An integral test on time dependent local extinction for super-coalescing Brownian motion with Lebesgue initial measure

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Abstract

This paper concerns the almost sure time dependent local extinction behavior for super-coalescing Brownian motion $X$ with $(1 + \beta)$-stable branching and Lebesgue initial measure on $\mathbb{R}$. We first give a representation of $X$ using excursions of a continuous state branching process and Arratia’s coalescing Brownian flow. For any nonnegative, nondecreasing and right continuous function $g$, put

$$
\tau := \sup\{t \geq 0 : X_t([-g(t), g(t)]) > 0\}.
$$

We prove that $\mathbb{P}\{\tau = \infty\} = 0$ or 1 according as the integral $\int_1^\infty g(t) t^{-1-1/\beta} dt$ is finite or infinite.

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1 Introduction

By a super-coalescing Brownian motion (SCBM in short) we mean a measure-valued stochastic process describing the time-space-mass evolution of a particle system in $\mathbb{R}$. In such a system the particles move according to (instantaneous) coalescing Brownian motions and the masses of those particles evolve according to independent continuous state branching processes (CSBP’s in short) with $(1 + \beta)$-stable branching law. Whenever two particles are in the same location their masses are added up with total mass continues

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the independent $(1 + \beta)$-stable branching. Note that this scheme is well-defined due to the additivity of the CSBP. For coalescence to happen we only consider SCBM on $\mathbb{R}$.

The SCBM has been studied in [4, 9, 10]. With arbitrary finite initial measure it can be obtained by taking a high-density/small-particle limit of the empirical measure process of coalescing-branching particle system with Poisson initial measure. Its probability law can be specified by the duality on coalescing Brownian motions.

Formally, the SCBM with Radon initial measure $\mu$ can be constructed by taking a monotone limit of SCBM's with initial measures $\mu$ truncated over increasing finite intervals. In this paper we will present a direct construction of the SCBM using excursions of the CSBP and Arratia’s coalescing Brownian flow following Dawson et al. [4]. A similar construction was proposed in Dawson and Li [3] for superprocess with dependent spatial motion. This procedure allows us to construct the SCBM simultaneously for all $t$. It turns out to be handy for the later coupling arguments in proving our main results.

Almost sure local extinction for super-Brownian motion on $\mathbb{R}^n$ says that, given any bounded Borel set in $\mathbb{R}^n$, almost surely the measure-value process does not charge on it after a time long enough. For super-Brownian motion with Lebesgue initial measure it was first studied in Iscoe [8] via analyzing the super-Brownian occupation time using nonlinear PDE. By time dependent local extinction we mean the local extinction behavior where the size of the above-mentioned set also depends $t$. Almost sure time dependent local extinction was discussed in Fleischmann et al [6]. An integral test was found in Zhou [11] on the almost sure time dependent local extinction for super-Brownian motion with $(1+\beta)$-stable branching and Lebesgue initial measure. Its proof is a Borel-Cantelli argument based on estimates of extinction probabilities. The additivity for super-Brownian motion plays a crucial role in the proof there since it allows us to decompose one super-Brownian motion into independent super-Brownian motions for different purposes and then treat them separately.

The SCBM often shares similar asymptotic properties as super-Brownian motion. In this paper we are going to show that the same integral test in Zhou [11] is also valid for the SCBM on $\mathbb{R}$. More precisely, for any nonnegative, nondecreasing and right continuous function $g$ on $\mathbb{R}^+$, we are going to show that the probability of seeing any mass over interval $[-g(t), g(t)]$ for time $t$ large enough is either 0 or 1 depending on whether the integral $\int g(t)^{-1-1/\beta} dt$ is finite or not.

The main difficulty for our current work on SCBM is that the SCBM is no longer additive due to the dependence of coalescing spatial motion. As a result we have to adopt strategies that are quite different from Zhou [11] to tackle this problem.

In one direction of our proof we propose a duality between two system of coalescing Brownian motions with boundary conditions, where in one system the Brownian motions are stopped at some barriers and in the other system the Brownian motions are reflected at the same barriers, which generalizes a boundary free dual relationship on coalescing Brownian motions in Zhou [10]. This duality first results in a duality on SCBM with absorbing barriers. Then it further leads to an estimate from below on the vanishing probability of the occupation time for SCBM. Such an estimate in terms of reflecting Brownian motions can be applied iteratively to show that the time dependent local extinction (with respect to $g$) occurs whenever function $g$ does not increase too fast, which is characterized by the integral condition.

In the other direction of our proof, when $g$ increases fast enough we first choose
a sequence of times \((t_n)\) strictly increasing to \(\infty\) and the associated disjoint intervals \([l_n, r_n], n = 1, 2, \ldots\) in \(\mathbb{R}^+\). We then consider an SCBM \(\bar{X}\) starting from Lebesgue measure restricted to the region \(\bigcup_{n=1}^{\infty} [-r_n, -l_n] \cup \bigcup_{j=1}^{\infty} [l_j, r_j]\). We are able to choose the spacings between intervals properly to satisfy the following constrains. On one hand, the spacing is not too small so that for each \(n\), up to time \(t_n\) the mass started from interval \([-r_n, -l_n] \cup [l_n, r_n]\) at time 0 is very unlikely to interact with masses initiated from other intervals. On the other hand, the spacing is also not too large so that the process \(\bar{X}\) still has enough initial mass to start with and by time \(t_n\) the probability \(\mathbb{P}\{\bar{X}_{t_n}([0, g(t_n)]) > 0\}\) is not too small. Then the proof can be carried out by coupling arguments together with several Borel-Cantelli arguments.

The approaches developed in this paper can be modified to study the almost sure time dependent local extinction for SCBM with Lévy branching mechanism other than stable branching. But we do not expect the result to be as clean.

The rest of the paper is arranged as follows. In Section 2 we present the construction of SCBM using Arratia’s flow and the branching excursion law. In Section 3 we find a duality between coalescing Brownian motions with either absorbing or reflecting boundary conditions, which then leads to a dual relationship between SCBM with absorbing barriers and reflecting Brownian motions in Section 4. Our main results, Theorem 4.4 and Theorem 4.5 and their proofs are presented in Section 4.

## 2 A construction of SCBM with excursions and Arratia’s flow

Let \(\gamma \geq 0\) and \(0 < \beta \leq 1\) be fixed constants. A **continuous state critical branching process** (CSBP) with \((1 + \beta)\)-stable branching is a right continuous Markov process taking values in \([0, \infty)\) whose transition semigroup \((Q_t)_{t \geq 0}\) is determined by

\[
\int_0^\infty e^{-yz}Q_t(x, dy) = \exp\{-x\psi_t(z)\}, \quad t, x, z \geq 0,
\]

where \(\psi_t(z)\) is the unique solution of

\[
\frac{\partial}{\partial t} \psi_t(z) = \frac{1}{1 + \beta} \gamma \psi_t(z)^{1+\beta}, \quad \psi_0(z) = z.
\]

It is easy to find that

\[
\psi_t(z) = z \left(\frac{1 + \beta}{1 + \beta + \gamma \beta t z^\beta}\right)^{1/\beta}.
\]

In the sequel of the paper, we shall always assume \(\gamma > 0\) unless otherwise specified. Then for any \(t > 0\) we have

\[
\lim_{z \to \infty} \psi_t(z) = \left(\frac{1 + \beta}{\gamma \beta t}\right)^{1/\beta} =: \psi_t(\infty) < \infty.
\]

Letting \(z \to \infty\) in (2.1) yields

\[
Q_t(x, \{0\}) = \exp\{-x\psi_t(\infty)\}, \quad t > 0, \quad x \geq 0.
\]
From (2.1) it follows that
\[ Q_t(x_1 + x_2, \cdot) = Q_t(x_1, \cdot) * Q_t(x_2, \cdot), \quad t, x_1, x_2 \geq 0. \]

In view of this infinite divisibility and (2.2), we have
\[ \kappa \text{ for a family of finite diffuse measures (} \kappa_t \text{)} \]
for every \( t \geq 0 \), \( a \geq 0 \).

\[ \psi_t(z) = \int_0^\infty (1 - e^{-zy}) \kappa_t(\text{d}y), \quad t > 0, \ z \geq 0, \]
(2.3)

for a family of finite diffuse measures (\( \kappa_t \)) on \((0, \infty)\); see, e.g., Bertoin and Le Gall [2].

A coalescing Brownian flow \( \{\phi(a, t) : a \in \mathbb{R}, t \geq 0\} \) is by definition an \( \mathbb{R} \)-valued two-parameter process such that for every \( a \in \mathbb{R}, t \mapsto \phi(a, t) \) is continuous; for every \( t \geq 0, a \mapsto \phi(a, t) \) is nondecreasing and right continuous; and for any \( n \geq 1 \) and \( (a_1, \ldots, a_n) \in \mathbb{R}^n \), the probability law of \( \{ \phi(a_1, t), \ldots, \phi(a_n, t) \} \) follows that of the coalescing Brownian motion starting at \((a_1, \ldots, a_n)\); see Arratia [1] for more details.

Let \( \mathcal{M} \) be the space of Radon measures on \( \mathbb{R} \) endowed with the topology of vague convergence. Let \( \mathcal{M}_a \) be the subset of \( \mathcal{M} \) consisting of purely atomic Radon measures. Let \( \mathcal{B}_0 \) be the space of bounded Borel functions on \( \mathbb{R} \) with bounded supports. Suppose that \( \{(\phi_1(t), \phi_2(t), \ldots) : t \geq 0\} \) is a countable system of coalescing Brownian motions and \( \{(\xi_1(t), \xi_2(t), \ldots) : t \geq 0\} \) is a countable system of independent CSBP’s with \( (1 + \beta) \)-stable branching law. We assume the two systems are defined on a complete probability space and are independent of each other. In addition, we assume \( \{\phi_1(0), \phi_2(0), \ldots\} \cap [-l, l] \) is a finite set for every finite \( l \geq 1 \). Then we define the \( \mathcal{M}_a \)-valued process
\[ X_t = \sum_{i=1}^\infty \xi_i(t) \delta_{\phi_i(t)}, \quad t \geq 0. \]
(2.4)

For any \( t \geq 0 \) let \( \mathcal{G}_t = \sigma(\mathcal{F}_t^\phi \cup \mathcal{F}_t^\xi) \), where
\[ \mathcal{F}_t^\phi := \sigma(\{\phi_i(s) : 0 \leq s \leq t; i = 1, 2, \ldots\}) \]
and
\[ \mathcal{F}_t^\xi := \sigma(\{\xi_i(s) : 0 \leq s \leq t; i = 1, 2, \ldots\}). \]

**Theorem 2.1** The process \( \{X_t : t \geq 0\} \) defined by (2.4) is a right continuous \((\mathcal{G}_t)\)-Markov process with transition semigroup \((P_t)_{t \geq 0}\) given by
\[ \int_{\mathcal{M}_a} e^{-\langle \mu, f \rangle} P_t(\mu, \text{d}\nu) = \mathbb{E} \left[ \exp \left\{ - \int_{\mathbb{R}} \psi_t(f(\phi(a, t))) \mu(\text{d}a) \right\} \right] \]
(2.5)
for \( \mu \in \mathcal{M}_a \) and \( f \in \mathcal{B}_0 \), where \( \phi(a, t) \) is a coalescing Brownian flow.

**Proof.** By the additivity of the CSBP’s it is easy to see \( \{X_t : t \geq 0\} \) a right continuous \((\mathcal{G}_t)\)-Markov process. By the independence of the two systems \( \{\phi_1(t), \phi_2(t), \ldots\} \) and \( \{\xi_1(t), \xi_2(t), \ldots\} \) we have
\[ \mathbb{E}[\exp\{-\langle X_t, f \rangle\}] = \mathbb{E} \left[ \exp \left\{ - \sum_{i=1}^\infty \xi_i(t) f(\phi_i(t)) \right\} \right] \]
(2.6)
Let \( W \) and let \( \{ M_t \}_{t \geq 0} \) for \( 0 < t \). The restriction of \( Q \) to \( W \) follows from the general theory of Markov processes; see, e.g., Getoor and Glover [7].

By a super-coalescing Brownian motion (SCBM) we mean a Markov process with state space \( \mathcal{M}_a \) and transition semigroup \( (P_t)_{t \geq 0} \) given by (2.5). Then the process constructed by (2.4) is a special case. We can also give a formulation of the SCBM with an arbitrary initial state \( \mu \in \mathcal{M} \). To this end, let us review some basic facts on CSBP’s. Let \( Q_t^0(x, \cdot) \) denote the restriction of the measure \( Q_t(x, \cdot) \) to \( (0, \infty) \). Since the origin 0 is a trap for the CSBP, the family of kernels \( (Q_t^0)_{t \geq 0} \) also constitutes a semigroup. Based on (2.4) and (2.3) one can check that

\[
\int_0^\infty (1 - e^{-zy}) \kappa_{s+t}(dy) = \int_0^\infty \kappa_s(dx) \int_0^\infty (1 - e^{-zy}) Q_t^0(x, dy), \quad s, t > 0, \quad z \geq 0.
\]

Then \( \kappa_s Q_t^0 = \kappa_{s+t} \). Therefore, \( (\kappa_t)_{t \geq 0} \) is an entrance law for \( (Q_t^0)_{t \geq 0} \).

Let

\[
W := \{ w : w \text{ is a right continuous positive function on } [0, \infty) \}
\]

and let

\[
\tau_0(w) := \inf\{ s > 0 : w(s) = 0 \} \quad \text{for } w \in W.
\]

Let \( W_0 \) be the set of paths \( w \in W \) such that \( w(0) = w(t) = 0 \) for \( t \geq \tau_0(w) \). Let \( \mathcal{B}(W_0) \) and \( \mathcal{B}_t = \mathcal{B}_t(W_0) \) denote the natural \( \sigma \)-algebra on \( W_0 \) generated by \( \{ w(s) : s \geq 0 \} \) and \( \{ w(s) : 0 \leq s \leq t \} \), respectively. Then there exits a unique \( \sigma \)-finite measure \( Q_\kappa \) on \( (W_0, \mathcal{B}(W_0)) \) such that

\[
Q_\kappa(w(t_1) \in dy_1, \ldots, w(t_n) \in dy_n)
= \kappa_{t_1}(dy_1) Q_{t_2 - t_1}^0 (y_1, dy_2) \cdots Q_{t_n - t_{n-1}}^0 (y_{n-1}, dy_n)
\]

for \( 0 < t_1 < t_2 < \ldots < t_n \) and \( y_1, y_2, \ldots, y_n \in (0, \infty) \). The existence of the measure \( Q_\kappa \) follows from the general theory of Markov processes; see, e.g., Getoor and Glover [2]. This measure is known as the excursion law of the CSBP. For \( r > 0 \), let \( Q_{\kappa,r} \) denote the restriction of \( Q_\kappa \) to \( W_r := \{ w \in W_0 : \tau_0(w) > r \} \). Note that

\[
Q_\kappa(W_r) = Q_{\kappa,r}(W_r) = \kappa_r(0, \infty) = \psi_r(\infty) < \infty.
\]

**Lemma 2.2** The coordinate process \( \{ w(t+r) : t \geq 0 \} \) under \( Q_{\kappa,r} \{ \cdot | \mathcal{B}_r \} \) is a CSBP with transition semigroup \( (Q_t)_{t \geq 0} \).
Proof. This follows from (2.6) by standard arguments. I here give a detailed proof for the convenience of the reader. For any $A \in \mathcal{B}((0, \infty))$ we can use (2.6) to see

$$Q_{\kappa,r}(w_r \in A, w_{r+t} > 0) = \kappa_r(A) - \int_A \kappa_r(dx)Q^0_t(x, (0, \infty))$$

for $t > 0$. Since $0$ is a trap, we have

$$Q_{\kappa,r}(w_r \in dy, w(t_1) \in dy_1, \ldots, w(t_n) \in dy_n)$$

for $0 < r < t_1 < t_2 < \ldots < t_n$ and $y > 0, y_1, y_2, \ldots, y_n \in [0, \infty)$. Thus for $A \in \mathcal{B}((0, \infty))$ and $A_1, \ldots, A_n \in \mathcal{B}([0, \infty))$,

$$\int_{\{w_r \in A\}} Q_{\kappa,r}\{w(t_1) \in A_1, \ldots, w(t_n) \in A_n\} Q_{\kappa,r}(dw)$$

Then an application of monotone class theorem yields the desired result. □

We now consider an arbitrary initial measure $\mu \in \mathcal{M}$. Suppose we have on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we have a coalescing Brownian flow \{\phi(a, t) : a \in \mathbb{R}, t \geq 0\} and a Poisson random measure $N(da, dw)$ on $\mathbb{R} \times \mathcal{W}_0$ with intensity measure $\mu(da)Q_{\kappa}(dw)$. We assume that $\{\phi(a, t)\}$ and $\{N(da, dw)\}$ are independent of each other. Denote the support of $N$ by $\text{supp}(N) = \{(a_i, w_i) : i \geq 1\}$. For $t \geq 0$ let $\mathcal{G}_t = \sigma(\mathcal{F}^N_t \cup \mathcal{F}^\phi_t)$, where

$$\mathcal{F}^N_t := \sigma(\{w_i(s) : 0 \leq s \leq t; i \geq 1\})$$

and

$$\mathcal{F}^\phi_t := \sigma(\{\phi(a, s) : 0 \leq s \leq t; a \in \mathbb{R}\}).$$

Then we define the $\mathcal{M}_a$-valued process

$$X_t = \int_\mathbb{R} \int_{\mathcal{W}_0} w(t)\delta_{\phi(a,t)}N(da, dw), \quad t > 0.$$  \hline (2.7)

**Theorem 2.3** The process \{\textit{X}_t : t > 0\} defined by (2.7) is a right continuous $(\mathcal{G}_t)$-Markov process with transition semigroup $(P_t)_{t \geq 0}$ and

$$\mathbb{E}[\exp\{-\langle X_t, f \rangle\}] = \mathbb{E} \left[ \exp \left\{ - \int_\mathbb{R} \psi_t(f(\phi(a,t)))\mu(da) \right\} \right]$$  \hline (2.8)

for $t > 0$ and $f \in \mathcal{B}_0$. 


Proof. Note that for any \( l \geq 1 \) and \( r > 0 \) we have a.s. \( m(l, r) := N([-l, l] \times W_r) < \infty \). In fact, we have

\[
\mathbb{E}[m(l, r)] = \mu([-l, l])\mathcal{Q}_r(W_r) = \mu([-l, l])\kappa_r(0, \infty) < \infty.
\]

Then, since \( \kappa_r(dx) \) is a diffuse measure, given \( G_r \) we can re-enumerate the set supp\((N)\) into \( \{(a_k, w_k) : i \geq 1\} \) so that: (i) \( |a_k| \leq |a_{k+1}| \leq \ldots \); and (ii) \( |a_k| = |a_{k+1}| \implies w_k(r) < w_{k+1}(r) \). Note that this enumeration only uses information from \( G_r \). As in the proof of Lemma 3.4 of Dawson and Li [3] one can see that \( \{w_k(r+t) : t \geq 0; i \geq 1\} \) under \( \mathbb{P}\{\cdot|G_r\} \) are independent CSBP’s which are independent of \( \{\phi(a, r+t) : t \geq 0; a \in \mathbb{R}\} \). Observe that

\[
X_{r+t} = \sum_{i=1}^{\infty} w_k(r+t)\delta_{\phi(a_k, r+t)}, \quad t \geq 0.
\]

Then Theorem 2.1 implies that \( \{X_{r+t} : t \geq 0\} \) under \( \mathbb{P}\{\cdot|G_r\} \) is a right continuous \((G_{r+t})\)-Markov process with transition semigroup \((P_t)_{t \geq 0}\). Thus \( \{X_t : t > 0\} \) under the non-conditioned probability \( \mathbb{P} \) is a right continuous \((G_t)\)-Markov process with transition semigroup \((P_t)_{t \geq 0}\). On the other hand, we have

\[
\mathbb{E}[\exp\{-\langle X_t, f \rangle\}] = \mathbb{E}\left[ \exp\left\{ -\int_{\mathbb{R}} \int_{\mathbb{W}_0} w(t)f(\phi(a, t))N(da, dw) \right\} \right] \\
= \mathbb{E}\left[ \exp\left\{ -\int_{\mathbb{R}} \mu(da) \int_{0}^{\infty} \left( 1 - e^{-uf(\phi(a, t))} \right) \kappa_t(da) \right\} \right] \\
= \mathbb{E}\left[ \exp\left\{ -\int_{\mathbb{R}} \psi_t(f(\phi(a, t)))\mu(da) \right\} \right].
\]

That proves (2.8). \( \square \)

Remark 2.4 Based on (2.8) one can show \( X_t \to \mu \) in probability (in fact almost surely with a little more work) as \( t \to 0 \). Then we may think the process as an SCBM starting from \( \mu \in \mathcal{M} \). From the above construction we see that starting from an arbitrary initial state in \( \mathcal{M} \), the SCBM collapses immediately into a purely atomic random measure with a countable support. Then the masses located at different points evolve according to independent CSBP’s with the supporting points evolving according to coalescing Brownian motions. The construction (2.7) of the SCBM generalizes that of Dawson et al. [4].

We can also give a useful alternate characterization of the SCBM following Zhou [10]. For \( y_1 \leq \ldots \leq y_{2n} \in \mathbb{R}^{2n} \), we write \( (Y_1(t), \ldots, Y_{2n}(t)) \) for a system of coalescing Brownian motion starting at \((y_1, \ldots, y_{2n})\). Given \( \{a_1, \ldots, a_n\} \subset \mathbb{R}^n \), throughout this paper we always put

\[
h_t(x) := \sum_{j=1}^{n} a_j 1_{[V_{2j-1}(t), V_{2j}(t)]}(x), \quad t \geq 0, x \in \mathbb{R}.
\]

By applying Theorem 2.1 to \( f = h_0 \) we obtain

Theorem 2.5 Let \( \{X_t : t > 0\} \) be the SCBM defined by (2.7). Then for any \( t > 0 \) we have

\[
\mathbb{E}[\exp\{-\langle X_t, h_0 \rangle\}] = \mathbb{E}\left[ \exp\left\{ -\int_{\mathbb{R}} \psi_t(h_t(a))\mu(da) \right\} \right].
\]
The above theorem shows that, the SCBM constructed in Zhou [10] using approximation is actually the special case with \( \beta = 1 \) of the SCBM defined by (2.7).

3 A duality on coalescing Brownian motions with either absorbing or reflecting boundary conditions

To obtain a useful estimate on occupation time for SCBM we proceed to recover a duality between coalescing Brownian motions with absorbing barriers and coalescing Brownian motions with reflecting barriers. A similar duality between coalescing Brownian motions with no boundary conditions had been found in Zhou [10]. In this section we want to derive the desired duality by modifying the previous arguments in Zhou [10].

Put \( Z := \{\ldots, -1, 0, 1, \ldots \} \) and \( Z' := Z + 1/2 = \{i + 1/2 : i \in Z\} \). An \( m \)-dimensional (symmetric) simple coalescing random walk on \( Z \) (respectively, \( Z' \)) describes the evolution of a collection of coalescing particles, in which each particle follows a continuous time, \( Z \) (respectively, \( Z' \))-valued, symmetric simple random walk jumping at rate one. Throughout this paper we always index the \( m \) “particles” in such a coalescing random walk from 1 to \( m \) according to the order of their initial values from the smallest to the largest. We can assume that after each coalescence with two particles involved the particle with higher index follows the movement of the one with lower index. To keep track of those indices involving in coalescence we introduce the following notion on the set of indices.

Let \( \mathcal{P}_m \) denote the set of interval partitions of the totality of indices \( [m] := \{1, \ldots, m\} \). That is, an element \( \pi \) of \( \mathcal{P}_m \) is a collection \( \pi = \{A_1(\pi), \ldots, A_h(\pi)\} \) of disjoint subsets of \([m]\) such that \( \bigcup_k A_k(\pi) = [m] \) and \( a < b \) for all \( a \in A_i \) and \( b \in A_j \), \( i < j \). The sets \( A_1(\pi), \ldots, A_h(\pi) \) consisting of consecutive indices are the intervals of the partition \( \pi \). The integer \( h \) is the length of \( \pi \) and is denoted by \( l(\pi) \). We write \( i \sim_\pi j \) if and only if \( i \) and \( j \) belong to the same interval of \( \pi \in \mathcal{P}_m \). Given \( \pi \in \mathcal{P}_m \), define

\[
\hat{Z}_m^\pi := \{(x_1, \ldots, x_m) \in Z^m : x_1 \leq \ldots \leq x_m \text{ and } x_i = x_j \text{ for } i \sim_\pi j\}.
\]

Write \( X = (X_1, \ldots, X_m) \) for a \( Z^m \)-valued simple coalescing random walk. If \( X(t) \in \hat{Z}_m^\pi \), then at time \( t \) those particles with indices \( \alpha_1(\pi) := \min A_1(\pi), \ldots, \alpha_l(\pi) := \min A_l(\pi) \) are still “free” and for \( i \notin \{\alpha_1(\pi), \ldots, \alpha_l(\pi)\} \) the \( i \)th particle at time \( t \) is “attached” to the free particle with index \( \min\{j : 1 \leq j \leq m, j \sim_\pi i\} = \max\{\alpha_k(\pi) : \alpha_k(\pi) \leq i\} \).

For \( \pi \in \mathcal{P}_m \), define a map \( K_\pi : \hat{Z}_m^\pi \to Z^{l(\pi)} \) by

\[
K_\pi(x) = K_\pi(x_1, \ldots, x_m) := (x_{\alpha_1(\pi)}, \ldots, x_{\alpha_l(\pi)})
\]

Notice that \( K_\pi \) is a bijection between \( \hat{Z}_m^\pi \) and the subspace \( \{x \in Z^{l(\pi)} : x_1 \leq x_2 \leq \ldots \leq x_{l(\pi)}\} \). We write \( K_\pi^{-1} \) for the inverse of \( K_\pi \). For brevity we also write \( x_\pi \) for \( K_\pi(x) \).

Write \( B(Z^m) \) for the collection of all bounded functions on \( Z^m \). The infinitesimal generator \( G \) for simple coalescing random walk on \( Z \) is the operator \( G : B(Z^m) \to B(Z^m) \) defined as follows. For any \( f \in B(Z^m), \pi \in \mathcal{P}_m \) and \( x \in \hat{Z}_m^\pi \), let \( \{e_i : 1 \leq i \leq l(\pi)\} \) be
the collection of coordinate vectors in \( \mathbb{Z}^{l(\pi)} \); that is, \( \mathbf{e}_i \) is the vector whose \( i^{th} \) coordinate is 1 and all the other coordinates are 0. Then

\[
G_f(x) := \frac{1}{2} \sum_{i=1}^{l(\pi)} f \circ K^{-1}_\pi(x + \mathbf{e}_i) + \frac{1}{2} \sum_{i=1}^{l(\pi)} f \circ K^{-1}_\pi(x - \mathbf{e}_i)
- l(\pi)f \circ K^{-1}_\pi(x).
\] (3.1)

We can define the generator for simple coalescing random walk on \( \mathbb{Z}' \) in the obvious way.

Intuitively, for \( a \in \mathbb{Z} \) a \textit{simple coalescing random walk absorbed at} \( a \) evolves just like a simple coalescing random walk except that any particle in the coalescing random walk stays at \( a \) after it first hits \( a \). The generator \( G_a \) for simple coalescing random walk absorbed at \( a \) is defined in the same way as (3.1) for \( x \in \mathbb{Z}_\pi^m \) with \( \{x_1, \ldots, x_m\} \cap \{a\} = \emptyset \).

For \( x \in \mathbb{Z}_\pi^m \) with \( x_{\alpha_j(\pi)} = a \) for some \( 1 \leq j \leq l(\pi) \), we define

\[
G_a f(x) := \frac{1}{2} \sum_{i \neq j} f \circ K^{-1}_\pi(x + \mathbf{e}_i) + \frac{1}{2} \sum_{i \neq j} f \circ K^{-1}_\pi(x - \mathbf{e}_i)
- (l(\pi) - 1)f \circ K^{-1}_\pi(x).
\]

A simple coalescing random walk absorbed at more than one points can be defined similarly.

Intuitively, for \( a \in \mathbb{Z} \) an \( m \)-dimensional \textit{simple coalescing random walk on} \( \mathbb{Z}' \) \textit{reflected at} \( a \) evolves like a simple coalescing random walk with each particle always staying on the same side of \( a \); whenever a particle reaches \( a - 1/2 \) (respectively, \( a + 1/2 \)) it stays at \( a - 1/2 \) (respectively, \( a + 1/2 \)) with rate 1/2 and gets reflected to \( a - 1/2 - 1 \) (respectively, \( a + 1/2 + 1 \)) with rate 1/2.

Write \( G_r \) for the generator of the above simple coalescing random walk with reflection. Then for \( f \in B(\mathbb{Z}'^m) \), \( G_r f(x) \) is defined in the same way as (3.1) for \( x \in \mathbb{Z}_\pi^m \) with \( \{x_1, \ldots, x_m\} \cap \{a - 1/2, a + 1/2\} = \emptyset \). For any \( x \in \mathbb{Z}_\pi^m \) such that \( x_{\alpha_j(\pi)} = a - 1/2 \) and either \( x_{\alpha_j+1(\pi)} > a + 1/2 \) or \( \alpha_j(\pi) = l(\pi) \), we define

\[
G_r f(x) := \frac{1}{2} \sum_{i} f \circ K^{-1}_\pi(x_i - \mathbf{e}_i) + \frac{1}{2} \sum_{i \neq j} f \circ K^{-1}_\pi(x_i + \mathbf{e}_i)
- \left(l(\pi) - \frac{1}{2}\right)f \circ K^{-1}_\pi(x).
\]

for any \( x \in \mathbb{Z}_\pi^m \) such that \( x_{\alpha_j(\pi)} = a + 1/2 \) and either \( x_{\alpha_j-1(\pi)} < a - 1/2 \) or \( \alpha_j(\pi) = 1 \), we define

\[
G_r f(x) := \frac{1}{2} \sum_{i} f \circ K^{-1}_\pi(x_i + \mathbf{e}_i) + \frac{1}{2} \sum_{i \neq j} f \circ K^{-1}_\pi(x_i - \mathbf{e}_i)
- \left(l(\pi) - \frac{1}{2}\right)f \circ K^{-1}_\pi(x).
\]
for any \( x \in \mathbb{Z}_\pi^m \) with \( x_{\alpha_j(\pi)} = a - 1/2 \) and \( x_{\alpha_j+1(\pi)} = a + 1/2 \),

\[
G_r f(x) := \frac{1}{2} \sum_{i \neq j} f \circ K_{\pi}^{-1}(x_i + e_i) + \frac{1}{2} \sum_{i \neq j+1} f \circ K_{\pi}^{-1}(x_i - e_i) - (l(\pi) - 1) f \circ K_{\pi}^{-1}(x_\pi);
\]

The previous definitions can be generalized to a simple coalescing random walk reflected at two different points \( a, b \) for any \( a < b \). Let \( a < b \) and \( b \) random array \( (\leq) \) and array \( (\geq) \).

Proposition 3.1 For each \( t \geq 0 \) the joint distribution of the \( m \times (n - 1) \)-dimensional random array \( (I_{ij}^{-}(t, Y(0))) \) coincides with that of the \( m \times (n - 1) \)-dimensional random array \( (I_{ij}^{-}(t, X(0))) \).

Proof. In this proof we adopt a standard martingale duality argument for which we refer to Section 4.4 of Ethier and Kurtz [5].

Write \( G_a \) and \( G_r \) for the generators for \( X \) and \( Y \), respectively. Given a function \( g : \{0, 1\}^{m(n-1)} \to \mathbb{R} \), for any vector \( x \in \mathbb{Z}^m \) with \( x_1 \leq \ldots \leq x_m \), and any vector \( y \in \mathbb{Z}^m \) with \( y_1 \leq \ldots \leq y_n \), set

\[
\hat{g}(x; y) := g(1_{[y_1, y_2]}(x_1), \ldots, 1_{[y_{n-1}, y_n]}(x_1), \ldots, 1_{[y_1, y_2]}(x_m), \ldots, 1_{[y_{n-1}, y_n]}(x_m)).
\]

We further put \( \hat{g}_x(\cdot) := \hat{g}(x; \cdot) \) and \( \hat{g}_y(\cdot) := \hat{g}(\cdot; y) \). Then in order to show that

\[
\mathbb{P}[\hat{g}(X(t); Y(0))] = \mathbb{P}[\hat{g}(X(0); Y(t))],
\]

we only have to show that

\[
G_a(\hat{g}_y)(x) = G_r(\hat{g}_x)(y)
\]

by considering all the possible values of \( x \) and \( y \) in relation to \( a \) and \( b \) and those plain facts such as for \( x_i \in [y_j, y_{j+1}] \), \( x_i + 1 \notin [y_j, y_{j+1}] \) if and only if \( x_i \notin [y_j, y_{j+1} - 1] \); see similar arguments in Zhou [10] for duality on coalescing simple random walk.

Taking a time-space scaling, the duality in Proposition 3.1 also holds for coalescing Brownian motions with absorbing or reflecting barriers.

Proposition 3.2 Given \( a, b \in \mathbb{R} \) with \( a < b \), let \( X \) be an \( m \)-dimensional coalescing Brownian motion absorbing at either \( a \) or \( b \); let \( Y \) be an \( n \)-dimensional coalescing Brownian motion starting at \( Y(0) := (y_1, \ldots, y_n) \) such that \( \{y_1, \ldots, y_n\} \cap \{a, b\} = \emptyset \) and \( Y \) is reflected at both \( a \) and \( b \). Then for each \( t \geq 0 \) the joint distribution of the \( m \times (n - 1) \)-dimensional random array \( (I_{ij}^{-}(t, Y(0))) \) coincides with that of the \( m \times (n - 1) \)-dimensional random array \( (I_{ij}^{-}(t, X(0))) \).
4 An integral text on almost sure local extinction for SCBM

We begin with a dual relationship between the SCBM with absorbing barriers and the coalescing Brownian motion with reflecting barriers. Intuitively, given a finite subset $A$ of $\mathbb{R}$ the SCBM with branching rate $\gamma \geq 0$ and absorption barriers at $A$ evolve just like a SCBM except that each particle involved is stopped as soon as it hits one of the points in $A$. But the branchings of those stopped particles are not affected. The SCBM with absorption at $A$ can be defined using integral representation (2.7) where we impose the absorbing condition at $A$ to the underlying Arratia flow $\phi$. Then the following duality follows from Proposition 3.2 and arguments similar to Theorem 2.3. Note that the random function $h_t$ is defined in (2.9).

**Theorem 4.1** Let $Z$ be an SCBM with absorption at $a$ and $b$ with $a < b$. For any $(a_i) \in \mathbb{R}^n$ and for any $y_1 < ... < y_{2n}$ with $\{y_1, \ldots, y_{2n}\} \cap \{a, b\} = \emptyset$, we have

$$\mathbb{P}[\exp\{-\langle Z_t, h_0 \rangle\}] = \mathbb{P}[\exp\{-\langle Z_0, \psi_t(h_t) \rangle\}],$$

(4.1)

where $\gamma \geq 0$ is the branching rate for $Z$ and $(Y_1, \ldots, Y_{2n})$ is a coalescing Brownian motion starting at $(y_1, \ldots, y_{2n})$ and reflected at both $a$ and $b$.

**Remark 4.1** For $\gamma = 0$, we put $h_0 := \lambda_{1, [a-\epsilon, b+\epsilon]}$ in Theorem 4.1. By first letting $\lambda \to \infty$ and then letting $\epsilon \to 0^+$ in (4.1) we can express the extinction probability of $Z$ at time $t$ in terms of reflecting Brownian motions and obtain

$$\mathbb{P}\{Z_t([a, b]) = 0\} = \mathbb{P}\{Z_0([Y_1(t), Y_2(t)]) = 0\},$$

(4.2)

where $Y_1$ is a Brownian motion starting and reflected at $a$ such that $Y_1(t) \leq a$ for all $t \geq 0$, and $Y_2$ is another independent Brownian motion starting and reflected at $b$ such that $Y_2(t) \geq b$ for all $t \geq 0$.

The duality in Theorem 4.1 leads immediately to the following estimate on the occupation time for SCBM.

**Proposition 4.2** Let $Z$ be an SCBM with $\sigma$-finite initial measure and branching rate $\gamma$. Then for $y_1 < y_2$ we have

$$\mathbb{P}\left\{\int_0^t Z_s([y_1, y_2])ds = 0\right\} \geq \mathbb{P}\{Z_0([Y_1(t), Y_2(t)]) = 0\},$$

(4.3)

where $Y_1$ and $Y_2$ are independent Brownian motions such that $(Y_1(0), Y_2(0)) = (y_1, y_2)$, and $Y_1$ and $Y_2$ are reflected at $y_1$ and $y_2$, respectively so that $Y_1(t) \leq y_1$ and $Y_2(t) \geq y_2$ for all $t \geq 0$.

The equality in (4.3) holds if $\gamma = 0$. 
Write $Z^\gamma$ for the SCBM with branching rate $\gamma \geq 0$. Let $Z^*$ be an SCBM with branching rate $\gamma = 0$ and with absorbing barriers at $y_1$ and $y_2$ such that $Z_0^* = Z_0^0$. A key observation is that without branching

$$\mathbb{P}\{Z_s^0([y_1, y_2]) = 0, \forall 0 < s \leq t\} = \mathbb{P}\{Z_s^0([y_1, y_2]) = 0, \forall 0 < s \leq t\} = \mathbb{P}\{Z_t^0([y_1, y_2]) = 0\}.$$ 

Then by (4.2)

$$\mathbb{P}\{Z_s^0([y_1, y_2]) = 0, \forall 0 < s \leq t\} = \mathbb{P}\{Z_0^0([Y_1(t), Y_2(t)]) = 0\}.$$ 

For $\gamma > 0$, using the construction of SCBM in Section 2 we can construct $Z^\gamma$ and $Z^0$ on the same probability space by letting

$$Z_t^0 := \sum_{x \in \{\phi(a,t): a \in \mathbb{R}\}} L(\{a : \phi(a,t) = x\}) \delta_x,$$

where $L$ denotes the Lebesgue measure on $\mathbb{R}$. Then $\mathbb{P}$ a.s. the support of $Z_t^\gamma$ is smaller or equal to the support of $Z_t^0$ for any $t \geq 0$. Therefore, inequality (4.3) follows readily. □

**Corollary 4.3** Given an SCBM $Z$ with branching rate $\gamma > 0$, for any $a > 0$ and $0 < s_1 < s_2$, we have

$$\mathbb{P}\{Z_t([-a,a]) = 0, \forall s_1 < t \leq s_2\} \geq \frac{2}{\pi(s_2 - s_1)} \int_0^\infty dx \int_0^\infty dy \exp \left\{ -\frac{x^2 + y^2}{2(s_2 - s_1)} - \psi_{s_1}(\infty) Z_0([Y_1(s_1), Y_2(s_1)]) \right\}, \quad (4.4)$$

where $(Y_1, Y_2)$ is a coalescing Brownian motion starting at $(-x - a, a + y)$.

**Proof.** By Proposition 4.2, Markov property at time $s_1$ and Theorem 2.3, we have

$$\mathbb{P}\{Z_t([-a,a]) = 0, \forall s_1 < t \leq s_2\} \geq \mathbb{P}\{Z_{s_1}([Y_1^*(s_2 - s_1), Y_2^*(s_2 - s_1)]) = 0\} = \mathbb{P}\{\exp \{-Z_0([Y_1(s_1), Y_2(s_1)])\psi_{s_1}(\infty)\}\}, \quad (4.5)$$

where $Y_1^*$ with $Y_1^*(t) \leq -a$ for all $t$ and $Y_2^*$ with $Y_2^*(t) \geq a$ for all $t$ are independent Brownian motions starting at $-a$ and $a$, and reflected at $-a$ and $a$, respectively; $(Y_1, Y_2)$ is a coalescing Brownian motion starting at $(Y_1^*(s_2 - s_1), Y_2^*(s_2 - s_1))$. Since for $y > 0$, by reflection principle

$$\mathbb{P}\{Y_2(0) - a \in dy\} = \mathbb{P}\{Y_2^*(s_2 - s_1) - a \in dy\} = \sqrt{2/\pi(s_2 - s_1)} e^{-y^2/(s_2 - s_1)} dy,$$
estimate (4.4) then follows.

Throughout this paper let $g(t), t > 0$, be any nonnegative, nondecreasing and right continuous function on $[0, \infty)$. For such a function $g$ we define the extinction time as

$$
\tau := \sup \{ t \geq 0 : X_t([-g(t), g(t)]) \neq 0 \}
$$

with the convention $\sup \emptyset = 0$.

**Theorem 4.4** Let $Z$ be an SCBM such that $Z_0(dx) = dx$ and $\gamma > 0$. If

$$
\int_1^\infty g(y)^{-1-1/\beta} dy < \infty, \quad (4.6)
$$

then

$$
P\{\tau < \infty\} = 1. \quad (4.7)
$$

**Proof.** We first want to show that (4.7) holds given $g(t) \geq t^\delta$ for some constant $1/2 < \delta < 1$ and for $t$ large enough. Put $t_n := e^n$. For $m$ large enough and $n \geq m$, by (4.5) we have

$$
P\{Z_t([-g(t_{n+1}), g(t_{n+1})]) = 0, \forall t_n < t \leq t_{n+1}\}
$$

$$
\geq P\{\exp \{ -Z_0([Y_1(t_n), Y_2(t_n)])\psi_{t_n}(\infty)\}\}
$$

$$
\geq P\{\exp \{ -Z_0([Y_1(t_n), Y_2(t_n)])\psi_{t_n}(\infty)\}; -2g(t_{n+1}) \leq Y_1(t_n) \leq Y_2(t_n) \leq 2g(t_{n+1})\}
$$

$$
\geq \exp \{ -4g(t_{n+1})\psi_{t_n}(\infty)\} - P\{\{Y_1(t_n) < -2g(t_{n+1})\} \cup \{Y_2(t_n) > 2g(t_{n+1})\}\}
$$

$$
\geq \exp \{ -4g(t_{n+1})\psi_{t_n}(\infty)\} - P\{Y_1(t_n) < -2g(t_{n+1})\} - P\{Y_2(t_n) > 2g(t_{n+1})\}
$$

$$
= \exp \{ -4g(t_{n+1})\psi_{t_n}(\infty)\} - 2P\{Y(t_{n+1}) > g(t_{n+1})\}
$$

$$
\geq \exp \{ -4g(t_{n+1})\psi_{t_n}(\infty)\} - 2P\{Y(t_{n+1}) > t_{n+1}^\delta\},
$$

where $(Y_1, Y_2)$ is defined as in (4.5) with $a = g(t_{n+1})$ and the process $Y$ is defined such that between times $0$ and $t_{n+1} - t_n$, $Y$ taking non-negative values is a Brownian motion starting at $0$ and reflected at $0$, and after time $t_{n+1} - t_n$, $Y$ evolves according to a Brownian motion with initial value $Y(t_{n+1} - t_n)$.

Then

$$
P\{\exists t_n < t \leq t_{n+1}, Z_t([-g(t_{n+1}), g(t_{n+1})]) \neq 0\}
$$

$$
\leq 1 - \exp \{ -4g(t_{n+1})\psi_{t_n}(\infty)\} + 2P\{Y(t_{n+1}) > t_{n+1}^\delta\}
$$

$$
\leq 4g(t_{n+1})\psi_{t_n}(\infty) + 2P\{Y(t_{n+1}) > t_{n+1}^\delta\}.
$$

Clearly, there exists a constant $c > 0$ such that for any large $t$

$$
P\{Y(t) > t^\delta\} \leq ce^{-t^{2\delta-1}/2},
$$

and then

$$
\sum_{n=1}^\infty P\{Y(t_n) > t_n^\delta\} < \infty.
$$
Moreover,

\[
\sum_{n=m}^{\infty} g(t_{n+1}) \psi_{t_n}(\infty) = \sum_{n=m}^{\infty} g(t_{n+1}) \left( \frac{1 + \beta}{\gamma \beta t_n} \right)^{1/\beta} \\
\leq c(\gamma, \beta) \int_{m+1}^{\infty} \frac{g(e^x)}{e^{(x-2)/\beta}} dx \\
\leq e^{2/3} c(\gamma, \beta) \int_{t_{m+1}}^{\infty} \frac{g(y)}{y^{1+1/\beta}} dy \\
< \infty,
\]

(4.8)

where \( c(\gamma, \beta) = \frac{1}{\beta} \left( \frac{1 + \beta}{\gamma \beta} \right)^{1/\beta} \). Therefore, the desired result follows from Borel-Cantelli lemma.

To show the desired result for any \( g \) satisfying (4.6), we can consider function \( g(t) + t^\delta \) instead. It follows from the previous result that (4.7) holds for function \( g(t) + t^\delta \). Then plainly, it also holds for \( g(t) \).

\[\square\]

**Theorem 4.5** Let \( Z \) be an SCBM such that \( Z_0(dx) = dx \) and \( \gamma > 0 \). If

\[
\int_1^{\infty} g(y)y^{-1-1/\beta} dy = \infty,
\]

then

\[\mathbb{P}\{\tau = \infty\} = 1.\]

**Proof.** Delayed.

Before proceeding with the proof for Theorem 4.5, we first define the positive and strictly increasing sequences \( (t_n), (l_n) \) and \( (r_n) \) as follows. Set \( t_0 = 1 \) and

\[t_{n+1} := \inf\{t \geq t_n : g(t) \geq 3g(t_n)\}, \quad n \geq 1.\]

Then

\[g(t_n) \leq g(t_{n+1}^-) \leq 3g(t_n) \leq g(t_{n+1}). \quad \text{(4.9)}\]

Define

\[r_n := \frac{9}{10} g(t_n) \quad \text{and} \quad l_n := \frac{31}{30} g(t_{n-1}).\]

By (4.9),

\[r_n - l_n \geq \frac{1}{10} (g(t_{n+1}^-) - g(t_n^-)). \quad \text{(4.10)}\]

Let \( I_n := [-r_n, -l_n] \cup [l_n, r_n] \) and \( I := \bigcup_{n=0}^{\infty} I_n \). We need two coalescing Brownian systems:

\[C_n = \{C_n(x, t) ; x \in I_n, t \geq 0\}\]
and
\[ C = \{ C(x, t); x \in I, t \geq 0 \} \]
such that, for any finite set \( A \subset \mathbb{R} \), both \( \{ C_n(x, \cdot); x \in A \cap I_n \} \) and \( \{ C(x, \cdot); x \in A \cap I \} \) are coalescing Brownian motions starting from \( A \cap I_n \) and \( A \cap I \), respectively.

Define
\[ \tau(x, y) := \inf\{ t \geq 0 : C(x, t) = C(y, t) \}. \]

According to the construction of coalescing Brownian motions on \( \mathbb{R} \) in [1], we may construct \( \{ C_n; n \geq 1 \} \) and \( C \) from a countable family of independent Brownian motions starting from \( Q := \{ i/2^n; i, n \in \mathbb{Z} \} \) such that \( C_n \) and \( C_m \) are independent for \( n \neq m \) and
\[ C_n(x, t) = C(x, t) \quad \text{for} \ x \in I_n \text{ and } t \leq \tau_n, \]
where \( \tau_n := \tau(0, -l_{n+1}) \land \tau(-l_n, 0) \land \tau(r_{n-1}, l_n) \land \tau(r_n, l_{n+1}) \).

In the following of this paper, for \( N(da, dw) \) we always mean a Poisson random measure on \( \mathbb{R} \times W_0 \) with intensity measure \( da_{Q_n}(dw) \). For Lebesgue measure \( L \) let \( \{ X^n_t : t \geq 0 \} \) and \( \{ \bar{X}_t : t \geq 0 \} \) be defined by \( X^n_0 = L|_{I_n}, \bar{X}_0 = L|_{I} \) and for \( t \geq 0 \),
\[ X^n_t = \int_{I_n} \int_{W_0} w(t) \delta_{C_n(x,t)} N(da, dw), \quad \bar{X}_t = \int_I \int_{W_0} w(t) \delta_{C(x,t)} N(da, dw). \]

Then by Theorem 4.3, \( \{ X^n_t : t \geq 0 \} \) and \( \{ \bar{X}_t : t \geq 0 \} \) are SCBM’s starting from \( L|_{I_n} \) and \( L|_{I} \), respectively. We first prove the following lemma of key estimates. Theorem 4.5 will be easily deduced from the lemma.

**Lemma 4.6** Set \( B_n := [-g(t_n), g(t_n)] \). Assume that \( t^{1/2+\varepsilon} \leq g(t) \leq 3t \) for \( t \geq 1 \) and some \( 0 < \varepsilon < \frac{1}{2} \). Then we have
\[
\sum_{n=0}^{\infty} \mathbb{P} \left\{ X^n_{t_n}(B_n^c) > 0 \right\} < \infty; \tag{4.11}
\]
\[
\sum_{n=0}^{\infty} \mathbb{P} \left\{ X^n_{t_n}(1) > 0 \right\} = \infty; \tag{4.12}
\]
\[
\sum_{n=0}^{\infty} \mathbb{P} \left\{ X^n_{t_n}(B_n^c) > \bar{X}_{t_n}(B_n) \right\} < \infty. \tag{4.13}
\]

**Proof.** Proof for (4.11). Set
\[ \Gamma_n := \left\{ \inf_{0 \leq s \leq t_n} C_n(-r_n, s) \geq -g(t_n) \right\} \cap \left\{ \sup_{0 \leq s \leq t_n} C_n(r_n, s) \leq g(t_n) \right\}. \]

Then
\[ \mathbb{P}\{\Gamma_n^c\} \leq \mathbb{P}\left\{ \inf_{0 \leq s \leq t_n} C_n(-r_n, s) \leq -g(t_n) \right\} + \mathbb{P}\left\{ \sup_{0 \leq s \leq t_n} C_n(r_n, s) \geq g(t_n) \right\} \leq 2 \exp \left\{ -\frac{g^2(t_n)}{100t_n} \right\} \leq 2 \exp \left\{ -\frac{t_n^{2\varepsilon}}{100} \right\}. \]
By (4.9) and the assumption that \( g(t) \leq 3^t \) for \( t > 1 \), we have

\[
\sum_{n=1}^{\infty} \mathbb{P}(\Gamma_n^c) < \infty.
\]

Then (4.11) follows from

\[
\mathbb{P}\{X_{\infty}^n(B_n^c) > 0\} = \mathbb{P}\{X_{\infty}^n(B_n^c) > 0; \Gamma_n \} + \mathbb{P}\{X_{\infty}^n(B_n^c) > 0; \Gamma_n^c\} \leq \mathbb{P}(\Gamma_n^c).
\]

**Proof for (4.12).** Since \( X^n(1) \) is a branching process starting from \( 2(r_n - l_n) \),

\[
\mathbb{P}\{X_{\infty}^n(1) > 0\} = 1 - \exp\{-2(r_n - l_n)\psi_{l_n}(\infty)\} \geq \min\left\{ \frac{\psi_{l_n}(\infty)}{15} (g(t_{n+1}) - g(t_n)) , 1 - e^{-1} \right\},
\]

where the inequality is deduced from (4.10) and the elementary inequality \( 1 - e^{-x} \geq x/3 \) for \( 0 \leq x \leq 1 \). Recall \( c(\gamma, \beta) \) in (4.8). Then (4.12) follows from

\[
\sum_{n=1}^{\infty} \psi_{l_n}(\infty)(g(t_{n+1}) - g(t_n)) = -\psi_{l_1}(\infty)g(t_1) - \sum_{n=2}^{\infty} g(t_{n+1})(\psi_{l_n}(\infty) - \psi_{l_{n+1}}(\infty)) \leq \frac{-1}{t_1} g(t_1) - \sum_{n=2}^{\infty} g(t_{n+1}) \int_{t_n}^{t_{n+1}} \frac{\partial \psi_{l_n}(\infty)}{\partial t} dt \geq \frac{-1}{t_1} g(t_1) + c(\gamma, \beta) \sum_{n=2}^{\infty} g(t) t^{-1-1/\beta} dt = \infty.
\]

**Proof for (4.13).** Set

\[
\Gamma_1(x, t, y) := \left\{ \inf_{0 \leq s \leq t} C(x, s) \geq y \right\} \quad \text{and} \quad \Gamma_2(x, t, y) := \left\{ \sup_{0 \leq s \leq t} C(x, s) \leq y \right\}.
\]

Define

\[
\Omega_1 := \Gamma_1(-r_n, t_n, -g(t_n)) \cap \Gamma_2(-l_n, t_n, -g(t_{n-1}));
\]
\[
\Omega_2 := \Gamma_1(-r_n-1, t_n, -g(t_{n-1})) \cap \Gamma_2(r_n, t_n, g(t_{n-1}));
\]
\[
\Omega_3 := \Gamma_1(l_n, t_n, g(t_{n-1})) \cap \Gamma_2(r_n, t_n, g(t_n));
\]
\[
\Omega_4 := \Gamma_1(l_{n+1}, t_n, g(t_{n})) \cap \Gamma_2(-l_{n+1}, t_n, -g(t_n)).
\]

Then

\[
\mathbb{P}\{\cup \Omega_i^c\} \leq 4 \exp\left\{ -\frac{g^2(t_n)}{100 l_n} \right\} + 4 \exp\left\{ -\frac{g^2(t_{n-1})}{100 l_n} \right\} \leq 4 \exp\left\{ -\frac{t_n^2}{100} \right\} + 4 \exp\left\{ -\frac{9t_n^2}{100} \right\},
\]
where the last inequality follows from (4.9). Now, we are ready to deduce (4.13). Note that on \( \cap \Omega_i \),
\[
C_n(x, t) = C(x, t) \quad \text{for} \quad t \leq t_n, \quad x \in I_n.
\]
It follows that
\[
P\{X_{t_n}(B_n) > \bar{X}_{t_n}(B_n)\}
\leq P\{X_{t_n}(B_n) > \bar{X}_{t_n}(B_n) ; \cap \Omega_i\} + P\{\cup \Omega_i\}
\leq 4 \exp \left\{ -\frac{t_n^2}{100} \right\} + 4 \exp \left\{ -\frac{9t_n^2}{100} \right\}.
\]
We thus obtain (4.13) by applying (4.9) again and by the assumption that \( g(t) \leq 3t \) for \( t > 1 \).

**Proof for Theorem 4.5.** First, suppose that \( t_{1/2+\epsilon} \leq g(t) \leq 3t \) for \( t \geq 1 \) and some \( 0 < \epsilon < \frac{1}{2} \). Define
\[
\tau_0 := \sup\{t \geq 0 : \bar{X}_t([g(t), g(t)]) > 0\}.
\]
Note that \( \{X^n_n; n = 1, 2, \ldots\} \) are independent SCBM’s. In addition, the sequence \( (t_n) \)
defined in Lemma 4.6 satisfies \( t_n \rightarrow \infty \) since \( g \) is increasing and \( g(t) \leq e^t \) for all \( t > 0 \).
By Lemma 4.6, three applications of Borel-Cantelli lemma yield
\[
P\{\tau_0 = \infty\} = 1.
\]
Let \( X_t \) be defined as (2.7) with \( X_0 = L \). Define \( \tilde{X}_t : t \geq 0 \) with \( \tilde{X}_0 = L|_I \) and
\[
\tilde{X}_t := \int_I \int_{\mathcal{W}_0} w(t)\delta_{\phi(a,t)}N(da,dw), \quad t > 0.
\]
Then \( \tilde{X} \) is an SCBM starting from \( L|_I \). Obviously,
\[
X_t([g(t), g(t)]) > \tilde{X}_t([g(t), g(t)]).
\]
Then the fact that \( \tilde{X} \) has the same distribution with \( \tilde{X} \) yields
\[
P\{\tau = \infty\} = 1.
\]
For more general \( g \) satisfying \( \int_1^\infty g(y)y^{-1-1/\beta}dy = \infty \), we can consider function
\[
g_0(y) := (g(y) \land 3^y) \lor y^{\frac{1}{\beta} + \epsilon}.
\]
First, one can check that
\[
\int_1^\infty (g(y) \land 3^y)y^{-1-1/\beta}dy = \infty.
\]
So,
\[
\sup\{t \geq 0 : X_t([g_0(t), g_0(t)]) > 0\} = \infty, \quad \text{a.s.}
\]
Meanwhile, according to Theorem 4.4
\[
\sup\{t \geq 0 : X_t([-t^{1/\beta + \epsilon}, t^{1/\beta + \epsilon}) > 0\} < \infty \quad \text{a.s.}
\]
Then
\[
\sup\{t \geq 0 : X_t([-g(t) \wedge 3', g(t) \wedge 3']) > 0\} = \infty \quad \text{a.s.}
\]
Then the desired result follows from \(g(t) \geq g(t) \wedge 3'\). We have thus finished the proof. \(\square\)

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