Stochastic Burgers equation with fractional derivative driven by multiplicative noise

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Abstract

This article is devoted to the study of the existence and uniqueness of mild solution to time- and space-fractional stochastic Burgers equation perturbed by multiplicative white noise. The required results are obtained by stochastic analysis techniques, fractional calculus and semigroup theory. We also proved the regularity properties of mild solution for this generalized Burgers equation.

Keywords: Stochastic Burgers equation, fractional derivative, mild solution, regularity properties.

1. Introduction

Stochastic Burgers equation (SBE) plays an important role in the modeling of many phenomena in different fields, such as fluid dynamics, nonlinear acoustics, hydrodynamics, cosmology, astrophysics and statistical physics, and so on. In the last decade, SBE has gained a great development in both theory and application and a large volume of literature is available on this subject (see e.g.\cite{1-4} and references therein). It is particularly mentioned that when the Laplacian operator $\Delta$ in SBE is replaced by fractional derivative, which can be used to describe anomalous diffusion processes in fractal flow and acoustic waves propagation in porous media \cite{6,22,23}. Sugimoto \cite{5} have studied the generalized Burgers-type equation with a fractal power of Laplacian in the principal part, which described the unidirectional propagation of acoustic waves through a gas-filled tube with a boundary layer. Besides, the space-fractional SBE also can be used to study the acoustic waves propagation in tunnels during the passage of the trains, which may yield a memory effect and other types of resonance phenomena \cite{6}. On the other hand, time-fractional differential equations are found to be quite effective in modelling anomalous diffusion processes as its can characterize the long memory processes \cite{14-16,21,24}. Hence, Burgers equation with time-fractional can be adapted to describe the memory effect of the wall friction through the boundary layer \cite{7}. Furthermore, the analytical solutions of the time- and space-fractional Burgers equations have been investigated by variational iteration method \cite{7} and Adomian decomposition method \cite{8}.

In this study, we focus on the following generalized SBE with time-space fractional derivative on a bounded domain $D \subset \mathbb{R}^d (1 \leq d \leq 3)$:

$$C^D_\alpha \frac{\partial u}{\partial t} - u \cdot \nabla u + (-\Delta)^{\frac{\alpha}{2}} u = g(u) \dot{W}(t), (t, x) \in (0, T] \times D,$$

(1.1)

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subject to the initial condition:

\[ u(0, x) = u_0(x), x \in D, \]  

and the Dirichlet boundary conditions:

\[ u(t, x)|_{\partial D} = 0, t \in [0, T], \]  

in which the term \( g(u)\dot{W}(t) = g(u)\frac{dW(t)}{dt} \) describes a state dependent random noise, where \( W(t)_{t \in [0, T]} \) is a \( \mathcal{F}_t \)-adapted Wiener process defined on a completed probability space \( (\Omega, \mathcal{F}, P) \) with the expectation \( E \), and associate with the normal filtration \( \mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\} \); The operator \( (-\triangle)_{\alpha} \), \( \alpha \in (1, 2) \) stands for the fractional power of the Laplacian (see [25]); We denote by \( C^D_{\beta} \) the Caputo derivative of order \( \beta \), which is defined by (see [9])

\[ C^D_{\beta}u(t, x) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(s, x)}{\partial s} \frac{ds}{(t-s)^\beta}, & 0 < \beta < 1, \\ \frac{\partial u(t, x)}{\partial t}, & \beta = 1, \end{cases} \]  

where \( \Gamma(\cdot) \) stands for the gamma function \( \Gamma(\beta) = \int_0^\infty t^{\beta-1}e^{-t}dt \).

Eq.(1.1) might be used to model anomalous diffusion processes in disordered media, and describe the acoustic wave propagation in porous media with memory effect and with random effects. Notice that the study of space-fractional SBE can be found in some literatures. For details, Brzeźniak and Debdi [10] proved existence and uniqueness of a mild global solution to the Cauchy problem for the stochastic fractional Burgers equation. Brzeźniak et al.[6] studied the ergodic properties of the solution for space-fractional SBE. Yang [11] proposed some estimates on the solution of space-fractional SBE and given the invariant measure. Lv and Duan [12] investigated the existence of martingale solutions and weak solutions for space-fractional SBE on a bounded domain. However, to the best of our knowledge, there are no existing works for the time- and space-fractional SBE, which is a fascinating and useful problem.

The main contribution of this paper is to establish the existence, uniqueness, and regularity properties of mild solution to time-space fractional SBE driven by multiplicative noise, which generalizes many previous works [6,11,12]. The rest of the paper is organized as follows. In Section 2, we will introduce some notations and preliminaries, which play a crucial role in our theorem analysis. In Section 3, the existence and uniqueness of mild solution to the problems of time-space fractional SBE are obtained by stochastic analysis techniques, fractional calculus and semigroup theory. Finally, the spatial and temporal regularity properties of mild solution to this time-space fractional SBE are proved.

2. Notations and preliminaries

Denote the basic functional space \( L^p(D), 1 \leq p < \infty \) and \( H^s(D) \) by the usual Lebesgue and Sobolev spaces, respectively. We assume that \( A \) is the negative Laplacian \( -\Delta \) in a bounded domain \( D \) with zero Dirichlet boundary conditions in a Hilbert space \( H = L^2(D) \), which are given by

\[ A = -\Delta, \quad \mathcal{D}(A) = H^3_0(D) \cap H^2(D). \]

Since the operator \( A \) is self-adjoint on \( H \) with discrete spectral, i.e., there exists the eigenvectors \( e_n \) with corresponding eigenvalues \( \lambda_n \) such that

\[ Ae_n = \lambda_n e_n, e_n = \sqrt{2} \sin(n\pi), \lambda_n = \pi^2n^2, n \in \mathbb{N}^+. \]
For any $s > 0$, let $\dot{H}^s$ be the domain of the fractional power $A^\frac{s}{2} = (-\triangle)^\frac{s}{2}$, which can be defined by

$$A^\frac{s}{2}e_n = \lambda_n^\frac{s}{2}e_n, n = 1, 2, \ldots,$$

and

$$\dot{H}^s = \mathcal{D}(A^\frac{s}{2}) = \{ v \in L^2(D), s.t. \|v\|_{H^s}^2 = \sum_{n=1}^\infty \lambda_n^\frac{s}{2}v_n^2 < \infty \},$$

where $v_n := \langle v, e_n \rangle$ with the inner product $\langle \cdot, \cdot \rangle$ in $L^2(D)$. We denote that $\|v\|_{H^s} = \|A^\frac{s}{2}v\|$, and the corresponding dual space $\dot{H}^{-s}$ with the inverse operator $A^{-\frac{s}{2}}$. We also denote $A_s$ for $A^\frac{s}{2}$ and the bilinear operator $B(u, v) = u \cdot \nabla v$, and $\mathcal{D}(B) = H_0^1(D)$ with a slight abuse of notation $B(u) := B(u, u)$. Then the Eqs.(1.1)-(1.3) can be rewritten as the following abstract formulation:

$$\begin{align*}
&CD^\beta_t u(t) = -A_s u(t) + B(u(t)) + g(u(t)) \frac{dW(t)}{dt}, t > 0, \\
&u(0) = u_0,
\end{align*}$$

(2.1)

where $\{W(t)\}_{t \geq 0}$ is a $Q$-Wiener process with linear bounded covariance operator $Q$ such that $\text{Tr}(Q) < \infty$. Further, there exists the eigenvalues $\lambda_n$ and corresponding eigenfunctions $e_n$ satisfy $Qe_n = \lambda_n e_n$, $n = 1, 2, \ldots$, then the Wiener process is given by

$$W(t) = \sum_{n=1}^\infty \lambda_n^{1/2} \beta_n(t) e_n,$$

in which $\{\beta_n\}_{n \geq 1}$ is a sequence of real-valued standard Brownian motions.

Let $L^2_0 = L^2(Q^{1/2}(H), H)$ denote the space of Hilbert-Schmidt operators from $Q^{1/2}(H)$ to $H$ with the norm $\|\phi\|_{L^2_0} := \|\phi Q^{1/2}\|_H = (\sum_{n=1}^\infty \phi Q^{1/2} e_n)^{1/2}$, i.e., $L^2_0 = \{ \phi \in L(H) : \sum_{n=1}^\infty \|\phi Q^{1/2} e_n\|^2 < \infty \}$, where $L(H)$ is the space of bounded linear operators from $H$ to $H$.

For an arbitrary Banach space $B$, we denote $\|\cdot\|_{L^p(\Omega; B)}$ by the norm in $L^p(\Omega, \mathcal{F}, P; B)$, which defined as

$$\|v\|_{L^p(\Omega; B)} = (\mathbb{E}[\|v\|_B^p])^{\frac{1}{p}}, \forall v \in L^p(\Omega, \mathcal{F}, P; B),$$

for any $p \geq 2$.

We shall also need the following result with respect to the fractional operator $A_s$ (see Ref.[11]).

**Lemma 2.1.** For any $\alpha > 0$, an analytic semigroup $S_\alpha(t) = e^{-tA_\alpha}$, $t \geq 0$ is generated by the operator $-A_s$ on $L^p$, and for any $\nu \geq 0$, there exists a constant $C_{\alpha, \nu}$ dependent on $\alpha$ and $\nu$ such that

$$\|A_s S_\alpha(t)\|_{\mathcal{L}(L^p)} \leq C_{\alpha, \nu} t^{-\frac{\alpha}{2}}, t > 0,$$

(2.2)

in which $\mathcal{L}(B)$ denotes the Banach space of all linear bounded operators from $B$ to itself.

Next, we will introduce the following lemma to estimate the stochastic integrals, which contains the Burkholder-Davis-Gundy’s inequality.

**Lemma 2.2.**([13]) For any $0 \leq t_1 < t_2 \leq T$ and $p \geq 2$, and for any predictable stochastic process $v : [0, T] \times \Omega \to L_0^2$, which satisfies

$$\mathbb{E}[\int_0^T \|v(s)\|_{L_0^2}^2 ds] < \infty,$$
then we have
\[ E[\| \int_0^t v(s) dW(s) \|^p] \leq C(p) E[\| \int_0^t v(s) \|^\frac{p}{2} ds] \tag{2.3} \]
where \( C(p) = \left( \frac{(p-1)^\frac{1}{p}}{p} \right) (\frac{\xi}{p-1})^\frac{p}{p-1} \) is a constant.

Inspired by the definition of the mild solution to the time-fractional differential equations (see Refs. [14-18]), we give the following definition of mild solution for our time-space fractional stochastic Burgers equation.

**Definition 2.1.** A \( \mathcal{F}_t \)-adapted process \((u(t))_{t \in [0, T]} \) is called a mild solution to (2.1), if \((u(t))_{t \in [0, T]} \in C([0, T]; H^\nu) \) \( \mathbb{P} \)-a.e., and it holds
\[
u(t) = E^\beta(t) u_0 + \int_0^t (t-s)^{\beta-1} E^\beta(t-s) B(u(s)) ds
+ \int_0^t (t-s)^{\beta-1} E^\beta(t-s) g(u(s)) dW(s),
\tag{2.4}
\]
for a.s. \( \omega \in \Omega \), where the generalized Mittag-Leffler operators \( E^\beta(t) \) and \( E^\beta(t) \) are defined as
\[
u(t) = \int_0^\infty M^\beta(\theta) S_\alpha(t^\theta) d\theta,
\]
and
\[
u(t) = \int_0^\infty \beta \theta M^\beta(\theta) S_\alpha(t^\theta) d\theta,
\]
which contain the Mainardi’s Wright-type function with \( \beta \in (0, 1) \) given by
\[ M^\beta(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^n}{n! \Gamma(1-\beta(1+n))}, \]
in which the Mainardi function \( M^\beta(\theta) \) act as a bridge between the classical integral-order and fractional derivatives of differential equations, for more details see [19,20]. Here, the derivation of mild solution (2.4) can be found in Appendix A.

Firstly, let us state the property of the special Mainardi function \( M^\beta(\theta) \). Further, the properties of generalized Mittag-Leffler operators \( E^\beta(t) \) and \( E^\beta(t) \) are proved.

**Lemma 2.3.** (see [15]) For any \( \beta \in (0, 1) \) and \( -1 < \nu < \infty \), it is not difficult to verify that
\[
u^\beta(\theta) \geq 0, \text{ and } \int_0^\infty \theta^\nu M^\beta(\theta) d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\beta\nu)} \tag{2.5}
\]
for all \( \theta \geq 0 \).

**Theorem 2.1.** For any \( t > 0 \), \( E^\beta(t) \) and \( E^\beta(t) \) are linear and bounded operators. Moreover, for \( 0 \leq \nu < \alpha < 2 \), there exist constants \( C_\alpha = C(\alpha, \beta, \nu) > 0 \) and \( C_\beta = C(\alpha, \beta, \nu) > 0 \) such that
\[
u^\beta(t) \leq C_\alpha t^{-\frac{\alpha \nu}{2}} \|v\|, \|E^\beta(t) v\|_{H^\nu} \leq C_\beta t^{-\frac{\alpha \nu}{2}} \|v\|. \tag{2.6}
\]

**Proof.** For \( t > 0 \) and \( 0 \leq \nu < \alpha < 2 \), by means of the Lemma 2.1 and Lemma 2.3, we have
\[
u^\beta(t) \leq \int_0^\infty M^\beta(\theta) \|A_\nu S_\alpha(t^\theta) v\| d\theta
\leq \int_0^\infty C_\alpha t^{-\frac{\alpha \nu}{2}} \theta^{-\frac{\nu}{\alpha}} M^\beta(\theta) \|v\| d\theta
= \frac{C_\alpha}{\Gamma(1-\frac{\alpha \nu}{2})} t^{-\frac{\alpha \nu}{2}} \|v\|, v \in L^2(D),
\]
and

\[
\|E_{\beta}(t)v\|_{H^\nu} \leq \int_0^\infty \beta \theta M_\beta(\theta) \|A_\nu S_\alpha(t^3 \theta)v\|d\theta \\
\leq \int_0^\infty C_{\alpha, \nu} \beta t^{\frac{\alpha}{\omega}} \theta^{1-\frac{\nu}{\alpha}} M_\beta(\theta)\|v\|d\theta \\
= \frac{C_{\alpha, \nu} \Gamma(2-\frac{\nu}{\alpha})}{\Gamma(1+\beta(1-\frac{\nu}{\alpha}))} t^{\frac{\alpha}{\omega}}\|v\|, \; v \in L^2(D),
\]

which imply that the estimates (2.6) hold, so it is easy to know that \(E_\beta(t)\) and \(E_{\beta, \beta}(t)\) are linear and bounded operators.

**Theorem 2.2.** For any \(t > 0\), the operators \(E_\beta(t)\) and \(E_{\beta, \beta}(t)\) are strongly continuous. Moreover, for any \(0 \leq t_1 < t_2 \leq T\) and for \(0 < \nu < \alpha < 2\), there exist constants \(C_{\alpha, \nu} = C(\alpha, \beta, \nu) > 0\) and \(C_{\beta, \nu} = C(\alpha, \beta, \nu) > 0\) such that

\[
\|(E_\beta(t_2) - E_\beta(t_1))v\|_{H^\nu} \leq C_{\alpha, \nu}(t_2 - t_1)^{\frac{\alpha}{\omega}}\|v\|, \tag{2.7}
\]

and

\[
\|(E_{\beta, \beta}(t_2) - E_{\beta, \beta}(t_1))v\|_{H^\nu} \leq C_{\beta, \nu}(t_2 - t_1)^{\frac{\alpha}{\omega}}\|v\|. \tag{2.8}
\]

**Proof.** For any \(0 < T_0 \leq t_1 < t_2 \leq T\), it is easy to deduce that

\[
\int_{t_1}^{t_2} \frac{dS_\alpha(t^3 \theta)}{dt}dt = S_\alpha(t^3 \theta) \big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \beta \theta t^{\beta-1} A_\alpha S_\alpha(t^3 \theta)dt.
\tag{2.9}
\]

For \(0 < \nu < \alpha < 2\), making use of the above expression, the Lemma 2.1 and Lemma 2.3, we can arrive at

\[
\|(E_\beta(t_2) - E_\beta(t_1))v\|_{H^\nu} = \|A_\nu (E_\beta(t_2) - E_\beta(t_1))v\| \\
= \| \int_0^\infty M_\beta(\theta)A_\nu (S_\alpha(t^3 \theta) - S_\alpha(t^3 \theta))v\| \\
= \int_0^\infty \beta \theta M_\beta(\theta) \|A_\nu S_\alpha(t^3 \theta)v\|d\theta \\
\leq \int_0^\infty \beta \theta M_\beta(\theta) \int_{t_1}^{t_2} t^{\beta-1} \|A_\alpha+S_\alpha(t^3 \theta)v\|_{L^2}dt d\theta \\
\leq \int_0^\infty C_{\alpha, \nu} \beta \theta^{\frac{\alpha}{\omega}} M_\beta(\theta) \left( \int_{t_1}^{t_2} t^{\beta-1} dt \right)\|v\|d\theta \\
= \frac{\alpha C_{\alpha, \nu}}{\nu \Gamma(1-\frac{\alpha}{\omega})(t_1^{\frac{\alpha}{\omega}} - t_2^{\frac{\alpha}{\omega}})} \|v\| \\
\leq \frac{\alpha C_{\alpha, \nu}}{\nu T_0^{\frac{\alpha}{\omega}} \Gamma(1-\frac{\alpha}{\omega})} (t_2 - t_1)^{\frac{\omega}{2\nu}} \|v\|, \; v \in L^2(D),
\]

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and

\[
\|(E_{\beta,\beta}(t_2) - E_{\beta,\beta}(t_1))v\|_{H^\nu} = \|A_v(E_{\beta,\beta}(t_2) - E_{\beta,\beta}(t_1))v\|
\]

\[
= \| \int_0^\infty \beta \theta M_{\beta}(\theta)A_\alpha(S_\alpha(t_2^\beta \theta) - S_\alpha(t_1^\beta \theta))v d\theta \|
\]

\[
\leq \int_0^\infty \beta^2 \theta^2 M_{\beta}(\theta) \int_1^{t_2} t^{\beta-1} \|A_{\alpha+v}S_\alpha(t^\beta \theta)v\| dt d\theta
\]

\[
\leq \int_0^\infty C_{\alpha,\nu} \beta^2 \theta^2 M_{\beta}(\theta) (\int_1^{t_2} t^{-1} \|v\| dt) \|v\| d\theta
\]

\[
= \frac{\alpha \beta C_{\alpha,\nu} \Gamma(2 - \frac{\nu}{2})}{\nu \Gamma(1 + \beta(1 - \frac{\nu}{2}))} (t_2 - t_1)^{\frac{\nu}{2}} \|v\|, v \in L^2(D).
\]

It is obviously to see that the term \(\|(E_{\beta}(t_2) - E_{\beta}(t_1))v\|_{H^\nu} \to 0\) and \(\|(E_{\beta,\beta}(t_2) - E_{\beta,\beta}(t_1))v\|_{H^\nu} \to 0\) as \(t_1 \to t_2\), which mean that the operators \(E_{\beta}(t)\) and \(E_{\beta,\beta}(t)\) are strongly continuous.

**Remark.** Assume \(\nu = 0\) in Theorem 2.2, then there exist constants \(C_\alpha = C(\alpha, \beta) > 0\) and \(C_\beta = C(\alpha, \beta) > 0\) such that

\[
\|(E_{\beta}(t_2) - E_{\beta}(t_1))v\| \leq C_\alpha (t_2 - t_1) \|v\|,
\]

(2.10)

and

\[
\|(E_{\beta,\beta}(t_2) - E_{\beta,\beta}(t_1))v\| \leq C_\beta (t_2 - t_1) \|v\|.
\]

(2.11)

**Proof.** For any \(0 < T_0 \leq t_1 < t_2 \leq T\), the same as the proof of Theorem 2.2, we get

\[
\|(E_{\beta}(t_2) - E_{\beta}(t_1))v\| = \| \int_0^\infty M_{\beta}(\theta)(S_\alpha(t_2^\beta \theta) - S_\alpha(t_1^\beta \theta))v d\theta \|_{L^2}
\]

\[
\leq \int_0^\infty \beta \theta M_{\beta}(\theta) \int_1^{t_2} t^{\beta-1} \|A_\alpha S_\alpha(t^\beta \theta)v\| dt d\theta
\]

\[
\leq \int_0^\infty C_{\alpha,\nu} \beta M_{\beta}(\theta) (\int_1^{t_2} t^{-1} dt) \|v\| d\theta
\]

\[
= \frac{C_{\alpha,\nu} \beta \ln t_2 - \ln t_1) ||v||}{T_0 (t_2 - t_1) \|v\|, v \in L^2(D),
\]

and

\[
\|(E_{\beta,\beta}(t_2) - E_{\beta,\beta}(t_1))v\| = \| \int_0^\infty \beta \theta M_{\beta}(\theta)(S_\alpha(t_2^\beta \theta) - S_\alpha(t_1^\beta \theta))v d\theta \|
\]

\[
\leq \int_0^\infty \beta^2 \theta^2 M_{\beta}(\theta) \int_1^{t_2} t^{\beta-1} \|A_\alpha S_\alpha(t^\beta \theta)v\| dt d\theta
\]

\[
\leq \int_0^\infty C_{\alpha,\nu} \beta^2 \theta^2 M_{\beta}(\theta) (\int_1^{t_2} t^{-1} dt) \|v\| d\theta
\]

\[
= \frac{C_{\alpha,\nu} \beta^2 \Gamma(2)}{\Gamma(1 + \beta)} (\ln t_2 - \ln t_1) ||v||
\]

\[
\leq \frac{C_{\alpha,\nu} \beta^2 \Gamma(2)}{\Gamma(1 + \beta)} (t_2 - t_1) \|v\|, v \in L^2(D).
\]

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This completes the proof.

3. Existence and uniqueness of mild solution

Our main purpose of this section is to prove the existence and uniqueness of mild solution to the problem (2.1). To do this, the following assumptions are imposed.

**Assumption 3.1.** The measurable function $g : \Omega \times H \to L^2_0$ satisfies the following global Lipschitz and growth conditions:

$$
\|g(v)\|_{L^2_0} \leq C\|v\|, \quad \|g(u) - g(v)\|_{L^2_0} \leq C\|u - v\|
$$

for any $u, v \in H$.

**Assumption 3.2.** Let $C > 0$ be a real number, then the bounded bilinear operator $B : L^2(D) \to \dot{H}^{-1}(D)$ satisfies the following properties:

$$
\|B(u)\|_{\dot{H}^{-1}} \leq C\|u\|^2,
$$

and

$$
\|B(u) - B(v)\|_{\dot{H}^{-1}} \leq C(\|u\| + \|v\|)\|u - v\|
$$

for any $u, v \in L^2(D)$.

**Assumption 3.3.** Assume that the initial value $u_0 : \Omega \to \dot{H}^\nu$ is a $\mathcal{F}_0$-measurable random variable, it holds that

$$
\|u_0\|_{L^p(\Omega; \dot{H}^\nu)} < \infty,
$$

for any $0 \leq \nu < \alpha < 2$.

**Theorem 3.1.** Let Assumptions 3.1 to 3.3 be satisfied for some $p \geq 2$, then there exists a unique mild solution $\{u(t)\}_{t \in [0,T]}$ in the space $L^p(\Omega; \dot{H}^\nu)$ with $0 \leq \nu < \alpha < 2$.

**Proof.** We fix an $\omega \in \Omega$ and use the standard Picard’s iteration argument to prove the existence of mild solution. To begin with, the sequence of stochastic process $\{u_n(t)\}_{n \geq 0}$ is constructed as

$$
\begin{align*}
\begin{cases}
u_{n+1}(t) = E_{\beta}(t)u_0 + N_1(u_n(t)) + N_2(u_n(t)), \\
u_0(t) = u_0,
\end{cases}
\end{align*}
$$

where

$$
\begin{align*}
N_1(u_n(t)) &= \int_0^t (t - s)^{\beta - 1}E_{\beta,\beta}(t - s)B(u_n(s))ds, \\
N_2(u_n(t)) &= \int_0^t (t - s)^{\beta - 1}E_{\beta,\beta}(t - s)g(u_n(s))dW(s).
\end{align*}
$$

The proof will be split into three steps.

**Step 1:** For each $n \geq 0$, we show that

$$
\sup_{t \in [0,T]} \mathbb{E}[\|u_n(t)\|^p_{\dot{H}^\nu}] < \infty.
$$

Note that

$$
\mathbb{E}[\|u_{n+1}(t)\|^p_{\dot{H}^\nu}] \leq 3^{p-1}\mathbb{E}[\|E_{\beta}(t)u_0\|^p_{\dot{H}^\nu}] + 3^{p-1}\mathbb{E}[\|N_1(u_n(t))\|^p_{\dot{H}^\nu}] \\
+ 3^{p-1}\mathbb{E}[\|N_2(u_n(t))\|^p_{\dot{H}^\nu}].
$$
The application of the Lemma 2.1 gives

\[ E[\|E_\beta(t)u_0\|_{H^r}] \leq E\left[ \int_0^\infty M_\beta(\theta)(\|A_\nu S_\alpha(t^\beta \theta)u_0\|^2)^{\frac{1}{2}} d\theta \right] \]

\[ = E\left[ \int_0^\infty M_\beta(\theta) \left( \sum_{n=1}^\infty (A_{\nu,e_n} e^{t^\beta t \theta} u_0, e_n)^2 \right)^{\frac{1}{2}} d\theta \right] \]

\[ = E\left[ \int_0^\infty M_\beta(\theta) \left( \sum_{n=1}^\infty (A_{\nu} u_0, e^{t^\beta t \theta} e_n)^2 \right)^{\frac{1}{2}} d\theta \right] \]

\[ \leq E\left[ \int_0^\infty M_\beta(\theta) \|u_0\|_{H^r} d\theta \right] = E[\|u_0\|_{H^r}] \tag{3.8} \]

Applying the following Hölder inequality to the second term of the right-hand side of (3.7)

\[ \int_a^b |f(t)g(t)|dt \leq (\int_a^b |f(t)|^p dt)^{\frac{1}{p}} (\int_a^b |g(t)|^q dt)^{\frac{1}{q}} \]

where \( p, q \in (1, \infty) \), we infer

\[ E[\|N_1(u_n(t))\|_{H^r}^p] \leq E\left[ \int_0^t \|t-s\|^p A_{\nu} E_{\beta,\beta}(t-s) A_{\nu-1} B(u_n(s))ds \right]^p \]

\[ \leq C_\beta^p \left( \int_0^t (t-s)^{p(\beta-1)\frac{\beta-1}{p-1}} ds \right)^{p-1} \int_0^t E[\|A_{\nu-1} B(u_n(s))\|_{L^2}^p]ds \]

\[ \leq K_1 \int_0^t E[\|u_n(s)\|_{H^r}^p]ds, \tag{3.9} \]

where \( K_1 = C_\beta^p C_{\nu}^p \left[ \frac{p-2}{(p\beta-1)\frac{\beta-1}{p-1}} \right] t^{p-1} T^{p(\beta-1)\frac{\beta-1}{p-1}} \max_{t\in[0,T]} E[\|u_n(t)\|_{H^r}^p] \).

Making use of the Hölder inequality and Lemma 2.2 to the third term of the right-hand side of (3.7), we get

\[ E[\|N_2(u_n(t))\|_{H^r}^p] \leq C(p) E\left[ \int_0^t \|t-s\|^p E_{\beta,\beta}(t-s) \|A_{\nu}\|_{L^2}^2 \|g(u_n(s))\|_{L^2}^2 ds \right]^\frac{p}{2} \]

\[ \leq C(p) C_{\nu}^p \left( \int_0^t (t-s)^{p(\beta-1)\frac{\beta-1}{p-1}} ds \right)^{\frac{p-2}{2}} \left( \int_0^t E\|A_{\nu}\|_{L^2}^2 ds \right)^{\frac{p}{2}} \]

\[ \leq K_2 \int_0^t E[\|u_n(s)\|_{H^r}^p]ds \tag{3.10} \]

where \( K_2 = C(p) C_{\nu}^p C_{\nu}^p \left[ \frac{p-2}{p(2\beta-1)\frac{\beta-1}{p-1}} \right] t^{p-1} T^{p(2\beta-1)\frac{\beta-1}{p-1}} \).

Using the above estimates (3.7)-(3.10), we have

\[ E[\|u_{n+1}(t)\|_{H^r}^p] \leq 3^{p-1} E[\|u_0\|_{H^r}^p] + 3^{p-1} (K_1 + K_2) \int_0^t E[\|u_n(s)\|_{H^r}^p]ds. \]

By means of the extension of Gronwall’s lemma, it holds that

\[ \sup_{t\in[0,T]} E[\|u_n(t)\|_{H^r}^p] < \infty, \]

for each \( n \geq 0 \).
Step 2: Show that the sequence \( \{u_n(t)\} \) is a Cauchy sequence in the space \( L^p(\Omega; \dot{H}^\nu) \). For any \( n \geq m \geq 1 \), applying the similar arguments employed to obtain (3.9) and (3.10), we get

\[
E[\|u_n(t) - u_m(t)\|_{\dot{H}^\nu}^p] \leq 2^{p-1} E[\|N_1(u_{n-1}(t)) - N_1(u_{m-1}(t))\|_{\dot{H}^\nu}^p] + 2^{p-1} E[\|N_2(u_{n-1}(t)) - N_2(u_{m-1}(t))\|_{\dot{H}^\nu}^p] \\
\leq K \int_0^T E[\|u_{n-1}(s) - u_{m-1}(s)\|_{\dot{H}^\nu}^p] ds,
\]

in which

\[
K = 2^{p-1} \left\{ C_\beta^p C_\nu^p \left[ \frac{p-1}{p(\beta - \frac{\alpha}{2}) - 1} \right] T^{p(\beta - \frac{\alpha}{2}) - 1} \left( \max_{t \in [0,T]} E[\|u_{n-1}(t)\|_{\dot{H}^\nu}^p] \right) + \max_{t \in [0,T]} E[\|u_{m-1}(t)\|_{\dot{H}^\nu}^p] + C(p) C_\beta^p \left[ \frac{p-2}{p(2\beta - 1) - 2} \right] T^{p(\beta - 1) - 2} \right\}.
\]

A direct application of Gronwall’s lemma yields

\[
\sup_{t \in [0,T]} E[\|u_n(t) - u_m(t)\|_{\dot{H}^\nu}^p] = 0.
\]

As a result, the sequence \( \{u_n(t)\} \) is a Cauchy sequence in the space \( L^p(\Omega; \dot{H}^\nu) \). Further, there exists a \( u(t) \in L^p(\Omega; \dot{H}^\nu) \) such that

\[
\sup_{t \in [0,T]} E[\|u_n(t) - u(t)\|_{\dot{H}^\nu}^p] = 0,
\]

for all \( T > 0 \).

Taking limits to the stochastic sequence \( \{u_n(t)\} \) in (3.5) as \( n \to \infty \), we finish the proof of the existence of mild solution to (2.1).

Step 3: We show the uniqueness of mild solution.

Assume \( u \) and \( v \) are two mild solutions of the problem (2.1), using the similar calculations as in Step 2, we can obtain

\[
\sup_{t \in [0,T]} E[\|u(t) - v(t)\|_{\dot{H}^\nu}^p] = 0,
\]

for all \( T > 0 \), which implies that \( u = v \), it follows that the uniqueness of mild solution.

Obviously, when \( \nu = 0 \), the above three steps still work. Thus the proof of Theorem 3.1 is completed.

4. Regularity of mild solution

In this section, we will prove the spatial and temporal regularity properties of mild solution to time-space fractional SBE based on the analytic semigroup.

Theorem 4.1. Let Assumptions 3.1 to 3.3 hold with \( 1 \leq \nu < \alpha < 2 \) and \( p \geq 2 \), let \( u(t) \) be a unique mild solution of the problem (2.1) with \( \mathbb{P}(u(t) \in \dot{H}^\nu) = 1 \) for any \( t \in [0,T] \), then there exists a constant \( C \) such that

\[
\sup_{t \in [0,T]} \|u(t)\|_{L^p(\Omega; \dot{H}^\nu)} \leq C(\|u_0\|_{L^p(\Omega; H)} + \sup_{t \in [0,T]} \|u(t)\|_{L^p(\Omega; H^\nu)})
\]
Proof. For any $0 \leq t \leq T$ and $1 \leq \nu < \alpha < 2$, we have
\[
\|u(t)\|_{L^p(\Omega; H^\nu)} = (\mathbb{E}[\|u(t)\|_{H^\nu}^p])^{1/p} = \|A_\nu u(t)\|_{L^p(\Omega; H)}
\leq \|A_\nu E_\beta(t)u_0\|_{L^p(\Omega; H)}
+ \|A_\nu \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(t-s)B(u(s))ds\|_{L^p(\Omega; H)}
+ \|A_\nu \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(t-s)g(u(s))dW(s)\|_{L^p(\Omega; H)}
= I + II + III.
\] (4.2)

Using Theorem 2.1, the first term can be estimated by
\[
I = \|A_\nu E_\beta(t)u_0\|_{L^p(\Omega; H)} \leq C_\alpha t^{-\frac{\alpha}{\alpha - \beta \nu}} \|u_0\|_{L^p(\Omega; H)} < \infty.
\] (4.3)

It is easy to know that
\[
\int_0^T C_\alpha t^{-\frac{\alpha}{\alpha - \beta \nu}} \|u_0\|_{L^p(\Omega; H)} dt = \frac{\alpha C_\alpha}{\alpha - \beta \nu} \|u_0\|_{L^p(\Omega; H)}.
\] (4.4)

The application of Theorem 2.1 and Assumptions 3.2, we get
\[
(II)^p \leq \mathbb{E}[\left(\int_0^t \| (t-s)^{\beta-1} A_\nu E_{\beta,\beta}(t-s)B(u(s)) \| ds \right)^p]
\leq C_\beta^p \left( \int_0^t (t-s)^{p(\beta-1) - \frac{p(\nu+1)}{\nu}} ds \right)^{p-1} \int_0^t \mathbb{E}[\|A_{\nu^{-1}} B(u(s))\|_{H^1}^p] ds
\leq C_1 \sup_{t \in [0,T]} \mathbb{E}[\|u(s)\|_{H^1}^p],
\] (4.5)

where $C_1 = C_\beta^p C_\beta \left\{ \frac{p-1}{p(p-\nu-1)} \right\}^{p-1} T^{p(\beta-1) - \frac{p(\nu+1)}{\nu}} \max_{t \in [0,T]} \mathbb{E}[\|u(t)\|_{H^1}]$.

By means of Theorem 2.1, Assumptions 3.1 and Lemma 2.2, we can deduce
\[
(III)^p \leq \mathbb{E}[\left( \int_0^t \| (t-s)^{\beta-1} A_{\nu^{-1}} E_{\beta,\beta}(t-s)g(u(s)) \|_{L^2}^p ds \right)^p]
\leq C(p) C_\beta^p \left( \int_0^t (t-s)^{2p(\beta-1) - \frac{2p(\nu+1)}{\nu}} ds \right)^{p-1} \int_0^t \mathbb{E}[\|A_{\nu^{-1}} g(u(s))\|_{L^2}^p] ds
\leq C_2 \sup_{t \in [0,T]} \mathbb{E}[\|u(s)\|_{H^1}^p],
\] (4.6)

where $C_2 = C(p) C_\beta^p C_\beta \left\{ \frac{p-1}{p(2p-\nu-1)} \right\}^{p-1} T^{p(2p-1) - \frac{4p(\nu+1)}{\nu}} \int_0^T \mathbb{E}[\|u(t)\|_{H^1}^{2p}] dt$.

Thus, we conclude the proof of Theorem 4.1 by combining with the estimates (4.2)-(4.6).

Next, we will devote to the temporal regularity of the mild solution. **Theorem 4.2.** Let Assumptions 3.1 to 3.3 be fulfilled with $0 < \nu < \alpha < 2$ and $p \geq 2$, for any $0 \leq t_1 < t_2 \leq T$, the unique mild solution $u(t)$ to the problem (2.1) is Hölder continuous with respect to the norm $\| \cdot \|_{L^p(\Omega; H^\nu)}$ and satisfies
\[
\|u(t_2) - u(t_1)\|_{L^p(\Omega; H^\nu)} \leq C(t_2 - t_1)^\gamma.
\] (4.7)
**Proof.** For any $0 < t_1 < t_2 < T$, from the mild solution (2.4), we have
\[
\begin{align*}
    u(t_2) - u(t_1) &= E_\beta(t_2)u_0 - E_\beta(t_1)u_0 \\
    &+ \int_0^{t_2} (t_2 - s)^{\beta-1}E_{\beta,\beta}(t_2 - s)B(u(s))ds \\
    &- \int_0^{t_1} (t_1 - s)^{\beta-1}E_{\beta,\beta}(t_1 - s)B(u(s))ds \\
    &+ \int_0^{t_2} (t_2 - s)^{\beta-1}E_{\beta,\beta}(t_2 - s)g(u(s))dW(s) \\
    &- \int_0^{t_1} (t_1 - s)^{\beta-1}E_{\beta,\beta}(t_1 - s)g(u(s))dW(s) \\
    &=: I_1 + I_2 + I_3, \tag{4.8}
\end{align*}
\]
where
\[
I_1 = E_\beta(t_2)u_0 - E_\beta(t_1)u_0,
\]
\[
I_2 = \int_0^{t_2} (t_2 - s)^{\beta-1}E_{\beta,\beta}(t_2 - s)B(u(s))ds \\
- \int_0^{t_1} (t_1 - s)^{\beta-1}E_{\beta,\beta}(t_1 - s)B(u(s))ds \\
= \int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}]E_{\beta,\beta}(t_2 - s)B(u(s))ds \\
+ \int_0^{t_2} (t_2 - s)^{\beta-1}E_{\beta,\beta}(t_2 - s)B(u(s))ds \\
=: I_{21} + I_{22} + I_{23}. \tag{4.9}
\]
and
\[
I_3 = \int_0^{t_2} (t_2 - s)^{\beta-1}E_{\beta,\beta}(t_2 - s)g(u(s))dW(s) \\
- \int_0^{t_1} (t_1 - s)^{\beta-1}E_{\beta,\beta}(t_1 - s)g(u(s))dW(s) \\
= \int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}]E_{\beta,\beta}(t_2 - s)g(u(s))dW(s) \\
+ \int_0^{t_2} (t_2 - s)^{\beta-1}E_{\beta,\beta}(t_2 - s)g(u(s))dW(s) \\
=: I_{31} + I_{32} + I_{33}. \tag{4.10}
\]
For any $0 < \nu < \alpha < 2$ and $p \geq 2$, by virtue of Theorem 2.2, it follows that
\[
E[|I_1|_p^\nu] = E[|A_{\nu}[E_\beta(t_2) - E_\beta(t_1)]u_0|^\nu] \\
\leq C_{\nu,\alpha}(t_2 - t_1)^{\frac{\nu\alpha}{2}} E[|u_0|^\nu]. \tag{4.11}
\]
For the first term $I_{21}$ in (4.9), applying the Assumptions 3.2 and Theorem 2.2 and Hölder’s inequality, we have

$$
E[\|I_{21}\|_{H_r}] = E[\| t^t_0 (t_1 - s)^{\beta - 1} \mathcal{A}_p [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] B(u(s)) ds \|^p] \\
\leq C_{\beta}^p \| t^t_0 (t_1 - s)^{\beta - 1} \mathcal{A}_p [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] B(u(s)) ds \|^p \\
\leq \frac{p - 1}{p\beta - 1} p^{-1}( \sup_{t \in [0, T]} \mathbb{E}[\|u(s)\|_{H^1}]^2)(t_2 - t_1) \frac{\beta(\nu + 1)}{\alpha}.
$$

(4.12)

Using the Assumptions 3.2, Theorem 2.1 and Hölder’s inequality, we get

$$
E[\|I_{22}\|_{H_r}] = E[\| t^t_0 [(t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1}] \mathcal{A}_p E_{\beta,\beta}(t_2 - s) B(u(s)) ds \|^p] \\
\leq C_{\beta}^p (t^t_0 [(t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1}] \times (t_2 - s)^{-\frac{\beta(\nu + 1)}{\alpha}} \mathcal{A}_p ds)^{p-1} \\
\times \sup_{t \in [0, T]} \mathbb{E}[\|u(s)\|_{H^1}]^2 (t_2 - t_1) \frac{\beta \alpha (p - 1)}{\alpha - \beta \nu}.
$$

(4.13)

and

$$
E[\|I_{23}\|_{H_r}] = E[\| t^t_1 (t_2 - s)^{\beta - 1} \mathcal{A}_p E_{\beta,\beta}(t_2 - s) B(u(s)) ds \|^p] \\
\leq C_{\beta}^p (t^t_1 [(t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1}] \mathcal{A}_p ds)^{p-1} \sup_{t \in [0, T]} \mathbb{E}[\|u(s)\|_{H^1}]^2 (t_2 - t_1) \frac{\beta \alpha (p - 1)}{\alpha - \beta \nu}.
$$

(4.14)

Next, by following the similar arguments as in the proof of (4.12)-(4.14) and using the Lemma 2.2, there holds

$$
E[\|I_{31}\|_{H_r}] = E[\| t^t_1 (t_2 - s)^{\beta - 1} \mathcal{A}_p [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] g(u(s)) dW(s) \|^p] \\
\leq C(p) E(\| t^t_1 (t_1 - s)^{\beta - 1} \mathcal{A}_p [E_{\beta,\beta}(t_2 - s) - E_{\beta,\beta}(t_1 - s)] \|^2 \| g(u(s)) \|^2 L^2 W ds)^{\frac{p}{2}} \\
\leq C(p) C_{\beta}^p \mathcal{A}_p (t_2 - t_1)^{\frac{p}{2}} \mathcal{A}_p (t_0)^{\frac{2p}{p - 2}} ds \mathcal{A}_p (t_0)^{\frac{2p}{p - 2}} \sup_{t \in [0, T]} \mathbb{E}[\|u(t)\|_{H^1}]^2 (t_2 - t_1) \frac{\beta \alpha (p - 1)}{\alpha - \beta \nu}.
$$

(4.15)
and
\[
E[\|I_{a2}\|_{L^p}^p] = E[\| \int_{t_1}^{t_2} [(t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1}] A_{\nu} E_{\beta, \alpha}(t_2 - s) g(u(s)) dW(s) \|^p]
\leq C(p) E[\int_{t_1}^{t_2} ||(t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1}] A_{\nu} E_{\beta, \alpha}(t_2 - s) \|^2 \|g(u(s))\|_{L^p}^2 ds]^{\frac{p}{2}}
\leq C(p) C_\beta^p \int_{t_1}^{t_2} \|[(t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1}] \times (t_2 - s)^{-\frac{\alpha}{\nu}} \|^{\frac{p}{2}} ds
\times \int_{t_1}^{t_2} E[\|g(u(s))\|_{L^p}^p] ds
\leq C(p) C^p C_\beta^p T \alpha(p - 2) 2p\beta(\alpha - \nu) - (p + 2)\alpha \frac{2}{\nu} \left( \sup_{t \in [0, T]} E[\|u(t)\|_p] \right) (t_2 - t_1)^{\frac{2p\beta(\alpha - \nu) - (p + 2)\alpha}{2\nu}}.
\tag{4.16}
\]

and
\[
E[\|I_{a3}\|_{L^p}^p] = E[\| \int_{t_1}^{t_2} (t_2 - s)^{\beta - 1} A_{\nu} E_{\beta, \alpha}(t_2 - s) g(u(s)) dW(s) \|^p]
\leq C(p) E[\int_{t_1}^{t_2} (t_2 - s)^{\beta - 1} A_{\nu} E_{\beta, \alpha}(t_2 - s) \|^2 \|g(u(s))\|_{L^p}^2 ds]^{\frac{p}{2}}
\leq C(p) C_\beta^p \int_{t_1}^{t_2} (t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1} \|^{\frac{p}{2}} ds \int_{t_1}^{t_2} E[\|g(u(s))\|_{L^p}^p] ds
\leq C(p) C^p C_\beta^p T \alpha(p - 2) 2p\beta(\alpha - \nu) - (p + 2)\alpha \frac{2}{\nu} \left( \sup_{t \in [0, T]} E[\|u(t)\|_p] \right) (t_2 - t_1)^{\frac{2p\beta(\alpha - \nu) - (p + 2)\alpha}{2\nu}}.
\tag{4.17}
\]

Taking expectation on both sides of (4.8), and in view of the estimates (4.11)-(4.17), we conclude that
\[
\|u(t_2) - u(t_1)\|_{L^p((t_1, t_2]; H^p)} \leq C(t_2 - t_1)^\gamma,
\tag{4.18}
\]
in which we take $\gamma = \min\left\{ \frac{\alpha}{\nu}, \frac{p\beta(\alpha - \nu) - (p + 2)\alpha}{2p\alpha}, \frac{2p\beta(\alpha - \nu) - (p + 2)\alpha}{2p\alpha} \right\}$ when $0 < t_2 - t_1 < 1$. Otherwise, if $t_2 - t_1 \geq 1$, then we set $\gamma = \max\left\{ \frac{2(\nu + 1)}{\alpha}, \frac{\beta(\alpha - \nu - 1)}{\alpha}, \frac{2p\beta(\alpha - \nu) - p\alpha}{2p\alpha} \right\}$.

This completes the proof of Theorem 4.2.

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Appendix A

Considering the following abstract formulation of time-space fractional stochastic Burgers equation:

\[
\begin{cases}
\mathcal{C}D_t^\beta u(t) = -A_\alpha u(t) + B(u(t)) + g(u(t)) \frac{dW(t)}{dt}, t > 0, \\
u(0) = u_0.
\end{cases}
\] (A1)

We derive the mild solution to (A1) by means of Laplace transform, which denoted by \(\hat{\cdot}\). Let \(\lambda > 0\), and we define that

\[
\hat{u}(\lambda) = \int_0^\infty e^{-\lambda s} u(s) ds, \quad \hat{B}(\lambda) = \int_0^\infty e^{-\lambda s} B(u(s)) ds,
\]

and

\[
\hat{G}(\lambda) = \int_0^\infty e^{-\lambda s} [g(u(s)) \frac{dW(s)}{ds}] ds = \int_0^\infty e^{-\lambda s} g(u(s)) dW(s).
\]

Upon Laplace transform, using the formula \(\mathcal{C}D_t^\beta u = \lambda^\beta \hat{u} - \lambda^{\beta-1} u_0\). Then applying the Laplace transform to (A1), we obtain

\[
\hat{u}(\lambda) = \frac{1}{\lambda} u_0 + \frac{1}{\lambda^\beta} (-A_\alpha) \hat{u}(\lambda) + \frac{1}{\lambda^\beta} [\hat{B}(\lambda) + \hat{G}(\lambda)]
\]

\[
= \lambda^{\beta-1} (\lambda^\beta I + A_\alpha)^{-1} u_0 + (\lambda^\beta I + A_\alpha)^{-1} [\hat{B}(\lambda) + \hat{G}(\lambda)]
\]

\[
= \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) u_0 ds + \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) [\hat{B}(\lambda) + \hat{G}(\lambda)] ds,
\] (A2)

in which \(I\) is the identity operator, and \(S_\alpha(t) = e^{-tA_\alpha}\) is an analytic semigroup generated by the operator \(-A_\alpha\).

We introduce the following one-sided stable probability density function:

\[
\omega_\beta(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\beta n-1} \frac{\Gamma(\beta n + 1)}{n!} \sin(n\pi\beta), \theta \in (0, +\infty),
\] (A3)

whose Laplace transform is given by

\[
\int_0^\infty e^{-\lambda \theta} \omega_\beta(\theta) d\theta = e^{-\lambda^\beta}, 0 < \beta < 1.
\] (A4)

Making use of above expression (A4), then the terms on the right-hand side of (A2) can be
written as
\[
\lambda^{3-1} \int_0^\infty e^{-\lambda^3 s} S_\alpha(s) u_0 ds
= \int_0^\infty \lambda^{3-1} e^{-\lambda^3 t^3} S_\alpha(t^3) u_0 dt(t^3)
= \int_0^\infty \beta(\lambda t)^{3-1} e^{-\lambda t^3} S_\alpha(t^3) u_0 dt
= \int_0^\infty \frac{1}{\lambda} \frac{d}{dt}[e^{-\lambda t^3} S_\alpha(t^3)] u_0 dt
= \int_0^\infty \int_0^\infty \theta \omega(\theta) e^{-\lambda t^3} S_\alpha(t^3) u_0 d\theta dt
= \int_0^\infty e^{-\lambda t} \theta \omega(\theta) S_\alpha(\frac{t^3}{\theta^3}) u_0 d\theta dt, \tag{A5}
\]
and
\[
\int_0^\infty e^{-\lambda^3 s} S_\alpha(s) \tilde{B}(\lambda) ds
= \int_0^\infty \beta t^{3-1} e^{-\lambda t^3} S_\alpha(t^3) \tilde{B}(\lambda) dt
= \int_0^\infty \int_0^\infty \beta t^{3-1} e^{-\lambda t^3} S_\alpha(t^3) e^{-\lambda s} B(u(s)) ds dt
= \int_0^\infty \int_0^\infty \int_0^\infty \beta \omega(\theta) e^{-\lambda t^3} S_\alpha(t^3) e^{-\lambda s} B(u(s)) d\theta ds dt
= \int_0^\infty \int_0^\infty \int_0^\infty \beta \omega(\theta) e^{-\lambda (t+s)} S_\alpha(\frac{t^3}{\theta^3}) \frac{t^{3-1}}{\theta^{3-1}} B(u(s)) d\theta ds dt
= \int_0^\infty e^{-\lambda t} \beta \int_0^t \int_0^\infty \omega(\theta) S_\alpha(\frac{(t-s)^3}{\theta^3}) \frac{(t-s)^{3-1}}{\theta^{3-1}} B(u(s)) d\theta ds dt. \tag{A6}
\]
and
\[
\int_0^\infty e^{-\lambda^3 s} S_\alpha(s) \tilde{G}(\lambda) ds
= \int_0^\infty \beta t^{3-1} e^{-\lambda t^3} S_\alpha(t^3) \tilde{G}(\lambda) dt
= \int_0^\infty \int_0^\infty \beta t^{3-1} e^{-\lambda t^3} S_\alpha(t^3) e^{-\lambda s} g(u(s)) dW(s) dt
= \int_0^\infty \int_0^\infty \beta \omega(\theta) e^{-\lambda t^3} S_\alpha(t^3) e^{-\lambda s} g(u(s)) d\theta dW(s) dt
= \int_0^\infty \int_0^\infty \int_0^\infty \beta \omega(\theta) e^{-\lambda (t+s)} S_\alpha(\frac{t^3}{\theta^3}) \frac{t^{3-1}}{\theta^{3-1}} g(u(s)) d\theta dW(s) dt
= \int_0^\infty e^{-\lambda t} \beta \int_0^t \int_0^\infty \omega(\theta) S_\alpha(\frac{(t-s)^3}{\theta^3}) \frac{(t-s)^{3-1}}{\theta^{3-1}} g(u(s)) d\theta dW(s) dt. \tag{A7}
\]
Together with (A2) and (A5)-(A7) helps us to get
\[
\hat{\mu}(\lambda) = \int_0^\infty e^{-\lambda t} \int_0^\infty \omega_\beta(\theta) S_\alpha \left( \frac{t^\beta}{\theta^\beta} \right) u_0 d\theta dt \\
\quad + \int_0^\infty e^{-\lambda t} [\beta \int_0^t \int_0^\infty \omega_\beta(\theta) S_\alpha \left( \frac{(t-s)^\beta}{\theta^\beta} \right) \frac{(t-s)^{\beta-1}}{\theta^\beta} B(u(s)) d\theta ds] dt \\
\quad + \int_0^\infty e^{-\lambda t} [\beta \int_0^t \int_0^\infty \omega_\beta(\theta) S_\alpha \left( \frac{(t-s)^\beta}{\theta^\beta} \right) \frac{(t-s)^{\beta-1}}{\theta^\beta} g(u(s)) d\theta dW(s)] dt. \tag{A8}
\]

Now, by means of inverse Laplace transform to (A8), we have achieved that
\[
u(t) = \int_0^\infty \omega_\beta(\theta) S_\alpha \left( \frac{t^\beta}{\theta^\beta} \right) u_0 d\theta \\
\quad + \beta \int_0^t \int_0^\infty \omega_\beta(\theta) S_\alpha \left( \frac{(t-s)^\beta}{\theta^\beta} \right) \frac{(t-s)^{\beta-1}}{\theta^\beta} B(u(s)) d\theta ds \\
\quad + \beta \int_0^t \int_0^\infty \omega_\beta(\theta) S_\alpha \left( \frac{(t-s)^\beta}{\theta^\beta} \right) \frac{(t-s)^{\beta-1}}{\theta^\beta} g(u(s)) d\theta dW(s) \\
\quad = \int_0^\infty \frac{1}{\beta} \theta^{-\frac{1}{\beta}} \omega_\beta(\theta^{-\frac{1}{\beta}}) S_\alpha \left( \frac{t^\beta}{\theta^\beta} \right) u_0 d\theta \\
\quad + \int_0^t \int_0^\infty \theta^{-\frac{1}{\beta}} \omega_\beta(\theta^{-\frac{1}{\beta}}) S_\alpha ((t-s)^\beta (t-s)^{\beta-1} B(u(s)) d\theta ds \\
\quad + \int_0^t \int_0^\infty \theta^{-\frac{1}{\beta}} \omega_\beta(\theta^{-\frac{1}{beta}}) S_\alpha ((t-s)^\beta (t-s)^{\beta-1} g(u(s)) d\theta dW(s). \tag{A9}
\]

Here, we also introduce the Mainardi’s Wright-type function
\[
M_\beta(\theta) = \sum_{n=0}^\infty \frac{(-1)^n \theta^n}{n! \Gamma(1-\beta(1+n))} \\
\quad = \frac{1}{\pi} \sum_{n=1}^\infty \frac{(-1)^n \theta^{n-1}}{(n-1)!} \Gamma(n\beta) \sin(n \pi \beta),
\]
where \(0 < \beta < 1\) and \(\theta \in (0, +\infty)\). Further, the relationships between the probability density function \(\omega_\beta(\theta)\) and Mainardi’s Wright-type function \(M_\beta(\theta)\) are shown that
\[
M_\beta(\theta) = \frac{1}{\beta} \theta^{-\frac{1}{\beta}} \omega_\beta(\theta^{-\frac{1}{\beta}}).
\]

We denote the generalized Mittag-Leffler operators \(E_\beta(t)\) and \(E_{\alpha,\beta}(t)\) as
\[
E_\beta(t) = \int_0^\infty M_\beta(\theta) S_\alpha (t^\beta \theta) d\theta,
\]
and
\[
E_{\alpha,\beta}(t) = \int_0^\infty \beta \theta M_\beta(\theta) S_\alpha (t^\beta \theta) d\theta.
\]
Therefore, the equation (A9) can be written as

$$u(t) = E^\beta(t)u_0 + \int_0^t (t - s)^{\beta - 1} E^\beta,\beta(t - s)B(u(s))ds + \int_0^t (t - s)^{\beta - 1} E^\beta,\beta(t - s)g(u(s))dW(s).$$

(A10)

Up to now, we have deduced the mild solution (A10) to the time-space fractional stochastic Burgers equation (A1).
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