Abstract. In [22], the authors proved that every complete intersection smooth projective variety $Y$ is a Fano visitor, i.e., its derived category $D^b(Y)$ is equivalent to a full triangulated subcategory of the derived category $D^b(X)$ of a smooth Fano variety $X$, called a Fano host of $Y$. They also introduced the notion of Fano dimension of $Y$ as the smallest dimension of a Fano host $X$ and obtained an upper bound for the Fano dimension of each complete intersection variety.

In this paper, we provide a Hodge-theoretic criterion for the existence of a Fano host which enables us to determine the Fano dimensions precisely for many interesting examples, such as low genus curves, quintic Calabi-Yau 3-folds and general complete intersection Calabi-Yau varieties.

Next we initiate a systematic search for more Fano visitors. We generalize the methods of [22] to prove that smooth curves of genus at most 4 are all Fano visitors and general curves of genus at most 9 are Fano visitors. For surfaces and higher dimensional varieties, we find more examples of Fano visitors and raise natural questions.

An interesting recent discovery is the existence of quasi-phantom subcategories in derived categories of some surfaces of general type with $p_g = q = 0$ ([8, 11, 31, 32, 33]). But no examples of Fano with quasi-phantom have been found. Applying a result of Orlov and Cayley’s trick used in [22], we construct a Fano orbifold whose derived category contains a quasi-phantom category.

1. Introduction

If one were to write up a list of keywords that describe recent development in algebraic geometry, it would be hard to miss the words like “derived category” or “categorification” on the top part. The derived category $D^b(X)$ of bounded complexes of coherent sheaves of a projective variety $X$ was found to be a sophisticated invariant which categorifies geometric invariants such as Hochschild homology, Hochschild cohomology and Grothendieck groups of algebraic varieties (cf. [26]). Many geometric statements were categorified which means that a deep categorical origin or explanation was discovered.

One basic problem in algebraic geometry is to study how a variety can be embedded in other varieties. In 2011, Bondal categorified the embedding problem and raised the following question (cf. [5]).

**Question 1.1.** (Fano visitor problem)

Let $Y$ be a smooth projective variety. Is there a Fano variety $X$ equipped with a fully faithful embedding $D^b(Y) \hookrightarrow D^b(X)$?

If the answer is yes, we call $Y$ a Fano visitor and $X$ a Fano host of $Y$. 

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From the categorical point of view, Fano varieties are of particular interest because they admit canonical semiorthogonal decompositions and many examples have been explicitly calculated (cf. [5, 9, 24, 25, 47]). If the answer to Question 1.1 is yes for all smooth projective varieties, the study of derived categories may be effectively reduced to those of Fano varieties.

Bondal and Orlov in [9] proved that the derived category of a hyperelliptic curve $Y$ of genus $g$ is embedded into the derived category of the intersection of two quadrics in $\mathbb{P}^{2g+1}$. Kuznetsov in [24] proved that the derived categories of some K3 surfaces are embedded into special cubic 4-folds. He also discovered some Fano 3-folds that contain the derived categories of certain smooth projective curves. Bernardara, Bolognesi and Faenzi in [5] proved that every smooth plane curve is a Fano visitor. Segal and Thomas in [47] proved that a general quintic 3-fold is a Fano visitor by finding an 11-dimensional Fano host.

In [22], the authors proved the following.

**Theorem 1.2.** [22, Theorem 4.1] All smooth projective complete intersections are Fano visitors.

Moreover, they defined the *Fano dimension* of a smooth projective variety $Y$ as the minimum dimension of Fano hosts $X$ of $Y$. The Fano dimension is defined to be infinite if no Fano hosts exist. It was also proved that an arbitrary complete intersection Calabi-Yau variety $Y$ of codimension $\leq 2$ or a general complete intersection Calabi-Yau variety of codimension $\geq 3$ has Fano dimension at most $\dim Y + 2$.

In this paper, we first provide a Hodge-theoretic criterion for the existence of a Fano host.

**Proposition 1.3.** (Proposition 4.7) Let $Y$ be a Fano visitor and $X$ be a Fano host of $Y$. Then we have the inequality of Hodge numbers

$$
\sum_{p-q=i} h^{p,q}(Y) \leq \sum_{p-q=i} h^{p,q}(X) \quad \text{for all } i.
$$

As a direct consequence, we obtain the following.

**Corollary 1.4.** (Corollary 4.8) If $h^{n,0}(Y) \neq 0$ for $n = \dim Y > 0$, then the Fano dimension of $Y$ is at least $n + 2$.

Combining this corollary with the Fano host construction in [22], we obtain the following.

**Corollary 1.5.** (Corollary 4.9 and Proposition 4.10)
The Fano dimension of a smooth projective curve of positive genus is at least 3. The Fano dimension of an arbitrary complete intersection Calabi-Yau variety $Y \subset \mathbb{P}^{n+c}$ of codimension $c \leq 2$ or a general complete intersection Calabi-Yau variety of codimension $c \geq 3$ is precisely $\dim Y + 2$.

For instance, every smooth quintic 3-fold has Fano dimension 5 and the Fano host constructed in [22] has the minimal possible dimension.

Next we initiate a systematic search for more Fano visitors. We generalize the construction and technique of [22] for complete intersections in more general varieties such as Grassmannians (cf. Theorem 3.1). Using this, we prove that smooth curves of genus at most 4 are all Fano visitors (cf. 5.2) and general curves of genus at most 9 are Fano visitors (cf. 5.3). For surfaces and higher dimensional varieties, we find more examples of Fano visitors and raise natural questions.
An interesting recent discovery is the existence of quasi-phantom subcategories in derived categories of some surfaces of general type with \( p_g = q = 0 \) \((6, 15, 31, 32, 33)\). But no examples of Fano with quasi-phantom have been found. Applying a result of Orlov and Cayley’s trick used in \([22]\), we construct a Fano orbifold whose derived category contains a quasi-phantom category (cf. Example 6.11). More precisely, we construct a smooth projective Fano variety \( X \) together with an action by a finite group \( G \) such that the derived category \( D^b(\mathcal{X}/\mathcal{G}) \) of the Fano orbifold \( \mathcal{X}/\mathcal{G} \) contains the derived category \( D^b(S) \) of the classical Godeaux surface \( S \) as a full triangulated subcategory. Since \( D^b(S) \) contains a quasi-phantom category by \([7]\), we find that the derived category \( D^b(\mathcal{X}/\mathcal{G}) \) contains a quasi-phantom category. As far as we know, this is the first discovery of a quasi-phantom category in the realm of Fano (orbifolds). However we do not know if there is a smooth Fano variety with a quasi-phantom category.

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Notation. In this paper, all varieties are defined over the complex number field \( \mathbb{C} \). For a vector bundle \( E \) on \( S \), the projectivization \( \mathbb{P}E := \text{Proj}(\text{Sym} E^\vee) \) of \( E \) parameterizes one-dimensional subspaces in fibers of \( E \). For a variety \( X \), \( D^b(X) \) denotes the bounded derived category of coherent sheaves on \( X \). The zero locus \( s^{-1}(0) \) of a section \( s : \mathcal{O}_X \to E \) of a vector bundle \( E \) over a scheme \( X \) is the closed subscheme of \( X \) whose ideal is the image of \( s^\vee : E^\vee \to \mathcal{O}_X \).

2. PRELIMINARIES

In this section we recall several definitions and facts which we will use later.

2.1. Fano varieties. Let us recall several definitions on Fano varieties.

Definition 2.1. A smooth projective variety \( X \) is called Fano (resp. weak Fano) if its anticanonical line bundle \( K^\vee_X \) is ample (resp. nef and big).

It is well known that the Picard group of a Fano variety is a free abelian group (cf. \([21\text{ Proposition } 2.1.2]\)).

Definition 2.2. Let \( X \) be a Fano variety. The largest positive integer \( i \) which divides \( K_X \) in \( \text{Pic}(X) \) is called the index of \( X \).

Fano varieties have many nice properties.

Theorem 2.3. For any positive integer \( n \), there are only finitely many deformation equivalence classes of Fano varieties of dimension \( n \).

Theorem 2.4. \([13 \text{ Theorem } 2.2]\) Fano varieties are rationally connected.

Theorem 2.5. \([11 \text{ Corollary } 4.18]\) Every smooth projective rationally connected variety is simply connected.

Therefore we see that every smooth projective Fano variety is simply connected. Mori cones of weak Fano varieties are also very special.

Theorem 2.6. \([13 \text{ Theorem } 2.3]\) \([48 \text{ Theorem } 1.4]\) The Mori cone of a weak Fano variety is a rational polyhedral cone generated by classes of rational curves.
If $X$ is Fano, $K_X^\vee$ is ample and hence
\[ H^{p,0}(X) \cong H^{0,p}(X) = H^p(X, \mathcal{O}_X) = H^p(X, K_X \otimes K_X^\vee) = 0 \quad \text{for } p > 0 \]
by the Kodaira vanishing theorem. So we obtain the following vanishing of Hodge numbers.

**Lemma 2.7.** If $X$ is a smooth projective Fano variety, $h^{p,0}(X) = 0$ for $p > 0$.

### 2.2. Semiorthogonal decomposition.
We recall the definition of semiorthogonal decompositions of derived categories of coherent sheaves.

**Definition 2.8.** Let $\mathcal{T}$ be a triangulated category. A semiorthogonal decomposition of $\mathcal{T}$ is a sequence of full triangulated subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_n$ satisfying the following properties:

(1) $\text{Hom}_{\mathcal{T}}(a_i, a_j) = 0$ for any $a_i \in \mathcal{A}_i$, $a_j \in \mathcal{A}_j$ with $i > j$;

(2) the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is $\mathcal{T}$.

We will write $\mathcal{T} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ to denote the semiorthogonal decomposition.

Let $E$ be a vector bundle of rank $r \geq 2$ over a smooth projective variety $S$ and let $Y = s^{-1}(0) \subset S$ denote the zero locus of a regular section $s \in H^0(S, E)$ such that $\dim Y = \dim S - \text{rank} E$. Let $X = w^{-1}(0) \subset \mathbb{P}E^\vee$ be the zero locus of the section $w \in H^0(\mathbb{P}E^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(1))$ determined by $s$ under the natural isomorphisms
\[ H^0(\mathbb{P}E^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(1)) \cong H^0(S, q_* \mathcal{O}_{\mathbb{P}E^\vee}(1)) \cong H^0(S, E) \]
where $q : \mathbb{P}E^\vee \to S$ is the projection map of the projective bundle.

Orlov proved in [44] that $D^b(X)$ has the following semiorthogonal decomposition which was subsequently generalized to higher degree hypersurface fibrations by Ballard, Deliu, Favero, Isik and Katzarkov in [2].

**Theorem 2.9.** [44] Proposition 2.10] There is a natural semiorthogonal decomposition
\[ D^b(X) = \langle q^* D^b(S), \ldots, q^* D^b(S) \otimes \mathcal{O}_X(r-2), D^b(Y) \rangle \]

**Remark 2.10.** Orlov proved in particular that there is a fully faithful exact functor from $D^b(Y)$ to $D^b(X)$ (cf. [44] Proposition 2.2). When a finite group $G$ acts on $S$ and $E$ compatibly and $s$ is a $G$-invariant section, there is an induced action of $G$ on $X$ and $Y$. His proof also works for this equivariant setting to give us a fully faithful exact functor from $D^b([Y/G])$ to $D^b([X/G])$. See [44] Remark 2.9.

### 2.3. Fano visitor problem.
We learned the following definition from [5].

**Definition 2.11.** A smooth projective variety $Y$ is called a Fano visitor if there is a smooth projective Fano variety $X$ together with a fully faithful (exact) embedding $D^b(Y) \to D^b(X)$. We call such a Fano $X$ a Fano host of $Y$.

Bondal’s question (Question [1]) asks if a smooth projective variety is a Fano visitor. It is easy to see that a Fano host $X$ of a smooth projective variety $Y$ is not unique because for instance the product $X \times \mathbb{P}^1$ is also a Fano host of $Y$. So we may ask for a Fano host of minimal dimension.

**Definition 2.12.** [22] The Fano dimension of a smooth projective variety $Y$ is the minimum among the dimensions $\dim X$ of Fano hosts $X$ of $Y$.

See [22] for more discussions and questions related to Fano visitors.
3. Fano visitors and Fano dimension

In this section, we recall and generalize the main construction and result in [22].

3.1. Cayley’s trick. Let $S$ be a smooth projective variety and $s \in H^0(S, E)$ be a regular section of a vector bundle of rank $r \geq 2$ such that $Y = s^{-1}(0)$ is smooth of dimension $\dim S - r$. Let $\mathbb{P}E^\vee = \text{Proj}(\text{Sym } E)$ denote the projectivization of $E^\vee$. Then we have an isomorphism

$$H^0(S, E) \cong H^0(\mathbb{P}E^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(1))$$

which gives us a section $w$ of $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ corresponding to $s$. Let $X = w^{-1}(0)$. Since $Y$ is smooth, $X$ is also smooth by local computation.

By Orlov’s theorem (cf. Theorem 2.9), there is a fully faithful embedding of $D^b(Y)$ into $D^b(X)$. Therefore if $X$ is Fano, then $X$ is a Fano host of $Y$ and $Y$ is a Fano visitor.

3.2. Complete intersections in projective space. When $Y \subset \mathbb{P}^m$ is a smooth complete intersection defined by a section $s'$ of $\bigoplus_{i=1}^l \mathcal{O}_{\mathbb{P}^m}(a_i)$ with $a_i > 0$ and $l \geq 0$, we enlarge the ambient space $\mathbb{P}^m$ to $\mathbb{P}^{m+c} = S$ and extend the vector bundle $\bigoplus_{i=1}^l \mathcal{O}_{\mathbb{P}^m}(a_i)$ to

$$\bigoplus_{i=1}^l \mathcal{O}_{\mathbb{P}^m+c}(a_i) \oplus \mathcal{O}_{\mathbb{P}^m+c}(1)^{\oplus c} = E$$

for $c \geq 0$. The section $s'$ together with a choice of defining linear equations for $\mathbb{P}^m \subset \mathbb{P}^{m+c}$ gives us a section $s$ of $E$ with $s^{-1}(0) = s'^{-1}(0) = Y$. Applying Cayley’s trick above, we obtain a hypersurface $X = w^{-1}(0)$ of $\mathbb{P}E^\vee$ whose dimension is $m + 2c + l - 2 = \dim Y + 2c + 2l - 2$.

The authors proved in [22, §4.2] that if $c$ is greater than $\sum_{i=1}^l a_i - m - l$ and $1 - l$, then $X$ is Fano. This proves the main result (Theorem 1.2) of [22] because $X$ is a Fano host of $Y$ by the discussion in §5.1.

3.3. A generalization. We can capture the essence of the proof of Theorem 1.2 in [22] as follows.

**Theorem 3.1.** We use Cayley’s trick in §3.1. Suppose that

1. $E$ is ample and $K_S^\vee \otimes \det E^\vee$ is nef, or
2. there is a nef line bundle $H$ such that $F := E \otimes H^\vee$ is a nef vector bundle and that $K_S^\vee \otimes \det E^\vee \otimes H^{-1}$ is ample.

Then $X = w^{-1}(0)$ is a Fano host of $Y = s^{-1}(0)$.

**Proof.** By Theorem 2.9 it suffices to show that $X$ is Fano. For (1), see [22, Lemma 3.1]. For (2), let $q : \mathbb{P}E^\vee \to S$ denote the canonical projection. Let us compute $K_X$. From the relative Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}E^\vee} \to q^*E^\vee \otimes \mathcal{O}_{\mathbb{P}E^\vee}(1) \to T_{\mathbb{P}E^\vee/S} \to 0,$$

we have $K_{\mathbb{P}E^\vee/S} = (q^* \det E^\vee) \otimes \mathcal{O}_{\mathbb{P}E^\vee}(r)$. From $K_{\mathbb{P}E^\vee} = q^*K_S \otimes K_{\mathbb{P}E^\vee/S}$ we have

$$K_{\mathbb{P}E^\vee}^\vee \cong q^*(K_S^\vee \otimes \det E^\vee) \otimes \mathcal{O}_{\mathbb{P}E^\vee}(r).$$

Therefore we get

$$K_X = K_{\mathbb{P}E^\vee}^\vee \otimes \mathcal{O}(-1)|_X \cong q^*(K_S^\vee \otimes \det E^\vee) \otimes \mathcal{O}_{\mathbb{P}E^\vee}(r-1)|_X$$
By the binomial expansion formula, we see that $(r^q \cdot \det E^q \otimes H^{r-1}) \otimes \mathcal{O}_{\mathbb{P}^q}(r-1)|_X$.

By assumption, both $q^*(K^\vee_S \otimes \det E^q \otimes H^{r-1})$ and $\mathcal{O}_{\mathbb{P}^q}(r-1)$ are nef line bundles, and so is $K^\vee_X$. To see that $K^\vee_X$ is big, let us compute the intersection number $(K^\vee_X)^{\dim X}$ as follows:

$$(K^\vee_X)^{\dim X} = (q^*(K^\vee_S \otimes \det E^q \otimes H^{r-1}) \otimes \mathcal{O}_{\mathbb{P}^q}(r-1)|_X)^{\dim X}$$

$$= (q^*(K^\vee_S \otimes \det E^q \otimes H^{r-1}) \otimes \mathcal{O}_{\mathbb{P}^q}(r-1))^{\dim X} \cdot (q^*H \otimes \mathcal{O}_{\mathbb{P}^q}(1))|_X$$

By the binomial expansion formula, we see that $(K^\vee_X)^{\dim X}$ is positive since every term is a multiple of a nef line bundle and $q^*(K^\vee_S \otimes \det E^q \otimes H^{r-1}) \otimes \mathcal{O}_{\mathbb{P}^q}(1)^{r-1}$ is strictly positive by our assumption. Therefore $K^\vee_X$ is nef and big, i.e. $X$ is a weak Fano variety. Then the Mori cone of $X$ is rational polyhedral and the extremal rays are generated by rational curves by Theorem 2.6.

Finally we claim that $K^\vee_X$ intersects positively with all irreducible curves. Let $C$ be an irreducible curve in $\mathbb{P}^q = \mathbb{P}^q$. If $q(C)$ is a point, then the degree of $\mathcal{O}_{\mathbb{P}^q}(r-1)|_C$ is positive because $\mathcal{O}_{\mathbb{P}^q}(1)$ is ample on each fiber of $q : \mathbb{P}^q \rightarrow S$. If $q(C)$ is a curve, then the degree of $q^*(K^\vee_S \otimes \det E^q \otimes H^{r-1})|_C$ is positive. Therefore we find that the degree of the line bundle $q^*(K^\vee_S \otimes \det E^q \otimes H^{r-1}) \otimes \mathcal{O}_{\mathbb{P}^q}(r-1)|_C$ is always positive. Since the Mori cone is polyhedral, this implies that $K^\vee_X$ is ample and $X$ is a Fano variety.

\[\square\]

**Remark 3.2.** In the proof of Theorem 4.1 in [22] §4.2], we used $H = \mathcal{O}_{\mathbb{P}^3}(1)$ and chose sufficient large $c$ as written in §4.2. However when the degrees of defining equations of $Y$ are large enough, then the above theorem tells us that we can choose larger $H$ and smaller $c$. This often gives a Fano host of smaller dimension as in the following example.

**Example 3.3.** Let $C$ be a non-hyperelliptic curve of genus 4. Then $C$ is the complete intersection of a quadric and a cubic in $\mathbb{P}^3$, i.e. $C$ is the zero locus of a regular section $s$ of $E = \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(3)$ over $S = \mathbb{P}^3$ and let $F = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ with $H = \mathcal{O}_{\mathbb{P}^3}(2)$. From the above theorem, we find that $X = w^{-1}(0)$ in Cayley’s trick (cf. [3.1]) is a 3-dimensional Fano host of $C$ because $F$ is nef and $K^\vee_S \otimes \det E^q \otimes H^{r-1} = \mathcal{O}_{\mathbb{P}^3}(1)$ is ample. Note that if we insist on using $H = \mathcal{O}_{\mathbb{P}^3}(1)$ instead, we have to enlarge $\mathbb{P}^3$ to $\mathbb{P}^4$ and extend $\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(3)$ to $\mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3) \oplus \mathcal{O}_{\mathbb{P}^4}(1)$, so that the Fano host is 5 dimensional.

By Example 3.3, we find that a non-hyperelliptic curve $C$ of genus 4 has Fano dimension at most 3. We will see below that indeed 3 is the Fano dimension of $C$.

## 4. Fourier-Mukai Transforms and an Embeddability Criterion

In this section, we use the Fourier-Mukai transform to give a Hodge-theoretic criterion for the existence of a fully faithful functor $D^b(Y) \rightarrow D^b(X)$ for smooth projective varieties $X$ and $Y$.

We first recall a fundamental result of Orlov that says all fully faithful exact functors are Fourier-Mukai.

**Theorem 4.1.** ([13] Theorem 5.14] Let $X$ and $Y$ be two smooth projective varieties and let

$$F : D^b(Y) \rightarrow D^b(X)$$
be a fully faithful exact functor. If $F$ admits right and left adjoint functors, then there exists an object $K \in D^b(Y \times X)$ unique up to isomorphism such that $F$ is isomorphic to the Fourier-Mukai transform

$$
\Phi_K = \pi_{X,*}(\pi_Y^*(-) \otimes K) : D^b(Y) \rightarrow D^b(X)
$$

where $\pi_X$ and $\pi_Y$ denote the projection morphisms from $X \times Y$ to $X$ and $Y$ respectively.

**Remark 4.2.** The assumption that $F$ admits right and left adjoint functors can be dropped by a theorem of Bondal and van den Bergh [8].

The Fourier-Mukai kernel $K$ also defines the cohomological Fourier-Mukai transform $\Phi^H_K : H^*(Y, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$.

**Definition 4.3.** [19] Let $K$ be a Fourier-Mukai kernel of $\Phi_K$. Then the cohomological Fourier-Mukai functor is the linear map $\Phi^H_K : H^*(Y, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$ defined by

$$
\Phi^H_K(-) = \pi_{X,*}(ch(K) \cdot \sqrt{td(X \times Y)} \cdot \pi_Y^*(-)).
$$

When a Fourier-Mukai functor gives an equivalence between derived categories of two smooth projective varieties, the induced cohomological Fourier-Mukai functor preserves the Hochschild homology groups.

**Proposition 4.4.** [19 Proposition 5.39] If $\Phi_K : D^b(Y) \rightarrow D^b(X)$ is an equivalence, then the induced cohomological Fourier-Mukai transform $\Phi^H_K : H^*(Y, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$ yields an isomorphism

$$
\bigoplus_{p-q=i} H^{p,q}(Y) \cong \bigoplus_{p-q=i} H^{p,q}(X) \quad \text{for all } i.
$$

We next generalize the above result to the case where $\Phi_K$ is a fully faithful functor.

**Proposition 4.5.** [19 Corollary 1.22] Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a fully faithful functor that admits a left adjoint $G \dashv F$ (resp. right adjoint $F \dashv H$). Then the natural morphism

$$
G \circ F \rightarrow id_{\mathcal{A}} \quad \text{(resp. } id_{\mathcal{A}} \rightarrow H \circ F)\quad
$$

is an isomorphism.

For Fourier-Mukai transforms, we always have left and right adjoint functors by a result of Mukai.

**Proposition 4.6.** [19 Proposition 5.9] A Fourier-Mukai transform admits left and right adjoint functors which are also Fourier-Mukai transforms.

Now we can give a criterion for the existence of a fully faithful functor $D^b(Y) \rightarrow D^b(X)$.

**Proposition 4.7.** If a Fourier-Mukai transform $\Phi_K : D^b(Y) \rightarrow D^b(X)$ is fully faithful, then the induced cohomological Fourier-Mukai transform $\Phi^H_K : H^*(Y, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$ yields an injective homomorphism

$$
\bigoplus_{p-q=i} H^{p,q}(Y) \subset \bigoplus_{p-q=i} H^{p,q}(X).
$$
Hence, we have the inequality

\[ \sum_{p-q=i} h^{p,q}(Y) \leq \sum_{p-q=i} h^{p,q}(X) \quad \text{for all } i. \]

Proof. We will follow the arguments in [19]. There exists a right adjoint \( \Phi_K^R \) of \( \Phi_K \) and \( \Phi_K^R \circ \Phi_K \equiv \text{id} \approx \Phi_{O^2} \) from the uniqueness of the Fourier-Mukai kernel. Then we get \( \Phi_K^H \circ \Phi_K^R \equiv \text{id} \) (cf. [19, Proposition 5.33]). Therefore \( \Phi_K^H \) induces an inclusion \( \Phi_K^H : H^*(Y, \mathbb{C}) \to H^*(X, \mathbb{C}) \) which satisfies

\[ \Phi_K^H(H^{p,q}(Y)) \subset \bigoplus_{r-s=p-q} H^{r,s}(X) \]

by the arguments in [19] Proposition 5.39]. □

Proposition 4.7 may also be obtained from [26, Theorem 7.6].

A first consequence of Proposition 4.7 is the following lower bound.

**Corollary 4.8.** Let \( Y \) be an \( n \)-dimensional smooth projective variety with \( h^{n,0}(Y) > 0 \) for \( n > 0 \). Then its Fano dimension is at least \( n+2 \).

Proof. Suppose that there is a Fano variety \( X \) of dimension at most \( n+1 \) and a fully faithful exact functor \( F : D^b(Y) \to D^b(X) \). By Proposition 4.7 we have the inequality

\[ 0 < h^{n,0}(Y) \leq \bigoplus_{p-q=n} h^{p,q}(X). \]

Obviously the right hand side is zero unless \( \dim X \) is \( n \) or \( n+1 \). By Lemma 2.7 \( h^{n,0}(Y) = 0 \). When \( \dim X = n+1 \), \( h^{n+1,1}(X) = h^{n,0}(X) = 0 \). Hence the right hand side is always zero if \( \dim X \leq n+1 \). This proves the proposition. □

When \( \dim Y = 1 \) and \( Y \) is not rational, \( h^1,0(Y) > 0 \) and so we obtain the following.

**Corollary 4.9.** The Fano dimension of a smooth projective curve which is not rational is at least 3.

We will see below that the Fano dimension of a curve \( Y \) is exactly 3 when the genus is 1 or 2 or when \( Y \) is a general curve of genus 4.

Combining Corollary 4.8 with the construction of Fano hosts in [22], we can determine the Fano dimension of a general complete intersection Calabi-Yau variety.

**Proposition 4.10.** Let \( Y \subset \mathbb{P}^{n+c} \) be a smooth projective complete intersection Calabi-Yau variety of dimension \( n \) defined by the vanishing of homogeneous polynomials \( f_1, \ldots, f_c \). Suppose \( c \leq 2 \) or \( Y \) is general in the sense that we can choose the defining polynomials such that the projective variety \( S \) defined by the vanishing of \( f_3, \ldots, f_c \) is smooth. Then the Fano dimension of \( Y \) is precisely \( n+2 \).

Proof. By [22, Proposition 3.6], the Fano dimension of \( Y \) is at most \( n+2 \). By Corollary 4.8 the Fano dimension is at least \( n+2 \). This proves the proposition. □

For instance, the Fano dimension of an arbitrary quintic 3-fold is 5 and the Fano host constructed in [22] is of minimal dimension.

5. Curves

In this section we search for Fano visitors among smooth projective curves. Curves in this section mean smooth projective curves.
5.1. Hyperelliptic curves. Bondal and Orlov proved that every hyperelliptic curve is a Fano visitor.

Theorem 5.1. Let $C$ be a hyperelliptic curve of genus $g$. Then there are two quadric hypersurfaces in $\mathbb{P}^{2g+1}$ whose intersection is a Fano host of $C$.

Corollary 5.2. A hyperelliptic curve $C$ of genus $g$ is a Fano visitor whose Fano dimension is at most $2g - 1$.

This corollary indicates that the Fano dimension of a curve of genus $g$ might increase as $g$ increases. Indeed the Fano dimension of a curve of genus $g$ may grow arbitrarily large as $g$ increases.

Proposition 5.3. Let $fd(g)$ be the minimum among the Fano dimensions of curves of genus $g$. Then $\lim_{g \to \infty} fd(g) = \infty$.

Proof. For any natural number $n$, there are only finitely many deformation equivalence classes of Fano varieties of dimension $n$. Therefore there are only finitely many possible values of $\sum_{i-j=1} h^1(X)$ for $n$-dimensional Fano varieties $X$. When the genus $g = h^{1,0}(C)$ of a curve $C$ is greater than all these possible values, there can be no $n$-dimensional Fano host of $C$. Therefore for any integer $n > 0$ there is an integer $g_0$ such that any curve of genus $g \geq g_0$ has Fano dimension greater than $n$. This proves the proposition. \qed

5.2. Low genus curves. In this subsection we prove that all curves $C$ of genus $g \leq 4$ are Fano visitors. If $g = 0$, $C = \mathbb{P}^1$ itself is a Fano variety. If $g = 1$, $C \subset \mathbb{P}^2$ is a complete intersection Calabi-Yau variety of codimension 1 and hence its Fano dimension is 3 by Proposition 4.10.

Corollary 5.4. Every elliptic curve is a Fano visitor and its Fano dimension is 3.

If $g = 2$, $C$ is a hyperelliptic curve and hence the Fano dimension is at most 3 by Corollary 5.2. By Corollary 1.9 the Fano dimension of $C$ is at least 3. So we proved the following.

Corollary 5.5. Every curve of genus 2 is a Fano visitor with Fano dimension 3.

If $g = 3$, it is well known that $C$ is either a plane quartic or a hyperelliptic curve. In the former case, we use the construction in §3.2 with $l = 1$, $m = 2$, $a_1 = 4$, $c = 2$ to obtain a Fano host $X$ of dimension 5. In the latter case, Theorem 5.1 gives a Fano host of dimension 5. So we obtain the following.

Corollary 5.6. Every curve of genus 3 is a Fano visitor and the Fano dimension is at most 5.

If $g = 4$, it is well known that $C$ is either the complete intersection of a quadric and a cubic in $\mathbb{P}^3$ or a hyperelliptic curve. In the former case, the Fano dimension is exactly 3 by Example 3.3. In the latter case, the Fano dimension is at most 7.

Corollary 5.7. Every curve $C$ of genus 4 is a Fano visitor with Fano dimension at most 7. If $C$ is non-hyperelliptic, then its Fano dimension is 3.
5.3. **General curves of genus** \( \leq 9 \). In this subsection, we use Mukai’s description of general curves \( C \) of genus \( g \leq 9 \) as complete intersections in homogeneous varieties and prove that they are Fano visitors.

A general curve \( C \) of genus 5 has canonical embedding into \( \mathbb{P}^4 \) whose image is the intersection of three general quadrics. Let \( S \) be one of the quadric hypersurfaces and let \( s \) be the section of \( E = \mathcal{O}_{\mathbb{P}^4}(2)|_S \) defined by the remaining two quadrics, so that \( C = s^{-1}(0) \). Then \( PE^\vee \cong S \times \mathbb{P}^1 \) is a Fano variety and hence the Mori cone of \( PE^\vee \) is rational polyhedral. Let \( H = \mathcal{O}_{\mathbb{P}^4}(2)|_S \). Then \( F = E \otimes H^{-1} = \mathcal{O}_S^{\oplus 2} \) is nef and \( K_S \otimes \det E^\vee \otimes H = \mathcal{O}_{\mathbb{P}^4}(1)|_S \) is ample. Therefore we obtain the following from Theorem 3.1.

**Corollary 5.8.** A general curve of genus 5 is a Fano visitor and its Fano dimension is 3.

For higher genus curves, we recall some results of Mukai.

**Definition 5.9.** A curve \( C \) has a \( g^r_d \) if there is a line bundle \( L \) on \( C \) with \( \deg L = d \) and \( h^0(C, L) \geq r + 1 \).

**Proposition 5.10.** [38, Proposition 1.9] The anticanonical line bundle of the Grassmannian \( Gr(k, n) \) is \( \mathcal{O}(n) \) where \( \mathcal{O}(1) \) is the very ample line bundle which gives the Plücker embedding.

Mukai proved that a general curve of genus 6 can be embedded into \( Gr(2, 5) \) as a complete intersection.

**Theorem 5.11.** [38] A curve \( C \) of genus 6 is the complete intersection of \( Gr(2, 5) \subset \mathbb{P}^9 \) and a 4-dimensional quadric in \( \mathbb{P}^5 \subset \mathbb{P}^9 \) if \( C \) is not bi-elliptic and has no \( g^1_3 \) or \( g^2_2 \).

**Proposition 5.12.** [39, Proposition 2.1] The anticanonical line bundle of the 10-dimensional orthogonal Grassmannian variety \( X_{10}^{12} \) is \( \mathcal{O}(8) \) where \( \mathcal{O}(1) \) is the very ample line bundle which gives the Plücker embedding.

Mukai proved that a general curve of genus 7 can be embedded in \( X_{12}^{10} \) as a complete intersection.

**Theorem 5.13.** [39] A curve \( C \) of genus 7 is a transversal linear section of \( X_{10}^{12} \subset \mathbb{P}^{15} \) if and only if \( C \) has no \( g^1_4 \).

Mukai proved that a generic curve of genus 8 can be embedded in \( Gr(2, 6) \) as a complete intersection.

**Theorem 5.14.** [38] A curve \( C \) of genus 8 is a transversal linear section of \( Gr(2, 6) \subset \mathbb{P}^{14} \) if and only if \( C \) has no \( g^2_2 \).

**Proposition 5.15.** [41, Proposition 2.3] The symplectic Grassmannian \( SpGr(n, 2n) \) is a smooth projective variety of dimension \( n(n+1)/2 \) whose anticanonical line bundle is \( \mathcal{O}(n+1) \) where \( \mathcal{O}(1) \) is the very ample line bundle which gives the Plücker embedding.

Mukai proved that a general curve of genus 9 can be embedded in \( SpGr(3, 6) \) as a complete intersection.

**Theorem 5.16.** [41] A curve \( C \) of genus 9 is a transversal linear section of \( SpGr(3, 6) \subset \mathbb{P}^{13} \) if and only if \( C \) has no \( g^2_3 \).
Corollary 5.17. A general curve of genus $g$ with $1 \leq g \leq 9$ is a complete intersection in a homogeneous variety.

Theorem 5.18. General curves of genus $g \leq 9$ are Fano visitors.

Proof. We already proved that general curves of genus $\leq 5$ are Fano visitors by using their canonical embeddings. Let $C \subset Z \subset \mathbb{P}^N$ be a curve of genus $6 \leq g \leq 9$ which is a complete intersection in a homogeneous variety $Z$ embedded in $\mathbb{P}^N$ via the Plücker embedding. From the adjunction formula we see that $K_C \cong \mathcal{O}_{\mathbb{P}^N}(1)|_C$. In each case, we can find varieties $C \subset S \subset Z \subset \mathbb{P}^N$ where $S$ is a 4-dimensional complete intersection in $Z$ and $C$ is the zero locus of a section of a rank 3 vector bundle on $S$. We then find that the variety $S$ and the rank 3 vector bundle satisfy the assumptions of Theorem 3.1. Therefore $C$ is a Fano visitor. Moreover we see that the Fano dimensions of general curves of genus $6 \leq g \leq 9$ are at most 5. \hfill $\square$

Theorem 5.18 enables us to provide many more examples of curves of genus $\geq 10$ which are Fano visitors.

Remark 5.19. After we finished writing this paper, we received a manuscript from M. S. Narasimhan [42] in which he proves that all curves of genus at least 6 are Fano visitors. He also proves that all non-hyperelliptic curves of genus 3, 4 or 5 are Fano visitors. Combined with Theorem 5.1 these results prove that all curves are Fano visitors.

It is well known that the moduli space of rank 2 stable vector bundles over a curve with fixed odd determinant is Fano. Narasimhan proves that the Fourier-Mukai transform defined by the universal bundle is fully faithful. In particular, the Fano dimension of an arbitrary curve of genus $g \geq 2$ is at most $3g-3$. But our discussion above for curves of low genus indicates that this upper bound is far from being optimal.

We end this section with the following question.

Question 5.20. What is the stratification on the moduli space $M_g$ of smooth curves of genus $g$, defined by the Fano dimension?

It will be interesting to compare the stratification by Fano dimension with other known stratifications on $M_g$.

6. Surfaces

In this section we discuss the Fano visitor problem for surfaces. Surfaces in this section always mean smooth projective surfaces. Unfortunately, we do not know much and so we raise more questions than give results. Let $Y$ be a surface and $\kappa$ denote its Kodaira dimension.

First, one can ask whether it is enough to consider the Fano visitor problem for minimal surfaces only.

Question 6.1. Let $Y$ be a smooth projective surface and $\hat{Y}$ denote the blowup of $Y$ at a point. Is $\hat{Y}$ a Fano visitor if $Y$ is a Fano visitor? More generally, is a variety birational to a Fano visitor a Fano visitor?

Many algebraic surfaces have fibration structures. Therefore the following question makes sense.
Question 6.2. Suppose that $Y$ is a fiber bundle over a variety $B$ with fiber $F$. Is $Y$ a Fano visitor if $B$ and $F$ are Fano visitors?

When the fibration is trivial, the answer to this question is a direct consequence of the following.

Proposition 6.3. [19, Corollary 7.4] Let $\Phi_K : D^b(A) \to D^b(X)$ and $\Phi_{K'} : D^b(A') \to D^b(X')$ be two fully faithful Fourier-Mukai transforms. Then
\[ \Phi_{K \boxtimes K'} : D^b(A \times A') \to D^b(X \times X') \]
is also fully faithful.

Therefore we have the following.

Corollary 6.4. If $B$ and $F$ are Fano visitors then $B \times F$ is a Fano visitor.

6.1. $\kappa = -\infty$ case. If the answer to Question 6.1 is yes, then we may assume $Y$ is either $\mathbb{P}^2$, a Hirzebruch surface or a ruled surface. If the answer to Question 6.2 is also yes, then the all surfaces with $\kappa = -\infty$ are Fano visitors by Remark 6.19.

6.2. $\kappa = 0$ case. The following is a consequence of Theorem 3.1 for K3 surfaces.

Corollary 6.5. Let $Y$ be a K3 surface which is the zero locus of a section of an ample vector bundle $E$ of rank $r$ on a Fano variety $S$ of dimension of $r + 2$ where $r \geq 2$. Then $Y$ is a Fano visitor. The Fano dimension of $Y$ is at most $2r$.

Example 6.6. Let $V$ be a Fano 3-fold and let $Y$ be a smooth divisor in $|K_V|$ which is a K3 surface by adjunction. When $V$ is the zero locus of a regular section of an ample vector bundle on another Fano manifold $W$ and the line bundle $K_V$ is the restriction of an ample line bundle on $W$, we find that $Y$ is a Fano visitor by Theorem 6.1. For example, general K3 surfaces of genus $6 \leq g \leq 10$ satisfy these conditions (cf. [35]). Therefore general K3 surfaces of genus $6 \leq g \leq 10$ are Fano visitors and their Fano dimensions are 4.

An Abelian surface which is the product of two elliptic curves is a Fano visitor by Corollary 6.4.

Example 6.7. Let $Y = E \times E'$ be the product of two elliptic curves. Then $Y$ is a Fano visitor.

6.3. $\kappa = 1$ case. Because every minimal surfaces with $\kappa = 1$ is an elliptic surface, it is natural to ask the following.

Question 6.8. Let $Y$ be an elliptic surface over a curve $C$. Is $Y$ a Fano visitor if $C$ is a Fano visitor?

Again we do not know the answer to this question unless $Y \to C$ is a trivial fibration.

6.4. $\kappa = 2$ case. By Theorem 3.1 we can provide many examples of surfaces of general type which are Fano visitors. However we do not know whether all surfaces of general type are Fano visitors or not.

Recently, interesting new categories in the derived categories of surfaces of general type with $p_g = q = 0$ were discovered (cf. [17, 6, 15, 31, 32, 33]). Their Grothendieck groups are finite torsion and their Hochschild homology groups vanish. We call them quasi-phantom categories. On the other hand, no Fano variety is known to have a quasi-phantom subcategory. Therefore the following question seems interesting.
Question 6.9. Is there a Fano variety $X$ whose derived category contains a quasi-phantom category?

Obviously this question is closely related to the Fano visitor problem.

Question 6.10. Let $Y$ be a surface of general type with $p_g = q = 0$. Is there a Fano host of $Y$?

For example, a Fano host of the Barlow surface will give us a Fano variety containing a phantom category.

Although we do not know the answer to Question 6.9, we can construct a Fano orbifold whose derived category contains a quasi-phantom category. Recall that a Fano orbifold refers to the quotient stack $[X/G]$ of a smooth Fano variety $X$ acted on by a finite group $G$.

Example 6.11. Let $Y \subset \mathbb{P}^3$ be the variety defined by Fermat quintic $f = z_0^5 + z_1^5 + z_2^5 + z_3^5 = 0$ and let $G = \mathbb{Z}_5 = \langle \xi \rangle$ act on $Y$ by $\xi \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : \xi z_1 : \xi^2 z_2 : \xi^3 z_3]$ where $\xi = e^{2\pi \sqrt{-1}/5}$ is a primitive fifth root of unity. The $G$-action on $Y$ is free and $Y/G$ is the classical Godeaux surface. Let $X = w^{-1}(0) \subset \mathbb{P}E^\vee$ be a Fano host of $Y = s^{-1}(0) \subset \mathbb{P}^5$ obtained by the construction in §3.2 where $s$ is the section of $E = \mathcal{O}_{\mathbb{P}^5}(5) \oplus \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 2}$ defined by the Fermat quintic $f$ and two linear polynomials $z_4, z_5$ that cut out $\mathbb{P}^3$ in $\mathbb{P}^5$. Let $G$ act on $z_4$ and $z_5$ trivially. Then $G$ acts on $\mathbb{P}^5$ and $E$ compatibly. Moreover the section $s = (f, z_4, z_5)$ is $G$-invariant.

By Orlov’s theorem (Remark 2.10), we see that there is a fully faithful embedding $D^b(Y/G) \to D^b([X/G])$ of the derived category of the classical Godeaux surface into the derived category of the Fano orbifold $[X/G]$. Since the derived category of the classical Godeaux surface contains a quasi-phantom category (cf. [7]), $D^b([X/G])$ also contains a quasi-phantom category.

It seems that similar argument should provide more examples of Fano orbifolds containing quasi-phantom categories (cf. [15, 31, 32, 33]). One may try to use the McKay correspondence to obtain varieties close to Fano whose derived categories contain quasi-phantom categories (cf. [10]). It will be a very nice application of higher dimensional McKay correspondence if one can provide examples of log Fano varieties containing quasi-phantom categories.

7. Higher dimensional varieties

Theorem 6.1 provides us with lots of examples of Fano visitors. In this section we discuss higher dimensional Fano visitors.

7.1. Complete intersections in Fano varieties. Many varieties can be described as the zero loci of regular sections of vector bundles on Fano varieties. When they satisfy the assumption of Theorem 6.1 we know that these varieties are Fano visitors and we can give upper bounds for their Fano dimensions.

The numerical conditions of Theorem 5.1 are particularly easy to check for Calabi-Yau varieties. Many Calabi-Yau varieties arise as complete intersections in homogeneous varieties. For these Calabi-Yau varieties, the construction in §3 gives the following proposition.

Proposition 7.1. Let $Y$ be an $n$-dimensional Calabi-Yau variety. Suppose that there is an embedding of $Y$ into a smooth projective Fano variety $S$ of dimension $\geq n + 2$ as the zero locus of a regular section of an ample vector bundle $E$ whose
rank coincides with the codimension of \( Y \). Let \( m \) be the smallest dimensions of such an \( S \). Then the Fano dimension of \( Y \) is at most \( 2m - n - 2 \). In particular if \( m = n + 2 \), then the Fano dimension of \( Y \) is \( n + 2 \).

For example let us consider Calabi-Yau 3-folds and general type varieties in some Fano 4-folds. Kürhle in [23] classified Fano 4-folds of index 1 which are zero loci of vector bundles on homogeneous varieties. Then we can consider transversal linear sections of these Fano 4-folds. For instance, let us consider Fano 4-folds of (b7) type which are zero loci of global sections of \( \mathcal{O}(1)^{\oplus 6} \) in \( Gr(2, 7) \). If the linear sections of these Fano 4-folds are smooth then it is easy to check that they are Fano visitors. The same argument works for transversal linear sections of many other Fano 4-folds described in [23]. It seems interesting to study derived categories of linear sections of these Fano 4-folds. See [23, 28, 34] for the geometry of these varieties.

7.2. Toric varieties. It seems interesting to consider the Fano visitor problem for toric varieties. Because many interesting problems about toric varieties can be described and solved via combinatorics of fans or polytopes, we expect that there should be a combinatorial approach to this problem.

**Question 7.2.** Let \( Y \) be a smooth toric variety. Is there a Fano host of \( Y \) constructed by an explicit combinatorial method? Can we compute its Fano dimension?

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