A New Formulation of General Relativity - Part I: Pre-Radar Charts as Generating Functions for Metric and Velocity

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Abstract

In this paper (Part I of a series of three papers) an axiomatic formulation of GR is given, here use is made of the concept of pre-radar charts. These charts have "infinitesimally" the same properties as the true radar charts used in space-time theory. Their existence in GR has far-reaching consequences which are discussed throughout the papers. For the sake of simplicity and convenience I consider only such material systems the state of which is defined by a velocity field, a mass density and a temperature field. But the main results hold also for more complex systems. It follows from the axiomatics that the pre-radar charts define an atlas for the space-time manifold and that, in addition, they generate the metric, the velocity field and the displacement of the matter. Therefore, they are called generating functions. They act like "potentials". At the end of the paper it is shown that the existence of pre-radar charts allows to simplify the original axiomatics drastically. But the two versions of GR are equivalent.

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1 Introduction

1.1: The subject of this treatise is the general theory of relativity (GR) in its classical form. In a strict sense, GR is not a single physical theory, rather it is a class of theories which share common features. From this point of view Schwarzschild space-time, Robertson-Walker space-times etc. are counted as separate relativistic theories. The common features of all these theories are “principles” they obey. More properly these principles should be called "axioms" because they have the same status as the axioms in mathematical theories. An inspection of relativistic theories reveals that their axioms can be grouped into five classes as follows.

GK: Axioms concerning geometry and kinematics.
EM: Axioms concerning matter and its motion, e.g. equations of motion and constitutive equations.
ED: Axioms concerning electromagnetism, e.g. Maxwells equations and constitutive equations.
EE: Einstein equation and constitutive equations.
AC: Additional conditions, e.g. initial conditions.

Clearly, in vacuum theories the axioms EM and ED are empty. The above classification of axioms possibly includes redundancies.

1.2: Among the many forms of presenting physical theories there are two extreme forms which are of special interest in our context.

(i) The first form is characterized by the property that the fundamental terms as in our case a set of events $M$, an atlas $A$, a metric $g$, a velocity field etc. are implicitly determined solely by axioms. Examples of such formulations of theories are well known in many branches of physics.

(ii) The second form of a physical theory can be characterized as "model theory", also widely known under the label "solution". In this case the letters $M$, $A$, $g$ etc. are replaced by explicit terms of mathematical analysis, and the axioms of case (i) occur as theorems, i.e. the axioms are satisfied by these explicit terms. Also this type of a theory is well known in physics. Clearly, mixed forms are on the market, too. In what follows, I will consider relativistic theories according to the first form. Formulating their axioms, I will make use of some results of the space-time theory (STT) which is developed in [1], [2], [3] and which is reviewed in [4]. A detailed account of this STT can be found in [5]. More specific, I will take some features of radar coordinates in order to define a weaker form of them which I call pre-radar coordinates. Using these coordinates in the context of GR is the
essential new aspect of this treatise. They can be comprised into one function \( \Psi \) depending on two events \( p, q \) which is a generating function for the atlas \( \mathcal{A} \), the metric \( g \) and the velocity \( v \), and which, in addition, determines the integral curves of \( v \).

1.3: A few remarks may illustrate the notion of radar coordinates as it is used in [1] to [5]. Let \( A \) be an observer, \( b \) an event and \( t_1, t_2 \) times measured by the clock of \( A \). Finally, let \( e = (e^1, e^2, e^3) \) be the direction of a light signal which leaves \( A \) at instant \( t_1 \) and comes back to \( A \) at \( t_2 \) after being reflected at \( b \). Then, if

\[
 t := \frac{1}{2}(t_2 + t_1), \quad r := \frac{1}{2}(t_2 - t_1)
\]

the radar coordinates \( x \) of \( b \) are given by \( x = (re^1, re^2, re^3, t) \). Here and in the sequel all quantities are dimensionless, and \( c = 1 \).

If \( A \) describes this situation with the help of its own radar coordinates he or she will get in two dimensions of \( \mathbb{R}^4 \) the picture shown in Fig. 1. Here the outgoing light signal is a straight line by definition, whereas the incoming signal is not necessarily straight, but it is a curve which is a subset of a Minkowskian backward light cone. Intuitively, a pre-radar chart of an observer \( A \) has the properties of a radar chart only in an "infinitesimal neighborhood" of the worldline of \( A \). In each case, radar coordinates are also pre-radar coordinates as introduced in Definition (3.2) of Section 3.1. It is to be emphasized that the term radar coordinate is not uniformly used in literature. Two examples may illustrate it. Coleman and Korté [7] define radar charts which are similar to those introduced above. The difference is that the measurement of the direction \( e \) is performed on the incoming part of the radar signal. The radar coordinates used in EPS axiomatics [6] are more different from ours. Here the radar coordinates \( (u, v, u', v') \) of an event \( a \) are defined by two observers \( A \) and \( A' \) which send out radar signals starting at the times \( u, u' \) and arriving at \( v, v' \) after being reflected at \( a \).

1.4: Though the class of theories I want to consider are continuum theories the notions observer, particle or real point are employed. This does not contradict the continuum point of view. Rather it reflects only the fact that we have two possibilities to describe continuum systems, namely in the way of Lagrange as systems of particles or in the way of Euler by fields. Later on I will use a mixture of both these descriptions.
1.5: Throughout this paper I will use the following strategy: each material point, i.e. each point which contributes to gravitation, is a part of the system of pre-radar observers. However it is possible that there are gravitationally irrelevant test particles which are pre-radar observers.

1.6: As indicated in Section 1.2 I intend to give a new axiomatic formulation of General Relativity. There is already a considerable amount of work done in axiomatics of relativity and space-time theory which cannot be reviewed here. Rather I refer to the paper of Schelb [5] where the reader will find an almost complete list of relevant papers. One of the most recent treatises in this field is that of Hehl and Obukhov [8] where a Lorentz metric on a manifold is constructed using electrodynamics without metric. But in what follows I do not adopt this general supposition, rather I will take into account only some special features of electrodynamics by using light signals as a basic concept. (Cf. also [9] and the literature quoted there.)
2 Description of the Systems to be Considered

2.1: In what follows, I will not treat the most general continuum systems, rather I consider only material systems which can be described by one velocity field $v$, one mass density $\eta$ and one empirical temperature $\vartheta$. This means that mixtures of fluids, especially plasmas are not taken into account. (Mixtures of fluids are treated e.g. in [10], p. 131). The above restriction is only a matter of convenience because the main results of this treatise are also valid for more complex systems.

Though each system considered is supposed not to have electromagnetic interaction it is assumed that all observers, i.e. particles of the material system and test particles can exchange light signals. It is assumed that these signals are irrelevant with respect to any kind of interaction. The only thing they can transport is information. They have the same status as test particles. The reason to take into account light signals is that we want to introduce pre-radar-coordinates. This can be effected most simply if we have light signals at our disposal.

2.2: In order to formulate a physical theory in the sense of (i) of Section 1.2 which is adequate to treat the systems described in Section 2.1, the fundamental mathematical terms have to be specified with the help of which the whole theory can be formulated and which are implicitly determined by the axioms of the theory. There are two kinds of fundamental terms, the so-called base sets and the so-called structural terms. The first ones contain the signs for the objects to be treated, whereas the latter, the structural terms define the basic properties of these objects. Expressed mathematically, the structural terms are relations which are elements of sets constructed from the base sets solely with the help of the operations ”power set” and ”Cartesian product”.

Let us first specify the base sets. The most fundamental term in any relativistic theory is the set of signs for events. This is indispensable! In addition, we want to speak about particles some of which bear gravitationally active masses, and we take into account (light) signals. Finally, we need the real numbers because we have to express the values of some quantities by numbers.
Let us now come to the structural terms. They are the metric, the velocity field, the mass density and the (empirical) temperature field. As usual in GR the set of events should be a manifold. This means that there are coordinates defined on certain sets of events. The structural term which is introduced in the present context is a relation which assigns four coordinates to each event in some neighborhood of an observer.

It is convenient to assign a common name to the theories treated in this paper. Notation (2.1): The physical theories which are determined by the above mentioned (and in Section 2.3 precisely described) base sets and structural terms which in turn are ruled by the axioms of Sections 3 and 4 are denoted \( \Phi_R \). Since \( \Phi_R \) represents a class of physical theories (in a strict sense) it is called a frame theory.

2.3: Summing up the considerations of Section 2.2 we arrive at the following result: The base sets of \( \Phi_R \) are \( M, P, S, \mathbb{R} \):
- \( M \) is the set of signs \( a, b, \ldots \) etc. for events
- \( P \) is the set of signs (i.e. indices) \( A, B, \ldots \) etc. for particles
- \( S \) is the set of signs (i.e. indices) \( s, s', \ldots \) etc. for signals
- \( \mathbb{R} \) is the set of the real numbers as usual.

The structural terms of \( \Phi_R \) are \( \hat{\Psi}, g, v, \eta, \vartheta \):
- \( \hat{\Psi} \) determines the pre-radar coordinates, i.e. \( (A, b, x) \in \hat{\Psi} \) means that observer \( A \) coordinatizes event \( b \) by \( x \)
- \( g \) is the metric
- \( v \) is the velocity field
- \( \eta \) is the mass density
- \( \vartheta \) is the temperature field.

3 Geometric and Kinematic Axioms

Following the notation of Section 1.1 the axioms of this section are denoted GK. Throughout the Sections 3, 4, and 5 the same natural numbers \( k \) resp. \( r = k - 1 \) are meant in phrases like "\( \ldots \in C^k, k \geq 3 \)" or "\( \ldots \in C^r, r \geq 2 \)."
3.1 Pre-radar charts

Intuitively, a pre-radar chart is a chart which has some (or perhaps all) of the properties of a true radar chart. Therefore, the following axioms can be motivated by pointing to the fact that true radar charts have a certain property. In Section 1.3 it was outlined how a radar observer coordinatizes a neighborhood of its worldline: he or she needs clocks and devices for measuring directions. In what follows it is assumed that each observer uses one and only one clock and one and only one directional measuring instrument. Therefore, the radar charts of observers thus equipped are unique. Hence the following axiom is self-evident:

\textbf{GK 1.1:} The structural term $\hat{\Psi}$ is a function: $\hat{\Psi} : \bigcup_{A \in P} \{A\} \times V_A \rightarrow \mathbb{R}^4$, where $V_A \subset M$ and $V_A \neq \emptyset$.

It is useful to introduce some notation.

\textbf{Definition (3.1):} For short we write $\psi_A := \hat{\Psi}(A, \cdot)$ and $O_A := \text{ran} \psi_A$; by definition $\text{dom} \psi_A = V_A$. Then let $\mathcal{A} = \{(V_A, \psi_A) : A \in P\}$.

The next axiom expresses that $\hat{\Psi}$ determines a manifold structure on $M$.

\textbf{GK 1.2: $\mathcal{A}$ is a $C^k$-atlas, $k \geq 3$ such that $(M, \mathcal{A})$ is a connected Hausdorff manifold.}

A motivation for further axioms comes from the fact that each radar observer $A$ coordinatizes himself or herself by $(0, 0, 0, t)$, the parameter $t$ being the time $A$ measures. In addition, the fourth component of each quadruple of radar coordinates measured by $A$ is a time at $A$.

\textbf{GK 1.3:} For each $A \in P$ there are two real numbers $u_1, u_2$ with $-\infty \leq u_1 < u_2 \leq \infty$ such that for each $\tau \in ]u_1, u_2[ =: J_A$ the relation $(0, 0, 0, \tau) \in \mathcal{O}_A$ holds; moreover, if $y_1 < u_1$ or $u_2 < y_2$, then $\{(0, 0, 0, \rho) : \rho \in ]y_1, y_2[ \} \not\subseteq \mathcal{O}_A$, i.e. $J_A$ is maximal.

Finally, some additional notation is introduced by
**Definition (3.2):** 1. The term $\mathcal{D}$ is the differential structure containing all charts which are $C^k$-compatible, $k \geq 3$ with $\mathcal{A}$.
2. The coordinates determined by the charts $(V_A, \psi_A)$ of $\mathcal{A}$ will be called for short $A$-coordinates etc. whereas the others are denoted by their coordinate functions $\chi$ etc.
3. Within the theory $\Phi_R$ the charts of $\mathcal{A}$ are called pre-radar charts.

### 3.2 Worldlines of particles

In our context there are two possibilities to define the worldline of an observer. First, the worldline of $A$ is the set of events which occur at $A$. Second, the worldline of $A$ is the set of events which $A$ coordinatizes by $(0,0,0,t)$. This leads to the following

**Definition (3.3):** For each $A \in \mathcal{P}$ the (surjective) function $\gamma_A : J_A \to W_A \subset M$ is defined by $\gamma_A(t) = \psi_A^{-1}(0,0,0,t), t \in J_A$. The set $W_A := \text{ran} \gamma_A$ is called the worldline of $A$. As usual $\dot{\gamma}_A$ denotes the velocity of $A$.

**Definition (3.4):** For each $A \in \mathcal{P}$ the clock $\mathcal{U}_A$ of $A$ is defined by $\mathcal{U}_A(a) = \psi_A^4(a), a \in W_A$.

**Remark (3.5):** 1. From Definition (3.3) it follows that $\psi_A \circ \gamma_A(t) = (0,0,0,t)$. Therefore, the function $\gamma_A$ is of class $C^k, k \geq 3$ and bijective.
2. From Definition (3.4) one concludes that

$$\mathcal{U}_A(\gamma_A(t)) = \psi_A^4 \circ \psi_A^{-1}(0,0,0,t) = t = \gamma_A^{-1}(\gamma_A(t)).$$

Hence $\mathcal{U}_A(a) = \gamma_A^{-1}(a)$ for all $a \in W_A$.

Since we want to describe a continuum, the set of particles should be a continuum. In addition, we want to describe the system by a smooth velocity field. Therefore, the worldlines of different particles cannot have common events. Hence the next two axioms are self-evident.

**GK 2.1:** The cardinality of $\mathcal{P}$ is that of a continuum and $\bigcup_{A \in \mathcal{P}} W_A = M$.

**GK 2.2:** For all $A, B \in \mathcal{P}$: if $A \neq B$ then $W_A \cap W_B = \emptyset$. 
In other words this axiom expresses that the set \( P \) of particles represents a congruence.

**Remark (3.6):** It follows directly from the Axioms GK 2.1 and 2.2 that there is a function \( F : M \to P \) which is surjective and which is given by \( F(a) = A \) for all \( a \in W_A := \text{ran}\, \gamma_A \).

With the help of \( F \) we are now able to define a function \( \Psi \) which later on turns out to be a generating function for the metric \( g \) and the velocity \( v \).

**Definition (3.7):** The function \( \Psi : \bigcup_{q \in M} V_{F(q)} \times \{q\} \to \mathbb{R}^4 \) is defined by \( \Psi(p,q) = \hat{\Psi}(F(q),p) = \psi_A(p) \) for \( A = F(q) \). The values of \( \Psi \) and \( \hat{\Psi} \) are mostly written as row vectors: \( \Psi = (\Psi_1, \ldots, \Psi_4) \) etc. But occasionally it is more convenient to write them as column vectors: \( \Psi = (\Psi_1, \ldots, \Psi_4)^T \) etc.

At this point, \( \Psi \) is nothing but another form of \( \hat{\Psi} \) which has the advantage that it allows to express the intuitively reasonable property of pre-radar coordinates that ”neighboring” observers attribute ”neighboring” coordinates to the same event. This means that \( \Psi \) is ”smooth” with respect to all of its arguments. Therefore, the following axiom is natural.

**GK 2.3:** \( \Psi \) is of class \( C^k \) with \( k \geq 3 \).

### 3.3 Axioms for the metric

Since the metric \( g \) is defined in the usual way it suffices to write down the axioms governing \( g \).

**GK 3.1:** The structural term \( g \) is a function \( g : \bigcup_{a \in M} \{a\} \times (T_aM \times T_aM) \to \mathbb{R} \), where \( T_aM \) is the tangent space at \( a \in M \).

It is convenient to introduce the

**Notation (3.8):** \( g(a,w,w') = g(a)(w,w') \) and \( g(a) = g_a = g(a)(\cdot,\cdot) \).
GK 3.2: \( g = g(\cdot) \) is a \( (0) \)-tensor field of class \( C^r \), \( r \geq 2 \).

GK 3.3: For each \( a \in M \): \( g_a \) is symmetric, non-degenerate and of signature 2.

With the help of \( g \) the clocks used by the observers now can be specified: they are to show proper time. This is the content of the next axiom which joins particles and metric.

GK 3.4: For each \( A \in P \) and each \( t \in J_A \): \( g(\gamma_A(t))(\dot{\gamma}_A(t), \dot{\gamma}_A(t)) = -1 \).

### 3.4 Properties of signals

In this subsection the relation between particles and signals is studied. The motivation for the following axioms comes from space-time theory (cf. [1],[2],[3],[4],[5]) where proper radar charts are treated in great detail. It is not possible to repeat all arguments of these papers here. One result may suffice. A true radar observer \( A \) describes outgoing light signals within his or her coordinate system by straight lines leaving the worldline of \( A \) by an angle of 45 degrees. In what follows one should also have in mind that \( S \) is merely a set of indices for signals.

GK 4.1: For each \( s \in S \) there is a function \( \sigma_s : K_s \to W_s^* \subset M \), \( K_s \subset \mathbb{R} \) with the following properties:

\( \sigma_s \) is of class \( C^k \), \( k \geq 3 \) and a null geodesic; moreover there is an \( A \in P \) such that

\[
K_s = [t_{s0}, t_{s1}] \subset J_A;
\]

\[
\psi_A^A(\sigma_s(t)) = t \quad \text{for all} \quad t \in K_s;
\]

there is an \( a \in W_A \) such that \( \sigma_s(t_{s0}) = a \);
there is an \( e = (e^1, e^2, e^3) \in S^2 \) (the 2-sphere) such that
\[
\frac{d}{dt} \psi_A(\sigma_s(t_{s0})) = (e^1, e^2, e^3, 1).
\]

To a certain extent also the converse of this axiom is needed.

**GK 4.2:** For each \( A \in P \), for each \( a \in W_A \) and for each \( (e^1, e^2, e^3) \in S^2 \) there is an \( s \in S \) such that the function \( \sigma_s \) determined by axiom GK 4.1 has the following properties:

\[
\sigma_s(t_{s0}) = a;
\]

\[
\frac{d}{dt} \psi_A(\sigma_s(t_{s0})) = (e^1, e^2, e^3, 1).
\]

### 3.5 Velocity

According to Definition (3.3) the velocity of particle \( A \) is \( \dot{\gamma}_A \). Therefore, the field \( v \) is determined by \( \dot{\gamma}_A \). This is the content of Axiom GK 5.

**GK 5:** The structural term \( v \) is a function \( v : M \to TM \) which is defined for each \( b \in M \) by \( v(b) = \dot{\gamma}_{F(b)}(\gamma_{F(b)}^{-1}(b)) \) and which is of class \( C^r, r \geq 2 \).

### 4 Further Axioms

#### 4.1 Equations of motion

Up to now only the geometric and the kinematic part of the theory \( \Phi_R \) was treated, and this part can be characterized, roughly speaking, by the phrase: everything is smooth. For the formulation of the further axioms we have to take matter into account. The basic quantities describing matter can again be smooth, but there are many interesting systems which show material discontinuities. It is not possible to treat these different classes of systems by the same kind of axioms. Therefore I restrict the further studies to the most simple class of systems which are those with smooth \( \eta \) and \( \vartheta \). In this case the following axiom is obvious.
**EM1**: The structural terms $\eta$ and $\vartheta$ are functions $\eta : M \to \mathbb{R}$, $\vartheta : M \to \mathbb{R}$ which are of class $C^r$, $r \geq 2$.

In a next step the constitutive equations have to be specified. In a theory of type $\Phi_R$ as treated in this paper there is only one constitutive element, the energy-momentum tensor $T$, and the constitutive equation relates $T$ to the structural terms $g, v, \eta, \vartheta$. This is the content of the next axiom where the bundle of $(^{n}_{m})$-tensors is denoted $T^m_n$.

**EM2**: $T$ is a function $T : M \to T^2_0 M$ which is of class $C^r, r \geq 2$ and which is defined by the functional $\mathcal{T}$ of $g, v, \eta, \vartheta$ for all $q \in M$ by $T(q) = \mathcal{T}(g, v, \eta, \vartheta)(q) \in T^2_0 M$. If $\eta(q) = 0$ for $q \in N \subset M$ then $\mathcal{T}(g, v, \eta, \vartheta)(q) = 0$.

This axiom is the point why $\Phi_R$ is a frame theory, for $\mathcal{T}$ is not explicitly specified. Each such specification of $\mathcal{T}$ defines a subclass of the systems governed by the (frame) theory $\Phi_R$. The usual forms of the energy-momentum tensor for systems of dust or for general Euler fluids are examples of classifying systems with the help of $\mathcal{T}$.

The next two axioms are the equation of continuity and the balance of energy and momentum. Though the latter is a consequence of Einstein’s equation it is written down here.

**EM3**: Throughout $M : \text{div}(\eta v) = 0$.

**EM4**: Throughout $M : \text{div}(T) = 0$.

### 4.2 Einstein equation

In our context there is no need to comment on Einstein’s equation or to motivate it. It suffices to write it down. So let, as usual, $R$ be the Ricci tensor, $\bar{R}$ the Ricci scalar, $\Lambda_0$ the (unspecified) cosmological constant and finally $\kappa_0$ Einstein’s gravitational constant. Moreover, $T^b$ denotes the covariant energy-momentum tensor.

**EE**: Throughout $M$: $R - \frac{1}{2} g \bar{R} + \Lambda_0 g = \kappa_0 T^b$. 
4.3 Additional conditions

The general term additional conditions (AC) comprises all those axioms which have to be imposed in order to get physically relevant and uniquely determined classes of models. In this context, by a model (or a "solution") the following is to be understood:

**Notation (4.1):** Let $M', P', S', \Psi', g', v', \eta', \vartheta'$ be terms defined within mathematical analysis or, more precisely, within the theory of sets such that the axioms GK, EM, EE and AC for a specified functional $T$ and a specified cosmological constant $\Lambda_0$ are satisfied. Then we say that these terms define an analytical or a set theoretical model of the frame theory $\Phi_R$.

A model is a theory of the form described in (ii) of Section 1.2. In this sense Robertson-Walker space-times, the Schwarzschild space-time etc. are models of $\Phi_R$. But this is not proved here.

There is a great variety of AC. Three examples may illustrate the role AC play: (i) Initial conditions, e.g. for solving a Cauchy problem. (ii) Boundary conditions, e.g. for space-times which are to be asymptotically flat in a certain sense. (iii) Symmetry conditions by which e.g. a general ansatz can be restricted to a more special form.

A more detailed discussion of this subject is outside the scope of this paper.

5 Some Consequences of the Axioms

5.1 Metric and velocity in B-coordinates

First of all let us fix some

**Notation (5.1):** The components of $v$ and $g$ with respect to B-coordinates (cf. Definition (3.2)) for each $B \in P$ are denoted $v_B^\alpha(y)$ and $g_B^{\alpha\beta}(y)$ where $y \in \psi_B[V_B]$. 
For the general coordinates $\chi$ with $(W, \chi) \in \mathcal{D}$ we write
\[ v^\alpha_{\chi}(x) \quad \text{and} \quad g^\chi_{\alpha\beta}(x) \quad \text{where} \quad x \in \chi[W]. \]

Then the following simple lemma holds:

**Lemma (5.2):** For each $B \in P$ and for all $y \in \psi_B[W_B]$ it holds that
\[ v_B^\alpha(y) = \delta^\alpha_4. \]

**Proof:** By Axiom GK5 and by making use of Definition (3.4) for all $b \in W_B$ one finds that $v(b) = \gamma_B(t)$ where $t = \mathcal{U}_B(b)$. Let $\gamma_B(t) = w_B^\alpha(t) \partial_{\psi_B^\alpha}$. Then
\[ w_B^\alpha(t) = \frac{d}{dt}(\psi_B^\alpha \circ \gamma_B)(t). \]

From Remark (3.5) it follows that $w_B^\alpha(t) = \delta^\alpha_4$ for all $t = \mathcal{U}_B(b)$ and $b \in W_B$, i.e. for all $t \in J_B$. Let $y = \psi_B(b)$ and $t = \mathcal{U}_B(b)$. Then for each $b \in W_B$ we have $v_B^\alpha(y) = w_B^\alpha(t)$, so that the proposition is proved.

A similar result holds for the metric $g$:

**Lemma (5.3):** For each $B \in P$ and for all $y \in \psi_B[W_B]$ it holds that
\[ g_B^{\alpha\beta}(y) = \eta_{\alpha\beta} \]
where $(\eta_{\alpha\beta}) := \text{diag}(1, 1, 1, -1)$ is the Minkowski matrix.

**Proof:** In what follows, for the sake of simplicity the argument $(y)$ is omitted.
1. From the Axioms GK 3.4 and GK 5 it follows that $g_B^{\alpha\beta}v_B^\alpha v_B^\beta = -1$.

By Lemma (5.2) we have $v_B^\alpha = \delta^\alpha_4$ so that $g_B^{\alpha\beta}v_B^\alpha v_B^\beta = g^{\beta}_{44} = \eta_{44}$.

2. From the Axioms GK 4.1 and 4.2 one concludes that for all $(e^1, e^2, e^3)$ the relation
\[ \sum_{j,k} g_j^k e^j e^k + 2 \sum_j g_{j4} e^j - 1 = 0 \]
holds. Now let $e^1 = \pm 1$, $e^2 = 0$, $e^3 = 0$. Then $g^B_{11} \pm 2g^B_{14} = 1$. Hence $g^B_{11} = 1 = \eta_{11}$, $g^B_{14} = g^B_{41} = 0 = \eta_{14}$. Similarly we find that $g^B_{22} = 1 = \eta_{22}$, $g^B_{24} = g^B_{42} = \eta_{24}$, $g^B_{33} = 1 = \eta_{33}$, and $g^B_{34} = g^B_{43} = \eta_{34}$.

Finally let $e^1 = \frac{1}{\sqrt{2}}$, $e^2 = \frac{1}{\sqrt{2}}$, $e^3 = 0$. Then $g^B_{12} = g^B_{21} = 0 = \eta_{12} = \eta_{21}$.

By analogous arguments one finds that $g^B_{jk} = 0 = \eta_{jk}$, $k = 1, 2, 3$, $j \neq k$.

**Remark (5.4):** 1. Lemma (5.3) is remarkable in so far as it states that for each observer $B$ the metric in $B$–coordinates, i.e. $g^B_{\alpha \beta}$ is Minkowskian not only for one point but for each point of the whole worldline $W_B$. Therefore, by Axiom GK 2.1 the metric $g$ as well as the velocity $v$ is completely determined on $M$ once for all $B \in P$ the worldlines $W_B$ are known. These in turn are determined by the coordinate function $\Psi$ resp. $\Psi$.

2. As already mentioned in Section 1.3, the proper radar coordinates are also pre-radar coordinates, i.e. $B$-coordinates for some observer $B$. Also the Fermi coordinates introduced by Synge in [13] p. 84 can be used to define pre-radar coordinates. But there are pre-radar charts which are neither radar charts nor Fermi charts.

3. It is stated without proof that a pre-radar observer $B$ is freely falling exactly if

$$\frac{\partial}{\partial y^j} g^B_{44} = 0$$

for all $y \in \psi_B[W_B]$ and $j = 1, 2, 3$.

### 5.2 Coordinate Representations for metric and velocity

In this section we are looking for explicit expressions for $g^\chi_{\alpha \beta}$ and $v^\alpha$ where $\chi$ are arbitrary coordinates. First of all we have to fix some notation.

**Definition (5.5):** Let $(W, \chi) \in D$ and $(V_B, \psi_B) \in A$. Then:

1. $\phi_{\chi B} := \psi_B \circ \chi^{-1}$, $\phi_{B \chi} := \phi^{-1}_{\chi B}$, $\phi_{AB} := \psi_B \circ \psi^{-1}_A$.

2. $G_{\chi} := F(\chi^{-1})$, where $F$ is defined in Remark (3.6).

3. $\phi_{\chi} := \Psi(\chi^{-1}, \chi^{-1})$ (cf. Definition (3.7)).

4. $\Lambda_{\chi} = ((\Lambda_{\chi}^\alpha_{\beta}))$ with $\Lambda_{\chi}^\alpha_{\beta}(x) := \frac{\partial \phi^\alpha_{\beta}(x,z)}{\partial x^\gamma} \bigg|_{z=x}$ where $\alpha$ denotes the rows.
and $\beta$ the columns.

**Remark (5.6):** Since $\Psi(p,q) = \psi_{F(q)}(p)$ it follows that

$$\phi_\chi(x,z) = \psi_{G_\chi(z)}(\chi^{-1}(x)) = \phi_\chi G_\chi(z)(x).$$

Next the representation for the metric is deduced.

**Proposition (5.7):** If $(W, \chi) \in \mathcal{D}$ then for all $x \in \text{ran}\chi$ we have

$$(5.1) \quad g^\chi_{\alpha\beta}(x) = \Lambda_\chi^\kappa \alpha (x) \Lambda_\chi^\lambda \beta (x) \eta_{\kappa\lambda}.$$  

**Proof:** Let $x = \chi(p)$ and $y = \psi_B(p)$. Then $y = \phi_\chi B(x)$. The matrix elements $g^\chi_{\alpha\beta}(x)$ and $g^B_{\kappa\lambda}(y)$ are related by

$$g^\chi_{\alpha\beta}(x) = \frac{\partial \phi^\kappa_\chi B(x)}{\partial x^\alpha} \frac{\partial \phi^\lambda_\chi B(x)}{\partial x^\beta} g^B_{\kappa\lambda}(y).$$

Therefore, if $B = G_\chi(z)$ it follows from Remark (5.6) that

$$g^\chi_{\alpha\beta}(x) = \frac{\partial \phi^\kappa_\chi (x,z)}{\partial x^\alpha} \frac{\partial \phi^\lambda_\chi (x,z)}{\partial x^\beta} g^B_{\kappa\lambda}(y).$$

Now let $z = x$. Then $B = G_\chi(x) = F(\chi^{-1}(x)) = F(p)$, so that $p \in W_B$ and $y \in \psi_B[W_B]$. Hence by Lemma (5.3) we have $g^B_{\kappa\lambda}(y) = \eta_{\kappa\lambda}$ for all $y \in \psi_B[W_B]$, so that by use of Definition (5.5) the proposition holds.

Also the components of velocity $v$ can be expressed by $\Lambda_\chi$ in the following way.

**Proposition (5.8):** If $(W, \chi) \in \mathcal{D}$ then for all $x \in \text{ran}\chi$ we get:

$$(5.2) \quad v_\chi^\alpha(x) = \Lambda_\chi^{-1} \alpha_4 (x).$$

**Proof:** Let $x = \chi(p)$, $y = \psi_B(p)$ and $x = \phi_B \chi(y)$. Then
\[ v^\alpha_x(x) = \frac{\partial \phi^\alpha_{B\chi}(y)}{\partial y^\beta} \, v^\beta_B(y). \]

By Definition (5.5) we have \( \Phi^-_{B\chi} = \Phi^-_{\chi B}. \) Therefore
\[
v^\alpha_x(x) = \left[ \left( \frac{\partial \phi_{\chi B}(x)}{\partial x} \right)^{1\!\!\!1}_\beta \right] \, v^\beta_B(y).
\]

If \( B = G\chi(z) \) then from Remark (5.6) it follows that
\[
v^\alpha_x(x) = \left[ \left( \frac{\partial \phi_{\chi}(x, z)}{\partial x} \right)^{1\!\!\!1}_\beta \right] \, v^\beta_B(y).
\]

Finally let \( x = z. \) Then \( B = G\chi(x) = F(p), \) hence \( p \in W_B \) and \( y \in \psi_B[W_B]. \) Therefore, from Lemma (5.2) we have \( v^\alpha_B(y) = \delta^\alpha_4. \) Using Definition (5.5) the proposition is proved.

**Corollary (5.9):** For the covariant components of the velocity the relation
\[(5.3)\]
\[ v^\chi_\alpha(x) = -\Lambda_{\chi \alpha}^4(x). \]
holds. For using the Propositions (5.7) and (5.8), we have
\[ v^\chi_\alpha = \Lambda_{\chi}^{-1\!\!\!1} \Lambda_{\chi \alpha}^\kappa \Lambda_{\chi \beta}^{\lambda} \eta_{\kappa \lambda} = -\Lambda_{\chi \alpha}^4. \]

Similarly, the contravariant components of the metric are
\[(5.4)\]
\[ g^{\alpha \beta}_\chi = \Lambda_{\chi}^{-1\!\!\!1} \Lambda_{\chi \alpha}^\kappa \Lambda_{\chi \beta}^{-1\!\!\!1} \eta^{\kappa \lambda}. \]

This follows directly from \((g^{\alpha \beta}_\chi)^{-1} = ((g^{\chi}_{\alpha \beta}))^{-1}\) and \( \eta^{\kappa \lambda} = \eta_{\kappa \lambda}. \)
5.3 Global representations

In this section the results of Section 5.2 will be shaped in a form which is independent of coordinates. These considerations show again the role the function $\Psi$ plays. We start with some notation.

**Definition (5.10):** Let $\Psi$ be the function introduced in Definition (3.7). Then

\[
e_{q\alpha}(p) := \partial_{\Psi^\alpha(\cdot, q)}\bigg|_p,
\]

\[
\Theta^\beta_q(p) := d\Psi^\beta(\cdot, q)\bigg|_p.
\]

**Remark (5.11):** Since $\Psi(\cdot, q)$ is a coordinate function, $(e_{q1}(p), \ldots, e_{q4}(p))$ is a tetrad in $T_pM$ and $(\Theta^1_q(p), \ldots, \Theta^4_q(p))$ is the dual tetrad in $T^*_pM$. Hence for all $p \in V_{F(q)}$ we have:

\[
\Theta^\beta_q(p)(e_{q\alpha}(p)) = \delta^\beta_\alpha = e_{q\alpha}(p)(\Theta^\beta_q(p)).
\]

In what follows we need only a special form of these bases.

**Notation (5.12):** 1. For all $p \in M$ we write:

\[
e_\alpha(p) := e_{p\alpha}(p), \quad \Theta^\beta(p) := \Theta^\beta_p(p).
\]

2. To simplify notation, the arguments $(p), (x)$ etc. and the index $\chi$ indicating a coordinate system are mostly omitted in the sequel.

**Proposition (5.13):** Let $(W, \chi) \in \mathcal{D}$. Then for all $p \in W$, $x = \chi(p)$ and $\Lambda = \Lambda_\chi$:

\[
\Theta^\beta(p) = \Lambda^\beta_\kappa(x)dx^\kappa,
\]

\[
e_\alpha(p) = \Lambda^{-1\alpha}_\lambda(x)\partial_x^\lambda.
\]
Proof: Let $q$ be fixed and $z = \chi(q)$. Then $y = \Psi(\chi^{-1}(x), q) = \phi(x, z)$ is the transformation between $\chi$-coordinates $x$ and pre-radar coordinates $y$. Therefore

$$dy^\beta = \frac{\partial \phi^\beta}{\partial x^\kappa} dx^\kappa, \quad \partial x^\alpha = \frac{\partial \phi^\lambda}{\partial x^\alpha} \partial y^\lambda.$$ 

Hence

$$\partial y^\lambda = \left[ \left( \frac{\partial \phi}{\partial x} \right)^{-1} \right]_\lambda^\alpha \partial x^\alpha.$$ 

With $z = x$ and Definition (5.5) the proposition is seen to hold.

Remark (5.14): The matrix elements $\Lambda^\beta_\kappa, \kappa = 1, \ldots, 4$ are the $\chi$-components of $\Theta^\beta$ and the $\Lambda^{-1}_\alpha^\lambda, \lambda = 1, \ldots, 4$ are the $\chi$-components of $e_\alpha$, i.e. $\Lambda^\beta_\kappa = \Theta^\beta(\partial x^\kappa)$ and $\Lambda^{-1}_\alpha^\lambda = e_\alpha(dx^\lambda)$. Hence under transformation of coordinates they transform like components of covectors and vectors.

Proposition (5.13) has an immediate consequence for $g$ and $v$. Inserting the formulae (5.9) and (5.10) into formulae (5.1) to (5.4) we arrive at the following result.

Proposition (5.15): Throughout $M$ we have:

$$g = \eta_{\alpha\beta} \Theta^\alpha \otimes \Theta^\beta, \quad g^\sharp = \eta^{\kappa\lambda} e_\kappa \otimes e_\lambda, \quad v^\flat = -\Theta^4, \quad v = e_4.$$ 

Therefore, it is justified to say that $\Psi$ generates $g$ and $v$.

The result can be stated thus: with respect to $(\Theta^1, \ldots, \Theta^4)$ the tetrad components of $g$ are $\eta_{\alpha\beta}$, and with respect to $(e_1, \ldots, e_4)$ the tetrad components of $v$ are $\delta^4_\lambda$.

Remark (5.16): Using Proposition (5.15) and Equation (5.7) we find the orthogonality relations

$$g(e_\kappa, e_\lambda) = \eta_{\kappa\lambda}, \quad g^\sharp(\Theta^\alpha, \Theta^\beta) = \eta^{\alpha\beta},$$

(5.11)
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and

\[(5.13) \quad \eta^{\kappa\lambda} g(e_\lambda, \cdot) = \Theta^\kappa, \quad \eta_{\alpha\beta} g^\flat(\Theta^\beta, \cdot) = e_\alpha.\]

Roughly speaking, the result of chapter 5 is this: the function \(\Psi\) is a "potential" such that the metric \(g\) and the velocity \(v\) are determined by the derivates of \(\Psi\). Moreover, \(\Psi\) itself has a physical meaning, namely \(\Psi(\cdot, q)\) is a coordinate function for each \(q \in M\).

The existence of the fields \(\Theta^\alpha, e_\beta, \alpha, \beta = 1, \cdots, 4\), hence the existence of the function \(\Psi\) which determines \(\Theta^\alpha\) and \(e_\beta\), has a consequence concerning time:

**Remark (5.17):**

1. From the Axioms GK 3.4 and GK 5 and from Proposition (5.15) one concludes that \(e_4 = v\) is a timelike \(C^r\)-vector field, \(r \geq 2\) on \(M\) so that it nowhere vanishes. Therefore, the space-time manifold \((M, A, g)\) is time orientable (cf. e.g. [11] p. 26). Already from Definition (3.4) where the clock \(U_A\) of a particle \(A \in P\) is introduced we conclude that \((W_A, U_A)\) is a one-dimensional manifold with a global chart for each \(A \in P\). Therefore, \(W_A\) cannot be a closed curve.
2. Following Geroch [12], in a noncompact space time the existence of a smooth global field of tetrads is a necessary condition of a spinor structure.

**Remark (5.18):** The function \(\Psi\) determines not only \(g\) and \(v\) but also a worldfunction \(\Omega\) (cf. [13]) by

\[\Omega(p, q) = \eta^{\kappa\lambda}(\Psi^\kappa(p, q) - \Psi^\kappa(q, q))(\Psi^\lambda(p, q) - \Psi^\lambda(q, q)).\]

Then it is easily seen that for \(\Omega(x, z) := \Omega(\chi^{-1}(x), \chi^{-1}(z))\) the relation

\[\frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \Omega(x, z)\bigg|_{z=x} = g^\chi_{\alpha\beta}\]

holds. The same result is obtained if one takes the covariant derivatives.

6 Alternative Axiomatics

In the investigation so far the set \(S\) of signals was introduced solely to ensure that Lemma (5.3) is provable. Thus the question arises if it is possible to avoid \(S\) by changing some axioms. This is indeed the case. To see this, let us introduce some
Notation (6.1): Let $\Phi^*_R$ be the frame theory which has the base sets $M, P, \mathbb{R}$ and the structural terms $\hat{\Psi}, g, v, \eta, \vartheta$ which are subject to the following axioms: GK 1.1 to 1.4, GK 2.1 to 2.3, EM, EE and AC together with GK*3 and 4 which read:

**GK*3:** For all $p \in M$:

$$g(p) = \eta_{\alpha\beta} \left. d\Psi^\alpha(p, q) \right|_{q=p} \otimes \left. d\Psi^\beta(p, q) \right|_{q=p}.$$  

**GK*4:** For all $p \in M$:

$$v(p) = \left. \partial_{\Psi^4(p, q)} \right|_{q=p},$$

Then the following theorem holds.

**Proposition (6.2):** The theories $\Phi_R$ and $\Phi^*_R$ are equivalent in the following sense. The theory $\Phi_R$ is stronger than $\Phi^*_R$, i.e. all axioms of $\Phi^*_R$ are theorems in $\Phi_R$. Conversely, there is a term $S^*$ defined in $\Phi^*_R$ such that all axioms of $\Phi_R$ are theorems in $\Phi^*_R$ if the letter $S$ is replaced by the term $S^*$.

**Proof:** From Proposition (5.15) one concludes that $\Phi_R$ is stronger than $\Phi^*_R$. To see the converse, one has to show that the Axioms GK 3 to GK 5 of $\Phi_R$ are theorems in $\Phi^*_R$. The term $g$ in Axiom GK* 3 is a $(0,2)$-tensor field throughout $M$ and $g(p)$ acts on $T_pM \times T_pM$ for all $p \in M$. Moreover, $g$ is of class $C^r, r \geq 2$, because $\Psi$ is of class $C^k, k \geq 3$. Finally $g$ is symmetric, non-degenerate and of signature 2 because $\hat{\eta} = \text{diag}(1,1,1,-1)$ has these properties. Therefore, the axioms GK 3.1 to 3.3 are theorems in $\Phi^*_R$.

Within $\Phi^*_R$ the function $\gamma_A$ is defined like in $\Phi_R$. Consequently, by Axiom GK*4 together with Definition (5.10) and Notation (5.12) one has $v(p) = e_4(p)$. Moreover, from Definition (3.3) it follows that $\dot{\gamma}_A(t) = e_4(\gamma_A(t))$. Therefore,

\[(6.1) \quad v(p) = \dot{\gamma}_{F(p)}(t), \quad t = \gamma_{F(p)}^{-1}(p)\]

so that Axiom GK 5 holds in $\Phi^*_R$. Because of $\Theta^a(e_4) = \delta^a_4$ also GK 3.4 is a theorem in $\Phi^*_R$.

In order to prove the Axioms GK 4.1 and 4.2 in $\Phi^*_R$ one has to define a term $S^*$ so that these axioms can be verified in $\Phi^*_R$ if $S$ is replaced by $S^*$. The
definition of $S^*$ runs as follows. For each $A \in P$ let us consider the set of all null geodesics $\sigma$ which start at $W_A$, and let $\zeta = \psi_A \circ \sigma$. Then $\sigma$ obeys the equations

\begin{align}
\nabla_{\dot{\sigma}} \dot{\sigma} &= r \dot{\sigma} \\
g(\dot{\sigma}, \dot{\sigma}) &= 0.
\end{align}

Let the parameter of $\sigma$ be denoted $\lambda$. Then in $\psi_A$-coordinates let $\zeta(\lambda_0) = x$, where $x = \psi_A(p)$, $p \in W_A$. (Generally $\lambda_0$ depends on $p$!) Now for each $p \in W_A$, (6.3) reads

\begin{align}
\eta_{\alpha\beta} \dot{\zeta}^\alpha(\lambda_0) \dot{\zeta}^\beta(\lambda_0) &= 0.
\end{align}

Hence $\dot{\zeta}^4(\lambda_0) > 0$ because only starting null geodesics are considered. This means we can choose $t = \zeta^4(\lambda)$ as a new parameter. Thus, using the same symbol $\zeta$, we have $\dot{\zeta}^4(t_0) = 1$. From (6.4) it follows that $\dot{\zeta}(t_0) = (e^1, e^2, e^3, 1)$ where $(e^1, e^2, e^3) \in S^2(S^2$ being the 2-sphere). Now we define $\hat{S}_A$ to be the set of all null geodesics $\sigma$, i.e. solutions of (6.2) and (6.3) which in $\psi_A$-coordinates obey the initial conditions $\zeta(t_0) = x_0$, $t_0 = x_0^4$ for any $x_0 = \psi_A(p)$, $p \in W_A$ and $\dot{\zeta}(t_0) = (e^1, e^2, e^3, 1)$ for any $(e^1, e^2, e^3) \in S^2$.

Finally, let $\hat{S} = \bigcup_{A \in P} \hat{S}_A$ and let $S^*$ be any set of the same cardinality as $\hat{S}$ which may serve as a set of indices for $\hat{S}$.

Then it is easily seen that the Axioms GK 4 are satisfied if $S$ is replaced by $S^*$.

This axiomatics is the simplest one for the considered systems.

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