Mathematical models of intergroup conflicts

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Abstract

The human society today is far from perfection and conflicts between groups of humans are frequent events. One example for such conflicts are armed intergroup conflicts. The collective behavior of the large number of cooperating participants in these conflicts allows us to describe the conflict on the basis of models containing only few variables. In this paper we discuss several cases of conflicts without use of weapons of non-conventional kind. In the ancient times the Chinese writer Sun Tsu mentioned that the war is an art. We can confirm that the conflict is an art but with much mathematics at the background.

Key words: conflict, attrition, ambush, combat, mathematical models

1 Introduction

Today we are excited by the fast advance of the physics and applied mathematics in the area of research of complex systems in Nature and society [1-8]. An important part of the ground for this success was created more than 60 years ago when L. F. Richardson and L. W. Lanchester, both famous British scientists, were the first who rose and applied the idea for mathematical modeling of arms races and military combats [9-12]. For many years the research on wars and other military conflicts was concentrated in the military universities and academies. In the last 20 years and especially after the terrorist attack on 11th of September 2001 this research became actual for many physicists and applied mathematicians too [13-15].

In this paper we shall follow the terminology used by Epstein [16] who applied ecological models of Lotka-Volterra kind for description of combats. Let us have two conflicting groups named the ”Red group” $R$ and the ”Blue group” $B$. A general form of model equations of an armed conflict between these groups is

$$ \frac{dB}{dt} = F(B, R; b, r), \quad \frac{dR}{dt} = G(B, R; b, r) $$

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where $R(t)$ and $B(t)$ are the numbers of armed members of the two groups; $b$ and $r$ are the "firing effectiveness" (technology level) of the groups; and $F$ and $G$ are linear or nonlinear functions, depending on the character of the conflict. Epstein [16] proposed the following class of models of the conflict

$$
\frac{dR}{dt} = -bB^{c_1}R^{c_2}, \quad \frac{dB}{dt} = -rR^{c_3}B^{c_4}
$$

where $c_{1,2,3,4}$ are real nonnegative coefficients. If these coefficients are constants the models are called hard models. If the coefficients depend on the the number of participants or on the parameters of the environment the models are called soft ones.

Important characteristics of the model (2) is the casualty exchange ratio or state equation (the ratio of eliminated members of the "red" and "blue" groups). For (2) the ratio is

$$
\eta = \frac{dR}{dB} = \frac{b}{r} \frac{B^{\lambda_b}}{R^{\lambda_r}}
$$

where $\lambda_b = c_1 - c_4$ and $\lambda_r = c_3 - c_2$. The integral form of the above casualty exchange ratio is

$$
I(B, R) = \frac{b}{\lambda_b + 1} B^{\lambda_b + 1} - \frac{r}{\lambda_r + 1} R^{\lambda_r + 1} = I_0(B_0, R_0)
$$

where $B_0$ and $R_0$ are the numbers of the members of the two groups at the beginning of the conflict (at $t = 0$).

For developing of intuition and decision skills it is of interest to know $R(t)$ and $B(t)$ in closed form. Such analytical solutions are possible only in small number of cases. Section 2 contains the solution of the system of model equations for selected values of the parameters $c_i$. Several concluding remarks are summarized in section 3.

2 Analytically solvable models

2.1 The attrition and the square law of Lanchester

The linear model of Lanchester describes position conflicts such as the battles for Somna and Verdune in 1916. The coefficients in (2) are $c_1 = c_3 = 1$, $c_2 = c_4 = 0$. The equation of state is quadratic

$$
bB^2(t) - rR^2(t) = bB_0^2 - rR_0^2 = \kappa_0
$$
where \( \kappa_0 \) can be positive, negative, or 0. For the case \( \kappa_0 = 0 \) we obtain \( B_0 = \sqrt{\frac{r}{b}} R_0 \) and the solutions are

\[
R(t) = R_0 e^{-at}, \quad B(t) = B_0 e^{-at}, \quad a = \sqrt{br}
\]

Evidently \( R(\infty) = B(\infty) = 0 \) which means that after endless position conflict the two groups are destroyed and no one of them wins. However the situation changes if \( \kappa_0 \neq 0 \).

Let us first assume that \( \kappa_0 > 0 \). From the state equation (5) \( B^2(t) = \frac{1}{b}(\kappa_0 + r R^2(t)) \) and the solutions of the model system of equations are

\[
R(t) = \frac{c_0^2 \exp(-2at) - \kappa_0}{2\sqrt{r c_0} \exp(-at)}, \quad B(t) = \frac{c_0^2 \exp(-2at) + \kappa_0}{2\sqrt{b c_0} \exp(-at)}
\]

where \( c_0 = \sqrt{r} R_0 + \sqrt{\kappa_0 + r R_0^2} = \sqrt{r} R_0 + \sqrt{b} B_0 \). The obtained solutions satisfy the initial conditions \( R(0) = R_0, B(0) = B_0 \). In addition at \( T_0 = \frac{1}{2a} \ln \left( \frac{c_0^2}{|\kappa_0|} \right) > 0 \) we obtain \( R(T_0) = 0 \) and \( B(T_0) = \sqrt{\kappa_0/b} \neq 0 \). In other words the Blue group will win the conflict if it lasts long enough. The Red group commanders have to change the strategy if they want to escape the defeat. If this does not happen after the time \( T_0 \) from the beginning the Red group will be completely destroyed - Fig. 1. We note again that this happens when \( \kappa_0 > 0 \), i.e., when

\[
B_0 > \sqrt{\frac{r}{b}} R_0
\]

The condition (8) (but with = instead of > ) for military combats is known as the square law of Lanchester: to stalemate and adversay army two times as numerous as yours, your army must be four times as effective. But in this case our army will be also destroyed. Thus the correct statement of the square law is as follows: to stalemate and adversay army two times as numerous as yours, your army must be more than four times as effective. In other words: in position war, in order to stop army that is \( n \) time larger than yours your army has to be more than \( n^2 \) technologically better (to have more than \( n^2 \) times larger firepower).

We now discuss the case \( \kappa_0 = - | \kappa_0 | < 0 \). The solutions of the model equations are

\[
R(t) = \frac{c_0^2 \exp(-2at) + |\kappa_0|}{2\sqrt{r c_0} \exp(-at)}, \quad B(t) = \frac{c_0^2 \exp(-2at) - |\kappa_0|}{2\sqrt{b c_0} \exp(-at)}
\]

Now the Red group wins if \( T_0 = \frac{1}{2a} \ln \left( \frac{c_0^2}{|\kappa_0|} \right) > 0 \) i.e. when \( R(T_0) = \sqrt{|\kappa_0|/r}, B(T_0) = 0 \).
2.2 The concentrated attack model

In this case the coefficients in the general model are \( c_1 = c_2 = c_3 = c_4 = 1 \) and the equation of state is

\[
bb(t) - rR(t) = bB_0 - rR_0 = a_0
\]

Epstein assumes that stalemate occurs when \( B(t) = R(t) = 0 \), i.e., when \( rR_0 = bB_0 \). Let us discuss this in more details.

If \( a_0 = 0 \) (Epstein case) then the solution of the model system of equations is

\[
R(t) = \frac{R_0}{1 + rR_0 t}, \quad B(t) = \frac{B_0}{1 + rR_0 t}
\]

thus at \( R(\infty) = B(\infty) = 0 \), none of the groups wins, the attack is stopped but the two groups are completely destroyed. This of course is not of favor for the group leaders. In order to consider more realistic scenarios we have
to set \(a_0 \neq 0\). In this case the solutions of the model equations are

\[
R(t) = \frac{a_0}{n_0 \exp(a_0 t) - r}, \quad B(t) = \frac{a_0}{b - m_0 \exp(-a_0 t)}
\]

(12)

where \(m_0 = rR_0/B_0\) and \(n_0 = bB_0/R_0\). Now the sign of \(a_0\) determines the asymptotic behavior of the number of members of the two groups. If \(a_0 > 0\) then \(B(\infty) = a_0/b = B_0 - (r/b)R_0\) and \(R(\infty) = 0\). Thus the Blue groups wins and the results of the attack is that the Red group is defeated -Fig.2

Now let \(a_0 = -|a_0|\). Then

\[
B(t) = \frac{-|a_0|}{b - m_0 \exp(|a_0| t)}, \quad R(t) = \frac{|a_0|}{r - n_0 \exp(-|a_0| t)}
\]

(13)

In this case as winner from the attack scenario is the Red group as \(B(\infty) = 0\), \(R(\infty) = |a_0|/r = R_0 - (b/r)B_0\).

Now let us discuss the following detail. Let us assume that at the beginning of the attack the Blue group has more soldiers than the Red group: \(B_0 > R_0\) but the firepower of the Red group is larger: \(r > b\). Then there must be a moment \(T_1\) where the two groups will have equal number of armed
members: \( R(T_1) = B(T_1) \). This moment \( T_1 \) can be determined from the equation of state (10)\]

\[(b - r)B(T_1) = a_0 \]  

What is interesting that when \( T_1 > 0 \) a solution exists only if \( a_0 = -\left| a_0 \right| < 0 \) and it is (see Fig.3) 

\[ T_1 = \frac{1}{\left| a_0 \right|} \ln \left( \frac{B_0}{R_0} \right) \]  

2.3 The ambush  

For this case \( c_1 = c_2 = c_3 = 1, c_4 = 0 \). The equation of state is 

\[ bb^2(t) - 2r R(t) = BB_0^2 - 2rR_0 = s_0 \]  

What is interesting here is that the large group does not win in any case. Let for an example \( s_0 = 0 \). Then the solutions of the model equations are 

\[ R(t) = \frac{R_0}{\left(1 + \frac{1}{2}bb_0t\right)^2}, \quad B(t) = \frac{B_0}{\left(1 + \frac{1}{2}bb_0t\right)} \]
which means that none wins \( R(\infty) = B(\infty) = 0 \) and the groups are destroyed. This of course is not acceptable for both sides.

More realistic are situations where \( s_0 \neq 0 \). Let first \( s_0 > 0 \). The solution of the model equations is

\[
(18) \quad B(t) = \sqrt{\frac{s_0}{b}} \frac{1 - \exp(-\omega_0 t)}{1 + \exp(-\omega_0 t)}, \quad R(t) = -\frac{2s_0}{r} \frac{\exp(-\omega_0 t)}{(1 + \exp(-\omega_0 t))^2}
\]

where \( \omega_0 = \sqrt{bs_0} \) and \( E_0 = \frac{\sqrt{s_0 - \sqrt{bb_0}}}{\sqrt{s_0 + \sqrt{bb_0}}} < 0 \). Thus in the asymptotic case \( B(\infty) = \sqrt{s_0/b} \) and \( R(\infty) = 0 \), i.e. the Blue groups wins.

Quite interesting is the case \( s_0 = -|s_0| < 0 \). In this case the solution of the model system is

\[
(19) \quad B(t) = B_0 \frac{1 - \Delta^{-1} \tan(\sigma_0 t)}{1 + \Delta \tan(\sigma_0 t)}, \quad R(t) = R_0 \frac{1 + \tan^2(\sigma_0 t)}{(1 + \Delta \tan(\sigma_0 t))^2}
\]

where \( \sigma_0 = \frac{1}{2} \sqrt{b |s_0|}, \Delta = B_0 \sqrt{b / |s_0|} \). In this case at \( T_0 = \frac{1}{\sigma_0} \arctan(\Delta) \) we obtain \( B(T_0) = 0, \quad R(T_0) = \frac{|s_0|}{2r} \neq 0 \). Thus at \( t = T_0 \) this conflict ends. Blue group is completely destroyed and the Red group wins.

### 2.4 Two cubic models

For these models \( c_1 = c_3 = 2, \ c_2 = c_4 = 1 \) and \( c_1 = c_3 = 1, \ c_2 = c_4 = 2 \). Epstein [16] argues that the exponents \( c_i \) in the general model (2) should be kept in the interval \([0, 1]\). However there is no evidence to sustain such an assertion. Here we investigate two models for which some of the exponents \( c_i \) equal 2.

Let first \( c_1 = c_3 = 2, \ c_2 = c_4 = 1 \). The model equations are

\[
(20) \quad \frac{dR}{dt} = -bB^2 R, \quad \frac{dB}{dt} = -rR^2 B
\]

The equation of state is

\[
(21) \quad bB^2(t) - rR^2(t) = bB_0^2 - rR_0^2 = \kappa_0 = \text{const}
\]

If \( \kappa_0 = 0 \) then the equation of the state together with one of the model equations yield

\[
(22) \quad R(t) = \frac{R_0}{\sqrt{1 + 2rR_0^2 t}}, \quad B(t) = \frac{B_0}{\sqrt{1 + 2bB_0^2 t}}
\]
No one of the groups wins at $t \to \infty$. If $\kappa_0 > 0$ we obtain

$$R(t) = \frac{\sqrt{\kappa_0} \exp(-\kappa_0 t)}{\sqrt{\gamma_0 - r \exp(-2\kappa_0 t)}}, \quad B(t) = \frac{1}{\sqrt{b}} \sqrt{\kappa_0 + r R^2(t)},$$

where $\gamma_0 = (\kappa_0 + r R_0^2)/R_0^2$. Thus $R(\infty) = 0$ and $B(\infty) = \sqrt{\kappa_0/b}$. If $\kappa_0 < 0$

$$R(t) = \frac{\sqrt{|\kappa_0|}}{\sqrt{r + \gamma_0 \exp(-2|\kappa_0| t)}}, \quad B(t) = \frac{1}{\sqrt{b}} \sqrt{r R^2(t) - |\kappa_0|},$$

Thus $R(\infty) = \sqrt{|\kappa_0|}/r$ and $B(\infty) = 0$.

Let now $c_1 = c_3 = 1$ and $c_2 = c_4 = 2$. The equilibrium condition (4) becomes

$$B(t) = B_0 \left[ \frac{R(t)}{R_0} \right]^{\epsilon}, \quad \epsilon = \frac{r}{b}$$

The final solution is

$$R(t) = R_0 \left(1 + \frac{t}{\tau_0}\right)^{-\frac{b}{r+b}}, \quad B(t) = B_0 \left(1 + \frac{t}{\tau_0}\right)^{-\frac{r}{r+b}}$$

where $\tau_0 = 1/((r+b)R_0 B_0)$ is a typical time of decay.

Let us form the ratio

$$\frac{B(t)}{R(t)} = \frac{B_0}{R_0} \left(1 + \frac{t}{\tau_0}\right)^{\frac{b}{r+b}}$$

where $b > r$ or $b < r$. Assuming $B_0 > R_0$ but $r > b$ and letting $B(T_1) = R(T_1)$ we obtain

$$T_1 = \tau_0 \left[ \left(\frac{B_0}{R_0}\right)^{\kappa} - 1 \right], \quad \kappa = \frac{r + b}{r - b}$$

Thus $B(t) < R(t)$ at $t > T_1$. Nevertheless, both groups, fight to the end ($B(\infty) = R(\infty) = 0$).

### 2.5 A model accounting for epidemic events

It is known from the history of the conflicts that epidemic events had frequently occurred particularly in case of attrition and ambush conflicts. The simplest way to account for this effect (removing of conflict participants because of sickness) is to modify the general model as follows

$$\frac{dB}{dt} = F(B, R; b, r) - H_B, \quad H_B = k_B B, \quad k_B \geq 0$$

$$\frac{dR}{dt} = G(B, R; b, r) - H_R, \quad H_R = k_R R, \quad k_R \geq 0$$

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where $k_B$ and $k_R$ are coefficients of morbility (sick rate) removal. Generally $k_B \neq k_R$.

We choose to demonstrate the effect of epidemics on the model (16) in the form

$$\frac{dB}{dt} = -rR; \quad \frac{dR}{dt} = -bBR - kR, \quad k > 0 \quad (30)$$

Hence

$$bB^2(t) - 2rR(t) + 2kB(t) = \tilde{s}_0 = \text{const} \quad (31)$$

where

$$\tilde{s}_0 = bB_0^2 - 2rR_0 + 2kB_0 \quad (32)$$

$$\tilde{s} = bB^2_0 > 0 \quad \text{if} \quad kB_0 = rR_0 \quad (33)$$

$$\tilde{s} = 2kB_0 > 0 \quad \text{if} \quad bB^2_0 = 2rR_0 \quad (34)$$

In the general case (32) the quantity $\tilde{s}_0$ can be zero, positive or negative. For an example if $\tilde{s}_0 = 0$ the solution of the model system is

$$B(t) = \frac{2\epsilon_0k\exp(-kt)}{1 - \epsilon_0b\exp(-kt)}; \quad R(t) = \frac{2\epsilon_0k^2\exp(-kt)}{r(1 - \epsilon_0b\exp(-kt)^2} \quad (35)$$

where

$$\epsilon_0 = \frac{B_0}{2k + bB_0} = \frac{B_0^2}{2rR_0}$$

These solutions degenerate into (17) at $k \to 0$.

### 3 Summary and conclusion

A class of mathematical models of armed conflicts (2) was investigated. The purpose was to identify particular cases with exact simple solutions in analytical form $B = B(t)$ and $R = R(t)$, where $B$ and $R$ were the armies’ numbers. It was found that these requirements were met by the linear (Lanchester’s) model known from long ago in the form (5) as well as by several nonlinear models described in this paper. These models demonstrate rich behavior in the time.

The predictions of the discussed models fall in two groups

- No one of the two groups $B$ and $R$ wins after endless ($t \to \infty$) attrition conflict- fighting to the finish (6), (26)
- one of the groups wins after limited in time or after an endless conflict (see Table 1)
\[ \frac{dR}{dt} = -bB - bBR - bBR - bB^2R - bBR^2 \]
\[ \frac{dB}{dt} = -rR - rBR - rR - rR^2B - rRB^2 \]
\[ I_0 = \begin{align*}
\kappa_0 (\ldots) & a_0 (\ldots) & s_0 (\ldots) & \kappa_0 (\ldots), (\ldots) & Q_0 (\ldots) \\
B(\infty) = 0 & B(\infty) = 0 & B(\infty) = 0 & B(\infty) = 0 & B(\infty) = 0 \\
R(\infty) = 0 & R(\infty) = 0 & R(\infty) = 0 & R(\infty) = 0 & R(\infty) = 0 \\
I_0 > 0 & B(T_0) = \sqrt{\kappa_0/b} & B(\infty) = a_0/b & B(\infty) = \sqrt{s_0/b} & B(\infty) = \sqrt{\kappa_0/b} & B(\infty) = 0 \\
R(T_0) = 0 & R(\infty) = 0 & R(\infty) = 0 & R(\infty) = 0 & R(\infty) = 0 \\
I_0 < 0 & R(T_0) = \sqrt{\kappa_0/b} & B(\infty) = 0 & B(\infty) = 0 & B(\infty) = 0 & B(\infty) = 0 \\
B(T_0) = 0 & R(\infty) = a_0/\sqrt{r} & R(T_0) = s_0/(2r) & R(\infty) = \sqrt{\kappa_0/b} & R(\infty) = 0 & B(\infty) = 0 \\
T_0 = T_0 = \frac{1}{2a} \ln \left( \frac{\Delta}{\kappa_0} \right) > 0 & T_0 = \infty & T_0 = \frac{1}{\sigma_0} \arctan(\Delta) & T_0 = \infty & T_0 = \infty & T_0 = \infty \\
c_0 = \sqrt{rR_0 + \sqrt{b}B_0} & \Delta = B_0\sqrt{b} - s_0 &
\]

Table 1: Summary of model predictions.

Both conflicting groups are defined by their initial numbers \( B_0, R_0 \) and respective firing effectivenesses per shot. The character of the conflict is modeled by the form of the functions \( F \) and \( G \). All models can be extended to account for occurring of epidemic events in the fighting groups. The model (30) is an example.

References

[1] Grossman, S., G. Mayer-Kress. Nature, 337, 1989, 701-704.

[2] Coleman, S. J. Math. Sociology, 18, 1993, 47 -64.

[3] Richards, D. Int. Studies Quarterly, 37, 1993, 55-72.

[4] Elliot, E. D. L. Kiel. Proc. Natl. Acad. Sci. USA , 99, 2002, 7193-7194.

[5] Troisi, A., V. Wong, M. A. Rather. Proc. Natl. Acad. Sci.USA, 102, 2005, 255 - 260.

[6] Dimitrova Z., N. K. Vitanov. Theoretical Population Biology, 66, 2004, 1-12.

[7] Vitanov, N. K., Z. I. Dimitrova, H. Kantz. Phys. Lett. A, 349, 2006, 350-355.
[8] PANCHEV S., T. SPASSOVA, N. K. VITANOV. Chaos Solitons & Fractals, 33, 2007, 1658-1671.

[9] RICHARDSON, L. F. Nature, 135, 1935, 830-831.

[10] RICHARDSON, L. F. Nature, 148, 1941, 598-598.

[11] RICHARDSON, L. F. J. Roy. Stat. Soc., 107, 1944, 242-250.

[12] RICHARDSON, L. F. J. Roy. Stat. Soc. A, 109, 1946, 130-156.

[13] EPSTEIN, J. M. Int. Security, 12, 1998, 154-165.

[14] KUZMAR, L. A. American Anthropology, 109, 2007, 318 - 329.

[15] LONGINI, I. M., M. E. HALLORAH, A. NIZAM, Y. YANG, S. F. XU, D. S. BURKE, D. A. T. CUMMINGS, J. M. EPSTEIN, J. M. (2007). Int. J. Infectious Diseases, 11, 2007, 98-108.

[16] EPSTEIN, J. M. Nonlinear Dynamics, Mathematical Biology and Social Sciences. Addison-Wesley, Readings, MA, 1997.

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