Alfvén waves and current relaxation: attenuation at high frequencies and large resistivity

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Abstract. The dispersion relations of Alfvén waves propagating in a resistive plasma are explored by assuming a finite relaxation time for the current density. It is shown that the proposed approach is consistent with the hydromagnetic approximation. An extension for the equation governing the space and time evolution of Alfvén waves is provided. New results are found at high values of the wave frequency \( \omega \): for a small resistivity, the wavelength increases as the cube of the equilibrium magnetic field but decreases with the cube of \( \omega \); for a large resistivity, the wave attenuation does not depend on \( \omega \), saturating to a finite value which is fully determined by the relaxation time of the current density. A transition frequency, \( \omega_t \), between two sharply distinct regimes of the perturbation is identified: for \( \omega < \omega_t \), the disturbance propagates in the resistive plasma as an attenuated oscillation; for \( \omega > \omega_t \), the wave ceases very rapidly to oscillate (in space), its amplitude saturating to a finite value. The results presented here may be relevant for investigations of some transient phenomena in plasma physics such as the reconnection of magnetic field lines.

1. Introduction

A simple kind of hydromagnetic wave, first explored by Alfvén [1], develops in an initially static, homogeneous and isotropic infinite plasma, subjected to a constant and uniform magnetic field, when a small disturbance of the fluid flow is produced in a direction perpendicular to the field. Such a perturbation gives rise to an infinitesimal magnetic field in the same direction of the flow, which, however, propagates parallel to the equilibrium field. This phenomenon is commonly referred to as an Alfvén wave.

Within the classical approach to the problem, the current density \( \vec{J} \), induced in a plasma of finite conductivity \( \sigma \), is related to the electric \( \vec{E} \) and magnetic \( \vec{B} \) fields, and to the fluid velocity \( \vec{v} \), through the standard Ohm’s law,

\[
\vec{J} = \sigma \left( \vec{E} + \vec{v} \times \vec{B} \right).
\]

Such an expression describes satisfactorily the dissipative processes which occur in the conductive plasma, provided the characteristic frequency of the perturbation is low enough. Particularly, the inertia of the charged species can be neglected.

At higher frequencies, however, inertial effects should become important. In the simple case of a plasma with two species (electron and ion), the standard Ohm’s law may be extended to (actually, there are three other terms in (2) to follow, proportional to the electron and ion...
pressure gradients, and to the Lorentz force density - the so called Hall effect -, but they do not play any relevant role within our model) \[2\]

\[
\frac{m_e m_i}{Z (n_e m_e + n_i m_i)} e^2 \frac{\partial \vec{J}}{\partial t} = \vec{E} + \vec{v} \times \vec{B} - \eta \vec{J},
\]

(2)

where \(\eta = \sigma^{-1}\) is the plasma resistivity, \(e\) is the electron charge, \(Z\) is the charge number, \(m_e\) \((m_i)\) and \(n_e\) \((n_i)\) are the electron (ion) mass and number density, respectively.

To see the main consequence of the introduction of the left hand side of (2) into the problem, we consider the simple case of a plasma with a singly ionized ion, \(Z = 1\). Then, if the electromagnetic field is suddenly removed from the presence of the fluid, the neutrality condition, \(n_e = Z n_i\), together with the approximation \(m_e \ll m_i\) imply

\[
\vec{J} = \vec{J}_0 e^{-t/\tau}.
\]

(3)

This means that the current density relaxes in the plasma, from some initial value, \(\vec{J}_0\) say, in a time scale of the order of

\[
\tau = \frac{m_e}{n_e e^2 \eta}.
\]

(4)

In this case, the so called ideal plasma approximation (a perfectly conductive fluid) should be properly achieved only by fully neglecting the electron mass, that is, by taking the limits \(\eta \to 0\) \((\sigma \to \infty)\) and \(m_e \to 0\) simultaneously. As a result, the relaxation time \(\tau\) in (4) would be always finite.

The considerations above suggest strongly that, at sufficiently high frequencies, for which inertial effects of charged species can not be neglected, the dissipative processes that occur in conductive plasmas should be described appropriately by the following extension of (1):

\[
\left(1 + \tau \frac{\partial}{\partial t}\right) \vec{J} = \sigma \left(\vec{E} + \vec{v} \times \vec{B}\right).
\]

(5)

where \(\tau\) is the relaxation time of the current density. A previous work had explored the attenuation and damping of electromagnetic fields in rigid conductive media, with basis on a generalization similar to that given by (5) \[3\]. Subsequently, the skin effect and dissipative processes in rigid conductors were investigated \[4\]. Recently, the metallic absorptivity at normal incidence was examined, thereby leading to an extension of Hagen-Rubens relation up to the near infrared \[5\]. More recently, the resistive heating and magnetic diffusion in good conductors have been rediscussed \[6\].

This work explores the dispersion for Alfvén waves, propagating in a plasma of finite resistivity, with basis on the extended Ohm’s law (5). First, the proposed approach is shown to be consistent with the hydromagnetic approximation. Then, an extension for the equation governing the space and time evolution of Alfvén waves is provided. As a consequence, new results for the wavelength and wave attenuation are found at high values of the wave frequency. Much interestingly, a transition frequency between two sharply distinct regimes of the perturbation is identified.

2. Hydromagnetic approximation

So far, the importance of inertial effects at high frequencies has been stressed. However, the following question arises naturally: how high should be the frequency such that the hydromagnetic approximation could be still valid? To address this issue, we start by considering the Ampère-Maxwell law,

\[
\frac{1}{\mu} \nabla \times \vec{B} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t},
\]

(6)
where \( \epsilon \) and \( \mu \) are the electric permittivity and magnetic permeability, respectively. By combining (5) and (6), we get

\[
\left( 1 + \tau \frac{\partial}{\partial t} \right) \frac{1}{\mu} \nabla \times \vec{B} - \sigma \vec{v} \times \vec{B} = \sigma \vec{E} + \left( 1 + \tau \frac{\partial}{\partial t} \right) \epsilon \frac{\partial \vec{E}}{\partial t}.
\]

For our purposes, it proves convenient to assume that the electric and magnetic fields vary harmonically in time, as \( \sim e^{-i\omega t} \) say. In this case, (7) leads to

\[
\frac{1}{\mu} \nabla \times \vec{B} - \frac{\sigma}{1 - i\omega \tau} \vec{v} \times \vec{B} = \left( \frac{\sigma}{1 - i\omega \tau} - i\epsilon \omega \right) \vec{E}.
\]

Now, if we ask what condition should be satisfied by the frequency \( \omega \) such that the displacement current \( \epsilon \partial_t \vec{E} \) could be neglected in (6), the obvious answer is

\[
\left| \frac{\sigma}{1 - i\omega \tau} \right| \gg \epsilon \omega.
\]

As one may easily check, condition (9) leads to

\[
\omega \ll \frac{1}{\tau \sqrt{2}} \left[ \sqrt{1 + \left( \frac{2\sigma\tau}{\epsilon} \right)^2} - 1 \right]^{1/2}.
\]

Condition (10) determines the domain of validity of the hydromagnetic approximation by defining an upper limit for the characteristic frequency of the electromagnetic field. In the limit \( \tau \to 0 \), inertial effects can be neglected and condition (10) recovers the standard upper limit for the frequency [7],

\[
\omega \ll \frac{\sigma}{\epsilon},
\]

which does not depend on the relaxation time.

3. Perturbative mechanism

\textbf{Figure 1.} An initially static, homogeneous and isotropic infinite plasma, subjected to a constant and uniform magnetic field \( B_0 \hat{z} \). The small rectangular plasma section ABCD, extending along the \( y \)-direction, suddenly acquires an infinitesimal velocity \( v_1 \hat{y} \). Then, due to the forces \( \sim v_1 B_0 \hat{x} \), the segment AB becomes negatively charged and CD, positively charged. Since we are dealing with a conductive medium, the plasma external to ABCD completes the electric circuit. Typical (solenoidal) current lines are shown.

A simple mechanism for the production of Alfvén waves can be described by following standard lines [8]. We start by considering an initially static, homogeneous and isotropic infinite plasma, subjected to a constant and uniform magnetic field \( B_0 \hat{z} \) (see figure 1). Then,
if the small rectangular plasma section ABCD, extending along the $y$-direction, acquires an infinitesimal velocity $v_1 \hat{y}$, the charged species experience forces $\sim v_1 B_0 \hat{y}$, which tend to separate electrons and ions. As a result, the segment AB becomes negatively charged and CD, positively charged. However, since we are dealing with a conductive medium, the plasma external to ABCD completes the electric circuit. Typical (solenoidal) current lines are shown in figure 1. Now, the induced current density $J_1 \hat{z}$ interacts with the equilibrium magnetic field $B_0 \hat{z}$, such that the force density $-J_1 B_0 \hat{y}$ decelerates ABCD while the plasma external to it is accelerated by $J_1 B_0 \hat{y}$. Eventually, ABCD transfers its motion to the neighboring external plasma. However, the original mechanism still operates and the whole process continues to develop, thereby propagating the disturbance along the $z$-direction. Therefore, if $E_1 \hat{x}$ and $B_1 \hat{y}$ are the induced electric and magnetic fields, respectively, satisfying Faraday’s law - see the second of (12) in the next section - one may explore the most simple wave motion, characterized by the perturbative quantities $v_1 (z) \hat{y}$, $J_1 (z) \hat{x}$, $E_1 (z) \hat{x}$, and $B_1 (z) \hat{y}$, provided the remaining components, all vanish. This is the Alfvén wave.

4. Alfvén waves

Given the perturbative mechanism, as shown in the preceding section, we pass now to describe the propagation of Alfvén waves in a plasma with finite conductivity. The set of basic equations can be recast by the extended Ohm’s law (5), together with the Ampère (calculations are carried out within the hydromagnetic approximation, see Sec. 2) and Faraday laws,

$$\nabla \vec{B} = \mu_0 \vec{J}, \quad \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}, \tag{12}$$

respectively, where $\mu_0$ is the vacuum magnetic permeability (we restrict ourselves to non magnetic plasmas), and the continuity and Euler (an inviscid fluid is assumed) equations,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v}), \quad \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla p}{\rho} + \frac{\vec{J} \times \vec{B}}{\rho}, \tag{13}$$

respectively, where $\rho$ is the mass density and $p$ is the fluid pressure.

According to (5), (12), and (13), we see that the first order perturbative quantities $v_1$, $J_1$, $E_1$, and $B_1$ satisfy the relations

$$\left(1 + \tau \frac{\partial}{\partial t}\right) J_1 = \sigma (E_1 + v_1 B_0), \tag{14}$$

$$-\frac{\partial B_1}{\partial z} = \mu_0 J_1, \quad \frac{\partial B_1}{\partial t} = -\frac{\partial E_1}{\partial z}, \tag{15}$$

$$\frac{\partial v_1}{\partial t} = -\frac{J_1 B_0}{\rho_0}, \tag{16}$$

where we have assumed that the fluid is incompressible (the mass density is constant and uniform, $\rho_0$ say, so that the velocity field is solenoidal, $\nabla \cdot \vec{v} = 0$). Particularly, the pressure gradient is determined by $\partial_z p = J_1 B_1$, which is a second order perturbative term, such that it may be excluded of the foregoing discussion.

By combining (14), (15), and (16), we get

$$v_A^2 \frac{\partial^2 B_1}{\partial z^2} = \frac{\partial^2 B_1}{\partial t^2} - \left(1 + \tau \frac{\partial}{\partial t}\right) \frac{1}{\mu_0 \sigma} \frac{\partial \vec{J}}{\partial t} \cdot \vec{B}, \tag{17}$$

where we have introduced the Alfvén speed $v_A = B_0 (\mu_0 \rho_0)^{-1/2}$. In the limit $\tau \to 0$, inertial effects can be neglected and (17) recovers the standard equation governing the propagation of Alfvén waves in a plasma with finite conductivity [8].
5. Dispersion relations

If we assume that the perturbative magnetic field varies harmonically in space and time, as
\[ \sim \exp(ikz - i\omega t) \] say, (17) leads to the general dispersion relation
\[ v_A^2 k^2 = \omega^2 \left[ 1 + i \left( 1 - i\omega \tau \right) \frac{k^2}{\mu_0 \sigma \omega} \right]. \quad (18) \]

5.1. Small resistivity

When the resistive correction term \( \sim \sigma^{-1} \) in (18) is small, the wave number \( k \) satisfies
\[ v_A k = \omega \left[ \left( 1 + \frac{\omega^2 \tau}{2\mu_0 \sigma v_A^2} \right) + i \frac{\omega}{2\mu_0 \sigma v_A^2} \right]. \quad (19) \]

5.1.1. Low frequencies

At low values of the wave frequency \( \omega \), the relaxation term \( \sim \tau \) in (19) just slightly corrects the wavelength (proportional to the inverse of the real part of the wave number) that increases as \( \sim B_0 \) but decreases with \( \sim \omega \). In the limit \( \tau \to 0 \), the inertial correction vanishes and (19) recovers the standard wave number describing the propagation through a slightly resistive plasma [7],
\[ k = \frac{\omega}{v_A} + i \frac{\omega^2}{2\mu_0 \sigma v_A^3}, \quad (20) \]

for which the wave attenuation (the imaginary part of the wave number) increases as \( \sim \omega^2 \) but decreases with \( \sim B_0^3 \).

5.1.2. High frequencies

At high frequencies, the unit can be neglected in (19) and the wave number is given by
\[ k = \frac{\omega^3 \tau}{2\mu_0 \sigma v_A^3} + i \frac{\omega^2}{2\mu_0 \sigma v_A^2}. \quad (21) \]

As it appears, the wave attenuation is the same as that of (20) at low frequencies. However, the wavelength increases as \( \sim B_0^3 \) but decreases with \( \sim \omega^3 \) now. This is the fundamental, new limiting result of (18) for a slightly resistive plasma, at high frequencies, due to the introduction of the finite relaxation time \( \tau \) for the current density in (5).

5.2. Large resistivity

In the opposite extreme for which the resistive term dominates, the wave number is given by

\[ (1 - i\omega \tau) k^2 = i\mu_0 \sigma \omega. \quad (22) \]

Note that the wave number does not depend on \( B_0 \) now.

5.2.1. Low frequencies

At low frequencies, the wave number in (22) is given by
\[ k = \sqrt{\frac{\mu_0 \sigma \omega}{2}} \left( 1 - \frac{\omega \tau}{2} \right) + i \sqrt{\frac{\mu_0 \sigma \omega}{2}} \left( 1 + \frac{\omega \tau}{2} \right). \quad (23) \]

This shows that the imaginary part of the wave number is slightly larger (by the small relaxation term \( \sim \tau \)) than the real one. Therefore, both parts increase approximately as \( \omega^{1/2} \). In the limit \( \tau \to 0 \), the inertial correction vanishes and the wave number acquires equal real and imaginary parts. In this case, both parts increase exactly with \( \omega^{1/2} \) and (23) recovers the standard wave number describing the propagation through a strongly resistive plasma [7].
5.2.2. High frequencies

At high frequencies, the wave number in (22) is given by

\[ k = \frac{1}{2\omega\tau} \sqrt{\frac{\mu_0\sigma}{\tau}} + i \sqrt{\frac{\mu_0\sigma}{\tau}}. \]  

(24)

As it appears, the wave attenuation does not depend on \( \omega \) any longer, saturating to a finite value now, which is fully determined by \( \tau \). Additionally, the real part of the wave number becomes much smaller (by the large factor \( \sim \omega\tau \)) than the imaginary one. Particularly, since the wavelength increases with no limit as \( \sim \omega \), for all practical purposes, we can consider that the perturbation does not oscillate (in space), being purely attenuated as \( \sim \tau^{-1/2} \). This is the fundamental, new limiting result of (18) for a strongly resistive plasma, at high frequencies, due to the introduction of the finite relaxation time \( \tau \) for the current density in (5).

6. Transition from low to high frequencies

![Image of graphs showing normalized propagation constant and attenuation coefficient as functions of normalized frequency and conductivity.]

**Figure 2.** The normalized propagation constant \( \hat{\beta} \) (dashed lines) and attenuation coefficient \( \hat{\alpha}/2 \) (full lines) as functions of the normalized frequency \( \hat{\omega} \), for selected values of the normalized conductivity \( \hat{\sigma} \): (a) \( \hat{\sigma} = 1000 \); (b) \( \hat{\sigma} = 500 \); (c) \( \hat{\sigma} = 200 \); (d) \( \hat{\sigma} = 100 \). Note the tendency for saturation of \( \hat{\alpha}/2 \) when both \( \hat{\omega} \) and \( \hat{\sigma}^{-1} \) attain large values, as predicted by (24).

As it has been shown by (24) in the preceding section, the attenuation for an Alfvén wave propagating through a strongly resistive plasma tends to a finite asymptotic value at high frequencies, due to inertial effects of charged species. Such a result enables us to rewrite the general dispersion relation (18) in the form

\[ \left( \hat{\sigma} - \hat{\omega}^2 \right) \hat{k}^2 = \hat{\omega}^2, \]  

(25)

where we have introduced the normalized quantities

\[ \hat{\omega} = \omega\tau, \quad \hat{\sigma} = \sigma\tau\mu_0v_A^2, \quad \hat{k} = k\sqrt{\frac{\tau}{\mu_0\sigma}}. \]  

(26)

Note that all normalization constants in (26) are fully determined by the relaxation time \( \tau \) for the current density. This means that such a procedure is possible only within the framework of the extended Ohm’s law (5).
By introducing the propagation constant $\beta$ and attenuation coefficient $\alpha/2$ in the usual way, that is, as the real and imaginary parts, respectively, of the wave number $k$, we can write $\hat{k} = \hat{\beta} + i\hat{\alpha}/2$ for the corresponding normalized quantities. In this case, (25) leads to (see figure 2)

$$\left\{ \hat{\beta}; \frac{\hat{\alpha}}{2} \right\} = \frac{\hat{\omega}}{\sqrt{2}} \left[ \frac{\sqrt{(\hat{\sigma} - \hat{\omega}^2)^2 + \hat{\omega}^2 \pm (\hat{\sigma} - \hat{\omega}^2)}}{(\hat{\sigma} - \hat{\omega}^2)^2 + \hat{\omega}^2} \right]^{1/2},$$

where the $\pm$ signs stand for $\hat{\beta}$ and $\hat{\alpha}/2$, respectively.

Figure 2 shows $\hat{\beta}$ and $\hat{\alpha}/2$ as functions of the normalized frequency $\hat{\omega}$, for some selected values of the normalized conductivity $\hat{\sigma}$. Note the tendency for saturation of $\hat{\alpha}/2$ when both $\hat{\omega}$ and $\hat{\sigma}^{-1}$ attain large values, as predicted by (24).

Now, a much interesting result of the presently developed model is also shown by figure 2: for each fixed $\hat{\sigma}$, $\hat{\beta}$ and $\hat{\alpha}/2$ both intercept at the same finite $\hat{\omega}$. From (27), we see that the condition $\hat{\beta} = \hat{\alpha}/2$ is satisfied for $\hat{\omega} = \hat{\sigma}^{1/2}$. By plugging back the physical quantities, as given by (26), this means that the propagation constant and attenuation coefficient coalesce at

$$\beta = \frac{\alpha}{2} = \left( \frac{\mu_0 \sigma}{\tau} \right)^{3/4} \sqrt{\frac{2v_A \tau}{3}},$$

when the wave frequency attains the value

$$\omega_t = v_A \sqrt{\frac{2\mu_0 \sigma}{\tau}}.$$  

The quantity $\omega_t$, given by (29), can be physically interpreted as the transition frequency between two sharply distinct regimes of the perturbation: as long as $\omega < \omega_t$, the disturbance propagates in the resistive plasma as an attenuated oscillation; as soon as $\omega > \omega_t$, the wave ceases very rapidly to oscillate (in space), its amplitude saturating to a finite value. In the limit $\tau \to 0$, inertial effects can be neglected, all transition quantities, as given by (28) and (29), diverge, and the perturbation exhibits a unique regime, the former one, for which the condition $\omega < \infty$ is trivially satisfied for the whole frequency range that is admissible within the hydromagnetic approximation (see section 2).

7. Conclusion

The dispersion relations of Alfvén waves propagating in a resistive plasma have been explored by assuming a finite relaxation time for the current density. The proposed approach has been shown to be consistent with the hydromagnetic approximation - see condition (10). An extension for the equation governing the space and time evolution of Alfvén waves has been provided - see (17). New results have been found at high values of the wave frequency $\omega$:

- for a small resistivity, the wavelength increases as the cube of the equilibrium magnetic field but decreases with the cube of the wave frequency - see (21);
- for a large resistivity, the wave attenuation does not depend on the wave frequency, saturating to a finite value which is fully determined by the relaxation time for the current density - see (24).

A transition frequency, $\omega_t$, between two sharply distinct regimes of the perturbation has been identified - see (29) -:

- for $\omega < \omega_t$, the disturbance propagates in the resistive plasma as an attenuated oscillation;
- for $\omega > \omega_t$, the wave ceases very rapidly to oscillate (in space), its amplitude saturating to a finite value.
It should be interesting to apply the results presented here for investigations of some transient phenomena in plasma physics such as the reconnection of magnetic field lines. Actually, a lower limit for the thickness of a reconnection layer can be determined by the inverse of the square root of the Lundquist number, a generally very large quantity, which is proportional to the plasma conductivity $\sigma$ [9]. Additionally, within one of the most basic approaches to this phenomenon, the Sweet-Parker model, the magnetic field lines may be shown to slip through the plasma to reconnect with a speed that is also proportional to the inverse of the square root of $\sigma$ [10].

Now, the time Fourier transform of the extended Ohm’s law (5) is readily seen to be given by

$$\vec{J} = \frac{\sigma}{1 - i\omega\tau} \left( \vec{E} + \vec{v} \times \vec{B} \right).$$

(30)

This shows that the complex coefficient of proportionality between the electromagnetic field and the current density can be physically interpreted as an effective conductivity depending on the wave frequency. Therefore, we argue that relevant corrections to space and time scales describing the reconnection of magnetic field lines could be expected by applying the presently developed model at sufficiently high frequencies.

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