Entropy-Based Approximation of the Secretary Problem

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Abstract

In this work we consider the well-known Secretary Problem – a number \( n \) of elements, each having an adversarial value, are arriving one-by-one according to some random order, and the goal is to choose the highest value element. The decisions are made online and are irrevocable – if the algorithm decides to choose or not to choose the currently seen element, based on the previously observed values, it cannot change its decision later regarding this element. The measure of success is the probability of selecting the highest value element, minimized over all adversarial assignments of values. Despite of a large volume of study, the following question is still far from being answered: how much randomness in selection of the arrival order is needed for the algorithm to achieve good approximation? We show existential and constructive upper bounds on approximation of the success probability in this problem, depending on the entropy of the randomly chosen arrival order, including the lowest possible entropy \( O(\log \log(n)) \) for which the probability of success could be constant. We show that below entropy level \( H < 0.5 \log \log n \), all algorithms succeed with probability 0 if random order is selected uniformly at random from some subset of permutations, while we are able to construct in polynomial time a non-uniform distribution with entropy \( H \) resulting in success probability of at least \( \Omega \left( \frac{1}{\log \log n + 3 \log \log \log n - H} \right) \), for any constant \( \epsilon > 0 \). We also prove that no algorithm using entropy \( H = O((\log \log n)^a) \) can improve our result by more than polynomially, for any constant \( 0 < a < 1 \). For entropy \( \log \log(n) \) and larger, our analysis precisely quantifies both multiplicative and additive approximation of the success probability. In particular, we improve more than doubly exponentially on the best previously known additive approximation guarantee for the secretary problem.

Keywords: Secretary problem, adversarial values, random order, online algorithms, approximation algorithms, entropy.
1 Introduction

Optimal stopping problems are classical and widely studied problems in statistics and probability, c.f., the reviews by Freeman [8] and Ferguson [7]. Among them is the secretary problem (also known as the marriage or the dowry problem), introduced by several renown statisticians in 1960’s: Lindley [19], Dynkin [5], Chow et al. [4] and Gilbert and Mosteller [9]. It has since attracted a lot of attention not only in the statistics and probability theory community, but also for the last decade or so, in the algorithms and algorithmic game theory community [11]. This problem finds applications in convex hulls and linear programming [24] as well as in online algorithms, algorithmic mechanism design, and specifically, in auction design [1, 12, 17].

The basic secretary problem is, given a positive integer \( n \), we have a sequence of \( n \) distinct adversarial numbers (also called values) arriving one by one in random order. The goal is to design an algorithm, also called a stopping rule, that with substantial success probability would find the largest among those numbers. The rule of this online game is such that when \( i \)-th number arrives, the algorithm, based on its knowledge about the previous \( i-1 \) numbers, has to decide whether to accept this \( i \)-th number. If the algorithm decides to accept then the game ends, and otherwise it continues to the next number. Note that if the algorithm rejects the first \( n-1 \) numbers, the last, \( n \)-th number, has to be accepted. This decision is irrevocable and has to be made without knowledge of the future numbers.

An execution of a secretarial algorithm is its run against fixed (but a priori unknown to the algorithm) adversarial order of values, which are shuffled by a random order. We say that the algorithm is successful in an execution if it accepts the largest of the adversarial values. This event is probabilistic, as it depends on the random order of adversarial numbers, and thus for a given algorithm and adversarial numbers, we could compute the probability of success of the algorithm for that adversarial assignment of values. The minimum of these probabilities, over all possible adversarial assignments (there are \( n! \) of them), defines the probability of success of the algorithm.

While a vast majority of previous work concentrated on analysis of the success probability in idealistic case of perfect randomness, i.e., when random orders are drawn from the set of all \( n \)-element permutations uniformly at random (c.f., [7, 11]), recently Kesselheim, Kleinberg and Niazadeh [14] initiated a study on the impact of amount of randomness (mainly, the bounded entropy) on the probability of success and on the design of efficient algorithms. Pursuing this direction, we study the following two questions:

1. How the probability of success depends on the entropy of the random order distribution; and
2. How to design efficient algorithms and corresponding random order distributions in polynomial time.

1.1 Previous and related work

The secretary problem was introduced and first analyzed in [19, 5, 4, 9]. In particular, asymptotically optimal algorithm with success probability \( \frac{1}{e} \) was proposed, when perfect randomness is available (i.e., random orders are chosen uniformly at random from the set of all \( n! \) permutations). Gilbert and Mosteller [9] showed that with perfect randomness in selection of random arrival order, no algorithm could achieve better probability of success than some simple wait-and-pick algorithm with specific threshold \( m \in [n-1] \) (which can be proved to be in \( \lfloor n/e \rfloor, \lceil n/e \rceil \)). Wait-and-pick algorithms are such that they keep observing values until some pre-defined threshold step \( m \in [n-1] \), and after that they accept the first value that is larger than all the previously observed ones (or the last occurring value otherwise).

Recently, Kesselheim, Kleinberg and Niazadeh [14] initiated a study on the impact of non-uniform random orders on the probability of success. They proved that if its distribution has entropy \( o(\log \log n) \) then no algorithm achieves a constant probability of success. They also presented a polynomial-time construction of a set of polylog \( n \) permutations such that wait-and-pick algorithm choosing a random order uniformly
from this set (i.e., with entropy $O(\log \log n)$) achieves probability of success $\frac{1}{2} - \omega\left(\frac{1}{\log \log \log (n)}\right)$, for any positive constant $c < 1$. Their construction includes several reduction steps, uses composition of three Reed-Solomon codes and auxiliary composition functions – see their full ArXiv version in [15] and a slightly simplified digest created by us in Section 10 for a better understanding of their approach compared to ours.

**Applications and related problems.** One of the first applications of the secretary approach in online algorithms is by Karp, Vazirani and Vazirani [13], where they used a random order approach to give an optimal randomized online algorithm for the maximum size bipartite matching problem. A randomized $(1/2 + \Omega(1))$-competitive algorithm that uses optimal entropy of $(1+o(1)) \log \log n$ for (the vertex-weighted version of) this problem has been recently designed by Buchbinder, Naor and Wajc [2] by using lossless online rounding of fractional matching algorithms. Online algorithms have been studied for weighted bipartite matching and combinatorial auctions, e.g., [18, 16].

Secretary problems find many applications in algorithmic mechanism design, in particular, in the design and analysis of truthful online auctions for various settings, see, e.g., survey paper of Babaioff et al. [1] and references therein. Truthful online mechanisms with bidders arriving in a random order have been designed for various matching markets and combinatorial auction markets, e.g., [6, 22]. For an extensive overview of the secretary models, their applications and extensions to related models (for instance matroid secretary, or prophet inequalities), see the recent survey by Gupta and Singla [11].

**Related work on useful techniques.** Oblivious rounding technique based on linear programming and Chernoff arguments introduced by Young [28] is not derandomization per se, but inspired our derandomization approach. One can also apply derandomized version of the Lovász Local Lemma [20, 3] to obtain our derandomization, however our approach and the proof are simpler.

### 1.2 Our results and contributions

**Motivation.** The secretary problem is among the most fundamental optimal stopping problems that has been extensively researched by diverse scientific communities, e.g., [1, 4, 5, 7, 8, 9, 11, 14, 17, 19, 23, 25]. This problem also finds numerous applications in: online algorithms, approximation algorithms, algorithmic mechanism design, and optimization. In the existing papers the best possible success probability of secretarial algorithms has been usually studied by specifying the optimal success probability under uniform full support distribution which is asymptotically close to $1/e$, e.g., at least $1/e - 1/n$ [19, 5, 9]. We study this problem under various degrees of available amount of randomness, measured by entropy, with a focus on low entropy. Under various degrees of entropy we show in a precise sense how the success probability changes and, in particular, how close it is to $1/e$, both above and below this optimal threshold. We demonstrate the whole spectrum of behavior, ranging from precise values above $1/e$ for the maximum entropy $\Theta(n \log(n))$, and entropy $\Theta(\log(n))$ which still suffices for success probability above $1/e$, down to $\Theta(\log \log(n))$ entropy that is necessary for a constant success probability, see [14], which we show is in fact slightly below $1/e$, finally down to entropy strictly less than $0.5 \log \log(n)$ where we prove that success probability rapidly declines to 0 for uniform distributions with a small support. We also study which of these distributions can be constructed in polynomial time to guarantee such close success probabilities.

We study in particular uniform distributions with not necessarily full support, the class extensively studied before, e.g., in [14]. Our algorithmic constructions, similarly to [14], output such distributions. But, interestingly, we prove that any secretarial algorithm that uses a uniform distribution with support strictly smaller than $n$, e.g., with entropy less than $\log(n)$, has success probability strictly smaller than $1/e$. Motivated by these results we then study precisely how close the success probability can be to $1/e$ for such distributions. Our study introduces and is driven by a host of new mathematical and algorithmic techniques.
that we believe are of independent interest. These techniques include new probabilistic analysis of the secretary problem, dimension reductions, and derandomization of Chernoff arguments via pessimistic estimators.

**Narrative flow of our paper.** We structure our results and the paper following the linear order of how the success probability of secretary algorithms decreases as entropy decreases. At the moderate entropy values of $\Theta(\log \log n)$ we have our existential results by the probabilistic method, which are the transition point to very small entropy below $\Theta(\log \log n)$. Finally, we present our main algorithmic results – derandomization of Chernoff bounds for moderate entropy, i.e., for which a constant success probability is feasible.

**Success probability and existence under different entropy (Section 2).** We study the optimal success probability of secretary algorithms under various degrees of available amount of randomness, measured by the Shannon entropy. Starting from the maximum entropy, $\Theta(n \log(n))$, of the uniform distribution with support $n!$, we find the precise formula for the optimal success probability of any secretary algorithm, see $OPT_n$ in Part 1 of Theorem [1]. We prove that any secretary algorithm that uses any, not necessarily uniform distribution, has success probability at most $OPT_n$ (Part 2, Theorem [1]). This improves on the result of Samuels [23], who proved that under the uniform distribution no secretary algorithm can achieve success probability of $1/e + \varepsilon$, for any constant $\varepsilon > 0$. We then prove that even entropy $\Theta(\log(n))$ suffices for a probability success above $1/e$ (see Corollary [1]). But, interestingly, no uniform probability distribution with small support and entropy strictly less than $\log(n)$ can have success probability above $1/e$ (Part 3, Theorem [1]). Turning to very small entropy of $\Theta(\log(\log(n)))$ we precisely characterize the success probability of uniform distributions that is below, and close to, $1/e$, and show how to construct such distributions in polynomial-time (Theorems [2] [7] and [8]). We finally prove a precise threshold of entropy, $0.5 \log \log(n)$, below which the success probability of any secretary algorithm declines to 0 for uniform distributions with a small support (Proposition [1]). Interestingly, however, we present polynomial time constructions of non-uniform probability distributions with entropy $H$ smaller than $0.5 \log \log(n)$ for which there exist secretary algorithms with small but positive success probability $\Omega \left( \frac{1}{(\log \log n + 3 \log \log \log n - H)^{\frac{1}{\varepsilon a + 1}}} \right)$, for any constant $\varepsilon > 0$, c.f., Theorem [4]. In this case we also show in Proposition [2] that no algorithm with entropy $H = o(\log \log n)$ can have success probability larger than $\Theta \left( \frac{H}{\log \log n} \right)$; in particular, for entropy $H = O((\log \log n)^{a})$, where $0 < a < 1$ is a constant, our constructive solution could be improved at most polynomially.

**Existential result (Section 2.2).** In Theorem [1] Part 1 we found a precise formula $\frac{1}{e} + \left( 1 - \frac{1}{e} \right) \frac{1}{2n} + \Theta \left( \frac{1}{n^{2/3}} \right)$ for the maximal success probability $OPT_n$ of secretarial algorithms. In the following result we present a new probabilistic analysis of the secretary problem. Towards this aim we identify a useful parameterization of this problem, denoted by $k \in \{2, 3, \ldots, n\}$, which is interpreted as corresponding to $k$ largest adversarial values. This parameter is crucial in our analysis – we entangle it carefully with the aimed entropy ($k$ is a function of $2^{\text{entropy}}$) and estimate tightly how this set of $k$ largest adversarial values is located with respect to the threshold $m_0$ of a wait-and-pick algorithm, and consequently, we obtain:

**Theorem 2** (Restated) There exists a small entropy permutation distribution, i.e., a small set of $O \left( \frac{k \log \log n}{(e)^{a/3 \varepsilon}} \right)$ permutations from which random arrival order is chosen u.a.r., such that wait-and-pick algorithm with threshold $m_0 = \left\lceil n/e \right\rceil$ achieves additive approximation to $OPT_n$ roughly $\varepsilon' + \frac{2}{k} \left( n - \frac{1}{e} \right)^k$, for any $3 \leq k \leq n - m_0$ and $\varepsilon' \in (0, 1)$.

Setting $k = O(\log \log n)$ and $\varepsilon' = \frac{1}{\log e \cdot n}$, for any fixed constant $C > 1$, the number of permutations is $O(C \cdot \log^{3C}(n) \cdot \log \log(n))$ (so the entropy is $\Theta(\log \log n)$ – the minimal needed for constant success probability) with additive approximation $O \left( \frac{1}{\log e \cdot n} \right)$. This improves on the constructive additive approximation
of $\omega(\frac{1}{\log \log log(n)})$ by Kesselheim et al. [14, 15], which holds for any positive constant $c < 1$. Our proof is based on the probabilistic method via Chernoff bound, but it differs from that of Kesselheim et al. [14, 15] in that our parameter $k$ allows us to specify more precisely how close the success probability is to $OPT_n$ by directly relating to $OPT_n$. Some technical proofs of Section 2 are deferred to Section 6.

**Derandomization of existential result (Section 3).** We then design a derandomization method to derandomize our existential proof. We derive from our existential proof a pessimistic estimator for the failure probability, c.f., [21]. This estimator is inspired by the oblivious rounding approach of Young [28]. To make it work we reduce the domain of a single step of derandomization to polynomial size, by generating each permutation position-by-position, using independent but sequentially related “index” random variables. This process is guided by a special algorithm to compute conditional failure probabilities. The resulting derandomization unfortunately leads to $O(n^k)$ running time, which is not polynomial for non-constant $k$. This leads to the following result, which fully preserves the guarantees of our existential proof:

**Theorem 3** (Restated) There exists a deterministic algorithm with running time $O(k \cdot \ell \cdot n^{k+2} \cdot \text{poly log}(n))$ that finds a multi-set $\mathcal{L} = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ of permutations $\pi_j \in \Pi_n$, from which random arrival order is chosen u.a.r., such that the wait-and-pick algorithm with threshold $m_0 = \lceil n/e \rceil$ achieves additive approximation to $OPT_n$ roughly $\epsilon' + \frac{k}{\ell} \left(1 - \frac{1}{\ell}\right)^k$, where $\ell = \Theta\left(\frac{k \log n}{(\epsilon')^2}\right)$, for $n > k \geq 3$, and $\epsilon' > 0$.

This result compares to that of Kesselheim et al. [14, 15] in the same way as our Theorem 2 above, but the running time of the algorithm is not polynomial. Some proofs of Section 3 are deferred to Section 7.

**Improved and simplified dimension reductions (Section 4).** To make our algorithm from Theorem 4 polynomial, we propose two dimension reduction methods based on a refined use of Reed-Solomon codes. On a high level, we construct a family of functions that have bounded number of collisions and their pre-images are of almost same sizes. Our first construction of such function family is based on a generalized Reed-Solomon code (see Lemma 4), where we managed to enforce same sizes of preimages (up to additive 1) by using algebraic properties of polynomials. Our second construction of such function family is based on a direct product of two Reed-Solomon codes (see Section 4.2). Both these constructions are inspired by Kesselheim at al. [14, 15], but they significantly improve their result by adding the constraint on sizes of preimages. This constraint, precisely tailored for the secretary problem, allows us to apply more direct techniques of finding permutations distributions over a set with reduced dimension. Our constructions are also much simpler, and use at most two codes, instead of three, and do not need auxiliary composition functions. Both constructions are computable in polynomial time and we believe that they are of independent interest. We augment these dimension reduction techniques with a detailed analysis of how they relate to the success probability of the secretarial algorithms, see Section 4 for more details, and deferred proofs in Section 8.

**Efficient algorithms for optimal entropy (Section 5).** To put together our derandomization, the dimension reduction methods and success probability analyses we need to deal with an additional technicality to avoid indices of adversarial $k$-tuples falling close to the threshold $m_0 = \lfloor n/e \rfloor$ in the random permutation. We show in Theorem 5 that this only slightly reduces success probability.

The first technique with composition of two Reed-Solomon codes lets us reduce the problem dimension from $n$ to $\frac{\log \log(n)}{\log \log \log(n)}$, and the second technique with one code and our derandomization from Theorem 4 reduces the dimension from $n$ to $\log^C(n)$ for any constant $C > 0$, and they lead to the following:

**Theorem 7 and 8** (Restated) A permutation set $\mathcal{L}$ of size $|\mathcal{L}| = O(\log(n))$ ($|\mathcal{L}| = O(\log^{4C}(n))$, resp.) can be computed in deterministic time polynomial in $n$, from which random arrival order is chosen u.a.r., such
that the wait-and-pick algorithm with threshold \( m_0 = \lfloor n/e \rfloor \) achieves additive approximation to \( \text{OPT}_n \) roughly \( \Theta \left( \frac{(\log \log n)^{3/2}}{\sqrt{\log \log n}} \right) \) \( \Theta \left( \frac{(\log \log n)^2}{\log^{3/2} \frac{n}{\log n}} \right) \), resp., for any fixed constant \( C > 0 \).

Some deferred proofs of Section 5 can be found in Section 9. The original analysis in [19, 5] shows that this algorithm’s success probability with full \( \Theta(n \log n) \) entropy is at least \( 1/e - 1/n \), implying an additive approximation \( \Theta(1/n) \). The second result in Theorem 7 has better approximation, but is more complex and builds on the first result. The second algorithm guarantees almost the same success probability as that in our existential proof, and the entropy of these distributions is optimal \( O(\log \log(n)) \). It also improves, over doubly-exponentially, on the additive approximation of \( \omega \left( \frac{1}{(\log \log \log(n))^c} \right) \) of Kesselheim et al. [14, 15], which holds for any positive constant \( c < 1 \).

Consequences and additional observations. See the text in Section 11.

1.3 Preliminaries

Denote a set \( \{1, 2, \ldots, i\} \) by \([i]\). Let \( n \) be the number of arriving elements/items. Each of them has a unique index \( i \in [n] \), and corresponding unique value \( v(i) \) assigned to it by an adversary. The adversary knows the algorithm and the distribution of random arrival orders. Given \( k \in \{2, 3, \ldots, n-1\} \), let \( K \) be the set of all ordered \( k \)-element subsets of \([n]\), called \( k \)-tuples; we have \(|K| = \binom{n}{k} \cdot k! < n^k\).

Let \( \Pi_n \) denote the set of all \( n! \) permutations of the sequence \((1, 2, \ldots, n)\). A probability distribution \( p \) over \( \Pi_n \) is a function \( p : \Pi_n \rightarrow [0, 1] \) such that \( \sum_{\pi \in \Pi_n} p(\pi) = 1 \). Shannon entropy, or simply, entropy, of the probability distribution \( p \) is defined as \( H(p) = -\sum_{\pi \in \Pi_n} p(\pi) \cdot \log(p(\pi)) \), where \( \log \) has base 2, and if \( p(\pi) = 0 \) for some \( \pi \in \Pi_n \), then we assume that \( 0 \cdot \log(0) = 0 \). A special case of a distribution, convenient to design efficiently, is when we are given a set \( \mathcal{L} \subseteq \Pi_n \) of permutations, and random order is selected uniformly at random (u.a.r. for short) from this set. In this case, it is easy to prove that the entropy of such distribution is \( \log |\mathcal{L}| \). We call an algorithm with such an associated probabilistic distribution \emph{uniform}, and otherwise \emph{non-uniform}.

For any positive integers \( k < n \), let \( [n]_k \) denote the set of all \( k \)-element subsets of set \([n]\). Given a sequence of (not necessarily sorted) values \( v(1), v(2), \ldots, v(n) \in \mathbb{R} \), we also denote by \( \text{ind}(k') \) the index of the element with the \( k' \)th largest value, that is, the \( k' \)th largest value is \( v(\text{ind}(k')) \).

Wait-and-pick algorithms, also called threshold or classic secretarial algorithms, are parametrized by threshold \( m \in [n-1] \), usually denoted by \( m_0 \). They work as follows: they apply random order to the adversarial assignment of values, keep observing the first \( m \) values in the order and then select the first \( i \)-th arriving element, for \( i > m \), whose value is larger than all the previously observed values. If such \( i \) does not exist, then the last, \( n \)-th, element is selected. It has been shown that some wait-and-pick algorithms are optimal in case of perfect randomness in selection of random arrival order, c.f., [9].

2 Success probability and existence under different entropy

We will show an existence of \( O(k \log n) \) permutations, such that if we choose one of them u.a.r. from this set, then the optimal secretarial algorithm with threshold \( \left\lfloor \frac{n}{2} \right\rfloor \) achieves nearly-optimal success probability.

2.1 Understanding optimal success probability

In classic approximation algorithms, with multiplicative or additive approximation guarantees, one studies polynomial time approximation algorithms for NP-hard optimization problems. There, the valuable re-
source is an algorithm’s running time, which we restrict to be polynomial in the problem size. Because obtaining even the value of the optimal solution is NP-hard for such problems, the design and analysis of such algorithms usually first identifies a polynomial-time computable bound on the optimum, see, e.g., [27].

In this paper we study polynomial time algorithms which construct secretarial algorithms with the probability of success that is optimal with an additive or multiplicative approximation guarantees. Additive approximation guarantees are achieved for entropy around $\Theta(\log \log n)$, and multiplicative approximation guarantees are achieved for entropy below $\Theta(\log \log n)$. In this study the valuable resource for such approximation algorithms is the amount of randomness, measured by entropy, of the probability distribution on random orders (permutations) used by the secretarial algorithm. In analogy to the classic approximation algorithms, we will first show how to relate to the optimum, i.e., the optimal success probability $OPT_n$ of any secretarial algorithm, in particular, when ideal randomness with maximum entropy is available (i.e., random orders are permutations in $\Pi_n$ drawn uniformly at random).

Let $f(k, m) = \frac{m}{e} (H_{k-1} - H_{m-1})$, where $H_k$ is the $k$-th harmonic number, $H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$. It is easy to prove that $f(n, m_0)$ is the exact success probability of the secretarial algorithm with threshold $m_0$ when random order is given by choosing u.a.r. a permutation $\pi \in \Pi_n$, see [11].

**Lemma 1** The following asymptotic behavior holds, if $k \to \infty$ and $j \leq \sqrt{k}$ is such that $m = k/e + j$ is an integer in $[k]$:

$$f\left(k, \frac{k}{e} + j\right) = \frac{1}{e} - \left(\frac{1}{2e} - \frac{1}{2} + \frac{e j^2}{2k}\right) \frac{1}{k} + \Theta\left(\left(\frac{1}{k}\right)^{3/2}\right).$$

The proof of Lemma 1 is in Section 6.1. We will now precisely characterize the maximum of function $f$. Recall that, $f(k, m) = \frac{m}{e} (H_{k-1} - H_{m-1})$, and note that $1 \leq m \leq k$. We have the discrete derivative of $f$: $h(m) = f(k, m + 1) - f(k, m) = \frac{1}{k} (H_{k-1} - H_m - 1)$, which is positive for $m \leq m_0$ and negative otherwise, for some $m_0 = \max\{m > 0 : H_{k-1} - H_m - 1 > 0\}$.

**Lemma 2** There exists an absolute constant $c > 1$ such that for any integer $k \geq c$, we have that $h\left(\left\lfloor\frac{k}{e}\right\rfloor - 1\right) > 0$ and $h\left(\left\lfloor\frac{k}{e}\right\rfloor + 1\right) < 0$. Moreover, function $f(k, \cdot)$ achieves its maximum for $m \in \{\lfloor\frac{k}{e}\rfloor, \lfloor\frac{k}{e}\rfloor + 1\}$, and is monotonically increasing for smaller values of $m$ and monotonically decreasing for larger values of $m$.

Lemma 2 is proved in Section 6.2. Theorem 1 below shows a characterization of the optimal success probability $OPT_n$ of secretarial algorithms, which will be complemented by existential result in Theorem 2.

**Theorem 1**

1. The optimal success probability of any secretarial algorithm for the problem with $n$ items which uses a uniform random order from $\Pi_n$ is $OPT_n = 1/e + c_0/n + \Theta((1/n)^{3/2})$, where $c_0 = 1/2 - 1/(2e)$.

2. The success probability of any secretarial algorithm for the problem with $n$ items which uses any probabilistic distribution on $\Pi_n$ is at most $OPT_n = 1/e + c_0/n + \Theta((1/n)^{3/2})$.

3. There exists an infinite sequence of integers $n_1 < n_2 < n_3 < \ldots$, such that the success probability of any secretarial algorithm for the problem with $n \in \{n_1, n_2, n_3, \ldots\}$ items which uses any uniform probabilistic distribution on $\Pi_n$ with support $\ell < n$ is strictly smaller than $1/e$.

**Proof.** Part 1 Gilbert and Mosteller [9] proved that under maximum entropy, the probability of success is maximized by wait-and-pick algorithm with some threshold. Another important property, used in many
papers (c.f., Gupta and Singla [1]), is that function \( f(n, m) \) describes the probability of success of the wait-and-pick algorithm with threshold \( m \).

Consider wait-and-pick algorithms with threshold \( m \in [n-1] \). By Lemma 2 function \( f(n, \cdot) \) achieves is maximum for threshold \( m \in \{\lfloor \frac{n}{e} \rfloor, \lfloor \frac{n}{e} \rfloor + 1 \} \), and by Lemma 1 taken for \( k = n \), it could be seen that for any admissible value of \( j \) (i.e., such that \( n/e + j \) is an integer and \( |j| \leq 1 \), thus also for \( j \in \{\lfloor \frac{n}{e} \rfloor - n/e, \lfloor \frac{n}{e} \rfloor + 1 - n/e \} \) for which \( f(n, m) \) achieves its maximum), and for \( c_0 = 1/2 - 1/(2e) \):

\[
f(n, \frac{n}{e} + j) = \frac{1}{e} + \frac{c_0}{n} + \Theta \left( \left( \frac{1}{n} \right)^{3/2} \right).
\]

Part 2 Consider a probabilistic distribution on set \( \Pi_n \), which for every permutation \( \pi \in \Pi_n \) assigns probability \( p_{\pi} \) of being selected. Suppose that the permutation selected by the adversary is \( \sigma \in \Pi_n \). Given a permutation \( \pi \in \Pi_n \) selected by the algorithm, let \( \chi(\pi, \sigma) = 1 \) if the algorithm is successful on the adversarial permutation \( \sigma \) and its selected permutation \( \pi \), and \( \chi(\pi, \sigma) = 0 \) otherwise.

Given a specific adversarial choice \( \sigma \in \Pi_n \), the total weight of permutations resulting in success of the secretarial algorithm is \( \sum_{\pi \in \Pi_n} p_{\pi} \cdot \chi(\sigma, \pi) \).

Suppose now that the adversary selects its permutation \( \sigma \) uniformly at random from \( \Pi_n \). The expected total weight of permutations resulting in success of the secretarial algorithm is \( \sum_{\sigma \in \Pi_n} q_{\sigma} \cdot \left( \sum_{\pi \in \Pi_n} p_{\pi} \cdot \chi(\sigma, \pi) \right) \), where \( q_{\sigma} = 1/n! \) for each \( \sigma \in \Pi_n \). The above sum can be rewritten as follows

\[
\sum_{\sigma \in \Pi_n} q_{\sigma} \cdot \left( \sum_{\pi \in \Pi_n} p_{\pi} \cdot \chi(\sigma, \pi) \right) = \sum_{\pi \in \Pi_n} p_{\pi} \cdot \left( \sum_{\sigma \in \Pi_n} q_{\sigma} \cdot \chi(\sigma, \pi) \right),
\]

and now we can treat permutation \( \pi \) as fixed and adversarial, and permutation \( \sigma \) as chosen by the algorithm uniformly at random from \( \Pi_n \), we have by Part 1 that \( \sum_{\sigma \in \Pi_n} q_{\sigma} \cdot \chi(\sigma, \pi) = OPT_n \). This implies that the expected total weight of permutations resulting in success of the secretarial algorithm is at most

\[
\sum_{\pi \in \Pi_n} p_{\pi} \cdot \chi(\sigma, \pi) \leq \sum_{\pi \in \Pi_n} p_{\pi} \cdot OPT_n = OPT_n.
\]

Therefore, there exists a permutation \( \sigma \in \Pi_n \) realizing this adversarial goal. Thus it is impossible that there is a secretarial algorithm that for any adversarial permutation \( \sigma \in \Pi_n \) has success probability \( > OPT_n \).

Part 3 Let \( \ell_i = 10^i \) and \( n_i = 10^i \ell_i \) for \( i \in \mathbb{N}_{\geq 1} \). Let us take the infinite decimal expansion of \( 1/e = 0.367879441171442... \) and define \( d_i > 1 \) the integer that is build from the first \( i \) digits in this decimal expansion after the decimal point, that is, \( d_1 = 3, d_2 = 36, d_3 = 367 \), and so on. The sequence \( d_i/\ell_i \) has the following properties: \( \lim_{i \to +\infty} d_i/\ell_i = 1/e \), for each \( i = 1, 2, ... \) we have that \( d_i/\ell_i < 1/e < (d_i + 1)/\ell_i \) and, moreover, \( j/\ell_i \notin [1/e, 1/e + 1/n_i] \) for all \( j \in \{0, 1, 2, \ldots, \ell_i\} \).

Let us now take any \( n = n_i \) for some (large enough) \( i \in \mathbb{N}_{\geq 1} \) and consider the secretary problem with \( n = n_i \) items. Consider also any secretarial algorithm for this problem that uses any uniform probability distribution on the set \( \Pi_{ni} \) with support \( \ell_i \). By Part 2 the success probability of this algorithm using this probability distribution is at most \( OPT_{ni} = 1/e + c_0/n_i + \Theta((1/n_i)^{3/2}) \). All possible probabilities in this probability distribution belong to the set \( \{ j/\ell_i : j \in \{0, 1, 2, \ldots, \ell_i\} \} \). We observe now that \( j/\ell_i \notin [1/e, 1/e + c_0/n_i + \Theta((1/n_i)^{3/2})] \) for \( j \in \{0, 1, 2, \ldots, \ell_i\} \). This fact holds by the construction and by the fact that constant \( c_0 \in (0, 1) \), and we may also need to assume that \( i \in \mathbb{N}_{\geq 1} \) is taken to be large enough to deal with the term \( \Theta((1/n_i)^{3/2}) \). Thus the success probability of this algorithm is strictly below \( 1/e \). \( \square \)

2.2 Existence via probabilistic method

Theorem 2 Given any integer parameter \( 3 \leq k \leq n - m_0 \), there exists a multi-set \( \mathcal{L} \subseteq \Pi_n \) (that is, it may contain multiple copies of some permutations) of the set of all permutations of size \( |\mathcal{L}| \leq O \left( \frac{k \log n}{(e^3)^2} \right) \)
such that if we choose one of these permutations u.a.r. from \( \mathcal{L} \), then the optimal secretarial algorithm with threshold \( m_0 \) achieves a success probability of at least

\[
(1 - \varepsilon')\rho_k, \text{ where } \rho_k = OPT_n - \frac{2}{k} \left( \frac{n - m_0}{n - 1} \right)^k,
\]

and any parameter \( 0 < \varepsilon' < 1 \). The value \( OPT_n \) denotes the probability of success of the secretarial algorithm with threshold \( m_0 \) when a permutation is chosen u.a.r. from set \( \Pi_n \). We assume here that \( m_0 = \alpha n \) for a constant \( \alpha \in (0, 1) \) such that \( \rho_3 = \Theta(1) \) (this holds, e.g., when \( \alpha = 1/e \) and \( n \) are large enough).

**Proof sketch.** The complete proof is deferred to Section 6.3. We will use the probabilistic method to show existence of the set \( \mathcal{L} \subseteq \Pi_n \). First, consider a random experiment that is choosing u.a.r. a single permutation \( \pi \) from \( \Pi_n \). We estimate the probability of success of the secretarial algorithm with threshold \( m_0 \). (Below, \( ind(i) \) refers to the position of the \( i \)-th largest adversarial value from \( v(1), \ldots, v(n) \) in permutation \( \pi \), i.e., \( \pi(ind(i)) \).) This probability is lower bounded by the probability of the following events \( E_i, i = 2, 3, \ldots, k \), where \( E_i = A_i \cap B_i \cap C_i \), \( A_i = \{ \pi(ind(i)) \in \{1, 2, \ldots, m_0\} \} \), \( B_i = \bigcap_{j=1}^{i-1} \{ind(j) \in \{m_0 + 1, \ldots, n\}\} \), and \( C_i = \{ \forall j = 2, 3, \ldots, i - 1: \text{ind}(1) < \text{ind}(j) \} \). We say that \( \pi \) covers the ordered \( i \)-tuple \( S = \{\pi(ind(i)), \pi(ind(1)), \pi(ind(2)), \ldots, \pi(ind(i - 1))\} \) if event \( E_i \) holds. We end up having:

\[
\Pr[E_i] = \Pr \left[ A_i \cap \left( \bigcap_{j=1}^{i-1} \{\text{ind}(j) \in \{m_0 + 1, \ldots, n\}\} \right) \cap C_i \right] = \frac{m_0}{n} \cdot \left( \prod_{j=1}^{i-1} \left( \frac{n - m_0 - (j - 1)}{(n - 1) - (j - 1)} \right) \frac{(i - 2)!}{(i - 1)!} \cdot \frac{n - 1}{m_0 - 1} \right).
\]

It follows that

\[
OPT_n = \sum_{2 \leq i \leq n - m_0 + 1} \Pr[E_i] \leq \sum_{2 \leq i \leq k} \Pr[E_i] + \frac{m_0}{n} \cdot \frac{m_0}{n - 1} \cdot \frac{n - 1}{m_0 - 1} \cdot \frac{n - m_0}{m_0 - 1} - \frac{2}{k} \left( \frac{n - m_0}{n - 1} \right)^k.
\]

Let us fix a particular ordered \( k \)-tuple \( \hat{S} = \{j_1, j_2, \ldots, j_k\} \subseteq [n] \). We say that an independently and u.a.r. chosen permutation \( \pi \in \Pi_n \) is successful for \( \hat{S} \) if event \( \bigcup_{i=2}^{k} E_i \) holds, where \( E_i = \{ \pi \text{ covers } i \text{-tuple } \{j_1, j_2, j_3, \ldots, j_{i-1}\} \} \), for \( i \in \{2, 3, \ldots, k\} \). By the argument above \( \pi \) is successful with probability \( \rho_k \).

Next, we choose independently \( \ell = c \log(n^k) \) permutations \( \pi_1, \pi_2, \ldots, \pi_{\ell} \) from \( \Pi_n \) u.a.r., for a fixed constant \( c \geq 1 \). These permutations will comprise the multi-set \( \mathcal{L} \). Let \( X_1^S, \ldots, X_{\ell}^S \) be random variables such that \( X_i^S = 1 \) if the corresponding random permutation \( \pi_i \) is successful for \( k \)-tuple \( S \), and \( X_i^S = 0 \) otherwise, for \( t = 1, 2, \ldots, \ell \). Then for \( X^S = X_1^S + \cdots + X_{\ell}^S \) we have that \( \operatorname{E}[X^S] = \rho_k \ell = c \rho_k \log(n^k) \) and by the Chernoff bound \( \Pr[X^S < (1 - \varepsilon') \cdot \rho_k \ell] \leq 1/n(\varepsilon')^2c\rho_kk/2 \), for any constant \( 0 < \varepsilon' < 1 \). Now, the probability that there exists a \( k \)-tuple \( \hat{S} = \{j_1, j_2, \ldots, j_k\} \) for which there does not exists a \( (1 - \varepsilon')\rho_k \) fraction of successful permutations among these \( \ell = c \log(n^k) \) random permutations is

\[
\Pr[\exists \text{ successful } \hat{S} : X^S < (1 - \varepsilon')c \rho_k \log(n^k)] \leq \frac{n}{k} \cdot k! / n(\varepsilon')^2c\rho_kk/2 \leq \frac{1}{n(\varepsilon')^2c\rho_kk/2},
\]

by the union bound. Thus all \( \binom{n}{k} \) \( k \)-tuples \( \hat{S} \) are covered with probability at least \( 1 - \frac{1}{n(\varepsilon')^2c\rho_kk/2} > 0 \) for \( c > 4/(\rho_3(\varepsilon')^2) = \Theta(1/\varepsilon'^2) \). This means that there exist \( \Theta(\log(n^k)/(\varepsilon')^2) \) permutations such that if we choose one among these permutations u.a.r., then for any \( k \)-tuples \( \hat{S} \), this permutation will be successful with probability \( (1 - \varepsilon')\rho_k \), which is success probability of the algorithm with threshold \( m_0 \). □
In this proof we have a multi-set \( L \) of \( \Theta(\log(n^k)/(\varepsilon')^2) \) permutations, and taking \( k = \Theta(\log(n)) \) and \( \varepsilon' = \Theta(1/n) \), we see by Theorem 2 and Part 1 in Theorem 1 that with entropy \( O(\log(n)) \) we can achieve the success probability above \( 1/e \):

**Corollary 1** There exists a multi-set \( L \subseteq \Pi_n \) of size \( |L| \leq O\left(\log_2(n) \cdot n^2\right) \) such that if we choose one of these permutations u.a.r. from \( L \), then the optimal secretarial algorithm with threshold \( m_0 = \lfloor n/e \rfloor \) achieves a success probability of at least \( 1/e + \Theta\left(\frac{1}{n}\right) \).

### 2.3 Very low entropy – advantage of non-uniform algorithms

#### 2.3.1 Upper bounds on success probability – limitations of uniform algorithms

In this section we prove that secretarial algorithms with uniform distributions are inherently limited – their probability of success drops to zero once the entropy gets smaller than \( 0.5 \log \log n - 1 \). Recall that for a slightly higher entropy, we can construct in polynomial time a uniform distribution of such entropy achieving almost optimal (with respect to negligible additive component) success probability, c.f., Section 5.

Let \( H \) denote an entropy of a given probabilistic distribution of permutations in \( \Pi_n \).

**Proposition 1** If the distribution is uniform on a set \( L \) of \( \ell \) permutations in \( \Pi_n \), then in order to have non-zero probability of success, \( \ell \) must be bigger than \( \sqrt{\log n} - 1 \).

**Proof.** By Lemma 6 in [15], \( \ell + 1 \log n \) is an upper bound on the success probability of any algorithm on the elements in set \( L \) distributed uniformly at random. Combining it with the fact that in case of such probabilistic space every event holds with probability either 0 or at least \( 1/\ell \), we obtain:

\[
\frac{\ell + 1}{\log n} > \frac{1}{\ell},
\]

which yields \( \ell > \sqrt{\log n} - 1 \). It implies that there is no algorithm with uniform distribution having entropy smaller than \( 0.5 \log \log n \) and non-zero probability of success. \( \square \)

In case of non-uniform distributions, we could prove only the following upper bound on the probability of success. In the next section we will try to match it by a secretarial algorithm with some non-uniform distribution, c.f., Theorem 3.

**Proposition 2** The probability of success of any secretarial algorithm run on set \( \Pi_n \) with any probability distribution of entropy \( H < \frac{1}{7} \log \log n - 1 \) is, for any integer \( \ell \), \( 4 < \ell \leq n \), at most:

\[
\frac{8H}{\log(\ell - 3)} + \left(1 - \frac{8H}{\log(\ell - 3)}\right) \frac{\ell + 1}{\log n},
\]

and thus is at most \( O\left(\frac{H \log n}{\log \log n}\right) \) when applying the above formula for \( \ell = \Theta\left(\frac{H \log n}{\log \log n}\right) \).

**Proof.** Consider the formula for some \( 4 < \ell \leq n \). Let \( L \) be the set of \( \ell \) elements of highest probability in the given distribution of entropy \( H \). By Lemma 7 in [15], the first summand in the formula is an upper bound on the probability of not selecting an element in \( L \) (and taking into account the fact that on such elements the probability of success of a given secretarial algorithm is at most 1). The other summand is an upper bound on the probability of success of the algorithm on the elements in set \( L \), by Lemma 6 in [15].
Observe that the formula is minimized when both summands are (asymptotically) equal, modulo rounding (as we assume that $\ell$ is an integer). Thus,

\[
\frac{8H}{\log(\ell - 3)} = \left(1 - \frac{8H}{\log(\ell - 3)}\right) \frac{\ell + 1}{\log n}
\]

for some $\ell = \Theta(\frac{H \log n}{\log \log n})$. Indeed, we may assume that $\frac{8H}{\log(\ell - 3)}$ is smaller than some arbitrary small constant, and thus it is enough to compare it directly with $\frac{\ell + 1}{\log n}$, which results in $\ell = O(\log n \log \log n)$, yielding $\log \ell = O(\log \log n)$, and thus we could $\ell = \Theta(\frac{H \log n}{\log \log n})$ from $(\ell + 1) \log(\ell - 3) = 8H \log n$. After plugging this $\ell$ into the formula, we get the probability of success upper bounded by $O(\frac{H}{\log \log n})$.

\[\square\]

### 2.3.2 Efficient non-uniform algorithm with very low entropy

In view of Proposition 1, we ask: is there an algorithm with a non-uniform distribution on $\Pi_n$ with any entropy $H < 0.5 \log \log n$ and a positive, preferably not too small, probability of success? We answer this question in affirmative, showing a substantial performance gap for secretarial algorithms run on uniform and non-uniform distributions.

**Theorem 3** For any entropy $H = O(\log \log n)$ and any $\epsilon > 0$, there is a polynomially-constructed secretarial algorithm associated with a non-uniform probabilistic distribution over $\Pi_n$ of entropy $H$ achieving probability of success $\Omega\left(\frac{1}{(\log \log n + 3 \log \log \log n - H)^{2+\epsilon}}\right)$.

**Proof.** Full proof of this theorem is in Section 6.4. We use the wait-and-pick algorithm with threshold $\lfloor n/e \rfloor$. In the remainder we construct and analyze a suitable non-uniform probabilistic distribution of a given entropy $H$ on set $\Pi_n$, on which the algorithm achieves probability of success $\Omega\left(\frac{1}{(\log \log n + 3 \log \log \log n - H)^{2+\epsilon}}\right)$.

Let $\zeta(\cdot)$ be the Riemann zeta function and let $\zeta^*(x, y) = \sum_{\ell=1}^{y} \frac{1}{\ell^2} \in (1, \zeta(x))$. Consider an integer parameter $\ell$ – it will denote the number of elements with highest probability in the constructed distribution, and, as we show later, it will also correspond to $2^H$, where $H$ will be the entropy of that distribution. We choose sets $X_i$ of size $|X_i| = x_i = \left\lfloor \frac{\ell^2}{y^{2+\epsilon} \zeta^*(2+\epsilon, y)} \right\rfloor$, for $1 \leq i \leq y$ and some arbitrary constant $0 < \epsilon < 1$, where $y > 0$ is an integer to be chosen later, and allocate probability $p_i = \frac{1}{2\ell}$ to each element in set $X_i$. It can be shown that this is indeed a probabilistic distribution with entropy $\ell + \Theta(1)$ and support of size $\ell^* = \Theta\left(\frac{2^H}{y^{2+\epsilon}}\right)$.

Taking $y = \log \frac{\log n}{\ell} + 3 \log \log \log n$ we get support of size at least logarithmic. Thus we could construct the support in polynomial time, using the support of the distribution constructed in Section 5 in Theorem 7 and we assign its elements (permutations) to sets $X_i$ arbitrarily. It follows that, in the worst case, for some adversarial order all these “good” permutations (i.e., guaranteeing selecting largest element from the adversarial order) may have smallest possible probability $p_y = \frac{1}{2\ell^*}$. Hence, the probability of success is at least

\[
\ell^* \cdot \left(\frac{1}{e} - \Theta\left(\frac{(\log \log n)^{5/2}}{\sqrt{\log \log n}}\right)\right) \cdot p_y \geq \Omega\left(\frac{2^y \cdot \ell}{y^{2+\epsilon} \cdot \frac{1}{2\ell}}\right) = \Omega\left(\frac{1}{y^{2+\epsilon}}\right)
\]

which is $\Omega\left(\frac{1}{(\log \log n + 3 \log \log \log n - H)^{2+\epsilon}}\right)$ as the entropy $H = \log \ell + \Theta(1) < c \log \log n$ for some constant $0 < c < 1$. \[\square\]
3 Derandomization via Chernoff Bound

**Theorem 4** Suppose that we are given integers $n$ and $k$, such that $n \geq 1$ and $n > k \geq 3$, and an error parameter $\varepsilon' > 0$. Define $\rho_k = OPT_n - \frac{2}{k} (1 - \frac{1}{e})^k$. Then for $\ell = \Theta\left(\frac{2k \log n}{\rho_k (\varepsilon')^2}\right)$ there exists a deterministic algorithm (Algorithm 1) that finds a multi-set $\mathcal{L} = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ of $n$-element permutations $\pi_j \in \Pi_n$, for $j \in [\ell]$, such that for every $k$-tuple there are at least $(1 - \varepsilon') \cdot \rho_k \ell$ successful permutations from $\mathcal{L}$. The running time of this algorithm is $O(k \cdot \ell \cdot n^{k+2} \cdot \text{poly} \log(n))$.

Missing details in this section and the full proof of Theorem 3 can be found in Section 7.

**Proof preparation.** To derandomize Chernoff argument of Theorem 2 we will use a special conditional expectations method with a pessimistic estimator. We will model an experiment to choose u.a.r. a permutation $\pi_j \in \Pi_n$ by independent “index” r.v.’s $X^j_1; \Pr[X^j_i = \{1, 2, \ldots, n - i + 1\}] = 1/(n - i + 1)$, for $i \in [n]$, to define $\pi = \pi_j \in \Pi_n$ “sequentially”: $\pi(1) = X^j_1, \pi(2) = \pi(2)$ is the $X^j_2$-th element in $I_1 = \{1, 2, \ldots, n\} \setminus \{\pi(1)\}$, $\pi(3)$ is the $X^j_3$-th element in $I_2 = \{1, 2, \ldots, n\} \setminus \{\pi(1), \pi(2)\}$, etc, where elements are increasingly ordered. Suppose random permutations $\mathcal{L} = \{\pi_1, \ldots, \pi_\ell\}$ are generated using $X^j_1, X^j_2, \ldots, X^j_n$ for $j \in [\ell]$. Given a $k$-tuple $\hat{S} \in \mathcal{K}$, recall definition of r.v. $X^\hat{S}_j$ for $j \in [\ell]$ from proof of Theorem 2. For $X^\hat{S} = X^S_1 + \ldots + X^S_\ell$ and $\varepsilon' \in (0, 1)$, we have $\mathbb{E}[X^\hat{S}] = \rho_k \ell$, and $\Pr[X^\hat{S} < (1 - \varepsilon') \cdot \rho_k \ell] \leq 1/\exp((\varepsilon')^2 \rho_k \ell/2) = (1/n^{\varepsilon'})^2 c \rho_k k/2$, where $\ell = c \log(n^k)$. We call the $k$-tuple $\hat{S} \in \mathcal{K}$ not well-covered if $X^\hat{S} < (1 - \varepsilon') \cdot \rho_k \ell$ (then a new r.v. $Y^\hat{S} = 1$), and well-covered otherwise (then $Y^\hat{S} = 0$). Let $Y = \sum_{\hat{S} \in \mathcal{K}} Y^\hat{S}$. By the above argument $\mathbb{E}[Y] = \sum_{\hat{S} \in \mathcal{K}} \mathbb{E}[Y^\hat{S}] < 1$ if $c \geq 1/(\varepsilon')^2$. We will maintain the expectation $\mathbb{E}[Y]$ below 1 in each step of our derandomization, where these steps will sequentially define these permutations for set $\mathcal{L}$.

**Outline of derandomization.** Let $\pi_1$ be identity permutation. For some $s \in [\ell - 1]$ let permutations $\pi_1, \ldots, \pi_s$ have already been chosen (“fixed”). We will chose a “semi-random” permutation $\pi_{s+1}$ position by position using $X^j_{s+1}$. Suppose that $\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)$ are already chosen for some $r \in [n - 1]$, where all $\pi_{s+1}(i)$ ($i \in [r - 1]$) are fixed and final, except $\pi_{s+1}(r)$ which is fixed but not final yet. We will vary $\pi_{s+1}(r) \in [n] \setminus \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1)\}$ to choose the best value for $\pi_{s+1}(r)$, assuming that $\pi_{s+1}(r + 1), \pi_{s+1}(r + 2), \ldots, \pi_{s+1}(n)$ are random. Permutations $\pi_{s+2}, \ldots, \pi_n$ are “fully-random”.

**Deriving a pessimistic estimator.** Given $\hat{S} \in \mathcal{K}$, observe that $X^\hat{S}_{s+1}$ depends only on $\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)$. We will show how to compute the conditional probabilities (see Algorithm 2 in Section 7.3.1) $\Pr[X^\hat{S}_{s+1} = 1 | \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)] (= \Pr[X^\hat{S}_{s+1} = 1] if r = 0)$, where randomness is over random positions $\pi_{s+1}(r + 1), \pi_{s+1}(r + 2), \ldots, \pi_{s+1}(n)$. Theorem 5 is proved in Section 7.3.1.

**Theorem 5** Suppose that values $\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)$ have already been fixed for some $r \in \{0\} \cup [n]$. There exist a deterministic algorithm (Algorithm 2 in Section 7.3.1) to compute $\Pr[X^\hat{S}_{s+1} = 1 | \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)]$, where the random event is the random choice of the semi-random permutation $\pi_{s+1}$ conditioned on its first $r$ elements already being fixed. The running time of this algorithm is $O(k \cdot n \cdot \text{poly} \log(n))$. Here $m_0 \in \{2, 3, \ldots, n - 1\}$ is the threshold of the secretarial algorithm.

**Pessimistic estimator.** Let $\hat{S} \in \mathcal{K}$. Denote $\mathbb{E}[X^\hat{S}_j] = \Pr[X^\hat{S}_j = 1] = \mu_j$ for each $j \in [\ell]$, and $\mathbb{E}[X^\hat{S}] = \sum_{j=1}^{\ell} \mu_j = \mu$. By Theorem 4 $f(n, \frac{n}{\varepsilon'}) = OPT_n = \frac{1}{e} + \frac{m_0}{n} + \Theta\left(\left(\frac{1}{n}\right)^{3/2}\right)$, where $c_0 = 1/2 - 1/(2e)$. By (1) in the proof of Theorem 2 and by Lemma 1 we obtain that $\mu_j \geq \rho_k \geq \frac{1}{e} - \Theta(1/k)$, for each $j \in [\ell]$. We will now use Raghavan’s proof of the Chernoff bound, see [28], for any $\varepsilon' > 0$, using that $\mu_j \geq \rho_k$ (see
more details in Section 7.3.2:

\[
\Pr \left[ X^\hat{S} < (1 - \varepsilon') \cdot \ell \cdot \rho_k \right] \leq \prod_{j=1}^{\ell} \frac{1 - \varepsilon' \cdot \mathbb{E}[X^\hat{S}]}{(1 - \varepsilon')(1 - \varepsilon')\rho_k} < \prod_{j=1}^{\ell} \frac{\exp(-\varepsilon'\mu_j)}{(1 - \varepsilon')(1 - \varepsilon')\rho_k} \leq \prod_{j=1}^{\ell} \frac{\exp(-\varepsilon'\rho)}{(1 - \varepsilon')(1 - \varepsilon')\rho_k}
\]

where last inequality follows by \( b(-x) > x^2/2 \), see, e.g., [28]. Thus, the union bound implies:

\[
\Pr \left[ \exists \hat{S} \in \mathcal{K} : X^\hat{S} < (1 - \varepsilon') \cdot \ell \cdot \rho_k \right] \leq \sum_{\hat{S} \in \mathcal{K}} \prod_{j=1}^{\ell} \frac{1 - \varepsilon' \cdot \mathbb{E}[X^\hat{S}]}{(1 - \varepsilon')(1 - \varepsilon')\rho_k}.
\] (2)

Let \( \phi_j(\hat{S}) = 1 \) if \( \pi_j \) is successful for \( \hat{S} \), and \( \phi_j(\hat{S}) = 0 \) otherwise, and failure probability (2) is at most:

\[
\sum_{\hat{S} \in \mathcal{K}} \prod_{j=1}^{\ell} \frac{1 - \varepsilon' \cdot \mathbb{E}[\phi_j(\hat{S})]}{(1 - \varepsilon')(1 - \varepsilon')\rho_k}.
\] (3)

\[
= \sum_{\hat{S} \in \mathcal{K}} \prod_{j=1}^{\ell} \frac{1 - \varepsilon' \cdot \phi_j(\hat{S})}{(1 - \varepsilon')(1 - \varepsilon')\rho_k} \cdot \left( \frac{1 - \varepsilon' \cdot \mathbb{E}[\phi_{s+1}(\hat{S})]}{(1 - \varepsilon')(1 - \varepsilon')\rho_k} \right)^{\ell - s - 1}
\] (4)

\[
\leq \sum_{\hat{S} \in \mathcal{K}} \prod_{j=1}^{\ell} \frac{1 - \varepsilon' \cdot \phi_j(\hat{S})}{(1 - \varepsilon')(1 - \varepsilon')\rho_k} \cdot \left( \frac{1 - \varepsilon' \cdot \mathbb{E}[\phi_{s+1}(\hat{S})]}{(1 - \varepsilon')(1 - \varepsilon')\rho_k} \right)^{\ell - s - 1}
\]

\[
= \Phi(\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)),
\] (5)

where equality (4) is conditional expectation under: (fixed) permutations \( \pi_1, \ldots, \pi_s \) for some \( s \in [\ell - 1] \), the (semi-random) permutation \( \pi_{s+1} \) currently being chosen, and (fully random) permutations \( \pi_{s+2}, \ldots, \pi_{\ell} \).

The first term (3) is less than \( |\mathcal{K}|/\exp((\varepsilon')^2\ell\rho_k/2) \), which is strictly smaller than 1 for large \( \ell \). Let us denote \( \mathbb{E}[\phi_{s+1}(\hat{S})] = \mathbb{E}[\phi_{s+1}(\hat{S}) | \pi_{s+1}(r) = \tau] = \Pr[X^\hat{S}_{s+1} = 1 | \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1), \pi_{s+1}(r) = \tau] \), where positions \( \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r) \) were fixed in the semi-random permutation \( \pi_{s+1} \) and \( \pi_{s+1}(r) \) was fixed in particular to \( \tau \in [n] \setminus \{ \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1) \} \), and it can be computed by using the algorithm from Theorem 5. This gives our pessimistic estimator \( \Phi \). Because \( s \) is fixed for all steps where the semi-random permutation is being decided, \( \Phi \) is uniformly proportional to \( \Phi_1 \):

\[
\Phi_1 = \sum_{\hat{S} \in \mathcal{K}} \left( \prod_{j=1}^{s} (1 - \varepsilon' \cdot \phi_j(\hat{S})) \right) \cdot (1 - \varepsilon' \cdot \mathbb{E}[\phi_{s+1}(\hat{S})]), \quad \Phi_2 = \sum_{\hat{S} \in \mathcal{K}} \left( \prod_{j=1}^{s} (1 - \varepsilon' \cdot \phi_j(\hat{S})) \right) \cdot \mathbb{E}[\phi_{s+1}(\hat{S})].
\] (6)

Recall \( \pi_{s+1}(r) \) in semi-random permutation was fixed but not final. To make it final, we choose \( \pi_{s+1}(r) \in [n] \setminus \{ \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1) \} \) that minimizes \( \Phi_1 \), which is equivalent to maximizing \( \Phi_2 \).

Proof of Lemma 3 can be found in Section 7.3.2.

Lemma 3 The above potential \( \Phi(\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)) \) is a pessimistic estimator if \( \ell \geq \frac{2k\ln(n)}{\rho(\varepsilon')^2} \).

Proof. (of Theorem 4) See Section 7.3.2. \qed
Algorithm 1: Find permutations distribution

**Input:** Positive integers \( n \geq 2, 2 \leq k \leq n, \ell \geq 2 \).

**Output:** A multi-set \( L \subseteq \Pi_n \) of \( \ell \) permutations.

1 /* This algorithm uses Function \( \text{Prob} (E_i, \hat{S}) \) from Algorithm 2 in Section 7.3.1 */

2 \( \pi_1 \leftarrow (1,2,\ldots,n) /* \text{Identity permutation} */ \)

3 \( L \leftarrow \{\pi_1\} \)

4 Let \( \mathcal{K} \) be the set of all ordered \( k \)-element subsets of \([n]\).

5 for \( \hat{S} \in \mathcal{K} \) do

6 \( w(\hat{S}) \leftarrow 1 - \epsilon' \cdot \phi_1(\hat{S}) \)

7 for \( s = 1 \ldots \ell - 1 \) do

8 for \( r = 1 \ldots n \) do

9 for \( \hat{S} \in \mathcal{K} \) do

10 \( \begin{cases} \Pr[E_i | \pi_{s+1}(1), \ldots, \pi_{s+1}(r) = \tau] \leftarrow \text{Prob}(E_i, \hat{S}), \text{for } i = 2 \ldots k. \\ E[\phi_{s+1}(\hat{S}) | \pi_{s+1}(r) = \tau] \leftarrow \sum_{i=2}^k \Pr[E_i | \pi_{s+1}(1), \ldots, \pi_{s+1}(r-1), \pi_{s+1}(r) = \tau] \end{cases} \)

11 Choose \( \pi_{s+1}(r) = \tau \) for \( \tau \in [n] \setminus \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r-1)\} \) to maximize \( \sum_{\hat{S} \in \mathcal{K}} w(\hat{S}) \cdot E[\phi_{s+1}(\hat{S}) | \pi_{s+1}(r) = \tau] \).

12 \( L \leftarrow L \cup \{\pi_{s+1}\} \)

13 for \( \hat{S} \in \mathcal{K} \) do

14 \( w(\hat{S}) \leftarrow w(\hat{S}) \cdot (1 - \epsilon' \cdot \phi_{s+1}(\hat{S})) \)

15 return \( L \)

4 Dimension reductions via Reed-Solomon codes

A set \( \mathcal{F} \) of functions \( f : [n] \rightarrow \ell \) is called a dimensionality-reduction set with parameters \( (n, \ell, d) \) if it satisfies the following:

1. the number of functions that have the same value on any element of the domain is bounded:
   \[ \forall_{i,j \in [n], i \neq j} : \{f \in \mathcal{F} : f(i) = f(j)\} \leq d; \]

2. for each function, the elements of the domain are almost uniformly partitioned into the elements of the image:
   \[ \forall_{i \in [\ell], f \in \mathcal{F}} : |f^{-1}(i)| \leq \frac{n}{\ell} + o(\ell). \]

The dimensionality-reduction set of functions is key in our approach to find probability distribution that guarantees a high success probability for wait-and-pick secretarial algorithms. When applied once, it reduces the size of permutations needed to be considered for optimal success probability from \( n \)-elements to \( \ell \)-elements. The above conditions (1) and (2) are intended to ensure that the found set of \( \ell \)-element permutations can be reversed into \( n \)-element permutations without much loss of success probability.
4.1 A polynomial time construction of the functions distribution

We first show a general pattern for constructing a set of functions that reduce the dimension of permutations from \( n \) to \( q < n \). The key idea behind the construction is to use refined Reed-Solomon codes. Reed-Solomon code is a set of polynomials of degree \( d \) over some finite field \( \mathbb{F} \) of size \( q \) (this requires an additional condition on \( q \) to be a prime number). Assume that we are given \( n \) polynomials: \( p_1, \ldots, p_n \) of degree exactly \( d \) over field \( \mathbb{F} \) that together constitute the Reed-Solomon code.

We define \( q \) functions \( f_i : [n] \rightarrow [q], i \in [q] \) as follows. Each function \( f_i \) is a mapping that maps \( j \in [n] \) to the image of \( j \)-th polynomial from the set \( p_1, \ldots, p_n \) taken in \( i \)-th element of \( \mathbb{F} \). Namely, we have \( f_i(j) = p_j(i) \), where \( p_j \) denotes the \( j \)-th polynomial. To this point, the construction has already been proposed by Kesselheim at. al [14]. Our new contribution is that we can also require, by carefully choosing the polynomials \( p_1, \ldots, p_n \) we used to define \( f_i \), that each function \( f_i \) satisfies \(|f_i^{-1}(k)| \sim \frac{n}{q}, k \in [q] \). In the next Lemma 4 we give the formal statement of this new proved result. The complete proof is given in Section 8. Lemma 4 immediately leads to our first construction of dimensionality-reduction set of functions, which we sum up in Corollary 2.

Lemma 4 There exists a set \( \mathcal{F} \) of functions \( f : [n] \rightarrow [q], \) for some prime integer \( q \geq 2 \), such that for any two distinct indices \( i, j \in [n], i \neq j \), we have

\[
|\{ f \in \mathcal{F} : f(i) = f(j) \}| \leq d
\]

and

\[
\forall q' \in [q] : |f^{-1}(q')| \in \left\{ \left\lfloor \frac{n}{q} \right\rfloor, \left\lceil \frac{n}{q} \right\rceil + 1 \right\},
\]

where \( 1 \leq d < q \) is an integer such that \( n \leq q^{d+1} \). Moreover, \(|\mathcal{F}| = q \) and set \( \mathcal{F} \) can be constructed in deterministic polynomial time in \( n, q, d \).

Corollary 2 Observe that setting \( q \in \Omega(\log n), d \in \Theta(q) \) in Lemma 4 results in a dimensionality-reduction set of functions \( \mathcal{F} \) with parameters \((n, q, \sqrt{q})\). Moreover, set \( \mathcal{F} \) has size \( q \) and as long as \( q \in \tilde{O}(n) \), it can be computed in polynomial time in \( n \).

4.2 Product of two Reed-Solomon’s codes

Reducing dimension from \( n \) to \( \log n \) may be sometimes insufficient, e.g., one may want to enumerate the set of all permutations, but the set of all \( \log n \)-element permutations is superpolynomial in \( n \). To tackle this problem we propose a construction based on composition of two dimensionality-reduction set of functions, \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \). Roughly speaking, the goal of the first set is to reduce the dimension from \( n \) to \( \log n \), while the goal of the second set is to reduce the dimension from \( \log n \) to \( \log \log n \). Such composition is not a totally new concept. It was proposed by Kesselheim et al. [14]. Our contribution, however, is in showing that each function that belongs to the composition of those two sets, \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), can preserve the property of equal preimage sizes, that is, it satisfies condition (2) from the definition of dimensionality-reduction set of functions. The complete reduction is provided in Section 8. The concise and formal summary is given below.

Corollary 3 For any \( \epsilon > 0 \) and \( q \geq (\log \log n)^\epsilon \) there exists a dimensionality-reduction set of functions with parameters \((n, q, \sqrt{q})\). Moreover, such set has size \( q^{\sqrt{q}} \) and can be computed in polynomial time in \( q^{\sqrt{q}} \).
5 Two low-entropy permutations distributions

We will start with a general theorem showing how from a dimensionality-reduction set of functions and a set of permutations over smaller dimensions \( \ell \), which guarantees high probability of success for threshold algorithms, generate a set of permutations over a larger dimension \( n \) that preserves the success probability.

Let us set a dimensionality-reduction set of functions \( \mathcal{F} \) and a set \( \mathcal{L} \) of \( \ell \)-element permutations such that \( \rho_k \) fraction of these permutations are successful for each \( k \)-tuple. Consider the following random experiment: first we draw u.a.r a function \( f \) from \( \mathcal{F} \) and then we draw u.a.r a permutation \( \pi \) from \( \mathcal{L} \). We can relate an \( n \)-element permutation to such an experiment as follows. First, function \( f \) determines for each element \( u \in [n] \) number of the block \( f(u) \in [\ell] \) to which this element is assigned. The permutation \( \pi \) sets the order of these blocks. Ultimately, the \( n \)-element permutation is created as first ordering blocks according to the permutation \( \pi \), and then listing numbers from each block in one sequence preserving the order of blocks. The order of numbers inside a single block is irrelevant. The key properties in this construction are twofold. First, if the probability that a pair of fixed elements \( i, j \) will end up in the same block is \( \frac{k^2d}{\ell} \), then from union bound we conclude that the probability that elements of a \( k \)-tuple will be assigned to different blocks is at least \( 1 - \frac{k^2d}{\ell} \). On the other hand, if the blocks are roughly the same size, the relative order of the blocks assigned to \( k \) numbers in the smaller permutations will be the same as the relative order of these \( k \) numbers in the larger permutation. Moreover, the order of blocks assigned to the \( k \) numbers with respect to the threshold in the smaller permutation will be the same as the order of these numbers with respect to the threshold in the larger permutation. Those properties allow us to carry smoothly properties of \( k \)-tuples, such as being successful, from the smaller \( \ell \)-element permutation to the larger \( n \)-element permutations. The idea described above is formalised in the following Theorem 6. The proof of this theorem and all other theorems from this section are given in Section 9.

**Theorem 6** Assume that we are given a set of dimensionality-reduction functions \( \mathcal{F} \) with parameters \((n, \ell, d)\) satisfying \( \ell^2 < \frac{n}{\ell^d} \), and a multiset \( \mathcal{L} \) of \( \ell \)-element permutations such that at least a fraction \( \rho_k \) of these permutations from \( \mathcal{L} \) are successful for each \( k \)-tuple, for some constant \( \rho_k \). Then there exists a set of \( n \)-element permutations \( \mathcal{L}' \) such that the classic secretarial algorithm executed on uniform distribution over the set \( \mathcal{L}' \) works with probability of success at least

\[
\left( 1 - \frac{k^2d}{\ell} \right) \rho_k.
\]

The set \( \mathcal{L}' \) can be computed in time \( O(|\mathcal{F}| \cdot |\mathcal{L}|) \).

5.1 Double dimensional reduction conjugated with a set of all permutations

First application of Theorem 6 is straightforward. The set \( \mathcal{F} \) is a set of functions that reduce the dimension from \( n \) to \( \ell = \frac{\log \log n}{\log \log \log n} \) with parameter \( d \) equal to \( \sqrt{\ell} \). Then we create set \( \mathcal{L} \) by enumerating all permutations of \( \ell \) elements. Provided that set \( \mathcal{L} \) is successful with sufficiently large ratio \( \rho_k \) for every \( k \)-tuple, which is guaranteed by Lemma 5 we obtain the final distribution of \( n \)-element permutations by putting sets \( \mathcal{F} \) and \( \mathcal{L} \) to Theorem 6. This leads to Theorem 7 below whose full proof can be found in Section 9.

**Lemma 5** Assume we are given integers \( k, n, k < n \). Consider the set \( \mathcal{L} \) of all \( n \)-elements permutations. For every \( k \)-tuple there are at least \( \rho_k |\mathcal{L}| \) successful permutations from \( \mathcal{L} \), where we define \( \rho_k := \text{OPT}_n - \frac{1}{k} \left( 1 - \frac{1}{e} \right)^k \).
Proof. The proof is repetition of the argument given in the analysis of Theorem 2.

\[ \square \]

**Theorem 7** There exists a permutations distribution \( D_n \) of entropy \( O(\log \log n) \) such that the classic secretarial algorithm executed on \( D_n \) picks the highest element with probability at least
\[
\frac{1}{e} - 3^{(\log \log \log n)^{5/2}}. 
\]
The distribution \( D_n \) can be computed in time polynomial in \( n \).

5.2 Single dimensional reduction conjugated with the pessimistic estimator

Our main result is based on a single dimensional reduction from \( n \) to \( \ell = \log n \). Because a single reduction results in a bigger set of permutations over the reduced dimension \( \ell \), in this case enumerating all permutations is not computationally feasible. Indeed, in order to save computational time, but also to generate sparser subset of \( \log n \) elements permutations, that guarantees high probability of success for wait-and-pick algorithms, we use a subtle method of pessimistic estimator, as described in Section 5. Finally, we conjugate the dimensionality-reduction set of functions with the permutation set using Theorem 6. This allows us to obtain the main result of our paper whose full proof can be found in Section 9.

**Theorem 8** There exists a permutation distribution \( D_n \) that can be computed in time \( O(n) \) and has entropy \( O(\log \log n) \) such that the classic secretarial algorithm on the permutation drawn from \( D_n \) picks the best element with probability of at least
\[
\frac{1}{e} - (C_1 \log \log n)^{\frac{5}{2}} - o\left(\frac{\log \log n}{\log n}\right), \quad \text{where} \quad C > 0 \text{ can be any fixed constant and} \quad C_1 = \frac{\log(e/(e-1))}{\log\left(\frac{e}{e-1}\right)}.
\]

6 Proofs from Section 2

6.1 Proof of Lemma 1

Proof. To prove this expansion we extend the harmonic function \( H_n \) to real numbers. Namely, for any real number \( x \in \mathbb{R} \) we use the well known definition:
\[
H_x = \psi(x + 1) + \gamma,
\]
where \( \psi \) is the digamma function and \( \gamma \) is the Euler-Mascheroni constant. Digamma function is just the derivative of the logarithm of the gamma function \( \Gamma(x) \), pioneered by Euler, Gauss and Weierstrass. Both functions are important and widely studies in real and complex analysis.

For our purpose, it suffices to use the following inequalities that hold for any real \( x > 0 \) (see Theorem 5 in [10]):
\[
\ln(x) - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120(x + 1/8)^4} < \psi(x) < \ln(x) - \frac{1}{2x} + \frac{1}{12x^2} + \frac{1}{120x^4}.
\]

Now we use these estimates for \( f(k, m) \) for \( \psi(k) \) and \( \psi(m) \) with \( m = k/e + j \):
\[
f(k, m) = \frac{m}{k}(\psi(k) - \psi(m)) =
\]
\[
= \frac{m}{k} \left( \ln(k) - \frac{1}{2k} + \frac{1}{12k^2} + \frac{1}{120(k + \theta(k))^4} - \ln(m) + \frac{1}{2m} + \frac{1}{12m^2} - \frac{1}{120(m + \theta(m))^4} \right)
\]
\[
= \frac{m}{k} \left( 1 + \ln\left(\frac{k}{em}\right) - \frac{1}{2k} + \frac{1}{2m} - \frac{1}{12k^2} + \frac{1}{12m^2} + \frac{1}{120(k + \theta(k))^4} - \frac{1}{120(m + \theta(m))^4} \right),
\]

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where \( \theta(x) \in (0, 1/8) \). Now, taking into account that \( m \in [k] \), we can suppress the low order terms under \( \Theta \left( \frac{1}{k^2} \right) \) to obtain

\[
f(k, m) = \frac{m}{k} \left( 1 + \ln \left( \frac{e}{km} \right) - \frac{1}{2k} + \frac{1}{2m} \right) + \Theta \left( \frac{1}{m^2} \right)
\]

\[
= \frac{m}{k} - \frac{m}{k} \ln \left( \frac{e}{k} \right) - \frac{m}{2k^2} + \frac{1}{2k} + \Theta \left( \frac{1}{m^2} \right)
\]

\[
= \frac{1}{e} + \frac{j}{k} - \left( \frac{1}{e} + \frac{j}{k} \right) \ln \left( \frac{k + je}{k} \right) - \frac{k/e + j}{2k^2} + \frac{1}{2k} + \Theta \left( \frac{1}{k^2} \right)
\]

\[
= \frac{1}{e} + \frac{j}{k} - \left( \frac{1}{e} + \frac{j}{k} \right) \ln \left( \frac{k + je}{k} \right) - \frac{1}{2ek} + \frac{1}{2k} + \Theta \left( \frac{1}{k^{3/2}} \right).
\]

We will now use the following well known Taylor expansion

\[
\ln \left( \frac{k + je}{k} \right) = \left( \frac{k + je}{k} - 1 \right) \frac{1}{2} \left( \frac{k + je}{k} - 1 \right)^2 + \frac{1}{3} \left( \frac{k + je}{k} - 1 \right)^3 - \ldots
\]

\[
= \frac{je}{k} - \frac{1}{2} \left( \frac{je}{k} \right)^2 + \frac{1}{3} \left( \frac{je}{k} \right)^3 - \ldots = \frac{je}{k} - \frac{1}{2} \left( \frac{je}{k} \right)^2 + \Theta \left( \frac{1}{k^{3/2}} \right).
\]

Using this expansion, we can continue from above as follows

\[
f(k, m) = \frac{1}{e} + \frac{j}{k} - \left( \frac{1}{e} + \frac{j}{k} \right) \ln \left( \frac{k + je}{k} \right) - \frac{1}{2ek} + \frac{1}{2k} + \Theta \left( \frac{1}{k^{3/2}} \right)
\]

\[
= \frac{1}{e} + \frac{1}{2e} \left( \frac{je}{k} \right)^2 - \frac{j^2e^2}{k^2} + \frac{j^3e^2}{k^3} - \frac{1}{2ek} + \frac{1}{2k} + \Theta \left( \frac{1}{k^{3/2}} \right)
\]

\[
= \frac{1}{e} + \frac{1}{2e} \left( \frac{je}{k} \right)^2 - \frac{j^2e^2}{k^2} + \frac{1}{2ek} + \Theta \left( \frac{1}{k^{3/2}} \right)
\]

\[
= \frac{1}{e} - \frac{j^2e}{2k^2} - \frac{1}{2ek} + \frac{1}{2k} + \Theta \left( \frac{1}{k^{3/2}} \right)
\]

\[
= \frac{1}{e} - \left( \frac{1}{2e} + \frac{j^2e}{2k^2} - \frac{1}{2} \right) \frac{1}{k} + \Theta \left( \frac{1}{k^{3/2}} \right).
\]

\[\square\]

### 6.2 Proof of Lemma

**Proof.** We first argue that function \( f(n, \cdot) \) has exactly one local maximum, which is also global maximum. To see it, observe that function \( h(m) \) is positive until \( H_m + 1 \) gets bigger than \( H_{k-1} \), which occurs for a single value \( m \) (as we consider function \( h \) for discrete arguments) and remains negative afterwards. Thus, function \( f(n, \cdot) \) is monotonically increasing until that point, and decreasing afterwards. Hence, it has only one local maximum, which is also global maximum.

It remains to argue that the abovementioned argument \( m \) in which function \( f(n, \cdot) \) achieves maximum is in \( \left\{ \frac{k}{e}, \frac{k}{e} + 1 \right\} \). We will make use of the following known inequalities.
Lemma 6 The following bounds hold for the harmonic and logarithmic functions:

1. \( \frac{1}{2(x+1)} < H_x - \ln x < \frac{1}{2x} \),

2. \( \frac{1}{24(x+1)} < H_x - \ln(x + 1/2) - \gamma < \frac{1}{24x^2} \),

3. \( \frac{x}{1+x} \leq \ln(1 + x) \leq x, \) which holds for \( x > -1 \).

Using the first bound (1) from Lemma 6, we obtain the following:

\[
k \cdot h \left( \left\lceil \frac{k}{e} \right\rceil - 1 \right) = H_{k-1} - H_{\left\lfloor \frac{k}{e} \right\rfloor - 1} - 1
\]

\[
> \ln(k-1) + \frac{1}{2k} - \ln \left( \left\lceil \frac{k}{e} \right\rceil - 1 \right) - \frac{1}{2(\left\lceil \frac{k}{e} \right\rceil - 1)} - 1
\]

\[
= \ln \left( \frac{e(k-1)}{e(\left\lceil \frac{k}{e} \right\rceil - 1)} \right) + \frac{1}{2k} - \frac{1}{2(\left\lceil \frac{k}{e} \right\rceil - 1)} - 1
\]

\[
= \ln \left( \frac{k-1}{e(\left\lceil \frac{k}{e} \right\rceil - 1)} \right) + \frac{1}{2k} - \frac{1}{2(\left\lceil \frac{k}{e} \right\rceil - 1)}
\]

Rewriting inequality (3) in Lemma 6 as \( \ln(y) \geq 1 - 1/y \), we obtain:

\[
> 1 - \frac{e(\left\lceil \frac{k}{e} \right\rceil - 1)}{k-1} + \frac{1}{2k} - \frac{1}{2(\left\lceil \frac{k}{e} \right\rceil - 1)}
\]

\[
= 1 - \frac{e(\left\lceil \frac{k}{e} \right\rceil - 1)}{k-1} - \frac{1}{2(\left\lceil \frac{k}{e} \right\rceil - 1)}
\]

\[
> 0,
\]

where the last inequality holds because it is equivalent to

\[
2(k-1)(\left\lceil k/e \right\rceil - 1) > 2e(\left\lceil k/e \right\rceil - 1)^2 + k-1 \iff 2k\left\lceil k/e \right\rceil + (4e-2)\left\lceil k/e \right\rceil > 2e(\left\lceil k/e \right\rceil)^2 + 3k + 2e - 3
\]

\[
\iff 2k\left\lceil k/e \right\rceil > 2e(\left\lceil k/e \right\rceil)^2 \text{ and } (4e-2)\left\lceil k/e \right\rceil \geq 3k+2e-3,
\]

\[
\iff k/e > \left\lceil k/e \right\rceil \text{ and } (4e-2)(k/e - 1) \geq 3k+2e-3,
\]

where the first inequality is obvious and the second holds for \( k = \Omega(1) \).

For the second part we again use the first bound (1) from Lemma 6 to obtain:

\[
k \cdot h \left( \left\lceil \frac{k}{e} \right\rceil + 1 \right) = H_{k-1} - H_{\left\lfloor \frac{k}{e} \right\rfloor + 1} - 1
\]

\[
< \ln(k-1) + \frac{1}{2(k-1)} - \ln \left( \left\lceil \frac{k}{e} \right\rceil + 1 \right) - \frac{1}{2(\left\lceil \frac{k}{e} \right\rceil + 2)} - 1
\]

\[
= \ln \left( \frac{e(k-1)}{e(\left\lceil \frac{k}{e} \right\rceil + 1)} \right) + \frac{1}{2(k-1)} - \frac{1}{2(\left\lceil \frac{k}{e} \right\rceil + 2)} - 1
\]

\[
= \ln \left( \frac{k-1}{e(\left\lceil \frac{k}{e} \right\rceil + 1)} \right) + \frac{1}{2(k-1)} - \frac{1}{2(\left\lceil \frac{k}{e} \right\rceil + 2)}
\]

\[
< 0,
\]
where the last inequality holds because it follows from
\[
\ln \left( \frac{k - 1}{e \left( \left\lfloor \frac{k}{e} \right\rfloor + 1 \right)} \right) < 0 \quad \text{and} \quad \frac{1}{2(k - 1)} - \frac{1}{2 \left( \left\lfloor \frac{k}{e} \right\rfloor + 2 \right)} < 0
\]

\[\iff k/e < \left\lfloor \frac{k}{e} \right\rfloor + 1 + 1/e \quad \text{and} \quad \left\lfloor \frac{k}{e} \right\rfloor < k - 3,\]
\[\iff k/e < \left\lfloor \frac{k}{e} \right\rfloor + 1 + 1/e \quad \text{and} \quad \left\lfloor \frac{k}{e} \right\rfloor \leq k/e \leq k - 3,\]
where the first inequality is obvious and the second holds for \(k \geq \frac{e - 1}{3e} \).

The second part of the lemma that function \(f\) achieves its maximum for \(m \in \{\frac{k}{e}, \left\lfloor \frac{k}{e} \right\rfloor + 1\}\) follows directly from the first part of that lemma and from the definition of the discrete derivative \(h(\cdot)\).

### 6.3 Complete proof of existential upper bound (Theorem 2)

**Proof.** (of Theorem 2) Let us consider an instance of the secretarial problem, where an adversary has already assigned the values \(v(1), v(2), \ldots, v(n)\) to all items \(1, 2, \ldots, n\). We will refer by parameter \(k' \in \{1, 2, \ldots, k\}\) to the \(k'\)th largest among these values. Thus, \(k' = 1\) refers to the largest value among \(v(1), v(2), \ldots, v(n)\), \(k' = 2\) refers to the second largest value among \(v(1), v(2), \ldots, v(n)\), and so on. Note that the \(k'\)th and \((k' + 1)\)st largest values might be the same.

We also denote by \(\text{ind}(k') \in \{1, 2, \ldots, n\}\) the index of the element with the \(k'\)th largest value, that is, the \(k'\)th largest value is \(v(\text{ind}(k'))\).

We will use the probabilistic method to show existence of the set \(\mathcal{L} \subseteq \Pi_n\).

Let us first consider a random experiment that is choosing u.a.r. a single permutation \(\pi\) from \(\Pi_n\). We will now estimate the probability of success of the secretarial algorithm with threshold \(m_0\). (Below, \(\text{ind}(i)\) refers to the position of the \(i\)th largest adversarial value from \(v(1), \ldots, v(n)\) in permutation \(\pi\), i.e., \(\pi(\text{ind}(i))\).)

This probability is lower bounded by the probability of union of the following disjoint events \(E_i, i = 2, 3, \ldots, k\), where

\[
E_i = A_i \cap B_i \cap C_i, \quad A_i = \{\text{ind}(i) \in \{1, 2, \ldots, m_0\}\}, \quad B_i = \bigcap_{j=1}^{i-1} \{\text{ind}(j) \in \{m_0 + 1, \ldots, n\}\},
\]

\[
C_i = \{\forall j = 2, 3, \ldots, i-1 : \text{ind}(1) < \text{ind}(j)\}.
\]

We say that \(\pi\) covers the ordered \(i\)-tuple \(\hat{S} = \{\pi(\text{ind}(i)), \pi(\text{ind}(1)), \pi(\text{ind}(2)), \ldots, \pi(\text{ind}(i-1))\}\) if event \(E_i\) holds. Now using the Bayes’ formula on conditional probabilities, we obtain, following precisely the order of the events which are connected by the intersection sign “\(\cap\)”, that is, when we compute the probability of the \(j\)th event, we condition on all previous events \(0, 1, 2, \ldots, j-1\), where \(j = 0\)’th event is \(A_i\), \(j = 1\)’st event is \(\{\text{ind}(1) \in \{\lfloor n/e \rfloor + 1, \ldots, n\}\}\), \(j = 2\)’nd event is \(\{\text{ind}(2) \in \{\lfloor n/e \rfloor + 1, \ldots, n\}\}\), and so forth, and, finally, \(j = i\)’th event is \(C_i\):

\[
\mathbb{P}[E_i] = \mathbb{P} \left[ A_i \cap \left( \bigcap_{j=1}^{i-1} \{\text{ind}(j) \in \{m_0 + 1, \ldots, n\}\} \right) \cap C_i \right]
\]
\[
= \frac{m_0}{n} \cdot \left( \prod_{j=1}^{i-1} \frac{n - m_0 - (j - 1)}{(n-1) - (j - 1)} \right) \cdot \frac{(i - 2)!}{(i - 1)!}.
\]
On the other hand, we observe that the union of events $\bigcup_{i \in \{2, \ldots, n-m_0+1\}} E_i$ is precisely the event that the secretarial algorithm with threshold $m_0$ will be successful on a random permutation from uniform distribution. By Theorem [1] we know that the latter probability is $\text{OPT}_n = \frac{1}{e} + \frac{c_0}{n} + \Theta((\frac{1}{n})^{3/2})$, where $c_0 = \frac{1}{2} - \frac{1}{2e}$.

Observe now, that that for every $j \geq 1$ we have

$$\frac{n - m_0 - (j - 1)}{(n - 1) - (j - 1)} \leq \frac{n - m_0}{n - 1},$$

therefore the following inequalities hold

$$\text{OPT}_n = \sum_{2 \leq i \leq n-m_0+1} \Pr[E_i] \leq \sum_{2 \leq i \leq k} \Pr[E_i] + \sum_{k+1 \leq i \leq n-m_0+1} m_0 \left( \frac{n-m_0}{n-1} \right)^{i-1} \frac{1}{i-1} \leq \sum_{2 \leq i \leq k} \Pr[E_i] + m_0 \frac{n-k}{n} \sum_{k \leq i \leq m_0} \left( \frac{n-m_0}{n-1} \right)^i \leq \sum_{2 \leq i \leq k} \Pr[E_i] + \frac{m_0}{n-k} \left( \frac{n-m_0}{n-1} \right)^{k} \cdot \frac{n-1}{m_0-1} \cdot \text{OPT}_n - \frac{2}{k} \left( \frac{n-m_0}{n-1} \right)^{k}. \quad (7)$$

Let $i \in \{2, 3, \ldots, k\}$. If $\pi \in \Pi_n$ is an independently and u.a.r. chosen permutation, then the fact that the event $E_i$ holds means that $\pi$ covers an $i$-tuple $\hat{S} = \{\pi(\text{ind}(i)), \pi(\text{ind}(1)), \pi(\text{ind}(2)), \ldots, \pi(\text{ind}(i-1))\}$.

Let us fix a particular ordered $k$-tuple $\hat{S} = \{j_1, j_2, \ldots, j_k\} \subseteq [n]$. We say that an independently and u.a.r. chosen permutation $\pi \in \Pi_n$ is successful for $\hat{S}$ iff event $\bigcup_{i=2}^{k} E_i$ holds, where

$$E_i = \{\pi \text{ covers } i \text{-tuple } \{j_i, j_1, j_2, \ldots, j_{i-1}\}\},$$

for $i \in \{2, 3, \ldots, k\}$. By the argument above $\pi$ is successful with probability $\rho_k$.

Let us now choose independently and uniformly at random $\ell = c \log(n^k)$ permutations $\pi_1, \ldots, \pi_\ell$ from $\Pi_n$, for a fixed constant $c \geq 1$. These permutations will comprise the multi-set $\mathcal{L}$.

Then, let $X_1^\hat{S}, \ldots, X_\ell^\hat{S}$ be random variables such that $X_i^\hat{S} = 1$ if the corresponding random permutation $\pi_i$ is successful for $\hat{S}$, and $X_i^\hat{S} = 0$ otherwise, for $t = 1, 2, \ldots, \ell$. Note that “successful” refers here to the above fixed $k$-tuple $\hat{S} = \{j_1, j_2, \ldots, j_k\}$.

Then for $X^\hat{S} = X_1^\hat{S} + \cdots + X_\ell^\hat{S}$ we have that $\mathbb{E}[X^\hat{S}] = \rho_k \ell = c \rho_k \log(n^k)$ and by the Chernoff bound

$$\Pr[X < (1 - \epsilon') \cdot \rho \ell] \leq \frac{1}{e^{(\epsilon')^2 c \rho_k \log(n^k)/2}} = \frac{1}{n^{(\epsilon')^2 c \rho_k / 2}},$$

for any constant $\epsilon'$ such that $0 < \epsilon' < 1$.

This means that for the specific fixed ordered $k$-tuple $\hat{S} = \{j_1, j_2, \ldots, j_k\}$, with probability at least $1 - 1/n^{(\epsilon')^2 c \rho_k / 2}$, at least $(1 - \epsilon') \cdot c \rho_k \log(n^k)$ among these $\ell = c \log(n^k)$ random permutations, that is, a $(1 - \epsilon') \rho_k$ fraction of them, are successful for the $k$-tuple $\hat{S}$. Now, the probability that there exists a $k$-tuple
\[ \hat{S} = \{j_1, j_2, \ldots, j_k\} \] for which there does not exists a \((1 - \varepsilon')\rho_k\) fraction of successful permutations among these \(c \log(n^k)\) random permutations is by the union bound

\[
\Pr[\exists k\text{-tuple } \hat{S} : X^\hat{S} < (1 - \varepsilon')c\rho_k \log(n^k)] \leq \binom{n}{k} \cdot \frac{k!}{n^{(\varepsilon')^2c\rho_kk/2}} \leq \frac{n^k}{n^{(\varepsilon')^2c\rho_kk/2}} \leq \frac{1}{n^{(\varepsilon')^2c\rho_kk/2 - k}}.
\]

So all \(\binom{n}{k} k\!) ordered \(k\)-tuples \(\hat{S}\) are covered with probability at least \(1 - \frac{1}{n^{(\varepsilon')^2c\rho_kk/2 - k}} > 0\). Thus, it suffices to choose \(c\) such that \((\varepsilon')^2c\rho_kk/2 > 2\), that is, \(c > 4/(\rho_k(\varepsilon')^2)\). In fact we will choose a slightly larger \(c\) such that \(c > 4/(\rho_3(\varepsilon')^2) = \Theta(1/(\varepsilon')^2)\).

This means that there exist \(\Theta((\log(n^k))/(\varepsilon')^2)\) permutations such that if we choose one among these permutations u.a.r., then for any \(k\)-tuples \(\hat{S}\), this permutation will be successful with probability at least \((1 - \varepsilon')\rho_k\). This will be the success probability of the secretarial algorithm with threshold \(m_0\).

We will now argue about the ordered \(k'\)-tuples for \(k' \in \{2, 3, \ldots, k - 1\}\).

**Claim 1** If a random permutation \(\pi \in \Pi_n\) is successful for all \(k\)-tuples, for a given \(k\) such that \(k \in \{2, 3, \ldots, n\}\), then \(\pi\) is successful for all \(k'\)-tuples for all \(k' \in \{2, 3, \ldots, k\}\).

**Proof.** Suppose that \(\pi\) is successful for all \(k\)-tuples and let \(\hat{S}' = \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{k'}\} \subseteq [n]\) be any \(k'\)-tuple, for some \(k' \in \{2, 3, \ldots, k - 1\}\). Let us now define a new \(k\)-tuple \(\hat{S}\) from \(\hat{S}'\) by adding to the end of \(\hat{S}'\) any \(k - k'\) distinct elements from \([n] \setminus \hat{S}'\). More formally, we have that \(\hat{S} = \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{k'}, \hat{s}_{k'+1}, \ldots, \hat{s}_k\}\), where \(\{\hat{s}_{k'+1}, \ldots, \hat{s}_k\}\) is any \(k - k'\) element subset of \([n] \setminus \hat{S}'\).

Permutation \(\pi\) is successful for \(\hat{S}\), i.e., event \(E_i = \{\pi \text{ covers } i\text{-tuple } \{\hat{s}_i, \hat{s}_{i+1}, \ldots, \hat{s}_{i+k'-1}\}\}\) holds for any \(i = 2, 3, \ldots, k\). But then, in particular, event \(E_i\) also holds for \(i = 2, 3, \ldots, k'\) because \(k' < k\), that is, \(\pi\) is also successful for \(\hat{S}'\).

This claim and the definition of events \(E_i\) imply for every such \(k'\)-tuple with \(k' \in \{2, 3, \ldots, k - 1\}\), at least a fraction \((1 - \varepsilon')\rho_{k'}\) of permutations from set \(\mathcal{L}\) will also be successful for that \(k'\)-tuple.

We finally argue that with set \(\mathcal{L}\) the secretarial algorithm can deal with all \(k'\)-tuples with the success probability \((1 - \varepsilon')\rho_{k'}\) (note that we have here \(\rho_k\), not \(\rho_{k'}\)), for any \(k' \in \{k + 1, k + 2, \ldots, n\}\). The argument is as follows. Let us take any such \(k'\)-tuple \(\hat{S}\) with \(k' \in \{k + 1, k + 2, \ldots, n\}\). Then observe that the set of permutations \(\mathcal{L}\) might not have permutations for which the algorithm is successful for \(\hat{S}\). In particular, observe, that none of the permutations in \(\mathcal{L}\) may cover \(\hat{S} = \{j_1, j_2, \ldots, j_{k+1}\}\). For instance, the event \(E_{k+1} = \{\pi \text{ covers } (k + 1)\text{-tuple } \{j_{k+1}, j_{k+2}, \ldots, j_k\}\}\) might not hold at all for any \(\pi \in \mathcal{L}\). However, we can take the \(k\)-tuple \(\hat{S}' = \{j_1, j_2, \ldots, j_k\} \subseteq \hat{S}\) and we know that a random permutation from \(\mathcal{L}\) will be successful for \(\hat{S}'\) with probability at least \((1 - \varepsilon')\rho_{k}\) by the above arguments. This implies that the algorithm is successful with a random permutation from \(\mathcal{L}\) also for \(\hat{S}\), with probability at least \((1 - \varepsilon')\rho_{k}\). We finally note that we can disregard the contributions to the probability of success from such \(k'\)-tuples for \(k' \geq k + 1\) because its probability mass, as shown above, is at most \(2^{k} \left(\frac{n - m_0}{n - 1}\right)^{k}\), and thus it is very tiny. \(\square\)

### 6.4 Full proof of Theorem 3

**Proof.** (of Theorem 3) We use the wait-and-pick algorithm with threshold \([n/e]\). In the remainder we construct and analyze a suitable non-uniform probabilistic distribution of a given entropy \(\mathcal{H}\) on set \(\Pi_n\),
on which the algorithm achieves probability of success $\Omega \left( \frac{1}{(\log \log n + 3 \log \log n - \mathcal{H})^{2+\epsilon}} \right)$. Let $\zeta(\cdot)$ be the Riemann zeta function and let $\zeta^*(x, y) = \sum_{i=1}^{y} \frac{1}{i^x} \in (1, \zeta(x))$. Consider an integer parameter $\ell$ – it will denote the number of elements with highest probability in the constructed distribution, and, as we show later, it will also correspond to $2^\mathcal{H}$, where $\mathcal{H}$ will be the entropy of that distribution. We choose sets $X_i$ of size $|X_i| = x_i = \frac{\ell 2^i}{i^{1+\epsilon} \zeta(2+\epsilon, y)}$, for $1 \leq i \leq y$ and some arbitrary constant $0 < \epsilon < 1$, where $y > 0$ is an integer to be chosen later, and allocate probability $p_i = \frac{1}{2^\ell}$ to each element in set $X_i$. It is indeed a probabilistic distribution, as 

$$\sum_{i=1}^{y} x_i p_i = \sum_{i=1}^{y} \frac{1}{i^{2+\epsilon} \cdot \zeta^*(2+\epsilon, y)} = 1.$$ 

The entropy of this distribution is 

$$\sum_{i=1}^{y} x_i p_i \log \frac{1}{p_i} = \sum_{i=1}^{y} \frac{i + \log \ell}{i^{2+\epsilon} \cdot \zeta^*(2+\epsilon, y)} = \zeta^*(1+\epsilon, y) + \log \ell = \log \ell + \Theta(1).$$

The support $\mathcal{L}^*$ of the distribution is of size 

$$\ell^* = \sum_{i=1}^{y} x_i = \frac{\ell}{\zeta^*(2+\epsilon, y)} \sum_{i=1}^{y} \frac{2^i}{i^{2+\epsilon}} = \Theta \left( \frac{\ell}{\zeta^*(2+\epsilon, y)} \cdot \frac{2^y}{y^{2+\epsilon}} \right).$$

The goal is to choose $y$ in such a way that the support has size which is “constructible”, for instance, $\ell \cdot \frac{2^y}{y^{2+\epsilon}}$ is logarithmic in $n$. It implies $y = \log \frac{\log n}{\ell} + 3 \log \log n$. Now, two challenges remain. First, assuming that we constructed the support of the distribution, how to split it among sets $X_i$ to maximize the probability of success of a suitable secretarial algorithm. Second, what should be such an algorithm. Our approach to them is as follows. We distribute the elements from the constructed support set, e.g., using the construction from Section 5 summarized in Theorem 7 arbitrarily among sets $X_i$. Since we rely on Theorem 7 a natural choice of efficient algorithm would be wait-and-pick with threshold $\lfloor n/e \rfloor$. We now estimate from below the probability of success of this algorithm and the distribution. First, Theorem 7 guarantees that a fraction of $\frac{1}{e} - \Theta \left( \frac{(\log \log n)^{5/2}}{\sqrt{\log \log n}} \right)$ of all permutations in the selected support $\mathcal{L}^*$ of size $O(\log n)$ guarantee selecting the largest element, for any adversarial order. Since we assign arbitrarily permutations in $\mathcal{L}^*$ to sets $X_i$ having different probabilities, in the worst case, for some adversarial order all these “good” permutations (i.e., guaranteeing selecting largest element from the adversarial order) may have smallest possible probability $p_y = \frac{1}{2^\ell}$. Hence, the probability of success is at least 

$$\ell^* \cdot \left( \frac{1}{e} - \Theta \left( \frac{(\log \log n)^{5/2}}{\sqrt{\log \log n}} \right) \right) \cdot p_y \geq \Omega \left( \frac{\ell}{\zeta^*(2+\epsilon, y)} \cdot \frac{2^y}{y^{2+\epsilon}} \cdot \frac{1}{2^\ell} \right) = \Omega \left( \frac{1}{y^{2+\epsilon}} \right),$$

which is $\Omega(\frac{1}{(\log \log n)^{2+\epsilon}})$ as the entropy $\mathcal{H} = \log \ell + \Theta(1) < c \log \log n$ for some constant $0 < c < 1$. □

7 Proofs from Section 3

Remark for Theorem 4. Note, that classic secretarial algorithm with threshold $\frac{1}{e'}$ executed on the uniform distribution over the set of permutations $\mathcal{L}$ achieves $(1 - e') \cdot \left( OPT - \frac{2}{e} \left( 1 - \frac{1}{e} \right)^k \right)$ success probability.
However, unless \( k = O(1) \), the construction time of multi-set \( \mathcal{L} \) is superpolynomial in \( n \), which makes this result inefficient as is. We will show how to use it efficiently in the next section in conjunction with a dimension reduction technique.

### 7.1 Proof preparation

Before formally proving Theorem 4, we will first outline the main ideas behind the proof and introduce some preliminaries. Towards this aim we will use Chernoff bound argument from the proof of Theorem 2 and a specially tailored method of conditional expectations to derandomize it. We will also need to derive a special pessimistic estimator for our derandomization.

In the proof of Theorem 2 we have random variables \( X_1, \ldots, X_\ell \), where random variable \( X_j \) corresponds to choosing independently and u.a.r. a permutation \( \pi_j \in \Pi_n \). Each random variable \( X_j \) has \( \Pi_n \) as its domain and the size of the domain is exponential in \( n \). For our derandomization we need that these domains are of polynomial size.

To achieve this, we will simulate this random experiment of choosing u.a.r. a permutation from \( \Pi_n \) by introducing for each \( X_j \) additional “index” random variables \( X_j^i \) for \( i = 1, 2, \ldots, n \). Each \( X_j^i \) chooses independently (from all other random variables) and u.a.r. an integer from \( \{1, 2, \ldots, n - i + 1\} \), that is, \( X_j^i \) is equal to one of the integers from \( \{1, 2, \ldots, n - i + 1\} \) with probability \( 1/(n - i + 1) \). Although the random variables \( X_j^1, X_j^2, \ldots, X_j^n \) are mutually independent, they define a random permutation \( \pi \in \Pi_n \) by the following sequential interpretation: \( \pi(1) = X_j^1, \pi(2) \) is the \( X_j^2 \)-th element from set \( I_1 = \{1, 2, \ldots, n\} \setminus \{\pi(1)\} \), \( \pi(3) \) is the \( X_j^3 \)-th element from set \( I_2 = \{1, 2, \ldots, n\} \setminus \{\pi(1), \pi(2)\} \), etc, where the elements are ordered in an increasing order in sets \( I_1, I_2 \), respectively. In general, \( \pi(i) \) is the \( X_j^i \)-th element from the set \( I_{i-1} = \{1, 2, \ldots, n\} \setminus \{\pi(1), \pi(2), \ldots, \pi(i-1)\} \), for \( i = 2, 3, \ldots, n \), where the elements are ordered in an increasing order in set \( I_{i-1} \). Observe that the last value \( \pi(n) \) is uniquely determined by all previous values \( \pi(1), \pi(2), \ldots, \pi(n-1) \), because \( |I_{n-1}| = 1 \).

Since the probability of choosing the index \( \pi(i) \) for \( i = 1, 2, \ldots, n \) is \( 1/(n - i + 1) \), and these random choices are independent, the final probability of choosing a specific random permutation is

\[
\frac{1}{n} \cdot \frac{1}{n - 1} \cdot \ldots \cdot \frac{1}{n - n + 1} = \frac{1}{n!}.
\]

Thus, this probability distribution is uniform on the set \( \Pi_n \), as we wanted. To summarise, we generate the set of random permutations, \( \pi_1, \ldots, \pi_\ell \), where permutation \( \pi_j \) is generated by the index random variables \( X_j^1, X_j^2, \ldots, X_j^n \).

Suppose we are given the random permutations \( \pi_1, \ldots, \pi_\ell \). Given an \( k \)-tuple \( \mathcal{S} = \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_k\} \subseteq [n] \), i.e., an ordered \( k \)-element subset, let us also introduce a random variable \( X_j^{\hat{S}} \) such that \( X_j^{\hat{S}} = 1 \) if permutation \( \pi_j \) is successful for \( k \)-tuple \( \mathcal{S} \) (with probability \( \rho_k \)), and \( X_j^{\hat{S}} = 0 \) otherwise, for \( j \in [\ell] \). Then for \( X^{\hat{S}} = X_1^{\hat{S}} + \ldots + X_\ell^{\hat{S}} \) we have that \( \mathbb{E}[X^{\hat{S}}] = \rho_k \ell \). Given any \( \varepsilon' \) such that \( 0 < \varepsilon' < 1 \), by Chernoff bound

\[
\mathbb{P}[X^{\hat{S}} < (1 - \varepsilon') \cdot \rho_k \ell] \leq 1/ \exp((\varepsilon')^2 \rho_k \ell/2) = 1/n((\varepsilon')^2c_\rho k/2),
\]

where \( \ell = c \log(n^k) \).

Let \( \mathcal{K} \) be the set of all ordered \( k \)-element subsets of \([n]\). Notice that \( |\mathcal{K}| = \binom{n}{k} \cdot k! < n^k \). We call the \( k \)-tuple \( \mathcal{S} \in \mathcal{K} \) not well-covered if \( X^{\hat{S}} < (1 - \varepsilon') \cdot \rho_k \ell \) and well-covered otherwise. Note that the \( k \)-tuple \( \mathcal{S} \) is well-covered if at least \((1 - \varepsilon') \cdot \rho_k \ell \) permutations from set \( \mathcal{L} \) are successful for \( \mathcal{S} \). We define a random
variable $Y^S$ such that $Y^S = 1$ if $S$ is not well-covered, and $Y^S = 0$ otherwise. Let $Y = \sum_{\hat{S} \in \mathcal{K}} Y^\hat{S}$. By the above argument the expected number of not well-covered $k$-tuples is:

$$
\mathbb{E}[Y] = \sum_{\hat{S} \in \mathcal{K}} \mathbb{E}[Y^\hat{S}] < 1,
$$

provided that $c \geq 1/(\epsilon')^2$. We will maintain the expectation $\mathbb{E}[Y]$ below 1 in each step of our derandomization, where these steps will sequentially define these permutations for set $\mathcal{L}$ permutation by permutation.

### 7.2 Outline of derandomization

We will choose permutations $\{\pi_1, \pi_2, \ldots, \pi_\ell\}$ sequentially, one by one. As the first permutation $\pi_1$, because of symmetry, we choose any, e.g., identity, permutation. Now, suppose that we have already chosen permutations $\pi_1, \ldots, \pi_s$ for some $s \leq \ell$; for $s = 1$, $\pi_1$ is the identity permutation. We will now describe how to choose the next permutation $\pi_{s+1}$, conditioning on permutations $\pi_1, \ldots, \pi_s$ being fixed and assuming that permutations $\pi_{s+2}, \ldots, \pi_n$ are fully random, i.e., chosen independently and u.a.r. from $\Pi_n$. The next permutation $\pi_{s+1}$ will be generated by choosing its indices $\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(n)$ sequentially one by one. In a general case, we will condition on a prefix of its indices $\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)$ up to $r$ being already chosen, where elements $\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r-1)$ are fixed and final, i.e., they will not change, but element $\pi_{s+1}(r)$ is also fixed but not final yet, i.e., we will vary $\pi_{s+1}(r) \in [n] \setminus \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r-1)\}$ to choose the best value for $\pi_{s+1}(r)$. We will do this choice assuming that the rest of the elements $\pi_{s+1}(r+1), \pi_{s+1}(r+2), \ldots, \pi_{s+1}(n)$ are random; when computing probabilities this randomness can be viewed as choosing a random permutation from $\Pi_{n-(r+1)+1}$. After the value $\pi_{s+1}(r)$ is finally decided it will be fixed and final. Then, we will decide the value $\pi_{s+1}(r+1)$ in the same way, and so on.

For any fixed $k$-tuple $\hat{S}$ we will first show how to compute the conditional probability that $\hat{S}$ is not well-covered. This for the fixed permutations $\pi_1, \ldots, \pi_s$ is an easy check if they cover $\hat{S}$ or not. For the “fully random” permutations $\pi_{s+2}, \ldots, \pi_n$ this probability will be obtained by the above Chernoff bound argument, we will derive and use here a special pessimistic estimator. For the “semi-random” permutation $\pi_{s+1}$, we will compute the conditional probability that it covers the tuple $\hat{S}$ or not, conditioning on positions $\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)$ being fixed, where $\pi_{s+1}(r)$ is not final but the rest are final. The probability over the fully random permutations $\pi_{s+2}, \ldots, \pi_n$ can easily be computed without conditioning because they are independent from the fixed and semi-random permutations.

### 7.3 Deriving a pessimistic estimator

We will use an approach inspired by Young [28] to derive a pessimistic estimator for our derandomization. We will use for this purpose part of the proof of Chernoff bound from Raghavan [21]. From the above argument we have

$$
\Pr[X^\hat{S} < (1 - \epsilon') \cdot \rho_k \ell] \leq 1/\exp((\epsilon')^2 \rho_k \ell/2) = 1/n((\epsilon')^2 \exp(k/2),
$$

where $\ell = c \log(n^k)$. Recall that $X^\hat{S} = X_1^\hat{S} + \ldots + X_s^\hat{S}$. In our derandomization, we will have the fixed Bernoulli r.v.’s $X_1^\hat{S}, \ldots, X_s^\hat{S}$ for the first $s$ fixed permutations $\pi_1, \ldots, \pi_s$, where for each $j \in [s]$ we have $X_j^\hat{S} = 1$ with probability 1, when permutation $\pi_j$ is successful for $\hat{S}$, and $X_j^\hat{S} = 0$ with probability 1, otherwise. For the fully random Bernoulli r.v.’s $X_{s+2}^\hat{S}, \ldots, X_n^\hat{S}$, for each $j \in \{s + 2, \ldots, n\}$ we have
We have also shown that \( \Pr[X^S_j = 1] = \rho_k \). And finally, we will also derive an expression (algorithm) for \( \Pr[X^S_{s+1} = 1] \) for the semi-random permutation \( \pi_{s+1} \).

### 7.3.1 Conditional probabilities and proof of Theorem 5

Let \( \hat{S} = \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_k\} \subseteq [n] \) be a \( k \)-tuple, i.e., an ordered \( k \)-element subset. Recall the process of generating a random permutation \( \pi_j \) by the index random variables \( X^1_j, X^2_j, \ldots, X^n_j \), which generate elements \( \pi_j(1), \pi_j(2), \ldots, \pi_j(n) \) sequentially, one-by-one, in this order.

We will define an algorithm to compute \( \Pr[X^S_{s+1} = 1 \mid \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)] \) for the semi-random permutation \( \pi_{s+1} \), by using an approach from the proof of Theorem 2. We will slightly abuse the notation and let for \( r = 0 \) to have that \( \Pr[X^S_{s+1} = 1 \mid \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)] = \Pr[X^S_{s+1} = 1] \). In this case, we will also show below how to compute \( \Pr[X^S_{s+1} = 1] \) when \( \pi_{s+1} \) is fully random.

**Proof of Theorem 5.** We will present the proof of Theorem 5. Let \( \hat{S} = \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_k\} \subseteq [n] \) be a \( k \)-tuple. Recall that an independently and u.a.r. chosen permutation \( \pi \in \Pi_n \) is successful for \( \hat{S} \) iff event \( \bigcup_{i=2}^k E_i \) holds, where

\[
E_i = \{ \pi \text{ covers } i \text{-tuple } \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{i-1}\} \},
\]

for \( i \in \{2, 3, \ldots, k\} \). By the argument in Theorem 2, \( \pi \) is successful with probability \( \rho_k \).

Recall the definitions of events \( E_i \), for \( i = 2, 3, \ldots, k \) from the proof of Theorem 2:

\[
E_i = A_i \cap B_i \cap C_i, \quad A_i = \{ \pi^{-1}(\hat{s}_i) \in \{1, 2, \ldots, m_0\} \}, \quad B_i = \bigcap_{j=1}^{i-1} \{ \pi^{-1}(\hat{s}_j) \in \{m_0 + 1, \ldots, n\} \}, \quad C_i = \{ \forall j = 2, 3, \ldots, i-1 : \pi^{-1}(\hat{s}_1) < \pi^{-1}(\hat{s}_j) \}.
\]

We have also shown that

\[
\Pr[E_i] = \Pr\left[ A_i \cap \bigcap_{j=1}^{i-1} \{ \pi^{-1}(\hat{s}_j) \in \{m_0 + 1, \ldots, n\} \} \cap C_i \right] = \frac{m_0}{n} \cdot \left( \prod_{j=1}^{i-1} \frac{n - m_0 - (j - 1)}{(n - j - 1)} \right) \cdot \frac{(i - 2)!}{(i - 1)!}.
\]

(8)

If \( r = 0 \) then \( \Pr[X^S_{s+1} = 1] = \sum_{i=2}^k \Pr[E_i] \), where each \( \Pr[E_i] \), for \( i \in \{2, 3, \ldots, k\} \), is computed by the above formula. Assume from now on that \( r \geq 1 \).

Suppose now that values \( \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r) \) have already been chosen for some \( r \in [n] \), i.e., they all are fixed and final, except that \( \pi_{s+1}(r) \) is fixed but not final. The algorithm will be based on an observation that the random process of generating the remaining values \( \pi_{s+1}(r+1), \pi_{s+1}(r+2), \ldots, \pi_{s+1}(n) \) can be viewed as choosing u.a.r. a random permutation of values in the set \([n] \setminus \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)\}\); so this random permutation has length \( n - r \).

To compute

\[
\Pr[X^S_{s+1} = 1 \mid \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)] = \sum_{i=2}^k \Pr[E_i \mid \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)],
\]
we proceed as follows. For simplicity, we will write below \( \Pr[E_i] \) instead of \( \Pr[E_i | \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)] \); we will use analogous convention for \( \Pr[A_i], \Pr[B_i] \) and \( \Pr[C_i] \). Below, we will only show how to compute probabilities \( \Pr[E_i] \), and to obtain \( \Pr[X^{s+1}_{s+1} = 1] \) one needs to compute \( \sum_{i=2}^6 \Pr[E_i] \).

**Algorithm 2: Conditional probabilities**

```plaintext
1 Function \( \text{Prob} (E_i, \hat{S}) : \\
2 \quad \text{if } \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{i-1}\} \cap \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(\min(r, m_0))\} \neq \emptyset \text{ then} \\
3 \quad \Pr[E_i] = 0 \\
4 \text{else} \\
5 \quad /* \text{Now } \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{i-1}\} \cap \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(\min(r, m_0))\} = \emptyset */ \\
6 \quad \text{if } r \leq m_0 \text{ then} \\
7 \quad \quad \Pr[E_i] = \left( \prod_{j=1}^{i-1} \frac{n-m_0-(j-1)}{(n-r)-(j-1)} \right) \cdot \frac{(i-2)!}{(i-1)!} \\
8 \quad \quad \text{if } \hat{s}_i \notin \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)\} \text{ then} \\
9 \quad \quad \quad \Pr[E_i] = \frac{m_0-n-r}{m_0} \cdot \left( \prod_{j=1}^{i-1} \frac{n-m_0-(j-1)}{(n-r)-(j-1)} \right) \cdot \frac{(i-2)!}{(i-1)!} \\
10 \quad \text{else} \\
11 \quad \quad /* \text{We have now } r > m_0 */ \\
12 \quad \quad \text{if } \hat{s}_i \notin \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(m_0)\} \text{ then} \\
13 \quad \quad \quad \Pr[E_i] = 0 \\
14 \quad \quad \text{else} \\
15 \quad \quad \quad /* \text{We have here } \hat{s}_i \in \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(m_0)\} */ \\
16 \quad \quad \quad \text{Let } T = \{\pi_{s+1}(m_0+1), \pi_{s+1}(m_0+2), \ldots, \pi_{s+1}(r)\}. \\
17 \quad \quad \quad \text{if } \hat{s}_i \in T \text{ and } |\{\hat{s}_2, \hat{s}_3, \ldots, \hat{s}_{i-1}\} \cap T| \geq 1 \text{ then} \\
18 \quad \quad \quad \quad \text{if } \exists \xi \in \{\hat{s}_2, \hat{s}_3, \ldots, \hat{s}_{i-1}\} \cap T : \pi_{s+1}^{-1}(\xi) < \pi_{s+1}^{-1}(\hat{s}_i) \text{ then} \\
19 \quad \quad \quad \quad \quad \Pr[E_i] = 0 \\
20 \quad \quad \quad \quad \text{if } \forall \xi \in \{\hat{s}_2, \hat{s}_3, \ldots, \hat{s}_{i-1}\} \cap T : \pi_{s+1}^{-1}(\xi) > \pi_{s+1}^{-1}(\hat{s}_i) \text{ then} \\
21 \quad \quad \quad \quad \quad \Pr[E_i] = 1 \\
22 \quad \quad \quad \text{if } \hat{s}_i \notin T \text{ then} \\
23 \quad \quad \quad \quad \text{if } \{\hat{s}_2, \hat{s}_3, \ldots, \hat{s}_{i-1}\} \cap T \neq \emptyset \text{ then} \\
24 \quad \quad \quad \quad \quad \Pr[E_i] = 0 \\
25 \quad \quad \quad \quad \text{if } \{\hat{s}_2, \hat{s}_3, \ldots, \hat{s}_{i-1}\} \cap T = \emptyset \text{ then} \\
26 \quad \quad \quad \quad \quad \Pr[E_i] = \frac{(i-2)!}{(i-1)!} \\
27 \quad \quad \quad \text{return } \Pr[E_i]
```

**Lemma 7** Algorithm \( \text{Prob}(E_i, \hat{S}) \) correctly computes \( \Pr[E_i] = \Pr[E_i | \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)] \) in time \( O(n \cdot \text{poly log}(n)) \).

**Proof.** In Case/step [2] of the algorithm at least one of the elements \( \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{i-1}\} \) is before position \( m_0 \) in permutation \( \pi_{s+1} \) which falsifies event \( B_i \). Therefore \( \Pr[B_i] = 0 \), implying \( \Pr[E_i] = 0 \). This means that
The algorithm considers the following cases: all elements from now on we can assume that
\[ \{s_1, s_2, \ldots, s_{i-1}\} \cap \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(\text{min}(r, m_0))\} = \emptyset. \quad (9) \]

To analyse Case/step 6 let us assume that \( r \leq m_0 \). If \( s_i \in \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)\} \) then element \( s_i \) is before position \( m_0 \) in \( \pi_{s+1} \), meaning that event \( A_i \) holds, thus \( \Pr[A_i] = 1 \). By assumption \( \mathcal{G} \) all elements \( \{s_1, s_2, \ldots, s_{i-1}\} \) are in the fully random part of permutation \( \pi_{s+1} \) of length \( n - r \), thus calculation \( \mathcal{B} \) of events \( B_i \) and \( C_i \) implies that
\[
\Pr[E_i] = \frac{m_0 - r}{n - r} \cdot \left( \prod_{j=1}^{i-1} \frac{n - m_0 - (j - 1)}{(n - r) - (j - 1)} \right) \cdot \frac{(i - 2)!}{(i - 1)!}.
\]
If \( s_i \notin \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)\} \) then element \( s_i \) is after last fixed position \( r \) in \( \pi_{s+1} \), so the only chance for event \( A_i \) to hold is if the position of \( s_i \) is chosen in the random part between positions \( r + 1 \) and \( m_0 \). The probability of this event is \( \Pr[A_i] = \frac{m_0 - r}{n - r} \), and therefore by \( \mathcal{B} \) we have
\[
\Pr[E_i] = \frac{m_0 - r}{n - r} \cdot \left( \prod_{j=1}^{i-1} \frac{n - m_0 - (j - 1)}{(n - r - 1) - (j - 1)} \right) \cdot \frac{(i - 2)!}{(i - 1)!}.
\]

Let us now assume for Case/step 12 that \( r > m_0 \). If \( s_i \notin \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(m_0)\} \) then event \( A_i \) is false, which implies that \( \Pr[A_i] = 0 \), and consequently \( \Pr[E_i] = 0 \). Suppose otherwise that \( s_i \in \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(m_0)\} \), which makes event \( A_i \) to hold and so \( \Pr[A_i] = 1 \). Under this assumption, the algorithm considers the following cases:

- Let us assume first that \( s_1 \in \{\pi_{s+1}(m_0 + 1), \pi_{s+1}(m_0 + 2), \ldots, \pi_{s+1}(r)\} \) and \(|\{s_2, s_3, \ldots, s_{i-1}\} \cap \{\pi_{s+1}(m_0 + 1), \pi_{s+1}(m_0 + 2), \ldots, \pi_{s+1}(r)\}| = r' \) for some \( r' \in \mathbb{N}_{\geq 1} \), then
  - If there is \( \kappa \in \{s_2, s_3, \ldots, s_{i-1}\} \cap \{\pi_{s+1}(m_0 + 1), \pi_{s+1}(m_0 + 2), \ldots, \pi_{s+1}(r)\} \) such that \( \kappa \) is before \( s_1 \) in \( \pi_{s+1} \), i.e., \( \pi_{s+1}^{-1}(\kappa) < \pi_{s+1}^{-1}(s_1) \), then this falsifies event \( C_i \), thus \( \Pr[C_i] = 0 \), which implies that \( \Pr[E_i] = 0 \).
  - Let now for any \( \kappa \in T = \{s_2, s_3, \ldots, s_{i-1}\} \cap \{\pi_{s+1}(m_0 + 1), \pi_{s+1}(m_0 + 2), \ldots, \pi_{s+1}(r)\} \) we have that \( \pi_{s+1}^{-1}(\kappa) > \pi_{s+1}^{-1}(s_1) \). This means that all elements from set \( T \) are after element \( s_1 \) in permutation \( \pi_{s+1} \). In case if \( T \neq \{s_2, s_3, \ldots, s_{i-1}\} \), let us denote \( T' = \{s_2, s_3, \ldots, s_{i-1}\} \) \( \setminus \) \( T \). By assumption \( \mathcal{G} \) all elements from set \( \{s_1, s_2, s_3, \ldots, s_{i-1}\} \) are after position \( m_0 \) in permutation \( \pi_{s+1} \), implying that \( \Pr[B_i] = 1 \). This also means that when finally all positions in permutation \( \pi_{s+1} \) will be fixed, elements from set \( T' \) will certainly be put after position \( r \). We also have that element \( s_1 \) is before position \( r \) in \( \pi_{s+1} \). This reasoning shows that all elements from set \( T \) are after element \( s_1 \) in permutation \( \pi_{s+1} \), implying that \( \Pr[C_i] = 1 \). It also implies that all elements from and also \( \Pr[C_i] = 1 \). And because we also work under the assumption that \( \Pr[A_i] = 1 \), we finally have that \( \Pr[E_i] = 1 \).

- Suppose now that \( s_1 \notin \{\pi_{s+1}(m_0 + 1), \pi_{s+1}(m_0 + 2), \ldots, \pi_{s+1}(r)\} \). This by assumption \( \mathcal{G} \) implies that element \( s_1 \) is after position \( r \) in \( \pi_{s+1} \).
  - If \( \{s_2, s_3, \ldots, s_{i-1}\} \cap \{\pi_{s+1}(m_0 + 1), \pi_{s+1}(m_0 + 2), \ldots, \pi_{s+1}(r)\} = \emptyset \) then there exists at least one element from set \( \{s_2, s_3, \ldots, s_{i-1}\} \) that is before position \( r \), that is, before element \( s_1 \) in \( \pi_{s+1} \), which falsifies event \( C_i \), so \( \Pr[C_i] = 0 \). This implies that \( \Pr[E_i] = 0 \).
Finally, in Case/step 26 we assume that
\[ \{\hat{s}_2, \hat{s}_3, \ldots, \hat{s}_{i-1}\} \cap \{\pi_{s+1}(m_0 + 1), \pi_{s+1}(m_0 + 2), \ldots, \pi_{s+1}(r)\} = \emptyset. \]
This, by assumption 3 means that all elements \( \{\hat{s}_1, \hat{s}_2, \hat{s}_3, \ldots, \hat{s}_{i-1}\} \) are in the fully random part \( \{\pi_{s+1}(r + 1), \pi_{s+1}(r + 2), \ldots, \pi_{s+1}(n)\} \) of permutation \( \pi_{s+1} \). Also, recall that we work under the assumption that \( \Pr[A_i] = 1 \), and by a similar reasoning as above we have that \( \Pr[B_i] = 1 \). Therefore, we have
\[ \Pr[E_i] = \Pr[C_i] = \frac{(i-2)!}{(i-1)!}. \]
Note that the case in Step 26 is only possible if \( |\{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{i-1}\}| = i - 1 \leq n - r \). If \( i - 1 > n - r \) then some of the elements from \( \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{i-1}\} \) must belong to \( \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)\} \), but such case have already been considered as one of the previous cases in the algorithm.

We now argue about the implementation of the algorithm. The main kind of operations are operations on subsets of set \([n]\), which are set membership and set intersections, which can easily be implemented in time \( O(n) \). The other kind of operations in computing \( \Pr[E_i] \) are divisions of numbers from the set \([n]\) and multiplications of the resulting rational expressions. Observe that, in particular, when computing \( \frac{(i-2)!}{(i-1)!} \) we simply use \( \frac{(i-2)!}{(i-1)!} = \frac{1}{i-1} \). Clearly, each of these arithmetic operations can be performed in time \( O(poly \log(n)) \).

The proof of the above lemma finishes the proof of Theorem 5.

### 7.3.2 Pessimistic estimator

Let \( \hat{S} = \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_k\} \subseteq [n] \) be a \( k \)-tuple, i.e., an ordered \( k \)-element subset. Recall that \( X_{\hat{S}} = X_{\hat{1}} + \ldots + X_{\hat{k}} \). Denote also \( \mathbb{E}[X_{\hat{S}}] = \Pr[X_{\hat{S}} = 1] = \mu_j \) for each \( j \in [\ell] \), and \( \mathbb{E}[X_{\hat{S}}] = \sum_{j=1}^{\ell} \mu_j = \mu \). We will now use Raghavan’s proof of the Chernoff bound, see [28], for any \( \varepsilon' > 0 \):
\[
\Pr \left[ X_{\hat{S}} < (1 - \varepsilon') \cdot \mu \right] = \Pr \left[ \prod_{j=1}^{\ell} \frac{(1 - \varepsilon') X_{\hat{j}}}{(1 - \varepsilon') \mu_j} \geq 1 \right]
\leq \mathbb{E} \left[ \prod_{j=1}^{\ell} \frac{1 - \varepsilon' \cdot X_{\hat{j}}}{(1 - \varepsilon') \mu_j} \right]
= \prod_{j=1}^{\ell} \frac{1 - \varepsilon' \cdot \mathbb{E}[X_{\hat{j}}]}{(1 - \varepsilon') \mu_j}
< \prod_{j=1}^{\ell} \frac{\exp(-\varepsilon' \mu_j)}{(1 - \varepsilon') \mu_j}
= \frac{1}{\exp(b(-\varepsilon') \mu)},
\]
where \( b(x) = (1 + x) \ln(1 + x) - x \), and the second step uses Bernoulli’s inequality \((1 + x)^r \leq 1 + rx\), that holds for \( 0 \leq r \leq 1 \) and \( x \geq -1 \), and Markov’s inequality, and the last inequality uses \( 1 - x \leq \exp(-x) \), which holds for \( x \geq 0 \) and is strict if \( x \neq 0 \).
By Theorem 1, \( f(n, \frac{\mu}{n}) = OPT_n = \frac{1}{\varepsilon} + \Theta \left( \frac{1}{n} \right)^{3/2} \), where \( c_0 = \frac{1}{2} - 1/(2c) \). By (7) in the proof of Theorem 2 and by Lemma 1 we obtain that \( \mu_j \geq \rho_k \geq \frac{1}{\varepsilon} - \Theta(1/k) \), for each \( j \in [\ell] \).

Then we can further upper bound the last line of Raghavan’s proof to obtain \( \Pr \left[ X^\hat{S} < (1 - \varepsilon') \cdot \ell \cdot \rho_k \right] \).

Now, repeating the Raghavan’s proof with each \( L \) and by the union bound we obtain that:

\[
\Pr \left[ X^\hat{S} < (1 - \varepsilon') \cdot \ell \cdot \rho_k \right] \leq \prod_{j=1}^{\ell} \frac{1 - \varepsilon' \cdot E[X^\hat{S}]}{(1 - \varepsilon')(1 - \varepsilon')\rho_k} < \prod_{j=1}^{\ell} \frac{\exp(-\varepsilon' \mu_j)}{(1 - \varepsilon')(1 - \varepsilon')\rho_k} \leq \prod_{j=1}^{\ell} \frac{\exp(-\varepsilon' \rho_k)}{(1 - \varepsilon')(1 - \varepsilon')\rho_k} = \frac{1}{\exp(b(-\varepsilon')\ell\rho_k)} < \frac{1}{\exp((\varepsilon')^2\ell\rho_k/2)},
\]

where the last inequality follows by a well known fact that \( b(-x) > x^2/2 \), see, e.g., 28. By this argument and by the union bound we obtain that:

\[
\Pr \left[ \exists S \in \mathcal{K} : X^\hat{S} < (1 - \varepsilon') \cdot \ell \cdot \rho_k \right] \leq \sum \prod_{j=1}^{\ell} \frac{1 - \varepsilon' \cdot E[X^\hat{S}]}{(1 - \varepsilon')(1 - \varepsilon')\rho_k},
\]

where \( \mathcal{K} \) is the set of all ordered \( k \)-element subsets of the set \([n]\).

Let us define a function \( \phi_j(\hat{S}) \) which is equal to 1 if permutation \( \pi_j \) is successful for the \( k \)-tuple \( \hat{S} \), and it is equal to 0 otherwise. The above proof upper bounds the probability of failure by the expected value of

\[
\sum \prod_{j=1}^{\ell} \frac{1 - \varepsilon' \cdot \phi_j(\hat{S})}{(1 - \varepsilon')(1 - \varepsilon')\rho_k},
\]

the expectation of which is less than \( |\mathcal{K}| / \exp((\varepsilon')^2\ell\rho_k/2) \), which is strictly smaller than 1 for appropriately large \( \ell \).

Suppose that we have so far chosen the (fixed) permutations \( \pi_1, \ldots, \pi_s \) for some \( s \in \{1, 2, \ldots, \ell - 1\} \), the (semi-random) permutation \( \pi_{s+1} \) is currently being chosen, and the remaining (fully random) permutations, if any, are \( \pi_{s+2}, \ldots, \pi_\ell \). The conditional expectation is then

\[
\sum_{S \in \mathcal{K}} \left( \prod_{j=1}^{s} \frac{1 - \varepsilon' \cdot \phi_j(\hat{S})}{(1 - \varepsilon')(1 - \varepsilon')\rho_k} \right) \cdot \left( 1 - \varepsilon' \cdot E[\phi_{s+1}(\hat{S})] \right) \cdot \left( 1 - \varepsilon' \cdot \rho \right) \left( 1 - \varepsilon' \cdot (1 - \varepsilon')\rho_k \right)^{\ell-s-1} = \Phi(\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)),
\]

where

\[
\Phi(\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r)) \leq \sum_{S \in \mathcal{K}} \left( \prod_{j=1}^{s} \frac{1 - \varepsilon' \cdot \phi_j(\hat{S})}{(1 - \varepsilon')(1 - \varepsilon')\rho_k} \right) \cdot \left( 1 - \varepsilon' \cdot E[\phi_{s+1}(\hat{S})] \right) \cdot \left( 1 - \varepsilon' \cdot \rho \right) \left( 1 - \varepsilon' \cdot (1 - \varepsilon')\rho_k \right)^{\ell-s-1}.
\]
Recall that the value of \( \pi \) and final, we simply choose the value \( \pi \)
derandomization. this last expression, which is equivalent to maximizing

\[ \text{Proof. (of Lemma 3)} \] This follows from the following

\( (a) \) and \( (b) \) follow easily by the above arguments and by the assumption about

\( \pi \) the partially fixed semi-random permutation

\( \text{Part (c) follows because } \Phi \) is an expected value conditioned on the choices made so far. For the precise

argument let us observe that

\[ \mathbb{E}[\phi_j(\hat{S})] \geq \rho_k. \] Note, that

\[ \mathbb{E}[\phi_{s+1}(\hat{S})] = \mathbb{E}[\phi_{s+1}(\hat{S}) | \pi_{s+1}(r) = \tau] \]

\[ = \mathbb{P}[X_{s+1}^\mathbf{S} = 1 | \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1), \pi_{s+1}(r) = \tau], \]

where positions \( \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r) \) have already been fixed in the semi-random permutation

\( \pi_{s+1}, \pi_{s+1}(r) \) has been fixed in particular to \( \tau \in [n] \setminus \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1)\} \), and this value

can be computed by using the algorithm from Theorem 5. This gives the pessimistic estimator \( \Phi \) for our
derandomization.

Because \( s \) is fixed for all steps where the semi-random permutation is being decided, this pessimistic

estimator is uniformly proportional to

\[ \sum_{\tilde{S} \in K} \left( \prod_{j=1}^s \left( 1 - \varepsilon' \cdot \phi_j(\hat{S}) \right) \right) \cdot \left( 1 - \varepsilon' \cdot \mathbb{E}[\phi_{s+1}(\hat{S})] \right). \]

Recall that the value of \( \pi_{s+1}(r) \) in the semi-random permutation was fixed but not final. To make it fixed

and final, we simply choose the value \( \pi_{s+1}(r) \in [n] \setminus \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1)\} \) that minimizes

this last expression, which is equivalent to maximizing

\[ \sum_{\tilde{S} \in K} \left( \prod_{j=1}^s \left( 1 - \varepsilon' \cdot \phi_j(\hat{S}) \right) \right) \cdot \mathbb{E}[\phi_{s+1}(\hat{S})]. \] (14)

**Proof.** This follows from the following 3 properties: (a) it is an upper bound on the conditional

probability of failure; (b) it is initially strictly less than 1; (c) some new value of the next index variable in

the partially fixed semi-random permutation \( \pi_{s+1} \) can always be chosen without increasing it.

Property (a) follows from (10) and (12). To prove (b) we see by (11) and (12) that

\[ \mathbb{P}(\exists \tilde{S} \in K : X^S < (1 - \varepsilon') \cdot \ell \rho_k) \leq |K|/ \exp((\varepsilon')^2 \ell \rho_k/2). \]

Observe further that \( |K| = \binom{n}{k} k! = \frac{n!}{(n-k)!} k! < n^k \), for \( 2 \leq k \leq n \). Therefore we obtain the following

condition on \( \ell \)

\[ \frac{n^k}{\exp((\varepsilon')^2 \ell \rho_k/2)} \leq 1 \iff \ell \geq \frac{2k \ln(n)}{\rho_k (\varepsilon')^2}. \]

(a) and (b) follow easily by the above arguments and by the assumption about \( \ell \).

Part (c) follows because \( \Phi \) is an expected value conditioned on the choices made so far. For the precise

argument let us observe that

\[ \mathbb{P}(X_{s+1}^\mathbf{S} = 1 | \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1)) \]

\[ = \sum_{\pi \in T} \frac{1}{n - r + 1} \cdot \mathbb{P}(X_{s+1}^\mathbf{S} = 1 | \pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1), \pi_{s+1}(r) = \tau], \]

where \( T = [n] \setminus \{\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1)\} \). Then by (13) we obtain

\[ \Phi(\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1)) \]

\[ = \sum_{\pi \in T} \frac{1}{n - r + 1} \cdot \Phi(\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1), \pi(r) = \tau) \]

\[ \geq \min\{\Phi(\pi_{s+1}(1), \pi_{s+1}(2), \ldots, \pi_{s+1}(r - 1), \pi(r) = \tau) : \tau \in T\}, \]

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which implies part (c).

Proof. (of Theorem 4) The computation of the conditional probabilities \( \text{Prob}(E_i, \hat{S}) \) by Algorithm 2 is correct by Theorem 5. Algorithm 1 is a direct translation of the optimization of the pessimistic estimator \( \Phi \). In particular, observe that the correctness of the weight initialization in Line 6 of Algorithm 1 and of weight updates in Line 16 follow from the form of the pessimistic estimator objective function in (14).

The value of the pessimistic estimator \( \Phi \) is strictly smaller than 1 at the beginning and in each step, it is not increased by properties of the pessimistic estimator (Lemma 3). Moreover, at the last step all values of all \( \ell \) permutations will be fixed, that is, there will be no randomness in the computation of \( \Phi \). Observe that \( \Phi \) is an upper bound on the expected number of not well-covered \( k \)-tuples from \( \mathcal{K} \). So at the end of the derandomization process the number of such \( k \)-tuples will be 0, implying that all these \( k \)-tuples will be well-covered, as desired.

A straightforward analysis of the running time of Algorithm 1 and Lemma 7 imply that its running time can be bounded by \( O(k \cdot \ell \cdot n^{k+2} \cdot \text{poly} \log(n)) \). \( \square \)

8 Proofs from Section 4

Proof. (of Lemma 4) Let us take any finite field \( \mathbb{F} \) of size \( q \geq 2 \). It is known that \( q \) must be of the following form: \( q = p^r \), where \( p \) is any prime number and \( r \geq 1 \) is any integer; this has been proved by Galois, see [26 Chapter 19]. We will do our construction assuming that \( \mathbb{F} = \mathbb{F}_q \) is the Galois field, where \( q \) is a prime number.

Let us take the prime \( q \) and the integer \( d \geq 1 \) such that \( q^{d+1} \geq n \). We want to take here the smallest such prime number and an appropriate smallest \( d \) such that \( q^{d+1} \geq n \).

Let us now consider the ring \( \mathbb{F}[x] \) of univariate polynomials over the field \( \mathbb{F} \) of degree \( d \). The number of such polynomials is exactly \( |\mathbb{F}[x]| = q^{d+1} \). By the field \( \mathbb{F}_q \) we chose, we have that \( \mathbb{F}_q = \{0, 1, \ldots, q-1\} \). We will now define the following \( q^{d+1} \times q \) matrix \( M = (M_i,g_i')_{i \in [q^{d+1}],g_i' \in \{0,1,\ldots,q-1\}} \) whose rows correspond to polynomials from \( \mathbb{F}[x] \) and columns – to elements of the field \( \mathbb{F}_q \).

Let now \( \mathbb{G} \subset \mathbb{F}[x] \) be the set of all polynomials from \( \mathbb{F}[x] \) with the free term equal to 0, that is, all polynomials of the form \( \sum_{i=1}^d a_ix^i \in \mathbb{F}[x] \), where all coefficients \( a_i \in \mathbb{F}_q \), listed in any fixed order: \( \mathbb{G} = \{g_1(x), g_2(x), \ldots, g_q(x)\} \). To define matrix \( M \) we will list all polynomials from \( \mathbb{F}[x] \) in the following order \( \mathbb{F}[x] = \{f_1(x), f_2(x), \ldots, f_{q^{d+1}}(x)\} \), defined as follows. The first \( q \) polynomials \( f_1(x), f_2(x), \ldots, f_{q}(x) \) are \( f_i(x) = g_i(x) + i - 1 \) for \( i \in \{1, \ldots, q\} \); note that here \( i - 1 \in \mathbb{F}_q \). The next \( q \) polynomials \( f_{q+1}(x), f_{q+2}(x), \ldots, f_{2q}(x) \) are \( f_{q+i}(x) = g_{q+i}(x) + i - 1 \) for \( i \in \{1, \ldots, q\} \), and so on. In general, to define polynomials \( f_{q(j+1)}(x), f_{q(j+2)}(x), \ldots, f_{qj+q}(x) \), we have \( f_{q(j+1)}(x) = g_{q(j+1)}(x) + i - 1 \) for \( i \in \{1, \ldots, q\} \), for any \( j \in \{0, 1, \ldots, q^d - 1\} \).

We are now ready to define matrix \( M \): \( M_i,g_i' = f_i(q') \) for any \( i \in [q^{d+1}], q' \in \{0, 1, \ldots, q-1\} \). From matrix \( M \) we define the set of functions \( \mathcal{F} \) by taking precisely \( n \) first rows of matrix \( M \) (recall that \( q^{d+1} \geq n \)) and letting the columns of this truncated matrix define functions in the set \( \mathcal{F} \). More formally, \( \mathcal{F} = \{h_{q'} : q' \in \{0, 1, \ldots, q-1\}\} \), where each function \( h_{q'} : [n] \rightarrow [q] \) for each \( q' \in \{0, 1, \ldots, q-1\} \) is defined as \( h_{q'}(i) = f_i(q') \) for \( i \in \{1, 2, \ldots, n\} \).

We will now prove that \( |h_{q'}^{-1}(q'')| \in \left\{ \left\lfloor \frac{n}{q} \right\rfloor, \left\lfloor \frac{n}{q} \right\rfloor + 1 \right\} \) for each function \( h_{q'} \in \mathcal{F} \) and for each \( q'' \in \{0, 1, \ldots, q-1\} \). Let us focus on column \( q' \) of matrix \( M \). Intuitively the property that we want to prove follows from the fact that when this column is partitioned into \( q^{d+1}/q \) “blocks” of \( q \) consecutive elements, each such block is a permutation of the set \( \{0, 1, \ldots, q-1\} \) of elements from the field \( \mathbb{F}_q \). More formally, the \( j \)th such “block” for \( j \in \{0, 1, \ldots, q^d - 1\} \) contains the elements \( f_{q(j+i)}(q') \) for all \( i \in \{1, 2, \ldots, n\} \). But
by our construction we have that \( f_{aq+i}(q') = g_{aq+i}(q') + i - 1 \) for \( i \in \{1, \ldots, q\} \). Here, \( g_{aq+i}(q') \in \mathbb{F}_q \) is a fixed element from the Galois field \( \mathbb{F}_q \) and elements \( f_{aq+i}(q') \) for \( i \in \{1, \ldots, q\} \) of the “block” are obtained by adding all other elements \( i - 1 \) from the field \( \mathbb{F}_q \) to \( g_{aq+i}(q') \in \mathbb{F}_q \). This, by properties of the field \( \mathbb{F}_q \) imply that \( f_{aq+i}(q') \) for \( i \in \{1, \ldots, q\} \) are a permutation of the set \( \{0, 1, \ldots, q-1\} \).

Claim. For any given \( j \in \mathbb{F}_q = \{0, 1, \ldots, q-1\} \) the values \( j + i \), for \( i \in \{0, 1, \ldots, q - 1\} \), where the addition is in the field \( \mathbb{F}_q \) modulo \( q \), are a permutation of the set \( \{0, 1, \ldots, q-1\} \), that is, \( \{j + i : i \in \{0, 1, \ldots, q - 1\}\} = \{0, 1, \ldots, q-1\} \).

Proof. In this proof we assume that addition and substraction are in the field \( \mathbb{F}_q \). The multiset \( \{j + i : i \in \{0, 1, \ldots, q - 1\}\} \subseteq \mathbb{F}_q \) consists of \( q \) values, thus it suffices to show that all values from the multiset are distinct. Assume contrary the there exists two different elements \( i, i' \in \mathbb{F}_q \) such that \( j + i = j + i' \). It follows that \( i' - i = 0 \). This cannot be true since \( |i'|, |i| < q \) and \( i' \) and \( i \) are different.

The property that \( |h_q^{-1}(q''')| \in \left\{ \left[ \frac{n}{q} \right], \left[ \frac{n}{q} \right] + 1 \right\} \) now follows from the fact that in the definition of the function \( h_q \) all the initial “blocks” \( \{f_{aq+i}(q') : i \in \{1, \ldots, q\}\} \) for \( j \in \{0, 1, \ldots, \left[ \frac{n}{q} \right] - 1\} \) are fully used, and the last “block” \( \{f_{aq+i}(q') : i \in \{1, \ldots, q\}\} \) for \( j = \left[ \frac{n}{q} \right] \) is only partially used.

Finally, we will prove now that \( |\{f \in \mathbb{F} : f(i) = f(j)\}| \leq d \). This simply follows form the fact that for any two polynomials \( g, h \in \mathbb{F}[x] \), they can assume the same values on at most \( d \), their degree, number of elements from the field \( \mathbb{F}_q = \{0, 1, \ldots, q-1\} \). This last property is true because the polynomial \( g(x) - h(x) \) has degree \( d \) and therefore it has at most \( d \) zeros in the field \( \mathbb{F}[x] \).

Let us finally observe that the total number of polynomials, \( q^d + 1 \), in the field \( \mathbb{F}[x] \) can be exponential in \( n \). However, this construction can easily be implemented in polynomial time in \( n, q, d \), because we only need the initial \( n \) of these polynomials. Thus we can simply disregard the remaining \( q^d+1 - n \) polynomials.

8.1 Complete description of product of two Reed-Solomon codes.

In the following we show that Reed-Solomon codes composed twice can produce a set of dimensionality-reduction functions with parameters \((n, (\log \log n)^\epsilon, (\log \log n)^\epsilon/2)\) for any \( \epsilon > 0 \).

Assume we are given an integer \( n \). Let \( \ell_2 \) be a prime number and \( d_2 = \lceil \sqrt{\ell_2} \rceil \). Choose \( \ell_1 \) to be a prime number in the interval \( \left[ \frac{1}{2} \ell_2, \ell_2 \right] \) and \( d_1 = \lceil \sqrt{\ell_1} \rceil \). The number \( \ell_1 \) exists due to the distribution of prime numbers. Additionally, the choice of numbers \( \ell_1 \) and \( \ell_2 \) must be such that

a) \( \ell_1 \cdot d_1 \geq n \), and b) \( \ell_2 \cdot d_2 = O(poly(n)) \)

If those two conditions are satisfied, Lemma \( \text{[4]} \) ensures we can construct a set \( \mathcal{F}_1 \) of functions \( f : [n] \to [\ell_1] \) with parameters \( n, q := \ell_1, d := d_1 \) in time \( O(poly(n)) \). Let \( \mathcal{F}_2 \) be another set of functions: \( f : [\ell_1] \to [\ell_2] \) specified by Lemma \( \text{[4]} \) with parameters \( n, q := \ell_2, d := d_2 \). The set \( \mathcal{F}_2 \) can also be constructed in polynomial time in \( n \).

We compose a set \( \mathcal{F} \) of functions \( f : [n] \to [\ell_2] \) from sets \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) in the following way: \( \mathcal{F} = \{f_2 \circ f_1 : f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2\} \). Observe, that \( |\mathcal{F}| = |\mathcal{F}_1| \cdot |\mathcal{F}_2| = \ell_1 \ell_2 \). Next, we show that properties obtained from Lemma \( \text{[4]} \) for sets \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) lift to the set \( \mathcal{F} \).
Lemma 8  For any two distinct numbers $i, j \in [n]$ we have:

$$|\{f \in \mathcal{F} : f(i) = f(j)\}| \leq \ell_2 d_1 + \ell_1 d_2$$

**Proof.** Take two distinct $i, j \in [n]$. Consider a function $f \in \mathcal{F}$. From the construction of $\mathcal{F}$ we know that $f = f_2 \circ f_1$ for some $f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2$. Now, if $f(i) = f(j)$ then $f_2(f_1(i)) = f_2(f_1(j))$, which means that either $f_1(i) = f_1(j)$ or $f_2(i') = f_2(j')$, where $i' = f_1(i), j' = f_1(j)$ and $i' \neq j'$. For the fixed pair of indices $i, j$ the number of functions $f_1 \in \mathcal{F}_1$ such that $f_1(i) = f_1(j)$ is at most $d_1$, therefore the first case can happen at most $|\mathcal{F}_2|d_1 = \ell_2 d_1$ times. Similarly, the second case can happen at most $|\mathcal{F}_1|d_2 = \ell_1 d_2$ times. The sum of these two bounds gives us the desired estimation. $\square$

Lemma 9  For any function $f \in \mathcal{F}$ we have:

$$\forall \ell' \in [\ell_2] : |f^{-1}(\ell')| \leq \left\lceil \frac{n}{\ell_2} \right\rceil + 3 \left\lceil \frac{n}{\ell_1} \right\rceil$$

**Proof.** Let us fix an integer $\ell' \in [\ell_2]$ and a function $f \in \mathcal{F}$. Observe, that the function $f$ has a unique decomposition $f = f_2 \circ f_1, f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2$, thus $|f^{-1}(\ell')| = |(f_2 \circ f_1)^{-1}(\ell')|$. From Lemma 8 we have that the set $f_2^{-1}(\ell')$ has either $\left\lceil \frac{\ell}{\ell_2} \right\rceil$ or $\left\lceil \frac{\ell}{\ell_2} \right\rceil + 1$ elements. Similarly, for fixed $\ell'' \in [\ell_1]$ the set $f_1^{-1}(\ell'')$ has either $\left\lceil \frac{n}{\ell_1} \right\rceil$ or $\left\lceil \frac{n}{\ell_1} \right\rceil + 1$ elements. These two bounds combined give us

$$|(f_2 \circ f_1)^{-1}(\ell')| \in \left[ \left\lceil \frac{n}{\ell_2} \right\rceil, \left\lceil \frac{n}{\ell_1} \right\rceil \right] + \left[ \left\lceil \frac{\ell}{\ell_2} \right\rceil + \left\lceil \frac{\ell}{\ell_2} \right\rceil + 1 \right]$$

$$|(f_2 \circ f_1)^{-1}(\ell')| \leq \left\lceil \frac{n}{\ell_2} \right\rceil + \left\lceil \frac{n}{\ell_1} \right\rceil + \left\lceil \frac{\ell}{\ell_2} \right\rceil + \left\lceil \frac{\ell}{\ell_2} \right\rceil + 1$$

where the last implication follows from the fact that $\left\lceil \frac{n}{\ell_2} \right\rceil + \left\lceil \frac{\ell}{\ell_2} \right\rceil + 1 = \left\lceil \frac{\ell}{\ell_2} \right\rceil + \left\lceil \frac{n}{\ell_1} \right\rceil + 1$. $\square$

Let us restate Corollary 3.

**Corollary 4** For any $\epsilon > 0$ and $q \geq (\log \log n)^{t}$ there exists a dimensionality-reduction set of functions with parameters $(n, q, \sqrt{q})$. Moreover, such set has size $q^{\sqrt{q}}$ and can be computed in polynomial time in $q^{\sqrt{q}}$.

**Proof.** Consider the above construction for parameters $\ell_2 := q$ and $\ell_1 := q^{\sqrt{q}}$. We can easily check that these parameters satisfy the conditions a) and b) of the above constructions. Sets $\mathcal{F}_1$ and $\mathcal{F}_2$ can be computed in time $O(q^{\sqrt{q}})$ and $O(q)$, respectively, due to Lemma 9. The correctness follows from Lemmas 8 and 9. $\square$

9  Proofs from Section 5

**Proof.** (of Theorem 6) For a given function $f \in \mathcal{F}$ and a permutation $\pi \in \mathcal{L}$ we denote by $\pi \circ f : [n] \to [n]$ any permutation $\sigma$ over set $[n]$ satisfying the following: $\forall i,j \in [n], i \neq j$ if $ind_\sigma(f(i)) < ind_\sigma(f(j))$ then $ind_\sigma(i) < ind_\sigma(j)$. The aforementioned formal definition has the following natural explanation. The function $f \in \mathcal{F}, f : [n] \to \ell$ may be interpreted as an assignment of each element from set $[n]$ to one of $\ell$ blocks. Next, permutation $\pi \in \mathcal{L}$ determines the order of those blocks. So, the final permutation is obtained
by listing the elements from the blocks in the order given by $\pi$. The order of elements inside the blocks is irrelevant.

The set $\mathcal{L}'$ of permutations over $[n]$ is defined as $\mathcal{L}' = \{ \pi \circ f : \pi \in \mathcal{L}, f \in \mathcal{F} \}$, and its size is $|\mathcal{L}'| = |\mathcal{L}| \cdot |\mathcal{F}|$. It is easy to observe that $\mathcal{L}'$ can be computed in $O(|\mathcal{F}| \cdot |\mathcal{L}|)$ time.

Let us set an adversarial order (permutation) of elements $\pi$. We will show that with probability $(1 - \frac{k^2d}{\ell})\rho_k$ a successful permutation from $\mathcal{L}'$ for the $k$-tuple $\hat{S} = \{ \pi_1(1), \pi_1(2), \ldots, \pi_1(k) \}$ is picked. This will conclude the theorem.

Observe, that the random experiment of choosing $\pi \circ f \in \mathcal{L}'$ can be seen as choosing random $f \in \mathcal{F}$ and random $\pi \in \mathcal{L}$ independently. That is true, because $\mathcal{L}'$ consists of compositions of each function from $\mathcal{F}$ with each permutation from $\mathcal{L}$.

Denote $f(\hat{S}) = \{ f(\pi_1(1)), \ldots, f(\pi_1(k)) \}$ a random variable being an image of the $k$-tuple under random function $f \in \mathcal{F}$. Assumed that $\mathcal{F}$ is a dimensionality-reduction set of functions with parameters $(n, \ell, d)$ we have that for any two indices $i, j \in [k], i \neq j$ the probability that $f(\pi_1(i)) = f(\pi_1(j))$ is at most $\frac{d}{\ell}$. By the union bound argument we conclude that the probability that for all $i, j \in [k], i \neq j$ it holds $f(\pi_1(i)) \neq f(\pi_1(j))$ is at least $1 - \frac{k^2d}{\ell}$.

Assume now, that the $k$-tuple $f(\hat{S})$ consists of pair-wise different elements. Quite naturally, we will show, that if $\pi \in \mathcal{L}$ is successful for $f(\hat{S})$, then the classic algorithm picks the highest element executed on permutation $\pi \circ f$. This will prove the claimed result since the probability that a uniform random permutation $\pi \in \mathcal{L}$ is successful for any $k$-tuple is at least $\rho_k$.

If a permutation $\pi \in \mathcal{L}$ is successful for $f(\hat{S}) = \{ f(\pi_1(1)), \ldots, \pi_1(k) \}$, then the following properties hold for some constant $c$ and some integer $k' \in \{2, \ldots, k\}$:

$$\text{ind}_\pi(f(\pi_1(2))) < \lfloor \frac{\ell}{e} \rfloor - c,$$
$$\forall i \in [k'] \setminus \{2\} : \text{ind}_\pi(f(\pi_1(i))) > \lfloor \frac{\ell}{e} \rfloor + c,$$

and

$$\forall 3 \leq i \leq k' : \text{ind}_\pi(f(\pi_1(1))) < \text{ind}_\pi(f(\pi_1(i))).$$

In other words, the permutation $\pi$ covers some $k'$-tuple that is contained in $k$-tuple $f(\hat{S})$. Without loss of generality, we can assume that $k' = k$. Otherwise, we can repeat the same reasoning for a tuple of the smaller size $k'$.

Now, the permutation $\pi \circ f$ is any permutation $\sigma$ of $n$-elements such that if we have that $\text{ind}_\sigma(f(\pi_1(i))) < \text{ind}_\sigma(f(\pi_1(j)))$, then $\text{ind}_\sigma(\pi_1(i)) < \text{ind}_\sigma(\pi_1(j))$. From this and the assumption that $k$-tuple $f(\hat{S})$ consists of pair-wise different elements, we have that the order of elements of $k$-tuple $f(\hat{S})$ in $\pi$ is the same as the order of elements of $k$-tuple $\hat{S}$ in $\pi \circ f$. It has left to show, that element $\pi_1(2)$ is placed before the threshold $\lfloor n/e \rfloor$ in permutation $\pi \circ f$ and all other elements of $\hat{S}$ are placed after. However, the definition of dimensionality-reduction set guarantees that

$$\forall j \in [\ell] : |f^{-1}(j)| \leq \frac{n}{\ell} + o(\ell).$$

Since $\text{ind}_\sigma(f(\pi_1(2))) < \lfloor \frac{\ell}{e} \rfloor - c$ and the operation $\pi \circ f$ preserves the order of elements from $[n]$ that are in different blocks after $f$ is applied, therefore we see that if $\sigma$ is the $n$-element permutation $\pi \circ f$ then

$$\text{ind}_\sigma(\pi_1(2)) < (\lfloor \frac{\ell}{e} \rfloor - c) \left( \frac{n}{\ell} + o(\ell) \right) \leq \lfloor n/e \rfloor - \frac{cn}{\ell} + o(\ell^2/e).$$

Since we assumed that $\frac{n}{\ell} > \ell^2$, we conclude that

$$\text{ind}_\sigma(\pi_1(2)) < \lfloor n/e \rfloor.$$
The same reasoning shows that every other element than $\pi_1(2)$ of the $k$-tuple $\hat{S}$ is placed on position greater than $\lfloor n/e \rfloor$ in any permutation $\pi \circ f$. It is clear then, that the classic algorithm executed on permutation $\pi \circ f$ encounters element $\pi_1(2)$ before it passes the threshold. It stores this element as the maximum of elements before the threshold, and then picks element $\pi_1(1)$ as the first element greater than $\pi_1(2)$ after the threshold. Thus $\pi \circ f$ is successful for $\hat{S}$, which proves the theorem.

Proof. (of Theorem 7) Let us set $\ell = \frac{\log \log n}{\log \log \log n}$ and $k = \log \ell$. Consider a dimensionality-reduction set of functions $\mathcal{F}$ given by Corollary 4 with parameters $(n, \ell, \sqrt{\ell})$. Since $\frac{\log \log n}{\log \log \log n} > (\log n)^{1/2}$, the construction given in Corollary 3 is valid. Note also, that the size of set $\mathcal{F}$ is $O(\ell^{4/3}) = O(\log n)$. Let $\mathcal{L}$ be the set of all $\ell$ elements permutations. By Lemma 5, we see that every $k$-tuple has at least a $\rho_k = OPT_{\ell} - \frac{1}{\ell} \left(1 - \frac{1}{e}\right)^k$ fraction of successful permutations in set $\mathcal{L}$. From Stirling’s approximation we obtain that $\log(|\mathcal{L}|) = \log(\ell!) = O(\ell \log \ell) = O(\log(\log n))$, thus $|\mathcal{L}| = O(\log(\log n))$ and we can enumerate all permutations in $\mathcal{L}$ in time polynomial in $n$.

Finally, we use Theorem 6 to conjugate set $\mathcal{F}$ with set $\mathcal{L}$ to obtain a set of $n$-element permutations $\mathcal{L}'$ on which we define a uniform distribution $D_n$. Since we considered the uniform distribution on $\mathcal{L}'$, we obtain that the entropy of $D_n$ is $\log |\mathcal{L}'| = \log |\mathcal{F}| + \log |\mathcal{L}| = O(\log \log n)$. The probability of success of the classic secretarial algorithm is, according to Theorem 6, the following:

$$\left(1 - \frac{k^2}{\sqrt{\ell}}\right) \left(OPT_{\ell} - \frac{1}{k} \left(1 - \frac{1}{e}\right)^k\right).$$

Substituting $\ell = \frac{\log \log n}{\log \log \log n}$, $k = \log \ell$ and skipping some basic calculations we finally obtain that the probability can be lower bounded by

$$\frac{1}{e} - \frac{3(\log \log \log n)^{5/2}}{\sqrt{\log \log n}},$$

which proves the theorem.

Proof. (of Theorem 8) Let us set $\ell := \log^C n$, $k := C \log_{e^c/(e-1)} \log n$ and $\epsilon' := \frac{1}{\log^C n}$, for any constant $C > 0$. Consider a dimensionality-reduction set $\mathcal{F}$ constructed as in Corollary 2 with parameters $(n, \ell, \sqrt{\ell})$. Observe that set $\mathcal{F}$ has size $O(\ell) = O(\log^C n)$ and can be computed in time polynomial in $n$. Next, let $\mathcal{L}$ be a set of $\ell$-element permutations obtained from Theorem 4 with parameters $n := \ell$, $k$ and $\epsilon'$, such that least $(1 - \epsilon') \left(OPT_{\ell} - \frac{2}{k} \left(1 - \frac{1}{e}\right)^k\right)$ fraction of permutations from $\mathcal{L}$ are successful for every $k$-tuple. From Theorem 4 we have that $|\mathcal{L}| = O(\log^{4C} n)$ and $\mathcal{L}$ can be computed in polynomial time in $n$. The last follows because $k^\ell = O\left(e^{(\log \log n)^3}\right)$ is polynomial in $n$.

We define the final permutations distribution $D_n$ to be a uniform distribution over set of $n$ elements permutations set $\mathcal{L}'$ provided by Theorem 5 from the conjugation of set $\mathcal{F}$ and set $\mathcal{L}$. Clearly, size of $\mathcal{L}'$ is polynomial in $\log n$, thus the entropy of $D_n$ is $O(\log \log n)$. Also, Theorem 6 implies that the classic secretarial algorithm executed on $D_n$ achieves a probability of success of at least

$$\left(1 - \frac{k^2}{\sqrt{\ell}}\right) \left(1 - \epsilon'\right) \left(OPT_{\ell} - \frac{2}{k} \left(1 - \frac{1}{e}\right)^k\right).$$

From Theorem 1 we obtain that $OPT_{\ell} = \frac{1}{e} + \frac{c_0}{e} + O\left(\left(\frac{1}{e}\right)^{3/2}\right)$, where $c_0 = 1/2 - 1/(2e)$. Therefore, we
can calculate the success probability of the classic algorithm as follows:

\[
\left(1 - \frac{k^2}{\sqrt{p}}\right)\left(1 - \epsilon'\right) \left(\text{OPT}_t - \frac{2}{k} \left(1 - \frac{1}{e}\right)^k\right)
\]

\[
\geq \left(1 - \frac{(C_1 \log \log n)^2}{\log^{C/2} n}\right) \left(1 - \frac{1}{\log^C n}\right) \left(\frac{c_0}{e} + \frac{2}{(\log^C n) \cdot C_1 \log \log n} + o\left(\frac{1}{\log^C n}\right)\right)
\]

\[
\geq \frac{1}{e} - \frac{(C_1 \log \log n)^2}{\log^{C/2} n} + o\left(\frac{(\log \log n)^2}{\log^{C/2} n}\right),
\]

where \(C_1 = \frac{C}{\log(e/3)}\), and \(C\) can be any positive constant.

\[\Box\]

10 The Kesselheim et al.’s [14, 15] construction

The following construction appeared in the work of Kesselheim et. al [14] for construction of distribution of permutations of \(n\) elements with entropy \(O((\log^2(n))\). As the original construction goes through several intermediate steps, see also the full version [15], below we extracted major facts.

Set an integer \(k, k \leq \sqrt{\log^{(3)}(n)}\). From Theorem 7 (p. 8, [14]) construct a uniform distribution \(D\) over set of \(O((\log^2 n))!\) permutations with \(n\) elements such that the distribution satisfies \((k, \delta_k)\)-UIOP condition, where \(\delta_k = \frac{1}{\sqrt{\log^{(3)}(n)}}\).

Let \(p_k, q_k\) be any integers such that \(p_k \in o(k^{1/2})\), \(q_k \in O(k^{1/2})\), respectively as \(k \to \infty\). Let \(d_k = \frac{k}{2p_k}\). Consider a partition of \([n]\) into \(q_k\) consecutive blocks of size between \([n/q]\) and \([n/q]\), denoted \(B_1, \ldots, B_{q_k} \subseteq [n]\). Let \(x_1, \ldots, x_{p_k} \in [n]\) be set of \(p_k\) distinct numbers. According to Lemma 2 (p. 6, [14]) the following lower bound is true considering a random permutation \(\pi\) from the distribution \(D\).

\[
\Pr \left[ \bigwedge_{j \in [p_k]} \pi(x_j) \in B_{b_j} \right] \geq q_k^{-p_k}(1 - \delta_k) - \frac{7p_k}{d_k^{1/4}}.
\]

This can be recalculated in the following way

\[
\Pr \left[ \bigwedge_{j \in [p_k]} \pi(x_j) \in B_{b_j} \right] \geq q_k^{-p_k}(1 - \delta_k) - \frac{7p_k}{d_k^{1/4}} \iff
\]

\[
\Pr \left[ \bigwedge_{j \in [p_k]} \pi(x_j) \in B_{b_j} \right] \geq q_k^{-p_k}(1 - \delta_k) - \frac{14p_k^{5/4}}{k^{1/4}} \iff
\]

\[
\Pr \left[ \bigwedge_{j \in [p_k]} \pi(x_j) \in B_{b_j} \right] \geq q_k^{-p_k}\left(1 - \delta_k - q_k^{-p_k}\frac{14p_k^{5/4}}{k^{1/4}}\right).
\]

This proves that the distribution \(D\) satisfies \((p_k, q_k, \delta'_k)\)-BIP condition (see the definition in [14]), where \(\delta'_k := \delta_k + q_k^{-p_k}\frac{14p_k^{5/4}}{k^{1/4}}\). Given a distribution satisfying \((p, q, \delta)\)-BIP, the classic secretary algorithm works with success probability at least \(\frac{1}{e} - \frac{e+1}{q} - \delta - (1 - \frac{1}{e})^{p-1}\), see Theorem 1, p. 4 in [14].
Therefore, the construction proposed by Kesselheim et. al \[14\] leads to an algorithm with success probability at least
\[
\frac{1}{e} - \Omega \left( \frac{1}{q_k} \right) - \frac{q_k^{p_k}}{k^{3/4}} - (1 - 1/e)^{p_k},
\]
as \(k \to \infty\). To have the error arbitrary small we must ensure that \(q_k^{p_k} = o(k)\). In particular \(q_k = o(k)\), since \(p > 1\). Given that \(k\) can be at most \(\sqrt{\log(3)n}\) we can express the probability of success of the classic algorithm being asymptotically \(\frac{1}{e} - \omega\left(\frac{1}{\log \log \log(n)}\right)^{p}\), for any positive constant \(c < 1\).

11 Discussion and conclusions

Similar upper bound as in Theorems \[7\] and \[8\] from Section \[5\] can be obtained by using derandomization via the Lovász Local Lemma \[20\], instead of the Chernoff bound; however, it would be more complicated, thus we decided to design a derandomization based on Chernoff bound. The reason we present both existential and constructive upper bounds on additive approximation is because on one hand, the existential result is tighter, but on the other – the constructive one is more practical. Additionally, the existential proof via probabilistic arguments forms a base for our derandomization and, later on, the constructive result. Finally, it gives a hint how the formula for a lower bound for additive approximation could look like, which is an interesting and challenging open problem, especially for low entropy down to \(\Theta(\log \log n)\). Similarly, although our estimates of the probability of success of (non-uniform) algorithms for entropy \(o(\log \log n)\) are reasonably close, there is still at least a polynomial multiplicative gap between upper and lower estimate.

It is also intriguing that, according to our existential result for \(k = \Omega(\log n)\) and \(\epsilon' < (1 - 1/e)^{\frac{1}{2\epsilon n}}\), there is a set of polynomial number of permutations \(L\) such that the wait-and-pick algorithm with threshold \(m \in \{\lfloor n/e \rfloor, \lceil n/e \rceil\}\), choosing random order uniformly at random from the set \(L\) has the probability of success converging to \(\frac{1}{e}\) from above, with \(n\) going to infinity. Unfortunately, the probability of success of our constructive algorithm (and all previous constructive solutions) converges to \(\frac{1}{e}\) from below, with \(n\) going to infinity. This could be caused by some complexity hardness of close approximation or by the fact that constructive algorithms were designed only for small entropy, for which it may be impossible to prove any (even existential) algorithm to have the success probability converging from above. We addressed this issue partially in Theorem \[1\] Part 3, proving that convergence from above is impossible for uniform distributions with support smaller than \(n\) (and thus, entropy smaller than \(\log n\)), but proving convergence from below for uniform or at least non-uniform cases remains open.

Extending our techniques to other problems with a (limited) random order components, and further study of computational complexity aspects of constructions of random order distributions with bounded entropy (especially for entropy of medium size), are examples of other challenging future directions arising from our work.

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