MAXIMAL OPERATORS OF VILENKIN-NÖRLUND MEANS

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Abstract. In this paper we prove and discuss some new \((H_p, \text{weak} - L_p)\) type inequalities of maximal operators of Vilenkin-Nörlund means with monotone coefficients. We also apply these results to prove a.e. convergence of such Vilenkin-Nörlund means. It is also proved that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

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1. Introduction

The definitions and notations used in this introduction can be found in our next Section. In the one-dimensional case the first result with respect to the a.e. convergence of Fejér is due to Fine [13]. Later, Schipp [25] for Walsh series and Pál, Simon [24] for bounded Vilenkin series showed that the maximal operator of Fejér means \(\sigma^*\) is of weak type \((1,1)\), from which the a.e. convergence follows by standard argument [16]. Fujji [14] and Simon [27] verified that \(\sigma^*\) is bounded from \(H_1\) to \(L_1\). Weisz [38] generalized this result and proved boundedness of \(\sigma^*\) from the martingale space \(H_p\) to the space \(L_p\), for \(p > 1/2\). Simon [26] gave a counterexample, which shows that boundedness does not hold for \(0 < p < 1/2\). A counterexample for \(p = 1/2\) was given by Goginava [9] (see also [31]). In [30] it was proved even stronger result than the maximal operator \(\sigma^*\) is unbounded. In fact, it was proved that there exists a martingale \(f \in H_{1/2}\), such that Fejér means of Vilenkin-Fourier series of the martingale \(f\) are not uniformly bounded in the space \(L_{1/2}\). Moreover, Weisz [40] proved that the following is true:

**Theorem W1.** The maximal operator of the Fejér means \(\sigma^*\) is bounded from the Hardy space \(H_{1/2}\) to the space \(weak - L_{1/2}\).

Riesz’s logarithmic means with respect to the trigonometric system was studied by several authors. We mention, for instance, the papers by Szász [29] and Yabuta [36]. These means with respect to the Walsh and Vilenkin systems were investigated by Simon [26] and Gát [5]. Blahota and Gát [3] considered norm summability of Nörlund logarithmic means and showed that Riesz’s logarithmic means \(R_n\) have better approximation properties on some unbounded Vilenkin groups, than the Fejér means. In [33] it was proved that the maximal operator of Riesz’s means \(R^*\) is not bounded from the Hardy space \(H_{1/2}\) to the space \(weak - L_{1/2}\), but is not bounded from the Hardy space \(H_p\) to the space \(L_p\), when \(0 < p \leq 1/2\). Since the set of

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Vilenkin polynomials is dense in $L_1$, by well-known density argument due to Marcinkiewicz and Zygmund [16], we have that $R_n f \to f$, a.e. for all $f \in L_1$.

Móricz and Siddiqi [19] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of $L_p$ function in norm. The case when $q_k = 1/k$ was excluded, since the methods of Móricz and Siddiqi are not applicable to Nörlund logarithmic means.

In [6] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means in the class of continuous functions and in the Lebesgue space $L_1$. Among other things, they gave a negative answer to a question of Móricz and Siddiqi [19]. Gát and Goginava [7] proved that for each measurable function $\phi(u) = \circ(u\sqrt{\log u})$, there exists an integrable function $f$ such that

$$\int_{G_m} \phi(|f(x)|) \, d\mu(x) < \infty$$

and there exists a set with positive measure, such that the Walsh-logarithmic means of the function diverges on this set. In [32] it was proved that there exists a martingale $f \in H_p$, $(0 < p \leq 1)$, such that the maximal operator of Nörlund logarithmic means $L^*$ is not bounded in the space $L_p$.

In [11] Goginava investigated the behaviour of Cesáro means of Walsh-Fourier series in detail. In the two-dimensional case approximation properties of Nörlund and Cesáro means was considered by Nagy (see [20]-[23]). The a.e. convergence of Cesáro means of $f \in L_1$ was proved in [12]. The maximal operator $\sigma_{\alpha,*} (0 < \alpha < 1)$ of the $(C,\alpha)$ means of Walsh-Paley system was investigated by Weisz [11]. In his paper Weisz proved that $\sigma_{\alpha,*}$ is bounded from the martingale space $H_p$ to the space $L_p$ for $p > 1/(1+\alpha)$. Goginava [10] gave a counterexample, which shows that boundedness does not hold for $0 < p \leq 1/(1+\alpha)$.

Recently, Weisz and Simon [28] showed the following statement:

**Theorem SW1.** The maximal operator $\sigma_{\alpha,*} (0 < \alpha < 1)$ of the $(C,\alpha)$ means is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space weak $-L_{1/(1+\alpha)}$.

In this paper we derive some new $(H_p, L_p)$-type inequalities for the maximal operators of Nörlund means, with monotone coefficients.

The paper is organized as following: In Section 3 we present and discuss the main results and in Section 4 the proofs can be found. Moreover, in order not to disturb our discussions in these Sections some preliminaries are given in Section 2.

## 2. Preliminaries

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of the positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo $m_k$.

Define the group $G_m$ as the complete direct product of the groups $Z_{m_i}$ with the product of the discrete topologies of $Z_{m_j}$’s.

The direct product $\mu$ of the measures

$$\mu_k\{j\} := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on $G_m$ with $\mu(G_m) = 1$. 

In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_n m_n < \infty$.

The elements of $G_m$ are represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots), \ (x_j \in Z_{m_j}).$$

It is easy to give a base for the neighborhood of $G_m$:

$$I_0(x) := G_m, \ I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\},$$

where $x \in G_m$, $n \in \mathbb{N}$.

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}^+$, and $I_n := G_m \setminus I_n$.

If we define the so-called generalized number system based on $m$ in the following way:

$$M_0 := 1, \ M_{k+1} := m_k M_k \ (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j} \ (j \in \mathbb{N}^+)$ and only a finite number of $n_j$’s differ from zero.

Next, we introduce on $G_m$ an orthonormal system which is called the Vilenkin system. At first, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp (2\pi i x_k / m_k), \ (i^2 = -1, x \in G_m, \ k \in \mathbb{N}).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k} (x), \ (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley system when $m \equiv 2$.

The norm (or quasi-norm) of the space $L_p(G_m) \ (0 < p < \infty)$ is defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu.$$

The space weak $- L_p(G_m)$ consists of all measurable functions $f$, for which

$$\|f\|_{weak-L_p}^p := \sup_{\lambda > 0} \lambda^p \mu (f > \lambda) < +\infty.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [35]).

Now, we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\hat{f}(n) := \int_{G_m} f \overline{\psi_n} \, d\mu \quad (n \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \ D_n := \sum_{k=0}^{n-1} \psi_k, \ (n \in \mathbb{N}^+)$$

respectively.
Recall that

\[ D_M (x) = \begin{cases} 
M, & \text{if } x \in I_n, \\
0, & \text{if } x \notin I_n.
\end{cases} \]

The \( \sigma \)-algebra generated by the intervals \( \{ I_n (x) : x \in G_m \} \) will be denoted by \( F_n (n \in \mathbb{N}) \). Denote by \( f = (f^{(n)}, n \in \mathbb{N}) \) a martingale with respect to \( F_n (n \in \mathbb{N}) \). (for details see e.g. [37]).

The maximal function of a martingale \( f \) is defined by

\[ f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|. \]

For \( 0 < p < \infty \) the Hardy martingale spaces \( H^p (G_m) \) consist of all martingales for which

\[ \| f \|_{H^p} := \| f^* \|_p < \infty. \]

If \( f = (f^{(n)}, n \in \mathbb{N}) \) is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

\[ \hat{f} (i) := \lim_{k \to \infty} \int_{G_m} f^{(k)} \overline{\psi_i} d\mu. \]

Let \( \{ q_k : k \geq 0 \} \) be a sequence of nonnegative numbers. The \( n \)-th Nörlund means for a Fourier series of \( f \) is defined by

\[ t_n f = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \]

where \( Q_n := \sum_{k=0}^{n-1} q_k. \)

We always assume that \( q_0 > 0 \) and \( \lim_{n \to \infty} Q_n = \infty. \) In this case it is well-known that the summability method generated by \( \{ q_k : k \geq 0 \} \) is regular if and only if

\[ \lim_{n \to \infty} \frac{q_{n-1}}{Q_n} = 0. \]

Concerning this fact and related basic results, we refer to [18].

If \( q_k \equiv 1 \), we get the usual Fejér means

\[ \sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f. \]

The \((C, \alpha)\)-means of the Vilenkin-Fourier series are defined by

\[ \sigma_n^\alpha f = \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f, \]

where

\[ A_0^\alpha = 0, \quad A_n^\alpha = \frac{(\alpha + 1) \cdots (\alpha + n)}{n!}, \quad \alpha \neq -1, -2, \ldots \]
The $n$-th Riesz's logarithmic mean $R_n$ and the Nörlund logarithmic mean $L_n$ are defined by

\[
R_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} S_k f, \quad L_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} S_k f
\]

respectively, where $l_n := \sum_{k=1}^{n-1} 1/k$.

For the martingale $f$ we consider the following maximal operators:

\[
t^* f := \sup_{n \in \mathbb{N}} |t_n f|, \quad \sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|, \quad \sigma^{\alpha,*} f := \sup_{n \in \mathbb{N}} |\sigma^\alpha_n f|,
\]

\[
R^* f := \sup_{n \in \mathbb{N}} |R_n f| \quad \text{and} \quad L^* f := \sup_{n \in \mathbb{N}} |L_n f|.
\]

A bounded measurable function $a$ is a $p$-atom, if there exists an interval $I$, such that

\[
\int_I ad\mu = 0, \quad \|a\|_\infty = \mu(I)^{-1/p}, \quad \text{supp} (a) \subset I.
\]

We also need the following auxiliary results:

**Lemma 1.** [39] A martingale $f = (f^{(n)}, n \in \mathbb{N})$ is in $H_p$ ($0 < p \leq 1$) if and only if there exists sequence $(a_k, k \in \mathbb{N})$ of $p$-atoms and a sequence $(\mu_k, k \in \mathbb{N})$, of real numbers, such that, for every $n \in \mathbb{N},$

\[
\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)},
\]

\[
\sum_{k=0}^{\infty} |\mu_k|^p < \infty.
\]

Moreover,

\[
\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p},
\]

where the infimum is taken over all decomposition of $f$ of the form (3).

**Lemma 2.** [39] Suppose that an operator $T$ is $\sigma$-linear and for some $0 < p \leq 1$ and

\[
\sup_{\rho > 0} \rho^p \mu \{ x \in I : |Ta| > \rho \} \leq c_p < \infty,
\]

for every $p$-atom $a$, where $I$ denotes the support of the atom. If $T$ is bounded from $L_\infty$ to $L_\infty$, then

\[
\|Tf\|_{\text{weak} - L_p} \leq c_p \|f\|_{H_p}.
\]

and if $0 < p < 1$, then $T$ is of weak type $(1,1)$:

\[
\|Tf\|_{\text{weak} - L_1} \leq c \|f\|_1.
\]
3. Main Results

Our first main result reads:

**Theorem 1.** a) Let \( \{q_k : k \geq 0\} \) be a sequence of nonnegative numbers, \( q_0 > 0 \) and

\[
\lim_{n \to \infty} Q_n = \infty.
\]

The summability method (2) generated by \( \{q_k : k \geq 0\} \) is regular if and only if

\[
\lim_{n \to \infty} \frac{q_{n-1}}{Q_n} = 0.
\]

Next, we state our main result concerning the maximal operator of the summation method (2), which we also show is in a sense sharp.

**Theorem 2.** a) The maximal operator \( t^* \) of the summability method (2) with nondecreasing sequence \( \{q_k : k \geq 0\} \), is bounded from the Hardy space \( H_{1/2} \) to the space \( \text{weak} - L_{1/2} \).

The statement in a) is sharp in the following sense:

b) Let \( 0 < p < 1/2 \) and \( \{q_k : k \geq 0\} \) is nondecreasing sequence, satisfying the condition

\[
\frac{q_0}{Q_n} \geq \frac{c}{n}, \quad (c > 0).
\]

Then there exists a martingale \( f \in H_p \), such that

\[
\sup_{n \in \mathbb{N}} \|t_n f\|_{\text{weak} - L_p} = \infty.
\]

Our next result shows that the statement in b) above hold also for nonincreasing sequences and now without any restriction like (5).

**Theorem 3.** Let \( 0 < p < 1/2 \). Then, for all Nörlund means with nonincreasing sequence \( \{q_k : k \geq 0\} \), there exists a martingale \( f \in H_p \), such that

\[
\sup_{n \in \mathbb{N}} \|t_n f\|_{\text{weak} - L_p} = \infty.
\]

Up to now we have considered the case \( 0 < p < 1/2 \), but in our final main result we consider the case when \( p = 1/(1 + \alpha) \), \( 0 < \alpha \leq 1 \), so that \( 1/2 \leq p < 1 \). Also this result is sharp in two different important senses.

**Theorem 4.** a) Let \( 0 < \alpha \leq 1 \). Then the maximal operator \( t^* \) of summability method (2) with non-increasing sequence \( \{q_k : k \geq 0\} \), satisfying the condition

\[
\frac{n^\alpha q_0}{Q_n} = O(1), \quad \frac{|q_n - q_{n+1}|}{n^{\alpha-2}} = O(1), \quad \text{as} \quad n \to \infty.
\]

is bounded from the Hardy space \( H_{1/(1+\alpha)} \) to the space \( \text{weak} - L_{1/(1+\alpha)} \).

The parameter \( 1/(1 + \alpha) \) in a) is sharp in the following sense:

b) Let \( 0 < p < 1/(1 + \alpha) \), where \( 0 < \alpha \leq 1 \) and \( \{q_k : k \geq 0\} \) be a non-increasing sequence, satisfying the condition

\[
\frac{q_n}{Q_n} \geq \frac{c}{n^\alpha}, \quad (c > 0).
\]
Then
\[ \sup_{n \in \mathbb{N}} \| t_n f \|_{\text{weak-}L_p} = \infty. \]

Also the condition (4) is "sharp" in the following sense:
c) Let \( \{ q_k : k \geq 0 \} \) be a non-increasing sequence, satisfying the condition
\[ \lim_{n \to \infty} \frac{q_0 n^\alpha}{Q_n} = \infty, \quad (0 < \alpha \leq 1). \]

Then
\[ \sup_{n \in \mathbb{N}} \| t_n f \|_{\text{weak-}L_{1/(1+\alpha)}} = \infty. \]

A number of special cases of our results are of particular interest and give both well-known and new information. We just give the following examples of such Corollaries:

**Corollary 1.** (See [32]) The maximal operator of the Fejér means \( \sigma^* \) is bounded from the Hardy space \( H_{1/2} \) to the space weak-\( L_{1/2} \) but is not bounded from \( H_p \) to the space weak-\( L_p \), when \( 0 < p < 1/2 \).

**Corollary 2.** (See [33]) The maximal operator of the Riesz’s means \( R^* \) is bounded from the Hardy space \( H_{1/2} \) to the space weak-\( L_{1/2} \) but is not bounded from \( H_p \) to the space weak-\( L_p \), when \( 0 < p < 1/2 \).

**Corollary 3.** The maximal operator of the \((C, \alpha)\)-means \( \sigma^{\alpha,*} \) is bounded from the Hardy space \( H_{1/(1+\alpha)} \) to the space weak-\( L_{1/(1+\alpha)} \) but is not bounded from \( H_p \) to the space weak-\( L_p \), when \( 0 < p < 1/(1 + \alpha) \).

**Corollary 4.** (See [32]) The maximal operator of the Nörlund logarithmic means \( L^* \) is not bounded from the Hardy space \( H_p \) to the space weak-\( L_p \), when \( 0 < p < 1 \).

**Corollary 5.** Let \( f \in L_1 \) and \( t_n \) be the Nörlund means, with nondecreasing sequence \( \{ q_k : k \geq 0 \} \). Then
\[ t_n f \to f, \quad \text{a.e., as } n \to \infty. \]

**Corollary 6.** Let \( f \in L_1 \) and \( t_n \) be Nörlund means, with non-increasing sequence \( \{ q_k : k \geq 0 \} \) and satisfying condition (4). Then
\[ t_n f \to f, \quad \text{a.e., as } n \to \infty. \]

**Corollary 7.** Let \( f \in L_1 \). Then
\[ \sigma_n f \to f, \quad \text{a.e., as } n \to \infty, \]
\[ R_n f \to f, \quad \text{a.e. as } n \to \infty, \]
\[ \sigma_n^{\alpha} f \to f, \quad \text{a.e., as } n \to \infty, \quad \text{when } 0 < \alpha < 1. \]

**Remark 1.** The statements in Corollary 7 are known (see [12], [13], [28], [16], [34]), but this unified approach to prove them is new.
4. PROOFS OF THE THEOREMS

**Proof of Theorem 1.** The proof is similar as in the case with Walsh system (see [18]), so we omit the details.

**Proof of Theorem 2.** By using Abel transformation we obtain that

\[ Q_n := \sum_{j=0}^{n-1} q_j = \sum_{j=1}^{n} q_{n-j} \cdot 1 = \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n, \]

and

\[ t_n f = \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j \sigma_j f + q_0 n \sigma_n f \right). \]

Let the sequence \( \{q_k : k \geq 0\} \) be non-decreasing. By combining (9) with (10) and using Abel transformation we get that

\[ |t_n f| \leq \left| \frac{1}{Q_n} \sum_{j=1}^{n} q_{n-j} S_j f \right| \]

\[ \leq \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| j |\sigma_j f| + q_0 n |\sigma_n f| \right) \]

\[ \leq \frac{c}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n \right) \sigma^* f \leq c \sigma^* f \]

so that

\[ t^* f \leq c \sigma^* f. \]

If we apply (11) and Theorem W1 we can conclude that the maximal operators \( t^* \) are bounded from the Hardy space \( H_{1/2} \) to the space \( \text{weak} - L_{1/2} \). It follows that (see Theorem W1) \( t^* \) has weak type-(1,1) and \( t_n f \to f \), a.e.

In the proof of the second part of Theorem 2 we mainly follow the method of Blahota, Gát and Goginava (see [1], [2]).

Let \( 0 < p < 1/2 \) and \( \{\alpha_k : k \in \mathbb{N}\} \) be an increasing sequence of positive integers such that:

\[ \sum_{k=0}^{\infty} 1/\alpha_k^p < \infty, \]

(12)

\[ \lambda \sum_{\eta=0}^{k-1} M_{\alpha_\eta}^{1/p} / \alpha_\eta < M_{\alpha_k}^{1/p} / \alpha_k, \]

(13)

\[ 32 \lambda M_{\alpha_{k-1}}^{1/p} / \alpha_{k-1} < M_{\alpha_k}^{1/p-2} / \alpha_k, \]

(14)

where \( \lambda = \sup_n m_n \).
We note that such an increasing sequence \( \{ \alpha_k : k \in \mathbb{N} \} \) which satisfies conditions (12)-(14) can be constructed.

Let

\[
(15) \quad f^{(A)} = \sum_{\{k : \lambda_k < A\}} \lambda_k \theta_k,
\]

where

\[
(16) \quad \lambda_k = \frac{\lambda}{\alpha_k}
\]

and

\[
(17) \quad \theta_k = \frac{M_{\alpha_k}^{1/p-1}}{\lambda} \left( D_{M_{\alpha_k+1}} - D_{M_{\alpha_k}} \right).
\]

It is easy to show that the martingale \( f = (f^{(1)}, f^{(2)}, \ldots f^{(A)}, \ldots) \in H_{1/2} \). Indeed, since

\[
S_{MA} \theta_k = \begin{cases} \ a_k, & \text{if } \alpha_k < A, \\ 0, & \text{if } \alpha_k \geq A, \end{cases}
\]

\[
\text{supp}(\theta_k) = I_{\alpha_k}, \quad \int_{I_{\alpha_k}} \theta_k d\mu = 0, \quad \|\theta_k\|_{\infty} \leq M_{\alpha_k}^{1/p} = (\text{supp } a_k)^{1/p},
\]

if we apply Lemma 1 and (12) we can conclude that \( f \in H_p, \ (0 < p < 1/2) \).

Moreover, it is easy to show that

\[
(18) \quad \hat{f}(j) = \begin{cases} \ M_{\alpha_k}^{1/p-1} \frac{\lambda}{\alpha_k}, & \text{if } j \in \{M_{\alpha_k}, \ldots, M_{\alpha_k+1} - 1\}, \ k = 0, 1, 2, \ldots, \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{M_{\alpha_k}, \ldots, M_{\alpha_k+1} - 1\}. \end{cases}
\]

We can write

\[
t_{M_{\alpha_k+1}}f = \frac{1}{Q_{M_{\alpha_k+1}}} \sum_{j=0}^{M_{\alpha_k}} q_j S_j f + \frac{q_0}{Q_{M_{\alpha_k+1}}} S_{M_{\alpha_k+1}} f := I + II.
\]

Let \( M_{\alpha_s} \leq j \leq M_{\alpha_{s+1}}, \) where \( s = 0, \ldots, k - 1. \) Moreover,

\[ |D_j - D_{M_{\alpha_s}}| \leq j - M_{\alpha_s} \leq \lambda M_{\alpha_s}, \quad (s \in \mathbb{N}) \]
so that, according to (11) and (18), we have that

\[
|S_j f| = \left| \sum_{v=0}^{M_{\alpha s+1} - 1} \hat{f}(v) \psi_v + \sum_{v=M_{\alpha s}}^{j-1} \hat{f}(v) \psi_v \right|
\]

\[
\leq \sum_{\eta=0}^{s-1} \sum_{v=M_{\alpha s}}^{M_{\alpha s+1} - 1} \frac{M_{\alpha s}^{1/p} - 1}{\alpha_q} \psi_v + \frac{M_{\alpha s}^{1/p} - 1}{\alpha_s} \left| (D_j - D_{M_{\alpha s}}) \right|
\]

\[
= \sum_{\eta=0}^{s-1} \frac{M_{\alpha s}^{1/p} - 1}{\alpha_q} \left( D_{M_{\alpha s+1}} - D_{M_{\alpha s}} \right) + \frac{M_{\alpha s}^{1/p} - 1}{\alpha_s} \left| (D_j - D_{M_{\alpha s}}) \right|
\]

\[
\leq \lambda \sum_{\eta=0}^{s-1} \frac{M_{\alpha s}^{1/p} - 1}{\alpha_q} + \frac{\lambda M_{\alpha s+1}^{1/p} - 1}{\alpha_s} \leq \frac{2 \lambda M_{\alpha s+1}^{1/p} - 1}{\alpha_{s-1}} + \frac{4 \lambda M_{\alpha s+1}^{1/p} - 1}{\alpha_{k-1}}
\]

Let \(M_{\alpha s+1} - 1 \leq j \leq M_{\alpha s}\), where \(s = 1, \ldots, k\). Analogously to (19) we can prove that

\[
|S_j f| = \left| \sum_{v=0}^{M_{\alpha s+1} - 1} \hat{f}(v) \psi_v \right| = \sum_{\eta=0}^{s-1} \sum_{v=M_{\alpha s}}^{M_{\alpha s+1} - 1} \frac{M_{\alpha s}^{1/p} - 1}{\alpha_q} \psi_v
\]

\[
= \sum_{\eta=0}^{s-1} \frac{M_{\alpha s}^{1/p} - 1}{\alpha_q} \left( D_{M_{\alpha s+1}} - D_{M_{\alpha s}} \right) \leq \frac{2 \lambda M_{\alpha s+1}^{1/p} - 1}{\alpha_{s-1}} \leq \frac{4 \lambda M_{\alpha s+1}^{1/p} - 1}{\alpha_{k-1}}
\]

Hence

\[
|I| \leq \frac{1}{Q_{M_{\alpha k+1}}} \sum_{j=0}^{M_{\alpha k}} q_j \left| S_j f \right| \leq \frac{4 \lambda M_{\alpha k+1}^{1/p} - 1}{\alpha_{k-1}} \frac{1}{Q_{M_{\alpha k+1}}} \sum_{j=0}^{M_{\alpha k}} q_j \leq \frac{4 \lambda M_{\alpha k+1}^{1/p} - 1}{\alpha_{k-1}}
\]

If we now apply (18) and (19) we get that

\[
|II| = \frac{q_0}{Q_{M_{\alpha k+1}}} \left| \frac{M_{\alpha k}^{1/p} - 1}{\alpha_k} \psi_{M_{\alpha k}} + S_{M_{\alpha k}} f \right|
\]

\[
= \frac{q_0}{Q_{M_{\alpha k+1}}} \left| \frac{M_{\alpha k}^{1/p} - 1}{\alpha_k} \psi_{M_{\alpha k}} + S_{M_{\alpha k-1}+1} f \right|
\]

\[
\geq \frac{q_0}{Q_{M_{\alpha k+1}}} \left( \left| \frac{M_{\alpha k}^{1/p} - 1}{\alpha_k} \psi_{M_{\alpha k}} \right| - \left| S_{M_{\alpha k-1}+1} f \right| \right)
\]

\[
\geq \frac{q_0}{Q_{M_{\alpha k+1}}} \left( \frac{M_{\alpha k}^{1/p} - 4 \lambda M_{\alpha k-1}^{1/p} - 1}{\alpha_k} \right) \geq \frac{q_0}{Q_{M_{\alpha k+1}}} \frac{M_{\alpha k}^{1/p} - 1}{4 \alpha_k}.
\]
Without lost the generality we may assume that $c = 1$ in (5). By combining (20) and (21) we get
\[
|t_{M_{\alpha k} + 1} f| \geq |II| - |I| \geq \frac{q_0 M_{\alpha k}^{1/p} - 4\lambda M_{\alpha k - 1}^{1/p}}{4\alpha_k} - 4\lambda M_{\alpha k - 1}^{1/p}
\]
\[
\geq \frac{M_{\alpha k}^{1/p - 2}}{4\alpha_k} - 4\lambda M_{\alpha k - 1}^{1/p} \geq \frac{M_{\alpha k}^{1/p - 2}}{8\alpha_k}.
\]

On the other hand
\[
\mu \left\{ x \in G_m : |t_{M_{\alpha k} + 1} f(x)| \geq \frac{M_{\alpha k}^{1/p - 2}}{8\alpha_k} \right\} = \mu(G_m) = 1.
\]

Let $0 < p < 1/2$. Then
\[
\frac{M_{\alpha k}^{1/p - 2}}{8\alpha_k} \cdot \mu \left\{ x \in G_m : |t_{M_{\alpha k} + 1} f(x)| \geq \frac{M_{\alpha k}^{1/p - 2}}{8\alpha_k} \right\}
\]
\[
= \frac{M_{\alpha k}^{1/p - 2}}{8\alpha_k} \to \infty, \quad \text{as } k \to \infty.
\]

The proof is complete.

**Proof of Theorem 3.** To prove Theorem 3 we use the martingale (15), where $\lambda_k$ are defined by (16), for which $\alpha_k$ satisfies conditions (12)-(14) and $\theta_k$ are given by (17). It is easy to show that for every non-increasing sequence $\{q_k : k \geq 0\}$ it automatically holds that
\[
q_0/Q_{M_{\alpha k} + 1} \geq 1/(M_{\alpha k} + 1).
\]

By combining (20) and (21) we see that
\[
|t_{M_{\alpha k} + 1} f| \geq |II| - |I| \geq \frac{M_{\alpha k}^{1/p - 2}}{8\alpha_k}.
\]

Analogously to (22) and (23) we then get that
\[
\sup_k \|t_{M_{\alpha k} + 1} f\|_{\text{weak-}L^p} = \infty.
\]

The proof is complete.

**Proof of Theorem 4.** Let $\{q_k : k \geq 0\}$ be a sequence of non-increasing numbers, which satisfies conditions of theorem 4. In this case (see [4], [31]) it was proved that
\[
|F_n| \leq \frac{c(\alpha)}{n^\alpha} \sum_{j=0}^{|n|} M_j^n |K_M|,
\]
where
\[
F_n = \frac{1}{Q_n} \sum_{k=1}^n q_{n-k} D_k f
\]
is Kernel of Nörlund means.
Weisz and Simon [28] (see also Gát, Goginava [8]) proved that the maximal operator $\sigma^{\alpha,*}$ of $(C, \alpha)$ ($0 < \alpha < 1$) means is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha)\infty}$. Their proof was depend on the following inequality

$$|K_n^\alpha| \leq c(\alpha) \sum_{j=0}^{n} M_J^\alpha |K_{M_j}|,$$

where $K_n^\alpha$ is Kernel of $(C, \alpha)$ means. Since our estimation of $F_n$ is the same, it is easy to see that the proof of first part of Theorem 4 will be quiet analogously to the Theorem SW1.

To prove the second part of Theorem 4 we use the martingale (15), where $\lambda_k$ is defined by (16), for which $\alpha_k$ satisfies conditions (12), (13) and (24)

$$32\lambda M_{\alpha_k-1}^{1/p} < M_{\alpha_k}^{1/p-1-\alpha}/\alpha_k,$$

where $\theta_k$ is given by (17).

We note that such an increasing sequence $\{\alpha_k : k \in \mathbb{N}\}$, which satisfies conditions (12), (13) and (24), can be constructed.

Let $0 < p < 1/(1 + \alpha)$. By combining (12) and (13) we get that

$$|t_{M_{\alpha_k}+1}f| \geq |II| - |I| = \frac{M_{\alpha_k}^{1/p-1}q_0}{4\alpha_k} Q_{M_{\alpha_k}+1} - 4\lambda M_{\alpha_k-1}^{1/p}.$$

Without lost the generality we may assume that $c = 1$ in (7). Since $1/p - 1 - \alpha > 0$ by using (24) we find that

$$|t_{M_{\alpha_k}+1}f| \geq \frac{M_{\alpha_k}^{1/p-1-\alpha}}{8\alpha_k}$$

and

$$\frac{M_{\alpha_k}^{1/p-1-\alpha}}{8\alpha_k} \cdot \mu \left\{ x \in G_m : |t_{M_{\alpha_k}+1}f| \geq \frac{M_{\alpha_k}^{1/p-1-\alpha}}{8\alpha_k} \right\}$$

$$= \frac{M_{\alpha_k}^{1/p-1-\alpha}}{8\alpha_k} \rightarrow \infty, \text{ as } k \rightarrow \infty.$$

Finally, we prove the third part of Theorem 4. Under condition (8) there exists an increasing sequence $\{\alpha_k : k \in \mathbb{N}\}$, which satisfies the conditions

$$\sum_{k=0}^{\infty} \left( \frac{Q_{M_{\alpha_k}+1}}{q_0 M_{\alpha_k}} \right)^{1/(1+\alpha)} = \infty,$$

$$\lambda \sum_{\eta=0}^{k-1} \frac{Q_{M_{\alpha_k}+1} M_{\alpha_k/2+1}}{q_0 M_{\alpha_k}^{1/2}} \leq \frac{Q_{M_{\alpha_k}+1} M_{\alpha_k/2+1}}{q_0 M_{\alpha_k}^{1/2}};$$

$$32\lambda Q_{M_{\alpha_k-1}+1} M_{\alpha_k-1}^{\alpha/2+1}/q_0^{1/2} < \left( \frac{q_0 M_{\alpha_k}^{\alpha}}{Q_{M_{\alpha_k}+1}} \right)^{1/2}.$$
where \( \lambda = \sup_n m_n \).

We note that such an increasing sequence \( \{ \alpha_k : k \in \mathbb{N} \} \), which satisfies conditions (25)-(27), can be constructed.

Let

\[
 f^{(A)} = \sum_{\{ k : \lambda_k < A \}} \lambda_k \theta_k,
\]

where

\[
 \lambda_k = \lambda \cdot \left( \frac{Q_{M\alpha k+1}}{q_0 M_{\alpha_k}^\alpha} \right)^{1/2}
\]

and \( \theta_k \) are given by (17) for \( p = 1/(1+\alpha) \). If we apply Lemma 1 and (25) analogously to the proof of the second part of Theorem 2 we can conclude that \( f = (f^{(1)}, f^{(2)} ... f^{(A)}...) \in H_{1/(1+\alpha)} \).

It is easy to show that

\[
 \hat{f}(j) = \begin{cases} 
 \left( \frac{Q_{M\alpha k+1} M_{\alpha k}^\alpha}{q_0} \right)^{1/2}, & \text{if } j \in \{ M\alpha_k, ..., M\alpha_{k+1} - 1 \}, k = 0, 1, 2, ..., \\
 0, & \text{if } j \notin \bigcup_{k=1}^\infty \{ M\alpha_k, ..., M\alpha_{k+1} - 1 \},
\end{cases}
\]

and

\[
 t_{M\alpha k+1} f = \frac{1}{Q_{M\alpha k+1}} \sum_{j=0}^{M\alpha_k} q_j S_j f + \frac{q_0}{Q_{M\alpha k+1}} S_{M\alpha k+1} f := III + IV.
\]

Let \( M_{\alpha s} \leq j \leq M_{\alpha s+1} \), where \( s = 0, ..., k-1 \). Analogously to (19) from (28) we have that

\[
 |S_j f| \leq \left| \sum_{\eta=0}^{s-1} \left( \frac{Q_{M\alpha_{\eta+1}} M_{\alpha_{\eta}}^\alpha}{q_0} \right)^{1/2} \left( D_{M\alpha_{\eta+1}} - D_{M\alpha_{\eta}} \right) \right| + \left( \frac{Q_{M\alpha s+1} M_{\alpha s}^\alpha}{q_0} \right)^{1/2} \left| (D_j - D_{M\alpha s}) \right|
\]

\[
 \leq 4 \lambda Q_{M\alpha_{s+1}}^{1/2} M_{\alpha_{s+1/2}}^{\alpha/2} q_0^{1/2}.
\]

Let \( M_{\alpha_{s+1}} + 1 \leq j \leq M_{\alpha s} \), where \( s = 1, ..., k \). Then

\[
 |S_j f| = \left| \sum_{\eta=0}^{s-1} \left( \frac{Q_{M\alpha_{\eta+1}} M_{\alpha_{\eta}}^\alpha}{q_0} \right)^{1/2} \left( D_{M\alpha_{\eta+1}} - D_{M\alpha_{\eta}} \right) \right| \leq 2 \lambda Q_{M_{\alpha_{s-1}}+1} M_{\alpha_{s-1/2}}^{\alpha/2} q_0^{1/2} \leq 4 \lambda Q_{M_{\alpha_{s-1}}+1} M_{\alpha_{s-1/2}}^{\alpha/2} q_0^{1/2},
\]

and
\begin{align}
|III| \leq \frac{1}{Q_{M_{\alpha k}} + 1} \sum_{j=0}^{M_{\alpha k}} q_j |S_j f| \leq \frac{4\lambda Q_{M_{\alpha k-1} + 1}^{\alpha/2 + 1} M_{\alpha k}^{\alpha/2 - 1}}{q_0^{1/2}} \sum_{j=0}^{M_{\alpha k}} q_j \leq \frac{4\lambda Q_{M_{\alpha k-1} + 1}^{\alpha/2 + 1} M_{\alpha k}^{\alpha} + 1}{q_0^{1/2}}.
\end{align}

If we apply (28) and (29) we get that
\begin{align}
|IV| \geq \left( \frac{Q_{M_{\alpha k} + 1}^{\alpha}}{q_0} \right)^{1/2} \frac{q_0}{Q_{M_{\alpha k} + 1}} \left| D_{M_{\alpha k} + 1} - D_{M_{\alpha k}} \right| - \frac{q_0}{Q_{M_{\alpha k} + 1}} |S_{M_{\alpha k} + 1} f|
\end{align}

By combining (27), (30) and (31) we get that
\begin{align}
|t_{M_{\alpha k} + 1} f| \geq |IV| - |III| \geq \frac{1}{4} \left( \frac{q_0 M_{\alpha k}^{\alpha}}{Q_{M_{\alpha k} + 1}} \right)^{1/2} - \frac{4\lambda Q_{M_{\alpha k-1} + 1}^{\alpha/2 + 1} M_{\alpha k}^{\alpha} + 1}{q_0^{1/2}} \frac{q_0}{Q_{M_{\alpha k} + 1}} \geq \frac{1}{8} \left( \frac{q_0 M_{\alpha k}^{\alpha}}{Q_{M_{\alpha k} + 1}} \right)^{1/2}.
\end{align}

Hence, it yields that
\begin{align}
\frac{1}{8} \left( \frac{q_0 M_{\alpha k}^{\alpha}}{Q_{M_{\alpha k} + 1}} \right)^{1/2} \mu \left\{ x \in G_m : \left| t_{M_{\alpha k} + 1} f(x) \right| \geq \frac{1}{8} \left( \frac{q_0 M_{\alpha k}^{\alpha}}{Q_{M_{\alpha k} + 1}} \right)^{1/2} \right\}
= \frac{1}{8} \left( \frac{q_0 M_{\alpha k}^{\alpha}}{Q_{M_{\alpha k} + 1}} \right)^{1/2} \mu (G_m) = \frac{1}{8} \left( \frac{q_0 M_{\alpha k}^{\alpha}}{Q_{M_{\alpha k} + 1}} \right)^{1/2} \to \infty, \text{ as } k \to \infty.
\end{align}

The proof is complete.

**A final remark:** Several of the operators considered in this paper, e.g. those described by the Nörlund means, Riesz’s logarithmic means and Nörlund logarithmic means are called Hardy type operators in the literature. The mapping properties of such operators, especially between weighted Lebesgue spaces, is much studied in the literature, see e.g. the books [15] and [17] and the references there. Such complimentary information can be of interest for further studies of the problems considered in this paper.

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