Asymptotic Nash Equilibrium for the $M$-ary Sequential Adversarial Hypothesis Testing Game

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Abstract—In this paper, we consider a novel $M$-ary sequential hypothesis testing problem in which an adversary is present and perturbs the distributions of the samples before the decision maker observes them. This problem is formulated as a sequential adversarial hypothesis testing game played between the decision maker and the adversary. This game is a zero-sum and strategic one. We assume the adversary is active under all hypotheses and knows the underlying distribution of observed samples. We adopt this framework as it is the worst-case scenario from the perspective of the decision maker. The goal of the decision maker is to minimize the expectation of the stopping time to ensure that the test is as efficient as possible; the adversary’s goal is, instead, to maximize the stopping time. We derive a pair of strategies under which the asymptotic Nash equilibrium of the game is attained. We also consider the case in which the adversary is not aware of the underlying hypothesis and hence is constrained to apply the same strategy regardless of which hypothesis is in effect. Numerical results corroborate our theoretical findings.

Index Terms—Game theory, Nash Equilibrium, $M$-ary Sequential Hypothesis Testing, Adversary

I. INTRODUCTION

Hypothesis testing is a fundamental problem in statistics and information theory. There are many works that have laid firm theoretical foundations for the fundamental limits of hypothesis testing. In this paper, we consider a new setting in which there exists an adversary that deliberately acts in a malicious way to cause a sequential test implemented by a decision maker to fail [1]. We term this new setting as the adversarial sequential hypothesis testing game. We are motivated by security and trustworthy issues of modern machine learning algorithms. Such issues have been studied extensively in the past decade. Machine learning algorithms can be shown to be highly vulnerable to adversarial perturbations [2]. For example, in image classification problem, there may be adversarial samples that adversely affect the performance of classification tasks. The adversary may adopt different attack strategies for images in different classes. In this case, it is important to identify the true class of images even under the perturbation of the adversary. In all these examples, when the distributions of observed samples are known, we can consider this problem under a game-theoretic framework and formulate the problem as a hypothesis testing game played by the decision maker and the adversary.

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A. Related Works

The works that are closely related to the present paper are those by Barni and Tondi [1], [3]–[5]. In these works, the authors considered a general framework to analyze binary hypothesis testing by taking into account the presence of an adversary who aims to impede the making of a correct decision. They introduced and analyzed an adversarial version of the Neyman–Pearson setup in which a defender and an adversary face off against each other. Given a null hypothesis $H_0$ and a test sequence $Z^n$, the defender must decide whether to accept hypothesis $H_0$ characterized by a distribution $P_0$. As in the classical Neyman–Pearson scenario, the defender must ensure that the type-I error probability (i.e., the probability of rejecting $H_0$ when $H_0$ is true) is no larger than a prescribed constant $\alpha \in (0, 1)$. In turn, the adversary observes a sequence $Y^n$ generated under an alternative hypothesis $H_1$, characterized by a different distribution $P_1$, and transforms it into a modified sequence so that when presented with the modified sequence, the defender still accepts $H_0$. In other words, the adversary aims at maximizing the type-II error probability (i.e., the probability that the defender accepts $H_0$ when $H_1$ holds), while the defender’s goal is to minimize it by taking into account the presence of the adversary. In the setting of [1], [3]–[5], adversarial hypothesis testing is modeled as a zero-sum game. In [1], the authors consider the case in which $P_0$ and $P_1$ are both known to the defender and the adversary. They showed that under certain assumptions, the game admits asymptotic Nash equilibrium and obtained the optimum strategies for the decision maker and the adversary at the equilibrium. In [3], the authors extended their previous works by considering a scenario in which $P_0$ is known only through one or more training sequences. They also derive the asymptotic Nash equilibrium of this setting. In [4], the authors also assume $P_0$ is known through training sequences but in our paper, the training data is corrupted by an adversary.

While [1], [3], [4] characterize the adversarial hypothesis testing problem when the adversary is only active in one of the two hypotheses, it is also reasonable to consider the case when the adversary is active under all hypotheses. Tondi, Barni, and Merhav [6] extended the game-theoretic formulation of the defender-adversary interaction to the case where the attacker acts under both hypotheses. Under this setting, a dominant (i.e., optimal regardless of what the defence strategy is) and
universal (i.e., not dependent on the underlying sources) adversary strategy can be obtained. Furthermore, Jin and Lai [7] also focus on this setting but they formulated it as a minimax problem. They obtain a nonasymptotic saddlepoint solution which reveals the optimal attack and defense strategies.

Instead of directly perturbing the observed sequence of samples, there are also works that permit the adversary to perturb the underlying distributions. In Yasodharan and Loiseau [8], the adversary chooses any distribution from a set of distributions and assigns each choice of distribution a cost function. They then considered non-zero-sum hypothesis testing games in both the Bayesian and the Neyman–Pearson frameworks. The authors showed that these games admit mixed strategy Nash equilibria. Zhang and Zou [9] extended the non-zero-sum hypothesis testing games in [8] to the sequential case and obtain the asymptotic Nash equilibrium. They first guessed the strategy $s_a$ that the adversary adopts and then designed a strategy $s_d(s_a)$ of the decision maker based on adversary’s strategy. However, their methods cannot be extended to the case when the adversary is active in all hypotheses.

Another line of work that is similar to our setting is the robust hypothesis testing problem. A robust binary hypothesis test is a minimax test for two hypotheses where the actual probability distributions of the observations are located in neighborhoods of a nominal density. The actual and nominal distributions are constrained in terms of a certain distance measure such as the relative entropy [10], the $\alpha$-divergence [11], and the Wasserstein distance [12]. The above-mentioned works show that the minimax solution is an optimal test based on the least favorable distributions (LFDs), i.e., a test that optimally separates the closest feasible distributions. For the $M$-ary case, Faúß, Zoubir, and Poor [13] considered a sequential $M$-ary robust hypothesis testing problem. They showed that the minimax solution is also an optimal test for the LFDs, but now the LFDs depend on the previous observations. This results in the sequence of samples being no longer i.i.d., but rather being a Markov process. In a follow-up work [14], the same authors obtain sufficient conditions for strict minimax optimality of sequential tests for multiple hypotheses under mild Markov assumptions. The differences between robust hypothesis testing and our problem are discussed in more detail in Remark 2 in Section III.

B. Main Contributions

In this paper, we focus on the $M$-ary sequential adversarial hypothesis testing game. There are $M$ hypotheses $H_i$, $i \in [M]$ and they are characterized by $M$ different distributions $P_i$, $i \in [M]$ respectively. Samples are collected sequentially and they are perturbed by the adversary. For the most part of the paper, we assume the adversary knows the underlying distribution of the observed samples, and has the ability to perturb the samples based on which hypothesis is in effect. There are four distinct contributions in our paper.

- We formulate a sequential adversarial $M$-ary hypothesis testing game and state our objective in terms of finding an asymptotic Nash equilibrium between the player and the adversary. The player’s objective is a linear combination of error exponents; this is in contrast to other works in robust hypothesis testing (see Remark 2 for details). The adversary is assumed to be powerful; it knows the true distributions, which hypothesis is in effect, and can perturb the player’s observations under both hypotheses.
- We derive optimal strategies for the player and the adversary that yield an asymptotic Nash equilibrium for this two-player sequential game using information-theoretic tools. Different from [9] in which the decision maker first estimates the strategy $s_a^*$ that the adversary will adopt, and then designs its strategy $s_d(s_a^*)$ based on the estimates, in our work, both strategies are executed simultaneously.
- We discuss the case when the adversary is incognizant of the underlying distributions of the observations. This is a weaker form of the adversary. Even though we are unable to obtain the pair of strategies that achieves the asymptotic Nash equilibrium, we show that the decision maker can achieve larger error exponents compared to the adversary-aware setting.
- Numerical results corroborate our theoretical findings. Specifically, we show on synthetic and real datasets that the empirical performance of the proposed strategies converge to their promised fundamental limits.

C. Paper Outline

The rest of the paper is structured as follows. In Section II, we introduce some preliminary knowledge on the $M$-ary sequential hypothesis testing and two player games. In Section III, we formulate the $M$-ary sequential adversarial hypothesis testing problem formally and introduce the definition of asymptotic Nash equilibrium. In Section IV, we present our main theorem (Theorem 1) about the set of strategies at which the asymptotic Nash Equilibrium can be obtained and the proof of our main theorem. In Section V, we consider a weaker form of the adversary who does not know the underlying distributions and derive bounds on the performance of the decision maker. In Section VI, we provide some numerical simulations. We conclude the paper in Section VII and propose some directions for future researches.

II. Preliminaries

A. $M$-ary Sequential Hypothesis Testing

In this section, we discuss the $M$-ary sequential hypothesis testing setup [15]. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed (i.i.d.) random variables with distribution $P$, and let $H_i$ be the hypothesis that $P = P_i$ for $i = 1, 2, \ldots, M$. We assume that $P_i \neq P_j$ for all $i \neq j$. The objective of this problem is to uncover the true hypothesis with a desired accuracy as quickly as possible (i.e., using the fewest number of samples). In this problem, there is a fundamental tradeoff between the number of samples and the error probabilities.

We will use sequential tests to learn the underlying hypothesis. Such tests consist of stopping rules and final decision rules. The stopping rule determines the number of samples that are collected until a decision is made and the final decision rule decides which of the $M$ hypotheses is the true one.
For $n \geq 1$, we define the log-likelihood ratio between distributions $P_i$ and $P_j$ as
\[
S_{ij}(n) = \sum_{k=1}^{n} \log \frac{P_i(X_k)}{P_j(X_k)}.
\]
For a threshold or boundary matrix $B = [B_{ij}]$, with $B_{ij} > 0$ and the $B_{ii} = 0$, define the matrix sequential probability ratio test (MSPRT) [16] $\delta_M^* = (T_M^*, d_M^*)$ that is constructed based on $(M + 1)/2$ one-sided SPRTs between hypotheses $H_i$ and $H_j$ as follows: The stopping rule is
\[
\text{Stop at the first } n \geq 1 \text{ s.t. } \exists i \in [M] \text{ s.t. } S_{ij}(n) \geq B_{ij} \forall j \neq i.
\]
Accept the unique $i \in [M]$ that satisfies these inequalities. Note that for $M = 2$ this test coincides with Wald’s sequential probability ratio test (SPRT) [17]. It can be shown that the MSPRT with proper thresholds is first-order asymptotically optimal in the sense of minimizing the expected sample sizes for all hypotheses [16, Chapter 4.3], i.e. for all tests $\delta_M = (T_M, d_M)$ with all error probabilities upper bounded by $\alpha_{\max} \in (0, 1)$, we have
\[
\lim_{\alpha_{\max} \to 0} \inf_{\delta_M^*} E_i[T_M] = E_i[T_M^*], \text{ for all } i = 1, 2, \ldots, M,
\]
where $T_M^*$ is the optimal stopping time.

B. Two Player Games

We now provide a brief introduction to two-player games. For a more detailed exposition, the reader is referred to [18]. A two-player game is defined as a quadruple $(S_1, S_2, u_1, u_2)$, where $S_1, S_2$ are the sets of strategies (actions) the first and the second player can choose from, and $u_1(s_1, s_2)$ (where $s_1 \in S_1$ and $s_2 \in S_2$) is the payoff (i.e., the gain) for player $i \in \{1, 2\}$, when the first player chooses the strategy $s_1 \in S_1$ and the second choose $s_2 \in S_2$. A pair of strategies $(s_1, s_2)$ is called a profile. In a zero-sum competitive game, the sum of the two payoffs is equal to 0, i.e., $u_1(s_1, s_2) + u_2(s_1, s_2) = 0$ for all $(s_1, s_2) \in S_1 \times S_2$. In other words, the gain of a player is equal to the loss of the other. We define the payoff function for a zero-sum game as $u = u_1 = -u_2$. A strategic game is a model of interaction in which each player chooses an action not having been informed of the other player’s action. We can think of the players’ action as being taken “simultaneously”. One common goal is to obtain a Nash equilibrium [19] of a zero-sum, strategic game, which is defined as follows. A profile $(s_1^*, s_2^*)$ is a Nash equilibrium if:
\[
\begin{align*}
    u(s_1^*, s_2^*) \geq u(s_1^*, s_2) & \quad \forall s_1 \in S_1, \text{ and} \\
    u(s_1^*, s_2^*) \leq u(s_1, s_2^*) & \quad \forall s_2 \in S_2.
\end{align*}
\]
In other words, a profile is a Nash equilibrium if no player can increase his/her payoff by changing his/her strategy unilaterally.

III. Problem Formulation

In this section, we first formulate the sequential $M$-ary adversarial hypothesis testing game. Let $\mathcal{X} = \{a_1, a_2, \ldots, a_K\}$ be the finite alphabet of the source and $\mathcal{P}(\mathcal{X})$ be the set of probability mass functions (also called distributions) supported on $\mathcal{X}$. There are $M$ hypotheses. We use $[M]$ to denote the finite set $\{1, \ldots, M\}$. Under hypothesis $H_i$, the underlying distribution is $P_i$ for $i \in [M]$. We also assume that the distributions $P_i, i \in [M]$ are known to both the decision maker and the adversary. Here the adversary perturbs the distribution; this has the effect of passing the original samples $\{X_t\}_{t=1}^{\infty}$ through a discrete memoryless channel, which we denote by a channel $A$ where the entries $[A]_{ij} = \Pr(Y = a_j|X = a_i)$ for $i, j \in [K]$. We assume that the channels $[A]$ are chosen such that $\sum_{i=1}^{K} P_i(X = a_i)[A]_{ij} > 0$ for $j \in [K]$ and $i \in [M]$, which means that the distribution of $Y$ (i.e., $Y \sim P_iA, i \in [M]$) has full support. Besides, motivated by the fact that the adversary’s power is bounded, we impose a distance constraint between the input distribution and output distribution of the adversary. This is characterized by distance measure/metric $d$. Here we do not specify the choice of measure $d$ for now and we aim to obtain results under some specific conditions on $d$. Then the adversary’s constraint is
\[
\delta(P, P_iA) \leq \Delta, \quad \forall i \in [M],
\]
where $\Delta > 0$ is a prescribed maximum distance between $P_i$ and $P_iA$ and $\Delta$ should be small to ensure that $\min_{i,j \in [M], i \neq j} \delta_i = \delta_i[P_i, P_j] \geq \epsilon > 0$. This means the KL divergences between perturbed distributions is positive. This constraint is important as it ensures that the true hypothesis can be learned uniquely. Besides, we assume $\Delta$ is known to the decision maker.

When the adversary is active under all hypotheses, there are two different scenarios we can consider. Firstly, the adversary knows underlying hypothesis $H_i, i \in [M]$ and secondly, the adversary does not. For the majority of the paper, we consider the awareness case as it is the worst-case scenario from the perspective of the decision maker. Later in Section V, we discuss the non-awareness case, i.e., the adversary is not aware of the underlying distribution.

Fig. 1 shows the $M$-ary sequential adversarial hypothesis testing game when the adversary knows the underlying distribution of $X$. At each time $n \in \mathbb{N}$, a sample $X_k$ is generated from $P_i$ and given to the adversary. The adversary modifies $X_k$ to $Y_k$ using the attack strategy. Here we note that the adversaries are different for $H_i, i \in [M]$. We denote the adversary’s strategy/channel under $H_i$ as $A_i$ for $i \in [M]$. Based on the adversary strategy, the distribution of $Y$ is $P_iA_i, i \in [M]$. Then the objective of decision maker is to decide which hypothesis is true based on the sequence up to the current time $\{Y_k\}_{k=1}^{n}$.

We define the integer-valued random variable $T \in \mathbb{N}$ as the stopping time with respect to the filtration $\{\mathcal{F}_n = \sigma(Y_1, Y_2, \ldots, Y_n)\}_{n \in \mathbb{N}}$ generated by the samples up to time $n$. To achieve the goal, the decision maker at each time $n$ can take one of two actions:
\begin{itemize}
    \item Stop drawing a new sample and declare that one of $H_i, i \in [M]$ is true.
    \item Continue to draw a new sample.
\end{itemize}

We denote the expectation of the stopping time under $H_i$ as $\mathbb{E}_i[T]$ for $i \in [M]$. The decision rule $\delta$ is a $M$-valued $\mathcal{F}_T$-measurable function. A test is a pair $\Phi = (T, \delta)$. To avoid
In the definition of MSEQ-AHT($S_D(\alpha), S_A(\Delta), u_\lambda(\alpha)$), the set of strategies for the adversary is comprised of all transition matrices that satisfy the distortion constraints. The set of strategies for the decision maker is comprised of all tests that the test error probabilities are upper bounded by a common $\alpha$. Besides, the payoff is a linear combination of the error exponents of the error probabilities $\alpha_i, i \in [M]$, and the decision maker wants to maximize it to make the detection more accurate and efficient, while the adversary wants to minimize it. For MSEQ-AHT($S_D(\alpha), S_A(\Delta), u_\lambda(\alpha)$), our goal is to obtain a profile ($\Phi^*, (A_1^*, \ldots, A_M^*)$) that achieves the asymptotic Nash equilibrium as $\alpha$ tends to zero, which is defined as follows.

**Definition 2** (Asymptotic Nash Equilibrium). We say that the (family of) profile(s) ($\Phi^*, (A_1^*, \ldots, A_M^*)$) (indexed by $\alpha > 0$) satisfies the asymptotic Nash equilibrium as $\alpha \to 0^+$ if

$$\lim_{\alpha \to 0^+} u_\lambda(\alpha)(\Phi^*, (A_1^*, \ldots, A_M^*)) \geq \lim_{\alpha \to 0^+} \sup_{\Phi \in S_D(\alpha)} u_\lambda(\alpha)(\Phi, (A_1^*, \ldots, A_M^*)) \tag{1}$$

and

$$\lim_{\alpha \to 0^+} -u_\lambda(\alpha)(\Phi^*, (A_1^*, \ldots, A_M^*)) \geq \lim_{\alpha \to 0^+} \sup_{(A_1, \ldots, A_M) \in S_A(\Delta)} -u_\lambda(\alpha)(\Phi^*, (A_1, \ldots, A_M)) \tag{2}$$

**Remark 1.** Our problem setting is similar to that of the sequential composite hypothesis testing [20] framework in which samples are generated i.i.d. by a distribution from a known set of distributions. However, in our work, the set of distributions is determined by the adversary and there is a payoff function that controls the choice of the strategies the adversary and the decision maker.

**Remark 2.** Our problem is, however, different from robust hypothesis testing [10]. In robust hypothesis testing problems, the true probability distributions are located in the neighborhoods of a nominal distribution. Instead, in our setting, we assume that the actual distribution is formed by the adversary’s perturbation by transition matrices. Besides, in minimax $M$-ary sequential hypothesis tests, e.g., in [14], the authors typically consider finding a sequential test $\Phi$ that minimizes the maximum of expectation of the stopping times over different distributions with the constraints that the error probabilities $\alpha_i, i \in [M]$ are upper bounded by fixed constants $\bar{\alpha}_i \in [0, 1]$ for all $i \in [M]$, i.e.,

$$\min_{\Phi} \max_{i \in [M]} \mathbb{E}_i[T] \quad \text{s.t.} \quad \max_{i \in [M]} \alpha_i \leq \bar{\alpha}_i.$$

In our problem setting, we consider a linear combination of the exponents $\log^{1/(1-\alpha)} \mathbb{E}_i[T], i \in [M]$ as the decision maker’s payoff function.

**IV. MAIN RESULTS**

To obtain the asymptotic Nash equilibrium of MSEQ-AHT($S_D(\alpha), S_A(\Delta), u_\lambda(\alpha)$), we first propose strategies for the decision maker and adversary. Then we prove that this pair of strategies achieves the asymptotic Nash equilibrium.
Define the type or empirical distribution of the sequence $x^m \in \mathcal{X}^m$ as

$$\hat{Q}_{x^m}(a) := \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{x_i = a\}, \quad \forall a \in \mathcal{X}.$$ 

Denote $\mathcal{A}_i(\Delta) := \{A_i : d(P_i, P_i A_i) \leq \Delta\}$ for $i \in [M]$. Then $\mathcal{S}_\Delta(\Delta) = A_1(\Delta) \times \ldots \times A_M(\Delta)$. For simplicity, we abbreviate $\mathcal{A}_i(\Delta)$ as $A_i$ for $i \in [M]$. Let $\zeta = 0.85$ from now on. Define a threshold

$$\gamma_n := \frac{\log C}{n} + \frac{1}{n^5} + \frac{|\mathcal{X}| \log(n + 1) + \log(M - 1)}{n},$$

where $C := \sum_{n=1}^{\infty} e^{-n^{1-\zeta}} < \infty$ is a finite constant. Define

$$Z_i^{(n)} := \min_{j \in [M], j \neq i} \left[ \min_{A_j \in A_j} D(\hat{Q}_{x^m} \| P_j A_j) \right].$$

Now we define a stopping time as

$$T^* = T^*_n := \inf \left\{ n \geq 1 : \exists i \in [M] \text{ s.t. } Z_i^{(n)} \geq \gamma_n \right\}, \quad (3)$$

and for $i \in [M],

$$T_i := \inf \left\{ n \geq 1 : Z_i^{(n)} \geq \gamma_n \right\}.$$ 

We also define the decision rule as for $i \in [M],

$$\delta^* := i \text{ if } T^* = T_i. \quad (4)$$

Finally, define

$$\Phi^*_i := \arg \min_{A_i \in A_i} \left\{ \min_{j \in [M] \setminus \{i\}} \left[ \min_{A_j \in A_j} D(\hat{Q}_{x^m} \| P_j A_j) \right] \right\}. \quad (5)$$

We note that $\Phi^*_i$ may not be unique.

Then the test used by the decision maker is $\Phi^* = (T^*, \delta^*)$.

Now we have the following theorem:

**Theorem 1.** If $\mathcal{S}_\Delta(\Delta)$ is a compact set, then for any $\lambda$ in which all elements are positive, $(\Phi^*, (A^*_1, \ldots, A^*_M))$ defined in (3)-(5) is the profile that attains the asymptotic Nash equilibrium as $\alpha \rightarrow 0^+$. Besides, the payoff at the asymptotic Nash equilibrium is

$$u^\alpha(\Phi^*, (A^*_1, \ldots, A^*_M)) = \sum_{i=1}^{M} \lambda_i \min_{A_j \in A_j} \left[ \min_{j \in [M] \setminus \{i\}} D(\hat{Q}_{x^m} \| P_j A_j) \right]. \quad (6)$$

Theorem 1 shows that as $\alpha \rightarrow 0^+$, the decision maker cannot increase the payoff function (i.e., the linear combination of error exponents) by changing its strategy $\Phi^*$ without the adversary changing its strategy $(A^*_1, \ldots, A^*_M)$. This is the implication of (1). Similarly, as $\alpha \rightarrow 0^+$, the payoff function cannot be increased by the adversary changing its strategy $(A^*_1, \ldots, A^*_M)$ when the strategy of decision maker is fixed to be $\Phi^*$. This is the implication of (2). We can also find that the strategies at the asymptotic Nash Equilibrium is independent of the choice of $\lambda_i$ for all $i \in [M]$.

For the optimal strategy $(A^*_1, \ldots, A^*_M)$ of the adversary, they can be obtained by solving the optimization problems

$$\max_{A_1, \ldots, A_M} \left\{ \min_{j \in [M] \setminus \{i\}} \left[ \min_{A_j \in A_j} D(\hat{Q}_{x^m} \| P_j A_j) \right] \right\}.$$ 

The constant 0.85 for $\zeta$ is somewhat arbitrary; any number in (0, 1) works for our analyses. We found that $\zeta = 0.85$ works best in our numerical experiments.
to convergence in mean. For the first step, we need to derive some properties of the stopping time $T^*_i$, $i \in [M]$ when $0 < \alpha \leq 1$ and $\alpha \to 0^+$, respectively. For the second step, we need to prove that the family of random variables $\left\{ \frac{T_i}{\log(1/\alpha)} \right\}_{0 < \alpha \leq 1}$ is uniformly integrable. We start with a basic lemma.

**Lemma 3** (Li, Nitinawarat, and Veeravalli [23]). Let $B(Q_0, Q_1)$ be the Bhattacharyya distance between two distributions $Q_0$ and $Q_1$, i.e.,

$$B(Q_0, Q_1) := - \log \left( \sum_{x \in X} Q_0(x) Q_1(x) \right).$$

If $Q_0$ and $Q_1$ are fully supported on $X$, then

$$2B(Q_0, Q_1) = \min_{P \in P(X)} D(P\|Q_0) + D(P\|Q_1).$$

It holds that

$$B^* := \min_{i \neq j} \left[ \min_{A_i \in A_i, A_j \in A_j} B(P_i A_i, P_j A_j) \right].$$

We have $B^* > 0$ as $\min_{i \neq j} D(P_i A_i \| P_j A_j) > 0$ for any $A_i \in A_i, i \in [M]$, which is the condition stated in the choice of $\Delta$ in Section III. We can now control the probability that the stopping time exceeds a certain deterministic value $n$.

**Lemma 4.** For every $n \geq 1$ and $i \in [M]$, we have

$$P_i(T^* \geq n) \leq \frac{1}{\alpha} e^{-(n-1)2B^*} (M - 1)n^2|X| e^{(n-1)c}.$$ 

**Proof.** Without loss of generality, we consider the case $i = 1$. We have

$$P_1(T^* \geq n)$$

$$\leq P_1 \left( \bigcap_{i=1}^{M} \left\{ Z_i \leq \gamma_{n-1} \right\} \right)$$

$$\leq P_1 \left( \min_{j \neq 1} \left\{ A_j \in A_j : \min_{A_i \in A_i} D(\hat{Q}_{Y^{n-1}} \| P_i A_j) \leq \gamma_{n-1} \right\} \right)$$

$$= P_1 \left( D(\hat{Q}_{Y^{n-1}} \| P_1 A_1) \geq D(\hat{Q}_{Y^{n-1}} \| P_1 \hat{A}_1) \right)$$

$$+ \gamma_{n-1} - \min_{j \neq 1} \left[ \min_{A_j \in A_j} D(\hat{Q}_{Y^{n-1}} \| P_j A_j) \right]$$

$$\leq P_1 \left( D(\hat{Q}_{Y^{n-1}} \| P_1 A_1) \right) \geq \gamma_{n-1} + 2B^*$$

$$\leq \frac{1}{\alpha} (M - 1)n^2|X| e^{(n-1)c},$$

where (a) is because when $j \neq 1$,$$
\min_{A_j \in A_j} D(\hat{Q}_{Y^{n-1}} \| P_j A_j) + D(\hat{Q}_{Y^{n-1}} \| P_1 \hat{A}_1) \geq 2B^*$$

and (b) is from Lemma 2.

Based on Lemma 4, under $H_i$, for every $0 < \alpha \leq 1$, we have

$$P_i(T^* = \infty) \leq \lim_{n \to \infty} P_i(T^* \geq n) = 0. \quad (8)$$

This means that the stopping time $T^*_i, i \in [M]$ are almost surely finite when $0 < \alpha \leq 1$. Thus, based on (3), (4) and (8), we have that there exists $i \in [M]$, such that

$$\min_{j \neq i} \left[ \min_{A_i \in A_i} D(\hat{Q}_{Y^{T^*_i}} \| P_i A_j) \right] \geq \gamma_{T^*_i}, \quad (9)$$

$$\min_{j \neq i} \left[ \min_{A_j \in A_j} D(\hat{Q}_{Y^{T^*_i-1}} \| P_j A_j) \right] \leq \gamma_{T^*_i-1}. \quad (10)$$

Define $\hat{Q}_i = P_i \hat{A}_i$ for $i \in [M]$. Next, by observing that for any distribution (probability mass function) $Q$, $D(Q \| \hat{Q}_i) \leq - \log \min_{y \in X} \hat{Q}_i(y)$. Denote $Q_{\text{max}} := \max_{i \in [M]} \left\{ - \log \min_{y \in X} \hat{Q}_i(y) \right\}$, we get from (9) that

$$P_i(T^* \leq n)$$

$$\leq \sum_{j=1}^{M} P_i \left( T^*_j \leq \log \frac{1}{\alpha}, T^*_j \leq n \right)$$

$$\leq MP_i \left( nQ_{\text{max}} > \log \frac{1}{\alpha}, T^*_j \leq n \right) = 0, \quad \forall n < \frac{\log \frac{1}{\alpha}}{Q_{\text{max}}}$$

which yields that $T^*_i \to \infty$ as $\alpha \to 0^+, P_i$-a.s.

Since we assumed that the adversary’s strategy is $(\hat{A}_1, \ldots, \hat{A}_M)$, the true distribution of $Y^n$ is $P_i \hat{A}_i$ under $H_i$. Thus, under $H_i$, by the strong law of large numbers, we have that $\hat{Q}_{Y^n} \to P_i \hat{A}_i$ a.s. as $n \to \infty$. Consequently, we conclude from the continuity of $D(\| P_i A_j \| P_j A_j)$ on the finite alphabet $X$ that under $H_i$, $D(\hat{Q}_{Y^n} \| P_i A_j) \to D(\hat{Q}_{Y^n} \| P_i \hat{A}_i)$ a.s. as $\alpha \to 0^+$ for each $A_j \in A_j$. Thus, we have shown the pointwise convergence for each $A_j \in A_j$. Now we prove the uniform almost sure convergence of $\min_{A_i \in A_i} D(\hat{Q}_{Y^{T^*_i}} \| P_i A_j)$.

Recall that $A_j$ is assumed to be a compact set. Note that $D(\hat{Q}_i \| Q_j)$ is strongly convex with respect to $Q_j$. Hence there is a unique $P_i A_j$ that minimizes $D(P_i A_i \| P_j A_j)$. We also need to show that $D(\hat{Q}_{Y^{T^*_i}} \| P_i A_j)$ is stochastically equicontinuous. That is, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\lim_{\alpha \to 0^+} P_i \left( \sup_{Q_i, Q'_j \in A_i : \|Q_i - Q'_j\| \leq \delta} \left| D(\hat{Q}_{Y^{T^*_i}} \| Q_j) - D(\hat{Q}_{Y^{T^*_i}} \| Q'_j) \right| > \epsilon \right) = 0.$$ 

At this point, we note that for every $\epsilon > 0$ and for $0 < \delta < \min_{Q_i \in A_i} \min_{y \in X} Q_i(y)$,

$$P_i \left( \sup_{\|Q_i - Q'_j\| \leq \delta} \left| D(\hat{Q}_{Y^{T^*_i}} \| Q_j) - D(\hat{Q}_{Y^{T^*_i}} \| Q'_j) \right| > \epsilon \right)$$

$$\leq P_i \left( \sup_{\|Q_i - Q'_j\| \leq \delta} \left| \sum_{a \in A} \hat{Q}_{Y^{T^*_i}}(a) \left| \frac{Q_j(a) - Q'_j(a)}{\min_{y \in X} Q_j(y)} \right| \right| > \epsilon \right)$$

$$\leq P_i \left( \sup_{\|Q_i - Q'_j\| \leq \delta} \left| \sum_{a \in A} \frac{|Q_j(a) - Q'_j(a)|}{\min_{y \in X} Q_j(y)} \right| > \epsilon \right)$$

$$\leq P_i \left( \sup_{\|Q_i - Q'_j\| \leq \delta} \frac{\|Q_i - Q'_j\|}{\min_{Q_i \in A_i} \min_{y \in X} Q_j(y)} > \epsilon \right) \leq 0,$$
where (a) is because for any \( x, y \geq \beta, |\log x - \log y| \leq \frac{1}{\beta}|x - y| \) and (b) follows from the choice of \( \delta \). Therefore we show that \( D(\tilde{Q}_{Y^T} || P_j A_j) \) is stochastically equicontinuous. Then based on the stochastic Arzelà–Ascoli lemma [24, Theorem 14.3.2], we have

\[
\lim_{\alpha \to 0^+} \inf_{A_j \in A_j} D(\tilde{Q}_{Y^T} || P_j A_j) \equiv \inf_{A_j \in A_j} D(\hat{P}_i A_i || P_j A_j),
\]

and

\[
\lim_{\alpha \to 0^+} \inf_{A_j \in A_j} D(\tilde{Q}_{Y^T} || P_j A_j) \equiv \inf_{A_j \in A_j} D(\hat{P}_i A_i || P_j A_j).
\]

Combining above results with (9) and (10), we deduce that under hypothesis \( H_i \),

\[
\lim_{\alpha \to 0^+} \frac{T^*}{\log(1/\alpha)} = \inf_{A_j \in A_j} D(\hat{P}_i A_i || P_j A_j).
\]

To go from a.s. convergence above to converge in mean, it suffices to prove that there exists an \( \epsilon_0 > 0 \) such that a family of random variables \( \{\frac{T^*}{\log(1/\alpha)}\}_{0 < \alpha \leq \epsilon_0} \) is uniformly integrable. That is, there exists an \( \epsilon_0 > 0 \) such that for all \( \alpha \in (0, \epsilon_0] \),

\[
\lim_{\eta \to \infty} \mathbb{E}_i \left[ \frac{T^*}{\log(1/\alpha)} \mathbb{I}_{\{T^*/\log(1/\alpha) \geq \eta\}} \right] = 0.
\]

Here we choose an \( \epsilon_0 \in (0, 1) \) such that \( x \in (0, \infty) \mapsto x \log(1/x) \) is increasing on \( (0, \infty) \) and \( \frac{\log(1/\epsilon_0)}{\epsilon_0} \leq 1 \). We now choose \( \eta > 0 \) such that \( \eta 2^{\eta} > 2|X| + 2 \). Then for any \( 0 < \alpha \leq \epsilon_0 \), we have the derivation shown in (11) (on the top of next page), where \( C_1 := 2^{2i|X| - 1} \alpha^{4B^*} \sum_{i=1}^{\infty} 2^{i|X|} e^{-B^*}, C_2 := e^{4B^*} \), and \( C_3 := 2^{2i|X| - 1} e^{4B^*} \sum_{i=1}^{\infty} 1 e^{-B^*} \). In the derivation of (11), (a) follows from the inequality \( (x + y)^k \leq 2^{k-1}(x^k + y^k) \) for any \( x, y > 0 \) and any integer \( k \), (b) follows from the choice of \( \epsilon_0 \). As (11) tends to 0 as \( \eta \to \infty \), we have proved the uniform integrability of the family of random variables \( \{\frac{T^*}{\log(1/\alpha)}\}_{0 < \alpha \leq \epsilon_0} \). Thus, under \( H_i \), we have

\[
\lim_{\alpha \to 0^+} \mathbb{E}_i \left[ \frac{T^*}{\log(1/\alpha)} \right] = \inf_{A_j \in A_j} D(\hat{P}_i A_i || P_j A_j).
\]

Therefore, when the adversary’s strategy is \( (\tilde{A}_1, \ldots, \tilde{A}_M) \), the asymptotic payoff function for the test \( \Phi^* \) as \( \alpha \to 0^+ \) is

\[
\lim_{\alpha \to 0^+} u^{(0)}_i(\Phi^*, (\tilde{A}_1, \ldots, \tilde{A}_M)) = \sum_{i=1}^M \lambda_i \inf_{A_j \in A_j} D(\hat{P}_i A_i || P_j A_j).
\]

### Part 3: Proof of the asymptotic Nash equilibrium at the profile \( (\Phi^*, (\tilde{A}_1^*, \ldots, \tilde{A}_M^*)) \)

We first prove (1) by deriving a lower bound on the expected sample size for a general sequential test. For a fixed pair \( (\tilde{A}_1^*, \ldots, \tilde{A}_M^*) \), we consider the multiplicity hypothesis testing problem in which under \( H_j \) the observations are generated from \( P_j A_i, i \in [M] \). Let \( \delta = (d, T) \) be a sequential test for the multiple hypothesis testing problem. Let \( \alpha_{ij} \) be the probability that \( H_j \) is accepted when \( H_i \) is the underlying hypothesis. Let \( \alpha_i \) be the probability that \( H_i \) is rejected when \( H_i \) is the true hypothesis. Note that \( \alpha_i = 1 - \alpha_{ii} = \sum_{j \neq i} \alpha_{ij} \).

Upper bound for the error probabilities \( \alpha_i, i \in [M] \). Based on [16, Lemma 4.3.1], then we have that for all \( i \in [M] \),

\[
\mathbb{E}_i[T^*] \geq \max_{j \in [M], j \neq i} \frac{1}{D(\hat{P}_i A_i || P_j A_j)} \times \left( \sum_{k \neq i} \alpha_{ik} \log \frac{\alpha_{ik}}{\alpha_{jk}} + (1 - \alpha_{i}) \log \frac{1 - \alpha_{i}}{\alpha_{ij}} \right)
\]

(b)

\[
\geq \max_{j \in [M], j \neq i} \frac{1}{D(\hat{P}_i A_i || P_j A_j)} \times \left( \sum_{k \neq i} \alpha_{ik} \log \frac{\alpha_{ik}}{\alpha_{jk}} + (1 - \alpha_{i}) \log \frac{1 - \alpha_{i}}{\alpha_{ij}} \right)
\]

(c)

\[
\geq \max_{j \in [M], j \neq i} \frac{1}{D(\hat{P}_i A_i || P_j A_j)} \times \left( \alpha_i \log \frac{\alpha_{i}}{1 - \alpha_{i}} + (1 - \alpha_{i}) \log \frac{1 - \alpha_{i}}{\alpha_{ij}} \right)
\]

\[
\mathbb{E}_i[T^*] \geq \max_{j \in [M], j \neq i} \frac{1}{D(\hat{P}_i A_i || P_j A_j)} \times \left( \alpha_i \log \frac{\alpha_{i}}{1 - \alpha_{i}} + (1 - \alpha_{i}) \log \frac{1 - \alpha_{i}}{\alpha_{ij}} \right)
\]

\[
\mathbb{E}_i[T^*] \geq \max_{j \in [M], j \neq i} \frac{1}{D(\hat{P}_i A_i || P_j A_j)} \times \left( \alpha_i \log \frac{\alpha_{i}}{1 - \alpha_{i}} + (1 - \alpha_{i}) \log \frac{1 - \alpha_{i}}{\alpha_{ij}} \right)
\]

In (15), the left-hand side \( \mathbb{E}_i[T^*] \) does not depend on \( \tilde{A}_j : j \neq i \). We recall that \( A_i \) is a compact set and \( X \) is finite and \( Q_i \) and \( Q_j \) have full support on \( X \). So we can maximize the right-hand side with respect to \( A_j \) and obtain that

\[
\mathbb{E}_i[T^*] \geq \max_{j \in [M], j \neq i} \frac{1}{D(\hat{P}_i A_i || P_j A_j)} \times \left( \alpha_i \log \frac{\alpha_{i}}{1 - \alpha_{i}} + (1 - \alpha_{i}) \log \frac{1 - \alpha_{i}}{\alpha_{ij}} \right)
\]

Thus, based on (16), we have

\[
\lim_{\alpha \to 0^+} \mathbb{E}_i[T^*] \geq \max_{j \in [M], j \neq i} \frac{1}{D(\hat{P}_i A_i || P_j A_j)} \times \left( \alpha_i \log \frac{\alpha_{i}}{1 - \alpha_{i}} + (1 - \alpha_{i}) \log \frac{1 - \alpha_{i}}{\alpha_{ij}} \right)
\]

As (16) holds for any test \( \Phi \in S_D(\alpha) \), we have

\[
\lim_{\alpha \to 0^+} \sup_{\Phi \in S_D(\alpha)} u^{(0)}_i(\Phi, (\tilde{A}_1^*, \ldots, \tilde{A}_M^*)) \leq \sum_{i=1}^M \lambda_i \inf_{A_j \in A_j} D(\hat{P}_i A_i || P_j A_j)
\]

\[
= \lim_{\alpha \to 0^+} u^{(0)}_i(\Phi^*, (\tilde{A}_1^*, \ldots, \tilde{A}_M^*))
\]

which completes the proof of (1).

To complete the proof of (2), we leverage a technical lemma.
It is obvious from the definition of $A$.

Lemma 5. For all $i \in [M]$, it holds that the family of numbers
$$\left\{ \log(1/\alpha) \right\}_{0<\alpha \leq 1}$$
converges uniformly on $A_i$ as $\alpha \to 0^+$. The proof of Lemma 5 is presented in Appendix A. From (13), we have that
$$\lim_{\alpha \to 0^+} -u^*_\alpha(\Phi^*, (\hat{A}_1, \ldots, \hat{A}_M))$$
$$= -\sum_{i=1}^M \lambda_i \min_{j \in [M] \setminus \{i\}} \left[ \min_{A_j \in A_j} D(P_i \hat{A}_i || P_j A_j) \right].$$

It is obvious from the definition of $A^*_i, i \in [M]$ in (5) that
$$\lim_{\alpha \to 0^+} -u^*_\alpha(\Phi^*, (A^*_1, \ldots, A^*_M))$$
$$= -\sum_{i=1}^M \lambda_i \min_{j \neq i} \left[ \min_{A_j \in A_j} D(P_i A^*_i || P_j A_j) \right]$$
$$\geq \sum_{i=1}^M \lambda_i \min_{j \neq i} \left[ \min_{A_j \in A_j} D(P_i \hat{A}_i || P_j A_j) \right]$$
$$= \lim_{\alpha \to 0^+} -u^*_\alpha(\Phi^*, (\hat{A}_1, \ldots, \hat{A}_M)).$$

V. EXTENSION TO THE ADVERSARY NON-AWARENESS SETTING

In this section, we consider the case when the adversary does not know the underlying distribution of observed samples in a binary hypothesis test. In this case, referring to Fig. 1, the adversary can only apply a common perturbation mechanism $A$ to the two hypotheses.

We define the expectation of the stopping time under $H_i$ as $E_i[\tau]$ for $i \in \{0, 1\}$ and a (non-aware) test is a pair $\Phi_{NA} = (\tau, \delta_{NA})$. We also define the adversary’s and decision maker’s strategy sets as
$$\mathcal{S}_A(\Delta) := \left\{ A : \max_{i \in \{0, 1\}} d(P_i, P_A) \leq \Delta \right\},$$
and
$$\mathcal{S}_D(\alpha) := \left\{ \Phi_{NA} : \max_{i \in \{0, 1\}} \sup_{A \in \mathcal{S}_A(\Delta)} \alpha_i(\Phi_{NA}) \leq \alpha \right\},$$
respectively, and the payoff function of the decision maker as
$$\hat{u}_\alpha(\Phi_{NA}, A) := \frac{\log(1/\alpha)}{E_0[\tau]} + \frac{\lambda \log(1/\alpha)}{E_1[\tau]}.$$

We define
$$S_n := \min_{A \in \mathcal{S}_A(\Delta)} \max_{i \in \{0, 1\}} \{D(\hat{Q}_{Y^*} || P_i A), D(\hat{Q}_{Y^*} || P_i A)\}.$$
and the decision rule as
\[
\delta_{\text{NA}}(Y^n) := \begin{cases} 
0, & \text{if } \min_{A \in \mathcal{S}_A(\Delta)} D(\hat{Q}_{Y^n}||P_1 A) \geq \gamma_n, \\
1, & \text{if } \min_{A \in \mathcal{S}_A(\Delta)} D(\hat{Q}_{Y^n}||P_0 A) \geq \gamma_n.
\end{cases}
\]

Then our adversary non-aware test is \( \Phi_{\text{NA}}^* = (\tau^*, \delta_{\text{NA}}) \).

We obtain the following two propositions which present the achievable and converse results respectively.

**Proposition 6.** If \( \mathcal{S}_A(\Delta) \) is a compact set, then for any \( \lambda > 0 \), we have that
\[
\lim_{\alpha \to 0^+} \hat{u}_\lambda^*(\Phi_{\text{NA}}, \tilde{A}) = \min_{A \in \mathcal{S}_A(\Delta)} \max \left\{ D(P_0 \tilde{A}||P_1 A), D(P_0 A || P_0 A) \right\} + \lambda \min_{A \in \mathcal{S}_A(\Delta)} \max \left\{ D(P_1 \tilde{A}||P_0 A), D(P_1 A || P_1 A) \right\}.
\]

**Proof sketch of Proposition 6.** Based on the definition of \( \delta_{\text{NA}} \), following the same procedure as in the first part of proof in Section IV, we can prove that \( \Phi_{\text{NA}}^* \in \hat{S}_D(\Delta) \). To obtain the error exponents for the test \( \Phi_{\text{NA}} \) and any \( \tilde{A} \in \hat{S}_A(\Delta) \), we first prove a result similar to Lemma 4. Set \( M = 2 \) in \( \gamma_n - 1 \). Then we have
\[
P_0(\tau^* > n) \leq P_0 \left( \min_{A \in \mathcal{S}_A(\Delta)} D(\hat{Q}_{Y^n}||P_0 A) \leq \gamma_n \right)
\leq P_0 \left( D(\hat{Q}_{Y^n}||P_1 \tilde{A}) \geq -\gamma_n + 2\hat{B}^* \right)
\leq \frac{1}{\alpha} e^{-(n-1)2\hat{B}^* n^n |\gamma(n-1)|^\xi},
\]
where \( \hat{B}^* := \min_{A \in \mathcal{S}_A(\Delta)} B(P_0 A, P_1 A) \). Then following the same procedure as in the second part of proof in Section IV but now the limit is
\[
\lim_{\alpha \to 0^+} \frac{1}{P_0(\tau^*)} = \min_{A \in \mathcal{S}_A(\Delta)} \max \left\{ D(P_0 \tilde{A}||P_1 A), D(P_0 A || P_0 A) \right\}.
\]

We can obtain an analogous result for \( \lim_{\alpha \to 0^+} \frac{1}{P_0(\tau^*)} \). So combining the above two results, we can obtain (18). \( \square \)

Based on Proposition 6, we see that the adversary can choose the strategy that minimizes the achievable bound of the decision maker:
\[
A^* = \arg \min_{A \in \hat{S}_A(\Delta)} \left\{ \min_{A \in \mathcal{S}_A(\Delta)} \max \left\{ D(P_0 \tilde{A}||P_1 A), D(P_0 A || P_0 A) \right\} + \lambda \min_{A \in \mathcal{S}_A(\Delta)} \max \left\{ D(P_1 \tilde{A}||P_0 A), D(P_1 A || P_1 A) \right\} \right\}.
\]

Using this strategy, we find that
\[
\lim_{\alpha \to 0^+} \hat{u}_\lambda^*(\Phi_{\text{NA}}^*, A^*) \geq \lim_{\alpha \to 0^+} \hat{u}_\lambda^*(\Phi_{\text{NA}}^*, (A^*, A^*))
\]
which means that the decision maker can obtain a better (no worse, to be precise) performance than the adversary awareness case. In Proposition 7, we prove a converse bound for any pair of strategies.

**Proposition 7.** For any test \( \Phi_{\text{NA}} \in \hat{S}_D(\alpha) \) and any \( \tilde{A} \in \hat{S}_A(\Delta) \), we have
\[
\lim_{\alpha \to 0^+} \hat{u}_\lambda^*(\Phi_{\text{NA}}, \tilde{A}) \leq D(P_0 \tilde{A}||P_1 \tilde{A}) + \lambda D(P_1 \tilde{A}||P_0 \tilde{A}).
\]

**Proof.** As the adversary adopts the same strategy on both hypotheses, the problem is equivalent to the following hypothesis testing problem: \( H_0 : P_0 \tilde{A}, \text{ v.s. } H_1 : P_1 \tilde{A} \). However, for this problem the decision maker have no knowledge of \( \tilde{A} \).

According to the optimality of sequential probability ratio test (SPRT) [16], the upper bound on the error exponents that the decision maker can obtain for any test is
\[
\lim_{\alpha \to 0^+} \hat{u}_\lambda^*(\Phi_{\text{NA}}, A) \leq D(P_0 \tilde{A}||P_1 \tilde{A}) + \lambda D(P_1 \tilde{A}||P_0 \tilde{A}),
\]
as desired. \( \square \)

**Remark 3.** Observe that the achievable and converse bounds in (18) and (19) respectively do not match in the adversary non-awareness setting. Thus, the pair of strategies \( (\Phi_{\text{NA}}^*, A^*) \) cannot, in general, achieve the asymptotic Nash equilibrium. However, comparing (18) to the adversary aware case in (6), we see that the decision maker can attain larger error exponents, which implies that the decision maker can perform better. This is aligned with our intuition, since now the adversary is weaker as it has to use the same \( A \) under both hypotheses.

## VI. Numerical Experiments

In this section, we provide two sets of experiments to corroborate the theory developed in the previous sections. The first uses synthetic data on a binary hypothesis testing problem with Bernoulli distributions to show that empirical stopping time converges to its theoretical counterpart. The second set of experiments shows that empirical stopping time converges to its theoretical counterpart on the MNIST dataset.

### A. Binary Test for Bernoulli Distributions

Let the distributions be \( \text{Bern}(p_0) \) under \( H_0 \) and \( \text{Bern}(\frac{1}{2}) \) under \( H_1 \), respectively. Without loss of generality, we assume \( 0 < p_0 < \frac{1}{2} \). We set the distortion measure \( d \) to be the total variation distance and the distortion level to be \( \Delta \). For the Bernoulli distribution, the adversary’s strategy takes the form
\[
A_i = \begin{bmatrix}
1 - b_i & 1 - a_i \\
1 & b_i
\end{bmatrix}, \quad i \in \{0, 1\}.
\]

Using the distortion constraints, we can obtain the relationship between \( a_i \) and \( b_i \) for \( i \in \{0, 1\} \) as follows:
\[
|1 - 2p_0 - b_0 + p_0(a_0 + b_0)| \leq \frac{\Delta}{2}, \quad |a_1 - b_1| \leq \Delta.
\]

Then based on Theorem 1, we can calculate the optimal adversary’s strategy by solving the optimization problem in (5). In the Bernoulli case, the perturbed distributions by the optimal
adversary strategy are attained on the boundary (shown in Fig 2 with red crosses), which means that

\[ 1 - 2p_0 - b_n^* + p_0(a_0^* + b_0^*) = \frac{\Delta}{2}, \]

\[ b_1^* - a_1^* = \Delta. \]

The payoff function at the asymptotic Nash equilibrium is

\[ \lambda_1 D_b \left( 0.5 - \frac{\Delta}{2} \| p_0 + \frac{\Delta}{2} \right) + \lambda_2 D_b \left( p_0 + \frac{\Delta}{2} \| 0.5 - \frac{\Delta}{2} \right), \]

where \( D_b(a \| b) := a \log \left( \frac{a}{b} \right) + (1 - a) \log \left( \frac{1 - a}{1 - b} \right) \) is the binary KL divergence between two Bernoulli distributions with parameters \( a, b \in (0, 1) \).

Now we set \( p_0 = 0.38 \) and \( \Delta = 0.05 \). We can calculate \( A_n^*, i \in \{0, 1\} \) numerically. Note that the optimizing matrices are not unique. One of the optimizing pairs is

\[ A_n^* = \begin{bmatrix} 0.5 & 0.5 \\ 0.3419 & 0.6581 \end{bmatrix} \quad \text{and} \quad A_1^* = \begin{bmatrix} 0.15 & 0.85 \\ 0.8 & 0.2 \end{bmatrix}. \]

Then in this case, the payoff function at the asymptotic Nash equilibrium is 0.0109\( \lambda_1 + 0.0108\lambda_2 \). To corroborate Theorem 1, now we simulate the sequential adversarial hypothesis testing procedure.

We set \( \alpha \) to different values and run the strategy defined in (3)–(4) a total of 50,000 times for each \( \alpha \) to observe the stopping times and hence, the convergence of the payoff function as \( \alpha \to 0^+ \) under \( H_0 \) and \( H_1 \), respectively. The results are shown in Figs. 3 and 4, respectively. The horizontal line is the theoretical payoff function at the Nash equilibrium, i.e., \( \min_{(A_0, A_1) \in \mathcal{S}_1(\Delta)} D(P_0 A_0 \| P_1 A_1) \) or \( \min_{(A_0, A_1) \in \mathcal{S}_1(\Delta)} D(P_1 A_1 \| P_0 A_0) \). The dotted line is the estimated payoff function, i.e., \( \frac{\log(1/\alpha)}{\epsilon_1(V)} \) for \( i = 0, 1 \). We observe that as \( \alpha \to 0^+ \), i.e., \( \log(1/\alpha) \to \infty \), \( \frac{\log(1/\alpha)}{\epsilon_1(V)} \) converges to \( D_b(p_0 + \Delta/2 \| 0.5 - \Delta/2) \) under \( H_0 \) and \( \frac{\log(1/\alpha)}{\epsilon_1(V)} \) tends to \( D_b(0.5 - \Delta/2 \| p_0 + \Delta/2) \) under \( H_1 \).

### B. Binary Test for the MNIST dataset

In the previous section, we applied our sequential hypothesis testing strategy to synthetic data. To demonstrate the utility of our strategy on real-world data, we now apply it to the MNIST dataset. For simplicity, we choose to test two classes from the MNIST dataset—digits 1 and 4. We also binarize the MNIST data by choose a threshold (here we choose the threshold to be 50). When the pixel value greater than the threshold, we set the value to 255 and otherwise, we set the value to 0. Fig. 2 shows a representative original image and its binarized version.
Denote the respective distributions for digits 1 and 4 respectively as \( I_1 = [f_{1.0}, f_{1.255}] \) and \( I_4 = [f_{4.0}, f_{4.255}] \). We use the training dataset to obtain an empirical estimates of their distributions. We find that

\[
I_1 = [0.9061, 0.09395] \quad \text{and} \quad I_4 = [0.8481, 0.1519].
\]

We set the distortion measure to be the KL distance to ensure the adversary’s optimal strategy by solving (5) numerically using MATLAB’s convex optimization toolbox. This yields

\[
A_1^* = \begin{bmatrix} 0.94 & 0.06 \\ 0.461 & 0.539 \end{bmatrix} \quad \text{and} \quad A_4^* = \begin{bmatrix} 0.9 & 0.1 \\ 0.645 & 0.355 \end{bmatrix}.
\]

Fig. 6 shows two examples of perturbed images produced the adversary. Note that as we use the pixel values to estimate the distribution, the adversary perturbs only the fraction of white and black pixels.

Now we use the test dataset to perform the sequential test. Due to the limited number of samples in the training and test data sets (there are 7877 images for digit 1 and 6824 images for digit 4), during the testing process, we use a resampling procedure to obtain more test images. Fig. 7 and Fig. 8 show the change of payoff function as \( \alpha \to 0^+ \) when the true digit is 4 and the true digit is 1, respectively. We observe that when \( \alpha \to 0^+ \), \( \frac{\log(1/\alpha)}{E_1[T^*]} \rightarrow D(I_4A_1^* \| I_4A_4^*) \) when the true digit is 1 and \( \frac{\log(1/\alpha)}{E_1[T^*]} \rightarrow D(I_4A_1^* \| I_1A_4^*) \) when the true digit is 4. Thus the conclusion here is the same as that for synthetic data, i.e., the promised fundamental limit is attained as \( \alpha \to 0^+ \).

**VII. Conclusion**

In this work, we consider the \( M \)-ary sequential adversarial hypothesis testing problem. Different from the traditional \( M \)-ary sequential test, in this problem, an adversary is active and tries to perturbed the distributions of observed samples. Our objective is to obtain a pair of strategies for the adversary and the decision maker, in which no party can increase its payoff by unilaterally changing its strategies, i.e., we wish to find the Nash equilibrium. In this paper, we obtain a pair of strategies at which the asymptotic Nash equilibrium is attained. The adversary’s strategy in the asymptotic Nash equilibrium is the transition matrices that minimize the Kullback–Leibler divergence between perturbed distributions, and the decision maker’s strategy at the asymptotic Nash equilibrium is analogous to the sequential version of Hoeffding’s test [25].

In the future, several directions could be considered. First, in this paper, when consider the case that the adversary is not aware the underlying distribution of observed samples, the achievable and converse bounds do not match. This means that the pair of strategies we propose can not achieve the Nash Equilibrium. We have endeavored to solve this problem but failed to find the pair of strategies attaining the Nash equilibrium. Thus, we could consider to find the pair of strategies attaining the mixed Nash equilibrium [18]. Second, as we only obtain the asymptotic Nash equilibrium when \( \alpha \to 0^+ \), one extension of our work is to consider the non-asymptotic Nash equilibrium for some fixed \( \alpha \in (0, 1) \). Third, we could also consider the case where the distribution of each hypothesis is unknown and we only access them through training sequences of each hypothesis. This case is analogue to sequential adversarial classification problem.

**Appendix**

**A. Proof of Lemma 5**

From the definitions of \( T^* \) and \( T_1 \), we can see that \( T_1 \geq T^* \). Similar to the proof of Eqn. (12), we also can prove that

\[
\lim_{\alpha \to 0^+} \frac{E_1[T_1]}{\log(1/\alpha)} = \frac{1}{\min_{j \neq 1} \left[ \min_{A_j \in A_j} D(P_{1A_j} \| P_jA_j) \right]}.
\]

(20)
Now we want to show that the convergence above is uniform on $A_1$, which allows us to establish (17).

According to the definition of $T_1$, we have

$$\min_{j \neq 1} \left[ \min_{A_j \in A_j} D(\hat{Q}_{Y,T_1} \| P_j A_j) \right] \geq \frac{\log \left( \frac{1}{\alpha} \right) T_1}{T_1} + \frac{1}{T_1^2} + \frac{|X| \log(T_1 + 1) + \log(M - 1)}{T_1},$$

and

$$\min_{j \neq 1} \left[ \min_{A_j \in A_j} D(\hat{Q}_{Y,T_1-1} \| P_j A_j) \right] \leq \frac{\log \left( \frac{1}{\alpha} \right) T_1}{T_1} + \frac{1}{(T_1 - 1)^2} + \frac{|X| \log(T_1) + \log(M - 1)}{T_1 - 1}. $$

Then, we have that

$$\left| \frac{\log \left( \frac{1}{\alpha} \right)}{T_1} - \min_{j \neq 1} \left[ \min_{A_j \in A_j} D(\hat{Q}_{Y,T_1} \| P_j A_j) \right] \right| \leq \frac{c_0}{T_1^2},$$

where $c_0$ does not depend on $\hat{A}_1$. Then, we define

$$D_{T_1} := \min_{j \neq 1} \left[ \min_{A_j \in A_j} D(\hat{Q}_{Y,T_1} \| P_j A_j) \right],$$

and

$$D_1 := \min_{j \neq 1} \min_{A_j \in A_j} D(P_j \hat{A}_1 \| P_j A_j).$$

We have

$$\mathbb{E}_1 \left[ \left( \frac{\log \left( \frac{1}{\alpha} \right) T_1}{T_1} - D_{T_1} \right) \right] = \mathbb{E}_1 \left[ \left( \frac{\log \left( \frac{1}{\alpha} \right) T_1}{T_1} - D_{T_1} + D_{T_1} - D_1 \right) \right] \leq \mathbb{E}_1 \left[ \left( \frac{\log \left( \frac{1}{\alpha} \right) T_1}{T_1} - D_{T_1} \right) \right] + \mathbb{E}_1 \left[ \left| D_{T_1} - D_1 \right| \right] \leq \mathbb{E}_1 \left[ \frac{c_0}{T_1^2} \right] + \mathbb{E}_1 \left[ \left| D_{T_1} - D_1 \right| \right].$$

Define $c_1 := \min_{j \neq 1} \left[ \min_{A_j \in A_j} (1 - \log \min_{y \in X} Q_j(y)) \right]$. For the first term, because

$$P_1(T_1 \leq n) \leq P_1 \left( \min_{j \neq 1} \left[ \min_{A_j \in A_j} D(\hat{Q}_{Y,T_1} \| P_j A_j) \right] \geq \log \left( \frac{1}{\alpha} \right), T_1 \leq n \right) \leq P_1(c_1 n > \log(1/\alpha), T_1 \leq n) = 0, \quad \forall n < \frac{\log(1/\alpha)}{c_1},$$

we have that

$$P_1 \left( T_1 < \frac{\log(1/\alpha)}{c_1} \right) = 0,$$

This means that

$$T_1 \geq \frac{\log(1/\alpha)}{c_1}, \quad \text{a.s.}$$

Thus,

$$\mathbb{E}_1 \left[ \frac{1}{T_1^2} \right] \leq \left( \frac{\log(1/\alpha)}{c_1} \right)^{-\zeta},$$

where $c_1$ does not depend on $\hat{A}_1$. For the second term, we define $c_2 := -\log \min_{\hat{A}_1 \in A_1} \min_{y \in X} \hat{Q}_1(y)$. Let $\varepsilon$ be an arbitrary fixed positive number. Then we have that (22) (on the top of next page), where (a) follows from that $| \min f(x) - \min g(x) | \leq \max | f(x) - g(x) |$, (b) follows from Pinsker’s inequality [22, Lemma 11.6.1] and $c_2, c_3$ do not depend on $\hat{A}_1$. We also have

$$P_1 \left( D(\hat{Q}_{Y,T_1} \| P_1 \hat{A}_1) \geq \varepsilon \right) \leq \sum_{k \geq \log(1/\alpha)/c_1} P_1 \left( D(\hat{Q}_{Y,T_1} \| P_1 \hat{A}_1) \geq \varepsilon \right) \leq \sum_{k \geq \log(1/\alpha)/c_1} c_4 e^{-ke} \leq c_5 e^{-\frac{\log(1/\alpha)}{c_1}},$$

where $c_5$ depends only on $|X|$. Thus,

$$\mathbb{E}_1 \left[ |D_{T_1} - D_1| \right] \leq \varepsilon + (c_2 |X| + \sqrt{c_2 |X|}) c_5 e^{-\frac{\log(1/\alpha)}{c_1}} + c_3 \sqrt{\varepsilon}. \quad (23)$$

Therefore, combining (21) and (23), we have

$$\mathbb{E}_1 \left[ -\frac{\log(1/\alpha)}{T_1} - D_1 \right] \leq \varepsilon + (c_2 |X| + \sqrt{c_2 |X|}) c_5 e^{-\frac{\log(1/\alpha)}{c_1}}$$

$$+ c_3 \sqrt{\varepsilon} + c_0 \left( \frac{\log(1/\alpha)}{c_1} \right)^{-\zeta},$$

As $c_i$ for $i = 0, 1, \ldots, 5$ do not depend on $\hat{A}_i$, the convergence in (20) is uniform over $A_i$. Now we show that the uniform convergence over $A_1$ also holds for $\left( \frac{\mathbb{E}_1[T]}{\log(1/\alpha)} \right)_{0 < q \leq 1}$ as $\alpha \to 0^+$. For $i \in [M]$, let $B_i$ be the event that $T_i < T_1$. Note that $P_1(\bigcup_{i \neq 0} B_i)$ is the error probability $\alpha_1$. Conditioned on the events $B_i, i \neq 1$, we have $T^* < T_1$ and conditioned on the event $B_1$, we have $T^* = T_1$.

$$\mathbb{E}_1[T^*] = \mathbb{E}_1[T^* \mathbb{I}_{B_2 \cup \cdots \cup B_M}] + \mathbb{E}_1[T^* \mathbb{I}_{B_1}] = \mathbb{E}_1[T_1] + \mathbb{E}_1[(T^* - T_1) \mathbb{I}_{B_2 \cup \cdots \cup B_M}] \geq \mathbb{E}_1[T_1] - \mathbb{E}_1[T^* \mathbb{I}_{B_2 \cup \cdots \cup B_M}], \quad (24)$$

From Eqn. (11) in the proof of uniform integrability, it follows that for the given $\varepsilon > 0$, there exists a finite constant $K$ that does not depend on $\hat{A}_1$ such that for any $0 < \alpha \leq \alpha_0$ and any $(\hat{A}_1, \ldots, \hat{A}_M)$,

$$\mathbb{E}_1 \left[ \frac{T^*}{\log(1/\alpha)} \mathbb{I}_{\{T^*(\alpha)/\log(1/\alpha) \geq K\}} \right] \leq \varepsilon.$$

Therefore, we have that

$$\mathbb{E}_1[T^* \mathbb{I}_{B_2 \cup \cdots \cup B_M}] = \mathbb{E}_1 \left[ \frac{T^*}{\log(1/\alpha)} \mathbb{I}_{\{T^*(\alpha)/\log(1/\alpha) \geq K\}} \mathbb{I}_{B_2 \cup \cdots \cup B_M} \right] \log \left( \frac{1}{\alpha} \right)$$

$$+ \mathbb{E}_1 \left[ \frac{T^*}{\log(1/\alpha)} \mathbb{I}_{\{T^*(\alpha)/\log(1/\alpha) \leq K\}} \mathbb{I}_{B_2 \cup \cdots \cup B_M} \right] \log \left( \frac{1}{\alpha} \right) \leq \varepsilon \log \left( \frac{1}{\alpha} \right) + K \mathbb{P}(B_2 \cup \cdots \cup B_M) \log \left( \frac{1}{\alpha} \right) \leq \varepsilon \log \left( \frac{1}{\alpha} \right) + K \log \left( \frac{1}{\alpha} \right).$$
\[
E_1 \left[ |D_{T_1} - D_1| \right] \leq E_1 \left[ D(\hat{Q}_{Y,T_1} || P_1 \tilde{A}_1) + \max_{j \neq 1} \max_{A_j \in \mathcal{A}_j} \sum_{a \in \mathcal{X}} (\hat{Q}_{Y,T_1}(a) - \hat{Q}_1(a)) \log \frac{\hat{Q}_1(a)}{\hat{Q}_j(a)} \right]
\]

\[
\leq E_1 \left[ D(\hat{Q}_{Y,T_1} || P_1 \tilde{A}_1) + c_1 |X| E_1 \left[ \sum_{a \in \mathcal{X}} (\hat{Q}_{Y,T_1}(a) - \hat{Q}_1(a)) \right] \right]
\]

\[
= E_1 \left[ D(\hat{Q}_{Y,T_1} || P_1 \tilde{A}_1) \right] + c_3 E_1 \left[ \sqrt{D(\hat{Q}_{Y,T_1} || P_1 \tilde{A}_1)} \right]
\]

\[
\leq \epsilon + (c_2 |X| + c_3 |X|) P_1 \left( D(\hat{Q}_{Y,T_1} || P_1 \tilde{A}_1) \geq \epsilon \right) + c_3 \sqrt{\epsilon},
\]

where (a) follows because \( P_1(B_2 \cup \cdots \cup B_M) \) is exactly the error probability \( \alpha \), which is upper bounded by \( \alpha \). From (24), we have

\[
E_1 \left[ \frac{T^*_1}{\log(1/\alpha)} \right] - E_1 \left[ \frac{T^*}{\log(1/\alpha)} \right] \leq E_1 \left[ \frac{T^*}{\log(1/\alpha)} \right] \leq \epsilon + K_\alpha,
\]

which, together with the arbitrariness of \( \epsilon \), implies that

\[
\lim_{\alpha \to 0^+} \sup_{A_1 \in \mathcal{A}_1} \left( E_1 \left[ \frac{T^*_1}{\log(1/\alpha)} \right] - E_1 \left[ \frac{T^*}{\log(1/\alpha)} \right] \right) = 0.
\]

Then it follows from the uniform convergence of \( E_1 \left[ \frac{T^*_1}{\log(1/\alpha)} \right] \) over \( \mathcal{A}_1 \) and (25) that

\[
\lim_{\alpha \to 0^+} \sup_{A_1 \in \mathcal{A}_1} \left( E_1 \left[ \frac{T^*}{\log(1/\alpha)} \right] - \frac{1}{D_1} \right) = 0,
\]

as desired. The arguments for other \( i \in [M] \) proceed similarly.

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