Longitudinal Impact of Straight Rod against Elastic Support

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Abstract. The paper implements a series method for numerical analyze a stress state of a straight rod in the process of its longitudinal impact on an elastic support at a given initial free rod movement before its impact with a given speed. The problem solution is based on entering a displacement function in trigonometric series. As a method implementation result, a computer program was created that allows multivariate calculations to be performed with high accuracy and the results output in a clear graphic form.

1. Introduction

Impact is a common phenomenon often found in technology, that leads to either positive or negative results depending on the situation. The positive thing is machines using an impact can act on a work piece with force many times greater than other machines of the same size using only quasi-static force could create. On the negative side, stresses significantly exceed the permissible operating stresses occurring in structures under impact that leads to significant wear and premature failure of structural parts.

Due to this fact, the target of method development for studying mechanical phenomena during impact is currently very urgent.

The unsteady deformed rod state under an impact is described by a partial differential wave equation. [2, 3, 4, 5, 6, 7, 8]. Navier had been solving the wave equation in series using the Fourier method [1, 9], and Saint-Venant [10] had been using the D'Alembert method by presenting the solution in the form of two unknown functions, where one of them describes a direct longitudinal wave and the other a backward wave.

To date, a large number of problems on the longitudinal impact of the rod under various boundary conditions have been solved [11–20]. The D'Alembert method was mainly used in solving these problems, since it is believed that the Navier method in the series has insufficient convergence.

The using of modern computer technology makes it possible to obtain fairly accurate solutions based on the Navier method, if the large number of members of the series are taken into consideration. At the same time, it is also important that the solution in the series makes it easy enough to analyze the change in the stress-strain state of an oscillating rod for an unlimited time interval after an initial short-term impact action.

This paper implements a series method for the numerical analysis of a straight rod stress state in the process of its longitudinal impact on an elastic support at given initial free rod movement prior to the impact at a certain given speed.
This problem is very common in modern technology, and in particular when analyzing the phenomena occur at drilling wells for gas and oil. For example, there is a sharp vertical and uncontrolled fall of the drill pipe, that leads to the impact of its coupling on the elevator. In this case, destruction or unacceptable deformation of structural elements that inhibit the unacceptable fall of the pipe into the well is possible. In this regard, the problem arises for determining the stresses in the pipe sections and the impact force of the pipe against the support hinder the pipe from falling into the well.

2. Problem definition

We set the task to develop and numerically test the methodology for calculating a straight rod stress state under its longitudinal impact on an elastic support after a free rod fall from a certain height without an initial velocity.

The analytical model of this process is presented in Fig. 1.

Fig. 1 shows a rod with a length $l$ that has an absolutely solid platform attached to its upper end. The rod is made of a material with an elastic modulus $E$, density $\rho$ and has a cross-sectional area $A$. At first, the rod freely falls without initial speed from a height $H$. After the moment of taking a contact with an elastic element having rigidity $K$, the rod has a speed defined by the formula:

$$V_0 = \sqrt{2gH}.$$

We superpose the reference point of the fixed coordinate axis on the interaction beginning point of the rod platform with the elastic element (Fig. 1). The displacement of the rod's cross-sections $u$ is set relative to the $X$-axis and is further considered as a function of coordinate $x$ and time $t$:

$$u = u(x, t).$$

We write the differential equation describing the dynamic processes in the $X$-axis direction [1]:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0.$$

![Figure 1. Analytical model of the process](image)
Where
\[ c^2 = \frac{E}{\rho}. \] (4)

We consider the general solution of equation (3) as an infinite series of the two functions product:
\[ u(x, t) = \sum_{k=1}^{\infty} U_k(x)V_k(t). \] (5)

where
\[ U_k(x) = B_k \cos a_k x + D_k \sin a_k x, \quad k = 1, 2, 3, \ldots \] (6)
\[ V_k(t) = M_k \cos p_k t + N_k \sin p_k t, \quad k = 1, 2, 3, \ldots \] (7)

Substituting (5) with (6) and (7) into (3), we obtain:
\[ \sum_{k=1}^{\infty} (c^2 a_k^2 - p_k^2)(B_k \cos a_k x + D_k \sin a_k x)(M_k \cos p_k t + N_k \sin p_k t) = 0. \] (8)

From (8) we obtain the system of equalities
\[ c^2 a_k^2 - p_k^2 = 0, \quad k = 1, 2, 3, \ldots \] (9)

From (9) we obtain:
\[ p_k = c a_k = a_k \sqrt{\frac{E}{\rho}}, \quad k = 1, 2, 3, \ldots \] (10)

3. Accounting for boundary conditions

The reaction acts on the rod from the side of the elastic support under a positive displacement \( u(0, t) > 0 \) of the rod's cross-section together with the coordinate \( x = 0 \) (Fig. 1):
\[ N = K \cdot u(0, t) = K \sum_{k=1}^{\infty} U_k(0)V_k(t) > 0 \quad \text{at} \quad u(0, t) > 0. \] (11)

This reaction causes tensile stress in the rod \( \sigma(0, t) > 0 \).

Moreover, taking into account (11), the equality:
\[ K \sum_{k=1}^{\infty} U_k(0)V_k(t) = A\sigma(0, t). \] (12)

We determine the stress in the rod's material on the base of (5), (6) and (7).

Taking into account (1) and Fig. 1 we write:
\[ \sigma = E \frac{\partial u}{\partial x} = E \sum_{k=1}^{\infty} \frac{\partial U_k(x)}{\partial x} V_k(t) = \sum_{k=1}^{\infty} a_k (D_k \cos a_k x - B_k \sin a_k x)V_k(t). \] (13)

Substituting (6), (13) into (12), we obtain:
\[ K \sum_{k=1}^{\infty} B_k V_k(t) = AE \sum_{k=1}^{\infty} a_k D_k V_k(t). \] (14)

The right rod's end at \( x = l \) is free, so taking into account (13) the boundary condition for it has the form:
\[
\sigma(l, t) = E \frac{\dddot{u}(l, t)}{\ddot{c}x} = E \sum_{k=1}^{\infty} a_k (D_k \cos a_k l - B_k \sin a_k l) V_k(t) = 0. \tag{15}
\]

Combining (14) and (15) into a single system, we obtain the equalities:
\[
\begin{cases}
KB_k = AEa_k D_k, \\
D_k \cos a_k l - B_k \sin a_k l = 0, \quad k = 1, 2, 3, \ldots.
\end{cases} \tag{16}
\]

We introduce the notation
\[
\xi_k = a_k l, \quad k = 1, 2, 3, \ldots. \tag{17}
\]

Next, we denote by \( C \) the rod stiffness under tension-compression:
\[
C = \frac{AE}{l} \tag{18}
\]

and introduce a coefficient determined by the ratio of the rod stiffness \( C \) to the elastic bond stiffness \( K \):
\[
f = \frac{C}{K}. \tag{19}
\]

After a series of transformations of relations (16), taking into account (17), (18) and (19), we obtain the system of equations
\[
\cos \xi_k - f \xi_k \sin \xi_k = 0, \quad k = 1, 2, 3, \ldots \tag{20}
\]
we find their roots
\[
\xi_k, \quad k = 1, 2, 3, \ldots. \tag{21}
\]

Knowing \( \xi_k \) from the first equation of system (16) we obtain:
\[
B_k = f \xi_k D_k. \tag{22}
\]

Next, using (17), we calculate the parameters:
\[
a_k = \frac{1}{l} \xi_k, \quad k = 1, 2, 3, \ldots, \tag{23}
\]

and then, using (10), we also determine the circular frequencies of natural vibrations:
\[
p_k = a_k \sqrt{\frac{E}{\rho}} = \xi_k \frac{1}{l} \sqrt{\frac{E}{\rho}}, \quad k = 1, 2, 3, \ldots. \tag{24}
\]

4. Accounting for initial conditions

At time \( t=0 \), the rod is not deformed, that is \( u(x, 0)=0 \), and all its points have the same speed \( V_0 \). This is taken into account by the expressions defining the initial conditions:
\[
u(x, 0) = 0, \quad \frac{\dddot{u}(x, 0)}{\ddot{c}t} = V_0. \tag{25}
\]

We write the first initial condition from (25) based on expression (5) with allowance for (7) for the initial time \( t=0 \):
\[
u(x, 0) = \sum_{k=1}^{\infty} U_k(x) V_k(0) = \sum_{k=1}^{\infty} U_k(x) M_k = 0. \tag{26}
\]
Equality (26) is valid under the conditions:

\[ M_k = 0, \quad k = 1, 2, 3, \ldots \]  

We determine the first time derivative of the function \( u(x, t) \) on the basis of (5), (7) taking into account (17) and (22) when designating \( R_k = D_k N_k, k=1, 2, 3, \ldots \):

\[
\frac{\partial u(x,t)}{\partial t} = \sum_{k=1}^{\infty} U_k(x) \frac{\partial V_k(t)}{\partial t} = \sum_{k=1}^{\infty} R_k p_k \left( f \xi_k \cos(\xi_k \frac{x}{l}) + \sin(\xi_k \frac{x}{l}) \right) \cos(p_k t). 
\]  

(28)

On the base of (28) for the initial time \( t=0 \) we write the second initial condition from (25):

\[
\frac{\partial u(x,0)}{\partial t} = \sum_{k=1}^{\infty} R_k p_k \left( f \xi_k \cos(\xi_k \frac{x}{l}) + \sin(\xi_k \frac{x}{l}) \right) = V_0. 
\]  

(29)

We introduce the notation

\[ T_k = R_k \frac{1}{l} \frac{1}{V_0} \frac{E}{\rho}, \quad k = 1, 2, 3, \ldots \]  

(30)

and we bring equation (29) to the form

\[
\sum_{k=1}^{\infty} T_k \left( f \xi_k^2 \cos(\xi_k \frac{x}{l}) + \xi_k \sin(\xi_k \frac{x}{l}) \right) = 1. 
\]  

(31)

As a result, we came to the problem to determine the coefficients \( T_k \) from equation (31).

In this study, to find the parameters \( T_k \) we form a grid of \( N_x \) nodes along \( X \)-axis that are the boundaries of the segments \( \Delta x_n \):

\[ \Delta x_n = x_n - x_{n-1}, \quad x_n = h(n-1), \quad h = \frac{l}{N_x - 1}, \quad n = 1, N_x - 1. \]  

(32)

Then, integrating equation (31) within the limits of segments \( \Delta x_n \), we write \( M = N_x - 1 \) equalities:

\[
\int_{x_{n-1}}^{x_n} \sum_{k=1}^{\infty} T_k \left( f \xi_k^2 \cos(\xi_k \frac{x}{l}) + \xi_k \sin(\xi_k \frac{x}{l}) \right) dx = \int_{x_{n-1}}^{x_n} \cdot 1 \cdot dx, \quad n = 1, N_x - 1. 
\]  

(33)

wherefrom we determine the expansion coefficients \( T_k \), \( k = 1, M \).

In this study, a personal computer made it possible to perform calculations at \( M = 2 \cdot 10^4 \) without any particular problems, and therefore to achieve high accuracy in satisfying the initial conditions specified by the mathematical equation (31).

Having determined the quantities \( T_k \) and using (30) with taking into account (4) we determine the coefficients \( R_k \):

\[
R_k = T_k l \frac{V_0}{\sqrt{E\rho}} = \frac{V_0}{c} T_k, \quad k = 1, 2, 3, \ldots \]  

(34)

As a result, after all the transformations based on (13), we obtain the final expression for calculating the stress in an arbitrary section of the rod with a coordinate \( x \) at an arbitrary time \( t \):

\[
\sigma = V_0 \sqrt{E \rho} \sum_{k=1}^{\infty} T_k \xi_k \left( \cos(\xi_k s) - f \xi_k \sin(\xi_k s) \right) \sin(p_k t) \quad \text{at} \quad s = x / l, \quad x \in [0, l]. 
\]  

(35)

We introduce the notation
\[
\begin{align*}
\sigma_{\infty} &= V_0 \sqrt{Ep}, \\
\bar{\sigma}(s, t) &= \sum_{k=1}^{\infty} T_k \xi_k \left( \cos(\xi_k s) - f \xi_k \sin(\xi_k s) \right) \sin p_k t.
\end{align*}
\]  

(36)

and write:

\[
\sigma = \sigma_{\infty} \bar{\sigma}(s, t) \quad \text{at} \quad s = x/l, \quad x \in [0, l].
\]

(37)

In (36), (37), a notation \(\sigma_{\infty}\) was introduced for the maximum stress in the rod under impact against an absolutely rigid support \((K = \infty)\), and \(\bar{\sigma}(s, t)\) is considered as a function of \(s\) and \(t\) relative to stress for a selected ratio of the stiffness of the rod and support \(f\).

5. Results

5.1. Stresses and impact forces in the upper rod section

The maximum stresses occur in the upper rod section at \(s = 0\) directly interacting with the elastic support. For this section Fig. 2a shows the dependences of the relative stresses \(\bar{\sigma}\) on time \(t\) for various relative stiffness in time intervals from the onset of rod interaction with elastic support to the moments when contact between the rod and the elastic support occurs at \(\bar{\sigma} < 0\).

![Figure 2](image-url)

**Figure 2.** a) Dependences of the relative stresses \(\bar{\sigma}\) on time \(t\); b) Connection breaking time between the rod and the elastic support.

Fig. 2a allows us to quantitatively evaluate the influence of the rod’s and the support stiffness ratio \(f\) on the law of stress variation in time \(t\) in the most dangerous upper rod section \((s=0)\) upon impact. Hence, knowing the stress in the upper rod section, we can determine the time force dependence of rod’s impact \(F_u\) on the elastic support:

\[
F_u(t) = A \sigma_{\infty} \bar{\sigma}(s = 0, t).
\]

(38)

5.2. The duration of the impact interaction for the rod with the support

The resulting solution is valid only in the time interval \([t_0, t_{br}]\) from the moment of the interaction beginning of the upper rod section with the elastic support \(t_0=0\) until the contact breaks between the rod and the elastic support \(t_{br}\).
The contact breaking time $t_{br}$ between the rod and the elastic support was determined for various values of relative stiffness $f = C/K$, defined as the transition time in the fixed section of the rod at $s=0$, from positive to zero, and then to negative relative rated stress $\sigma \leq 0$.

The calculated dependence of $t_{br}$ on $f$ for the considered rod is presented in Fig. 2b, wherefrom it is seen that with a fixed rod's stiffness $C$ with elastic support stiffness $K$ decreasing, the time $t_{br}$ increases until the contact between the rod and the elastic support breaks.

5.3. Dependences of the relative stress on the ratio of the rod’s and the elastic support’s stiffness in the selected sections of the rod

When designing and operating rods under longitudinal impact, it is necessary to be able to analyze the influence of relative stiffness $f$ on the value of the maximum relative stress $\tilde{\sigma}_{\text{max}}$ in various rod sections determined by the parametric coordinate $s$. In this case, the maximum stress determined by the formula

$$\sigma_{\text{max}} = \sigma_c \tilde{\sigma}_{\text{max}} \text{ at } \sigma_c = V_0 \sqrt{Ep}.$$ (39)

According to (39), it is necessary to know the relative maximum stress to estimate the maximum stress $\tilde{\sigma}_{\text{max}}$. The stress dependences $\tilde{\sigma}_{\text{max}}$ on the relative stiffness $f$ in various rod sections are shown in Fig. 3a.

From Fig. 3a it is clear a noticeable decrease in the relative stress is observed with an increase in the $f$ coefficient. It can be seen that the greatest decrease takes place at the points (at $s \to 1$) farthest from the contact zone of the rod section with the elastic support.

The practical graph value is that it allows quickly and with sufficient accuracy evaluate the stress in various rod sections with different ratios between the stiffness of the impacted rod and the elastic support.

5.4. Dependences of the relative stress on the coordinate of the rod cross-section for the selected ratios of the rod’s stiffness and the elastic support stiffness

Fig. 3b shows changes in the maximum relative stress $\tilde{\sigma}_{\text{max}}$ along the rod’s axis for various ratio values of the rod’s stiffness and the elastic support stiffness $f$. 

![Figure 3](image-url)
It can be seen (Fig. 3b) for small values \( f < 0.2 \) at a sufficiently large distance from the fulcrum, the maximum relative stresses \( \sigma_{\text{max}} \) take almost the same maximum value equal to 1. A significant decrease in stresses \( \sigma_{\text{max}} \) is observed along the entire rod's length at a further \( f \) increase (Fig. 3b).

The graphs shown in Fig. 3b make it possible to objectively understand the effect of the elastic support stiffness on the stress in the impacted rod.

6. Conclusion
- The problem of the stress state of a straight rod with its longitudinal impact on an elastic support is solved. The solution is based on the writing of the displacement function by using the trigonometric series,
- as a method implementation result, a computer program was created that allows multivariate calculations to be performed with high accuracy and the results output in a clear graphic form,
- the stress state in the rod was calculated for various relations between the rod stiffness and the stiffness of the elastic support. Constructed graphs allow you to assess quickly and quantitatively the stress magnitude in various rod sections and the impact force of the rod against the support for various parameters of the falling rod (length, cross-sectional area, elastic modulus of the material, speed of the rod before impact).

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