Abstract

Koornwinder polynomials are \(q\)-orthogonal polynomials equipped with extra five parameters and the \(BC_n\)-type Weyl group symmetry, which were introduced by Koornwinder (1992) as multivariate analogue of Askey-Wilson polynomials. They are now understood as the Macdonald polynomials associated to the affine root system of type \((C_\vee^n, C_n)\) via the Macdonald-Cherednik theory of double affine Hecke algebras. In this paper we give explicit formulas of Littlewood-Richardson coefficients for Koornwinder polynomials, i.e., the structure constants of the product as invariant polynomials. Our formulas are natural \((C_\vee^n, C_n)\)-analogue of Yip’s alcove-walk formulas (2012) which were given in the case of reduced affine root systems.

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## 1 Introduction

### 1.1 Koornwinder polynomials

Askey-Wilson polynomials [AW85] are \(q\)-orthogonal polynomials of one variable equipped with extra parameters \((a, b, c, d)\), which recover various \(q\)-analogues of Jacobi polynomials by specialization of the parameters. In [K92], Koornwinder introduced \(n\)-variable analogue of Askey-Wilson polynomials, which are today called Koornwinder polynomials. In the \(n = 1\) case they coincide with Askey-Wilson polynomials, and in the case of \(n \geq 2\) they are equipped with extra five parameters \((a, b, c, d, t)\). By specializing these parameters, one can recover Macdonald polynomials [MSS, M13] of types \(BC_n, B_n\) and \(C_n\).

Let us give a brief explanation on Macdonald polynomials. Let \(S\) be an affine root system in the sense of [M13, Chap. 1]. If \(S\) is reduced, then \(S = S(R)\) or \(S = S(R)\), where \(S(R)\) is the affine root system associated to an irreducible finite root system \(R\), and \(S(R)\) is the dual of \(S(R)\). In the reduced case, if \(R\) is a finite root system of type \(X\) \((X = A_n, B_n, C_n, D_n, BC_n, E_6, E_7, E_8, F_4, G_2)\), then we call \(S\) an affine root system of type \(X\) and call \(S(R)\) an affine root system of type \(X\). We also call a reduced \(S\) an untwisted affine root system. We leave the other case, if \(S\) is non-reduced, then it is of the form \(S = S_1 \cup S_2\), where \(S_1\) and \(S_2\) are reduced affine root systems. In the non-reduced case, if \(S_1\) and \(S_2\) are of type \(X\) and \(Y\) respectively, then we call \(S\) an affine root system of type \((X.Y)\).

The Macdonald polynomial \(P_\lambda(x)\) is a \(q\)-orthogonal polynomial which is a simultaneous eigenfunction of a family of \(q\)-difference operators associated to an affine root system \(S\). Today Macdonald polynomials are formulated by the Macdonald-Cherednik theory, which is based on the representation theory of affine Hecke algebras. This theory was first developed for untwisted affine root systems. Below we call \(P_\lambda(x)\) Macdonald polynomials of type \(X\) if the corresponding untwisted affine root system \(S\) is of type \(X\).

Let us go back to Koornwinder polynomials. By the works of Noumi [N95], Sahi [S00], Stokman [S00] and others, it is clarified that one can apply Macdonald-Cherednik theory to the non-reduced affine root system of type \((C_n', C_n)\) in the sense of [M13, Chap. 1], and that one can recover Koornwinder polynomials as Macdonald polynomials of type \((C_n', C_n)\). As a result, Koornwinder polynomials are characterized as the ones having most parameters in the family of Macdonald polynomials.

For the convenience of the following explanation, let us give a brief account on the notations used in this paper. First, we introduce the notations for the root system \(R\) of type \(C_n\). See [2.1.1] for details. Let \(h_2 := \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i\) be a lattice of rank \(n\). We denote the set of roots by \(R := \{x \epsilon_i, x \epsilon_i | i \neq j\} \cup \{\pm 2 \epsilon_i | i = 1, \ldots, n\} \subset h_2^\vee\) and denote simple roots by \(\alpha_i \in R (i = 1, \ldots, n)\). We define the inner product on \(h_2^\vee\) by \((\epsilon_i, \epsilon_j) := \delta_{i,j}\), and define the fundamental weights \(\omega_i \in h_2^\vee\) by \((\omega_i, \alpha_j) = \delta_{i,j}\). Note that the weight lattice is \(P := \mathbb{Z} \omega_1 \oplus \cdots \oplus \mathbb{Z} \omega_n = h_2^\vee\). We denote the set of dominant weights by \(h_2^\vee_+ := \{\mu \in h_2^\vee | (\alpha_i, \mu) \geq 0, i = 1, \ldots, n\} \subset h_2^\vee\). Here \(\alpha_i^\vee\) is the coroot corresponding to \(\alpha_i\). We also denote by \(W_0\) the finite Weyl group.

Next we introduce the notations for the affine root system \(S\) of type \((C_n', C_n)\), and explain the parameters of Koornwinder polynomials. See [2.1.2] and [2.1.1] for details. By considering the extension \(h_2^\vee := h_2^\vee \oplus \mathbb{Z} \delta\) of the lattice \(h_2^\vee\), we have the affine root system \(S := \{\pm \epsilon_i + \frac{1}{2} \delta, \pm 2 \epsilon_i + k \delta \mid k \in \mathbb{Z}, i = 1, \ldots, n\} \cup \{\pm \epsilon_i + \epsilon_j + k \delta \mid k \in \mathbb{Z}, 1 \leq i < j \leq n\} \subset h_2^\vee \otimes \mathbb{Q}\) of type \((C_n', C_n)\) and the extended affine Weyl group \(W\). By using the group ring \(t\) of the weight lattice \(P = h_2^\vee\), we can present the group \(W = t(P) \rtimes W_0\) in the case of rank \(n \geq 2\), there are five orbits for the action of \(W\) on \(S\), and we consider the parameters associated to these orbits, denoting them by \((t_0, t, t, u_0, u_0, u_0)\). By adding the parameter \(q\) and the square root of each parameter, we define the base field \(\mathbb{K}\) by

\[
\mathbb{K} := \mathbb{Q}(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, t_0^{\frac{1}{2}}, t, u_0^{\frac{1}{2}}, u_0^{-\frac{1}{2}}).
\]

Koornwinder polynomials have these five plus one parameters \((q, t_0, t, u_0, u_0)\). In the case of rank \(n = 1\), there are four \(W\)-orbits, and the parameters are \((q, t_0, u_0, u_0)\). In this case Koornwinder polynomials are equivalent to Askey-Wilson polynomials as mentioned before. By [N95, §3] and [S00, (5.2)], we have the following correspondence to the original parameters \((q, a, b, c, d)\) of Askey-Wilson polynomials.

\[
(q, a, b, c, d) = (q, q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}}, -q^{-\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}}, t, u_0^{\frac{1}{2}}, -u_0^{-\frac{1}{2}}).
\]  

For a family \(x = (x_1, \ldots, x_n)\) of commutative variables, we denote the Laurent polynomial ring of \(x_i\’s\) by \(\mathbb{K}[x^{\pm 1}]\). The finite Weyl group \(W_0\) acts on \(\mathbb{K}[x^{\pm 1}]\) naturally, and we denote the invariant ring by
equipped with the coloring of folding steps by either black or gray. We denote by $\Gamma$ \textit{colored alcove walks} whose steps belong to the dominant chamber $C$ that is the structure constants of the product in the invariant ring $c_{\mathfrak{c}_t}$ as the multiplicity of the irreducible decomposition of the tensor representation. For Hall-Littlewood polynomials as the irreducible characters of the general linear group, we can interpret the coefficient explicitly for the corresponding LR coefficient $c_{\lambda,\mu}$.

Chap. VI, the one-column type $(1_k)$. Hereafter we call $c_{\lambda,\mu}$ \textit{LR coefficients} for simplicity.

Let us recall what is known in the case of type $A$. The classical LR coefficients are the structure constants of the product $s_\lambda s_\mu = \sum_\nu c_{\lambda,\mu}^{\nu} s_\nu$ of Schur polynomials $s_\lambda$ in the ring of symmetric polynomials. We have explicit formulas for the classical LR coefficients via Young tableaux. Regarding Schur polynomials as the irreducible characters of the general linear group, we can interpret the coefficient $c_{\lambda,\mu}^{\nu}$ as the multiplicity of the irreducible decomposition of the tensor representation. For Hall-Littlewood polynomials, which are $t$-deformations of Schur polynomials, we can also consider the LR coefficients $c_{\lambda,\mu}^{\nu}$, and some explicit formulas are known. See [M95] Chap. II, (4.11) for example.

Although Macdonald polynomial of type $A$ is a $q$-deformation of Hall-Littlewood polynomial, no explicit formula for the corresponding LR coefficient $c_{\lambda,\mu}^{\nu}$ had been unknown for a long time. In [M95] Chap. VI, [6], Macdonald derived some combinatorial formulas for Pieri coefficients using arms and legs of Young diagrams. Here Pieri coefficients mean the LR coefficients $c_{\lambda,\mu}^{\nu}$ with $\lambda$ the one-row type $(k)$ or the one-column type $(1^k)$, where the weights are identified with Young diagrams or the partitions.

On the LR coefficients of Macdonald polynomials, Yip [Y12] made a great progress. Using \textit{alcove walks}, an explicit formula of $c_{\lambda,\mu}^{\nu}$ is given in [Y12] Theorem 4.4 for the Macdonald polynomials of untwisted affine root systems. Also a simplified formula [Y12] Corollary 4.7 is derived in the case $\lambda$ is equal to a minuscule weight. In particular, this simplified formula recovers Macdonald’s formula for Pieri coefficients of type $A$ [Y12] Theorem 4.9. In Yip’s study, the key ingredient is the notion of alcove walk, originally introduced by Ram [R06]. We will explain the relevant notations and terminology in 2.1.3.

1.2 Littlewood-Richardson coefficients

The understanding of Macdonald polynomials has been rapidly advanced since the emergence of the Macdonald-Cherednik theory. Currently Macdonald polynomials, in particular those of type $A$, appear in various fields in mathematics, and have increasing importance. However, the study of Koornwinder polynomials seems to be less advanced than the Macdonald polynomials of the other root systems, and there are many pending problems for the $(C_n, C_n)$-type.

In this paper, we consider \textit{Littlewood-Richardson coefficients} $c_{\lambda,\mu}^{\nu}$ of Koornwinder polynomials $P_\lambda$, that is the structure constants of the product in the invariant ring $\mathbb{K}[x^\pm 1]^{W_0}$:

$$P_\lambda P_\mu = \sum_\nu c_{\lambda,\mu}^{\nu} P_\nu.$$  

Hereafter we call $c_{\lambda,\mu}^{\nu}$ \textit{LR coefficients} for simplicity.

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1.3 Main result

The main result of this paper is the following Theorem 1, which is a natural $(C_n, C_n)$-type analogue of Yip’s alcove walk formulas for LR coefficients in [Y12] Theorem 4.4. Let us prepare the necessary notations and terminology for the explanation.

Let $A$ be the fundamental alcove of the extended affine Weyl group $W$ (see 2.1.3). Given an element $w \in W$, we take a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Given a bit sequence $b = (b_1, \ldots, b_r) \in \{0, 1\}^r$ and another element $z \in W$, we call a sequence of alcoves of the form

$$p = (p_0 := zA, p_1 := zs_{i_1} A, p_2 := zs_{i_1} s_{i_2} A, \ldots, p_r := zs_{i_1} \cdots s_{i_r} A)$$

an \textit{alcove walk} of type $\overrightarrow{wz} := (i_1, \ldots, i_r)$ beginning at $zA$. We denote by $\Gamma(\overrightarrow{wz}, z)$ the set of such alcove walks. See Example 2.1.1 for examples of alcove walks.

For an alcove walk $p$, we call the transition $p_k \to p_{k+1}$ the $k$-th step of $p$. The $k$-th step of $p$ is called a \textit{folding} if $b_k = 0$ where the bit sequence $b$ corresponds to the alcove walk $p$ (see Table 2.1.1).

In our main result, we use a \textit{colored alcove walk} introduced by Yip [Y12]. It is an alcove walk equipped with the coloring of folding steps by either black or gray. We denote by $\Gamma(\overrightarrow{wz}, z)$ the set of colored alcove walks whose steps belong to the dominant chamber $C \subset \mathfrak{h}_n^\ast := \mathfrak{h}_n^\ast \otimes \mathbb{R}$. 

3
Then we have

Here the term \( \text{factorized} \), and we have \( W \) of the stabilizer

properties. Koornwinder polynomials correspond to the end of the colored alcove walks

p

\( p \) is obtained by symmetrizing \( x \) is actually depends only on \( p \) actually depends only on \( v \) in \( W^\mu \), which corresponds to the beginning of the colored alcove walk \( p \).

Let us explain the outline of proof of Theorem 1. We denote by \( \psi_\lambda(\mu) \) the non-symmetric Koornwinder polynomials \([899] \ [900]\), which will be explained in \([2.2.2]\). Here we need the following two properties.

1. \( \{ E_\mu(x) \mid \mu \in (h^*_2)_+ \} \) is a \( \mathbb{K} \)-basis of \( \mathbb{K}[x^{\pm 1}] \).
2. \( P_\mu(x) \) is obtained by symmetrizing \( E_\mu(x) \) (Fact \([2.2.2]\)). More precisely, using the symmetrizer \( U \) in \([2.2.33]\), we have

The outline of proof is a straight \((C_n', C_n)\)-type analogue of Yip's derivation in \([Y12]\). The argument can be divided into four steps, and below we explain them abbreviating some coefficients and ranges of summations.

(i) For dominant weights \( \lambda, \mu \in (h^*_2)_+ \), we derive an expansion formula

\[
x^\mu E_\lambda(x) = \sum_{p \in \Gamma^C} c_p E_{w(\mu)}(x)
\]
of the product of the non-symmetric Koornwinder polynomial $E_\lambda(x)$ and the monomial $x^\mu$ (Corollary 3.3.5). Here the index set $\Gamma^C$ consists of alcove walks belonging to the dominant chamber $C$. The symbol $w(p) \in (h^*_2)_+$ will be given in (3.1.4).

(i) We use Ram-Yip type formula (Fact 2.2.2), an expansion formula for the non-symmetric Koornwinder polynomials in terms of monomials:

$$E_\mu(x) = \sum_{p \in \Gamma} f_p t_{d(p)}^p x^{\text{wt}(p)}.$$  

This formula was derived by Orr and Shimozono [OS18], based on the work of Ram and Yip [RY11] on the same type formula for the untwisted affine root systems.

(ii) Using (i) and (ii), we can calculate the product of the non-symmetric Koornwinder polynomial $E_\mu(x)$ and the Koornwinder polynomial $P_\lambda(x)$ in an extension $\overline{DH}(W)$ of the double affine Hecke algebra $DH(W)$, and express it as a sum over alcove walks (3.3.4). Then we can rewrite it as a sum over colored alcove walks and have (Proposition 3.3.2):

$$E_\mu(x)P_\lambda(x) = \sum_{v \in W^+} \sum_{p \in \Gamma^2} A_p C_p E_{w(p)}(x).$$

(iii) Using (i) and (ii), we can calculate the product of the non-symmetric Koornwinder polynomial $E_\mu(x)$ and the Koornwinder polynomial $P_\lambda(x)$ in an extension $\overline{DH}(W)$ of the double affine Hecke algebra $DH(W)$, and express it as a sum over alcove walks (3.3.4). Then we can rewrite it as a sum over colored alcove walks and have (Proposition 3.3.2):

(iv) Theorem 1 is obtained by symmetrizing $E_\mu(x)$ in (iii) and switching $\lambda \leftrightarrow \mu$.

1.4 Organization and notation

Organization

We explain the organization of this paper.

In §2, we explain Koornwinder polynomials based on the Macdonald-Cherednik theory. In §2.1, we explain the root system and alcove walks. We introduce the root system $R$ of type $C_n$ in §2.1.1, the affine root system $S$ of type $(C_n^\vee, C_n)$ in §2.1.2, and alcove walks in §2.1.3. In the next §2.2, we explain affine Hecke algebras and Koornwinder polynomials. We introduce the affine Hecke algebra $H(W)$ of type $(C_n^\vee, C_n)$ in §2.2.1 and review the basic representation constructed by Noumi [N95]. Then we introduce the double affine Hecke algebra $DH(W)$ of type $(C_n^\vee, C_n)$ in §2.2.2 and explain the non-symmetric Koornwinder polynomials $E_\lambda$ (Fact 2.2.2). Finally, we introduce Koornwinder polynomials $P_\lambda$ in §2.2.3 (Fact 2.2.4).

In §3, we derive our main Theorem 3.4.2. The outline of the discussion is given by the four steps (1) (iv) previously explained, and the organization of §3 follows that.

In §4, we derive several corollaries of the main Theorem 3.4.2. In §4.1, we discuss the case of rank $n = 1$, that is the case of Askey-Wilson polynomials. In particular, we give a simplified formula for the Pieri coefficient (Proposition 4.1.1), and recover the recurrence formula of Askey-Wilson polynomials in [AW85] from our Pieri formula (Remark 4.1.4). In §4.2, we discuss the specialization to Macdonald polynomials of type $C_n$ (Proposition 4.2.1), and recover Yip’s result [Y12] in the case of $C_n$-type (Remark 4.2.2). In §4.3, we discuss the Hall-Littlewood limit $q \to 0$, and show that LR coefficients are somewhat simplified (Proposition 4.3.1). In §4.4, we display examples of LR coefficients in the case of rank $n = 2$.

Notation and terminology

Here are the notations and terminology used throughout in this paper.

- We denote by $\mathbb{Z}$ the ring of integers, by $\mathbb{N} = \mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}$ the set of non-negative integers, by $\mathbb{Q}$ the field of rational numbers, and by $\mathbb{R}$ the field of real numbers.
- We denote by $e$ the unit of a group.
- We denote an action of a group $G$ on a set $S$ by $g.s$ for $g \in G$ and $s \in S$, and denote the $G$-orbit of $s$ by $G.s$ or by $Gs$.
- For a commutative ring $k$ and a family of commutative variants $x = (x_1, x_2, \ldots)$, we denote by $k[x^\pm 1]$ the Laurent polynomial ring $k[x_1^\pm 1, x_2^\pm 1, \ldots]$.
- We denote by $\delta_{i,j}$ the Kronecker delta.
2 Koornwinder polynomials

2.1 Root systems

2.1.1 Root systems for type $C_n$

Let $(R, b^*_Z, R', b_Z)$ be the root data of type $C_n$. Thus $b^*_Z = \bigoplus_{i=1}^n Z\varepsilon_i$ and $b_Z = \bigoplus_{i=1}^n Z\epsilon_i$ are lattices of rank $n$, and we have the non-degenerate bilinear form $\langle \cdot, \cdot \rangle : b^*_Z \times b^*_Z \to Z$, $\langle \epsilon_i', \epsilon_j \rangle = \delta_{i,j}$. We identify $b^*_Z = b_Z$ and $\epsilon_i = \epsilon_i'$ by this bilinear form $\langle \cdot, \cdot \rangle$. The set $R$ of roots and the set $R'$ of coroots are given by

\[ R = \{ \pm \epsilon_i \pm \epsilon_j \mid i \neq j \} \cup \{ \pm 2\epsilon_i \mid i = 1, \ldots, n \} \subset b^*_Z, \]
\[ R' = \{ \pm \epsilon_i \pm \epsilon_j \mid i \neq j \} \cup \{ \pm \epsilon_i \mid i = 1, \ldots, n \} \subset b_Z. \]

We use the following choice of the subset $R_+ \subset R$ of positive roots and the subset $R'_+ \subset R'$ of positive coroots.

\[ R_+ := \{ \epsilon_i \pm \epsilon_j \mid i < j \} \cup \{ 2\epsilon_i \mid i = 1, \ldots, n \}, \quad R'_+ := \{ \epsilon_i \pm \epsilon_j \mid i < j \} \cup \{ \epsilon_i \mid i = 1, \ldots, n \}, \]

We have $R = R_+ \cup -R_+$ and $R' = R'_+ \cup -R'_+$. The simple roots $\alpha_i \in R$ ($i = 1, \ldots, n$) are given by

\[ \alpha_1 := \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1} := \epsilon_{n-1} - \epsilon_n, \quad \alpha_n := 2\epsilon_n. \]

For each root $\alpha \in R$, we denote the associated coroot by $\alpha' := 2\alpha/\langle \alpha, \alpha \rangle \in b^*_Z = b_Z$. The correspondence $\alpha \mapsto \alpha'$ is a bijection, and we have $\langle \alpha', \alpha \rangle = 2$. The coroots for simple roots are $\alpha_i' = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1}' = \epsilon_{n-1} - \epsilon_n$, and $\alpha_n' = \epsilon_n$. We call $\alpha_i'$ simple coroots.

For $\alpha \in R$, we write $s_\alpha$ the reflection by the hyperplane $H_\alpha := \{ x \in b^*_Z \mid \langle \alpha', x \rangle = 0 \}$ in $b^*_R = b^*_Z \otimes \mathbb{Z} \mathbb{R}$. That is,

\[ s_\alpha x := x - \langle \alpha', x \rangle \alpha, \quad x \in b^*_Z. \]

We write $s_i := s_{\epsilon_i}$ for $i = 1, \ldots, n$. The finite Weyl group $W_0$ is defined to be the subgroup of $GL(b^*_Z)$ generated by $s_1, \ldots, s_n$. As an abstract group, $W_0$ is a Coxeter group with generators $s_1, \ldots, s_n$ and relations

\[ s_i^2 = 1 \quad (i = 1, \ldots, n), \]
\[ s_is_j = s_js_i \quad (|i - j| > 1), \]
\[ s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad (i = 1, \ldots, n - 2), \]
\[ s_{n-1}s_{n}s_{n-1}s_n = s_ns_{n-1}s_{n-1}s_n. \]

Next we introduce notation for weights of the root system of type $C_n$. For $i = 1, \ldots, n$, we define $\omega_i := \epsilon_1 + \cdots + \epsilon_i \in b^*_Z$, and call them the fundamental weights. Then we have $\langle \alpha_i', \omega_j \rangle = \delta_{i,j}$ for $i, j = 1, \ldots, n$. We define the root lattice $Q$ and the weight lattice $P$ by

\[ Q := Z\omega_1 \oplus \cdots \oplus Z\omega_n \subset b^*_Z = P := Z\omega_1 \oplus \cdots \oplus Z\omega_n \subset b^*_Z. \]

The action of $W_0 \subset GL(b^*_Z)$ on $b^*_R$ preserves the weight lattice $P = b^*_Z$. We denote this action by $\lambda \mapsto w.\lambda$ for $w \in W_0$ and $\lambda \in P$.

2.1.2 Affine root system of type $(C^*_n, C_n)$

Let $t(P)$ be the group algebra of the weight lattice $P = b^*_Z$. Denoting by $t(\lambda) \in t(P)$ the element associated to $\lambda \in P$, we have $t(P) = \{ t(\lambda) \mid \lambda \in b^*_Z \}$ and $t(\lambda)t(\mu) = t(\lambda + \mu)$ ($\lambda, \mu \in b^*_Z$). Let us consider the lattice extension $\tilde{b}^*_Z := b^*_Z \oplus \mathbb{Z}\delta$ of $b^*_Z$ and the coefficient extension $\tilde{b}^*_R := \tilde{b}^*_Z \otimes \mathbb{Z} \mathbb{R}$. We define the action of $t(P)$ on $\tilde{b}^*_R$ by

\[ t(\lambda).(\mu + m\delta) := \mu + (m - \langle \mu, \lambda \rangle)\delta, \quad \mu + m\delta \in \tilde{b}^*_R = \tilde{b}^*_Z \oplus \mathbb{R}\delta. \]

The relation of $w \in W_0$ and $t(\lambda) \in t(P)$ in the group $GL(\tilde{b}^*_R)$ is then given by $wt(\lambda)w^{-1} = t(w.\lambda)$. The subgroup $W \subset GL(\tilde{b}^*_R)$ generated by $t(P)$ and $W_0$ is called the extended affine Weyl group. That is,

\[ W := t(P) \rtimes W_0 \subset GL(\tilde{b}^*_R). \]
The action of the element $s := t(\epsilon_1)s_{2\epsilon_1} \in W$ on $P = h^*_R$ is given by $s\epsilon_i = \delta - \epsilon_i$ and $s\epsilon_i = \epsilon_{i+1}$ ($i = 2, \ldots, n$), which is the same as the reflection $s_0 := s_{\alpha_0}$ with respect to the hyperplane $H_{\alpha_0} := \{x \in h^*_R | \langle \alpha_0, x \rangle = 0\}$ for the affine root $\alpha_0 := \delta - 2\epsilon_1 \in h^*_R$. Here we set $\alpha_0^\vee := \frac{1}{2}c - \epsilon_1$, where $c$ is the basis element in the one-dimensional extension $h^*_R := h^*_R \oplus \mathbb{R}c$ of $h^*_R := h^*_R \otimes_{\mathbb{Z}} \mathbb{R}$. We also set $\langle c, x \rangle = 1$ for all $x \in h^*_R$.

As an abstract group, $W$ is a Coxeter group with generators $s_0, s_1, \ldots, s_n$ and relations

$$\begin{align*}
s_i^2 &= 1 \quad (i = 0, \ldots, n), \\
s_is_j &= s_js_i \quad (|i - j| > 1), \\
s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} \quad (i = 1, \ldots, n - 2), \\
s_is_{i+1}s_is_{i+1} &= s_{i+1}s_is_{i+1} \quad (i = 0, n - 1).
\end{align*}$$

We define the subset $S_+ \subset S$ of positive roots by

$$S_+ := \{\pm \epsilon_i + \frac{k}{2}\delta, \pm 2\epsilon_i + k\delta \mid k \in \mathbb{Z}, \ i = 1, \ldots, n\} \cup \{\pm \epsilon_i \pm \epsilon_j + k\delta \mid k \in \mathbb{Z}, 1 \leq i < j \leq n\} \subset h^*_R.$$

We also denote by $\lesssim_R$ the corresponding Bruhat order. The reduced expressions of $t(\epsilon_i)$ ($i = 1, \ldots, n$) are given by

$$\begin{align*}
t(\epsilon_1) &= s_0s_1 \cdots s_{n-1}s_ns_{n-1} \cdots s_2s_1, \\
t(\epsilon_2) &= s_1s_0s_1 \cdots s_{n-1}s_ns_{n-1} \cdots s_2, \\
t(\epsilon_i) &= s_{i-1} \cdots s_is_0s_1 \cdots s_{n-1} \cdots s_i, \\
t(n) &= s_{n-1} \cdots s_is_0s_1 \cdots s_n.
\end{align*}$$

Now we define the affine root system $S$ of type $(C_n^\vee, C_n)$ in the sense of \cite[1.18]{M03} and \cite{S00} by

$$S := \{\pm \epsilon_i + \frac{k}{2}\delta, \pm 2\epsilon_i + k\delta \mid k \in \mathbb{Z}, \ i = 1, \ldots, n\} \cup \{\pm \epsilon_i \pm \epsilon_j + k\delta \mid k \in \mathbb{Z}, 1 \leq i < j \leq n\} \subset h^*_R.$$ (2.1.4)

We also define the subset $S_+ \subset S$ of positive roots by

$$S_+ := \{\alpha + k\delta, \alpha^\vee + \frac{k}{2}\delta \mid \alpha \in R_+, \alpha^\vee \in R^\vee, k \in \mathbb{N}\} \cup \{\alpha + k\delta, \alpha^\vee + \frac{k}{2}\delta \mid \alpha \in R_-, \alpha^\vee \in R^\vee, k \in \mathbb{N}\}. $$

(2.1.5)

We then have $S = S_+ \cup S_-$ with $S_- := -S_+$. We also set $\tilde{R} := R \cup R^\vee$. Then any $\beta \in S$ can be uniquely written as $\beta = \alpha + k\delta \in S$ with $\alpha \in \tilde{R}$ and $k \in \frac{1}{2}\mathbb{Z}$. We denote the corresponding projection $S \to \tilde{R}$ by

$$\bar{\beta} := \beta - \alpha \quad (\beta = \alpha + k\delta \in S, \ \alpha \in \tilde{R}, \ k \in \frac{1}{2}\mathbb{Z}).$$

(2.1.6)

We also denote

$$\tilde{R}_+ := R_+ \cup R^\vee, \quad \tilde{R}_- := -\tilde{R}_+.$$ (2.1.7)

Hereafter the case of type $(C_n^\vee, C_n)$ is also called the rank $n$ case.

### 2.1.3 Alcove walks

**Alcove walks** are introduced by Ram \cite{R06} as analogue of Littelmann paths for affine Hecke algebras. They are valuable combinatorial objects, and used in Ram-Yip type formula \cite{RT11, OS18} for non-symmetric Macdonald-Koornwinder polynomials, and in Yip’s formula \cite{Y12} for Littlewood-Richardson rules of Macdonald polynomials in the untwisted affine root systems. In this part we introduce the notation of alcove walks which will be used throughout in the text. Basically we follow the notations in \cite[2.2]{Y12}, but make slight modifications.

Let us regard an affine root $\beta = \alpha + k\delta \in S$ ($\alpha \in \tilde{R}, \ k \in \frac{1}{2}\mathbb{Z}$) as a affine linear function on $h^*_R$ by

$$\beta(v) = \langle \alpha, v \rangle + k \quad (v \in h^*_R).$$

An alcove is defined to be a connected component of the complement $h^*_R \setminus \bigcup_{\alpha \in S} H_\alpha$ of the hyperplanes $H_\alpha := \{x \in h^*_R \mid \alpha(x) = 0\}$. The fundamental alcove $A$ is the alcove given by

$$A := \{x \in h^*_R \mid \alpha_i(x) > 0 \mid i = 0, \ldots, n\}. $$

(2.1.8)
Its boundary consists of the hyperplanes $H_{\alpha_0}, H_{\alpha_1}, \ldots, H_{\alpha_n}$. Note that the mapping

$$W \ni w \mapsto wA \in \pi_0(b_\alpha^w \setminus \bigcup_{\alpha \in S} H_\alpha)$$

is a bijection. An alcove $wA$ is surrounded by $n + 1$ hyperplanes, say $H_{\gamma_i}$ ($i = 0, \ldots, n$). We call the intersection $H_{\gamma_i} \cap wA$ an edge of the alcove $wA$, where $wA$ denotes the closure with respect to the Euclidean topology. Note that each hyperplane $H_{\gamma_i}$ separates $wA$ and another alcove $vA$, which can be written as $v = ws_j$ for some $j = 0, \ldots, n$. Then the edge $H_{\gamma_i} \cap wA$ is just the intersection $wA \cap ws_jA$, and has two sides, which we call the $wA$-side and the $ws_jA$-side.

Given an alcove $wA$, we give a sign $\pm$ to each of the two sides on an edge of $wA$. Let $H_{\gamma_i}$, ($i = 0, \ldots, n$) be the hyperplanes surrounding $wA$. By renaming the indices $i$ if necessary, we can assume that the hyperplane $H_{\gamma_i}$ separates $wA$ and $ws_iA$. Then using the projection $\gamma_i \mapsto \overline{\gamma_i}$ in (2.1.6) and the symbols $\overline{R}_\pm$ in (2.1.10), we set the signs by the following rule:

- If $\overline{\gamma_i} \in \overline{R}_+$, then the $wA$-side of $H_{\gamma_i} \cap wA$ is assigned by $+$ and the $ws_iA$-side is by $-$. 
- If $\overline{\gamma_i} \in \overline{R}_-$, then $wA$-side is assigned by $-$ and the $ws_iA$-side is by $+$.

See Figure 2.1.1 for the assignment in the rank 2 case.

![Figure 2.1.1: Signs for the edges of the fundamental alcove A in the rank 2 case](image)

Given an element $w \in W$ and a reduced expression $w = s_{i_1} \cdots s_{i_r}$, we define a subset $\mathcal{L}(w) \subset S$ by

$$\mathcal{L}(w) := \{ \alpha_{i_1}, s_{i_1}\alpha_{i_2}, \ldots, s_{i_1} \cdots s_{i_{r-1}} \alpha_{i_r} \}.$$  \hspace{1cm} (2.1.9)

The set $\{H_\beta \mid \beta \in \mathcal{L}(w)\}$ consists of the hyperplanes separating $A$ and $wA$. Given elements $v, w \in W$ and their reduced expressions, we also set

$$\mathcal{L}(v, w) := (\mathcal{L}(v) \cup \mathcal{L}(w)) \setminus (\mathcal{L}(v) \cap \mathcal{L}(w)).$$ \hspace{1cm} (2.1.10)

The set $\{H_\beta \mid \beta \in \mathcal{L}(v, w)\}$ consists of the hyperplanes separating $vA$ and $wA$. If $v \preceq_B w$, then we have

$$\mathcal{L}(v, w) = v.\mathcal{L}(v, v^{-1}w) = v.\mathcal{L}(v^{-1}w).$$ \hspace{1cm} (2.1.11)

Let us again given $w \in W$ and a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Then the mapping

$$\{0, 1\}^r \ni (b_1, \ldots, b_r) \mapsto s_{i_1}^{b_1} \cdots s_{i_r}^{b_r} \in \{ v \in W \mid v \preceq_B w \}$$

is a bijection. Let us given extra $z, w \in W$ such that $v \preceq_B w$. We can write $v = s_{i_1}^{b_1} \cdots s_{i_r}^{b_r}$ with $b = (b_1, \ldots, b_r) \in \{0, 1\}^r$. We then consider the following sequence $p$ of alcoves.

$$p = (p_0 := zA, p_1 := zs_{i_1}^{b_1}A, p_2 := zs_{i_1}^{b_1}s_{i_2}^{b_2}A, \ldots, p_r := zs_{i_1}^{b_1} \cdots s_{i_r}^{b_r}A).$$

The sequence $p$ is called an \textit{alcove walk of type } $\overrightarrow{w}$ \textit{beginning at } $zA$, and we denote by $\Gamma(\overrightarrow{w}, z)$ the set of alcove walks of this kind. The symbol $\overrightarrow{w}$ emphasizes that we choose a reduced expression $w = s_{i_1} \cdots s_{i_r}$.  

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Example 2.1.1 (Alcove walks in the rank 2 case). For \( w = s_1s_2s_1s_0 \) and \( z = e \in W \), the two alcove walks

\[
p_1 := (A, A, s_2A, s_2s_1A, s_2s_1s_0A), \quad p_2 := (A, s_1A, s_1s_2A, s_1s_2s_1A, s_1s_2s_1s_0A) \in \Gamma(\mathbf{w}, z)
\]

are shown in Figure 2.1.2, where the gray region is the fundamental alcove \( A \), and the number \( i = 0, 1, 2 \) on a hyperplane means that it belongs to the \( W \)-orbit of \( H_{s_i} \).

![Figure 2.1.2: Alcove walks \( p_1 \) and \( p_2 \)](image)

For an alcove walk \( p \in \Gamma(\mathbf{w}, z) \) and \( k = 1, \ldots, r \), the transition \( p_{k-1} \to p_k \) is called the \( k \)-th step of \( p \). The \( k \)-th step is called a crossing if \( b_k = 1 \), and called a folding if \( b_k = 0 \). The correspondence between the bit \( b_k \) and the \( k \)-th step is shown in Table 2.1.1, where we denote by \( v_k-1 \in W \) the element such that \( p_{k-1} = v_k-1A \).

\[
\begin{array}{c|c|c|c}
 b_k & \text{1} & \text{0} \\
 \hline
 p_{k-1} & \text{crossing} & p_k & \text{folding} \\
 v_{k-1}s_iA \\
\end{array}
\]

Table 2.1.1: Correspondence between bits and steps

Let us again given \( z, w \in W \) with a reduced expression \( w = s_{i_1} \cdots s_{i_r} \). For an alcove walk \( p = (zA, \ldots, zs_{i_1}^{b_1} \cdots s_{i_r}^{b_r}A) \in \Gamma(\mathbf{w}, z) \), we define \( e(p) \in W \) by

\[
e(p) := zs_{i_1}^{b_1} \cdots s_{i_r}^{b_r}.
\] (2.1.12)

Thus \( e(p) \) corresponds to the end of \( p \). We also define \( h_k(p) \in S \) for \( k = 1, \ldots, r \) by the following rule. Denote \( v := s_{i_1}^{b_1} \cdots s_{i_{k-1}}^{b_{k-1}} \) for simplicity, so that we have \( p_{k-1} = vA \). Then we define

\[
h_k(p) := \text{the affine root such that the corresponding hyperplane } H_{h_k(p)} \text{ separates } vA \text{ and } vA.
\] (2.1.13)

Furthermore, we call the \( k \)-th step of \( p \in \Gamma(\mathbf{w}, z) \) an ascent if \( zs_{i_1}^{b_1} \cdots s_{i_{k-1}}^{b_{k-1}} \leq_B zs_{i_1}^{b_1} \cdots s_{i_k}^{b_k} \), and call it a descent if \( zs_{i_1}^{b_1} \cdots s_{i_{k-1}}^{b_{k-1}} \geq_B zs_{i_1}^{b_1} \cdots s_{i_k}^{b_k} \). We denote the set of descent steps of \( p \) by

\[
\text{des}(p) := \{ k = 1, \ldots, r \mid \text{the } k \text{-th step is a descent}\}.
\] (2.1.14)

Recalling the sign on an edge of an alcove (see Figure 2.1.1 for an example), we can classify each step of an alcove walk \( p \) into four types as in Table 2.1.2, where we used the symbol \( v_k-1 \in W \) such that \( p_{k-1} = v_k-1A \).
Using this classification, we define $\varphi_{\pm}(p) \subset \{1, \ldots, r\}$ by
\[
\varphi_+(p) := \{k \mid \text{the } k\text{-th step of } p \text{ is a positive folding}\},
\]
\[
\varphi_-(p) := \{k \mid \text{the } k\text{-th step of } p \text{ is a negative folding}\},
\]
and define $\xi_{\text{des}}(p) \subset \{1, \ldots, r\}$ by
\[
\xi_{\text{des}}(p) := \{k \mid \text{the } k\text{-th step of } p \text{ is a crossing and } k \in \text{des}(p)\}.
\]
Note that we fix a reduced expression $w = s_{i_1} \cdots s_{i_r}$ in the definitions of $\varphi_{\pm}(p)$ and $\xi_{\text{des}}(p)$.

### Table 2.1.2: Classification of steps in alcove walks

| positive crossing | negative crossing | positive folding | negative folding |
|-------------------|-------------------|-----------------|-----------------|
| $-+\mapsto pk$   | $+\mapsto pk$    | $+\mapsto pk$  | $-\mapsto pk$  |
| $p_{k-1}$         | $pk$             | $p_{k-1}$       | $vk_{k-1}s_{i_k}A$ |

\[\xi_{\text{des}}(p) := \{k \mid \text{the } k\text{-th step of } p \text{ is a crossing and } k \in \text{des}(p)\}.
\] (2.1.16)

### 2.2 Affine Hecke algebras and Koornwinder polynomials

In this subsection, we explain the realization of non-symmetric Koornwinder polynomials via the polynomial representation of the affine Hecke algebra type $\mathcal{C}_n$, and introduce Koornwinder polynomials by their symmetrization.

#### 2.2.1 Affine Hecke algebras of type $(\mathcal{C}_n', \mathcal{C}_n)$ and polynomial representations

Recall the affine root system $S$ of type $(\mathcal{C}_n', \mathcal{C}_n)$ and the extended affine Weyl group $W$ explained in \[2.1.2\]. Let $\{t_\alpha \mid \alpha \in S\}$ be parameters satisfying the condition $t_\alpha = t_\beta$ for $\beta \in W.\alpha$. Since the $W$-orbits in $S$ are given by $W.\alpha_i = W.\alpha'_i (i = 1, \ldots, n - 1), W.\alpha_n, W.\alpha'_n, W.\alpha_0, W.\alpha'_0,$ we can replace the family $\{t_\alpha\}$ by
\[(t_{\alpha_0}, t_{\alpha_1} = t_{\alpha'_1}, t_{\alpha_2}, t_{\alpha'_2}) = (t_0, t_n, t_0, u_0, u_n).
\] (2.2.1)

We will also denote $t_1, \ldots, t_{n-1} := t$. Now we set the base field $\mathbb{K}$ as
\[\mathbb{K} := \mathbb{Q}(q^{\frac{1}{r}}, t_0^\frac{1}{r}, t_1^\frac{1}{r}, t_0^\frac{1}{r}, u_0^\frac{1}{r}, u_1^\frac{1}{r}),
\] (2.2.2)

and all the linear spaces, their tensor products, and the algebras will be those over $\mathbb{K}$ unless otherwise stated.

The affine Hecke algebra $H(W)$ is the associative algebra generated by $T_0, T_1, \ldots, T_n$ subject to the following relations.

\[
(T_i - t_i^\frac{1}{r})(T_i + t_i^\frac{1}{r}) = 0 \quad (i = 0, \ldots, n),
\]
\[
T_i T_j = T_j T_i \quad (|i - j| > 1, (i, j) \notin \{(n, 0), (0, n)\}),
\] (2.2.3)
\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (i = 1, \ldots, n - 2),
\] (2.2.4)
\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (i = 0, n - 1).
\] (2.2.5)

The relations (2.2.3)–(2.2.5) are called the braid relations.

Given an element $w \in W$ together with a reduced expression $w = s_{i_1} \cdots s_{i_r}$, we consider the alcove walk $(A, s_{i_1}A, \ldots, s_{i_r}A, A = wA) \in \Gamma(\overline{w}, e)$, and define $Y^w \in H(W)$ by
\[Y^w := T_{i_1}^{e_i} \cdots T_{i_r}^{e_i},
\] (2.2.6)
where we set $\epsilon_k := 1$ if the $k$-th step of $p$ is a positive crossing, and set $\epsilon_k := -1$ if the $k$-th step is a negative crossing according to the classification in Figure 2.1.2. The decomposition of $Y^w$ by $T_i$’s is independent of the choice of a reduced expression of $w$. By the relations of $H(W)$, we find that the family \( \{ Y^w \mid w \in W \} \) is mutually commutative \[95\] §2.

As explained in \[103\] §3, we can calculate $Y^{t(\epsilon_i)}$ using the reduced expression of $t(\epsilon_i)$ in \[2.2.10\]. The result is

\[
Y^{t(\epsilon_1)} = T_0 \cdots T_n T_{n-1} \cdots T_1, \\
Y^{t(\epsilon_2)} = T_1^{-1} T_0 \cdots T_{n-1} T_n T_{n-1} \cdots T_2, \\
Y^{t(\epsilon_3)} = T_{n-1}^{-1} \cdots T_1^{-1} T_0 \cdots T_{n-1} T_n T_{n-1} \cdots T_1, \\
Y^{t(\epsilon_n)} = T_{n-1}^{-1} \cdots T_1^{-1} T_0 T_1 \cdots T_n.
\] (2.2.7)

Now we denote by

\[ \mathbb{K}[Y^{\pm 1}] = \mathbb{K}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}] \subset H(W), \quad Y_i := Y^{t(\epsilon_i)} \quad (i = 1, \ldots, n) \]

the ring of Laurent polynomials in $Y_1, \ldots, Y_n$. Then we have an isomorphism $H(W) \simeq H(W_0) \otimes \mathbb{K}[Y^{\pm 1}]$, where $H(W_0)$ is the Hecke algebra of the finite Weyl group $W_0$. The latter is the subalgebra of $H(W)$ generated by $T_1, \ldots, T_n$.

Next we review the basic representation of the affine Hecke algebra $H(W)$ introduced by Noumi \[95\]. Let $\mathbb{K}(x) = \mathbb{K}(x_1, \ldots, x_n)$ be the field of rational functions with $n$ variables. Then the mapping

\[
T_i \mapsto t_i^\frac{1}{2} + t_i^{-\frac{1}{2}} \frac{1 - t_i x_i/x_{i+1}}{1 - x_i/x_{i+1}} (s_i - 1) \quad (i = 1, \ldots, n - 1), \\
T_0 \mapsto t_0^\frac{1}{2} + t_0^{-\frac{1}{2}} \frac{(1 - u_0^{-1} t_0^\frac{1}{2} q^2 x_1^{-1})(1 + u_0^\frac{1}{2} t_0 q^2 x_1^{-1})}{1 - q x_1^{-2}} (s_0 - 1), \\
T_n \mapsto t_n^\frac{1}{2} + t_n^{-\frac{1}{2}} \frac{(1 - u_n^{-1} t_n q x_n)(1 + u_n^\frac{1}{2} t_n^{-1} x_n)}{1 - x_n^2} (s_n - 1)
\] (2.2.8)
defines a ring homomorphism $\rho : H(W) \to \text{End}(\mathbb{K}(x))$. Moreover its image is contained in the endomorphism algebra $\text{End}_\mathbb{K}(\mathbb{K}[x^{\pm 1}]) \subset \text{End}_\mathbb{K}(\mathbb{K}(x))$ of the Laurent polynomials. We call $\rho$ the basic representation of $H(W)$. Hereafter we identify $H(W)$ and its image under $\rho$, and regard $H(W)$ as a subalgebra of $\text{End}_\mathbb{K}(\mathbb{K}[x^{\pm 1}])$. The right hand sides of (2.2.8) are $q$-difference operators called Dunkl operators of type $(C_\alpha^{\pm}, C_n)$.

Let us give a simplified description of (2.2.8). Using

\[
u_i := \begin{cases} 1 & (i = 1, \ldots, n - 1) \\ u_0 & (i = 0) \\ u_n & (i = n) \end{cases}, \quad \alpha_i := \begin{cases} \xi_i/x_{i+1} & (i = 1, \ldots, n - 1) \\ q x_1^{-2} & (i = 0) \\ x_n^2 & (i = n) \end{cases},
\]

we can rewrite $T_i$’s as

\[
T_i = t_i^\frac{1}{2} + t_i^{-\frac{1}{2}} \frac{(1 - u_i^{-1} t_i^\frac{1}{2} x_i^{\alpha_i})(1 + u_i^\frac{1}{2} t_i^{-1} x_i^{\alpha_i})}{1 - x_i^{\alpha_i}} (s_i - 1),
\] (2.2.9)

where we identified the left and right hand sides in (2.2.8) as claimed before. Let us further define the rational functions $c_i(z), d_i(z) \in \mathbb{K}(z)$ by

\[
c_i(z) := t_i^\frac{1}{2} \frac{(1 - u_i^{-1} t_i^\frac{1}{2} z^{\mp 1})(1 + u_i^\frac{1}{2} t_i^{-1} z^{\mp 1})}{1 - z}, \quad d_i(z) := \frac{t_i^\frac{1}{2} - c_i(z)}{1 - z} + \frac{(u_i^{-1} - u_i^{-\frac{1}{2}}) z^{\mp 1}}{1 - z}.
\] (2.2.10)

Then we can rewrite (2.2.8) or (2.2.9) as

\[
T_i = t_i^\frac{1}{2} + c_i(x^{\alpha_i})(s_i - 1) = t_i^\frac{1}{2} s_i + d_i(x^{\alpha_i})(1 - s_i) = c_i(x^{\alpha_i}) s_i + d_i(x^{\alpha_i}).
\] (2.2.11)
For later use, we calculate the action of the element $Y^β$ on 1 in the basic representation for an affine root $β = α + kδ ∈ S (α ∈ ˆR, k ∈ Z)$. Let us define

$$q^{h(α + kδ)} := q^{-k}, \quad h^{t(α + kδ)} := \prod_{γ ∈ R_+} l_{t}^1(γ^γ, α) \prod_{γ ∈ R'_+} (t_0 t_n)^{1/2}(γ^γ, α).$$  \hspace{1cm} (2.2.12)

Here $R'_+ := \{ε_i ± ε_j \mid 1 ≤ i < j ≤ n\}$ denotes the set of positive short roots, and $R'_+ := \{2ε_i \mid 1 ≤ i ≤ n\}$ denotes the set of positive long roots. Then we can check

$$Y^β 1 = q^{h(β)} h^{t(β)}. \hspace{1cm} (2.2.13)$$

See also [S00, Proposition 4.5] for a more general formula.

Finally we recall the Lusztig relations in the basic representations of affine Hecke algebra. For each weight $λ = (λ_1, \ldots, λ_n) ∈ P = h^n_{-pt}$, we define $x^λ ∈ \mathbb{K}[x^{±1}]$ by

$$x^λ := x^λ_1 \cdots x^λ_n ∈ \mathbb{K}[x^{±1}]. \hspace{1cm} (2.2.14)$$

**Fact 2.2.1** (Lusztig relations, [LS99 Proposition 3.6]). For $i = 0, \ldots, n$ and $λ ∈ h^n_{pt}$, we have

$$T_i x^λ - x^{s_i λ} T_i = d_i(z^{α_i})(x^λ - x^{s_i λ}),$$

where the rational function $d_i(z)$ is defined by (2.2.10).

### 2.2.2 Double affine Hecke algebras and non-symmetric Koornwinder polynomials

Next we review the double affine Hecke algebra $DH(W)$ of type $(C_n^\vee, C_n)$ and the non-symmetric Koornwinder polynomials $E_λ(x)$, following [M03, Sa99] and [S00].

As in the previous §2.2.1, we regard $H(W)$ as an $\mathbb{K}$-subalgebra of $\text{End}_\mathbb{K}(\mathbb{K}[x^{±1}])$ by the basic representation (2.2.8). We define the double affine Hecke algebra $DH(W) ⊂ \text{End}_\mathbb{K}(\mathbb{K}[x^{±1}])$ as the $\mathbb{K}$-subalgebra generated by $\mathbb{K}[x^{±1}]$, $H(W_0)$ and $\mathbb{K}[Y^{±1}]$. Thus

$$DH(W) := \langle \mathbb{K}[x^{±1}], H(W_0), \mathbb{K}[Y^{±1}] \rangle \subset \text{End}_\mathbb{K}(\mathbb{K}[x^{±1}]).$$

As in the case of untwisted affine root systems, the algebra $DH(W)$ has the Cherednik anti-involution $φ$ ([Sa99 §3]):

$$φ(x_i) = Y_i^{-1}, \quad φ(Y_i) = x_i^{-1}, \quad φ(T_i) = T_i \quad (i = 1, \ldots, n),$$

$$φ(u_n) = t_0, \quad φ(t_0) = u_n. \hspace{1cm} (2.2.15)$$

On the element $T_0$ the anti-involution acts as $φ(T_0) = T_{s_{2i}}^{-1} x_{-1}^{-1}$. In fact, we have $T_0 = Y_1 T_{s_{2i}}^{-1}$ and $T_{s_{2i}} = T_1 \cdots T_n T_{n-1} \cdots T_1$ by (2.2.7). Hereafter we denote

$$T_i^{\vee} := φ(T_i) \quad (i = 0, \ldots, n). \hspace{1cm} (2.2.16)$$

Next we introduce the $x$- and $Y$-intertwiners for $DH(W)$ following [M03 §5.6]. Let $\barDH(W)$ be the coefficient extension of $DH(W)$ by rational functions of $x$’s and $Y$’s. In other words, we set

$$\barDH(W) := \langle \mathbb{K}(x), H(W_0), \mathbb{K}(Y) \rangle \subset \text{End}_\mathbb{K}(\mathbb{K}(x)). \hspace{1cm} (2.2.17)$$

Here $\mathbb{K}(x)$ and $\mathbb{K}(Y)$ are the fields of rational functions of $x_i$ and $Y_i$ ($i = 1, \ldots, n$) respectively. For $i = 0, \ldots, n$, we define $S_i^x ∈ \barDH(W)$ by

$$S_i^x := T_i + φ_i^+(x^{α_i}) = T_i^{-1} + φ_i^-(x^{α_i}), \hspace{1cm} (2.2.18)$$

where

$$φ_i^±(z) := \frac{(t_1^± - t_i^±) + z^{±1/2}(u_1^± - u_i^±)}{1 - z^{±1}} ∈ \mathbb{K}(z). \hspace{1cm} (2.2.19)$$

We call $S_i^x$ the $x$-intertwiners.
Let us explain some basic properties of $x$-intertwiners. Recalling the rational function $d_i(z)$ in (2.2.19) and the expression of $T_i$ in (2.2.11), we have

$$\varphi_i^+(z) = d_i(z), \quad S_i^x = T_i - d_i(x^{\alpha_i}) = c_i(x^{\alpha_i}) s_i.$$  \hspace{1cm} (2.2.20)

For each weight $\lambda \in h_+^*$, we have

$$S_i^x x^\lambda = x^{\nu_i(\lambda)} S_i^x,$$  \hspace{1cm} (2.2.21)

by the Lusztig relations (Fact 2.2.1). Moreover, by [M03, (5.5.2)], the $x$-intertwiners $S_i^x$ ($i = 0, \ldots, n$) satisfy the same braid relations as (2.2.3)–(2.2.5):

$$S_i^x S_j^x = S_j^x S_i^x \quad (|i - j| > 1),$$  \hspace{1cm} (2.2.22)

$$S_i^x S_{i+1}^x S_i^x = S_{i+1}^x S_i^x S_{i+1}^x \quad (i = 1, \ldots, n - 2),$$  \hspace{1cm} (2.2.22)

$$S_i^x S_{i+1}^x S_i^x S_{i+1}^x = S_{i+1}^x S_i^x S_{i+1}^x S_i^x \quad (i = 0, n - 1).$$  \hspace{1cm} (2.2.22)

Given an element $w \in W$, choose a reduced expression $w = s_{i_1} \cdots s_{i_p}$, and set

$$S_w^x := S_{i_1}^x \cdots S_{i_p}^x \in \overline{DH}(W).$$  \hspace{1cm} (2.2.23)

By the braid relations, $S_w^x$ is independent of the choice of a reduced expression of $w$.

Next we introduce $Y$-intertwiners. First, note that the anti-involution $\phi$ can be extended to $\overline{DH}(W)$. In fact, $\overline{DH}(W)$ is the Ore localization of the non-commutative algebra $DH(W)$ by the commutative subalgebras $\mathbb{K}[x^{\pm 1}]$ and $\mathbb{K}[Y^{\pm 1}]$, and $\phi$ is an isomorphism on these commutative subalgebras. We denote the extension of $\phi$ to $\overline{DH}(W)$ by the same symbol $\phi$. Now we define the $Y$-intertwiners $S_i^Y \in \overline{DH}(W)$ by

$$S_i^Y := \phi(S_i^x) = T_i + \psi_i^+(Y^{-\alpha_i}) - T_i^{-1} + \psi_i^-(Y^{-\alpha_i}) \quad (i = 1, \ldots, n),$$

$$S_0^Y := \phi(S_0^x) = T_0 + \psi_0^+(qY_0^2) - T_0^{-1} + \psi_0^-(qY_0^2),$$  \hspace{1cm} (2.2.24)

where the symbols $\psi_i^\pm(z)$ denote the images of $\varphi_i^\pm(z)$ given in (2.2.19) under the extended anti-involution $\phi$. That is, we have

$$\psi_i^\pm(z) := \varphi_i^\pm(z) = \pm \frac{t_0^\pm - t_i^\pm}{1 - z^{\pm 1}} \quad (i = 1, \ldots, n - 1),$$

$$\psi_0^\pm(z) := \pm \frac{t_0^\pm - u_0^\pm + z^{\pm 1}(u_0^\pm - u_i^\pm)}{1 - z^{\pm 1}},$$  \hspace{1cm} (2.2.25)

$$\psi_i^+(z) := \pm \frac{t_i^+ - t_0^+ + z^{1/2}(t_0^+ - t_i^+)}{1 - z^{1/2}}.$$  \hspace{1cm} (2.2.25)

We can deduce properties of $S_i^Y$’s from those of $S_i^x$’s. For example, applying the anti-involution $\phi$ to the relation (2.2.21), we have

$$S_i^Y Y^\lambda = Y^{s_i^x \lambda} S_i^Y$$  \hspace{1cm} (2.2.26)

for each $i = 0, \ldots, n$ and $\lambda \in h_+^*$. We can also see that $S_i^Y$’s satisfy the same braid relations as $S_i^x$’s (2.2.22). For an element $w \in W$, we can define $S_w^Y \in \overline{DH}(W)$ by choosing a reduced expression $w = s_{i_1} \cdots s_{i_p}$ and

$$S_w^Y := S_{i_1}^Y \cdots S_{i_p}^Y \in \overline{DH}(W).$$  \hspace{1cm} (2.2.27)

It is well-defined by the braid relations of $S_i^Y$’s.

Finally we explain the non-symmetric Koornwinder polynomials. For each weight $\mu \in P = h_+^*$, we regard $t(\mu)W_0 \subset W$ by the decomposition $W = t(\mu) \times W_0$ in (2.1.2). Then we define $w(\mu) \in W$ by the following description:

$$w(\mu)$$ is the shortest element among $t(\mu)W_0 \subset W$.  \hspace{1cm} (2.2.28)

Now we have:
Fact 2.2.2 ([Sa99, §6]. [SC00] Theorem 4.8]). For \( \mu \in h_{\mathbb{Z}}^* \), the element
\[
E_{\mu}(x) := S_{w(\mu)}^{Y} 1
\]
belongs to \( K[x^{\pm 1}] \). We call it the non-symmetric Koornwinder polynomial associated to \( \mu \).

By (2.2.26), \( E_{\mu}(x) \) is a simultaneous eigenfunction of the family \( \{ Y_{\lambda} \mid \lambda \in h_{\mathbb{Z}}^* \} \) of Dunkl operators. Note that our normalization of \( E_{\mu}(x) \) is different from that in [Sa99, SC00]. In loc. cit., the coefficient of \( x^{\mu} \) is normalized to be 1.

### 2.2.3 Koornwinder polynomials

Now we introduce Koornwinder polynomials by symmetrizing non-symmetric Koornwinder polynomials.

First, we define the set \((h^*_\mathbb{Z})_+ \subset h^*_\mathbb{Z} \) of dominant weights by
\[
(h^*_\mathbb{Z})_+ := \{ \mu \in h^*_\mathbb{Z} \mid \langle \alpha^\vee, \mu \rangle \geq 0, \ i = 1, \ldots, n \}.
\]
For a dominant weight \( \mu \in (h^*_\mathbb{Z})_+ \), we denote the stabilizer of \( \mu \) in the finite Weyl group \( W_0 \) by
\[
W_\mu := \{ w \in W_0 \mid w.\mu = \mu \} \subset W_0,
\]
and denote the longest element among \( W_\mu \) by
\[
w_\mu \in W_\mu.
\]
Next, using the notations in (2.1.2) and (2.2.1) we define \( t_w \in K \) for each \( w \in W \) by
\[
t_w := \prod_{\beta \in L(w)} t_\beta \in K.
\]
Here \( \{ t_\alpha \mid \alpha \in S \} \) is the \( W \)-invariant family of parameters (2.2.1), \( K \) is the base field (2.2.2), and \( L(w) \subset S \) is given by (2.1.1). If \( w = s_{i_1} \cdots s_{i_r} \in W \) is the shortest element, then we have \( t_w = t_{i_1} \cdots t_{i_r} \).

For a dominant weight \( \mu \in (h^*_\mathbb{Z})_+ \), we define the Poincaré polynomial \( \mu \) in \( K \) of the stabilizer \( W_\mu \) by
\[
W_\mu(t) := \sum_{u \in W_\mu} t_u.
\]

**Lemma 2.2.3.** For each element \( \mu \in (h^*_\mathbb{Z})_+ \), we have
\[
\sum_{u \in W_\mu} \left( \prod_{\alpha \in L(1,u)} t_\alpha^{\frac{1}{2}} \frac{1 - t^{ht(-\alpha)} t_\alpha^{-1}}{1 - t^{ht(\alpha)} t_\alpha} \right) \left( \prod_{\alpha \in L(u,w_\mu)} t_\alpha^{\frac{1}{2}} \frac{1 - t^{ht(-\alpha)} t_\alpha}{1 - t^{ht(\alpha)} t_\alpha} \right) = t_{w_\mu}^{-\frac{1}{2}} W_\mu(t).
\]
For a proof, see [Y12, Lemma 3.4].

Next we define the symmetrizer \( U \) by
\[
U := \sum_{w \in W_0} t_{w_{\mu} w}^{-\frac{1}{2}} T_w.
\]

By [M03 (5.5.9)], we then have
\[
U T_i = U t_i^{\frac{1}{2}}, \quad T_i U = t_i^{\frac{1}{2}} U \quad (i = 1, \ldots, n).
\]

Hereafter we denote the ring of \( W_0 \)-invariant Laurent polynomials by
\[
K[x^{\pm 1}]_{W_0} := \{ f \in K[x^{\pm 1}] \mid w. f = f, w \in W_0 \}.
\]
Here \( W_0 \) acts on \( x^\lambda \) by (2.2.14) by the action on the weight \( \lambda \). Also recall that for each \( \mu \in (h^*_\mathbb{Z})_+ \subset h^*_\mathbb{Z} \) we defined \( w(\mu) = t(\mu) W_0 \subset W \) by (2.2.25).

**Fact 2.2.4 ([SC00] Theorem 6.6).** For each dominant weight \( \lambda \in (h^*_\mathbb{Z})_+ \), the element
\[
P_\lambda(x) := \frac{1}{t_{w_\lambda}^{\frac{1}{2}} W_\lambda(t)} U S_{w(\lambda)}^{Y} 1 = \frac{1}{t_{w_\lambda}^{\frac{1}{2}} W_\mu(t)} U E_{\lambda}(x) \in DH(W)
\]
belongs to \( K[x^{\pm 1}]_{W_0} \). We call \( P_\lambda(x) \) the (monic) Koornwinder polynomial associated to \( \lambda \).

Note that the coefficient of \( x^{\lambda} \) in \( P_\lambda(x) \) is 1 since the coefficient of the top term \( x^\lambda \) in \( U S_{w(\lambda)}^{Y} 1 \) is \( t_{w_\lambda}^{\frac{1}{2}} W_\lambda(t) \). To emphasize the root system \((C'_n, C_n)\), we call \( P_\lambda(x) \) the Koornwinder polynomial of rank \( n \) or of type \((C'_n, C_n)\).
3 Littlewood-Richardson coefficients

Yip [Y12] Theorem 4.4] derived a combinatorial explicit formula of LR coefficients for Macdonald polynomials $P_\lambda(x)$ in the case of untwisted affine root systems. In this section, we derive a $(C_n, C_n)$-analogue of Yip’s formula. The outline of the derivation is quite similar to Yip’s proof [Y12 §3.1–4.1], but we need non-trivial adjustments in each step.

3.1 Products of non-symmetric Koornwinder polynomials and monomials

In [Y12] Theorem 3.3, Yip derived an expansion formula for the product of the monomial $x^\nu$ and the non-symmetric Macdonald polynomial $E_\lambda(x)$ in the case of untwisted affine root systems. In this subsection, we give its $(C_n, C_n)$-type analogue (Corollary 3.1.3).

We will use the notations in [2.2.17] and [2.2.21] In particular, $\widetilde{DH}(W)$ is the extension [2.2.17] of the double affine Hecke algebra $DH(W)$ of type $(C_n, C_n)$, $S^\nu_w \in \widetilde{DH}(W)$ is the $W$-intertwiner [2.2.21], and $S^\nu_w$ for $w \in W$ is the product of $S^\nu_w$’s [2.2.27]. We also denote the Bruhat order in $W$ by $\preceq_B$.

As a preparation of Proposition 3.1.1, we derive a product formula of the $W$-intertwiners.

Proposition 3.1.1. For $w \in W$ and $i = 0, \ldots, n$, we have the following relations in $\widetilde{DH}(W)$.

(i) If $w \preceq_B s_i w$, then $S^\nu_i S^\nu_w = S^\nu_{i w}$, where

\[
n_0(Y^\beta) := \frac{(1 - u_n^{-\frac{1}{2}} u_0^\frac{1}{2} Y^\frac{2}{2})(1 + u_n^{-\frac{1}{2}} u_0^\frac{1}{2} Y^\frac{2}{2})}{1 - Y^\beta} \quad (\beta \in W.\alpha_0),
\]

\[
n_i(Y^\beta) := 1 - t Y^\beta \frac{1 - t^{-1} Y^\beta}{1 - Y^\beta} \quad (\beta \in W.\alpha_i, 0 < i < n),
\]

\[
n_n(Y^\beta) := \frac{(1 - t_n^{-\frac{1}{2}} l_0^\frac{1}{2} Y^\frac{2}{2})(1 + t_n^{-\frac{1}{2}} l_0^\frac{1}{2} Y^\frac{2}{2})}{1 - Y^\beta} \quad (\beta \in W.\alpha_n).
\]

Proof. Fix $w \in W$ and choose a reduced expression $w = s_{i_1} \cdots s_{i_r}$. By the definitions [2.2.27], [2.2.21] and the equation [2.2.20], we have

\[
S^\nu_w = S^\nu_i \cdots S^\nu_i = (T^\nu_{i_1} + \psi^\nu_i(Y^{-\alpha_{i_1}})) \cdots (T^\nu_{i_r} + \psi^\nu_i(Y^{-\alpha_{i_r}})) = c^\nu_{i_1}(Y^{-\alpha_{i_1}}) s_{i_1} \cdots c^\nu_{i_r}(Y^{-\alpha_{i_r}}) s_{i_r} = c^\nu_i(Y^{-\alpha_i}) \cdots c^\nu_i(Y^{-\alpha_i}) w.
\]

Here we set $\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$ $(k = 1, \ldots, r)$ and

\[
c^\nu_i(c(z)) = \begin{cases} u_n^{-\frac{1}{2}}(1 - u_n^{\frac{1}{2}} u_0^{-\frac{1}{2}} z^{-\frac{1}{2}})(1 + u_n^{\frac{1}{2}} u_0^{-\frac{1}{2}} z^{-\frac{1}{2}}) & (i = 0), \\
(1 - t z \frac{1 - t^{-1} z}{1 - z}) & (0 < i < n), \\
(1 - t_n^{-\frac{1}{2}} l_0^\frac{1}{2} z^{-\frac{1}{2}})(1 + t_n^{-\frac{1}{2}} l_0^\frac{1}{2} z^{-\frac{1}{2}}) & (i = n). \end{cases}
\]

Since $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression, we have $\beta_k \in S_+$ for $k = 1, \ldots, r$, where $S_+ \subset S$ denotes the set of positive affine roots [2.1.5]. The product $S^\nu_i$ $(i = 0, \ldots, n)$ and $S^\nu_w$ is now calculated as

\[
S^\nu_i S^\nu_w = c^\nu_i(Y^{-\alpha_i}) s_{i_1} c^\nu_i(Y^{-\beta_1}) \cdots c^\nu_i(Y^{-\beta_r}) w. \quad (3.1.1)
\]

If $\ell(s_i w) = \ell(w) + 1$, then the equation [3.1.1] becomes $S^\nu_i S^\nu_w = S^\nu_{i w}$. If $\ell(s_i w) = \ell(w) - 1$, then there exists $k \in \{1, \ldots, r\}$ such that $s_k(\beta_k - 1) \in S_+$ and $s_i(\beta_k) \in S_-$. Since we have $\beta_k = \alpha_i$, the equation [3.1.1] becomes

\[
S^\nu_i S^\nu_w = c^\nu_i(Y^{-\alpha_i}) s_{i_1} c^\nu_i(Y^{-\beta_1}) \cdots c^\nu_i(Y^{-\beta_r}) w = c^\nu_i(Y^{-\alpha_i}) c^\nu_i(Y^{-\alpha_i}) \cdots c^\nu_i(Y^{-\alpha_i}) s_{i_1} c^\nu_i(Y^{-\beta_1}) \cdots c^\nu_i(Y^{-\beta_r}) w = c^\nu_i(Y^{-\alpha_i}) c^\nu_i(Y^{-\alpha_i}) c_{i_1}(Y^{-\alpha_i}) \cdots c_{i_k}(Y^{-\alpha_i}) c_{i_1}(Y^{-\alpha_i}) \cdots c_{i_k}(Y^{-\alpha_i}) s_{i_1} c^\nu_i(Y^{-\beta_1}) \cdots c^\nu_i(Y^{-\beta_r}) w.
\]

Here the symbol $\sim$ denotes skipping the term. Then the consequence follows from the equality $c^\nu_i(Y^{-\alpha_i}) c^\nu_i(Y^{-\alpha_i}) = n_i(Y^{-\alpha_i})$, which can be checked by a direct calculation. \hfill $\square$
The same discussion shows the following statement.

**Corollary 3.1.2.** For \( w \in W \) and \( i = 0, \ldots, n \), we have the following relations in \( \widetilde{DH}(W) \)

(i) If \( w \preceq_B w_{s_i} \), then \( S^Y_{w} S^Y_{w_{s_i}} = S^Y_{w_{s_i}} S^Y_{w} \),

(ii) If \( w \succeq_B w_{s_i} \), then \( S^Y_{w} S^Y_{w_{s_i}} = S^Y_{w_{s_i}} n_i(Y^{-\alpha_i}) \), where \( n_i(Y^{-\alpha_i}) \) is given in Proposition 3.1.1.

Next we recall the notations on alcove walks in \([2.1.3]\). Given \( z, w \in W \) together with a reduced expression \( z = s_{i_1} \cdots s_{i_r} \), we defined the set \( \Gamma(\overrightarrow{z}, w) \) of alcove walks of type \( \overrightarrow{z} = (i_r, \ldots, i_1) \) beginning at \( wA \). For an alcove walk \( p = (p_0, \ldots, p_r) \in \Gamma(\overrightarrow{z}, w) \), the \( k \)-th step means the transition from \( p_{k-1} \) to \( p_k \), which is classified into the four types in Table 2.1.2.

Now we define \( x^z \in \widetilde{DH}(W) \) for \( z \in W \) with a chosen reduced expression \( z = s_{i_1} \cdots s_{i_r} \). Let \( q \) be the alcove walk given by

\[
q := (zA, z_{s_1}A, z_{s_1}s_{i_2}A, \ldots, z_{s_1} \cdots s_{i_r}A = A) \in \Gamma(\overrightarrow{z}, z).
\]

Here \( \overrightarrow{z}^{-1} := (i_1, \ldots, i_r) \). Then we define \( x^z \) by

\[
x^z := (T^\gamma_{i_r})^r \cdots (T^\gamma_{i_1})^s,
\]

where \( T^\gamma_{i} := \phi(T_{i}) \in DH(W) \) as in \([2.2.10]\), and we set \( \epsilon_k := 1 \) if the \( k \)-th step is a positive crossing, and \( \epsilon_k := -1 \) if the \( k \)-th step is a negative crossing according to the classification in Table 2.1.2.

**Proposition 3.1.3.** Given \( z, w \in W \) with a chosen reduced expression \( z = s_{i_1} \cdots s_{i_r} \), we have

\[
x^z S^Y_w = \sum_{p \in \Gamma(\overrightarrow{z}, w^{-1})} S^Y_{e(p)} g_p(Y) n_p(Y)
\]

in \( \widetilde{DH}(W) \), where \( e(p) \in W \) is the element \([2.1.12]\), and the terms \( g_p(Y) \) and \( n_p(Y) \) are given by

\[
g_p(Y) := \prod_{k \in \varphi_-(p)} (-\psi_{i_k}^-(Y^{-h_k}(p))) \prod_{k \in \varphi_+(p)} (-\psi_{i_k}^+(Y^{-h_k}(p))),
\]

\[
n_p(Y) := \prod_{k \in \xi_{\text{des}}(p)} n_{i_k}(Y^{-h_k}(p)).
\]

Here \( h_k(p) \) is given by \([2.1.13]\), \( \varphi_+(p) \) and \( \varphi_-(p) \) are by \([2.1.14]\), \( \xi_{\text{des}}(p) \) is by \([2.1.10]\), \( \psi_{i_k}^\pm(z) = \phi(\varphi_{i_k}^\pm(z)) \) is by \([2.2.26]\), and \( n_{i_k}(z) \) is given in Proposition 3.1.1.

**Proof.** We show the statement by induction on the length of \( z \in W \). If \( \ell(z) = 0 \), that is \( z = e \), then the right hand side consists only of the term for \( p = (p_0 = wA) \), so that it is equal to \( S^Y_w \), and we have the relation.

Next we assume \( z \neq e \) and that the result holds for any element \( w \in W \) such that \( w < \ell(z) \).

Fix a reduced expression of \( z \), and write it as \( z = s_{i_r} \zeta = s_{i_r} \cdots s_{i_1} \). By the hypothesis, we can write

\[
x^z S^Y_w = (T^\gamma_{i_r})^r \cdots (T^\gamma_{i_1})^s \sum_{p \in \Gamma(\overrightarrow{z}, w^{-1})} (T^\gamma_{i_r})^s S^Y_{e(p)} g_p(Y) n_p(Y).
\]

(3.1.3)

Here \( \epsilon \in \{\pm 1\} \) is the sign determined by \( z \). Let us calculate the rightmost side. Take an element

\[
p = (w^{-1}A, w^{-1}s_{i_r}A, \ldots, w^{-1}s_{i_1}A) \in \Gamma(\overrightarrow{z}, w^{-1}).
\]

Since we have \( (T^\gamma_{i_r})^\pm = S^Y_{i_r} - \psi_{i_r}^\pm(Y^{-\alpha_i}) \) by the definition \([2.2.10]\) of \( T^\gamma_{i} \), the term contributed by \( p \) becomes

\[
(T^\gamma_{i_r})^s S^Y_{e(p)} g_p(Y) n_p(Y) = (S^Y_{i_r} - \psi_{i_r}^\pm(Y^{-\alpha_i})) S^Y_{e(p)} g_p(Y) n_p(Y) = S^Y_{i_r} S^Y_{e(p)} g_p(Y) n_p(Y) + (\psi_{i_r}^\pm(Y^{-\alpha_i})) S^Y_{e(p)} g_p(Y) n_p(Y) = S^Y_{i_r} S^Y_{e(p)} g_p(Y) n_p(Y) + S^Y_{e(p)} g_p(Y) n_p(Y).
\]

In the last equality we used \([2.2.26]\). We treat the two terms in the last line separately.

For the first term \( S^Y_{i_r} S^Y_{e(p)} g_p(Y) n_p(Y) \), we further divide the argument into two cases according to the Bruhat order.
(i) The case $e(p)^{-1} \preceq_B s_i e(p)^{-1}$. By Proposition 3.1.1, we have $S_Y^Y S_{e(p)^{-1}} = S_{s_i e(p)^{-1}} = S_{e(p_i)^{-1}}$, where the alcove walk

$$p_1 = (w^{-1}A, w^{-1}s_i^{-1}A, \ldots, w^{-1}s_i^{-1} \cdots s_i^{-1} A) \in \Gamma(\mathcal{Z}, w^{-1})$$

is an extension of $p$ by a crossing (Table 2.1.2). By the hypothesis $e(p)^{-1} \preceq_B s_i e(p)^{-1}$, the last step of $p_1$ is an ascent, and we have $\varphi_+ (p_1) = \varphi_+ (p), \varphi_- (p_1) = \varphi_- (p)$ and $\xi_{\text{des}}(p_1) = \xi_{\text{des}}(p)$. Thus we have $g_p(Y) n_p(Y) = g_{p_1}(Y) n_{p_1}(Y)$ and $S_Y S_{e(p)^{-1}} g_p(Y) n_p(Y) = S_{e(p_i)^{-1}} g_{p_1}(Y) n_{p_1}(Y)$.

(ii) The case $e(p)^{-1} \succeq_B s_i e(p)^{-1}$. By Propositions 3.1.4 we have

$$S_Y^Y S_{e(p)^{-1}} = n_i (Y^{\alpha_i}) S_{s_i e(p)^{-1}} = n_i (Y^{\alpha_i}) S_{e(p_1)^{-1}} = S_{e(p_i)^{-1}} n_i (Y^{-e(p_i)\alpha_i}).$$

Here $p_1 \in \Gamma(\mathcal{Z}, w^{-1})$ is the same as (3.1.4), but in this case the last step is a descent crossing, and the hyperplane crossed by the last step is $H_{e(p_i)\alpha_i}$, since

$$h_{r+1}(p_1) = -e(p)(\alpha_i) = -e(p_1) s_i(\alpha_i) = e(p_1) \alpha_i.$$ We then have $\xi_{\text{des}}(p_1) = \xi_{\text{des}}(p) \cup \{r+1\}$ and $n_{p_1}(Y) = n_p(Y) n_i(Y^{-h_{r+1}(p_1)})$. Combining them with $\varphi_+(p_1) = \varphi_+(p)$ and $\varphi_-(p_1) = \varphi_-(p)$, we have $n_i(Y^{-e(p)\alpha_i}) g_p(Y) n_p(Y) = g_{p_1}(Y) n_{p_1}(Y)$. Hence also in this case, we have $S_Y^Y S_{e(p)^{-1}} g_p(Y) n_p(Y) = S_{e(p_i)^{-1}} g_{p_1}(Y) n_{p_1}(Y)$.

Taking the summation over $p$, we therefore have

$$\sum_{p_1 \in \Gamma(\mathcal{Z}, w^{-1})} S_Y^Y S_{e(p)^{-1}} g_{p_1}(Y) n_{p_1}(Y) = \sum_{p_1 \in \Gamma(\mathcal{Z}, w^{-1})} S_Y^Y S_{e(p)^{-1}} g_{p_1}(Y) n_{p_1}(Y).$$

Next we consider the term $S_Y S_{e(p)^{-1}} (-\psi^Y_i(Y^{-e(p)\alpha_i})) g_p(Y) n_p(Y)$. We make a similar argument as in the first term, and here we use the alcove walk $p_2 \in \Gamma(\mathcal{Z}, w^{-1})$ which is an extension of $p$ by a folding. We have $e(p_2) = e(p), \varphi_+(p_2) = \varphi_+(p) \cup \{r+1\} \cup \xi_{\text{des}}(p_2) = \xi_{\text{des}}(p)$. Using $p_2$ we have $S_Y S_{e(p)^{-1}} (-\psi^Y_i(Y^{-e(p)\alpha_i})) g_p(Y) n_p(Y) = S_Y S_{e(p_2)^{-1}} g_{p_2}(Y) n_{p_2}(Y)$. We therefore have

$$\sum_{p_2 \in \Gamma(\mathcal{Z}, w^{-1})} S_Y S_{e(p_2)^{-1}} (-\psi^Y_i(Y^{-e(p)\alpha_i})) g_p(Y) n_p(Y) = \sum_{p_2 \in \Gamma(\mathcal{Z}, w^{-1})} S_Y S_{e(p_2)^{-1}} g_{p_2}(Y) n_{p_2}(Y).$$

By (3.1.5) and (3.1.6), we have $x^\alpha S_Y^Y = \sum_{p \in \Gamma(\mathcal{Z}, w^{-1})} S_Y S_{e(p)^{-1}} g_p(Y) n_p(Y)$. Hence the induction step is proved.

The definition $3.1.2$ of $x^\pm$ for $z \in W$ and the definition $2.2.4$ of $x^\mu$ for $\mu \in P = h\mathbb{Z}$ are consistent in the following sense. Recall that we denote by $t(\mu) \in t(P) \subset W$ the element associated to $\mu \in P = h\mathbb{Z}$.

**Lemma 3.1.4.** We have $x^t(\mu) = x^\mu$ for $\mu \in P = h\mathbb{Z}$. In particular, we have $x^t(i) = x_i$ for $i = 1, \ldots, n$.

**Proof.** It is enough to show the latter half. By (2.2.7), we have

$$Y_i^{-1} = T_i^{-1} \cdots T_n^{-1} T_{n-1}^{-1} \cdots T_1^{-1} T_0^{-1} T_1 \cdots T_{i-1} \quad (i = 1, \ldots, n).$$

Applying the anti-involution $\phi$ (2.2.15) to these, we have

$$x_i = \phi(Y_i^{-1}) = T_i^\vee \cdots T_n^\vee (T_0^\vee)^{-1} \cdots (T_{n-1}^\vee)^{-1} (T_n^\vee)^{-1} \cdots (T_i^\vee)^{-1}.$$ On the other hand, we can calculate $x^t(i)$ directly by Definition 3.1.2, and can check $x_i = x^t(i)$. □

We denote the dominant chamber for the weight lattice by

$$C := \{ x \in h\mathbb{Z} \mid \langle \alpha^\vee, x \rangle > 0, \alpha \in R_+ \}.$$

As for the fundamental alcove $A$ (2.1.8), we have $A \subset C$.  

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Let \( v, w \in W \), and choose a reduced expression \( v = s_{i_1} \cdots s_{i_r} \) of \( v \). If an alcove walk \( p \in \Gamma(\overline{v}, w) \) satisfies \( e(p)^{-1}A \subset C \), where \( e(p) \in W \) is the element (2.1.12), then using the \( W \)-valued function \( w(\cdot) \) in (3.2.3), we define \( \varpi(p) \in (B^+_n) \) by the relation
\[
e(p)^{-1} = w(\varpi(p)).
\] (3.1.7)

Also we define \( \Gamma^C(\overline{v}, w) \subset \Gamma(\overline{v}, w) \) by
\[
\Gamma^C(\overline{v}, w) := \{ p = (p_0, \ldots, p_r) \in \Gamma(\overline{v}, w) \mid p_i \in C, \ \forall i = 0, \ldots, r \}. \tag{3.1.8}
\]

Using these symbols, we have the following corollary of Proposition 3.1.3.

**Corollary 3.1.5** (c.f. [Y12 Corollary 4.1]). Let \( \lambda, \mu \in B_n^+ \), and fix a reduced expression \( t(\lambda) = s_{i_1} \cdots s_{i_r} \). Then we have
\[
x^\lambda E_\mu(x) = \sum_{p \in \Gamma^C(t(\lambda), w(\mu)^{-1})} g_p n_p E_{\varpi(p)}(x),
\]
where
\[
g_p := \prod_{k \in \varphi(-p)} (-\psi_{(n)}^k(q^{sh(-h_k(p))}ht(-h_k(p)))) \prod_{k \in \varphi(\cdot)(p)} (-\psi_{(n)}^k(q^{sh(-h_k(p))}ht(-h_k(p)))),
\]
and
\[
n_p := \prod_{k \in \xi(u)(p)} n_{i_k}(q^{sh(-h_k(p))}ht(-h_k(p))).
\]

**Proof.** We apply \( x^\lambda S^Y_w = \sum_{p \in \Gamma(t^{-1}, w^{-1})} S^Y_e(p)^{-1} g_p(Y) n_p(Y) \) in Proposition 3.1.3 to \( z = t(\lambda) \) and \( w = w(\mu) \). Since \( x^t(\lambda) = x^\lambda \) by Lemma 3.1.4 we have
\[
x^\lambda S^Y_w(\mu) = \sum_{p \in \Gamma(t(\lambda), w(\mu)^{-1})} S^Y_e(p)^{-1} g_p(Y) n_p(Y).
\]

Taking the product of each side with 1 and using the definition of the non-symmetric Koornwinder polynomial \( E_\mu(x) \) (Fact 2.2.2) and the equality \( Y_1^2 = q^{sh(\beta)}ht(\beta) \) in (2.2.13), we have
\[
x^\lambda E_\mu(x) = \sum_{p \in \Gamma(t(\lambda), w(\mu)^{-1})} g_p n_p S^Y_e(p)^{-1}.1.
\]

Next we consider the condition under which the factor \( n_{i_k}(q^{sh(-h_k(p))}ht(-h_k(p))) \) in \( n_p \) vanishes. By the definition of the factor (Proposition 3.1.3), the condition is \( q^{sh(-h_k(p))}ht(-h_k(p)) = t^{i_k} - 1 \) (\( i_k = 1, \ldots, n - 1 \)) and \( q^{sh(-h_k(p))}ht(-h_k(p)) = t^{i_k} - 1 \) (\( i_k = n \)). Then by the definition (2.1.13) of \( h_k(p) \), the alcove walk \( p \) that contributes to the summation is contained in the dominant chamber \( C \). Now the consequence follows from the definition of \( E_\mu(x) \) and that (3.1.7) of \( \varpi(p) \).

### 3.2 Some lemmas

In this subsection we prepare some lemmas for the symmetrizer \( U \) and the Koornwinder polynomials \( P_\lambda(x) \), which are \((C_n^+, C_n^-)\)-type analogue of [Y12 Proposition 3.6].

**Lemma 3.2.1** (c.f. [Y12 Proposition 3.6 (a)])]. The symmetrizer \( U \) (2.2.30) has the following expression.
\[
U = \sum_{w \in W_0} S^Y_w \prod_{\alpha \in L(w^{-1}, w_0^{-1})} b(Y^{-\alpha}),
\]
where
\[
b(Y^{-\alpha}) := \begin{cases} \frac{t^{\frac{1}{2} - t^{-1}Y^{-\alpha}}}{1 - Y^{-\alpha}} & (\alpha \not\in W_0, \alpha_n) \\ \frac{\frac{1}{2} - t^{\frac{1}{2} - \frac{1}{2}Y^{-\alpha}}(1 - t^{\frac{1}{2} - \frac{1}{2}Y^{-\alpha}})}{1 - Y^{-\alpha}} & (\alpha \in W_0, \alpha_n) \end{cases}.
\] (3.2.1)

Here \( L(v, w) \subset S \) is given by (2.1.10), and \( w_0 \in W_0 \) is the longest element (2.2.30).

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Proof. By the definition of $U$ and the definition \((2.2.7)\) of the $Y$-intertwiner $S^Y_w$, we can expand $U$ as

$$U = \sum_{w \in W_0} S^Y_w b_w(Y), \quad b_w(Y) \in \mathbb{K}(Y).$$

For the longest element $w_0 \in W_0$, the coefficient of $T_{w_0}$ in $U$ is 1, and thus we have $b_{w_0}(Y) = 1$. We calculate the term $b_w(Y)$ for $w \in W_0 \setminus \{w_0\}$ by induction on the length $\ell(w)$. Assume $b_w(Y) = \prod_{\alpha \in \mathcal{L}(v^{-1}, w_0^{-1})} b(Y^{-\alpha})$ for any element $v \in W_0$ satisfying $\ell(v) > \ell(w)$. By the equality $UT_i = U t_i^{\frac{1}{2}}$ \((i = 1, \ldots, n)\) in \((2.2.24)\) and the definition \((2.2.21)\) of $S_i^Y$, we have

$$\sum_{w \in W_0} S^Y_w b_w(Y) t_i^{\frac{1}{2}} = U t_i^{\frac{1}{2}} = U T_i = \sum_{w \in W_0} S^Y_w b_w(Y) T_i = \sum_{w \in W_0} S^Y_w b_w(Y)(S^Y_i - \psi_i^+(Y^{-\alpha_i})) \quad (3.2.2)$$

Now note that for $w \neq w_0$ there exists an index $i = 1, \ldots, n$ such that $w \not\equiv_B v := ws_i$. Taking this index $i$ and comparing the coefficients of $S^Y_w$ in the equality \((3.2.2)\) with the help of \((2.2.25)\) and Corollary \(3.1.2\) we have $b_i(Y) t_i^{\frac{1}{2}} = b_w(s_i Y) - b_i(Y) \psi_i^+(Y^{-\alpha_i})$. Here $b_w(s_i Y)$ is obtained from $b_w(Y)$ by replacing $Y^\lambda$ with $Y^{s_i \cdot \lambda}$. Then by the definition \((2.2.25)\) of $\psi_i^+(z)$ we have

$$b_w(Y)/b_w(s_i Y) = t_i^{\frac{1}{2}} \frac{1}{\psi_i^+(Y^{-\alpha_i})} t_i^{\frac{1}{2}} + \psi_i^+(Y^{-\alpha_i}) = t_i^{\frac{1}{2}} + \psi_i^+(Y^{-\alpha_i})$$

so that it is equal to $b(Y^{-\alpha_i})$. On the other hand, by \((2.1.11)\) we have $\mathcal{L}(w^{-1}, w_0^{-1}) = s_i \mathcal{L}(v^{-1}, w_0^{-1}) \cup \{\alpha_i\}$. Thus we have $b_w(Y) = b_w(s_i Y) b(Y^{-\alpha_i}) = \prod_{\alpha \in \mathcal{L}(w^{-1}, w_0^{-1})} b(Y^{-\alpha})$.

\[
\square
\]

We can apply the argument of the proof to the stabilizer $W_\mu \subset W_0$ for a dominant weight $\mu \in (h^*_+)$. As a result, we have the following claim.

Corollary 3.2.2. For each $\mu \in (h^*_+)\uparrow$, we have

$$\sum_{u \in W_\mu} t_{w_0 u} T_u = \sum_{w \in W_\mu} S^Y_w \prod_{\alpha \in \mathcal{L}(w^{-1}, w_0^{-1})} b(Y^{-\alpha}).$$

Here $b(Y^{-\alpha}) \in \mathbb{K}(Y)$ is given by \((3.2.1)\).

For a dominant weight $\mu \in (h^*_+)\uparrow$, we denote by

$$W^\mu \subset W_0 \quad (3.2.3)$$

the complete system of representatives of the quotient set $W_0/W_\mu$ consisting of the shortest elements. We also denote by $v_\mu \in W^\mu$ its longest element.

Now let us recall the element $w(\mu) \in t(\mu) W_0 \subset W$ in the \((2.2.28)\). We then have the following lemma for the Koornwinder polynomial $P_\lambda(x)$ \((Fact \ 2.2.1)\) and the non-symmetric Koornwinder polynomial $E_\mu(x)$ \((Fact \ 2.2.2)\).

Lemma 3.2.3 \((c.f. \ [Y12 \ Proposition \ 3.6 \ (b)]\)). For $\lambda \in (h^*_+)\uparrow$ we have

$$P_\lambda(x) = \sum_{w \in W^\lambda} \left[ \prod_{\alpha \in \mathcal{L}(\lambda, -1) \subset \mathcal{L}(v^{-1}, w_0^{-1})} \rho(\alpha) \right] E_{w, \lambda}(x),$$

$$\rho(\alpha) := \begin{cases} \frac{t^{\frac{1}{2}} - 1 - t^{-1} q^{sh(-\alpha) - \text{ht}(-\alpha)}}{1 - q^{sh(-\alpha) \text{ht}(-\alpha)}} & (\alpha \not\in \mathcal{L}(\mu)) \\
\frac{1}{t^{\frac{1}{2}}} \frac{1}{t^{\frac{1}{2}} - 1} \frac{1}{t^{\frac{1}{2}} - 1} & (\alpha \in \mathcal{L}(\mu)) \end{cases} \quad (\alpha \in W.\alpha_n)$$

where $sh(\beta)$ and $\text{ht}(\beta)$ for $\beta \in S$ are given by \((2.2.12)\).
Proof. We write Lemma 5.2.1 as
\[ U = \sum_{w \in W_0} S^Y_w b_{(w^{-1}, w^{-1})}(Y), \quad b_{(w^{-1}, w^{-1})}(Y) := \prod_{\alpha \in \mathcal{L}(w^{-1}, w^{-1})} b(Y^{-\alpha}). \]

Since \( W^{\lambda} \) consists of representatives of \( W_0/W_{\lambda} \), there exist \( v \in W^{\lambda} \) and \( u \in W_{\lambda} \) uniquely such that \( w = vu \). Using Corollary 3.2.2, we have
\[
U = \sum_{w \in W_0} S^Y_w b_{(w^{-1}, w^{-1})}(Y) = \left[ \sum_{v \in W^{\lambda}} S^Y_v b_{(v^{-1}, v^{-1})}(Y) \right] \left[ \sum_{w \in W_{\lambda}} S^Y_w b_{(w^{-1}, w^{-1})}(Y) \right]
= \left[ \sum_{v \in W^{\lambda}} S^Y_v b_{(v^{-1}, v^{-1})}(Y) \right] \left[ \sum_{w \in W_{\lambda}} t_{w^{-1}u} T_u \right].
\]
The product with \( S^Y_{v(\lambda)}1 \) gives
\[
US^Y_{v(\lambda)}1 = \left[ \sum_{v \in W^{\lambda}} S^Y_v b_{(v^{-1}, v^{-1})}(Y) \right] \left[ \sum_{w \in W_{\lambda}} t_{w^{-1}u} T_u \right] S^Y_{v(\lambda)}1 = t_{v^{-1}u} W_{\lambda}(t) \sum_{w \in W_{\lambda}} S^Y_w b_{(v^{-1}, v^{-1})}(Y) S^Y_{w(\lambda)}1,
\]
where in the second equality we used the Poincaré polynomial \( 2.2.22 \) and the relation \( (T_u f)(1) = t^u f \) for \( u \in W_0 \) and \( f \in \mathbb{K}[x^{\pm 1}] \) satisfying \( u(f) = f \). The latter relation is shown as follows. If \( s_i f = f \) for some \( i = 1, \ldots, n \), then we have \( (T_i - t_i^u f = c_i(x^{\alpha_i})(s_i - 1)f = 0) \), and \( T_i f = t_i^u f \). Now the relation follows by induction on the length of \( u \in W_0 \).

Let us continue the calculation \( 3.2.3 \). Note that we have \( vw(\lambda) = w(v, \lambda) \) for \( v \in W^{\lambda} \). By this relation and \( 2.2.20 \), each term in the right hand side of \( 3.2.3 \) becomes
\[
S^Y_v b_{(v^{-1}, v^{-1})}(Y) S^Y_{v(\lambda)}1 = S^Y_v S^Y_{w(\lambda)} b_{(v^{-1}, v^{-1})}(w(\lambda)^{-1}Y) 1 = (S^Y_{w(\lambda)}1)(b_{(v^{-1}, v^{-1})}(w(\lambda)^{-1}Y) 1).
\]
Here \( b_{(v^{-1}, v^{-1})}(w(\lambda)^{-1}Y) \) is obtained from \( b_{(v^{-1}, v^{-1})}(Y) \) by replacing \( Y^\mu \) with \( Y^{w(\lambda)^{-1} \mu} \). Now let us recall the equality \( Y^{\alpha} 1 = q^{\lambda(\alpha)} t^{\lambda(\alpha)} \) in \( 2.2.13 \). Then we have \( b(Y^{-\alpha}) 1 = \rho(\alpha) \), and therefore
\[
b_{(v^{-1}, v^{-1})}(w(\lambda)^{-1}Y) 1 = \prod_{\alpha \in \mathcal{L}(v^{-1}, v^{-1})} (b(Y^{-w(\lambda)^{-1} \alpha}) 1) = \prod_{\alpha \in \mathcal{L}(v^{-1}, v^{-1})} \rho(\alpha).
\]
By summing over \( v \in W^{\lambda} \) we have
\[
US^Y_{v(\lambda)}1 = t_{v^{-1}u} W_{\lambda}(t) \sum_{w \in W_{\lambda}} \prod_{\alpha \in \mathcal{L}(v^{-1}, v^{-1})} \rho(\alpha) E_{v, \lambda}(x),
\]
Now the result follows from the definition of \( P_{\lambda}(x) \) (Fact 2.2.1). \( \square \)

3.3 Ram-Yip type formula and its application

In [Y12] Theorem 4.2, Yip derived an expansion formula \( E_{\mu}(x) P_{\lambda}(x) = \sum_{\nu} a^\nu_{\mu} E_{\nu}(x) \) for the product of the non-symmetric Macdonald polynomial \( E_{\mu}(x) \) and the Macdonald polynomial \( P_{\lambda}(x) \) in the case of untwisted affine root systems. In this subsection, we give its \((C_n', C_n')\)-type analogue, i.e., an expansion formula for the product of the non-symmetric Koornwinder polynomial and the Koornwinder polynomial (Proposition 3.3.2).

As a preparation, we cite the explicit formula of the non-symmetric Koornwinder polynomial via alcove walks derived by Orr and Shimozono [OS18]. It is a \((C_n', C_n')\)-analogue of the explicit formula of the non-symmetric Macdonald polynomial in the untwisted affine root systems derived by Ram and Yip [RY11]. Let us call these formulas Ram-Yip type formulas.

We prepare the necessary notations for the explanation. Let us given \( v, w \in W \) and a reduced expression of \( w \). For an alcove walk \( p \in \Gamma(\theta^w, z) \), we denote the decomposition of the element \( e(p) \in W \) \( 2.4.12 \) with respect to the presentation \( W = t(P) \rtimes W_0 \) by
\[
e(p) = t(wt(p)) d(p), \quad d(p) \in W_0, \ wt(p) \in b^+_1.
\]
}(3.3.1)
Fact 3.3.1 ([RY11 Theorem 3.1], [OS18 Theorem 3.13]). For $\mu \in \mathfrak{h}_+^*$, let $w(\mu)$ be the shortest element among $t(\mu)W_0 \subset W$, and fix its reduced expression $w(\mu) = s_{i_1} \cdots s_{i_r}$. Then we have

$$E_\mu(x) = \sum_{p \in \Gamma(w(\mu), e)} f_p \psi_p(x)^{w(\mu)},$$

$$f_p := \prod_{k \in \varphi^+(p)} \psi^+_k(q^{sh(-\beta_k)}l^{ht(-\beta_k)}) \prod_{k \in \varphi^-(p)} \psi^-_k(q^{sh(-\beta_k)}l^{ht(-\beta_k)}),$$

where we set $\beta_k := s_{i_1} \cdots s_{i_k+1}(\alpha_{i_k})$ for $k = 1, \ldots, r$.

Next we introduce some notations necessary for Proposition 3.3.2 which are basically the ones in [Y12 §4.1]. Let us given $v, w \in W$ and a reduced expression $v = s_{i_1} \cdots s_{i_r}$. Recall the set $\Gamma^C(\overline{v}, w)$ of alcove walks belonging to the dominant chamber $C$ as in (3.1.8). Consider an alcove walk in $\Gamma^C(\overline{v}, w)$ together with coloring of all the folding steps by either black or gray. We call such a data a colored alcove walk, and denote by

$$\Gamma^C_2(\overline{v}, w) \quad (3.3.2)$$

the set of colored alcove walks arising from alcove walks in $\Gamma^C(\overline{v}, w)$.

For a colored alcove walk $p \in \Gamma^C_2(\overline{v}, w)$, we denote by

$$p^* \in \Gamma(\overline{v}^{-1}, w^{-1}e(p)) \quad (3.3.3)$$

the uncolored alcove walk obtained by straightening all the gray folding steps of $p$ and by translation so that it ends at $e(p^*) = e \in W$. More explicitly, for a colored positive walk $p \in \Gamma^C_2(\overline{v}, w)$ with

$$p = (wA, ws_{i_1}^1 A, \ldots, ws_{i_r}^1 \cdots s_{i_r}^r A),$$

we define $\tilde{p}_k$ for $k = 1, \ldots, r$ as follows, according to whether the $k$-th step $p_{k-1} = ws_{i_1}^{b_1} \cdots s_{i_k-1} A \rightarrow p_k = ws_{i_1}^{b_1} \cdots s_{i_k}^{b_k} A$ is a gray folding step or not:

$$\tilde{p}_k := \begin{cases} ws_{i_1}^{b_1} \cdots s_{i_k-1} A & (p_{k-1} \rightarrow p_k \text{ is a gray folding step}) \\
p_k & (\text{otherwise}) \end{cases}$$

Thus we obtain a new uncolored alcove walk $\tilde{p} = (\tilde{p}_0, \ldots, \tilde{p}_r) \in \Gamma(\overline{v}, w)$, which was called the one obtained “by straightening all the gray foldings”. Next we denote by $(c_1, \ldots, c_r) \in \{0, 1\}^r$ the bit sequence corresponding to $\tilde{p}$. In other words, we have $\tilde{p} = (wA, ws_{i_1}^{c_1}, \ldots, s_{i_r}^{c_r} A)$. Now the alcove walk $p^*$ is obtained by reversing the order of $\tilde{p}$ and translating the start to $w^{-1}e(\tilde{p})$. Explicitly, we have

$$p^* := (s_{i_1}^{c_1}, \ldots, s_{i_r}^{c_r} A, s_{i_1}^{c_1}, \ldots, s_{i_{r-1}}^{c_{r-1}} A, \ldots, s_{i_1}^{c_1} A, A).$$

Proposition 3.3.2 (c.f. [Y12 Theorem 4.2]). For a weight $\mu \in \mathfrak{h}_+^*$, we take a reduced expression $w(\mu) = s_{i_1} \cdots s_{i_r}$ of $w(\mu) \in t(\mu)W_0 \subset W$. Then for any dominant weight $\lambda \in (\mathfrak{h}_+^*)^*$, we have

$$E_\mu(x)P_\lambda(x) = \sum_{v \in W^\lambda} \sum_{p \in \Gamma^C_2(w(\mu)^{-1}, (v\lambda)(-1))} A_p C_p E_{\pi(p)}(x).$$

Here $W^\lambda$ is given by (3.2.3), and the term $A_p$ is given with the help of $\rho(\alpha)$ in Lemma 3.2.3 by

$$A_p := \prod_{\alpha \in w(\lambda)^{-1}L(w^{-1}e_{s_{i_1}^{-1}})} \rho(\alpha),$$

$$\rho(\alpha) := \begin{cases} \left( \frac{1 - t^{-1}q^{sh(-\alpha)}l^{ht(-\alpha)}}{1 - q^{sh(-\alpha)}l^{ht(-\alpha)}} \right) & (\alpha \not\in W.\alpha_n) \\
\left( \frac{1 + t^{-1}t_n^{-\frac{1}{2}}q^{-\frac{1}{2}}q^{sh(-\alpha)}l^{ht(-\alpha)}(1 - t_0^{-\frac{1}{2}}t_n^{-\frac{1}{2}}q^{-\frac{1}{2}}q^{sh(-\alpha)}l^{ht(-\alpha)})}{1 - q^{sh(-\alpha)}l^{ht(-\alpha)}} \right) & (\alpha \in W.\alpha_n) \end{cases}.$$
We also used $\beta$ on the Ram-Yip type formula

**Proof.** On the Ram-Yip type formula $E_\mu(x) = \sum_{h \in \Gamma_{(w(\mu),c)}} f_h t_{d(h)}^\frac{1}{2} \lambda(w(Y)) S^Y_{w(\lambda)} 1$ (Fact 3.3.1), let us act $US_{w(\lambda)}$ from the left. Then we have

$$E_\mu(x)US_{w(\lambda)}^Y = \sum_{h \in \Gamma_{(w(\mu),c)}} f_h t_{d(h)}^\frac{1}{2} \lambda(w(Y)) S^Y_{w(\lambda)} 1.$$ 

Here the second equality follows from the definition of $\lambda(w(Y))$ and $d(h)$, as well as from the relation $T_i U = t_i^\frac{1}{2} U$ in 2.2.24. Moreover, by Lemma 3.3.23 and using the notation in its proof, we have

$$E_\mu(x)US_{w(\lambda)}^Y = \sum_{h \in \Gamma_{(w(\mu),c)}} f_h x^{e(h)} \sum_{v \in W^\lambda} S_v b_{(v^{-1},v^{-1})} 1 \sum_{h \in \Gamma_{(w(\mu),c)}} f_h x^{e(h)} S_{w(\lambda)} h_{(v^{-1},v^{-1})} 1.$$ 

Here we set $A_p := b_{(\alpha(w(\lambda))^{-1},\alpha(w(\lambda)))} = \prod_{\lambda \in W(\lambda)} \rho(\alpha)$. As for the factor $f_h x^{e(h)} S_{w(\lambda)}^Y 1$ in the final line of 3.3.4, denoting $z := (\alpha(w(\lambda)))^{-1}$ and using Proposition 3.1.3 and Corollary 3.1.5 we have

$$f_h x^{e(h)} S_{w(\lambda)}^Y 1 = f_h \sum_{q \in \Gamma^C(e^{-1})} n_q g_q E_{w(z)}(x).$$

We will rewrite this sum over uncolored alcove walks in $\Gamma^C(e^{-1},z)$ as a sum over colored alcove walks in $\Gamma_{(w(\mu))}^{\frac{1}{2}}(e^{-1},z)$. Let us given an uncolored alcove walk $q \in \Gamma^C(e^{-1}), z$. Since $q$ is an alcove walk of type $e^{-1} = w^{-1}$, we can compare the bit sequence of $q$ with the bit sequence of $h$. In this comparison, if the $k$-th step of $q$ is a folding and the $k$-th step of $h$ is a crossing, then we color the $k$-th folding step of $q$ by gray. Otherwise we color it by black. Thus we obtain a colored alcove walk, which is denoted by $p$. Note that we have $p \in \Gamma_{(w(\mu))}^{\frac{1}{2}}(e^{-1},z)$. Then each term of the right hand side in 3.3.5 is equal to

$$f_h n_q g_q E_{w(z)}(x) = f_p^* n_q g_p E_{w(p)}(x),$$

where $p^*$ is given by (3.3.3). We can also express $f_p^*$ using $\beta_k = s_i_1 \cdots s_{i_{k-1}} (\alpha_{i_k})$ as

$$f_p^* = \prod_{k \in \psi_{w_*}(p^*)} \psi_{i_{k}^*}(q^{h(-\beta_k)} t_{l}(\beta_k)) \prod_{k \in \phi_1(p^*)} \psi_{i_{k}^*}(q^{h(-\beta_k)} t_{l}(\beta_k)).$$
As a result, the last line of (3.3.4) is rewritten by a sum over \( p \in \Gamma^C_w(\overline{w(\mu)^{-1}}) \).

Divided by the factor \( t_{w,\mu}^\pm W_\lambda(t) \), the left hand side of (3.3.3) is equal to \( P_\lambda(x) \). Thus we have

\[
E_\mu(x)P_\lambda(x) = \sum_{v \in W_\lambda} A_p \sum_{h \in \Gamma_P(\overline{w(\mu)^{-1}}) \cap P^C_\varepsilon} f_h \sum_{q \in \Gamma_P^C(\overline{w(\mu)^{-1}},z)} n_{q,\delta} E_{w(q)}(x)
= \sum_{v \in W_\lambda} A_p \sum_{p \in \Gamma^C_w(\overline{w(\mu)^{-1}},z)} f_p g_p n_p E_{w(p)}(x).
\]

We obtain the result by collecting the terms from \( f_p \), \( g_p \) and \( n_p \) which depend only on the \( k \)-th step of \( p \in \Gamma^C_w(\overline{w(\mu)^{-1}}) \) and denoting them by \( C_{p,k} \).

\[\square\]

### 3.4 Littlewood-Richardson coefficients for Koornwinder polynomials

In this subsection, we derive our main Theorem 3.4.2 on LR coefficients of Koornwinder polynomials.

We start with a preliminary lemma. Recall the complete system \( W_\lambda \) of representatives of \( W_0/W_\lambda \) in (3.2.3) and the element \( w(\lambda) \in t(\lambda)W_0 \) in (2.2.28).

**Lemma 3.4.1** (c.f. [Y12], Proposition 3.7). Let \( \lambda \in (h^+_\mathbb{Z})_+ \). If \( v \in W_\lambda \) satisfies \( vw(\lambda) \triangleright_B w(\lambda) \), then we have

\[
US_v^Y S^Y_{w(\lambda)}^1 = \left[ \prod_{\alpha \in \mathcal{L}(w(\lambda)^{-1},(vw(\lambda))^{-1})} \rho(\alpha) \right] US^Y_{w(\lambda)} 1,
\]

where \( \rho(\alpha) \) is defined in Lemma 3.2.3.

**Proof.** Recall the equality \( UT_i = Ut^\frac{1}{2}_i \) for \( i = 1, \ldots, n \). Therefore we have

\[
US_v^Y S^Y_{w(\lambda)}^1 = U(T_i^Y + \psi_i^+(Y^{-\alpha_i}))S^Y_{w(\lambda)}^1 = US^Y_{w(\lambda)}(t_i^Y + \psi_i^+(q^{\delta-h(-w^{-1}\alpha_i)}t^{\lambda-h(-w^{-1}\alpha_i)})).
\]

Assume that \( v \in W_\lambda \) satisfies \( vw(\lambda) \triangleright_B w(\lambda) \) and take a reduced expression \( v = s_{i_1} \cdots s_{i_r} \). Using the above relation, we expand the product \( US^Y_{v} S^Y_{w(\lambda)}^1 = US^Y_{v_1} \cdots S^Y_{v_r} S^Y_{w(\lambda)}^1 \) in order. We have

\[
US^Y_{v} S^Y_{w(\lambda)}^1 = U(t_i^Y + \psi_i^+(Y^{-\alpha_i}))S^Y_{s_{i_1}} S^Y_{w(\lambda)}^1 = US^Y_{s_{i_1}} (t_i^Y + \psi_i^+(Y^{-s_{i_1} \cdots s_{i_2} \alpha_i}))S^Y_{w(\lambda)}^1
= \cdots
= US^Y_{w(\lambda)} \prod_{j=1}^r (t_{j}^Y + \psi_{j}^+(Y^{-w(\lambda)^{-1} s_{j_1} \cdots s_{j_{j+1}} \alpha_j})) S^Y_{w(\lambda)}^1
\]

\[
= US^Y_{w(\lambda)} \left[ \prod_{j=1}^r (t_{j}^Y + \psi_{j}^+(Y^{-w(\lambda)^{-1} s_{j_1} \cdots s_{j_{j+1}} \alpha_j})) \right] 1 = US^Y_{w(\lambda)} \left[ \prod_{\alpha \in \mathcal{L}(w(\lambda)^{-1},(w(\lambda))^{-1})} \rho(\alpha) \right] 1.
\]

Therefore the claim is obtained. \[\square\]

We prepare some symbols to state the main theorem. For \( \mu \in h^+_\mathbb{Z} \), the orbit \( W_0 / \mu \) contains a unique dominant weight. We denote it by

\[
\mu_+ \in W_0 / \mu \cap (h^+_\mathbb{Z})_+.
\]

(3.4.1)

Let us also recall the set \( \Gamma^C_w(\overline{w}, w) \) of colored alcove walks defined in (3.3.2).

**Theorem 3.4.2.** Let us give dominant weights \( \lambda, \mu \in (h^+_\mathbb{Z})_+ \). Choose a reduced expression \( w(\lambda) = s_{i_r} \cdots s_{i_1} \) of the shortest element \( w(\lambda) \in t(\lambda)W_0 \) in (2.2.28). Then we have

\[
P_\lambda(x)P_\mu(x) = \frac{1}{t_{w,\mu}^\pm W_\lambda(t)} \sum_{v \in W_\mu} \sum_{p \in \Gamma^C_w(\overline{w(\mu)^{-1}},(w(\mu))^{-1})} A_p B_p C_p P_{-w_0 \cdot wt(p)}(x).
\]

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Here $A_p := \prod_{\alpha \in w(\mu)^{-1} L(v^{-1}, v^{-1})} \rho(\alpha)$ with $\rho(\alpha)$ given in Proposition 3.3.2. The term $C_p$ is the same as that in Proposition 3.3.2 and $w(p) \in B_2^+$ is defined by 3.3.4. The term $B_p$ is defined by

$$B_p := \prod_{\alpha \in L(v(t(w(p))w_0, e(p))} \rho(\alpha).$$

Proof. The strategy is to calculate the product of Koornwinder polynomials by acting the symmetrizer $U$ to each side of the equation in Proposition 3.3.2.

For a colored alcove walk $p \in \Gamma_0^+(w(\lambda)^{-1}, (w(\mu))^{-1})$, let $z$ be the shortest element among $\{z \in W_0 \mid z \cdot \pi(p)_+ = \pi(p)\}$. Note that we have $w(\pi(p)_+)^{-1} = w(-w_0 \pi(p)_+)$. Since $e(p) \in t(w(p))W_0$ by the definition of $w(p)$, $w(-w_0 \pi(p)_+)$ is the shortest element among $t(w(p))W_0$. By Lemma 3.4.1 we then have

$$UE_{\pi(p)}(x)1 = US_{\pi(p)}^YS_{\pi(p)}^Y1 = \left[ \prod_{\alpha \in L(t(w(p))w_0, e(p))} \rho(\alpha) \right] P_{-w_0, wt(p)}(x).$$

By Proposition 3.3.2 and this equality, we have

$$P_\lambda(x)P_\mu(x) = \frac{1}{t_{e_+}^\pm W_\lambda(t)} U E_{\lambda}(x) P_\mu(x)1
= \frac{1}{t_{e_+}^\pm W_\lambda(t)} \sum_{v \in W_\mu} \sum_{p \in \Gamma_0^+(w(\lambda)^{-1}, (w(\mu))^{-1})} A_p B_p C_p P_{-w_0, wt(p)}(x).$$

Hence the claim is obtained.

\[\square\]

4 Special cases of Littlewood-Richardson coefficients

In Theorem 3.3.2 we derived an explicit formula of the LR coefficient $c^\lambda_{\nu, \mu}$ in the product $P_\lambda(x)P_\mu(x) = \sum_{\nu} c^\nu_{\lambda, \mu} P_\nu(x)$ of Koornwinder polynomials using alcove walks. In this section, we discuss several specializations of the formula.

4.1 Askey-Wilson polynomials

As mentioned in 4.1, Koornwinder polynomials in the rank one case are nothing but Askey-Wilson polynomials. In this case LR coefficients of Askey-Wilson polynomials are expected to be simpler than the general rank case in Theorem 3.3.2.

As a preparation, we summarize the data of the root system of rank 1. We consider the Euclidean space $V = \mathbb{R}^\vee$ of dimension 1 and its dual space $V^* = \mathbb{R}e$. The root system of type $C_1$ is $R = \{ \pm 2e \} \subset V^*$, the simple root is $\alpha_1 = 2e$, and the fundamental weight is $\omega = e$. The weight lattice is $P = \mathbb{Z}e \subset V^*$, and the set of dominant weights is $(\mathbb{H}_2)^+ = \mathbb{Z}e$. The finite Weyl group $W_0$ is the group of order two generated by $s_1 := s_{\alpha_1}$, and the longest element of $W_0$ is $w_0 = s_1$. The affine root system of rank 1 is $S = \{ \pm 2e + k\delta, \pm e + \frac{k}{2}\delta \mid k \in \mathbb{Z} \}$ with $\alpha_0 = \delta - 2e$, and the extended affine Weyl group $W$ is the group generated by $s_1$ and $s_0 := s_{\alpha_0}$. The decomposition $W = t(P) \rtimes W_0$ is the semi-direct product of $t(P) = \langle t(e_1) = s_0s_1 \rangle \simeq \mathbb{Z}^2$ and $W_0 = \langle s_1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$.

We denote by

$$P_l(x) = P_l(x; q, t_0, t_1, u_0, u_1)$$

the Askey-Wilson polynomial associated to the dominant weight $\lambda = t\omega = te \ (l \in \mathbb{N})$. Note that it has five parameters.

First, we consider the simplest case. Following the case of type $A$ (see 1.2), we call the LR coefficients $c^\nu_{\lambda, \mu}$ with $\lambda$ or $\mu$ equal to a minuscule weight Pieri coefficients. Since the weight $\omega_1$ is the unique minuscule weight in the root system of type $C_n$, we consider the case $\lambda = \omega$ for the rank one case.

Let us write down explicitly the Askey-Wilson polynomial $P_1(x) = P_2(x)$. In the following calculation, we need an explicit form of the term $\rho(\alpha)$ ($\alpha \in S$) in Proposition 3.3.2 and Theorem 3.4.2. The result is:

$$\rho(\alpha) := \frac{1}{t_1^{\frac{1}{2}} - q^\frac{1}{2}t_0^{\frac{1}{2}}t_1^{\frac{1}{2}}(t_0t_1)^{-\frac{1}{2}}(1 - q^\frac{1}{2}t_0^{\frac{1}{2}}t_1^{\frac{1}{2}}(t_0t_1)^{-\frac{1}{2}})} \quad (\alpha = 2je + k\delta \in S). \quad (4.1.1)$$

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Lemma 4.1.1. The Askey-Wilson polynomial associated to the minuscule weight $\omega$ is

$$P_1(x) = x + x^{-1} + \rho(2\delta - \alpha_1)\psi_0(qt_0t_1) + t_1^\pm \psi_1^+(qt_0t_1) + \psi_0^+(q^2t_0t_1)\psi_0^-(qt_0t_1).$$

Here $\psi_k^\pm(z)$ ($k = 0, 1$) is given by (2.2.23) with $n = 1$. Explicitly, we have

$$\psi_0^\pm(z) := \pm(\frac{u_1^\pm - u_1^-}{1 - z^{\pm1}}) + z^{\pm1}\frac{1}{1 - z^{\pm1}}(\frac{1}{1 - z^{\pm1}} - \frac{1}{1 - z^{\pm1}}).$$

(4.1.2)

**Proof.** Below we use the word *non-symmetric Askey-Wilson polynomials* to mean non-symmetric Koornwinder polynomials (Fact 2.2.2) in the rank 1 case. By Lemma 3.2.3, we can rewrite $P_1(x)$ as a linear combination of non-symmetric Askey-Wilson polynomials $E_k(x) = E_k\omega(x)$, $k \in \mathbb{Z}$. The result is

$$P_1(x) = \rho(2\delta - \alpha_1)E_1(x) + E_{-1}(x).$$

Next, using the Ram-Yip type formula (Fact 3.3.1), we can expand $E_1(x)$ and $E_{-1}(x)$ by monomials. The results are

$$E_1(x) = t_1^\pm x + \psi_0^-(qt_0t_1), \quad E_{-1}(x) = x^{-1} + t_1^\pm \psi_0^+(qt_0t_1) + t_1^\pm \psi_1^-(q^2t_0t_1)x + \psi_1^+(q^2t_0t_1)\psi_0^-(qt_0t_1).$$

By these formulas, we have

$$P_1(x) = x^{-1} + t_1^\pm \rho(2\delta - \alpha_1) + t_1^\pm \psi_0^+(q^2t_0t_1)x + \psi_0^-(qt_0t_1)\rho(2\delta - \alpha_1) + t_1^\pm \psi_1^+(qt_0t_1) + \psi_1^+(q^2t_0t_1)\psi_0^-(qt_0t_1).$$

By a direct calculation, the coefficients of $x$ is shown to be

$$t_1^\pm \rho(2\delta - \alpha_1) + t_1^\pm \psi_1^+(q^2t_0t_1) = 1.$$

Therefore the claim is obtained.

**Remark 4.1.2.** Let us replace the parameters $(q, t_0, t_1, u_0, u_1)$ with the original parameters $(q, a, b, c, d)$ of Askey-Wilson polynomials in [AW85]. The correspondence (4.1.1) of parameters can be rewritten as

$$(q, t_0, t_1, u_0, u_1) = (q, -q^{-1}ab, -cd, -a/b, -c/d).$$

Using this correspondence and the relation $abcd = qt_0t_1$, we can rewrite $P_1(x)$ as

$$P_1(x) = x + x^{-1} + \frac{\pi s - s'}{1 - \pi}, \quad \pi := abcd, \quad s := a + b + c + d, \quad s' := a^{-1} + b^{-1} + c^{-1} + d^{-1}.$$ 

We can then compare $P_1(x)$ with the original Askey-Wilson polynomials $p_\pi(z)$ in [AW85] p.5. By loc. cit., we have $p_\pi(z) = 2(1 - \pi)z + \pi s - s'$, and thus

$$(1 - \pi)P_1(x) = p_\pi((x + x^{-1})/2).$$

Therefore they coincide up to the normalization factor.

**Proposition 4.1.3.** For a dominant weight $\lambda = \omega \in (\mathfrak{h}_2^+)\_+$, $l \in \mathbb{N}$, we have

$$P_l(x) = P_{l+1}(x) + F_lP_1(x) + G_lP_{l-1}(x),$$

where

$$F_l := -\rho(2\delta - \alpha_1)\psi_0^-(q^{2l+1}t_0t_1) + \psi_0^+(qt_0t_1) + \psi_0^+(q^{2l-1}t_0t_1) + \psi_0^-(qt_0t_1),$$

$$G_l := -\rho(2\delta - \alpha_1)\rho(2l - 1)\frac{\pi s - s'}{1 - \pi} + \frac{\pi s - s'}{1 - \pi}n_0^-(q^{2l-1}t_0t_1).$$

Here $\rho(\alpha)$ is given by (4.1.1), $\psi_k^\pm(z)$ is given by (4.1.2), and $n_0^-(z)$ is given in Proposition 3.1.3 with $n = 1$. Explicitly, the last one is given by

$$n_0^-(z) := \frac{(1 - u_0^\pm u_0^\pm z\pm)}{1 - z}(1 + u_0^\pm u_0^\pm z\pm)(1 - u_0^\pm u_0^\pm z\pm)(1 - u_0^\pm u_0^\pm z\pm).$$
Proof. By Theorem 3.4.2 we have

\[ P_\mu(x)P_\lambda(x) = \frac{1}{t_{w_0}W_\mu(t)} \sum_{v \in W^\lambda} \sum_{\mu \in \Gamma_V^\mu(w(\mu)^{-1} (v w(\lambda))^{-1})} A_\mu B_\mu C_\mu P_{-w_0 \cdot \omega(t)}(x) \]

for dominant weights \( \lambda, \mu \in (\mathfrak{h}_r^+) \). We apply this equation to the case \( \mu = \omega \) and \( \lambda = \omega \). In this case the stabilizer \( W_\mu \subset W_0 \) in (2.2.29) is \( W_\omega = \{ e \} \), and the complete system \( W^\lambda \) (2.2.3) of representatives of \( W_0/W_\lambda \) is \( W^\omega = \{ e, s_1 \} \). As for the shortest element \( w(\nu) \in t(\nu)W_0 \) given in (2.2.28) we have by \( t(\omega) = s_0 s_1 \) that \( w(\omega) = s_0 \) and \( w(l \omega) = (s_0 s_1)^{l-1} s_0 \).

First, we calculate the denominator \( t_{w_0}W_\omega(t) \). As for the longest element \( w_0 \in W^\mu \) in (2.2.30), we have \( w_\omega = e \). Thus, by recalling the definition (2.2.31) of \( t_w \) \( (w \in W) \), we have \( t_{w_0}W_\omega(t) = t_0^* t_e = 1 \).

Next, as for the sum in the right hand side, we calculate the case \( v = s_1 \). The set of alcove walks is then \( \Gamma_V^\omega(w(\mu)^{-1}, (v w(\lambda))^{-1}) = \Gamma_V^\omega(s_0, t(l \omega)) \). In the upper half of Table 4.1.1 we display the alcove walks \( p \) therein together with the corresponding terms \( A_\mu, B_\mu \) and \( C_\mu \). In the table we denote by \( H_0 \) and \( H_1 \) the hyperplanes in the \( W \)-orbits of \( H_{\alpha_0} \) and \( H_{\alpha_1} \) respectively. We also denote a black folding by a solid line, and a gray folding by a dotted line.

Next we study the case \( v = e \). The set of alcove walks is \( \Gamma_V^\omega(w(\mu)^{-1}, (v w(\lambda))^{-1}) = \Gamma_V^\omega(s_0, w(l \omega)^{-1}) \), and in the lower half of Table 4.1.1 we display the alcove walks \( p \) therein together with the corresponding terms \( A_\mu, B_\mu \) and \( C_\mu \).

| \( v = s_1 \) | \( p^* \) | \( p \) | \( A_\mu \) | \( B_\mu \) | \( C_\mu \) |
|---|---|---|---|---|---|
| \( e \) | \( t(l \omega) \) | \( H_1 \) | \( H_0 \) | 1 | 1 | 1 |
| \( H_1 \) | \( H_0 \) | \( \rho(2l \delta - \alpha_1) \) | \( -\psi_0^{-}(q^{2l-1} t_0 t_1) \) |

| \( v = e \) | \( p^* \) | \( p \) | \( A_\mu \) | \( B_\mu \) | \( C_\mu \) |
|---|---|---|---|---|---|
| \( e \) | \( t((l-1) \omega) \) | \( H_1 \) | \( H_0 \) | \( \rho(2l \delta - \alpha_1) \) | \( \rho((2l-2) \delta - \alpha_1) \) | \( n_0(q^{2l-1} t_0 t_1) \) |
| \( H_1 \) | \( H_0 \) | \( \rho(2l \delta - \alpha_1) \) | \( -\psi_0^{+}(q^{2l-1} t_0 t_1) \) |
| \( e \) | \( t((l-1) \omega) \) | \( H_1 \) | \( H_0 \) | \( \rho(2l \delta - \alpha_1) \) | \( 1 \) | \( -\psi_0^{+}(q^{2l-1} t_0 t_1) \) |

Table 4.1.1: Colored alcove walks in Proposition 4.1.3

The claim is obtained by to summaries the above calculation. \( \square \)
Remark 4.1.4. Continuing Remark 4.1.2, we rewrite the result in Proposition 4.1.3 in terms of the original parameters \((q, a, b, c, d)\) of Askey-Wilson polynomials. The result is

\[ P_l(x)p_l(x) = P_{l+1}(x) + F_lP_l(x) + G_lP_l(x), \]

(4.1.3)

where the factors \(F_l\) and \(G_l\) are given by

\[ F_l := \frac{f_l + (\pi s' - s)}{1 - q} \quad \text{and} \quad G_l := \frac{g_l - \gamma l^{-1}}{\gamma l^{-1}}. \]

\(G_l\) is given in Theorem 3.4.2 to the case

Proof. As for the denominator

In the case \((\text{Corollary} 4.1.5). \)

\[ \text{polynomials.} \]

\[ \text{series.} \]

Continuing Remark 4.1.2, we rewrite the result in Proposition 4.1.3 in terms of the \(\pi := abcd, \quad s := a + b + c + d, \quad s' := a^{-1} + b^{-1} + c^{-1} + d^{-1}, \quad \gamma := (q l^{-1}, q l^{-1}) (1 - q l^{-1}) \cdots (1 - q l^{-2} \pi). \]

In the case \(l = 0\), we have \(\rho (-\alpha_1) = 0\), and thus \(F_0 = 0\). If we define \(p_l(z)\) by the relation \(P_l(x) = \gamma_l^{-1} p_l((x + x^{-1})/2)\), then the relation (4.1.3) can be rewritten as

\[ 2\pi p_l(z) = h_l p_{l+1}(z) + f_l p_l(z) + g_l p_{l-1}(z), \quad h_l := \frac{1 - \pi q^l_1}{(1 - \pi q^l_1)(1 - \pi q^l_2)}, \quad p_0(z) = 1, \quad p_{-1}(z) = 0. \]

This recurrence formula is nothing but the one in [AWS5] (1.24)–(1.27). Thus \(p_l\) coincides with the original Askey-Wilson polynomial in [AWS5], and in particular, it can be expressed as a \(q\)-hypergeometric series.

So far we studied Pieri coefficients. Next we study the general LR coefficients for Askey-Wilson polynomials.

Corollary 4.1.5. For dominant weights \(\lambda, \omega, m, \omega \in (h^*_\mathbb{Z})_+, \quad \lambda, m \in \mathbb{N}\), we have

\[ P_\omega(x)P_{m\omega}(x) = \sum_{v \in W_{\omega}, p \in \Gamma^+_{\omega}(t((l-1)\omega)_{s_0}, t(m\omega)_{s_1}v)} A^\text{AW}_p B^\text{AW}_p C_p P_{\nu\omega, \omega \nu}(x), \]

where the terms \(A^\text{AW}_p\) and \(B^\text{AW}_p\) are given by

\[ A^\text{AW}_p := \begin{cases} \rho(2m\delta - \alpha_1) & (v = e) \\ 1 & (v = s_1) \end{cases}, \quad B^\text{AW}_p := \begin{cases} \rho(\ell(e(p))\delta - \alpha_1) & (\ell(e(p)) \in 2\mathbb{Z}) \\ 1 & (\ell(e(p)) \notin 2\mathbb{Z}) \end{cases} \]

with \(\rho(\alpha)\) in Proposition 4.1.3 and \(C_p\) is given in Theorem 3.4.2.

Proof. We apply the formula

\[ P_\lambda(x)P_\mu(x) = \frac{1}{t^e_{\lambda, \omega} W_\lambda(t)} \sum_{v \in W_\omega, p \in \Gamma^+_{\omega}(w(\lambda)^{-1}, (vw(\mu)))^{-1}} A_p B_p C_p P_{-w_0, \nu \omega}(x), \]

in Theorem 3.4.2 to the case \(\lambda = \omega\) and \(mu = m\omega\). Similarly as in Proposition 4.1.3, we have \(W_{\omega}(\lambda) = \{e\}\) and \(W^m_{\omega} = \{e, s_1\} = W_0\). Using \(t(\lambda) = (s_0 s_1)^l\) and \(t(m\omega) = (s_0 s_1)^m\), we have \(w(\omega) = t((l-1)\omega)s_1 = (s_0 s_1)^{l-1} s_0\) and \(w(\mu) = (s_0 s_1)^{m-1} s_0\). Therefore the range of the sum of alcove walks in the right hand side becomes

\[ \Gamma^+_{\omega}(w(\lambda)^{-1}, (vw(\mu))^{-1}) = \Gamma^+_{\omega}(t((l-1)\omega)s_0, t(m\omega)_{s_1}v) \quad (v \in W_0). \]

As for the denominator \(t^e_{\omega, \omega} W_\omega(t)\), we have by \(w_\omega = e\) that \(t^e_{\omega, \omega} W_\omega(t) = t^e_{e, e} = 1\).

Now we study the factors \(A_p\) and \(B_p\), and want to reduce the ranges of the products. First, as for the product \(A_p = \prod_{\alpha \in \mathcal{L}(vw(\mu))^{-1}, t(-\omega_0\mu)} \rho(\alpha)\), the longest element \(w_0 \in W_0\) is \(s_1\) and \(t(\mu) = (s_0 s_1)^l = w(\mu)^{-1} s_0\). Thus, in the case \(v = e\), we have

\[ \mathcal{L}(vw(\mu))^{-1}, t(-\omega_0\mu) = \mathcal{L}(w(\mu)^{-1}, t(\mu)) = \{2m\delta - \alpha_1\}. \]
In the case \(v = s_1\), we have
\[
\mathcal{L}((vw(\mu))^{-1}, t(-w_0\mu)) = \mathcal{L}(w(\mu)s_1, t(\mu)) = \mathcal{L}(t(\mu), t(\mu)) = \emptyset.
\]
Hence \(A_p\) is equal to \(A_p^{AW}\) in the claim.

Next we consider the product \(B_p = \prod_{\alpha \in \mathcal{L}(t(w_0p)v_1w_0\mu))} \rho(\alpha)\). We separate the argument according to whether the length \(\ell(\ell(p))\) of \(p\) is even or odd. In the case \(\ell(p)\) is even, there is \(k \in \mathbb{N}\) such that \(e(p) = (s_k s_1)^n = t(k\omega), 0 \leq k \leq m\). In this case, the range of the product is
\[
\mathcal{L}(t(w_0p)v_0, e(p)) = \mathcal{L}(t(k\omega)s_1, t(k\omega)) = \{2k\delta - \alpha_1\}.
\]
In the case \(\ell(p)\) is odd, there is \(k \in \mathbb{N}\) such that we can write \(e(p) = (s_k s_1)^{k-1} s_0 = t(k\omega)s_1, 1 \leq k \leq m\) Therefore the range of the product is
\[
\mathcal{L}(t(w_0p)v_0, e(p)) = \mathcal{L}(t(k\omega)s_1, t(k\omega)s_1) = \emptyset.
\]
Therefore \(B_p\) is equal to \(B_p^{AW}\) in the claim. \(\square\)

### 4.2 Macdonald polynomials of type \(C_n\)

By the specialization \(u_0 = u_n = 1\) in the parameters \((q, t, t_0, t_n, u_0, u_n)\) of Koornwinder polynomials, we obtain Macdonald polynomials of type \(C_n\):

\[
P_{\lambda}^{C_n}(x) := P(x; q, t, t_0, t_n, u_0 = 1, u_n = 1).
\]

In this subsection, we derive the LR coefficients for \(P_{\lambda}^{C_n}(x)\) from Theorem 3.4.2 which recovers Yip’s result [Y12] Theorem 4.4] in the particular case of type \(C_n\) (Remark 1.2.2).

We prepare some terminology for the following argument. If the \(k\)-th step of an alcove walk \(p \in \Gamma(v, w)\) is of the form \(p_{k-1} = uA \Rightarrow p_k = us_k A (b \in \{0, 1\}, u \in W)\), then we call it a 0-step.

Recalling the sets \(\varphi_+(p)\) of foldings in (2.1.15), we denote by \(\varphi_0(p) \subset \varphi_+(p) \cup \varphi_-(p)\) the set of 0-steps.

**Proposition 4.2.1.** Let us given dominant weights \(\lambda, \mu \in (\mathfrak{h}_n^+)\) and a reduced expression \(w(\lambda) = s_{i_1} \cdots s_{i_1}\) of the shortest element \(w(\lambda)\) given in (2.2.28). Then we have
\[
P_{\lambda}^{C_n}(x)P_{\mu}^{C_n}(x) = \frac{1}{(t_{w_0}^{\lambda})^{-1} W_{\lambda}^{C_n}(t)} \sum_{v \in W_{\mu}} \sum_{p \in \Gamma_{2,\mu}(w(\lambda)^{-1}, (vw(\mu))^{-1})} A_p B_p C_{p^{C_n}_{w_0, w_0}} P_{\mu}^{C_n}(x).
\]

Here \(t_{w_0}^{\lambda}\) and \(W_{\lambda}^{C_n}(t)\) are obtained by specializing \(u_0 = u_n = 1\) in \(t_{w_0}\) and \(W_{\lambda}(t)\) respectively, and
\[
\Gamma_{2,\mu}(w(\lambda)^{-1}, (vw(\mu))^{-1}) := \{ p \in \Gamma_{2}(w(\lambda)^{-1}, (vw(\mu))^{-1}) \mid \varphi_0(p) = \emptyset \}.
\]

As for the factor \(C_{p^{C_n}_{w_0, w_0}}\), recall the term
\[
\prod_{k \in \Delta_{\mu}(p)} n_{i_k} (q^{h_k(p)}t^{h_k(p)})
\]
corresponding to a negative crossing in \(C_p\) (Proposition 3.3.2). Then \(C_{p^{C_n}_{w_0, w_0}}\) is obtained by replacing \(n_{i_k}(q^{h_k(p)}t^{h_k(p)})\) with 1 if \(i_k = 0\).

**Proof.** We specialize \(u_0 = u_n = 1\) in the formula
\[
P_{\lambda}(x)P_{\mu}(x) = \frac{1}{t_{w_0}^{\lambda} W_{\lambda}(t)} \sum_{v \in W_{\mu}} \sum_{p \in \Gamma_{2}(w(\lambda)^{-1}, (vw(\mu))^{-1})} A_p B_p C_{p} P_{\mu}(x)
\]
of Theorem 3.4.2. Since \(u_0\) and \(u_n\) don’t appear in \(A_p\) and \(B_p\), we discuss the specialization of \(C_p\) only.

By Proposition 3.3.2, the parameters \(u_0\) and \(u_n\) appear in \(C_p = \prod_{k=1} C_{p,k}\) when \(p\) has a 0-step. If the \(k\)-th step of \(p\) is a 0-step and a negative crossing, then we have
\[
C_{p,k} = \prod_{k \in \Delta_{\mu}(p)} n_{i_k} (q^{h_k(p)}t^{h_k(p)}).
\]

28
Here the parameters $u_0, u_n$ appear only in $n_0(q^{ah(-h_k(p))}t^{ht(-h_k(p))})$. By specializing $u_0 = u_n = 1$ in the factor

$$n_0(z) = \frac{(1 - u_n^{-\frac{1}{2}}u_0^{-\frac{1}{2}}z^\frac{1}{2})(1 + u_n^{-\frac{1}{2}}u_0^{-\frac{1}{2}}z^{-\frac{1}{2}})(1 + u_n^{-\frac{1}{2}}u_0^{-\frac{1}{2}}z^\frac{1}{2})(1 - u_n^{-\frac{1}{2}}u_0^{-\frac{1}{2}}z^{-\frac{1}{2}})}{1 - z},$$

we have $n_0(z) = 1$.

Since we have $\psi_1^2(z) = 0$ by the specialization $u_0 = u_n = 1$, if an alcove walk $p$ has a folding that is also a 0-step, then we have $C_p = 0$. Thus $\Gamma_{C,\infty}^C$ in the claim is the effective set of alcove walks, and we obtain the conclusion. \hfill \Box

**Remark 4.2.2.** Let us explain how to recover Yip’s formula [Y12, Theorem 4.2] of type $C_n$ from Proposition 4.2.1. The factor $\Gamma_{C,\infty}^C((w(\lambda))^{-1}, (w(\mu))^{-1})$ in Proposition 4.2.1 coincides with the set of the colored alcove walks of type $C_n$ in [Y12, Theorem 4.2]. We can factor the term $C_{p,k}$ in Proposition 4.3.2 according to whether the hyperplane corresponding to a gray folding belongs to the $W$-orbit of $H_{\alpha_k}$.

Then by collecting the signs in $C_{p,k}$’s, we recover Yip’s formula.

**Remark 4.2.3.** As mentioned in [K92] one can also obtain Macdonald polynomials of types $BC_n$ and $B_n$, other than $C_n$, by specializing parameters in Koomwinder polynomials as shown in [K92]. Thus we expect to make a similar argument as in this subsection for these types. However, there are subtleties in the specialization for these types since our basic representation of $H(W)$ uses the weight lattice $P = h_Z^*$ of type $C_n$. We hope to discuss this point in a future work.

### 4.3 Hall-Littlewood limit

In the case of type $A_n$, the specialized Macdonald polynomials $P_{\lambda}(x; q = 0, t)$ coincide with Hall-Littlewood polynomials. Motivated by this fact, Yip calls in [Y12 §4.5] the specialized Macdonald polynomials in the untwisted cases at $q = 0$ Hall-Littlewood polynomials, and derived a simplified formula of LR coefficients. Following Yip’s terminology, let us call the specialized Koomwinder polynomials $P_\lambda(x; t) := P_\lambda(x; q = 0, t, t_n, u_0, u_n)$ the Hall-Littlewood limit.

**Proposition 4.3.1** (c.f. [Y12 Corollary 4.13]). Let us given dominant weights $\lambda, \mu \in (h_Z^*)_+$ and a reduced expression $w(\lambda) = s_{i_1}\cdots s_{i_l}$ of the shortest element $w(\lambda)$ (2.2.28). Then we have

$$P_\lambda(x; t)P_\mu(x; t) = \frac{1}{t_{w(\lambda)}W(\lambda)} \sum_{p \in \Gamma_{+}^C((w(\lambda))^{-1}, (w(\mu))^{-1})} F_p(t)P_{-w_0, wt(p)}(x; t),$$

$$F_p(t) := \prod_{\alpha \in \mathcal{L}((w(\mu))^{-1}, t(-w_0))} t_\alpha^\frac{1}{2} \prod_{\alpha \in \mathcal{L}(t(wt(p))w_0, c(p))} t_\alpha^\frac{1}{2} \times \prod_{k \in \varphi_+(p), \ a_k \notin W, a_0} (t_\alpha^{-\frac{1}{2}} - t_\alpha^\frac{1}{2}) \prod_{k \in \varphi_+(p), \ a_k \in W, a_0} (u_n^{-\frac{1}{2}} - u_n^\frac{1}{2}).$$

Here $\Gamma_{+}^C((w(\lambda))^{-1}, (w(\mu))^{-1})$ is the subset of $\Gamma_{+}^C((w(\lambda))^{-1}, (w(\mu))^{-1})$ consisting of alcove walks whose foldings are positive.

**Proof.** We denote the coefficient in Theorem 4.3.2 by

$$a_{p}(q, t) := A_p B_p C_p.$$

First, we show that if $a_{p}(0, t) \neq 0$ for a colored alcove walk $p \in \Gamma_{+}^C((w(\lambda))^{-1}, (w(\mu))^{-1})$, then all the foldings of $p$ are gray and positive. We assume that the $k$-th step of $p$ is a gray negative folding. Then, as for the factor $C_{p,k} = -\psi_{i_k}(q^{ah(-h_k(p))}t^{ht(-h_k(p))})$ we have $C_{p,k}|_{q=0} = 0$. In fact, we have

$$\psi_{i_k}(z) = \frac{(t_{i_k}^\frac{1}{2} - t_{i_k}^{-\frac{1}{2}}) + z^\frac{1}{2}(u_{i_k}^\frac{1}{2} - u_{i_k}^{-\frac{1}{2}})}{1 - z} = \frac{z(t_{i_k}^\frac{1}{2} - t_{i_k}^{-\frac{1}{2}}) + z^\frac{1}{2}(u_{i_k}^\frac{1}{2} - u_{i_k}^{-\frac{1}{2}})}{1 - z}.
and by substituting $z = q^{\text{sh}(-h_\kappa(p))} t^{\text{ht}(-h_\kappa(p))}$ and $q = 0$ we have $C_{p,k|q=0} = 0$. Thus we showed that no gray negative folding contributes to $a_p(0,t)$.

Next we show that black foldings of $p$ don’t contribute to $a_p(0,t)$. Note that there exists an alcove walk $l \in \Gamma(w(\lambda),e)$ whose steps are crossings since we fixed a reduced expression of $w(\lambda)$. Moreover all the steps of $l$ are positive. Then we find that any alcove walk in $\Gamma^C_p(w(\lambda),e) \setminus \{l\}$ has a negative folding. In other words, if an alcove walk $p \in \Gamma^C_p(w(\lambda)^{-1},(vw(\mu))^{-1})$ has a black folding, then $p^* \in [3.3.3]$ has at least one negative folding. Then, as for the factor $C_{p,k} = -\psi_{p,k}(q^{\text{sh}(-h_k)}) t^{\text{ht}(-h_k)}$, we have $C_{p,k|q=0} = 0$ by a direct calculation. Thus, no black folding contributes to $a_p(0,t)$.

By the discussion so far, we find that neither colored folding contributes to $a_p(0,t)$. Thus, the set of alcove walks effective to the sum is $\{p \in \Gamma^C_p(w(\mu)^{-1},(v.w(\lambda))^{-1}) \mid \varphi(p) = \varphi(p)\}$.

Specializing $q = 0$ in $A_p$, $B_p$ and $C_p$, we have

\[
A_p|_{q=0} = \prod_{\alpha \in L((v.w(\mu)))} t^{\frac{1}{3}}_{\alpha}, \quad B_p|_{q=0} = \prod_{\alpha \in L(t(wt(p)))} t^{\frac{1}{3}}_{\alpha}, \quad C_p|_{q=0} = \prod_{k \in \varphi_p(p), \alpha_k \notin W.\alpha_0} (t^{\frac{1}{2}}_{\alpha_k} - t^{\frac{1}{2}}_{\alpha_k}) \prod_{k \in \varphi_p(p), \alpha_k \in W.\alpha_0} (u_n^{-\frac{1}{2}} - u_n^{\frac{1}{2}}).
\]

Therefore the claim is obtained. \hfill \Box

### 4.4 Examples in rank 2

Finally, as explicit examples of LR coefficients in Theorem 3.4.2 we calculate the product $P_\lambda(x)P_\mu(x)$ of Koornwinder polynomials of rank 2.

We write down the root system of rank 2. The root system of type $C_2$ is

\[ R := \{ \pm \epsilon_1 \pm \epsilon_2 \} \cup \{ \pm 2 \epsilon_1, \pm 2 \epsilon_2 \} \subset V^* := \mathbb{R} \epsilon_1 \oplus \mathbb{R} \epsilon_2, \]

the simple roots are $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = 2 \epsilon_2$, and the fundamental weights are $\omega_1 = \epsilon_1$ and $\omega_2 = \epsilon_1 + \epsilon_2$. The weight lattice is $P = h_{\epsilon_1}^\mathbb{Z} = \mathbb{Z} \epsilon_1 \oplus \mathbb{Z} \epsilon_2 \subset V^*$, and the set of dominant weights is $(h_{\epsilon_1}^\mathbb{Z})_+ = \{ \lambda \epsilon_1 + \lambda_2 \epsilon_2 \in h_{\epsilon_1}^\mathbb{Z} \mid \lambda_1 \geq \lambda_2 \geq 0 \}$. The finite Weyl group $W_0$ is the hyper-octahedral group of order 8 generated by $s_1 := s_{\epsilon_1}$ and $s_2 := s_{\epsilon_2}$. The longest element of $W_0$ is $w_0 = s_1 s_{21} s_2 = s_2 s_1 s_{21}$.

The affine root system of type $(C_2^2, C_2)$ is

\[ S = \{ \pm 2 \epsilon_i + k \delta, \pm \epsilon_i + \frac{1}{2} k \delta \mid k \in \mathbb{Z}, i = 1, 2 \} \cup \{ \pm \epsilon_i \pm \epsilon_j + k \delta \mid k \in \mathbb{Z} \}, \]

and the affine simple root is $\alpha_0 = \delta - 2 \epsilon_1$. The extended affine Weyl group $W$ is generated by $s_1, s_2$ and $s_0 := s_{\alpha_0}$, and the decomposition $W = t(P) \rtimes W_0$, \cite[2.1.2] is a semi-direct product of $t(P) = (t(\epsilon_1), t(\epsilon_2)) \simeq \mathbb{Z}^2$ and $W_0 = (s_1, s_2) \simeq \{ \pm 1 \}^2 \rtimes \mathbb{Z}_2$. The elements $t(\epsilon_1)$ and $t(\epsilon_2)$ have reduced expressions $t(\epsilon_1) = s_0 s_1 s_2 s_1$ and $t(\epsilon_2) = s_1 s_0 s_1 s_2$ respectively.

In this setting we apply Theorem 3.4.2 to the case $\lambda = \omega_1$ and $\mu = \omega_2$. The result is as follows.

**Proposition 4.4.1.** For Koornwinder polynomials of rank 2, we have

\[
P_{\omega_1}(x)P_{\omega_2}(x) = \frac{1}{t^{\frac{3}{2}}_{\lambda} + t^{\frac{3}{2}}_{\mu}} \left( FP_{\omega_1+w_2}(x) + GP_{\omega_2}(x) + H P_{\omega_1}(x) \right),
\]

where

\[
F := 2\rho(2\delta - 2\epsilon_1),
\]

\[
G := -2\rho(2\delta - 2\epsilon_2)\rho(2\delta - (\epsilon_1 + \epsilon_2))\rho(2\delta - 2\epsilon_1)\rho(\epsilon_1 - \epsilon_2)
\]

\[
\times \left( \psi_0^-(ql_0 t_2) + \psi_0^+(ql_0 t_2) + \psi_0^-(ql_0 t_2) + \psi_0^+(ql_0 t_2) \right).
\]

\[
H := 2\rho(2\delta - 2\epsilon_1)\rho(2\delta - (\epsilon_1 + \epsilon_2))^2 \rho(2\delta - 2\epsilon_2)\rho(2\epsilon_2)\rho(ql_0 t_2).
\]

**Proof.** Applying Theorem 3.4.2 to $n = 2$, $\lambda = \omega_1$ and $\mu = \omega_2$, we have

\[
P_{\omega_1}(x)P_{\omega_2}(x) = \frac{1}{t^{\frac{3}{2}}_{\lambda} + t^{\frac{3}{2}}_{\mu}} \sum_{w \in W^{\omega_2}} \sum_{p \in \Gamma^C_p(w(\omega_1)^{-1},(vw(\omega_2))^{-1})} A_p B_p C_p P_{-w_0,w_0}(x).
\]
We have $W_{\omega_1} = \{e, s_2\}$, $W_{\omega_2} = \{e, s_2, s_1s_2, s_2s_1s_2\}$ and $w(\omega_1) = s_0$, $w(\omega_2) = s_0s_1s_0$. The denominator $t_{w_{\omega_1}} W_{\omega_1}(t)$ can be calculated with the help of $w_{\omega_1} = s_2$ as $t_{w_{\omega_1}} W_{\omega_1}(t) = t_{s_2}^{-\frac{1}{2}} (t_e + t_{s_2}) = t_2^{-\frac{1}{2}} + t_2^\frac{1}{2}$.

Next we consider the term $A_p B_p C_p$. The alcove walk $p^*$ associated to $p \in \Gamma_2^s(w(\omega_1))^{-1}, (vw(\omega_2))^{-1}$ is given by either $p_1^*$ or $p_2^*$ in Table 4.4.1.

![Diagram](image)

Table 4.4.1: Classification of $p^*$

Let us calculate the term $A_p = \prod_\alpha \rho(\alpha)$. The range of the product is

$$w(\mu)^{-1} \mathcal{L}(v^{-1}, v^{-1}) = \mathcal{L}((vw(\omega_2))^{-1}, t(-w_0\omega_2)),$$

and according to $v \in W_{\omega_2} = \{e, s_2, s_1s_2, s_2s_1s_2\}$ it is given by

$$\mathcal{L}((vw(\omega_2))^{-1}, t(-w_0\omega_2)) = \begin{cases} 
\{2\delta - 2\epsilon_1, 2\delta - (\epsilon_1 + \epsilon_2), 2\delta - 2\epsilon_2\} & (v = e) \\
\{2\delta - (\epsilon_1 + \epsilon_2), 2\delta - 2\epsilon_2\} & (v = s_2) \\
\{2\delta - 2\epsilon_2\} & (v = s_1s_2) \\
\emptyset & (v = s_2s_1s_2).
\end{cases}$$

Then we have

$$A_p = \begin{cases} 
\rho(2\delta - 2\epsilon_1)\rho(2\delta - (\epsilon_1 + \epsilon_2))\rho(2\delta - 2\epsilon_2) & (v = e) \\
\rho(2\delta - (\epsilon_1 + \epsilon_2))\rho(2\delta - 2\epsilon_2) & (v = s_2) \\
\rho(2\delta - 2\epsilon_2) & (v = s_1s_2) \\
1 & (v = s_2s_1s_2).
\end{cases}$$

For each $v \in W_{\omega_2}$ and the corresponding colored alcove walks $p$, we calculate $B_p$ and $C_p$. The results are shown in Tables 4.4.2 and 4.4.3. The symbol in the column of $p$ such as $X_{11}$ and $X_{12}$ refers to the corresponding picture in Figure 4.4.1.

| $p^*$ | $p$ | $B_p$ | $C_p$ | $-uw_0 wt(p)$ |
|-------|-----|-------|-------|---------------|
| $p_1^*$ | $X_{11}$ | $\rho(2\epsilon_2)\rho(\delta - (\epsilon_1 + \epsilon_2))$ | $n_0(ql_0 t_1)$ | $\omega_1$ |
|       | $X_{12}$ | $\rho(\epsilon_1 - \epsilon_2)$ | $-\psi_0(ql_0 t_1)$ | $\omega_2$ |
| $p_2^*$ | $X_2$ | $\rho(\epsilon_1 - \epsilon_2)$ | $-\psi_0(ql_0 t_1)$ | $\omega_2$ |

Table 4.4.2: Colored alcove walks in the case $v = e$

| $p^*$ | $p$ | $B_p$ | $C_p$ | $-uw_0 wt(p)$ |
|-------|-----|-------|-------|---------------|
| $p_1^*$ | $Y_{11}$ | $\rho(2\epsilon_2)\rho(2\delta - 2\epsilon_1)\rho(\delta - (\epsilon_1 + \epsilon_2))$ | $n_0(ql_0 t_1)$ | $\omega_1$ |
|       | $Y_{12}$ | $\rho(\epsilon_1 - \epsilon_2)\rho(2\delta - 2\epsilon_1)$ | $-\psi_0(ql_0 t_1)$ | $\omega_2$ |
| $p_2^*$ | $Y_2$ | $\rho(\epsilon_1 - \epsilon_2)\rho(2\delta - 2\epsilon_1)$ | $-\psi_0(ql_0 t_1)$ | $\omega_2$ |

Table 4.4.3: Colored alcove walks in the case $v = s_2$

The claim is now obtained by summing the terms $A_p B_p C_{-wu wt(p)}(x)$. $\square$
Table 4.4.4: Colored alcove walks in the case \( v = s_2 s_1 s_2 \)

Figure 4.4.1: Colored alcove walks in Proposition 4.4.1
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