An exact standard radiation-dominated era solution residing in higher-order gravity

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Abstract

We show that the standard evolution of radiation-dominated era (RDE) universe $a \propto t^{1/2}$ is a sufficient condition to solve a sixth order gravitational field equation derived from the Lagrangian containing $BR_{ab}R_{ab} + CRR_{c}^{c}$ as well as a polynomial $f(R)$ for a spatially flat radiation FLRW universe. By virtue of the similarity between $R_{ab}R_{ab}$ and $R^2$ models up to the background order and of the vanishing property of $R_{c}^{c}$ for $H = 1/(2t)$, the analytical solution can be obtained from a special case to general one. This actually proves that, even within modified gravitational theory involving higher-order terms, the standard cosmic evolution is valid in Einstein gravity with the higher-order terms, which are generally expected to have small impacts. An application of this background solution to the tensor-type perturbation reduces the complicated equation to the standard second order equation of gravitational wave.

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I. INTRODUCTION

Even though Einstein’s general relativity (GR) has successfully passed the observational tests in the solar system scale, one of efforts to generalize GR for cosmology is introducing additional higher-order derivative terms due to both theoretical and phenomenological reasons. For instance, sixth order $R \square R$ \[1\] as well as fourth order $R^{ab}R_{ab}$ terms \[2-4\] are considered here with the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi} (f(R) + BR^{ab}R_{ab} + CR\square R) + L_m \right],$$

(1)

where $f(R)$ is a polynomial function of the Ricci scalar $R$

$$f(R) = \sum_{n=1}^{N} A_{(n)} R^n = R + A_2 R^2 + A_3 R^3 + ..., \quad A_{(1)} \equiv 1, \quad A_{(2)} \equiv A,$$

(2)

$A_{(n)}, B, C$ are constants, $R_{ab}$ is the Ricci tensor. The d’Alembertian of $R$ is $\square R \equiv g^{ab} R_{,a}^{\ ,b}$ where the semicolon denotes covariant derivative, and the matter part Lagrangian is defined as $\delta (\sqrt{-g} L_m) \equiv \frac{1}{2\sqrt{-g}} g^{ab} T_{(m)}^{ab} \delta g_{ab}$. In this paper we follow the Hawking-Ellis [5] convention, but adopt (Planckian) natural units $\hbar \equiv 1 \equiv G_N \equiv k_B$. As special cases of general $f(R)$ gravity [6-8], $N = 1$ and $N = 2$ in Eq. (2) correspond to the gravity theory of Einstein and Starobinsky [9], respectively. The $R^2$ theory is not only favored by Planck Collaboration [10] as an inflationary model that predicts successfully some observables such as the spectral index and tensor-to-scalar ratio, but also recently proposed as scalaron dark matter model that estimates its mass [11]. Moreover, there have been attempts to extend the $R^2$ theory. The model with the next $N = 3$ term as a small contribution to $N = 2$ theory was also studied [12-14], recently, $N = 4$ case was also investigated [15]. A fourth order $BR^{ab}R_{ab}$ theory, that is neither of $f(R)$ model nor conformally equivalent to Einstein gravity, was also introduced in the literature and textbooks [2, 6, 16]. For the physical meaning or motivation of $CR\square R$ theory, e.g., a conformal equivalence to two interacting scalar fields causing inflation, we refer to Ref. [1]. According to a review for higher order gravity theory [17], $\square R$ model was pioneered by Buchdahl [18] (1951), and quantum gravitational higher order corrections to Einstein-Hilbert action was the idea of Sakharov [19] (1967), prior to Starobinsky. The case including $N = 3$ and $CR\square R$ without the $B$-term for an inflationary regime was studied using phase diagrams and conformal transformation [12].

In cosmology, modified gravity models are usually applied to the inflationary epoch or to the present age in order to describe the accelerated expansion of the universe. The effects
of \( f(R) \) gravity was considered also in the radiation-dominated era (RDE) for the study of baryogenesis \[20\] or big bang nucleosynthesis \[21, 22\]. In particular, besides the numerical solutions for the given gravity models, a standard solution for a scale factor describing an evolution of RDE Friedmann-Lemaître-Robertson-Walker (FLRW) universe was found for a theory with a generic Lagrangian containing almost arbitrary function of \( R \) \[23\]. In this paper, we show that a standard RDE solution is still viable for the gravity models involving the fourth or sixth order differential equations, by the \( B \) and \( C \) terms in Eq. (7) which were proposed in the previous literature. This actually proves that, even within modified gravitational theory involving higher-order terms, the standard cosmic evolution is valid in Einstein gravity with the higher-order terms, which are generally expected to have small impacts.

In Section II, we introduce the generalized Einstein field equation for our model. In Section III, starting with some standard cosmological assumptions, we apply the gravity models from specific one with basic power-law ansatz of the scale factor to general case with simple logic and find a common standard RDE solution in Eq. (19), which is our main result. The final section is for the mathematical conclusion and discussions about the assumptions, etc.

II. GRAVITATIONAL FIELD EQUATIONS

The gravitational field equation (GFE) \[1, 2, 12\] from the metric variation \[24, 25\] of the action (1) is

\[
\left[ g_{ab} \left( \frac{\partial f}{\partial R} \right)^c \right]_{,c} - \left( \frac{\partial f}{\partial R} \right)_{,ab} R_{ab} - \frac{1}{2} f g_{ab} - 8\pi \left( T^{(B)}_{ab} + T^{(C)}_{ab} \right) = 8\pi T^{(m)}_{ab},
\]

where

\[
T^{(B)}_{ab} = \frac{B}{8\pi} \left( \frac{1}{2} R^{cd} R_{cd} g_{ab} + R_{ab} - 2 R^{cd} R_{acbd} - \frac{1}{2} g_{ab} R_{cd}^{;c} - R_{ab}^{;c} \right),
\]

\[
T^{(C)}_{ab} = \frac{C}{8\pi} \left[ 2 R^{ce}_{;cab} - 2 R_{ab} R^{ce}_{;c} + R_{a} R_{b} - g_{ab} \left( 2 R^{ce}_{;d} R_{c}^{d} + \frac{1}{2} R^{ce} R^{;c} \right) \right].
\]

For a derivation of the GFE, we use the following relation

\[
\delta (\Box R) = -\delta g_{ab} R^{abc}_{;c} - \delta g_{ab} R^{ab}_{;c} - g^{ab} \delta g_{abc} R^{cd}_{;d} + \delta g_{ab}^{;bac} - R^{ab}_{;c} \delta g_{ab}^{;c} - R^{a} R^{bc}_{;b} + \frac{1}{2} R^{ce} g_{ab} \delta g_{abc},
\]
referring to the helpful work by Barth and Christensen [25]. The contraction of the GFE in Eq. (3) with an inverse metric $g^{ab}$ will be useful for RDE cosmology, which is given as

$$\left[3\left(\frac{\partial f}{\partial R}\right)^{\alpha}_{c} + \frac{\partial f}{\partial R} R - 2f\right] + 2BR^{c}_{d} + C\left[6R^{c}_{d} R^{d}_{c} + 2RR^{c}_{c} + R^{c} R_{c}\right] = 8\pi T^{(m)c}_{c}. \tag{7}$$

For later convenience, we also introduce an alternative equivalent form of the GFE after arranging the directly varied GFE in Eq. (3) [2, 26]

$$G_{ab} = \frac{1}{F}\left[8\pi T^{(m)ab}_{ab} + F_{ab} - g_{ab}\left(F^{c}_{c} - \frac{f - RF}{2}\right) + C\left(R_{a}R_{b} - \frac{1}{2}g_{ab}R^{c}R_{c}\right) + B\left(\frac{2}{3}RR_{ab} + \frac{1}{2}R_{ab}^{cd}g_{cd} + \frac{1}{3}R_{ab} - 2R_{abcd} + \frac{1}{6}g_{ab}R^{c}_{c} - R_{ab}^{c}_{c}\right)\right], \tag{8}$$

$$F \equiv f_{,R} + \frac{2}{3}BR + 2CR^{c}_{c}, \quad f_{,R} \equiv \frac{\partial f}{\partial R}, \tag{9}$$

and the trace of Eq. (8)

$$-R = \frac{1}{F}\left[8\pi T^{(m)c}_{c} - 3F^{c}_{c} + 2f - 2FR + \frac{2}{3}BR^{2} - CR^{c}R_{c}\right]. \tag{10}$$

### III. BACKGROUND UNIVERSE DURING THE RDE

We adopt a spatially flat FLRW metric (for a detailed review, see Ch.13 of Weinberg’s textbook [24]) representing a homogeneous and isotropic background universe

$$ds^2 = a^2(\eta)\left(-d\eta^2 + \delta_{\alpha\beta}dx^\alpha dx^\beta\right), \tag{11}$$

where $a(\eta)$ is the cosmic scale factor, $\eta$ is the conformal time ($dt \equiv ad\eta$), and $\dot{a}/a \equiv \frac{1}{a} \frac{da}{dt} \equiv H$ is the Hubble expansion rate. We adopt a perfect fluid in standard cosmology, whose energy-momentum tensor is composed of time-dependent energy density $\mu(t)$ and pressure $p(t)$

$$T^{(m)0}_{0} = -\mu(t), \quad T^{(m)0}_{\alpha} = 0, \quad T^{(m)\alpha}_{\beta} = p(t)\delta^{\alpha}_{\beta}. \tag{12}$$

In the RDE the cosmological constant is negligible and the equation of state (EoS) for radiation-like is simply given with the RDE density $\mu$

$$p = \frac{1}{3}\mu. \tag{13}$$
A. Case $N = 2, C = 0$

In this subsection, a special case with $f(R) = R + AR^2$ (Starobinsky model of $f(R)$ gravity) and $C = 0$ (no sixth order gravity) is considered. Then the temporal and spatial components of the GFE in Eq. (3), respectively, become

\begin{align*}
8\pi\mu &= 3H^2 + 6(3A + B)(2\dot{H}H - \dot{H}^2 + 6\ddot{H}H^2), \quad (14) \\
8\pi p &= -(2\dot{H} + 3H^2) - 2(3A + B)
\left(2\frac{d^3H}{dt^3} + 12\ddot{H}H + 9\dot{H}^2 + 18\dot{H}H^2\right), \quad (15)
\end{align*}

where it is notable that $A$- and $B$-terms play qualitatively the same role as far as the evolution of the background universe is concerned. The modified Friedmann equation in Eq. (14) and Eq. (15) can be checked by substituting them into the following continuity equation:

\[ \dot{\mu} + 3H(\mu + p) = 0. \quad (16) \]

A system of three independent ordinary differential equations (ODEs) in Eqs. (14), (15) (or (16)), and (13), including 3 unknown functions, $a(t), \mu(t)$, and $p(t)$. Putting Eqs. (14) and (13) into Eq. (16), the set of ordinary differential equations can be reduced as an ODE for $H(t)$ that has various solutions:

\[ \dot{H} + 2H^2 + 2(3A + B)
\left(\frac{d^3H}{dt^3} + 7\ddot{H}H + 4\dot{H}^2 + 12\dot{H}H^2\right) = 0. \quad (17) \]

This is a nonlinear third order ODE for $H(t)$ (fourth order eq. for $a(t)$); however, an exact analytical solution can be easily found by a power-law ansatz:

\[ a(t) \propto t^\alpha, \quad H(t) = \frac{\alpha}{t}, \quad \dot{H} = -\frac{\alpha}{t^2}, \quad \ddot{H} = 2\frac{\alpha}{t^3}, \quad \frac{d^3H}{dt^3} = -6\frac{\alpha}{t^4}. \quad (18) \]

Among two mathematical solutions $\alpha = 1/2$ or $\alpha = 0$ for Eq. (17), the former as a standard is selected rather than a non-expanding solution.

Consequently, the flat FLRW universe model under $R + AR^2$ as well as $BR_{ab}R_{ab}$ gravity has a standard RDE solution

\[ a(t) = (const)t^{1/2}, \quad H(t) = \frac{1}{2t}, \quad \dot{H}(H) = -2H^2, \quad (19) \]

where the third one describes a flipped parabola (a 2D phase diagram) on a $H-\dot{H}$ plane in which only quadrant IV is physically acceptable. We can also notice that the RDE solution (19) is still viable in more generalized models such as any polynomial $f(R)$ or a sixth order gravity.
B. Case $B = 0 = C$, any natural number $N$

For a polynomial $f(R)$ (Eq. (2)) model setting $B = C = 0$, we use the same method with the previous case to get an ODE for $H(t)$ whose counterpart is Eq. (17)

$$
\left[ \frac{\dot{R}}{2} - 3(\dot{H} + 2\dot{H}H) \right] f_{,R} - 3(\dot{H} + H^2) \frac{d}{dR} f_{,R} + 3\dot{R} \frac{d}{dR} \left( H \dot{R} \frac{d}{dR} f_{,R} \right) + 4H \left[ \frac{\dot{f}}{2} - 3(\dot{H} + H^2) f_{,R} + 3H \dot{R} \frac{d}{dR} f_{,R} \right] = 0,
$$

(20)

where the Ricci scalar $R$ from the metric in Eq. (11) and its time derivative are, respectively

$$
R = 6(\dot{H} + 2H^2), \quad \dot{R} = 6(\ddot{H} + 4\dot{H}H),
$$

(21)

and $\dot{R} \frac{d}{dR} X(t) = \frac{d}{dt} X(t)$. Substituting Eq. (18) into Eq. (21), one can obtain

$$
R(t) = \frac{6\alpha(2\alpha - 1)}{t^2}.
$$

(22)

This result simply shows how the standard RDE solution in Eq. (19) from the Einstein gravity ($N = 1$) is also a solution of the GFE in Eq. (20) involving any polynomial $f(R)$ gravity because the condition of $\alpha = 1/2$ yields

$$
R = 0 = \dot{R} = \ddot{R} = f, \quad f_{,R} = 1.
$$

(23)

According to Barrow and Ottewill [23], the standard RDE evolution given in Eq. (19) is a solution of not only a polynomial $f(R)$, but also any $f(R)$ theory in which $f(0) = 0$ and $f_{,R}(0) \neq 0$. Those exceptional examples are $f(R) \sim R^{-N}, \ln R$, and so on.

Meanwhile the trace in Eq. (7) in this case ($B = C = 0$) is useful [6], especially for the perfect fluid governed by the RDE EoS in Eq. (13),

$$
\left[ 3 \left( \frac{\partial f}{\partial R} \right)^c + \frac{\partial f}{\partial R} R - 2f \right] = T^{(m)c} = 0.
$$

(24)

This differential equation equivalent to Eq. (20) within the flat FLRW model tells us [6] that $f(R)$ theories have more various solutions (the exact solution (19) is just one of them) than Einstein’s theory ($f = R$) relating $R$ with $T^{(m)c}$ not differentially but algebraically, and that the function (19) is not necessary but sufficient to be a solution of the ODE in Eq. (24).
C. General case

In analogous to the previous cases, it is now easy to see the trace in Eq. (10) (or Eq. (7)) for a RDE flat FLRW universe

\[-R = \frac{1}{F}[-3F^e_c + 2f - 2FR + \frac{2}{3}BR^2 - CR^e_c R_{;e}].\] (25)

has an analytical solution of \(H(t) = 1/(2t)\) because the solution \((\alpha = 1/2)\) implies that

\[R = 0 = R^c_{;e} = -(\ddot{R} + 3H\dot{R}) = f = \dot{R}, \quad F \equiv f_{;R} + \frac{2}{3}BR + 2CR^e_c = 1 = f_{;R}.\] (26)

The traced Eq. (25) is a complicated fifth order ODE for \(H(t)\) admitting various solutions, but the standard RDE solution is a sufficient condition to satisfy Eq. (25) regardless of the constants \(A_{(n \geq 2)}, B,\) and \(C\). Among other various numerical solutions of Eq. (25) depending on those constants, it would be an intriguing problem which one could be a candidate as a physical solution describing the evolution of a RDE universe affected by the modified gravity models.

IV. CONCLUSION AND DISCUSSIONS

We conclude that the standard RDE solution of \(a(t) \propto t^{1/2}\) obtained from the Einstein gravity \((N = 1)\) is also a RDE solution in a spatially flat FLRW universe filled with perfect fluid under a generalized gravity model whose Lagrangian is \[
\frac{1}{16\pi} [\sum_{n=1}^{N} A_{(n)} R^n + BR^a_b R_{ab} + CR^e_c + L_m].
\]
Besides the polynomial \(f(R)\) theory \((A_{(n)}\)-terms) as a subset of more general \(f(R)\) gravity \([23]\), the fourth order \(BR^a_b R_{ab}\) (\(B\) gravity) and sixth order \(CRR^e_c\) (\(C\) gravity) theories in the FLRW model also allow the same solution in Eq. (19) by virtue of the qualitative sameness of the \(R^a_b R_{ab}\) with the \(R^2\) theory (see Eqs. (14, 15, 17)) and of the vanishing property of \(\Box R\) (Eq. (26)) with the solution. This analytical solution can be some fiducial curves (regardless of the coefficients \(A_{(n \geq 2)}, B,\) and \(C\)) that can test the numerical computations to find other numerical solution curves (depending on the coefficients).

At first we made efforts to obtain some numerical RDE solutions of Eq. (17) instead of the exact analytical solution in the special case \((R^a_b R_{ab}\) model added to Starobinsky theory for \(N = 2)\), we inductively have a hypothesis that the solution in Eq. (19) is also a solution in any polynomial \(f(R)\) model and deductively tried to prove it, to which the discovery for general \(f(R)\) gravity by Barrow and Ottewill was prior \([23]\).
Even if terms involved by $A$ and $B$ representing the four-derivative theory in the action Eq. (1) share the same solution at the background order in the homogeneous and isotropic model, the two terms behave quite differently in the perturbed universe with gravitational wave [2, 27] or density fluctuation [3]. A model with an action containing a general function of a contracted quantity $R_{ab} R_{ab}$ [28] is not likely to have the same RDE solution in Eq. (19) since the quantity $R_{ab} R_{ab} = 12(\dot{H}^2 + 3\dot{H}H^2 + 3H^4)$ is non-vanishing even if $H = 1/(2t)$.

While the generic $f(R)$ and $RR^{c_c}$ gravity can be conformally transformed into Einstein gravity with single minimally coupled scalar field (MSF) [4, 6, 29, 30] and with two interacting MSFs [1] respectively, $R_{ab} R_{ab}$ gravity does not have such a symmetry [2, 3].

How about the standard matter-dominated era (MDE) solution $a(t) \propto t^{2/3}$? This MDE solution from the Einstein gravity is hardly an exact solution in a polynomial $f(R)$ gravity as well as in the $B$ or $C$ gravity unless the effects of the $A_{(n \geq 2)}$, $B$ and $C$ theories are small enough during the MDE because $R(t)$ cannot become zero with $\alpha = 2/3$. However, if the universe evolved like Eq. (18) during the RDE, the correction terms from the $B$ or $C$ gravity might decay enough to converge to the standard curve $a(t) \propto t^{2/3}$. The MDE in $f(R)$ gravity was investigated in Ref. [31]. It is also introduced that semiclassical analysis of the Wheeler-DeWitt equation describing the universe before the inflationary epoch allows a radiation-like solution $a(t) \propto \sqrt[4]{t}$ [32].

One of our main assumptions is a spatially flat FLRW universe (three space curvature $K = 0$), beyond which our argument is unlikely established since the Ricci scalar

$$R = 6(\dot{H} + 2H^2 + \frac{K}{a^2})$$

from the FLRW metric including the closed or open universe model [33, 36]

$$ds^2 = a^2\left(-d\eta^2 + \frac{dr^2}{(1-Kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right),$$

is non-vanishing with $a(t) \propto t^{1/2}$ unless $K/a^2$ term is negligible. For simplicity, our arguments are based on the homogeneous and isotropic assumption, so called the cosmological principle. However, there is a debate on the principle [37, 39] and researchers [40, 41] introduce alternative metrics beyond FLRW, e.g. to explain the current accelerating expansion of the universe. Comparison between $A$ and $B$ models using those metrics may be another issue.

Cosmological perturbations for a case for the $f(R)$ and $RR^{c_c}$ gravity ($B = 0 = L_m$ and general $f(R)$ in Eq. (11)) were investigated [26], where the arranged GFE in Eq. (8) is more
convenient than the directly varied GFE in Eq. (3) since each component of the Einstein tensor $G^a_b$ up to the linearly perturbed order is already calculated and listed, e.g. in Ref. [42]. The equation for a tensor-type (tracefree and transverse) perturbation variable $C^{(t)}_{\alpha\beta}$ in Fourier space for the model was derived as

$$
\ddot{C}^{(t)}_{\alpha\beta} + \left(3H + \frac{\dot{F}}{F}\right)\dot{C}^{(t)}_{\alpha\beta} + \frac{k^2 + 2K}{a^2}C^{(t)}_{\alpha\beta} = 0,
$$

(29)

where $F = f_{,R} + 2CR^c_c$, and $k$ is the wavenumber of the perturbations. Interestingly, this complicated gravitational wave (GW) equation can be simplified to a standard form when the evolution of a flat background universe is described by the solution in Eq. (19) and $f_{,R} = 1$ during the relating era so that $F$ becomes unity. Further, the simplified GW equation can become the Bessel equation whose exact solutions are known [26, 42–44].

Future investigations with a part of this higher-order Lagrangian would be e.g. finding other kinds of physical RDE evolution of the universe beyond the standard solution found here, and trying applications to physics of neutron star(s), to alternative cosmological metrics or to quantum cosmology where modified gravity effects may be significant.

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