THE SUBSUMS OF ZERO-SUM FREE SEQUENCES IN FINITE CYCLIC GROUPS

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Abstract. Let $\mathbb{Z}_n$ be the cyclic group of order $n \geq 3$ additively written. S. Savchev & F. Chen (2007) proved that for each zero-sum free sequence $S = a_1 \cdot \cdot \cdot a_t$ over $\mathbb{Z}_n$ of length $t > n/2$, there is an integer $g$ coprime to $n$ such that, if $r$ denotes the least positive integer in the congruence class $r$ modulo $n$, then $\sum_{i=1}^{t} \frac{a_i}{g} < n$. Under the same hypothesis, in this paper we show that

$$\left\{ \sum_{i \in \Lambda} \frac{a_i}{g} \mid \Lambda \subset \{1, 2, \ldots, t\} \right\} = \{1, 2, \ldots, \sum_{i=1}^{t} \frac{a_i}{g} \}.$$ 

It simplifies many calculations on inverse zero-sum problems.

1. Introduction

Given a finite group $G$, the Zero-Sum Problems study conditions to ensure that a given sequence over $G$ has a non-empty subsequence with prescribed properties (such as length, repetitions, weights) such that the product of its elements is equal to the identity of $G$. This class of problems have been extensively studied for abelian groups (see, for example, the surveys [5, 8]), and since little more than a decade there are some results for non-abelian groups (see, for example, [1, 3, 4, 11, 13, 15, 16, 19, 22]).

Denote by $[a, b]$ the interval $\{n \in \mathbb{N}; a \leq n \leq b\}$. By a sequence $S$ over $G$ we mean an element $S$ (finite and unordered) of the free abelian monoid $\mathcal{F}(G)$, equipped with the sequence concatenation product denoted by $\cdot$. A sequence $S \in \mathcal{F}(G)$ has the form

$$S = g_1 \cdot \cdot \cdot g_k = \prod_{i=1}^{k} g_i \in \mathcal{F}(G)$$

where $g_i \in G$ are the terms of $S$ and $k = |S| \geq 0$ is the length of $S$. Since the sequences are unordered,

$$S = \prod_{i=1}^{k} g_i = \prod_{i=1}^{k} g_{\tau(i)} \in \mathcal{F}(G)$$

for any permutation $\tau : [1, k] \to [1, k]$. Given $g \in G$ and $t \geq 0$, we abbreviate

$$g^{(t)} = g \cdot \cdot \cdot g,$$

$t$ times

For $g \in G$, the multiplicity of the term $g$ in $S$ is denoted by $v_g(S) = \#\{i \in [1, k]; g_i = g\}$, therefore our sequence $S$ may also be written as

$$S = \prod_{g \in G} g^{v_g(S)}.$$

A sequence $T$ is called a subsequence of $S$ if $T \subseteq S$ in $\mathcal{F}(G)$, or equivalently, $v_g(T) \leq v_g(S)$ for all $g \in G$. From now on, we assume that $G$ is abelian and additively written.

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We also define:

\[ \sigma(S) = g_1 + \cdots + g_k \in G \] the sum of \( S \);

\[ \Sigma(S) = \bigcup_{|T| \geq 1} \{ \sigma(T) \} \subset G \] the set of subsequence sums of \( S \);

\[ S \cdot T^{-1} = \prod_{g \in G, \ \nu_g(T) > 0} g \] the subsequence of \( S \) formed by the terms that do not lie in \( T \);

\[ S \cap K = \prod_{g \in S} g \] the subsequence of \( S \) that lie in a subset \( K \) of \( G \).

The sequence \( S \) is called

- zero-sum free if \( 0 \notin \Sigma(S) \);
- zero-sum sequence if \( \sigma(S) = 0 \).

An important type of zero-sum problem is to determine the \textit{Davenport constant} of a finite group \( G \): This constant, denoted by \( d(G) \), is the maximal integer such that there exists a sequence over \( G \) (repetition allowed) of length \( d(G) \) which is zero-sum free, i.e,

\[ d(G) = \sup\{|S| > 0; S \in \mathcal{F}(G) \text{ is zero-sum free}. \} \]

Using Pigeonhole Principle on the partial sums \( g_1, g_1 + g_2, \ldots, g_1 + \cdots + g_n \), it is easy to show that if \( S \in \mathcal{F}(\mathbb{Z}_n) \) and \( |S| = n \) then \( S \) is not zero-sum free. Furthermore, if \( \gcd(a, n) = 1 \) then the sequence \( a^{[n-1]} \) is zero-sum free. Thus \( d(\mathbb{Z}_n) = n - 1 \). The zero-sum free sequence \( S = (1, 0, \ldots, 0)_{[n_1-1]} \cdots (0, \ldots, 0, 1)_{[n_k-1]} \) over \( \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \) shows that

\[ d(\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}) \geq \sum_{i=1}^{k} (n_i - 1). \tag{1.1} \]

Olson \cite{17, 18} proved that the equality holds for \( p \)-groups and in the case of rank two. It is conjectured that the equality also holds for groups of the form \( \mathbb{Z}_n^k \). However, the equality does not hold for a general abelian group of order \( k \geq 4 \) (see, for example, \cite{12, 13}).

Denote by \( \nu \) the least positive integer in the congruence class \( a \mod n \) and let \( n \geq 2k+1 \geq 3 \). Bovey, Erdős & Niven \cite{2} proved that if \( S \in \mathcal{F}(\mathbb{Z}_n) \) is zero-sum free and \( |S| = n - k \), then

\[ v_a(S) \geq n - 2k + 1 \]

for some \( a \mid S \). Moreover, they showed that this inequality is the best possible whether \( n \geq 3k - 2 \). Savchev & Chen \cite{20} proved that if \( 2k+1 \leq n \leq 3k - 2 \) then there exists \( a \mid S \) such that

\[ v_a(S) \geq n - k - \left\lfloor \frac{n - 1}{3} \right\rfloor < \begin{cases} \frac{2n+1}{3} - k & \text{if } n \equiv 0 \pmod{3}; \\ \frac{2n+1}{3} - k & \text{if } n \equiv 1 \pmod{3}; \\ \frac{2n+2}{3} - k & \text{if } n \equiv 2 \pmod{3}. \end{cases} \]

In addition, they proved that for each zero-sum free sequence \( S = a_1 \cdots a_{n-k} \in \mathcal{F}(\mathbb{Z}_n) \) there is an integer \( g \) with \( \gcd(g, n) = 1 \) such that

\[ \sum_{i=1}^{n-k} g_{a_i} \leq n - 1. \]

In this paper, we show that if \( n \geq 2k + 1 \geq 3 \) then

\[ \left\{ \sum_{i \in \Lambda} g_{a_i} \mid \Lambda \subset [1, t] \right\} = \left[ 1, \sum_{i=1}^{t} g_{a_i} \right]. \]
The proof is done in Section 3. In Section 2 we present some auxiliary lemmas, while in Section 4 we present an application that simplifies some calculations on inverse problems.

2. Preliminary

Our proof is based on the following results:

Lemma 2.1. Let \( S \in \mathcal{F}(\mathbb{Z}) \) of the form \( S = 1^{v_1} \bullet 2^{v_2} \), where \( v_1, v_2 \geq 1 \). Then \( \Sigma(S) = [1, v_1 + 2v_2] \).

Proof: The inclusion \( \subset \) is clear. For the opposite \( \supset \), let \( N \in [1, v_1 + 2v_2] \). Then \( N = \alpha + 2\beta \), where \( 0 \leq \alpha \leq v_1 \) and \( 0 \leq \beta \leq v_2 \) (but not \( \alpha = \beta = 0 \)), which implies \( N = \sum_{i=1}^{\alpha} 1 + \sum_{i=1}^{\beta} 2 \in \Sigma(S) \). \( \Box \)

Lemma 2.2. Let \( S \in \mathcal{F}(\mathbb{Z}) \) of the form \( S = 1^{v_1} \bullet 3^{v_3} \), where \( v_1 \geq 2 \) and \( v_3 \geq 1 \). Then \( \Sigma(S) = [1, v_1 + 3v_2] \).

Proof: The inclusion \( \subset \) is clear. For the opposite \( \supset \), let \( N \in [1, v_1 + 3v_2] \). Then \( N = \alpha + 3\gamma \), where \( 0 \leq \alpha \leq v_1 \) and \( 0 \leq \gamma \leq v_2 \) (but not \( \alpha = \gamma = 0 \)), which implies \( N = \sum_{i=1}^{\alpha} 1 + \sum_{i=1}^{\gamma} 3 \in \Sigma(S) \). \( \Box \)

Lemma 2.3. Let \( G \) be a finite abelian group and let \( S \in \mathcal{F}(G) \). If \( T|S \) then \( \sigma(T) + \sigma(S \bullet T^{-1}) = \sigma(S) \).

In particular, if \( \sigma(T) = t \) then \( \sigma(S \bullet T^{-1}) = \sigma(S) - t \).

Proof: Trivial. \( \Box \)

3. The subsums of a zero-sum free sequence

Theorem 3.1. Let \( n, k \) be positive integers with \( n \geq 2k + 1 \geq 3 \) and let \( S = a_1 \bullet \cdots \bullet a_{n-k} \in \mathcal{F}(\mathbb{Z}_n) \) be a zero-sum free sequence with

\[
\sum_{i=1}^{n-k} a_i \leq n - 1.
\]  # (3.1)

Then

\[ \Sigma(S) = [1, \sigma(S)] \]

In other words, for all \( 1 \leq t \leq \sum_{i=1}^{n-k} a_i \) there exist \( a_{i_1} \bullet \cdots \bullet a_{i_r} \parallel S \) such that \( \sum_{j=1}^{r} a_{i_j} = t \).

Proof: The inclusion \( \subset \) is clear. For the opposite \( \supset \), by Lemma 2.3 and inequality 3.1 it is enough show that

\[
[1, \left\lfloor \frac{n}{2} \right\rfloor] \subset \Sigma(S).
\]  # (3.2)

For simplicity, we denote \( v_i(S) \) by \( v_i \). Suppose that \( v_1 \leq 1 \). Then

\[
\sum_{i=1}^{n-k} a_i \geq 1 + 2(n - k - 1) = 2n - 2k - 1 \geq n,
\]

since \( n \geq 2k + 1 \), which is an absurd. Therefore, \( v_1 \geq 2 \).

Let \( v_\ell = \max \{v_i\} \) (that is, \( \ell \) is the term of greatest multiplicity in \( S \)). By the results of Bovey, Erdős & Niven [2] and Savchev & Chen [20], we split into two cases:

(i) Case \( n \geq 3k - 2 \): we have \( v_\ell \geq n - 2k + 1 \geq \left\lceil \frac{n-1}{3} \right\rceil \).

(i.1) If \( \ell \geq 3 \) then

\[
\sum_{i=1}^{n-k} a_i \geq 3v_\ell \geq 3 \left( \frac{n-1}{3} \right) = n - 1, \quad \text{contradiction!}
\]
(i.2) If \( \ell = 2 \), since
\[
\sum_{i=1}^{n-k} \overline{\alpha_i} > 2v_2 \geq 2 \left\lceil \frac{n-1}{3} \right\rceil \geq \frac{n}{2}
\]
and \( v_1 \geq 2 \), Lemma 2.1 implies that \( [1, \left\lceil \frac{n}{2} \right\rceil] \subset \Sigma(S) \). Therefore, inclusion (3.2) holds and we are done in this subcase.

(i.3) If \( \ell = 1 \), define \( 0 \leq \alpha \leq k - 1 \) by equation \( v_1 = n - 2k + 1 + \alpha \). By Pigeonhole Principle, the average of the elements \( \overline{\alpha_i} \) without the terms 1 and 2 must be at least 3 (otherwise, there would be \( \overline{\alpha}_i < 3 \) that could be removed before). Hence,
\[
3 \leq \sum_{i=1}^{n-k} \overline{\alpha_i} \geq v_1 + 2v_2 \geq n - 1 - 3 \left( \frac{k-1}{2} \right) \geq \frac{n}{2},
\]
which implies
\[
v_2 \geq \begin{cases} 
  k - 1 - 2\alpha & \text{if } \alpha \leq \frac{k-1}{2} \\
  0 & \text{if } \alpha > \frac{k-1}{2}
\end{cases}
\]
If \( \alpha \leq \frac{k-1}{2} \) then we have
\[
\sum_{i=1}^{n-k} \overline{\alpha_i} \geq v_1 + 2v_2 \geq n - 1 - 3 \left( \frac{k-1}{2} \right) \geq \frac{n}{2},
\]
and we are done by Lemma 2.1 and inclusion (3.2).
Otherwise, if \( \alpha > \frac{k-1}{2} \) then we have
\[
v_1 \geq n - 2k + 1 + \frac{k}{2} \geq \frac{n}{2},
\]
thus every element in \( [1, \left\lceil \frac{n}{2} \right\rceil] \) can be obtained as sum of 1’s and we are done in this case by inclusion (3.2).

(ii) Case \( 2k + 1 \leq n < 3k - 2 \): we have \( v_\ell \geq n - k - \left\lfloor \frac{n-1}{3} \right\rfloor \geq \frac{n}{6} \).

(ii.1) If \( \ell \geq 4 \) then
\[
\sum_{i=1}^{n-k} \overline{\alpha_i} \geq 4v_\ell \geq 4 \left( \frac{n+5}{6} \right) + \left\lceil \frac{n-1}{3} \right\rceil \geq n - 1, \text{ contradiction!}
\]

(ii.2) If \( \ell = 3 \), since
\[
\sum_{i=1}^{n-k} \overline{\alpha_i} > 3v_3 \geq \frac{n+5}{2} \geq \frac{n}{2}
\]
and \( v_1 \geq 2 \), Lemma 2.2 implies that \( [1, \left\lceil \frac{n}{2} \right\rceil] \subset \Sigma(S) \). Therefore, inclusion (3.2) holds and we are done in this subcase.

(ii.3) If \( \ell = 2 \), since \( v_1 \geq 2 \) and
\[
\sum_{i=1}^{n-k} \overline{\alpha_i} \geq 2v_2 + \sum_{i=1}^{n-k} \overline{\alpha_i} \geq 2 \left( \frac{n+5}{6} \right) + \left\lceil \frac{n-1}{3} \right\rceil \geq \frac{n-1}{2},
\]
then Lemma 2.1 implies that \( [1, \left\lceil \frac{n}{2} \right\rceil] \subset \Sigma(S) \). Therefore, inclusion (3.2) holds and we are done in this subcase.

(ii.4) If \( \ell = 1 \), define \( 0 \leq \alpha \leq \left\lfloor \frac{n-1}{3} \right\rfloor \) by equation \( v_1 = n - k - \left\lfloor \frac{n-1}{3} \right\rfloor + \alpha \). Using Pigeonhole Principle again, the average of the elements \( \overline{\alpha_i} \) without the terms 1 and 2 must be at least 3,
In other words, for all $a$ zero-sum free sequence satisfying Equation 3.1. Then

The hypothesis $F$ be a finite non-abelian group multiplicatively written, and $S$ be a finite non-abelian group generated by $\{x,y\}$, $x$, $y$. Zhuang & Gao \cite{Zhuang22} proved in inclusion (3.2).

Remark 3.2. By the results of Savchev & Chen, there is no loss of generality in the assumption (3.1).

Remark 3.3. The hypothesis $n \geq 2k + 1 \geq 3$ can not be removed. In fact, if $n = 5$, $k = 3$, $S_1 = \{1 \cdot 3 \in \mathcal{F}(\mathbb{Z}_5)$ and $S_2 = \{2\} \in \mathcal{F}(\mathbb{Z}_5)$, then $2 \notin \Sigma(S_1) = \{1, 3, 4\}$ and $1, 3 \notin \Sigma(S_2) = \{2, 4\}$.

Corollary 3.4. Let $n, k$ be positive integers with $n \geq 2k + 1 \geq 3$ and let $S = a_1 \cdot \cdots \cdot a_{n-k} \in \mathcal{F}(\mathbb{Z}_n)$ be a zero-sum free sequence satisfying Equation (3.1). Then

\[ [1, n-k] \subset \Sigma(S). \]

In other words, for all $1 \leq t \leq n-k$ there exist $a_{i_1} \cdot \cdots \cdot a_{i_r} \mid S$ such that $\sum_{j=1}^r \overline{a_{i_j}} = t$.

4. Simplifying calculations

In this section, we present an example of application that simplify some calculations on inverse zero-sum problems. Let $G$ be a finite non-abelian group multiplicatively written, and $S = g_1 \cdot \cdots \cdot g_k \in \mathcal{F}(G)$. Define:

\[ \pi(S) = \{\tau \in [1,k]! : \text{the set of products of } S; \] \[ \Pi(S) = \bigcup_{|T| \geq 1} \pi(T) \subset D_{2n} \text{ the set of subsequence products of } S. \]

The sequence $S$ is called product-one free if $1 \notin \Pi(S)$. The small Davenport constant $d(G)$ can be defined analogously for $G$:

\[ d(G) = \sup\{|S| > 0; S \in \mathcal{F}(G) \text{ is product-one free}\}. \]

Let $D_{2n}$ be the Dihedral Group, ie, the group generated by $x$ and $y$ satisfying $x^2 = y^n = 1$ and $yx = y^{-1}$ xy^{-1}$. Zhuang & Gao \cite{Zhuang22} proved $d(D_{2n}) = n$. In a joint work with Brochero Martínez \cite{Brochero}, we exhibit all the extremal length product-one free sequences over $D_{2n}$, showing that:

Theorem 4.1 (\cite{Brochero}, Theorem 1.3). Let $n \geq 3$ and $S \in \mathcal{F}(G)$ such that $|S| = n$.

(1) If $n \geq 4$ then $S$ is product-one free if and only if for some $1 \leq t \leq n-1$ with $\gcd(t, n) = 1$ and $0 \leq s \leq n-1$, $S = (y^t)^{[n-1]} \cdot xy^s$.

(2) If $n = 3$ then $S$ product-one free if and only if either $S = x \cdot xy \cdot xy^2$ or $S = (y^t)^{[2]} \cdot xy^n$ for $t \in \{1, 2\}$ and $n \in \{0, 1, 2\}$. 
In the original paper, the proof was obtained considering the cases

- $|S \cap \langle y \rangle| = n$,
- $|S \cap \langle y \rangle| = n - 1$,
- $|S \cap \langle y \rangle| = n - 2$,
- $|S \cap \langle y \rangle| = n - 3$,
- $n - 2 \lfloor \log_2 n \rfloor - 1 \leq |S \cap \langle y \rangle| \leq n - 4$, and
- $|S \cap \langle y \rangle| \leq n - 2 \lfloor \log_2 n \rfloor - 2$,

besides considering the initial cases $3 \leq n \leq 7$ separately. Only the third and fourth cases resemble each other. The case $n$ prime had already been done in [3], which reduced the handwork at least in the initial steps.

**Proof:** In our new approach, there is no need to consider any initial steps, and in addition, the first two cases keep identical to the original paper. We propose to consider only the following

- $|S \cap \langle y \rangle| = n$,
- $|S \cap \langle y \rangle| = n - 1$,
- $\frac{n}{2} < |S \cap \langle y \rangle| \leq n - 2$, and
- $|S \cap \langle y \rangle| \leq \frac{n}{4}$.

For the third case, we notice that $x y^\alpha \cdot x y^\beta \cap S$ for some $\alpha \neq \beta \pmod{n}$ (otherwise, $x y^\alpha \cdot x y^\beta = 1$).

Assume that $\alpha - \beta \in [1, \lceil \frac{n}{2} \rceil]$ (otherwise, switch $\alpha$ and $\beta$). Using the result of Savchev & Chen if needed, we may assume that if $S \cap \langle y \rangle$ does not satisfy the hypotheses of Corollary 3.4 then $S$ is not product-one free. This corollary ensures that $\alpha - \beta$ can be obtained as sum of the exponents of the elements in $S \cap \langle y \rangle$.

Let $S = S \cdot \langle S \cap \langle y \rangle \rangle^{-1}$. Then $|T| = k$, which implies that $T$ produces $\lceil \frac{k}{2} \rceil$ new elements in $S \cap \langle y \rangle$. Let $x y^\alpha \cdot x y^\beta y = y^{\beta - \alpha}$ one of these, where $1 \leq \alpha - \beta \leq \frac{n}{2}$. If $n \geq k + 1$ then $n - k + \lceil \frac{k}{2} \rceil - 1 \geq \lceil \frac{n}{2} \rceil$, thus Corollary 3.4 ensures that the $n-k$ terms of $S \cap \langle y \rangle$ joint with the $\lceil \frac{k}{2} \rceil - 1$ new elements from $T$ yields a product of the form $P = y^{\beta - \alpha}$. Therefore, $P \cdot x y^\alpha \cdot x y^\beta = 1$, and $S$ is not product-one free. It only remains the case $n = k$.

But in this case the elements must be all distinct (otherwise the product of a identical pair would be 1), therefore $S = x \cdot x y^\alpha \cdot x y^\beta \cdots \cdot x y^{\beta - 1}$. If $n \geq 4$ then $x \cdot x y^\alpha \cdot x y^\beta \cdot x y^\delta = 1$ and $S$ is not product-one free. If $n = 3$, the sequence $x \cdot x y^\alpha \cdot x y^\beta$ over $D_6$ is product-one free, and we are done.

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