Thermodynamic Bethe Ansatz for the
Spin-1/2 Staggered XXZ- Model

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Abstract

We develop the technique of Thermodynamic Bethe Ansatz to investigate the ground state and the spectrum in the thermodynamic limit of the staggered XXZ models proposed recently as an example of integrable ladder model. This model appeared due to staggered inhomogeneity of the anisotropy parameter $\Delta$ and the staggered shift of the spectral parameter. We give the structure of ground states and lowest lying excitations in two different phases which occur at zero temperature.
1 Introduction

The quasi-one-dimensional spin ladder models have attracted much attention in recent years; besides the mathematical interest, these models are believed to describe many real magnetic materials invented relatively recently [1].

In connection with this the construction and exact solution of exactly integrable ladder models became very actual. In particular the Bethe Ansatz exact solution can provide a deep insight into the properties of these models. Recently in a series of articles [2, 3, 4, 5], it have been proposed a technique for constructing integrable models where the model parameter $\Delta$ (the anisotropy parameter of the $XXZ$ and anisotropic $t-J$ models) have a staggered disposition of the sign along both chain and time directions. Due to the staggered shift of the spectral parameter also considered there, these models have led to Hamiltonians formulated on two leg zig-zag quasi-one dimensional chains.

The technique is based on an appropriate modification of the Yang-Baxter equations ($YBE$) ([6, 7, 8]) for the $R$-matrices which are the conditions for commutativity of the transfer matrices at different values of the spectral parameter. The transfer matrix is defined as the staggered product of $R$-matrices, which have staggered sign of the anisotropy parameter $\Delta$ and staggered shift of the spectral parameter of the model along the chain. The shift of spectral parameter in a product of $R$-matrices was considered in earlier works of several authors [9, 10], by use of which they have analyzed the ordinary $XXZ$ model in the infinite limit of the spectral parameter. Later, in the article [11], this modified transfer matrices has been used in order to construct integrable models on the ladder.

Various integrable models on ladders were also constructed in articles [12, 13, 14, 15, 16]. The above described construction essentially differs from those due to the inhomogeneity of anisotropy parameter along the chain.

The subject of present paper is to study the exactly soluble spin-$1/2$ two-leg zig-zag ladder model proposed in [2], in the thermodynamic limit. We use the technique of Thermodynamic Bethe Ansatz developed by M. Gaudin [25] and M.Takahashi and M.Suzuki [26] for the Heisenberg $XXZ$ model. This method is based on the study of Bethe Ansatz equations (BAE). Another powerful method in theoretical study of spin ladder models which we partially use is the bosonization method [17, 18, 19]. The bosonization have been used very often to find out and to understand many physical aspects of these systems. This method yields a low-energy effective field theory of a spin system, which can be treated by powerful methods of quantum field theory.

In the model under consideration, the $R$-matrix is a product of two samples of $XXZ$ $R$-matrices with the staggered sign of anisotropy parameter $\Delta$ and the shift of spectral parameter by a new model parameter. The Hamiltonian

$$
H = \sum_{j=1}^{N} \sum_{s=0,1} \left\{ (-1)^s/2 \left[ \sigma_{j,s}^{x} \sigma_{j+1,s}^{x} + \sigma_{j,s}^{y} \sigma_{j+1,s}^{y} - \sigma_{j,s}^{z} \sigma_{j+1,s}^{z} \right] + J_1 \left( -1 \right)^s \left[ \sigma_{j,s}^{+} \left( \sigma_{j,s+1}^{+} \sigma_{j+1,s}^{-} - \sigma_{j,s+1}^{-} \sigma_{j+1,s}^{+} \right) - \sigma_{j+1,s}^{+} \left( \sigma_{j,s}^{+} \sigma_{j,s+1}^{-} - \sigma_{j,s}^{-} \sigma_{j,s+1}^{+} \right) \right] - J_2 \left[ \sigma_{j,s}^{+} \sigma_{j,s+1}^{+} \sigma_{j+1,s}^{-} - \sigma_{j,s}^{-} \sigma_{j+1,s}^{+} \right] + \mu_0 H \sigma_{j,s} \right\} ;
$$

$$
\sigma_{N+1,s} = \sigma_{1,s}, \quad \sigma_{j,2} = \sigma_{j+1,0}, \quad J_1 = \frac{\sin \lambda}{\sin \theta}, \quad J_2 = \tan \lambda \cot \theta.
$$
describes the interaction of $2N$ spins located on the sites of zig-zag ladder (see Fig.1), in the external "magnetic" field $H$ along the $z$-axis. Parameterizations of coupling constants $J_1$ and $J_2$ are given by the two combinations

1. $\lambda \sim \text{real}, \quad \theta \sim \text{imaginary}, \quad \text{then} \quad 0 \leq |J_1| \leq |J_2| \leq \infty,$

2. $\lambda \sim \text{imaginary}, \quad \theta \sim \text{real}, \quad \text{then} \quad 0 \leq |J_2| < |J_1| \leq \infty.$

which fix the Hamiltonian to be Hermitian and give all the possible (imaginary) values of $J_1$ and $J_2$.

By construction, this model can be solved exactly through the Bethe Ansatz technique \cite{2}. Starting from the ferromagnetic eigenstate with all spins up, each $M$-particle eigenstate is obtained by adding $M$ magnons which are parameterized by unequal complex rapidities $x_1, \ldots, x_M$; the roots of the Bethe Ansatz equations \cite{2}

$$\left[ \frac{\sinh \left( \frac{\lambda}{2}(x_k - i\frac{\theta}{\lambda} - i) \right) \sinh \left( \frac{\lambda}{2}(x_k + i\frac{\theta}{\lambda} - i) \right)}{\sinh \left( \frac{\lambda}{2}(x_k - i\frac{\theta}{\lambda} + i) \right) \sinh \left( \frac{\lambda}{2}(x_k + i\frac{\theta}{\lambda} + i) \right)} \right]^N = -\prod_{i=1}^{M} \frac{\sinh \left( \frac{\lambda}{2}(x_k - x_i - 2i) \right)}{\sinh \left( \frac{\lambda}{2}(x_k - x_i + 2i) \right)}.$$

The corresponding state has total spin projection $S_z = N - M$ and energy

$$E(x_1, \ldots, x_M) = \frac{\sin(\lambda + \theta) \sin(\lambda - \theta) \sin \lambda}{\sin^2 \theta \cos \lambda} \times$$

$$\sum_{j=1}^{M} \left\{ \frac{\sin \lambda}{\sinh \left( \frac{\lambda}{2}(x_j + i\frac{\theta}{\lambda} - i) \right) \sinh \left( \frac{\lambda}{2}(x_j + i\frac{\theta}{\lambda} + i) \right)} - \frac{\sin \lambda}{\sinh \left( \frac{\lambda}{2}(x_j - i\frac{\theta}{\lambda} - i) \right) \sinh \left( \frac{\lambda}{2}(x_j - i\frac{\theta}{\lambda} + i) \right)} \right\}.$$
In section 3, the spectral equations are considered in zero “magnetic” field $H = 0$. We have reduced the system of spectral equations in zero temperature limit $T \to 0$ to a single integral equation and sketched out the structure of ground states and lowest lying excitations of the model, for two different sectors $\{1\}$ and $\{2\}$. We shown that the spectrum is gapless in $\{1\}$ and has both gapped and gapless excitations in $\{2\}$. Two special limits $J_1 = 0$, $J_2 \neq 0$ and $J_1 = J_2$ are considered in section 4. We have shown that when $J_1 = 0$, in the thermodynamic limit our model is equivalent to two species of Heisenberg XXZ models with different signs of hopping constants. We studied the interaction between the two species, which is of next to leading order, by use of both the finite size correction and the bosonization methods. We recognized that the limit $J_1 = J_2$ inherits some features from both $|J_1| < |J_2|$ and $|J_1| > |J_2|$ regions. We have summarized our results in the last section.

2 The Bethe Ansatz equations in the thermodynamic limit and the spectral equations

In what follows we shall be interested in the thermodynamic limit of the theory. Namely, the size of the system $N$ and the number of elementary excitations in the energy interval $[\varepsilon, \varepsilon + d\varepsilon]$, $M(\varepsilon)$, will go to infinity by the finite ratio

$$N \to \infty, \quad M(\varepsilon) \to \infty, \quad M(\varepsilon)/N = \text{const}$$

Then, according to [25, 26], general solutions of (3) are grouped in so-called strings. The character of the string solution strongly depends on whether the proper value of complex parameter $\lambda$ is real or imaginary. Thus in two regions $\{1\}$ and $\{2\}$ the model exhibits different properties which we shall analyze separately.

In spite of possible shortcomings of the string hypothesis (see for instance [27]), we base our analysis on the string solution approach, which we believe renders correct results in this thermodynamic regime, as it is in the case of Heisenberg spin chains.

It is worth to mention that we take the parameter $\theta$ to be finite. In the absence of magnetic field, all the spectral functions typically go to zero at (finite or infinite) Fermi points. Then we expect that in the region $\{1\}$ there will be no effective restrictions on the variation range of spectral parameters, and the spectrum will be gapless.

In this section we shall bring the basic properties of Bethe Ansatz exact solution. This solution will allow to derive the spectral equations of the theory which describe the spectrum of physical excitations and provide a systematic method for constructing the physical vacuum and the equilibrium properties of the system.

2.1 The case of real $\lambda$ and imaginary $\theta$ ($|J_1| \leq |J_2|$)

2.1.1 The properties of Bethe Ansatz equations

In this case we change the parameter $\theta \to i\theta$ in order to deal with real $\theta$.

Equations (3) and eigenvalues (4) have periodic property with respect to the shift of parameters $x_i \to x_i + 2p_0i$, where the parameter $p_0 = \frac{x}{\lambda}$ is introduced. In the thermodynamic limit, string solutions have the form [26]

$$x_{\alpha,i}^{n,k} = x_{\alpha,i}^n + i(n + 1 - 2k) + \frac{p_0}{2}(1 - v), \quad (\text{mod } 2p_0i), \quad k = 1, 2, \ldots, n; \quad (5)$$
characterized by the parity $v = \pm 1$, the real part (the center of string) $x^n_{\alpha, \pm}$ and the order $n$. The allowed values of $n$ are restricted by the conditions

$$v \sin[k\lambda] \sin[(n-k)\lambda] > 0,$$

which follows from the normalizability conditions for the wave functions [28].

Due to the symmetry $\lambda \rightarrow \pi - \lambda$ of the eqs. (3), we shall first consider the case $\lambda < \pi/2$. Solutions of (6) for arbitrary values of parameter $p_0$ are given in [26]. The starting point is the representation of $p_0$ in terms of continued fraction

$$p_0 = \nu_1 + \frac{1}{\nu_2 + \frac{1}{\nu_3 + \ldots}},$$

where $\nu_i$ are integers. Now, in order to describe strings which satisfy (6), one defines series of real numbers $p_i$ and series of integers $m_i$ and $y_i$ as follows:

$$p_0 = \pi/\lambda, \quad p_1 = 1, \quad p_i = p_{i-2} - p_{i-1}\nu_{i-1},$$
$$m_0 = 0, \quad m_i = \sum_{k=1}^{i} \nu_k,$$
$$y_{-1} = 0, \quad y_0 = 1, \quad y_1 = \nu_1, \quad y_i = y_{i-2} + \nu_iy_{i-1}.$$

The order and parity of all strings can be expressed by these series:

$$n_j = y_{i-1} + (j - m_i)y_i \quad \text{for} \quad m_i \leq j < m_{i+1}, \quad j = 1, 2, \ldots,$$
$$v_1 = +1, \quad v_{m_i} = -1, \quad v_j = \exp(\pi i [(n_j - 1)/p_0]),$$

where "[ ]" is the Gauss’ symbol. Suppose that there are $M_j$ bound states of parity $v_j$ and order $n_j$. Then we write the coordinate of the center $x^n_{\alpha}$ as $x^n_{\alpha}$. Eqs. (3) can be written in terms of the centers of strings

$$Np_j(x^n_{\alpha}) = 2\pi J_j^i + \sum_{n=1}^{\infty} \sum_{\beta=1}^{M_n} \Theta_{jn}(x^n_{\alpha} - x^n_{\beta}).$$

Here the function $p_j(x)$ is the bare $j-$ string momentum

$$p_j(x) = f(x + \theta/\lambda, n_j, v_j) + f(x - \theta/\lambda, n_j, v_j),$$

and the bare scattering phase between the $j-$ string and the $n-$ string equals

$$\Theta_{jk}(x) = \sum_p [f(x, p, v_jv_k) + f(x, p + 2, v_jv_k)],$$

where the sum is over $p = |n_j - n_k|, |n_j - n_k| + 2, \ldots, n_j + n_k - 2$. In eqs. (11) and (12) the function

$$f(x, k, v) = i \ln \left[ \frac{\sinh \left( \frac{x}{2} - ik \right) - i\pi \left( 1 - v \right) \tanh \left( \frac{\pi x}{2p_0} \right)}{\sinh \left( \frac{x}{2} + ik \right) + i\pi \left( 1 - v \right) \sec \left( \frac{\pi x}{2p_0} \right)} \right] = 2v \tan^{-1} \left( \left( \frac{\pi n_k}{2p_0} \right) \frac{x}{\csc \left( \frac{2p_0}{\pi x} \right)} \right)$$

is introduced.
The total momentum of a given configuration is $P = \sum_k p_k(x)$. The energy can be expressed as $E = A \sum_k h_k(x)$, where

$$h_k(x) = \frac{d}{dx} [f(x - \theta/\lambda, n_k, v_k) - f(x + \theta/\lambda, n_k, v_k)]$$

(14)

is the bare-particle energy, and the constant

$$A = \frac{\cosh(2\theta) - \cos(2\lambda) \tan \lambda}{\sinh^2 \theta}$$

is introduced.

The quantities $J^\alpha_j$ are integers or half-integers then $N$ is odd or even respectively. The set $\{J^\alpha_j\}$ defines a unique solution $\{x^\alpha_j\}$ of (3). A set of numbers $\tilde{J}^\alpha_j$ omitted in $\{J^\alpha_j\}$ defines the holes in the distribution of $x^\alpha_j$. In the thermodynamic limit, the string "particles" and "holes" are continuously distributed with densities $\rho_j$ and $\tilde{\rho}_j$. Then eqs. (3) in the leading order in $1/N$ acquire the form of integral equations as follows:

$$b_j(x) = (-1)^{r(j)}(\rho_j(x) + \tilde{\rho}_j(x)) + \sum_k T_{jk} * \rho_k(x),$$

(15)

where

$$T_{jk}(x) = \frac{1}{2\pi} \frac{d}{dx} \Theta_{jk}(x),$$

and

$$b_k(x) = \frac{1}{2\pi} \frac{d}{dx} [f(x + \theta/\lambda, n_k, v_k) + f(x - \theta/\lambda, n_k, v_k)].$$

The convolution "*" is defined as

$$f * g(x) = \int_{-\infty}^{+\infty} f(x - y)g(y)dy.$$

The sign-factor $(-1)^{r(j)} = \text{sign}(J^\alpha_j - J^\alpha_{j-1})$ indicates the signature of the derivative $f'(x, j, v_j)$. It equivalently can be defined from the inequality (see Ref. 26)

$$m_{r(j)} \leq j < m_{r(j)+1}.$$

The energy and momentum are of the form

$$E/N = \sum_{j=1}^\infty \int_{-\infty}^\infty (Ah_j(x) + 2n_j\mu_0H) \rho_j(x)dx - \mu_0H,$$

(16)

$$P/N = \sum_{j=1}^\infty \int_{-\infty}^\infty p_j(x)\rho_j(x)dx.$$  

(17)

Following to 29, we can express the entropy by

$$S/N = \sum_{j=1}^\infty \int_{-\infty}^\infty ((\rho_j + \tilde{\rho}_j) \ln(\rho_j + \tilde{\rho}_j) - \rho_j \ln \rho_j - \tilde{\rho}_j \ln \tilde{\rho}_j) dx.$$  

(18)
To treat the case \( \lambda > \pi/2 \), one can use the above mentioned symmetry \( \lambda \rightarrow \pi - \lambda \) and replace \( \lambda \) by \( \pi - \lambda \) in (10)-(15). This replacement preserves the previous classification of strings, with the new parameter \( p_0 = \frac{\pi}{x} \). Eqs. (10) retain their form, but the bare energy and momentum change their signs in (11) and (14):

\[
p_j(x, \lambda) \rightarrow -p_j(x, \pi - \lambda), \quad h_j(x, \lambda) \rightarrow -h_j(x, \pi - \lambda).
\] (19)

### 2.1.2 The spectral equations

Let the system is in the state characterized by densities \( \rho_j(x) \) and \( \tilde{\rho}_j(x) \). The equilibrium dynamics of the system at some temperature \( T \) can be extracted by minimizing the free energy

\[
F = E - TS
\]

with respect to independent \( \rho_j \). This yields the following non-linear integral equations for \( \eta_j = \tilde{\rho}_j/\rho_j \) (the main spectral equations):

\[
\ln \eta_j = \frac{1}{T} (A h_j + 2n_j \mu_0 H) + \sum_{k=1}^{\infty} (-1)^{r(k)} T_{j,k} * \ln(1 + \eta_k^{-1}).
\] (20)

It is more convenient to deal with a symmetric integral operator

\[
A_{jk}(x - y) = (-1)^{r(j)} \delta_{j,k} \delta(x - y) + T_{j,k}(x - y).
\]

Its inverse operator has the form

\[
A^{-1}_{jl} = (-1)^{i+1} (\delta_{j,l} \delta(x - y) - s_i [(1 - 2\delta_{m_i-1,j}) \delta_{j-1,l} + \delta_{j+1,l}]) \quad m_i-1 \leq j \leq m_i - 2,
\]

\[
A^{-1}_{jl} = (-1)^{i+1} (\delta_{j,l} \delta(x - y) - s_i [(1 - 2\delta_{m_i-1,j}) \delta_{j-1,l} - d_i \delta_{j,l} - s_{i+1} \delta_{j+1,l}]) \quad j = m_i - 1,
\] (21)

where

\[
s_i(x) = \frac{1}{4p_i} \frac{1}{\cosh(\frac{\pi x}{2p_i})}, \quad d_i(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega x}}{2 \cosh(p_i \omega) \cosh(p_{i+1} \omega)}.
\]

This operator acts on \( h_j \) and \( b_j \) in very simple way:

\[
\sum_l A^{-1}_{jl} * h_l = 2\pi \delta_{j,1} (s_1(x - \theta/\lambda) - s_1(x + \theta/\lambda)) \equiv \varepsilon_0(x).
\] (22)

\[
\sum_l A^{-1}_{jl} * b_l = \delta_{j,1} (s_1(x + \theta/\lambda) + s_1(x - \theta/\lambda)),
\]

The free energy can be expressed in transparent form

\[
F = -T \int_{-\infty}^{\infty} (s_1(x + \theta/\lambda) + s_1(x - \theta/\lambda)) \ln(1 + \eta_1) dx.
\] (23)

Here we used (10), (20) and the properties (22).
In the zero temperature limit $T \to 0$ the system goes to its vacuum state. For further analysis of this limit it is useful to introduce the so-called pseudo-energies by $\eta_j(x) = \exp \left( \frac{\varepsilon_j(x)}{T} \right)$ and transform (20) in terms of these functions as follows:

$$\varepsilon_j^+(x) = Ah_j(x) + 2n_j \mu_0 H - \sum_{k=1}^{\infty} (-1)^{r(k)} A_{j,k} \varepsilon_k^-(x),$$

(24)

or, using the properties of inverse operator (22),

$$\varepsilon_1(x) = A\varepsilon_0(x) + s_1 * \varepsilon_2^+(x),$$

(25)

$$\varepsilon_j(x) = s_i * [(1 - 2\delta_{m_{i-1},j})\varepsilon_{j-1}^+(x) + \varepsilon_{j+1}^+(x)], \quad m_{i-1} \leq j \leq m_i - 2$$

$$\varepsilon_j(x) = s_i * (1 - 2\delta_{m_{i-1},j})\varepsilon_{j-1}^+(x) + d_i * \varepsilon_j^+(x) + s_{i+1} * \varepsilon_{j+1}^+(x), \quad j = m_i - 1,$$

$$\lim_{j \to \infty} \frac{\varepsilon_j(x)}{n_j} = 2\mu_0 H.$$

Here the dagger means positive parts of corresponding functions:

$$\varepsilon_j^+(x) = \begin{cases} \varepsilon_j(x), & \text{if } \varepsilon_j(x) \geq 0, \\ 0, & \text{if } \varepsilon_j(x) < 0, \end{cases} \quad \varepsilon_j^-(x) = \varepsilon_j(x) - \varepsilon_j^+(x)$$

From eqs. (16) and (24) one can represent the energy in a transparent form

$$E = E_0 + E_{exc},$$

where

$$E_0/N = \sum_j (-1)^{r(j)} \int \varepsilon_j^-(x)b_j(x)dx$$

(26)

$$= - \int_{-\infty}^{\infty} [s_1(x + \theta/\lambda) + s_1(x - \theta/\lambda)] \varepsilon_1^+(x)dx,$$

(27)

$$E_{exc}/N = \sum_j \int \left[ \varepsilon_j^+(x)\rho_j(x) - \varepsilon_j^-(x)\bar{\rho}_j(x) \right] dx.$$  

(28)

By definition, (28) is non-negative. Therefore, $E_{min} = E_0$ is the energy of vacuum configuration, which can be realized then the following two conditions are satisfied:

$$\varepsilon_j^+(x)\rho_j(x) = 0, \quad \varepsilon_j^-(x)\bar{\rho}_j(x) = 0.$$

(29)

The expression for vacuum energy (27) follows from (15), (21), (24) and agrees with (28) in the limit $T \to 0$.

2.2 The case of imaginary $\lambda$ and real $\theta$ ($|J_1| > |J_2|$)

2.2.1 The properties of the Bethe Ansatz equations

In order to ease notations and deal with real spectral parameter and real $\lambda$, we change $x \to \frac{1}{\lambda} \varphi$ and $\lambda \to i\lambda$. The corresponding Bethe Ansatz equations and all the physical quantities are periodic in each $\varphi_k$ with a real period $2\pi$. 

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The string solutions are of the form
\[ \varphi^{n,k}_\alpha = \varphi^n_\alpha + i\lambda(n + 1 - 2k), \quad (\text{mod } 2\pi), \quad k = 1, 2, \ldots, n; \] (30)
where \( \varphi^n_\alpha \) is the real part (the center of string) and the order \( n \) is not bounded from above
[25].

The bare particle momentum \( p_n(\varphi) \) and energy \( h_n(\varphi) \) are defined like in (11) and (14):
\[ p_n(x) = f(\varphi + \theta, n) + f(\varphi - \theta, n), \] (31)
\[ h_k(x) = \frac{d}{d\varphi} [f(\varphi + \theta, n) - f(\varphi - \theta, n)]. \] (32)

Here the function \( f \) is defined as
\[ f(\varphi, k) = -i \ln \left[ \frac{\sin \frac{1}{2}(\varphi - i\lambda k)}{\sin \frac{1}{2}(\varphi + i\lambda k)} \right]. \] (33)

Due to the mentioned periodicity, one can restrict the interval \( \varphi_k \in [-\pi, \pi] \) once fixing the logarithmic branch in (33).

The bare two-particle scattering phase \( \Theta_{jk} \) is also given by (12) with the function (33).

The densities of the \( n \)-strings and holes \( \rho_n(\varphi) \) and \( \tilde{\rho}_n(\varphi) \), defined in the region \([−\pi, \pi]\), obey the following integral equations:
\[ b_j(\varphi) = \rho_j(\varphi) + \tilde{\rho}_j(\varphi) + \sum_k T_{jk} \ast \rho_k(\varphi), \] (34)

where
\[ T_{jk}(\varphi) = \frac{1}{2\pi} \frac{d}{d\varphi} \Theta_{jk}(\varphi), \quad b_k(\varphi) = \frac{1}{2\pi} \frac{d}{d\varphi} [f(\varphi + \theta, n_k, v_k) + f(\varphi - \theta, n_k, v_k)]. \]

The symbol "\( \ast \)" means
\[ f \ast g(\varphi) = \int_{-\pi}^{\pi} f(\varphi - \psi)g(\psi)d\psi. \]

The energy, momentum and entropy of a given configuration \( \rho_n(\varphi), \tilde{\rho}_n(\varphi) \) are
\[ E/N = \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (Ah_n(\varphi) + 2n\mu_0 H) \rho_n(\varphi)d\varphi - \mu_0 H, \] (35)
\[ P/N = \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} p_n(\varphi)\rho_n(\varphi)d\varphi. \] (36)
\[ S/N = \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} ((\rho_n + \tilde{\rho}_n) \ln(\rho_n + \tilde{\rho}_n) - \rho_n \ln \rho_n - \tilde{\rho}_n \ln \tilde{\rho}_n) d\varphi. \] (37)
2.2.2 The spectral equations

In this case, the same steps as in previous one are appropriate. First we minimize the free energy and get the following spectral equations:

\[
\ln \eta_n = \frac{1}{T} (Ah_n + 2n\mu_0 H) + \sum_{m=1}^{\infty} T_{n,m} \ln(1 + \eta_n^{-1}). \tag{38}
\]

Then we pass to zero temperature limit, where (38) is equivalent to the following integral equations for pseudo-energies \( \eta_n(\varphi) = \exp \left( \frac{\varepsilon_n(\varphi)}{T} \right) \):

\[
\varepsilon_n^+(\varphi) = Ah_n(\varphi) + 2n\mu_0 H - \sum_{m=1}^{\infty} A_{n,m} \varepsilon_m^-(\varphi), \tag{39}
\]

where we introduced the operator

\[
A_{n,m}(\varphi) = \delta_{n,m}\delta(\varphi) + T_{n,m}(\varphi).
\]

Its inverse will have a form

\[
A_{n,m}^{-1}(\varphi) = \delta_{n,m}\delta(\varphi) - \frac{1}{2\pi} D(\varphi)(\delta_{n,m-1} + \delta_{n,m+1}), \tag{40}
\]

with

\[
D(\varphi) = \sum_k \frac{e^{ik\varphi}}{2\cosh(k\lambda)}
\]

which is positive valued function with period \(2\pi\). Inverting the system (39), one gets:

\[
\varepsilon_1(\varphi) = A\varepsilon_0(\varphi) + \frac{1}{2\pi} D * \varepsilon_2^+(\varphi), \tag{41}
\]

\[
\varepsilon_n(\varphi) = \frac{1}{2\pi} D * \left( \varepsilon_{n-1}^+(\varphi) + \varepsilon_{n+1}^+(\varphi) \right), \quad n \geq 2
\]

\[
\lim_{n \to \infty} \frac{\varepsilon_n(\varphi)}{n} = 2\mu_0 H.
\]

Here the property

\[
A_{n,m}^{-1} * h_m(\varphi) = \delta_{1,n}\varepsilon_0(\varphi) = \delta_{1,n}(D(\varphi + \theta) - D(\varphi - \theta))
\]

have been used.

This system of coupled equations is simpler than the previous system (25). In particular, one can immediately conclude that the functions \( \varepsilon_n \) are non-negative for \( n \geq 2 \). This means that the vacuum is formed by 1-strings only.

For the energy of given configuration \( \rho_n, \tilde{\rho}_n \), the equations analogous to (26)–(28) are valid:

\[
E = E_0 + E_{exc},
\]
where
\[ E_{0}/N = \sum_{n} \int \varepsilon_{n}^{-}(\varphi)b_{n}(\varphi)d\varphi \]  
\[ = -\frac{1}{2\pi} \int_{-\pi}^{\pi} [D(\varphi + \theta) + D(\varphi - \theta)] \varepsilon_{1}^{+}(\varphi)d\varphi; \]  
\[ E_{\text{exc}}/N = \sum_{n} \int [\varepsilon_{n}^{+}(\varphi)\rho_{n}(\varphi) - \varepsilon_{n}^{-}(\varphi)\tilde{\rho}_{n}(\varphi)] d\varphi. \]  

3 Ground state and excitation spectrum in zero "magnetic" field

Here we shall study the structure of the ground state and excitation spectrum at \( H = 0 \).

First consider the case of real \( \lambda \), (1), in the region \( 0 < \lambda < \pi/2 \). By definition,
\[ \eta_{j}(x) = \frac{\tilde{\rho}_{j}(x)}{\rho_{j}(x)} = \exp \left( \frac{\varepsilon_{j}(x)}{T} \right). \]

In the limit \( T \to 0 \) survive only the states whose rapidities obey \( \varepsilon_{j}(x) < 0 \). Relations (29) also indicate that, in the ground state, the occupied rapidities obey \( \varepsilon_{j} < 0 \). Let us denote this region by \( \Delta_{j} \) (the Dirac sea):
\[ \varepsilon_{j}(x) < 0, \quad x \in \Delta_{j}. \]

According to this, the structure of the ground state is governed by the system of coupled equations (25).

It can be shown that in zero "magnetic" field \( H = 0 \) eqs. (25) have solutions with the following properties:
\[ \varepsilon_{j}(x) = \int_{\varepsilon_{1} \geq 0} G_{j}(x-y)\varepsilon_{1}(y)dy \geq 0, \quad 2 \leq j \leq m_{1} - 1, \]  
\[ G_{j}(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x} \frac{\cosh(p_{0} - j - 1)\omega}{\cosh(p_{0} - 2)\omega} \geq 0, \quad 2 \leq j \leq m_{1} - 1, \]  
\[ \varepsilon_{m_{1}}(x) = -s_{2} \varepsilon_{m_{1} - 1}(x) \leq 0, \]  
\[ \varepsilon_{j}(x) = 0, \quad j > m_{1}. \]  

These relations constitute the solution of (25) in terms of unknown function \( \varepsilon_{1}(x) \). Inserting the integral representation (45) for non-negative \( \varepsilon_{2} \) into the first equation of (25), we arrive to the following integral equation with respect to \( \varepsilon_{1}(x) \):
\[ \varepsilon_{1}(x) = A\varepsilon_{0}(x) + \int_{\varepsilon_{1} \geq 0} R(x-y)\varepsilon_{1}(y)dy, \]  
\[ R(x) = s_{1} * G_{2}(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x} \frac{\cosh[(p_{0} - 3)\omega]}{2 \cosh[(p_{0} - 2)\omega] \cosh[\omega]} \geq 0. \]
Here one of course needs to solve eq. (48). One could assume that the Wiener-Hopf method would be effective. This method would nicely work in the case when the region $\varepsilon_1 \geq 0$ is spanned from infinity to some finite point. Our analysis shows that this is not the case. Then we employed both perturbative Wiener-Hopf and numerical methods. In order to be shorter, we will present the detailed analysis and related results elsewhere. Meanwhile we bring here some speculative analysis which make the general picture rather clear.

As defined by (22), the so-called disturbance function $\varepsilon_0$ is an odd function with positive values at positive arguments, $\varepsilon_0(x) > 0$, $x > 0$. This means that $\varepsilon_1$ is also positive at non-negative values of $x$: $\varepsilon_1(x) > 0$, $x > 0$ ($A > 0$). Further analysis of eq. (48) shows that $\varepsilon_1(x)$ takes negative values in some region in $(-\infty, 0)$. This region which we denoted by $\Delta_1$, is the Dirac sea of $1$-strings. Furthermore, $\varepsilon_{m_1}$ is non-positive within the whole real axis $-\infty < x < +\infty$. So the Dirac sea of $m_1$-strings $\Delta_{m_1}$ is $(-\infty, +\infty)$. The rest pseudo-energies are non-negative. Thus the vacuum is filled by the $1$ and $m_1$ strings:

$$
\rho_1^0(x) \geq 0, \quad \bar{\rho}_1^0(x) \geq 0, \quad \Delta_1 \subset (-\infty, 0),
$$

$$
\rho_{m_1}^0(x) \geq 0, \quad \bar{\rho}_{m_1}^0(x) = 0, \quad \Delta_{m_1} = (-\infty, +\infty),
$$

$$
\rho_j^0(x) = 0, \quad j \neq 1, m_1, \quad \Delta_j = \emptyset,
$$

$$
\Delta = \Delta_1 \oplus \Delta_{m_1}.
$$

where $\rho^0$, $\bar{\rho}^0$ are vacuum distribution functions, $\Delta$ is the whole Dirac sea. Integrating (15) for $j = m_1$ we get that the vacuum configuration is $M_1^0 + M_{m_1}^0 = N$, $M_j^0 = 0$, $j \neq 1, m_1$, where $M_i^0$ are numbers of vacuum $i$-strings. So we have the half-filled band and $S_z = 0$, i.e., we deal with an antiferromagnetic case.

Let us turn to considerations of excitation spectrum. The possible excited states can be formed by adding $l_k$ excited $k$-strings with rapidities $z^k_\alpha$, $\alpha = 1, \ldots, l_k$ and subtracting $v_j$ vacuum $j$- strings with the rapidities $z^h_{\alpha j}$. This state will have a total spin

$$
S_z = \sum_j n_j v_j - \sum_k n_k l_k \geq 0,
$$

and energy

$$
\mathcal{E} = \sum_k \sum_{\alpha=1}^{l_k} \varepsilon_k(z^k_\alpha) - \sum_j \sum_{\alpha=1}^{v_j} \varepsilon_j(z^h_{\alpha j}), \quad z^k_\alpha \in (-\infty; +\infty) \setminus \Delta_k, \quad z^h_{\alpha j} \in \Delta_j.
$$

The BAE poses restrictions on the possible combinations of excitation parameters $\{z^k_\alpha\}$ and $\{z^h_{\alpha j}\}$. We miss here details of the treatment of this problem, which would involve the analysis of Eqs. (10) in the next to leading order (15), i.e., the $1/N$ corrections to (15). This is based on the known methods [30, 31, 32] and will be reported elsewhere. Instead we bring here some results for lowest-lying excitations with $|M_i - M_i^0| \leq 1$.

- $M_i = M_i^0$: The particle-hole excitations:

$$
E = -\varepsilon_1(z^{h,1}) + \varepsilon_1(z^1), \quad S_z = 0.
$$

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• $M_1 = M_1^0 - 1$, $M_i = M_i^0$, $i > 1$; The two-hole excitations:

$$E = -\varepsilon_1(z_1^{h,1}) - \varepsilon_1(z_2^{h,1}), \quad S_z = 1. \quad (53)$$

• $M_{m_1} = M_{m_1}^0 - 1$, $M_i = M_i^0$, $i \neq m_1$; The particle-hole excitations:

$$E = -\varepsilon_1(z^{h,1}) + \varepsilon_1(z^1), \quad S_z = 1. \quad (54)$$

• $M_1 = M_1^0 - 1$, $M_{m_1} = M_{m_1}^0 + 1$, $M_i = M_i^0$, $i \neq 1, m_1$; The two-hole excitations:

$$E = -\varepsilon_1(z_1^{h,1}) - \varepsilon_1(z_2^{h,1}), \quad S_z = 0. \quad (55)$$

• $M_1 = M_1^0 + 1$, $M_{m_1} = M_{m_1}^0 - 1$, $M_i = M_i^0$, $i \neq 1, m_1$; The two-particle excitations:

$$E = \varepsilon_1(z_1^1) + \varepsilon_1(z_2^1), \quad S_z = 0. \quad (56)$$

All the rest excitations have $|M_i - M_i^0| > 1$ for some $i$. Then, it is possible to have some combinations of excitations given above, as well as a new ones. For example, when $p_0 > M_{m_1}^0 - M_{m_1} > p^0_0 (> 1)$, $M_i = M_i^0$, $i \neq m_1$, together with other possible excitations it appears a new branch of two $m_1$-hole excitation with the energy

$$E = -\varepsilon_{m_1}(z_1^{h,2}) - \varepsilon_{m_1}(z_2^{h,2}). \quad (57)$$

Notice that in any case the particle-hole excitation (52) can be repeatedly excited: they can appear in any number as well as in combination with any other excitations.

Besides these branches of the spectrum, there also exist bound states of magnons whose energies are given by $\varepsilon_j$, $j = 1, 2, \ldots, m_1 - 1$. These bound states are single-parametric, spin-singlet elementary excitations. Indeed, to construct a single bound state of $s$ magnons, we must choose

$$l_k = \delta_{k,s}, \quad \rho_1(x) = 0, \quad x \notin \Delta_1, \quad \tilde{\rho}_1(x) = 0, \quad x \in \Delta_1, \quad \tilde{\rho}_{m_1} = 0.$$

Then the $j = 1$ and $j = m_1$ eqs. (15) acquire the form

$$b_1(x) = \rho_1(x) + \tilde{\rho}_1(x) + T_{11} * \rho_1(x) + T_{1m_1} * \rho_{m_1}(x) + \frac{1}{N} T_{1s}(x - z_s) \quad (58)$$

$$b_{m_1}(x) = -\rho_{m_1}(x) + T_{m_11} * \rho_1(x) + T_{m_1m_1} * \rho_{m_1}(x) + \frac{1}{N} T_{m_1s}(x - z_s),$$

where $z_s$ parameterizes the excitation. From the last equation it follows that this configuration has a zero spin,

$$S_z = v_1 + v_{m_1} - s l_s = 0.$$

It can be shown that (58) has a unique solution for $z_s \in (-\infty; +\infty)$. It constitutes elementary excitation with the energy

$$E = \varepsilon_s(z_s). \quad (59)$$

As seen from (28) and (17), strings with $j > m_1$ have vanishing energies and become important only when one classifies excited states with respect to the total spin $S_z$. 

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Due to the properties of spectral functions $\varepsilon$ \textsuperscript{15} - \textsuperscript{17}, excitation energies can take any positive values close to zero, then excitation parameters $\{z^1\}$ and $\{z^h\}$ come close to Fermi points or $z_s \to \pm \infty$, i.e., the spectrum is massless, as it was expected.

In the region $\pi/2 < \lambda < \pi$, we use transformation \textsuperscript{19} which produces a mapping of this region onto the previous one. From \textsuperscript{19} and due to the fact that $s_i(x)$ and $d_i(x)$ in \textsuperscript{21} are even functions of $x$ whereas $h_j(x)$ and $\varepsilon_0(x)$ in \textsuperscript{22} are odd functions of $x$, one gets that all of our solutions and results remain valid in this region also, when we change $\lambda$ by $\pi - \lambda$ and the variable $x$ by $-x$, in the corresponding functions.

In the second case (2) of imaginary $\lambda$ we deal with the spectral system \textsuperscript{41}. It can be easily shown that

$$
\varepsilon_n = \int_{-\pi}^{\pi} \frac{\sinh[(n-1)\lambda]}{\cosh[(n-1)\lambda] - \cos(\varphi - \psi)} \varepsilon_1^+(\psi) d\psi > 0 \quad \text{for} \quad n \geq 2 \quad (60)
$$

obeys \textsuperscript{41} if $\varepsilon_1(\varphi)$ is solution to the integral equation

$$
\varepsilon_1(\varphi) = A\varepsilon_0(\varphi) + \int_{-\pi}^{\pi} r(\varphi - \psi) \varepsilon_1^+(\psi) d\psi, \quad (61)
$$

where the real positive valued kernel $r(\varphi) = \frac{1}{2\pi} \sum_k \frac{e^{-|\lambda k|}}{2\cosh(\lambda k)} e^{ik\varphi}$ is periodic with $2\pi$. As in the previous case, from this integral equation we conclude that $\varepsilon_1(\varphi)$ takes negative values in some region $\Delta \subset (-\infty, 0)$. The vacuum is formed by 1-strings filled with rapidities from $\Delta_1$; the vacuum densities satisfy

$$
\rho_1^0(\varphi) = 0, \quad \varphi \notin \Delta_1, \quad \rho_1^0 = 0, \quad \varphi \in \Delta_1,
$$

$$
\rho_n^0(\varphi) = 0, \quad \varphi \in [-\pi, \pi], \quad n = 2, 3, \ldots.
$$

By the integration of \textsuperscript{34} for $n = 1$ one gets for the vacuum configuration

$$
M_1 + \frac{1}{2} \tilde{M}_1 = N, \quad M_1, \tilde{M}_1 \neq 0, \quad M_n = 0, \quad n \geq 2. \quad (62)
$$

This means that our vacuum in this case has spontaneous spin $S_z \neq 0$. The band-filling is $M_1 > 1/2$ (the exact value of band filling can be extracted from the exact solution of \textsuperscript{61} and \textsuperscript{34} only).

The excitation spectrum includes massless 1-string particle-hole excitations with energies

$$
E = \sum_p \varepsilon_1(\psi_p) - \sum_q \varepsilon_1(\psi_q^h), \quad \psi_p \notin \Delta_1, \quad \psi_q^h \in \Delta, \quad (63)
$$

and massive breathers of order $n$, $n \geq 2$ with energies

$$
E_n = \sum_p \varepsilon_n(\psi_p). \quad (64)
$$
4 Some Special Limits

4.1 $J_1 = 0, \quad J_2 \neq 0$

This case is also in the region $[1]$. It corresponds to the limit $\theta \to \pm i \infty$, when $J_1 = 0$ and $J_2 = \mp i \tan \lambda$. Consider the integral equation $[22]$ for $\theta \gg 1$ (we are changing $\theta \to i \theta$). By the definition $[22]$, $\varepsilon_0$ depends on the combinations $\pm \theta/\lambda$ and is essentially non-zero in the vicinity of $\pm \theta/\lambda$. One can expect the solution $\varepsilon_1$ has the same property and expand $[22]$ near to these points: the mutual influence of these regions will be of order $O(e^{-|\varepsilon_0|^2})$. Then $[22]$ goes to the following two decoupled integral equations:

$$
\varepsilon_1^R(x) = \frac{\pi \tan \lambda/\lambda}{\cosh(\frac{\pi}{2} x)} + \int_{\varepsilon_1^l \geq 0} R(x-y) \varepsilon_1^R(y) dy,
$$

$$
\varepsilon_1^L(x) = -\frac{\pi \tan \lambda/\lambda}{\cosh(\frac{\pi}{2} x)} + \int_{\varepsilon_1^l \geq 0} R(x-y) \varepsilon_1^L(y) dy,
$$

where $\varepsilon_1^L(x) = \varepsilon_1(x - \theta/\lambda)$, $\varepsilon_1^R(x) = \varepsilon_1(x + \theta/\lambda)$.

By their forms, Eqs. (65) are the same as corresponding integral equation for the spectral function $\varepsilon_1$ of Heisenberg XXZ model (see for example $[20]$): from this naive point of view, here we have two samples of XXZ chains with different signs of coupling constants.

It is natural to try to confirm this double chain structure starting from the Hamiltonian, which on the other hand would be seen a self-consistency check for our results. In the case under consideration, the Hamiltonian $[11]$ takes the form

$$
H = \sum_{j=1}^{N} \sum_{s=0,1} (-1)^s/2 \left[ \sigma_{j,s}^x \sigma_{j+1,s}^x + \sigma_{j,s}^y \sigma_{j+1,s}^y - \sigma_{j,s}^z \sigma_{j+1,s}^z \right]
$$

$$
+ \tan \lambda i \left[ \sigma_{j,s+1}^z (\sigma_{j,s}^+ \sigma_{j+1,s}^- - \sigma_{j,s}^- \sigma_{j+1,s}^+) \right]
$$

(66)

Being rewritten as follows:

$$
H = \frac{1}{\cos \lambda} \sum_{j=1}^{N} \sum_{s=0,1} (-1)^s \left[ \exp\{-i \lambda \sigma_{j,1-s}^z \} \cdot \sigma_{j,s}^+ \sigma_{j+1,s}^- + h.c. \right] - \frac{\cos \lambda}{2} \sigma_{j,s}^z \sigma_{j+1,s}^z,
$$

(67)

it reveals the underlying double chain structure, which is more evident under the following gauge-like unitary transformation:

$$
H = \frac{1}{\cos \lambda} e^{-i \frac{\lambda}{2} \sum_{N \geq n > k \geq 1} \sigma_{n,0}^z \sigma_{k,1}^z} \left[ H_0^{xz} + \left( \frac{i \lambda}{e} \sum_{k=1}^{N} \sigma_{k,1}^z \sigma_{N,0}^z + h.c. \right) \right]
$$

$$
- H_1^{xz} - \left( \frac{-i \lambda}{e} \sum_{n=1}^{N} \sigma_{n,0}^z \sigma_{N,1}^z + h.c. \right) \right] i \frac{\lambda}{2} \sum_{N \geq n > k \geq 1} \sigma_{n,0}^z \sigma_{k,1}^z;
$$

where

$$
H_s^{xz} = H_{s,open}^{xz} = \sum_{l=1}^{N-1} \sigma_{l,s}^+ \sigma_{l+1,s}^- + \sigma_{l,s}^- \sigma_{l+1,s}^+ + \frac{1}{2} \cos \lambda \sigma_{l,s}^z \sigma_{l+1,s}^z.
$$
So the system can be described by the unitarily equivalent Hamiltonian

$$\hat{H} = \frac{1}{\cos \lambda} \left\{ H_0^{xxz} + (e^{i\lambda} \sum_{k=1}^{N} \sigma_{x,k}^z \sigma_{y,0}^z + h.c.) - H_1^{xxz} - (e^{-i\lambda} \sum_{n=1}^{N} \sigma_{x,n}^z \sigma_{y,1}^z + h.c.) \right\}$$ (68)

By the further application of Bethe Ansatz technique, one can diagonalize this Hamiltonian. Then one parameterizes the eigenvalues of (66) by means of two sets of complex rapidities \(\{u^s_{k}\}\), as attached to the chains \(H_s^{xxz}\), \(s = 0, 1\). The eigenvalue problem leads to the Bethe Ansatz equations

$$\left[ \frac{\sinh \left( \frac{i}{2} (u_k^s - i) \right)}{\sinh \left( \frac{i}{2} (u_k^s + i) \right)} \right]^N = -e^{i\lambda(N-2M_0)} \prod_{i=1}^{M_1} \frac{\sinh \left( \frac{i}{2} (u_0^s - u_i^s - 2i) \right)}{\sinh \left( \frac{i}{2} (u_0^s - u_i^s + 2i) \right)}$$ (69)

$$\left[ \frac{\sinh \left( \frac{i}{2} (u_0^s - i) \right)}{\sinh \left( \frac{i}{2} (u_0^s + i) \right)} \right]^N = -e^{-i\lambda(N-2M_1)} \prod_{i=1}^{M_0} \frac{\sinh \left( \frac{i}{2} (u_0^s - u_i^s - 2i) \right)}{\sinh \left( \frac{i}{2} (u_0^s - u_i^s + 2i) \right)}.$$ (69)

The phase factors reflect the twisted boundary terms in the Hamiltonian \(\hat{H}\). On the other hand, these phase factors reflect an interaction between the two sets; one could assume, that the \(s = 0\) spins are inside a "gauge field" generated by the \(s = 1\) spins and vice versa. The energy eigenvalues are given by the formula

$$E = E^0 + E^1, \quad E^s(\{u^s_{k}\}) = (-1)^s 2\tan \lambda \sum_{i=1}^{M_s} \frac{\sin \lambda}{\sinh(\frac{i}{2}u^s_{i})\sinh(\frac{i}{2}u^s_{i} + i\frac{\pi}{2})}.$$ (70)

which does not affect the mentioned interaction explicitly.

Let us mention that the same equations one would get from BAE (3). Indeed, in the limit \(\theta \gg 1, (\theta \rightarrow i\theta)\), the roots of BAE (3) can be divided into two groups gathered close to the points \(\pm \theta/\lambda\): \(u^0 = u + \theta/\lambda\) and \(u^1 = u - \theta/\lambda\). Taking the limit \(\theta \rightarrow \infty\) in the BAE (3), one would get (69).

It is intuitively clear that, in the thermodynamic limit \(N \rightarrow \infty\), boundary terms does not affect the bulk behavior and the Hamiltonian (66) will have the same critical properties as two non-interacting XXZ- chains with usual periodic boundary conditions and opposite signs of coupling constants. This is exactly what we had in (65); the \(s = 0\) and \(s = 1\) chains correspond to \(L\) and \(R\) sectors respectively.

The interaction effect is of the next to leading order (actually of order \(1/N^2\)). It can be studied using the known finite size correction methods [30, 31, 32]. The idea is to consider two XXZ chains, each with additional phase shifts in the Bethe Eqs. (14). The phase shift of one of the chains is described by the total spin of another chain. This requires rather long calculations. We bring here only the result for the energy difference between the ground state and the one with \(n_s^R (n_s^L)\) particles added at the right (left) Fermi points of the \(s = 0, 1\) chains:

$$\Delta E = \frac{2\pi}{4N^2} \sum_s v_{0,s} \left[ \left( C_{2}^s + C_{1}^s \delta_{1-s}^2 \right) N_s^2 + C_{2}^s J_s^2 + 2(-1)^s \delta_{1}^2 C_{2}^s J_s^2 N_{1-s} + K_s^R + K_s^L - \frac{1}{3} \right],$$ (71)

where \(v_0\) is the Fermi velocity and \(\xi\) is the dressed charge introduced in [33],

$$v_{0,s} = \frac{\pi}{2\lambda_s} \sin \lambda_s, \quad \xi_{s}^2 = \frac{\pi}{2(\pi - \lambda_s)}; \quad \lambda_0 \equiv \lambda, \quad \lambda_1 \equiv \pi - \lambda.$$ (72)
and $K^{L,R}$ are some integer quantum numbers. The interaction parameter $\delta$ comes from the shifted phases. It is defined as
\[ N_{1-s}\delta = \left\{ \frac{\lambda}{2\pi}N_{1-s} \right\}, \quad s = 0, 1, \tag{73} \]
where $\{x\}$ is the fractional part of $x$. Quantum numbers $K$ are unimportant here, because are factored from the remaining interaction. We also used notations $N_s = n_s^R + n_s^L$, $J_s = n_s^R - n_s^L$. Thus we get the “effective Luttinger liquid” description of our model [34].

A similar integrable model containing two species of XXZ chains with the same signs of coupling constants is discussed by Schulz and Shastry in [35]. There the equations equivalent to (69) are derived in terms of momenta of Bethe excitations. The fact that the hopping parameters are of the same signs makes the physics different from our case.

Following Ref. [17], we finish this section with the bosonization study of (66). In order to obtain a right continuum theory, we first apply the Jordan-Wigner transformation along the zigzag links. The continuum fermionic Hamiltonian is obtained by the replacement $\psi_{n,s} \rightarrow \sqrt{a_0}(R_s(x) + (-1)^nL_s(x))$, $x = na_0$, where $\psi_{n,s}$ is the discrete Fermion field, $R(x)$ and $L(x)$ are the continuous right and left Dirac fields respectively, $a_0$ is the lattice spacing. In the leading order we get:
\[ H = H_0 + H_{int}, \]
\[ H_0 = v_0i \int dx \sum_s (R_s^+(x)\partial R_s(x) - L_s^+(x)\partial L_s(x)); \]
\[ H_{int} = v_0g \int dx \sum_s (-1)^s \left\{ \frac{1}{2a_0}(J_s^R - J_s^L) + (J_s^R + J_s^L)(J_s^{1-s} - J_s^{1-s}) + (R_s^+L_s + L_s^+R_s)^2 \right\}. \]
Here $J_s^R = R_s^+R_s$, $J_s^L = L_s^+L_s$ are the right and left fermion currents respectively, $v_0$ is the Fermi velocity and $g = 2\cot\lambda\tanh\theta$ is the coupling constant. Then, using the Bosonization prescriptions, we arrive at the bosonic Hamiltonian
\[ H = v_0 \int dx \left\{ (1 - \frac{g}{2\pi})(\Pi_0^2 + (\partial\Phi_1)^2) + (1 + \frac{g}{2\pi})(\Pi_1^2 + (\partial\Phi_0)^2) + \frac{g}{\pi} \Pi_0\partial\Phi_1 - \Pi_1\partial\Phi_0 \right\}. \tag{74} \]
Here we introduced Bose fields $\Phi_s(x)$ and canonical conjugate momenta $\Pi_s(x)$ with the commutations
\[ [\Phi_s(x), \Pi_{s'}(y)] = i\delta_{ss'}\delta(x - y), \]
In deriving (73) we have dropped strongly irrelevant interaction terms like $\cos(4\sqrt{\pi}\Phi_s(x))$. It is not hard to check that in the first order in small $\lambda$ the Hamiltonian (74) gives the energy difference equivalent to (71). The latter allows also to extract an exact values for Fermi velocities.

The canonical transformation
\[ \partial Q_+ = \frac{1}{\sqrt{K_+}} \frac{\partial\Phi_0 + \Pi_1}{\sqrt{2}}, \quad \partial Q_- = \frac{1}{\sqrt{K_-}} \frac{\partial\Phi_1 + \Pi_0}{\sqrt{2}}; \]
\[ P_+ = \sqrt{K_+} \frac{\Pi_0 - \partial\Phi_1}{\sqrt{2}}, \quad P_- = \sqrt{K_-} \frac{\Pi_1 - \partial\Phi_0}{\sqrt{2}}, \]
\[ K_\pm = \sqrt{1 \pm g/\pi}, \]
brings (74) into the Gaussian form

$$H = v_0 K_+ \int dx \left( P_+^2 + \partial Q_+^2 \right) + v_0 K_- \int dx \left( P_-^2 + \partial Q_-^2 \right). \tag{75}$$

We thus see that the interaction between the two chains effects on the renormalization of Fermi velocities only.

### 4.2 $J_1 = J_2$

This special limit is a crossing of two regions (1) and (2). It takes place when $\lambda \to 0$ and $\theta \to 0$ with finite ratio $\frac{\theta}{\lambda} = \text{const}$. In order to consider this limit, it is convenient to redefine $\theta \to i\lambda \theta$ in the region (1), and $\lambda \to i\lambda$, $\theta \to -i\lambda \theta$ and $\phi \to \lambda \phi$, in the region (2) (here $\phi$ is the rapidity variable introduced in the section (2.2)). In both cases, we will have

$$J_1 = J_2 = -\frac{i}{\theta}.$$

Consider this limit within (1). Here one has $p_0 \to \infty$ and $\nu_1 = m_1 \to \infty$. This makes the system of coupled spectral equations (25) simpler; the pseudo-energies $\varepsilon_j$ become non-negative for $j \geq 2$, like in the case of (41). Then (24) for $j = 1$ goes into the integral equation for $\varepsilon_1(x)$ as follows:

$$\varepsilon_1(x) = A_0 h(x) + \int_{\varepsilon_1 \leq 0} T_0(x - y) \varepsilon_1(y) dy, \tag{76}$$

where $A_0 = 2 \left( 1 + 1/\theta^2 \right)$, $T_0(x) = \frac{1}{2\pi (x/2)^2 + 1}$, $h(x) = \frac{2}{(x-\theta)^2 + 1} - \frac{2}{(x+\theta)^2 + 1}$. The rest $\varepsilon_j$ for $j \geq 2$ can be expressed in terms of the solution of (76) through (45), where

$$G_j(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x} e^{-|\omega|} = \frac{1}{2\pi} \frac{2(j-1)}{x^2 + (j-1)^2}.$$

Let us mention that one in principle could take the limit in (48). Then of course the obtained integral equation will be equivalent to (76); instead of our case, it will contain an integral over the region $\varepsilon_1 \geq 0$.

The same equation could be found in the region (2). Indeed, the rescaling $\varphi \to \lambda \varphi$ destroys the periodicity $\varphi \to \varphi + 2\pi$ and the new $\varphi$ varies from $-\infty$ to $+\infty$. Taking this in account, one can make sure that (24) for $n = 1$ goes into (76). This proves that we indeed get the same pictures coming from two different regions (1) and (2).

From (76) one can conclude that $\varepsilon_1$ takes negative values in some region $\Delta_0 \subset (-\infty, 0)$. Thus the vacuum in this case is filled by 1-strings with the rapidities from this region; the corresponding vacuum densities with the properties

$$\rho_v(x) = 0, \quad x \notin \Delta_0, \quad \tilde{\rho}_0(x) = 0, \quad x \in \Delta_0 \tag{77}$$

obey the integral equation

$$b(x) = \rho_v(x) + \tilde{\rho}_0(x) + \int_{\Delta_0} T_0(x - y) \rho_v(y) dy, \tag{78}$$
where \( b(x) = \frac{1}{2\pi} \left[ \frac{2}{(x+\theta)^2 + 1} + \frac{2}{(x-\theta)^2 + 1} \right] \). This integral equation together with (77) uniquely defines the vacuum densities. The vacuum configuration have properties very similar to the one from the section (2.2), given in (62).

Another phase transition occurs when one in addition sends \( \theta \to \pm \infty \). It corresponds to \( J_1 = J_2 = 0 \). This case can be treated similar to the special limit from the previous section. It can be checked that (76) splits into two integral equations which are equivalent to corresponding spectral equations of isotropic XXX-chain. The two integral equations will differ by opposite signs of the free term (disturbance function), which means that the corresponding isotropic chains are one ferromagnetic and another antiferromagnetic. The same follows from the Hamiltonian straightforwardly, when one puts \( J_1 = J_2 = 0 \). This is another consistency proof of our results.

5 Summary and Conclusion

To summarize we have applied the method introduced by Gaudin in [25] and Takahashi and Suzuki in [26] to the exactly solvable two-leg ladder model with zigzag like interaction constructed in [2]. We have established two distinct zero temperature phases, corresponding to two regions in coupling constant space \( |J_1/J_2| < 1 \), (1) and \( |J_1/J_2| > 1 \), (2). We considered the model without magnetic field. When \( |J_1/J_2| < 1 \), we found the model has gapless excitations above the antiferromagnetic ground state formed by filling all the possible 1-strings (even magnons) with rapidities from some region \( \Delta_1 \subset (-\infty, 0) \) and \( m_1 \)-strings (odd magnons) with rapidities from \( (-\infty, +\infty) \). So this phase is conformal. In the region (2), we have seen the model possesses a ferromagnetic ground state with gapless particle-hole excitations and massive breathers of length \( n \geq 2 \). The intersection of these two phases \( |J_1| = |J_2| \) was considered in the last section. In the last section, we have considered also the special limit when \( J_1 = 0 \). We have shown that in this case our spectral equations are equivalent to analogous equations for two XXZ chains with opposite signs of coupling constants. This double chain structure was confirmed by a unitary transformation of the Hamiltonian. We would like to mention that it naturally arises a question about the scaling dimensions of fields when the conformal phase occurs. These calculations could be done by use of the method introduced in [36] and [37].

The problem which did not allow to get a precise values of Fermi points, Fermi velocities, momenta of states, as well as conformal dimensions is to solve a Fredholm type equations as (48) and (61) are. As we already mentioned, the perturbative Wiener-Hopf method could of course be effective, though it does not usually give an explicit solutions. The numerical analysis is also appropriate. We plan to present this analysis and related results elsewhere.

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