Exact Solution for Chameleon Field, 
Self-Coupled Through the Ratra–Peebles Potential with $n = 1$
and Confined Between Two Parallel Plates

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We calculate the chameleon field profile, confined between two parallel plates, filled with air at pressure $P = 10^4$ mbar and room temperature and separated by the distance $L$, in the chameleon field theory with Ratra–Peebles self–interaction potential with index $n = 1$. We give the exact analytical solution in terms of Jacobian elliptic functions, depending on the mass density of the ambient matter. The obtained analytical solution can be used in qBounce experiments, measuring transition frequencies between quantum gravitational states of ultracold neutrons and also for the calculation of the chameleon field induced Casimir force for the CANNEX experiment. We show that the chameleon–matter interactions with coupling constants $\beta \leq 10^4$ can be probed by qBounce experiments with sensitivities $\Delta E \leq 10^{-18}$ eV. At $L = 30.1 \mu m$ we reproduce the result $\beta < 5.8 \times 10^8$, obtained by Jenke et al. Phys. Rev. Lett. 112, 151105 (2014) at sensitivity $\Delta E \sim 10^{-14}$ eV. In the vicinity of one of the plates our solution coincides with the solution, obtained by Brax and Pignol (Phys. Rev. Lett. 107, 111301 (2011)) (see also Ivanov et al. Phys. Rev. D 87, 105013 (2013)) above a plate at zero density of the ambient matter.

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I. INTRODUCTION

The chameleon field, changing its mass in dependence of the density of its environment [1, 2], has been invented to avoid the problem of the equivalence principle violation [3]. Nowadays it is accepted that the chameleon field, identified with quintessence [3, 4], i.e. a canonical scalar field, can be useful for the explanation of the late–time acceleration of the Universe expansion [7–10] (see, for example, [11]). In addition, the chameleon field may shed light on dark energy with quintessence [4–6], i.e. a canonical scalar field, can be useful for the explanation of the late–time acceleration of the Universe expansion [7–10] (see, for example, [11]). In this paper we give the analytical solution for the chameleon field, confined between two parallel plates, in the chameleon field theory with Ratra–Peebles self–interaction potential with index $n = 1$ [31, 32]. Unfortunately, there is still no plausible explanation why the chameleon field couples to neutrons as test particles, whereas to $^{133}$Cs test particles with $\beta < 10^4$. The new run of qBounce experiments with ultracold neutrons, using Gravity Resonance Spectroscopy and invented for measurements of the transitions frequencies between quantum gravitational states of ultracold neutrons bouncing above a mirror [33] should give new upper bounds on the chameleon–neutron couplings.

In this paper we give the analytical solution for the chameleon field, confined between two parallel plates, in the chameleon field theory with Ratra–Peebles self–interaction potential with index $n = 1$. Such a potential is very
popular in atom interferometry \cite{29,32}. In section II we write the equation of motion for the chameleon field and define the effective potential of the chameleon–matter and chameleon self–interactions. In section III we solve the equation of motion for the chameleon field profile in terms of Jacobian elliptic functions. In section IV we use the boundary conditions for the chameleon field and derive the equation for the definition of the parameter $\phi_0$, which is the maximum value of the chameleon field between two parallel plates. We make numerical calculations of the parameters of the chameleon field profile. In section V we calculate the contributions of the chameleon field profile, derived in section III, to the transition frequencies of quantum gravitational states of ultracold neutrons. In section VI we discuss the obtained results.

II. EQUATION OF MOTION FOR CHAMELEON FIELD, CONFINED BETWEEN TWO PLATES

In this section we search for the solution of the chameleon equation of motion for the chameleon field, confined between two parallel plates parallel the $(x,y)$ plane, separated by a length $L$ and localized at $z = 0$ and $z = L$, respectively. Following \cite{28} the corresponding equation of motion for the chameleon field $\phi(z)$ is given by

$$\frac{d^2 \phi}{dz^2} = \frac{\partial V_{\text{eff}}(\phi)}{\partial \phi},$$

where $V_{\text{eff}}(\phi)$ is the effective potential defined by

$$V_{\text{eff}}(\phi) = \frac{\Lambda^n}{\phi} + \beta \frac{\rho}{M_{P1}} \phi.$$ \hspace{1cm} (2)

Here $\Lambda^n/\phi$ is the Ratra–Peebles potential of the self–interaction of the chameleon field with index $n = 1$, whereas the term proportional to the matter density $\rho$ is the potential of the chameleon–matter interaction. Then, $\Lambda = \sqrt{3M_{P1}^2}H^2\Omega_M = 2.24(2) \times 10^{-3}$ eV is the dark energy scale parameter, $M_{P1} = 1/\sqrt{8\pi G_N} = 2.435 \times 10^{17}$ eV and $G_N$ are the reduced Planck mass and the Newtonian gravitational constant, respectively, and $\Omega_M \simeq 0.685$ is the relative dark energy density in the Universe \cite{34}.

III. SOLUTION TO EQUATION OF MOTION FOR CHAMELEON FIELD, CONFINED BETWEEN TWO PLATES

The solution of Eq. (1) we obtain from the first integral of Eq. (1) equal to

$$\frac{1}{2} \left( \frac{d\phi}{dz} \right)^2 = V_{\text{eff}}(\phi) - V_{\text{eff}}(\phi_0),$$

where $V_{\text{eff}}(\phi_0)$ is the integration constant, given by

$$V_{\text{eff}}(\phi_0) = \frac{\Lambda^n}{\phi_0} + \beta \frac{\rho}{M_{P1}} \phi_0.$$ \hspace{1cm} (4)
and \( \phi_0 \) is the value of the chameleon field at \( z = L/2 \), i.e. \( \phi(L/2) = \phi_0 \) and \( d\phi(z)/dz|_{z=L/2} = 0 \) \[28\]. The solution to Eq. \((3)\) we obtain in the form of the integral \[28\]

\[
\frac{\phi_0^{3/2}}{\Lambda^{5/2}} \int_{x}^{1} \frac{\sqrt{x'} \, dx'}{(1-x')(1-k^2x')} = \sqrt{2} \left( \frac{L}{2} - z \right),
\]

(5)

where \( x = \phi/\phi_0 \) and \( k^2 = \phi_0^2/\phi_0^2 \) with \( \phi_v \) equal to \( \phi_v = \sqrt{\Lambda^3 M_{\text{Pl}}/\beta \rho} \) \[28\]. Here \( \phi_v \) is a minimum of the chameleon field, which it can reach in the unrestricted space interval. Making a change of variables \( x' = \sin^2 \varphi \) we arrive at the expression

\[
\int_{\text{arcsin} \sqrt{x}}^{\pi/2} \frac{\sin^2 \varphi \, d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = \frac{2}{L_S} \left( \frac{L}{2} - z \right),
\]

(6)

where \( L_S = \sqrt{2} \phi_0^{3/2}/\Lambda^{5/2} \) is the slope parameter. The right–hand–side (r.h.s.) of Eq. \((6)\) we give in the form of the derivative with respect to the parameter \( k^2 \). Replacing then \( k^2 \to p^2 \) we get

\[
\frac{\partial}{\partial p^2} \int_{\text{arcsin} \sqrt{x}}^{\pi/2} \frac{1-p^2 \sin^2 \varphi \, d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = -\frac{1}{L_S} \left( \frac{L}{2} - z \right).
\]

(7)

Integrating Eq. \((7)\) over \( p^2 \) in the limits \( 0 \leq p^2 \leq k^2 \) we arrive at the expression

\[
\int_{\text{arcsin} \sqrt{x}}^{\pi/2} \sqrt{1-k^2 \sin^2 \varphi} \, d\varphi = \text{arccos} \sqrt{x} = -\frac{k^2}{L_S} \left( \frac{L}{2} - z \right).
\]

(8)

The integral over \( \varphi \) can be given in terms of the elliptic integral of the second kind \[35\]

\[
\int_{\text{arcsin} \sqrt{x}}^{\pi/2} \sqrt{1-k^2 \sin^2 \varphi} \, d\varphi = E\left( \frac{\pi}{2}, k \right) - E(\text{arcsin} \sqrt{x}, k).
\]

(9)

This allows to rewrite Eq. \((8)\) as follows

\[
E(\text{arcsin} \sqrt{x}, k) + \text{arccos} \sqrt{x} = E\left( \frac{\pi}{2}, k \right) + \frac{k^2}{L_S} \left( \frac{L}{2} - z \right).
\]

(10)

Then, we use the following properties of the elliptic integral of the second kind \[36\]

\[
E(u, k) = u - k^2 \int_{0}^{u} \sin^2(t) \, dt = u - k^2 \text{Sn}(u, k),
\]

(11)

where \( \text{sn}(t, k) \) is the Jacobian elliptic function \[35\]. Plugging Eq. \((11)\) into Eq. \((10)\) we get

\[
\text{Sn}\left( \text{arcsin} \sqrt{\frac{\phi(z)}{\phi_0}}, k \right) = \text{Sn}\left( \frac{\pi}{2}, k \right) - \frac{1}{L_S} \left( \frac{L}{2} - z \right),
\]

(12)

where we have set \( x = \phi(z)/\phi_0 \) (see Eq. \((9)\)). The chameleon field \( \phi(z) \) is given by

\[
\phi(z) = \phi_0 \sin^2 \left( \text{Sn}^{-1} \left\{ \left[ \text{Sn}\left( \frac{\pi}{2}, k \right) - \frac{1}{L_S} \left( \frac{L}{2} - z \right) \right], k \right\} \right),
\]

(13)

where \( \text{Sn}^{-1}(x, k) \) is the inverse function of the Jacobian elliptic function \( \text{Sn}(u, k) = x \). Since at \( z = L/2 \) we get \( \text{Sn}^{-1}(\text{Sn}(\pi/2, k), k) = \pi/2 \), the solution Eq. \((13)\) gives \( \phi(L/2) = \phi_0 \).

Practically, the solution Eq. \((13)\) is valid in the space interval \( 0 \leq z \leq L/2 \). For the definition of the solution valid in the space interval \( 0 \leq z \leq L \) we follow \[28\] and get

\[
\phi(z) = \phi_0 \sin^2 \left( \text{Sn}^{-1} \left\{ \left[ \text{Sn}\left( \frac{\pi}{2}, k \right) - \frac{L}{2L_S} \right] - 1 - \frac{2z}{L} \right\}, k \right), \quad 0 \leq z \leq L.
\]

(14)
Let us show that the solution Eq. (14) is continuous in the vicinity of $z = L/2$. For this aim we define the chameleon field as follows $\phi(z) = \phi_0 - \delta \phi(z)$, where $\delta \phi(z)$ is a small deviation of the chameleon field $\phi(z)$ from $\phi_0$ in a small vicinity of $z = L/2$ such as $\phi_0 \gg \delta \phi(z)$. Using then the expansion

$$\text{Sn} \left( \arcsin \sqrt{\frac{\phi(z)}{\phi_0}} \right) \rightarrow \text{Sn} \left( \arcsin \sqrt{1 - \frac{\delta \phi(z)}{\phi_0}} \right) = \text{Sn} \left( \frac{\pi}{2} k \right) - \text{Sn}^2 \left( \frac{\pi}{2} k \right) \sqrt{\frac{\delta \phi(z)}{\phi_0}}$$

and Eq. (12) we get

$$\delta \phi(z) = \frac{\phi_0}{L^2} \left( \frac{L}{2} - z \right)^2,$$

where $L_S = L_S \text{sn}^2(\frac{\pi}{2}, k)$. Thus, in the vicinity of $z = L/2$ the chameleon field profile is defined by the function

$$\phi(z) = \phi_0 \left( 1 - \frac{L^2}{4L^2} \left( 1 - \frac{2z}{L} \right)^2 \right).$$

This confirms a continuity of the chameleon field profile Eq. (14) in the vicinity of $z = L/2$. Now we may proceed to the definition of the parameter $\phi_0$. For this aim we have to use the boundary conditions.

**IV. BOUNDARY CONDITIONS AND DEFINITION OF $\phi_0$**

The main parameter of the solution Eq. (13) is $\phi_0$, which defines also $k = \phi_0/\phi_v$. Following [28] we determine it from boundary conditions:

$$\phi(z) \big|_{z=0-} = \phi(z) \big|_{z=0+} = \phi_b, \quad \phi(z) \big|_{z=L} = \phi(z) \big|_{z=L+} = \phi_b,$$

$$\left( \frac{d\phi}{dz} \right)^2 \big|_{z=0} = \left( \frac{d\phi}{dz} \right)^2 \big|_{z=L-} = \left( \frac{d\phi}{dz} \right)^2 \big|_{z=L+} = \left( \frac{d\phi}{dz} \right)^2 \big|_{z=L} = k^2.$$

Skipping intermediate calculations (see [28]) we arrive at the relation

$$\phi_b = \frac{2\phi_m - \rho}{\rho_m} \phi_0 - \frac{\phi_m^2}{\phi_0} = \frac{2\phi_m - \phi_v^2}{\phi_v} \phi_0 - \frac{\phi_m^2}{\phi_0},$$

where $\rho_m$ is the mass density of the plates and $\phi_v$ is the minimum of the chameleon field inside the plates $\phi_v = \sqrt{\Lambda^4 \rho M/\beta \rho_m}$. At $\rho \to 0$ or $\phi_v \to \infty$ Eq. (19) reduces to Eq. (39) of Ref. [28]. One more equation we obtain by using Eq. (13) at $z = 0$ (or Eq. (14) at $z = L$), which can be rewritten as follows

$$\text{Sn} \left( \arcsin \sqrt{\frac{\phi_b}{\phi_v}} \right) = \text{Sn} \left( \frac{\pi}{2} \frac{\phi_b}{\phi_v} \right) = \frac{\Lambda}{2v^2} \left( \frac{v^2}{\phi_v} \right)^{3/2} \left( \phi_v / \phi_0 \right)^{3/2}.$$

Plugging Eq. (19) into Eq. (20) we obtain the equation, which defines the parameter $\phi_0$ through the dark energy scale $\Lambda$, the spatial scale of the experimental setup, and matter densities $\rho$ and $\rho_m$ between two parallel plates and inside the plates, respectively. For numerical calculations we use $L = 20 \text{ cm}$, $\rho = 1.188 \times 10^{-10} \text{ g/cm}^3 = 5.123 \times 10^8 \text{ eV}^4$, corresponding the air density and pressure $P = 10^{-4} \text{ mbar}$, $\rho_m = 2.51 \text{ g/cm}^3 = 1.082 \times 10^{19} \text{ eV}^4$ [33], and $\Lambda = 2.24 \times 10^{-3} \text{ eV}$ and $\beta = 10^4$. However, in order to reproduce the result $\beta < 10^4$, obtained in [32], we have to use $L = 1 \text{ cm}$. The value of the chameleon–matter coupling constant $\beta = 10^4$ is chosen in agreement with recent results in the atom interferometry [51, 32]. Since for $\beta = 10^4$, $\Lambda = 2.24 \times 10^{-3} \text{ eV}$ and $\rho_m = 1.082 \times 10^{19} \text{ eV}^4$ we get $\phi_m = \sqrt{\Lambda^4 \rho M/\beta \rho_m} = 3.56 \times 10^{-5} \text{ eV}$, for the accepted parameters of the experimental setup [32] we may set $\phi_b = 0$. As a result, we arrive at the equation for the parameter $k$, given by

$$k^{3/2} \text{Sn} \left( \frac{\pi}{2} k \right) = \frac{\Lambda}{2v^2} \left( \frac{v^2}{\phi_v} \right)^{3/2}.$$

For $\beta = 10^4$, $\Lambda = 2.24 \times 10^{-3} \text{ eV}$ and $\rho = 5.123 \times 10^8 \text{ eV}^4$ we obtain $\phi_v = \sqrt{\Lambda^4 \rho M/\beta \rho} = 5.177 \text{ eV}$, and for $L = 20 \text{ cm}$ the numerical solution of Eq. (21) gives $k = 0.070$ and $\phi_0 = 0.360 \text{ eV}^4$.

In Fig. 4 we give the profile of the chameleon field, defined by Eq. (14), in the spatial region $0 \leq z \leq L = 20 \text{ cm}$ and calculated for $\beta = 10^4$, air density $\rho = 1.188 \times 10^{-10} \text{ g/cm}^3$ at pressure $P = 10^{-4} \text{ mbar}$ and room temperature.
FIG. 1: The profile of the chameleon field, defined by Eq. (14) in the spatial region $0 \leq z \leq L = 20 \text{ cm}$ between two parallel plates and calculated for $\beta = 10^4$, air density $\rho = 1.188 \times 10^{-10} \text{ g/cm}^3$ at pressure $P = 10^{-4} \text{ mbar}$ and room temperature.

V. TRANSITION FREQUENCIES BETWEEN QUANTUM GRAVITATIONAL STATES OF ULTRACOLD NEUTRONS

Because of a very large value $L = 20 \text{ cm}$ the wave functions of the quantum gravitational states of ultracold neutrons we may take in the form, corresponding to ultracold neutrons bouncing in the gravitational field of the Earth above a mirror 

$$
\psi_{k'}(z) = \frac{1}{\sqrt{\ell_0}} \frac{\text{Ai}(\xi_{k'} + \xi)}{\sqrt{\int_0^{\infty} |\text{Ai}(\xi_{k'} + \xi)|^2 d\xi}},
$$

(22)

where $\xi = z/\ell_0$, $\text{Ai}(x)$ is the Airy function, $\ell_0 = (2m^2 g)^{-1/3} = 5.87 \mu\text{m}$, $\xi_{k'}$ are the roots of the equation $\text{Ai}(\xi_{k'}) = 0$, caused by the boundary condition $\psi_{k'}(0) = 0$. The roots $\xi_{k'}$ define the energy spectrum of quantum gravitational states $E_{k'} = -mg\ell_0\xi_{k'}$ for $k' = 1, 2, \ldots$, where $mg\ell_0 = 0.602 \text{ peV}$. According \[33\], the transitions $|1\rangle \rightarrow |3\rangle$ and $|1\rangle \rightarrow |4\rangle$ have been only observed. The contributions $\delta\omega_{k'1}$ of the chameleon field to the transition frequencies of the transitions $|1\rangle \rightarrow |k'\rangle$ are equal to

$$
\delta\omega_{k'1} = \beta \frac{m}{M_{\text{Pl}}} \int_0^{L/\ell_0} d\xi \phi(\xi_{\ell_0}) \left( |\psi_{k'}(\xi)|^2 - |\psi_{1}(\xi)|^2 \right)
$$

(23)

for $k' = 3, 4$ \[33\], where $\xi = z/\ell_0$ and the chameleon field profile $\phi(\xi_{\ell_0})$ is given by Eq. (14). The numerical calculations, carried out with the chameleon field profile Eq. (14), give the following results: $\delta\omega_{31} = 3.72 \times 10^{-18} \text{ eV}$ and $\delta\omega_{41} = 4.98 \times 10^{-18} \text{ eV}$. Thus, the contribution of the chameleon field to the transition frequencies of quantum gravitational states of ultracold neutrons, bouncing above a mirror in the gravitational field of the Earth, with the chameleon–matter coupling constant $\beta = 10^4$ are at the level of sensitivities $\Delta E \sim 10^{-18} \text{ eV}$.

For the comparison we propose the following approximation. One may assert that because of the wave functions of the gravitational states of ultracold neutrons the contributions to the transition frequencies $\delta\omega_{31}$ and $\delta\omega_{41}$ come from the spatial region or order of a few micrometers. This means that we may use an approximation for the chameleon field profile in the vicinity of the lower plate. Assuming that $\phi_0 \gg \phi(z)$ we transform Eq. (14) into the form

$$
\phi(z) = \Lambda \left( \frac{3}{\sqrt{2}} \frac{\Lambda \ell_0}{\ell_0} \right)^{2/3} \left( \frac{z}{\ell_0} \right)^{2/3} = 6.078 \times 10^{-4} \left( \frac{z}{\ell_0} \right)^{2/3} \text{ eV},
$$

(24)

which agrees well with the results, obtained in \[27, 28\] for the chameleon field profile above a mirror in the space with density $\rho = 0$. Plugging Eq. (24) into Eq. (23) and integrating over the region $0 \leq \xi < \infty$ we obtain $\delta\omega_{31} = 2.34 \times 10^{-18} \text{ eV}$ and $\delta\omega_{41} = 3.14 \times 10^{-18} \text{ eV}$ at $\beta = 10^4$.

VI. ALLOWED REGION OF THE CHAMELEON–MATTER COUPLING CONSTANT $0 \leq \beta < \beta_{\text{max}}$ IN DEPENDENCE OF THE PARAMETERS OF THE EXPERIMENTAL SETUP

According to Eq. (3), the right–hand–side (r.h.s.) $V_{\text{eff}}(\phi) - V_{\text{eff}}(\phi_0)$ of Eq. (3) is not positive–definite. Since the chameleon field is a real scalar field, the integrand of Eq. (3) should be a real function. This implies that the parameter

$$
\Delta \phi = \sqrt{\frac{2}{\sqrt{2}} \frac{\Lambda \ell_0}{\ell_0} \left( \frac{z}{\ell_0} \right)^{2/3}}
$$

should be a real function. This implies that the parameter
\[ k^2 = \frac{\phi_0^2}{\phi_c^2} \] should be restricted from above by unity, i.e. \( k^2 < 1 \) or \( \phi_0 < \phi_c^2 \) (see Eqs. (6) - (9)). Together with the boundary condition Eq. (21) this gives the constraint on the allowed region of the chameleon–matter coupling constant \( \beta \)

\[ 0 \leq \beta < \beta_{\text{max}} = \frac{\Lambda^4 M_{\text{Pl}}}{\rho \phi_0^2} \]  

(25)

For the air density \( \rho = 1.188 \times 10^{-10} \text{g/cm}^3 \), corresponding to air density at pressure \( P = 10^{-4} \text{mbar} \) and room temperature, and \( \phi_0 = 0.277 \text{eV} \), caused by the solution of the boundary condition Eq. (21), we get \( \beta_{\text{max}} = 3.5 \times 10^6 \). For the Ratra–Peebles potential with index \( n \) the allowed region of the chameleon–matter coupling constant is restricted by

\[ 0 \leq \beta < \beta_{\text{max}} = \frac{\Lambda^{4+n} M_{\text{Pl}}}{\rho \phi_0^{n+1}}. \]  

(26)

We would like to note that \( \beta_{\text{max}} \) does not define the upper bound of the chameleon–matter coupling constant, but it defines a maximal value of the chameleon–matter coupling constant, which can be measured for given density \( \rho \) and the distance \( L \), caused by the experimental setup.

We would like to emphasize that the constraint on the maximal value of the chameleon–matter coupling constant exists only for the chameleon field, confined between two parallel infinitely large plates. In this case the second order differential equation of motion of the chameleon field possesses the first integral, having the shape of Eq. (3), where the r.h.s. is not positive–definite. For any 2D– or 3D–profile of the chameleon field, when the equation of motion

\[ \Delta \phi = \frac{\partial V_{\text{eff}}(\phi)}{\partial \phi}, \]  

(27)

where \( \Delta \) is the Laplacian in the 2D– or 3D–dimensional space, cannot be reduced to the form of Eq. (3), the constraint on the maximal value of the chameleon–matter coupling constant \( \beta_{\text{max}} \) does not exist. The latter concerns the results, obtained in the neutron interferometry by Brax et al. [24] and Lemmel et al. [22], where the 2D– and 3D–profiles of the chameleon field have been used for the analysis of the phase shift of the neutron wave function, respectively.

VII. CONCLUSION

We have found the exact solution for the chameleon field, confined between two parallel plates and described by the chameleon field theory with the Ratra–Peebles potential with index \( n = 1 \). Such a potential is very popular for the measurements of the chameleon–matter coupling constant in atom interferometry [29, 32]. We have applied the obtained chameleon field profile to the analysis of the transition frequencies of the quantum gravitational states of ultracold neutrons, bouncing above a mirror in the gravitational field of the Earth [33]. We have shown that in the vicinity of the plate at \( z = 0 \) our exact solution Eq. (14) reduces to the form

\[ \phi(z) = \Lambda(3\Lambda \phi_0)^{\frac{2}{3}}(z/\ell)_0^{\frac{2}{3}}, \]  

which agrees well with the chameleon field profile, calculated in [27, 28] for zero density of ambient matter. Then, we have shown that the contribution of the chameleon field profile to the transition frequencies between the states \( |1 \rangle \rightarrow |3 \rangle \) and \( |1 \rangle \rightarrow |4 \rangle \), caused by the chameleon–neutron interaction with the chameleon–matter coupling constant \( \beta \leq 10^4 \) [31, 32], can be observed only at the level of sensitivities \( \Delta E \leq 10^{-18} \text{eV} \).

Replacing \( L = 20 \text{cm} \) by \( L = 30.1 \mu\text{m} \) the obtained profile of the chameleon field Eq. (14) can be used for the qBounce experiments with ultracold neutrons, bouncing between two mirrors [25] and described by the wave functions [28], and the measurement of the Casimir force, caused by the chameleon field, in the CANNEX experiments [30].

We would like to emphasize that the allowed region \( 0 \leq \beta < \beta_{\text{max}} \) for a measurement of the chameleon–matter coupling constant \( \beta \) depends on the ambient matter density \( \rho \) and the distance \( L \) between two parallel plates.

For example, for the air density \( \rho = 1.188 \times 10^{-10} \text{g/cm}^3 \) at pressure \( P = 10^{-4} \text{mbar} \) and room temperature and the distance between two plates \( L = 30, 1 \mu\text{m} [23] \) the constraint \( k < 1 \) and the boundary condition Eq. (21) define the following allowed region of the chameleon–matter coupling constant \( 0 \leq \beta < \beta_{\text{max}} = 4 \times 10^{11} \). This does not contradict the result \( \beta < 5.8 \times 10^8 \), obtained by Jenke et al. [23]. For the chameleon–matter coupling constant \( \beta = 5.8 \times 10^8 \) and density \( \rho = 1.188 \times 10^{-10} \text{g/cm}^3 \) we get \( \phi_0 = 1.022 \times 10^{-3} \text{eV}, \phi_v = 2.150 \times 10^{-2} \text{eV} \) and \( k = \phi_0/\phi_v = 0.048 \). This gives the following contributions of the chameleon field to the transition frequencies \( \delta \omega_{31} = 5.20 \times 10^{-14} \text{eV} \) and \( \delta \omega_{41} = 6.67 \times 10^{-14} \text{eV} \), which agree well with the results obtained by Jenke et al. [23]. For the calculation of the transition frequencies \( \delta \omega_{31} = 5.20 \times 10^{-14} \text{eV} \) and \( \delta \omega_{41} = 6.67 \times 10^{-14} \text{eV} \) we have used the wave functions of the ultracold neutrons, confined between two parallel plates and defined in [28].
For $L = 100 \mu \text{m}$ and $\rho = 1.188 \times 10^{-10} \text{g/cm}^3$ we get the allowed region of the chameleon–matter coupling constants $0 \leq \beta < \beta_{\text{max}} = 8.5 \times 10^9$. Then, for $\beta = 10^7$ we obtain $\phi_0 = 2.271 \times 10^{-3} \text{eV}$, $\phi_v = 0.164 \text{eV}$ and $k = \phi_0/\phi_v = 0.014$. The contributions of the chameleon field to the transition frequencies are equal to $\delta \omega_{21} = -2.69 \times 10^{-17} \text{eV}$ and $\delta \omega_{41} = +5.45 \times 10^{-16} \text{eV}$. These results become observably at sensitivity $\Delta E < 10^{-17} \text{eV}$.\[18\]

For $L = 1 \text{cm}$ and $\rho = 1.188 \times 10^{-11} \text{g/cm}^3$ we get the allowed region of the chameleon–matter coupling constants $0 \leq \beta < \beta_{\text{max}} = 1.6 \times 10^5$. Then, for $\beta = 10^5$ we obtain $\phi_0 = 4.825 \times 10^{-2} \text{eV}$, $\phi_v = 0.164 \text{eV}$ and $k = \phi_0/\phi_v = 0.029$. The contributions of the chameleon field to the transition frequencies amount to $\delta \omega_{21} = 1.23 \times 10^{-15} \text{eV}$ and $\delta \omega_{41} = 1.65 \times 10^{-15} \text{eV}$. These results can be observed at sensitivity $\Delta E < 5 \times 10^{-15} \text{eV}$.\[33\]

Finally, we would like to mention the paper by Burrage, Copeland, and Stevenson\[40\], where the authors solve the problem of the chameleon field, confined between two parallel plates. Unlike our solution Eq. 14, calculated for a non-vanishing matter density $\rho \neq 0$ between two parallel plates, the solution Eq. 5 is calculated in\[10\] at zero matter density $\rho = 0$. It is important to emphasize that because of the constraint $k^2 = \phi_0^2/\phi_v^2 < 1$ together with Eq. 21, caused by boundary conditions, the account for a non-vanishing matter density between two parallel plates restricts from above $0 \leq \beta < \beta_{\text{max}}$ the values of the chameleon–matter coupling constant $\beta$. The coupling constant $\beta_{\text{max}}$ depends on a matter density and a distance between plates and characterizes a maximal chameleon–matter coupling constant, which can be measured for given experimental conditions.

VIII. ACKNOWLEDGEMENTS

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[1] J. Khoury and A. Weltman, Phys. Rev. Lett. 93, 171104 (2004); Phys. Rev. D 69, 044026 (2004).
[2] D. F. Mota and D. J. Shaw, Phys. Rev. D 75, 063501 (2007); Phys. Rev. Lett. 97, 151102 (2007).
[3] Ci. M. Will, in Theory and Experiment in Gravitational Physics, Cambridge University Press, Cambridge 1993.
[4] I. Zlatev, L. Wang, and P. J. Steinhardt, Phys. Rev. Lett. 82, 896 (1999); P. J. Steinhardt, L. Wang, and I. Zlatev, Phys. Rev. D 59, 123504 (1999).
[5] S. Tsujikawa, Class. Quantum Grav. 30, 214003 (2013).
[6] A. N. Ivanov and M. Wellenzohn, Can Chameleon Field be identified with Quintessence ?, arXiv: 1607.00884 [gr-qc].
[7] S. Perlmutter et al., Astron. Soc. 29, 1351 (1997).
[8] A. G. Riess et al., Astron. J. 116, 1009 (1998).
[9] S. Perlmutter et al., Astron. J. 517, 565 (1999).
[10] A. Goobar et al., Physica Scripta T 85, 47 (2000).
[11] Ph. Brax, C. van de Bruck, A.–Ch. Devis, J. Khoury, and A. Weltman, Phys. Rev. D 70, 123518 (2004).
[12] P. E. Peebles and Bh. Ratra, Rev. Mod. Phys. 75, 559 (2003).
[13] E. J. Copeland, M. Sami, S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006).
[14] J. A. Frieman, M. S. Turner, and Dr. Huterer, Annu. Rev. Astron. Astrophys. 46, 385 (2008).
[15] Bh. Jain et al., ArXiv:1309.3359 [astro-ph.CO].
[16] Ph. Brax and A.–C. Davis, Phys. Rev. D 91, 063503 (2015).
[17] G. Pignol, Int. J. Mod. Phys. A 30, 1530048 (2015).
[18] H. Abele, T. Jenke, H. Leeb, and J. Schmiedmayer, Phys.Rev. D 81, 065019 (2010).
[19] T. Jenke, P. Geltenbort, H. Lemmel, and H. Abele, Nature Physics 7, 468 (2011).
[20] H. Abele, T. Jenke, D. Stadler, and P. Geltenbort, Nucl. Phys. A 827, 593c (2009).
[21] T. Jenke, D. Stadler, H. Abele, and P. Geltenbort, Nucl. Instr. and Meth. in Physics Res. A 611, 318 (2009).
[22] H. Abele and H. Leeb, New J. Phys. 14, 055010 (2012).
[23] T. Jenke, G. Cronenberg, J. Bürgdorfer, L. A. Chizhova, P. Geltenbort, A. N. Ivanov, T. Lauer, T. Lins, S. Rotter, H. Saul, U. Schmidt, and H. Abele, Phys. Rev. Lett. 112, 151105 (2014).
[24] Ph. Brax, G. Pignol, and D. Roulier, Phys. Rev. D 88, 083004 (2013).
[25] H. Lemmel, Ph. Brax, A. N. Ivanov, T. Jenke, G. Pignol, M. Pitschmann, T. Potocar, M. Wellenzohn, M. Zawisky, and H. Abele, Phys. Lett. B 743, 310 (2015).
[26] K. Li, M. Arif, D. G. Cory, R. Haun, B. Heacock, M. G. Huber, J. Nsofni, D. A. Pushin, P. Sagg, D. Sarenac, C. B. Shahi, V. Skavysh, W. M. Snow, and A. R. Young, Phys. Rev. D 93, 062001 (2016).
[27] Ph. Brax and G. Pignol, Phys. Rev. Lett. 107, 111301 (2011).
[28] A. N. Ivanov, R. Höllwieser, T. Jenke, M. Wellenzohn, and H. Abele, Phys. Rev. D 87, 105013 (2013).
[29] C. Burrage, E. J. Copeland, and E. A. Hinds, JCAP 03, 042 (2015); [arXiv:1408.1309v3 [astro-ph.CO]].
[30] C. Burrage and E. J. Copeland, Contemporary Physics 57, 164 (2015); [arXiv:1507.07493 [astro-ph.CO]].
[31] P. Hamilton, M. Jaffe, Ph. Haslinger, Q. Simmons, H. Müller, and J. Khoury, Science 349, 849 (2015).
[32] B. Elder, J. Khoury, Ph. Haslinger, M. Jaffe, H. Müller, and P. Hamilton, Chameleon Dark Energy and Atom Interferometry e-Print: [arXiv:1603.06587 [astro-ph.CO]].
[33] G. Cronenberg, Invited talk at workshop on “Dark energy in the laboratory” held on 20th - 22nd of April 2016 at Chicheley Hall, home of the Kavli Royal Society International Centre, United Kingdom: http://nottingham.ac.uk/~ppzphy7/webpages/conferences/deitl
[34] K. A. Olive et al. (Particle Data Group), Chin. Phys. A 38, 090001 (2014).
[35] Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, edited by M. Abramowitz and I. A. Stegun, Third Printing with corrections, National Bureau of Standards, Applied Mathematics Series • 55, p. 589 eq. 17.2.8, Washington 1972.
[36] see Ref. [35] and p. 573 eq. 16.9.1 and p. 576 eq. 16.26.1 and eq. 16.25.1.
[37] see Ref. [35] p. 591 eq. 17.3.12.
[38] R. L. Gibbs, Am. J. Phys. 43, 25 (1975).
[39] R. Sedmik, Invited talk at workshop on “Dark energy in the laboratory” held on 20th - 22nd of April 2016 at Chicheley Hall, home of the Kavli Royal Society International Centre, United Kingdom; http://nottingham.ac.uk/~ppzphy7/webpages/conferences/deitl
[40] Cl. Burrage, E. J. Copeland, and J. A. Stevenson, [arXiv:1604.00342 [astro-ph.CO]].