Relic gravitational waves produced after preheating

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Abstract

We show that gravitational radiation is produced quite efficiently in interactions of classical waves created by resonant decay of a coherently oscillating field. For simple models of chaotic inflation in which the inflaton interacts with another scalar field, we find that today’s ratio of energy density in gravitational waves per octave to the critical density of the universe can be as large as $10^{-12}$ at the maximal wavelength of order $10^5$ cm. In the pure $\lambda\phi^4$ model, the maximal today’s wavelength of gravitational waves produced by this mechanism is of order $10^6$ cm, close to the upper bound of operational LIGO and TIGA frequencies. The energy density of waves in this model, though, is likely to be well below the sensitivity of LIGO or TIGA at such frequencies. We discuss possibility that in other inflationary models interaction of classical waves can lead to an even stronger gravitational radiation background.

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I. INTRODUCTION

Recent research in inflationary cosmology has attracted attention to highly non-equilibrium states created in a decay of a coherently oscillating field after the end of inflation. These states could support a number of non-equilibrium phenomena, such as non-thermal symmetry restoration \cite{1} and baryogenesis \cite{1,2} shortly after or during the decay of the oscillating field.

In this paper we want to show that non-equilibrium states produced by the decay of coherent oscillations of a field are a quite efficient source of a stochastic background of gravitational waves.

There are several possible processes in the early universe capable to produce a stochastic background of relic gravitational waves. One is the parametric amplification of vacuum graviton fluctuations during inflation \cite{4}. This process is efficient on all frequency scales. Waves with lowest frequencies cause inhomogeneities of cosmic microwave background \cite{5}. In conventional scenarios, this restricts the amplitude of high-frequency gravitational waves to be far below \cite{6} the experimental limits accessible for direct detection experiments in near future; this conclusion changes in superstring motivated cosmologies \cite{7}. Another source of gravitational radiation is classical emission that accompanies collisions of massive bodies. A natural source in this class in the early universe is a strongly first-order phase transition when gravitational waves are produced in collisions of bubbles of a new phase \cite{8–10}, in particular the phase transition that terminates first-order inflation \cite{9}. Gravitational radiation is also emitted during the decay of a cosmic string network \cite{11}.

In this paper we discuss a new source of relic gravitational waves. Decay of coherent oscillations of a scalar field can produce large, essentially classical fluctuations via parametric amplification (resonance) \cite{12}. These classical fluctuations, which can be viewed either as classical waves traveling through the universe or, at least qualitatively, as quantum “particles” in states with large occupation numbers, interact with the oscillating background and each other. This interaction, which we call rescattering \cite{13}, is accompanied by gravitational radiation. That is the effect we want to estimate.

Favorable conditions for an effective parametric resonance in cosmology naturally appear in inflationary models \cite{14,15}. We consider two types of simple inflationary models here. One type of models has two scalar fields with an interaction potential of the form $g^2\phi^2X^2/2$, and the resonance produces mostly fluctuations of a scalar field $X$ other than the field $\phi$ that oscillates (although subsequent rescattering processes produce large fluctuations of the field $\phi$ as well). We consider a range of moderate values of the coupling $g^2$ (see below); in this case fluctuations are not suppressed too strongly by non-linear effects (cf. Refs. \cite{16,18}). In simplest models of this type, $\phi$ is the inflaton itself. For these models, we find that typically
\(\sim 10^{-5}\) of the total energy of the universe go into gravitational waves, at the time of their production. The minimal today’s frequency \(f_{\text{min}}\) of these waves is typically of order \(10^5\) Hz, and today’s spectral density at this frequency can be as large as \(10^{-12}\) of the critical density.

Another type of model we considered was the pure \(\lambda \phi^4\) model of chaotic inflation. In this model, the minimal today’s frequency \(f_{\text{min}}\) is of order \(10^4\) Hz, close to the upper bound of the operational LIGO and TIGA frequencies [3]: \(10 \text{ Hz} \lesssim f_{\text{LIGO}} \lesssim 10^4\) Hz. We do not yet have efficient means of extrapolating our numerical results for today’s spectral intensity to the minimal frequency of this model, but we do not expect it to be above \(10^{-11}\) of the critical density. That would be well below the sensitivity of LIGO or TIGA at frequencies of order \(10^4\) Hz.

We believe, however, that in the absence of a commonly accepted specific inflationary model, it is premature to rule out possibility of experimental detection, even already by LIGO or TIGA, of gravity waves produced by the mechanism we consider here. At the end of this paper, we discuss possibility of a stronger background of gravitational waves in models with more fields or more complicated potentials.

II. GENERAL FEATURES OF POST-INFLATIONARY DYNAMICS

In scenarios where inflation ends by resonant decay of coherent field oscillations, post-inflationary dynamics has two rapid stages. At the first of these, called preheating [14], fluctuations of Bose fields interacting with the oscillating field grow exponentially fast, as a result of parametric resonance, and achieve large occupation numbers. At the second stage, called semiclassical thermalization [13], rescattering of produced fluctuations smears out the resonance peaks in power spectra and leads to a slowly evolving state, in which the power spectra are smooth [13,16,17]. The system begins to exhibit chaotic behavior characteristic of a classical non-linear system with many degrees of freedom. In the course of subsequent slow evolution, the power spectra propagate to larger momenta; we expect that eventually this process will lead to a fully thermalized state (which does not admit a semiclassical description).

In many models, we find that during the stage of semiclassical thermalization, or chaotization, fluctuations grow somewhat beyond their values at the end of the resonance stage. In such cases, the most effective graviton production takes place at the end of and shortly after the chaotization stage.

In order not to confine ourselves to any particular type of inflationary scenario, we will not assume that the oscillating field \(\phi\) is the inflaton itself. (It is distinct from the inflaton, for example, in hybrid inflationary models [19].) Let us denote the amplitude of the oscillating
field $\phi$ at the end of the chaotization stage as $\phi_{ch}$, the frequency of its oscillations as $m$, and the Hubble parameter at that time as $H_{ch}$. These same parameters at the end of inflation, when the oscillations start, will be denoted as $\phi(0)$, $m(0)$, and $H(0)$. In simplest models of chaotic inflation, in which $\phi$ is the inflaton, $\phi(0) \sim M_{Pl}$, and $H(0) \sim m(0)$. Although we will use such models for illustrative purposes, our general formulas do not assume these conditions. Oscillations of $\phi$ cannot start unless $m(0) \gtrsim H(0)$, but on the other hand, they can start at $m(0) \gg H(0)$, if they have to be triggered by some other field. Even if $H(0) \sim m(0)$, as in simple models of chaotic inflation, we still have

$$H_{ch} < H_r \ll m$$

where $H_r$ is the Hubble parameter at the end of the resonance stage. This is because the frequency of oscillations redshifts slower (if at all) than the Hubble parameter. We will use the condition (1) in what follows.

We will consider models, in which oscillating field $\phi$ interacts with a massless scalar field $X$. The Lagrangians for these models are of the form

$$L = \frac{1}{2}g^{\mu
u} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2}g^{\mu\nu} \partial_\mu X \partial_\nu X - V(\phi, X)$$

where

$$V(\phi, X) = V_\phi(\phi) + \frac{1}{2}g^2 \phi^2 X^2$$

and $V_\phi$ is a potential for the field $\phi$. We will consider two types of $V_\phi$: $V_\phi = \frac{1}{2}m^2 \phi^2$ (massive $\phi$) and $V_\phi = \frac{1}{4}\lambda \phi^4$ (massless $\phi$). The effect we discuss is not limited to these particular models and exists in a wide variety of inflationary models that have a preheating stage. Note that in the model with massless $\phi$, the frequency $m$ of oscillations of $\phi_0$ at the end of chaotization is $m \sim \sqrt{}\lambda \phi_{ch}$.

The oscillating $\phi$ amplifies fluctuations of $X$ via parametric resonance. An important parameter in the problem is the resonance parameter $q$. Depending on the choice of the potential for $\phi$ (see above), $q = g^2 \phi^2(0)/4m^2$ (massive $\phi$) or $q = g^2/4\lambda$ (massless $\phi$).

In the case of massive $\phi$, it is useful to introduce, in addition to the resonance parameter $q$, the redshifted resonance parameter at the end of the resonance stage

$$q_r = \frac{\phi^2(t_r)}{\phi^2(0)}$$

where $t_r$ is the time corresponding to the end of the resonance stage, and $\phi(t_r)$ is the amplitude of oscillations of $\phi_0$ at that time. In the model with massive $\phi$ and massless $X$, parametric resonance can fully develop in an expanding universe if $q_r \gtrsim 1$. Similarly, we can introduce
For uniformity of notation, we will sometimes use $q_r$ or $q_{ch}$ instead of $q$ in the case of massless $\phi$; there is no difference between the three of these in that case.

Resonant production is most effective for fluctuations of fields that couple to $\phi$ not too weakly but also not too strongly, those with $q_r \sim 1$. For $q_r \gg 1$, the maximal size of $X$ fluctuations is significantly suppressed by non-linear effects. Because, for instance, superstring models predict a plethora of scalar fields, we expect that some of those will have couplings in the optimal range. So, in what follows we consider moderate values of $q_r$, $1 \lesssim q_r \leq 100$.

In the model with massive $\phi$, the oscillating zero-momentum mode $\phi_0$ drops rapidly at the end of the chaotization stage, and all its energy at that time is transferred to fluctuations. The variance of $\phi$, $\langle (\delta \phi)^2 \rangle$ at the end of chaotization is thus much larger than $\phi_{ch}^2$. This does not happen in the model with massless $\phi$. So, in what follows we consider two cases: $\langle (\delta \phi)^2 \rangle \lesssim \phi_{ch}^2$ and $\langle (\delta \phi)^2 \rangle \gg \phi_{ch}^2$.

For $q \gtrsim 1$ in the model where a massless $\phi$ interacts with $X$, parametric resonance for $\phi$ itself is insignificant. To study a case different in that respect, we consider also the pure $\lambda \phi^4/4$ model, in which $\phi$ decays solely due to self-coupling ($g^2 = 0$ in Eq. (3)).

III. CALCULATIONAL PROCEDURES

Because the time scale of processes that give rise to gravitational radiation is much smaller than the time scale of the expansion of the universe, $H_{ch}^{-1}$, the energy of gravitational waves can be approximately computed starting from the well-known formulas for flat space-time. Total energy of gravitational waves radiated in direction $\mathbf{n} = \mathbf{k}/\omega$ in flat space-time is

$$\frac{dE}{d\Omega} = 2G\Lambda_{ij,lm}(\mathbf{n}) \int_0^\infty \omega^2 T^{ij*}(\mathbf{k}, \omega) T^{lm}(\mathbf{k}, \omega) d\omega$$

(6)

where $T^{ij}(\mathbf{k}, \omega)$ are Fourier components of the stress tensor, and $\Lambda_{ij,lm}$ is a projection tensor made of the components of $\mathbf{n}$ and Kronecker’s deltas.

For models with two fields ($\phi$ and $X$) with potentials and moderate $q_r$, we will present both analytical estimates and numerical calculations based on Eq. (3). For analytical estimates, we choose a specific process—gravitational bremsstrahlung that accompanies creation and annihilation of fluctuations of the field $\phi$. This is not the only process that can produce a significant amount of gravitational radiation. For example, because the mass of $X$ fluctuations oscillates due to interaction with $\phi_0$, the collection of $X$ fluctuations works
(until $\phi_0$ decays completely) as a gravitational antenna. The reason why we concentrate on bremsstrahlung from $\phi$ is that it is a significant source of gravity waves with small frequencies, which have the best chance to be detected today. Indeed, we will see that both the frequency dependence and the overall magnitude of the effect are reasonably close to those of the full intensity for small frequencies that we obtain numerically. So, bremsstrahlung from $\phi$ appears to be at least one of the main sources of gravitational radiation with small frequencies in the states we consider here.

Numerical calculations were done for simple models of chaotic inflation, in which $\phi$ is the inflaton itself. The calculations were done in conformal time, in which the evolution of the system is Hamiltonian (for more detail, see Ref. [16]), and then rescaled back to the physical time. We computed directly Fourier transforms $T^{ij}(k, \omega)$ over successive intervals of conformal time, the duration of which was taken small compared to the time scale of the expansion yet large enough to accommodate the minimal frequency of gravity waves we had in these calculations. Intensities from different intervals were summed up with the weight that takes into account expansion of the universe (see more on that below). Being a finite size effect, the minimal frequency of our numerical calculations was much larger than the actual minimal frequency of gravity waves, $\omega_{\text{min}} \sim H_{ch}$. In the models with two fields ($\phi$ and $X$), the spectrum obtained numerically is approximated reasonably well by the bremsstrahlung spectrum at small frequencies. This allows us to extrapolate the numerical results to $\omega \sim \omega_{\text{min}}$ using the frequency dependence of bremsstrahlung.

For the pure $\lambda \phi^4/4$ model, the analytical method that we use to estimate the intensity of radiation does not apply at frequencies for which we have numerical data. So, we do not have efficient means of extrapolating our numerical results to $\omega_{\text{min}}$. For this model, we contented ourselves with numerical simulations.

**IV. ANALYTICAL ESTIMATES**

In general, $X$ and $\phi$ fluctuations produced by parametric resonance scatter off the homogeneous oscillating background (condensate) of $\phi$, knocking $\phi$ out of the condensate and into modes with non-zero momenta. In cases when resonance amplifies mostly fluctuations of $X$, the scattering process can be viewed as a two-body decay of $X$, $X_k \rightarrow X'_k + \phi_p$, in the time-dependent background field of the condensate. It can also be thought of as evaporation of the inflaton condensate. There is also the inverse process (condensation).

As we noted in the introduction, one can describe the states we are considering either as collections of interacting classical waves, or as collections of “particles” in modes with large occupation numbers. For estimating the intensity of bremsstrahlung that accompanies...
creation and annihilation of fluctuations of the field $\phi$, the description of these fluctuations in terms of particles is more convenient. For the notion of particle, i.e an entity moving freely between collisions, to have anything but purely qualitative meaning, the energy of a freely moving “particle” should not be modulated too strongly by its interactions with the background.

In the models (2) with $q_r \gtrsim 1$, fluctuations of $\phi$ are reasonably well described as particles, even in the case of massless $\phi$ where they are coupled to the oscillating $\phi_0$. Indeed, typical momenta $p$ of $\phi$ fluctuations at the end of the chaotization stage are of order $m$, the frequency of oscillations of $\phi_0$ at that time. The frequency of oscillations of a mode with momentum $p \sim m$ (the would-be energy of a particle) is only moderately modulated by its coupling to the oscillating $\phi_0$. Note also that for $q_r \gtrsim 1$, the Hartree correction to the frequency squared of $\phi$, $g^2X^2$, does not exceed $m^2$ itself [17], so interaction with $X$ also does not modulate frequencies of fluctuations of $\phi$ too much.

So, let us consider fluctuations of $\phi$ as particles, neglecting modulations of their energy, and estimate the intensity of gravitational bremsstrahlung emitted by these particles in the scattering processes described above. We will assume that bremsstrahlung from $\phi$ is at least one of the main sources of gravity waves with small frequencies (we will specify the notion of “small” frequency below). For estimation purposes, we will neglect its interference with possible small-frequency radiation from other sources.

To describe radiation with small enough frequencies, we can regard the particles—quanta of $\phi$—as semiclassical wave packets. This can be seen by considering the Fourier transform of the stress tensor of a single classical particle. Using identity transformations, we write that stress tensor as

$$\omega T^{ij}_{\omega}(k, \omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{p^j p^i}{p^0 (1 - n \mathbf{v})} \frac{\partial}{\partial t} e^{i\omega t - i\mathbf{k} \cdot \mathbf{r}} dt$$

where $p^\mu$, $\mathbf{r}$, and $\mathbf{v}$ are the particle’s time-dependent four-momentum, position, and velocity. Uncertainty that we can tolerate in the position of a particle, to be able to use this formula, is of order $1/k$. Uncertainty we can tolerate in momentum is of order of the momentum spread among the particles, $\Delta p$. Thus, the semiclassical approximation applies when $\omega = k \ll \Delta p$. In the models with two fields, at the end of chaotization, $\Delta p \sim m$, so Eq. (7) applies for $\omega \ll m$.

On the other hand, in the pure $\lambda \phi^4/4$ model, fluctuations do not grow during chaotization, so for the most efficient graviton production we should consider the end of the resonance stage. The power spectrum of fluctuations at the end of resonance, and long into chaotization, is concentrated in rather narrow peaks, so that $\Delta p \ll m$. Then, Eq. (7) applies only at quite low frequencies, much lower than those we had in numerical simulations.
As a result, for this model, we could not check our analytical estimates against numerical simulations.

In the rest of this section we consider the models with two fields. For \( \omega \ll m \), not only we can use the semiclassical expression for the stress tensor of \( \phi \), but in addition each act of evaporation (or condensation) of a \( \phi \) particle can be regarded as instantaneous. So, we can use the small frequency approximation, familiar from similar problems in electrodynamics [21]. It amounts to integrating by parts in (7) and then replacing the exponential with unity. After that, the integral is trivially taken and depends only on the particle’s final (or initial) momentum.

To obtain the power \( P \) radiated by a unit volume, we substitute the small-frequency limit of Eq. (4) into Eq. (3), multiply the result by the rate at which \( \phi \) fluctuations are created (or destroyed), and integrate with the occupation numbers \( n_\phi(p) \) of \( \phi \) fluctuations. The rates of the creation and annihilation processes are almost equal (see below). With both processes taken into account, we obtain

\[
\frac{dP}{d\omega} \approx 4G \int_0^\infty n_\phi(p)R(p)F(v)\frac{p^4 dp}{(2\pi)^3}
\]

where \( P_\phi(p) \) is the power spectrum of \( \phi \), \( R(p) \) is the evaporation (condensation) rate, \( v = p/p^0 \), and the function

\[
F(v) = \frac{4}{v^2} \left( 2 - \frac{4}{3}v^2 - \frac{1 - v^2}{v} \ln \frac{1 + v}{1 - v} \right)
\]

arises after integration over direction of particle’s momentum; we have assumed that both the power spectrum and the rate depend only on the absolute value of momentum. Notice that the spectrum (8) is \( \omega \) independent, as characteristic of small-frequency bremsstrahlung.

To estimate the rate \( R \), we will consider, qualitatively, \( X \) fluctuations also as “particles” characterized by their own occupation numbers \( n_X(k) \). This is not a strictly defensible view, as \( X \) fluctuations are strongly coupled to the oscillating background, but it should do for estimation purposes.

Consider first the case \( \langle (\delta \phi)^2 \rangle \lesssim \phi_{ch}^2 \). The rate \( R(p) \) can then be estimated as

\[
R(p) \sim \frac{g^4 \phi_{ch}^2}{m} \int \frac{d^3k}{(2\pi)^3} \frac{n_X(k)n_X(k-p)}{8\omega_k \omega_{k-p} p^0}
\]

where a factor of \( 1/m \) appears instead of the usual energy delta function because energy of the “particles” participating in the process is not conserved, due to time-dependence of \( \phi_0 \). This factor estimates the time scale at which energy non-conservation sets in; in our case, it is the period of the oscillations of \( \phi_0 \).
Notice that the rate (10) is larger than the net kinetic rate, which would enter a kinetic equation for $\phi$, by a factor of order of a typical occupation number of $X$. The net kinetic rate is the difference between two (large) numbers—the rate at which collisions supply particles to a given mode and the rate at which they remove them. But each collision is accompanied by bremsstrahlung, so $\mathcal{R}$ is not the net kinetic rate but a rate of the “in” and “out” processes separately.

For moderate $q$, we can use the following estimates (cf. Refs. [13,16,17]): $p \sim k \sim \omega \sim m$, $n_\phi \sim n_X \sim 1/g^2$. Using these estimates, we obtain $\mathcal{R}(p) \sim \phi^2_{ch}/m$, for $p \sim m$, and

$$\frac{d\mathcal{P}}{d\omega} \sim \frac{m^4\phi^2_{ch}}{g^2M^2_{Pl}} \sim \frac{m^2\phi^4_{ch}}{M^2_{Pl}}$$

(11)

where in the last relation we have used $g^2 \sim q_{ch}m^2/\phi^2_{ch}$ and dropped a factor of $1/q_{ch}$.

An estimate for the total energy density of gravitational waves is obtained by multiplying the power $\mathcal{P}$ by the time $\Delta t$ during which the radiation was substantial. For the ratio of $\rho_{GW}$ per octave to the total energy density of the universe after chaotization, we obtain

$$\frac{1}{\rho_{tot}} \frac{d\rho_{GW}}{d\ln \omega} \sim \frac{m^2\phi^4_{ch}}{M^2_{Pl}\rho_{tot}} \sim \frac{\phi^2_{ch}}{M^2_{Pl}} \omega \Delta t$$

(12)

where in the last relation we used $\rho_{tot} \sim m^2\phi^2_{ch}$. The time during which the radiation was substantial is determined by the redshift, $\Delta t \sim H^{-1}_{ch}$.

For the opposite case $\langle(\delta \phi)^2 \rangle \gg \phi^2_{ch}$, we need to replace $\phi^2_{ch}$ in the estimate (10) by $\langle(\delta \phi)^2 \rangle$. Now using $\langle(\delta \phi)^2 \rangle \sim m^2/g^2$ and $\rho_{tot} \sim m^2 \langle(\delta \phi)^2 \rangle$, we see that we need to make the same replacement in Eq. (12).

These estimates apply at the end of the chaotization stage and extend into the infrared up to frequencies of the order of the horizon scale at that time, $\omega \sim H_{ch}$. (We will present estimates for today’s $\rho_{GW}$ below.)

As an example, consider the model with massive $\phi$ in which $\phi$ is the inflaton itself, and $m = 10^{-6}M_{Pl}$. Take $q_r \sim 1$, which in this case corresponds to $q \sim 10^4$ [10]. The inflaton completely decays into fluctuations during the chaotization stage, and at the end of it $\langle(\delta \phi)^2 \rangle \sim 10^{-6}M_{Pl}$ [17]. Using $\langle(\delta \phi)^2 \rangle$ in place of $\phi^2_{ch}$ in Eq. (12), we obtain $\rho_{tot}^{-1}d\rho_{GW}/d\ln \omega \sim 10^{-6}$ at $\omega \sim H_{ch}$.

Our analytical estimates are rather crude, as we were somewhat cavalier with numerical factors. They do bring out, however, not only the frequency dependence of the effect but to some extent also its overall magnitude. Notice, for example, that the last estimate in Eq. (12) does not contain powers of $g^2$ or $m^2$ (these two can be expressed through each other and $\phi^2_{r}$, using the condition $q_r \sim 1$). These quantities, if present, would change the estimate by many orders of magnitude, in drastic disagreement with our numerical results.
V. BACKGROUND GRAVITATIONAL RADIATION TODAY

Let us now translate the above estimates into estimates for gravitational background radiation today. It is important that gravitational radiation produced after preheating has not interacted with matter since then [22].

First, let us estimate the physical wavelength today, \(l = 2\pi/k_0\), that corresponds to a wave vector \(k_{ch}\) at the end of the chaotization stage. We have \(k_0 = k_{ch}a_{ch}/a_0\), where \(a_0\) is the scale factor today. Thermal equilibrium was established at some temperature \(T_*\), and the universe was radiation dominated at that time. This happened when relevant reaction rates became equal to the expansion rate. We can find the Hubble parameter at that time as \(H_* = 8\pi G\rho_*/3\), where \(\rho_* = g_*\pi^2T_*^4/30\), and \(g_* \equiv g(T_*)\); \(g(T)\) stands for the effective number of ultrarelativistic degrees of freedom at temperature \(T\). The expansion factor from \(T = T_*\) down to \(T = T_0 \approx 3K\) is given by \(a_*/a_0 = (g_0/g_*)^{1/3}(T_0/T_*) = (g_0^{1/3}/g_*^{1/12})(8\pi^3/90)^{1/4}(T_0/\sqrt{H_*M_{Pl}})\), where \(g_0 \equiv g(T_0)\). Depending upon model parameters, the universe could expand from \(T = T_{ch}\) to \(T = T_*\) as matter or radiation dominated. We can write \(H_* = H_{ch}(a_{ch}/a_0)^\alpha\), where \(\alpha = 2\) for the radiation dominated case and \(\alpha = 3/2\) for a matter dominated universe. We obtain \(k_0 = k_{ch}a_{ch}/a_0 = k_{ch}(a_{ch}/a_0)(a_*/a_0) \approx 1.2k_{ch}(H_{ch}M_{Pl})^{-1/2}(a_{ch}/a_0)^{1-\alpha/2}T_0\). The model dependent factor \((a_{ch}/a_0)\) does not enter this relation in the radiation dominated case and enters in power 1/4 for a matter dominated universe, i.e. this factor gives a not very important correction (at most an order of magnitude). For today’s wavelength, corresponding to wave vector \(k_{ch}\) at the end of chaotization, we find \(l = 2\pi/k_0 \approx 0.5(M_{Pl}H_{ch})^{1/2}k_{ch}^{-1}(a_*/a_{ch})^{1-\alpha/2}\) cm.

The smallest wave vector of radiation that could be produced at the end of chaotization is of order \(H_{ch}\). The corresponding maximal today’s wavelength \(l_{max}\) will fall into the range of the LIGO detector, i.e. \(l_{max} > 3 \times 10^6\) cm, when \(H_{ch} < 10^5-10^6\) GeV.

For illustration, let us consider simple models of chaotic inflation with potentials \(\lambda \phi^4/4\), where the oscillating field \(\phi\) is the inflaton itself. The Hubble parameter at the end of the chaotization stage in these models can be extracted from numerical integrations of Refs. \[13,16,17\]. For example, for massless inflaton we get \(H_{ch}/M_{Pl} = (3\lambda/2\pi)^{1/2}\tau_{ch}^{-2}\), where \(\tau_{ch}\) is conformal time at the end of the chaotization stage. For \(\lambda = 10^{-13}\) and, say, \(q = 30\) we get (see Fig. 1) \(\tau_{ch} \approx 125\). This gives \(H_{ch}/M_{Pl} \approx 1.4 \times 10^{-11}\) and \(l_{max} \approx 1.3 \times 10^5\) cm. Another example is the case of massive inflaton with \(m = 10^{-6}M_{Pl}\) and \(q = 10^4\). Here we find \(H_{ch}/M_{Pl} \approx 2 \times 10^{-9}\). This gives \(l_{max}\) in the range \(10^{4}\text{–}10^{5}\text{ cm}.\)

An interesting case is that of the pure \(\lambda \phi^4/4\) model (\(g^2 = 0\) in \[3\]). In this case, fluctuations do not grow during chaotization (they actually decrease due to redshift), so the most efficient production of gravity waves takes place after the end of the resonance
(preheating) stage. So, instead of $H_{ch}$ in the above formulas we use $H_r$, the Hubble parameter at the end of resonance. We have $H_r/M_{Pl} = (3\lambda/2\pi)^{1/2} \tau_{r}^{-2}$, where $\tau_r$ is conformal time at the end of resonance. In this model, resonance develops slower than in models with $q_r \sim >1$. As a result, $H_r$ is smaller, and today’s maximal wavelength is larger. For $\lambda = 10^{-13}$, we get $\tau_r \approx 500$ \cite{13}, and $H_r/M_{Pl} \approx 9 \times 10^{-13}$, which gives, for radiation produced at that time, $l_{max} \approx 5 \times 10^{5}$ cm. Note, that by the time $\tau \approx 1000$ the inflaton still has not decayed in this model, fluctuations are still highly non-equilibrium and gravity waves generation will continue even at later time. This will increase $l_{max}$ by at least another factor of two bringing it close to the upper boundary of operational LIGO and TIGA frequencies.

Now, let us estimate today’s intensity of radiation. The today’s ratio of energy in gravity waves to that in radiation is related to $(\rho_{GW}/\rho_{tot})_{ch}$ via

$$\left(\frac{\rho_{GW}}{\rho_{rad}}\right)_0 = \left(\frac{\rho_{GW}}{\rho_{tot}}\right)_{ch} \left(\frac{a_{ch}}{a_*}\right) 4^{-2\alpha} \left(\frac{g_0}{g_*}\right)^{1/3} \quad \text{(13)}$$

For the often used parameter $\Omega_g(\omega) \equiv (\rho_{crit}^{-1}d\rho_{GW}/d\ln\omega)_0$, where $\rho_{crit}$ is the critical energy density, we obtain, using (12),

$$\Omega_g(\omega) h^2 \sim \Omega_{rad} h^2 \frac{g_{ch}^2}{M_{Pl}^2} \omega \left(\frac{a_{ch}}{a_*}\right) 4^{-2\alpha} \left(\frac{g_0}{g_*}\right)^{1/3} \quad \text{(14)}$$

where $\Omega_{rad}$ is today’s value of the ratio $\rho_{rad}/\rho_{crit}$: $\Omega_{rad} h^2 = 4.31 \times 10^{-5}$ \cite{24}. When the oscillating zero-momentum mode decays completely during the stage of chaotization, $\phi_{ch}$ in (14) is replaced by $\langle (\delta\phi)^2 \rangle_{ch}$.

For example, in the case of massless inflaton with $\lambda = 10^{-13}$ and $q = 30$, we have $\phi_{ch}^2 \sim 10^{-5} M_{Pl}^2$, and with $g_*/g_0 \sim 100$ the estimate (14) gives $\Omega_g h^2 \sim 10^{-10}$ at $l_{max} \sim 10^5$ cm.

VI. NUMERICAL RESULTS

We have studied numerically gravitational radiation in the model where $\phi$ is the massless inflaton with $V_\phi = \lambda \phi^4/4$, $\lambda = 10^{-13}$, interacting with a massless field $X$ according to Eq. (3). We have used the fully non-linear method of Ref. \cite{13}. We have solved equations of motion on a lattice $128^3$ in a box with periodic boundary conditions.

The model is classically conformally invariant, so by going to conformal variables, the equations of motion can be reduced to those in flat space-time.

Time evolution of the variances $\langle X^2 \rangle$ and $\langle (\delta\phi)^2 \rangle$ (in rescaled units) for resonance parameter $q = 30$ is shown in Fig. 1. The rescaled conformal time $\tau$ is related to time $t$ by $\sqrt{\lambda}\phi(0)dt = a(\tau)d\tau/a(0)$. The rescaled conformal fields $\chi$ and $\varphi$ are related to the original
fields by $X = \chi \phi(0)a(0)/a(\tau)$ and $\varphi = \phi \phi(0)a(0)/a(\tau)$. In this model, $\phi(0) \approx 0.35M_{Pl}$ and $a(\tau)/a(0) \approx 0.51\tau + 1$ \[10\].

We see that for the above values of parameters, parametric resonance ends at $\tau \approx 73$. The resonance stage is followed by a plateau (cf. Refs. [13,17]). At the plateau, the variances of fluctuations do not grow, but an important restructuring of the power spectrum of $X$ takes place. The power spectrum of $X$ changes from being dominated by a resonance peak at some non-zero momentum to being dominated by a peak near zero. (For some $q$, though, the strongest peak is close to zero already during the resonance.) When the peak near zero becomes strong enough, the growth of variances resumes (in Fig. 1, that happens at $\tau \approx 84$), and the system enters the chaotization stage.

The power spectrum of $\phi$ during the time interval of Fig. 1 is shown in Fig. 2. The rescaled comoving momentum $k$ is related to physical momentum $k_{\text{phys}}$ as $k_{\text{phys}} = \sqrt{\lambda \phi(0)}a(0)k/a(\tau)$. Note that in the range $1 \lesssim k \lesssim 10$, the power spectrum is approximately a power law, as characteristic of Kolmogorov spectra [24].

Interaction of the fields with gravity is not conformal, and the flat-space-time formula (3) for the energy of gravity waves can be used only after we make an approximation. The approximation replaces the actual expanding universe with a sequence of static universes. Specifically, conformal time was divided into steps of $\Delta \tau = 2\pi L$, where $L$ is the size of the integration box, and at each step the physical variables (fields, frequency, and momenta) were obtained from the conformal ones, using for $a(\tau)$ the actual scale factor taken at the middle point of the step. The energy of gravitational waves was computed at each step using (3), with the corresponding physical variables. Then, energies from all the steps were summed up.

Today’s $\Omega_g$, in physical units, that had been accumulated by conformal time $\tau = 200$ is shown in Fig. 3 for two values of $q$: $q = 30$ (dashed line) and $q = 105$ (dotted line). The solid line corresponds to the pure $\lambda \phi^4$ model (no interaction with $X$ field) and includes contributions from times up to $\tau = 1600$. In the latter case, the peaks in $\Omega_g$ at frequencies $f \gtrsim 4 \times 10^7$ Hz, seen in the figure, are correlated with peaks in the power spectrum of $\phi$ at early stages of chaotization, cf. Ref. [13]. Since we included contributions to $\Omega_g$ from times long after the time when the peaks in the power spectrum disappeared, and yet the peaks in $\Omega_g$ had not been washed out, we suggest that in this model the peaks in $\Omega_g$ are a feature potentially observable at present. Observation of such peaks could select a particular model of inflation.

We see also that in the model with two interacting fields ($\phi$ and $X$), the linear $\omega$ dependence at small $\omega$, characteristic of bremsstrahlung, is overall well born out, both for $q = 30$ and $q = 105$. The minimal today’s frequency in this model is $f_{\text{min}} \sim 10^5$ Hz. Extrapolating
the results of Fig. 3 to that minimal frequency using the linear law, we obtain the magnitude of $\Omega_g h^2$ of order $10^{-12}$ for $q = 30$ and of order $10^{-13}$ for $q = 105$. Recall that the analytical estimate (12) (using $\phi_{ch}^2 \sim 10^{-5} M_{Pl}^2$) gave $\Omega_g h^2$ of order $10^{-10}$ at the minimal frequency.

In the pure $\lambda \phi^4/4$ model, we could not discern any simple pattern of frequency dependence for $\Omega_g$ at small frequencies. This is consistent with our discussion of this model in Sect. IV: if the linear frequency dependence sets in at all in this model, that happens only at frequencies much smaller than those in Fig. 3. This makes it difficult for us to extrapolate our numerical results for this model to the maximal today’s wavelength, $l_{max} \sim 10^6$ cm, or, equivalently, minimal frequency $f_{\text{min}}$ of order $10^4$ Hz. We have no reason to suppose, however, that $\Omega_g h^2$ can actually increase to small frequencies, beyond its value of order $10^{-11}$ at $f \sim 10^6$ Hz.

To confirm that Fig. 3 is not a numerical artifact, we have made runs in which the interaction was switched off, i.e. $g^2$ was set to zero, at $\tau = 100$, for the case $q = 30$. To exclude also the effect of self-interaction, the term $\lambda \phi^4$ was replaced at that time by $m^2 \phi^2$. The system then becomes a collection of free particles (plus the oscillating zero-momentum background) and, if solved exactly in infinite space, should not radiate. In our simulations, the intensity of radiation during the interval from $\tau = 100$ to $\tau = 125$ was three orders of magnitude smaller than it was during the same time interval with the interaction present.

VII. CONCLUSION

We have shown that gravitational radiation is produced quite efficiently in interactions of classical waves created by resonant decay of a coherently oscillating field. For simple models of chaotic inflation in which the inflaton interacts with another scalar field, we find that today’s ratio of energy density in gravitational waves per octave to the critical density of the universe can be as large as $10^{-12}$ at the maximal wavelength of order $10^5$ cm. In the pure $\lambda \phi^4$ model, the maximal today’s wavelength of gravitational waves produced by this mechanism is of order $10^6$ cm, close to the upper bound of operational LIGO and TIGA frequencies. The energy density of waves in this model likely to be well below the sensitivity of LIGO or TIGA at such frequencies.

In other types of inflationary models (or even with other parameters) the effect can be much stronger. We do not exclude that among these there are cases in which it can be observable already by LIGO or TIGA. The relevant situations are:

1. At some values of the coupling constant $g^2$ (or the resonance parameter $q$), the most resonant momenta are close to $k = 0$. In the model with massless inflaton this happens, for example, for $q = 100$ (and does not happen for $q = 30$ or $q = 105$, which we
discussed so far; in these cases, the most resonant momenta are at \( k \sim 1 \). The lowest frequency that we had in the box in our numerical simulations for \( q = 100 \) was \( f \approx 8 \times 10^6 \) Hz. At that frequency, \( \Omega_g \) for \( q = 100 \) was almost two orders of magnitude larger than for \( q = 105 \). In addition, the entire spectrum of gravitational waves appears to be shifted to the left with respect to the spectra shown in Fig. 3. Cases when resonance is “tuned” to be close to \( k = 0 \) deserve further study. Question remains, how large, in such cases, the intensity of gravitational waves can be at the horizon scale, \( H_{ch} \).

2. It is important to consider in detail models of hybrid inflation [19], where the oscillating field need not be the inflaton itself, and so the frequency of the oscillations may be unrelated to the inflaton mass.

3. In models where large fluctuations produced at preheating cause non-thermal phase transitions, as suggested in Ref. [1], domains or strings can form. A large amount of gravitational radiation can be produced in collisions of domain walls, in a way somewhat similar to how it happens [9,10] in models of first-order inflation, or in decays of a string network, cf. Refs. [11]. In particular, in cases when domains are formed, intensity of gravitational radiation at the horizon scale, at the moment when the domain structure disappears, is expected to be much larger than in cases without domains.

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REFERENCES

[1] L. Kofman, A. Linde, and A. A. Starobinsky, Phys. Rev. Lett. 76, 1011 (1996); I. I. Tkachev, Phys. Lett. B376, 35 (1996).

[2] E. W. Kolb, A. D. Linde, and A. Riotto, hep-ph/9606260; M. Yoshimura, hep-ph/9605246.

[3] See e.g., K. S. Thorne, In: Particle and Nuclear Astrophysics and Cosmology in the Next Millennium, Snowmass 94: Proceedings, Eds.: E. W. Kolb and R. D. Peccei (World Scientific, Singapore, 1995).

[4] L. P. Grishchuk, Sov. Phys. JETP 40, 409 (1975).

[5] V. A. Rubakov, M. V. Sazhin, A. V. Veryaskin, Phys. Lett. 115B, 189 (1982); R. Fabbri and M. D. Pollock, Phys. Lett. 125B, 445 (1983); R. L. Davis, H. M. Hodges, G. F. Smoot, P. J. Steinhardt, and M. S. Turner, Phys. Rev. Lett. 69, 1856 (1992).

[6] M. S. Turner, astro-ph/9607066.

[7] R. Brustein, M. Gasperini, M. Giovannini, and G. Veneziano, Phys. Lett. B361, 45 (1995); M. Gasperini, hep-th/9607146.

[8] C. J. Hogan, Mon. Not. Roy. Astron. Soc. 218, 629 (1986); E. Witten, Phys. Rev. D30, 272 (1984); L. M. Krauss, Phys. Lett. B284, 229 (1992).

[9] M. S. Turner and F. Wilczek, Phys. Rev. Lett. 65, 3080 (1990).

[10] A. Kosowsky, M. S. Turner, and R. Watkins, Phys. Rev. Lett. 69, 2026 (1992); Phys. Rev. D45, 4514 (1992); A. Kosowsky and M. S. Turner, Phys. Rev. D47, 4372 (1993); M. Kamionkowski, A. Kosowsky, and M. S. Turner, Phys. Rev. D49, 2837 (1994).

[11] T. Vachaspati and A. Vilenkin, Phys. Rev. D31, 3052 (1985); B. Allen and E. P. S. Shellard, Phys. Rev. D45, 1898 (1992); R. R. Caldwell and B. Allen, Phys. Rev. D45, 3447 (1992); R. R. Caldwell, R. A. Battye, and E. P. S. Shellard, astro-ph/9607130.

[12] Comprehensive discussion of parametric resonance in quantum field theory was given by A. A. Grib, S. G. Mamaev, and V. M. Mostepanenko, Quantum Effects in Strong External Fields [in Russian] (Atomizdat, Moscow, 1980).

[13] S. Khlebnikov and I. Tkachev, Phys. Rev. Lett. 77, 219 (1996).

[14] L. Kofman, A. Linde, and A. A. Starobinsky, Phys. Rev. Lett. 73, 3195 (1994).

[15] J. H. Traschen and R. H. Brandenberger, Phys. Rev. D 42, 2491 (1990); Y. Shtanov,
J. Traschen, and R. Brandenberger, Phys. Rev. D 51, 5438 (1995); D. Boyanovsky, H. J. de Vega, R. Holman, D.-S. Lee, and A. Singh, Phys. Rev. D 51, 4419 (1995); D. Boyanovsky, M. D’Attanasio, H. J. de Vega, R. Holman, and D.-S. Lee, Phys. Rev. D 52, 6805 (1995).

[16] S. Khlebnikov and I. Tkachev, hep-ph/9608458, Phys. Lett. B390, 80 (1997).

[17] S. Khlebnikov and I. Tkachev, hep-ph/9610477.

[18] T. Prokopec and T. G. Roos, hep-ph/9610400.

[19] A. D. Linde, Phys. Lett. B259, 38 (1991); Phys. Rev. D49, 748 (1994).

[20] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972), Sect. 10.4.

[21] L. D. Landau and E. M. Lifshitz, Classical Theory of Fields (Pergamon, Oxford, 1975), Sect. 69; J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1975), Sect. 15.1.

[22] Ya. B. Zel’dovich and I. D. Novikov. Relativistic Astrophysics, Volume 2, The Structure and Evolution of the Universe. (U. Chicago Press, Chicago, 1983), Sect. 7.2.

[23] E. W. Kolb and M. S. Turner, The Early Universe (Addison-Wesley, Redwood City, California, 1990), p. 76.

[24] V. E. Zakharov, Kolmogorov Spectra of Turbulence, In: Handbook of Plasma Physics, Eds.: M. M. Rosenbluth and R. Z. Sagdeev (Elsevier, 1984), Vol. 2, p. 3.
FIG. 1. Variances of fields $X$ (solid curve) and $\phi$ (dotted curve) as functions of conformal time in the model with massless inflaton for $\lambda = 10^{-13}$ and $q = 30$.

FIG. 2. Power spectrum of the field $\phi$ for the same model as in Fig. 1, output every period at the maxima of $\varphi_0(\tau)$; $k$ is rescaled comoving momentum (see text).
FIG. 3. Today’s spectral density of gravitational waves in the pure $\lambda\phi^4$ model (solid line) and in the model where interaction $g^2\phi^2 X^2$ with a massless scalar field $X$ is added; the dashed line corresponds to $q = 30$, and the dotted line corresponds to $q = 105$, where $q = g^2/4\lambda$. We used $g^*/g_0 = 100$. 