Nonlocal effects induced by the phase of the Schrödinger wavefunction for a particle in a cavity with moving boundaries

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Abstract: We investigate the dynamics of a particle in a confined periodic system — a time-dependent oscillator confined by infinitely high and moving walls — and focus on the evolution of the phase of the wavefunction. It is shown that for some specific initial states in this potential, the phase evolves nonlocally. We further elaborate a thought experiment devised to detect this form of single-particle nonlocality. We point out that within the non-relativistic formalism based on the Schrödinger equation (SE), detecting this form of nonlocality can give rise to signaling. We believe this effect is an artifact, but the standard relativistic corrections to the SE do not appear to fix it. Specific illustrations are given, with analytical results in the adiabatic approximation, and numerical computations to show that contributions from high-energy states (corresponding to superluminal velocities) are negligible.

I. INTRODUCTION

While quantum nonlocality based on multi-particle entanglement is well-recognized, single particle nonlocality remains controversial. The primary candidate of the latter would be the Aharonov-Bohm effect [1]. In the Aharonov-Bohm effect, the wavefunction phase evolution is deemed to be nonlocal. The phase contains dynamical and geometric components. Both the dynamical [2, 3] and the geometric [4] components are ascribed a nonlocal origin. Nevertheless, the nonlocal character of the Aharonov-Bohm effect has been disputed on the ground that electromagnetic forces might be able to account for the AB phase [5, 6]. The nonlocal character of the quantum phase therefore remains controversial.

In this work, we investigate the same issue of phase nonlocality of a single quantum particle but we focus on an entirely different system. The system we will be dealing with is a time-dependent linear oscillator confined by infinitely high walls, one of which is moving. Put differently, our system is an infinite well subjected to a time-dependent harmonic potential and in which one of the well’s walls has an oscillatory motion. The first reason for choosing this system is that analytic solutions of the time-dependent Schrödinger equation are known [7]. The second, less mundane reason, is that such systems, and in particular their simplest variant (a box with a linearly moving wall) have long been suspected of manifesting some form of nonlocality [8, 11].

It was indeed conjectured [8, 9] that the moving wall could nonlocally change the phase
of a wavepacket at the center of the box that remained localized far from the wall. While this conjecture proved to be incorrect [12], it was recently noted [11] that when a quantum state had a non-zero probability amplitude near the wall, a linearly expanding wall induced instantaneously a current density at any point of the box. We note for completeness that systems with moving boundaries are of current interest in practical schemes in the field of quantum engines or in atomic spectroscopy [13–15].

In the present paper, we will be interested in the phase evolution of a quantum state in a confined oscillator with a moving wall. More precisely, we will focus on a particle in a cavity whose left wall is fixed (say at $x = 0$) but whose right wall oscillates, while inside the cavity the particle is subjected to an oscillator potential. We will require the initial state to be spread throughout the cavity but the phase will only be measured near the static wall (at $x = 0$) of the cavity. For an appropriately chosen initial state $\psi(t_0)$ and Hamiltonian parameters, the time evolved state after one period of oscillation is simply $\psi(x, t_0 + T) = e^{-i\mu}\psi(x, t_0)$, where $\mu$ is a global phase. The same initial state evolving in an identical cavity but with a different motion for the moving wall will not yield the same global phase, although the cause of the different phase evolution is due to a potential that is different only in a small region near the moving wall.

Naturally these phases are not observable, but we elaborate a thought experiment that allows an observer located near the fixed wall to nonlocally infer the phase at time $t = T$ by splitting the state into a spatial superposition at $t = 0$ and observing the resulting interference after the parts are recombined. Since the non-relativistic Schrödinger equation does not impose an upper bound for energies or velocities, the observer near the fixed wall can even infer the phase before a light signal has the time to propagate from the moving wall. This feature could give rise to some form of signaling (although our observer needs to accumulate ensemble statistics in order to deduce the phase difference and cannot infer anything in a single shot). We should stress that we do not believe this signaling is physical, but rather that it is an artifact of employing a non-relativistic formalism. However, we will show that it is not clear how the tiny contributions from high-energy eigenstates (with superluminal velocities) can account for the observed change in the global phase.
II. RESULTS

We will start by introducing the Hamiltonian of the system and the solutions to the Schrödinger equation (Sec. II A). We will then focus on the phase evolution in particular by comparing the cases in which the Hamiltonian is identical except at the moving boundary (Sec. II B). In the latter case, we will give the solution in the adiabatic approximation. Sec. II C deals first with the nonlocal nature of the quantum phase, and then explains how the formalism can result in signaling. A specific protocol will be given. We will then illustrate our results by choosing a specific potential and boundary motion (Sec. II D).

A. Time-dependent oscillators with moving walls

1. Time-dependent linear oscillators

A quantum time-dependent linear oscillator is a system comprising a particle of mass $m$ subjected to the Hamiltonian, 

$$H_{TDLO} = \frac{P^2}{2m} + \frac{1}{2}m\Omega^2(t)x^2,$$  

(1)

where we will assume in this work $\Omega^2(t)$ to be $T$-periodic, $\Omega^2(t + T) = \Omega^2(t)$. The solutions $\phi(x, t)$ of the Schrödinger equation,

$$i\hbar\partial_t\phi(x, t) = H_{TDLO}\phi(x, t),$$  

(2)

depend on an initial condition $\phi(x, t = 0) = \phi_0(x)$ and on appropriate boundary conditions, eg $\phi(x \to \pm\infty, t) \to 0$. These solutions can be obtained in semi-analytical form. Several methods have been developed, from the one relying on obtaining the eigenfunctions of dynamical invariants (see [16] and Refs therein), pioneered by Lewis and Riesenfeld [17], to more general Lie system based approaches [18].

A time-dependent linear oscillator can be confined in a box bounded by infinitely high walls. Let $L_0$ denote the width of the well, ie the distance between the two walls. The corresponding Hamiltonian is given by,

$$H_{FW} = \frac{P^2}{2m} + V$$  

(3)

$$V(x) = \begin{cases} 
\frac{1}{2}m\Omega^2(t)x^2 & \text{for } 0 \leq x \leq L_0 \\
+\infty & \text{otherwise}
\end{cases},$$  

(4)
and the solutions $\phi(x, t)$ of the Schrödinger equation must respect the boundary conditions $\phi(x = 0, t) = \phi(x = L_0, t) = 0$. We have chosen here a non-symmetric box on $[0, L_0]$, hence only odd-symmetry wavefunctions of the entire problem will come into play. There are no general methods to solve the confined time-dependent (nor the confined standard harmonic) oscillator (see [19, 20] and Refs therein for specific cases).

2. Confined oscillator with time-dependent boundary conditions

We now confine the time-dependent oscillator in a box with a moving outer wall, i.e. the box has length $L(t)$, with boundaries at $x = 0$ and $x = L(t)$. The Hamiltonian is similar to Eqs. (3)-(4):

$$H = \frac{p^2}{2m} + V$$

$$V(x) = \begin{cases} 
\frac{1}{2}m\Omega^2(t)x^2 & \text{for } 0 \leq x \leq L(t) \\
+\infty & \text{otherwise}
\end{cases}$$

This system is mathematically intricate as it involves a different Hilbert space at each time $t$ and standard quantities such as the the time derivative are ill-defined. Nevertheless, solutions of the Schrödinger equation for specific boundary conditions $L(t)$ and related frequencies $\Omega_L(t)$ are known [7]. While a proper [21] approach involves using time-dependent unitary transformations mapping the time-dependent boundary conditions to fixed ones, it is more straightforward to verify by direct substitution that,

$$\psi_n(x, t) = \sqrt{2/L(t)} \exp \left( -i \frac{\hbar \pi^2 n^2}{2m} \int_{t_0}^{t} \frac{1}{L^2(t')} dt' \right) \exp \left( i \frac{m}{2\hbar L(t)} L(t) x^2 \right) \sin \frac{n\pi x}{L(t)},$$

obeys the Schrödinger equation,

$$i\hbar \partial_t \psi_n(x, t) = -\frac{\hbar^2}{2m} \partial_x^2 \psi_n(x, t) + \frac{1}{2}m\Omega^2(t)x^2 \psi_n(x, t),$$

with the boundary conditions,

$$\psi_n(0, t) = \psi_n(L(t), t) = 0,$$

provided that,

$$\Omega^2(t) = -\frac{\partial_t^2 L(t)}{L(t)}.$$
Eq. (7) gives the even solutions (with \( n \) a natural integer); the odd solutions are readily obtained. The \( \psi_n(x,t) \) are not instantaneous eigenstates, but form a set of orthogonal basis functions that can be used to determine the evolution of an arbitrary initial state. We will choose in this work \( L(t) \) and therefore \( \Omega^2(t) \) to be \( T \)-periodic functions. An important property is the phase increment after a full period. Indeed, comparing \( \psi_n(x,t_0) \) and \( \psi_n(x,t_0+T) \) leads immediately to,

\[
\psi_n(x,t_0+T) = e^{-i\mu_n} \psi_n(x,t_0),
\]

(11)

with,

\[
\mu_n = \frac{\hbar\pi^2n^2}{2m} \int_{t_0}^{t_0+T} \frac{1}{L^2(t')} dt'.
\]

(12)

Following Aharonov and Anandan [22], the total phase \( \mu \) can be parsed into a “dynamical” part \( \delta_n \) encapsulating the usual phase increment due to the instantaneous expectation value of the Hamiltonian and a “geometric” part \( \gamma_n \). The dynamical phase,

\[
\delta_n = \frac{\hbar}{L^2(t')} \int_{t_0}^{t_0+T} \langle \psi_n(t') | H | \psi_n(t') \rangle dt',
\]

(13)

is readily computed [12] and the nonadiabatic geometric phase, \( \gamma_n \) is then obtained as,

\[
\gamma_n = \mu_n - \delta_n = \frac{m}{12\hbar\pi^2n^2} \left( 2\pi^2n^2 - 3 \right) \int_{t_0}^{t_0+T} \left( L(t) \partial_t^2 L(t) - \partial_L(t) \right)^2 dt.
\]

(14)

Hence after a full cycle, a time-dependent oscillator with moving walls in state \( \psi_n \) returns to its initial state except for a phase increment \( \mu_n \).

B. The wavefunction phase evolution

1. Phase and walls motions

The phase \( \mu_n \) [Eq. (12)] is a property of the entire wavefunction, although part of the phase increment is due to the walls’ motion. To see this, we will compare two oscillators that have exactly the same potential everywhere except in the vicinity of the outer wall. To this end we will examine three different cases:

1. an oscillator with the outer wall moving according to some function \( L_1(t) \) and with a time-dependent frequency obeying Eq. (10), \( \Omega^2_1(t) = -\partial_t^2 L_1(t)/L_1(t) \);
2. an oscillator with the outer wall moving according to a function $L_2(t)$ and with a
time-dependent frequency also obeying Eq. (10), $\Omega_2^2(t) = -\partial_t^2 L_2(t)/L_2(t)$;

3. an oscillator with the outer wall moving according to the function $L_2(t)$ but evolving
in the potential of the first case, $\Omega_1^2(t) = -\partial_t^2 L_1(t)/L_1(t)$.

We will take $L_2(t)$ to be very close to $L_1(t)$ and with the same period $T$. The idea is to
compare the phase in cases 1 and 3, which evolve in the same potential $\Omega_1(t)$ except in
the region near the walls, since the boundary conditions, depending respectively on $L_1(t)$
and $L_2(t)$, are slightly different. We will start from the same initial state $\psi(x, t = 0)$
of the type given by Eq.(7). In order to have the same initial state, we must impose
$L_1(t = 0) = L_2(t = 0)$ and $\partial_t L_1(t = 0) = \partial_t L_2(t = 0)$. Picking functions for which
$\partial_t L_1(t = 0) = \partial_t L_2(t = 0) = 0$ leads to an initial state,

$$\psi(x, t = 0) = \sqrt{2/L_0} \sin \frac{n \pi x}{L_0}, \quad (15)$$

with $L_0 = L_1(t = 0) = L_2(t = 0)$.

The time-evolved wavefunction in cases 1 and 2 is obtained directly from Eq. (7). The
total phase increment after one period $\mu_1^n$ and $\mu_2^n$ is given by Eq. (12). Case 3 however does
not respect Eq. (10) and therefore does not fit in the framework developed in Sec. II A 2.

We will look for a perturbative solution in this case. Note that in case 3 there is no reason
to expect the existence of a global phase increment, though due to the continuity of the
wavefunction, we can expect that if $L_1(t)$ and $L_2(t)$ are close enough, the phase increment
after one full cycle will not deviate far from a constant value in the vicinity of a given point.
We will be interested in the phase in the vicinity of $x = 0$, which is the farthest region from
the moving walls.

2. Phase increment in case 3

Quantum state evolution: Let $\phi(x, t)$ denote the solution of the Schrödinger equation for
the case 3 mentioned in the preceding subsection. Let $\psi_2^2(x, t)$ denote the basis functions of
Eq. (7) with $L(t) = L_2(t)$. Since $\phi(x, t)$ obeys the boundary condition $\phi(L_2(t), t) = 0$, we
will look for a solution in the form,

$$\phi(x, t) = \sum_{k=1}^{\infty} a_k(t) \psi_2^2(x, t), \quad (16)$$
with the initial condition being,

$$\phi(x, 0) = \psi_n^2(x, 0) \text{ or } a_k(0) = \delta_{kn} . \quad (17)$$

**Adiabatic approximation:** To obtain the coefficients $a_k(t)$ in closed form, approximations need to be made. The simplest regime is the “adiabatic” approximation which involves neglecting the contribution of basis states other than the initial one, as specified by the initial condition (17). Hence $a_k(t) = \delta_{nk} a_n(t)$ and $a_n(t)$ is obtained from Eq. (27) (see Methods) by keeping only the diagonal contribution, yielding,

$$a_k(t) = \delta_{nk} \exp \left( -\frac{i}{\hbar} \frac{m}{12n^2\pi^2} (2\pi^2n^2 - 3) \int_0^t \left( L_2(t') \partial^2_{t'} L_2(t') - \frac{\partial^2_{t'} L_1(t')}{L_1(t')} L_2^2(t') \right) dt' \right) . \quad (18)$$

Plugging this back into Eq. (16), we see that within this approximation the total phase after one full cycle is,

$$\mu_{ad}^n = \frac{\hbar \pi^2 n^2}{2m} \int_0^T \frac{1}{L_2^2(t')} dt' + \frac{m}{12n^2 \pi^2 h} \left( 2\pi^2n^2 - 3 \right) \int_0^T \left( L_2(t') \partial^2_{t'} L_2(t') - \frac{\partial^2_{t'} L_1(t')}{L_1(t')} L_2^2(t') \right) dt' . \quad (19)$$

The first term is the case 2 phase $\mu_n^2$, and the second term appears as a correction. Part of this correction is due to the dynamical phase, which is different from case 2 since the potential in case 3 is the one from case 1, leading to the dynamical phase given in Eq. (33). The remaining part of this correction, proportional to $\int_0^T \left( L_2(t') \partial^2_{t'} L_2(t') - (\partial_{t'} L_2(t'))^2 \right) dt'$ is a geometric term due to the boundary conditions.

The adiabatic approximation is expected to hold when $L_1(t)$ and $L_2(t)$ are almost identical. Then, in order to discriminate the phase of the different cases mentioned above, it is crucial that $\mu_n^1$ and $\mu_n^2$ differ significantly. Indeed, $\mu_{ad}^n$ appears as a correction to $\mu_n^2$ [Eq. (19)], so that ensuring that $\mu_{ad}^n \approx \mu_n^2 \neq \mu_n^1$ while still having $L_1(t) \approx L_2(t)$ typically implies high values of $n$ and/or small values of $m$ [see Eq. (12)].

**Generic case:** In general, the adiabatic approximation will of course not be valid. The generic case is characterized by a set of non-negligible coefficients $a_k(t)$ with $k$ lying in the interval $[n - N, n + N]$. Indeed, initially only $a_n(t = 0)$ is non-vanishing, and the coupling between $a_n(t = 0)$ and the different coefficients $a_k(t)$ falls off as $1/k^3$ for large $k$ (see Eqs. (27) and (28) in the Methods section). Since here we are interested in short time evolutions, we can expect that only a few basis states $\psi_k$ centered on $k = n$ will contribute in the
expansion Eq. (16). While there is no simple analytical formula giving the phase increment in this generic situation, it is straightforward to compute numerically the wavefunction in case 3 and from there extract the phase increment after one full cycle. Note that in general there is no reason to expect that similarly to Eq. (11), the wavefunction after one period will be equal to the initial wavefunction up to a global phase. We can however expect, for reasonable choices of $L_1(t)$ and $L_2(t)$, the phase to be slowly varying (as a function of $x$) and approximately constant in the neighborhood of $x = 0$, which will be our region of interest in the protocols described below.

C. Nonlocality and signaling

1. Nonlocal origin of the quantum phase

Let us go back to the three cases described in Sec. II B 1, from the point of view of an observer placed in the neighborhood of $x = 0$. Let the system be initially prepared in a state $\psi_n(t = 0)$ given by Eq. (7). Since we want $\psi_n(t = 0)$ to be independent of whether the system will evolve with boundary functions $L_1(t)$ or $L_2(t)$, we must enforce $L_1(t = 0) = L_2(t = 0)$ and for simplicity we set $\partial_t L_1(t = 0) = \partial_t L_2(t = 0) = 0$. Let us assume the observer, say Alice, can determine the phase difference between the initially prepared state $\psi_n(t = 0)$ and the state after one period $\psi_n(t = T)$. Alice can therefore discriminate case 1 from case 2, since the phases at $t = T$, $\mu_1^n$ and $\mu_2^n$ [Eq. (12)] will be different. But the Hamiltonian is also different in these two cases, a point Alice could have checked by making local measurements in her vicinity, so she won’t be surprised by finding different phases depending on whether the system evolved in case 1 or case 2.

However, when comparing cases 1 and 3, the Hamiltonian is identical except in the vicinity of the wall’s position: only the wall’s motion differs in cases 1 and 3. Nevertheless, the total phase at $t = T$ will be different, including in the region where Alice is standing. Hence the phase difference between cases 1 and 3 must be attributed to the potential in the region near the opposite wall. Since the wall can be arbitrarily far from the $x = 0$ region, we can say that the phase difference appears to be due to local potentials varying in an arbitrarily remote region: the phase increment, as it appears in the region near the static wall at $x = 0$, has a nonlocal origin.
2. Signaling

Let us still assume that Alice has access to the phase difference between the wavefunctions at $t = 0$ and $t = T$ (such a protocol is given immediately below), and further assume that the walls are sufficiently far away so that the time it takes for a light signal emitted from the moving (right) wall to reach her position near the fixed left wall, $\tau \approx L(t = 0)/c$ is larger than the period $T$ of the wall’s motion. By measuring the phase difference, she can determine whether the wall is moving according to $L_1(t)$ or $L_2(t)$. Alice can thus discriminate case 1 from case 3 before a signal sent from the moving wall, say by Bob, reaches her. In principle, by choosing different functions $L_2(t)$, Bob could send signals to Alice superluminally.

3. Protocol

We describe here one possible protocol that formally leads to signaling. Indeed, by changing the motion of the wall at $x = L$, Bob is able to change the global phase evolution, which is an instantaneous and measurable effect that occurs everywhere in the cavity. Because of this, Bob can send a message by choosing how the wall moves after $t = 0$, and then Alice can detect this choice by making local measurements near $x = 0$. To make sure the message is clearly resolved, Alice and Bob need to share a large ensemble of identical cavities, and Bob must make the same choice of motion for all of them. Provided that Alice completes her measurements well before $t = \tau \approx L(0)/c$, this message is sent faster than $c$.

In order for Alice to perform her measurement, the cavity will need to have a bit more structure (see Fig. [1]). Suppose that in the $y$-direction, the cavity has the potential of a fixed infinite square well of width $W \ll L$, with one wall at $y = 0$, the other at $y = W$, and a beam splitter at $y = W/2$, which runs the full length from $x = 0$ to $x = L$. The harmonic potential is initially turned off and the walls are stationary.

The initial state of the particle is $\psi(x, t = 0)g(y, t = 0)$, where $\psi(x, t = 0)$ is the cavity state we have been discussing, and $g(y, t = 0)$ is a Gaussian wave-packet with average $y$-velocity $v_y > 0$ which begins well-localized at $y = W/4$, as displayed in Fig. [1]. We assume that this experiment will finish quickly enough that the spreading of this wave-packet can be ignored.

When the particle strikes the beam splitter, half of the wavefunction is reflected and half
FIG. 1: A thought experiment allowing communication via the nonlocal phase of the wavefunction.

The walls of the cavity are initially at rest, the harmonic potential is turned off, and the detectors are not yet in place. The particle begins in the state \( \psi(x, t = 0)g(y, t = 0) \), where \( \psi(x, = 0) \) is an excited \( x \)-mode of the cavity, and \( g(y, t = 0) \) is a narrow Gaussian initially centered at \( W/4 \) with average velocity \( v_y > 0 \). When the packet has divided at the beam splitter, the harmonic potential \( (1/2)\Omega_1^2(t)x^2 \) is turned on throughout the cavity, and the wall segment at \( (x = L, y < W/2) \) begins to move as \( L_1(t) \). At the same moment, Bob chooses whether the wall segment at \( (x = L, y > W/2) \) begins to move as \( L_1(t) \) (message 0) or \( L_2(t) \) (message 1). The two half-packets propagate along \( y \), then bounce off their respective walls and meet back at the beam splitter, a period \( T = W/v_y \) after they left it. Bob choosing \( L_1(t) \) results in perfect interference, so the entire pulse recombines on the \( y > W/2 \) side of the beam splitter, whereas choosing \( L_2(t) \) results in a phase difference so that the interference is no longer perfect. Alice can detect Bob’s choice at \( t = 3T/2 = 3L/2v_y \) by inserting detector \( D_1 \) at \( W/4 \) and detector \( D_2 \) at \( 3W/4 \) and measuring the relative intensity for the entire ensemble of cavities. \( L_1(t) \), \( L_2(t) \), and \( \Omega_1(t) \) are all periodic function of time, with period \( T \).

is transmitted. At the moment the packets have passed the beam splitter, the harmonic potential is turned on throughout the cavity, and the wall segment at \( x = L \) and \( y < W/2 \) begins to move according to \( L_1(t) \). Bob also chooses at this moment whether the wall segment at \( x = L \) and \( y > W/2 \) begins to move according to \( L_1(t) \) (message 0) or \( L_2(t) \)
The two half-packets propagate along $y$ then bounce off their respective walls, and meet back at the beam splitter, a period $T = W/v_y$ after they left it — which is incidentally when $L_1(t) = L_2(t)$ again. The harmonic potential is then turned off once more.

The cavity is tuned so that if Bob chose $L_1(t)$, then when the two half-pulses meet, they interfere destructively for $y < W/2$ and constructively for $y \geq W/2$, and thus the particle always ends up in the region $y \geq W/2$. However, if Bob chose $L_2(t)$, then the two half-pulses would have accumulated a phase difference of $\mu$ before they meet again, and the interference would no longer be perfectly constructive/destructive.

At $t = 3T/2 = 3W/2v_y$, Alice places two detectors near $x = 0$, $D_1$ at $y = W/4$ and $D_2$ at $y = 3W/4$, and measures the total particle counts to determine probabilities $P_1$ and $P_2$. Because the detectors have a small width $\Delta x$, there is only a small probability that either one will fire in a given cavity, but this can be overcome by the large ensemble size. As a result, provided $3W/2L \ll v_y/c$, Alice is able to receive Bob’s message before a signal traveling at $c$ could reach her.

D. Illustration

1. System Hamiltonian

For the purpose of illustration, let us choose the following wall motion functions,

$$L_j(t) = L_0 + q_j(\cos \omega t - 1), \quad (20)$$

where $L_0 = L(t = 0)$ and $q_j \ll L_0$. Both walls move according to Eq. 20, but with the different amplitudes, $q_j$. Index $j = 1$ is case 1 from section II B 1, with $\Omega_2^2(t) = -\partial^2 L_1(t)/L_1(t)$, and $j = 2$ is case 3, with the same $\Omega_2^2(t) = \Omega_1^2(t)$ and an independent $L_2(t)$. The Hamiltonian for cases 1 and 3 is obtained, from Eqs. (5)-(6) and (20) as,

$$H_j = \frac{p^2}{2m} + V_j \quad (21)$$

$$V_j(x) = \begin{cases} \frac{1}{2} m \frac{q_j \omega^2 \cos \omega t}{L_0 + q_j(\cos \omega t - 1)} x^2 & \text{for } 0 \leq x \leq L_j(t) \\ +\infty & \text{otherwise} \end{cases} \quad (22)$$

that is the potential differs only in the interval between $L_1(t)$ and $L_2(t)$. 

12
In case 1, the phase increment $\mu_1^n$ is calculated from Eq. (12) as,

$$\mu_1^n = \frac{\hbar \pi n^2 (L_0 - q_1)}{m \omega (L_0(L_0 - 2q_1))^{3/2}}. \quad (23)$$

In case 3 the phase increment can be obtained in the adiabatic approximation from Eq. (19), giving explicitly,

$$\mu_{ad}^n = \frac{\hbar \pi n^2 (L_0 - q_2)}{m \omega (L_0(L_0 - 2q_2))^{3/2}} + \frac{m}{\hbar} \frac{1}{12n^2 \pi^2} (2\pi^2 n^2 - 3) \frac{2\pi \omega L_0^2 \left( \sqrt{L_0(L_0 - 2q_1)} - L_0 + q_1 \right) (q_1 - q_2)^2}{q_1^2 \sqrt{L_0(L_0 - 2q_1)}} . \quad (24)$$

The adiabatic approximation requires here $q_2 = q_1 + \varepsilon$, with $\varepsilon \ll q_1$, and $L_0 \gg q_1, q_2$.

2. Examples displaying nonlocality

We now give a couple of numerical examples based on the Hamiltonians (22)-(22) with parameters giving rise to nonlocality. As stated above, this is defined when the period, now $T = 2\pi/\omega$ is smaller than the time it takes a light signal to travel from the initial moving wall’s position to the static leftward wall, $\tau = L_0/c$. The numerical computations for the case 3 wavefunctions $\phi(x,t)$ are carried out by solving the truncated version of the coupled system defined by Eqs. (16)-(27). This is similar to numerical methods used in previous related works [11, 23] except that the expansion basis is taken to be the solutions of case 2 rather than the instantaneous eigenstates (ie, those obeying $H\xi_n(x,t) = E_n(t)\xi_n(x,t)$).

For both examples, we take the initial state to be the ground state, see Eq. (17) with $n = 2$. The first illustration concerns an instance for which the adiabatic approximation holds. This is confirmed by the results obtained numerically for the coefficients $a_2(t)$ and $a_{1,3}(t)$ plotted in Fig. 2. $|a_2(t)|^2$ is almost always equal to unity for all times $0 < t < T$, while $|a_1(t)|^2$ is small and $|a_3(t)|^2$ (as well as $|a_k(t)|^2$ with $k > 3$) are negligible. In the second illustration, we keep the same parameters as for the first illustration except for the amplitude of the oscillating boundary, $q_2$, that is significantly increased. While $|a_2(t)|^2$ is still the dominant term, $|a_1(t)|^2$ and $|a_3(t)|^2$ are not negligible, as seen from the numerical results shown in Fig. 3.

The phase difference $\Delta \mu = \mu_3 - \mu_1$ at $t = T$ between the case 3 and the case 1 cavities is shown in Fig. 4. Recall that in case 3, Eq. (11) is not exactly verified – it holds approximately when $q_1 \approx q_2$ (and in particular when the adiabatic approximation holds).
FIG. 2: Time evolution of the coefficients $a_k(t)$ of Eq. (16) in the adiabatic case, corresponding to the Hamiltonian given by Eqs. (21)-(22) with the following parameters (in natural atomic units): $m = 1/75, L_0 = 37, q_1 = 7, q_2 = 7.04, \omega = 25$ (verifying $(T = 2\pi/\omega) < (\tau = L_0/c)$ where $c$ is the light velocity). The initial state is $\phi(x,0) = \psi_{n=2}^2(x,0)$. The thick solid blue line represents the numerically computed values of $|a_2(t)|^2$ while the inset shows the corrections to the adiabatic approximation stemming from exact numerical computations: $|a_k(t)|^2$ for $k = 1, 3, 4$ are represented by the blue dot-dashed, solid red and dashed green curves, respectively.

but $\mu^3$ varies with $x$ as $q_1$ becomes significantly different from $q_2$. For our present purposes the important feature is that $\Delta \mu$ varies slowly with $x$ in the spatial region in which the phase difference will be measured (i.e., in the vicinity of $x = 0$). This is the case in the illustrations shown here.

A crucial feature for our argument on nonlocality concerns the absence of superluminal velocities. As we have already mentioned, the Schrödinger equation does not impose any bounds on energy eigenstates and admits solutions of arbitrarily high energies, hence involving superluminal velocities. We therefore need to check that no such states are needed in order to account for the effect we observe in our illustrations. From a numerical standpoint, it is straightforward to compute the average velocity and its standard deviation as a function of time and check it lies below the light velocity. As a rule of thumb, each function $\psi_n(x,t)$ of Eq. (7) has a velocity component obtained by decomposing the standing wave $\sin \left( \frac{n\pi x}{L(t)} \right)$ into the forward and backward traveling waves. The average velocity of, say the forward traveling wave $\psi_n^+(x,t)$, is obtained as,

$$
<v_n> = \frac{\langle \psi_n^+ | P | \psi_n^+ \rangle}{m} = \frac{n\pi \hbar}{mL(t)} + \frac{1}{2} \frac{\partial L(t)}{\partial t}.
$$

(25)
FIG. 3: Same as Fig. 2 in the nonadiabatic case. The Hamiltonian parameters are those given for Fig. 2 except we set here $q_2 = 7.33$. The initial state is again $\phi(x,0) = \psi_{n=2}^2(x,0)$. The coefficients $|a_k(t)|^2$ are shown by the curves in thick solid blue ($k = 2$), dot-dashed ($k = 1$), solid red ($k = 3$) and green dashed ($k = 4$). The coefficients $a_k(t)$ for $k \geq 5$ are of smaller and decreasing magnitude.

Alternatively, $< v >$ and $< v^2 - < v >^2 >^{1/2}$ can be obtained straightforwardly for the standing wave $\psi_n(x,t)$. In both cases, when time-averaged over a period this gives a velocity of the order of $n\pi\hbar/mL_0$.

We have seen that in the adiabatic case working with the sole function $\psi_{n_0}^2(x,t)$ is sufficient to account for the phase difference $\Delta \mu$. In the example given in Fig. 2 we have $n_0 = 2$ and $v/c \approx 0.09$. In the nonadiabatic example, more basis states need to be included; rather than setting a cut-off value in the sum (16) as a function of the value of $|a_k(t)|$, we use a stricter criterion requiring that the numerically computed phase $\mu^3$ displays variations negligible compared to $\Delta \mu$ and becomes constant as additional basis states are included in the expansion. This is illustrated in Fig. 3 where it is seen that states up to $k = 15$ must be included in the expansion (16); this state corresponds to a velocity $v/c = 15\pi\hbar/mcL_0 \approx 0.70$ (note however that for the overall state $\phi(x,t)$, the time averages over a period for $< v >$ and $< v^2 - < v >^2 >^{1/2}$ are much lower, resp. 0 and 0.06c).

III. DISCUSSION

We have investigated a system in which different boundary conditions induce different cyclic global phases on the total wavefunction, although the system evolves in identical
FIG. 4: Phase difference $\Delta \mu = \mu^3 - \mu^1$ at $t = T$ between the case 3 and the case 1 cavities obtained from numerical computations. Top: the case 3 cavity is in the adiabatic regime (corresponding to the parameters given in Fig. [2]). Applying the analytical expression (24), the phase difference is constant and computed to be $\mu^{ad} - \mu^1 = -2.2 \times 10^{-3}$. Bottom: the case 3 cavity is in a nonadiabatic regime (the parameters are those given in Fig. [3]).

potentials except near the boundary. This global phase is acquired by the entire wavefunction instantaneously, without relying on a dynamical process in space. In this sense it appears to be a nonlocal effect, and we have devised an interferometer protocol to detect it, and to send fast-than-light signals.

We assume this appears possible only because of some artifact of non-relativistic quantum mechanics; it is indeed well-known that the Schrödinger equation admits eigenstates of arbitrarily high energies, implying arbitrarily high velocities. In a recent work [11] a similar type of nonlocality in a related system (a cavity in free space expanding linearly in time) was seen on the current density. Signaling could be obtained by making weak measurements of the momentum. Discarding the contribution of high energy basis states (that would propagate superluminally) was somewhat subtle: several hundred basis states needed to be
FIG. 5: The phase $\mu^3$ (blue dots) in the nonadiabatic case of Fig. 3 obtained from the numerically computed wavefunction $\phi(x,t)$ as the basis size increases, cf. Eq. (16). It is seen that states $\psi_k^2$ up to at least $k = 15$ are necessary in order for the computed value of $\mu^3$ to converge. The solid red line represents $\mu^1$, the phase increment in case 1, and the relevant quantity to display nonlocality here is $\Delta \mu = \mu^3 - \mu^1$.

included in order to compute the current density, and while a basis restricted to subluminal states achieved the nonlocal effect, convergence of the current density demanded the inclusion of a more complete basis, containing superluminal states.

Here instead nonlocality arises through a periodic evolution of the phase. Weak measurements are not required to observe signaling. Only a very low number of basis states are involved in the computation, and in the adiabatic limit, only a single state contributes. Nevertheless, although it looks unlikely that high energy superluminal states are responsible for the form of nonlocality we have investigated here, it remains impossible to totally discard their role. Indeed, from a formal standpoint a complete basis in a non-relativistic setting will necessarily include superluminally propagating states. And the adiabatic limit is of course an approximation.

A second question that comes up concerns possible specific features in the system employed. Indeed the present system combines two features, each of which is known to lead to difficulties. First the confining potential is modeled as an infinite wall, whereas inside the cavity the potential is a time-dependent oscillator. Introducing infinite discontinuities is known to lead to peculiar dynamical features [24]. Second, dealing with time-dependent boundary conditions involves formally [12] [21] a different Hilbert space at each time $t$. The time-dependent unitary mapping to a standard problem – a problem with fixed boundary
conditions defined in a single Hilbert space – yields a Hamiltonian with a time-dependent mass, so that the (local) boundary conditions are mapped to a (delocalized) time-dependent parameter.

Conversely, one may wonder whether the nonlocal effect mentioned here could be generic, in the sense that the wavefunction phase has a global character even if it includes dynamical effects due to a potential that is non-vanishing only in a small region over which the wavefunction is defined. This would be analogous to the nonlocal features of the Aharonov-Bohm phase [1, 3, 4]. However the AB effect is significantly different from the features characterizing the present problem: it is an electromagnetic effect, the vector potential is non-vanishing over the entire region in which the wavefunction is defined, the individual phases are gauge-dependent, and there is no signaling [25, 26].

To conclude, we have investigated the evolution of the phase of the wavefunction of a particle trapped in a confined time-dependent oscillator with a moving boundary and found the phase to be nonlocal. We have also seen that in some circumstances this form of nonlocality can give rise to signaling, an apparent nonphysical artifact. Further work is needed to pinpoint the different sources accounting for this behavior within the formalism and to see if it is possible to discriminate the effects due to dynamical nonlocality from the artifacts leading to signaling.

IV. METHODS

Computation of the phase increment in case 3

Applying the Schrödinger equation with \( V = \frac{1}{2} m \Omega_1^2(t) x^2 \) for \( 0 < x < L_2(t) \) to the right hand-side of Eq. (16) and recalling that \( i \hbar \partial_t \psi_k^2(x, t) = \left( \frac{p_x^2}{2m} + \frac{1}{2} m \Omega_2^2(t) x^2 \right) \psi_k^2(x, t) \) leads to,

\[
\sum_k i \hbar \partial_t a_k(t) \psi_k^2(x, t) = \sum_k \frac{1}{2} m \left( \Omega_1^2(t) - \Omega_2^2(t) \right) x^2 a_k(t) \psi_k^2(x, t),
\]

and by projecting \( \langle \psi_j^2 \rangle \) we obtain,

\[
\partial_t a_j(t) = -\frac{i}{\hbar} \sum_k a_k(t) \frac{1}{2} m \left( \Omega_1^2(t) - \Omega_2^2(t) \right) \int_0^{L_2(t)} x^2 \left[ \psi_j^2(x, t) \right]^* \psi_k^2(x, t) dx.
\]

The integral can be readily computed, yielding,

\[
8k_j (-1)^{k+j} L_2(t)^2 \exp \left( -i \hbar \pi^2 (k^2 - j^2) \int L_2(t')^{-2} dt' / (2m) \right),
\]

(28)
for $k = j$, the integral becomes,
\[
\frac{(2j^2\pi^2 - 3) \, L_2(t)^2}{6j^2\pi^2}.
\] (29)

In the adiabatic case, only the coefficient of the initial state is non-zero, $a_j(t) = a_k(t) = \delta_{nk}\delta_{nj}a_n(t)$. There is no summation in Eq. (27) and $a_n(t)$ is obtained by solving analytically the differential equation, yielding Eq. (18). The integral in that equation may be obtained analytically for specific choices of $L_1(t)$ and $L_2(t)$; otherwise a simple numerical integration is in order.

In the generic (nonadiabatic) case there are no analytical solutions. We obtain numerical solutions by writing first Eq. (26) as
\[
\sum_{k=1}^{\infty} i\hbar \partial_t \left[ a_k(t) \psi_k^2(x,t) \right] = \sum_{k=1}^{\infty} a_k(t) \left[ -\frac{\hbar^2}{2m} \partial_x^2 \psi_k^2(x,t) + \frac{1}{2} m\Omega_1^2(t)x^2 \psi_k^2(x,t) \right],
\] (30)
and then multiplying this equation by $(\psi_j^2(x,t))^*$ and integrating. The basis functions are orthogonal, and the remaining non-trivial integrals
\[
\int_0^{L_2(t)} \left[ \psi_j^2(x,t) \right]^* \partial_t \psi_k^2(x,t) dx \text{ and } \int_0^{L_2(t)} \left[ \psi_j^2(x,t) \right]^* x^2 \psi_k^2(x,t) dx
\] (31)
can be integrated analytically without difficulty. This leaves us with a system of linear first-order (in time) coupled equations for $a_k(t)$, subject to the initial condition $a_k(t = 0) = \delta_{nk}$. This formally infinite system is truncated by setting an upper bound $k_{max}$ in the sum of Eq. (30). The choice of $k_{max}$ depends on the desired accuracy of the solutions: since $|a_k(t)| \to 0$ as $k$ increases, it should be checked that $|a_{k_{max}}(t)\psi_{k_{max}}^2(x,t)| < z$ throughout the $x$ and $t$ intervals over which the solutions are computed ($z$ is the numerical zero). In the illustration displayed in Fig. 3, a basis size with $k_{max} = 100$ leads to $z = 1.5 \times 10^{-8}$ for the numerically obtained wavefunction.

Note that the dynamical phase for case 3 can be obtained by computing,
\[
\delta_n = \hbar^{-1} \int_0^T \left\langle \psi_n^2(t') \left| \left( \frac{P^2}{2m} + \frac{m}{2} \Omega_1^2(t)X^2 \right) \right| \psi_n^2(t') \right\rangle dt'
= \frac{\hbar \pi^2 n^2}{2m} \int_0^T \frac{1}{L_2^2(t')} dt' + \frac{m}{12n^2\pi^2\hbar} (2\pi^2 n^2 - 3) \int_0^t \left( \partial_{t'}^2 L_2(t') \right)^2 \left( \frac{\partial_{t'} L_2(t')}{L_2(t')} \right) \left( \frac{\partial_{t'} L_1(t')}{L_1(t')} \right) \left( \frac{\partial_{t'}^2 L_1(t')}{L_1(t')} \right). \] (32)
(33)

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