LIVŠIĆ MEASURABLE RIGIDITY THEOREM FOR $C^1$ GENERIC VOLUME-PRESEVING ANOSOV SYSTEMS

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Abstract. In this paper, we prove that for $C^1$ generic volume-preserving Anosov diffeomorphisms of a compact Riemannian manifold, Livšić measurable rigidity theorem holds. We also prove that for $C^1$ generic volume-preserving Anosov flows of a compact Riemannian manifold, Livšić measurable rigidity theorem holds.

1. Introduction

Let $T : M \to M$ be a diffeomorphism on a compact Riemannian manifold $M$. We consider a cocycle $\mathcal{A} : \mathbb{Z} \times M \to \mathbb{R}$; that is, a map satisfying the cocycle relation

$$\mathcal{A}(n_1 + n_2, x) = \mathcal{A}(n_1, T^{n_2}(x)) + \mathcal{A}(n_2, x),$$

for every $n_1, n_2 \in \mathbb{Z}$ and every $x \in M$. Following the definition in cohomological algebra, we call a cocycle $\mathcal{A}$ a coboundary if it satisfies the cohomological equation:

$$(1) \quad \mathcal{A}(n, x) = \Phi(T^n(x)) - \Phi(x),$$

where $\Phi : M \to \mathbb{R}$ is a function. Furthermore, two cocycles are called cohomologous if their difference is a coboundary.

It is easy to see that coboundary $\mathcal{A}$ must have trivial periodic data, i.e.

$$\mathcal{A}(n, x) = 0, \quad \forall x \in M, \quad T^n(x) = x.$$  \hspace{1cm} (2)

One has three natural questions to propose.

1. Is the necessary condition, trivial periodic data, also a sufficient condition?
2. Measurable rigidity: If the cocycle $\mathcal{A} : \mathbb{Z} \times M \to \mathbb{R}$ is Hölder continuous, can we get a Hölder continuous solution $\Phi$ to equation (1) from a measurable solution?
3. Higher regularity: If the cocycle $\mathcal{A} : \mathbb{Z} \times M \to \mathbb{R}$ is $C^r$ for some $1 \leq r \leq \infty$ or $r = \omega$, is a continuous solution to equation (1) also $C^r$?

Livšić took the lead in considering these three questions for the case when $f$ is a transitive Anosov diffeomorphism on a compact Riemannian manifold $M$ \cite{Livsic}. Thus, we call results answering the above questions Livšić theorems. Current research is usually concerned with two variations on this subject, namely altering the base system $T$ and altering the group $\mathbb{R}$. Some of the highlights are \cite{Klingenberg, Kunita, Katznelson, Katznelson2, Katznelson3, Katznelson4, Katznelson5, Katznelson6, Katznelson7, Katznelson8}. The following theorems are some classical results.

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Theorem 1.1. \[1, 2, 18, 21\] Let \( T : M \to M \) be a \( C^1 \) transitive Anosov diffeomorphism on a compact Riemannian manifold \( M \) and let \( \phi : M \to \mathbb{R} \) be a Hölder continuous function.

1. Existence of solutions. \( \phi = \Phi(T) - \Phi \) has a Hölder continuous solution \( \Phi \) if and only if \( \sum_{x \in O} \phi(x) = 0 \), for every \( T \)-periodic orbit \( O \).

2. Measurable rigidity. For any Gibbs measure \( \mu \) with Hölder continuous potential, if there exists a \( \mu \)-measurable solution \( \Phi \) to \( \phi = \Phi(T) - \Phi \), then there is a continuous solution \( \Psi \), with \( \Phi = \Psi \), a.e. \( \mu \).

Theorem 1.2. \[1, 2, 18, 21\] Let \( \{ T^t \} \) be a \( C^1 \) transitive Anosov flow on a compact Riemannian manifold \( M \) generated by the vector field \( \xi \) and let \( \phi : M \to \mathbb{R} \) be a Hölder continuous function.

1. Existence of solutions. \( \phi = \Phi'_\xi \) for a Hölder continuous function \( \Phi \) differentiable along the flow if and only if \( \int_{p_0}^{t_p} \phi(T^s(p)) \, ds = 0 \) for every periodic point \( p \) with period \( t_p \).

2. Measurable rigidity. For any Gibbs measure \( \mu \) with Hölder continuous potential, if there exists a \( \mu \)-measurable solution \( \Phi \) differentiable along the flow such that \( \phi = \Phi'_\xi \mu \)-almost everywhere, then there is a continuous solution \( \Psi \), with \( \Phi = \Psi \), a.e. \( \mu \).

In this paper, we are only concerned with measurable rigidity. The proof of measurable rigidity in \[18\] is based on the Markov partitions and Livšic type theorems for cocycles over shifts of finite type \[16\], which depend heavily on the equipment of Gibbs measures. For the definition of Gibbs measures, we refer the reader to a classical and short book \[19\] by Bowen. For other measures, measurable rigidity may not hold.

In this paper, we consider the measurable rigidity for the special measure, volume measure \( m \). It is known that for \( C^2 \) volume-preserving Anosov diffeomorphisms, the volume measure is a Gibbs measure with the Hölder continuous potential

\[ \varphi = -\log \det(DT|E^u) . \]

However, the volume measure for \( C^1 \) volume-preserving Anosov diffeomorphism may not be a Gibbs measure with Hölder continuous potential.

Under a \( C^1 \) generic hypothesis, we have the following result.

Theorem 1.3. There exists a residual subset \( G \) of \( C^1 \) Anosov volume-preserving diffeomorphisms on a compact Riemannian manifold \( M \) such that for any \( T \in G \) and any Hölder continuous function \( \phi : M \to \mathbb{R} \), the following three conditions are equivalent:

1. \( \sum_{x \in O} \phi(x) = 0 \), for every \( T \)-periodic orbit \( O \),
2. \( \phi(x) = \Phi(T(x)) - \Phi(x) \) has a continuous solution,
3. \( \phi(x) = \Phi(T(x)) - \Phi(x) \), a.e. for some measurable function \( \Phi \).

We also get a parallel result for Anosov flows.

Theorem 1.4. There exists a residual subset \( G \) of \( C^1 \) Anosov volume-preserving flows on a compact Riemannian manifold \( M \) such that for any flow \( \{ T^t \} \in G \) and any Hölder continuous function \( \phi : M \to \mathbb{R} \), the following three conditions are equivalent:

1. \( \int_0^{t_p} \phi(T^s(p)) \, ds = 0 \) for every periodic point \( p \) with period \( t_p \),
(2) $\phi = \Phi'_\xi$ for a Hölder continuous function $\Phi$ differentiable along the flow,
(3) $\phi = \Phi'_\xi$ almost everywhere for a measurable function $\Phi$ differentiable along
the flow.

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2. Preliminaries

2.1. Anosov diffeomorphisms. Assume $M$ to be a compact Riemannian man-
ifold. Recall that a diffeomorphism $T : M \to M$ is called Anosov if there is a
$T$-invariant splitting
$$TM = E_s \oplus E_u$$
and constants $C, \rho < 1$, such that
$$\forall v \in E_s, \|DT^n v\| \leq C\rho^n \|v\|,$$
$$\forall v \in E_u, \|DT^{-n} v\| \leq C\rho^n \|v\|.$$ 
Now we formulate the Central Limit Theorem for $C^2$ volume-preserving Anosov
diffeomorphisms. Its proof involves the construction of Markov partition of Anosov
diffeomorphisms and the corresponding statistical property of subshifts of finite
type.

**Theorem 2.1** (Central Limit Theorem). [19] Let $T$ be a $C^2$ Anosov volume-preserving diffeomorphism on compact Riemannian manifold $M$. Let $m$ be the volume measure on $M$. Let $\phi$ be a Hölder continuous function on $M$ with no measurable solution $\Phi$ to the equation:
$$\phi(x) - \int \phi(x) dx = \Phi(T(x)) - \Phi(x).$$
Then $\phi$ satisfies the Central Limit Theorem with respect to $T$, i.e. there exists a constant $\sigma > 0$ such that for any $-\infty < \alpha < +\infty$,
$$\lim_{n \to +\infty} m \left\{ x \in M : \frac{\sum_{i=0}^{n-1} \phi(T^i(x)) - n\overline{\phi}}{\sqrt{n}} < \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{u^2}{2}} du,$$
where $\overline{\phi} = \int_M \phi(x) dm$. What’s more, $\sigma^2 = \lim_{n \to +\infty} \frac{\int (\phi_n)^2}{n} \frac{dx}{m}$, where $\phi_n = \sum_{i=0}^{n-1} (\phi(T^i(x)) - \overline{\phi})$ and $\sigma$ is called the variance with respect to $\phi$.

2.2. Anosov flows. Let us recall the definition of Anosov flow first. Let $M$ be a compact Riemannian manifold. Let $T : \mathbb{R} \times M \to M$ be a $C^1$ flow on $M$ generated
by the vector field $\xi = \frac{d}{dt}(T^t)|_{t=0}$ where $T^t(\cdot)$ denotes $T(t, \cdot)$. Flow $T : \mathbb{R} \times M \to M$
is called Anosov if there is a continuous splitting $TM = E^+ \oplus E^0 \oplus E^-$ with $E^0$
spanned by $\xi$ and there are positive constants $c_1, c_2$ and $\gamma$ such that
$$\|D(T^t)(\eta)\| \geq c_1 \cdot e^{t\gamma} \cdot \|\eta\|, \forall \eta \in E^+ \text{ and } t \geq 0,$$
$$\|D(T^t)(\eta)\| \leq c_2 \cdot e^{-t\gamma} \cdot \|\eta\|, \forall \eta \in E^- \text{ and } t \geq 0.$$ 
We use the notation $\{T^t\}$ as the flow $T(t, \cdot)$ in this paper. We give the $C^r$ topology
of flows in the following definition.
Definition 2.2 (C^r Topology for Flows). Let \( \mathcal{F}^r(M) \) be the space of \( C^r \)-flows on \( M \). Every flow \( \{T^t\} \in \mathcal{F}^r(M) \) restricts to a \( C^r \) map \( T^{[t]} : [0, t_0] \times M \to M \). Since \([0, t_0] \times M\) is compact, we may take the usual \( C^r \) topology on \( C^r \) maps \([0, t_0] \times M \to M\), and thereby define a \( C^r \) topology on \( \mathcal{F}^r(M) \). Using the one parameter group property of flows, it is easy to see that the \( C^r \) topology we have defined on \( \mathcal{F}^r(M) \) is independent of \( t_0 > 0 \).

In the interest of proving Theorem 1.4, we will also use the Central Limit Theorem for \( C^2 \) volume-preserving Anosov flows, which is proved by taking advantage of the Markov partition for Anosov flows.

Theorem 2.3 (Central Limit Theorem for Anosov flows). Let \( \{T^t\} \) be a \( C^2 \) Anosov volume-preserving flow on a compact Riemannian manifold \( M \) generated by vector field \( \xi \) and let \( \phi : M \to \mathbb{R} \) be a Hölder continuous function on \( M \). Let \( m \) be the volume measure on \( M \). If there is no measurable function \( \Phi : M \to \mathbb{R} \) differentiable along the flow \( \{T^t\} \) such that

\[
\phi = \Phi' \xi, \ a.e.
\]

then \( \phi \) satisfies the Central Limit Theorem relative to \( \{T^t\} \), i.e. there exists a constant \( \sigma > 0 \) such that for any \(-\infty < \alpha < +\infty\)

\[
\lim_{t \to +\infty} m \left\{ x \in M : \frac{\int_0^t (\phi(T^s(x)) - \bar{\phi}) \, ds}{\sigma \sqrt{t}} < \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}u^2} \, du
\]

where \( \bar{\phi} = \int_M \phi(x) \, dm \). Moreover,

\[
\sigma^2 = \lim_{t \to +\infty} \frac{\int_M (\phi_t(x))^2 \, dm}{t}
\]

where \( \phi_t \) denotes \( \int_0^t (\phi(T^s(x)) - \bar{\phi}) \, ds \) and \( \sigma \) is called the variance to function \( \phi \).

We formulate the definition of cocycles for Anosov flows, introducing some useful terminology along the way. In particular, let \( T : \mathbb{R} \times M \to M \) be a \( C^1 \) Anosov flow, a continuous map \( A : \mathbb{R} \times M \to \mathbb{R} \) is a cocycle over \( T \) if

\[
A(t_1 + t_2, x) = A(t_1, T^{t_2}(x)) + A(t_2, x)
\]

for every \( t_1, t_2 \in \mathbb{R} \) and every \( x \in M \). Cocycle \( A : \mathbb{R} \times M \to \mathbb{R} \) is a coboundary if there exists a map \( \Phi : M \to \mathbb{R} \) such that

\[
A(t, x) = \Phi(T^t(x)) - \Phi(x).
\]

From now on, we only consider cocycles which are differentiable along the flow. Namely, \( A(t, p) \) is a \( C^1 \) function of \( t \) for all \( p \in M \). A cocycle \( A : \mathbb{R} \times M \to \mathbb{R} \) is called Hölder continuous of exponent \( \alpha \in (0, 1) \) if the map

\[
x \mapsto \lim_{t \to 0} \frac{1}{t} A(t, x)
\]

is Hölder continuous of exponent \( \alpha \). Thus not only the cocycle \( A \) is differentiable along the flow, but also the derivative of \( A \) along the flow are Hölder continuous functions on the manifold \( M \).

There is a natural bijection between cocycles and functions on \( M \). A cocycle \( A(t, x) \) is said to be based on a function \( \phi : M \to \mathbb{R} \) if

\[
A(t, x) = \int_0^t \phi(T^s(x)) \, ds.
\]
The function $\phi: M \to \mathbb{R}$ is the **infinitesimal generator** $\xi(A)$ of cocycle $A$. The existence of this generator is due to the differentiability of the cocycle $A$ along the flow. The cocycle $A$ is Hölder continuous with exponent $\alpha$ if and only if the function $\phi: M \to \mathbb{R}$ is a Hölder continuous function with exponent $\alpha$. Moreover, if the equation

$$A(t, x) = \Phi(T^t(x)) - \Phi(x)$$

holds, then $\phi = \Phi_T := d\Phi(\xi)$.

### 3. Proof of Theorem

We now begin the proof of Theorem. First we state an essential definition.

**Definition 3.1.** Let $T$ be a $C^1$ Anosov volume-preserving diffeomorphism on a compact Riemannian manifold $M$. For any given constants $C > 0, \tilde{C} > 0, \varepsilon > 0$ and any given periodic point $p \in M$ with period $P(p)$, set

$$F_T(\tilde{C}, \varepsilon, p) = \left\{ \phi \mid \phi \text{ is an } \alpha \text{-Hölder continuous function on } M, \int_M \phi \, dx = 0, \|\phi\|_{\alpha} \leq \tilde{C}, \sum_{i=0}^{P(p)-1} \phi(T^i(p)) \geq \varepsilon \right\}.$$  

We say $T$ is of $(C, \tilde{C}, \varepsilon, p)$-type, if there exists a common time $N$, such that for any $p \in F_T(\tilde{C}, \varepsilon, p)$, there exists at least one moment $1 \leq k \leq N$ such that,

$$\mu \{ x \in M : \phi_k(x) > C \} > \frac{1}{2} - \varepsilon,$$

where $\phi_k(x) = \sum_{i=0}^{k-1} \phi(T^i(x))$.

In the following proposition, we use the Central Limit Theorem to prove that for any $(C, \tilde{C}, \varepsilon, p)$, $C^2$ Anosov volume-preserving diffeomorphisms are $(C, \tilde{C}, \varepsilon, p)$-type.

**Proposition 3.2.** Let $T$ be a $C^2$ Anosov volume-preserving diffeomorphism on a compact manifold $M$. For any $(C, \tilde{C}, \varepsilon, p)$, $T$ is $(C, \tilde{C}, \varepsilon, p)$-type.

**Proof.** Fix constants $(C, \tilde{C}, \varepsilon)$ and a periodic point $p$ arbitrarily. According to Theorem, for any $\phi \in F_T(\tilde{C}, \varepsilon, p)$, there exists $\sigma > 0$, such that for any $\alpha_0 > 0$, there exists $N_0 \in \mathbb{N}$ satisfying for any $n \geq N_0$,

$$m \left\{ x \in M : \frac{\sum_{i=0}^{n-1} \phi(T^i(x))}{\sigma \sqrt{n}} > \alpha_0 \right\} \geq \frac{1}{\sqrt{2\pi}} \int_{\alpha_0}^{+\infty} e^{-\frac{u^2}{2}} \, du - \frac{\varepsilon}{2} \geq \frac{1}{2} e^{-\frac{1}{2} \alpha_0^2} - \frac{\varepsilon}{2}.$$
Choose $\alpha_0$ small enough such that $\frac{1}{2}e^{-\frac{1}{2}2^{\alpha_0}} \geq \frac{1}{2} - \frac{\varepsilon}{2}$. Assume $N_1$ to be an integer satisfying $\frac{C}{\sigma N_1} \leq \alpha_0$. Let $N(\phi) := \max\{N_0, N_1\}$. Then, for any $n \geq N(\phi)$,
\[
m \left\{ x \in M : \sum_{i=0}^{n-1} \phi(T^i(x)) > C \right\} \geq m \left\{ x \in M : \frac{\sum_{i=0}^{n-1} \phi(T^i(x))}{\sigma \sqrt{n}} > \alpha_0 \right\} > \frac{1}{2}e^{-\frac{1}{2}2^{\alpha_0}} - \frac{\varepsilon}{2} > \frac{1}{2} - \varepsilon.
\]

For this fixed time $N(\phi)$, there exists a small neighborhood $U(\phi)$ of $\phi$ such that for any function $\tilde{\phi} \in U(\phi)$, we have
\[
m \left\{ x \in M : \phi_{N(\phi)}(x) > C \right\} > \frac{1}{2} - \varepsilon,
\]
where $\phi_{N(\phi)}(x) = \sum_{i=0}^{N(\phi)-1} \phi(T^i(x))$.

Due to the compactness of the set $\mathcal{F}(\tilde{C}, \varepsilon, p)$, there exists a finite cover $\mathcal{P} = \{U(\phi_i)\}_{i=0}^{K}$ of $\mathcal{F}(\tilde{C}, \varepsilon, p)$ and thus a common time
\[N = \max_{1 \leq i \leq K} N(\phi_i).
\]
This common time $N$ satisfies the condition we want. \hfill \square

Now we prove that $(C, \tilde{C}, \varepsilon, p)$-type implies Livšic measurable rigidity.

**Proposition 3.3.** Let $T$ be a $C^1$ Anosov volume-preserving diffeomorphism on a compact Riemannian manifold $M$. Assume that for any $C > 0$, $\tilde{C} > 0$, $\varepsilon > 0$ and any periodic point $p$, $T$ is $(C, \tilde{C}, \varepsilon, p)$-type. Then for any $\alpha$-Hölder continuous function $\phi : M \to \mathbb{R}$, we have three equivalent properties as follows:

1. $\phi(x) = \Phi(T(x)) - \Phi(x)$ has a continuous solution;
2. $\sum_{x \in \mathcal{O}} \phi(x) = 0$, for every $T$-periodic orbit $\mathcal{O}$;
3. $\phi(x) = \Phi(T(x)) - \Phi(x)$, a.e. for some measurable function $\Phi$.

**Proof.** By Theorem 1.1 and the fact that $C^1$ volume-preserving Anosov diffeomorphisms are transitive, we only need to check measurable rigidity, i.e. proving (1) from (3). Assume $\Phi$ is a measurable solution to $\phi(x) = \Phi(T(x)) - \Phi(x)$, a.e., then $\Phi$ is finite almost everywhere. For any small number $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that
\[m\{x \in M : \Phi(x) \leq C_\varepsilon\} > 1 - \frac{\varepsilon}{2}.
\]
Thus, by the identity $\phi_n(x) = \Phi(T^n(x)) - \Phi(x)$, it follows that
\[m\{x \in M : \phi_n(x) \leq 2C_\varepsilon\} > 1 - \frac{\varepsilon}{2}, \quad \forall \ n \geq 1.
\]
If there is no continuous solution for $\phi(x) = \Phi(T(x)) - \Phi(x)$, then there must exist a periodic point $p$ and $\varepsilon > 0$ such that $\sum_{i=0}^{T(p)-1} \phi(T^i(p)) \geq \varepsilon$. However, as $T$ is $(C, \tilde{C}, \varepsilon, p)$-type, for $C > 2C_\varepsilon$ and function $\phi$, there exists a time $1 \leq k \leq N$, such that
\[
(1 - \frac{\varepsilon}{2}) + \left(\frac{1}{2} - \varepsilon\right) \leq m\{x \in M : \phi_k(x) \leq 2C_\varepsilon\} + m\{x \in M : \phi_k(x) \geq C\}
\]
\[
\leq 1,
\]
which is a contradiction. So there exists continuous solution \( \tilde{\Phi} : M \to \mathbb{R} \) such that 
\[
\phi_n(x) = \tilde{\Phi}(T^n(x)) - \tilde{\Phi}(x) \quad \text{and moreover} \quad \tilde{\Phi} = \Phi, \text{a.e.}
\]
\( \square \)

From Proposition 3.2, we have \( C^2 \) Anosov volume-preserving diffeomorphisms are good type. According to A. Avila’s result [22] about the regularization of volume-preserving maps:

**Theorem 3.4.** [22] Smooth maps are \( C^1 \) dense in \( C^1 \) volume-preserving maps.

By Theorem 3.4 and the \( C^1 \) stability of Anosov systems, we can get \( C^2 \) Anosov volume-preserving diffeomorphisms are dense in \( C^1 \) Anosov volume-preserving diffeomorphisms, which is a key point in our proof.

**Theorem 3.5.** There exists a residual subset \( \mathcal{G} \) of \( C^1 \) Anosov volume-preserving diffeomorphisms on a compact Riemannian manifold \( M \) such that for any \( T \in \mathcal{G} \) and any \( \phi : M \to \mathbb{R} \) Hölder continuous, the Livšic theorem holds, i.e. the following three conditions are equivalent:

1. \( \phi(x) = \Phi(T(x)) - \Phi(x) \) has a continuous solution \( \Phi \);
2. \( \sum_{x \in \mathcal{O}} \phi(x) = 0 \), for every \( T \)-periodic orbit \( \mathcal{O} \);
3. \( \phi(x) = \Phi(T(x)) - \Phi(x) \), a.e. for some measurable function \( \Phi \).

**Proof.** We only need to prove (1) from (3) generically. Take a countable basis \( V = \{ V_1, V_2, \ldots \} \) of \( M \). Let \( A^1_m \) be the set of \( C^1 \) Anosov volume-preserving diffeomorphisms and let \( A^2_m \) be the set of \( C^2 \) Anosov volume-preserving diffeomorphisms.

Denote \( H_k = \{ G \in A^1_m \mid \text{there exists a neighborhood } U(G) \subset A^1_m \text{ such that for any } G_1 \in U(G), \text{ for any } C > 0, \tilde{C} > 0, \varepsilon > 0 \text{ and for any periodic point of } G_1, p \in V_k \in V \text{ with period } P(p) \leq k, G_1 \text{ is } (C, \tilde{C}, \varepsilon, p) - \text{type} \} \).

It is easy to see that \( H_k \) is open. Set
\[
\mathcal{G} := \bigcap_{k \in \mathbb{N}} H_k.
\]

Now we prove \( \mathcal{G} \) is the generic set we want.

Let \( A^2_m \) be the set of \( C^2 \) Anosov volume-preserving diffeomorphisms. In order to proof the density of \( H_k \), we prove the following lemma first.

**Lemma 3.6.** The set \( A^2_m \) is contained in \( H_k \), for all \( k \geq 1 \).

**Proof.** Fix any \( T \in A^2_m \) and \( V_k \). We finish our proof by choosing smaller and smaller neighborhoods of \( T \). By Proposition 3.2, for any \( T \in A^2_m \) and any \( (C, \tilde{C}, \varepsilon, p) \), \( T \) is \( (C, \tilde{C}, \varepsilon, p) \)-type. Thus, there exists \( N \) such that : for any \( \phi \in \mathcal{F}_T(\tilde{C}, \varepsilon, p) \), there exists \( 1 \leq i \leq N \) such that,
\[
m\{ x \in M : \phi_i(x) > C \} > \frac{1}{2} - \varepsilon.
\]

Consider the set
\[
\mathcal{F}_G(\tilde{C}, \varepsilon, p) = \left\{ \phi \mid \phi \text{ is an } \alpha \text{-Hölder continuous function on } M, \int_M \phi \, dx = 0, \right. \]
\[
\left. \| \phi \|_\alpha \leq \tilde{C}, \quad \sum_{i=0}^{P(p)-1} \phi(G^i(p)) \geq \varepsilon \right\},
\]
where $G$ is a $C^1$ Anosov volume-preserving diffeomorphism $C^1$-close to $T$ and $p \in \mathcal{V}_k$ is a periodic point with period $P(p) \leq k$ for $G$.

There exists a small neighborhood $W(T)$ of $T$ such that for any $G \in W(T)$, $\mathcal{F}_G(\hat{C}, \varepsilon, p) \subset \mathcal{F}_T(\hat{C}, \frac{\varepsilon}{2}, q)$, where $q$ is the continuation of $p$ given by structure stability with the same period $P(p) \leq k$. Thus, there exists $N$ such that for any $\phi \in \mathcal{F}_G(\hat{C}, \varepsilon, p) \subset \mathcal{F}_T(\hat{C}, \frac{\varepsilon}{2}, q)$, we have a time $1 \leq i \leq N$ such that,

$$m\{x \in M : \phi_i \cdot T(x) > C\} > \frac{1}{2} - \frac{\varepsilon}{2},$$

where $\phi_i = \sum_{j=0}^{i-1} \phi(T^j(x))$. Next, there exists a smaller neighborhood $V(T) \subset W(T)$ of $T$ such that for any $G \in V(T)$, there exists $N$ such that for any $\phi \in \mathcal{F}_G(\hat{C}, \varepsilon, p) \subset \mathcal{F}_T(\hat{C}, \frac{\varepsilon}{2}, q)$, we have there exists $1 \leq i \leq N$ such that,

$$m\{x \in M : \phi_i \cdot T(x) > \frac{C}{2}\} > \frac{1}{2} - \frac{\varepsilon}{2}.$$

Taking the uniform hyperbolicity of $T$ into account, there are only finite $p \in \mathcal{V}_k$ with period $P(p) \leq k$ for every $G \in V(T)$. Thus we get another smaller neighborhood $U(T) \subset V(T)$, such that for any $G \in U(T)$, and any constants $C > 0, \hat{C} > 0, \varepsilon > 0$ and any periodic point $p \in \mathcal{V}_k$ with period $P(p) \leq k$, $G$ is $(C, \hat{C}, \varepsilon, p)$-type.

Thus, $T \in H_k$. This completes the proof of this lemma. \hfill \Box

So Lemma 5.6 implies that $H_k$ is $C^1$ dense in $C^1$ Anosov volume-preserving diffeomorphisms and then we get that $\mathcal{G}$ is a generic set.

It is easy to see from the definition that for every diffeomorphism $G \in \mathcal{G}$ and tuple $(C, \hat{C}, \varepsilon, p)$, $G$ is $(C, \hat{C}, \varepsilon, p)$-type. By Proposition 5.3 we finish the proof. \hfill \Box

4. Proof of Theorem 1.4

The argument for Anosov flows proceeds in an almost identical fashion as in the previous section, mutatis mutandis. Theorem 1.2 and Theorem 2.3 instead of Theorem 1.1 and Theorem 2.1 are needed in the proof of Theorem 1.4.

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