Large Arrays of Linearly Coupled Josephson Junctions: A Survey

D. G. Aronson
School of Mathematics
University of Minnesota
Minneapolis MN 55455
email: arons001@umn.edu

Abstract: In this note I survey the extensive literature on the dynamics of large series arrays of identical current biased Josephson junctions coupled through various shared loads. The equations describing the dynamics of these arrays are invariant under permutation of the junctions so that, in addition to the usual dynamical systems and numerical methods, group theoretic methods can also be applied. In practice it is desirable to operate these circuits at a stable in-phase oscillation. The works summarized here are devoted to the study of where in parameter space the in-phase oscillations are stable and how the stability is lost. In particular, they focus on the variety of states produced through bifurcation as the in-phase oscillation loses stability. These states include, among others, discrete rotating waves and semirotors.

Hadley, Beasley & Wisenfeld [10] studied numerically large series arrays of current biased Josephson junctions coupled through a variety of shared loads. In practice it is desirable to operate these circuits at a stable in-phase oscillation. The works summarized here are devoted to the study of where in parameter space the in-phase oscillations are stable and how the stability is lost. Here we focus on the variety of states produced through bifurcation as the in-phase oscillation loses stability. A common feature of the models considered by Hadley et al. is that the junctions are all equally coupled to one another through the load so that the equations describing their dynamics are invariant under permutation of the junctions. Thus, in addition to the usual dynamical systems and numerical methods, group theoretic methods can also be applied [4].

Let \( \varphi_k (k = 1, ..., N) \) denote the difference in phase of the quasiclassical superconducting wavefunctions on the two sides of the \( k \)-th junction and let \( Q \) denote the current flowing through the load. The evolution of the \( \varphi_k \) and \( Q \) is governed by the system of equations:

\[
\beta \ddot{\varphi}_k + \dot{\varphi}_k + \sin(\varphi_k) + Q = I, \tag{1}
\]

where \( \beta \) is the dimensionless measure of the intrinsic capacity of the junctions, \( I \) is the bias current and \( \mathcal{F} \) is an integro-differential operator which depends on the particular load considered. Note that the system is invariant under permutation of the junctions. Here we will concentrate on the load equation:

\[
L \ddot{Q} + R \dot{Q} + C^{-1} Q = \frac{1}{N} \sum_{k=1}^{N} \dot{\varphi}_k, \tag{2}
\]
where $L, R$ and $C$ are, respectively, the inductance, resistance and capacity of the load.

A running solution of the system (1) is an $(N + 1)$-tuple

$$\{\varphi_1, \varphi_2, \ldots, \varphi_N, Q\}$$

for which there is a minimum $T > 0$ such that

$$\varphi_k(t + T) = \varphi_k(t) + 2\pi$$

for $k = 1, \ldots, N$ and $Q, \dot{Q}$ is a $T$-periodic function if $C < \infty$ or $C^{-1} = 0$ respectively. In the sequel we will ignore this distinction and simply say that $Q$ is $T$-periodic. An in-phase or symmetric running solution $\{\varphi_1, \varphi_2, \ldots, \varphi_N, Q\}$ is characterized by

$$\varphi_1 = \varphi_2 = \ldots = \varphi_N .$$

If we define $\varphi = \varphi_1$ then the system (1), (2) reduces to the system

$$\beta \ddot{\varphi} + \dot{\varphi} + \sin(\varphi) + \dot{Q} = I \quad L\dot{Q} + R\dot{Q} + C^{-1}Q = \varphi . \quad (3)$$

[4] is devoted to the special cases of a pure capacitive load

$$Q = \frac{3}{N} \sum_{k=1}^{N} \varphi_k \quad (4_{cap})$$

and a pure resistive load

$$Q = \frac{1}{N} \sum_{k=1}^{N} \varphi_k \quad (4_{rea})$$

with an appropriate scaling suggested by [10]. In these two cases we can eliminate $Q$ from the system (!), (2) to obtain the system

$$\beta \ddot{\varphi}_k + \dot{\varphi}_k + \sin(\varphi_k) + \frac{A}{N} \sum_{j=1}^{N} \varphi_j + \frac{B}{N} \sum_{j=1}^{N} \sin(\varphi_j) = CI, \quad k = (1, \ldots, N), \quad (5)$$

where for a capacitive load

$$A = B = \frac{3}{3 + \beta} \quad C = \frac{\beta}{3 + \beta}$$

and

$$A = C = 1, \quad B = 0$$

for a resistive load. Symmetric in-phase solutions to (5) are characterized by $\varphi = \varphi_1 = \varphi_2 = \ldots = \varphi_N$. In the capacitive load case $\varphi$ satisfies the pendulum equation

$$(3 + \beta)\ddot{\varphi} + \dot{\varphi} + \sin(\varphi) = I$$
while for a resistive load

\[ \beta \ddot{\varphi} + 2\dot{\varphi} + \sin(\varphi) = I. \]

The symmetric in-phase oscillations in the pure capacitive and resistive load cases are analyzed in detail in [4]. It is shown that these oscillations are born in a homoclinic connection and can only lose stability through a fixed-point or a period-doubling bifurcation. The \( S_N \) symmetry of the equations governing the evolution of an array of \( N \) Josephson junctions is inherited by the Poincaré map associated with the in-phase solutions. A large portion of [4] is devoted to the study of generic \( S_N \) symmetric fixed-point and period-doubling bifurcations for maps. We classify all possible symmetry breaking period-doubling bifurcations which can arise in generic period-two states and discuss their stability. We also show that generic fixed-point symmetry breaking bifurcations can only produce unstable fixed points. Note that the group theoretic analysis only tells us which states cannot occur and which states are possible. For any specific symmetric system such as the junction arrays considered in [4] the question of which states actually do occur can only be answered by detailed analysis of the particular system.

At a period-doubling bifurcation the period-two points of the Poincaré map correspond to states where the junctions divide into two or three groups. Inside each group the junctions oscillate in phase. When the number of junctions in each group is different, the oscillations associated with each group are distinct and their periods are approximately twice the period of the original synchronous solution. When the groups are of the same size both groups follow the same waveform, but there is a half period phase shift between groups. If there is a third group of oscillators then, generically, its size will differ from that of the two equal groups and its period of oscillation will be half that of the other groups. Thus some junctions will have period doubled oscillations while others do not.

Determination of the stability of each of the possible period-two points requires detailed calculations. Essentially the only period-two points which can be asymptotically stable are those that correspond to two groups of junctions of approximately equal size. Specifically, if the groups consist of \( k \) and \( N - k \) junctions, then generically stability is possible when \( N/3 \leq k \leq N/2 \). Generic \( S_N \)-symmetric fixed-point bifurcations produce only asymptotically unstable points. Thus generically for each \( k \) between 1 and \( N/2 \) the fixed-point bifurcations produce fixed points in which the junctions are divided into two groups of \( k \) and \( N - k \) junctions. Within each group the junctions oscillate synchronously. There are no other fixed points. These theoretic results are born out by detailed numerical simulations carried out using Doedel’s [9] path following software AUTO as reported in [4]. The numerical results are all consistent with the theoretical predictions, that is, the solutions found and those not found are consistent with the theory.

The numerical investigations also led to the discovery of interesting points on the curve in parameter space along which the homoclinic connections which give rise to the in-phase solutions occur. These points are homoclinic twist points.
At such points, as the parameters are varied along the curve of homoclinic points through it, the tangent flow along the homoclinic trajectory begins to twist vectors in transverse directions. Three homoclinic twist points were found in the resistive load case and a unique homoclinic twist point was found in the capacitive load case. The latter point is of particular interest since it is a codimension-two bifurcation point where curves of fixed-point and period-doubling bifurcations intersect and it is called a homoclinic twist bifurcation point. Detailed abstract study of homoclinic twist bifurcations can be found in [2] and [3]. In [3] we analyze bifurcations occurring in the vicinity of a homoclinic twist bifurcation point for a generic two-parameter family of $Z_2$ equivariant ODEs in four dimensions. Generically in the two-parameter space there exists a region of stability of the symmetric periodic solution bounded by a curve of period-doubling bifurcations and a curve of pitchfork bifurcations. These bifurcation curves terminate at the twist point. The period-doubling leads to the creation of a single doubly winding periodic solution while the pitchfork bifurcation produces two symmetry-related periodic orbits. Moreover there is a curve in the parameter space, also terminating at the twist point, where there exists a pair of symmetry-related homoclinic loops. The periodic orbits born in the period-doubling and pitchfork bifurcations continue in the parameter space to the line of the two homoclinic loops and terminate there in an infinite-period bifurcation. There also exists a branch of doubly winding homoclinic loops. No homoclinic twist bifurcation points have been found in the resistive load case. However two the twist points are very close together and a more refined numerical investigation could show that they actually coincide. The resulting point would then be a homoclinic twist bifurcation point.

The dynamics of a pair of identical Josephson junctions coupled through a shared purely capacitive load are governed by the system

$$\beta \dot{\varphi}_1 + \varphi_1 + \sin \varphi_1 + \frac{C}{2(C + \beta)}(\dot{\varphi}_1 + \sin \varphi_1 + \dot{\varphi}_2 + \sin \varphi_2) = \frac{\beta I}{C + \beta}$$

$$\beta \dot{\varphi}_2 + \varphi_2 + \sin \varphi_2 + \frac{C}{2(C + \beta)}(\dot{\varphi}_1 + \sin \varphi_1 + \dot{\varphi}_2 + \sin \varphi_2) = \frac{\beta I}{C + \beta}.$$ 

Numerical simulations show that this system possesses a variety of different running and periodic solutions. Continuation studies using AUTO indicate that many of these solution branches are generated by a codimension-2 connection which occurs at a particular parameter point which is not a homoclinic twist point. These branches are described in detail in [1]. There we also study a general two-parameter system whose properties reflect many of those found in the numerical studies of the Josephson junction system. In particular, the model system is assumed to possess an appropriate codimension-two connection and it is proved that its unfolding generates a large variety of codimension-1 curves. These results, combined with the particular symmetry and periodicity properties of junction equations account for all the numerically observed solution branches in [1]. Indeed, the theoretical analysis predicted the existence of branches which were not initially observed. Some of these branches were subsequently found,
but most of them are beyond the reach of numerical simulation. The branch of doubly winding homoclinic loops found by [4] originates from the homoclinic twist bifurcation point and terminates in the codimension-2 point found by [1].

In addition to the in-phase or symmetric running solutions, other types of running solutions occur and have important effects on the overall dynamics of the full system. Of particular interest are the discrete rotating wave solutions. A running solution is said to be a discrete rotating wave solution if there exists a running function \( \psi \) of period \( T \) such that

\[
\phi_j = \psi \left( t - \frac{(j-1)T}{N} \right) \quad j = 1, 2, ..., N
\]

(6)

that is, such that every junction follows the same waveform \( \psi \) but with a delay \( T/N \).

Discrete rotating wave solutions were first observed numerically by Hadley et al. in [10] and subsequently in [4] were they are called pony-on-a-merry-go-round (POM) solutions. If \( N = KM \) then POMs can be formed by grouping the junctions into \( M \) blocks of \( K \) synchronous junctions with a delay of \( T/M \) between successive blocks. Substitution of (6) in (5) yields the delay-differential equation

\[
\beta \ddot{\psi}(t) + \dot{\psi}(t) + \sin \psi(t) + \frac{A}{N} \sum_{k=0}^{N-1} \psi \left( t - \frac{kT}{N} \right) + \frac{B}{N} \sum_{k=0}^{N-1} \sin \left( \psi(t - \frac{kT}{N}) \right) = CI.
\]

(7)

This delay equation is special in that the delays \( kT/N \) are coupled to the period \( T \) of the running solution \( \psi \). To use global techniques it is necessary to decouple the period \( T \) and introduce a delay parameter \( \tau \) leading to the equation

\[
\beta \ddot{\psi}(t) + \dot{\psi}(t) + \sin \psi(t) + \frac{A}{N} \sum_{k=0}^{N-1} \psi \left( t - \frac{k\tau}{N} \right) + \frac{B}{N} \sum_{k=0}^{N-1} \sin \left( \psi(t - \frac{kT}{N}) \right) = CI.
\]

(8)

with \( 0 \leq \tau < N \). Note that for \( \tau = 0 \) the running solution of (8) corresponds to the in-phase running solution of (5). In [5] a homotopy and degree argument is used to extend the known solution of (8) for \( \tau = 0 \) to prove the existence of the desired solution to (7) for \( \tau = T \).

Various authors have studied POMs under different names. Mirollo [14], Strogatz & Mirollo [17], Nichols & Wiesenfeld, among others, call them splay phase solutions. A compromise position has been agreed upon and these solutions are now universally known as discrete rotating wave (DRW) solutions and I will use that terminology from here on. In [14] Mirollo proves the existence of DRW solutions (which he calls splay phase solutions) for a pure resistive load in the special case \( \beta = 0 \). Instead of the homotopy argument used to prove existence in [5] his proof is based on the Lefschitz trace formula.

So far we have considered junction arrays with only pure capacitive or pure resistive loads. Arrays with general LCR loads are studied in [8], [12], [11] and [15]. In [8] Aronson & Huang consider the system

\[
\beta \ddot{\varphi}_k + \dot{\varphi}_k + \sin \varphi_k + \dot{Q} = I \text{ for } k = 1, 2, ..., N \quad \text{and} \quad L\dot{Q} + R\dot{Q} + C^{-1}Q = \frac{1}{N} \sum_{j=1}^{N} \dot{\varphi}_j,
\]

(9)
where \( L, R \) and \( C \) are, respectively, the inductance, resistance and capacitance of the load. Note that \( C^{-1} = 0 \) in case \( C = \infty \). We were interested in running solutions of this system which follow a single waveform. A running solution \( \{ \varphi_1, \varphi_2, \ldots, \varphi_N, Q \} \) of period \( T \) is said to be a single waveform solution (SWFS) if there exists a running function \( \phi \) of period \( T \) and numbers \( 0 = \rho_1 \leq \rho_2 \leq \ldots \leq \rho_N < 1 \), not necessarily distinct, such that \( \varphi_j(t) = \phi(t - \rho_j T) \) for \( j = 1, 2, \ldots, N \). An in-phase solution is an SWFS with \( \phi = \varphi_1 \) and \( \rho_j = 0 \) for all \( j \). A DRW solution is a SWFS with \( \rho_j = (j - 1)/N \) for all \( j \).

In [8] we define a general class of SWFS for \( \beta > 0 \) which includes in-phase and DRW solutions and prove their existence. Let \( \{ \varphi_1, \varphi_2, \ldots, \varphi_N, Q \} \) be a SWFS with waveform \( \phi \), period \( T \) and delays \( \{ \rho_1, \ldots, \rho_N \} \). Since \( \phi \) is a \( T \)-periodic running solution we have

\[
\beta \ddot{\phi} + \dot{\phi} + \sin \phi(t) + \dot{Q}(t + T) = I.
\]

Moreover, for each \( j \in \{1, 2, \ldots, N\} \) it follows from the definition of SWFS that

\[
\beta \ddot{\phi}(t) + \dot{\phi}(t) + \sin \phi(t) + \dot{Q}(t + \rho_j T) = I.
\]

Thus

\[
\dot{Q}(t) = \dot{Q}(t + \rho_2 T) = \ldots = \dot{Q}(t + \rho_N T) = \dot{Q}(t + T)
\]

and \( \phi \) satisfies

\[
\beta \ddot{\phi}(t) + \dot{\phi}(t) + \sin \phi(t) + \dot{Q}(t) = I.
\]

Let \( \eta(t) \) denote the right hand side of the \( Q \)-equation in (9). Then

\[
\eta(t) = \frac{1}{N} \sum_{j=1}^{N} \phi(t - \rho_j T)
\]

and it follows that

\[
\eta(t) = \eta(t + \rho_2 T) = \ldots = \eta(t + \rho_N T) = \eta(t + T).
\]

Suppose that \( M \leq N \) of the delays \( \rho_j \) are distinct. Label them

\[
0 = r_1 < r_2 < \ldots < r_M
\]

and let \( l_1, l_2, \ldots, l_M \) denote their multiplicities. Then \( l_j \in \mathbb{N} \) and

\[
\sum_{j=1}^{M} l_j = N.
\]

For SWFS the pair \( \{ \phi, Q \} \) is a solution to the system

\[
\beta \ddot{\phi} + \dot{\phi} \sin \phi + \dot{Q} = I \quad \dot{LQ} + R\dot{Q} + C^{-1}Q = \eta(t),
\]

where

\[
\eta(t) = \frac{1}{N} \sum_{j=1}^{M} l_j \phi(t - r_j T).
\]

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Thus $\phi$ is a $T$-periodic running function, and
\[ \dot{Q} \text{ and } \eta \text{ have periods } r_2 T, \ldots, r_M T, \text{ and } T. \tag{12} \]

Assume that $M \geq 2$. Since $\dot{Q}$ is $T$-periodic, real valued and smooth it has a convergent Fourier representation
\[ \dot{Q} = \sum_{m} q_m e^{2\pi i m(t/T)}, \]
where $q_m = \overline{q_m}$. On the other hand, $\dot{Q}$ is also $r_T T$-periodic for every $r \in \{r_2, \ldots, r_M\}$. Therefore we must have
\[ q_m (1 - e^{2\pi i mr}) = 0 \text{ for all } m \in \mathbb{Z}. \]

If $r$ is irrational then $1 - e^{2\pi i mr} \neq 0$ for any $m \neq 0$ and it follows that $\dot{Q}(t) \equiv q_0$. Thus we conclude that if $Q \neq \text{constant}$ then all of the delays $r_2, \ldots, r_M$ are rational numbers. In particular:

If $Q \neq \text{constant}$ then $r_j = \frac{p_j}{K}$ for $j = 2, \ldots, M$, where $K$ is the least common multiple of the denominators of the delays. \tag{13}

Note that it is not necessarily the case that any of the delays in the above representation are in lowest terms. Write $r_j = p'_j / K_j$ where $(rp'_j, K_j) = 1$. Then $1 - e^{2\pi i m r_j} = 0$ only if $m \equiv 0 \pmod{K_j}$. Since $K$ is the least common multiple of the $K_j$, it follows that $1 - e^{2\pi i m r_j} \neq 0$ for at least one value of $j \in \{2, \ldots, M\}$ for every $m \neq 0 \pmod{K}$. Therefore $q_m = 0$ if $m \neq 0 \pmod{K}$.

In this general case we cannot reduce the existence problem to the study of a single delay equation as was done in the pure capacitive and resistive load cases. Instead, to prove existence of a SWFS we introduce a homotopy parameter $\lambda \in [0, 1]$ in the system (10)
\[ \beta \ddot{\phi} + \dot{\phi} \sin \phi + \lambda \dot{Q} = I \quad L\ddot{Q} + R\dot{Q} + C^{-1}Q = \eta(t) \]
where $\eta(t)$ is given by (11) and the delays satisfy (13). For each $\lambda \in [0, 1]$ we seek a pair $\{\phi, Q\}$ which satisfies (10), where $\phi$ is a running solution of period $T(\lambda)$, and $Q$ and $\eta$ satisfy (12) with $T = T(\lambda)$.

For $\lambda = 0$ the $\phi$-equation is uncoupled from the $Q$-equation and is just the simple damped driven pendulum equation. It is well known \cite{?} that there exists a function $I(\beta) : \mathbb{R}^+ \rightarrow [0, 1]$ such that for $I > I(\beta)$ the pendulum equation
\[ \ddot{\phi} + \dot{\phi} + \sin \phi = I \]
possesses a running solution $\phi_0$ with period $T_0 > 0$. Moreover, $\dot{\phi}(t) > 0$ for all $t$, and is uniquely determined by the phase condition $\phi_0(0) = 0$. Thus to start the homotopy we need to have a solution $Q_0$ of the equation
\[ L\ddot{Q} + R\dot{Q} + C^{-1}Q = \frac{1}{N} \sum_{j=1}^{M} I_j \phi_0(t - r_j T) \]
such that \( Q_0 \) has periods \( r_2 T_0, r_3 T_0, \ldots, r_M T_0 \) and \( T_0 \). In the Appendix A of [AH] it is shown that generically this can only occur if \( l_j = l \) for all \( j \in \{1, 2, \ldots, M\} \), \( lM = N \) and

\[
  r_j = \frac{j - 2}{M} \text{ for } j = 1, 2, \ldots, M.
\]

Hence we restrict attention to delays which satisfy this necessary condition.

For any positive integers \( l, M \) and \( N \) such that \( lM = N \) we define an \((l, M, N)\)-SWFS of period \( T \) to be the pair \( \{\phi, Q\} \) such that \( \phi \) is a running function of period \( T \) and \( Q \) is a \( T/M \)-periodic function such that \( \{\phi_1, \phi_2, \ldots, \phi_N, Q\} \) is a running solution to (1), (2) with

\[
  \phi_j(t) = \phi \left( t - \frac{h}{M} T \right) \text{ for } j = hl + 1, \ldots, (h+1)l \text{ and } h = 0, 1, \ldots, M - 1.
\]

Note that an \((N, 1, N)\)-SWFS is just an in-phase solution, an \((1, N, N)\)-SWFS is an ordinary discrete rotating wave solution and an \((l, M, N)\)-SWFS for \( 1 < l < M - 1 \) is a discrete rotating wave solution with \( M \) clusters each consisting of \( l \) junctions. The waveform \( \phi \) and the load current \( Q \) for an \((l, M, N)\)-SWFS satisfy the system

\[
  \beta \phi + \phi \sin \phi + \dot{Q} = I \quad L\dot{Q} + R\dot{Q} + C^{-1}Q = \frac{l}{N} \sum_{j=1}^{M} \phi(t - r_j T).
\]

In [AH] we prove:

**Theorem 1:** Assume that \( M \) is a divisor of \( N \), \( I \in (1, \infty) \), \( \beta \in (0, \infty) \), \( R \in [0, \infty) \), \( L \in [0, \infty) \) and \( C \in (0, \infty) \) with \( R + C^{-1} > 0 \). Then if \( l = N/M \) there exists an \((l, M, N)\)-SWFS to (1), (2).

The theorem is proved by homotopy and degree theory arguments when \( LC^{-1} = 0 \) or \( LRC^{-1} > 0 \). Limit arguments are used to settle the remaining cases. Observe that Theorem 1 only asserts the existence of \((l, M, N)\)-SWFSs and, in particular, does not say anything about the existence or nonexistence of SWFSs of any other description. Indeed, any SWFS outside the \((l, M, N)\)-class will be invisible to the homotopy argument. However, it is conjectured that the only SWFSs are those in \((l, M, N)\)-class.

In practice, one is usually interested in large arrays of Josephson junctions and so it is of interest to know how the DRW solutions depend on the number \( N \) of junctions. The existence of a limit of the DRW solutions and their periods as \( N \to \infty \) is proved in [7] in the pure capacitive and resistive load cases. The limiting waveform is a solution of a pendulum equation. In both cases the DRWs are analytic functions of \( t \) uniformly in \( N \). Thus the rate of convergence is exponential and the continuum limit provides an excellent approximation even for moderate values of \( N \). Running solutions to (5) are not unique, since any time-shift of a given solution is again a solution. There is however, an additional source of non-uniqueness. If a single DRW exists then there are in fact \( N! \) of them obtained by permutation of the indices. Of these, \((N - 1)!\) are genuinely different and the remaining ones can be generated by cyclic permutations which
are equivalent to time-shifts. For general $N$ using the uniqueness (up to time-shifts) of the running solutions of the pendulum equation, it is shown in [7] that the DRWs of (5) are unique up to permutation and time-shift for sufficiently large $N$.

In [12] Huang & Aronson generalized the results of [7] on the limiting behavior as $N \to \infty$ of DRW solutions to arrays with general LRC loads governed by the system (9). Index the waveform $\psi$ and the charge current $Q$ with $N$ and normalize by the condition $\psi_N(0) = 0$ for all $N$. We prove:

**Theorem 2:** Suppose $I > 1, \beta > 0, C^{-1} \neq 0$ or $R \neq 0$ and let $\{\psi_N, Q_N\}$ be a normalized running solution of (9) with period $T_N >$

(i) There exist functions $\psi$ and $Q$, and a positive constant $T$ such that

$$\lim_{N \to \infty} T_N = T, \quad \lim_{N \to \infty} Q_N(t) = Q(t) \quad \text{and} \quad \lim_{N \to \infty} \psi_N(t) = \psi(t).$$

The convergence is exponential. Moreover all the derivatives of $\psi_N$ converge to the corresponding derivatives of $\psi$.

(ii) The limiting waveform $\psi$ is a running solution of the pendulum equation

$$\beta \ddot{\psi} + \dot{\psi} + \sin \psi + \frac{2\pi p}{T} = I,$$

where $p = \begin{cases} 0 & \text{if } C^{-1} \neq 0 \\ R^{-1} & \text{if } C^{-1} = 0 \text{ and } R \neq 0 \end{cases}$.

The limiting charge $Q$ or the limiting load current $\dot{Q}$ will be constant if $C < \infty$ or $C^{-1} = 0$ respectively.

(iii) There exists an integer $N_0$ such that for $N > N_0$ the running solution of (9) is unique.

Mirollo & Rosen also consider arrays with general LRC loads. Rather than the homotopy and degree theoretic arguments of [8], they base their proofs on formulating an SWFS as a fixed point equation in an appropriate Hilbert space. Thus, in particular, they do not make any assumptions about the solution of the pendulum equation and, indeed, derive the existence directly. Specifically, they prove existence for $I > 1$ as well as the uniqueness of the DRWs for sufficiently large $N$. They also prove

(iv) There exists an $I_0$ depending only on $L, R$ and $C$ such that the Josephson junction system (9) admits a unique SWFS for all $I > I_0$.

Finally Mirollo & Rosen show that, in general, the waveform is not unique. For this purpose they applied Newton’s method to their functional equation in very precise calculations and found various parameter sets for which there are multiple waveforms. For example, when

$$\beta = 4, L = 100, R = C = 0.01, N = 2 \quad \text{and} \quad I = 0.7034$$

they found three distinct waveforms as well as a solution to the pendulum equation.

Another class of solutions found numerically by Aronson et al. in [4] are the so-called *semirotors*. Here the junctions are split into two blocks, consisting of $k$ and $N-k$ identical junctions. The junctions in one block each follow
the same $T$-periodic running solution $\psi_1$ while each junction in the other block follows a genuinely $T$-periodic function $\psi_2$. A semirotor is said to be \textit{simple} if $\psi_1(t) > 0$. In [6] Aronson, Krupa & Ashwin study semirotors in the case of pure resistive and pure capacitive loads. We prove that for sufficiently large $\beta$ the system (5) has two families of simple semirotors. By exploiting the symmetries of the system the number of degrees of freedom is reduced to 4. Numerical studies suggest the form of the solution for small $\epsilon = 1/\beta$ and the implicit function theorem is used to verify the actual existence of solution of this form. In the resistive case it is shown that if $I < (N + k)/N$ for fixed $k \in \{1, ..., N - 1\}$ there exists $\beta(I)$ such that for $\beta > \beta(I)$ the system (5) has two simple semirotor solutions $\{\psi_j^1, \psi_j^2\}$ ($j = 1, 2$) depending smoothly on $\beta$. The same result holds for the capacitive load case if $I < 1$. In both cases, as $\beta \to \infty$ the periodic component of the semirotor shrinks to a single point while the running component approaches a horizontal line. The stability of the semirotors is also considered in [6]. We compute the asymptotic expansions of the Floquet multipliers of the linearized system and prove:

**Theorem 3:** Let \( \Lambda \) be defined by

\[
\sin \Lambda = \begin{cases} 
\frac{NI}{N+k} & \text{for a resistive load} \\
1 & \text{for a capacitive load}
\end{cases}
\]

Then for sufficiently small $\epsilon$ the semirotors corresponding to $\cos \Lambda > 0$ are asymptotically stable and those corresponding to $\cos \Lambda < 0$ are unstable.

the two families of semirotors, one consists of stable solutions and the other of solutions of saddle–node type. It is conjectured that as $I$ is increased the two families merge in a saddle-node bifurcation. This conjecture has been verified numerically (via AUTO) for the case of $N = 2$ junctions.

Existence and non-existence of semirotors for arrays with general LRC loads is studied by Huang [11]. She also considers the limiting behavior of semirotors as $\beta \to \infty$, but she does not take up the stability question. She proves:

**Theorem 4:** (i) Assume that $R^2 - 4LC^{-1} \neq 0$. Then for every fixed $k \in \{1, 2, ..., N - 1\}$

when $C^{-1} \neq 0$ and $I < 1$ or when $C^{-1} = 0$ and $I < \frac{k}{NR}$

there exists a function $\beta(I)$ such that the system (9) has two simple semirotor solutions for every $\beta > \beta(I)$. These solutions depend smoothly on $\beta$.

(ii) As $\beta \to \infty$ the phase curve of the running waveform for a simple semirotor approaches a horizontal line while the phase curve of the periodic waveform shrinks to a single point.

The existence proof is based on an extension of the implicit function theorem method of [6] and does not rule out the possibility of more than two semirotor solutions.

In the pure resistive and pure capacitive load cases Huang proves a non-existence result for sufficiently large values of $I$. Specifically:

**Theorem 5:** There exist constants $I(R)$ and $I(\beta, C)$ such that for $I > I(R)$ there is no semirotor solution to the system (5) with pure resistive load and
no semirotor solution to system (5) with pure capacitive load if $I > I(\beta, C)$.

The main thrust of the proof is showing that the inequalities guarantee that all of the $\phi_j(t) > 0$ for large enough $t$. Actually

$$I(R) = \frac{2}{R} + 1 \text{ and } I(\beta, C) = 1 + \frac{2C}{\beta + C}$$

suffice. Huang conjectures that a similar result holds for general LRC loads.

Everything which we have discussed so far has concerned arrays of identical Josephson junctions, however it is practically impossible to produce exactly identical junctions. In [10] Hadley, Beasly & Wiesenfeld briefly consider non-identical arrays subject to external noise and present the results of small-scale numerical simulations. Their simulations of 100 junctions show that the in-phase solution remains essentially stable when modest junction mismatches and thermal noise are included. Dhamala & Wiesenfeld [9] present a more detailed analysis and show (numerically) the onset of frequency locking as a function of the statistical disorder in the bias current. Wiesenfeld, Colet & Strogatz [20] obtain very precise results on phase locking for zero-capacity junctions by mapping the junction equations onto the equations of Sahagachi & Kuramoto [18] in the limit of weak disorder and weak coupling. The latter equations are explicitly solvable.

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