Quantum fluctuations in the Dvali–Gabadadze–Porrati model and the size of the crossover scale

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Abstract. The Dvali–Gabadadze–Porrati model introduces a parameter, the crossover scale $r_c$, setting the scale where higher dimensional effects are important. In order to agree with observations and to explain the current acceleration of the Universe, $r_c$ must be of the order of the present Hubble radius. We discuss a mechanism for generating a large $r_c$, assuming that it is determined by a dynamical field and exploiting the quantum effects of the graviton. For simplicity, we consider a scalar field $\Psi$ with a kinetic term on the brane instead of the full metric perturbations. We compute the Green function and the one-loop expectation value of the stress tensor $\langle T_{\mu\nu} \rangle$ of $\Psi$ on the background defined by a flat bulk and an inflating brane (self-accelerated or not). We also include the flat brane limit. The quantum fluctuations of the bulk field $\Psi$ provide an effective potential for $r_c$. For a flat brane, the one-loop effective potential $V_{\text{eff}}(r_c)$ is of the Coleman–Weinberg form, and admits a minimum for large $r_c$ without fine-tuning. When we take into account the brane curvature, a sizable contribution at the classical level changes this picture. In this case, the potential can develop a minimum (maximum) for the non-self-accelerated (self-accelerated) branch.

Keywords: extra dimensions, cosmology with extra dimensions, quantum field theory on curved space, cosmological applications of theories with extra dimensions

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1. Introduction

One of the most startling observations in cosmology in the last few years is that our universe is currently in a phase of accelerated expansion [1, 2]. One possibility for explaining this phenomenon is modifying the gravitational sector at large distances. An interesting realization of this idea was proposed by Dvali, Gabadadze and Porrati (DGP) [3], and arises within the brane world scenario. In the DGP model, there are two curvature terms, one in the bulk and the other on the brane. The ratio between the respective Newton constants $1/M^3$ and $1/m_P^2$ defines an important quantity, the crossover scale $r_c$. At distances shorter than $r_c$, gravity behaves as in four dimensions whereas for larger distances it behaves as in five dimensions.

This model admits a ‘self-accelerated’ solution, where the brane accelerates at late times even with vanishing brane tension or bulk cosmological constant [4]. In this solution, the acceleration arises as a purely gravitational effect, and the Hubble rate is $1/r_c$. On the other hand, $r_c$ must be of the order of the present Hubble radius in order not to conflict with observations [5]–[8] (see also [9] for a recent review on the phenomenology of the DGP model). Thus, this model provides an appealing explanation of the current acceleration of the Universe. There exists an abundant literature concerning the stability of the self-accelerated solution [10]–[15]. So far, there seems to be agreement that perturbatively, the DGP model has ghosts for the self-accelerating solution. However, this mode is strongly coupled, so it is not clear whether it remains a ghost at the full non-perturbative level. Here, we shall leave this issue aside and, instead, we discuss another essential ingredient.
for the model to account for the current accelerated expansion, namely, whether it is possible to have a large $r_c$ naturally.

One argument for obtaining a large $r_c$ is that the quantum effects from the numerous matter fields on the brane would generate the localized kinetic term for the graviton with a large coefficient $[3, 16]$. To this end, though, one would need an enormous number of fields. Another argument is that the cut-off on the brane and in the bulk could actually be different $[17]$. We shall also mention that in the model of $[18]$, the corresponding crossover scale is naturally large. Here, we shall take the approach that $r_c$ is determined by some (scalar) field, whose dynamics drives $r_c$ to a large value. Some mechanisms of this sort were already introduced in $[16]$. Our aim is to elaborate on this idea. We shall concentrate on the possibility that the quantum effects of the bulk fields (the graviton) provide such a mechanism, given that this requires a minimal extension of the model. The main assumption is that the Newton constant in bulk (or on the brane) is promoted to a Brans–Dicke-type field $\Phi$, which determines the value of the crossover scale $r_c$. The idea is then that the quantum effects from the graviton induce an effective potential for $\Phi$ that drives $r_c$ to large values. As we shall see, the one-loop effective potential induced by the bulk graviton is of the form

$$V_{\text{eff}}^{\text{1-loop}}(r_c) = \frac{\alpha - \beta \log(\Lambda r_c)}{r_c^4}$$

where $\beta$ is a constant of order $10^{-3}$ (see below for details), $\Lambda$ the cut-off of the theory and $\alpha$ is an order-1 constant. This potential has a minimum at $r_c \simeq e^{\alpha/\beta}/\Lambda$, which can be naturally very large. This is a very appealing way to generate a large $r_c$, but is not the end of story. The reason is that in the most interesting case when the brane is accelerating, there is a large contribution at the classical level due to the curvature terms in the bulk and on the brane. In the simplest situation, the minimum of the potential occurs at a value of $r_c$ too small for phenomenological purposes.

As mentioned before, here we discuss the quantum fluctuations in the DGP model. Specifically, the aim is to estimate the vacuum fluctuations of the graviton on the background defined by a 4D inflating brane embedded in a flat 5D space. We shall consider the two possible branches described in $[4]$, and also the case when the brane is flat. To simplify the analysis, instead of the full graviton we shall consider a 5D massless minimally coupled scalar field $\Psi$. This replacement is motivated because in an appropriate gauge, the spin-2 metric fluctuations obey the same equation as a massless field, so the analyses of the Green function do not differ much. In practice, then, we are ignoring the brane bending mode of the true DGP model. The discussion of this sector is quite a bit more complicated because it is strongly coupled $[10, 11]$, aside from the fact that it is a ghost in the self-accelerated branch $[10]–[13]$. On the other hand, quantities like the vacuum expectation value of the stress tensor $\langle T_{\mu\nu} \rangle$ may certainly differ for the true graviton and the scalar considered here, but still we shall restrict to the latter for illustrative purposes.

On the more technical side, we shall mention that the above result for the potential seems badly divergent in the $r_c \to 0$ limit. As we show in the body of the paper, this is only an artefact of the renormalization procedure used. Schemes based on analytic continuation (dimensional regularization, $\zeta$-function) lead to the expression above. Using a more physical scheme, for instance introducing a cut-off, one unveils divergences of the form $\Lambda^k/r_c^{4-k}$ with $k = 1, 2, 3, 4$, which are implicitly ‘renormalized away’ by the schemes.
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based on analytic continuation. Including these terms, the $r_c \to 0$ limit is regular as expected. Still, for large $r_c$, it makes sense to use the usual rules of renormalization in terms of the effective field $\varphi = 1/r_c$. As we shall see, this is already suggested by the fact that the classical (‘boundary’) effective action [10] for the DGP model is analytic in $1/r_c$ (for large $r_c$).

The quantum effects from bulk fields with brane-localized kinetic terms have been considered previously in a slightly different context. The authors of [19] computed the Casimir energy present in a system of two flat branes and showed that when the weights of the localized kinetic terms are different, the Casimir force could stabilize the distance between the branes $L$. The use of the Casimir effect to stabilize the size of the extra dimension was first discussed in [20,21] and in the brane world set-up in [22]–[24]. The work of [19] is relevant here because in the one-brane limit (when one of the branes is sent to infinity), their result must be compatible with the effective potential found here in the limit when the brane is flat. The $L \to \infty$ limit of the result quoted in [19] vanishes, apparently suggesting that the potential $V_{\text{eff}}(r_c)$ is trivial, which is of course not the case. The reason for this apparent discrepancy is that reference [19] deals with the $L$-dependent part of the potential (the Casimir energy).

This paper is organized as follows. In section 2 we introduce the scalar version of the DGP model. In section 3 we work out the fluctuations of the model in the background defined by the de Sitter brane (in both branches). We compute the Green function in section 3.1 and $\langle T_{\mu\nu} \rangle$ in section 3.2. In section 3.3 we discuss the limit when the brane is flat, and section 3.4 deals with the stress tensor on the brane. The reader not interested in the technical details might skip these sections, and go to section 4 where the effective potential for $r_c$ is derived and section 4.1 where the mechanism for obtaining a large $r_c$ is discussed.

2. The DGP model

The action for the DGP model is

$$S = \int d^5x \sqrt{g} M^3 R(5) + \int d^4x \sqrt{h} \left( m_P^2 R(4) - \tau \right),$$

(2.1)

where $M$ and $m_P$ are the five- and four-dimensional Planck masses, $R(5)$ is the bulk Ricci scalar, $g$ and $h$ denote the determinants of the induced metrics in the bulk and on the brane, $R(4)$ is the Ricci scalar of the brane geometry and $\tau$ is the brane tension. This model admits two solutions with a line element of the form [4]

$$ds^2 = dy^2 + a^2(y) ds^2_{(4)}$$

(2.2)

where $ds^2_{(4)}$ is the metric on a de Sitter space of unit curvature radius and with

$$a(y) = H^{-1} \pm |y|$$

(2.3)

where $H$ is the Hubble constant on the brane (located at $y = 0$). Throughout this paper, the upper sign corresponds to the self-accelerated branch (or simply the ‘+’ branch) [4]. The − or ‘normal’ branch can be visualized as the interior of the space bounded by a hyperboloid (the brane), whereas the + branch corresponds to the exterior. For each
branch, the Israel junction conditions require that
\[ \pm m_P^2 \frac{H}{r_c} = m_P^2 \frac{H^2}{2} - \frac{\tau}{6}, \tag{2.4} \]
where we introduced the crossover scale
\[ r_c \equiv \frac{m_P^2}{2M^3}. \]
Equation (2.4) fixes \( H \) in terms of \( r_c \) and \( \tau \). In the literature, the term ‘self-accelerated’ is usually reserved for the solution with \( \tau = 0 \) (for which \( H r_c = 1 \)). Here, we shall consider all values of \( H r_c \), which can be obtained by appropriately choosing \( \tau \) and we call any solution in the + branch ‘self-accelerated’.

2.1. A dynamical crossover scale

We now extend the original induced gravity term to the most general Brans–Dicke-type modification of the DGP model,
\[ S = \int d^5 x \sqrt{g} \left( \Phi_5 R(5) + \text{kin} \right) + \int d^4 x \sqrt{h} \left( \Phi_4 R(4) + \text{kin} \right), \tag{2.5} \]
where \( \Phi_{5,4} \) are unrelated bulk and brane fields and we have omitted their kinetic terms, as we are interested in the configurations with constant fields only. The important point is that each BD field controls the value of the Newton constant on the brane or in the bulk so that \( \langle \Phi_5 \rangle = M^3 \) and \( \langle \Phi_4 \rangle = m_P^2 \). In terms of them the effective crossover scale is given by
\[ r_c = \frac{\Phi_4}{2\Phi_5}. \tag{2.6} \]

The equations of motion in the model (2.5) become
\[ V_{\text{eff}} \equiv -12\Phi_4 H^2 \pm 12\Phi_5 H + \tau + \delta V(\Phi_5, \Phi_4) = 0, \tag{2.7} \]
together with \( V'_{\text{eff}} = 0 \), where a prime denotes here differentiation with respect to \( \Phi_5 \) or \( \Phi_4 \). In equation (2.7), \( \delta V \) represents the potential responsible for fixing them (which might include bulk and brane contributions). Note that the condition (2.7) is nothing but the extension of the Friedmann equation (2.4). The first two terms in (2.7) are classical contributions due the brane and the bulk curvature terms in the action. The former induces a negative runaway potential for \( \Phi_4 \), while the latter is positive for the self-accelerated branch. As we shall see, for large \( r_c \), the one-loop effective potential is analytic in \( 1/r_c \propto 1/\Phi_4 \). Thus, it seems problematic that the quantum effects can induce a minimum at a large \( \Phi_4 \). From now on we shall consider that \( \Phi_4 = m_P^2 \) is fixed, and instead we try to ‘stabilize’ \( \Phi_5 \) using the quantum effect from the graviton. Note also that this choice automatically leads us to the four-dimensional Einstein frame, which is the physical one. See [30] for a recent discussion of the cosmology of a DGP scenario with a dynamical \( \Phi_4 \) (and constant \( \Phi_5 \)).
2.2. Scalar toy model

We are interested in the fluctuations of the graviton propagating in the background (2.2). One can show that the spin-2 polarizations of the graviton perturbations obey the same equation of motion as a massless minimally coupled scalar field (see e.g. [12]) with a kinetic term on the brane. Hence, we shall simplify the problem and rather consider a scalar field (even under $Z_2$ symmetry) propagating on the background (2.2). Its action is

$$S = -\frac{1}{2} \int d^5 x \sqrt{g} (\partial_\mu \Psi)^2 - \int d^4 x \sqrt{h} r_c (\partial_i \Psi)^2,$$  \hspace{1cm} (2.8)

where $g$ and $h$ denote the determinants of the background metrics on the bulk and on the brane (2.2).

2.3. Classical boundary effective action

Let us now discuss how the crossover scale appears in the effective four-dimensional theory. Consider for simplicity the solution with a flat brane, and we have in mind a situation with large $r_c$ (compared to any other scale in the problem). A simple way to obtain the effective theory of the model (2.8) is to compute the effective boundary action. Following [10], once some data on the brane are specified for the bulk field $\Psi|_0 = \psi$, the equation of motion in the bulk is solved by $\Psi = e^{-\mu \Delta} \psi$, where $\Delta = (-\Box_{(4)})^{1/2}$ and $\Box_{(4)}$ is the 4D D’Alembertian operator. Inserting this solution back into the bulk action, we can integrate it explicitly and obtain a surface term. Thus, the boundary effective action is

$$S_{\text{eff}} = \int d^4 x \sqrt{h} \left(r_c \Box_{(2)} - \Delta\right) \psi.$$  \hspace{1cm} (2.9)

For large values of $r_c$, the four-dimensional term dominates. Hence, it is preferable to express this as the action for a 4D canonical field. Restricting our attention to constant configurations of $r_c$, we can always rescale the field as $\hat{\psi} = \sqrt{2r_c} \psi$ in the canonical form, so the effective action is

$$S_{\text{eff}} = \frac{1}{2} \int d^4 x \sqrt{h} \left(\hat{\psi} \Box_{(4)} \hat{\psi} - \varphi \hat{\psi} \Delta \hat{\psi}\right),$$  \hspace{1cm} (2.9)

where we introduced

$$\varphi \equiv \frac{1}{r_c}.$$  

This effective action describes the resonant or metastable mode of the DGP model. This mode propagates as in four dimensions for a typical distance $r_c$ after which it behaves as in five dimensions. Put another way, the width of decay into five-dimensional modes is of order $1/r_c$. Hence, at distances shorter than $r_c$ this 4D effective representation is accurate.

The point that we want to emphasize here is that in this regime, the effective action (2.9) is analytic with respect to $\varphi$. This is a rather special feature of this model. As we shall see, all the explicit expressions computed below are regular in the $\varphi \to 0$ limit. Finally, let us note that in terms of the model of section 2.1 we have $\varphi = 2\Phi_5/m_p^2$, so we can interpret $\varphi$ as the bulk Brans–Dicke field (restricted on the brane).

Before switching to the explicit computation of the Green function, we shall briefly discuss the generalization of the boundary effective action to the curved brane case.
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It is straightforward to show from equations (3.1) and (3.3) that when the brane is inflating, the solution for $\Psi$ in the bulk with data on the brane specified by $\psi$ is given by

$$\Psi = a^{-3/2}e^{-|z|}$, where $z$ is the conformal coordinate ($dz = dy/a$), $\hat{\rho} \equiv (-\Box(4) + (3/2)^2)^{1/2}$ and now $\Box(4)$ is the covariant D’Alembertian in de Sitter space. Using the same logic as above, the boundary effective action in this case is

$$S_{\text{eff}} = \int d^4x \sqrt{h} \psi \left( r_c \Box(4) + \frac{3}{2} H - \sqrt{\left( \frac{3H}{2} \right)^2 - \Box(4)} \right) \psi. \quad (2.10)$$

From the analytic structure of this action as a function of $\Box(4)$, we can already identify the main features of the effective theory. For both branches, there is a branch cut starting at $\Box(4) = (3H/2)^2$, which corresponds to the continuum of Kaluza–Klein modes. Aside from this, we can also see the presence of one lighter mode. Performing the same rescaling of $\psi$ as above, we can identify the mass of this mode as the non-derivative term in (2.10). For the non-self-accelerated branch, this mode is massless. As we will see shortly, this agrees perfectly with the Kaluza–Klein decomposition. For the self-accelerated branch, we read from (2.10) a mass squared given by $3H/r_c$ which agrees with the analysis of the KK spectrum (3.13) to leading order (in the large $Hr_c$ limit).

3. Quantum fluctuations

In this section, we shall take the bulk spacetime to be $(n + 2)$-dimensional. The Klein–Gordon equation on the space (2.2),

$$[\Box(n+2) + Hr_c \delta(z) \Box(n+1)] \Psi = 0, \quad (3.1)$$

is separable into modes of the form

$$\Psi(z,x^\mu) = \sum_p U_p(z) \psi_p(x^\mu). \quad (3.2)$$

In the previous equation, $\Box(n+2)$ and $\Box(n+1)$ are the bulk and brane D’Alembertians. For clarity, we omit the set of indices labelling the angular momentum of the four-dimensional modes $\psi_p(x^\mu)$. The $(n + 1)$-dimensional modes obey $[\Box(n+1) - m^2] \psi_p = 0$ and represent four-dimensional fields with mass

$$m^2 \equiv [(n/2)^2 + p^2]H^2.$$

The mass spectrum is determined by the radial equation, which can be cast in the Schrödinger form

$$[-\partial_z^2 + V(z)] \tilde{U}_p = p^2 \tilde{U}_p \quad (3.3)$$

with $\tilde{U}_p = a^{n/2}U_p$, and

$$V(z) = \left( \frac{a^{n/2}}{a^{n/2}} \right)^{''} - \frac{n^2}{4} - 2r_c m^2 \delta(z) = \left( \pm nH - 2r_c m^2 \frac{H}{H} \right) \delta(z), \quad (3.4)$$

where a prime denotes differentiation with respect to $z$, and we introduced the conformal coordinate $dz = dy/a$. Hence, the modes are combinations of $e^{\pm ip|z|}$, and the spectrum is
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composed of a continuum of Kaluza–Klein modes with masses \( m_{kk} > nH / 2 \), plus possibly one discrete (or ‘bound’) state with mass \( m_{bs} < nH / 2 \).

In terms of the mode decomposition, the boundary condition on the brane reduces to

\[
[\partial_z - \nu_\pm(p)] \tilde{U}_p |_{0+} = 0
\]

where \( |_{0+} \) stands for \( \lim_{z \to 0} \) with \( z > 0 \) and

\[
\nu_\pm(p) \equiv \pm \frac{n}{2} - \left( p^2 + \left( \frac{n}{2} \right)^2 \right) H r^c.
\]

Note that (3.5) is similar to the Dirichlet boundary condition for high KK mass modes, whereas it recovers the usual Neumann-type condition for light modes. This is indeed the key ingredient for obtaining a four-dimensional behaviour at short distances in the DGP model, since it implies that high energy modes are suppressed on the brane.

We shall normalize the modes according to

\[
(\Psi_1, \Psi_2) \equiv \int d\Sigma \xi^\mu i (\Psi_1^\dagger \partial_\mu \Psi_2 - \Psi_2 \partial_\mu \Psi_1^\dagger) + 2 r_c \int d\sigma \xi^\mu i (\Psi_1^\dagger \partial_\mu \Psi_2 - \Psi_2 \partial_\mu \Psi_1^\dagger) |_0
\]

where \( \xi^\mu \) is a future-directed timelike unit vector and \( d\Sigma \) (\( d\sigma \)) is the volume element of the surface \( \Sigma \) (\( \sigma \)) normal to \( \xi^\mu \) in the bulk (on the brane). For any pair of modes satisfying the equation of motion (3.1), the norm (3.7) is independent of the choice of \( \Sigma \). We shall take \( \xi^\mu \) to be along the time coordinate of the observers on the brane. It is convenient to introduce the Rindler coordinates \( \{a, t\} \), related to the rectangular coordinates in the bulk \( \{T, X\} \) through

\[
R = a \cosh t
\]

\[
T = a \sinh t
\]

where \( R = |X| \). In terms of these coordinates, \( \xi^\mu = \delta^\sigma / a \) and the metric on the de Sitter slices \( y = \text{const} \) is \( d\sigma^2_{(n+1)} = a^2 [-dt^2 + \cosh^2 t d\Omega^2_n] \) where \( d\Omega^2_n \) is the metric on an \( n \)-dimensional sphere.

For a mode of the form \( \Psi_{p,l} = U_p \psi_{p,l} \) where \( l \) is a collective index of the dS part, we have

\[
(\Psi_{p,l}, \Psi_{p,l}) = \int d\sigma d\Omega |(1 + 2r_c H \delta(z)) |U_p|^2 (\psi_{p,l}, \psi_{p,l})_{(n+1)},
\]

where \( (\psi_1, \psi_2)_{(n+1)} \) is the usual four-dimensional Klein–Gordon product, \( \int d\sigma \xi^\mu i (\psi_1^\dagger \partial_\mu \psi_2 - \psi_2 \partial_\mu \psi_1^\dagger) \). The continuum of KK modes is normalized as \( (\Psi^{kk}_{p,l}, \Psi^{kk}_{p,l}) = \delta(p - p') \delta_{l,l'} \), which leads to

\[
\int_{-\infty}^{\infty} dz (1 + 2r_c H \delta(z)) |U_{kk}|^2 = \delta(p - p'),
\]

while for the bound state we shall impose

\[
\int_{-\infty}^{\infty} dz (1 + 2r_c H \delta(z)) |U_{bs}|^2 = 1.
\]

A bound (or ‘localized’) state is a mode with pure imaginary \( p = i\lambda \) and \( \lambda > 0 \), so that its wavefunction \( \tilde{U}_{bs} \sim e^{-\lambda |z|} \) is normalizable. The boundary condition implies that
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\( \lambda \) must satisfy

\[- \lambda - \nu_{\pm} (i \lambda) = 0 \tag{3.10} \]

where \( \nu_{\pm} \) is given in (3.6). The two roots are, for each branch,

\[ \lambda_1 = \frac{\pm n}{2}, \]
\[ \lambda_2 = \frac{n}{2} - \frac{1}{H r_c}. \tag{3.11} \]

It is convenient to introduce the notation \( \lambda_\uparrow \) and \( \lambda_\downarrow \) as the larger and smaller of \( \lambda_{1,2} \).

Only \( \lambda_\uparrow \) is positive, so this root corresponds to the bound state in both branches. Its wavefunction is \( \tilde{U}_{bs} = N_{bs} e^{-\lambda_\uparrow z} \) with (see (3.7))

\[ N_{bs}^2 = \frac{\lambda_\uparrow}{1 + 2 H r_c \lambda_\uparrow} = \frac{\lambda_\uparrow}{n H r_c + 1}. \tag{3.12} \]

In the \(-\) branch, the bound state is massless (\( \lambda_\uparrow = n/2 \)). In the self-accelerated branch, it is normalizable only for \( H r_c > 2/n \), and its mass is

\[ m_{bs(+)}^2 = \frac{n H r_c - 1}{r_c^2}. \tag{3.13} \]

The behaviour of this mode as a function of \( r_c \) is as follows. For \( H r_c \gg 2/n \), the bound state is light. For smaller \( r_c \), it acquires a mass until at \( r_c = 2/n H \) it is degenerate with the KK. At this point, this becomes unnormalizable and it should be rather described as a resonance, or quasi-normal mode. Its mass acquires an imaginary part \( \rho = \rho_0 - i \Gamma \) that is interpreted as the width of decay into KK modes [31,32]. In this model, this actually turns out to be a purely decaying quasi-normal mode (\( \rho_0 = 0 \)), with decay width \( \Gamma = 1/r_c - 3 H/2 \).

The other mode, at \( \lambda_\downarrow \), is not normalizable, so strictly speaking it is not included in the spectrum. However, one can think that it corresponds to a resonance, or equivalently that the whole KK continuum in some sense behaves like this mode. For the \(-\) branch, its decay width is \( 1/r_c + 3 H/2 \), and for the \(+\) branch it is \( 3 H/2 \). Note that in the latter case, we can have two different resonances for \( H r_c < 2/n \). The spectra of normalizable and resonant modes are sketched in figure 1.

3.1. The Green function

Before starting the computation of the Green function on the backgrounds (2.2), we shall mention that for a massless minimally coupled field, there is an ambiguity in the definition of the Green function in an overall additive constant. The shift symmetry \( \Psi \to \Psi + \text{const} \) allows us to ‘gauge away’ any constant contribution. This turns out to be important for the non-self-accelerated branch because in this case there is a massless 4D mode, and as is well known massless minimally coupled scalar fields enjoy an (IR) divergent Green function in spaces with closed spacelike sections. In any case, and precisely thanks to the shift symmetry, this divergence does not appear in the stress tensor \( \langle T_{\mu \nu} \rangle \). This issue has been discussed at length for four dimensions [33,34] and in the brane world context [25].

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1 To be precise, here we are assuming that \( H r_c \) is large enough.

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**Figure 1.** We represent the spectrum of modes in the $p$ plane, related to the mass by $m^2 = (n^2/4 + p^2)H^2$. The continuum of KK modes have $p$ real and are represented by the thick black line, both for the self-accelerated branch, (a), and for the non-self-accelerated branch, (b). The crosses denote the positions of the other two discrete modes in the spectrum for some representative value of $Hr_c$. In the upper half-plane, these modes are normalizable and correspond to the bound state. In the lower half-plane, they are not normalizable, and represent a resonance. The shaded line indicates the possible values that the pole at $p = i\lambda_2$ can take as $Hr_c$ is varied from 0 to $\infty$. Note that in the self-accelerated branch and for $Hr_c < 2/n$ there is no bound state. Rather, there are two resonances.

These explicit computations further show that both $\langle T_{\mu\nu} \rangle$ and the Green function are everywhere regular.

We can compute the (Hadamard) Green function as a mode sum,

\[
\begin{align*}
G^{(1)} &= G^{kk} + G^{bs}, \\
G^{bs} &= \theta(\lambda_>) \mathcal{U}_{bs}^{bs}(z) \mathcal{U}_{bs}^{bs}(z') \mathcal{G}^{(1)}_{\text{Mink}}(\text{dS}) \\
G^{kk} &= \int_0^\infty dp \mathcal{U}_p^{kk}(z) \mathcal{U}_p^{kk}(z') \mathcal{G}^{(1)}_{\text{Mink}}(\text{dS}),
\end{align*}
\]

where $G^{(1)}_{\text{Mink}}$ denotes the massive Bunch–Davies Green function in $(n+1)$-dimensional de Sitter space. The $\theta(\lambda_>)$ factor in $G^{bs}$ ensures that it is only included when it is normalizable.

One easily finds [25, 26] (for the ‘right’ half of the space, $z > 0$) that the normalized radial KK modes are

\[
\mathcal{U}_p^{kk}(z) = \sqrt{\frac{1}{\pi a^n(z)(1 + (\nu/p)^2)}} \left( \cos (pz) + \frac{\nu}{p} \sin (pz) \right).
\]

It is convenient to introduce the *subtracted* Green function

\[
\overline{G}^{(1)} \equiv G^{(1)} - G^{(1)}_{\text{Mink}},
\]

where $G^{(1)}_{\text{Mink}}$ is the Green function in Minkowski space, as it is free from ultraviolet
divergences in the bulk. In terms of the mode expansion,
\[
\overline{G}^{(1)} = G^{\text{bs}} + \int_{-\infty}^{\infty} dp \left[ \mathcal{U}_p^{kk}(z) \mathcal{U}_p^{kk}(z') \right]^{\text{sub}} G_p^{(dS)} \tag{3.17}
\]
with
\[
\left[ \mathcal{U}_p^{kk}(z) \mathcal{U}_p^{kk}(z') \right]^{\text{sub}} = \frac{1}{4\pi (a(z)a(z'))^{n/2}} \frac{p - i\nu}{p + i\nu} e^{i(z+z')p}. \tag{3.18}
\]
One can now perform the \( p \) integral in (3.17) closing the contour in the complex \( p \) plane by the upper half-plane and summing the residues of the enclosed poles. Using
\[
\frac{p - i\nu}{p + i\nu} = -1 - 2i \frac{\lambda_\alpha + \lambda_\beta}{\lambda_\alpha - \lambda_\beta} \left( \frac{\lambda_\alpha}{p - i\lambda_\alpha} - \frac{\lambda_\beta}{p - i\lambda_\beta} \right) \tag{3.19}
\]
we see that the only pole from \( \left[ \mathcal{U}_p^{kk}(z) \mathcal{U}_p^{kk}(z') \right]^{\text{sub}} \) contributing to (3.17) is at \( p = i\lambda_\alpha \), the bound state location. Given that
\[
\frac{\lambda_\alpha + \lambda_\beta}{\lambda_\alpha - \lambda_\beta} = -\frac{1}{nH\zeta_\alpha \mp 1},
\]
we find that this pole in the contribution from the KK modes equals the contribution from the bound state \( G^{\text{bs}} \) (see equation (3.12)), with the opposite sign\(^2\). The remaining poles in (3.17) arise from \( G_p^{(1)} \). Performing the sum over them (see the appendix for details) we find
\[
\overline{G}^{(1)} = -G_0 - \frac{2\lambda_\alpha + \lambda_\beta}{\lambda_\alpha - \lambda_\beta} (\lambda_\alpha G_{\lambda_\alpha} - \lambda_\beta G_{\lambda_\beta}), \tag{3.20}
\]
where
\[
G_0 \equiv \frac{2H^n}{nS_{(n+1)}} \left( \frac{1}{1 + (aa'H^2)^2 - 2aa'H^2 \cos \zeta} \right)^{n/2}, \tag{3.21}
\]
\[
G_{\lambda} \equiv \frac{2H^n}{nS_{(n+1)}} \left( \frac{aa'H^2}{nS_{(n+1)}^{-(n(1±1)/2)}} \right)^{n/2 - \lambda} \times F_1 \left( \frac{n}{2} - \lambda, \frac{n}{2}; \frac{n}{2} - \lambda + 1; (aa'H^2)^{±1}e^{i\zeta}, (aa'H^2)^{±1}e^{-i\zeta} \right). \tag{3.22}
\]
Here, \( \zeta \) is the invariant distance in dS space and \( S_{(n+1)} = 2\pi^{1+n/2}/\Gamma(1+n/2) \) is the volume of a unit \((n+1)\)-dimensional sphere, and
\[
F_1 (a, b_1, b_2; z_1, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a)_{k+l}(b_1)_l(b_2)_k z_1^l z_2^k}{(c)_{k+l} l! k!}, \tag{3.23}
\]
is the Appell hypergeometric function of two variables (here, \( (a)_k \equiv \Gamma(a + k)/\Gamma(a) \)).

In the \(-\) branch \( \lambda_\alpha = n/2 \), so it is clear from (3.22) and (3.23) that the \( \lambda_\alpha \) term in (3.20) induces a divergence, as previously anticipated. This divergence arises from the constant mode of the bound state, which is massless in this branch. It is well known that massless scalar fields in de Sitter space produce this type of infrared divergence.

\(^2\) Note that any normalization of the modes other than (3.7) would not lead to this cancellation, with the corresponding wrong short distance behaviour of the Green function.
See [33, 34] for a discussion for in dS space and [25] for a similar brane model. Here, we will simply ignore all constant contributions to the Green function, given that they do not contribute to any observable associated with \( \Psi \). This is a consequence of the shift symmetry of the model, \( \Psi \rightarrow \Psi + \text{const} \), which in other words allows us to gauge away the constant contributions. Hence, the leading order behaviour of (3.20) near the light cone \( (aa' \rightarrow 0) \) is

\[
\overline{G}^{(1)}(y, y', x, x') \sim H^{n+2}aa' \cos \zeta.
\]

For a source point \( x' \) on the brane with \( t = 0 \), and \( x \) close to the horizon \( (U \equiv R - T = 0) \), then \( \zeta \) is large and pure imaginary, so we have \( \overline{G}^{(1)} \sim H^{n+1}T \), which is perfectly regular on the horizon.

For the + branch, we find that at infinity \( (aa' \rightarrow \infty) \) the leading order terms in (3.20) cancel each other, and we have

\[
\overline{G}^{(1)}(y, y', x, x) \sim H^n (aa'H^2)^{-(n+1)} \cos \zeta.
\]

The disturbance generated by a source on the brane at \( t' = 0 \) at spatial infinity decays slightly faster than for a flat brane, \( \overline{G}^{(1)} \sim 1/(Hy^{n+1}) \). See the appendix for the form of \( \overline{G}^{(1)} \) in the bulk in the coincidence limit.

### 3.2. Vacuum expectation value of the stress tensor

The stress tensor for a classical field configuration can also be split into bulk and surface parts as

\[
T^\mu{}^\nu_{\text{bulk}} = \partial_\mu \Psi \partial_\nu \Psi - \frac{1}{2} (\partial_\nu \Psi)^2 g_{\mu\nu},
\]

\[
T^i{}^j_{\text{brane}} = 2r_c \left[ \partial_i \Psi \partial_j \Psi - \frac{1}{2} (\partial_k \Psi)^2 h_{ij} \right] \delta(y),
\]

where \( h_{ij} \) is the induced metric on the brane, and the equation of motion has been used. In point splitting regularization, the v.e.v. \( \langle T_{\mu\nu} \rangle \) is computed as

\[
\langle T_{\mu\nu} \rangle_{\text{bulk}} = \frac{1}{2} \lim_{y',x' \rightarrow y,x} \left\{ \partial_\mu \partial_\nu \Psi - \frac{1}{2} g_{\mu\nu} \partial^\lambda \partial_\lambda \Psi \right\} \overline{G}^{(1)}(y, y', x, x')
\]

and

\[
\langle T^i{}^j \rangle_{\text{brane}} = r_c \delta(y) \lim_{x' \rightarrow x} \left\{ \partial_i \partial_j \Psi - \frac{1}{2} h_{ij} \partial^k \partial_k \Psi \right\} \overline{G}^{(1)}(0, x, x'),
\]

where \( \partial_\mu \equiv \partial/\partial x^\mu \).

For the – branch, we find

\[
\langle T^y \rangle_{\text{bulk}} = \frac{n}{2S_{(n+1)}} H^{n+2} \left\{ \frac{1}{(1 - (aH)^2)^{n+1}} - \frac{1}{1 + nHr_c(aH)^2} \left( \frac{1 - (aH)^2}{1 - (aH)^2} \right)^n - \frac{nHr_c + 2}{Hr_c(1 + nHr_c)} (aH)^{2(n+1)/Hr_c} B_{(aH)^2} \left( n + 1 + \frac{1}{Hr_c}, -n \right) \right\}.
\]

From the local conservation of the stress tensor, the remaining components \( \langle T^i_j \rangle \equiv -\rho h^i_j \) are given by

\[
\rho_{\text{bulk}} = - \left( 1 + \frac{1}{n+1} a \partial_a \right) \langle T^y \rangle_{\text{bulk}}.
\]

\(^3\) We omit the anomaly term since it vanishes in the bulk for odd dimension, and on the brane it can be absorbed in a renormalization of the brane tension.
Note that, on the light cone \((a = 0)\), \(\rho \simeq - \langle T^y_y \rangle\) and approaches a constant value of order \(H^{n+2}\). Thus, as expected, \(\langle T^\mu_\nu \rangle\) is completely regular on the horizon.

For the \(+\) branch, we find

\[
\langle T^y_y \rangle^{\text{bulk}} = \frac{n}{2S_{(n+1)}} H^{n+2} \left\{ \frac{1}{((aH)^2 - 1)^{n+1}} - \frac{1}{1 - nHr_c(aH)^2} + \frac{1}{(1 - 1/(aH)^2)^n} \right\} \nonumber
\]

\[
- \frac{nHr_c - 2}{nHr_c(1 - nHr_c)} (aH)^{-2(n+1-1/Hr_c)} B_1/(aH)^2 \left( \frac{1}{Hr_c}, -n \right) .
\]

Note that despite appearances, this expression is finite for \(Hr_c = 1/n\). At infinity \((a \simeq y \to \infty)\), the leading order terms of the three summands in (3.27) cancel each other, so \(\langle T^y_y \rangle^{\text{bulk}}\) and \(\rho^{\text{bulk}}\) decay like \(H^{n+2}(aH)^{-2(n+2)} \simeq 1/(Hy^2)^{(n+2)}\).

Close to the brane \((y \to 0)\),

\[
\langle T^y_y \rangle^{\text{bulk}} \simeq \pm \frac{n}{2S_{(n+1)}} \frac{H}{y^{n+1}} (1 + {\cal O}(Hy)) ,
\]

so \(\langle T^y_y \rangle^{\text{bulk}}\) vanishes in the \(H \to 0\) limit, as required by the local conservation of the stress tensor. As for the components parallel to the brane \((n = 3)\), we find

\[
\langle T^i_j \rangle^{\text{bulk}} = \frac{9}{256\pi^2} \left( \frac{1}{y^5} + \frac{2 + 3Hr_c + 9(Hr_c)^2}{2(1 + 3Hr_c)} \frac{1}{rcy^4} + \cdots \right) h^i_j .
\]

The \(r_c \to 0\) limit of this expression is not well defined because it holds only for \(y\) smaller than \(H\) and \(r_c\).

### 3.3. Flat brane limit

To recover the flat brane case, we take \(\zeta = H\Delta x\), where \(\Delta x\) is the distance between points \(x\) and \(x'\), in \((n + 1)\)-dimensional flat space. If we take \(H \to 0\) and keep \(Hr_c\) (and hence \(\lambda_2\)) fixed, this means that we are letting \(r_c \to \infty\) and will obtain the limit with Dirichlet boundary conditions. Recalling that in terms of the proper bulk coordinate \(y\) we have \(aH = e^{y^2} = 1 \pm Hy\) and taking the limit \(H \to 0\) in (3.20), (3.21) and (3.22), one easily obtains

\[
\overline{G}^{(1)}(y, y'; x, x') = -\frac{2}{nS_{(n+1)}} \frac{1}{(Y^2 + \Delta x^2)^{n/2}}
\]

where \(Y \equiv y + y'\), which indeed corresponds to the renormalized Hadamard function in flat space with Dirichlet boundary conditions.

To recover the \(H \to 0\) limit keeping \(r_c\) finite, we have to take into account that in this limit\(^4\), \(\lambda_2 \to -\infty\). Using the representation

\[
\frac{1}{a} F_1(a, b, b, a + 1; z_1, z_2) = \int_{0}^{\infty} d\tau \frac{e^{-\tau}}{((1 - z_1 e^{-\tau/a})(1 - z_2 e^{-\tau/a}))^b},
\]

we obtain

\[
\lim_{\lambda \to -\infty} -\lambda G_{\lambda} = \frac{2}{nS_{(n+1)}} \frac{1}{r_c^n} \int_{0}^{\infty} d\tau \frac{e^{-\tau}}{(\tau^2 + (Y^2 + \Delta x^2)/r_c^2 - 2Y/r_c)^{n/2}} .
\]

\(^4\) The contribution from \(\lambda_1\) in (3.20) vanishes in the limit \(H \to 0\).
Note that in the limit as required by the local conservation of the stress tensor, and the result for an ordinary scalar with Neumann boundary conditions. Conversely, in the limit, the field $\Psi$ is subject to Neumann or Dirichlet boundary conditions. As is well known, the stress tensor on the brane diverges even after subtracting the boundary conditions.

On the other hand, for $Y \neq 0$ and $\Delta x = 0$, we have

$$\lim_{H \to 0} G_{i j}^{(1)} (0, 0; x, x') = \frac{1}{4\pi^2} \left( -\frac{1}{\Delta x^2} Y_i (\Delta x/r_c) + H_{-1} (\Delta x/r_c) \right),$$

where $Y_i(z)$ is the Bessel function of the second kind and $H_{-1}(z)$ is the Struve function. In those limits, the field $\Psi$ is subject to Neumann or Dirichlet boundary conditions.

As is well known, the stress tensor on the brane diverges even after subtracting the boundary conditions. The maximal symmetry of the $y = \text{const}$ surfaces ensures that

$$\langle \partial^i \Psi \partial_j \Psi \rangle = \frac{1}{n+1} \langle \partial^k \Psi \partial_k \Psi \rangle h_{ij}.$$  

Hence, we can compute the v.e.v. of the surface stress tensor induced by the bulk field $\Psi$ as

$$\langle T_{ij} \rangle_{\text{brane}} = -(n-1)r_c \langle \partial_i \Psi \partial_j \Psi \rangle \delta(y) = (n-1)r_c h_{ij} \delta(y) \lim_{\zeta \to 0, y \to 0} \frac{\partial \zeta G_{1}^{(1)} (y, y'; \zeta)}{2\zeta}.$$  

As is well known, the stress tensor on the brane diverges even after subtracting the Minkowski part. So, we have to introduce some alternative or additional method to
regulate it. One option is to use dimensional regularization. Expressions like $1/y^{n+2}$ are regularized by taking the $y \to 0$ limit for negative enough $n$ and then continuing back to $n = 3$. Using this prescription, we obtain

$$\langle T^i_j \rangle^\text{brane} = \frac{2(n-1) \Gamma (-1-n)}{S_{(n+1)}} \delta(y) h^i_j = \frac{1}{16\pi^2} \left( \frac{1}{r_c^4} \frac{1}{n-3} - \frac{\log (r_c \mu)}{r_c^4} \right) \delta(y) h^i_j \quad (3.34)$$

where we introduced an arbitrary renormalization scale $\mu$. The divergence requires a renormalization of $\varphi^4$, where $\varphi = 1/r_c$ is the relevant field for the effective field theory, as argued in section 2.1.

To obtain a more physical picture, we shall show the form of the brane stress tensor regularized using a cut-off, which we can assume is related to the brane thickness $d$. Again, dimensional regularization leads to a result valid for large $r_c$.

$$\langle T^i_j \rangle^\text{brane} = -\frac{3}{2\pi^2} \frac{e^{d/r_c}}{r_c^4} \Gamma (-4, d/r_c) \delta(y) h^i_j \quad (3.35)$$

where the dots denote positive powers of $d$ and the expansion (3.36) holds for $d \ll r_c$. In the opposite limit, equation (3.35) is perfectly regular, and in fact it vanishes for $r_c \to 0$. Several comments are now in order. First of all, note that the finite parts of equations (3.34) and (3.36) agree. Since equation (3.36) only holds for $r_c \gg d$, we observe that the dimensional regularization result correctly captures the large $r_c$ form of $\langle T^i_j \rangle^\text{brane}$, and should not be trusted for small $r_c$. We shall interpret this as indicating that the divergent terms in the expansion (3.36) should be absorbed in a (finite set of) counter-terms involving negative powers of $r_c$. This follows along the same lines as section 2.3, where we saw that the field associated with $r_c$ appearing in the 4D effective theory (in the large $r_c$ regime) is nothing but $\varphi = 1/r_c$.

Before discussing equations (3.34)–(3.36) and the application of these results to the computation of the effective potential for $r_c$, we shall display the form of the brane stress tensor when the brane is inflating. Again, dimensional regularization leads to a result valid for large $r_c$. We shall not include here the form of $\langle T^i_j \rangle^\text{brane}$ using a cut-off distance. The expressions are lengthy and do not illustrate essentially anything more than is already contained in (3.36). For the $-\text{branch}$, we obtain

$$\langle T^i_j \rangle^\text{brane} = \frac{(n-1)H^{n+1}}{S_{(n+1)}} \frac{2 + nHr_c}{1 + nHr_c} B \left( n + 2 + \frac{1}{Hr_c}, -n - 1 \right) \delta(y) h^i_j$$

$$= -\frac{(1 + Hr_c)(2 + 3Hr_c)(1 + 4Hr_c)}{32\pi^2 r_c^4} \left\{ \frac{1}{n-3} + \psi \left( 5 + \frac{1}{Hr_c} \right) + \log (H/\mu) - \frac{Hr_c}{(1 + 3Hr_c)(2 + 3Hr_c)} \right\} \delta(y) h^i_j \quad (3.37)$$

up to a rescaling of $\mu$.

\[5\] Note that once we introduce a cut-off distance, in order to compute $\langle T^i_j \rangle$ we should use the full Green function instead of the subtracted one (3.16).
For the + branch, we obtain

\[ (T^i_j)_0 = \left( \frac{(n-1)H^{n+1}2-nHr_c}{S(n+1)} \right) \frac{1}{1-nHr_c} B \left( 2+\frac{1}{Hr_c}, -n-1 \right) \delta(y) h^i_j \]

\[ = \frac{32\pi^2 r_c^4}{(1-Hr_c)(1-2Hr_c)(2-3Hr_c)(1-4Hr_c)} \times \left\{ \frac{1}{n-3} + \psi \left( \frac{1}{Hr_c} - 4 \right) + \log \left( \frac{H}{\mu} \right) + \frac{Hr_c}{(1-3Hr_c)(2-3Hr_c)} + \cdots \right\} \]

\[ \times \delta(y) h^i_j. \]  

(3.38)

There are five kinds of divergent terms in (3.37) and (3.38), proportional to \( H^4, H^3/r_c, H^2/r_c^2, H/r_c^3 \) and \( 1/r_c^4 \). They can be renormalized with counter-terms of the type \( R^2(4), K^3, R(4)\varphi^2, K\varphi^3 \) and \( \varphi^4 \), where \( K \) is the extrinsic curvature and as before \( \varphi = 1/r_c \).

Note as well that we can readily identify two of the special cases when the divergent part of (3.38) vanishes: one corresponds to the truly self-accelerated solution (with \( \tau = 0 \) and \( Hr_c = 1 \)); the other corresponds to the only case when equation (2.4) has only one solution, \( Hr_c = 1/2 \).

In the \( H \to 0 \) limit, the finite part reduces to

\[ \langle T^i_j \rangle_0 = \mp \frac{1}{16\pi^2} \frac{\log (\mu r_c)}{r_c^4} h^i_j \delta(y), \]

which agrees with the flat brane case for the self-accelerated branch only. This also happens in the expressions for the components of the bulk stress tensor equations (3.28) and (3.33). This might seem strange at first sight, but this might be a consequence of the fact that in the – branch the bound state is present for all non-vanishing values of \( H \), so this limit is discontinuous in this sense.

4. One-loop effective potential for \( r_c \)

This section is divided into two parts. In the first, we obtain the one-loop effective potential \( V_{\text{eff}} \) for \( r_c \) in the flat brane case, by three methods. The first one uses the expression that we obtained in section 3.4 for the (expectation value of) the stress tensor on the brane. We re-derive it using the usual Kaluza–Klein decomposition and, finally, using the boundary effective theory of section 2.3. Then, we discuss the application of this potential to attempt to generate a large value of the crossover scale \( r_c \) in section 4.1.

\[ V_{\text{eff}} \text{ obtained from the stress tensor on the brane, equation (3.34). From (2.8), the coupling between the (canonical) field } \varphi \text{ associated with } r_c \text{ and } \Psi \text{ is of the form } r_c(\varphi)(\partial \Psi)^2. \]

\[ \text{The equation of motion for } \varphi \text{ is } \]

\[ \Box_4 \varphi = r'_c(\varphi)(\partial \Psi)^2. \]  

(4.1)

To take into account the quantum effects, one can replace \( (\partial \Psi)^2 \) by its vacuum expectation value. Then, we can identify the one-loop effective potential \( V_{\text{eff}}(\varphi_c) \) induced by \( \Psi \) as

\[ V_{\text{eff}}(r_c) = \langle (\partial \Psi)^2 \rangle, \]  

(4.2)
and using equation (3.36),
\[ V_{\text{eff}}(r_c) = A\Lambda^4 + B\frac{\Lambda^3}{r_c} + C\frac{\Lambda^2}{r_c^2} + D\frac{\Lambda}{r_c^3} - \frac{1}{64\pi^2} \frac{\log(r_c\Lambda)}{r_c^4} + \cdots, \]  
(4.3)
where \( A, B, C, D \) are constants, we identified \( \Lambda \) as the cut-off scale which we can associate with the brane thickness, and the dots indicate suppressed terms. As argued before, the relevant field in the 4D effective action is \( \varphi = 1/r_c \). We discuss the renormalization conditions and the application of this result below. Before that, we shall re-derive the same result with two other methods.

**Kaluza–Klein decomposition.** We can obtain a slightly better understanding of the result (4.3) by working in terms of the dimensional or Kaluza–Klein (KK) reduction. Introducing the KK decomposition \( \Psi = \int d^m U_m \psi_m \) as in equation (3.2), we see from equation (4.2) that
\[ V'_{\text{eff}}(r_c) = \int d^m U_m^2 \langle (\partial \psi_m)^2 \rangle, \]  
(4.4)
where \( \psi_m \) are 4D scalar fields with mass \( m \). In the previous equation, we have assumed that modes with different \( m \) are independent, so that one has \( \langle \psi_m \psi_{m'} \rangle = \langle \psi_m^2 \rangle \delta(m - m') \). We can easily compute \( \langle (\partial \psi_m)^2 \rangle \) e.g. using point splitting. One finds
\[ \langle (\partial \psi_m)^2 \rangle = \frac{1}{8\pi^2} \left\{ 16\Lambda^4 - 4m^2\Lambda^2 - m^4 \log \left( \frac{\Lambda}{m} \right) + \cdots \right\}, \]  
(4.5)
where \( \Lambda \) is the cut-off and the dots stand for suppressed terms. For a Minkowski brane, the wavefunction of the KK modes on the brane is (see equation (3.15))
\[ U_m^2|_0 = \frac{1}{\pi \lambda + (r_c m)^2}. \]
The integral over the KK continuum of (4.5) with this ‘weight’ contains a few divergent terms. The first term in (4.5) gives a quartic divergence, while the rest give at most cubic divergences. As we shall see below this is neatly captured by the boundary effective theory of section 2.3. Aside from this, we shall only comment that this results in a potential with the same features as (4.3). In particular, we reproduce the same coefficient for the finite term. This agreement might not be surprising since, after all, the Green function (and so the stress tensor) have also been obtained as a mode sum. However, the regularization procedure is slightly different. Equation (4.3) is obtained by point splitting the stress tensor, that is, evaluating the wavefunction of the modes slightly off the brane. Instead, in the method described here, the sum over the modes is cut off at a certain frequency.

**Boundary effective action description.** In the language of the boundary effective action of section 2.3, the decay of the resonant mode into higher dimensional modes is described by a (non-local) coupling of the form \( \varphi \tilde{\psi} \Delta \tilde{\psi} \), where \( \Delta = \sqrt{-\Box(4)} \) and \( \varphi = 1/r_c \). Then, we can obtain the one-loop effective potential as the sum over all one-loop 1PI diagrams with vanishing the external momentum. One easily obtains
\[ V_{\text{eff}} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \log \left( 1 + \frac{k/r_c}{k^2} \right), \]  
(4.6)
where \( k^i \) is the Euclidean momentum flowing in the loop. This integral diverges. Introducing a cut-off \( \Lambda \) to regulate it, we obtain

\[
V_{\text{eff}} = \frac{1}{64\pi^2} \left\{ \frac{4\Lambda^3}{3r_c} - \frac{\Lambda^2}{r_c^2} + \frac{4\Lambda}{3r_c^3} - \frac{\log(r_c\Lambda) + 1/4}{r_c^4} + \cdots \right\},
\]

where the dots denote terms suppressed by powers of \( \Lambda \).

### 4.1. A large crossover scale?

We shall regard equation (4.3) as the result for the one-loop effective potential \( V_{\text{eff}} \) when the brane is flat (which holds for large \( r_c \)). The constants \( A, B, \) etc are unknown to the effective theory, and must be fixed by means of a set of renormalization conditions. We shall assume that \( \phi = 1/r_c \) has a quartic potential \( \alpha \phi^4 \) at tree level. Accordingly, we impose as renormalization conditions that the coefficients \( A, B, C \) and \( D \) in (4.3) vanish. Then, the potential at one-loop level is of the Coleman–Weinberg \( [35] \) form

\[
V_{\text{eff}}(\phi) = \phi^4 (\alpha + \beta \log(\phi/\Lambda))
\]

where \( \beta = 1/64\pi^2 \). The minimum of this potential is at \( r_c \simeq e^{\alpha/\beta}/\Lambda \), which can be very large without fine-tuning. This seems quite encouraging as it might provide a natural explanation for why \( r_c \) is so large in the DGP model, and ultimately why the current acceleration of the universe is so small. This potential gives a very small mass, of order \( 1/r_c \). Aside from being potentially unstable under quantum corrections, this would require a very suppressed coupling of \( \phi \) to matter in order not to generate an observable fifth force \( [36] \). Rather than elaborating on these issues, we shall now address how this picture changes when we include the curvature of the brane—an essential ingredient if we are concerned about the accelerated expansion.

For an inflating brane, the one-loop contribution (which can be obtained from (4.2) and (3.37), (3.38)) is slightly more complicated than (4.8). The most important difference is now the classical contribution \( \sim \pm H m_P^2 \phi \) discussed in section 2.1, which clearly dominates over (4.8). In the following, we shall assume that \( \phi \) is not protected by any symmetry. It is then ‘natural’ to have a potential of the form

\[
V_{\text{eff}}(\phi) = \pm H m_P^2 \phi + A \Lambda^4 + B \Lambda^3 \phi + C \Lambda^2 \phi^2 + \cdots,
\]

where \( A, B, \ldots \) are order-1 coefficients and \( \Lambda \) is the cut-off. Given that the piece linear in \( \phi \) represents the term due to the background curvature we shall take the renormalization condition \( B = 0 \) (which amounts to a redefinition of \( m_P \)). Then, the potential has a minimum (maximum) for the normal (self-accelerated) branch at \( r_c \sim (\Lambda/m_P)^2 C/H \).

There are different estimates available in the literature \( [3,10,11,17,37] \) for the cut-off \( \Lambda \) for the bulk graviton, but all of them are much lower than \( m_P \). Hence, the expectation value generated for \( r_c \) is too small for phenomenological purposes.

As an aside, we shall mention that an intriguing situation arises if the mass term for \( \phi \) is Planckian. This is the case, for instance, if this field couples directly to matter and the cut-off in the matter loops is of order \( m_P \). One should still keep the linear term, as it encodes the coupling to the background curvature. Then, the extremum occurs

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\(^6\) We could also include the loops of \( \phi \) itself, but the argument would not change much.
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at $r_c \sim 1/H$,\(^7\) which seems quite suggestive for phenomenology. Unfortunately, this corresponds to a maximum for the self-accelerated branch. In the normal branch, instead, $r_c \sim 1/H$ is a minimum, and hence $r_c$ does indeed ‘track’ the Hubble radius. In this case, the Friedmann equation (equation (2.4)) looks conventional except that the effective Newton constant is smaller. This happens because for $r_c \sim 1/H$ the brane is kept on the verge of the five-dimensional regime. A complete analysis of the cosmology in this kind of model (which should include the effect of the kinetic term for $r_c$) goes beyond the scope of this paper (see [30] for the case of a time-dependent Brans–Dicke field on the brane) and is left for future research. It would also be of special interest to clarify whether this mechanism can be applied to the model of [15], where a self-accelerated phase can take place in the ‘normal’ branch.

5. Conclusions

We have discussed the quantum effects from the bulk fields in the DGP model, in a toy version of it where instead of the graviton we considered a bulk scalar field with a kinetic term on the brane. We have computed the Green function and the stress tensor induced by quantum fluctuations. We have shown that they are regular everywhere except on the brane, and described their form in several limits of interest.

The crossover scale $r_c$ in the DGP model is usually assigned a large value—of the order of the present Hubble radius—in a somewhat ad hoc way. We used our results to clarify whether the quantum effects can provide a natural mechanism for generating such a large value. Once $r_c$ is promoted to a dynamical field, this problem translates to finding the potential that drives it to large values. Here, we considered the one-loop effective potential $V_{\text{eff}}(r_c)$ induced by the bulk fields.

With a flat brane in a flat bulk, we showed that this is possible by a quite simple argument. The fluctuations of the five-dimensional ‘graviton’ provide a Coleman–Weinberg-type potential, which can induce a minimum at very large $r_c$ without fine-tuning. When the curvature of the brane is included, the potential is substantially modified due to a tree-level contribution originating from the background curvature. In the present set-up, the value thus generated for $r_c$ is too small to be compatible with observations. However, simple extensions of the model suggest that it is possible to generate a minimum where $r_c$ is of the order of the Hubble radius $1/H$ in the normal branch, indicating the self-consistency of this case at one-loop level.

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\(^7\) Note that the one-loop contribution alone also admits local minima at $r_c \sim 1/H$. 
Appendix. Some formulae for the Green function

In order to perform the integration in (3.17), it is convenient to use the representation for the massive Hadamard function in dS space [25]

\[
G_p^{(dS)(1)}(x, x') = \frac{2}{(n - 1)S_{(n)}} \frac{2^{-(n-1)/2}}{(1 - \cos \zeta)^{(n-1)/2}} \left\{ 2F_1 \left( -ip + \frac{1}{2}, ip + \frac{1}{2}; -n - 3; 1 - \cos \zeta \right) \right.
\]

\[
+ \frac{\Gamma ((n/2) - ip) \Gamma ((n/2) + ip) \Gamma (-1/2)}{\Gamma ((1/2) - ip) \Gamma ((1/2) + ip) \Gamma ((n - 1)/2)} \times \left( \frac{1 - \cos \zeta}{2} \right)^{(n-1)/2} 2F_1 \left( -ip + \frac{n}{2}, ip + \frac{n + 1}{2}; 1 - \cos \zeta \right) \right\},
\]

where \(2F_1\) is the hypergeometric function and \(\zeta\) is the invariant distance in dS space.

Including the bound state contribution, we obtain

\[
G^{(1)} = \frac{1}{S_{(n)}\Gamma ((n + 1)/2)} \left( \frac{1}{4aa'} \right)^{n/2}
\]

\[
\times \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left\{ -1 - 2 \frac{\lambda_> + \lambda_<}{\lambda_> - \lambda_<} \left( \frac{\lambda_>}{n/2 + k + j - \lambda_>} - \frac{\lambda_<}{n/2 + k + j - \lambda_<} \right) \right\}
\]

\[
\times \frac{(-1)^k \Gamma (n + 2k + j)}{j! k! \Gamma ((n + 1)/2 + k)} (aa'H^2)^{(n/2+k+j)} \left( \frac{1 - \cos \zeta}{2} \right)^k
\]

(A.1)

where \(\zeta\) is the invariant distance in dS space and \(S_{(n+1)} = 2\pi^{1+n/2}/\Gamma(1 + n/2)\) is the volume of a unit \((n + 1)\)-dimensional sphere. This summation can be readily performed for the first term in the curly brackets, leading to minus \(G_0\) as defined in (3.21). The two remaining terms in (A.1) can be obtained from \(G_0\) as

\[
G_\lambda = \int_0^\infty d\alpha \, e^{\lambda \alpha} \, G_0 \left( H \to e^{\pm \alpha/2}H \right),
\]

(A.2)

which leads to (3.22) quite immediately.

For the sake of completeness, we include here the expression of the Green function when \(x\) and \(x'\) are ‘aligned’, that is, when their four-dimensional coordinates coincide, \(\zeta = 0\). We obtain

\[
G^{(1)} = \frac{H^n (aa'H^2)^{\pm n/2}}{nS_{(n+1)}}
\]

\[
\times \left\{ \frac{(aa'H^2)^n}{[(aa'H^2)^2 - 1]^n} + 2 \frac{\lambda_> + \lambda_<}{\lambda_> - \lambda_<} \left[ A \left( \lambda_> (aa'H^2)^{\mp 1} \right) A \left( \lambda_< (aa'H^2)^{\mp 1} \right) \right] \right\},
\]

(A.3)

where we introduced

\[
A(\lambda, x) \equiv x^\lambda \, B_x \left( \frac{n}{2} - \lambda, 1 - n \right)
\]
and $B_x(a,b)$ is the incomplete beta function. For small $x$,

$$A(\lambda,x) \simeq \frac{\lambda}{n/2 - \lambda} x^{n/2} (1 + \mathcal{O}(x)). \quad (A.4)$$

Let us now elaborate on the IR divergence present in (A.3) for the $-\,$branch, due to the $\lambda_\times = n/2$ contribution (the massless bound state). Using

$$B_x(a,b) = (-1)^{-a} B(1 - a - b, a) + (-1)^b B_{1/x}(1 - a - b, b)$$

where $B(a,b) = B_1(a,b)$ is the Euler beta function, we can find that the expansion around $\lambda = n/2$,

$$\frac{e^{n z/2}}{\lambda} A(\lambda,x) \simeq -\frac{1}{\lambda - n/2} \log x + (-1)^{n+1} B_x(n,1-n) - \psi(n)$$

$$- \gamma + i \pi + \mathcal{O}(\lambda - n/2),$$

up to an irrelevant constant, and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. In this equation, we kept the digamma term in order to obtain a finite limit for $n = 3$. For small $x$,

$$(-1)^{n+1} B_{1/x}(n,1-n) - \psi(n) \sim \gamma - i \pi + \log x + nx + \cdots$$

up to constant terms and where the dots denote higher powers of $x$. The terms linear in $\log x$ cancel, and the leading order term is $\mathcal{O}(x)$. The other two contributions to the Green function behave like $\text{const} + \mathcal{O}(x)$. Hence, the Green function is well behaved at infinity (which is represented by the light cone in the $-\,$branch).

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