From Operator Algebras to Superconformal Field Theory

YASUYUKI KAWAHIGASHI
Department of Mathematical Sciences
University of Tokyo, Komaba, Tokyo, 153-8914, Japan
E-mail: yasuyuki@ms.u-tokyo.ac.jp

March 16, 2010

Abstract

We survey operator algebraic approach to (super)conformal field theory. We discuss representation theory, classification results, full and boundary conformal field theories, relations to supervertex operator algebras and Moonshine, connections to subfactor theory of Jones and certain aspects of noncommutative geometry of Connes.

1 Introduction

Quantum field theory is a physical theory and its mathematical aspects have been connected to many branches of contemporary mathematics. Particularly (super)conformal field theory has attracted much attention over the last 25 years since [4]. In this paper we make a review on the current status of the operator algebraic approach to (super)conformal field theory.

We start with general ideas of the operator algebraic approach to quantum field theory, which is called algebraic quantum field theory. We then specialize conformal field theories on the 2-dimensional Minkowski space, its light rays and their compactifications. We deal with mathematical axiomatization of physical ideas, and study the mathematical objects starting from the axioms. Our main tool for studying them is representation theory, and we present their basics, due to Doplicher-Haag-Roberts. We further introduce the induction machinery, called $\alpha$-induction, and present classification results.

We then present another mathematical axiomatization of the same physical ideas. It is a theory of (super)vertex operator algebras. After giving basics, we study the Moonshine phenomena from an operator algebraic viewpoint.

*Supported in part by the Grants-in-Aid for Scientific Research, JSPS.
The theory of subfactors initiated by Jones has opened many surprising connections of operator algebras to various branches of mathematics and physics such as 3-dimensional topology, quantum groups and statistical mechanics. We present aspects of subfactor theory in connection to conformal field theory.

Noncommutative geometry initiated by Connes is a new approach to geometry based on operator algebras and it is also related to many branches in mathematics and physics from number theory to the standard model. We present new aspects of noncommutative geometry in the operator algebraic approach to superconformal field theory.

We emphasize the operator algebraic aspects of (super)conformal field theory. Still, we apologize any omission in our discussions caused by our bias and ignorance.

2 Chiral superconformal field theory

2.1 An idea of algebraic quantum field theory

We deal with a chiral superconformal field theory, a kind of quantum field theory, based on operator algebraic methods in this section. We start with a very naive idea on what a quantum field is. A classical field is simply a function on a spacetime, mathematically speaking. In quantum mechanics, numbers are replaced by operators. So instead of ordinary functions, we study operator-valued functions on a spacetime. It turns out that we also have to deal with something like a $\delta$-function, we thus deal with operator-valued distributions, and they should be mathematical formulations of quantum fields. An ordinary Schwartz distribution assigns a number to each test function, by definition. An operator-valued distribution should assign an operator, which may be unbounded, to each test function. There is a precise mathematical definition of this notion of an operator-valued distribution, and we can further axiomatize a physical idea of what a quantum field should be. Such an axiomatization is known as a set of Wightman axioms.

We work on a Minkowski space $\mathbb{R}^4$. For $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$, their Minkowski inner product is $x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$. When a linear transform $\Lambda$ on $\mathbb{R}^4$ preserves the Minkowski inner product, it is called a Lorentz transform. The set of such a $\Lambda$ satisfying $\Lambda_{00} > 0$ and $\det \Lambda = 1$ is called the restricted Lorentz group. The universal cover of the restricted Lorentz group is naturally identified with $SL(2, \mathbb{C})$. The set of transformations of the form $x \mapsto \Lambda x + a$, where $\Lambda$ is an element of the restricted Lorentz group, is called the restricted Poincaré group. The universal cover of the restricted Poincaré group is naturally written as

$$\{(A, a) \mid A \in SL(2, \mathbb{C}), a \in \mathbb{R}^4\}.$$ 

We say that two regions $O_1, O_2$ are spacelike separated, if for any $x = (x_0, x_1, x_2, x_3) \in O_1$ and $y = (y_0, y_1, y_2, y_3) \in O_2$, we have $(x_0 - y_0)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2 < 0$.

We now briefly explain the Wightman axioms, but we do not go into full details here, since they are not our main objects here. See [74] for details.
1. We have closed operators $\phi_1(f), \phi_2(f), \ldots, \phi_n(f)$ on a Hilbert space $H$ for each test function $f$ on $\mathbb{R}^4$ in the Schwartz class.

2. We have a dense subspace $D$ of $H$ which is contained in the domains of all $\phi_i(f)$, $\phi_i(f)^*$, and we have $\phi_i(f)D \subset D$ and $\phi_i(f)^*D \subset D$ for all $i = 1, 2, \ldots, n$. For $\Phi, \Psi \in D$, the map $f \mapsto (\phi_i(f)\Phi, \Psi)$ is a Schwartz distribution.

3. We require that there exists a unitary representation $U$ of the universal cover of the restricted Poincaré group satisfying the following, where $S$ is an $n$-dimensional representation of $SL(2, \mathbb{C})$ and $\Lambda(A)$ is the image of $A \in SL(2, \mathbb{C})$ in the restricted Lorentz group.

$$U(A,a)D = D,$$

$$U(A,a)\phi_i(f)U(A,a)^* = \sum_j S(A^{-1})_{ij}\phi_j(f(A,a)),$$

$$f(A,a) = f(\Lambda(A)^{-1}(x - a)).$$

Here the second line means that the both hand sides are equal on $D$.

4. If the supports of $C^\infty$-functions $f$ and $g$ are compact and spacelike separated, then we have $[\phi_i(f), \phi_j(g)]_\pm = 0$, where the symbol $[~,~]_-$ and $[~,~]_+$ denote the commutator and the anticommutator, respectively, and it means that the left hand side is zero on $D$. The choice of $\pm$ depends on $i, j$. We also have $[\phi_i(f), \phi_j(g)^*)_\pm = 0$.

5. We have a distinguished unit vector $\Omega \in D$ called the vacuum vector, unique up to phase, and it satisfies the following. We have $U(A,a)\Omega = \Omega$, and the spectrum of the four-parameter unitary group $U(I,a), a \in \mathbb{R}^4$, is in the closed positive cone

$$\{(p_0, p_1, p_2, p_3) \mid p_0 \geq 0, p_0^2 - p_1^2 - p_2^2 - p_3^2 \geq 0\}.$$

We can apply arbitrary $\phi_i(f)$ and $\phi_j(g)^*$ to $\Omega$ for finitely many times. We require that the linear span of such vectors is dense in $H$.

This is a nice formulation and many people have worked within this framework, but distributions and unbounded operators cause various technical difficulties in a rigorous treatment. Bounded linear operators are much more convenient for algebraic handling, and we seek for a mathematical framework using only bounded linear operators. There has been such a framework pursued by Araki, Haag and Kastler in early days, and such an approach is called algebraic quantum field theory today. The basic reference is Haag’s book [48] and we now explain its basic idea as follows.

Suppose we have a family of operator-valued distributions $\{\phi\}$ subject to the Wightman axioms as above. Take an operator-valued distribution $\phi$ and a test function $f$ on a certain spacetime. That is, $f$ is a $C^\infty$-function with a compact support and suppose that its support is contained in a bounded region $O$ in the spacetime.
Then $\phi(f)$ gives an (unbounded) operator and suppose it is self-adjoint. In quantum mechanics, observables are represented as (possibly unbounded) self-adjoint operators. We now regard $\phi(f)$ as an observable in the spacetime region $O$. We have a family of such quantum fields and many test functions with supports contained in $O$, so we have many unbounded operators for each $O$. Then we make spectral projections of these unbounded operators and consider a von Neumann algebra $\mathcal{A}(O)$ generated by these projections. In this way, we have a family of von Neumann algebras $\{\mathcal{A}(O)\}$ on the same Hilbert space parameterized by bounded spacetime regions. We now consider what kind of properties such a family is expected to satisfy, and then we forget the Wightman fields and just consider these properties as axioms for a family of von Neumann algebras $\{\mathcal{A}(O)\}$. Then we mathematically study such a family of von Neumann algebras satisfying the axioms. That is, we construct examples, study relations among various properties, classify such families, and so on, just as in a usual axiomatic mathematical theory.

We now list such “expected properties” which become axioms later. Our spacetime is again a Minkowski space $\mathbb{R}^4$, where the speed of light is set to be 1. Then as a bounded region $O$, it is enough to consider only double cones, which are of the form $(x + V_+) \cap (y + V_-)$, where $x, y \in \mathbb{R}^4$ and

$$V_\pm = \{z = (z_0, z_1, \ldots, z_3) \in \mathbb{R}^4 \mid z_0^2 - z_1^2 - z_2^2 - z_3^2 > 0, \pm z_0 > 0\}.$$

1. (Isotony) For a larger double cone $O_2$ than $O_1$, we have more test functions, and hence more operators. So we should have $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.

2. (Locality) Suppose two double cones $O_1$ and $O_2$ are spacelike separated. Then we cannot make any interaction between the two regions even with the speed of light. Then observables in the two regions mutually commute. We thus require that the elements in $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ commute. (This corresponds to the case of the commutator in the Wightman axioms.)

3. (Poincaré Covariance) In the Minkowski space, the natural symmetry group is the restricted Poincaré group. We require that there exists a unitary representation $U$ of the universal cover of the restricted Poincaré group satisfying $\mathcal{A}(gO) = U_g \mathcal{A}(O) U_g^*$, where $gO$ is the image of $O$ under the action of the quotient image of $g$ in the Poincaré group.

4. (Vacuum) We have a distinguished unit vector $\Omega \in H$, unique up to phase, satisfying $U_g \Omega = \Omega$ for all elements $g$ in the restricted Poincaré group,

5. (Cyclicity of the vacuum) We require that $\bigcup_O \mathcal{A}(O) \Omega$ is dense in $H$.

6. (Spectrum Condition) If we restrict the representation $U$ to the translation subgroup, its spectrum is contained in the closure of $V_+$.

It is clear that the above set of axioms is very similar to that of Wightman. The assignment of $\mathcal{A}(O)$ to $O$ is traditionally called a net (of von Neumann algebras).
It is very hard to construct an example satisfying the above axioms. In the 4-dimensional Minkowski space, we have only a family of examples known under the name of free field models.

### 2.2 Local conformal nets

Now we work on a little bit different version of a net of von Neumann algebras explained in the previous subsection.

In the above framework, we have chosen the Minkowski space $\mathbb{R}^4$ as our spacetime and its symmetry group has been the restricted Poincaré group, but it is also possible to replace the spacetime and the symmetry group in this framework. We now choose the 2-dimensional Minkowski space and the conformal symmetry group. We will describe more details of this setting in Subsection 2.6 but for a moment, we simply regard it as a net $\{\mathcal{A}(O)\}$ of von Neumann algebras satisfying some axioms, where $O$ is a double cone in the 2-dimensional Minkowski space.

Then it is possible to “restrict” the theory on two light rays, $\{(t, x) \mid t = \pm x\}$, where $t, x$ are the time and space coordinates of the 2-dimensional Minkowski space now. A double cone is projected to an interval on the light ray, and in this way, it is possible to have a net of von Neumann algebras parameterized by intervals on the real line. We have a high symmetry group of conformal transformations, and it is natural and convenient to consider the one-point compactification $S^1$ of the real line, where $t \in \mathbb{R}$ corresponds to $(-t + i)/(t + i) \in S^1$, where $S^1$ is a unit circle on the complex plane and $\infty$ corresponds to $-1 \in S^1$. We now have a net of von Neumann algebras parameterized by intervals contained in $S^1$. (See [56, Section 2] for more details on this “restriction” procedure. See [39] and reference there for this in other approaches to conformal field theory.) We now write down a precise set of axioms.

An interval $I \subset S^1$ means a non-empty, non-dense open connected subset of $S^1$. That is, $S^1$ itself nor $S^1$ minus one point is not an interval. We have a family $\{\mathcal{A}(I)\}$ of von Neumann algebras on a fixed Hilbert space $H$ parameterized by intervals $I \subset S^1$.

Note that $PSL(2, \mathbb{R})$ acts on $S^1$ through fractional linear transformation on $\mathbb{R}$.

This group is also called the Möbius group.

1. (Isotony) For intervals $I_1 \subset I_2$, we have $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.

2. (Locality) For intervals $I_1, I_2$ with $I_1 \cap I_2 = \emptyset$, we have $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = 0$

3. (Möbius covariance) There exists a unitary representation $U$ of $PSL(2, \mathbb{R})$ on $H$ satisfying $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$ for any $g \in PSL(2, \mathbb{R})$ and any interval $I$.

4. (Positivity of energy) The generator of the one-parameter rotation subgroup of $U$, called the conformal Hamiltonian, is positive.

5. (Conformal covariance) There exists a projective unitary representation $U$ of $Diff(S^1)$ on $H$ extending the unitary representation of $PSL(2, \mathbb{R})$ such that for
all intervals $I$, we have
\[
U(g)A(I)U(g)^* = A(gI), \quad g \in \text{Diff}(S^1),
\]
\[
U(g)AU(g)^* = A, \quad A \in A(I), \quad g \in \text{Diff}(I'),
\]
where $\text{Diff}(S^1)$ is the group of orientation-preserving diffeomorphisms of $S^1$ and $\text{Diff}(I')$ is the group of diffeomorphisms $g$ of $S^1$ with $g(t) = t$ for all $t \in I$. (Here $I'$ is the interior of the complement of an interval $I$.)

6. (Vacuum vector) There exists a unit $U$-invariant vector $\Omega$ in $H$, called the vacuum vector, which is cyclic for the von Neumann algebra generated by $\bigcup I A(I)$.

7. (Irreducibility) The von Neumann algebra $\bigvee_{I \subset S^1} A(I)$ generated by all $A(I)$’s is $B(H)$. 

It is clear that the above axioms basically correspond to the axioms for the Minkowski space $\mathbb{R}^4$ and the Poincaré group. Note that the locality now takes a very simple form.

Now the set of intervals on $S^1$ is not directed with respect to inclusions, so the name net is not mathematically appropriate, and the correct terminology should be a cosheaf, for example, but the name “net” has been widely used and we also use it here. The net $\{A(I)\}$ satisfying the above axioms is called a local conformal net.

We now list some consequences of the above axioms. (See [55] and references there for more details.)

The vacuum vector $\Omega$ is cyclic and separating for each von Neumann algebras $A(I)$. This is called the Reeh-Schlieder theorem.

Let $I_1 \subset S^1$ be the upper semicircle. Let $C : S^1 \to \mathbb{R} \cup \{\infty\}$ be the Cayley transform given by $C(z) = -i(z-1)(z+1)^{-1}$. We define a one-parameter group $\Lambda_{t_1}(s)$ of diffeomorphisms by $C\Lambda_{t_1}(s)C^{-1} : x \mapsto e^{s}x$. We also set $r_{t_1}$ by $r_{t_1}(z) = \bar{z}$ for $z \in S^1$. For a general interval $I$, we choose $g \in PSL(2, \mathbb{R})$ with $I = gI_1$. Then we set $\Lambda_I = g\Lambda_{t_1}g^{-1}$ and $r_I = gr_{t_1}g^{-1}$. These are independent of the choice of $g$, thus well-defined. The action of $r_{t_1}$ on $PSL(2, \mathbb{R})$ gives a semi-direct product $PSL(2, \mathbb{R}) \rtimes \mathbb{Z}_2$. Let $\Delta_I$ and $J_I$ be the modular operator and the modular conjugation with respect to $(A(I), \Omega)$. We now have an extension of the representation $U$ appearing in the Möbius covariance property, still denoted by $U$, such that $U(g)$ is unitary [resp. anti-unitary] when $g$ is orientation preserving [resp. reversing]. This $U$ satisfies $U(\Lambda_I(2\pi t)) = \Delta_I^t$ and $U(r_I) = J_I$. This is called the Bisognano-Wichmann property [11, 40].

We have $A(I') = A(I)'$ and this is called the Haag duality. This follows directly from the above Bisognano-Wichmann property and it means that locality holds maximally.

It also follows that each von Neumann algebra $A(I)$ is automatically a factor, and it is of type $\text{III}_1$, except for the trivial case $A(I) = C$. Actually, for all known cases, each $A(I)$ is the unique Araki-Woods factor of type $\text{III}_1$. So each algebra $A(I)$ does not contain any information on conformal field theory. It is relative relations of factors $A(I)$ that contain information on conformal field theory.
If we have a family \( \{I_i\} \) of intervals with \( I \subset \bigcup_i I_i \), then the von Neumann algebra \( A(I) \) is contained in the von Neumann algebra generated by \( \bigcup_i A(I_i) \). This property is called *additivity* \([35]\).

We have a notion of a stronger version of this additivity as follows. This holds only for some examples, and does not hold in general. Let \( I \) be an interval and \( x \in I \). Then \( I \setminus \{x\} \) is a union of two intervals \( I_1 \) and \( I_2 \). The *strong additivity* means that \( A(I) \) is generated by \( A(I_1) \) and \( A(I_2) \). This has some similarity to amenability of a single operator algebra, and gives a kind of amenability type condition for a family of von Neumann algebras.

We have another related property called the *split property* \([30]\). Let \( I_1, I_2 \) be two intervals in \( S^1 \) with \( \overline{I_1} \cap \overline{I_2} = \emptyset \). Then we require that \( x \otimes y \mapsto xy \) extends to an isomorphism from \( A(I_1) \otimes A(I_2) \) to the von Neumann algebra generated by \( A(I_1) \) and \( A(I_2) \). (Note that these the two von Neumann algebras act on \( H \otimes H \) and \( H \), respectively.) It is known that if \( e^{-tL_0} \) is of trace class for all \( t > 0 \), where \( L_0 \) is the conformal Hamiltonian, then the split property holds \([25]\). This trace class condition is easy to verify for concrete examples. This holds for all known examples, but may not hold in general.

It is not easy to construct an example of a local conformal net (except for the trivial one with \( A(I) = \mathbb{C} \) for all \( I \)). A family of examples has been constructed by Buchholz-Mack-Todorov \([13]\) from the \( U(1) \)-currents.

Another important family has been constructed by A. Wassermann \([76]\), and the outline of the construction is as follows. Let \( L(SU(N)) \) be the loop group, that is, the set of \( C^\infty \)-functions on \( S^1 \) with values in \( SU(N) \), with pointwise multiplication. For an interval \( I \subset S^1 \), we set \( L_I(SU(N)) \) to be the set of \( C^\infty \)-functions \( f \) on \( S^1 \) with values in \( SU(N) \) such that \( f(z) = \text{Id} \in SU(N) \) for \( z \notin I \). Then for each positive integer \( k \), we have a vacuum positive energy representation \( \pi \) of \( L(SU(N)) \), which is a projective unitary representation on some Hilbert space \( H \) having the vacuum vector \( \Omega \). Then we set \( A(I) \) to be the von Neumann algebra generated by \( \pi(L_I(SU(N))) \). A similar construction has been studied for other Lie groups by various people \([75]\).

Projective unitary representation of \( \text{Diff}(S^1) \) has been constructed by \([44, 75]\).

There is also a general construction of a local conformal net from an even lattice. We will mention this in Section 4.

We finally list some methods to construct new examples from known examples (with some property).

1. If we have two local conformal nets \( \{A(I)\} \) on \( H \) and \( \{B(I)\} \) on \( K \), we have another local conformal net \( \{A(I) \otimes B(I)\} \) on \( H \otimes K \). This is the tensor product construction. It is easy to see that this gives a local conformal net.

2. For a local conformal net \( \{A(I)\} \) with a nice representation theory, we can make a crossed product construction by a finite abelian group \( G \). This is called a *simple current extension* of \( \{A(I)\} \). (See \([6]\) Part II for a concrete example, though it is not called a simple current extension there.)
3. If a group $G$ is contained in the automorphism group of a local conformal net $\{A(I)\}$, we can make a fixed point net $\{A(I)^G\}$. (The notion of an automorphism will be explained in Section 4 in detail.) This is well-studied for finite groups $G$, and in such a case, this construction is called the orbifold construction. (See [80] for details.)

4. If we have two local conformal nets $\{A(I)\}$ and $\{B(I)\}$ with $A(I) \subset B(I)$, we can construct a new local conformal net $\{A(I) \cap B(I)\}$. This is called the coset construction. (See [79] for details.)

In the constructions 2, 3 and 4 in the above, we actually have to specify the Hilbert space appropriately.

### 2.3 Representation theory

The fundamental tool to study local conformal nets is their representation theory.

We take a local conformal net $A$. A representation $\pi$ of $A$ is a family $\{\pi_I\}$ of representations of $A(I)$ on the same Hilbert space $H_\pi$ with the condition that $\pi_{I_1}$ extends $\pi_{I_2}$ if $I_1 \supset I_2$.

We say that a representation $\pi$ on $H_\pi$ is diffeomorphism covariant if there exists a projective unitary representation $U_\pi$ of the universal cover of $\mathrm{Diff}(S^1)$ on $H_\pi$ such that

$$\pi_{gI}(U(g) x U(g)^*) = U_\pi(g) \pi_I(x) U_\pi(g)^*$$

for all $x \in A(I)$ and all $g$ in the universal cover, where $gI$ is the image of $I$ by the natural image of $g$ in $\mathrm{Diff}(S^1)$. A M"obius covariant representation is also defined in a similar way. We sometimes use the terminology DHR representation, which means a diffeomorphism covariant or M"obius covariant representation depending on the context, where DHR stands for Doplicher-Haag-Roberts [29]. The identity map gives a DHR representation of $A$ on the initial Hilbert space $H$. This is called the vacuum representation. We also have a natural notion of irreducibility for DHR representations.

We now introduce a numerical invariant of the net $A$ called the central charge.

The Virasoro algebra is the infinite dimensional Lie algebra generated by elements $\{L_n \mid n \in \mathbb{Z}\}$ and a central element $c$ with relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

and $[L_n, c] = 0$. It is the (complexification of) the unique, non-trivial one-dimensional central extension of the Lie algebra of $\mathrm{Diff}(S^1)$.

For this infinite dimensional Lie algebra, we have notions of a unitary representation which means that we have $L_n^* = L_{-n}$ in the representation space and positive energy which means the image of $L_0$ is positive. A projective unitary representation of $\mathrm{Diff}(S^1)$ gives a unitary representation of the Virasoro algebra. In any irreducible unitary representation the Virasoro algebra, the element $c$ is mapped to a scalar and
its value is called the central charge. We denote this value by the same symbol $c$, and it is known that all the possible values of the central charge is the set

$$\{1 - 6/m(m + 1) \mid m = 2, 3, 4, \ldots\} \cup [1, \infty)$$

by \cite{38} and \cite{43}. The unitary representation of the Virasoro algebra arising from the projective unitary representation of $\text{Diff}(S^1)$ decomposes into irreducible representations, all with the same central charge $c > 0$. We define the central charge of the local conformal net $\mathcal{A}$ to be this value.

The irreducible unitary representations of the Virasoro algebra produces Wightman fields on $S^1$ and they provide local conformal nets $\text{Vir}_c$ for all possible values of $c$ \cite{14}. (See \cite{55} for more details.)

We say that a representation $\pi$ is localized in an interval $I_0$ if we have $H_\pi = H$ and $\pi I_0 = \text{id}$. For a given interval $I_0$ and a representation $\pi$ on a separable Hilbert space, there is a representation $\tilde{\pi}$ unitarily equivalent to $\pi$ and localized in $I_0$. This is because all representations of a type III factor $\mathcal{A}(I_0')$ on Hilbert spaces are unitarily equivalent. If $\pi$ is a representation localized in $I_0$, then by Haag duality implies that $\pi I$ is an endomorphism of $\mathcal{A}(I)$ if $I \supset I_0$. The endomorphism $\pi$ is called a DHR endomorphism localized in $I_0$ \cite{29}. The (Jones) index of a representation $\pi$ is the Jones index $[\mathcal{A}(I) : \pi I(\mathcal{A}(I))]$ of $\pi_I$, if $I \supset I_0$. (This number is independent of $I$. See Section 5 for more on the Jones index.) The statistical dimension $d(\pi)$ of $\pi$ is the square root of this index. The unitary equivalence $[\pi]$ class of a representation $\pi$ of $\mathcal{A}$ is called a (superselection) sector of $\mathcal{A}$.

We now introduce a notion of a tensor product of two DHR representations. Note that we have an obvious notion of a tensor product for representations of a group, but not for DHR representations of a local conformal net. Two DHR representations give two DHR endomorphisms. Then they can be composed and the result is still a DHR endomorphism. This gives a proper definition of a tensor product of DHR representations, and with this notion, we have a tensor category of DHR endomorphisms \cite{29}.

For group representations $\pi$ and $\sigma$, the tensor products $\pi \otimes \sigma$ and $\sigma \otimes \pi$ are obviously unitarily equivalent, but for DHR endomorphisms $\lambda$ and $\mu$ of $\mathcal{A}$, it is not clear to see the relation between $\lambda \mu$ and $\mu \lambda$. It turns out they are unitarily equivalent, and we have a natural choice of unitary operator $\text{Ad}(\varepsilon(\lambda, \mu))\lambda \mu = \mu \lambda$. The choices of $\varepsilon(\lambda, \mu)$ give a braiding and the tensor category of the DHR endomorphisms gives a braided tensor category \cite{39}. (Also see \cite{6} for an exposition.)

A finite group has only finitely many irreducible unitary representations up to unitary equivalence. In theory of quantum groups, it sometimes similarly happens that a quantum group has only finitely many irreducible unitary representations up to unitary equivalence. Such finiteness property is sometimes called rationality. We have an operator algebraic counterpart of this notion for a local conformal net $\mathcal{A}$ as follows and the property is called complete rationality.

Take a local conformal net $\mathcal{A}$ with split property. We split the circle $S^1$ into four intervals $I_1, I_2, I_3, I_4$ in the counterclockwise order. Then we have a subfactor
\((\mathcal{A}(I_1) \vee \mathcal{A}(I_3)) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'\). We say that the local conformal net \(\mathcal{A}\) is \textit{completely rational} if the index of this subfactor is finite. We also call the index value of this subfactor \(\mu\)-index of the local conformal net \(\mathcal{A}\). (In the original definition in [59], strong additivity was also assumed, but it has been shown in [65] that it follows automatically from the above definition.)

In [59], we have shown that complete rationality implies that we have only finitely many irreducible DHR representations up to unitary equivalence and that all these have finite Jones indices. Furthermore, the braiding we have for DHR representations is nondegenerate [69]. That is, we have a finite dimensional unitary representation of \(SL(2, \mathbb{Z})\) arising from this braiding [73]. (The dimension of this representation is the number of irreducible DHR representations of a local conformal net up to unitary equivalence.) The \(\mu\)-index of a local conformal net is equal to the square sum of the dimensions of the unitary equivalence classes of the irreducible DHR representations of the local conformal net [59]. This number measure the size of the braided tensor category of the DHR representations.

Wassermann’s examples [76] arising from \(SU(N)\) at level \(k\) are completely rational by Xu’s computation of the \(\mu\)-index [78]. The Virasoro nets \(\text{Vir}_c\) with \(c < 1\) are also completely rational [55].

### 2.4 \(\alpha\)-induction and classification

If we have two groups \(H \subset G\), we can obviously restrict a representation of \(G\) to \(H\), and furthermore, we have a notion of an induced representation which gives a representation of \(G\) from that of \(H\). We have a similar notion of induction for DHR representations of inclusions of local conformal nets. This induction procedure was first defined in [66], further studied in [77] with many interesting examples. It was named as \(\alpha\)-\textit{induction} in [6], and further studied in [7], [8] in connection to the methods in [70].

Suppose we have an inclusion of local conformal nets \(\mathcal{A} \subset \mathcal{B}\). (Note that the Hilbert space associated with \(\mathcal{A}\) is a subspace of that associated with \(\mathcal{B}\).) Suppose the Jones index of \(\mathcal{A}(I) \subset \mathcal{B}(I)\), which is independent of \(I\), is finite.

Take a DHR endomorphism \(\lambda\) of \(\mathcal{A}\) which is localized in a fixed interval \(I\). For a while, we regard \(\lambda\) simply as an endomorphism of a single factor \(\mathcal{A}(I)\). We now would like to extend it to a larger factor \(\mathcal{B}(I)\) as an endomorphism. The inclusion map of \(\mathcal{A} \subset \mathcal{B}\) naturally defines a DHR representation of \(\mathcal{A}\), so it gives a DHR endomorphism \(\theta\) localized in \(I\), and we again regard \(\theta\) as an endomorphism of \(\mathcal{A}(I)\). The braiding structure of the DHR endomorphisms produces unitary operators \(\epsilon^\pm(\lambda, \theta)\) with \(\text{Ad}(\epsilon^\pm(\lambda, \theta)) \lambda \theta = \theta \lambda\). (Note that a positive braiding and a negative one automatically come in a pair.)

By a general theory of type III subfactors, we have an isometry \(v \in \mathcal{B}(I)\) with \(vx = \theta(x)v\) for all \(x \in \mathcal{A}(I)\) and we also have \(\mathcal{B}(I) = \mathcal{A}(I)v\). We can then extend an endomorphism \(\lambda\) of \(\mathcal{A}(I)\) to \(\mathcal{B}(I)\) by setting \(\alpha^\pm_\lambda(v) = \epsilon^\pm(\lambda, \theta)^*v\), where \(\epsilon^\pm\) means a positive and a negative braiding operators. It is not difficult to see that this indeed gives an extension of an endomorphism. The extended endomorphism \(\alpha^\pm_\lambda\) does not
extend to a DHR endomorphism of \( B \) in general, and it only gives a slightly weaker version called a soliton endomorphism, but we do not go into details on this here.

Suppose that a local conformal net \( A \) is completely rational in the above setting. Let \( \lambda, \mu \) be representatives of unitary equivalence classes of irreducible DHR representations, and regard them as endomorphisms of \( A(I) \) for a fixed interval \( I \). Now \( \alpha^+_\lambda \) is an endomorphism of \( B(I) \), and we set

\[
Z_{\lambda \mu} = \dim(\text{Hom}(\alpha^+_\lambda, \alpha^-_\mu)),
\]

where we define

\[
\text{Hom}(\rho_1, \rho_2) = \{ a \in M \mid a \rho_1(x) = \rho_2(x) a \text{ for all } x \in M \}
\]

for two endomorphisms \( \rho_1, \rho_2 \) of a von Neumann algebra \( M \). In this way, we have a square matrix \( Z \) whose size is the number of unitary equivalence classes of the irreducible DHR representations of \( A \) and whose entries are nonnegative integers. We denote the vacuum representation of \( A \) by \( 0 \), and we then have \( Z_{00} = 1 \). Now one of the main results in [7], Theorem 5.7 there, says that the matrix \( Z \) commutes with the image of the unitary representation of \( \text{SL}(2, \mathbb{Z}) \) arising from the nondegenerate braiding of \( A \). Actually, it is shown in [7] that this property of \( Z \) holds in a more general situation where we have a (possibly degenerate) braiding not necessarily arising from a local conformal net. The graphical methods used for this proof are based on [70] and are also used in [8].

In general, when we have a unitary representation of \( \text{SL}(2, \mathbb{Z}) \) arising from a nondegenerate braiding as above, a matrix \( Z \) is called a modular invariant if it satisfies the following three conditions, where \( 0 \) is the vacuum representation.

1. \( Z_{00} = 1 \).
2. \( Z_{\lambda \mu} \in \{0, 1, 2, \cdots\} \).
3. The matrix \( Z \) commutes with the image of the unitary representation of \( \text{SL}(2, \mathbb{Z}) \).

In general, for a given such representation of \( \text{SL}(2, \mathbb{Z}) \), we have only finitely many modular invariants \( Z \). For the \( SU(N) \) nets with level \( k \) constructed by A. Wassermann [76], the representations of \( \text{SL}(2, \mathbb{Z}) \) have been explicitly known and coincide with the previously known ones in the context of loop groups or Kac-Moody algebras. The representations of \( \text{SL}(2, \mathbb{Z}) \) for the Virasoro nets \( \text{Vir}_c \) with \( c < 1 \) follow from [79]. (See [55] for explicit descriptions.)

Summarizing the above, we produce a modular invariant matrix \( Z \) from an inclusion of local conformal nets \( A \subset B \) where \( A \) is completely rational and the Jones index is finite. (Then \( B \) is automatically completely rational by [59].) Suppose only \( A \) is fixed. Then the above gives a map from the set of extensions \( B \) with finite indices to the set of modular invariant matrices \( Z \). (The finite index property of \( B \) automatically holds if we have irreducibility condition \( A(I)' \cap B(I) = \mathbb{C} \) by [55] based on [51].) This map is not injective nor surjective in general, but for an explicitly known unitary
representation of $SL(2,\mathbb{Z})$ arising from the DHR representations of $\mathcal{A}$, the number of modular invariants is often small and we can explicitly write down all of them. Then this severely restricts possibility of an extension $\mathcal{B}$. In the case of $SU(2)$ nets of level $k$ and the Virasoro nets $\text{Vir}_c$ with $c < 1$, complete lists of modular invariants have been given in Cappelli-Itzykson-Zuber [16]. Gannon has such classification lists for many other cases. See [12] for more details.

If $\mathcal{B}$ is a local conformal net with $c < 1$, then it is automatically an irreducible extension of the Virasoro net $\text{Vir}_c$. From the known classification of modular invariants for $\text{Vir}_c$, we can finally classify all possible extensions $\mathcal{B}$. This has been done in [55] and the classification list is as follows.

1. The Virasoro nets $\text{Vir}_c$ with $c < 1$.
2. The simple current extensions of the Virasoro nets with index 2.
3. Four exceptionals at $c = 21/22, 25/26, 144/145, 154/155$.

We refer to [39] and reference therein for the role of modular invariants in other approaches to conformal field theory.

### 2.5 Superconformal nets

We now extend the above theory to superconformal case. Recall that in the above set of the Wightman axioms, we have seen $\pm$ in the (anti) commutator.

Let $\mathcal{A}$ be a local conformal net acting on a Hilbert space $H$ with the vacuum vector $\Omega$. A unitary operator $U$ on $H$ is said to be an automorphism of $\mathcal{A}$ if we have $U\Omega = \Omega$ and $U\mathcal{A}(I)U^\dagger = \mathcal{A}(I)$ for all intervals $I \subset S^1$. Such a unitary $U$ is also called a gauge unitary.

A $\mathbb{Z}_2$-grading on $\mathcal{A}$ is $\gamma = \text{Ad}(U)$, where $U$ is an involutive gauge unitary. For such a $\mathbb{Z}_2$-grading $\gamma$, an element $x \in \mathcal{A}(I)$ for some $I$ is said to be homogeneous if $\gamma(x) = \pm x$. For such $x$, we say the parity $p(x)$ is 0 [resp. 1] if $\gamma(x) = x$ [resp. $\gamma(x) = -x$]. Any element $x$ of $\mathcal{A}(I)$ for some $I$ is uniquely decomposed as $x = x_0 + x_1$ with $p(x_k) = k$.

A Möbius covariant Fermi net $\mathcal{A}$ on $S^1$ is a $\mathbb{Z}_2$-graded net whose the symmetry group is the covering of the Möbius group and which satisfies the following property, which is called graded locality.

[Graded locality] For $x \in \mathcal{A}(I_1), y \in \mathcal{A}(I_2)$ with $I_1 \cap I_2 = \emptyset$, we have $[x, y] = 0$, where $[\ , \ ]$ is a graded commutator. That is, we have $[x, y] = xy - (-1)^{p(x)p(y)}yx$ for homogeneous $x, y$.

Note the Bose subnet $\mathcal{A}_b$, namely the $\gamma$-fixed point subnet $\mathcal{A}^\gamma$ of degree zero elements, is local. If we define a unitary $Z$ by $Z = (1 - iU)/(1 - i)$, then we have $Z\Omega = \Omega$, $Z^2 = U$ and $\mathcal{A}(I') \subset Z\mathcal{A}(I)Z^*$. 

A Fermi conformal net is a Möbius covariant Fermi net with an extra property on the representation of the symmetry group, which is the covering of $\text{Diff}(S^1)$. We also
have a natural notion of a DHR representation for a Fermi conformal net. See [18] for details.

We say a Fermi conformal net is completely rational when its Boson part is completely rational.

Now we study the two $N = 1$ super Virasoro algebras. They have even generators $c, L_n, n \in \mathbb{Z}$, and odd generators $G_r, r \in \mathbb{Z} + 1/2$ or $r \in \mathbb{Z}$, with the following relations. Here $c$ is the central charge and the elements $L_n, n \in \mathbb{Z}$, are the usual Virasoro generators.

\[
[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0}, \quad (2)
\]

\[
[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}, \quad (3)
\]

\[
[G_r, G_s] = 2 L_{r+s} + \frac{c}{3} \left(r^2 - \frac{1}{4}\right) \delta_{r+s,0}. \quad (4)
\]

If $r \in \mathbb{Z} + 1/2$, then the resulting infinite dimensional Lie algebra is called the Neveu-Schwarz algebra, and if $r \in \mathbb{Z}$, then the Lie algebra is called the Ramond algebra. They together make $N = 1$ super Virasoro algebras.

As in the Virasoro algebra case, we have a notion of a unitary (positive energy) representation. In an irreducible unitary representation, the central charge $c$ is mapped to a scalar, and its value is also called the central charge. The set of the possible values of the central charge is now

\[
\left\{ \frac{3}{2} \left(1 - \frac{8}{m(m+2)}\right) \Big| m = 3, 4, 5, \ldots \right\} \cup \left[ \frac{3}{2}, \infty \right)
\]

by [38]. Furthermore, in an irreducible unitary representation, $\{L_n\}$ and $\{G_r\}$ define operator-valued distributions as in the Virasoro case, and they produce a Fermi conformal net $S_{\text{Vir}_c}$. It is called a super Virasoro net with central charge $c$. A Fermi conformal net is called a superconformal net when it is an extension of a super Virasoro net $S_{\text{Vir}_c}$.

Superconformal nets with $c < 3/2$ are classified again with modular invariants listed by Cappelli [15] as follows [18].

1. The super Virasoro net with $c = \frac{3}{2} \left(1 - \frac{8}{m(m+2)}\right)$, labeled with $(A_{m-1}, A_{m+1})$.

2. Index 2 extensions of the above (1), labeled with $(A_{4m'-1}, D_{2m'+2}), (D_{2m'+2}, A_{4m'+3})$.

3. Six exceptionals labeled with $(A_9, E_6), (E_6, A_{13}), (A_{27}, E_8), (E_8, A_{31}), (D_6, E_6), (E_6, D_8)$.

**2.6 Full conformal field theory**

We now consider a conformal field theory on the $(1+1)$-dimensional Minkowski space $\mathcal{M}$.
A local Möbius covariant net $\mathcal{A}$ on $\mathcal{M}$ is an assignment of a von Neumann algebra $\mathcal{A}(O)$ on $H$ to each double cone $O \subset \mathcal{M}$ with the following properties.

- (Isotony) For $O_1 \subset O_2$, we have $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- (Locality) If $O_1$ and $O_2$ are spacelike separated, then $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ mutually commute.
- (Möbius covariance) There exists a unitary representation $U$ of the direct product of the universal cover of $PSL(2, \mathbb{R})$ and its another copy on $H$ satisfying $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$ for a double cone $O$ and $g \in U$, where $U$ is a connected neighborhood of the identity of the direct product of the universal cover of $PSL(2, \mathbb{R})$ and its another copy satisfying $gO \subset \mathcal{M}$ for all $g \in U$.
- (Vacuum vector) There exists a unit $U$-invariant vector $\Omega$ which is cyclic for the $\bigcup_O \mathcal{A}(O)$.
- (Positive energy) The one-parameter unitary subgroup of $U$ corresponding to the time translations has a positive generator.

Let $G$ be the quotient of the direct product of the universal cover of $PSL(2, \mathbb{R})$ and its another copy module the relation $(r_{2\pi}, r_{-2\pi}) = (id, id)$. Then as in [56, Section 2], the net $\mathcal{A}$ extends to a local $G$-covariant net $\mathcal{A}$ on $\mathbb{R} \times S^1$. By extending the above definition, we can also define a diffeomorphism covariant net on $\mathbb{R} \times S^1$ as in [56, Section 2]. Such a net is a mathematical realization of a full conformal field theory.

We then have representation theory and classification theory for such local conformal nets [56]. We again refer to [39] and reference therein for studies of full conformal field theory in other approaches.

### 2.7 Boundary conformal field theory

We can also formulate boundary conformal field theory in our framework. In the above setting of the $(1+1)$-dimensional Minkowski space $\{(x, t) \mid x, t \in \mathbb{R}\}$, we now restrict our consideration to the half space $\{(x, t) \mid t \in \mathbb{R}, x > 0\}$. We consider only double cones contained in this half space. Then we can formulate a local conformal net in this setting as in [67].

We have a classification result for such boundary conformal field theories for small central charges [60].

### 3 Supervertex operator algebras

We have seen the notion of superconformal net, which is a mathematical axiomatization of a physical idea of superconformal field theory. However, a superconformal net does not give the unique possible axiomatization, and we have another mathematical
axiomatization of the same physical idea. It is the notion of (super)vertex operator algebra and we present it here.

As we mentioned at the beginning, a quantum field should be some kind of operator-valued distribution on a spacetime. In our setting, the spacetime is now the 1-dimensional circle \( S^1 \). So an operator-valued distribution has a Fourier series expansion \( \sum_{n \in \mathbb{Z}} a_n z^n \). In our previous formulation of a chiral (super)conformal field theory with a (super)conformal net, we have the conformal Hamiltonian \( L_0 \), and the Hilbert space \( H \) has an eigenspace decomposition \( H = \bigoplus_{n \geq 0} H_n \), where \( H_n \) is the eigenspace for \( L_0 \) with the eigenvalue \( n \) and the direct sum means an \( L^2 \)-direct sum. Now we consider only an algebraic direct sum of \( H_n \), and we assume that each vector \( u \) in this direct sum produces an operator-valued distribution \( Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1} \), where \( a_n \) is an operator acting on \( H \). (It is customary to use the number \( n \) in \( u_n \) for the coefficient of \( z^{-n-1} \).) This is called a state-field correspondence, because a vector, called a state, gives an operator-valued distribution, called a field, and \( Y(u, z) \) is called a vertex operator.

Based on the above idea, a set of axiomatization of a supervertex operator algebra is given as follows. (There are several variations of the axioms, but we take one of the simplest forms, which is parallel to that of our graded local net.) (See [37], [54] for full details.)

1. The space \( V \) is a vector space over \( \mathbb{C} \) and it has a superspace decomposition \( V^{(0)} \oplus V^{(1)} \), where \( V^{(0)} \) [resp. \( V^{(1)} \)] is an even [resp. odd] space. When \( v \in V^{(0)} \) [resp. \( v \in V^{(1)} \)], we write \( p(v) = 0 \) [resp \( p(v) = 1 \)].

2. The map \( Y \) is from \( V \otimes V \to V((z)) \) and the image of \( u \otimes v \) by \( Y \) is written as \( Y(u, z)v = \sum_{n \in \mathbb{Z}} u_nvz^{-n-1} \). For \( u \in V^{(j)} \) and \( v \in V^{(k)} \), we have \( u_nv \in V^{(j+k)} \).

3. We have a vacuum vector \( 1 \in V \) with \( Y(1, z)u = u \) and \( Y(u, z)1|_{z=0} = u \) for all \( u \in V \).

4. We have a conformal element \( \omega \in V \) such that we have \( Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \) and \( L_n \)'s satify the Virasoro relation \([11]\) with some \( c \in \mathbb{C} \).

5. The action of \( L_0 \) on \( V \) is diagonalizable as \( V = \bigoplus_{n \geq 0, n \in \mathbb{Z}/2} V_n \), where \( V_n \) is the eigenspace of \( L_0 \) with eigenvalue \( n \) and each \( V_n \) is finite dimensional.

6. (Translation) We have \([L_{-1}, Y(v, z)] = D_z Y(v, z)\).

7. (Locality) For \( u, v \in V \), there is a sufficiently large integer \( N \) satisfying

\[
(z - w)^N Y(u, z) Y(v, w) = (-1)^{p(u)p(v)} (z - w)^N Y(v, w) Y(u, z).
\]

A conformal element is also called a Virasoro element. We say \( V \) is a vertex operator algebra when \( V^{(1)} = 0 \).

When \( V/V_{2V} \) is finite dimensional, we say that \( V \) is \( C_2 \)-cofinite. This has some formal similarity to complete rationality of a local conformal net. See [40], [47], [82] for more details on this \( C_2 \)-cofiniteness and its consequences.
We also consider a special element $\tau \in V_{3/2}$ called a superconformal element. Its defining property is that the coefficients $G_r = \tau_{r+1/2}$, $r \in \mathbb{Z} + 1/2$, of the corresponding supervertex operator satisfy the Neveu-Schwarz relations as in (2). We have $\tau_0 \tau/2 = \omega$. A supervertex operator algebra with a fixed choice of a superconformal elements is called an $N = 1$ supervertex operator algebra.

The underlying space $V$ should be a Hilbert space $H$ (before completion) in the framework of superconformal nets, but in the above set of axioms, we have nothing on the inner products. If we have an appropriate positive definite inner product on $V$, then we say that the supervertex operator algebra has unitarity. From a viewpoint of relations to operator algebras, this is the case we are interested in.

We also have a notion of a module over a vertex operator algebra. Basically we consider $v_n w$, where $v$ is in a vertex operator algebra and $w$ is in another vector space. We require certain conditions similar to the above axioms.

The Kac-Moody algebras and integral lattices are two basic sources to construct examples. See [37] and other papers for details.

It is expected that superconformal nets and $N = 1$ supervertex operator algebras are in a bijective correspondence, at least under some nice additional assumptions, since they give different axiomatizations of the same physical objects, but no such general correspondences have been known. Still, if one has some idea, technique or construction for one of the two, we can often “translate” it to the other theory. We explain some of them below.

4 Moonshine and its generalizations

We now start with the following very general problem.

Problem 4.1. Suppose a group $G$ is given. Realize it as the automorphism group of some algebraic structure in an interesting way.

This formulation is too vague, needless to say. One classical concrete formulation of the above is the inverse Galois problem, which asks for a realization of a given finite group as the Galois group over $\mathbb{Q}$, and is still open today.

Our main objects of interest here are operator algebras. So we should take operator algebras as the “algebraic structure” in the above problem, but an infinite dimensional operator algebra always has a rather large group of inner automorphisms. One way to kill such inner automorphisms is to consider $\text{Out}(M) = \text{Aut}(M)/\text{Int}(M)$ for $M$, say, von Neumann algebra $M$. Popa and Vaes [72] has constructed a $\text{II}_1$ factor $M$ with $\text{Out}(M) = \text{Aut}(M)/\text{Int}(M) \cong G$ for any given finitely presented group $G$.

Another operator algebraic formulation of the above problem is to ask for a realization of $G$ as the Galois group for an inclusion $N \subset M$, where the Galois group means

$$\text{Aut}(M \mid N) = \{ \alpha \in \text{Aut}(M) \mid \alpha(x) = x \text{ for all } x \in N \}.$$ 

Such a realization has been classically known for any finite group $G$ as follows. Take a free action of $G$ on the hyperfinite $\text{II}_1$ factor $M$. (For example, realize $M$ as the tensor
product of $|G|$ copies of a hyperfinite II$_1$ factor and let $G$ act on it as permutations of the tensor copies.) Then setting $N = M^G$, the fixed point subalgebra, realizes $G = \text{Aut}(M \mid N)$.

In this section, we present a different formulation of the above problem based on operator algebras and some realization examples, but before doing so, we need to review a development of the Moonshine conjecture and its solution in the context of vertex operator algebras.

We have so far discussed some realization of given groups, and we are here interested in finite groups. Among the finite groups, the simple ones are obviously basic objects to study. Today, classification of finite simple groups is complete, and the classification list consists of the following groups. (See [37], [42] and references there for details.)

1. Cyclic groups of prime order.
2. Alternating groups of degree 5 or higher.
3. 16 series of Lie type groups over finite fields.
4. 26 sporadic groups.

The groups in the third category are matrix groups such as $PSL(n, \mathbb{F}_q)$. The 26 groups in the last category are the exceptional objects in the classification, and the first such example was found by Mathieu in the 19th century. Among these 26 groups, the largest group in terms of the order is called the Monster group, and its order is

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71,$$

which is around $8 \times 10^{53}$. This group was first constructed by Griess [45] as the automorphism group of a certain 196884-dimensional commutative nonassociative algebra. The smallest dimension of Monster’s non-trivial irreducible representation is known to be 196883.

We now recall the definition and properties of the classical $j$-function. It is a function of a complex number $\tau$ with $\text{Im} \, \tau > 0$ and defined as

$$j(\tau) = \frac{(1 + 240 \sum_{n>0} \sigma_3(n) q^n)^3}{q \prod_{n>0} (1 - q^n)^24} = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots,$$

where $\sigma_3(n)$ is the sum of the cubes of the divisors of a positive integer $n$ and $q = \exp(2\pi i \tau)$.

We have the modular invariance property, $j(\tau) = j \left( \frac{a\tau + b}{c\tau + d} \right)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$, and this is the only function satisfying this property and starting with $q^{-1}$, up to freedom of the constant term. The constant term 744 above is rather arbitrary, and we here use $J(\tau) = j(\tau) - 744$. 

17
McKay noticed $196884 = 196883 + 1$ for the first nontrivial coefficient of the function $J(\tau)$ and the first nontrivial dimension of the irreducible representations of the Monster group. This might look purely accidental, but similar relations for the other coefficients of the $J$-function and the dimensions of the irreducible representations of the Monster group have been subsequently found. Based on these observations, Conway-Norton [21] formulated the Moonshine conjecture roughly as follows, which has been now proved by Borcherds [10].

1. We have a “natural” infinite dimensional graded vector space $V = \bigoplus_{n=0}^{\infty} V_n$ with $\dim V_n < \infty$ having some algebraic structure whose automorphism group is the Monster group.

2. For each element $g$ in the Monster, the power series $\sum_{n=0}^{\infty} (\text{Tr} g|_{V_n}) q^{n-1}$ is a special function called a Hauptmodul for some discrete subgroup of $SL(2, \mathbb{R})$. When $g$ is the identity element, we obtain the $J$-function.

The power series in the second part above is called the McKay-Thompson series. The discrete subgroups appearing in the second part have a special property called genus zero property. The statement in the first part is vague, since it does not specify “some algebraic structure”. It was in response to this problem that Frenkel-Lepowsy-Meurman [37] gave an axiomatization of vertex operator algebras. They also gave a realization of $V$ in the above part (1) and called it the Moonshine vertex operator, which is written as $V^\natural$, since it should be a natural structure. We sometimes call the property of the vertex operator algebra in the second part above the Moonshine property.

Their construction roughly goes as follows. They start with the exceptional lattice in dimension 24 called the Leech lattice $\Lambda$. This is a special embedding of $\mathbb{Z}^{24}$ into $\mathbb{R}^{24}$, and the inner product of any two vectors in the image is always an even integer. (There is a deep theory on such lattices. See [22] for example.) Then they have a general construction of a vertex operator algebra $V_\Lambda$ for this lattice $\Lambda$. Physically speaking, it corresponds to a theory of strings living on $\mathbb{R}^{24}/\Lambda$. The involution sending $x \to -x$ on $\Lambda$ induces an automorphism of $V_\Lambda$ of order 2. We take a fixed point vertex operator algebra of this automorphism, then it turns out that this has a nontrivial extension, and this extended vertex operator algebra is the Moonshine vertex operator algebra $V^\natural$. (This extension at the last step is given by the simple current extension.) This construction is called a twisted orbifold construction, where the name “orbifold” refers to the fixed points of an action of a finite group.

Miyamoto [68] has a new construction of $V^\natural$ based on the fact that it is an extension of the 48th tensor power of the Virasoro vertex operator algebra $L(1/2, 0)$, which was found by [27]. This kind of extension of tensor powers of the Virasoro vertex operator algebra $L(1/2, 0)$ has been studied in the name of framed vertex operator algebra by [26].

We have constructed an operator algebraic counterpart $A^\natural$ of the Moonshine vertex operator algebra based on this idea of framed vertex operator algebra in [58] as follows.
The Virasoro vertex operator algebra $L(1/2,0)$ has a direct counterpart $\text{Vir}_{c=1/2}$ as a local conformal net. The representation theory of $\text{Vir}_{c=1/2}^{\otimes k}$ is well-understood, so we can make their extensions as local conformal nets. In this way, we can construct $\mathcal{A}^2$ naturally. (Dong-Xu [28] has a general construction of local conformal nets from lattices, as a counterpart of general lattice vertex operator algebras.)

The Hilbert space on which the local conformal net $\mathcal{A}^2$ acts is simply a Hilbert space completion of $V^2$ with its natural positive definite inner product, and the Virasoro algebra has a unitary representation with $c = 24$ on it. The Virasoro generator $L_0$ is the generator of the rotation group and it has the eigenspace decomposition $H = \bigoplus_{n \geq 0} H_n$ where $H_n$ is the eigenspace of $L_0$ with eigenvalue $n$.

The group of all gauge unitary operators is the automorphism group of the net $\mathcal{A}$ and it is also called the gauge group of the net $\mathcal{A}$. It is always a compact group. Such a unitary operator $u$ automatically commutes with the action of the Möbius group, so in particular, it preserves the decomposition $H = \bigoplus_{n \geq 0} H_n$. (Note that such a unitary operator automatically acts on the Virasoro subnet $\text{Vir}_r$ trivially if the local conformal net is strongly additive by [19], which is the case we are mainly interested.) So the McKay-Thompson series for the local conformal net $\mathcal{A}^2$ are identified with those for the vertex operator algebra $V^2$.

Now we have to prove that the automorphism group of $V^2$ and that of $\mathcal{A}^2$ are identified. It is easy to see that the former is contained in the latter, but the converse inclusion is nontrivial. The vertex operator corresponding to the conformal element is called the stress-energy tensor, and it is naturally interpreted as an operator-valued distribution on the circle. In the case of $V^2$ as an extension of $L(1,2,0) \otimes \mathbb{C}^{48}$, we have 48 copies of such stress-energy tensors with $c = 1/2$ and their automorphic images under the Monster group action are also nice operator-valued distributions, since each such automorphism is a unitary operator acting on our Hilbert space. In this way, we have “sufficiently many” operator-valued distributions, and from this fact, we can prove that each automorphism of the net $\mathcal{A}^2$ indeed arises from an automorphism of the vertex operator algebra $V^2$ as in [58, Theorem 5.4].

We now discuss other finite simple groups. Among the 26 sporadic finite simple groups, we have three groups with Conway’s name attached. They are $Co_1$, $Co_2$, $Co_3$ and $Co_1$ has the largest order, around $4.2 \times 10^{18}$. It is isomorphic to the automorphism group of the Leech lattice divided by its center of order 2. Duncan constructed an “super” analogue of the Moonshine vertex operator algebra and showed that its automorphism group, as an $N = 1$ supervertex operator algebra, is this group $Co_1$ in [31]. We now present its operator algebraic counterpart. We study this object within a general operator algebraic framework for super conformal field theory.

Duncan considered an $N = 1$ supervertex operator algebra in this setting. An automorphism of a supervertex operator algebra fixing the superconformal element is said to be an automorphism of an $N = 1$ supervertex operator algebra.

Duncan constructed two $N = 1$ supervertex operator algebras $A^{f_1}$ and $V^{f_1}$, and showed they are isomorphic in [31]. He then showed its group of automorphisms of $N = 1$ supervertex operator algebra is Conway’s group $Co_1$. Its character $\text{Tr}(q^{\Delta_0-c/24})$
is
\[ q^{-1/2} + 276q - 1/2 + 2048q + \cdots = \frac{\theta_{E_8} \tau \eta(\tau)}{\eta(\tau/2)^8 \eta(2\tau)^8} - 8, \] (5)
where \( q = \exp(2\pi i \tau), \) \( \text{Im} \, \tau > 0. \) The construction of \( V^{f_2} \) is a twisted \( \mathbb{Z}_2 \)-orbifold construction from a \( N = 1 \) supervertex operator algebra arising from the lattice \( \mathbb{Z}^4 \oplus E_8 \) as in discussions after Theorem 6.1 in [31]. It is an extension of \( L(1/2, 0)^{24} \), where \( L(1/2, 0) \) is the Virasoro vertex operator algebra with \( c = 1/2. \) We first have an analogue of [58, Lemma 5.1]. That is, we first consider a vertex operator subalgebra of \( V^{f_2} \) generated by \( g(L(1/2, 0)^{24}) \) for all \( g \in \text{Co}_1. \) Then it turns out that this is the even part of the supervertex operator algebra \( V^{f_2}. \) We next consider the supervertex operator algebra generated by the even part of the supervertex operator algebra \( V^{f_2} \) and its superconformal element. Since the even part is a fixed point of an automorphism of order 2, the Galois correspondence shows that this supervertex operator algebra must be equal to \( V^{f_2} \) itself.

The representation theories of the vertex operator algebra \( L(1/2, 0) \) and the local conformal net \( \text{Vir}_{1/2} \) are identified on the level of Hilbert spaces as in [58, Section 3] based on [79]. Let \( H \) be the Hilbert space completion of \( V^{f_2} \) with respect to the natural inner product on the extension of \( L(1/2, 0)^{24}. \) Then \( k \)th stress-energy tensor \( T_k(z), \) \( k = 1, 2, \ldots, 24, \) with \( c = 1/2 \) acts on \( H \) as in the arguments after Lemma 5.1 in [58]. Let \( G(z) \) be the superstress-energy tensor with \( c = 12 \) arising from the superconformal element. As in [58, Lemma 5.2], [18, Section 6], [17, Sections 4–5], the family of Wightman fields
\[ \{gT_k(z) \mid g \in \text{Co}_1, k = 1, 2, \ldots, 24\} \cup \{G(z)\} \]
are strongly graded local, where the definition of strong locality in [58, Section 5] is extended to the strongly graded local case. Note that each \( g \in \text{Co}_1 \) gives a unitary operator on \( H \) and \( gG(z)g^{-1} = G(z) \) by the definition of the automorphism group of an \( N = 1 \) supervertex operator algebra.

Now as in [58, Lemma 5.2], we have a graded local net \( \{A^{f_2}(I)\} \) having subnet \( \{SVir_{c=12}(I)\}. \) As in [58, Theorem 5.4], we can show that the automorphism group of the graded local net \( \{SVir_{c=12}(I)\} \) and the automorphism group of the supervertex operator algebra \( V^{f_2} \) leaving the natural inner product invariant are identified. From the above construction, the subgroup of the latter fixing the superconformal element is identified with the subgroup of the former fixing the subnet \( \{SVir_{c=12}(I)\} \) pointwise. (Each element of \( \text{Co}_1 \) fixes the natural inner product of \( V^{f_2} \) by the construction [31].)

**Theorem 4.2.** The superconformal net \( A^{f_2} \) constructed above is a completely rational graded local net with \( c = 12 \) having \( \text{SVir}_{c=12} \) as a subnet. Its character is given by (5) and the group of automorphisms of \( A^{f_2} \) fixing \( \text{SVir}_{c=12} \) pointwise is Conway’s group \( \text{Co}_1. \)

Note that the McKay-Thompson series for each element in \( \text{Co}_1 \) has been computed by Duncan in [31, Section 7], and we have the same series in the operator algebraic approach.
A similar structure has been pursued for other sporadic finite simple groups also by Duncan [32]. It is known that 20 of the 26 sporadic finite simple groups are “involved” in the Monster in the sense that they are quotients of subgroups of the Monster group. The Conway groups $Co_1, Co_2, Co_3$ are among these 20, and the other six are called “pariah” groups. One of the “pariah” groups is the Rudvalis group $Ru$, and its order is around $1.5 \times 10^{11}$. It is closely related to the Conway-Wales lattice of rank 28 over $\mathbb{Z}[i]$.

Duncan constructed two supervertex operator algebras with automorphic actions of the Rudvalis group with certain analogue of the Moonshine property on two-variable power series arising from group elements of the Rudvalis group in [32]. We now construct an operator algebraic counterpart for one of the two. The other supervertex operator algebra of Duncan in [32] has no unitarity, so it has no operator algebraic counterpart. (That is, we do not have a positive definite inner product, so we cannot construct a Hilbert space from the very beginning.)

Duncan [32] has an “enhanced supervertex operator algebra” $A_{Ru}$. Now we first ignore the “enhanced structure”, then it is simply an $N = 1$ supervertex operator algebra with $c = 28$ containing $L(1/2,0)^{\otimes 56}$ as in the above case of $A^{f\natural}$. The above machinery to construct a graded local net from 56 copies of stress-energy tensors with $c = 1/2$ and their automorphic images under the action of the Rudvalis group together with a single super stress-energy tensor with $c = 28$ produces a completely rational superconformal net $A_{Ru}$, with the Rudvalis group $Ru$ acting as the automorphisms fixing the $N = 1$ super Virasoro subnet $\{SVir_{c=28}(I)\}$ elementwise. Let $\{\mathcal{B}(I)\}$ be the fixed point net of $\{A_{Ru}(I)\}$ with the action of the Rudvalis group. Then by the classical Galois correspondence, the group of automorphisms of $\{A_{Ru}(I)\}$ fixing $\{\mathcal{B}(I)\}$ elementwise is the Rudvalis group. The subnet $\{\mathcal{B}(I)\}$ should correspond to the mysterious enhanced structure in [32], but the meaning is not understood well yet.

The analogue of the Moonshine property of Duncan [32] makes sense together with $\forall_{Ru}$, but this supervertex operator algebra does not have unitarity, so the operator algebraic interpretation of this property is rather incomplete, unfortunately.

In the above constructions, we have dealt with separate groups separately. From traditional ideas in operator algebras, all amenable objects should have some unified constructions. In our setting, the groups are finite, so they are amenable, needless to say, and von Neumann algebras are amenable, that is, injective if we have a split property which is known to hold in all the above examples. So we expect some uniform construction which works for all finite groups at once, and it would give a much deeper understanding on vertex operator algebras, but such a construction seems far from today’s understanding, unfortunately.

5 Subfactor theory

Now we discuss relations of the above framework to general theory of subfactors. Jones initiated his theory of subfactors [52] first for type $II_1$ factors. Kosaki [61]
extended it to arbitrary factors, and Longo [62] showed that the statistical dimension of a superselection sector in the Doplicher-Haag-Roberts theory [29] is identified with the square root of the Jones index for the image of an endomorphism of a type III factor. This is how subfactor theory is related to quantum field theory, and a great deal of interactions have been worked our over many years. Here we make a quick review on classification theory.

By Popa’s deep analytic results in [71], classification of subfactors of the hyperfinite II_1 factor \( M \) with finite Jones index is reduced to classification of certain representation theoretic invariants if we have certain amenability condition on the subfactor. The case of finite depth, where we have certain finiteness of irreducible objects in a tensor category of representations, has caught much attention. This gives a special case of amenable subfactors, and roughly similar to the rational case of a conformal field theory and related to the theory of quantum groups at roots of unity. Such representation theoretic data are characterized by various methods such as Ocneanu’s paragroup [69] and Jones’ planar algebras [53]. A fundamental invariant is the principal graph of a subfactor, which is a finite graph for the case of finite depth.

From the beginning of the subfactor theory [52], it has been known that the index value 4 is the first special value. Classification of subfactors with index less than 4 was found by Ocneanu [69] and the case with index equal to 4 was also worked out by various people. If the index is less than 4, the principal graph must be one of the \( A_n-D_{2n}-E_{6,8} \) Dynkin diagrams. For each of \( A_n \) and \( D_{2n} \) graphs, we have a unique subfactor, and for each of \( E_6 \) and \( E_8 \), we have two subfactors. We refer to [34] for details.

Haagerup considered a problem of listing subfactors up to index \( 3 + \sqrt{3} \) in [49]. Up to this index value, if a subfactor does not have a finite depth, then the principal graph must be \( A_\infty \). He gave a list of countable graphs, and showed that if we have a subfactor with index value in \( (4, 3 + \sqrt{3}) \), then the principal graph must be one in the list. He and Asaeda gave realization of two in the list in [2]. One infinite series in [49] were shown to be impossible in Bisch [9], and we had no progress on the remaining cases for some years.

Then Etingof, Nikshych and Ostrik [33, Theorem 8.51] proved that the Jones index of a subfactor with finite depth must be a cyclotomic integer. That is, the index value is an algebraic integer contained in the field \( \mathbb{Q} (\zeta) \), where \( \zeta \) is some root of unity. Asaeda and Yasuda [1, 3] proved that this kills all of the remaining graphs in the Haagerup list [49] except for one. This final remaining case has been realized recently by Bigelow, Morrison, Peters and Snyder [5].

The proof of Etingof, Nikshych and Ostrik [33] is not easy to understand for operator algebraists, so here we present a version of their proof in a style more familiar to operator algebraists.

The starting point is the following result of Coste-Gannon [23], where \( N_{ij}^k \) is a nonnegative integer defined by the Verlinde formula

\[
N_{ij}^k = \sum_l \frac{S_{i l} S_{j l} S_{k l}}{S_{0 l}}
\]

(6)
Theorem 5.1 (Coste-Gannon). Let \((S_{ij})_{i,j=0,1,...,m}\) be a symmetric unitary matrix with the following properties.

\[
N^k_{ij} \equiv \sum_l S_{il} S_{jl} \tilde{S}_{kl} / S_{0l}
\]

is rational for all \(i, j, k\),

\[S_{0j} > 0, \quad \text{for all } j.\]

Then we have a cyclotomic field \(F\) containing all \(S_{ij}\).

The proof of the above result actually shows commutativity of the Galois group for the Galois extension of \(\mathbb{Q}\) containing all \(S_{ij}\). Then the classical Kronecker-Weber theorem gives that the Galois extension is contained in some cyclotomic field. (A proof for this is also included in the appendix of [33].)

We now start a proof of the statement that the Jones index is a cyclotomic integer, if we have a finite depth. Let \(N \subset M\) be a subfactor with finite index and finite depth. We may assume that \(N\) and \(M\) are of type III, by tensoring a common type III factor if necessary. (This is not essential. The following arguments can be easily translated into the bimodule language for type II\(_1\) subfactors.) Suppose that \(\{\rho_i\}_{i=1}^n\) is a system of irreducible endomorphisms of \(M\) arising from the subfactor \(N \subset M\). Then we obtain the Longo-Rehren subfactor \(\mathbb{D}_{(\rho_i)}\), \(M \otimes M^{\text{opp}} \subset R\), where we have \(\bigoplus_{i=1}^n \rho_i \otimes \rho_i^{\text{opp}}\) as the dual canonical endomorphism for this subfactor. This is a “quantum double subfactor” for the system \(\{\rho_i\}_{i=1}^n\) and we follow the description in [50]. Note that we have a system \(\{\lambda_k\}_{k=1}^m\) of irreducible endomorphisms of \(R\) arising from \(\{\rho_i \otimes \rho_i^{\text{opp}}\}_{i,j}\) and the subfactor \(M \otimes M^{\text{opp}} \subset R\) as in [50] Section 4. By [50] Theorem 5.5, the system \(\{\lambda_k\}_{k=1}^m\) gives a modular tensor category and we have the Verlinde formula as above by [73]. Then by the above theorem of Coste-Gannon, we have a cyclotomic field \(F\) which contains all \(S_{ij}\) arising from the system \(\{\lambda_k\}_{k=1}^m\). In particular, we have \(d(\lambda_k) = S_{0k} / S_{00} \in F\) for all \(k\), where \(d(\lambda_k)\) stands for the statistical dimension, which is equal to \(\left[ R : \lambda_k(R) \right]^{1/2}\). Let \(\iota\) be the inclusion map for the subfactor \(M \otimes M^{\text{opp}} \subset R\). Denote the index value \(\left[ R : M \otimes M^{\text{opp}} \right]\) by \(w\). Note that \(w\) is a sum of some \(d(\lambda_k)\)'s with multiplicity, so it is in the field \(F\). Then for any \(j\), we have a decomposition \([\iota(\rho_j \otimes \text{id})\tilde{\iota}] = \bigoplus l_{jk} \lambda_k\), where \(l_{jk}\) is the multiplicity. Then we have

\[
d(\rho_j) = d(\rho_j \otimes \text{id}) = \frac{\sum_k l_{jk} d(\lambda_k)}{w} \in F.
\]

Now the canonical endomorphism \(\gamma_M\) for the subfactor \(N \subset M\) decomposes as \(\bigoplus n_i \rho_i\), where \(n_i\) is the multiplicity. Then we have

\[
[M : N] = \sum n_i d(\rho_i) \in F,
\]

which gives the desired conclusion.
6 Noncommutative geometry

Now we discuss relations to noncommutative geometry of Connes [20].

A commutative unital $C^*$-algebra is isomorphic to $C(X)$ where $X$ is a compact Hausdorff space. So a general $C^*$-algebra is regarded as a noncommutative analogue of a compact Hausdorff space, but in order to study geometry, we need more structure than just a compact Hausdorff space.

Let $M$ be a closed Riemannian manifold. From the $C^*$-algebra $C(M)$, we can recover $M$ only as a topological space, so even in the commutative case, we need additional structures. If the manifold has an extra structure called a spin structure, we have a spinor bundle on $M$, and its $L^2$-sections give a Hilbert space $H$. The smooth function algebra $C^\infty(M)$ acts on this Hilbert by a pointwise multiplication, and we have an unbounded self-adjoint operator $D$ on this $H$, called the Dirac operator, which is a kind of a “square root” of the Laplacian on $M$. From the triple $(C^\infty(M), H, D)$, we can recover complete geometric information on $M$. From these, the Connes axiomatization of a noncommutative compact Riemannian spin manifold is given as a triple $(A, H, D)$ of a $\ast$-subalgebra $A$ of $B(H)$ for a Hilbert space and a self-adjoint operator $D$ on $H$. Such a triple is called a spectral triple.

1. All the resolvents of $D$ are compact operators.

2. We have $[D, a] \in B(H)$ for all $a \in A$.

The commutator $[D, a]$ has a domain naturally, and we mean that it has a bounded extension.

If a spectral triple arises from a compact Riemannian spin manifold as above, then the condition that the dimension of the manifold is less than $p$ is expressed in terms of the eigenvalues of the Laplacian as the condition that the $n$th eigenvalue of $L_0$ is $O(n^{-1/p})$. We have an infinite dimensional version of this condition called $\theta$-summability which is defined by the condition $\text{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$.

Longo [64] suggested relations between superselection sectors in conformal field theory and elliptic operators. Conformal field theory should give an infinite dimensional version of a noncommutative manifold, because it has infinite degree of freedom. Also see [41] for connections of superconformal field theory and noncommutative geometry.

For an ordinary Riemannian manifold, the asymptotic behavior of $\text{Tr}(e^{-2\pi t \Delta})$ as $t \to 0^+$ contains geometric information on the manifold. In [57], we have pursued a similar study for the asymptotic behavior of $\log(\text{Tr} e^{-2\pi t L_0})$ for a local conformal net. These two asymptotic behaviors analogous, but note that we have “log” for the latter. This comes from the “infinite dimensionality” of our noncommutative structure. Anyway, here we have some correspondence between the Dirac operator $\Delta$ of a Riemannian manifold and the conformal Hamiltonian $L_0$ of a local conformal net. So we expect that the Dirac operator is somehow analogous to a square root of $L_0$. One of the Ramond relations [2] gives $G_0^2 = L_0 - c/24$, and $-c/24$ simply gives
a scalar in a representation, so if we have an $N = 1$ supersymmetry, we expect that the image of $G_0$ gives an analogue of the Dirac operator as a part of a spectral triple.

Based on this analogy, nets of spectral triples $(A(I), H, D)$ parametrized by intervals $I \subset S^1$ have been constructed in [17]. We first have a graded local net $\{A(I)\}$ first from a representation of the Ramond algebra on $H$, but we have to drop the axiom on the vacuum vector, since representations of the Ramond algebra do not have a vacuum vector. Then $G_0$ gives the “Dirac operator” on the same Hilbert space $H$. We now need a $\ast$-algebra for a spectral triple. We have a super derivation $\delta = [\cdot, D]$, where the bracket means the supercommutator. Let $\text{Dom}(\delta)$ be the set of operators $x \in B(H)$ with $\delta(x) \in B(H)$ in an appropriate sense. Then we have $C^\infty(\delta) = \bigcap_{n=1}^\infty \text{Dom}(\delta^n)$.

For each interval $I \subset S^1$, we can set $A(I) = A(I) \cap C^\infty(\delta)$, and we do obtain a net of spectral triples, but the problem is that this $A(I)$ may be too small, e.g., it may be that we have $A(I) = \mathbb{C}$. It has been actually shown in [17] that $A(I)$ is strongly dense in $\mathcal{A}(I)$ for each interval, so each $A(I)$ is certainly nontrivial.

Acknowledgements. The authors thanks John F. Duncan for explanations on his work, and Sebastiano Carpi and the referee for comments on this paper.

References

[1] M. Asaeda, Galois groups and an obstruction to principal graphs of subfactors, Internat. J. Math. 18 (2007), 191–202.

[2] M. Asaeda, & U. Haagerup, Exotic subfactors of finite depth with Jones indices $(5 + \sqrt{13})/2$ and $(5 + \sqrt{17})/2$, Commun. Math. Phys. 202 (1999), 1–63.

[3] M. Asaeda, & S. Yasuda, On Haagerup's list of potential principal graphs of subfactors, Commun. Math. Phys. 286 (2009), 1141–1157.

[4] A. A. Belavin, A. M. Polyakov & A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. 241 (1984), 333–380.

[5] S. Bigelow, S. Morrison, E. Peters & N. Snyder, Constructing the extended Haagerup planar algebra, arXiv:0909.4099

[6] J. Böckenhauer & D. E. Evans, Modular invariants, graphs and $\alpha$-induction for nets of subfactors, I Commun. Math. Phys. 197 (1998), 361–386; II 200 (1999), 57–103; III 205 (1999), 183–228.

[7] J. Böckenhauer, D. E. Evans & Y. Kawahigashi, On $\alpha$-induction, chiral projectors and modular invariants for subfactors, Commun. Math. Phys. 208 (1999), 429–487.

[8] J. Böckenhauer, D. E. Evans & Y. Kawahigashi, Chiral structure of modular invariants for subfactors, Commun. Math. Phys. 210 (2000), 733–784.

[9] D. Bisch, Principal graphs of subfactors with small Jones index, Math. Ann. 311 (1998), 223–231.

[10] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, Invent. Math. 109 (1992), 405–444.

[11] R. Brunetti, D. Guido & R. Longo, Modular structure and duality in conformal quantum field theory, Commun. Math. Phys. 156 (1993), 201–219.
[12] D. Buchholz & H. Grundling, *Algebraic supersymmetry: A case study*, Commun. Math. Phys. 272 (2007), 699-750.

[13] D. Buchholz, G. Mack & I. Todorov, *The current algebra on the circle as a germ of local field theories*, Nucl. Phys. B, Proc. Suppl. 5B (1988), 20–56.

[14] D. Buchholz & H. Schulz-Mirbach, *Haag duality in conformal quantum field theory*, Rev. Math. Phys. 2 (1990), 105–125.

[15] A. Cappelli, *Modular invariant partition functions of superconformal theories*, Phys. Lett. B 185 (1987), 82–88.

[16] A. Cappelli, C. Itzykson & J.-B. Zuber, *The A-D-E classification of minimal and $A_1^{(1)}$ conformal invariant theories*, Commun. Math. Phys. 113 (1987), 1–26.

[17] S. Carpi, R. Hillier, Y. Kawahigashi & R. Longo, *Spectral triples and the super-Virasoro algebra*, preprint 2008. arXiv:0811.4128.

[18] S. Carpi, Y. Kawahigashi & R. Longo, *Structure and classification of superconformal nets*, Ann. Henri Poincaré. 9, 1069–1121 (2008).

[19] S. Carpi & M. Weiner, *On the uniqueness of diffeomorphism symmetry in conformal field theory*, Comm. Math. Phys. 258 (2005), 203–221.

[20] A. Connes, “Noncommutative Geometry” Academic Press (1994).

[21] J. H. Conway & S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. 11 (1979), 308–339.

[22] J. H. Conway & N. J. A. Sloane, “Sphere packings, lattices and groups” (third edition), Springer (1998).

[23] A. Coste, T. Gannon, *Remarks on Galois symmetry in rational conformal field theories*, Phys. Lett. B 323 (1994), 316–321.

[24] P. Di Francesco, P. Mathieu & D. Sénéchal, “Conformal Field Theory”, Springer (1996).

[25] C. D’Antoni, R. Longo & F. Radulescu, *Conformal nets, maximal temperature and models from free probability*, J. Operator Theory 45 (2001), 195–208.

[26] C. Dong, R. L. Griess, Jr. & G. Höhn, *Framed vertex operator algebras, codes and the Moonshine module*, Comm. Math. Phys. 193 (1998), 407–448.

[27] C. Dong, G. Mason & Y. Zhu, *Discrete series of the Virasoro algebra and the moonshine module*, Proc. Symp. Pure. Math., Amer. Math. Soc. 56 II (1994), 295–316.

[28] C. Dong & F. Xu, *Conformal nets associated with lattices and their orbifolds*, Adv. Math. 206 (2006), 279–306.

[29] S. Doplicher, R. Haag & J. E. Roberts, *Local observables and particle statistics*, I. Commun. Math. Phys. 23 (1971), 199–230; II. 35 (1974), 49–85.

[30] S. Doplicher & R. Longo, *Standard and split inclusions of von Neumann algebras*, Invent. Math. 73 (1984) 493–536.

[31] J. F. Duncan, *Super-moonshine for Conway’s largest sporadic group*, Duke Math. J. 139 (2007), 255–315.

[32] J. F. Duncan, *Moonshine for Rudvalis’s sporadic group I, math.RT/0609449 II, math.RT/0611355*

[33] P. Etingof, D. Nikshych, V. Ostrik, *On fusion categories*, Ann. of Math. 162 (2005) 581–642.
[34] D. E. Evans & Y. Kawahigashi, “Quantum Symmetries on Operator Algebras”, Oxford University Press, Oxford (1998).

[35] K. Fredenhagen & M. Jörß, Conformal Haag-Kastler nets, pointlike localized fields and the existence of operator product expansion, Commun. Math. Phys. 176 (1996), 541–554.

[36] K. Fredenhagen, K.-H. Rehren & B. Schroer, Superselection sectors with braid group statistics and exchange algebras, I. Commun. Math. Phys. 125 (1989), 201–226; II. Rev. Math. Phys. Special issue (1992), 113–157.

[37] I. Frenkel, J. Lepowsky & A. Meurman, “Vertex operator algebras and the Monster”, Academic Press (1988).

[38] D. Friedan, Z. Qiu & S. Shenker, Details of the non-unitarity proof for highest weight representations of the Virasoro algebra, Commun. Math. Phys. 107 (1986), 535–542.

[39] J. Fuchs, I. Runkel & C. Schweigert, Twenty-five years of two-dimensional rational conformal field theory, arXiv:0910.3145.

[40] J. Fröhlich & F. Gabbiani, Operator algebras and conformal field theory, Commun. Math. Phys. 155 (1993), 569–640.

[41] J. Fröhlich & K. Gawedzki, Conformal field theory and geometry of strings, in “Mathematical quantum theory. I. Field theory and many-body theory (Vancouver, BC, 1993)”, 57–97, CRM Proc. Lecture Notes, 7, Amer. Math. Soc. 1994.

[42] T. Gannon, “Moonshine Beyond The Monster: The Bridge Connecting Algebra, Modular Forms And Physics”, Cambridge University Press (2006).

[43] P. Goddard, A. Kent & D. Olive, Unitary representations of the Virasoro and super-Virasoro algebras, Commun. Math. Phys. 103 (1986) 105–119.

[44] R. Goodman & N. R. Wallach, Projective unitary positive-energy representations of Diff(S^1), J. Funct. Anal. 63 (1985), 299–321.

[45] R. L. Griess, Jr., The friendly giant, Invent. Math. 69 (1982), 1–102.

[46] Y.-Z. Huang, Vertex operator algebras and the Verlinde conjecture, Commun. Contemp. Math. 10 (2008), 103–154.

[47] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, Commun. Contemp. Math. 10 (2008), 871–911.

[48] R. Haag, “Local Quantum Physics”, Springer (1996).

[49] U. Haagerup, Principal graphs of subfactors in the index range $4 < 3 + \sqrt{2}$, in Subfactors — Proceedings of the Taniguchi Symposium, Katata —, (ed. H. Araki, et al.), World Scientific (1994), 1–38.

[50] M. Izumi, The structure of sectors associated with the Longo-Rehren inclusions, Commun. Math. Phys. 213 (2000) 127–179.

[51] M. Izumi, R. Longo & S. Popa A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras, J. Funct. Anal. 10 (1998), 25–63.

[52] V. F. R. Jones, Index for subfactors, Invent. Math. 72 (1983), 1–25.

[53] V. F. R. Jones, Planar algebras I, to appear in New Zealand J. Math., arXiv:math.QA/9909027.

[54] V. Kac, “Vertex algebras for beginners” (Second edition), University Lecture Series, 10, American Mathematical Society (1998).

[55] Y. Kawahigashi & R. Longo, Classification of local conformal nets. Case $c < 1$, Ann. of Math. 160 (2004), 493–522.
Y. Kawahigashi & R. Longo, *Classification of two-dimensional local conformal nets with $c < 1$ and 2-cohomology vanishing for tensor categories*, Commun. Math. Phys. 244 (2004), 63–97.

Y. Kawahigashi & R. Longo, *Noncommutative spectral invariants and black hole entropy*, Commun. Math. Phys. 257 (2005), 193–225.

Y. Kawahigashi & R. Longo, *Local conformal nets arising from framed vertex operator algebras*, Adv. Math. 206 (2006), 729–751.

Y. Kawahigashi, R. Longo & M. Müger, *Multi-interval subfactors and modularity of representations in conformal field theory*, Commun. Math. Phys. 219 (2001), 631–669.

Y. Kawahigashi, R. Longo, U. Pennig, U & K.-H. Rehren, *The classification of non-local chiral CFT with $c < 1$*, Commun. Math. Phys. 271 (2007), 375–385.

H. Kosaki, *Extension of Jones’ theory on index to arbitrary factors*, J. Funct. Anal. 66 (1986), 123–140.

R. Longo, *Index of subfactors and statistics of quantum fields*, I. Commun. Math. Phys. 126 (1989), 217–247; II. 130 (1990), 285–309.

R. Longo, *A duality for Hopf algebras and for subfactors*, Commun. Math. Phys. 159 (1994), 133–150.

R. Longo, *Notes for a quantum index theorem*, Commun. Math. Phys. 222 (2001) 45-96.

R. Longo & F. Xu, *Topological sectors and a dichotomy in conformal field theory*, Commun. Math. Phys. 251 (2004), 321–364.

R. Longo & K.-H. Rehren, *Nets of subfactors*, Rev. Math. Phys. 7 (1995), 567–597.

R. Longo & K.-H. Rehren, *Local fields in boundary conformal QFT*, Rev. Math. Phys. 16 (2004), 909–960.

M. Miyamoto, *A new construction of the moonshine vertex operator algebra over the real number field*, Ann. of Math. 159 (2004), 535–596.

A. Ocneanu, *Quantized group, string algebras and Galois theory for algebras*, in *Operator algebras and applications, Vol. 2 (Warwick, 1987)*, (ed. D. E. Evans and M. Takesaki), London Mathematical Society Lecture Note Series 36, Cambridge University Press, Cambridge (1988), 119–172.

A. Ocneanu, *Paths on Coxeter diagrams: from Platonic solids and singularities to minimal models and subfactors*, (Notes recorded by S. Goto), in *Lectures on operator theory*, (ed. B. V. Rajarama Bhat et al.), The Fields Institute Monographs, AMS Publications (2000), 243–323.

S. Popa, *Classification of amenable subfactors of type II*, Acta Math. 172 (1994), 163–255.

S. Popa & S. Vaes, *Strong rigidity of generalized Bernoulli actions and computations of their symmetry groups*, Adv. Math. 217 (2008), 833–872.

K.-H. Rehren, *Braid group statistics and their superselection rules*, in “The algebraic theory of superselection sectors” (ed. D. Kastler), Palermo 1989, Singapore, World Scientific (1990), 333–355.

R. F. Streater & A. S. Wightman, “PCT, spin and statistics, and all that”, Princeton Landmarks in Physics, Princeton University Press (2000).

V. Toledano Laredo, *Integrating unitary representations of infinite-dimensional Lie groups*, J. Funct. Anal. 161 (1999), 478–508.

A. Wassermann, *Operator algebras and conformal field theory III: Fusion of positive energy representations of SU(N) using bounded operators*, Invent. Math. 133 (1998), 467–538.
[77] F. Xu, New braided endomorphisms from conformal inclusions, Commun. Math. Phys. 192 (1998), 347–403.

[78] F. Xu, Jones-Wassermann subfactors for disconnected intervals, Commun. Contemp. Math. 2 (2000), 307–347.

[79] F. Xu, Algebraic coset conformal field theories I, Commun. Math. Phys. 211 (2000), 1–44.

[80] F. Xu, Algebraic orbifold conformal field theories, Proc. Nat. Acad. Sci. U.S.A. 97 (2000) 14069–14073.

[81] F. Xu, Mirror extensions of local nets, Commun. Math. Phys. 270 (2007) 835-847.

[82] Y. Zhu, Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc. 9 (1996) 237–302.