Scalar one-loop Feynman integrals with complex internal masses revisited

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Abstract

In this paper, we study systematically scalar one-loop two-, three-, four-point Feynman integrals with complex internal masses. The analytic results presented in this work are valid for real as well as complex internal masses. The calculations are then implemented into a Mathematica (version 9) package. Furthermore, one compares numerically the package with LoopTools (version 2.14) in both real and complex internal masses. We find a perfect agreement between the numerical results generated in this work and LoopTools in all cases. What is more, different from traditional approach proposed by G. ’t Hooft and M. Veltman, the method used in this report can be extended to evaluate tensor one-loop integrals directly. Therefore, this may open a new approach to solve the inverse Gram determinant problem analytically.

Keywords: One-loop Feynman integrals, Numerical methods for quantum field theory.

1. Introduction

Higher-order corrections to many processes of interest in perturbative quantum field theory play an important role in matching high precision data at future colliders, e.g. the Large Hadron Collider (LHC) and the International Linear Collider (ILC) [1, 2, 3].

In general framework for computing cross sections (decay rate) at one-loop corrections, evaluating tensor one-loop integrals are one of significant steps. These tensors are reduced frequently to the basic scalar functions which are known as scalar one-loop one-, two-, three- and four-point integrals. Following the traditional tensor reductions, one may encounter the inverse Gram determinant problem [4, 5]. Consequently, this leads to numerical instabilities. Analytic solutions for this problem are ambitious to gain numerical stability. Moreover, with regard to the evaluation for multi-particle processes at future colliders in which Feynman diagrams including internal unstable particles that can be on-shell, one has to resume their propagators by introducing complex masses. In other words, we have to work in complex mass scheme (see Ref. [6] for more detail). Therefore, the calculations for tensor (and scalar) one-loop integrals with complex internal masses are also of great interest.

There are available many papers for evaluating the scalar (and tensor) one-loop integrals in space-time dimension $D = 4 - 2\varepsilon$ at $\varepsilon$-expansion. In Refs. [7, 8], the authors provided the analytic formulas for scalar one-loop one-, two-, three-point functions with real/complex internal masses and four-point functions with real internal masses. Later, a more compact expression for scalar four-point functions with real internal masses was presented in Ref. [9]. Recently, scalar
one-loop four-point functions with complex internal masses were reported in Refs. [10, 11]. Other calculations for one-loop integrals were carried out in Refs. [12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23]. Furthermore, various packages were built for the numerical evaluation of one-loop integrals such as FF [17], LoopTools [24], XLOOPS-GiNaC [25], Golem95 [20, 21] and others [18, 19, 26, 27, 28, 29, 30]. Along with the analytic calculations, several numerical approaches have been developed for computing one-loop and higher-loop integrals [31, 32, 33, 34, 35, 36, 37, 38]. To the best our knowledge, all the above calculations and packages could not solve analytically and completely the inverse Gram determinant problem. Besides that, many of the aforementioned works did not provide the analytical results (and packages) for evaluating one-loop integrals with complex internal masses.

It is unquestionable that solving the inverse Gram determinant problem analytically and proving an alternative method for evaluating one-loop Feynman integrals with including complex internal masses are mandatory. There exists several methods which can solve the Gram determinant problem analytically. One of these approaches is using scalar one-loop integrals at higher space-time dimensions \( D \geq 4 \) in tensor reduction, as pointed out in Refs. [39, 40, 41, 42, 43]. Alternatively, one considers to evaluate tensor one-loop integrals directly. Our interest focuses on tackling the preceding problem by following the second method. We will base on the approach developed in Refs. [12, 13, 14] for evaluating tensor one-loop integrals. In the early stage of this project, we first study systematically scalar one-loop Feynman integrals. We extend previous works in [12, 13, 14, 16] to compute scalar one-loop integrals at general external momentum assignments and with complex internal masses. Last but not least, we implement the results in this paper into a Mathematica package. One also checks numerically this work with LoopTools [24] in both real and complex mass cases.

The layout of the paper is as follows: In section 2, we present the method for evaluating scalar one-loop functions in detail. In section 3, we show the numerical checks this work with LoopTools. Conclusions and plans for future work are presented in section 4. Several useful formulas used in this calculation are shown in the appendixes.

2. The calculation

In this section, based on the method in Refs. [12, 13, 14], we present in detail the calculations for scalar one-loop functions with complex internal masses. It should be noted that the analytic results for scalar one-loop two-, three-point functions with real internal masses were reported in Refs. [12, 13]. Meanwhile scalar one-loop four-point functions with real internal masses were discussed in Ref. [14]. Later, the author of Ref. [16] extended the work in Ref. [14] for evaluating the four-point functions with complex masses. However, in all above papers, the calculations were limited to compute one-loop integrals in the case of at least one time-like external momentum. In this paper, we are going to extend the previous works to calculate scalar one-loop functions with complex internal masses and at general case of external momentum assignments. In the following subsections, we keep the same conventions in Refs. [12, 13, 14, 16].

2.1. One-loop one-point functions

We first arrive at the simplest case which is scalar one-loop one-point functions. Feynman integrals for these functions are defined as

\[
J_1(m^2) = \int d^Dl \frac{1}{l^2 - m^2 + i\rho}. \tag{1}
\]
Here, the loop-momentum is $l$ in space-time dimension $D$. In complex mass scheme, the square of internal mass takes the form

$$m^2 = m_0^2 - i m_0 \Gamma,$$

with $\Gamma$ is decay width of unstable particle. The Feynman prescription is $i \rho$. In this paper, dimensional regularization is performed within space-time dimension $D = 4 - 2 \varepsilon$. We are only interested in the results for expansion of $J_1$ at $\varepsilon^0$.

Performing Wick rotation as $l_0 \to il_0$, one then converts $J_1$ from Minkowski into Euclid space. Finally, the resulting reads

$$J_1(m^2) = -i \int d^Dl_E \frac{1}{l_E^2 + m^2 - i \rho}$$

$$= -2\pi^{D/2} i \frac{1}{\Gamma(D/2)} \int dl_E \frac{l_E^{D-1}}{l_E^2 + m^2 - i \rho}.$$ (4)

Applying the formula (138) in appendix A, we arrive at

$$J_1(m^2) = -i \pi^{D/2} \Gamma \left( \frac{2 - D}{2} \right) (m^2 - i \rho)^{\frac{D-2}{2}}.$$ (5)

This result has been presented in many papers, e.g. [7, 14], etc. Because $m^2 \in \mathbb{R}^+$ or $m^2 \in \mathbb{C}$, one can take $\rho \to 0$ in Eq. (5).

In space-time dimension $D = 4 - 2 \varepsilon$, we expand $J_1(m^2)$ in terms of $\varepsilon$ up to $\varepsilon^0$. The expansion is written as

$$J_1(m^2) = m^2 \left( \frac{1}{\varepsilon} - \gamma_E - \ln(\pi) \right) + m^2 \left( 1 - \ln m^2 \right).$$ (6)

This formula is valid not only for real mass but also for complex mass.

### 2.2. One-loop two-point functions

We are now going to consider scalar one-loop two-point functions. The Feynman integrals for these functions are defined as follows:

$$J_2(q^2, m_1^2, m_2^2) = \int d^Dl \frac{1}{\mathcal{P}_1 \mathcal{P}_2}.$$ (7)

Where the inverse Feynman propagators are given by

$$\mathcal{P}_1 = (l + q)^2 - m_1^2 + i \rho,$$

$$\mathcal{P}_2 = l^2 - m_2^2 + i \rho.$$ (9)

Here, $q$ is external momentum and $m_1, m_2$ are internal masses, as described in Fig. (1). As mentioned in previous subsection, the internal masses are

$$m_k^2 = m_{0k}^2 - i m_{0k} \Gamma_k,$$ (10)

for $k = 1, 2$. $\Gamma_k$ are decay widths of unstable particles. $J_2$ is a function of $q^2, m_1^2, m_2^2$ and its symmetric under the interchange of $m_1^2 \leftrightarrow m_2^2$. 

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2.2.1. \( q^2 > 0 \)

For time-like momentum, e.g. \( q^2 > 0 \), we rederive the results in Ref. [12]. By working in the rest frame of \( q \), one has

\[
q = q(q_{10}, \vec{0}_{D-1}),
\]

(11)

In parallel space spanned by external momentum and its orthogonal space, as explained in Ref. [12], \( J_2 \) takes the form of

\[
J_2(q_{10}, m_1^2, m_2^2) = \frac{2\pi^{D-1}}{\Gamma \left( \frac{D-1}{2} \right)} \int_{-\infty}^{\infty} dl_0 \int_0^\infty dl_\perp \frac{l_\perp^{D-2}}{P_1 P_2},
\]

(12)

in which the inverse Feynman propagators are now obtained as

\[
P_1 = (l_0 + q_{10})^2 - l_\perp^2 - m_1^2 + i\rho,
\]

(13)

\[
P_2 = l_0^2 - l_\perp^2 - m_2^2 + i\rho.
\]

(14)

Partitioning the integrand is as follows

\[
\frac{1}{P_1 P_2} = \frac{1}{P_1(P_2 - P_1)} + \frac{1}{P_2(P_1 - P_2)}.
\]

(15)

We then make a shift \( l'_0 = l_0 + q_{10} \), and the resulting reads

\[
J_2(q_{10}, m_1^2, m_2^2) = \frac{2\pi^{D-1}}{\Gamma \left( \frac{D-1}{2} \right)} \frac{1}{2q_{10}} \int_{-\infty}^{\infty} dl_0 \int_0^\infty dl_\perp \times
\]

\[
\times \left\{ \frac{l_\perp^{D-2}}{\left[ l_0^2 - l_\perp^2 - m_1^2 + i\rho \right] \left[ l_0 + \left( \frac{m_2^2}{2} - M_d \right) \right]} - \frac{l_\perp^{D-2}}{\left[ l_0^2 - l_\perp^2 - m_2^2 + i\rho \right] \left[ l_0 - \left( \frac{m_1^2}{2} + M_d \right) \right]} \right\}.
\]

(16)

The \( l_\perp \)-integration is taken easily by applying Eq. (138) in appendix A. We arrive at

\[
\frac{J_2(q_{10}, m_1^2, m_2^2)}{\Gamma \left( \frac{3-D}{2} \right)} = -\frac{\pi^{D-1}}{2q_{10}} e^{i\pi(D-3)/2} \int_{-\infty}^{\infty} dl_0 \left\{ \frac{(l_0^2 - m_2^2 + i\rho)_{D-3}}{l_0 + \left( \frac{m_1^2}{2} - M_d \right)} - \frac{(l_0^2 - m_1^2 + i\rho)_{D-3}}{l_0 - \left( \frac{m_2^2}{2} + M_d \right)} \right\}.
\]

(17)

The analytic formula for \( J_2 \) in this case is expressed in terms of \( \mathcal{R} \)-functions [44] as

\[
\frac{J_2(q_{10}, m_1^2, m_2^2)}{\Gamma \left( 2 - \frac{D}{2} \right)} = -\frac{\pi^{D/2} e^{i\pi(3-D)/2}}{q_{10}} \times
\]

(18)
with \( M_d = \frac{m_1^2 - m_2^2}{2q_{10}} \). It is easy to verify that the result in Eq. (18) is symmetric under the interchange of \( m_1^2 \leftrightarrow m_2^2 \). This reflects the symmetry of \( J_2 \). We have just derived again Eq. (7) in Ref. [12]. In \( D = 4 - 2\varepsilon \), applying expansion formula for \( \mathcal{R}_{-\varepsilon} \) in appendix B, (see Eq. (140) for more detail), we get

\[
\frac{J_2}{i\pi^2} = \frac{1}{\varepsilon} - \gamma_E - \ln \pi + \left\{ 1 + \left( 1 + \frac{M_d}{q_{10}} \right) \left[ Z \ln \left( \frac{Z - 1}{Z + 1} \right) - \ln m_1^2 \right] \right\} + \{ m_1^2 \leftrightarrow m_2^2 \} \text{ term},
\]

(19)

with

\[
Z = \sqrt{1 - \frac{4(m_1^2 - i\rho)q_{10}^2}{(q_{10}^2 + m_1^2 - m_2^2)^2}} = \sqrt{\lambda \frac{(q_{10}^2, m_1^2, m_2^2)}{(q_{10}^2 + m_1^2 - m_2^2)^2}}.
\]

(20)

Where \( \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz \) is K\"allen function. Here, in the function \( \lambda(q_{10}^2, m_1^2, m_2^2) \), \( m_i^2 \) are understood as \( m_i^2 - i\rho \) for \( i = 1, 2 \). Hence, in the case of real internal masses, the arguments of logarithm functions in (19) are never in the negative real axis. This gives total agreement with Eq. (8) in Ref. [12].

2.2.2. \( q^2 < 0 \)

In this article, we are also interested in calculating two-point functions with \( q^2 < 0 \). In this case, we work in configuration of external momentum as follows

\[
q = q(0, q_{11}, \overrightarrow{0}_{D-2}).
\]

(21)

In parallel and orthogonal space, the integral \( J_2 \) in Eq. (7) takes the form of

\[
J_2(q_{11}, m_1^2, m_2^2) = \frac{\pi D-2}{\Gamma(D-2)} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{-\infty}^{\infty} dl_\perp \frac{l_{D-3}^{\perp}}{\mathcal{P}_1 \mathcal{P}_2}.
\]

(22)

Where the inverse Feynman propagators now become

\[
\mathcal{P}_1 = l_0^2 - (l_1 + q_{11})^2 - l_\perp^2 - m_1^2 + i\rho,
\]

(23)

\[
\mathcal{P}_2 = l_0^2 - l_1^2 - l_\perp^2 - m_2^2 + i\rho.
\]

(24)

We realize that the \( l_0 \)-poles of the \( J_2 \)’s integrand locate in the second and fourth quarters of \( l_0 \)-complex plane. As a result, if we close the integration contour in the first and third quarters of \( l_0 \)-complex plane, there are no residue contributions to \( l_0 \)-integration from these poles. Subsequently, one arrives at the following relation for \( l_0 \)-integration

\[
\int_{-\infty}^{\infty} dl_0 = \int_{-\infty}^{+\infty} dl_0.
\]

(25)
With the help of this relation, combining with the Wick rotation $l_0 \rightarrow il_0$. After that, we apply a further transformation as

$$ l_0 \rightarrow l_\perp \cos \theta, \quad (26) $$

$$ l_\perp \rightarrow l_\perp \sin \theta. \quad (27) $$

The $J_2$ is then obtained as

$$ J_2(q_{11}, m_1^2, m_2^2) = \frac{2i \pi^{\frac{D-1}{2}}}{\Gamma \left( \frac{D-1}{2} \right)} \int_{-\infty}^{\infty} dl_1 \int_{0}^{\infty} dl_\perp \frac{l_\perp^{D-2}}{\mathcal{P}_1 \mathcal{P}_2}, \quad (28) $$

with

$$ \mathcal{P}_1 = (l_1 + q_{11})^2 + l_\perp^2 + m_1^2 - i\rho, \quad (29) $$

$$ \mathcal{P}_2 = l_1^2 + l_\perp^2 + m_2^2 - i\rho. \quad (30) $$

Following the same previous procedure, we first apply the partition for the integrand as Eq. (15). One makes a shift $l_1 \rightarrow l_1 + q_{11}$, and we arrive at

$$ J_2(q_{11}, m_1^2, m_2^2) = \frac{2i \pi^{\frac{D-1}{2}}}{\Gamma \left( \frac{D-1}{2} \right)} \frac{1}{2q_{11}} \int_{-\infty}^{\infty} dl_1 \int_{0}^{\infty} dl_\perp \times \left\{ \frac{l_\perp^{D-2}}{[l_1^2 + l_\perp^2 + m_1^2 - i\rho][l_1 + \frac{q_{11}}{2} + M_d]} - \frac{l_\perp^{D-2}}{[l_1^2 + l_\perp^2 + m_2^2 - i\rho][l_1 - \frac{q_{11}}{2} + M_d]} \right\}, \quad (31) $$

with $M_d = \frac{m_1^2 - m_2^2}{2q_{11}}$.

The $l_\perp$-integration can be carried out by using Eq. (138) in appendix A. The resulting reads

$$ J_2(q_{11}, m_1^2, m_2^2) = \frac{\pi^{\frac{D-1}{2}}}{2q_{11}} \int_{-\infty}^{\infty} dl_1 \left\{ \left( l_1^2 + m_1^2 - i\rho \right)^{\frac{D-3}{2}} \frac{l_1^{D-3}}{l_1 + \left( \frac{q_{11}}{2} + M_d \right)} - \left( l_1^2 + m_2^2 - i\rho \right)^{\frac{D-3}{2}} \frac{l_1^{D-3}}{l_1 - \left( \frac{q_{11}}{2} + M_d \right)} \right\}. \quad (32) $$

By comparing Eq. (32) with Eq. (17), one realizes that this integral has the same form with $J_2$ in (17). Thus, we can also present $J_2$ in this case as $\mathcal{R}$-functions

$$ J_2(q_{11}, m_1^2, m_2^2) = -\frac{i\pi^{D/2} e^{-i\pi(D-3)/2}}{2q_{11}} \times \frac{l_1^{D-3}}{l_1 + \left( \frac{q_{11}}{2} + M_d \right)} \mathcal{R}_D \left( 1, \frac{3 - D}{2}, -\left( \frac{q_{11}}{2} - M_d \right)^2, -m_1^2 + i\rho \right) $$

$$ + \left( \frac{q_{11}}{2} + M_d \right) \mathcal{R}_D \left( 1, \frac{3 - D}{2}, -\left( \frac{q_{11}}{2} + M_d \right)^2, -m_2^2 + i\rho \right) \right\}. $$

This result also shows the symmetry of $J_2$ under the interchange of $m_1^2 \leftrightarrow m_2^2$.

In $D = 4 - 2\varepsilon$, with the help of $\varepsilon$-expansion for $\mathcal{R}_{-\varepsilon}$ in Eq. (140) in appendix B, one gets

$$ J_2(q_{11}, m_1^2, m_2^2) \frac{1}{i\pi^2} = \frac{1}{\varepsilon} - \gamma_E - \ln \pi + \left\{ 1 + \left( 1 - \frac{M_d}{q_{11}} \right) \left[ Z \ln \left( \frac{Z - 1}{Z + 1} \right) - \ln m_1^2 \right] \right\} $$

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\[ + \{m_1^2 \leftrightarrow m_2^2\} \text{ term,} \]  

with
\[
Z = \sqrt{1 + \frac{4(m_1^2 - i\rho)q_{11}^2}{(q_{11}^2 + m_1^2 - m_2^2)^2}} = \sqrt{\frac{\lambda(-q_{11}^2, m_1^2, m_2^2)}{(q_{11}^2 + m_1^2 - m_2^2)^2}}. \tag{35}
\]

It should be reminded that \( m_i^2 \rightarrow m_i^2 - i\rho \) for \( i = 1, 2 \) in the argument of Källen function.

2.2.3. \( q^2 = 0 \)

In this case, one may take the limit of \( q_{10} \rightarrow 0 \) in Eq. (12). \( J_2 \) is then decomposed into two \( J_1 \). In particular, \( J_2 \) is expressed as
\[
J_2(0, m_1^2, m_2^2) = \Gamma \left( 1 - \frac{D}{2} \right) \frac{(m_2^2)^{\frac{D}{2} - 1} - (m_1^2)^{\frac{D}{2} - 1}}{m_1^2 - m_2^2}. \tag{36}
\]

If \( m_1 = m_2 \), one presents \( J_2 \) as
\[
J_2 = \Gamma \left( 2 - \frac{D}{2} \right) (m^2)^{\frac{D}{2} - 2}. \tag{37}
\]

In \( D = 4 - 2\varepsilon \), series expansion of \( J_2 \) up to \( \varepsilon^0 \) are as
\[
J_2(0, m_1^2, m_2^2) = \frac{1}{\varepsilon} - \gamma_E + 1 - \sum_{i=1}^{2} (-1)^i m_i^2 \ln(m_i^2) \left( \frac{1}{m_1^2 - m_2^2} \right). \tag{38}
\]

2.3. One-loop three-point functions

Scalar one-loop three-point functions with complex internal masses are calculated in this subsection. The Feynman integrals for these functions are defined as follows
\[
J_3(p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2) = \int d^Dp_{1,2,3} \frac{1}{p_1^2 p_2^2 p_3^2}. \tag{39}
\]

Here, the inverse Feynman propagators are given by
\[
p_1 = (l + p_1)^2 - m_1^2 + i\rho, \tag{40}
\]
\[
p_2 = (l + p_1 + p_2)^2 - m_2^2 + i\rho, \tag{41}
\]
\[
p_3 = l^2 - m_3^2 + i\rho. \tag{42}
\]

All external momenta flow inward as described in Fig. 2. The internal masses have the same form as Eq. (10). The \( J_3 \) is a function of six independent parameters such as \( p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2 \). The symmetry of \( J_3 \) is given in Table 1.

\[
\begin{array}{cccccc}
  p_1^2 & p_2^2 & p_3^2 & m_1^2 & m_2^2 & m_3^2 \\
p_1^2 & p_2^3 & p_3^2 & m_1^2 & m_2^2 & m_3^2 \\
p_2^2 & p_3^2 & p_1^2 & m_2^2 & m_3^2 & m_1^2 \\
p_2^2 & p_3^2 & p_1^2 & m_2^2 & m_3^2 & m_1^2 \\
p_3^2 & p_1^2 & p_2^2 & m_3^2 & m_1^2 & m_2^2 \\
p_3^2 & p_1^2 & p_2^2 & m_3^2 & m_1^2 & m_2^2 \\
\end{array}
\]

Table 1: Symmetry of \( J_3 \).
2.3.1. At least one time-like external momentum, e.g. $p_1^2 > 0$

In this case, working in rest frame of $p_1$, external momenta take the following configuration

$$q_1 = p_1 = q_1(q_{10}, 0, \vec{0}_{D-2}), \quad q_2 = p_1 + p_2 = q_2(q_{20}, q_{21}, \vec{0}_{D-2}).$$

In parallel and orthogonal space, the integral $J_3$ gets the form of

$$J_3(q_{10}, q_{20}, q_{21}, m_1^2, m_2^2, m_3^2) = \frac{2\pi^{D-2}}{\Gamma\left(\frac{D-2}{2}\right)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{l_1^{D-3}}{P_1P_2P_3},$$

in which the Feynman propagators are written explicitly as

$$P_1 = (l_0 + q_{10})^2 - l_1^2 - l_2^2 - m_1^2 + i\rho,$$
$$P_2 = (l_0 + q_{20})^2 - (l_1 + q_{21})^2 - l_2^2 - m_2^2 + i\rho,$$
$$P_3 = l_2^2 - l_1^2 - l_2^2 - m_3^2 + i\rho.$$

Applying the same procedure for $J_2$, we first perform the partition for $J_3$’s integrand as follows

$$\frac{1}{P_1P_2P_3} = \sum_{k=1}^{3} \frac{1}{\mathcal{P}_k \prod_{l=1 \atop l \neq k}^{3} (\mathcal{P}_l - \mathcal{P}_k)}.$$

One then makes a shift $l_i = l_i + q_{ki}$, for $i = 0, 1$ and $k = 1, 2, 3$. The resulting reads

$$J_3 = \frac{2\pi^{D-2}}{\Gamma\left(\frac{D-2}{2}\right)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{l_1^{D-3}}{\prod_{k=1 \atop k \neq l}^{3} \left[ l_0^2 - l_1^2 - l_2^2 - m_k^2 + i\rho \right] \prod_{l=1 \atop k \neq l}^{3} \left( a_{lk}l_0 + b_{lk}l_1 + c_{lk} \right)}.$$

Figure 2: One-loop three-point Feynman diagrams.
In the above formula, we have introduced new kinematic variables which are defined as

\[ a_{lk} = 2(q_{l0} - q_{k0}), \quad \text{(51)} \]
\[ b_{lk} = -2(q_{l1} - q_{k1}), \quad \text{(52)} \]
\[ c_{lk} = (q_k - q_l)^2 + m_k^2 - m_l^2. \quad \text{(53)} \]

It is important to note that \( a_{lk}, b_{lk} \in \mathbb{R} \) and \( c_{lk} \in \mathbb{C} \).

Using Eq. (138) in appendix A, the \( l_\perp \)-integration is performed first. It reads

\[
\int_0^\infty dl_\perp \left[ \frac{D-3}{l_\perp^2 - l_{l1}^2 - l_\perp^2 + i\rho} \right] = -\frac{\Gamma \left( \frac{D-2}{2} \right) \Gamma \left( \frac{2-D}{2} \right)}{2} \left( -l_{l0}^2 + l_{l1}^2 + m_k^2 - i\rho \right)^{\frac{D-2}{2}}. \quad \text{(54)}
\]

After integrating over \( l_\perp \), the \( J_3 \) then reads

\[
\frac{J_3}{\Gamma \left( \frac{2-D}{2} \right)} = -\pi \left( \frac{D-2}{2} \right) \int_{-\infty}^\infty dl_{l0} \int_{-\infty}^\infty dl_{l1} \sum_{k=1}^{3} \left[ \prod_{l=1\atop k \neq l}^3 \right] (a_{lk}l_{l0} + b_{lk}l_{l1} + c_{lk}) \quad \text{(55)}
\]

We have just arrived at the two-fold integral (55). It will be integrated by applying the residue theorem. In fact, we plan for carrying out the \( l_0 \)-integration first. For this purpose, we first linearize \( l_0 \) by performing a shift \( \tilde{l}_1 = l_1 + l_0 \). Under this transformation, the Jacobian is 1 and the integration region is unchanged. By relabeling \( \tilde{l}_1 \) by \( l_1 \), \( J_3 \) is casted into the form of

\[
\frac{J_3}{\Gamma \left( \frac{2-D}{2} \right)} = -\pi \left( \frac{D-2}{2} \right) \sum_{k=1}^{3} \int_{-\infty}^\infty dl_{l0} \int_{-\infty}^\infty dl_{l1} \left[ \prod_{l=1\atop k \neq l}^3 \right] \left[ AB_{lk}l_{l0} + b_{lk}l_{l1} + c_{lk} \right] \quad \text{(56)}
\]

Where new notation has been introduced as

\[ AB_{lk} = a_{lk} - b_{lk} \in \mathbb{R}. \quad \text{(57)} \]

The integrand now depends linearly on \( l_0 \). Thus, the \( l_0 \)-integration can now be taken easily. In order to apply the residue theorem, we first analyze \( l_0 \)-poles of \( J_3 \)'s integrand. The first pole has already mentioned as

\[ l_0 = \frac{l_{l1}^2 + m_k^2 - i\rho}{2l_1}, \quad \text{(58)} \]

with its imaginary part reads

\[ \text{Im}(l_0) = -\frac{m_{0k} \Gamma_k + i\rho}{2l_1}. \quad \text{(59)} \]

From Eq. (59), we confirm that the location of this pole on the \( l_0 \)-complex plane is determined by the sign of \( l_1 \). For example, it is located in upper (lower) half-plane of \( l_0 \) when \( l_1 > 0 \) (\( l_1 < 0 \)) (see Fig. (3) for more detail).

Two other poles of the \( l_0 \)-integrand are the roots of the following equation:

\( (AB_{lk})l_0 + b_{lk}l_1 + c_{lk} = 0 \). \quad \text{(60)}
These roots are written explicitly as

\[ l_0 = \frac{-b_{lk} l_1 + c_{lk}}{A B_{lk}}, \]  

which have the following imaginary parts

\[ \text{Im}(l_0) = -\frac{\text{Im}(c_{lk})}{A B_{lk}}. \]  

Figure 3: The contour for the \( l_0 \)-integration.

We first split the \( l_1 \)-integration into two domains which are \([-\infty, 0]\) and \([0, \infty]\). One then closes the integration contour in the lower (upper) half-plane of \( l_0 \) for \( l_1 < 0 \) (for \( l_1 > 0 \)). Following the residue theorem, we take into account the residue contributions from the poles in Eq. (61). Applying the theorem and labeling \( l_1 = z \) give

\[ \frac{J_3}{\Gamma \left( 2 - \frac{D}{2} \right)} = -\pi^{\frac{D}{2}} i \sum_{k=1}^{3} \sum_{l=1}^{3} \left[ 1 - \delta(AB_{lk}) \right] \frac{[1 - \delta(AB_{lk})]}{A_{mlk}} \times \]

\[ \left\{ f_1^+ \int_0^\infty dz \left[ \frac{\left( 1 + \frac{2b_{lk}}{A B_{lk}} \right) z^2 + 2 \frac{c_{lk}}{A B_{lk}} z + m_k^2 - i\rho}{(z + F_{mlk})} \right]^{\frac{D-2}{2}} \right. \]

\[ + f_1^- \int_0^\infty dz \left[ \frac{\left( 1 + \frac{2b_{lk}}{A B_{lk}} \right) z^2 - 2 \frac{c_{lk}}{A B_{lk}} z + m_k^2 - i\rho}{(z - F_{mlk})} \right]^{\frac{D-2}{2}} \}

for \( m \neq l \). In this equation, we have introduced new notations

\[ A_{mlk} = -A B_{km} b_{lk} + A B_{lk} b_{km} \in \mathbb{R}, \]  

\[ C_{mlk} = -A B_{km} c_{lk} + A B_{lk} c_{km} \in \mathbb{C}, \]  

\[ F_{mlk} = \frac{C_{mlk}}{A_{mlk}} \pm i\rho' \in \mathbb{C}, \]  

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for \( l, k = 1, 2, 3, l \neq k \) and \( m \neq l \) and \( \rho' \to 0^+ \).

The \( f_{ik}^+ \) and \( f_{ik}^- \) functions indicate the locations of \( l_0 \)-poles in Eq. (61) which contribute to the \( l_0 \)-integration. These functions are defined as

\[
f_{ik}^+ = \begin{cases} 2, & \text{if } \text{Im } \left( -\frac{c_{ik}}{AB_{ik}} \right) > 0, \\ 1, & \text{if } \text{Im } \left( -\frac{c_{ik}}{AB_{ik}} \right) = 0, \quad \text{and } f_{ik}^- = \begin{cases} 2, & \text{if } \text{Im } \left( -\frac{c_{ik}}{AB_{ik}} \right) < 0, \\ 0, & \text{if } \text{Im } \left( -\frac{c_{ik}}{AB_{ik}} \right) = 0, \quad (67) \\ 0, & \text{if } \text{Im } \left( -\frac{c_{ik}}{AB_{ik}} \right) > 0. \end{cases}
\]

If \( AB_{ik} = 0 \), there are no residue contributions from \( l_0 \) to the integration. Taking this point into account in the general formula for \( J_3 \), we introduce the \( \delta \)-function as follows

\[
\delta(x) = \begin{cases} 0, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases} \quad (68)
\]

It is also important to verify that

\[
\text{Im } \left( 1 + \frac{2b_{ik}}{AB_{ik}} \right) z^2 + \frac{2c_{ik}}{AB_{ik}} z + m_k^2 - i\rho = \mp 2 \text{ Im } \left( -\frac{c_{ik}}{AB_{ik}} \right) z - m_0 \Gamma_k - \rho \leq 0. \quad (69)
\]

We have just arrived at one-fold integral, as shown in Eq. (63). It will be expressed in terms of \( R \)-functions. We first consider the general case which is

\[
\alpha_{ik} = 1 + \frac{2b_{ik}}{AB_{ik}} = \frac{a_{ik} + b_{ik}}{a_{ik} - b_{ik}} \neq 0. \quad (70)
\]

In this case, let \( Z_{ik}^{(1)} \) and \( Z_{ik}^{(2)} \) be the roots of the equations

\[
z^2 - 2\frac{c_{ik}}{a_{ik} + b_{ik}} z + m_k^2 - i\rho = 0. \quad (71)
\]

These roots are written explicitly as

\[
Z_{ik}^{(1, 2)} = \frac{c_{ik}}{a_{ik} + b_{ik}} \pm \sqrt{\left( \frac{c_{ik}}{a_{ik} + b_{ik}} \right)^2 - m_k^2 - i\rho} / \alpha_{ik}. \quad (72)
\]

With this definition, in real mass case, we decompose numerator of \( J_3 \)'s integrand in Eq. (63) as follows

\[
\left[ \left( 1 + \frac{2b_{ik}}{AB_{ik}} \right) z^2 + 2\frac{c_{ik}}{AB_{ik}} z + m_k^2 - i\rho \right]^{\frac{D-4}{2}} = \left[ \alpha_{ik} \left( z^2 + 2\frac{c_{ik}}{a_{ik} + b_{ik}} z + m_k^2 - i\rho \right) - i\rho \right]^{\frac{D-4}{2}}. \quad (73)
\]

Following Eq. (136) in appendix A, the resulting equation reads

\[
\left[ \alpha_{ik} \left( z^2 + 2\frac{c_{ik}}{a_{ik} + b_{ik}} z + m_k^2 - i\rho \right) - i\rho \right]^{\frac{D-4}{2}} = (\alpha_{ik} - i\rho)^{\frac{D-4}{2}} \frac{\partial^{D-4}}{2} \left( z \mp Z_{ik}^{(1)} \right)^{\frac{D-4}{2}} \left( z \mp Z_{ik}^{(2)} \right)^{\frac{D-4}{2}}. \quad (74)
\]
In complex mass case, applying Eq. \((135)\), we arrive at another relation
\[
\left[ \alpha_{lk} \left( z^2 + 2 \frac{c_{lk}}{a_{lk} + b_{lk}} z + \frac{m^2_k}{\alpha_{lk}} \right) - i\rho \right]^{\frac{D-4}{2}} = (\alpha_{lk} - i\rho)^{\frac{D-4}{2}} \left[ (z + Z^{(1)}_{ik})(z + Z^{(2)}_{ik}) \right]^{\frac{D-4}{2}}
\]
\[
= S^+_{ik} \left( z + Z^{(1)}_{ik} \right)^{\frac{D-4}{2}} \left( z + Z^{(2)}_{ik} \right)^{\frac{D-4}{2}}. \quad (75)
\]

We have already introduced \(S^+_ik\) which they are defined as
\[
S^+_ik = (\alpha_{lk} - i\rho)^{\frac{D-4}{2}} \times \exp \left[ 2\pi i \theta (-\alpha_{lk}) \theta(\pm \text{Im}(Z^{(1)}_{ik})) \theta(\pm \text{Im}(Z^{(2)}_{ik})) \left( \frac{D-4}{2} \right) \right] \times \exp \left[ -2\pi i \theta (\alpha_{lk}) \theta(\mp \text{Im}(Z^{(1)}_{ik})) \theta(\mp \text{Im}(Z^{(2)}_{ik})) \left( \frac{D-4}{2} \right) \right].
\]

The decomposition of the numerator of \(J_3\)-integrand in Eq. \((75)\) also covers the relation in Eq. \((74)\). In fact, when all masses are real, one can check that \(\text{Im}(Z^{(1)}_{ik}) = -\text{Im}(Z^{(2)}_{ik})\), Eq. \((75)\) gets back to Eq. \((74)\).

The integral in Eq. \((63)\) becomes
\[
\frac{J_3}{\Gamma (2 - \frac{D}{2})} = -\pi \frac{D}{2} i \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{[1 - \delta(AB_{lk})]}{A_{mlk}} \times \int_0^\infty dz \left\{ S^+_ik f^+_ik \int_0^\infty dz \left\{ S^-ik f^-ik \right\} \right\}, \quad (77)
\]

for \(m \neq l\).

The integral in Eq. \((77)\) can be presented in terms of \(R\)-functions \([44]\) as follows
\[
\frac{J_3}{\Gamma (2 - \frac{D}{2})} = -\pi \frac{D}{2} i B(4 - D, 1) \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{[1 - \delta(AB_{lk})]}{A_{mlk}} \times \left\{ S^+_ik f^+_ik R_{D-4} \left( 2 - \frac{D}{2}, 2 - \frac{D}{2}, 1; Z^{(1)}_{ik}, Z^{(2)}_{ik}, F_{mlk} + i\rho \right) \right\}
\]
\[
+ S^-ik f^-ik R_{D-4} \left( 2 - \frac{D}{2}, 2 - \frac{D}{2}, 1; -Z^{(1)}_{ik}, -Z^{(2)}_{ik}, -F_{mlk} - i\rho \right) \right\}, \quad (78)
\]

for \(m \neq l\). Based on the results for \(J_3\) with real masses in Refs. \([12, 13, 14]\), we can consider Eq. \((78)\) as new contributions in this report. In fact, \(f^+_ik = f^-ik = 1\) and \(S^\pm ik = 1\) for real internal masses, we then confirm again Eq. \((11)\) in \([13]\).

We emphasize that Eq. \((77)\) is valid if \(A_{mlk} \neq 0\). For the case of \(A_{mlk} = 0\), the \(J_3\) integral becomes
\[
\frac{J_3}{\Gamma (2 - \frac{D}{2})} = -\pi \frac{D}{2} i \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{[1 - \delta(AB_{lk})]}{C_{mlk}} \times \int_0^\infty dz \left\{ \frac{(a_{lk} + b_{lk})}{a_{lk} - b_{lk}} z^2 + 2 \frac{c_{lk}}{AB_{lk}} z + m^2_k - i\rho \right\}^{\frac{D-4}{2}} \times \left\{ f^+_ik \int_0^\infty \left\{ f^-ik \right\} \right\}, \quad (79)
\]
This integral can be also written in terms of $\mathcal{R}$-function as follows

$$\frac{J_3}{\Gamma \left(2 - \frac{D}{2}\right)} = -\pi \frac{D}{2} i B(3 - D, 1) \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{[1 - \delta(AB_{lk})]}{A_{mlk}} \times$$

$$\times \left\{ S_{lk}^+ f_{lk}^+ \mathcal{R}_{D-3} \left(2 - \frac{D}{2}, 2 - \frac{D}{2}; Z_{lk}^{(1)}, Z_{lk}^{(2)}\right) + S_{lk}^- f_{lk}^- \mathcal{R}_{D-3} \left(2 - \frac{D}{2}, 2 - \frac{D}{2}; -Z_{lk}^{(1)}, -Z_{lk}^{(2)}\right) \right\},$$

for $m \neq l$.

We now consider a special case which is $\alpha_{lk} = 1 - \frac{2b_{lk}}{AB_{lk}} = 0$. In this case, $J_3$ in Eq. (63) can be reduced to the following integral

$$\frac{J_3}{\Gamma \left(2 - \frac{D}{2}\right)} = -\pi \frac{D}{2} i \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{[1 - \delta(AB_{lk})]}{A_{mlk}} \times$$

$$\times \left\{ f_{lk}^+ \int_{0}^{\infty} dz \left( \frac{\mathcal{R}_{\frac{D}{2}-2} \left(1, 2 - \frac{D}{2}; F_{mlk} + i\rho, \frac{m_k^2 - i\rho}{2c_{lk}} AB_{lk}\right)}{(z + F_{mlk})} \right) + f_{lk}^- \int_{0}^{\infty} dz \left( \frac{-\mathcal{R}_{\frac{D}{2}-2} \left(1, 2 - \frac{D}{2}; -F_{mlk} - i\rho, \frac{m_k^2 - i\rho}{2c_{lk}} AB_{lk}\right)}{(z - F_{mlk})} \right) \right\}.$$

This will be formulated with the help of $\mathcal{R}$-function as

$$\frac{J_3}{\Gamma \left(2 - \frac{D}{2}\right)} = \pi \frac{D}{2} i \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{[1 - \delta(AB_{lk})]}{A_{mlk}} B \left(2 - \frac{D}{2}, 1\right) \times$$

$$\times \left\{ f_{lk}^+ \mathcal{R}_{\frac{D}{2}-2} \left(1, 2 - \frac{D}{2}; F_{mlk} + i\rho, \frac{m_k^2 - i\rho}{2c_{lk}} AB_{lk}\right) + f_{lk}^- \mathcal{R}_{\frac{D}{2}-2} \left(1, 2 - \frac{D}{2}; -F_{mlk} - i\rho, \frac{m_k^2 - i\rho}{2c_{lk}} AB_{lk}\right) \right\},$$

for $m \neq l$.

The $\varepsilon$-expansion for the $\mathcal{R}$-functions which appears in this subsection is performed in appendix B (see Eq. (140) to Eq. (142) for more detail).

### 2.3.2 All $p_i^2 < 0$ for $i = 1, 2, 3$

In this article, we are also interested in all $p_i^2 < 0$ for $i = 1, 2, 3$. In this case, external momenta are given by

$$q_1 = p_1 = q_1(0, q_{11}, \overrightarrow{0}_{D-2}), \quad q_2 = p_1 + p_2 = q_2(q_{20}, q_{21}, \overrightarrow{0}_{D-2}).$$

In parallel and orthogonal space, $J_3$ then becomes

$$J_3(q_{11}, q_{20}, q_{21}, m_1^2, m_2^2, m_3^2) = -\frac{2\pi \frac{D}{2}}{\Gamma \left(\frac{D}{2}-2\right)} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{0}^{\infty} dl_2 l^{D-3} \frac{l^{D-3}}{P_1 P_2 P_3},$$

\[85\]
with
\[ \mathcal{P}_1 = (l_1 + q_{11})^2 - l_0^2 + l_1^2 + m_1^2 - i\rho, \]
\[ \mathcal{P}_2 = (l_1 + q_{21})^2 - (l_0 + q_{20})^2 + l_1^2 + m_2^2 - i\rho, \]
\[ \mathcal{P}_3 = l_1^2 - l_0^2 + l_1^2 + m_3^2 + i\rho. \]

Repeating the previous calculation, we arrive at the following relation
\[ J_3(q_{11}, q_{20}, q_{21}, m_1^2, m_2^2, m_3^2) = -J_3^{(p_i^2 > 0)}(q_{11}, q_{20}, q_{21}, -m_1^2, -m_2^2, -m_3^2). \]

Hence, the analytic results for \( J_3 \) in this case are obtained as the same form with Eq. (78) by replacing \( q_{10} \rightarrow q_{11}, q_{20} \leftrightarrow q_{21} \) and \( m_i^2 - i\rho \rightarrow -m_i^2 + i\rho \) for \( i = 1, 2, 3 \).

We note that Eqs. (78, 89) also hold for cases of one and two light-like momenta. For example, if \( p_i^2 = 0 \), we then rotate \( J_3 \) (see Table (1)) towards \( q_1 = p_i^2 \neq 0 \) for \( i = 2 \) or \( i = 3 \). For all light-like momenta, e.g. \( p_1^2 = 0, p_2^2 = 0, p_3^2 = (p_1 + p_2)^2 = 0 \), we do not consider in this paper.

2.4. One-loop four-point functions

In this section, we apply the same method for evaluating scalar one-loop four-point functions with complex internal masses. We extend the analytic results for these functions in [14, 16] to the case of all space-like momenta, e.g. all \( p_i^2 < 0 \) for \( i = 1, \cdots, 4 \).

The Feynman integrals for these functions are given by
\[ J_4(p_1^2, p_2^2, p_3^2, p_4^2, s, t, m_1^2, m_2^2, m_3^2, m_4^2) = \int \frac{d^d l}{\mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_3 \mathcal{P}_4}. \]

Where the inverse Feynman propagators are
\[ \mathcal{P}_1 = (l + p_1)^2 - m_1^2 + i\rho, \]
\[ \mathcal{P}_2 = (l + p_1 + p_2)^2 - m_2^2 + i\rho, \]
\[ \mathcal{P}_3 = (l + p_1 + p_2 + p_3)^2 - m_3^2 + i\rho, \]
\[ \mathcal{P}_4 = l_1^2 - m_4^2 + i\rho. \]

In general, \( J_4 \) is a function of \( p_1^2, p_2^2, p_3^2, p_4^2, s, t, m_1^2, m_2^2, m_3^2, m_4^2 \) with \( s = (p_1 + p_2)^2, t = (p_2 + p_3)^2 \).

All internal masses have the same form of Eq. (10) in complex mass scheme.

In this calculation, we are not going to deal with infrared divergence using dimensional regularization in \( D = 4 + 2\varepsilon \). Instead, we introduce fictitious mass for photon. Thus, we can work directly in space-time dimension \( D = 4 \). Let us define the momenta \( q_i = \sum_{j=1}^{i} p_j \) for \( i, j = 1, 2, \cdots, 4 \) (see Fig. (4)).

In the case of all kinematic variables \( p_i^2 \) for \( i = 1, \cdots, 4 \) and \( s, t \) that are greater than 0, we should work in the configuration for external momenta as in Eqs. (7 – 10) in Ref. [16]. The analytic results for \( J_4 \) in this case were shown in Ref. [16]. If some of these kinematic variables are greater or smaller than 0, we can evaluate \( J_4 \) by two following ways. First, we can perform the rotation \( J_4 \) as Table 1 in Ref. [10] in order to satisfy the condition \( q_i^2 > 0 \). The analytic results for that case were also reported in Ref. [16]. In the second direction which is studied in this paper, we can work in the configuration of the external momenta as in Eq. (95). In this paper, we also checked the internal consistency of both ways using numerical computation.

In the case of all \( p_i^2 < 0 \) for \( i = 1, \cdots, 4 \) and \( s < 0, t < 0 \) (we call this is unphysical configuration), we work in the configuration of the external momenta given by
\[ q_1 = (0, q_{11}, 0, 0), \]
It is important to note that $J_4$ in unphysical configuration for external momenta may appear when we reduce higher-point functions to four-point integrals. They may occur, especially when we consider one-loop corrections to multi-particle processes like $2 \rightarrow 5, 6$, etc.

In parallel and orthogonal space, $J_4$ is given by

$$J_4 = 2 \int_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int_{0}^{\infty} dl_\perp \frac{1}{[l_0^2 - (l_1 + q_{12})^2 - l_2^2 - l_\perp^2 - m_4^2 + i\rho]} \frac{1}{[(l_0 + q_{20})^2 - (l_1 + q_{21})^2 - l_2^2 - l_\perp^2 - m_5^2 + i\rho]} \frac{1}{[(l_0 + q_{30})^2 - (l_1 + q_{31})^2 - (l_2 + q_{32})^2 - l_\perp^2 - m_6^2 + i\rho]} \frac{1}{[l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\rho]}.$$

The calculation is similar to that of Ref. [16]. We refer to paper [16] for more details of the method. Here, we just rewrite the analytic expressions for $J_4$ in Ref. [16] which are valid for this case. In particular, the $J_4$ is presented in one-fold integral as follows

$$\frac{J_4}{i\pi^2} = \bigoplus_{nmkl} \int_0^\infty dz G(z) \left\{ \Omega_{nmkl}^+ - f_{ik}^+ g_{mlk}^+ \ln \left( \frac{1 - \beta \varphi}{\beta} z + \frac{E}{\beta} \right) - f_{ik}^- g_{mlk}^- \ln \left( \frac{1 - \beta \varphi}{\beta} z - \frac{E}{\beta} \right) - (f_{ik}^+ g_{mlk}^+ + f_{ik}^- g_{mlk}^-) \ln \left( \frac{-P z}{\beta} + \frac{P Z_{1\varphi}}{\beta} \right) - (f_{ik}^+ g_{mlk}^+ + f_{ik}^- g_{mlk}^-) \ln \left( z - Z_{2\varphi} \right) + f_{ik}^+ g_{mlk}^+ \ln (-P \varphi z + P \varphi Z_{1\varphi}) + f_{ik}^+ g_{mlk}^+ \ln (P \varphi z - P \varphi Z_{1\varphi}) + f_{ik}^- g_{mlk}^- \ln (z - Z_{2\varphi}) + (f_{ik}^+ g_{mlk}^- - f_{ik}^- g_{mlk}^+) \ln (P z + Q) \right\}$$
The kinematic variables with 

\[ f_{ik} g_{mlk}^- \ln \left( \frac{P_z}{\beta} - \frac{PZ_1}{\beta} \right) + (f_{ik}^- g_{mlk} + f_{ik}^+ g_{mlk}) \ln \left( \frac{1 - \beta \varphi}{\beta} z + \frac{F}{\beta} \right) \]

\[ f_{ik} g_{mlk}^- \ln \left( -\frac{P_z}{\beta} + \frac{PZ_1}{\beta} \right) - (f_{ik}^- g_{mlk} + f_{ik}^+ g_{mlk}) \ln (-P \varphi z + P \varphi Z_1 \varphi) \]

\[ f^- g^- \ln (z - Z_2^\varphi) + (f_{ik}^- g_{mlk}^+ - f_{ik}^+ g_{mlk}^-) \ln (P z + Q) \]

\[ -(f_{ik}^- g_{mlk}^+ + f_{ik}^+ g_{mlk}^-) \ln (z - Z_2^\varphi) \]

\[ + 2\pi i \bigoplus_{nmlk} \left( f_{ik}^+ g_{mlk}^- \theta [\text{Im}(Q)] + f_{ik}^- g_{mlk}^+ [-\text{Im}(Q)] \right) \int_{-\infty}^{\infty} dz G(z) \left[ -\text{Im} \left( S(\sigma, z) \right) \right]. \]

In this formula, the related kinematic variables are given as follows.

\[
\begin{align*}
    a_{lk} &= 2(q_{l0} - q_{k0}), \\
    b_{lk} &= -2(q_{l1} - q_{k1}), \\
    c_{lk} &= -2(q_{l2} - q_{k2}), \\
    d_{lk} &= (q_l - q_k)^2 - (m_i^2 - m_k^2),
\end{align*}
\]

with \( a_{lk}, b_{lk}, c_{lk} \in \mathbb{R} \) and \( d_{lk} \in \mathbb{C} \). Then the kinematic variable \( AC_{lk} \) is

\[ AC_{lk} = a_{lk} + c_{lk} \in \mathbb{R}. \]  

The \( f_{ik}^+, f_{ik}^- \) functions contributing to the integrals are defined as

\[
\begin{align*}
    f_{ik}^+ &= \begin{cases} 
        0, & \text{if } \text{Im} \left( -\frac{d_{lk}}{AC_{lk}} \right) < 0; \\
        1, & \text{if } \text{Im} \left( -\frac{d_{lk}}{AC_{lk}} \right) = 0; \\
        2, & \text{if } \text{Im} \left( -\frac{d_{lk}}{AC_{lk}} \right) > 0.
    \end{cases} \\
    f_{ik}^- &= \begin{cases} 
        0, & \text{if } \text{Im} \left( -\frac{d_{lk}}{AC_{lk}} \right) > 0; \\
        1, & \text{if } \text{Im} \left( -\frac{d_{lk}}{AC_{lk}} \right) = 0; \\
        2, & \text{if } \text{Im} \left( -\frac{d_{lk}}{AC_{lk}} \right) < 0.
    \end{cases}
\end{align*}
\]

The kinematic variables \( A_{mlk}, B_{mlk} \) and \( C_{mlk} \) are defined as

\[
\begin{align*}
    A_{mlk} &= a_{mk} - \frac{a_{lk}}{AC_{lk}} AC_{mk}, \\
    B_{mlk} &= b_{mk} - \frac{b_{lk}}{AC_{lk}} AC_{mk}, \\
    C_{mlk} &= c_{mk} - \frac{c_{lk}}{AC_{lk}} AC_{mk}.
\end{align*}
\]

The \( g_{mlk}^+, g_{mlk}^- \) functions are also given by

\[
\begin{align*}
    g_{mlk}^+ &= \begin{cases} 
        2, & \text{if } \text{Im} \left( -\frac{C_{mlk}}{B_{mlk}} \right) > 0; \\
        1, & \text{if } \text{Im} \left( -\frac{C_{mlk}}{B_{mlk}} \right) = 0; \\
        0, & \text{if } \text{Im} \left( -\frac{C_{mlk}}{B_{mlk}} \right) < 0;
    \end{cases} \\
    g_{mlk}^- &= \begin{cases} 
        0, & \text{if } \text{Im} \left( -\frac{C_{mlk}}{B_{mlk}} \right) > 0; \\
        1, & \text{if } \text{Im} \left( -\frac{C_{mlk}}{B_{mlk}} \right) = 0; \\
        2, & \text{if } \text{Im} \left( -\frac{C_{mlk}}{B_{mlk}} \right) < 0.
    \end{cases}
\end{align*}
\]
Other related kinematic variables are

\[ \alpha_{lk} = \frac{b_{lk}}{AC_{lk}} , \] (105)

\[ D_{mlk} = 1 - \frac{2\alpha_{lk}}{AC_{lk}} + \frac{b_{lk}^2}{AC_{lk}^2} = -4\left(\frac{q_l - q_k}{AC_{lk}}\right)^2 , \] (106)

\[ Q_{mlk} = -2\left(\frac{C_{mlk}}{B_{mlk}} + \frac{d_{lk}}{AC_{lk}}\beta_{mlk}\right) , \] (107)

\[ P_{mlk} = -2\left[\left(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\right)(1 + \beta_{mlk}\varphi_{mlk}) - D_{mlk}\beta_{mlk} - \varphi_{mlk}\right] , \] (108)

\[ E_{mlk} = -2\left(\frac{d_{lk}}{AC_{lk}} + \frac{C_{mlk}}{B_{mlk}}\varphi_{mlk}\right) , \] (109)

\[ Z_{mlk} = D_{mlk} - 2\left(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\right)\varphi_{mlk} + \varphi_{mlk}^2 , \] (110)

\[ F_{nmfk} = \frac{C_{nmk}B_{mlk} - B_{nmk}C_{mlk}}{A_{nmk}B_{mlk} - B_{nmk}A_{mlk}} . \] (111)

Where \( \beta_{mlk}^{(1,2)} \) and \( \varphi_{mlk}^{(1,2)} \) are calculated as

\[ \beta_{mlk}^{(1,2)} = \frac{\left(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\right)\pm \sqrt{(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk})^2 - D_{mlk}}}{D_{mlk}} , \] (112)

\[ \varphi_{mlk}^{(1,2)} = \frac{\left(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\right)\pm \sqrt{(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk})^2 - D_{mlk}}}{D_{mlk}} . \] (113)

The \( G(z) \) in these integrands are as follows

\[ G(z) = \frac{1}{-P(z - T_1)(z - T_2)} , \] (114)

with

\[ T_1 = \frac{(Q + PF - \beta E)}{-2P} + \frac{\sqrt{(Q + PF - \beta E)^2 - 4P(QF + \beta m_k^2 - i\rho)}}{-2P} , \] (115)

\[ T_2 = \frac{(Q + PF - \beta E)}{-2P} - \frac{\sqrt{(Q + PF - \beta E)^2 - 4P(QF + \beta m_k^2 - i\rho)}}{-2P} . \] (116)

The \( S(\sigma, z) \) function in the last line in these integrals are given by

\[ S(\sigma, z) = P\sigma z^2 + (E + Q\sigma)z - m_k^2 + i\rho = P\sigma(z - Z_{1\sigma})(z - Z_{2\sigma}) , \] (117)

with \( \sigma = -\frac{1}{\beta_{mlk}}, -\varphi_{mlk} \) and

\[ Z_{1\varphi} = \frac{(E - Q\varphi) + \sqrt{(E - Q\varphi)^2 - 4P\varphi(m_k^2 - i\rho)}}{2P\varphi} , \] (118)

\[ Z_{2\varphi} = \frac{(E - Q\varphi) - \sqrt{(E - Q\varphi)^2 - 4P\varphi(m_k^2 - i\rho)}}{2P\varphi} . \] (119)
\[
Z_{1\beta} = \frac{(E - Q/\beta) + \sqrt{(E - Q/\beta)^2 - 4P/\beta(m_k^2 - i\rho)}}{2P/\beta},
\tag{120}
\]
\[
Z_{2\beta} = \frac{(E - Q/\beta) - \sqrt{(E - Q/\beta)^2 - 4P/\beta(m_k^2 - i\rho)}}{2P/\beta}.
\tag{121}
\]

The kinematic variables \(\Omega_{nmlk}^\pm\) are given by
\[
\Omega_{nmlk}^+ = 2\pi i f_{lk}^+ g_{mlk}^+ \theta \left[ -\text{Im}(P\varphi Z_{1\varphi}) \right] \theta \left[ \text{Im}(Z_{2\varphi}) \right] -2\pi i (f_{lk}^+ g_{mlk}^+ \theta) \left[ -\text{Im}(P\varphi Z_{1\varphi}) \right] \theta \left[ -\text{Im}(Z_{2\varphi}) \right) + (f_{lk}^+ g_{mlk}^+ + f_{lk}^- g_{mlk}^-) \ln \left( \frac{F}{\beta} \right) -2\pi i (f_{lk}^+ g_{mlk}^+ + f_{lk}^- g_{mlk}^-) \theta \left[ \text{Im} \left( -\frac{PZ_{1\beta}}{\beta} \right) \right] \theta \left[ \text{Im}(Z_{2\beta}) \right],
\tag{122}
\]
\[
\Omega_{nmlk}^- = -2\pi i f_{lk}^+ g_{mlk}^- \theta \left[ \text{Im} \left( -\frac{PZ_{1\beta}}{\beta} \right) \right] \theta \left[ -\text{Im}(Z_{2\beta}) \right] +2\pi i f_{lk}^- g_{mlk}^- \theta \left[ \text{Im} \left( -\frac{PZ_{1\beta}}{\beta} \right) \right] \theta \left[ \text{Im}(Z_{2\beta}) \right] -f_{lk}^+ g_{mlk}^- \ln \left( \frac{F}{\beta} \right) -f_{lk}^- g_{mlk}^+ \ln \left( \frac{F}{\beta} \right) -2\pi i (f_{lk}^- g_{mlk}^- + f_{lk}^+ g_{mlk}^+) \theta \left[ -\text{Im}(P\varphi Z_{1\varphi}) \right] \theta \left[ \text{Im}(Z_{2\varphi}) \right).\tag{123}
\]

Finally, we have used the following notation
\[
\bigoplus_{nmlk} = \sum_{k=1}^{4} \sum_{l=1}^{4} \sum_{m=1}^{4} \sum_{m \neq l}^{4} \frac{1}{AC_{lk}(A_{mlk} B_{mlk} - A_{mlk} B_{mlk})} \left[ 1 - \delta(AC_{lk}) \right] \left[ 1 - \delta(B_{mlk}) \right].
\tag{124}
\]

The \(z\)-integrals now are splitted into three basic integrals which are
\[
\mathcal{R}_1 = \int_{0}^{\infty} G(z) dz,
\tag{125}
\]
\[
\mathcal{R}_2 = \int_{0}^{\infty} \ln(az + b)G(z) dz,
\tag{126}
\]
\[
\mathcal{R}_3 = \int_{-\infty}^{\infty} \theta \left[ -\text{Im} \left( \frac{S(\sigma, z)}{P z + Q} \right) \right] G(z) dz.
\tag{127}
\]

These integrals are calculated in detail in the appendix C. It is noted that the formula in (97) is valid for \(D_{mlk} < 0\) or \(0 < D_{mlk} < \left( \frac{A_{mlk} B_{mlk} - \alpha_{lk}}{B_{mlk}} \right)^2\). For other cases, one can refer to Ref. [16] for more detailed.
3. Numerical checks

XLOOPS-GiNaC written in $C^{++}$ using the GiNaC library [25] can handle tensor (and scalar) one-loop two-, three-point functions with real internal masses. In our previous work [16], part of program for evaluating scalar one-loop four-point functions has been implemented into $C^{++}$, ONELOOP4PT.CPP. It can evaluate numerically $J_4$ with real/complex masses in the case of at least one time-like external momentum. Here, we rewrite ONELOOP4PT.CPP by including with new extension for $J_4$ and implement $J_1$, $J_2$, $J_3$ in this report into Mathematica (version 9) package. The program now can evaluate numerically scalar one-loop integrals with real/complex masses at general external momentum assignments. The syntax of these functions are as follows

\begin{align}
\text{ONELOOP1PT}(m^2, \rho), \\
\text{ONELOOP2PT}(p^2, m_1^2, m_2^2, \rho), \\
\text{ONELOOP3PT}(p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, \rho), \\
\text{ONELOOP4PT}(p_1^2, p_2^2, p_3^2, p_4^2, s, t, m_1^2, m_2^2, m_3^2, m_4^2, \rho),
\end{align}

with $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$.

In this section, we compare the finite parts of $J_1$, $J_2$, $J_3$, $J_4$ in this report with using LoopTools (version 2.14) in both real and complex internal masses. In Table 2, the finite parts of $J_1$ are cross-checked with LoopTools. One finds a perfect agreement between this work and LoopTools in all cases.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$m^2$ & This work & LoopTools \\
\hline
100 & $-360.51701859880916$ & $-360.51701859880916$ \\
100 $- 5i$ & $-360.39207063012879 + 23.027932702631205i$ & $-360.39207063012879 + 23.027932702631201i$ \\
\hline
\end{tabular}
\caption{Comparing $J_1$ in this work with LoopTools. We set $\rho = 10^{-15}$.}
\end{table}

In Table 3 (Table 4), we compare the finite parts of $J_2$ in this paper with LoopTools in real (complex) internal masses respectively. We find that the results from this work and LoopTools are in good agreement in both cases. In Table 3, when $q^2$ is above threshold, or $q^2 > (m_1 + m_2)^2$, the imaginary part of $J_2$ is non-zero. While in the region (below threshold) $q^2 < (m_1 + m_2)^2$, the imaginary part of $J_2$ is zero. It is understandable and the data demonstrates these points clearly in Table 3.
In Tables 3, 4, 5, we check the finite parts of $J_3$ with LoopTools in real/complex internal masses. In Table 3, one changes the external momentum configuration. While the internal masses are varied in Table 4. We finally consider the numerical checks for $J_3$ with complex internal masses which the numerical results are presented in Table 5. We obtain a good agreement between the results from our work and LoopTools in all cases.

In Table 6, we observe the contribution of the imaginary part of $J_3$ in the region of $p_i^2 > (m_j + m_k)^2$. While in the region of $p_i^2 < (m_j + m_k)^2$ for $i \neq j \neq k$, the imaginary part of $J_3$ is zero. We observe these points clearly in this Table.
| \( m_1^2, m_2^2, m_3^2 \) | This work | LoopTools |
|------------------|-------------|-------------|
| \((0, 0, 0)\)    | \(-0.04743237875039833 - 0.055989586160261064 \, i\) | \(-0.047432337875041190 - 0.055989586160261083 \, i\) |
| \((10, 20, 30)\) | \(-0.016727876585043306 - 0.030266162066846561 \, i\) | \(-0.016727876585043308 - 0.030266162066846562 \, i\) |
| \((50, 50, 50)\) | \(-0.014058959690622575 - 0.016727876585043306 \, i\) | \(-0.014058959690622577 - 0.016727876585043308 \, i\) |

Table 6: In case of \((p_1^2, p_2^2, p_3^2) = (10, 150, -30)\) and \(\rho = 10^{-15}\).

In the Table 7, we also use Golam95C for cross-checking \(J_3\) with complex internal masses in our work and LoopTools. We find three programs give a perfect agreement results.

| \((p_1^2, p_2^2, p_3^2)\) | This work | LoopTools | Golam95C |
|------------------|-------------|-------------|-----------|
| \((100, 200, -300)\) | \(0.000302117943631926 - 0.022175834817012830 \, i\) | \(0.000302117943631917 - 0.022175834817012831 \, i\) | \(0.000302117943631906 - 0.022175834817012827 \, i\) |
| \((100, -200, -300)\) | \(-0.012274730929707654 - 0.005253630073729939 \, i\) | \(-0.012274730929707652 - 0.005253630073729934 \, i\) | \(-0.012274730929707645 - 0.005253630073729934 \, i\) |
| \((-100, -2000, -300)\) | \(-0.003332905358821172 - 0.00146109020218081 \, i\) | \(-0.003332905358821172 - 0.00146109020218078 \, i\) | \(-0.003332905358821173 - 0.00146109020218077 \, i\) |

Table 7: In case of \((m_1^2, m_2^2, m_3^2) = (10 - 3 \, i, 20 - 4 \, i, 30 - 5 \, i)\), and \(\rho = 10^{-15}\).

Lastly, we compare \(J_4\) in this work with LoopTools. One focuses on the case of all space-like momenta. In Table 8, we vary the external momentum configuration and consider real internal masses. In Table 9, one examines all \(p_1^2, p_2^2, p_3^2, p_4^2\), \(s, t < 0\) with complex mass cases. In Table 9, we change \(m_4^2\). We find a good agreement between the results in this work and LoopTools in all cases.
Table 8: In case of \((m_1^2, m_2^2, m_3^2) = (10, 20, 30, 40)\), and \(\rho = 10^{-15}\).

| \((p_1^2, p_2^2, p_3^2, p_4^2, s, t)\) | \((\text{This work}) \times 10^{-4}\) | \((\text{LoopTools}) \times 10^{-4}\) | \((\text{Golem95C}) \times 10^{-4}\) |
|----------------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| \((-10, 60, 10, 90, 200, 5)\)          | \(-11.268763555152268 + 14.171199111711949 i\) | \(-11.268763555152266 + 14.171199111711946 i\) |
| \((-10, -60, -10, -90, 200, 5)\)      | \(2.1724718865815314 + 2.1645195914946021 i\) | \(2.1724718865815296 + 2.1645195914946018 i\) |
| \((-10, -60, -10, -90, 200, -5)\)     | \(2.0640147463938176 + 2.1335567055149013 i\) | \(2.0640147463938226 + 2.1335567055149037 i\) |
| \((-10, -60, -25, -90, 200, -5)\)     | \(1.5152693508494305 + O(10^{-24}) i\) | \(1.5152693508494312 + O(10^{-19}) i\) |

Table 9: In case of \((p_1^2, p_2^2, p_3^2, p_4^2, s, t) = (-10, -70, -20, -100, -15, -5)\), and \(\rho = 10^{-15}\).

| \((m_1^2, m_2^2, m_3^2, m_4^2)\) | \((\text{This work}) \times 10^{-4}\) | \((\text{LoopTools}) \times 10^{-4}\) | \((\text{Golem95C}) \times 10^{-4}\) |
|------------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| \((10 - 2 i, 20, 30 - 3 i, 40)\)   | \(1.4577467887809371 + 0.13004659190213070 i\) | \(1.4577467887809479 + 0.13004659190214078 i\) | \(1.4577467887809603 + 0.13004659190215162 i\) |
| \((10 - 2 i, 20, 30 - 3 i, 80)\)   | \(1.0235403166014101 + 0.0874193853007884 i\) | \(1.0235403166014069 + 0.0874193853007809 i\) | \(1.0235403166014161 + 0.0874193853008030 i\) |
| \((10 - 2 i, 20, 30 - 3 i, 120 - 10 i)\) | \(0.79634677966095624 + 0.1085714569206661 i\) | \(0.79634677966095526 + 0.1085714569206717 i\) | \(0.79634677966097001 + 0.1085714569206592 i\) |

4. Conclusions

In this paper, we have studied systematically scalar one-loop two-, three-, four-point integrals with real/complex internal masses. The calculations are considered at general case of external momentum assignments. In the numerical checks, one has compared this work with LoopTools. We found a good agreement between the results generated from this work and those from LoopTools. Moreover, this work provides a framework which can be extended to calculate tensor integrals directly. It may provide a new method to solve analytically the inverse determinant problem. In future work, based on this complete calculation, we will proceed to the evaluation for tensor one-loop integrals [45].

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Appendix A

In Appendix A, we present here several useful formulas. The first ones that we mention are

\[
\ln(ab) = \ln(a) + \ln(b) + \eta(a, b), \tag{132}
\]

\[
\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b) + \eta\left(a, \frac{1}{b}\right). \tag{133}
\]

The \(\eta\)-function is defined as

\[
\eta(a, b) = 2\pi i \left\{ \theta\left[-\text{Im}(a)\right] \theta\left[-\text{Im}(b)\right] \theta(\text{Im}(ab)) - \theta\left[\text{Im}(a)\right] \theta\left[\text{Im}(b)\right] \theta\left[-\text{Im}(ab)\right] \right\}. \tag{134}
\]

Furthermore, one can derive the following relation

\[
(ab)^\alpha = e^{\alpha \ln(ab)} = e^{\alpha \eta(a,b)} a^\alpha b^\alpha. \tag{135}
\]

In the case of \(a, b \in \mathbb{R}\), one has the following relation

\[
(ab \pm i\rho)^\alpha = (a \pm i\rho')^\alpha \left(b \pm \frac{i\rho}{a}\right)^\alpha \quad \text{for} \quad a, b \in \mathbb{R}. \tag{136}
\]

Here, \(\rho'\) has the same sign with \(\rho\).

The Beta function is also given by

\[
\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{137}
\]

In this paper we also use this basic integral

\[
\int_0^\infty \frac{z^{\alpha-1}}{Z_k + z} dz = \mathcal{B}(1-\alpha, \alpha)(Z_k)^{\alpha-1}. \tag{138}
\]

Appendix B

The \(\mathcal{R}\)-function [44] is defined as

\[
\int_r^\infty (x-r)^{\alpha-1} \prod_{i=1}^k (z_i + w_i x)^{-b_i} dx
\]

\[
= \mathcal{B}(\beta - \alpha, \alpha) \mathcal{R}_{\alpha-\beta} \left(b_1, \ldots, b_k, r + \frac{z_1}{w_1}, \ldots, r + \frac{z_k}{w_k}\right) \prod_{i=1}^k w_i^{-b_i}, \tag{139}
\]

with \(\beta = \sum_{i=1}^k b_i\).

Expansion for \(\mathcal{R}\)-functions in terms of \(\varepsilon\) are devoted in the following paragraphs.
1. $\mathcal{R}_{-\varepsilon} \left( -\frac{1}{2} + \varepsilon, 1, z_1, z_2 \right)$: 
   Expansion for this function was presented in Refs. [12, 14].
   \[
   \mathcal{R}_{-\varepsilon} \left( -\frac{1}{2} + \varepsilon, 1, z_1, z_2 \right) = \frac{1}{2} \left( z_1 \right)^{-\varepsilon} \left( z_2 \right)^{-\varepsilon} \left( u_2 u_1^{2\varepsilon} + u_1 u_2^{2\varepsilon} \right),
   \]
   with $u_{1,2} = 1 \pm \sqrt{1 - \frac{z_1}{z_2}}$.

2. $\mathcal{R}_{-2\varepsilon} \left( 1, \varepsilon, \varepsilon, x, y, z \right)$: 
   Expansion for this function was presented in Refs. [13, 14].
   \[
   \mathcal{R}_{-2\varepsilon} \left( \varepsilon, \varepsilon, 1; x, y, z \right) = 1 - 2\varepsilon \ln z + 2\varepsilon^2 \left\{ \text{Li}_2 \left( 1 - \frac{x}{z} \right) + \ln \left( 1 - \frac{x}{z} \right) \eta \left( \frac{x}{z} \right) \right. \\
   + \ln(z) \left[ \eta \left( x - z, \frac{1}{1 - z} \right) - \eta \left( x - z, -\frac{1}{z} \right) \right] + \ln(z)^2 \\
   + \text{Li}_2 \left( 1 - \frac{y}{z} \right) + \ln \left( 1 - \frac{y}{z} \right) \eta \left( \frac{y}{z} \right) \\
   + \ln(z) \left[ \eta \left( y - z, \frac{1}{1 - z} \right) - \eta \left( y - z, -\frac{1}{z} \right) \right] \left. \right\},
   \]

3. $\mathcal{R}_{1-2\varepsilon} \left( \varepsilon, \varepsilon, x, y \right)$: 
   Expansion for this function was presented in Refs. [14].
   \[
   \mathcal{R}_{1-2\varepsilon} \left( \varepsilon, \varepsilon; x, y \right) = \frac{1}{2} (x + y) - [y \ln(y) + x \ln(x)] \varepsilon.
   \]

4. $\mathcal{R}_{-\varepsilon} \left( 1, \varepsilon; y, z \right)$: 
   Expansion for this function was presented in Refs. [14].
   \[
   \mathcal{R}_{-\varepsilon} \left( \varepsilon, \varepsilon; y, z \right) = 1 - \varepsilon \ln y + \varepsilon^2 \left\{ \text{Li}_2 \left( 1 - \frac{z}{y} \right) + \frac{1}{2} (\ln y)^2 \right. \\
   + \ln y \left[ \eta \left( z - y, \frac{1}{1 - y} \right) - \eta \left( z - y, -\frac{1}{y} \right) \right] \\
   + \ln \left( 1 - \frac{z}{y} \right) \eta \left( \frac{z}{y} \right) \left. \right\}.
   \]

**Appendix C**

We consider three basic integrals in the following paragraphs. Here, all the shown formulas are taken from [16].

1. Basic integral $I$:
   The basics integral $I$ is defined as
   \[
   \mathcal{R}_1(x, y) = \int_0^\infty \frac{1}{(z + x)(z + y)} dz = \frac{\ln(x) - \ln(y)}{x - y},
   \]
   with $x, y \in \mathbb{C}$.
2. Basic integral II:
The basic integral II is
\[
\mathcal{R}_2(r, x, y) = \int_0^\infty \frac{\ln(1 + rz)}{(z + x)(z + y)} \, dz = -\frac{1}{x - y} \left[ \text{Li}_2(1 - rx) - \text{Li}_2(1 - ry) \right] - \frac{1}{x - y} \left[ \eta(x, r) \ln(1 - rx) - \eta(y, r) \ln(1 - ry) \right],
\]
with \( r, x, y \in \mathbb{C} \).

3. Basic integral III:
The basic integral III has the form of
\[
\mathcal{R}_3 = \int_{-\infty}^\infty dz \, G(z) \theta \left[ \text{Im} \left( \frac{S(\sigma, z)}{Pz + Q} \right) \right],
\]
with \( G(z) \) and \( S(\sigma, z) \) are defined in Eqs. (114, 117) respectively. We know that \( \text{Im} \left( \frac{S(\sigma, z)}{Pz + Q} \right) \) is independent of \( \sigma \), we then can expand the integrand as
\[
\int_{-\infty}^\infty dz \, G(z) \theta \left[ \text{Im} \left( \frac{S(\sigma, z)}{Pz + Q} \right) \right] = \int_{-\infty}^\infty dz \, G(z) \theta \left[ A_0 z^2 + B_0 z + C_0 \right].
\]
Where \( A_0, B_0, C_0 \) are given by
\[
A_0 = P \text{ Im}(E), \quad B_0 = P \Gamma_k + \rho P + \text{Re}(Q) \text{Im}(E) - \text{Im}(Q) \text{Re}(E), \quad C_0 = \text{Im}(Q) \text{Re}(m_k^2) + \text{Re}(Q)(\Gamma_k + \rho)
\]
For \( \text{Im} \left( \frac{S(\sigma, z)}{Pz + Q} \right) \geq 0 \) in the region \( \Omega \subset \mathbb{R} \), one then has
\[
\int_{-\infty}^\infty dz \, G(z) \theta \left[ \text{Im} \left( \frac{S(\sigma, z)}{Pz + Q} \right) \right] = \int_\Omega dz \, G(z).
\]
The integral in right-hand side of this equation can be reduced to basic integral I.

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