The quantum jump approach — applications to quantum optics and to spin-boson detector models

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Abstract. The quantum jump approach (QJA), a discretized version of continuous observations of systems coupled to a bath, such as photons, is outlined and it is shown how it avoids the quantum Zeno effect by temporal coarse-graining. Applications to quantum counting processes and spin-boson models for detecting a particle are given.

1. Introduction

What happens when an excited atom is waiting to decay and to emit a photon? When does this happen? How is one sure it is happening at a particular time, \( t \) say, and not before? Clearly, to be sure of this one must monitor the atom and watch for the first appearance of a photon. This monitoring should, in principle, be done continuously because there is a difference between the observation of a photon at time \( t \) and the first observation of a photon at \( t \), i.e. no photon observation before \( t \) and then eventually an observation at \( t \). The quantum mechanical Hamiltonian does not contain explicit emission times, and the state of the complete system — atom plus radiation field (photons) — develops into a complicated superposition which also does not contain an explicit time for the emission of a photon. Thus, instead of speaking of the “emission of a photon” as an objective event one should rather speak of the “(first) detection of a photon”.

The difficulty with this, though, is that the quantum mechanical description of continuous measurements is notoriously complicated in practical applications. This difficulty was highlighted by the widely publicized so-called quantum Zeno effect (QZE) which was derived in a general form by Misra and Sudarshan [1]. The quantum Zeno effect states that, under the assumption of the von Neumann-Lüders projection rule\(^1\), of a semi-bounded Hamiltonian and of time-reversal invariance, rapidly repeated measurements will slow down the time development of a system, with an eventual freezing of the state in the limit. It has been pointed out by Lüders [5] that the von Neumann-Lüders rule can only yield an approximate description of a measurement and that taking it at face value is an over-idealization. In consequence, the “freezing of the state”

\(^1\) The projection rule as currently used has been formulated by Lüders [2]. For observables with degenerate eigenvalues his formulation differs from that of von Neumann [3]. It is noteworthy that in the first edition of his book Dirac [4] defines observations which cause minimal disturbance and which seem to be related to Lüders’ prescription; curiously, though, in later editions this passage has been omitted.
alluded to in the formulation of the QZE is an unreachable limit. However, the slow-down of the time development has been experimentally verified [6]. The von Neumann-L"{u}ders projection rule successfully tries to describe the actions of measurements which interfere as little as possible with the system, and since it is simple and easy to apply it is widely used in modern developments of quantum mechanics like quantum teleportation and quantum information theory.

In the first part of this article we will outline an approach how to describe the watching for the first photon, then the next and so on. This is the so-called quantum jump approach (QJA)[7, 8, 9, 10]. It is based on the simple idea of replacing continuous measurements by discrete measurements separated by a small time interval $\Delta t$. Ideally one would like to take the limit $\Delta t \to 0$, but in the calculation one then immediately runs into the QZE and finds an early example of freezing to a subspace, not a freezing of the state (see Section 2.3). Keeping $\Delta t$ small, but finite, one arrives at a description which, on a coarse-grained time scale, can be regarded as continuous for all practical purposes. In the approach, everything can be calculated explicitly, and the results are independent of the particular choice of $\Delta t$. The use of a coarse-grained time scale by-passes the QZE. This approach is similar in spirit to the Monte-Carlo wavefunction approach [11] and to quantum trajectories [12]. The QJA can be carried over to other systems which are coupled to a continuous number of degrees of freedom. Moreover, as will be seen in an example, it can also be applied to systems coupled to a finite or discrete number of degrees of freedom where it can yield considerable simplifications in the otherwise very involved numerical treatment, simplifications which are valid until revivals occur. As will be explained later, mathematically the underlying reason why the QJA works is the validity or, in the discrete case, the approximate validity of the Markov property.

There are numerous applications of the QJA. These include applications to electron shelving [13], to quantum beats [14], to quantum counting processes, to efficient solutions of Bloch equations by simulation[11], to an analysis of the widely discussed experiment in [6] relating to the quantum Zeno effect, to the problem of arrival times of particles, to double jumps of dipole-dipole interacting atoms and to spin-boson detector models for quantum mechanical particles [15, 16].

In the application to the QZE experiment of Itano et al. [6] the QJA helped to resolve a controversy in connection with the interpretation of this experiment. It had been argued that everything could be described by a Hamiltonian dynamics [17] or by Bloch equations [18], without invoking measurements. In my opinion this is indeed true. However, in [19] the experiment of Itano et al. was analyzed using the QJA and it was shown that, at least in an approximate way, it could also be described in terms of measurements and by the von Neumann-L"{u}ders rule, with the ensuing applicability of the QZE, and thus explaining the slow-down of the time development. Moreover, this latter viewpoint was shown to be more fruitful in that it yielded a quick intuitive understanding of the dynamics involved, without recourse to lengthy numerical calculations involving Bloch equations.

After the overview of the QJA we will indicate in the remainder applications to quantum counting processes and describe recent results for spin-boson detector models. Quantum counting processes were introduced axiomatically by Davies and Srinivas [20]. It will be shown here that these axioms are not needed if one goes to a coarse-grained time-scale and uses the QJA. Moreover, in this case the counting operators do not have to be arrived at heuristically but can be calculated explicitly. The spin-boson detector models were originally introduced in [21]. The application of the QJA approach yields considerable simplifications in the continuum limit of these models and leads to a connection with fluorescence detector models, i.e. models based on interactions with lasers.
2. Overview of quantum jump approach

2.1. The conditional Hamiltonian $H_{\text{cond}}$

The Hamiltonian for an $N$-level atom plus quantized electromagnetic field $\mathbf{E} = \mathbf{E}^{(+)} + \mathbf{E}^{(-)}$ is given in the Schrödinger picture, in the limit of long wave-lengths and in the rotating-wave approximation, by

$$ H = \sum_i \hbar \omega_i |i\rangle \langle i| + H_P^0 + \left[ e \mathbf{D}^{(-)} \cdot \mathbf{E}^{(+)}(t) + \text{h.c.} \right] + \left[ e \mathbf{D}^{(-)} \cdot \mathbf{E}^{(+)} + \text{h.c.} \right] $$

$$ \equiv H_A^0 + H_P^0 + H_{AL}(t) + H_{AF} $$

(1)

where $\mathbf{E}_c(t)$ is an external field and

$$ \mathbf{D}^{(-)} = \sum_{i > j} \mathbf{D}_{ij} |i\rangle \langle j|, \quad \mathbf{D}_{ij} = \langle i| \mathbf{X} |j\rangle, \quad \mathbf{E}^{(+)} = \sum_{k\lambda} \nu \left\{ \frac{\hbar \omega_k}{2\nu V} \right\}^{1/2} \varepsilon_{k\lambda} a_{k\lambda} $$

and a frequency cutoff is included. $V$ is the quantization volume, later taken to infinity, and $i > j$ means $\omega_i > \omega_j$.

Let photon measurements be performed at times $t_1 = \Delta t, t_2 = 2\Delta t, \cdots$ on an ensemble of systems which, at time $t_0 = 0$, is supposed to be in the initial state $|0_{ph}\rangle |\psi\rangle$. By the von Neumann-Lüders projection rule the subensemble for which no photons are detected until time $t_n$ is described, with $P_0 \equiv |0_{ph}\rangle \langle 0_{ph}|$, by

$$ P_0 U(t_n, t_n - \Delta t) P_0 \cdots P_0 U(t_1, 0) |0_{ph}\rangle |\psi\rangle \equiv |0_{ph}\rangle U_{\text{cond}}(t_n, 0) |\psi\rangle. $$

(2)

The norm squared of this is the probability of finding no photons for the measurements between 0 and $t_n$. $U_{\text{cond}}$ gives the time development of an atom under the condition that no photon is observed until time $t$, and it will now be determined explicitly by simple second order perturbation theory. Going over to the interaction picture with respect to $H_A^0 + H_P^0$ one has

$$ H_I(t) = H_{AL}(t) + H_{AF}(t) $$

(3)

which is obtained by replacing $|i\rangle \langle j|$ and $a_{k\lambda}$ in the original interaction Hamiltonian by $|i\rangle \langle j| e^{i\omega_{ij} t}$ and $a_{k\lambda} e^{-i\omega_{ij} t}$, respectively, with

$$ \omega_{ij} \equiv \omega_i - \omega_j. $$

We now calculate, for $t_i \leq t' < t_{i+1}$,

$$ \langle 0_{ph}| \frac{d}{dt'} U_I(t', t_i) |0_{ph}\rangle. $$

(4)

In the first-order contribution only $H_{AL}^I(t)$ remains since

$$ \langle 0_{ph}| H_{AF}^I(t) |0_{ph}\rangle = 0. $$

(5)

The second order is, by Eq. (5),

$$ -\hbar^{-2} \int_{t_i}^{t'} dt'' \left\{ \langle 0_{ph}| H_{AF}^I(t') H_{AF}^I(t'') |0_{ph}\rangle + H_{AL}^I(t') H_{AL}^I(t'') \right\}. $$

(6)

Now, if the external field $\mathbf{E}_c(t)$ is smooth in time, then the second part in Eq. (6) contributes a term of higher order in $\Delta t$ and can therefore be omitted. Thus the second-order contribution

$^2$ For thermal or a chaotic external field, however, this part may give rise to a contribution linear in $\Delta t$ and then has to be retained. In this case one can no longer work with state vectors or wavefunctions but has to use (conditional) density matrices. A particular example of this is treated in Ref. [14].
becomes

\[ -\hbar^{-2} \int_{t_i}^{t_f} dt' \sum_{ij\ell_{m}} |i\rangle \langle j| (\ell) \sum_{k\lambda} \frac{e^{2\hbar \omega_k}}{2\varepsilon_0 V} (D_{ij} \cdot \varepsilon_{k\lambda})(\varepsilon_{k\lambda} \cdot D_{m\ell}) e^{-i(\omega_k - \omega_{ij})t' + i(\omega_k - \omega_{im})t''} \]

\[ = -\hbar^{-2} \sum_{ij\ell} \frac{e^{2\hbar \omega_k}}{2\varepsilon_0 V} (D_{ij} \cdot \varepsilon_{k\lambda})(\varepsilon_{k\lambda} \cdot D_{j\ell}) e^{-i(\omega_k - \omega_{ij})t'} \]  

One can now use properties of the correlation function

\[ \kappa_{ij\ell m}(\tau) \equiv \sum_{k\lambda} \frac{e^{2\hbar \omega_k}}{2\varepsilon_0 V} (D_{ij} \cdot \varepsilon_{k\lambda})(\varepsilon_{k\lambda} \cdot D_{\ell m}) e^{-i(\omega_k - \omega_{\ell m})\tau} . \]  

(8)

With \( V^{-1} = \Delta^3 k/(2\pi)^3 \) one can perform the limit \( V \to \infty \), and the sum over \( k \) becomes an integral over \( \omega \), with a suitable frequency cutoff, and an integral over the unit sphere. The correlation function has an effective width of the order of \( \omega_{\ell m}^{-1} \) around \( \tau = 0 \), and for \( t' - t_i \gg \omega_{\ell m}^{-1} \) one can therefore extend the \( \tau \) integration in Eq. (7) to infinity. This is equivalent to the approximation

\[ \int_0^{t' - t_i} d\tau e^{i(\omega_{\ell m} - \omega_{k})\tau} \approx \pi \delta(\omega_k - \omega_{\ell m}) + iP\frac{1}{\omega_k - \omega_{\ell m}} \]  

and corresponds to the usual Markov approximation in the derivation of the Bloch equations.

For the second-order contribution one then obtains

\[ -\sum_{ij\ell} \frac{e^{2\hbar \omega_k}}{2\varepsilon_0 V} D_{ij} \cdot \varepsilon_{k\lambda} \cdot D_{j\ell} \left( \pi \delta(\omega_k - \omega_{\ell m}) + iP\frac{1}{\omega_k - \omega_{\ell m}} \right) . \]

The principal-value term is analogous to a level shift and is, as usual, omitted. The last integral equals \( \Gamma_{i\ell j\ell} \) where

\[ \Gamma_{i\ell j\ell} \equiv \frac{e^{2\hbar \omega_k}}{6\pi\varepsilon_0 \hbar^3 c^3} D_{ij} \cdot D_{k\ell} |\omega_{k\ell}|^3 + \text{principal value term} . \]  

(10)

Hence, integrating over \( t' \) from \( t_i \) to \( t_{i+1} \) and using \( 1 + \delta \approx e^\delta \) for small \( \delta \), we obtain

\[ \langle 0_{ph} | U(t_{i+1}, t_i) | 0_{ph} \rangle = \exp \left\{ -\frac{i}{\hbar} \int_{t_i}^{t_{i+1}} dt' \left\{ H_{AA}'(t') - i\hbar \sum_{ij\ell} \Gamma_{ij\ell} e^{i\omega_{ij\ell}t'} |i\rangle \langle \ell| \right\} \right\} . \]  

(11)

For small \( \Delta t \) this can be replaced by a time-ordered exponential and thus, with \( t \equiv t_n \),

\[ \prod_{1}^{n} \langle 0_{ph} | (U(t_i, t_{i-1}) | 0_{ph} \rangle \cong \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t dt' \left\{ H_{AA}'(t') - i\hbar \sum_{ij\ell} \Gamma_{ij\ell} e^{i\omega_{ij\ell}t'} |i\rangle \langle \ell| \right\} \right\} \]  

(12)

where the product sign on the l.h.s. includes an ordering in an obvious way. Since

\[ \langle 0_{ph} | U(t_i, t_{i-1}) | 0_{ph} \rangle = e^{-iH_A^0 t_i/\hbar} \langle 0_{ph} | U(t_i, t_{i-1}) | 0_{ph} \rangle e^{iH_A^0 t_{i-1}/\hbar} \]  

(13)
and since, for \( t \equiv t_n = n\Delta t \),

\[
U_{\text{cond}}(t, 0) = \prod_{i=1}^{n} (0_{\text{ph}}|U(t_i, t_{i-1})|0_{\text{ph}}),
\]

we obtain, on a coarse-grained time scale, from Eqs. (12) and (13)

\[
U_{\text{cond}}(t, 0) = T \exp \left\{ -\frac{i}{\hbar} \int_{0}^{t} dt' \left\{ H^0_A + H_{AL}(t') - i\hbar \sum_{i, \ell > j} \Gamma_{ijj\ell} |i\rangle \langle \ell | \right\} \right\} \tag{14}
\]

which is the transformation of Eq. (12) back to the Schrödinger picture.

Thus, with the atomic operator \( \Gamma \) defined as

\[
\Gamma = \sum_{i, \ell > j} \Gamma_{ijj\ell} |i\rangle \langle \ell |
\]

the conditional Hamiltonian for an \( N \)-level atom with no photon emission until time \( t \) is, on the coarse-grained time scale, given by

\[
H_{\text{cond}}(t) = H^0_A + H_{AL}(t) - i\hbar \Gamma \tag{16}
\]

For initial atomic state \( |\psi\rangle \) the probability to find no photon until time \( t \) is thus given by

\[
\|U_{\text{cond}}(t, 0)|\psi\rangle\|^2, \tag{17}
\]

and the probability to find the first photon in \( (t, t + \Delta t) \) is the difference of this expression for \( t \) and \( t + \Delta t \). Thus, on the coarse grained time scale, the probability density \( w(t) \) for the first photon is the negative derivative of Eq. (16),

\[
w(\tau) = -\frac{d}{dt} \|U_{\text{cond}}(t, 0)|\psi\rangle\|^2 = \langle \psi|\Gamma + \Gamma^\dagger|\psi\rangle. \tag{18}
\]

2.2. Connection with the quantum Zeno effect

When one lets \( \Delta t \) become smaller and smaller the above derivation shows very nicely how and where the quantum Zeno effect turns up in a very natural way. If \( \Delta t \) is chosen much smaller than the inverse optical frequencies, the last exponential in Eq. (7) can be replaced by 1, and the integral becomes proportional to \( t' - t_i \). Eq. (11) is then replaced by

\[
(0_{\text{ph}}|U_{I}(t_{i+1}, t_i)|0_{\text{ph}}) = \exp \left\{ -\frac{i}{\hbar} \left\{ H_{AL}(t_i)\Delta t - i\hbar \text{const} \sum_{i'j' \ell \ell'} \Gamma_{ijj'\ell'} e^{i\omega_{i'j'}\Delta t} |i\rangle \langle \ell |(\Delta t)^2 \right\} \right\}. \tag{19}
\]

The product of these operators then becomes, for \( \Delta t \to 0 \)

\[
T \exp \left\{ -\frac{i}{\hbar} \int_{0}^{t} dt' \left\{ H_{AL}(t') \right\} \right\}. \tag{19}
\]

This is a purely atomic operator, and hence for \( \Delta t \to 0 \) the time development takes place in the atomic subspace only. The atomic state does not freeze, only the field becomes frozen. This is an example of freezing to a subspace the possibility of which was already pointed out in [1]. For this reason one cannot choose \( \Delta t \) arbitrarily small in the quantum jump approach.
2.3. The reset operation $\mathcal{R}$

We consider an ensemble where no photons are present at time $t_0$ and assume that a short time $\Delta t$ later a photon has been found. In this section we going to determine the state (or density matrix) of an atom after a such a broadband detection of a photon. The initial ensemble is thus described by $\rho = |0\rangle_A \langle 0|_A$ where $\rho_A$ is the atomic density matrix. The state of the subensemble for which photons are detected by a non-absorptive measurement at time $t_1 = t_0 + \Delta t$ is given, in view of the von Neumann-Lüders projection rule, by

$$P_1 \rho(t_1) P_1 / \text{tr}(\cdot)$$

where

$$P_1 = 1 - |0\rangle_A \langle 0|_A.$$  \hspace{1cm} (20)

Note that Eq. (20) still contains the photons.

After a photon measurement by absorption no photons are present any longer and it was argued in Ref. [8] that the resulting reset state is obtained from Eq. (20) by a partial trace over the photons, i.e., by

$$|0\rangle_A \langle 0|_A \text{tr}_{ph}(P_1 \rho(t_1) P_1) / \text{tr}(\cdot).$$

The physical reason for this is that for the atomic description alone it should make no difference in infinite space whether or not the photons are absorbed, as long as they are sufficiently far away from the atom and no longer interacting with it. Eq. (22) can be calculated by perturbation theory for $U_I(\Delta t, 0)$, as in Section 2.1. Going up to second order in the interaction one obtains in a straightforward way for the reset state

$$\mathcal{R}\rho_A \Delta t \equiv \text{tr}_{ph}(P_1 U(\Delta t, 0)|0\rangle_A \langle 0|_A U(\Delta t, 0)^\dagger P_1 = e^{-i\hat{H}_A^0 \Delta t / \hbar} \sum_{i, j, \ell, m} |j\rangle \langle i| \langle \ell| \langle m| e^{i\hat{H}_A^0 \Delta t / \hbar} \times \int_0^{\Delta t} dt' \int_0^{\Delta t} dt'' e^{i\omega_k(t'-t'')} - i\omega_j t' + i\omega_m t'' \sum_{k, \lambda} e^{2\omega_k / 2\varepsilon_0 \hbar} V_k e^{i\varepsilon_k \lambda} \cdot (\varepsilon_{k\lambda} \cdot \hat{D}_{\ell m}) \cdot .$$

Note that the atomic trace $\text{tr}_A \hat{\mathcal{R}}\rho_A \Delta t$ gives the probability for a photon to be found at time $t_0 + \Delta t$. The external field drops out since its action is of second order in $\Delta t$. We decompose the rectangular integration domain over $t'$ and $t''$ in Eq. (23) into two triangles, and using the Markov property as in Eqs. (7) and (9) one obtains in the limit $V \to \infty$

$$\mathcal{R}\rho_A \Delta t = e^{-i\hat{H}_A^0 \Delta t / \hbar} \sum_{i, j, \ell, m} \{\Gamma_{ji\ell m} + \Gamma_{\ell m ji}\} |j\rangle \langle i| \langle \ell| \langle m| e^{i\hat{H}_A^0 \Delta t / \hbar} \int_0^{\Delta t} dt' e^{-i(\omega_j - \omega_m) t'} \cdot$$

with $\Gamma_{ijkl}$ given by Eq. (10). For simplicity we now assume that the atomic transition frequencies are well separated. Then the last integral contributes only for $ij = \ell m$ and one obtains

$$\mathcal{R}\rho_A = \sum_{i, j, \ell, m} \{\Gamma_{ji\ell m} + \Gamma_{\ell m ji}\} |j\rangle \langle i| \langle \ell| \langle m| .$$

(24)

For the general case one can use the same expression, as shown in [9]. Up to normalization this is the state of an atom after a photon detection at time $t_0 + \Delta t$, under the condition that no photon was present at time $t_0$. Its trace, multiplied by $\Delta t$, gives the probability for this event.
3. Application to quantum counting processes

In [20] Davies-Srinivas proposed axioms of for ‘homogeneous quantum counting processes’. These axioms imply the existence of two superoperators $J$ and $S_t$ which map trace class operators to trace class operators and give the probability density for finding exactly $n$ photons at times $t_1 < t_2 \cdots < t_n$ in $[t_0, t]$ in the form

$$w(t_1, \cdots, t_n; [t_0, t]) = \operatorname{tr} \left( \hat{S}_{t-t_n} \hat{J} \hat{S}_{t_{n-1}-t_n} \cdots \hat{J} \hat{S}_{t_1-t_0} \rho_A(t_0) \right).$$

The explicit form of the operators is not known but have to be obtained by an educated guess. In the QJA one can obtain the same structure, but all operators can be calculated explicitly.

In the QJA we define the superoperator in atomic space

$$S_{t, t'} \rho_A := U_{\text{cond}}(t, t') \rho_A U_{\text{cond}}(t, t')^\dagger$$

Then the probability density for finding exactly $n$ photons at times $t_1 < t_2 \cdots < t_n$ in $[t_0, t]$ can be calculated to be given by

$$w(t_1, \cdots, t_n; [t_0, t]) = \operatorname{tr} \left( S_{t-t_n} \mathcal{R} S_{t_n-t_{n-1}} \cdots \mathcal{R} S_{t_1-t_0} \rho_A(t_0) \right).$$

The superoperators $\mathcal{R}$ and $S_{t,t'}$ of the QJA not only automatically satisfy the properties of the operators $\hat{J}$ and $\hat{S}_{t,t'}$ of Davies-Srinivas, but are also explicitly known. The only difference is that in the QJA time is course-grained while the axioms of Davies-Srinivas assume continuous time from the start.

4. Application to spin-boson detector models

In this model [21] there are spins (in a meta-stable state) which are weakly coupled to a bath of bosons (small $\gamma_j$ in (25)). In addition there is a moving particle, and where its wavefunction overlaps with a spin there is a high spin flip probability (large $g_j$ in (25)). First there will be a single spin flip due to the passing particle (accompanied by a boson emission). This leads to a large energetic gain for flipping neighbors i.e. a large flip probability. This then leads to a sudden flip of all spins from metastability, a sort of domino effect (together with a burst of photons) is conserved. Therefore, the first detection of the particle, taken as the first spin flip, is equivalent to the detection of the first boson.

Note that in this model there is no direct measurement on the particle. The Hamiltonian is of the form

$$\begin{align*}
H &= \frac{\hat{p}^2}{2m} + \sum_j \frac{e_j^{(j)}}{2} \sigma_z^{(j)} - \sum_{j<j'} \frac{J_{jj'}}{2} \sigma_z^{(j)} \otimes \sigma_z^{(j')} + \hbar \sum_\ell \omega_\ell \hat{a}_\ell^\dagger \hat{a}_\ell \\
&\quad + \hbar \sum_j \sum_\ell \left( \gamma_j^{(j)} e^{i J_{j\ell}} \hat{a}_\ell^\dagger \sigma_-^{(j)} + \text{h.c.} \right) + \hbar \sum_j \chi_j^{(j)}(\hat{x}) \sum_\ell \left( g_j^{(j)} e^{i J_{j\ell}} \hat{a}_\ell^\dagger \sigma_-^{(j)} + \text{h.c.} \right)
\end{align*}$$

(25)

where $\chi_j^{(j)}(x)$ is a position dependent sensitivity function for the increased coupling due to the particle. It should be noted that the excitation number (i.e. ± for spins plus number of photons) is conserved. Therefore, the first detection of the particle, taken as the first spin flip, is equivalent to the detection of the first boson.

In the corresponding continuum limit ($N \to \infty$) and with the simplifying assumption that all sensitivity functions $\chi_j^{(j)}$ are equal, i.e. $\chi_j^{(j)}(x) \equiv \chi(x)$, an analysis similar to that in Section 2.1 yields that the time development under the condition of no spin flip (no boson) is given by the conditional Hamiltonian

$$H_{\text{cond}} \equiv \frac{\hat{p}^2}{2m} + \frac{\hbar}{2} (\delta_{\text{shift}} - i A) \chi(\hat{x})^2$$
where the constants $A$ and $\delta_{\text{shift}}$, corresponding to a decay (Einstein coefficient) and level shift, respectively, are obtained from the correlation function

$$\kappa(\tau) \equiv \sum \ell |g_{\ell}|^2 e^{-i(\omega_{\ell}-\omega_0)\tau}.$$  

Here the Markov property is needed, i.e. $\kappa(\tau) \approx 0$ if $\tau$ is larger than some small correlation time $\tau_c$. In $H_{\text{cond}}$ the spin-boson detector enters mainly through an imaginary term which decreases the norm (since the non-detection probability decreases) and which mathematically mimics a position dependent absorption. This is similar to the fluorescence model [22].

For illustration we consider a 1d example with a single spin and $N$ discrete boson modes $\omega_{\ell} = \omega_{\text{M}} n/N$, $n = 1, \ldots, N$, coupling constants $\gamma^{(j)}_l = 0$ and $g_{\ell} = -i G \sqrt{\omega_{\ell}/N}$ where $\omega_{\text{M}}$ is a maximal boson frequency. A particle wave packet is coming from the left. The probability of finding the detector spin in state $|\downarrow\rangle$ at time $t$ is given by

$$P_{\text{discrete}}^1(t) = \sum \ell \int_{-\infty}^{\infty} dx \left| \langle x \downarrow | 1_{\ell} | \Psi_t \rangle \right|^2 \equiv 1 - P_{\text{discrete}}^0(t),$$

As long as no recurrences occur due to the discrete nature of the bath (i.e. no transitions $|\downarrow 1_{\ell} \rangle \leftrightarrow |\uparrow 0 \rangle$) one can regard

$$w_{\text{discrete}}^1(t) = \frac{d}{dt} P_{\text{discrete}}^1(t) = -\frac{d}{dt} P_{0\text{discrete}}^1(t).$$

as the probability density for a spin flip (i.e. for a detection) at time $t$. The numerical evaluation is extremely time consuming. An example is given in Fig. 1 (dots) for $N = 40$, $\chi(x) = \theta(x)$ (Heaviside function) and an incoming Gaussian wave packet. The solid line is the result of the QJA for the continuum limit $N \to \infty$ which is much easier to evaluate.

**Reset operation:** Let the complete system (bath, detector, particle) be described at a particular time by a density matrix of the form

$$\rho \equiv |0\rangle \langle 0| \otimes (\uparrow_1 \cdots \uparrow_D) \rho_p \langle \uparrow_1 \cdots \uparrow_D | \langle 0|,$$

(with $\rho_p$ the particle density matrix), i.e., no boson and all spins up. If a boson is found in a broadband boson measurement a time $\Delta t$ later, the density matrix for the corresponding subensemble is obtained by sandwiching the above expression with

$$P_1 \equiv \sum \ell |1_{\ell}\rangle \langle 1_{\ell}|,$$  

by the von Neumann-Lüders projection rule, and the trace gives the probability. The subsystem consisting solely of the particle is then described after the detection of a boson by a partial trace,

$$\text{tr}_{\text{det}} \text{tr}_{\text{bath}} P_1 U(\Delta t, 0) \rho U^\dagger(\Delta t, 0) P_1 \equiv \mathcal{R} \rho_p \Delta t,$$

which defines the operation $\mathcal{R}$. This can be calculated by second-order perturbation theory. Note that even if $\rho_p$ is a pure state the reset state will in general not be pure.

In the continuum limit, $N \to \infty$, one can proceed similarly to Section 2.3. With the Markov property and $\chi^{(j)}(x) \equiv \chi(x)$, one obtains to first order in $\Delta t$

$$\mathcal{R} \rho_p = A \chi(\hat{x}) \rho_p \chi(\hat{x})$$

(29)
for the state immediately after a detection. If $\varrho_p$ is a pure state, then the reset state is also a pure state. In particular, if $\varrho_p = |\psi_{\text{cond}}^t\rangle \langle \psi_{\text{cond}}^t|$, then the reset state is given by the wave function

$$|\psi_{\text{reset}}^t\rangle = A^{1/2} \chi(\hat{x}) |\psi_{\text{cond}}^t\rangle.$$  (30)

This result is physically very reasonable since it means that right after a detection of the particle by a detector located in a specific region the particle is localized there. In particular, if $\chi$ is the characteristic function of the region the state is just projected onto this region by the detection.

We again consider the above 1d example with $N$ (discrete) boson modes and $\chi(x) = \theta(x)$, but now $N = 15$. In Fig. 2 (dots) we have plotted $\langle x | R_\varrho_p | x \rangle$ from Eq. (28) as a function of $x$ for a Gaussian wave packet. For the corresponding continuum limit the solid line of Fig. 2 depicts

$$\langle x | R_\varrho_p | x \rangle = A \left| \langle x | \Theta(\hat{x}) | \psi \rangle \right|^2$$  (31)

from Eq. (30). Up to small deviations around $x = 0$, the reset states from the discrete and the continuum model are in very good agreement.

In the present model, with $f^{(j)}_x$ independent of $x$, there is no explicit recoil on the particle from the created boson. This is in line with the original idea of a minimally invasive measurement. The absence of an explicit recoil distinguishes the present detector model from other models which are based on the direct interaction with the particle's internal degrees of freedom, such as interaction with laser light. It appears reasonable that in the present model no such recoil on the particle occurs: After all, the boson is emitted not by the particle but by the spin lattice. Hence, the recoil should be experienced by this lattice, rather than by the particle, similar to what occurs in the Mössbauer effect. Of course, the projection of the wave packet onto the detector region by means of the reset operation also changes the momentum distribution of the wave packet.
5. Discussion and summary

The quantum jump approach is applicable also to other systems which are coupled to a bath. As the analysis has shown one needs that the second order perturbation theory is proportional to $\Delta t$, which is essentially the Markov property. The QJA is not applicable for times in which revivals occur, such as for discrete bath modes. The temporal coarse graining in the approach is due to the use of the measurement theory of von Neumann-Luders and their projection rule. Without coarse-graining this would lead to the quantum Zeno effect. The angular distribution of photons or other bosons can be included included in the approach [10]. In general there are no quantum mechanical predictions for a single measurement on a single system, but statements may be possible for repeated measurements on a single system.

In summary, we have shown the following.

- Repeated, semi-continuous, selective null (photon or boson) measurements lead to a nonhermitean conditional time-development operator between photon or boson detections.
- With this conditional time development the probability density of first photon or boson is given by the decrease of the norm-square of the state.
- After a photon or boson detection one has to reset the state (‘jump’), then there is again the conditional time development until the next detection, and so on. This leads to the so-called quantum trajectories.
- Applications of this approach include light and dark periods, photon statistics, quantum Zeno effect, quantum beats, quantum counting processes, and quantum arrival times.

Acknowledgments

I would like to dedicate this paper to George Sudarshan on the occasion of his 75th birthday.

References

[1] Misra B and Sudarshan E C G 1977 *J. Math. Phys.* **18** 756
[2] Lüders G 1951 *Ann. Phys. (Leipzig)* **443** 322
[3] von Neumann J 1932 *Mathematische Grundlagen der Quantenmechanik* (Berlin: Springer) Chapter V.1. (English edition: 1955 *Mathematical Foundations of Quantum Mechanics* (Princeton: University Press))
[4] Dirac P A M 1930 *The Principles of Quantum Mechanics* 1st Ed. (Oxford: Clarendon Press) p 49
[5] Lüders G, private communication
[6] Itano W M, Heinzen D J, Bollinger J J, and Wineland D J 1990 *Phys. Rev. A* **41** 2295
[7] Hegerfeldt G C and Wilser T S 1992 *Classical and Quantum Systems. Proc. of the II. Intern. Wigner Symposium, Goslar 1991* ed Doebner H D, Scherer W, and Schroek F (Singapur: World Scientific) p 104
[8] Hegerfeldt G C 1993 *Phys. Rev. A* **47** 449.
[9] Hegerfeldt G C and Sondermann D 1996 *Quantum Semicl. Opt.* **8** 121
[10] Hegerfeldt G C 2003 *Lecture Notes in Physics* vol 622 *Irreversible Quantum Dynamics* ed F Benatti and R Floreanini (Berlin Heidelberg: Springer) p 233
[11] Dalibard J, Castin Y, and Molmer K 1992 *Phys. Rev. Lett.* **68** 580
[12] Carmichael H 1993 *Lecture Notes in Physics* vol m 18 *An Open Systems Approach to Quantum Optics* (Berlin Heidelberg: Springer)
[13] Dehmelt H G 1975 *Bull. Am. Phys. Soc.* **20** 60; Bergquist J C, Hulet R G, Itano W M and Wineland D J 1986 *Phys. Rev. Lett.* **57** 1699; Sauter T, Blatt R, Neuhauser W and Toschek P E 1986 *Opt. Comm.* **60** 287; Nagourney W, Sandberg J, and Dehmelt H 1986 *Phys. Rev. Lett.* **56** 2797
[14] Hegerfeldt G C and Plenio M B 1993 *Phys. Rev. A* **47** 2186.
[15] Hegerfeldt G C, Neumann J T, and Schuman L S 2006 *J. Phys. A: Math. Gen.* **39** 14447
[16] Hegerfeldt G C, Neumann J T, and Schulan L S 2006 *quant-ph/0610041* (to be published in *Phys. Rev. A*)
[17] Petrosky T, Tasaki S, and Prigogine I 1990 *Phys. Lett. A* **151** 109
[18] Freichs V and Schenze A 1991 *Phys. Rev. A* **44** 1962
[19] Beige A and Hegerfeldt G C 1996 *Phys. Rev. A* **53** 53
[20] Srinivas M D and Davies E B 1981 *Opt. Acta* **28** 981
[21] Gaveau B and Schuman L S 1990 *J. Stat. Phys.* **58** 1209
[22] Ruschhaupt A, Damborenea J A, Navaro B, Muga J G and Hegerfeldt G C 2004 *Europhys. Lett.* **67** 1