The Dynamic Behavior of Quantum Statistical Entropy in 5D Ricci-flat Black String with Thin-layer Approach

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In this paper, the statistical-mechanical entropies of 5D Ricci-flat black string is calculated through the wave modes of the quantum field with improved thin-layer brick-wall method. The modes along the fifth dimension are semi-classically quantized by Randall-Sundrum mass relationship. We use the two-dimensional area to describe this black string's entropy which, in the small-mass approximation, is a linear sum of the area of the black hole horizon and the cosmological horizon. The proportionality coefficients of entropy are discretized with quantized extra dimensional modes. It should be noted that the small-mass approximation used in our calculation is naturally justified by the assumption that the two branes are located far apart.

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I. INTRODUCTION

In 1970s, Hawking [1] justified that the black holes should be treated as a thermodynamic system which contributes the temperature associated with its surface gravity, viz

\[ T = \frac{\kappa}{2\pi}, \]

where \( \kappa \) is the surface gravity at the horizon. Shortly before this discovery, the concept of black hole entropy is originally developed by Bekenstein [2]. The famous result is that the entropy of black hole is proportional to the area of its horizon. The quantitative entropy is

\[ S = \frac{1}{4} A, \]

where \( A \) is the area of horizon. In this way, there are many literatures studying the origin of black hole entropy by various approaches. One of them is the famous brick wall method (BWM) shown by 't Hooft. In this model, there is a brick wall around event horizon. In order to avoid the occurrence of divergent entropy (or free energy), the matter field is assumed to be vanished beyond a mini-distance outside black hole. Then brick wall and black hole construct a thermodynamic system. The entropy of black hole is identified with the statistical mechanical entropy created by the excitation of external quantum field. However, the application of BWM must satisfy the condition that the external field should be in thermal equilibrium with black hole in a large spatial region, i.e., BWM should not be used directly.
to the nonequilibrium system such as a black hole with multi-horizon. Recently, this confinement was solved by an improved BWM — the thin-layer approach [5]. In the thin-layer model, the large thermodynamics equilibrium is replaced by the local equilibrium on a microscopic scale. The mathematical difference of integration range implies a distinct physical significance. As an effective approach to nonequilibrium systems, the thin-layer model is usually employed to study many types of multi-horizon spaces, for instance, Schwarzschild-de Sitter black hole [6], Kerr-de Sitter black hole [7], Vaidya black hole [5] and so on [8]. In this paper, we try to use the thin-layer method to study the entropy of a 5D nonequilibrium black string, which there are two horizons that one is event horizon and the other is cosmological horizon.

Here the Space-Time-Matter (STM) [9] [10] theory is mentioned as the foundation of this work. In STM, the extra dimension is non-compacted. The 5D manifold is Ricci-flat while the 4D hypersurface is curved by the 4D induced matter. Mathematically, this approach is supported by Campbell’s theorem [11]. Recently an accelerating universe has proposed to be a way to interpret the astronomical data of type Ia supernovae [12]. Combining this astonishing observation in the standard cosmology leads to that our universe approaches de Sitter geometries in both past and future [13]. Hence we consider a Schwarzschild-de Sitter black hole embedded into a 5D Ricci-flat space. A class of 5D Ricci-flat black hole solutions containing a 4D de Sitter space are shown by Mashhoon et al [9] [14] [15]. In our previous work [16], a system with double Randall-Sundrum (RS) branes is constructed using these solutions. If matter trapped on the brane undergoes gravitational collapse, a black hole will form naturally and its horizon extends into the extra dimension which is transverse to brane. Such higher dimensional object looked like a black hole on the brane is actually a black string in the higher dimensional brane world. Hence a 5D Ricci-flat black string is obtained naturally. As one candidate of higher dimensional black holes, its thermodynamical aspect is needed to be studied.

On the other hand, in the last decade of 20th century the additional spacelike dimensional ADD model [17] and RS model [18] [19] raise the upsurge of study higher dimensional brane world. Based on these brane world model, various black holes are studied [20] and their entropy also becomes an interesting topic [21], and the STM theory is equivalent to the brane world model [22] [23] [24] [25]. Therefore it is worth to study the entropy of 5D Ricci-flat black string with a 4D effective cosmological constant.

This paper is organized as follows: In Section II, the 5D Ricci-flat black string metric and the surface gravity near horizons are presented. In section III, by semi-classical method the quantized modes are obtained with WKB approximation. In section VI, the entropy of 5D Ricci-flat black string is calculated with the assumption of far apart two branes. We adopt the signature (+, −, −, −, −) and put $\hbar$, $C$, and $G$ equal to unity. Greek indices $\mu, \nu, \ldots$ will be taken to run over 0, 1, 2, 3 as usual, while capital indices A, B, C, ... run over all five coordinates (0, 1, 2, 3, 4).

II. 5D RICCI-FLAT BLACK STRING WITH AN EFFECTIVE COSMOLOGICAL CONSTANT

A static, three-dimensional spherically symmetric 5D line element having a 4D effective cosmological constant takes the form [9] [14] [15]

$$ds^2 = \frac{\Lambda \xi^2}{3} \left[ f(r) dt^2 - \frac{1}{f(r)} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] - d\xi^2,$$

which actually is a black brane solution in brane world. The metric function is

$$f(r) = 1 - 2M \frac{r}{r} - \frac{\Lambda}{3} r^2,$$
where $\xi$ is an open non-compact extra dimension coordinate and $M$ is the central mass. One should note that the above $\Lambda$ is an induced cosmological constant which is obtained by the reduction from 5D to 4D. In another words, since this metric is Ricci-flat $R_{AB} = 0$, there is no cosmological constant in 5D space. So one can actually deal with this $\Lambda$ as a parameter which comes from the fifth dimension. The part of this metric inside the square bracket is exactly the same line-element as the 4D Schwarzschild-de Sitter solution, which is bounded by two horizons — an inner horizon (event horizon) and an outer horizon (cosmological horizon). This solution has been studied in many works focusing mainly on the induced constant $\Lambda$, the extra force and so on.

In the work of [16], the binary Randall-Sundrum branes system is constructed. Now we briefly list the nontrivial results below. By a coordinate transformation $\xi = \sqrt{3}/\Lambda \exp(\sqrt{\Lambda/3} y)$ [16], the metric (3) then takes a manifestly conformal form

$$ds^2 = e^{2\sqrt{\Lambda} y} \left[ f(r) dt^2 - \frac{1}{f(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - dy^2 \right].$$

(5)

Then we use this metric to construct a RS type brane model in which the first brane is at $y = 0$, and the second brane is at $y = y_1$. In this way the double brane model is obtained, and the fifth dimension becomes finite. It could be very small as in RS 2-brane model [18] or very large as in RS 1-brane model [19], and it is a black string intersecting the brane world, that is to say, on the hypersurface of fixed extra dimension this metric describes a SdS black hole. However, when viewed from 5D, the horizon does not form a 4D sphere — it looks like a black string lying along the fifth dimension. Hence we call the solution black string. Arnowitt-Deser-Misner (ADM) mass $\tilde{M}$ of the black string measured on the second brane is

$$\tilde{M} = M e^{\sqrt{\Lambda/3} y_1},$$

(6)

while its ADM mass $\tilde{M} = M$ locating at the first brane.

The metric function (4) can be recomposed as follows

$$f(r) = \frac{\Lambda}{3r} (r - r_e)(r_c - r)(r - r_o).$$

(7)

The singularity of the metric is determined by $f(r) = 0$. Here we only consider the real solutions. The solutions to this equation are black hole event horizon $r_e$, cosmological horizon $r_c$ and a negative solution $r_o = -(r_e + r_c)$. The last one has no physical significance, and $r_c$ and $r_o$ are given as

$$\left\{ \begin{array}{l}
  r_e = \frac{2}{\sqrt{\Lambda}} \cos \chi, \\
  r_c = \frac{2}{\sqrt{\Lambda}} \cos(\frac{2\pi}{3} - \chi),
\end{array} \right.$$  

(8)

where $\chi = \frac{1}{3} \arccos(-3M\sqrt{\Lambda})$ with $\pi/6 \leq \chi \leq \pi/3$. The real physical solutions are accepted only if $\Lambda$ satisfy $\Lambda M^2 \leq \frac{1}{9}$ [15].

So according to the metric function, the tortoise coordinate is

$$x = \frac{1}{2M} \int \frac{dr}{f(r)}.$$  

(9)

Integration of this equation shows that $x$ can be expressed explicitly in the following form:

$$x = \frac{1}{2M} \left[ \frac{1}{2K_e} \ln \left( \frac{r}{r_e} - 1 \right) - \frac{1}{2K_e} \ln \left( 1 - \frac{r}{r_e} \right) + \frac{1}{2K_o} \ln \left( 1 - \frac{r}{r_o} \right) \right].$$

(10)
where

\[ K_i = \frac{1}{2} \left| \frac{df}{dr} \right|_{r=r_i} \]  \hspace{1cm} (11)

That is

\[ K_e = \frac{(r_e - r_c)(r_e - r_o)}{6r_e} \Lambda, \]  \hspace{1cm} (12)

\[ K_c = \frac{(r_c - r_e)(r_c - r_o)}{6r_c} \Lambda, \]  \hspace{1cm} (13)

\[ K_o = \frac{(r_o - r_e)(r_o - r_c)}{6r_o} \Lambda. \]  \hspace{1cm} (14)

According to Hawking temperature formula (1), there are two distinct temperatures near event horizon \( r_e \) and cosmological horizon \( r_c \). So the 5D black string space is a nonequilibrium system apparently. Therefore, the BWM can not be applied directly to this multi-horizon space. So the proper selection is thin-layer approach — improved BWM method.

Being a minimally coupled quantum scalar field \( \Phi \) with mass \( m_0 \), the field equation on the background \( \xi \) is

\[ \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^A} \left( \sqrt{g} g^{AB} \frac{\partial}{\partial x^B} \right) \Phi - m_0^2 \Phi = 0. \]  \hspace{1cm} (15)

The modes of the scalar field can be decomposed as the separable form,

\[ \Phi = e^{-iEt} R_\omega(r) L(y) Y_{lm}(\theta, \phi), \]  \hspace{1cm} (16)

where \( E \) is the particle energy and \( Y_{lm}(\theta, \phi) \) is the usual spherical harmonic function. Then the equations for \( L(y) \) and \( R_\omega(r) \) read as follows,

\[ e^{-3\sqrt{3}y} \frac{d}{dy} \left( e^{3\sqrt{3}y} \frac{d}{dy} \right) L(y) + \left( e^{2\sqrt{3}y} m_0^2 + \mu^2 \right) L(y) = 0; \]  \hspace{1cm} (17)

\[ E^2 \frac{1}{f(r)} R_\omega(r) + \frac{1}{r^2} \frac{d}{dr} \left( r^2 f(r) \frac{d}{dr} \right) R_\omega(r) - \left( \mu^2 + \frac{l(l+1)}{r^2} \right) R_\omega(r) = 0. \]  \hspace{1cm} (18)

The eigenvalue \( \mu^2 \) is the effective mass on the brane and Eq. (18) is exactly the same as the usual radial equation of massive scalar particle around 4D SdS black hole. The similar effective mass can be found in the entropy of RS black string [21]. According to the above ADM mass relationship (6), one can get the effective mass \( \mu \) located on the two branes,

\[ \mu = \begin{cases} m_0, & \text{the first brane;} \\ m_0 e^{\sqrt{3}y_1}, & \text{the second brane.} \end{cases} \]  \hspace{1cm} (19)

III. SEMI-CLASSICAL QUANTIZED MODES ALONG EXTRA DIMENSION

In the interest of the entropy of this black string, we investigate the modes along the fifth dimension \( y \) before the other four dimensions \( (t, r, \theta, \phi) \). It is expected to find the definite expression of \( \mu \) which plays a role of the effective mass on the brane. So Eq. (17) is simplified as follows,

\[ \frac{d^2 L(y)}{dy^2} + \sqrt{3} \Lambda \frac{dL(y)}{dy} + \left( e^{2\sqrt{3}y} m_0^2 + \mu^2 \right) L(y) = 0. \]  \hspace{1cm} (20)
We assume the modes of the fifth dimension is \( L(y) = e^{i \gamma(y)} \) and the wave number \( k_y \) of wave function \( L(y) \) is
\[
k_y^2 = \left( \frac{\partial \gamma}{\partial y} \right)^2.
\]
Here we employ Wentzel-Kramers-Brillouin (WKB) approximation [4] in which the wave function is expanded in a series of Planck constant \( \hbar \) (small quantity). So the result can be obtained by stagewise approximation.

By WKB approximation [4], the wave number \( k_y \) can be written as
\[
k_y^2 = e^{2 \sqrt{\frac{\pi}{3} y^1 m_0^2}} + \mu^2. \tag{21}
\]

The extra dimension wave number \( n_y \) satisfies the semi-classical quantization condition
\[
\pi n_y(\mu) = \int_0^{y_1} dy k_y(y, \mu) = \sqrt{\frac{3}{\Lambda}} \left\{ - \sqrt{(m_0 - \mu)(m_0 + \mu)} + \sqrt{e^{2 \sqrt{\frac{\pi}{3} y_1} m_0^2} - \mu^2} \right. \\
+ \mu \cot^{-1} \left( \frac{\mu}{\sqrt{m_0^2 - \mu^2}} \right) - \mu \cot^{-1} \left( \frac{\mu}{\sqrt{e^{2 \sqrt{\frac{\pi}{3} y_1}} - 1}} \right) \left\}, \tag{22}
\right.
\]
where the tensions of branes are not restricted. It is very hard to directly solve \( \mu \) from the above function. In order to simplify calculation, the effective ADM mass relationship \( (19) \) is adopted here. If the first brane \( y = 0 \) is our real world, the RS relation is reduced to \( \mu = m_0 \). Substituting this ADM mass relationship into the quantization condition \( (22) \), the expected expression of effective ADM mass can be obtained as
\[
\mu = \frac{\pi n_y}{\alpha} \quad (n_y = 1, 2, 3, \ldots), \tag{23}
\]
where
\[
\alpha = \sqrt{e^{2 \sqrt{\frac{\pi}{3} y_1}} - 1} - \cot^{-1} \left( \frac{1}{\sqrt{e^{2 \sqrt{\frac{\pi}{3} y_1}} - 1}} \right). \tag{24}
\]

So the modes of extra dimension is quantized and the mass is discretized naturally. Apparently, \( \alpha \) can be used to determine the relationship between ADM mass and the position of brane. We call parameter \( \alpha \) the ADM mass factor whose function is drawn in Fig. 1. It is clear that if \( y_1 \to \infty \), the limit of \( \alpha \to \infty \) exists and the modes spectrum is continuous. Similar behavior can be found in the work of the RS black string [21].

![FIG. 1: The ADM mass factor versus the brane position \( y_1 \) (\( \Lambda = 0.1 \) is assumed).](image)
IV. ENTROPY OF THE RICCI-FLAT BLACK STRING

We assume the radial wave function satisfies $R_\omega(r) \sim \exp(iS(r))$. Making use of the WKB approximation, we can obtain a $r$-dependent radial wave number $k_r(r)$ by

$$k_r^2 = \left( \frac{\partial S(r)}{\partial r} \right)^2 = \left[ E^2 f^{-1}(r) - \left( \mu^2 + \frac{l(l+1)}{r^2} \right) \right] f^{-1}(r).$$  \hspace{1cm} (25)

In usual 4D space, the mass $\mu$ is always treated as a small mass approximation and sometimes is ignored during calculation. However, it is not necessary to do this and the $\mu$ is kept as the effective ADM mass in higher dimensional gravity such as in Randall-Sundrum black string [21]. The number of radial wave $n_r$ is also obtained via the semi-classical quantized condition,

$$\pi n_r = \int_r k_r(r) dr.$$  \hspace{1cm} (26)

So the total number of modes $N_r$ with energy less than or equal to $E$ is given by

$$\pi N_r = \int_r (2l + 1) \pi n_r dl.$$  \hspace{1cm} (27)

According to the canonical assembly theory with definite particles number $N$, volume $V$ and temperature $T$, the free energy of the quantum scalar field at inverse temperature $\beta$ is written as

$$\beta F = \sum_E (1 - e^{-\beta E})$$

$$= \int_E g(E) \ln \left( 1 - e^{-\beta E} \right) dE = \int_0^{+\infty} \ln \left( 1 - e^{-\beta E} \right) dN_r,$$  \hspace{1cm} (28)

where $g(E) = dN_r(E)/dE$ is the density of states and the summation is substituted by integration with semiclassical view. Substituting state function $N_r(E)$ into Eq. (28), we get

$$F = -\frac{1}{\pi} \int_0^{+\infty} dE \int_r dr \int_1^{2l + 1} dl \frac{k_r(r, E, l)}{e^{\beta E} - 1}.$$  \hspace{1cm} (29)

The $l$ integral can be simplified by restricting integral range, and the $F$ can be explicitly written as

$$F = \frac{2}{3\pi} \int_0^{+\infty} \frac{dE}{e^{\beta E} - 1} \int_r dr \frac{r^2}{f^2(r)} \left[ E^2 - f(r) \mu^2 \right]^{3/2}.$$  \hspace{1cm} (30)

In order to separate the variable $\mu$ and metric function $f(r)$, the integrand can be expanded in a series. Here we assume the two branes are very far apart each other. A new parameter $\epsilon = \mu^2 = \pi^2 n_g^2/\alpha^2$ is introduced. When the second brane is sent to infinite, i.e. $\alpha \rightarrow \infty$, the parameter $\epsilon$ can be treated naturally as small quantity according to the effective ADM mass [23]. So using the relationship $(E^2 - f(r)\epsilon)^{3/2} \approx E^3 - 3/2 f(r) E \epsilon$, the free energy [30] is rewritten as

$$F = \frac{2}{3\pi} \int_r dr \int_0^{+\infty} \frac{r^2}{f^2(r)} \frac{E^3}{e^{\beta E} - 1} dE + \frac{1}{\pi} \int_r^{+\infty} \frac{r^2 \epsilon}{f(r)} \frac{E}{e^{\beta E} - 1} dE$$

$$= -\frac{2\pi^3}{45\beta^4} \int_r \frac{r^2}{f^2(r)} dr + \frac{\pi}{6\beta^2} \int_r \frac{r^2 \epsilon}{f(r)} dr.$$  \hspace{1cm} (31)

The first term is exactly the same as the usual 4D SdS case [27] and the second shows the effect of extra dimension in brane world. Then the integral range of radial direction is determined by the improved thin-layer BWM boundary
\[ \Phi(t, r, \theta, \phi, y) = 0 \quad \text{for} \quad r_c + \varepsilon_c \leq r \leq r_c + \varepsilon_c + \delta_c; \quad (32) \]
\[ \Phi(t, r, \theta, \phi, y) = 0 \quad \text{for} \quad r_c - \varepsilon_c - \delta_c \leq r \leq r_c - \varepsilon_c, \quad (33) \]

where \( \varepsilon_c \) and \( \varepsilon_c \) are the infinitesimal cutoff factors \( (\varepsilon_c, \varepsilon_c \ll r_c \text{ or } r_c) \); \( \delta_c \) and \( \delta_c \) are the thickness of thin-layer. Then the main contributions of the integration near horizons \( r_c \) and \( r_c \) are given respectively by

\[
F_c = \frac{-2\pi^3}{45 \beta} \int_{r_c+\varepsilon_c+\delta_c}^{r_{+\varepsilon_c}} \frac{r^2}{f^2(r)} dr + \frac{\pi}{6 \beta^2} \int_{r_c+\varepsilon_c}^{r_{+\varepsilon_c+\delta_c}} r^2 e^f(r) dr
\]
\[
= \frac{-2\pi^3}{45 \beta} \frac{r_c^2}{f_c^2(r_c)} + \frac{\pi \varepsilon_c}{6 \beta^2} \frac{r_c^2}{(r_c - r_o) r_c - r_c} \ln \frac{r_c - r_o}{r_c - r_c}, \quad (34) \]
\[
F_c = \frac{-2\pi^3}{45 \beta} \int_{r_c-\varepsilon_c-\delta_c}^{r_{-\varepsilon_c}} \frac{r^2}{f^2(r)} dr + \frac{\pi}{6 \beta^2} \int_{r_c-\varepsilon_c}^{r_{-\varepsilon_c-\delta_c}} r^2 e^f(r) dr
\]
\[
= \frac{-2\pi^3}{45 \beta} \frac{r_c^2}{f_c^2(r_c)} + \frac{\pi \varepsilon_c}{6 \beta^2} \frac{r_c^2}{(r_c - r_o) r_c - r_c} \ln \frac{r_c - r_o}{r_c - r_c}. \quad (35) \]

Here the total free energy is \( F = F_c + F_c \). Hence, the entropy of 5D Ricci-flat black string is given by

\[
S = \beta^2 \frac{\partial F_c}{\partial \beta} \bigg|_{\beta = \beta_c} + \beta^2 \frac{\partial F_c}{\partial \beta} \bigg|_{\beta = \beta_c}
\]
\[
= \eta_c A_c + \eta_c A_c, \quad (36) \]

with

\[
\eta_c = \frac{\delta_c}{90 \beta_c \varepsilon_c (\varepsilon_c + \delta_c)} - \frac{\pi n_c^2 \Lambda^{3/2}}{72 \alpha^2 \cos(2\pi/3 - \chi)} \ln \frac{\varepsilon_c + \delta_c}{\varepsilon_c}; \quad (37) \]
\[
\eta_c = \frac{\delta_c}{90 \beta_c \varepsilon_c (\varepsilon_c + \delta_c)} - \frac{\pi n_c^2 \Lambda^{3/2}}{72 \alpha^2 \cos \chi} \ln \frac{\varepsilon_c}{\varepsilon_c + \delta_c}. \quad (38) \]

where \( A_c = 4\pi r_c^2 \) and \( A_c = 4\pi r_c^2 \) are the areas of two horizons; \( \chi \) is determined by effective cosmological constant, namely, \( \chi = \frac{1}{3} \cos^{-1}(-3M \sqrt{\Lambda}) \). The above simplified representation is obtained by using the surface gravity expressions \((12) \sim (14)\) and horizons expressions \((8)\). The relationships between inverse Hawking temperature \( \beta \) and horizons are

\[
\beta_c = \frac{2\pi}{\kappa_c} = \frac{12\pi}{\Lambda} \cdot \frac{r_c}{(r_c - r_o)(r_c - r_o)}; \quad (39) \]
\[
\beta_c = \frac{2\pi}{\kappa_c} = \frac{12\pi}{\Lambda} \cdot \frac{r_c}{(r_c - r_o)(r_c - r_o)} \quad (40) \]

It is clear that the 5D black string entropy \((36)\) is a linear sum of the area of the black hole horizon and that of the cosmological horizon. Comparing with 4D case, the 5D Ricci-flat black string’s entropy is modified by the second item of proportionality coefficients \( \eta_c \) and \( \eta_c \), which are discretized by the quantized extra dimensional modes. Furthermore, it is well known that the small mass approximation in 4D case is considered as the contribution of the vacuum surrounding black hole in brick-wall method \((24)\), which is very unnatural and is tried to be solved by many people. However, in this 5D Ricci-flat black brane solution, the small mass is obtained naturally by the topological property of far apart binary branes. When the second brane is sent far away, the ADM mass factor \( \alpha \) tends to infinity. Hence, the effective ADM mass on the first brane is naturally small. If extra dimension does exist and is visible near black hole, this nontrivial entropy may can show something interesting.
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