THE EARLY EVOLUTION OF THE RANDOM GRAPH PROCESS IN PLANAR GRAPHS AND RELATED CLASSES

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ABSTRACT. We study the random planar graph process introduced by Gerke, Schlatter, Steger, and Taraz [The random planar graph process, Random Structures Algorithms 32 (2008), no. 2, 236-261; MR2387559]: Begin with an empty graph on \( n \) vertices, consider the edges of the complete graph \( K_n \) one by one in a random ordering, and at each step add an edge to a current graph only if the graph remains planar. They studied the number of edges added up to step \( t \) for ‘large’ \( t = \omega(n) \). In this paper we extend their results by determining the asymptotic number of edges added up to step \( t \) in the early evolution of the process when \( t = O(n) \). We also show that this result holds for a much more general class of graphs, including outerplanar graphs, planar graphs, and graphs on surfaces.

1. INTRODUCTION AND RESULTS

1.1. Motivation. Erdős and Rényi [8,9] introduced the classical random graph process \( \{G_{n,t}\}_{t=0}^{N} \), where one starts with an empty graph on vertex set \( [n] := \{1, \ldots, n\} \) and adds the \( \binom{n}{2} \) many edges of the complete graph \( K_n \) one after another in a random order. Since then, many exciting results on \( G_{n,t} \) have been obtained (see e.g., [5,12,16] for an overview), and \( G_{n,t} \) is also known as the Erdős-Rényi random graph process, because it has the same distribution as the uniform random graph on \( [n] \) with exactly \( t \) edges.

A variant of the Erdős-Rényi random graph process is the \( \mathcal{P} \)-constrained random graph process, where an edge is added only when a certain graph property \( \mathcal{P} \) is preserved. More formally, given \( n \in \mathbb{N} \) and a graph property \( \mathcal{P} \), i.e., a class of graphs with specific properties, we choose a random ordering \( e_1, \ldots, e_N \) of the edges of the complete graph \( K_n \). Then we let \( P_{n,0} \) be the empty graph on vertex set \( [n] \). For \( t \in [N] \), we set \( P_{n,t} = P_{n,t-1} + e_t \) and say that \( e_t \) is accepted if \( P_{n,t-1} + e_t \in \mathcal{P} \); otherwise, we set \( P_{n,t} = P_{n,t-1} \) and say that \( e_t \) is rejected. Furthermore, we say that the edge \( e_t \) is queried at step \( t \). We denote by \( \delta(t) := t - e(P_{n,t}) \) the number of rejected edges.

Prominent examples of a graph property \( \mathcal{P} \) for which the \( \mathcal{P} \)-constrained random graph process has been extensively studied include triangle-freeness [2,10,11], and more generally \( H \)-freeness for a fixed graph \( H \) [8,16,25], or having bounded maximum degree [28,29]; these are all ‘local’ properties. More ‘global’ properties have also been considered, such as planarity [13], \( k \)-colourability [21], \( k \)-matching-freeness [19], and the König property [17]. Most of the obtained results are on properties of the final graph \( P_{n,N} \) and much less is known about the ‘evolution’ of these processes.

Gerke, Schlatter, Steger, and Taraz [13] considered the \( \mathcal{P} \)-constrained random graph process for the property \( \mathcal{P} \) of being planar. Among other interesting results, they showed the following.

Theorem 1.1 ([13 Theorem 1.1]). Let \( \mathcal{P} \) be the class of planar graphs and \( \{P_{n,t}\}_{t=0}^{N} \) the \( \mathcal{P} \)-constrained random graph process. For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[ \mathbb{P}[e(P_{n,\delta n}) \leq (1 + \varepsilon) n] < e^{-n}. \]
Theorem 1.1 immediately implies the following result on the asymptotic number of edges accepted until a superlinear step. Throughout the paper, we will use standard Landau notation for asymptotics and all the asymptotics are taken as \( n \to \infty \). We say that an event holds with high probability (whp for short) if it holds with probability tending to one as \( n \to \infty \).

**Corollary 1.2.** Let \( \mathcal{P} \) be the class of planar graphs. Let \( \{P_{n,t}\}_{t=0}^N \) be the \( \mathcal{P} \)-constrained random graph process and \( t = t(n) \in [N] \) be such that \( n \ll t \ll n^2 \). Then whp \( e(P_{n,t}) = (1 + o(1)) n \).

We note that the upper bound in Corollary 1.2, i.e., \( e(P_{n,t}) \leq (1 + o(1)) n \), follows directly from Theorem 1.1. Furthermore, it is well known that whp the largest component of \( G_{n,t} \) has \( (1 + o(1)) n \) vertices if \( t \gg n \) (see e.g., [9]). Together with the simple fact that the number of vertices in the largest components of \( G_{n,t} \) and \( P_{n,t} \) coincide (see Remark 2.5), this implies the lower bound on \( e(P_{n,t}) \) in Corollary 1.2.

Gerke, Schlatter, Steger, and Taraz asked the asymptotic behaviour of \( e(P_{n,t}) \) in the earlier stage when \( t = O(n) \). In this paper we answer this question in a more general setting: We determine \( e(P_{n,t}) \) in the case \( t = O(n) \) for a wide range of graph classes, including outerplanar graphs, planar graphs, and graphs on surfaces (see Theorem 1.7).

Gerke, Schlatter, Steger, and Taraz also studied a random graph \( P_{n,m=m_0} \), which is a graph obtained from the \( \mathcal{P} \)-constrained random graph process when \( m_0 \) many edges have actually been accepted: In other words,

\[
P_{n,m=m_0} = P_{n,t_0}, \quad \text{where} \quad t_0 := \min \{t | e(P_{n,t}) = m_0\}.
\]

Equivalently, \( P_{n,m=m_0} \) can be obtained by the so-called random greedy process. There we start with an empty graph on \( n \) vertices and in each step we add an edge chosen uniformly at random from those which are not yet in the graph and do not violate the property \( \mathcal{P} \). They showed that in the ‘dense’ regime when \( m_0 = cn/2 \) for \( 2 < c < 6 \), whp \( P_{n,m=m_0} \) is connected.

**Theorem 1.3 (Theorem 1.2)).** Let \( \mathcal{P} \) be the class of planar graphs and \( P_{n,m=m_0} \) be as defined in 1.1. If \( m_0 = m_0(n) \) is such that \( m_0 = cn/2 \) for \( 2 < c < 6 \), then whp \( P_{n,m=m_0} \) is connected.

Another well-known random graph model is the uniform random planar graph. More generally, let \( P(n,m_0) \) denote a graph that is chosen uniformly at random from all graphs in \( \mathcal{P} \) having vertex set \( [n] \) and \( m_0 \) edges. A classical question is whether or not a random graph \( P_{n,m=m_0} \) ‘behaves’ like the uniform random graph \( P(n,m_0) \). For example, Giménez and Noy [14] showed that the probability that the uniform random planar graph \( P(n,m_0) \) is connected is bounded away from one if \( m_0 = cn/2 \) for \( 2 < c < 6 \), i.e., a statement as in Theorem 1.3 is not true for \( P(n,m_0) \). A consequence of our results will be that the largest components of the two random graphs \( P_{n,m=m_0} \) and \( P(n,m_0) \) behave ‘differently’ also in the sparse regime when \( m_0 = cn/2 \) for \( 1 < c < 2 \) (see [9]).

Another natural property which was considered in [13] is the number of edges one has to query until \( m_0 \) of them have been accepted. We denote this number by \( q(m_0) \), i.e.,

\[
q(m_0) := \min \{t | e(P_{n,t}) = m_0\}.
\]

Gerke, Schlatter, Steger, and Taraz [13] asked the order of \( q(3n/4) \) when \( \mathcal{P} \) is the class of planar graphs. In Corollary 1.9 we will determine \( q(3n/4) \) as a special case of our general result.

**1.2. Main results.** In the following definition we extract the properties of planar graphs which are essential for our proof, but are satisfied by other well-known classes of graphs that can be characterised by ‘forbidden minors’ (see Proposition 1.5).

**Definition 1.4.** Throughout the paper, let \( \mathcal{P} \) be a class of graphs fulfilling the following properties:

(a) it is not equal to the class of all graphs;
(b) it contains all edgeless graphs;
(c) it is closed under taking isomorphism;
(d) it is closed under taking minors;
(e) it is weakly addable, i.e., it is closed under adding an edge between two components;
(f) it is closed under adding an edge in a tree component.

Throughout the paper, we consider only vertex-labelled simple undirected graphs. It is straightforward to check the properties in Definition 1.4 for the following general class of graphs.

**Proposition 1.5.** For any \( r \in \mathbb{N} \) let \( H_1, \ldots, H_r \) be 2-edge-connected graphs that contain at least two cycles. Then the class \( \mathcal{H} \) of all graphs that contain none of \( H_1, \ldots, H_r \) as a minor fulfils the properties (a)–(f) in Definition 1.4.

Prominent examples for the class \( \mathcal{H} \) in Proposition 1.5 are the following:
- the class of all cactus graphs (\( H_1 = \text{‘diamond graph’}, \) that is, \( K_4 \) minus one edge);
- the class of all outerplanar graphs (\( H_1 = K_4, H_2 = K_{2,3} \));
- the class of all series-parallel graphs (\( H_1 = K_4 \));
- the class of all planar graphs (\( H_1 = K_5, H_2 = K_{3,3} \));
- the class of all graphs embeddable on an orientable surface of genus \( g \in \mathbb{N} \) (only the existence of graphs \( H_1, \ldots, H_r \) is known, see [27]).

To state our main results, we need also the following definition.

**Definition 1.6.** Given \( c > 0 \) let \( \beta(c) \) be the unique positive solution of the equation \( 1 - x = e^{-cx} \) and define
\[
f(x) := 2\beta(c) + c(1 - \beta(c))^2.
\]
Denote by \( f^{-1} \) the inverse function of \( f \).

Note that \( \beta(c) \) is equal to the survival probability of a Galton-Watson process with offspring distribution Poisson with mean \( c \). Basic properties of the function \( f : (1, \infty) \to (1, 2) \), including the existence of the inverse function \( f^{-1} \), can be found in Lemma A.1.

In the following theorem we provide the asymptotic order of the number of accepted edges \( e(P_{n,t}) \) when \( t = O(n) \) for any class of graphs \( \mathcal{P} \) satisfying the properties in Definition 1.4. As \( e(P_{n,t}) \) is ‘quite close’ to \( t \) in this early stage of the evolution, it is more convenient to state the asymptotic order of the number of rejected edges \( r(t) = t - e(P_{n,t}) \) instead of \( e(P_{n,t}) \).

**Theorem 1.7.** Let \( \mathcal{P} \) be a class of graphs satisfying the properties (a)–(f) in Definition 1.4 and \( (P_{n,t})_{t=0}^N \) be the \( \mathcal{P} \)-constrained random graph process. Let \( h = h(n) = o(1) \) be a function which tends to \( \infty \) arbitrarily slowly as \( n \to \infty \). Let \( t = t(n) \in [N] \) and \( s = s(n) \). Then
\[
r(t) = \begin{cases} 0 & \text{if } t = n/2 - s \text{ for } s \gg n^{2/3}; \\
O(h) & \text{if } t = n/2 + s \text{ for } s = O(n^{2/3}); \\
\Theta(s^3/n^2) & \text{if } t = n/2 + s \text{ for } n^{2/3} \ll s \ll n; \\
(c - f(c) + o(1))n/2 & \text{if } t = cn/2 \text{ for } c > 1.
\end{cases}
\]

When \( t = n/2 + s \) for \( s = O(n^{2/3}) \), the statement that whp \( r(t) = O(h) \) can be equivalently formulated as follows: For each \( \lambda > 0 \) there exists a \( C = C(\lambda) \) such that \( \mathbb{P}(r(t) < C) > 1 - \lambda \) for all sufficiently large \( n \in \mathbb{N} \), i.e., the statement is ‘slightly weaker’ than having whp \( r(t) = O(1) \).

In the case \( t = cn/2 \) for \( c > 1 \) the statement that whp \( r(t) = (c - f(c) + o(1))n/2 \) can be simplified to whp \( e(P_{n,t}) = (f(c) + o(1))n/2 \). Using Theorem 1.7, we obtain the following nice, alternative description of a \( \mathcal{P} \)-constrained random graph process: In the very early stage of the process when \( t \leq n/2 + o(n) \), ‘almost’ all edges are accepted. More formally, whp the ‘next’ edge \( e_{t+1} \) will be accepted as long as \( t \leq n/2 + o(n) \). However, this changes when \( t = cn/2 \) for \( c > 1 \). The acceptance mainly depends whether or not both of the endpoints of \( e_{t+1} \) lie in the largest component of \( P_{n,t} \).

**Corollary 1.8.** Let \( \mathcal{P} \) be a class of graphs satisfying the properties (a)–(f) in Definition 1.4 and \( (P_{n,t})_{t=0}^N \) be the \( \mathcal{P} \)-constrained random graph process. Let \( t = t(n) \in [N] \) be such that \( t = cn/2 \) for a constant \( c > 1 \), and let \( L = L(P_{n,t}) \) denote the largest component of \( P_{n,t} \). Then whp the following hold.
Figure 1. The fraction \(e(P_{n,t})/t\) of edges that are accepted up to step \(t = cn/2\) (solid line) and the fraction \(r(t)/t\) of rejected edges (dotted line). For \(c > 1\) we have \(e(P_{n,t})/t \sim f(c)/c\) and \(r(t)/t \sim 1 - f(c)/c\).

(a) If \(e_{t+1} \subseteq V(L)\), then \(e_{t+1}\) is rejected.
(b) If \(e_{t+1} \nsubseteq V(L)\), then \(e_{t+1}\) is accepted.

Using Theorem \[1.7\] we can determine the asymptotic number of queried edges \(q(m_0)\) until \(m_0\) of them have been accepted in the case \(m_0 = cn/2\) for \(1 < c < 2\). In particular, this answers the open problem on \(q(3n/4)\) for the property \(\mathcal{P}\) of being planar from [13].

Corollary 1.9. Let \(\mathcal{P}\) be a class of graphs satisfying the properties (a)-(f) in Definition 1.4 and \(q(m_0)\) be as defined in (2). If \(m_0 = cn/2\) for a constant \(c \in (1,2)\), then whp \(q(m_0) = (f^{-1}(c) + o(1))n/2\).

In particular, whp \(q(3n/4) = (f^{-1}(3/2) + o(1))n/2\), where \(f^{-1}(3/2) = 1.6188\ldots\).

We note that Corollary 1.9 follows directly from Theorem 1.7 and the observation that \(f\) is strictly increasing (see Lemma [1.10]).

Our next main result provides the asymptotic order of the largest component of \(P_{n,m = m_0}\).

Theorem 1.10. Let \(\mathcal{P}\) be a class of graphs satisfying the properties (a)-(f) in Definition 1.4 and \((P_{n,t})_{t=0}^N\) be the \(\mathcal{P}\)-constrained random graph process. Let \(m_0 = m_0(n) \in [N]\) and \(s = s(n)\). Let \(P_{n,m = m_0}\) be defined as in (1) and let \(v(L(P_{n,m = m_0}))\) denote the number of vertices in the largest component of \(P_{n,m = m_0}\). Then whp

\[
v(L(P_{n,m = m_0})) = \begin{cases} 
\Theta(n) & \text{if } m_0 = cn/2 \text{ for } c < 1; \\
(1/2 + o(1))n^2/s^2 \log(s^3/n^2) & \text{if } m_0 = n/2 - s \text{ for } n^{2/3} < s < n; \\
(2/3)n^2/\Theta^2(s) & \text{if } m_0 = n/2 + s \text{ for } s = O(n^{2/3}); \\
(4 + o(1))s & \text{if } m_0 = n/2 + s \text{ for } n^{2/3} < s < n; \\
(\beta(f^{-1}(c) + o(1)))n & \text{if } m_0 = cn/2 \text{ for } 1 < c < 2; \\
(2 + o(1))n & \text{if } m_0 = cn/2 \text{ for } c = 2; \\
n & \text{if } m_0 = cn/2 \text{ for } c > 2.
\end{cases}
\]

For the property \(\mathcal{P}\) of being planar Theorem 1.10 reveals a different behaviour of \(P_{n,m = m_0}\) in the 'sparse' regime than that of the uniform random planar graph \(P(n,m_0)\). More formally, if \(m_0 = cn/2\) for \(1 < c < 2\), then whp,

\[
v(L(P_{n,m = m_0})) = (\beta(f^{-1}(c) + o(1)))n > (c - 1 + o(1))n = v(L(P(n,m_0))),
\]

where the last equality follows from \[18\]. We refer to Figure 2 for an illustration of \(v(L(P_{n,m = m_0}))\) and \(v(L(P(n,m_0)))\).

1.3. Outline of the paper. The rest of the paper is structured as follows. After providing the necessary definitions and concepts in Section 2, we prove Theorem 1.7 in Section 3. In Section 4 we use so-called ‘addable’ and ‘forbidden’ edges to prove Corollary 1.8. Finally in Section 5 we prove Theorem 1.10.
2. Preliminaries

2.1. Notations for graphs. We begin with some notations for graphs that will be used in the rest of the paper.

Definition 2.1. Given a graph $H$ we denote by

- $V(H)$ the vertex set of $H$ and $\nu(H)$ the order of $H$, i.e., the number of vertices in $H$;
- $E(H)$ the edge set of $H$ and $e(H)$ the size of $H$, i.e., the number of edges in $H$;
- $\Delta(H)$ the maximum degree of $H$;
- $L(H)$ the largest component of $H$;
- $C(H)$ the 2-core of $H$, which is the maximal subgraph of $H$ with minimum degree at least two;
- $\text{ex}(H) := e(H) - \nu(H) + \#t(H)$ the excess of $H$, where $\#t(H)$ is the number of tree components in $H$.

Definition 2.2. Given a class $\mathcal{P}$ of graphs, we write $\mathcal{P}(n)$ for the subclass of $\mathcal{P}$ containing the graphs on vertex set $[n]$ and $\mathcal{P}(n, m)$ for the subclass of $\mathcal{P}$ containing the graphs on vertex set $[n]$ with $m$ edges, respectively.

2.2. Properties of the Erdős-Rényi random graph. In this section we state the properties of the Erdős-Rényi random graph $G_{n,t}$ which we will use in our proofs. First we consider the case $t = n/2 + s$ for $n^{2/3} \ll s \ll n$ and then the case $t = cn/2$ for $c > 1$.

Theorem 2.3 (\cite{4,22,23,26}). Let $t = t(n) \in [N]$ and $s = s(n)$ be such that $t = n/2 + s$ for $n^{2/3} \ll s \ll n$ and $L = L_G(n,n,t)$ be the largest component of the Erdős-Rényi random graph $G_{n,t}$. Furthermore, let $C = C(L)$ be the 2-core of $L$ and $F$ be the forest obtained from $L$ by deleting the edges of $C$. Then whp

(a) $\nu(L) = (4 + o(1)) s$;
(b) $\text{ex}(G_{n,t}) = \Theta(s^2/n^2)$;
(c) each tree component of $F$ is of order $o(s)$;
(d) $\Delta(G_{n,t}) = (1 + o(1)) \log n/\log\log n$;
(e) $\Delta(C) = 3$ if in addition $s \ll n^{3/4}$.

We note that (a) and (b) are shown in \cite{22,4}, (d) in \cite{4}, and (c) in \cite{23}, respectively. Furthermore, (c) follows by the fact from \cite{26} that conditioned on fixed values of $\nu(L)$ and $\nu(C)$, whp all tree components of $F$ are of order $o(\nu(L))$ as long as $\nu(C)^2 = \omega(\nu(L))$. Furthermore, whp $G_{n,t}$ satisfies this condition, because whp $\nu(C) = \Theta(s^2/n)$ by \cite{23} and $\nu(L) = (4 + o(1)) s$ by (a).

Next we collect some properties of the Erdős-Rényi random graph $G_{n,t}$ when $t = cn/2$ for $c > 1$. 
**Theorem 2.4 ([9]).** Let \( t = t(n) \in [N] \) be such that \( t = cn/2 \) for \( c > 1 \). Let \( G = G_{n,t} \) be the Erdős-Rényi random graph, and \( L = L(G) \) the largest component of \( G \). Furthermore, let \( \beta(c) \) and \( f(c) \) be as in Definition [7,6]. Then whp (see e.g., [24]).

(a) \( v(L) = (\beta(c) + o(1))n \);
(b) all components of \( G \) apart from \( L \) are of order \( o(n) \);
(c) \( \text{ex}(G) = (c - f(c) + o(1))n/2 \).

2.3. **Properties of \( P_{n,t} \) and \( \mathcal{P} \).** We will often use the following simple observation.

**Remark 2.5.** Due to properties (b) and (e) of Definition [1,4] there is a path between two vertices in \( P_{n,t} \) if and only if there is one in \( G_{n,t} \).

Next, we show that the number of rejected edges up to step \( t \) is bounded above by the excess of \( G_{n,t} \). This will be a main ingredient to obtain the upper bounds in Theorem 1.7.

**Lemma 2.6.** For all \( t \in [N] \) we have

\[ r(t) \leq \text{ex}(G_{n,t}). \]

**Proof.** We consider an edge \( e_i \) which is rejected in the \( \mathcal{P} \)-constrained random graph process. By Definition [1,4] the two endpoints of \( e_i \) lie in the same component of \( P_{n,t-1} \), which is not a tree component. Together with the fact \( P_{n,t-1} \subseteq G_{n,t-1} \) it implies that adding \( e_i \) to \( G_{n,t-1} \) increases the excess by one, i.e., \( \text{ex}(G_{n,t}) = \text{ex}(G_{n,t-1}) + 1 \). As \( \text{ex}(G_{n,t}) \) is non-decreasing in \( t \), this implies the statement.

A graph class \( \mathcal{A} \) for which there exists a constant \( c > 0 \) such that \( |\mathcal{A}(n)| \leq n!c^n \) for all \( n \in \mathbb{N} \) is often called small (see e.g., [24]). The following statement shows that the class \( \mathcal{P} \) in Definition [1,4] is small.

**Theorem 2.7 ([24]).** Let \( \mathcal{P} \) be a class of graphs satisfying the properties (a),(b),(c), and (d) in Definition [1,4]. Then there exists a constant \( c > 0 \) such that \( |\mathcal{P}(n)| \leq n!c^n \) for all \( n \in \mathbb{N} \).

2.4. **Decomposition of graphs.** In the proof of Theorem 1.7 we will split the largest component of \( P_{n,t} \) into connected parts of roughly equal size. To that end, we will use the following lemma, which is an extension of [20] Proposition 4.5) to vertex-weighted graphs.

**Lemma 2.8.** Let \( H \) be a connected graph with maximum degree at most \( \Delta \geq 1 \). We assign each vertex \( x \in V(H) \) a vertex-weight \( w(x) > 0 \). Assume that \( \max_{x \in V(H)} w(x) \leq M \) for some \( M > 0 \). Then, given \( a > 0 \) there exist disjoint vertex sets \( V_1, \ldots, V_r \subseteq V(H) \) such that

- \( H[V_i] \) is connected for each \( i \in [r] \);
- \( a \leq W_i \leq a\Delta + M \) for each \( i \in [r] \);
- \( W(H) - \sum_{i=1}^r W_i < a \).

where \( W(H) := \sum_{x \in V(H)} w(x) \) denotes the total vertex-weight of \( H \) and \( W_i := \sum_{x \in V_i} w(x) \) the total vertex-weight of \( V_i \), respectively.

**Proof.** We proceed by induction on \( v(H) \). For \( v(H) = 1 \) let \( x \) denote the single vertex in \( H \). We set \( V_1 = \{x\} \) if \( w(x) \geq a \) and let \( r = 0 \) otherwise.

Now assume \( v(H) > 1 \). If \( W(H) < a \), we set \( r = 0 \). Assume otherwise that \( W(H) \geq a \). We perform a breadth-first search (BFS) starting at some arbitrary vertex \( y \). For each vertex \( x \in V(H) \) let \( D_x \) be the set of vertices consisting of \( x \) and all descendants of \( x \) in the BFS-tree and let \( W(x) := \sum_{u \in D_x} w(u) \) be the total vertex-weight of \( D_x \). Let \( z \in V(H) \) be a vertex such that \( W(z) \) is minimal among all vertices \( x \) with \( W(x) \leq a \). We note that such a vertex exists, as \( W(y) = W(H) \geq a \). Let \( z_1, \ldots, z_k \) be the neighbours of \( z \) that are contained in \( D_z \). By minimality of \( z \) and the fact \( W(z_i) < W(z) \) we have \( W(z_i) < a \) for all \( i \in [k] \). Thus, we get \( W(z) = \sum_{i=1}^k w(z_i) + w(z) \leq a\Delta + M \). Therefore, we choose \( V_1 = D_z \). By construction \( H[V_1] \) is connected and we have \( a \leq W_1 \leq a\Delta + M \). Furthermore, the graph obtained from \( H \) by deleting \( V_1 \) is connected and has less vertices than \( H \). Hence, we can apply the induction hypothesis to this graph to obtain the remaining sets \( V_2, \ldots, V_r \) of our desired decomposition.

We will show that if we have a ‘suitable’ decomposition of \( P_{n,t} \), we cannot add too ‘many’ further edges without creating a minor of the complete graph \( K_f \) for an appropriate \( f \in \mathbb{N} \). To that end, we will use the following lemma, which is a ‘weighted’ version of the well-known Turán’s theorem.
Lemma 2.9. For fixed $n \in \mathbb{N}$ we assign a vertex-weight $w(x) > 0$ to each vertex $x$ of $K_n$. Furthermore, we define the edge-weight $w(xy) := w(x)w(y)$ for each edge $xy \in E(K_n)$. Let $S_n := \sum_{x \in V(K_n)} w(x)$ be the total vertex-weight of $K_n$ and $M_n := \max_{x \in V(K_n)} w(x)$ be the maximum vertex-weight of $K_n$. For each subgraph $H \subseteq K_n$ denote by $w(H) := \sum_{e \in E(H)} w(e)$ the total edge-weight of $H$. Then for each $\ell \geq 2$ we have
\[
  w(K_n) - \min \{ w(H) \mid H \subseteq K_n, K_\ell \not\subseteq H \} \geq \frac{S_n}{2} \left( \frac{S_n}{\ell - 1} - M_n \right).
\]

Proof. To ease notation, let $A_n := \max \{ w(H) \mid H \subseteq K_n, K_\ell \not\subseteq H \}$. In [1] Bennett, English, and Taland-Fisher showed that
\[
  A_n = \frac{1}{2} \cdot \max_{Q \not\in \mathcal{Q}_\ell \cdot 2} \sum_{Q \in \mathcal{Q}_\ell \cdot 2} W(Q)W(Q'),
\]
where the maximum is taken over all partitions $\mathcal{Q}$ of $V(K_n)$ into $\ell - 1$ parts and $W(Q) := \sum_{x \in Q} w(x)$ denotes the total vertex-weight of $Q \in \mathcal{Q}$. Now let $\mathcal{Q}^\ast$ be some partition for which the maximum is attained. Note that $\sum_{Q \in \mathcal{Q}_\ell \cdot 2} W(Q) = \sum_{x \in V(K_n)} w(x) = S_n$. Furthermore, we have
\[
  A_n = \frac{1}{2} \sum_{Q \not\in \mathcal{Q}_\ell \cdot 2} W(Q)W(Q') = \frac{1}{2} \left( \sum_{Q \in \mathcal{Q}_\ell} W(Q) \right)^2 - \frac{1}{2} \sum_{Q \in \mathcal{Q}} W(Q)^2 = \frac{S_n^2}{2} - \frac{1}{2} \sum_{Q \in \mathcal{Q}^\ast} W(Q)^2.
\]
Using the 'AM-QM inequality', in other words, $\sum_{Q \in \mathcal{Q}^\ast} W(Q)^2 \geq \frac{1}{(\ell - 1)} \left( \sum_{Q \in \mathcal{Q}^\ast} W(Q) \right)^2$, we obtain
\[
  A_n \leq \frac{S_n^2}{2} - \frac{1}{2} \cdot \frac{1}{\ell - 1} \left( \sum_{Q \in \mathcal{Q}^\ast} W(Q) \right)^2 = \frac{S_n^2}{2} \left( 1 - \frac{1}{\ell - 1} \right).
\]
Finally, the total edge-weight of $K_n$ satisfies
\[
  w(K_n) = \sum_{e \in E(K_n)} w(e) = \frac{1}{2} \sum_{x \not\in \mathcal{V}(K_n)} w(x)w(y) = \frac{S_n^2}{2} - \frac{1}{2} \sum_{x \in V(K_n)} w(x)^2 \geq \frac{S_n^2}{2} - \frac{S_nM_n}{2}.
\]
This together with (4) implies the statement. \qed

3. Rejected Edges: Proof of Theorem 1.7

The upper bounds follow immediately from Lemma 2.6.

3.1. Proof of upper bounds. Due to [7, 15] and Theorem 2.3[b] and 2.4[c] we have that whp
\[
  \text{ex}(G_{n,t}) = \begin{cases} 
    0 & \text{if } t = n/2 - s \text{ for } s \gg n^{2/3}; \\
    O(h) & \text{if } t = n/2 + s \text{ for } s = O(n^{2/3}); \\
    \Theta(s^3/n^2) & \text{if } t = n/2 + s \text{ for } n^{2/3} \ll s \ll n; \\
    (c - f(o(1)))n/2 & \text{if } t = cn/2 \text{ for } c > 1.
  \end{cases}
\]
Together with Lemma 2.6 this implies the upper bounds in Theorem 1.7.

Before proving the lower bounds we first sketch the main ideas.

3.2. Proof idea for lower bounds. First we consider the case $t = n/2 + s$ for $n^{2/3} \ll s \ll n$. We use a consequence of the properties [a] and [c] in Definition 1.4 that there exists an $\ell \in \mathbb{N}$ such that no graph in $\mathcal{P}$ contains the complete graph $K_\ell$ as a minor. We then apply a 'sprinkling' type argument: Let $P = P_{n,t}$ and let $P' = P_{n,t'}$ for $t' = n/2 + s/2$. We first reveal the edges $e_1, \ldots, e_t$. Given the realisation of $P'$, we split the vertex set $V(L(P'))$ of the largest component of $P'$ into disjoint sets $V_1, \ldots, V_\ell$ of 'almost' equal sizes such that $P'[V_i]$ is connected for each $i \in [\ell]$, where $\ell \geq \ell$. Next we reveal the remaining edges $e_{t+1}, \ldots, e_\ell$ up to step $t$ and show that for each pair $i \neq j$ there are 'many' edges between $V_i$ and $V_j$ which are queried up to step $t$. As $K_\ell$ is not a minor of $P'$, there are some pairs of which all of these edge are rejected. This provides a lower bound on the number of rejected edges. The precise way of decomposing the largest component of $P'$ differs in the cases $s \ll n^{3/4}$ and $s \gg n^{17/24}$. We note that it is sufficient to deal with these two cases, because the general case $n^{2/3} \ll s \ll n$ follows by considering appropriate subsequences.
The starting point for the case \( t = cn/2 \) for \( c > 1 \) is Theorem 2.7. Roughly speaking, it says that only a very small number of all graphs on \( n \) vertices lie in \( \mathcal{P}(n) \). Using that we show that for each \( \delta > 0 \), whp there is no graph \( H \in \mathcal{P}(n,(1+\delta)n) \) such that all edges of \( H \) are already queried before step \( t := n \cdot \log n \). In particular, this shows that whp \( e(P_{n,t}) \leq (1+o(1))n \). It is well known that whp \( G_{n,t} \) and therefore also \( P_{n,t} \) are connected. Thus, we obtain that whp \( ex(P_{n,t}) \leq ex(P_{n,t}) = o(n) \).

Furthermore, we have that
\[
t - e(P_{n,t}) = e(G_{n,t}) - e(P_{n,t}) \geq ex(G_{n,t}) - ex(P_{n,t}).
\]

Hence, we get a lower bound by using whp \( ex(P_{n,t}) = o(n) \) and \( ex(G_{n,t}) = (c - f(c) + o(1))n/2 \) from Theorem 2.4(c).

3.3. Proof of lower bounds. (i) We start with the case \( t = n/2 + s \) for \( n^{3/2} \ll s \ll n \).

Take \( t' = n/2 + s/2 \) and let \( P' = P_{n,t'}, G' = G_{n,t'}, P = P_{n,t}, \) and \( G = G_{n,t} \). Furthermore, let \( L(P') \) and \( L(G') \) be the largest components of \( P' \) and \( G' \), respectively.

Due to the properties [a] and [d] in Definition 1.4 there exists an \( \ell \in \mathbb{N} \) such that there is no graph in \( \mathcal{P} \) having the complete graph \( K_{\ell} \) as a minor. Now we distinguish two cases.

Case 1: \( s \ll n^{3/4} \). First reveal the edges \( e_1, \ldots, e_r \).

Let \( C = C(L(G')) \) be the 2-core of \( L(G') \) and \( F(G') \) the forest obtained from \( L(G') \) by deleting the edges of \( C \). Moreover, for a vertex \( x \in V(C) \) let \( T_x \) be the tree component of \( F(G') \) containing \( x \). By Definition 1.4(b) and [e] (we have \( V(L(P')) = V(L(G')) \)) and \( E(L(P')) \subseteq E(L(G')) \); furthermore, each edge of \( F(G') \) is also contained in \( L(P') \). Thus, there is a connected and spanning subgraph \( C' \subseteq C \) such that \( L(P') \) can be obtained by replacing each vertex \( x \) in \( C' \) by the tree \( T_x \).

We apply Lemma 2.8 to \( C' \), where we define the vertex-weight of a vertex \( x \in V(C') \) by \( w(x) := |V(T_x)| \). Then due to Theorem 2.3(a), (c), and (e) whp the total vertex-weight of \( C' \) satisfies
\[
W(C') := \sum_{x \in V(C')} w(x) = \sum_{x \in V(C')} |V(T_x)| = v(L(P')) = v(L(G')) = (2 + o(1))s,
\]
the maximum vertex-weight of \( C' \) satisfies
\[
\max_{x \in V(C')} w(x) = \max_{x \in V(C')} |V(T_x)| = o(s),
\]
and the maximum degree of \( C' \) is bounded by \( \Delta(C') \leq \Delta(C) = 3 \). Assuming this whp event holds, we apply Lemma 2.8 to \( C' \) with \( \Delta = 3 \) and \( a = M = s/2(3\ell) \) and obtain disjoint vertex sets \( V_1, \ldots, V_r \subseteq V(C') \) such that

- \( C'[\tilde{V}_i] \) is connected for each \( i \in [r] \);
- \( a = s/2(3\ell) \leq W_i \leq a\Delta + M = 4s/3(3\ell) \) for each \( i \in [r] \);
- \( W(C') - \sum_{i=1}^r W_i < a = s/2(3\ell) \),

where \( W_i := \sum_{x \in \tilde{V}_i} w(x) = \sum_{x \in \tilde{V}_i} |V(T_x)| \).

For \( i \in [r] \) let \( V_i \) be the set of vertices that lie in some \( T_x \) for a \( x \in \tilde{V}_i \). Then \( V_1, \ldots, V_r \subseteq L(P') \) are pairwise disjoint and satisfy the following properties:

- \( P'[V_i] \) is connected for each \( i \in [r] \);
- \( s/2(3\ell) \leq |V_i| \leq 4s/3(3\ell) \) for each \( i \in [r] \);
- \( \sum_{k=1}^{r} |V_i| \leq v(L(P')) - s/3(3\ell) \geq 4s/3 \).

Note that \( \ell \leq r = \Theta(1) \). Next we reveal the edges \( e_{r+1}, \ldots, e_r \). We claim that whp for each pair \( i \neq j \in [r] \) there are at least \( A := s^2/(18\ell^2 n^2) \) many edges between points in \( V_i \) and \( V_j \) which have been queried up to step \( i \). Assume that for some \( i \neq j \in [r] \) this is not the case and let \( k \in [N] \) be such that \( t' < k \leq t \). Then in the step of revealing \( e_k \) there were at least \( |V_i||V_j| - A \geq s^2/(10\ell^2) \) many edges going between \( V_i \) and \( V_j \) which had not been queried yet. Hence, we have
\[
\mathbb{P} \{ e_k \text{ has one endpoint in } V_i \text{ and one in } V_j \} \geq s^2/(5\ell^2 n^2) =: p.
\]

Letting \( X \) be the number of edges going between \( V_i \) and \( V_j \) that have been queried up to step \( t \) it implies
\[
\mathbb{P} \{ X \leq A \} \leq \mathbb{P} \{ \text{Bin} \left( s/2, p \right) \leq A \} = O\left( n^2/s^3 \right) = o(1).
\]
As \( r = \Theta(1) \) this shows the claim. By the choice of \( \ell \) we know that \( K_\ell \) is not a minor of \( P = P_{n,t} \). Hence, there is a pair \( i \neq j \in [r] \) such that there is no edge in \( P \) going between \( V_i \) and \( V_j \). Together with the claim this yields that whp at least \( A = \Theta(s^3/n^2) \) many edges have been rejected up to step \( t \). This concludes the case \( s < n^{3/4} \).

**Case 2:** \( s \gg n^{17/24} \). First we reveal again only the edges \( e_1, \ldots, e_{\ell} \).

Using Theorem 2.3.a and (d) we have that whp \( v(L(P')) = v(L(G')) = (2 + o(1))s \) and \( \Delta(L(P')) \leq \Delta(G') = (1 + o(1))\log n/\log \log n \). Assuming this whp event holds, we apply Lemma 2.8 to \( L(P') \) where we assign a vertex-weight \( w(x) = 1 \) to each \( x \in V(L(P')) \), \( \Delta = \log n \), \( M = 1 \), and \( \alpha = s/\ell (\log n) \). This leads to disjoint sets \( V_1, \ldots, V_r \subseteq V(L(P')) \) such that \( L(P')|V_i| \) is connected for each \( i \in [r] \),

\[
\frac{s}{\ell (\log n)} \leq |V_i| \leq \frac{s}{\ell} + 1 \quad \text{for each } i \in [r], \quad \text{and}
\]

\[
\sum_{i \in [r]} |V_i| \geq \ell (P') - \frac{s}{\ell (\log n)} = (2 + o(1))s. \tag{6}
\]

Next reveal the remaining edges \( e_{\ell+1}, \ldots, e_t \). We claim that whp for each pair \( i \neq j \in [r] \) there are at least \( B = B(i, j) := |V_i||V_j|s/(2n^2) \) edges that have been queried up to step \( t \). To prove the claim, let \( i \neq j \in [r] \) be fixed and denote by \( X \) the number of edges between \( V_i \) and \( V_j \) that have been queried up to step \( t \). Analogous to (5) we obtain, with \( q := |V_i||V_j|/(2n^2) \),

\[
\mathbb{P}[X \leq B] \leq \mathbb{P}[\sin(s/2, q) \leq B] = O \left( \frac{n^2}{|V_i||V_j|s} \right) = O \left( \frac{n^2 \log n^2}{s^3} \right) = o(n^{-1/9}),
\]

where we used Chebyshev’s inequality, (6), and \( s \gg n^{17/24} \). As \( r = O(\log n) \), the claim follows by the union bound. Next, let \( H \) be the graph with (super)vertex set \( V(H) = \{V_1, \ldots, V_r\} \) and two vertices \( V_i \) and \( V_j \) are connected if and only if there is an edge in \( P \) going between \( V_i \) and \( V_j \). We assign each vertex \( V_i \) in \( H \) the vertex-weight \( w(V_i) := |V_i| \). Due to (6) and (7) we have that the maximum vertex-weight and the total vertex-weight of \( V(H) \) satisfy

\[
M_r := \max_{i \in [r]} w(V_i) \leq \frac{s}{\ell} + 1 \quad \text{and} \quad S_r := \sum_{i \in [r]} w(V_i) = (2 + o(1))s.
\]

Let \( I \) be the set of unordered pairs \( i \neq j \) such that there is no edge in \( H \) going between \( V_i \) and \( V_j \). We note that \( K_\ell \not\subseteq H \), as \( K_\ell \) is not a minor of \( P \). Then by Lemma 2.9 we obtain

\[
\sum_{(i,j) \in I} |V_i||V_j| \geq \frac{S_r}{2} \left( \frac{S_r}{s-1} - M_r \right) = \Theta(s^3).
\]

By definition of \( I \) there is no edge in \( P \) going between \( V_i \) and \( V_j \) for each unordered pair \( \{i, j\} \in I \). Furthermore, whp for each of these pairs at least \( B = |V_i||V_j|s/(2n^2) \) many edges between \( V_i \) and \( V_j \) have been queried up to step \( t \). Thus, the number of rejected edges satisfies that whp

\[
r(t) \geq \frac{s}{2n^2} \sum_{(i,j) \in I} |V_i||V_j| \geq \Theta(s^3/n^2).
\]

This concludes the case \( s \gg n^{17/24} \) and therefore also the case \( t = n/2 + s \) for \( n^{2/3} \ll s \ll n \).

(ii) We consider the case where \( n = cn^{2} \) for \( c > 1 \). Let \( \bar{t} := n \cdot \log n \), \( \bar{P} := P_{n, \bar{t}} \), and \( P := P_{n,t} \). We claim that whp

\[
e(\bar{P}) \leq (1 + o(1))n. \tag{8}
\]

To show the claim, we use an idea from [13]: Let \( \delta > 0 \), \( m := (1 + \delta)n \), \( H \in \mathcal{P}(n, m) \), \( E(H) = \{f_1, \ldots, f_m\} \), and as in Section 1.1 \( \mathcal{N} := \binom{m}{2} \). We have

\[
\mathbb{P}[H \subseteq \bar{P}] \leq \mathbb{P}\left[ \left\{ f_1, \ldots, f_m \right\} \subseteq \left\{ e_1, \ldots, e_{\bar{t}} \right\} \right] \leq \prod_{i=1}^{m} \mathbb{P}[f_i \in \left\{ e_1, \ldots, e_{\bar{t}} \right\}] = \left( \frac{\bar{t}}{N} \right)^m = \left( \frac{2 \log n}{n - 1} \right)^m.
\]
By Theorem 2.7 there exists a \( c > 0 \) such that \( |\mathcal{P}(n)| \leq n!c^n \) for all \( n \in \mathbb{N} \). Thus, we obtain by taking the union bound

\[
\mathbb{P} \left[ e(\tilde{P}) \geq m \right] \leq |\mathcal{P}(n)| \left( \frac{2\log n}{n-1} \right)^m \leq n!c^n \left( \frac{2\log n}{n-1} \right)^m
\]

\[
\leq n^n c^n \left( \frac{2\log n}{n-1} \right)^m = \left( \frac{nc(2\log n)^{1+\delta}}{(n-1)^{1+\delta}} \right)^n = o(1),
\]

which gives \( \ref{18} \). Next we use the well-known fact that whp \( G_{n,\ell} \) is connected (see e.g., \( \ref{8} \)). By Remark 2.5 this is also true for \( \tilde{P} \). Together with \( \ref{18} \) this implies that whp \( e(\tilde{P}) = o(n) \). As \( P \subseteq \tilde{P} \), we obtain whp \( e(P) = o(n) \). Due to \( P \subseteq G \) we have \( e(G) - e(P) \geq e(G) - e(P) \). Hence, we have that whp

\[
t - e(P) = e(G) - e(P) \geq e(G) - e(P) = (c - f(c) + o(1)) n/2 - o(n),
\]

where we used Theorem 2.4(c) for the last equality. This shows the lower bound in the case \( t = cn/2 \) for \( c > 1 \) and concludes the proof of Theorem 1.7.

4. ADDABLE AND FORBIDDEN EDGES AND PROOF OF COROLLARY 1.8

How likely is it that the ‘next’ edge \( e_{t+1} \) gets accepted? Equivalently, what is the number of potential edges that can be added to \( P_{n,t} \) without violating property \( \mathcal{P} \)?

**Definition 4.1.** Let \( \mathcal{P} \) be a class of graphs and let \( H \in \mathcal{P}(n) \). Then we call an edge \( e \in E(K_n) \setminus E(H) \) **addable** to \( H \) if \( H + e \in \mathcal{P} \) and **forbidden** in \( H \) otherwise, i.e., if \( H + e \notin \mathcal{P} \). Furthermore, we set

\[
\text{add}(H) := ||e \in E(K_n) \setminus E(H) | e \text{ is addable to } H||;
\]

\[
\text{forb}(H) := ||e \in E(K_n) \setminus E(H) | e \text{ is forbidden in } H||.
\]

In the next theorem we determine the number of forbidden edges in \( P_{n,t} \). Combining it with Theorem 1.7 one can also compute \( e(P_{n,t}) \), because

\[
\text{add}(P_{n,t}) = N - \text{forb}(P_{n,t}) - e(P_{n,t}).
\]

**Theorem 4.2.** Let \( \mathcal{P} \) be a class of graphs satisfying the properties (a–f) in Definition 1.4 and \( (P_{n,t})_{t=0}^N \) be the \( \mathcal{P} \)-constrained random graph process. Let \( h = h(n) = o(1) \) be a function which tends to \( \infty \) arbitrarily slowly as \( n \to \infty \). Let \( t = t(n) \in \mathbb{N} \) and \( s = s(n) \). Then whp

\[
\text{forb}(P_{n,t}) = \begin{cases} 
O \left( \frac{hn^2}{s} \right) & \text{if } t = n/2 - s \text{ for } s \gg n^{2/3}; \\
O \left( \frac{hn^{1/3}}{} \right) & \text{if } t = n/2 + s \text{ for } s = O \left( n^{2/3} \right); \\
\Theta \left( s^2 \right) & \text{if } t = n/2 + s \text{ for } n^{2/3} \ll s \ll n; \\
\beta (c)^2 + o(1) n^{2/3} & \text{if } t = cn/2 \text{ for } c > 1.
\end{cases}
\]

We note that for planar graphs (and many other graph classes mentioned in Proposition 1.5) we actually have that whp

\[
\text{forb}(P_{n,t}) = 0 \text{ if } t = n/2 - s \text{ for } s \gg n^{2/3}.
\]

However, this is not true in general for a class \( \mathcal{P} \) that satisfies the properties (a–f) in Definition 1.4. In order to prove Theorem 4.2 we will use the following lemma. We recall that for fixed \( n \in \mathbb{N} \) we denote by \( r(t) = t - e(P_{n,t}) \) the number of rejected edges up to step \( t \).

**Lemma 4.3.** Let \( \mathcal{P} \) and \( (P_{n,t})_{t=0}^N \) be as in Theorem 4.2. Let \( t_1 = t_1(n), t_2 = t_2(n) \in \mathbb{N} \) and \( \alpha = \alpha(n) > 0 \) be such that \( \alpha \cdot (t_2 - t_1) = o(1) \) and \( t_2 = o \left( \alpha \cdot n^2 \right) \).

(a) If whp \( r(t_2) - r(t_1) \leq \alpha \cdot (t_2 - t_1) \), then whp \( \text{forb}(P_{n,t_1}) \leq (1 + o(1)) \alpha \cdot N \).

(b) If whp \( r(t_2) - r(t_1) \geq \alpha \cdot (t_2 - t_1) \), then whp \( \text{forb}(P_{n,t_2}) \geq (1 + o(1)) \alpha \cdot N \).
Proof. Due to property $[d]$ of Definition 1.4 a forbidden edge in $P_{n,t}$ stays forbidden in $P_{n,t’}$ for all $t’ > t$. Thus, $\text{forb}(P_{n,t})$ is non-decreasing in $t$. Let $\epsilon > 0$ be fixed and we denote by $\mathcal{F}$ the event that $\text{forb}(P_{n,t}) \geq (1 + \epsilon) \alpha \cdot N$. Then we have that for each $t \in \{ t_1 + 1, \ldots, t_2 \}$ and $n$ large enough,
\[
P \left[ e_t \text{ is rejected } | \mathcal{F} \right] \geq \frac{(1 + \epsilon) \alpha \cdot N - t_2}{N} \geq (1 + \epsilon/2) \alpha,
\]
where we used $t_2 = o(\alpha \cdot n^2)$ in the last inequality. Hence, we obtain
\[
P \left[ r(t_2) - r(t_1) \leq \alpha \cdot (t_2 - t_1) \right] \leq \frac{\alpha \cdot (t_2 - t_1)}{N} \leq (1 + \epsilon/2) \alpha.
\]
Together with the fact that whp $r(t_2) - r(t_1) \leq \alpha \cdot (t_2 - t_1)$ it implies that $\mathbb{P} [ \mathcal{F} ] = o(1)$. As $\epsilon > 0$ was arbitrary, statement $[a]$ follows. Statement $[b]$ can be obtained similarly. 

Combining Theorem 1.7 with Lemma 4.3 we can prove Theorem 4.2.

**Proof of Theorem 4.2** (i) We start with the case $t = n/2 + s$ for $n^{2/3} \ll s \ll n$.

To prove the upper bound, we set $t_2 = n/2 + 2s$. By Theorem 1.7 we have that whp $r(t_2) - r(t) = O(s^2/n^2)$. Hence, there exists $\alpha = \alpha(n)$ such that $\alpha = \Theta(s^2/n^2)$ and whp $r(t_2) - r(t) \leq \alpha \cdot (t_2 - t)$. Hence, Lemma 4.3(a) yields that whp
\[
\text{forb}(P_{n,t}) \leq (1 + o(1)) \alpha \cdot N = \Theta(s^2).
\]

Similarly, we obtain the lower bound: Taking $t_1 = n/2$ we have by Theorem 1.7 that whp $r(t) = \Theta(s^2/n^2)$ and $r(t_1) = O(h)$ for $h = o(1)$, and thus $\frac{r(t)-r(t_1)}{t-n} = \Theta(s^2/n^2)$. Together with Lemma 4.3(b) it implies that whp $\text{forb}(P_{n,t}) \geq \Theta(s^2)$. 

(ii) The assertions in the cases $t = n/2 - s$ for $s \gg n^{2/3}$ and $t = n/2 + s$ for $s = O(n^{2/3})$ can be shown similarly.

(iii) Finally, we consider the regime $t = cn/2$ for $c > 1$. Let $c_2 > c$ and $t_2 = c_2 n/2$. By Theorem 1.7 we have that whp
\[
r(t_2) - r(t_1) = (c_2 - f(c_2) - c + f(c) + o(1)) n/2.
\]
Hence, Lemma 4.3(a) implies that whp
\[
\text{forb}(P_{n,t}) \leq (1 + o(1)) \left( 1 - \frac{f(c_2) - f(c)}{c_2 - c} \right) n^2/2.
\]
With $c_2 \upharpoonright c$ we obtain that whp
\[
\text{forb}(P_{n,t}) \leq (1 + o(1)) \left( 1 - f'(c) \right) n^2/2 = (1 + o(1)) \left( \beta(c)^2 \right) n^2/2,
\]
where we used Lemma A.4(a) in the last equality. Using Lemma 4.3(b) we obtain in a similar way that whp $\text{forb}(P_{n,t}) \geq (1 + o(1)) \left( \beta(c)^2 \right) n^2/2$. This completes the proof.

We conclude this section by showing Corollary 1.8.

**Proof of Corollary 1.8**. Let $C_1, \ldots, C_r$ be the components of $P_{n,t}$ such that $\nu(C_1) \geq \ldots \geq \nu(C_r)$. By Theorem 2.4(a) and (b) and Remark 2.5 we have whp $\nu(C_1) = (\beta(c) + o(1)) n$ and $\nu(C_2) = o(n)$. Furthermore, let $E_1 \subseteq E(K_n) \setminus E(P_{n,t})$ be the subset of edges with both endpoints in $L$ and $E_2$ the remaining edges of $E(K_n) \setminus E(P_{n,t})$. We have that whp
\[
|E_1| = (\beta(c)^2 + o(1)) n^2/2; \\
|E_2| = (1 - \beta(c)^2 + o(1)) n^2/2.
\]
Due to property $[e]$ of Definition 1.4 the two endpoints of a forbidden edge lie in the same component. Hence, the number of forbidden edges in $E_2$ is whp at most
\[
\nu(C_2)^2 + \ldots + \nu(C_r)^2 \leq \nu(C_2) (\nu(C_2) + \ldots + \nu(C_r)) \leq \nu(C_2) n = o(n^2).
\]
Together with (10) it shows assertion $[b]$. Furthermore, it implies that the number of forbidden edges in $E_1$ is whp
\[
\text{forb}(P_{n,t}) - o(n^2) = (\beta(c)^2 + o(1)) n^2/2.
\]
Combining this with $[b]$ yields statement $[a]$. 

\[\square\]
5. The random graph \( P_{n,m=m_0} \): proof of Theorem 1.10

Throughout this section, let \( \mathcal{P} \) be a class of graphs satisfying the properties (a)–(f) in Definition 1.4 and \( (P_{n,t})^N \) be the \( \mathcal{P} \)-constrained random graph process. Recall that \( P_{n,m=m_0} \) denotes the graph in which exactly \( m_0 \) edges have actually been added. We assume that \( m_0 = m_0(n) \in [N] \) is such that \( P_{n,m=m_0} \) always exists, i.e., for any ordering of the potential edges, at least \( m_0 \) of them have been accepted at the end of the process.

**Proof of Theorem 1.10.** By Remark 2.5, we have
\[
P \leq \beta \text{ for 0 and } \beta\text{ that whp of this paper.}
\]
We have that
\[
P(t_1) = m_0 \leq e(P_{n,t_2}) \quad \text{whp} 
\]
for 0 \( \leq t_1 \leq t_2 \).

(i) First consider the case \( m_0 = cn/2 \) for \( 1 < c < 2 \). Let \( \epsilon > 0 \) be small, \( t_1 = (f^{-1}(c) - \epsilon)n/2 \), and \( t_2 = (f^{-1}(c) + \epsilon)n/2 \). By Theorem 1.7, we have that whp \( e(P_{n,t_1}) \leq m_0 \leq e(P_{n,t_2}) \). Thus, (11) implies that whp
\[
\beta(f^{-1}(c) - \epsilon + o(1))n \leq v(L(P_{n,m=m_0})) \leq \beta(f^{-1}(c) + \epsilon + o(1))n,
\]
where we used Theorem 2.4(a). As \( \beta \) is continuous, we obtain with \( \epsilon \downarrow 0 \) that whp \( v(L(P_{n,m=m_0})) = (\beta(f^{-1}(c) + o(1))n \)

(ii) The four cases where \( m_0 \leq n/2 + o(n) \) can be shown similarly.

(iii) Next, we observe that \( v(L(P_{n,m=m_0})) \) is non-decreasing in \( m_0 \), \( \lim_{c \downarrow 1/2} f^{-1}(c) = \infty \), and \( \lim_{c \to \infty} \beta(c) = 1 \). Thus, in case of \( m_0 = n \) the statement follows by taking \( c \downarrow 2 \) in the previously shown fact that whp
\[
v(L(P_{n,m=m_0})) = (\beta(f^{-1}(c) + o(1))n
\]
if \( m_0 = cn/2 \) for \( 1 < c < 2 \).

(iv) Finally, we consider the case \( m_0 = cn/2 \) for \( 2 < c \) and let \( t_1 = n \cdot \log n \). By (8) we know that whp \( e(P_{n,t_1}) \leq m_0 \). Thus, using (11) yields
\[
v(L(P_{n,m=m_0})) \geq v(L(G_{n,t_1})) = n,
\]
because whp \( G_{n,t_1} \) is connected. This finishes the proof. \( \square \)

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Lemma A.1. For $c > 1$ let $\beta(c)$ be the unique positive solution of the equation $1 - x = e^{-cx}$ and let $f(c) = 2\beta(c) + c(1 - \beta(c))^2$ (as in Definition 1.6). Then the following hold:

(a) $f'(c) = 1 - \beta(c)^2$;
(b) $f$ is strictly increasing;
(c) $f$ is continuous;
(d) $\lim_{c \downarrow 1} f(c) = 1$;
(e) $\lim_{c \to \infty} f(c) = 2$;
(f) $f : (1, \infty) \to (1, 2)$ is bijective and therefore, has an inverse $f^{-1} : (1, 2) \to (1, \infty)$.

Proof. By definition we have $1 - \beta(c) = e^{-c\beta(c)}$. Taking the derivative of both sides yields

$$-\beta'(c) = (-\beta(c) - c\beta'(c)) e^{-c\beta(c)} = (-\beta(c) - c\beta'(c))(1 - \beta(c)).$$

Rearranging this gives

$$\beta'(c) = \frac{\beta(c) - \beta(c)^2}{1 - c + c\beta(c)},$$

Using it we obtain

$$f'(c) = 2\beta'(c) + (1 - \beta(c))^2 - 2c(1 - \beta(c))\beta'(c)$$

$$= \frac{2\beta(c) - 2\beta(c)^2 + (1 - \beta(c))^2 (1 - c + c\beta(c)) - 2c(1 - \beta(c)) (\beta(c) - \beta(c)^2)}{1 - c + c\beta(c)}$$

$$= 1 - \beta(c)^2,$$

which shows (a). As $\beta(c) \in (0, 1)$ this implies also (b). By definition $\beta$ is continuous and therefore so is $f$, proving (c). We observe that $\lim_{c \to 1} \beta(c) = 0$. Hence, we obtain that

$$f(c) = 2\beta(c) + c(1 - \beta(c))^2 \to 2 \cdot 0 + 1 \cdot 1 = 1 \quad \text{as} \quad c \downarrow 1,$$

which shows (d).

For (e) we note that $\lim_{c \to \infty} \beta(c) = 1$ and for $c$ large enough we have

$$ce^{-2\beta(c)c} \leq ce^{-c} \to 0 \quad \text{as} \quad c \to \infty.$$

Thus, we obtain

$$f(c) = 2\beta(c) + ce^{-2\beta(c)c} \to 2 \cdot 1 + 0 = 2 \quad \text{as} \quad c \to \infty,$$

which shows (e).

Finally, (f) follows by combining (b), (c), (d) and (e).