A FLOW METHOD TO THE ORLICZ-ALEKSANDROV PROBLEM

JINRONG HU, JIAQIAN LIU, AND DI MA

Abstract. In this paper, we obtain an existence result of smooth solutions to the Orlicz-Aleksandrov problem from the perspective of geometric flow. Furthermore, a special uniqueness result of solutions to this problem shall be discussed.

1. Introduction

The Brunn-Minkowski theory, the core content of convex geometry, was developed by Minkowski, Aleksandrov, Fenchel, and many others. The Minkowski problem, involving surface area measure is a classical problem in the Brunn-Minkowski theory, which was proposed and solved by Minkowski himself [30, 31]. As an important counterpart of the Minkowski problem, the classical Aleksandrov problem was introduced by Aleksandrov in [1]. It is a characterization problem for the Aleksandrov integral curvature, which was originally solved by Aleksandrov via a topological argument, see [1]. Different from Aleksandrov’s method, Oliker [33] gave a new proof based on a variational technique inspired by optimal transport theory, and recently Bertrand [3] also provided an alternative approach to treat this problem using optimal mass transport.

The $L^p$ Brunn-Minkowski theory, as an analogue of the Brunn-Minkowski theory, which was initiated by Firey [11] in the 1950s, but this theory gained a new life when Lutwak enriched it with the concept of the $L^p$ surface area measure in [26] in the early 1990s. The $L^p$ Minkowski problem involving the $L^p$ surface area measure is a fundamental problem in this theory, which has been the breeding ground equipped with many different results in a series of papers [4, 5, 9, 19, 26, 27, 38]. With the development of the $L^p$ Brunn-Minkowski theory, as a parallelism of the $L^p$ Minkowski problem, Huang-LYZ [20] introduced the $L^p$ Aleksandrov problem for the $L^p$ Aleksandrov integral curvature, and completely established the existence result for $p > 0$. At the same time, Zhao [37] presented a solution to the $L^p$ Aleksandrov problem for origin-symmetric polytopes in the case of $-1 < p < 0$.

The Orlicz-Brunn-Minkowski theory, as a new generation of the $L^p$ Brunn-Minkowski theory, was launched by Lutwak, Yang and Zhang [28, 29]. Based on their work, Haberl-LYZ [17] first proposed and solved the Orlicz Minkowski problem. Since then, the relevant Orlicz-Minkowski
type problems have been widely studied, see for instance [14, 18, 21, 36]. Naturally, apart from the Orlicz Minkowski problem, the corresponding measure characterization problem for the Aleksandrov problem has been sought for in the Orlicz-Brunn-Minkowski theory. In the recent excellent work, Feng-He [10] defined the Orlicz integral curvature and posed the Orlicz Aleksandrov problem finding the conditions of a given finite Borel measure \( \mu \) on \( \mathbb{S}^{n-1} \) as a multiple of the Orlicz integral curvature of a convex body in \( \mathbb{R}^n \), they also gave the necessary and sufficient conditions for the existence of even solutions to the Orlicz Aleksandrov problem via a variational argument.

It is worth mentioning that for the special case in which the given measure \( \mu \) has a positive density \( f \) on \( \mathbb{S}^{n-1} \), the existence of the solution of the Orlicz Aleksandrov problem amounts to solving the following Monge-Ampère type equation,

\[
\gamma \varphi \left( \frac{1}{h} \right) h \left( |\nabla h|^2 + h^2 \right)^{-\frac{2}{p}} \det(\nabla^2 h + h I) = f \quad (1.1)
\]

for some positive constant \( \gamma > 0 \). It is clear to see that, in the case \( \varphi(s) = s^p \) with \( p \in \mathbb{R} \), the Orlicz Aleksandrov problem reduces to the \( L_p \) Aleksandrov problem, and the constant \( \gamma \) can be merged to \( h \) due to the homogeneousness of the \( L_p \) Aleksandrov integral curvature. In particular, when equation (1.1) corresponds to the regularity of solutions to the Aleksandrov problem, this topic has been intensively studied. For example, Oliker [32] and Pogorelov [34] independently investigated it in the circumstance that \( f \) is a smooth positive function, and Guan-Li [15] further dealt with certain degenerate Monge-Ampère type equations in the setting that \( f \) is smooth but only nonnegative. Note that, for general \( \varphi \), the uniqueness of solutions to equation (1.1) is open.

The main purpose of this paper concerns the existence of the solution of the Orlicz Aleksandrov problem associated with the solvability of equation (1.1) from the perspective of geometric flow. The geometric flow generated by Gauss curvature was first studied by Firey [12]. From then on, there appeared various Gauss curvature flows, see for instance, Andrews-Guan [2], Chen-Huang-Zhao [6], Chou-Wang [8], Li-Sheng-Wang [23, 24] and their references therein. In the spirit of their work, we consider a family of strictly convex hypersurfaces \( \partial \Omega_t \) parameterized by smooth map \( X(\cdot, t) : \mathbb{S}^{n-1} \to \mathbb{R}^n \) satisfying the following flow equation,

\[
\begin{cases}
\frac{\partial X(x, t)}{\partial t} = -\frac{f(\nu)(x)\kappa(X, t)}{\varphi(1/\nu(x))} + X(x, t); \\
X(x, 0) = X_0(x),
\end{cases} \quad (1.2)
\]

where \( \kappa \) is the Gauss curvature of the hypersurface \( \partial \Omega_t \) at \( X(\cdot, t) \), \( \nu = x \) is the unit outer normal vector of \( \partial \Omega_t \) at \( X(\cdot, t) \), and \( \eta(t) \) is defined by

\[
\eta(t) = \frac{\int_{\mathbb{S}^{n-1}} du}{\int_{\mathbb{S}^{n-1}} \frac{f(x)}{\varphi(1/\nu(x))} dx}.
\]
With the aid of the flow equation (1.2), we are devoted to solving equation (1.1). Before elaborating the main results of this paper, we do some preparation with setting
\[
\Phi_0 = \left\{ \Re : \lim_{s \to 0} \Re(s) = 0, \lim_{s \to \infty} \Re(s) = \infty \right\},
\]
where \( \Re : (0, \infty) \to (0, \infty) \) is continuously differentiable and strictly increasing function. On the other hand, we set
\[
\Psi_0 = \left\{ \Re : \lim_{s \to 0} \Re(s) = \infty, \lim_{s \to \infty} \Re(s) = 0 \right\},
\]
where \( \Re : (0, \infty) \to (0, \infty) \) is continuously differentiable and strictly decreasing function.

We are now in a position to state that the main aim of current work is to obtain the long time existence and convergence results of the flow (1.2). It is shown in the following theorem.

**Theorem 1.1.** Let \( \Omega_0 \) be a smooth, origin symmetric and strictly convex body in \( \mathbb{R}^n \). Suppose \( f : \mathbb{S}^{n-1} \to (0, \infty) \) is smooth and even, and \( \varphi : (0, \infty) \to (0, \infty) \) is smooth. If there is

(i) \( \Re \in \Phi_0 \) satisfying \( \Re(s) = \int_0^s \frac{1}{\varphi(t)} dt \) with \( \Re' \neq 0 \) on \( (0, \infty) \), for some \( \hat{C} > 0 \), we have
\[
\int_{\mathbb{S}^{n-1}} f(x) \Re(1/|x \cdot \theta|) dx \leq \hat{C}, \quad \forall \theta \in \mathbb{S}^{n-1},
\]
and choosing a \( \Omega_0 \) such that
\[
\int_{\mathbb{S}^{n-1}} f(x) \Re(1/h(\Omega_0, x)) dx > \hat{C};
\]

or

(ii) \( \Re \in \Psi_0 \) satisfying \( \Re(s) = \int_s^{\infty} \frac{1}{\varphi(t)} dt \) with \( \Re' \neq 0 \) on \( (0, \infty) \),

then there exists a smooth, origin symmetric, strictly convex solution \( \Omega_t \) to flow equation (1.2) for all time \( t > 0 \), and it converges along a sequence in \( C^\infty \) to a smooth, origin symmetric, strictly convex solution to equation (1.1) for some positive constant \( \gamma \).

In view of Theorem 1.1, it should be remarked that, since \( \Re \) is decreasing in condition (ii), we automatically have
\[
\int_{\mathbb{S}^{n-1}} f(x) \Re(1/|x \cdot \theta|) dx < \infty, \quad \forall \theta \in \mathbb{S}^{n-1}.
\]
Hence, the integral conditions in (i) compared to the condition (ii) may be not more restrictive.

The organization of this paper goes as follows: In Section 2, we collect some basic knowledge about convex bodies. In Section 3, we introduce the geometric flow and functional associated with the Orlicz Aleksandrov problem. In Section 4 and 5, we obtain the priori estimates of the solution to the relevant flow. In Section 6, we complete the proof of Theorem 1.1. At last, we shall provide a special uniqueness result to the solution of equation (1.1) under mild monotonicity assumption on \( \varphi \).
2. Preliminaries

In this section, we list some basic facts regarding convex bodies. For quick and good references, please refer to the books of Gardner [13] and Schneider [35].

Our setting will put on the \(n\)-Euclidean space \(\mathbb{R}^n\). Denote by \(S^{n-1}\) the unit sphere. A convex body is a compact convex set of \(\mathbb{R}^n\) with non-empty interior. For \(Y, Z \in \mathbb{R}^n\), \(Y \cdot Z\) stands for the standard inner product. For \(Y \in \mathbb{R}^n\), we denote by \(|Y| = \sqrt{Y \cdot Y}\) the Euclidean norm.

Let \(\Omega\) be a convex body containing the origin in \(\mathbb{R}^n\), and let \(h(\Omega, \cdot)\) be the support function of \(\Omega\) (with respect to the origin), i.e., for any \(x \in S^{n-1}\),

\[
h(\Omega, x) = \max\{x \cdot Y : Y \in \Omega\}.
\]

The map \(g : \partial \Omega \to S^{n-1}\) denotes the Gauss map of \(\partial \Omega\). Meanwhile, for \(\omega \subset S^{n-1}\), the inverse of Gauss map \(g\) is expressed as

\[
g^{-1}(\omega) = \{Z \in \partial \Omega : g(Z) \text{ is defined and } g(Z) \in \omega\}.
\]

For simplicity in the subsequence, we abbreviate \(g^{-1}\) as \(X\). For a convex body \(\Omega\) being of class \(C^2_{+}\), i.e., its boundary is of class \(C^2\) and of positive Gauss curvature, the support function of \(\Omega\) can be written as

\[
h(\Omega, x) = x \cdot X(x) = g(Z) \cdot Z, \text{ where } x \in S^{n-1}, \ g(Z) = x \text{ and } Z \in \partial \Omega.
\]

In fact, we can parametrize \(\partial \Omega\) by \(X(x)\). Let \(\{e_1, e_2, \ldots, e_{n-1}\}\) be a local orthonormal frame on \(S^{n-1}\), \(h_i\) be the first order covariant derivatives of \(h(\Omega, \cdot)\) on \(S^{n-1}\) with respect to a local orthonormal frame. Differentiating (2.1) with respect to \(e_i\), we get

\[
h_i = e_i \cdot X(x) + x \cdot X_i(x).
\]

Since \(X_i\) is tangent to \(\partial \Omega\) at \(X(x)\), we obtain

\[
h_i = e_i \cdot X(x).
\]

Combining (2.1) and (2.2), we have

\[
X(x) = \sum_i h_i(\Omega, x)e_i + h(\Omega, x)x = \nabla h(\Omega, x) + h(\Omega, x)x.
\]

Here \(\nabla\) is the spherical gradient. On the other hand, since we can extend \(h(\Omega, x)\) to \(\mathbb{R}^n\) as a 1-homogeneous function \(h(\Omega, \cdot)\), then restrict the gradient of \(h(\Omega, \cdot)\) on \(S^{n-1}\), it yields that

\[
\nabla h(\Omega, x) = X(x), \ \forall x \in S^{n-1},
\]

where \(\nabla\) is the gradient operator in \(\mathbb{R}^n\). Let \(h_{ij}\) be the second order covariant derivatives of \(h(\Omega, \cdot)\) on \(S^{n-1}\) with respect to a local orthonormal frame. Then, applying (2.3) and (2.4), we have the
following equalities:
\[
\nabla h(\Omega, x) = \sum_i h_i e_i + hx, \quad X_i(x) = \sum_j (h_{ij} + h \delta_{ij}) e_j. \tag{2.5}
\]

The Gauss curvature of \(\partial \Omega\) at \(X(x)\) is given by
\[
\kappa = \frac{1}{\det(\nabla^2 h + hI)},
\]
where \(\nabla^2 h\) denotes the spherical Hessian matrix of \(h\), and \(I\) is the identity matrix.

The radial function \(\rho\) of \(\Omega\) is given by
\[
\rho(u) = \max\{\lambda > 0 : \lambda u \in \Omega\}, \quad \forall u \in S^{n-1}.
\]
It is clear to see that \(\rho(u)u \in \partial \Omega\) for any \(u \in S^{n-1}\), then the Gauss map \(g\) of \(\partial \Omega\) can be expressed as
\[
g(\rho(u)u) = \rho(u)u - \nabla \rho \sqrt{\rho^2 + |\nabla \rho|^2}.
\]
Let \(u\) and \(x\) be related by the following equality:
\[
\rho(u)u = \nabla h(\Omega, x) = X(x) = \nabla h + hx, \tag{2.6}
\]
then we obtain (see, e.g., [16])
\[
x = \frac{\rho(u)u - \nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}, \quad u = \frac{\nabla h + hx}{\sqrt{|\nabla h|^2 + h^2}}, \tag{2.7}
\]
and the following formula is clear (see for example [24])
\[
\frac{h(x)}{\kappa(x)} dx = \rho''(u) du. \tag{2.8}
\]

3. The geometric flow and relevant functional

In this section, we are in the place to introduce the geometric flow and the relevant functional.

Let \(\Omega_0\) be a smooth, origin symmetric and strictly convex body in \(\mathbb{R}^n\), \(f : S^{n-1} \to (0, \infty)\) be a smooth function, as presented above, we are concerned with a family of convex hypersurfaces \(\partial \Omega_t\) parameterized by smooth map \(X(\cdot, t) : S^{n-1} \to \mathbb{R}^n\) satisfying the following flow equation,
\[
\begin{aligned}
\frac{\partial X(\cdot, t)}{\partial t} &= -\frac{f(\nu)X(\cdot, t) \nu(\cdot, t)}{\varphi(1/(\varphi \cdot (X(\cdot, t) \cdot \nu)))} + X(\cdot, t); \\
X(\cdot, 0) &= X_0(\cdot),
\end{aligned} \tag{3.1}
\]
where \(\kappa\) is the Gauss curvature of the hypersurface \(\partial \Omega_t\) at \(X(\cdot, t)\), \(\nu = x\) is the unit outer normal vector of \(\partial \Omega_t\) at \(X(\cdot, t)\), and for any \(u \in S^{n-1}\), \(\eta(t)\) is defined by
\[
\eta(t) = \frac{\int_{S^{n-1}} du}{\int_{S^{n-1}} \frac{f(x)}{\varphi(1/(\varphi \cdot (X(\cdot, t) \cdot \nu)))} dx}. \tag{3.2}
\]
where \( \rho(u, t) \) is the radial function of \( \Omega \), for any \( u \in \mathbb{S}^{n-1} \) satisfying (2.6), this together with the definition of support function (2.1), then the following relation between \( x \) and \( u \) can be deduced:

\[
\rho(u, t)(u \cdot x) = h(x, t).
\]  

(3.4)

Applying (3.4), as shown in [6], we obtain

\[
\frac{\partial \rho(u, t)}{\partial t} = \frac{\rho(u, t)}{h(x, t)} \frac{\partial h(x, t)}{\partial t} + \rho(u, t);
\]

(3.3)

where \( \rho \) is the support function as

\[
\{ \text{the support function as} \}
\]

(3.1)

\[
\text{Combining (3.3) with (3.5), it is not hard to see that} \ \rho(u, t) \ \text{satisfies the following flow equation,}
\]

\[
\begin{align*}
\frac{\partial \rho(u, t)}{\partial t} &= -f(x) \psi^{1/n}(1/h(x, t)) + \rho(u, t); \\
\rho(u, 0) &= \rho_0(u).
\end{align*}
\]

(3.6)

In the study of the flow (3.1), it is essential to derive the following results.

**Lemma 3.1.** Under the assumptions of Theorem 1.1. Let \( \Omega \) be a smooth, origin symmetric, strictly convex solution to the flow (3.1). Then

\[
\int_{\mathbb{S}^{n-1}} \log \rho(u, t) du = \text{constant}
\]

for \( t \geq 0 \). Here \( \rho(\cdot, t) \) is the radial function of \( \Omega \).

**Proof.** Using (2.8), (3.2) and (3.6), we have

\[
\frac{d}{dt} \int_{\mathbb{S}^{n-1}} \log \rho(u, t) du = \int_{\mathbb{S}^{n-1}} \frac{1}{\rho(u, t)} \frac{d\rho(u, t)}{dt} du
\]

\[
= \int_{\mathbb{S}^{n-1}} \frac{1}{\rho(u, t)} \left( \rho(u, t) - f(x) \psi^{1/n} \right) h(x, t) du
\]

\[
= \int_{\mathbb{S}^{n-1}} du - \int_{\mathbb{S}^{n-1}} \frac{f(x)}{\psi^{1/n} h(x, t)} \rho(u, t) du
\]

\[
= \int_{\mathbb{S}^{n-1}} du - \int_{\mathbb{S}^{n-1}} \frac{f(x)}{\psi^{1/n} h(x, t)} du
\]

\[
= \int_{\mathbb{S}^{n-1}} du - \int_{\mathbb{S}^{n-1}} du
\]

\[
= 0.
\]

This completes the proof. \( \square \)
Lemma 3.2. If the condition (i) of Theorem 1.1 holds, then along the flow (3.1), $P(t)$ is non-decreasing.

Proof. Due to $\mathcal{K}'(1/h) = -\frac{h}{\varphi(1/h)}$, by virtue of (3.2) and (3.3), we have

$$P'(t) = \int_{\mathbb{R}^{n}} \nabla \mathcal{K}(1/h) \cdot \nabla f(x) dx = -\int_{\mathbb{R}^{n}} f(x) \frac{\partial h(x,t)}{\partial t} dx$$

$$= \int_{\mathbb{R}^{n}} -\frac{f(x)}{h(x,t)\varphi(1/h)} \left( h(x,t) - \frac{f \rho^k \kappa(t)}{\varphi(1/h)} \right) dx$$

which turns into

$$P'(t) \int_{\mathbb{R}^{n}} \nabla \mathcal{K}(1/h) \cdot \nabla \varphi(1/h) dx = -\int_{\mathbb{R}^{n}} \left( \int_{\mathbb{R}^{n}} \frac{f(x)}{\varphi(1/h)} dx \right)^2 + \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{f^2 \rho^k}{h \varphi(1/h)^2} dx \int_{\mathbb{R}^{n}} \frac{h}{\kappa \rho^n} dx$$

$$= -\int_{\mathbb{R}^{n}} \left( \int_{\mathbb{R}^{n}} \frac{f(x)}{\varphi(1/h)} dx \right)^2 + \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{f^2 \rho^k}{h \varphi(1/h)^2} dx \int_{\mathbb{R}^{n}} \frac{h}{\kappa \rho^n} dx$$

$$\geq 0,$$

where the last inequality is due to the Hölder inequality. Therefore, $P(t)$ is non-decreasing. \hfill \Box

Lemma 3.3. If the condition (ii) of Theorem 1.1 holds, then along the flow (3.1), $P(t)$ is non-increasing.

Proof. The proof is similar to Lemma 3.2. Since $\mathcal{K}'(1/h) = -\frac{h}{\varphi(1/h)}$, we have

$$P'(t) \int_{\mathbb{R}^{n}} \nabla \mathcal{K}(1/h) \cdot \nabla \varphi(1/h) dx = \left( \int_{\mathbb{R}^{n}} \frac{f(x)}{\varphi(1/h)} dx \right)^2 - \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{f^2 \rho^k}{h \varphi(1/h)^2} dx \int_{\mathbb{R}^{n}} \frac{h}{\kappa \rho^n} dx$$

$$= \left( \int_{\mathbb{R}^{n}} \frac{f(x)}{\varphi(1/h)} dx \right)^2 - \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{f^2 \rho^k}{h \varphi(1/h)^2} dx \int_{\mathbb{R}^{n}} \frac{h}{\kappa \rho^n} dx$$

$$\leq 0.$$
Therefore, $P(t)$ is non-increasing. \hfill \Box

4. \textit{C}^0, \textit{C}^1 \textit{estimates}

In this section, we shall obtain the \textit{C}^0, \textit{C}^1 estimates of solutions to the flow (3.1). Let us begin with completing the \textit{C}^0 estimate.

\textbf{Lemma 4.1.} \textit{Under the assumptions of Theorem [1,1], let} $\Omega_t$ \textit{be a smooth, origin symmetric, and strictly convex solution to the flow (3.1). Then, there exists a positive constant} $C$, \textit{independent of} $t$, \textit{such that}

\begin{equation}
\frac{1}{C} \leq h(x,t) \leq C, \quad \forall (x,t) \in S^{n-1} \times (0, \infty),
\end{equation}

\textit{and}

\begin{equation}
\frac{1}{C} \leq \rho(x,t) \leq C, \quad \forall (x,t) \in S^{n-1} \times (0, \infty).
\end{equation}

\textit{Here} $h(x,t)$ \textit{and} $\rho(x,t)$ \textit{are the support function and the radial function of} $\Omega_t$. \textit{Furthermore,} $\eta(t)$ \textit{is uniformly bounded above, and below away from zero.}

\textit{Proof.} \textit{In view of [7] Lemma 2.6}, we know

\begin{equation}
\min_{\mathbb{S}^{n-1}} h(x,t) = \min_{\mathbb{S}^{n-1}} \rho(u,t), \quad \max_{\mathbb{S}^{n-1}} h(x,t) = \max_{\mathbb{S}^{n-1}} \rho(u,t).
\end{equation}

(4.3) illustrates that (4.1) and (4.2) are equivalent. So, for upper bound (or lower bound), we only need to establish (4.1) or (4.2).

Our aim is to prove (4.1). We first prove the upper bound of (4.1), on the one hand, if the condition (i) of Theorem [1,1] holds, assume that $R(t) := \max_{\mathbb{S}^{n-1}} h(x,t)$ is attained at the north pole $\tilde{x} = (0, \cdots, 0, 1)$ at $t > 0$. By virtue of the convexity of $\Omega_t$ and the definition of $h(x,t)$, it follows that (see e.g., [7] Lemma 2.6)

\begin{equation}
h(x,t) \geq \max_{\mathbb{S}^{n-1}} \rho(u,t)|u \cdot x| = \max_{\mathbb{S}^{n-1}} h(x,t)|\tilde{x} \cdot x| = R(t)|x_n|,
\end{equation}

where $x_n$ is $n$-th coordinate component of $x$. Let $\varepsilon$ be a small positive constant, then we set

$\mathcal{O}^* = \{x \in \mathbb{S}^{n-1} : |x_n| < \varepsilon\}$.

Suppose $R(t) > 1$, as $P$ is non-decreasing and $R \geq 0$ is strictly increasing, from (4.4), we obtain

\begin{equation}
P(0) \leq P(t)
= \int_{\mathbb{S}^{n-1} \setminus \mathcal{O}^*} R(1/h(x,t)) f(x) dx + \int_{\mathcal{O}^*} R(1/h(x,t)) f(x) dx
\leq (\max_{\mathbb{S}^{n-1}} f) R(1/(\varepsilon R(t))) |S^{n-1} \setminus \mathcal{O}^*| + \int_{\mathcal{O}^*} R(1/(R(t)|x_n|)) f(x) dx
\leq (\max_{\mathbb{S}^{n-1}} f) R(1/(\varepsilon R(t))) |S^{n-1} \setminus \mathcal{O}^*| + \int_{\mathcal{S}^{n-1}} f(x) R(1/|x_n|) dx.
\end{equation}
Note that \( \lim_{s \to 0} \Re(s) = 0 \), in view of (4.5), as \( R \to \infty \), we obtain

\[
P(0) \leq \hat{C}. \tag{4.6}
\]

By our choice of \( \Omega_0 \), satisfying \( P(0) > \hat{C} \), then (4.6) is violated. So the upper bound of \( h(x, t) \) is obtained.

On the other hand, if the condition \((ii)\) of Theorem 1.1 holds. Supposing again \( R(t) \) is attained at the north pole \( \tilde{x} = (0, \cdots, 0, 1) \) at \( t > 0 \). With the aid of again the convexity of \( \Omega_t \), we have

\[
h(x, t) \geq R(t)|x_n|. \tag{4.7}
\]

Denote \( \tilde{O} = \{ x \in \mathbb{S}^{n-1} : |x_n| \geq \frac{1}{2} \} \), then (4.7) becomes

\[
P(0) \geq (\min_{\mathbb{S}^{n-1}} f) \int_{\{|x_n| \geq \frac{1}{2}\} \cap \mathbb{S}^{n-1}} \Re(2/R(t)) \, dx
\]

By virtue of \( \lim_{s \to 0} \Re(s) = \infty \), it illustrates that \( h(x, t) \) is uniformly bounded above. So we obtain the upper bound of (4.1).

For the uniform lower bound of \( h \), we argue by contradiction. Let \( \{t_k\} \subset [0, \infty) \) be a sequence such that \( h(x, t_k) \) is not uniformly bounded away from 0, i.e.,

\[
\min_{\mathbb{S}^{n-1}} h(\cdot, t_k) \to 0 \quad \text{as} \quad k \to \infty.
\]

With the aid of the upper bound of (4.1), using Blaschke selection theorem (see [35]), then there is a sequence in \( \{\Omega_{t_k}\} \), which is still denoted by \( \{\Omega_{t_k}\} \), such that

\[
\Omega_{t_k} \to \tilde{\Omega} \quad \text{as} \quad k \to \infty.
\]

Since \( \Omega_{t_k} \) is an origin-symmetric convex body, \( \tilde{\Omega} \) is also origin-symmetric. Then, we have

\[
\min_{\mathbb{S}^{n-1}} h_{\tilde{\Omega}} = \lim_{k \to \infty} \min_{\mathbb{S}^{n-1}} h_{\Omega_{t_k}} = 0.
\]

This implies that \( \tilde{\Omega} \) is contained in a hyperplane in \( \mathbb{R}^n \). Then, we have

\[
\rho_{\tilde{\Omega}} = 0 \quad \text{a.e. in} \ \mathbb{S}^{n-1}. \tag{4.8}
\]
By virtue of (4.8) and Lemma 3.1, for any \( \varepsilon > 0 \), we have
\[
\int_{\mathbb{S}^{n-1}} \log \rho_{\Omega_0} \, du = \int_{\mathbb{S}^{n-1}} \log \rho_{\Omega_k} \, du
\leq \lim_{k \to \infty} \int_{\mathbb{S}^{n-1}} \log[\rho_{\Omega_k} + \varepsilon] \, du
= \int_{\mathbb{S}^{n-1}} \log \varepsilon \, du
\to -\infty \quad \text{as} \quad \varepsilon \to 0,
\]
which is a contradiction. Then, we have
\[
\min_{\mathbb{S}^{n-1} \times (0, \infty)} h(x, t) \geq C
\]
for some positive constant \( C \), independent of \( t \). The lower bound of (4.1) follows. The lower and upper bounds on \( h \) imply bounds on \( \eta(t) \). Hence, we complete the proof. \( \square \)

The \( C^1 \) estimate is as follows.

**Lemma 4.2.** Under the assumptions of Theorem 1.1. Let \( \Omega_t \) be a smooth, origin symmetric, and strictly convex solution to the flow (3.1). Then, there exists a positive constant \( C \), independent of \( t \), such that
\[
|h'(u, t)| \leq C, \quad \forall \,(u, t) \in \mathbb{S}^{n-1} \times (0, \infty)
\]
and
\[
|\nabla \rho(u, t)| \leq C, \quad \forall \,(u, t) \in \mathbb{S}^{n-1} \times (0, \infty)
\]
for some \( C > 0 \), independent of \( t \).

**Proof.** By means of (2.1), (2.6) and (2.7), we obtain the following equalities:
\[
\rho^2 = h^2 + |\nabla h|^2, \quad h = \frac{\rho^2}{\sqrt{p^2 + |\nabla \rho|^2}}.
\]
(4.9)

Then (4.9) and Lemma 4.1 lead to this lemma directly. \( \square \)

5. \( C^2 \) estimate

Utilizing the above \( C^0 \), \( C^1 \) estimates, the upper and lower bounds of principal curvatures will be derived. It is shown in the following result.

**Theorem 5.1.** Under the assumptions of Theorem 1.1. Let \( \Omega_t \) be a smooth, origin symmetric and strictly convex solution to the flow (3.1). Then there exists a positive constant \( C \), independent of \( t \), such that the principal curvatures \( \kappa_i \) of \( \Omega_t \), \( i = 1, \ldots, n-1 \), are bounded from above and below, satisfying
\[
\frac{1}{C} \leq \kappa_i \leq C, \quad \forall \,(x, t) \in \mathbb{S}^{n-1} \times (0, \infty).
\]
(5.1)
Proof. First, we shall prove the upper bound of Gauss curvature $\kappa$. It is essential to construct the following auxiliary function,

$$Q(x, t) = \frac{f(x)^{\kappa}}{\varphi(1/h)} \frac{\int_{\mathbb{R}^n-1} du}{\int_{\mathbb{R}^n-1} \varphi(1/h) dx} - h(x, t) = \frac{-h_t}{h - \varepsilon_0},$$  \quad (5.2)

where

$$\varepsilon_0 = \frac{1}{2} \min_{\mathbb{R}^n \times (0, \infty)} h(x, t) > 0.$$  

For any fixed $t \in (0, \infty)$, assume that the maximum of $Q(x, t)$ is achieved at $x_0$. Rotate the axes so that $\{w_{ij}\}$ is diagonal at $x_0$, where $w_{ij} := h_{ij} + h\delta_{ij}$. Thus, we obtain that at $x_0$,

$$0 = \nabla_i Q = \frac{-h_{ii}}{h - \varepsilon_0} + \frac{h_i h_i}{(h - \varepsilon_0)^2}.  \quad (5.3)$$

Then, by virtue of (5.3), at $x_0$, we also obtain

$$0 \geq \nabla_{ii} Q = \frac{-h_{iii}}{h - \varepsilon_0} + \frac{2h_i h_{ii} + h_i h_{ii}}{(h - \varepsilon_0)^2} - \frac{2h_i h_i^2}{(h - \varepsilon_0)^3}
\quad = \frac{-h_{iii}}{h - \varepsilon_0} + \frac{h_i h_{ii}}{(h - \varepsilon_0)^2}.  \quad (5.4)$$

Then, (5.4) tells

$$-h_{iii} - h_i \leq \frac{h_i h_{ii}}{h - \varepsilon_0} - h_i
\quad = \frac{-h_i}{h - \varepsilon_0} [h_{ii} + (h - \varepsilon_0)]
\quad = Q(w_{ii} - \varepsilon_0).  \quad (5.5)$$

Furthermore, applying (3.3) and (5.2), we have

$$\partial_t Q = \frac{-h_{ii}}{h - \varepsilon_0} + \frac{h_i^2}{(h - \varepsilon_0)^2}$$

$$= \left[ \frac{f}{h - \varepsilon_0} \right] \frac{\partial \{ \text{det} (\nabla^2 h + hI) \}^{-1}}{\partial t} \rho^n \varphi(1/h) \int_{\mathbb{R}^n-1} \frac{f}{\varphi(1/h)} dx + \rho^n \frac{\partial}{\varphi(1/h)} \left[ \frac{\int_{\mathbb{R}^n-1} f dx}{\int_{\mathbb{R}^n-1} \varphi(1/h) dx} \right]$$

$$+ \kappa \left[ \frac{\varphi^n}{\varphi(1/h)} \right] \frac{\int_{\mathbb{R}^n-1} f dx}{\int_{\mathbb{R}^n-1} \varphi(1/h) dx} + Q + Q^2.  \quad (5.6)$$
In light of (5.6), by using (5.5), at $x_0$, one has

$$
\frac{\partial[\det(\nabla^2_{\mathbb{S}^{n-1}} h + hI)]}{\partial t} = -[\det(\nabla^2_{\mathbb{S}^{n-1}} h + hI)]^{-2} \sum_i \frac{\partial[\det(\nabla^2_{\mathbb{S}^{n-1}} h + hI)]}{\partial w_{ii}} (h_{ii} + h_i) \\
\leq [\det(\nabla^2_{\mathbb{S}^{n-1}} h + hI)]^{-2} \sum_i \frac{\partial[\det(\nabla^2_{\mathbb{S}^{n-1}} h + hI)]}{\partial w_{ii}} Q(w_{ii} - \varepsilon_0) \\
= \kappa Q[(n - 1) - \varepsilon_0 \sum_i w^{ij}],
$$

where $\{w^{ij}\}$ is the inverse matrix of $\{w_{ij}\}$. Recall the fact that the eigenvalue of $\{w_{ij}\}$ and $\{w^{ij}\}$ are respectively the principal radii and principal curvature of $\partial \Omega_i$ (see for example [39]). Then we have

$$
\frac{\partial[\det(\nabla^2_{\mathbb{S}^{n-1}} h + hI)]}{\partial t} = \kappa Q[(n - 1) - \varepsilon_0 H] \\
\leq \kappa Q[(n - 1) - \varepsilon_0(n - 1)\kappa^{1/\kappa}],
$$

where $H$ denotes the mean curvature of $\partial \Omega_i$, and the last inequality stems from $H \geq (n - 1)(\Pi w^{ij})^{1/n} = (n - 1)\kappa^{1/\kappa}$.

In addition,

$$
\frac{\partial}{\partial t} \left[ \int_{\mathbb{S}^{n-1}} \frac{du}{f(1/h) dx} \right] = -\frac{\int_{\mathbb{S}^{n-1}} du}{(\int_{\mathbb{S}^{n-1}} f(1/h) dx)^2} \frac{f}{h^2 \varphi(1/h)^2} \varphi'(1/h) \frac{\partial h}{\partial t} dx \\
= \frac{\int_{\mathbb{S}^{n-1}} du}{(\int_{\mathbb{S}^{n-1}} f(1/h) dx)^2} \frac{f}{h^2 \varphi(1/h)^2} \varphi'(1/h)(h - \varepsilon_0) Q dx \\
\leq (\max_{h \in I(0,\infty)} |\varphi'(1/h)|) Q(x_0, t) \frac{\int_{\mathbb{S}^{n-1}} du}{(\int_{\mathbb{S}^{n-1}} f(1/h) dx)^2} \int_{\mathbb{S}^{n-1}} \frac{f(h - \varepsilon_0)}{h^2 \varphi(1/h)^2} dx,
$$

where $I(0,\infty)$ is a bounded interval depending only on the upper and lower bounds of $h$ on $(0, \infty)$, and $\varphi'(1/h)$ denotes $\frac{\partial \varphi(1/h)}{\partial(1/h)}$, then using the first equality of (4.9) and (5.3), we have

$$
\frac{\partial}{\partial t} \left( \frac{\rho^n}{\varphi(1/h)} \right) = \frac{\rho^{n-2}}{\varphi(1/h)} \left( hh_t + \sum_k h_k h_{t_k} \right) + \frac{\rho^n}{h^2 \varphi(1/h)^2} \varphi'(1/h) \frac{\partial h}{\partial t} \\
= \frac{\rho^{n-2}}{\varphi(1/h)} Q(\varepsilon_0 h - \rho^2) - \frac{\rho^n}{h^2 \varphi(1/h)^2} \varphi'(1/h)(h - \varepsilon_0) Q \\
\leq \frac{\rho^{n-2}}{\varphi(1/h)} Q(\varepsilon_0 h - \rho^2) + (\max_{h \in I(0,\infty)} |\varphi'(1/h)|) \frac{\rho^n}{h^2 \varphi(1/h)^2} (h - \varepsilon_0) Q.
$$
Substituting (5.7), (5.8) and (5.9) into (5.6), we obtain that at $x_0$,
\[
\partial_t Q \leq \frac{f}{h - \varepsilon_0} \left[kQ((n - 1) - \varepsilon_0(n - 1)\kappa) + \frac{\rho^\mu}{\varphi(1/h)} \int_{\mathbb{S}^{n-1}} \frac{f}{\varphi(1/h)} \, dx \right]
\]
\[
+ \frac{\rho^\mu}{\varphi(1/h)} \kappa(\max_{h \in I_{(0,0)}} |\varphi'(1/h)|)Q \frac{\int_{\mathbb{S}^{n-1}} \frac{f}{\varphi(1/h)} \, dx}{(\int_{\mathbb{S}^{n-1}} \frac{f}{\varphi(1/h)} \, dx)^2} \int_{\mathbb{S}^{n-1}} \frac{f(h - \varepsilon_0)}{h^2\varphi(1/h)^2} \, dx
\]
\[
+ \kappa \left(\frac{n\rho^{n-2}}{\varphi(1/h)} Q(\varepsilon_0 h - \rho^2) + (\max_{h \in I_{(0,0)}} |\varphi'(1/h)|)\frac{\rho^\mu}{h^2\varphi(1/h)^2} (h - \varepsilon_0)Q \right) \frac{\int_{\mathbb{S}^{n-1}} \frac{f}{\varphi(1/h)} \, dx}{\int_{\mathbb{S}^{n-1}} \frac{f}{\varphi(1/h)} \, dx} + Q + Q^2.
\]

The a priori estimates in Section 4 and equation (5.2) allow us to assume $\kappa \approx Q \gg 1$. Then, using (5.10), we conclude
\[
\partial_t Q \leq C_0 Q^2 (C_1 - \varepsilon_0 Q^{\frac{n-1}{n}}) < 0 \tag{5.11}
\]
for some $C_0, C_1$ only depending on $\min_{\mathbb{S}^{n-1}} f, \max_{\mathbb{S}^{n-1}} f$, $\|\varphi\|_{C^1(I_{(0,0)})}$, $\|h\|_{C^0(\mathbb{S}^{n-1} \times (0,\infty))}$, $\|\rho\|_{C^0(\mathbb{S}^{n-1} \times (0,\infty))}$, $\min_{(0,\infty)} h$ and $\min_{(0,0)} \varphi$. Hence, the ODE (5.11) tells that
\[
Q(x_0, t) \leq C \tag{5.12}
\]
for some $C > 0$, independent of $t$.

Making use of the priori estimates in Section 4 and Section 5 (5.2) and (5.12), for any $(x, t)$, we obtain
\[
\kappa = \frac{(h - \varepsilon_0)Q(x, t) + h}{\int_{\mathbb{S}^{n-1}} \frac{f(x)}{\varphi(1/h)} \frac{\int_{\mathbb{S}^{n-1}} \frac{f}{\varphi(1/h)} \, dx}{\int_{\mathbb{S}^{n-1}} \frac{f}{\varphi(1/h)} \, dx}} \leq \frac{(h - \varepsilon_0)Q(x_0, t) + h}{\int_{\mathbb{S}^{n-1}} \frac{f(x)}{\varphi(1/h)} \frac{\int_{\mathbb{S}^{n-1}} \frac{f}{\varphi(1/h)} \, dx}{\int_{\mathbb{S}^{n-1}} \frac{f}{\varphi(1/h)} \, dx}} \leq C \tag{5.13}
\]
for some $C > 0$, independent of $t$. So, the upper bound of Gauss curvature is established.

We are now in a position to prove the lower bound of (5.1). We introduce the auxiliary function
\[
E(x, t) = \log \lambda_{\max}(\{w_{ij}\}) - d \log h(x, t) + \|\nabla h\|^2, \tag{5.14}
\]
where $d$ and $l$ are positive constants to be specified later, $\lambda_{\max}(\{w_{ij}\})$ is the maximal eigenvalue of $\{w_{ij}\}$. As showed in above, one can know that the eigenvalue of $\{w_{ij}\}$ and $\{w_{ij}\}$ are respectively the principal radii and principal curvature of $\partial \Omega_t$.

For any fixed $t \in (0, \infty)$, suppose that the maximum of $E(x, t)$ is attained at $x_0$ on $\mathbb{S}^{n-1}$. By a rotation of coordinates, we may assume that $\{w_{ij}(x_0, t)\}$ is diagonal, and $\lambda_{\max}(\{w_{ij}(x_0, t)\}) = w_{11}(x_0, t)$. To obtain the lower bound of principal curvature, it is necessary to get the upper bound of $w_{11}$. By means of the above assumption, we transform (5.14) into
\[
\widetilde{E}(x, t) = \log w_{11} - d \log h(x, t) + \|\nabla h\|^2. \tag{5.15}
\]
Utilizing again the above assumption, thus, for any fixed $t \in (0, \infty)$, $\tilde{E}(x, t)$ has a local maximum at $x_0$, which implies that, at $x_0$, it yields
\begin{equation}
0 = \nabla_i \tilde{E} = w^{11} \nabla_i w_{11} - \frac{d}{h} + 2l \sum_j h_j h_{ji},
\end{equation}
(5.16)
and
\begin{equation}
0 \geq \nabla_{ii} \tilde{E} = w^{11} \nabla_{ii} w_{11} - (w^{11})^2 (\nabla_i w_{11})^2 - d \left( \frac{h_i}{h} - \frac{h_i^2}{h^2} \right) + 2l \left[ \sum_j h_j h_{jii} + h_{ii}^2 \right].
\end{equation}
(5.17)
Furthermore, we have
\begin{equation}
\partial_t \tilde{E} = w^{11} \partial_t w_{11} - d \frac{h_t}{h} + 2l \sum_j h_j h_{jt} = w^{11} (h_{11t} + h_t) - d \frac{h_t}{h} + 2l \sum_j h_j h_{jt}.
\end{equation}
(5.18)
On the other hand, applying (3.2) and (3.3), we have
\begin{align}
\log(h - h_t) &= \log \left( f \frac{\rho^n}{\varphi(1/h)} \int_{S^{n-1}} \frac{du}{\varphi(1/h)} \right) \\
&= -\log \det(\nabla^2 h + hI) + \chi(x, t),
\end{align}
(5.19)
where
\begin{align}
\chi(x, t) := \log \left( f \frac{\rho^n}{\varphi(1/h)} \int_{S^{n-1}} \frac{du}{\varphi(1/h)} \right).
\end{align}
(5.20)
Now, taking the covariant derivative of (5.19) with respect to $e_j$, it follows that
\begin{align}
\frac{h_j - h_{jt}}{h - h_t} &= - \sum_{i,k} w^{ik} \nabla_j w_{ik} + \nabla_j \chi \\
&= - \sum_i w^{ij} (h_{jii} + h_0 \delta_{ij}) + \nabla_j \chi,
\end{align}
(5.21)
and
\begin{align}
\frac{h_{11} - h_{11t}}{h - h_t} - \frac{(h_1 - h_{1t})^2}{(h - h_t)^2} \\
&= - \sum_i w^{ii} \nabla_{11} w_{ii} + \sum_{i,k} w^{ii} w^{ik} (\nabla_{1} w_{ik})^2 + \nabla_{11} \chi.
\end{align}
(5.22)
On the other hand, the Ricci identity on sphere reads
\begin{align}
\nabla_{11} w_{ij} = \nabla_{ij} w_{11} - \delta_{ij} w_{11} + \delta_{11} w_{ij} - \delta_{1i} w_{1j} + \delta_{1j} w_{1i}.
\end{align}
This together with (5.17), (5.18), (5.21) and (5.22), by a direct computation (see also [6]), we have at \( x_0 \),

\[
\frac{\partial_t \overline{E}}{h-h_t} = w^{11} \left[ \frac{(h_{11t} - h_{11} + h - h + h_t)}{h-h_t} \right] - \frac{1}{h} \frac{h_t - h + h}{(h-h_t)} + 2l \frac{\sum_j h_j h_{jj}}{h-h_t} \\
= w^{11} \left[ \frac{(h_1 - h_{1t})^2}{(h-h_t)^2} + \sum_i w^{ij} \nabla_{ij} w_{11} - \sum_{i,k} w^{ij} w^{kk} (\nabla_i w_{1k})^2 - \nabla_{11} \chi \right] \\
+ \frac{1}{h-h_t} - w^{11} + \frac{d}{h} - \frac{d}{h-h_t} + 2l \frac{\sum_j h_j h_{jj}}{h-h_t} \\
\leq w^{11} \left[ \sum_i w^{ij} (\nabla_{ij} w_{11} - w_{11} + w_{ii}) - \sum_{i,k} w^{ij} w^{kk} (\nabla_i w_{1k})^2 \right] \\
+ \frac{1}{h-h_t} (1-d) - w^{11} \nabla_{11} \chi + \frac{d}{h} + 2l \frac{\sum_j h_j h_{jj}}{h-h_t} \\
\leq \sum_i w^{ii} \left[ (w^{11})^2 (\nabla_i w_{11})^2 + d \left( \frac{h_{ij}}{h} - \frac{h_{ii}^2}{h^2} \right) - 2l \left( \sum_j h_j h_{jj} + h_{ii}^2 \right) \right] \\
- w^{11} \sum_{i,k} w^{ij} w^{kk} (\nabla_i w_{1k})^2 - w^{11} \nabla_{11} \chi + \frac{d}{h} + 2l \frac{\sum_j h_j h_{jj}}{h-h_t} + \frac{1}{h-h_t} (1-d) \tag{5.23}
\]

\[
\leq \sum_i w^{ii} \left[ (h_{ij} + h - h_t) - \frac{h_{ii}^2}{h^2} \right] - 2l \sum_i w^{ii} h_{ii}^2 + 2l \sum_i h_{jj} \left[ - \sum_i w^{ii} h_{ji} + \frac{h_{jj}}{h-h_t} \right] \\
- w^{11} \nabla_{11} \chi + \frac{d}{h} + \frac{1}{h-h_t} (1-d) \\
\leq -d \sum_i w^{ii} - 2l \sum_i w^{ii} (w_{ii}^2 - 2w_{ii} h_t) - w^{11} \nabla_{11} \chi + \frac{nd}{h} + \frac{1}{h-h_t} (1-d) \\
+ 2l \sum_j h_{jj} \left[ \frac{h_{jj}}{h-h_t} + \sum_i w^{jj} h_{ij} - \nabla_{ji} \chi \right] \\
\leq -w^{11} \nabla_{11} \chi - 2l \sum_j h_{jj} \nabla_{11} \chi + 2l [\nabla h_{ii}^2 - d] \sum_i w^{ii} - 2l \sum_i w^{ii} \\
+ \frac{2l [\nabla h_{ii}^2 + 1 - d]}{h-h_t} + 4(n-1)hd + \frac{nd}{h}.
\]

On the other hand, differentiating (5.20), by the first equality of (4.9), at \( x_0 \), we obtain

\[
\nabla_{ji} \chi = \frac{f_j}{f} + n \frac{h_{jj} + h_j h_{jj}}{\rho^2} + \frac{\phi'(1/h) h_j}{h^2 \phi(1/h)}, \tag{5.24}
\]
and

\[
\nabla_{11} \chi = \frac{ff_{11} - f^2_i}{f^2} + n \frac{hh_{11} + h_i^2 + h_{11}^2 + \sum h_j h_{j11}}{\rho^2} - 2n \frac{(hh_1 + h_1 h_{11})^2}{\rho^4} - 2 \frac{\varphi'(1/h)h_i^2}{h^2 \varphi(1/h)} - \frac{\varphi''(1/h)h_i^2}{h^4 \varphi(1/h)} + \frac{\varphi'(1/h)h_{11}}{h^2 \varphi(1/h)} + \frac{[\varphi'(1/h)]^2 h_i^2}{h^4 \varphi(1/h)^2},
\]

(5.25)

where \( \varphi''(1/h) \) denotes \( \frac{\partial^2 \varphi(1/h)}{\partial (1/h)^2} \). Using \( C^0 \), \( C^1 \) estimates in Section 4, (5.16), (5.24) and (5.25), we get

\[
-2l \sum_j h_j \nabla_j \chi - w^{11} \nabla_{11} \chi
\]

\[
= -2l \sum_j h_j \left[ \frac{f f_j - f^2_j}{f^2} + \frac{hh_j + h_j h_{jj}}{\rho^2} + \frac{\varphi'(1/h)h_j}{h^2 \varphi(1/h)} \right]
- w^{11} \left[ \frac{ff_{11} - f^2_i}{f^2} + n \frac{hh_{11} + h_i^2 + h_{11}^2 + \sum h_j h_{j11}}{\rho^2} - 2n \frac{(hh_1 + h_1 h_{11})^2}{\rho^4} - \frac{\varphi'(1/h)h_i^2}{h^2 \varphi(1/h)} - \frac{\varphi''(1/h)h_i^2}{h^4 \varphi(1/h)} + \frac{\varphi'(1/h)h_{11}}{h^2 \varphi(1/h)} + \frac{[\varphi'(1/h)]^2 h_i^2}{h^4 \varphi(1/h)^2} \right]
\leq C_1 l + n \sum_j \frac{h_j}{\rho^2} (-2lh_j h_{jj} - w^{11} h_{j11}) + C_2 w^{11} + 4n w^{11} \frac{hh^2_i h_{11}}{\rho^4}
+ w^{11} \left[ \frac{hh_{11} + h_i^2}{\rho^2} + \frac{2nh_i^2 h_{11}}{\rho^4} + \frac{\varphi'(1/h)h_{11}}{h^2 \varphi(1/h)} \right]
\leq C_1 l + n \sum_j \frac{h_j}{\rho^2} \left( w^{11} h_1 \delta_{j1} - \frac{h_j}{h} \right) + C_2 w^{11} + 4n w^{11} \frac{hh^2_i (w_{11} - h)}{\rho^4}
+ w^{11} \left[ \frac{h(w_{11} - h) + (w_{11} - h)^2}{\rho^2} + \frac{2nh_i^2 (w_{11} - h)^2}{\rho^4} + \frac{\varphi'(1/h)(w_{11} - h)}{h^2 \varphi(1/h)} \right],
\]

(5.26)

where \( C_1 \) is a positive constant depending on \( \|f\|_{C^1_1(G^{n-1})} \), \( \|h\|_{C^1_1(G^{n-1} \times (0,\infty))} \), \( \|\varphi\|_{C^1_1(I(0,\infty))} \), \( \min(\mathbb{R}^{n-1} \times (0,\infty)) \), \( h \), \( \min_{\mathbb{R}^{n-1} f} \) and \( \min_{I(0,\infty)} \varphi \), and \( C_2 \) is a positive constant depending on \( \|f\|_{C^2_1(G^{n-1})} \), \( \|h\|_{C^1_1(G^{n-1} \times (0,\infty))} \), \( \|\varphi\|_{C^1_1(I(0,\infty))} \), \( \min(\mathbb{R}^{n-1} \times (0,\infty)) \), \( \rho \), \( \min_{\mathbb{R}^{n-1} f} \), \( \min_{I(0,\infty)} \varphi \) and \( \min(\mathbb{R}^{n-1} \times (0,\infty)) \) \( h \). To proceed further, (5.26) reduces to

\[
-2l \sum_j h_j \nabla_j \chi - w^{11} \nabla_{11} \chi
\leq C_3 l + C_4 w^{11} + C_5 + n \frac{\rho^2 + 2h_i^2}{\rho^4} w_{11},
\]

(5.27)

where \( C_3, C_4, C_5 \) are positive constants, depending only on \( \|f\|_{C^1_1(G^{n-1})} \), \( \|h\|_{C^1_1(G^{n-1} \times (0,\infty))} \), \( \|\varphi\|_{C^1_1(I(0,\infty))} \), \( \min(\mathbb{R}^{n-1} \times (0,\infty)) \), \( \rho \), \( \min_{\mathbb{R}^{n-1} f} \), \( \min_{I(0,\infty)} \varphi \), and \( \min(\mathbb{R}^{n-1} \times (0,\infty)) \) \( h \), \( \min_{\mathbb{R}^{n-1} f} \) and \( \min_{I(0,\infty)} \varphi \).
We substitute (5.27) into (5.23), and choose \( d = 2l \max_{S^{n-1} \times (0,\infty)} |\nabla h|^2 + 1 \), with
\[
l = n \frac{\max_{S^{n-1} \times (0,\infty)} \rho^2 + 2 \max_{S^{n-1} \times (0,\infty)} |\nabla h|^2}{\min_{S^{n-1} \times (0,\infty)} \rho^4} + 1.
\]

Then, at \( x_0 \), we have
\[
\frac{\partial_t \tilde{E}}{h - h_i} \leq C_3 l + C_4 w^{11} + C_5 \sum_i w_{ii} + 4(n - 1)hl + \frac{nd}{h} < 0 \tag{5.28}
\]
provided \( w_{11} \gg 1 \). Hence, (5.28) tells
\[
E(x_0, t) = \tilde{E}(x_0, t) \leq C \tag{5.29}
\]
for some \( C > 0 \), independent of \( t \). The inequality in (5.29) implies that the principal radii is bounded from above. So, we complete the proof of Theorem 5.1.

\[ \square \]

6. Existence of smooth solution to this problem

In this section, we first focus on obtaining the long time existence and convergence results of flow (3.1), which completes the proof of Theorem 1.1.

Making use of the uniform estimates in Section 4 and Section 5, this allows us to assert that equation (3.1) is uniformly parabolic in \( C^2 \) norm space. Then, by virtue of the standard Krylov’s regularity theory \[22\], we demonstrate the long-time existence and regularity of the solution of equation (3.1). Furthermore,
\[
\|h\|_{C^{l,j}_{\infty}(S^{n-1} \times (0,\infty))} \leq C \tag{6.1}
\]
for some \( C > 0 \), independent of \( t \), and for each pairs of nonnegative integers \( i \) and \( j \).

With the aid of the Arzelà-Ascoli theorem and a diagonal argument, there exists a sequence of \( t \), denoted by \( \{t_k\}_{k \in \mathbb{N}} \subset (0,\infty) \), and a smooth function \( h(x) \) such that
\[
\|h(x, t_k) - h(x)\|_{C^l(S^{n-1})} \to 0
\]
uniformly for any nonnegative integer \( i \) as \( t_k \to \infty \). This illustrates that \( h(x) \) is a support function. Let \( \Omega \) be the convex body determined by \( h(x) \). We conclude that \( \Omega \) is smooth, origin symmetric and strictly convex.

We are now in position to prove the solvability of the following equation
\[
\gamma \varphi(1/h)h(|\nabla h|^2 + h^2)^{-\frac{4}{n}} \det(\nabla^2 h + hI) = f
\]
for some positive constant \( \gamma \).

On the one hand, if the condition (i) holds, Lemma 3.2 tells that, for any \( t > 0 \), we have
\[
P'(t) \geq 0. \tag{6.2}
\]
In view of the $C^0$ estimate on $h$ showed in Lemma 4.1 for any $t \geq 0$, there exists a positive constant $C$ which is independent of $t$, such that

$$P(t) \leq C. \quad (6.3)$$

Combining (6.2) and (6.3), we obtain

$$\int_0^t P'(s)ds = P(t) - P(0) \leq P(t) \leq C,$$

which implies

$$\int_0^\infty P'(t)dt \leq C.$$

This reveals that there exists a sequence of $t_k \rightarrow \infty$ such that

$$P'(t_k) \rightarrow 0 \quad \text{as} \quad t_k \rightarrow \infty. \quad (6.4)$$

On the other hand, recall (3.8), it yields

$$P'(t_k) \int_{\mathbb{R}^{n-1}} f \varphi(1/h) dx \bigg|_{t=t_k} = - \left( \int_{\mathbb{R}^{n-1}} \sqrt{f^2 \rho^h \kappa} \sqrt{h \varphi(1/h)} dx \bigg|_{t=t_k} \right)^2 + \left( \int_{\mathbb{R}^{n-1}} \frac{f^2 \rho^h \kappa}{h \varphi(1/h)^2} dx \int_{\mathbb{R}^{n-1}} \frac{h}{\kappa \rho^n} dx \right|_{t=t_k}. \quad (6.5)$$

Taking the limit in (6.5), using (6.4), we have

$$\left( \int_{\mathbb{R}^{n-1}} \frac{f^2 \rho^h \kappa}{h \varphi(1/h)^2} dx \right)^2 = \int_{\mathbb{R}^{n-1}} \frac{f^2 \rho^h \kappa}{h \varphi(1/h)^2} dx \int_{\mathbb{R}^{n-1}} \frac{h}{\kappa \rho^n} dx. \quad (6.6)$$

In light of (3.8) and (6.6), by the equality condition of the Hölder inequality, and using the a priori estimates in Section 4 and Section 5, we conclude that there exists a positive constant $\gamma$ such that

$$\frac{f^2 \rho^h \kappa}{h \varphi(1/h)^2} = \gamma^2 \frac{h}{\kappa \rho^n},$$

which implies that

$$\gamma \varphi(1/h)h(\nabla h)^2 + h^2)\frac{2}{2} \det(\nabla^2 h + hI) = f$$

has a solution.

On the other hand, if the condition (ii) holds, Lemma 3.3 tells that, for any $t > 0$,

$$P'(t) \leq 0,$$

and

$$\int_0^t (-P'(s))ds = P(0) - P(t) \leq P(0).$$

This gives

$$\int_0^\infty (-P'(t))dt \leq P(0). \quad (6.7)$$
Equation (6.7) tells us that there exists a sequence of $t_k \to \infty$ such that

$$-\dot{P}'(t_k) \to 0 \text{ as } t_k \to \infty.$$ 

Following the similar line as above, we conclude that

$$\gamma \varphi(1/h)h(|\nabla h|^2 + h^2)^{-\frac{4}{n}} \det(\nabla^2 h + hI) = f$$

has a solution for some positive constant $\gamma$. Hence, we complete the proof of Theorem 1.1.

As presented in the introduction, for general $\varphi$, there may be no uniqueness of solutions to equation (1.1). Here, following the similar lines as in [9, 25], we shall give a special uniqueness result of solutions to equation (1.1) based on mild monotonicity assumption on $\varphi$ in the case of $\gamma = 1$. It is shown in the following.

**Theorem 6.1.** Under the assumptions of Theorem 1.1, assume moreover that $\varphi$ is increasing on $(0, \infty)$. Then the solutions to equation

$$\varphi(1/h)h(|\nabla h|^2 + h^2)^{-\frac{4}{n}} \det(\nabla^2 h + hI) = f \quad (6.8)$$

is unique.

**Proof.** Assume $h_1$ and $h_2$ be two solutions of equation (6.8). To prove $h_1 = h_2$, on the one hand, we take by contradiction and assume that $\max_{S^n-1} \frac{h_1}{h_2} > 1$. Suppose $\frac{h_1}{h_2}$ achieves its maximum at $x_0 \in S^{n-1}$. It follows $h_1(x_0) > h_2(x_0)$. Let $G = \log \frac{h_1}{h_2}$. So, at $x_0$, one has

$$0 = \nabla G = \frac{\nabla h_1}{h_1} - \frac{\nabla h_2}{h_2}, \quad (6.9)$$

and applying (6.9), we deduce

$$0 \geq \nabla^2 G$$

$$= \frac{\nabla^2 h_1}{h_1} \frac{\nabla h_1}{h_1} - \frac{\nabla^2 h_1}{h_2} \frac{\nabla h_1}{h_2} + \frac{\nabla^2 h_2}{h_2} \frac{\nabla h_2}{h_2} \quad (6.10)$$

Since $h_1$ and $h_2$ are solutions of equation (6.8), using (6.8) and (6.10), at $x_0$, we obtain

$$1 = \varphi(1/h_2)h_2(|\nabla h_2|^2 + h_2^2)^{-\frac{4}{n}} \det(\nabla^2 h_2 + h_2I) \varphi(1/h_1)h_1(|\nabla h_1|^2 + h_1^2)^{-\frac{4}{n}} \det(\nabla^2 h_1 + h_1I)$$

$$= \varphi(1/h_2)h_2 \frac{h_2^{-n} \left( \frac{\nabla h_1}{h_2}^2 + 1 \right)^{-\frac{n}{2}}} \varphi(1/h_1)h_1 \frac{h_1^{-n} \left( \frac{\nabla h_2}{h_1}^2 + 1 \right)^{-\frac{n}{2}}} \det \left( \frac{\varphi h_2}{h_2} + I \right) \det \left( \frac{\varphi h_1}{h_1} + I \right) \quad (6.11)$$

$$\geq \varphi(1/h_2) \varphi(1/h_1).$$
In light of the assumption in Theorem 6.1, together with (6.11), it implies that \( h_1(x_0) \leq h_2(x_0) \), which is a contradiction. This reveals

\[
\max_{g=-1} \frac{h_1}{h_2} \leq 1. \tag{6.12}
\]

On the other hand, interchanging the role of \( h_1 \) and \( h_2 \), applying the same argument as above, we have

\[
\max_{g=-1} \frac{h_2}{h_1} \leq 1. \tag{6.13}
\]

Combining (6.12) and (6.13), this illustrates that \( h_1 = h_2 \). So, we complete the proof.

\[\square\]

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School of Mathematics, Hunan University, Changsha, 410082, Hunan Province, China

Email address: hujinrong@hnu.edu.cn

School of Mathematics, Hunan University, Changsha, 410082, Hunan Province, China

Email address: liujiaqian@hnu.edu.cn

School of Mathematics, Hunan University, Changsha, 410082, Hunan Province, China

Email address: madi@hnu.edu.cn