Abstract

In this paper, we study the realizability problem for retarded functional differential equations near an equilibrium point undergoing a nonlinear mode interaction between a saddle-node bifurcation and a non-resonant multiple Hopf bifurcation. In contrast to the case of transcritical/multiple Hopf interaction which was studied in an earlier paper [4], the analysis here is complicated by the presence of a nilpotency which introduces a non-compact component in the symmetry group of the normal form. We present a framework to analyse the realizability problem in this non-semisimple case which exploits to a large extent our previous results for the realizability problem in the semisimple case. Apart from providing a solution to the problem of interest in this paper, it is believed that the approach used here could potentially be adapted to the study of the realizability problem for toroidal normal forms in the general case of repeated eigenvalues with Jordan blocks.
1 Introduction

Retarded functional differential equations (RFDEs) are frequently used as models for various phenomena [1, 2, 16, 17, 18, 19, 20]. While the phase space for the resulting dynamical system is infinite dimensional, the existence of finite-dimensional invariant center manifolds near bifurcation points imply that much of the machinery developed for the analysis of finite codimension bifurcations in ordinary differential equations (e.g., normal forms, unfolding theory) are portable to RFDEs. Indeed, there is ample evidence in the literature, e.g., [1, 16, 19], that these tools and techniques of local analysis can give much valuable information about the dynamics of RFDEs.

In this context, one of the fundamental questions concerns the characterization of the range of dynamics accessible near a bifurcation point for a given RFDE model. This is not a trivial question, since even for scalar RFDEs, it is the possible to have bifurcations of equilibria with large dimensional center manifolds. In this case, if there are not enough independent delay terms in the nonlinear part of the RFDE, there may be severe restrictions on the possible dynamics which can be realized in the center manifold equations. The study of these types of questions is known generally as the realizability problem for RFDEs, and we refer the reader to the Introduction of [4] and to [8, 9, 14] for more details.

In a previous paper [4], we studied the realizability problem for scalar RFDEs in two important cases: the case of multiple non-resonant Hopf bifurcation, and the case of the interaction between a transcritical steady-state bifurcation and a multiple non-resonant Hopf bifurcation. In particular, we used the fact that these bifurcations admit normal forms with toroidal symmetries, and that these symmetries allow for a decoupling of the center manifold normal form equations into a radial part and an angular part. The uncoupled radial equations are a crucial part of the dynamics near the bifurcation, and it is thus very reasonable to consider the problem of realizability of the class of radial equations within a given RFDE. This problem was solved in general in [4] for the two important bifurcation scenarios described above. Moreover, it was shown in [4] that our solution to this realizability problem, that is, our estimate on the number of independent delays sufficient to achieve complete realizability, is optimal. The results in [4] considerably generalize previous results of Faria and Magalhães [9] and of Buono and Bélair [3].

However, in [4] the case of the interaction between a saddle-node steady-state bifurcation and a multiple non-resonant Hopf bifurcation was not studied, because there is a nilpotency associated to this bifurcation which is such that the approach used in [4] is not applicable. This nilpotency arises from the fact that the normal form for the saddle-node bifurcation contains an unfolding parameter which appears as an affine linear perturbation of the singular
vector field, i.e.

\[ \dot{\rho} = \nu + a \rho^2 + O(\rho^3) \]
\[ \dot{\nu} = 0. \]

A consequence of this nilpotency is that the normal form for the saddle-node/multiple Hopf interaction admits a symmetry group which is a direct product of a compact torus group, and a non-compact group isomorphic to the additive group of real numbers. This latter non-compact portion does not occur in the cases studied in [4]. It thus becomes important to develop a framework which allows for the characterization of the effects of this non-compact symmetry on the normal form before being able to address the realizability problem for this bifurcation in RFDEs.

In this paper, we adapt the Faria and Magalhães normal form procedure [6, 7] to the saddle-node/multiple Hopf interaction in scalar RFDEs, and use it to develop a framework suitable to studying the realizability problem for the radial part of the normal form in this nilpotent (non-semisimple) case. The framework is carefully constructed so that in the end, we may use our solution to the realizability problem in the semisimple case in [4] to the fullest possible extent.

2 Preliminaries

In this section we will briefly recall some standard results and terminology in the bifurcation theory of RFDEs in order to establish the notation. For more details, see [4, 6, 7, 15].

2.1 Infinite dimensional parameterized ODE

Suppose \( r > 0 \) is a given real number, \( n \geq 1 \) is a given integer and \( C_n \equiv C([-r, 0], \mathbb{R}^n) \) is the Banach space of continuous functions from \([-r, 0]\) into \( \mathbb{R}^n \) with supremum norm. We define \( u_t \in C_n \) as \( u_t(\theta) = u(t + \theta), -r \leq \theta \leq 0 \). Let us consider the following parameterized family of scalar \((n = 1)\) nonlinear retarded functional differential equations

\[ \dot{z}(t) = L(\alpha)z_t + F(z_t, \alpha), \]  

(2.1)

where \( L : C_1 \times \mathbb{R}^{s+1} \to \mathbb{R} \) is a parameterized family \((s \geq 0)\) of bounded linear operators from \( C_1 \) into \( \mathbb{R} \) and \( F \) is a smooth function from \( C_1 \times \mathbb{R}^{s+1} \) into \( \mathbb{R} \). In the “prequel” [4] to this paper, we assumed that \( F(0, 0) = 0 \), and \( DF(0, 0) = 0 \). While this hypothesis included the cases of multiple Hopf bifurcation and of a mode-interaction between a multiple Hopf and a transcritical type steady-state bifurcation, it excluded the case of a mode interaction in which the steady-state was of saddle-node type. Therefore, in this paper, we assume the following weaker hypothesis.
Hypothesis 2.1 $F(0, 0) = 0$, $D_1 F(0, 0) = 0$, and $D_\alpha F(0, 0) \neq 0$.

Performing a linear change of parameters and relabeling the parameters if necessary, we may then rewrite (2.1) as

$$
\dot{z}(t) = L_0 z_t + \nu + \hat{F}(z_t, \nu, \mu),
$$

(2.2)

where we have set $\alpha \equiv (\nu, \mu) \in \mathbb{R}^1 \times \mathbb{R}^s$, $L_0 \equiv L(0)$, and $\hat{F}(z_t, \nu, \mu) = (L(\nu, \mu) - L_0)z_t + F(z_t, \nu, \mu) - D_\nu F(0, 0, 0)$. It follows from Hypothesis 2.1 that

$$
\hat{F}(0, 0, 0) = 0, \quad D\hat{F}(0, 0, 0) = 0.
$$

Clearly, the parameter $\nu$ plays a distinguished role in (2.2) in comparison to the other parameters $\mu$.

Spectral hypothesis

Suppose we set $(\nu, \mu) = (0, 0)$ in (2.2). If we then write $L_0$ as

$$
L_0 \phi = \int_{-r}^{0} d\eta(\theta) \phi(\theta),
$$

(2.3)

where $\eta$ is a real-valued function of bounded variation in $[-r, 0]$ and we let $A_0$ be the infinitesimal generator of the semi-flow associated with the linear RFDE $\dot{z}(t) = L_0 z_t$, then it is well-known that the spectrum $\sigma(A_0)$ of $A_0$ is equal to the point spectrum of $A_0$, and $\lambda \in \sigma(A_0)$ if and only if $\lambda$ satisfies the characteristic equation

$$
det \Delta (\lambda) = 0, \quad \Delta (\lambda) = \lambda - \int_{-r}^{0} d\eta(\theta) e^{\lambda \theta}.
$$

(2.4)

Denote by $\Lambda_0$ the set of eigenvalues of $\sigma(A_0)$ with zero real part.

Hypothesis 2.2 Throughout the rest of the paper, we assume the following hypotheses on $\Lambda_0$. Each element of $\Lambda_0$ is a simple eigenvalue of $A_0$, and $\Lambda_0$ has the following form:

$$
\Lambda_0 = \{0, \pm i\omega_1, \ldots, \pm i\omega_p\},
$$

where $\omega_1, \ldots, \omega_p$, are independent over the rationals, i.e. if $r_1, \ldots, r_p$ are rational numbers such that $\sum_{j=1}^{p} r_j \omega_j = 0$, then $r_1 = \cdots = r_p = 0$. We further assume that the rest of the spectrum of $A_0$ is bounded away from the imaginary axis.
Phase space decomposition

In order to properly analyse the role of the parameters in equation (2.2) we need to augment this equation by considering the following system

\[
\begin{align*}
\dot{z}(t) &= L_0 z_t + \nu(0) + \mathcal{F}\left(\begin{pmatrix} z_t \\ \nu_t \end{pmatrix}, \mu_t\right), \\
\dot{\nu}(t) &= 0,
\end{align*}
\]

(2.5a)

(2.5b)

where \(\mathcal{F}((z_t, \nu_t)^T, \mu) = \hat{F}(z_t, \nu(0), \mu(0))\). The above system can thus be viewed as an \(s\)-dimensional family (parameterized by \(\mu\)) of 2-dimensional RFDEs.

Taking into account Hypothesis 2.2, the linearized equation \((\dot{z}(t), \dot{\nu}(t)) = (L_0 z_t + \nu(0), 0)\) associated to (2.5a) has simple non-resonant characteristic values \(\pm i\omega_1, \ldots, \pm i\omega_p\), and a characteristic value at 0 of multiplicity 2. We then let \(P \subset C_2\) designate the \(2p + 2\)-dimensional center subspace which is spanned by the columns of the following matrix

\[
\hat{\Phi} = \begin{pmatrix}
1 & e^{i\omega_1 \theta} & e^{-i\omega_1 \theta} & \cdots & e^{i\omega_p \theta} & e^{-i\omega_p \theta} & \theta \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix}.
\]

(2.6)

We note that \(\hat{\Phi}\) satisfies the linear differential equation \(\frac{d\hat{\Phi}}{d\theta} = \hat{\Phi}\hat{B}\), where \(\hat{B}\) is the \((2p + 2) \times (2p + 2)\) matrix \(\hat{B} = \text{diag}(0, i\omega_1, -i\omega_1, \ldots, i\omega_p, -i\omega_p, 0) + \hat{N}\), and \(\hat{N}\) is the nilpotent matrix whose entries are all zero except the entry at the intersection of the first row and last column, whose value is 1.

We decompose \(C_2\) as

\[
C_2 = P \oplus Q.
\]

(2.7)

Defining \(C_2^* \equiv C([0, r], \mathbb{R}^{2*})\), where \(\mathbb{R}^{2*}\) is the 2-dimensional space of row vectors, we introduce the adjoint bilinear form on \(C_2^* \times C_2^*\):

\[
(\psi, \phi) = \psi(0)\phi(0) - \int_{-r}^0 \int_0^\theta \psi(\xi - \theta) d\tilde{\eta}(\theta) \phi(\xi) d\xi,
\]

(2.8)

where \(d\tilde{\eta}(\theta)\) is of the form

\[
d\tilde{\eta}(\theta) = \begin{pmatrix}
d\eta(\theta) \\
0
\end{pmatrix},
\]

\[
d\delta(\theta) d\theta
\]

where \(\delta(\theta)\) is the Dirac delta function.
$d\eta(\theta)$ is as in (2.3) and $\delta(\theta) d\theta$ is such that

$$\int_{-r}^{0} f(\theta) \delta(\theta) d\theta = f(0).$$

We may then choose a basis $\{\psi_1, \ldots, \psi_{2p+2}\}$ of the dual space $P^*$ such that if $\hat{\Psi} = \text{col}(\psi_1, \ldots, \psi_{2p+2})$ then $(\hat{\Psi}, \hat{\Phi}) = I_{2p+2}$, where throughout the paper we use the convention that for a given integer $q \geq 1$, $I_q$ is the $q \times q$ identity matrix. The following result will be useful later.

**Lemma 2.3** Let $\hat{\Phi}$ be as in (2.6) and $\hat{\Psi}(\xi) = (\psi_{k,\ell}(\xi))$ be a $(2p+2) \times 2$ matrix whose rows form a basis for $P^*$, and such that $(\hat{\Psi}, \hat{\Phi}) = I_{2p+2}$. Then

$$\psi_{2p+2,1}(0) = 0, \quad \text{and} \quad \psi_{k,1}(0) \neq 0, \quad 1 \leq k \leq 2p+1.$$

**Proof** It follows from Hypothesis 2.2 that

$$\int_{-r}^{0} d\eta(\theta) = 0, \quad \int_{-r}^{0} d\eta(\theta) e^{\pm i\omega_j \theta} = \pm i\omega_j, \quad j = 1, \ldots, p$$

and

$$\int_{-r}^{0} d\eta(\theta) \theta \neq 1, \quad \int_{-r}^{0} d\eta(\theta) \theta e^{\pm i\omega_j \theta} \neq 1, \quad j = 1, \ldots, p.$$

Using (2.8), the equation $(\Psi, \Phi) = I_{2p+2}$ leads to

$$\psi_{2p+2,1}(0) \left(1 - \int_{-r}^{0} d\eta(\theta) \theta\right) = 0,$$

$$\psi_{k,1}(0) \left(1 - \int_{-r}^{0} d\eta(\theta) \theta e^{i\omega_j \theta}\right) = 1, \quad k = 2, 4, \ldots, 2p,$$

and

$$\psi_{k,1}(0) \left(1 - \int_{-r}^{0} d\eta(\theta) \theta e^{-i\omega_j \theta}\right) = 1, \quad k = 3, 5, \ldots, 2p+1,$$

so we get the desired conclusion.

Consider now the Banach space $BC_2$ of functions from $[-r, 0]$ into $\mathbb{R}^2$ which are uniformly continuous on $[-r, 0)$ with a jump discontinuity at 0. Elements of $BC_2$ are written as

$$\phi + X_0 \lambda$$
where φ ∈ C², λ ∈ ℝ² and X₀ is the 2 × 2 matrix-valued function

\[
X₀(θ) = \begin{cases} 
I₂ & \text{if } θ = 0 \\
0 & -r \leq θ < 0.
\end{cases}
\]

Let π : BC² → P denote the projection

\[π(φ + X₀λ) = ˚Φ [(˚Ψ, φ) + ˚Ψ(0)λ],\]

where ( , ) is the bilinear form (2.8). We may then extend the splitting (2.7) to

\[BC² = P ⊕ \text{ker } π,\] (2.9)

with the property that Q ⊆ ker π.

This structure now allows for a decomposition of the phase space which facilitates the implementation of the normal form procedure with parameters for RFDEs developed by Faria and Magalhães in [7]. Specifically, it follows that (2.5a) is equivalent to the parameterized family

\[
\begin{align*}
\left( \begin{array}{c} \dot{x} \\ \dot{ν} \end{array} \right) &= ˚B \left( \begin{array}{c} x \\ ν \end{array} \right) + ˚Ψ(0) \left( \mathcal{F} \left( ˚Φ \left( \begin{array}{c} x \\ ν \end{array} \right) + y, μ \right) \right) \quad (2.10a) \\
\frac{d}{dt}y &= A_{Q¹}y + (I - π)X₀ \left( \mathcal{F} \left( ˚Φ \left( \begin{array}{c} x \\ ν \end{array} \right) + y, μ \right) \right) \quad (2.10b)
\end{align*}
\]

where ˚Ψ(0) is as in Lemma 2.3, x ∈ ℝ²⁺, ν ∈ ℝ, μ ∈ ℝ, y ∈ Q¹ = Q ∩ C²₁, (C²₁ is the subspace of C² consisting of continuously differentiable functions), and A_{Q¹} is the operator from Q¹ into ker π defined by

\[A_{Q¹} = ˚Φ + X₀ \left[ \int_{-r}^{0} d\tilde{η}(θ) φ(θ) - ˚Φ(0) \right].\]

2.2 Faria and Magalhães normal form

Consider the formal Taylor expansion of the nonlinearity F terms in (2.8)

\[\mathcal{F}(u) = \sum_{j \geq 2} \frac{1}{j!} \hat{F}_j(u), \quad u ∈ C_{2+s},\]
where \( \hat{F}_j(w) = H_j(w, \ldots, w) \), with \( H_j \) belonging to the space of continuous multilinear symmetric maps from \( C_{2+s} \times \cdots \times C_{2+s} \) (\( j \) times) to \( \mathbb{R} \). If we denote \( x = (x, \nu)^T \) and \( f_j = (f_j^1, f_j^2) \), where

\[
\begin{align*}
f_j^1(x, y, \mu) &= \hat{\Psi}(0) \left( \frac{\hat{F}_j(\hat{\Phi}x + y, \mu)}{0} \right), \\
f_j^2(x, y, \mu) &= (I - \pi) X_0 \left( \frac{\hat{F}_j(\hat{\Phi}x + y, \mu)}{0} \right),
\end{align*}
\] (2.11)

then (2.10) can be written as

\[
\begin{align*}
\dot{x} &= \hat{B}x + \sum_{j \geq 2} \frac{1}{j!} f_j^1(x, y, \mu) \quad (2.12a) \\
\frac{d}{dt} y &= A_{Q_1} y + \sum_{j \geq 2} \frac{1}{j!} f_j^2(x, y, \mu) \quad (2.12b)
\end{align*}
\]

The spectral hypotheses we have specified in Hypothesis 2.2 are sufficient to conclude that the non-resonance condition of Faria and Magalhães \cite{7} holds. Consequently, using successively at each order \( j \) a near identity change of variables of the form

\[
(x, y) = (\hat{x}, \hat{y}) + U_j(\hat{x}, \mu) = (\hat{x}, \hat{y}) + (U_j^1(\hat{x}, \mu), U_j^2(\hat{x}, \mu)),
\] (2.13)

(where \( U_j^{1,2} \) are homogeneous degree \( j \) polynomials in the indicated variables, with coefficients respectively in \( \mathbb{R}^{2p+2} \) and \( Q_1 \)) system (2.12) can be put into formal normal form

\[
\begin{align*}
\dot{x} &= \hat{B}x + \sum_{j \geq 2} \frac{1}{j!} g_j^1(x, y, \mu) \quad (2.14a) \\
\frac{d}{dt} y &= A_{Q_1} y + \sum_{j \geq 2} \frac{1}{j!} g_j^2(x, y, \mu) \quad (2.14b)
\end{align*}
\]

such that the center manifold is locally given by \( y = 0 \) and the local flow of (2.11) on this center manifold is given by

\[
\dot{x} = \hat{B}x + \sum_{j \geq 2} \frac{1}{j!} g_j^1(x, 0, \mu).
\] (2.15)

The nonlinear terms \( g_j^1 \) in (2.15) are in normal form in the classical sense with respect to the matrix \( \hat{B} \).
3 Non-semisimple Equivariant Normal Form

The matrix $\tilde{B}$ which appears in the previous section has a nilpotency associated with it which somewhat complicates the computation of the normal form (2.13). In particular, the analysis which was presented in [4] in the semisimple case does not carry over here. Nevertheless, in this section we will show how to generalize the normal form analysis presented in [4] to the present non-semisimple case. Apart from solving the problem which interests us in this paper, this approach may also shed some light on the general case where Hypothesis 2.2 is relaxed to include repeated eigenvalues with Jordan blocks.

3.1 $\mathbb{T}^p \times \mathbb{R}$ normal forms

Let $\Psi(0)$ denote the $(2p + 1) \times 1$ matrix obtained from the first $2p + 1$ elements of the first column of $\Psi(0)$ in Lemma 2.3 and let $B$ be the $(2p + 1) \times (2p + 1)$ matrix

$$B = \text{diag}(0, i\omega_1, -i\omega_1, \ldots, i\omega_p, -i\omega_p).$$

(3.1)

It is easy to see from Lemma 2.3 that $f^1_j$ in (2.11) is of the form

$$f^1_j(x, 0, \mu) = \begin{pmatrix} \Psi(0) \tilde{F}_j(\tilde{\Phi} x, \mu) \\ 0 \end{pmatrix},$$

and we will thus consider normal forms for the following class of formal vector fields on $\mathbb{R}^{2p+2+s}$

$$\begin{pmatrix} \dot{x} \\ \dot{\nu} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} Bx + \nu e_0 \\ 0 \\ 0 \end{pmatrix} + \sum_{j \geq 2} \begin{pmatrix} f_j(x, \nu, \mu) \\ 0 \\ 0 \end{pmatrix},$$

(3.2)

where we will use the convention that $e_k$ is the unit row vector in $\mathbb{R}^q$ with all entries equal to zero except the entry in the row $k + 1$ whose value is 1, and where $q$ will depend on the context.

Using a mixture of complex and real coordinates, we identify

$$\mathbb{R}^{2p+2+s} = \{(x_0, x_1, \bar{x}_1, \ldots, x_p, \bar{x}_p, \nu, \mu_1, \ldots, \mu_s) \mid x_0, \nu, \mu_j \in \mathbb{R}, x_k \in \mathbb{C}, j = 1, \ldots, s, k = 1, \ldots, p\}.$$

Define the following $(2p + 2 + s) \times (2p + 2 + s)$ matrix

$$\tilde{B} = \text{diag}(\tilde{B}, \mathbf{0}_s),$$

where $\mathbf{0}_s$ is the $s \times s$ zero matrix, and let $\tilde{B}^T$ denote the transpose of $\tilde{B}$. Let

$$\Gamma = \{e^{s\tilde{B}^T} \mid s \in \mathbb{R}\}$$

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where the closure is taken in the space of \((2p + 2 + s) \times (2p + 2 + s)\) matrices. Note that \(\Gamma\) is an abelian connected Lie group isomorphic to \(T^p \times \mathbb{R}\) where \(T^p\) is the \(p\)-torus:

\[
\mathbb{T}^p = \{ \text{diag}(1, e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_p}, e^{-i\theta_p}, 1, I_s) \mid \theta_j \in \mathbb{S}^1, j = 1, \ldots, p \}
\]

(3.3)

and \(\mathbb{R}\) is the one-parameter group parameterized as

\[
\mathbb{R} = I_{2p+2+s} + \Theta \tilde{N}^T, \quad \Theta \in \mathbb{R}
\]

(3.4)

where \(\tilde{N}^T\) is the transpose of the nilpotent matrix

\[
\tilde{N} = \tilde{B} - \text{diag}(0, i\omega_1, -i\omega_1, \ldots, i\omega_p, -i\omega_p, 0, 0_s).
\]

**Definition 3.1** For a given integer \(\ell \geq 2\) and a given normed space \(X\), we denote by \(H^p_{2p+2+s}(X)\) the linear space of homogeneous polynomials of degree \(\ell\) in the \(2p+2+s\) variables \(x = (x_0, x_1, \ldots, x_p, \nu, \mu)\), \(\nu = (\mu_1, \ldots, \mu_s)\), with coefficients in \(X\). For \(X = \mathbb{R}^{2p+2+s}\), define \(H^p_{2p+2+s}(\mathbb{R}^{2p+2+s}, \Gamma) \subset H^p_{2p+2+s}(\mathbb{R}^{2p+2+s})\) to be the subspace of \(\Gamma\)-equivariant polynomials, i.e.

\[
\tilde{f} \in H^p_{2p+2+s}(\mathbb{R}^{2p+2+s}, \Gamma) \iff \tilde{f} \in H^p_{2p+2+s}(\mathbb{R}^{2p+2+s}) \quad \text{and} \quad \gamma \tilde{f}(\gamma^{-1}w) = \tilde{f}(w), \quad \forall \ w = (x, \nu, \mu) \in \mathbb{R}^{2p+2+s}, \quad \forall \ \gamma \in \Gamma.
\]

Normal forms for (3.2) are computed using the homological operator

\[
\mathcal{L}_{\tilde{B}} : H^p_{2p+2+s}(\mathbb{R}^{2p+2+s}) \longrightarrow H^p_{2p+2+s}(\mathbb{R}^{2p+2+s})
\]

\[
\tilde{f} \mapsto (\mathcal{L}_{\tilde{B}} \tilde{f})(w) = D\tilde{f}(w)\tilde{B}w - \tilde{B}\tilde{f}(w).
\]

To specify the normal form, we must find a complement in \(H^p_{2p+2+s}(\mathbb{R}^{2p+2+s})\) to the range of \(\mathcal{L}_{\tilde{B}}\). The following is a very well-known result in the theory of normal forms [5, 10, 11].

**Proposition 3.2**

\[
H^p_{2p+2+s}(\mathbb{R}^{2p+2+s}) = H^p_{2p+2+s}(\mathbb{R}^{2p+2+s}, \Gamma) \oplus \text{range} \mathcal{L}_{\tilde{B}}
\]

It is straightforward to compute the general element of \(H^p_{2p+2+s}(\mathbb{R}^{2p+2+s}, \Gamma)\).
Lemma 3.3 A smooth vector field \( \tilde{f} : \mathbb{R}^{2p+2+s} \rightarrow \mathbb{R}^{2p+2+s} \) is \( \mathbb{T}^p \)-equivariant if and only if \( \tilde{f} \) is of the form

\[
\tilde{f}(x, \nu, \mu) = \begin{pmatrix}
a_0(x_0, x_1 \bar{x}_1, \ldots, x_p \bar{x}_p, \nu, \mu) \\
a_1(x_0, x_1 \bar{x}_1, \ldots, x_p \bar{x}_p, \nu, \mu) x_1 \\
a_1(x_0, x_1 \bar{x}_1, \ldots, x_p \bar{x}_p, \nu, \mu) x_1 \\
\vdots \\
a_p(x_0, x_1 \bar{x}_1, \ldots, x_p \bar{x}_p, \nu, \mu) x_p \\
a_p(x_0, x_1 \bar{x}_1, \ldots, x_p \bar{x}_p, \nu, \mu) x_p \\
b(x_0, x_1 \bar{x}_1, \ldots, x_p \bar{x}_p, \nu, \mu) \\
c_1(x_0, x_1 \bar{x}_1, \ldots, x_p \bar{x}_p, \nu, \mu) \\
\vdots \\
c_s(x_0, x_1 \bar{x}_1, \ldots, x_p \bar{x}_p, \nu, \mu),
\end{pmatrix}
\] (3.5)

where \( a_1, \ldots, a_p \) are smooth and complex-valued, and \( a_0, b, c_1, \ldots, c_s \) are smooth and real-valued. Furthermore, a vector field of the form (3.5) is \( \Gamma \)-equivariant (where \( \Gamma \cong \mathbb{T}^p \times \mathbb{R} \)) if and only if \( a_0, a_1, \ldots, a_p, c_1, \ldots, c_s \) are \( \nu \)-independent, and

\[
a_0 = x_0 g_0(x_0, x_1 \bar{x}_1, \ldots, x_p \bar{x}_p, \mu), \quad b = \nu g_0(x_0, x_1 \bar{x}_1, \ldots, x_p \bar{x}_p, \mu) + g_1(x_0, x_1 \bar{x}_1, \ldots, x_p \bar{x}_p, \mu)
\]

for some smooth functions \( g_0 \) and \( g_1 \).

Proof The vector field \( \tilde{f} \) is \( \Gamma \)-equivariant if and only if \( \tilde{f} \) is both \( \mathbb{T}^p \)-equivariant and \( \mathbb{R} \)-equivariant. For the general form of the \( \mathbb{T}^p \)-equivariant vector field (3.5), see [4]. We then further require that (3.5) commute with all matrices of the form \( I_{2p+2+s} + \Theta N^T, \Theta \in \mathbb{R} \). The result follows after a straightforward computation. ■

The vector field \( \tilde{f} \) in (3.2) has the special form \( \tilde{f} = (f, 0, 0) \), which we want our normal form changes of variables to preserve. Since we are only interested in the first \( 2p + 1 \) components of (3.2), we would like to obtain a splitting of \( H^{2p+2+s}_t(\mathbb{R}^{2p+1}) \) akin to the splitting of \( H^{2p+2+s}_t(\mathbb{R}^{2p+2+s}) \) in Proposition 3.2. For this purpose, we will need the following
Definition 3.4

(a) We define $H^{2p+2+s}_t(\mathbb{R}^{2p+1}, \mathbb{T}^p)$ to be the subset of $H^{2p+2+s}_t(\mathbb{R}^{2p+1})$ consisting of mappings $f: \mathbb{R}^{2p+2+s} \rightarrow \mathbb{R}^{2p+1}$ whose components are of the form of the first $2p+1$ components of (3.2).

As the notation suggests, note that $H^{2p+2+s}_t(\mathbb{R}^{2p+1}, \mathbb{T}^p)$ consists precisely of elements of $H^{2p+2+s}_t(\mathbb{R}^{2p+1})$ which are equivariant under an action of $\mathbb{T}^p$ on $\mathbb{R}^{2p+1}$:

$$f \in H^{2p+2+s}_t(\mathbb{R}^{2p+1}, \mathbb{T}^p) \iff f(\gamma_0 x, \nu, \mu) = \gamma_0 f(x, \nu, \mu), \quad \forall \gamma_0 \in \Gamma_0, \forall (x, \nu, \mu) \in \mathbb{R}^{2p+2+s},$$

where $\Gamma_0$ is the group of $(2p+1) \times (2p+1)$ matrices which is isomorphic to $\mathbb{T}^p$, and is parameterized as

$$\Gamma_0 = \{ \text{diag}(1, e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_p}, e^{-i\theta_p}) \mid \theta_j \in \mathbb{S}^1, j = 1, \ldots, p \} \quad (3.6)$$

(b) We define $H^{2p+1+s}_t(\mathbb{R}^{2p+1}, \mathbb{T}^p)$ to be the subspace of $H^{2p+2+s}_t(\mathbb{R}^{2p+1}, \mathbb{T}^p)$ consisting of $\nu$-independent and $\mathbb{T}^p$-equivariant polynomials.

(c) We define the following operator

$$\mathcal{L}_{B, \nu} : H^{2p+2+s}_t(\mathbb{R}^{2p+1}) \rightarrow H^{2p+2+s}_t(\mathbb{R}^{2p+1})$$



where $B$ is as in (3.1), and note that $\mathcal{L}_{B, \nu}$ is the usual homological operator associated to the $\dot{x}$ component of (3.2). Furthermore, we have $\mathcal{L}_B(f, 0, 0) = (\mathcal{L}_{B, \nu} f, 0, 0)$.

Proposition 3.5

$$H^{2p+2+s}_t(\mathbb{R}^{2p+1}) = H^{2p+1+s}_t(\mathbb{R}^{2p+1}, \mathbb{T}^p) \oplus \text{range } \mathcal{L}_{B, \nu}.$$
3.2 Equivariant projection

We will now construct an appropriate linear projection associated with the splitting of $H^{2p+2+s}_\ell(\mathbb{R}^{2p+1})$ given in Proposition 3.5.

**Definition 3.6** Let $\int_{\Gamma_0} d\gamma$ denote the normalized Haar integral on $\Gamma_0 \cong \mathbb{T}^p$ (see (3.6)). We define the linear operator

$$\hat{A} : H^{2p+2+s}_\ell(\mathbb{R}^{2p+1}) \longrightarrow H^{2p+2+s}_\ell(\mathbb{R}^{2p+1})$$

$$f \mapsto (\hat{A} f)(x, \nu, \mu) = \int_{\Gamma_0} \gamma f(\gamma^{-1}x, 0, \mu) d\gamma.$$

**Proposition 3.7** $\hat{A}$ is a projection. Furthermore,

$$\text{range } \hat{A} = H^{2p+1+s}_\ell(\mathbb{R}^{2p+1}, \mathbb{T}^p) \quad (3.7)$$

and

$$\text{ker } \hat{A} = \text{range } \mathcal{L}_{B,\nu}. \quad (3.8)$$

**Proof** The proof is given in the appendix. □

For any $f \in H^{2p+2+s}_\ell(\mathbb{R}^{2p+1})$, write

$$f = \hat{A} f + (I - \hat{A}) f,$$

and note that $\hat{A} f$ is $\mathbb{T}^p$-equivariant and $\nu$-independent and that $(I - \hat{A}) f \in \text{ker } \hat{A}$. From Proposition 3.7, there exists $h \in H^{2p+2+s}_\ell(\mathbb{R}^{2p+1})$ such that $\mathcal{L}_{B,\nu} h = (I - \hat{A}) f$.

3.3 Phase decoupling

Elements of the space $H^{2p+1+s}_\ell(\mathbb{R}^{2p+1}, \mathbb{T}^p)$ are $\nu$-independent and equivariant with respect to the torus group $\Gamma_0 \cong \mathbb{T}^p$ defined in (3.6). As seen in (3), this toroidal equivariance can be used to achieve a decoupling of the normal form. Specifically, we have

**Proposition 3.8** Consider the following differential equation on $\mathbb{R}^{2p+1}$:

$$\dot{x} = B x + \nu e_0 + f(x, \mu),$$
where \( f \) is smooth, \( \nu \)-independent, satisfies \( f(0,0) = 0 \), \( Df(0,0) = 0 \), and is \( \Gamma_0 \cong \mathbb{T}_p \)-equivariant, i.e. \( f \) has the form

\[
f(x, \mu) = \begin{pmatrix}
a_0(x_0, x_1, \ldots, x_p, \mu) \\
a_1(x_0, x_1, \ldots, x_p, \mu) x_1 \\
\vdots \\
a_p(x_0, x_1, \ldots, x_p, \mu) x_p
\end{pmatrix}
\]

where \( a_1, \ldots, a_p \) are smooth and complex-valued, and \( a_0 \) is smooth and real-valued. Then under the change of variables \( x_0 = \rho_0 \), \( x_j = \rho_j e^{i\theta_j}, j = 1, \ldots, p \), this differential equation transforms into

\[
\begin{align*}
\dot{\rho}_0 &= \nu + a_0(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu) \\
\dot{\rho}_j &= \text{Re}(a_j(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu)) \rho_j, \quad j = 1, \ldots, p
\end{align*}
\]

and

\[
\dot{\theta}_j = i\omega_j + \text{Im}(a_j(\rho_0, \rho_1^2, \ldots, \rho_p^2, \mu)), \quad j = 1, \ldots, p.
\]

**Proof** This is a simple computation.

As in [4], we will call the subsystem \((3.10)\) the *uncoupled radial part* of the normal form \((3.9)\). Recall that this uncoupled radial part has some residual reflectional symmetry. Denote by \( \mathbb{Z}_{2,p} \) the group whose action on \( \mathbb{R}^{2p+1} \) is given by

\[
(\rho_0, \rho_1, \ldots, \rho_p) \rightarrow (\rho_0, \lambda_1 \rho_1, \ldots, \lambda_p \rho_p),
\]

where \( \lambda_j \in \{1, -1\}, j = 1, \ldots, p \).

**Definition 3.9** For a given integer \( \ell \geq 2 \) and a given normed space \( X \), we denote by \( H^{\ell+1+s}_p(X) \) the linear space of homogeneous polynomials of degree \( \ell \) in the \( p+1+s \) variables \( \rho = (\rho_0, \rho_1, \ldots, \rho_p) \) and \( \mu = (\mu_1, \ldots, \mu_s) \) with coefficients in \( X \). Denote by \( H^{\ell+1+s}_p(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}) \subset H^{\ell+1+s}_p(\mathbb{R}^{p+1}) \) the subspace of \( H^{\ell+1+s}_p(\mathbb{R}^{p+1}) \) consisting of \( \mathbb{Z}_{2,p} \)-equivariant polynomials.
It is easy to show (see [4]) that the most general element of $H^{p+1+s}_\ell(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p})$ has the form of the right-hand side of (3.10). We then define the following surjective linear mapping

$$ \Pi : H^{2p+1+s}_\ell(\mathbb{R}^{2p+1}, \mathbb{T}^p) \longrightarrow H^{p+1+s}_\ell(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}) $$

(3.13)

which is defined by sending the general element (3.9) of $H^{2p+1+s}_\ell(\mathbb{R}^{2p+1}, \mathbb{T}^p)$ to the following element of $H^{p+1+s}_\ell(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p})$:

$$ \begin{pmatrix}
  a_0(\rho_0, \rho^2_1, \ldots, \rho^2_p, \mu) \\
  \text{Re}(a_1(\rho_0, \rho^2_1, \ldots, \rho^2_p, \mu)) \rho_1 \\
  \vdots \\
  \text{Re}(a_p(\rho_0, \rho^2_1, \ldots, \rho^2_p, \mu)) \rho_p
\end{pmatrix}. $$

(3.14)

### 3.4 Parameter splitting

As seen in [4], it is useful when considering unfoldings to be able to refine Propositions 3.5 and 3.7 in order to make explicit the roles of $(x, \nu)$ as primary variables and $\mu$ as unfolding parameters in (2.14a). For this purpose, we define the following spaces.

**Definition 3.10** Let $d \geq 1$ be a given integer (for our purposes, $d$ will be equal to either $2p + 1$ or to $p + 1$), $\ell \geq 2$ be an integer, $X$ be a normed linear space, and $G$ be a linear group acting on $X$. Let $H^{d+1+s}_\ell(X)$ be the linear space of homogeneous polynomials of degree $\ell$ in the variables $(\xi_1, \ldots, \xi_d)$, $\nu$ and $\mu_1, \ldots, \mu_s$, and $H^{d+s}_\ell(X, G) \subset H^{d+1+s}_\ell(X)$ be the subspace of $\nu$-independent and $G$-equivariant polynomials.

(a) We define $H^{d+1}_\ell(X) \subset H^{d+1+s}_\ell(X)$ and $H^{d+1}_\ell(X, G) \subset H^{d+s}_\ell(X, G)$ to be the subspaces of $\mu$-independent polynomials.

(b) We define $P^{d+1+s}_\ell(X) \subset H^{d+1+s}_\ell(X)$ and $P^{d+s}_\ell(X, G) \subset H^{d+s}_\ell(X, G)$ to be the subspaces of polynomials which vanish at $\mu = 0$.

It follows from these definitions that

$$ H^{2p+2+s}_\ell(\mathbb{R}^{2p+1}) = H^{2p+2}_\ell(\mathbb{R}^{2p+1}) \oplus P^{2p+2+s}_\ell(\mathbb{R}^{2p+1}) $$

$$ H^{2p+1+s}_\ell(\mathbb{R}^{2p+1}, \mathbb{T}^p) = H^{2p+1}_\ell(\mathbb{R}^{2p+1}, \mathbb{T}^p) \oplus P^{2p+1+s}_\ell(\mathbb{R}^{2p+1}, \mathbb{T}^p) $$

(3.15)

$$ H^{p+1+s}_\ell(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}) = H^{p+1}_\ell(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}) \oplus P^{p+1+s}_\ell(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}). $$

Furthermore, the various operators $\mathcal{L}_{B,\nu}$, $\hat{A}$ and $\Pi$ previously defined preserve these decompositions. We then get the following refinement of Propositions 3.5 and 3.7.
Proposition 3.11
\[ L_{B,\nu}(H^{2p+2}_{2p+1}(\mathbb{R}^{2p+1})) \subset H^{2p+2}_{2p+1}(\mathbb{R}^{2p+1}), \quad L_{B,\nu}(P_\ell^{2p+2+s}(\mathbb{R}^{2p+1})) \subset P_\ell^{2p+2+s}(\mathbb{R}^{2p+1}), \]
\[ \hat{A}(H^{2p+2}_{2p+1}(\mathbb{R}^{2p+1})) = H^{2p+1}_{2p+1}(\mathbb{R}^{2p+1}, \mathbb{T}^p), \quad \hat{A}(P_\ell^{2p+2+s}(\mathbb{R}^{2p+1})) = P_\ell^{2p+1+s}(\mathbb{R}^{2p+1}, \mathbb{T}^p), \]
\[ \Pi(H^{2p+1}_{2p+1}(\mathbb{R}^{2p+1}, \mathbb{T}^p)) = H^{p+1}_{2p+1}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}), \quad \Pi(P_\ell^{2p+1+s}(\mathbb{R}^{2p+1}, \mathbb{T}^p)) = P_\ell^{p+1+s}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}) \]

Consequently, if we define \( \hat{A}|_1 \) and \( \hat{A}|_2 \) to be respectively the restrictions of \( \hat{A} \) on \( H^{2p+2}_{2p+1}(\mathbb{R}^{2p+1}) \) and on \( P_\ell^{2p+2+s}(\mathbb{R}^{2p+1}) \), then \( \hat{A}|_1 \) and \( \hat{A}|_2 \) are projections. Similarly define \( L_{B,\nu}|_1 \) and \( L_{B,\nu}|_2 \) to be respectively the restrictions of \( L_{B,\nu} \) on \( H^{2p+2}_{2p+1}(\mathbb{R}^{2p+1}) \) and on \( P_\ell^{2p+2+s}(\mathbb{R}^{2p+1}) \), then
\[ \text{range } \hat{A}|_1 = H^{2p+1}_{2p+1}(\mathbb{R}^{2p+1}, \mathbb{T}^p), \quad \text{range } \hat{A}|_2 = P_\ell^{2p+1+s}(\mathbb{R}^{2p+1}, \mathbb{T}^p), \]
\[ \ker \hat{A}|_1 = \text{range } L_{B,\nu}|_1, \quad \ker \hat{A}|_2 = \text{range } L_{B,\nu}|_2, \]
\[ H^{2p+2}_{2p+1}(\mathbb{R}^{2p+1}) = H^{2p+1}_{2p+1}(\mathbb{R}^{2p+1}, \mathbb{T}^p) \oplus \text{range } L_{B,\nu}|_1, \]
\[ P_\ell^{2p+2+s}(\mathbb{R}^{2p+1}) = P_\ell^{p+1+s}(\mathbb{R}^{p+1}, \mathbb{T}^p) \oplus \text{range } L_{B,\nu}|_2. \]

We now combine the results of this section with the Faria and Magalhães normal form procedure outlined in Section 2 in order to obtain the following version of Theorem 5.8 of [6] and Theorem 2.16 of [7].

Theorem 3.12 Consider the system (2.12)
\[ \dot{x} = \tilde{B} x + \sum_{j \geq 2} f_j^1(x, y, \mu) \]
\[ \frac{d}{dt} y = A_{Q^1} y + \sum_{j \geq 2} f_j^2(x, y, \mu), \] (3.16)
where \( x = \begin{pmatrix} x \\ \nu \end{pmatrix} \). Write
\[ f_j^1(x, 0, \mu) = \begin{pmatrix} (\Psi(0)\tilde{F}_j(\Phi x, \mu)) \\ 0 \end{pmatrix} = \begin{pmatrix} h_j(x) + q_j(x, \mu) \\ 0 \end{pmatrix}, \] (3.17)
where \( h_j \in H^{2p+2}_{2p+1}(\mathbb{R}^{2p+1}) \) and \( q_j \in P_\ell^{2p+2+s}(\mathbb{R}^{2p+1}) \). Then there is a formal near-identity change of variables
\[ (x, y) \rightarrow (\hat{x}, \hat{y}) + (U^1(x), U^2(x)) + (W^1(x, \mu), W^2(x, \mu)) \]
(where \( W^1(x, 0) = 0 \) and \( W^2(x, 0) = 0 \)) which transforms (3.16) into system (2.14) (upon dropping the hats), and the flow on the invariant local center manifold \( y = 0 \) is given by

\[
\begin{align*}
\dot{x} &= Bx + \nu e_0 + \sum_{j \geq 2} ((\dot{A}_1(h_j + Y_j))(x) + (\dot{A}_2(q_j + Z_j))(x, \mu)), \\
\dot{\nu} &= 0 \\
\dot{\mu} &= 0,
\end{align*}
\]

(3.18)

where \( Y_2 = 0 \), \( Z_2 = 0 \), and for \( j \geq 3 \), \( Y_j = Y_j(x, \nu) \) and \( Z_j = Z_j(x, \nu, \mu) \) are the extra contributions to the terms of order \( j \) coming from the transformation of the lower order \( (< j) \) terms, and \( Z_j \) vanishes at \( \mu = 0 \).

4 Main results

In the semisimple case of [4], the main realizability results are a consequence of the surjectivity of a certain linear operator between suitable spaces of polynomials, which is proven in Proposition 4.3 of [4]. In this section, we define the corresponding linear operator in the present non-semisimple case, and prove that its surjectivity follows from the surjectivity in the semisimple case. Once this has been achieved, we will state our main realizability results for the class of uncoupled radial equations (3.14) which are obtained from writing the normal form center manifold equations (3.18) in polar coordinates.

4.1 Linear analysis

**Definition 4.1** For a given integer \( \ell \geq 2 \), let \( H^{p+2+s}_{\ell}(\mathbb{R}) \) denote the linear space of homogeneous degree \( \ell \) polynomials in the \( 2p + 2 \) variables

\[
\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_p) = \left( \left( \begin{array}{c} \mathbf{v}_0 \\ w_0 \end{array} \right), \left( \begin{array}{c} \mathbf{v}_1 \\ w_1 \end{array} \right), \ldots, \left( \begin{array}{c} \mathbf{v}_p \\ w_p \end{array} \right) \right),
\]

and \( s \) parameters \( \mu = (\mu_1, \ldots, \mu_s) \) with real coefficients. Denote by \( H^{p+1+s}_{\ell}(\mathbb{R}) \) the linear space of homogeneous degree \( \ell \) polynomials in the variables \( v = (v_0, v_1, \ldots, v_p) \) and \( \mu = (\mu_1, \ldots, \mu_s) \). Note that we may decompose these spaces as in (3.15) as follows

\[
H^{2p+2+s}_{\ell}(\mathbb{R}) = H^{2p+2}_{\ell}(\mathbb{R}) \oplus P^{2p+2+s}_{\ell}(\mathbb{R})
\]

and

\[
H^{p+1+s}_{\ell}(\mathbb{R}) = H^{p+1}_{\ell}(\mathbb{R}) \oplus P^{p+1+s}_{\ell}(\mathbb{R})
\]

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where $H_{\ell}^{2p+2}(\mathbb{R})$ and $H_{\ell}^{p+1}(\mathbb{R})$ are the $\mu$-independent polynomials, and $P_{\ell}^{2p+2+s}(\mathbb{R})$ and $P_{\ell}^{p+1+s}(\mathbb{R})$ are the polynomials which vanish at $\mu = 0$. Finally, define the surjective linear mapping
\[
\mathcal{R} : H_{\ell}^{2p+2+s}(\mathbb{R}) \longrightarrow H_{\ell}^{p+1+s}(\mathbb{R})
\]
as
\[
(\mathcal{R}(h))(v, \mu) = h \left( \left( \begin{array}{c} u_0 \\ 0 \\ \vdots \\ 0 \end{array} \right), \left( \begin{array}{c} v_1 \\ 0 \\ \vdots \\ 0 \end{array} \right), \ldots, \left( \begin{array}{c} v_p \\ 0 \\ \vdots \\ 0 \end{array} \right), \mu \right).
\]

Let $\hat{\Phi}$ be as in (2.9). We will define $\Phi$ to be the $1 \times (2p + 1)$ matrix obtained from the first $2p + 1$ elements of the first row of $\hat{\Phi}$, i.e.
\[
\Phi = \left( 1 \ e^{i\omega_1 \theta} \ e^{-i\omega_1 \theta} \ \cdots \ e^{i\omega_p \theta} \ e^{-i\omega_p \theta} \right).
\]
From Lemma 2.3 and the fact that $\Psi(0)$ denotes the $(2p + 1) \times 1$ matrix obtained from the first $2p + 1$ elements of the first column of $\hat{\Psi}(0)$ in Lemma 2.3, it follows that
\[
\Psi(0) = \text{col}(u_0, u_1, \overline{u}_1, \ldots, u_p, \overline{u}_p)
\]
where $u_0 \neq 0$ is real and $u_j \neq 0$ are complex, $j = 1, \ldots, p$.

Let $\tau = (\tau_1, \ldots, \tau_{p+1}) \in \mathbb{R}^{p+1}$, and define
\[
\hat{E}_{\tau} = \begin{pmatrix} \hat{\Phi}(\tau_1) \\ \vdots \\ \hat{\Phi}(\tau_{p+1}) \end{pmatrix} = \begin{pmatrix} 1 & e^{i\omega_1 \tau_1} & e^{-i\omega_1 \tau_1} & \cdots & e^{i\omega_p \tau_1} & e^{-i\omega_p \tau_1} & \tau_1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & e^{i\omega_1 \tau_2} & e^{-i\omega_1 \tau_2} & \cdots & e^{i\omega_p \tau_2} & e^{-i\omega_p \tau_2} & \tau_2 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & e^{i\omega_1 \tau_{p+1}} & e^{-i\omega_1 \tau_{p+1}} & \cdots & e^{i\omega_p \tau_{p+1}} & e^{-i\omega_p \tau_{p+1}} & \tau_{p+1} \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \end{pmatrix}
\]
\[
E_{\tau} = \begin{pmatrix} \Phi(\tau_1) \\ \vdots \\ \Phi(\tau_{p+1}) \end{pmatrix} = \begin{pmatrix} 1 & e^{i\omega_1 \tau_1} & e^{-i\omega_1 \tau_1} & \cdots & e^{i\omega_p \tau_1} & e^{-i\omega_p \tau_1} \\ 1 & e^{i\omega_1 \tau_2} & e^{-i\omega_1 \tau_2} & \cdots & e^{i\omega_p \tau_2} & e^{-i\omega_p \tau_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{i\omega_1 \tau_{p+1}} & e^{-i\omega_1 \tau_{p+1}} & \cdots & e^{i\omega_p \tau_{p+1}} & e^{-i\omega_p \tau_{p+1}} \end{pmatrix}.
\]
As in [4], we define the $\ell$-mappings associated to $\hat{E}_{\tau}$ and to $E_{\tau}$ respectively as follows:
\[
\hat{\mathcal{E}}_{\ell}^t : H_{\ell}^{2p+2+s}(\mathbb{R}) \longrightarrow H_{\ell}^{2p+2+s}(\mathbb{R}^{2p+1})
\]
\[
(\hat{\mathcal{E}}_{\ell}^t(h))(x, \mu) \equiv \Psi(0) h(\hat{E}_{\tau} x, \mu)
\]
Now, let $\Pi : H^{2p+1+s}(\mathbb{R}^{2p+1}, T^p) \rightarrow H^{p+1+s}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p})$ be the mapping defined in (3.13)-(3.14), let $\tilde{A} : H^{2p+2+s}(\mathbb{R}^{2p+1}) \rightarrow H^{2p+1+s}(\mathbb{R}^{2p+1}, T^p)$ be the projection operator defined in Definition 3.6 and define

$$A : H^{2p+1+s}(\mathbb{R}^{2p+1}) \rightarrow H^{2p+1+s}(\mathbb{R}^{2p+1}, T^p)$$

by

$$A g(x, \mu) = \int_{\Gamma_0} \gamma g(\gamma^{-1} x, \mu) d\gamma,$$

as in [4]. Our main result in this section is the following:

**Proposition 4.2** For an open and dense set $U \subset \mathbb{R}^{p+1}$, the following linear mapping is surjective for all $\tau \in U$:

$$\Pi \circ \tilde{A} \circ \tilde{E}_\tau : H^{2p+2+s}(\mathbb{R}) \rightarrow H^{p+1+s}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}).$$

**Proof** It is a simple computation to show that

$$\Pi \circ \tilde{A} \circ \tilde{E}_\tau = (\Pi \circ A \circ \tilde{E}_\tau) \circ \mathcal{R},$$

where $\mathcal{R}$ is the surjective linear operator defined in (4.1). In [4], it was shown that there exists an open and dense set $U \subset \mathbb{R}^{p+1}$ such that for all $\tau \in U$, the mapping

$$\Pi \circ A \circ \tilde{E}_\tau : H^{p+1+s}(\mathbb{R}) \rightarrow H^{p+1+s}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p})$$

is surjective. Thus, for all $\tau \in U$, $\Pi \circ \tilde{A} \circ \tilde{E}_\tau$ is surjective.}

### 4.2 Main results on realizability

With Proposition 4.2 in hand, we obtain realizability results analogous to the semisimple case. Since the proofs are almost identical, we merely give the statements of these results here. First, we will define the following linear spaces of (non-homogeneous) polynomials
Definition 4.3

\[ \mathcal{H}_\ell^{p+1+s}(\mathbb{R}) \equiv \bigoplus_{j=2}^{\ell} H_j^{p+1+s}(\mathbb{R}), \quad \mathcal{H}_\ell^{p+1}(\mathbb{R}) \equiv \bigoplus_{j=2}^{\ell} H_j^{p+1}(\mathbb{R}), \]
\[ \mathcal{P}_\ell^{p+1+s}(\mathbb{R}) \equiv \bigoplus_{j=2}^{\ell} P_j^{p+1+s}(\mathbb{R}), \quad \mathcal{H}_\ell^{p+1+s}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}) \equiv \bigoplus_{j=2}^{\ell} H_j^{p+1+s}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}), \]
\[ \mathcal{H}_\ell^{p+1}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}) \equiv \bigoplus_{j=2}^{\ell} H_j^{p+1}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}), \quad \mathcal{P}_\ell^{p+1+s}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}) \equiv \bigoplus_{j=2}^{\ell} P_j^{p+1+s}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}). \]

Theorem 4.4 Consider the RFDE (4.4), and let \( \Lambda_0 \) denote the set of solutions of (4.4) with zero real part. Suppose that Hypothesis 2.2 is satisfied. Let \( \ell \geq 2 \) be a given integer. For each \( h \in \mathcal{H}_\ell^{p+1}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}) \) and each \( q \in \mathcal{P}_\ell^{p+1+s}(\mathbb{R}^{p+1}, \mathbb{Z}_{2,p}) \) there are \( p+1 \) distinct points \( \tau_1, \ldots, \tau_{p+1} \in [-r, 0] \), an \( \eta \in \mathcal{H}_\ell^{p+1}(\mathbb{R}) \) and a \( \xi \in \mathcal{P}_\ell^{p+1+s}(\mathbb{R}) \) such that if

\[ \hat{F}(z_\ell, \nu, \mu) = \eta(z(t + \tau_1), \ldots, z(t + \tau_d)) + \xi(z(t + \tau_1), \ldots, z(t + \tau_d), \mu) \]

in (4.2), then in polar coordinates, the radial part of the center manifold equations (3.18) in \( \mathbb{T}^p \times \mathbb{R} \)-equivariant normal form up to degree \( \ell \) reduces to \( \dot{\rho} = \nu e_0 + h(\rho) + q(\rho, \mu) \), where \( \rho \equiv (\rho_0, \rho_1, \ldots, \rho_p) \). In fact, \( \tau \equiv (\tau_1, \ldots, \tau_{p+1}) \) can be chosen in an open and dense set of \([-r, 0]^{p+1} \), independently of the particular \( h \) and \( q \) to be realized (i.e. only \( \eta \) and \( \xi \) must be changed in order to account for different jets to be realized).

As in the semisimple case of [4], we can show that the number of delays \( p+1 \) shown above to be sufficient to solve the realizability problem for the radial part is optimal i.e. surjectivity is violated beyond some finite order \( \ell_0 \) if the number of delays is less than \( p+1 \) (see Theorem 5.4 of [4]).

For general RFDEs, we have the following result on realization of unfoldings:

Theorem 4.5 Consider the general nonlinear RFDE

\[ \dot{z}(t) = L_0 z_\ell + \nu + N(z_\ell) \quad (4.7) \]

where \( \nu \in \mathbb{R}, L_0 : C_1 \to \mathbb{R} \) is a bounded linear operator from \( C_1 \equiv \mathcal{C} \left([-r, 0], \mathbb{R}\right) \) into \( \mathbb{R} \), and \( N \) is a smooth function from \( C_1 \) into \( \mathbb{R} \), with \( N(0) = 0 \), \( DN(0) = 0 \). Let \( \Lambda_0 \) denote the set of solutions of (4.2) with zero real part and suppose that Hypothesis 2.2 is satisfied. Then the local dynamics of (4.2) near the origin on an invariant center manifold can be described by a system of ordinary differential equations on \( \mathbb{R}^{2p+1} \). Moreover, this ODE system can be brought into \( \mathbb{T}^p \times \mathbb{R} \)-equivariant normal form to any desired order \( \ell \), and the resulting
(truncated at order \( \ell \)) normal form can be uncoupled into a \( p + 1 \)-dimensional system and a \( p \)-dimensional system

\[
\dot{\rho} = \nu e_0 + h(\rho; N) \tag{4.8}
\]

\[
\dot{\theta} = k(\rho; N), \tag{4.9}
\]

where for given \( N \), \( h(\cdot; N) \) is some element of \( \mathcal{H}_p^{p+1}(\mathbb{R}^{p+1}, \mathbb{Z}_2^p) \), and \( k(\cdot; N) : \mathbb{R}^p \to \mathbb{R}^p \).

Let \( \tilde{h}(\rho, \mu) \) be an \( s \)-parameter equivariant unfolding of \( h \) of degree at most \( \ell \), i.e. \( \tilde{h} \in \mathcal{H}_p^{p+1+s}(\mathbb{R}^{p+1}, \mathbb{Z}_2^p) \) and \( \tilde{h}(\cdot, 0) = h(\cdot; N) \). Then there exists an \( s \)-parameter unfolding of (4.7) of the form

\[
\dot{z}(t) = L_0(z_t) + \nu + N(z_t) + \xi(z(t + \tau_1), \ldots, z(t + \tau_{p+1}), \mu) \tag{4.10}
\]

(where \( \tau = (\tau_1, \ldots, \tau_{p+1}) \in \mathbb{R}^{p+1} \), and \( \xi \in \mathcal{P}_{p+1+s}(\mathbb{R}) \) vanishes at \( \mu = 0 \)) which realizes the unfolded radial equations

\[
\dot{\rho} = \nu e_0 + \tilde{h}(\rho, \mu)
\]

on an invariant center manifold for (4.10).

### 4.3 Generic saddle-node/Hopf interaction

The following example is an immediate consequence of our results.

**Example 4.6** Consider the RFDE (2.2) in the case \( \mu = 0 \),

\[
\dot{z}(t) = L_0(z_t) + \nu + \hat{F}(z_t), \tag{4.11}
\]

such that the characteristic equation (2.4) has simple purely imaginary roots \( \pm i\omega \neq 0 \), a simple root at 0, and no other roots on the imaginary axis. If

\[
\hat{F}(z_t) = A_{20}(z(t + \tau_1))^2 + A_{11}z(t + \tau_1)z(t + \tau_2) + A_{02}(z(t + \tau_2))^2 \\
A_{30}(z(t + \tau_1))^3 + A_{21}z((t + \tau_1))^2z(t + \tau_2) + A_{12}z(t + \tau_1)(z(t + \tau_2))^2 + A_{03}(z(t + \tau_2))^3, \tag{4.12}
\]

where \( \tau_1, \tau_2 \in [-r, 0] \), then the uncoupled radial part of the center manifold equations to cubic order are the following Guckenheimer [12, 13] normal form

\[
\dot{\rho}_0 = \nu + a_1\rho_0^3 + a_2\rho_0^2 + a_3\rho_0 + a_4\rho_0^2 \tag{4.13}
\]

\[
\dot{\rho}_1 = b_1\rho_0\rho_1 + b_2\rho_1^3 + b_3\rho_1\rho_0^2,
\]

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where the coefficients $a_{1,2,3,4}$ and $b_{1,2,3}$ are functions of $A_{20}$, $A_{11}$, $A_{02}$, $A_{30}$, $A_{21}$, $A_{12}$, $A_{03}$, $\tau_1$ and $\tau_2$. From our results, it follows that generically, any values of $a_{1,2,3,4}$ and $b_{1,2,3}$ can be achieved by appropriate choice of $A_{20}$, $A_{11}$, $A_{02}$, $A_{30}$, $A_{21}$, $A_{12}$, $A_{03}$. Also, the following versal unfolding of (4.13)

\[
\dot{\rho}_0 = \nu + a_1 \rho_0^2 + a_2 \rho_1^2 + a_3 \rho_0^3 + a_4 \rho_0 \rho_1^2 \\
\dot{\rho}_1 = \mu \rho_1 + b_1 \rho_0 \rho_1 + b_2 \rho_1^3 + b_3 \rho_1 \rho_0^2,
\]

is generically realized by the following unfolding of (4.11)

\[
\dot{z}(t) = L_0 z + \nu + \tilde{F}(z) + \mu \left( A_{10} z(t + \tau_1) + A_{01} \mu_2 z(t + \tau_2) \right)
\]

for appropriate choice of $A_{10}$ and $A_{01}$.

5 Concluding remarks

Our solution here to the realizability problem for the non-semisimple saddle-node/multiple Hopf interaction in scalar RFDEs complements our results in [4] for the semisimple cases of transcritical/multiple Hopf interaction and non-resonant multiple Hopf bifurcation. We note that our results here on realizability are generic, optimal in the number of delays required to guarantee realizability, and applicable to any finite order expansion and truncation of the normal form. Therefore, nonlinear degeneracies and their unfoldings for the saddle-node/multiple Hopf interaction are covered by our theory.

As in [4], we have not considered general $n > 1$ dimensional systems of RFDEs. This is not based on any deep theoretical issues associated to the $n > 1$ case, but rather to complications arising out of notation and messy algebraic computations which would make the exposition extremely cumbersome. We have, however, every reason to believe that the realizability problem for each of the bifurcations studied in [4] and in the present paper could be solved for systems using the techniques and framework we have developed.

More subtle is the issue of relaxing Hypothesis 2.2 to include repeated eigenvalues with Jordan blocks, and rational resonances in the purely imaginary eigenvalues. Apart from affecting the dimension of the torus group admitted by the normal form (and consequently the dimension of the uncoupled radial equations), Jordan blocks would introduce an additional non-compact component to the normal form symmetry. In this case, our present analysis in this paper may shed some valuable light on a suitable approach to tackle the general problem, i.e. first solve the semisimple case, and then try to exploit the semisimple solution.
as much as possible in order to prove realizability in the associated non-semisimple problem.

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A Proof of Proposition 3.5

Let $f$ be a given element of $H_{2p+2+s}^2(\mathbb{R}^{2p+1})$, and consider $\tilde{f} = (f, 0, 0) \in H_{2p+2+s}^2(\mathbb{R}^{2p+2+s})$. From Proposition 3.2, there exists $h = (h^x, h^\nu, h^\mu) \in H_{2p+2+s}^2(\mathbb{R}^{2p+2+s})$ and a unique $g = (g^x, g^\nu, g^\mu) \in H_{2p+2+s}^2(\mathbb{R}^{2p+2+s}, \Gamma)$ such that

$$(f, 0, 0) = L_B(h^x, h^\nu, h^\mu) + (g^x, g^\nu, g^\mu). \quad (A.1)$$

Note that

$$L_B(h^x, h^\nu, h^\mu) = \left( L_{B,\nu} h^x - h^\nu e_0, D_x h^\nu B x + \nu \frac{\partial h^\nu}{\partial x_0}, D_x h^\mu B x + \nu \frac{\partial h^\mu}{\partial x_0} \right).$$

Thus, from (A.1), we get

$$D_x h^\nu B x + \nu \frac{\partial h^\nu}{\partial x_0} + g^\nu = 0 \quad (A.2)$$

and

$$f = L_{B,\nu} h^x - h^\nu e_0 + g^x. \quad (A.3)$$

From Lemma 3.3, $g^\nu$ is of the form

$$g^\nu = \nu r_1(x_0, x_1 \overline{x_1}, \ldots, x_p \overline{x_p}, \mu) + r_2(x_0, x_1 \overline{x_1}, \ldots, x_p \overline{x_p}, \mu),$$

where $r_1$ is such that the component of $g^x$ along $e_0$, $g_0^x$, is of the form

$$g_0^x = x_0 r_1(x_0, x_1 \overline{x_1}, \ldots, x_p \overline{x_p}, \mu).$$

Therefore, the $\nu$ part of equation (A.2) reduces to

$$\nu \frac{\partial h^\nu}{\partial x_0} + \sum_{j=1}^p i \omega_j \left( x_j \frac{\partial h^\nu}{\partial x_j} - \overline{x_j} \frac{\partial h^\nu}{\partial x_j} \right) + \nu r_1 + r_2 = 0. \quad (A.4)$$

Before we proceed any further, we will need the following two lemmas.
Lemma A.1 Let \( h \) be a real-valued smooth function of \( x_0, x_1, \ldots, x_p, \nu, \mu \) such that the function \( g(x_0, x_1, \ldots, x_p, \nu, \mu) \) defined by
\[
g(x_0, x_1, \ldots, x_p, \nu, \mu) \equiv \sum_{j=1}^{p} i\omega_j \left( x_j \frac{\partial h}{\partial x_j} - \overline{x_j} \frac{\partial h}{\partial \overline{x_j}} \right) \quad \text{(A.5)}
\]
is \( \mathbb{T}^p \) invariant, i.e.
\[
g(x_0, e^{i\theta_1} x_1, e^{-i\theta_1} \overline{x_1}, \ldots, e^{i\theta_p} x_p, e^{-i\theta_p} \overline{x_p}, \nu, \mu) = g(x_0, x_1, \ldots, x_p, \overline{x_p}, \nu, \mu) \quad \text{(A.6)}
\]
for all \( \theta_1, \ldots, \theta_p \in \mathbb{R} \), and for all \( (x, \nu, \mu) \in \mathbb{R}^{2p+2+s} \). Then \( h \) is also \( \mathbb{T}^p \) invariant, and \( g = 0 \).

**Proof** A simple computation using (A.5) and (A.6) leads to
\[
\frac{d}{d\xi} h(x_0, e^{i\omega_1 \xi} x_1, e^{-i\omega_1 \xi} \overline{x_1}, \ldots, e^{i\omega_p \xi} x_p, e^{-i\omega_p \xi} \overline{x_p}, \nu, \mu) = g(x_0, x_1, \ldots, x_p, \overline{x_p}, \nu, \mu).
\]
Integrating this equation gives
\[
h(x_0, e^{i\omega_1 \xi} x_1, e^{-i\omega_1 \xi} \overline{x_1}, \ldots, e^{i\omega_p \xi} x_p, e^{-i\omega_p \xi} \overline{x_p}, \nu, \mu) - h(x_0, x_1, \ldots, x_p, \overline{x_p}, \nu, \mu) = \xi g(x_0, x_1, \ldots, x_p, \overline{x_p}, \nu, \mu).
\]
For any given \( (x, \nu, \mu) \in \mathbb{R}^{2p+2+s} \), the left hand side of the previous equation is bounded in \( \xi \), and it follows that \( g \) must equal 0 and that
\[
h(x_0, e^{i\omega_1 \xi} x_1, e^{-i\omega_1 \xi} \overline{x_1}, \ldots, e^{i\omega_p \xi} x_p, e^{-i\omega_p \xi} \overline{x_p}, \nu, \mu) \equiv h(x_0, x_1, \ldots, x_p, \overline{x_p}, \nu, \mu).
\]
The conclusion follows from the non-resonance condition on \( \omega_1, \ldots, \omega_p \) specified in Hypothesis 2.2 using density and continuity.

Lemma A.2 Equation (A.4) implies that \( h' \) must be of the form
\[
h' = h(x_0, x_1 \overline{x_1}, \ldots, x_p \overline{x_p}, \nu, \mu) = - \int_0^{x_0} r_1(t, x_1 \overline{x_1}, \ldots, x_p \overline{x_p}, \nu, \mu) dt - m(x_1 \overline{x_1}, \ldots, x_p \overline{x_p}, \nu, \mu).
\]
Proof Write $h''(x,\nu,\mu) = \sum_{j=0}^{\ell} a_j(x,\mu)\nu^j$. Equation \((A.4)\) becomes
\[
\sum_{j=0}^{\ell} \frac{\partial a_j}{\partial x_0} \nu^{j+1} + \sum_{j=0}^{\ell} \left( \sum_{j=1}^{p} i\omega_j \left( x_j \frac{\partial a_j}{\partial x_j} - \overline{x_j} \frac{\partial a_j}{\partial \overline{x_j}} \right) \right) \nu^j = g, \tag{A.7}
\]
where $g = -(\nu r_1 + r_2)$ is $T^p$ invariant as in Lemma \ref{lem:tnp_invariant}. Applying Lemma \ref{lem:tnp_invariant} successively to the coefficient of $\nu^0$, $\nu^1$, $\ldots$, $\nu^{\ell+1}$ in \((A.7)\), we get that $r_2 = 0$, $\frac{\partial a_0}{\partial x_0} = -r_1$, $\frac{\partial a_j}{\partial x_0} = 0$, $j = 1, \ldots, \ell$, and $a_0, \ldots, a_\ell$ are $T^p$-invariant. The conclusion of the lemma then follows immediately upon integration.
\[\blacksquare\]

The component of \((A.3)\) along $e_0$ now has the form
\[
f_0 = \nu \frac{\partial h_0^x}{\partial x_0} + \sum_{j=1}^{p} i\omega_j \left( x_j \frac{\partial h_0^x}{\partial x_j} - \overline{x_j} \frac{\partial h_0^x}{\partial \overline{x_j}} \right) + \int_0^{x_0} r_1(t, x_1, \ldots, x_p, \mu) \, dt + m(x_1, \ldots, x_p, \nu, \mu) + x_0 r_1(x_0, x_1, \ldots, x_p, \mu). \tag{A.8}
\]
Using Taylor’s theorem, we write
\[
m(x_1, \ldots, x_p, \nu, \mu) = m(x_1, \ldots, x_p, 0, \mu) + \nu \hat{m}(x_1, \ldots, x_p, \nu, \mu),
\]
and note that
\[
\nu \hat{m}(x_1, \ldots, x_p, \nu, \mu) = \nu \frac{\partial}{\partial x_0} x_0 \hat{m}, \text{ and}
\]
\[
\sum_{j=1}^{p} i\omega_j \left( x_j \frac{\partial}{\partial x_j} x_0 \hat{m} - \overline{x_j} \frac{\partial}{\partial \overline{x_j}} x_0 \hat{m} \right) = 0.
\]
Consequently, if we denote $\hat{h}_0^x = h_0^x + x_0 \hat{m}$, then \((A.8)\) reduces to
\[
f_0 = \nu \frac{\partial \hat{h}_0^x}{\partial x_0} + \sum_{j=1}^{p} i\omega_j \left( x_j \frac{\partial \hat{h}_0^x}{\partial x_j} - \overline{x_j} \frac{\partial \hat{h}_0^x}{\partial \overline{x_j}} \right) + \int_0^{x_0} r_1(t, x_1, \ldots, x_p, \mu) \, dt + \hat{m}(x_1, \ldots, x_p, 0, \mu) + x_0 r_1(x_0, x_1, \ldots, x_p, \mu). \tag{A.9}
\]
Together, (A.9) and the last 2 \( p \) components of (A.3) imply that \( f \) can be written as a sum of an element in range \( \mathcal{L}_{B,\nu} \) and an element in \( H_{\ell}^{2p+1+s}(\mathbb{R}^{2p+1}, \mathbb{T}^p) \), i.e.

\[
H_{\ell}^{2p+2+s}(\mathbb{R}^{2p+1}) = H_{\ell}^{2p+1+s}(\mathbb{R}^{2p+1}, \mathbb{T}^p) + \text{range} \mathcal{L}_{B,\nu}.
\]

We now must show that the above sum is, in fact, a direct sum. Suppose that \( h^x \in H_{\ell}^{2p+2+s}(\mathbb{R}^{2p+1}) \) and \( g^x \in H_{\ell}^{2p+1+s}(\mathbb{R}^{2p+1}, \mathbb{T}^p) \) are such that

\[
\mathcal{L}_{B,\nu} h^x + g^x = 0. \tag{A.10}
\]

Write the component of \( g^x \) along \( e_0 \) as

\[
\hat{g}^0(x_0, x_1\overline{x_1}, \ldots, x_p\overline{x_p}, \mu) = \int_0^{x_0} \left( t \hat{g}^0(t, x_1\overline{x_1}, \ldots, x_p\overline{x_p}, \mu) + t^2 \frac{\partial \hat{g}^0}{\partial x_0}(t, x_1\overline{x_1}, \ldots, x_p\overline{x_p}, \mu) \right) dt.
\]

Note that \( r_1 \) is regular at \( x_0 = 0 \), since both the integral term above and its derivative with respect to \( x_0 \) vanish at \( x_0 = 0 \). A simple computation verifies that

\[
x_0 \hat{g}^0(x_0, x_1\overline{x_1}, \ldots, x_p\overline{x_p}, \mu) = \int_0^{x_0} r_1(t, x_1\overline{x_1}, \ldots, x_p\overline{x_p}, \mu) dt + x_0 r_1(x_0, x_1\overline{x_1}, \ldots, x_p\overline{x_p}, \mu). \tag{A.12}
\]

Consequently, if we define \( g^\nu = 0 \), \( h^\mu = 0 \),

\[
\hat{g}^\nu = \nu r_1(x_0, x_1\overline{x_1}, \ldots, x_p\overline{x_p}, \mu)
\]

and

\[
h^\nu = - \int_0^{x_0} r_1(t, x_1\overline{x_1}, \ldots, x_p\overline{x_p}, \mu) dt - g^\nu(0, x_1\overline{x_1}, \ldots, x_p\overline{x_p}, \mu),
\]

then from Lemma 3.3 we see that \( ((x_0 r_1, g_1^x, \ldots, g_{2p}^x), g^\nu, g^\mu) \in H_{\ell}^{2p+2+s}(\mathbb{R}^{2p+2+s}, \Gamma) \). It now follows from (A.10) that

\[
(0, 0, 0) = \mathcal{L}_{B}(h^x, h^\nu, h^\mu) + ((x_0 r_1, g_1^x, \ldots, g_{2p}^x), g^\nu, g^\mu).
\]

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From Proposition 3.2, we get that \( r_1 = g_1^\tau = \cdots = g_{2p}^\tau = 0 \) and \( \mathcal{L}_B(h^\tau, h^\nu, h^\mu) = (0,0,0) \), from which it follows that

\[
\mathcal{L}_{B,\nu} h^\tau + g_0^\tau(0, x_1x_1, \ldots, x_px_p, \mu) \mathbf{e}_0 = 0.
\]

The component of the above equation along \( \mathbf{e}_0 \), is

\[
\nu \frac{\partial h_0^x}{\partial x_0} + \sum_{j=1}^p i \omega_j \left( x_j \frac{\partial h_0^x}{\partial x_j} - x_j \frac{\partial h_0^x}{\partial x_j} \right) = -g_0^x(0, x_1x_1, \ldots, x_px_p, \mu).
\]

Using Lemma A.1, we conclude that \( g_0^x(0, x_1x_1, \ldots, x_px_p, \mu) = 0 \), \( \mathcal{L}_{B,\nu} h^\tau = 0 \), and from (A.11), that \( g^x = 0 \). Therefore, we conclude that

\[
H_{t}^{2p+2+}(\mathbb{R}^{2p+1}) = H_{t}^{2p+1+s}(\mathbb{R}^{2p+1}, \mathbb{T}^p) \oplus \text{range } \mathcal{L}_{B,\nu}.
\]

\[\blacksquare\]

**B Proof of Proposition 3.7**

The proof that \( \hat{A} \) is a projection is similar to the proof given for the projection operator \( A \) used in the semisimple case in \[4\].

Now, let \( f \in \text{range } \hat{A} \), then \( \hat{A} f = f \), i.e.

\[
f(x, \nu, \mu) = \int_{\Gamma_0} \gamma f(\gamma^{-1}x, 0, \mu) \, d\gamma.
\]

So, for any \( \sigma \in \Gamma_0 \), we have

\[
\sigma f(\sigma^{-1}x, \nu, \mu) = \sigma \int_{\Gamma_0} \gamma f(\gamma^{-1}\sigma^{-1}x, 0, \mu) \, d\gamma = \int_{\Gamma_0} \sigma \gamma f((\sigma \gamma)^{-1}x, 0, \mu) \, d\gamma
\]

\[
= \int_{\Gamma_0} \gamma f(\gamma^{-1}x, 0, \mu) \, d\gamma = f(x, \nu, \mu) = f(x, 0, \mu),
\]

and thus \( f \in H_{t}^{2p+2+s}(\mathbb{R}^{2p+1}, \mathbb{T}^p) \). On the other hand, if \( f \in H_{t}^{2p+s+1}(\mathbb{R}^{2p+1}, \mathbb{T}^p) \), then

\[
(\hat{A} f)(x, \nu, \mu) = \int_{\Gamma_0} \gamma f(\gamma^{-1}x, 0, \mu) \, d\gamma = \int_{\Gamma_0} f(x, 0, \mu) \, d\gamma
\]

\[
f(x, 0, \mu) = f(x, \nu, \mu),
\]

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so \( f \in \text{range } \hat{A} \). This establishes (3.7). We now establish (3.8). Since \( \hat{A} \) is a projection, then

\[
H^2p+2+s\ell(R^{2p+1}) = \text{range } \hat{A} \oplus \ker \hat{A}.
\]

From Proposition 3.6 we conclude that \( \dim \ker \hat{A} = \dim \text{range } \mathcal{L}_{B,\nu} \). Thus, we need only show that \( \text{range } \mathcal{L}_{B,\nu} \subset \ker \hat{A} \). Recall the following lemma which was proved in [4].

**Lemma B.1** Let \( g : \Gamma_0 \longrightarrow \mathbb{R}^{2p+1} \) be a continuous function, then

\[
\int_{\Gamma_0} g(\gamma) \, d\gamma = \lim_{T \to \infty} \frac{1}{T} \int_0^T g(e^{Bs}) \, ds.
\]

Now, let \( f \in \text{range } \mathcal{L}_{B,\nu} \), then there exists \( g \in H^2p+2+s\ell(R^{2p+1}) \) such that

\[
D_x g(x, \nu, \mu) Bx - Bg(x, \nu, \mu) + \nu \frac{\partial g}{\partial x_0}(x, \nu, \mu) = f(x, \nu, \mu), \quad \forall (x, \nu, \mu) \in \mathbb{R}^{2p+2+s}.
\]

Therefore, using Lemma B.1 we get

\[
(\hat{A}f)(x, \nu, \mu) = \int_{\Gamma_0} \gamma f(\gamma^{-1}x, 0, \mu) \, d\gamma = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{Bs} f(e^{-Bs}x, 0, \mu) \, ds
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{Bs} \left( D_x g(e^{-Bs}x, 0, \mu) B e^{-Bs}x - Bg(e^{-Bs}x, 0, \mu) \right) \, ds
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{d}{ds} \left( -e^{Bs} g(e^{-Bs}x, 0, \mu) \right) \, ds
\]

\[
= \lim_{T \to \infty} \frac{-e^{BT} g(e^{-BT}x, 0, \mu) + g(x, 0, \mu)}{T}
\]

and this last limit is equal to 0, since the numerator is bounded in \( T \) for any given \( (x, \mu) \in \mathbb{R}^{2p+1+s} \). So we conclude that \( f \in \ker A \), and thus that \( \ker A = \text{range } \mathcal{L}_{B,\nu} \). This establishes (3.8), and concludes the proof of Proposition 3.7.

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