Pricing Employee Stock Options (ESOs) with Random Lattice

E Chendra¹, L Chin², A Sukmana³
Department of Mathematics, Parahyangan Catholic University, Bandung

¹erwinna@unpar.ac.id, ²chin@unpar.ac.id, ³asukmana@unpar.ac.id

Abstract. Employee Stock Options (ESOs) are stock options granted by companies to their employees. Unlike standard options that can be traded by typical institutional or individual investors, employees cannot sell or transfer their ESOs to other investors. The sale restrictions may induce the ESO’s holder to exercise them earlier. In much cited paper, Hull and White propose a binomial lattice in valuing ESOs which assumes that employees will exercise voluntarily their ESOs if the stock price reaches a horizontal psychological barrier. Due to nonlinearity errors, the numerical pricing results oscillate significantly so they may lead to large pricing errors. In this paper, we use the random lattice method to price the Hull-White ESOs model. This method can reduce the nonlinearity error by aligning a layer of nodes of the random lattice with a psychological barrier.

1. Introduction
Employee Stock Option (ESO) is a call option on company’s stock granted by the company to its employees. ESO has become very popular in the last twenty years, especially for start-up companies. These companies do not have the resources to pay key employees and they solve this problem by supplementing the salaries with ESOs. If the company does well and shares are sold to the public in an IPO, the ESO is likely become very valuable. Many companies also attract and keep the best employees by offering them very attractive stock option packages.

ESO has unique characteristics that distinguish it with standard options. ESO often last as long as 10 to 15 years. It cannot be exercised immediately because there is a vesting period. When employees leave their jobs during the vesting period, the ESO becomes worthless. But if they leave after the vesting period, employees forfeit ESOs that are out-of-the-money and they have to exercise options that are in-the-money. Employees also are not permitted to sell their ESOs. The only way employees can realize a cash benefit from the ESO is by exercising its and selling the underlying shares. This leads to a tendency for ESOs to be exercised earlier.

In practice the early exercise behavior of employees varies widely from company to company. In a much cited paper, Hull and White [1] suggest a simple model to capture the earlier exercise behavior. They assume that employees will exercise voluntarily their ESO as soon as the option has vested and the stock price reaches a certain multiple of a strike price. In other words, there is a single psychological barrier to induce an earlier exercise in the model. Then they use a binomial tree method to value the ESO. However, this method has a disadvantage to price options that has a barrier feature. Figlewski and Gao [2] argue that the pricing results oscillate significantly due mainly to the nonlinearity error. The nonlinearity error occurs at certain critical locations such as a certain point, a price level, or a time point. For ESO, the critical price level occurs along the psychological barrier price. To overcome this, we use a random lattice method proposed by Das [3] to reduce the
nonlinearity error by restructuring the tree to make the critical price level coincide with the tree’s nodes.

2. The Hull-White ESO Model
The Hull-White ESO model assumes that employees will exercise their options if the ratio of the stock price to the strike price is above a certain level \((M)\). Suppose there are \(N\) time steps of length \(\Delta t\) in the tree, \(S_{i,j}\) is the stock price at the \(j^{th}\)-node of the CRR Binomial tree at time \(i\Delta t\), and \(f_{i,j}\) is the value of the ESO at this node. Define \(K\) as the strike price of the option, \(v\) as the time when vesting period ends, \(\lambda\) as the employee exit rate, and \(r\) as the riskless interest rate. Assume that the probability of employees leaving the company for each period of time \(\Delta t\) as \(1 - e^{-\lambda \Delta t}\) [4]. At the maturity date \(T\), the payoff of the ESO is given by

\[
 f_{N,j} = \max(S_{N,j} - K, 0)
\]

and for \(0 \leq i \leq N - 1\):

- If \(i\Delta t < v\) then
  \[
  f_{i,j} = (e^{-\lambda \Delta t}).e^{-r\Delta t}[pf_{i+1,j+1} + (1-p)f_{i+1,j}]
  \]

- If \(i\Delta t \geq v\) and \(S_{i,j} \geq MK\) then
  \[
  f_{i,j} = S_{i,j} - K
  \]

- If \(i\Delta t \geq v\) and \(S_{i,j} < MK\) then
  \[
  f_{i,j} = (1 - e^{-\lambda \Delta t}).\max(S_{i,j} - K, 0) + (e^{-\lambda \Delta t}).e^{-r\Delta t}[pf_{i+1,j+1} + (1-p)f_{i+1,j}]
  \]

Then the value of the ESO is \(f_{0,0}\).

3. Random Lattice
The random lattice, also known in the literature as the stochastic-mesh method, is first proposed by Das [3] to price a European option and an Asian option. This method combines the computational benefits of Monte Carlo simulation with the ability to implement dynamic programming on a lattice. The basic idea for the lattice is to restrict the stochastic process to a finite number of discrete values, which defines a set of “buckets” into which every stock price is assigned.

Denote the number of buckets as \(m\), which is the width of the tree, and the number of level in the tree as \(d\), its depth. Then the random lattice takes on a matrix structure of size \(m \times d\). To generate the lattice, we conduct \(n\) root-leaf random walks through it, starting with the same value at the root for each path. Assume that the stock price follows a lognormal diffusion process, then at each node, the next stock price is randomly generated by

\[
 S_{\Delta t} = S_0 \exp[(r - 0.5\sigma^2)\Delta t + \sigma\varepsilon \sqrt{\Delta t}]
\]

where \(\varepsilon \sim N(0,1)\), \(\sigma\) is the volatility, and \(S_0\) is the initial stock price. Once the next stock price is obtained, then we round it up or down to the nearest bucket and proceed to generate the next stock price and so on. For every edge on the lattice from node \(i\) at time \(t\) to node \(j\) at time \(t + 1\), we maintain a count of number of traversals made via the edge, denote as \(c_t(i,j)\), and keep track of the total number of out-traversals from node \(i\) at time \(t\), denote as \(c_t(i)\). Therefore

\[
 c_t(i) = \sum_{j=1}^{m} c_t(i,j)
\]

From this, we derive the transition probabilities \(p_{t,j}(i)\) on the lattice, i.e.

\[
 p_t(i,j) = \frac{c_t(i,j)}{c_t(i)}
\]
Once we have the probabilities for the lattice, the stochastic process is in place. Then we can compute the final payoffs at the last level of the lattice and the price of the ESO at the root node is determined by dynamic programming on the lattice.

4. Numerical Results
Assume that $S_0 = K = 50$, $r = 7.5\%$, $\sigma = 30\%$, $T = 10$ years, $m = 51$, $M = 2$, $\nu = 3$ years, $\lambda = 6\%$, $d = 300$, and $n = 10^5$ then the ESO price with random lattice method is 19.071 while the ESO price with binomial method is 18.718. Figure 1 shows the convergence of ESO prices with random lattice method for various values of $d$.

Figure 1. ESO Price vs Number of Time Steps

5. Conclusions
We have presented the random lattice method for valuing employee stock options. The method is easy to implement and flexible. For further research, we will compare the ESO price with other lattice methods, like the Bino-Trinomial Tree (BTT) method proposed by Dai and Lyuu [5].

References
[1] Hull J and White A 2004 How to value employee stock options *Financial Analysts Journal* **60** 114-9.
[2] Figlewski S and Gao B 1999 The adaptive mesh model: a new approach to efficient option pricing *Journal of Financial Economics* **53**(3) 313-51.
[3] Das SR 2011 Random lattices for option pricing problems in finance *Journal of Investment Management* **9**(2) 134-52.
[4] Ammann M and Seiz R 2004 Valuing employee stock options: does the model matter? *Financial Analysts Journal* **60**(5) 21-37.
[5] Dai TS and Lyuu YD 2010 The Bino-Trinomial Tree: a simple model for efficient and accurate option pricing *Journal of Derivatives* **17**(4) 7-24.