TRAVELLING WAVES IN THE BOUSSINESQ TYPE SYSTEMS

EVGENI DINVAY

Abstract. Considered herein are a number of variants of the Boussinesq type systems modelling surface water waves. Such equations were derived by different authors to describe the two-way propagation of long gravity waves. A question of existence of special solutions, the so called solitary waves, is of a particular interest. There are a number of studies relying on a variational approach and a concentration-compactness argument. These proofs are technically very demanding and may vary significantly from one system to another. Our approach is based on the implicit function theorem, which makes the treatment easier and more unified.

1. Introduction

Consideration is given to the following one-dimensional Boussinesq-type system

\[
\begin{aligned}
K_b \partial_t \eta + K_a \partial_x v + \partial_x (\eta v) &= 0, \\
K_d \partial_t v + K_c \partial_x \eta + \partial_x v^2/2 &= 0,
\end{aligned}
\]

where \(K_a, \ldots, K_d\) are Fourier multiplier operators in the space of tempered distributions \(S'(\mathbb{R})\). The space variable is \(x \in \mathbb{R}\) and the time variable is \(t \in \mathbb{R}\). The unknowns \(\eta, v\) are real valued functions of these variables, representing the free surface elevation and velocity at a different level in the fluid layer, respectively. A classical example is the so called \((a, b, c, d)\)-system with \(K_a = 1 + a\partial_x^2, K_b = 1 - b\partial_x^2, K_c = 1 + c\partial_x^2, K_d = 1 - d\partial_x^2\), introduced in [2]. Other examples with non-local operators will be given in Conclusion.

Abusing slightly notations and exploiting the travelling wave ansatz \(\eta(x, t) = \eta(x - \omega t), v(x, t) = v(x - \omega t)\), with \(\omega \in \mathbb{R}\), we rewrite System (1.1) in the form

\[
\begin{aligned}
-\omega K_b \eta + K_a v + \eta v &= 0, \\
-\omega K_d v + K_c \eta + v^2/2 &= 0.
\end{aligned}
\]

(1.2)

Note that we assume \(\eta, v\) vanishing at infinity. If \(v\) is known then obviously \(\eta\) can be found through the second equation as

\[\eta = \omega K_c^{-1} K_d v - K_c^{-1} v^2/2.\]

Thus the problem reduces to investigation of a single equation on \(v\) of the form

\[(\omega^2 - M^2) v = \omega F v^2 + \omega G(vHv) + T(v, v, v),\]

where \(F, G, H\) are Fourier multipliers and \(T\) is a trilinear bounded operator in some Sobolev space \(H^s_\varepsilon(\mathbb{R})\) of even functions. Assigning, in particular, \(M = \sqrt{K_a K_b^{-1} K_c K_d^{-1}}, F = K_d^{-1}/2, G = K_b^{-1} K_c K_d^{-1}, H = K_c^{-1} K_d\) and \(T(v, v, v) = K_b^{-1} K_c K_d^{-1}(vK_c^{-1} v^2/2)\), we arrive back to the initial problem (1.2). The general equation (1.3) is of main interest below.

Different particular versions of System (1.1) appeared in [1 2 7 11 13]. For some of them existence of solitary wave solutions was proved in [4 5 8 15]. These results are obtained by reformulation of (1.2) as a variational problem, different in each case, and by appealing to the concentration-compactness principle. The corresponding proofs are technically very demanding and vary significantly from one particular system to another. Another shortcoming of this approach is that uniqueness of minimizers needs in general additional studies. On the other hand such
treatment can sometimes be advantageous, as for example in [1], where also stability for the set of minimizers was obtained.

The variational approach has been extensively used to prove existence of solitary wave solutions to the single unidirectional travelling wave equation

$$\omega v = Mv + n(v)$$

as well, see [10] [19], for instance. An alternative elegant proof with an entirely different approach was given by Stefanov and Wright [18]. They have rescaled the travelling wave equation (1.4) introducing a small parameter that lead in the limit to the KdV travelling wave equation. Now existence together with uniqueness come from appeal to the implicit function theorem. In the current work we extend their approach to Equation (1.3). Note that it provides us with a unified treatment for all systems (1.1), including those of them that have not been proved possessing solitary waves so far. We regard concrete examples in Conclusion. One can also add, that the implicit function argument normally gives small solutions, whereas the variational method may produce big ones as well.

2. Notations, Assumptions and Main result

We start this section by recalling all the necessary standard notations. For any positive numbers $a$ and $b$ we write $a \lesssim b$ if there exists a constant $C$ independent of $a,b$ such that $a \leq Cb$. The Fourier transform is defined by the usual formula

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int f(x)e^{-i\xi x}dx$$
on Schwartz functions. By the Fourier multiplier operator $\varphi(D)$ with symbol $\varphi$ we mean the line $\mathcal{F}(\varphi(D)f) = \varphi(\xi)\hat{f}(\xi).$ In particular, $D = -i\partial_x$ is the Fourier multiplier associated with the symbol $\varphi(\xi) = \xi$. For any $\alpha \in \mathbb{R}$ the Bessel potential of order $-\alpha$ is the Fourier operator $J^\alpha = (D)^\alpha$, where we exploit the symbol notation $J(\xi) = (\xi) = \sqrt{1 + \xi^2}$. Note that below we stick to this abuse of notations, namely we notate Fourier multipliers and their symbols by the same letters, for example, $F$ appeared as an operator in (1.3) can stand for the corresponding symbol $F(\xi)$ in other context. The $L^2$-based Sobolev space $H^\alpha(\mathbb{R})$ is defined by the norm $\|f\|_{H^\alpha} = \|J^\alpha f\|_{L^2}$, whereas $H^\alpha_0(\mathbb{R})$ is its subspace of even functions.

In order to study Equation (1.3) we will need to rescale it by a scalar parameter $\varepsilon \in \mathbb{R} \setminus \{0\}$. For any function $\varphi : \mathbb{R} \to \mathbb{R}$ we use notation $\varphi_\varepsilon(x) = \varphi(\varepsilon x).$ For a Fourier multiplier $F$ by $F_\varepsilon$ we mean the operator with the symbol $F_\varepsilon(\xi) = F(\varepsilon \xi).$ Notation $F_0$ stands for the constant $F(0).$ In general, for any $n$-linear operator $B$ we define operator $B_\varepsilon$ by the line

$$B_\varepsilon(f^1,\ldots, f^n) = (B(f^1_\varepsilon,\ldots, f^n_\varepsilon))_\frac{1}{\varepsilon}, \quad \varepsilon \neq 0.$$

We deliberately use the same notation for rescaled general operators and Fourier multipliers, since obviously rescaling of a Fourier multiplier (by rescaling the symbol) coincides with the composition of rescaling on the physical side by $\varepsilon$ and $1/\varepsilon$.

Solvability of (1.3) is investigated by approximation with the Korteweg–de Vries equation

$$-\partial_x^2 v + v - \gamma v^2 = 0, \quad (2.1)$$

where $\gamma \neq 0$ is to be given below. It has a unique solution $v = \sigma$ with

$$\sigma(x) = \frac{3}{2\gamma} \text{sech}^2 \left(\frac{x}{2}\right) \quad \text{for } \gamma \neq 0. \quad (2.2)$$

Assumption 1. The multiplier symbols $M, F, G, H : \mathbb{R} \to \mathbb{R}$ are even and there exist $\xi_1 > 0$ and $\beta < 1$ such that

1. $M \in C^{0,1}(\mathbb{R}|[-\xi_1,\xi_1])$ that is, it has Lipschitz continuous third derivative, with

$$m_1 = \sup_{|\xi| < \xi_1} M(\xi) < M(0), \quad (2.3)$$
\[ m_2 = \sup_{|\xi| \leq \xi_1} M''(\xi) < 0; \quad (2.4) \]

(2) \( M \) is bounded from below as
\[ m_3 = \inf_{\xi \in \mathbb{R}} (M(\xi) + M(0)) > 0; \quad (2.5) \]

(3) there exist derivatives \( M', F', G', H' \) on \( \mathbb{R} \) satisfying
\[ |M'(\xi)|, |F'(\xi)|, |G'(\xi)|, |H'(\xi)| \lesssim (\xi)^\beta; \quad (2.6) \]

(4) there exist bounded derivatives \( F'', G'', H'' \) on \([-\xi_1, \xi_1]\);

(5) the multiplier symbol \( F \) is bounded on \( \mathbb{R} \);

(6) \( G \) is smoothing \( H \), that is there exists \( s_h \geq 0 \) satisfying
\[ |G(\xi)| \lesssim (\xi)^{-s_h}, \quad |H(\xi)| \lesssim (\xi)^{s_h}; \quad (2.7) \]

(7) there is a relation between \( F_0, G_0 \) and \( H_0 \) of the form
\[ \gamma = -\frac{1}{M''(0)} [F(0) + G(0)H(0)] \neq 0. \]

Operators \( M, F, G, H \) are defined in subspaces of even functions, due to evenness of the corresponding symbols. One can easily check that there is no loss of generality in assuming that \( \xi_1 > 0 \) and \( \beta < 1 \) are unique for all symbols \( M, F, G, H \), since one can always take the minimum such \( \xi_1 \) and the maximum such \( \beta \). Clearly, \( s_h \leq 1 + \beta < 2 \). Note that for \( s > 1/2 \) and \( s \geq s_h/2 \) the bilinear part in Equation (1.3) is bounded in \( H^s(\mathbb{R}) \times H^s(\mathbb{R}) \), as will be shown below.

**Assumption 2.** Let \( s \in \mathbb{R} \) be such that \( T \) is a trilinear bounded operator in \( H^s(\mathbb{R}) \) and for some \( s_t < 2 \) it holds true that
\[ \| T_\varepsilon \| \lesssim |\varepsilon|^{-s_t}, \quad \| T_\varepsilon(\sigma, \sigma, \sigma) \|_{H^s} = o\left( \frac{1}{\varepsilon} \right) \]
as \( \varepsilon \to 0 \). Here \( \sigma \) is defined in (2.2).

Both inequalities are easy to check for the concrete examples regarded in the last section. Note that \( \| T_\varepsilon \| \leq |\varepsilon|^{-s-1} \| T \| \) for \( 0 < |\varepsilon| \leq 1 \). The first inequality is guaranteed automatically provided \( s < 1 \), and the second one provided \( s < 0 \).

In the next theorem we prove existence of small travelling waves corresponding to velocities slightly above \( M(0) \). We introduce the wave speed parametrised by \( \varepsilon \in \mathbb{R} \) as follows
\[ \omega_\varepsilon = M(0) - \frac{1}{2} M''(0) \varepsilon^2. \quad (2.8) \]

**Theorem 1.** Let Assumptions 1 and 2 be met with \( s \geq 1 \). Then there exists \( \varepsilon_0 > 0 \), so that for every \( \varepsilon \in (0, \varepsilon_0) \), there is a unique travelling wave solution to Equation (1.3), associated with the phase velocity \( \omega = \omega_\varepsilon \) given by (2.8), of the form \( v(x) = \varepsilon^s V^{\varepsilon}(\varepsilon x) \) with \( V^{\varepsilon} \in H^s(\mathbb{R}) \) satisfying
\[ V^{\varepsilon} = \sigma + o_{H^s}(\varepsilon). \]

3. Preliminaries

We state an estimate, firstly appeared in [14] in a weaker form, and later sharpened in [17].

**Lemma 1.** Suppose \( a, b, c \in \mathbb{R} \). Then for any \( f \in H^a(\mathbb{R}), g \in H^b(\mathbb{R}) \) and \( h \in H^c(\mathbb{R}) \) the following inequality holds
\[ \|fgh\|_{L^1} \lesssim \|f\|_{H^a} \|g\|_{H^b} \|h\|_{H^c} \quad (3.1) \]
provided that
\[ a + b + c > \frac{1}{2}, \quad a + b \geq 0, \quad a + c \geq 0, \quad b + c \geq 0. \]

An immediate corollary of this Lemma and duality argument is the following.
Lemma 2. Let \( \varphi \) be a real-valued even function defined on \( \mathbb{R} \) such that

1. it has derivative \( \varphi' \) satisfying \( |\varphi'(\xi)| \leq |\xi|^{\beta} \) for some \( \beta < 1 \) and every \( \xi \in \mathbb{R} \);
2. it has a bounded second derivative \( \varphi'' \) on some interval \((-\xi_1, \xi_1)\) around zero.

Then

\[
\| J^{-2}(\varphi(D) - \varphi(0)) f \|_{H^s} \lesssim |\varepsilon|^{\frac{3-2\beta}{2}} \| f \|_{H^s}
\]

for any \( f \in H^s(\mathbb{R}) \) provided \( |\varepsilon| \leq 1 \).

Proof. Let \( R > 0 \) be large to be chosen below dependent on \( \varepsilon \). One can assume that \( \varepsilon > 0 \) in addition. We calculate the square of norm by integrating separately over the ball \( B_R = (-R, R) \) and its complement \( B_R^c \) as follows

\[
N_f^2 = \left\| J^{-2} \frac{\varphi(D) - \varphi(0)}{\varepsilon} f \right\|_{H^s}^2 = \int |\varphi'(\xi)|^2 \left| \frac{\theta(\varepsilon \xi)}{\varepsilon} \right|^2 \| f \|_{H^s}^2 d\xi
\]

\[
\leq \sup_{\xi \in B_R} |\varphi'|^2 \int_{B_R} \left| \frac{\theta(\varepsilon \xi)}{\varepsilon} \right|^2 \| f \|_{H^s}^2 d\xi + C \int_{B_R^c} \| \theta(\varepsilon \xi) \|_{H^s}^2 \| f \|_{H^s}^2 d\xi.
\]

Clearly, one can take \( R \) in a way so that \( \varepsilon R \leq \xi_1 \), and so

\[
\sup_{\xi \in B_R} |\varphi'| = \sup_{|\xi| \leq \varepsilon R} |\varphi'(\xi)| - \varphi'(0) \leq \varepsilon R \| \varphi'' \|_{L^\infty(-\xi_1, \xi_1)}.
\]

To estimate the second integral notice that for \( |\xi| \geq R \) we have

\[
|\theta(\varepsilon \xi) - M(\varepsilon \xi)|^2 \lesssim |\varepsilon x|^{2\beta} \| \xi \|^{2\beta - 2} \lesssim R^{2\beta - 2}.
\]

Thus \( N_f \lesssim (\varepsilon R + R^{3-1}) \| f \|_{H^s} \), and so it is left to assign \( R = \varepsilon^{-1/(2-\beta)} \xi_1 \) in order to conclude with the proof. \( \square \)

This Lemma is used below to approximate the operators \( F_{\varepsilon}, G_{\varepsilon} \) and \( H_{\varepsilon} \), whereas the operator \( M_{\varepsilon} \) is approximated with the help of the following two results.

Corollary 3. Given Assumption 1, for any \( f \in H^s(\mathbb{R}) \) and \( |\varepsilon| \leq 1 \) it holds true that

\[
\left\| J^{-2} \left( \frac{1}{\omega_{\varepsilon} + M_{\varepsilon}} - \frac{1}{2M_0} \right) f \right\|_{H^s} \lesssim |\varepsilon|^{\frac{3-2\beta}{2-\beta}} \| f \|_{H^s}
\]

Proof. For any \( \xi \in \mathbb{R} \) we have

\[
\left| \frac{1}{\omega_{\varepsilon} + M(\varepsilon \xi)} - \frac{1}{2M_0} \right| \leq \frac{|\omega_{\varepsilon} - M_0 + M(\varepsilon \xi) - M_0|}{2M_0(\omega_{\varepsilon} + M(\varepsilon \xi))} \leq \frac{1}{2|M_0|3} \left( \frac{1}{2} |M''(0)| \varepsilon^2 + |M(\varepsilon \xi) - M_0| \right),
\]

which after implementation of Lemma 2 finishes the proof. \( \square \)

Lemma 3. Let \( M \) satisfy Assumption 1 Then there is \( C > 0 \) such that for any \( \varepsilon \neq 0 \) it holds that

\[
\sup_{\xi \in \mathbb{R}} \left| \frac{\varepsilon^2}{M(0) - \frac{1}{2} M''(0) \varepsilon^2} - M(\varepsilon \xi) + \frac{2}{M''(0)(1 + \xi^2)} \right| \leq C \varepsilon^2.
\]
A proof is given in [18]. This lemma implies that there exists the inverse bounded operator
\[ \varepsilon^2 (\omega_{\varepsilon} - M_{\varepsilon})^{-1} = \frac{2}{M''(0)}(1 - \partial_x^2)^{-1} + O_B(H^s)(\varepsilon^2) \] (3.2)
as \( \varepsilon \to 0 \). We provide the following lemma with a complete proof, though the idea can be found in
[12, 18], for example.

**Lemma 4.** Let \( s \geq 0, \gamma \neq 0 \) and \( \sigma(x) \) be defined by (2.2). Then the operator
\[ K = 1 - 2\gamma J^{-2}(\sigma) \] (3.3)
is bounded and has a bounded inverse in \( H^s_+ (\mathbb{R}) \).

**Proof.** The fact that \( K \in B(H^s_+ (\mathbb{R})) \) is obvious. By the bounded inverse theorem it is enough to prove that \( K \) is invertible. Firstly, we regard the case \( 0 \leq s < 3/2 \). The composition \( (\xi)^{-2}F \sigma \) is a Hilbert-Schmidt operator from \( L^2(\mathbb{R}, dx) \) to \( L^2(\mathbb{R}, (\xi)^{2s}d\xi) \). Then from continuity of the injection \( H^s(\mathbb{R}) \to L^2(\mathbb{R}) \) and of the inverse Fourier transform \( F^{-1} : L^2(\mathbb{R}, (\xi)^{2s}d\xi) \to H^s(\mathbb{R}) \) we deduce that \( K - 1 \) is compact in \( H^s(\mathbb{R}) \). Now let us assume that \( K \) is not invertible in the subspace of even functions. Then by the Fredholm alternative there is a non-trivial \( f_0 \in \ker K \subset H^s_+ (\mathbb{R}) \), and so \( f_0 = 2\gamma J^{-2}(\sigma f_0) \). It implies that \( f_0 \in H^s_+ (\mathbb{R}) \) and \( (-\partial_x^2 + 1 - 2\gamma \sigma) f_0 = 0 \). However, it is known about the last operator that its kernel is spanned by \( \sigma' \). This implies that \( f_0 \) is odd, which is a contradiction. Thus \( K \) is invertible.

Finally, regarding the case \( s \geq 3/2 \), for any \( g \in H^s_+ (\mathbb{R}) \) one can find a unique \( f \in H^1_+ (\mathbb{R}) \) such that \( Kf = g \). Hence \( f = 2\gamma J^{-2}(\sigma f) + g \) implying \( f \in H^\text{min}(3,s) (\mathbb{R}) \). After several repetitions we arrive to \( f \in H^s_+ (\mathbb{R}) \), which proves invertibility of \( K \).

## 4. **Proof**

We start by introducing a mapping from \( H^s_+ (\mathbb{R}) \times \mathbb{R} \) to \( H^s_+ (\mathbb{R}) \) as follows. For \( \varepsilon \neq 0 \) we define
\[ \Phi(v, \varepsilon) = v - \varepsilon^2 (\omega_{\varepsilon}^2 - M_{\varepsilon}^2)^{-1} \left[ \omega_{\varepsilon} F_{\varepsilon} v_2 + \omega_{\varepsilon} G_{\varepsilon}(vH_{\varepsilon}v) + \varepsilon^2 T_{\varepsilon}(v, v, v) \right] \] (4.1)
and \( \Phi(v, 0) \) is defined as a limit when \( \varepsilon \to 0 \) provided it exists. Firstly, we analyse \( \Phi \) around the hyperplane \( \varepsilon = 0 \) and prove that \( \Phi \) is continuous at \( (\sigma, 0) \) with \( \Phi(\sigma, 0) = 0 \). In order to simplify the presentation we restrict the domain of \( \Phi \) to \( B_1(\sigma) \times (-1, 1) \). Due to (3.2) we need to consider an expression of the form
\[ J^{-2} G_{\varepsilon}(vH_{\varepsilon}v) = G_0 H_0 J^{-2} v^2 + H_0 J^{-2}(G_{\varepsilon} - G_0) v^2 + G_{\varepsilon}(J^{-2} v^2) J^{-2} (H_{\varepsilon} - H_0) v. \]
Applying Lemma 2 and Corollary 1 we obtain
\[ J^{-2} G_{\varepsilon}(vH_{\varepsilon}v) = G_0 H_0 J^{-2} v^2 + O_{H^s}(\|v\|_{H^s}^{3/2 - \beta}). \]
Similarly, by Lemma 2 we estimate
\[ J^{-2} F_{\varepsilon} v^2 = F_0 J^{-2} v^2 + O_{H^s}(\|v\|_{H^s}^{3/2 - \beta}). \]
Finally, making use of Corollary 3 we obtain
\[ J^{-2} (\omega_{\varepsilon} + M_{\varepsilon})^{-1} \left[ \omega_{\varepsilon} F_{\varepsilon} v_2 + \omega_{\varepsilon} G_{\varepsilon}(vH_{\varepsilon}v) \right] = \frac{1}{2} [F_0 + G_0 H_0] J^{-2} v^2 + O_{H^s}(\|v\|_{H^s}^{3/2 - \beta}) \]
which together with (4.1) and (3.2) lead, as we show below, to
\[ \Phi(v, \varepsilon) = v - \gamma J^{-2} v^2 + o_{H^s}(\varepsilon) + O_{H^s}(\|v\|_{H^s}^{2 - \max(s_h, s_t)}) \varepsilon^2 ||v - \sigma||_{H^s} \] (4.2)
Indeed, as one can easily see \( ||H_{\varepsilon} \sigma||_{H^s} \leq C ||J_{\varepsilon}^h(\sigma)||_{H^s} \leq C ||\sigma||_{H^{s+h} + s_h} \), and so
\[ \varepsilon^2 G_{\varepsilon}(vH_{\varepsilon}v) = \varepsilon^2 G_{\varepsilon}(vH_{\varepsilon} \sigma) + \varepsilon^2 G_{\varepsilon}(vH_{\varepsilon}(v - \sigma)) = O_{H^s}(\varepsilon^2 + ||v||_{H^s}^2 ||v - \sigma||_{H^s}) \]
Appealing to Assumption 2 one obtains
\[ \varepsilon^2 T_{\varepsilon}(v, v, v) = \varepsilon^2 T_{\varepsilon}(\sigma, \sigma, \sigma) + O_{H^s}(\|v - \sigma\|^2_{H^s}) = o_{H^s}(\varepsilon) + O_{H^s}(\|v - \sigma\|^2_{H^s}) \]
which finishes the proof of (4.2).

The first conclusion that we make from Equality (4.2) is that \( \Phi(v, 0) \) is well defined as a limit, namely
\[
\Phi(v, 0) = v - \gamma J^{-2} v^2,
\]
and in particular \( \Phi(\sigma, 0) = 0 \). Secondly, we conclude that
\[
\Phi(v, \varepsilon) = K(v - \sigma) + O_{H^s} \left( \|v - \sigma\|^2_{H^s} \right) + o_{H^s}(\varepsilon) + O_{H^s} \left( |\varepsilon|^{2 - \max(s_h, s_t)} \|v - \sigma\|_{H^s} \right)
\]
with \( K \) defined in Lemma 4, which means in particular that \( \Phi \) is differentiable at the point \((\sigma, 0)\).

In order to appeal to the implicit function theorem the following assumptions should be fulfilled

1. \( \Phi \) is continuous at the point \((\sigma, 0)\);
2. \( \Phi(\sigma, 0) = 0 \);
3. there exists \( \partial_v \Phi \) on the domain of \( \Phi \) continuous at the point \((\sigma, 0)\);
4. there exists \((\partial_v \Phi(\sigma, 0))^{-1} = K^{-1} \) bounded in \( H^s_\varepsilon(\mathbb{R}) \).

Note that the last assertion is proved in Lemma 4 and so it is left only to check the third statement.

A straightforward calculation for any \( \varepsilon \neq 0 \) gives
\[
\partial_v \Phi(v, \varepsilon)w = w - \varepsilon^2 \left( \omega_\varepsilon^2 - M_\varepsilon^2 \right)^{-1} \left[ 2 \omega_\varepsilon F_\varepsilon(vw) + \omega_\varepsilon G_\varepsilon(wH_\varepsilon v + vH_\varepsilon w) \right.
\]
\[
+ \varepsilon^2 \left( T_\varepsilon(w, v, v) + T_\varepsilon(v, w, v) + T_\varepsilon(v, v, w) \right) \]
and otherwise
\[
\partial_v \Phi(v, 0)w = w - 2\gamma J^{-2}(vw)
\]
for any \( w \in H^s_\varepsilon(\mathbb{R}) \). Repeating the analysis given above one can obtain
\[
\partial_v \Phi(v, \varepsilon) - K = O_{H^s} \left( \|v - \sigma\|^3_{H^s} + \|v\|_{H^s}^{3 - 2\beta} + |\varepsilon|^{2 - \max(s_h, s_t)} \right)
\]
as \((v, \varepsilon) \to (\sigma, 0)\) in \( H^s_\varepsilon(\mathbb{R}) \times \mathbb{R} \). Thus by the implicit function theorem there exist \( \varepsilon_0 > 0 \) and a unique \( V^\varepsilon \) defined for \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \) and continuous at \( \varepsilon = 0 \) such that \( \Phi(V^\varepsilon, \varepsilon) = 0 \) for each \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \).

In order to obtain the asymptotic expression we make use of (4.2) again to deduce
\[
V^\varepsilon - \sigma - \gamma K^{-1} J^{-2}(V^\varepsilon - \sigma)^2 = o_{H^s}(\varepsilon) + O_{H^s} \left( |\varepsilon|^{2 - \max(s_h, s_t)} \|V^\varepsilon - \sigma\|_{H^s} \right)
\]
and so
\[
V^\varepsilon - \sigma = o_{H^s}(\varepsilon) + O_{H^s} \left( |\varepsilon|^{2 - \max(s_h, s_t)} \|V^\varepsilon - \sigma\|_{H^s} \right),
\]
which proves the asymptotic expression and concludes the proof of Theorem 1.

5. Conclusion

5.1. \((a, b, c, d)\)-Boussinesq systems. Setting \( K_a = 1 + a\partial_\xi^2, K_b = 1 - b\partial_\xi^2, K_c = 1 + c\partial_\xi^2, K_d = 1 - d\partial_\xi^2 \) in (1.1), where \( a, b, c, d \in \mathbb{R} \), we arrive to a system derived in [2]. Its Cauchy problem was studied in [3]. It exhibits solitary wave solutions, as was shown in [4] by the variational method. Using Theorem 1 we can extend significantly their result at least for small surface tension, namely, to a not necessary Hamiltonian case corresponding to \( b = d \). Indeed, let us shorten the range of coefficients \( a, b, c, d \) in a way that System (1.1) is well posed (see [3]) and \( K_c \) is invertible, so that we obtain the first restriction
\[
b, d \geq 0, \quad a, c \leq 0.
\]
Analyzing the symbol \( M \) around zero
\[
M(\xi) = \sqrt{\frac{(1 - a\xi^2)(1 - c\xi^2)}{(1 + b\xi^2)(1 + d\xi^2)}} = 1 - \frac{1}{2}(a + b + c + d)\xi^2 + O(\xi^4)
\]
and taking into account (2.4) we arrive to the second necessary restriction
\[
a + b + c + d > 0.
\]
Condition \((2.3)\) implies either
\[ bd > ac \text{ or } bd = ac = 0. \tag{5.3} \]
Finally, the fact that the loss of derivative \(s_h < 2\), associated with \(H(\xi) = (1 + d\xi^2)/(1 - c\xi^2)\), leads to either
\[ c < 0 \text{ or } d = 0. \tag{5.4} \]
One claims that Conditions \((5.1)-(5.4)\) are sufficient to fulfil both Assumptions \([1, 2]\) and so to prove the existence and uniqueness of small solitary waves. Indeed,
\[ T_\varepsilon(f, g, h) = \frac{1}{2} G_\varepsilon \left( f (1 - c\varepsilon^2 D^2)^{-1} (gh) \right), \]
where \(G_\varepsilon = (1 + b\varepsilon^2 D^2)^{-1} (1 - c\varepsilon^2 D^2) (1 + d\varepsilon^2 D^2)^{-1}\) is uniformly bounded due to \((5.1), (5.2)\) and so \(\|T_\varepsilon\| \lesssim 1\), which implies Assumption \([2]\). The rest is obvious.

5.2. Aceves-Sánchez-Minzoni-Panayotaros model. Let us introduce a notation
\[ K = \frac{\tanh D}{D} \tag{5.5} \]
that we use in subsequent. Assigning \(K_a = K\) and \(K_b = K_c = K_d = 1\) we arrive to a system introduced in \([1]\). Its Cauchy problem was studied in \([16]\). It exhibits solitary wave solutions, as was shown in \([15]\) by the variational method. Note that \(M = \sqrt{K}\), \(F = 1/2\), \(G = H = 1\), and \(T(f, g, h) = fgh/2\). Thus we immediately obtain solitary waves by Theorem \([1]\).

5.3. Hur-Pandey model. Assigning \(K_c = K\) and \(K_a = K_b = K_d = 1\) we arrive to a system introduced in \([13]\). Note that \(M = \sqrt{K}\), \(F = 1/2\), \(G = K\), \(H = K^{-1}\), and \(T(f, g, h) = K (fK^{-1}(gh)) / 2\). Assumption \([1]\) is straightforward to check. In order to proceed note that
\[ T_\varepsilon(f, g, h) = \frac{1}{2} G_\varepsilon(f H_\varepsilon(gh)) \]
and so appealing to Corollary \([2]\) one obtains Assumption \([2]\) with \(s_t = s_h = 1\). Thus Theorem \([1]\) is applicable again.

5.4. Dinvay-Dutykh-Kalisch model. Setting eventually \(K_c = 1\) and \(K_a = K_b = K_d = K^{-1}\) we arrive to a system firstly appeared in \([7]\). It was proved later to be globally well-posed in \([9]\). It also admits existence of solitary waves \([8]\) by the variational method. It demonstrates a good performance even in simulations of big waves \([6, 7]\). Let us show that Theorem \([1]\) is applicable. We have \(M = \sqrt{K}\), \(F = K/2\), \(G = K^2\), \(H = K^{-1}\), and \(T(f, g, h) = K^2(fgh)/2\). Assumptions \([1]\) and \([2]\) are obviously satisfied.

As a final remark let us point out that Theorem \([1]\) can be applied to a more general Whitham-Boussinesq type system as one introduced in Remark 5.3 of \([11]\). It is possible to show that a linearisation of System \((1.1)\) around a solitary wave solution leads to an operator having both positive and negative essential spectrum. This is a serious difficulty, so we leave investigation of stability to a future work.

Acknowledgments. The author acknowledges the support of the ERC EU project 856408-STUOD.

References

[1] Aceves-Sánchez, P., Minzoni, A., and Panayotaros, P. Numerical study of a nonlocal model for water-waves with variable depth. Wave Motion 50, 1 (2013), 80–93.
[2] Bona, J. L., Chen, M., and Saut, J.-C. Boussinesq Equations and Other Systems for Small-Amplitude Long Waves in Nonlinear Dispersive Media. I: Derivation and Linear Theory. Journal of Nonlinear Science 12, 4 (Aug 2002), 283–318.
[3] Bona, J. L., Chen, M., and Saut, J.-C. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media: II. The nonlinear theory. Nonlinearity 17, 3 (feb 2004), 925–952.
[4] Chen, M., Nguyen, N. V., and Sun, S.-M. Solitary-wave solutions to Boussinesq systems with large surface tension. Discrete & Continuous Dynamical Systems 26, 4 (2010), 1153–1184.
[5] Chen, M., Nguyen, N. V., and Sun, S.-M. Existence of traveling-wave solutions to Boussinesq systems. Differential Integral Equations 24, 9/10 (09 2011), 895–908.
Dinvay, E. On well-posedness of a dispersive system of the Whitham–Boussinesq type. *Applied Mathematics Letters* 88 (2019), 13–20.

Dinvay, E., Dutykh, D., and Kalisch, H. A comparative study of bi-directional Whitham systems. *Applied Numerical Mathematics* 141 (2019), 248–262. Nonlinear Waves: Computation and Theory-X.

Dinvay, E., and Nilsson, D. Solitary wave solutions of a whitham–boussinesq system. *Nonlinear Analysis: Real World Applications* 60 (2021), 103280.

Dinvay, E., Selberg, S., and Tesfahun, A. Well-Posedness for a Dispersive System of the Whitham–Boussinesq Type. *SIAM Journal on Mathematical Analysis* 52, 3 (2020), 2353–2382.

Ehrnström, M., Groves, M. D., and Wahlén, E. On the existence and stability of solitary-wave solutions to a class of evolution equations of Whitham type. *Nonlinearity* 25, 10 (sep 2012), 2903–2936.

Emerald, L. Rigorous Derivation from the Water Waves Equations of Some Full Dispersion Shallow Water Models. *SIAM Journal on Mathematical Analysis* 53, 4 (2021), 3772–3800.

Friesecke, G., and Pego, R. L. Solitary waves on FPU lattices: I. qualitative properties, renormalization and continuum limit. *Nonlinearity* 12, 6 (oct 1999), 1601–1627.

Hur, V. M., and Pandey, A. K. Modulational instability in a full-dispersion shallow water model. *Studies in Applied Mathematics* 142, 1 (2019), 3–47.

Klainerman, S., and Selberg, S. Bilinear estimates and applications to nonlinear wave equations. *Communications in Contemporary Mathematics* 04, 02 (2002), 223–295.

Nilsson, D., and Wang, Y. Solitary wave solutions to a class of Whitham–Boussinesq systems. *Zeitschrift für angewandte Mathematik und Physik* 70, 3 (Apr 2019), 70.

Pei, L., and Wang, Y. A note on well-posedness of bidirectional Whitham equation. *Applied Mathematics Letters* 98 (2019), 215–223.

Selberg, S., and Tesfahun, A. Low regularity well-posedness of the Dirac-Klein-Gordon equations in one space dimension. *Communications in Contemporary Mathematics* 10, 02 (2008), 181–194.

Stefanov, A., and Wright, J. D. Small Amplitude Traveling Waves in the Full-Dispersion Whitham Equation. *Journal of Dynamics and Differential Equations* (Oct 2018).

Weinstein, M. I. Existence and dynamic stability of solitary wave solutions of equations arising in long wave propagation. *Communications in Partial Differential Equations* 12, 10 (1987), 1133–1173.

*Email address:* Evgueni.Dinvay@inria.fr

**Inria Rennes - Bretagne Atlantique, Campus universitaire de Beaulieu Avenue du Général Leclerc, 35042 Rennes Cedex, France**