Some matrix inequalities related to $J_S$– normal matrices

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Abstract
In this paper, we introduced the concept of # partial ordering and derived some results related to $J_S$– Normal matrices.

Keywords
# partial ordering, $J_S$– normal, $J_S$– unitary matrices, $J_S$– eigen values.

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1. Introduction

The $J_S$– Normal matrices and results related to $J_S$ Normal matrices was introduced and discussed in [1-4]. Here we extended this results of [5] in this context of $J_S$– Normal matrices.

Definition 1.1. The # partial order denoted by $\leq#$ is a relation on $\mathbb{C}$ defined by $A \leq# B$ if there exists a $A^#A = A^#B$.

Notation 1.2. Let $M_n(\mathbb{C})$ be the $n \times n$ matrices over the complex field; $\mathcal{N}_{J_S}$ be the set of all $J_S$ normal matrices; $\mathcal{U}_{J_S}$ be the set of all $J_S$ unitary matrices; $\mathcal{D}$ be the set of all diagonal matrix and throughout this paper $D$ is a nonsingular diagonal matrix.

2. Main Results

Theorem 2.1. Let $A,B \in \mathcal{N}_{J_S}$ with $1 \leq \text{rank}(A) \leq \text{rank}(B)$. Then the following conditions are equivalent.

1. $A \leq# B$

2. There is a matrix $U \in \mathcal{U}_{J_S}$ such that $U^#AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, U^#BU = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix}$ with $E \neq 0$, is a diagonal matrix.

3. There is a matrix $U \in \mathcal{U}_{J_S}$ such that $U^#AU = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, U^#BU = \begin{pmatrix} F & 0 \\ 0 & G \end{pmatrix}$, where $F$ is a nonsingular square matrix and $G \neq 0$.

4. If a matrix $U \in \mathcal{U}_{J_S}$ satisfies $U^#AU = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$, $U^#BU = \begin{pmatrix} F' & 0 \\ 0 & G \end{pmatrix}$, where $F$ is a square matrix with $|F| \neq 0$, $F'$ is a square matrix of the same order, and $G \neq 0$ then $F = F'$.

5. If a matrix $U \in \mathcal{U}_{J_S}$ satisfies $U^#AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, $U^#BU = \begin{pmatrix} D' & 0 \\ 0 & E \end{pmatrix}$, where $D' \in \mathcal{D}$ and of the same order and $E \neq 0$, is a diagonal matrix, then $D = D'$.

6. If a matrix $U \in \mathcal{U}_{J_S}$ satisfies $U^#AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, then $U^#BU = \begin{pmatrix} D & 0 \\ 0 & G \end{pmatrix}$, where $G \neq 0$.

7. All $J_S$– eigen vectors corresponding to nonzero $J_S$– eigen values of $A$ are $J_S$– eigen vector of $B$ corresponding to the same $J_S$– eigen values.
Proof. We split the proof of the theorem into four parts.

Part 1: (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2): Assume (1). Then by normality, \( A^*B = BA^* \) and therefore simultaneously diagonalizable. Since \( A^\# \) and \( A \) have the same \( J_5 \) eigen vectors, also \( A \) and \( B \) are simultaneously diagonalizable, then \( AB = BA \).

Suppose, let \( D' \in D \) of \( B \) there exist a matrix \( U \in \mathcal{U}_{J_5} \) such that \( A = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} U^*, B = U \begin{pmatrix} D' & 0 \\ 0 & E \end{pmatrix} U^* \), where \( D \in D \) and \( |D| \neq 0, E \neq 0 \). Now, \( A^* = U \begin{pmatrix} D^* & 0 \\ 0 & 0 \end{pmatrix} U^* \). Therefore,

\[
A^*A = U \begin{pmatrix} D^*D & 0 \\ 0 & 0 \end{pmatrix} U^* \tag{2.1}
\]

and

\[
AB = U \begin{pmatrix} D^*D' & 0 \\ 0 & 0 \end{pmatrix} U^* \tag{2.2}
\]

From 2.1 and 2.2 we get \( A^*A = A^*B \) implies \( D^*D = D^*D' \) implies \( D = D' \). Therefore, \( D \in D \) with \( |D| \neq 0 \) and \( D' \in D \) and of the same order and \( E \neq 0 \).

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Direct calculation

Part 2: (1) ⇒ (4) ⇒ (5) ⇒ (1)

This is a trivial modification of part-1.

Part 3: (2) ⇒ (6).

(2) ⇒ (6): Assume (2),

Let \( U \in \mathcal{U}_{J_5} \) satisfies \( U^*AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \).

By (b), \( \exists V \in \mathcal{U}_{J_5} \) such that \( V^*AV = \begin{pmatrix} D' & 0 \\ 0 & E \end{pmatrix} \). Interchanging the columns of \( V \) if necessary, we assume \( D = D' \). Let \( U = (U_1U_2) \) be such a partition that,

\[
U^*AU = \begin{pmatrix} U_1^*AU_1 & U_1^*AU_2 \\ U_2^*AU_1 & U_2^*AU_2 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \tag{2.3}
\]

Then, for the corresponding partition \( V = (V_1V_2) \) we have

\[
V^*AV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \tag{2.4}
\]

and

\[
V^*BV = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} \tag{2.5}
\]

Noting that, \( (4) \Rightarrow A = V \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^* \)

\[
(5) \Rightarrow B = V \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} V^*
\]

\[
U^*BU = U^*V \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} V^*U
\]

\[
U^*BU = U^*V \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} V^*U
\]

\[
U^*BU = \begin{pmatrix} U_1^*V_1 & U_2^*V_1 \\ U_1^*V_2 & U_2^*V_2 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} V_1^*V_2 \tag{2.2}
\]

By the normality condition,

\[
U^*BU = \begin{pmatrix} U_1^*V_1 & U_2^*V_1 \\ U_1^*V_2 & U_2^*V_2 \end{pmatrix} V_1^*V_2 \tag{2.2}
\]

Since \( U_1^*AU_1 = D \) let us take \( U_2^*V_2E_2V_2U_2 = G, G \neq 0 \).

Therefore, \( U^*BU = \begin{pmatrix} D & 0 \\ 0 & G \end{pmatrix} \), where \( D \) is a nonsingular and \( G \neq 0 \).

Hence the proof (6).

(6) ⇒ (2): Assume(6), Let \( U \in \mathcal{U}_{J_5} \) such that \( U^*AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \). Then by (6) \( U^*BU = \begin{pmatrix} D & 0 \\ 0 & G \end{pmatrix} \), where \( G \neq 0 \). Since \( G \in \mathcal{N}_{J_5} \), \( \exists W \in \mathcal{U}_{J_5} \) such that \( E = W^*GW \in D \).

Let \( V = U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow V^* = V \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \)

Then, \( V^*AV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \) and \( V^*BV = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} \)

Hence the proof (2).

Part 4: (1) ⇔ (7).

Proof is obvious.

Theorem 2.2. Let \( A, B \in \mathcal{N}_{J_5} \) and \( A \preceq B \), then \( A \) and \( B \) commutes.

Proof. Let \( A, B \in \mathcal{N}_{J_5} \). Then \( AA^* = A^*A \) and \( BB^* = B^*B \).

Let, \( U^*AU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \) and \( U^*BV = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} \).

\( \Rightarrow A = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} U^* \) and \( B = U \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} U^* \)

\( \Rightarrow A^* = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} U^* \)

Therefore, \( A^*A = A*B = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} U^* \Rightarrow A \preceq B \).

Similarly, we can prove \( AB = BA \).

Theorem 2.3. Let \( A, B \in \mathcal{N}_{J_5} \) with \( 1 \leq \text{rank}(A) < \text{rank}(B) \). Then
1. \( A \preceq^h B \) is equivalent to the following.

2. \( A^2 \preceq^h B^2 \) and if \( A \) and \( B \) have nonzero \( J_S \) eigen values \( \alpha \) and respectively \( \beta \) such that \( \alpha^2 \) and \( \beta^2 \) are \( J_S \) eigen values of \( A^2 \) and respectively \( B^2 \) with common \( J_S \) eigenvector of \( A \) and \( B \).

Proof. Assume (1) \( A \preceq^h B \Rightarrow A^h A = A^h B \).

Let \( U \in U_{2n} \) such that
\[
U^h A U = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad U^h B U = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix}
\]
and also, by the (2).1 of (2), \( U^h A^2 U = \begin{pmatrix} D^2 & 0 \\ 0 & 0 \end{pmatrix} \) and \( U^h B^2 U = \begin{pmatrix} D^2 & 0 \\ 0 & E^2 \end{pmatrix} \)

\( \alpha \) and \( \beta \) have nonzero \( J_S \) eigen values of \( A \) and \( B \) respectively. Therefore \( \alpha^2 \) and \( \beta^2 \) nonzero \( J_S \) eigen values of \( A^2 \) and \( B^2 \) respectively. Suppose, \( X \) be the common \( J_S \)-eigenvector of \( A^2 \) and \( B^2 \), then \( \alpha = \beta \) and \( X \) is a common \( J_S \) eigenvector of \( A \) and \( B \).

Conversely, Assume (2). Then \( U^h A^2 U = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} \) and
\[
U^h B^2 U = \begin{pmatrix} \Delta & 0 \\ 0 & \Gamma \end{pmatrix}, \quad \text{where} \quad U, \Delta, \Gamma \text{ are matrices obtained by applying (2) of Theorem 2.1 to} \ A^2 \text{ and } B^2.
\]

Let \( u_{(i)} \) by the column vectors of \( U \) and denote \( r = \text{rank}(A) \).

For \( i = 1, 2, 3, \ldots, r \), we have \( A^2 u_{(i)} = B^2 u_{(i)} = \delta_{(i)} u_{(i)} \), where \( \delta_{(i)} = \text{diag} A \).

So by the second part of (1), there exist complex numbers \( d_{(i)} \) such that for all \( i = 1, 2, \ldots, r \). We have \( d_{(i)}^2 = \delta_{(i)} \) and \( A u_{(i)} = B u_{(i)} \).

Let \( D \) be the diagonal matrix with \( d_{(i)} = \text{diag} D \).

For \( i = r + 1, \ldots, n \), we have \( B^2 u_{(i)} = \gamma_{(i-r)} u_{(i)} \), where \( \gamma_{(i)} = \text{diag} G \). Take complex numbers \( e_{(1)}, \ldots, e_{(n-r)} \) satisfying \( e_{(i)}^2 = \gamma_{(i)} \), for \( i = 1, 2, \ldots, n - r \).

Let \( E \in D \) with \( e_{(i)} = \text{diag} E \).

Then \( U^h A U = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \) and \( U^h B U = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} \).

This equation satisfies condition (1). Therefore, \( A^2 \preceq^h B^2 \Rightarrow A \preceq^h B \).

\( \square \)

Corollary 2.4. Let \( A, B \in N_{J_S} \) whose all \( J_S \) eigen values have nonnegative real parts. Then \( A^2 \preceq^h B^2 \) if and only if \( A \preceq^h B \).

Theorem 2.5. Let \( A, B \in N_{J_S} \) with \( 1 \leq \text{rank}(A) < \text{rank}(B) \).

Then

1. \( A \preceq^h B \) is equivalent to the following.

2. \( A^2 \preceq^h B^2 \) and if \( U^h A U = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \) and \( U^h B U = \begin{pmatrix} DH & 0 \\ 0 & E \end{pmatrix} \), where \( U \in U_{2n} \), \( D \in D \) with \( |D| \neq 0, H \in U_{2n} \) diagonal matrix and \( E \neq 0 \) is a diagonal matrix, then \( H = I \).

Proof. For (1)\( \Rightarrow \) the first part of (2), see the proof of Theorem 2.3. For (1)\( \Rightarrow \) the second part of (2), see (5) of Theorem 2.1.