Large-$N$ Eigenvalue Distribution of Randomly Perturbed Asymmetric Matrices

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Short title: Eigenvalue Distribution of Random Asymmetric Matrices

PACS number(s): 05.40.+j

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Abstract

The density of complex eigenvalues of random asymmetric $N \times N$ matrices is found in the large-$N$ limit. The matrices are of the form $H_0 + A$ where $A$ is a matrix of $N^2$ independent, identically distributed random variables with zero mean and variance $N^{-1}v^2$. The limiting density $\rho(z, z^*)$ is bounded. The area of the support of $\rho(z, z^*)$ cannot be less than $\pi v^2$. In the case of $H_0$ commuting with its conjugate, $\rho(z, z^*)$ is expressed in terms of the eigenvalue distribution of the non-perturbed part $H_0$. 
Random Hermitean and real symmetric matrices have been extensively studied since the 50’s, the time when Wigner introduced them into theoretical physics. A lot of results concerning these matrices and respective techniques are known now. In contrary to this, random complex and real asymmetric matrices are much less studied. Although they have already proved to be useful. We mention here only two examples (but see a discussion in [1]). These are: i) quantum chaotic scattering and decaying processes, where complex eigenvalues of random non-Hermitean matrices are used to analyse statistical properties of resonances [1, 2, 3], and ii) neural network dynamics where synaptic matrices are in general asymmetric and the distribution of their eigenvalues is important for the understanding of network dynamics [4, 5].

In this letter we consider random real asymmetric matrices of the form

\[ H = H_0 + A. \]

\( A = [a_{jk}]_{j,k=1}^N \) is a matrix of \( N^2 \) independent, identically distributed random variables such that

\[ \langle a_{jk} \rangle = 0, \quad \langle a_{jk} a_{lm} \rangle = N^{-1} v^2 \delta_{jl} \delta_{km}. \]  

The angle brackets \( \langle \ldots \rangle \) denote average over the random variables \( a_{jk} \). For simplicity we assume that \( a_{jk} \) are Gaussian but our results remain valid for a wider class of distributions. We treat \( A \) as a perturbation and \( H_0 \) as the non-perturbed part and our aim is to determine the large-\( N \) limit of the averaged density of complex eigenvalues of \( H_0 + A \).

If \( A \) and \( H_0 \) are symmetric (or Hermitean) and \( A \) obeys the GOE (GUE) statistics, then \( H_0 + A \) is known as the deformed GOE (GUE) [6]. In this ensemble eigenvalues are real and, hence, their density is completely determined by the imaginary part of the Green’s function \( G(E + i0) = \langle N^{-1} \text{tr} (E + i0 - H)^{-1} \rangle \). It appears that in the large-\( N \) limit the Green’s function of the deformed GOE (GUE) is related to that of \( H_0 \) by the so-called Pastur’s equation [7]:

\[ G(z) = G_0(z - v^2 G(z)). \]  

Although this equation cannot be solved explicitly (except of a few cases) it
provides useful information about the density of eigenvalues. For instance, one can prove that the density of eigenvalues in the deformed GOE (GUE) is bounded and generically decays as the square root in the vicinity of the spectrum boundaries [8].

Eigenvalues of asymmetric matrices are complex and their average density $\rho(z, z^*)$ is determined by the electrostatic potential

$$\Phi(\kappa, z, z^*) = -N^{-1}\langle \log \det[(zI - H)^* (zI - H) + \kappa^2 I]\rangle$$

by means of Poisson’s equation

$$\rho(z, z^*) = -\frac{1}{\pi} \left. \frac{\partial^2 \Phi(\kappa, z, z^*)}{\partial z \partial z^*} \right|_{\kappa=0}$$

$I$ is the identity matrix. Positive infinitesimal $\kappa$ is introduced in order to regularize the potential. Provided $\kappa = 0$, $\Phi$ as a function of complex $z$ has a singularity whenever $z$ equals one of the eigenvalues of $H$.

Anticipating an important role of positive semi-definite matrices $\mathcal{H} = (zI - H)^* (zI - H)$ in studying complex eigenvalues of $H$ we introduce the following Green’s function

$$R(\kappa) = \langle N^{-1}\text{tr} (\mathcal{H} + \kappa^2 I)^{-1}\rangle$$

(3)

corresponding to $\mathcal{H}$. $R(\kappa)$ as a function of $\kappa$ is analytic in the right half of the complex plane and obviously determines the density of eigenvalues of $\mathcal{H}$. We show that in the large-$N$ limit the Green’s functions of $\mathcal{H}$ and $\mathcal{H}_0 = (zI - H_0)^* (zI - H_0)$ are related by the equation (10) which can be thought as generalization of Pastur’s result to the case of positive semi-definite random matrices. In passing we find derivatives of the electrostatic potential. This allows us to derive an expression for the average density of complex eigenvalues of $H$, $\rho(z, z^*)$, and for the domain of their distribution. The respective expressions (12) - (14) are given in terms of $H_0$. Actually they set up the only restriction to $H_0$: quantities entering (12) - (14) must be well defined in the large-$N$ limit. We do not specify $H_0$ further. It can be real or complex and either deterministic or random. In the latter case it is assumed that $H_0$ is statistically independent.
of $A$ and it is understood that the average over realizations of $H_0$ has been taken.

At this point it is worth mentioning that in the specific case of $H_0$ commuting with its conjugate $H_0^*$ (i.e., $H_0$ can be symmetric, skew-symmetric, Hermitean, skew-Hermitean, etc) $\rho(z, z^*)$ can be explicitly expressed in terms of the density of eigenvalues of $H_0$ (see (15)-(17)). This should be compared with the case of deformed GOE (GUE) where only the relation (2) between Green’s functions is known.

Our last remark concerns matrices studied in [1, 3]. They are of the form $iVV^\top + B$, where $V$ is an $N \times M$ matrix of $NM$ independent Gaussian variables and $B$ obeys the GOE statistics. These random matrices differ from those considered here in that aspect that $B$ is symmetric while $A$ is asymmetric. The eigenvalue distribution of $VV^\top$ is known [10] and it seems interesting to recover the results of [1, 3], which were obtained by means of the replica trick [1] and supersymmetry calculations [3], in the framework of our approach. But this problem goes beyond the aim of the present letter.

Introducing the notation $G(\kappa)$ for the inverse to $\mathcal{H} + \kappa^2 I$ we rewrite the following obvious matrix identity $I = \langle G(\kappa)(\mathcal{H} + \kappa^2 I)^{-1}\rangle$ as

$$\kappa^2 \langle G(\kappa) \rangle = I - (zI - H_0)^* \langle (zI - H)G(\kappa) \rangle + \langle A^*(zI - H)G(\kappa) \rangle.$$  \hfill (4)

$zI - H_0$ is statistically independent of $G(\kappa)$ but $A$ which enters the $(zI - H)$ term in the r.h.s. of (4) is not. In order to decouple $\langle A G(\kappa) \rangle$ and $\langle A^* A G(\kappa) \rangle$ we first notice that each of the entries $G_{pq}$ of the matrix $G(\kappa)$ is a function of the Gaussian variable $a_{lm}$. Therefore

$$\langle a_{lm} G_{pq} \rangle = \langle a_{lm}^2 \rangle \langle \partial G_{pq}/\partial a_{lm} \rangle = N^{-1} v^2 \langle \partial G_{pq}/\partial a_{lm} \rangle.$$ \hfill (5)

This is the only place where Gaussian distribution of $a_{lm}$ is used. In the non-Gaussian case it can be shown that (5) holds up to the $1/N^2$ order if $a_{lm} = N^{-1/2} \alpha_{lm}$ and the random variables $\alpha_{lm}$ possess several first moments.
Straightforward application of (5) and the following rule for differentiating matrix elements of $G(\kappa)$ with respect to those of $A$

$$\frac{\partial G_{pq}}{\partial a_{lm}} = [G(zI - H)^*]_{pl} G_{mq} + G_{pm} [(zI - H)G]_{lq}.$$  \hfill (6)

gives

$$\langle (zI - H)G(\kappa) \rangle = (zI - H_0)\langle G(\kappa) \rangle - v^2 \langle (zI - H)G(\kappa) \cdot N^{-1}\text{tr}G(\kappa) \rangle + O(1/N).$$

One can readily check (6) making use of $\partial G_{pq}/\partial H_{km} = -G_{pk} G_{mq}$ and of the chain rule. The normalized trace of $G(\kappa)$ is a self-averaging extensive quantity \textsuperscript{8}. That is it becomes non-random in the large-$N$ limit: $N^{-1}\text{tr}G(\kappa) = R(\kappa) + O(1/N)$, where $R(\kappa) = \langle N^{-1}\text{tr}G(\kappa) \rangle$. Therefore we conclude that

$$\langle (zI - H)G(\kappa) \rangle = (zI - H_0)\langle G(\kappa) \rangle [1 + v^2 R(\kappa)]^{-1} + O(1/N).$$  \hfill (7)

Similar reasoning leads to

$$\langle A^*(zI - H)G(\kappa) \rangle = -v^2\kappa^2 R(\kappa)\langle G(\kappa) \rangle + O(1/N).$$  \hfill (8)

Collecting (4) and (7)-(8) we find that in the leading order

$$\langle G(\kappa) \rangle = \frac{1 + v^2 R(\kappa)}{(zI - H_0)^* (zI - H_0) + \kappa^2 [1 + v^2 R(\kappa)]^2 I}.$$  \hfill (9)

Introducing the notation $G_0(\kappa)$ for the inverse to $H_0 + \kappa^2 I$ one can write (9) in the form $\langle G(\kappa) \rangle = (1 + v^2 R(\kappa))G_0(\kappa[1 + v^2 R(\kappa)])$. $R(\kappa)$ is to be determined from the self-consistency equation

$$R(\kappa) = [1 + v^2 R(\kappa)]R_0(\kappa[1 + v^2 R(\kappa)]) ,$$  \hfill (10)

where $R_0(\kappa) = N^{-1}\text{tr}G_0(\kappa)$.

Since $-\partial \Phi/\partial z^* = \langle N^{-1}\text{tr}(zI - H)G(\kappa) \rangle$ one can use (9) and (10) to calculate $\rho(z, z^*)$. Indeed,

$$-\frac{\partial \Phi(\kappa, z, z^*)}{\partial z^*} = N^{-1}\text{tr}(zI - H_0)G_0(\kappa[1 + v^2 R(\kappa)]) + O(1/N).$$
Simple analysis of (10) shows that in the leading order
\[ \left. \frac{\partial \Phi(\kappa, z, z^*)}{\partial z^*} \right|_{\kappa=0} \]
in given by
\[ - \left. \frac{\partial \Phi(\kappa, z, z^*)}{\partial z^*} \right|_{\kappa=0} = N^{-1} \mathrm{tr}(zI - H_0)G_0(\gamma(z, z^*)) , \]
where \( \gamma(z, z^*) = \lim_{\kappa \to 0^+} \kappa[1 + v^2 R(\kappa)] \) is the solution of \( R_0(\gamma) = v^{-2} \) if \( z \) lies inside the domain \( D \) determined by the inequality
\[ R_0(0) = N^{-1} \mathrm{tr}[(zI - H_0)^*(zI - H)]^{-1} \geq v^{-2} \]
and \( \gamma(z, z^*) = 0 \) otherwise. Since in the latter case \( \left. \frac{\partial \Phi}{\partial z^*} \right|_{\kappa=0} \) does not depend on \( z \) we conclude immediately that \( \rho(z, z^*) = 0 \) outside \( D \). On the other hand, differentiating (11) with respect to \( z \) one finds that inside \( D \)
\[ \rho(z, z^*) = (\pi v^2)^{-1} - \pi^{-1} I(z, z^*) \]
where \( I(z, z^*) \) is the large-\( N \) limit of
\[ -N^{-1} \mathrm{tr} (zI - H_0)G_0(\gamma(z, z^*))((zI - H_0)^*G_0(\gamma(z, z^*)) \]
\[ - \left[ N^{-1} \mathrm{tr} (zI - H_0)G_0^2(\gamma(z, z^*)) \right]^2 \left[ N^{-1} \mathrm{tr} G_0^2(\gamma(z, z^*)) \right]^{-1} . \]
For any two matrices \( P \) and \( Q \) \( |\mathrm{tr} P Q^*|^2 \leq \mathrm{tr} PP^* \mathrm{tr} QQ^* \). Therefore \( I(z, z^*) \geq 0 \) and \( \rho(z, z^*) \) is bounded by \( (\pi v^2)^{-1} \). This fact which is interesting in its own has also an important consequence: the area of \( D \), the support of \( \rho(z, z^*) \), is not less than the area of a disk with radius \( v \).

In order to illustrate the formulas derived we consider a few examples. If \( H_0 = 0 \), then the equation \( R_0(0) = v^{-2} \) which determines the boundary of \( D \) takes the form \( |z|^2 = v^2 \) and \( I(z, z^*) \) obviously vanishes. Thus, we recover the circular distribution [11, 12, 13]: \( \rho(z, z^*) = (\pi v^2)^{-1} \) inside the disk \( |z| \leq v \) and zero outside.

In our next example \( H_0 \) is the Jordan block \( J = [h_0\delta_{j,k}]_{j,k=1}^{N} \), \( h_0 > 0 \). \( J \) has only one eigenvalue \( z = 0 \) which is defective and highly sensitive to perturbations. On replacing zero in the left lower corner of \( J \) by small positive \( \varepsilon \), one gets \( N \) distinct eigenvalues \( h_0(\varepsilon/h_0)^{1/N} \exp(2\pi ik/N) \). For fixed \( N \) the
perturbed eigenvalues approach zero as the parameter $\varepsilon/h_0$ vanishes but the rate of convergence is extremely slow if $N$ is large. For instance if $N = 50$ one needs $\varepsilon/h_0 \propto 10^{-50}$ in order to confine the eigenvalues into the disk $|z|/h_0 \leq 0.1$. Therefore, if not exponentially small, perturbation splits zero eigenvalue of $J$ into the circle $|z| = h_0$. As can be seen from (12) this phenomenon manifests itself in the large-$N$ limit. Indeed, in the case of $H_0 = J$ (12) reduces to $||z|^2 - h_0^2| \leq v^2$. Therefore if $v < h_0$ the eigenvalues of $J + A$ are distributed in the annulus $1 - \frac{v^2}{h_0^2} \leq \left| \frac{z}{h_0} \right|^2 \leq 1 + \frac{v^2}{h_0^2}$ which degenerates into a circle as $v$ vanishes. When $v \geq h_0$ the eigenvalues are distributed in the disk $\left| \frac{z}{h_0} \right|^2 \leq 1 + \frac{v^2}{h_0^2}$. In Fig.1 we present results of numerical diagonalization of random matrices $J + A$. As can be seen, the correspondence between the numerical results and our analytical predictions (for $N \to \infty$) is quite good.

![Figure 1](image_url)

Figure 1: Distribution of numerically computed eigenvalues of the random matrices $J + A$ in the complex plane $z/h_0$. In each of the plots $N = 50$ and the number of samples is 40. (a) $v^2/h_0^2 = 1/2$, (b) $v^2/h_0^2 = 1$. The full lines show the boundary of the support of $\rho(z, z^*)$ in the large-$N$ limit.

If $H_0$ commutes with its conjugate $H_0^*$, our formulas (12)-(14) become simpler. Let us assume for certainty that the eigenvalues of $H_0$ are real. Then
the boundary of $D$ is determined by
\[
\int \frac{n(\lambda)d\lambda}{|z - \lambda|^2} = \frac{1}{v^2},
\]
where $n(\lambda)$ is the density of eigenvalues of $H_0$. $\rho(z, z^*)$ is given by the same expression as before but now $I(z, z^*)$ is
\[
\int \frac{|z - \lambda|^2n(\lambda)d\lambda}{||z - \lambda|^2 + \gamma^2(z, z^*)|^2} - \left[ \int \frac{(z - \lambda)n(\lambda)d\lambda}{||z - \lambda|^2 + \gamma^2(z, z^*)|^2} \right]^2 \left[ \int \frac{n(\lambda)d\lambda}{||z - \lambda|^2 + \gamma^2(z, z^*)|^2} \right]^{-1}
\]
and $\gamma(z, z^*)$ has to be found from
\[
\int \frac{n(\lambda)d\lambda}{|z - \lambda|^2 + \gamma^2} = \frac{1}{v^2}
\]

This work was supported in part by the Deutsche Forschungsgemeinschaft under Grant No SFB237 and by the International Science Foundation under Grant No U2S000.
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