COMMUTATOR FORMULAS FOR GRADIENT RICCI SHRINKER AND THEIR APPLICATION TO LINEAR STABILITY

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Abstract. In this paper we have found some commutator formulas between $\text{div} f, \Delta f, L, \text{div}^\dagger f, \Delta f$ on closed orientable $GRS^+$ metrics (Gradient Ricci Shrinks), then with them we have generalized a Theorem of Cao and Zhu about necessary condition for linear stability of a $GRS^+$.

1. Introduction

A complete Riemannian manifold is called a Ricci Soliton if there exist a complete vector field $X$ and a real constant $\lambda$ such that

$$Ric + L_X g = \lambda g$$

where $Ric$ is Ricci tensor, a soliton is called expanding, steady or shrinking if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively, for degenerate case $X = 0$ soliton is called trivial, therefore Einstein manifolds are special cases of Ricci solitons. Whenever $X = \nabla f$ for some smooth function $f \in C^\infty(M)$ the soliton is called gradient, in this case for shrinking soliton we have

$$(1.1) \quad Ric + \nabla^2 f = \frac{1}{2\tau} g$$

where $\tau$ is a positive constant, $f$ is called the potential function of soliton. In this paper we denote gradient Shrinking soliton metrics briefly with $GRS^+$. For more information on Ricci soliton see [2].

The concept of Ricci soliton was invented by Richard Hamilton in mid 80’s as Riemannian metrics that up to diffeomorphism and scaling are fixed points of the Ricci flow equation. Perelman in a remarkable paper [18] discovered several variational structures for Ricci flow, one of them is the $\nu$-entropy. He proved that critical points of the $\nu$-entropy are $GRS^+$ metrics, therefore this question arose that is this true that $GRS^+$ metrics are the local maximum of $\nu$-entropy, trivially if the second variation of the $\nu$-entropy is positive, then soliton can not be the local maximum of $\nu$-entropy but suppose that the second variation of the $\nu$-entropy always nonpositive, then is this true that soliton is the local maximum of $\nu$-entropy ? Therefore it was necessary to calculate the second variation of the $\nu$-entropy. Hamilton, Ilmanen and Cao in [4] calculated the second variation of the $\nu$-entropy for positive Einstein metrics. They defined a $GRS^+$ which is linear stable whenever the second variation of the $\nu$-entropy is nonpositive and otherwise linear unstable. Hamilton conjectured that at least in dimension four, only linear stable $GRS^+$
are Einstein metrics with positive scalar curvature. Cao and Zhu in [3] calculated
the second variation of the $\nu$-entropy for nontrivial $GRS^+$, they proved that for
linear stability of $GRS^+$ it is necessary that the first eigenvalue of the weighted
Lichnerowicz Laplace operator $\Delta_{f,L}$ restricted to transversal tensor (i.e $\text{div} f h = 0$)
is not greater than zero and only eigentensor of zero should be Ricci tensor $\text{Ric}$.

Finally Kröncke in [12] proved that if every infinitesimal solitonic deformation
of soliton is integrable, then $GRS^+$ is the local maximum of $\nu$-entropy if and only
if the second variation of the $\nu$-entropy is nonpositive.

Kröncke proved that this condition is failed for complex projective space $\mathbb{C}P^n$
with Fubiny-Study metric and although complex projective space $\mathbb{C}P^n$ with Fubiny-
Study metric is a linear stable $GRS^+$, but this $GRS^+$ is not the local maximum of
$\nu$-entropy in the space of all of Riemannian metrics on $\mathbb{C}P^n$.

Subject of this paper is closed orientable gradient Ricci shrinking soliton (briefly
$GRS^+$). In the second section we will introduce our conventions and necessary
definitions for our work, then we bring the necessary formulas and theorems without
proofs from other papers. In the third section we will get several commutator
formulas between some differential operators. We prove that

**Theorem 1.1.** For a closed orientable $GRS^+ (M^n, g, f, \tau)$ and $h \in C^\infty(S^2(T^*M)), \omega \in \Omega^1(M)$ and $a \in C^\infty(M)$ we have

$$
\Delta_f da = d\Delta_f a + \frac{1}{2\tau} da \\
\text{div}_f \Delta_f \omega = \Delta_f \text{div}_f \omega + \frac{1}{2\tau} \text{div}_f \omega \\
\mathcal{L}_{\#(\Delta_f \omega)} g = \Delta_{f,L}(\mathcal{L}_{\#} g) + \frac{1}{2\tau} \mathcal{L}_{\#} g
$$

Note that these formulas can be extended to other kinds of the Ricci soliton and
noncompact Ricci soliton, but this is not the subject of this paper. These commutator
formulas are used to investigate relations between Ricci curvature bounds, spectrum of the weighted Lichnerowicz Laplacian and properties of eigentensors of the weighted Lichnerowicz Laplacian. In the fourth section inspired by work of Cao and He[5], we will prove that the stability operator $N$ of a $GRS^+$ on $\text{Im}(\text{div}_f)$
equals to zero pointwisely.

**Theorem 1.2.** For a closed orientable $GRS^+ (M^n, g, f, \tau)$, the stability operator (Jacobi field operator) of the $\nu$-entropy $N$ on $\text{Im}(\text{div}_f)$ pointwisely equals to zero, in other word for every vector field $X \in \chi(M)$

$$
N(\mathcal{L}_X g) = 0
$$

In fifth section we will extend Theorem (1.3) of Cao and Zhu in [3], we find a
weaker bound other than zero $(-\frac{1}{2\tau})$. We will prove two Theorems

**Theorem 1.3.** A necessary condition for linear stability of closed orientable
$GRS^+$ is that the first eigenvalue of the weighted Lichnerowicz Laplacian $\Delta_{f,L}$ (except zero with one multiplicity and Ricci tensor $\text{Ric}$ as eigentensor) is not greater
than $-\frac{1}{2\tau}$.

**Theorem 1.4.** Suppose that for a closed orientable $GRS^+$ the first eigenvalue of the weighted Lichnerowicz Laplacian $\Delta_{f,L}$ is not greater than $-\frac{1}{\tau}$ (except zero with one multiplicity and Ricci tensor $\text{Ric}$ as eigentensor), then soliton is linear stable.
2. Preliminaries

2.1. Conventions and Notations. We define Riemannian curvature as

\[ Rm(X, Y, Z, W) = - < R(X, Y)Z, W > \]

with this convention we have these commutator formulas

\[
\begin{align*}
\nabla_i \nabla_j \omega_k &= \nabla_j \nabla_i \omega_k - R_{ijkp} \omega_p = \nabla_j \nabla_i \omega_k + R_{ijkp} \omega^p \\
\nabla_i \nabla_j T_{pq} &= \nabla_j \nabla_i T_{pq} - R_{ijpq} T_{mq} - R_{ijqm} T_{pq} \\
&= \nabla_j \nabla_i T_{pq} + R_{ijpm} T_{mq} + R_{ijqm} T_{pq}
\end{align*}
\]

As explained in introduction, we need the second variation of the \( \nu \)-entropy. Here we introduce some conventions (our notations is similar to \([3]\) and \([4]\)).

For any symmetric covariant two tensor \( h, (h_{ij}) \) and any 1-form \( \omega, (\omega_i) \) we denote

\[
\begin{align*}
\text{div} \omega &= g^{pq} \nabla_p \omega_q \\
\text{div}_f \omega &= e^f \text{div}(e^{-f} \omega) \\
\text{div}_f h &= e^f \text{div}(e^{-f} h) \\
\text{div}_f \omega &= g^{pq} (\nabla_p \omega_q - \omega_p \nabla_q f) \\
\Delta_f &= \text{div}_f \nabla = \Delta - \nabla \nabla f \\
(Rm(h, -))_{ij} &= R_{pqij} h^{pq} \\
\text{div}_f^1 \omega &= - \frac{1}{2} L_{\#} \omega_g, \quad (\text{div}_f^1 \omega)_{ij} = - \frac{1}{2} (\nabla_i \omega_j + \nabla_j \omega_i)
\end{align*}
\]

In this paper we use the weighted \( L^2 \)-inner product with respect to measure \( dm \):

\[
< -, - >_{dm} = \int_M < -, - > (4\pi)^{-\frac{n}{2}} e^{-f} dv, \quad dm = (4\pi)^{-\frac{n}{2}} e^{-f} dv, \quad \int_M dm = 1
\]

where \( < -, - > \) is inner product induced from Riemannian metric of \( M \) on arbitrary tensor bundle, \( dv \) is Riemannian volume form and \( f \) is the potential function of soliton. Therefore it is important that in our notation \( < -, > \) is pointwise inner product on tensors and \( < -, >_{dm} \) is global inner product for tensor fields and \( |T| \) is norm of some tensor \( T \) in some point and \( |T|_{dm} \) is norm of tensor field \( T \). Note that Cauchy–Schwarz inequality is valid for \( < -, >_{dm} \) and \( | - |_{dm} \).

With this \( L^2 \)-inner product we have the weighted divergence theorem for arbitrary 1-form \( \omega \in \Omega^1(M) \) and smooth function \( a \in C^\infty(M) \) and \( dm = (4\pi)^{-\frac{n}{2}} e^{-f} dv \), we have

\[
\int_M \text{div}_f (\omega) dm = 0, \quad \int_M \Delta_f dm = 0
\]

Here we need to find formal-adjoint of the defined operators with respect to the weighted \( L^2 \)-inner product. We have

\[ \text{div}_f (aw) = a \text{div}_f \omega + < \omega, da > \]

therefore formal-adjoint of \( \text{div}_f \) on 1-forms is \(-d\). On the other hand we have

\[ \text{div}_f (h(\omega, -)) = < \text{div}_f h, \omega > + < h, \text{div}_f \omega > \]

therefore formal-adjoint of \( \text{div}_f \) on symmetric covariant two tensors is \( \text{div}_f^1 \), formal-adjoint of \( \nabla \) on 1-forms and symmetric covariant two tensors is \(-\text{div}_f \) and \( \Delta_f \) is a
self-adjoint operator. Therefore if we denote the formal adjoint with respect to the weighted $L^2$ inner product of a differential operator $D$ with $D^\dagger$, then we have
\[ d^\dagger = -\text{div}_f, \nabla^\dagger = -\text{div}_f, (\text{div}_f)^\dagger = \text{div}^\dagger_f. \]
i.e
\[
\int_M < \text{div}_f \omega, a > dm = \int_M - < \omega, da > dm \\
\int_M < \text{div}_f h, \omega > dm = \int_M - < h, \text{div}^\dagger_f \omega > dm \\
\int_M < \Delta_f h, k > dm = \int_M - < h, \Delta_f k > dm
\]
Considering Bakry-Emery-Ricci tensor and Ricci solitons, Lott in [15], defined a weighted version of Lichnerowicz Laplacian which we work with that
\[ \Delta_{f,L}h = \Delta_f h + 2Rm(h,-) - (Ric + \nabla^2 f).h - h.(Ric + \nabla^2 f) \]
such that $Rm(h,-)_{ij} = R_{pqij}h^{pq}$, $(A.B)_{ij} = g^{pq}A_{ip}B_{qj}$. For special case of Ricci solitons we have
\[ \Delta_{f,L}h = \Delta_f h + 2Rm(h,-) - \frac{1}{\tau}h \]
Trivially $\Delta_{f,L}$ is a self-adjoint operator with respect to the measure $dm$, very important note is that
\[ \text{tr} \Delta_{f,L}h \neq \Delta_f \text{tr}h \]

2.2. **W-entropy.** For a closed orientable Riemannian manifold $(M^n, g)$ we define Perelman’s W-entropy as
\[
W(g, f, \tau) = \int_M [\tau (R + |\nabla f|^2) + f - n] dm \\
= (4\pi \tau)^{-\frac{2}{n}} e^{-f} dv
\]
such that $g$ is Riemannian metric, $f$ is a smooth function on $M$, $\tau$ is a positive real number, $R$ is scalar curvature and $dv$ is volume form of manifold, with these notations for any positive real number $c$ and diffeomorphism $\varphi \in Diff(M)$ we have
\[
W(cf, c\tau) = W(g, f, \tau) = W(\varphi^* g, \varphi^* f, \tau),
\]
where $\varphi^* f(p) = f(\varphi(p))$.

Then we define Perelman’s $\nu$-entropy as
\[
\nu(g) = \inf \{ W(g, f, \tau) : f \in C^\infty(M), \tau > 0 \}
\]
\[
\int_M (4\pi \tau)^{-\frac{2}{n}} e^{-f} dv = 1
\]
Note that $\nu$-entropy may be infinite and therefore minimizing pair $(f, \tau)$ does not exist. Anyway it is proved that [13] p.5, if $\nu$-entropy exists, then the minimizing pair $(f, \tau)$ should satisfies these conditions.
\[(2.4) \quad \tau (-2\Delta f + |\nabla f|^2 - R) - f + n + \nu = 0\]
\[(2.5) \quad \int_M f dm = \frac{n}{2} + \nu\]

It is proved that for a closed GRS\(^+\), \(\nu\)-entropy is finite([12] Remark (3.4)) and in the \(C^2\)-neighborhood of GRS\(^+\) in the space of all Riemannian metrics on \(M\), \(\nu\)-entropy exists and is finite and there exists an unique minimizing pair \((f, \tau)\) ([12] Remark 3.2 and [3]).

**Theorem 2.1 (Fist Variation Formula).** The first variation of the \(\nu\)-entropy for a closed Riemannian manifold \((M^n, g)\) in the perturbation direction \(h \in C^\infty(S^2(T^*M))\) is given by
\[(2.6) \quad \nu_g'(h) = \int_M \left( -\tau < h, Ric + \nabla^2 f - \frac{1}{2\tau} g > dm \right)\]

**Proof.** See [3] Lemma (2.2) \(\Box\)

From this theorem we conclude that if minimizing pair \((f, \tau)\) satisfies solitonic equation (1.1), then \(g\) is a critical point for \(\nu\)-entropy in the space of all Riemannian metrics on \(M\). Conversely suppose that for a closed Riemannian manifold \((M^n, g)\), a smooth function \(a \in C^\infty(M)\) and positive real number \(c\) we have \(Ric + \nabla^2 a = cg\), then from Theorem (2.1) and because \(\nu\)-entropy is invariant with respect to the action of diffeomorphism group and scaling of metric it follows that \(\nu_g'(\frac{1}{2} L_n a g - cg) = 0\) therefore \(\nu_g'(\nabla^2 a - cg) = 0\) and

\[0 = \nu_g'(0) = \nu_g'(Ric + \nabla^2 a - cg) = \nu_g'(Ric) = \nu_g'(Ric + \nabla^2 f - \frac{1}{2\tau} g)\]

and we conclude that
\[\int_M |Ric + \nabla^2 f - \frac{1}{2\tau} g|^2 dm = 0\]
and finally
\[Ric + \nabla^2 f = \frac{1}{2\tau} g\]

therefore \(g\) is critical point for \(\nu\)-entropy in the space of all of Riemannian metrics on \(M\) if and only if minimizing pair \((f, \tau)\) satisfies solitonic equation (1.1). On the other hand Theorem (2.1) says that under Ricci flow as well as \(\nu\)-entropy is finite, \(\nu\)-entropy is monotone increasing and is constant if and only if initial metric is a GRS\(^+\) and minimizing pair \((f, \tau)\) satisfies solitonic equation. Here we can state Theorem of Cao and Zhu in [3] wich states the exact expression of the second variation of the \(\nu\)-entropy.

**Theorem 2.2 (Second Variation Formula).** For a closed orientable GRS\(^+\) \((M^n, g, f, \tau)\) the second variation of the \(\nu\)-entropy for any perturbation direction \(h \in C^\infty(S^2(T^*M))\) is given by
\[(2.7) \quad \nu_g''(h) = \int_M < N(h), h > dm\]

Here
\[N(h) = \frac{1}{2} \Delta f h + Rm(h, -) + div^1 div_f h + \frac{1}{2} \nabla^2 v_f h - Ric \frac{\int_M < Ric, h > dm}{\int_M R dm}\]
\[
= \frac{1}{2} \Delta f, L h + \frac{1}{2} h + \text{div}^\dagger \text{div} f h + \frac{1}{2} \nabla^2 v h - \text{Ric} \int_M \text{Ric}, h > dm \int_M R dm
\]

in this expression, \( v_h \) is unique solution of the equation
\[
(2.8) \quad \Delta f v_h + \frac{1}{2\tau} v_h = \text{div} f \text{div} f h, \int_M v_h dm = 0
\]
Here \( N \) is a self-adjoint degenerate elliptic operator.

Proof. see [3] Theorem(1.1) and [4]. \( \square \)

2.3. Useful Formulas.

**Theorem 2.3.** For a closed orientable GRS\( ^+ \) \((M^n, g, f, \tau)\) we have
\[
\text{Ric}(\nabla f, -) = \frac{1}{2} dR \quad \text{i.e.} \quad \text{div} f \text{Ric} = 0
\]
\[
g^{pq} \nabla_f R_{qjkl} = g^{pq} R_{pjkl} \nabla f \quad \text{i.e.} \quad \text{div} f \text{Rm} = 0
\]
\[
\Delta f \text{Ric} + 2 \text{Rm}(h, -) = \frac{1}{\tau} \text{Ric} \quad \text{i.e.} \quad \Delta f, L \text{Ric} = 0
\]
\[
\Delta f R = \frac{1}{\tau} R - 2 |\text{Ric}|^2
\]
\[
\Delta f f = - \frac{1}{\tau} f + \text{Const}
\]
\[
\int_M R dm = 2\tau \int_M |\text{Ric}|^2 dm
\]

Proof. see [20] Lemma(2.1). \( \square \)

**Theorem 2.4.** For a closed orientable GRS\( ^+ \) \((M^n, g, f, \tau)\), the first eigenvalue of the weighted Laplacian on functions \( \Delta f \) is strictly less than \(-\frac{1}{2\tau}\).
\[
(\lambda_1 < -\frac{1}{2\tau})
\]

Proof. see [3] page 9. \( \square \)

From this theorem it follows that \( v_h \) function in the second variation formula of the \( \nu \)-entropy exists and is unique.

3. Commutator Formulas

In this section we will prove commutator formulas between \( \text{div}^\dagger f, \text{div} f, \Delta f, \Delta f, L \) on a closed orientable GRS\( ^+ \). Stability operator of the \( \nu \)-entropy has complicated formula, this operator is related to the weighted Lichnerowicz Laplacian. One of the difficulties in computation of the second variation for a given perturbation direction, is computation of an unknown function \( v_h \) whose Hessian is in the stability operator. This function satisfies the second order differential equation
\[
\Delta f v_h + \frac{1}{2\tau} v_h = \text{div} f \text{div} f h
\]
note that from Theorem [23] this function is unique, in some situation for example relation between linear stability and dynamical stability, from Ebin-Berger decomposition Theorem (see [1] Collorary(4.1)), tangent space to metric \( g \), i.e \( C^\infty(S^2(T^*M)) \) decomposes to two orthogonal subspaces
\[
C^\infty(S^2(T^*M)) = \text{Ker}(\text{div} f) \oplus \text{Im}(\text{div} f)
\]
Then because \( \nu \)-entropy is invariant under the scaling and action of diffeomorphism group \( \text{Diff}(M) \) on metric \( g \) and since tangent vector to action of diffeomorphism group on metric is \( L_\# \omega = -2\text{div}_f^1 \omega, \omega \in \Omega^1(M) \), therefore the second variation of the \( \nu \)-entropy on \( \text{Im} (\text{div}_f^1) \) is zero, hence in this situation we can assume that \( \text{div}_f h = 0 \) so \( \nu_h = 0 \). In general except for a few special cases we have to find unstability direction case by case and state by state, therefore we have to find \( \nu_h \).

Now because it is difficult to find \( \nu_h \), we understand that if we take \( \nu_h = \text{div}_f \text{div}_f k \) for unknown tensor \( k \in C^\infty(S^2(T^*M)) \), then we have

\[
\Delta_f \text{div}_f \text{div}_f k + \frac{1}{2\tau} \text{div}_f \text{div}_f k = \text{div}_f \text{div}_f h
\]

now if we find commutator between \( \text{div}_f \text{div}_f \) and \( \Delta_f \), then we get a better understanding of relation between \( \nu_h \) and \( h \). After this we find our commutator formulas and specially we find \( \text{div}_f \Delta_f L_k = \Delta_f \text{div}_f \text{div}_f k \) for \( k \in C^\infty(S^2(T^*M)) \) (indeed this is why that we found our commutator formulas).

Now if \( \nu_h = \text{div}_f \text{div}_f k \), then we have

\[
\text{div}_f \Delta_f L k + \frac{1}{2\tau} k = \text{div}_f \text{div}_f h
\]

Therefore if \( h = \Delta_f L k + \frac{1}{2\tau} k \), then \( \nu_h = \text{div}_f \text{div}_f k \). Now whenever we work with \( k \) instead of \( h \) i.e given \( k \in C^\infty(S^2(T^*M)) \), then \( h = \Delta_f L k + \frac{1}{2\tau} k \) and \( \nu_h \) is found. In the other hand we start with \( \nu_h \) as \( \nu_h = \text{div}_f \text{div}_f k \), and then we find an \( h \in C^\infty(S^2(T^*M)) \) for that. After we have found our commutator formulas we found that Deruelle, Alix [7] already obtained one of our commutator formulas with only a time derivative difference (Theorem 3.4), indeed the order and method of our initial proof of our results is almost exactly the same as the method of Deruelle.

Here we give another order and proof for our commutator formulas. The point is that we have found them without knowing that Deruelle had already reached to these formulas.

**Theorem 3.1.** For a closed orientable \( \text{GRS}^+ (M^n, g, f, \tau) \) and any smooth function \( a \in C^\infty(M) \) we have

\begin{equation}
\Delta_f da = d\Delta_f a + \frac{1}{2\tau} da
\end{equation}

**Proof.**

\[
\Delta_f da - d\Delta_f a = \Delta da - \nabla \nabla_f da - (d\Delta a - d < \nabla a, \nabla f >) =
\]

\[
= \Delta da - \nabla^2 a(\nabla f, -) - d\Delta a + \nabla^2 a(\nabla f, -) + \nabla^2 f(\nabla a, -) =
\]

\[
= \Delta da - d\Delta a + \nabla^2 f(\nabla a, -) = (\text{Ric} + \nabla^2 f)(\nabla a, -) =
\]

\[
= \frac{1}{2\tau} da
\]

\[\square\]

**Theorem 3.2.** For a closed orientable \( \text{GRS}^+ (M^n, g, f, \tau) \) and \( \omega \in \Omega^1(M) \) we have

\[
\text{div}_f \Delta_f \omega = \Delta_f \text{div}_f \omega + \frac{1}{2\tau} \text{div}_f \omega
\]
Proof. Take $a = \text{div} f \Delta f \omega - \Delta f \text{div} f \omega - \frac{1}{2\tau} \text{div} f \omega$. Now we have
\[
\int_M a^2 dm =
\]
\[
= \int_M |\text{div} f \Delta f \omega - \Delta f \text{div} f \omega - \frac{1}{2\tau} \text{div} f \omega|^2 dm
\]
\[
= \int_M < \text{div} f \Delta f \omega - \Delta f \text{div} f \omega - \frac{1}{2\tau} \text{div} f \omega, a > dm
\]
\[
= \int_M < \omega, -\Delta f da + d\Delta f a + \frac{1}{2\tau} da > dm
\]
\[
= 0
\]

Now since $\int_M a^2 dm = 0$ and $M$ is compact, therefore we conclude that $a = 0$ i.e $\text{div} f \Delta f \omega = \Delta f \text{div} f \omega + \frac{1}{2\tau} \text{div} f \omega$.

\[\square\]

**Theorem 3.3.** For a closed orientable $\text{GRS}^+(M^n, g, f, \tau)$ and $\omega \in \Omega^1(M)$ we have

\[
\Delta_{f,l}(\mathcal{L}_\#g) + \frac{1}{2\tau} \mathcal{L}_\#g = \mathcal{L}_\#(\Delta f \omega) g
\]

Proof.

\[
\Delta f (\mathcal{L}_\#g) = \Delta (\mathcal{L}_\#g) - \nabla f \mathcal{L}_\#g
\]

Now for the first term of the right hand side we have
\[
\Delta (\mathcal{L}_\#g)_{ij} = g^{pq} [\nabla_p \nabla_q (\nabla_i \omega_j + \nabla_j \omega_i)]
\]
\[
= g^{pq} [\nabla_p (\nabla_i \nabla_q \omega_j + R_{qij} \omega^s + \nabla_j \nabla_i \omega_q + R_{qji} \omega^s)]
\]
\[
= g^{pq} [\nabla_p (\nabla_i \nabla_q \omega_j + \nabla_p \nabla_j \omega_i)]
\]
\[
+ (\nabla_p R_{qij} \omega^s + \nabla_p R_{qji} \omega^s) + (R_{qij} + R_{qji}) \nabla_p \omega^s
\]
\[
= g^{pq} [\nabla_i \nabla_p \nabla_q \omega_j + \nabla_j \nabla_p \nabla_i \omega_q + (\nabla_p R_{qij} \omega^s + \nabla_p R_{qji} \omega^s)]
\]
\[
- 2(R_{qij} + R_{qji}) \nabla_p \omega^s + g^{\alpha\beta} (R_{qij} \nabla_p \omega_\beta + R_{qji} \nabla_p \omega_\beta)
\]
\[
= (\nabla_i \Delta \omega_j + \nabla_j \Delta \omega_i) + g^{pq}(\nabla_p R_{qij} \omega^s + \nabla_p R_{qji} \omega^s)
\]
\[
- 2g^{pq}(R_{qij} + R_{qji}) \nabla_p \omega^s + g^{\alpha\beta} (R_{qij} \nabla_p \omega_\beta + R_{qji} \nabla_p \omega_\beta)
\]

And for the second term we have
\[
(\nabla f \mathcal{L}_\#g)_{ij} = g^{pq} [\nabla_p (\nabla_i \omega_j + \nabla_j \omega_i) \nabla_q f]
\]
\[
= g^{pq} [\nabla_i \nabla_p \omega_j + R_{pji} \omega^s + \nabla_j \nabla_p \omega_i + R_{pj} \omega^s] \nabla_q f
\]
\[
= g^{pq} [(R_{pji} + R_{pj}) \nabla_q f \omega^s + (\nabla_i \nabla_p \omega_j + \nabla_j \nabla_p \omega_i) \nabla_q f]
\]
\[
= g^{pq} [(R_{qij} + R_{qji}) \nabla_p f \omega^s + (\nabla_i (\nabla_p \omega_j + \nabla_p \omega_j) \nabla_q f) - \nabla_i \nabla_q f \nabla_p \omega_j]
\]
\[
= \nabla_j (\nabla_p \omega_i \nabla_q f) - \nabla_j \nabla_q f \nabla_p \omega_i]
\]
\[
= g^{pq} [(R_{qij} + R_{qji}) \nabla_p f \omega^s + (\nabla_i (\nabla_p \omega_j \nabla_q f) + \nabla_j (\nabla_p \omega_i \nabla_q f)]
\]
\[
- \nabla_i \nabla_q f \nabla_p \omega_j + \nabla_j \nabla_q f \nabla_p \omega_i]
\]
Note that
\[ \nabla_i \nabla_p \omega_j \nabla_q f = (\nabla_i (\nabla_p \omega_j \nabla_q f) - \nabla_i \nabla_p f \nabla_q \omega_j) \]

Now if we substitute these two calculated terms and considering
\[ \Delta f (\mathcal{L}_{\#} g)_{ij} = (\nabla_i \Delta \omega_j + \nabla_j \Delta \omega_i) - (\nabla_i (\nabla_v f \omega) + \nabla_j (\nabla_v f \omega)) \]
\[ + \left[ (g^{\alpha\beta} (R_{i\alpha} + \nabla_i \nabla f) \nabla_\beta \omega_j + g^{\alpha\beta} (R_{j\alpha} + \nabla_j \nabla f) \nabla_\beta \omega_i) \right] \]
\[ - 2g^{pq} (R_{qisj} + R_{sijq}) \nabla_p \omega^s + g^{pq} (\nabla_p R_{qisisj} - R_{qisis} \nabla_q f) \nabla_p \omega^s \]
\[ + g^{pq} (\nabla_p R_{qjis} - R_{qjis} \nabla_q f) \nabla_p \omega^s \]

Now according to Theorem (2.3) we have \( \text{div}_i Rm = 0 \) therefore
\[ \Delta f (\mathcal{L}_{\#} g)_{ij} = (\nabla_i \Delta \omega_j + \nabla_j \Delta \omega_i) - (\nabla_i (\nabla_v f \omega) + \nabla_j (\nabla_v f \omega)) \]
\[ + \left[ (g^{\alpha\beta} (R_{i\alpha} + \nabla_i \nabla f) \nabla_\beta \omega_j + g^{\alpha\beta} (R_{j\alpha} + \nabla_j \nabla f) \nabla_\beta \omega_i) \right] \]
\[ - 2g^{pq} (R_{qisj} + R_{sijq}) \nabla_p \omega^s = 2g^{pq} g^{rs} (R_{qris} + R_{sirq}) \nabla_p \omega^r \]
\[ = 2g^{pq} g^{rs} R_{qris} (\nabla_p \omega^r + \nabla_r \omega_p) \]
\[ = 2Rm (\mathcal{L}_{\#} g, -)_{ij} \]

On the other hand we have
\[ g^{pq} (R_{qisj} + R_{sijq}) \nabla_p \omega^s = g^{pq} g^{rs} (R_{qris} + R_{sirq}) \nabla_p \omega^r \]
\[ = g^{pq} g^{rs} R_{qris} (\nabla_p \omega^r + \nabla_r \omega_p) \]
\[ = 2Rm (\mathcal{L}_{\#} g, -)_{ij} \]

and finally
\[ \Delta f (\mathcal{L}_{\#} g) = \mathcal{L}_{\#} \Delta \omega g - \mathcal{L}_{\#} \nabla_v f \omega g + \frac{1}{2 \tau} \mathcal{L}_{\#} \omega g - 2Rm (\mathcal{L}_{\#} g, -) \]
\[ = \mathcal{L}_{\#} \Delta f \omega g + \frac{1}{2 \tau} \mathcal{L}_{\#} \omega g - 2Rm (\mathcal{L}_{\#} g, -) \]

Therefore we conclude that
\[ (\Delta f (\mathcal{L}_{\#} g) + 2Rm (\mathcal{L}_{\#} g, -) - \frac{1}{\tau} \mathcal{L}_{\#} \omega g) + \frac{1}{2 \tau} \mathcal{L}_{\#} \omega g = \mathcal{L}_{\#} \Delta \omega \omega g \]

And finally it follows that
\[ \Delta_{f,L} (\mathcal{L}_{\#} \omega g) + \frac{1}{2 \tau} \mathcal{L}_{\#} \omega g = \mathcal{L}_{\#} (\Delta f \omega) g \]

\[ \square \]

**Corollary 3.1.** For a closed orientable GRS\(^+\) and \( \omega \in \Omega^1(M) \) we have
\[ (3.3) \quad \text{div}_i \Delta f \omega = \Delta_{f,L} \text{div}_i \omega + \frac{1}{2 \tau} \text{div}_i \omega \]

**Theorem 3.4.** For a closed orientable GRS\(^+\) \((M^n, g, f, \tau)\) and \( h \in C^\infty(S^2(T^*M)) \) we have
\[ (3.4) \quad \Delta f \text{div}_f h = \text{div}_f \Delta_{f,L} h + \frac{1}{2 \tau} \text{div}_f h \]

**Proof.** Take \( \omega = \Delta f \text{div}_f h - \text{div}_f \Delta_{f,L} h - \frac{1}{2 \tau} \text{div}_f h \), now we have
\[ \int_M |\omega|^2 \, dm = \int_M < \Delta_f \text{div}_f h - \text{div}_f \Delta_{f,L} h - \frac{1}{2\tau} \text{div}_f h, \omega > \, dm \]
\[ = \int_M < h, \text{div}_f \Delta_f \omega - \Delta_{f,L} \text{div}_f \omega - \frac{1}{2\tau} \text{div}_f \omega > \, dm \]
\[ = 0 \]

Now \( \int_M |\omega|^2 \, dm = 0 \), therefore because \( M \) is compact it follows that \( \omega = 0 \).

\[ \Delta_f \text{div}_f h = \text{div}_f \Delta_{f,L} h \quad \text{(3.5)} \]

**Proof.** By Theorem (3.4) it follows that
\[ \Delta_f \text{div}_f h = \text{div}_f \Delta_{f,L} h + \frac{1}{2\tau} \text{div}_f h \]
and
\[ \text{div}_f \Delta_f \text{div}_f h = \text{div}_f \text{div}_f \Delta_{f,L} h + \frac{1}{2\tau} \text{div}_f \text{div}_f h \]

On the other hand because \( \text{div}_f h \) is a differential form therefore according to Theorem (3.2) if we take \( \omega = \text{div}_f h \) we have
\[ \text{div}_f \Delta_f \text{div}_f h = \Delta_f \text{div}_f \text{div}_f h + \frac{1}{2\tau} \text{div}_f \text{div}_f h \]
and finally \( \Delta_f \text{div}_f \text{div}_f h = \text{div}_f \text{div}_f \Delta_{f,L} h \)

**Theorem 3.6.** For a closed orientable \( \text{GRS}^+ (M^n, g, f, \tau) \) and \( h \in C^\infty(S^2(T^*M)) \)

\[ \text{div}_f \text{div}_f \Delta_{f,L} h = \Delta_{f,L} \text{div}_f \text{div}_f h \quad \text{(3.6)} \]

**Proof.** From Theorem (3.4) we have
\[ \text{div}_f \text{div}_f \Delta_{f,L} h = \text{div}_f \text{div}_f (\Delta_{f,L} h) - \frac{1}{2\tau} \text{div}_f \text{div}_f (\text{div}_f h) \]
\[ = \text{div}_f \Delta_{f,L} \text{div}_f h - \frac{1}{2\tau} \text{div}_f \text{div}_f \text{div}_f h \]

Now from Corollary (3.1) we have
\[ \text{div}_f \text{div}_f \Delta_{f,L} h = \Delta_{f,L} \text{div}_f \text{div}_f h + \frac{1}{2\tau} \text{div}_f \text{div}_f \text{div}_f h - \frac{1}{2\tau} \text{div}_f \text{div}_f h \]
\[ = \Delta_{f,L} \text{div}_f \text{div}_f h \]

### 4. Kernel of Stability Operator

We know that \( \nu \)-entropy is invariant with respect to action of diffeomorphism group \( \text{Diff}(M) \), now because tangent vector to action of diffeomorphism group on metrics
is $L_{\#} g = -2 \text{div}^i_j \omega, \omega \in \omega(M)$, therefore the second variation of the $\nu$-entropy on $\text{Im}(\text{div}^i_j)$ is zero

$$< N(\text{div}^i_j \omega), \text{div}^i_j \omega > dm = \int_M < N(\mathcal{L}_X g), \mathcal{L}_X g > dm = 0$$

But is this true that for $h \in \text{Im}(\text{div}^i_j)$, $N(h) = 0$? Cao and He in [5] proved that for trivial $\text{GRS}^+$ (Einstein metric with positive scalar curvature) $\text{Im}(\text{div}^i_j) \subset \text{Ker} N$. In this section we extend this result to nontrivial $\text{GRS}^+$.

For computation of the stability operator on $\text{Im}(\text{div}^i_j)$ we need to compute $\nu_h, \text{div} \text{div}^i_j$ on $\text{Im}(\text{div}^i_j)$. For this purpose we need to prove some theorems.

**Theorem 4.1.** For a closed orientable $\text{GRS}^+$ $(M^n, g, f, \tau)$ we have

$$\text{div}_f(\mathcal{L}_{\#} g) = \Delta_f \omega + d(\text{div}_f \omega) + \frac{1}{2}\tau \omega $$

**Proof.** Because $\text{div}_f(\mathcal{L}_{\#} g) = \text{div}(\mathcal{L}_{\#} g) - \mathcal{L}_{\#} g(\nabla f, -)$ therefore

$$\text{div}_f(\mathcal{L}_{\#} g)_i = g^{pq} [\nabla_p (\nabla_i \omega_q + \nabla_i \omega_q) - (\nabla_p \omega_i + \nabla_i \omega_p) \nabla_q f]$$

we have

$$\nabla_p \nabla_i \omega_q = \nabla_i \nabla_p \omega_q + R_{piqs} \omega^s$$

Therefore

$$\text{div}_f(\mathcal{L}_{\#} g)_i = g^{pq} [\nabla_p \nabla_i \omega_q + \nabla_i \nabla_p \omega_q + R_{piqs} \omega^s - \nabla_p \omega_i \nabla_q f - \nabla_i \omega_p \nabla_q f]$$

Now since

$$\nabla_i \omega_p \nabla_q f = \nabla_i (\nabla_q f \omega_p) - \nabla_i \nabla_q f \omega_p$$

therefore

$$\text{div}_f(\mathcal{L}_{\#} g)_i = g^{pq} [\nabla_p \nabla_i \omega_q + \nabla_i \nabla_p \omega_q + R_{piqs} \omega^s - \nabla_p \omega_i \nabla_q f - \nabla_i (\omega_p \nabla_q f)$$

$$+ \nabla_i \nabla_q f \omega_p]$$

And finally we conclude that

$$\text{div}_f(\mathcal{L}_{\#} g) = \Delta_f \omega + d(\text{div}_f \omega) + \frac{1}{2}\tau \omega $$

**Corollary 4.1.** For a closed orientable $\text{GRS}^+$ and $\omega \in \Omega^1(M)$ we have

$$\text{div}_f \text{div}^i_j \omega = -\frac{1}{2} (\Delta_f \omega + d(\text{div}_f \omega) + \frac{1}{2}\tau \omega)$$

**Lemma 4.1.** For a closed orientable $\text{GRS}^+$ $(M^n, g, f, \tau)$ and $\omega \in \Omega^1(M)$ we have

$$\text{div}^i_j \text{div}_f(\text{div}^i_j \omega) = -\frac{1}{2} \Delta_f L \text{div}^i_j \omega - \frac{1}{2} \text{div}^i_j \omega + \frac{1}{2} \nabla^2 \text{div}_f \omega$$
Proof. From Corollary (4.1) we have
\[
div_f \div f (\div f \omega) = -\frac{1}{2} \div f (\Delta_f \omega + d(div_f \omega) + \frac{1}{2}\tau \omega)
\]
\[
= -\frac{1}{2} \div f \Delta_f \omega + \frac{1}{2} \nabla^2 \div f \omega - \frac{1}{4\tau} \div f \omega
\]
From Corollary (3.1) we have
\[
div_f \Delta_f \omega = \Delta_{f,L} \div f \omega + \frac{1}{2} \div f \omega
\]
Therefore
\[
div_f \div f (\div f \omega) = -\frac{1}{2} \Delta_{f,L} \div f \omega + \frac{1}{2}\tau \omega
\]

Lemma 4.2. For a closed orientable GRS \((M^n, g, f, \tau)\) and \(\omega \in \Omega^1(M)\) we have
\[
(4.4) \quad div_f \div f (\div f \omega) = -(\Delta_f \div f \omega + \frac{1}{2}\tau \div f \omega)
\]
Proof. From Corollary (4.1) we have
\[
div_f \div f (\div f \omega) = -\frac{1}{2} \div f (\Delta_f \omega + d(div_f \omega) + \frac{1}{2}\tau \omega)
\]
Now according to Theorem (3.2) we have
\[
div_f \div f (\div f \omega) = -\frac{1}{2} (\Delta_f \div f \omega + \frac{1}{2}\tau \div f \omega + \Delta_f \div f \omega + \frac{1}{2}\tau \div f \omega)
\]
\[
= -\frac{1}{2} \Delta_f \div f \omega + \frac{1}{2}\tau \div f \omega
\]

Theorem 4.2. For a closed orientable GRS \((M^n, g, f, \tau)\) and \(\omega \in \Omega^1(M)\) we have
\[
(4.5) \quad N(\div f \omega) = 0
\]
Proof. Stability operator of the \(\nu\)-entropy has four non trivial terms \(\Delta_{f,L} \div f \omega, \nabla^2 v_h\) and finally coefficient of Ricci tensor, at first we calculate all of these terms on \(I_m(\div f \omega)\) separately, for this we need to calculate
\[
\Delta_{f,L} \div f \omega, \div f \div f (\div f \omega), \div f \div f (\div f \omega)
\]
For computation of \(v_h\), from Lemma (4.2) it follows that
\[
div f \div f (\div f \omega) = -(\Delta_f \div f \omega + \frac{1}{2}\tau \div f \omega)
\]
therefore according to uniqueness of \(v_h\), we conclude that for \(h = \div f \omega, v_h = -\div f \omega\), therefore we have
\[
\nabla^2 v_h = -\nabla^2 \div f \omega
\]
For \(\div f \div f h\) from Lemma (4.1) it follows that
\[
\div f \div f (\div f \omega) = -\frac{1}{2} \Delta_{f,L} \div f \omega - \frac{1}{2}\tau \div f \omega + \frac{1}{2}\nabla^2 \div f \omega
\]
Furthermore we have
\[
\int_M < \text{Ric}, \text{div} f^j \omega > dm = \int_M < \text{div}_f (\text{Ric}), \omega > dm = 0
\]

Now if we substitute these terms in the expression of the stability operator, then we have
\[
N(h) = \frac{1}{2} \Delta_{f,L} \text{div} f^j \omega + \frac{1}{2} \text{div} f^j \omega + \text{div}_f \text{div} f^j \text{div}_f \text{div} f^j \omega - \frac{1}{2} \nabla^2 v_h - \text{Ric} \int_M < \text{Ric}, \text{div} f^j \omega > dm \int_M R dm
\]
\[
= \frac{1}{2} \Delta_{f,L} \text{div} f^j \omega + \frac{1}{2} \text{div} f^j \omega + \text{div}_f \text{div} f^j \text{div}_f \text{div} f^j \omega - \frac{1}{2} \nabla^2 v_h
\]
\[
= \frac{1}{2} \Delta_{f,L} \text{div} f^j \omega + \frac{1}{2} \text{div} f^j \omega + \frac{1}{2} \Delta_{f,L} \text{div} f^j \omega - \frac{1}{2} \frac{\tau}{\tau} \text{div} f^j \omega + \frac{1}{2} \nabla^2 \text{div}_f \omega - \frac{1}{2} \nabla^2 \text{div}_f \omega
\]
\[
= 0
\]

□

5. Main Results

Cao and Zhu in [3] using similarity between shrinking soliton of mean curvature flow and GRS$^+$ of Ricci flow and work of Colding and Minicozzi in linear stability of shrinking soliton of mean curvature flow, have shown that for linear stability of GRS$^+$ metrics it is necessary that the first eigenvalue of the weighted Lichnerowicz Laplace operator (restricted to transversal tensor i.e $\text{div} f^j h = 0$) is zero with multiplicity one and with Ric being an eigentensor $\text{Ric}$. In this section we will extend their result. First we eliminate condition $(h \in \text{Ker}(\text{div} f))$ and we replace zero bound with a weaker bound. Secondly we find the sufficient condition for linear stability of a GRS$^+$. Finally we find some relations between eigentensors of the stability operator $N$ and the weighted Lichnerowics Laplacian $\Delta_{f,L}$.

**Theorem 5.1.** A necessary condition for linear stability of closed orientable GRS$^+$ $(M^n, g, f, \tau)$ is that the first eigenvalue of the weighted Lichnerowicz Laplacian $\Delta_{f,L}$ (except zero with one multiplicity and Ricci tensor $\text{Ric}$ as eigentensor) is not greater than $-\frac{1}{2\tau}$.

**Proof.** Suppose that $\Delta_{f,L} h = \lambda h$ for $h \in C^\infty(S^2(T^* M))$, $\lambda \in \mathbb{R}$ and $\lambda > -\frac{1}{2\tau}$. First suppose that $\lambda \neq 0$, we compute the second variation of the $\nu$-entropy in the direction $h$, for this purpose we should compute $N(h)$. Similar to previous section we have to compute four non trivial terms $\Delta_{f,L} \text{div} f^j \text{div}_f \text{div} f^j \omega$ and finally coefficient of Ricci tensor, first we compute $v_h$. Because $\Delta_{f,L} h = \lambda h$, therefore by Theorem [5.5] we have
\[
\Delta_f \text{div}_f \text{div} f^j h = \text{div}_f \text{div}_f \Delta_{f,L} h = \lambda \text{div}_f \text{div} f^j h
\]
And therefore
\[
\Delta_f \text{div}_f \text{div} f^j h + \frac{1}{2\tau} \text{div} f^j h = (\lambda + \frac{1}{2\tau}) \text{div}_f \text{div} f^j h
\]
so $\lambda + \frac{1}{2\tau}$ is an eigenvalue of $\Delta_f$ and $\text{div}_f \text{div} f^j h$ is an eigentfunction of $\Delta_f$, but according to espectral estimate of Theorem [2.4] it follows that $\lambda < -\frac{1}{2\tau}$, therefore
div$^2$div$^2 h = 0$ and according to uniqueness of $\nu_h$, $\nu_h = 0$. On the other hand we assume that $\lambda \neq 0$ therefore

$$
\int_M <\text{Ric}, h > dm = \int_M <\text{Ric}, \frac{\lambda}{\lambda} h > dm
$$

$$
= \int_M <\text{Ric}, \frac{1}{\lambda} \Delta_{f,L} h > dm
$$

$$
= \int_M <\Delta_{f,L} \text{Ric}, \frac{1}{\lambda} h > dm
$$

$$
= 0
$$

From Theorem (2.3) we have $\Delta_{f,L} \text{Ric} = 0$, therefore the coefficient of Ricci tensor in the expression of the stability operator is zero.

$$
\int_M <\text{Ric}, h > dm = 0
$$

Now we have

$$
\nu_{g''}(h) = \int_M <N(h), h > dm
$$

Here

$$
N(h) = \frac{1}{2} \Delta_{f,L} h + \frac{1}{2\tau} h + \text{div}^2 \text{div} h + \frac{1}{2} \nabla^2 v_h - \text{Ric} \int_M <\text{Ric}, h > dm \frac{1}{\int M R dm}
$$

In this expression the coefficient of Ricci is zero thus

$$
N(h) = \frac{1}{2} \left( \frac{\lambda}{2} + \frac{1}{2\tau} \right) h + \text{div}^2 \text{div} h + \frac{1}{2} \nabla^2 v_h
$$

Therefore

$$
\nu_{g''}(h) = \int_M <N(h), h > dm
$$

$$
= \int_M \left( \frac{\lambda}{2} + \frac{1}{2\tau} \right) h + \text{div}^2 \text{div} h, h > dm
$$

$$
= \int_M \left( \frac{\lambda}{2} + \frac{1}{2\tau} \right) |h|^2 dm + \int_M <\text{div}^2 \text{div} h, h > dm
$$

$$
= \int_M \left( \frac{\lambda}{2} + \frac{1}{2\tau} \right) |h|^2 dm + \int_M |\text{div} h|^2 dm
$$

$$
\geq \frac{1}{4\tau} \int_M |h|^2 dm > 0
$$

Therefore $h$ is an unstability direction and $GRS^+$ is linear unstable.

Now suppose that $\Delta_{f,L} h = 0, \ h \neq \mu \text{Ric}, \mu \in \mathbb{R}, \mu \neq 0$, by Theorem (3.5) it follows that

$$
\Delta_f \text{div}_f \text{div}_f h = \text{div}_f \text{div}_f \Delta_{f,L} h = 0
$$
from the maximum principle it follows that \(\text{div}_f \text{div}_f h = 0\) and according to uniqueness of \(v_h\), we have \(v_h = 0\). Therefore we conclude that

\[
\nu''_g(h) = \int_M <N(h), h > \, \text{dm} = \int_M \frac{1}{2\tau} |h|^2 \, \text{dm} - \frac{\left(\int_M <\text{Ric}, h > \, \text{dm}\right)^2}{\int_M |\text{Ric}|^2 \, \text{dm}}.
\]

From Theorem 2.3 we know that \(\int_M \text{Rdm} = \int_M 2\tau |\text{Ric}|^2 \, \text{dm}\), therefore

\[
\nu''_g(h) \geq \int_M \frac{1}{2\tau} |h|^2 \, \text{dm} - \frac{\left(\int_M <\text{Ric}, h > \, \text{dm}\right)^2}{\int_M |\text{Ric}|^2 \, \text{dm}} \geq \left(\int_M \frac{1}{2\tau} |h|^2 \, \text{dm} \int_M 2\tau |\text{Ric}|^2 \, \text{dm} - \left(\int_M <\text{Ric}, h > \, \text{dm}\right)^2\right) \frac{1}{2\tau \int_M |\text{Ric}|^2 \, \text{dm}}.
\]

But according to Cauchy–Schwarz inequality we know that last term is positive and is zero if and only if \(h = \mu \text{Ric}, \mu \in \mathbb{R}, \mu \neq 0\) but we assume that \(h \neq \mu \text{Ric}, \mu \in \mathbb{R}, \mu \neq 0\) therefore \(\nu''_g(h) > 0\) and soliton is unstable.

\[\square\]

**Theorem 5.1.** Suppose that for a closed oriantable \(\text{GRS}^+ (M^n, g, f, \tau)\) the first eigenvalue of the weighted Lichnerowicz Laplacian \(\Delta_{f,L}\) is not greater than \(-\frac{1}{2}\) (except zero with one multiplicity and Ricci tensor Ric as eigentensor), then soliton is linear stable.

**Proof.** The main idea of the proof is that for arbitrary \(h \in C^\infty(S^2(T^*M))\) for simplification of computation, we write \(h\) in a common orthonormal eigenbasis of the two operators \(\Delta_{f,L}\) and \(\Delta_{f,L} + \text{div}_f \text{div}_f\) (we prove that this eigenbasis exists), then we prove that the second variation of the \(\nu\)-entropy in the direction \(h\) is nonpositive.

First we prove that two operators \(\Delta_{f,L}\) and \(\Delta_{f,L} + \text{div}_f \text{div}_f\) have common orthogonal eigenbasis, for this purpose we prove that these two operators are diagonalizable.

These two operators are strongly elliptic. Indeed suppose that we have a differential operator \(L : C^\infty(E) \rightarrow C^\infty(F)\) between two vector bundles \((E, F)\), and suppose that the principal symbol of \(L\) in the direction \(\omega \in \Omega^1(M)\) is denoted by \(\sigma_\omega(D)\) (i.e. we have \(\sigma_\omega : E_\rho \rightarrow F_\rho\)). Now trivially the operator \(\Delta_{f,L}\) which is a twisted Lichnerowicz Laplacian, is strongly elliptic with principal symbol

\[
\sigma_\omega(\Delta_{f,L})h = |\omega|^2 h,
\]

for principal symbol of \(\Delta_{f,L} + \text{div}_f \text{div}_f\), a simple computation shows that

\[
\sigma_\omega(\Delta_{f,L} + \text{div}_f \text{div}_f)h = |\omega|^2 h + h(\omega, -) \otimes \omega + \omega \otimes h(\omega, -)
\]
such that $h \in C^{\infty}(S^2(T^*(M)))$ and $\omega \in \Omega^1(M)$. Now we have

$$< \sigma(\Delta_{f,L} + \text{div}^f_\dagger \text{div} f)h, h >$$

$$= < |\omega|^2 h + h(\#\omega, -) \otimes \omega + \omega \otimes h(\#\omega, -), h >$$

$$= |\omega|^2 |h|^2 + 2|\omega(\#\omega, -)|^2$$

$$\geq |\omega|^2 |h|^2$$

Therefore $(\Delta_{f,L} + \text{div}^f_\dagger \text{div} f)$ is strongly elliptic.

Now $\Delta_{f,L}$ and $\Delta_{f,L} + \text{div}^f_\dagger \text{div} f$ are self-adjoint strongly elliptic operators and by compactness of $M$ (see [12] page 15) and according to spectral theory have a discrete set of eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$ ... and $\lambda_n \to -\infty$ and any eigenvalue has finite multiplicity and eigentensors of different eigenvalues are orthogonal. These two operators extend to two continuous linear maps on completion of $C^{\infty}(S^2(T^*(M)))$ and from elliptic regularity all eigentensors of these operators are smooth, but by Theorem (3.6) these two operators commute, therefore should have a common basis of eigentensors (from functional analysis we know that any two diagonalizable commutative continuous maps diagonalizable simultaneously), with Gram–Schmidt process we can construct an orthonormal common basis of eigentensors of these operators.

Suppose that an arbitrary tensor $h \in C^{\infty}(S^2(T^*(M)))$ may be written, in an unique way in this basis, as

$$h = \Sigma_{i=1}^{\infty} \lambda_i h_i$$

Now we compute the second variation of the $\nu$-entropy in the direction $h$. First suppose that for an eigentensor $h_i \in C^{\infty}(S^2(T^*(M)))$ of $\Delta_{f,L}$ and $\Delta_{f,L} + \text{div}^f_\dagger \text{div} f$ we have

$$\Delta_{f,L} h_i = \lambda_i h_i$$

$$\Delta_{f,L} + \text{div}^f_\dagger \text{div} f h_i = \mu_i h_i$$

Therefore $\text{div}^f_\dagger \text{div} f = (\lambda_i - \mu_i) h_i$. Now if $\lambda_i \neq \mu_i$, then

$$h_i = (\lambda_i - \mu_i)^{-1} \text{div}^f_\dagger \text{div} f h_i$$

Therefore $h_i \in \text{Im} (\text{div}^f_\dagger)$, and by Theorem (4.2),

$$N(h_i) = 0$$

Now suppose that $\lambda_i = \mu_i$, hence $\text{div}^f_\dagger \text{div} f h_i = (\lambda_i - \mu_i) h_i = 0$, therefore we conclude that

$$\int_M |\text{div} f h_i|^2 dm = \int_M < \text{div}^f_\dagger \text{div} f h_i, h_i > dm = 0$$

Therefore it follows that $\text{div} f h_i = 0$ and hence $\nu h_i = 0$ i.e.

$$\text{div}^f_\dagger \text{div} f h_i + \frac{1}{2} \nabla^2 v h_i = 0$$
For this reason that we assume that \( \lambda \leq -\frac{1}{\tau} \) and \( \text{Ric} \) is only a generator of eigenspace of the zero eigenvalue of \( \Delta_{f,L} \), therefore if \( (h_i \neq \lambda \text{Ric}, \lambda \in \mathbb{R}, \lambda \neq 0) \) we conclude that

\[
\int_M < \text{Ric}, h_i > \, dm = \int_M < \text{Ric}, \frac{\lambda_i}{\lambda} h_i > \, dm
\]

\[
= \int_M < \text{Ric}, \frac{1}{\lambda_i} \Delta_{f,L} h_i > \, dm
\]

\[
= \int_M < \Delta_{f,L} \text{Ric}, \frac{1}{\lambda_i} h_i > \, dm
\]

\[
= 0
\]

Now for the stability operator of \( h_i \) we have

\[
N(h_i) = \left( \frac{\lambda_i}{2} + \frac{1}{2\tau} \right) h_i
\]

then we can write \( h \in C^\infty(S^2(T^*(M))) \) in a new way in this basis, as

\[
h = c_0 \text{Ric} + \sum_{\lambda_i = \mu} c_i h_i + \sum_{\lambda_i \neq \mu} c_j h_j
\]

Therefore it follows that

\[
N(h) = N(c_0 \text{Ric} + \sum_{\lambda_i = \mu} c_i h_i + \sum_{\lambda_i \neq \mu} c_j h_j)
\]

\[
= \sum_{\lambda_i = \mu} c_i N(h_i)
\]

And for the second variation of the \( \nu \)-entropy in the direction \( h \) we have

\[
\nu_\nu^\prime\prime(h) = \int_M < N(h), h > \, dm
\]

\[
= \int_M < \sum_{\lambda_i = \mu} c_i(\frac{\lambda_i}{2} + \frac{1}{2\tau}) h_i, c_0 \text{Ric} + \sum_{\lambda_i = \mu} c_i h_i + \sum_{\lambda_i \neq \mu} c_j h_j > \, dm
\]

\[
= \int_M < \sum_{\lambda_i = \mu} c_i(\frac{\lambda_i}{2} + \frac{1}{2\tau}) h_i, \sum_{\lambda_i = \mu} c_i h_i > \, dm
\]

In the last equation we have used the fact that \( N \) is a self-adjoint operator. Now because our basis is orthonormal therefore

\[
\nu_\nu^\prime\prime(h) = \int_M \sum_{\lambda_i = \mu} c_i^2(\frac{\lambda_i}{2} + \frac{1}{2\tau}) |h_i|^2 \, dm
\]

But we assume that except the zero eigenvalue with the \( \text{Ric} \) as only generator of eigenspace, \( \lambda_i \leq -\frac{1}{\tau} \) therefore \( \lambda_i + \frac{1}{2\tau} \leq 0 \) so \( \nu_\nu^\prime\prime(h) \leq 0 \) i.e soliton is stable. \( \square \)

Unfortunately for case \( \lambda \in (-\frac{1}{\tau}, -\frac{1}{2\tau}) \) we have not any information about stability of soliton. Here we give some information about the eigenvalues of the stability operator \( N \).

**Theorem 5.2.** For a closed orientable \( \text{GRS}^+ (M^n, g, f, \tau) \) all eigentensors of the stability operator except eigentensors of the zero eigenvalue are eigentensors of the weighted Lichnerowicz Laplacian \( \Delta_{f,L} \).
Proof. Suppose that \( h_i \in C^\infty(S^2(T^*(M))) \) is an eigentensor of \( \Delta_{f,L} \) with eigenvalue \( \lambda_i \neq 0 \). First we have \( \text{div}_f h_i = 0 \)

We have

\[
\int_M |\text{div}_f N(h_i)|^2 dm = \int_M \langle N(h_i), \text{div}^\dagger_f \text{div}_f N(h_i) \rangle dm = 0
\]

therefore \( \text{div}_f N(h_i) = 0 \). Indeed for every \( h \in C^\infty(S^2(T^*(M))) \) such that \( \Delta_{f,L} h \neq 0 \) we have \( \text{div}_f N(h) = 0 \). Secondly since \( \lambda_i \neq 0 \) so

\[
\int_M \langle \text{Ric}, h_i \rangle dm = \int_M \langle \text{Ric}, \frac{\lambda_i}{\lambda_i} h_i \rangle dm = \int_M \langle N(\text{Ric}), \frac{1}{\lambda_i} h_i \rangle dm = 0
\]

Therefore for the stability operator of \( h_i \) we have

\[
N(h_i) = \frac{1}{2} \Delta_{f,L} h_i + \frac{1}{2\tau} h_i = \lambda_i h_i
\]

And finally

\[
\Delta_{f,L} h_i = 2(\lambda_i - \frac{1}{2\tau}) h_i
\]

\( \square \)

Hall and Morphy in [9] proved that any compact Kähler \( GRS^+ \) with \( \text{dim}H^{1,1}(M) \geq 2 \) is linear unstable. For this purpose they define the map

\[
S : \Omega^{(1,1)}(M) \to C^\infty(S^2(T^*M))
\]

such that for twisted harmonic 1-1 form \((\omega, \Delta_{f,H} \omega = 0)\), we have \( \text{div}_f S(\omega) = 0 \) and \( \Delta_{f,L} S(\omega) = S(\Delta_{f,H} \omega) = 0 \). Then they consider a linear combination of two twisted harmonic forms \( \omega = a.\omega_1 + b.\omega_2 \) such that image of this linear combination is perpendicular to Ricci tensor \( (\int_M < S(\omega), \text{Ric} > dm = 0) \). Now the image of this linear combination under map \( S \) is an unstability direction. Here we extend their result.

**Theorem 5.3.** Any compact orientable Kähler \( GRS^+ (M^n, g, f, \tau) \) with \( \text{dim}H^{1,1}(M) = 1 \) is linear unstable unless for twisted harmonic form \((\omega, \Delta_{f,H} \omega = 0)\) and complex structure \( J \), we have \( S(\omega) = \omega(J, -) = \lambda \text{Ric} \) i.e \( S(\omega)_{ij} = g^{rs} \omega_{ir} J_{sj} = \lambda R_{ij}, \lambda \in \mathbb{R}, \lambda \neq 0 \)

**Proof.** Similar to case \( \Delta_{f,L} h = 0 \) in Theorem 5.1 and because \( \Delta_{f,L} S(\omega) = 0 \) and \( \text{div}_f S(\omega) = 0 \) (see [9]Lemma(4.1),Proposition(4.2),Lemma(4.3)) for \( h = S(\omega) \) we conclude that
\[ \nu_g''(h) \geq \int_M \frac{1}{|h|^2} dm - \frac{(\int_M \langle Ric, h \rangle dm)^2}{\int_M |Ric|^2 dm} \]

\[ \geq \left( \int_M \frac{1}{|h|^2} dm \int_M |Ric|^2 dm - (\int_M \langle Ric, h \rangle dm)^2 \right) \frac{1}{2\tau \int_M |Ric|^2 dm} \]

\[ \geq \left( \int_M |h|^2 dm \int_M |Ric|^2 dm - (\int_M \langle Ric, h \rangle dm)^2 \right) \frac{1}{2\tau \int_M |Ric|^2 dm} \]

But according to Cauchy–Schwarz inequality we know that the last term is positive and is zero if and only if \( h = \mu Ric, \mu \in \mathbb{R}, \mu \neq 0 \) but we assume that \( h \neq \mu Ric, \mu \in \mathbb{R}, \mu \neq 0 \), therefore \( \nu_g''(h) > 0 \) and soliton is unstable. \( \square \)

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