On non-projective small resolutions

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Abstract
We construct a large class of projective threefolds with one node (aka non-degenerate quadratic singularity) such that their small resolutions are not projective.

Keywords Small resolution · Picard–Lefschetz theory · Extremal ray

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1 Introduction

One of the main results of this note is the following

Proposition 1.1 Suppose that \( X \subset \mathbb{P}^n \) is a smooth 4-dimensional projective variety over \( \mathbb{C} \) such that \( H^0(X, \omega_X(1)) \neq 0 \), where \( \omega_X \) is the canonical sheaf. Then a general singular hyperplane section \( Y_0 \subset X \) has precisely one singular point \( a \in Y_0 \), this point is a node, and small resolutions of the point \( a \) cannot be projective varieties.

Corollary 1.2 Suppose that \( X \subset \mathbb{P}^n \) is an arbitrary smooth 4-dimensional projective variety over \( \mathbb{C} \). Then there exists a natural number \( m_0 \) such that, for any \( m \geq m_0 \), a general singular intersection of \( X \) with a hypersurface of degree \( m \) has precisely one singular point, this point is a node, and small resolutions of this node cannot be projective varieties.

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Proof Take an \( m_0 \) such that \( H^0(\omega_X(m)) \neq 0 \) for all \( m \geq m_0 \) and apply Proposition 1.1 to the \( m \)-th Veronese image of \( X \).

\[ \square \]

Remark 1.3 In view of Proposition 1.6 below, this corollary follows also from Proposition 1.1 and [4, Exposé XVIII, Corollaire 6.4].

Remark 1.4 For the case in which the fourfold \( X \subset \mathbb{P}^n \) is a complete intersection, the result of Corollary 1.2 is contained in Theorem B from the paper [13] by Polizzi, Rapagnetta, and Sabatino. Indeed, this theorem from [13] implies that if \( X \) is a complete intersection and \( Y \) is the intersection of \( X \) with a hypersurface of degree \( \geq 3 \) and if the only singular point of \( Y \) is a node, then the projective variety \( Y \) is factorial. On the other hand, the factoriality, and even \( \mathbb{Q} \)-factoriality, of a nodal threefold implies that it has no projective small resolution (see [3] or Remark 2.2 below). Thus, for complete intersections this result from [13] provides an explicit bound \( m_0 = 3 \) in Corollary 1.2.

Besides, [13, Theorem B] provides for the case of finitely many ordinary singularities of higher multiplicity on \( Y \).

We derive Proposition 1.1 from a stronger result. To state it, recall some terminology from SGA7.

Definition 1.5 Suppose that \( X \subset \mathbb{P}^n \) is a smooth projective variety over \( \mathbb{C} \). We will say that Condition (A) is satisfied for the variety \( X \) if either \( X^* \subset (\mathbb{P}^n)^* \) is not a hypersurface or \( X^* \) is a hypersurface and the restriction homomorphism \( H^{d-1}(X, \mathbb{Q}) \rightarrow H^{d-1}(Y, \mathbb{Q}) \), where \( d = \dim X \) and \( Y \subset X \) is some (equivalently, any) smooth hyperplane section, is not a surjection.

(The restriction \( H^{d-1}(X, \mathbb{Q}) \rightarrow H^{d-1}(Y, \mathbb{Q}) \) is always an injection by virtue of the Lefschetz hyperplane section theorem.)

It follows from [4, Exposé XVIII, Theorem 6.3] and the hard Lefschetz theorem that in characteristic zero Definition 1.5 is equivalent to the definition from Section 5.3.5 of [4, Exposé XVIII], where Condition (A) was originally stated.

Recall also (loc. cit.) that Condition (A) can be violated only if \( d = \dim X \) is even and that, for an even-dimensional variety \( X \subset \mathbb{P}^n \) such that \( X^* \) is a hypersurface, Condition (A) is equivalent to the non-triviality of the monodromy group acting on \( H^{d-1}(Y, \mathbb{Q}) \) (if \( X^* \) is not a hypersurface, this monodromy group is automatically trivial).

Taking all this into account, here is the promised stronger result.

Proposition 1.6 Suppose that \( X \subset \mathbb{P}^n \) is a smooth 4-dimensional projective variety over \( \mathbb{C} \).

If Condition (A) holds for \( X \) and \( X^* \subset (\mathbb{P}^n)^* \) is a hypersurface, then a general singular hyperplane section \( Y_0 \subset X \) has precisely one singular point \( a \in Y_0 \), this point is a node, and small resolutions of \( Y_0 \) cannot be projective varieties.

If Condition (A) does not hold for \( X \), in which case \( X \) is automatically a hypersurface, then a general singular hyperplane section \( Y_0 \subset X \) has precisely one singular point \( a \in Y_0 \), this point is a node, and there exists a small resolution for \( Y_0 \) in the category of projective varieties.
It is believed (see [4, Exposé XVIII, 5.3.5]) that Condition (A) holds for “most” varieties of even dimension. In small dimensions one can hope for a complete classification of the exceptions. In dimension 2, varieties for which Condition (A) is violated were classified by Zak [15]. As far as we know, no classification in dimension 4 has been found yet.

The paper is organized as follows. In Sect. 2 we state and prove several well-known properties of small resolutions for which we did not manage to find references; thus, the results in this section do not claim too much novelty. In Sect. 3 we establish the link between Condition (A) and properties of small resolutions of singular hyperplane sections of a smooth fourfold. Finally, we prove Propositions 1.1 and 1.6 in Sect. 4.

Notation and conventions

Base field will be the field \( \mathbb{C} \) of complex numbers.

By node on an algebraic threefold \( W \) we mean a point \( a \in W \) such that \( \mathcal{O}_a,W \cong \mathbb{C}[[x,y,z,t]]/(x^2 + y^2 + z^2 + t^2) \) (a non-degenerate quadratic singularity). If \( \sigma : \tilde{W} \to W \) is the blowup of \( a \) and \( Q = \sigma^{-1}(a) \), then \( \tilde{W} \) is non-singular along \( Q \), \( Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \), and the normal sheaf \( \mathcal{O}_Q(Q) \) is isomorphic to \( \mathcal{O}_Q(-1, -1) = \text{pr}^*_1 \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \text{pr}^*_2 \mathcal{O}_{\mathbb{P}^1}(-1) \).

If \( X \) is a topological space and \( Y \subset X \), then \( X/Y \) stands for the topological space obtained from \( X \) by contraction of \( Y \) to a point.

If \( X \) is a smooth projective variety, then \( N_1(X) = \text{Num}(X) \otimes \mathbb{R} \), where \( \text{Num}(X) \) is the group of 1-cycles modulo numerical equivalence; \( \text{NE}(X) \subset N_1(X) \) (resp. \( \overline{\text{NE}}(X) \subset N_1(X) \)) is the cone (resp. the closed cone) of effective 1-cycles on \( X \) modulo numerical equivalence. If \( C \subset X \) is an effective curve, then its class in \( N_1(X) \) is denoted by \( [C] \).

The notation for the Betti numbers of a topological space \( X \) is \( b_i(X) \).

If \( W \) is a 3-dimensional algebraic variety with isolated singularities then by a small resolution we mean a proper morphism of complex varieties \( \pi : W_1 \to W \) such that \( W_1 \) is smooth, \( \pi \) is an isomorphism over the smooth locus of \( W \), and the fibers over singular points of \( W \) have dimension at most 1.

2 Small resolutions

The results of this section seem to be well known, but we failed to find appropriate references.

Proposition 2.1 Suppose that \( W \) is a projective threefold with a unique singular point \( a \) that is a node, and let \( \sigma : \tilde{W} \to W \) be the blowup of the point \( a \). Put \( Q = \sigma^{-1}(a) \subset \tilde{W} \) and let \( i_* : H_2(Q, \mathbb{Q}) \to H_2(\tilde{W}, \mathbb{Q}) \) be the homomorphism induced by the embedding \( Q \hookrightarrow \tilde{W} \). In this setting, either rank \( i_* = 1 \) and \( b_2(W) = b_4(W) \), or rank \( i_* = 2 \) and \( b_2(W) = b_4(W) - 1 \).

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Proof Since \( Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \) is a projective subvariety of \( \widetilde{W} \), rank \( i_{\ast} \geq 1 \). Since \( W \) is homeomorphic to \( \widetilde{W}/Q \), we may consider the following fragments of the exact sequence of the pair \((\widetilde{W}, Q)\):

\[
H_4(Q, \mathbb{Q}) \to H_4(\widetilde{W}, \mathbb{Q}) \to H_4(W, \mathbb{Q}) \to H_3(Q, \mathbb{Q}), \quad (1)
\]

\[
H_2(Q, \mathbb{Q}) \xrightarrow{i_{\ast}} H_2(\widetilde{W}, \mathbb{Q}) \to H_2(W, \mathbb{Q}) \to H_1(Q, \mathbb{Q}), \quad (2)
\]

It follows from (1) that \( b_4(W) = b_4(\widetilde{W}) - 1 \), and it follows from (2) that \( b_2(W) = b_2(\widetilde{W}) - \text{rank } i_{\ast} \). Since the smoothness of \( \widetilde{W} \) implies that \( b_2(\widetilde{W}) = b_4(\widetilde{W}) \), the proposition follows.

Remark 2.2 The condition \( b_2(W) = b_4(W) \) for nodal projective threefolds \( W \) was considered by Cheltsov in [3]. In particular, Cheltsov observes that this condition is equivalent, at least in some important cases, to the \( \mathbb{Q} \)-factoriality of \( W \) and that the \( \mathbb{Q} \)-factoriality of a nodal threefold implies that it has no projective small resolutions.

Again, let \( W \) be a projective threefold with a unique singular point \( a \), and suppose that \( a \) is a node.

Proposition 2.3 In the above setting, let \( \sigma : \widetilde{W} \to W \) be the blowup of \( W \) at \( a \), and put \( Q = \sigma^{-1}(a) \subset \widetilde{W} \). Let \( i_{\ast} : H_2(Q, \mathbb{Q}) \to H_2(\widetilde{W}, \mathbb{Q}) \) be the homomorphism induced by the embedding \( Q \subset \widetilde{W} \). Then the following implications hold:

(i) If \( \text{rank } i_{\ast} = 1 \), then the point \( a \) does not admit a small resolution in the category of projective varieties.

(ii) If \( \text{rank } i_{\ast} = 2 \), then the point \( a \) admits a small resolution in the category of projective varieties.

We begin with some lemmas.

Lemma 2.4 Suppose that \( W \) is a 3-dimensional algebraic variety with a unique singular point \( a \) that admits a small resolution \( \pi : W_1 \to W \), where \( W_1 \) is an algebraic variety; put \( C = \pi^{-1}(a) \) (set-theoretically). Besides, suppose that the singularity of \( W \) at the point \( a \) is rational. Then \( H^1(C, \mathcal{O}_C) = 0 \).

Proof Arguing by contradiction, assume that \( H^1(C, \mathcal{O}_C) \neq 0 \). Denote by \( \widetilde{C} = \pi^*a \) the scheme-theoretical fiber over \( a \), so that \( \widetilde{C}_{\text{red}} \cong C \). Since there exists a surjection \( \mathcal{O}_{\widetilde{C}} \to \mathcal{O}_C \) and since the functor \( H^1(\cdot) \) is right exact on one-dimensional schemes, one has \( H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) \neq 0 \). If \( j \subset \mathcal{O}_{W_1} \) is the ideal sheaf of the closed subscheme \( \widetilde{C} \subset W_1 \), then, by the same right exactness, \( H^1(\mathcal{O}_{W_1}/j^n) \) surjects onto \( H^1(\mathcal{O}_{W_1}/j^n) \) for any natural \( n \). Hence, \( \lim\nolimits_{n} H^1(\mathcal{O}_{W_1}/j^n) \) surjects onto \( H^1(\mathcal{O}_{\widetilde{C}}) \) \( \neq 0 \). So, \( \lim\nolimits_{n} H^1(\mathcal{O}_{W_1}/j^n) \neq 0 \). By the theorem on formal functions (see for example [6, Theorem 11.1]) this implies that the \( m_{a} \)-adic completion of the stalk of \( R^1\pi_{\ast}\mathcal{O}_{W_1} \) at \( a \) is not zero. Hence, the stalk itself is not zero either, which contradicts the hypothesis that the singularity of \( W_1 \) at \( a \) is rational.

Lemma 2.5 Suppose that \( W \) is a projective threefold with a unique singular point \( a \) and that \( \pi : W_1 \to W \) is its small resolution in the category of complex spaces. Then the singular cohomology \( H^k(\pi^{-1}(a), \mathbb{Q}) \) is independent of the small resolution \( \pi \), for all \( k \).
**Proof** This proof is an adaptation of a less elementary (but quicker) proof suggested by Braverman.

In the proof we will work with sheaves and complex varieties in the classical topology. Suppose that \( \pi : W_1 \to W \) is a small resolution of \( W \) and put \( C = \pi^{-1}(a) \). The morphism \( \pi \) induces an isomorphism between \( W_1 \setminus C \) and \( W \setminus \{a\} \); we denote this complex manifold by \( U \). Let \( i : U \hookrightarrow W \), \( j : U \hookrightarrow W_1 \) be the natural embeddings. Let \( Q_U \) (resp. \( Q_{W_1} \)) be the constant sheaf with the stalk \( Q \) on \( U \) (resp. \( W_1 \)). Since \( \pi \circ j = i \), there exists a first quadrant spectral sequence

\[
E_2^{pq} = R^p\pi_* (R^q j_* Q_U) \implies R^{p+q} i_* Q_U
\]

(see [5, Theorem III.8.3e], where we put \( \mathcal{R}_X = \mathbb{Z} \) (the constant sheaf)). Since \( C \) is a 1-dimensional complex subspace in the 3-dimensional smooth complex manifold \( W_1 \), one has \( R^0 j_* Q_U = Q_{W_1} \) and \( R^q j_* Q_U = 0 \) for \( q = 1, 2 \), so \( E_2^{pq} = 0 \) for \( 0 < q < 3 \). Taking these vanishing into account, one infers from (3) that \( R^k \pi_* Q_{W_1} \cong R^k i_* Q_U \) for \( 0 \leq k \leq 2 \). Besides, \( R^k \pi_* Q_{W_1} = 0 \) for \( k > 2 \) since the fibers of \( \pi \) are at most (complex) 1-dimensional. Thus, the sheaves \( R^k \pi_* Q_{W_1} \) are independent of the choice of \( \pi \).

It remains to observe that for any natural \( k \) the stalk of \( R^k \pi_* Q \) at \( a \) is isomorphic to \( H^k(\pi^{-1}(a), \mathbb{Q}) \), so this cohomology is independent of the choice of \( \pi \) as well. \( \square \)

**Remark 2.6** Actually, it is the direct image \( R\pi_* Q_{W_1} \) that is independent of the choice of \( \pi \). To wit, the latter direct image is isomorphic (in the derived category of constructible sheaves on \( W \)) to \( \tau_{\leq 2} R\pi_* Q_U \), where \( \tau_{\leq 2} \) is the truncation functor.

**Lemma 2.7** Let \( W \) be a three-dimensional projective variety, smooth except for a point \( a \in W \) that is a node, and suppose that \( f : Z \to W \) is a projective birational morphism such that \( Z \) is smooth and \( f \) induces an isomorphism between \( Z \setminus f^{-1}(a) \) and \( W \setminus \{a\} \). Put \( f^{-1}(a) = F \) (set-theoretically). If \( F \cong \mathbb{P} (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (-n)) \) for some \( n \geq 0 \), then \( F \cong \mathbb{P}^1 \times \mathbb{P}^1 \) and the mapping \( f \) is isomorphic to the blowup \( \sigma : \tilde{W} \to W \) of the point \( a \in W \).

**Proof** Let \( \mathcal{I}_a \subset \mathcal{O}_W \) be the ideal sheaf of the point \( a \). Since \( Z \) is smooth and the zero locus of the ideal \( f^{-1}\mathcal{I}_a \) coincides with the surface \( F \subset Z \), one has \( f^{-1}\mathcal{I}_a \cdot \mathcal{O}_Z = \mathcal{O}_Z(-mF) \) for some \( m > 0 \), which is an invertible ideal sheaf. Now it follows from the universality of blowup [6, Proposition II.7.14] that there exists a morphism \( g : Z \to \tilde{W} \), where \( \sigma : \tilde{W} \to W \) is the blowup of \( W \) at \( a \), such that \( \sigma \circ g = f \).

Let \( Q \subset \tilde{W} \) be the exceptional divisor of the blowup \( \sigma \). Since \( f \) induces an isomorphism between \( Z \setminus f^{-1}(a) \) and \( W \setminus \{a\} \) and \( \sigma \) induces an isomorphism between \( \tilde{W} \setminus Q \) and \( W \setminus \{a\} \), one concludes that \( g \) induces an isomorphism between \( Z \setminus F \) and \( \tilde{W} \setminus Q \). Since \( g(Z \setminus F) = \tilde{W} \setminus Q \) and \( g(Z) \) is closed in \( W \), one has \( g(F) = Q \). We claim that \( n = 0 \). Arguing by contradiction, suppose that \( n > 0 \).

Identify \( Q \) with the smooth quadric in \( \mathbb{P}^3 \). Let \( \ell, m \subset Q \) be generators of the quadric, from different families, and let \( g' : F \to Q \) be the restriction of \( g \) to \( Q \). Put \( L = (g')^* \ell \) and \( M = (g')^* m \) (the pullback divisors on \( F \)). Since the self-intersections \( (\ell, \ell) = (m, m) = 0 \) and the complete linear systems \( |\ell| \) and \( |m| \) on \( Q \) have no
basepoints, we conclude that \((L, L) = (M, M) = 0\) and the complete linear systems \(|L|\) and \(|M|\) have no basepoints either.

Since \(F \cong \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-n))\), one has \(\text{Pic}(F) = \mathbb{Z}s \oplus \mathbb{Z}f\), where \(f\) is the class of a fiber and the intersection indices are \((s, s) = -n\), \((s, f) = 1\), \((f, f) = 0\) (see [6, Proposition 2.9]). Now suppose that \(D \sim as + bf\) is an effective divisor on \(F\) such that \((D, D) = 0\) and \(|D|\) has no basepoints. We claim that such a \(D\) is linearly equivalent to a multiple of a fiber. Indeed, it follows from our assumptions that

\[
\begin{align*}
(D, D) &= -na^2 + 2ab = 0, \\
(D, s) &= -na + b \geq 0, \\
(D, f) &= a \geq 0.
\end{align*}
\]

If \(a > 0\), then it follows from (4) that \(b = na/2\) and \(-na/2 \geq 0\), whence \(n = 0\), which contradicts our assumption \(n > 0\); if \(a = 0\), then \(D \sim bf\), which proves our claim.

It follows from what we have just proved that both the divisors \(L\) and \(M\) are linearly equivalent to a multiple of the fiber, and so is \(L + M\). Since \(\mathcal{O}_F(L + M) = (g')^*\mathcal{O}_Q(1)\), this contradicts the fact that \(g'(F)\) is two-dimensional.

Thus, \(n = 0\) and \(F\) is isomorphic to the smooth two-dimensional quadric. It follows from the fact that the two-dimensional quadric does not contain exceptional curves that \(g': F \to Q\) has finite fibers, so the morphism \(g\) has finite fibers, so \(g\) is finite. Since \(g\) is birational and \(\widetilde{Y}\) is smooth, \(g\) is an isomorphism.

**Proof of assertion (i) of Proposition 2.3** Given that \(i_s = 1\), we have to prove that a projective small resolution of the point \(a \in W\) does not exist.

Arguing by contradiction, suppose that \(\pi: W_1 \to W\), where \(W_1\) is a projective variety, is a small resolution, and put \(C = \pi^{-1}(a)\) (set-theoretically). Lemma 2.4 implies that \(H^1(C, \mathcal{O}_C) = 0\). There exists, in the category of complex spaces, a small resolution of the node \(a\) for which the fiber over \(a\) is isomorphic to \(\mathbb{P}^1\) (see for example [1, Lemma 4] or [14]). Now Lemma 2.5 implies that \(H^2(C, \mathcal{O}) \cong H^2(\mathbb{P}^1, \mathcal{O})\) (singular cohomology), so \(C\) is irreducible. Putting these two facts together, one concludes that \(C \cong \mathbb{P}^1\).

Let \(s: Z \to W_1\) be the blowup of the curve \(C\) in \(W_1\). Since \(C \cong \mathbb{P}^1\), one has \(s^{-1}(C) \cong F = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-n))\) for some \(n \geq 0\). The composition \(\pi \circ s: Z \to W\) blows down the surface \(F\), so, by Lemma 2.7 above, \(F\) is isomorphic to the smooth two-dimensional quadric and \(\pi \circ s: Z \to W\) is isomorphic to \(\sigma : \widetilde{W} \to W\), where \(\widetilde{W}\) is the blowup of \(a\); we identify \(Z\) with \(\widetilde{W}\) and \(F\) with \(Q\) via this isomorphism. In the quadric \(F \subset Z\), let \(\ell \subset F\) be a fiber of the projection \(\sigma|_F: F \to C\), and let \(m \subset F\) be a “line of the ruling” that is mapped onto \(C\) by \(\sigma\). We supposed that \(W_1\) is a projective variety; if \(H\) is an ample divisor on \(W_1\), then \((s^*H, \ell) = 0\) and \((s^*H, m) > 0\), so the images of the fundamental classes of \(\ell\) and \(m\) in \(H_2(Z, \mathbb{Q}) \cong H_2(\widetilde{W}, \mathbb{Q})\) are not proportional. Hence, the rank of the mapping \(H_2(Q, \mathbb{Q}) \to H_2(\widetilde{W}, \mathbb{Q})\) induced by the embedding \(Q \hookrightarrow \widetilde{W}\) is equal to 2, which contradicts our hypothesis.

To prove assertion (ii) of Proposition 2.3 we will need yet another lemma.
Lemma 2.8 Suppose that $X$ is a smooth projective threefold and $F \subset X$ is an irreducible surface. Let $\NE_1 \subset N_1(X)$ be the cone of effective 1-cycles on $X$ such that all their components lie in $F$, and let $\NE_2 \subset N_1(X)$ be the cone of effective 1-cycles on $X$ such that none of their components lies in $F$. In this setting, if $\overline{\NE}_i$, $i = 1, 2$, is the closure of $\NE_i$, then $\overline{\NE}(X) = \overline{\NE}_1 + \overline{\NE}_2$.

Proof Observe first that $\overline{\NE}_1 + \overline{\NE}_2$ is closed in $N_1(X)$. Indeed, if $A, B \subset \mathbb{R}^n$ are closed convex cones such that $A \cap (-B) = \{0\}$, then their sum $A + B \subset \mathbb{R}^n$ is closed (see for example [7, Theorem 2.1] or [2, Theorem 3.2]), so to prove this assertion it suffices to show that $\overline{\NE}_1 \cap (-\overline{\NE}_2) = \{0\}$. Now if $u \in \overline{\NE}_1 \cap (-\overline{\NE}_2)$ and if $H$ is an ample divisor on $X$, then $(u, H) \geq 0$ since $u \in \overline{\NE}_1$, and $(u, H) \leq 0$ since $u \in -\overline{\NE}_2$, whence $(u, H) = 0$. Since $u \in \overline{\NE}(X)$, Kleiman’s criterion for amplitude (see [11, Theorem 1.4.29]) implies that $u = 0$, and we are done.

Since, by the very construction, $\overline{\NE}(X) = \overline{\NE}_1 + \overline{\NE}_2 \subset \overline{\NE}_1 + \overline{\NE}_2$ and since $\overline{\NE}_1 + \overline{\NE}_2$ is closed, one has $\overline{\NE}(X) \subset \overline{\NE}_1 + \overline{\NE}_2$. Taking into account that $\overline{\NE}_1, \overline{\NE}_2 \subset \overline{\NE}(X)$, we are done. \hfill \Box

Proof of assertion (ii) of Proposition 2.3 It is clear that one can embed $\tilde{W}$ in a projective space so that the “lines” of two rulings on the quadric $Q = \sigma^{-1}(a) \subset \tilde{W}$ are embedded as actual lines; to that end, it suffices to consider a projective embedding $W \subset \mathbb{P}^N$, put

$$\tilde{\mathbb{P}}^N = \{(x, L) \in \mathbb{P}^N \times \mathbb{P}^{N-1} : x \in L\},$$

where $\mathbb{P}^{N-1}$ is the set of lines in $\mathbb{P}^N$ passing through $a$, and observe that

$$\tilde{W} = p^{-1}(W \setminus \{a\}) \subset \tilde{\mathbb{P}}^N,$$

where $p : \tilde{\mathbb{P}}^N \to \mathbb{P}^N$ is induced by the projection on the first factor. Till the end of the proof we will fix a projective embedding of $\tilde{W}$ with these properties.

Let $\ell, m \subset Q$ be lines of two different rulings. We claim that $\mathbb{R}_+[\ell]$ is an extremal ray in $\overline{\NE}(\tilde{W})$.

Observe that, since $\emptyset_Q(Q) \cong \emptyset_Q(-1, -1)$, one has $\omega_{\tilde{W}} \otimes \emptyset_Q \cong \emptyset_Q(-1, -1)$, so $(\ell, K_{\tilde{W}}) = -1 < 0$. Thus, to prove that $\mathbb{R}_+[\ell]$ is an extremal ray it remains to show that if $[\ell] = u + v$, where $u, v \in \overline{\NE}(\tilde{W})$, then $u$ and $v$ are multiples of $[\ell]$.

Suppose now that $[\ell] = u + v$, $u, v \in \overline{\NE}(\tilde{W})$, and put $X = \tilde{W}$ and $F = Q$ in Lemma 2.8. One concludes that $u = u_1 + u_2$ and $v = v_1 + v_2$, where, using the notation of the above-mentioned lemma, $u_1, v_1 \in \overline{\NE}_1$ and $u_2, v_2 \in \overline{\NE}_2$. Putting $w_1 = u_1 + v_1 \in \overline{\NE}_1$, $w_2 = u_2 + v_2 \in \overline{\NE}_2$, one has $[\ell] = w_1 + w_2$.

Let $H$ be the divisor class of a hyperplane section of $\tilde{W}$. Intersecting $\ell$ with $H$ and with $Q$, one has

$$1 = (w_1, H) + (w_2, H),$$

$$-1 = (w_1, Q) + (w_2, Q).$$

(5)

Since $w_1$ is a linear combination of curves lying in $Q$ and $\emptyset_Q(Q) \cong \emptyset_Q(-H)$, one has $(w_1, Q) = -(w_1, H)$. Adding equation (5) one obtains $(w_2, H) + (w_2, Q) = 0$. \hfill \Box
However, \((w_2, H) \geq 0\) since \(H\) is ample and \((w_2, Q) \geq 0\) since no component of \(w_2\) lies in \(Q\). Hence, \((w_2, H) = (w_2, H) + (v_2, H) = 0\). Since \(H\) is ample, \((u_2, H) \geq 0\) and \((v_2, H) \geq 0\), whence \(u_2 = v_2 = 0\) by Kleiman's criterion [11, Theorem 1.4.29].

Thus, \(u = u_1, v = v_1, \) so \(u, v \in \overline{\text{NE}}_1\). Since \(\text{Pic}(Q)\) is generated by the classes of \(\ell\) and \(m\), one has \(\overline{\text{NE}}_1 = \{a[\ell] + b[m] : a, b \geq 0\}\). Observe now that \([\ell]\) and \([m]\) are linearly independent in \(N_1(X)\). Indeed, image of the fundamental class of a projective curve \(C \subset \tilde{W}\) in the second rational homology group of \(\tilde{W}\) is uniquely determined by intersection indices of \(C\) with divisors. Now if \([\ell]\) and \([m]\) are linearly dependent, then images of fundamental classes of \(\ell\) and \(m\) are proportional in \(H_2(\tilde{W}, \mathbb{R})\), so rank \(i_\ast = 1\), which contradicts our hypothesis. Thus, if \(u = a[\ell] + b[m], v = a'[\ell] + b'[m]\), and \([\ell] = u + v, \) one has \(b + b' = 0\). Since \(b, b' \geq 0\), this implies that \(b = b' = 0\), so \(u\) and \(v\) are multiples of \(v, \) as required.

Thus, \(\mathbb{R}_+[\ell]\) is an extremal ray. Since \([\ell]\) is not a multiple of \([m]\), this ray is of the type \((1.2.1)\) in the notation of [9]. Hence, there exists a birational mapping \(s : \tilde{W} \to W_1, \) where \(W_1\) is smooth and \(s^{-1}\) defines the blowing up of a curve \(C \subset W_1\) such that \(C \cong \mathbb{P}^1\) and \(N_{W_1|C} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\). The rational mapping \(s_1 = \sigma \circ s^{-1} : W_1 \to W\) is continuous in the classical topology; since \(W_1\) is smooth, the mapping \(s_1\) is a morphism. Hence, \(s_1 : W_1 \to W\) is the desired small resolution. \(\square\)

### 3 Some consequences of Condition (A)

Suppose that \(X \subset \mathbb{P}^N\) is a smooth projective variety of dimension 4 for which \(X^* \subset (\mathbb{P}^N)^*\) is a hypersurface. Let \(Y_0\) be a hyperplane section with a single singular point which is a node.

If \(\tilde{Y}\) is the blowup of \(Y_0\) at the singularity, let \(Q \subset \tilde{Y}\) be the inverse image of the singularity; \(Q\) is isomorphic to the smooth two-dimensional quadric. Let \(i_\ast : H_2(Q, \mathbb{Q}) \to H_2(\tilde{Y}, \mathbb{Q})\) be the homomorphism induced by the embedding \(Q \hookrightarrow \tilde{Y}\).

**Proposition 3.1** In the above setting, if Condition (A) holds for \(X\), then rank \(i_\ast = 1\), and if Condition (A) does not hold for \(X\), then rank \(i_\ast = 2\).

**Proof** A general smooth hyperplane section \(Y \subset X\) can be included in a Lefschetz pencil one of the fibers of which is \(Y\). By \(S \subset Y\) we denote the vanishing cycle (homeomorphic to the sphere \(S^3\)) that is contracted to an \(A_1\) singularity, so that \(Y_0\) is homeomorphic to \(Y/S\). It is well known (see [10] or [4, Exposés XVII and XVIII]) that the monodromy group acting in \(H^3(Y, \mathbb{Q})\) is generated by (pseudo)reflections in the classes of vanishing cycles corresponding to singular fibers of the Lefschetz pencil, and that all these classes are conjugate by the action of the monodromy group. Thus, Condition (A) for the variety \(X\) is violated if and only if the vanishing cycle \(S\) is homologous to zero in \(H_3(Y, \mathbb{Q})\).

Consider the following two fragments of the exact sequence of the pair \((Y, S)\):

\[
H_2(S, \mathbb{Q}) \to H_2(Y, \mathbb{Q}) \to H_2(Y_0, \mathbb{Q}) \to H_1(S, \mathbb{Q}),
\]

\[
H_4(S, \mathbb{Q}) \to H_4(Y, \mathbb{Q}) \to H_4(Y_0, \mathbb{Q}) \to H_3(S, \mathbb{Q}) \xrightarrow{j_*} H_3(Y, \mathbb{Q})
\]
where \( j : S \to Y \) is the natural embedding.

It follows from (6) that \( b_2(Y) = b_2(Y_0) \), and it follows from (7) and Poincaré duality for \( Y \) that \( b_2(Y_0) = b_2(Y) \) if Condition (A) holds and that \( b_4(Y_0) = b_2(Y_0) + 1 \) if Condition (A) does not hold. Now the desired result follows from Proposition 2.1.

\[ \square \]

4 Conclusion

Now we can prove Propositions 1.1 and 1.6.

**Proof of Proposition 1.6** Suppose that \( X \subset \mathbb{P}^n \) is a smooth 4-dimensional projective variety over \( \mathbb{C} \) such that \( X^* \subset (\mathbb{P}^n)^* \) is a hypersurface. It follows from [8, Theorem 17] that a general singular hyperplane section of \( X \) has precisely one singular point, and this point is a node.

Suppose now that Condition (A) holds for \( X \); let \( Y_0 \subset X \) be a hyperplane section with a unique singularity \( a \in Y_0 \) which is a node. It follows from Proposition 3.1 that \( \text{rank } i_* = 1 \), where \( i : Q \to \tilde{Y}_0 \) is the natural embedding, \( \sigma : \tilde{Y}_0 \to Y_0 \) is the blowup of \( a \), and \( Q = \sigma^{-1}(a) \). Then Proposition 2.3 implies the non-existence of a small resolution for \( Y_0 \).

On the other hand, if Condition (A) does not hold for \( X \), then, keeping the previous notation, it follows from Proposition 3.1 that \( \text{rank } i_* = 2 \), whence, by virtue of Proposition 2.3, a projective small resolution for \( Y_0 \) exists.

\[ \square \]

**Proof of Proposition 1.1** Suppose that \( X \subset \mathbb{P}^n \) is a smooth projective fourfold such that \( H^0(X, \omega_X(1)) \neq 0 \). If \( Y \subset X \) is a smooth hyperplane section, then it follows from the implication iii) \( \Rightarrow \) iv) in [12, Proposition 6.1] that \( b_3(Y) > b_3(X) \), in particular, the restriction homomorphism \( H^3(X, \mathbb{Q}) \to H^3(Y, \mathbb{Q}) \) cannot be surjective. A theorem of A. Landman (see [8, Theorem 22]) asserts that if \( X \subset \mathbb{P}^n \) is a smooth \( d \)-dimensional projective manifold such that \( X^* \subset (\mathbb{P}^n)^* \) is not a hypersurface, then \( b_{d-1}(Y) = b_{d-1}(X) \). Thus, if \( H^0(X, \omega_X(1)) \neq 0 \) then \( X^* \) is a hypersurface and Condition (A) is satisfied for \( X \). Now Proposition 1.6 applies.

\[ \square \]

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