A Small and Non-simple Geometric Transition

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Abstract Following notation introduced in the recent paper (Rossi Int. J. Geom. Methods Mod. Phys. 12(5), 2015), this paper is aimed to present in detail an example of a small geometric transition which is not a simple one i.e. a deformation of a conifold transition. This is realized by means of a detailed analysis of the Kuranishi space of a Namikawa cuspidal fiber product, which in particular improves the conclusion of Y. Namikawa in Remark 2.8 and Example 1.11 of Namikawa (Topology 41(6), 1219–1237, 2002). The physical interest of this example is presenting a geometric transition which can’t be immediately explained as a massive black hole condensation to a massless one, as described by Strominger (Nucl. Phys. B451, 97–109, 1995).

Keywords Fiber products of rational elliptic surfaces · Smoothing of singularities · Resolution of singularities · Calabi–Yau threefolds · Calabi–Yau web · Geometric transition · Conifold transition · Deformation of a Calabi–Yau threefold · Deformation of a geometric transition

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1 Introduction

Let $X$ be a complex projective threefold with terminal singularities and admitting a small resolution $\hat{X} \xrightarrow{\phi} X$ such that $\hat{X}$ is a Calabi–Yau threefold (in the sense of Definition 1), where “small” means that the exceptional locus $\text{Exc}(\phi)$ has codimension greater than or equal to two. Then it is well known that $\text{Exc}(\phi)$ consists of a finite disjoint union of trees of rational curves of A–D–E type [7, 21, 26, 32, 36]. In his paper [28], Remark 2.8, Y. Namikawa considered the following

**Problem** When does $\hat{X}$ have a flat deformation such that each tree of rational curves splits up into mutually disjoint $(-1, -1)$–curves?

Let us recall that a $(-1, -1)$–curve is a rational curve in $X$ whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. It arises precisely as exceptional locus of the resolution of an ordinary double point (a node) also called a conifold point since it is an isolated hypersurface singularity whose tangent cone is a non–degenerate quadratic cone.

Namikawa’s problem is interesting in the context of H. Clemens type problems of cycle deformations (see e.g. [7], Corollary (4.11)). Moreover, it is of significant interest in the context of (deformations of) geometric transitions and therefore in the study of the moduli space for Calabi–Yau threefolds. Let us recall that a geometric transition (gt) between two Calabi–Yau threefolds is the process obtained by “composing” a birational contraction to a normal threefold with a complex smoothing (see Definition 4). If the normal intermediate threefold has only nodal singularities then the considered gt is called a conifold transition. The interest in geometric transitions goes back to the ideas of H. Clemens [5] and M. Reid [37] which gave rise to the so called Calabi–Yau Web Conjecture (see also [12] for a revised and more recent version) stating that (more or less) all Calabi–Yau threefolds can be connected to each other by means of a chain of geometric transitions, giving a sort of (unexpected) “connectedness” of the moduli space for Calabi–Yau threefolds. There is also a considerable physical interest in geometric transitions owing to the fact that they connect topologically distinct models of Calabi–Yau vacua: the physical version of the Calabi–Yau Web Conjecture is a sort of (in this case expected) “uniqueness” of a space–time model for supersymmetric string theories (see e.g. [4] and references therein).

In this context, Namikawa’s problem can then be rephrased as follows

**Problem** (for small geometric transitions) When does a small gt have the same “deformation type” (see Definition 5) of a conifold transition?

Since the geometry of a general gt can be very intricate, while the geometry of a conifold transition is relatively easy and well understood as a topological surgery [5], the mathematical interest of such a problem is evident.

On the other hand, conifold transitions were the first (and among the few) geometric transitions to be physically understood as massive black holes condensation to massless ones, by A. Strominger [43]. Answering the given problem would then
give a significant improvement in the physical interpretation of (at least the small) geometric transitions bridging topologically distinct Calabi–Yau vacua.

Unfortunately in [28], Remark 2.8, Namikawa observed that a flat deformation positively resolving the given problem “does not hold in general” and produced an example of a cuspidal fiber self–product of an elliptic rational surface with sections whose resolution admits exceptional trees, composed of couples of rational curves intersecting at one point, which should not deform to a disjoint union of \((-1, -1)\)–curves. Nevertheless, Example 1.11 in [28], supporting such a conclusion, did not give a correct argument since the proposed deformation is actually a trivial one.

In the present paper we will overcome this argument by giving a detailed analysis of the Kuranishi space of the Namikawa cuspidal fiber product, allowing us to go far beyond his conclusion in [28], Remark 2.8. Moreover this will give rise to an explicit example of a small gt which is not a simple one i.e. it has not the same deformation type of a conifold transition: such an example has already been sketched in §9.2 of [40], without any proof. Here all the needed details will be given.

The paper is organized as follows. In Section 2 we introduce notation, preliminaries and main facts needed throughout the paper. Section 3 is then devoted to present the Namikawa construction of a fiber self–product of a particular elliptic rational surface with sections and singular “cuspidal” fibers (which will be called cuspidal fiber product). These are threefolds admitting six isolated singularities of Kodaira type \(I\!I \times I\!I\) which have been rarely studied in either the pioneering work of C. Schoen [41] or the recent [16]. For this reason, their properties, small resolutions and local deformations are studied in detail. In particular, all the local deformations induced by global versal deformations are studied in Proposition 6, while all the local deformations of a cuspidal singularity to three distinct nodes are studied in Proposition 7. They actually do not lift globally to the given small resolution, as stated by Theorem 4, revising the Namikawa considerations of [28], Remark 2.8 and Example 1.11: see Remark 3 and Theorem 5. The last Section 4 is dedicated to apply Theorem 4 to deformations of geometric transitions: for further details the interested reader is referred to [40] §7.1.

2 Preliminaries and Notation

**Definition 1** (Calabi–Yau 3–folds) A smooth, complex, projective 3–fold \(X\) is called Calabi–Yau if

1. \(K_X \cong \mathcal{O}_X\),
2. \(h^{1,0}(X) = h^{2,0}(X) = 0\).

The standard example is the smooth quintic threefold in \(\mathbb{P}^4\). The given definition is equivalent to require that \(X\) has holonomy group a subgroup of \(SU(3)\) (see [15] for a complete description of equivalences and implications).
2.1 Deformations of complex spaces

Let us start by recalling that a complex space is a ringed space \((X, \mathcal{O}_X)\) where \(X\) is a Hausdorff topological space locally isomorphic to a locally closed analytic subset of some \(\mathbb{C}^n\) and \(\mathcal{O}_X\) is the induced sheaf of holomorphic functions. A pointed complex space is a pair \((X, x)\) consisting of a complex space \(X\) and a distinguished point \(x \in X\). A morphism \(f : (X, x) \to (Y, y)\) of pointed complex spaces is a morphism \(f : X \to Y\) of complex spaces such that \(f(x) = y\).

Complex space germs are pointed complex spaces whose morphisms are given by equivalence classes of morphisms of pointed complex spaces defined in some open neighborhood of the distinguished point. Let \((X, x)\) be a complex space germ and \(U \subset X\) an open neighborhood of \(x\): then the inclusion map \(U \hookrightarrow X\) gives an isomorphism of complex space germs \((U, x) \cong (X, x)\). Then \(U\) is called a representative of the germ \((X, x)\).

A complex space germ is also called a singularity.

**Definition 2** (Deformation of a complex space, [30] §5, [1]§ XI.2) Let \(X\) be a complex space. A deformation of \(X\) is a flat, holomorphic map \(f : \mathcal{X} \to (B, o)\) from the complex space \(\mathcal{X}\) over a complex space germ \((B, o)\), endowed with an isomorphism \(X \cong \mathcal{X}_o := f^{-1}(o)\) on the central fibre, for short denoted by \(f : (\mathcal{X}, X) \to (B, o)\). If \(X\) is compact then we will also require that \(f\) is a proper map. If \(X\) is singular and the fibre \(X_b = f^{-1}(b)\) is smooth, for some \(b \in B\), then \(X\) is called a smoothing family of \(X\). With a slight abuse of notation, if \(b \neq o\) then the fiber \(X_b\) is called either a deformation or a smoothing of \(X\) when \(X\) is a deformation or a smoothing family of \(X\), respectively.

Let \(\Omega_X\) be the sheaf of holomorphic differential forms on \(X\) and consider the Lichtenbaum–Schlessinger cotangent sheaves [22] of \(X\), \(\Theta^i_X = \text{Ext}^i(\Omega_X, \mathcal{O}_X)\). Then \(\Theta^0_X = \text{Hom}(\Omega_X, \mathcal{O}_X) =: \Theta_X\) is the “tangent” sheaf of \(X\) and \(\Theta^i_X\) is supported over \(\text{Sing}(X)\), for any \(i > 0\). Consider the associated local and global deformation objects

\[
T^i_X := H^0(X, \Theta^i_X), \quad \mathcal{T}^i := \text{Ext}^i\left(\Omega^1_X, \mathcal{O}_X\right), \quad i = 0, 1, 2.
\]

Then by the local to global spectral sequence relating the global Ext and sheaf \(\text{Ext}\) (see [13] and [8] II, 7.3.3) we get

\[
E^{p,q}_2 = H^p\left(X, \Theta^q_X\right) \Longrightarrow \mathcal{T}^{p+q}_X
\]

giving that

\[
\mathcal{T}^0_X \cong T^0_X \cong H^0(X, \Theta_X), \quad \text{if } X \text{ is smooth then } \mathcal{T}^i_X \cong H^i(X, \Theta_X), \quad \text{if } X \text{ is Stein then } T^i_X \cong \mathcal{T}^i_X.
\]
Given a deformation family $X \xrightarrow{f} B$ of $X$ for each point $b \in B$ there is a well defined linear (and functorial) map

$$D_{bf} : T_{bB} \to T_{1Xb}$$

(Generalized Kodaira–Spencer map)

where $T_{bB}$ denotes the Zariski tangent space to $B$ at $b$ ([11] Lemma II.1.20, [31] §2.4).

For the following terminology the reader is referred to [23] §6.C, [11] §1.3, [30] Def. 5.1, [31] §2.6, among many others. A deformation $f : (\mathcal{X}, X) \to (B, o)$ of a complex space $X$ is called \emph{versal} (some authors say \emph{complete}) if, for any deformation $g : (\mathcal{Y}, X) \to (C, p)$ of $X$, there exists a holomorphic map of germs of complex spaces $h : (C, p) \to (B, o)$ such that $g = h^*(f)$ i.e. the following diagram commutes

$$\begin{array}{ccc}
\mathcal{Y} = U \times_B \mathcal{X} & \xrightarrow{g=h^*(f)} & \mathcal{X} \\
\downarrow & & \downarrow f \\
C & \xrightarrow{h} & B
\end{array}$$

In particular the generalized Kodaira–Spencer map $\kappa(f)$ turns out to be surjective ([31], §2.6). Moreover the deformation $f$ is said to be an \emph{effective versal} (or \emph{miniversal}) deformation of $X$ if it is versal and $\kappa(f)$ is injective, hence an isomorphism. Finally the deformation $f$ is said to be \emph{universal} if it is versal and $h$ is uniquely defined. This suffices to imply that $f$ is an effective versal deformation of $X$ ([31], §2.7.1).

The following result is a central one in the theory of deformation of complex spaces: it is due to many authors as A. Douady [6], H. Grauert [9] and [10], V.P. Palamodov [29], among many others.

**Theorem 1** (Existence of a versal deformation) 1. Any representative $U$ of an isolated singularity $(X, x)$ has a miniversal deformation

$$f : (U, U) \to (B, o)$$

(see [9, 11]).

2. Any (compact) complex space $X$ has an effective versal deformation

$$f : (\mathcal{X}, X) \to (B, o)$$

(see [29] for the non compact case and [6, 10, 30] for the compact case).

3. The germ of complex space $(B, o)$ obtained by the previous parts (1) and (2) is isomorphic to the germ $(q_X^{-1}(0), 0)$, where $q_X : T^1_X \to T^2_X$ is a suitable holomorphic map (the obstruction map) such that $q_X(0) = 0$ (see [11] II.1.5, [31] §2.5)

In particular if $q_X \equiv 0$ (e.g. when $T^2_X = 0$) then $(B, o)$ turns out to be isomorphic to the complex space germ $(T^1_X, 0)$.

**Definition 3** (Kuranishi space and number) The germ of complex space $(B, o)$ of parts (1) and (2) of the previous Theorem 1 is called the \emph{Kuranishi space of either $(X, x)$ or $X$, respectively}: in the first case one can easily check that it does not depend
on the choice of the representative $U$. The Kuranishi space is said to give an analytic representative of the deformation functors $\text{Def}(U)$ and $\text{Def}(X)$, respectively, allowing one to set the identifications

$$\text{either } (\text{Def}(X, x) :=) \text{ Def}(U) = (B, o) \text{ or } \text{Def}(X) = (B, o) , \text{ respectively.}$$

The Kuranishi numbers $\text{def}(X, x)$ of $(X, x)$ and $\text{def}(X)$ of $X$, are then the maximum dimensions of irreducible components of $\text{Def}(X, x)$ and $\text{Def}(X)$, respectively.

$\text{Def}(X, x)$ and $\text{Def}(X)$ are said to be smooth, or unobstructed, if the obstruction map $q_X$ is the constant map $q_X \equiv 0$. This means that any first order deformation arises to give a deformation of either $U$ or $X$, respectively. In this case either $\text{Def}(X, x) \cong (T^1_U, 0)$ or $\text{Def}(X) \cong (T^1_X, 0)$, respectively, giving either $\text{def}(X, x) = \dim_\mathbb{C} T^1_U$ or $\text{def}(X) = \dim_\mathbb{C} T^1_X$, respectively. By a slight abuse of notation, we will write

$$\text{either } \text{Def}(X, x)\cong T^1_U \text{ or } \text{Def}(X)\cong T^1_X , \text{ respectively.}$$

**Theorem 2** ([30] Thm. 5.5) *Let $X$ be a compact complex space such that $T^0_X = 0$. Then the versal effective deformation of $X$, given in part (2) of Theorem 1, is actually a universal deformation of $X$.***

**Example 1** (The germ of complex space of an isolated hypersurface singularity) Let $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}, n \geq 0$, be a holomorphic map admitting an isolated critical point in the origin $0 \in \mathbb{C}^{n+1}$ and consider the local ring $\mathcal{O}_0$ of germs of holomorphic function of $\mathbb{C}^{n+1}$ in the origin. It is a well known fact that $\mathcal{O}_0$ is isomorphic to the ring of convergent power series $\mathbb{C}\{x_1, \ldots, x_{n+1}\}$. A germ of hypersurface singularity $(U_0, 0)$ is defined by means of the Stein complex space

$$U_0 := \text{Spec}(\mathcal{O}_{f,0}) \quad (4)$$

where $\mathcal{O}_{f,0} := \mathcal{O}_0/(\overline{f})$ and $\overline{f}$ is the germ represented by the series expansion of the above given holomorphic function $f$. Let $J_f \subset \mathbb{C}\{x_1, \ldots, x_{n+1}\}$ be the jacobian ideal of $f$ i.e. the ideal generated by the partial derivatives of $f$. Then the following facts hold:

- since $U_0$ is Stein, (3) gives that $T^i_{U_0} \cong T^i_{U_0}$ ,

- since we are dealing with an isolated hypersurface singularity,

$$T^i_{U_0} \cong T^i_{U_0} = 0 , \quad \text{for } i \geq 2 \quad (5)$$

(for the case $i = 2$, which is all what is useful in the following, see e.g. [42] §3 Example on pg. 26; for $i \geq 2$ see e.g. [11] Prop. C.4.6(3)),

- the Kuranishi space of an isolated hypersurface singularity is then completely described as follows

$$\text{Def}(U_0, 0) \cong T^1_{U_0} \cong \mathcal{O}_{f,0}/J_f \cong \mathbb{C}\{x_1, \ldots, x_{n+1}\}/((\overline{f}) + J_f) \quad (6)$$

(the first isomorphism follows by Theorem 1 and the previous (5); for the second isomorphism see e.g. [42] §3 Example on pg. 24, [11] Corollary II.1.17).
Example 2 (The deformation theory of a Calabi–Yau threefold) Let us now consider the case of a Calabi–Yau threefold $X$. A central result in the deformation theory of Calabi–Yau manifolds is the well–known Bogomolov–Tian–Todorov–Ran Theorem \cite{3, 34, 44, 45} asserting that the Kuranishi space $\text{Def}(X)$ is smooth, hence $\text{Def}(X) \cong T^1_X$ and (2) gives that

$$
def(X) = \dim \mathbb{C} T^1_X = h^1(X, \Theta_X) = h^{2,1}(X)$$

(7)

where the last equality on the right is obtained by the Calabi–Yau condition $K_X \cong \mathcal{O}_X$. Applying the Calabi–Yau condition once again gives $h^0(\Theta_X) = h^{2,0}(X) = 0$. Therefore (1) and Theorem 2 give the existence of a universal effective family of Calabi–Yau deformations of $X$. In particular $h^{2,1}(X)$ turns out to be the dimension of the complex moduli space of $X$.

2.2 Deformation of a morphism

Let us quickly recall the concept of deformation of a morphism as defined by Z. Ran in \cite{33}.

Consider a morphism $\phi : Y \to X$ of complex spaces and let $B$ be a connected complex space with a special point $o \in B$ such that $g : (\mathcal{Y}, Y) \to (B, o)$ and $f : (\mathcal{X}, X) \to (B, o)$ are deformation families of $Y$ and $X$, respectively. Then a deformation family of the morphism $\phi$ is a morphism $\Phi : \mathcal{Y} \to \mathcal{X}$ such that the following diagram commutes

$$
\begin{array}{ccc}
Y & \xrightarrow{\phi} & \mathcal{Y} \\
\downarrow \phi & & \downarrow \Phi \\
X & \xrightarrow{f} & \mathcal{X} \\
\downarrow g & & \downarrow \\
o \in B & &
\end{array}
$$

(8)

with $Y = g^{-1}(o)$, $X = f^{-1}(o)$ and $\phi = \Phi|_{g^{-1}(o)}$.

Given two distinct points $b_1, b_2 \in B$ the morphism $\phi_2 := \Phi|_{g^{-1}(b_2)}$ is called a deformation of the morphism $\phi_1 := \Phi|_{g^{-1}(b_1)}$ and viceversa.

2.3 The Friedman diagram

Let $\phi : Y \to X$ be a birational contraction of a Calabi–Yau manifold $Y$, of dimension $n \geq 3$, to a normal variety $X$ with isolated rational singularities and assume that the codimension of the exceptional locus $E := \text{Exc}(\phi)$ is greater than or equal to 2: hence $\phi$ is what is usually called a small resolution of $X$. This latter assumption ensures that the Friedman argument of Lemma (3.1) in \cite{7} still applies to give that
\[ R^0 \phi_* \Theta_Y \cong \Theta_X. \] Then we get the following commutative diagram, to which we refer as the Friedman diagram (see [7] (3.4)),

\[
\begin{array}{cccccc}
H^1(R^0 \phi_* \Theta_Y) & \longrightarrow & T^1_Y & \longrightarrow & H^0(R^1 \phi_* \Theta_Y) & \longrightarrow & H^2(R^0 \phi_* \Theta_Y) \longrightarrow T^2_Y \\
\delta_1 \downarrow & & \lambda_p \downarrow & & \delta_{loc} \downarrow & & \delta_2 \\
H^1(\Theta_X) & \longrightarrow & T^1_X & \longrightarrow & T^1_X & \longrightarrow & H^2(\Theta_X) \longrightarrow T^2_X \\
\end{array}
\]

(9)

where:

- the first row is the lower terms exact sequence of the Leray spectral sequence of \( \phi_* \Theta_Y \), where we can use (2) as \( Y \) is smooth;
- the second row is the lower terms exact sequence of the Local to Global spectral sequence converging to \( T^*_X = \text{Ext}^n(\Omega^1_X, \mathcal{O}_X) \);
- the vertical maps \( \delta_i \), \( \delta_{loc} \)

arise from the natural map \( \phi^* \Omega_X \rightarrow \Omega_Y \) just recalling that \( X \) has rational singularities, hence giving

\[ \text{Ext}^i(\Omega_X, \mathcal{O}_X) = \text{Ext}^i(\phi^* \Omega_X, \mathcal{O}_X) = \text{Ext}^i(\phi^* \Omega_X, \mathcal{O}_Y); \]

**Proposition 1** ([7] Prop. (2.1)) Let \( U_p \) be a Stein neighborhood of the singular point \( p \in P = \text{Sing}(X) \) and set \( V_p := \phi^{-1}(U_p) \). Let Def(\( U_p \)) and Def(\( V_p \)) be the Kuranishi spaces defined in parts 1 and 2 (the non-compact case) of Thm. 1, respectively. Then, under hypothesis given above, one gets:

1. \( T^1_{V_p} \cong H^1(V_p, \Theta_{V_p}) \cong H^0(R^1 \phi_* \Theta_{V_p}) \) and \( H^0(R^1 \phi_* \Theta_Y) \cong \bigoplus_{p \in P} T^1_{V_p} \) is the tangent space to \( \prod_{p \in P} \text{Def}(V_p) \),
2. \( T^1_{U_p} \cong T^1_{V_p} \) and \( T^1_{X} \cong \bigoplus_{p \in P} T^1_{U_p} \) is the tangent space to \( \prod_{p \in P} \text{Def}(U_p) \),
3. morphisms \( \delta_{loc} \) and \( \delta_1 \) in the Friedman diagram (9) are injective.

**Proof** Parts 1 and 2 follow immediately by Prop. (2.1), part 1, in [7], recalling the previously displayed formulas (2) and (3). To prove part 3 notice, on the one hand, that the injectivity of \( \delta_1 \) follows from the injectivity \( \delta_{loc} \), by an easy diagram chase. On the other hand, \( \delta_{loc} \) is injective by Prop. (2.1), part 2, in [7]. \( \square \)

**Remark 1** Since \( Y \) is a Calabi–Yau 3-fold, the Bogomolov–Tian–Todorov–Ran Theorem gives that \( \text{Def}(Y) \cong H^1(\Theta_Y) \). Moreover \( \phi \) is a small resolution giving that \( \text{Sing}(X) \) is composed at most by terminal singularities of index 1: in [27], Theorem A, Y. Namikawa proved an extension of the Bogomolov–Tian–Todorov–Ran Theorem allowing to conclude that \( \text{Def}(X) \) is smooth also in the present situation, hence giving \( \text{Def}(X) \cong \mathbb{T}^1_X \). By the previous Prop. 1, the localization near to a singular point...
\( p \in \text{Sing}(X) \) of the second square on the left of the Friedman diagram (9) can then be rewritten as follows

\[
\begin{align*}
\text{Def}(Y) & \cong H^1(\Theta_Y) & \rightarrow_{\lambda_{E_p}} & \text{Def}(V_p) \cong H^0(R^1\phi_*\Theta_{V_p}) \\
\downarrow_{\delta_1} & & \downarrow_{\delta_{\text{loc},p}} & \\
\text{Def}(X) & \cong T^1_X & \rightarrow_{\lambda_p} & \text{Def}(U_p) \cong T^1_{U_p}
\end{align*}
\] (10)

where \( E_p = \phi^{-1}(p) \) is the exceptional locus over \( p \). In the following we will refer to this diagram as the \textit{local Friedman diagram}. Notice that all the maps involved in this diagram are compatible with corresponding natural transformations between the associated deformation functors. For a deeper understanding of this fact the interested reader is referred to [46, §1], [7, (3.4)], [20, (11.3-4)] and references therein.

### 2.4 Fiber products of rational elliptic surfaces with sections

In the present subsection we will review some well known facts about rational elliptic surfaces with section and their fiber products. For further details the reader is referred to [24, 25] and [41].

Let \( Y \) and \( Y' \) be \textit{rational elliptic surfaces with sections} i.e. rational surfaces admitting elliptic fibrations over \( \mathbb{P}^1 \):

\[
r : Y \longrightarrow \mathbb{P}^1, \quad r' : Y' \longrightarrow \mathbb{P}^1
\]

with distinguished sections \( \sigma_0 \) and \( \sigma'_0 \), respectively (notation as in [41] and [25]). Define

\[
X := Y \times_{\mathbb{P}^1} Y'.
\] (11)

Write \( S \) (resp. \( S' \)) for the images of the singular fibers of \( Y \) (resp. \( Y' \)) in \( \mathbb{P}^1 \).

**Proposition 2**

1. The fiber product \( X \) is smooth if and only if \( S \cap S' = \emptyset \). In particular, if smooth, \( X \) is a Calabi–Yau threefold ([41] §2) and \( \chi(X) = 0 \).
2. \( Y \) (resp. \( Y' \)) is the blow–up of \( \mathbb{P}^2 \) at the base locus of a rational map \( \varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \) (resp. \( \varphi' : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \)) ([25] Prop. 6.1).
3. If \( Y = Y' \) is sufficiently general and \( r (= r') \) admits at most nodal fibers, then there always exists a small projective resolution \( \widetilde{X} \) of \( X \) ([41] Lemma (3.1)). Moreover if \( Y \) has exactly \( v \geq 0 \) nodal fibers then \( \chi(X) = v \) and \( \chi(\widetilde{X}) = 2v \).

**Theorem 3** (Weierstrass representation of an elliptic surface with section, [17] Thm. 1, [24] Thm. 2.1, [14] §2.1 and proof of Prop. 2.1) Let \( r : \tilde{Y} \longrightarrow C \) be a relatively minimal elliptic surface over a smooth base curve \( C \), whose generic fibre is smooth and admitting a section \( \sigma : C \longrightarrow \tilde{Y} \) (then \( \tilde{Y} \) is algebraic [18]). Let \( \mathcal{L} \) be the co–normal sheaf of \( \sigma(C) \subset \tilde{Y} \).

Then \( \mathcal{L} \) is invertible and there exists \( A \in H^0(C, \mathcal{L}^{\otimes 4}) \), \( B \in H^0(C, \mathcal{L}^{\otimes 6}) \).
such that \( \tilde{Y} \) is the minimal resolution of the closed subscheme \( Y \) of the projectivized bundle \( \mathbb{P}(E) := \mathbb{P}(L^\oplus 3 \oplus L^\oplus 2 \oplus O_C) \) defined by the zero locus of the homomorphism

\[
\begin{array}{ccc}
(A, B) : E = L^\oplus 3 \oplus L^\oplus 2 \oplus O_C & \longrightarrow & L^\oplus 6 \\
(x, y, z) & \longmapsto & -x^2z + y^3 + Ayz^2 + Bz^3
\end{array}
\]  

(12)

The pair \( (A, B) \) (hence the homomorphism (12)) is uniquely determined up to the transformation \( (A, B) \mapsto (c^4A, c^6B) \), \( c \in \mathbb{C}^\ast \) and the discriminant form

\[ \delta := 4A^3 + 27B^2 \in H^0(C, L^\oplus 12) \]

vanishes at a point \( \lambda \in C \) if and only if the fiber \( Y_\lambda := r^{-1}(\lambda) \) is singular.

**Remark 2** Assume that the elliptic surface \( r : Y \longrightarrow C \) is rational. Then \( C \cong \mathbb{P}^1 \) is a rational curve and the section \( \sigma(C) \) is a \( (-1) \)-curve in \( Y \) (see [24] Proposition (2.3) and Corollary (2.4)). In particular \( \mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(1) \) and (12) is a homomorphism

\[
\begin{array}{ccc}
\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(A, B) & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(6) \\
(x, y, z) & \longmapsto & -x^2z + y^3 + Ayz^2 + Bz^3
\end{array}
\]  

(13)

Consider the fiber product

\[ X := Y \times_{\mathbb{P}^1} Y \]

of the Weierstrass fibration defined as the zero locus \( Y \subset \mathbb{P}(E) \) of the bundle homomorphism (13). Hence, for generic \( A, B \), the rational elliptic surface \( Y \) has smooth generic fiber and a finite number of distinct singular fibers associated with the zeros of the discriminant form \( \delta = 4A^3 + 27B^2 \). In general the singular fibers are nodal and \( \text{Sing}(X) \) is composed by a finite number \( \nu = 12 \) of distinct nodes. We can then apply Proposition 2(3) to guarantee the existence of a small resolution \( \hat{X} \longrightarrow X \) whose exceptional locus is the union of disjoint \( (-1, -1) \)-curves, i.e. rational curves \( C \cong \mathbb{P}^1 \) in \( X \) whose normal bundle is \( N_{C|X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \).

Anyway, if either \( A \) and \( B \) have a common root or \( A \equiv 0 \), the Weierstrass fibration \( Y \) may admit cuspidal fibers: in this case the existence of a small resolution for \( X \) is no more guaranteed by Proposition 2(3).

### 3 The Namikawa Fiber Product

In [28], §0.1, Y. Namikawa considered the Weierstrass fibration associated with the bundle homomorphism

\[
\begin{array}{ccc}
(0, B) : E = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(6) & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(6) \\
(x, y, z) & \longmapsto & -x^2z + y^3 + B(\lambda) z^3
\end{array}
\]  

(14)

i.e. its zero locus \( Y \subset \mathbb{P}(E) \). The associated discriminant form is \( \delta(\lambda) = 27B(\lambda)^2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(12)) \) whose roots are precisely those of \( B \in H^0(\mathcal{O}_{\mathbb{P}^1}(6)) \). Hence, for a generic \( B \), the rational elliptic surface \( Y \longrightarrow \mathbb{P}^1 \) has smooth generic fiber and six distinct cuspidal fibers.
Proposition 3 The fiber product \( X := Y \times_{\mathbb{P}^1} Y \) is a threefold admitting 6 threefold cups whose local type is described by the singularity
\[
X^2 - U^2 - Y^3 + V^3 \in \mathbb{C}[X, Y, U, V].
\] (15)
In the standard Kodaira notation these are singularities of type II \( \times \) II ([18], Theorem 6.2).

Proof Our hypothesis give
\[
\mathbb{P}(E) \times \mathbb{P}^1[\lambda] \supset Y : x^2 z = y^3 + B(\lambda) z^3.
\] (16)
Then its fiber self–product can be represented as follows
\[
\mathbb{P} := \mathbb{P}(E) \times \mathbb{P}(E) \times \mathbb{P}^1[\lambda] \supset X : \begin{cases} x^2 z = y^3 + B(\lambda) z^3 \\ u^2 w = v^3 + B(\lambda) w^3. \end{cases}
\] (17)
Since the problem is a local one, let us consider the open subset \( \mathcal{A} \subset \mathbb{P} \) defined by setting
\[
\mathcal{A} := \{(x : y : z) \times (u : v : w) \times (\lambda_0 : \lambda_1) \mid z \cdot w \cdot \lambda_1 \neq 0\} \cong \mathbb{C}^5(X, Y, U, V, t) \] (18)
where \( X = x/z, Y = y/z, U = u/w, V = v/w \) and \( t = \lambda_0/\lambda_1 \). Then \( \mathcal{A} \cap X \) can be locally described by equations
\[
\begin{cases} X^2 = Y^3 + B(t) \\ U^2 = V^3 + B(t), \end{cases}
\] (19)
where \( B(\lambda) = \lambda^6 B(t) \). If \( t_0 \) is a zero of the discriminant \( \delta(t) = 27 B(t)^2 \) then \( p_{t_0} = ((0 : 0 : 1), (0 : 0 : 1), (t_0 : 1)) \in X \) is a singular point, whose local equations are obtained from (19) by replacing \( B(t) \) with its Taylor expansion in a neighborhood of \( t_0 \), giving
\[
\begin{cases} X^2 = Y^3 + t^i B^{(i)}(t_0)/i! + o(t^i) \\ U^2 = V^3 + t^i B^{(i)}(t_0)/i! + o(t^i), \end{cases}
\] (20)
where \( i \) is the minimum order of derivatives such that \( B^{(i)}(t_0) = dB^i(t_0)/dt^i \) does not vanish. Then the germ described by (20) turns out to be the same described by the local equation \( X^2 - U^2 - Y^3 + V^3 = 0 \).

As already observed above, Proposition 2(3) can then no more be applied to guarantee the existence of a small resolutions \( \hat{X} \longrightarrow X \). In this case Y. Namikawa proved the following

Proposition 4 ([28] Example in §0.1) The cuspidal fiber product \( X = Y \times_{\mathbb{P}^1} Y \) associated with the Weierstrass fibration \( Y \), defined as the zero locus in \( \mathbb{P}(E) \) of the bundle homomorphism (14), admits six small resolutions which are connected to each other by flops of \((-1, -1)\)–curves. The exceptional locus of any such resolution is given by six disjoint couples of \((-1, -1)\)–curves intersecting in one point.
the projectivized bundle $\mathbb{P} := \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1$ in which $X$ is embedded. Then the cusps of $X$ are firstly resolved to nodes and then finally resolved to smooth points. The key point of the construction is that $\text{Sing}(X)$ is contained in the diagonal locus $\Delta$ of $\mathbb{P}$; moreover $X$ turns out to be invariant under the action of a cyclic group of order 6 acting on $\mathbb{P}$. Then the six resolutions of $X$ are constructed by a successive blow up of suitable couples of images of $\Delta$ under the action of this cyclic group.

Give $X$ by (17) and consider the following cyclic map on $\mathbb{P}$

$\tau : \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1 \rightarrow \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1$

\[ ((x : y : z), (u : v : w), \lambda) \mapsto ((x : y : z), (-u : \epsilon v : w), \lambda), \]

where $\epsilon$ is a primitive cubic root of unity. The second equation in (17) ensures that $\tau X = X$. Since $\tau$ generates a cyclic group of order 6, the orbit of the codimension 2 diagonal locus $\Delta := \{(x : y : z), (u : v : w), \lambda) \in \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1 | (x : y : z) = (u : v : w)\}$ is given by six distinct codimension 2 cycles $\{\tau^i \Delta \mid 0 \leq i \leq 5\}$. For any $i$, $\tau^i \Delta$ cuts on $X$ a Weil divisor $D_i := \tau^i \Delta \cap X$ containing $\text{Sing}(X)$: in fact

$\text{Sing}(X) = \{((0 : 0 : 1), (0 : 0 : 1), \lambda) \in \mathbb{P} \mid B(\lambda) = 0\} \subset D_i = \left\{(x : y : z), (-1)^i x :\epsilon^i y : z), \lambda) \in \mathbb{P} \mid x^2 z - y^3 - B(\lambda) z^3 = 0\right\}.$

Then

$\text{Sing}(X) = \bigcap_{i=0}^{5} D_i.$

Let $\mathbb{P}_i$ be the blow–up of $\mathbb{P}$ along $\tau^i \Delta$: the exceptional divisor is a $\mathbb{P}^1$–bundle over $\tau^i \Delta$. Let $X_i$ be the strict transform of $X$ in the blow–up $\mathbb{P}_i \rightarrow \mathbb{P}$. Since $\text{Sing}(X)$ is entirely composed by singularities of type (15), $X_i$ is singular and $\text{Sing}(X_i)$ contains only nodes. Moreover $X_i \rightarrow X$ turns out to be a small partial resolution whose exceptional locus is a union of disjoint $(-1, -1)$–curves, one over each singular point of $X$.

Consider the strict transform $(\tau^i+1)\Delta$ of $\tau^i+1 \Delta$ in the blow–up $\mathbb{P}_i \rightarrow \mathbb{P}$. Let $\hat{\mathbb{P}}_i$ be the blow–up of $\mathbb{P}_i$ along $(\tau^i+1)\Delta$ and $\hat{X}_i$ be the strict transform of $X_i$ in $\hat{\mathbb{P}}_i$. Then

$\hat{X}_i \rightarrow X$ is a smooth small resolution satisfying the statement, for any $0 \leq i \leq 5$.

To prove this fact we have to check that:

1. the exceptional locus of the resolution $\hat{X}_i \rightarrow X$ is actually composed by disjoint couples of $(-1, -1)$–curves intersecting in one point,
2. the resolutions $\hat{X}_i \rightarrow X$ are to each other connected by flops of $(-1, -1)$–curves.

Let us prove 1 locally, by explicitly computing the induced resolution of a singular point of type (15) over the open subset $A \subset \mathbb{P}$ defined in (18). Up to an isomorphism
we may always assume that $B(1 : 0) \neq 0$, implying that $\text{Sing}(X) \subset \mathcal{A} \cap X$. Let us assume that $B(t_0 : 1) = 0$, then

$$p_{t_0} := ((0 : 0 : 1), (0 : 0 : 1), (t_0 : 1)) \in \text{Sing}(X)$$

is a threefold cusp whose local equation (15) can be factored as follows

$$(X - U)(X + U) = (Y - V)(Y - \epsilon V)(Y - \epsilon^2 V).$$

Let $\mathcal{A}'$ be the section of $\mathcal{A}$ with the hyperplane $t = t_0$. Then

$$\mathcal{A}' \cap \tau^i \Delta = \{(X, Y, U, V, t_0) \in \mathbb{C}^5 \mid X - (-1)^i U = Y - \epsilon^i V = 0\} \cong \mathbb{C}^2.$$  

Rewrite (22) as $xy = uv[(1 + \epsilon)v - \epsilon u]$, where

$$x = X - (-1)^i U, \quad y = X - (-1)^{i+1} U,$$

$$u = Y - \epsilon^i V, \quad v = Y - \epsilon^{i+1} V, \quad w = Y - \epsilon^{i+2} V.$$ 

The blow up $\mathbb{P}_i \rightarrow \mathbb{P}$ of $\tau^i \Delta$ induces over $\mathcal{A}'$ the blow up $\mathcal{A}'_i \rightarrow \mathcal{A}'$ of the plane $x = u = 0$. The strict transform $X_i$ is then locally given by

$$\mathcal{A}'_i \cap X_i = \begin{cases} \mu_1 x = \mu_0 u \\ \mu_0 y = \mu_1 v[(1 + \epsilon)v - \epsilon u] \end{cases}$$

where $\mathbb{P}^1[\mu_0, \mu_1]$ is the small exceptional locus of $\mathcal{A}'_i \cap X_i \rightarrow \mathcal{A}' \cap X$. Notice that $\mathcal{A}'_i \cap X_i$ is still singular admitting a node in the point $((0, t_0), (0 : 1)) \in \mathcal{A}'_i \cap \mathbb{P}^1$. On the other hand, the strict transform $(\tau^{i+1} \Delta)_i$ of $\tau^{i+1} \Delta$ is locally given by

$$\begin{cases} \mu_1 x = \mu_0 u \\ y = v = 0 \end{cases} \quad (23)$$

The blow up $\widehat{\mathbb{P}}_i \rightarrow \mathbb{P}_i$ of $\mathbb{P}$ along $(\tau^{i+1} \Delta)_i$ induces over $\mathcal{A}'_i$ the blow up $\widehat{\mathcal{A}}'_i \rightarrow \mathcal{A}'_i$, along (23). Then the strict transform $\widehat{X}_i$ of $X$ is locally described as the following codimension three closed subset of $\mathcal{A}' \times \mathbb{P}^1[\mu] \times \mathbb{P}^1[v]$

$$\widehat{\mathcal{A}}'_i \cap \widehat{X}_i = \begin{cases} \mu_1 x = \mu_0 u \\ v_1 y = v_0 v \\ \mu_0 v_0 = \mu_1 v_1[(1 + \epsilon)v - \epsilon u] \end{cases}.$$ 

Observe that:

- $\widehat{\mathcal{A}}'_i \cap \widehat{X}_i$ is smooth,
- $\widehat{\mathcal{A}}'_i \cap \widehat{X}_i \rightarrow \mathcal{A}' \cap X$ is an isomorphism outside of $(0, t_0) \in \mathcal{A}' \cap X$, which locally represents $p_{t_0} \in \text{Sing}(X)$,
- the exceptional fiber over $(0, t_0) \in \mathcal{A}' \cap X$ is described by the closed subset $\{\mu_0 v_0 = 0\} \subset \mathbb{P}^1[\mu] \times \mathbb{P}^1[v]$, which is precisely a couple of $\mathbb{P}^1$’s meeting in the point $((0, t_0), (0 : 1), (0 : 1)) \in \widehat{\mathcal{A}}'_i \cap \widehat{X}_i \subset \mathcal{A}' \times \mathbb{P}^1 \times \mathbb{P}^1$,
- by construction any exceptional $\mathbb{P}^1$ is a $(-1, -1)$–curve.
To prove \( ii \), it suffices to show that:

- for any \( 0 \leq i \leq 5 \) the following flops of \((-1, -1)\)-curves exist:

\[
\begin{align*}
X_i \leftarrow & \rightarrow X_{i+2}, \\
X_i \leftarrow & \rightarrow X_{i+3}.
\end{align*}
\]

As before rewrite the local equation (22) as

\[
xy = uv[(1 + \epsilon)v - \epsilon u] \quad \text{in } \mathbb{C}[x, y, u, v].
\]

Then locally \( X_i \) corresponds to blow up the plane \( x = u = 0 \) of \( \mathbb{C}^4 \) while \( X_{i+2} \) corresponds to blow up the plane \( x = v = 0 \). Ignore the term \([ (1 + \epsilon)v - \epsilon u] \): then our situation turns out to be similar to the well known \textit{Kollár quadric} ([19] Example 3.2) giving a flop

\[
\begin{align*}
X_i \leftarrow & \rightarrow X_{i+2}.
\end{align*}
\]

Analogously \( X_{i+3} \) corresponds to blow up \( y = u = 0 \) still getting a flop

\[
\begin{align*}
X_i \leftarrow & \rightarrow X_{i+3}.
\end{align*}
\]

3.1 Deformations and resolutions

Let \( X = Y \times_{\mathbb{P}^1} Y \) be the Namikawa fiber product defined above, starting from the bundle’s homomorphism (14). For a general \( b \in H^0(\mathcal{O}_{\mathbb{P}^1}(6)) \), the singular locus \( \text{Sing}(X) \) is composed by six cusps of type (15). Let us rewrite the local equation of this singularity as follows

\[
x^2 - y^3 = z^2 - w^3. \quad (24)
\]

It is a singular point of Kodaira type \( III \times III \). Moreover it is a \textit{compound Du Val singularity} of \( cA_2 \) type i.e. a threefold point \( p \) such that, for a hyperplane section \( H \) through \( p \) (in the present case assigned e.g. by \( w = 0 \)), \( p \in H \) is a Du Val surface singularity of type \( A_2 \) (see [35], §0 and §2, and [2], chapter III).

Recalling (6), the Kuranishi space of the cusp (24) is the \( \mathbb{C} \)-vector space

\[
T^1 \cong T^1 \cong \mathcal{O}_{\mathbb{F},0}/J_F \cong \mathbb{C}[x, y, z, w]/(F + J_F) \cong \langle 1, y, w, yw \rangle_{\mathbb{C}} \quad (25)
\]

where \( F = x^2 - y^3 - z^2 + w^3 \) and \( J_F \) is the associated Jacobian ideal. Then a miniversal deformation of (24) is given by the zero locus of

\[
F_{\Lambda} : x^2 - y^3 - z^2 + w^3 + \lambda + \mu y - v w + \sigma y w \in \mathbb{C}[x, y, z, w], \quad \Lambda = (\lambda, \mu, -v, \sigma) \in T^1. \quad (26)
\]
The fibre $X_\Lambda = \{ F_\Lambda = 0 \}$, of the miniversal deformation family $X \to T^1$, is singular if and only if the Jacobian rank of the polynomial function $F_\Lambda$ is not maximum at some zero point of $F_\Lambda$. Singularities are then given by $(0, y, 0, w) \in \mathbb{C}^4$ such that

$$\begin{align*}
3y^2 - \sigma w - \mu &= 0 \\
3w^2 + \sigma y - \nu &= 0 \\
\sigma yw + 2\mu y - 2\nu w + 3\lambda &= 0
\end{align*}$$

(27)

where the first two conditions come from partial derivatives of $F_\Lambda$ and the latter is obtained by applying the first two conditions to the vanishing condition $F_\Lambda(0, y, 0, w) = 0$.

**Proposition 5** A fibre of the miniversal deformation family $X \to T^1$ of the cusp (24) admits at most three singular points.

**Proof** It is a direct consequence of conditions (27). Fix a point

$$\Lambda = (l, \mu, -\nu, \sigma) \in T^1.$$

If $\sigma = 0$ then conditions (27) become conditions (31) below. The argument given in the proof of Proposition 6 shows that common solutions of (31) cannot be more than two.

Let us then assume that $\sigma \neq 0$. The first equation in (27) gives

$$\sigma w = 3y^2 - \mu$$

(28)

and the third equation multiplied by $\sigma$ gives

$$\sigma^2 yw + 2\mu \sigma y - 2\nu \sigma w + 3\lambda \sigma = 0.$$  

(29)

Put (1) in (3) to get

$$3\sigma y^3 - 6\nu y^2 + \mu \sigma y + 2\mu \nu + 3\lambda \sigma = 0.$$  

Fixing $\Lambda = (l, \mu, -\nu, \sigma) \in T^1$, the latter is a cubic polynomial in the unique unknown variable $y$. Then the common solutions of (27) cannot be more than 3. □

For any $p \in \text{Sing}(X)$ let $U_p = \text{Spec } \mathcal{O}_{F,p}$ be a representative of the complex space germ locally describing the singularity $p \in X$. By Theorem 1, Definition 3, (25) and (26)

$$\text{Def}(U_p) \cong T^1 \cong \langle 1, y, w, yw \rangle \subset \mathbb{C}.$$  

(30)

Recalling the local Friedman diagram (10), consider the localization map

$$\lambda_p : \text{Def}(X) \cong T^1_X \longrightarrow \text{Def}(U_p) \cong T^1.$$  

**Proposition 6** The deformations of the cusp (24) induced by localizing a deformation of the fiber product $X$ are based on a hyperplane of the Kuranishi space $T^1$ in (25). More precisely, using coordinates introduced in (26),

$$\forall p \in \text{Sing}(X) \quad \text{Im} (\lambda_p) = S := \{ \sigma = 0 \} \subset T^1.$$  

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In particular every deformation parameterized by $S$ may admit at most 2 singular points which are

- ordinary double points if $\mu \cdot v \neq 0$,
- compound Du Val of type $cA_2$ if $\mu \cdot v = 0$ and precisely of Kodaira type
  
  $II \times II$ if $\mu = v = 0$,
  $I_1 \times II$ otherwise.

Proof Let $X = Y \times_{\mathbb{P}^1} Y$ be the Namikawa fiber product given by (17). A deformation family $\mathcal{Y}$ of the rational elliptic surface $r : Y \to \mathbb{P}^1$ comes equipped with a natural morphism $R : \mathcal{Y} \to \mathbb{P}^1$ such that $R|_Y = r$. Then a deformation family of $X$ is obtained as the fiber product $X = \mathcal{Y} \times_{\mathbb{P}^1} \mathcal{Y}$ of two deformation families of $Y$ (this fact is explained in several points of the Namikawa’s paper [28], e.g. in the Introduction and in Remark 2.8). The Kuranishi space of the cusp $\{x^2 - y^3 = 0\}$ is given by

$$T^1_{\text{cusp}} \cong \mathbb{C}[x, y]/(x^2 - y^3, x, y^2) \cong (1, y)\mathbb{C}$$

meaning that a deformation of $Y$ can be locally given by the germ

$$x^2 - y^3 + \lambda + \mu y - t^i B^i(t_0) = 0 \quad , \quad (\lambda, \mu) \in T^1_{\text{cusp}} ,$$

where $t_0$ is a zero of $B(t)$ and $B^i(t_0)$ has the same meaning as in (20). Therefore, by applying an analogous local analysis as that given in the proof of Proposition 3 and recalling (26), it turns out that a deformation of the threefold cusp (24) induced by localizing a deformation of $X$, is given by the germ of complex space

$$\begin{cases} x^2 - y^3 + \lambda_1 + \mu y - t^i B^i(t_0) = 0 \\ z^2 - w^3 + \lambda_2 + \nu w - t^i B^i(t_0) = 0 \end{cases}$$

giving the following local equation

$$x^2 - y^3 + \lambda_1 + \mu y = z^2 - w^3 + \lambda_2 + \nu w .$$

Then all such deformations span the subspace

$$\{(\lambda_1 - \lambda_2, \mu, -\nu, 0)\} = \{\sigma = 0\} \subset T^1_{\text{cusp}} .$$

Setting $\sigma = 0$ in conditions (27) gives the following equations

$$\begin{align*}
3y^2 - \mu &= 0 \\
3w^2 - \nu &= 0 \\
2\mu y - 2\nu w + 3\lambda &= 0
\end{align*}$$

which can be visualized in the $y, w$–plane as follows:

- the first condition as two parallel and symmetric lines with respect to the $y$–axis; they may coincide with the $y$–axis when $\mu = 0$;
- the second condition as two parallel and symmetric lines with respect to the $w$–axis; they may coincide with the $w$–axis when $\nu = 0$;
- the last condition as a line in general position in the $y, w$–plane.
Clearly, fixing the point $\Lambda = (l, \mu, -\nu, \sigma) \in \mathbb{T}^1$, it is not possible to have more than two distinct common solutions of (31) with respect to the variables $y, w$.

To analyze the singularity type, let $p = (0, y, 0, w)$ be a singular point of $F = x^2 - y^3 - z^2 + w^3 + \lambda \mu y - \nu w = 0$ and translate $p$ to the origin by replacing $y \mapsto y + y\mu$, $w \mapsto w + w\nu$.

Then conditions (31) impose that the translated $F$ is

$$\tilde{F} = x^2 - y^3 - z^2 + w^3 - 3y^2 + 3w^2$$

giving the classification above.

**Corollary 1** If the deformation $X_{\Lambda}$ of the cusp (24), associated with $\Lambda \in \mathbb{T}^1$, admits three distinct singular points then $\Lambda \in \mathbb{T}^1 \setminus S$, which is

$$\Lambda = (\lambda, \mu, -\nu, \sigma) \text{ with } \sigma \neq 0.$$
it turns out to be 0 if and only if \( \Lambda \) satisfies the following conditions

\[
\begin{align*}
4\nu^2 - \mu\sigma^2 &= 0 \\
\sigma^4 - 36\mu\nu - 27\lambda\sigma &= 0 \\
3\mu^2\sigma^2 - \nu\sigma^4 - 36\mu\nu^2 - 54\lambda\nu\sigma &= 0
\end{align*}
\]  

(32)

Then the first equation gives

\[
\mu = 4\frac{\nu^2}{\sigma^2}.
\]

Putting this in the second equations we get

\[
\lambda = \frac{\sigma^3}{27} - \frac{16\nu^3}{3\sigma^3}.
\]

Then, from the third equation in (32), we get the following factorization

\[
\nu(4\nu - \sigma^2)(4\nu - \epsilon\sigma^2)(4\nu - \epsilon^2\sigma^2) = 0
\]

where \( \epsilon \) is a primitive cubic root of unity. All the solutions of (32) are then the following

\[
\Lambda_0 = (\frac{1}{27}\sigma^3, 0, 0, \sigma) , \quad \Lambda_1 = (-\frac{5}{108}\sigma^3, \frac{1}{4}\sigma^2, -\frac{1}{4}\sigma^2, \sigma) , \quad \Lambda_2 = (-\frac{5}{108}\sigma^3, \frac{\epsilon^2}{4}\sigma^2, -\frac{\epsilon}{4}\sigma^2, \sigma) , \quad \Lambda_3 = (-\frac{5}{108}\sigma^3, \frac{\epsilon}{4}\sigma^2, -\frac{\epsilon^2}{4}\sigma^2, \sigma).
\]

Let us first consider the second solution \( \Lambda_1 \). In this particular case \( R_2 \) becomes

\[
R_2 = 3\sigma y^3 - \frac{3}{2}\sigma^2 y^2 + \frac{1}{4}\sigma^3 y - \frac{1}{72}\sigma^4 = 3\sigma \left( y - \frac{\sigma}{6} \right)^3,
\]

meaning that \( \Lambda_1 \) is actually the base of a trivial deformation of (24) since the fiber associated with \( \sigma \) admits the unique singular point \((0, \sigma/6, 0, -\sigma/6)\) which is still a cusp of type (24). Moreover solutions \( \Lambda_2 \) and \( \Lambda_3 \) give trivial deformations too, since they can be obtained from \( \Lambda_1 \) by replacing

\[
\text{either } y \mapsto \epsilon y, \quad w \mapsto \epsilon^2 w \quad \text{(giving } \Lambda_2 \text{)} \\
\text{or } y \mapsto \epsilon^2 y, \quad w \mapsto \epsilon w \quad \text{(giving } \Lambda_3 \text{)}.
\]

It remains to consider the first solution \( \Lambda_0 \). In this case \( R_2 \) becomes

\[
R_2 = y^3 + \frac{\sigma^3}{27} = \left( y + \frac{\sigma}{3} \right) \left( y + \frac{\epsilon\sigma}{3} \right) \left( y + \frac{\epsilon^2\sigma}{3} \right),
\]

then the deformation \( X_\sigma, \sigma \neq 0 \), turns out to admit three distinct nodes given by

\[
(0, -\frac{\sigma}{3}, 0, \frac{\sigma}{3}) , \quad \left( 0, -\frac{\epsilon\sigma}{3}, 0, \frac{\epsilon^2\sigma}{3} \right) , \quad \left( 0, -\frac{\epsilon^2\sigma}{3}, 0, \frac{\epsilon\sigma}{3} \right).
\]

(34)

Notice that the base curve \( \Lambda_0 \subset \mathbb{P}^1 \cong \mathbb{C}^4 \) is actually the plane smooth curve \( C = \{ \sigma^3 - 27\lambda = \mu = \nu = 0 \} \) meeting the hyperplane \( S = \{ \sigma = 0 \} \) only in the origin, where they are transversal since a tangent vector to \( C \) in the origin is a multiple of \((0, 0, 0, 1)\). The statement is then proved by thinking \( \mathbb{P}^1 \) as the tangent space in
the origin to the germ of complex space Def\((U_0)\) and representing the functor of 1st–order deformation of the cusp (24).

Let \(\hat{X} \xrightarrow{\phi} X\) be one of the six small resolutions constructed in Proposition 4 and consider the localization near to \(p \in \text{Sing}(X)\)

\[
\hat{U}_p := \phi^{-1}(U_p) \xrightarrow{\phi} \hat{X} \\
U_p := \text{Spec} \mathcal{O}_{F,p} \xrightarrow{\phi} X
\]

and consider the associated local Friedman diagram (10), which is the following commutative diagram between Kuranishi spaces

\[
\begin{array}{ccc}
\text{Def}(\hat{X}) & \xrightarrow{\lambda_{\hat{E}p}} & \text{Def}(\hat{U}_p) \\
\downarrow \delta_1 & & \downarrow \delta_{\text{loc},p} \\
\text{Def}(X) & \xrightarrow{\lambda_p} & \text{Def}(U_p)
\end{array}
\]

**Theorem 4** The image of the map \(\delta_{\text{loc},p}\) in diagram (36) is the plane smooth curve \(C \subset T^1\) defined in Proposition 7. In particular this means that

(a) \(\text{def}(\hat{U}_p) = 1\),
(b) \(\text{Im}(\lambda_p) \cap \text{Im}(\delta_{\text{loc},p}) = 0\),
(c) \(\text{Im}(\lambda_{\hat{E}p}) = 0\).

**Proof** By the construction of the resolution \(\hat{X} \xrightarrow{\phi} X\), \(\text{Im}(\delta_{\text{loc},p}) \subset T^1\) parameterizes all the deformations of \(U_p\) induced by deformations of \(\hat{U}_p\). Then a fiber of the versal deformation family of \(U_p\) over a point in \(\text{Im}(\delta_{\text{loc},p}) \subset T^1\) has local equation respecting the factorization (22) of the local equation (15). A general deformation respecting such a factorization can be written as follows

\[(X - U + \xi)(X + U + \nu) = (Y - V + \alpha)(Y - \epsilon V + \beta)(Y - \epsilon^2 V + \gamma)\]

for \((\xi, \nu, \alpha, \beta, \gamma) \in \mathbb{C}^5\). By means of the translation

\[X \mapsto X - \frac{\xi + \nu}{2}, \quad U \mapsto U + \frac{\xi - \nu}{2}\]

the left part of (37) becomes \(X^2 - U^2\). After some calculation on the right part, (37) can then be rewritten as follows

\[
F_a := F - (\alpha + \beta + \gamma)Y^2 - (\alpha + \epsilon^2 \beta + \epsilon \gamma)V^2 - (\alpha + \epsilon \beta + \epsilon^2 \gamma)YV - (\alpha \gamma + \alpha \beta + \beta \gamma)Y + (\beta \gamma + \epsilon \alpha \gamma + \epsilon^2 \alpha \beta)V - \alpha \beta \gamma = 0
\]
where $F := X^2 - U^2 - Y^3 + V^3$ and $a := (\alpha, \beta, \gamma) \in A \cong \mathbb{C}^3$. Consider the deformation $f : \mathcal{U} \rightarrow (A, 0)$ defined by setting
\[ \forall a \in A \quad \mathcal{U}_a := f^{-1}(a) = \{ F_a = 0 \} . \]

The following facts then occur:

1. the fibre $\mathcal{U}_a$ is isomorphic to the central fibre $\mathcal{U}_0$ if and only if $a$ is a point of the plane $\pi = \{ \alpha + \epsilon\beta + \epsilon^2\gamma = 0 \} \subset A$; in particular $\mathcal{U}|_\pi \rightarrow \pi$ is a trivial deformation;
2. the open subset $V := A \setminus \pi$ is the base of a deformation of the cusp $\mathcal{U}_0 = \{ F = 0 \}$ to 3 distinct nodes;
3. there exists a morphism of germs of complex spaces $g : (A, 0) \rightarrow (\mathbb{T}^1, 0)$ to the Kuranishi space $\mathbb{T}^1$ described in (25), such that $\text{Im} g$ turns out to be precisely the plane smooth curve $C$ defined in Proposition 7 and parameterizing the deformation of the cusp (24) to three distinct nodes.

Let us postpone the proof of these facts to observe that fact (3) means that the deformation $\mathcal{U} \rightarrow A$ is the pull–back by $g$ of the versal deformation $\mathcal{V} \rightarrow \mathbb{T}^1$ i.e. $\mathcal{U} = A \times_{\mathbb{T}^1} \mathcal{V}$. Then $C = \text{Im}(\delta_{\text{loc},p})$ proves the first part of the statement. Part (a) of the statement then follows by recalling that $\delta_{\text{loc},p}$ is injective. Moreover Propositions 6 and 7 allow one to prove part (b). Finally part (c) follows by (b), the injectivity of $\delta_{\text{loc},p}$ and the commutativity of diagram (36).

Let us then prove facts (1), (2) and (3) stated above.

(1), (2) : this facts are obtained analyzing the common solutions of
\[ F_a = \partial_X F_a = \partial_Y F_a = \partial_U F_a = \partial_V F_a = 0 . \]
Since $\partial_X F_a = 2X$ and $\partial_U F_a = 2U$, we can immediately reduce to look for the common solutions $(0, Y, 0, V)$ of
\[ F_a(0, Y, 0, V) = \partial_Y F_a = \partial_V F_a = 0 . \] (39)
After some calculations one finds that these common solutions are given by
\[ p_1 = \left( 0, \frac{\epsilon\beta - \gamma}{1-\epsilon}, 0, \frac{\beta - \gamma}{\epsilon(1-\epsilon)} \right) \]
\[ p_2 = \left( 0, \frac{\epsilon^2\alpha - \gamma}{1-\epsilon^2}, 0, \frac{\alpha - \gamma}{1-\epsilon^2} \right) \]
\[ p_3 = \left( 0, \frac{\epsilon\alpha - \beta}{1-\epsilon}, 0, \frac{\alpha - \beta}{1-\epsilon} \right) \]
which have to be necessarily distinct since
\[ \frac{\beta - \gamma}{\epsilon(1-\epsilon)} = \frac{\alpha - \gamma}{1-\epsilon^2} \iff \frac{\alpha - \gamma}{1-\epsilon^2} = \frac{\alpha - \beta}{1-\epsilon} \iff \frac{\alpha - \beta}{1-\epsilon} = \frac{\beta - \gamma}{\epsilon(1-\epsilon)} \]
\[ \iff \alpha + \epsilon\beta + \epsilon^2\gamma = 0 . \]
On the other hand, if $\alpha + \epsilon\beta + \epsilon^2\gamma = 0$ then we get the unique singular point $p_1 = p_2 = p_3$ which is still a threefold cusp.
(3): Look at the definition (38) of $F_a$ and construct $g$ as a composition $g = i \circ p$ where

- $p : (A, 0) \cong (\mathbb{C}^3, 0) \to (A, 0) \cong (\mathbb{C}^3, 0)$ is a linear map of rank 1 whose kernel is the plane $\pi \subset A$ defined in (1),
- $i : (A, 0) \cong (\mathbb{C}^3, 0) \to (\mathbb{T}^1, 0) \cong (\mathbb{C}^4, 0)$ is the map $(\alpha, \beta, \gamma) \mapsto (\lambda, \mu, -\nu, \sigma)$ given by
  
  \[
  \lambda = -\alpha \beta \gamma, \quad \mu = -\alpha \gamma - \alpha \beta - \beta \gamma, \\
  \nu = -\beta \gamma - \epsilon \alpha \gamma - \epsilon^2 \alpha \beta, \quad \sigma = -\alpha - \epsilon \beta - \epsilon^2 \gamma;
  \]

then, by (2) and Proposition 7, necessarily $\text{Im } i = C$ and $i|_{\text{Im } p}$ is the rational parameterization $\Lambda_0$ given in (33).

The linear map $p$ has to annihilate the coefficients of $Y^2$ and $V^2$ in (38) i.e.

\[
\alpha + \beta + \gamma = \alpha + \epsilon^2 \beta + \epsilon \gamma = 0.
\]

Then we get the following conditions

\[
\text{Im } p = \left\{ (\epsilon, 1, \epsilon^2) \right\}_{C} \subset A, \quad \ker p = \pi = \left\{ (-\epsilon, 1, 0), (-\epsilon^2, 0, 1) \right\}_{C} \subset A
\]

which determine $p$, up to a multiplicative constant $k \in \mathbb{C}$, as the linear map represented by the rank 1 matrix

\[
\begin{pmatrix}
1 & \epsilon & \epsilon^2 \\
\epsilon^2 & 1 & \epsilon \\
\epsilon & \epsilon^2 & 1
\end{pmatrix}
\]

Then

\[
p(a) = k(\epsilon^2 \alpha + \beta + \epsilon \gamma) \cdot \left( \epsilon, 1, \epsilon^2 \right)
\]

and

\[
g(a) = i \circ p \left( a \right) = \left( -k^3(\alpha + \epsilon \beta + \epsilon^2 \gamma)^3, 0, 0, -3k(\alpha + \epsilon \beta + \epsilon^2 \gamma) \right)
\]

which satisfies equations $\sigma^3 - 27\lambda = \mu = \nu = 0$ of $C \subset \mathbb{T}^1$.

\[\square\]

**Remark 3** Propositions 6 and 7 and Theorem 4 give a detailed and revised version of what observed by Y. Namikawa in [28] Examples 1.10 and 1.11 and Remark 2.8. In fact point (c) of Theorem 4 proves the following

**Theorem 5** In the notation introduced above, every global deformation of the small resolution $\widehat{X}$ induces only trivial local deformations of a neighborhood of the exceptional fibre $\phi^{-1}(p)$ over a cusp $p \in \text{Sing}(X)$.

### 4 A Small and Non-simple Geometric Transition

Finally we propose an example of a small geometric transition which is not a simple gt i.e. it is not a deformation of a conifold transition, as defined in the following. This example has been already sketched in §9.2 of [40]. Thanks to the detailed analysis of
the Kuranishi space of a Namikawa fiber product presented above, all the following statements are now completely proved.

Let us first of all recall the main definitions.

**Definition 4** (see [38] and references therein) Let \( \hat{X} \) be a Calabi–Yau threefold and \( \phi : \hat{X} \to X \) be a birational contraction onto a normal variety. Assume that there exists a Calabi–Yau smoothing \( \tilde{X} \) of \( X \). Then the process of going from \( \hat{X} \) to \( \tilde{X} \) is called a geometric transition (for short transition or gt) and denoted either by \( T(\hat{X}, X, \tilde{X}) \) or by the diagram

\[
\begin{array}{c}
\hat{X} \\
\downarrow \phi \\
X \\
\downarrow \phi \\
\tilde{X}
\end{array}
\]

A gt \( T(\hat{X}, X, \tilde{X}) \) is called conifold if \( X \) admits only ordinary double points (nodes or o.d.p.’s) as singularities. Moreover a gt \( T(\hat{X}, X, \tilde{X}) \) will be called small if \( \text{codim}_X \text{Exc}(\phi) > 1 \), where \( \text{Exc}(\phi) \) denotes the exceptional locus of \( \phi \).

The most well known example of a gt is given by a generic quintic threefold \( X \subset \mathbb{P}^4 \) containing a plane. One can check that \( \text{Sing}(X) \) is composed by 16 nodes. Look at the strict transform of \( X \), in the blow–up of \( \mathbb{P}^4 \) along the contained plane, to get the resolution \( \hat{X} \), while a generic quintic threefold in \( \mathbb{P}^4 \) gives the smoothing \( \tilde{X} \). Due to the particular nature of \( \text{Sing}(X) \) the gt \( T(\hat{X}, X, \tilde{X}) \) is actually an example of a conifold transition and hence of a small gt, too.

**Remark 4** Let \( T(\hat{X}, X, \tilde{X}) \) be a small gt. Then \( \text{Sing}(X) \) is composed at most by terminal singularities of index 1 which turns out to be isolated hypersurface singularities (actually of compound Du Val type [35, 36]). The exceptional locus \( \text{Exc}(\phi : \tilde{X} \to X) \) is then composed by a finite number of trees of transversally intersecting rational curves, dually represented by ADE Dynkin diagrams [7, 21, 26, 32].

Moreover [27, Thm. A] allows us to conclude that \( \text{Def}(X) \) is smooth. Therefore

\[
\text{def}(X) = \dim_{\mathbb{C}} T^1_X .
\]

Moreover the Leray spectral sequence of the sheaf \( \phi_* \Theta_{\tilde{X}} \) gives

\[
h^0(\Theta_X) = h^0(\phi_* \Theta_{\tilde{X}}) = h^0(\Theta_{\tilde{X}}) = h^2(\tilde{X}) = 0
\]

where the first equality on the left is a consequence of the isomorphism \( \Theta_X \cong \phi_* \Theta_{\tilde{X}} \) (see [7] Lemma (3.1)) and the last equality on the right is due to the Calabi–Yau condition for \( \tilde{X} \). Then Theorem 2 and (1) allow to conclude that \( X \) admits a unique smoothing component giving rise to a universal effective family of Calabi–Yau deformations.

In the case of small geometric transitions we may then set the following

**Definition 5** Two small geometric transitions

\[
T_1(\tilde{X}_1, X_1, \hat{X}_1), T_2(\tilde{X}_2, X_2, \hat{X}_2)
\]
are direct deformation equivalent (i.e. direct def-equivalent) if the associated birational morphisms $\phi_i : \hat{X}_i \rightarrow X_i$, $i = 1, 2$, are deformations of each other as defined in Section 2.2.

The equivalence relation of small geometric transitions generated by direct def-equivalence is called def-equivalence (or deformation type) of small geometric transitions.

In particular a small gt $T(\hat{X}, X, \tilde{X})$ is called simple if it is def-equivalent to a conifold transition.

Actually def-equivalence of geometric transitions is a more complicated concept which reduces to the given Definition 5 in the case of small geometric transitions. The interested reader is referred to [40] §8.1 for any further detail.

A direct consequence of Theorem 4 is then the following

**Corollary 2** There exist small geometric transitions which are not def-equivalent to a conifold transition.

**Proof** Consider the Namikawa fiber product $X := Y \times_{\mathbb{P}^1} Y$, a smooth resolution $\phi : \hat{X} \rightarrow X$, as defined in Proposition 4 and a fiber product of generic rational elliptic surfaces $\tilde{X} := Y' \times_{\mathbb{P}^1} Y''$. Then $T(\hat{X}, X, \tilde{X})$ is a small gt which can’t be direct def-equivalent to a conifold transition for what observed in Remark 3.

Assume now that $T(\hat{X}, X, \tilde{X})$ is def-equivalent to a conifold transition. This means that there exists a finite chain of morphism deformations connecting the resolution $\phi : \hat{X} \rightarrow X$ with a conifold resolution. Hence there exists at least one of those morphism deformations locally inducing a non-trivial deformation of the exceptional locus, so violating condition (c) of Theorem 4. \(\square\)

**Remark 5** (The Friedman diagram of the Namikawa fiber product) As a final result let us give a complete account of the Friedman diagram (9) in the case of the small resolution $\phi : \hat{X} \rightarrow X$ of a cuspidal Namikawa fiber product $X = Y \times_{\mathbb{P}^1} Y$.

Let us start by computing the Kuranishi number def$(X)$, by an easy moduli computation. In fact the moduli of the elliptic surface $Y$ are 8 since they are given by the moduli of an elliptic pencil in $\mathbb{P}^2$. Moreover the 6 cuspidal fibers are parameterized by the roots in $\mathbb{P}^1$ of a general element in $H^0(\mathcal{O}_{\mathbb{P}^1}(6))$ up to the action of the projective group $\mathbb{P} GL(2)$. Then

$$\text{def}(X) = 2 \cdot \text{def}(Y) + h^0(\mathcal{O}_{\mathbb{P}^1}(6)) - \dim \text{GL}(2, \mathbb{C}) = 2 \cdot 8 + 7 - 4 = 19.$$ Recalling the existence of the small geometric transition $T(\hat{X}, X, \tilde{X})$, described in the proof of Corollary 2, it turns out that

$$\text{def}(X) = \text{def}(\hat{X}) = h^{1,2}(\hat{X}) = 19 = h^{1,1}(\hat{X})$$ since $\hat{X}$ is a Calabi–Yau 3–fold with $\chi(X) = 0$ ([41] §2). Recalling that

- the small resolution $\phi : \hat{X} \rightarrow X$ is obtained as a composition of 2 blow-ups,
- the exceptional locus Exc($\phi$) has 12 irreducible components,
- every cusp of $X$ has Milnor (and Tyurina) number 4,
then Calabi–Yau conditions on $\hat{X}$ and Theorem 7 in [39] gives that

$$\text{def}(\hat{X}) = h^1(\Theta_{\hat{X}}) = h^{1,2}(\hat{X}) = h^{1,2}(\hat{X}) - 16 = 3$$

$$\dim T^2_{\hat{X}} = h^2(\Theta_{\hat{X}}) = h^{1,1}(\hat{X}) = h^{1,1}(\hat{X}) + 2 = 21$$

Moreover recall that:
- by Proposition 1(1) and Theorem 4(a), $h^0(R^1\phi_*\Theta_{\hat{X}}) = 6$,
- by Theorem 4(c) the localization map $l_E$ in (9) is trivial,
- by Proposition 1(2) and relation (25), $\dim T^1_{\hat{X}} = 24$,
- the last horizontal maps on the right in the Friedman diagram (9) are, in this case, surjective since both related with spectral sequences having $E^{1,1}_{\infty} = E^{0,2}_{\infty} = 0$.

Putting all together, the Friedman diagram (9) becomes the following one

\[ \begin{array}{cccccccc}
0 & \rightarrow & \mathbb{C}^3 & \cong & \mathbb{C}^3 & \xrightarrow{\lambda_E = 0} & \mathbb{C}^6 & \xrightarrow{d_2} & \mathbb{C}^{27} & \rightarrow & 0 \\
& & \downarrow\delta_1 & & \downarrow\delta_{\text{loc}} & & \downarrow\delta_2 & & & & \\
0 & \rightarrow & \mathbb{C}^3 & \rightarrow & \mathbb{C}^{19} & \rightarrow & \mathbb{C}^{24} & \rightarrow & \mathbb{C}^{27} & \rightarrow & \mathbb{C}^{19} & \rightarrow & 0 \\
\end{array} \]

In particular notice that

$$h^1(X, \Theta_X) = h^1(\hat{X}, \Theta_{\hat{X}}) = 3$$

contradicting the sufficient condition (13) in Proposition 32 of [40] which has to be satisfied by a simple gt. This is consistent with the previous Corollary 2.

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