A NONLOCAL TOY MODEL OF PATTERNS FORMATION

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ABSTRACT. We study a pattern formation model described by certain nonlocal evolution equation. This evolution equation is obtained by a modification of a model introduced by Shigeru Kondo to explain colour patterns on a skin of the guppy fish. We prove the existence of stationary solutions in the linear and nonlinear cases either using the bifurcation theory or the Schauder fixed point theorem. We also present numerical studies of this model and show that it exhibits patterns similar to those obtained by Kondo.

1. Introduction. It is well-known that reaction-diffusion equations can be widely used to describe behaviour of many biological phenomena, such as forming patterns in an organism. Japanese biologist Shigeru Kondo studied in his works [6–8] a process of a formation of colour patterns on a guppy fish (Poecilia reticulata). He observed that a dye present in a fish organism influence itself which leads to a diversified patterns on a fish skin. Kondo claimed that the process of patterns formation is a result of a single dye self-interaction. It is known that patterns can be obtained by reaction-diffusion systems, where at least two substances interact with themselves. Kondo proposed an explanation of this phenomena using an alternative model to classical reaction-diffusion systems, which may be insufficient in this case. The detailed description of that model is presented in Subsection 1.1. Other nonlocal models are explained in Subsection 1.2. Numerical simulations indicate that under specific conditions we may obtain diversified patterns, namely nonconstant stationary solutions. Model proposed by Kondo was not analysed mathematically, thus it is not clear whether results obtained numerically correspond to nonconstant stationary solutions.

The original Kondo model seems to be difficult to study mathematically, thus we propose its modification which perhaps is not well justified from a biological point of view but it describes pattern formation phenomena similar to those in the Kondo model. In Section 3 we consider the linear version of this toy model. We prove the existence and stability of stationary solutions under particular conditions. We also explain how a stationary solution depends on the dye self-interaction. The reasoning is extended in Section 5, where we prove the existence of nonconstant stationary solutions.

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solutions of the nonlinear version of the toy model. This solution is small and it is obtained using a bifurcation theorem. In Section 6 we prove the existence of large nonconstant stationary solutions using the Schauder fixed point theorem. This method allows us to construct reasonable nonconstant stationary solutions that correspond to patterns observed in biology. Our reasoning is restricted to the one dimensional case, since the construction of appropriate invariant spaces seems to be much more complicated in higher dimensions. In Section 7 we present numerical simulations obtained for that model. We discuss results obtained for various parameter range.

1.1. Biological motivation. Mathematical model proposed by Kondo is a differential equation that describes the concentration of a specific substance in a fish skin. Kondo considered the fish surface not as a two dimensional manifold embedded in $\mathbb{R}^3$ but as a bounded and connected subset of plane $\Omega \subset \mathbb{R}^2$. Let $u$ denote the concentration of a substance. Biologists claim that the substance production rate results from destruction law and cell synthesis law. The destruction law states that production rate is negatively impacted by the substance density. Cell synthesis law states that production rate is influenced by the substance distribution over the surface. There, the following model is used

$$\frac{\partial u}{\partial t} = (S - a)u,$$

where $a > 0$ is a constant cell destruction rate and $S$ corresponds to the cell synthesis. The cell synthesis is a process of sophisticated cell interactions, dependent on stimulation operator. Kondo in his work [6] claimed that cell influence on neighbours production rate depends only on distance between cells. Mathematically cell synthesis is approximated as a convolution with a radial kernel $K$

$$Stim(x, y) = \int \int u(x - \xi, y - \eta)K(\sqrt{\xi^2 + \eta^2})d\xi d\eta.$$

To ensure that the cell density will not be divergent to infinity, the cell synthesis follows the saturation law. The saturation function is introduced to ensure that the density of a substance is tempered and is given by the following formula

$$S = \begin{cases} 
0, & Stim < 0, \\
Stim, & 0 < Stim \leq M, \\
M, & Stim > M.
\end{cases}$$

The saturation formula states that the impact of cell density on a production rate is less than $M$ and cannot be negative. In this model function $u$ is bounded by $\frac{M}{a}$. This implies that, biologically, the value of $u$ can be interpreted as cell density between 0 and $\frac{M}{a}$ rather than 0 and 1.

In biological models, the kernel is designed to have positive and negative parts as well. This states that cells can increase or decrease the neighbours production rate. The positive impact is called activation and negative is called inhibition. The type of impact depends on distance. Notice that because of the saturation function, the kernel impact on cell production cannot be negative. However, the positive impact can be reduced by the negative part of the kernel. Numerical simulations exhibit, that without inhibition or activation part no diversified patterns can be achieved in Kondo model, namely dynamic described by it is trivial.
Let $K$ be a convolution kernel and $f$ be a saturation function. Kondo model can be rewritten as

$$u_t = -au + f(K * u).$$

(4)

In this work we will analyse the toy model based on Kondo model (4), where we no longer claim that $f$ is 0 for negative arguments. We will assume that $f$ is twice differentiable in the neighbourhood of 0 and $f'(0) \neq 0$. This modification is crucial to apply basic mathematical tools such as the bifurcation theorem and Schauder theorem to obtain the existence of nonconstant stationary solutions. We call this a toy model, since this modification alters many biological properties, for example the solution is no longer positive for positive initial conditions. Moreover, unlike in the Kondo model there exist patterns for positive convolution kernels and negative kernels.

1.2. Other nonlocal models. The nonlocal models with convolution kernels are widely used in various fields such as genetics, neurology and ecology. For example, Amari [1] modelled the dynamics of neuron fields in the brain using the following equation

$$\tau u_t = -u + w * H(u) + s, \quad \text{for } x \in \mathbb{R},$$

where $u(x,t)$ is the membrane potential of the neurons, $w$ is the convolution kernel, $s$ describes the external stimuli and $H$ is the Heaviside function. The convolution operator represents the influence of cells in the neighbourhood on the membrane potential.

Another of the mathematical models for nonlocal spatial dispersal is the following equation

$$u_t = k * u - bu + f(u),$$

where $u(x,t)$ is the population density, $k$ is the convolution kernel, $b$ is the positive constant and $f$ is the nonlinear function. The kernel $k$ corresponds to the transition possibility and $b$ describes the degradation rate. This model was studied by Hutson et al. [5]. Despite the behaviour of the model is similar to the corresponding reaction diffusion system, authors claimed that the nonlocal model is more suitable to describe the single species dispersal.

Berestycki et al. [2] analysed the nonlocal Fisher-KPP equation. The equation is obtained from classical Fisher equation by introducing the convolution operator. This model corresponds to the population dynamics with nonlocal interactions

$$u_t = \Delta u + \mu u(1 - k * u), \quad x \in \mathbb{R}^d.$$  

Authors discovered that there may exists travelling waves in this model, similarly as in classical Fisher choice of equation, for an appropriate coefficients. Moreover, they proved that there are no nonnegative nonconstant stationary solutions for particular domain dimension.

An important mathematical result comes from the work by Ninomiya et al. [9], who studied the general extension of reaction-diffusion system to the nonlocal evolution equation on the one dimensional torus

$$u_t = \Delta u + g(u, J * u), \quad \text{in } T \times (0, \infty),$$

$$u_0(x) = u(x,0), \quad \text{on } T.$$  

They proved that under particular conditions, this nonlocal model can be approximated by classical reaction-diffusion systems.
2. Preliminaries. Before we begin our analysis we need to introduce the convolution operation presented in the previous section. Since \( \Omega \) is a bounded domain, the convolution of two functions supported on \( \Omega \) does not have to be supported on \( \Omega \). Thus we understand a convolution in the following sense. Let \( f \in L^2(\Omega_1) \), \( g \in L^2(\Omega_2) \) and let \( \tilde{f}, \tilde{g} \in L^2(\mathbb{R}^n) \) be the extensions of \( f, g \) by 0 outside of \( \Omega_1 \) and \( \Omega_2 \). We define the convolution of these functions as
\[
\tilde{f} \ast \tilde{g} = \tilde{f} \ast \tilde{g} |_{\Omega_2}.
\]
In particular, let \( K \in L^2(\mathbb{R}^n) \) be a radial compactly supported convolution kernel. In the subsequent part of this work we use the following notation
\[
K \ast \Omega u = K \ast \tilde{u} |_{\Omega}
\]
where \( \Omega \) is the domain of \( u \).

**Lemma 2.1.** The convolution (5) has the property \( \|K \ast \Omega u\|_{L^\infty} \leq \|K\|_{L^2} \|u\|_{L^2} \).

**Proof.** Let \( \tilde{K} \) be the extension of \( K \) with 0 outside of \( \Omega \). It follows from (5) that
\[
\|K \ast \Omega u\|_{L^\infty} = \left\| \int_{\mathbb{R}^n} K(x - y)\tilde{u}(y)dy \right\|_{L^\infty} \leq \left\| \int_{\mathbb{R}^n} K(x - y)\tilde{u}(y)dy \right\|_{L^\infty} \leq \left\| \int_{\mathbb{R}^n} |K(x - y)\tilde{u}(y)dy| \right\|_{L^\infty} \leq \|K\|_{L^2} \|\tilde{u}\|_{L^2} = \|K\|_{L^2} \|u\|_{L^2}.
\]
\[\square\]

**Remark 1.** There exists \( H(x, y) \in L^2(\Omega \times \Omega) \), such that the operator \( Tu \) can be represented as
\[
Tu(x) = \int_{\Omega} K(x - y)\tilde{u}(y)dy = \int_{\Omega} H(x, y)u(y)dy.
\]
The kernel \( H \) is given by the explicit formula \( H(x, y) = K(x - y) \) on \( \Omega \times \Omega \). It follows from the evenness of the kernel \( K \) that \( H(x, y) = H(y, x) \) and \( H \) is a constant on \( y = x + k \) lines.

In this work we analyse the following problem. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded, connected and compact domain with smooth boundary. Let \( f \in C^1(\mathbb{R}) \) be a bounded function. Let \( u : \mathbb{R} \times \Omega \to \mathbb{R} \) be \( C^1 \) with respect to the first coordinate. Consider the following initial value problem
\[
\begin{align*}
  u_t &= -au + f \left( \int_{\Omega} H(x, y)u(y)dy \right), \\
  u(x, 0) &= u_0(x).
\end{align*}
\]
First, we prove the existence of global solutions for this problem.

**Lemma 2.2.** For every initial condition \( u_0(x) \in L^2(\Omega) \) there exists a unique global solution \( u(x, t) \in C([0, \infty), L^2(\Omega)) \) of problem (7).

**Proof.** Integrating the equation over time domain yields
\[
u(x, t) - u(x, 0) = -a \int_0^t u(x, s)ds + \int_0^t f \left( \int_{\Omega} H(x, y)u(y, s)dy \right)ds.
\]
Let us define the operator \( G : L^2(\Omega) \to L^2(\Omega) \)
\[
Gu = u(x, 0) - a \int_0^t u(x, s)ds + \int_0^t f \left( \int_{\Omega} H(x, y)u(y, s)dy \right)ds.
\]
We will prove that $G$ is a contraction for appropriately small $t$. For each $u_1$ and $u_2$, such that $u_1, u_2 \in C([0, t], L^2(\Omega))$ we have

\[
\|Gu_1 - Gu_2\|_2 = \| -a \int_0^t (u_1(x, s) - u_2(x, s))ds + \int_0^t f \left( \int_{\Omega} H(x, y)(u_1(y, s) - u_2(y, s))dy \right) ds \|_2 \\
\leq \|a \int_0^t (u_1(x, s) - u_2(x, s))ds\|_2 + \| \int_0^t f \left( \int_{\Omega} H(x, y)(u_1(y, s) - u_2(y, s))dy \right) ds \|_2.
\]

Thus, by the integral Minkowski inequality we obtain

\[
\|a \int_0^t (u_1(x, s) - u_2(x, s))ds\|_2 = \left( a \int_{\Omega} \left( \int_0^t (u_1(x, s) - u_2(x, s))ds \right) dx \right)^{\frac{1}{2}} \\
\leq a \int_0^t \left( \int_{\Omega} (u_1(x, s) - u_2(x, s))^2 dx \right)^{\frac{1}{2}} dt \\
= at \cdot \|u_1 - u_2\|_2. \tag{10}
\]

The function $f$ fulfils the global Lipschitz condition, therefore by the Minkowski and Schwarz inequalities we obtain

\[
\| \int_0^t f \left( \int_{\Omega} H(x, y)(u_1(y, s) - u_2(y, s))dy \right) ds \|_2 \\
= \left( \int_0^t \left( \int_{\Omega} f \left( \int_{\Omega} H(x, y)(u_1(y, s) - u_2(y, s))dy \right) ds \right)^2 dx \right)^{\frac{1}{2}} \\
\leq \|f\|_{C^1} \int_0^t \left( \int_{\Omega} \left( \int_{\Omega} H(x, y)(u_1(y, s) - u_2(y, s))dy \right)^2 dx \right)^{\frac{1}{2}} ds \\
\leq \|f\|_{C^1} \int_0^t \int_{\Omega} \left( \int_{\Omega} (H(x, y)(u_1(y, s) - u_2(y, s))^2 dx \right)^{\frac{1}{2}} dyds \\
\leq \|f\|_{C^1} \int_0^t \int_{\Omega} \|u_1 - u_2\|_{L^2(\Omega)} \|H\|_{L^2(\Omega \times \Omega)} ds \\
= \|f\|_{C^1} \|H\|_{L^2(\Omega \times \Omega)} \cdot t \|u_1 - u_2\|_2. \tag{11}
\]

It follows that for sufficiently small $t$ the operator $G$ is a contraction. By the Banach fixed point theorem we obtain that there exists a unique solution on an interval $[0, t]$. Since $f$ is globally Lipschitz, we can iteratively obtain the existence on all $\mathbb{R}_+$. \hfill \Box

Now we will present some fundamental properties of convolution operators. Let us recall well-known properties of integral operators. The detailed proof of Proposition 1 can be found e.g. in [3, page 168].

**Proposition 1.** The operator $T : L^2(\Omega) \to L^2(\Omega)$ given by the formula

\[ Tu(x) = \int_{\Omega} H(x, y)u(y)dy \]

with $H \in L^2(\Omega \times \Omega)$ is compact. Furthermore, there exists a sequence of nonzero eigenvalues $\{\lambda_j\}_{j=1}^\infty$ fulfilling $\lim_{j \to \infty} \eta_j = 0$ and an orthonormal basis $\{e_i\}_{i=1}^\infty$ of $\text{Im}(T)$ such that

\[ Te_j = \lambda_j e_j. \]
3. Linearized model. In this section we will analyse the linearised version of the model (7). Results presented in this section allows us to apply the bifurcation theorem in the nonlinear case. Let \( \Omega \subseteq \mathbb{R}^2 \) be an open, bounded, connected domain. We consider the problem

\[
\begin{align*}
  u_t &= -au + b \cdot Tu, \quad x \in \Omega, \quad t > 0 \\
  u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]  

where \( a > 0, b \in \mathbb{R} \setminus \{0\} \).

3.1. Existence of stationary solutions.

Lemma 3.1. Assume that \( a > 0, b \neq 0 \). There exists a nonzero stationary solution of problem (13) if and only if

\[
\frac{a}{b} = \lambda_k \quad \text{for some } k \in \mathbb{N},
\]

where \( \lambda_k \) is an eigenvalue of operator \( T \) (see Theorem 1). Moreover each such stationary solution is of the form

\[
\tilde{u} = \sum_{l=1}^{n_k} C_l \epsilon_{jl},
\]

where \( \epsilon_{jl} \) are eigenfunctions corresponding to the eigenvalue \( \lambda_k \), the number \( n_k \) denotes the multiplicity of \( \lambda_k \) eigenvalue and \( C_l \) are arbitrary constants.

Proof. For a stationary state, we have

\[
0 = -au + b \cdot Tu
\]

Thus we immediately obtain that the solution exists if and only if \( \frac{a}{b} \) is an eigenvalue for some \( k \). Moreover the solution is a combination of all eigenfunctions corresponding to above-mentioned eigenvalue.

3.2. Stability of stationary solutions. We are now ready to formulate and prove a stability theorem for linear model (13).

Theorem 3.2. Assume that \( a > 0, b \neq 0 \). Let \( \tilde{u} \) be a nonzero stationary solution of problem (13), corresponding to the eigenvalue \( \lambda_k \). The solution \( \tilde{u} \) is stable if and only if \( b \lambda_j \leq a \) for each \( j \in \mathbb{N} \), where \( \{\lambda_j\}_{j=1}^\infty \) are eigenvalues mentioned in Proposition 1.

Proof. From lemma 3.1 the stationary solution is given by

\[
\tilde{u} = \sum_{l=1}^{k} C_l^1 \epsilon_{jl}.
\]

Consider the problem with perturbed initial condition \( u_0 = \tilde{u} + u_0' \). A solution of problem (13) can be expressed using orthogonality properties of eigenfunctions

\[
u(t, x) = \tilde{u}(x) + \sum_{j \neq k} a_j(t) \tilde{\epsilon}_j(x)
\]

with suitably chosen \( a_j \)'s. Substituting those functions into equation (13) exhibits, that it is enough to analyse the stability of the trivial solution \((0, 0)\). Hence,

\[
\sum_j a_j'(t) \tilde{\epsilon}_j(x) = -a \left( \tilde{u}(x) + \sum_{j \neq k} a_j(t) \tilde{\epsilon}_j(x) \right) + b \left( \lambda_k \tilde{u}(x) + \sum_{j \neq k} a_j(t) \lambda_j \tilde{\epsilon}_j(x) \right)
\]

\[
= -a \sum_{j \neq k} a_j(t) \tilde{\epsilon}_j(x) + b \sum_{j \neq k} a_j(t) \lambda_j \tilde{\epsilon}_j(x).
\]
Since \( \{ \breve{e}_j \}_{j=1}^\infty \) is an orthonormal basis, we obtain
\[
a'_j(t) = a_j(t) (b \cdot \lambda_j - a)
\] (18)
Thus, the stability of solutions to problem (13) is equivalent to the stability of the solution of system (18) for each \( j \in \mathbb{N} \). It follows that this solution is stable if and only if
\[
b \cdot \lambda_j - a \leq 0
\]
for each \( j \in \mathbb{N} \). Since \( \lambda_j \to 0 \) as \( j \to \infty \), we obtain that solution is stable if \( \frac{a}{b} \) is equal to the maximal eigenvalue for \( b > 0 \) and minus minimal eigenvalue for \( b < 0 \).

We will now prove the following lemma which will be useful in the subsequent part of this work.

\textbf{Lemma 3.3.} If the assumptions of Theorem 3.2 hold, then every stable solution of problem (13) with initial condition \( u_0 \) fulfills
\[
\| u(t) \|_2 \leq \| u_0 \|_2 \quad \text{for all} \quad t \geq 0.
\] (19)
Moreover, for each \( a_j(t) \) holds
\[
a_j^2(t) \leq e^{2\mu_j t} a_j^2(0),
\] (20)
with \( \mu_j = b \cdot \lambda_j - a \).

\textbf{Proof.} Let us express \( u_0 \) in the orthonormal basis \( \{ \breve{e}_j \} \). We obtain
\[
u_0 = \sum_j a_j \breve{e}_j, \quad \| u_0 \|^2 = \sum_j a_j^2
\] (21)
The second part of the claim follows from the form of solution of the linear problem for each \( a_j \). The solution is given by the explicit formula
\[
a_j(t) = a_j e^{\mu_j t},
\] (22)
where \( \mu_j = b \cdot \lambda_j - a \). \( \mu_j \) is a bounded negative sequence with \( \sup_j \{ \mu_j \} = 0 \). Since \( \breve{e}_j \) is the orthonormal basis, the following inequality holds for each \( t \)
\[
\| u(t) \|_2 = \sum_j a_j^2(t) = \sum_j a_j^2 e^{2\mu_j t} \leq \sum_j a_j^2 e^{2\sup_j \mu_j} = \| u_0 \|^2_2.
\] (23)

4. \textbf{Nonlinear model.} Let \( \Omega \subseteq \mathbb{R}^2 \) be an open, bounded, connected domain. Let \( T : L^2(\Omega) \to L^2(\Omega) \) be an integral operator
\[
Tu(x) = \int_\Omega H(x,y)u(y)dy
\] (24)
with the kernel \( H \in L^\infty(\Omega \times \Omega) \). Consider the nonlinear model
\[
u_t = -au + f(Tu), \quad x \in \Omega, \quad t > 0,
\]
\[
u(x,0) = u_0(x), \quad x \in \Omega.
\] (25)
For our purpose we assume that \( f(0) = 0 \), \( f \in C^2(\mathbb{R}) \), namely \( f, f', f'' \) are continuous and bounded. We will also assume that \( a > 0 \). The stationary solutions of problem (25) satisfies
\[
0 = -au + f(Tu), \quad x \in \Omega
\] (26)
In this section, we prove the existence of nonconstant stationary solutions of problem (25). Such stationary solutions can be classified into two distinct classes. The first class consists of stationary solutions obtained in a neighbourhood of zero
solution. We prove their existence by a local linearisation of the equation and by application of the bifurcation theory. We apply methods from the theory for elliptic equations thus we need to claim additional assumptions for eigenvalues of operator $T$. These solutions are relatively small and do not use the saturation property of function $f$. Furthermore, the existence can be proved only under particular conditions for model coefficients thus it does not explain the general mechanism of pattern formation observed in numerical simulations.

Second class contains stationary solutions that are obtained through an application of the fixed point theory. Under specific assumptions on the function $f$ and the kernel $K$ we are able to construct solutions in one dimensional case only. These stationary solutions hit saturation level, and are supported by numerical simulations for wide range of model coefficients. It is expected that the reasoning presented in one dimensional case can be generalized to higher dimensions, but it would require much more technical details for a specific space construction.

In this section we impose the following assumptions.

**Assumption 1.** We assume that operator $T : L^2(\Omega) \to L^2(\Omega)$ is given by the convolution with radial, compactly supported bounded kernel $K$

$$Tu = K * \Omega u = \int_{\mathbb{R}^n} K(x-y)u(y)dy \bigg|_{\Omega} = \int_{\Omega} H(x,y)u(y)dy.$$  (27)

**Assumption 2.** By Proposition 1, there exists an orthonormal basis $\{e_i\}_{i=1}^\infty$ of $\text{Im}(t)$ such that $Te_j = \lambda_j e_j$. We assume, moreover, that $\lambda_j \neq 0$ and only a finite number of $\lambda_j$ are negative.

If we define, formally, the operator $T^{-1} : Te_j = \frac{1}{\lambda_j} e_j$ for each $j \in \mathbb{N}$ then equation (26) takes the form

$$0 = -aT^{-1}(v) + f(v).$$  (28)

Using the variational method, we are going to find a solution to equation

$$aT^{-1}(v) + dv = dv + f(v).$$  (29)

where fixed $d$ satisfies $d + a\lambda_j > 0$ for each $j \in \mathbb{N}$.

**Definition 4.1.** We define the bilinear form $Q(\cdot,\cdot)$ associated with the operator $aT^{-1} + bI$ and its domain by the formula

$$Q(u,v) = \sum_{j=1}^{\infty} a_j b_j \left( \frac{a}{\lambda_j} + d \right),$$

for each

$$u, v \in D(Q) = \left\{ u = \sum_{j=1}^{\infty} a_j e_j : \sum_{j=1}^{\infty} a_j^2 \left( \frac{a}{\lambda_j} + d \right) < \infty \right\}.$$

**Remark 2.** Note that,

$$\langle e_j, v \rangle_Q = b_j \left( \frac{a}{\lambda_j} + d \right) = \left( \frac{a}{\lambda_j} + d \right) (e_j, v) = \left( (aT^{-1} + d)e_j, v \right).$$

**Remark 3.** Since $\frac{a}{\lambda_j} + d > 0$ for each $j \in \mathbb{N}$ the bilinear form $\langle \cdot, \cdot \rangle_Q$ is a scalar product on $D(Q)$. We denote by $\| \cdot \|_Q$ the corresponding norm.
Lemma 4.2. Let $K$ be a bounded radial kernel with compact support. The image of operator $T$ given by (24) is a subset of $L^\infty(\Omega)$. Moreover $\text{Im}(T)$ is dense in $D(Q)$ and $D(Q)$ is dense in $L^2(\Omega)$.

Proof. Let $u \in L^2(\Omega)$. We have

$$\|K \ast_{\Omega} u\|_\infty = \left\| \int_{\mathbb{R}^n} \tilde{K}(x-y)\tilde{u}(y)dy \right\|_{\Omega} \leq C \int_{\mathbb{R}^n} |\tilde{u}(y)| = C \int_\Omega |u(y)| \leq C|\Omega|^{1/2}\|u\|_2 < \infty$$

To show the density, observe that finite sums $u_N = \sum_{j=1}^N a_j e_j \in \text{Im}(T)$, thus

$$\|u - u_N\|_Q = \sum_{j=N+1}^\infty a_j^2 \left( \frac{a}{\lambda_j} + d \right)^{N\to\infty} 0 \quad \text{and} \quad \|u - u_N\|_2 = \sum_{j=N+1}^\infty a_j^2 \frac{N\to\infty}{N\to\infty} 0.$$  

\[\square\]

Remark 4. Notice that $\| \cdot \|_{L^2} \leq C \cdot \| \cdot \|_Q$ for all $u \in D(Q)$ which is a consequence of the relation $\lambda_j \to 0$.

Definition 4.3 (Weak solution). Let the Assumptions (1)-(2) hold true. The function $v \in D(Q)$ is a weak solution of equation (26) if

$$(u, v)_Q = d \int_\Omega v\varphi + \int_\Omega f(v)\varphi \quad \text{for each } \varphi \in D(Q). \quad (30)$$

We study problem (30) by variational methods. For $F'(v) = f(v)$ we define a functional $J : D(Q) \to \mathbb{R}$, by the formula

$$J(v) = \frac{d}{2} \int_\Omega v^2 + \int_\Omega F(v), \quad \text{for each } v \in D(Q). \quad (31)$$

First, we prove basic properties of the functional $J$.

Lemma 4.4. The functional

$$J(v) = \frac{d}{2} \int_\Omega v^2 + \int_\Omega F(v) \quad (32)$$

satisfies $J \in C^2(D(Q), \mathbb{R})$.

Proof. It is an elementary calculation to show

- $J \in C(D(Q), \mathbb{R})$,
- it is differentiable in the Fréchet sense and for each $v \in D(Q)$
  $$\langle DJ(v), \varphi \rangle_Q = d \int_\Omega v\varphi + \int_\Omega f(v)\varphi.$$
- $DJ \in C(D(Q), \text{Lin}(D(Q), \mathbb{R}))$.

The second Fréchet derivative as the point $v \in D(Q)$ is represented by the bilinear form

$$\langle D^2J(v)\varphi, \psi \rangle_Q = d \int_\Omega \varphi\psi + \int_\Omega f'(v)\varphi\psi. \quad (33)$$

Let us show that $D^2J \in C\left(D(Q), \text{Lin}(D(Q), \text{Lin}(D(Q), \mathbb{R}))\right)$. We use the $3\varepsilon$ argument. For $v_n \to v$ in $D(Q)$, $\varphi, \psi \in D(Q)$, $\varphi_m \in \text{Im}(T)$ and $\varphi_m \to \varphi$ in $D(Q)$ we
estimate
\[ |(D^2\mathcal{J}(v) - D^2\mathcal{J}(v_n))\varphi, \psi| = \left| \int_\Omega \left( f'(v) - f'(v_n) \right) \varphi \psi \right| \]
\[ \leq \int_\Omega |f'(v)| |\varphi - \varphi_n| |\psi| + \int_\Omega |f'(v_n)| |\varphi_n| |\psi| + \int_\Omega |f'(v_n)| |\varphi - \varphi_n| |\psi|. \]  (34)

The derivative of \( f' \) is bounded, thus the norms of first and third integral are less then \( \epsilon \) for sufficiently large \( m \).

\[ \int_\Omega |f'(v)||\varphi - \varphi_n||\psi| \leq \|f'\|_\infty \|\varphi_n - \varphi\|_2 \|\psi\|_2 \leq C \|\varphi_n - \varphi\|_Q \|\psi\|_Q \leq \epsilon. \]  (35)

For each \( m \), function \( \varphi_m \) is bounded, thus the norm of the integral is less then \( \epsilon \) for sufficiently large \( n \).

\[ \int_\Omega |f'(v) - f'(v_n)||\varphi_m||\psi| \leq \|f''\|_\infty \int_\Omega |v_n - v||\varphi_m||\psi| \leq C \|v_n - v\|_Q \|\psi\|_Q \leq \epsilon. \]  (36)

It follows that for each \( \epsilon \) there exists \( m, n \in \mathbb{N} \) such that

\[ |(D^2\mathcal{J}(v) - D^2\mathcal{J}(v_n))\varphi, \psi| \leq 3\epsilon \]

and the second derivative is continuous.

\[ \square \]

5. Existence using a bifurcation theorem. Now recall a classification result from the bifurcation theory. Let \( X \) be a real Hilbert space, \( \Omega \in X \) be a neighbourhood of 0. Let \( L : \Omega \to X \) be a linear continuous operator and let \( H \in C(\Omega, X) \). Set \( H(v) = o(\|v\|) \) as \( v \to 0 \). Consider equation

\[ Lv + H(v) = \lambda v. \]  (37)

Obviously, there exists a trivial solution \( (\lambda, 0) \in \mathbb{R} \times X \) for each \( \lambda \).

**Definition 5.1.** A point \((\mu, 0) \in \mathbb{R} \times X\) is called a bifurcation point for equation (37) if every neighbourhood of \((\mu, 0)\) contains a nontrivial solution of (37).

**Lemma 5.2.** If \((\mu, 0)\) is a bifurcation point then \( \mu \) belongs to the spectrum of operator \( L \).

**Proof.** Let \((\mu, 0)\) be a bifurcation point for equation (37). Consider the family of balls

\[ B_n \left( (\mu, 0), \frac{1}{n} \right) \subset \mathbb{R} \times X. \]

For each \( n \) there exists \( (\mu_n, v_n) \in B_n \) satisfying

\[ Lv_n + H(v_n) = \mu_n v_n. \]  (38)

Obviously, we have \( \mu_n \to \mu \). Divide both sides of this equation by the norm of \( v_n \) and consider the weak solution

\[ \int_\Omega \frac{Lv_n}{\|v_n\|} \varphi + \int_\Omega \frac{H(v_n)}{\|v_n\|} \varphi = \int_\Omega \mu_n \frac{v_n}{\|v_n\|} \varphi. \]

By assumption for \( H \), if \( n \to \infty \) we obtain \( \int \frac{H(v_n)}{\|v_n\|} \varphi \to 0 \). Put \( w_n = \frac{v_n}{\|v_n\|} \). Sequence \( \{w_n\}_{n=1}^\infty \) is bounded, hence it is weakly compact. Let \( w_n \to w \). It follows

\[ \int_\Omega L(w) \varphi = \int_\Omega \mu w \varphi. \]

The equality holds for each \( v \in X \), hence \( \mu \) belongs to the spectrum of the operator \( L \).
Theorem 5.3 (Rabinowitz Bifurcation Theorem). Let $X$ be a real Hilbert space, $U$ a neighbourhood of 0 in $X$ and $I \in C^2(U, \mathbb{R})$ with $I'(v) = Lv + H(v)$, $L$ be linear and $H(v) = o(\|v\|)$ at $v = 0$. If $\mu$ is an isolated eigenvalue of $L$ of finite multiplicity, then $(\mu, 0)$ is a bifurcation point for (37). Moreover, at least one of the following occurs:

1. $(\mu, 0)$ is an isolated solution of (37) in $\{\mu\} \times X$
2. There is one-sided neighbourhood, $\Lambda$ of $\mu$ such that for all $\lambda \in \Lambda \setminus \{\mu\}$, equation (37) possesses at least two distinct nontrivial solutions.
3. There is a neighbourhood $I$ of $\mu$ such that for all $\lambda \in I \setminus \{\mu\}$, equation (37) possesses at least one nontrivial solution.

The detailed proof of 5.3 can be found in [10, page 412]. We are now ready to prove the existence of stationary nontrivial solutions for problem (30). First, we need to transform the equation into the form of

$$Lv + H(v) = \lambda v.$$  \hfill (39)

for some eigenvalue of operator $L$. We define operators $L$ and $H$

$$L(v) = \left(\frac{a}{\lambda} + d\right)v$$ and $$H(v) = f(v) - a\frac{1}{\lambda}v.$$  

The equation (30) can be rewritten as

$$\langle v, \varphi \rangle_Q = \int_\Omega L(v) + H(v)$$  \hfill (40)

Let us define the nonlinear functional $I(v) : D(B) \to \mathbb{R}$

$$I(v) = \frac{\lambda}{2} \int_\Omega v^2 + \int_\Omega \left(F(v) - \frac{a}{2\lambda_k} \cdot v^2\right).$$  \hfill (41)

Theorem 5.4. Consider system (25). Let Assumptions (1)-(2) hold true. If $f \in C^2(\mathbb{R})$ fulfills the global Lipschitz condition, $f(0) = 0$ and $f'(0) = a\frac{1}{\lambda}$, then there exist a sequence $\{v_n\}_{n=1}^\infty \subset \mathbb{R}$, which converges to 1 and a sequence of nonconstant functions $\{v_n\}_{n=1}^\infty \subset D(Q)$, such that $(v_n)$ is a weak solution of

$$0 = -ac_n v_n + f(Tv_n) + d(1-c_n)Tv_n,$$  \hfill (42)

for each $n \in \mathbb{N}$ and $\psi \in L^2(\Omega)$.

Proof. Let $I(v)$ be as in (41) then $DI(v) = Lv + H(v)$. From Lemma 4.4 we obtain that $I \in C^2(D(Q), \mathbb{R})$. To prove that $H(u) = o(\|v\|)_{D(Q)}$, we need to check that if $\|v\|_{D(Q)} \to 0$, then

$$\frac{\|H(v)\|_{D(Q)}}{\|v\|_{D(Q)}} \to 0.$$  \hfill (43)

Since $f(0) = 0$ we obtain

$$\frac{\|f(v) - a\frac{1}{\lambda}v\|_{D(Q)}}{\|v\|_{D(Q)}} = \frac{\|f(v) - f(0) - a\frac{1}{\lambda}v\|_{D(Q)}}{\|v\|_{D(Q)}}.$$  \hfill (44)

The right hand side tends to the derivative of $f$ if $\|v\|_{D(Q)} \to 0$. It follows from the assumption that the numerator tends to 0.
Now we can apply Theorem 5.3 to obtain that \((1, 0)\) is a bifurcation point for system \((40)\). It follows that there exist a sequence of \(\{c_n\}_{n=1}^{\infty}\) convergent to 1 and a sequence of nonconstant functions \(\{v_n\}_{n=1}^{\infty} \subset D(Q)\) such that
\[
c_n(v_n, \varphi)_Q = \int_{\Omega} \left( \frac{a}{ \lambda_j} + d \right) v_n \varphi + \int_{\Omega} \left( f(v_n) - \frac{a}{ \lambda_j} v_n \right) \varphi
\]
for each \(n \in \mathbb{N}\) and each \(\varphi \in D(Q)\). The equation \((45)\) is a weak solution of
\[
0 = -ad_n v_n + f(Tu_n) + d(1-d_n)Tu_n.
\]

**Remark 5.** If function \(f\) is a linear function equal to \(f(v) = bv\) then the condition in Theorem 5.4 is equal to the condition in Theorem 3.1.

The stability results obtained in the linear case cannot be easily extended for stationary solutions obtained through bifurcation theorem. Numerical simulations point out that the stability of solutions strongly depend on the value of second derivative of function \(f\) and in some cases solutions may become unstable. This topic requires further investigation.

### 6. Existence using a Schauder theorem

In this section, we prove the existence of stationary solutions using the fixed point theory in one dimensional case. This assumption allows us to omit technical assumptions on the kernel \(H\), in the construction of an invariant subspace for \(T\). The two dimensional construction is similar, but the assumptions for convolution kernels are different. We consider the equation, when either
\[
-u + f(Tu) = 0, \quad x \in \Omega = [-r, r] \quad \text{or} \quad \Omega = \mathbb{R}.
\]
with function \(f\) given by
\[
f(x) = \begin{cases} 
1, & x > 1, \\
x, & x \in [-1, 1], \\
-1, & x < -1.
\end{cases}
\]

We denote by \(K_+\) the positive part of \(K\) and by \(K_-\) its negative part. In this section we assume that \(\text{supp}(K_+) = [-2, 2]\) and \(\text{supp}(K_-) = [-2 - s, -2] \cup [2, 2 + s]\) for certain \(s > 0\). We denote the \(\text{supp}(K) = [-2 - s, 2 + s]\) as \(\Omega_K\). We simplify the equation by omitting the degradation coefficient and by fixing the kernel support. In the general case, all lemmas hold true, but the assumptions need to be modified. Observe that a solution of \((47)\) exists if \(u\) is a fixed point of the operator
\[
f(T) : L^2(\Omega) \to L^2(\Omega).
\]

In Theorem 6.5 below, we prove the existence of fixed point using the Schauder fixed point theorem (see e.g. [4, page 538]). Obviously \(u \equiv 0\) is a fixed point of this operator. To ensure the existence of non-trivial solutions, we construct a particular invariant set \(B\) for this mapping. The choice of this set is strongly dependent on the properties of convolution kernel \(K\). We introduce the appropriate sets for main three classes of convolution kernels: positive kernels, sign changing kernels, and negative kernels.
6.1. **Positive kernels.** In this subsection we construct the invariant set for positive kernels. We assume that the domain $\Omega = [-r, r]$ is large, namely $r > 4$.

**Lemma 6.1.** Let

$B = \{ u : u(x) = 1 \text{ for } x > 2, \ u(x) = -1 \text{ for } x < -2, \ u \text{ is odd and monotone} \}$.  \hfill (50)

If $\int K \geq 2$, $K$ is symmetric and positive then $f(T) : B \to B$.

**Proof.** If $x > 1$ then we have

$$Tu(x) = \int_{x-2}^{x+2} K(x-y)u(y)dy \geq \int_{x}^{x+2} K(x-y)u(y)dy = \int_{x}^{x+2} K(x-y)dy \geq 1. \hfill (51)$$

Analogously for $x < 1$ we obtain the result. If $|x| < 1$ then the convolution of odd and monotone function with positive and even function is odd and monotone. Thus the truncation is also odd and monotone. \hfill \Box

Notice that, the width of the gap between 1 and −1 is equal to the support of the kernel. In the next step, we prove the existence for smaller gaps, under appropriate assumptions for kernel $K$. This allows us to generalize the result for kernels with negative parts. We define the following set

$B = \{ u : u(x) = 1 \text{ for } x > 1, u(x) = -1 \text{ for } x < -1, \ u \text{ is odd and monotone} \}$. \hfill (52)

**Lemma 6.2.** Let $B$ be given by (52). If $\int K \geq 2$, $K$ is symmetric, positive and a decreasing function for positive $x$ then $f(T) : B \to B$.

**Proof.** If $x > 1$ then we have

$$Tu(x) = \int_{x-2}^{x+2} K(x-y)u(y)dy = \int_{x}^{x+2} K(x-y)dy + \int_{x-2}^{x} K(x-y)u(y)dy. \hfill (53)$$

Since $K$ is an increasing function for negative arguments, and $u$ is an increasing function we have

$$\int_{x-2}^{x} K(x-y)u(y)dy \geq 0. \hfill (54)$$

Analogously for $x < 1$ we obtain the result. If $|x| < 1$ then the convolution of odd and monotone function with positive function is odd and monotone. Thus, the truncation is also odd and monotone. \hfill \Box

6.2. **Small inhibition.** In this subsection we construct appropriate invariant space for sign changing kernels with sufficiently small negative part. We assume that the domain $\Omega = [-r, r]$ is large, namely $r > 4 + 2s$.

**Lemma 6.3.** Let $B$ be given by (52). If $\int_{0}^{2} K_{+} \geq 1 + \int K_{-}$, $K$ is symmetric, $K_{+}$ is a decreasing function for positive $x$ then $f(T) : B \to B$.

**Proof.** If $|x| < 1$ then the negative part of the kernel is integrated with +1 and −1 respectively. It follows, that the negative part can be omitted. The convolution of odd and monotone function with positive function is odd and monotone. Thus, the truncation is odd and monotone. If $x > 1$ we have

$$Tu(x) = \int_{x-2}^{x+2+s} K(x-y)u(y)dy \geq \int_{x}^{x+2} K(x-y) + 2 \int_{x+2}^{x+2+s} K(x-y)dy \geq 1. \hfill (55)$$

The case $x < -1$ is proved analogously. \hfill \Box
Lemma 6.4. Let $B$ be given by (52). If $\int_2^K K > 1 + \int_2^K K$, $K$ is symmetric and $K_+$ is a decreasing function for positive $x$. Moreover, if for each $x \in [0, 2]$ we have $\int_0^x K_+(y)dy \geq \int_2^x K_-(y-2)dy$ then $f(T) : B \to B$.

Proof. If $|x| < 1$ then the negative part of the kernel is integrated with $+1$ and $-1$ respectively. It follows that the negative part can be omitted. The convolution of odd and monotone function with positive function is odd and monotone. Thus, the truncation is odd and monotone. If $x > 1$ we have

$$T u(x) = \int_{x-2-s}^{x+2+s} K(x-y)u(y)dy$$

$$\geq \int_x^{x+2+s} K(x-y)dy + \int_1^1 K(x-y)u(y)dy$$

$$+ \int_{x-2-s}^{x-1} K_-(x-y)dy + \int_1^0 K_+(x-y)dy$$

The first integral is positive by assumption. The second integral is positive from monotone property of $K_+$. The last integral is positive from the last assumption. The case $x < -1$ is proved analogously.

Theorem 6.5. Let $\Omega = [-r, r] \subset \mathbb{R}$ and $K$ be a kernel that fulfils assumptions of either Lemma 6.1 or Lemma 6.2 or Lemma 6.3 or Lemma 6.4. Let $f$ be a saturation function (48). There exists a nonconstant stationary solution of (47).

Proof. Let $B$ be an invariant convex set. The operator $T$ is compact, thus $f(T)$ is compact. From Lemma 6.1, Lemma 6.2, Lemma 6.3 or Lemma 6.4 we have $f(T) : B \to B$. By the Schauder fixed point theorem we obtain that there exists a fixed point in $B$, that is a nonconstant solution in $L^2(\Omega)$ of equation (47).

By the same reasoning we obtain immediately the family of nonconstant solutions.

Corollary 1. Let $\{\Omega_j\}_{j=1}^N$, $\Omega_j \subset \Omega$ be the family of disjoint intervals such that

- for each $j$, $\text{diam}(\Omega_j) > 4 + 2s$,
- for each $k \neq j$, $\text{dist}(\Omega_j, \Omega_k) > 2$.

Then for each sequence $\{i_j\}_{j=1}^N$, $i_j \pm 1$ there exists a stationary solution $u$ such that $u|\Omega_j = i_j$.

Remark 6. The results can be generalized to the two dimensional case. Under modified assumptions for kernel $K$, the function $u(x, y) = u(x)$ is a two dimensional nonconstant stationary solution.

6.3. Negative kernels. In this subsection we prove the existence of stationary solutions for negative kernels $K$. Under appropriate assumptions there exist nonconstant stationary solutions for kernels without activation part. Obtained solutions are significantly different than in previous cases. The solutions are periodic function, thus we assume $\Omega = \mathbb{R}$.

Lemma 6.6. Let $\Omega = \mathbb{R}$, $K < 0$, $\text{supp}(K) = [-2-s, -2] \cup [2, 2+s]$, with $s < 2$, $K$ be symmetric on $[2, 2+s]$ and $[-2-s, -2]$. Define set $B$ of periodic functions $u$ with period $4 + 2s$ such that

- $u(x) = 1$ if $x \in [0, 2] + k(4+2s)$ for $k \in \mathbb{N}$,
- $u(x) = -1$ if $x \in [2+s, 4+s] + k(4+2s)$ for $k \in \mathbb{N}$,
- $u$ is monotone and even on each section $[2, 2+s] + k(2+s)$ for $k \in \mathbb{N}$.
If $\int K \leq -2$ then $f(T) : B \to B$.

Proof. We consider two cases. If $x \in [0, 2]$, (similarly for $x \in [2 + s, 4 + s]$) we have

$$Tu(x) = \int_{x+2}^{x+2+s} K(x-y)u(y)dy + \int_{2+s}^{x+2+s} K(x-y)u(y)dy$$

$$+ \int_{-s}^{x-2} K(x-y)u(y)dy + \int_{x-2-s}^{-s} K(x-y)u(y)dy$$

(53)

From the symmetry of $u$ we have

$$\int_{x+2}^{x+2+s} K(x-y)u(y)dy \geq 0 \quad \int_{-s}^{x-2} K(x-y)u(y)dy \geq 0$$

(54)

By assumption negative parts of kernel $K$ are symmetric. Function $u$ is equal to $-1$ on integration set and thus we have

$$\int_{x-2-s}^{-s} K(x-y)u(y)dy + \int_{2+s}^{x+2+s} K(x-y)u(y)dy = -\frac{1}{2} \int K \geq 1$$

(55)

If $x \in [2, 2 + s]$ then we have

$$Tu(x) = \int_{x-2-s}^{-s} K(x-y)u(y)dy + \int_{x+2}^{x+2+s} K(x-y)u(y)dy$$

(56)

The convolution of a symmetric and a monotone function with negative function is symmetric and negative. We have that the first integral is a positive decreasing function and the second is a negative decreasing function. Moreover, from the symmetry of $u$ we have that the sum of the integrals is symmetric which completes the proof.

Theorem 6.7. Let $\Omega = \mathbb{R}$. If the convolution kernel is negative and $\int K \leq -2$ then there exists a nonconstant periodic solution of (47).

Proof. Let $B$ be an invariant convex set and denote

$$C_{per}(\mathbb{R}) = \{ u \in C(\mathbb{R}) , \quad u(x + 4 + 2s) = u(x) \}.$$ 

The operator $T : C_{per}(\mathbb{R}) \to C_{per}(\mathbb{R})$ is compact, thus $f(T)$ is compact. Since $B$ is a bounded and closed subset of image of $f(T)$ we have that $B$ is compact. From lemma 6.6 we have $f(T) : B \to B$. By the Schauder fixed point theorem we obtain that there exists a fixed point in $B$ that is the nonconstant solution in $L^2(\Omega)$ of equation (47).

Numerical simulations indicate that stationary solutions should be stable but this topic requires further investigation.
7. Numerical simulations. In this section we present numerical simulation of solution to the following model

\[ u_t = -u + f(Tu) \quad x \in \Omega, \quad t > 0 \]
\[ u_0 = u(0,x), \quad x \in \Omega. \]

(57)

We assume that degradation rate is equal to 1, \( f \) is odd and monotone. We use the explicit Euler method with fixed step time \( dt \)

\[ u_{n+1} = u_n + dt(-au_n + f(Tu_n)). \]

(58)

The operator \( T \) is a convolution operator with a kernel \( K \) defined at a finite set of points, namely \( \{k_i\}_{i=-N}^N \) in one dimension, or \( \{k_{i,j}\}_{i,j=-N}^N \) in two dimensions. Let \( \tilde{u}_n \) be the extension of \( u_n \) with 0 outside of \( \Omega \). We have

\[ Tu_n(x) = \sum_{i,j=-N}^N K(k_{i,j})\tilde{u}_n(x - k_{i,j}). \]

(59)

The function \( f \) is either a linear function \( f(x) = bx \) or a saturation function

\[ f(x) = \begin{cases} -1, & bx < -1, \\ bx, & -1 \leq bx \leq 1, \\ 1, & bx > 1, \end{cases} \]

(60)

with an arbitrary constant \( b \).

We present simulation results obtained in one dimension and two dimensions. The simulations in one dimension are used to present the solutions for linear function \( f \) (see section 3) and for Schauder theorem (see section 6). Patterns obtained in linear case are eigenfunctions of operator \( Tu \), and thus is is more convenient to

![Figure 1. Four basic types of one dimensional convolution kernels. Kernel \( K_1 \) corresponding to long range inhibition and local activation concentrated at 0. Kernel \( K_2 \) corresponding to long range inhibition and long range activation. Activating kernel \( K_3 \) and inhibiting kernel \( K_4 \).](image-url)
present them in one dimension. Two dimensional eigenfunctions are very diversified and they cannot be easily represented as two dimensional plot. We analyse different types of obtained solutions and their behaviour near the transition gap. Two dimensional simulations are used to present patterns obtained in the nonlinear model for different kernels and model coefficients. One dimensional numerical simulations are conducted for four basic convolution kernels. The following kernels $K_1$, $K_2$, $K_3$ and $K_4$ are presented on Fig. 1. Each kernel corresponds to one of the fundamental class. First class consists of kernels with positive and negative parts and with positive part concentrated near 0. Those kernels corresponds to the local activation and long range inhibition. Second class includes kernels with positive and negative part as well, but positive part is not in the neighbourhood of 0. We show that, this property can modify the character of obtained patterns. Third class consists kernels with only positive part. The inhibition process is absent. Fourth class includes kernels with only negative part. We assume that there is only the inhibition process. Biologically, kernels from this class are not relevant, since nothing stimulates $u$ to rise but we proved that they still produce certain patterns.

We use the following simulation parameters: Grid size 600 (1-D), 600x600 (2-D), Domain size $[-50, 50]$ (1-D), $[-50, 50]^2$ (2-D), Kernel grid size 49 (1-D), 49x49 (2-D), Kernel size $[-4, 4]$ (1-D), $[-4, 4]^2$ (2-D), Step size - varies, Initial condition - varies.

### 7.1. One dimensional simulations.

#### 7.1.1. Linear case. Simulation results for system (57) with the linear function $f(u) = bu$ are presented in Fig. 2. Each figure shows pattern obtained for kernels $K_1$, $K_2$, $K_3$ and $K_4$. Initial condition is set randomly with a uniform distribution from $-1$ to 1. In order to have stationary solution we set $b = \frac{1}{\lambda_j}$, where $\lambda_j$ is the maximal

![Figure 2](image-url)
eigenvalue. Exact values of $\{\lambda_j\}$ are not known. There are obtained through numerical approximation. From the analysis of the linear system (see section 3) we know that the pattern is a certain eigenfunction whose coefficient is constant, while all other eigenfunctions coefficients converge to zero. This eigenfunction is presented in the Fig. 2. The maximal eigenvalue depends on the shape of the involved kernel. We can observe that for kernels with constant sign, the first eigenvalue is also the maximal eigenvalue and thus pattern are simpler than patterns obtained for kernels $K_1$ and $K_2$. Notice that the value of stationary solution is small and significantly smaller than the initial condition. In linear case the eigenfunction coefficients cannot increase and the value of stationary solution is equal to the expansion of initial condition in the orthonormal basis.

7.1.2. Non-linear case. Fig. 3 represents patterns whose existence was proved with the Schauder fixed point theorem (see Section 6). We present numerical simulations for kernels $K_1$, $K_2$, $K_3$ and $K_4$. The aim of this subsection is to discuss the behaviour of function $u$ in the neighbourhood of transient points. Because in case of kernels $K_1$, $K_2$ and $K_3$ the solution is constant almost everywhere we focused our attention on the neighbourhood of 0. The initial value of $u$ is a step function

![Figure 3](image-url)

**Figure 3.** Pattern formation in the nonlinear problem for operator $T$ given by the convolution (5) with kernels $K_1$, $K_2$, $K_3$ and $K_4$. Function $f$ is a saturation function with $b = 0.8$ ($K_1$), $b = 0.7$ ($K_2$), $b = 0.1$ ($K_3$) and $b = 1.0$ ($K_4$). Initial condition is a step function $(-1$ for $x < 0$ and $1$ for $x > 0$ (see subsection 7.1.2 for a discussion).
equal to $-1$ for $x < 0$ and $1$ for $x > 0$. Obtained solution are not small and their existence is strongly connected with the $f$ saturation property.

In section 6 we introduced particular invariant set $B$ for different types of convolution kernel. We assumed general condition for functions $u$ in the neighbourhood of 0, such that it is monotone and odd. We claimed that the width of the gap between $-1$ and $1$ is equal to the support of the positive part of convolution kernel. From numerical simulations we have that the shape of function $u$ in the gap can be different for different types of kernels, therefore a more accurate construction of $B$ space is complex. Solutions can be locally convex or concave. Moreover, for kernel $K_3$ there exists a neighbourhood of 0 where solution is zero. Notice, that the width of gap between $-1$ and $1$ is at least two times smaller than the assumed value for $B$. For kernel $K_4$ we obtained a periodic solution on $\mathbb{R}$. The width of the gap is negligible.

### 7.2. Two dimensional results.

In this section we present two dimensional results obtained for nonlinear model. We do not include simulations for linear model, since the plots are not as clear as one dimensional, though the mechanism of forming patterns is analogous. We analyse five different kernels in four configurations. It is easy to see, that there exists a particular constant, namely $b_{\text{critical}} = \frac{1}{\lambda}$ such that for any function coefficient $b < b_{\text{critical}}$ each solution for any initial condition converges to 0, therefore there exists no patterns. When $b = b_{\text{critical}}$ we have linear case. To obtain large nonconstant stationary solution we need to ensure that at least one eigenfunction becomes unstable, that means $b > b_{\text{critical}}$. The value of $b$ determines the character of pattern. More unstable eigenfunctions yields to more complex patterns. We present the results for large $b$ and for $b$ slightly bigger than $b_{\text{critical}}$. Secondly, we change initial condition $u_0$. We apply random initial condition uniformly distributed on $[-1, 1]$ and regular initial condition

$$u_0(x) = \begin{cases} 1, & x \in [-4, 4]^2, \\ -1, & x \notin [-4, 4]^2. \end{cases}$$

We call the first condition random $u_0$ and the second condition regular $u_0$. 

The kernel $K_1$. In this subsection we analyse stationary solutions obtained for operator $T$ with convolution kernel $K_1$ shown in the Fig. (4). This kernel is positive near 0, and negative otherwise, what corresponds to the local activation and the long range inhibition. The support of the negative part is set to ensure that the integral over the kernel is relatively small. It implies that the negative part is comparable to positive part what improves regularity of patterns.

Simulations obtained for operator $T$ given by convolution kernel with $K_1$ are presented in Fig. 5. Top left image shows the patterns obtained for random initial condition and large $b$ coefficient. The instability of many eigenfunctions yields the unsymmetrical and complex patterns. Notice that, these patterns are in some sense regular. An analysis of the white areas indicates, that width and distance of white areas depend on the support of the convolution kernel. In this case, the width and distance is equal to the half of the positive kernel support. Top right image represents the pattern obtained for regular initial condition and large $b$. As previously we have many unstable eigenfunction, however since the initial condition is symmetric, then all unstable eigenfunctions are symmetric and thus this pattern is symmetric. We obtain that the evolution of the pattern starts from the middle and then spreads across the domain. In the evolution process we can obtain various shapes, like X shaped patterns in the corners. They results from the property that the distance between white areas is fixed, and thus split paths cannot come close again. Bottom row represents the simulations for small $b$ coefficient, such that a few eigenfunctions become unstable and thus patterns become regular. In case of random $u_0$ we obtain patterns with long white lines, that are similar to the truncation of corresponding unstable eigenfunctions. In case of regular $u_0$, we have white rings in the middle of the domain. Near the boundary, circles become deformed, since we do not integrate over the whole kernel domain.

Figure 4. Convolution kernel $K_1$. The kernel corresponds to the local activation and long range inhibition. Activation part is concentrated at 0 and the activation part is relatively similar to the inhibition part.
Figure 5. Patterns in the nonlinear problem for operator \( T \) given by the convolution (5) with kernel \( K_1 \). Function \( f \) is a saturation function with \( b = 1 \) (upper row) and \( b = 0.0063 \) (lower row). Initial condition is random (left column) or a positive square in the middle of the domain (right column) (see subsection 7.2.1 for a discussion).
7.2.2. The kernel $K_2$. In this subsection we analyse stationary solutions obtained for operator $T$ with convolution kernel $K_2$ shown in the Fig. (6). This kernel is positive near 0 and negative otherwise, what corresponds to the local activation and long range inhibition. The support of the negative part is set to to ensure that the integral over the kernel is relatively large. It implies that the negative part is negligible compared to the positive part what disturbs the regularity of patterns.

Simulations performed for operator $T$ given by convolution kernel with $K_1$ are presented in Fig. 7. Top left image shows the patterns obtained for random initial condition and large $b$ coefficient. Notice that, the pattern is irregular. The width and distance between white areas is no longer preserved. The distance and width are no longer determined by the convolution kernel and can be arbitrary. Since negative part is small, large white and black areas are invariant for the operator $f(T)$ and thus we cannot expect any regularisation property. The shape of obtained pattern depends strongly on the initial condition. This phenomenon is exhibited in the top right figure. We set initial condition as the characteristic function of a square in the middle of the domain. Large positive kernel is insufficient to spread the positive solution across the domain, but it can preserve the existing one if they are eligibly large. We also observe the smoothing property of convolution operator, since the square become a circle.

Bottom row represents simulations for small $b$ coefficient, such that a few eigenfunctions become unstable. We set $b = 0.007$ approximately since exact eigenvalues are not known. As previously, patterns become regular. In the case of a random $u_0$ we obtain patterns with long white lines, that are similar to the truncation of corresponding unstable eigenfunctions. In case of regular $u_0$ we can observe an interesting phenomenon. For large $b$ the pattern is concentrated near the initial condition but for small $b$ the pattern covers the whole domain. The crucial difference is that for small $b$ the large white area is no longer $f(T)$ invariant. We see that, the initial white square vanishes. It follows that, for small $b$ we get the regularisation property and thus the pattern become governed by the unstable eigenfunctions.

Figure 6. Convolution kernel $K_2$. The kernel corresponds to the local activation and long range inhibition. Activation part is concentrated at 0 and the activation part is significantly larger than the inhibition part.
Figure 7. Patterns in the nonlinear problem for the operator $T$ given by the convolution $(5)$ with the kernel $K_2$. Function $f$ is a saturation function with $b = 1$ (upper row) and $b = 0.0070$ (lower row). Initial condition is random (left column) or a positive square in the middle of the domain (right column) (see subsection 7.2.2 for a discussion).
7.2.3. *The kernel* $K_3$. In this subsection we analyse stationary solutions obtained for operator $T$ with the convolution kernel $K_3$, shown in the Fig. (8). This kernel is zero near 0, positive at some distance from 0, and negative otherwise. This corresponds to the long range activation and long range inhibition. The support of the negative part is set to to ensure that the integral over the kernel is relatively small. It implies that the negative and the positive part are comparable, what ensures the regularity of patterns.

Simulations obtained for operator $T$ given by convolution kernel with $K_1$ are presented in Fig. 9. Top left image shows the patterns obtained for random initial condition and large $b$ coefficient. We get pattern, with general shape determined by the support of the convolution kernel. The width and distance between white paths are fixed and equal to the half of the positive kernel support. Notice, that locally the solution is distorted with a set of small humps distributed across the white paths. The humps result from the shape of the kernel, namely zero value in the neighbourhood of 0. The size of humps depends on the width of the gap between positive parts of the kernel, however we are not able to provide exact values. The location of humps is determined by the initial condition. Top right image represents pattern obtained for regular $u_0$. We obtain a X-shaped pattern. The evolution of the pattern starts near the initial square edges and spreads. The white paths are straight, the direction of the evolution does not change and the paths do not split. It is the result of lack of the local interaction. Secondly, the pattern does not cover the whole domain, but stops near the domain boundary. White path reaches the boundary exactly at square edge. Notice that, the white paths are distorted with uniformly distributed humps. Since there are no other paths in the neighbourhood, the humps are not smoothed.

The bottom row represents the simulations for small $b$ coefficient, such that a few eigenfunctions become unstable. We set $b = 0.0127$ approximately, since exact eigenvalues are not known. Patterns become regular but they are significantly different than those for large $b$. The white lines are thin. It follows that the maximal eigenvalue corresponds to the high order eigenfunction. Patterns for both cases, regular and random $u_0$ are similar. We obtain long regular white lines.

**Figure 8.** Convolution kernel $K_3$. The kernel corresponds to the local activation and the long range inhibition. Activation part is not concentrated at 0, there is no local activation and the activation part is relatively similar to the inhibition part.
Figure 9. Patterns in the nonlinear problem for operator $T$ given by the convolution \((5)\) with the kernel $K_3$. Function $f$ is a saturation function with $b = 1$ (upper row) and $b = 0.0127$ (lower row). Initial condition is random (left column) or a positive square in the middle of the domain (right column) (see subsection 7.2.3 for a discussion).
7.2.4. The kernels $K_4$ and $K_5$. In this subsection we analyse stationary solutions obtained for the operator $T$ with the convolution kernels $K_4$ and $K_5$, shown in the Fig. (10). We analyse both kernels in one subsection, since the mechanism of pattern formation is similar. Main feature of kernels $K_4$ and $K_5$ is that the integrals over the kernels are large and negative, and patterns are determined by the inhibition process.

Simulations obtained for operator $T$ given by the convolution kernel with $K_4$ and $K_5$ are presented in Fig. 11 and Fig. 11 respectively. Top left images shows the patterns obtained for random initial condition and a large $b$ coefficient. There are many unstable eigenfunctions, however the pattern seems to be regular. We have many parallel white paths. The patterns are determined with a few eigenfunctions, even for large $b$. There are relatively few eigenvalues in the neighbourhood of the maximal eigenvalue. Eigenfunction corresponding to those eigenvalues diverge faster, and thus the impact of the remaining unstable eigenfunctions is negligible. For small $b$ we improve the regularity even more. The width and the distance of white paths differs for patterns obtained for both kernels. Width of path for kernel $K_5$ is equal to half of the distance between the negative parts. Paths obtained for kernel $K_4$ are thin. This is a result of the local inhibition. The exact value of the path width cannot be easily obtained just by analysing the kernel support.

Simulations obtained for regular $u_0$ exhibits an interesting phenomenon. For large $b$ we obtain regular symmetric pattern on a domain, however the evolution does not start from the initial square. For large and negative value of integral over kernel, the constant solutions become unstable. This forms slightly perturbed rings around the initial square. The mechanism of forming rings seems to be similar as the mechanism of pattern formation for negative kernels (see theorem 6.7), but we are unable to prove it. For small $b$ the the integral over kernel is insufficiently large to ensure that rings are formed across the whole domain, thus we see rings in the neighbourhood of initial square, and the unstable eigenfunction near the boundary.

![Figure 10. Convolution kernels $K_4$ and $K_5$. The kernel $K_4$ corresponds to the local inhibition and long range activation. The inhibition part is concentrated at 0, and the activation part is significantly smaller than the inhibition part. The kernel $K_4$ corresponds to the long range inhibition, no activation, and there is no inhibition in the neighbourhood of 0.](image-url)
Figure 11. Patterns in the nonlinear problem for the operator $T$ given by the convolution (5) with kernel $K_4$. The function $f$ is a saturation function with $b = 1$ (upper row) and $b = 0.0352$ (lower row). Initial condition is random (left column) or a positive square in the middle of the domain (right column) (see subsection 7.2.4 for a discussion).
Figure 12. Patterns in the nonlinear problem for operator $T$ given by the convolution (5) with kernel $K$. Function $f$ is a saturation function with $b = 1$ (upper row) and $b = 0.0040$ (lower row). Initial condition is random (left column) or a positive square in the middle of the domain (right column) (see subsection 7.2.4 for a discussion).
8. Conclusion. In this work, we analysed the modified model based on the biological model proposed by Shigeru Kondo. We attempted to explain the process of patterns formation. In the linear case we proved that the stable stationary solution is a linear combination of eigenfunctions and the values of eigenfunctions are determined by the initial condition.

In the nonlinear case we proved the existence of small nonconstant stationary solutions using the Rabinowitz Bifurcation Theorem. We have obtained, that under specific assumptions for derivative of $f$ there may exist a set of coefficients $d_n$ such that bifurcation occurs and nonconstant stationary solution exists. However, proposed theorem is insufficient to decide whether there exists a stationary solution for a given $d$. We obtain only the discontinuous branch of bifurcation points. To obtain the complete theorem for any $d$ we would need to recall the bifurcation theorem with a continuous branch of bifurcation points. Notice that, saturation function used in numerical simulations does not satisfy the $C^2$ assumption. To tackle with this problem, one can approximate saturation function with appropriate smooth function. Notice that we did not use the fact that the dimension is equal to 1 or 2. All of those results hold in any dimension.

In the nonlinear one dimensional case, we proved the existence of nonconstant stationary solutions using the Schauder fixed point theorem. We obtained that under specific assumptions for the kernel $K$ there exists invariant subspace of the operator $T$. Using this method we can obtain diversified patterns, whenever the size of the transient gap is sufficiently large. Results obtained in one dimensional case can be employed to obtain pattern in two dimensional case, but this requires more complex assumptions for convolution kernel. Numerical simulations indicate that the type of the convolution kernel strongly influences the shape of the transient gap, however this topic requires further investigation.

Numerical simulations demonstrates existence of two different mechanism of forming stationary solutions. The first one, that corresponds to linear function and bifurcation theory depends on the vanishing of eigenfunctions. When model coefficients fulfil particular conditions, one eigenfunction does not converge to 0 and it forms a stationary solution. The second mechanism, strongly involve the saturation property of $f$ function. For particular value of $b$ there exists a set of eigenfunctions, that are unstable. As time passes, those eigenfunctions are growing exponentially, until first eigenfunction reaches saturation level. Then the solution concentrates near the values of $u_{max}$ and $u_{min}$ and subsequently stabilizes. Those patterns can be diversified, dependently on initial condition and model coefficients. Moreover, those solutions match patterns obtained in biological models.

Numerical analysis of kernels $K_1 - K_5$ in two dimensional case, represents four different mechanisms of pattern formation. When the integral of kernel $K$ is positive and large, we obtain patterns with large white or black areas. Those areas are invariant for operator $f(T)$, thus we do not observe the regularisation property. Moreover, the white area does not spread across the domain. When the integral of $K$ is positive and relatively small, then we obtain solutions with a fixed width of white and black paths. White area can spread across the domain and the solution covers the whole domain $\Omega$. When the integral of kernel $K$ is negative, the constant function is no longer the stationary solution. The evolution starts on the whole domain simultaneously. These solutions are regular even for large values of derivative of function $f$. 
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