Localization and tracking in mobile networks: Virtual convex hulls and beyond

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Abstract

In this paper, we discuss the problem of tracking the locations of an arbitrary number of agents moving in a bounded region. Assuming that each agent knows its motion precisely, and also its distances and angles to the nodes in its communication radius, we provide a geometric approach to continually update the distances and angles even when the agents move out of range. Based on this approach, we provide a simple linear update to track the locations of an arbitrary number of mobile agents when they follow some convexity in their deployment and motion, given at least \( m + 1 \) agents whose locations are always known (hereinafter referred to as anchors) in \( \mathbb{R}^m \). More precisely, this linear update requires the agents to intermittently move inside the convex hull of the anchors.

Since the agents are mobile, they may move inside and outside of the convex hull formed by the anchors and the above convexity in the deployment is not necessarily satisfied. To deal with such issues, we introduce the notion of a virtual convex hull with the help of the aforementioned geometric approach. Based on the virtual hulls, we provide another algorithm where no agent is required to lie in any convex hull at any given time. Each agent only keeps track of a virtual convex hull, which may not physically exist, of other nodes (agents and/or anchors), and updates its location with respect to its neighbors in the virtual hull. We show that the corresponding localization algorithm can be abstracted as a Linear Time-Varying (LTV) system that asymptotically tracks the true locations of the agents. Finally, we show that exactly one anchor suffices to track the locations of an arbitrary number of mobile agents in \( \mathbb{R}^m, m \geq 1 \), using the approach described in this paper.

I. INTRODUCTION

Localization is a well-studied problem, which refers to the collection of algorithms that estimate the location of nodes in a network. Relevant applications include traffic control, industrial automation, robotics, and environment monitoring [1]–[4]. The literature on localization
consists of centralized and distributed approaches. Despite the benefits of centralized algorithms [5]–[8], they are impractical in large networks where each node has limited power and communication capability. Some notable distributed localization techniques include successive refinements, [9]–[11], probabilistic approaches, [12], multilateration, [13]–[15], and graph theoretical methods, [16]–[18], some other relevant references include [19]–[22]. Of significant relevance to this work is DILOC, [23], which is a distributed algorithm based on the barycentric representation. DILOC requires all agents, with unknown locations, to satisfy global convexity, i.e., each agent lies inside the convex hull of (at least) \( m + 1 \) anchors, with known locations, in \( \mathbb{R}^m, m \geq 1 \). Since communication with anchors may not be possible, DILOC further assumes local convexity, i.e., each agent has \( m + 1 \) neighbors such that it lies in their convex hull; the location estimate is now updated as a linear-convex combination of such \( m + 1 \) neighbors. Assuming local and global convexity at each agent, DILOC converges to the true agent locations; communication noise, packet losses, and imperfect distances are also addressed in [23]–[26].

Indeed, DILOC requires somewhat stringent deployment implied by satisfying the two convexity conditions. In this context, Refs. [27], [28] extend DILOC by providing a barycentric representation that does not require convexity. The fundamental notion behind these barycentric-based methods, [23], [24], [27], [28], is that the agents implement linear (not linearized) iterations that are guaranteed to converge regardless of the initial conditions, unlike the nonlinear localization algorithms. In particular, DILOC can be abstracted as a Linear Time-Invariant (LTI) system whose system matrix is stable under the convexity conditions and thus forgets the initial conditions. We note that even if local and global convexity is exhibited by a static network, it is highly unlikely that a network of mobile agents, [29], [30], will follow any convexity. A related algorithm, [25], is developed for mobile agents but assumes that mobile agents satisfy both convexity conditions at each time instant. Localization algorithms specifically designed for mobile networks have been proposed in [31]–[36]. Most of these algorithms use Sequential Monte Carlo (SMC) due to its simplicity in implementation. However, SMC methods are time-consuming as they need to keep sampling and filtering until enough samples are obtained to represent the posterior distribution of a mobile agent’s position, [37].

In this paper, we consider localization and tracking of a mobile network, assuming that each agent knows its distances and angles to the nodes (agents and/or anchors) in its communication
radius. In addition, we assume that each agent knows the motion it undertook, e.g., by using an accelerometer. Similar to DILOC, we are interested in implementing linear-convex, distributed iterations such that the convergence is invariant to the agents’ initial conditions. However, the implementation is not as straightforward as DILOC, because: (i) the neighborhood at each agent is dynamic resulting into a Linear Time-Varying (LTV) system; (ii) global convexity may not be satisfied, i.e., the agents may move in and out of the convex hull formed by the anchors; (iii) local convexity may not be satisfied at all times; (iv) local convexity may not be satisfied at any given time. As we will show, the resulting LTV system comprises of system matrices that may be: identity–when no agent is able to satisfy local convexity; stochastic–when an agent finds $m+1$ neighboring agents passing local convexity; and, sub-stochastic–when this neighborhood includes at least one anchor. Clearly, establishing the asymptotic behavior of such an LTV system is non-trivial.

To address these issues, our approach can be described as follows. We first develop a geometric framework, where an agent tracks its distance and angle to a neighbor as soon as they communicate. Agent $i$ now needs $m+1$ nodes that satisfy local convexity to linearly update its location estimate. Since such neighbors may never exist, see (iv) above, we introduce the notion of a virtual convex hull: When an agent, say $i$, makes contact with a neighbor, it obtains the neighbor’s distance, angle, and location estimate. Agent $i$ then could move away from the neighbor but keeps track of the distance and angle to the former location of this node. As agent $i$ continues to move in the network, there comes a time when it has exchanged information with $m+1$ distinct nodes. Agent $i$ thus knows the distances (and angles) to each of these nodes, albeit to the locations where agent $i$ found them in the past. At this point, agent $i$ uses the distances and angles to test for local convexity, see Section II-A for this test. If the test fails, it continues to travel in the network and meets new nodes. Eventually, the local convexity is satisfied and agent $i$ updates its location w.r.t to the nodes in this virtual convex hull. We call this virtual as when the update is applied, say at time $k$, the nodes forming this hull may be at arbitrary locations. The entire process with all of the details is explained thoroughly in Section V.

Using the above approach, we show that agent locations are refined as the procedure continues and the algorithm tracks the true agent locations. A somewhat surprising result is that localization

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1In wireless networks, the distance and angles may be estimated with, e.g., Received Signal Strength (RSS), Time of Arrival (ToA), Time Difference of Arrival (TDoA), and/or Angle of Arrival (AoA) measurements. We further argue that if each agent is equipped with a camera and a compass, the distance and angle information can be obtained rather easily.
in mobile networks can be achieved with exactly one anchor when the total number of nodes (agents and anchors) is at least \( m + 2 \) in \( \mathbb{R}^m \). We now describe the rest of the paper. In Section II, we formulate the problem. We introduce the geometric approach to keep track of distances and angles in Section III and use this information towards some simple localization algorithms. We then provide a general algorithm that does not require either local or global convexity in Section IV, with convergence analysis in Section V; this algorithm relies on the notion of virtual convex hull. We discuss the results in Section VI and provide simulation results in Section VII. Finally, Section VIII concludes the paper.

II. Preliminaries and Problem Formulation

Consider a network of \( N \) mobile agents, in the set \( \Omega \), with unknown locations, and \( M \) mobile anchor(s), in the set \( \kappa \), with known locations, all located in \( \mathbb{R}^m, m \geq 1 \); let \( \Theta = \Omega \cup \kappa \) be the set of all nodes. Let \( x^i_k \in \mathbb{R}^m \) be an \( m \)-dimensional row vector that denotes the true location of the \( i \)th node, \( i \in \Theta \), at time \( k \), where \( k \geq 0 \) is the discrete-time index. We assume that each agent is able to compute its distance and angles to the nearby nodes (agents and/or anchors) by using RSSI, ToA, TDoA, or camera-based methods.\(^{38,39}\) The problem is to track the locations of the mobile agents in \( \Omega \). Below, we describe DILOC, which was originally introduced in \(^{23}\).

A. DILOC

DILOC assumes static nodes, i.e., \( x^i_k = x^i, \forall i \in \Theta \), and that each agent satisfies global convexity, i.e., it lies in the convex hull, denoted by \( C(\kappa) \), of the anchors. Using only the inter-node distances, each sensor, say \( i \), finds a triangulation set, \( \Theta_i \), of \( m + 1 \) neighbors that satisfy local convexity, i.e., \( i \in C(\Theta_i), |\Theta_i| = m + 1 \). In any arbitrary dimension, \( m \), a convex hull inclusion test that only uses inter-node distances is

\[
i \in C(\Theta_i), \quad \text{if } \sum_{j \in \Theta_i} A_{\Theta_i \cup \{i\} \setminus j} = A_{\Theta_i},
\]

where \( A_{\Theta_i} \) denotes the \( m \)-dimensional volume, area in \( \mathbb{R}^2 \) or volume in \( \mathbb{R}^3 \), of \( C(\Theta_i) \), see Fig. 1, and can be computed by using the Cayley-Menger determinant,\(^{23,40}\).

Given a triangulation set, \( \Theta_i \), for every \( i \in \Omega \), each sensor updates its location estimate, \( x^i_k \) at time \( k \), as follows:

\[
x^i_{k+1} = \sum_{j \in \Theta_i \cap \Omega} \frac{A_{\Theta_i \cup \{i\} \setminus j}}{A_{\Theta_i}} x^j_k + \sum_{j \in \Theta_i \cap \kappa} \frac{A_{\Theta_i \cup \{i\} \setminus j}}{A_{\Theta_i}} x^j^*.
\]
where the coefficients, \( \frac{A_{\Theta_i \cup \{i\} \setminus j}}{A_{\Theta_i}} \), of the above linear-convex combination are the barycentric coordinates, and are associated to Möbius, \([41]\). Assuming that each sensor successfully finds a triangulation set, and non-trivial configurations, i.e., \( A_\kappa \neq 0, A_{\Theta_i} \neq 0, \forall i \in \Omega \), it is shown in \([23]\) that DILOC converges to the true sensor locations, regardless of the initial conditions, and only requires \( m + 1 \) anchors. Refs. \([23]\)–\([26]\) further characterize the probability of successful triangulation, effect of imperfect communication, and noise on the distance measurements, among many other refinements.

**B. Assumptions**

We are interested in adapting DILOC to mobile agents. Particular challenges with mobile networks are that the agents may not satisfy the global and local convexity at any time. To this aim, we assume a random motion model on the nodes (agents and anchors); the true locations of the nodes evolve as

\[
x_{i,k+1}^* = x_{i,k}^* + \bar{x}_{i,k+1}^* , \quad i \in \Theta ,
\]

where \( \bar{x}_{i,k+1}^* \) is a random motion vector such that each node, \( i \in \Theta \), lies in a bounded region in \( \mathbb{R}^m \). We now enlist our assumptions:

**A0: Anchor locations**, \( x_{i,k}^* , i \in \kappa \), are perfectly known at all time.

**A1: Motion vectors**, \( \bar{x}_{i,k}^* , i \in \Omega \), are perfectly known to each agent, \( i \in \Omega \), at all times.

**A2: Distances and angles** to every node in a radius, \( r \), is known to each node.

All of the above assumptions are not uncommon in the related literature, e.g., see \([32]\), \([35]\). As a motivating scenario, consider a multi-agent network of aerial vehicles where each vehicle is equipped with a camera, wireless communication, a compass, and an accelerometer. Clearly, motion vectors, distances, and angles can be easily estimated at each agent in a certain visual or communication radius. Under the above assumptions, we are interested in tracking the true
locations of each agent without the presence of any central coordinator. Although our approach is applicable to arbitrary dimensions, $m > 2$, we use $\mathbb{R}^2$ in the remainder of the paper for simplicity and ease of notation. In $\mathbb{R}^2$, we may describe the random motion of the $i$th node, $i \in \Theta$, as follows. Let the true location in $\mathbb{R}^2$ be decomposed as $x_k^i = [x_k^i, y_k^i]$, then node $i$’s motion may be captured as:

$$
\begin{align*}
x_{k+1}^i &= x_k^i + d_{k+1}^i \cos(\theta_{k+1}^i), \\
y_{k+1}^i &= y_k^i + d_{k+1}^i \sin(\theta_{k+1}^i),
\end{align*}
$$

(4)

where $\theta_{k+1}^i$ and $d_{k+1}^i$ are the angle and distance traveled by node $i$, between time $k$ and $k+1$, and are random, e.g., uniformly distributed over the following intervals: $[0, 2\pi]$ and $[0, d_{\text{max}}]$, and are chosen such that each node, $i \in \Theta$, does not leave a bounded region of interest. The random motion is assumed to be statistically independent across the nodes.

### III. A Geometric Approach towards Localization

In this section, we present a geometric framework, which is the basis of our approach. Consider agent, $i \in \Omega$, to fall within a distance, $r$, of node, $j \in \Theta$, at some time $k_j$; by $A2$, agent $i$ knows its distance, $d_{k_j}^{ij}$, and angle, $\alpha_{k_j}^{ij}$, to node $j$. Once this information is acquired at time $k_j$, agent $i$ tracks the distance and angle to the true position, $x_{k_j}^{j*}$, for all $k \geq k_j$, even when they move apart. In other words, agent, $i$, has the following information for each node it has communicated with:

$$
\{j, k_j, d_{k_j}^{ij}, \alpha_{k_j}^{ij}, x_{k_j}^j\}, \quad k \geq k_j, j \in \Theta,
$$

(5)

where $k_j$ is the instant of the most-recent contact, $d_{k_j}^{ij}, \alpha_{k_j}^{ij}$, are the distance and angle between the true positions, $x_k^i$ and $x_{k_j}^{j*}$, $k \geq k_j$; and, finally, $x_{k_j}^j$ is the $j$th position estimate obtained at the most recent contact between $i$ and $j$—this information will be relevant later. We now summarize the procedure to track, $d_{k_j}^{ij}, \alpha_{k_j}^{ij}$, in different communication scenarios.

#### A. Distance and Angle tracking with direct communication

We now show how an agent tracks the distance and angle to another node it has directly communicated with, see Fig. 2.

**Lemma 1.** Consider agent, $i \in \Omega$, with true position, $x_{k_j}^i$, to communicate with node, $j \in \Theta$, at time $k_j$. Let $d_{k_j}^{ij}$ and $\alpha_{k_j}^{ij}$ be the distance and angle between $i$ and $j$. Suppose agent $i$ and $j$...
move apart at time $k_j + 1$, where the motion is given by $d_{k_j+1}^i$ and $\theta_{k_j+1}^i$, both known to agent $i$, see Eqs. (4).

The distance between the true positions, $x_{k_j+1}^{i*}$ and $x_{k_j}^{j*}$, is

$$d_{k_j+1}^{ij} = \left( (d_{k_j}^{ij})^2 + (d_{k_j+1}^i)^2 - 2d_{k_j}^{ij} d_{k_j+1}^i \cos(\pi - \theta_{k_j+1}^i + \alpha_{k_j}^{ij}) \right)^{1/2}, \quad (6)$$

whereas the corresponding angle is

$$\alpha_{k_j+1}^{ij} = \alpha_{k_j}^{ij} \pm \cos^{-1} \left( \frac{(d_{k_j}^{ij})^2 + (d_{k_j+1}^i)^2 - (d_{k_j+1}^{ij})^2}{2d_{k_j}^{ij} d_{k_j+1}^{ij}} \right). \quad (7)$$

Proof. Let us start with the distance update. Consider the triangle, with vertices at $x_{k_j}^{i*}$, $x_{k_j+1}^{i*}$ and $x_{k_j}^{j*}$. To find the current distance of agent $i$, from the position of agent $j$ at time $k_j$, we can use the law of cosines, which connects the length of an unknown side of a triangle to the lengths of the two other sides and the angle opposite to the unknown side. This triangle is depicted in Fig. 2 (Right), in which the two known side lengths are $d_{k_j}^{ij}$ and $d_{k_j+1}^i$. Thus, by finding the angle between these sides of the triangle, the length of the third side can be determined according to the following

$$(d_{k_j+1}^{ij})^2 = (d_{k_j}^{ij})^2 + (d_{k_j+1}^i)^2 - 2d_{k_j}^{ij} d_{k_j+1}^i \cos \angle(d_{k_j}^{ij}, d_{k_j+1}^i), \quad (8)$$

in which $\angle(d_{k_j}^{ij}, d_{k_j+1}^i)$ indicates the angle between the two sides of triangle with the lengths of $d_{k_j}^{ij}$ and $d_{k_j+1}^i$. It is straightforward to verify that

$$\angle(d_{k_j}^{ij}, d_{k_j+1}^i) = \pi - \theta_{k_j+1}^i + \alpha_{k_j}^{ij}, \quad (9)$$

which combined with Eq. (8), leads to Eq. (6).
On the other hand, when the lengths of all three sides of a triangle are known, each angle can be also computed. Let us denote the angle between the two sides of the triangle with the lengths $d_{k_j}^{i,j}$ and $d_{k_j+1}^{i,j}$ by $\alpha$. We can write

$$\alpha = \cos^{-1}\left(\frac{(d_{k_j}^{i,j})^2 + (d_{k_j+1}^{i,j})^2 - (d_{k_j}^{i})^2)}{2d_{k_j}^{i,j}d_{k_j+1}^{i,j}}\right),$$

which is the angle opposite to the side of the triangle with the length of $d_{k_j}^{i}$. It can be verified that

$$\alpha_{k_j+1}^{ij} = \begin{cases} 
\alpha_{k_j}^{ij} + \alpha, & \theta_{k_j+1}^{ij} \leq \alpha_{k_j}^{ij}, \\
\alpha_{k_j}^{ij} - \alpha, & \theta_{k_j+1}^{ij} > \alpha_{k_j}^{ij},
\end{cases}$$

which leads to Eq. (7), and completes the proof. \(\square\)

We now show how agent $i$ can use the distance/angle information to find the distances between any pair of agents it has previously visited.

**Lemma 2.** Suppose agent $i$ has previously visited two other nodes, $j$, and $\ell$, hence possesses the following information:

$$\{j, k_j, d_{k}^{ij}, \alpha_{k_j}^{ij}, x_{k_j}^{j}\}, \quad k \geq k_j,$$

$$\{l, k_\ell, d_{k}^{i\ell}, \alpha_{k_\ell}^{i\ell}, x_{k_\ell}^{\ell}\}, \quad k \geq k_\ell.$$

Agent $i$ uses this information at any time, $k \geq \max\{k_j, k_\ell\}$, to find the distance between $x_{k_j}^{i\ast}$ and $x_{k_\ell}^{\ell\ast}$ as follows:

$$\tilde{d}_{k}^{j\ell} = \sqrt{d_{k}^{ij}^2 + d_{k}^{i\ell}^2 - 2d_{k}^{ij}d_{k}^{i\ell}\cos(\tilde{\beta}_{k}^{j\ell})},$$

in which $\tilde{\beta}_{k}^{j\ell}$ is the angle between the two lines connecting $x_{k_j}^{i\ast}$ to $x_{k_j}^{j\ast}$ and $x_{k_\ell}^{\ell\ast}$, and is given by

$$\tilde{\beta}_{k}^{j\ell} = 2\pi - (\alpha_{k_j}^{ij} + \alpha_{k_\ell}^{i\ell}).$$

**Proof.** As illustrated in Fig. 3(Left), locations of agent $i$ at time $k$, agent $j$ at time $k_j$, and agent $\ell$ at time $k_\ell$ form the three vertices of a triangle with two known ($d_{k}^{ij}, d_{k}^{i\ell}$) and one unknown ($\tilde{d}_{k}^{j\ell}$) sides. Assuming that the angle between the sides with known lengths is given as $\tilde{\beta}_{k}^{j\ell}$, we can simply use the law of cosines to find the other side’s length, which leads directly to Eq. (12). Since agent $i$ knows its angle to both agents, $j$ and $\ell$, at time $k$, it can find $\tilde{\beta}_{k}^{j\ell}$ as follows. Consider the horizontal line at the position of agent $i$ in Fig. 3(Left). This line divides $\tilde{\beta}_{k}^{j\ell}$
into two acute angles, which can be easily verified to be equal to \( \pi - \alpha_{ij}^k \) (the upper angle), and \( \pi - \alpha_{i\ell}^k \) (the lower angle), and Eq. (13) follows.

The procedure described in the above Lemmas 1 and 2 leads to a simple localization scheme as described below. To avoid nonlinearity in the solution\(^2\) and in order to build towards DILOC-type linear iterations, we consider linear strategies to track the locations of mobile agents. To this aim, let agent, \( i \in \Omega \), move randomly, see Eqs. (4), in a bounded region of interest with \( m + 1 = 3 \) anchors in \( \mathbb{R}^2 \). Let agent, \( i \), maintain a flag

\[
f(k) = \{f_1, f_2, f_3\}, \quad f_{j \in \{1, 2, 3\}} \in \{0, 1\},
\]

which indicates whether or not it has visited each of the anchors up to time \( k \). The flag is initialized at \( \{0, 0, 0\} \), and the agent sets \( f_{j \in \{1, 2, 3\}} \) to one when it finds the \( j \)th anchor within a radius, \( r \), and subsequently acquires the angle and distance to the \( j \)th anchor; this information is continually updated by Lemma 1. As agent, \( i \), continues to move randomly, it visits each of the three anchors with probability 1 over time, i.e., \( f(\bar{k}) = \{1, 1, 1\} \) for some \( \bar{k} \geq 0 \), see Fig. 4 (Left). At this point agent, \( i \) possesses its current distance to each of the three anchors: \( d_{i1}^{\bar{k}}, d_{i2}^{\bar{k}}, d_{i3}^{\bar{k}} \), and angles: \( \beta_{i1}^{\bar{k}}, \beta_{i2}^{\bar{k}}, \beta_{i3}^{\bar{k}} \), see Fig. 4 (Left). With the help of Lemma 2, agent \( i \) thus also knows the inter-anchor distances: \( d_{12}, d_{23}, d_{31} \).

Recall that we are interested in implementing a linear update, i.e., Eq. (2) in Section II-A. To implement this update, agent \( i \) computes areas of the four corresponding triangles: \( A_{123}, A_{i12}, A_{i23}, A_{i31}, \)

\(^2\)When only distances to at least \( m + 1 \) anchors are known, trilateration, e.g., in \( \mathbb{R}^2 \), requires three anchors and solves three circle equations. Clearly, trilateration is nonlinear and coupled in the coordinates; while an iterative procedure built on these nonlinear updates does not converge in general.
e.g., using the Cayley-Menger determinant \cite{23}, \cite{40}, and checks for the convexity condition, i.e., Eq. (1). The procedure above leads to the following localization algorithm, see also Fig. 4 (Right).

![Localization Algorithm Diagram](image)

**Proposition 1.** Consider an agent with random motion model according to Eqs. (4). The agent can find its exact location **linearly** at time $k$, if it

(i) visits each anchor at least once prior to time $k$; and,

(ii) lies inside the convex hull of the anchors at time $k$.

The proposition can be easily verified. In order to implement a linear update, each agent must satisfy global convexity. Hence, when the flag becomes $\{1, 1, 1\}$, each agent keeps implements the update only when it moves inside the anchor convex hull. Note that Prop. 1 requires each agent to directly communicate with each anchor. In the following, we consider the case when this direct communication is not possible.

**B. Distance and Angle tracking with indirect communication**

We now show how an agent finds its distance and angle to another node without direct communication.

**Lemma 3.** Suppose agent $i$ communicates with agent $j$ at time $k$; by A2 it thus knows $d_i^{ij}$ and $\alpha_i^{ij}$. Suppose agent $j$ already knows its distance and angle, $d_j^{i\ell}$ and $\alpha_j^{i\ell}$, to node $\ell$. Agent $i$
computes the distance and angle to node $\ell$ as follows:

$$d_{ik}^\ell = \sqrt{(d_{ik}^j)^2 + (d_{ik}^\ell)^2 - 2d_{ik}^j d_{ik}^\ell \cos(\alpha_{ik}^j + \alpha_{ik}^\ell)},$$

$$\alpha_{ik}^\ell = \alpha_{ik}^j + \cos^{-1}\left(\frac{(d_{ik}^\ell)^2 + (d_{ik}^j)^2 - (d_{ik}^j)^2}{2d_{ik}^\ell d_{ik}^j}\right).$$

Proof. This procedure is detailed in Fig. 3 (Right). Consider the triangle whose vertices are the location of agents $i$, $j$, and $\ell$, at time $k$. The length of two sides of this triangle is known to agent $i$; $d_{ik}^j$ via direct communication with agent $j$, and $d_{ik}^\ell$ is communicated to agent $i$ by agent $j$. For agent $i$ to find the unknown length, $d_{ik}^\ell$, the distance between agent $i$ and agent $\ell$, it needs to find the angle, $\beta$. Since agent $i$ knows its angle, $\alpha_{ik}^j$, to agent $j$ and can find (via communication with agent $j$) the angle, $\alpha_{ik}^\ell$, between agents $j$ and $\ell$, it can subsequently find $\beta$ (similar to how we derived $\tilde{\beta}_{ik}^\ell$ in Eq. (13)) as

$$\beta = \alpha_{ik}^j + \alpha_{ik}^\ell. \quad (15)$$

In addition, in order to find its angle, $\alpha_{ik}^\ell$, to agent $\ell$, agent $i$ requires the angle opposite to $d_{ik}^j$. Since the lengths of all sides of the triangle are now known to agent $i$, this angle is

$$\cos^{-1}\left(\frac{(d_{ik}^\ell)^2 + (d_{ik}^j)^2 - (d_{ik}^j)^2}{2d_{ik}^\ell d_{ik}^j}\right)$$

and Eq. (15) follows.

From the above lemma, note that up to time $k$, agent $i$ may never have communicated directly with the agent $\ell$, i.e., $r < d_{ik}^\ell$, $\forall k$. However, agent $i$ knows the distance and angle to agent $\ell$ via another node, $j$, that communicated with agent $\ell$; from this time on, agent $i$ further possesses the following set: $\{l, k_j, d_{kj}^\ell, \alpha_{kj}^l, x_{k_j}^l\}$, $k \geq k_j \geq k_\ell$, where $k_j$ is when $i \leftrightarrow j$ and $k_\ell$ is when $j \leftrightarrow \ell$. We now provide a localization algorithm that is based on Lemma 3.

**Proposition 2.** Consider a network of $N$ mobile agents moving randomly and $m + 1$ anchors arbitrarily placed. Each agent finds its exact location linearly at time $k$, if it

(i) finds the angles and distances to each anchors prior to time $k$, by either direct or indirect communication; and,

(ii) lies inside the convex hull of the anchors at time $k$.

The proposition can be easily verified. Fig. 5 shows a network with $N = 3$ agents, 3 anchors, and random agent trajectories. In Fig. 5 (Left), agents are free to explore the entire region. In
Fig. 5 (Right), we restrict each mobile agent to a region such that each agent can communicate with at most one anchor. Thus, each agent has to communicate with other mobile agents to find the angles and distances to out-of-reach anchors. We note that Props. 1 and 2 require each agent to (directly or indirectly) update with the anchor information and eventually satisfy the global convexity. In the next section, we describe the main contribution of this paper, i.e., *when the local and global convexity may never be satisfied.*

![Networks of 3 mobile agents and 3 anchors (indicated by red triangles): (Left) Agents are free to explore the whole region; (Right) Agents have disjoint exploration areas.](image)

**Fig. 5.** Networks of 3 mobile agents and 3 anchors (indicated by red triangles): (Left) Agents are free to explore the whole region; (Right) Agents have disjoint exploration areas.

### IV. DISTRIBUTED MOBILE LOCALIZATION: ALGORITHM

We now consider a network of $N$ agents with unknown locations and $M$ anchors, according to the motion model introduced in Section II. Let $\mathcal{V}_i(k) \subseteq \Theta$ be the *i-visited set*, defined as the set of distinct nodes visited by agent, $i \in \Omega$, up to time $k$; and call an element in this set as *i-visited node*. We start by introducing the notion of a virtual convex hull.

#### A. Virtual convex hull

Suppose agent $i$ communicates with node $j$ at time $k_j$, and obtains the distance, $d_{kj}^{ij}$, and angle, $\phi_{kj}^{ij}$, to $j$, along with $j$’s current location estimate, $x_{kj}^i$, i.e., $j \in \mathcal{V}_i(k_j)$. At any time, $k > k_j$, agents, $i$ and $j$, may move apart but agent $i$ now knows $d_{k}^{ij}$, $\phi_k^{ij}$, $\forall k \geq k_j$, using the geometric framework discussed in Section III. At some later time, $k_\ell > k_j$, agent $i$ establishes a contact with another node, $\ell$, and thus obtains $x_{k\ell}^\ell$, and keeps track of $d_{k\ell}^{i\ell}$, $\phi_k^{i\ell}$, $\forall k \geq k_\ell$, thus $j, \ell \in \mathcal{V}_i(k), \forall k \geq k_\ell$. Finally, agent $i$ meets agent $n$ at some $k_n > k_\ell$, and thus now possesses the *past* location estimates: $x_{k_j}^j, x_{k_\ell}^\ell, x_{k_n}^n$; and the *current* distances and angles: $d_{k}^{jn}, \phi_k^{jn}, k \geq k_n$, with $q = j, \ell, n$. At this point $j, \ell, n \in \mathcal{V}_i(k), \forall k \geq k_n$, and agent $i$ can use the angle/distance
information (and related results from Section III) to test for local convexity. If local convexity is satisfied, the set, \( \Theta_i(k) \triangleq \{ j, \ell, n \} \subseteq V_i(k) \), forms the virtual convex hull; otherwise, agent \( i \) continues to add nodes in \( V_i(k) \) and some combination passes the local convexity.

Fig. 6 illustrates this process in four frames. Fig. 6 (a) shows the trajectories of four agents: \( \odot, \square, \odot, \odot \), over \( k = 1, \ldots, 9 \); the time-indices are marked inside the agent symbols. From the perspective of agent \( \odot \), see Fig. 6 (b): it first makes contact (communicates) with agent \( \square \), at time \( k_{\square} = 2 \), and then they both move apart; next, it makes contact with agents, \( \odot \) at \( k_{\odot} = 4 \), and \( \odot \) at \( k_{\odot} = 6 \). We have \( \mathcal{V}_\odot(2) = \{ \square \} \), \( \mathcal{V}_\odot(4) = \{ \square, \odot \} \), and \( \mathcal{V}_\odot(6) = \{ \square, \odot, \odot \} \), where a non-trivial convex hull becomes available at \( k = 6 \). However, agent \( \odot \) does not lie in the corresponding convex hull, \( C(\mathcal{V}_\odot(6)) \), and cannot update its location estimate with the past neighboring estimates: \( x_{k_{\square}}, x_{k_{\odot}}, x_{k_{\odot}} \). At this point, agent \( \odot \) must wait until it either moves inside the convex hull of \( \square, \odot, \odot \), or finds another agent with which the local convexity is satisfied. The former is shown in Fig. 6 (d), where agent \( \odot \) has moved inside \( C(\mathcal{V}_\odot(6)) \) at some later time, \( k = 9 \); we have \( \Theta_\odot(9) = \{ \square, \odot, \odot \} \). The notion of a virtual convex hull is evident from this discussion: an agent may only communicate with at most one agent at any given time; when the local convexity is satisfied eventually, the updating agent may not be in communication with the corresponding nodes. Once \( \Theta_\odot(k) \) is successfully formed, agent \( \odot \) updates its location linearly using the barycentric representation in Eq. (2), where the coefficients are computed using the distance equations in Lemmas 1, 2, and 3 and the Cayley-Menger determinant. After this update, agent \( \odot \) removes \( \Theta_\odot(k) \) from \( V_i(\odot) \), as the location estimates of the nodes in \( \Theta_\odot(k) \) are utilized\(^3\).

The following result will be useful in the sequel.

**Lemma 4.** For each \( i \in \Omega \), there exists a set, \( \Theta_i(k) \subseteq V_i(k) \), such that \( |\Theta_i(k)| = m + 1 \) and \( i \in C(\Theta_i(k)) \), for infinitely many \( k \)'s.

**Proof.** Since the agents move inside a bounded region and their motion is random, there is a non-zero probability that agent \( i \) (eventually) visits some other node. Thus,

\[
\lim_{k \to \infty} \mathcal{V}_i(k) = \Theta \setminus \{ i \}, \quad \forall i \in \Omega.
\]  

\(^3\)We choose this simple strategy to remove information from \( V_i(k) \) for convenience. Another candidate strategy is to use a forgetting factor, which chooses the past used nodes less frequently.
As an agent, $i$, visits at least $m+1$ nodes in distinct locations, it is able to perform the (convex-hull) inclusion tests. Again, there is a non-zero probability for agent $i$ to lie inside the convex hull formed by the first $m+1$ visited nodes. Now, more nodes are added to the set $V_i(k)$ as $k$ increases, and agent $i$ is more likely to find a virtual convex hull.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{virtual_convex_hull.png}
\caption{Virtual convex hull with four agents: $\bigcirc$, $\square$, $\triangle$, w.r.t agent $\bigcirc$: (a) Agent trajectories and time-indices; (b) $\bigcirc \leftrightarrow \square$ at $k = 4$, $\bigcirc \leftrightarrow \bigcirc$ at $k = 4$, $\bigcirc \leftrightarrow \bigcirc$ at $k = 6$; (c) Virtual convex hull of agents, $\square$, $\bigcirc$, $\triangle$, available at agent $\bigcirc$ at $k = 6$; (d) Trajectories at $k > 6$, test passed at $k = 9$.}
\end{figure}

\section{Algorithm}
Consider a network of $N$ agents and $M$ anchors in $m$-dimensional Euclidean space, $\mathbb{R}^m$. Each agent starts with a random guess of its locations estimate. When an agent finds a nearby node, it acquires the neighbor’s location estimate and measures its distance and angle to that node. At each subsequent iteration, the $i$th agent updates the angles and distances to all of the $i$-visited nodes, following the procedure described in Section III. However, the location estimate are only updated when two nodes communicate, i.e., the distance between them becomes less than $r$. We now describe the localization algorithm in this case according to the number, $|V_i(k)|$, of $i$-visited nodes in the $i$-visited set, $V_i(k)$. There are two different update scenarios for any arbitrary agent, $i$:

\begin{enumerate}
    \item $0 \leq |V_i(k)| < m + 1$: Agent, $i$, does not update its current location estimate.
\end{enumerate}
(ii) $|V_i(k)| \geq m + 1$: Agent, $i$, performs the inclusion test; if the test is passed the location update is applied.

Using the above, consider the following update:

$$ x_{k+1}^i = \alpha_k x_k^i + (1 - \alpha_k) \sum_{j \in \Theta_i(k)} a_{ij}(k)x_k^j + x_{k+1}^i, $$

where $x_k^i$ is the vector of the $i$th agent’s coordinates at time $k$, $x_{k+1}^i$ captures the motion, $a_{ij}(k)$ is the barycentric coordinate of node $i$ with respect to the nodes $j \in \Theta_i(k)$, and $\alpha_k$ is a design parameter such that

$$ \alpha_k = \begin{cases} 1, & \forall k \mid \Theta_i(k) = \emptyset, \\ \beta, & \forall k \mid \Theta_i(k) \neq \emptyset, \end{cases} $$

and $\Theta_i(k)$ is a virtual convex hull. Note that $\Theta_i(k) = \emptyset$ does not necessarily imply that agent $i$ has no neighbors at time $k$, but only that no set of neighbors meet the (virtual) convexity. Eq. (18) immediately implies that the self-weight at each agent is always lower bounded, i.e.,

$$ 0 < \beta \leq p_{ii}(k) \leq 1, \forall i, k. $$

We can rewrite Eq. (17) as

$$ x_{k+1}^i = \alpha_k x_k^i + (1 - \alpha_k) \left( \sum_{j \in \Theta_i(k) \cap \Omega} p_{ij}(k)x_k^j \right), $$

$$ + (1 - \alpha_k) \left( \sum_{m \in \Theta_i(k) \cap \kappa} b_{im}(k)u_k^m \right) + x_{k+1}^i, $$

where

$$ a_{ij} = \begin{cases} p_{ij}, & \text{if } j \in \Theta_i(k) \cap \Omega, \\ b_{im}, & \text{if } m \in \Theta_i(k) \cap \kappa. \end{cases} $$

The above algorithm can be written in matrix form as

$$ x_{k+1} = P_k x_k + B_k u_k + x_{k+1}, \quad k > 0, $$

where $x_k$ is the vector of agent coordinates evaluated at time $k$, and $x_{k+1}$ is the change in the location of agents at the beginning of the $k$th iteration according to the motion model. Also $P_k$, and $B_k$, contain the barycentric coordinates with respect to the agents with unknown locations, and anchors, respectively. We refer to $P_k$ as system matrix of the above LTV system and $B_k$.
as the input matrix at time $k$. Since accurate information is only injected via the anchors, it is reasonable to set a lower bound on the weights assigned to the anchor states. In particular, we make the following assumption.

**A3: Anchor contribution.** For any update that involves an anchor, i.e., for any $b_{im}(k) \neq 0$, we assume that

$$0 < \alpha \leq b_{im}(k), \quad \forall i \in \Omega, m \in \Theta_i(k) \cap \kappa, k.$$  

(23)

Equivalently, if agent $i$ updates with an anchor, we have

$$0 < \alpha \leq (B_k)_{ij}, \quad \forall j \in \Theta_i(k) \cap \kappa,$$

(24)

where $(B_k)_{ij}$ is the $(i,j)$th element of the matrix, $B_k$.

With the lower bounds on both the self-weights, Eq. (19), and the weights assigned to the anchors, Eq. (24), we note that at time $k$, the matrix of barycentric coordinates w.r.t. agents with unknown locations, i.e., the system matrix $P_k$, is either

(i) *identity*, when no update occurs; or,

(ii) *identity except a stochastic $i$th row*, when there is no anchor in the virtual convex hull, i.e., $\Theta_i(k) \cap \kappa = \emptyset$; or,

(iii) *identity except a sub-stochastic $i$th row*, when there is at least one anchor in the virtual convex hull.

In the next section, we provide sufficient conditions under which the iterative localization algorithm, Eq. (22), tracks the true locations agent locations.

**V. Distributed Mobile Localization: Analysis**

Before we proceed, we borrow the following result on the asymptotic stability of LTV systems from [42].

**A. Asymptotic stability of stochastic LTV systems**

Consider an LTV system: $x_{k+1} = P_k x_k$ such that the system matrix, $P_k$, is time-varying and random. The system matrix, $P_k$, represents at most one state update, say the $i$th state, for any $k$, i.e., at most one row, $i$, of $P_k$ is different from identity. In addition, assume the following on the update:
**B0:** If the updating row, \( i \), in \( P_k \) sums to 1, then

\[
0 < \beta_1 \leq (P_k)_{i,i}, \quad \beta_1 \in \mathbb{R},
\]

(25)

**B1:** If the updating row, \( i \), in \( P_k \) does not sum to 1, then

\[
\sum_j (P_k)_{i,j} \leq \beta_2 < 1, \quad \beta_2 \in \mathbb{R}.
\]

(26)

To analyze the asymptotic behavior of an LTV system with such system matrices, we introduce the notion of a slice, \( M_j \), which is the smallest product of consecutive system matrices,

\[
M_t = \prod_k P_k,
\]

such that the entire chain of systems matrices is covered by non-overlapping slices, i.e.,

\[
\prod_t M_t = \prod_k P_k,
\]

and each slice has a subunit infinity norm (max. row sum), i.e., \( \|M_t\|_\infty < 1, \forall t \). Slices are initiated by sub-stochastic system matrices, and terminated after all row sums are strictly less than one. Ref. [42] also shows that the upper bound on the infinity norm of a slice is further related to the length of the slice, i.e., the number of matrices forming the slice. The following theorem characterizes the asymptotic stability of the above LTV system.

**Theorem 1.** With assumptions, B0-B1, the system \( x_{k+1} = P_k x_k \), is absolutely asymptotically stable, i.e.,

\[
\lim_{k \to \infty} x_k = 0_N,
\]

if either one of the following is true:

(i) Each slices has a bounded length, i.e.,

\[
|M_j| \leq N < \infty, \quad \forall j, \quad N \in \mathbb{N};
\]

(27)

(ii) There exist a set, \( J_1 \), consisting of an infinite number of slices such that

\[
|M_j| \leq N_1 < \infty, \quad \forall M_j \in J_1,
\]

(28)

\[
|M_j| < \infty, \quad \forall M_j \notin J_1;
\]

(29)

(iii) There exists a set, \( J_2 \), of slices such that

\[
\exists M_j \in J_2 : \quad |M_j| \leq \frac{1}{\ln(\beta_1)} \ln \left( \frac{1 - e^{-\gamma_2 i - \gamma_1}}{1 - \beta_2} \right) + 1,
\]

(30)
for every $i \in \mathbb{N}$, and $|M_j| < \infty$, $j \not\in J$.

The proof is available in our prior work, [42]. Here we explain the intuition behind the above theorem.

Case (i): When each slice has a bounded length, its infinity norm is also bounded with a positive number strictly less than one. Therefore, the infinity norm of the infinite product of slices converges to zero.

Case (ii) The entire chain of slices can be partitioned into two sets: one including an infinite number of slices with bounded length and thus with a subunit infinity norm; the other including the remaining slices with finite but unbounded length. Analogously to the previous case, the product of slices in the first set converges to zero, which in turn leads to absolute asymptotic stability.

Case (iii) The asymptotic stability of the system is guaranteed even in the non-trivial case, where there exist an infinite subset of slices whose lengths are not bounded, but do not grow faster than the exponential growth in Eq. (30). As detailed in [42], if the slices are such that there exist a slice with length following Eq. (30) for every $i \in \mathbb{N}$, not necessarily in any order, the infinite product of slices goes to a zero matrix, and the system is absolutely asymptotically stable.

With the help of this theorem, we now analyze the convergence of Eq. (22) in the following section.

B. Convergence Analysis

We start with the following lemma.

**Lemma 5.** Under the conditions A0-A3, the product of system matrices, $P_k$’s, in the LTV system, Eq. (22), converges to zero.

**Proof.** First, we need to ensure Assumptions B0-B1. We now show that Eq. (22) follow these assumptions. First, note that B0 is immediately verified by Eq. (19), assuming $\beta_1 = \beta$. To ensure B1, we note that it is implied by A3. This is because if $\Theta_i \cap \kappa$ is not empty, we can write

$$\sum_{j \in \Theta_i(k) \cap \Omega} p_{ij}(k) = 1 - \sum_{m \in \Theta_i(k) \cap \kappa} b_{im}(k), \quad (31)$$

where we used the fact that barycentric coordinates sum to one due to the convexity. When there is only one anchor among the neighbors of the updating node, and the minimum weight is
assigned to this anchor, Eq. (23), the R.H.S. of Eq. (31) is maximized. This provides an upper bound on the $i$th row sum:

$$\sum_{j \in \Theta_i(k) \cap \Omega} p_{ij}(k) \leq 1 - \alpha < 1,$$

which ensures $B_1$ with $\beta_2 = 1 - \alpha$. In addition, Lemma 4 ensures that each node updates infinitely often with different neighbors. Subsequently, each slice is completed after all nodes receive anchor information (at least once) either directly or indirectly, and the asymptotic convergence of Eq. (22) follows under the conditions in Theorem 1.

We will explain the consequences of the A3, B0-B1 in the next section. The following theorem completes the localization algorithm for mobile multi-agent networks.

**Theorem 2.** Under the assumptions A0-A3, Eq. (22) asymptotically tracks the true agent locations.

**Proof.** In order to show the convergence to the true locations, we show that the error between the location estimate, $x_k$, and the true location, $x^*_k$, goes to zero. To find the error dynamics, note that the true agent locations follow:

$$x^*_k = P_k x^*_k + B_k u_k + x_{k+1}.$$

Subtracting Eq. (22) from Eq. (33), we get the network error

$$e_{k+1} \triangleq x^*_k - x_{k+1} = P_k (x^*_k - x_k) = P_k e_k,$$

which goes to zero when

$$\lim_{k \to \infty} \prod_{l=0}^{k} P_l = 0_{N \times N},$$

which follow from Lemma 5.

The following theorem characterizes the number of anchors.

**Theorem 3.** Under Assumptions A0-A3, Eq. (22) tracks the true agent locations in $\mathbb{R}^m$, when the number of agents and anchors follows:

$$M \geq 1,$$

$$N + M \geq m + 2.$$
**Proof.** Let's first consider the requirement of at least one anchor. Without an anchor, a sub-stochastic row never appears in $P_k$; with at least one anchor, sub-stochastic rows, following Eq. (32), appear in $P_k$’s, and absolute asymptotic stability of the error dynamics follows. Next, exactly $m + 1$ nodes (agents and/or anchors) are required to form a (virtual) convex hull in $\mathbb{R}^m$. Thus, when the total number of nodes, $N + M$, is $m + 1$ or less, no agent can find $m + 1$ other nodes to test for (virtual) convex hulls. With at least one anchor and at least $m + 2$ total nodes, any agent with unknown location infinitely finds itself in arbitrary (virtual) convex hulls, from Lemma 4. Thus, Theorem 1 is applicable and the proof follows.

**VI. DISCUSSION**

In this section, we provide some relevant remarks:

(i) In static DILOC, see Section II-A and almost every other localization algorithm, $m + 1$ anchors are required to localize an agent with unknown location in $\mathbb{R}^m$ without ambiguity. However, when the nodes are mobile, *exactly one anchor* is sufficient to inject valuable information and the motion provides the remaining degrees of freedom.

(ii) In Theorem 1, slice representation is closely related to information flow in the network. Each slice is initiated with a sub-stochastic update, i.e., when one agent with unknown location *directly* receives information from an anchor by having this anchor in its virtual convex hull. On the other hand, a slice is terminated after the information from the anchor(s) is propagated through the network and reaches every agent either directly or indirectly. Here, *directly* means that an agent has an anchor in its virtual convex hull; while *indirectly* means that an agent has a neighbor in its convex hull, which receives direct or indirect information from the anchor. Once the anchor information reaches every agent in the network, the slice notion and Theorem 1 provide the conditions on the rate, Eq. (30), at which this information should propagate for the absolute asymptotic stability of the error.

(iii) Assumption B0 and Eq. (19) imply that each agent remembers its past information. If a lower bound on the self-weights is not assigned, an agent may lose valuable information when it updates with other agents that have not previously communicated (directly or indirectly) with an anchor.

(iv) Assumption A3 implies that if there is an anchor in the (virtual) convex hull, a certain amount of information is always contributed to the anchor. In other words, the updating agent has to lie in an *appropriate* position inside the convex hull. Assumption A3 states that the
weight assigned to the anchor (which comes from barycentric coordinates) should be at least $\alpha$. Therefore, the area (in $\mathbb{R}^2$) of the triangle corresponding to the anchor must take an adequate portion of the area of the whole convex hull. This is illustrated in Fig. 7 where the updating agent lies in a virtual convex hull, consisting of an anchor (node $j$), and two other agents. Node $i$ communicated with the anchor and the two other agents (node $m$ and node $\ell$), at time instants, $k_j$, $k_m$, and $k_\ell$, respectively.

Fig. 7. (Left) Agent $i$ is located on the threshold boundaries, which assigns the minimum weight, $\alpha$, to the anchor. (Right) Agent $i$ is located in an inappropriate location inside the convex hull.

Fig. 7 (Left) illustrates the position of agent $i$ at time $k$, $x^*_k$, and the threshold boundaries, such that

$$b_{ij}(k) = \frac{A_{\Theta_i(k) \cup \{i\} \setminus j}}{A_{\Theta_i(k)}} = \alpha.$$  

Eq. (38)

Fig. 7 (Right) on the other hand shows that if the agent lies in $\bar{x}^*_k$ (or any other position within the triangle of $x^*_k$, $x^*_l$, and $x^*_m$), the left hand side of Eq. (38) is less than $\alpha$, and Eq. (23) does not hold. Since the corresponding update does not provide enough valuable information for agent $i$, no update occurs in this case.

(v) When there is no anchor in the network, all updates are stochastic, and we get $e_{k+1} = P_k e_k$, with $\rho(P_k) = 1$, $\forall k$, i.e., a neutrally-stable system, which leads to a bounded steady-state error in the location estimates, see Fig. 8. In other words, relative locations are tracked because the motion removes certain ambiguity from the locations. However, absolute tracking without any ambiguity requires at least one anchor.

VII. Simulations

In this section, we provide simulation results for the localization algorithm introduced in Section IV. In the beginning, all nodes (agents and anchors) are randomly deployed within
the region of \( x \in [-5, 15], \ y \in [-5, 15] \) in \( \mathbb{R}^2 \). All mobile agents follow the random motion described in Section II, where angle and distance distributions are uniform over the intervals of \([0, 2\pi]\) and \([0, 10]\), respectively. Each agent can only communicate with the nodes within its communication radius, which is set to \( r = 2 \) for all simulations. Since the motion is random, agents may not be able to communicate with any anchor at a given time. On the other hand, they may never lie inside the convex hull of \((m+1=3)\) neighboring nodes. If an agent does not find enough neighbors at a given time, or does not lie inside the convex hull of the neighbors, it begins to search for a virtual convex hull of the nodes it has previously visited. If an agent fails to find a virtual convex hull, it does not perform any update. However, if a virtual convex hull passes the inclusion test, Eq. (1), the agent updates its location estimate according to Eq. (20). The updating agent then removes all prior information corresponding to the nodes in the virtual hull and continues the motion. In all simulations, we set \( \alpha_k = 0.2 \) to ensure that the agents do not completely forget the past information, and \( \alpha = 0.1 \) to guarantee an adequate contribution from the anchor(s) when they are involved in an update.

Fig. 9 (Left) shows the random trajectories of \( N = 3 \) mobile agents for the first 25 iterations. There are \( M = 3 \) anchors with fixed positions in the network. We choose the 2-norm of the error vector, \( e_k \), to characterize the convergence, i.e., the algorithm converges when \( \|e_k\|_2 \to 0 \). As shown in Fig. 9 (Right), the localization algorithm, Eq. (22), tracks the true agent locations. Motion and convergence of \( N = 3 \) mobile agents in a network of size \( N + M = 4 \) (i.e., with \( M = 1 \) anchor) is illustrated in Fig. 10 for comparison. As expected, the rate of convergence decreases when there is only one anchor in the network. Finally, Fig. 11 (Left) shows the convergence of a network with \( N + M = 103 \) nodes, \( N = 100 \) mobile agents and \( M = 3 \)
Fig. 9. A network of 3 mobile agents and 3 fixed anchors; Red triangles indicate anchors, and blue circles show the initial position of agents. (Left) Motion model; (Right) Convergence.

Fig. 10. A network of 3 mobile agents and 1 fixed anchors. (Left) Motion model; (Right) Convergence.

anchors; whereas Fig. 11 (Right) shows the convergence of a network with $N + M = 101$ nodes, $N = 100$ mobile agents and $M = 1$ anchor.

Fig. 11. (Left) Convergence of a network of 100 mobile agents and 3 anchors; (Right) Convergence of a network of 100 mobile agents and 1 anchor.
VIII. Conclusions

In this paper, we provide two algorithms to localize a network of mobile agents moving randomly in a bounded region of interest. Assuming that each agent knows its distance and angle to the nodes in its communication radius, we provide a geometric framework, which allows the agents to keep track of the distances and angles to any previously visited node. Our first linear algorithm supersedes the traditional nonlinear localization approach, and requires at least $m + 1$ anchors to locate an arbitrary number of mobile agents in $\mathbb{R}^m$. In this method, agents are required to eventually lie inside the convex hull of the anchors to find their exact locations. To avoid this issue, we introduce the notion of a virtual convex hull, which forms the basis of a more general localization algorithm in mobile networks. We further show exactly one anchor suffices to localize an arbitrary number of mobile agents when each agent is able to find appropriate (virtual) convex hulls.

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