On Parameter Estimation of Hidden Telegraph Process

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Abstract

The problem of parameter estimation by the observations of the two-state telegraph process in the presence of white Gaussian noise is considered. The properties of estimator of the method of moments are described in the asymptotics of large samples. Then this estimator is used as preliminary one to construct the one-step MLE-process, which provides the asymptotically normal and asymptotically efficient estimation of the unknown parameters.

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1 Introduction

This work is devoted to the problem of parameter estimation by the observations in continuous time \(X_T = (X(t), 0 \leq t \leq T)\) of the following stochastic process

\[
dX_t = Y(t) \, dt + dW_t, \quad X_0,
\]

here \(W_t, 0 \leq t \leq T\) is a standard Wiener process, \(X(0) = X_0\) is the initial value independent of the Wiener process and \(Y(t), 0 \leq t \leq T\) is a two-state
(\(y_1\) and \(y_2\)) stationary Markov process with transition rate matrix
\[
\begin{pmatrix}
-\lambda & \lambda \\
\mu & -\mu
\end{pmatrix}
\].

We suppose that the values \(\lambda > 0\) and \(\mu > 0\) are unknown and we have to estimate the two-dimensional parameter \(\vartheta = (\lambda, \mu) \in \Theta\), here \(\Theta = (c_0, c_1) \times (c_0, c_1)\) by the observations \(X^t, 0 < t \leq T\), i.e., the estimator \(\hat{\vartheta}_{t,T}, 0 < t \leq T\) is stochastic process. Here \(c_0 < c_1\) are positive constants.

Therefore, our goal is to construct an on-line estimator-process \(\hat{\vartheta}_T^\star = (\hat{\vartheta}^\star_{t,T}, 0 < t \leq T)\), which can be sufficiently easy to evaluate and is asymptotically optimal in some sense, as \(T \to \infty\). This estimator-process we construct in two steps. First we introduce a learning interval \([0, T^\delta]\), here \(\delta \in (\frac{1}{2}, 1)\) and propose a \(T^\delta\)-consistent preliminary estimator constructed of the method of moments. Then we improve it up to asymptotically efficient one with the help of slightly modified one-step MLE procedure.

Such model of observations is called “Hidden Markov Model” (HMM) or partially observed system. There exists an extensive study of such type HMM for discrete time models of observations, see, for example, [5], [1], [2] and the references therein. For continuous time observation models this problem is not so-well studied. See, for example, Elliot et al. [5], here the Part III “Continuous time estimation” is devoted to such models of observations. One can find there the discussion of the problems of filtration and parameter estimation in the case of observations of finite-state Markov process in the presence of White noise. The problem most close to our statement was studied by Chigansky [3], who considered the parameter estimation in the case of hidden finite-state Markov process by continuous time observations. He showed the consistency, asymptotic normality and asymptotic efficiency of the MLE of one-dimensional parameter. The case of two-state hidden telegraph process studied in our work was presented there as example but there supposed that \(\lambda = \mu\). The problem of parameter estimation for similar models including the identification of partially observed linear processes (models of Kalman filtration) were studied by many authors. Let us mention here [10], [1], [17], [11]. The problem of asymptotically efficient estimation for the model of telegraph process observed in discrete times was studied in [6].

The proposed here one-step MLE-process is motivated by the work of Kamatani and Uchida [8] who introduced Newton-Raphson multi-step estimators in the problem of parameter estimation by discrete time observations of diffusion process. In particular, it was shown that multi-step Newton-Raphson procedure allows to improve the rate of convergence of preliminary
estimator up to asymptotically efficient one. Note that preliminary estimator is constructed by all observations and they studied estimator of the unknown parameter. Applied in this work estimator-process uses the preliminary estimator constructed by the observations on the initial learning interval and follows the similar construction as that one introduced in the work [13] (see as well [12]).

2 Problem statement and auxiliary results

We start with description of the MLE for this model of observations. The stochastic process \( \Pi \) according to innovation theorem (see [14], Theorem 7.12) admits the representation

\[
dX_t = m(t, \vartheta) \, dt + d\tilde{W}_t, \quad X_0, \quad 0 \leq t \leq T,
\]

here \( m(t, \vartheta) \) is the conditional expectation

\[
m(t, \vartheta) = E_\vartheta [Y(t) \mid \mathcal{F}_t^X] = y_1 P_\vartheta (Y(t) = y_1 \mid \mathcal{F}_t^X) + y_2 P_\vartheta (Y(t) = y_2 \mid \mathcal{F}_t^X).
\]

Here \( \mathcal{F}_t^X \) is the \( \sigma \)-algebra generated by the observations up to time \( t \), i.e., \( \mathcal{F}_t^X := \sigma (X_s, 0 \leq s \leq t) \) and \( \tilde{W}_t, 0 \leq t \leq T \) is an innovation Wiener process.

Let us denote

\[
\pi (t, \vartheta) = P_\vartheta (Y(t) = y_1 \mid \mathcal{F}_t^X), \quad P_\vartheta (Y(t) = y_2 \mid \mathcal{F}_t^X) = 1 - \pi (t, \vartheta).
\]

Hence

\[
m(t, \vartheta) = y_2 + (y_1 - y_2) \pi (t, \vartheta).
\]

The random process \( \pi (t, \vartheta), 0 \leq t \leq T \) satisfies the following equation (see [14], Theorem 9.1 and equation (9.23) there)

\[
d\pi (t, \vartheta) = [\mu - (\lambda + \mu) \pi (t, \vartheta) \\
+ \pi (t, \vartheta) (1 - \pi (t, \vartheta)) (y_2 - y_1) (y_2 + (y_1 - y_2) \pi (t, \vartheta))] \, dt \\
+ \pi (t, \vartheta) (1 - \pi (t, \vartheta)) (y_1 - y_2) \, dX_t.
\]

Denote by \( \{P_\vartheta(t), \vartheta \in \Theta\} \) the measures induced by the observations \( X^t = (X_s, 0 \leq s \leq t) \) of stochastic processes \( \Pi \) with different \( \vartheta \) in the space of realizations \( C[0, t] \) (continuous on \([0, t] \) functions). These measures are equivalent and the likelihood ratio function

\[
L(\vartheta, X^t) = \frac{dP_\vartheta(t)}{dP_0(t)} (X^t), \quad \vartheta \in \Theta, \quad 0 < t \leq T
\]
can be written as follows
\[
L (\vartheta, X^t) = \exp \left\{ \int_0^t m (s, \vartheta) \, dX_s - \frac{1}{2} \int_0^t m (s, \vartheta)^2 \, ds \right\}.
\]

Here \( \mathbf{P}_0 (t) \) is the measure corresponding to \( X^t \) with \( Y (s) \equiv 0 \).

The MLE-process \( \hat{\vartheta}_{t,T} \) is defined by the equation
\[
L \left( \hat{\vartheta}_{t,T}, X^t \right) = \sup_{\vartheta \in \Theta} L (\vartheta, X^t), \quad 0 < t \leq T. \tag{3}
\]

It is known that in the one-dimensional case \((d = 1, \lambda = \mu = \vartheta)\) the MLE \( \hat{\vartheta}_{T,T} = \hat{\vartheta}_T \) is consistent, asymptotically normal
\[
\sqrt{T} \left( \hat{\vartheta}_T - \vartheta \right) \Rightarrow \mathcal{N} (0, I (\vartheta)^{-1})
\]
and asymptotically efficient (see [3], [7]). Here \( I (\vartheta) \) is the Fisher information.

Note that the construction of the MLE-process \( \hat{\vartheta}_{t,T}, 0 < t \leq T \) according to (3) and (2) is computationally difficult problem because we need to solve the family of equations (2) for all \( \vartheta \in \Theta \) and (3) for all \( t \in (0, T] \).

We propose the following construction. First we study an estimator \( \bar{\vartheta}_T \) of the method of moments and show that this estimator is \( \sqrt{T} \)-consistent, i.e.,
\[
E_{\vartheta} \left| \sqrt{T} \left( \bar{\vartheta}_T - \vartheta \right) \right|^2 \leq C.
\]

Then using this estimator \( \bar{\vartheta}_{T^\delta} \) obtained by the observations on the learning interval \([0, T^\delta]\), here \( \frac{1}{2} < \delta < 1 \), we introduce the one-step MLE-process
\[
\hat{\vartheta}_{t,T}^{*} = \bar{\vartheta}_{T^\delta} + T^{-1/2} \mathbb{I}_t \left( \hat{\vartheta}_{T^\delta} \right)^{-1} \Delta_t \left( \hat{\vartheta}_{T^\delta}, X^t \right), \quad T^\delta \leq t \leq T.
\]

Here the empirical Fisher information matrix
\[
\mathbb{I}_t (\vartheta) = \frac{1}{t} \int_{T^\delta}^t \dot{m} (\vartheta, s) \dot{m} (\vartheta, s)^* \, ds \longrightarrow \mathbb{I} (\vartheta),
\]
as \( t \to \infty, T^\delta = o (t) \) and the vector score-function process is
\[
\Delta_t (\vartheta, X^t) = \frac{1}{\sqrt{t}} \int_{T^\delta}^t \dot{m} (\vartheta, s) \left[ dX_s - m (\vartheta, s) \, ds \right].
\]

Here and in the sequel dot means the derivation w.r.t. parameter \( \vartheta \), \( \dot{m} (\vartheta, t) \) is the vector-column of the derivatives \( \dot{m}_\lambda (\vartheta, t) \) and \( \dot{m}_\mu (\vartheta, t) \). The estimator
\( \vartheta_{t,T}^* \) is in some sense asymptotically efficient. In particular for \( \vartheta_{T,T}^* = \vartheta_T^* \) we have

\[
\sqrt{T} (\vartheta_T^* - \vartheta) \implies \mathcal{N} \left( 0, I(\vartheta)^{-1} \right),
\]
i.e., it is asymptotically equivalent to the MLE. Note that the calculation of the estimator \( \vartheta_{t,T}^* \) for all \( t \in [T^\delta, T] \) requires the solution of the equation \([2]\) for one value \( \vartheta = \bar{\vartheta}_{T^\delta} \) only.

Recall as well the well-known properties of the Telegraph (stationary) process \( Y(t), t \geq 0 \).

1. The stationary distribution of the process \( Y(t) \) is

\[
P_\vartheta \{ Y(t) = y_1 \} = \frac{\mu}{\lambda + \mu}, \quad P_\vartheta \{ Y(t) = y_2 \} = \frac{\lambda}{\lambda + \mu} \tag{4}
\]

2. Let us denote \( P_{ij}(t) = P_\vartheta \{ Y(t) = y_j | Y(0) = y_i \} \), then solving the Kolmogorov equation we obtain

\[
P_{11}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t}, \quad P_{12}(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t},
\]
\[
P_{21}(t) = \frac{\mu}{\lambda + \mu} - \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu)t}, \quad P_{22}(t) = \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu)t} \tag{5}
\]

It follows from \([4]\) and \([5]\) that

\[
K(s) = E_\vartheta Y(t) Y(t + s) = \left( \frac{y_1 \mu + y_2 \lambda}{\lambda + \mu} \right)^2 + (y_2 - y_1)^2 \frac{\lambda \mu}{(\lambda + \mu)^2} e^{-(\lambda + \mu)s} = (\bar{Y})^2 + D e^{-(\lambda + \mu)s}, \tag{6}
\]

here

\[
\bar{Y} = E_\vartheta Y(t) = \frac{y_1 \mu + y_2 \lambda}{\lambda + \mu}, \quad D = (y_2 - y_1)^2 \frac{\lambda \mu}{(\lambda + \mu)^2}. \tag{7}
\]

3. Let \( \mathcal{F}_t^Y \subset \mathcal{F} \) be a family of \( \sigma \)-algebras, induced by the events

\[
\{ Y(s) = y_i, 0 \leq s \leq t, i = 1, 2 \}.
\]

Then it follows from \([5]\) that for some constant \( K > 0 \) and \( A < T \) and for all \( s > A, t > 0 \) the inequality

\[
| E_\vartheta \{ Y(s + T) Y(t + T) | \mathcal{F}_A^Y \} - E_\vartheta [Y(s) Y(t)] | < Ke^{-(\lambda + \mu)(T - A)} \tag{8}
\]

holds.
3 Method of moments estimator

Let us consider the problem of the construction of $\sqrt{T}$-consistent estimators of the parameter $\vartheta$ by the method of moments. Recall that we observe in continuous time the stochastic process

$$dX_t = Y(t)\,dt + dW_t, \quad X_0, \quad 0 \leq t \leq T,$$

(9)

here $W_t, 0 \leq t \leq T$ is a standard Wiener process, $X_0$ is independent of $W_t, 0 \leq t \leq T$ initial value, $Y(t) = Y(t, \omega)$ is stationary Markov process with two states $y_1$ and $y_2$ and infinitesimal rate matrix

$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$  

The processes $Y(t), t \geq 0$ and $W_t, t \geq 0$ are independent.

We suppose for simplicity that $T$ is an integer number. Introduce the condition

$$\lambda \in [c_0, c_1], \mu \in [c_0, c_1]$$

(10)

here $c_0$ and $c_1$ are some positive constants.

To introduce the estimators we need the following notations.

- The function
  $$\Phi(x) = \frac{1}{x} - \frac{1}{x^2} \left(1 - e^{-x}\right).$$
  (11)

- The statistics
  $$\zeta_T = \frac{1}{T} \sum_{i=0}^{T-1} [X_{i+1} - X_i]^2 - 1.$$  
  (12)

- The random variable $\alpha_T$ is defined as a solution of the equation
  $$\zeta_T = \left(\frac{X_T}{T}\right)^2 + 2\eta_T \Phi(\alpha_T),$$
  (13)

  here
  $$\eta_T = \left(\frac{X_T}{T} - y_1\right) \left(y_2 - \frac{X_T}{T}\right).$$
  (14)

- The event $A_T$ that the equation (13) has a solution.
• The random variable
\[ \beta_T = \alpha_T \mathbb{I}_{\{A_T\}} + (c_0 + c_1) \mathbb{I}_{\{A_T\}} \]  

(15)

The method of moments estimator is \( \hat{\vartheta}_T = (\hat{\lambda}_T, \hat{\mu}_T) \), here
\[ \hat{\lambda}_T = \frac{X(T) - y_1}{y_2 - y_1} \beta_T; \quad \hat{\mu}_T = \frac{y_2 - X(T)}{y_2 - y_1} \beta_T. \]  

(16)

The properties of these estimators are given in the following theorem.

**Theorem 1** Let the condition (10) holds. Then for the estimators (16) and some constant \( C > 0 \) we have for all \( T > 0 \)
\[ E_\vartheta \left( \sqrt{T} \left( \hat{\lambda}_T - \lambda \right) \right)^2 < C, \quad E_\vartheta \left( \sqrt{T} (\hat{\mu}_T - \mu) \right)^2 < C. \]  

(17)

The proof is given in several steps.

The next lemma gives \( \sqrt{T} \)-consistent estimator for \( \bar{Y} \) (see (7)).

**Lemma 1** Let the condition (10) be fulfilled. Then the estimator \( X_T/T \) is uniformly consistent for \( \bar{Y} \) and for any \( T > 0 \)
\[ E_\vartheta \left( \frac{X_T}{T} - \bar{Y} \right)^2 \leq \frac{C}{T}, \]  

(18)

here the constant \( C > 0 \) does not depend on \( \vartheta \).

**Proof.** Making use (6) we obtain the following relations
\[ E_\vartheta \left( \frac{X_T}{T} - \bar{Y} \right)^2 = E_\vartheta \left| \frac{1}{T} \int_0^T [Y(t) - \bar{Y}] \, dt + \frac{W_T}{T} \right|^2 \]
\[ = \frac{1}{T} + \frac{1}{T^2} E_\vartheta \left| \int_0^T [Y(t) - \bar{Y}] \, dt \right|^2 \]
\[ \leq \frac{1}{T} \left( 1 + \frac{2\lambda \mu}{(\lambda + \mu)^2} (y_2 - y_1)^2 \right) \leq \frac{1}{T} \left( 1 + \frac{\epsilon_1^2}{4c_0^2} (y_2 - y_1)^2 \right). \]

**Corollary.** The existence of the consistent estimator for \( \frac{\lambda}{\lambda + \mu} \) and \( \frac{\mu}{\lambda + \mu} \) follows from (4) and Lemma 1. Indeed, from the equality
\[ \bar{Y} = \frac{\lambda}{\lambda + \mu} y_2 + \frac{\mu}{\lambda + \mu} y_1 \]
and Lemma 1 we obtain
\[
\begin{align*}
E_\vartheta \left[ \sqrt{T} \left( \frac{T^{-1}X_T - y_1}{y_2 - y_1} - \frac{\lambda}{\lambda + \mu} \right) \right]^2 &< C, \\
E_\vartheta \left[ \sqrt{T} \left( \frac{y_2 - T^{-1}X_T}{y_2 - y_1} - \frac{\mu}{\lambda + \mu} \right) \right]^2 &< C.
\end{align*}
\] (19)

The statistics
\[
\frac{X_T}{T} = \frac{1}{T} \int_0^T Y(t) \, dt + \frac{W_T}{T}
\]
is the sum of a bounded a.s. random variable and an independent of it gaussian random variable with parameters \((0, T^{-1})\). Hence for \(\eta_T\) defined in (14) we can write the estimate
\[
E_\vartheta \left[ \sqrt{T} (\eta_T - D) \right]^2 < C,
\] (20)
here the constant \(C > 0\) does not depend on \(T\) and \(\vartheta\). The constant \(D\) is defined in (7).

Note that from the condition (10) we have
\[
\frac{\lambda \mu}{(\lambda + \mu)^2} > \frac{c_0^2}{4c_1^2}
\]
and we easily obtain the estimate (20) for the estimator
\[
\tilde{\eta}_T = \max \left\{ \eta_T, \frac{c_0^2}{8c_1^2} \right\}
\] (21)

Lemma 2 The following equality holds
\[
E_\vartheta \zeta_T = \tilde{Y}^2 + 2D\Phi (\lambda + \mu)
\] (22)
and under the condition (10) we have as well
\[
E_\vartheta \left[ \sqrt{T} (\zeta_T - E_\vartheta \zeta_T) \right]^2 < C.
\] (23)

Proof. From stationarity of the process \(Y(t)\) and (6) we obtain
\[
E_\vartheta \zeta_T = E_\vartheta [X_1 - X_0]^2 - 1 = E_\vartheta \int_0^1 \int_0^1 Y(s) Y(t) \, ds \, dt + 1 - 1
= \tilde{Y}^2 + 2D\Phi (\lambda + \mu).
\] (24)
Denote
\[
\gamma_i = \int_{i}^{i+1} Y(t) \, dt; \quad \Delta W(i) = W_{i+1} - W_i.
\]

Further, from the equality
\[
\zeta_T - E_\theta \zeta_T = \frac{1}{T} \sum_{i=0}^{T-1} (\gamma_i^2 - E_\theta \gamma_i^2) + \frac{2}{T} \sum_{i=0}^{T-1} \gamma_i \Delta W(i) + \frac{1}{T} \sum_{i=0}^{T-1} (\Delta W(i)^2 - 1)
\]
follows the estimate
\[
E_\theta (\zeta_T - E_\theta \zeta_T)^2 \leq \frac{3}{T^2} E_\theta \left( \sum_{i=0}^{T-1} (\gamma_i^2 - E_\theta \gamma_i^2) \right)^2 + \frac{12}{T^2} E_\theta \left( \sum_{i=0}^{T-1} \gamma_i \Delta W(i) \right)^2 + \frac{3}{T^2} E_\theta \left( \sum_{i=0}^{T-1} (\Delta W(i)^2 - 1) \right)^2 := 3J_1 + 12J_2 + 3J_3. \tag{25}
\]

From stationarity of \(Y(t)\) we obtain
\[
J_1 = \frac{1}{T^2} E_\theta \left( \sum_{i=0}^{T-1} (\gamma_i^2 - E_\theta \gamma_i^2) \right)^2 = \frac{1}{T^2} \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} E_\theta (\gamma_i^2 - E_\theta \gamma_i^2) (\gamma_j^2 - E_\theta \gamma_j^2)
\]
\[
= \frac{1}{T^2} \sum_{i,j=0}^{T-1} \int_0^1 \int_0^1 \int_0^1 \int_0^1 E_\theta \{Y(s)Y(t) \}
\]
\[
E_\theta \left[ Y(|i-j| + s_1) Y(|i-j| + t_1) |\mathcal{F}_t^Y \} \right] dsdt ds_1 dt_1 - E_\theta \gamma_0^2 \right). \tag{26}
\]

The estimate (25) allows to write
\[
|E_\theta \left[ Y(|i-j| + s_1) Y(|i-j| + t_1) |\mathcal{F}_t^Y \} - E_\theta Y(s_1) Y(t_1) \right| \leq K e^{-(\lambda+\mu)|j-i|}.
\]

From this estimate, (26) and (10) we obtain
\[
J_1 \leq \frac{K}{T^2} \sum_{i,j=0}^{T-1} e^{-(\lambda+\mu)|j-i|} \leq \frac{K_1}{T}.
\]

The following estimates are evident
\[
J_2 = \frac{1}{T^2} E_\theta \left( \sum_{i=0}^{T-1} \gamma_i \Delta W(i) \right)^2 = \frac{1}{T^2} \sum_{i=0}^{T-1} E_\theta \gamma_i^2 = \frac{E_\theta \gamma_0^2}{T} \leq \frac{K}{T},
\]
\[
J_3 = \frac{1}{T^2} E_\theta \left( \sum_{i=0}^{T-1} [\Delta W(i)^2 - 1] \right)^2 = \frac{1}{T^2} \sum_{i=0}^{T-1} E_\theta [\Delta W(i)^2 - 1]^2 \leq \frac{K}{T}.
\]

The second proposition of the Lemma 2 follows from these estimates and (25).
Lemma 3 The function $\Phi (x)$ (see (11)) has the following properties

$$\lim_{x \to 0^+} \Phi (x) = \frac{1}{2},$$

$$\lim_{x \to \infty} \Phi (x) = 0,$$

$$\Phi' (x) < 0, \text{ for } x > 0.$$ (27) (28) (29)

Proof. From (12) we obtain the representations

$$\Phi (x) = \frac{1}{2} - \frac{x}{3!} + \frac{x^2}{4!} - \frac{x^3}{5!} + \ldots$$

$$\Phi' (x) = - \left( \frac{1}{3!} - \frac{2x}{4!} \right) - \left( \frac{3x^2}{5!} - \frac{4x^3}{6!} \right) - \ldots$$

which allow to verify the limits (27) and (28) and as well the estimate (29) for $x < 2$. For $x \geq 2$ this estimate follows from the explicit expression for this derivative

$$\Phi' (x) = \frac{1}{x^2} \left( \frac{2}{x} - 1 \right) - \left( \frac{2}{x^3} + \frac{1}{x^2} \right) e^{-x}.$$ (30)

Let us consider the equation (13) for $\alpha_T$, here $\zeta_T$ and $\eta_T$ are defined in (12) and (14) respectively. Due to Lemma 3 this equation has not more than one solution. Recall that $A_T$ is the following event: the equation (13) has solution and consider the statistics $\beta_T$ defined in (15) (here $c_0, c_1$ are the constants from the condition (10)).

Lemma 4 Under the condition (10) the estimate $\beta_T$ is $\sqrt{T}$-consistent for $\lambda + \mu$. Moreover, for some constant $C > 0$ which does not depend on $T$ and $\vartheta$ we have the property

$$E_{\vartheta} \left[ \sqrt{T} (\beta_T - (\lambda + \mu)) \right]^2 < C.$$ (30)

Proof. It follows from Lemmata 1 and 2 that

$$\zeta_T = \bar{Y}^2 + 2D\Phi (\lambda + \mu) + \varepsilon_1 (T).$$ (31)

Here and below we have for $\varepsilon_i (T), i = 1, 2 \ldots$ the estimates

$$E_{\vartheta} \left( \sqrt{T} \varepsilon_i (T) \right)^2 < C.$$ (30)
By Lemma 1, estimates (20), (21) and the boundedness of \( \Phi(x) \) we obtain as well the relation

\[
\zeta_T = \bar{Y}^2 + 2\bar{\eta}_T \Phi(\lambda + \mu) + \varepsilon_2(T).
\]

(32)

If we have the event \( \mathcal{A}_T \) then it follows from (13) and (32) that

\[
2\bar{\eta}_T \Phi(\alpha_T) = 2\bar{\eta}_T \Phi(\lambda + \mu) + \varepsilon_3(T).
\]

This relation, Lemma 3 and the separation from zero by a positive constant of the estimator \( \bar{\eta}_T \) (see Corollary to Lemma 1) yield for \( \omega \in \mathcal{A}_T \)

\[
\Phi(\alpha_T) - \Phi(\lambda + \mu) = \varepsilon_4(T).
\]

Therefore from Lemma 3 we obtain

\[
\mathbb{E}_\theta \left\{ \mathbb{I}_{\{\alpha_T\}} \sqrt{T} (\beta_T - (\lambda + \mu)) \right\}^2 < C.
\]

(33)

If \( \omega \in \mathcal{A}_T^c \) then the equation

\[
\bar{Y}^2 + 2D \Phi(\lambda + \mu) + \gamma_3(T) = \left( \frac{X(T)}{T} \right)^2 + 2\bar{\eta}_T \Phi(x)
\]

has no solution \( x \in [2c_0, 2c_1] \).

It follows from (34), Lemma 1 and the Corollary that the equation

\[
\Phi(x) = \Phi(\lambda + \mu) + \varepsilon_4(T)
\]

has no solution for \( x \in [2c_0, 2c_1] \).

Hence we can write \( \mathcal{A}_T^c \subset \{ |\varepsilon_4(T)| > \alpha \} = \mathcal{B}_T \) for some positive constant \( \alpha \) which does not depend on \( T \). This allow us to write

\[
P(\mathcal{A}_T^c) \leq P(\mathcal{B}_T) = P\left( |\varepsilon_4(T)| > \alpha \right) \leq P\left( \left| \sqrt{T}\varepsilon_4(T) \right|^2 > \alpha^2 T \right)
\]

\[
\leq \frac{\mathbb{E}_\theta \left| \sqrt{T}\varepsilon_4(T) \right|^2}{\alpha^2 T} \leq \frac{C}{T}.
\]

This estimate and (33) prove the Lemma 4.

Proof of the Theorem 1. The obtained results allow us to prove that the estimators defined in (17) are \( \sqrt{T} \)-consistent. Indeed, from the obvious equality

\[
\hat{\lambda}_T - \lambda = \beta_T \frac{X(T)}{T} - y_1 - \beta_T \frac{\lambda}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} (\beta_T - (\lambda + \mu))
\]
and \( (19) \) we obtain by Lemma 4 the estimate
\[
E_\varphi \left( \sqrt{T} \left( \hat{\lambda}_T - \lambda \right) \right)^2 \leq 2E_\varphi \left[ \sqrt{T} \beta_T \left( \frac{X(T)}{T} - \frac{y_1}{y_2 - y_1 - \frac{\lambda}{\lambda + \mu}} \right)^2 \right] + 2 \left( \frac{\lambda}{\lambda + \mu} \right)^2 E_\varphi \left[ \sqrt{T} \left( \beta_T - (\lambda + \mu) \right) \right]^2 < C.
\]
The second inequality in \((17)\) can be proved by the same way.

Therefore the estimator \( \hat{\varphi}_T = \left( \hat{\lambda}_T, \hat{\mu}_T \right) \) is \( \sqrt{T} \)-consistent.

4 One-step MLE

Our goal is to construct the asymptotically efficient estimator-process of the parameter \( \varphi = (\lambda, \mu) \in \Theta \). We do it in two steps. First we obtain by the observations \( X_T^T = (X_t, 0 \leq t \leq T) \) on the learning interval \([0, T]\) the method of moments estimator \( \hat{\varphi}_T = (\hat{\lambda}_T, \hat{\mu}_T) \) studied in the preceding section. Here \( \delta \in \left( \frac{1}{2}, 1 \right) \). This estimator by Theorem 1 satisfies the condition:
\[
\sup_{\varphi \in K} T^6 E_\varphi \left| \hat{\varphi}_T - \varphi \right|^2 \leq C,
\]
here the constant \( C > 0 \) does not depend on \( T \) and \( \varphi \in \Theta \). Remind that \( \Theta = (c_0, c_1) \times (c_0, c_1) \). Introduce the additional condition
\[
\mathcal{M} (N). \text{ For some } N \geq 2 \quad \frac{c_0}{(y_1 - y_2)^2} > \frac{2N + 9}{4}. \quad (35)
\]

Having this preliminary estimator \( \hat{\varphi}_T \), we propose one-step MLE which is based on one modification of the score-function
\[
\Delta_t (\varphi, X^T) = \frac{1}{\sqrt{t}} \int_0^t \dot{m}(\varphi, s) \left[ dX_s - m(\varphi, s) ds \right], \quad T^\delta \leq t \leq T
\]
as follows
\[
\varphi_{1,T}^* = \hat{\varphi}_T + t^{-1} I_t (\hat{\varphi}_T)^{-1} \int_{T^\delta}^t \dot{m}(\varphi_{T^\delta}, s) \left[ dX_s - m(\varphi_{T^\delta}, s) ds \right]. \quad (36)
\]
Here the vector
\[ \dot{m}(\vartheta, s) = (y_1 - y_2) \frac{\partial \pi(s, \vartheta)}{\partial \vartheta} = (y_1 - y_2) \left( \frac{\partial \pi(t, \vartheta)}{\partial \lambda}, \frac{\partial \pi(t, \vartheta)}{\partial \mu} \right)^\ast \]
and the empirical Fisher information matrix \( \mathbb{I}_t(\vartheta) \) is
\[ \mathbb{I}_t(\vartheta) = \frac{1}{t} \int_{t_0}^{t} \dot{m}(\vartheta, s) \dot{m}(\vartheta, s)^\ast ds \rightarrow \mathbb{I}(\vartheta) \]
as \( t \to \infty \) by the law of large numbers. Here \( \mathbb{I}(\vartheta) \) is the Fisher information matrix
\[ \mathbb{I}(\vartheta) = (y_1 - y_2)^2 \mathbb{E}_\vartheta \left( \frac{\partial \pi(s, \vartheta)}{\partial \vartheta} \frac{\partial \pi(s, \vartheta)^\ast}{\partial \vartheta} \right). \]

Let us change the variable \( \tau = tT^{-1} \in [0, 1] \) and introduce the random process \( \dot{\vartheta}_T^\ast(\tau), \tau_3 \leq \tau \leq 1 \), here \( \dot{\vartheta}_T^\ast(\tau) = \dot{\vartheta}_{\tau T, T}^\ast \) and \( \tau_3 = T^{-1} \to 0 \).

**Theorem 2** Suppose that \( \vartheta \in \Theta, \delta \in (\frac{1}{2}, 1) \) and the condition \( \mathcal{M}(2) \) holds, then the one-step MLE-process is consistent: for any \( \nu > 0 \) and any \( \tau \in (0, 1] \)
\[ \Pr_{\vartheta_0} \left\{ |\dot{\vartheta}_T^\ast(\tau) - \vartheta_0| > \nu \right\} \to 0 \quad (37) \]
and it is asymptotically normal
\[ \sqrt{\tau T} (\dot{\vartheta}_T^\ast(\tau) - \vartheta_0) \Rightarrow \mathcal{N}(0, \mathbb{I}(\vartheta_0)^{-1}) \quad (38) \]

**Proof.** Let us denote the partial derivatives
\[ \dot{\pi}_\lambda(t, \vartheta) = \frac{\partial \pi(t, \vartheta)}{\partial \lambda}, \quad \dot{\pi}_\mu(t, \vartheta) = \frac{\partial \pi(t, \vartheta)}{\partial \mu}, \quad \ddot{\pi}_{\lambda, \lambda}(t, \vartheta) = \frac{\partial^2 \pi(t, \vartheta)}{\partial \lambda^2}, \]
and so on.

**Lemma 5** Suppose that \( \vartheta \in \Theta \) and \( N > 1 \). If the condition
\[ \frac{c_0}{(y_1 - y_2)^2} > \frac{N + 1}{4} \quad (39) \]
holds, then
\[ \sup_{\vartheta \in \Theta} \mathbb{E}_{\vartheta_0} \left( |\dot{\pi}_\lambda(t, \vartheta)|^N + |\dot{\pi}_\mu(t, \vartheta)|^N \right) < C_1, \quad (40) \]
and if the condition
\[
\frac{c_0}{(y_1 - y_2)^2} > \frac{2N + 9}{4}
\]
holds, then
\[
\sup_{\vartheta \in \Theta} \mathbf{E}_{\vartheta_0} \left( |\hat{\pi}_{\lambda, \vartheta} (t, \vartheta)|^N + |\hat{\pi}_{\lambda, \mu} (t, \vartheta)|^N + |\hat{\pi}_{\mu, \vartheta} (t, \vartheta)|^N \right) < C_2.
\]

Here the constants \( C_1 > 0, C_2 > 0 \) do not depend on \( t \).

**Proof.** For simplicity of exposition we write
\[
\hat{\pi}_\lambda (t, \vartheta) = \hat{\pi}_\lambda, \quad \hat{\pi}_\mu (t, \vartheta) = \hat{\pi}_\mu, \quad \pi (t, \vartheta) = \pi.
\]

By the formal differentiation of
\[
\frac{d}{dt} \pi = [\mu - (\lambda + \mu) \pi - \pi (1 - \pi) (y_1 - y_2) (y_2 + (y_1 - y_2) \pi)] \, dt + \pi (1 - \pi) (y_1 - y_2) \, dX_t.
\]
we obtain the equations
\[
\begin{align*}
\frac{d\hat{\pi}_\lambda}{dt} &= -\pi \frac{d}{dt} \hat{\pi}_\lambda \left[ \lambda + \mu + (1 - 2\pi) (y_1 - y_2) (y_2 + (y_1 - y_2) \pi) \right] \\
&\quad + \pi (1 - \pi) (y_1 - y_2)^2 \frac{d}{dt} \hat{\pi}_\lambda (1 - 2\pi) (y_1 - y_2) \, dX_t,
\end{align*}
\]
\[
\begin{align*}
\frac{d\hat{\pi}_\mu}{dt} &= [1 - \pi] \frac{d}{dt} \hat{\pi}_\mu \left[ \lambda + \mu + (1 - 2\pi) (y_1 - y_2) (y_2 + (y_1 - y_2) \pi) \right] \\
&\quad + \pi (1 - \pi) (y_1 - y_2)^2 \frac{d}{dt} \hat{\pi}_\mu (1 - 2\pi) (y_1 - y_2) \, dX_t.
\end{align*}
\]

If we denote the true value of the parameters by \( \vartheta_0 \) and \( \pi (t, \vartheta_0) = \pi^o \) etc., then these equations for \( \vartheta = \vartheta_0 \) become
\[
\begin{align*}
\frac{d\hat{\pi}_\lambda^o}{dt} &= -\pi^o \frac{d}{dt} \hat{\pi}_\lambda^o \left[ \lambda_0 + \mu_0 + \pi^o (1 - \pi^o) (y_1 - y_2)^2 \right] \, dt \\
&\quad + \hat{\pi}_\lambda^o (1 - 2\pi^o) (y_1 - y_2) \, d\hat{W}_t, \\
\frac{d\hat{\pi}_\mu^o}{dt} &= [1 - \pi^o] \frac{d}{dt} \hat{\pi}_\mu^o \left[ \lambda_0 + \mu_0 + \pi^o (1 - \pi^o) (y_1 - y_2)^2 \right] \, dt \\
&\quad + \hat{\pi}_\mu^o (1 - 2\pi^o) (y_1 - y_2) \, d\hat{W}_t,
\end{align*}
\]
here
\[
\frac{d\pi^o}{dt} = [\mu_0 - (\lambda_0 + \mu_0) \pi^o] \, dt + \pi^o (1 - \pi^o) (y_1 - y_2) \, d\hat{W}_t.
\]

This system of linear for \( \hat{\pi}_\lambda \) and \( \hat{\pi}_\mu \) equations can be re-written as follows
\[
\begin{align*}
\frac{dx_t}{dt} &= [\mu_0 - ax_t] \, dt + bx_t (1 - x_t) \, d\hat{W}_t, \\
\frac{dy_t}{dt} &= -x_t \frac{d}{dt} \left[ a + b^2 x_t (1 - x_t) \right] y_t \, dt + b (1 - 2x_t) y_t \, d\hat{W}_t, \\
\frac{dz_t}{dt} &= [1 - x_t] \frac{d}{dt} \left[ a + b^2 x_t (1 - x_t) \right] z_t \, dt + b (1 - 2x_t) z_t \, d\hat{W}_t.
\end{align*}
\]
Note that as \( \lambda_0 > 0 \) and \( \mu_0 > 0 \) the process \( \pi(t, \vartheta_0) = x_t \in (0, 1) \) is ergodic with two reflecting borders 0 and 1. Therefore the process \( \pi^o(t, \vartheta_0) \) is ergodic with the invariant density

\[
f(\vartheta_0, x) = \frac{[x(1-x)]^{2(\mu_0-\lambda_0) - 2} G(\vartheta_0)}{2(\mu_0-\lambda_0) - 2} \exp \left\{ -\frac{2\mu_0 + 2(\lambda_0 - \mu_0) x}{(y_1 - y_2)^2 x (1-x)} \right\} \]

\[
= \frac{[x(1-x)]^{\gamma} G(\vartheta_0)}{\gamma} \exp \left\{ -\frac{\gamma \mu_0 - \gamma \lambda_0}{x} - \frac{\gamma \lambda_0}{1-x} \right\}
\]

here we denoted \( \gamma = 2(y_1 - y_2)^{-1} \) and \( G(\vartheta_0) \) is the normalizing constant

\[
G(\vartheta_0) = \int_0^1 [x(1-x)]^{\gamma} G(\vartheta_0) \exp \left\{ -\frac{\gamma \mu_0 - \gamma \lambda_0}{x} - \frac{\gamma \lambda_0}{1-x} \right\} dx.
\]

The processes \( y_t \) and \( z_t \) have explicit expressions

\[
y_t = -\int_0^t \exp \left\{ -\int_v^t \left[ a + b^2 x_s (1-x_s) - \frac{b^2}{2} (1-2x_s)^2 \right] ds \right\} x_s d\bar{W}_s \] 

\[
+ b \int_v^t \left( 1-2x_s \right) d\bar{W}_s \right\} x_v dv, \quad \text{(52)}
\]

\[
z_t = \int_0^t \exp \left\{ -\int_v^t \left[ a + b^2 x_s (1-x_s) - \frac{b^2}{2} (1-2x_s)^2 \right] ds \right\} x_s d\bar{W}_s \right\} \left[ 1-x_v \right] dv. \quad \text{(53)}
\]

Let us put \( x_s = \frac{1}{2} - x_s \). Then we have

\[
x_s (1-x_s) - \frac{1}{2} (1-2x_s)^2 = -3x_s^2 + \frac{1}{4}
\]

and

\[
y_t = \int_0^t \left( x_v - \frac{1}{2} \right) e^{-(a+4z^2)(t-v)} \exp \left\{ 3b^2 \int_v^t x_s^2 ds + 2b \int_v^t x_s d\bar{W}_s \right\} dv.
\]

To estimate the moments \( \mathbf{E}_{\vartheta_0} |y_t|^N \) we note that \( |x_v - \frac{1}{2}| \leq \frac{1}{2} \) and use the Hölder inequality

\[
\left( \int_0^t |f(v) g(v)| dv \right)^N \leq \left( \int_0^t |f(v)|^{\frac{N}{N-1}} dv \right)^{N-1} \int_0^t |g(v)|^N dv
\]
with \( f(v) = \exp \left\{ -a(t - v) \varepsilon \right\} \) and
\[
g(v) = \exp \left\{ -\left( a(1 - \varepsilon) + \frac{b^2}{4} \right)(t - v) + 3b^2 \int_v^t x_s^2 ds + 2b \int_v^t x_s d\bar{W}_s \right\},
\]
here \( \varepsilon > 0 \). This yields the estimate
\[
E_{\vartheta_0} |y_t|^N \leq C(N, \varepsilon) \int_0^t e^{-N\left( a(1-\varepsilon) + \frac{b^2}{4} \right)(t-v)} E_{\vartheta_0} e^{3N b^2 \int_v^t x_s^2 ds + 2N b \int_v^t x_s d\bar{W}_s} dv,
\]
here the constant \( C(N, \varepsilon) > 0 \) does not depend on \( t \). Further, we can write
\[
E_{\vartheta_0} \exp \left\{ 3N b^2 \int_v^t x_s^2 ds + 2N b \int_v^t x_s d\bar{W}_s \right\} = E_{\vartheta_0} \exp \left\{ 2N b \int_v^t x_s d\bar{W}_s - 2N b^2 \int_v^t x_s^2 ds \right\} \exp \left\{ Nb^2 (2N+3) \int_v^t x_s^2 ds \right\} \leq \exp \left\{ \frac{Nb^2}{4} (2N+3)(t-v) \right\}
\]
because \( x_s^2 \leq 1/4 \) and
\[
E_{\vartheta_0} \exp \left\{ 2N b \int_v^t x_s d\bar{W}_s - 2N b^2 \int_v^t x_s^2 ds \right\} = 1.
\]
Therefore,
\[
E_{\vartheta_0} |y_t|^N \leq C(N, \varepsilon) \int_0^t e^{-N\left( a(1-\varepsilon) + \frac{b^2}{4} \right)(2N+3)(t-v)} dv
\]
\[
= C(N, \varepsilon) \int_0^t e^{-N\left( a(1-\varepsilon) - \frac{b^2}{4} \right)(N+1)(t-v)} dv.
\]
We see that if
\[
\frac{\lambda_0 + \mu_0}{(y_1 - y_2)^2} > \frac{1}{2} + \frac{N}{2},
\]
then \( E_{\vartheta_0} |y_t|^N \leq C \). In particular, if in the condition (39) we put \( N = 2 \) and choose sufficiently small \( \varepsilon > 0 \), then we obtain the estimate
\[
\sup_{\vartheta_0 \in \Theta} E_{\vartheta_0} \left| \frac{\partial \pi(t, \vartheta_0)}{\partial \lambda} \right|^2 \leq C,
\]
(54)
We have the corresponding estimate \( C > 0 \) does not depend on \( t \).

We need as well to estimate the derivatives \((44), (45)\) for the values \( \vartheta \neq \vartheta_0 \). The equation for \( \dot{\pi}_\lambda \) becomes

\[
d\dot{\pi}_\lambda = -\pi \, dt - \dot{\pi}_\lambda \left[ \lambda + \mu + (1 - 2\pi) \left( y_1 - y_2 \right)^2 \left( \pi - \pi^0 \right) + \pi \left( 1 - \pi \right) \left( y_1 - y_2 \right)^2 \right] \, dt + \dot{\pi}_\lambda \left( 1 - 2\pi \right) \left( y_1 - y_2 \right) \, d\bar{W} (t). \tag{55}\]

Hence if we put \( a = \lambda + \mu, y_t = \dot{\pi}_\lambda \) and \( b = y_1 - y_2 \), then we obtain the equation

\[
dy_t = -x_t dt - \left[ a + b^2 \left( 1 - 2x_t \right) \left( x_t - x_t^0 \right) + b^2 x_t \left( 1 - x_t \right) \right] y_t dt + b \left( 1 - 2x_t \right) y_t \, d\bar{W}_t. \]

The solution of this equation can be written explicitly like \((52)\) but with additional term \( b^2 \left( 1 - 2x_t \right) \left( x_t - x_t^0 \right) \) in the exponent. This term satisfies the inequality

\[
\left( 1 - 2x_t \right) \left( x_t - x_t^0 \right) \geq -1.
\]

Hence if we repeat the evaluation of the \( E_{\vartheta_0} |y_t|^2 \) as it was done above, then for it boundness we obtain the condition

\[
\frac{\lambda + \mu}{\left( y_1 - y_2 \right)^2} > \frac{3}{2} + \frac{N}{2}.
\]

For the second derivative \( \ddot{\pi} = \ddot{\pi}_{\lambda, \lambda} (t, \vartheta) \) we obtain the similar estimates of the moments as follows. The equation for \( \ddot{\pi} \) is

\[
d\ddot{\pi} = -y_t \left[ 2 - 2b^2 y_t \left( x_t - x_t^0 \right) + 2b^2 y_t \left( 1 - 2x_t \right) \right] dt - 2by_t^2 \, d\bar{W}_t - \ddot{\pi} \left[ a + b^2 \left( 1 - 2x_t \right) \left( x_t - x_t^0 \right) + b^2 x_t \left( 1 - x_t \right) \right] dt + b\ddot{\pi} \left( 1 - 2x_t \right) \, d\bar{W}_t.
\]

Let us write it as

\[
d\ddot{\pi} = A (t) \, dt + B (t) \, d\bar{W}_t - \ddot{\pi} \left[ a + C (t) \right] \, dt + \ddot{\pi} \, D (t) \, d\bar{W}_t.
\]

in obvious notations. Hence the solution of it is

\[
\frac{\partial^2 \pi (t, \vartheta)}{\partial \lambda^2} = \int_0^t e^{-\int_0^s [a + C (s) - \frac{1}{2} \dot{D} (s)] \, ds} \int_0^s \frac{1}{2} D (s) \, d\bar{W}_v \left[ A (v) \, dv + B (v) \, d\bar{W}_v \right].
\]

We have the corresponding estimate

\[
C (s) - \frac{D (s)}{2} = b^2 \left( 1 - 2x_s \right) \left( x_s - x_s^0 \right) + \frac{b^2}{2} \left[ 2x_s \left( 1 - x_s \right) - (1 - 2x_s)^2 \right] \geq -b^2 - 3b^2 \left( x - \frac{1}{2} \right)^2 + \frac{b^2}{4} = -\frac{3b^2}{4} - \frac{3b^2}{2} \left( x - \frac{1}{2} \right)^2 \geq -\frac{3b^2}{2}.
\]

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because \((x - \frac{1}{2})^2 \leq \frac{1}{4}\). Therefore,

\[
a - \frac{3}{2}b^2 - \frac{2N + 3}{4}b^2 = a - \frac{9}{4}b^2 - \frac{N}{2}b^2
\]

and if

\[
\frac{\lambda + \mu}{(y_1 - y_2)^2} > \frac{9}{4} + \frac{N}{2},
\]

then we obtain

\[
\mathbb{E}_{\vartheta_0} \left| \frac{\partial^2 \pi (t, \vartheta)}{\partial \lambda^2} \right|^N < C.
\]

Hence under condition (41) we have

\[
\sup_{\vartheta \in \Theta} \mathbb{E}_{\vartheta_0} \left| \frac{\partial^2 \pi (t, \vartheta)}{\partial \lambda^2} \right|^N < C.
\]

The similar estimates can be obtained for the other derivatives. Lemma 5 is proven.

**Lemma 6** The solutions \((x_t, y_t, z_t)\) of the equations (49)-(51) have ergodic properties. In particular, we have the following mean square convergence

\[
\frac{1}{T} \int_0^T \dot{\vartheta}_0 \dot{m}_\lambda (t, \vartheta_0)^2 \, dt = \frac{b^2}{T} \int_0^T y_t^2 \, dt \to I_{11} (\vartheta_0),
\]

\[
\frac{1}{T} \int_0^T \dot{\vartheta}_0 \dot{m}_\lambda (t, \vartheta_0) \vartheta_0 \mu (t, \vartheta_0) \, dt = \frac{b^2}{T} \int_0^T y_t z_t \, dt \to I_{12} (\vartheta_0),
\]

\[
\frac{1}{T} \int_0^T \dot{\vartheta}_0 \dot{m}_\mu (t, \vartheta_0)^2 \, dt = \frac{b^2}{T} \int_0^T z_t^2 \, dt \to I_{22} (\vartheta_0).
\]

**Proof.** For the proof of the invariant measure existence see [3], section 4.2. Note that the equations (49)-(51) do not coincide with that of [3], because there it is supposed that \(\lambda = \mu\) and \(y_1 = 1, y_2 = 0\), but the arguments given there are directly applied to the system of equations (49)-(51) too.

Recall that the strong mixing coefficient \(\alpha (t)\) for ergodic diffusion process (49) satisfies the estimate

\[
\alpha (t) < e^{-c|t|}.
\]
For the proof see Theorem in [15]. To check the conditions of this theorem we change the variables in the equation (49)

\[ \xi_t = g(x_t), \quad g(x) = \int_{1/2}^{x} \frac{dv}{bv(1-v)}, \quad x \in (0, 1) \]

and obtain the stochastic differential equation

\[ d\xi_t = A(\xi_t) dt + dW_t, \quad \xi_0 = g(x_0), \quad 0 \leq t \leq T. \]

The process \( \xi_t, t \geq 0 \) has ergodic properties and the drift coefficient \( A(\cdot) \) satisfies the conditions of this theorem.

Now to verify the convergence

\[ \mathbb{E} \left( \int_0^T y_t^2 dt - \frac{1}{T} \int_0^T \mathbb{E}_{\theta_0} y_t^2 dt \right)^2 \leq C_3 T^k. \]

**Proof.** For proof see Lemma 2.1 in [9].

Therefore if we put \( Y_y = y^2_t - \mathbb{E}_{\theta_0} y_t^2, m = 3 \) and \( k = 1 \), then we obtain the convergence (56).

Let us verify the consistency of the one-step MLE-process. We can write

\[
\begin{align*}
\mathbb{P}_{\theta_0} \left\{ |\hat{\theta}_T^*(\tau) - \vartheta_0| > \nu \right\} & \leq \mathbb{P}_{\theta_0} \left\{ |\tilde{\theta}_{T^0} - \vartheta_0| > \frac{\nu}{2} \right\} \\
& + \mathbb{P}_{\theta_0} \left\{ \left| \mathbb{E}_T(\tilde{\theta}_{T^0})^{-1} \int_{T^0}^{\tau T} \hat{m}(\tilde{\theta}_{T^0}, s) \left[ dX_s - m(\tilde{\theta}_{T^0}, s) ds \right] \right| > \frac{\nu}{2} \right\}.
\end{align*}
\]
For the first probability by the Theorem 1, we have

\[
P_{\vartheta_0} \left\{ \left| \hat{\vartheta}_{T^s} - \vartheta_0 \right| > \frac{\nu}{2} \right\} \leq \frac{4}{\nu^2} \mathbb{E}_{\vartheta_0} \left| \hat{\vartheta}_{T^s} - \vartheta_0 \right|^2 \leq \frac{C}{\nu^2 T^s} \to 0.
\]

The second probability can be evaluated as follows

\[
P_{\vartheta_0} \left\{ \left| \hat{\vartheta}_{T^s} - \vartheta_0 \right| > \frac{\nu}{2} \right\} \leq \frac{4}{\nu^2} \mathbb{E}_{\vartheta_0} \left| \hat{\vartheta}_{T^s} - \vartheta_0 \right|^2 \leq \frac{C}{\nu^2 T^s} \to 0.
\]

Here \( \Delta m \left( \hat{\vartheta}_{T^s}, s \right) = m(\vartheta_0, s) - m(\hat{\vartheta}_{T^s}, s) \). We can write

\[
\left| \frac{\| \mathbb{I}_{T^s} \hat{\vartheta}_{T^s} \|^{-1}}{\tau T} \int_{T^s}^{\tau T} \dot{m}(\hat{\vartheta}_{T^s}, s) \, d\bar{W}_s \right| \leq \left| \frac{\| \mathbb{I}_{T^s} \hat{\vartheta}_{T^s} \|^{-1}}{T^\gamma} \int_{T^s}^{\tau T} \dot{m}(\hat{\vartheta}_{T^s}, s) \, d\bar{W}_s \right|,
\]

where \( \gamma \) is such that \( \delta - \gamma > \frac{1}{2} \). Hence

\[
P_{\vartheta_0} \left\{ \left| \frac{\| \mathbb{I}_{T^s} \hat{\vartheta}_{T^s} \|^{-1}}{\tau T} \int_{T^s}^{\tau T} \dot{m}(\hat{\vartheta}_{T^s}, s) \, d\bar{W}_s \right| > \frac{\nu}{4} \right\} \leq \frac{1}{T^{\delta-\gamma}} \left| \int_{T^s}^{\tau T} \dot{m}(\hat{\vartheta}_{T^s}, s) \, d\bar{W}_s \right| \to 0,
\]

as \( T \to \infty \), because

\[
P_{\vartheta_0} \left\{ \left| \int_{T^s}^{\tau T} \dot{m}(\hat{\vartheta}_{T^s}, s) \, d\bar{W}_s \right| > \frac{\sqrt{T}}{2} \right\} \leq \frac{1}{\nu T^{2\delta-2\gamma}} \left| \int_{T^s}^{\tau T} \dot{m}(\hat{\vartheta}_{T^s}, s) \right|^2 \, ds \leq \frac{C}{\nu T^{2\delta-2\gamma-1}} \to 0.
\]
Recall that $2\delta - 2\gamma - 1 > 0$. Further

$$
P_0 \left\{ \frac{\|\mathbb{I}_r T(\hat{\vartheta}_T s)^{-1}\|}{\tau T} \int_{T^3}^{T^4} \hat{m}(\vartheta_T s, s) \left[ m(\vartheta_0, s) - m(\hat{\vartheta}_T s, s) \right] ds \bigg| > \frac{\nu}{4} \right\}
$$

$$
\leq P_0 \left\{ \frac{1}{\tau T^{1-\gamma}} \left\| \int_{T^3}^{T^4} \hat{m}(\vartheta_T s, s) \int_{0}^{1} \hat{m}(\vartheta_v, s)^* dvds \left( \hat{\vartheta}_T s - \vartheta_0 \right) \bigg| > \frac{\sqrt{\nu}}{2} \right\}
$$

$$
+ P_0 \left\{ \frac{\|\mathbb{I}_r T(\hat{\vartheta}_T s)^{-1}\|}{T^\gamma} > \frac{\sqrt{\nu}}{2} \right\} \rightarrow 0
$$

as $T \to \infty$, because $\hat{\vartheta}_T s - \vartheta_0 = O(T^{-\delta/2})$ and other terms are bounded in probability. Here $\vartheta_v = \vartheta_0 + v(\hat{\vartheta}_T s - \vartheta_0)$.

To prove (38) we write

$$
\sqrt{\tau T} \left( \mathbb{P}_\vartheta T \left( \tau - \vartheta_0 \right) = \sqrt{\tau T} \left( \hat{\vartheta}_T s - \vartheta_0 \right) + \frac{\mathbb{I}_r T(\hat{\vartheta}_T s)^{-1}}{\sqrt{\tau T}} \int_{T^3}^{T^4} \hat{m}(\vartheta_T s, s) d\hat{W}_s
$$

$$
+ \frac{\mathbb{I}_r T(\hat{\vartheta}_T s)^{-1}}{\sqrt{\tau T}} \int_{T^3}^{T^4} \hat{m}(\vartheta_T s, s) \left[ m(\vartheta_0, s) - m(\hat{\vartheta}_T s, s) \right] ds.
$$

We have the estimate

$$
\mathbb{E}_0 \left\| \frac{1}{\sqrt{T}} \int_{T^3}^{T^4} \left[ \hat{m}(\vartheta_T s, s) - \hat{m}(\vartheta_0, s) \right] d\hat{W}_s \right\|^2
$$

$$
\leq \frac{1}{\tau T} \int_{T^3}^{T^4} \mathbb{E}_0 \left[ \hat{m}(\vartheta_T s, s) - \hat{m}(\vartheta_0, s) \right]^2 ds \rightarrow 0
$$

as $T \to \infty$, and by the central limit theorem the convergence in distribution

$$
\frac{1}{\sqrt{T}} \int_{T^3}^{T^4} \hat{m}(\vartheta_0, s) d\hat{W}_s \implies \mathcal{N}(0, \mathbb{I}(\vartheta_0)).
$$

Further, let us denote $\hat{\vartheta}_T s = \sqrt{\tau T} \left( \hat{\vartheta}_T s - \vartheta_0 \right)$, then we can write

$$
\hat{\vartheta}_T s + \frac{\mathbb{I}_r T(\hat{\vartheta}_T s)^{-1}}{\sqrt{\tau T}} \int_{T^3}^{T^4} \hat{m}(\vartheta_T s, s) \left[ m(\vartheta_0, s) - m(\hat{\vartheta}_T s, s) \right] ds
$$

$$
= \mathbb{I}_r T(\hat{\vartheta}_T s)^{-1} \left( \mathbb{I}_r T(\hat{\vartheta}_T s) - \frac{1}{\tau T} \int_{T^3}^{T^4} \hat{m}(\vartheta_T s, s) m(\vartheta_r, s)^* dr ds \right) \hat{\vartheta}_T s,
$$

here $\vartheta_r = \hat{\vartheta}_T s + r \left( \hat{\vartheta}_T s - \vartheta_0 \right)$. The presentation

$$
\hat{m}(\vartheta_r, s) = \hat{m}(\vartheta_T s, s) + \int_{0}^{1} \hat{m}(\vartheta_q, s) dq \left( \hat{\vartheta}_T s - \vartheta_0 \right).
$$
and the equality
\[ I_{rT}(\hat{\vartheta}_T) = \frac{1}{\tau T} \int_{T}^{T} \dot{m}(\hat{\vartheta}_T, s) \dot{m}(\hat{\vartheta}_T, s)^* ds \]
allows us to write
\[ \dot{v}_T + \frac{I_{rT}(\hat{\vartheta}_T)^{-1}}{\sqrt{\tau T}} \int_{T}^{T} \dot{m}(\hat{\vartheta}_T, s) \left[ m(\vartheta_0, s) - m(\hat{\vartheta}_T, s) \right] ds = \sqrt{\tau T} \left| \dot{\vartheta}_T - \vartheta_0 \right|^2 O(1) = T^{\frac{1}{2} - \delta} O(1) \to 0, \]
as \( T \to \infty \).

Let us verify that the Fisher information matrix is non degenerate. It is sufficient to show that the matrix
\[ \mathcal{J}(\vartheta_0) = \begin{pmatrix} E_{\vartheta_0} \dot{y}_t^2 & E_{\vartheta_0} \dot{y}_t \dot{z}_t \\ E_{\vartheta_0} \dot{y}_t \dot{z}_t & E_{\vartheta_0} \dot{z}_t^2 \end{pmatrix}, \]
is non degenerated. Here \( \dot{y}_t, \dot{z}_t \) are stationary solutions of (50) and (51) respectively. If this matrix is degenerated, then
\[ E_{\vartheta_0} \dot{y}_t^2 E_{\vartheta_0} \dot{z}_t^2 = \left( E_{\vartheta_0} \dot{y}_t \dot{z}_t \right)^2. \tag{57} \]
Recall that by Cauchy-Schwarz inequality
\[ \left( E_{\vartheta_0} \dot{y}_t \dot{z}_t \right)^2 \leq E_{\vartheta_0} \dot{y}_t^2 E_{\vartheta_0} \dot{z}_t^2 \]
with equality if and only if \( \dot{z}_t = c \dot{y}_t \) with some constant \( c \neq 0 \). Therefore in the case of equality we have \( E_{\vartheta_0} (c \dot{y}_t - \dot{z}_t)^2 = 0 \).

Introduce a new process \( \tilde{v}_t = c \dot{y}_t - \dot{z}_t \) as a solution of the equation
\[ d\tilde{v}_t = [\dot{x}_t (1 - c) - 1] dt - \left[ a + b^2 \dot{x}_t (1 - \dot{x}_t) \right] \tilde{v}_t dt + b (1 - 2 \dot{x}_t) \tilde{v}_t d\tilde{W}_t, \]
here \( \tilde{v}_t \) and \( \dot{x}_t \) are stationary solutions.

Further, following [3], Section 4, here the similar estimate was obtained, we write this solution as
\[ \tilde{v}_t = \tilde{v}_0 e^{-a t} + \int_{0}^{t} e^{-a(t-s)} [\dot{x}_s (1 - c) - 1] ds - b^2 \int_{0}^{t} e^{-a(t-s)} \dot{x}_s (1 - \dot{x}_s) \tilde{v}_s d\tilde{W}_s + \int_{0}^{t} e^{-a(t-s)} (1 - 2 \dot{x}_s) \tilde{v}_s d\tilde{W}_s. \]
Hence
\[
E_{\theta_0}\left(\int_0^t e^{-a(t-s)} \left[\tilde{x}_s (1 - c) - 1\right] ds\right)^2 \leq 4 \left(1 + e^{-2at}\right) E_{\theta_0} \tilde{v}_t^2 \\
+ \frac{4b^4}{a} \int_0^t e^{-a(t-s)} \frac{1}{16} E_{\theta_0} \tilde{v}_s^2 ds + 4b^2 \int_0^t e^{-2a(t-s)} E_{\theta_0} \tilde{v}_s^2 ds \leq C E_{\theta_0} \tilde{v}_t^2
\]
with some constant \(C > 0\) which does not depend on \(t\). Recall that \(E_{\theta_0} \tilde{v}_t^2\) does not depend on \(t\) too because \(\tilde{v}_t\) is stationary solution. Therefore if we show that for all \(c\)
\[
\lim_{t \to \infty} E_{\theta_0}\left(\int_0^t e^{-a(t-s)} \left[\tilde{x}_s (1 - c) - 1\right] ds\right)^2 > 0,
\]
then the matrix \(J(\theta_0)\) is non degenerate. The random process
\[
\zeta_t = \int_0^t e^{-a(t-s)} \left[\tilde{x}_s (1 - c) - 1\right] ds
\]
is the solution of the equation
\[
\frac{d\zeta_t}{dt} = -a\zeta_t + \tilde{x}_t (1 - c) - 1, \quad \zeta_0 = 0.
\]
The elementary calculations show that for all \(\theta_0\) and \(c\)
\[
\lim_{t \to \infty} E_{\theta_0} \zeta_t^2 = \frac{E_{\theta_0} [\tilde{\pi}_0 (1 - c) - 1]^2}{a^2} > 0,
\]
here \(\tilde{\pi}_0\) is the stationary distribution.

Therefore the Fisher information matrix \(I(\theta_0)\) is non degenerate for all \(\theta_0 \in \Theta\).

Note that the limit covariance matrix of the one-step MLE-process by the Theorem 2 coincides with the covariance of the asymptotically efficient MLE \[3\], therefore \(\theta_T^* (\tau)\) is asymptotically efficient too.

## 5 Discussions

The learning interval in one-step section is \([0, T^\delta]\), where \(\delta \in \left(\frac{1}{2}, 1\right]\), i.e., it is negligible with respect to the whole observations time \(T\). It can be done even shorter, if we use two-step MLE-process approach, as it was proposed...
in [13]. It corresponds to the learning interval \([0, T^\delta]\) with \(\delta \in (\frac{1}{4}, \frac{1}{2})\). The procedure is the follows. First we obtain the preliminary estimator \(\hat{\vartheta}_{T^\delta}\) as before. Then we introduce the second preliminary estimator

\[
\vartheta_{t,T}^* = \hat{\vartheta}_{T^\delta} + t^{-1}\mathbb{I}_t(\hat{\vartheta}_{T^\delta})^{-1}\Delta_t(\hat{\vartheta}_{T^\delta}, X^t), \quad t \in [T^\delta, T]
\]

and then we define two-step MLE-process

\[
\vartheta_{t,T}^{**} = \vartheta_{t,T}^* + t^{-1}\mathbb{I}_t(\hat{\vartheta}_{T^\delta})^{-1}\Delta_t(\hat{\vartheta}_{T^\delta}, \vartheta_{t,T}^*, X^t), \quad t \in [T^\delta, T]
\]

here

\[
\Delta_t(\vartheta_1, \vartheta_2, X^t) = \frac{1}{\sqrt{t}} \int_{T^\delta}^t \dot{m}(\vartheta_1, s) [dX_s - m(\vartheta_2, s) \, ds], \quad t \in [T^\delta, T].
\]

It can be shown that for all \(\tau \in (0, 1]\) and \(t = \tau T\) we have the asymptotic normality of the estimator \(\vartheta_{T^\tau}^{**}(\tau) = \vartheta_{T^\tau;T}^{**}\):

\[
\sqrt{\tau T} (\vartheta_{T^\tau}^{**}(\tau) - \vartheta_0) \Rightarrow \mathcal{N}(0, \mathbb{I}(\vartheta_0)^{-1}).
\]

See the details in [13].

Note that it can be shown that the one-step MLE-process converges in distribution to the limit Brownian motion. Let us introduce the random process

\[
\eta_T(\tau) = \tau \sqrt{T} (\mathbb{I}(\vartheta_0)^{-1/2} (\vartheta_{T^\tau}(\tau) - \vartheta_0)), \quad \tau_\delta \leq \tau \leq 1,
\]

here \(\tau_\delta = T^{\delta-1} \to 0\). More detailed analysis shows that the random process \(\eta_T(\tau), \tau_\star \leq \tau \leq 1\) converges to two-dimensional standard Wiener process \(W(\tau), \tau_\star \leq \tau \leq 1\) with any \(\tau_\star \in (0, 1]\). For the details see the proof of such convergence in similar problem in [13].

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