LEMMA POINCARÉ FOR $L_{\infty, \text{loc}}$-FORMS

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ABSTRACT. We show that every closed $L_{\infty, \text{loc}}$-form on $\mathbb{R}^n$ is exact. Differential is understood in the sense of currents. The proof does not use any explicit geometric constructions. De Rham theorem follows.

1. INTRODUCTION

We work with a class $\Omega^*_{\infty, \text{loc}}$ of differential forms on $\mathbb{R}^n$, those with coefficients locally in $L_{\infty}$, with differential satisfying the same condition. “Locally” will mean “on a neighborhood with compact closure”, as usual. Classes $\Omega^*_\infty$ and $\Omega^*_{\infty, \text{loc}}$ are invariant under bi-Lipschitz coordinate changes and therefore are naturally defined on any Lipschitz manifold. In context of his geometric theory of integration Whitney in his book [W] considered a class of so-called flat forms, invariant under Lipschitz maps, and proved a Poincaré lemma for them [W, Th IX.12A] using an elaborate geometric construction. We give a different shorter proof below, based on the techniques from [GKS]. Differential forms of class $\Omega^*_{\infty, \text{loc}}$ are nothing but locally flat forms in the sense of Whitney.

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2. CLASS $\Omega^*_{\infty, \text{loc}}$

The differential forms we consider are not the ones in the smooth category, they belong properly in the Lipschitz category and are defined as follows.

On an open domain $U \subset \mathbb{R}^n$ a $k$-form is an almost everywhere defined map

$$w : U \to \Lambda^k(\mathbb{R}^n)$$

Two maps are considered equivalent and define the same form if they coincide almost everywhere. Denote by $\mathcal{F}^k(U)$ a vector space of differential forms of degree $k$, defined on $U$. Vector spaces $\mathcal{F}^k(U)$ form a graded ring, $\mathcal{F}^*(U)$ with respect to addition and exterior product.

By $\mathcal{L}^k_{\infty}$ we denote the space of $k$-forms with bounded coefficients. The union of these for all $k$ make up a graded ring $\mathcal{L}^*_\infty$.

The ring of smooth forms with compact support in $U$ will be denoted $\mathcal{D}^*(U)$. According to de Rham [dR], $k$-currents on $U$ are linear continuous functionals on
\( \mathcal{D}^{n-k}(U) \), denoted \( \mathcal{E}^{k}(U) \). For each current \( w \in \mathcal{E}^{k}(U) \) one can define its exterior differential \( dw \in \mathcal{E}^{k+1}(U) \) by the formula:

\[ < dw, \alpha > = (-1)^{k+1} < w, d\alpha >, \quad \forall \alpha \in \mathcal{D}^{n-k-1}(U) \]

Define

\[ \Omega^{*}_{\infty}(U) = \{ w \mid w \in L^{*}_{\infty}(U), dw \in L^{*}_{\infty}(U) \} , \]

where by \( dw \in L^{*}_{\infty}(U) \) we mean that \( dw \) as a functional has an integral representation with an \( L^{*}_{\infty} \)-kernel:

\[ < dw, \alpha > = \int_{U} dw \wedge \alpha, \]

where we abuse notation: the \( dw \) under the integral means the form \( dw \in L^{*}_{\infty}(U) \), same notation as the original functional it represents. Using a multi-index

\[ \vec{i} = (i_1, i_2, ..., i_k) \quad i_1 < ... < i_k, \]

we shall write a \( k \)-form \( w \) as follows:

\[ w = \sum_{\vec{i}} a_{\vec{i}} dx^{\vec{i}}. \]

On every compact \( K \) in \( U \) consider a seminorm:

\[ |w|_{K, \infty} = \max \text{ess sup}_{x \in K} |a_{\vec{i}}(x)| \]

We will use sup to denote essential supremum from now on. In \( \Omega^{*}_{\infty} \) consider seminorms

\[ |w|_{K, \infty} = |w|_{K, \infty} + |dw|_{K, \infty} \]

With these seminorms \( L^{k}_{\infty}(U) \) and \( \Omega^{k}_{\infty}(U) \) become locally convex topological vector spaces.

We will compare now the forms of class \( \Omega^{*, \text{loc}}_{\infty}(U) \) with locally flat forms in the sense of Whitney \([W, \text{IX.6}]\). We recall some definitions.

If \( P \) is an \( s \)-dimensional plane in \( \mathbb{R}^{n} \), and \( A \) - a measurable set in \( P \), then \( |A|_s \) shall stand for the \( s \)-dimensional Lebesque measure of \( A \).

Given a form \( w \in \mathcal{E}^{k}(U) \) and a set \( Q \subset U \), a simplex \( \sigma^{s} \subset U (s \geq k) \) is said to be \( Q\text{-good} \) for \( w \), if all coefficients of \( w \) are measurable on \( \sigma^{s} \) and \( |\sigma^{s}\setminus Q|_s = 0 \). Simplex \( \sigma^{s} \) is said to be \( Q\text{-excellent} \) if \( \sigma^{s} \) and each of its faces of dimension \( \geq k \) is \( Q\text{-good} \).

A form \( w = \sum_{\vec{i}} a_{\vec{i}} dx^{\vec{i}} \in \mathcal{E}^{k}(U) \) is said to be \( \text{flat} \) if:

(a) coefficients \( a_{\vec{i}} \) are measurable in \( U \),

(b) \( \sup_{x \in U} |a_{\vec{i}}(x)| \leq N \) for some \( N \),

(c) there exist such \( Q \subset U \) and \( N' \), that \( |U\setminus Q|_n = 0 \), and for every \( Q\text{-excellent} \) for \( w \) simplex \( \sigma^{k+1} \subset U \):

\[ |\int_{\partial \sigma^{k+1}} w| \leq N'|\sigma^{k+1}|_{k+1}. \]

**Theorem 2.1.** Class \( \Omega^{*, \text{loc}}_{\infty}(U) \) coincides with the space of all locally flat forms in \( U \). The exterior differential of an arbitrary form \( w \in \Omega^{*, \text{loc}}_{\infty}(U) \) coincides with the differential of \( w \) in the sense of Whitney \([W, \text{IX.12}]\).

**Proof** See \([\text{GKS}, \text{Theorem 1.5}]\) \( \Box \)
3. \( \Omega^*_\infty, \text{loc} \)-Morphisms

A map \( f : U \to \mathbb{R}^m, U \subset \mathbb{R}^n \), is called locally Lipschitz if for every compact \( K \subset U \) there exists a constant \( C \), satisfying:

\[
|f(x) - f(x')| \leq C|x - x'|
\]

for any \( x, x' \in K \). Every locally Lipschitz function \( f : U \to \mathbb{R}^1 \) is differentiable almost everywhere on \( U \), its partial derivatives are generalized partial derivatives. Hence, the exterior differential \( df \) of a locally Lipschitz function \( f \), considered as a form of degree 0, has a coordinate representation

\[
df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n.
\]

Locally Lipschitz map \( \varphi : U \to V \) will be called an \( l \)-map, if for any point \( x \in U \) there exist a neighborhood \( W \) of \( x \) and a constant \( C \), such that

\[
||\varphi^{-1}(A) \cap W||_n \leq C|A|_m
\]

for any open \( A \subset V \). Preimage of a measure zero set under an \( l \)-map has measure zero. Hence for any \( l \)-map \( \varphi : U \to V \) there is a well-defined homomorphism \( \varphi^* : \mathcal{F}^*(V) \to \mathcal{F}^*(U) \).

According to Rademacher theorem [W, IX.11] any locally Lipschitz map is differentiable almost everywhere, therefore any \( l \)-map \( \varphi \) satisfies:

\[
\varphi^*(\mathcal{L}_\infty^*(V)) \subset (\mathcal{L}_\infty^*(U))
\]

**Theorem 3.1.** If \( \varphi : U \to V, U \subset \mathbb{R}^n, V \subset \mathbb{R}^m \) - an \( l \)-map, \( w \in \Omega^*_\infty, \text{loc}(V) \), then

\[
\varphi^* w \in \Omega^*_\infty, \text{loc}(U), \quad d(\varphi^* w) = \varphi^* dw.
\]

The map

\[
\varphi^* : \Omega^*_\infty, \text{loc}(V) \to \Omega^*_\infty, \text{loc}(U)
\]

is continuous.

**Proof** See [GKS, Theorem 2.2] □

**Remark 3.2.** A homeomorphism \( \varphi : U \to V, U, V \subset \mathbb{R}^n \) is called Lipschitz if \( \varphi \), \( \varphi^{-1} \) are locally Lipschitz. For Lipschitz homeomorphisms Theorem 3.1 is satisfied, since they are \( l \)-maps.

Whitney defines homomorphism \( \varphi^* \) on classes of flat forms for any locally Lipschitz \( \varphi : U \to V, U \subset \mathbb{R}^n, V \subset \mathbb{R}^m \) [W, X.9]. Due to Theorem 3.1 the construction of Whitney defines a homomorphism \( \varphi^* : \Omega^*_\infty, \text{loc}(V) \to \Omega^*_\infty, \text{loc}(U) \). If \( \varphi \) is a Lipschitz homeomorphism then \( \varphi^* \) in the sense of Whitney [W, X.9] coincides with the \( \varphi^* \) introduced above.

4. Lemma Poincaré for bounded forms

Lemma Poincaré for forms with \( p \)-summable coefficients for finite \( p \) is proven in [GKS, Lemma 3.4]. The case \( p = \infty \) (locally flat forms) is treated in [W] via nontrivial explicit geometric construction. We use analytic technique similar to that used in [GKS] to obtain a short alternative proof for \( p = \infty \) case. We start with the following key observation.
Lemma 4.1. Let $I$ be a cube in $\mathbb{R}^n$ and $w$ an arbitrary smooth $k$-form on $I$. There exists a smooth closed $w'$ ($dw' = 0$), such that:

$$|w - w'|_{I, \infty} \leq C|dw|_{I, \infty}$$

for some constant $C$ depending only on $n$, $k$ and $I$.

Proof. We can assume $I = I^n$ in $\mathbb{R}^n$. For $k \geq n$ the statement is true for $C = 0$, for other cases we will proceed by induction on $n$. For $n = 1$ we only need to consider $k = 0$. In this case $w = f(x)$ and we have:

$$\sup_{x \in I} |f(x) - f(0)| = \sup_{x \in I} \left| \int_0^x f'(\tau) d\tau \right| \leq \int_0^1 |f'(\tau)| d\tau \leq \sup_{x \in I} |f'(x)| .$$

In other words,

$$|f(x) - f(0)|_{I, \infty} \leq |df|_{I, \infty} .$$

Suppose now $n > 1$. An arbitrary smooth form $w$ can be represented in the following way:

$$w = w_1 \wedge dx^n + w_2 ,$$

where forms $w_1$ and $w_2$ do not contain $dx^n$ in their coordinate representations. Clearly:

$$|w|_{I, \infty} = \max(|w_1|_{I, \infty}, |w_2|_{I, \infty})$$

Forms $w_1$ and $w_2$ can be considered as those on the cube $I' = I^{n-1}$, depending on a parameter $x^n$. Differentials of these in $I'$ we will denote by $d'w_1$ and $d'w_2$. Then

$$dw = d'w_1 \wedge dx^n + (-1)^k \frac{\partial w_2}{\partial x^n} \wedge dx^n + d'w_2 ,$$

where by $\frac{\partial w_2}{\partial x^n}$ we mean a form, whose coefficients are the partial derivatives of $w_2$ with respect to $x^n$. We have:

$$(dw)_1 = d'w_1 + (-1)^k \frac{\partial w_2}{\partial x^n} , \quad (dw)_2 = d'w_2$$

Denoting

$$d'w_2 = \sum_{i_1 < \ldots < i_{k+1} < n} a_{i_1, \ldots, i_{k+1}} dx^{i_1} \ldots dx^{i_{k+1}}$$

we have:

$$|d'w_2|_{I', \infty} = \max_i \sup_{x \in I} |a_i(x)| = \max_i \sup_{x^n \in [0, 1]} \sup_{x \in I'} |a_i(x)| = \sup_{x^n \in [0, 1]} |d'w_2|_{I', \infty}$$

Hence for some $n^{th}$ coordinate value $\tau \in [0, 1]$:

$$|d'w_2(\tau)|_{I', \infty} \leq |d'w_2|_{I', \infty}$$

Consider a form

$$\tilde{w} = \int_{\tau}^{x^n} w_1(\sigma) d\sigma ,$$

with the integration carried out over the $n^{th}$ coordinate. Its differential is

$$(dw) = (dw)_1 dx^n + (-1)^k w_2(\tau) .$$

The induction step gives us a closed form $\tilde{w}$ on $I'$ such that:

$$|w_2(\tau) - \tilde{w}|_{I', \infty} \leq C|d'w_2|_{I', \infty} .$$
Let us define $w' = (-1)^{k-1} d\hat{w} + \hat{w}$. Here $\hat{w}$ is considered as a form on $I$, independent of $x^n$. According to (4.3)\]

\begin{equation}
(4.6) \quad w - w' = (-1)^k \int_{\tau}^{x^n} (dw)_1 dx^n + w_2(\tau) - \hat{w}.
\end{equation}

Let us estimate the norm of $\int_{\tau}^{x^n} (dw)_1 dx^n$. Since the integral is taken separately of each coefficient $a_i$ of the form $(dw)_1$ we have:

$$ | \int_{\tau}^{x^n} (dw)_1 dx^n |_{I, \infty} = \max_{i} \sup_{l} | \int_{\tau}^{x^n} a_i dx^n | $$

$$ \leq \max_{i} \sup_{l} \int_{0}^{1} | a_i | dx^n \leq \max_{i} \sup_{l} \sup_{x^n} | a_i | = | (dw)_1 |_{I, \infty} $$

Hence, from (4.6), (4.5), (4.3), (4.2), (4.1):

$$ |w - w'|_{I, \infty} \leq |(dw)_1|_{I, \infty} + C |(dw)_2|_{I, \infty} $$

$$ \leq C' \max( |(dw)_1|_{I, \infty}, |(dw)_2|_{I, \infty}) $$

where $C' = \max(1, C)$.

Now notice that

$$ |dw|_{I, \infty} = |(dw)_1 \wedge dx^n + (dw)_2|_{I, \infty} = \max( |(dw)_1|_{I, \infty}, |(dw)_2|_{I, \infty}) $$

Thus $|w - w'|_{I, \infty} \leq 2C' |dw|_{I, \infty}$ which concludes the proof. \(\square\)

**Corollary 4.2.** For any smooth closed $k$-form $w$ ($k > 0$) on any cube $I \subset \mathbb{R}^n$ there exists such smooth "primitive" $\theta$ ($d\theta = w$) that

$$ |\theta|_{I, \infty} \leq C |w|_{I, \infty} $$

for some constant $C$.

**Proof** According to the (usual) Lemma Poincaré for smooth forms, there exists a smooth $\theta_1$, such that $d\theta_1 = w$. Lemma 4.1 then provides for existence of a smooth closed $\theta_2$ satisfying $|\theta_1 - \theta_2|_{I, \infty} \leq C |w|_{I, \infty}$. The form $\theta = \theta_1 - \theta_2$ is the desired primitive. \(\square\)

**Lemma 4.3.** Any $x \in \mathbb{R}^n$ has a neighborhood $V$ such that any closed $\mathcal{L}_{\infty}^k$-form $w$ on $V$ is exact: for $k > 0$ $\exists \theta \in \mathcal{L}_{\infty}^{k-1}(V)$ : $d\theta = w$.

If $k = 0$, then $w$ is a locally constant function.

**Proof** A function which has all of its generalized derivatives vanish is locally constant. For $k > 0$ consider an open cube $I$, centered at $x$. According to [GKS Lemma 1.3] and the Remark following it, it is possible to choose a sequence of smooth closed $k$-forms that weakly converges to $w$.

Using Mazur's theorem [Y Theorem V.1.2] we can construct another sequence $\{w_s\}$ (of finite convex combinations of the elements of the original sequence) that will satisfy:

$$ |w - w_s|_{I, \infty} \leq \frac{1}{2^s} $$

and $dw_s = 0$ on $I$.

From Corollary 4.2 it follows that there exist such forms $\theta_s$ on the cube $I$, that

\begin{align*}
    d\theta_1 = w_1, & \quad d\theta_s = w_s - w_{s-1} \quad \text{for } s > 1 \quad \text{and} \quad |\theta_s|_{I, \infty} \leq C |w_s - w_{s-1}|_{I, \infty}.
\end{align*}
Since
$$|\theta_s|_{1,\infty} \leq C|w_s - w_{s-1}|_{1,\infty} \leq C(|w - w_s|_{1,\infty} + |w - w_{s-1}|_{1,\infty}) \leq \frac{C}{2s} + \frac{C}{2s-1},$$
the series \(\sum_{s=1}^{\infty} \theta_s\) converges in \(L^{k-1}(I)\). Let \(\theta = \sum_{s=1}^{\infty} \theta_s\).

Since
$$|w - \sum_{s=1}^{N} d\theta_s|_{1,\infty} = |w - w_N|_{1,\infty} \leq \frac{1}{2^N},$$
we see that \(d\theta = w\), as required. \(\square\)

Denote by \(H^{\ast}_{\infty,\text{loc}}(M)\) the cohomology of the cochain complex \(\{\Omega^{\ast}_{\infty,\text{loc}}, d\}\) on a manifold \(M\).

**Theorem 4.4** (De Rham Theorem). For any Lipschitz manifold \(M\) the cohomology groups \(H^{\ast}_{\infty,\text{loc}}(M)\) are canonically isomorphic to the cohomology groups \(H^{\ast}(M; \mathbb{R})\) of the manifold \(M\).

Denote by \(\lambda : H^{\ast}_{\infty,\text{loc}}(M) \rightarrow H^{\ast}(M; \mathbb{R})\) the canonical isomorphism above.

**Theorem 4.5.** If \(f : M \rightarrow N\) is a Lipschitz map then \(f^{\ast}\lambda = \lambda f^{\ast}\).

The last two theorems follow from Lemma 4.3, [GKS, Lemma 3.1] and the general theorems of sheaf theory [G, 11.4.6, 11.4.16].

It turns out that after relaxing the locality requirement, de Rham Theorem above is in general no longer valid for \(H^{\ast}_{\infty}(M)\). In other words \(H^{\ast}_{\infty}(M)\) provides a genuinely different (in fact metric rather than topological) invariant. De Rham theorem for \(H^{\ast}_{\infty}(M)\) still holds on compact manifolds. This leads us to ask the following questions.

Do compact manifolds form the maximal class of spaces \(M\) for which the de Rham theorem holds? When is \(0 < \dim H^{\ast}_{\infty}(M) < \infty\) for non-compact \(M\)?

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