Scalar mesons in the Nambu–Jona-Lasinio model with the ’t Hooft interaction.

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Abstract

We calculate the mass spectra of the pseudoscalar and scalar meson nonets in the Nambu–Jona-Lasinio model with the ’t Hooft interaction. We obtain satisfactory result for the pseudoscalar mesons. For the scalar mesons, the ’t Hooft interaction somewhat increases the values of the masses. However, it is not sufficient to explain the whole scalar mass spectrum. The situation could be improved for the $\sigma$ and $f_0$ mesons through mixing with the glueball state. For the description of the masses of $a_0$ and $K^*_0$ mesons, it is necessary to involve the other models. The strong decay widths of the scalar mesons are described.

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1. Introduction

During the last years noticeable progress has been achieved in both experimental and theoretical investigations of scalar mesons. The low-mass sigma meson appeared in the last Review of Particle Properties (1996) [1]. A new theoretical analysis of experimental data on the low-mass sigma meson has been completed in the papers [2, 3]. Scalar mesons in the NJL model with the ’t Hooft interaction were investigated in [4, 5]. In the papers [5] the coupling of the $\bar{q}q$ states to the two-pion continuum and a model of confinement were also included into consideration. Here we continue these investigations in the framework of the standard NJL model with the ’t Hooft interaction, following the papers [4]–[6].

Our work is organized as follows. In Sec. 2, we describe the characteristics of our NJL model with the ’t Hooft interaction. We define the main parameters of this model - the cut-off parameter $\Lambda$ and the constituent u-quark mass $m_u$ using the experimental values of the pion decay coupling constant $F_\pi=93$ MeV, the $\rho$-meson coupling constant $g_\rho = 6.14 \left(\frac{g_\rho^2}{4\pi} \approx 3\right)$, describing the decay $\rho \rightarrow 2\pi$, and the relation $g_\rho = \sqrt{6}g_\sigma$. In Sec. 3, we describe the masses of the isovector and strange mesons. In Sec. 4, we calculate the quark-loop contributions to the masses of the $\eta, \eta', \sigma$ and $f_0$ mesons. In Sec. 5, the numerical estimations of the quark-loop contributions to the meson masses are carried out. In Sec. 6 the strong decay widths of the scalar mesons are estimated. Sec. 7 contains discussion and the conclusion.

2. The NJL model with the ’t Hooft interaction

The U(3)×U(3) version of the NJL model supplemented by the ’t Hooft interaction takes the form

\[
L = \bar{q}(i\hat{\partial} - m^0)q + \frac{G}{2} \sum_{i=0}^{8} [(\bar{q}\lambda_i q)^2 + (\bar{q}i\gamma_5 \lambda_i q)^2] - K \{\text{det}[\bar{q}(1 + \gamma_5)q] + \text{det}[\bar{q}(1 - \gamma_5)q]\} \quad (1)
\]

where $\lambda_i$ (i=1,...,8) are the Gell-Mann matrices and $\lambda^0 = \sqrt{\frac{2}{3}} \mathbf{1}$, with $\mathbf{1}$ being the unit matrix; $m^0$ is a current quark mass matrix with diagonal elements $m^0_u$, $m^0_d$, $m^0_s$ ($m^0_u \approx m^0_d$). The Lagrangian (1) can be rewritten in the form (see [3])

\[
L = \bar{q}(i\hat{\partial} - m^0)q + \frac{1}{2} \sum_{i=1}^{9} [G_i^{(-)} (\bar{q}\tau_i q)^2 + G_i^{(+)} (\bar{q}i\gamma_5 \tau_i q)^2] + G_{us}^{(-)} (\bar{q}\lambda_u q)(\bar{q}\lambda_s q) + G_{us}^{(+)} (\bar{q}i\gamma_5 \lambda_u q)(\bar{q}i\gamma_5 \lambda_s q) \quad (2)
\]

where

$$
\tau_i = \lambda_i \quad (i = 1, ..., 7), \quad \tau_8 = \lambda_u = (\sqrt{2}\lambda_0 + \lambda_8)/\sqrt{3},
$$
\[ \tau_9 = \lambda_s = (-\lambda_0 + \sqrt{2\lambda_8})/\sqrt{3}, \]
\[ G_1^{(\pm)} = G_2^{(\pm)} = G_3^{(\pm)} = G \pm 4Km_s I_1(m_s), \]
\[ G_4^{(\pm)} = G_5^{(\pm)} = G_6^{(\pm)} = G_7^{(\pm)} = G \pm 4Km_u I_1(m_u), \]
\[ G_8^{(\pm)} = G \mp 4Km_s I_1(m_s), \quad G_9^{(\pm)} = G, \quad G_{10}^{(\pm)} = \pm 4\sqrt{2}Km_u I_1(m_u). \] (3)

Here \( m_u \) and \( m_s \) are the constituent quark masses and

\[ I_n^\Lambda(m_i) = \frac{N_c}{(2\pi)^4} \int d^4k \frac{\theta(\Lambda^2 - k^2)}{(k^2 + m_i^2)^n}. \] (4)

Here we have used the Euclidean space and the cut-off parameter \( \Lambda \).

Let us define the parameters \( m_u \) and \( \Lambda \), following the papers \[7, 8, 9\]. We shall use the four equations

1) the Goldberger-Treiman relation

\[ g_{\pi \bar{q}q} = \frac{m_u}{F_\pi}, \]

(5)

2) the relation between \( g_\rho \) and \( g_{\sigma \bar{q}q} \) \[7]-[10]

\[ g_\rho = \sqrt{6}g_{\sigma \bar{q}q}. \] (6)

3) the relation between \( g_{\sigma \bar{q}q} \) and \( g_{\pi \bar{q}q} \), which was obtained by taking into account the \( \pi - a_1 \) transitions (see \[8, 9\])

\[ g_{\pi \bar{q}q} = Z^{-1/2}g_{\sigma \bar{q}q}, \quad Z = \left(1 - \frac{6m_u^2}{M_{a_1}^2}\right)^{-1}, \] (7)

where \( M_{a_1} \) is the \( a_1 \)-meson mass;

4) the expression for \( g_{\sigma \bar{q}q} \) through a logarithmically divergent integral \[7, 8\]

\[ g_{\sigma \bar{q}q}^2 = [4I_2^\Lambda(m_u)]^{-1}. \] (8)

From equations (3)-(8) it is possible to express the constituent \( u \)-quark mass through the observable values \( F_\pi = 93 \text{ MeV}, \quad g_\rho \approx 6.14 \) and \( M_{a_1} = 1230 \text{ MeV} \)

\[ m_u^2 = \frac{M_{a_1}^2}{12} \left[ 1 - \sqrt{1 - \left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2} \right], \quad m_u = 280 \text{ MeV}. \] (9)

Then, from (8) and (8) we define

\[ \Lambda = 1250 \text{ MeV}. \] (10)
3. The masses of the isovector and strange mesons

Now let us first consider bosonization of the diagonal parts of the Lagrangian \( \mathcal{L} \) including the isovector and strange mesons. In Sec. 4, we complete the bosonization of the last part of the Lagrangian \( \mathcal{L} \) containing the nondiagonal terms.

After renormalization of the meson fields we obtain \[1, 2, 3\]

\[
L(\pi, K, a_0, K_0^*) = -\frac{g_\pi^2 \pi^2}{2G_\pi} - \frac{g_K^2}{G_K} K^2 - \frac{g_{a_0}^2 \phi^2}{2G_{a_0}} - \frac{g_{K_0}^2}{G_{K_0}} K_0^{'2} - \frac{1}{i\partial - M} \left\{ 1 + \sum_{i=1}^7 \left( g_{\phi_i} i\gamma_5 \lambda_i \phi_i + g_{\sigma_i} \lambda_i \sigma_i \right) \right\},
\]

(11)

where \( \phi_i \) and \( \sigma_i \) are the pseudoscalar and scalar fields, respectively, \( \pi^2 = \pi^0^2 + 2\pi^+\pi^- \), \( K^2 = K^0 K^0 + K^+ K^- \), \( a_0^2 = a_0^0 + 2a_0^+ a_0^- \), \( K_0^{'2} = K_0^* K_0^* + K_0^+ K_0^- \)

\[
G_\pi = G + 4K_m I_1(m_s), \quad G_K = G_\pi - 4K(m_s I_1(m_s) - m_u I_1(m_u)), \\
G_{a_0} = G_\pi - 8K_m I_1(m_s), \quad G_{K_0} = G_\pi - 4K(m_s I_1(m_s) + m_u I_1(m_u)),
\]

(12)

\[
g_{a_0}^2 = [4I_2(m_u)]^{-1}, \quad g_{K_0}^{'2} = [4I_2(m_u, m_s)]^{-1}, \quad I_2(m_u, m_s) = \frac{N_c}{(2\pi)^4} \int d^4k \frac{\theta(\Lambda^2 - k^2)}{(k^2 + m_s^2)(k^2 + m_u^2)} = \frac{3}{(4\pi)^2(m_s^2 - m_u^2)} \left[ m_s^2 \ln \left( \frac{\Lambda^2}{m_s^2} + 1 \right) - m_u^2 \ln \left( \frac{\Lambda^2}{m_u^2} + 1 \right) \right],
\]

(13)

\[
g_\pi = Z_\pi^{1/2} g_{a_0}, \quad g_K = Z_K^{1/2} g_{K_0}, \quad Z_\pi \approx Z_K \approx 1.44.
\]

Then, in the one-loop approximation, the following expressions for the meson masses are obtained \[3\]

\[
M_\pi^2 = g_\pi^2 \left[ \frac{1}{G_\pi} - 8I_1(m_u) \right], \\
M_K^2 = g_K^2 \left[ \frac{1}{G_K} - 4[I_1(m_u) + I_1(m_s)] \right] + Z(m_s - m_u)^2, \\
M_{a_0}^2 = g_{a_0}^2 \left[ \frac{1}{G_{a_0}} - 8I_1(m_u) \right] + 4m_u^2, \\
M_{K_0}^{'2} = g_{K_0}^{'2} \left[ \frac{1}{G_{K_0}} - 4[I_1(m_u) + I_1(m_s)] \right] + (m_u + m_s)^2.
\]

(14)

The first relation is used to define the parameter \( G_\pi \). For the experimental values \( M_{\pi^0} = 135 \text{ MeV} \) we have

\[
G_\pi = 4.92 \text{ GeV}^{-2}.
\]

(15)
4. The masses of the $\eta$, $\eta'$, $\sigma$ and $f_0$ mesons

The non-diagonal part of the Lagrangian (3) has the form

$$\Delta L = \frac{1}{2} \left\{ G^{(+)}_u(qi\gamma_5\lambda_u\eta) + 2G^{(+)}_{us}(\bar{q}\gamma_5\lambda_u\eta) + G^{(+)}_s(\bar{q}\gamma_5\lambda_s\eta) + \right.$$  
$$+ G^{(-)}_u(\bar{q}\lambda_u\eta) + 2G^{(-)}_{us}(\bar{q}\lambda_s\eta) + G^{(-)}_s(\bar{q}\lambda_s\eta) \} = (\bar{q}\gamma_5\lambda_u\eta)T^{P}_{\alpha\beta}(\bar{q}\gamma_5\lambda_\beta\eta), \ (\alpha = u, s) \ (\beta = u, s)$$  

where

$$G^{(+)}_u = G_\pi - 8Km_sI_1(m_s), \quad G^{(-)}_u = G_\pi,$$
$$G^{(+)}_s = G_\pi - 4Km_sI_1(m_s), \quad G^{(-)}_{us} = \pm 4\sqrt{2}Km_uI_1(m_u),$$  

$$T^{P(S)} = \frac{1}{2} \left( \begin{array}{ccc} G^{(\pm)}_u & G^{(\pm)}_{us} & G^{(\pm)}_s \\ G^{(\pm)}_{us} & G^{(\pm)}_u & G^{(\pm)}_s \\ G^{(\pm)}_s & G^{(\pm)}_s & G^{(\pm)}_s \end{array} \right)$$  

After bosonization we obtain

$$\Delta \bar{L} = -\frac{g_{\eta_u}g_{\eta_\beta}}{4}\eta_\alpha(T^{P})_{\alpha\beta}^{-1}\eta_\beta - \frac{g_{\sigma_u}g_{\sigma_\beta}}{4}\sigma_\alpha(T^{S})_{\alpha\beta}^{-1}\sigma_\beta -$$  
$$-i \operatorname{Tr} \ln \left\{ 1 + \frac{1}{i\partial - M} [g_{\eta_u}i\gamma_5\lambda_u\eta_\alpha + g_{\sigma_u}\lambda_\alpha\sigma_\alpha] \right\}$$  

where

$$(T^{P(S)})^{-1} = \frac{2}{G^{(\pm)}_uG^{(\pm)}_s - (G^{(\pm)}_{us})^2} \left( \begin{array}{ccc} G^{(\pm)}_u & -G^{(\pm)}_{us} \\ -G^{(\pm)}_{us} & G^{(\pm)}_s \end{array} \right),$$  

$$g_{\sigma_u} = g_{\sigma_\eta_u}, \quad g_{\sigma_s} = [4I^{N}_2(m_s)]^{-1/2}, \quad g_{\eta_u} = g_{\eta_\pi u}, \quad g_{\eta_s} = Z^{1/2}g_{\sigma_s}.$$  

From the Lagrangian (19), in the one-loop approximation, the following expressions for the mass terms are obtained

$$\Delta L^{(2)} = -\frac{g_{\eta_u}g_{\eta_\beta}}{4}\eta_\alpha(T^{P})_{\alpha\beta}^{-1}\eta_\beta - \frac{g_{\sigma_u}g_{\sigma_\beta}}{4}\sigma_\alpha(T^{S})_{\alpha\beta}^{-1}\sigma_\beta +$$  
$$+ 4I_1(m_u)(g^{2}_{\eta_u}\eta^2_u + g^{2}_{\sigma_u}\sigma^2_u) + 4I_1(m_s)(g^{2}_{\eta_s}\eta^2_s + g^{2}_{\sigma_s}\sigma^2_s) -$$  
$$-2(m^2_u\sigma^2_u + m^2_s\sigma^2_s) = \frac{1}{2} \left\{ \eta_\alpha M^{P}_{\alpha\beta}\eta_\beta + \sigma_\alpha M^{S}_{\alpha\beta}\sigma_\beta \right\}$$

where

$$M^{P}_{uu} = g^{2}_{\eta_u} \left( \frac{1}{2}(T^{P})^{-1}_{uu} - 8I_1(m_u) \right),$$
$$M^{P}_{ss} = g^{2}_{\eta_s} \left( \frac{1}{2}(T^{P})^{-1}_{ss} - 8I_1(m_s) \right),$$
$$M^{P}_{us} = \frac{1}{2}g_{\eta_u}g_{\eta_s}(T^{P})^{-1}_{us},$$
\[ M_{uu}^S = g_{\sigma_u}^2 \left( \frac{1}{2} (T^S)^{-1}_{uu} - 8 I_1(m_u) \right) + 4m_u^2, \]
\[ M_{ss}^S = g_{\sigma_s}^2 \left( \frac{1}{2} (T^S)^{-1}_{ss} - 8 I_1(m_s) \right) + 4m_s^2, \]
\[ M_{us}^S = \frac{1}{2} g_{\sigma_u} g_{\sigma_s} (T^S)^{-1}_{us}. \]  

After diagonalization of the Lagrangian (22) we find masses of the pseudoscalar and scalar mesons \( \eta, \eta', \sigma \) and \( f_0 \)
\[ M_{(\eta, \eta')}^2 = \frac{1}{2} \left[ M_{ss}^P + M_{uu}^P \mp \sqrt{(M_{ss}^P - M_{uu}^P)^2 + 4(M_{us}^P)^2} \right], \]
\[ M_{(\sigma, f_0)}^2 = \frac{1}{2} \left[ M_{ss}^S + M_{uu}^S \mp \sqrt{(M_{ss}^S - M_{uu}^S)^2 + 4(M_{us}^S)^2} \right]. \]

Let us define the mixing angle for the pseudoscalar mesons
\[ \eta_s = \eta \cos \bar{\theta} + \eta' \sin \bar{\theta}, \]
\[ \eta_u = -\eta \sin \bar{\theta} + \eta' \cos \bar{\theta}, \quad \bar{\theta} = \theta - \theta_0 \]  
where \( \theta_0 \approx 35.3^\circ \) is the ideal mixing angle (ctg \( \theta_0 = \sqrt{2} \)) and \( \theta \) is the singlet-octet mixing angle
\[ \tan 2\bar{\theta} = \frac{2M_{us}^P}{M_{ss}^P + M_{uu}^P}. \]

For the scalar mesons we use the relations
\[ \sigma_u = \sigma \cos \bar{\phi} + f_0 \sin \bar{\phi}, \]
\[ \sigma_s = -\sigma \sin \bar{\phi} + f_0 \cos \bar{\phi}, \quad \bar{\phi} = \theta_0 - \phi \]  
where \( \phi \) is the singlet-octet mixing angle and
\[ \tan 2\bar{\phi} = \frac{2M_{us}^S}{M_{ss}^S - M_{uu}^S}. \]

5. Numerical estimations of quark-loop contributions to meson masses

Using for the parameters \( m_s \) and \( K \) the values
\[ m_s = 425 \text{ MeV}, \quad K = 13.3 \text{ GeV}^{-5} \]
we obtain the following estimations for the masses of the pseudoscalar and scalar mesons
\[ M_{\pi} = 135 \text{ MeV}, \quad M_K = 495 \text{ MeV}, \]
\[ M_{\eta} = 520 \text{ MeV}, \quad M_{\eta'} = 1000 \text{ MeV}. \]
\[ M_\sigma = 550 \text{ MeV}, \quad M_{f_0} = 1130 \text{ MeV}, \]
\[ M_{a_0} = 810 \text{ MeV}, \quad M_{K^*_0} = 960 \text{ MeV}, \]
\[ \theta = -19^\circ, \quad \phi = 24^\circ. \]  

Note that the pion and kaon masses are the input parameters in our model. The experimental data are \[ \begin{align*}
M_{\pi^0} &= 134.9764 \pm 0.0006 \text{ MeV}, \quad M_{\pi^\pm} = 139.6 \text{ MeV}, \\
M_{K^+} &= 493.677 \pm 0.016 \text{ MeV}, \quad M_{K^0} = 497.672 \pm 0.031 \text{ MeV}, \\
M_\eta &= 547.45 \pm 0.19 \text{ MeV}, \quad M_{\eta'} = 957.77 \pm 0.14 \text{ MeV}, \\
\theta &\approx -20^\circ \quad [11].
\end{align*} \]  

Comparing the theoretical results with the experimental data we can see that we have obtained satisfactory results for the pseudoscalar mesons and for the octet-singlet mixing angle of the \((\eta\eta')\) mesons. However, for the scalar mesons we have got the masses smaller than the experimental data (except for the \(f_0\) meson).

6. **Strong decays of the scalar mesons.**

Now, let us show what the information concerning the strong decays of the scalar and pseudoscalar mesons we can obtain from the Lagrangian [11] and [19]. These Lagrangians allow us to get the following expressions for the scalar-pseudoscalar meson vertices, describing the corresponding strong decays of the scalar mesons

\[ g_{\sigma \pi \pi} = \frac{2m_\pi^2 Z^{1/2}}{F_\pi} \cos \phi \]
\[ g_{f_0 \pi \pi} = \frac{2m_\pi^2 Z^{1/2}}{F_\pi} \sin \phi \]
\[ g_{a_0 \eta \pi} = \frac{2m_\pi^2 Z^{1/2}}{F_\pi} \sin \theta \]
\[ g_{K^*_0 K^- \pi^0} = \frac{2m_\pi^2 Z^{1/2}}{F_\pi} \]
\[ g_{K^*_0 K^0 \pi^-} = \frac{2\sqrt{2}m_\pi m_\nu Z^{1/2}}{F_\pi} \]  

With these vertices one obtains the following decay widths

\[ \begin{align*}
\Gamma_{\sigma \rightarrow \pi \pi} &\approx 700 \text{ MeV}, \quad \Gamma_{f_0 \rightarrow \pi \pi} \approx 20 \text{ MeV}, \\
\Gamma_{a_0 \rightarrow \eta \pi} &\approx 130 \text{ MeV}, \quad \Gamma_{K^*_0 \rightarrow K \pi} \approx 330 \text{ MeV}.
\end{align*} \]
These values are in qualitative agreement with the experimental data

\[ \Gamma_{\sigma \rightarrow \pi \pi}^{\text{exp}} \sim (600 - 1000) \text{ MeV}, \quad \Gamma_{f_0 \rightarrow \pi \pi}^{\text{exp}} \sim (30 - 78) \text{ MeV}, \]

\[ \Gamma_{a_0 \rightarrow \eta \pi}^{\text{exp}} \sim (50 - 100) \text{ MeV}, \quad \Gamma_{K^*_0 \rightarrow K \pi}^{\text{exp}} \approx (263 \pm 26 \pm 21) \text{ MeV}. \]  

(38)

7. Conclusion

Our calculations have shown that the ’t Hooft interaction allows us to describe the masses of pseudoscalar meson’s masses and their singlet-octet mixing angle in satisfactory agreement with the experiment\[\parallel\]. For the scalar meson masses we have also obtained results more close to experimental data than in the NJL model without the ’t Hooft interaction. However, the masses of the \(a_0\) and especially the \(K^*_0\) mesons are noticeably less then the experimental ones.

In order to get satisfactory result for the \(f_0\) meson it is necessary to take into account the mixing of the \(f_0\) and \(\sigma\) mesons with the glueball state (see [12]). The problem with masses of \(a_0\) and \(K^*_0\) mesons could be solved in the framework of the four-quark \((\bar{q}q^2)\) MIT-bag model [13] or the \(\bar{K}K\) molecule model [14]. However, it is interesting to note that in spite of a very rough description of the scalar meson’s masses our model allows us to obtain a qualitatively true picture of the strong decay widths of the scalar mesons.

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