A survey on the blow-up method for fast-slow systems

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Abstract. In this document we review a geometric technique, called the blow-up method, as it has been used to analyze and understand the dynamics of fast-slow systems around non-hyperbolic points. The blow-up method, having its origins in algebraic geometry, was introduced in 1996 to the study of fast-slow systems in the seminal work by Dumortier and Roussarie [DR96], whose aim was to give a geometric approach and interpretation of canards in the van der Pol oscillator. Following [DR96], many efforts have been performed to expand the capabilities of the method and to use it in a wide range of scenarios. Our goal is to present in a concise and compact form those results that, based on the blow-up method, are now the foundation of the geometric theory of fast-slow systems with non-hyperbolic singularities. We cover fold points due to their great importance in the theory of fast-slow systems as one of the main topics. Furthermore, we also present several other singularities such as Hopf, pitchfork, transcritical, cusp, and Bogdanov-Takens, in which the blow-up method has been proved to be extremely useful. Finally, we survey further directions as well as examples of specific applied models, where the blow-up method has been used successfully.

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* Invited survey article.

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1. Introduction

Natural phenomena often occur at several time scales. Examples of these are: chemical reactions (such as enzymatic and autocatalytic reactions), neuron dynamics, population dynamics, network dynamics (such as social and power networks), biological processes (such as cell division and regulatory processes) [BL14, Hek10], among many others. Models in the form of Ordinary Differential Equations (ODEs) that are suitable to study and describe the aforementioned phenomena are called fast-slow systems or singularly perturbed ODEs.

From a mathematical perspective, the analysis and study of fast-slow systems has always been of interest to the community. Several techniques involving asymptotic and geometric methods [Eck11, KC12, Kue15, O'M91, Ver05], have been developed to understand the dynamics of fast-slow systems. In this document we take a geometric viewpoint.

In qualitative terms, the key idea is to separate the subprocesses acting at the different time scales, understand them, and then try to describe the full dynamics based on the simpler subsystems. This idea can be made rigorous and has proven to be quite powerful. However, it may happen that the interaction between the fast and the slow dynamics cannot be separated directly. At the level of the model, this obstacle is seen as singularities and thus more advanced methods are required. The presence or absence of singularities often determines what type of mathematical technique is suitable for analysis. Here we are interested in a technique, called the blow-up method, that allows us to study fast-slow systems around singularities.

Briefly speaking, the blow-up method is a geometric desingularization technique based on generalized polar and projective changes of coordinates. What we aim to gain by applying this method is simpler dynamical behaviour. Via the change of coordinates defined by the blow-up, we attempt to transform the analysis of fast-slow systems around singularities to a series of easier subproblems, which can be dealt with known methods of dynamical systems. Afterwards, we gather all the information obtained from the subproblems and draw conclusions on the dynamics of the full fast-slow system. It is important to remark that the blow-up (or also blowing up) method was initially developed for the resolution of singularities of algebraic curves, and more generally of varieties over fields of characteristic zero [Hir64a, Hir64b]. The idea of resolution consists on a certain process that provides a regular curve from a singular one. We will see that this idea is similar for ordinary differential equations. In the context of dynamical systems, the blow-up method has been used, for example, to desingularize, and classify degenerate singularities of (single-scale) vector fields [Tak74, Dum77, BM90, Dum91, ÁFJ11]. Here we present the use of blow-up for fast-slow multiscale systems.

Our goal with this survey is to gather in a concise and compact form the most relevant results in the theory of fast-slow systems for which the blow-up method has been fundamental. The results that we include in the main part of this survey have extended and complemented the general theory of fast-slow systems and are nowadays the foundation of many other studies. Our approach is structured as follows: In Section 2 we provide some necessary mathematical preliminaries, where we sketch Fenichel’s theorem and the blow-up method. Later, in Section 3 we present our survey which includes results on generic folds and canards. Then we recall some fundamental results outside the scope of folded singularities, and present several applications, where the blow-up method has been used. We conclude in Section 4 with a summary and an outlook for the blow-up method within the context of fast-slow systems.
Notation

Naturally, different notations are used across the whole body of literature regarding fast-slow systems. In this article we homogenize the notation used for the most important geometric objects that one encounters while studying fast-slow systems. Although the explicit definition of the geometric objects changes depending on the specific problem, their role in the analysis is often similar. Clarifying the way we decide to denote these objects allows us to recycle the notation, thus facilitating the exposition. The reader may refer back to the following conventions throughout:

- $\mathcal{C}_0$ denotes the critical manifold of a given fast-slow system to be specified when appropriate.
- $\mathcal{S}_0$ is a compact subset of $\mathcal{C}_0$. The attracting and repelling parts of $\mathcal{S}_0$ are denoted by $\mathcal{S}_a^0$ and $\mathcal{S}_r^0$ respectively. When further distinction is needed, we add appropriate subscripts.
- $\mathcal{S}_\varepsilon$ denotes a slow manifold, and $\mathcal{S}_a^\varepsilon$ (resp. $\mathcal{S}_r^\varepsilon$) denotes the attracting (resp. repelling) part of $\mathcal{S}_\varepsilon$.
- $\Sigma^{\text{en}}$ is a codimension 1 entry section of the phase-space transverse to $\mathcal{S}_a^0$. $\Sigma^{\text{ex}}$ is used to denote a codimension 1 exit section of the phase-space transverse to $\mathcal{S}_a^\varepsilon$ beyond a non-hyperbolic singularity. When more than one exit section is necessary, we add appropriate subscripts.
- $\Pi$ denotes the transition map between $\Sigma^{\text{en}}$ and $\Sigma^{\text{ex}}$ induced by the flow of a fast-slow system. When more than one transition map is necessary, we add appropriate subscripts.
- $K_*$ denotes a chart of the blow-up space. We distinguish the charts using a subscript depending the number of charts required for the analysis of a specific singularity.

2. Preliminaries

A fast-slow system is a singularly perturbed ordinary differential equation (ODE) of the form

$$
\begin{align*}
\varepsilon \dot{x} &= f(x, y, \varepsilon), \\
\dot{y} &= g(x, y, \varepsilon),
\end{align*}
$$

(2.1)

where the over-dot denotes the derivative with respect to the slow time $\tau$, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ denote the fast and the slow variables respectively, and $\varepsilon > 0$ is a small parameter accounting for the time scale separation. By defining the fast time $t := \tau/\varepsilon$, we can rewrite (2.1) as an $\varepsilon$-family of ODEs of the form

$$
\begin{align*}
x' &= f(x, y, \varepsilon), \\
y' &= \varepsilon g(x, y, \varepsilon),
\end{align*}
$$

(2.2)

where now the prime $'$ denotes the derivative with respect to the fast time $t$. We assume that $f(x, y, \varepsilon)$ and $g(x, y, \varepsilon)$ are $C^k$-functions, with $k$ sufficiently large. We observe that (2.1) and (2.2) are smoothly equivalent for $\varepsilon > 0$, meaning that their only difference is the time parametrization of the corresponding trajectories.

The term Geometric Singular Perturbation Theory (GSPT) \cite{Pen79, Jon95}, refers to the collection of geometric techniques with which fast-slow systems can be analyzed. Roughly speaking, the idea of GSPT is to study the limits of (2.1) and (2.2) as $\varepsilon \to 0$, analyze invariant objects in the two limits, and then use perturbation methods to describe the dynamics of (2.1) and (2.2) for $\varepsilon > 0$ sufficiently small. Accordingly, by setting $\varepsilon = 0$ in (2.1) we obtain

$$
\begin{align*}
0 &= f(x, y, 0), \\
\dot{y} &= g(x, y, 0),
\end{align*}
$$

(2.3)

which is called the reduced system (or slow subsystem), and the flow of (2.3) is called the slow flow. Structurally, the system (2.3) is a Constrained Differential Equation (CDE) \cite{Tak76} (or Differential
Algebraic Equation. From (2.1) we get for \( \varepsilon = 0 \) the ODE
\[
\begin{align*}
    x' &= f(x, y, 0), \\
    y' &= 0,
\end{align*}
\]
which is called the layer equation (or fast subsystem) and where we can view the slow variables \( y \) as parameters. The two singular limit systems (2.3) and (2.4) are not equivalent any more. However, we observe that the set \( \{ f(x, y, 0) = 0 \} \) defines the phase-space of solutions of the slow subsystem and the set of equilibrium points of the fast subsystem equation. This leads to the following natural definition:

**Definition 2.1 (Critical manifold).** The critical manifold is defined as the set
\[
C_0 = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \mid f(x, y, 0) = 0\}.
\]

Strictly speaking, solutions of the equation \( f(x, y, 0) = 0 \) do not necessarily define a manifold. For example, the solutions may self-intersect. However, generically, there are open regions in the phase-space where \( C_0 \) is indeed a manifold. So, as a convention in fast-slow systems one keeps referring to \( C_0 \) as the critical manifold. An important characteristic that critical manifolds may possess is normal hyperbolicity:

**Definition 2.2.** A point \( p = (x, y) \in C_0 \) is called hyperbolic if the matrix \( D_x f(p, 0) \in \mathbb{R}^{m \times m} \), where \( D_x \) stands for the total derivative with respect to \( x \), has no eigenvalues on the imaginary axis. The critical manifold \( C_0 \) is called normally hyperbolic if all points \( p \in C_0 \) are hyperbolic.

Note that equivalently to Definition 2.2, one can say that \( C_0 \) is normally hyperbolic if all its points are hyperbolic equilibrium points of the fast subsystem. We call a hyperbolic point \( p \) attracting if all eigenvalues have negative real part, repelling if all eigenvalues have positive real part, and saddle-type if there are negative and positive real part eigenvalues. On the contrary situation, we shall call \( p \in C_0 \) non-hyperbolic if the matrix \( D_x (p, 0) \) has at least one eigenvalue on the imaginary axis. Whether the critical manifold is, or is not, normally hyperbolic will distinguish the tools that are needed for the analysis of the corresponding fast-slow system. In the case of normal hyperbolicity we have Fenichel’s theorem (see Section 2.1); in the case of loss of normal hyperbolicity due to the presence of non-hyperbolic points, we may use the blow-up method (see Section 2.2).

**2.1. Fenichel’s theorem.** Fenichel’s theorem [Fen79] (see also [Jon95, Tik52]) provides geometric tools and techniques to analyze fast-slow systems with hyperbolic points:

**Theorem 2.3.** Suppose that \( S_0 \subseteq C_0 \) is a compact normally hyperbolic submanifold (possibly with boundary) of the critical manifold \( C_0 \) of (2.2). Then, for \( \varepsilon > 0 \) sufficiently small, the following hold:

1. **(M1)** There exists a locally invariant manifold \( S_\varepsilon \) diffeomorphic to \( S_0 \). Local invariance means that trajectories can enter or leave \( S_\varepsilon \) only through its boundaries.
2. **(M2)** \( S_\varepsilon \) has Hausdorff distance \( O(\varepsilon) \), as \( \varepsilon \to 0 \), from \( S_0 \).
3. **(M3)** The flow on \( S_\varepsilon \) converges to the slow flow as \( \varepsilon \to 0 \).
4. **(M4)** \( S_\varepsilon \) is \( C^k \)-smooth.
5. **(M5)** \( S_\varepsilon \) is normally hyperbolic and has the same stability properties, i.e., attracting, repelling or saddle-type, with respect to the fast variables as \( S_0 \).
\( \text{(M6) } S_\varepsilon \text{ is usually not unique. In regions that remain at a fixed distance from } \partial S_\varepsilon, \text{ all manifolds satisfying items (M1) – (M5) above lie at a Hausdorff distance of order } O(\exp(-C/\varepsilon)) \text{ from each other for some } C > 0, C \in O(1), \text{ as } \varepsilon \to 0. \)

In the past, one would refer to Fenichel’s theorem as Geometric Singular Perturbation Theory (GSPT). As a matter of fact, Fenichel’s article \[ \text{Fen79} \] is entitled “Geometric Singular Perturbation Theory for Ordinary Differential Equations”. However, nowadays GSPT includes many more geometric techniques such as the blow-up method and fast-slow normal form theory, among others. A few remarks are now in place.

**Remark 2.4.**

- The concept of normal hyperbolicity is applicable, under its appropriate modifications, to invariant manifolds in general and not only to sets of equilibria as in Definition 2.2, see [Fen71, HPS70] and [Kue15, Chapter 2.3]. It turns out, however, that normal hyperbolicity for sets of equilibria is simpler to define. For invariant manifolds in general, normal hyperbolicity is more involved.
- Although \( S_\varepsilon \) is usually not unique, any choice of such manifolds is called “the” slow manifold. In turn, one should check if the results one obtains are independent or not on the choice of slow manifold. It is often the case that the results do not depend on such a choice. Furthermore, even though the terms “critical manifold” and “slow manifold” are sometimes used interchangeably or even equivalently, it is important to make a distinction between them.\(^1\) One of the reasons for this is that, for example, slow manifolds may be extended beyond a non-hyperbolic point, and such extension may not have a relationship with the critical manifold. Many examples of this situation are shown below.
- In qualitative terms, Fenichel’s theorem shows that, under the assumptions of Theorem 2.3, a fast-slow system without non-hyperbolic points can be regarded as a regular perturbation of its corresponding singular systems near \( C_0 \).

The next section describes a geometric technique to locally analyze fast-slow systems around non-hyperbolic points.

### 2.2. The blow-up method.

In this section we provide a review on the blow-up method as used for the analysis of non-hyperbolic points of fast-slow systems. Although there are several (equivalent) versions and improvements, we restrict to the quasihomogeneous case as it is more commonly used nowadays. For further information see [Kue15, Chapter 7] and references therein.

In order to appropriately define the blow-up method for fast-slow systems we first rewrite the \( \varepsilon \)-family of vector fields (2.2) on \( \mathbb{R}^{m+n} \) as a single vector field on \( \mathbb{R}^{m+n+1} \) of the form

\[
\begin{align*}
x' &= f(x, y, \varepsilon) \\
y' &= \varepsilon g(x, y, \varepsilon) \\
\varepsilon' &= 0.
\end{align*}
\]

(2.6)

Let \( X : \mathbb{R}^{m+n+1} \to \mathbb{R}^{m+n+1} \) be the vector field, which in coordinates is defined by (2.6), and let \( S^N \) denote the \( N \)-th dimensional sphere. To use spheres, cylinders, or related spaces as blown-up spaces is often very convenient, yet not necessary [Kue16]. Next, we can formally define the blow-up transformation most commonly used in fast-slow systems:

\(^1\)In this document \( C_0 \) is a critical manifold and \( S_0 \) is a compact subset of \( C_0 \). A perturbation of \( S_0 \), denoted by \( S_\varepsilon \), is a slow manifold. Note that \( C_\varepsilon \) may not be well-defined since we do not impose compactness to \( C_0 \); but see [Eld13].
shown that \( X \) to-one mapping between trajectories of \( X \) exponents (appearing in (2.7) are not necessarily the same. We omit the term “quasihomogeneous” when all (quasihomogeneous) blow-down. The word quasihomogeneous reflects the fact that the exponents \( \alpha \) are equal to 1.

This greatly simplifies the analysis, since usually we find that \( \tilde{\alpha} \) does not vanish at \( X \). Thus, we do not vanishing any more along \( r = 0 \), we may try to infer the dynamics of \( \tilde{X} \) for \( r > 0 \) small from the restriction \( \tilde{X} \big|_{\{r = 0\}} \). This greatly simplifies the analysis, since usually we find that \( \tilde{X} \) has semi-hyperbolic singularities, hyperbolic singularities, or no singularities at all. Finally, due to the equivalences between \( X, \tilde{X} \), and \( X \), the flow of \( \tilde{X} \) for \( r > 0 \) sufficiently small provides a complete description of the flow of \( X \) for \( \varepsilon > 0 \) sufficiently small.

When we study high dimensional problems, say for \( m + n > 2 \), working with polar coordinates can become cumbersome. Then, we rather work in charts that cover the blow-up space. In each of the charts we can define local coordinates and a corresponding local vector field. In practice, what we do to define local coordinates in a chart is to fix one of the blow-up coordinates to \( \varepsilon = 1 \). This approach is called “directional blow-up”. For example, to perform a blow-up in the \( \varepsilon \)-direction we would define new coordinates according to \( \phi : \mathbb{R}^{n+m+1} \to \mathbb{R}^{n+m+1} \) given by

\[
\phi(\bar{x}, \bar{y}, \varepsilon, r) = (r^\alpha \bar{x}, r^\beta \bar{y}, r^\gamma, \varepsilon),
\]

that is by fixing \( \varepsilon = 1 \). Similarly, we can define blow-ups in any of the other directions.

**Remark 2.6.** The chart \( \tilde{K} := \{ \varepsilon = 1 \} \) is the most important one and it is called rescaling chart, the family chart or the central chart. The rest of the charts are often referred to as phase-directional charts.

Directional blow-ups induce local vector fields on each of the (Euclidean) charts. Once the analysis of the relevant local vector fields is performed, one can overlap suitable regions of the charts and match the flow on such charts via the so-called matching maps (or transition maps) to describe the dynamics all around \( \mathbb{S}^{m+n} \times I \). In particular, this process allows us to track invariant objects, principally centre manifolds, across the blow-up space. A schematic representation of the blow-up map is provided in Figure 1.
Figure 1. Sketch of the blown-up space and of some of the directional charts. In practice, via the blow-up method, we study local vector fields defined in the charts, and then "glue" trajectories and other invariant objects together to describe the dynamics in a small neighborhood of \( \mathcal{B}_0 \), which in turn provides the dynamics of a fast-slow system around the origin for \( \varepsilon > 0 \) sufficiently small.

The blow-up attempts to make the analysis of fast-slow systems near non-hyperbolic equilibrium points more accessible. Yet, the analysis of the vector fields obtained after blow-up is still highly non-trivial due to several difficult obstacles such as the nonlinearity of the local vector field, presence of semi-hyperbolic equilibria, and the appearance of resonances, just to mention a few. It may even happen that, in one of the charts, there is a singularity that is still non-hyperbolic, in which case the blow-up procedure may be applied again. Furthermore, obtaining the right weights \( (\alpha, \beta, \gamma) \) is not always trivial. One way in which one can obtain such weights is via the Newton Polyhedron [AFJ11, Dum77, Kue15].

In the next section we are going to review some of the most fundamental and (mathematically) influential works that have established the blow-up method as one preferred technique to study fast-slow systems in the neighborhood of non-hyperbolic points.

3. Survey

Due to their importance to the theory, we first focus on fold points, folded singularities and canards. Later on, we briefly discuss instances where the blow-up method has been used to study singularities beyond the fold such as Hopf, transcritical, pitchfork, cusp, and Bogdanov-Takens singularities.

3.1. The generic fold. The generic fold is the first natural situation, where a non-hyperbolic point appears in planar fast-slow systems. The exposition of this section mainly summarizes the results of [KS01a].
The manifold \( S \) is a contraction with contraction rate \( \varepsilon \). Hence, it is not possible to consider an asymptotic series in positive integer weights of \( \varepsilon \). In other words, it is not possible to consider an asymptotic series in positive integer weights of \( \varepsilon \) as an expansion of \( S \) in a small neighborhood of the fold point. Next, the second claim states that the exponential contraction towards \( S \), which before the fold point is guaranteed by Fenichel’s theorem, is not lost after crossing the fold point. We observe that this fact is not a regular perturbation argument derived from the corresponding layer equation, but can be proven via the blow-up analysis.

Let us now point-out the key steps of the proof of Theorem \( 3.2 \). The blow-up is defined by

\[
x = \bar{r} \bar{x}, \quad y = \bar{r}^2 \bar{y}, \quad \varepsilon = \bar{r}^3 \varepsilon.
\]

The weights of the blow-up map follow formally from the quasi-homogeneity \( \text{Dum91} \) of the function \(-y + x^2\). On a practical level, one can also guess good weights by observing that the
scaling \((x, y, t) \mapsto (\varepsilon^{1/3} x, \varepsilon^{2/3} y, \varepsilon^{-1/3} t)\) removes \(\varepsilon\) to leading-order from \((3.3)\). Using \((3.6)\), the blow-up analysis is carried over three charts, namely
\[(3.7) \quad K_1 = \{\bar{y} = 1\}, \quad K_2 = \{\bar{\varepsilon} = 1\}, \quad K_3 = \{\bar{x} = 1\}.
\]
The behaviour of the trajectories passing through \(\Sigma^m\) as they approach (but do not cross) a small neighbourhood of the origin is studied in chart \(K_1\). The main objects of study in this chart are 2-dimensional centre manifolds associated to semi-hyperbolic equilibrium points. To be more precise, in chart \(K_1\) one can show the existence of a 2-dimensional attracting and a 2-dimensional repelling centre manifolds that are denoted by \(\mathcal{M}_1^a\) and \(\mathcal{M}_1^r\) respectively. In fact, sections of these centre manifolds correspond to perturbations of \(S_0^a\) and \(S_0^r\), that is to slow manifolds, along normally hyperbolic parts of the critical manifold. The advantage of using the blow-up method is that it allows to track invariant objects (in this case centre manifolds) as they pass through all the charts.

Next, in chart \(K_2\), we study the behaviour of trajectories in a small neighbourhood of the origin. The main object of study in this chart is a Riccati differential equation \([\text{MR80}]\) pp. 68-72 of the form
\[
(3.8) \quad x' = -y^2 + x^2, \\
y' = -1, \\
r' = 0
\]
where we use the subscript 2 to indicate local coordinates in the chart \(K_2\). A detailed analysis of \((3.8)\) allows, in particular, to extend \(\mathcal{M}_2^a\) found in chart \(K_1\) to an invariant manifold \(\mathcal{M}_2^a\) beyond the fold point. The previous is due to the fact that sections of the centre manifold \(\mathcal{M}_1^a\) (in chart \(K_1\)) can be matched with solutions of \((3.8)\) with \(r_2\) constant (in chart \(K_2\)). Afterwards, in chart \(K_3\), we study the dynamics of the trajectories, as they leave a small neighbourhood of the fold point. More specifically, similar to the situation in the previous two charts, via a change of coordinates we can define an invariant object \(\mathcal{M}_3^a\) corresponding to \(\mathcal{M}_2^a\) but defined in the local coordinates of chart \(K_3\). Then we can pay special attention to trajectories on \(\mathcal{M}_3^a\). The main object of study in this chart is a 3-dimensional nonlinear ODE with a resonant hyperbolic saddle at the origin.

The fact that the origin is resonant is a major obstacle as it prevents to linearize the nonlinear system in order to have a detailed description of the flow near the origin. In fact, dealing with the aforementioned resonance provides that the next term in the expansion of \(h\) is of the order \(O(\varepsilon \ln \varepsilon)\). Once the analysis in these three charts is performed, the transition \(\Pi\) can be described by overlapping the individual transitions in each of the charts via the corresponding matching maps.

We emphasize that the order of the function \(h\) was already known in \([\text{MR80}]\). In fact, rigorous asymptotic expansions of the function \(h\) appear in \([\text{MR80}]\) equation (16.11) page 106. It is important to note that, although during the 80’s the blow-up method was not known per se in the context of fast-slow systems, the analysis performed in \([\text{MR80}]\) starts from a rescaling similar to the blow-up presented above, see (8.1) and (8.2) in \([\text{MR80}]\). More recently, in \([\text{VGKS05}]\) Theorem 2 a rigorous asymptotic expansion of the function \(h\) is obtained by combining the blow-up method with matched asymptotic analysis. The main idea of \([\text{VGKS05}]\) is to compute, in each chart, asymptotic expansions of the centre manifolds \(\mathcal{M}_i^a\) mentioned above, and then match such local expansions across the blow-up space.

---

\(^2\)We recall the definition of resonant equilibrium point: consider an ODE of the form \(\dot{x} = Ax\), where \(x \in \mathbb{R}^n\), and let \(A\) have all its eigenvalues \(\lambda_1, \ldots, \lambda_n\) with non-zero real part. We say that \(x = 0\) is resonant if there is a relation \(\lambda_j = \sum_{i=1}^{n} m_i \lambda_i\), where the \(m_i\)'s are non-negative integers with \(\sum_{i=1}^{n} m_i \geq 2\).
Figure 2. Schematic of a fast-slow system near a generic fold point. Up to leading order terms, the critical manifold $C_0$, shown in dashed, is given by $C_0 = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$. The (blue) lines with double arrows depict the dynamics of the layer equation. Thus, $C_0$ has an attracting ($S_{a0}$) and a repelling ($S_{r0}$) branch. Away from the fold point, Fenichel’s theorem shows that, for $\varepsilon > 0$ sufficiently small, $S_{a\varepsilon}$ and $S_{r\varepsilon}$ are smoothly perturbed to invariant manifolds (in this case trajectories) $S_{a\varepsilon}$ and $S_{r\varepsilon}$ respectively. The analysis (via the blow-up method) shows that $S_{a\varepsilon}$ can be extended beyond the fold point as depicted in the figure. In particular, one can show that the distance between the $x$-axis and the intersection $S_{a\varepsilon} \cap \Sigma^{ex}$ is of order $O(\varepsilon^{2/3})$.

3.2. Planar canards. In section 3.1 we have seen that trajectories near a generic fold point first follow the attracting part of the critical manifold, and then follow the fast direction. Perhaps counter-intuitively, for certain one-parameter families of planar fast-slow systems, there are trajectories near a fold singularity that can closely follow the unstable region of the critical manifold for time of order $O(1)$. Such type of trajectories are called canards, see Figure 3 (and the rest of the figures in this Section). In this section we summarize the results of [DR96, KS01c] and Section 3 of [KS01a] dealing with the description of canards in planar fast-slow systems.

Remark 3.3. The blow-up technique was first introduced for the analysis of fast-slow systems in [DR96] to explain, in a geometric way, the canard phenomenon in the van der Pol oscillator. Canards were already explained in [Ben81] with the language of non-standard analysis. Although [DR96] was the first work describing geometrically the canard phenomenon, we start from [KS01a] as it follows several arguments analogous to the generic fold point.

We start by considering a one-parameter family of planar fast-slow systems given by

$$
x' = f(x, y, \omega, \varepsilon),
\quad y' = \varepsilon g(x, y, \omega, \varepsilon),
$$

where $\omega \in \mathbb{R}$. Just as in the generic fold case of Section 3.1 we may assume that a fold point exists at the origin, say for $\omega = 0$. 
DEFINITION 3.4 (Canard point). Consider (3.9). We say that \((x, y) = (0, 0)\) is a canard point if
\[
(3.10) \quad f(0, 0, 0, 0) = 0, \quad \frac{\partial f}{\partial x}(0, 0, 0, 0) = 0, \quad g(0, 0, 0, 0) = 0
\]
hold together with the non-degeneracy conditions
\[
(3.11) \quad \frac{\partial^2 f}{\partial x^2}(0, 0, 0, 0) \neq 0, \quad \frac{\partial f}{\partial y}(0, 0, 0, 0) \neq 0, \quad \frac{\partial g}{\partial x}(0, 0, 0, 0) \neq 0, \quad \frac{\partial g}{\partial \omega}(0, 0, 0, 0) \neq 0.
\]

The main difference with the generic fold case of Section 3.1 is that there is an equilibrium point at the origin, which is imposed by \(g(0, 0, 0, 0) = 0\). The non-degeneracy conditions imply a transverse intersection at the origin of the nullclines of (3.9). The critical manifold of (3.9) is, up to leading order terms, the same as in Section 3.1. So, we use the same notation and do not repeat its definition here. Similarly to the generic fold case, it can be shown that a planar fast-slow system has a local expression near a canard point given by
\[
\begin{align*}
x' &= -y h_1(x, y, \omega, \varepsilon) + x^2 h_2(x, y, \omega, \varepsilon) + \varepsilon h_3(x, y, \omega, \varepsilon), \\
y' &= \varepsilon (x h_4(x, y, \omega, \varepsilon) - \omega h_5(x, y, \omega, \varepsilon) + y h_6(x, y, \omega, \varepsilon)),
\end{align*}
\]
where
\[
\begin{align*}
h_3 &= O(x, y, \omega, \varepsilon), \\
h_j &= 1 + O(x, y, \omega, \varepsilon), \quad j = 1, 2, 4, 5.
\end{align*}
\]

REMARK 3.5. Up to leading order terms (3.12) reads as
\[
\begin{align*}
x' &= -y + x^2, \\
y' &= \varepsilon (x - \omega).
\end{align*}
\]
Let, for a moment, \(\varepsilon > 0\). Then a Hopf bifurcation occurs at \(\omega = 0\). Moreover, the norm of the corresponding eigenvalues tends to zero (and to infinity in the slow time scale) as \(\varepsilon \to 0\). This situation is known as singular Hopf bifurcation [Bra98, BE86].

Define the constants
\[
\begin{align*}
a_1 &= \frac{\partial h_3}{\partial x}(0, 0, 0, 0), \\
a_5 &= h_6(0, 0, 0, 0), \\
A &= -\frac{\partial h_1}{\partial x}(0, 0, 0, 0) + 3\frac{\partial h_2}{\partial x}(0, 0, 0, 0) - 2\frac{\partial h_4}{\partial x}(0, 0, 0, 0) - 2a_5
\end{align*}
\]
and the sections
\[
\Sigma^j = \{(x, y) \in \mathbb{R}^2 \mid x \in I_j, y = \rho^2\}, \quad j = \text{en, ex},
\]
where \(\rho > 0\) and \(I_j\) is a suitable small interval, see Figure 3. Let \(q_{\text{en,ex}} = \Sigma^\text{en} \cap S_0^e, q_{\text{ex,ex}} = \Sigma^\text{ex} \cap S_0^e\) and denote by \(\Pi : \Sigma^\text{en} \to \Sigma^\text{ex}\) the transition induced by (3.9). The main result that describes the behavior of the slow manifolds \(S_0^e\) and \(S_0^e\) around the canard point is as follows.

THEOREM 3.6 [KS01a, Theorem 3.1]. Assume that (3.9) has a canard point at the origin, i.e., satisfies (3.10) and (3.11). Assume that the solution \(x_0(t)\) of the reduced problem connects \(S_0^e\) to \(S_0^e\). Then there exists \(\varepsilon_0 > 0\) and a smooth function \(\omega_c(\sqrt{\varepsilon})\) defined on \([0, \varepsilon_0]\) such that for \(\varepsilon \in [0, \varepsilon_0]\) the following assertions hold.

(C1) \(\Pi(q_{\text{en,ex}}) = q_{\text{ex,ex}}\) if and only if \(\omega = \omega_c(\sqrt{\varepsilon})\).
(C2) The function \( \omega_c \) has the expansion

\[
\omega_c(\sqrt{\varepsilon}) = -\left( \frac{a_1 + a_5}{2} + \frac{1}{8} A \right) \varepsilon + O(\varepsilon^{3/2})
\]

(C3) The transition map \( \Pi \) is defined only for \( \omega \) in an interval around \( \omega_c(\sqrt{\varepsilon}) \) of size \( O(\exp(-C/\varepsilon)) \)
for some \( C > 0 \).

(C4) There is a splitting condition at \( \omega = \omega_c(\sqrt{\varepsilon}) \) given by

\[
\frac{\partial}{\partial \omega} \left( \Pi(q_{\text{gen}, \varepsilon}) - q_{\text{ex}, \varepsilon} \right)|_{\omega = \omega_c(\sqrt{\varepsilon})} > 0.
\]

A schematic representation of Theorem [3.6] is given in Figure 3. The most important aspect of Theorem [3.6] is that it tells us that for \( \varepsilon > 0 \) there is exactly one value of the parameter \( \omega \) for which \( S^2_2 \) is extended to exactly \( S^2_1 \), that is, they coincide as stated in (C1). The previous trajectory is usually called maximal canard to distinguish it from others that may follow \( S^2_1 \) for time of order \( O(1) \). Another important point of Theorem [3.6] is its third item (C3), which tells us that canard orbits near planar folds occur only for values of \( \omega \) within an exponentially small interval. Related to the previous observation we further note that the intersection of \( S^2_2 \) and \( S^1_2 \) is not transversal and thus small \( C^1 \)-perturbations of the parameter \( \omega \) destroy the maximal canard; see (C4) and see also Section 3.3.

Let us now point out the key ingredients of the analysis to prove Theorem [3.6] In many ways, the blow-up analysis of a canard point is similar to that of a generic fold point. One of the differences is that now the parameter \( \omega \) is included in the blow-up. The appropriate blow-up transformation is defined by

\[
x = \bar{r} \bar{x}, \quad y = \bar{r}^2 \bar{y}, \quad \varepsilon = \bar{r}^2 \bar{\varepsilon}, \quad \omega = \bar{r} \bar{\omega}.
\]

Due to the definition of the \( \Sigma^{\infty} \) and \( \Sigma^m \) sections, the blow-up analysis is performed only in two charts, namely \( K_1 = \{ \tilde{y} = 1 \} \) and \( K_2 = \{ \tilde{\varepsilon} = 1 \} \). The analysis in chart \( K_1 \) for the canard point is analogous to the analysis in chart \( K_1 \) for the generic fold point. The only difference is that in the canard case, the consideration of \( \omega \) in the blow-up map increases by one the dimension of the centre manifolds. Thus, in \( K_1 \), the main object of study are 3-dimensional centre manifolds \( M_1^1 \) and \( M_1^2 \) that are of centre-stable and centre-unstable type respectively. These centre manifolds correspond to invariant manifolds \( M_2^1 \) and \( M_2^2 \) in the chart \( K_2 \). In chart \( K_2 \) the key object of study is a planar ODE of the form

\[
\begin{align*}
x'_2 &= -y_2 + x_2^2, \\
y'_2 &= x_2,
\end{align*}
\]

which is obtained after restriction to \( \{ r_2 = \omega_2 = 0 \} \); compare with (3.8). One may check that (3.20) is completely integrable, having a constant of motion denoted by \( H(x_2, y_2) \). This means that the trajectories of (3.20) are determined by the level curves of \( H \). Here \( (x_2, y_2, r_2, \omega_2) \), as usual, denote local coordinates in \( K_2 \).

There is a special curve \( \gamma_c \) given by the solution of \( H = 0 \). The major importance of \( \gamma_c \) is that it precisely connects \( S^2_2 \) with \( S^1_0 \). Naturally in the blow up space \( S^2_0 \) and \( S^1_0 \) correspond to \( M_2^1|_{r_2=0} \) and \( M_2^2|_{r_2=0} \) respectively.

The next step is to investigate how such a connection breaks for \( r_2 > 0, \omega_2 \neq 0 \), which will describe how the manifolds \( M_2^1 \) and \( M_2^2 \) are connected. For such a purpose an analysis based on the Melnikov method [GH13] is used. In essence, the Melnikov method provides an expression for the distance between \( M_2^1 \) and \( M_2^2 \), denoted by \( d = d(r_2, \omega_2) \). Therefore, for \( \varepsilon > 0 \) the manifolds
$S^a_\varepsilon$ and $S^r_\varepsilon$ are connected if and only if $d = 0$. Careful estimates of the function $d$ and a blow-down provide the expression for $\omega_c$ in Theorem 3.6 and the breaking argument proving (C4). We finalize the summary of the canard point by mentioning that the role of the constant $A$ has to do with the non-degeneracy of the Hopf bifurcation that occurs for $\omega = 0$, see for example [Kue15, Section 8.3], and Figure 7.

![Figure 3. Schematic representation of a maximal canard. Note that $S^a_\varepsilon$ extends precisely to $S^r_\varepsilon$ beyond the origin. Trajectories passing through $\Sigma^{en}$ exponentially close to $S^a_\varepsilon$ follow $S^r_\varepsilon$ for time of order $O(1)$ before being repelled from it.](image)

Next, let us discuss canards in the context of the van der Pol oscillator as they have a particular motivating relevance for the entire subject. Here the main reference is [DR96]. To be more precise, [DR96] is concerned with canard cycles. Canard cycles are closed orbits having “a canard piece”. The model of the van der Pol oscillator to be considered is

\[
x' = y - \frac{x^2}{2} - \frac{x^3}{3},
\]

\[
y' = \varepsilon(\omega - x).
\]

In the singular limit $\varepsilon = 0$, the phase portrait is as shown in Figure 4 and the critical manifold is a cubic curve given by

\[
C_0 = \left\{ (x, y) \in \mathbb{R}^2 \mid y = \frac{x^2}{2} + \frac{x^3}{3} \right\}.
\]

We note that $C_0$ has two fold points namely $(x, y) = \left\{ (-1, \frac{1}{6}), (0, 0) \right\}$. Furthermore, there is a canard point at the origin for $\omega = 0$. It is well known [LDMP77] that (3.21) has at most one limit cycle and that when such a limit cycle exists, it is hyperbolic and attracting. Assuming that a limit cycle $\Gamma_{\varepsilon, \omega}$ of (3.21) exists, the question posed by [DR96] can be formulated as follows: “what happens to $\Gamma_{\varepsilon, \omega}$ as $\varepsilon \to 0$?” The main contribution of [DR96] is the precise asymptotic description of the canard cycles that occur for (3.21). In other words, [DR96, Theorem 4] provides asymptotic estimates of the parameter $\omega$ for each of the canard cycles shown in Figure 5. All such cycles occur within an exponentially small interval of the parameter $\omega$. This means that the transition between small amplitude canard cycles towards relaxation oscillations occurs for an exponentially
Figure 4. The van der Pol oscillator: the critical manifold is the cubic curve $C_0$; the curve in this case is also called S-shaped curve, which is motivated by viewing it in the $(y,x)$-plane. The critical manifold has two fold points indicated by filled circles. The singular limit is represented by the blue lines (layer equation) and by the arrows along $C_0$ (CDE). The CDE has an equilibrium point at $x = \omega$ indicated by an empty circle. As it is described in this Section, the parameter $\omega$ is essential to distinguish several types of limit cycles. The curve $\Gamma_{\varepsilon,\omega}$ indicates a typical limit cycle that exists for $\varepsilon > 0$ sufficiently small.

small variation of the parameter. This effect is called canard explosion. The analysis presented in [DR96] is highly non-trivial: besides introducing the blow-up method for fast-slow systems, it uses techniques of nonlinear analysis such as normal form theory, centre manifold theory, asymptotic analysis, Abelian integrals [Bli33, DR06], foliations [CN13], among others.

Extensions and generalizations of the results of [DR96] can be found in [KS01c]. The setting of [KS01c] is not confined to the van der Pol oscillator, but it applies to general one parameter families of planar fast-slow systems (3.9) having and S-shaped critical manifold (a particular case being the van der Pol oscillator). Similar to [DR96] the main result of [KS01c] is the proof that the transition between small amplitude periodic orbits to relaxation oscillations occurs within an exponentially small interval of the parameter $\omega$, the precise statement is in [KS01c] Theorem 3.3], while results on the stability of the canard cycles are given in [KS01c] Theorems 3.4, 3.5 and 3.6]. We briefly summarize the strategy to prove such claims as follows. Given an $S$-shaped critical manifold, one assumes that one of the fold points is generic while the other is a canard point. Then, one can locally study the flow near the aforementioned fold points with the techniques reviewed in Section 3.1 and at the beginning of this Section. Afterwards, families of periodic orbits can be constructed by “gluing” together pieces of trajectories in the blow-up space with trajectories of the layer equation and trajectories near normally hyperbolic parts of the critical manifold. A schematic of this construction is shown in Figure 6, while a bifurcation diagram, also representing the canard explosion is shown in Figure 7.

So far, we have seen that a mechanism leading to canards in planar fast-slow systems is a singular Hopf bifurcation. Some higher codimension bifurcations leading to canards have also been studied via the blow-up method. For example, for the Bogdanov-Takens bifurcation, which is of codimension 2, we have e.g. [DMD11b, Chi11]. The degenerate case, that is a planar system for which normal hyperbolicity is lost up to arbitrary order, has been dealt with in e.g. [DMD06]. Another recent discovery is that canards can also be found in cases where the critical manifold
Figure 5. Singular canard cycles (or limit periodic sets) giving rise to canard cycles in the van der Pol oscillator, compare with Figure 4. For $\varepsilon > 0$ sufficiently small closed orbits exist as perturbations of these singular cycles.

Figure 6. Schematic of the families of periodic orbits, parametrized by $\omega$ in (3.21), constructed by gluing together trajectories in the blow-up space with trajectories in the singular limit. The flow near the fold and canard points (compare with Figure 4) have been replaced by the flow in the blow-up space. Appropriate combinations of orbits correspond to those depicted in Figure 4. Perturbations of these orbits lead to canard cycles for $\varepsilon > 0$ sufficiently small.

is of codimension zero [KM18]. Finally, we note that there is strong active research towards proving the finiteness claim of Hilbert’s 16th problem [Sma98] based on the tools (and their further generalizations) described in this Section. A non-exhaustive list of references dealing with the cyclicity of canard cycles is: for (fast-slow) Liénard systems [DMD08, DMD11a, DMDR11]
In particular, we have that for $A < 0$ all canard cycles are stable.

A proof of finite cyclicity for fast-slow Darboux systems is available in [BG16].

3.3. Folded singularities and canards in higher dimensions. In Section 3.2 we have seen that canards appear in one-parameter families of planar fast-slow systems having a folded critical manifold. However, canards are degenerate in such a case since they exist only for an exponentially small interval of parameter values and disappear after small perturbations of the parameter. This situation is “fixed” in fast slow-systems with at least two slow variables, where canards are generic.

Canards in $\mathbb{R}^3$ were first studied in [Ben83, Ben90, Ben01] with techniques from non-standard analysis; see also [BKR02] for a different approach in which the critical manifold is assumed to have a generic self-intersection. In this section we review some of the works that, based on the blow-up method, have provided a geometric description of canards in fast-slow systems with two, or more, slow variables.

Let us start with a fast-slow system with two slow and one fast variable given by

$$
\begin{align*}
    x' &= f(x, y_1, y_2, \varepsilon), \\
    y_1' &= \varepsilon g_1(x, y_1, y_2, \varepsilon), \\
    y_2' &= \varepsilon g_2(x, y_1, y_2, \varepsilon),
\end{align*}
$$

where $(x, y) = (x, y_1, y_2) \in \mathbb{R}^3$. Analogous to the planar case, the critical manifold

$$
C_0 = \{(x, y_1, y_2) \in \mathbb{R}^3 | f(x, y_1, y_2, 0) = 0\}
$$

is a non-degenerate folded surface if

$$
\begin{align*}
    f(0, 0, 0, 0) &= 0, \\
    \frac{\partial f}{\partial x}(0, 0, 0, 0) &= 0, \\
    \frac{\partial^2 f}{\partial x^2}(0, 0, 0, 0) &\neq 0, \\
    D_y f(0, 0, 0, 0) &\neq 0 \in \mathbb{R}^2.
\end{align*}
$$

Without loss of generality we may assume that $\frac{\partial f}{\partial y_1}(0, 0, 0, 0) \neq 0$. Note that there is a fold line along $\ell_f = \{(x, y) \in \mathbb{R}^3 | x = 0, y_1 = 0\}$ and that the phase-space of the slow subsystem is 2-dimensional.
Similar to the planar case, the corresponding critical manifold is a folded surface as shown in Figure 8.

Analogous to the planar case, we say that the origin is a generic fold point if $g_1(0, 0, 0, 0) \neq 0$. In this case, (3.23) can be rewritten as

$$
\begin{align*}
x' &= y_1 + x^2 + O(\varepsilon x, \varepsilon y_1, \varepsilon y_2, \varepsilon^2, x^3, xy_1 y_2, x^3 y_1), \\
y'_1 &= \varepsilon (1 + O(x, y_1, y_2, \varepsilon)), \\
y'_2 &= \varepsilon g_2(x, y_1, y_2, \varepsilon).
\end{align*}
$$

(3.26)

The analysis of (3.26) as its flow passes near $\ell_f$ is similar to that of the generic fold in Section 3.1, and the details can be found, for example, in [SW04 Theorem 1]. On the other hand, we say that the origin of (3.23) is a folded singularity if besides (3.25) we have

$$
\left( \frac{\partial f}{\partial y_1} g_1 + \frac{\partial f}{\partial y_2} g_2 \right) |_{(0,0,0,0)} = 0.
$$

(3.27)

If the origin is a folded singularity, then (3.23) can be rewritten as [SW01 Proposition 2.1]

$$
\begin{align*}
x' &= y_1 + x^2 + O(\varepsilon x, \varepsilon y_1, \varepsilon y_2, \varepsilon^2, x^3, xy_1 y_2, x^3 y_1), \\
y'_1 &= \varepsilon (ax + by_2 + O(y_1, \varepsilon, y_2^2, xy_2, x^2)), \\
y'_2 &= \varepsilon (c + O(x, y_1, y_2, \varepsilon)).
\end{align*}
$$

(3.28)

Since a folded singularity is necessary for the existence of canards, we now focus on (3.28). Note that due to the leading order terms of (3.28) we have that $C_0$ is attracting for $x < 0$ and repelling for $x > 0$. The relationship between the flow on the critical manifold and the fold line is of vital importance. A classification of the possible phase-portraits on $C_0$ near $\ell_f$ can be found in [Tak76 Theorem 5.1] and in [SW01 Lemma 2.1]. Such a classification makes use of the so-called desingularized system [Tak76 Section 4.10] which, up to leading order terms, is given by

$$
\begin{align*}
\dot{x} &= ax + by_2, \\
\dot{y}_2 &= -2cx.
\end{align*}
$$

(3.29)

Note that the origin is an equilibrium point of (3.29). Even though the origin is not an equilibrium point of (3.28), it is possible to relate the flow of (3.29) to the flow of (3.28) on $C_0$. Thus, a classification of the flow of (3.29) is useful and is summarized in Table 1.

| Condition | Eigenvalues | Name |
|-----------|-------------|------|
| $bc < 0$  | $\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 < 0 < \lambda_2$ | saddle |
| $0 < 8bc < a^2$ | $\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 \lambda_2 > 0$ | node |
| $a^2 < 8bc$ | $\lambda_{1,2} \in \mathbb{C}$, $\Re(\lambda_{1,2}) \neq 0$ | focus |
| $bc = 0$, $c \neq 0$ | $\lambda_1 = 0$, $\lambda_2 = a$ | saddle-node type I |
| $bc = 0$, $b \neq 0$ | $\lambda_1 = 0$, $\lambda_2 = a$ | saddle-node type II |
| $8bc = a^2$ | $\lambda_{1,2} = \frac{a}{2}$ | degenerate node |

Table 1. Topological Classification of (3.29).

We can see from the above classification that the saddle, node, and focus cases are generic, while the saddle-node and the degenerate node are both degenerate and of codimension 1. From (3.29) one can obtain the flow on $C_0$ as follows: away from $\ell_f$, the trajectories on $C_0|_{x<0}$ are given by trajectories of (3.29) restricted to $x < 0$, while trajectories on $C_0|_{x>0}$ are given by reversing the
trajectories of (3.29) restricted to \( x > 0 \), see the details in e.g. [Tak76] Section 4.10, and Figure 8 for a couple of examples.

Due to the above classification and relationship between the slow subsystem and the desingularized system, we call the origin of (3.28) a folded saddle, folded node, folded focus, folded saddle-node, or folded degenerate node according to the coefficients of Table 1. We now have the following definition of canards coming from the flow on \( C_0 \).

**Definition 3.7.** Solutions of the slow subsystem passing through a canard point from an attracting critical manifold to a repelling critical manifold are called **singular canards**. Solutions of the slow subsystem passing through a canard point from a repelling critical manifold to an attracting critical manifold are called **faux canards**.

See, for example Figure 8 (a) where \( \Gamma_1 \) is a singular canard and \( \Gamma_2 \) a faux canard.

It follows from the classification of Table 1 and the resulting reduced flow that there are singular canards for all cases, except for the folded focus case. We can now recall the main result of [SW01], stating under which conditions singular canards persist for \( \varepsilon > 0 \) sufficiently small.

**Theorem 3.8** ([SW01], Theorem 4.1). Assume (3.28). In the folded saddle and in the folded node case singular canards \( \Gamma_1 \) perturb to maximal canard solutions for sufficiently small \( \varepsilon \).

For a folded node with \( \lambda_1 < \lambda_2 < 0 \) a maximal canard solution corresponding to the weak eigendirection \( (\Gamma_2) \) exists for sufficiently small \( \varepsilon \) provided that the ratio \( \mu_2 = \frac{\lambda_1}{\lambda_2} \) is not a natural number.

See Figure 8 for a schematic representation of the trajectories \( \Gamma_1 \) and \( \Gamma_2 \) in each case. We shall denote the perturbation of \( \Gamma_1 \) and of \( \Gamma_2 \) by \( \Gamma_{1,\varepsilon} \) and \( \Gamma_{2,\varepsilon} \), respectively. The idea of the proof is similar to that in Section 3.2. The relevant charts are \( K_1 = \{ \bar{y}_1 = 1 \} \) and \( K_2 = \{ \bar{\varepsilon} = 1 \} \). It is precisely in chart \( K_2 \) where transversality of the intersection of the slow manifolds can be shown.

We now sketch the main procedure: first of all, the blown up vector field in chart \( K_2 \) has (up to appropriate conditions met by the hypothesis of Theorem 3.8) explicit algebraic solutions \( \gamma_1, \gamma_2 \) ([SW01], Lemma 4.2]. These are crucial as they connect the repelling and attracting parts of the slow manifolds (equivalently, and analogously to Section 3.2, of centre manifolds in chart \( K_1 \)). Moreover, using variational arguments, it is further shown by studying a Weber equation (see [AS65], Chapter 19 and [OLBC10], Chapter 12]) that the invariant manifolds in chart \( K_2 \) corresponding to the slow manifolds \( S^a_\varepsilon \) and \( S^r_\varepsilon \) intersect transversally along the \( \gamma_i \) solutions. Furthermore, the meaning of the ratio \( \mu_2 \) is quite interesting.

**Lemma 3.9** ([SW01], Lemma 4.4]). Suppose we are in the node scenario and that \( n - 1 < \mu_2 < n \). Then the slow manifolds \( S^a_\varepsilon \) and \( S^r_\varepsilon \) twist \( n - 1 \) times around the corresponding maximal canard in the neighborhood of the fold curve.

We have so far covered all generic cases according to Table 1 except for those involving resonances in the folded node case [Wec05]. The main effect that is observed due to resonances is that there are other maximal canards that bifurcate from the weak maximal canard. In terms of the slow manifolds, this means that \( S^a_\varepsilon \) and \( S^r_\varepsilon \) have further transversal intersections, called **secondary maximal canards**.

To better grasp what happens in this situation, let us first note that in the folded node case we can differentiate two types of singular canards: a) singular canards \( \Gamma_1 \) and \( \Gamma_2 \) corresponding to the eigendirections of (3.29) and b) all other singular canards within the shaded region of Figure 8. The latter type of singular canards may also correspond to maximal canards for \( \varepsilon > 0 \) sufficiently small, this motivates the next definition.
Definition 3.10. All maximal canards not obtained as perturbations of the singular canards $\Gamma_1, \Gamma_2$ using Theorem 3.8 are called "secondary maximal canards."

Skipping the technicalities, the main result of [Wec05] is as follows: "If $\mu_2 \in \mathbb{N}$, then for $\varepsilon > 0$ sufficiently small there are additional secondary maximal canards bifurcating from the weak maximal canard upon variation of the ratio $\mu_2$". The proof of the aforementioned bifurcation result requires a highly non-trivial analysis partially contained in [Wec05] and further completed in [MW17].

Remark 3.11. At this point we would like to emphasize the main difference between the folded node and the folded saddle cases: in the case of the folded saddle, the canards that persist are those related to the stable and unstable manifolds of the desingularized system (3.29), while the number of canards for the node case is much larger due to the existence of secondary canards. In particular, secondary maximal canards are produced at resonances, that is at positive integer values of the ratio $\mu_2$. Moreover, the number of generated secondary maximal canards is determined by $\mu_2$.

Visualizing the slow manifolds, especially as they twist in the folded node case, is difficult and the reader is referred to e.g. [DKO08, GH05] for further details. We finalize this section by briefly sketching some generalizations and referring to works regarding the degenerate saddle-node cases.

- The canard theory sketched above can be extended to fast-slow systems with $m \geq 2$ slow and $n \geq 1$ fast variables [Wec12]. The arguments use the assumption of $n - 1$ stable fast directions and centre manifold reduction.
- The codimension 1 folded saddle node cases are studied with the help of the blow-up method in [KW10] (FSN-II) and in [VW15] (FSN-I). One primary interest in studying such folded singularity is their relation to complex oscillatory motion such as Mixed Mode Oscillations (MMOs) [DGK+] and chaos. Furthermore, folded saddle-nodes appear in many models of applied interest such as the forced van der Pol oscillator [BEG03, GWY06, BDG+16], biochemical reactions [MSLC98, MS01, Moe02], and neuron dynamics [EW09, KPK08, DMW15], to mention a few. Unfolding such degenerate cases, that is considering one-parameter families of fast-slow systems with folded singularities, provides a much richer geometry. One difference to keep in mind between the two cases is that the FSN-II has an associated singular Hopf bifurcation with two slow variables [Guc08], while the FSN-I does not present a Hopf bifurcation but rather a true saddle-node. There are several surprising behaviors to be noted: the first is the possibility of canard trajectories along $S_r^{\varepsilon}$ to jump back towards $S_a^{\varepsilon}$; the second is the existence of oscillatory motion of faux canards around a maximal canard, see [MW17].

3.4. Delayed Hopf bifurcation. In dynamic bifurcation problems, that is where the bifurcation parameter is slowly changed, it can be observed that the transition to instability occurs for parameter values way beyond the prediction of the static bifurcation diagram [Nei09]. This effect is known as "delayed loss of stability". One landmark example of such a phenomenon occurs in the so-called "slow passage through a Hopf bifurcation" [BER89]. The aforementioned dynamical behavior is also highly relevant in many applications, e.g. [SM96, BC11]; see also [KV18] for a recent example of slow passage through a Hopf bifurcation in reaction-diffusion equations.

Although the problem of slow passage through a Hopf bifurcation has been long studied, [HKSW16] presents a new geometric proof based on the blow-up method, which we now sketch.
Figure 8. Phase-portraits for the folded saddle (first row) and folded node (second row) cases of Table 1. The planar phase-portraits (a),(c) correspond to the flow of the desingularized vector field (3.29), while the phase-portraits (b),(d) are the corresponding reduced flows on $C_0$. Note that trajectories on the stable (i.e. lower) part of $C_0$ are given by trajectories of the desingularized system for $x < 0$ while the trajectories on the unstable (i.e. upper) part of $C_0$ are given by reversing the direction of the trajectories of the desingularized system for $x > 0$. In the saddle case (a),(b) the trajectory $\Gamma_1$ is a singular canard while $\Gamma_2$ is a faux canard. Note that these are the only canard trajectories. In the node case (c),(d) $\Gamma_1$ is a singular canard tangent to the strong eigendirection while $\Gamma_2$ is a singular canard tangent to the weak eigendirection. Observe that, in contrast to the saddle case, in the node case we have many other singular canard trajectories indicated by the shaded region in $C_0$. 
Consider a fast-slow system (2.1)-(2.2) with \((m, n) = (2, 1)\) given by
\[
\begin{align*}
    x_1' &= yx_1 - x_2 - x_1(x_1^2 + x_2^2), \\
    x_2' &= x_1 + yx_2 - x_2(x_1^2 + x_2^2), \\
    y' &= \varepsilon.
\end{align*}
\] (3.30)

Note that for \(\varepsilon = 0\) we have that (3.30) corresponds to the normal form of a (supercritical) Hopf bifurcation with \(y\) the bifurcation parameter and \(y = 0\) the bifurcation point. This motivates the following definition.

**Definition 3.12.** A Hopf bifurcation that occurs in the layer equation of a fast-slow system in which the slow variable acts as the bifurcation parameter is called **Delayed Hopf Bifurcation**.

The idea now is to study the dynamics of (3.30) as the “dynamic parameter” \(y\) slowly passes through the Hopf bifurcation point \(y = 0\). It is straightforward to see that the critical manifold is
\[
C_0 = \{(x_1, x_2, y) \in \mathbb{R}^3 | x_1 = 0, x_2 = 0\},
\] (3.31)
i.e., it is just the \(y\)-axis. One can further show, for example by changing to polar coordinates, that \(C_0|_{y < 0}\) is attracting while \(C_0|_{y > 0}\) is repelling, and that the origin is a non-hyperbolic point of the layer equation. The next observation is of crucial importance: the linearization of the layer equation (3.30) at the non-hyperbolic origin is not nilpotent. In fact it is given by \(\dot{x} = Jx\), where
\[
J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\] (3.32)

This is a first obstacle to use the blow-up method. Furthermore, note that such linearization corresponds to a fast rotation of the fast variables. We will see below that the main idea of [HKSW16] is to “eliminate” such a fast rotation allowing the use of the blow-up method. It is convenient to consider a modified complex version
\[
\begin{align*}
    z' &= (y + i)x - |x|^2x + \varepsilon k_0, \\
    y' &= \varepsilon.
\end{align*}
\] (3.33)
System (3.33), with \(k_0 = 0\) is obtained from (3.30) via the change of coordinates \(z = x_1 + ix_2\) with \(i = \sqrt{-1}\). The term \(\varepsilon k_0\), \(k_0 \neq 0\), is added to break the invariance \(\{z \in C_0 \mid y = 0\}\) for \(\varepsilon > 0\). It is also clear from (3.33) that \(z = 0\) is a stable (resp. unstable) focus for \(y < 0\) (resp. \(y > 0\)) and that at \(z = 0\) an exchange of stability occurs. Let \(\delta > 0\) be small and, as usual, define
\[
\begin{align*}
    S_a^\varepsilon &= \{(z, y) \in C_0 | y < -\delta\}, \\
    S_r^\varepsilon &= \{(z, y) \in C_0 | y > \delta\},
\end{align*}
\] (3.34)
and we denote by \(S_a^0\) and \(S_r^0\) the corresponding slow manifolds. The manifolds \(S_a^\varepsilon\) and \(S_r^\varepsilon\) have analytic extensions to \(y = 0\), and the goal is to compute the distance between \(S_a^\varepsilon\) and \(S_r^\varepsilon\) at \(y = 0\) for \(\varepsilon > 0\) sufficiently small. Let such a distance be denoted by \(d(\varepsilon)\). The main result of [HKSW16] is a new geometric proof of the following well-known theorem.

**Theorem 3.13** ([HKSW16] Theorem 3.1) see also [Nei87, Shi73]. Let \(k_0 \neq 0\). For sufficiently small values of \(\varepsilon > 0\),
\[
d(\varepsilon) \leq \left(\sqrt{\varepsilon \sqrt{2\pi} \kappa} + O(\varepsilon)\right) \exp\left(-\frac{1}{2\varepsilon}\right),
\] (3.35)
\[^3\text{In principle a generic term } O(\varepsilon) \text{ can be added instead of } \varepsilon \kappa, \text{ but the results are similar in such a case.}\]
where $\kappa > 0$ is bounded away from 0, and is determined in part by $|k_0|$.

The implication of Theorem 3.13 is that, due to the fact that the separation between the slow manifolds $S^s_1$ and $S^u_1$ is exponentially small, solutions starting near $S^s_1$ stay close to $S^u_1$ for time $O(1/\varepsilon)$ after crossing the bifurcation point. This is observed as a delay in the onset of (large amplitude) oscillations. To prove Theorem 3.13 one first considers manifolds associated to integrate along $\Gamma$.

The main observation is that, since (3.33) is analytic, by Cauchy’s integral theorem it is the equivalent to integrate along $\Gamma_{-1}$ than to integrate along the real $y$-axis. The advantage, however, is that along $\Gamma_{-1}$, the fast rotation is eliminated, and the resulting non-hyperbolic equilibrium point is nilpotent. So, the fast-slow system (3.33) along $\Gamma_{-1}$ reads as

\begin{align}
\begin{aligned}
z' &= sz - |z|^2 z + \varepsilon k_0, \\
s' &= \varepsilon,
\end{aligned}
\end{align}

where we can see that $(z, s, \varepsilon) = (0, 0, 0)$ is indeed a nilpotent point. We now proceed to sketch the blow-up analysis. The blow-up map is given by

\begin{align}
\begin{aligned}
z &= r\hat{z}, \\
s &= r\hat{s}, \\
\varepsilon &= r^2 \hat{\varepsilon}.
\end{aligned}
\end{align}

The relevant charts are $K_1 = \{\hat{s} = -1\}$, $K_2 = \{\hat{\varepsilon} = 1\}$, and $K_3 = \{\hat{s} = 1\}$. Next, the strategy is similar to the planar fold case: in chart $K_1$ (resp. $K_3$) one finds a 2-dimensional centre-stable (resp. centre-unstable) manifold corresponding to $S^a_2$ (resp. $S^s_2$). Then, in chart $K_2$ one tracks such invariant manifolds. To compute the distance between the manifolds we take advantage of the fact that the blown-up vector field in $K_2$ can be integrated along appropriate paths. We remark that due to the cubic terms in (3.37) a secondary blow-up is performed. The delayed onset of oscillations due to a slow passage through a Hopf bifurcation can be observed in Figure 9.

3.5. Beyond fold and Hopf singularities. As we have reviewed so far, the understanding of codimension-one fold and Hopf points is of extreme importance due to their genericity in nonlinear fast-slow systems. However, there are many other singularities that have also been studied via the blow-up method, which we summarize here.

Transcritical and pitchfork singularities: Let us consider (2.2) with $(x, y) \in \mathbb{R}^2$ and assume that the layer equation undergoes either a transcritical or a pitchfork bifurcation at the origin where $y \in \mathbb{R}$ is the bifurcation parameter [KS01b]. The goal is to understand the dynamics for $\varepsilon > 0$ small in the neighbourhood of the aforementioned singularities.

In the case of the transcritical singularity, it can be shown (Lemma 2.1 in [KS01b]) that a canonical form is given as

\begin{align}
\begin{aligned}
x' &= x^2 - y^2 + \omega \varepsilon + O(x^3, x^2 y, xy^2, y^3, \varepsilon x, \varepsilon y, \varepsilon^2), \\
y' &= \varepsilon(1 + O(x, y, \varepsilon)),
\end{aligned}
\end{align}

where $\omega$ is considered as a continuous parameter. The role of the parameter $\omega$ is to distinguish two types of dynamics, as is stated below in Theorem 3.14. The critical manifold $C_0$ is given by $C_0 = \{(x, y) \in \mathbb{R}^2 \mid x = \pm y + O(y^2)\}$, which clearly has a self-intersection at the origin. From another perspective, the critical manifold can be regarded as the union of one dimensional manifolds $\{x = y + O(y^2)\}$ and $\{x = -y + O(y^2)\}$, which intersect transversally at the origin. Such intersection implies that the origin is a non-hyperbolic equilibrium point of the layer equation. In either
Figure 9. Simulation of a slow passage through a Hopf bifurcation of (3.30) with $\varepsilon = 0.05$. The critical manifold is given by $C_0 = \{(x_1, x_2) = (0, 0)\}$, it is attracting (solid black line) for $y < 0$ and repelling (dashed black line) for $y > 0$. The point $H = (0,0,0)$ denotes the bifurcation point and the wire-frame paraboloid represents the amplitude of oscillations according to the $y$-parameter value. We show two trajectories and observe that the onset of oscillations occurs way beyond the Hopf point.

case, it is convenient to assume that a compact subset of $C_0$ around the origin is given by the union of four branches, namely $S_{0,+}^a, S_{0,-}^a, S_{0,+}^r, S_{0,-}^r$, as shown in Figure 10. Thus, away from the origin, Fenichel's theorem guarantees that the previous critical manifolds persist as slow manifolds $S_{\varepsilon,+}^a, S_{\varepsilon,-}^a, S_{\varepsilon,+}^r, S_{\varepsilon,-}^r$ for $\varepsilon > 0$ sufficiently small. The idea of [KS01b] is to describe the fate of $S_{\varepsilon,-}^a$ as it crosses a small neighborhood of the origin. In order to state the main result regarding the transcritical singularity, let $\rho > 0$ be small and $I \in \mathbb{R}$ a small interval containing 0 $\in \mathbb{R}$. Define the sections

$$
\Sigma_{en} = \{(x, y) \in \mathbb{R}^2 \mid x = -\rho, y + \rho \in I\}, \\
\Sigma_{ex} = \{(x, y) \in \mathbb{R}^2 \mid x = \rho, y \in I\}, \\
\Sigma_{ea} = \{(x, y) \in \mathbb{R}^2 \mid x = -\rho, y - \rho \in I\}.
$$

Let $\Pi_e : \Sigma_{ex} \to \Sigma_{ea}$ and $\Pi_a : \Sigma_{en} \to \Sigma_{ex}$ denote the corresponding transitions.

**Theorem 3.14 (KS01b).** Consider (3.39) and fix $\omega \neq 1$. There exists $\varepsilon_0 > 0$ such that the following assertions hold for $\varepsilon \in (0, \varepsilon_0)$.

(T1) If $\omega > 1$ then the manifold $S_{\varepsilon,-}^a$ passes through $\Sigma_{ea}$ at a point $(\rho, h(\varepsilon))$ where $h(\varepsilon) \in O(\sqrt{\varepsilon})$.

The section $\Sigma_{en}$ is mapped by $\Pi_a$ to an interval containing $S_{\varepsilon,-}^a \cap \Sigma_{ea}$ of size $O(\exp(-C/\varepsilon))$, where $C$ is a positive constant.

(T2) If $\omega < 1$ then $\Sigma_{en}$ (including the point $S_{\varepsilon,-}^a \cap \Sigma_{en}$) is mapped by $\Pi_e$ to an interval about $S_{\varepsilon,+}^a$ of size $O(\exp(-C/\varepsilon))$, where $C$ is a positive constant.

See Figure 10 for a schematic of the results of Theorem 3.14. It is important to note that the constant $\varepsilon_0$ tends to 0 as $\omega \to 1$. In fact, the case $\omega = 1$ is degenerate and is not detailed in
However, it is argued in Remark 2.2 of [KS01b] that canard solutions occur in this case, the treatment being similar to that of [KS01a], see also the Example in Section 3 of [KS01b]. From another perspective, i.e. not using blow-up, a particular case of this degenerate scenario is investigated in [Sch85].

For the proof of Theorem 3.14 one employs a blow-up map defined by $\phi(r, \bar{x}, \bar{y}, \bar{z}) = (r\bar{x}, r\bar{y}, r^2\bar{z})$. The blow-up analysis is carried out in three charts, namely $K_1 = \{\bar{x} = -1\}$, $K_2 = \{\bar{z} = 1\}$ and $K_3 = \{\bar{x} = 1\}$. The analysis in charts $K_1$ and $K_3$ is similar and amounts to describing the organization of the dynamics according to two 2-dimensional centre manifolds, and which are tracked along the blow-up space. Another important remark is that in the charts $K_1$ and $K_3$ the parameter $\omega$ does not play any essential role. In chart $K_2$, the main object of study is the ODE

$$
x_2' = x^2 - y^2 + \omega,
$$$$
y_2' = 1,$$

where as usual we are using $(x_2, y_2)$ to denote local coordinates. The main goal in this chart is to connect the appropriate centre manifolds of charts $K_1$ and $K_3$.

A more general scenario compared to the one presented above is treated in [DM15], where in brief terms, unfoldings of Eq. 3.39 are considered. In fact, since [DM15] considers fast-slow systems with a transcritical singularity in its most generic context, more passages as compared to those shown in Figure 10 can occur, [DM15, Theorems 5 and 6]. The most important of such passages is, of course, the canard. A consequence of this analysis is the extension of known results on the stability of canard cycles [DMDR11] to the case including finite passages through a transcritical singularity [DM15, Theorem 4].

Regarding the *pitchfork singularity*, it is shown (Lemma 4.1 of [KS01b]) that a canonical form is given by

$$
x' = x(y - x^2) + \omega \varepsilon + a x^2 y + O(xy^2, y^3, \varepsilon x, \varepsilon y, \varepsilon^2),
$$$$
y' = \varepsilon(\pm 1 + bx + O(y, \varepsilon)),$$

where $\omega$ is assumed non-zero and $(a, b) \in \mathbb{R}^2$. In this case, the analysis and results depend on the sign in Eq. 3.42. Similar to the transcritical case, the critical manifold can be given as the union of four branches $S_{0, +}, S_{0, -}, S_{0, +}, S_{0, -}$. Away from a small neighbourhood of the origin and for $\varepsilon > 0$ sufficiently small, the previous four branches persist as the slow manifolds $S_{e, +}, S_{e, -}, S_{e, +}, S_{e, -}$ respectively, see Figure 10. Let $\rho > 0$ be fixed and $I \subset \mathbb{R}$ be a small interval around 0 $\in \mathbb{R}$. Define the sections

$$
\Sigma^I = \{(x, y) \in \mathbb{R}^2 \mid x \in I, y = -\rho\},
$$

$$
\Sigma^+ = \{(x, y) \in \mathbb{R}^2 \mid x = \rho, y - \rho^2 \in I\},
$$

$$
\Sigma^- = \{(x, y) \in \mathbb{R}^2 \mid x = -\rho, y - \rho^2 \in I\},
$$

and define transitions $\Pi^I : \Sigma^I \to \Sigma^+ \cup \Sigma^-$ and $\Pi^\pm : \Sigma^\pm \to \Sigma^I$. The former transition is well defined when the sign in Eq. 3.42 is positive while the latter is well defined when the sign in Eq. 3.42 is negative. Regarding the pitchfork singularity, the main result is:

**Theorem 3.15 ([KS01b])**. Consider Eq. 3.42 and fix $\omega \neq 0$. There exists $\varepsilon_0 > 0$ such that the following assertions hold for $\varepsilon \in (0, \varepsilon_0]$.

**P1** If the sign in Eq. 3.42 is positive and $\omega > 0$ then $\Sigma^I$ (including the point $\Sigma^I \cap S_{e, +}$) is mapped by $\Pi^I$ to an interval about $\Sigma^+ \cap S_{e, +}$ of size $O(\exp(-C/\varepsilon))$, where $C$ is a positive constant.
(P2) If the sign in (3.42) is positive and \( \omega < 0 \) then \( \Sigma^t \) (including the point \( \Sigma^t \cap S_{c,-}^a \)) is mapped by \( \Pi^t \) to an interval about \( \Sigma^- \cap S_{c,-}^a \) of size \( O(\exp(-C/\varepsilon)) \), where \( C \) is a positive constant.

(P3) If the sign in (3.42) is negative then \( \Sigma^+ \) and \( \Sigma^- \) are mapped by \( \Pi^+ \) and \( \Pi^- \), respectively, to intervals about \( S_{c,t}^a \cap \Sigma^t \) of size \( O(\exp(-C/\varepsilon)) \), where \( C \) is a positive constant.

See Figure 11 for a schematic of the claims of Theorem 3.15. In this case, the blow-up map is defined via \( \phi(r, \bar{x}, \bar{y}, \bar{\varepsilon}) = (r \bar{x}, r^2 \bar{y}, r^4 \bar{\varepsilon}) \). The blow-up analysis is carried out in five charts, namely \( K_1 = \{\bar{y} = -1\} \), \( K_2 = \{\bar{\varepsilon} = 1\} \), \( K_3 = \{\bar{y} = 1\} \), \( K_4 = \{\bar{x} = -1\} \) and \( K_5 = \{\bar{x} = 1\} \). As in the transcritical case, the idea is to track centre manifolds along the blow-up space. The relevant centre manifolds are first found in charts \( K_1 \) and \( K_3 \) and are related to normally hyperbolic parts of the critical manifold. Accordingly, \( M_{1,t}^a, M_{3,+,t}^a, M_{3,-t}^a \) and \( M_{3,t}^a \) denote centre manifolds (in the blow-up space) associated to the branches \( S_{0,t}^a, S_{0,+,t}^a, S_{0,-t}^a \) and \( S_{0,t}^a \) respectively. On the other hand, in charts \( K_4 \) and \( K_5 \) one finds hyperbolic equilibrium points.

The most important analysis occurs in chart \( K_2 \). In chart \( K_2 \) one must connect the aforementioned centre manifolds. We sketch the key argument: let \( M_{2, t}^a \) and \( M_{5, t}^a \) denote the centre manifolds \( M_{1, t}^a \) and \( M_{3, t}^a \), respectively, written in the coordinates of chart \( K_2 \). The main result of the analysis in chart \( K_2 \) is that the invariant manifolds \( M_{2, t}^a \) and \( M_{5, t}^a \) intersect transversally along \( (x_2, y_2, r_2) = (0, 0, t_2) \), where \( t_2 \in (-\infty, \infty) \). The proof of this fact is done via the analysis of a Melnikov integral, see Section 3 in [KS01b]. In terms of the slow manifolds, the previous result implies that for \( \varepsilon > 0 \) sufficiently small and \( \omega \neq 0 \), the distance between the slow manifolds \( S_{c,t}^a \) and \( S_{c,t}^r \) around the origin, is non-zero. On the other hand, when \( \omega = 0 \) the slow manifolds \( S_{c,t}^a \) and \( S_{c,t}^r \) are connected.

![Diagram](attachment:flow.png)

**Figure 10.** Flow of a fast-slow system near a transcritical singularity for \( \omega < 1 \) on the left and \( \omega > 1 \) on the right. The case \( \omega = 1 \) (not shown) corresponds to a maximal canard in which \( S_{c,-}^a \) and \( S_{c,+}^r \) coincide.

**Cusp singularity:** The cusp singularity is more degenerate than the fold. In fact, at the level of classification of singularities of generic smooth functions, the cusp singularity is the next in the list of singularities (or catastrophes) [Arn90]. This means that the cusp is of codimension 2, that is, it appears generically in 2-parameter families of smooth functions.
Figure 11. Phase-portraits of a fast-slow system near a pitchfork singularity for the cases of Theorem 3.15. Phase-portrait (a) (resp. (b)) corresponds to the case \( \omega < 0 \) (resp. \( \omega > 0 \)) with the sign of (3.42) positive, while phase-portrait (c) corresponds to the sign of (3.42) negative.

In the context of fast-slow systems, the cusp point was first studied in \[\text{BKK13}\]. Given a fast-slow system (2.1) with \((m, n) = (1, 2)\), the conditions for having a generic cusp point at the origin are

\[
\begin{align*}
f(0, 0, 0, 0) &= 0, \\
\frac{\partial f}{\partial x}(0, 0, 0, 0) &= 0, \\
\frac{\partial^2 f}{\partial x^2}(0, 0, 0, 0) &= 0
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial^3 f}{\partial x^3}(0, 0, 0, 0) &\neq 0 \\
\left[ \frac{\partial f}{\partial y_1}(0, 0, 0, 0) \quad \frac{\partial f}{\partial y_2}(0, 0, 0, 0) \right] \cdot g(0, 0, 0, 0) &\neq 0.
\end{align*}
\]
If a fast-slow system satisfies (3.44)-(3.45), then it can be written as [BKK13] Proposition 2.5
\begin{align}
x' &= -(x^3 + y_2x + y_1) + O(\varepsilon, xy_1, x^3y_2, x^4), \\
y'_1 &= \varepsilon(1 + O(x, y_1, y_2, \varepsilon)), \\
y'_2 &= \varepsilon O(x, y_1, y_2, \varepsilon).
\end{align}

We show in Figure 12 the corresponding critical manifold
\begin{equation}
C_0 = \{(x, y_1, y_2) \in \mathbb{R}^3 | -(x^3 + y_2x + y_1) + O(\varepsilon, xy_1, x^3y_2, x^4) = 0\}
\end{equation}
and the reduced flow near the cusp point for a particular choice of signs in (3.45). To state the main theorem of [BKK13] let us define the sections
\begin{align}
\Sigma^{en} &= \{(x, y_1, y_2) \in \mathbb{R}^3 | y_1 = -\delta\}, \\
\Sigma^{ex} &= \{(x, y_1, y_2) \in \mathbb{R}^3 | y_1 = \delta\},
\end{align}
where \(\delta > 0\) is small. Furthermore, let \(S_0^-\) and \(S_0^+\) be compact subsets of \(C_0\) satisfying\(^4\)
\begin{equation}
S_{0,-} = \{(x, y_1, y_2) \in C_0 | y_1 < -\delta\}, \\
S_{0,+} = \{(x, y_1, y_2) \in C_0 | y_1 > \delta\}.
\end{equation}
Accordingly we denote by \(S_{\varepsilon,-}\) and \(S_{\varepsilon,+}\) the respective slow manifolds. The main result describes the passage of trajectories starting near the slow manifold \(S_{\varepsilon,-}\) through a neighborhood of the cusp point:

**Theorem 3.16 ([BKK13] Theorem 2).** For system (3.46) there exists \(\varepsilon_0 > 0\) sufficiently small such that for all \(0 < \varepsilon \leq \varepsilon_0\) the following statements hold:

1. **The transition map** \(\Pi : \Sigma^{en} \to \Sigma^{ex}\) induced by the flow of (3.46) is a diffeomorphism mapping a rectangular neighborhood of \(S_{0,-}\) into \(\Sigma^{out}\).
2. The choice of \(S_{\varepsilon,+}\) can be made in such a way that \(\Pi(\Sigma^{en} \cap S_{\varepsilon,-}) \subset S_{\varepsilon,+}\).
3. The map \(\Pi\) is exponentially contracting in the \(x\) direction and the derivative in the \(y_2\) direction is bounded above and below. More precisely
\begin{equation}
\frac{1}{M} \leq \left| \frac{\partial \Pi}{\partial y_2} \right| \leq M.
\end{equation}

The proof of Theorem 3.16 follows the same ideas as for the generic fold case, so we point out some of the differences: first of all, due to the quasihomogeneity of the fast equation, the blow-up map is given by \((x, y_1, y_2, \varepsilon) = (\tilde{r}^{\frac{1}{2}} \tilde{x}, \tilde{r}^2 \tilde{y}_1, \tilde{r} \tilde{y}_2, \tilde{r}^3 \tilde{\varepsilon})\). These weights can be obtained by checking the quasi-homogeneity of the vector field. Next, due to the geometric properties of the critical manifold, it is now necessary to study the blown-up vector field in five charts, namely: \(K_{\pm \tilde{y}_1} = \{\tilde{y}_1 = \pm 1\}, \ K_{\pm \tilde{y}_2} = \{\tilde{y}_2 = \pm 1\}, \ K_{\pm \tilde{\varepsilon}} = \{\tilde{\varepsilon} = \pm 1\}\). The analysis starts in \(K_{-\tilde{y}_1}\). Then, depending on the initial conditions one transitions to either \(K_{+\tilde{y}_2}, K_{\pm \tilde{\varepsilon}}\), or to \(K_{-\tilde{y}_2}\). In \(K_{+\tilde{y}_2}\) we study trajectories passing sufficiently away from the cusp point and along the regular part of \(S_0\). In \(K_{\pm \tilde{\varepsilon}}\) we study trajectories passing through a small neighborhood of the cusp point. In \(K_{-\tilde{y}_2}\) we study trajectories passing sufficiently away from the cusp point and across the folded part of \(S_0\). It is important to remark that the flow is regular in the charts \(K_{+\tilde{y}_2}, K_{\pm \tilde{\varepsilon}}\) and \(K_{-\tilde{y}_2}\). Moreover, in \(K_{-\tilde{y}_2}\) one may invoke the

\(^4\)Note that both critical manifolds \(S_{0,-}\) and \(S_{0,+}\) are attracting, so we omit the superscript “a”.
results of the generic fold in Section 3.1. Finally, we further transition from $K_{+y_{1}}, K_{+\bar{y}_{1}}$ and $K_{-y_{2}}$ to $K_{+\bar{y}_{1}}$, completing the analysis. A schematic of Theorem 3.16 is shown in Figure 12.

Another contribution of [BKK13] is the study of trajectories with initial conditions below the cusp point. Interestingly, it can be shown that, under appropriate assumptions, two trajectories with exponentially close initial conditions cross $\Sigma_{ex}$ algebraically apart, see [BKK13, Proposition 3] and Figure 12 for a sketch.

There are some refinements to the results of Theorem 3.16 in the case where the higher order terms of (3.46) are of order $O(\varepsilon)$. In particular, [JKBR16] provides more accurate information on the transition map $\Pi : \Sigma_{en} \rightarrow \Sigma_{ex}$. The analysis performed in [JKBR16] uses a heavy amount of normal form theory, in particular results for quasihomogeneous vector fields [LS10], an appropriate normal form for fast-slow systems [JK15], as well as for semi-hyperbolic singularities [Tak71, Bon97, Bon96], and some other classical normal form techniques [Che63, Ste58].

Bogdanov-Takens: Fast-slow systems related to a standard Bogdanov-Takens bifurcation have been studied using the blow-up method in [DMD11b, Chi11]. Although the approaches are similar, the concerns and results are different, and here we sketch the most important ones. Let us start from [DMD11b] where a planar fast-slow system of the form

\begin{equation}
\begin{aligned}
x' &= y, \\
y' &= -xy + \varepsilon(b_{0} + b_{1}x + x^{2} + x^{3}G(x, \omega)) + y^{2}H(x, y, \omega),
\end{aligned}
\end{equation}
where $\omega$ denotes parameters in a compact subset of an Euclidean space (any finite dimension is allowed) and $b = (b_0, b_1) \sim 0 \in \mathbb{R}^2$, are considered. It is important to mention that (3.52) is a normal form, implying that the results of [DMD11b] apply to a much wider class of planar fast-slow systems which after changes of coordinates can be brought to the form (3.52). Note that for $\varepsilon > 0$, (3.52) indeed corresponds to (a versal unfolding of) the normal form of a Bogdanov-Takens bifurcation [Bog75] and [Tak01, Theorem 6.2]. The main result of [DMD11b] consists on a full description of the bifurcation independent of $\varepsilon$, in contrast with [Bog75, Tak01] for which “the validity of the results shrinks” as $\varepsilon \to 0$. The blow-up analysis is now considerably more involved than in the fold case, and is mostly performed in the family (or central) chart.

In [Chi11] a one-parameter family of fast-slow systems presenting a Bogdanov-Takens point is studied, but in this case in a system with two fast and one slow variables of the form

$$
\begin{align*}
x'_1 &= f_1(x_1, x_2, y, \varepsilon, \delta), \\
x'_2 &= f_2(x_1, x_2, y, \varepsilon, \delta), \\
y' &= \varepsilon g(x_1, x_2, y, \varepsilon, \delta).
\end{align*}
$$

(3.53)

Accordingly, the critical manifold is one-dimensional and is supposed to be “S-shaped”, similar to the one appearing in the van der Pol oscillator. As such, the critical manifold has two fold points. The main assumption is that such fold points coincide with a Bogdanov-Takens bifurcation point of the layer equation. Furthermore, it is assumed that the stable part of the critical manifold correspond to stable foci of the layer equation, while the unstable part of the critical manifold corresponds to saddle nodes, see Figure 13 for a schematic representation of the critical manifold and the flow of the layer equation.

![Figure 13. Schematic of the critical manifold of (3.53) considered in [Chi11], where the critical manifold is a 1-dimensional S-shaped curve. The parameter $\delta$, which controls the strength of attraction of $S_{0,\pm}$, plays a crucial role: for some pairs $(\varepsilon, \delta)$ (3.53) has a stable periodic orbit, while for other values of $(\varepsilon, \delta)$ (3.53) exhibits chaotic motion.](image)

The main results of [Chi11] prove and describe relaxation oscillations [Chi11, Theorem 2] and chaotic motion [Chi11, Theorem 3] of (3.53) according to the parameters $(\delta, \varepsilon)$. The parameter $\delta$ is used to control the strength of stability of the stable part of the critical manifold. The core
analysis heavily depends on studying a first Painlevé equation (see [OLBC10, Chapter 32] and [Inc56, Chapter 14.4]) in the blow-up space.

The above examples are particular instances of fast-slow dynamical systems, where singularities beyond the classical fold appear. As it is evident, the analysis of fast-slow systems becomes more complicated, for example, when the number of fast variables increases and/or the degeneracy of the non-hyperbolic points also increases. There are several other situations that fall out of the scope of folded or Hopf-type singularities, see Sections 3.6 and 3.7 for further references.

### 3.6. Further directions

In this survey we have reviewed research related to the foundations of the blow-up method as an important tool to analyze fast-slow systems. Naturally, there are particular topics that have not been covered in this survey, and moreover, there are new emerging and exciting research directions, mostly away from classical contexts, where the blow-up method is being employed. The following are just some examples of works concerning and/or related to fast-slow dynamics with non-hyperbolic singularities, where the blow-up method has proved useful already:

**Unbounded critical manifolds:** One usually considers loss of normally due to tangency of the critical manifold and the fast foliation along sets of lower dimension than the critical manifold itself. Clear examples are the fold points and lines reviewed in Sections 3.1, 3.2, and 3.3 and cusp points sketched in Section 3.5. However, it may occur that somewhere in the phase-space, the critical manifold aligns with the fast-foliation. These cases have been studied in [GS09, Kue14, Kri17].

**Codimension 0 critical manifolds:** In all examples provided so far, the critical manifold is at least of codimension 1. For example, for the generic fold in Section 3.1 the critical manifold is of codimension 1, while in the Bogdanov-Takens case of [Chi11], the critical manifold is of codimension 2. There are cases, however, where the critical manifold may have codimension 0. This situation has recently been considered in [KM18] motivated by hysteresis operators [KM17].

**Piecewise-smooth systems:** Canard theory (see Sections 3.2 and 3.3) has been extended to piecewise-linear systems [DGP+16, DFGK+18] and in a similar context the blow-up method has been used to analyze piece-wise smooth (fast-slow) systems [KH15a, KH15b, dMdS18, Jef16].

**Torus canards:** In all the fast-slow systems considered so far, the critical manifold is made of critical points of the fast dynamics. However, in high dimensional problems, the dynamics of the layer equation may converge to, for example, limit cycles instead of equilibrium points. In this context, canard theory has been extended to cases where the fast dynamics has limit cycles as limiting sets, leading to the so-called torus canards [BDG+16, Vo17], see also [JNS18].

**Discrete time systems:** The blow-up method has been recently used to study discrete fast-slow systems with a transcritical singularity [EK18] directly without using continuous approximations. This represents additional steps to extending the blow-up to the context of maps with singularities; see also [NS13, NSS09].

### 3.7. Applications

The main part of this document has been dedicated to reviewing important theoretical progress on the understanding of fast-slow dynamics using the blow-up method. As such, we have emphasized the usefulness of the method to, for example, explain the canard explosion phenomenon. However, the blow-up method has also been used in interesting particular applications. Here we briefly summarize some examples to illustrate the breadth of different areas, where the blow-up technique can be useful.
In neuroscience: The progress in the theory of fast-slow systems, particularly the understanding of canards, and the related mixed-mode oscillations [DGK+12], has had a great impact in neuroscience. As an example, the Fitzugh-Nagumo, the Hodgkin-Huxley, and other related models have been studied from several mathematical perspectives, especially from a multi-time scale approach [GK09, GK10, DKK13, RW07]. A recent instance where the blow-up method is instrumental is [CS18], where the transition between two distinct oscillatory patterns in the Fitzugh-Nagumo system is precisely described.

In systems biology: The analysis of a model for the division cycle of an embryonic cell, known as the Goldbeter model, is studied in [KS16]. The model is 3-dimensional with 2-slow and one fast direction, but is not written in the standard form (2.1). Nonetheless, it is shown that the model indeed has two timescales, and tools from geometric singular perturbation theory are then used. The main goal is to describe a regulatory phosphorylation-dephosphorylation cycle that occurs in two timescales; see also [KS11].

In chemistry: The Olsen model for the peroxidase-oxidase reaction is considered in [KS15]. There the model is 4-dimensional. Besides this evident obstacle, the Olsen model is not in standard form (2.1), has several small parameters, and has non-fold singularities. Even more interestingly, the Olsen model has no defined fast and slow variables, but presents three distinct regimes in which there are 1, 2, and 3 fast variables. The blow-up method is essential in this case to understand the dynamics in all the regimes and leads to the description of non-classical relaxation oscillations of the Olsen model.

In engineering: In [BBK17], a model describing an earthquake fault is considered. The model being studied is a Hamiltonian system, and possesses an unbounded critical manifold, see further information above. The blow-up method is used to show the existence of limit cycles associated to a degenerate Hopf bifurcation. Another recent application is found in [IPS18], where a Micro-Electro Mechanical System (MEMS) undergoing a singular effect known as “touchdown” is analyzed. In this case, the blow-up method allows a precise analysis of the steady states of a regularized model, and the description of the dynamics as a couple of small parameters tend to zero.

In control theory: Singular perturbation problems are classical in control theory. The case for which the critical manifold is normally hyperbolic is in fact well understood [KKO99]. The case where the critical manifold loses normal hyperbolicity had not received much attention. Recently, the blow-up method has allowed the design of controllers that stabilize non-hyperbolic points (of any degeneracy) of fast-slow systems with one fast variable [JKS18, JKSdPF19], and the model order reduction of fast-slow systems with hidden/degenerate normally hyperbolic critical manifolds [JKS17].

4. Summary and Outlook

This survey has been devoted to provide a concise recollection of the most influential progress in the blow-up theory of fast-slow systems. As such, we have principally covered folded singularities and in particular how the blow-up method allowed a geometric description of canards, canard cycles, and canard explosion. We have also outlined on the role of the blow-up method when analyzing other singularities beyond the fold, such as in the case of the Hopf, transcritical, pitchfork, cusp, and Bogdanov-Takens points. Moreover, we have briefly reviewed some recent theoretical and applied research in fast-slow systems, where the blow-up method is also one of the main mathematical tools for analysis.
It is evident that the blow-up method enables us to study and understand complicated fast-slow dynamics around non-hyperbolic points. However, it should also be clear that the difficulties do not end once we apply the method. In several cases, the dynamics within the charts are still quite intricate and, for example, one encounters special equations for which their asymptotics are important (e.g. Riccati, Weber, and Panilevě). Furthermore, one must trace invariant manifolds, compute their distance (Melnikov integral), control the obstacle of resonances, or deal with complex and/or complicated path integrals, just to mention a few of the difficulties.

We predict that for many of the theoretical and applied studies of fast-slow systems to come, the blow-up method will likely continue to be one of the techniques of choice when studying such dynamical systems. Furthermore, due to the everyday increase of complexity of systems and their models, the blow-up method will have to be extended and or adapted to such needs. We envision that, for instance, geometric studies of high dimensional systems and/or high codimension singularities are going to increase in relevance. Finally, although the blow-up method has been a great success for deterministic ODE systems, it is at this point a daunting challenge to extend it to a wider range of dynamical systems including partial and stochastic or even stochastic-partial differential equations.

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