The Darboux process and a noncommutative bispectral problem

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Abstract. The Darboux process, also known by many other names, played a very important role in some extremely enjoyable joint work that Hans and I did 25 years ago. I revisit a version of this problem in a case when scalars are replaced by matrices, i.e., elements of a non-commutative ring. Many of the issues studied here can be pushed to the case of a ring with identity, but my emphasis is on very concrete examples involving $2 \times 2$ matrices.

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1. The bispectral problem

About 30 years ago I was extremely lucky. I gave a talk in Berkeley where I mentioned the following problem:

Find all nontrivial instances where a function $\varphi(x, k)$ satisfies

$$L \left( x, \frac{d}{dx} \right) \varphi(x, k) \equiv (-D^2 + V(x))\varphi(x, k) = k\varphi(x, k)$$

as well as

$$B \left( k, \frac{d}{dk} \right) \varphi(x, k) \equiv \left( \sum_{i=0}^{M} b_i(k) \left( \frac{d}{dk} \right)^i \right) \varphi(x, k) = \Theta(x)\varphi(x, k).$$

All the functions $V(x), b_i(k), \Theta(x)$ are, in principle, arbitrary except for smoothness assumptions. Notice that here $M$ is arbitrary (finite).

I was fortunate that Hans was in the audience, and about a week later he came up with a tool to attack this problem. After a few months

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of intense work, mainly by slow mail, we found ourselves with a rather nice picture.

The complete answer to the problem is given as follows:

**Theorem 1** [DG]. If $M = 2$, then $V(x)$ is (except for translation) either $c/x^2$ or $ax$, i.e., we have a Bessel or an Airy case. If $M > 2$, there are two families of solutions.

a) $L$ is obtained from $L_0 = -D^2$ by a finite number of Darboux transformations $(L = AA^* \rightarrow \tilde{L} = A^*A)$. In this case $V$ is a rational solution of the Korteweg-deVries equation and all rational solutions of KdV decaying at infinity show up in this fashion.

b) $L$ is obtained from $L_0 = -D^2 + \frac{1}{4x^2}$ after a finite number of rational Darboux transformations.

It was later observed in [MZ] that in the second case we are dealing with rational solutions of the Virasoro or master symmetries of KdV.

In the first case the space of common solutions has dimension one, in the second it has dimension two. One refers to these as the rank one and rank two situations. In [DG] one finds several other equivalent descriptions of the solution such as those in terms of the monodromy group of the equation.

Observe that the “trivial cases” when $M = 2$ are self-dual in the sense that since the eigenfunctions $f(x,k)$ are functions either of the product $xk$ or of the sum $x + k$, one gets $B$ by replacing $k$ for $x$ in $L$.

My reasons for asking this question can be traced back to an effort to understand some work on “time-and-band-limiting” that had led me to isolate certain properties of well known special functions. For an example relating to orthogonal polynomials see [G1]. For a more updated versions of this connection between the bispectral problem and the issue of time-and-band-limiting see, [G2, G3, GY2].

The work with Hans gave rise to a large number of papers by other people, some of which can be found in the list at the end of this paper. This listing is far from complete and I apologize for the omissions.

It may be appropriate to observe that what we are calling the Darboux process has been reinvented many times, including in the work of some rather well known people, see for instance [Sc, IH]. Reference [YZ] talks about the Geronimus transformation, from 1940, and its inverse the Christoffel transform. It is clear that the first one (as noticed in [YZ]) has a lot in common with what we are calling the Darboux transformation. See also [SVZ, Z].
2. The Bochner Krall problem

In a series of papers with Luc Haine, [GH1, GH2, GH3] and then with Luc and Emil Horozov, see [GHH1, GHH2], we noticed that a large class of polynomials
\[ p_n(k) \]
that satisfy three term recursion relations in the variable \( n \), as well as differential equations in the variable \( k \) can be obtained by an application of a similar Darboux transformation starting from the so called classical orthogonal polynomials of Jacobi, Laguerre and Hermite. In this case one goes from a tridiagonal matrix
\[ L_0 = AB \]
to a new tridiagonal matrix \( L \) by factorizing the first one (or a function of it) as a product of two bidiagonal matrices. As indicated in [GH1] some form of this method is given in [MS1].

In this case one runs into the Toda flows and its master symmetries. Further work on these lines can be found in [GY1] and [GH4, GH5] and for a very nice survey of all this material see [H2].

The origins of this line of work is contained in papers such as [Bo, Koo, Kra1].

3. A matrix valued version of the Darboux process for a difference operator

Consider the block tridiagonal matrix \( L_0 \)
\[ L_0 = \begin{pmatrix} B_0 & I \\ A_1 & B_1 & I \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \]
where all the matrices \( A_i, B_i \) are of size \( N \times N \) and \( I \) denotes the \( N \times N \) identity matrix.

If we try to factorize this in the form
\[ L_0 = \alpha \beta \]
where
\[ \alpha = \begin{pmatrix} \alpha_0 & I \\ 0 & \alpha_1 & I \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \]
and
\[ \beta = \begin{pmatrix} I \\ \beta_1 & I \\ \vdots & \ddots \end{pmatrix} \]
with all the matrices $\alpha_i, \beta_i$ are of size $N \times N$, and then define the matrix

$$L = \beta \alpha$$

we have that

$$L = \begin{pmatrix}
\tilde{B}_0 & I \\
\tilde{A}_1 & \tilde{B}_1 & I \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}$$

where all the matrices $\tilde{A}_i, \tilde{B}_i$ are of size $N \times N$, and the following relations hold

$$\beta_n = B_{n-1} - \alpha_{n-1}, \alpha_n = A_n \beta_n^{-1}$$

which then gives

$$\tilde{B}_n = B_n - \beta_{n+1} + \beta_n = B_{n-1} - \alpha_{n-1} + \alpha_n$$

and

$$\tilde{A}_n = \beta_n A_{n-1} \beta_n^{-1} = \alpha_n^{-1} A_n \alpha_{n-1}.$$ 

These expressions are valid for $n = 2, 3, \ldots$ in the case of $\tilde{A}_n$ and for $n = 1, 2, \ldots$ in the case of $\tilde{B}_n$.

Above we take $\beta_0 = 0$, so that $\tilde{B}_0 = \alpha_0 = B_0 - \beta_1$. We also need to take $\tilde{A}_1 = (B_0 - \alpha_0) \alpha_0$.

A moment’s thought gives that once $L_0$ is given, the only free parameter is the matrix $\alpha_0$.

Just as in [GHI], and in spite of the fact that one is dealing here with a semi-infinite block tridiagonal matrix, it is possible to see the connection between this construction and that in [MST]. One puts

$$\beta_n = -\phi_n \phi_{n-1}^{-1}$$

and then notices that this amounts to choosing $\phi$ in the null-space of $L$. Since $L$ is not doubly-infinite we seem to have lost some freedom in picking this subspace, but this can be remedied as in [GHI] by considering $L$ as a limit of an appropriate doubly-infinite matrix with a rich null-space.

### 4. Fancier versions of the Darboux process

It is well known that it is useful to extend the original method of Darboux consisting of going from

$$L_0 = AB$$

to

$$L = BA$$

in an appropriate way. Notice that in the standard case we have

$$BL_0 = LB.$$
The new idea is to allow for an arbitrary banded matrix (or a differential operator) $U$ and to declare $L$ a Darboux transform of $L_0$ as long as we have

$$UL_0 = LU.$$ 

In several of the uses of the original Darboux’s method one needs to apply it repeatedly and this fancier version of the method takes care of that.

One should also keep in mind the results in [GHH1, GHH2, H2] where the usual factorization followed by a reversal of the factors is applied not directly to $L$ but to a constant coefficient polynomial in $L$.

I thank Jose Liberati for pointing out to me that at the very end in [GGRW] one finds an application of the theory of quasideterminants (a notion that goes back to Cayley) to obtain expressions for matrix valued orthogonal polynomials in terms of their matrix valued moments. Many of these results, as well as others, have been derived independently by making good use of the notion of Schur complements in L. Miranian’s Berkeley thesis. The main new results are contained in [M]. In the next section we give a very brief look into the theory of matrix valued orthogonal polynomials and a short guide to the literature that is relevant to us.

One should remark that this same theory of quasideterminants has been studied in connection with a certain Darboux process for a matrix Schroedinger equation in [GV]. In this case one needs to consider this fancier version. For a nice use of quasideterminants in our context see [BL].

The matrix version of the Darboux process for a difference operator discussed in the previous section could be extended in this fancier fashion too.

## 5. Matrix valued orthogonal polynomials

Given a self-adjoint positive definite matrix valued weight function $W(t)$, Krein, see [K1, K2], considers the skew symmetric bilinear form defined for any pair of matrix valued functions $P(t)$ and $Q(t)$ by the numerical matrix

$$\langle P, Q \rangle = \langle P, Q \rangle_W = \int_{\mathbb{R}} P(t)W(t)Q^*(t)dt,$$

where $Q^*(t)$ denotes the conjugate transpose of $Q(t)$.

Proceeding as in the case of a scalar valued inner product Krein proves that there exists a sequence $(P_n)_n$ of matrix polynomials, orthogonal with respect to $W$, with $P_n$ of degree $n$ and monic.
Krein goes on to prove that any sequence of monic orthogonal matrix valued polynomials \((P_n)_n\) satisfies a three term recurrence relation

\[
A_n P_{n-1}(t) + B_n P_n(t) + P_{n+1}(t) = tP_n(t),
\]

where \(P_{-1}\) is the zero matrix and \(P_0\) is the identity matrix. These coefficient matrices enjoy certain properties: in particular the \(A_n\) are nonsingular.

The equation above can be rewritten as

\[
\mathcal{L} P_n(t) = tP_n(t)
\]

with a matrix \(\mathcal{L}\) such as the one that has appeared in previous sections.

To place ourselves in the context of the \textbf{bispectral problem} we consider matrix valued polynomials \((P_n)_n\) satisfying not only the equation above but also “right hand side” differential equations of the form:

\[
P_n D = \Lambda_n P_n \quad \text{for all} \quad n \geq 0
\]

with \(\Lambda_n\) a matrix valued eigenvalue and \(D\) a differential operator of order \(s\) with matrix coefficients given by

\[
D = \sum_{i=0}^{s} \partial^i F_i(x), \quad \partial = \frac{d}{dx},
\]

which acts on \(P(x)\) by means of

\[
PD = \sum_{i=0}^{s} \partial^i (P)(x) F_i(x).
\]

This problem in the matrix case has been studied in \([D, DG1, G10, G1, GPT1, GPT2, GPT3, GT]\) and in a few other places.

One can see that the differential operators that correspond to a fixed family of polynomials form an associative algebra which in general is non-commutative, see \([DG1, GT]\). The problem of exhibiting elements of this algebra that have a minimal order will occupy us in a few examples in the next two sections.

6. A few examples

Here we consider in detail few examples of the matrix version of the basic Darboux process described above.

For \(\lambda > 3/2\) consider the monic matrix valued polynomials which are orthogonal with respect to the weight matrix

\[
W(x) = ((2 - x)x)^{\lambda - 3/2} \begin{pmatrix} 1 & x - 1 \\ x - 1 & 1 \end{pmatrix}, \quad x \in [0, 2].
\]
Let
\[ L_0 = \begin{pmatrix} B_0 & I & & & \\ A_1 & B_1 & I & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \]
be the corresponding block tridiagonal matrix with
\[ B_n = \frac{1}{2} \frac{\lambda - 1}{(n + \lambda)(n + \lambda - 1)} S + I \]
\[ A_n = \frac{n(n + 2\lambda - 2)}{4(n + \lambda - 1)^2} I. \]

Here \( S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

These polynomials can be seen to be joint eigenfunctions of a first order differential operator, an observation that was made for a special value of \( \lambda \) in [CG1, CG2].

If \( \alpha_0 \) is an arbitrary matrix we can consider the monic polynomials that result from one application of the Darboux process to the block tridiagonal matrix \( L_0 \) with free parameter \( \alpha_0 \).

We can see that for an invertible symmetric \( \alpha_0 \) the new orthogonality weight is given by
\[ \tilde{W}(x) = (2 - x)^{\lambda - 3/2} x^{\lambda - 5/2} \begin{pmatrix} 1 & x - 1 \\ x - 1 & 1 \end{pmatrix} \]
\[ - \frac{2^{2\lambda} Be \left( \frac{2\lambda - 1}{2}; \frac{2\lambda - 1}{2} \right)}{4(2\lambda - 3)} \begin{pmatrix} 2\lambda - 2 & -1 \\ -1 & 2\lambda - 2 \end{pmatrix} - (2\lambda - 3)\alpha_0^{-1} \delta_0(x). \]

Here \( Be \) stands for the usual Beta function.

Recall that the Darboux process played an important role in getting new bispectral situations in [DG].

We show below some examples that illustrate that for appropriate values of \( \lambda \) the new polynomials are joint eigenfunctions of some higher order differential operators, i.e., we get new bispectral situations. This appears to have little to do with \( \alpha_0 \) being symmetric.

Example 1.
\[ \lambda = 5/2 \quad \alpha_0 = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}. \]

Here we find one differential operator \( D \) satisfying
\[ P_n D = \Lambda_n P_n \]
with

\[ D = \sum_{r=0}^{4} \partial^r B_r \]

and

\[ B_4 = (t - 2)^2 t^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]
\[ B_3 = 4(t - 2)t(3t - 2) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]
\[ B_2 = \frac{24}{5} t(7t - 9) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]
\[ B_1 = \frac{8}{5} \begin{pmatrix} 5t + 6 & -8t \\ -2(5t + 3) & 13t \end{pmatrix} \]
\[ B_0 = \begin{pmatrix} -\frac{8 \times 11}{5} & -8 \\ -\frac{56}{5} & 0 \end{pmatrix} \]

\[ \Lambda_n = \left( \begin{array}{c} \frac{(n+2)(5n^2+20n^2+3n-44)}{5n^4+30n^3+43n^2+2n+32} \\ \frac{5n^4+30n^3+43n^2-14n+40}{n(5n^3+30n^2+43n+26)} \end{array} \right). \]

There are no operators of lower order in the algebra.

**Example 2.**

\[ \lambda = 7/2, \quad \alpha_0 = \begin{pmatrix} 3 & -1 \\ 5 & 7 \end{pmatrix}. \]

Here the corresponding operator is given by

\[ D = \sum_{r=0}^{6} \partial^r B_r \]
with
\[
B_6 = \frac{(t - 2)^3 t^3}{15} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]
\[
B_5 = \frac{2(t - 2)^2 t^2 (5t - 4)}{5} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]
\[
B_4 = 4(t - 2) t (5t^2 - 8t + 2) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]
\[
B_3 = 16t (5t^2 - 12t + 6) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]
\[
B_2 = \frac{16t(131t - 148)}{39} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]
\[
B_1 = \frac{32}{19} \begin{pmatrix} -4(2t - 7) & 3(t - 6) \\ 9t - 28 & -2(2t - 9) \end{pmatrix}
\]
\[
B_0 = \frac{1}{19} \begin{pmatrix} -13 \times 32 & -3 \times 64 \\ 11 \times 32 & 0 \end{pmatrix}
\]
and we have
\[
P_n D = \Lambda_n P_n
\]
with
\[
\Lambda_n = \left( \frac{19n^6 + 285n^5 + 1615n^4 + 4275n^3 + 2446n^2}{285} \right) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]
\[
+ \frac{1}{285} \begin{pmatrix} -12480n - 6240 & 10080n - 2880 \\ 12960n + 5280 & -10560n \end{pmatrix}
\]

**Example 3.**
\[
\lambda = \frac{9}{2} \quad \alpha_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

In this case there is one operator of order eight whose corresponding \( \Lambda_n \) is given below
\[
\Lambda_n = \begin{pmatrix} (n - 3)(n + 6)(n + 7)(n + 8) \alpha_n & -(n - 2)n(n + 7)(n + 8) \beta_n \\ -(n + 1)(n + 6) \gamma_n & (n - 1)n(n + 1)(n + 10) \delta_n \end{pmatrix}
\]
with
\[
\alpha_n = n^4 + 10n^3 + 59n^2 + 170n + 840
\]
\[
\beta_n = n^4 + 14n^3 + 95n^2 + 322n + 1080
\]
\[
\gamma_n = n^6 + 21n^5 + 169n^4 + 651n^3 + 1198n^2 + 840n - 20160
\]
\[
\delta_n = n^4 + 18n^3 + 143n^2 + 558n + 1512.
\]
Once again, this corresponds to the lowest order possible differential operator in the corresponding algebra.

7. A few Jacobi type examples

A different avenue for exploring the similarities as well as the differences between the use of the Darboux process in the scalar and the matrix valued case is given by the examples in this section.

First recall that in the scalar case it follows from results in KoKo1, Z, H2, GY1 that the polynomials orthogonal to the weight \( \mu(x) \) consisting of a Jacobi density plus two possible delta masses of nonnegative strengths \( W, V \) at the ends of the interval, i.e.,

\[
\mu(x) = (1 - x)^{\alpha}(1 + x)^{\beta} + W \delta_1(x) + V \delta_{-1}(x)
\]

satisfy differential equations when \( \alpha \) and \( \beta \) satisfy certain natural integrality conditions. The simplest example is given by the so called Koorwinder polynomials, corresponding to \( \alpha = \beta = 0 \). If the weight at 1 is the only one that is present then the order is \( 2\alpha + 4 \). If both delta weights are thrown in, then the order is \( 2\alpha + 2\beta + 6 \). The results can be obtained by an application of the Darboux process as shown in H2, GY1.

We consider now a small collection of situations analogous to the ones above.

The weight matrices will, as before, consist of a matrix weight density plus a pair of deltas at the endpoints weighted by certain matrices \( W, V \), i.e., we have

\[
\tilde{W}(x) = (1 - x)^{\alpha}(1 + x)^{\beta} \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} + W \delta_1(x) + V \delta_{-1}(x).
\]

For the first batch of examples we will assume that \( \alpha, \beta \) are both 0. If \( V \) and \( W \) coincide with the matrix \( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \) then we find two linearly independent operators of order 5 and one of order 6 as well as other operators of higher order. There are no other operators of lower order.

If \( V \) is the matrix \( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \) and \( W \) is the matrix \( \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \) or the matrix \( \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \) then we find two linearly independent operators of order 6 as well as other operators of higher order. There are no other operators of lower order.
More generally if $V$ is the matrix $\begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}$ and $W$ is the matrix $\begin{pmatrix} c^2 & cd \\ cd & d^2 \end{pmatrix}$ then we have the same situation as in the last example.

In general if $V$ and $W$ are of the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $\begin{pmatrix} d & e \\ e & f \end{pmatrix}$ then the lowest order operator in the algebra is just one operator of order 8.

We come now to a different sort of examples.

Assume that $\alpha$ and $\beta (> -1)$ are arbitrary, but insist in picking $W$ and $V$ to be arbitrary (and non-necessarily equal) nonnegative multiples of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

In this case there is a very nice second order differential operator in the algebra which is independent of the choice of the scalar factors that appear in front of the matrix above to give $W$ and $V$. There is no lower order operator in the algebra. When the deltas are both missing then the algebra contains an operator of order 1.

The right handed differential operator alluded to above is a scalar operator of the usual Jacobi type, with coefficients $(1 - x^2)$ and $(\alpha + \beta - 1) - x(\alpha + \beta + 3)$ multiplied on the right by the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

The eigenvalue is $-n(n+\alpha+\beta+2)$ multiplied by this same matrix.

8. An explicit differential operator

The paper [DG] contains a proof that in the continuous-continuous case, when all operators in question are differential operators, the so called ad-conditions

$$\text{ad} L^{m+1}(\Theta) = 0$$

are necessary and sufficient to have what has eventually been called a bispectral situation, i.e., a solution of the original problem tackled with Hans.

This condition gives a set of nonlinear equations that need to be solved in the unknowns $L, \Theta$.

It is important to see that this condition can be easily adapted to other situations, including the present noncommutative one. This approach was taken up in [GI] and in [GT]. In the second of these papers the "ad-conditions" are shown to be equivalent, once again, to bispectrality.

In general finding the differential operators of lowest possible order that appear in a bispectral situation is not easy. By repeated applications of the Darboux process one may end getting elements of the
corresponding algebra that are not necessarily of the lowest possible order. This issue has surfaced in several different papers, starting with [DG] and a nice account is given in [H2].

In [GT] one finds an explicit construction of a differential operator that results from the conditions

\[ \text{ad} L^{m+1}(\Lambda) = 0. \]

The operator \( D \) is given by

\[ D = \sum_{r=0}^{m} \partial^r (P) \frac{S_{m-r}}{r!} \]

with matrix coefficients \( S_k = S_k(x) \) given by

\[ S_k = ((L - xI)^{m-k} \Lambda P)_0. \]

In particular, we display some of the coefficients,

\[ S_0 = ((L - xI)^{m} \Lambda P)_0, \]

and, at the other end,

\[ S_m = (\Lambda P)_0 = \Lambda_0, \]

and this operator satisfies the desired condition

\[ P_n D = \Lambda_n P_n \quad n \geq 0. \]

The subindex 0 above refers to the first entry of the corresponding “vector” with matrix valued entries.

9. Toda flows with matrix valued time

As seen in [GH2], and certainly in other places too, repeated applications of the scalar Darboux process introduces “times and flows” that are related to the Toda flows. Since these times appear as the free parameters in each application of the process, it is only natural to raise the issue of “matrix valued times”.

10. Electrostatics: Heine, Stieltjes, Darboux

In a remarkable paper that follows on earlier work of Heine, Steiltjes came up with a nice electrostatic interpretation for the zeros of the Jacobi polynomials. Later work of Stieltjes as well as of I. Schur and D. Hilbert showed similar interpretations in the case of the Laguerre and Hermite polynomials.

In [G7, G8] I raise the possibility of some relation between the Darboux process where the orthogonality functional gets more and more
complicated with every application of the process and the corresponding electrostatic interpretation of the families of polynomials that appear along the way.

It would be interesting to see what if anything of this picture can be developed in the matrix valued case.

11. Markov chains

In \([G4, G5, G6, GdI]\) one finds examples of interesting quasi-birth-and-death processes that can be studied by exploiting their connection with certain specific examples of matrix valued orthogonal polynomials. In particular in \([G5, G6]\) one finds examples where the recurrence of the process is related to the presence of a matrix valued delta weight at 1. Since the appearance of these delta weights is one of the main characteristic of an application of the Darboux process one may wonder about a probabilistic interpretation of the relation that may exist between two Markov chains whose transition probability matrices are related by a Darboux transformation.

12. Things that appear before their time

One of the most surprising phenomena uncovered in \([DG]\) has to do with what was called “the cusps”, namely degenerate situations that correspond to degeneracies of “higher order operators” yielding “lower order” ones. To put this in the context of scalar valued orthogonal polynomials consider the so called Koornwinder polynomials which are orthogonal to Lebesgue measure in \([-1, 1]\) plus a pair of delta masses at the end points of the interval. In this case one knows that the corresponding orthogonal polynomials are the common eigenfunctions of a sixth order differential operator.

In the special case when the strength of the two delta masses agrees one gets an operator of order four, and one can say that in a search according to the order of these operators this example, just as “the cusps” in \([DG]\), appears before its time.

We made a tentative exploration of the situation in the matrix valued case and examples of this phenomenon are seen in section 7.

13. The multivariable case

In this section we mention that in \([G9]\) one finds a specific random walk introduced by Hoare and Rahman, see \([HR]\) which we show leads to a bispectral situation in terms of polynomials of two variables. I also want to mention that in the multivariable case one finds a version
of the Darboux process to obtain interesting deformations of the two dimensional Chebyshev measure, see [GI1].

14. Conclusion

It is clear that very little of the development that I have tried to summarize here could have happened were it not for my good fortune of teaming up with Hans at the beginning of this journey.

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