Research Article

A New Perspective to Algebraic Characterization on Controllability of Multiagent Systems

Bo Liu,1,2 Housheng Su,3 Licheng Wu,1 and Shengchao He4

1School of Information Engineering, Minzu University of China, Beijing 100081, China
2Artificial Intelligence School, Wuchang University of Technology, Wuhan 430223, China
3Key Laboratory of Imaging Processing and Intelligence Control, School of Artificial Intelligence and Automation, Huazhong University of Science and Technology, Wuhan 430074, China
4College of Science, North China University of Technology, Beijing 100144, China

Correspondence should be addressed to Bo Liu; boliu@ncut.edu.cn and Housheng Su; houshengsu@qq.com

Received 8 December 2019; Revised 15 February 2020; Accepted 27 February 2020; Published 27 March 2020

Academic Editor: Xianming Zhang

Copyright © 2020 Bo Liu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Recently, algebraic characterization of multiagent controllability through its topology has been widely concerned by the systems and control community. The controllability of leader-follower networked multiagent systems under the framework of generic linear dynamics is firstly discussed via $\lambda$-matrix. Some new algebra-theoretic necessary and/or sufficient conditions of the controllability for generic linear multiagent systems are established. Moreover, the controllable conditions for multiagent networks with special topological graphs through $\lambda$-matrix are presented.

1. Introduction

In recent years, cooperative and coordinated control of networked multiagent systems has become a surge of research activities. The controllability is a basic and important problem in modern control theory, which plays a key role in the analysis and synthesis of networked multiagent systems and has wide applications and advantages in formation control, pinning control, containment control and tracking control, etc [1–10].

A controllable multiagent system under a leader-follower framework means that all the followers can be governed to any desirable state (configuration) from any given initial state (configuration) in finite time by designing control protocol or algorithm exerted on leaders for every follower. Controllability problem for a group of systems was first put forward by Tanner [11] in 2004 from the viewpoint of algebra, in which single-integrator continuous-time model with a leader in terms of nearest neighbor rules was formulated, and an algebraic controllable criterion in view of eigenvalues and right eigenvectors of submatrices of Laplacian matrix was obtained under a fixed topology. Studies in this algebraic point of view have provided a theoretical basis for understanding internal relationships and interactions among graph structures, evolutionary protocol, and controllability. Afterwards, considerable efforts on the controllability of complex dynamical networks have been devoted from algebraic perspectives. Liu et al. [12] first proposed the concept of the controllability for a leader-follower dynamic network with discrete-time state and established some algebraic criteria on the controllability under different topologies, respectively. Immediately after that, the authors, respectively, investigated the switching controllability [13, 14] and the group controllability [15, 16] of discrete-time/continuous-time networked multiagent systems with/without multiple leaders and coupling time delays. Wang et al. [17] studied the controllability of multiagent systems with the consensus protocols under directed topologies for high-order-integrator dynamic agents and general linear dynamic intelligent agents, respectively, in which the authors illustrated that the controllability congruously depended on the interconnection topology among agents and dynamics. Guan and Wang [18] studied the controllability of a group of mobile autonomous agents based on absolute protocol under fixed and switching topologies, respectively. In addition, some other important necessary and/or sufficient conditions on the controllability in multiagent networks were...
studied for special topologies, such as path graphs and cycle graphs [19, 20], multichain graphs [21], star graphs and tree graphs [22], grid graphs [23], symmetric structures [24], circulant networks [25], and two-time-scale networks [26–28]. Controllability can also be analyzed by the eigenvectors of Laplacian as in [29]. Moreover, the authors discussed the controllability of dynamic-edge multiagent networks and established PBH-like conditions in [30].

A parallel research method is from graphical perspectives, which depicts the graph-theoretic features of making multiagent systems controllable under different models formulated for structural controllability. The work in [31] first put forward structural controllability for multiagent networks from the viewpoint of graph, where a single-leader multiagent network was introduced and graph-theoretic criterion on controllability was derived under an undirected topology, and these similar problems were extended to higher-order dynamic agents in [32]. Rahman and Ji [33] made use of the relaxed equitable partition for information communication topology to establish some controllable conditions for multiagent systems. The structural controllability was discussed for real-world complex networks by graph-theoretic technique [34]. The controllable characterizations of multiagent networks were given from graphical opinion [35]. Guan et al. studied the structural controllability for heterogeneous multiagent networks based on directed and weighted topology from algebraic and graphical perspectives [36].

At present, the controllability of multiagent networks is still in the initial stage of development. The problem of controllability, whether in the establishment of mathematical model or in theoretical analysis, will face the influence of many factors such as topological structure, evolution protocols, communication distance, information quantization, external disturbance, communication restriction, environmental noise, parameter uncertainty, and so on, which make the problem of controllability of multiagent networks present new characteristics and theoretical difficulties lacking more effective tools for theoretical analysis. In modern control theory, the PBH rank test plays an important role in judging whether a system is controllable, while $\lambda$-matrix exists in the PBH rank test. In this paper, we introduce $\lambda$-matrix to determine the controllability of the generic linear multiagent systems, while $\lambda$-matrix is derived from the PBH. It is well known that the PBH rank test is very useful because it only relies on the eigenvalues of the system matrices. In fact, for multiple delayed and high-dimensional multiagent systems, it is too complex and difficult to compute the rank of controllable matrices of such system. Yet, by Matlab, it is very easy to calculate the eigenvalues of the system via the PBH rank test. However, the existing results on controllability are seldom studied via $\lambda$-matrix, especially from the viewpoint of determinant factors and invariant factors. To the best of our knowledge, in higher algebra, the theory of $\lambda$-matrix is quite mature, while determinant factors and invariant factors can be used to judge whether two matrices are similar or not. Furthermore, if two matrices are equivalent to each other, then they will have the same determinant factors and invariant factors. Therefore, the controllability of multiagent networks via $\lambda$-matrix technique will be highlighted here. This paper concentrates on the controllability of generic linear multiagent networks with absolute protocol on fixed topology. Compared with the existing works, the main contributions of this paper are summed up as follows. Firstly, the model of generic linear multiagent networks based on absolute protocols under a leader-follower framework is proposed. Secondly, the definitions and properties of $\lambda$-matrix, determinant factors, and invariant factors are first introduced to investigate the controllability of multiagent networks by elementary transformation for $\lambda$-matrix. Finally, sufficient and/or necessary algebraic criteria on the controllability via determinant factors and invariant factors are established for generic linear multiagent networks. Furthermore, the controllable conditions for multiagent networks with special topological graphs by $\lambda$-matrix are also established. Compared with other existing methods, the $\lambda$-matrix method has the following advantages. (1) Intuitive form: $[\lambda I - A, B]$ for system $(A, B)$ is easy to obtain. (2) Simple calculation: it only requires elementary transformation for $[\lambda I - A, B]$ to get its Smith standard form, determinant factors and invariant factors, etc. (3) Various methods of judgement: for the same system, we can, respectively, use its Smith standard form, determinant factors, and invariant factors to determine whether the system can be controllable or not. In particular, for special graphs, their corresponding $\lambda$-matrix is simpler and special to judge the controllability. Moreover, Theorems 1–4 in the following are sufficient and effective to judge the controllability properties of a linear system, including a multiagent system, in the current literature. In this paper, we can get the $\lambda$-matrix from the PBH rank test and deal with $\lambda$-matrix to obtain Theorems 1–4 in the following by using some special skills and methods, so that our conclusions are not ordinary, which cannot be directly deduced from the existing conclusions. Compared with other traditional approaches, $\lambda$-matrix and determinant factors have some advantages such as intuitive form, simple calculation, and various judging methods. Therefore, some new algebra-theoretic necessary and/or sufficient conditions of the controllability for generic linear multiagent systems are established via $\lambda$-matrix. So, the introduction of $\lambda$-matrix is of practical significance.

The remainder of the paper is organized as follows. The problem formulation and mathematical preliminaries are given in Section 2. The main results for controllability of generic linear multiagent systems are presented in Section 3. Section 4 shows numerical simulation examples, and a conclusion is given in Section 5.

2. Problem Formulation and Preliminaries

Consider a generic linear multiagent network consisting of $n + n_l$ dynamic agents described as

$$\dot{x}_i = Ax_i + Bu_i, \quad i \in n + n_l, \quad (1)$$

where $x_i \in \mathbb{R}^m$ is the state of the $i$-th agent, $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times p}$, $n + n_l \triangleq \{1, 2, \ldots, n + n_l\}$, $n$ is the followers’ number, and $n_l$ denotes the number of leaders influenced by external control.
inputs, for example, in Figure 1, $u_i \in \mathbb{R}^p$ is the control input. The control agreement protocol similar to reference [37] from the perspective of the controllability is governed by

$$u_i = -K_1 x_i + \sum_{j \neq i} K_2 a_{ij} (x_j - x_i),$$

(2)

where the feedback gains $K_1, K_2 \in \mathbb{R}^{n \times m}$ are to be designed. $a_{ij} \geq 0$ represents the edge weight from $j$ to $i$, and $\mathcal{N}_i$ indicates the neighbor set of agent $i$. The information topology links between agents are described by an undirected weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, a)$ with $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and $\mathcal{E} = \{(v_i, v_j): v_i, v_j \in \mathcal{V}\}$, whose Laplacian is defined as $L = \Delta - a$, where $\Delta$ is the diagonal degree matrix, $a$ is the adjacency matrix, and $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ with

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{k = 1, k \neq i} a_{ik}, & i = j, \\ 0, & \text{otherwise}. \end{cases}$$

For such network in the leader-follower structure (see Figure 1), the Laplacian matrix can be divided into

$$L = \begin{bmatrix} L_1 & L_0 \\ L_0 & L_1 \end{bmatrix},$$

(4)

where $L_1$ and $L_0$ stand for the indices of followers and leaders, respectively, and $L_0$ and $L_{1f}$ stand for the communications from the leaders to the followers and from the followers to the leaders, respectively. More details can be seen in literature [38].

Denote $x_i = (x_{1i}, x_{2i}, \ldots, x_{ni})^T$ and $x_i = (x_{ni+1}, x_{ni+2}, \ldots, x_{ni+n})^T$; then, system (1) with protocol (2) follows that

$$\dot{x}_i = \bar{A} x_i + \bar{B} x_i,$$

(5)

where $\bar{A} = I \otimes (A - B K_1) - L_0 \otimes (B K_2) \in \mathbb{R}^{mn \times mn}, \bar{B} = -L_0 \otimes (B K_2) \in \mathbb{R}^{mn \times 1}.

Remark 1. It is easy to see that the dimension of system (5) is higher than that of system (1). Obviously, it is more difficult and complex to analyze the controllability of multiagent networks so that it is always a challenging task since such system is high dimensional. Next, we will concentrate on the controllability of generic linear networked multiagent systems with high dimension, which is theoretically shown from a new perspective to algebraic features on the controllability.

3. Controllability Analysis Using $\lambda$-Matrix

Here, some useful concepts and symbols about $\lambda$-matrix in the study are reviewed briefly (see references [38, 39] for details), and the mathematical definitions and the classical criterion of the controllability for multiagent systems are given.

Definition 1 (see [39]) ($\lambda$-matrix). If the element of a matrix is a polynomial of $\lambda$, that is, it is the element of $P(\lambda)$, where $P$ is a number field and $P(\lambda)$ is a polynomial ring under the number field $P$, then such matrix is called $\lambda$-matrix. Since $P \subset P[\lambda]$, digital matrix $A$ under the number field $P$ is also $\lambda$-matrix, denoted as $A(\lambda)$.

Definition 2 (see [39]) (the rank of $\lambda$-matrix). For $\lambda$-matrix $A(\lambda)$, if there is a $r$-order determinant $A(\lambda)$ (denoted as $|A(\lambda)|$) which is not zero and all $(r + 1)$-order determinant $A(\lambda)$ are zero (if there still exist), then the rank of $\lambda$-matrix is $r$.

Definition 3 (see [39]) (determinant factor). Suppose that the rank of polynomial matrix $A(\lambda)$ is $r(\geq 1)$; for positive integer $k; 1 \leq k \leq r$, and the maximum common factor of all $k$-order subdeterminants of $A(\lambda)$ with first coefficient being 1 is defined as the $k$-order determinant factor of $A(\lambda)$, denoted as $D_k(\lambda)$. As $k > r$, it is easy to know that $D_k(\lambda) = 0$ from the definition of the rank of $\lambda$-matrix. In addition, for the convenience of discussion, let $D_0(\lambda) = 1$.

Definition 4 (see [39]) (invariant factor). $d_i(\lambda) = D_{i-1}(\lambda) / D_i(\lambda)$ ($i = 1, 2, \ldots, r$) are called as invariant factors. Denote $d_0(\lambda) = 0$ ($n \geq i > r$).

Definition 5 (controllability). System (5) is said to be controllable if for any initial state $x_i(t_0)$ and any final state $x_i$ there exists a finite time $t > t_0$ such that $x_i(t) = x_i$ by adjusting the leaders’ movement.

Definition 6 (controllability matrix). The controllability matrix of system (5) is defined as

$$Q = \begin{bmatrix} \bar{B}, \bar{A} \bar{B}, \bar{A}^2 \bar{B}, \ldots, \bar{A}^{mn-1} \bar{B} \end{bmatrix},$$

(6)

where $Q \in \mathbb{R}^{mn \times mn}$.

From Definitions 5 and 6, we have the following propositions immediately.

Proposition 1 (rank test). System (5) is controllable if $\text{rank}(Q) = mn$.

Proposition 2 (PBH test). System (5) is controllable if one of the following statements is satisfied:

(i) $\text{rank}[sI - \bar{A}, \bar{B}] = mn, \forall s \in \Phi$

(ii) $\text{rank}[\lambda I - \bar{A}, \bar{B}] = mn, i = 1, 2, \ldots, mn$

where $\Phi$ stands for complex number field and $\lambda_i$ is the eigenvalue of $A$ for $i = 1, 2, \ldots, mn$. 

Figure 1: A schematic diagram of the networked topology.
According to Definitions 2–3, we know that the rank of \( A(\lambda) \) as well as its determinant factors and invariant factors can keep unchanged under elementary transformation. So, we will characterize the controllability of system (5) by using determinant factors and invariant factors.

**Theorem 1.** System (5) is controllable if there exists an \( mn \)-order determinant factor of \([sI - \bar{A}, \bar{B}]\) such that \( D_{mn}(s) = 1 \).

**Proof.** Necessity. By contradiction, if \( \text{rank}[sI - \bar{A}, \bar{B}] \neq mn \), then rank\([sI - \bar{A}, \bar{B}]\) = \( r \) \(< mn \). Therefore, \( D_{r}(s) \neq 0 \) and all \( D_{r+1}(s) = 0 \), where \( D_{r}(s) \) are \( r \)-order determinant factors of \([sI - \bar{A}, \bar{B}]\). So from Definition 3, \( D_{mn}(s) = 0 \), which contradicts the fact that \( D_{mn}(s) = 1 \). Therefore, the proof of the necessity is finished.

**Sufficiency 1.** Since system (5) is controllable, then rank\([sI - \bar{A}, \bar{B}]\) = \( mn \), \( \forall s \in \Phi \), and then from Definition 3, there must exist an \( mn \)-order determinant factor of \([sI - \bar{A}, \bar{B}]\), such that \( D_{mn}(s) \neq 0 \), where \( D_{mn}(s) \) is the maximum common factor of all \( mn \)-order subdeterminants of \( A(\lambda) \) with first coefficient being 1, denoted as \( D_{mn}(s) = (D_{mn}(s), D_{mn}(s), \ldots, D_{mn}(s)) \) with \( D_{mn}(s), D_{mn}(s), \ldots, D_{mn}(s) \) being \( mn \)-order subdeterminants of matrix \([sI - \bar{A}, \bar{B}]\), where \( D_{mn}(s), D_{mn}(s), \ldots, D_{mn}(s) \) are all co-prime and not all zero. Therefore, \( D_{mn}(s) = (D_{mn}(s), D_{mn}(s), \ldots, D_{mn}(s)) = 1 \); this gives the proof of the sufficiency.

**Remark 2.** Notice that for some polynomials, if two polynomials are co-prime, then all polynomials are co-prime.

**Theorem 2.** System (5) is controllable if there are two co-prime \( mn \)-order determinant factors of \([sI - \bar{A}, \bar{B}]\).

**Proof.** Since there exist two co-prime \( mn \)-order determinant factors of \([sI - \bar{A}, \bar{B}]\), without loss of generality, suppose that \( (D_{mn}(s), D_{mn}(s)) = 1 \), and we can easily have

\[
\begin{vmatrix}
D_{mn}(s) & D_{mn}(s) \\
D_{mn}(s) & D_{mn}(s)
\end{vmatrix} = \begin{vmatrix}
D_{mn}(s) & D_{mn}(s)
\end{vmatrix}
\]

for \( r \geq 2 \); therefore, \( (D_{mn}(s), D_{mn}(s), \ldots, D_{mn}(s)) = 1 \). From Theorem 1, system (5) is controllable. This completes the proof.

**Remark 3.** Theorem 2 provides a simple way to judge the controllability of multiagent networks with lower dimension. Especially, if there exists an \( mn \)-order determinant factor of \([sI - \bar{A}, \bar{B}]\), which is a nonzero constant, then the system must be controllable.

**Definition 7.** A square \( \lambda \)-matrix is unimodular if \( \det(A(\lambda)) = c \neq 0 \), where \( c \) is a constant.

From Definition 7, the following result can be obtained immediately.

**Theorem 3.** System (5) is controllable if there exists an \( mn \)-order unimodular matrix for \([sI - \bar{A}, \bar{B}]\).

**Remark 4.** Notice that the unimodular matrix is easier to check and calculate so that it can play a key role in judging the controllability of multiagent networks.

**Definition 8.** \( \lambda \)-matrix \( A(\lambda)_{n \times n} \) is reversible if there is \( \lambda \)-matrix \( B(\lambda)_{n \times n} \), such that \( A(\lambda)B(\lambda) = B(\lambda)A(\lambda) = E \), where \( E \) is \( n \times n \) identity matrix.

**Lemma 1** (see [39]). \( \lambda \)-matrix \( A(\lambda)_{n \times n} \) is reversible if \( \det(A(\lambda)) = c \neq 0 \), where \( c \) is a constant.

From Definition 8 and Lemma 1, the following results are easily available.

**Corollary 1.** \( \lambda \)-matrix \( A(\lambda)_{n \times n} \) is reversible if \( \lambda \)-matrix \( A(\lambda)_{n \times n} \) is unimodular.

**Corollary 2.** System (5) is controllable if there exists an \( mn \)-order reversible matrix for \([sI - \bar{A}, \bar{B}]\).

It is generally known that, in modern control theory, the nonsingular linear transformation does not change the controllability of the system, and the nonsingular linear transformation is the result of some elementary transformation synthesis.

**Proposition 3** (see [39]). If \( \lambda \)-matrix \( A(\lambda) \) can be changed into \( B(\lambda) \) by a series of elementary transformations, then \( A(\lambda) \) is equivalent to \( B(\lambda) \), denoted as \( A(\lambda) \sim B(\lambda) \).

**Lemma 2** (see [39]). Any \( \lambda \)-matrix \( A(\lambda) \in P[\lambda]_{n \times n} \) is equivalent to the following diagonal form:

\[
\Lambda = \begin{bmatrix}
d_1(\lambda) & & \\
& \ddots & \\
& & d_r(\lambda) \\
0 & & \ddots \\
& & & 0
\end{bmatrix},
\]

where \( \Lambda \) is called as the standard form of \( A(\lambda) \), where \( r \geq 1 \); an polynomial \( d_i(\lambda) \) is called as an invariant factor whose first coefficient is 1 for \( i = 1, 2, \ldots, r \) and satisfies \( d_i(\lambda) \mid d_{i+1}(\lambda) \) for \( i = 1, 2, \ldots, r - 1 \).

**Lemma 3** (see [39]). Equivalent \( \lambda \)-matrices have the same rank.

**Lemma 4** (see [39]). \( \lambda \)-matrices are equivalent if they have the same determinant factors and the same invariant factors.

The PBH test is an important method to judge the controllability of multiagent networks, but if \([sI - \bar{A}, \bar{B}]\) is extremely complex and high dimensional, then we can turn \([sI - \bar{A}, \bar{B}]\) into its standard form via elementary transformations, which is easier to judge the controllability of such
Step 1: calculate the rank of \( A(\lambda) \). Suppose that \( \text{rank}(A(\lambda)) = r \); then, there are \( r \) determinant factors in \( A(\lambda) \).

Step 2: by elementary transformation for \( A(\lambda) \), choose \( i_1, i_2, \ldots, i_r \) rows and \( i_1, i_2, \ldots, i_k \) columns \((1 \leq i_1 < i_2 < \cdots < i_k \leq r)\) from \( A(\lambda) \) to constitute a determinant of order \( k \), that is, \( d_1(\lambda), d_2(\lambda), \ldots, d_k(\lambda) \).

Step 3: when \( i_1 = 1, i_2 = 2, \ldots, i_k = k \), \( d_k(\lambda) \) is the greatest common factor of all the determinant of \( k \)-order in Step 2.

Step 4: by step 3, \( A(\lambda) \) turns into its Smith standard form \( A \), and it is easy to get the \( k \)-order determinant factor of \( A(\lambda) \) as \( D_k(\lambda) = d_1(\lambda) \times d_2(\lambda) \times \cdots \times d_k(\lambda) \), \( d(k = 1, 2, \ldots, r) \).

Step 5: by step 4, let \( d_1(\lambda) = D_1(\lambda), d_2(\lambda) = D_2(\lambda) \mid D_1(\lambda), \ldots, d_4(\lambda) = D_4(\lambda) \mid D_3(\lambda) \).

**Theorem 4.** System (5) is controllable if \( mn \) invariant factors of \( [sI - \bar{A}, \bar{B}] \) are all equal to 1.

**Proof.** Based on Theorem 1 and Lemma 3, we know that system (5) is controllable if an \( mn \)-order determinant factor \( D_{mn}(\lambda) = 1 \) of \( [sI - \bar{A}, \bar{B}] \). At the same time, \( d_1(\lambda) = D_1(\lambda) = 1, d_2(\lambda) = D_2(\lambda) = 1, \ldots, d_{mn}(\lambda) = D_{mn-1}(\lambda) \mid D_{mn}(\lambda) = 1 \). This completes the proof. \( \square \)

From Theorem 4 and Lemmas 2–4, the following simple and easy result can be obtained.

**Corollary 3.** System (5) is controllable if \( [sI - \bar{A}, \bar{B}] \sim A = \text{diag} \left\{ 1, 1, 1, 0, \ldots, 0 \right\} \).

Note that Theorem 4 is derived from invariant factors that make up the standard form of \( A(\lambda) \). Therefore, we will give a brief algorithm to get invariant factors by determinant factors (see Algorithm 1).

**Algorithm 1:** Algorithm for solving invariant factors.

Figure 2: Path graph.

3.1. Some Special Graphs. This section will present the controllable conditions for multiagent networks with special topological graphs through \( \lambda \)-matrix and find that it is more effective, simpler, and easier to compute, test, and verify the results.

From Theorem 2 of literature [17], with leaders selected in advance, the controllability for the generic linear multiagent network is decided by matrix pair \( (-L_f, -L_d) \), where \( L_f \) and \( L_d \) show the information flows among follower agents and those from leaders to followers, respectively.

**Lemma 5** (see [17]). System (5) is controllable if matrix pair \( (-L_f, -L_d) \) is controllable.

**Remark 5.** From Lemma 5, we can immediately know that \( [sI - \bar{A}, \bar{B}] \sim [sI + L_f, -L_d] \), so there must exist \( K_1, K_2 \) such that system (5) is controllable if matrix pair \( (-L_f, -L_d) \) is controllable. This provides more and better methods to make generic linear multiagent systems controllable.

**Theorem 5.** Path graph is controllable.

**Proof.** For Figure 2(a), we can have

\[
-L_f = \begin{bmatrix}
    a & -a & 0 & \cdots & 0 & 0 \\
    -2a & -a & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 2a & -a \\
    0 & 0 & \cdots & -a & a
\end{bmatrix}
\]

Then,

\[
G(s) = [sI + L_f, -L_d] = \begin{bmatrix}
    s - a & a & 0 & \cdots & 0 & 0 & 0 \\
    a & s - 2a & a & \cdots & 0 & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & \cdots & s - 2a & a & 0 \\
    0 & 0 & \cdots & a & s - a & c
\end{bmatrix}
\]

Removing the first column of \( G(s) \),
Easily compute $\det(G_1(s)) = a^{n-1}c \neq 0$. According to Theorem 3, path graph is controllable. The proof is similar for Figure 2(b), omitted here.

**Remark 6.** Notice that from path graph, it is more obvious to see that its $\lambda$-matrix is $n \times (n+1)$, in which we only need to delete the first column of $\lambda$-matrix to get an $n \times n$ lower triangular matrix with constant diagonal elements. It is easy to see that the determinant of the $n \times n$ lower triangular matrix is nonzero constant. Immediately, we can know that path graph is controllable. This is the simplest and most intuitive way of judging the controllability by the various methods available.

**Theorem 6.** Complete graph is uncontrollable.

**Proof.** For Figure 3, we can have

\[
-L_f = \begin{bmatrix}
(n-1)a & -a & -a & \cdots & -a & -a \\
-a & (n-1)a & -a & \cdots & -a & -a \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-a & -a & -a & \cdots & (n-1)a & -a \\
-a & -a & -a & \cdots & -a & (n-1)a \\
\end{bmatrix},
\]

and

\[
-L_{fi} = \begin{bmatrix}
c \\
c \\
c \\
c \\
\end{bmatrix}.
\]

Then,

\[
G(s) = [sI + L_f, -L_{fi}]
\]

Removing the $i$-th column of $G(s)$,

\[
G_i(s) = \begin{bmatrix}
s - (n-1)a & a & \cdots & a & c \\
\vdots & a & s - (n-1)a & \cdots & a & c \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a & a & a & \cdots & a & c \\
a & a & a & \cdots & a & \cdots & a & c \\
a & a & a & \cdots & s - (n-1)a & c \\
\end{bmatrix}.
\]
Figure 3: Complete graph.

Figure 4: Star graph.

Figure 5: Topology.
for $i = 1, 2, \ldots, n$; by computing, we can get that
\[
\det(G_i(s)) = (-1)^{n+1}c(s-na)^{n-1}
\]
is not a constant, and if we remove the $(n+1)$-th column, then

\[
G_{n+1}(s) = \begin{bmatrix}
    s - (n-1)a & a & a & \cdots & a & a \\
    a & s - (n-1)a & a & \cdots & a & a \\
    \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
    a & a & a & \cdots & s - (n-1)a & a \\
    a & a & a & \cdots & a & s - (n-1)a
\end{bmatrix}.
\]
By computing, we can get that \( \det(G_{n+1}(s)) = s(s-na)^{n-1} \) is also not a constant. Thus, from Theorem 3, complete graph is uncontrollable.

**Theorem 7.** Star graph is uncontrollable.

**Proof.** For Figure 4, we can have

\[
-L_i = \begin{bmatrix}
(n-1)a & -a & -a & \cdots & -a & -a \\
-a & a & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-a & 0 & 0 & \cdots & a & 0 \\
-a & 0 & 0 & \cdots & 0 & a
\end{bmatrix},
\]

(16)

\[
-L_i = \begin{bmatrix}
c \\
c \\
\vdots \\
c
\end{bmatrix}
\]

(17)

Removing the \( i \)-th column of \( G(s) \),

\[
G_i(s) = \begin{bmatrix}
s - (n-1)a & a & \cdots & a & a & a & c \\
a & s - a & \cdots & 0 & 0 & \cdots & 0 & c \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
a & 0 & \cdots & s - a & 0 & \cdots & 0 & c \\
a & 0 & \cdots & 0 & a & \cdots & 0 & c \\
i + 1 & a & 0 & \cdots & 0 & s - a & \cdots & 0 & c \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a & 0 & \cdots & 0 & 0 & \cdots & s - a & c
\end{bmatrix}
\]

(18)
for \( i = 1, 2, \ldots, n \); by computing, we can get that 
\[ \text{det}(G(s)) = (-1)^{n+1}(s - na) (s - a)^{n-2} \] 
is not a constant, and if we remove the \((n+1)\)-th column, then

\[
G_{n+1}(s) = \begin{bmatrix}
  s - (n-1)a & a & a & \cdots & a & a \\
  a & s - a & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  a & 0 & 0 & \cdots & s - a & 0 \\
  a & 0 & 0 & \cdots & 0 & s - a
\end{bmatrix}
\]

By computing, we can get \( \text{det}(G_{n+1}(s)) = s(s - na) (s - a)^{n-2} \) is also not a constant. Thus, from Theorem 3, star graph is uncontrollable.

Remark 7. In context, the system matrices with special topological graphs have simple and special forms, so the corresponding \( \lambda \)-matrix—an \( n \times (n + 1) \) matrix is also simple and special. Subsequently, it only needs to delete one column of \( \lambda \)-matrix to get an \( n \times n \) matrix, whose determinant is intuitive and easy to compute. Thus, by introducing \( \lambda \)-matrix, we can easily judge whether the system can be controllable or not. Therefore, the \( \lambda \)-matrix method is more intuitive and easier to compute, test, verify, and judge the controllability for generic linear multiagent systems.

4. Simulation Examples

4.1. Example 1. In this example, we consider a five-agent network with followers 1–3 and leaders 4–5 described by Figure 5, and the corresponding Laplacian matrices of Figure 5 can be given by

\[
L_f = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix},
\]

\[
L_l = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\]

Let

\[
A = \begin{bmatrix}
-2 & 1 \\
1 & -1
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\]

and easily calculate rank\([ -L_f, (L_f)^2, (L_f)^3 ]\) = 3; from Theorem 5, \((-L_f, -L_f)\) is completely controllable. Since \( A \in \mathbb{R}^{2 \times 2} \) and the number of followers is 3, then we can easily take

\[
E_m = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix},
\]

\[
F_m = \begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix},
\]

\[
I_n = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Design \( K_1 \) and \( K_2 \) as \( K_1 = B^{-1}(A - E_m) = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} \) and \( K_2 = CF_m = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \). Compute

\[
\tilde{A} = I \otimes (A - BK_1) - L_f \otimes (BK_2) = \begin{bmatrix}
0 & -2 & 0 & 0 & 0 \\
-1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & -2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
\tilde{B} = -L_f \otimes (BK_2) = \begin{bmatrix}
0 & -2 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0
\end{bmatrix}
\]

Then,

\[
G(s) = [sI - \tilde{A}, \tilde{B}] = \begin{bmatrix}
s & 1 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 0 \\
1 & s + 1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & -2 & s & 3 & 0 & -2 & 0 & 0 & 0 & 0 \\
-1 & -1 & 2 & s + 2 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & s & 1 & 0 & 0 & 0 & -2 \\
0 & 0 & -1 & -1 & 1 & s + 1 & 0 & -1 & -1 & 0
\end{bmatrix}
\]

Based on computing, we can get \( D_0(s) = 1 \) of \( G(s) = [sI - \tilde{A}, \tilde{B}] \). Based on Theorem 1, such system is controllable.

Figures 6 and 7 are the followers’ trajectories, where the stars are random initial positions (configuration) and the circular dots are desired positions (configuration), such as a straight-line configuration and a triangle configuration, respectively.

From this example, we can find that we can design appropriate \( K_1 \) and \( K_2 \) based on matrix pair \((-L_f, -L_f)\).

4.2. Example 2. In this example, we consider a four-agent network with followers 1–3 and leader 4 described by Figure 8, and the corresponding Laplacian matrices of Figure 8 can be given by
controllability of multiagent systems. This new perspective provides a new way to further explore the methods on the controllability of such system. The unimodular matrix can also provide simpler and easier multiagent networks. Moreover, invariant factors and factors play a key role in characterizing the controllability of λ-matrix. The results have shown that the determinant of generic linear multiagent networks from the perspective of this paper has studied the controllability problem of the multiagent systems. It is easy to see that the method using λ-matrix is easier to judge. Especially for low-order matrices, almost no calculation is required.

Remark 8. It is easy to see from the numerical examples 1-2 that the method using λ-matrix is easier to judge. Especially for low-order matrices, almost no calculation is required.

5. Conclusion

This paper has studied the controllability problem of the generic linear multiagent networks from the perspective of λ-matrix. The results have shown that the determinant factors play a key role in characterizing the controllability of multiagent networks. Moreover, invariant factors and unimodular matrix can also provide simpler and easier methods on the controllability of such system. This new perspective provides a new way to further explore the controllability of multiagent systems.

Data Availability

No data were used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant nos. 61773023, 61991412, 61773416, and 61873318, the Program for HUST Academic Frontier Youth Team under Grant no. 2018QYTD07, and the Frontier Research Funds of Applied Foundation of Wuhan under Grant no. 2019010701011421.

References

[1] H. Su, Y. Ye, Y. Qiu, Y. Cao, and M. Z. Q. Chen, “Semi-output consensus for discrete-time switching networked systems subject to input saturation and external disturbances,” IEEE Transactions on Cybernetics, vol. 49, no. 11, pp. 3934–3945, 2019.
[2] J. Zhang and H. Su, “Time-varying formation for linear multi-agent systems based on sampled data with multiple leaders,” Neurocomputing, vol. 339, pp. 59–65, 2019.
[3] X. Zhang, Q. Han, X. Ge et al., “Networked control systems: a survey of trends and techniques,” IEEE/CAA Journal of Automatica Sinica, vol. 7, no. 1, pp. 1–17, 2020.
[4] L. Ding, Q.-L. Han, X. Ge, and X.-M. Zhang, “An overview of recent advances in event-triggered consensus of multiagent systems,” IEEE Transactions on Cybernetics, vol. 48, no. 4, pp. 1110–1123, 2018.
[5] X. Wang and H. Su, “Consensus of hybrid multi-agent systems by event-triggered/self-triggered strategy,” Applied Mathematics and Computation, vol. 359, pp. 490–501, 2019.
[6] Y. Liu and H. Su, “Containment control of second-order multi-agent systems via intermittent sampled position data communication,” Applied Mathematics and Computation, vol. 362, Article ID 124522, 2019.
[7] B. Liu, N. Xu, H. Su, L. Wu, and J. Bai, “On the observability of leader-based multiagent systems with fixed topology,” Complexity, vol. 2019, Article ID 9487574, 10 pages, 2019.
[8] X. Wang and H. Su, “Self-triggered leader-following consensus of multi-agent systems with input time delay,” Neurocomputing, vol. 330, pp. 70–77, 2019.
[9] Y. Liu and H. Su, “Some necessary and sufficient conditions for containment of second-order multi-agent systems with sampled position data,” Neurocomputing, vol. 378, pp. 228–237, 2020.
[10] J. Zhang and H. Su, “Formation-containment control for multi-agent systems with sampled data and time delays,” Neurocomputing, 2019.
[11] H. G. Tanner, “On the controllability of nearest neighbor interconnections,” in Proceedings of the 43rd IEEE Conference on Decision and Control, vol. 3, pp. 2467–2472, Nassau, Bahamas, December 2004.
[12] B. Liu, T. Chu, L. Wang, and G. Xie, “Controllability of a leader-follower dynamic network with switching topology,” IEEE Transactions on Automatic Control, vol. 53, no. 4, pp. 1009–1013, 2008.
[13] B. Liu, T. Chu, L. Wang, Z. Zuo, G. Chen, and H. Su, “Controllability of switching networks of multi-agent systems,” International Journal of Robust and Nonlinear Control, vol. 22, no. 6, pp. 630–644, 2012.

[14] B. Liu, H. Su, R. Li, D. Sun, and W. Hu, “Switching controllability of discrete-time multi-agent systems with multiple leaders and time-delays,” Applied Mathematics and Computation, vol. 228, pp. 571–588, 2014.

[15] B. Liu, Y. Han, F. Jiang, H. Su, and J. Zou, “Group controllability of discrete-time multi-agent systems,” Journal of the Franklin Institute, vol. 353, no. 14, pp. 3524–3559, 2016.

[16] B. Liu, H. Su, F. Jiang, Y. Gao, L. Liu, and J. Qian, “Group controllability of continuous-time multi-agent systems,” IET Control Theory & Applications, vol. 12, no. 11, pp. 1665–1671, 2018.

[17] L. Wang, F. Jiang, G. Xie, and Z. Ji, “Controllability of multi-agent systems based on agreement protocols,” Science in China Series F: Information Sciences, vol. 52, no. 11, pp. 2074–2088, 2009.

[18] Y. Guan and L. Wang, “Structural controllability of multi-agent systems with absolute protocol under fixed and switching topologies,” Science in China Series F: Information Sciences, vol. 60, pp. 1–15, 2017.

[19] G. Parlangeli and G. Notarstefano, “On the reachability and observability of path and cycle graphs,” IEEE Transactions on Automatic Control, vol. 57, no. 3, pp. 743–748, 2012.

[20] X. Liu and Z. Ji, “Controllability of multiagent systems based on path and cycle graphs,” International Journal of Robust and Nonlinear Control, vol. 28, no. 1, pp. 296–309, 2018.

[21] M. Cao, S. Zhang, and M. K. Camlibel, “A class of uncontrollable diffusively coupled multiagent systems with multichain topologies,” IEEE Transactions on Automatic Control, vol. 58, no. 2, pp. 465–469, 2013.

[22] Z. Ji, H. Lin, and H. Yu, “Leaders in multi-agent controllability under consensus algorithm and tree topology,” Systems & Control Letters, vol. 61, no. 9, pp. 918–925, 2012.

[23] G. Notarstefano and G. Parlangeli, “Controllability and observability of grid graphs via reduction and symmetries,” IEEE Transactions on Automatic Control, vol. 58, no. 7, pp. 1719–1731, 2013.

[24] H. Su, M. Long, and Z. Zeng, “Controllability of two-time-scale discrete-time multiagent systems,” IEEE Transactions on Cybernetics, vol. 50, no. 4, pp. 1440–1449, 2018.

[25] M. Nabi-Abdolyousefi and M. Mesbahi, “On the controllability properties of circulant networks,” IEEE Transactions on Automatic Control, vol. 58, no. 12, pp. 3179–3184, 2013.

[26] M. Long, H. Su, and B. Liu, “Group controllability of two-time-scale multi-agent networks,” Journal of the Franklin Institute, vol. 355, no. 13, pp. 6045–6061, 2018.

[27] M. Long, H. Su, and B. Liu, “Second-order controllability of two-time-scale discrete-time multi-agent systems,” IET Control Theory & Applications, vol. 13, no. 15, pp. 2356–2364, 2019.

[28] M. Long, H. Su, and B. Liu, “Second-order controllability of two-time-scale multi-agent systems,” Applied Mathematics and Computation, vol. 343, pp. 299–313, 2019.

[29] B. Zhao, Y. Guan, and L. Wang, “Controllability improvement for multi-agent systems: leader selection and weight adjustment,” International Journal of Control, vol. 89, no. 10, pp. 2008–2018, 2016.

[30] Y. Wang, J. Xiang, Y. Li, and M. Z. Q. Chen, “Controllability of dynamic-edge multi-agent systems,” IEEE Transactions on Control of Network Systems, vol. 5, no. 3, pp. 857–867, 2018.

[31] M. Zamani and H. Lin, “Structural controllability of multi-agent systems,” in Proceedings of the American Control Conference, pp. 5743–5748, St. Louis, MO, USA, June 2009.

[32] P. Alireza, H. Lin, and Z. Ji, “Structural controllability of high order dynamic multi-agent systems,” in Proceedings of IEEE Conference on Automation and Mechatronics, pp. 327–332, Singapore, June 2010.

[33] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, “Controllability of multi-agent systems from a graph-theoretic perspective,” SIAM Journal on Control and Optimization, vol. 48, no. 1, pp. 162–186, 2009.

[34] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, “Controllability of complex networks,” Nature, vol. 473, no. 7346, pp. 167–173, 2011.

[35] Z. Ji and H. Yu, “A new perspective to graphical characterization of multiagent controllability,” IEEE Transactions on Cybernetics, vol. 47, no. 6, pp. 1471–1483, 2017.

[36] Y. Guan, Z. Ji, L. Zhang, and L. Wang, “Controllability of heterogeneous multi-agent systems under directed and weighted topology,” International Journal of Control, vol. 89, no. 5, pp. 1009–1024, 2016.

[37] Z. Ji, H. Lin, and H. Yu, “Protocols design and uncontrollable topologies construction for multi-agent networks,” IEEE Transactions on Automatic Control, vol. 60, no. 3, pp. 781–786, 2015.

[38] C. Godsil and G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, NY, USA, 2001.

[39] W. Qiu, Advanced Algebra, Tsinghua University Press, Beijing, China, 2013.