Non-adiabatic effect on Laughlin's argument of the quantum Hall effect

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Abstract. We have numerically studied a non-adiabatic charge transport in the quantum Hall system pumped by a magnetic flux, as one of the simplest theoretical realizations of non-adiabatic Thouless pumping. In the adiabatic limit, a pumped charge is quantized, known as Laughlin's argument in a cylindrical lattice. In a uniform electric field, we obtained a formula connecting quantized pumping in the adiabatic limit and no-pumping in the sudden limit. The intermediate region between the two limits is determined by the Landau gap. A randomness or impurity effect is also discussed.

In the paper by Laughlin\(^1\), the quantum Hall system on a cylinder with two edges penetrated by an Aharonov-Bohm (AB) flux \(\Phi\) is considered, where the flux changes from 0 to a flux quantum \(\Phi_0 = h/e\) adiabatically. The change of the AB flux enforces electrons to move form one edge to the other edge. In the adiabatic limit, net charge of transported electron is quantized due to a request from the gauge transformation, which is known as Laughlin’s argument in a cylindrical lattice. This topological aspect is the key feature of quantization of the Hall conductivity\(^2-4\).

Although Laughlin’s argument gives a theoretical explanation for the quantum Hall effect, quantized charge transport is realized in various situations as Thouless pumping\(^5\). Thouless argued that an electron system in time dependent potential such as right moving potential \(U(x, t) = \sin(2\pi(x/L - t/T))\) with period \(T\) and \(L\) can pump a quantized charge in analogy with water pumping by Archimedean screw\(^6\). Recently, in mesoscopic system, electron pumping by adiabatic change of a cyclic potential \(U(x, t)\) has attracted much attention both experimentally\(^7\) and theoretically\(^6\). Not only adiabatic pumping but also non-adiabatic pumping is also important and realized easily in experimental situation\(^8\), which needs a extension of original Thouless pumping theoretically\(^9\). To increase net current induced by successive pumping, non-adiabatic pumping is thought to be more efficient because fast pumping transports more electrons\(^10\). In this paper, going back to the cylindrical system of Laughlin’s argument, we change the AB flux \(\Phi\) non-adiabatically, i.e., non-adiabatic effect on edge-state pumping. For this purpose, we introduce a time-dependent flux \(\Phi(t) = \Phi_0 t/T\) with the period \(T\). This system is one of the simplest theoretical realizations of non-adiabatic Thouless pumping. In addition, we shall study a square lattice penetrated by the flux \(\Phi\) without boundary as shown in Fig. 1. There is no edge state at \(\Phi = 0\) and edge states induced by \(\Phi\). One of our motivations is to study how a edge state goes through \(\Phi = \Phi_0\) because the edge state comes across the second
Landau level as shown in Fig. 2. Here we suppose the lowest Landau level is filled at $\Phi = 0$.
Level crossing of left and right edge states at $\Phi = \Phi_0/2$ is easily understood by a Landau-Zener tunneling as studied massively in one-dimensional ring\[11].

We investigate the time evolution of the ground state in a square lattice with the lengths $L_x$ and $L_y$ under the magnetic field $Ba^2 = p/q\Phi_0$, where $a$ is the lattice constant. The time-dependent Schrödinger equation $i\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$ is solved numerically with time step $\Delta t$. To preserve the norm $\langle \Psi(t) | \Psi(t) \rangle$, the Suzuki Trotter decomposition of $H$ is used[12, 13]. Although $H$ is one-body Hamiltonian, the state $|\Psi(t)\rangle$ is a many-body state filled up to Fermi energy which is fixed around the center of the first Landau gap. To observe pumped charge, we calculate $\Delta N = \langle \Psi(t) | (N_R - N_L) | \Psi(t) \rangle$, where $N_L$ ($N_R$) is the number of electrons of left (right) system. $\Delta N(T) = 0$ in the sudden limit while $\Delta N(T) = -2$ in the adiabatic limit because the left edge state is occupied and the right edge state is empty at $t = T$ as shown in Fig. 2. In other words, charge is carried by extended states of the first Landau band as explained in Laughlin’s argument. Hereafter, we limit ourselves to the commensurate lattice with the periodic boundary condition (PBC): $L_y = q \times l_y$, $L_x = q \times l_x$ with integer numbers $l_x$ and $l_y$. We set the magnetic field $p/q = 1/7$, the unit of energy $t = 1$, $a = 1$, and the system size and time step are took large enough to obtain the thermodynamic limit. The maximum lattice size is $L_y = L_x = 70$.

The Hamiltonian is defined as $H(\Phi) = -t \sum_{mn} c_{m+1,n}^\dagger c_{mn} - t \sum_{mn} c_{m,n+1}^\dagger c_{mn} + h.c.$, where we take the Landau gauge $\theta^{x}_{mn} = 0$. When we consider the cylindrical system, a site index $(m, n)$ is limited to $m \in [1, L_x-1], n \in [1, L_y]$. For $\theta^y_{mn}$, we consider two Hamiltonians; $H_{uni}(t)$ under uniform electric field and $H_{imp}(t)$ under local electric field, where the electric field is given by $E := -\partial_t A$ and vector potential is related with the hopping phase through $\theta^{y}_{mn} = \int_{r_{mn}} A \cdot d\mathbf{r}$. Note that the dynamic electric field can be determined by the electron system but we suppose the static electric field to consider the electron system in two simple limits. The uniform electric field shown in Fig. 3(a) is realized by $\theta^{y}_{mn} = 2\pi (pm/q + \frac{r}{T L_y})$ and the local electric field shown in Fig. 4(a) is realized by $\theta^{y}_{mn} = 2\pi (pm/q + \delta_{n L_y} \frac{r}{T})$. Except in the adiabatic limit, there is no static gauge transformation between $H_{uni}$ and $H_{imp}$. Of course, one can find the function $\chi(t, r)$ of the time-dependent gauge transformation $A_{imp}(t, r) = A_{uni}(t, r) - \nabla \chi(t, r)$ with taking into account new scalar potential $\phi(t, r) = -\partial_t \chi(t, r)$. That is, $H_{uni}$ and $H_{imp} + \phi(t, r)$ are the same. Although $H_{uni}$ is more natural than $H_{imp}$, we shall also study $H_{imp}$ for comparison. Note that $H_{uni}(t + T) \neq H_{uni}$ and $H_{imp}(t + T) = H_{imp}$(t).

When we consider the cylindrical lattice described by $H_{uni}$, a wave number in the $y$ direction is a good quantum number and preserved for any $T$. Then, the possible diabatic transition occurs in each separated sector labeled by $k_y$. After the Fourier
transformation \( c_{m,n} = \frac{1}{r_y} \sum_{k_y} e^{i k_y n} c_m(k_y) \), we get \( H_{uni} = \frac{1}{r_y} \sum_{k_y} H(k_y) \) with \( H(k_y) = -t \sum_{m=1}^{L_x-1} \left( c_{m+1}^\dagger(k_y) c_m(k_y) + h.c. \right) - 2t \sum_{m=1}^{L_x-1} U(m, t) c_{m}^\dagger(k_y) c_m(k_y) \), where the on-site potential \( U(m, t) = \cos \left( k_y - 2\pi (pm/q + \frac{t}{T y}) \right) \) is periodic: \( U(m + q, t) = U(m, t + TL_y) = U(m, t) \). This Hamiltonian is one of the simplest theoretical realizations of Thouless pumping.

As a result, the spatial distribution of the charge density shows the quantized charge transport, i.e., \( \Delta N(T) = -2 \) in some parameter range. There arises the question when quantization of pumped charge breaks down, i.e., the limitation of Laughlin’s argument in a non-adiabatic process. Considering the Landau-Zener tunneling, the upper limit of \( T \) is given as \( T < \hbar/\Delta_e \), where \( \Delta_e \) is an energy gap of the overlapped edge states due to finite \( L_x \). Since \( \Delta_e \) is exponentially small due to large system size, the upper limit of \( T \) becomes infinite. On the other hand, the lower limit of \( T \) is \( 1/\omega_c \), where \( \hbar \omega_c \) is the Landau gap. Figure 3(b) shows \( \Delta N \) as a function of \( \Phi/\Phi_0 = t/T \). At large \( T > 1 \) pumped charge is quantized \( \Delta N(T) = -2 \), while \( \Delta N(T) \) becomes zero at small \( T \). Solid lines are fitted with the formula

\[
\Delta N = -\frac{2t}{T} + \frac{2}{\omega_c T} \sin(\omega_c t),
\]

which connects quantized pumping in the adiabatic limit \( T = \infty \) and no-pumping in the sudden limit \( T = 0 \) and actually shows good agreement as shown in Fig. 3(b) and (c). We note that \( \omega_c \) is about 1.4. However, Fig. 4 for \( H_{imp} \) shows disagreement with solid lines given by Eq. 1 especially for small \( T \). A major difference is shown in Fig. 3(c) and Fig. 4(c). Pumped charge per time can increase by fast pumping only in the uniform system.

We found the result of \( H_{imp} \) on the square lattice with the PBC is quite similar to that of \( H_{imp} \) on the cylindrical lattice. It means that topology of two systems can not affect edge-state

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**Figure 3.** (a) model of \( H_{uni} \) on the cylindrical lattice (b) \( \Delta N \) as a function of \( \Phi/\Phi_0 \). Solid lines are \( \Delta N \) of Eq. 1 with fixed \( \omega_c \) and respective \( T \). (c) \( \Delta N \) as a function of \( t \).

**Figure 4.** (a) model of \( H_{imp} \) on the cylindrical lattice (b) \( \Delta N \) as a function of \( \Phi/\Phi_0 \). Solid lines are \( \Delta N \) of Eq. 1 with fixed \( \omega_c \) and respective \( T \). (c) \( \Delta N \) as a function of \( t \).
pumping in the thermodynamic limit. Moreover, there is no singularity at \( \Phi/\Phi_0 = t/T = 1 \), which is expected from Fig. 2. Figure 5 shows that two regions, \( t/T < 1 \) and \( t/T > 1 \), are smoothly connected. This result is similar to that on the cylindrical lattice again.

Finally, we studied the effect of random on-site potential \( W \) as shown in Fig. 6. Disagreement with the solid line (Eq. 1) becomes large with increasing \( W \) and random-averaged \( \Delta N \) approach \( \Delta N = -2t/T \), which is the form in the adiabatic limit.

In summary, we have studied numerically the non-adiabatic effect on edge-state pumping in the quantum Hall system. The formula (Eq. 1) shows a good agreement with data of \( H_{uni} \) on the cylindrical lattice and connects quantized pumping \( \Delta N(T) = -2 \) in the adiabatic limit and no-pumping \( \Delta N(T) = 0 \) in the sudden limit. Non-adiabatic pumping can be efficient to increase net current but has the limitation due to inhomogeneity of electric field. We have observed clear steps of \( \Delta N(t) \) due to the Landau gap for \( H_{uni} \) and for \( H_{imp} \) at large \( T > 1/\omega_c \). However, this effect is weak against randomness.

We acknowledge discussions with M. Kinuhara. The computation in this work has been done using the facilities of the Supercomputer Center, Institute for Solid State Physics, University of Tokyo and Altix3700BX2 at YITP in Kyoto University. This work has been supported in part by Grants-in-Aid for Scientific Research, No. 20340098, 20654034 from JSPS and No. 220029004, 20046002 on Priority Areas from MEXT.

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