On the integrated mean squared error of wavelet density estimation for linear processes

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Abstract
Let \( \{X_n : n \in \mathbb{N}\} \) be a linear process with density function \( f(x) \in L^2(\mathbb{R}) \). We study wavelet density estimation of \( f(x) \). Under some regular conditions on the characteristic function of innovations, we achieve, based on the number of nonzero coefficients in the linear process, the minimax optimal convergence rate of the integrated mean squared error of density estimation. Considered wavelets have compact support and are twice continuously differentiable. The number of vanishing moments of mother wavelet is proportional to the number of nonzero coefficients in the linear process and to the rate of decay of characteristic function of innovations. Theoretical results are illustrated by simulation studies with innovations following Gaussian, Cauchy and chi-squared distributions.

Keywords Linear process · Wavelet method · Density estimation · Projection operator

Mathematics Subject Classification Primary 62G07 · Secondary 62G05 , 62M10

1 Introduction

In this paper we consider the linear process

\[
X_n = \sum_{i=0}^{\infty} a_i \epsilon_{n-i},
\]
with probability density function \( f(x) \), where the innovations \( \varepsilon_i \) are independent and identically distributed (i.i.d.) real-valued random variables in some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and \( \{a_i\} \) are real coefficients such that \( a_0 \neq 0 \) and \( \sum_{i=0}^{\infty} a_i \varepsilon_{n-i} \) is well defined. With observations \( \{X_k\}_{k=1}^n \), we introduce a wavelet based estimator of density function \( f(x) \) and study the order of its integrated mean squared error (IMSE).

Most of the estimators for such processes presented in the literature are the kernel estimators. In particular, under the condition that the innovations have bounded second moment, central limit theorems for kernel density estimators of linear processes were derived in Wu and Mielniczuk (2002). Hall and Hart (1990) proved that the IMSE of the kernel estimator for two-sided linear processes with given \( n \) observations can be decomposed into the sum of 2 terms: the IMSE of the same kernel estimator based on i.i.d. sample of size \( n \) following the same distribution and a term proportional to the variance of the sample mean. Hence, since the variance of the sample mean of short range dependent linear processes has the order \( O(1/n) \) (see, e.g., Priestley 1981), the IMSE of the kernel estimator for two-sided short range dependent linear processes has the same order as the IMSE of the kernel estimator for i.i.d. data. Analogous results for a larger class of processes are given by Mielniczuk (1997), where under certain conditions on the Fourier transform of the kernel estimator, a similar decomposition for the IMSE is obtained. For strictly stationary sequences, the rates of the IMSE of kernel density estimators are derived in Meloche (1990). Moving average processes \( z_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \cdots - \theta_q \varepsilon_{t-q} \) of order \( q \) (\( \geq 1 \)) has been studied in Saavedra and Cao (2000). Assuming that the density function of the innovation \( \varepsilon_1 \) has continuous and bounded derivatives of order up to 4, the parametric order 1/\( n \) for the IMSE of kernel density estimator was derived.

For independent sequences, the IMSE rates of wavelet density estimators have been studied in the works Donoho and Johnstone (1994, 1995), Donoho et al. (1995, 1996), Zhang and Zheng (1999), Lu (2013), and Giné and Madych (2014). However, compared to the kernel density estimators, for linear processes the IMSE rates of wavelet density estimators have been barely covered in the literature. Below we present some of the works where related topics are studied. First, a general overview of the use of wavelets in the theory of non-parametric function estimation is given in Johnstone (1999). For time series data, wavelet-based estimators of densities and marginal densities were obtained in Leblanc (1996) and Gannaz and Wintenberger (2010). Note that the random variables considered in those two works are assumed to have compactly supported distributions. Chesneau (2014) studied the wavelet hard thresholding density estimators for observations with \( \alpha \)-mixing dependence. Also, under some regular conditions on the strong mixing coefficient and on the densities, Kou and Guo (2018) gave an upper bound for the IMSE of linear wavelet density estimators of strong mixing sequences. In particular, if \( a_k = 2^{-k} \) and \( \varepsilon_k \sim N(0, \sigma^2) \), \( k \geq 0 \), then the linear process (1) is strong mixing. Wavelet based methods for various mixing, quadrant and \( m \)-dependence conditions have also been developed in the works Badaoui and Rhomari (2015), Chesneau (2012) and Chesneau et al. (2012), Chesneau et al. (2019) and Li and Zhang (2022). In particular, in the recent related works Chesneau et al. (2019) and Li and Zhang (2022), wavelet-based estimators of bounded, compactly supported densities of \( m \)-dependent processes and of spectral densities of non-Gaussian linear processes belonging to Besov spaces are proposed. The rates of convergence derived for those estimators coincide with the minimax optimal rates with additional logarithmic factors.

In the present work we study the IMSE of wavelet density estimation for linear processes of the form (1). The estimator is obtained by substituting the coefficients in wavelet series expansion of density function \( f \in L^2(\mathbb{R}) \) by the average values of corresponding translations...
and dyadic dilations of wavelets at the observations \( \{X_k\}_{k=1}^n \). Optimal choice of wavelets is determined by the rate of decay of the characteristic function of innovations \( \varepsilon_i \) and by the number of nonzero coefficients in the linear process. Note that those two features of the linear process also determine the order of smoothness of the density function \( f \). In particular, if the induced smoothness of \( f \) is of order \( m \), then the optimal choice of wavelets allows to achieve the minimax optimal nonparametric estimation rate \( n^{-\frac{2m}{2m+1}} \). In case \( m = \infty \), the IMSE rates of estimators can get arbitrarily close to \( n^{-1} \), and in this case the wavelets with higher number of vanishing moments induce better rates of estimation. Note that there is a similar phenomena for kernel density estimations: kernel functions with higher order allow to derive faster estimation rates (see, e.g., Marron 1994).

The paper is structured as follows: in Sect. 2 we present the general setting and the main results. We perform simulation studies in Sect. 3 to confirm the results of Sect. 2. Section 4 gives the proofs of several auxiliary results which lead to the proof of the main theorem. The proof is based on the decomposition of the IMSE of linear wavelet estimator into three parts, first of which corresponds to the scaling function and the other two are the partial sum and the tail of the series with coefficients corresponding to mother wavelet. Convergence rates of these parts are derived in Sect. 4 by rewriting them as sums of Fourier transforms at integers and applying the formula of Poisson. Section 4 also presents the properties of considered wavelets, the justification of application of Poisson summation formula and the proofs of several properties of characteristic functions of \( X_1 \) and \( \varepsilon_1 \).

We apply the following notations throughout the paper: for a function \( g(x) \in L^1(\mathbb{R}) \), we use \( \hat{g}(u) = \int_\mathbb{R} e^{-ixu} g(x) \, dx \) as the definition of its Fourier transform, where \( i \) is the imaginary unit. The characteristic function of linear process \( X_n = \sum_{i=0}^\infty a_i \varepsilon_{n-i} \) is denoted by \( \phi(\lambda) \), and \( \phi_\varepsilon(\lambda) \) denotes the characteristic function of innovations \( \varepsilon_i \). That is, \( \phi(\lambda) = \mathbb{E}[e^{i\lambda X_1}] \) and \( \phi_\varepsilon(\lambda) = \mathbb{E}[e^{i\lambda \varepsilon_1}] \). For each \( i, j \in \mathbb{N} \), we define \( H(X_i) := H(X_i)(\lambda) := e^{i\lambda X_i} - \phi(\lambda) \), and \( H_{ij}(u, v) := \mathbb{E} e^{iuX_i+ivX_j} - \phi(u)\phi(v) = \mathbb{E}[H(X_i)(u)H(X_j)(v)] \), \( \lambda, u, v \in \mathbb{R} \). For two sequences \( \{a_n\} \) and \( \{b_n\} \), the notation \( a_n = O(b_n) \) indicates the existence of a constant \( C \) such that \( a_n \leq C b_n \) for all \( n \in \mathbb{N} \). Finally, \( \lceil \cdot \rceil \) and \( \lfloor \cdot \rfloor \) denote, respectively, the usual ceiling and floor functions on \( \mathbb{R} \), and the positive constant \( c \) that appears in the proofs may vary from line to line, but is independent of \( i, j, k, n \) and other indices involved.

## 2 Main results

Before proceeding to the imposed assumptions and the main result, let us present several particular settings for which the linear process (1) is well-defined. If \( \{\varepsilon_i : i \in \mathbb{Z}\} \) is a sequence of i.i.d. random variables in \( L^p(\mathbb{R}) \) for some \( p > 0 \), \( \mathbb{E} \varepsilon_i = 0 \) when \( p \geq 1 \), and \( \{a_i\}_{i=0}^\infty \) is a sequence of real coefficients such that \( \sum_{i=0}^\infty |a_i|^{2\lambda \wedge p} < \infty \), by Kolmogorov’s three-series theorem, the linear process \( X_n \) given in (1) exists and is well-defined. As for linear processes with symmetric \( \alpha \)-stable innovations (\( 0 < \alpha < 2 \)), i.e., the law of innovations having characteristic function \( \mathbb{E}[e^{i\lambda \varepsilon_1}] = \exp(-c_\alpha |\lambda|^{\alpha}) \) for some positive constant \( c_\alpha \) only depending on \( \alpha \), by Kolmogorov’s three-series theorem, the series given in (1) converges almost surely if and only if \( \sum_{i=0}^\infty |a_i|^{\alpha} < \infty \) (see, e.g., Samorodnitsky and Taqqu 1994).

We now turn to the definition of the wavelet estimator \( \hat{f}_n \) of the density function \( f \) of linear process (1). As the density function \( f \) is assumed to be in \( L^2(\mathbb{R}) \), then \( f \) accepts the representation

\[
f(x) = \sum_{k \in \mathbb{Z}} \alpha_{0k} \varphi_{0k}(x) + \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} \beta_{jk} \psi_{jk}(x),
\]
where \( \psi_{jk}(x) = 2^{j/2} \psi(2^j x - k) \), \( \varphi_{jk}(x) = 2^{j/2} \varphi(2^j x - k) \), \( j, k \in \mathbb{Z} \), are the translations and dyadic dilations of mother wavelet \( \psi \) and the scaling function \( \varphi \). The orthonormality of functions \( \varphi_{0k}(x) \) and \( \psi_{jk}(x), \quad j \geq 0, k \in \mathbb{Z} \), in (2) implies that
\[
\alpha_{0k} = \int_{\mathbb{R}} f(x) \varphi_{0k}(x) \, dx = \mathbb{E} \varphi_{0k}(X_1),
\]
\[
\beta_{jk} = \int_{\mathbb{R}} f(x) \psi_{jk}(x) \, dx = \mathbb{E} \psi_{jk}(X_1).
\]
Therefore, given a sample \( X_1, \ldots, X_n \), we can approximate the coefficients \( \alpha_{0k} \) and \( \beta_{jk} \) by the quantities
\[
\hat{\alpha}_{0k} = \frac{1}{n} \sum_{i=1}^{n} \varphi_{0k}(X_i)
\]
and
\[
\hat{\beta}_{jk} = \frac{1}{n} \sum_{i=1}^{n} \psi_{jk}(X_i),
\]
and obtain an estimator
\[
\hat{f}_n(x) = \sum_{k \in \mathbb{Z}} \hat{\alpha}_{0k} \varphi_{0k}(x) + \sum_{j=0}^{j_n} \sum_{k \in \mathbb{Z}} \hat{\beta}_{jk} \psi_{jk}(x),
\]
(3) of \( f(x) \). The choice of considered wavelets is specified in the Assumption 2, while the selection of optimal truncation location \( j_n \) is given in the Theorem 2.1 below. Assumptions on the linear process (1) needed to derive the IMSE rate of the estimator (3) are presented in Assumption 1.

**Assumption 1** For the linear process (1), suppose that the density function \( f(x) \) is in \( L^2(\mathbb{R}) \), and the coefficients satisfy \( \sum_{i=0}^{\infty} |a_i|^{\gamma} < \infty \), for some \( \gamma \in (0, 1] \). Suppose that there are at least \( M \geq 1 \) non-zero coefficients, and, without loss of generality, let \( a_0 \neq 0 \). Assume that for some constant \( \beta \geq \gamma \) the characteristic function \( \phi_\varepsilon \) of innovations satisfies
\[
u^\beta \phi_\varepsilon(u) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}).
\]
(4)

**Assumption 2** We consider wavelets for which both \( \psi \) and \( \varphi \) have compact support and are twice continuously differentiable. Also, we assume that \( \psi \) has \( [M\beta] \) vanishing moments, where \( M \) and \( \beta \) are the parameters from Assumption 1.

Note that Daubechies wavelets of order \( 2[M\beta] \lor 8 \) and higher satisfy the Assumption 2 (see Daubechies 1992, Ch. 7).

The following theorem is the main result of the paper.

**Theorem 2.1** Assume that the derivative of the characteristic function \( \phi_\varepsilon \) of innovations has bounded modulus on \( \mathbb{R} \). In case \( \gamma \in (1/2, 1] \), assume that the innovations also satisfy
\[
\mathbb{E} |e^{i\lambda X_1} - \phi_\varepsilon(\lambda)|^2 \leq c (|\lambda|^{2\gamma} \wedge 1)
\]
(5)
for all \( \lambda \in \mathbb{R} \) and for some constant \( c > 0 \) which only depends on \( \gamma \). Then, under the Assumptions 1 and 2, for \( j_n = \left\lceil \frac{\log n}{2M\beta + 1} \right\rceil \),
\[
\mathbb{E} \int_{\mathbb{R}} \left( \hat{f}_n(x) - f(x) \right)^2 \, dx = O\left(n^{-\frac{2M\beta}{2M\beta + 1}}\right).
\]
Note that the derivative of $\phi_e$ has bounded modulus on $\mathbb{R}$ if, for example, $\mathbb{E} |\varepsilon_i| < \infty$ (see Lukacs 1970, p. 22). Also, the condition (5) is satisfied if, for example, $\mathbb{E} |\varepsilon|^2 \gamma < \infty$ (see Lemma 7.3 in Sang et al. 2018). We thus have the following corollary of Theorem 2.1:

**Corollary 2.1** Assume that the innovations $\varepsilon_i$ of the linear process (1) have finite second moment. Then, under the Assumptions 1 and 2, for $j_n = \lceil \log_2 \frac{n}{2M\beta + 1} \rceil$,

$$\mathbb{E} \int_{\mathbb{R}} \left[ \hat{f}_n(x) - f(x) \right]^2 dx = O\left( n^{-2M\beta} \right).$$

**Remark 2.1** If the linear processes have finitely many non-zero coefficients, say, $M$, then the condition (4) implies

$$\int_{\mathbb{R}} |u^{M\beta} \phi(u)| du < \infty,$$

where $\phi$ is the characteristic function of the linear process $X_t$. (6), in turn, implies (see Folland 1999, Theorem 8.22) that $f \in C^{M\beta}$.Remarkably, the rate $n^{-2M\beta^2 + 1}$, which is obtained by applying Daubechies wavelets of order $2^M \beta$, is the best possible mean square convergence rate even with given $n$ independent observations (see, e.g., Wahba 1975). If the linear processes have infinite number of non-zero coefficients, theoretically we can have IMSE rate arbitrarily close to $n^{-1}$. However, we shall apply wavelets with $m$ vanishing moments to reach the rate $n^{-2m^2 + 1}$, and the larger $m$ is chosen the closer we get to the optimal rate $n^{-1}$. Note that in kernel density estimation we have similar requirement on the kernel function: to attain the rate $n^{-2m^2 + 1}$ of estimation of density function $f \in C^m$, where $m$ is an even positive integer, we shall apply $m$-th order kernel function $K(\cdot)$ with $\int_{\mathbb{R}} x^p K(x) dx = 0, 1 \leq p < m$, (see, e.g., Marron 1994).

**Remark 2.2** Note that in Theorem 2.1 the estimator $\hat{f}_n$ depends on the value of $j_n$ which, in turn, is determined by the values of $M$ and $\beta$. Hence, the identification of optimal estimator assumes the knowledge of both $M$ and $\beta$.

**Remark 2.3** Note that if $\gamma \in (0, 1/2]$ then the condition (5) is automatically satisfied (see Lemma 4.1). If the innovations have non-degenerate distribution, the range $\gamma \in (0, 1]$ in inequality (5) is optimal and cannot be extended (see Lemma 7.4 in Sang et al. 2018). Also, it is easy to see that

$$\text{Var} (e^{i\lambda \varepsilon_1}) = \mathbb{E} |e^{i\lambda \varepsilon_1} - \phi_e(\lambda)|^2 = 1 - |\phi_e(\lambda)|^2.$$

Therefore, condition (5) together with $\sum_{i=1}^{\infty} |a_i|^\gamma < \infty$ imply

$$\sum_{i=1}^{\infty} \sqrt{\text{Var} (e^{i\lambda a_i \varepsilon_1})} = \sum_{i=1}^{\infty} \sqrt{\mathbb{E} |e^{ia_i \lambda \varepsilon_1} - \phi_e(a_i \lambda)|^2} \leq c|\lambda|^\gamma \sum_{i=1}^{\infty} |a_i|^\gamma < \infty.$$

Hence, according to the definition of short or long memory dependence for linear processes as in Sang et al. (2018), Theorem 2.1 provides an IMSE result for short memory linear processes.

To see some examples, it is easy to verify that for any $\beta > 0$ the condition (4) is satisfied for linear processes with innovations following either Gaussian distribution ($\alpha = 2$) or symmetric stable distribution with index $1 \leq \alpha < 2$. In particular, for the Gaussian case we have
\[ \mathbb{E} |e^{itx_0} - \phi_x(t)|^2 \leq c (|t|^2 + 1), \] and, therefore, we can take \( \gamma = 1 \) in condition (5). In the \( \alpha \)-stable case, \( \mathbb{E} |e^{it\lambda} - \phi_x(t)|^2 \leq c_\alpha (|\lambda|^\alpha \wedge 1) \), and the condition (5) is satisfied for \( \gamma = \alpha/2 \). Also, the derivative of the characteristic function \( \phi_x(u) = e^{-c|u|^\beta} \) \((1 < \alpha < 2)\) exists and is bounded on \( \mathbb{R} \). In the case of 1-stable distribution, the derivative of the characteristic function \( \phi_x(u) = e^{-c|u|} \) exists and is bounded everywhere except at 0. However, it still has bounded left and right derivatives at 0 which is sufficient for the proof of Theorem 2.1 for this case. If the linear processes have only finite number of nonzero coefficients, then the condition (4) is satisfied for all \( \gamma \in (0, 1/2] \) (see Lemma 4.1).

### 3 Simulation study

In this section we conduct a simulation study to evaluate the wavelet density estimator of linear processes with various coefficient sequences and with innovations following stable, Gaussian or chi-squared distributions. For each case, we shall compare the true density function with our wavelet density estimator. We first perform simulation study for the linear processes with infinitely many non-zero coefficients and with innovations following stable, Gaussian or Cauchy innovations and with coefficients determined by certain selected values of \( \nu \).

**Step 1** Here we use the method proposed in Sang et al. (2018) to find the density function \( f(x) \) of linear process \( X_n = \sum_{i=0}^{\infty} a_i \varepsilon_{n-i} \) with various coefficient sequences \( \{a_i\} \) and with various innovations \( \varepsilon_i \). For convenience, we sketch the method below.

The characteristic function of the linear process \( X_n \) has the form

\[
\phi(t) = \mathbb{E}[e^{itX_n}] = \prod_{i=0}^{\infty} \mathbb{E}[e^{it\varepsilon_{n-i}}] = \prod_{i=0}^{\infty} \phi_\varepsilon(a_it).
\]

For any \( d < \frac{1}{2}, d \notin \mathbb{Z} \), let \( a_i = \frac{\Gamma(i+d)}{\Gamma(i+1)} \), \( i \geq 0 \). By Stirling’s formula, \( \Gamma(x) \sim \sqrt{2\pi} e^{x} x^{-\frac{1}{2}} \) as \( x \to \infty \), and \( a_i \sim i^{d-1} / \Gamma(d) \) as \( i \to \infty \). We perform simulation study for linear processes with Gaussian or Cauchy innovations and with coefficients determined by certain selected values of \( d \).

**Case 1.** Suppose the innovations \( \{\varepsilon_i\} \) are i.i.d. standard normal random variables. Then, the characteristic function of the innovation is \( \phi_\varepsilon(t) = e^{-t^2/2} \), and, therefore, \( \phi(t) = e^{-t^2/2} \sum_{i=0}^{\infty} a_i^2 \).

In this case the linear process \( \{X_n\} \) is normally distributed with mean 0, variance \( A^2 := \sum_{i=0}^{\infty} a_i^2 \) and has density function \( f(x) = \frac{1}{\sqrt{2\pi} A} e^{-x^2/2A^2} \). By Gauss’s theorem for hypergeometric series (see Gauss 1866), \( A^2 = \sum_{i=0}^{\infty} a_i^2 = \frac{\Gamma(1-2d)}{\Gamma(1-d)} \) for any \( d < \frac{1}{2}, d \notin \mathbb{Z} \) [see also Bailey Bailey (1935) or its direct calculation in Sang and Sang (2017)]. In particular, if \( d = -0.5 \) then \( A^2 = 4/\pi \), and \( A^2 = 3.39531 \) if \( d = -1.5 \). Hence, for \( d = -0.5 \)

\[
f(x) = \frac{\sqrt{2}}{4} e^{-x^2/8}
\]

and for \( d = -1.5 \) we have

\[
f(x) = 0.216506 e^{-0.147262x^2}.
\]
Case 2. If the innovations \(\{\varepsilon_i\}\) have i.i.d. symmetric \(\alpha\)-stable distribution with \(0 < \alpha < 2\), then \(\phi_{\varepsilon}(t) = e^{-|t|^\alpha}\) and \(\phi(t) = e^{-|t|^{\alpha}} \sum_{i=0}^{\infty} |a_i|^{\alpha}\). In particular, if \(\alpha = 1\), we have the standard Cauchy distribution with \(\phi_{\varepsilon}(t) = e^{-|t|}\) and \(\phi(t) = e^{-|t|} \sum_{i=0}^{\infty} |a_i|\). In this case, with the help of the software Mathematica, we can approximate \(\sum_{i=0}^{\infty} |a_i|\) by \(\sum_{i=0}^{100,000} |a_i| = 1.99822\), if \(d = -0.5\), and by \(\sum_{i=0}^{100,000} |a_i| = 3\), if \(d = -1.5\). Then, the density of the linear process corresponding to the choice \(d = -0.5\) is given by

\[
f(x) = \frac{1}{1.99822\pi (1 + (x/1.99822)^2)},
\]

and for the case \(d = -1.5\) we have

\[
f(x) = \frac{3}{\pi(9 + x^2)}.
\]

As in this case \(\mathbb{E}[e^{2\lambda\varepsilon_1} - \phi_{\varepsilon}(\lambda)]^2 \leq c_\alpha (|\lambda|^{\alpha} \wedge 1)\), then we can choose \(\gamma = \alpha/2\) to fulfill the condition (5). As \(a_i = \Gamma(i+d) / \Gamma(d)|\lambda|^{\gamma} \sim i^{d-1} / \Gamma(d)\), then, to guarantee the convergence of the series \(\sum_{i=0}^{\infty} |a_i|^{\gamma}\), we should have \(d < 1 - \frac{1}{\gamma} = 1 - \frac{2}{\alpha}\). In particular, for innovations following Cauchy distribution we have \(\alpha = 1\), and, therefore, the convergence of the series is guaranteed for memory parameters \(d < 1\). In the following step we will see that in the case \(d < -1\) the performance of the estimator is indeed better than in the case when \(d > -1\).

Step 2. In the previous step we considered linear processes with Gaussian and Cauchy innovations and with memory parameters \(d = -0.5\) and \(d = -1.5\). By applying a modification of the MATLAB code from Faï et al. (2009), for each of these cases we produce a linear process \(\{X_i\}_{i=1}^{n}\) with \(n = 2^{16}\) observations. For each generated linear process, we apply the 1−D wavelet estimator from the Wavelet Analyzer App of MATLAB to estimate the true density function. As required in Assumption 2, we use the Daubechies wavelets of order 8 and we do not apply any soft or hard thresholding method in the procedure to obtain the estimator (3). Figures 1 and 2 below show the performance of wavelet density estimation of linear processes with Gaussian and Cauchy innovations for the cases \(d = -0.5\) and \(d = -1.5\). They confirm the result of Theorem 2.1 for the cases when innovations follow Gaussian distribution and \(d = -0.5\), \(-1.5\) and when the innovations follow Cauchy distribution and \(d = -1.5\). When the innovations follow Cauchy distribution and \(d = -0.5\), the performance of the estimator is worse than the other three cases. However, since \(d = -0.5 > -1\), then, as indicated in the last part of previous step, the result in this case is not guaranteed by Theorem 2.1.

We also perform a simulation study for the moving average process \(X_n = \sum_{i=0}^{3} \varepsilon_{n-i}\) of order 4 with coefficients \(a_i = 1\), \(0 \leq i \leq 3\), and with innovations following chi-squared distribution with 6 degrees of freedom. Then \(X_n\) follows chi-squared distribution with 24 degrees of freedom. Here we produce a MA(4) process of length \(n = 2^{16}\) and, as it was done in the previous cases, to estimate the true density function, we apply the 1−D wavelet estimator from MATLAB software with Daubechies wavelets of order 8. Performance of the estimator in this case is presented in Fig. 3 which again confirms the result of Theorem 2.1.

4 Proofs

The orthonormality of functions \(\varphi_{0k}(x)\) and \(\psi_{jk}(x), j \geq 0, k \in \mathbb{Z}\), together with the decompositions (2) and (3), allows to represent the integrated mean squared error (IMSE) of the estimator \(\hat{f}_n\) in the form
Fig. 1 Wavelet density estimation of simulated processes with Gaussian innovations and $d = -0.5$ (left), $-1.5$ (right).

Fig. 2 Wavelet density estimation of simulated processes with Cauchy innovations and $d = -0.5$ (left), $-1.5$ (right).

$$\mathbb{E} \int_{\mathbb{R}} \left( \hat{f}_n(x) - f(x) \right)^2 dx = I_1 + I_2 + I_3,$$

where

$$I_1 = \sum_{k \in \mathbb{Z}} \mathbb{E}(\hat{\alpha}_{0k} - \alpha_{0k})^2, \quad I_2 = \sum_{j=0}^{j_n} \sum_{k \in \mathbb{Z}} \mathbb{E}(\hat{\beta}_{jk} - \beta_{jk})^2$$

and

$$I_3 = \sum_{j=j_n+1}^{\infty} \sum_{k \in \mathbb{Z}} \beta_{jk}^2.$$

To prove Theorem 2.1, in Sect. 4.4 we estimate each of $I_1$, $I_2$ and $I_3$ by representing them as sums of the values of Fourier transforms of integrable functions at integers and applying the formula of Poisson. In Sect. 4.2 we use the Assumption 2 to derive the properties of considered wavelets that will be used in the estimations of $I_1$, $I_2$ and $I_3$, and the conditions of Poisson formula are verified in Sect. 4.3. We begin with the Sect. 4.1 below which presents two
Fig. 3 Wavelet density estimation of simulated moving average process with 4 non-zero coefficients. The innovations follow chi-squared distribution with 6 degrees of freedom and the moving average process follows chi-squared distribution with 24 degrees of freedom.

auxiliary lemmas regarding the characteristic functions of innovations and linear processes that will be used in the later proofs.

4.1 Estimation of $\sum_{1 \leq i, j \leq n} |H_{ij}(u, v)|$

Recall that for $i, j \in \mathbb{N}$ and for $\lambda, u, v \in \mathbb{R}$, we define $H(X_i)(\lambda) = e^{i\lambda X_i} - \phi(\lambda)$ and $H_{ij}(u, v) = \mathbb{E} e^{i(uX_i + vX_j)} - \phi(u)\phi(v) = \mathbb{E} \left[ H(X_i)(u)H(X_j)(v) \right]$, where $\phi(\lambda) = \mathbb{E} [e^{i\lambda X_1}]$ is the characteristic function of the linear process $X_n = \sum_{i=0}^{\infty} a_i \varepsilon_{n-i}$. To estimate the order of $\sum_{1 \leq i, j \leq n} |H_{ij}(u, v)|$, we extend the projection method applied in Sang et al. (2018). In their Lemma 7.1, they applied the projection method to estimate the upper bound of the quantity $\sum_{1 \leq i, j \leq n} |\mathbb{E} \left[ H(X_i)(\lambda)H(X_j)(\bar{\lambda}) \right]|$, where $H(X_j)(\bar{\lambda})$ is the conjugate of $H(X_j)(\lambda)$.

For each $i \in \mathbb{Z}$, let $\mathcal{F}_i$ be the $\sigma$-field generated by $\{\varepsilon_k : k \leq i\}$. Given an integrable complex-valued random variable $Y$, we define the projection operator $\mathcal{P}_i$ as

$$\mathcal{P}_i Y = \mathbb{E} [Y | \mathcal{F}_i] - \mathbb{E} [Y | \mathcal{F}_{i-1}].$$

It is easy to see that if $i \neq j$ then for any pair of integrable complex-valued random variables $Y$ and $W$ we have

$$\mathbb{E} [\mathcal{P}_i Y \mathcal{P}_j W] = 0.$$

In the estimation of $\sum_{1 \leq i, j \leq n} |H_{ij}(u, v)|$, we shall apply the condition (5) of Theorem 2.1. As in the theorem this condition is imposed only for the case $\gamma \in (1/2, 1]$, in the following auxiliary lemma we show that the condition also holds for the case $\gamma \in (0, 1/2]$.

**Lemma 4.1** If the derivative of $\phi_\varepsilon$ has bounded modulus on $\mathbb{R}$, then (5) holds for all $\gamma \in (0, 1/2]$, $\lambda \in \mathbb{R}$.
Proof Note that for all $\lambda \in \mathbb{R}$, $E |e^{i\lambda x} - \phi_e(\lambda)|^2 = 1 - |\phi_e(\lambda)|^2 \leq 1$. In particular, if $|\lambda| \geq 1$, then (5) holds for all $\gamma \in (0, 1]$. For $|\lambda| < 1$ and $\gamma \in (0, 1/2]$ we have

$$E |e^{i\lambda x} - \phi_e(\lambda)|^2 = 1 - |\phi_e(\lambda)|^2 = (1 + |\phi_e(\lambda)|)(1 - |\phi_e(\lambda)|) \leq 2(|\phi_e(0)| - |\phi_e(\lambda)|) \leq 2|\phi_e(0) - \phi_e(\lambda)| \leq c|\lambda| \leq c|\lambda|^2\gamma,$$

where the last inequality holds as $\gamma \in (0, 1/2]$.

Lemma 4.2 Suppose $\sum_{i=0}^{\infty} |a_i|^{\gamma} < \infty$ and (5) holds for some $\gamma \in (0, 1]$. Then there exists a positive constant $c$ such that

$$\sum_{1 \leq i \neq j \leq n} |H_{ij}(u, v)| \leq cn \left\{ |uv|^{\gamma} |\phi_e(ua_0)\phi_e(va_0)| + |u|^{\gamma} |\phi_e(ua_0)| + |v|^{\gamma} |\phi_e(va_0)| \right\} \quad (10)$$

and

$$|H_{11}(u, v)| = |E [H(X_1)(u)H(X_1)(v)]| \leq c|uv|^{\gamma} |\phi_e(ua_0)\phi_e(va_0)| + c(|uv|^{\gamma} \wedge 1) \prod_{\ell=1}^{\infty} |\phi_e(ua_\ell + va_\ell)|. \quad (11)$$

In particular, if the condition (4) is also satisfied, then

$$\sum_{1 \leq i, j \leq n} |H_{ij}(u, v)| \leq cn, \quad (12)$$

where $c > 0$ is a constant that only depends on $a_0$ and $\gamma$.

Proof Using the definition of projection operator $P_k$ in (8), applying (9), the telescoping technique and the triangle inequality, for $i \leq j$ we have

$$\left| E [H(X_i)(u)H(X_j)(v)] \right| \leq \sum_{k=-\infty}^{i} \left| E \left[ P_k H(X_i)(u)P_k H(X_j)(v) \right] \right|$$

$$= \sum_{k=-\infty}^{i} \left| E \left[ P_0 H(X_{i-k})(u)P_0 H(X_{j-k})(v) \right] \right|$$

$$= \sum_{k=-\infty}^{i} \left| E \left[ \prod_{\ell=1}^{i-k} |\phi_e(ua_\ell + va_\ell)| \prod_{\ell=0}^{j-k-1} |\phi_e(ua_\ell + va_\ell)| \right| \times E \left[ (e^{iu(au_0 + va_0)} - \phi_e(ua_0))(e^{i(au_0 + va_0)} - \phi_e(va_0)) \right] \right|. \quad (13)$$

Here $\prod_{\ell=0}^{m} |\phi_e(ua_\ell)| = 1$ if $m < 0$. Next, we decompose the sum $\sum_{k=-\infty}^{i} \phi_e(ua_0) \phi_e(va_0)$ corresponding to the cases $k < i$ and $k = i$. By Cauchy-Schwartz inequality, we have

$$\left| E [H(X_i)(u)H(X_j)(v)] \right| \leq \sum_{k=-\infty}^{i-1} |\phi_e(ua_0)\phi_e(va_0)| \sqrt{1 - |\phi_e(ua_{i-k})|^2} \sqrt{1 - |\phi_e(va_{j-k})|^2}$$

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Thus,\[ + \prod_{\ell=0}^{j-i-1} |\phi_{f}(va_{\ell})| \prod_{\ell=1}^{\infty} |\phi_{e}(uau_{\ell} + va_{\ell+j-i})| \sqrt{1 - |\phi_{e}(uau_{0})|^2} \sqrt{1 - |\phi_{e}(va_{j-i})|^2}.\]

Therefore, using the conditions of the lemma, for the case \( i < j \) we get
\[
|E[H(X_i)(u)H(X_j)(v)]| \leq c|uv|^\gamma \sum_{k=-\infty}^{i-1} |\phi_{e}(uau_{0})\phi_{e}(va_{0})||a_{i-k}|^\gamma |a_{j-k}|^\gamma \\
+ c|v|^\gamma |\phi_{e}(va_{0})||a_{j-i}|^\gamma.
\]

Hence,
\[
\sum_{1 \leq i \leq n-1} \sum_{i+1 \leq j \leq n} |E[H(X_i)(u)H(X_j)(v)]| \\
\leq c \sum_{1 \leq i \leq n-1} \sum_{i+1 \leq j \leq n} |uv|^\gamma \sum_{k=-\infty}^{i-1} |\phi_{e}(uau_{0})\phi_{e}(va_{0})||a_{i-k}|^\gamma |a_{j-k}|^\gamma \\
+ c \sum_{1 \leq i \leq n-1} \sum_{i+1 \leq j \leq n} |v|^\gamma |\phi_{e}(va_{0})||a_{j-i}|^\gamma \\
\leq cn|uv|^\gamma |\phi_{e}(uau_{0})\phi_{e}(va_{0})| + cn|v|^\gamma |\phi_{e}(va_{0})|.
\]

Similarly,
\[
\sum_{1 \leq j \leq n-1} \sum_{j+1 \leq i \leq n} |E[H(X_i)(u)H(X_j)(v)]| \\
\leq cn|uv|^\gamma |\phi_{e}(uau_{0})\phi_{e}(va_{0})| + cn|u|^\gamma |\phi_{e}(uau_{0})|.
\]

Thus,
\[
\sum_{1 \leq i \neq j \leq n} |E[H(X_i)(u)H(X_j)(v)]| \\
= \left\{ \sum_{1 \leq i \leq n-1} \sum_{i+1 \leq j \leq n} + \sum_{1 \leq j \leq n-1} \sum_{j+1 \leq i \leq n} \right\} |E[H(X_i)(u)H(X_j)(v)]| \\
\leq cn\{ |uv|^\gamma |\phi_{e}(uau_{0})\phi_{e}(va_{0})| + |u|^\gamma |\phi_{e}(uau_{0})| + |v|^\gamma |\phi_{e}(va_{0})| \}
\]

which proves (10). To prove (11) we note that when \( i = j \) then
\[
|E[H(X_i)(u)H(X_i)(v)]| \\
\leq \sum_{k=-\infty}^{i-1} |\phi_{e}(uau_{0})||\phi_{e}(va_{0})| \sqrt{1 - |\phi_{e}(uau_{i-k})|^2} \sqrt{1 - |\phi_{e}(va_{i-k})|^2} \\
+ \sum_{\ell=1}^{\infty} |\phi_{e}(uau_{\ell} + va_{\ell})| \sqrt{1 - |\phi_{e}(uau_{0})|^2} \sqrt{1 - |\phi_{e}(va_{0})|^2} \\
\leq c|uv|^\gamma |\phi_{e}(uau_{0})\phi_{e}(va_{0})| + c(|uv|^\gamma + 1) \prod_{\ell=1}^{\infty} |\phi_{e}(uau_{\ell} + va_{\ell})|.
\]
Remark 4.1 In the i.i.d. case, that is, in the case when \( a_0 \) is the only nonzero coefficient, we have that both \( \sum_{1 \leq i \neq j \leq n} |H_{ij}(u, v)| \) and \((13)\) vanish, while the term \((14)\) is equal to \( c |uv|^{\gamma} \). Thus, in this case we get \( \sum_{1 \leq i \leq n} \sum_{j \leq n} |H_{ij}(u, v)| = n |H_{11}(u, v)| \leq cn (|uv|^{\gamma} \wedge 1) \).

### 4.2 Properties of wavelets

In this part we present the properties of wavelets that will be used in the following proofs. According to Assumption 2, \( \varphi \) has compact support and is twice continuously differentiable. Therefore (see, e.g., Stein and Shakarchi 2003, pp. 132), its Fourier transform satisfies

\[
|\hat{\varphi}(u)| \leq \frac{c}{1 + u^2}. \tag{15}
\]

Moreover, as \((\hat{\varphi})' = -ix\hat{\varphi}(x)\) and \(x\varphi(x)\) also has compact support and is twice continuously differentiable, then

\[
|(\hat{\varphi}(u))'| \leq \frac{c}{1 + u^2}. \tag{16}
\]

Similarly, for the Fourier transform of \( \psi \) we have

\[
|\hat{\psi}(u)| \leq \frac{c}{1 + u^2} \tag{17}
\]

and

\[
|(\hat{\psi}(u))'| \leq \frac{c}{1 + u^2}. \tag{18}
\]

Also, as the mother wavelet function of the considered Daubechies wavelets has \([M\beta]\) vanishing moments \( \int_{\mathbb{R}} x^r \psi(x) \, dx = 0 \), then \((\hat{\psi})^{(r)}(0) = 0, r = 0, \ldots, [M\beta] - 1\). Since the derivative of \( \hat{\psi} \) of order \([M\beta]\) is bounded: \(|(\hat{\psi})^{([M\beta])}(u)| \leq \int_{\mathbb{R}} |t^{[M\beta]}| |\psi(t)| \, dt < \infty \), then the function \( \hat{\psi}(s)/s^{[M\beta]} \) is bounded on \( \mathbb{R} \). Here we shall use the L’Hopital’s Rule to obtain the boundedness of \( \hat{\psi}(s)/s^{[M\beta]} \) around 0. In particular, the function \( \hat{\psi}(s)/s^{[M\beta]} \) is also bounded on \( \mathbb{R} \).

### 4.3 Poisson summation formula

To prove Theorem 2.1 we will consider expressions of the form

\[
\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ik(u+v)} g(u, v) \, du \, dv,
\]

where \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is some integrable function. Defining

\[
h(t) = \int_{\mathbb{R}} g(t + s, -s) \, ds, \tag{19}
\]

we can rewrite the above sum as

\[
\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} e^{ikt} h(t) \, dt = \sum_{k \in \mathbb{Z}} \hat{h}(k),
\]

and the latter sum of the values of Fourier transform of the function \( h(t) \) at integers can be calculated using the following Poisson summation formula (see Zygmund 1966, Ch. II, §13):
**Theorem 4.1** (Poisson summation formula) If \( h \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) \) is an absolutely integrable function of bounded variation and \( 2h(x) = h(x + 0) + h(x - 0) \) for all \( x \in \mathbb{R} \), then
\[
\sum_{k \in \mathbb{Z}} h(k) = 2\pi \sum_{k \in \mathbb{Z}} h(2k\pi).
\]

In particular, if the function \( h \) defined by (19) satisfies the conditions of the above theorem, then
\[
\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(u+v)} g(u, v) du dv = 2\pi \sum_{k \in \mathbb{Z}} h(2k\pi). \tag{20}
\]

Throughout the proof of Theorem 2.1 we are going to apply the Poisson summation formula to functions
\[
h_1(t) = \sum_{1 \leq i, m \leq n} \int_{\mathbb{R}} \hat{\phi}(t + s)\hat{\phi}(-s) H_{im}(t + s, -s) ds,
\]
\[
h_2(t) = \sum_{1 \leq i, m \leq n} \int_{\mathbb{R}} \hat{\psi}(t + s)\hat{\psi}(-s) H_{im}(2^j(t + s), -2^j s) ds
\]
and
\[
h_3(t) = \int_{\mathbb{R}} \hat{\psi}(t + s)\hat{\phi}(2^j(t + s))\phi(-2^j s) ds,
\]
where \( \phi \) and \( \psi \) are, respectively, the scaling function and the mother wavelet discussed in the previous subsection, \( \phi \) is the characteristic function of linear process \( X_n \) and \( j \in \mathbb{N} \) is some number.

**Lemma 4.3** Under the conditions of Theorem 2.1, the functions \( h_1, h_2 \) and \( h_3 \) satisfy the conditions of Theorem 4.1.

**Proof** As \( H_{ij}(u, v) := \mathbb{E} e^{iuX_i + ivX_j} - \phi(u)\phi(v) \), then \( |H_{ij}(u, v)| \leq 2 \) for all \( i, j \in \mathbb{N} \), which together with (15) implies that \( h_1 \in L^1(\mathbb{R}) \). Let us now show that \( \int_{\mathbb{R}} |h_1(t)| dt < \infty \), which will imply (see Baernstein 2019, p. 125) that \( h_1 \in BV(\mathbb{R}) \). To bound the function \( h_1(t) \) we first estimate \( \left| \frac{\partial}{\partial u} H_{ij}(u, v) \right| \). As \( X_i = \sum_{p=0}^{\infty} a_p \epsilon_{i-p} = \sum_{m=-i}^{\infty} a_{i+m} \epsilon_{m} \) and \( X_j = \sum_{m=-j}^{\infty} a_{j+m} \epsilon_{m} \), then for given \( u, v \in \mathbb{R} \) and for \( i \leq j \) we have that
\[
u X_i + v X_j = \sum_{m=-i}^{\infty} u a_{i+m} \epsilon_{-m} + \sum_{m=-j}^{\infty} v a_{j+m} \epsilon_{-m}
\]
\[
= \sum_{m=-j}^{\infty} v a_{j+m} \epsilon_{m} + \sum_{m=-i}^{\infty} (u a_{i+m} + v a_{j+m}) \epsilon_{m}
\]
and, therefore,
\[
E_{ij}(u, v) := \mathbb{E} e^{iuX_i + ivX_j} = \prod_{m=-j}^{-i-1} \phi_e(v a_{j+m}) \prod_{m=-i}^{\infty} \phi_e(u a_{i+m} + v a_{j+m})
\]
(as usual, sums and products with negative number of summands and factors are assumed to be, respectively, 0 and 1). Thus,
\[
\frac{\partial}{\partial u} E_{ij}(u, v) = \sum_{m=-i}^{\infty} a_{i+m} \phi_e'(u a_{i+m} + v a_{j+m}) \frac{E_{ij}(u, v)}{\phi_e(u a_{i+m} + v a_{j+m})}. \tag{21}
\]
Also, as
\[ \phi(u) = \prod_{i=0}^{\infty} \phi_{\varepsilon}(ua_i), \]
then (see Giraitis et al. 1996, p. 322)
\[ \phi'(u) = \sum_{i=0}^{\infty} a_i \phi'_{\varepsilon}(ua_i) \prod_{j \geq 0 \atop j \neq i} \phi_{\varepsilon}(ua_j), \quad u \in \mathbb{R}, \]
and, therefore,
\[ \frac{\partial}{\partial u} \phi(u) \phi(v) = \sum_{m=-i}^{\infty} a_{i+m} \phi'_{\varepsilon}(ua_{i+m}) \frac{\phi(u) \phi(v)}{\phi_{\varepsilon}(ua_{i+m})}. \]
Hence, using the boundedness of \( \phi'_{\varepsilon} \), we get
\[
\sup_{u,v \in \mathbb{R}} \left| \frac{\partial}{\partial u} H_{ij}(u, v) \right| = \sup_{u,v \in \mathbb{R}} \left| \frac{\partial}{\partial u} E_{ij}(u, v) - \frac{\partial}{\partial u} \phi(u) \phi(v) \right| \\
= \sup_{u,v \in \mathbb{R}} \left| \sum_{m=-i}^{\infty} a_{i+m} \phi'_{\varepsilon}(ua_{i+m} + va_{j+m}) \frac{E_{ij}(u, v)}{\phi_{\varepsilon}(ua_{i+m} + va_{j+m})} \right. \\
\left. - \sum_{m=-i}^{\infty} a_{i+m} \phi'_{\varepsilon}(ua_{i+m}) \frac{\phi(u) \phi(v)}{\phi_{\varepsilon}(ua_{i+m})} \right| \\
\leq c \sum_{m=-i}^{\infty} |a_{i+m}| < \infty.
\]
Therefore, using also (16), we have
\[
\left| \frac{\partial}{\partial t} \left[ \hat{\phi}(t+s) \hat{\phi}(-s) H_{ij}(t+s, -s) \right] \right| \leq c \left| \hat{\phi}(-s) \right|,
\]
and since \( \int_{\mathbb{R}} |\hat{\phi}(-s)| ds < \infty \), we get by dominated convergence theorem, that
\[
\int_{\mathbb{R}} |h'_1(t)| dt \\
\leq \sum_{1 \leq i, j \leq n} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{\partial}{\partial t} \left[ \hat{\phi}(t+s) \hat{\phi}(-s) H_{ij}(t+s, -s) \right] ds \right| dt \\
\leq cn^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{1 + (t+s)^2} \frac{1}{1 + s^2} ds dt < \infty,
\]
where we also used the inequalities (15) and (16). Proofs for the functions \( h_2 \) and \( h_3 \) are identical. \( \square \)

### 4.4 Estimation of \( I_1, I_2 \) and \( I_3 \)

**Lemma 4.4** Under the conditions of Theorem 2.1
\[ |I_1| \leq \frac{c}{n}, \]
\( \square \) Springer
\textbf{Proof} We have that

\[
I_1 = \sum_{k \in \mathbb{Z}} \mathbb{E}(\hat{\alpha}_{0k} - \alpha_{0k})^2 = \sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \varphi_{0k}(X_i) - \mathbb{E} \varphi_{0k}(X_1) \right]^2
\]

\[
= \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}} \hat{\varphi}_{0k}(u) e^{iuX_i} du - \mathbb{E} \int_{\mathbb{R}} \hat{\varphi}_{0k}(u) e^{iuX_1} du \right]^2
\]

\[
= \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}} \hat{\varphi}_{0k}(u) e^{iuX_i} du - \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}} \hat{\varphi}_{0k}(u) \phi(u) du \right]^2
\]

\[
= \frac{1}{4\pi^2 n^2} \sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \int_{\mathbb{R}} \hat{\varphi}(u)e^{iku} \left( \sum_{i=1}^{n} [e^{iuX_i} - \phi(u)] \right) \left( \sum_{i=1}^{n} [e^{iuX_i} - \phi(u)] \right) du \right]^2
\]

Let

\[ g_1(u, v) = \hat{\varphi}(u)\hat{\varphi}(v) \sum_{1 \leq i, j \leq n} H_{ij}(u, v). \]

Then

\[ h_1(t) = \int_{\mathbb{R}} g_1(t + s, -s) ds = \int_{\mathbb{R}} \hat{\varphi}(t + s)\hat{\varphi}(-s) \sum_{1 \leq i, j \leq n} H_{ij}(t + s, -s) ds \]

and by Theorem 4.1 and Lemma 4.3 we have

\[
I_1 = \frac{1}{2\pi n^2} \sum_{k \in \mathbb{Z}} h_1(2k\pi)
\]

\[
= \frac{1}{2\pi n^2} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \hat{\varphi}(2k\pi + s)\hat{\varphi}(-s) \sum_{1 \leq i, j \leq n} H_{ij}(2k\pi + s, -s) ds.
\]

Hence, applying (12) and (15) we get

\[
|I_1| \leq \frac{1}{2\pi n^2} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{\varphi}(2k\pi + s)||\hat{\varphi}(-s)| \sum_{1 \leq i, j \leq n} |H_{ij}(2k\pi + s, -s)| ds
\]

\[
\leq \frac{c}{n} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{\varphi}(2k\pi + s)||\phi(s)| ds
\]

\[
\leq \frac{c}{n} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \frac{1}{1 + (2k\pi + s)^2} \frac{1}{1 + s^2} ds
\]

\[
\leq \frac{c}{n} \int_{\mathbb{R}} \frac{1}{1 + s^2} ds \leq \frac{c}{n}.
\]

\[
\square
\]
Lemma 4.5 Under the conditions of Theorem 2.1, for any \( j_n > 0 \),
\[ |I_2| \leq \frac{c_2j_n}{n}. \]

Proof We have that
\[ I_2 = \sum_{j=0}^{j_n} \sum_{k \in \mathbb{Z}} \mathbb{E}(\hat{\beta}_{jk} - \beta_{jk})^2 = \sum_{j=0}^{j_n} \sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \psi_{jk}(X_i) - \mathbb{E} \psi_{jk}(X_1) \right]^2 \]
\[ = \frac{1}{4\pi^2} \sum_{j=0}^{j_n} \sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \psi_{jk}(u)e^{iuX_i} du - \mathbb{E} \left( \psi_{jk}(u)e^{iuX_1} \right) \right]^2 \]
\[ = \frac{1}{4\pi^2} \sum_{j=0}^{j_n} \sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \psi_{jk}(u)e^{iuX_i} du - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left( \psi_{jk}(u)\phi(u) \right) \right]^2 \]
\[ = \frac{1}{4\pi^2} \sum_{j=0}^{j_n} \sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \int_{\mathbb{R}} \psi_{jk}(u) \sum_{i=1}^{n} (e^{iuX_i} - \phi(u)) du \right]^2 \]
\[ = \frac{1}{4\pi^2} \sum_{j=0}^{j_n} \sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \int_{\mathbb{R}} \psi_{jk}(u) \psi_{jk}(v) \sum_{1 \leq i, m \leq n} [e^{iuX_i} - \phi(u)][e^{ivX_m} - \phi(v)] dudv \right. \]
\[ \left. = \frac{2^{-j}}{4\pi^2} \sum_{j=0}^{j_n} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} e^{iku2^{-j}} \hat{\psi}(u2^{-j}) e^{ikv2^{-j}} \hat{\psi}(v2^{-j}) \sum_{1 \leq i, m \leq n} H_{im}(u,v) dudv \right. \]
\[ \left. = \frac{2^j}{4\pi^2} \sum_{j=0}^{j_n} \sum_{k \in \mathbb{Z}} \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\psi}(u) \hat{\psi}(v) e^{iu+v k} \sum_{1 \leq i, m \leq n} H_{im}(2^j u, 2^j v) dudv \right] := \sum_{j=0}^{j_n} I_{2,j}. \]

Let
\[ g_2(u, v) = \hat{\psi}(u) \hat{\psi}(v) \sum_{1 \leq i, m \leq n} H_{im}(2^j u, 2^j v). \]

Then
\[ h_2(t) = \int_{\mathbb{R}} g_2(t + s, -s) ds = \int_{\mathbb{R}} \hat{\psi}(t+s) \hat{\psi}(-s) \sum_{1 \leq i, m \leq n} H_{im}(2^j(t+s), -2^j s) ds. \]

By the Poisson formula (20) we have
\[ I_{2,j} = \frac{2^j}{2\pi n^2} \sum_{k \in \mathbb{Z}} h_2(2k\pi) = \frac{2^j}{2\pi n^2} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \hat{\psi}(2k\pi + s) \hat{\psi}(-s) \sum_{1 \leq i, m \leq n} H_{im}(2^j(2k\pi + s), -2^j s) ds. \]

Hence, applying (12) and (17), we get
\[ |I_{2,j}| \leq \frac{2^j}{2\pi n^2} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{\psi}(2k\pi + s)||\hat{\psi}(-s)| \sum_{1 \leq i, m \leq n} |H_{im}(2^j(2k\pi + s), -2^j s)| ds \]
\[ \leq \frac{c_2^j}{n} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{\psi}(2k\pi + s)||\hat{\psi}(s)| ds \leq \frac{c_2^j}{n} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \frac{1}{1 + (2k\pi + s)^2} |\hat{\psi}(s)| ds \]
\[ \frac{c2^j}{n} \int_{\mathbb{R}} |\hat{\psi}(s)| ds \leq \frac{c2^j}{n}. \]

Thus,
\[ |I_2| \leq \sum_{j=0}^{j_n} |I_{2,j}| \leq \frac{c2^j n}{n}. \]

**Lemma 4.6** Under the conditions of Theorem 2.1, for any \( j_n > 0 \),
\[ |I_3| \leq c2^{-2j_n M\beta}. \]

**Proof** Recall that \( \phi \) is the characteristic function of \( X_1 \). We then have that
\[
I_{3,j} := \sum_{k \in \mathbb{Z}} \beta_{jk}^2 = \sum_{k \in \mathbb{Z}} \left[ \mathbb{E} \hat{\psi}_{jk}(X_1) \right]^2
\]
\[
= \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}} \left[ \mathbb{E} \int_{\mathbb{R}} \hat{\psi}_{jk}(u) e^{iuX_1} du \right]^2
\]
\[
= \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}} \left[ \int_{\mathbb{R}} \hat{\psi}_{jk}(u) \phi(u) du \right]^2
\]
\[
= \frac{2^j}{4\pi^2} \sum_{k \in \mathbb{Z}} \left[ \int_{\mathbb{R}} e^{iku} \hat{\psi}(u) \phi(2^j u) du \right]^2
\]
\[
= \frac{2^j}{4\pi^2} \sum_{k \in \mathbb{Z}} \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(u+v)k} \hat{\psi}(u) \hat{\psi}(v) \phi(2^j u) \phi(2^j v) du dv \right].
\]

Let
\[ g_3(u, v) = \hat{\psi}(u) \hat{\psi}(v) \phi(2^j u) \phi(2^j v). \]

Then
\[ h_3(t) = \int_{\mathbb{R}} g_3(t + s, -s) ds = \int_{\mathbb{R}} \hat{\psi}(t + s) \hat{\psi}(-s) \phi(2^j (t + s)) \phi(-2^j s) ds. \]

Hence, by the Poisson formula (20),
\[
I_{3,j} = \frac{2^j}{2\pi} \sum_{k \in \mathbb{Z}} h_3(2k\pi) = \frac{2^j}{2\pi} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \hat{\psi}(2k\pi + s) \hat{\psi}(-s) \phi(2^j (2k\pi + s)) \phi(-2^j s) ds
\]
\[
= \frac{2^j}{2\pi} \int_{\mathbb{R}} \hat{\psi}(s) \hat{\psi}(-s) \phi(2^j s) \phi(-2^j s) ds
\]
\[
+ \frac{2^j}{2\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{-1}^{1} \hat{\psi}(2k\pi + s) \hat{\psi}(-s) \phi(2^j (2k\pi + s)) \phi(-2^j s) ds
\]
\[
+ \frac{2^j}{2\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{|s| > 1} \hat{\psi}(2k\pi + s) \hat{\psi}(-s) \phi(2^j (2k\pi + s)) \phi(-2^j s) ds
\]
\[ := I_{3,j}^{(1)} + I_{3,j}^{(2)} + I_{3,j}^{(3)}. \]
As $\phi(u) = \prod_{i=0}^{\infty} \phi_i(a_i u)$ and there are at least $M$ nonzero coefficients among $a_i$, $i \in \mathbb{N}_0$, then it follows from (4) that

$$\int_{\mathbb{R}} |u^{M\beta} \phi(u)|^2 du < \infty.$$ 

Using also the boundedness of $\hat{\psi}(s)/s^{M\beta}$, we get

$$|I_{3,j}| \leq c 2^j \int_{\mathbb{R}} |\hat{\psi}(s)\hat{\psi}(-s)\phi(2^j s)\phi(-2^j s)| ds \leq c 2^j \int_{\mathbb{R}} |s^{M\beta} \phi(2^j s)\phi(-2^j s)| ds \leq 2^{-2jM\beta}.$$ 

For $k \in \mathbb{Z} \setminus \{0\}$ and $s \in (-1, 1]$ we have $|2k\pi + s| > 1$, and, therefore, the condition (4) implies that $|\phi(2^j (2k\pi + s))| \leq c 2^{-jM\beta}$. Hence,

$$|I_{3,j}''| \leq c 2^j \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{-1}^{1} |\hat{\psi}(2k\pi + s)\hat{\psi}(-s)\phi(2^j (2k\pi + s))\phi(-2^j s)| ds$$

$$\leq c 2^{-jM\beta} \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{-1}^{1} \frac{1}{1 + (2k\pi + s)^2} |\hat{\psi}(-s)\phi(-2^j s)| ds$$

$$\leq c 2^{-jM\beta} \int_{-1}^{1} \int_{|s| > 1} |\hat{\psi}(s)| ds \leq c 2^{-jM\beta} \int_{-1}^{1} |s^{M\beta} \phi(-2^j s)| ds \leq c 2^{-2jM\beta}.$$ 

Similarly, for $|s| > 1$ we have $|\phi(-2^j s)| \leq c 2^{-jM\beta}$, and, therefore,

$$|I_{3,j}''| \leq c 2^j \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{|s| > 1} |\hat{\psi}(2k\pi + s)\hat{\psi}(-s)\phi(2^j (2k\pi + s))\phi(-2^j s)| ds$$

$$\leq c 2^{-jM\beta} \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{|s| > 1} |\hat{\psi}(2k\pi + s)\hat{\psi}(-s)\phi(2^j (2k\pi + s))| ds$$

$$= c 2^{-jM\beta} \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{\mathbb{R}} |\hat{\psi}(u)\psi(-u + 2k\pi)\phi(2^j u)| du$$

$$\leq c 2^{-jM\beta} \int_{\mathbb{R}} |\hat{\psi}(u)\phi(2^j u)| du \leq c 2^{-jM\beta} \int_{\mathbb{R}} |u^{M\beta} \phi(2^j u)| du \leq 2^{-2jM\beta} \leq c 2^{-2jM\beta}.$$ 

Hence,

$$|I_3| = \left| \sum_{j=j_n+1}^{\infty} I_{3,j} \right| \leq c 2^{-2j_nM\beta}.$$

$\square$
Proof of Theorem 2.1 Putting together the results of Lemmas 4.4, 4.5 and 4.6 and taking
\[ J_n = \lceil \frac{\log_2 n}{2M^2 \beta + 1} \rceil \]
proves the Theorem 2.1.

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References

Baernstein A (2019) Symmetrization in analysis, with David Drasin and Richard S. Laugesen. New Mathematical Monographs, 36. Cambridge University Press, Cambridge
Bailey WN (1935) General hypergeometric series. Cambridge tracts in mathematics and mathematical physics
Badaoui M, Rhomari N (2015) Blockshrink wavelet density estimator in \( \phi \)-mixing framework, Functional Statistics and Applications. Springer, Berlin, pp 29–50
Chesneau C (2012) Wavelet linear estimation of a density and its derivatives from observations of mixtures under quadrant dependence. ProbStat Forum 5:38–46
Chesneau C (2014) A general result on the mean integrated squared error of the hard thresholding wavelet estimator under \( \alpha \)-mixing dependence. J Probab Stat, Article ID: 403764
Chesneau C, Dewan I, Doosti H (2012) Wavelet linear density estimation for associated stratified size-biased sample. J Nonparametric Stat 24(2):429–45
Chesneau C, Doosti H, Stone L (2019) Adaptive wavelet estimation of a function from an \( m \)-dependent process with possibly unbounded \( m \). Commun Stat - Theory Methods 48:1123–1135
Daubechies I (1992) Ten lectures on wavelets. Springer, New York
Donoho DL, Johnstone IM (1994) Ideal spatial adaptation by wavelet shrinkage. Biometrika 81:425–455
Donoho DL, Johnstone IM (1995) Adapting to unknown smoothness via wavelet shrinkage. J Am Stat Assoc 90:1200–1224
Donoho DL, Johnstone IM, Kerkyacharian G, Picard D (1995) Wavelet shrinkage: asymptopia? J R Stat Soc B 57:301–369
Donoho DL, Johnstone IM, Kerkyacharian G, Picard D (1996) Density estimation by wavelet thresholding. Ann Stat 24:508–539
Faÿ G, Moulines E, Roueff F, Taqqu MS (2009) Estimators of long-memory: Fourier versus wavelets. J Econom 151:159–177
Folland G (1999) Real analysis: modern techniques and their applications. Wiley, New York
Gannaz I, Wintenberger O (2010) Adaptive density estimation with dependent observations. ESAIM Probab Stat 14:151–172
Gauss F (1866) Disquisitiones generales circa sericm infinitam, Ges. Werke, 3, 123–163 and 207–229
Giné E, Madych WR (2014) On wavelet projection kernels and the integrated squared error in density estimation. Stat Probab Lett 91:32–40
Giraitis L, Koul HL, Surgailis D (1996) Asymptotic normality of regression estimators with long memory errors. Stat Probab Lett 29:317–335
Hall P, Hart JD (1990) Convergence rates in density estimation for data from infinite-order moving average processes. Probab Theory Relat Fields 87:253–274
Johnstone IM (1999) Wavelets and the theory of non-parametric function estimation. Philos Trans R Soc Lond Ser A: Math Phys Eng Sci 357(1760):2475–2493
Kou JK, Guo HJ (2018) Wavelet density estimation for mixing and size-biased data. J Inequalities Appl, p 189
Leblanc F (1996) Wavelet linear density estimator for a discrete time stochastic process: Lp-losses. Stat Probab Lett 27:71–84
Li L, Zhang B (2022) Nonlinear wavelet-based estimation to spectral density for stationary non-Gaussian linear processes. Appl Comput Harmon Anal 60:176–204
Lu L (2013) On the integrated squared error of the linear wavelet density estimator. J Stat Plan Inference 143:1548–1565
Lukacs E (1970) Characteristic functions, 2nd edn. Griffin & Co., London
Marron JS (1994) Visual understanding of higher order kernels. J Comput Graph Stat 3:447–458
Meloche J (1990) Asymptotic behavior of the mean integrated squared error of kernel density estimators for dependent observations. Can J Statistics 18(3):205–211
Mielniczuk J (1997) On the asymptotic mean integrated squared error of a kernel density estimator for dependent data. Stat Probab Lett 34:53–58
Priestley MB (1981) Spectral analysis and time series. Academic Press, New York
Saavedra A, Cao R (2000) On the estimation of the marginal density of a moving average process. Can J Stat 28(4):799–815
Samorodnitsky G, Taqqu MS (1994) Stable non-Gaussian random processes. Chapman and Hall, New York
Sang H, Sang Y (2017) Memory properties of transformations of linear processes. Stat. Inference Stoch. Process. 20:79–103
Sang H, Sang Y, Xu F (2018) Kernel entropy estimation for linear processes. J Time Ser Anal 39(4):563–591
Stein E, Shakarchi R (2003) Complex analysis. Princeton University Press, Princeton
Wahba G (1975) Optimal convergence properties of variable knot, kernel and orthogonal series methods for density estimation. Annu Stat 3:15–29
Wu WB, Mielniczuk J (2002) Kernel density estimation for linear processes. Annu Stat 30(5):1441–1459
Zhang S, Zheng Z (1999) On the asymptotic normality for the \( L_2 \)-error of wavelet density estimators with application. Commun Stat - Theory Methods 28:1093–1104
Zygmund A (1966) Trigonometric series, vol I, II. Cambridge University Press, Cambridge

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