Two-dimensional isochronous nonstandard Hamiltonian systems

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Abstract. We identify a generic class of two dimensional nonstandard Hamiltonian systems which exhibit isochronous behaviour. This class of systems belongs to the two dimensional mixed Liénard-type equations and is obtained by generalizing the scalar modified Emden equation (MEE) to two dimensions. We show that the generalized class of equations admits a Hamiltonian description and exhibits periodic and quasi-periodic oscillations for suitable choice of parameters and also \( PT \) symmetric property.

1. Introduction

Certain nonstandard Lagrangian and Hamiltonian dynamical systems \([1,3]\) encompass very interesting classes of nonlinear oscillators and admit fascinating dynamical properties \([4,6]\) such as isochronous oscillations, linearization, nonlocal transformations and so on \([7,13]\). In particular, the Liénard class of oscillators appear in the study of a wide range of fields such as seismology \([14]\), biological regulatory systems \([15]\), in the study of a self gravitating stellar gas cloud \([16]\), optoelectronics, fluid mechanics \([17]\). Many of these Liénard class of equations admit limit cycle/periodic oscillations which are used to model many physical phenomenon. Identifying such classes of coupled Liénard-type equations admitting isochronous oscillations is an interesting area of research. Several procedures have been developed to construct and identify classes of isochronous oscillators. In particular, Calogero \([18]\) and Calogero and Leyvraz \([19,23]\) have developed many techniques to generate isochronous oscillators. In a recent paper a procedure to generate scalar isochronous systems recursively from a given Hamiltonian \([24]\) and a method to construct higher dimensional isochronous nonsingular Hamiltonian systems have been discussed \([25]\). In this paper we obtain a class of coupled Liénard type oscillator equations admitting isochronous oscillations by generalizing the nonstandard Lagrangian of the scalar system to coupled systems. In order to do so let us consider...
the scalar Liénard-type linear/ quadratic and mixed type (in velocities) systems. For example the Liénard equation with linear velocity term,
\[
\ddot{x} + F(x)\dot{x} + G(x) = 0,
\]
(1)
ads a class of interesting nonstandard type conserved Hamiltonian [7] and isochronous solutions for the choice [10],
\[
F(x) = 3kx, \quad G(x) = k^2 x^3 + \omega^2 x.
\]
(2)
For the choice \(\omega = 0\) the above equation reduces to the modified Emden equation which is well studied in the literature and it occurs in the study of equilibrium configurations of a spherical cloud acting under the mutual attraction of its molecules and subject to the laws of thermodynamics [11] and in the modelling of the fusion of pellets [12]. For the choice \(\omega \neq 0\) the above equation exhibits isochronous behaviour, that is it admits periodic oscillations with frequency of oscillation independent of the amplitude [7]. The general solution of this equation is given as
\[
x(t) = \frac{A \sin(\omega t + \delta)}{1 - \frac{kA}{\omega} \cos(\omega t + \delta)}.
\]
(3)
The corresponding system admits a nonstandard Hamiltonian
\[
H = \frac{1}{2} \hat{F}(p)x^2 + U(p),
\]
(4)
with
\[
\hat{F}(p) = \omega^2 \left(1 - \frac{2k}{\omega^2 p}\right), \quad U(p) = \frac{\omega^4}{2k^2} \left(\sqrt{1 - \frac{2k}{\omega^2 p}} - 1\right)^2,
\]
(5)
where the canonically conjugate momentum
\[
p = \frac{\omega^2}{2k} \left(1 - \frac{\omega^4}{(k\dot{x} + k^2\dot{x}^2 + \omega^2)^2}\right).
\]
(6)
System (4) is \(PT\) symmetric (\(x \rightarrow -x, \ t \rightarrow -t, \ \dot{x} \rightarrow \dot{x}\)) and is also exactly quantizable in momentum space [26]. The class of nonlinear oscillators containing both quadratic and linear terms in \(\dot{x}\) is called a mixed Liénard-type equation. It can be written in the form
\[
\ddot{x} + J(x)\dot{x}^2 + F(x)\dot{x} + G(x) = 0,
\]
(7)
where \(F(x), \ G(x)\) and \(J(x)\) are functions of \(x\). For example with the choice
\[
J(x) = \frac{f_x}{f}, \quad F(x) = \frac{(r+2)h_x}{(r+1)f}, \quad G(x) = \frac{hh_x}{(r+1)f^2}, \ r \neq -1
\]
(8)
the system admits a nonstandard Lagrangian and Hamiltonian functions of the form
\[
L = \frac{1}{(f(x)\dot{x} + h(x))^r},
\]
(9)
where \(f(x)\) and \(h(x)\) are arbitrary functions of \(x\), \(r\) is a suitably chosen real positive parameter and the Hamiltonian
\[
H = \frac{p}{f} \left(\left(-\frac{rf}{p}\right)^{\frac{r}{r+1}} - h\right) - \left(-\frac{rf}{p}\right)^{\frac{r}{r+1}},
\]
(10)
with the conjugate momentum
\[ p = -rf(f\dot{x} + h)^{(r+1)}. \]  
(11)

In the above, \( r \) is a real positive number so chosen such that \( H \) is real and in (9)-(11) only the principal branch is taken when fractional powers appear. Note that the MEE is a special case of the above system with \( f = 1, \ r = 1 \) and \( h = kx^2 \). We also note here that this class of Lagrangian and Hamiltonian systems are also drawing considerable interest in connection with supersymmetry related partner systems [27, 28] and \( PT \)-symmetric systems [26]. Now, it is of considerable importance to extend the study of above type nonstandard Lagrangian and Hamiltonian systems to higher degrees of freedom. Particularly we wish to identify two dimensional nonstandard Hamiltonian systems which are isochronous and in future to study their exact quantization as in the case of MEE [26].

A natural way of generalising the above one dimensional nonstandard Lagrangian is to modify (9) to the form,
\[ L = \frac{1}{(f\dot{x} + g\dot{y} + h)^r} \]  
(12)

where \( f = f(x, y), g = g(x, y), h = h(x, y) \). In this case, however the Lagrangian turns out to be singular or degenerate [3] which can be verified from the vanishing the Hessian,
\[ \Delta = \begin{vmatrix} \frac{\partial^2 L}{\partial \dot{x}^2} & \frac{\partial^2 L}{\partial \dot{y} \partial \dot{x}} \\ \frac{\partial^2 L}{\partial \dot{y} \partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{y}^2} \end{vmatrix} = 0. \]  
(13)

To obtain a nonsingular Lagrangian, a modification of the above Lagrangian with suitable terms is essential. For this purpose, we make a simple minded extension of the above form (12) judiciously and succeed to identify non-trivial coupled nonstandard and nonsingular Lagrangian type nonlinear evolution equations. We then show how from the associated equations of motion one can obtain coupled mixed Liénard-type equations, which may be considered as the natural extension of the above single degree of freedom mixed Liénard-type equation (7). Then the procedure is specifically illustrated for the MEE equation to obtain a two dimensional isochronous extension of the MEE equation.

The plan of the paper is as follows. In section 2, we introduce an arbitrary form of nonsingular Lagrangian and use it to obtain the corresponding Newton’s equation of motion. However the resultant equation of motion is not in the form of mixed Liénard-type class of oscillators. By demanding the resultant dynamical equations are of generalized coupled mixed Liénard-type equations, we deduce the functional form of the allowed nonstandard Lagrangian. In section 3, we show that the conditions obtained from the coupled mixed Liénard-type oscillators allow one to deduce another class of Liénard-type oscillators which exhibits periodic and quasiperiodic motions. Also we have shown that the corresponding Hamiltonian obtained from the coupled nonsingular Lagrangian can be transformed into a two dimensional harmonic oscillator Hamiltonian through appropriate canonical transformations. In section 4, we have shown that a special case of Liénard-type of oscillators leads to a two dimensional version of the
modified Emden equation which exhibits isochronous property. The general solution is shown to admit periodic as well as quasiperiodic solutions for suitable choices of parameters. Finally, in section 5, we present our conclusions.

2. Two dimensional coupled mixed Liénard-type equations

We wish to identify a coupled mixed Liénard-type system which possesses a nonstandard Hamiltonian and admits isochronous solution by a suitable generalization of the scalar Lagrangian \[ \Pi \] of MEE. As noted earlier, the Lagrangian \[ \Pi \] which is a natural generalization in two dimensions is singular. One can overcome this problem by suitably redefining the Lagrangian in the form,

\[
L = \sum_{i=1}^{2} \frac{1}{(f_i \dot{x} + g_i \dot{y} + h_i)^{r_i}}.
\]  

(14)

where \( f_i = f_i(x, y), \ g_i = g_i(x, y), \ h_i = h_i(x, y), \ i = 1, 2 \). With this choice of the Lagrangian we can show that Hessian in the present case is non-zero,

\[
\Delta = \left| \begin{array}{cc}
\frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} \\
\frac{\partial^2 L}{\partial y \partial x} & \frac{\partial^2 L}{\partial y^2}
\end{array} \right| \neq 0.
\]  

(15)

Consequently, the Lagrangian is nonsingular. From the above Lagrangian, the equation of motion can be obtained as

\[
\ddot{x} = \left( g_2(x, y)q_1(\dot{x}, \dot{y}, x, y) + g_1(x, y)q_2(\dot{x}, \dot{y}, x, y) + d_1(x, y)\dot{x}^2 \\
+ d_2(x, y)\dot{y}^2 + d_3(x, y)\dot{x} \dot{y} + s_1(x, y)\dot{x} + s_2(x, y)\dot{y} + u_1(x, y) \right),
\]  

(16)

\[
\dot{y} = -\left( f_2(x, y)q_1(\dot{x}, \dot{y}, x, y) + f_1(x, y)q_2(\dot{x}, \dot{y}, x, y) + d_4(x, y)\dot{x}^2 \\
+ d_5(x, y)\dot{y}^2 + d_6(x, y)\dot{x} \dot{y} + s_3(x, y)\dot{x} + s_4(x, y)\dot{y} + u_2(x, y) \right),
\]  

(17)

where

\[
q_1 = \frac{(r_2 + 1)}{\Delta r_1} \left( r_2(f_1 \dot{x} + g_1 \dot{y} + h_1)^{(r_1+2)}(f_2 \dot{x} + g_2 \dot{y} + h_2)^{-(r_2+1)} \right.
\]

\[
\times \left( f_2 h_{2y} - g_2 h_{2x} + (f_{2y} - g_{2x})(\dot{x} f_2 + \dot{y} g_2) \right),
\]  

(18)

\[
q_2 = \frac{(r_1 + 1)}{\Delta r_2} \left( r_1(f_1 \dot{x} + g_1 \dot{y} + h_1)^{-(r_1+1)}(f_2 \dot{x} + g_2 \dot{y} + h_2)^{(r_2+2)} \right.
\]

\[
\times \left( f_1 h_{1y} - g_1 h_{1x} + (f_{1y} - g_{1x})(\dot{x} f_1 + \dot{y} g_1) \right),
\]  

(19)

\[
d_1 = \frac{1}{\Delta^2} \left( f_1 f_2 g_1 (f_{2y} - g_{2x})(r_1 + 1) + g_2 (f_{1y} - g_{1x})(r_2 + 1) \right.
\]

\[
+ (r_1 + 1)(r_2 + 1)(g_1 g_2 (f_{2f_1x} + f_{1f_2x}) - f_2 f_{2x} g_1^2 - f_1 f_{1x} g_2^2),
\]  

(20)

\[
d_2 = \frac{1}{\Delta^2} \left( g_1^2 (r_1 + 1)(g_2 (f_{2y} - g_{2x}) - f_2 g_{2y}(r_2 + 1)) + g_2^2 (r_2 + 1) \right.
\]
$$\times (g_1(f_{1y} - g_{1x}) - f_1g_{1y}(r_1 + 1)) + g_1g_2(r_1 + 1)$$
$$\times (r_2 + 1)(f_{1g2y} + f_2g_{1y})\right),$$

$$d_3 = \frac{1}{\Delta^2}\left(g_1g_2(f_2(r_2 + 1)(f_{1y}(r_1 + 2) + g_{1x}r_1) + f_1(r_1 + 1)
\times (f_{2y}r_2 + 2) + g_{2x}r_2)) - g_1^2f_2(1 + 1)(f_{2y}r_2 + g_{2x}(r_2 + 2))
g_1^2f_1(r_2 + 1)(f_{1y}r_1 + g_{1x}(r_1 + 2))\right),$$

$$d_4 = \frac{1}{\Delta^2}\left(f_2^2(r_1 + 1)(f_2(f_{2y} - g_{2x}) + f_{2x}g_2(r_2 + 1)) + f_2^2(r_2 + 1)
\times (f_1(f_{1y} - g_{1x}) + f_{1x}g_1(r_1 + 1)) - f_1f_2(r_1 + 1)(r_2 + 1)
\times (f_{2x}g_1 + f_{1x}g_2)\right),$$

$$d_5 = \frac{1}{\Delta^2}\left(g_1g_2(f_1(f_{2y} - g_{2x})(r_1 + 1) + f_2(f_{1y} - g_{1x})(r_2 + 1))
+ (f_2g_1 - f_1g_2)(f_2g_{1y} - f_1g_{2y})(r_1 + 1)(r_2 + 1)\right),$$

$$d_6 = \frac{1}{\Delta^2}\left(f_1^2g_2(r_1 + 1)(g_{2x}r_2 + f_{2y}(r_2 + 2)) + f_2^2g_1(r_2 + 1)(g_{1x}r_1
+ f_{1y}(r_1 + 2)) - f_1f_2g_1(r_1 + 1)(f_{2y}r_2 + g_{2x}(r_2 + 2)) + g_1(r_2 + 1)
\times (f_{1y}r_1 + g_{1x}(r_1 + 2))\right),$$

$$s_1 = \frac{1}{\Delta^2}\left(g_1(r_1 + 1)(f_1(h_2(f_{2y} - g_{2x}) + f_2h_{2y}) - f_2g_1h_{2x}(r_2 + 2))
g_2(r_2 + 1)(f_2(h_1(f_{1y} - g_{1x}) + f_1h_{1y}) - f_1g_2h_{1x}(r_1 + 2))
+ g_1g_2(r_1 + 1)(r_2 + 1)(f_2h_{1x} + f_1h_{2x})\right),$$

$$s_2 = \frac{1}{\Delta^2}\left(g_1^2(r_1 + 1)(h_2(f_{2y} - g_{2x}) - g_2h_{2x} - f_2h_{2y}(r_2 + 1))
g_2^2(r_2 + 1)(h_1(f_{1y} - g_{1x}) - g_1h_{1x} - f_1h_{1y}(r_1 + 1)) + g_1g_2
\times (f_2h_{1y}(r_1 + 2)(r_2 + 1) + f_1h_{2y}(r_1 + 1)(r_2 + 2))\right),$$

$$s_3 = \frac{1}{\Delta^2}\left(f_1^2(r_1 + 1)(h_2(f_{2y} - g_{2x}) + f_2h_{2y} + g_2h_{2x}(r_2 + 1))
+ f_2^2(r_2 + 1)(h_1(f_{1y} - g_{1x}) + f_1h_{1y} + g_1h_{1x}(r_1 + 1))
- f_1f_2(g_2h_{1x}(r_1 + 2)(r_2 + 1) + g_1h_{2x}(r_1 + 1)(r_2 + 2))\right),$$

$$s_4 = \frac{1}{\Delta^2}\left(f_1g_1(r_1 + 1)(h_2(f_{2y} - g_{2x}) - g_2h_{2x}) + f_2g_2(r_2 + 1)
\times (h_1(f_{1y} - g_{1x}) - g_1h_{1x} - f_1f_2(r_1 + 1)(r_2 + 1)(g_2h_{1y} + g_1h_{2y})
+ f_1^2g_2h_{2y}(r_1 + 1)(r_2 + 2) + f_2^2g_1h_{1y}(r_1 + 2)(r_2 + 1)\right),$$

$$u_1 = \frac{1}{\Delta^2}\left[g_1h_2(f_{1h2y} - g_{1h2x})(r_1 + 1) + g_2h_1(f_{2h1y} - g_{2h1x})
\times (r_2 + 1)\right],$$
Solving equation (18) and (19), we get a set of partial differential equations for the system of mixed Liénard-type class of oscillators of the form

\[ u_2 = \frac{1}{\Delta^2} \left[ f_1 h_2 (f_1 h_{2y} - g_1 h_{2x}) (r_1 + 1) + f_2 h_1 (f_2 h_{1y} - g_2 h_{1x}) \right] \times (r_2 + 1), \]  
\[ \Delta = (f_2 g_1 - f_1 g_2) \sqrt{(r_1 + 1)(r_2 + 1)}. \]  

The mixed scalar Liénard-type equation (7) has only linear and quadratic terms in \( \dot{x} \). On the other hand the coupled system of second order equations of motion (16) and (17), obtained from the nonsingular Lagrangian (14) is not in the class of mixed Liénard-type oscillators because it contains higher/different powers of \( \dot{x} \) and \( \dot{y} \) than quadratic and linear powers. By equating these higher/different degree coefficients to zero, analyzing them and making use of the results in the original coupled equations (16) and (17), we can obtain the relevant evolution equations. For this purpose we equate the terms \( q_1 \) and \( q_2 \) in equations (16) and (17) to zero as they contain higher-degree terms in \( \dot{x} \), \( \dot{y} \) and their products. Therefore we take

\[ q_1(\dot{x}, \dot{y}, x, y) = 0, \quad q_2(\dot{x}, \dot{y}, x, y) = 0. \]  

Solving equation (18) and (19), we get a set of partial differential equations for the variables \( f_i \) and \( g_i \), \( i = 1, 2 \). We can easily see that from \( q_1(\dot{x}, \dot{y}, x, y) = 0 \), we get

\[ f_{2y} = g_{2x}, \quad f_2 = \frac{g_2 h_{2x}}{h_{2y}}, \]  
\[ \text{Similarly from } q_2(\dot{x}, \dot{y}, x, y) = 0, \text{ we get} \]

\[ f_{1y} = g_{1x}, \quad f_1 = \frac{g_1 h_{1x}}{h_{1y}}, \]  

On substituting the above forms in equations (16) and (17), we obtain the coupled system of mixed Liénard-type class of oscillators of the form

\[ \ddot{x} = \frac{-1}{\Delta g_1 g_2 r_{12}} \left[ g_1 g_2 \left( (r_1 + 1)(r_2 + 1) \left( (f_{2x} g_1 - f_{1x} g_2) \dot{x}^2 - (g_{1y} g_2 - g_1 g_{2y}) \dot{y}^2 \right) \right. \right. \]
\[ \left. \left. - 2 (g_{1y} g_2 - g_1 g_{2x}) \dot{x} \dot{y} \right) + (r_1 + 1)(r_2 + 2) (h_{2x} \dot{x} + h_{2y} \dot{y}) g_1 \right] \]
\[ \left. \left. - (r_1 + 2) (r_2 + 1) (h_{1x} \dot{x} + h_{1y} \dot{y}) g_2 \right) - g_2^2 h_1 h_{1y} (r_2 + 1) \right] \]
\[ + g_1^2 h_2 h_{2y} (r_1 + 1), \]  
\[ \ddot{y} = \frac{1}{\Delta g_1 g_2 r_{12} h_{1y} h_{2y}} \left[ g_1 g_2 \left( (r_1 + 1)(r_2 + 1) \left( (f_{2x} g_1 h_{1y} h_{2y} - f_{1x} g_2 h_{1y} h_{2x}) \dot{x}^2 \right) \right. \right. \]
\[ \left. \left. + (g_{2y} g_1 h_{1x} h_{2y} - g_{1y} g_2 h_{1y} h_{2x}) \dot{y}^2 + 2 (g_1 g_{2x} h_{1y} h_{2y} - g_1 g_{2y} h_{1y} h_{2x}) \dot{x} \dot{y} \right) \right. \]
\[ \left. \left. - (r_1 + 1)(r_2 + 1) g_2 h_{1y} h_{2x} (h_{1x} \dot{x} + h_{1y} \dot{y}) \right) + (r_1 + 1)(r_2 + 1) g_1 h_2 h_{1y} h_{2y} (r_1 + 1) \right] \]
\[ - g_2^2 h_1 h_{1y} h_{2y} (r_2 + 1). \]  


The Hamiltonian associated with (36) and (37) corresponding to the Lagrangian (14) can now be written down as

\[
H = \frac{r_1}{\Delta} \left[ \frac{g_1}{h_1}(p_2h_{1x} - p_1h_{1y}) \left( h_2 - \left( \frac{g_1(h_{1x}p_2 - h_{1y}p_1)r_{12}}{h_{1y}r_1\hat{\Delta}} \right)^{\frac{r_2}{r_1+1}} \right) \right. \\
+ \left. \frac{g_2}{h_2}(p_1h_{2y} - p_2h_{2x}) \left( h_1 - \left( \frac{g_2(h_{2y}p_1 - h_{2x}p_2)r_{12}}{h_{2y}r_1\hat{\Delta}} \right)^{\frac{r_1}{r_1+1}} \right) \right] \\
- \left( \frac{g_1(h_{1x}p_2 - h_{1y}p_1)r_{12}}{h_{1y}r_1\hat{\Delta}} \right)^{\frac{r_2}{r_1+1}} - \left( \frac{g_2(h_{2y}p_1 - h_{2x}p_2)r_{12}}{h_{2y}r_1\hat{\Delta}} \right)^{\frac{r_1}{r_1+1}}, \quad (38)
\]

where \( \hat{\Delta} = g_1g_2(h_{1y}h_{2x} - h_{1x}h_{2y})(h_{1y}h_{2y})^{-1}r_{12}, \quad r_{12} = [(r_1 + 1)(r_2 + 1)]^{\frac{1}{2}}. \) Here the conjugate momenta \( p_1 \) and \( p_2 \) are defined as

\[
p_1 = L_{\hat{x}} = -\frac{f_1}{(f_1\hat{x} + g_1\hat{y} + h_1)^{r_1+1}} - \frac{f_2}{(f_2\hat{x} + g_2\hat{y} + h_2)^{r_2+1}}, \quad (39)
\]
\[
p_2 = L_{\hat{y}} = -\frac{g_1}{(f_1\hat{x} + g_1\hat{y} + h_1)^{r_1+1}} - \frac{g_2}{(f_2\hat{x} + g_2\hat{y} + h_2)^{r_2+1}}. \quad (40)
\]

### 3. Reduction to a subclass exhibiting quasiperiodic motion

In our further analysis, for simplicity, we assume the parameters \( r_1 = r_2 = 1 \) in the Lagrangian given by (14). Now, let the quantities \( f_i\hat{x} + g_i\hat{y}, \quad i = 1, 2, \) be the total derivatives (when \( r_1 = r_2 = 1 \)) of certain functions \( \rho_i(x, y) \).

\[
f_i\hat{x} + g_i\hat{y} = \frac{dt}{dt} \left[ \rho_i(x, y) \right] = \rho_{ix}\hat{x} + \rho_{iy}\hat{y}. \quad (41)
\]

From the above equation, we find \( f_i = \rho_{ix}, \quad g_i = \rho_{iy}, \quad i = 1, 2. \) Substituting Eq. (41) in Eq. (14) we get

\[
L = \sum_{i=1}^{2} \frac{1}{(\rho_{ix}\hat{x} + \rho_{iy}\hat{y} + h_i)}. \quad (42)
\]

Similarly, the condition (35) reduces to

\[
\rho_{ix} = \frac{\rho_{iy}h_{ix}}{h_{iy}}. \quad (43)
\]

Equation (43) can also be written as

\[
\begin{vmatrix}
\rho_{ix} & \rho_{iy} \\
\h_{ix} & \h_{iy}
\end{vmatrix} = 0. \quad (44)
\]

Consequently the term \( h_i \) is functionally dependent on \( \rho_i \) that is \( h_i = Q_i(\rho_i) \).

Then the Lagrangian (42) can also be written as

\[
L = \sum_{i=1}^{2} \frac{1}{(\rho_{ix}\hat{x} + \rho_{iy}\hat{y} + Q_i(\rho_i))}. \quad (45)
\]
Now the modified Emden equation is a special case of Liénard type of oscillators \[7\] which is obtained for specific forms of \(Q_i(\rho_i) = \rho_i^2 + \lambda_i\), where \(\lambda_i\)'s are constants. From the Lagrangian \[45\], the corresponding equation of motion is

\[
\ddot{x} = \left(\rho_{1xx}\rho_{2y} - \rho_{1y}\rho_{2xy}\right)\dot{x}^2 + \left(\rho_{1yy}\rho_{2y} - \rho_{1y}\rho_{2yy}\right)\dot{y}^2 + 3\left(\rho_{1x}\rho_{1\rho_{2xy}}\rho_{2y} - \rho_{2x}\rho_{1y}\rho_{2xy}\rho_{2y}\right)\dot{x}\dot{y} + \rho_{1y}\rho_{2y}\left(\rho_1^2 + \lambda_1 - \rho_2^2 + \lambda_2\right)\left(\rho_{1y}\rho_{2x} - \rho_{1x}\rho_{2y}\right)^{-1},
\]

(46)

\[
\ddot{y} = \left(\rho_{1x}\rho_{2x} - \rho_{1xx}\rho_{2y}\right)\dot{x}^2 + \left(\rho_{1x}\rho_{2yy} - \rho_{1y}\rho_{2xy}\right)\dot{y}^2 + 3\rho_{1x}\rho_{2x}\rho_{2y} - \rho_{1y}\rho_{2x}\rho_{2y}\right)\dot{x}\dot{y} + \rho_{2x}\rho_{1\rho_{2xy}}\rho_{1y}\rho_{2xy}\rho_{2x}\left(\rho_2^2 + \lambda_2 - \rho_1^2 + \lambda_1\right)\left(\rho_{1y}\rho_{2x} - \rho_{1x}\rho_{2y}\right)^{-1}.
\]

(47)

The associated Hamiltonian becomes

\[
H = \left(\rho_{2y}\rho_{1x} - \rho_{2x}\rho_{2y}\right)\left(\rho_1^2 + \lambda_1 + \rho_{1x}\rho_{2y} - \rho_{1y}\rho_{2x}\right)^{2} - 2\sqrt{\left(\rho_{1y}\rho_{2x} - \rho_{1x}\rho_{2y}\right)\left(\rho_{1x}\rho_{2y} - \rho_{1y}\rho_{2x}\right)}\right)
\times \left(\rho_{1y}\rho_{2x} - \rho_{1x}\rho_{2y}\right)^{-1}.
\]

(48)

Here the conjugate momenta \(p_1\) and \(p_2\) are defined as

\[
p_1 = L_x = \frac{\rho_{1x}}{\left(\rho_{1x}\dot{x} + \rho_{1y}\dot{y} + \rho_1^2 + \lambda_1\right)^2} - \frac{\rho_{2x}}{\left(\rho_{2x}\dot{x} + \rho_{2y}\dot{y} + \rho_2^2 + \lambda_2\right)^2},
\]

(49)

\[
p_2 = L_y = \frac{\rho_{1y}}{\left(\rho_{1x}\dot{x} + \rho_{1y}\dot{y} + \rho_1^2 + \lambda_1\right)^2} - \frac{\rho_{2y}}{\left(\rho_{2x}\dot{x} + \rho_{2y}\dot{y} + \rho_2^2 + \lambda_2\right)^2}.
\]

(50)

The Hamiltonian \[48\] is connected to the Hamiltonian of a system of uncoupled linear harmonic oscillators \(\tilde{H} = \frac{1}{2}(P_1^2 + P_2^2 + \lambda_1U_1^2 + \lambda_2U_2^2)\) through the following canonical transformation,

\[
P_1 = \left[\lambda_1 + \left(\lambda_1^2 - 2\lambda_1 \left(\frac{p_{1}\rho_{2y} - p_{2}\rho_{2x}}{\rho_{2y}\rho_{1x} - \rho_{1y}\rho_{2x}}\right)\right)^{\frac{1}{2}}\right],
\]

(51)

\[
P_2 = \left[\lambda_2 + \left(\lambda_2^2 - 2\lambda_2 \left(\frac{p_{2}\rho_{1x} - p_{1}\rho_{1y}}{\rho_{2y}\rho_{1x} - \rho_{1y}\rho_{2x}}\right)\right)^{\frac{1}{2}}\right],
\]

(52)

\[
U_1 = -\frac{\rho_{1}}{\lambda_1} \left[\lambda_1^2 - 2\lambda_1 \left(\frac{p_{1}\rho_{2y} - p_{2}\rho_{2x}}{\rho_{2y}\rho_{1x} - \rho_{1y}\rho_{2x}}\right)\right]^{\frac{1}{2}},
\]

(53)

\[
U_2 = -\frac{\rho_{2}}{\lambda_2} \left[\lambda_2^2 - 2\lambda_2 \left(\frac{p_{2}\rho_{1x} - p_{1}\rho_{1y}}{\rho_{2y}\rho_{1x} - \rho_{1y}\rho_{2x}}\right)\right]^{\frac{1}{2}}.
\]

(54)

The general solution of \[46\] and \[47\] can be found after choosing the forms of \(f_1\) and \(f_2\) by using the relations \[51\] - \[54\] and the harmonic oscillator solution

\[
U_i = A_i \sin(\omega_i t + \delta_i)
\]

\[
P_i = A_i \omega_i \cos(\omega_i t + \delta_i), \quad \omega_i = \sqrt{\lambda_i}
\]

(55)
where $A_i$ and $\delta_i$, $(i = 1, 2)$ are arbitrary constants. The above canonical transformations are identified by generalizing the knowledge of the canonical transformation for the scalar equation which is discussed in the appendix.

3.1. Nonlocal transformation

The system of coupled mixed MEE equations (46) and (47) can also be related to a system of uncoupled simple harmonic oscillators,

\[ \ddot{u} + \lambda_1 u = 0, \quad \ddot{v} + \lambda_2 v = 0, \]  

through the nonlocal transformations,

\[ u = \rho_1(x, y)e^{\int \rho_1(x, y)\, dt}, \quad v = \rho_2(x, y)e^{\int \rho_2(x, y)\, dt}. \]  

Equations having such type of nonlocal transformations and their solutions have been studied in Ref. [29]. In addition to using the canonical transformation obtained in the previous subsection to find the solution of equation (45), one can also obtain the solution of it by solving the following set of coupled first order equations arising from (56) and (57),

\[ \dot{x} = \frac{uv(\rho_2^2\rho_{1y} - \rho_1^2\rho_{2y}) + \dot{w}\rho_1\rho_{2y} - u\dot{w}\rho_2\rho_{1y}}{uv(\rho_1\rho_{2y} - \rho_1\rho_{2x})}, \]  

\[ \dot{y} = \frac{uv(\rho_1^2\rho_{2x} - \rho_2^2\rho_{1x}) + \dot{w}\rho_2\rho_{1x} - u\dot{w}\rho_1\rho_{2x}}{uv(\rho_1\rho_{2y} - \rho_1\rho_{2x})}. \]

Here $u$ and $v$ are the solutions of the simple harmonic oscillator equations [56]. The difficulty in solving the equations (57) depends on the form of $\rho_1$ and $\rho_2$ (see [29]).

4. A system of coupled Liénard-type equations

Next we can obtain a special case of Liénard-type of oscillators (45) by choosing

\[ \rho_1 = k_1(\alpha_1x^{m_1} + \alpha_2y^{m_2}), \quad \rho_2 = k_2(\alpha_3y^{m_3} + \alpha_4x^{m_4}). \]  

where $k_i$’s and $\alpha_j$’s are arbitrary real parameters and $m_j$’s are positive integers and $i = 1, 2$, and $j = 1, 2, 3, 4$. In this case the specific form of the nonstandard and nonsingular Lagrangian with two degrees of freedom takes the form

\[ L = \frac{1}{\lambda_1 + k_1^2(\alpha_1x^{m_1} + \alpha_2y^{m_2})^2 + k_1(m_1\alpha_1x^{m_1+1} - m_2\alpha_2y^{m_2+1})} + \frac{1}{\lambda_2 + k_2^2(\alpha_3y^{m_3} + \alpha_4x^{m_4})^2 + k_2(m_3\alpha_3y^{m_3+1} - m_4\alpha_4x^{m_4+1}).} \]

The equation of motion of the mixed Liénard-type class of oscillators can be obtained from the above Lagrangian. It is of the form

\[ \ddot{x} = -\frac{1}{xy^2\delta_m} \left( x^2y^2 \left[ m_1m_3\alpha_1\alpha_3(m_1 - 1)x^{m_1}y^{m_3} - m_2m_4\alpha_2\alpha_4(m_4 - 1)x^{m_4}y^{m_2} \right] + 3k_1y^{m_3}(m_3\alpha_1x^{m_1}y\dot{x} + m_2\alpha_2y^{m_2}\dot{y}) \right. \]
where the form ways of generalizing the modified Emden equation to two dimensions, for example see

The associated Hamiltonian becomes

\[
\delta \dddot{y} = \frac{1}{y x^2 \delta_m} \left( \dot{y}^2 x^2 \left[ m_2 m_4 \alpha_2 \alpha_4 (m_2 - 1)x^{m_2} m_2 - \lambda_1 m_3 \alpha_3 \alpha_3 (m_3 - 1)x^{m_3} y^{m_3} \right] 
- \dot{x}^2 y^2 \left[ m_1 \alpha_3 \alpha_4 \alpha_4 (m_4 - m_1)x^{m_2} m_4 + \lambda_1 y^{m_4} \right] + \left[ 3 k_1 x^m \alpha_1 \alpha_2 y x + \lambda_2 \right] m_4 \alpha_4 \dot{y} \left( \alpha_4 m_4 + \alpha_3 m_3 \right) 
+ \lambda_2 \dot{y}^2 \left[ m_1 \alpha_1 \alpha_3 \alpha_3 \right] \right),
\]

(62)

where \( \delta_m = m_1 m_3 \alpha_3 \alpha_3 x^{m_3} y^{m_3} - \lambda_2 m_4 \alpha_4 \alpha_4 x^{m_4} y^{m_4} m_2. \)

The associated Hamiltonian becomes

\[
H = \frac{1}{k_1 k_2 \delta_m} \left[ \dot{x}^2 \left( m_4 \alpha_4 \alpha_2 x^{m_2} y - m_3 \alpha_3 \alpha_3 p_1 x^{m_3} \right) \left( k_2^2 \lambda_1 \right) \left( k_1^2 \lambda_1 \right) 
+ \left[ m_2 \alpha_3 \alpha_2 \alpha_4 \right] \left( m_2 \alpha_2 p_1 x^{m_2} y - m_1 \alpha_1 p_2 x^{m_1} y \right) \left( k_2^2 \lambda_2 \right) 
- \lambda_2 \left( m_1 \alpha_1 p_2 x^{m_1} y \right) \left( k_2 \lambda_1 \right) \left( k_1 \lambda_2 \right) \left( m_4 \alpha_4 \alpha_2 x^{m_4} y - m_3 \alpha_3 \alpha_3 p_1 x^{m_3} \right) \right].
\]

(64)

where \( X_m = \alpha_1 x^{m_1} + \alpha_2 y^{m_2}, Y_m = \alpha_4 x^{m_4} + \alpha_3 y^{m_3}. \)

4.1. Coupled Modified Emden Equation: A special case

The coupled system of equations (62) and (63) reduces to a coupled generalization of the modified Emden equation (1)–(2) for the choice \( m_1 = m_2 = m_3 = m_4 = 1 \) and is of the form

\[
\ddot{x} = -\frac{1}{\delta_1} \left[ 3 k_1 (\alpha_1 x + \alpha_2 y) + k_2^2 (\alpha_1 x + \alpha_2 y)^2 + \lambda_1 \right] (\alpha_3 \alpha_1 x + \alpha_2 y),
\]

(65)

\[
\ddot{y} = \frac{1}{\delta_1} \left[ 3 k_1 (\alpha_1 x + \alpha_2 y) + k_2^2 (\alpha_3 y + \alpha_4 x)^2 + \lambda_2 \right] (\alpha_4 \alpha_1 x + \alpha_2 y),
\]

(66)

where \( \delta_1 = (\alpha_1 \alpha_3 - \alpha_2 \alpha_4). \) Note that the above system of equations is PT symmetric under the combined transformations, \( (t \rightarrow -t, x \rightarrow -x, y \rightarrow -y). \) There are other ways of generalizing the modified Emden equation to two dimensions, for example see
Figure 1. (color online) Periodic oscillations with $\omega_1 : \omega_2 = 4 : 4$, $k_1 = k_2 = 1$, $\alpha_1 = \alpha_3 = 5.5$, $\alpha_2 = \alpha_4 = 3$ (a) Time series plot (b) Projected phase portrait. Similar plots can be given for the $y$ variable.

Ref. [9]. Even though the generalized version identified as the coupled modified Emden equation in Ref. [9] is isochronous, the system lacks a Hamiltonian description in order to quantize it. However, we find that the system (65) and (66) has the well defined Hamiltonian

$$H = \frac{1}{k_1 k_2 \delta_1} \left[ k_2 (p_2 a_4 - p_1 a_3) (k_1^2 X^2 + \lambda_1) + k_1 (p_1 a_2 - p_2 a_1) (k_2^2 Y^2 + \lambda_2) - 2 \sqrt{\delta_1 (k_1 k_2 (p_1 a_2 - p_2 a_1)^{\frac{3}{2}} + k_2 k_1 (p_2 a_4 - p_1 a_3)^{\frac{3}{2}})} \right],$$

(67)

where $X = (\alpha_1 x + \alpha_2 y)$ and $Y = (\alpha_3 y + \alpha_4 x)$. Here we note that one can use a variable transformation $X = (\alpha_1 x + \alpha_2 y)$ and $Y = (\alpha_3 y + \alpha_4 x)$ in equation (65) to obtain two uncoupled modified Emden equations.

In order to find the general solution of the equations (65) and (66) we use suitable canonical transformation to the above Hamiltonian to reduce it to a simpler form. We find that the above Hamiltonian is connected to the Hamiltonian of a two dimensional linear harmonic oscillator through the following canonical transformation, see (51)-(54),

$$P_1 = \lambda_1 + \left[ \lambda_1^2 - \frac{2 \lambda_1 (a_3 p_1 - a_4 p_2)}{k_1 (a_1 a_3 - a_2 a_4)} \right]^\frac{1}{2},$$

(68)

$$P_2 = \lambda_2 + \left[ \lambda_2^2 - \frac{2 \lambda_2 (a_1 p_2 - a_2 p_1)}{k_2 (a_1 a_3 - a_2 a_4)} \right]^\frac{1}{2},$$

(69)

$$U_1 = -\frac{k_1 (a_1 x + a_2 y)}{\lambda_1} \left[ \lambda_1^2 - \frac{2 \lambda_1 (a_3 p_1 - a_4 p_2)}{k_1 (a_1 a_3 - a_2 a_4)} \right]^\frac{1}{2},$$

(70)

$$U_2 = -\frac{k_2 (a_4 x + a_3 y)}{\lambda_2} \left[ \lambda_2^2 - \frac{2 \lambda_2 (a_1 p_2 - a_2 p_1)}{k_2 (a_1 a_3 - a_2 a_4)} \right]^\frac{1}{2},$$

(71)
The general solution of equations (65) and (66) can then be obtained by substituting the general solution of the two dimensional harmonic oscillator given by the expressions

\[ U_1 = A \sin(\omega_1 t + \delta_1), \quad U_2 = B \sin(\omega_2 t + \delta_2), \]
\[ P_1 = A\omega_1 \cos(\omega_1 t + \delta_1), \quad P_2 = B\omega_2 \cos(\omega_2 t + \delta_2), \] (72)

where \( \omega_j = \sqrt{\lambda_j}, j=1,2 \) into Eqs. (68)-(71) and solving the resultant equations. We obtain

\[ x = \frac{A\alpha_3\omega_1 \sin(\omega_1 t + \delta_1)}{k_1\delta_1 (\omega_1 - A \cos(\omega_1 t + \delta_1))} - \frac{B\alpha_2\omega_2 \sin(\omega_2 t + \delta_2)}{k_2\delta_1 (\omega_2 - B \cos(\omega_2 t + \delta_2))}, \] (73)

\[ y = -\frac{A\alpha_4\omega_1 \sin(\omega_1 t + \delta_1)}{k_1\delta_1 (\omega_1 - A \cos(\omega_1 t + \delta_1))} + \frac{B\alpha_1\omega_2 \sin(\omega_2 t + \delta_2)}{k_2\delta_1 (\omega_2 - B \cos(\omega_2 t + \delta_2))}, \] (74)

where \( A, B, \delta_1, \delta_2 \) are arbitrary constants. The above solution is oscillatory and is periodic and bounded for suitable choice of parameters namely \( 0 < A < \omega_1 \) and \( 0 < B < \omega_2 \). Here one can note that the frequency of oscillations is again independent of the amplitude for the two dimensional coupled modified Emden equation. Figures 1 and 2 show two types of oscillatory behaviour, namely periodic and quasiperiodic oscillations, depending upon the ratio of frequencies \( \omega_1 \) and \( \omega_2 \). Fig.1 shows periodic

\[ \begin{array}{cc}
\text{(a)} & \text{(b)} \\
\text{(c)} & \text{(d)}
\end{array} \]

**Figure 2.** (color online) Quasiperiodic oscillations with \( \omega_1 : \omega_2 = 4 : \sqrt{3}, k_1 = k_2 = 1, \alpha_1 = \alpha_3 = 5.5, \alpha_2 = \alpha_4 = 3 \). (a) Time series plot (b) Projected phase portrait (c) Poincaré section. Similar plots can be given for the \( y \) variable.
oscillations for $\omega_1 : \omega_2 = 4 : 4$, $k_1 = k_2 = 1$, $\alpha_1 = \alpha_3 = 5.5$, $\alpha_2 = \alpha_4 = 3$ and Fig.2 shows quasiperiodic oscillations for $\omega_1 : \omega_2 = 4 : \sqrt{3}$, $k_1 = k_2 = 1$, $\alpha_1 = \alpha_3 = 5.5$, $\alpha_2 = \alpha_4 = 3$.

5. Conclusion

In this paper we have identified a class of coupled mixed Liénard type nonlinear evolution equations. This class of equation is identified by generalizing the nonstandard Lagrangian of the scalar MEE to a suitable nonsingular two dimensional Lagrangian. Imposing the condition that the resultant Euler-Lagrange equation should be of mixed Liénard type form, we have identified a specific class of equations which admits isochronous solutions. The procedure is illustrated for the special case of two dimensional modified Emden equation and is found to be isochronous as well as $PT$ symmetric for suitable choice of parameters, exhibiting quasiperiodic and periodic oscillations. The above procedure can in principle be extended to higher degrees of freedom also. The problem of quantization of system [67] is also worth further consideration.

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Appendix

Let us consider the following mixed Liénard type equation,

$$\ddot{x} + x^2 \frac{u_{xx}}{u_x} + 3ku\dot{x} + k^2 u^2 + \lambda \frac{u}{u_x} = 0, \quad (75)$$

where $u(x)$ is an arbitrary function of $x$. This equation is obtained by applying the transformation $y = u(x)$ in the linearizable MEE

$$\ddot{y} + 3ky\dot{y} + k^2 y^3 + \lambda y = 0. \quad (76)$$

The resultant equation possess the Hamiltonian structure

$$H = -kp \frac{u^2}{u_x} - \frac{\lambda p}{k u_x} + 2 \sqrt{\frac{p}{k u_x}}, \quad (77)$$

which can be deduced from the Lagrangian [12]. This Hamiltonian can be transformed to the Hamiltonian of the simple harmonic oscillator equation through the canonical transformation

$$U = \frac{u(x) \sqrt{\lambda (\lambda - \frac{2kp}{u_x})}}{\lambda}, \quad P = \frac{\lambda - \sqrt{\lambda (\lambda - \frac{2kp}{u_x})}}{k}. \quad (78)$$
Generalizing the above canonical transformation to two dimensions we obtain the canonical transformations (vide Eqs. (51)-(54)) for the coupled MEE.

References

[1] E.C.G. Sudarshan and N. Mukunda, Classical Dynamics: A Modern Perspective (John Wiley and Sons, Inc., New York, 1974)
[2] V.I. Arnold, Mathematical Methods of Classical Mechanics (Springer, New York, 1978)
[3] R. Santilli, Foundations of Theoretical Mechanics I (Springer-Verlag, New York, 1978)
[4] Z. Musielak, J. Phys. A: Math. Theor. 41, 055205 (2008)
[5] J.F. Carinena, M.F. Ranada and M. Santander, J. Math. Phys. 46, 062703 (2005)
[6] J. Jose and E.J. Saletan, Classical Dynamics: A Contemporary Approach (Cambridge University Press, Cambridge, 2002)
[7] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, Phys. Rev. E 72, 066203 (2005)
[8] G. Gubbiotti and M.C. Nucci, J. Nonlinear Math. Phys. 21, 248 (2014)
[9] R. Gladwin Pradeep, V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, J. Phys. A: Math. Theor. 42, 135206 (2009)
[10] M. Lakshmanan and V.K. Chandrasekar, Eur. Phys. J. Special Topics 222, 665 (2013)
[11] S. Chandrasekar, An Introduction to the Study of Stellar Structure (Dover, New York, 1957)
[12] V.J. Erwin, W.F. Ames and E. Adams Wave Phenomena: Modern Theory and Applications ed. C. Rogers and J.B. Moodie (North-Holland, Amsterdam, 1984)
[13] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, J. Phys. A: Math. Theor. 40, 4717 (2007)
[14] J.H.E. Cartwright, V.M. Eguiluz, E. Hernandez-Garcia and O. Piro, Int. J. Bifurcat. Chaos 9, 2197 (1999)
[15] Nicolas Glade, Loic Forest and Jacques Demongeot, C.R. Acad. Sci. Paris, Ser. I 344, 253 (2007)
[16] S.L. Shapiro and S.A. Teukolsky, Black Holes, White Dwarfs, Neutron Stars (New York: Wiley, 1983)
[17] M.V. Kalashnik, V.O. Kakhiani, D.G. Lominadze, K.I. Patarashvili, P.N. Svirkunov and S.D. Taskazde, Fluid Dynamics, 39, 790 (2004)
[18] F. Calogero, Isochronous Systems (Oxford: Oxford University Press, New York, 2008)
[19] F. Calogero and F. Leyvraz, J. Phys. A: Math. Gen. 39, 11803 (2006)
[20] F. Calogero and F. Leyvraz, J. Phys. A: Math. Theor. 40, 12931 (2007)
[21] F. Calogero and F. Leyvraz, J. Phys. A: Math. Theor. 41, 175202 (2008)
[22] F. Calogero and F. Leyvraz, J. Nonlinear Math. Phys. 14, 612 (2007)
[23] P. Guha and A.G. Choudhury, J. Phys. A: Math. Theor. 42, 192001 (2009)
[24] V.K. Chandrasekar, A. Durga Devi and M. Lakshmanan, J. Nonlinear Math. Phys. 17, 251 (2010)
[25] A. Durga Devi, R. Gladwin Pradeep, V.K. Chandrasekar and M. Lakshmanan, J. Nonlinear Math. Phys. 20, 78 (2013)
[26] V. Chithiika Ruby, M. Senthilvelan and M. Lakshmanan, J. Phys. A : Math. Theor. 45, 382002 (2012)
[27] T.L. Curtright, C.K. Zachos, J. Phys. A : Math. Theor. 47, 145201 (2014)
[28] B. Bagchi, S. Modak and P.K. Panigrahi, Acta Polytechnica 54, 79 (2014)
[29] R. Gladwin Pradeep, V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, J. Math. Phys. 51, 103513 (2010)