Almost Perfect Metals in One Dimension

Chaitanya Murthy\textsuperscript{1} and Chetan Nayak\textsuperscript{1,2}

\textsuperscript{1}Department of Physics, University of California, Santa Barbara, CA 93106, USA
\textsuperscript{2}Microsoft Quantum, Station Q, University of California, Santa Barbara, CA 93106, USA

We show that a one-dimensional quantum wire with as few as 2 channels of interacting fermions can host metallic states of matter that are stable against all perturbations up to $q^\text{th}$-order in fermion creation/annihilation operators for any fixed finite $q$. The leading relevant perturbations are thus complicated operators that are expected to modify the physics only at very low energies, below accessible temperatures. The stability of these non-Fermi liquid fixed points is due to strong interactions between the channels, which can (but need not) be chosen to be purely repulsive. Our results might enable elementary physical realizations of these phases.

Introduction. Metallic states of matter are gapless and often unstable to either insulating behavior or superconductivity. This is especially true in one-dimensional systems, where the localizing effects of disorder are particularly strong [1]. For a single channel (i.e. a single propagating mode of each chirality at the Fermi energy), disorder-induced localization can only be avoided when the interaction is strongly attractive, while proximity-induced superconductivity can only be avoided when it is strongly repulsive [2]. The situation is more complicated—and much more interesting—when there are multiple channels. We will show that, surprisingly, even for $N = 2$ channels, it is possible to have a metallic state that is stable against all perturbations up to $q^\text{th}$-order in fermion creation/annihilation operators for any fixed finite $q$, which we call (absolute) $q$-stability. (Note, however, that the corresponding region of the phase diagram shrinks as $q$ increases, and it is unclear whether one can take $q \to \infty$ when $N < 23$).

Gapless phases of interacting fermions in one dimension are described at low energies by Luttinger liquid (LL) theory [3]. They exhibit a remarkable and universal phenomenology that distinguishes them from Fermi liquids, but this phenomenology is often obscured in experiments due to dimensional crossover, ordering, or localization [2]. Thus, a physically realizable stable LL is not only interesting as a matter of principle, but also for the practical reason that it would provide a useful experimental platform to study non-Fermi liquid physics.

For $N = \infty$, it was shown two decades ago in Refs. [4–10] that there exist “sliding Luttinger liquid” phases which are stable against many, but not all, low-order perturbations; it was argued that the relevant ones are likely to have small bare values [11]. More recently, it was discovered that it is possible for a one-dimensional metal to be stable against all non-chiral perturbations (without restriction on the order) [12]. An explicit construction was given for $N = 23$, which demonstrated that such a metallic state is stable against all $m^\text{th}$-order perturbations for all $m \leq q$.

Model and Definitions. Consider a system of interacting fermions in a 1D quantum wire. In the absence of disorder, quasi-momentum along the wire is a good quantum number, and the system is characterized by its band structure. The band index distinguishes different bulk bands of the material, as well as different sub-bands arising from confinement in the transverse direction, and includes spin (as it must if spin-orbit coupling is important). At low energies, the relevant degrees of freedom are those associated with the vicinity of each Fermi point in a partially filled band. If there are $2N$ Fermi points $k_{F,i}$ with associated Fermi velocities $v_f$ (the first $N$ of which are positive and the second $N$ negative), then the effective theory of the system involves $2N$ chiral spinless

The basic observation underlying the results of this paper and of Ref. [12] is that the possible perturbations of an $N$-channel LL can be represented as lattice points in a fictitious $2N$-dimensional space with two different metrics: the mixed-signature $(N, N)$ metric $\text{diag}(I_N, -I_N)$ and the Euclidean metric $\mathbb{I}_{2N}$, where $\mathbb{I}_N$ is the $N \times N$ identity matrix. The mixed-signature interval from the origin to a lattice point measures the chirality of the associated perturbation, while the Euclidean interval measures its scaling dimension; points sufficiently far from the origin are irrelevant in the renormalization group (RG) sense. The lattice is naturally graded into “shells” consisting of perturbations of a given order; low-order perturbations belong to the inner shells. The effect of interactions is to deform the lattice by an $SO(N, N)$ transformation [13]. For $N = 1$, the deformation is a Lorentz boost that is “aligned” with the lattice; such a boost unavoidably pulls one of the innermost lattice points closer to the origin, enhancing the susceptibility of the system to either localization or induced superconductivity. For $N \geq 2$, on the other hand, the boosts can be “misaligned” with the lattice planes in such a way that all lattice points in the innermost $q$ shells are pushed away from the origin, making the corresponding perturbations irrelevant.

Remarkably, these absolutely $q$-stable phases can occur even for purely repulsive interactions. Two-channel repulsive LLs can occur in a number of different contexts. One simple example, with sufficient generality to permit the phases described here, is a single-spinful-channel quantum wire with spin-orbit coupling and Zeeman field parallel to the spin-orbit field. In this case, the four Fermi points (two right-moving and two left-moving) have different Fermi momenta and velocities. Moreover, the interactions between the densities at the different Fermi points are not bound by any symmetries and can all be different. Our construction shows that, for any fixed finite $q$, there exist purely repulsive local interactions for which such a metallic state is stable against all $m^\text{th}$-order perturbations for all $m \leq q$. 
Dirac fermions $\psi_I$, where $\psi_I^\dagger (\psi_I^\dagger + N)$ creates a right-moving (left-moving) fermion excitation about $k_{F,I} (k_{F,I} + N)$. The effective action is given by $S_{\text{eff}} = S_0 + S_{\text{int}} + S_{\text{pert}}$, where

$$S_0 + S_{\text{int}} = \int dt dx \left[ \psi_I^\dagger i (\partial_t + v_I \partial_x) \psi_I + U_{IJ} \psi_I^\dagger \psi_I \psi_J^\dagger \psi_J \right].$$

The indices $I, J$ are both summed from 1 to $2N$, and the real symmetric $2N \times 2N$ matrix $U$ parameterizes all density-density interactions. All other interaction terms, as well as any quadratic terms accounting for dispersion nonlinearities, are packaged into $S_{\text{pert}}$. If the system is perturbed in any way, for instance by introducing disorder or by proximity-coupling the wire to an external 3D superconductor, the appropriate terms are also included in $S_{\text{pert}}$.

The first part of the action, $S = S_0 + S_{\text{int}}$, describes a gapless $N$-channel Luttinger liquid. $S_0$ can be treated non-perturbatively via the method of bosonization [14]. Introducing a chiral boson $\phi_I (\phi_I + N)$ for each chiral fermion $\psi_I (\psi_I + N)$, we obtain the bosonic representation

$$S = \frac{1}{4\pi} \int dt dx \left[ K_{IJ} \partial_t \phi_I \partial_x \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J \right]$$

with $K = I_N \otimes -I_N$ and $V_{IJ} = v_I \delta_{IJ} + \frac{1}{2} U_{IJ}$. The fermion operators are given in terms of the bosons by $\psi_I^\dagger = \frac{1}{\sqrt{2\pi}} e^{i\phi_I} \gamma_I$ and $\psi_I = \frac{1}{\sqrt{2\pi}} e^{-i\phi_I} \gamma_I$, where $a$ is a short-distance cutoff [15], and the Klein factors $\gamma_I$ satisfy $\gamma_I \gamma_J = -\gamma_J \gamma_I$ for $I \neq J$.

The LL action (2) is a fixed point under RG flow. Thus, Eq. (2) defines a manifold of RG fixed points $S = S[V]$, parameterized by the symmetric positive definite $2N \times 2N$ matrix $V$. Our results are based on a systematic linear stability analysis of these fixed points. A generic perturbation of $S[V]$ has the form

$$S' = \int dt dx \left[ \xi(x) O(t,x) + \xi^*(x) O^\dagger(t,x) \right],$$

where $O$ is a local bosonic operator and $\xi(x)$ is an appropriate function. It is natural to distinguish three types of perturbation: (i) global perturbations, in which $\xi(x) = g e^{i\alpha}$ is constant in space, (ii) random ones, in which $\xi(x)$ is a Gaussian random variable with $\xi(x) = 0$ and $\xi^*(x) \xi(x') = \sqrt{g} \delta(x - x')$, and (iii) local ones, in which $\xi(x) = g e^{i\alpha} \delta(x - x_0)$ acts only at a point. In each case, the strength of the perturbation is measured by a “coupling constant” $g$, whose value depends on the short-distance cutoff $a$ (equivalently, the energy scale $\Lambda$ at which we probe the system). The linearized RG equation specifying how $g$ changes under an infinitesimal change of the cutoff, $a \rightarrow e^\lambda a$, is

$$\frac{d \ln g}{dt} = d_{\text{eff}} - \Delta,$$

where $d_{\text{eff}} = 2, \frac{3}{2}, 1$ in the global, random, or local cases respectively, and where $\Delta$ is the scaling dimension of $O$. The perturbation is relevant if $\Delta < d_{\text{eff}}$, marginal (at tree-level) if $\Delta = d_{\text{eff}}$, and irrelevant if $\Delta > d_{\text{eff}}$.

The quadratic action (2) can be destabilized by localization or the opening of a gap, either of which can be caused by a perturbation (3) if $O$ is a vertex operator $O_m \equiv e^{imT \phi_I}$, where $m \in \mathbb{Z}^{2N}$ (we suppress cutoff factors for brevity). The operator $O_m$ is bosonic if and only if its conformal spin $K(m) = \frac{1}{2} m^T \gamma m$ is an integer. At the fixed point $S[V]$, the scaling dimension of $O_m$ is

$$\Delta(m) = \frac{1}{2} m^T \gamma m,$$

where $M = A^T A$, and $A \in SO(N, N)$ diagonalizes the interaction matrix, $A^T A = \text{diag}(u_i)$. Although $V$ does not uniquely determine $A$ by this criterion, it does uniquely determine $A^T A$, so the expression for $\Delta(m)$ is well-defined. Given any $M$, the set of corresponding interaction matrices can be parameterized as

$$V = M^{-1/2} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} M^{-1/2},$$

where $X$ and $Y$ are arbitrary symmetric positive definite $N \times N$ matrices, and $M^{-1/2}$ is the unique positive definite square root of $M^{-1}$ [16]. This parameterization of $V$ is closely related to, but distinct from, the one used in Refs. [17, 18].

We define two notions of stability of a LL fixed point $S[V]$. We say that it is $\infty$-stable if all non-chiral (i.e. $K(m) = 0$) perturbations are irrelevant at $S[V]$. We say that it is absolutely $\infty$-stable if all chiral (i.e. $K(m) \neq 0$) perturbations are irrelevant as well. Chiral perturbations cannot lead to an energy gap, but one might worry that such perturbations, if relevant, will grow large enough to cause a major re-arrangement of the ground state, thereby affecting the scaling dimensions of non-chiral operators. Note that the scaling dimensions are continuous functions of $V$, so each stable fixed point belongs to a stable Luttinger liquid phase. $\infty$-stable phases cannot exist when the LL has only $N = 1$ channel. They can be shown to exist—by explicit construction—when $N \geq 23$ [12]. In the intermediate range, $1 < N < 23$, the existence of $\infty$-stable phases remains an open question at this time. Meanwhile, upper bounds on the density of high-dimensional sphere packings [19] imply that absolutely $\infty$-stable phases cannot exist with $N < 11$ channels. It is again possible to show—by explicit construction—that they do exist when $N$ is sufficiently large. For completeness, we discuss these matters in more detail in the Supplemental Material [16].

From a physical point of view, however, the notions of stability introduced above are unnecessarily restrictive. If there are physical reasons to expect the bare value $g_0$ of a relevant coupling to be small, then although this coupling will eventually destabilize the metallic state, this will only happen at very low temperatures $T \sim \Lambda_0 g_0^{1/(d_{\text{eff}} - \Delta)}$. We expect $g_0$ to be small for perturbations that are sufficiently high-order in the fermion fields. This expectation is based on the assumption that such terms are not appreciably generated during RG flow from the underlying microscopic theory (which only has terms up to quartic order) to the intermediate-energy effective theory described by $S_{\text{eff}}$. 


Each vertex operator $O_m$ in the bosonic formulation corresponds to terms that are $|m|^n$-order in the fermion fields, where $|m| \equiv \sum_{i=1}^{2N} |m_i|$. We say that the fixed point $S[V]$ is $q$-stable if $q$ is the largest integer such that all non-chiral perturbations of $S[V]$ with $|m| \leq q$ are irrelevant. We say that it is absolutely $q$-stable if $q$ is the largest integer such that all perturbations with $|m| \leq q$ are irrelevant. Our earlier notions of stability are the limiting cases $q = \infty$. Based on the comments in the previous paragraph, it is plausible that, in any real system, there will be no observable difference between $q$-stability (absolute $q$-stability) and $\infty$-stability (absolute $\infty$-stability) at accessible temperatures if $q$ is sufficiently large [20].

Relation to Integral Quadratic Lattices. As described in the Introduction, there is a beautiful geometric picture associated with all of this. To any interaction matrix $V$ diagonalized by $A \in SO(N, N)$, we associate a lattice $AZ^{2N} = \{Am \mid m \in \mathbb{Z}^{2N}\}$ in a fictitious $\mathbb{R}^{2N}$ equipped with two metrics: the mixed-signature $(N, N)$ metric $\text{diag}(I_N, -I_N)$ and the Euclidean metric $I_{2N}$. For concreteness, we make the particular choice $A = M^{1/2}$. The scaling dimension of an operator is equal to half the Euclidean interval from the origin to the associated lattice point, $\Delta(m) = \frac{1}{2} \|Am\|^2$. There are three “spheres of relevance” centered at the origin, with Euclidean radii $\sqrt{2d_{\text{eff}}} = 2\sqrt{3}, \sqrt{2}$; any lattice point inside these spheres represents a perturbation that is relevant if it is global, random, or local, respectively. The chiral-ity (i.e. conformal spin) of an operator is equal to half the mixed-signature interval from the origin to the associated lattice point; chiral operators correspond to “spacelike” or “timelike” intervals, and non-chiral operators to “lightlike” (null) intervals. The lattice is naturally graded into “shells” of fixed $|m| \equiv \sum_{i=1}^{2N} |m_i|$, the Manhattan distance from the origin to the lattice point $m$ in the undeformed $\mathbb{Z}^{2N}$ lattice, which equals the order of the corresponding perturbation $O_m$ in the fermion fields. Bosonic operators have even $|m|$. The fixed point $S[V]$ is $q$-stable if no lightlike even lattice point in the innermost $q$ shells falls within the sphere of Euclidean radius $2$ centered at the origin. It is absolutely $q$-stable if the same also holds for spacelike and timelike even lattice points in these shells.

Let us briefly illustrate these ideas in the simplest case, that of $N = 1$ channel. The matrix $A \in SO(1, 1)$ then describes a boost (hyperbolic rotation) of the plane, and can be parameterized as $A = A(\lambda) = e^{-i/(2\lambda)^\sigma \mathbf{r}}$. At the free fixed point, $\lambda = 0$, the most relevant perturbations couple $O_{SC} \equiv \psi_R \psi_L \sim e^{i(\phi_1 + \phi_2)}$ to an external 3D superconductor, or $O_{CDW} \equiv \psi_R \psi_L^\dagger \sim e^{i(\phi_1 + \phi_2)}$ to a periodic potential. The corresponding lattice points are $m = (-1, 1)$ and $m = (1, 1)$ respectively; these lie along the null directions. Both operators have $\Delta = 1$ when $\lambda = 0$, so both perturbations are relevant at the free fixed point. Now turn on interactions, so that $\lambda \neq 0$. Under the boost $A(\lambda)$, points are “pushed away” from the origin along one null direction but simultaneously “pulled in” along the other, as shown in Figure 1. Thus, $\lambda < 0$ makes $O_{CDW}$ less relevant but $O_{SC}$ more relevant, while $\lambda > 0$ does the opposite. The interaction matrix $V$ can be parameterized as in Eq. (6), with $X = u_1 + 0$, $Y = u_2 + 0$, and $M^{1/2} = A(-\lambda) = e^{(\lambda/2)^\sigma \mathbf{r}}$; its off-diagonal matrix element is $V_{12} = \frac{1}{2}(u_1 + u_2)\sinh \lambda$. Thus, $\lambda < 0$ ($\lambda > 0$) corresponds to attractive (repulsive) interactions, and we reproduce the well-known phenomenology of the 1-channel Luttinger liquid [2] (the usual “Luttinger parameter” is equal to $e^{-\lambda}$). Clearly, stability is impossible with just $N = 1$ channel.

Stable Luttinger Liquids. We now turn to the general case of $N$ channels. Our approach is to study all possible scaling dimension matrices $M$, without worrying at first about which interaction matrices $V$ give rise to them. Having identified some $M$’s of interest, we can then reconstruct the corresponding $V$’s using Eq. (6).

A useful structure theorem for $SO(N, N)$, called the hyperbolic CS decomposition [21], ensures that $M = A^TA$ can be written as a product of independent boosts in orthogonal planes:

$$M = \begin{bmatrix} Q_1 \quad 0 \\ 0 \quad Q_2 \end{bmatrix} \begin{bmatrix} C & -S \\ -S & C \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix},$$ (7)

where $Q_1, Q_2 \in SO(N)$, $C = \text{diag}(\cosh \lambda_i)$, and $S = \text{diag}(\sinh \lambda_i)$, with $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$. A stable phase can only occur if the boost planes in this decomposition of $M$ are sufficiently “out of alignment” with the lattice planes of $\mathbb{Z}^{2N}$, so that different boosts can compensate one another—if not, some lattice points will be pulled close to the origin, as in the $N = 1$ case. This compensation between boosts presumably works best when all rapidities are equal, i.e. when...
$\lambda_i = \lambda$. If we assume this form, Eq. (7) reduces to

$$M = \begin{bmatrix} I_N \cosh \lambda & -Q \sinh \lambda \\ -Q^T \sinh \lambda & I_N \cosh \lambda \end{bmatrix},$$

where $Q = Q_1^T Q_2 \in SO(N)$.

We will now prove that, with $N \geq 2$ channels, absolutely $q$-stable phases exist for any finite $q$. Let $\mathbf{m} = (\mathbf{m}_R, \mathbf{m}_L)$, with $\mathbf{m}_{R/L} \in \mathbb{Z}^N$, and assume that $M$ is of the form (8) (recall that this can be arranged by taking, for instance, $V \propto M^{-1}$). If either $\mathbf{m}_R$ or $\mathbf{m}_L$ vanishes, then $\Delta(\mathbf{m}) = \frac{1}{2}||\mathbf{m}||^2 \cosh \lambda > 2$ for $\lambda > \arccosh 2$. If neither $\mathbf{m}_R$ nor $\mathbf{m}_L$ vanishes, we can rewrite the inequality $\Delta(\mathbf{m}) > 2$ as

$$\hat{m}_R^T Q \hat{m}_L < f\left(\frac{||\mathbf{m}_R||}{||\mathbf{m}_L||}\right) \coth \lambda - \frac{2 \csch \lambda}{||\mathbf{m}_R|| ||\mathbf{m}_L||},$$

where $\hat{m}_\nu = \mathbf{m}_\nu / ||\mathbf{m}_\nu||$ and $f(x) = \frac{1}{2}(x + x^{-1})$. There are a finite number of vectors $\mathbf{m} \in \mathbb{Z}^N$ that satisfy $||\mathbf{m}|| \leq q$, so the unit vectors $\hat{m}_{R/L}$ in Eq. (9) belong to a finite set $\Omega_q$. This set cannot fill the unit sphere densely, so there exists $Q \in SO(N)$ and $\epsilon > 0$ such that $\hat{m}_R^T Q \hat{m}_L < 1 - \epsilon$ for all $\hat{m}_{R/L} \in \Omega_q$. But $f(x) \coth \lambda > 1$ for any $x, \lambda > 0$, while $\csch \lambda \to 0$ as $\lambda \to \infty$. Therefore the right side of Eq. (9) is greater than $1 - \epsilon$ for sufficiently large $\lambda$.

In the $N = 2$ channel case, $M$ is parameterized, according to Eq. (7), by two rapidities ($\lambda_1, \lambda_2$) and two angles ($\theta_1, \theta_2$, where $\theta_i$ is the rotation angle of $\hat{Q}_i \in SO(2)$). It is convenient to write these as

$$\lambda_{1,2} = \lambda \pm \delta, \quad \theta_{1,2} = \frac{1}{2}(\alpha \mp \theta).$$

In the limit $\delta \to 0$, we recover the simpler form (8) with $Q \in SO(2)$ a rotation by angle $\theta$ (the angle $\alpha$ drops out in this limit). The full parameterization of $M$ is written down explicitly in the Supplemental Material [16].

We construct an “absolute $q$-stability phase diagram” for the $2$-channel LL by assigning to each point $(\lambda, \delta, \theta, \alpha)$ in the resulting parameter space its absolute $q$-stability value, $q$. Figure 2 shows the $\delta = 0$ slice of this diagram; other slices may be found in Ref. [16]. Each point in the phase diagram corresponds to a 6-parameter family of interaction matrices $V$, which can be obtained using Eq. (6). The resulting general expression for $V$ is given in Ref. [16]. Here, we concentrate on the particular case in which the diagonal blocks $V_{RR}$ and $V_{LL}$ are equal (up to a permutation of the modes), and $\delta = 0$. In this case,

$$V = \begin{bmatrix} v_+ & w \\ w & v_- \end{bmatrix} \begin{bmatrix} c_0 & c_+ \\ c_0 & c_- \end{bmatrix} \begin{bmatrix} v_+ & w \\ w & v_- \end{bmatrix},$$

where $v_\pm = v \pm u$,

$$c_0 = v \cos \theta \tanh \lambda, \quad c_\pm = (w \cos \theta \pm v \sin \theta) \tanh \lambda.$$
[1] P. A. Lee and T. V. Ramakrishnan, Disordered electronic systems, Rev. Mod. Phys. 57, 287 (1985).
[2] T. Giamarchi, Quantum physics in one dimension (Clarendon Press, Oxford, UK, 2003).
[3] F. D. M. Haldane, ‘Luttinger liquid theory’ of one-dimensional quantum fluids. I. Properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas, J. Phys. C 14, 2585 (1981).
[4] S. A. Kivelson, E. Fradkin, and V. J. Emery, Electronic liquid-crystal phases of a doped Mott insulator, Nature 393, 550 (1998), arXiv:cond-mat/9707327.
[5] L. Golubović and M. Golubović, Fluctuations of quasi-two-dimensional smectics intercalated between membranes in multilamellar phases of DNA-cationic lipid complexes, Phys. Rev. Lett. 80, 4341 (1998).
[6] C. S. O’Hern and T. C. Lubensky, Sliding columnar phase of DNA-lipid complexes, Phys. Rev. Lett. 80, 4345 (1998), arXiv:cond-mat/9712049.
[7] C. S. O’Hern, T. C. Lubensky, and J. Toner, Sliding phases in XY models, crystals, and cationic lipid-DNA complexes, Phys. Rev. Lett. 83, 2745 (1999), arXiv:cond-mat/9904415.
[8] V. J. Emery, E. Fradkin, S. A. Kivelson, and T. C. Lubensky, Quantum theory of the smectic metal state in stripe phases, Phys. Rev. Lett. 85, 2160 (2000), arXiv:cond-mat/0001077.
[9] A. Vishwanath and D. Carpentier, Two-dimensional anisotropic non-Fermi-liquid phase of coupled Luttinger liquids, Phys. Rev. Lett. 86, 676 (2001), arXiv:cond-mat/0003036.
[10] R. Mukhopadhyay, C. L. Kane, and T. C. Lubensky, Sliding Luttinger liquid phases, Phys. Rev. B 64, 045120 (2001), arXiv:cond-mat/0102163.
[11] We restrict attention in this paper to systems with short-ranged interactions. Long-ranged interactions can also stabilize a Luttinger liquid against a $2k_F$ potential and disorder [22].
[12] E. Plamadeala, M. Mulligan, and C. Nayak, Perfect metal phases of one-dimensional and anisotropic higher-dimensional systems, Phys. Rev. B 90, 1 (2014), arXiv:1404.4367.
[13] The Lie group $SO(N,N)$ consists of all matrices $A \in \mathbb{R}^{2N \times 2N}$ that satisfy $AKA^T = K$ and $\det A = 1$, where $K = I_N \oplus -I_N$.
[14] J. von Delft and H. Schoeller, Bosonization for beginners — Refermionization for experts, Annalen Phys. 7, 224 (1998), arXiv:cond-mat/9805275.
[15] Rigorously, $a$ is an infinitesimal regularization parameter, and is not related to the running RG scale. In order to streamline the exposition, we will ignore this technical distinction.
[16] See Supplemental Material.
[17] J. E. Moore and X.-G. Wen, Classification of disordered phases of quantum Hall edge states, Phys. Rev. B 57, 10138 (1998), arXiv:cond-mat/9710208.
[18] C. Xu and J. E. Moore, Stability of the quantum spin Hall effect: Effects of interactions, disorder, and $Z_2$ topology, Phys. Rev. B 73, 045322 (2006), arXiv:cond-mat/0508291.
[19] H. Cohn and N. Elkies, New upper bounds on sphere packings I, Ann. Math. 157, 689 (2003), arXiv:math/0110009.
[20] Even if this assumption turns out to be false, a $q$-stable phase can be expected to exhibit novel and exotic instabilities, since all the usual instabilities correspond to operators with small $|m|$.
[21] N. J. Higham, $J$-orthogonal matrices: Properties and generation, SIAM Rev. 45, 504 (2003).
[22] B. Dóra and R. Moessner, Luttinger liquid with complex forward scattering: Robustness and Berry phase, Phys. Rev. B 7, 075127 (2016), arXiv:1510.01519.
S1. PROPERTIES OF THE MAP FROM INTERACTION MATRICES TO SCALING DIMENSION MATRICES

Let $\mathcal{P}_N$ denote the set of real symmetric positive definite $N \times N$ matrices, and let $\mathcal{M}_N \equiv SO(N, N) \cap \mathcal{P}_N$. The map $\varphi$ from interaction matrices $V \in \mathcal{P}_2N$ to “scaling dimension matrices” $M \in \mathcal{M}_N$ is defined as

$$\varphi : \mathcal{P}_2N \to \mathcal{M}_N, \quad \varphi : V \mapsto A^T A,$$

where $A \in SO(N, N)$ and $AVA^T = D$ is diagonal.

The first and second lemmas below show that $\varphi$ is well-defined. The third, fourth and fifth lemmas characterize the inverse images $\varphi^{-1}(M)$, and yield the parameterization of $V$ matrices used in the main text. All of these results are elementary, but we record them here for completeness.

**Lemma 1.** If $V \in \mathcal{P}_2N$, then there exists $A \in SO(N, N)$ such that $AVA^T = D$ is diagonal.

**Proof.** (by construction). Let $V^{1/2}$ denote the unique symmetric positive definite square root of $V$, so that

$$V^{1/2} = (V^{1/2})^T, \quad V^{1/2} > 0, \quad (V^{1/2})^2 = V,$$

and let $V^{1/2} \equiv (V^{1/2})^{-1} = (V^{-1})^{1/2}$. The matrix $V^{-1/2} KV^{-1/2}$ (where $K = \mathbb{I}_N \oplus -\mathbb{I}_N$) is symmetric, and can therefore be diagonalized by some $Q \in SO(2N)$. Sylvester’s theorem of inertia [S1] ensures that $Q$ can be chosen so that $QV^{-1/2} KV^{-1/2} Q^T = D^{-1} K$, where $D$ is diagonal and positive definite. Then $A \equiv D^{1/2} Q V^{-1/2}$ satisfies $AKA^T = K$, det $A > 0$, and $AVA^T = D$, as desired. □

**Lemma 2.** If $A_i \in SO(N, N)$ and $A_i V A_i^T = D_i$ is diagonal for $i = 1, 2$, then $A_i^T A_1 = A_2^T A_2$.

**Proof.** Note that every $A \in SO(N, N)$ is invertible, with $A^{-1} = KA^T K$, where $K = \mathbb{I}_N \oplus -\mathbb{I}_N$ (this follows immediately from the condition $AKA^T = K$, and the fact that $K^2 = \mathbb{I}_2N$). Thus, to prove the lemma it suffices to prove the equivalent statement that $D_2 = AD_1 A^T$ implies $A^T A = \mathbb{I}_2N$, where $A \equiv A_2 A_1^{-1} \in SO(N, N)$.

Using $A^T = KA^{-1} K$, the equation $D_2 = AD_1 A^T$ can be rewritten as

$$D_2 K = A(D_1 K) A^{-1}.$$  \hspace{1cm} (S3)

Thus $D_2 K$ and $D_1 K$ are similar. But similar diagonal matrices can differ only by a permutation of the diagonal elements. Taking account of the sign structure due to $K$, one must have $D_2 = PD_1 P^{-1}$, with $P = P_1 \oplus P_2$, where the $P_i$ are $N \times N$ permutation matrices. Since $B \equiv P^{-1} A \in SO(N, N)$ satisfies $B^T B = \mathbb{I}_{2N}$ if and only if $A^T A = \mathbb{I}_{2N}$, we can assume without loss of generality that $D_2 = D_1 \equiv D$. Then Eq. (S3) implies that $A$ preserves each eigenspace of $DK$, so that (at the very least) $A \in O(N) \times O(N)$, and so $A^T A = \mathbb{I}_{2N}$. □

**Lemma 3.** $V \in \varphi^{-1}(M)$ if and only if

$$V = M^{-1/2} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} M^{-1/2}$$  \hspace{1cm} (S4)

for some $X, Y \in \mathcal{P}_N$, where $M^{-1/2}$ denotes the unique positive definite square root of $M^{-1}$.

**Proof.** Every $M \in \mathcal{M}_N = SO(N, N) \cap \mathcal{P}_2N$ has a unique positive definite symmetric square root $M^{-1/2} \in \mathcal{M}_N$. Furthermore, $M = A^T A$ for $A \in SO(N, N)$ iff $A = M^{1/2} R$, where $R \in O(N) \times O(N)$. Therefore, $V \in \varphi^{-1}(M)$ iff $(M^{1/2} R)^T V (M^{1/2} R) = D$ for some diagonal positive definite $D$; equivalently, $V = M^{-1/2} R D R^T M^{-1/2}$. The claim follows. □

The scaling dimension matrix $M \in \mathcal{M}_N$ can, by the hyperbolic CS decomposition, be written as

$$M = \begin{bmatrix} Q_1^T & 0 \\ 0 & Q_2^T \end{bmatrix} \begin{bmatrix} C & -S \\ -S & C \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix},$$  \hspace{1cm} (S5)

where $Q_1, Q_2 \in SO(N), C = \text{diag}(\cosh \lambda_i)$, and $S = \text{diag}(\sinh \lambda_i)$, with $\lambda_i \in \mathbb{R}, i = 1, 2, \ldots, N$.

**Lemma 4.** $V \in \varphi^{-1}(M)$ if and only if

$$V = \begin{bmatrix} X + F Y F^T & X F + F Y \\ F^T X + Y F^T & F^T X F + Y \end{bmatrix}$$  \hspace{1cm} (S6)

for some $X, Y \in \mathcal{P}_N$, where

$$F \equiv Q_1^T \text{diag}(\tanh(\lambda_i/2)) Q_2.$$  \hspace{1cm} (S7)

**Proof.** According to Lemma 3, $V \in \varphi^{-1}(M)$ iff $V = M^{-1/2} \tilde{X} \oplus \tilde{Y} M^{-1/2}$ for some $\tilde{X}, \tilde{Y} \in \mathcal{P}_N$. From Eq. (S5), it follows that

$$M^{-1/2} = \begin{bmatrix} \tilde{C}_1 & \tilde{S} \\ \tilde{S}^T & \tilde{C}_2 \end{bmatrix},$$  \hspace{1cm} (S8)

where $\tilde{C}_\nu = Q_\nu^T \text{diag}(\cosh(\lambda_i/2)) Q_\nu$ ($\nu = 1, 2$) and $\tilde{S} = Q_1^T \text{diag}(\sinh(\lambda_i/2)) Q_2$. Thus,

$$V = \begin{bmatrix} \tilde{C}_1 \tilde{X} \tilde{C}_1 + \tilde{S} \tilde{Y} \tilde{S}^T & \tilde{C}_1 \tilde{X} \tilde{S} + \tilde{S} \tilde{Y} \tilde{C}_2 \\ \tilde{S}^T \tilde{X} \tilde{C}_1 + \tilde{C}_2 \tilde{Y} \tilde{S}^T & \tilde{S}^T \tilde{X} \tilde{S} + \tilde{C}_2 \tilde{Y} \tilde{C}_2 \end{bmatrix}.$$  \hspace{1cm} (S9)
Now define $X = \tilde{C}_1 \tilde{X} \tilde{C}_1$ and $Y = \tilde{C}_2 \tilde{Y} \tilde{C}_2$. These maps from $\tilde{X}, \tilde{Y} \in \mathcal{P}_N$ to $X, Y \in \mathcal{P}_N$ are bijections, because $\tilde{C}_i \in \mathcal{P}_N$. Noting that $\tilde{S} \tilde{C}_2^{-1} = \tilde{C}_1^{-1} \tilde{S} = F$, we obtain the claimed result, Eq. (S6).

We now write the interaction matrix in block form as

$$V = \begin{bmatrix} V_{RR} & V_{RL} \\ V_{LR} & V_{LL} \end{bmatrix},$$

where $V_{RR}, V_{LL} \in \mathcal{P}_N$ and $V_{LR} = V_{RL}^T$.

**Lemma 5.** $V \in \varphi^{-1}(M)$ if and only if

$$V_{RR} \in \mathcal{P}_N, \quad V_{RL} = \mathcal{P}_N, \quad V_{LL} = \mathcal{P}_N,$$

where $F$ is defined in Eq. (S7) above.

**Proof.** Equations (S11a) and (S11b) are the Schur complement condition for positive definiteness of a symmetric matrix [S2]; $V \in \mathcal{P}_N$ iff these equations hold. By Lemma 3, $V \in \varphi^{-1}(M)$ iff $M^{1/2} V M^{1/2} = \tilde{X} \oplus \tilde{Y}$ for some $\tilde{X}, \tilde{Y} \in \mathcal{P}_N$. In the notation of Eq. (S8), one has

$$M^{1/2} = \begin{bmatrix} \tilde{C}_1 \tilde{S} & -\tilde{S} \\ -\tilde{S}^T & \tilde{C}_2 \end{bmatrix}.$$ (S12)

Conjugating the equation $M^{1/2} V M^{1/2} = \tilde{X} \oplus \tilde{Y}$ by the positive definite matrix $\tilde{C}_1^{-1} \oplus \tilde{C}_2^{-1}$, it becomes

$$\begin{bmatrix} \mathbb{I}_N & -F \\ -F^T & \mathbb{I}_N \end{bmatrix} \begin{bmatrix} V_{RR} & V_{RL} \\ V_{LR}^T & V_{LL} \end{bmatrix} \begin{bmatrix} \mathbb{I}_N & -F \\ -F^T & \mathbb{I}_N \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix},$$ (S13)

where $X = \tilde{C}_1^{-1} \tilde{X} \tilde{C}_1^{-1}$ and $Y = \tilde{C}_2^{-1} \tilde{Y} \tilde{C}_2^{-1}$. These maps from $\tilde{X}, \tilde{Y} \in \mathcal{P}_N$ to $X, Y \in \mathcal{P}_N$ are bijections. Therefore, $V \in \varphi^{-1}(M)$ iff Eq. (S13) holds for some $X, Y \in \mathcal{P}_N$. The off-diagonal block of Eq. (S13) yields Eq. (S11c). The diagonal blocks of Eq. (S13) are automatically satisfied, because the matrix on the left side is positive definite (it was constructed by conjugating $V \in \mathcal{P}_2$ by other matrices in $\mathcal{P}_2$).

To gain some intuition for the parametrization (S4) of $V$, first consider the limit $M = \mathbb{I}_2$. Then the interactions encoded in $X$ simply mix the right-movers amongst themselves, leading to new modes with renormalized velocities, while $Y$ does the same with the left-movers. All scaling dimensions (being determined by $M$ alone) remain equal to their values at the free fixed point. Next consider a different limit, $X = Y = \mathbb{I}_N$. Now $V = M^{-1}$ is itself in $SO(N, N)$, and its inverse gives the scaling dimensions directly. To connect these two limits, consider the Euclidean space of all symmetric $N \times N$ matrices, $\mathbb{R}^{N(N+1)/2}$. The positive definite matrices occupy the interior of a convex cone $\mathcal{P}_N \subset \mathbb{R}^{N(N+1)/2}$. The space of interaction matrices is $\mathcal{Y} = \mathcal{P}_2$. According to Eq. (S4), $\mathcal{Y}$ should be regarded as a bundle of lower-dimensional convex cones $\mathcal{P}_N \times \mathcal{P}_N$ (parameterized by $X, Y$) as fibers over the $\mathbb{R}^{N(N+1)/2}$.

$N^2$-dimensional submanifold $\mathcal{M}_N \equiv SO(N, N) \cap \mathcal{P}_2$ (parameterized by $M$). The scaling dimensions $\Delta(m)$, regarded as functions from $\mathcal{Y} \to \mathbb{R}$, are then constant on each fiber. Each Luttinger liquid phase, defined in terms of its instabilities (or lack thereof), thus extends over the interior of a solid cone emanating from the vertex of $\mathcal{Y}$.

The $N = 1$ channel case again provides a nice illustration of these general ideas. The set $\mathcal{Y} = \mathcal{P}_2$ consists of all $2 \times 2$ matrices

$$V = \begin{bmatrix} \alpha + \beta & \gamma \\ \gamma & \alpha - \beta \end{bmatrix}.$$ (S14)

with $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ and $\alpha > (\beta^2 + \gamma^2)^{1/2}$. This is quite clearly the interior of a circular cone in $\mathbb{R}^3$. The parameterization (S4), with $M = e^{-\lambda x} = \mathcal{M}$, corresponds to

$$\alpha = \frac{1}{2} (x + y) \cosh \lambda,$$ (S15a)

$$\beta = \frac{1}{2} (x - y),$$ (S15b)

$$\gamma = \frac{1}{2} (x + y) \sinh \lambda,$$ (S15c)

where $x, y > 0$. For fixed $\lambda$, the image of the resulting map $(x, y) \mapsto (\alpha, \beta, \gamma)$ is a slice of the cone $\mathcal{P}_2$, in a plane parallel to the $\beta$-axis and at an angle $\arctan(\tanh \lambda)$ from the $\alpha$-axis. Each such slice is the interior of a cone in $\mathbb{R}^2$, with an opening angle that decreases with increasing $|\lambda|$. In terms of stability with respect to clean SC and CDW perturbations, $\mathcal{Y} = \mathcal{P}_2$ splits into four regions: $\lambda < -\log 2 (\Delta_{SC} < \Delta_{CDW}$), $-\log 2 < \lambda < 0 (\Delta_{SC} < \Delta_{CDW} < \Delta_{SC})$, $0 < \lambda < \log 2 (\Delta_{CDW} < \Delta_{SC} < \Delta_{SC})$, and $\lambda > \log 2 (\Delta_{CDW} < \Delta_{SC} < \Delta_{CDW})$. These regions indeed take the form of solid cones emanating from the vertex of $\mathcal{P}_2$, as illustrated in Figure S1.
S2. EXPLICIT PARAMETERIZATION OF MATRICES FOR \( N = 2 \) CHANNEL LUTTINGER LIQUID

A. Scaling dimension matrix \( M \)

Let \( Q(\phi) \) denote the \( SO(2) \) rotation matrix

\[
Q(\phi) = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix},
\]

(S16)

let \( P(\phi) \) denote the \( O(2) \) reflection matrix

\[
P(\phi) = \begin{bmatrix} \cos \phi & \sin \phi \\ \sin \phi & -\cos \phi \end{bmatrix},
\]

(S17)

and let

\[
\Lambda = \begin{bmatrix} \lambda + \delta & 0 \\ 0 & \lambda - \delta \end{bmatrix}.
\]

(S18)

The parameterization of \( M \in M_2 \equiv SO(2,2) \cap P_4 \) described in Eqs. (7) and (10) of the main text corresponds to

\[
M = \begin{bmatrix} Q^T(\alpha - \theta/2) & 0 \\ 0 & Q^T(\alpha + \theta/2) \end{bmatrix} \begin{bmatrix} \cosh \Lambda & -\sinh \Lambda \\ -\sinh \Lambda & \cosh \Lambda \end{bmatrix} \begin{bmatrix} Q(\alpha - \theta/2) & 0 \\ 0 & Q(\alpha + \theta/2) \end{bmatrix},
\]

(S19)

where each entry is a \( 2 \times 2 \) matrix. Performing the matrix multiplication, we can write the result as

\[
M = \begin{bmatrix} I_2 \cosh \lambda & -Q(\theta) \sinh \lambda \\ -Q^T(\theta) \sinh \lambda & I_2 \cosh \lambda \end{bmatrix} \cosh \delta + \begin{bmatrix} P(\alpha - \theta) \sinh \lambda & -P(\alpha) \cosh \lambda \\ -P(\alpha) \cosh \lambda & P(\alpha + \theta) \sinh \lambda \end{bmatrix} \sinh \delta,
\]

(S20)

where \( I_2 \) is the \( 2 \times 2 \) identity matrix. In the limit \( \delta \to 0 \), the dependence on \( \alpha \) drops out and \( M \) reduces to the form given in Eq. (8) of the main text, with \( N = 2 \) and \( Q = Q(\theta) \).

B. Interaction matrix \( V \) (general expression)

We parameterize the interaction matrix \( V \in P_4 \) using Lemma 4. Let

\[
V = \begin{bmatrix} V_{RR} & V_{RL} \\ V_{LR} & V_{LL} \end{bmatrix}.
\]

(S21)

Lemma 4 states that \( V \in \varphi^{-1}(M) \) if and only if

\[
V_{RR} = X + FYF^T,
\]

(S22a)

\[
V_{LL} = Y + F^TXF,
\]

(S22b)

\[
V_{RL} = XF + FY
\]

(S22c)

for some \( X, Y \in P_2 \). The \( 2 \times 2 \) matrix \( F \) corresponding to the \( M \) given in Eqs. (S19) or (S20) is:

\[
F = Q^T(\frac{\alpha-\theta}{2}) \begin{bmatrix} \tanh(\frac{\lambda+\delta}{2}) & 0 \\ 0 & \tanh(\frac{\lambda-\delta}{2}) \end{bmatrix} Q(\frac{\alpha+\theta}{2}) = \frac{\sinh \lambda}{\cosh \lambda + \cosh \delta} Q(\theta) + \frac{\sinh \delta}{\cosh \lambda + \cosh \delta} P(\alpha),
\]

(S23)

where \( Q(\phi) \) and \( P(\phi) \) are defined in Eqs. (S16) and (S17).

The matrices \( X, Y \in P_2 \) can be conveniently parameterized as

\[
X = \begin{bmatrix} x_0 + x_1 & x_2 \\ x_2 & x_0 - x_1 \end{bmatrix},
\]

(S24a)

\[
Y = \begin{bmatrix} y_0 + y_1 & y_2 \\ y_2 & y_0 - y_1 \end{bmatrix},
\]

(S24b)
where \((x_0, x_1, x_2) \in \mathbb{R}^3, x_0 > (x_1^2 + x_2^2)^{1/2}\), and \((y_0, y_1, y_2) \in \mathbb{R}^3, y_0 > (y_1^2 + y_2^2)^{1/2}\). Using Eqs. (S23) and (S24) in Eq. (S22), we obtain:

\[
V_{RR} = \begin{bmatrix}
    x_0 + x_1 & x_2 \\
    x_2 & x_0 - x_1
\end{bmatrix} + \sinh^2 \lambda \left(\frac{2 \sinh \delta}{(\cosh \lambda + \cosh \delta)^2} \begin{bmatrix}
    y_0 + y_1 \cos 2 \theta + y_2 \sin 2 \theta & y_2 \cos 2 \theta - y_1 \sin 2 \theta \\
    y_2 \cos 2 \theta - y_1 \sin 2 \theta & y_0 - y_1 \cos 2 \theta - y_2 \sin 2 \theta
\end{bmatrix}
\right)
\]

\[
+ \frac{2 \sinh \lambda \sinh \delta}{(\cosh \lambda + \cosh \delta)^2} \begin{bmatrix}
    y_1 \cos (\alpha + \theta) + y_2 \sin (\alpha - \theta) + y_0 \cos (\alpha - \theta) - y_0 \sin (\alpha - \theta) \\
    y_1 \sin (\alpha + \theta) - y_0 \sin (\alpha + \theta) + y_2 \cos (\alpha + \theta) - y_0 \cos (\alpha + \theta)
\end{bmatrix}
\]

\[
+ \frac{\sinh^2 \delta}{(\cosh \lambda + \cosh \delta)^2} \begin{bmatrix}
    y_0 + y_1 \cos 2 \alpha + y_2 \sin 2 \alpha & y_1 \sin 2 \alpha - y_0 \sin 2 \alpha \\
    y_1 \sin 2 \alpha - y_0 \sin 2 \alpha & y_0 - y_1 \cos 2 \alpha + y_2 \sin 2 \alpha
\end{bmatrix},
\]

(S25a)

\[
V_{LL} = \begin{bmatrix}
    y_0 + y_1 & y_2 \\
    y_2 & y_0 - y_1
\end{bmatrix} + \sinh^2 \lambda \left(\frac{2 \sinh \delta}{(\cosh \lambda + \cosh \delta)^2} \begin{bmatrix}
    x_0 + x_1 \cos 2 \theta - x_2 \sin 2 \theta & x_1 \sin 2 \theta \\
    x_2 \sin 2 \theta & x_0 - x_1 \cos 2 \theta + x_2 \sin 2 \theta
\end{bmatrix}
\right)
\]

\[
+ \frac{2 \sinh \lambda \sinh \delta}{(\cosh \lambda + \cosh \delta)^2} \begin{bmatrix}
    x_1 \cos (\alpha - \theta) + x_2 \sin (\alpha - \theta) + x_0 \cos (\alpha + \theta) - x_0 \sin (\alpha + \theta) \\
    x_1 \sin (\alpha - \theta) - x_0 \sin (\alpha + \theta) + x_2 \cos (\alpha - \theta) - x_0 \cos (\alpha + \theta)
\end{bmatrix}
\]

\[
+ \frac{\sinh^2 \delta}{(\cosh \lambda + \cosh \delta)^2} \begin{bmatrix}
    x_0 + x_1 \cos 2 \alpha + x_2 \sin 2 \alpha & x_1 \sin 2 \alpha - x_0 \sin 2 \alpha \\
    x_1 \sin 2 \alpha - x_0 \sin 2 \alpha & x_0 - x_1 \cos 2 \alpha + x_2 \sin 2 \alpha
\end{bmatrix},
\]

(S25b)

\[
V_{RL} = \frac{\sinh \lambda}{\cosh \lambda + \cosh \delta} \left(\begin{bmatrix}
    (x_0 + y_0 + x_1 + y_1) \cos \theta - (x_2 + y_2) \sin \theta & (x_2 + y_2) \cos \theta + (x_1 - y_1 + x_0 + y_0) \sin \theta \\
    (x_2 + y_2) \cos \theta + (x_1 - y_1 - x_0 - y_0) \sin \theta & (x_0 + y_0 - x_1 - y_1) \cos \theta + (x_2 - y_2) \sin \theta
\end{bmatrix}
\right)
\]

\[
+ \frac{\sinh \delta}{\cosh \lambda + \cosh \delta} \left(\begin{bmatrix}
    (x_1 + y_1 + x_0 + y_0) \cos \alpha + (x_2 + y_2) \sin \alpha & (x_0 + y_0 + x_1 + y_1) \sin \alpha - (x_2 - y_2) \cos \alpha \\
    (x_0 + y_0 - x_1 - y_1) \sin \alpha + (x_2 - y_2) \cos \alpha & (x_1 + y_1 - x_0 - y_0) \cos \alpha + (x_2 + y_2) \sin \alpha
\end{bmatrix}
\right).
\]

(S25c)

Equation (S25) gives a complete and explicit parameterization of the possible interaction matrices \(V \in \mathcal{B}_4\) of a 2-channel Luttinger liquid, in terms of the ten real parameters \((\lambda, \delta, \theta, \alpha, x_0, x_1, x_2, y_0, y_1, y_2)\). Of these, only the first four \((\lambda, \delta, \theta, \alpha)\) affect scaling dimensions; they determine the scaling dimension matrix \(M\) via Eq. (S20). The remaining six parameters can be chosen arbitrarily, subject only to the constraints \(x_0 > (x_1^2 + x_2^2)^{1/2}\) and \(y_0 > (y_1^2 + y_2^2)^{1/2}\) (if either of these inequalities is violated, the resulting \(V\) will fail to be positive definite).

C. Interaction matrix \(V\) (in the special case \(\delta = 0\))

In the limit \(\delta \to 0\), only the first line of each of Eqs. (S25a–S25c) survives, and the dependence on \(\alpha\) drops out (as it must, since it also drops out of \(M\) when \(\delta \to 0\)). It is convenient to change parameters from \(x_i, y_i\) to \(\tilde{x}_i, \tilde{y}_i\), where

\[
(x_0, x_1, x_2) = (\tilde{x}_0, \tilde{x}_1 \cos \theta + \tilde{x}_2 \sin \theta, \tilde{x}_2 \cos \theta - \tilde{x}_1 \sin \theta),
\]

\[
y_0, y_1, y_2) = (\tilde{y}_0, \tilde{y}_1 \cos \theta - \tilde{y}_2 \sin \theta, \tilde{y}_2 \cos \theta + \tilde{y}_1 \sin \theta).
\]

(S26a)

(S26b)

In terms of the new parameters, one has

\[
V_{RR} = \begin{bmatrix}
    z^+_0(\lambda) + z^+_1(\lambda) \cos \theta + z^+_2(\lambda) \sin \theta & z^+_2(\lambda) \cos \theta - z^+_1(\lambda) \sin \theta \\
    z^+_2(\lambda) \cos \theta + z^+_1(\lambda) \sin \theta & z^+_0(\lambda) - z^+_1(\lambda) \cos \theta - z^+_2(\lambda) \sin \theta
\end{bmatrix},
\]

(S27a)

\[
V_{LL} = \begin{bmatrix}
    z^+_0(\lambda) + z^+_1(\lambda) \cos \theta - z^+_2(\lambda) \sin \theta & z^+_1(\lambda) \cos \theta + z^+_2(\lambda) \sin \theta \\
    z^+_2(\lambda) \cos \theta + z^+_1(\lambda) \sin \theta & z^+_0(\lambda) - z^+_1(\lambda) \cos \theta + z^+_2(\lambda) \sin \theta
\end{bmatrix},
\]

(S27b)

\[
V_{RL} = \begin{bmatrix}
    (\tilde{x}_0 + \tilde{y}_0) \cos \theta + (\tilde{x}_1 + \tilde{y}_1) (\tilde{x}_2 + \tilde{y}_2) + (\tilde{x}_0 + \tilde{y}_0) \sin \theta \\
    (\tilde{x}_2 + \tilde{y}_2) - (\tilde{x}_0 + \tilde{y}_0) \sin \theta (\tilde{x}_0 + \tilde{y}_0) \cos \theta - (\tilde{x}_1 + \tilde{y}_1)
\end{bmatrix} \tanh(\lambda/2).
\]

(S27c)

where

\[
z^+_i(\lambda) \equiv \tilde{x}_i + \tilde{y}_i \tanh^2(\lambda/2),
\]

(S28a)

\[
z^-_i(\lambda) \equiv \tilde{x}_i \tanh^2(\lambda/2) + \tilde{y}_i
\]

(S28b)
Equation (S27) gives an explicit parameterization, in terms of eight real parameters \((\lambda, \theta, \tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{y}_0, \tilde{y}_1, \tilde{y}_2)\), of the interaction matrices \(V \in \mathcal{R}_4\) that give rise to scaling dimension matrices \(M \in \mathcal{M}_2\) of the form

\[
M = \begin{bmatrix}
    \mathbb{I}_2 \cosh \lambda & -Q(\theta) \sinh \lambda \\
    -Q^T(\theta) \sinh \lambda & \mathbb{I}_2 \cosh \lambda
\end{bmatrix};
\]

\((S29)\)
equivalently, these are the \(V\) matrices that would correspond to each point \((\lambda, \theta)\) in Figure 2 of the main text. The parameters \(\tilde{x}_i\) and \(\tilde{y}_i\) must satisfy \(\tilde{x}_0 > (\tilde{x}_1^2 + \tilde{x}_2^2)^{1/2}\) and \(\tilde{y}_0 > (\tilde{y}_1^2 + \tilde{y}_2^2)^{1/2}\) in order for \(V\) to be positive definite.

\section{Interaction matrix \(V\) (for \(\delta = 0\) and \(V_{RR} \cong V_{LL}\))}

We now consider the particular choice of parameters

\[
(\tilde{y}_0, -\tilde{y}_1, \tilde{y}_2) = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2) = \zeta (1, a, b).
\]

\((S30)\)

Note that \(\zeta > 0\) is an irrelevant overall scale factor, and the condition for \(V\) to be positive definite is \(a^2 + b^2 < 1\). In terms of the new parameters, Eq. (S27) reduces to

\[
V_{RR} = \begin{bmatrix} v_+ & w \\ w & v_- \end{bmatrix}, \quad V_{LL} = \begin{bmatrix} v_- & w \\ w & v_+ \end{bmatrix}, \quad V_{RL} = \begin{bmatrix} c_0 & c_+ \\ c_- & c_0 \end{bmatrix},
\]

\((S31)\)

where

\[
v_\pm = v \pm u, \quad w = \zeta \cosh \lambda \sech^2(\lambda/2), \quad u = \zeta (b \sin \theta \cosh \lambda + a \cos \theta) \sech^2(\lambda/2), \quad w = \zeta (b \cosh \theta \cosh \lambda - a \sin \theta) \sech^2(\lambda/2),
\]

\((S32a-d)\)

\[
c_0 = 2 \zeta \cos \theta \tanh(\lambda/2), \quad c_\pm = 2 \zeta (b \pm \sin \theta) \tanh(\lambda/2).
\]

\((S32e-f)\)

We now restrict attention to \(\lambda > 0\) and \(\theta \in (0, \pi/2)\), and identify the values of \(a, b\) for which all matrix elements of \(V\) are nonnegative. The diagonal elements of a positive definite matrix are necessarily positive, so \(v_\pm > 0\). Since \(\lambda > 0\) and \(\theta \in (0, \pi/2)\), one also has \(c_0 \geq 0\). Nonnegativity of the remaining matrix elements, \(w\) and \(c_\pm\), requires

\[
b \cos \theta \cosh \lambda \geq a \sin \theta, \quad b \geq \sin \theta.
\]

\((S33a-b)\)

In fact, Eq. (S33a) is superfluous, because it follows from Eq. (S33b) and the positive definiteness condition; assuming the latter, we have \(a < (1 - b^2)^{1/2} \leq \cos \theta \leq \sin \theta \cot \theta \cosh \lambda \leq b \cot \theta \cosh \lambda\), which is equivalent to Eq. (S33a). We conclude that the \(V\) matrix given above is positive definite with all entries nonnegative if \(a^2 + b^2 < 1\) and \(b \geq \sin \theta\).

These results are completely equivalent to the ones stated in the main text. Indeed, using Eq. (S32), one can verify that the following linear relations hold:

\[
c_0 = v \cos \theta \tanh \lambda, \quad c_\pm = (w \cos \theta \pm v_\pm \sin \theta) \tanh \lambda.
\]

\((S34a-b)\)

These are precisely the relations that one obtains by using Eq. (S31) and \(F = Q(\theta) \tanh(\lambda/2)\) in Eq. (S11c) of Lemma 5. Therefore, we can regard \((v, w, u)\), instead of \((\zeta, a, b)\), as the independent variables parameterizing \(V\). From Eq. (S32), we have

\[
\begin{bmatrix} u/v \\ w/v \end{bmatrix} = \begin{bmatrix} \cos \theta \sech \lambda & \sin \theta \\ -\sin \theta \sech \lambda & \cos \theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.
\]

\((S35)\)

Inverting this linear system,

\[
\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos \theta \cosh \lambda & -\sin \theta \cosh \lambda \\ \sin \theta & \cosh \lambda \end{bmatrix} \begin{bmatrix} u/v \\ w/v \end{bmatrix}.
\]

\((S36)\)

Thus, the conditions \(a^2 + b^2 < 1\) and \(b \geq \sin \theta\) translate to:

\[
(a \cos \theta - w \sin \theta)^2 \cosh^2 \lambda + (w \cos \theta + u \sin \theta)^2 < v^2,
\]

\((S37a)\)

\[
w \cos \theta + u \sin \theta \geq v \sin \theta.
\]

\((S37b)\)
### S3. ABSOLUTE $q$-STABILITY PHASE DIAGRAM FOR $N = 2$ CHANNEL LUTTINGER LIQUID

The interaction matrix $V \in \mathcal{P}_4$ of a 2-channel Luttinger liquid depends on ten real parameters; these can be chosen according to Eq. (S25). Of the ten, only the four parameters $\xi \equiv (\lambda, \delta, \theta, \alpha)$ affect scaling dimensions; they determine the scaling dimension matrix $M(\xi)$ via Eq. (S20), which in turn determines $\Delta(m; \xi) \equiv \frac{1}{2} m^T M(\xi) m$.

Let $q(\xi)$ denote the absolute $q$-stability value of a Luttinger liquid with parameters $\xi$. By definition, $q(\xi)$ is the largest integer such that $\Delta(m; \xi) > 2$ for all nonzero $m \in \mathbb{Z}^4$ with $K(m) \in \mathbb{Z}$ and $|m| \leq q(\xi)$, where $K(m) = \frac{1}{2} m^T K m$, $K = I_2 \oplus -I_2$. Because of the inequality $\Delta(m) \geq |K(m)|$, one can restrict attention to those $m$ for which $K(m) = 0, \pm 1, \pm 2$. Thus, an equivalent definition of $q(\xi)$ to the one given above is: $q(\xi)$ is the smallest positive integer such that $\Delta(m; \xi) \leq 2$ for some $m \in \mathbb{Z}^4$ with $K(m) \in \{0, \pm 1, \pm 2\}$ and $|m| = q(\xi) + 1$.

We use the second definition to determine the phase diagram numerically. The algorithm is straightforward: at each point $\xi$, compute $\Delta(m; \xi)$ for all integer vectors $m$ with $K(m) \in \{0, \pm 1, \pm 2\}$ in shells of increasing $|m|$, until either a vector $m_*$ is found for which $\Delta(m_*; \xi) \leq 2$, or $|m|$ passes a specified cutoff value $q_*$. Set $q(\xi) = |m_*| - 1$ in the former case, and $q(\xi) = q_*$ in the latter. The shells of vectors with $|m| = 2, \ldots, q_*$ can be tabulated in advance, and the matrix $M(\xi)$ only needs to computed once at each point $\xi$. Figure 2 of the main text and Figure S2 below were obtained by this method, with cutoff $q_* = 22$.

![Figure S2](image-url)
By construction, $\tilde{K}_\nu (\nu = R/L)$ is a positive-definite integer matrix with determinant 1, and so the same is true of its inverse. Thus, $\tilde{K}_\nu^{-1}$ can be regarded as a *Gram matrix* of an $N$-dimensional unimodular integral lattice $\tilde{\Gamma}_\nu$ with positive definite inner product. Concretely, one can take the columns of $\tilde{K}_\nu^{-1/2}$ to form a basis for $\tilde{\Gamma}_\nu$, so that the lattice vectors are $\tilde{v}_\nu = \tilde{K}_\nu^{-1/2} \tilde{m}_\nu$, $\tilde{m}_\nu \in \mathbb{Z}^N$. The right/left scaling dimensions are equal to half the norm-squared of these lattice vectors,

$$\left( \tilde{\Delta}_m^R, \tilde{\Delta}_m^L \right) = \left( \frac{1}{2} |\tilde{v}_R|^2, \frac{1}{2} |\tilde{v}_L|^2 \right).$$

Non-chiral operators have $\tilde{\Delta}_m^R = \tilde{\Delta}_m^L$ and hence $|\tilde{v}_R| = |\tilde{v}_L|$. Thus, if all nonzero lattice vectors in $\tilde{\Gamma}_R$ or in $\tilde{\Gamma}_L$ have norm-squared $> 2$ (i.e. if at least one of the two lattices is “non-root”), then the corresponding Luttinger liquid phase is $\infty$-stable. There are of course chiral operators for which only one of $\tilde{v}_R$ or $\tilde{v}_L$ is nonzero. Therefore, to obtain an *absolutely* $\infty$-stable phase, the lattices $\tilde{\Gamma}_{R/L}$ must both have minimum norm-squared $> 4$.

Unimodular integral lattices are self-dual, so $\tilde{K}_\nu$ is also a Gram matrix of $\tilde{\Gamma}_\nu$ (possibly with respect to a different basis). Therefore, $\tilde{K} = \tilde{K}_R \oplus -\tilde{K}_L$ is a Gram matrix of the unimodular integral lattice $\tilde{\Gamma}_R \oplus \tilde{\Gamma}_L$ of signature $(N, N)$. Conjugating the Gram matrix $\tilde{K}$ by $W \in SL(2N, \mathbb{Z})$ corresponds merely to a basis change in this lattice. Thus, $\tilde{\Gamma}_R \oplus \tilde{\Gamma}_L \cong \mathbb{Z}^N N$, the signature $(N, N)$ lattice with Gram matrix $K = I_N \oplus -I_N$.

Let us summarize what we have accomplished so far. We have reduced the construction of $\infty$-stable (absolutely $\infty$-stable) phases of an $N$-channel Luttinger liquid to the identification of $N$-dimensional unimodular integral lattices $\tilde{\Gamma}_{R/L}$ with minimum norm-squared $> 2$ ($> 4$), subject to the constraint that $\tilde{\Gamma}_R \oplus \tilde{\Gamma}_L \cong \mathbb{Z}^N N$ as a lattice of signature $(N, N)$.

We now make use of two mathematical facts. The first fact is that there is a unique signature $(N, N)$ unimodular lattice of each parity (even/odd), where an integral lattice is *even* if the norm-squared of all lattice vectors is an even integer, and is *odd* otherwise [S3]. The lattice $\mathbb{Z}^N 2N$ with Gram matrix $K = \frac{1}{N} \mathbb{1}_N \oplus -\frac{1}{N} \mathbb{1}_N$ is clearly odd. Thus, $\tilde{\Gamma}_R \oplus \tilde{\Gamma}_L \cong \mathbb{Z}^N N$ if (and only if) at least one of $\tilde{\Gamma}_{R/L}$ is odd.

The second fact is that, for any positive integer $\mu$, there exists an $N$-dimensional positive definite unimodular lattice whose shortest nonzero vector has $|v|^2 = \mu$ [S4]. The required dimension $N$ increases with $\mu$; a theorem of Rains and Sloane [S5] states that

$$\mu \leq 2 \lfloor N/24 \rfloor + 2,$$

unless $N = 23$, in which case $\mu \leq 3$. Here $|x|$ denotes the integer part of $x$ (i.e. $x$ rounded down). Thus, to obtain $\mu = 3$ requires $N \geq 23$, and to obtain $\mu = 5$ requires $N \geq 48$.

In $N = 23$ dimensions, the shorter Leech lattice $\Lambda_{23}$ has minimum norm-squared $\mu = 3$. Correspondingly, there is an $\infty$-stable 23-channel Luttinger liquid with $\tilde{\Gamma}_R = \tilde{\Gamma}_L = \Lambda_{23}$, dubbed the “symmetric shorter Leech liquid” [12]. The “symmetric” modifier distinguishes this phase from the “asymmetric shorter Leech liquid” which has $\tilde{\Gamma}_R = \Lambda_{23}$ and $\tilde{\Gamma}_L = \mathbb{Z}^N 23$, and which is also $\infty$-stable. These phases are discussed in

\[ S4. CONSTRUCTION OF $\infty$-STABLE (ABSOLUTELY $\infty$-STABLE) LUTTINGER LIQUID PHASES WITH $N \geq 23$ ($N \geq 52$), FOLLOWING PLAMADELA ET AL. (2014) \]
more detail in Ref. [12], and the remarkable transport properties of the latter were analyzed in Ref. [S6].

In $N = 52$ dimensions, the lattice $G_{52}$ has $\mu = 5$ [S7], and there is a corresponding absolutely $\infty$-stable 52-channel Luttinger liquid with $\Gamma_R = \Gamma_L = G_{52}$.

S5. SPHERE PACKING BOUNDS AND THE NON-EXISTENCE OF ABSOLUTELY $\infty$-STABLE PHASES FOR $N < 11$

The sphere packing problem [S3] is to find the densest possible packing of non-overlapping spheres into $\mathbb{R}^n$. The density of a packing is simply the fraction of space that is contained inside the spheres. Given any lattice $\Gamma \subset \mathbb{R}^n$, we can obtain an associated sphere packing by placing spheres at each lattice point, with radii equal to half the length of the shortest lattice vector. If $\Gamma$ has a unit cell of volume $\Omega$ and shortest nonzero vector of length $2r$, then the density of the associated packing, $d_\Gamma$, simply equals the volume of an $n$-ball of radius $r$:

$$d_\Gamma = \frac{1}{\Omega} \frac{\pi^{n/2} r^n}{\Gamma(n/2 + 1)}.$$ (S48)

Hence, upper bounds on the density of sphere packings in $\mathbb{R}^n$ yield upper bounds on the length, $2r$, of the shortest nonzero vector in $\Gamma$.

For an $N$-channel LL, the scaling dimensions of bosonic vertex operators are given by $\Delta(m) = \frac{1}{2} |A m|^2$, with $A \in SO(N, N)$ and $m \in D_{2N}$, the “checkerboard lattice” $D_{2N} = \{ m \in \mathbb{Z}^{2N} : |m| \in 2\mathbb{Z} \}$. The checkerboard lattice has unit cell volume $\Omega = 2$. Since $\det A = 1$, the same holds for the deformed lattice $\Gamma \equiv A D_{2N} \subset \mathbb{R}^{2N}$. Absolute $\infty$-stability requires every nonzero vector in $\Gamma$ to have norm-squared $> 4$, which corresponds to $r > 1$. Thus, the corresponding sphere packing would have density

$$d_\Gamma > \frac{1}{2} \frac{\pi^{N}}{\Gamma(N + 1)}. \quad \text{(S49)}$$

For $N < 11$, this contradicts known upper bounds on the density of sphere packings due to Cohn and Elkies [19]. Hence, absolutely $\infty$-stable phases cannot exist with $N < 11$ channels.

[S1] J. J. Sylvester, A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares, Philos. Mag. 4, 138 (1852).

[S2] S. Boyd and L. Vandenberghe, Convex optimization (Cambridge University Press, Cambridge, UK, 2004).

[S3] J. H. Conway and N. J. A. Sloane, Sphere packings, lattices, and groups (Springer, New York, 1999).

[S4] J. Milnor and D. Husemoller, Symmetric bilinear forms (Springer, Berlin, 1973).

[S5] E. M. Rains and N. J. A. Sloane, The shadow theory of modular and unimodular lattices, J. Number Theory 73, 359 (1998), arXiv:math/0207294.

[S6] E. Plamadeala, M. Mulligan, and C. Nayak, Transport in a one-dimensional hyperconductor, Phys. Rev. B 93, 1 (2016), arXiv:1509.04280.

[S7] P. Gaborit, Construction of new extremal unimodular lattices, Eur. J. Comb. 25, 549 (2004).