Cosmology of non-Hermitian (C)PT-invariant scalar matter

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Abstract. We use the generalization of quantum theory for the case of non-Hermitian Hamiltonians with (C)PT symmetry to show how a classical cosmological model describes a smooth transition from ordinary dark energy to the phantom one. We consider the PT symmetric flat Friedmann model of two scalar fields with positive kinetic terms. The solution for the normal field is real while the solution for the second field is purely imaginary, realizing classically the “phantom” behavior. The energy density and pressure happen to be real and the corresponding geometry is well-defined. The Lagrangian for the linear perturbations leads to positive energy densities for both the fields, so that the problem of stability does not arise. The phantom phase in the cosmological evolution appears to be transient and the Big Rip never occurs.

1. Introduction
The discovery of cosmic acceleration [1] has triggered an intensive building of dark energy models [2] accounting for its origin. Dark energy is characterized by a negative pressure whose ratio to the energy density $w = p/\varepsilon$ should be less than $-1/3$. If it is less than $-1$
such type of dark energy is usually entitled as “phantom” dark energy [3]. It implies the so-called superacceleration effect, which in some models culminates approaching a new type of cosmological singularity called Big Rip [4]. While the cosmological constant \( w = -1 \) is still a possible candidate for the role of dark energy, there are observations that give some indications in favor of the models where the equation of state parameter \( w \) changes with time and moreover is less than \(-1\) now [5, 6]. A way to introduce the phantom energy is to consider a scalar field with the negative sign of the kinetic energy term. However such a model has been reasonably criticized insofar as it is unstable with respect to linear perturbations [7].

Here we propose a cosmological model inspired by PT symmetric theory [8] of scalar fields in flat space-time, selecting complex potentials which possess classical phantom solutions for homogeneous and isotropic universe. Complex (non-Hermitian) Hamiltonians with PT symmetry have been extensively investigated in quantum mechanics and quantum field theory [9]. We use a particular complex scalar field Lagrangian, which has real solutions of the classical equations of motion. Thereby we provide a cosmological model describing in a natural way an evolution involving the transition from normal matter to phantom matter, crossing smoothly the phantom divide line. Meanwhile quantum fluctuations have positive energy density and this ensures the stability around a classical background configuration. Thus we explore the classical phantom behavior among complex potentials providing the real energy spectrum bounded from below [8, 9, 10, 11].

The PT symmetric approach to the extension of quantum physics consists in the weakening of the requirement of Hermiticity, while keeping all the physical observables real. It was shown that the axioms of quantum theory are maintained if the complex extension preserves (C)PT symmetry [9], accordingly the energy spectrum remains real and bounded from below and physical observables are properly identified using a homogeneity transformation relating a non-Hermitian quantum system to a Hermitian one. Some attempts to apply the PT symmetric formalism to cosmology were discussed in [12]. We shall consider the complex extension of matter Lagrangians requiring the reality of all the physically measurable quantities and the well-definiteness of geometrical characteristics. We start with the flat Friedmann model of two scalar fields with positive kinetic terms. The potential of the model is additive. While the potential of one (normal) field is real, that of the other field is complex. We find a classical complex solution of the system of the two Klein-Gordon equations together with the Friedmann equation. The solution for the “normal” field is real while the solution for the second field is purely imaginary realizing classically the “phantom” behavior. The geometry is well-defined and the energy density is always positive. Moreover, the effective Lagrangian for the linear perturbations has the correct potential signs for both the fields and provides the positive energy density, so that the problem of the stability does not arise reflecting, in fact, that the energy spectrum of quantum states remains real and bounded from below. Nevertheless, the background (homogeneous Friedmann) dynamics is determined by an effective action including two real fields one normal and one phantom.

Remarkably, in this approach the phantom phase in the cosmological evolution is inevitably transient. The number of phantom divide line (PDL) crossings, (i.e. events such that the ratio \( w \) between pressure and energy density passes through the value \(-1\)) can be only even and the Big Rip never occurs. It makes it drastically different from the genuine “quintom”-like models [13] in which the Big Rip seems to be unavoidable and quantum perturbations destabilize the universe.

2. Complex Lagrangians in classical field theory and cosmology
Let us consider a non-Hermitian (complex) Lagrangian of a scalar field

\[
L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi^* - V(\phi, \phi^*),
\]
with the corresponding action,

\[ S(\phi, \phi^*, g) = \int d^4x \sqrt{-g}(L + \frac{1}{6}R(g)), \]  

(2)

where \( \|g\| \) stands for the determinant of a metric \( g^{\mu\nu} \) and \( R(g) \) is the scalar curvature term and the Newton gravitational constant is normalized to \( 3/8\pi \) to simplify the Friedmann equations further on.

We employ potentials \( V(\Phi, \Phi^*) \) satisfying the invariance condition

\[ (V(\Phi, \Phi^*))^* = V(\Phi^*, \Phi), \]  

(3)

while the condition

\[ (V(\Phi, \Phi^*))^* = V(\Phi, \Phi^*), \]  

(4)

is not satisfied. This condition represents a generalized requirement of \((C)PT\) symmetry.

Let’s define two real fields,

\[ \phi \equiv \frac{1}{2}(\Phi + \Phi^*), \quad \chi \equiv \frac{1}{2i}(\Phi - \Phi^*). \]  

(5)

Then, for example, such a potential can have a form

\[ V(\Phi, \Phi^*) = \tilde{V}(\Phi + \Phi^*, \Phi - \Phi^*) = \tilde{V}(\phi, i\chi), \]  

(6)

where \( \tilde{V}(x, y) \) is a real function of its arguments. In the last equation one can recognize the link to the so called \((C)PT\) symmetric potentials if to supply the field \( \chi \) with a discrete charge or negative parity.

Here, the functions \( \phi \) and \( \chi \) appear as the real and the imaginary parts of the complex scalar field \( \Phi \), however, in what follows, we shall treat them as independent spatially homogeneous variables depending only on the time parameter \( t \) and, when necessary, admitting the continuation to complex values.

We shall consider a flat spatially homogeneous Friedmann universe with the metric

\[ ds^2 = dt^2 - a^2(t)dl^2, \]  

(7)

satisfying the Friedmann equation

\[ h^2 = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\chi}^2 + \tilde{V}(\phi, i\chi). \]  

(8)

The variable \( a(t) \) represents a cosmological radius of the universe and the Hubble variable \( h(t) \equiv \dot{a}/a \) characterizes the velocity of expansion of the universe. The Friedmann equation (8) is nothing but Einstein equation, for the universe filled by scalar fields.

The equations of motion for fields \( \phi \) and \( \chi \) have the form

\[ \ddot{\phi} + 3h\dot{\phi} + \frac{\partial \tilde{V}(\phi, i\chi)}{\partial \phi} = 0, \]  

(9)

\[ i\ddot{\chi} + 3ih\dot{\chi} - \frac{\partial \tilde{V}(\phi, i\chi)}{i\partial \chi} = 0. \]  

(10)

Equations (8), (9) and (10) are obtained by variation of the action (2) with the lagrangian (1) and the potential (6) with respect to the metric, and the scalar field variables \( \phi \) and \( \chi \). Evidently, a purely imaginary solution exists for eq. (10) which makes both the Lagrangian and the energy real and therefore fits well the reality of energy density and the pressure in the Friedmann universe.
3. Advent of phantom and stability

Now let us introduce the two-field scalar Lagrangian with the special type of $(C)PT$ invariant complex potential

\[ L = \frac{\dot{\phi}^2}{2} + \frac{\dot{\chi}^2}{2} - Ae^{\alpha \phi} + Be^{\beta \chi}, \quad (11) \]

where $A$ and $B$ are real positive constants. This Lagrangian is the sum of two terms. The term representing the scalar field $\phi$ is a standard matter one, and it can generate a power-law cosmological expansion [14]. The kinetic term of the scalar field $\chi$ looks also standard, but its potential is complex. Dynamical systems of this type have been studied in the framework of PT symmetric quantum mechanics [8] in the flat space-time and, in particular, the exponential potential has been analyzed in great detail in [11]. Therein the most important feature of this potential has been proven that the spectrum of the corresponding Hamiltonian is real and bounded from below, provided correct boundary conditions are assigned. Inspired by this fact we shall look for a classical complex solution of the system, consisting of two Klein-Gordon equations for the fields $\phi$ and $\chi$, (9) and (10), and the Friedmann equation (8).

As it has been noticed a suitable classical solution which we are looking for should provide the reality and positivity of the right-hand side of the Friedmann equation (8). The solution where the scalar field $\phi$ is real, while the scalar field $\chi$ is purely imaginary $\chi = i\xi$, uniquely satisfies this condition. Accordingly, the Lagrangian (11) evaluated on this solution is real as well. This is important because on homogeneous solutions the Lagrangian coincides with the pressure, which indeed should be real.

Substituting $\chi = i\xi$ into the Friedmann equation (8) we shall have

\[ h^2 = \frac{\dot{\phi}^2}{2} - \frac{\dot{\xi}^2}{2} + Ae^{\alpha \phi} - Be^{-\beta \xi}, \quad (12) \]

Hence, classically we have the Friedmann equation with two fields: one normal and one phantom. In the next section we shall study the cosmological dynamics of the system, including (9), (12) and

\[ \ddot{\xi} + 3h \dot{\xi} - B\beta e^{-\beta \xi} = 0. \quad (13) \]

The distinguishing feature of such an approach to the construction of effective phantom Lagrangians becomes clear when one calculates the linear perturbations around the classical solutions. Indeed the second variation of the action for the field $\chi$ gives the quadratic part of the effective Lagrangian of perturbations:

\[ L_{eff}^{(2)} = \frac{1}{2} \delta \dot{\chi}^2 - B\beta^2 e^{-\beta \xi_0}(\delta \chi)^2, \quad (14) \]

where $\xi_0$ is a homogeneous classical solution of the dynamical system under consideration. It is easy to see that the effective Lagrangian (14) is real and its potential term has a sign providing the stability of the background solution with respect to linear perturbations as the related Hamiltonian is positive,

\[ H_{eff}^{(2)} = \frac{1}{2} \delta \pi^2 + B\beta^2 e^{-\beta \xi_0}(\delta \chi)^2; \quad \delta \pi \Leftrightarrow \delta \dot{\chi}. \quad (15) \]

Let us list the main differences between our Lagrangian and corresponding solutions vs. some similar models, using two fields (normal scalar and phantom) and exponential potentials [13]. First, we begin with two normal (non-phantom) scalar fields, with normal kinetic terms, but one of these fields is associated to a complex exponential potential. Second, the sign of the real constant multiplying this exponential potential is negative. Third, the background classical
solution of the dynamical system consisting of two Klein-Gordon equations and the Friedmann equation is such that the second field is purely imaginary, while all the geometric characteristics are well-defined, i.e. real. Fourth, the interplay between transition to the purely imaginary solution of the equation for the field \( \chi \) and the negative sign of the corresponding potential provides us with the effective Lagrangian for the linear perturbations of this field which have correct sign for both the kinetic and potential terms and in such a way the problem of stability of the our effective phantom field is resolved.

In the next section we shall describe the cosmological solutions for our system of equations.

4. Cosmological evolution without Big Rip

First of all notice that our dynamical system permits the existence of cosmological trajectories which cross PDL. Indeed, the crossing point is such that the time derivative of the Hubble parameter

\[
\dot{h} = -\frac{3}{2}(\dot{\phi}^2 - \dot{\xi}^2)
\]

is equal to zero. We always can choose \( \dot{\phi} = \pm \dot{\xi} \), at \( t = t_{PDL} \) provided the values of the fields \( \phi(t_{PDL}) \) and \( \xi(t_{PDL}) \) are chosen in such a way, that the general potential energy \( A e^{\alpha \phi} - B e^{-\beta \xi} \) is non-negative. Obviously, \( t_{PDL} \) is the moment of PDL crossing. However, the event of the PDL crossing cannot happen only once. Indeed, the fact that the universe has crossed phantom divide line means that it was in effectively phantom state before or after such an event, i.e. the effective phantom field \( \xi \) dominated over the normal field \( \phi \). However, if this dominance lasts for a long time it implies that non only the kinetic term \(-\dot{\xi}^2/2\) dominates over the kinetic term \(-\dot{\phi}^2/2\) but also the potential term \(-B \exp(-\beta \xi)\) should dominate over \( A \exp(\alpha \phi)\); but it is impossible, because contradicts to the Friedmann equation (12). Hence, the period of the phantom dominance should finish and one shall have another point of PDL crossing. Generally speaking, only the regimes with even number of PDL crossing events are possible. Numerically, we have found only the cosmological trajectories with the double PDL crossing. Naturally, the trajectories which do not experience PDL crossing at all also exist and correspond to the permanent domination of the normal scalar field. Obviously there is no place for the Big Rip singularity in this picture, because such a singularity is connected with the drastically dominant behavior of the effective phantom field, which is impossible as was explained above. In other words, the impossibility of approaching the Big Rip singularity can be argued as follows. Approaching the Big Rip, one has a growing behavior of the scale factor \( a(t) \) of the type \( a(t) \sim (t_{BR} - t)^{-q} \), where \( q > 0 \). Then the Hubble parameter is

\[
h(t) = \frac{q}{t_{BR} - t}
\]

and its time derivative is \( \dot{h}(t) = q(t_{BR} - t)^{-2} \). Then according to eq. (16)

\[
\frac{\dot{\xi}^2}{2} - \frac{\dot{\phi}^2}{2} = \frac{q}{3(t_{BR} - t)^2}.
\]

Substituting eq. (18) into the Friedmann equation (12), we notice that the potential of the scalar field \( \phi \) should behave as \( 1/(t_{BR} - t)^2 \). Hence the field \( \phi \) should be

\[
\phi = \phi_0 - \frac{2}{\alpha} \ln(t_{BR} - t),
\]

where \( \phi_0 \) is an arbitrary constant. Now substituting eqs. (17) and (19) into the Klein-Gordon equation for the scalar field \( \phi \) (9), the condition of the cancellation of the most singular terms in this equation which are proportional to \( 1/(t_{BR} - t)^2 \) reads

\[
2 + 6q + A \alpha^2 \exp(\alpha \phi_0) = 0.
\]
This condition cannot be satisfied because all the terms in the left-hand side of eq. (20) are positive. This contradiction demonstrates that it is impossible to reach the Big Rip.

Now we describe briefly some cosmologies contained in our model.

**Figure 1.** The evolution starts from a Big Bang-type singularity and goes through a transient phase of superaccelerated expansion (“phantom era”), which lies between two crossings of PDL (points $A$ and $B$). Then the universe expands infinitely.

**Figure 2.** The evolution starts with a contraction in the infinitely remote past. At the point $A$ the contraction becomes superdecelerated and turns in a superaccelerated expansion ($B$). In $C$ the second PDL crossing ends the ”phantom era”; the decelerated expansion continues till the universe begins contracting. In a finite time a Big Crunch-type singularity is reached.

**Figure 3.** The cosmological evolution begins with a contraction in the infinitely remote past. At point $A$ the model crosses PDL: the contraction becomes superdecelerated until the universe stops ($h = 0$) and starts expanding. At $B$ the “phantom era” ends and the expansion continues infinitely.

**Figure 4.** Evolution from a Big Bang-type singularity to an infinite expansion, without any crossing of PDL. This evolution is thus guided by the ”normal” field $\phi$.

5. Conclusion

As is known the data are compatible with the presence of the phantom energy, which can be in a most natural way realized by the phantom scalar field with a negative kinetic term. Such a field suffers from the instability problem, which makes it vulnerable. Inspired by the development of PT symmetric quantum theory [9] we introduced the PT symmetric two-field cosmological model where both the kinetic terms are positive, but the potential of one of the fields is exponential with complex phase. This type of quantum mechanics is known to possess a real spectrum bounded from below [11] when it is realized in flat space-time. We extended
our study to a classical background solution of two Klein-Gordon equations together with the Friedmann equation, when one of this fields (normal) is real while the other is purely imaginary. The scale factor in this case is real and positive — compatible with positivity of the energy density and the pressure. The background dynamics of the universe is determined by two effective fields - one normal and one phantom, while the Lagrangian of the linear perturbations determines the real positive Hamiltonian. Thus, the problem of instability is absent. Whereas in such models the rigorous proof of the real energy spectrum bounded from below has been done in one-dimensional quantum mechanics embedded in flat space-time our results make it plausible that the full quantum field theory of two scalar field with a (C)PT invariant complex interaction in the Friedmann metric background may well be consistent, at least, for certain types of interactions justified in quantum mechanics.

As a by product of the structure of the model, the phantom dominance era is transient, the number of the phantom divide line crossings is even and the Big Rip singularity is excluded.

Acknowledgments
A.A. is grateful to the Organizing Committee of the Symposium on Prospects in the Physics of Discrete Symmetries DISCRETE'08 for the hospitality. This work was partially supported by Grants RFBR 08-02-00923 and LSS-4899.2008.2. The work of A.A. was supported by grants 2005SGR00564, 2009SGR, FPA2007-66665, by the Consolider-Ingenuo 2010 Program CPAN (CSD2007-00042), by Grant RFBR and Program RNP 2.1.1.10046.

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