VERY SINGULAR SOLUTIONS
FOR THIN FILM EQUATIONS WITH ABSORPTION

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Abstract. The large-time behaviour of weak nonnegative and sign changing solutions of the thin film equation (TFE) with absorption

$$u_t = -\nabla \cdot (|u|^{n} \nabla u) - |u|^{p-1}u,$$

where $$n \in (0,3)$$ and the absorption exponent $$p$$ belongs to the subcritical range

$$p \in (n+1, p_0), \quad \text{with} \quad p_0 = 1 + n + \frac{4}{n},$$

is studied. Firstly, the standard free-boundary problem with zero-height, zero contact angle and zero-flux conditions at the interface and bounded compactly supported initial data is considered. Very singular similarity solutions (VSSs) have the form

$$u_s(x,t) = t^{\frac{1}{p-1}} f(y), \quad y = x/t^{\beta}, \quad \beta = \frac{n+1}{4(p-1)} > 0.$$ 

Here $$f$$ solves the quasilinear degenerate elliptic equation

$$-\nabla \cdot (|f|^{n} \nabla f) + \beta \nabla f \cdot y + \frac{1}{p-1} f - |f|^{p-1}f = 0$$

that becomes an ODE for $$N = 1$$ or in the radial setting in $$\mathbb{R}^N$$. By a combination of analytical, asymptotic, and numerical methods, existence of various branches of similarity profiles $$f$$ parameterized by $$p$$ is established. Secondly, in parallel, changing sign VSSs of the Cauchy problem are described.

This study is motivated by the detailed VSS results for the second-order porous medium equation with critical absorption ($$u \geq 0$$)

$$u_t = \nabla \cdot (u^{n} \nabla u) - u^p \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad p = 1 + n + \frac{2}{N}, \quad n \geq 0,$$

which have been known since the 1980s.

1. Introduction: very singular similarity solutions

1.1. On the PME with critical absorption. Second-order quasilinear diffusion equations with absorption are typical for combustion theory. The most well-known model, which became a canonical object of intensive investigation in the 1970–90s, is the porous medium equation (PME) with absorption

$$(1.1) \quad u_t = \nabla \cdot (u^{n} \nabla u) - u^p \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad (u \geq 0),$$

where $$n \geq 0$$ and $$p \in \mathbb{R}$$ are given parameters. Mathematical theory of such PDEs was founded by A.S. Kalashnikov in the 1970; see his fundamental survey [30] for the
full PME history in the 1950–80s. Besides new phenomena of localization and interface propagation, for more than twenty years, the PME with absorption (1.1) was a crucial model for determining various asymptotic patterns, which can occur for large times $t \gg 1$ or close to finite-time extinction as $t \to T^-$ (for $p < 1$). For (1.1), there are several parameter ranges with different asymptotics of solutions such as

$$
p > p_0 = 1 + n + \frac{2}{N}, \quad p = p_0, \quad 1 + n < p < p_0, \quad p = 1 + n,$$

$$1 < p < 1 + n, \quad p = 1, \quad 1 - n < p < 1, \quad p = 1 - n, \quad p < 1 - n,$$

etc.; see details, references, and history in the book [22, Ch. 5, 6]. A transitional behaviour for (1.1) occurs precisely at the first critical absorption exponent

$$(1.2) \quad p_0 = 1 + n + \frac{2}{N},$$

where logarithmically perturbed asymptotic patterns occur. VSSs, as special key solutions, are known to appear in the subcritical range $p \in (n + 1, p_0)$, while for $p > p_0$, the asymptotics of solutions as $t \to \infty$ correspond to pure diffusive PME, where the absorption term becomes negligible.

We will consider a new quasilinear parabolic model obtained by adding to the standard thin film operator an extra absorption term. This creates a non-conservative evolution PDE with the differential operator that is only “partially” divergent. We are going to show that, nevertheless, this higher-order nonlinear PDE exhibits several evolution features that are surprisingly similar to those for the PME with absorption (1.1). Moreover, we also claim that mathematical features and properties of both equations are also similar but, indeed, the results are much more difficult to justify for the higher-order case. Several conclusions of the present paper remain interesting and difficult open mathematical problems, which we would like to stress upon.

We claim that many of the presented results and ideas cannot and will not be proved rigorously in a few years to come, but a reasonably detailed qualitative understanding of such popular nowadays models and nonlinear PDEs is essential.

1.2. The TFE with absorption. Our goal is to show that VSS phenomena exist in the new model that is the thin film equation (TFE) with absorption

$$(1.3) \quad u_t = -\nabla \cdot (|u|^n \nabla u) - |u|^{p-1} u,$$

where $n > 0$ and $p > 1$ are fixed exponents. For convenience, it is written for solutions of changing sign to be also studied. Extra absorption terms in such TFEs are to model the effects of evaporation or permeability of the surface. We refer to [35] and [1, 27] for derivation of various TFE models including the non-conservative cases. Other related details and references on such PDEs can be found in the previous paper [21] on the TFE (1.3) devoted to the delicate case of the critical absorption exponent

$$(1.4) \quad p_0 = 1 + n + \frac{4}{N}.$$

In the present paper we study equation (1.3) in the complementary subcritical range

$$(1.5) \quad n + 1 < p < p_0 = 1 + n + \frac{4}{N}.$$
All necessary key references on various results on modern TFE theory that are necessary for justifying principal regularity and other assumptions on solutions will be presented below and are partially available in [21]. See also [23, Ch. 3], where further references are given and several properties of TFEs with absorption are discussed.

We will consider two main problems for the TFE (1.3):

(I) the standard free-boundary problem (FBP) admitting nonnegative solutions, and

(II) the Cauchy problem with another functional setting and solutions of changing sign.

(I) For the FBP, equation (1.3) is equipped with zero-height, zero contact angle, and zero-flux (conservation of mass) free-boundary conditions

\[ u = \nabla u = \nu \cdot (u^n \nabla \Delta u) = 0 \]

at the singularity surface (interface) \( \Gamma_0[u] \), which is the lateral boundary of \( \text{supp} \, u \) with the outward unit normal \( \nu \). Bounded, smooth, and compactly supported initial data

\[ u(x, 0) = u_0(x) \quad \text{in} \quad \Gamma_0[u] \cap \{ t = 0 \} \]

are added to complete a suitable functional setting of the FBP. We then assume that these three free-boundary conditions give a correctly specified problem for the fourth-order parabolic equation, at least for sufficiently smooth and bell-shaped initial data, e.g., in the radial setting; see references below.

(II) For the Cauchy problem (CP), we need solutions exhibiting the maximal regularity at interfaces, and this demands oscillatory character of such solutions. We refer to the book [23, Ch. 3] and [15, 17] for details concerning definitions and details on oscillatory changing sign solutions of various TFEs and other nonlinear higher-order PDEs.

1.3. On the pure TFE and similarity solutions. In what follows we will need well-known similarity solutions of the usual divergent TFE with mass conservation

\[ u_t = -\nabla \cdot (|u|^n \nabla \Delta u). \]

Earlier references on derivation of such fourth-order TFE and related models can be found in [28, 36], where first analysis of some self-similar solutions for \( n = 1 \) was performed. Source-type similarity solutions of (1.8) for arbitrary \( n \) were studied in [6] for \( N = 1 \) and in [18] for the equation in \( \mathbb{R}^N \). More information on similarity and other solutions can be found in [4, 3, 7]. TFEs admit non-negative solutions constructed by “singular” parabolic approximations of the degenerate nonlinear coefficients. We refer to the pioneering paper [2] and to various extensions in [13, 14, 33, 38] and the references therein. See also [29] for mathematics of solutions of the FBP and CP of changing sign (such a class of solutions of the CP will be considered later on).

In both the FBP and the CP, the source-type solutions of the TFE (1.8) take the form

\[ u_s(x, t) = t^{-\beta N} F(y), \quad y = x/t^\beta, \quad \text{with exponent} \quad \beta = \frac{1}{4+nN}, \]

where, in the FBP, \( F(y) \geq 0 \) is a radially symmetric compactly supported solution of the ODE [6, 18]

\[ B(F) \equiv -\nabla \cdot (F^n \nabla \Delta F) + \beta \nabla F \cdot y + \beta NF = 0. \]
In the case $n = 1$, the similarity profile for the FBP is given explicitly,

\begin{equation}
F(y) = c_0(a^2 - |y|^2)^2, \quad c_0 = \frac{1}{8(N + 2)(N + 4)}, \quad a > 0,
\end{equation}

which was first constructed in [36]. Most advanced results of asymptotic stability for the TFE (1.8) were obtained for the similarity solutions with profiles (1.11) for $n = 1$, where the rescaled equation is a gradient system. See a full account of such studies in [11] ($L^1$-convergence), [10] ($H^1$-convergence) and also [25] for more general properties including existence and uniqueness for the FBP (cf. [15], § 6.2, where these questions were treated in the von Mises variables). For other values of $n \neq 1$, the results are much weaker and are not complete. This emphasizes how difficult the non fully divergent TFEs with non-monotone operators are for the qualitative asymptotic study.

For the Cauchy problem, there exists a unique (up mass-scaling) oscillatory similarity profile of (1.10) for not that large $n \in (0, n_h)$, $n_h = 1.759...$ [15], while for $n \in (0, 1)$, existence and uniqueness are straightforward consequences of the result in [5] on oscillatory solutions. See also some details in [23], § 3.7 and interesting oscillatory features of similarity dipole solutions of the TFE (1.8) observed in [8]. Stability of such sign changing similarity solutions is quite plausible but was not proved rigorously being an open challenging problem.

1.4. On main results: comparison of logarithmically perturbed asymptotics for $p = p_0$ and VSSs for $p < p_0$. The critical case $p = p_0$ was studied in [21], where it was shown that the asymptotic behaviour of solutions is governed by a $\ln t$-perturbed similarity solution (1.9). For instance, for $n = 1$, relying on the explicit representation (1.11) and good spectral properties of the corresponding self-adjoint linearised rescaled operator, it was shown that, for $p = p_0 = 2 + \frac{4}{N}$, the TFE with absorption (1.3) admits asymptotic patterns of the following form as $t \to \infty$:

\begin{equation}
\begin{aligned}
&u(x, t) \sim (t \ln t)^{-\beta N} F_\ast (x/t^\beta (\ln t)^{-\beta N/4}) \quad (\beta = \frac{1}{1+\frac{4}{N}}), \\
\end{aligned}
\end{equation}

where $F_\ast$ is a fixed rescaled profile from the family (1.11) with a uniquely chosen parameter $a = a_\ast (N) > 0$. For the semilinear case $n = 0$, i.e., for the fourth-order parabolic equation written for solutions of changing sign

\begin{equation}
\begin{aligned}
&u_t = -\Delta^2 u - |u|^{p-1} u, \\
\end{aligned}
\end{equation}

the critical behaviour like (1.12) is known to occur at the critical exponent $p = 1 + \frac{4}{N}$ [19], which is precisely (1.4) with $n = 0$. In this case, the centre manifold analysis also uses spectral properties of a non self-adjoint linear operator studied in [12].

It turns out that, in the subcritical range (1.5), the generic behaviour of compactly supported solutions needs a different similarity description leading to the VSSs. In the next Section 2, we describe the VSSs in the subcritical range $p \in (n + 1, p_0)$ for the FBP. In the final Section 3, we discuss VSS structures for the Cauchy problem admitting maximal regularity solutions of changing sign. In general, we show the branching of the VSS profiles from the similarity patterns $F$ of the pure TFE.
2. VSSs for the FBP in the subcritical range $p \in (n + 1, p_0)$

In the range $(1.5)$, the TFE with absorption $(1.3)$ admits the self-similar very singular solution (VSS) of the standard form

\begin{equation}
    u_s(x, t) = t^{-\frac{1}{p-1}} f(y), \quad y = x/t^\beta, \quad \beta = \frac{n-(n+1)}{4(p-1)} > 0,
\end{equation}

where $f$ solves the nonlinear elliptic equation

\begin{equation}
    -\nabla \cdot (|f|^n \nabla f) + \beta \nabla f \cdot y + \frac{1}{p-1} f - |f|^{p-1} f = 0.
\end{equation}

For $n = 0$ and $p \in (1, 1 + \frac{4}{N})$, the existence of a finite number of oscillatory profiles $f$ in $\mathbb{R}^N$ corresponding to the Cauchy problem for $(1.13)$ was detected in [24] by using a $p$-bifurcation analysis.

For the current FBP, the solvability of $(2.2)$ and existence of a nonnegative compactly supported $f$ is unknown even in radial geometry, and we are going to present some analytic and numerical evidence concerning this. In 1D, $(2.2)$ is an ODE,

\begin{equation}
    -(|f|^n f''')' + \beta f' y + \frac{1}{p-1} f - |f|^{p-1} f = 0 \quad \text{for } y \in (0, y_0), \quad f'(0) = f'''(0) = 0,
\end{equation}

where we put two symmetry boundary conditions at the origin. At the interface $y = y_0$, the solution expansion is as follows (see [6, 15, 18]):

\begin{equation}
    f(y) = C_0(y_0 - y)^2 + \frac{C_0^4 - \beta y_0}{(3-2n)(4-2n)(5-2n)} (y_0 - y)^{5-2n}(1 + o(1)),
\end{equation}

where $y_0 > 0$ and $C_0 > 0$ are arbitrary two parameters. Therefore, in general, matching the 2D bundle $(2.3)$, comprising two positive parameters $y_0$ and $C_0$, with two symmetry conditions in $(2.3)$ cannot give more than a countable number of similarity FBP profiles \{f\} (provided the parameter dependence is analytic).

Figure 1 shows typical first positive symmetric VSS profiles $f_0(y)$ constructed numerically by bvp4-solver in MATLAB, by shooting in the $y_0$-parameter with conditions

\begin{equation}
    f'(y_0) = f'''(y_0) = 0.
\end{equation}

The correct choice of the interface location $y_0 > 0$ is obtained from the zero-height condition $f(y_0) = 0$, so in the limit, at $y_0$, all three free-boundary conditions $(1.6)$ are valid. For numerics, we use the regularization in the fourth-order operator in $(2.3)$ by replacing

\begin{equation}
    |f|^n \mapsto (\varepsilon^2 + f^2)^{\frac{n}{2}},
\end{equation}

with typically $\varepsilon = 10^{-3}$, and similar tolerances of the method. Notice a big “almost flat” part of the profile $f_0(y)$ in (c) for $p = 2$ and $n = 0.8$. Here, unlike other three cases, $p = 2$ is sufficiently close for $n + 1 = 1.8$, which is another critical exponent for the ODE $(2.3)$, since $\beta = 0$ then.

In Figure 2, we explain further details and the results of the actual shooting procedure for the last two previous patterns. For the reason of comparison, in (b) by dashed line we show the FBP profile for the semilinear case $n = 0$ that was studied in [24] in the case of the Cauchy (not an FBP) setting. It is curious that interfaces for $n = 0$ and $n = 1$ are close to each other, but not the profiles. It is worth observing that Figure 2(a) reveals some oscillatory character of typical solutions near interfaces. This indicates that there
Figure 1. FBP profiles satisfying \((2.3), (2.4)\) for various values of \(p\) and \(n\).

exist other patterns \(f_l\) for the FBP, which admit a few oscillations near interfaces and form in the limit the solution of the Cauchy problem; cf. [21, Prop. 5.1].

Let us mention again that existence and multiplicity of VSS profiles for \((2.3)\) are open problems. In the next section, we will discuss the question of existence of a finite number of profiles in the subcritical range in the case of the CP.

3. VSSs in the Cauchy problem for \(p \in (n + 1, p_0)\)

3.1. On local oscillatory structure near interfaces. The VSSs take the same self-similar form \((2.1)\), where the radial rescaled profile \(f\) of changing sign solves the ODE \((2.2)\) in \(\mathbb{R}^N\). We refer to [21, § 5] and [15] for details on the oscillatory structure of such similarity profiles close to finite interfaces, and also to [23, Ch. 3], where “homotopy”-like approaches to the Cauchy problem for various TFEs are presented.

Namely, it was shown that the asymptotic behaviour of \(f(y)\) satisfying equation \((2.3)\) near the interface point as \(y \to y_0^- > 0\) is given by the expansion

\[
(3.1) \quad f(y) = (y_0 - y)^\mu \varphi(s), \quad s = \ln(y_0 - y), \quad \mu = \frac{3}{n},
\]
where, after scaling $\varphi \mapsto \beta^{\frac{1}{n}} \varphi$, the oscillatory component $\varphi$ satisfies the following autonomous ODE, where exponentially small as $s \to -\infty$ terms are omitted:

$$
\varphi''' + 3(\mu - 1)\varphi'' + (3\mu^2 - 6\mu + 2)\varphi' + \mu(\mu - 1)(\mu - 2)\varphi + \frac{\varphi}{|\varphi|^{n}} = 0.
$$

According to this singularity analysis, for $n \in (0, n_h)$, where

$$
n_h = 1.759...
$$

is the heteroclinic bifurcation point for the ODE (3.2), there exists a stable (as $s \to +\infty$; at $s \to -\infty$ all solutions are unstable in view of shifting in $y_0$) changing sign periodic solution $\varphi(s)$ of (3.2). According to (3.1), this gives similarity profiles of changing sign, which being extended by $f(y) \equiv 0$ for $y > y_0$ forms a compactly supported solution $f \in C^\alpha$ in a neighbourhood of $y = y_0$, with $\alpha \sim \frac{3}{n}$. Notice that $\alpha \to +\infty$ as $n \to 0^+$, so the regularity at $y = y_0$ improves to $C^\infty$ at $n = 0^+$. These functions are oscillatory near the interfaces. The first results on the oscillatory behaviour of similarity profiles for fourth-order ODEs related to the source-type solutions of the divergent parabolic PDE

$$
(3.3) \quad u_t = -(|u|^{m-1}u)_{xxxx} \quad (m > 1),
$$

were obtained in [5]. It turns out that these results can be applied to the rescaled TFE (1.10), but for $n \in (0, 1)$ only (the ODEs for (1.8) and (3.3) then coincide after change). Some existence and multiplicity of periodic solutions of ODEs such as (3.2) are known [15], and often lead to a number of open problems. Therefore, numerical and some analytic evidence remain key, especially for sixth and higher-order TFEs, [23, Ch. 3].

Thus, we express the above results as follows: there exists a 2D bundle of asymptotic solutions of (2.3) in $\mathbb{R}$ having the expansion (3.1),

$$
(3.4) \quad f(y) = (y_0 - y)^{\frac{2}{n}} \varphi(s + s_0), \quad \text{with two parameters } y_0 > 0 \text{ and } s_0 \in \mathbb{R},
$$

where we also take into account the phase shift $s_0$ of the periodic orbit $\varphi(s)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.pdf}
\caption{Results of shooting of the profiles in Figure 1 (c) and (d).}
\end{figure}
Therefore, matching the 2D bundle (3.4) with two symmetry conditions at the origin in (2.3) leads to a reasonable well-posed problem of 2D matching, which remains essentially open still. In the case of analytic dependence on parameters involved, such a problem cannot possess more than countable set of isolated solutions. Actually, we are going to show that the number of VSS profiles is always finite.

3.2. Global behaviour of VSS profiles. In Figure 3 we present the first even VSS profile \( f_0(y) \) (and some \( f_2(y) \)) satisfying (2.2) for \( N = 1, p = 2, \) or 3 and various \( n. \) Here we fix the same regularization (2.6) where \( \varepsilon \) and ToIs are about \( 10^{-2} \), which is sufficient accuracy. Notice that in 1D,

\[
\begin{align*}
&f_0, f_2, f_4, \ldots \text{ are even functions, and} \\
&f_1, f_3, f_5, \ldots \text{ are odd.}
\end{align*}
\]

As we have mentioned, the case \( n = 0 \) corresponds to smooth VSSs for the semilinear parabolic equation (1.13), which were studied in [24], so we can always compare the results with those for the TFE (1.3) with small \( n > 0. \) For \( n = 0, \) the VSS profiles \{\( f_l, \ l \geq 0 \)\} are known to appear at subcritical (for \( p < p_l \)) pitchfork bifurcations at critical exponents

\[
p_l = 1 + \frac{4}{N+l}, \quad l = 0, 1, 2, ...
\]

Figures 3(a) and (b) show a strong similarity of the corresponding VSS profiles \( f_0 \) for various \( n, \) and, moreover, confirm that the profiles can be continuously deformed to each other as \( n \to 0^+. \) This is related to a general homotopy approach to the Cauchy problem for TFEs and other degenerate PDEs with non-monotone operators; see [23, Ch. 3] and [15, 17].

In Figure 3(a) for \( p = 2, \) we also show the smaller VSS profile \( f_2(y) \) for \( n = 0 \) (the dashed line), and also a couple of profiles for negative \( n = -0.1 \) and \( -0.2. \) These are \( f_2(y), \) and not \( f_0(y). \) In (b), we also calculate \( f_0 \) for the negative \( n = -0.2. \) Recall that for \( n < 0, \) (1.3) demonstrates typical features of a fast-diffusion problem. There is no finite propagation in this case, but solutions are equally oscillatory as \( y \to +\infty, \) thus inheriting this property from \( n = 0. \) Notice that for \( n = 0, \) according to (3.5),

\[
p_2 = 1 + \frac{4}{3} < 3,
\]

so that \( f_2(y) \) does not exist for \( p = 3. \) But \( f_2 \) exists for positive \( n = 0.5 \) (the dotted line in (b)) and is sufficiently small, meaning that the corresponding critical bifurcation exponent \( p_2(n) \) is slightly above 3.

3.3. On a boundary layer as \( p \to n + 1. \) For \( n = 0 \) and any \( p \in (1, p_0), \) there exists a finite number

\[
M \sim \frac{p_0-p}{p-1} \to +\infty \quad \text{as} \quad p \to 1^+ \quad (n = 0),
\]

of different VSS profiles, which are obtained by standard bifurcation theory, [24]. We expect that a similar multiplicity property remains valid for the TFE for \( n > 0, \) though in this case bifurcation branches are governed by the linearised TFE operator (see [21] § 2 and the results below), so that bifurcation points \{\( p_l \)\} are not given explicitly by a
Figure 3. First oscillatory VSS profiles $f_0$ and $f_2$ of the CP satisfying (2.2) in $\mathbb{R}$.

A typical strong oscillatory behaviour of the VSS profiles for $p \approx (n + 1)^+$ is shown in Figure 4 for $p = 2$ and $n = 0.95$. We present here first six even VSS profiles from different $p_{2r}$-branches; see further explanations below. Notice formation of an interesting “boundary layer” as $p \to n + 1$, where the VSS profiles $f(y)$ become more and more oscillatory reflecting the fact that a suitable solution satisfying at $p = n + 1$ the ODE (2.3) for $N = 1$, does not exist. Numerics in Figure 4 suggest that solutions of (3.6) are highly oscillatory and are not compactly supported. Recall that the oscillatory structure (3.1) near interfaces demands the linear term $+\beta f'y$ in the ODE (2.3) with $\beta > 0$. For $\beta = 0$, such solutions do not exist. Moreover, for $\beta < 0$, there exist positive solutions near interfaces, which correspond to blow-up problems [16].

3.4. Bifurcation of the first $p$-branch at $p = p_0^-$: a nonlinear version. We now study the behaviour of the first $p$-branch of the VSS profiles $f = f_0(y)$, when $p$ approaches from below the critical exponent (1.4). For $n = 0$, such a behaviour was studied in [24] by a standard bifurcation approach showing that the VSS profiles vanish with the rate:

$$\|f\|_\infty \sim (p_0 - p)^{\frac{N}{4}} \to 0 \quad \text{as} \quad p \to p_0^-.$$  

The bifurcation analysis in [24] was based on known point spectrum (3.19) and other spectral properties of a non-self-adjoint linear operator $B$, which is (1.10) for $n = 0$. The operator $B$ appears in the ODE (1.10) for $n = 0$ generating the rescaled kernel of the fundamental solution of the bi-harmonic operator $D_t + \Delta^2$.

Local bifurcation from $p_0$. We now perform a formal *nonlinear* version of such a bifurcation (branching) analysis for $n > 0$. As usual, according to classic branching
theory [32, 37], a justification (if any) is performed for the equivalent quasilinear integral equation with compact operators. For simplicity, we present basic computations for the differential version.

We introduce the small parameter \( \varepsilon = p_0 - p \), so that, as \( \varepsilon \to 0 \),
\[
\frac{1}{p-1} = \frac{N}{4+nN} + \varepsilon \frac{N^2}{(4+nN)^2} + O(\varepsilon^2)
\quad \text{and} \quad
\beta = \frac{1}{4+nN} - \varepsilon \frac{nN}{4(4+nN)^2} + O(\varepsilon^2).
\]

Substituting this expansion into (2.2) and performing the standard linearization yields
\[
(3.8) \quad B(f) + \varepsilon(L_1 f - |f|^{n+\frac{4}{N}} f \ln |f|) - |f|^{n+\frac{4}{N}} f + \ldots = 0,
\]
where \( L_1 = \frac{N^2}{(4+nN)^2} (NI - \frac{n}{4} y \cdot \nabla) \)
is a linear operator, and \( B \) is the rescaled operator (1.10) of the pure TFE. Notice that the fact that the operator \( B \) in (3.8) occurs in the rescaled pure TFE correctly describes the essence of a “nonlinear bifurcation phenomenon” to be revealed.

Next, we use an extra invariant scaling of operator \( B \) by setting
\[
(3.9) \quad f(y) = b\tilde{F}(y/b^{n/4}),
\]
where \( b = b(\varepsilon) > 0 \) is a small parameter, \( \beta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), to be determined. Substituting (3.9) into (3.8) and omitting higher-order terms yields
\[
(3.10) \quad B(\tilde{F}) + \varepsilon L_1 \tilde{F} - b^{n+\frac{4}{N}} |\tilde{F}|^{n+\frac{4}{N}} \tilde{F} + \ldots = 0.
\]

Finally, we perform linearization \( \tilde{F} = F + Y \), where \( F \) is the “fundamental”, supported in \( B_1 \), similarity profile of the Cauchy problem for the pure TFE, i.e., satisfying (1.10) for \( N = 1 \). This yields the non-homogeneous problem
\[
(3.11) \quad B'(F)Y + \varepsilon L_1 F - b^{n+\frac{4}{N}} |F|^{n+\frac{4}{N}} F + \ldots = 0.
\]
Here the derivative is given by
\[ B'(F)Y = -\nabla \cdot |F|^n \left( \frac{\nabla}{F}(\nabla \Delta F)Y + \nabla \Delta Y) \right) + \beta y \cdot \nabla Y + \beta NY. \]

The rest of the analysis depends on assumed good spectral properties of the linearised operator \( B'(F) \). We follow the lines of a similar analysis performed for the FBP case in [21, § 2], where, in the FBP for \( n = 1 \) and \( F \) given by (1.11), the operator \( B'(F) \) turns out to possess a (Friedrichs) self-adjoint extension with compact resolvent and discrete spectrum. Such a self-adjoint extension does not exist for the oscillatory \( F(y) \). Here we need to use general theory of non-self-adjoint operators; see e.g., [26]. A proper functional setting of this operator is more straightforward for \( N = 1 \) (and in the radial setting), where, using the behaviour of \( F(y) \rightarrow 0 \) as \( y \rightarrow 1 \), it is possible to check whether the resolvent is compact in a suitable weighted \( L^2 \) space. In general, this is a difficult problem; see below.

We assume that such a proper functional setting is available for \( B \), so we deal with operators having solutions with “minimal” singularities at the boundary \( S_1 \), where the operator is degenerate and singular. Namely, we find the first eigenfunction \( \psi_0 \) with \( \lambda_0 = 0 \) of \( B'(F) \). Let \( \psi_0^* \) be the corresponding first eigenfunction of the adjoint operator \( (B')^*(F) \) defined in a natural way using the topology of the dual space \( L^2(B_1) \) and having the same point spectrum (the latter is true for compact operators in a suitable space [31, Ch. 4]). Moreover, it can be seen from the divergent form of the linearised operator \( B'(F) \) that, after bi-orthonormalisation,
\[ \langle \psi_\alpha, \psi_\beta^* \rangle = \delta_{\alpha\beta}, \]
the corresponding first eigenfunction of \( (B')^*(F) \) can be taken as
\[ \psi_0^*(y) \equiv 1. \]
This simplifies the rest of computations, though one can restore these similarly for arbitrary \( \psi_l^* \), as suggested later on for finding other critical bifurcation points \( \{p_l\} \).

Further, we assume that there exists the orthogonal subspace \( \text{Span}\{\psi_l, l \geq 1\} \perp \psi_0 \) of eigenfunctions of \( B'(F) \), and we look for solutions of (3.11) in the form
\[ Y = C\psi_0 + w, \]
with a constant \( C = C(\varepsilon) \) and a function \( w \perp \psi_0 \), i.e., \( \langle w, \psi_0^* \rangle = 0 \). Recall again that, in doing so, we need to transform (3.11) into an equivalent integral equation with compact operators, but for convenience, we continue our computations using the differential version; see some details in [24, § 3].

Thus, multiplying (3.11) by \( \psi_0^* \) in \( L^2(B_1) \) and integrating by parts in the differential term \( y \cdot \nabla F \) in \( L^1 F \), we obtain the following orthogonality condition of solvability (Lyapunov-Schmidt’s branching equation [37, § 27]):
\[ \varepsilon \frac{N^2}{4(4+nN)} \int F = b^{n+\frac{4}{N}} \int |F|^{n+\frac{4}{N}} F. \]
Therefore, the parameter \( b(\varepsilon) \) in (3.19) for \( p \approx p_0 \) is given by (cf. (3.7) for \( n = 0 \))
\[ b(\varepsilon) = [\gamma_0(p_0 - p)]^\frac{N}{4+nN}, \quad \text{with} \quad \gamma_0 = \frac{N^2}{4(4+nN)} \int F / \int |F|^{n+\frac{4}{N}} F, \]
provided that $\int |F|^{n+\frac{4}{N}}F > 0$ (not an easy inequality that can be checked numerically).

For $n = 0$, a rigorous justification of this bifurcation analysis can be found in [24, § 6], where a countable number of $p$-branches originated at bifurcation points (3.5) was detected on the basis of known spectral properties of the corresponding linear operator (3.18); see details in [12]. For $n > 0$, the justification needs spectral properties of the linearised operator $B'(F)$ and the corresponding adjoint one $(B'(F))^*$, which remain an open problem. In particular, it would be important to know that the bi-orthonormal eigenfunction subset $\{\psi_l\}$ of the operator $B'(F)$ is complete and closed in a weighted $L^2$-space (for $n = 0$, such results are available [12]). We expect that for $n \approx 0$, there exist critical exponents for the TFE with absorption that are close to those in (3.5) at $n = 0$. This can be checked by standard branching-type calculus; see Appendix in [20], where nonlinear eigenfunctions of the rescaled PME in $\mathbb{R}^N$ were studied by a branching approach.

**On global extension of $p$-branches.** For comparison, we begin with Figure 5 that presents the first monotone branch of VSS profiles $f_0(y)$ in the semilinear case $n = 0$ that exists for all $p \in (1, p_0 = 1 + \frac{4}{N})$; cf. [24]. On the vertical axis, we put $\|f\|_\infty$ that, for such profiles, is simply $f(0)$.

Figure 6 show that such monotone branches persist until $n = 0.11$ (a), while, for a slightly larger $n = 0.13$, we first observe a non-monotone branch of patterns $f_0(y)$ (b), and this persists for most of larger $n$’s. This is a *quasilinear* phenomenon to be discussed in greater detail below. Moreover, we observe a typical *turning* point of the $p$-branch at $p = 2.45...$, which, as usual, is characterized by existence of a non-trivial centre subspace
of the linearized operator,

\[ 0 \in \sigma(B'(f)). \]

These global bifurcation diagrams are calculated with the enhanced Tols = 10^{-4} and small step sizes \( \Delta p \sim 10^{-3} \). We claim that the turning, saddle-node bifurcation for \( N = 1 \) occurs above the critical exponent \( n_{s-n} \approx 0.12 \).

Figure 7 illustrates the vanishing behaviour of the VSS profiles \( f_0(y) \) as \( p \to p_0 = n + 5 \) for \( N = 1 \) and \( n = 1 \). Notice that, according to (3.14), the rate of decay is fast,

\[ \|f\|_{\infty} \sim (6 - p)^{1/5} \quad \text{as} \quad p \to 6^- . \]

In Figure 8 we show the corresponding first \( p \)-bifurcation branch that is originated at \( p = 6^- \). Such a behaviour of this \( p \)-branch is similar to that for the semilinear parabolic equation for \( n = 0 \); cf. [24]. The global behaviour of this branch is unusual: the branch exhibits a definite non-monotonicity and turning for \( p \approx 3.46 \ldots \)

Such non-monotonicity branching phenomena were consistent in numerical experiments. Similar features are shown for \( n = 0.5 \) in Figure 9. Let us discuss possible consequences of such a behaviour that was not available for \( n = 0 \), and hence exists in a strongly quasilinear case \( n > 0 \) sufficiently large (at least for \( n > 0.11 \) as Figure 6 suggests). It follows from principles of general branching theory [37] that if such a \( p \)-branches vanishes, it must end up at bifurcation points only.

On the other hand, we know that the profiles \( f_0 \) persist until the critical value \( p = n + 1 \); see Figure 4 where \( f_0(y) \) is available for \( n = 0.95 \) and \( p = 2 > n + 1 = 1.95 \). Therefore, if a \( p_0 \)-branch disappear at some bifurcation point \( p_k > n + 1 \), it must appear at another (possibly, saddle-node) bifurcation point \( \hat{p}_l < p_k \). As we will explain, at standard pitchfork bifurcations from 0 at \( p = p_l \) with \( l = 1, 2, \ldots \), other types of VSS profiles \( f_l \) appear, so that the new appearance of \( f_0 \) can be associated with new bifurcations and branching.
Therefore, $f_0$ must appear at some subcritical saddle-node bifurcation, most probably embracing branches of profiles $f_0$ and $f_2$ that have a similar geometric structure.

We now answer the question posed above. In Figure 10, we show the $p_{0,2}$-branch for $n = 1$ in a neighbourhood of the bifurcation point $p_2 = 3.333...$ and of the turning point shown in Figure 8. This branch has two turning points which are saddle-node bifurcations. It follows that the even profiles $f_0(y)$ and $f_2(y)$ belong to the same $p_{0,2}$-branch, i.e., can be continuously deformed to each other as solutions of the ODE (2.3). Note that the pitchfork bifurcation at $p = p_2$ is now supercritical (the branch is originated for $p > p_2 = 3.333...$). Figure 11 shows three different “$f_0$” profiles for $p = 3.6$, which is shown by the vertical dashed line in Figure 10. The smallest profile is actually $f_2(y)$, which thus is homotopically equivalent to $f_0$ (i.e., admits a homotopic path via a family of operators).

We expect that such saddle-node bifurcations can occur on other $p_l$-branches creating necessary profiles in different $p$-intervals. For instance, we observed an evidence that the next $p_{1,3}$-branch occurs.

Thus, the $p_{0,2}$-branch is a closed curve. This type of closed $\mu$-bifurcation branches were earlier found in [24, § 6.4] for VSSs with another type of parameterization in the ODEs like (2.3), $n = 0$ (so that $\beta = \frac{1}{4}$), with the change

$$\frac{1}{4} f' y \mapsto \mu f' y,$$

with parameter $\mu \in (0, \frac{1}{4}]$. Then a $\mu$-branch was shown to appear at a pitchfork bifurcation point and was continued until another, smaller one, i.e., a global continuation of such branches up to $\mu = \frac{1}{4}^+$ was impossible. In addition, essentially non-monotone bifurcation branches were detected [9, p. 1802] in rather similar fourth-order ODEs associated with blow-up in higher-order reaction-diffusion equations such as

$$u_t = -u_{xxxx} + |u|^{p-1}u \quad \text{in} \quad \mathbb{R} \times (0, T).$$

**On other $p$-bifurcation branches.** A similar “nonlinear” bifurcation analysis can be performed on the basis of any suitable similarity profiles $F_i(y)$ of the pure TFE (1.8). Namely, this profile appears in the similarity solution of (1.8),

$$u_t(x, t) = t^{-\alpha_l} F_i(y), \quad y = x/t^{\beta_l}, \quad \text{where} \quad \beta_l = \frac{1-n\alpha_l}{4} > 0,$$

and $\alpha_l \in (0, \frac{1}{4})$ is a parameter. Instead of (1.10), the function $F_i$ solves the following elliptic equation:

$$B_i(F) \equiv -\nabla \cdot (|F|^{p-1} \nabla F) + \beta_i \nabla F \cdot y + \alpha_l F = 0 \quad \text{in} \quad \mathbb{R}^N.$$

According to the principles of self-similarity of the second kind (a notion introduced by Ya.B. Zel’dovich in 1956 [39]), the acceptable values of the parameter $\beta_l$ are chosen from the solvability of the problem (3.16), i.e., from existence of a nontrivial compactly supported solution $F_i(y)$ in $\mathbb{R}^N$. We expect existence of a countable (up to obvious scaling) set of such solutions $\Phi = \{\alpha_l, F_i(y)\}$, with most of them not being radially symmetric. In a certain sense, this looks like a nonlinear extension of the linear eigenvalue problem that occurs for $n = 0$, where $\beta_l = \frac{1}{4}$ and (3.16) reads

$$B_i F \equiv -\Delta^2 F + \frac{1}{4} \nabla F \cdot y + \alpha_l F = 0 \quad \text{in} \quad \mathbb{R}^N,$$
so that $\lambda_l \equiv \frac{N}{4} - \alpha_l$ are eigenvalues of the non self-adjoint operator

\begin{equation}
B = -\Delta^2 + \frac{1}{4} y \cdot \nabla + \frac{N}{4} I.
\end{equation}

This spectrum is discrete and is given by

\begin{equation}
\lambda_l = -\frac{l}{4}, \quad l = 0, 1, 2, \ldots,
\end{equation}
Figure 9. $N = 1$, $n = 0.5$: the first $p$-bifurcation branch originated at the first critical exponent $p = 5.5$.

Figure 10. The enlarged non-monotonicity part of the bifurcation $p_{0,2}$-diagram for the ODE (2.3) in $\mathbb{R}$ for $n = 1$. The behaviour close to $p_2 = 3.333...$ with two turning (saddle-node) points.

and each eigenvalue has finite multiplicity. This determines all possible values of the parameters of self-similarity

$$
\alpha_l = \frac{N+l}{4}, \quad l = 0, 1, 2, \ldots
$$
For small $n > 0$, the linear eigenvalue problem (3.17) can predict the nonlinear eigenfunctions of (3.16) by the $n$-branching approach in the lines similar to that in Appendix A in [20], though the justification in the fourth-order case is much more difficult.

For larger $n > 0$, the nonlinear eigenfunctions $\{F_l\}$ are unknown even in the ODE case $N = 1$ or in radial setting in $\mathbb{R}^N$. Recall that, even in the simplest case $l = 1$ meaning the 1-dipole solution $F_1(y)$, the existence of such a profile for not small $n$ is still unclear mathematically; see references and results in [7] and more recent paper [8].

Finally, once the nonlinear eigenfunction subset $\{F_l\}$ of the pure TFE (1.8) is known (say, for small $n > 0$), for each function $F_l(y)$ supported in $B_1$ after rescaling, one can develop the above bifurcation approach taking in (3.9) $\tilde{F} = F_l$ for any $l = 0, 1, 2, \ldots$. This gives the sequence of critical exponents

$$\beta \equiv \frac{p -(n+1)}{4(p-1)} = \beta_l = \frac{1-n\alpha_l}{4} \implies p_l = 1 + \frac{1}{\alpha_l}.$$  

For $n = 0$, (3.21) and (3.20) yield precisely the known critical bifurcation points (3.5) studied in [24].

Therefore, a more rigorous bifurcation analysis is available for $p \approx p_l$ and $n \approx 0$, so this means the branching approach with the parameter $\mu = (p, n)^T \approx (1 + \frac{N}{N+1}, 0)^T$ in $\mathbb{R}^2$.

3.5. On other VSS profiles. As we have mentioned, for $n = 0$, for any $p$ in the subcritical range $p \in (1, p_0)$, there exists a finite number of VSS profiles $\{f_l\}$, and each $p$-branch is originated at bifurcation points (3.5); see [24] § 6. We claim that similar VSS patterns also exist for $n > 0$ and each $f_{l+1}(y)$ has a more complicated form than the previous one $f_l(y)$, i.e., has “more oscillatory” structure with more non-exponentially small oscillations and “essential” zeros. Figure 12 shows two VSS profiles $f_0$ and $f_1$ for $n = \frac{1}{2}$, $p = 2$ and

![Figure 11. Three $f_{0,2}(y)$ profiles from Figure 10 for $p = 3.6$.](image)
\( m = 2, N = 1, n = 0.5, p = 2 \): two VSS profiles

\( m = 2, N = 1, n = 1, p = 2.5 \): two VSS profiles

\( (a) \ n = \frac{1}{2}, \ p = 2 \)

\( (b) \ n = 1, \ p = 2.5 \)

**Figure 12.** First two VSS profiles for \( n = \frac{1}{2}, \ N = 1, \ p = 2 \) (a) and for \( n = 1, \ N = 1, \ p = 2.5 \) (b).

\( n = 1, \ p = 2.5 \) in the 1D case (they look similarly emphasizing continuity in \( n \)). It is difficult to check numerically whether other profiles exist.

For comparison, in Figure 13 we present five VSS profiles for the semilinear PDE for \( n = 0, \) with \( N = 1 \) and \( p = 1.7, \) which are computed much easier. According to (3.5), there exist five bifurcation points above 1.7,

\[
(3.22) \quad p_0 = 5, \ p_1 = 3, \ p_2 = \frac{7}{3}, \ p_3 = 2, \ p_4 = \frac{9}{5} = 1.8 \quad (p_5 = \frac{5}{3} = 1.66... < 1.7).
\]

Therefore, for \( p = 1.7, \) bearing in mind that the \( p \)-bifurcation branches are monotone decreasing in \( p, \) there exist precisely five VSS profiles indicated in this Figure. By continuity, we expect that these profiles do not essentially change and exist for all small enough \( n > 0, \) where these become compactly supported with the interface dependence [15, § 10]

\[
y_0(n) \sim n^{-\frac{3}{4}} \rightarrow +\infty \quad \text{as} \quad n \rightarrow 0^+.
\]

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