KÁRMÁN VORTEX STREET FOR THE GENERALIZED SURFACE QUASI-GEOSTROPHIC EQUATION

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Abstract. We are concerned with the existence of periodic travelling-wave solutions for the generalized surface quasi-geostrophic (gSQG) equation (including incompressible Euler equation), known as von Kármán vortex street. These solutions are of $C^1$ type, and are obtained by studying a semilinear problem on an infinite strip whose width equals to the period. By a variational characterization of solutions, we also show the relationship between vortex size, travelling speed and street structure. In particular, the vortices with positive and negative intensity have the same or different scaling size in our construction, which constitutes the regularization for Kármán point vortex street.

Keywords: Kármán vortex street; the gSQG equation; $C^1$ type solutions; Lyapunov-Schmidt reduction.

1. Introduction and main results

When a two-dimensional bluff body is placed in a uniform stream moving at certain velocities, vortices with opposite intensity will arise along two parallel staggered rows, which is observed as water flow going through a pipe, or wind passing an obstacle. The best-known event caused by this pattern is the fall of Tacoma narrows bridge in 1940. Experimental study of periodic vortex shedding can be traced back to 1870s in [30, 36], while the theoretical model was proposed by von Kármán [21, 22], and hence this phenomenon is known as von Kármán vortex street nowadays in literatures. For the reason that the exact problem is complex from a theoretical point of view, some simplified models were investigated in [2, 26, 32]. The main idea is using different kinds of solutions to approximate Kármán point vortex street, since the latter is the basic and simplest pattern of periodic vortex shedding.

It is notable that although viscosity and bluff body are involved in the generation of Kármán vortex street, they seem not to influence anymore the evolution of the vortex street (For more details on the effect of Reynolds number and shape of bluff body, we refer to [13, 18, 29] and references therein). This fact indicates that an inviscid incompressible fluid model can be used to describe the vortex dynamics in Kármán vortex street. In [34, 35], Saffman and Schatzman studied Kármán vortex street for Euler flow. Under the assumption that the support of each vortex has finite area, they conducted a series of numerical simulation to show the existence of one-directional periodic vortex shedding.

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travelling at a constant speed. Moreover, they obtained a linear stability of Kármán vortex street where the size of vortices and street width satisfy a special condition.

We are going to study the existence of $C^1$ type Kármán vortex street for the generalized surface quasi-geostrophic (gSQG) equation, which can be written as follows

$$\begin{aligned}
\begin{cases}
\partial_t \vartheta + v \cdot \nabla \vartheta = 0 & \text{in } \mathbb{R}^2 \times (0, T), \\
v = \nabla_\perp \psi, \ \psi = (\Delta)^{-s} \vartheta & \text{in } \mathbb{R}^2 \times (0, T), \\
\vartheta|_{t=0} = \vartheta_0 & \text{in } \mathbb{R}^2,
\end{cases}
\end{aligned}$$

(1.1)

with $0 < s \leq 1$, where $(x_1, x_2) \perp = (x_2, -x_1)$, $\nabla_\perp = (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$, $\vartheta(x, t) : \mathbb{R}^2 \times (0, T) \to \mathbb{R}$ the active scalar being transported by the velocity field $v(x, t) : \mathbb{R}^2 \times (0, T) \to \mathbb{R}^2$ generated by $\vartheta$, and $\psi$ the corresponding stream function. The operator $(-\Delta)^{-s}$ is defined by

$$(-\Delta)^{-s} \vartheta(x) = \int_{\mathbb{R}^2} G_s(x - y) \vartheta(y) dy,$$

where $G_s$ is the fundamental solution of $(-\Delta)^{-s}$ in $\mathbb{R}^2$ given by

$$G_s(x) = \begin{cases}
\frac{1}{2\pi} \ln \frac{1}{|x|}, & \text{if } s = 1, \\
\frac{c_s}{|x|^{2-2s}}, & \text{if } 0 < s < 1,
\end{cases}$$

with $\Gamma$ the Euler gamma function.

When $s = 1$, (1.1) is the vorticity formulation of 2D incompressible Euler equation. When $s = \frac{1}{2}$, (1.1) is the surface quasi-geostrophic (SQG) equation, which is relevant to the atmosphere circulation and ocean dynamics [10]. The gSQG model (1.1) with $0 < s < 1$ was proposed by Córdoba et al. in [11], and was taken as a generalization of the Euler equation and the SQG equation.

In 1963, Yudovich [40] proved the global well-posedness of (1.1) with the initial data in $L^1 \cap L^\infty$ for $s = 1$. However, the global well-posedness for the general case $0 < s < 1$ remains unknown due to the loss of regularity for velocity field. In [10], Constantin et al. established local well-posedness of the gSQG equation for classical solutions, which is known for sufficiently regular initial data by [8, 14, 25]. The study of local existence in different function spaces can be found in [7, 27, 38, 39]. Resnick [31] proved global existence for weak solutions to the SQG equations with any initial data in $L^2$. This remarkable result was then improved by Marchand [28] to any initial data belonging to $L^p$ with $p > 4/3$. On the other hand, Kiselev and Nazarov [24] constructed solutions of the gSQG equations with arbitrary Sobolev growth.

As concrete examples for the gSQG flow, various kinds of global solutions to (1.1) are constructed. There are mainly two kinds of global solutions: The rotating solutions and the travelling-wave solutions. The rotating solutions are also known as the V-states, and the first explicit non-trivial V-state is Kirchhoff ellipse given in [23] for $s = 1$. In the past decades, different methods were developed to construct solutions of this type, and we refer to [11, 16, 19, 20, 37] for more discussion. As for the travelling-wave solutions, the early example is the Lamb dipole or Chaplygin-Lamb dipole [26], which is a travelling vortex pair in the case $s = 1$. In [11, 14, 15, 17], several kinds of travelling-wave solutions were given...
by a similar approach for rotating solutions. We shall bring the attention of readers to that Kármán vortex street is a special kind of travelling-wave solution other than vortex pairs, which consists of infinite vortices and has a periodic structure. Furthermore, different from vortex pairs which can only travel along their axis of symmetry, the uniform travelling speed of a vortex street can be chosen in other directions by adjusting the phase difference of two sides of the street. We will show these properties later in our main theorem.

To explain the problem we are to address and state our results, we need to introduce notations for convenience: \( \delta_z \) is the Dirac measure located at \( z \in \mathbb{R}^2 \), \( \chi_\Omega \) denotes the characteristic function of \( \Omega \subset \mathbb{R}^2 \), \( e_i \) is the unit vector of \( x_i \) axis for \( i = 1, 2 \); \( O_\varepsilon(1) \) will be used to denote quantities which stay bounded as \( \varepsilon \) goes to zero and \( o_\varepsilon(1) \) to denote quantities which go to zero as \( \varepsilon \) goes to zero. \( O_\varepsilon(1) \) and \( o_\varepsilon(1) \) only depend on \( \varepsilon \).

As a preliminary, we cast an eye on the most singular type of Kármán vortex street, where solutions to (1.1) are composed of two parallel rows of point vortices. If we denote \( p = (-d, -a) \), \( q = (d, a) \) with \( d \geq 0 \) as half of street width and \( a \in [0, l/2) \) as half of the phase(The case \( d = a = 0 \) must be ruled out), then these \( x_2 \)-directional periodic travelling-wave solutions take the form

\[
\vartheta(x, t) = \sum_{k \in \mathbb{Z}} \delta_p(x + kl e_2 - tU_{d,l,a}) - \sum_{k \in \mathbb{Z}} \delta_q(x + kl e_2 - tU_{d,l,a}),
\]

where \( l > 0 \) is the period length, and \( U_{d,l,a} \in \mathbb{R}^2 \) is the uniform travelling speed. According to the dynamic formula for point vortex model given by Rosenzweig [33], the travelling speed \( U_{d,l,a} \) can be computed directly as

\[
U_{d,l,a} = -C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{(p - q + kl e_2)}{|p - q + kl e_2|^{4-2s}},
\]

where

\[
C_s = \begin{cases} 
1/2\pi, & \text{if } s = 1, \\
(2 - 2s)c_s, & \text{if } 0 < s < 1.
\end{cases} \tag{1.2}
\]

In particular, when \( a = 0 \) or \( l/4 \) we can use symmetry of the solution with respect to \( x_2 = kl \) or \( kl \pm l/4 \) to derive that

\[
U_{d,0} = W_1 e_2, \quad U_{d,l/4} = W_2 e_2,
\]

where

\[
W_1(d) = C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{2d}{(4d^2 + k^2l^2)^{2-s}}, \tag{1.3}
\]

and

\[
W_2(d) = C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{2d}{(4d^2 + (kl + \frac{l}{2})^2)^{2-s}}. \tag{1.4}
\]
Recently, García [13] constructed a family of patch type solutions to approximate Kármán point vortex street with $1/2 < s \leq 1$. These solutions have the following explicit expression

$$\vartheta_\varepsilon(x) = \frac{1}{\varepsilon^2 \pi} \sum_{k \in \mathbb{Z}} \chi_{D_\varepsilon + (-d,kl-a)}(x - t \mathbf{U}_\varepsilon) - \frac{1}{\varepsilon^2 \pi} \sum_{k \in \mathbb{Z}} \chi_{-D_\varepsilon + (d,kl+a)}(x - t \mathbf{U}_\varepsilon),$$

where $D_\varepsilon$ is a perturbation of the disc $B_\varepsilon(0)$ centered at the origin with sufficiently small radius $\varepsilon$, and $\mathbf{U}_\varepsilon$ is the uniform travelling speed. The approach in [13] highly relies on the patch structure: Employing Biot-Savart law, the author obtained the contour dynamic equation for vortex boundary, and calculated its linearization at point vortex solutions. The key point of the construction is to choose $\mathbf{U}_\varepsilon = \mathbf{U}_{d,l,a} + o(1)$ properly, so that an isomorphism condition is satisfied for linearized operator. Then a family of nontrivial solutions can be obtained by implicit function theorem.

In the present paper, we will focus on the construction of $C^1$ type Kármán vortex street. To be more precise, we will prove the existence of travelling-wave solutions to (1.1) with the formulation

$$\vartheta_\varepsilon(x,t) = \vartheta_{0,\varepsilon}(x - t \mathbf{U}_\varepsilon),$$

where $\mathbf{U}_\varepsilon \in \mathbb{R}^2$ is the uniform travelling speed, $\varepsilon$ is some size parameter, and the initial data $\vartheta_{0,\varepsilon}(x)$ is given by

$$\vartheta_{0,\varepsilon}(x) = \sum_{k \in \mathbb{Z}} \vartheta_{1,\varepsilon}(x + k\mathbf{e}_2) - \sum_{k \in \mathbb{Z}} \vartheta_{2,\varepsilon}(x + k\mathbf{e}_2).$$

Here, we assume that $\vartheta_{1,\varepsilon}(x), \vartheta_{2,\varepsilon}(x) \in C^1(\mathbb{R}^2)$ and satisfy

$$\text{supp}(\vartheta_{1,\varepsilon}) \subset B_{L\varepsilon}((-d,-a)), \quad \text{supp}(\vartheta_{2,\varepsilon}) \subset B_{L\sigma(\varepsilon)}((d,a)),$$

where $L > 0$ is some large constant, $\sigma(\varepsilon) > 0$ is the size function in the sense that $L\varepsilon$ and $L\sigma(\varepsilon)$ are the upper bounds for the diameters of supports of vortices with positive vorticity and vortices with negative vorticity respectively. A novelty of our construction is that vortices on the right hand side may have a different size function compared with those on the left hand side, that is, we make the following assumption on $\sigma(\varepsilon)$:

(H) As $\varepsilon \to 0$, $\sigma(\varepsilon)/\varepsilon \leq C$ for fixed $C > 0$, and $\varepsilon^\tau/\sigma(\varepsilon) = o(1)$ for some $1 < \tau \leq 2$.

There are several difficulties in the construction of $C^1$ type solutions mentioned above. Firstly, we do not impose any symmetry with respect to $x_2$-axis, and $\vartheta_{1,\varepsilon}(x), \vartheta_{2,\varepsilon}(x)$ may have different profiles apart from the difference in vortex size. Secondly, due to the general $C^1$ type vorticity, the velocity of flow can not be recovered by vortex boundary alone, and the method by studying contour dynamic equation is invalid. To achieve our goal, we will take another approach, which is from a new angle of view but also reduces the construction into a finite-dimensional problem. We will briefly explain our strategy. For easy understanding, we first assume $a = 0$, $\vartheta_{0,\varepsilon}$ is symmetric with respect to $x_2 = kl$; or $a = l/4$, $\vartheta_{0,\varepsilon}$ is symmetric with respect to $x_2 = kl \pm l/4$, so that the travelling speed is in $x_2$ direction and we can write $\mathbf{U}_\varepsilon = W_\varepsilon \mathbf{e}_2$ for some scalar $W_\varepsilon$. 
According to (1.5), by introducing the $x_2$-directional periodic stream function $\tilde{\psi}_\varepsilon$, (1.1) can be rewritten as
\[
(\nabla^2 \tilde{\psi}_\varepsilon - W_\varepsilon e_2) \cdot \nabla \vartheta_{0,\varepsilon}(x) = 0, \quad \tilde{\psi}_\varepsilon = (-\Delta)^{-s} \vartheta_{0,\varepsilon} \quad \text{in} \ \mathbb{R}^2,
\]
which means $\vartheta_{0,\varepsilon}$ is functional related to $\tilde{\psi}_\varepsilon + W_\varepsilon x_1$. It is natural to impose $\vartheta_{0,\varepsilon} = f(\tilde{\psi}_\varepsilon + W_\varepsilon x_1)$ for some $C^1$ monotone $f$, and transform (1.7) into a semilinear elliptic equation
\[
(-\Delta)^s \tilde{\psi}_\varepsilon = f(\tilde{\psi}_\varepsilon + W_\varepsilon x_1) \quad \text{in} \ \mathbb{R}^2. \tag{1.8}
\]
One can easily verify that (1.8) provides a family of classical solutions to (1.7) by theory of regularity for elliptic equations. We will follow the framework in [1, 5] to construct desired solutions to (1.8) by a Lyapunov-Schmidt reduction.

However, there are several new ideas in our construction: since $\tilde{\psi}_\varepsilon$ and $\vartheta_{0,\varepsilon}$ are periodic over $\mathbb{R}^2$, the energy of Kármán vortex street is infinite, which leads to a difficulty for variational characterization of solutions. Inspired by [3] on one-dimensional periodic problem, we will study (1.8) restricted in an infinite strip whose width equals the corresponding typical period. We will give the explicit formula for (1.8) in Section 2. Moreover, our construction needs much more careful estimate compared with [1] due to the different sizes of positive and negative vortices.

When the solvability of (1.9) is considered, another problem arises from the fundamental solution of $(-\Delta)^s$: $G_\varepsilon(x)$ is of order $|x|^{2s-2}$ when $0 < s < 1$, and $\ln(1/|x|)$ when $s = 1$. But $2 - 2s$ is unsatisfactorily less than 1 if $s > 1/2$. This fact may cause the divergence for $L^\infty$ norm of $\psi_\varepsilon$ when we deal with the influence of infinite vortices. Thanks to the unique structure of Kármán vortex street, where each positive vortex matches a negative vortex with equal intensity, we observe that the influence of two equally distant vortex pairs is actually of order $|x|^{2s-4}$. As a result, $\psi_\varepsilon$ has a convergent $L^\infty$ norm, and our method does work as desired.

Having made the preparations, we are now in the position to state our first result.

**Theorem 1.1.** Suppose $s \in (0, 1]$. Then there exist $\varepsilon_0 > 0$ and $\tau$ in (H) such that for any $\varepsilon \in (0, \varepsilon_0)$, (1.1) has a $x_2$-directional periodic travelling-wave solution $\vartheta_\varepsilon(x, t) = \vartheta_{0,\varepsilon}(x - tW_\varepsilon e_2)$, where the initial data $\vartheta_{0,\varepsilon}(x) \in C^1(\mathbb{R}^2)$ is symmetric with respect to $x_2 = kl$ for $k \in \mathbb{Z}$, $l > 0$, and has the form
\[
\vartheta_{0,\varepsilon}(x) = \sum_{k \in \mathbb{Z}} \vartheta_{1,\varepsilon}(x + kle_2) - \sum_{k \in \mathbb{Z}} \vartheta_{2,\varepsilon}(x + kle_2),
\]
with $l > 0$, supp$(\vartheta_{1,\varepsilon}) \subset B_{L_\varepsilon}((-d, 0))$, supp$(\vartheta_{2,\varepsilon}) \subset B_{L_\sigma(\varepsilon)}((d, 0))$ for $\sigma(\varepsilon)$ satisfying (H), $d > 0$, and some large $L > 0$. The scalar $W_\varepsilon$ satisfies
\[
W_\varepsilon = C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{2d}{(4d^2 + k^2l^2)^{2-s}} + o_\varepsilon(1),
\]
where $C_s$ is given in (1.2). Moreover, it holds in the sense of measure

$$\vartheta_{0,\varepsilon}(x) \to \sum_{k \in \mathbb{Z}} \delta_{(-d,kl)}(x) - \sum_{k \in \mathbb{Z}} \delta_{(d,kl)}(x) \quad \text{as} \quad \varepsilon \to 0,$$

Remark 1.2. When $0 < s < 1$, the function $W_1(d)$ given in (1.3) is monotonically decreasing whose range is $(0, +\infty)$. As a result, $W_\varepsilon$ can take any positive values by adjusting $d$. When $s = 1$, there is an explicit formula $W_1(d) = \frac{1}{2} \coth\left(\frac{\pi d}{2}\right)$. Since the range of coth on $\mathbb{R}_+$ is $(1, \infty)$, we deduce that $W_\varepsilon > 0$ if $\varepsilon$ is sufficiently small.

From (1.3), we see that Theorem 1.1 corresponds to the Kármán point vortex street for $a = 0$ when we let $\varepsilon \to 0$. As a counterpart of (1.4), the result for the case $a = l/4$ can be stated as follows.

**Theorem 1.3.** Suppose $s \in (0, 1]$. Then there exist $\varepsilon_0 > 0$ and $\tau$ in (H) such that for any $\varepsilon \in (0, \varepsilon_0)$, (1.1) has a $x_2$-directional periodic travelling-wave solution $\vartheta_\varepsilon(x, t) = \vartheta_{0,\varepsilon}(x - tW_\varepsilon e_2)$, where the initial data $\vartheta_{0,\varepsilon}(x) \in C^1(\mathbb{R}^2)$ is symmetric with respect to $x_2 = kl \pm l/4$ for $k \in \mathbb{Z}$, $l > 0$, and has the form

$$\vartheta_{0,\varepsilon}(x) = \sum_{k \in \mathbb{Z}} \vartheta_{1,\varepsilon}(x + kl e_2) - \sum_{k \in \mathbb{Z}} \vartheta_{2,\varepsilon}(x + kl e_2).$$

with $\text{supp}(\vartheta_{1,\varepsilon}) \subset B_{L\varepsilon}((-d, -l/4))$, $\text{supp}(\vartheta_{2,\varepsilon}) \subset B_{L\varepsilon}((d, l/4))$ for $\sigma(\varepsilon)$ satisfying (H), $d \geq 0$, and some large $L > 0$. The scalar $W_\varepsilon$ satisfies

$$W_\varepsilon = C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{2d}{4d^2 + (kl + \frac{l}{2})^2} + o_\varepsilon(1)$$

with $C_s$ given in (1.2). Moreover, it holds in the sense of measure

$$\vartheta_{0,\varepsilon}(x) \to \sum_{k \in \mathbb{Z}} \delta_{(-d,kl-l/4)}(x) - \sum_{k \in \mathbb{Z}} \delta_{(d,kl+l/4)}(x) \quad \text{as} \quad \varepsilon \to 0.$$

Remark 1.4. When $0 < s < 1$, $W_2(d)$ in (1.4) will first increase and then decrease to 0 on $\mathbb{R}_+$. Hence we have $W_\varepsilon < \sup_{\mathbb{R}_+} W_2 + 1$ if $\varepsilon$ is sufficiently small. While for $s = 1$, it holds $W_1(d) = \frac{1}{2} \tanh\left(\frac{\pi d}{2}\right)$. Since the range of tanh on $\mathbb{R}_+ \cup \{0\}$ is $[0, 1)$, we deduce that $W_\varepsilon < 1/l$ as long as $\varepsilon$ is sufficiently small.

Notice that $d$ can be 0 in Theorem 1.3. In this special case, the vortex street is located along $x_2$-axis and nearly stagnating, namely, travelling speed is almost zero. In particular, if we assume $\vartheta_{0,\varepsilon}$ is even in $x_1$-direction, then the solution $\vartheta_\varepsilon$ is stationary, which gives another example for nontrivial stationary solution to (1.1) with $0 < s \leq 1$ besides the one constructed in [10].

More generally, we have the following result for arbitrary phase $a \in (0, l/2)$, where the uniform travelling speed $U_\varepsilon$ can have different directions other than $x_2$-direction.

**Theorem 1.5.** Suppose $s \in (0, 1]$, $p = (-d, -a)$ and $q = (d, a)$ with $d \geq 0$, $a \in (0, l/2)$. Then there exist $\varepsilon_0 > 0$ and $\tau$ in (H) such that for any $\varepsilon \in (0, \varepsilon_0)$, (1.1) has a $x_2$-directional periodic travelling-wave solution $\vartheta_\varepsilon(x, t) = \vartheta_{0,\varepsilon}(x - tU_\varepsilon)$, where the initial data
\[ \vartheta_0,\varepsilon(x) \in C^1(\mathbb{R}^2) \] has the form
\[ \vartheta_0,\varepsilon(x) = \sum_{k \in \mathbb{Z}} \vartheta_{1,\varepsilon}(x + k\mathbf{e}_2) - \sum_{k \in \mathbb{Z}} \vartheta_{2,\varepsilon}(x + k\mathbf{e}_2). \]
with \( l > 0 \), \( \text{supp}(\vartheta_{1,\varepsilon}) \subset B_{Le}(p) \), \( \text{supp}(\vartheta_{2,\varepsilon}) \subset B_{Le(\varepsilon)}(q) \) for \( \sigma(\varepsilon) \) satisfying (H) and some large \( L > 0 \). The uniform travelling speed \( U_\varepsilon \in \mathbb{R}^2 \) satisfies
\[ U_\varepsilon = -C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{(p - q + k\mathbf{e}_2)^\perp}{|p - q + k\mathbf{e}_2|^{3-2s}} + o(1) \]
with \( C_s \) given in (1.2). Moreover, it holds in the sense of measure
\[ \vartheta_0,\varepsilon(x) \rightharpoonup \sum_{k \in \mathbb{Z}} \delta_p(x + k\mathbf{e}_2) - \sum_{k \in \mathbb{Z}} \delta_q(x + k\mathbf{e}_2) \quad \text{as} \quad \varepsilon \to 0. \]

The solutions constructed above actually constitute the regularization for Kármán point vortex street. Recall that a vortex dynamic system is called a vortex-wave system, if it is composed of highly concentrated vortices known as “vortex”, and relatively scattered vortices known as “wave”. Suppose \( \sigma(\varepsilon) \) satisfies (H) with \( \sigma(\varepsilon) = o(\varepsilon)(1) \). Then compared with the positive vortices, the size of negative vortices in Theorem 1.1.3 and 1.5 has a sharper shrinking rate. So, in this situation, our result can be regarded as the regularization of foresaid vortex-wave system with “vortex” on the right and “wave” on the left.

Our proof will begin with the relatively simple case \( a = 0 \) or \( l/4 \). In Section 2, we consider the gSQG equation with \( 0 < s < 1 \), and give proofs for Theorem 1.1 and 1.3 for this case.

2. Construction for the gSQG Equation with \( 0 < s < 1 \)

In this section we consider the gSQG equation with \( 0 < s < 1 \), and give proofs for Theorem 1.1 and 1.3 for this case.

2.1. Approximate solutions. To regularize Kármán point vortex street, we are going to construct a family of solutions \( \vartheta_{0,\varepsilon} \) to (1.6) such that in the sense of measure
\[ \vartheta_{0,\varepsilon}(x) \rightharpoonup \sum_{k \in \mathbb{Z}} \delta_p(x + k\mathbf{e}_2) - \sum_{k \in \mathbb{Z}} \delta_q(x + k\mathbf{e}_2) \quad \text{as} \quad \varepsilon \to 0. \]

We say a function is \( l \)-periodic, if it takes \( l \) as a period in \( x_2 \) direction. To ensure that the energy of solution is finite, we are to consider the problem in some typical period. For this purpose, we denote \( (-\Delta)^s_x \) as \( (-\Delta)^s \) acting on \( l \)-periodic functions and restricted in the
typical infinite strip domain $\mathbb{R} \times (-l/2, l/2)$ corresponding to the period, which is given by
the explicit formula

$$(-\Delta)^s \psi(x) = \int_{\mathbb{R} \times (-l/2, l/2)} J_s(x-z) (\psi(x) - \psi(z)) \, dz,$$

(2.1)

where

$$J_s(x) = \sum_{k \in \mathbb{Z}} \frac{C_s}{|x + kle_2|^{2s+2}}, \quad C_s = \frac{2^{2s} \Gamma(1-s)}{\pi |\Gamma(-s)|},$$

and $\psi(x)$ is some $l$-periodic restricted in $\mathbb{R} \times (-l/2, l/2)$. We can also denote the inverse
of $(-\Delta)^s$ as $(-\Delta)^{-s}$ with the integral representation

$$(-\Delta)^{-s} \vartheta(x) = \int_{\mathbb{R} \times (-l/2, l/2)} K_s(x-z) \vartheta(z) \, dz,$$  

(2.2)

for scalar function $\vartheta(x)$ with $\text{supp}(\vartheta(x)) \subset \mathbb{R} \times (-l/2, l/2)$.

By the deduction in Section 1, we will consider the following semilinear elliptic problem

$$\begin{cases}
(-\Delta)^s \psi = \varepsilon^{(2-2s)\gamma_1-2}(\psi + W_\varepsilon x_1 - \varepsilon^{2s-2} \lambda_+ \chi_{B_r(p)}) \\
-\sigma(\varepsilon)^{2(2s)\gamma_2-2}(-\psi - W_\varepsilon x_1 - \sigma(\varepsilon)^{2s-2} \lambda_- \chi_{B_r(q)})
\end{cases} \quad \text{in } \mathbb{R} \times (-l/2, l/2),$$

(2.3)

where $\lambda_+$ and $\lambda_-$ are undetermined parameters and will be suitably chosen, $W_\varepsilon$ is the
traveling speed of Kármán vortex street determined by location of $p$, $q$ and $\varepsilon$, $1 < \gamma_1, \gamma_2 < \frac{2+2s}{2-2s}$ ($\gamma_1 \neq \frac{1}{1-s}$), $\sigma(\varepsilon)$ satisfies assumption (H) with $\tau = \min\{\gamma_2, 2\}$, and $r > 0$ is a
small constant such that $B_r(p)$ and $B_r(q)$ are disjoint. Moreover, we assume

$$p = (-d, -a), \quad q = (d, a),$$

where $d > 0$ for $a = 0$; or $d > 0$ for $a = l/4$.

Before giving the approximate solutions to (2.3), we introduce the following fractional
plasma problem, which can be regarded as the limit problem locally.

$$\begin{cases}
(-\Delta)^s u = (u - 1)^\gamma_+ \quad \text{in } \mathbb{R}^2, \\
u(x) \to 0 \quad \text{as } |x| \to \infty,
\end{cases}$$

(2.4)

where $0 < s < 1$ and $1 < \gamma < \frac{2+2s}{2-2s}$. In view of [9], (2.4) has a unique radial solution $U(x)$
known as the ground state with following asymptotic behavior

$$\lim_{|x| \to \infty} U(x) = c_s M_\gamma |x|^{-2+2s}, \quad \lim_{|x| \to \infty} U''(|x|) = -C_s M_\gamma |x|^{-3+2s},$$

where $M_\gamma = \int_{\mathbb{R}^2} (U - 1)^\gamma_+ \, dx > 0$. 
Let radial functions $U_1(x), U_2(x)$ be the ground states of (2.4) with exponent $\gamma = \gamma_1$ and $\gamma = \gamma_2$ respectively. A suitable approximate solution to (2.3) is

$$\Psi_\varepsilon(x) = \varepsilon^{2s-2} \sum_{k \in \mathbb{Z}} \mu_+^{-\frac{2s}{\gamma_1-1}} U_1 \left( \frac{x-p + k\ell e_2}{\varepsilon \mu_+} \right) - \sigma(\varepsilon)^{2s-2} \sum_{k \in \mathbb{Z}} \mu_-^{-\frac{2s}{\gamma_2-1}} U_2 \left( \frac{x-q + k\ell e_2}{\sigma(\varepsilon) \mu_-} \right),$$

where $\mu_+, \mu_-\,\varepsilon$ are positive parameters to be chosen later. To make (2.5) convergent for every $x \in \mathbb{R} \times (-l/2, l/2)$, the above sum is understood in the sense

$$\Psi_\varepsilon(x) = \varepsilon^{2s-2} \mu_+^{-\frac{2s}{\gamma_1-1}} U_1 \left( \frac{x-p}{\varepsilon \mu_+} \right) - \sigma(\varepsilon)^{2s-2} \mu_-^{-\frac{2s}{\gamma_2-1}} U_2 \left( \frac{x-q}{\sigma(\varepsilon) \mu_-} \right) + \lim_{N \to \infty} \sum_{k=1}^{N} \left( \varepsilon^{2s-2} \sum_{m=\pm k} \mu_+^{-\frac{2s}{\gamma_1-1}} U_1 \left( \frac{x-p + m\ell e_2}{\varepsilon \mu_+} \right) - \sigma(\varepsilon)^{2s-2} \sum_{m=\pm k} \mu_-^{-\frac{2s}{\gamma_2-1}} U_2 \left( \frac{x-q + m\ell e_2}{\sigma(\varepsilon) \mu_-} \right) \right).$$

Since as $\varepsilon \to 0$, in the sense of measure,

$$(-\Delta)^s \Psi_\varepsilon(x) \to M_{\gamma_1} \mu_+^{-\frac{2s}{\gamma_1-1}} \delta_p(x) - M_{\gamma_2} \mu_-^{-\frac{2s}{\gamma_2-1}} \delta_q(x),$$

we require that $\mu_+$ and $\mu_-$ satisfy

$$M_{\gamma_1} \mu_+^{-\frac{2s}{\gamma_1-1}} = 1, \quad M_{\gamma_2} \mu_-^{-\frac{2s}{\gamma_2-1}} = 1,$$

which can always be achieved since $\gamma_1, \gamma_2 \neq \frac{1}{1-8}$. For simplicity, we will denote

$$U_{1,\varepsilon}(x) = \mu_+^{-\frac{2s}{\gamma_1-1}} U_1 \left( \frac{x-p}{\varepsilon \mu_+} \right), \quad U_{2,\varepsilon}(x) = \mu_-^{-\frac{2s}{\gamma_2-1}} U_2 \left( \frac{x-q}{\sigma(\varepsilon) \mu_-} \right).$$

Then by direct computation, for $x \in B_r(p)$ we have

$$(-\Delta)^s \Psi_\varepsilon - \varepsilon^{2s-2} (\Psi_\varepsilon + W_\varepsilon x_1 - \varepsilon^{2s-2} \lambda_+)^{\gamma_1} \chi_{B_r(p)} + \sigma(\varepsilon)^{2s-2} (-\Psi_\varepsilon - W_\varepsilon x_1 - \sigma^{2s-2} \lambda_-)^{\gamma_2} \chi_{B_r(q)}$$

$$= \varepsilon^{-2} \left( \left( U_{1,\varepsilon}(x) - \mu_+^{-\frac{2s}{\gamma_1-1}} \right)^{\gamma_1} + \left( \sum_{k \in \mathbb{Z}} U_{1,\varepsilon}(x + k\ell e_2) - \frac{\varepsilon^{2s-2}}{\sigma^{2s-2}} \sum_{k \in \mathbb{Z}} U_{2,\varepsilon}(x + k\ell e_2) + W_\varepsilon \varepsilon^{2s-2} x_1 - \lambda_+ \right)^{\gamma_1} + \right).$$
Similarly, for $x \in B_\epsilon(q)$ it holds
\[
(-\Delta)^s \Psi_\epsilon - \varepsilon^{(2-2s)\gamma_1-2}(\Psi_\epsilon + W_\epsilon x_1 - \varepsilon^{2-2s}{\lambda_+})^{\gamma_1} \chi_{B_\epsilon(p)}
+ \sigma^{(2-2s)}\gamma_2^{-2}(-\Psi_\epsilon - W_\epsilon x_1 - \sigma^{2s-2}{\lambda_-})^{\gamma_2} \chi_{B_\epsilon(q)}
= \sigma^{-2} \left(- \left(U_{2,\epsilon}(x) - \mu_{-\gamma_2} \right) \right)_{+} 
+ \left( \sum_{k \in \mathbb{Z}} U_{2,\epsilon}(x + k\epsilon e_2) - \sigma^{2-2s} \sum_{k \in \mathbb{Z}} U_{1,\epsilon}(x + k\epsilon e_2) - W_\epsilon \sigma^{2s} x_1 - \lambda_+ \right)^{\gamma_2}.
\]

To ensure that $\Psi_\epsilon(x)$ is a good approximation to the solution to (2.3), we choose $\lambda_+$ and $\lambda_-$ in such a way that
\[
\lim_{N \to \infty} \sum_{k=1}^{N} \left( \sum_{m=\pm k} U_{1,\epsilon}(p + m\epsilon e_2) - \frac{\varepsilon^{2-2s}}{\sigma^{2-2s}} \sum_{m=\pm k} U_{2,\epsilon}(p + m\epsilon e_2) \right),
\]
\[
\lim_{N \to \infty} \sum_{k=1}^{N} \left( \sum_{m=\pm k} U_{2,\epsilon}(q + m\epsilon e_2) - \frac{\varepsilon^{2-2s}}{\sigma^{2-2s}} \sum_{m=\pm k} U_{1,\epsilon}(q + m\epsilon e_2) \right),
\]
\[
\sum_{m=\pm k} U_{1,\epsilon}(p + m\epsilon e_2) - \varepsilon^{2-2s} \sum_{m=\pm k} U_{2,\epsilon}(p + m\epsilon e_2) \approx C|kl|^{2s-4},
\]
and
\[
\sum_{m=\pm k} U_{2,\epsilon}(q + m\epsilon e_2) - \sigma^{2s} \sum_{m=\pm k} U_{1,\epsilon}(q + m\epsilon e_2) \approx C|kl|^{2s-4}.
\]

Hence $\lambda_+$ and $\lambda_-$ have the following asymptotic behavior
\[
\lambda_+ = \mu_{+\gamma_1} + O(\varepsilon^{2-2s}), \quad \lambda_- = \mu_{-\gamma_2} + O(\sigma^{2-2s}),
\]
and the error of the approximation by $\Psi_\epsilon$ is
\[
(-\Delta)^s \Psi_\epsilon - \varepsilon^{(2-2s)\gamma_1-2}(\Psi_\epsilon + W_\epsilon x_1 - \varepsilon^{2s-2}\lambda_+)^{\gamma_1} \chi_{B_\epsilon(p)}
+ \sigma^{(2-2s)}\gamma_2^{-2}(-\Psi_\epsilon - W_\epsilon x_1 - \sigma^{2s-2}\lambda_-)^{\gamma_2} \chi_{B_\epsilon(q)}
= O(\varepsilon^{1-2s}) \chi_{B_{L\epsilon}(p)} + O(\sigma^{1-2s}) \chi_{B_{L\epsilon}(q)},
\]
where $L > 0$ is some large constant.

Notice that we are to construct $\hat{\vartheta}_{0,\epsilon}(x)$ which is symmetric with respect to $x_2 = kl$ when $a = 0$, or to $x_2 = kl \pm l/4$ when $a = l/4$. For further use, we denote this symmetry
restricted in the typical strip $\mathbb{R} \times (-l/2, l/2)$ as $l$-symmetry. We will focus on $l$-symmetric solutions to (2.3), which are small perturbations around $\Psi_\varepsilon(x)$ and can be written as

$$\psi_\varepsilon(x) = \Psi_\varepsilon(x) + \omega_\varepsilon(x), \quad x \in \mathbb{R} \times (-l/2, l/2),$$

where $\omega_\varepsilon(x)$ is a family of $l$-symmetric perturbation terms. Actually, by this decomposition we can transform (2.3) into an equation for $\omega_\varepsilon(x)$, and we will discuss this issue in the rest of this section.

2.2. The linear theory. To find suitable $\omega_\varepsilon(x)$ such that $\psi_\varepsilon(x)$ are solutions to (2.3), it is necessary to study the linearized operator of (2.3) at $\Psi_\varepsilon(x)$, which is given by

$$L_\varepsilon w = (-\Delta)^s w - f_u(x, \Psi_\varepsilon)w \quad \text{in} \quad \mathbb{R} \times (-l/2, l/2),$$

where and hereafter in this section $f(x, u)$ is the (nonlinear) function in the left hand side of (2.3), that is

$$f(x, u) = \varepsilon^{(2-2s)\gamma_1-2}(u + W_\varepsilon x_1 - \varepsilon^{2s-2}\lambda_+ \gamma_1 \chi_{B_r(p)},$$

$$- \sigma(\varepsilon)^{(2-2s)\gamma_2-2}(-u - W_\varepsilon x_1 - \sigma(\varepsilon)^{2s-2}\lambda_- \gamma_2 \chi_{B_r(q)},$$

so its Fréchet derivative at $\Psi_\varepsilon$ is

$$f_u(x, \Psi_\varepsilon) = \varepsilon^{(2-2s)\gamma_1-2}(\Psi_\varepsilon + W_\varepsilon x_1 - \varepsilon^{2s-2}\lambda_+ \gamma_1 \chi_{B_r(p)},$$

$$+ \sigma(\varepsilon)^{(2-2s)\gamma_2-2}(\Psi_\varepsilon - W_\varepsilon x_1 - \sigma(\varepsilon)^{2s-2}\lambda_- \gamma_2 \chi_{B_r(q)}.$$}

Therefore we obtain the following equation for $\omega$ which is equivalent to (2.3)

$$L_\varepsilon \omega_\varepsilon = -E_\varepsilon + R_\varepsilon(\omega_\varepsilon) \quad \text{in} \quad \mathbb{R} \times (-l/2, l/2),$$

where

$$E_\varepsilon = (-\Delta)^s \Psi_\varepsilon - f(x, \Psi_\varepsilon)$$

and

$$R_\varepsilon(\omega_\varepsilon) = f(x, \Psi_\varepsilon + \omega_\varepsilon) - f(x, \Psi_\varepsilon) - f_u(x, \Psi_\varepsilon)\omega_\varepsilon.$$
where
\[ Z_{1,\varepsilon}(x) = \varepsilon^{2s-2} \sum_{k \in \mathbb{Z}} \partial_{x_1} U_{1,\varepsilon}(x + k\ell e_2), \quad Z_{2,\varepsilon}(x) = \sigma^{2s-2} \sum_{k \in \mathbb{Z}} \partial_{x_1} U_{2,\varepsilon}(x + k\ell e_2). \]

Hence we are to consider the following projected linear problem:
\[
\begin{cases}
\mathbb{L}_\varepsilon \omega_\varepsilon = h(x) + \alpha_\varepsilon f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) & \text{in } \mathbb{R} \times (-l/2, l/2), \\
\int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) \omega_\varepsilon(x) dx = 0, \\
\omega_\varepsilon(x) \to 0 & \text{as } |x_1| \to \infty.
\end{cases}
\tag{2.13}
\]

Moreover, we assume that \( h(x) \) is \( l \)-symmetric, and satisfies
\[
\text{supp}(h(x)) \subset B_{l\varepsilon}(p) \cup B_{l\varepsilon}(q)
\tag{2.14}
\]
for some large constant \( L > 0 \). The norms we will use to deal with \ref{2.13} are
\[
\|\omega_\varepsilon\|_* = \sup_{x \in \mathbb{R} \times (-l/2, l/2)} \rho(x)^{-1} |\omega_\varepsilon(x)|,
\]
where
\[
\rho(x) = \left| \frac{1}{\varepsilon^{2s-2} + |x-p|^{2s-2}} - \frac{1}{\sigma^{2s-2} + |x-q|^{2s-2}} \right| + \lim_{N \to \infty} \sum_{k=1}^{N} \frac{1}{|x + k\ell e_2|^{4-2s}},
\]
and
\[
\|h\|_{**} = \sup_{x \in \mathbb{R} \times (-l/2, l/2)} \varepsilon^{2} |h(x)| + \sup_{x \in \mathbb{R} \times (-l/2, l/2)} \sigma^{2} |h(x)|.
\]

We have the following a priori estimate for \ref{2.13}.

**Lemma 2.2.** Assume that \( h(x) \) is \( l \)-symmetric, which satisfies \ref{2.14} and \( \|h\|_{**} < \infty \). Then there exists a small \( \varepsilon_0 > 0 \) and a positive constant \( C \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and solution pair \((\omega_\varepsilon, \alpha_\varepsilon)\) to \ref{2.13}, it holds
\[
\|\omega_\varepsilon\|_* + (\sigma(\varepsilon))^{-1} |\alpha_\varepsilon| \leq C \|h\|_{**}.
\tag{2.15}
\]

**Proof.** First, let us estimate the second term of the left hand side of \ref{2.13} and prove
\[
(\sigma(\varepsilon))^{-1} |\alpha_\varepsilon| \leq C (\|h\|_{**} + o_\varepsilon(1) \|\omega_\varepsilon\|_*).
\tag{2.16}
\]

From \ref{2.13}, the coefficient \( \alpha_\varepsilon \) is given by
\[
\alpha_\varepsilon \int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon) Z_\varepsilon^2 dx = \int_{\mathbb{R} \times (-l/2, l/2)} Z_\varepsilon \mathbb{L}_\varepsilon \omega_\varepsilon dx - \int_{\mathbb{R} \times (-l/2, l/2)} h Z_\varepsilon dx.
\]

According to the expansion of \( f_u(x, \Psi_\varepsilon) \) in \ref{2.11}, we have
\[
\int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon) Z_\varepsilon^2 dx = (1 + o_\varepsilon(1)) \varepsilon^{2s-4} \int_{\mathbb{R} \times (-l/2, l/2)} \gamma_1(U_{1,\varepsilon} - \mu_{+ \gamma_{1-1}^{2s}}) \gamma_{1-1} \left( \frac{\partial U_{1,\varepsilon}}{\partial y_1} \right)^2 dy^1
\]
\[
+ (1 + o_\varepsilon(1)) \sigma^{2s-4} \int_{\mathbb{R} \times (-l/2, l/2)} \gamma_2(U_{2,\varepsilon} - \mu_{- \gamma_{2-1}^{2s}}) \gamma_{2-1} \left( \frac{\partial U_{2,\varepsilon}}{\partial y_1} \right)^2 dy^2
\]
\[
= c_1 (1 + o_\varepsilon(1)) \varepsilon^{2s-4} + c_2 (1 + o_\varepsilon(1)) \sigma^{2s-4},
\tag{2.17}
\]
where $y^1 = \frac{x}{\sigma \mu_+}$, $y^2 = \frac{x}{\sigma \mu_-}$, and $c_1, c_2 > 0$ are some constants. On the other hand, it holds

$$\int_{\mathbb{R} \times (-1/2,1/2)} Z_\varepsilon (-\Delta)^s \omega_\varepsilon dx = \int_{\mathbb{R} \times (-1/2,1/2)} \omega_\varepsilon (-\Delta)^s Z_\varepsilon dx$$

$$= \int_{\mathbb{R} \times (-1/2,1/2)} \omega_\varepsilon \left( \varepsilon^{-2s} \gamma_1 \left( U_{1,\varepsilon}(x) - \mu_+ \frac{2s}{\gamma_1 - 1} \right) \right)^{\gamma_1 - 1} Z_{1,\varepsilon} - \sigma^{-2s} \gamma_2 \left( U_{2,\varepsilon}(x) - \mu_+ \frac{2s}{\gamma_1 - 1} \right)^{\gamma_2 - 1} Z_{2,\varepsilon} dx.$$ 

For $x_1 < 0$, we have

$$\left| \varepsilon^{-2s} \gamma_1 \left( U_{1,\varepsilon}(x) - \mu_+ \frac{2s}{\gamma_1 - 1} \right) \right|^{\gamma_1 - 1} Z_{1,\varepsilon} - \varepsilon^{-2s} \gamma_1 \left( U_{1,\varepsilon}(x) - \mu_+ \frac{2s}{\gamma_1 - 1} \right)^{\gamma_1 - 1} Z_{1,\varepsilon} - \varepsilon^{-2s} \gamma_2 \left( U_{2,\varepsilon}(x) - \mu_+ \frac{2s}{\gamma_1 - 1} + O(\varepsilon^{3-2s}) \right)^{\gamma_2 - 1} Z_{2,\varepsilon}$$

$$\leq C\varepsilon^{-3+(3-2s)\min\{\gamma_1-1,1\}} \chi_{B_{L\varepsilon}(p)}.$$ 

Similarly, for $x_1 > 0$, the term is

$$-\sigma^{-2s} \gamma_2 \left( U_{2,\varepsilon}(x) - \mu_+ \frac{2s}{\gamma_1 - 1} \right)^{\gamma_2 - 1} Z_{2,\varepsilon} + \sigma^{-2s} \gamma_2 \left( U_{2,\varepsilon}(x) - \mu_+ \frac{2s}{\gamma_1 - 1} + O(\sigma^{3-2s}) \right)^{\gamma_2 - 1} Z_{2,\varepsilon}$$

$$\leq C\sigma^{-3+(3-2s)\min\{\gamma_2-1,1\}} \chi_{B_{L\sigma}(\sigma)}.$$ 

Hence we derive from Hölder inequality that

$$\left| \int_{\mathbb{R} \times (-1/2,1/2)} Z_\varepsilon \mathbb{L}_\varepsilon \omega_\varepsilon dx \right| \leq o_\varepsilon(1) \cdot \|\omega_\varepsilon\|\mathbb{L}_s \sigma^{2s-3}. \quad (2.18)$$

By the definition of norm $\|h\|_{\mathbb{L}_s}$, we also have

$$\left| \int_{\mathbb{R} \times (-1/2,1/2)} hZ_\varepsilon dx \right| \leq \|h\|_{\mathbb{L}_s} \sigma^{2s-3}. \quad (2.19)$$

Then (2.16) follows directly from (2.17) (2.18) and (2.19).

Next, we estimate the first term of the left hand side of (2.15) and prove

$$\|\omega_\varepsilon\| \leq C \|h\|_{\mathbb{L}_s}. \quad (2.20)$$

To this end, we will argue by contradiction. Suppose that there exists a sequence $\{\varepsilon_n\}$ satisfying $\varepsilon_n \to 0$, solution pairs $(\omega_{\varepsilon_n}, \alpha_{\varepsilon_n})$ to (2.13) for some $h_{\varepsilon_n}$, such that

$$\|\omega_{\varepsilon_n}\| = 1, \quad \|h_{\varepsilon_n}\| \to 0 \quad \text{as} \quad n \to \infty. \quad (2.21)$$

We want to show that for any $L > 0$, it always holds

$$\varepsilon_n^{2-2s} \|\omega_{\varepsilon_n}\|_{L^\infty(B_{L\varepsilon_n}(p))} + \sigma_n^{2-2s} \|\omega_{\varepsilon_n}\|_{L^\infty(B_{L\sigma_n}(\sigma))} \to 0 \quad \text{as} \quad n \to \infty, \quad (2.22)$$

where $\sigma_n = \sigma(\varepsilon_n)$. 
Suppose (2.22) is not true, without loss of generality, we assume that the first term satisfies for some constant \( \Lambda_0 > 0 \)
\[
\varepsilon_n^{2-2s} \| \omega_n \|_{L^\infty(B_{L^\infty(p)})} \geq \Lambda_0.
\]
Set
\[
\tilde{\omega}_n(y) = \varepsilon_n^{2-2s} \mu_+^{2s} \omega_n(\varepsilon_n \mu + y).
\]
From (2.13), on every compact set \( \tilde{\omega}_n(y) \) satisfies
\[
(-\Delta)^s \tilde{\omega}_n(y) - \gamma_1(U_1 - 1 + O(\varepsilon_n^{3-2s})) \tilde{\omega}_n(y) + o_\varepsilon(1) = \varepsilon_n^{2-2s} \mu_+^{2s} h_\varepsilon(\varepsilon_n \mu + y) + \varepsilon_n \alpha_\varepsilon \gamma_1(U_1 - 1 + O(\varepsilon_n^{3-2s})) \tilde{\omega}_n(y) + o_\varepsilon(1),
\]
which is equivalent to
\[
(-\Delta)^s \tilde{\omega}_n(y) - \gamma_1(U_1 - 1) \tilde{\omega}_n(y) + o_\varepsilon(1) = \mathcal{R}_n(y),
\]
where
\[
\mathcal{R}_n(y) = \varepsilon_n^{2-2s} \mu_+^{2s} h_\varepsilon(\varepsilon_n \mu + y) + o_\varepsilon(1) \tilde{\omega}_n(y) + \varepsilon_n \alpha_\varepsilon \gamma_1(U_1 - 1 + o_\varepsilon(1)) \tilde{\omega}_n(y) + o_\varepsilon(1).
\]
Since \( \varepsilon_n^{2-2s} \mu_+^{2s} h_\varepsilon(y) \to 0 \), and \( \varepsilon_n \alpha_\varepsilon \leq \sigma_n^{-1} \alpha_\varepsilon \leq C(\|\cdot\|_{L^\infty} + o_\varepsilon(1) \|\omega_n\|_{L^\infty}) = o_\varepsilon(1) \) on every compact set by (2.16) and (2.21), we have \( \mathcal{R}_n(y) \to 0 \) as \( n \to \infty \).

Let \( n \to \infty \), we may assume that \( \tilde{\omega}_n \) converge uniformly on compact sets to a function \( \tilde{\omega} \) satisfying
\[
\| \tilde{\omega} \|_{L^\infty(B_{L^\infty(0)})} \geq \Lambda_0 \mu_+^{2s}.
\]
However, \( \tilde{\omega} \) is even in \( y_2 \) direction, and is a solution to
\[
(-\Delta)^s \tilde{\omega}(y) - \gamma_1(U_1 - 1) \tilde{\omega}(y) = 0
\]
with \( \tilde{\omega}(y) \to 0 \) as \( |y| \to \infty \). Furthermore, it satisfies the orthogonality condition
\[
\int_{\mathbb{R}^2} \gamma_1(U_1 - 1) \tilde{\omega} \frac{\partial U_1}{\partial y_1} dy = 0.
\]
According to Theorem 2.21 it must hold \( \tilde{\omega} \equiv 0 \), which is a contradiction to (2.23). Hence we deduce that \( \varepsilon_n^{2-2s} \| \omega_n \|_{L^\infty(B_{L^\infty(p)})} \to 0 \). For the second term in (2.22), we can use a similar method to prove \( \sigma_n^{2-2s} \| \omega_n \|_{L^\infty(B_{L^\infty(q)})} \to 0 \). Hence we have proved (2.22).

In view of (2.13), \( \omega_n \) satisfies
\[
(-\Delta)^s \omega_n = f_u(x, \Psi_0) \omega_n + h_\varepsilon + \alpha_\varepsilon f_u(x, \Psi_\varepsilon) Z_\varepsilon
\]
Using the explicit formulation (2.2) of \( (-\Delta)^s \), we have
\[
\omega_n(x) = \int_{\mathbb{R} \times (-1/2,1/2)} K_s(x - z) (f_u(x, \Psi_0) \omega_n(x) + h_\varepsilon + \alpha_\varepsilon f_u(x, \Psi_\varepsilon) Z_\varepsilon(x)) dz
\]
for the kernel \( K_s \) given in (2.1), which implies
\[
\rho(x)^{-1} \| \omega_n(x) \| \leq C(\varepsilon_n^{2-2s} \| \omega_n \|_{L^\infty(B_{L^\infty(p)})} + \sigma_n^{2-2s} \| \omega_n \|_{L^\infty(B_{L^\infty(q)})} + h_\varepsilon).
Hence if we combine (2.16) (2.21) (2.22), we can obtain $\|\omega_n\|_\ast \to 0$ as $n \to \infty$, which is a contradiction to (2.21) and yields (2.20). To finish our proof, we notice that (2.15) is the consequence of (2.16) and (2.20). \hfill \Box

Using the a priori estimate given in Lemma 2.2, we have the following result for (2.13).

**Lemma 2.3.** Assume that $h(x)$ is $l$-symmetric, which satisfies (2.14) and $\|h\|_{\ast\ast} < \infty$. Then there exists a small $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, (2.13) has a unique solution $\omega_\varepsilon = T_\varepsilon h$, where $T_\varepsilon$ is a linear operator of $h$. Moreover, there exists a constant $C > 0$ independent of $\varepsilon$ such that

$$\|\omega_\varepsilon\|_\ast \leq C\|h\|_{\ast\ast}.$$ 

**Proof.** Denote the Hilbert space

$$H := \{g \in \dot{H}^s(\mathbb{R} \times (-l/2, l/2)) : g \text{ is } l\text{-symmetric, } \int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon)Z_\varepsilon(x)g(x)dx = 0 \}$$

endowed with the inner product

$$[u, g] = \int_{\mathbb{R} \times (-l/2, l/2)} \int_{\mathbb{R} \times (-l/2, l/2)} J_s(x - z) (u(x) - u(z)) (g(x) - g(z)) dxdz,$$

where the kernel $J_s(x)$ is defined in (2.1). Then we can express (2.13) in a weak form, namely, to find $\omega_\varepsilon \in H$ such that

$$[\omega_\varepsilon, g] = \langle f_u(x, \Psi_\varepsilon)\omega_\varepsilon + h, g \rangle, \quad \forall \ g \in H.$$

According to Riesz’s representation theorem, the above equation has an equivalent operational form

$$\omega_\varepsilon = (-\Delta)_s^{-s}(f_u(x, \Psi_\varepsilon)\omega_\varepsilon) + (-\Delta_s)^{-s}h.$$

Notice that $(-\Delta)_s^{-s}(f_u(x, \Psi_\varepsilon)(\cdot))$ is a compact operator on $H$. By Fredholm’s alternative, this equation has a unique solution for any $h$ if the homogeneous equation

$$\omega_\varepsilon = (-\Delta)_s^{-s}(f_u(x, \Psi_\varepsilon)\omega_\varepsilon)$$

has only trivial solution in $H$, which can be obtained by Lemma 2.2 The estimate $\|\omega_\varepsilon\|_\ast \leq C\|h\|_{\ast\ast}$ follows from (2.20). Hence the proof is complete. \hfill \Box

2.3. **The reduction.** To solve (2.12), we will first solve (2.13) for

$$h(x) = -E_\varepsilon + R_\varepsilon(\omega_\varepsilon). \quad (2.24)$$

Then we will deal with a one-dimensional problem so that $\alpha_\varepsilon = 0$, which can be achieved by choosing suitable travelling speed $W_\varepsilon$. This process is known as the Lyapunov-Schmidtl reduction. For the solvability of (2.13) and (2.24), we have the following lemma.

**Lemma 2.4.** There are $\varepsilon_0 > 0$ and $r_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ there exists a unique solution $\omega_\varepsilon$ to (2.13) and (2.24) in the ball $\|\omega_\varepsilon\|_\ast \leq r_0$. Moreover, it holds

$$\|\omega_\varepsilon\|_\ast \leq C\varepsilon^{3-2s} \quad (2.25)$$

for some constant $C > 0$, and $\omega_\varepsilon$ is continuous with respect to $\varepsilon$. 


Proof. Since $h_\varepsilon(x)$ is given in (2.24), it is easy to verify that $h(x)$ is $l$-symmetric and satisfies (2.14). Hence from Lemma 2.3 for the $h(x)$, we have the estimate
\[ \|T_\varepsilon h\|_* \leq C\|h\|_{**}. \] (2.26)

Denote
\[ X := \{ u \in L^\infty(\times(-l/2,l/2)) : u \text{ is } l\text{-symmetric, } \|u\|_* < \infty, \} \]
endowed with the norm $\| \cdot \|_*$, and $A_\varepsilon : X \rightarrow X$ the operator given by
\[ A_\varepsilon \omega_\varepsilon := T_\varepsilon(-E_\varepsilon + R_\varepsilon(\omega_\varepsilon)). \]
If we let
\[ B_{r_0} := \{ \omega_\varepsilon \in X : \|\omega_\varepsilon\|_* \leq r_0 \} \]
be a closed neighborhood of the origin in $X$, then solve equation (2.13) is equivalent to find a fixed point of $A_\varepsilon$ in $B_{r_0}$,
\[ A_\varepsilon \omega_\varepsilon = \omega_\varepsilon. \]

In the following we prove that $A_\varepsilon$ does have a fixed point in $B_{r_0}$ by showing that $A_\varepsilon$ is a contraction map in $B_{r_0}$. First we show that $A_\varepsilon$ maps $B_{r_0}$ into itself. From (2.9), we have
\[ \|E_\varepsilon\|_{**} \leq C\varepsilon^{-3-2s}. \] (2.27)
Then we are to estimate $\|R_\varepsilon(\omega_\varepsilon)\|_{**}$. To this aim, we can split $R_\varepsilon(\omega_\varepsilon)$ into two terms
\[ R_\varepsilon(\omega_\varepsilon) = R_{1,\varepsilon}(\omega_\varepsilon) + R_{2,\varepsilon}(\omega_\varepsilon), \]
where
\[ R_{1,\varepsilon}(\omega_\varepsilon) = \varepsilon^{2-2s}\gamma_1^{-2}\left((\Psi_\varepsilon + \omega_\varepsilon + W_\varepsilon x_1 - \varepsilon^{2-2s}\lambda_+^\gamma_1\right)
- (\Psi_\varepsilon + W_\varepsilon x_1 - \varepsilon^{2-2s}\lambda_+^\gamma_1 - \gamma_1(\Psi_\varepsilon + W_\varepsilon x_1 - \varepsilon^{2-2s}\lambda_+^\gamma_1 - \omega_\varepsilon)\chi_{B_{L\varepsilon}}(p)\right) \]
and
\[ R_{2,\varepsilon}(\omega_\varepsilon) = \sigma^{2-2s}\gamma_2^{-2}\left(-\Psi_\varepsilon - \omega_\varepsilon - W_\varepsilon x_1 - \sigma^{2-2s}\lambda_+^\gamma_2\right)
- (-\Psi_\varepsilon - W_\varepsilon x_1 - \sigma^{2-2s}\lambda_+^\gamma_2 + \gamma_2(-\Psi_\varepsilon - W_\varepsilon x_1 - \sigma^{2-2s}\lambda_+^\gamma_2 - \omega_\varepsilon)\chi_{B_{L\varepsilon}}(p).\]

By the choice of $\lambda_+$ in (2.7), we have
\[ R_{1,\varepsilon}(\omega_\varepsilon) = \varepsilon^{-2}\left(\sum_{k \in \mathbb{Z}} U_{1,\varepsilon}(x + k\varepsilon e_2) - \frac{\varepsilon^{2-2s}}{\sigma^{2-2s}} \sum_{k \in \mathbb{Z}} U_{1,\varepsilon}(x + k\varepsilon e_2) + \varepsilon^{2-2s}\omega_\varepsilon - \mu_+^{\frac{2s}{\gamma_1-1}} + O(\varepsilon^{3-2s})\right)^\gamma_1
- (-\sum_{k \in \mathbb{Z}} U_{1,\varepsilon}(x + k\varepsilon e_2) - \frac{\varepsilon^{2-2s}}{\sigma^{2-2s}} \sum_{k \in \mathbb{Z}} U_{1,\varepsilon}(x + k\varepsilon e_2) - \mu_+^{\frac{2s}{\gamma_2-1}} + O(\varepsilon^{3-2s})\right)^\gamma_2
- \gamma_1\left(-\sum_{k \in \mathbb{Z}} U_{1,\varepsilon}(x + k\varepsilon e_2) - \frac{\varepsilon^{2-2s}}{\sigma^{2-2s}} \sum_{k \in \mathbb{Z}} U_{1,\varepsilon}(x + k\varepsilon e_2) - \mu_+^{\frac{2s}{\gamma_1-1}} + O(\varepsilon^{3-2s})\right)^\gamma_1 - \varepsilon^{2-2s}\omega_\varepsilon\chi_{B_{L\varepsilon}}(p),\]
which yields \( \|R_{1,\varepsilon}(\omega_\varepsilon)\|_{**} \leq C\|\omega_\varepsilon\|_{*}^{\min\{\gamma_1,2\}} \). Using a similar method, we can also prove \( \|R_{2,\varepsilon}(\omega_\varepsilon)\|_{**} \leq C\|\omega_\varepsilon\|_{*}^{\min\{\gamma_1,2\}} \). So we conclude that
\[
\|R_{\varepsilon}(\omega_\varepsilon)\|_{**} \leq C\|\omega_\varepsilon\|_{*}^{\min\{\gamma_1,\gamma_2,2\}}. \tag{2.28}
\]

If we combine (2.26) (2.27) and (2.28), we deduce that for \( \omega_\varepsilon \in B_{r_0} \), it holds
\[
\|A_\varepsilon\omega_\varepsilon\| \leq C\varepsilon^{3-2s} + C r_0^{\min\{\gamma_1,\gamma_2,2\}}.
\]

As a result, \( A_\varepsilon \) maps \( B_{r_0} \) into itself if we choose \( \varepsilon \) and \( r_0 \) sufficiently small. On the other hand, for \( \omega_\varepsilon^1, \omega_\varepsilon^2 \in B_{r_0} \) we have
\[
\|R_{\varepsilon}(\omega_\varepsilon^1) - R_{\varepsilon}(\omega_\varepsilon^2)\|_{**} \leq C(\|\omega_\varepsilon^1\|_{*}^{\min\{1-\gamma_2,1\}} + \|\omega_\varepsilon^2\|_{*}^{\min\{1-\gamma_2,1\}})\|\omega_\varepsilon^1 - \omega_\varepsilon^2\|_{*}.
\]

Hence it holds
\[
\|A_\varepsilon\omega_\varepsilon^1 - A_\varepsilon\omega_\varepsilon^2\|_{**} \leq C r_0^{\min\{1-\gamma_2,1\}}\|\omega_\varepsilon^1 - \omega_\varepsilon^2\|_{*},
\]

and \( A_\varepsilon \) is a contraction mapping from \( B_{r_0} \) into itself if \( r_0 \) is sufficiently small. Thus (2.13) and (2.24) admits a unique solution \( \omega_\varepsilon \in B_{r_0} \).

According to the estimate (2.27), we derive
\[
\|\omega_\varepsilon\|_{*} \leq C\|E_\varepsilon\|_{**} \leq C\varepsilon^{3-2s}. \tag{2.29}
\]

Since \( E_\varepsilon \) and \( R_{\varepsilon}(\omega_\varepsilon) \) depend continuously on \( \varepsilon \), we see that \( \omega_\varepsilon \) is continuous with respect to \( \varepsilon \) by the fixed point characterization. So we have finished the proof. \( \square \)

We have already obtained a solution \( \psi_\varepsilon(x) = \Psi_\varepsilon(x) + \omega_\varepsilon(x) \) to
\[
\begin{align*}
(-\Delta)^s\psi_\varepsilon &= f(x, \psi_\varepsilon) + \alpha_\varepsilon f_\varepsilon(x, \Psi_\varepsilon)Z_\varepsilon(x) \quad \text{in } \mathbb{R} \times (-l/2, l/2), \\
\psi_\varepsilon(x) &\to 0 \quad \text{as } |x| \to \infty.
\end{align*} \tag{2.30}
\]

If we multiply the first equation of (2.30) by \( Z_\varepsilon \) and integrate over \( \mathbb{R} \times (-l/2, l/2) \), we deduce that
\[
\alpha_\varepsilon \int_{\mathbb{R} \times (-l/2, l/2)} f_\varepsilon(x, \Psi_\varepsilon)Z_\varepsilon^2 dx = \int_{\mathbb{R} \times (-l/2, l/2)} ((-\Delta)^s\psi_\varepsilon - f(x, \psi_\varepsilon)) Z_\varepsilon dx.
\]

To make the right hand side of above equality being zero, we will use the following lemma later.

**Lemma 2.5.** It holds
\[
\int_{\mathbb{R} \times (-l/2, l/2)} ((-\Delta)^s\psi_\varepsilon - f(x, \psi_\varepsilon)) Z_\varepsilon dx = C \left( C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{p_1 - q_1}{q - p + k(1 - 2s)} + W_{\varepsilon} \right) + o_\varepsilon(1),
\]

where \( C > 0 \) is a constant independent of \( \varepsilon \), and \( p_1, q_1 \) denote the first coordinates of \( p, q \) respectively.

**Proof.** From (2.13) and (2.24), we have
\[
(-\Delta)^s\psi_\varepsilon - f(x, \psi_\varepsilon) = \mathbb{L}_\varepsilon \psi_\varepsilon + E_{\varepsilon} - R_{\varepsilon}(\omega_\varepsilon) \quad \text{in } \mathbb{R} \times (-l/2, l/2).
\]
From Taylor’s formula, it holds
\[ (\Delta)^{\frac{s}{2}} \psi_\varepsilon - f(x, \psi_\varepsilon) \] \[ \int_{\mathbb{R} \times (-l/2, l/2)} \left((-\Delta)^{\frac{s}{2}} \psi_\varepsilon - f(x, \psi_\varepsilon)\right) Z_\varepsilon \, dx \]
\[ = \int_{\mathbb{R} \times (-l/2, l/2)} \mathbb{1}_E \omega_\varepsilon Z_\varepsilon \, dx + \int_{\mathbb{R} \times (-l/2, l/2)} E_\varepsilon Z_\varepsilon \, dx - \int_{\mathbb{R} \times (-l/2, l/2)} R_\varepsilon(\omega_\varepsilon) Z_\varepsilon \, dx. \]

We first deal with the term \( \int_{\mathbb{R} \times (-l/2, l/2)} E_\varepsilon Z_\varepsilon \, dx \). By our choice of \( \lambda_+ \) and \( \lambda_- \) in (2.7) and (2.8), we can split \( E_\varepsilon \) as
\[ E_\varepsilon = E_{1, \varepsilon} + E_{2, \varepsilon}, \]
where
\[ E_{1, \varepsilon} = \varepsilon^{-2} \chi_{B_L}(p) \left( (U_{1, \varepsilon}(x) - \mu_+^{-\frac{2s}{\gamma_1-1}})^{\gamma_1} - (U_{1, \varepsilon}(x) - \mu_+^{-\frac{2s}{\gamma_1-1}}) \right) \]
\[ + \sum_{k \neq 0} U_{1, \varepsilon}(x + kl e_2) - \frac{\varepsilon^{2-2s}}{\sigma^{2-2s} \sum_{k \in \mathbb{Z}} U_{2, \varepsilon}(x + kl e_2) \sum_{k \neq 0} U_{1, \varepsilon}(p + kl e_2) \sum_{k \neq 0} U_{1, \varepsilon}(p + kl e_2) \]
\[ + \frac{\varepsilon^{2-2s}}{\sigma^{2-2s} \sum_{k \in \mathbb{Z}} U_{2, \varepsilon}(p + kl e_2) + W_{\varepsilon} \varepsilon^{2-2s}(x_1 - d)^{\gamma_1})_+ \],
and
\[ E_{2, \varepsilon} = \sigma^{-2} \chi_{B_{L^*}}(q) \left( -(U_{2, \varepsilon}(x) - \mu_-^{-\frac{2s}{\gamma_2-1}})^{\gamma_2} + (U_{2, \varepsilon}(x) - \mu_-^{-\frac{2s}{\gamma_2-1}}) \right) \]
\[ + \sum_{k \neq 0} U_{2, \varepsilon}(x + kl e_2) - \frac{\varepsilon^{2-2s}}{\sigma^{2-2s} \sum_{k \in \mathbb{Z}} U_{1, \varepsilon}(x + kl e_2) \sum_{k \neq 0} U_{2, \varepsilon}(q + kl e_2) \sum_{k \neq 0} U_{1, \varepsilon}(q + kl e_2) \]
\[ + \frac{\varepsilon^{2-2s}}{\sigma^{2-2s} \sum_{k \in \mathbb{Z}} U_{1, \varepsilon}(q + kl e_2) - W_{\varepsilon} \varepsilon^{2-2s}(x_1 - d)^{\gamma_2})_+ \].

Since
\[ |E_{1, \varepsilon}| \leq C \varepsilon^{-2} \varepsilon^{3-2s} \quad \text{and} \quad |E_{2, \varepsilon}| \leq C \sigma^{-2} \sigma^{3-2s}, \]
we can show that
\[ \int_{\mathbb{R} \times (-l/2, l/2)} E_{1, \varepsilon} Z_{2, \varepsilon} \, dx \leq C \varepsilon^{3-2s}, \quad \int_{\mathbb{R} \times (-l/2, l/2)} E_{2, \varepsilon} Z_{1, \varepsilon} \, dx \leq C \sigma^{3-2s}. \]

From Taylor’s formula, it holds
\[ E_{1, \varepsilon} = -\varepsilon^{-2} \chi_{B_L}(p) (U_{1, \varepsilon}(x) - \mu_+^{-\frac{2s}{\gamma_1-1}})^{\gamma_1-1} \left( \sum_{k \neq 0} U_{1, \varepsilon}(x + kl e_2) \right) \]
\[ - \frac{\varepsilon^{2-2s}}{\sigma^{2-2s} \sum_{k \in \mathbb{Z}} U_{2, \varepsilon}(x + kl e_2) \sum_{k \neq 0} U_{1, \varepsilon}(p + kl e_2) + \frac{\varepsilon^{2-2s}}{\sigma^{2-2s} \sum_{k \in \mathbb{Z}} U_{2, \varepsilon}(p + kl e_2) + W_{\varepsilon} \varepsilon^{2-2s}(x_1 - d) + \varepsilon^{-2} \varepsilon^{(3-2s)(\gamma_1-1)} \chi_{B_L}(p).} \]
To compute \( \int_{\mathbb{R} \times (-l/2,l/2)} E_{1,\varepsilon} Z_{1,\varepsilon} dx \), we can integrate by parts and use the asymptotic behavior of \( U_{1,\varepsilon}, U_{2,\varepsilon} \) to obtain

\[
\int_{\mathbb{R} \times (-l/2,l/2)} E_{1,\varepsilon} Z_{1,\varepsilon} dx = \varepsilon^{-2} \int_{\mathbb{R} \times (-l/2,l/2)} \left( U_{1,\varepsilon}(x) - \mu_+ \frac{-2s}{\gamma_1} \right) dx
\]

\[
\left( \varepsilon^{2s-2} \sum_{k \neq 0} \partial_{x_1} U_{1,\varepsilon}(x + k\varepsilon e_2) - \sigma^{2s-2} \sum_{k \in \mathbb{Z}} \partial_{x_1} U_{2,\varepsilon}(x + k\varepsilon e_2) + W_\varepsilon + O(\varepsilon) \right) dx
\]

\[
= \varepsilon^{-2} \int_{\mathbb{R} \times (-l/2,l/2)} \left( U_{1,\varepsilon}(x) - \mu_+ \frac{-2s}{\gamma_1} \right) dx
\]

\[
\left( (2 - 2s)c_1 \lim_{N \to \infty} \sum_{|k| \leq N} \frac{p_1 - q_1}{|p - q + k\varepsilon e_2|^{4-2s}} + W_\varepsilon + O(\varepsilon) \right) dx
\]

\[
= C_1 \left( C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{p_1 - q_1}{|p - q + k\varepsilon e_2|^{4-2s}} + W_\varepsilon \right) + O(\varepsilon),
\]

where we have used symmetry to derive \( \lim_{N \to \infty} \sum_{|k| \leq N} \partial_{x_1} U_{1,\varepsilon}(p + k\varepsilon e_2) = 0 \), \( C_1 > 0 \) is a constant, and \( p_1, q_1 \) denote the first coordinates of \( p, q \) respectively. Similarly, for some \( C_2 > 0 \) we can get

\[
- \int_{\mathbb{R} \times (-l/2,l/2)} E_{2,\varepsilon} Z_{2,\varepsilon} dx = C_2 \left( C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{-q_1 + p_1}{|q - p + k\varepsilon e_2|^{4-2s}} + W_\varepsilon \right) + O(\sigma).
\]

Hence we have

\[
\int_{\mathbb{R} \times (-l/2,l/2)} E_\varepsilon Z_\varepsilon dx = C \left( C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{p_1 - q_1}{|q - p + k\varepsilon e_2|^{4-2s}} + W_\varepsilon \right) + o_\varepsilon(1). \quad (2.31)
\]

Recall that \( \sigma(\varepsilon) \) satisfies assumption (H) with \( \tau = \min\{\gamma_2, 2\} \). Since \( \|\omega_\varepsilon\|_s \leq C\varepsilon^{-2s} \), for the term \( \int_{\mathbb{R} \times (-l/2,l/2)} R_\varepsilon(\omega_\varepsilon) Z_\varepsilon dx \) we have

\[
\int_{\mathbb{R} \times (-l/2,l/2)} R_\varepsilon(\omega_\varepsilon) Z_\varepsilon dx = \int_{\mathbb{R} \times (-l/2,l/2)} R_\varepsilon(\omega_\varepsilon) Z_\varepsilon dx + \int_{\mathbb{R} \times (-l/2,l/2)} R_\varepsilon(\omega_\varepsilon) Z_\varepsilon dx
\]

\[
\leq C \|\omega_\varepsilon\|_{\min\{\gamma_1, 2\}} \varepsilon^{-3+2s} + C \|\omega_\varepsilon\|_{\min\{\gamma_2, 2\}} \sigma^{-3+2s} \quad (2.32)
\]

\[
\leq C\varepsilon^{-2s} \min\{\gamma_1-1, 1\} + C\varepsilon^{3-2s} \min\{\gamma_2, 2\} \sigma^{-3+2s} = o_\varepsilon(1).
\]
To deal with the last term $\int_{\mathbb{R} \times (-l/2,l/2)} \mathbb{L}_\varepsilon \omega_\varepsilon Z_\varepsilon dx$, we can use the estimate in the proof of Lemma 2.2 to deduce that

$$
\int_{\mathbb{R} \times (-l/2,l/2)} \mathbb{L}_\varepsilon \omega_\varepsilon Z_\varepsilon dx = \int_{\mathbb{R} \times (-l/2,l/2)} \mathbb{L}_\varepsilon \omega_\varepsilon Z_\varepsilon dx + \int_{\mathbb{R} \times (-l/2,l/2)} \mathbb{L}_\varepsilon \omega_\varepsilon Z_\varepsilon dx
\leq C \|\omega_\varepsilon\|_s \varepsilon^{(3-2s) \min\{\gamma_1-2,0\}} + C \|\omega_\varepsilon\|_s \sigma(3-2s) \min\{\gamma_2-2,0\}
\leq C \varepsilon^{(3-2s) \min\{\gamma_1-1,1\}} + C \varepsilon^{3-2s} \sigma(3-2s) \min\{\gamma_2-2,0\} = o_\varepsilon(1).
$$

Finally, if we combine (2.31), (2.32) and (2.33), then the proof is complete. \(\square\)

Now we are ready to given proofs for Theorem 1.1 and 1.3 with $0 < s < 1$.

**Proof of Theorem 1.1 and 1.3 with $0 < s < 1$**: In view of (2.30), to obtain a family of desired solutions to (2.3), we only need to find suitable $W_\varepsilon$ so that the corresponding $\alpha_\varepsilon = 0$. Notice that we already have

$$
\int_{\mathbb{R} \times (-l/2,l/2)} f_u(x, \Psi_\varepsilon) Z_\varepsilon^2 dx > 0.
$$

By Lemma 2.5, $\alpha_\varepsilon = 0$ is equivalent to the variational characterization

$$
W_\varepsilon = C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{-p_1 + q_1}{|q - p + k\ell_2|^4 - 2s} + o_\varepsilon(1).
$$

Recall the definitions of $p$ and $q$. When $a = 0$, the condition for $W_\varepsilon$ is

$$
W_\varepsilon = C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{2d}{(4d^2 + k^2 l^2)^2 - s} + o_\varepsilon(1);
$$

while for $a = l/4$, it must hold

$$
W_\varepsilon = C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{2d}{(4d^2 + (l + k/2)^2)^2 - s} + o_\varepsilon(1).
$$

By (2.3) and the periodic setting, the weak convergence of solutions is obvious. By the choice of $p, q$, the existence of $W_\varepsilon$ follows and the corresponding $\alpha_\varepsilon = 0$. The $C^1$ property of $\vartheta_{0,\varepsilon}$ can be deduced from the standard regularity theory for elliptic equations. Hence we have completed the proof. \(\square\)

**Remark 2.6.** In [1], the condition $\alpha_\varepsilon = 0$ is described as $\psi_\varepsilon$ being a critical point of some energy functional $E(\psi)$, which degenerates in $x_2$ direction. Since the vortex pair constructed in [1] has an odd symmetry, this description is appropriate and vivid. In our situation, vortices on different sides of the street have a different energy blow up rate if $\delta(\varepsilon) = o_\varepsilon(1)$. However, our description for $\alpha_\varepsilon = 0$ does make sense, because assumption (H) ensures that the small terms caused by $R_\varepsilon$ or $\mathbb{L}_\varepsilon \omega_\varepsilon$ are of order $o_\varepsilon(1)$, and can not exceed the secondary term in energy functional, which is of order $O_\varepsilon(1)$ and determines $W_\varepsilon$. 
3. Construction for the Euler equation

In this section we consider the remaining case $s = 1$ for gSQG equation, namely the Euler equation and give proofs for Theorem 1.1 and 1.3 in this case.

3.1. Approximate solutions. As we have done in Section 2, we are going to obtain a series of $l$-symmetric solutions to the following semilinear elliptic problem

\[
\begin{cases}
-\Delta \psi = \varepsilon^{-2} \left( \frac{\lambda_+}{2\pi} + \frac{\lambda_+}{2\pi} \ln \frac{x}{\varepsilon} \right) \chi_{B_r(p)} \\
-\sigma(\varepsilon)^{-2} \left( -\psi - W_\varepsilon x_1 - \frac{\lambda_+}{2\pi} \ln \frac{1}{\sigma(\varepsilon)} \right) \chi_{B_r(q)} \quad \text{in } \mathbb{R} \times (-l/2, l/2), \\
\psi(x) \to 0 \quad \text{as } |x_1| \to \infty,
\end{cases}
\]

(3.1)

where $\lambda_+$ and $\lambda_-$ are undetermined parameters and will be suitably chosen, $W_\varepsilon$ is the travelling speed of Kármán vortex street determined by location of $\infty$, and $\sigma(\varepsilon)$ satisfies assumption (H) with $\tau = \min(\gamma_1, \gamma_2)$, and $r > 0$ is a small constant such that $B_r(p)$ and $B_r(q)$ are disjoint. We still assume

\[
p = (-d, -a), \quad q = (d, a),
\]

where $d > 0$ for $a = 0$; or $d \geq 0$ for $a = l/4$.

Suppose $V(x) = V(|x|)$ is the unique radial solution of

\[-\Delta V = V^\gamma, \quad V \in H_0^1(B_1(0)), \quad V > 0 \quad \text{in } B_1(0).\]

(3.2)

For $s_+, s_-$ undetermined, let

\[
V_{1,\varepsilon}(x) = \begin{cases}
\frac{1}{2\pi} \ln \frac{1}{\varepsilon} + \varepsilon^{-2} \gamma_1^{-1} x_1 \ln \frac{\varepsilon}{s_+} V_1(\frac{|x-p|}{s_+}) & |x - p| \leq s_+,
\frac{1}{2\pi} \ln \frac{1}{\varepsilon} \cdot \ln \frac{|x-p|}{\ln s_+} & |x - p| > s_+,
\end{cases}
\]

where $V_1$ is the solution to (3.2) with exponent $\gamma = \gamma_1$, and

\[
V_{2,\varepsilon}(x) = \begin{cases}
\frac{1}{2\pi} \ln \frac{1}{\sigma} + \sigma^{-2} \gamma_2^{-1} x_1 \ln \frac{\varepsilon}{s_-} V_2(\frac{|x-q|}{s_-}) & |x - q| \leq s_-,
\frac{1}{2\pi} \ln \frac{1}{\sigma} \cdot \ln \frac{|x-q|}{\ln s_-} & |x - q| > s_-,
\end{cases}
\]

where $V_2$ is solution to (3.2) with exponent $\gamma = \gamma_2$. To make $V_{1,\varepsilon}, V_{2,\varepsilon}(x) \in C^1$, we need to choose $s_+, s_-$ such that

\[
\varepsilon^{-2} \gamma_1^{-1} s_+^{-1} |V_1'(1)| = \frac{1}{2\pi} \frac{|\ln \varepsilon|}{|\ln s_+|}, \quad \sigma^{-2} \gamma_2^{-1} s_-^{-1} |V_2'(1)| = \frac{1}{2\pi} \frac{|\ln \sigma|}{|\ln s_-|},
\]

which is equivalent to

\[
s_+ = \mu_+ \varepsilon, \quad s_- = \mu_- \sigma
\]

for some constants $\mu_+, \mu_- > 0$. Then a suitable approximate solution to (3.1) is

\[
\Psi_\varepsilon(x) = \sum_{k \in \mathbb{Z}} V_{1,\varepsilon}(x + k\ell_2) - \sum_{k \in \mathbb{Z}} V_{2,\varepsilon}(x + k\ell_2)
\]

(3.3)
for \( x \in \mathbb{R} \times (-l/2, l/2) \). Similar to (2.5), to make (3.3) convergent, we assume the sum is understood in the sense

\[
\Psi_\varepsilon(x) = V_{1,\varepsilon}(x) - V_{2,\varepsilon}(x)
\]

\[
+ \lim_{N \to \infty} \sum_{k=1}^{N} \left( \sum_{m=\pm k} V_{1,\varepsilon}(x + ml_2) - \sum_{m=\pm k} V_{2,\varepsilon}(x + ml_2) \right).
\]

From (3.1), for \( x \in B_r(p) \) we have

\[
- \Delta \Psi_\varepsilon - \varepsilon^{-2} \left( \Psi_\varepsilon + W_\varepsilon x_1 - \frac{\lambda_+}{2\pi} \ln \frac{1}{\varepsilon} \right)^{\gamma_1} + \chi_{B_r(p)}
\]

\[
+ \sigma^{-2} \left( -\Psi_\varepsilon - W_\varepsilon x_1 - \frac{\lambda_-}{2\pi} \ln \frac{1}{\sigma} \right)^{\gamma_2} + \chi_{B_r(q)}
\]

\[
= \varepsilon^{-2} \left( \left( V_{1,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} \right)^{\gamma_1} + \right.
\]

\[
- \left( \sum_{k \in \mathbb{Z}} V_{1,\varepsilon}(x + k\varepsilon e_2) - \sum_{k \in \mathbb{Z}} V_{2,\varepsilon}(x + k\varepsilon e_2) + W_\varepsilon x_1 - \frac{\lambda_+}{2\pi} \ln \frac{1}{\varepsilon} \right)^{\gamma_1} + \chi_{B_r(p)}
\]

\[
- \left( V_{2,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} \right)^{\gamma_2} + \right.
\]

\[
+ \left. \left( \sum_{k \in \mathbb{Z}} V_{2,\varepsilon}(x + k\varepsilon e_2) - \sum_{k \in \mathbb{Z}} V_{1,\varepsilon}(x + k\varepsilon e_2) - W_\varepsilon x_1 - \frac{\lambda_-}{2\pi} \ln \frac{1}{\sigma} \right)^{\gamma_2} \right) + \chi_{B_r(q)}.
\]

Similarly, for \( x \in B_r(q) \) it holds

\[
- \Delta \Psi_\varepsilon - \varepsilon^{-2} \left( \Psi_\varepsilon + W_\varepsilon x_1 - \frac{\lambda_+}{2\pi} \ln \frac{1}{\varepsilon} \right)^{\gamma_1} + \chi_{B_r(p)}
\]

\[
+ \sigma^{-2} \left( -\Psi_\varepsilon - W_\varepsilon x_1 - \frac{\lambda_-}{2\pi} \ln \frac{1}{\sigma} \right)^{\gamma_2} + \chi_{B_r(q)}
\]

\[
= \sigma^{-2} \left( - \left( V_{2,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\sigma} \right)^{\gamma_2} + \right.
\]

\[
+ \left. \left( \sum_{k \in \mathbb{Z}} V_{2,\varepsilon}(x + k\varepsilon e_2) - \sum_{k \in \mathbb{Z}} V_{1,\varepsilon}(x + k\varepsilon e_2) - W_\varepsilon x_1 - \frac{\lambda_-}{2\pi} \ln \frac{1}{\sigma} \right)^{\gamma_2} \right) + \chi_{B_r(q)}.
\]

To ensure that \( \Psi_\varepsilon(x) \) is a good approximation of the solution to (3.1), we choose \( \lambda_+ \) and \( \lambda_- \) such that

\[
\lim_{N \to \infty} \sum_{k=1}^{N} \left( \sum_{m=\pm k} V_{1,\varepsilon}(p + ml_2) - \sum_{m=\pm k} V_{2,\varepsilon}(p + ml_2) \right)
\]

\[
- V_{2,\varepsilon}(p) - W_\varepsilon d - \frac{\lambda_+}{2\pi} \ln \frac{1}{\varepsilon} = - \frac{\lambda_-}{2\pi} \ln \frac{1}{\varepsilon},
\]

(3.4)
\[
\lim_{N \to \infty} \sum_{k=1}^{N} \left( \sum_{m=\pm k} V_{2,\varepsilon}(q + m\mathbf{e}_2) - \sum_{m=\pm k} V_{1,\varepsilon}(q + m\mathbf{e}_2) \right) - V_{1,\varepsilon}(q) - W_\varepsilon d - \frac{\lambda_-}{2\pi} \ln \frac{1}{\sigma} = -1 - \frac{1}{2\pi} \ln \frac{1}{\sigma}.
\]

(3.5)

Notice that the sums in the above two equalities are convergent, since as \(|kl| \to \infty|
\sum_{m=\pm k} V_{1,\varepsilon}(p + m\mathbf{e}_2) - \sum_{m=\pm k} V_{2,\varepsilon}(p + m\mathbf{e}_2) \approx C|kl|^{-2},
and
\sum_{m=\pm k} V_{2,\varepsilon}(q + m\mathbf{e}_2) - \sum_{m=\pm k} V_{1,\varepsilon}(q + m\mathbf{e}_2) \approx C|kl|^{-2}.

By (3.4) and (3.5), for \(\lambda_+\) and \(\lambda_-\) we have the following asymptotic estimate
\[
\lambda_+ = 1 + O\left(\frac{\varepsilon}{\ln \varepsilon}\right), \quad \lambda_- = 1 + O\left(\frac{\sigma}{\ln \sigma}\right).
\]

Using Pohozaev identity \(\int_{B_1(0)} V^p = 2\pi|V'(1)|\), one can easily verify that it holds in the sense of measure that as \(\varepsilon \to 0\),
\[
-\Delta \Psi_\varepsilon(x) \to \delta_p(x) - \delta_q(x) \quad \text{in } \mathbb{R} \times (-l/2, l/2).
\]
The error of the approximation by \(\Psi_\varepsilon\) is
\[
- \Delta \Psi_\varepsilon - \varepsilon^{-2} \left( \Psi_\varepsilon + W_\varepsilon x_1 - \frac{\lambda_+}{2\pi} \ln \frac{1}{\varepsilon} \right)^{\gamma_1} \chi_{B_{1}(p)}
+ \sigma^{-2} \left( -\Psi_\varepsilon - W_\varepsilon x_1 - \frac{\lambda_-}{2\pi} \ln \frac{1}{\sigma} \right)^{\gamma_2} \chi_{B_{1}(q)}
= O(\varepsilon^{-1})\chi_{B_{L\varepsilon}(p)} + O(\sigma^{-1})\chi_{B_{L\sigma}(q)},
\]
where \(L > 0\) is some large constant.

Similarly to the case \(0 < s < 1\), the desirable \(l\)-symmetric solutions to (3.1) has the form
\[
\psi_\varepsilon(x) = \Psi_\varepsilon(x) + \omega_\varepsilon(x),
\]
with \(x \in \mathbb{R} \times (-l/2, l/2)\), and \(\omega_\varepsilon(x)\) a family of \(l\)-symmetric perturbation terms. Hence we are going to study the equation for \(\omega_\varepsilon(x)\).

3.2. The linear theory. The linearized operator of (3.1) at \(\Psi_\varepsilon(x)\) is
\[
\mathbb{L}_\varepsilon w = -\Delta w - f_\varepsilon(x, \Psi_\varepsilon)w \quad \text{in } \mathbb{R} \times (-l/2, l/2),
\]
(3.7)
where \( f(x,u) \) is the (nonlinear) function in the right hand side of (3.1) with \( \psi \) being replaced by \( u \). So
\[
f_u(x, \Psi_\varepsilon) = \varepsilon^{-2\gamma_1} \left( \Psi_\varepsilon + W_\varepsilon x_1 - \frac{\lambda_+}{2\pi} \ln \frac{1}{\varepsilon} \right)^{\gamma_1 - 1} + \chi_{B_r(p)}
\]
\[
+ \sigma(\varepsilon)^{-2\gamma_2} \left( -\Psi_\varepsilon - W_\varepsilon x_1 - \frac{\lambda_-}{2\pi} \ln \frac{1}{\sigma(\varepsilon)} \right)^{\gamma_2 - 1} + \chi_{B_r(q)}.
\]

(3.8)

Hence we can write (3.1) as
\[
\mathbb{L}_\varepsilon \omega_\varepsilon = -E_\varepsilon + R_\varepsilon(\omega_\varepsilon) \quad \text{in } \mathbb{R} \times (-l/2, l/2),
\]
(3.9)

where
\[
E_\varepsilon = -\Delta \Psi_\varepsilon - f(x, \Psi_\varepsilon)
\]
and
\[
R_\varepsilon(\omega_\varepsilon) = f(x, \Psi_\varepsilon + \omega_\varepsilon) - f(x, \Psi_\varepsilon) - f_u(x, \Psi_\varepsilon)\omega_\varepsilon.
\]

Let
\[
\tilde{V} = \left\{ \begin{array}{ll}
V(|x|) & \text{if } |x| \leq 1, \\
|V'(1)| \ln \frac{1}{|x|} & \text{if } |x| > 1,
\end{array} \right.
\]
where \( V \) is the unique solution of (3.2). Then the locally linearized operator for our problem is
\[
\mathbb{L}_0 w = -\Delta w - \gamma \tilde{V}_+^{1-\gamma} w \quad \text{in } \mathbb{R}^2.
\]

The following nondegeneracy theorem can be found in [5, 12]:

**Theorem 3.1.** If \( \varphi \) is in the kernel of \( \mathbb{L}_0 \), then \( \varphi \) is a linear combination of \( \partial_{x_1} \tilde{V} \) and \( \partial_{x_2} \tilde{V} \).

Hence the kernel of \( \mathbb{L}_0 \) is spanned by
\[
Z_\varepsilon(x) = Z_{1,\varepsilon}(x) - Z_{2,\varepsilon}(x),
\]
where
\[
Z_{1,\varepsilon}(x) = \sum_{k \in \mathbb{Z}} \partial_{x_1} V_{1,\varepsilon}(x + ke_2), \quad Z_{2,\varepsilon}(x) = \sum_{k \in \mathbb{Z}} \partial_{x_1} V_{2,\varepsilon}(x + ke_2).
\]

We will study the following projected linear problem:
\[
\begin{aligned}
\mathbb{L}_\varepsilon \omega_\varepsilon &= h(x) + \alpha_\varepsilon f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) \quad \text{in } \mathbb{R} \times (-l/2, l/2), \\
\int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) \omega_\varepsilon(x) dx &= 0, \\
\omega_\varepsilon(x) &\to 0 \quad \text{as } |x_1| \to \infty,
\end{aligned}
\]
(3.10)

where we assume \( l \)-symmetric \( h(x) \) satisfies
\[
supp(h(x)) \subset B_{L_\varepsilon}(p) \cup B_{L_\sigma}(q)
\]
for some large constant \( L > 0 \). The norms we will use for the case \( s = 1 \) are
\[
\|
\omega_\varepsilon
\|_* = \sup_{x \in \mathbb{R} \times (-l/2, l/2)} \rho(x)^{-1} |\omega_\varepsilon(x)|,
\]
Proof. Similar to the proof of Lemma 2.2, we first prove

$$\alpha \varepsilon$$

then there exists a small $$\varepsilon_0 > 0$$ such that for any $$\varepsilon \in (0, \varepsilon_0)$$ and solution pair $$(\omega_\varepsilon, \alpha_\varepsilon)$$ to (2.13), and for some constant $$C > 0$$

$$\|\omega_\varepsilon\|_* + (\sigma(\varepsilon))^{-1}|\alpha_\varepsilon| \leq C\|h\|_*.$$ (3.12)

The following a priori estimate is the counterpart of Lemma 2.2 in Section 2.

**Lemma 3.2.** Assume that $$h(x)$$ is l-symmetric, which satisfies (3.11) and $$\|h\|_* < \infty$$. Then there exists a small $$\varepsilon_0 > 0$$ such that for any $$\varepsilon \in (0, \varepsilon_0)$$ and solution pair $$(\omega_\varepsilon, \alpha_\varepsilon)$$ to (2.13), and for some constant $$C > 0$$

$$\|\omega_\varepsilon\|_* + (\sigma(\varepsilon))^{-1}|\alpha_\varepsilon| \leq C\|h\|_*.$$ (3.12)

**Proof.** Similar to the proof of Lemma 2.2, we first prove

$$(\sigma(\varepsilon))^{-1}|\alpha_\varepsilon| \leq C(\|h\|_* + o_\varepsilon(1)\|\omega_\varepsilon\|_*).$$ (3.13)

The coefficient $$\alpha_\varepsilon$$ is given by

$$\alpha_\varepsilon \int_{\mathbb{R} \times (-l/2,l/2)} f_u(x, \Psi_\varepsilon)Z_\varepsilon^2 dx = \int_{\mathbb{R} \times (-l/2,l/2)} Z_\varepsilon \bar{\omega}_\varepsilon dx - \int_{\mathbb{R} \times (-l/2,l/2)} hZ_\varepsilon dx.$$ (3.13)

Notice that $$s_+ = \varepsilon \mu_+$$ and $$s_- = \sigma \mu_-$$.

For the left hand side we have

$$\int_{\mathbb{R} \times (-l/2,l/2)} f_u(x, \Psi_\varepsilon)Z_\varepsilon^2 dx = (1 + o_\varepsilon(1))\varepsilon^2 \int_{\mathbb{R} \times (-l/2,l/2)} \gamma_1 \left( V_{1,\varepsilon} - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} + \left( \frac{\partial V_{1,\varepsilon}}{\partial y_1} \right)^2 dy_1 \right) + (1 + o_\varepsilon(1))\sigma^2 \int_{\mathbb{R} \times (-l/2,l/2)} \gamma_2 \left( V_{2,\varepsilon} - \frac{1}{2\pi} \ln \frac{1}{\sigma} + \left( \frac{\partial V_{2,\varepsilon}}{\partial y_1} \right)^2 dy_2 \right)$$

$$= c_1(1 + o_\varepsilon(1))\varepsilon^2 + c_2(1 + o_\varepsilon(1))\sigma^2,$$ (3.14)

where $$y^1 = \frac{x}{s_+},$$ $$y^2 = \frac{x}{s_-},$$ and $$c_1, c_2 > 0$$ are some constants. For the right hand side, it holds

$$\int_{\mathbb{R} \times (-l/2,l/2)} Z_\varepsilon(-\Delta \omega_\varepsilon) dx = \int_{\mathbb{R} \times (-l/2,l/2)} \omega_\varepsilon(-\Delta Z_\varepsilon) dx$$

$$= \int_{\mathbb{R} \times (-l/2,l/2)} \omega_\varepsilon \left( -\varepsilon^2 \gamma_1 \left( V_{1,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} \right)^2 + Z_{1,\varepsilon} \right) - \sigma^2 \gamma_2 \left( V_{2,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\sigma} \right)^2 Z_{2,\varepsilon} dx.$$ (3.14)
For \( x_1 < 0 \), we have
\[
\left| \varepsilon^{-2}\gamma_1 \left( V_{1,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} \right)^{\gamma_1-1} Z_{1,\varepsilon} - \varepsilon^{-2}\gamma_2 \left( V_{2,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} \right)^{\gamma_2-1} Z_{2,\varepsilon} - f_n(x, \Psi_{\varepsilon}) Z_\varepsilon \right|
\]
\[
= \varepsilon^{-2}\gamma_1 \left( V_{1,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} \right)^{\gamma_1-1} Z_{1,\varepsilon} - \varepsilon^{-2}\gamma_1 \left( V_{1,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} + O(\varepsilon) \right)^{\gamma_1-1} Z_\varepsilon
\]
\[
\leq C\varepsilon^{-3+\min\{\gamma_1-1,1\}} \chi_{B_L(p)}.
\]
For \( x_1 > 0 \), the term is
\[
-\sigma^{-2}\gamma_2 \left( V_{2,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\sigma} \right)^{\gamma_2-1} Z_{2,\varepsilon} + \sigma^{-2}\gamma_2 \left( V_{2,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\sigma} + O(\sigma) \right)^{\gamma_2-1} Z_\varepsilon
\]
\[
\leq C\sigma^{-3+\min\{\gamma_2-1,1\}} \chi_{B_L(q)}.
\]
Using Hölder’s inequality, we have
\[
\left| \int_{\mathbb{R} \times (-1/2,1/2)} Z_\varepsilon \mathbb{I}_{\mathbb{E}_\varepsilon} \omega_\varepsilon \, dx \right| \leq o_\varepsilon(1) \cdot \|\omega_\varepsilon\|_{\sigma^{-1}},
\]
and
\[
\left| \int_{\mathbb{R} \times (-1/2,1/2)} h Z_\varepsilon \, dx \right| \leq \|h\|_{\sigma^{-1}}.
\]
Then (3.13) is a direct consequence of (3.14), (3.15) and (3.16).

Then we will prove
\[
\|\omega_\varepsilon\|_{\sigma} \leq C\|h\|_{\sigma^-},
\]
which is achieved by deriving a contradiction. We assume that there exists a sequence \( \{\varepsilon_n\} \) satisfying \( \varepsilon_n \to 0 \), and solution pairs \( (\omega_{\varepsilon_n}, \alpha_{\varepsilon_n}) \) to (3.10) for some \( h_{\varepsilon_n} \), such that
\[
\|\omega_{\varepsilon_n}\|_{\sigma} = 1, \quad \|h_{\varepsilon_n}\|_{\sigma} \to 0 \quad \text{as} \quad n \to \infty.
\]
We want to show that for any \( L > 0 \), it always holds as \( n \to \infty, \)
\[
\|\omega_{\varepsilon_n}\|_{L^\infty(B_{L\xi_n}(p))} + \|\omega_{\varepsilon_n}\|_{L^\infty(B_{L\xi_n}(q))} \to 0,
\]
where \( \alpha_n = \sigma(\varepsilon_n) \). Indeed, if (3.19) is false, then we can suppose the first term satisfies for some \( \Lambda_0 > 0 \)
\[
\|\omega_{\varepsilon_n}\|_{L^\infty(B_{L\xi_n}(p))} \geq \Lambda_0 > 0.
\]
Let
\[
\tilde{\omega}_{\varepsilon_n}(y) = \mu_+^{2\gamma_1} \omega_{\varepsilon_n}(s+y+p),
\]
then in every compact set \( \tilde{\omega}_{\varepsilon_n}(y) \) satisfies
\[
-\Delta \tilde{\omega}_{\varepsilon_n}(y) - \gamma_1(\tilde{V}_1 + O(\varepsilon_n))^{\gamma_1-1} \tilde{\omega}_{\varepsilon_n}(y) + o_{\varepsilon_n}(1)
\]
\[
= \varepsilon_n^{2}\mu_+^{2\gamma_1} \tilde{h}_{\varepsilon_n}(s+y+p) + \varepsilon_n^{-1}\alpha_{\varepsilon_n} \gamma_1(\tilde{V}_1 + O(\varepsilon_n))^{\gamma_1-1} \left( \frac{\partial \tilde{V}_1}{\partial y_1} + o_{\varepsilon_n}(1) \right),
\]
which is equivalent to
\[-\Delta \tilde{\omega}_{\varepsilon_n}(y) - \gamma_1 V_1^{0 \gamma_1 - 1} \tilde{\omega}_{\varepsilon_n}(y) + o_{\varepsilon_n}(1) = \mathcal{R}_n(y),\]
where
\[\mathcal{R}_n(y) = \varepsilon_n^2 \mu_+^{\gamma_1 - 1} h_{\varepsilon_n}(s + y + p) + o_{\varepsilon_n}(1) \cdot \tilde{\omega}_{\varepsilon_n}(y) + \varepsilon_n^{-1} \tilde{\alpha}_{\varepsilon_n} (\tilde{V}_1 + o_{\varepsilon_n}(1))^{\gamma_1 - 1} \left( \frac{\partial \tilde{V}_1}{\partial y_1} + o_{\varepsilon_n}(1) \right).\]

Since \(\varepsilon_n^2 \mu_+^{\gamma_1 - 1} h_{\varepsilon_n}(y) \to 0\), and \(\varepsilon_n^{-1} \tilde{\alpha}_{\varepsilon_n} \leq \sigma^{-1} \alpha_{\varepsilon_n} \leq C(\|h\|_{**} + o_{\varepsilon_n}(1)\|\omega_{\varepsilon_n}\|_*) = o_{\varepsilon_n}(1)\) on every compact set by (3.13) and (3.18), we have \(\mathcal{R}_n(y) \to 0\). Let \(n \to \infty\), we may assume that \(\tilde{\omega}_{\varepsilon_n}\) converge uniformly on compact sets to a function \(\tilde{\omega}\) satisfying
\[\|\tilde{\omega}\|_{L^\infty(B_{\mu_+}^1(0))} \geq \Lambda_0 \mu_+^{\gamma_1 - 1}.\] (3.20)

However, \(\tilde{\omega}\) is even in \(y_2\) direction, and is a solution to
\[-\Delta \tilde{\omega}(y) - \gamma_1 V_1^{0 \gamma_1 - 1} \tilde{\omega}(y) = 0.\]

By (3.10), it also satisfies the orthogonality condition
\[\int_{\mathbb{R}^2} \gamma_1 V_1^{0 \gamma_1 - 1} \tilde{\omega} \frac{\partial \tilde{V}_1}{\partial y_1} dy = 0.\]

According to Theorem 3.1, it must hold \(\tilde{\omega} \equiv 0\), which is a contradiction to (3.20). Hence we deduce that \(\|\omega_{\varepsilon_n}\|_{L^\infty(B_{\mu_+}^1(\varepsilon_n))} \to 0\). For the second term in (3.19), we can use a similar method to prove \(\|\omega_{\varepsilon_n}\|_{L^\infty(B_{\mu_+}^1(\varepsilon_n))} \to 0\).

In view of (2.13), \(\omega_{\varepsilon_n}\) satisfies
\[-\Delta \omega_{\varepsilon_n} = f_u(x, \Psi_0) \omega_{\varepsilon_n} + h_{\varepsilon_n} + \alpha_{\varepsilon_n} f_u(x, \Psi_{\varepsilon_n}) Z_{\varepsilon_n}.\]

So we have
\[\omega_{\varepsilon_n}(x) = \int_{\mathbb{R}^2 \times (-1,1)/2} K_1(x - z) \left( f_u(x, \Psi_0) \omega_{\varepsilon_n}(x) + h_{\varepsilon_n}(x) + \alpha_{\varepsilon_n} f_u(x, \Psi_{\varepsilon_n}) Z_{\varepsilon_n}(x) \right) dz,\]
where
\[K_1(x) = \sum_{k \in \mathbb{Z}} G_1(x + ke_2).\]

This implies
\[\rho(x)^{-1} |\omega_{\varepsilon_n}(x)| \leq C \left( \|\omega_{\varepsilon_n}\|_{L^\infty(B_{\mu_+}^1(\varepsilon_n))} + \|\omega_{\varepsilon_n}\|_{L^\infty(B_{\mu_+}^1(\varepsilon_n))} + \|h_{\varepsilon_n}\|_{**} + \sigma_n^{-1} \alpha_{\varepsilon_n} \right).\]

Now, from (3.13) (3.18) (3.19), we can obtain \(\|\omega_{\varepsilon_n}\|_* \to 0\) as \(n \to \infty\), which is a contradiction to (3.18). Thus (3.17) is obvious, and (3.12) is the consequence of (3.13) and (3.17).

We have the following lemma for the projective problem (3.10).
**Lemma 3.3.** Assume that \( h(x) \) is \( l \)-symmetric, which satisfies (3.11) and \( \| h \|_{**} < \infty \). Then there exists a small \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \), (3.10) has a unique solution \( \omega_\varepsilon = T_\varepsilon h \), where \( T_\varepsilon \) is a linear operator of \( h \). Moreover, there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that
\[
\| \omega_\varepsilon \|_* \leq C \| h \|_{**}.
\]

**Proof.** Denote the Hilbert space
\[
H := \left\{ g \in \dot{H}^1(\mathbb{R} \times (-l/2, l/2)) : g \text{ is } l \text{-symmetric}, \int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) g(x) dx = 0 \right\}
\]
endowed with the inner product
\[
[u, g] = \int_{\mathbb{R} \times (-l/2, l/2)} \nabla u(x) \nabla g(x) dx.
\]
Then we can use Riesz’s representation theorem and Fredholm’s alternative to derive the desired conclusion, just as what we did in the proof of Lemma 2.3. \( \square \)

### 3.3. The reduction.

To solve (3.1), we will deal with (3.10) for \( h(x) = -E_\varepsilon + R_\varepsilon(\omega_\varepsilon) \).

Using Lemma 3.3 and a contraction mapping argument, the existence and uniqueness for solutions to (3.10) and (3.21) can be established in the following lemma. Since it is similar to that of Lemma 2.4, we omit its proof.

**Lemma 3.4.** There are \( \varepsilon_0 > 0 \) and \( r_0 > 0 \) such that for \( \varepsilon \in (0, \varepsilon_0) \) there exists a unique \( \omega_\varepsilon \) to (3.10) and (3.21) in the ball \( \| \omega_\varepsilon \|_* \leq r_0 \). Moreover, it holds
\[
\| \omega_\varepsilon \|_* \leq C \varepsilon
\]
for some constant \( C > 0 \), and \( \omega_\varepsilon \) is continuous with respect to \( \varepsilon \).

We can easily verify that \( \psi_\varepsilon(x) = \Psi_\varepsilon(x) + \omega_\varepsilon(x) \) satisfies
\[
-\Delta \psi_\varepsilon = f(x, \psi_\varepsilon) + \alpha_\varepsilon f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) \quad \text{in } \mathbb{R} \times (-l/2, l/2).
\]

To eliminate the second term on the right hand side, we multiply (3.23) by \( Z_\varepsilon \) and integrate over \( \mathbb{R} \times (-l/2, l/2) \) to get
\[
\alpha_\varepsilon \int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon) Z_\varepsilon^2 dx = \int_{\mathbb{R} \times (-l/2, l/2)} (-\Delta \psi_\varepsilon - f(x, \psi_\varepsilon)) Z_\varepsilon dx.
\]

The following lemma gives a precise condition to ensure that \( \alpha_\varepsilon = 0 \).

**Lemma 3.5.** It holds
\[
\int_{\mathbb{R} \times (-l/2, l/2)} (-\Delta \psi_\varepsilon - f(x, \psi_\varepsilon)) Z_\varepsilon dx = C \left( \frac{1}{2\pi} \lim_{N \to \infty} \sum_{|k| \leq N} \frac{p_1 - q_1}{|q - p + k| e_2^2 + W_\varepsilon} \right) + o_\varepsilon(1),
\]
where \( C > 0 \) is a constant independent of \( \varepsilon \), and \( p_1, q_1 \) denote the first coordinates of \( p, q \) respectively.
Proof. By (3.10) and (3.21), we have
\[-\Delta \psi_\varepsilon - f(x, \psi_\varepsilon) = L_\varepsilon \psi_\varepsilon + E_\varepsilon - R_\varepsilon(\omega_\varepsilon) \quad \text{in } \mathbb{R} \times (-l/2, l/2).\]
Multiplying this equality by $Z_\varepsilon$ and integrating over $\mathbb{R} \times (-l/2, l/2)$, we obtain
\[
\int_{\mathbb{R} \times (-l/2, l/2)} (-\Delta \psi_\varepsilon - f(x, \psi_\varepsilon)) Z_\varepsilon \, dx = \int_{\mathbb{R} \times (-l/2, l/2)} L_\varepsilon \omega_\varepsilon Z_\varepsilon \, dx + \int_{\mathbb{R} \times (-l/2, l/2)} E_\varepsilon Z_\varepsilon \, dx - \int_{\mathbb{R} \times (-l/2, l/2)} R_\varepsilon(\omega_\varepsilon) Z_\varepsilon \, dx.
\]
By the choice of $\lambda_+$ and $\lambda_-$ in (3.4) and (3.5), $E_\varepsilon$ can be split into \[E_\varepsilon = E_{1,\varepsilon} + E_{2,\varepsilon},\]
where
\[
E_{1,\varepsilon} = \varepsilon^{-2} \chi_{B_{L_\varepsilon}}(p) \left( \left( V_{1,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} \right)_+ - \left( V_{1,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} \right)_- + \sum_{k \neq 0} V_{1,\varepsilon}(x + k\ell e_2) - \sum_{k \in \mathbb{Z}} V_{2,\varepsilon}(x + k\ell e_2) - \sum_{k \neq 0} V_{1,\varepsilon}(p + k\ell e_2) + \sum_{k \in \mathbb{Z}} V_{2,\varepsilon}(p + k\ell e_2) + W_\varepsilon(x_1 - d) \right)_+ \right),
\]
and
\[
E_{2,\varepsilon} = \sigma^{-2} \chi_{B_{L_\varepsilon}}(q) \left( \left( V_{2,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\sigma} \right)_+ + \left( V_{2,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\sigma} \right)_- + \sum_{k \neq 0} V_{2,\varepsilon}(x + k\ell e_2) - \sum_{k \in \mathbb{Z}} V_{1,\varepsilon}(x + k\ell e_2) - \sum_{k \neq 0} V_{2,\varepsilon}(q + k\ell e_2) + \sum_{k \in \mathbb{Z}} V_{1,\varepsilon}(q + k\ell e_2) - W_\varepsilon(x_1 - d) \right)_+ \right).
\]
Since \[|E_{1,\varepsilon}| \leq C\varepsilon^{-2} \cdot \varepsilon \quad \text{and} \quad |E_{2,\varepsilon}| \leq C\sigma^{-2} \cdot \sigma,\]
It holds
\[
\int_{\mathbb{R} \times (-l/2, l/2)} E_{1,\varepsilon} Z_{2,\varepsilon} \, dx \leq C\varepsilon, \quad \int_{\mathbb{R} \times (-l/2, l/2)} E_{2,\varepsilon} Z_{1,\varepsilon} \, dx \leq C\sigma.
\]
By Taylor’s formula, we have
\[
E_{1,\varepsilon} = -\varepsilon^{-2} \chi_{B_{L_\varepsilon}}(p) \left( V_{1,\varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} \right)_+ \left( \sum_{k \neq 0} V_{1,\varepsilon}(x + k\ell e_2) - \sum_{k \in \mathbb{Z}} V_{2,\varepsilon}(x + k\ell e_2) - \sum_{k \neq 0} V_{1,\varepsilon}(p + k\ell e_2) + \sum_{k \in \mathbb{Z}} V_{2,\varepsilon}(p + k\ell e_2) + W_\varepsilon(x_1 - d) \right) + \varepsilon^{-2} \cdot \varepsilon^{\gamma_1-1} \chi_{B_{L_\varepsilon}(p)}.
\]
We then integrate by parts and use the asymptotic behavior of $V_{1, \varepsilon}, V_{2, \varepsilon}$ to obtain

$$
\int_{\mathbb{R} \times (-l/2, l/2)} E_{1, \varepsilon} Z_{1, \varepsilon} dx = \varepsilon^{-2} \int_{\mathbb{R} \times (-l/2, l/2)} \left( V_{1, \varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} \right) + \left( \sum_{k \neq 0} \partial_{x_1} V_{1, \varepsilon}(x + kle_2) - \sum_{k \in \mathbb{Z}} \partial_{x_1} V_{2, \varepsilon}(x + kle_2) + W_\varepsilon + O(\varepsilon) \right) dx
$$

$$
= \varepsilon^{-2} \int_{\mathbb{R} \times (-l/2, l/2)} \left( V_{1, \varepsilon}(x) - \frac{1}{2\pi} \ln \frac{1}{\varepsilon} \right) + \left( \frac{1}{2\pi} \lim_{N \to \infty} \sum_{|k| \leq N} \frac{p_1 - q_1}{|p - q + kle_2|^2} + W_\varepsilon + O(\varepsilon) \right) dx
$$

$$
= C_1 \left( \frac{1}{2\pi} \lim_{N \to \infty} \sum_{|k| \leq N} \frac{p_1 - q_1}{|p - q + kle_2|^2} + W_\varepsilon \right) + O(\varepsilon),
$$

where $p_1, q_1$ denotes the first coordinates of $p, q$. Similarly, we can obtain

$$
- \int_{\mathbb{R} \times (-l/2, l/2)} E_{2, \varepsilon} Z_{2, \varepsilon} dx = C_2 \left( \frac{1}{2\pi} \lim_{N \to \infty} \sum_{|k| \leq N} \frac{-q_1 + p_1}{|q - p + kle_2|^2} + W_\varepsilon \right) + O(\varepsilon).
$$

Hence we can derive

$$
\int_{\mathbb{R} \times (-l/2, l/2)} E_\varepsilon Z_\varepsilon dx = C \left( \frac{1}{2\pi} \lim_{N \to \infty} \sum_{|k| \leq N} \frac{p_1 - q_1}{|q - p + kle_2|^2} + W_\varepsilon \right) + o_\varepsilon(1). \quad (3.24)
$$

Recall that $\sigma(\varepsilon)$ satisfies assumption (H) with $\tau = \min\{\gamma_2, 2\}$. We also have

$$
\int_{\mathbb{R} \times (-l/2, l/2)} R_\varepsilon(\omega_\varepsilon) Z_\varepsilon dx = \int_{\mathbb{R} \times (-l/2, l/2)} R_\varepsilon(\omega_\varepsilon) Z_\varepsilon dx + \int_{\mathbb{R} \times (-l/2, l/2)} R_\varepsilon(\omega_\varepsilon) Z_\varepsilon dx
$$

$$
\leq C\|\omega_\varepsilon\|_{\text{min}\{\gamma_1, 2\}} \varepsilon^{-1} + C\|\omega_\varepsilon\|_{\text{min}\{\gamma_2, 2\}} \sigma^{-1}
\leq C\varepsilon^{\text{min}\{\gamma_1 - 1, 1\}} + C\varepsilon^{\text{min}\{\gamma_2, 2\}} \sigma^{-1} = o_\varepsilon(1), \quad (3.25)
$$

and

$$
\int_{\mathbb{R} \times (-l/2, l/2)} \mathbb{L}_\varepsilon \omega_\varepsilon Z_\varepsilon dx = \int_{\mathbb{R} \times (-l/2, l/2)} \mathbb{L}_\varepsilon \omega_\varepsilon Z_\varepsilon dx + \int_{\mathbb{R} \times (-l/2, l/2)} \mathbb{L}_\varepsilon \omega_\varepsilon Z_\varepsilon dx
$$

$$
\leq C\|\omega_\varepsilon\|_{\text{min}\{\gamma_1 - 2, 0\}} + C\|\omega_\varepsilon\|_{\sigma}^{\text{min}\{\gamma_2 - 2, 0\}}
\leq C\varepsilon^{\text{min}\{\gamma_1 - 1, 1\}} + C\varepsilon \cdot \sigma^{\text{min}\{\gamma_2 - 2, 0\}} = o_\varepsilon(1). \quad (3.26)
$$

Using (3.24), (3.25) and (3.26), we arrive at the conclusion.
Having made all the necessary preparation, now we are in a position to prove Theorem 1.1 and 1.3 for the case $s = 1$.

**Proof of Theorem 1.1 and 1.3 with $s = 1$**: We are to solve $\alpha \varepsilon = 0$ directly. Since it holds
\[
\int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi \varepsilon) Z^2 \varepsilon dx > 0,
\]
the condition $\alpha \varepsilon = 0$ is equivalent to
\[
W \varepsilon = \frac{1}{2\pi} \lim_{N \to \infty} \sum_{|k| \leq N} \frac{-p_1 + q_1}{|q - p + k\varepsilon e_2|^2} + o_\varepsilon(1)
\]
by Lemma 3.5. Hence if $a = 0$, $W \varepsilon$ should satisfy
\[
W \varepsilon = \frac{1}{2\pi} \lim_{N \to \infty} \sum_{|k| \leq N} \frac{2d}{4d^2 + k^2} + o_\varepsilon(1);
\]
while for $a = l/4$, the condition turns to be
\[
W \varepsilon = \frac{1}{2\pi} \lim_{N \to \infty} \sum_{|k| \leq N} \frac{2d}{4d^2 + (kl + \frac{l}{4})^2} + o_\varepsilon(1).
\]
The weak convergence of $\vartheta_{0, \varepsilon}$ is obvious as $\varepsilon \to 0$. By the choice of $p, q$ we get the existence of $W \varepsilon$ so that the corresponding $\alpha \varepsilon = 0$. $C^1$ property follows from regularity theory for Laplacian. Thus the proof is complete. $\square$

4. **General result for $a \in (0, l/2)$**

In this section, we will investigate the general case $a \in (0, l/2)$. Due to the loss of symmetry with respect to $x_1$-direction, the travelling speed $U \varepsilon$ is no longer $x_2$-directional when $a \neq 0$ and $a \neq l/4$ hold simultaneously. For simplicity, we still denote the $l$-periodicity restricted in $\mathbb{R} \times (-l, l)$ as $l$-symmetry, and make the following decomposition for the uniform travelling speed
\[
U \varepsilon = W \varepsilon e_1 + W \varepsilon e_2. \tag{4.1}
\]
Then for $0 < s < 1$, the aim is to find a family of $l$-symmetry solutions to
\[
\begin{cases}
(-\Delta)^{s} \psi = \varepsilon^{(2-2s)\gamma_{1}-2}(\psi + W \varepsilon x_1 - \overline{W \varepsilon x_2} - \varepsilon^{2s-2}\lambda_{+})^{\gamma_{1}} \chi_{B \varepsilon(p)} \\
-\sigma^{(2-2s)\gamma_{2}-2}(-\psi - W \varepsilon x_1 + \overline{W \varepsilon x_2} - \sigma^{2s-2}\lambda_{-})^{\gamma_{2}} \chi_{B \varepsilon(q)} & \text{in } \mathbb{R} \times (-l/2, l/2), \\
\psi(x) \to 0 & \text{as } |x_1| \to \infty.
\end{cases} \tag{4.2}
\]
For $s = 1$, the equation comes to be
\[
\begin{cases}
-\Delta \psi = \varepsilon^{-2} \left(\psi + W \varepsilon x_1 - \overline{W \varepsilon x_2} - \frac{\lambda_{+}}{2\pi} \ln \frac{1}{\varepsilon} \right)^{\gamma_{1}} \chi_{B \varepsilon(p)} \\
-\sigma^{-2} \left(-\psi - W \varepsilon x_1 + \overline{W \varepsilon x_2} - \frac{\lambda_{-}}{2\pi} \ln \frac{1}{\sigma} \right)^{\gamma_{2}} \chi_{B \varepsilon(q)} & \text{in } \mathbb{R} \times (-l/2, l/2), \\
\psi(x) \to 0 & \text{as } |x_1| \to \infty.
\end{cases} \tag{4.3}
\]
Here, \( p \) and \( q \) are also given by
\[
p = (-d, -a), \quad q = (d, a).
\]
However, different from our assumption in Section 2 and 3, the parameter \( a \) can be chosen in \((0, l/2)\) arbitrarily in this situation, and \( d \) can take any nonnegative values.

Readers can easily verify that the construction of approximate solutions is almost the same as before, and an \( l \)-symmetric solution to \((4.2) \) or \((4.3) \) has the form
\[
\psi_\varepsilon(x) = \Psi_\varepsilon(x) + \omega_\varepsilon(x),
\]
with \( x \in \mathbb{R} \times (-l/2, l/2) \), and \( \omega_\varepsilon(x) \) an \( l \)-symmetric error term. Using the same notations as in previous sections, we are to consider the equation
\[
\mathbb{L}_\varepsilon \omega_\varepsilon = -E_\varepsilon + R_\varepsilon(\omega_\varepsilon) \quad \text{in} \quad \mathbb{R} \times (-l/2, l/2),
\]
where the definition of \( \mathbb{L}_\varepsilon, E_\varepsilon \) and \( R_\varepsilon \) are given in Section 2 and 3. Compared with the case \( a = 0 \) or \( l/4 \), the essential difference for general \( a \in (0, l/2) \) on linear theory is: The kernel of \( \mathbb{L}_\varepsilon \) is 2-dimensional due to the loss of symmetry. Besides \( Z_\varepsilon \) introduced before, there exists another basis of the kernel. When \( 0 < s < 1 \), it is
\[
Z_\varepsilon = \varepsilon^{2s-2} \sum_{k \in \mathbb{Z}} \partial_{x_2} U_{1,\varepsilon}(x + k\varepsilon e_2) - \sigma^{2s-2} \sum_{k \in \mathbb{Z}} \partial_{x_2} U_{2,\varepsilon}(x + k\varepsilon e_2);
\]
while for \( s = 1 \), it turns to be
\[
Z_\varepsilon = \sum_{k \in \mathbb{Z}} \partial_{x_2} V_{1,\varepsilon}(x + k\varepsilon e_2) - \sum_{k \in \mathbb{Z}} \partial_{x_2} V_{2,\varepsilon}(x + k\varepsilon e_2).
\]
Recall the nondegeneracy results in Theorem 2.1 and Theorem 3.1. Using the radial symmetry of ground states, \( Z_\varepsilon \) and \( Z_\varepsilon \) satisfy the approximate orthogonal condition
\[
\int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) Z_\varepsilon(x) dx = O(1) \cdot (\sigma(\varepsilon))^{-1} \quad (4.5)
\]
Hence in general, the projected linear problem is
\[
\begin{cases}
\mathbb{L}_\varepsilon \omega_\varepsilon = h(x) + \alpha_{1,\varepsilon} f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) + \alpha_{2,\varepsilon} f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) & \text{in} \quad \mathbb{R} \times (-l/2, l/2)\\
\omega_\varepsilon(x) \to 0 & \text{as} \quad \left| x_1 \right| \to \infty.
\end{cases}
\]
\[\text{(4.6)}\]
Together with orthogonal conditions
\[
\int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) \omega_\varepsilon(x) dx = 0 \quad \text{and} \quad \int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) \omega_\varepsilon(x) dx = 0.
\]
\[\text{(4.7)}\]
Employing the approximate orthogonal condition \((4.5)\), we can derive the linear theory for \((4.6) \) and \((4.7) \), just as we did in previous two sections. Then, using a contraction mapping argument, a family of solutions \( \omega_\varepsilon \) to \((4.6) \) and \((4.7) \) is obtained, which satisfies the estimate
\[
\| \omega_\varepsilon \|_* \leq C \varepsilon^{3-2s}.
\]
We leave out the details of proof for readers.
By the discussion given in Section 2 and 3, we only need to find suitable conditions such that \( \alpha_{1,\varepsilon} = 0 \) and \( \alpha_{2,\varepsilon} = 0 \) hold simultaneously. In view of (4.6), we have

\[
(-\Delta)^s \psi_\varepsilon = f(x, \psi_\varepsilon) + \alpha_{1,\varepsilon} f_u(x, \Psi_\varepsilon) Z_\varepsilon(x) + \alpha_{2,\varepsilon} f_u(x, \Psi_\varepsilon) \overline{Z_\varepsilon(x)}
\]

in \( \mathbb{R} \times (-l/2, l/2) \), where \( (-\Delta)^s \) equals \( -\Delta \) when \( s = 1 \). To apply energy method, we multiply the above equality by \( Z_\varepsilon(x) \), \( \overline{Z_\varepsilon(x)} \) separately. Using the approximate orthogonal condition (4.5) once more, and noticing that \( \alpha_{i,\varepsilon} \leq \sigma(\varepsilon) \cdot \|h\|_{**} \leq \sigma(\varepsilon) \cdot \varepsilon^{3-2s} \), we deduce

\[
\alpha_{1,\varepsilon} \int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon) Z_\varepsilon^2 dx + o_\varepsilon(1) = \int_{\mathbb{R} \times (-l/2, l/2)} \left((-\Delta)^s \psi_\varepsilon - f(x, \psi_\varepsilon)\right) Z_\varepsilon dx.
\]

and

\[
\alpha_{2,\varepsilon} \int_{\mathbb{R} \times (-l/2, l/2)} f_u(x, \Psi_\varepsilon) \overline{Z_\varepsilon}^2 dx + o_\varepsilon(1) = \int_{\mathbb{R} \times (-l/2, l/2)} \left((-\Delta)^s \psi_\varepsilon - f(x, \psi_\varepsilon)\right) \overline{Z_\varepsilon} dx.
\]

Then, by a similar procedure as in the proofs of Lemma 2.5 and 3.5, we can derive that

\[
\int_{\mathbb{R} \times (-l/2, l/2)} \left((-\Delta)^s \psi_\varepsilon - f(x, \psi_\varepsilon)\right) Z_\varepsilon dx = C \left( C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{p_1 - q_1}{|q - p + k e_2|^{4-2s}} + W_\varepsilon \right) + o_\varepsilon(1),
\]

and

\[
\int_{\mathbb{R} \times (-l/2, l/2)} \left((-\Delta)^s \psi_\varepsilon - f(x, \psi_\varepsilon)\right) \overline{Z_\varepsilon} dx = C \left( C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{-p_2 + q_2}{|q - p + k e_2|^{4-2s}} + \overline{W_\varepsilon} \right) + o_\varepsilon(1),
\]

where \( p_i, q_i \) denote the \( i \)-th coordinate of \( p, q \) respectively for \( i = 1, 2 \). Thus conditions \( \alpha_{1,\varepsilon} = 0 \) and \( \alpha_{2,\varepsilon} = 0 \) turn to be

\[
W_\varepsilon = C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{2d}{(4d^2 + (k l + 2a)^2)^{2-s}} + o_\varepsilon(1), \tag{4.8}
\]

and

\[
\overline{W_\varepsilon} = C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{k l + 2a}{(4d^2 + (k l + 2a)^2)^{2-s}} + o_\varepsilon(1). \tag{4.9}
\]

Combining (4.8) and (4.9), we see that the reduction conditions are equivalent to

\[
U_\varepsilon = -C_s \lim_{N \to \infty} \sum_{|k| \leq N} \frac{(p - q + k e_2)^+}{|p - q + k e_2|^{4-2s}} + o_\varepsilon(1).
\]

The weak convergence of \( \psi_\varepsilon \) is obvious, and \( C^1 \) regularity is from a standard theory. So we have actually proved Theorem 1.5.
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