a) \subset A$ (note that $v_a \neq v_t$). We choose an
extension \( \tilde{v}_a \) of \( v_a \) to \( K^s \). Let \( R \subset K^s \) denote the valuation ring corresponding to \( \tilde{v}_a \). By Condition 2.18 (b), the branch locus \( S \subset \mathbb{P}^1_{K^s} \) of \( f \) extends to a subscheme \( S_R \subset \mathbb{P}^1_R \) such that \( S_R \) is étale over \( \text{Spec} \, R \). It follows that \( f \) has good reduction at \( \tilde{v}_a \). Let \( f_a : Y_a \to \mathbb{P}^1_{k_c} \) be the reduction of \( f \) at \( \tilde{v}_a \). We call \( f_a \) the specialization of \( f \) at \( t = a \). By definition, \( f_a \) is a \( G \)-cover defined over \( k^s \), and \( k \) is a field of moduli for \( f_a \).

The following theorem states that one obtains the f.o.d.-obstruction for \( f_a \) by ‘specializing’ the f.o.d.-obstruction for \( f \) at \( t = a \). Moreover, this specialization procedure is compatible with computing the residue. The essential idea behind the theorem is that the class \( \omega_t \) behaves just as one would expect it to behave if \( k \) were a local field of equal characteristic, with field of coefficients \( k_0 \), and the class \( \omega_t \) arises from pull-back via the inclusion \( k_0((t)) \to k((t)) \).

**Theorem 2.20.** Assume that Conditions 2.18 (a), (b) and (c) hold. Let \( \omega_t \in H^2(K_t, \mathbb{C}) \) be the f.o.d.-obstruction for \( f \) and \( \omega_a \in H^2(k, C) \) the f.o.d.-obstruction for \( f_a \). Then

(i) The residue \( r_v(\omega_t) \in H^1(k(C(-1))) \) of \( \omega_t \) at \( v_t \) is unramified at \( v \), i.e. is induced by a class \( \tilde{r}_v(\omega_t) \in H^1(k_0(C(-1))) \).

(ii) If \( H^1(k_0(C(-1))) \), we have the formula

\[
\omega_v(a) = v(a) \cdot \tilde{r}_v(\omega_t).
\]

(iii) If \( r_v(\omega_t) = 0 \), then \( \omega_t \) lies in the submodule \( H^2(k_0, C) \), and the equality \( \omega_v = \omega_t \) holds.

**Proof.** Let \( k'^{\sigma} \) be the maximal unramified extension of \( k \) and \( \sigma^{\sigma} \) the ring of integers of \( k'^{\sigma} \). We denote by \( S_\mathcal{A} \) the set of discrete valuations of \( K \) which dominate \( A \). Define \( L'^{\sigma} \subset K^s \) as the maximal algebraic extension of \( K \) which is unramified over each valuation \( w \in S_\mathcal{A} \). By Purity of Branch Locus, we can identify \( \text{Gal}(L'^{\sigma}/K) \) with \( \pi_1(\text{Spec} \, A) \), see [13]. Since \( A \) is a henselian local ring with residue field \( k_0 \), \( \pi_1(\text{Spec} \, A) \) is canonically isomorphic to \( \Gamma_{k_0} \). Moreover, \( B'^{\sigma} := \sigma^{\sigma}[[t]] \) is the integral closure of \( A \) in \( L'^{\sigma} \). Similarly, let \( L \subset K^s \) be the maximal algebraic extension of \( K \) which is unramified over each valuation \( w \in S_\mathcal{A} - \{v_1\} \). We may identify \( \text{Gal}(L/K) \) with \( \pi_1(V) \), where \( V = \text{Spec} \, A[1/t] \). Let \( B \) be the integral closure of \( A \) in \( L \). Abhyankar’s Lemma together with Purity shows that \( B = \bigcup_n \sigma^{\sigma}[[t^{1/n}]] \), where \( n \) runs over all integers prime to \( p \) (compare with the proof of Lemma 2.19). Therefore, \( \text{Gal}(L'^{\sigma}/L) = \hat{\mathbb{Z}}'(1) \). We obtain the following commutative diagram:

\[
\begin{array}{cccc}
1 & \rightarrow & \hat{\mathbb{Z}}'(1) & \rightarrow & \Gamma_{K_t} & \rightarrow & \Gamma_k & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \hat{\mathbb{Z}}'(1) & \rightarrow & \pi_1(V) & \rightarrow & \Gamma_{k_0} & \rightarrow & 1.
\end{array}
\]

The vertical arrows are induced by the embedding of \( K^s \) into \( K_t^s \). By [13], we have canonical isomorphism \( H^i_{\mathcal{A}}(V, C) \cong H^i(\pi_1(V), C) \), for \( i \geq 0 \). Applying [23, II.A.1, §1 and §2] to (2.22), we obtain the following commutative diagram:

\[
\begin{array}{cccc}
0 & \rightarrow & H^2(k_0, C) & \rightarrow & H^2_{\mathcal{A}}(V, C) & \rightarrow & H^1(k_0(C(-1))) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^2(k, C) & \rightarrow & H^2(K_t, C) & \rightarrow & H^1(k, C(-1)) & \rightarrow & 0.
\end{array}
\]
The lower row is nothing else then the residue sequence (2.4) for the field $K_t$. The vertical arrows are the restriction homomorphisms from Galois cohomology.

Let $\phi : V \to H_r(G)$ be the morphism given by Lemma 2.19 and $\omega_V := \phi^*\tilde{\omega} \in H^2_c(V, C)$ the pull back of the global f.o.d.-obstruction $\tilde{\omega} \in H^2_c(H_r(G), C)$, see Remark 1.3 (iii) and 25. By functoriality and (2.21), the morphism $H^2_c(V, C) \to H^2(K_t, C)$ in (2.23) maps $\omega_V$ to $\omega_t$, the f.o.d.-obstruction for $f_t$. Therefore, Part (i) of Theorem 2.21 follows from the commutativity of diagram (2.24).

Let $K_a$ denote the completion of $K$ with respect to the discrete valuation $v_a$ and $K_a^u$ its maximal unramified extension. We may identify $K_a$ with $k((t-a))$ and $K_a^u$ with $k^s((t-a))$. Moreover, we may identify $\text{Gal}(K_a^u/K_a)$ with $\Gamma_k$, the absolute Galois group of $k$. Since $v_a$ is unramified in the extension $L/K$, there exists an embedding $L \hookrightarrow K_a^u = k^s((t-a))$. Explicitly, we choose a compatible system $a^{1/n} \in k^s$ of $n$th roots of $a \in k$ (where $n$ runs over the integers prime to $p$), and define $L \hookrightarrow K_a^u$ by

\[
t^{1/n} \mapsto a^{1/n}(1 + \frac{t-a}{na})^{1/n} = a^{1/n}(1 + \frac{t-a}{na} + \cdots).
\]

Let $q_0 \in \hat{Z}'(1)$ be a topological generator, corresponding to a compatible system $(\zeta_n)$ of $n$th roots of unity, where $p \nmid n$. Let $\hat{Z}'(1) \hookrightarrow \Gamma_k$ be the natural morphism; it sends $q_0$ to the automorphism of $k$ which maps $\pi t^{1/n}$ to $\zeta_n \pi t^{1/n}$, where $\pi$ is a uniformizer of $k$. Similarly, let $\hat{Z}'(1) \hookrightarrow \pi_1(V) = \text{Gal}(L/K)$ be the morphism that sends $q_0$ to the automorphism of $L$ which maps $t^{1/n}$ to $\zeta_n t^{1/n}$. It follows from (2.24) that the restriction homomorphism $\Gamma_k^u \to \text{Gal}(L/K) = \pi_1(V)$ induces a the diagram

\[
\begin{array}{cccccc}
1 & \to & \hat{Z}'(1) & \to & \Gamma_k & \to & \Gamma_{k_0} & \to & 1 \\
 & v(a) & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \hat{Z}'(1) & \to & \pi_1(V) & \to & \Gamma_{k_0} & \to & 1,
\end{array}
\]

where the left vertical arrow sends $q_0$ to $q_0^{v(a)}$. The vertical arrow on the right is clearly the identity. We apply again II.A.1, §1 and §2, this time to (2.25), and we obtain the following diagram.

\[
\begin{array}{cccccc}
0 & \to & H^2(k_0, C) & \to & H^2_c(V, C) & \xrightarrow{\bar{r}_t} & H^1(k_0, C(-1)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^2(k_0, C) & \to & H^2(k, C) & \xrightarrow{r_a} & H^1(k_0, C(-1)) & \to & 0.
\end{array}
\]

The upper row is the same as in (2.23), the lower row is the residue sequence for the field $k$, see (2.3). The left and middle vertical arrows are the restriction homomorphism from Galois cohomology; in particular, the left vertical arrow is the identity. Checking the definition of the residue map (see the proof of Proposition 2.21), one finds that the vertical arrow on the right is multiplication by $v(a)$ and that the diagram (2.26) commutes. This proves Part (ii) and (iii) of the theorem.

**Example 2.21.** Let $f : Y \to \mathbb{P}^1$ be the $A_5$-covers with branch locus $S = \{\pm \sqrt{3} \pm \sqrt{17}\}$ and Hurwitz description $|g| = \{g_1, g_1^{-1}, g_3, g_3^{-1}\} \in \text{Ni}^u(3A^4)$ considered in Proposition 2.17, Case 1. We showed that $\mathbb{Q}((t))$ is a field of moduli for $f$ and
that \( r_t(\omega) = (-d) \in \mathbb{Q}^*/\mathbb{Q}^{*2} \), where \( \omega \in H^2(\mathbb{Q}(t), C) \) is the f.o.d.-obstruction and \( r_t \) denotes the residue map corresponding to the valuation with parameter \( t \).

Let \( p > 5 \) be a prime and assume that \( p \nmid d \). Choose an embedding of \( \mathbb{Q} \) into \( \bar{\mathbb{Q}}_p \). We will consider \( f \) as a cover over \( \mathbb{Q}_p((t)) \), with field of moduli \( \mathbb{Q}_p((t)) \). The discriminant of \( S \) is

\[
\delta(S) = (2^{12}(d^2 + 2dt + t^2)d^2t^2) = (t^2) \vartriangleleft \mathbb{Z}_p[[t]],
\]

and Condition 2.18 holds. We may therefore specialize \( f \) at \( t = p \), and we obtain an \( A_5 \)-cover \( f_p : Y_p \to \mathbb{P}^1 \) over \( \bar{\mathbb{Q}}_p \), with branch locus \( S_p = \{ \pm \sqrt{d} \pm \sqrt{p} \} \) and field of moduli \( \mathbb{Q}_p \). Since \( p \nmid d \), the residue \( r_1(\omega) = (-d) \in H^1(\mathbb{Q}_p, \pm 1) \) is unramified, as predicted by Part (i) of Theorem 2.20. By Part (ii), \( r_p(\omega_p) = (-d) \in H^1(\mathbb{F}_p, \pm 1) = \mathbb{F}_p^*/\mathbb{F}_p^{*2} \), where \( \omega_p \in H^2(\mathbb{Q}_p, \pm 1) = \text{Br}_2(\mathbb{Q}_p) \) is the f.o.d.-obstruction for \( f_p \). In terms of the norm residue symbol, we have

\[
\omega_p = (-d, -p)_p = \begin{cases} 
-1 & \text{if } -d \equiv x^2 \pmod{p}, \\
1 & \text{if } -d \not\equiv x^2 \pmod{p}.
\end{cases}
\]

It is not hard to extend the computations of Section 2.4 to show that the f.o.d.-obstruction \( \omega_t \in H^2(\mathbb{Q}(t), \pm 1) = \text{Br}_2(\mathbb{Q}(t)) \) is given by the Hilbert symbol \((-d, -t)\). Thus, to specialize \( \omega_t \) to \( \omega_p \), one had to “plug in” \( t = p \).

References

1. P. Bailey and M. D. Fried, Hurwitz monodromy, spin separation and higher levels of a modular tower, this volume.
2. S. Beckmann, Ramified primes in the field of moduli of branched coverings of curves, J. of Algebra 125 (1989), 236–255.
3. K.S. Brown, Cohomology of groups, Graduate texts in Mathematics, no. 87, Springer-Verlag, 1994.
4. K. Coombes and D. Harbater, Hurwitz families and arithmetic Galois groups, Duke Math. J. 52 (1985), no. 4, 821–839.
5. P. Dèbes and J.-C. Douai, Algebraic covers: Field of moduli versus field of definition, Annales Sci. E.N.S. 30 (1997), 303–338.
6. Local-global principles for algebraic covers, Israel J. Math. 103 (1998), 237–257.
7. P. Dèbes, J.-C. Douai, and M. Emsalem, Familles de Hurwitz et cohomologie non abélienne, Ann. Inst. Fourier 50 (2000), no. 1, 113–149.
8. P. Dèbes and D. Harbater, Fields of definition of \( p \)-adic covers, J. reine angew. Math. 498 (1998), 223–236.
9. M. Dettweiler, Plane curve complements and curves on Hurwitz spaces, Preprint, 2000.
10. M. D. Fried and H. Völklein, The inverse Galois problem and rational points on modular spaces, Math. Ann. 290 (1991), 771–800.
11. A. Fröhlich, Orthogonal representations of Galois groups, Stiefel-Whitney classes and Hasse-Witt invariants, J. Reine Angew. Math. 360 (1985), 84–123.
12. L. Gerritzen, F. Herrlich, and M. van der Put, Stable \( n \)-pointed trees of projective lines, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen 91 (1988), 131–163.
13. A. Grothendieck, Revêtement étalés et groupe fondamental (SGA I), LNM, no. 224, Springer-Verlag, 1971.
14. B. Huppert, Endliche Gruppen I, Grundlehren, no. 134, Springer-Verlag, 1967.
15. Y. Ihara, On the embedding of \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) into \( \text{GT} \), The Grothendieck Theory of Dessins d’Enfants (Leila Schneps, ed.), Lond. Math. Soc. Lecture Note Series, no. 200, Camb. Univ. Press, 1994, pp. 289–321.
[16] Y. Ihara and M. Matsumoto, *On Galois actions on profinite completions of braid groups*, Proceedings AMS-NSF Summer Conference, vol. 186, 1995, Cont. Math series, Recent Developments in the Inverse Galois Problem, 173–200.
[17] G. Malle and B. H. Matzat, *Inverse Galois theory*, Monographs in Mathematics, Springer, 1999.
[18] H. Nakamura, *Limits of Galois representations in fundamental groups along maximal degeneration of marked curves*, I. Amer. J. Math. 121 (1999), 315–358.
[19] F. Pop, *Riemann Existence Theorem with Galois action*, Algebra and Number Theory (G. Frey and J. Ritters, eds.), W. de Gruyter, Berlin, 1994, pp. 193–218.
[20] M. Saïdi, *Revêtements modérés et groupe fondamental de graphe de groupes*, Comp. Math. 107 (1997), 321–340.
[21] J.-P. Serre, *Relèvements dans \( \hat{\mathbb{A}}_n \)*, C. R. Acad. Sci. Paris 311 (1990), 477–482.
[22] J.-P. Serre, *Topics in Galois theory*, Research notes in mathematics, 1, Jones and Bartlett Publishers, 1992, Lecture notes prepared by Henri Darmon.
[23] ———, *Cohomologie Galoisienne*, 5. ed., LNM, no. 5, Springer-Verlag, 1994.
[24] H. Völklein, *Groups as Galois groups*, Cambridge Studies in Adv. Math., no. 53, Cambridge Univ. Press, 1996.
[25] S. Wewers, *Construction of Hurwitz spaces*, Thesis, Preprint No. 21 of the IEM, Essen, 1998.
[26] ———, *Deformation of tame admissible covers of curves*, Aspects of Galois Theory (H. Völklein, ed.), London Math. Soc. Lecture Note Series, no. 256, Cambridge Univ. Press, 1999, pp. 239–282.
[27] ———, *Rational boundary points on Hurwitz spaces*, preprint 1999, at http://www.math.upenn.edu/~wewers.

UNIVERSITY OF PENNSYLVANIA

E-mail address: wewers@math.upenn.edu