Spherical accretion of matter by charged black holes on $f(T)$ Gravity

M.E. Rodrigues1 · E.L.B. Junior2

Abstract We studied the spherical accretion of matter by charged black holes on $f(T)$ Gravity. Considering the accretion model of a isentropic perfect fluid we obtain the general form of the Hamiltonian and the dynamic system for the fluid. We have analysed the movements of an isothermal fluid model with $p = \omega e$ and where $p$ is the pressure and $e$ the total energy density. The analysis of the cases shows the possibility of spherical accretion of fluid by black holes, revealing new phenomena as cyclical movement inside the event horizon.

Keywords $f(T)$ Gravity · Spherical accretion

1 Introduction

Gravitation and Cosmology modern are based on General Relativity (GR), which in turn is based on the Riemannian Differential Geometry. The structure of the so-called spacetime, a four-dimensional differentiable manifold, is completely characterized by a metric, because the connection, Riemann and Ricci tensors and scalar curvature depend only of the metric components, its inverse and their derivatives (Wald 1984). It is a fact that this structure can only describe the current evolution of our universe, being consistent with the observational data, introducing a fluid model where the pressure must necessarily be negative today, model known by $\Lambda$CDM (Frieman et al. 2008).

An alternative description for the accelerated evolution of our universe is one in which modifies the Einstein equation, for a generalized such that at some threshold parameters theory falls in GR. The simplest way to do this is to directly modify the Einstein–Hilbert action to another which has the limit of GR, such as of the $f(R)$ Gravity (Nojiri and Odintsov 2007; Sotiriou and Faraoni 2010; Felice and Tsujikawa 2010; Clifton et al. 2012), which can be made the limit for small curvatures and get $f(R) \sim a_0 + a_1 R + O(R^2)$, for example. It is common generalize GR by modifying the Einstein–Hilbert action, using a function of a scalar theory, as the theories $f(R, T)$ Gravity (Alvarenga et al. 2013; Houndjo et al. 2013; Jamil et al. 2012; Houndjo and Piattella 2012; Houndjo 2012; Harko et al. 2011), $f(G)$ (Bamba et al. 2010a,b; Houndjo et al. 2014; Rodrigues et al. 2014; Nojiri et al. 2010) and $f(R, G)$ (Cognola et al. 2006; Elizalde et al. 2010; Myrzakulov et al. 2011; Felice and Suyama 2009a,b, 2011; Felice et al. 2010a,b; Felice and Tsujikawa 2009; Felice and Tanaka 2010; Nojiri and Odintsov 2005, 2003) Graiveties, for example, where $T$ is the trace of the energy-momentum tensor and $G$ the Gauss–Bonnet term.

An alternative description, but equivalent, of gravity can be done now using the torsion of space-time, rather than curvature. Considering now the identically zero curvature, we can assign the gravitational interaction effects to torsion. The theory that represents such an alternative is commonly called Teleparallel Theory (TT) (Aldrovandi and Pereira 2010; Aldrovandi et al. 2004; Maluf 2013; Hehl et al. 1995), where are now tetrads that make the role of dynamic fields, which completely determining the geometric objects such as connection, torsion, contorsion and Tor-
sion scalar. As this theory is equivalent to GR, the equations of motion can only describe the accelerated evolution of the current phase of our universe through again the introduction of dark energy. Then we can think analogously to generalize this theory to one that contains terms of order higher torsion scalar, as $f(T)$ Gravity (Harko et al. 2014a; Basilakos et al. 2013; Sadjadi 2012; Cardone et al. 2012; Bamba et al. 2011, 2012, 2013; Xu et al. 2012; Karami and Abdolmaleki 2012; Wei et al. 2011, 2012; Boehmer et al. 2012, 2011; Wu and Geng 2012; Capozziello et al. 2011; Miao et al. 2011; Meng and Wang 2011; Li et al. 2011a,b; Ferraro and Fiorini 2011; Zhang et al. 2011; Wang 2011; Sotiriou et al. 2011; Zheng and Huang 2011; Dent et al. 2011; Yang 2011a,b; Karami and Abdolmaleki 2013; Wu and Yu 2010, 2011; Bengochea 2011; Chen et al. 2011; Myrzakulov 2011; Nashed and Hanafy 2014; Ferraro and Fiorini 2007; Tamanini and Boehmer 2012; Daouda et al. 2012; Gonzalez et al. 2012a). We also have the same possibilities for generalization for $f(T, T)$ (Kiani and Nozari 2014; Harko et al. 2014b) and $f(T, T_G)$ (Kofinas and Saridakis 2014a,b) Gravities, or more generally as in Gonzalez and Vasquez (2015). In this paper, we will restrict ourselves to $f(T)$ Gravity.

In the case of the study of Cosmology by $f(T)$ Gravity, we have the most varied results. The study by local phenomena such as black holes solutions, we still have much work ahead of us. The first hole black solution theory is charged solutions on $3D$ (Gonzalez et al. 2012b) and $D$ dimensions (Capozziello et al. 2013). Later appears more charged solutions in $4D$, re-obtaining thus solutions of Reisnner–Nordstrom, Reisnner–Nordstrom-AdS (dS), Schwarzschild-AdS (dS) e Schwarzshild (Rodrigues et al. 2013). Another paper discusses the formulation of the Kerr solution for $f(T)$ Gravity (Bejarano et al. 2015). The solutions of charged black holes, resulting in non-linear electrodynamics are obtained in (Junior et al. 2015a), which we will study here in this work. We also have regular black holes solutions in Junior et al. (2015b). Even with few solutions found in this theory, we have very few studies of local phenomena such as bending of light (Ruggiero 2016) and solar tests (Farrugia et al. 2016; Iorio and Saridakis 2012). The theory has been successfully tested with the most recent observational data (Nunes et al. 2016, 2017).

The study of accretion of matter by a black hole was first performed by Bondi (1952), but in a Newtonian form. Generalization to account for relativistic effects was formulated by Michel on 1971 (Michel 1972), so is called accretion type-Michel for this approach. In this paper we will use the accretion type-Michel. The description of spherical accretion made by Michel is the phenomenon critically, having a certain radius value where the system becomes critical. Considering effects of curvature, we have several results of spherical accretion by black holes (Debnath 2015; Frolov 2004; Anslyn et al. 2013; Jawad and Shahzad 2016; Bahamonde and Jamil 2015; Das et al. 2007; Peirani and Pacheco 2008; Chattopadhyay and Ryu 2009; Malec 1999; Mach 2015; Chaverra et al. 2016; Flammang 1982; Sharif and Abbas 2013; Chaverra and Sarbach 2015; Mach et al. 2013; Babichev et al. 2005, 2008; Das 2004), and some are for modified gravity (Anslyn 2016; Ahmed et al. 2016b; Ainou 2017). The only pioneer work in considering the zero curvature and the torsion effects on $f(T)$ Gravity was recently published (Ahmed et al. 2016a). We will follow the same methodology of this work but generalizing the accretion analysis for a general spherically symmetric metric.

So our motivation is to treat the spherical accretion of matter by charged black hole of the $f(T)$ theory. In Sect. 2 we discussed the spherical accretion of matter by a black hole, giving the expression of the Hamiltonian of the fluid and dynamic system. In Sect. 2 we restrict our analysis to the model of isothermal isentropic fluids, particularizing for ultra-stiff cases, ultra-relativistic, radiation and sub-relativistic fluids. In Sect. 4 we present our final considerations.

## 2 Spherical accretion

The spherical accretion study of matter by black holes is usually based on the movement of fluid in the neighboring region to the event horizon. This can be addressed as follows. Let’s take two equations that completely characterizes the accretion (ejection) spherical of a perfect isotropic fluid; one is the equation of conservation of source material and the other is the equation that describes the conservation of energy. Let’s start by the equation similar to that of continuity in fluid mechanics, which is described by Rezzolla and Zanotti (2013)

\[ \nabla_\mu J^\mu \equiv 0, \]  

(1)

where $J^\mu = nu^\mu$, $n = \rho$ is the number of the baryonic density and $u^\mu = dx^\mu / d\tau$ is the four-speed fluid.

Analogously to the theorem of Noether of fluid mechanics, Bernoulli’s theorem in hydrodynamics relativistic have to spherical symmetry, a certain amount that is conserved in the co-moving frame to fluid flow. This law is represented by the equation (Rezzolla and Zanotti 2013)

\[ u^\nu \nabla_\nu [hu_\mu \xi^\mu] \equiv 0, \]  

(2)

where $h$ is the enthalpy of the fluid and $\xi^\mu$ a Killing vector of the temporal symmetry of space-time.

To simplify our study and make the equations having analytical solutions, we will restrict ourselves to the case where the space-time has spherical symmetry, which implies in the following general form of the line element

\[ dS^2 = A(r)dt^2 - B(r)dr^2 - C(r)(d\theta^2 + \sin^2 \theta d\phi^2). \]  

(3)
For the line element (3) we have the following metric determinant \( g = -ABC^2 \).

The fluid movement follows the same symmetries of space-time, i.e. spherical symmetry. So we can then take a fluid that moves in temporal and radial directions, and its four-speed given by \([u^\mu] = [u^0, u, 0, 0]\). Thus, because of the spherical symmetry, the continuity equation (1), integrated, gives us

\[
C\sqrt{AB}u = c_1. \tag{4}
\]

This is one of the equations principles that govern the movement of the fluid.

We can then normalize the four-speed for \( g_{\mu\nu}u^\mu u^\nu = 1 \), which gives us the following identities

\[
u^0 = \sqrt{A^{-1}(1 + Bu^2)}, \quad u_0 = g_{00}u^0 = \sqrt{A(1 + Bu^2)}. \tag{5}\]

Because of spherical symmetry, it is common we take the following vector of Killing \([\xi^\mu] = [1, 0, 0, 0]\) to integrate (2), resulting in

\[
h\sqrt{A(1 + Bu^2)} = c_2. \tag{6}\]

We may relate the three-velocity of the fluid with the radial and temporal components of the four-velocity through the line element to the equatorial plane \( \theta = \pi/2 \)

\[
dS^2(\theta = \pi/2) = (\sqrt{A}dt)^2 - (\sqrt{B}dr)^2. \tag{7}\]

Now we define the three-velocity of the fluid by

\[
v = \frac{\sqrt{B}dr}{\sqrt{A}dt}, \tag{8}\]

that replacing \( u = dr/d\tau \) and \( u^0 = dt/d\tau \), we have

\[
v^2 = \frac{B}{A} \left( \frac{u}{u^0} \right)^2. \tag{9}\]

Using (5) and isolating \( u^2 \), we have

\[
u^2 = \frac{v^2}{B(1 - v^2)}, \quad u^2_0 = \frac{A}{1 - v^2}. \tag{10}\]

Now through (10) we can rewrite our equation (4) as

\[
\frac{A(Cmv)^2}{1 - v^2} = c_1^2. \tag{11}\]

Bernoulli’s theorem of relativistic hydrodynamics says that every symmetry of space-time we have a conserved quantity associated with this symmetry. For the temporal translational symmetry, the conserved quantity is the square of the integration constant \( c_2 \) in (6), and should be proportionate to the Hamiltonian of the fluid. Using again (10), we define the Hamiltonian fluid as

\[
H(r, v) = \frac{h^2(r, v)A(r)}{1 - v^2}. \tag{12}\]

The intensive and extensive thermodynamic quantities of the fluid are related by equations

\[
dp = n(dh - Tds), \quad de = hdn + nTdS, \tag{13}\]

where \( T \) and \( s \) are specific entropy and the temperature of the fluid, and \( e \) is the total energy density of the fluid. In general, the study of fluid motion becomes very complicated, and it is due to this reason that we must assume two simplifications for perfect fluid our case. The first is considered that a fluid does not heat exchange with the outside, then the fluid is classified as adiabatic fluid, and \( u^\mu \nabla_\mu s = 0 \). The second is when the fluid keeping its entropy constant during movement, then on the fluid thus characterized by isentropic where \( \nabla_\mu s = 0 \) and therefore \( s = s_0 \in \mathbb{R}^+ \).

Considering that our case study is a fluid adiabatic and isentropic, this implies \( ds \equiv 0 \) and then

\[
dp = nh, \quad de = hdn. \tag{14}\]

We can divide \( dp \) by \( de \) to define the speed of sound as

\[
a^2 = \frac{dp}{de} = \frac{d\ln h}{d\ln n}. \tag{15}\]

Now we will establish a dynamic system through Hamiltonian (12)

\[
\frac{dr}{dt} = \frac{\partial H}{\partial v}, \quad \frac{dv}{dt} = -\frac{\partial H}{\partial r}. \tag{16}\]

Now do the same steps contained in Ahmed et al. (2016a), remembering that our metric is generalized to (3), what gives us the following dynamic system

\[
\dot{r} = \frac{2Ah^2}{v(1 - v^2)^2}(v^2 - a^2), \tag{17}\]

\[
\dot{v} = -\frac{h^2}{1 - v^2}\left[\frac{dA}{dr} - 2a^2A\frac{d\ln(\sqrt{AC})}{dr}\right]. \tag{18}\]

We have here some possibilities of critical points for this dynamic system, but we will focus only on single physical possibility, where

\[
\nu_c = a, \quad A' = 2a^2A(\ln(\sqrt{AC})), \tag{19}\]

which defines a critical point for the system (17)–(18).

We will now specify a famous model for the perfect fluid for the next section.
3 Brief introduction on $f(T)$ gravity

Symmetries have always played an important role in Physics, for example, Galilean symmetry in Newtonian physics. With the unification of space and time by Special Relativity, other symmetries have come to call attention of the researchers, namely, the Lorentz symmetry that relativizes the notion of time and space; Poincaré symmetry that is associated with translation; symmetry of Weyl, which is a symmetry of scale; symmetry, local scale; and Super Symmetry (SUSY). The Lorentz symmetry is crucial for unification theories. In the standard model, the Lorentz symmetry is assumed, however, in the context of Bosonic strings where the condensation of tensor fields is dynamically possible, one must consider the breaking of this symmetry (Bonetti et al. 2017).

In Belich et al. (2015) it was shown that the Lorentz symmetry breaking naturally induces the SUSY violation, thus SUSY reveals that the Lorentz symmetry breaking is performed with a bosonic background, along with a whole set of fermions that condense in the process.

In the search for a quantization of gravitation is the breaking of Lorentz symmetry is necessary, therefore, standard model extensions include Lorentz invariance and is due to the fact that the usual standard model does not include a quantum theory of gravity, in addition, it is necessary to overcome the theoretical difficulties of quantum gravity.

The Lorentz symmetry breaking has also been studied in the context of nonlinear electromagnetic Lagrangian, such as BI, in Bufalo (2015), where the energy of the electromagnetic field of a point charge at the origin of the coordinate system is finite and in a certain limit of a parameter lies in Maxwell’s theory.

Since the generalization of the Teleparallel theory to $f(T)$ Gravity has gained strength in the last few years, a crucial problem of it has deserved enough attention: the breakdown of invariance by local Lorentz transformations. This comes from the formulation of theory through a torsion scalar that is not invariant by local Lorentz transformations. This peculiarity implies a theory with second order differential equations, but this is not beneficial to this theory, as it is a consequence of this fact on local symmetry. In Junior and Rodrigues (2016) was proposed a generalization of teleparallel that keeps the invariance under transformations of local Lorentz, as well as general transformations of coordinates. In this formulation, the theory lies in the generalization of GR ($f(R)$) to a threshold of a parameter contained in it, and this crucial parameter for consideration of any cosmological phenomenon. With the advancement of $f(t)$ Gravity arises interest in the electromagnetic coupling and thus this theory, based on BI, we can formulate a Lagrangian in $f(t)$ for non-linear electrodynamic.

The geometry used in the theory called $f(T)$ Gravity is that of Weitzenbock. We first define the line element through

$$dS^2 = g_{μν}dx^μ dx^ν = η_{μν}e^μ e^ν, e^μ = e^μ_μ dx^μ,$$  \hspace{1cm} (20)

where $e^μ_μ$ are the tetrad matrices. We can define their inverse through the relation $dx^μ = e^μ_ν e^ν$, so we have $e^μ_μ e^ν_μ = δ^ν_δ, e^μ_ν e^ν_μ = δ^ν_ν$. Taking the spin connection null and requiring that spacetime have identically zero curvature we have the Weitzenbock connection

$$Γ^α_μν = e^α_α ∂_ν e^μ_α = e^α_α ∂_ν e^μ_α.$$  \hspace{1cm} (21)

So now is the torsion that accounts for the gravitational interaction. We define the torsion tensor through the Weitzenbock connection as

$$T^σ_μν = Γ^σ_νμ − Γ^σ_μν.$$  \hspace{1cm} (22)

We can also define the contortion tensor and another tensor that will help in the definition of the torsion scalar

$$K^μ_α = \frac{1}{2}(T^μ_α − T^μ_α − T^α_μ).$$  \hspace{1cm} (23)

$$S^μ_α = \frac{1}{2}(K^μ_α + δ^μ_α T^ν_ρ − δ^ν_μ T^ρ_α).$$  \hspace{1cm} (24)

Now we can define the torsion scalar as follows

$$T = T^α_μν S^μ_α.$$  \hspace{1cm} (25)

The Lagrangian of the theory is given as a function of the torsion scalar coupled with Non-Linear Electrodynamical (NED)

$$L = e[f(T) + 2κ^2 L_{NED}(F)],$$  \hspace{1cm} (26)

where $F = (1/4)F_μ_ν F^μ_ν$ and $F_μ_ν$ are the components of Maxwell’s 2-form, $κ = 8πG/c^4$. Taking the Euler–Lagrange equation in relation to the tetrad for Lagrangian (26) we have the following equation of motion

$$S^μ_ρ δ^ρ_α T f(T) + [e^{-1} e^μ_ρ δ^ρ_α (ee_α S^μ_α)] + T^α_β S^μ_α f(T) = \frac{1}{4} δ^μ_ρ f(T) = \frac{κ^2}{2} T^μ_ρ,$$  \hspace{1cm} (27)

$$T^μ_ρ = −\frac{2}{κ^2} \left[δ^μ_ρ L_{NED}(F) − \frac{∂L_{NED}(F)}{∂F} F_β_α F^μ_α \right].$$

with $e = det(e^μ_α)$. Now taking the Euler–Lagrange equation in relation to the potential $A_μ$, with $F_μ_ν = ∂_μ A_ν − ∂_ν A_μ$, for the Lagrangian (26) we have the equation of motion

$$\nabla_μ \left[ F^μ_ν \frac{∂L_{NED}}{∂F} \right] = 0.$$  \hspace{1cm} (28)
Choosing the following tetrads
\[
[e^\alpha_i] = \begin{bmatrix}
\sqrt{A} & 0 & 0 & 0 \\
0 & \sqrt{B} \sin \theta \cos \phi & \sqrt{C} \cos \theta \cos \phi & -\sqrt{C} \sin \theta \sin \phi \\
0 & \sqrt{A} \sin \theta \sin \phi & \sqrt{C} \cos \theta \sin \phi & \sqrt{C} \sin \theta \cos \phi \\
0 & \sqrt{A} \cos \theta & -\sqrt{C} \sin \theta & 0 \\
\end{bmatrix}.
\] (29)

we can reconstruct the spherically symmetric line element (3). Inserting the tetrads (29) on the equations of motion (27) and (28), making several considerations Junior et al. (2015a), we obtain the following solutions

\[
dS_i^2 = \left(1 - \frac{2r_0}{r^2}\right) dt^2 - \left(1 - \frac{2r_0}{r^2}\right)^{-2} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\] (30)

\[
dS_2^2 = \frac{1}{r^2} \sqrt{r^4 - 4e^{\omega_0} r^2} - \frac{r_0}{r^4 - 4e^{\omega_0} r^2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] (31)

\[
dS_3^2 = \frac{1}{r^2} \sqrt{r^4 - 8e^{\omega_0} r^2} - \frac{r_0}{r^4 - 2r_0^2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] (32)

\[
dS_4^2 = (e^{\omega_0/2} r - 1)^{-2} (d^2 t - dr^2) - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] (33)

\[
dS_5^2 = \frac{1}{r^2} \sqrt{r^4 - 4e^{\omega_0} r^2} - \frac{r_0}{r^4 - 4e^{\omega_0} r^2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] (34)

where \((i) F^{10}\) with \(i = 1, \ldots, 4\) are the Maxwell tensor components. The event horizons are determined by \(B^{-1}(r) = 0\), that through (30)–(33) are given by

\[
r_{h(1)} = \sqrt{2r_0}, \quad r_{h(2)} = \sqrt{2e^{\omega_0/4}}, \quad r_{h(3)} = (2r_0)^{1/6}, \quad r_{h(4)} = e^{-\omega_0/2}.
\] (35)

These are four solutions of charged black holes arising from \(f(T)\) theory, with non-linear electrodynamics source obtained in Junior et al. (2015a). We have studied here cases on what the functional form of \(f(T)\) differs from reference (Ahmed et al. 2016a), so new phenomena of accretion for certain fluids arise. These solutions are just toy models, since we know that astrophysical black holes usually emerge with rotation.

### 4 Isothermal fluids

An important model that is well in agreement with the reality is when the fluid keeps a constant temperature in the thermodynamic process and thus the speed of sound is a constant, \(a^2 = \omega = dp/de\). This is another very important simplification to analytically solve the equations of motion of the fluid. Thus, integrating, we have what the pressure is proportional to total energy density

\[
p = \omega e.
\] (36)

We define the enthalpy of the fluid by Rezzolla and Zanotti (2013), Ahmed et al. (2016a)

\[
h = \frac{e + p}{n},
\] (37)

Here we can find an ordinary differential equation for the total density of the fluid, for this, we take (14), (36) and (37), we integrate with respect to \(n\), which provides us

\[
e(n) = \frac{e_c}{n_c^{\omega+1}} n^{1+\omega}.
\] (38)

The enthalpy (37) is then given by

\[
h = \frac{(1 + \omega)e_c}{e_c^{\omega+1} - n_c^{\omega}}.
\] (39)

We want to work with the thermodynamic variables \((r, v)\), then we must put the other thermodynamic quantities on the basis of these two variables. We then replace the dependence on \(n\) for \(n(r, v)\), for this, we will use (4)

\[
n = \frac{c_1}{vC} \sqrt{\frac{1 - v^2}{A}}.
\] (40)

We now take (39) and (12), considering (40), to rewrite as

\[
\mathcal{H} = \mathcal{H}_0 \frac{1}{(vC)^{2\omega}} \left[\frac{A}{1 - v^2}\right]^{1-\omega},
\] (41)

\[
\mathcal{H}_0 = \left[\frac{(1 + \omega)e_c e_c^{\omega}}{n_c^{\omega+1}}\right]^2.
\]

We can then define \(\mathcal{H}\) by a transformation in the time coordinate (Killing vector of this symmetrical), or \(t \to \mathcal{H}_0 t\).
It is clear from the Fig. 1, where we have no intersection between the cur- 
}eves for solution (31) are obtained for $r < r_c(4)$ and for solution (32) by the expression $H_4(r, v) = \langle H_{c(3)}^6 - 10^{-1}, H_{c(3)}^6, H_{c(3)}^6 + 2 \rangle$. 
We represent the phase space of the solutions in Fig. 2. In the diagram at the top of the left side is the phase space of the first solution (30), we have the following types of movement of the fluid: a) Red curves. Those who are before the critical radius, $r_{c(1)} = 1.73205$, it begins with a
supersonic ejection movement ($1 > v > v_c$), going to subsonic ejection ($v_c > v > 0$), coming close to the horizon with zero speed, then passing to the movement of a subsonic accretion ($0 < v < -v_c$), ending in a supersonic accretion ($-v_c > v > -1$). The curve in the lower part of the diagram begins as an supersonic accretion, passing the accretion for subsonic, ending with zero velocity away from the event horizon. The curve at the top of the diagram begins far from the horizon like a subsonic ejection, reaching supersonic ejection away from the horizon to near the speed of light. b) The blue curves. We have three possibilities here. The first is the curve of the region smaller than the critical radius. This movement begins as a supersonic ejection, it passes through a critical point of bifurcation unstable, becoming subsonic ejection, which reaches a zero speed on the horizon, becoming subsonic accretion, passing again to an unstable bifurcation, finishing as accretion supersonic. The second is that the bottom of the diagram, which begins as a supersonic accretion, through a bifurcation ending as a subsonic accretion. The third is the curve of the upper diagram, which begins as a subsonic ejection, through a bifurcation and ending as a supersonic ejection. c) The green curves.

We have three types of movements here. At the bottom we have a purely super sonic accretion movement. At the top we have a purely supersonic ejection movement. Far from the horizon, near zero speed, fluid movement begins as subsonic ejection, increasing the speed to maximum, then decreasing the radial velocity to zero by passing the motion to subsonic accretion, and the farther away the horizon speed tends to zero.

The movement of fluid to the solutions (31) and (32) is identical to the first solution. The only differences are the values of the horizon and the critical radius, which are $\{r_c(2) = 2.00961, r_h(2) = 1.81589\}$ and $\{r_c(3) = 1.20094, r_h(3) = 1.12246\}$. The movement of the fluid for solution (33) is quite different from other solutions. The phase space for this solution is shown in Fig. 2 at the bottom right. The critical radius and the horizon are located in $\{r_c(4) = 0.404354, r_h(4) = 0.606531\}$. We note that the value of the horizon this time is greater than the critical radius, making it clear in the diagram. We can see that we have the following movements for the fluid: a) the red curves, the fluid begins with supersonic accretion, passing for a subsonic accretion. Then, the top of the diagram begins as subsonic ejection,
Fig. 3 Representation of the phase space for solutions (30) (top left), (31) (top right), (32) (lower left) and (33) (bottom right). The sound speed value is given by $a^2 = \omega = 1/3$.

and ending supersonic ejection. b) The blue curves, whose movement is identical to the red curves. c) The green curves. The curves in the region $r > r_h$ have the same movement of fluid to the curves in red. But the fluid exhibits a cyclic motion within the horizon, where there is a stable critical point. This movement can not be observed by an observer outside the event horizon. This is the first time it appears such a phenomenon.

4.3 Radiation

An important model for a fluid is when its total energy density is equivalent to one third pressure, then classify as radiation, and $\omega = 1/3$. The Hamiltonians, arising from (42), for this case are

$$
\mathcal{H}_1(r, v) = \left[ \frac{(r^2 - 2r_0)}{r^4 v(1 - v^2)} \right]^{2/3},
$$

$$
\mathcal{H}_2(r, v) = \left[ \frac{r^4 - 4e^{\gamma_0}}{r^4 v(1 - v^2)} \right]^{2/3},
$$

(47)

$$
\mathcal{H}_3(r, v) = \left[ \frac{(r^6 - 2r_0)^{1/3}}{r^4 v(1 - v^2)} \right]^{2/3},
$$

$$
\mathcal{H}_4(r, v) = \left[ \left( \frac{e^{\gamma_0/2} - r - 1}{2} \right)^2 (1 - v^2)r^2 v \right]^{-2/3},
$$

(48)

The horizons are given by (35). The critical rays are determined by (19)

$$
r_{c(1)} = 2\sqrt{r_0}, \quad r_{c(2)} = (8e^{\gamma_0})^{1/4}, \quad r_{c(3)} = (2\sqrt{r_0})^{1/3}, \quad r_{c(4)} = e^{-\gamma_0/2}/2.
$$

(49)

The critical speed of sound is given by $v_c = \sqrt{\omega} = 0.57735$.

The movements of the fluid for the solutions (31) and (33) are similar to the previous section, obtained the contour of the equations $\mathcal{H}_3(r, v) = \{H_{c(2)}^3 - 10^{-2}, H_{c(2)}^3, H_{c(2)}^3 + 3 \times 10^{-2}\}$ and $\mathcal{H}_4(r, v) = \{H_{c(4)}^3 - 10^4, H_{c(4)}^3, H_{c(4)}^3 + 11 \times 10^3\}$, with $H_{c(2)}^3 = 0.155199, H_{c(4)}^3 = 12768.3$. The Fig. 3 shows the phase space of these solutions on the right side in the top and lower. As the movements are similar to the previous section, we will not comment on them here. The main difference is the critical radius for each movement.
In the case of the left side of the phase diagram at the top of the Fig. 3, concerning the equation $H^1_3(r, v) = \{H^3_{c(1)} - 10^{-1}, H^3_{c(1)} + 3 \times 10^{-1}\}$ where $H^3_{c(1)} = 0.105469$, the movements are divided as follows: a) curves on red. The movement begins as supersonic ejection, passing to subsonic ejection, reducing the ejection speed to zero, then going to subsonic accretion, and ending as supersonic accretion. Interestingly, there is a movement inside the event horizon inversely to the outside, starting as supersonic accretion and ending as ejection (inside) supersonic. b) Curves in blue, we have the beginning as supersonic accretion, through an unstable critical point, becoming accretion subsonic. At the top is similar. The movement begins as subsonic ejection, goes through an unstable critical point, ends as ejection supersonic. A novelty is the movement that appears inside the event horizon inversely to the outside, starting as supersonic accretion and ending as ejection (inside) supersonic. c) Curves on green. There is an accretion of movement and other purely supersonic ejection. Two other movements, an accretion and other ejection purely subsonic. Similarly to other curves, there is a movement within the horizon that begins as supersonic accretion and ends as supersonic ejection. In the case of movements to the equation $H^3_9(r, v) = \{H^3_{c(3)} - 1, H^3_{c(3)} + 2\}$, where $H^3_{c(3)} = 4.80542$, we have a lot of similarity with the previous case, in which the main difference is the existence of motion where the radial coordinate is greater than the critical radius, for curves in red, as shown in Fig. 3.

### 4.4 Sub-relativistic fluid

Now finally, we discuss the case where the pressure is equal to $p = \rho/4$, then $a^2 = \omega = 1/4$. This value of the speed of sound characterizes fluid as sub-relativistic. In this case, the Hamiltonian for the solutions (30)–(33) are given by

$$
\begin{align*}
H_1(r, v) &= \frac{1}{r^2 \sqrt{v}} \left[ \frac{r^2 - 2r_0}{r^2(1 - v^2)} \right]^{3/4} \\
H_2(r, v) &= \frac{1}{r^2 \sqrt{v}} \left[ \frac{\sqrt{r^4 - 4v^6}}{r^2 v(1 - v^2)} \right]^{3/4}, \\
H_3(r, v) &= \frac{1}{r^2 \sqrt{v}} \left[ \frac{(r^6 - 2r_0 v)^{1/3}}{r^2 v(1 - v^2)} \right]^{3/4},
\end{align*}
$$

Fig. 4 Representation of the phase space for solutions (30) (top left), (31) (top right), (32) (lower left) and (33) (bottom right). The sound speed value is given by $a^2 = \omega = 1/4$.
\[ H_4(r, v) = \frac{1}{r^2 \sqrt{E}} \left( e^{r_0^2/2} r - 1 \right)^2 \left( 1 - v^2 \right)^{-3/4}. \]  

(51)

The critical speed of sound is given by \( c_0 = \sqrt{\frac{\omega}{\omega}} = 0.5 \). The Fig. 4 shows the phase space of Hamiltonians (50) and (51). The contours were obtained for the equations \( H_4(r, v) = \{ H_4^{(1)}, H_4^{(2)}, H_4^{(3)} \} \) where \( H_4^{(1)}(r, v) = 0.87912; \ H_4^{(2)}(r, v) = 0.87912 - 10^{-3}; \ H_4^{(3)}(r, v) + 3 \times 10^{-2} \) where \( H_4^{(2)} = 0.0262795; \ H_4^{(3)}(r, v) = 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.87912 - 0.8

(52)

5 Conclusion

The spherical accretion of matter by charged black holes on \( f(T) \) Gravity is studied and the conclusion is that it is possible, resulting in several movements for the fluid.

Considering a perfect isentropic and isothermal fluid, we analyzed the cases of ultra-stiff, ultra-relativistic, radiation and sub-relativistic fluids. We consider the contour for each specific value of the Hamiltonian as a constant movement, and represent the phase space for the four analyzed solutions. The conclusion follows that it is perfectly possible accretion and ejection of matter spherical type-fluid by the black holes studied. The movement of the fluid more deserves emphasis here is the cyclic appearing within the horizon for a given energy of the fluid.

Our perspective is that these solutions are stable for both a thermodynamic system (thermodynamic stability) and for small perturbations in the geometry (geometric stability). This should be the next test for these solutions in future work.

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