Rotating Strings with B-field

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Abstract

Some of the recent important developments in understanding string/gauge dualities are based on the idea of highly symmetric motion of “string solitons” in $AdS_5 \times S^5$ geometry originally suggested by Gubser, Klebanov and Polyakov. In this paper we study symmetric motion of short strings in the presence of antisymmetric closed string B field. We compare the values of the energy and the spin in the case of non-vanishing B field with those obtained in the case of B=0. The presence of NS-NS antisymmetric field couples the fluctuation modes that indicates changes in the quantum corrections to the energy spectrum.

1 Introduction

The main developments and research efforts in string theory in the last years were focused on the understanding of string/gauge duality and especially AdS/CFT correspondence. AdS/CFT correspondence is based on the conjecture that type II B string theory in $AdS_5 \times S^5$ background with large number of fluxes turned on $S^5$ is dual to four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM). Actually this is the best studied case where the spectrum of a state of string theory on $AdS_5 \times S^5$ corresponds to the spectrum of single trace operators in the gauge theory. The great interest

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in AdS/CFT correspondence is inspired by the simple reason that it can be used to make predictions about $\mathcal{N} = 4$ SYM at strong coupling.

Until recently this conjecture has been mostly tested in the supergravity approximation. In this case it was found that the supergravity modes on $AdS_5 \times S^5$ are in one to one correspondence with the chiral operators in the gauge theory. However, this restriction means that the curvature is small and all $\alpha' R$ corrections are neglected ($R$ is the radius of $AdS$). These truncated considerations are not enough to verify the correspondence in its full extent and to extract useful information about the gauge theory at strong coupling. It is important therefore to go beyond the supergravity limit and to consider at least $\alpha' R$ correction.

The recent progress in string propagation in pp-wave background attracted much interest [1, 2, 3]. This background possesses at least two very attractive features: it is maximally supersymmetric and at the tree level Green-Schwarz superstring is exactly solvable [4, 5]. Since this background can be obtained by taking the Penrose limit [6] of $AdS_5 \times S^5$ geometry, it is natural then to ask about AdS/CFT correspondence in these geometries. In an important recent development, Berenstein, Maldacena and Nastase (BMN) [7] have been able to connect the tree level string theory and the dual SYM theory in a beautiful way. Starting from the gauge theory side, BMN have been able to identify particular string states with gauge invariant operators with large R-charge $J$, relating the energy of the string states to the dimension of the operators. It has been shown that these identifications are consistent with all planar contributions [8] and non-planar diagrams for large $J$ [9, 10].

Although this is a very important development, it describes however only one particular corner of the full range of gauge invariant operators in SYM corresponding to large $J$. It is desirable therefore to extend these results to wider class of operators. Some ideas about the extension of AdS/CFT correspondence beyond the supergravity approximation were actually suggested by Polyakov [11] but until recently they wasn’t quite explored.

In an important recent development, Gubser, Klebanov and Polyakov [12] (GKP) suggested another way of going beyond the supergravity approximation. The main idea is to consider a particular configuration of closed string in $AdS_5 \times S^5$ background executing a highly symmetric motion. The theory of a generic string in this background is highly non-linear, but semi-classical treatment will ensure the existence of globally conserved quantum numbers. AdS/CFT correspondence will allow us to relate these quantum
numbers to the dimensions of particular gauge invariant operators in SYM. Gubser, Klebanov and Polyakov investigated three examples:

a) rotating “string soliton” on $S^5$ stretched along the radial direction $\rho$ of AdS part. This case represents string states carrying large R-charge and GKP have been able to reproduce the results obtained from string theory in pp-wave backgrounds [7].

b) rotating “string soliton” on $AdS_5$ stretched along the radial direction $\rho$. This motion represents string states carrying spin $S$. The corresponding gauge invariant operators suggested in [12] are important for deep inelastic scattering amplitudes as discussed in [13].

c) strings spinning on $S^5$ and stretched along an angular direction at $\rho = 0$.

All these states represent highly excited string states and since the analysis is semiclassical, the spin $S$ is assumed quantized.

Shortly after [12], interesting generalizations appeared. In [14, 15, 16] a more general solution which includes the above cases was investigated. Beside the solution interpolating between the cases considered by GKP, the authors of [14, 15, 16] have been able to find a general formula relating the energy $E$, spin $S$ and the R-charge $J$. In [17] a more general solution interpolating between all the above cases was suggested. Further progress has been made for the case of black hole AdS geometry [18] and confining AdS/CFT backgrounds [19, 20]. The study of the operators with large R-charge and twist two have been initiated in [22, 23]. Further interesting developments of this approach can be found also in [24, 21]. We would note also the solution for circular configurations suggested in [25]. The string soliton in this case can be interpreted as a pulsating string.

From the preceding discussion it is clear that the extension of the semiclassical analysis to wider class of moving strings in $AdS_5 \times S^5$ background is highly desirable. The purpose of this note is to consider a more general sigma model of closed strings moving in the above background. One of the possibilities is to include closed string with antisymmetric B-field turned on. We will consider the case of short strings in B-field and will analyze the relation between the conserved quantum numbers in the theory - in our case the energy $E$ and the spin $S$.

The paper is organized as follows. In the next section we present a brief review of rotating “string soliton” in $AdS_5 \times S^5$ background. In Section 3, we start by considering short strings with B-field turned on. In analogy with [12], we find the relation between the energy $E$ and spin $S$. In Section 4,
we study the quadratic fluctuations around the classical solution of Sec. 3. We conclude with summary and comments on the results and discuss some further directions of development.

2 Rotating strings in $AdS_5 \times S^5$

In this section we will review the semiclassical analysis of rotating strings [12] following mainly [14, 15, 16].

The idea is to consider rotating and boosted closed “string soliton” stretched in particular directions of $AdS_5 \times S^5$ background. The general form of the non-linear sigma model Lagrangian (bosonic part) can be written as:

$$\mathcal{L}_B = \frac{1}{2} \sqrt{-g} g^{\alpha \beta} \left[ G^{(AdS_5)}_{ab} \partial_\alpha X^a \partial_\beta X^b + G^{(S^5)}_{nm} \partial_\alpha X^n \partial_\beta X^m \right]$$

The supergravity solutions of $AdS_5 \times S^5$ in global coordinates has the form:

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega^2_{S_3} + d\psi_1^2 + \cos^2 \psi_1 \left( d\psi_2^2 + \cos^2 \psi_2 \ d\Omega^2_{\tilde{S}_3} \right)$$

where,

$$d\Omega_{S_3}^2 = d\beta_1^2 + \cos^2 \beta_1 \left( d\beta_2^2 + \cos^2 \beta_2 \ d\beta_3^2 \right)$$

$$d\Omega_{\tilde{S}_3}^2 = d\psi_3^2 + \cos^2 \psi_3 \left( d\psi_4^2 + \cos^2 \psi_4 \ d\psi_5^2 \right)$$

(we have incorporated $R^2$ factor in the overall constant multiplying the action, i.e. $1/\alpha \rightarrow R^2/\alpha$). Looking for solutions with conserved energy and angular momentum, we assume that the motion is executed along $\phi = \beta_3$ direction on $AdS_5$ and $\theta = \psi_5$ direction on $S^5$. Solution for closed string folded onto itself can be found by making the following ansatz:

$$t = \kappa \tau; \quad \phi = \omega \tau; \quad \theta = \nu \tau \quad \rho = \rho(\sigma); \quad \beta_1 = \beta_2 = 0; \quad \psi_i = 0 \ (i = 1, \ldots, 4)$$

where $\omega, \kappa$ and $\nu$ are constants. We note that one can always use the reparametrization invariance of the worldsheet to set the time coordinate of space-time $t$ proportional to the worldsheet proper time $\tau$. This means that our rod-like string is rotating with constant angular velocity in the corresponding spherical parts of $AdS_5$ and $S^5$. 


Using the ansatz (5) and the lagrangian (1) one can easily find the classical string equations, which in this case actually reduces to the following equation for $\rho(\sigma)$:

$$\frac{d^2 \rho}{d\sigma^2} = (\kappa^2 - \omega^2) \cosh \rho \sinh \rho$$

(6)

and the Virasoro constraints take the form:

$$\rho'^2 = \kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho - \nu^2.$$  

(7)

One can obtain also the induced metric which turns out to be conformally flat as expected:

$$g_{\alpha\beta} = \rho'^2 \eta_{\alpha\beta},$$

(8)

where prime on $\rho$ denotes derivative with respect to $\sigma$. It is straightforward to obtain the constants of motion of the theory under consideration:

$$E = \sqrt{\lambda} \int_0^{2\pi} d\sigma \left( -\frac{\partial L_B}{\partial t} \right) = \sqrt{\lambda} \kappa \int_0^{2\pi} d\sigma \cosh^2 \rho$$

(9)

$$S = \sqrt{\lambda} \int_0^{2\pi} d\sigma \left( \frac{\partial L_B}{\partial \phi} \right) = \sqrt{\lambda} \omega \int_0^{2\pi} d\sigma \sinh^2 \rho$$

(10)

$$J = \sqrt{\lambda} \int_0^{2\pi} d\sigma \left( \frac{\partial L_B}{\partial \theta} \right) = \sqrt{\lambda} \nu$$

(11)

where we denoted $R^2/\alpha'$ as $\sqrt{\lambda}$. An immediate but important relation following from (9) and (10) connect $E$ and $S$:

$$E = \sqrt{\lambda} \kappa + \frac{\kappa}{\omega} S$$

(12)

Assuming that $\rho(\sigma)$ has a turning point at $\pi/2$ and using the periodicity condition\(^4\), one can relate the three free parameters $\kappa, \omega$ and $\nu$ as follows:

$$\coth^2 \rho_0 = \frac{\omega^2 - \nu^2}{\kappa^2 - \nu^2} = 1 + \eta$$

(13)

\(^3\)We follow the notations of [12] and [14]  
\(^4\)Since the closed string is folded onto itself $\rho(\sigma) = \rho(\sigma + \pi)$.  

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where $\rho_0$ is the maximal value of $\rho(\sigma)$. A new parameter $\eta$ introduced in (13) is useful for studying the limiting cases of short strings ($\eta$ large) and long strings ($\eta$ small). Using the constraint (7), the following expressions for the periodicity condition and the constants of motion can be found:

$$\sqrt{\kappa^2 - \nu^2} = \frac{1}{\sqrt{\eta}} F\left(\frac{1}{2}, \frac{1}{2}, 1; -\frac{1}{\eta}\right),$$  \hspace{1cm} (14)

$$E = \frac{\sqrt{\lambda\kappa}}{\sqrt{\kappa^2 - \nu^2}} \frac{1}{\sqrt{\eta}} F\left(-\frac{1}{2}, \frac{1}{2}, 1; -\frac{1}{\eta}\right),$$  \hspace{1cm} (15)

$$S = \frac{\sqrt{\lambda\omega}}{\sqrt{\kappa^2 - \nu^2} 2\eta^{3/2}} F\left(\frac{1}{2}, \frac{1}{2}, 2; -\frac{1}{\eta}\right),$$  \hspace{1cm} (16)

One can study several limiting cases. The simplest one is when $\nu << 1$ and therefore $S << 1$. In this case

$$E^2 \approx J^2 + 2\sqrt{\lambda} S,$$

which can be interpreted as a string moving in flat space along a circle with angular momentum $J$ and rotating in a plane with spin $S$.

The second case is when the boost energy is much smaller than the rotational energy: $\nu^2 << S$. The expression (18) then reduces to the flat space Regge trajectory [12, 14]:

$$E \approx \sqrt{2S} + \frac{\nu}{2\sqrt{2S}},$$  \hspace{1cm} (19)

The last and most interesting case is when the boost energy is large, i.e. $\nu >> 1$. In this case one can find:

$$E \approx J + S + \frac{\lambda S}{2J^2} + \ldots$$  \hspace{1cm} (20)
which coincides with the leading order terms of BMN formula:

\[ E = J + \sum_{n=-\infty}^{\infty} \sqrt{1 + \frac{\lambda n^2}{J^2}} N_n + O \left( \frac{1}{\sqrt{\lambda}} \right) \]  

(21)

Similar analysis can be done in the case of long strings. In this case very interesting relations between the constants of motion were found. We give below the results of [12, 14, 16]

a) when \( \nu << \ln 1/\eta \)

\[ E \approx S + \frac{\sqrt{\lambda}}{\pi} \ln \left( S/\sqrt{\lambda} \right) + \frac{\pi J^2}{2\sqrt{\lambda} \ln \left( S/\sqrt{\lambda} \right)} \]

b) when \( \nu >> \ln 1/\eta \)

\[ E \approx S + J + \frac{\lambda}{2\pi^2 J} \ln^2 \left( S/J \right) \]

We conclude this section by noting that more general solutions can be obtained by taking \( \psi_1 = \psi_1(\sigma) \) [17]. In this case the solutions interpolate smoothly between the cases described above.

3 Classical solutions with B-field

In this Section we consider rotating short strings in the presence of B-field. The general sigma model action in this case is given by (see for instance [26]):

\[ S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left\{ \left[ \sqrt{g} g^{\alpha\beta} G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \right] + \varepsilon^{\alpha\beta} B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' \sqrt{g} \Phi(X) R^{(2)} \right\} \]  

(22)

where the target space metric \( G_{\mu\nu} \) is given by (2,3,4), \( g_{\alpha\beta} \) is the worldsheet metric and \( \varepsilon^{\alpha\beta} \) is the 2-d antisymmetric tensor density. It is a simple exercise to check that the following ansatz is compatible with the classical equations of motion:

\[ \beta_3 = \phi = \omega \tau; \quad \beta_1 = \beta_2 = 0; \quad t = \kappa \tau \]
\[ \psi_5 = \theta, \quad \theta = \theta(\sigma) = \theta(\sigma + \pi); \quad \rho = \rho(\sigma) = \rho(\sigma + \pi) \]  

(23)
\[ B = b(\theta)d\rho \wedge d\phi \]  

(24)
We will restrict our considerations to the simplest non-trivial case of

\[ H_{\theta\phi} = b = \text{const} \]

which means that \( B_{\rho\phi} \) is linear in \( \theta \)

\[ B_{\rho\phi} = \theta(\sigma) b \] (25)

The configuration defined by (23) and (24) describes a "string soliton" stretched along \( \rho \) and \( \theta \), and moving on a circle along \( \phi \) direction. With the choice (23-25) the lagrangian reduces to

\[
\mathcal{L} = -\kappa^2 \cosh^2 \rho + (\partial_\sigma \rho)^2 + \omega^2 \sinh^2 \rho + (\partial_\sigma \theta)^2 + 2\omega B_{\rho\phi}(\theta) \rho' \\
= -\kappa^2 \cosh^2 \rho + \rho'^2 + \omega^2 \sinh^2 \rho + \theta'^2 + 2\omega b \rho' \theta. \quad (26)
\]

Straightforward calculations leads to the following string equation of motion,

\[
\frac{d^2 \rho}{d\sigma^2} + b\omega \frac{d\theta}{d\sigma} = (\kappa^2 - \omega^2) \sinh \rho \cosh \rho \]
\[
\frac{d^2 \theta}{d\sigma^2} - b\omega \frac{d\rho}{d\sigma} = 0. \quad (27)
\]

(28)

Note that in contrast to the previous cases and the more general solution of [17], the equations for \( \rho \) and \( \theta \) are coupled, as expected. Beside the equations of motion, we must ensure the conformal invariance of the model, or in other words, impose the Virasoro constraints. The only non-trivial constraint can be read off from the lagrangian (26)

\[
(\rho')^2 + (\theta')^2 = \kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho \] (29)

A straightforward check shows that the equations of motion (27, 28) are compatible with (29). Equations (ref2.6-29) can also be derived from Nambu-Goto action:

\[
S_{NG} = \frac{1}{4\pi \alpha'} \int d^2 \sigma \left[ \sqrt{\tilde{g}} + \varepsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \right] \] (30)

where

\[
\tilde{g}_{\alpha\beta} = G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \] (31)
is the induced metric on the worldsheet. Substituting the expressions for $G_{\mu\nu}$ from (23-25) and using the ansatz (23,24) for $X^\mu$ we find that

$$
\tilde{g}_{\tau\tau} = -\kappa^2 \cosh^2 \rho + \omega^2 \sinh^2 \rho
$$

$$
\tilde{g}_{\sigma\sigma} = \left( \rho'^2 + \theta'^2 \right)
$$

Substituting in the NG action (30), we get for the Lagrangian

$$
\mathcal{L} \propto \sqrt{\kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho} \sqrt{\rho'^2 + \theta'^2} - \omega \beta \theta(\sigma) \rho'(\sigma)
$$

Equations of motion following from the Lagrangian (33) are

$$
\frac{d}{d\sigma} \left( \theta' \sqrt{\kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho} \sqrt{\rho'^2 + \theta'^2} \right) - b\omega \rho' = 0
$$

and

$$
\frac{d}{d\sigma} \left( \rho' \sqrt{\kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho} \sqrt{\rho'^2 + \theta'^2} \right) - \frac{\left( \kappa^2 - \omega^2 \right) \sinh \rho \cosh \rho \sqrt{\rho'^2 + \theta'^2}}{\sqrt{\kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho}} + b\omega \theta' = 0
$$

Now the induced metric on the worldsheet can always be put in the conformal gauge. This requires $g_{\tau\tau} = -g_{\sigma\sigma}$. Thus we require that

$$
\rho'^2 + \theta'^2 = \kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho
$$

It can be readily seen that (34) and (35) reduce to (28) and (27) respectively.

Before we proceed further, we have to first eliminate the $\theta$ dependence in (29) so that to be able to express all the quantities in terms of $\rho$ only (actually we need $\rho'$ only). Integrating (28) once we find:

$$
\theta' = \alpha + b\omega \rho(\sigma)
$$

As in the previous Section, we make two assumptions. First, we consider $\rho$ periodic in $\sigma$ with turning point at $\pi/2$. Secondly, we choose the minimal value of $\rho$ at $\sigma = 0$ to be vanishing. With this choice the integration constant $\alpha$ is fixed to be 0:

$$
\theta' = b\omega \rho(\sigma)
$$
Now we can substitute for $\theta'$ into the Virasoro constraint (29) to find:

$$\rho^2 = \kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho - b^2 \omega^2 \rho^2$$  \hspace{1cm} (39)

Since we consider rotating short strings, there are restrictions on the range of the parameters $\kappa$ and $\omega$, coming from the periodicity of $\rho$. At the turning point ($\sigma = \pi/2$) $\rho'$ vanishes. This gives the relation between $\kappa, \omega$ and $b$:

$$0 = \kappa^2 \cosh^2 \rho_0 - \omega^2 \sinh^2 \rho_0 - b^2 \omega^2 \rho_0^2$$  \hspace{1cm} (40)

or

$$\coth^2 \rho_0 = \frac{\omega^2}{\kappa^2} \left(1 + \frac{\rho_0^2}{\sinh^2 \rho_0}\right)$$  \hspace{1cm} (41)

For short strings $\rho_0$ is very small, so we can use the following approximation:

$$\sinh^2 \rho_0 \approx \rho_0^2$$  \hspace{1cm} (42)

and then (41) becomes

$$\coth^2 \rho_0 \approx \frac{\omega^2}{\kappa^2} (1 + b^2)$$  \hspace{1cm} (43)

In order for (43) to be valid for generic $b$ the ratio $\omega^2/\kappa^2$ must be large (as $\coth^2 \rho_0$ is large for small $\rho_0$). As in Section 2, we define a new parameter $\eta$ through

$$\frac{\omega^2}{\kappa^2} = 1 + \eta$$  \hspace{1cm} (44)

Within this approximation eq. (43) can be written as:

$$\cosh^2 \rho_0 - \frac{\omega^2}{\kappa^2} \sinh^2 \rho_0 - b^2 \omega^2 \rho_0^2 \approx 1 - \left[b^2 + \eta(1 + b^2)\right] \rho_0^2 = 0$$  \hspace{1cm} (45)

so that:

$$\rho_0 \approx \frac{1}{\sqrt{\xi}} ; \quad \xi = b^2 + \eta(1 + b^2).$$  \hspace{1cm} (46)

We expect that the periodicity condition on $\rho(\sigma)$ will give us a relation between $\kappa$ and $\xi$. Indeed, using the approximation (42) we find:

$$d\sigma \approx \frac{d\rho}{\kappa \sqrt{1 - \xi \rho^2}}$$  \hspace{1cm} (47)
or, integrating over $\sigma$

$$2\pi = \frac{4}{\kappa} \int_{0}^{\rho_0} \frac{d\rho}{\sqrt{1 - \xi \rho^2}} \quad (48)$$

In other words, the periodicity condition reduces to:

$$1 \approx \sin \frac{\pi}{2} \kappa \sqrt{\xi} \quad (49)$$

and therefore

$$\kappa \approx \frac{1}{\sqrt{\xi}} = \frac{1}{\sqrt{b^2 + \eta(1 + b^2)}} \quad (50)$$

Now we proceed with the constants of motion. The boost energy can be defined in a standard way:

$$S = \sqrt{\lambda} \int_{0}^{2\pi} \frac{d\sigma}{2\pi} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \sqrt{\lambda} \omega \int_{0}^{2\pi} \frac{d\sigma}{2\pi} \sinh^2 \rho. \quad (51)$$

It is useful to define a $\lambda$ independent variable $S = S/\sqrt{\lambda}$. In short string approximation the expression for $S$ greatly simplifies and takes the form:

$$S \approx \frac{2\omega}{\pi \kappa} \int_{0}^{\rho_0} \frac{\rho^2 d\rho}{\sqrt{1 - \xi \rho^2}} \quad (52)$$

The integral in (52) can be easily evaluated and the final expression for $S$ in terms of the constant parameters of the model is

$$S \approx \frac{\omega}{2\kappa \xi \sqrt{\xi}}. \quad (53)$$

We see that the expression for $S$ is similar to that obtained in [12, 14, 16], but with "deformed" parameter $\eta \rightarrow \xi$. One can use now the relation between $\kappa$ and $\omega$ (44) (and taking into account that $\kappa \approx 1/\sqrt{\xi}$) to obtain

$$\omega^2 \approx (1 + \eta) \frac{1}{\xi}. \quad (54)$$

For short strings, $\eta$ is very large and therefore

$$S \approx \frac{1}{2\eta(1 + b^2)^{3/2}} \quad (55)$$
or

\[ \eta \approx \frac{1}{2S(1 + b^2)^{3/2}} \gg 1. \]  

(56)

Note that in this case \( S \) has to be very small: \( S \ll 1 \).

Since in AdS/CFT correspondence, the dimensions of the operator in the gauge theory side are determined by the energy, we want to find a relation between the spin \( S \) and the energy. The latter can be obtained from the lagrangian (26) in a standard way

\[ E = \sqrt{\lambda} \left( \frac{d\sigma}{2\pi} \left( -\frac{\partial L}{\partial \dot{t}} \right) \right) = \sqrt{\lambda \kappa} \int_0^{2\pi} d\sigma \cosh^2 \rho \]  

(57)

Defining \( \lambda \) independent quantity

\[ \mathcal{E} = \kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2 \rho \]  

(58)

and using (51) we obtain the same relation between \( \mathcal{E} \) and \( S \) as in [12, 14]

\[ \mathcal{E} = \kappa + \frac{\kappa}{\omega} S \]  

(59)

It is not necessary to calculate \( \mathcal{E} \) explicitly but instead simply use the above relation between \( \mathcal{E} \) and \( S \). The final expression then takes the form:

\[ \mathcal{E} = \sqrt{2S} \sqrt{\frac{1 + b^2}{1 + 2b^2 \sqrt{1 + b^2} S}} + \frac{\sqrt{2S(1 + b^2)^{3/2}}}{\sqrt{1 + 2S(1 + b^2)^{3/2}}} \]  

(60)

Since, when \( b = 0 \) (60) reproduces the corresponding case of [12, 14]

\[ \mathcal{E} = \sqrt{2S} + \frac{\sqrt{2S}}{\sqrt{1 + 2S}} \]  

(61)

one can interpret (60) as a smooth deformation of the latter. For small boost energy we recover again the flat space Regge trajectory

\[ \mathcal{E} \approx \sqrt{2S} \]  

(62)
In the case of small $\rho_0$ the model looks as a point-like string moving in the presence of closed string B-field. In this approximation the solutions for $\rho$ and $\theta$ are:

\[
\rho \approx \frac{1}{\sqrt{b^2 + \eta(1 + b^2)}} \sin \sigma \tag{63}
\]

\[
\theta \approx \theta_0 - \frac{1}{\sqrt{b^2 + \eta(1 + b^2)}} \cos \sigma \tag{64}
\]

and since $\eta$ is very large the amplitudes of $\rho$ and $\theta$ are very small. Again, when $b = 0$, we find the solution of [12, 14, 16]

\[
\rho \approx \frac{1}{\sqrt{\eta}} \sin \sigma \tag{65}
\]

We conclude this Section with the following remark. One can choose the B field to be oriented along $(\theta, \phi)$ plane and depending on $\rho$. If we impose the condition of constancy of the field strength, we end up with

\[
B_{\theta\phi} = b\rho(\sigma). \tag{66}
\]

The eqs. of motion in this case are slightly modified

\[
\rho'' - b\omega \theta' = (\kappa^2 - \omega^2) \sinh \rho \cosh \rho \tag{67}
\]

\[
\theta'' + b\omega \rho' = 0 \tag{68}
\]

with the same constraint

\[
(r')^2 + (\theta')^2 = \kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho. \tag{69}
\]

Eliminating $\theta$ from (69) we arrive at the same expression for $\rho'$ as in (39)

\[
(r')^2 + b^2 \omega^2 \rho^2 = \kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho \tag{70}
\]

From here on the analysis proceeds in the very same way as in the above and with the same conclusions.

### 4 Quadratic fluctuations

In this Section we would like to study the quadratic fluctuations around the classical solutions (24, 25). The study of the fluctuations around given particular string configuration means semiclassical approximation of the theory
and therefore it allows us to study the leading quantum corrections to the energy spectrum.

Let us start with some string configuration $\bar{X}^\mu$ satisfying the classical equations of motion. A standard method for studying small fluctuations around $\bar{X}^\mu$ is the use of Riemann normal coordinates. Our starting point will be the action:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left\{ \sqrt{g} \alpha^{\alpha\beta} G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu + \varepsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \right\}$$  \hspace{1cm} (71)

The expansion of the entries of (71) is given by (see for instance [26])

$$\partial_\alpha X^\mu = \partial_\alpha \bar{X}^\mu + D_\alpha \xi^\mu + \frac{1}{3} R^\mu_{\lambda\nu\kappa} \xi^\lambda \xi^\kappa \partial_\alpha \bar{X}^\nu + \cdots$$

$$G_{\mu\nu}(X) = G_{\mu\nu}(\bar{X}) - \frac{1}{3} R_{\mu\nu\rho\lambda} \xi^\rho \xi^\lambda + \cdots$$

$$B_{\mu\nu}(X) = B_{\mu\nu}(\bar{X}) + D_\rho B_{\mu\nu}(\bar{X}) \xi^\rho + \frac{1}{2} D_\lambda D_\rho B_{\mu\nu}(\bar{X}) \xi^\lambda \xi^\rho - \frac{1}{6} R^\lambda_{\rho\mu\nu}(\bar{X}) B_{\lambda\nu}(\bar{X}) \xi^\rho \xi^\kappa + \frac{1}{6} R^\lambda_{\rho\nu\kappa}(\bar{X}) B_{\lambda\mu}(\bar{X}) \xi^\rho \xi^\kappa + \cdots$$

where

$$D_\alpha \xi^\mu = \partial_\alpha \xi^\mu + \Gamma^\mu_{\lambda\nu} \xi^\lambda \partial_\alpha \bar{X}^\nu$$

and $R^\mu_{\nu\rho\lambda}$ is the usual Riemann tensor for the target space. Now we can use the above expansion to obtain the quadratic part of (71)

$$S^{(2)} = S_G^{(2)} + S_B^{(2)}$$

where

$$S_G^{(2)} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g} \alpha^{\alpha\beta} \left[ G_{\mu\nu}(\bar{X}) D_\alpha \xi^\mu D_\beta \xi^\nu + A_{\mu\nu;\alpha\beta} \xi^\mu \xi^\nu \right]$$  \hspace{1cm} (72)

$$S_B^{(2)} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \varepsilon^{\alpha\beta} \left[ \partial_\alpha \bar{X}^\delta H_{\delta\mu\nu} \xi^\nu D_\beta \xi^\mu + \frac{1}{2} D_\lambda H_{\delta\mu\nu} \xi^\delta \xi^\tau \partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}^\tau \right]$$  \hspace{1cm} (73)

The explicit form of $A_{\mu\nu;\alpha\beta}$ is given by$^5$

$$A_{\mu\nu;\alpha\beta} = R_{\mu\nu\rho\kappa}(\bar{X}) \partial_\alpha \bar{X}^\delta \partial_\beta \bar{X}^\kappa$$

$^5$The expressions are analogous to those in [27] for the open superstring and the mass term $A_{\mu\nu;\alpha\beta}$ term was also obtained there.
One can rewrite (72) in Lorentz frame by using vielbeins defined by

\[ G_{\mu\nu} = E^A_{\mu} E^B_{\nu} \eta_{AB} \]
\[ \xi^A = E^A_{\mu} \xi^{\mu}. \]

After some straightforward calculations the fluctuation part \( S_G^{(2)} \) is found to be

\[ S_G^{(2)} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left[ \mathcal{D}_\alpha \xi^A \mathcal{D}_\beta \xi^B + (g^{\alpha\beta} R_{ACBD} Y^C_{\alpha} Y^D_{\beta}) \xi^A \xi^B \right] \]

where

\[ \mathcal{D}_\alpha \xi^A = \partial_\alpha \xi^A + \omega^{AB}_\alpha \xi^B \]
\[ Y^A_{\alpha} = E^A_{\mu} \partial_\alpha \bar{X}^\mu \]

and the projected spin connection is given by

\[ \omega^A_{\alpha} = E^A_{\mu} \left( \partial_\alpha E^\nu_{\beta} + \Gamma^\nu_{\mu\lambda} E^\lambda_B \right) \]
\[ \omega^{AB}_\alpha = \omega^{AB}_{\mu} \partial_\alpha \bar{X}^\mu. \]

It is straightforward to show that the only non-vanishing projected spin connection coefficients are\(^6\)

\[ \omega^0_{01} = \kappa \sinh \rho, \quad \omega^2_{01} = \omega \cosh \rho. \]

and it is also useful to introduce projected vielbeins

\[ e^A_{\alpha} = E^A_{\mu} \partial_\alpha \bar{X}^\mu \]

\[ e^0_0 = \kappa \cosh \rho; \quad e^1_1 = \rho'; \quad e^2_0 = \omega \sinh \rho; \quad e^3_0 = \theta'; \]

In these notations the massive term for the non-trivial \( AdS \) part becomes

\[ A_{AB} = g^{\alpha\beta} \eta_{CD} e^C_{\alpha} e^D_{\beta} \eta^{AB} - g^{\alpha\beta} e^A_{\alpha} e^B_{\beta} \]

\(^6\)The spin connection coefficients and the projected vielbeins were also used in [14].
We will use the fact that the induced metric is conformally flat to replace \( g_{\alpha\beta} \) by \( \eta_{\alpha\beta} \). The first term in (79) then becomes
\[
\eta^{\alpha\beta} \eta_{CDE} \epsilon^C_\alpha \epsilon^D_\beta \eta^{AB} = 2 \left( \rho' + \theta' \right)^2 \eta^{AB}.
\]
The total mass term then becomes
\[
A_{AB} \xi^A \xi^B = \left( \rho' \xi^0 - \theta' \xi^3 \right)^2 + 2 \left( \rho' + \theta' \right) \left( -\xi^0 \xi^2 + \xi^2 \right) + \left( \kappa \cosh \rho \xi^0 - \omega \sinh \rho \xi^2 \right)^2 + \left( \rho' \xi^1 - \theta' \xi^3 \right)^2.
\] (80)

Now we give the final expression for the quadratic fluctuations of \( S_G^{(2)} \):
\[
S_G^{(2)} = -\frac{1}{4\pi\alpha'} \int d^2 \sigma \left[ \left( \partial_0 \xi^0 + \kappa \sinh \rho \xi^1 \right)^2 - \left( \partial_0 \xi^1 + \kappa \sinh \rho \xi^0 - \omega \cosh \rho \xi^2 \right)^2 \right.
\]
\[
- \left( \partial_0 \xi^2 + \omega \cosh \rho \xi^1 \right)^2 - \left( \partial_1 \xi^0 \right)^2 + \left( \partial_1 \xi^1 \right)^2 + \left( \partial_1 \xi^2 \right)^2 + \left( \partial_1 \xi^3 \right)^2
\]
\[
+ \left( \rho' \xi^0 - \theta' \xi^3 \right)^2 + 2 \left( \rho' + \theta' \right) \left( -\xi^0 \xi^2 + \xi^2 \right) + \left( \kappa \cosh \rho \xi^0 - \omega \sinh \rho \xi^2 \right)^2 + \left( \rho' \xi^1 - \theta' \xi^3 \right)^2 \left. \right]
\] (81)

Now we proceed with the quadratic fluctuations for B-part of the action (73). In Lorentz frame it takes the form
\[
S_B^{(2)} = -\frac{1}{4\pi\alpha'} \int d^2 \sigma \varepsilon^{\alpha\beta} \left[ \left( \partial_0 \vec{X}^\lambda \delta_{\lambda}^\mu E^\nu E^\mu \right) \xi^A \partial_\alpha \xi^B \right.
\]
\[
+ \frac{1}{2} \left( D_\lambda \delta_{\lambda}^\mu \xi^A \partial_\alpha \vec{X}^\nu \partial_\nu \vec{X}^\mu \right) \xi^A \xi^B \left. \right]
\] (82)

Lengthy but straightforward calculations lead to the following final result for \( S_B^{(2)} \)
\[
S_B^{(2)} = -\frac{1}{4\pi\alpha'} \int d^2 \sigma \left[ \omega \left( \xi^\rho \partial_1 \xi^\phi - \xi^\phi \partial_1 \xi^\rho \right) + \rho' b \left( \xi^\rho \partial_0 \xi^\theta - \xi^\theta \partial_0 \xi^\rho \right) \right.
\]
\[
+ \left. b \theta' \left( \xi^\rho \partial_0 \xi^\phi - \xi^\phi \partial_0 \xi^\rho + \kappa \sinh \rho \xi^t \xi^\phi \right) \right]
\] (83)

The quadratic fluctuations for G-part of the action, \( S_G^{(2)} \), looks very similar to those obtained in [14, 16], but the additional part \( S_B^{(2)} \) adds new feature. Due to (83) the equations for different modes are coupled and therefore the presence of B-field will drastically change the energy spectrum. We will address the detailed study of this issue to future investigations.
5 Conclusions

In present paper we considered a rotating “string soliton” in the presence of NS-NS B-field. Using suitable choice of the string configuration and constant field strength for $B_{\mu\nu}$, we analyse (in the case of short strings) the classical values of the energy $E$ and the spin $S$ of the system. The obtained values for the constants of motion deviate smoothly from those obtained without B-field:

$$E = \frac{\sqrt{2S\sqrt{1+b^2}}}{\sqrt{1+2b^2\sqrt{1+b^2}S}} + \frac{\sqrt{2S(1+b^2)^{3/2}}}{\sqrt{1+2S(1+b^2)^{3/2}}}$$  \hspace{1cm} (84)

$$S \approx \frac{\omega}{2\kappa \xi \sqrt{\xi}}$$  \hspace{1cm} (85)

One can analyse two limiting cases for the value of the field strength $b$.

a) First case obviously is when $b$ is very small (compared to $S$). In this case the contribution of the B-field can be neglected and the smooth limit $b \to 0$ reduces to the case studied in [12, 14].

b) The second possibility is when $bS$ is very large (note that according to (56) $S$ is small). From (84) then one can find that

$$E = \frac{\sqrt{2S\sqrt{1+b^2}}}{\sqrt{1+2b^2\sqrt{1+b^2}S}} + \frac{\sqrt{2S(1+b^2)^{3/2}}}{\sqrt{1+2S(1+b^2)^{3/2}}} \propto 1 + \frac{1}{b} + \ldots$$  \hspace{1cm} (86)

and the dependence on $S$ drops out from the leading contributions.

In Section 4 we considered the fluctuations around the classical solutions of Section 3. The contributions due to the presence of NS-NS antisymmetric B-field couple the fluctuation modes in a non-trivial way. This indicates changes in the quantum corrections to the energy spectrum compared to the case without B-field.

This paper is incomplete in several ways. First of all we analysed the case of short strings only. Actually the long string case investigated in [12] leads to a more involved relation between the energy and the spin:

$$E = S + \frac{\sqrt{\lambda}}{\pi} \log(S/\sqrt{\lambda}) + \ldots$$
In our case the energy $E$ and the spin $S$ are defined by
\begin{align*}
E &= \frac{\kappa \sqrt{\lambda}}{2\pi} \int_0^{\rho_0} \frac{cosh^2 \rho d\rho}{\sqrt{\kappa^2 c^2 - \omega^2 sinh^2 \rho - b^2 \omega^2 \rho^2}} \\
S &= \frac{\omega \sqrt{\lambda}}{2\pi} \int_0^{\rho_0} \frac{sinh^2 \rho d\rho}{\sqrt{\kappa^2 c^2 - \omega^2 sinh^2 \rho - b^2 \omega^2 \rho^2}}
\end{align*}

Although we do not have explicit solution so far, one can speculate that since the integral is dominated by the contributions around the turning point $\sigma = \pi/2$, the behavior of $E$ and $S$ should be qualitatively the same. One can expect then that the relation between $E$ and $S$ will be approximately the same, however the impact of B-field on this speculation remain unclear and needs more rigorous study. We will return to this question in the near future.

The second question we did not consider in this paper is the fermionic part. This however is important for several reasons. Besides supersymmetry, we should note that the fermions are important also for divergence cancellations of the 2d (induced metric) curvature. Indeed, the contribution of the fluctuation modes to the logarithmic divergences is proportional to the trace of the mass matrix $A_{AB}$ (74) We expect that these divergences will be cancelled by the contributions coming from the fermions in the same way as in [14] and the theory then will remain conformally invariant. This point however needs a rigorous clarification which we leave for separate study.

The last, but maybe the most important question is the identification of the energy with the dimension of certain operators from SYM theory side. Since this is an important question we will return to this issue in the near future.

One more remark is in order. After suitable redefinitons, the equations for the fluctuations can be brought to a form very similar to those obtained in [28]. One can wonder if there is a deeper relation beyond the similarity between the case studied here and pp-wave limit of Pilch-Warner geometries studied in [28].

We hope that further studies of rotating strings in presence of NS-NS

\footnote{See for detailed study in the case of open superstrings [27]}
B-field will shed more light on AdS/CFT correspondence.

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