About a theorem of Wiener on the Bessel-Kingman Hypergroup

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A theorem of Wiener on the circle group was strengthened and extended by Fournier in [2] to locally compact abelian groups and extended further to the Bessel-Kingman hypergroup with parameter $\alpha = \frac{1}{2}$ by Bloom/Fournier/Leinert in [1]. We further extend this theorem to Bessel-Kingman hypergroups with parameter $\alpha > \frac{1}{2}$.

1 The Bessel-Kingman Hypergroup

In this paper we will prove a theorem of Fournier (4The Bessel-Kingman Hypergroup-pro.4) on the Bessel-Kingman Hypergroup with parameter $\alpha \geq \frac{1}{2}$. We will use the proof in [1] for the case $\alpha = \frac{1}{2}$ as a guideline, which will be altered where necessary. 4The Bessel-Kingman Hypergrouppro.4 is based upon a theorem of Wiener but treats a more general case. Following [3] and [1] we define:

Definition 1:
The Bessel-Kingman Hypergroup with parameter $\alpha$ is defined as $K = (\mathbb{R}_+, \ast_\alpha)$, where

$$\varepsilon_x \ast_\alpha \varepsilon_y(f) = \int_{|x-y|}^{x+y} K_\alpha(x, y, z) f(z) z^{2\alpha+1} \, dz$$

with

$$K_\alpha(x, y, z) = \frac{\Gamma(\alpha + 1)}{\Gamma(\frac{1}{2}) \Gamma(\alpha + \frac{1}{2}) 2^{2\alpha-1}} \frac{[(z^2 - (x - y)^2)((x + y)^2 - z^2)]^{\alpha-\frac{1}{2}}}{(xyz)^{2\alpha}}$$

and identity involution $(x^- = x)$. The Haar measure $\omega_\alpha(dz)$ is of the form $\omega_\alpha(dz) = z^{2\alpha+1}dz$. The characters are given by $\chi_\lambda(x) := j_\alpha(\lambda x), \ x \in \mathbb{R}_+$ where $j_\alpha$ denotes the modified Bessel function of order $\alpha$:

$$j_\alpha(x) := \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha + 1)}{2^{2k} k! \Gamma(\alpha + k + 1)} x^{2k}, \ \forall x \in \mathbb{R}.$$
One has $\chi_0 \equiv 1$. Furthermore $K$ is a Pontryagin hypergroup. In fact $K \cong K^\wedge$, where the isomorphism is given by $\lambda \mapsto \chi_\lambda$. We note that $(\mathbb{R}_+, \ast_n)$ is commutative because $K(x, y, z) = K(y, x, z)$ and $\int_{|x-y|} f_{x+y} \cdots = f_{|y-x|} \cdots$.

As a convention, we denote

- $I_n := [n-1, n)$,
- $C_\Gamma := \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2) \Gamma(\alpha+1/2) 2^{2\alpha-1}}$,
- $\omega_n := \omega_n(I_n)$.

Furthermore let $\alpha \geq 1/2$. The case $-1/2 < \alpha < 1/2$ will not be treated.

**Definition 2:**
For a hypergroup $(\mathbb{R}_+, \ast)$ with Haar measure $\omega$, the discrete amalgam norm is given by

$$\|f\|_{p,q} := \left( \sum_{n=1}^{\infty} \omega_n \left( \frac{1}{\omega_n} \int_{I_n} |f|^p \, d\omega \right)^{q/p} \right)^{1/q}.$$  

In the case $p$ or $q$ equal to $\infty$ we set by convention

$$\|f\|_{\infty, q} := \left( \sum_{n=1}^{\infty} \omega_n \sup_{x \in I_n} |f(x)|^q \right)^{1/q},$$

$$\|f\|_{p, \infty} := \sup_{n \in \mathbb{N}} \left( \frac{1}{\omega_n} \int_{I_n} |f|^p \, d\omega \right)^{1/p} \quad \text{and}$$

$$\|f\|_{\infty, \infty} := \sup_{n \in \mathbb{N}} \left( \sup_{x \in I_n} |f(x)| \right) = \|f\|_{\infty}.$$  

The function spaces $\{f \text{ measurable} \mid \|f\|_{p,q} < \infty\}$ will be denoted as $(L^p, \ell^q)(\mathbb{R}_+, \ast)$. For these spaces the following properties hold:

$$\|f\|_{p_1,q} \leq \|f\|_{p_2,q}, \quad \text{if } p_1 \leq p_2.$$  

$$\|f\|_{p,q_1} \leq C \|f\|_{p,q_2}, \quad \text{if } q_1 \geq q_2.$$  

particularly, it holds for $p_1 \leq p_2$ and $q_1 \geq q_2$

$$(L^{p_2}, \ell^{q_2})(\mathbb{R}_+, \ast) \subset (L^{p_1}, \ell^{q_1})(\mathbb{R}_+, \ast)$$

and

$$(L^p, \ell^q)(\mathbb{R}_+, \ast) \subset L^p(\mathbb{R}_+, \ast) \cap L^q(\mathbb{R}_+, \ast) \quad \text{for } p \geq q,$$

$$L^p(\mathbb{R}_+, \ast) \cup L^q(\mathbb{R}_+, \ast) \subset (L^p, \ell^q)(\mathbb{R}_+, \ast) \quad \text{for } p \leq q.$$
Definition 3:
Because we are situated on a hypergroup, we can also form amalgam spaces by shifting the unit interval $I_1$ using the left-translation $\tau_y$ defined as

$$
\tau_y f(x) = f(y *_\alpha x) = \int_{|x-y|}^{x+y} K_\alpha(x, y, z) f(z) z^{2\alpha+1} dz.
$$

For the Bessel-Kingman hypergroup $(\mathbb{R}_+, *_\alpha)$ the continuous $(p, \infty)$-amalgam norm is given by

$$
\sup_{y \in \mathbb{R}_+} \left( \int |f|^{p} \tau_y 1_{[0,1)} \, d\omega \right)^{1/p}.
$$

Now we are able to state our more general version of the theorem of Fournier [2, Theorem 3.1]. The proof of this theorem is the main part of this paper. As said before it is closely related to a theorem of Wiener which implies that an integrable function with non-negative Fourier transform on the unit circle which is square integrable on a unit neighborhood is already square integrable on the whole circle. In [2] this theorem was extended to $\mathbb{R}$ and to LCA-Groups.

Theorem 4 (Theorem of Fournier):
For $f \in L^1(\mathbb{R}_+, *_\alpha)$ with $\hat{f} \geq 0$ the following statements are equivalent:

i) $f$ is square integrable on a neighborhood of 0.

ii) $\hat{f} \in (L^1, \ell^2)(\mathbb{R}_+, *_\alpha)$.

iii) $f \in (L^2, \ell^\infty)(\mathbb{R}_+, *_\alpha)$.

To prove this we follow [1]. So we have to check the following properties of $(\mathbb{R}_+, *_\alpha)$ with $\alpha \geq 1/2$.

Equivalence of the discrete and the continuous amalgam norms. We show that the norms defined in [2]The Bessel-Kingman Hypergroup[2] and [3]The Bessel-Kingman Hypergroup[3] are equivalent.

Uniform boundedness of the translation operator on $(L^p, \ell^q)(\mathbb{R}_+, *_\alpha)$. We show that

$$
\|\tau_y f\|_{p,q} \leq C \cdot \|f\|_{p,q}, \quad \forall y \in \mathbb{R}_+
$$

with a constant $C$ independent of $y$. For this we will first prove the uniform boundedness on $(L^\infty, \ell^1)$. Then by invoking duality and interpolation arguments we get the general case.
Properties of the convolution. For the convolution a generalized version of the Young inequality on amalgams must hold:

**Theorem 5 (Young inequality on amalgams):**
For \( f_1 \in (L^{p_1}, \ell^{q_1}) \) and \( f_2 \in (L^{p_2}, \ell^{q_2}) \), where
\[
\left( \frac{1}{p}, \frac{1}{q} \right) = \left( \frac{1}{p_1}, \frac{1}{q_1} \right) + \left( \frac{1}{p_2}, \frac{1}{q_2} \right) - (1, 1)
\]
we have \( f_1 \ast f_2 \in (L^{p}, \ell^{q}) \) and
\[
\|f_1 \ast f_2\|_{p,q} \leq C \cdot \|f_1\|_{p_1,q_1} \cdot \|f_2\|_{p_2,q_2}.
\]

Properties of the Fourier transformation. For the Fourier transform of a function \( f \) we need a more generalized form of the Hausdorff-Young theorem:

**Theorem 6 (Hausdorff-Young theorem on amalgams):**
For \( f \in (L^{p}, \ell^{q})(\mathbb{R}^+, \ast_\alpha) \) with \( 1 \leq p, q \leq 2 \) it holds, that \( \hat{f} \in (L^{q'}, \ell^{p'})(\mathbb{R}^+, \ast_\alpha) \).

Here too, we will consider the special cases \((p, q) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}\). The result then follows with interpolation arguments.

1.1 Equivalence of the discrete and the continuous amalgam norms

**Proposition 7:**
We have
\[
\|f\|_{p,\infty} \leq C \sup_{y \in \mathbb{R}^+} \left( \int |f|^p \tau_y 1_{[0,1]} \ d\omega_\alpha \right)^{1/p}.
\]

**Proof:** Let us concentrate on the term \( \tau_y 1_{[0,1]} \):
\[
\tau_y 1_{[0,1]}(x) = \frac{C_\tau}{(xy)^{2\alpha}} \int_{|x-y|}^{x+y} 1_{[0,1]}(z) \left[ (z^2 - (x-y)^2) (x+y)^2 - z^2 \right]^{\alpha-1/2} dz.
\]
Like in [1], it will be sufficient to look at the supremum in (1.20) taken only over all \( y \) of the form \( y = n + 1/2, \ 1 \leq n \in \mathbb{Z} \). We therefore consider \( \tau_{n+1/4} 1_{[0,1]}(x) \) for \( x \in I_{n+1} \).

The domain of the integration is
\[
[x-y, x+y] \cap [0, 1] = [x-n-1/2, x+n+1/2] \cap [0, 1] = [x-n-1/2, 1] \supset [1/2, 1] \quad \text{for } x \in I_{n+1}.
\]
By using $x^{2a} = x \cdot (x^2)^{a-1/2}$ together with $|x - n - 1/2| \leq 1/2$ we get

$$\frac{C \cdot n^{2a-1}}{(n + 1)(n + 1/2)^{2a}} \cdot \int_{1/2}^{1} z \left[ \left( z^2 - (x - n - 1/2)^2 \right) \left( (x + n + 1/2)^2 - z^2 \right) \right]^{a-1/2} dz$$

$$\geq \frac{C \cdot n^{2a-1}}{(n + 1)(n + 1/2)^{2a}} \cdot \int_{1/2}^{1} \frac{1}{x} \left[ \frac{2}{x} \right]^{a-1/2} dz.$$ 

(\ast) = 1 + \frac{2(n + 1/2)}{x} + \frac{(n + 1/2)^2 - z^2}{x^2} and $\frac{1}{x}$ are decreasing in $x$.

Therefore it follows that $\tau_{n+1/2}(x) \geq \tau_{n+1/2}(n + 1) \forall x \in I_{n+1}$. Further we will show that

$$\tau_{n+1/2}(n + 1) \geq C \cdot \frac{1}{n^{2a+1}}.$$ 

$$\tau_{n+1/2}(n + 1) = \frac{C \cdot n^{2a-1}}{(n + 1)(n + 1/2)^{2a}} \cdot \int_{1/2}^{1} \left[ z^2 - \frac{1}{4} \left( 2n + \frac{3}{2} \right) \right]^{a-1/2} dz$$

(sor by powers of $n$)

$$= \frac{C \cdot n^{2a-1}}{(n + 1)(n + 1/2)^{2a}} \cdot \int_{1/2}^{1} \left[ n^2 z^2 - 1 + n \left( 6z^2 - \frac{3}{2} \right) + \left( \frac{10}{4} z^2 - \frac{3}{2} - \frac{9}{16} \right) \right]^{a-1/2} dz$$

(place $n^2$ outside the brackets)

$$= \frac{C \cdot n^{2a-1}}{(n + 1)(n + 1/2)^{2a}} \cdot \int_{1/2}^{1} \left[ z(4z^2 - 1) + n^{-1} \left( 6z^2 - \frac{3}{2} \right) + n^{-2} \left( \frac{10}{4} z^2 - \frac{3}{2} - \frac{9}{16} \right) \right]^{a-1/2} dz$$

$$\geq \frac{C \cdot n^{2a-1}}{(n + 1)(n + 1/2)^{2a}} \cdot \int_{1/2}^{1} z(4z^2 - 1)^{a-1/2} dz = C' \cdot \frac{n^{2a-1}}{(n + 1)(n + 1/2)^{2a}}$$

$$\geq C' \cdot \frac{n^{2a-1}}{(n + (n + 1))^{2a}} = C' \cdot \frac{n^{2a-1}}{(4n^2)^{2a}} = C \cdot \frac{1}{n^{2a+1}}.$$ 

Above we used that both $f_1(z) := \left( 6z^2 - \frac{3}{2} \right)$ and $f_2(z) := \left( \frac{10}{4} z^2 - \frac{3}{2} - \frac{9}{16} \right)$ are greater than 0 on the interval $[1/2, 1]$. 

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Figure 1: $f_2(z)$

\begin{align*}
f_1(z) &\geq 0 \iff 6z^2 - \frac{3}{2} \geq 0 \iff z^2 \geq \frac{1}{4} \iff |z| \geq \frac{1}{2} \\
f_2(z) &\geq 0 \iff \frac{10}{4} z^2 - z^4 - \frac{9}{16} \geq 0 \\
&\iff -\left(z^2 - \frac{5}{4}\right)^2 + 1 \geq 0 \iff 1 \geq |z^2 - \frac{5}{4}| \\
&\iff \frac{3}{2} \geq |z| \geq \frac{1}{2}.
\end{align*}

Further we have

$$\omega_{n+1} = \int_n^{n+1} z^{2\alpha + 1} \, dz \geq n^{2\alpha + 1}.$$  

So

$$\tau_{n+1/2} 1_{[0,1]}(n+1) \geq C \cdot \frac{1}{n^{2\alpha + 1}} \\ \geq C \cdot \frac{1}{\omega_{n+1}} \text{ for } n \geq 1.$$ 

Now we can summarize.

\begin{align*}
\sup_{y \in [1,\infty)} \left( \int |f|^p \tau_y 1_{[0,1]} \, d\omega_\alpha \right)^{1/p} &\geq \sup_{n \geq 1} \left( \int_{n-1/2}^{n+3/2} |f|^p \tau_{n+1/2} 1_{[0,1]} \, d\omega_\alpha \right)^{1/p} \\
&\geq \sup_{n \geq 1} \left( \int_{I_{n+1}} |f|^p \tau_{n+1/2} 1_{[0,1]} \, d\omega_\alpha \right)^{1/p} \\
&\geq \sup_{n \geq 1} \left( \tau_{n+1/2} 1_{[0,1]}(n+1) \int_{I_{n+1}} |f|^p \, d\omega_\alpha \right)^{1/p} \\
&\geq \sup_{n \geq 1} \left( C \frac{1}{\omega_{n+1}} \int_{I_{n+1}} |f|^p \, d\omega_\alpha \right)^{1/p}.
\end{align*}
moreover for \(x, y \in I_1\) it obviously holds that \(\sup_{y \in I_1} \tau_y 1_{[0,1]}(x) \geq \tau_0 1_{[0,1]}(x) = 1\). So

\[
\sup_{y \in \mathbb{R}^+} \left( \int |f|^p \tau_y 1_{[0,1]} \, d\omega_\alpha \right)^{1/p} \geq C \cdot \|f\|_{p,\infty}.
\]

\[\square\]

**Proposition 8:**

*It holds that*

\[
\|f\|_{p,\infty} \geq C \sup_{y \in \mathbb{R}^+} \left( \int |f| \tau_y 1_{[0,1]} \, d\omega_\alpha \right)^{1/p}.
\]

**Proof:**

i) Let \(y \in [0,1)\). Then one has

\[
\tau_y 1_{[0,1]}(x) = \begin{cases} 
1, & \text{if } x + y \leq 1, \\
\leq 1, & \text{if } |x - y| \leq 1 \text{ and } x + y \geq 1, \\
0, & \text{if } |x - y| \geq 1.
\end{cases}
\]

Note that \(\tau_y 1_{[0,1]}(x) = 1_{[0,1]}(x + y) = \varepsilon_x \ast \varepsilon_y(1_{[0,1]})\). Therefore if \(\text{supp}(\varepsilon_x \ast \varepsilon_y) = [|x - y|, x + y] \subset [0,1]\), then \(\varepsilon_x \ast \varepsilon_y(1_{[0,1]}) = 1\) because \(\varepsilon_x \ast \varepsilon_y \in M^1(K)\).

It follows now that \(\tau_y 1_{[0,1]} \leq 1_{[0,2]}\) and therefore

\[
\int |f|^p \tau_y 1_{[0,1]} \, d\omega_\alpha \leq \int |f|^p 1_{[0,1]} \, d\omega_\alpha + \int |f|^p 1_{[1,2]} \, d\omega_\alpha \\
\leq \int \frac{1}{\omega_1} 1_{[0,1]} |f|^p \, d\omega_\alpha + 2^{2\alpha + 1} \int \frac{1}{\omega_2} 1_{[1,2]} |f|^p \, d\omega_\alpha
\]

because \(\omega_n = \int_{n-1}^n 2^{2\alpha + 1} \, dz \leq n^{2\alpha + 1}\), hence \(1 \leq \frac{n^{2\alpha + 1}}{\omega_n}\).

So

\[
\left( \int |f|^p \tau_y 1_{[0,1]} \, d\omega_\alpha \right)^{1/p} \leq \left( \int I_{I_1} \frac{1}{\omega_1} |f|^p \, d\omega_\alpha \right)^{1/p} + 2^{2\alpha + 1} \left( \int I_{I_2} \frac{1}{\omega_2} |f|^p \, d\omega_\alpha \right)^{1/p} \\
\leq (1 + 2^{2\alpha + 1}) \|f\|_{p,\infty}.
\]

ii) If \(y \in [1, 2)\) then it follows, analogously to the calculation above, that \(\tau_y 1_{[0,1]} \leq 1_{[0,3]}\).

\[
\int |f|^p \tau_y 1_{[0,1]} \, d\omega_\alpha \leq \int |f|^p 1_{[0,1]} \, d\omega_\alpha + \int |f|^p 1_{[1,2]} \, d\omega_\alpha + \int |f|^p 1_{[2,3]} \, d\omega_\alpha \\
\leq \int \frac{1}{\omega_1} 1_{[0,1]} |f|^p \, d\omega_\alpha + 2^{2\alpha + 1} \int \frac{1}{\omega_2} 1_{[1,2]} |f|^p \, d\omega_\alpha \\
+ 3^{2\alpha + 1} \int \frac{1}{\omega_3} 1_{[2,3]} |f|^p \, d\omega_\alpha,
\]

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therefore we have
\[
\left( \int |f|^p \tau_y 1_{[0,1]} \, d\omega_k \right)^{1/p} \leq \left( \int 1_{\omega_1} |f|^p \, d\omega_\alpha \right)^{1/p} + 2 \frac{2\alpha+1}{p} \left( \int 1_{\omega_2} |f|^p \, d\omega_\alpha \right)^{1/p} + 3 \frac{2\alpha+1}{p} \left( \int 1_{\omega_3} |f|^p \, d\omega_\alpha \right)^{1/p} \leq (1 + 2 \frac{2\alpha+1}{p} + 3 \frac{2\alpha+1}{p}) \|f\|_{p,\infty}.
\]

iii) Let \( k \geq 2 \). We consider \( y \in I_{k+1} = [k, k+1] \), i.e. \( \text{supp} \, \tau_y 1_{[0,1]} \subset [k-1, k+2] \).
Now it follows
\[
\tau_y 1_{[0,1]}(x) = \frac{C_\Gamma}{x^{2\alpha} y^{2\alpha}} \int_{|x-y|}^1 z \left[ (z^2 - (x-y)^2)((x+y)^2 - z^2) \right]^{\alpha-1/2} \, dz
\]
\[
= \frac{C_\Gamma}{xy} \int_{|x-y|}^1 z \left[ \frac{(z^2 - (x-y)^2)}{x^2 y^2} \right]^{\alpha-1/2} \, dz
\]
\[
\leq \frac{C_\Gamma}{xy} \int_{|x-y|}^1 z \left( \frac{4}{xy} \right)^{\alpha-1/2} \, dz \leq \frac{C_\Gamma \cdot 4^{\alpha-1/2}}{(xy)^{\alpha+1/2}}.
\]
The last inequality is due to \( 1 - |x-y| \leq 1 \) and the standard estimate. By using \( k-1 \leq x < k+2 \) and \( k \leq y < k+1 \), we get
\[
\leq \frac{C_\Gamma 4^{\alpha-1/2}}{(k-1)^{\alpha+1/2} k^{\alpha+1/2}} \leq \frac{C_\Gamma 4^{\alpha-1/2} \left( \frac{k}{k-1} \right)^{\alpha+1/2}}{k^{\alpha+1/2} k^{\alpha+1/2}} \leq \frac{C_\Gamma 4^{\alpha-1/2} 2^{\alpha+1/2}}{k^{2\alpha+1}} \leq \frac{C \cdot 1}{k^{2\alpha+1}}.
\]
We note that
\[
\omega_{k+2} = \int_{k+1}^{k+2} z^{2\alpha+1} \, dz \leq (k + 2)^{2\alpha+1} \leq 2^{2\alpha+1} k^{2\alpha+1}.
\]
Hence we get
\[
\tau_y 1_{[0,1]}(x) \leq \frac{C'}{\omega_{k+2}} \leq \frac{C'}{\omega_{k+1}} \leq \frac{C'}{\omega_k} \quad \text{for} \quad y \in I_k, \, k \geq 2.
\]
Altogether we obtain
\[
\int |f|^p \tau_y 1_{[0,1]} \, d\omega_\alpha \leq C' \cdot \int |f|^p \, d\omega_\alpha \quad (j = k, k + 1, k + 2)
\]
\[
\Rightarrow \left( \int |f|^p \tau_y 1_{[0,1]} \, d\omega_\alpha \right)^{1/p} \leq C'^{1/p} \cdot \sum_{j=k}^{k+2} \left( \int |f|^p \, d\omega_\alpha \right)^{1/p}
\]
\[
\leq 3 \cdot C'^{1/p} \cdot \|f\|_{p,\infty} \quad \square
\]
Choosing \(C''\) as the maximum of the constants in i) - iii) concludes the proof.

1.2 Uniform boundedness of the translation operator on \((L^p, \ell^q)(\mathbb{R}_+, *_g)\)

Proposition 9 (The case \((L^\infty, \ell^1)(\mathbb{R}_+, *_g)\)):
For \(f \in (L^\infty, \ell^1)(\mathbb{R}, *_g)\) and \(y \in \mathbb{R}_+\) it holds that
\[
\|\tau_y f\|_{\infty,1} \leq C \cdot \|f\|_{\infty,1}
\]
with a constant \(C\) independent of \(y\).

Proof: Like in [1] it is enough to consider only functions \(f_n := 1_{I_n} = 1_{[n, n+1]}\). That means we show only that
\[
\|\tau_y f_n\|_{\infty,1} \leq C \cdot \|f_n\|_{\infty,1} = C \cdot \omega_n
\]
with \(C\) independent of \(n\) and \(y\). For the sake of completeness, we repeat the proof for the correctness of our constraint.

Let \(c_n := \|P_n f\|_\infty\) (\(P_n\) denotes the restriction to the interval \(I_n\)) and let \(g = \sum_n c_n f_n\).

\[
\|\tau_y f\|_{\infty,1} \leq \|\tau_y g\|_{\infty,1} \leq \sum_n c_n \|\tau_y f_n\|_{\infty,1}
\]
\[
\leq \sum_n c_n C \|f_n\|_{\infty,1} = C \|f\|_{\infty,1}.
\]

Fix \(y\) and \(n\). We denote a \(k \in \mathbb{Z}_+\) as exceptional if \(k = 1\) or if there exists \(x \in I_k\) so that \(|x - y|\) or \(x + y\) lies in \(I_n\). The set of all exceptional indices will be denoted as \(E\). An index \(k \in \mathbb{Z}_+\) which is not exceptional will be called generic. \(G := \mathbb{Z}_+ \setminus E\).

For \(k \in G\) the intersection of \([|x - y|, x + y]\) and \(I_n\) is either empty or all of \(I_n\) for all \(x \in I_k\). Then \(\tau_y f_n\) either vanishes on all of \(I_k\) or is in the form of
\[
\tau_y f_n(x) = \frac{C_T}{(xy)^{2\alpha}} \int_{n-1}^n z \left( z^2 - (x - y)^2 \right) \left( (x + y)^2 - z^2 \right)^{\alpha - 1/2} \, dz, \quad \text{for} \ x \in I_k.
\]
So it is easy to see that we can claim the following statements about \( x \in \{ \tau_y f_n > 0 \} \) if \( x \in I_k \) and \( k \) generic.

\[
|x - y| < n - 1 < n \leq x + y \quad \Rightarrow \quad x - y < n - 1 \quad \text{and} \quad y - x < n - 1 \quad \text{and} \quad n \leq x + y
\]

\[
\Rightarrow \quad x < n + y - 1 \quad \text{and} \quad y - n + 1 < x \quad \text{and} \quad n - y \leq x.
\]

Taken together it holds that \( \text{sup} \{ 0, y - n + 1, n - y \} \). Equating the two terms (\( \neq 0 \)) of the lower bound yields \( y = n - 1/2 \). Thus we can distinguish two cases:

i) \( y \leq n - 1/2 \) \( \Rightarrow \) \( n - y \leq x < n + y - 1 \)

ii) \( y \geq n - 1/2 \) \( \Rightarrow \) \( y - n + 1 < x < n + y - 1 \).

Using \( \omega_k \leq k^{2\alpha + 1} \) and

\[
\frac{1}{x^{2\alpha}} \leq \frac{1}{(k-1)^{2\alpha}} \leq C \cdot \frac{1}{k^{2\alpha}}
\]

we get for \( x \in I_k \), \( k \geq 2 \)

\[
\frac{1}{\omega_n} \sum_{k \in G} \omega_k \| P_k(\tau_y f_n) \|_\infty
\]

\[
= C \Gamma \cdot \sum_{k \in G} \frac{\omega_k}{\omega_n} \sup_{x \in I_k} \int_{n-1}^{n} \frac{z}{(xy)^{2\alpha}} \left[ (z^2 - (x - y)^2) \left( (x + y)^2 - z^2 \right) \right]^{\alpha - 1/2} dz
\]

\[
\leq C' \cdot \sum_{k \in G} \frac{k^{2\alpha + 1}}{\omega_n y^{2\alpha} k^{2\alpha}} \sup_{x \in I_k} \int_{n-1}^{n} z \left[ (z^2 - (x - y)^2) \left( (x + y)^2 - z^2 \right) \right]^{\alpha - 1/2} dz
\]

\[
\leq C' \cdot \sum_{k \in G} \frac{\omega_n y^{2\alpha}}{\omega_n y^{2\alpha}} \sup_{x \in I_k} \int_{n-1}^{n} z \left[ (z - x + y)(z + x - y)(x + y - z)(x + y + z) \right]^{\alpha - 1/2} dz.
\]

\[
(*)
\]

We consider \( n \) and \( y \) as in the first case and substitute, according to the sign, \( z \) by \( n \) resp. \( n - 1 \) and \( x \) by \( n - y \) resp. \( n + y - 1 \). It follows that

\[
[\text{I}] \leq C' \cdot \sum_{k \in G} \frac{k}{\omega_n y^{2\alpha}} \sup_{x \in I_k} (n - n + y + y)(n + n + y - 1 - y)
\]

\[
(n + y - 1 - n + 1 + y)(n + y - 1 + y + n) \right)^{\alpha - 1/2}
\]

\[
\leq C' \cdot \sum_{k \in G} \frac{k}{\omega_n y^{2\alpha}} n \left[ 2y(2n - 1)2y(2n + 2n - 2) \right]^{\alpha - 1/2} \quad \text{(because} \ 2y \leq 2n - 1)\]

\[
\leq C'' \cdot \sum_{k \in G} \frac{k n y^{2\alpha - 1} y^{2\alpha - 1}}{n^{2\alpha + 1} y^{2\alpha}} = C'' \cdot \sum_{k \in G} \frac{k}{n y}
\]

\[
\leq C'' \cdot \sum_{k \in G} \frac{2}{y} \leq 4C''.
\]

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In the last row it was used that $k \leq n + y - 1 \leq 2n$ and that the sum consists of fewer than $2y$ terms: $n + y - 1 - (n - y) = 2y - 1$.

Now let $y$ and $n$ be as in the second case. Then we can estimate $\odot$ analogously.

\[
\odot \leq \left[(n - y + n - 1 + y)(n + n + y - 1 - y)\right]^{a-1/2} \\
\leq \left[(2n - 1)(2n - 1)2y(2y + 2y)\right]^{a-1/2} \quad \text{(because $2n - 1 \leq 2y$)} \\
\leq C \cdot n^{2a-1} y^{2a-1} \\
\Rightarrow (\star) \leq C'' \cdot \sum_{k \in G} \frac{k n^{2a} y^{2a-1}}{y^{2a} n^{2a+1}} = C'' \cdot \sum_{k \in G} \frac{k}{n y} \\
\leq C'' \cdot \frac{2}{n} \leq 4C''.
\]

Similar to the first case we used here that $k \leq n + y - 1 \leq 2y$ and that the sum consists of fewer than $n + y - 1 - (y - n + 1) = 2n - 2$ terms.

If $k$ is now an exceptional index, we have to estimate $\omega_k \| P_k(\tau_y f_n) \|_\infty$. For exceptional indices we have by definition either $x + y \in I_n$ for $x \in I_k$, i.e. $y + I_k$ intersects the interval $I_n$, or $I_n - y$ intersects $I_k$. There are at most two such indices $k \in E$. The other cases of exceptional indices derive from the cases where $I_n + y$ or $y - I_n$ intersect the interval $I_k$, or where $k = 1$. Each of these cases, with the exception of $k = 1$, yields at most 2 exceptional indices. Thus there are at most 7. By looking closer one can see that there are in fact only 5 of them.

If $k \leq 3n$ then one has, with the use of

\[
\tau_y f_n(x) = \int_{|x-y|}^{x+y} f_n(x) K(x, y, z) \omega_\alpha(dz) \leq \int_{|x-y|}^{x+y} K(x, y, z) \omega_\alpha(dz) = 1
\]

and

\[
\omega_{3n} = \int_{3n-1}^{3n} z^{2\alpha+1} dz = \frac{1}{2\alpha + 2} \left((3n)^{2\alpha+2} - (3n - 1)^{2\alpha+2}\right) \\
= \frac{2^{2\alpha+2}}{2^{2\alpha+2}} \left(n^{2\alpha+2} - (n - \frac{1}{3})^{2\alpha+2}\right) \\
\leq \frac{2^{2\alpha+2}}{2^{2\alpha+2}} \left(n^{2\alpha+2} - (n - 1)^{2\alpha+2}\right) = 2^{2\alpha+2} \int_{n-1}^{n} z^{2\alpha+1} dz \\
= 3^{2\alpha+2} \omega_n,
\]

the following estimate:

\[
\omega_k \| P_k(\tau_y f_n) \|_\infty \leq \omega_k \leq \omega_{3n} \leq 3^{2\alpha+2} \omega_n = 3^{2\alpha+2} \| f_n \|_{\infty,1}. \quad (1.70)
\]
If $k$ is exceptional and $k > 3n$, then one of the intervals $y \pm I_n$ must intersect $I_k$. The smallest value for $y$ must therefore satisfy $y + n = k - 1$. That implies that
\[ y + \frac{1}{3}k > k - 1 \Rightarrow y > \frac{2}{3}k - 1 > \frac{1}{3}k, \]  
(1.71)
because $k > 3$. Particularly we have $y > \frac{1}{3}x$ for all $x \in I_k$ in these cases. Now we can find an upper bound for the remaining exceptional indices.

\[ \tau_y f_n(x) = \frac{C_T}{(xy)^{2\alpha}} \int_{|x-y|}^{n} z \left[ (z^2 - (x - y)^2) \left( (x + y)^2 - z^2 \right) \right]^{\alpha - \frac{1}{2}} \, dz \]
\[ \leq \frac{C_T}{(xy)^{2\alpha}} \left( n - |x - y| \right) n \cdot \left[ (n^2 - (x - y)^2) \frac{4xy}{4} \right]^{\alpha - \frac{1}{2}} \]
\[ \leq \frac{C_T4^{\alpha - \frac{1}{2}}}{(xy)^{\alpha + \frac{1}{2}}} n \left[ n^2 - (x - y)^2 \right]^{\alpha - \frac{1}{2}} \]
\[ \leq \frac{C_T4^{\alpha - \frac{1}{2}}}{\left( \frac{4}{3}x * x \right)^{\alpha + \frac{1}{2}}} n \left( 2n - 1 \right)^{\alpha - \frac{1}{2}} \]
\[ \leq \frac{C_T4^{\alpha - \frac{1}{2}}3^{\alpha + \frac{1}{2}}}{x^{2\alpha + 1}} n^{\alpha + \frac{1}{2}} \]
\[ \leq \frac{C_T4^{\alpha - \frac{1}{2}}3^{\alpha + \frac{1}{2}}}{(k - 1)^{2\alpha + 1}}. \]

Hence we can estimate the remaining terms of the norm.
\[ \omega_k \| P_k(\tau_y f_n) \|_\infty \leq C \cdot k^{2\alpha + 1} n^{\alpha + \frac{1}{2}} \leq C \cdot \left( \frac{4}{3} \right)^{2\alpha + 1} n^{\alpha + \frac{1}{2}} \leq C' \cdot \omega_n = C' \cdot \| f_n \|_{\infty, 1} \]

Using duality and complex interpolation as in [1], the boundedness of translation on $(L^\infty, \ell^1)(\mathbb{R}^+, \ast_\alpha)$ can be extended to the general amalgam spaces $(L^p, \ell^q)(\mathbb{R}^+, \ast_\alpha)$.

### 1.3 Fourier transformation on $(L^p, \ell^q)(\mathbb{R}^+, \ast_\alpha)$

We have yet to prove the following theorem:

**Theorem 10 (Hausdorff-Young theorem for amalgams):**

For $f \in (L^p, \ell^q)(\mathbb{R}^+, \ast_\alpha)$ with $1 \leq p, q \leq 2$ holds, that $\hat{f} \in (L^q', \ell^p')(\mathbb{R}^+, \ast_\alpha)$.

The special case $p = q = 2$ is already known (see Theorem 2.2.22 in [3]). We will consider the other extreme cases $p, q \in \{1, 2\}$ first. Therefore we will have to show that
\[ ||\mathbf{1}_{I_1}||_{\infty,2} < \infty. \]

\[ \mathbf{1}_{I_1}(\lambda) = \int_{\mathbb{R}^+} 1_{I_1}(x) \chi_{\lambda}(x) \, d\omega_\alpha(x) \]

\[ = \int_{0}^{1} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha + 1)}{2^{2k} k! \Gamma(\alpha + k + 1)} (\lambda x)^{2k} \, d\omega_\alpha(x) \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha + 1) \lambda^{2k}}{2^{2k} k! \Gamma(\alpha + k + 1)} \int_{0}^{1} x^{2k} x^{2\alpha + 1} \, dx \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha + 1) \lambda^{2k}}{2^{2k} k! \Gamma(\alpha + k + 1) 2k + 2\alpha + 2} \]

\[ = \frac{\Gamma(\alpha + 1) 2^\alpha}{\lambda^{\alpha + 1}} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k + \alpha + 1}}{2^{2k+\alpha+1} k! \Gamma(\alpha + k + 2)} \]

\[ = \frac{\Gamma(\alpha + 1) 2^\alpha}{\lambda^{\alpha + 1}} J_{\alpha + 1}(\lambda). \]

Here \( J_{\alpha + 1} \) denotes the Bessel function of degree \( \alpha + 1 \). According to 9.2.1 in [4] it holds for \( \lambda \to \infty \) that

\[ J_{\alpha + 1}(\lambda) \approx \sqrt{\frac{2}{\pi \lambda}} \cos(\lambda - \frac{1}{2}(\alpha + 1)\pi - \frac{1}{4}\pi). \]

Altogether for large enough \( \lambda \) we have

\[ \left| \mathbf{1}_{I_1}(\lambda) \right| \approx \left| \frac{\Gamma(\alpha + 1) 2^\alpha}{\lambda^{\alpha + 1}} \sqrt{\frac{2}{\pi \lambda}} \cos(\lambda - \frac{1}{2}(\alpha + 1)\pi - \frac{1}{4}\pi) \right| \leq C \cdot \lambda^{-\alpha - 3/2}. \]

Now we can show that \( ||\mathbf{1}_{I_1}||_{\infty,q} < \infty \Leftrightarrow q > 2 \frac{\alpha + 1}{\alpha + 3/2} \). Particularly for \( q \geq 2 \)

\[ ||\mathbf{1}_{I_1}||_{\infty,q} = \left( \sum_{k=0}^{\infty} \omega_k \sup_{\lambda \in I_k} |\mathbf{1}_{I_1}(\lambda)|^q \right)^{\frac{1}{q}} \]

\[ \leq C \cdot \left( C' + \sum_{k=N_\alpha}^{\infty} k^{2\alpha + 1} q(-\alpha - 3/2) \right)^{\frac{1}{q}} \]

\[ = C'' + C \cdot \left( \sum_{k=N_\alpha}^{\infty} k^{2\alpha + 1 - q(\alpha + 3/2)} \right)^{\frac{1}{q}}, \]

where \( N_\alpha \) and \( C \) be chosen such that \( |\mathbf{1}_{I_1}(\lambda)| \leq C \cdot \lambda^{-\alpha - 3/2} \) for \( \lambda \geq N_\alpha \). The series converges iff \( 2\alpha + 1 - q(\alpha + 3/2) < -1 \Leftrightarrow q > 2 \frac{\alpha + 1}{\alpha + 3/2} \). One can easily see that \( \mathbf{1}_{I_1} \) does not lie in \( (L^p, \ell^q)(\mathbb{R}^+, *_\alpha) \) if \( q \leq 2 \frac{\alpha + 1}{\alpha + 3/2} \) and for all \( p \). Now we proceed again as in
Let \( g_1 = \frac{1}{\omega_1} \cdot 1_{I_1} *_{\alpha} 1_{[0,2]} \) and \( g_n = \frac{1}{\omega_1} \cdot 1_{I_1} *_{\alpha} 1_{[n-2,n+1]} \) for \( n \geq 2 \). We show that \( g_n(x) = 1 \) \( \forall x \in I_n \). For this let \( x \in I_1 \).

\[
g_1(x) = \frac{1}{\omega_1} \int_{\mathbb{R}_+} 1_{I_1}(y) \tau_x 1_{[0,2]}(y) \, d\omega_\alpha(y)
= \frac{1}{\omega_1} \int_0^1 \int_{|x-y|}^{x+y} 1_{[0,2]}(z) K(x,y,z) \, d\omega_\alpha(z) \, d\omega_\alpha(y)
= \frac{1}{\omega_1} \int_0^1 \int_{[|x-y|,x+y] \cap [0,2]} K(x,y,z) \, d\omega_\alpha(z) \, d\omega_\alpha(y)
= \frac{1}{\omega_1} \int_0^1 1 \, d\omega_\alpha(y) = 1,
\]

because for \( x, y \in [0,1] \Rightarrow [|x-y|, x+y] \subset [0,2] \Rightarrow [|x-y|, x+y] \cap [0,2] = [|x-y|, x+y] \) and thus the inner integral yields the value 1. Entirely analog it holds for \( x \in I_n \), that

\[
g_n(x) = \frac{1}{\omega_1} \int_{\mathbb{R}_+} 1_{I_1}(y) \tau_x 1_{[n-2,n+1]}(y) \, d\omega_\alpha(y)
= \frac{1}{\omega_1} \int_0^1 \int_{[|x-y|,x+y] \cap [n-2,n+1]} K(x,y,z) \, d\omega_\alpha(z) \, d\omega_\alpha(y)
= \frac{1}{\omega_1} \int_0^1 1 \, d\omega_\alpha(y) = 1,
\]

because for \( x \in [n-1,n] \) and \( y \in [0,1] \Rightarrow [|x-y|, x+y] \subset [n-2, n+1] \Rightarrow [|x-y|, x+y] \cap [n-2, n+1] = [|x-y|, x+y] \). The rest of the proof runs exactly as in [1], with 3 replaced by \( \frac{1}{\omega_1} \) and \(*_{1/2} \) replaced by \(*_{\alpha} \). Note that Young’s inequality can be obtained as in [1] as well. \( \square \)
2 Equivalence on compact/discrete hypergroups

As an addendum, if $K$ is a compact or discrete Hypergroup, let us note the equivalence of the continuous amalgam norm (here denoted as $\| \cdot \|_{p,\infty}^*$) with the discrete amalgam norm.

We first describe the discrete case. The discrete amalgam norm on a discrete hypergroup is defined as one would expect as

$$\|f\|_{p,q} = \left( \sum_{k \in K} \omega(\{k\}) \left( \frac{1}{\omega(\{k\})} |f(k)|^p \omega(\{k\}) \right)^{q/p} \right)^{1/q} = \left( \sum_{k \in K} \omega(\{k\}) |f(k)|^q \right)^{1/p} = \|f\|_q,$$

For the equivalence of the norms we compute

$$\|f\|_{p,\infty}^* = \sup_{n \in K} \left( \sum_{k \in K} \omega(\{k\}) |f(k)|^p \omega(\{k\}) \right)^{1/p} = \sup_{n \in K} \left( \sum_{k \in K} 1_{\{0\}}(n \cdot k) |f(k)|^p \omega(\{k\}) \right)^{1/p}$$

so, using $0 \in \text{supp} \epsilon_n \Leftrightarrow k = n^-$, we get

$$\|f\|_{p,\infty} = \sup_{n \in K} \left( \epsilon_n * \epsilon_n(\{0\}) |f(n)|^p \omega(\{n\}) \right)^{1/p}$$

and finally, with Theorem 1.3.26 in [3]: $\omega(\{n\}) = \epsilon_n * \epsilon_n(\{0\})^{-1}$, we have

$$\|f\|_{p,\infty} = \sup_{n \in K} |f(n)| = \|f\|_\infty.$$

On the other hand, in the compact case, the discrete amalgam norm shrinks to

$$\left( \frac{1}{\omega(K)} \int_K |f|^p \ d\omega \right)^{q/p} = \left( \int_K |f|^p \ d\omega \right)^{1/p} = \|f\|_p.$$

The equivalence now follows:

$$\|f\|_{p,\infty}^* = \sup_{y \in K} \left( \int \tau_y 1_K |f|^p \ d\omega \right)^{1/p} = \sup_{y \in K} \left( \int \int K 1_K \ d(\epsilon_y * \epsilon_x) |f|^p(x) \ \omega(dx) \right)^{1/p} = \|f\|_p.$$

So, in both cases, we obtained not only equivalence, but equality of norms.
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