Circular orders, ultra-homogeneous order structures and their automorphism groups

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Abstract. We study topological groups $G$ for which the universal minimal $G$-system $M(G)$, or the universal irreducible affine $G$-system $IA(G)$ are tame. We call such groups “intrinsically tame” and “convexly intrinsically tame”, respectively. These notions, which were introduced in [13], are generalized versions of extreme amenability and amenability, respectively. When $M(G)$, as a $G$-system, admits a circular order we say that $G$ is intrinsically circularly ordered. This implies that $G$ is intrinsically tame.

We show that given a circularly ordered set $X_0$, any subgroup $G \leq \text{Aut}(X_0)$ whose action on $X_0$ is ultrahomogeneous, when equipped with the topology $\tau_p$ of pointwise convergence, is intrinsically circularly ordered. This result is a “circular” analog of Pestov’s result about the extreme amenability of ultrahomogeneous actions on linearly ordered sets by linear order preserving transformations. We also describe, for such groups $G$, the dynamics of the system $M(G)$, and show that it is extremely proximal (whence $M(G)$ coincides with the universal strongly proximal $G$-system), and deduce that the group $G$ must contain a non-abelian free group.

In the case where $X$ is countable, the corresponding Polish group of circular automorphisms $G = \text{Aut}(X_0)$ admits a concrete description. Using the Kechris-Pestov-Todorcevic construction we show that $M(G) = \text{Split}(T; Q_0)$, a circularly ordered compact metric space (in fact, a Cantor set) obtained by splitting the rational points on the circle $T$. We show also that $G = \text{Aut}(Q_0)$ is Roelcke precompact, satisfies Kazhdan’s property $T$ (using results of Evans-Tsankov) and has the automatic continuity property (using results of Rosendal-Solecki).

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1. Introduction

The universal minimal $G$-system $M(G)$ of a topological group $G$ can serve as a beautiful link between the theories of topological groups and topological dynamics. In general, $M(G)$ may be very large. For instance, it is always nonmetrizable for locally compact noncompact groups. The question whether $M(G)$ is small in some sense or another is pivotal in this theory. Recall that $G$ is said to be extremely amenable (or to have the fixed point on compacta property) when the system $M(G)$ is trivial. An important and now well studied (see [4, 26, 37]) question is: when is $M(G)$ metrizable?

Another interesting new direction is to determine when is $M(G)$ dynamically small or non-chaotic. A closely related question is: when is the universal irreducible affine $G$-system $IA(G)$ (dynamically) small? It is well known that $G$ is amenable iff $IA(G)$ is trivial.

In [13] we raised the question: when are $M(G)$ and $IA(G)$ tame dynamical systems? In the present work we give a new sufficient condition which guarantees that $M(G)$ and $IA(G)$ are circularly ordered $G$-systems (hence, by our previous work [12], tame).

Recall that an order preserving effective action $G \shortrightarrow X$ on a linearly ordered infinite set $X_<$ is said to be ultrahomogeneous if every order isomorphism between two finite subsets can be extended to an order automorphism $g \in G$ of $X_<$. We say that $X_<$ is ultrahomogeneous when the tautological action of $\text{Aut}(X_<)$ on $X$ is ultrahomogeneous.

As was shown by Pestov [27], for any subgroup $G \leq \text{Aut}(X_<)$ whose action on $X_<$ is ultrahomogeneous, the topological group $(G, \tau_p)$, in its pointwise convergence topology, is extremely amenable. That is, every continuous action of $G$ on a compact Hausdorff space has a fixed point. For example the Polish group $\text{Aut}(Q_<)$ of automorphisms of the linearly ordered set $\mathbb{Q}$ is extremely amenable in its pointwise convergence topology. Ultrahomogeneous structures, Ramsey theory and Fraïssé limits play a major role in many modern works, [27, 13, 15, 19, 28, 25, 23].

Our aim here is to examine the role of circular orders (c-order, for short). We say that an effective action of a group $G$ of c-order automorphisms on a c-ordered infinite set $X_\circ$ is ultrahomogeneous if every c-order isomorphism between two finite subsets can be extended to a c-order automorphism of $X_\circ$ in $G$. We say that $X_\circ$ is ultrahomogeneous when the tautological action of $\text{Aut}(X_\circ)$ on $X$ is ultrahomogeneous.

An important particular case is $X = Q_\circ = \mathbb{Q}/\mathbb{Z}$, the rational points of the circle $T = \mathbb{R}/\mathbb{Z}$. The corresponding automorphism group $G := \text{Aut}(Q_\circ)$, equipped with the topology of pointwise convergence $\tau_p$, is a Polish nonarchimedean topological group (hence a closed subgroup of the symmetric group $S_\infty$).
Let $G$ be a topological group and $P$ a property of $G$-dynamical systems, we say $G$ has the intrinsic property $P$ when the universal minimal $G$-system $M(G)$ has this property. Of course when $P$ is a property which is preserved by factors, this is the same as saying that every minimal $G$-system has the property $P$. With $P$ the property of being the trivial one point system, having the intrinsic $P$ property reduces to extreme amenability. Our general idea is to relax this extreme case by choosing $P$ to be a “small” class of $G$-systems (see [13]).

In Theorem 4.3 we prove that when a group $G$ acts ultrahomogeneously on an infinite circularly ordered set $X$, then $G$, with the pointwise convergence topology it inherits from this action, is intrinsically c-ordered. That is, the universal minimal $G$-system $M(G)$ is a circularly ordered $G$-system. In particular this applies to $\text{Aut}(X_c)$ when $X_c$ is ultrahomogeneous. We do not know whether this implies that every minimal $G$-system is circularly ordered.

Furthermore, in the case where $X$ is countable, it follows that $M(G)$ is a metrizable circularly ordered $G$-system. In particular this is true for $G = \text{Aut}(\mathbb{Q}_c)$. This theorem also applies to Thompson’s (finitely generated, nonamenable) circular group $T$, which acts ultrahomogeneously on $D_c$, the set of dyadic rationals in the circle.

Again the idea is to regard the requirement that $M(G)$ be a c-ordered set as a relaxation of the requirement that $M(G)$ be a trivial system. In this sense Theorem 4.3 is indeed an analog of Pestov’s theorem because every linearly ordered minimal compact $G$-system is necessarily trivial.

In contrast to linear orders, the class of minimal compact circularly ordered $G$-spaces is quite large. For instance, the study of minimal subsystems of the circle $S$ with an action defined by a c-order preserving homeomorphism goes back to classical works of Poincaré, Denjoy and Markley. In symbolic dynamics we have extensive studies of Sturmian like systems which are c-ordered. Finally, by another theorem of Pestov, the universal minimal system $M(H_+(\mathbb{T}))$ of the Polish group $H_+(\mathbb{T})$, of c-order preserving homeomorphisms of the circle $\mathbb{T}$, is $\mathbb{T}$ itself with the tautological action, [28].

In Theorem 4.9 we describe, for groups $G$ as in Theorem 4.3, the dynamics of the system $M(G)$, and show that it is extremely proximal (whence $M(G)$ is the universal strongly proximal $G$-system). We also deduce that the group $G$ must contain a non-abelian free group.

One of the reasons we think of a c-ordered dynamical system as being a relaxation of being trivial is the fact we proved in [12], that every c-ordered compact, not necessarily metrizable, $G$-space $X$ is tame. In fact we prove there a stronger result, namely that such a system can always be represented on a Rosenthal Banach space. We refer to [11, 12, 13] for more information about tame dynamical systems. See also Remark 2.5.

In Section 2 we prepare the ground for our main results which are proved in Section 4. In Section 5 we present an alternative proof of Theorem 4.3 in the countable case. It employs the notion of Fraissé classes and the Kechris-Pestov-Todorcevic theory [19] and has the advantage that it automatically yields an explicit description of $M(G)$. Namely, in terms of [12], $M(\text{Aut}(\mathbb{Q}_c))$ is the system $\text{Split}(\mathbb{T};\mathbb{Q}_c)$ obtained from the circle by splitting in two the rational points.
In Section 6 we examine some other properties of the Polish group \( \text{Aut}(\mathbb{Q}_\circ) \). This topological group has the automatic continuity property (Lemma 6.1). This easily follows from the automatic continuity property of \( \text{Aut}(\mathbb{Q}_\circ) \) established by Rosendal and Solecki [33]. As an application we get that the discrete group \( \text{Aut}(\mathbb{Q}_\circ) \) is metrically intrinsically c-ordered (see Definition 2.2 below). That is, every action of \( G := \text{Aut}(\mathbb{Q}_\circ) \) by homeomorphisms on a metric compact space, admits a closed \( G \)-invariant subsystem which is a circularly ordered \( G \)-subsystem, Corollary 6.2.

We also note that the group \( \text{Aut}(\mathbb{Q}_\circ) \) is Roelcke precompact (Proposition 6.3); and finally, a result of Evans and Tsankov [5] implies that it has Kazhdan’s property (T) (see Corollary 6.5).

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2. Some generalizations of (extreme) amenability

An action \( G \times X \to X \) of a group \( G \) on a set \( X \) is effective if \( gx = x \ \forall x \in X \) is possible only for \( g = e \), where \( e \) is the identity of \( G \). Below all compact spaces are Hausdorff. A compact space \( X \) with a given continuous action \( G \curvearrowright X \) of a topological group \( G \) on \( X \) is said to be a dynamical \( G \)-system.

Recall the classical definition from [34]: a sequence \( f_n \) of real valued functions on a set \( X \) is said to be independent if there exist real numbers \( a < b \) such that

\[
\bigcap_{n \in P} f_n^{-1}(-\infty, a) \cap \bigcap_{n \in M} f_n^{-1}(b, \infty) \neq \emptyset
\]

for all finite disjoint subsets \( P, M \) of \( \mathbb{N} \).

By A. Köhler’s [17] definition, a dynamical \( G \)-system \( X \) is tame if for every continuous real valued function \( f : X \to \mathbb{R} \) the family of functions \( fG := \{fg\}_{g \in G} \) does not contain an independent sequence.

The following result is a dynamical analog of a well known Bourgain-Fremlin-Talagrand dichotomy [1].

**Theorem 2.1.** [9] Let \( X \) be a compact metric dynamical \( S \)-system and let \( E = E(X) \) be its enveloping semigroup. We have the following alternative. Either

1. \( E \) is a separable Rosenthal compact (hence \( E \) is Fréchet and \( \text{card} \ E \leq 2^{|\mathbb{N}|} \)); or
2. the compact space \( E \) contains a homeomorphic copy of \( \beta\mathbb{N} \) (hence \( \text{card} \ E = 2^{|\mathbb{N}|} \)).

The first possibility holds iff \( X \) is a tame \( S \)-system.

A dynamical \( G \)-system \( X \) is said to be circularly (linearly) ordered if \( X \) is a circularly (linearly) ordered space and each element of \( G \) preserves the circular (linear) order [12]. In Section 3 we give some background about circular order.

A topological group \( G \) is said to be nonarchimedean if it has a base of open neighbourhoods of the identity consisting of (clopen) subgroups. Equivalently, the groups which can be embedded into the symmetric groups \( S_X \).

As usual, \( M(G) \) denotes the universal minimal \( G \)-system of a topological group \( G \). By \( IA(G) \) we denote the universal irreducible affine \( G \)-system, [8]. Recall that
A topological group $G$ is amenable (extremely amenable) iff $IA(G)$ (respectively, $M(G)$) is trivial. The following definition proposes some generalizations.

**Definition 2.2.**\[13\] A topological group $G$ is said to be:

1. **Intrinsically tame** if the universal minimal $G$-space $M(G)$ is tame. Equivalently, if every continuous action of $G$ on a compact space $X$ has a closed $G$-subspace $Y$ which is tame.
2. **Intrinsically c-ordered** if $M(G)$ is a c-ordered $G$-system.
3. **Convexly intrinsically tame** if the universal irreducible affine $G$-system $IA(G)$ is tame. Equivalently, if every continuous affine action on a compact convex space $X$ has a closed (not necessarily affine) $G$-subspace $Y$ which is tame.
4. **Convexly intrinsically c-ordered** if every continuous affine action on a compact convex space $X$ has a closed $G$-subspace $Y$ which is c-ordered.
5. For brevity we use the following short names: int-tame, int-c-ord, conv-int-tame, conv-int-c-ord.
6. If in (1) we consider only metrizable spaces $X$ then we say that $G$ is metrically intrinsically tame (in short: metr-int-tame). Similarly can be defined also the notions of metrically intrinsically c-ordered, and metrically convexly intrinsically ordered groups.

**Remark 2.3.** By results of \[12\] every c-ordered $G$-system is tame. Thus, we have the following diagram of implications:

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extr. amenability ───── int-c-ord ───── int-tame ───── metr-int-tame
       \                     \                     \                     \
amenability ───── conv-int-c-ord ───── conv-int-tame ───── metr-conv-int-tame
```

**Examples 2.4.**\[13\]

1. $SL_n(\mathbb{R})$, $n > 1$ (more generally, any semisimple Lie group $G$ with finite center and no compact factors) are conv-int-tame nonamenable topological groups which are not int-tame. Moreover, $SL_2(\mathbb{R})$ is int-c-ordered.

   Sketch: by Furstenberg’s result \[6\] the universal minimal strongly proximal system $M_{sp}(G)$ is the homogeneous space $X = G/P$, where $P$ is a minimal parabolic subgroup (see \[8\]). Results of Ellis and Akin (Example \[13\] Example 6.2.1) show that the enveloping semigroup $E(G, X)$ in this case is a Rosenthal compact space, whence the system $(G, X)$ is tame by the dynamical BFT dichotomy (Theorem 2.1).

   Note that in the case of $G = SL_2(\mathbb{R})$ it follows that in any compact affine $G$-space one can find either a 1-dimensional real projective $G$-space (a copy of the circle) or a fixed point. For general $SL_n(\mathbb{R})$ – flag manifolds and their $G$-quotients.

2. The Polish group $H_+(\mathbb{T})$ is an int-c-ord nonamenable topological group. Note that for $G = H_+(\mathbb{T})$ every compact $G$-space $X$ contains either a copy of $\mathbb{T}$, as a $G$-subspace, or a $G$-fixed point.

   Sketch: This follows from Pestov’s theorem \[27\], which identifies $(G, M(G))$ for $G = H_+(\mathbb{T})$ as the tautological action of $G$ on $\mathbb{T}$.

3. The Polish groups $\text{Aut}(\mathcal{S}(2))$ and $\text{Aut}(\mathcal{S}(3))$ of automorphisms of the circular directed graphs $\mathcal{S}(2)$ and $\mathcal{S}(3)$, are intrinsically c-ordered (hence, also
int-tame). The universal minimal $G$-systems for $\text{Aut}(S(2))$ and $\text{Aut}(S(3))$ are computed by L. Nguyen van Thé in [25].

(4) The Polish group $H(C)$, of homeomorphisms of the Cantor set, is not conv-int-tame.

(5) The Polish group $G = S_\infty$, of permutations of the natural numbers, is amenable, hence conv-int-tame but not int-tame.

Given a class $P$ of compact $G$-systems one can define the notions “intrinsically $P$ group” and “convexly intrinsically $P$ group” in a manner analogous to the one we adopted for $P = \text{Tame}$.

Recall that the following inclusion relations are valid

$$\text{AP} \subset \text{WAP} \subset \text{HNS} \subset \text{Tame}$$

where, $\text{AP} =$ almost periodic (equivalently, equicontinuous) $G$-systems, $\text{WAP} =$ weakly almost periodic systems, $\text{HNS} =$ hereditarily nonsensitive. For the definitions of $\text{Asp}(G)$, $\text{HNS}$ and $\text{WAP}$ see for example [13]. By the dynamical BFT dichotomy [11, 13], the class $\text{Tame}$ of dynamical systems, plays a special role being, in a sense, the largest class of all “small” systems.

Note also that if $P$ is the class of all metrizable $G$-systems then $G$ is int-$P$ if and only if $M(G)$ is metrizable.

REMARK 2.5. It turns out that in this terminology a topological group is convexly intrinsically HNS (and, hence, also conv-int WAP) iff it is amenable. This follows from the fact that every HNS minimal $G$-system is almost periodic; see [22, Prop. 7.18] and [9, Lemma 9.2.3]. Thus we have

$$\text{int-AP} = \text{int-WAP} = \text{int-HNS}.$$  

Also, by the left amenability of the algebra $\text{Asp}(G)$, which corresponds to the class of HNS systems, [10], we get

$$\text{amenability} = \text{conv-int-AP} = \text{conv-int-WAP} = \text{conv-int-HNS}.$$  

This “collapsing effect” inside HNS and the exceptional role of tameness in the dynamical BFT dichotomy suggest that the notion of convex intrinsic tameness is a natural generalization of amenability. This is also supported by several natural examples (see Examples 2.4 and Corollary 4.6).

3. Circular order, topology and inverse limits

In this section we give some technical results about circular order which we use in Section 3. For more information and properties we refer to [12].

DEFINITION 3.1. [20, 3] Let $X$ be a set. A ternary relation $R \subset X^3$ on $X$ is said to be a circular (or, sometimes, cyclic) order if the following four conditions are satisfied. It is convenient sometimes to write shortly $[a, b, c]$ instead of $(a, b, c) \in R$.

1. Cyclicity: $[a, b, c] \Rightarrow [b, c, a]$;
2. Asymmetry: $[a, b, c] \Rightarrow (b, a, c) \notin R$;
3. Transitivity: \[ \begin{cases} [a, b, c] & \Rightarrow [a, b, d]; \\ [a, c, d] & \Rightarrow [a, b, d]; \end{cases} \]
4. Totality: if $a, b, c \in X$ are distinct, then $[a, b, c]$ or $[a, c, b]$.

Observe that under this definition $[a, b, c]$ implies that $a, b, c$ are distinct.
For \( a, b \in X \) define the (oriented) intervals:
\[
(a, b)_R := \{ x \in X : \{a, x, b\} \}, \quad [a, b)_R := (a, b) \cup \{a\}, \quad (a, b]_R := (a, b) \cup \{b\}.
\]
Sometimes we drop the subscript, or write \((a, b)_R\) when context is clear. Clearly, \( X \setminus [a, b)_R = (b, a)_R \) for \( a \neq b \) and \( X \setminus [a, b]_R = X \setminus \{a\} \).

**Remark 3.2.** ([R page 35])

(1) Every linear order \(<\) on \( X \) defines a standard circular order \( R_< \) on \( X \) as follows: \([x, y, z]\) iff one of the following conditions is satisfied:
\[
x < y < z, \quad y < z < x, \quad z < x < y.
\]

(2) (cuts) Let \((X, R)\) be a c-ordered set and \( z \in X \). For every \( z \in X \) the relation
\[
z <_z a, \quad a <_z b \iff [z, a, b] \quad \forall a \neq b \neq z \neq a
\]
is a linear order on \( X \) and \( z \) is the least element. This linear order restores the original circular order. Meaning that \( R_\leq = R \).

The following two technical results are easy to verify.

**Proposition 3.3.**

(1) For every c-order \( R \) on \( X \) the family of subsets
\[
\mathcal{B}_1 := \{ X \setminus [a, b)_R : a, b \in X \} \cup \{X\}
\]
forms a base for a topology \( \tau_R \) on \( X \) which we call the interval topology of \( R \).

(2) If \( X \) contains at least three elements then the (smaller) family of intervals
\[
\mathcal{B}_2 := \{ (a, b)_R : a, b \in X, a \neq b \}
\]
forms a base for the same topology \( \tau_R \) on \( X \).

(3) The interval topology \( \tau_R \) of every circular order \( R \) is Hausdorff.

**Lemma 3.4.** Let \( R \) be a circular order on \( X \) and \( \tau_R \) the induced (Hausdorff) topology. Then for every \( [a, b, c] \) there exist neighborhoods \( U_1, U_2, U_3 \) of \( a, b, c \) respectively such that \([a', b', c']\) for every \( (a', b', c') \in U_1 \times U_2 \times U_3 \).

Denote by \( C_n := \{1, 2, \cdots, n\} \) the standard \( n \)-cycle with the natural circular order. Let \((X, R)\) be a c-ordered set. We say that a vector \((x_1, x_2, \cdots, x_n) \in X^n\) is a cycle in \( X \) if it satisfies the following two conditions:

(1) For every \([i, j, k]\) in \( C_n \) and distinct \( x_i, x_j, x_k \) we have \([x_i, x_j, x_k]\);

(2) \( x_i = x_k \Rightarrow (x_i = x_{i+1} = \cdots = x_{k-1} = x_k) \lor (x_k = x_{k+1} = \cdots = x_{i-1} = x_i) \).

Injective cycle means that all \( x_i \) are distinct.

**Definition 3.5.** Let \((X_1, R_1)\) and \((X_2, R_2)\) be c-ordered sets. A function \( f : X_1 \to X_2 \) is said to be c-order preserving if \( f \) moves every cycle to a cycle. Equivalently, if it satisfies the following two conditions:

(1) For every \([a, b, c]\) in \( X \) and distinct \( f(a), f(b), f(c) \) we have \([f(a), f(b), f(c)]\);

(2) If \( f(a) = f(c) \) then \( f \) is constant on one of the closed intervals \([a, c], [c, a]\).
Lemma 3.6. Let $X_\infty := \lim_{\leftarrow} (X_i, I)$ be the inverse limit of the inverse system

$$\{f_{ij}: X_j \rightarrow X_i, \quad i \leq j, \quad i, j \in I\}$$

where $(I, \leq)$ is a directed poset. Suppose that every $X_i$ is a c-ordered set with the c-order $R_i \subset X_i^3$ and each bonding map $f_{ij}$ is c-order preserving. On the inverse limit $X_\infty$ define a ternary relation $R$ as follows. An ordered triple $(a, b, c) \in X_\infty^3$ belongs to $R$ iff $[p_i(a), p_i(b), p_i(c)]$ is in $R_i$ for some $i \in I$. Then $R$ is a c-order on $X_\infty$ and each projection map

$$p_i: X_\infty \rightarrow X_i, \quad p_i(a) = a_i$$

is c-order preserving.

Proof. We start with

Claim 1: For every three distinct elements $a, b, c \in X_\infty$ there exists a separating projection $p_i: X_\infty \rightarrow X_i$; that is, $a_i = p_i(a), b_i = p_i(b), c_i = p_i(c)$ are distinct.

Indeed, since $a, b, c \in X_\infty$ are distinct there exist indexes $j(a, b), j(a, c), j(b, c) \in I$ such that

$$a_{j(a, b)} \neq b_{j(a, b)}, \quad a_{j(a, c)} \neq c_{j(a, c)}, \quad b_{j(b, c)} \neq c_{j(b, c)}.$$

Since $I$ is directed we may choose $i \in I$ which dominates all three indexes $j(a, b), j(a, c), j(b, c)$. Then $a_i, b_i, c_i$ are distinct.

Claim 2: If $[a_i, b_i, c_i]$ for some $i \in I$ and $a_j, b_j, c_j$ are distinct in $X_j$ for some $j \in J$ then $[a_j, b_j, c_j]$.

Indeed, choose an index $k \in I$ such that $i \leq k, j \leq k$ then $a_k, b_k, c_k$ are distinct. Necessarily $[a_k, b_k, c_k]$. Otherwise, $[b_k, a_k, c_k]$ by the Totality axiom. Then also $[a_i, b_i, c_i]$ because the bonding map $f_{jk}: X_k \rightarrow X_i$ is c-order preserving and $a_i, b_i, c_i$ are distinct in $X_i$ (since $[a_i, b_i, c_i]$).

Since $[a_k, b_k, c_k]$ it follows that $[a_j, b_j, c_j]$ because the bonding map $f_{jk}: X_k \rightarrow X_j$ is c-order preserving.

Now we show that $R$ is a c-order (Definition 3.1) on $X_\infty$.

The Cyclicity axiom is trivial.

Asymmetry axiom is easy by Claim 2.

Transitivity: by Claims 1 and 2 there exists $k \in I$ such that $[a_k, b_k, c_k]$ and $[a_k, c_k, d_k]$. Hence, $[a_k, b_k, d_k]$ by the transitivity of $R_k$. Therefore, $[a, b, d]$ in $X_\infty$ by the definition of $R$.

Totality: if $a, b, c \in X$ are distinct, then $a_j, b_j, c_j$ are distinct for some $j \in I$ by Claim 1. By the totality of $R_j$ we have $[a_j, b_j, c_j] \lor [a_j, c_j, b_j]$, hence also $[a, b, c] \lor [a, c, b]$ in $R$.

So, we proved that $R$ is a c-order on $X_\infty$.

Now we show that each projection $p_i: X_\infty \rightarrow X_i$ is c-order preserving. Condition (1) of Definition 3.3 is satisfied for every $i \in I$ by Claim 2 and the definition of $R$. In order to verify condition (2) of Definition 3.3 assume that $p_i(a) = p_i(b)$ for some distinct $a, b \in X_\infty$. We have to show that $p_i$ is constant on one of the closed intervals $[a, b], [b, a]$. If not then there exist $u, v \in X_\infty$ such that $[a, u, b], [b, v, a]$ but $p_i(u) \neq p_i(a) \neq p_i(v)$. As in the proof of Claim 1 one may choose an index $k \in I$ such that the elements $p_k(a), p_k(b), p_k(u), p_k(v)$ are distinct in $X_k$. Then we get
that the bonding map $f_{ik}: X_k \to X_i$ does not satisfy condition (2) of Definition 3.7. This contradiction completes the proof. □

Lemma 3.7. In terms of Lemma 3.7 assume in addition that every $X_i$ is a compact $c$-ordered space and each bonding map $f_{ij}$ is continuous. Then the topological inverse limit $X_\infty$ is also a $c$-ordered (nonempty) compact space.

Proof. Let $\tau_\infty$ be the usual topology of the inverse limit $X_\infty$. It is well known that the inverse limit $\tau_\infty$ of compact Hausdorff spaces is nonempty and compact Hausdorff. Let $\tau_c$ be the interval topology (see Proposition 3.3) of the $c$-order $R$ on $X_\infty$, where $(X_\infty, R)$ is defined as in Lemma 3.6. We have to show that $\tau_\infty = \tau_c$. Since $\tau_c$ is Hausdorff it is enough to show that $\tau_\infty \supseteq \tau_c$. This is equivalent to showing that every interval $(u, v)_o$ is $\tau_\infty$-open in $X_\infty$ for every distinct $u, v \in X_\infty$, where

\[(u, v)_o := \{x \in X | [u, x, v]\}.

Let $w \in (u, v)_o$; that is, $[u, w, v]$. By our definition of the $c$-order $R$ of $X_\infty$ we have $[u_i, w_i, v_i]$ in $X_i$ for some $i \in I$. The interval $O_i := (u_i, v_i)_o$ is open in $X_i$. Then its preimage $p_i^{-1}(O_i)$ is $\tau_\infty$-open in $X_\infty$. On the other hand,

\[w \in p_i^{-1}(O_i) \subseteq (u, v)_o.

Indeed, if $x \in p_i^{-1}(O_i)$ then $p_i(x) \in (u_i, v_i)_o$. This means that $[u_i, x, v_i]$ in $X_i$. By the definition of $R$ we get that $[u, x, v]$ in $X_\infty$. So, $x \in (u, v)_o$. □

Lemma 3.8.

(1) Let $\hat{G}$ be the two-sided completion of a topological group $G$. Then $G$ is int-tame (respectively, int-$c$-ordered) iff $\hat{G}$ is int-tame (respectively, int-$c$-ordered).

(2) Let $h: G_1 \to G_2$ be a continuous dense homomorphism. Then if $G_1$ is int-tame (respectively, int-$c$-ordered), so is $G_2$.

Proof. (1) Let $G \curvearrowright X$ be a continuous action on a compact Hausdorff space $X$. Then we have the continuous extended action $\hat{G} \curvearrowright X$ induced by the continuous homomorphism $\gamma: G \to H(X)$, where $H(X)$ (the full homeomorphism group) is always two-sided complete. Also, $M(G) = M(\hat{G})$. Now observe that $G \curvearrowright X$ is tame ($c$-ordered)iff $\hat{G} \curvearrowright X$ is tame ($c$-ordered). Indeed, in the case of tameness, observe that $cl_p(f\hat{G}) = cl_p(fG)$ for every $f \in C(X)$ and use the characterization (see [11] Proposition 5.6) of tame functions.

For the case of $c$-order, let $X$ be a $c$-ordered $G$-system. It is enough to show that $X$ is a $c$-ordered $\hat{G}$-system. Let $t \in \hat{G}$ and assume $[x, y, z]$. We have to show that $[tx, ty, tz]$. Assuming the contrary, by the Totality axiom (Definition 3.1) we have $[ty, tx, tz]$. Choose a net $g_i$ in $G$ which tends to $t$. Clearly, $[gi x, gix, giz]$. Finally apply Lemma 3.7 and the Totality axiom.

(2) Follows easily from (1) because $h(G_1)$ and $G_2$ have the “same” two-sided completion. □

4. Ultrahomogeneous actions on circularly ordered sets

Now we introduce the following definition, a natural circular version of the ultrahomogeneity for linear orders.
Definition 4.1. We say that an effective action of a group $G$ of \(c\)-order automorphisms on a \(c\)-ordered infinite set $X$ is ultrahomogeneous if every \(c\)-order isomorphism between two finite subsets can be extended to a \(c\)-order automorphism of $X$. Let us say that a circularly ordered set $X$ is ultrahomogeneous if the action $H_+(X) \curvearrowright X$ is ultrahomogeneous.

Lemma 4.2. If a group $G$ acts ultrahomogeneously on an infinite circularly ordered set $X$, then the order type of $X$ is dense; i.e. there are no vacuous intervals $(a, b)_R$.

Proof. Let $z \in X$ be an arbitrary point and consider the linearly ordered set $(X, <_z)$ (the “cut” at $z$, Remark 3.2.2). Then $X \setminus \{z\}$ is a ultrahomogeneous linearly ordered set, whence is of the dense type. As $z$ is arbitrary our assertion follows. □

Let $H$ be a closed subgroup of a topological group $G$. Denote by $q: G \to G/H$, $g \mapsto gH = [g]$ the natural (open) projection on the coset $G$-space $G/H$ endowed with the quotient topology. Recall that the topological space $G/H$ admits a natural right uniformity $\mu_r(G/H)$. A uniform basis of $\mu_r(G/H)$ is a family of all entourages of the form $\tilde{V} := \{(xH, yH) : xy^{-1} \in V\}$ where $V \in N_e(G)$ is a neighborhood of the identity $e$ in $G$. Then $q$ is uniformly continuous and the Samuel compactification of $\mu_r(G/H)$ induces the maximal $G$-compactification $G/H \hookrightarrow \beta G/H$ of the $G$-space $G/H$ (which, in this case, is a topological embedding).

Recall also that every uniform structure can be defined by uniform coverings. In the case of $\mu_r(G/H)$ the corresponding basis is the following family of uniform coverings of $G/H$

$$\nu_V := \{V[x] : [x] = xH \in G/H\}$$

where $V \in N_e(G)$.

The following result is a circular analog of Pestov’s result. As we already mentioned the “intrinsically linearly ordered” groups (that is, the groups with linearly ordered $M(G)$) are exactly the extremely amenable groups. About the structure and properties of $M(G)$ see Theorem 4.9.

Theorem 4.3. Let a group $G$ act ultrahomogeneously on an infinite circularly ordered (discrete) set $X$. Then $G$, with the pointwise topology, is intrinsically \(c\)-ordered (i.e., $M(G)$ is a \(c\)-ordered $G$-system). If $X$ is countable then $M(G)$ is metrizable.

Proof. Choose any point $z \in X$. The corresponding stabilizer $H := St(z)$ is a clopen subgroup of $G$.

As in [12] Proposition 2.3] consider the canonical linear order on $X$ defined by the rule:

$z <_z a, \quad a <_z b \Leftrightarrow [z, a, b] \quad \forall a \neq b \neq z \neq a.$

By construction (the smallest element) $z$ is $H$-fixed.

Claim. The induced action $H \curvearrowright X \setminus \{z\}$ on the linearly ordered set $X \setminus \{z\}$ is linear order preserving and ultrahomogeneous.
Lemma 4.2) intervals $(t, z, y)$ in $X \setminus \{z\}$. Then $[z, x, y]$. For every $h \in H$ we have \[ [h(z), h(x), h(y)] = [z, h(x), h(y)]. \] This means that $h(x) < z, h(y)$. Furthermore, the action $H \curvearrowright X \setminus \{z\}$ is ultrahomogeneous. First of all the action $St(z) \curvearrowright X \setminus \{z\}$ is also effective. For every pair of $k$-element chains
\[ x_1 < z, x_2 < z, \ldots < z, x_k \quad y_1 < z, y_2 < z, \ldots < z, y_k \]
in $X_{<z} \setminus \{z\}$ consider the pair of $(k+1)$-chains by adding the least element $z$. That is, consider
\[ z < z, x_1 < z, x_2 < z, \ldots < z, x_k \quad z < z, y_1 < z, y_2 < z, \ldots < z, y_k. \]
We can treat them as a pair of cycles in the circularly ordered set $X_{<z}$. The bijection $x_i \mapsto y_i, z \mapsto z$ is an isomorphism between finite circularly ordered sets. Therefore there exists an extension $g : X \rightarrow X, g \in G$. Since $g(z) = z$ we have $g \in H$ and $g$ preserves the order $<z$ on $X$ and hence also on $X \setminus \{z\}$. \[ \square \]

By Pestov's theorem 27 mentioned above, $H$, equipped with the pointwise topology on the discrete space $X \setminus \{z\}$ is extremely amenable. Note that this topology on $H = St(z)$ is the same as the topology of simple convergence on $X$ with respect to the action of $H$ on the discrete set $X$ (since $z$ is fixed under $H$). So, we may treat $H$ as a topological subgroup of $G$.

We have the natural isomorphism of $G$-actions
\[ i : G/H \rightarrow X, \ gH \mapsto gz. \]

For convenience sometimes we identify below the discrete $G$-sets $G/H$ and $X$. We can treat $i$ also as an isomorphism of uniform spaces, where $G/H$ us endowed with its right uniformity $\mu_r(G/H)$ and $X$ carries the natural uniformity $\mu_X$ such that the coverings of the form $\{Vx : x \in X\}$, with $V$ a neighborhood of the identity in $G$, is a base of the uniformity.

Let $\beta \gamma(G/H)$ be the maximal $G$-compactification of the discrete coset $G$-space $G/H := \{gH : g \in G\}$. Since $H$ is extremely amenable one may apply another result of Pestov 28. Theorem 6.2.9] which asserts that any minimal compact $G$-subsystem of $\beta \gamma(G/H)$ is isomorphic to the universal system $M(G)$. Therefore, it is enough to show that $\beta \gamma(G/H)$ is c-ordered (it is easy to show that a closed subspace of a c-ordered compact space is again c-ordered). Below we will see that $G/H$, as a uniform space with respect to the usual right uniformity, is precompact (Lemma 4.4). So, in this case $\beta \gamma(G/H)$ is just the completion $G/H$ (and the Samuel compactification) of the precompact uniform $G$-space $G/H$.

So our aim is to show that the compact space $G/H$ is a c-ordered $G$-system. We show this in Lemma 4.5 below. For these purposes we will need some preparations.

Let $F := \{t_1, t_2, \ldots, t_m\}$ be an $m$-cycle on $X$. That is, a $c$-order preserving injective map $F : C_m \rightarrow X$, where $t_i = F(i)$ and $C_m := \{1, 2, \ldots, m\}$ with the natural circular order. We have a natural equivalence "modulo-$m$" between $m$-cycles (with the same support).

For every given cycle $F := \{t_1, t_2, \ldots, t_m\}$ define the corresponding finite disjoint covering $cov \! F$ of $X$, by adding to the list: all points $t_i$ and (nonempty by Lemma 1.2) intervals $(t_i, t_{i+1})_o$ between the cycle points. More precisely we consider the following disjoint cover which can be think of an equivalence relation on $X$.
\[ cov \! F := \{t_1, (t_1, t_2)_o, t_2, (t_2, t_3)_o, \ldots, t_m, (t_m, t_1)_o\}. \]
Moreover, \( \text{cov}_F \) naturally defines also a finite c-ordered set \( X_F \) by “gluing the points” of the interval \( (t_i, t_{i+1})_o \) for each \( i \). So, the c-ordered set \( X_F \) is the factor-set of the equivalence relation \( \text{cov}_F \) and it contains \( 2m \) elements. In the extremal case of \( m = 1 \) (that is, for \( F = \{t_1\} \)) we define \( \text{cov}_F := \{t_1, X \setminus \{t_1\}\} \).

We have the following canonical c-order preserving onto map
\[
\pi_F : X \to X_F, \quad \pi_F(x) = \begin{cases} 
t_i & \text{for } x = t_i 
(t_i, t_{i+1})_o & \text{for } x \in (t_i, t_{i+1})_o,
\end{cases}
\]

**Lemma 4.4.** The family \( \{\text{cov}_F\} \) where \( F \) runs over all finite injective cycles
\[
F : \{1, 2, \cdots, m\} \to X = G/H
\]
on \( X \) is a bases of the natural uniformity of the coset right uniform space \( G/H \). This uniform structure is precompact.

**Proof.** First note the following general fact. If \( G \) is a nonarchimedean (NA) topological group and \( H \) is its arbitrary subgroup. Then the uniform space \( G/H \) is NA. That is, uniformly continuous equivalence relations on \( G/H \) form a uniform bases of the right uniformity. Indeed, for every open subgroup \( P \) of \( G \) we have an equivalence relation on \( G \) – partition by double cosets of the form \( P \chi H \), where \( x \in S(P, H) \) and \( S(P, H) \subset G \) is a subset of representatives. This partition induces an equivalence relation on \( G/H \) which is an element of the right uniformity that majorizes the basic uniform covering \( \nu_P = \{P[x] : [x] \in G/H\} \).

Back to our \( G \) from Theorem 4.3 which acts on the discrete set \( X \). By the definition of pointwise topology \( \tau_p \), basic neighborhoods of the identity \( e \in G \) are subgroups
\[
V_F = \text{St}(F) := \cap_{x \in F} \text{St}(x),
\]
where \( F \) is a finite subset of \( X \). Accordingly, we have a basic uniform covering \( \{V_F : x \in X\} \) of \( X \). We can suppose that \( F = \{t_1, t_2, \cdots, t_m\} \) is a cycle of \( X \). Since the action is ultrahomogeneous it follows that \( V_F x = (t_i, t_{i+1})_o \) for every \( x \in (t_i, t_{i+1})_o \) (and \( V_F t_i = t_i \)). Hence we obtain that \( \text{cov}_F = \{V_F x : x \in X\} \) is, in fact, a finite covering. It follows immediately from the definitions that \( G/H = X \), as a uniform structure, is precompact meaning that its completion \( \hat{G}/H \) is compact. \( \square \)

Let \( \text{Cycl}(X) \) be the set of all finite injective cycles. Every finite \( m \)-element subset \( A \subset X \) defines a cycle \( F_A : \{1, \cdots, m\} \to X \) (perhaps after some reordering) which is uniquely defined up to the natural cyclic equivalence and the image of \( F_A \) is \( A \).

\( \text{Cycl}(X) \) is a poset if we define \( F_1 \leq F_2 \) whenever \( F_1 : C_{m_1} \to X \) is a sub-cycle of \( F_2 : C_{m_2} \to X \). This means that \( m_1 \leq m_2 \) and \( F_1(C_{m_1}) \subset F_2(C_{m_2}) \). This partial order is directed. Indeed, for \( F_1, F_2 \) we can consider \( F_3 = F_1 \cup F_2 \) whose support is the union of the supports of \( F_1 \) and \( F_2 \).

For every \( F \in \text{Cycl}(X) \) we have the disjoint finite \( \mu_X \)-uniform covering \( \text{cov}_F = \{V_F x : x \in X\} \) of \( X \). As before we can look at \( \text{cov}_F \) as a c-ordered (finite) set \( X_F \). Also, as in equation (1.1) we have a c-order preserving natural map \( \pi_F : X \to X_F \) which are uniformly continuous into the finite (discrete) uniform space \( X_F \). Moreover, if \( F_1 \leq F_2 \) then \( \text{cov}_{F_2} \subset \text{cov}_{F_1} \). This implies that the equivalence relation \( \text{cov}_{F_2} \) is sharper than \( \text{cov}_{F_1} \). We have a c-order preserving (continuous) onto bonding map \( f_{F_1, F_2} : X_{F_2} \to X_{F_1} \) between finite c-ordered sets. Moreover, \( f_{F_1, F_2} \circ \pi_{F_2} = \pi_{F_1} \).
In this way we get an inverse system
\[
\{f_{F_1,F_2}: X_{F_2} \to X_{F_1}, \ F_1 \leq F_2, \ I\}
\]
where \((I, \leq) = Cylc(X)\) be the directed poset defined above. It is easy to see that 
\(f_{F,F} = id\) and \(f_{F_1,F_3} = f_{F_1,F_2} \circ f_{F_2,F_3}\) for every \(F_1 \leq F_2 \leq F_3\).

Denote by
\[
X_\infty := \lim_{\longleftarrow}\{X_F, I\} \subset \prod_{F \in I} X_F
\]
the corresponding inverse limit. Its typical element is \(\{x_F : F \in Cylc(X)\} \in X_\infty\),
where \(x_F \in X_F\). The set \(X_\infty\) carries a circular order \(R\) as in Lemma 3.3.

For every given \(g \in G\) (it is c-order preserving on \(X\)) we have the induced isomorphism \(X_F \to X_{gF}\) of c-ordered sets, where \(t_i \mapsto gt_i\) and \((t_i, t_{i+1})_o \mapsto (gt_i, gt_{i+1})_o\) for every \(t_i \in \text{cov}_F\). For every \(F_1 \leq F_2\) we have \(f_{F_1,F_2}(x_{F_2}) = x_{F_1}\).
This implies that \(f_{gF_1,gF_2}(x_{gF_2}) = x_{gF_1}\). So, \((gx_F) = (x_{gF}) \in X_\infty\).
Therefore \(g: X \to X\) can be extended canonically to a map 
\(g_\infty: X_\infty \to X_\infty, \ g_\infty(x_F) := (x_{gF})\).
This map is a c-order automorphism. Indeed, if \([x, y, z]\) in \(X_\infty\) then there exists \(F \in I\) such that \([x_F, y_F, z_F]\) in \(X_F\). Since \(g: X \to X\) is a c-order automorphism we obtain that \([gx_F, gy_F, gz_F]\) in \(X_{gF}\).

One may easily show that we have a continuous action \(G \times X_\infty \to X_\infty\), where \(X_\infty\) carries the compact topology of the inverse limit as a closed subset of the topological product \(\prod_{F \in I} X_F\) of finite discrete spaces \(X_F\).

**Lemma 4.5.** \(\beta(G/H) = \widehat{G/H} \simeq X_\infty\) as compact \(G\)-spaces. Furthermore, if \(X\) is countable then \(G/H\) is a metrizable compact.

**Proof.** Recall that the map \(i: G/H \to X, \ gH \mapsto gz\) identifies the discrete \(G\)-spaces \(G/H\) and \(X\). Moreover, as our discussion above shows, \(i\) is a uniform isomorphism of the precompact uniform spaces \((G/H, \mu_r)\) and \((X, \mu_X)\), where \(\mu_X\) can be treated as the weak uniformity with respect to the family of maps \(\{\pi_F: X \to X_F : F \in Cylc(X)\}\) (into the finite uniform spaces \(X_F\)).

Observe that \(f_{F_1,F_2} \circ \pi_{F_2} = \pi_{F_1}\) for every \(F_1 \leq F_2\). By the universal property of the inverse limit we have the canonical uniformly continuous map \(\pi_\infty: X \to X_\infty\). It is easy to see that it is an embedding of uniform spaces and that \(\pi_\infty(X)\) is dense in \(X_\infty\). Since \(X = G/H\) is a precompact uniform space we obtain that its uniform completion is a compact space and can be identified with \(X_\infty\). The uniform embedding \(G/H \to X_\infty\) is a G-map. It follows that the uniform isomorphism \(\widehat{G/H} \to X_\infty\) is also a G-map.

On the other hand this inverse limit \(X_\infty\) is c-ordered as it follows from Lemmas 3.6 and 3.7. Furthermore, as we have already mentioned the action of \(G\) on \(X_\infty\) is c-order preserving. Therefore \(X_\infty = \widehat{G/H}\) is a compact c-ordered \(G\)-system.

Since \(M(G)\) is its subsystem this completes the proof of the first part of Theorem 4.3. Note that \(G\) is always nonarchimedean, being a topological subgroup of the symmetric group \(S_X\). When \(X\) is countable \(S_X \cong S_N\) is a Polish group and the group \((G, \tau_p)\) is separable and metrizable. Moreover, analyzing the proof, we see that in that case \(\widehat{G/H}\) and \(M(G)\) are metrizable.
In fact, \( M(G) \) coincides with \( \beta_G(G/H) \), as it follows from Theorem 4.9 below.

Let \( T := \mathbb{R}/\mathbb{Z} \) be the circle and \( q : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \) is the canonical projection. Denote by \( \mathbb{Q}_o \) the set \( q(\mathbb{Q}) \) of all rationals into the circle \( \mathbb{T} \) (endowed with the natural c-order). Let \( D_o = D/\mathbb{Z} \) be its subset \( q(D) \), where \( D \) is a set of all dyadic rationals. Denote by \( H_+(\mathbb{T}) \) the group of all circular order preserving homeomorphisms of the circle \( \mathbb{T} \).

**Corollary 4.6.** The following topological groups are intrinsically circularly ordered. Hence, also (convexly) intrinsically tame.

1. Polish group \( G := \text{Aut}(\mathbb{Q}_o) \). Furthermore, \( M(G) \) is metrizable.
2. Thompson’s circular group \( T \) with the pointwise convergence topology (acting on \( D_o \));
3. Topological group (not Polish) \( H_+(\mathbb{T}_{\text{discr}}) \) in the pointwise topology \( \tau_p \) with respect to the action \( H_+(\mathbb{T}) \rhd \mathbb{T}, (\tau_{\text{discr}}) \), where \( \mathbb{T} \) carries the discrete topology;
4. Polish group \( H_+(\mathbb{T}) \) in the usual compact open topology.

**Proof.** (1) \( \mathbb{Q}_o \) is ultrahomogeneous as a c-ordered set.
(2) The action of the Thompson’s group \( T \) on \( D_o \) is ultrahomogeneous.
(3) The action \( H_+(\mathbb{T}) \rhd \mathbb{T} \) is ultrahomogeneous.
(4) \( (H_+(\mathbb{T}, \tau_{\text{discr}}), \tau_p) \) is intrinsically c-ordered by (3). Now observe that we have a well defined continuous onto injective homomorphism \( H_+(\mathbb{T}, \tau_{\text{discr}}) \to H_+(\mathbb{T}) \) and apply Lemma 3.3.2.

**Remark 4.7.** It is a standard fact that Thompson group \( F \) acts ultrahomogeneously on the linearly ordered set \( D \), [2 Lemma 4.2]. So, by Pestov’s theorem \( F \), as a topological group in the pointwise topology, is extremely amenable. Recall that it is an open problem whether the discrete group \( F \) is amenable.

It seems also to be well known (or, to follow easily from [2 Lemma 4.2], as was observed by G. Golan) that Thompson’s circular group \( T \) acts ultrahomogeneously on \( D/\mathbb{Z} \). See also [7 Remark 6].

We need to discuss more properties of the \( G \)-system \( X_\infty = \beta_G(G/H) \), the inverse limit system constructed in Theorem 4.3. By our construction \( X \) is a discrete space densely embedded into the compact space \( X_\infty \). First we observe that \( u \in X_\infty \) is an isolated point of \( X_\infty \) if and only if \( u \in X \). Indeed, it is well known that every dense locally compact subspace of a compact space is open. Therefore, \( X \) is an open subset of \( X_\infty \). It follows that every \( x \in X \) is an isolated point of \( X_\infty \). On the other hand, no other point \( y \) of \( X_\infty \) is not isolated. Indeed, if yes, then the open subset \( \{y\} \) intersects the dense subset \( X \).

Let \( Z = X_\infty \setminus X \). Then \( Z \) is a closed (hence, compact) \( G \)-invariant subset of \( X_\infty \). Recall that

\[
X_\infty := \lim_{\leftarrow} (X_F, I) \subset \prod_{F \in I} X_F
\]

is an inverse limit. Its elements \( u \in X_F \) are “threads” \( u = \{(u_F) : F \in \text{Cycl}(X)\} \), where \( u_F \in X_F, F := \{t_1, t_2, \ldots, t_m\} \subset (X, o) \) is a cycle, \( X_F \) is a finite disjoint (cyclically ordered) cover of \( X \)

\[
X_F := \{t_1, (t_1, t_2)_o, t_2, (t_2, t_3)_o, \ldots, t_m, (t_m, t_1)_o\}
\]
and \( q_F : X_\infty \rightarrow X_F \) is the natural c-order preserving projection.

A thread \( u \in X_\infty \) represents an element \( x \in X \) iff there exists \( F \in C_{\text{uccl}}(X) \) such that \( u_F = t_i = x \) for some \( t_i \in F \) (it is the only case when \( u \) is isolated).

A typical (basic) neighborhood of \( u \in X_\infty \) is
\[
O_F(u) := \{ v \in X_\infty : v_F = u_F \}.
\]

Here \( u_F \) is one of the members in \( X_F \). If \( u \in Z \) (i.e., \( u \notin X \)) then necessarily \( u_F = (t_i, t_{i+1})_0 \) for some \( i \) and then \( u \in (t_i, t_{i+1}) \) (that is, \( [t_i, u, t_{i+1}] \) in \( X_\infty \)). In fact, taking into account that \( q_F : X_\infty \rightarrow X_F \) is c-order preserving, we have
\[
(4.2) \quad O_F(u) = (t_i, t_{i+1})_\infty.
\]

For every \( x \in X \) define \( x^- \in X_\infty \) as a thread \( u = (u_F) \in X_\infty \) such that for every cycle \( F \) which contains \( t_i = x \), we have \( u_F = (t_{i-1}, t_i) \). This defines one and only one element from \( X_\infty \) having this property. The map \( X \rightarrow X^- \), \( x \mapsto x^- \) is a bijection. Dually (replacing \( t_i \) by \( t_{i-1} \)) can be defined \( x^+ \). Set \( X^- = \{ x^- : x \in X \} \), \( X^+ = \{ x^+ : x \in X \} \) and \( X^* = X^- \cup X \cup X^+ \). It is easy to see that all three subsets are pairwise disjoint.

**Definition 4.8.** \([S]\) A compact \( G \)-system \( Z \) is called:

1. **Strongly proximal** if for every \( \mu \in \mathcal{P}(Z) \), the compact space of probability measures on \( Z \) equipped with its weak-star topology, there are a net \( g_\alpha \) in \( G \) and a point \( z \in Z \) such that \( \lim g_\alpha \mu = \delta_z \), the point mass at \( z \).
2. **Extremely proximal** if \( Z \) is infinite and for every nonempty closed subset \( A \subseteq Z \) with \( A \neq Z \), there is a net \( g_\alpha \) in \( G \) and a point \( z \in Z \) such that, in the space \( 2^Z \) of closed subsets of \( Z \) equipped with the Vietoris topology, \( \lim g_\alpha A = \{ z \} \).

It is shown in \([S]\) Proposition VII, 3.5] that if a group \( G \) admits a nontrivial minimal extremely proximal action on a compact space \( Z \) then \( G \) contains a free subgroup on two generators.

**Theorem 4.9.** Let a group \( G \) act ultrahomogeneously on an infinite circularly ordered set \( X \). Then the \( G \)-system \( Z = X_\infty \setminus X \) is:

1. **Minimal**, hence \( M(G, \tau_p) = X_\infty \setminus X \).
2. **Extremely proximal**.
3. The universal strongly proximal minimal system \( M_{sp}(G) \) of \( G \).

Moreover,

4. \( G \) contains a free subgroup on two generators.
5. The universal irreducible affine \( G \)-system \( IA(G) \) is \( P(X_\infty \setminus X) \) (the affine \( G \)-system of all probability measures on \( X_\infty \setminus X \)).

**Proof.** (1) Let \( u, v \in Z \). Consider a basic neighborhood \( O_F(u) \) in \( X_\infty \). We will show that there exists \( g \in G \) such that \( gv \in O_F(u) \) (then necessarily \( gv \in O_F(u) \cap Z \)). By our construction, \( u \in (t_i, t_{i+1})_\infty \). Take any pair \( a, b \in X \) such that \( v \in (a, b)_\infty \). By the ultrahomogeneity there exists \( g \in G \) such that \( ga = t_i, gb = t_{i+1} \). Then (since the \( g \)-translation preserves the circular order) \([a, v, b] \) implies \([t_i, gv, t_{i+1}] \). This means that \( gv \in (t_i, t_{i+1})_\infty = O_F(u) \), as desired.

(2) Let \( A \) be a closed subset of \( Z \) such that \( A \neq Z \). Then there exists a pair \( a, b \in X \) such that \( A \subset [a, b]_\infty \). Indeed, choose \( u \in Z \setminus A \). Then \( Z \setminus A \) is an open neighborhood of \( u \) in \( Z \). There exists a basic neighbourhood \( O_F(u) \) in \( X_\infty \).
(where \( O_F(u) := \{ v \in X_\infty : v_F = u_F \} \)), such that \( O_F(u) \cap Z \subset Z \setminus A \). Since \( u \notin X \), we have \( u_F = (b, a)_\infty \) for some \( a, b \in X \). So, \( (b, a)_\infty \subset Z \setminus A \). Since \( (b, a)_\infty \cup [a, b]_\infty = X_\infty \), we obtain \( A \subset [a, b]_\infty \).

**Claim:** Let \( F \in Cycl(X, o) \) and \( O_F(u) = (x_1, x_2)_\infty \) be a basic neighborhood of \( u = x^-_2 \in X_\infty \) (where \( x_1, x_2 \in X \)). Then there exists \( a_F \in X \) such that
\[
a_F \in (x_1, x^-_2)_\infty, \quad x^-_2 \in (a_F, x_2)_\infty.
\]

**Proof.** By definitions of \( x^+_1, x^-_2 \) and \( [x_1, x^-_2, x_2] \), there exists a sufficiently large \( F = \{ t_1, t_2, \ldots, t_m \} \in Cycl(X) \) such that for some subcycle \( [i, k, j] \) we have
\[
t_i = x_1, \quad u_F = (t_k, t_j)_o, \quad t_j = x_2.
\]
Now, define \( a_F := t_k \). \( \square \)

Choose any \( x_0 \in X \). Then \( x^-_2 \in Z \). By Claim there exists a net
\[
\{ a_F : F \in Cycl(X) \} \quad \text{where} \quad a_F \in X
\]
such that \( \cap F(a_F, x_0)_\infty = \{ x^-_0 \} \) and \( \lim F a_F = x^-_0 \). By the ultrahomogeneity, there is a net \( \{ g_F \in G \} \) such that \( g_F(a) = a_F, g_F(b) = x_0 \). Then it follows that \( \lim g_F(A) = \{ x^-_0 \} \) in the Vietoris topology.

(3) Let \( \mu \) be a probability measure on \( X_\infty \setminus X \). Let \( \mu = \mu_a + \mu_c \) be the decomposition of \( \mu \) as a sum of an atomic and continuous measures. By proximality (which certainly follows from extreme proximality) there is a net in \( G \) which shrinks \( \mu_a \) to a single point mass (with the appropriate weight). So we now assume that \( \mu \) is continuous. By regularity, given an \( \varepsilon > 0 \) there is a compact set \( A \subset X_\infty \) with \( 1 - \varepsilon < \mu(A) < 1 \). Applying extreme proximality to \( A \) we can find a net \( g_0 \in G \) with \( \lim_A g_0 = \{ z \} \). It is not hard to see that this implies that \( \nu = \lim g_0 \mu \) (which by compactness we can assume exists) has the property that \( \nu(V) \geq 1 - \varepsilon \) for any open neighborhood \( V \) of \( z \). As \( \varepsilon \) is arbitrary this concludes the proof of strong proximality.

Since the universal system \( M(G) = X_\infty \setminus X \) is strongly proximal, then it is the universal strongly proximal minimal system of \( G \).

(4) By (2) we may apply [8 Prop. VII, 3.5].

(5) Apply results of [8] making use of (3). \( \square \)

**Corollary 4.10.** In the context of Theorem 4.9 suppose that \( X \) is countable. Then any nontrivial factor map \( \pi : M(G) \to Y \) is an almost one to one extension; i.e. for a dense \( G \) subset \( Z_0 \subset M(G) \) we have \( \pi^{-1}(\pi(z)) = \{ z \} \) for every \( z \in Z_0 \).

**Proof.** In that case \( M(G) \) is metric and if \( \pi : M(G) \to Y \) is a nontrivial factor map and \( R_\pi = \{ (z, z') : \pi(z) = \pi(z') \} \), then the set-valued map \( \pi^{-1} : Y \to 2^{M(G)} \), being upper-semi-continuous, has a dense \( G \) subset \( Y_0 \subset Y \) of continuity points. Let \( Z_0 = \pi^{-1}(Y_0) \), a dense \( G \) subset of \( M(G) \). Suppose \( z_0 \in Z_0 \) and let \( z_1 \) be an arbitrary point of \( M(G) \). Then \( R_\pi[z_1] \) is a proper closed subset of \( M(G) \) and, by minimality and extreme proximality of \( M(G) \), there is a sequence \( g_i \in G \) with \( \lim g_i z_1 = z_0 \) and \( \lim g_i R_\pi[z_1] = \{ z_0 \} \). However, as \( z_0 \) is a continuity point, we also have \( \lim g_i R_\pi[z_1] = R_\pi[z_0] \), whence \( R_\pi[z_0] = \{ z_0 \} \). \( \square \)
5. The Fraïssé class of finite circularly ordered systems and the KPT theory

In this section we present an alternative proof of Theorem 4.3 in the countable case. It employs the notion of Fraïssé classes and the Kechris-Pestov-Todorcevic theory and has the advantage that it automatically yields an explicit description of $M(G)$.

In the sequel we will freely use the notations and results from this seminal work. We begin by citing the following theorem proved in [19] Theorem 7.5.

**Theorem 5.1.** Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$, $\mathcal{K}$ a reasonable Fraïssé order class in $L$. Let $\mathcal{K}_0 = \mathcal{K}|L_0$, and $\mathbf{F} = \text{Flim}(\mathcal{K})$, $\mathbf{F}_0 = \text{Flim}(\mathcal{K}_0)$. Let $G = \text{Aut}(\mathbf{F})$, $G_0 = \text{Aut}(\mathbf{F}_0)$. Let $X_\mathcal{K}$ be the set of linear orderings $\prec$ on $F(=F_0)$ which are $\mathcal{K}$-admissible.

1. If $\mathcal{K}$ has the Ramsey property, the $G_0$-ambit $(X_\mathcal{K},\prec_0)$ is the universal $G_0$-ambit with the property that $G$ stabilizes the distinguished point, i.e. it can be mapped homomorphically to any $G_0$-ambit $(X,x_0)$ with $G \cdot x_0 = \{x_0\}$. Thus any minimal subflow of $X_\mathcal{K}$ is the universal minimal flow of $G_0$. In particular, the universal minimal flow of $G_0$ is metrizable.

2. If $\mathcal{K}$ has the Ramsey and ordering properties, $X_\mathcal{K}$ is the universal minimal flow of $G_0$.

If $(A,\lbrack\cdot,\cdot\rbrack)$ is a finite circularly ordered set we say that a linear order on $A$ is compatible when it is obtained from $(A,\lbrack\cdot,\cdot\rbrack)$ by a cut (see [12] Remark 2.4.). In the context of the following theorem this is the same as being admissible in the sense of [19] Definition 7.2. Thus $X_\mathcal{K}$ can be identified with the orbit closure of $\prec_0$, $G_0 \cdot \prec_0 \subset \text{LO}$, the compact set of linear orders on $F$, the universe of $\mathbf{F}$.

For the definition and properties of $\text{Split}(X;\mathbb{Q}_0)$ see [12] Lemma 2.11.

**Theorem 5.2.** Let $L = \{\lbrack\cdot,\cdot\rbrack,\prec\}$ be a signature and $L_0 = L \setminus \{\prec\}$. Let $\mathcal{K}$ be the Fraïssé order class in $L$ consisting of all the finite circularly ordered sets with compatible linear order. Let $\mathbf{F} = \text{Flim}(\mathcal{K})$, $\mathbf{F}_0 = \text{Flim}(\mathcal{K}_0)$ and set $G = \text{Aut}(\mathbf{F})$, $G_0 = \text{Aut}(\mathbf{F}_0)$. Then

1. $\mathcal{K}$ is a reasonable order class and it satisfies the Ramsey and ordering properties.

2. Let $\mathbb{Q}_0$ denote the set $\mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} = \mathbb{T}$ of roots of unity, equipped with the cyclic order relation $\lbrack\cdot,\cdot\rbrack$ it inherits from $\mathbb{R}/\mathbb{Z}$. Then $\mathbf{F} = \text{Flim}(\mathcal{K}) = \langle \mathbb{Q}_0,\lbrack\cdot,\cdot\rbrack,\prec_0 \rangle$, where $\prec_0$ is the dense linear order on $\mathbb{Q}_0$ induced by the linear order obtained from the $\alpha$-cut $\mathbb{Q}_0 \cap (\alpha,\alpha+1) \pmod{1}$ for some (any) $\alpha \in \mathbb{R} \setminus \mathbb{Q}_0$.

3. $M(G_0) = X_\mathcal{K} \cong \text{Split}(\mathbb{T};\mathbb{Q}_0)$ is the universal minimal $G_0$-flow.

**Proof.** (1) It is easy to see that $\mathcal{K}$ is a reasonable order class. It is also easy to check that it has the ordering property. The fact that $\mathcal{K}$ has the Ramsey property follows by the classical Ramsey theorem.

(2) Clearly $(\mathbb{Q}_0,\lbrack\cdot,\cdot\rbrack,\prec_0)$ is ultrahomogeneous and $\mathcal{K} = \text{Age}(\mathbf{F})$. Now apply Fraïssé’s theorem [19] Theorem 2.2 to conclude that $\mathbf{F} = (\mathbb{Q}_0,\lbrack\cdot,\cdot\rbrack,\prec_0)$.

(3) By Theorem 5.1 we have

$$M(G_0) = X_\mathcal{K} = \mathcal{O}_0 \cdot \prec_0 \subset \text{LO}(\mathbb{Q}_0).$$

Define a map $\Phi : \text{Split}(X;\mathbb{Q}_0) \to X_\mathcal{K}$ as follows. For $\xi \in \mathbb{T} \setminus \mathbb{Q}_0$ let $\Phi(\xi) = \prec_\xi$ be the linear order defined by the $\xi$-cut $\mathbb{Q}_0 \cap (\xi,\xi+1) \pmod{1}$. For $\eta \in \mathbb{Q}_0$ let
\( \Phi(\eta^-) = \prec_\eta^- \) be the linear order defined by \( \mathbb{Q}_0 \cap [\eta, \eta + 1) \mod 1 \), i.e. the dense order with \( \eta \) as a first element; and let \( \Phi(\eta^+) = \prec_\eta^+ \) be the linear order defined by \( \mathbb{Q}_0 \cap (\eta, \eta + 1] \mod 1 \), i.e. the dense order with \( \eta \) as the last element. It is easy to check that \( \Phi \) defines a \( G_0 \)-flow isomorphism. □

**Remark 5.3.** Here \( G_0 \) can be identified with \( \text{Aut}(\mathbb{Q}_0) \). With this explicit presentation of \( M(G_0) \) it is easy to see that the canonical map \( \text{Split}(T; \mathbb{Q}_0) \to T \) is the only nontrivial factor of the \( G_0 \)-flow \( M(G_0) \). It follows that every continuous action of the Polish group \( (\text{Aut}(\mathbb{Q}_0), \tau_p) \) on a compact space \( X \) has a closed \( G \)-subspace \( Y \) which is c-ordered.

**Remark 5.4.** It is not hard to check that, in the case that \( X \) is countable, the collection \( \text{Cycl}(X) \) of finite injective cycles on \( X \) forms a countable projective Fraïssé family of finite topological \( L \)-structures in the sense of Irvin and Solecki [16] (here \( L \) is the relational language \( L = \{R\} \)). Of course then \( X_\infty = \hat{G}/H \) is the corresponding projective Fraïssé limit of \( \text{Cycl}(X) \).

### 6. Automatic continuity and Roelcke precompactness

#### 6.1. Automatic continuity and metr-int-c-ord.

**Lemma 6.1.** The Polish group \( \text{Aut}(\mathbb{Q}_0) \) has the automatic continuity property. That is, every group homomorphism \( h: \text{Aut}(\mathbb{Q}_0) \to G \) to a separable topological group \( G \) is continuous.

**Proof.** It is a well known theorem by Rosendal and Solecki [33] that the Polish group \( \text{Aut}(\mathbb{Q}_<) \) has the automatic continuity property. The same is true for the larger group \( \text{Aut}(\mathbb{Q}_0) \) because the Polish group \( \text{Aut}(\mathbb{Q}_<) \) is an open subgroup of \( \text{Aut}(\mathbb{Q}_0) \). □

**Corollary 6.2.** For every action of \( G := \text{Aut}(\mathbb{Q}_0) \) by homeomorphisms on a metric compact space there exists a closed c-ordered \( G \)-subsystem. That is, the discrete group \( \text{Aut}(\mathbb{Q}_0) \) is metrically intrinsically c-ordered.

**Proof.** This follows from Theorem 4.3 and Lemma 6.1 taking into account Remark 5.3. □

#### 6.2. Roelcke precompactness and Kazhdan’s property (T).

**Proposition 6.3.** The Polish group \( G := \text{Aut}(\mathbb{Q}_0) \) is Roelcke precompact.

**Proof.** It is well known that the Polish subgroup \( H := \text{Aut}(\mathbb{Q}_<) \) is Roelcke precompact. By our construction in Theorem 4.3 the coset uniform space \( (G/H, \mu_r) \) is precompact. Now apply Proposition 6.4. See also Proposition 6.6 for another proof in a more general case. □

The following result generalizes [30] Prop. 9.17.

**Proposition 6.4.** Let \( H \) be a subgroup of a topological group \( G \) such that \( H \) is Roelcke precompact as a topological group and the coset space \( G/H \) is precompact in the standard right uniformity. Then \( G \) is Roelcke precompact.
Proof. We have to show that for every neighborhood $U$ of $e$ in $G$ there exists a finite subset $F \subset G$ such that $G = UF U$. In fact, it is enough to show that $G = U^2 FU$. Since $G/H$ is precompact with respect to the standard right uniformity there exist finitely many points $a_1 H, a_2 H, \ldots, a_n H$ in the coset space $G/H = \{[g] = gH \}_{g \in G}$ such that $E := \{a_1, a_2, \ldots, a_n\}$ is a subset of $G$ and

$$G/H = \cup \{U[a_i] : a_i \in E\}.$$

This implies that $G = U EH$.

Now, for $G = U^2 FU$ it suffices to show that $EH \subset UFU$ for some finite $F \subset G$.

By our assumption $H$ is Roelcke precompact as a topological group. That is, $(H, R_H)$ is precompact. It follows that $H$ is a precompact subset of $G$ with respect to the Roelcke uniformity $R_G$ of $G$. Then the same is true for each subset $a_i H \subset G$. Hence the finite union $EH$ is also Roelcke precompact subset of $G$. Therefore, for given $U$ there exists a finite subset $F$ in $G$ such that $EH \subset UFU$,

as required.

\[ \Box \]

Corollary 6.5. $\text{Aut}(\mathbb{Q}_o)$ satisfies the Kazhdan's property (T).

Proof. By a result of Evans and Tsankov [5] every nonarchimedean Roelcke precompact Polish group has the Kazhdan’s property (T).

\[ \Box \]

Proposition 6.6. Let $G \rtimes X$ be an ultrahomogeneous action on a circularly (linearly) ordered set $X$. Then the topological group $(G, \tau_p)$ is Roelcke precompact.

Proof. First consider the case of linearly ordered $X$. Then Roelcke precompactness of $G$ can be proved as in the proof of Rosendal [31, Theorem 5.2] (for countable $X = \mathbb{Q}$). See also Tsankov [35]. One may show that for every open subgroup $U$ of $G$ there are only finitely many different double cosets $UxU$ in $G$.

In the case of circularly ordered $X$ we combine Lemma 6.4. and Theorem 4.3 which asserts that the coset space $G/H$ is right precompact, where $H$ is a stabilizer subgroup $St(z)$ which is Roelcke precompact by the linear order case.

\[ \Box \]

Remark 6.7.

1) Propositions 6.3 and 6.6 seemingly can be derived also by results of Tsankov [35].

2) The Polish group $G := \text{Aut}(\mathbb{Q}_o)$ has strong uncountable cofinality as was proved by Rosendal [31, Theorem 7.1]. Hence (again by the results of Rosendal [32]) it has properties (OB) and (ACR). The latter means that any affine continuous action of $G$ on a separable reflexive Banach space has a fixed point.

7. Some perspectives and questions

7.1. Conv-int-tame nonamenable discrete groups. For nondiscrete topological groups we have several examples of conv-int-tame nonamenable groups. Among others, the Lie groups $SL_n(\mathbb{R})$ and the Polish group $H_+(\mathbb{T})$ (see Example 2.3). However, the discrete case is open.
Question 7.1. Is there a nonamenable convexly intrinsically tame (countable) discrete group?

Note that a (topological) group $G$ is conv-int-tame iff its universal minimal strongly proximal $G$-system $M_{sp}(G)$ is tame. Thinking about possible candidates we observe that the free group $F_2$ is not conv-int-tame, because $M_{sp}(F_2)$ is not tame. One way to see this is as follows. Let $X$ be the Cantor set, $(X, T)$ be the Morse minimal system and $S: X \to X$ a self homeomorphism such that the cascade $(X, S)$ is strongly proximal. Let $F_2$, the free group on generators $a, b$, act on $X$ via the map $F_2 \to \text{Homeo}(X)$, defined by $a \mapsto T, b \mapsto S$. Clearly then the $F_2$-system $X$ is minimal, strongly proximal but not tame, since already the subaction $(X, T)$ is not tame.

7.2. Topological subgroups of $S_\infty$.

Problem 7.2. Kechris-Pestov-Todorcevic result [19, Theorem 4.8] characterizes extremely amenable subgroups of $S_\infty$ in terms of Fraïssé limits. Namely, $\text{Aut}(A)$ is extremely amenable (where $A$ is the Fraïssé limit of a class $K$) if and only if the Fraïssé order class $K$ has the Ramsey property.

It would be interesting to study (or, even characterize) some other classes of Polish groups (or of closed subgroups of $S_\infty$) $G$, for which $M(G)$ is:

1. int-c-ordered.
2. int-tame.
3. conv-int-c-ordered.
4. conv-int-tame.
5. metric versions of the previous concepts (as in Definition 2.2).

The class (1) contains the Polish group $\text{Aut}(Q_\circ)$ (which is not extremely amenable). The class (4) contains all countable discrete amenable groups.

7.3. Other structures. Pestov’s theorem can be reformulated by saying that if $G$ acts ultrahomogeneously on a linearly ordered set then for the topological group $(G, \tau_p)$ in its pointwise convergence topology the minimal universal $G$-system $M(G)$ is also linearly ordered. Indeed, note that every compact minimal linearly ordered $G$-space is trivial.

As we have seen Pestov’s theorem can be naturally generalized to ultrahomogeneous circularly ordered sets. Namely, $M(G)$ is again circularly ordered. It is natural to wonder if there are some analogs for other structures.

Question 7.3. Let $G$ act on a set $X$ with some structure $R$. Under which reasonable conditions the universal minimal $G$-system $M(G, \tau_p)$ also admits a structure of the same type?

8. Appendix: Large ultrahomogeneous circularly ordered sets

It is well known that for every infinite cardinal $\tau$ there exists a ultrahomogeneous linearly ordered set $X$ of cardinality $\tau$ (see for example [27]). As expected, the same is true for circularly ordered sets. For the countable case we have the unique (up to isomorphism) model $Q_\circ$. For the cardinality $2^{\aleph_0}$ we have, at least, the circle $\mathbb{T}$. As to the general case, very recently, responding to our question, J.K. Truss [36] and V.G. Pestov [29] informed us that, according to their (unpublished) notes, the following result holds (their approaches are essentially different).
Proposition 8.1. (Pestov [29], Truss [36]) For any prescribed infinite cardinal \( \tau \) there exists an ultrahomogeneous circularly ordered set of cardinality \( \tau \).

With his permission we reproduce Pestov’s proof of Proposition 8.1.

Proof. Step 1:
For every infinite cardinal \( \tau \), there exists a linearly ordered field having \( \tau \) as its cardinality, this is well known and was rediscovered a number of times [18, 24].

The simplest construction is as follows (borrowed from [21]). Given a field \( k \), denote \( k(\alpha) \) a simple transcendental extension of \( k \) with variable \( \alpha \). In other words, \( k(\alpha) \) consists of all rational functions \( p(\alpha)/q(\alpha) \) where \( p, q \) are polynomials in \( \alpha \) with coefficients in \( k \). If now \( k \) is an ordered field, then \( k(\alpha) \) becomes an ordered field with \( \alpha \) as a positive infinitesimal, that is, \( 0 < \alpha < x \) for all \( x \in k \), \( x > 0 \). The sign of a polynomial \( p(\alpha) = a_0 + a_1\alpha + \ldots + a_n\alpha^n \) is the sign of the non-zero coefficient of the lowest degree. This extends to the rational functions in an obvious way. Clearly, \( k \) is an ordered subfield of \( k(\alpha) \).

If \( \tau \) is an infinite cardinal, we construct recursively in \( \beta \leq \tau \) an increasing transfinite sequence \( k_\beta \) of ordered fields, where \( k_0 = \mathbb{Q} \) (or any other fixed ordered field), \( k_{\beta+1} = k_\beta(\alpha_\beta) \) with \( \alpha_\beta \) being a positive infinitesimal over \( k_\beta \), and for limit cardinals \( \beta \) we set \( k_\beta = \bigcup_{\gamma < \beta} k_\gamma \). It is easy to see that the ordered field \( k_\tau \) has the required cardinality \( \tau \).

Step 2:
Every ordered field \( k \) has characteristic zero. The absolute value in \( k \) is defined as \( |x| = \max\{x, -x\} \). Given an ordered field \( k \), an element \( x \in k \) is finite if for some natural number \( n \) one has \( |x| \leq n \). The subset \( \text{fin}(k) \) of all finite elements of an ordered field forms a convex subring, in which \( \mathbb{Z} \) is a cofinal and coinitial ordered subring.

On the additive factor-group \( \text{fin}(k)/\mathbb{Z} \) one may define now a circular order as a ternary relation \( R \). The argument is similar to the case of the circle \( \mathbb{R}/\mathbb{Z} \). A triple \((a, b, c)\) is in \( R \) if and only if one can write \( a = a' + \mathbb{Z}, b = b' + \mathbb{Z}, c = c' + \mathbb{Z} \) with \( a' \leq b' \leq c' \).

Clearly, the cardinality of \( \text{fin}(k)/\mathbb{Z} \) equals that of \( k \).

Step 3:
To verify ultrahomogeneity of \( \text{fin}(k)/\mathbb{Z} \), let \( a_1, \ldots, a_n \), and \( b_1, \ldots, b_n \), be two \( n \)-tuples of elements of the group which are positively cyclically ordered, that is, whenever \([i, j, k]\) within the finite group \( \mathbb{Z}_k \), one has \([a_i, a_j, a_k]\) and \([b_i, b_j, b_k]\). Rotating both sets (and thus preserving the cyclic order), one can assume without loss in generality that \( a_1 = b_1 = 0 \). Identifying the group \( \text{fin}(k)/\mathbb{Z} \) with the interval \([0, 1] \) in \( k \), one gets two sets of representatives of the \( n \)-tuples within \( k \), \( 0 = a'_1 < a'_2 < \ldots < a'_n < 1 \) and \( 0 = b'_1 < b'_2 < \ldots < b'_n < 1 \). Now, like in the proof of Assertion 5.1 in [27], apply a piecewise linear, order preserving bijective transformation of \([0, 1] \) onto itself sending \( a'_i \mapsto b'_i, i = 1, 2, \ldots, n \):

\[
f(x) = \begin{cases} 
\frac{b'_1}{a'_1}x, & \text{if } 0 \leq x \leq a'_1, \\
\frac{b'_{i+1} - b'_i}{a'_{i+1} - a'_i}(x - a'_i), & \text{if } a'_i \leq x \leq a'_{i+1}, i = 1, 2, \ldots, n - 1, \\
\frac{1}{1 - a'_n}(x - a'_n), & \text{if } a'_n \leq x < 1.
\end{cases}
\]
The resulting map lifts to a cyclical-order preserving self-bijection of the group, taking the first \( n \)-tuple to the second (strictly speaking, we have to compose this map with the two rotations).

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