A Refined Similarity Hypothesis for Transverse Structure Functions

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We argue on the basis of empirical data that Kolmogorov’s refined similarity hypothesis (RSH) needs to be modified for transverse velocity increments, and propose an alternative. In this new form, transverse velocity increments bear the same relation to locally averaged enstrophy (squared vorticity) as longitudinal velocity increments bear in RSH to locally averaged dissipation. We support this hypothesis by analyzing high-resolution numerical simulation data for isotropic turbulence. RSH and its proposed modification for transverse velocity increments (RSHT) appear to represent two independent scaling groups.

In all small-scale turbulence research driven by expectations of universality, one deals with so-called velocity increments, which are differences of velocities between two spatial positions separated by a fixed distance. Of special interest are velocity increments when the separation distances belong to the inertial range, that is, length scales which are small compared to a typical large-scale motion but large compared to a typical viscous cut-off scale. It is fair to say that much of the phenomenological work on this subject, summarized in Refs. \textsuperscript{1,2} and K62 \textsuperscript{3}, is based to some degree or another on two sets of similarity hypotheses proposed by Kolmogorov, abbreviated as K41 \textsuperscript{4} and K62 \textsuperscript{5}. The latter, also called the refined similarity hypotheses (RSH), have been verified both in real experiments and numerical simulations of turbulence—at least to an extent that certifies them as reasonable. The verification has focused, largely for historical reasons of experimental convenience, on longitudinal velocity increments, that is, velocity increments for which the separation distance is aligned with the velocity component considered. The theme of this Letter is that RSH is inadequate for transverse velocity increments, for which the separation distance is transverse to the direction of the velocity component considered, and that a non-trivial modification (RSHT) is necessary to account for experimental facts. We conclude that RSH and RSHT form two independent scaling groups in small-scale turbulence.

We need a few definitions before proceeding further. The rate of energy dissipation is given by \( \varepsilon = \nu (\partial u_i/\partial x_j + \partial u_j/\partial x_i)^2 \), where \( \nu \) is the fluid viscosity, and its local average is defined as \( \varepsilon_r(x,t) = \int \varepsilon \, dV \), where the integration volume \( V(r) \), with the characteristic linear dimension \( r \), is centered at the spatial position \( x \). The longitudinal velocity increment is defined as \( \delta u_r = u(x+r) - u(x) \), where the velocity component \( u \) and the separation distance \( r \) are both in the same direction, say \( x \). The form of RSH that has been verified extensively \textsuperscript{1,2,6} is given by the relation

\[
\delta u_r = \beta_1 (r \varepsilon_r)^{1/3},
\]

where \( \beta_1 \) is a stochastic variable independent of \( r \) and \( \varepsilon_r \). Assume now that the \( p \)-th order longitudinal structure function \( S^L_p(r) \equiv \langle (\delta u_r)^p \rangle \sim r^{\zeta^L_p} \) and \( \langle \varepsilon_r^{p} \rangle \sim r^{\tau_p} \). Here the angular brackets denote suitable averages. It then follows from Eq. (1) that

\[
\zeta^L_p = \frac{p}{3} + \frac{\tau_p}{3}.
\]

This equation connects the scaling exponents of the longitudinal structure functions with those of the locally averaged dissipation function. If a further assumption about the statistics of \( \varepsilon_r \) can be made, such as log-normal \textsuperscript{6}, multifractal \textsuperscript{7}, or log-Poisson \textsuperscript{8}, \( \tau_p \) can be obtained analytically via Eq. (2).

The thinking in K62 is that Eq. (1) holds equally well if the longitudinal velocity increment is replaced by a transverse velocity increment, namely \( \delta v_r = v(x+r) - v(x) \). The separation distance \( r \) is transverse to the velocity component \( v \). If true, this would imply that the scaling exponents \( \zeta^T_p \) and \( \zeta^L_p \), for longitudinal and transverse structure functions, respectively, are equal. A kinematic constraint from isotropy \textsuperscript{9,11} assures us that \( \zeta^T_p = \zeta^L_p \). Two sets of measurements \textsuperscript{12} appear to suggest (or imply) that the equality holds, within experimental uncertainty, for larger \( p \) as well. On the other hand, more recent numerical \textsuperscript{13,14} and experimental work \textsuperscript{15} has revealed that transverse velocity increments are more intermittent than longitudinal ones, and that \( \zeta^T_p \) are measurably smaller than \( \zeta^L_p \), at least for \( p > 4 \). These results imply that the RSH cannot be equally true for both longitudinal and transverse velocity increments. With this in mind, we first obtain the scaling exponents of longitudinal and transverse structure functions. We employ the numerical data from simulations of isotropic turbulence, carried out using 512\(^3\) mesh points in a periodic
verse velocity increments as functions of $r$ were used in the data analysis. Figure 1 shows the flatnesses of longitudinal and transverse velocity increments as functions of $r$. For all values of $r$ except perhaps those comparable to the box size, the velocity increment along the transverse direction has a larger flatness, suggesting that the transverse velocity increment is more intermittent [14]. This shows that significantly more averaging is required for transverse quantities than for longitudinal quantities.

\[ S_p^T(r) = S_p^L(r) \]

In Fig. 2, the transverse structure functions $S_p^T(r)$ are plotted as functions of $r$ for $p = 2, 4, 6, 8, 10$. An inertial-range power-law scaling can be identified for $0.2 \leq r \leq 0.6$ (the whole box size being $2\pi$). This is also the scaling range determined [13] from Kolmogorov’s 4/5 law for the third-order structure function [17].

\[ S_p^L(r) = \frac{1}{2} \]

The exponents $\zeta_p^L$, determined by least-square fits within the inertial range just mentioned, are plotted in Fig. 3. The use of Extended Self-Similarity (ESS) technique [13] yields very similar results [19]. The results for $\zeta_p^T$, obtained earlier [20], have also been plotted for comparison. These results demonstrate that while both exponents show considerable anomaly due to intermittency effects (as evidenced by the deviation from the dotted line given by K41), the transverse exponents are systematically smaller than the longitudinal ones for $p > 3$. Typical scaling exponents for $S_p^T$ are: $\zeta_p^T = 0.71 \pm 0.04$, $\zeta_p^T = 1.25 \pm 0.067$, $\zeta_p^T = 1.63 \pm 0.079$ and $\zeta_p^T = 1.87 \pm 0.078$. While the longitudinal exponents agree quite well with existing scaling models [9,10,21], the difference between the transverse exponents and the models increases with $p$. For brevity, we show only a comparison with the log-Poisson model [10], which is typical.

\[ \zeta_p^L = 2.00 \]

FIG. 1. Flatnesses of velocity increments, $S_p^L(r)/S_p^L(2)^2$ and $S_p^T(r)/S_p^T(2)^2$, as functions of $r$.

\[ \zeta_p^T \]

FIG. 2. The transverse structure functions $S_p^T(r)$ as functions of $r$ for $p = 2, 4, 6, 8, 10$.

\[ \delta u_r = \beta_2(r) \Omega_r^{1/3}, \]

where $\Omega = \nu_\omega^2$, $\omega = \nabla \times u$ is the vorticity, and $\Omega_r$ is the local average of $\Omega$ in the same way as $\varepsilon_r$ is the local average of $\varepsilon$ in Eq. (1). A necessary condition for the above equation to be plausible is that $\varepsilon_r$ should scale differently from $\Omega_r$, and that the difference must be compatible in sign with that of the difference between $\zeta_p^L$ and $\zeta_p^T$. That
this is indeed so was demonstrated recently \[24\]; that is, if \(⟨Ω r^2⟩ \sim r^p\), \(op < τ_p\). Thms, Eq. (3) is a priori worth exploring as a candidate for the refined similarity hypothesis for transverse velocity increments (RSHT).

We now test the validity of Eq. (3). In particular, we test whether the relation

\[
ζ_p^T = \frac{p}{3} + op/3, \tag{4}
\]

which follows from Eq. (3), is true. We should note that the equations

\[
δv_r = β_3(rε_r)^{1/3}, \tag{5}
\]

\[
δu_r = β_4(rΩ_r)^{1/3} \tag{6}
\]

are both dimensionally plausible, and would yield

\[
ζ_p^L = \frac{p}{3} + op/3, \tag{7}
\]

\[
ζ_p^T = \frac{p}{3} + τ_p/3. \tag{8}
\]

Equation (4) would derive greater support if the alternative relations (7) and (8) can be shown to be unsatisfactory.

In Figs. 4 (a) and (b), we show comparisons of numerical results for \(p\) up to 10 with predictions of (3) and (4) for the transverse case, and with predictions of (3) and (5) for the longitudinal case. (For odd order, we have used absolute values of the appropriate velocity differences.) The exponents \(τ_p/3\) and \(op/3\) used in the above relations were taken from previously published results for the same data \[24\]. For \(p < 3\), the differences between the implied relations (2), (4), and (7), (8) and numerical results are relatively small, and cannot be distinguished easily. On the other hand, with increasing \(p\), only \(ζ_p^L - op/3\) and \(ζ_p^T - τ_p/3\) coincide with \(p/3\), showing that Eqs. (3) and (8) are valid to a good approximation. The numerical evidence that \(ζ_p^L - op/3\) and \(ζ_p^T - τ_p/3\) depart from \(p/3\) for large \(p\) effectively disqualifies (3) and (8). Note that the error bars in Figs. 4 are quite small and do not affect this conclusion.

A more detailed test of Eq. (3) is given in Fig. 5, where we plot the normalized probability density function (PDF) of \(β_2 \equiv Δv_r/(rΩ_r)^{1/3}\) with \(r = 0.2, 0.34\) and 0.44, all of which lie in the inertial range. The three PDFs collapse quite well. It is seen that in the region \(p/3 < \langle β_2 \rangle < 3\), the PDFs agree with the standard Gaussian distribution. For larger values of \(p/3\), the PDFs seem to depart slightly from Gaussian, with a tendency perhaps towards exponential. This trend has also been seen for the longitudinal counterpart, \(β_1\), investigated by Wang et al. \[8\]. One difference between the PDFs of \(β_1\) and \(β_2\), which follows from Kolmogorov’s 4/5ths law \[1\], is that \(β_2\) is not expected to be skewed, whereas \(β_1\) should have a third moment equal to \(-4/5\). The PDFs of \(β_3\) and \(β_4\) have also been obtained but not shown here. They do not collapse for inertial range separations; nor do they agree with the Gaussian distribution even in the core region; and they possess larger departures from Gaussian near the tails.

The results in Figs. 4 and Fig. 5 suggest that RSHT proposed in (3) is a good working approximation connecting the transverse velocity increment with the enstrophy field. In addition, the differences of the intermittency properties between \(δu_r\) and \(δv_r\) as well as \(ε_r\) and \(Ω_r\) strongly suggest the possibility of two independent scaling groups. Note that \(ε_r\) and \(Ω_r\) are the symmetric and antisymmetric parts of the strain-rate tensor, respectively; so we expect that this difference, reflecting the difference in the inertial-range physics of the dissipation and vortex dynamics, may quite plausibly carry over to the high Reynolds number limit as well.

It should be pointed out that a more general refined similarity hypothesis, which encompasses results for both velocity components, cannot be ruled out. One possible scenario could be

\[
Δv_r = β_5(rΩ_r^{α}ε_r^{1-α})^{1/3}, \tag{9}
\]

where \(β_5\) is again a universal stochastic variable and \(α\) could, in general, be a function of the order index \(p\) \[23\]. Equation (9) is more complex than (3) and (8), since the scalings of the cross-correlation functions \(⟨Ω_r^pε_r⟩\) have to be taken into account.
In summary, we have proposed a refined similarity hypothesis for the transverse direction (RSHT); it connects the statistics of the transverse structure function in the inertial range with the locally averaged enstrophy. Data analysis of DNS for the Navier-Stokes equations at moderate Reynolds numbers demonstrate that RSHT is valid for the transverse structure functions in the inertial range, while Kolmogorov’s RSH has been shown previously to be essentially correct for the longitudinal case \[ \frac{\beta_2}{(\langle \beta_2 \rangle^2)^{1/2}} \] for \( r = 0.2, 0.34 \) and 0.44. The important implication of RSHT is the possibility of two independent scaling groups which may correspond to different intermittent physics in fluid turbulence: these are related to the symmetric part of the strain-rate, or the dissipation physics, and those related to the antisymmetric part of the strain-rate, or the vortex dynamics. This view also suggests that no more than two sets of independent exponents are required for describing the scaling of all small-scale features as longitudinal and transverse velocity increments, dissipation, enstrophy, and circulation [4].

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