COMPACT KÄHLER MANIFOLDS WITH NONPOSITIVE BISECTIONAL CURVATURE

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Abstract. Let \((M^n, g)\) be a compact Kähler manifold with nonpositive bisectional curvature. We show that a finite cover is biholomorphic and isometric to a flat torus bundle over a compact Kähler manifold \(N^k\) with \(c_1 < 0\). This confirms a conjecture of Yau. As a corollary, for any compact Kähler manifold with nonpositive bisectional curvature, the Kodaira dimension is equal to the maximal rank of the Ricci tensor. We also prove a global splitting result under the assumption of certain immersed complex submanifolds.

1 Introduction

The uniformization theorem of Riemann surfaces says the sign of curvature could determine the conformal structure in some sense. Explicitly, if the curvature is positive, it is covered by \(\mathbb{P}^1\) or \(\mathbb{C}\). On the other hand, if the curvature is less than a negative constant, it is covered by the unit disk \(D^2\).

It is natural to wonder whether there are generalizations in higher dimensions. For the compact case, the famous Frankel conjecture says if a compact Kähler manifold has positive holomorphic bisectional curvature, then it is biholomorphic to \(\mathbb{C}P^n\). This conjecture was solved by Mori [M79] and Siu-Yau [SY80] independently. In fact, Mori proved the stronger Hartshorne conjecture. Later, Mok [M88] solved the generalized Frankel conjecture. The result says that, if a compact Kähler manifold has nonnegative holomorphic bisectional curvature, then the universal cover is isometric-biholomorphic to \((\mathbb{C}^k, g_0) \times (\mathbb{P}^{n_1}, \theta_1) \times \cdots \times (\mathbb{P}^{n_l}, \theta_l) \times (M_1, g_1) \times \cdots \times (M_i, g_i)\), where \(g_0\) is flat; \(\theta_k\) are metrics on \(\mathbb{P}^{n_k}\) with nonnegative holomorphic bisectional curvature; \((M_j, g_j)\) are compact irreducible Hermitian symmetric spaces.

If the curvature is negative, the current knowledge is much less satisfactory. For example, a famous conjecture of Yau says if a complete simply connected Kähler manifold has sectional curvature between two negative constants, then it is a bounded domain in \(\mathbb{C}^n\). So far, it is not even known whether there exists a nontrivial bounded holomorphic function on such manifolds.

As in the Riemannian case, it is often important to understand the difference between the negatively curved case and the nonpositive case. The former tends to be hyperbolic in some sense, while the latter usually possesses some rigidity properties.
For compact Kähler manifolds with nonpositive holomorphic bisectional curvature, there is a conjecture of Yau (page 2 of [WZ02], also [WZ03]):

**Conjecture.** Let $M^n$ be a compact Kähler manifold with nonpositive holomorphic bisectional curvature. Then there exists a finite cover $M'$ of $M$ such that $M'$ is a holomorphic and metric fibre bundle over a compact Kähler manifold $N$ with nonpositive bisectional curvature and $c_1(N) < 0$, and the fiber is a flat complex torus.

Recall that a fiber bundle is called a metric bundle, if for any $p \in N$, there is some neighborhood $p \in U \subset N$ such that the bundle over $U$ is isometric to the product of the fiber with $U$. In [Y74], Yau proved the following.

**Theorem 1.** Let $M$ be a compact complex submanifold of a complex torus $T^n$. Then $M$ is a torus bundle over a complex submanifold $N$ in $T^n$ such that the induced Kähler metric on $N$ has negative definite Ricci tensor in an open dense set of $N$.

Since complex submanifolds in $T^n$ has nonpositive holomorphic bisectional curvature, Yau’s theorem confirms the conjecture when $M$ is a complex submanifold of $T^n$. Zheng [Z02] proved this conjecture under the extra assumption that $M$ has nonpositive sectional curvature and the metric is real analytic. In [WZ02], Wu and Zheng proved this conjecture by only assuming that the metric is real analytic. They first proved a local splitting result by a careful study of the foliation at the points where the Ricci tensor has maximal rank. By real analyticity, the foliation could be extended to the whole manifold. In this note we confirm the conjecture above.

**Theorem 2.** Let $(M^n, g)$ be a compact Kähler manifold with nonpositive holomorphic bisectional curvature. Then there exists a finite cover $M'$ of $M$ such that $M'$ is a holomorphic and metric fiber bundle over a compact Kähler manifold $N^k$ with nonpositive bisectional curvature and the Ricci curvature is strict negative in an open set on $N$. Thus $c_1(N) < 0$. The fiber is a flat complex torus $T$. Furthermore, $M'$ is diffeomorphic to $T \times N$. Finally, if $r$ is the maximal rank of the Ricci curvature of $g$, then $r = k = \text{Kod}(M)$ where $\text{Kod}(M)$ is the Kodaira dimension of $M$.

**Corollary 1.** For any compact Kähler manifold with nonpositive bisectional curvature, the Kodaira dimension is equal to the maximal rank of the Ricci tensor.

**Corollary 2.** Let $M^n$ be a compact Kähler manifold with nonpositive bisectional curvature. If the Ricci tensor degenerates everywhere, i.e., the maximal rank of the Ricci tensor is strictly less than $n$, then the universal cover splits off a nontrivial complex Euclidean factor holomorphically and isometrically.

One can ask a question similar to Corollary 2 in the Riemannian setting. Namely, for a compact Riemannian manifold with nonpositive sectional curvature, if the Ricci tensor degenerates everywhere, is it true that the universal cover has a nontrivial Euclidean factor? In Guler and Zheng’s paper [GZ11], a counterexample (due to