On the global existence and stability of 3-D viscous cylindrical circulatory flows

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Abstract

In this paper, we are concerned with the global existence and stability of a 3-D perturbed viscous circulatory flow around an infinite long cylinder. This flow is described by 3-D compressible Navier-Stokes equations. By introducing some suitably weighted energy spaces and establishing a priori estimates, we show that the 3-D cylindrical symmetric circulatory flow is globally stable in time when the corresponding initial states are perturbed suitably small.

Keywords: Compressible Navier-Stokes equations, cylindrical symmetric, circulatory flow, weighted energy space, global existence

Mathematical Subject Classification 2000: 35L70, 35L65, 35L67, 76N15

1 Introduction

In this paper, we are concerned with the global stability problem of cylindrical symmetric circulatory flows for the three-dimensional compressible Navier-Stokes equations (see Figure 1 below). The compressible Navier-Stokes equations in three space dimensions are

\begin{equation}
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\rho \partial_t u + \rho u \cdot \nabla u + \nabla P(\rho) = \nu_1 \Delta u + \nu_2 \nabla \text{div} u,
\end{cases}
\end{equation}

where $\rho > 0$ is the density, $u = (u_1, u_2, u_3)$ is the velocity, $\nu_1 > 0$ and $\nu_1 + \nu_2 > 0$ hold, and the state equation is given by $P(\rho) = A \rho^\gamma$ with the constants $A > 0$ and $\gamma > 1$.

We now give a mathematical description on the 3-D viscous cylindrical flow around an infinite long cylinder $\{x = (x_1, x_2, z) \in \mathbb{R}^3 : r = \sqrt{x_1^2 + x_2^2} \leq 1, z \in \mathbb{R}\}$. Set $\Omega = \{(r, z) : r > 1, z \in \mathbb{R}\}$

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and \((\rho(t, x), u(t, x)) = (\rho(t, r, z), u_r(t, r, z)\frac{x'}{r} + u_\theta(t, r, z)\frac{x'\perp}{r}, u_z(t, r, z))\), where \(x' = (x_1, x_2)\) and \(x'\perp = (-x_2, x_1)\). In this case, (1.1) has the following equivalent form in \([0, \infty) \times \Omega\):

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{1}{r} \partial_r (r \rho u_r) + \partial_z (\rho u_z) &= 0, \\
\rho \frac{\partial u_r}{\partial t} + \rho (u_r \partial_r u_r + u_z \partial_z u_r - \frac{u^2_\theta}{r}) + \partial_z \frac{\partial P}{\partial z} &= \nu_1 \left( \partial_r \left( \frac{1}{r} \partial_r (ru_r) \right) + \partial^2_z u_r \right), \\
\rho \frac{\partial u_\theta}{\partial t} + \rho (u_r \partial_r u_\theta + u_\theta \partial_z u_r + \frac{u_\theta u_r}{r}) &= \nu_1 \left( \partial_r \left( \frac{1}{r} \partial_r (ru_\theta) \right) + \partial^2_z u_\theta \right), \\
\rho \frac{\partial u_z}{\partial t} + \rho (u_r \partial_r u_z + u_z \partial_z u_z) + \partial_z \frac{\partial P}{\partial z} &= \nu_1 \left( \partial^2_r u_z + \partial^2_z u_z + \frac{1}{r} \partial_r u_z \right) + \nu_2 \partial_z \left( \frac{1}{r} \partial_r (ru_\theta) + \partial_z u_z \right).
\end{align*}
\]

(1.2)-(1.5)

Figure 1: Cylindrical circulatory flow around a cylinder

We start to look for a special steady circulatory flow \((\bar{\rho}(r), \bar{u}_r(r), \bar{u}_\theta(r), 0)\) of (1.2)-(1.5) around the cylinder \(\{(r, z) : r \leq 1, z \in \mathbb{R}\}\). Such a flow is called a background solution of (1.2)-(1.5) in the whole paper. By (1.2)-(1.5), one knows that \((\bar{\rho}(r), \bar{u}_r(r), \bar{u}_\theta(r))\) satisfies

\[
\begin{align*}
\frac{1}{r} \partial_r (r \bar{\rho} \bar{u}_r) &= 0, \\
\bar{\rho}(\bar{u}_r \partial_r \bar{u}_r - \frac{\bar{u}^2_\theta}{r}) + \partial_z \frac{\partial P}{\partial z} &= (\nu_1 + \nu_2) \partial_r \left( \frac{1}{r} \partial_r (r \bar{u}_r) \right), \\
\bar{\rho}(\bar{u}_r \partial_r \bar{u}_\theta + \frac{\bar{u}_\theta \bar{u}_r}{r}) &= \nu_1 \partial_r \left( \frac{1}{r} \partial_r (r \bar{u}_\theta) \right).
\end{align*}
\]

(1.6)
On the other hand, in order to solve (1.6), one naturally poses a boundary condition on \( \Sigma' = \{ x' : r = 1 \} \) as follows

\[
(\bar{\rho}, \bar{u}_r, \bar{u}_\theta)|_{r=1} = (\bar{\rho}_0, 0, M_0),
\]

(1.7)

where \( \bar{\rho}_0 > 0 \) and \( M_0 > 0 \) are given constants. In addition, one also requires that at infinity

\[
\lim_{r \to \infty} \bar{u}_r(r) = 0, \quad \lim_{r \to \infty} \bar{u}_\theta(r) = 0, \quad \lim_{r \to \infty} \bar{u}_z(r) = 0.
\]

(1.8)

As illustrated in [3] or [22], it is easy to know that (1.6) with (1.7)-(1.8) has a unique solution for \( r \geq 1 \)

\[
(\bar{\rho}(r), \bar{u}_r(r), \bar{u}_\theta(r)) = \left( \left( \bar{\rho}_0^\gamma - 1 + \frac{(\gamma - 1)M_0^2}{2A\gamma(1 - \frac{1}{r^2})} \right) \frac{1}{r^\gamma}, 0, \frac{M_0}{r} \right).
\]

(1.9)

In the paper, we focus on the global stability problem of the background solution \((\bar{\rho}(r), \bar{u}_r(r), \bar{u}_\theta(r), 0)\). Namely, the global solution problem of (1.2)-(1.5) in the domain \([0, \infty) \times \Omega\) will be studied under the following perturbed initial value conditions:

\[
(\rho, u_r, u_\theta, u_z)(0, r, z) = (\bar{\rho}(r) + \rho_0(r, z), u_\theta^0(r, z), u_\theta^0(r, z), u_z^0(r, z)),
\]

(1.10)

\[
\lim_{r \to \infty} u_r(t, r, z) = \lim_{r \to \infty} u_\theta(t, r, z) = \lim_{r \to \infty} u_z(t, r, z) = 0,
\]

(1.11)

\[
(u_r(t, r, z), u_\theta(t, r, z), u_z(t, r, z))|_{r=1} = (0, M_0, 0),
\]

(1.12)

where \( \rho_0 \in H_0^2(\Omega) \) and \((u_0^r, u_0^\theta, u_0^z) \in H_0^2(\Omega)\).

Let \( \rho(t, r, z) = \bar{\rho}(r) + \phi(t, r, z), u_r(t, r, z) = v_r(t, r, z), u_\theta(t, r, z) = \bar{u}_\theta(r) + v_\theta(t, r, z), u_z(t, r, z) = v_z(t, r, z)\), and \( v = (v_r, v_\theta, v_z) \). Then equations (1.2)-(1.5) together with (1.10)-(1.12) can be written as

\[
\partial_t \phi + \frac{1}{r} \partial_r (r \bar{\rho} v_r) + \partial_z (\bar{\rho} v_z) = f,
\]

(1.13)

\[
\partial_t v_r - \frac{2M_0}{r^2} v_\theta + \gamma \partial_z (\bar{\rho}^\gamma - 2 \phi) - \frac{\nu_1}{\bar{\rho}} \left( \partial_r \left( \frac{1}{r} \partial_r (r v_r) \right) + \partial_z^2 v_r \right) - \frac{\nu_2}{\bar{\rho}} \partial_r \left( \frac{1}{r} \partial_r (r v_r) + \partial_z v_z \right) = g_1,
\]

(1.14)

\[
\partial_t v_\theta - \frac{\nu_1}{\bar{\rho}} \left( \partial_r \left( \frac{1}{r} \partial_r (r v_\theta) \right) + \partial_z^2 v_\theta \right) = g_2,
\]

(1.15)

\[
\partial_t v_z + \gamma \partial_z (\bar{\rho}^\gamma - 2 \phi) - \frac{\nu_1}{\bar{\rho}} \left( \partial_r^2 v_z + \partial_z^2 v_z + \frac{1}{r} \partial_r v_z \right) - \frac{\nu_2}{\bar{\rho}} \partial_z \left( \frac{1}{r} \partial_r (r v_r) + \partial_z v_z \right) = g_3
\]

(1.16)

with the initial-boundary value conditions

\[
(\rho, v)|_{t=0} = (\phi_0, v_0) \equiv (\rho_0(r, z), u_0^r(r, z), u_0^\theta(r, z), u_0^z(r, z)),
\]

(1.17)

\[
v|_{r=1} = (0, 0, 0), \quad \lim_{r^2 + z^2 \to \infty} v = (0, 0, 0),
\]

(1.18)

where

\[
f = -\frac{1}{r} \partial_r (r \phi v_r) - \partial_z (\phi v_z),
\]
\[ g_1 = \frac{v_0^2}{r} - v_r \partial_r v_r - v_z \partial_z v_r - \partial_z Q(\bar{r}, \phi) - \frac{\nu_1 \phi}{(\phi + \bar{\rho})\bar{\rho}} \left( \partial_r \left( \frac{1}{r} \partial_r (r v_r) \right) + \partial_z^2 v_r \right) \]
\[ \quad - \frac{\nu_2 \phi}{(\phi + \bar{\rho})\bar{\rho}} \partial_z \left( \frac{1}{r} \partial_r (r v_r) + \partial_z v_z \right), \]
\[ g_2 = -v_r \partial_r v_\theta - v_z \partial_z v_\theta - \frac{v_\theta v_r}{r} - \frac{\nu_1 \phi}{(\phi + \bar{\rho})\bar{\rho}} \left( \partial_r \left( \frac{1}{r} \partial_r (r v_\theta) \right) + \partial_z^2 v_\theta \right), \]
\[ g_3 = -v_r \partial_r v_z - v_z \partial_z v_z - \partial_z Q(\bar{r}, \phi) - \frac{\nu_1 \phi}{(\phi + \bar{\rho})\bar{\rho}} \left( \partial_z^2 v_z + \partial_z^2 v_z + \frac{1}{r} \partial_r v_r \right) \]
\[ \quad - \frac{\nu_2 \phi}{(\phi + \bar{\rho})\bar{\rho}} \partial_z \left( \frac{1}{r} \partial_r (r v_r) + \partial_z v_z \right), \]
and
\[ Q(\bar{r}, \phi) = \frac{\gamma(\gamma - 2)}{2} \bar{\phi}^2 \int_0^1 (\bar{\rho} + s\phi)^{\gamma - 3} ds. \]

To state our main results conveniently, we now introduce the following notations: for \( w_1, w_2 \in L^2(\Omega) \), set
\[ (w_1, w_2) = \int_\Omega w_1 w_2 d\tau dz. \]

In addition, \( w \in L^p(\Omega) \) \((1 \leq p < \infty)\) means that
\[ \|w\|_{L^p} = \|w\|_p = \left( \int_\Omega |w|^p d\tau dz \right)^{\frac{1}{p}} < +\infty. \]

And define
\[ L^p_0(\Omega) = \{ w \in \mathcal{D}'(\Omega) : \|w\|_{L^p} < +\infty \}, \]

where
\[ \|w\|_{L^p} = \left( \int_\Omega |w|^p r d\tau dz \right)^{\frac{1}{p}}. \]

Set \( D = (\partial_r, \partial_z) \) and define for \( k \in \mathbb{N} \cup \{0\} \)
\[ \|w\|^2_{H^k} = \sum_{j=0}^k \|\sqrt{r} D^j w\|^2_2. \]

Denote by
\[ \tilde{H}^k = \{ w \in \mathcal{D}'(\Omega) : \|w\|_{\tilde{H}^k} < +\infty \}. \]

The main conclusion in the paper is:

**Theorem 1.1** There exists a constant \( \varepsilon > 0 \) such that if \( \|\phi_0\|_{\tilde{H}^2} + \|v_0\|_{\tilde{H}^3} \leq \varepsilon \), then problem (1.13)-(1.16) together with (1.17)-(1.18) has a unique global solution \((\tilde{\phi}, v) \in C([0, \infty), \tilde{H}^2 \times \tilde{H}^3)\) satisfying
\[ \|\phi\|^2_{H^2} + \|v\|^2_{H^3} + \int_0^\infty (\|D(\tilde{\phi} - 2\phi)\|_{H^1} + \|Dv\|_{H^3}) d\tau \leq C(\|\phi_0\|^2_{H^2} + \|v_0\|^2_{H^3}). \]
Remark 1.1. For the original problem (1.2)-(1.5) with (1.10)-(1.12), one knows from Theorem 1.1 that the perturbed cylindrical symmetric circulatory flows are globally stable.

Remark 1.2. So far there have been extensive results on the global spherically symmetric (or helically symmetric) weak/strong/classical solutions to the compressible Navier-Stokes equations (in this case, the solution admits a form \((\rho(t,x), u(t,x)) = (\rho(t,r), U(t,r)\frac{x}{r})\) or \((\rho(t,x), u(t,x)) = (\rho(t,r,z), u_r(t,r,z)\frac{x}{r}, u_z(t,r,z))\) with \(x' = (x_1, x_2)\) and \(x'^\perp = (-x_2, x_1)\), one can see [2], [4], [7], [14-15], [19], [21] and the references therein. Here, we point out that our initial data in (1.10) has no bounded energy (due to \(u(0, x) \sim \frac{1}{r}\)), which is a little different from the cases in the aforementioned references.

Remark 1.3. For the Cauchy problem or initial-boundary value problem in exterior domain of 3-D compressible Navier-Stokes equations, when the initial data are in some suitably weighted energy spaces or are of small perturbations with respect to the constant states, many authors have established the local/global existence of weak/strong/classical solutions in appropriate function spaces, one can find the details in [1], [5-6], [8], [13], [16-18], [20] and so on. If we intend to study the general (not cylindrical symmetric) global perturbation problem of 3-D circulatory flows for (1.1), the methods applied in the above references cannot be applied directly since our perturbed initial data are different from those (for examples, our initial data have not finite energies or are not of the small perturbations of constant states). On the other hand, motivated by the results and methods in [9-10] and [11-12], where the global stabilities and large time behaviors of the perturbed constant equilibrium on the half space, and of the perturbed plane Couette flow are studied respectively when the Reynolds and Mach numbers are sufficiently small, we hope that the global stability of generally perturbed viscous circulatory flows can be established in our future research.

Let’s recall some previous works which are related to our results. For the initial-boundary value problem of (1.1), the local classical solution is obtained in [20] with \(\rho_0\) being positive and bounded. Applying the energy methods in Sobolev spaces, the authors in [17] established the global existence of classical solutions to (1.1) when the initial data are of small perturbations for a non-vacuum constant state and no slip boundary conditions are posed. Recently, for the case that the initial density is allowed to vanish and even has compact support and the smooth initial data are of small total energy, the authors in [8] established the global existence and uniqueness of classical solutions whose corresponding far fields are vacuum or non-vacuum. For the arbitrary initial data with finite total energies, the global existence of weak solutions to 3-D compressible Navier-Stokes equations has been established by P. L. Lions in [13] for suitably large adiabatic exponent \(\gamma\), and subsequently this result was improved to the cases of \(\gamma > \frac{4}{3}\) for the general solutions in [5] and \(\gamma > 1\) for cylindrically symmetric solutions in [15] respectively. In addition, D. Hoff in [7] showed the existence of spherically symmetric weak solutions for \(\gamma = 1\) and discontinuous initial data. Here we point out that the corresponding background solution of (1.2)-(1.5) is not a constant state, which is different from those situations in the aforementioned references; in addition, compared with reference [22], where the global existence and stability of a 2-D perturbed viscous symmetric circulatory flow around a disc are established, the analysis in the present paper is more involved due to the multi-dimensional spaces.

To prove Theorem 1.1, we require to establish some global weighted energy estimates of the solution \((\phi, v)\) to (1.13)-(1.16). Thanks to delicate analysis, the uniform weighted estimates of \((\phi, v)\) are obtained.
by making full use of the properties (for instance, $\partial_r \bar{\rho} \sim \frac{1}{r^2}$) of the background solution and choosing suitable multipliers. Based on this and the local existence result of classical solution to (1.13)-(1.16) with (1.17)-(1.18), Theorem 1.1 is shown by the continuity argument.

The paper is organized as follows: In §2, we derive some uniform energy estimates from the linearized parts of (1.13)-(1.16). From this, some uniform weighted energy inequalities of $(\phi, v)$ are obtained and subsequently the proof of Theorem 1.1 is completed in §3.

## 2 Some Elementary Estimates

In this section, we establish some basic weighted energy inequalities on the solution $(\phi, v)$ of (1.13)-(1.16) with (1.17)-(1.18).

**Lemma 2.1 (Weighted $L^2$-estimate of $(\phi, v)$).** For the solution $(\phi, v) \in C([0, \infty), \tilde{H}^2 \times \tilde{H}^3)$ of problem (1.13)-(1.16) with (1.17)-(1.18), we have

\[
\|\sqrt{\rho}v_r\|_2^2 + \|\rho^{-2}\phi\|_2^2 + \int_0^t \left( \|v_r\|_{\sqrt{r}}^2 + \|v_\theta\|_{\sqrt{r}}^2 + \|\sqrt{r}Dv\|_2^2 + \|\sqrt{r}(\frac{1}{r}\partial_r(rv_r) + \partial_z v_z)\|_2^2 \right) dt
\leq C(\|\sqrt{r}v_{0r}\|_2^2 + \|\sqrt{r}v_{0\theta}\|_2^2) + C \int_0^t A_1 \, dt,
\]

where and below $C > 0$ stands for a generic constant, and $A_1 = |(g, \rho rv)| + |(f, \rho^{-2}\phi)|$ with $g = (g_1, g_2, g_3)$.

**Proof.** It follows from $\int \Omega (1.14) \times \rho rv_r \, drdz$, $\int \Omega (1.15) \times \rho rv_\theta drdz$ and $\int \Omega (1.16) \times \rho rv_\theta drdz$ that

\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho}v_r\|_2^2 + \gamma(\partial_r(\rho^{-2}\phi), \rho rv_r) + \nu_1(\|v_r\|_{\sqrt{r}}^2 + \|\sqrt{r}\partial_r v_r\|_2^2 + \|\sqrt{r}\partial_z v_z\|_2^2)
+ \nu_2(\frac{1}{r}\partial_r(rv_r) + \partial_z v_z, r \partial_r v_r) = (g_1, \rho rv_r) + \left(\frac{2M_0}{r}v_\theta, \bar{\rho} v_r\right),
\]

(2.1)

\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho}v_\theta\|_2^2 + \nu_1(\|v_\theta\|_{\sqrt{r}}^2 + \|\sqrt{r}\partial_r v_\theta\|_2^2 + \|\sqrt{r}\partial_z v_z\|_2^2) = (g_2, \rho rv_\theta)
\]

(2.2)

and

\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho}v_z\|_2^2 + \gamma(\partial_z(\rho^{-2}\phi), \rho rv_z) + \nu_1(\|\sqrt{r}\partial_r v_z\|_2^2 + \|\sqrt{r}\partial_z v_z\|_2^2)
+ \nu_2(\frac{1}{r}\partial_r(rv_r) + \partial_z v_z, r \partial_r v_r) = (g_3, \rho rv_z).
\]

(2.3)

Adding (2.1) and (2.3) yields

\[
\frac{1}{2} \frac{d}{dt}(\|\sqrt{\rho}v_r\|_2^2 + \|\sqrt{\rho}v_z\|_2^2) - \gamma(\rho^{-2}\phi, \partial_r(\rho rv_r) + \partial_z(\rho rv_z))
+ \nu_1(\|v_r\|_{\sqrt{r}}^2 + \|\sqrt{r}\partial_r v_r\|_2^2 + \|\sqrt{r}\partial_z v_z\|_2^2)
+ \nu_2(\|\sqrt{r}v_z\|_2^2) + \nu_2(\frac{1}{r}\partial_r(rv_r) + \partial_z v_z)\|_2^2
\]
By \( |(2M_0 \overline{v}_g, \overline{p} \overline{v}_r)| \leq \frac{\nu_1}{2} \frac{\overline{v}_r}{\sqrt{r}} \| \frac{\overline{v}_g}{\sqrt{r}} \|_2 + C \| \frac{\overline{v}_g}{\sqrt{r}} \|_2 \), one has that from \((2.2)\) and \((2.4)\),

\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{r} \overline{p} \overline{v} \|^2_2 - \gamma (\overline{p}^{-2} \phi, \partial_r (\overline{p} \overline{v}_r) + \partial_z (\overline{p} \overline{v}_z)) + \nu_1 (\| \frac{v_r}{\sqrt{r}} \|^2_2 + \| \sqrt{r} \partial_r \overline{v}_r \|^2_2 + \| \sqrt{r} \partial_z \overline{v}_z \|^2_2) + \nu_2 \sqrt{r} (\frac{1}{r} \partial_r (\overline{v}_r + \partial_z \overline{v}_z)) \|^2_2 \\
\leq C(g_1, \overline{p} \overline{v}_r) + C(g_2, \overline{p} \overline{v}_r) + C(g_3, \overline{p} \overline{v}_z).
\]

\[\tag{2.5}\]

In addition, it follows from \( \int_{\Omega} (1.13) \times \gamma \overline{p}^{-2} r \phi \overline{d} r \overline{d} z \) that

\[
\frac{\gamma}{2} \frac{d}{dt} \| \sqrt{r} \overline{p} \overline{v} \|^2_2 + \gamma (\overline{p}^{-2} \phi, \partial_r (\overline{p} \overline{v}_r) + \partial_z (\overline{p} \overline{v}_z)) = \gamma (f, \overline{p}^{-2} r \phi).
\]

\[\tag{2.6}\]

Consequently, adding \((2.5), (2.6)\) and integrating with respect to the time variable \( \tau \) over \((0, t)\) yields Lemma 2.1.

\section*{Lemma 2.2 (Weighted \(L^2\)–estimate of \((\partial_t \phi, \partial_t v, D v)\)). For the solution \((\phi, v) \in C([0, \infty), \tilde{H}^2 \times \tilde{H}^3)\) of problem (1.13)-(1.16) with (1.17)-(1.18), we have

\[
\| \sqrt{r} D v \|_2^2 + \| \sqrt{r} (\frac{1}{r} \partial_r (r v_r) + \partial_z v_z) \|^2_2 + \int_0^t \left( \| \sqrt{r} \partial_t \overline{v}_r \|^2_2 + \| \sqrt{r} \overline{p}^{-2} \partial_t \phi \|^2_2 \right) d \tau \leq C \| (\phi_0, v_0) \|^2_{H^1} + C \int_0^t (A_1 + A_2) d \tau,
\]

where \( A_1 \) is defined in Lemma 2.1, and \( A_2 = \| (g, \overline{p} \partial_t v) \| + \| (f, r \overline{p}^{-2} \partial_t \phi) \| \).

\section*{Proof.} By computing \( \int_{\Omega} (1.14) \times \overline{p} \partial_t \overline{v}_r \overline{d} r \overline{d} z, \int_{\Omega} (1.15) \times \overline{p} \partial_t \overline{v}_r \overline{d} r \overline{d} z \) and \( \int_{\Omega} (1.16) \times \overline{p} \partial_t v_z \overline{d} r \overline{d} z \), we obtain that

\[
(\partial_t v_r, \overline{p} \partial_t v_r) - (2M_0 \frac{v_r}{r}, \overline{p} \partial_t v_r) + \gamma (\partial_r (\overline{p}^{-2} \phi), \overline{p} \partial_t v_r) \\
- \nu_1 (\partial_t \overline{v}_r (1 \overline{r} \partial_r (r v_r)) + \partial_z^2 v_r, r \partial_t v_r) - \nu_2 (\partial_t \overline{v}_r (1 \overline{r} \partial_r (r v_r) + \partial_z v_z), r \partial_t v_r) \\
= (g_1, \overline{p} \partial_t v_r), \tag{2.7}
\]

\[
(\partial_t v_9, \overline{p} \partial_t v_9) - \nu_1 (\partial_t \overline{v}_9 (1 \overline{r} \partial_r (r v_9)) + \partial_z^2 v_9, r \partial_t v_9) = (g_2, \overline{p} \partial_t v_9), \tag{2.8}
\]

and

\[
(\partial_t v_z, \overline{p} \partial_t v_z) + \gamma (\partial_z (\overline{p}^{-2} \phi), \overline{p} \partial_t v_z) \\
- \nu_1 (\partial_z^2 v_z + (1 \overline{r} \partial_z v_z) + \partial_z^2 v_z, r \partial_t v_z) - \nu_2 (\partial_z (1 \overline{r} \partial_r (r v_r) + \partial_z v_z), r \partial_t v_z).
\]
Then adding (2.10) and (2.11), and using (2.12), we arrive at

\[
= (g_3, \bar{p}\bar{r}\partial_t v_z).
\]

Note that

\[
-(\frac{\nu_1}{\rho} \bar{r}_t (\frac{1}{r} \bar{r}_r (r \bar{v}_r)), \bar{p}\bar{r}\partial_t v_r) = \frac{\nu_1}{2} \frac{d}{dt} (\| \frac{v_r}{r} \|^2 + \| r^{1/2} \bar{r}_r (r^{1/2} \bar{v}_r) \|^2).
\]

This, together with (2.7)-(2.9), derives

\[
\frac{d}{dt} \left( \nu_1 (\frac{v_r}{r})^2 + \nu_2 (\frac{v_\theta}{\sqrt{r}})^2 + \sqrt{r} Dv_r \right) + \nu_2 \sqrt{r} \left( \frac{1}{r} \bar{r}_t (r \bar{v}_r) + \bar{z}_r (\bar{v}_z) \right) \leq C \left( r \bar{r}\partial_t v_r + r \bar{v}_r \right).
\]

Computing \( \int_{\Omega} \frac{1}{r} \bar{r}_t \partial_t \rho d\bar{r} d\bar{z} \) yields

\[
\gamma \| \sqrt{r} \bar{r}\partial_t \rho \|^2 + \gamma \left( \frac{1}{r} \bar{r}_t (r \bar{v}_r) + \bar{z}_r (\bar{v}_z), r \bar{r}\partial_t \rho \right) = \gamma \left( f, r \bar{r}\partial_t \rho \right).
\]

In addition, we see that

\[
\gamma \left( \frac{1}{r} \bar{r}_t (r \bar{v}_r) + \bar{z}_r (\bar{v}_z), r \bar{r}\partial_t \rho \right) - \gamma \left( \frac{1}{r} \bar{r}_t (r \bar{v}_r) + \bar{z}_r (\bar{v}_z), r \bar{r}\partial_t \rho \right) = -\gamma \left( \frac{1}{r} \bar{r}_t (r \bar{v}_r) + \bar{z}_r (\bar{v}_z), r \bar{r}\partial_t \rho \right).
\]

Then adding (2.10) and (2.11), and using (2.12), we arrive at

\[
\frac{d}{dt} \left( \nu_1 \| \frac{v_r}{r} \|^2 + \nu_2 \left( \frac{v_\theta}{\sqrt{r}} \right)^2 \right) + C \left( r \bar{r}\partial_t v_r + \bar{v}_z \right) \leq C \left( r \bar{r}\partial_t v_r + \bar{v}_z \right).
\]

Since

\[
\gamma \left( r \bar{r}\partial_t \rho, \bar{v}_z \right) \leq C \| \sqrt{r} \left( \frac{1}{r} \bar{r}_t (r \bar{v}_r) + \bar{z}_r (\bar{v}_z) \right) \|^2 + C \left( \| r^{1/2} \bar{r}_r \bar{v}_r \|^2 + \| \frac{v_r}{r} \|^2 \right) + \frac{C}{4} \| \sqrt{r} \bar{r}\partial_t \rho \|^2,
\]

and

\[
\left( \| r \bar{r}\partial_t \rho, \bar{v}_z \| \right) \leq C \left( \| r \bar{r}\partial_t \rho, \bar{v}_z \| \right) + C \left( \| r \bar{r}\partial_t \rho, \bar{v}_z \| \right) + C \left( \| r \bar{r}\partial_t \rho, \bar{v}_z \| \right),
\]

(2.13) and

\[
\left( \| r \bar{r}\partial_t \rho, \bar{v}_z \| \right) \leq C \left( \| r \bar{r}\partial_t \rho, \bar{v}_z \| \right) + C \left( \| r \bar{r}\partial_t \rho, \bar{v}_z \| \right),
\]

(2.14) and

\[
\left( \| r \bar{r}\partial_t \rho, \bar{v}_z \| \right) \leq C \left( \| r \bar{r}\partial_t \rho, \bar{v}_z \| \right) + C \left( \| r \bar{r}\partial_t \rho, \bar{v}_z \| \right),
\]

(2.15)
integrating (2.13) with respect to the time variable $\tau$ over $(0, t)$ and combining (2.14)-(2.15) yield

\[
\frac{1}{2} \int_0^t \left( \| \sqrt{r} \partial_r v \|_2^2 + \frac{\gamma}{2} \| \sqrt{r} \rho \gamma^{-2} \partial_t \phi \|_2^2 \right) d\tau + \left( \frac{\nu_1}{4} \left( \frac{\nu_1}{\sqrt{r}} \| \frac{\partial_v}{\sqrt{r}} \|_2^2 + \| \sqrt{r} D v \|_2^2 \right) \right)
\]

\[
+ \frac{\nu_2}{4} \| \sqrt{r} (\frac{1}{r} \rho (r v_r) + \partial_z v_z) \|_2^2 - C \| \sqrt{r} \rho \gamma^{-2} \phi \|_2^2 - C \| \frac{\nu_1}{\sqrt{r}} \|_2^2 \right)
\]

\[
\leq C \|(\phi_0, v_0)\|_{H^1_0}^2 + \int_0^t \left( \| (g, \rho \partial_t v) \| + \| (2M_0 \nu_0, \rho \partial_t v_r) \| + | \gamma(f, r \rho \gamma^{-2} \partial_t \phi) | + C \| \sqrt{r} \frac{1}{r} \rho \partial_t (r v_r) + \partial_z v_z \|_2^2 + C \| \frac{\nu_1}{\sqrt{r}} \|_2^2 \right) d\tau.
\]  

(2.16)

Together with Holder’s inequality and Lemma 2.1, this yields

\[
\| \sqrt{r} D v \|_2^2 + \| \sqrt{r} \frac{1}{r} \rho \partial_t (r v_r) + \partial_z v_z \|_2^2 + \int_0^t \left( \| \sqrt{r} \partial_t v \|_2^2 + \| \sqrt{r} \rho \gamma^{-2} \partial_t \phi \|_2^2 \right) d\tau
\]

\[
\leq C \|(\phi_0, v_0)\|_{H^1_0}^2 + C \int_0^t (A_1 + A_2) d\tau,
\]  

(2.17)

which completes the proof of Lemma 2.2.

Taking $\partial_t^k \partial_z^j$ ($k = 0, 1$ and $j = 0, 1, 2$) on both sides of equations (1.13)-(1.16), we then have

\[
\partial_t (\partial_t^k \partial_z^j \phi) + \frac{1}{r} \partial_r (r \rho \partial_t^k \partial_z^j v_r) + \rho \partial_z \partial_t^k \partial_z^j v_z = \partial_t^k \partial_z^j f,
\]  

(2.18)

\[
\partial_t (\partial_t^k \partial_z^j v_r) - \frac{2M_0}{r^2} \partial_t^k \partial_z^j v_\theta + \gamma \partial_r (\rho \gamma^{-2} \partial_t^k \partial_z^j \phi) - \frac{\nu_1}{\rho} \left( \partial_r (\frac{1}{r} \rho \partial_r (r \partial_t^k \partial_z^j v_r)) + \partial_z \partial_t^k \partial_z^j v_r \right)
\]

\[
- \frac{\nu_2}{\rho} \partial_r (\frac{1}{r} \rho \partial_t^k \partial_z^j v_r) + \rho \partial_z \partial_t^k \partial_z^j v_z = \partial_t^k \partial_z^j g_1,
\]  

(2.19)

\[
\partial_t (\partial_t^k \partial_z^j v_\theta) - \frac{\nu_1}{\rho} \left( \partial_r (\frac{1}{r} \rho \partial_r (r \partial_t^k \partial_z^j v_\theta)) + \partial_z \partial_t^k \partial_z^j v_\theta \right) = \partial_t^k \partial_z^j g_2,
\]  

(2.20)

\[
\partial_t (\partial_t^k \partial_z^j v_z) + \gamma \partial_z (\rho \gamma^{-2} \partial_t^k \partial_z^j \phi) - \frac{\nu_1}{\rho} \left( \partial_t^2 \partial_z^k \partial_z^j v_z + \partial_z \partial_t^k \partial_z^j v_z + \frac{1}{r} \partial_r \partial_t^k \partial_z^j v_z \right)
\]

\[
- \frac{\nu_2}{\rho} \partial_r (\frac{1}{r} \rho \partial_t^k \partial_z^j v_z) + \rho \partial_z \partial_t^k \partial_z^j v_z = \partial_t^k \partial_z^j g_3.
\]  

(2.21)

Adding $\int_{\Omega} \gamma \rho^{-2} r \partial_t^k \partial_z^j \phi \times (2.18) d\tau d\rho$ and $\int_{\Omega} \rho \partial_t^k \partial_z^j v_r \times (2.19) d\tau d\rho$ and $\int_{\Omega} \rho \partial_t^k \partial_z^j v_\theta \times (2.20) d\tau d\rho$ and $\int_{\Omega} \rho \partial_t^k \partial_z^j v_z \times (2.21) d\tau d\rho$, then as in Lemma 2.1-Lemma 2.2, we can obtain

**Lemma 2.3** (Weighted $L^2$-estimate of $(\partial_t^k \partial_z^j \phi, \partial_t^k \partial_z^j v)$ with $k = 0, 1$ and $j = 0, 1, 2$). For the solution $(\phi, v) \in C([0, \infty), H^2 \times H^3)$ of problem (1.13)-(1.16) with (1.17)-(1.18), we have

\[
\| \sqrt{r} \partial_t^k \partial_z^j v \|_2 + \| \sqrt{r} \rho^{-2} r \partial_t^k \partial_z^j \phi \|_2 + \int_0^t \left( \| \frac{\partial_t^k \partial_z^j v_r \|_2^2 + \| \frac{\partial_t^k \partial_z^j v_\theta \|_2^2 + \| \frac{\partial_t^k \partial_z^j v_z \|_2^2}{\sqrt{r}^2} \right)
\]

\[
\leq C \|(\phi_0, v_0)\|_{H^1_0}^2 + C \int_0^t (A_1 + A_2) d\tau,
\]  

(2.22)
where

\[ A \]

of problem (1.13)-(1.16) with (1.17)-(1.18), we have

\[ f, r \]

By

\[ C \]

\[ \sum_{k,j=0}^{1} \int_0^t \left( (\partial_{i_1}^k \partial_{i_2}^j g, \bar{\rho} \partial_{i_1}^k \partial_{i_2}^j v) \right) d\tau. \]

On the other hand, by equations (2.18)-(2.21) and Lemma 2.3 with \( k = 1, j = 0 \), as in Lemma 2.2, we can also obtain

**Lemma 2.4 (Weighted L^2-estimate of \((\partial_t^2 \phi, \partial_t^2 v, \partial_t Dv)\)).** For the solution \((\phi, v) \in C([0, \infty), \tilde{H}^2 \times \tilde{H}^3)\) of problem (1.13)-(1.16) with (1.17)-(1.18), we have

\[
\left\| \sqrt{r} \partial_t Dv \right\|_2^2 + \left\| \sqrt{r} \left( \frac{1}{r} \partial_r (r \partial_r v_r + \partial_{zz}^2 v_z) \right) \right\|_2^2 + \int_0^t \left( \left\| \sqrt{r} \partial_t^2 \phi \right\|_2^2 + \left\| \sqrt{r} \partial_t \phi \right\|_2^2 \right) d\tau
\]

\[
\leq C \left( \left\| \phi_0 \right\|_{H^2}^2 + \left\| v_0 \right\|_{H^3}^2 + C \int_0^t (A_1 + A_2 + A_3) d\tau, \right.
\]

where \( A_1 \) and \( A_2 \) are defined in Lemma 2.1 and Lemma 2.2 respectively, and \( A_3 = \left| (\partial_t g, \bar{\rho} \partial_t v) \right| + \left| (\partial_t f, r \bar{\rho}^{-2} \partial_t \phi) \right| + \left| (\partial_t g, \bar{\rho} \partial_t^2 v) \right| + \left| (\partial_t f, r \bar{\rho}^{-2} \partial_t^2 \phi) \right| \).

Next we start to derive the a priori estimates of \( \phi \). By \((\nu_1 + \nu_2) \partial_r (\bar{\rho}^\gamma \times (1.13))\) and direct computation, one arrives at

\[
(\nu_1 + \nu_2) \partial_{tt} \bar{\rho}^\gamma - (\nu_1 + \nu_2) \bar{\rho}^\gamma \partial_r (\frac{1}{r} \partial_r (r v_r))
\]

\[
= (\nu_1 + \nu_2) \left\{ \partial_r (\bar{\rho}^\gamma \partial_r (\bar{\rho}^\gamma \partial_r v_r)) - \partial_r (\bar{\rho}^\gamma \partial_r (\bar{\rho}^\gamma \partial_r v_r)) \right\}
\]

\[
= \nu_1 + \nu_2 \left\{ \partial_r (\bar{\rho}^\gamma \partial_r (\bar{\rho}^\gamma \partial_r v_r)) - \partial_r (\bar{\rho}^\gamma \partial_r (\bar{\rho}^\gamma \partial_r v_r)) \right\}
\]

Adding (2.22) and \( \bar{\rho}^\gamma \times (1.13) \) yields

\[
(\nu_1 + \nu_2) \partial_{tt} \bar{\rho}^\gamma - \gamma \bar{\rho}^\gamma \partial_r (\bar{\rho}^\gamma \partial_r \phi) = h,
\]

where

\[
\frac{1}{2} \left\| \sqrt{r} \partial_t \phi \right\|_2^2 + \left\| \sqrt{r} \partial_t \phi \right\|_2^2 \leq \left( \partial_r (\bar{\rho}^\gamma \partial_r \phi), rh \right).
\]

By \( \int_\Omega r \partial_r (\bar{\rho}^\gamma \times (2.23)) dr dz \), we have
Also, we obtain that from \( \int_{\Omega} r \partial_t^2 (\bar{\rho}^{-2} \phi) \times \partial_r (2.23) dr dz. \)

\[
\frac{\nu_1 + \nu_2}{2} \frac{d}{dt} \| r^{1/2} \partial_t^2 (\bar{\rho}^{-2} \phi) \|_2^2 + \| \sqrt{\rho} r \partial_t^2 (\bar{\rho}^{-2} \phi) \|_2^2 \\
\leq (\partial_r^2 (\bar{\rho}^{-2} \phi), r \partial_r h) + (\partial_r^2 (\bar{\rho}^{-2} \phi), r \partial_r (\bar{\rho}) \partial_t (\bar{\rho}^{-2} \phi)). 
\]

(2.25)

Next, we show that

**Lemma 2.5 (Weighted higher order energy estimate).** For the solution \((\phi, v) \in C([0, \infty), \bar{H}^2 \times \bar{H}^3)\) of problem (1.13)-(1.16) with (1.17)-(1.18), we have

\[
\| r^{1/2} \partial_t (\bar{\rho}^{-2} \phi) \|_2^2 + \| r^{1/2} \partial_r (\bar{\rho}^{-2} \partial_x \phi) \|_2^2 + \| r^{1/2} \partial_t^2 (\bar{\rho}^{-2} \phi) \|_2^2 + \int_0^t \left( \| r^{1/2} \partial_t (\bar{\rho}^{-2} \phi) \|_2^2 \\
+ \| r^{1/2} \partial_t^2 v_r \|_2^2 + \| r^{1/2} \partial_r (\bar{\rho}^{-2} \phi) \|_2^2 + \| r^{1/2} \partial_t^2 \partial_x v_r \|_2^2 + \| \sqrt{r} \partial_r^2 (\bar{\rho}^{-2} \phi) \|_2^2 \right) d\tau \\
\leq C(\| \phi_0 \|_{\bar{H}^2}^2 + \| v_0 \|_{\bar{H}^3}^2) + C \int_0^t \left( A_1 + A_2 + \| g \|_{\bar{H}^1}^2 + \| f \|_{\bar{H}^1}^2 + \| (r \partial_t (\bar{\rho}^{-2} \partial_x \phi), \bar{\rho}^{-2} \partial_r^2 f) | \\
+ |(\partial_r^2 (\bar{\rho}^{-2} \phi), r \bar{\rho}^{-2} \partial_r^2 f)| + |(\partial_r^2 f, \bar{\rho}^{-2} r \partial_r^2 f)| \right) d\tau. 
\]

**Proof.** We see that

\[
(\partial_r (\bar{\rho}^{-2} \phi), r h) \leq \frac{1}{2} \| \sqrt{\bar{\rho}} r \partial_r (\bar{\rho}^{-2} \phi) \|_2^2 + C \| r^{1/2} h \|_2^2. 
\]

In addition,

\[
\| r^{1/2} h \|_2^2 \leq C \left\{ \| r^{1/2} f \|_2^2 + \| r^{-1/2} v_r \|_2^2 + \| r^{1/2} \partial_r v_r \|_2^2 + \| r^{1/2} \partial_x v_r \|_2^2 \\
+ \| r^{1/2} \partial_x v_z \|_2^2 + \| r^{1/2} \partial_x^2 v_r \|_2^2 + \| r^{1/2} \partial_x v_r \|_2^2 + \| r^{1/2} g_1 \|_2^2 + \| r^{1/2} \partial_r f \|_2^2 \right\}.
\]

\[
\leq C \left\{ \| f \|_{\bar{H}^1}^2 + \| g \|_{\bar{H}^1}^2 + \| r^{-1/2} v_r \|_2^2 + \| r^{1/2} Dv \|_2^2 + \| r^{1/2} \partial_x Dv \|_2^2 + \| r^{1/2} \partial_x v_r \|_2^2 \right\}.
\]

Together with (2.24), this yields

\[
\frac{\nu_1 + \nu_2 d}{2} \| r^{1/2} \partial_t (\bar{\rho}^{-2} \phi) \|_2^2 + \frac{\gamma}{2} \| \sqrt{\rho} r \partial_t (\bar{\rho}^{-2} \phi) \|_2^2 \\
\leq C \left\{ \| f \|_{\bar{H}^1}^2 + \| g \|_{\bar{H}^1}^2 + \| r^{-1/2} v_r \|_2^2 + \| r^{1/2} Dv \|_2^2 + \| r^{1/2} \partial_x Dv \|_2^2 + \| r^{1/2} \partial_x v_r \|_2^2 \right\}. 
\]

(2.27)

Note that

\[
(\partial_r^2 (\bar{\rho}^{-2} \phi), r \partial_r h) \leq \frac{1}{8} \| \sqrt{\rho} r \partial_r^2 (\bar{\rho}^{-2} \phi) \|_2^2 + C \| r^{1/2} \partial_r h_2 \|_2^2 + (\partial_r^2 (\bar{\rho}^{-2} \phi), r \partial_r h_1), \\
(\partial_r^2 (\bar{\rho}^{-2} \phi), r \partial_r (\bar{\rho}) \partial_r (\bar{\rho}^{-2} \phi)) \leq \frac{1}{4} \| \sqrt{\rho} r \partial_r^2 (\bar{\rho}^{-2} \phi) \|_2^2 + C \| r^{1/2} \partial_r (\bar{\rho}) \partial_r (\bar{\rho}^{-2} \phi) \|_2^2.
\]
Combining (2.27), (2.31) with \( j \) and together with (2.25), this yields

\[
\|r^{1/2} \partial_r h_2 \|^2 \leq C \left( \|r^{-1/2} v_r \|^2 + \|r^{-1/2} v_\theta \|^2 + \|r^{1/2} Dv \|^2 + \|r^{1/2} \partial_r v_r \|^2 + \|r^{1/2} \partial_r^2 v_r \|^2 \right)
\]

and

\[
(\partial_r^2 (\tilde{\rho}^{-2} \phi), r \partial_r h_1) = (\nu_1 + \nu_2) (\partial_r^2 (\tilde{\rho}^{-2} \phi), r \partial_r^2 (\tilde{\rho}^{-2} \phi) f + r \partial_r^2 (\tilde{\rho}^{-2} \phi) f) \leq \frac{1}{8} \|r^{1/2} \partial_r^2 (\tilde{\rho}^{-2} \phi) \|^2 + C \|f \|^2_{L^1} + C (\partial_r^2 (\tilde{\rho}^{-2} \phi), r \tilde{\rho}^{-2} \partial_r^2 f).
\]

Together with (2.25), this yields

\[
\frac{d}{dt} \|r^{1/2} \partial_r^2 (\tilde{\rho}^{-2} \phi) \|^2 + \|r^{1/2} \partial_r^2 (\tilde{\rho}^{-2} \phi) \|^2 \leq C \left( \|r^{-1/2} v_r \|^2 + \|r^{-1/2} v_\theta \|^2 + \|r^{1/2} Dv \|^2 + \|r^{1/2} \partial_r v_r \|^2 + \|r^{1/2} \partial_r^2 v_r \|^2 \right)
\]

and integrating

\[
(\nu_1 + \nu_2) \partial_r^2 \partial_r^2 v_r = \tilde{\rho} \partial_r \partial_r^2 v_r - \frac{2M}{r^2} \partial_r^2 v_\theta + \gamma \tilde{\rho} \partial_r (\tilde{\rho}^{-2} \partial_r \phi) - (\nu_1 + \nu_2) \partial_r (\frac{\partial_r v_r}{\rho}) - \nu_1 \partial_r^2 + \nu_2 \partial_r \partial_r^2 v_r - \tilde{\rho} \partial_r^2 g_1.
\]

This derives that

\[
(\nu_1 + \nu_2) \|r^{1/2} \partial_r^2 \partial_r^2 v_r \|^2 \leq C \left( \|r^{1/2} \partial_r \partial_r^2 v_r \|^2 + \|r^{1/2} \partial_r \partial_r^2 v_\theta \|^2 + \|r^{1/2} \partial_r (\tilde{\rho}^{-2} \partial_r \phi) \|^2 + \|r^{1/2} \partial_r (\frac{\partial_r v_r}{\rho}) \|^2 \right)
\]

Combining (2.27), (2.31) with \( j = 0 \), Lemma 2.1-2.2 and Lemma 2.3 with \( k = 0, j = 1 \) and integrating with respect to the time variable \( \tau \) over \((0, t)\), we have

\[
\|r^{1/2} \partial_r (\tilde{\rho}^{-2} \phi) \|^2 + \int_0^t \left( \|r^{1/2} \partial_r (\tilde{\rho}^{-2} \phi) \|^2 + \|r^{1/2} \partial_r v_r \|^2 \right) d\tau \leq C \|\phi(0, v_0)\|^2_{L^2} + C \int_0^t \left( A_1 + A_2 + \|g\|^2_{L^1} + \|f\|^2_{L^1} \right) d\tau.
\]
Note that we have from (2.23)

$$(\nu_1 + \nu_2)\partial_{t_r}^2 (\rho^{-2} \partial_z \phi) + \rho^{-2} \partial_r (\rho^{-2} \partial_z \phi) = \partial_z h.$$  \hspace{1cm} (2.33)

Then it follows from $\int_{\Omega} \partial_t (\rho^{-2} \partial_z \phi) \times (2.33)rdrdz$ that

$$\frac{d}{dt} \|r^{1/2} \partial_r (\rho^{-2} \partial_z \phi)\|^2_2 + \|r^{1/2} \partial_r (\rho^{-2} \partial_z \phi)\|^2_2$$

$$\leq C(\partial_{t_r} h, r \partial_r (\rho^{-2} \partial_z \phi))$$

$$\leq C \left( \|r^{1/2} \partial_z g\|^2_2 + \|r^{1/2} \partial_r f\|^2_2 + (\rho^{-2} r \partial_r (\rho^{-2} \partial_z \phi), \rho^{-2} \partial^2_r f) + \|r^{1/2} \partial^2_r v_r\|^2_2 ight.$$ 

$$+ \|r^{1/2} \partial_z v_r\|^2_2 + \|r^{1/2} \partial_r v_r\|^2_2 + \|r^{1/2} \partial^2_r v_r\|^2_2 + \|r^{1/2} \partial^2_z v_r\|^2_2 + \|r^{1/2} \partial_z v_0\|^2_2$$

$$\left. + \|r^{1/2} \partial^2_z v_0\|^2_2 \right).$$  \hspace{1cm} (2.34)

This, together with (2.31) with $j = 1$, yields

$$\|r^{1/2} \partial_r (\rho^{-2} \partial_z \phi)\|^2_2 \leq \left( \int_0^t \left( \|r^{1/2} \partial_r (\rho^{-2} \partial_z \phi)\|^2_2 + \|r^{1/2} \partial^2_r v_r\|^2_2 \right) d\tau \right.$$ 

$$\leq C(\|\phi_0\|^2_{H^2} + v_0^2_{H^3}) + \left( \int_0^t C \left( \|r \partial_r (\rho^{-2} \partial_z \phi), \rho^{-2} \partial^2_r f\| + \|r^{1/2} \partial_r g\|^2_2 + \|r^{1/2} \partial_r f\|^2_2 ight.$$ 

$$+ \|r^{1/2} \partial^2_r v_r\|^2_2 + \|r^{1/2} Dv\|^2_2 + \|\partial^2_r v_z\|^2_2 \right) d\tau.$$  \hspace{1cm} (2.35)

Consequently, by (2.29), (2.32), (2.35), Lemma 2.1-2.2, and Lemma 2.3 with $k = 0, j = 1, 2$, we complete the proof of Lemma 2.5.

3 Global estimates and proof of Theorem 1.1

For the solution $(\phi, v) \in C([0, \infty), \tilde{H}^2 \times \tilde{H}^3)$ of problem (1.13)-(1.16) with (1.17)-(1.18), by Lemma 2.1-2.5, we obtain that

$$\|v\|^2_{H^1} + \|\partial_t v\|^2_{H^1} + \|\sqrt{r} \partial_t \phi\|^2_2 + \|\sqrt{r} \partial_r \phi\|^2_2 + \|r^{1/2} \partial_r (\rho^{-2} \phi)\|^2_2 + \|r^{1/2} \partial_r D(\rho^{-2} \phi)\|^2_2$$

$$\leq \left( \int_0^t \left( \frac{v_r}{\sqrt{r}}\|^2_2 + \|v_\theta\|^2_2 + \|\sqrt{r} Dv\|^2_2 + \|\partial_t v\|^2_{H^1} + \|\sqrt{r} \partial_t \phi\|^2_2 + \|\sqrt{r} \partial_r \phi\|^2_2 + \|\sqrt{r} \partial^2_r \phi\|^2_2 + \|\sqrt{r} \partial^2_t \phi\|^2_2 ight) d\tau \right.$$ 

$$\left. + \|r^{1/2} \partial_r (\rho^{-2} \phi)\|^2_2 + \|r^{1/2} \partial^2_r v_r\|^2_2 + \|r^{1/2} \partial_r (\rho^{-2} \partial_z \phi)\|^2_2 + \|r^{1/2} \partial^2_r \partial_z v_r\|^2_2 + \|\sqrt{r} \partial^2_r (\rho^{-2} \phi)\|^2_2 \right) d\tau$$

$$\leq C(\|\phi_0\|^2_{H^2} + v_0^2_{H^3}) + C \left( \int_0^t (A_1 + A_2 + A_3 + \|g\|^2_{H^1} + \|f\|^2_{H^1} + (r \partial_r (\rho^{-2} \partial_z \phi), \rho^{-2} \partial^2_r f) \right.$$ 

$$+ \|\partial^2_r (\rho^{-2} \phi, \rho^{-2} \partial^2_r f)\| + \|\partial^2_r g, \rho \partial^2_r v\| + \|\partial^2_r f, \rho^{-2} r \partial^2_r \phi\|) d\tau.$$  \hspace{1cm} (3.1)
Taking \( k = 0, j = 1 \) and \( k = 0, j = 2 \) in Lemma 2.3 respectively, and then adding them yields

\[
\|\sqrt{r} \partial_z v\|_{L^2}^2 + \|\sqrt{r} \partial^2_z v\|_{L^2}^2 + \|\sqrt{r} \partial \phi\|_{L^2}^2 + \|\sqrt{r} \partial^2 \phi\|_{L^2}^2 + \int_0^t (\|\sqrt{r} \partial_z Dv\|_{L^2}^2 + \|\sqrt{r} \partial^2 Dv\|_{L^2}^2) \, dt
\]

\[
\leq C(\|\phi_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + C \sum_{j=1}^2 \int_0^t (|(\partial^2_z g, \bar{\rho} \rho \partial_z^2 v)| + |(\partial^2_j f, \bar{\rho} \gamma^{-2} r \partial_z^2 \phi)|) \, dt. \tag{3.2}
\]

Set

\[
M_1 = A_1 + A_2 + A_3 + \|g\|_{L^2}^2 + \|f\|_{L^2}^2 + \|(r D^2(\bar{\rho} \gamma^{-2}) \phi, D^2(\bar{\rho} \gamma^{-2} f))\| + |(D^2 g, \bar{\rho} r D^2 v)|. \tag{3.3}
\]

It follows from (3.1) and (3.2) that

\[
\begin{align*}
\|v\|_{L^2}^2 + \|\partial_t v\|_{L^2}^2 + \|\sqrt{r} \partial \phi\|_{L^2}^2 + \|\sqrt{r} \partial_z v\|_{L^2}^2 + \|\sqrt{r} \partial^2 \phi\|_{L^2}^2 + \|\sqrt{r} \partial^2_z v\|_{L^2}^2 & \\
+ \int_0^t \left( \|\frac{v}{\sqrt{r}}\|_{L^2}^2 + \|\frac{v}{\sqrt{r}}\|_{L^2}^2 + \|\sqrt{r} Dv\|_{L^2}^2 + \|\partial_t v\|_{L^2}^2 + \|\sqrt{r} \partial \phi\|_{L^2}^2 + \|\sqrt{r} \partial^2 \phi\|_{L^2}^2 + \|\sqrt{r} \partial^2_z Dv\|_{L^2}^2 \right) \, dt & \\
\leq C \left( \|\phi_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \int_0^t M_1 \, dt \right). \tag{3.4}
\end{align*}
\]

In addition, by (1.13), we see that

\[
D \partial_t \phi = D(-\frac{1}{r} \partial_r (r \bar{\rho} v_r) - \partial_z (\bar{\rho} v_z) + f).
\]

Rewriting (1.15) and (1.16) as follows

\[
\begin{align*}
\nu_1 \partial_t^2 v & = \bar{\rho} \partial_t v - \nu_1 \partial_r \left( \frac{v}{r} \right) - \bar{\rho} g_2, \tag{3.5} \\
\nu_1 \partial^2_t v & = \bar{\rho} \partial_t v + \gamma \bar{\rho} \gamma^{-1} \partial_z \phi - \nu_1 \partial^2_z v - \nu_1 \frac{1}{r} \partial_r v - \nu_2 \partial_z \left( \frac{1}{r} \partial_r (r v_r) + \partial_z v_z \right) - \bar{\rho} g_3. \tag{3.6}
\end{align*}
\]

Then by (3.4)-(3.6) and (2.31) with \( j = 0 \), we have

\[
\begin{align*}
\|v\|_{L^2}^2 + \|\partial_t v\|_{L^2}^2 + \|\bar{\rho} \gamma^{-2} \phi\|_{L^2}^2 + \|\partial \phi\|_{L^2}^2 & \\
\leq C \left( \|\phi_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \int_0^t M_1 \, dt \right) + C \sup_{0 \leq t \leq T} (\|f\|_{L^2}^2 + \|\sqrt{r} g\|_{L^2}^2). \tag{3.7}
\end{align*}
\]

Furthermore, by (3.5)-(3.6), (2.31) with \( j = 0, \) (3.4) and (3.7), we also have

\[
\begin{align*}
\|v\|_{L^2}^2 + \|\partial_t v\|_{L^2}^2 + \|\bar{\rho} \gamma^{-2} \phi\|_{L^2}^2 + \|\partial \phi\|_{L^2}^2 & \\
+ \int_0^t \left( \|\frac{v}{\sqrt{r}}\|_{L^2}^2 + \|\frac{v}{\sqrt{r}}\|_{L^2}^2 + \|\sqrt{r} Dv\|_{L^2}^2 + \|\partial_t v\|_{L^2}^2 + \|\sqrt{r} \partial \phi\|_{L^2}^2 + \|\sqrt{r} \partial^2 \phi\|_{L^2}^2 \right) \, dt & \leq C \left( \|\phi_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \int_0^t M_1 \, dt \right) + C \sup_{0 \leq t \leq T} (\|f\|_{L^2}^2 + \|\sqrt{r} g\|_{L^2}^2).
\end{align*}
\]
\[ +\|\sqrt{r}\partial_t^2 v\|_2^2 + \|D(\rho^{-2}\phi)\|_{H^1}^2 \right) d\tau \\
\leq C \left( \|\phi_0\|_{H^2}^2 + \|v_0\|_{H^3}^2 + \int_0^t M_1 d\tau \right) + C \sup_{0 \leq \tau \leq t} \left( \|f\|_{H^1}^2 + \|g\|_{H^1}^2 \right). \tag{3.8} \]

Define the energy functional

\[ N(t) = \|v\|_{H^3}^2 + \|\partial_t v\|_{H^1}^2 + \|\rho^{-2}\phi\|_{H^2}^2 + \|\partial_t \phi\|_{H^1}^2 + \int_0^t \left( \|\frac{v_r}{\sqrt{r}}\|_2^2 + \|\frac{v_\theta}{\sqrt{r}}\|_2^2 + \|\sqrt{r}Dv\|_{\tilde{H}^3} \right) d\tau \\
+ \|\partial_t v\|_{H^2}^2 + \|\sqrt{r}\partial_t \phi\|_{H^1}^2 + \|\sqrt{r}\partial_t^2 \phi\|_{\tilde{H}^3}^2 + \|\sqrt{r}\partial_t^2 v\|_{\tilde{H}^3}^2 + \|D(\rho^{-2}\phi)\|_{H^1}^2 \right) d\tau. \]

Next, we show that \( N(t) \) is uniformly bounded for any \( t \geq 0 \). In the proof procedure, we will employ the following Gagliardo-Nirenberg’s inequality repeatedly:

**Lemma 3.1** (i) Assume \( 2 \leq p \leq +\infty \). Let \( j \) and \( k \) be integers satisfying \( 0 \leq j < k, k > j + 2(\frac{1}{2} - \frac{1}{p}) \).

Then there exists a constant \( C > 0 \) such that

\[ \|D^j w\|_{L^p(\Omega)} \leq C\|w\|_{L^2(\Omega)}^{1-a} \|D^k w\|_{L^2(\Omega)}^a, \]

where \( a = \frac{1}{k}(j + 1 - \frac{2}{p}) \).

(ii) Assume \( w(x) = w(r, z) \) and \( x \in \mathbb{R}^3 \). Let \( 2 \leq p \leq +\infty \) and let \( j \) and \( k \) be integers satisfying

\[ 0 \leq j < k, k > j + 3(\frac{1}{2} - \frac{1}{p}). \]

Then there exists a constant \( C > 0 \) such that

\[ \|D^j w\|_{L^p_r(\Omega)} \leq C\|w\|_{L^2_r(\Omega)}^{1-a} \|D^k w\|_{L^2_r(\Omega)}^a, \]

where \( a = \frac{1}{k}(j + 3 - \frac{3}{p}) \).

**Proposition 3.1** Assume \( N(t) \leq 1 \), we then have

\[ N(t) \leq C(\|v_0\|_{H^3}^2 + \|\phi_0\|_{H^2}^2) + CN(t)^{3/2}. \]

**Proof.** At first, we deal with \( A_i \ (i = 1, 2, 3) \) in the expression of \( M_1 \) in (3.8). In order to estimate the term \( \int_0^t A_1 d\tau \), by the expression of \( A_1 \), we require to treat \( \int_0^t (g_1, \tilde{\rho}rv_r) d\tau \) and \( \int_0^t (f, \tilde{\rho}^2 r^2 \phi) d\tau \).

Note that

\[ \int_0^t (g_1, \tilde{\rho}rv_r) d\tau \]
Next, we treat each term in the right hand side of (3.9). One has that

\[
\int_0^t \int_\Omega v^2 \rho v_r dr dz d\tau \leq C \int_0^t \int_\Omega |r^{1/3} v_\theta| \frac{|v_\theta|}{r^{1/2}} |r^{1/6} v_r| dr dz d\tau
\]

\[
\leq C \int_0^t \|r^{1/3} v_\theta\|_{L^2} \|v_\theta\|_{L^3} \|v_r\|_{L^6} d\tau.
\]

(3.10)

By Lemma 3.1 (i) with \(j = 0, p = 3\) and \(k = 1\), we have

\[
\|v_\theta\|_{L^3} \leq C \|v_\theta\|_{L^2}^{2/3} \|D(v_\theta\|_{L^2})^{1/3}.
\]

This, together with Lemma 3.1(ii) with \(j = 0, p = 6\) and \(k = 1\), we obtain that from (3.10)

\[
\int_0^t \int_\Omega \frac{v_\theta^2}{r} \rho v_r dr dz d\tau
\]

\[
\leq C \sup_{0 \leq \tau \leq t} \|r^{1/2} v_\theta\|_2 \int_0^t \|r^{1/3} \partial_r v_r\|_2 \|r^{1/2} v_r\|_2 \|v_r\|_{L^6} d\tau
\]

\[
\leq C N(t)^{3/2}.
\]

(3.11)

And we also have that

\[
\int_0^t \int_\Omega v_r \partial_r v_r \rho v_r dr dz d\tau
\]

\[
\leq C \sup_{0 \leq \tau \leq t} \|r^{1/2} v_r\|_2 \int_0^t \|r^{1/3} \partial_r v_r\|_2 \|r^{1/2} v_r\|_2 \|v_r\|_{L^6} d\tau
\]

\[
\leq C \sup_{0 \leq \tau \leq t} \|r^{1/2} v_r\|_2 \int_0^t \|r^{1/3} \partial_r v_r\|_2 \|v_r\|_{L^6} d\tau
\]

\[
\leq C N(t)^{3/2}
\]

(3.12)

and

\[
\int_0^t \int_\Omega \partial_r Q(\bar{\rho}, \phi) \rho v_r dr dz d\tau
\]

\[
\leq C \int_0^t \int_\Omega \left(|\phi \partial_r (\bar{\rho}^{-2} \phi) v_r| + |\phi^2 \partial_r \rho v_r|\right) dr dz d\tau
\]

\[
\leq C \int_0^t \|r^{1/3} \phi\|_3 \|r^{1/2} \partial_r (\bar{\rho}^{-2} \phi)\|_2 \|r^{1/6} v_r\|_6 d\tau + \int_0^t \|\phi\|_3 \|r^{1/6} \bar{\rho}^{-2} \phi\|_6 \|v_r\|_2 d\tau
\]
Analogously, Combining (3.9) and (3.11)-(3.15), we eventually obtain

$$N \leq CN(t)^{3/2},$$  \hspace{1cm} (3.13)

here we point out that we have used the crucial fact of $\partial_r \bar{\rho} = O(r^{-3})$ in (3.13), and $\|v_r\|_{L^6} \leq C \|Dv_r\|_{L^2} = C \|r^{1/2}Dv_r\|_2$ by Lemma 3.1(ii) with $j = 0, p = 6$ and $k = 1$.

On the other hand, it follows from direct computation that

$$\int_0^t \int_\Omega \left( \frac{\phi}{(\phi + \bar{\rho})} \left( \nu_1 (\partial_r (\frac{1}{r} \partial_r (rv_r)) + \partial_z^2 v_r) + \nu_2 \partial_r (\frac{1}{r} \partial_r (rv_r) + \partial_z v_z) \right) \right) r \bar{\rho} v_r dr dz d\tau$$

$$\leq C \int_0^t \int_\Omega \left( \frac{\phi}{(\phi + \bar{\rho})} \left( |rv_r v_r| + |\sqrt{r} \sqrt{r} D^2 v_r v_r| \right) dr d\tau$$

$$\leq CN(t)^{3/2}. \hspace{1cm} (3.14)$$

Similarly to (3.13), we arrive at

$$\int_0^t (f, \bar{\rho} \gamma^2 r \phi) dr d\tau = - \int_0^t \int_\Omega \left( \frac{1}{r} \partial_r (r \phi v_r) + \partial_z (\phi v_z) \right) \bar{\rho} \gamma^2 r \phi dr dz d\tau$$

$$= \int_0^t \int_\Omega \left( r \phi v_r \partial_r (\bar{\rho} \gamma^2 \phi) + r \phi v_r \partial_z (\bar{\rho} \gamma^2 \phi) \right) dr dz d\tau$$

$$\leq C \int_0^t \int_\Omega \left( |r \phi^2 v_r \partial_r \bar{\rho}| + |r \phi v_r D (\bar{\rho} \gamma^2 \phi) | \right) dr dz d\tau$$

$$\leq CN(t)^{3/2}. \hspace{1cm} (3.15)$$

Combining (3.9) and (3.11)-(3.15), we eventually obtain

$$\int_0^t A_1 d\tau \leq CN(t)^{3/2}. \hspace{1cm} (3.16)$$

Analogously, $A_i \ (i = 2, 3)$ and the terms such as $\|g\|_{H^1}$, $\|f\|_{H^1}$ and $(D^2 g, \bar{\rho} r D^2 v)$ can be treated like $A_1$. For examples, we treat the terms $(\partial_t g_3, \bar{\rho} r \partial_t^2 v_z)$ and $(r D^2 \bar{\rho} \gamma^2 f, D^2 (\bar{\rho} \gamma^2 f))$ in $A_3$. Note that

$$\int_0^t (\partial_t g_3, \bar{\rho} r \partial_t^2 v_z) dr d\tau$$

$$= \int_0^t \int_\Omega \left( \partial_t \left( -v_r \partial_r v_z - v_z \partial_z v_z - \partial_z Q(\bar{\rho}, \phi) - \frac{\nu_1 \phi}{(\phi + \bar{\rho})} \right) \partial_t^2 v_z + \partial_z^2 v_z + \frac{1}{r} \partial_r v_r ight)$$

$$\leq C \int_0^t \int_\Omega \left( \left( \|v_r\|_{L^2} \|\partial_r v_z\|_{\infty} + \|r^{1/2} \partial_r^2 v_z\| \|v_r\|_{L^2} + \|r^{1/2} \partial_t^2 v_z\|_2 \right) dr d\tau$$

$$\leq CN(t)^{3/2}. \hspace{1cm} (3.17)$$
By Lemma 3.1 (i) with $j = 0$, $p = 4$, we can obtain
\begin{align*}
\int_0^t \int_\Omega \partial_t^2 Q(\bar{\rho}, \phi) \bar{\rho} r \partial_r^2 v_z dr dz d\tau \\
\leq C \int_0^t \left( \|r^{1/2} \partial_r^2 \phi\|_2 \|\phi\|_\infty + \|r^{1/2} \partial_t \phi\|_4 \|\partial_z \phi\|_4 \right) \|r^{1/2} \partial_t^2 v_z\|_2 d\tau \\
\leq C \left( \|\bar{\rho}^{\gamma-2} \phi\|_{\dot{H}^1} + \|\bar{\rho}^{\gamma-2} \partial_t \phi\|_{\dot{H}^1} \right) \int_0^t \left( \|r^{1/2} \partial_t \phi\|_{\dot{H}^1} + \|r^{1/2} \partial_z \phi\|_2 \right) \|r^{1/2} \partial_t^2 v_z\|_2 d\tau \\
\leq CN(t)^{3/2}.
\end{align*}

By Lemma 3.1 (i) with $j = 1$, $p = 4$ and $j = 0$, $p = 4$, one has
\begin{align*}
\int_0^t \int_\Omega \partial_t \left\{ - \frac{\phi}{(\phi + \bar{\rho})^2} \left( \nu_1 \left( \partial_r^2 v_z + \partial_z^2 v_z + \frac{1}{r} \partial_r v_r \right) + \nu_2 \partial_z \left( \frac{1}{r} \partial_r (r v_r) + \partial_z v_z \right) \right) \right\} \bar{\rho} r \partial_r^2 v_z dr dz d\tau \\
\leq C \int_0^t \left( \|\partial_t \phi\|_4 \|Dv\|_{\dot{H}^2} + \|\phi\|_\infty \|\partial_t v\|_{\dot{H}^2} \right) \|r^{1/2} \partial_t^2 v_z\|_2 d\tau \\
\leq CN(t)^{3/2}.
\end{align*}

Then collecting (3.17)-(3.19) yields
\begin{align*}
\int_0^t \left( \partial_t g_3, \bar{\rho} r \partial_r^2 v_z \right) d\tau &\leq CN(t)^{3/2}.
\end{align*}

In the end, we deal with the term \( \int_0^t \langle r D^2(\rho^{\gamma-2} \phi), D^2(\bar{\rho}^{\gamma-2} f) \rangle d\tau \). We see that
\begin{align*}
\int_0^t \langle r D^2(\rho^{\gamma-2} \phi), D^2(\bar{\rho}^{\gamma-2} f) \rangle d\tau \\
= \int_0^t \int_\Omega r D^2(\rho^{\gamma-2} \phi) D^2(\bar{\rho}^{\gamma-2} \left( \frac{1}{r} \partial_r (r \phi v_r) - \partial_z (\phi v_z) \right)) dr dz d\tau.
\end{align*}

In addition, we observe that
\begin{align*}
D^2 \left( \rho^{\gamma-2} \left( \frac{1}{r} \partial_r (r \phi v_r) - \partial_z (\phi v_z) \right) \right) \\
= -v_r \partial_r (D^2(\rho^{\gamma-2} \phi)) - v_z \partial_z (D^2(\rho^{\gamma-2} \phi)) + \mathcal{R}
\end{align*}

with
\begin{align*}
\mathcal{R} = D^2 \left( \rho^{\gamma-2} \left( \frac{1}{r} \partial_r (r \phi v_r) - \partial_z (\phi v_z) \right) \right) + v_r \partial_r (D^2(\rho^{\gamma-2} \phi)) + v_z \partial_z (D^2(\rho^{\gamma-2} \phi)).
\end{align*}

It follows from direct computation that
\begin{align*}
\int_0^t \int_\Omega r D^2(\rho^{\gamma-2} \phi) \mathcal{R} dr dz d\tau
\end{align*}
\[ \leq C \left( \| r^{1/2} D^2 ( \rho^{-2} \phi ) \|_2 + \| v \|_{H^1} \right) \int_0^t \left( \| r^{1/2} D^2 ( \rho^{-2} \phi ) \|_2 + \| Dv \|_{H^1} \right) \| Dv \|_{H^3} d\tau \]
\[ \leq CN(t)^{3/2}. \quad (3.22) \]

For the remainder term in (3.21), one sees that
\[
\left| \int_0^t \int_0^t \int_0^1 \int_0^1 \left( \| r^{1/2} D^2 ( \rho^{-2} \phi ) \right) \left( -v_r \partial_r (D^2 ( \rho^{-2} \phi )) - v_z \partial_z (D^2 ( \rho^{-2} \phi )) \right) d r d z d \tau \right| \\
= \frac{1}{2} \left| \int_0^t \int_0^1 \left( (D^2 ( \rho^{-2} \phi ))^2 (\partial_z (r v_r) + r \partial_z v_z) \right) d r d z d \tau \right| \\
\leq CN(t)^{3/2}. \quad \text{(Analogously as in (3.22))} \quad (3.23) \]

Thus
\[ \int_0^t (r D^2 ( \rho^{-2} \phi ), D^2 (\rho^{-2} f)) d \tau \leq CN(t)^{3/2}. \quad (3.24) \]

Consequently, by (3.8), (3.16), (3.20), (3.23)-(3.24), and analogous treatments for \( A_1 \), we obtain that
\[ \sum_{i=1}^3 \left| \int_0^t A_i d \tau \right| \leq CN(t)^{3/2} \quad (3.25) \]

and
\[ \sup_{0 \leq \tau \leq t} \left( \| f \|_{H^1}^2 + \| g \|_{H^1}^2 \right) \leq CN(t)^{3/2}. \quad (3.26) \]

Substituting (3.25)-(3.26) into (3.8) yields the proof of Proposition 3.1. \[ \blacksquare \]

Based on Proposition 3.1, we now start to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 3.1, when \( \| v_0 \|_{H^3}^2 + \| \rho_0 \|_{H^2}^2 \leq \varepsilon^2 \) and \( \varepsilon > 0 \) is small, then \( N(t) \leq C \varepsilon^2 \) holds uniformly for any \( t \geq 0 \). This, together with the local existence of classical solution to (1.13)-(1.16) with (1.17)-(1.18) (one can see [20]) and continuity argument, yields the global solution of problem (1.13)-(1.16) with (1.17)-(1.18). Thus the proof of Theorem 1.1 is completed. \[ \blacksquare \]

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