A NEW REFORMULATION OF THE MUSKAT PROBLEM WITH SURFACE TENSION

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Abstract. Two formulas that connect the derivatives of the double layer potential and of a related singular integral operator evaluated at some density \( \vartheta \) to the \( L^2 \)-adjoints of these operators evaluated at the density \( \vartheta' \) are used to recast the Muskat problem with surface tension and general viscosities as a system of equations with nonlinearities expressed in terms of the \( L^2 \)-adjoints of these operators. An advantage of this formulation is that the nonlinearities appear now as a derivative. This aspect and abstract quasilinear parabolic theory are then exploited to establish a local well-posedness result in all subcritical Sobolev spaces \( W^{s,p}(\mathbb{R}) \) with \( p \in (1,\infty) \) and \( s \in (1 + 1/p, 2) \).

1. Introduction

In this paper we consider the two-dimensional Muskat problem describing the dynamics in an unbounded two fluids system which moves with constant speed \( V \) in a horizontal/vertical Hele-Shaw cell or in a porous medium. The fluids are assumed to fill the entire plane and the free interface between the fluids is parameterized as the graph
\[
\{(x, f(t, x) + tV) : x \in \mathbb{R}\} \quad \text{for} \ t \geq 0.
\]
We take into account both gravity and surface tension effects. Let therefore \( \kappa(f(t)) \) denote the curvature of the free interface and let \( \sigma > 0 \) be the surface tension coefficient. The subscript \(-/+\) is used to denote the fluid located below/above the interface, \( g \geq 0 \) is the Earth’s gravity, and \( k \) is the permeability of the homogeneous porous medium. Moreover, the positive constants \( \mu_- \) and \( \rho_\pm \) are the viscosity and the density of the fluids. Introducing a further unknown \( \omega := \omega(t, x) \), with \( 2(1 + (\partial_x f)^2)^{-1/2} \omega \) measuring the jump of the velocity field in tangential direction at the interface, the Muskat problem can be expressed in a compact form as the following coupled system
\[
\begin{align*}
\frac{df}{dt}(t) &= B(f(t))[\omega(t)], \quad t > 0, \\
(1 - a_\mu \mathcal{A}(f(t)))[\omega(t)] &= b_\mu (\sigma \kappa(f(t)) - \Theta f(t))', \quad t > 0, \\
f(0) &= f_0,
\end{align*}
\]
(1.1)
cf., e.g., [1,4,29,36]. The constants in (1.1) are given by the relations
\[
b_\mu := \frac{k}{\mu_- + \mu_+} > 0, \quad a_\mu := \frac{\mu_- - \mu_+}{\mu_- + \mu_+} \in (-1,1), \quad \Theta := g(\rho_- - \rho_+) + \frac{\mu_- - \mu_+}{k} V \in \mathbb{R}.
\]
Throughout the paper \((\cdot)'\) denotes differentiation with respect to the spatial variable \( x \). Furthermore, given a Lipschitz continuous map \( f : \mathbb{R} \to \mathbb{R} \), the singular integral operators \( \mathcal{A}(f) \) and \( B(f) \)

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in (1.1) are given by

\[
A(f)[\varpi](x) := \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{f'(x) - (\delta_{[x,y]} f)/y \varpi(x - y)}{1 + [(\delta_{[x,y]} f)/y]^2} dy,
\]

\[
B(f)[\varpi](x) := \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{1 + f'(x)(\delta_{[x,y]} f)/y \varpi(x - y)}{1 + [(\delta_{[x,y]} f)/y]^2} dy,
\]

(1.2)

for \(\varpi \in L^2(\mathbb{R})\). We use \(\text{PV}\) to denote the principal value and the shorthand notation

\[\delta_{[x,y]} f := f(x) - f(x - y), \quad x, y \in \mathbb{R}.
\]

These operators are bounded, that is \(A(f), B(f) \in \mathcal{L}(L^p(\mathbb{R}))\) for all \(p \in (1, \infty)\), see Lemma 2.1 (i) below. Let \(A(f)^*, B(f)^* \in \mathcal{L}(L^2(\mathbb{R}))\) denote their \(L_2\)-adjoint operators. We point out that the operator \(A(f)^*\) is the double layer potential for the Laplace operator corresponding to the unbounded hypersurface \(\{y = f(x)\} \subset \mathbb{R}^2\).

A key point in our analysis are the following identities

\[(A(f)^*[\vartheta])' = -A(f)[\vartheta] \quad \text{and} \quad (B(f)^*[\vartheta])' = -B(f)[\vartheta],\]

(1.3)

which are satisfied provided that \(f \in W^2_p(\mathbb{R})\) and \(\vartheta \in W^1_p(\mathbb{R})\), see Proposition 2.3 below. Using (1.3), we show in Section 3 that (1.1) can be formulated as an evolution problem for the unknown \((f, \vartheta)\) which reads as

\[
\begin{align*}
    \frac{df}{dt}(t) &= -B(f(t))^*[\vartheta(t)]', \quad t > 0, \\
    (1 + a_p A(f(t))^*[\vartheta(t)] &= b_p (\sigma k(f(t)) - \Theta f(t)), \quad t > 0, \\
    f(0) &= f_0.
\end{align*}
\]

(1.4)

The connection between \(\varpi(t)\) and \(\vartheta(t)\) is through the relation

\[\varpi(t) = (\vartheta(t))', \quad t > 0.
\]

We shall take advantage of the new formulation (1.4) to establish the well-posedness of the Muskat problem with surface tension in all subcritical Sobolev spaces \(W^s_p(\mathbb{R})\), \(s \in (1 + 1/p, 2)\) and \(p \in (1, \infty)\). Compared to (1.1), the formulation (1.4) has several advantages. On the one hand, it enables us to consider an equation related to (1.4)\(_2\) in the Sobolev space \(W^{s-1}_p(\mathbb{R})\), \(s \in (1 + 1/p, s)\), see equation (4.4), whereas in the context of (1.1)\(_2\) one more derivative appears on the right of this equation (and the right hand side is in the latter case a distribution). On the other hand, the equation (1.4)\(_1\) will be considered in a Sobolev space with a negative index, more precisely in \(W^{s-2}_p(\mathbb{R})\), but its right side is the derivative of function that lies in \(W^{s-1}_p(\mathbb{R})\), and this is very useful when establishing estimates.

Exploiting also the quasilinear character of the curvature operator \(\kappa(f)\), the core of our analysis is to show that (1.4) can be recast as a quasilinear parabolic evolution problem for \(f\). Based on these properties an on the abstract quasilinear parabolic theory presented in [2] (see also [34]), we prove the following local well-posedness result.
Theorem 1.1. Let $p \in (1, \infty)$, $1 + 1/p < s < 2$, and chose (an arbitrary small) $\zeta \in (0, (s-\infty)/3]$. Then, given $f_0 \in W^s_p(\mathbb{R})$, there exists a unique maximal solution $f := f(\cdot ; f_0)$ to (1.4) such that

\[
 f \in C((0, T^+), W^s_p(\mathbb{R})) \cap C((0, T^+), W^s_{p+1}(\mathbb{R})) \cap C^1((0, T^+), W^{-2}_{p-2}(\mathbb{R}))
\]

and

\[
 f \in C^\zeta([0, T^+), W^s_p(\mathbb{R})),
\]

where $T^+ = T^+(f_0) \in (0, \infty]$ is the maximal existence time. Moreover, $[(t, f_0) \mapsto f(t; f_0)]$ defines a semiflow on $W^s_p(\mathbb{R})$ which is smooth in the open set

\[
 \{(t, f_0) : f_0 \in W^s_p(\mathbb{R}), \ 0 < t < T^+(f_0) \} \subset \mathbb{R} \times W^s_p(\mathbb{R})
\]

and

\[
 f \in C^\infty((0, T^+) \times \mathbb{R}, \mathbb{R}) \cap C^\infty((0, T^+), W^k_p(\mathbb{R})) \text{ for all } k \in \mathbb{N}.
\]

There is a vast, partially quite recent, mathematical literature on the Muskat problem with surface tension and its one-phase version, the Hele-Shaw problem with surface tension. The studies investigate different important aspects of the models such as the well-posedness, cf. e.g. [4, 5, 7, 13, 14, 16, 17, 19, 20, 26, 37–39], the existence of global (weak or strong) solutions [9, 23, 24, 27], the stability properties of the stationary solutions [11, 13, 14, 16, 22, 23, 31, 34, 37, 38], the zero surface tension limit of the problems [4, 21], and the singular limit when the thickness of the layers (or a nondimensional parameter) vanishes [6, 15, 25, 32].

The formulation (1.1) of the unconfined Muskat problem with surface tension considered herein is derived from the classical formulation [35] by using potential theory (see [10] for a first result in this direction). The advantage of this formulation compared to the classical one is that now the equations of motion can be studied under quite general assumptions on the function $\mathcal{f}$ and this leads to quite optimal results. Indeed, in the references [29–31] the well-posedness of the problem is established for $H^{2+\varepsilon}$-initial data, with $\varepsilon \in (0, 1)$ arbitrarily small, and these results were improved in the very recent papers [21, 36] where the initial data are taken from $H^{1+\frac{\varepsilon}{d}}(\mathbb{R}^d)$, with $\varepsilon > 0$ arbitrarily small and $d \geq 1$. It is important to point out that $W^{1+\frac{\varepsilon}{d}}(\mathbb{R}^d)$ is a critical space for (1.1), see [28, 36]. For the restrictive range $p \in (1, 2)$ the well-posedness of (1.1) in $W^{1+\frac{1}{p+\varepsilon}}(\mathbb{R}^d)$, again with $\varepsilon > 0$ arbitrarily small, was established recently in [28] in the particular case of fluids with equal viscosities (by using a different approach than in this paper). Finally, the stability properties of equilibria to the periodic version of (1.1) have been studied in [31, 34].

Our main result in Theorem 1.1 extends the well-posedness theory to all $L_p$-based subcritical Sobolev spaces $W^s_p(\mathbb{R})$, $s \in (1 + 1/p, 2)$ and $p \in (1, \infty)$ in the general case $\rho_- = \rho_+, \mu_- = \mu_+ \in \mathbb{R}$. An important aspect in the analysis is the invertibility of the operator $\lambda - \lambda(f)$, $\lambda \in \mathbb{R} \setminus (-1, 1)$ and $f \in W^s_p(\mathbb{R})$, in $L_p(\mathbb{R})$. In the case of a bounded Lipschitz domain, when $f$ is merely Lipschitz continuous, this property is a deep result of harmonic analysis, see [41]. In the unbounded setting considered herein we establish this property directly by a using different strategy than in the bounded case [41]. We also mention that there are not so many references that consider the Muskat problem in a classical $L_p$-setting with $p \neq 2$ and, apart from the references [1, 28], we only add the paper [8] where the particular case $\sigma = 0$ and $\mu_- = \mu_+$ is considered.

Notation. Given $n \in \mathbb{N}$ and Banach spaces $E, E_1, \ldots, E_n, F$, $n \in \mathbb{N}$, we write $\mathcal{L}^n(\prod_{i=1}^n E_i, F)$ to denote the Banach space of bounded $n$-linear maps from $\prod_{i=1}^n E_i$ to $F$, and $\mathcal{L}^n_{sym}(E, F)$ stands for the space of $n$-linear, bounded, and symmetric maps $A : E^n \rightarrow F$. Moreover, $C^{-1}(E, F)$
we introduce a family of singular integral operators that is needed in the analysis and we establish the relations (1.9) and the equivalence of the formulations (1.1) and (1.4). Finally, in Section 4 we formulate (1.4) as a quasilinear parabolic evolution equation for $f$ and we prove our main result Theorem 1.1.

2. A FAMILY OF SINGULAR INTEGRAL OPERATORS AND THE PROOF OF (1.3)

The main goal of this section is to establish the relations (1.3). To this end we first introduce a family of multilinear singular integral operators which play a key role in the analysis of the unconfined Muskat problem and also of the unconfined quasi-stationary Stokes problem [33]. Given $n, m \in \mathbb{N}$, Lipschitz continuous functions $a_1, \ldots, a_m, b_1, \ldots, b_n : \mathbb{R} \to \mathbb{R}$, and $\vartheta \in L_p(\mathbb{R})$, we set

$$B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \vartheta](x) := \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{\vartheta(x-y)}{y} \prod_{i=1}^{n} (\delta_{[x,y]} b_i / y) \prod_{i=1}^{n} [1 + (\delta_{[x,y]} a_i / y)] \, dy, \quad x \in \mathbb{R}. $$

(2.1)

If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous we further set

$$B_{n,m}^0(f)[\vartheta] := B_{n,m}(f, \ldots, f)[f, \ldots, f, \vartheta]. $$

(2.2)
These operators have been considered in the $L_p$-setting with $p \in (1, \infty)$ in [1, 28]. We recall the following results.

**Lemma 2.1.** Let $p \in (1, \infty)$, $n$, $m \in \mathbb{N}$, and $s \in (1 + 1/p, 2)$.

(i) Let $a_1, \ldots, a_m, b_1, \ldots, b_n : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Then, there exists a positive constant $C = C(n, m, \max_{i=1,\ldots,m} \|a_i\|_{\infty})$ such that

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \cdot]\|_{L(L_p(\mathbb{R}))} \leq C \prod_{i=1}^{n} \|b'_i\|_{\infty}.$$  

Moreover, $B_{n,m} \in C^1(W^1_{\infty}(\mathbb{R})^m, L_{sym}(W^1_{\infty}(\mathbb{R}), L(L_p(\mathbb{R}))))$.

(ii) Given $a_1, \ldots, a_m \in W^s_p(\mathbb{R})$, there exists a constant $C = C(n, \mathbb{R}, s, \max_{1 \leq i \leq m} \|a_i\|_{W^{s}_{p}})$ such that

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \vartheta]\|_{W^{s-1}_{p}} \leq C \|\vartheta\|_{W^{s-1}_{p}} \prod_{i=1}^{n} \|b'_i\|_{W^{s-1}_{p}} \tag{2.3}$$

for all $b_1, \ldots, b_n \in W^s_p(\mathbb{R})$ and $\vartheta \in W^{s-1}_{p}(\mathbb{R})$.

Moreover, $B_{n,m} \in C^1(W^s_p(\mathbb{R})^m, L^n_{sym}(W^s_p(\mathbb{R}), L(W^{s-1}_{p}(\mathbb{R}))))$.

(iii) Let $n \geq 1$. Given $a_1, \ldots, a_m \in W^s_p(\mathbb{R})$, there exists $C = C(n, m, s, \max_{1 \leq i \leq m} \|a_i\|_{W^{s}_{p}})$ such that

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \vartheta]\|_{p} \leq C \|b'_1\|_{p} \|\vartheta\|_{W^{s-1}_{p}} \prod_{i=2}^{n} \|b'_i\|_{W^{s-1}_{p}} \tag{2.4}$$

for all $b_1, \ldots, b_n \in W^s_p(\mathbb{R})$ and $\vartheta \in W^{s-1}_{p}(\mathbb{R})$.

Moreover, $B_{n,m} \in C^1(W^s_p(\mathbb{R})^m, L^n_{sym}(W^1_p(\mathbb{R}) \times W^s_p(\mathbb{R})^{n-1}, L(W^{s-1}_{p}(\mathbb{R}), L_p(\mathbb{R}))))$.

**Proof.** See [1, Lemma 2] for the proof of (i), [1, Lemma 5] for the proof of (ii), and [1, Lemma 4] for the proof of (iii). \hfill \Box

Before establishing (1.3), we prove in Lemma 2.2 below that $B_{n,m}$ maps into $W^1_p(\mathbb{R})$ provided that its arguments are more regular. The proof uses the following algebraic property

$$(B_{n,m}(\bar{a}_1, \ldots, \bar{a}_m) - B_{n,m}(a_1, \ldots, a_m))[b_1, \ldots, b_n, \vartheta] = \sum_{i=1}^{m} B_{n+2,m+1}(\bar{a}_1, \ldots, \bar{a}_i, a_i, \ldots, a_m)[b_1, \ldots, b_n, a_i + \bar{a}_i, a_i - \bar{a}_i, \vartheta]. \tag{2.5}$$

**Lemma 2.2.** Let $n, m \in \mathbb{N}$, $a_1, \ldots, a_m, b_1, \ldots, b_n \in W^2_p(\mathbb{R})$, and $\vartheta \in W^1_p(\mathbb{R})$ be given. The function $\varphi := B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \vartheta]$ belongs then to $W^1_p(\mathbb{R})$ and

$$\varphi'(x) = \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{\partial (x - y)}{\partial x} \left( \frac{\delta(x - y)}{y} \prod_{i=1}^{n} \left[ \delta(x,y)[b_i]/y \right]^{2} \prod_{i=1}^{m} \left[ 1 + \left[ \delta(x,y)[a_i]/y \right]^{2} \right] \right) dy. \tag{2.6}$$

**Proof.** Recalling that the generator of the $C_0$-group $\{\tau_{\xi}\xi \in \mathbb{R} \subset L(W^s_p(\mathbb{R}))$, $s \geq 0$, is the linear operator $[f \mapsto f] \in L(W^{s+1}_p(\mathbb{R}), W^s_p(\mathbb{R}))$, it suffices to show that $D\xi \varphi := (\tau_{\xi \varphi} - \varphi)/\xi$ converges
in $L_p(\mathbb{R})$ in the limit $\xi \to 0$. To this end we infer from (2.5) that
\[
D_\xi \varphi = \sum_{i=1}^{n} B_{n,m}(\tau_\xi a_1, \ldots, \tau_\xi a_m)[b_1, \ldots, b_{i-1}, D_\xi b_i, \tau_\xi b_{i+1}, \ldots, \tau_\xi b_n, \tau_\xi \vartheta] \\
+ B_{n,m}(\tau_\xi a_1, \ldots, \tau_\xi a_m)[b_1, \ldots, b_n, D_\xi \vartheta] \\
- \sum_{i=1}^{m} B_{n+2,m+1}(\tau_\xi a_1, \ldots, \tau_\xi a_i, a_i, \ldots, a_m)[b_1, \ldots, b_n, D_\xi a_i, \tau_\xi a_i + a_i, \vartheta].
\]
In view of Lemma 2.1 (i) and (iii) we may pass to the limit $\xi \to 0$ in $L_2(\mathbb{R})$ in the latter equality to conclude that $\varphi \in W_p^1(\mathbb{R})$ with
\[
\varphi' = B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \vartheta] \\
+ \sum_{i=1}^{n} B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_{i-1}, b'_i, b_{i+1}, \ldots, b_n, \vartheta] \\
- 2 \sum_{i=1}^{m} B_{n+2,m+1}(a_i, a_i, \ldots, a_m)[b_1, \ldots, b_n, a'_i, a_i, \vartheta].
\]
(2.7)

The relation (2.6) follows now directly from (2.7) and the definition (2.1) of the operator $B_{n,m}$.

We are now in a position to prove (1.3).

**Proposition 2.3.** Given $f \in W_p^2(\mathbb{R})$ and $\vartheta \in W_p^1(\mathbb{R})$, the functions $\mathbb{A}(f)^*[\vartheta]$ and $\mathbb{B}(f)^*[\vartheta]$ belong to $W_p^1(\mathbb{R})$ and
\[
(\mathbb{A}(f)^*[\vartheta])' = -\mathbb{A}(f)[\vartheta'] \quad \text{and} \quad (\mathbb{B}(f)^*[\vartheta])' = -\mathbb{B}(f)[\vartheta'].
\]

**Proof.** The operators $\mathbb{A}(f)^*$ and $\mathbb{B}(f)^*$ are given by the formulas
\[
\mathbb{A}(f)^*[\vartheta](x) = \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{\partial(x-y)}{y} \frac{(\delta_{[x,y]} f)/y - f'(x-y)}{1 + [(\delta_{[x,y]} f)/y]^2} dy,
\]
\[
\mathbb{B}(f)^*[\vartheta](x) = -\frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{\partial(x-y)}{y} \frac{1 + f'(x-y)(\delta_{[x,y]} f)/y}{1 + [(\delta_{[x,y]} f)/y]^2} dy,
\]
(2.8)
cf. [29]. Recalling (2.1) and (2.2), we have
\[
\mathbb{A}(f)[\vartheta] = f' \mathcal{B}_{0,1}^0(f)[\vartheta] - \mathcal{B}_{1,1}^0(f)[\vartheta], \quad \mathbb{A}(f)^*[\vartheta] = \mathcal{B}_{1,1}^0(f)[\vartheta] - \mathcal{B}_{0,1}^0(f)[f'\vartheta],
\]
\[
\mathbb{B}(f)[\vartheta] = \mathcal{B}_{0,1}^0(f)[\vartheta] + f' \mathcal{B}_{1,1}^0(f)[\vartheta], \quad \mathbb{B}(f)^*[\vartheta] = -\mathcal{B}_{0,1}^0(f)[\vartheta] - \mathcal{B}_{1,1}^0(f)[f'\vartheta].
\]
(2.9)

Since $W_p^1(\mathbb{R})$ is an algebra, we immediately obtain from Lemma 2.2 that the functions in (2.9) belong all to $W_p^1(\mathbb{R})$. Taking now advantage of the identity
\[
\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\delta_{[x,y]} f - y f'(x-y)}{y^2 + (\delta_{[x,y]} f)^2} = \frac{\partial}{\partial y} \frac{-y \delta_{[x,y]} f'}{y^2 + (\delta_{[x,y]} f)^2}, \quad x, y \in \mathbb{R}, y \neq 0,
\]
Lemma 2.2, (2.9), and integration by parts lead to

\[ (\mathbb{A}(f)^* [\vartheta])'(x) = \mathbb{A}(f)^* [\vartheta]'(x) + \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \vartheta(x-y) \frac{\partial}{\partial x} \frac{\delta_{[x,y]} f - y f'(x-y)}{y^2 + (\delta_{[x,y]} f)^2} \, dy \]

\[ = \mathbb{A}(f)^* [\vartheta]'(x) - \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \vartheta'(x-y) \frac{y (\delta_{[x,y]} f')}{y^2 + (\delta_{[x,y]} f)^2} \, dy \]

\[ = \mathbb{A}(f)^* [\vartheta]'(x) - \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \vartheta'(x-y) \frac{y (\delta_{[x,y]} f')}{y^2 + (\delta_{[x,y]} f)^2} \, dy \]

\[ = (\mathbb{A}(f)^* [\vartheta] - f'^* B_{0,1}^0 (f)[\vartheta] + B_{0,1}^0 (f)[f' \vartheta])(x) \]

\[ = -\mathbb{A}(f)[\vartheta]'(x) \]

for almost all \( x \in \mathbb{R} \). Arguing similarly, we obtain in view of the relation

\[ \frac{\partial}{\partial x} \frac{y + f'(x-y)(\delta_{[x,y]} f)}{y^2 + (\delta_{[x,y]} f)^2} = \frac{\partial}{\partial y} \frac{y + f'(x-y)(\delta_{[x,y]} f)}{y^2 + (\delta_{[x,y]} f)^2}, \quad x, y \in \mathbb{R}, \ y \neq 0, \]

that

\[ (\mathbb{B}(f)^* [\vartheta])'(x) = \mathbb{B}(f)^* [\vartheta]'(x) - \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \vartheta(x-y) \frac{\partial}{\partial x} \frac{y + f'(x-y)(\delta_{[x,y]} f)}{y^2 + (\delta_{[x,y]} f)^2} \, dy \]

\[ = \mathbb{B}(f)^* [\vartheta]'(x) - \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \vartheta'(x-y) \frac{y (\delta_{[x,y]} f)(\delta_{[x,y]} f')}{y^2 + (\delta_{[x,y]} f)^2} \, dy \]

\[ = \mathbb{B}(f)^* [\vartheta]'(x) - \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \vartheta'(x-y) \frac{\delta_{[x,y]} f (\delta_{[x,y]} f')}{y^2 + (\delta_{[x,y]} f)^2} \, dy \]

\[ = (\mathbb{B}(f)^* [\vartheta] - f'^* B_{0,1}^0 (f)[\vartheta] + B_{1,1}^0 (f)[f' \vartheta])(x) \]

\[ = -\mathbb{B}(f)[\vartheta]'(x) \]

for almost all \( x \in \mathbb{R} \). This completes the proof. \( \square \)

3. On the Spectrum of \( \mathbb{A}(f)^* \) and the Equivalent Formulation (1.4) of (1.1)

The core of the analysis in this section addresses the invertibility of the operator \( \lambda - \mathbb{A}(f)^* \) with \( \lambda \in \mathbb{R} \setminus (-1, 1) \). This property is used below when establishing the equivalence of the two formulations (1.1) and (1.4), but also in Section 4 when formulating (1.4) as an evolution problem for \( f \).

Let \( p \in (1, \infty) \) and \( s \in (1 + 1/p, 2) \) be fixed in the following. We provide three invertibility results. Given \( f \in W_p^s(\mathbb{R}) \), we show that \( \lambda - \mathbb{A}(f)^* \) belongs to \( \text{Isom}(L_p(\mathbb{R})) \) and \( \text{Isom}(W_p^{s-1}(\mathbb{R})) \) for each \( \lambda \in \mathbb{R} \setminus (-1, 1) \), see Theorem 3.1 and Proposition 3.4 below. Moreover, under the assumption that \( f \in W_p^3(\mathbb{R}) \), we prove additionally that \( \lambda - \mathbb{A}(f)^* \in \text{Isom}(W_p^1(\mathbb{R})) \), see Corollary 3.5.

These invertibility properties together with the the corresponding invertibility results for \( \lambda - \mathbb{A}(f) \) established in [1] immediately provide the equivalence of the formulations (1.1) and (1.4) in the setting of solutions which satisfy \( f(t) \in W_p^3(\mathbb{R}) \) for all \( t > 0 \) (our solutions all share this regularity,
see Theorem 1.1). Indeed, Corollary 3.5, the relation (1.3), [1, Theorem 3 and Theorem 4], and the fact that \( a_\mu \in (-1, 1) \) combined show that, given \( f \in W^3_p(\mathbb{R}) \), the unique solution \( \vartheta \in W^1_p(\mathbb{R}) \) to
\[
(1 + a_\mu A(f)^*)(\vartheta) = b_\mu (\sigma \kappa(f) - \Theta f)
\]
has the property that \( \varpi := \vartheta' \in L_p(\mathbb{R}) \) is the unique solution to
\[
(1 - a_\mu A(f))(\varpi) = b_\mu (\sigma \kappa(f) - \Theta f)'.
\]
The equivalence of (1.1) and (1.4) is now a direct consequence of this property and of (1.3).

It remains to establish the invertibility results mentioned above. The main step is to prove the invertibility in \( \mathcal{L}(L_p(\mathbb{R})) \), see Theorem 3.1 below. When establishing this property in \( \mathcal{L}(W^{s-1}_p(\mathbb{R})) \) and \( \mathcal{L}(W^1_p(\mathbb{R})) \) we use to a large extent this result.

**Theorem 3.1.** Given \( p \in (1, \infty) \), \( s \in (1 + 1/p, 2) \), and \( f \in W^s_p(\mathbb{R}) \), we have
\[
\lambda - A(f)^* \in \text{Isom}(L_p(\mathbb{R})) \quad \text{for all } \lambda \in \mathbb{R} \setminus (-1, 1).
\]

The proof of Theorem 3.1 is presented later on in this section as it necessitates some preparation.

In the case of a bounded Lipschitz domain, the invertibility issue for the layer potentials for Laplace’s equation has been addressed in [41]. In the unbounded graph geometry considered herein, it was recently shown in [1], by using different arguments than in [41], that, under the hypotheses of Theorem 3.1, \( \lambda - A(f) \in \text{Isom}(L_p(\mathbb{R})) \), see [1, Theorem 3 and Theorem 4]. Theorem 3.1 is in the particular case \( p \in (1, 2] \) a direct consequence of these results. Indeed, if \( p \in (1, 2] \), Sobolev’s embedding ensures that \( f \in W^s_p(\mathbb{R}) \hookrightarrow W^{s'}_{p'}(\mathbb{R}) \), where \( p' \) is the adjoint exponent to \( p \), that is \( 1/p + 1/p' = 1 \), and \( s' = s - 1/p + 1/p' \in (1 + 1/p', 2) \). [1, Theorem 3 and Theorem 4] then ensure that \( \lambda - A(f) \in \text{Isom}(L_{p'}(\mathbb{R})) \), and, by duality, we conclude that \( \lambda - A(f)^* \in \text{Isom}(L_p(\mathbb{R})) \).

This argument is clearly valid only when \( p \in (1, 2] \). For \( p > 2 \) we argue differently, the strategy of proof being similar as that of [1, Theorem 3 and Theorem 4]. To start, we choose for each \( \varepsilon \in (0, 1) \), a finite \( \varepsilon \)-localization family, that is a family
\[
\{(\pi^\varepsilon_j, x^\varepsilon_j) : -N + 1 \leq j \leq N\} \subset \text{BUC}^\infty(\mathbb{R}, [0, 1]) \times \mathbb{R},
\]
with \( N = N(\varepsilon) \in \mathbb{N} \) sufficiently large, such that \( x^\varepsilon_j \in \text{supp} \pi^\varepsilon_j, -N + 1 \leq j \leq N \), and

- \( \text{supp} \pi^\varepsilon_j \subset \{ |x| \leq \varepsilon + 1/\varepsilon \} \) is an interval of length \( \varepsilon \) for \( |j| \leq N - 1 \), \( \text{supp} \pi^\varepsilon_N \subset \{ |x| \geq 1/\varepsilon \} \);
- \( \pi^\varepsilon_j \cdot \pi^\varepsilon_l = 0 \) if \( \max\{|j|, |l|\} \leq N - 1 \) or \( |l| \leq N - 2, j = N \);
- \( \sum_{j=-N+1}^{N} (\pi^\varepsilon_j)^2 = 1 \);
- \( \| (\pi^\varepsilon_j)^{(k)} \|_\infty \leq C \varepsilon^{-k} \) for all \( k \in \mathbb{N}, -N + 1 \leq j \leq N \).

To each finite \( \varepsilon \)-localization family we associate a second family
\[
\{x^\varepsilon_j : -N + 1 \leq j \leq N\} \subset \text{BUC}^\infty(\mathbb{R}, [0, 1])
\]
with the following properties

- \( \chi_j^\varepsilon = 1 \) on \( \text{supp} \pi_j^\varepsilon \);
- \( \text{supp} \chi_j^\varepsilon \) is an interval of length \( 3\varepsilon \) and with the same midpoint as \( \text{supp} \pi_j^\varepsilon \), \( j \leq N - 1 \);
- \( \text{supp} \chi_N^\varepsilon \subset \{ |x| \geq 1/\varepsilon - \varepsilon \} \) and \( \xi + \text{supp} \pi_N^\varepsilon \subset \text{supp} \chi_N^\varepsilon \) for \( |\xi| < \varepsilon \).

As stated in [1, Lemma 9], the norm

\[
\left[ f \mapsto \sum_{j=-N+1}^{N} \| \pi_j^\varepsilon f \|_{W_p^r} \right] : W_p^r(\mathbb{R}) \to [0, \infty), \quad p \in (1, \infty), \ r \geq 0,
\]

is equivalent to the standard norm on \( W_p^r(\mathbb{R}) \).

In order to prove the injectivity of \( \lambda - A(f)^* \), we need to establish the unique solvability of the equation

\[
(\lambda - A(f)^*)[\vartheta] = 0
\]

in \( L_p(\mathbb{R}) \). Since (3.1) is equivalent to the system

\[
(\lambda - \chi_j^\varepsilon A(f)^*\pi_j^\varepsilon)[\vartheta] = \chi_j^\varepsilon(\pi_j^\varepsilon A(f)^*\vartheta - A(f)^*\pi_j^\varepsilon \vartheta), \quad -N + 1 \leq j \leq N, \tag{3.2}
\]

we next consider the operators on the right of (3.2) and prove they are all compact.

**Lemma 3.2.** Let \( p \in [2, \infty), \ s \in (1 + 1/p, 2), \) and \( f \in W^s_p(\mathbb{R}) \). Given \(-N + 1 \leq j \leq N\), the linear operator \( K_j : L_p(\mathbb{R}) \to L_p(\mathbb{R}) \) defined by

\[
K_j[\vartheta] := \chi_j^\varepsilon(\pi_j^\varepsilon A(f)^*\vartheta - A(f)^*\pi_j^\varepsilon \vartheta), \quad \vartheta \in L_p(\mathbb{R}), \tag{3.3}
\]

is compact.

**Proof.** According to the Riesz-Fréchet-Kolmogorov theorem, it suffices to show that

\[
\sup_{\|\vartheta\|_p \leq 1} \int_{\{ |x| > R \}} |K_j[\vartheta]|(x)^p \, dx \to 0 \quad \text{and} \quad \sup_{\|\vartheta\|_p \leq 1} \|\pi^\varepsilon(K_j[\vartheta]) - K_j[\vartheta]\|_p \to 0. \tag{3.4}
\]

Step 1. In order to prove the first convergence in (3.4), we set \( a_\varepsilon := \varepsilon + 1/\varepsilon \) and we note that \( \pi_N^\varepsilon = 1 \) for \( |x| > a_\varepsilon \). Let \( R > 2a_\varepsilon \). We then have \( \chi_j^\varepsilon(x) = 0 \) for all \( |x| > R \) and \( |j| \leq N - 1 \). Hence, it remains to establish the convergence for \( j = N \). Since for \( |y| < R/2 \) we have \( |x - y| \geq |x| - R/2 > a_\varepsilon \), it holds that \( \delta_{[x,y]} \pi_N^\varepsilon = 0 \) for all \( |x| > R \) and \( |y| < R/2 \), and therefore

\[
\left( \int_{\{ |x| > R \}} |K_N[\vartheta]|(x)^p \, dx \right)^{1/p} \leq \left( \int_{\{ |x| > R \}} \int_{\{ |y| > R/2 \}} |\vartheta(x - y)| \frac{\delta_{[x,y]}f - yf'(x - y)}{y^2} \delta_{[x,y]} \pi_N^\varepsilon \, dy \right)^{1/p} dx \leq T_1 + T_2 + T_3,
\]
where, in view of $|\delta_{[x,y]}^N| \leq 1$ and $||\vartheta||_p \leq 1$, we have

$$T_1 := \left( \int_{|x| > R} |f(x)|^p \left( \int_{|y| > R/2} \frac{|w(x-y)|}{y^2} dy \right)^{p} dx \right)^{1/p} \leq \frac{C}{R^{(p+1)/p}} ||f||_p \rightarrow 0,$$

$$T_2 := \left( \int_{|x| > R} \left( \int_{|y| > R/2} \frac{|(f \vartheta)(x-y)|}{y^2} dy \right)^{p} dx \right)^{1/p} \leq \int_{|y| > R/2} \frac{1}{y^2} \left( \int_{|x| > R} |(f \vartheta)(x-y)|^p dx \right)^{1/p} dy \leq \frac{C}{R} ||f||_\infty \rightarrow 0,$$

$$T_3 := \left( \int_{|x| > R} \left( \int_{|y| > R/2} \frac{|(f' \vartheta)(x-y)|}{y} |\delta_{[x,y]}^N| dy \right)^{p} dx \right)^{1/p} .$$

We used Hölder’s inequality to estimate $T_1$ and Minkowski’s integral inequality when we considered $T_2$. With respect to $T_3$ we note that, given $|x| > R$, the term $\delta_{[x,y]}^N$ can be different from 0 only if $y \in (x - \varepsilon, x + \varepsilon)$. Using Hölder’s inequality, we then get

$$T_3 \leq \left( \int_{|x| > R} \left( \int_{x-a\varepsilon}^{x+a\varepsilon} \frac{|(f' \vartheta)(x-y)|}{y} dy \right)^{p} dx \right)^{1/p}$$

$$\leq ||f'\vartheta||_p \left( \int_{|x| > R} \left( \int_{x-a\varepsilon}^{x+a\varepsilon} \frac{1}{y} dy \right)^{p/p'} dx \right)^{1/p}$$

$$\leq C ||f'||_\infty \left( \int_{|x| > R} \left| \frac{1}{x-a\varepsilon} - \frac{1}{x+a\varepsilon} \right|^{p/p'} dx \right)^{1/p} ,$$

where, as before, $p' \in (1,2]$ is the adjoint exponent to $p$. Since $p' - 1 \in (0,1]$, the inequality $a^{*'} - b^{*'} \leq (a - b)^r$, which holds for all $0 < b < a$ and $r \in (0,1)$, together with the estimate $|x \pm a\varepsilon| \geq |x|/2$ for $|x| > R$ leads us to

$$T_3 \leq C ||f'||_\infty \left( \int_{|x| > R} \frac{1}{x^2} dx \right)^{1/p} = \frac{C}{R} ||f'||_\infty \rightarrow 0 .$$

Step 2. We now establish the second convergence in (3.4). Let therefore $||\vartheta||_p \leq 1$ and $\xi \in \mathbb{R}$ be arbitrary with $|\xi| < \min\{\varepsilon, 1/2\}$. Given $-N + 1 \leq j \leq N$, we have

$$||\tau^\xi(J_j[\vartheta]) - K_j[\vartheta]||_p \leq \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3 ,$$

where, after a suitable change of variables, the right side of the latter inequality may be expressed as

$$\tilde{T}_1 := ||\tau^\xi x_j^\varepsilon - x_j^\varepsilon ||_\infty ||\tau^\xi \Lambda(f)^* [\vartheta] - \Lambda(f)^* [\tau^\xi \vartheta]||_p \leq C|\xi| \rightarrow 0,$$

$$\tilde{T}_2 := \left( \int_\mathbb{R} \left( \int_\mathbb{R} K_1^\varepsilon(x,y,\xi)(f' \vartheta)(x-y) dy \right)^p dx \right)^{1/p},$$

$$\tilde{T}_3 := \left( \int_\mathbb{R} \left( \int_\mathbb{R} K_2^\varepsilon(x,y,\xi) \vartheta(x-y) dy \right)^p dx \right)^{1/p} .$$
with
\[
K_1^\varepsilon(x, y, \xi) := \frac{(\delta_{[x+\xi, y+\xi]} \pi_j^\varepsilon)/(y + \xi)}{1 + [(\delta_{[x+\xi, y+\xi]} f)/(y + \xi)]^2} - \frac{(\delta_{[x, y]} \pi_j^\varepsilon)/y}{1 + [(\delta_{[x, y]} f)/y]^2},
\]
\[
K_2^\varepsilon(x, y, \xi) := \frac{[(\delta_{[x+\xi, y+\xi]} f)/(y + \xi)] \cdot [(\delta_{[x+\xi+\xi]} \pi_j^\varepsilon)/(y + \xi)] - [(\delta_{[x, y]} f)/y] \cdot [(\delta_{[x, y]} \pi_j^\varepsilon)/y]}{1 + [(\delta_{[x+\xi, y+\xi]} f)/(y + \xi)]^2}.
\]

Let now \( a_\varepsilon := \varepsilon + 1/\varepsilon + 1/2 \). Then, \( \sup \pi_j^\varepsilon, |j| \leq N - 1 \), as well as \( \sup (1 - \pi_j^\varepsilon) \) are subintervals of \( \{|x| \leq a_\varepsilon - 1/2\} \). Since \( |\xi| < 1/2 \) and \( W^s_p(\mathbb{R}) \hookrightarrow \text{BUC}^{s-1/p}(\mathbb{R}) \), we have
\[
\left|\frac{\delta_{[x+\xi, y+\xi]} \pi_j^\varepsilon}{y + \xi} - \frac{\delta_{[x, y]} \pi_j^\varepsilon}{y}\right| \leq C|\xi|\left[\left(\frac{1}{|y|}\mathbf{1}_{\{|x| \leq a_\varepsilon\}}(x) + \frac{1}{y^2}\right)\mathbf{1}_{\{|y| > 1\}}(y) + \mathbf{1}_{\{|y| < 1\}}(y)\right],
\]
\[
\left|\frac{\delta_{[x+\xi, y+\xi]} f}{y + \xi} - \frac{\delta_{[x, y]} f}{y}\right| \leq C|\xi|^{s-1-1/p}\left[\frac{1}{|y|}\mathbf{1}_{\{|y| > 1\}}(y) + \mathbf{1}_{\{|y| < 1\}}(y)\right)
\]
for \( x, y \in \mathbb{R} \) and \( -N + 1 \leq j \leq N \). The relations (3.5) lead us to
\[
|K_1^\varepsilon(x, y, \xi)| \leq C|\xi|^{s-1-1/p}\left[\frac{1}{|y|}\mathbf{1}_{\{|x| \leq a_\varepsilon\}}(x)\mathbf{1}_{\{|y| > 1\}}(y) + \frac{1}{y^2}\mathbf{1}_{\{|y| > 1\}}(y) + \mathbf{1}_{\{|y| < 1\}}(y)\right],
\]
\[
|K_2^\varepsilon(x, y, \xi)| \leq C|\xi|^{s-1-1/p}\left[\frac{1}{y^2}\mathbf{1}_{\{|y| > 1\}}(y) + \mathbf{1}_{\{|y| < 1\}}(y)\right].
\]

Hölder’s inequality and Minkowski’s inequality lead, in view of the latter estimates, to
\[
\left(\int_{\{|x| \leq a_\varepsilon\}} \left(\int_{\{|y| > 1\}} \left|\frac{f^r\vartheta(x - y)}{|y|}\right| dy\right)^p dx\right)^{1/p} \leq \|f^r\vartheta\|_p \left(\int_{\{|y| > 1\}} \frac{1}{|y|^p} dy\right)^{1/p} (2a_\varepsilon)^{1/p} \leq C \|f^r\|_\infty,
\]
\[
\left(\int_\mathbb{R} \left(\int_{\{|y| > 1\}} \frac{1}{y^2}(f^{rk}\vartheta)(x - y) dy\right)^p dx\right)^{1/p} \leq \|f^{rk}\vartheta\|_p \left(\int_{\{|y| > 1\}} \frac{1}{y^2} dy\right)^{1/p} \leq C \|f^{rk}\|_\infty, \quad k = 0, 1,
\]
\[
\left(\int_\mathbb{R} \left(\int_{\{|y| < 1\}} |f^{rk}\vartheta(x - y)| dy\right)^p dx\right)^{1/p} \leq \|f^{rk}\vartheta\|_p \left(\int_{\{|y| < 1\}} 1 dy\right)^{1/p} \leq C \|f^{rk}\|_\infty, \quad k = 0, 1,
\]
and we conclude that
\[
\tilde{T}_2 + \tilde{T}_3 \leq C|\xi|^{s-1-1/p} \rightarrow 0, \quad \xi \to 0,
\]
which completes our arguments. \( \square \)

In the next lemma we consider the left side of (3.2) and prove that, if \( \varepsilon \) is chosen sufficiently small, the operator on the left of (3.2) is invertible in \( \mathcal{L}(L_p(\mathbb{R})) \) for all \( \lambda \in \mathbb{R} \setminus (-1, 1) \).

**Lemma 3.3.** Let \( p \in [2, \infty), s \in (1 + 1/p, 2), \) and \( f \in W^s_p(\mathbb{R}) \). If \( \varepsilon \) is sufficiently small, then
\[
\|\chi_j^\varepsilon A(f)\chi_j^\varepsilon\|_{\mathcal{L}(L_p(\mathbb{R}))} < 1 \quad \text{for all } -N + 1 \leq j \leq N.
\]
Hence, if \( j \) and, using Minkowski’s inequality, we get

\[
\|\chi_j^N A(f)[\chi_j^N \vartheta]\|_p \leq \left( \int_R \left( \int_R (\chi_j^N(x) (x-y) f(x-y) \delta(x,y) f(y) - f'(x-y) ) y^2 + \left( \delta(x,y) f(y) \right)^2 \right)^{p-1} dy \right)^{1/p} dx \right)^{1/p} \]

\[
\leq [f']_{s-1-1/p} \int_R |y|^{s-2-1/p} \left( \int_{\text{supp } \chi_j^N} (\chi_j^N(x) (x-y) \vartheta(x-y))^{p=1} dy \right)^{1/p} dx \]

\[
\leq C \|f\|_{R^p} \|\vartheta\|_p \int_{\text{supp } \chi_j^N} |y|^{s-2-1/p} dy \]

\[
\leq C \|f\|_{R^p} \|\vartheta\|_p \varepsilon^{s-1-1/p}. \]

Hence, if \( \varepsilon \) is sufficiently small, then \( \|\chi_j^N A(f)[\chi_j^N \vartheta]\|_{L_p(\mathbb{R})} < 1 \).

Step 2. Let now \( j = N \). Since \( \text{supp } \chi_j^N \subset \{|x| \geq 1/\varepsilon - 1\} \) we have \( \|\chi_j^N A(f)[\chi_j^N \vartheta]\|_p \leq T_1 + T_2 \), where

\[
T_1 := \left( \int_{\{|x| > 1/\varepsilon - 1\}} \left( \int_{\{|y| < 1\}} \chi_j^N(x) \delta(x,y) f(y) - f'(x-y) (\chi_j^N \vartheta) \right)^{p=1} \right)^{1/p} \]

\[
T_2 := \left( \int_{\{|x| > 1/\varepsilon - 1\}} \left( \int_{\{|y| > 1\}} \chi_j^N(x) \delta(x,y) f(y) - f'(x-y) (\chi_j^N \vartheta) \right)^{p=1} \right)^{1/p}. \]

With respect to \( T_1 \) we note that for \( |x| > 1/\varepsilon - 1 \) and \( |y| < 1 \) the mean value theorem implies that

\[
|\delta(x,y) f(y) - f'(x-y) | \leq 2\|f'\|_{L_{\infty}(\{|x| > 1/\varepsilon - 2\})} f'(x-y) |y|^{s/2+1/2-1/2p}, \]

and, using Minkowski’s integral inequality, we get

\[
T_1 \leq 2\|f'\|_{L_{\infty}(\{|x| > 1/\varepsilon - 2\})} f'(x-y) \left( \int_{\{|y| < 1\}} \right)^{1/p} dx \right)^{1/p} \]

\[
\leq C \|f\|_{R^p} \|f'\|_{L_{\infty}(\{|x| > 1/\varepsilon - 2\})} \int_{\{|y| < 1\}} \left( \int_{\{|y| < 1\}} \right)^{1/p} dx \right)^{1/p} \]

\[
\leq C \|f\|_{R^p} \|f'\|_{L_{\infty}(\{|x| > 1/\varepsilon - 2\})} \|\vartheta\|_p. \]

In order to estimate \( T_2 \), we note that \( T_2 \leq T_{2a} + T_{2b} + T_{2c} \), where

\[
T_{2a} := \left( \int_{\{|x| > 1/\varepsilon - 1\}} |f(x)|^{p=1} \left( \int_{\{|y| > 1\}} |\vartheta(x-y)| dy \right)^{p=1} \right)^{1/p} \leq C \|f\|_{L_p(\{|x| > 1/\varepsilon - 1\})} \|\vartheta\|_p, \]

\[
T_{2b} := \left( \int_{\{|x| > 1/\varepsilon - 1\}} |f\chi_j^N \vartheta(x-y)| \left( \int_{\{|y| > 1\}} \frac{1}{y^{3/2+1/2p-s/2}} \right)^{p=1} \right)^{1/p} \]

\[
\leq C \|f\|_{L_{\infty}(\{|x| > 1/\varepsilon - 1\})} \|\vartheta\|_p, \]

\[
T_{2c} := \left( \int_{\{|x| > 1/\varepsilon - 1\}} \left( \int_{\{|y| > 1\}} \frac{1}{y^{3/2+1/2p-s/2}} \right)^{p=1} \right)^{1/p} \]

\[
\leq C \|f\|_{L_{\infty}(\{|x| > 1/\varepsilon - 1\})} \|\vartheta\|_p. \]
We used Hölder’s inequality to estimate $T_{2a}$ and Minkowski’s integral inequality for $T_{2b}$. With respect to $T_{2c}$, a simple algebraic manipulation reveals that

$$T_{2c} \leq \left( \int_{\mathbb{R}} \left( \int_{\{|y|>1\}} \left( \frac{\delta_{[x,y]} f}{y^2} \frac{f' \chi^\varepsilon_N \vartheta}{y} (x-y) dy \right) dx \right)^p dx \right)^{1/p} + \pi \| H_1[f' \chi^\varepsilon_N \vartheta] \|_p,$$

where $H_1$ denotes the truncated Hilbert transform

$$H_1[\vartheta](x) = \frac{1}{\pi} \int_{\{|y|>1\}} \frac{\vartheta(x-y)}{y} dy.$$

Similarly as in the case of $T_{2b}$ we get

$$\left( \int_{\mathbb{R}} \left( \int_{\{|y|>1\}} \left( \frac{\delta_{[x,y]} f}{y^2} \frac{f' \chi^\varepsilon_N \vartheta}{y} (x-y) dy \right) dx \right)^p dx \right)^{1/p} \leq C \| f \|_L^p \| f' \|_{L_{\infty}([|x|>1/\varepsilon-1])} \| \vartheta \|_p.$$

Moreover, it is well-known that $H_1 \in \mathcal{L}(L_p(\mathbb{R}))$, hence

$$\| H_1[f' \chi^\varepsilon_N \vartheta] \|_p \leq C \| f' \chi^\varepsilon_N \vartheta \|_p \leq C \| f \|_{L_{\infty}([|x|>1/\varepsilon-1])} \| \vartheta \|_p.$$

Due to the fact that $f \in W_p^s(\mathbb{R})$, $s > 1 + 1/p$, for $\varepsilon \to 0$ we have $\| f^{(k)} \|_{L_{\infty}([|x|>1/\varepsilon-2])} \to 0$, $k = 0, 1$, and $\| f \|_{L_p([|x|>1/\varepsilon-1])} \to 0$. The estimates established above now ensure that the claim holds true also for $j = N$.

We are now in a position to establish the aforementioned invertibility property in $\mathcal{L}(L_p(\mathbb{R}))$.

**Proof of Theorem 3.1.** As mentioned in the discussion following Theorem 3.1, it remains to consider the case when $p \in (2, \infty)$. We devise the proof in two steps.

Step 1. In this step we show that $\lambda - \mathbb{A}(f)^*$ is injective. To start, we fix for each $\varepsilon \in (0, 1)$ a function $a_\varepsilon \in \text{BUC}^\infty(\mathbb{R}, [0, 1])$ such that $a_\varepsilon = 0$ in $\{ |x| < 1/\varepsilon \}$, $a_\varepsilon = 1$ in $\{ |x| > 1/\varepsilon + 1 \}$, and $\| a'_\varepsilon \|_\infty \leq 2$. Recalling (2.9), we may decompose $\mathbb{A}(f)^*$ as the sum

$$\mathbb{A}(f)^* = A_1^\varepsilon + A_2^\varepsilon,$$

where

$$A_1^\varepsilon := B_{1,1}(f)[a_\varepsilon f, \cdot] - B_{0,1}(f)[a_\varepsilon f]^\cdot,$$

$$A_2^\varepsilon := B_{1,1}(f)[(1-a_\varepsilon) f, \cdot] - B_{0,1}(f)[((1-a_\varepsilon)f)^\cdot].$$

Lemma 2.1 (i) ensures that $A_i^\varepsilon \in \mathcal{L}(L_q(\mathbb{R}))$, $i = 1, 2$, for all $q \in (1, \infty)$ and

$$\| A_1^\varepsilon \|_{\mathcal{L}(L_q(\mathbb{R}))} \leq C \| (a_\varepsilon f)' \|_\infty \leq C \| f \|_{W_{\infty}^1([|x|>1/\varepsilon])} \to 0.$$

Consequently, for sufficiently small $\varepsilon$ we have

$$\lambda - A_1^\varepsilon \in \text{Isom}(L_2(\mathbb{R})) \cap \text{Isom}(L_p(\mathbb{R})) \quad \text{for all } \lambda \in \mathbb{R} \setminus (-1, 1). \tag{3.6}$$

Let now $\lambda \in \mathbb{R} \setminus (-1, 1)$ be fixed and let $\vartheta \in L_p(\mathbb{R})$ be a solution of $(\lambda - \mathbb{A}(f)^*)[\vartheta] = 0$, or equivalently

$$(\lambda - A_1^\varepsilon)[\vartheta] = A_2^\varepsilon[\vartheta],$$

where $\varepsilon \in (0, 1/4)$ is fixed such that (3.6) holds true. Our goal is to prove that $\vartheta \in L_2(\mathbb{R})$, which in view of (3.6) is equivalent to showing that $A_2^\varepsilon[\vartheta] \in L_2(\mathbb{R})$. Then, since $\lambda - \mathbb{A}(f)^* \in \text{Isom}(L_2(\mathbb{R}))$ for all $f \in \text{BUC}^1(\mathbb{R})$ and $\lambda \in \mathbb{R} \setminus (-1, 1)$, see the arguments in the proof of of [29, Theorem 3.5], we conclude that $\vartheta = 0$, hence $\lambda - \mathbb{A}(f)^* \in \mathcal{L}(L_p(\mathbb{R}))$ is injective.
In order to show that $A_2^\varepsilon[\vartheta] \in L_2(\mathbb{R})$ we write

$$A_2^\varepsilon[\vartheta] = A_2^\varepsilon[a_2 \vartheta] + A_2^\varepsilon[(1 - a_2) \vartheta],$$

where, due to the fact that $(1 - a_2) = 0$ in $\{|x| > 1/\varepsilon^2 + 1\}$ and $\vartheta \in L_p(\mathbb{R})$ with $p \in (2, \infty)$, we have $(1 - a_2)f \vartheta \in L_2(\mathbb{R})$. Lemma 2.1 (i) now yields $A_2^\varepsilon[(1 - a_2) \vartheta] \in L_2(\mathbb{R})$, and it remains to prove that $A_2^\varepsilon[a_2 \vartheta] \in L_2(\mathbb{R})$. To this end we set $g_\varepsilon := (1 - a_\varepsilon)f$ and note that

$$\|A_2^\varepsilon[a_2 \vartheta]\|_2 \leq \left( \int_\mathbb{R} \left( \int_{|y| > 1} \frac{(a_\varepsilon \vartheta)(x - y)}{y} \left( \delta_{|[x,y]|} g_\varepsilon |y| - g_\varepsilon(x - y) \right)^2 dy \right)^2 dx \right)^{1/2}.$$

Indeed, $\varepsilon \in (0, 1/4)$ implies that $1/\varepsilon^2 - 1 > 1/\varepsilon + 2$. Consequently, if $x \in \mathbb{R}$, then either $|x| > 1/\varepsilon + 2$ or $|x| < 1/\varepsilon^2 - 1$. Moreover, in the inner integral we integrate only over the set $\{|y| > 1\}$, since the integrand is zero on $\{|y| < 1\}$. Assume first that $|x| < 1/\varepsilon^2 - 1$. Then $|x - y| < 1/\varepsilon^2$ and by the definition of $a_2$ we get that $a_2(x - y) = 0$ and the integrand is zero. In the case when $|x| > 1/\varepsilon + 2$ we obtain that $|x - y| > 1/\varepsilon + 1$, hence $g_\varepsilon(x) = g_\varepsilon(x - y) = g_\varepsilon(x - y) = 0$ and the integrand is again zero.

It thus remains to estimate the latter integral. Because of the relation $(1 - a_\varepsilon)^{(k)} a_\varepsilon = 0$, $k = 0, 1$, we have that $g_\varepsilon^{(k)} a_2 = 0$, $k = 0, 1$, and together with Hölder’s inequality we get

$$\|A_2^\varepsilon[\vartheta]\|_2 \leq \left( \int_\mathbb{R} |g_\varepsilon(x)|^2 \left( \int_{|y| > 1} \frac{|(a_\varepsilon \vartheta)(x - y)|}{y} dy \right)^2 dx \right)^{1/2} \leq C \|a_\varepsilon \vartheta\|_p \|g_\varepsilon\|_2 \leq C \|\vartheta\|_p \|g_\varepsilon\|_2.$$

Due to the fact that $g_\varepsilon \in L_p(\mathbb{R})$ with $p \in (2, \infty)$ satisfies $g_\varepsilon = 0$ in $\{|x| > 1/\varepsilon + 1\}$, we deduce that $g_\varepsilon \in L_2(\mathbb{R})$, and therewith we may conclude $A_2^\varepsilon[\vartheta] \in L_2(\mathbb{R})$. Summarizing, we have established the injectivity of $\lambda - \mathcal{A}(f)^*$.

Step 2. We now prove there exists a constant $C_0 > 0$ such that

$$\| (\lambda - \mathcal{A}(f)^*)[\vartheta] \|_p \geq C_0 \|\vartheta\|_p \quad \text{for all } \vartheta \in L_p(\mathbb{R}) \text{ and } \lambda \in \mathbb{R} \setminus (-1, 1). \quad (3.7)$$

To this end, we argue by contradiction and assume that the claim (3.7) is false. Then, there exists a sequence $(\vartheta_n)_n \subset L_p(\mathbb{R})$ and a bounded sequence $(\lambda_n)_n \subset \mathbb{R}$ such that $|\lambda_n| \geq 1$, $\|\vartheta_n\|_p = 1$ for all $n \in \mathbb{N}$, and with $(\lambda_n - \mathcal{A}(f)^*[\vartheta_n] =: \varphi_n \to 0$ in $L_p(\mathbb{R})$. After possibly extracting some subsequences, we may assume that $\lambda_n \to \lambda$ in $\mathbb{R}$ and $\vartheta_n \to \vartheta$ in $L_p(\mathbb{R})$. Given $h \in L_p'(\mathbb{R})$, it holds, in view of $\mathcal{A}^*(f) \in \mathcal{L}(L_{p'}(\mathbb{R}))$, cf. (1.2) and Lemma 2.1 (i), that

$$\langle (\lambda - \mathcal{A}(f)^*)[\vartheta], h \rangle_{L_2} = \langle \vartheta, (\lambda - \mathcal{A}(f)[h]) \rangle_{L_2} = \lim_{n \to \infty} \langle \vartheta, (\lambda_n - \mathcal{A}(f)[h]) \rangle_{L_2} = \lim_{n \to \infty} \langle \varphi_n, h \rangle_{L_2} = 0.$$

Hence, $(\lambda - \mathcal{A}(f)^*)[\vartheta] = 0$, and the result established in Step 1 implies that $\vartheta = 0$.

Since $|\lambda_n| \geq 1$ for all $n \in \mathbb{N}$, we may choose in virtue of Lemma 3.3 a constant $\varepsilon > 0$ such that $(\lambda_n - \chi_j^\varepsilon \mathcal{A}(f)^* \chi_j^\varepsilon), (\lambda - \chi_j^\varepsilon \mathcal{A}(f)^* \chi_j^\varepsilon) \in \text{Isom}(L_p(\mathbb{R}))$ for all $-N + 1 \leq j \leq N$ and all $n \in \mathbb{N}$. Owing to

$$1 = \|\vartheta_n\|_p \leq \sum_{j=-N+1}^{N} \|\pi_j^\varepsilon \vartheta_n\|_p,$$

there exists an integer $j_\varepsilon$ with $-N + 1 \leq j_\varepsilon \leq N$ and a subsequence of $(\vartheta_n)_n$ (not relabeled) such that

$$\|\pi_j^\varepsilon \vartheta_n\|_p \geq (2N)^{-1} \quad \text{for all } n \in \mathbb{N}. \quad (3.8)$$
Recalling the definition (3.3) of the operators $K_j$, $-N+1 \leq j \leq N$, we deduce from the identity $(\lambda_n - \mathcal{A}(f)^* [\vartheta_n]) = \varphi_n$ the following formula

$$
\pi_j [\vartheta_n] = (\lambda_n - \chi_j A(f)^* \chi_j)^{-1} [K_j [\vartheta_n]] + (\lambda_n - \chi_j A(f)^* \chi_j)^{-1} [\pi_j [\vartheta_n]]
$$

(3.9)

for all $n \in \mathbb{N}$. Since $K_j^*$ is compact, cf. Lemma 3.2, it follows that $K_j^* [\vartheta_n] \to 0$ in $L_p(\mathbb{R})$. Also using the convergences $\lambda_n - \chi_j A(f)^* \chi_j \to \lambda - \chi_j A(f)^* \chi_j$ in $L(L_p(\mathbb{R}))$ and $\varphi_n \to 0$ in $L_p(\mathbb{R})$, we now infer from (3.9) that $\pi_j [\vartheta_n] \to 0$ in $L_p(\mathbb{R})$, which contradicts (3.8). This proves (3.7).

Since $\lambda - A(f)^* \in \text{Isom}(L_p(\mathbb{R}))$ for all $|\lambda| > \|A(f)^*\|_{L(L_p(\mathbb{R}))}$, the estimate (3.7) and the method of continuity, cf. [3, Proposition I.1.1.1], imply that $\lambda - A(f)^* \in \text{Isom}(L_p(\mathbb{R}))$ for all $\lambda \in \mathbb{R} \setminus (-1, 1)$, and the proof is complete. \hfill \Box

Before proceeding, let us emphasize, as a straightforward consequence of Theorem 3.1, that the estimate (3.7) holds for each $p \in (1, \infty)$ (possibly with a different constant $C_0$). Using this estimate, we now provide the following invertibility result.

**Proposition 3.4.** Given $p \in (1, \infty)$, $s \in (1 + 1/p, 2)$, and $f \in W^s_p(\mathbb{R})$, we have

$$
\lambda - A(f)^* \in \text{Isom}(W^{s-1}_p(\mathbb{R})) \quad \text{for all } \lambda \in \mathbb{R} \setminus (-1, 1).
$$

**Proof.** Given $\vartheta \in W^{s-1}_p(\mathbb{R})$ and $\lambda \in \mathbb{R} \setminus (-1, 1)$, we set $\varphi := (\lambda - A(f)^*)[\vartheta]$. The formula (2.9), Lemma 2.1 (ii), and the algebra property of $W^{s-1}_p(\mathbb{R})$ entail that $\varphi \in W^{s-1}_p(\mathbb{R})$. Letting $C_0 > 0$ denote the constant in (3.7) and recalling (1.7), we have

$$
[\vartheta]^{p}_{W^{s-1}_p} = \int_{\mathbb{R}} \frac{|\vartheta - \tau \xi \vartheta|^p}{|\xi|^{1+(s-1)p}} d\xi \leq C_0 \int_{\mathbb{R}} \frac{|(\lambda - A(f)^*)[\vartheta - \tau \xi \vartheta]|^p}{|\xi|^{1+(s-1)p}} d\xi
$$

$$
\leq 2^p C_0 \int_{\mathbb{R}} \frac{|\varphi - \tau \xi \varphi|^p}{|\xi|^{1+(s-1)p}} d\xi + 2^p C_0 \int_{\mathbb{R}} \frac{|(A(f)^* - A(\tau \xi f)^*)[\tau \xi \vartheta]|^p}{|\xi|^{1+(s-1)p}} d\xi
$$

$$
= 2^p C_0 ([\lambda - A(f)^*][\vartheta])^{p}_{W^{s-1}_p} + 2^p C_0 \int_{\mathbb{R}} \frac{|(A(f)^* - A(\tau \xi f)^*)[\tau \xi \vartheta]|^p}{|\xi|^{1+(s-1)p}} d\xi.
$$

(3.10)

In order to estimate the integrand in last term of the latter inequality we infer from (2.9) that

$$
\|A(f)^* - A(\tau \xi f)^*)[\tau \xi \vartheta] \|_p \leq \|(B_{1,1}^0(f) - B_{1,1}^0(\tau \xi f))[\tau \xi \vartheta] \|_p
$$

$$
+ \|B_{0,1}(f)[f' \tau \xi \vartheta] - B_{0,1}(\tau \xi f)[\tau \xi (f' \vartheta)] \|_p.
$$

Let now $s' \in (1 + 1/p, s)$ be chosen. The relation (2.5), Lemma 2.1 (i) and (iii) (with $s = s'$), and the embedding $W^{s'}_p(\mathbb{R}) \hookrightarrow \text{BUC}(\mathbb{R})$ lead us to

$$
\|(B_{1,1}^0(f) - B_{1,1}^0(\tau \xi f))[\tau \xi \vartheta] \|_p \leq \|B_{1,1}(\tau \xi f)[f - \tau \xi f, \tau \xi \vartheta] \|_p
$$

$$
+ \|B_{3,2}(f, \tau \xi f)[f + \tau \xi f, f - \tau \xi f, f, \tau \xi \vartheta] \|_p
$$

$$
\leq C \|f' - \tau \xi f'\|^p \|\vartheta\|_{W^{s'}_p}.
$$

Therefore, the proposition is proved.

\hfill \Box
and
\[
\|B_{0,1}(f)[f'\tau_\xi \vartheta] - B_{0,1}(\tau_\xi f)[\tau_\xi (f'\vartheta)]\|_p \\
\leq \|B_{0,1}(\tau_\xi f)[(f' - \tau_\xi f')\tau_\xi \vartheta]\|_p + \|B_{2,2}(f, \tau_\xi f)[f + \tau_\xi f, f - \tau_\xi f, f'\tau_\xi \vartheta]\|_p \\
\leq C\|f' - \tau_\xi f'\|_p \|\vartheta\|_{W^{s-1}_p}
\]
for all \(\xi \in \mathbb{R}\), where \(C > 0\) depends only on \(f\). These estimates together with (3.10) and (3.7) imply there exists a constant \(C_1\) (which depends only on \(f\)) such that
\[
\|\vartheta\|_{W^{s-1}_p} \leq C_1((\lambda - \Lambda(f)^*)[\vartheta]\|_{W^{s-1}_p} + \|\vartheta\|_{W^{s-1}_p}), \quad \vartheta \in W^{s-1}_p(\mathbb{R}).
\] (3.11)
The interpolation property (1.8) and Young’s inequality now imply there exists \(C > 0\) with
\[
\|\vartheta\|_{W^{s-1}_p} \leq \frac{1}{2C_1}\|\vartheta\|_{W^{s-1}_p} + C\|\vartheta\|_p, \quad \vartheta \in W^{s-1}_p(\mathbb{R}).
\]
In view of this estimate and relying also on (3.11) and (3.7), we may find a constant \(C\) such that
\[
\|\vartheta\|_{W^{s-1}_p} \leq C((\lambda - \Lambda(f)^*)[\vartheta]\|_{W^{s-1}_p}, \quad \vartheta \in W^{s-1}_p(\mathbb{R}) \text{ and } \lambda \in \mathbb{R} \setminus (-1, 1),
\]
The method of continuity [3, Proposition I.1.1.1] leads now, similarly as in the proof of Theorem 3.1, to the desired conclusion. \(\Box\)

As a final result of this section we establish the following corollary.

**Corollary 3.5.** Given \(f \in W^{2}_p(\mathbb{R})\), we have \(\lambda - \Lambda(f)^* \in \text{Isom}(W^{1}_p(\mathbb{R}))\) for all \(\lambda \in \mathbb{R} \setminus (-1, 1)\).

**Proof.** Given \(\vartheta \in W^{1}_p(\mathbb{R})\), we infer from Proposition 2.3 that \(\Lambda(f)^*[\vartheta]\) belongs to \(W^{2}_p(\mathbb{R})\) and its derivative satisfies \((\Lambda(f)^*[\vartheta])' = -\Lambda(f)[\vartheta']\). This property, [1, Theorem 3 and Theorem 4], and (3.7) imply there exists a constant \(C > 0\) such that
\[
||\vartheta||_{W^{s-1}_p}^{p} = ||\vartheta||_{W^{2}_p}^{p} = \|((\lambda - \Lambda(f)^*\vartheta)\|_{W^{s-1}_p}^{p} + ||(\lambda - \Lambda(f)^*\vartheta')\|_{W^{s-1}_p}^{p} \\
\geq C||\vartheta||_{W^{s}_p}^{p}
\]
for all \(\lambda \in \mathbb{R} \setminus (-1, 1)\) and all \(\vartheta \in W^{1}_p(\mathbb{R})\). The claim follows now from this estimate via the method of continuity. \(\Box\)

4. The quasilinear evolution problem and the proof of the main result

As a first step we take advantage of the quasilinear character of the curvature operator to reformulate, in virtue of Proposition 3.4, the evolution problem (1.4) as a quasilinear evolution problem for \(f\) in a suitable functional analytic setting. More precisely, we recast (1.4) as the evolution problem
\[
\frac{df}{dt}(t) = \Phi(f(t))[f(t)], \quad t > 0, \quad f(0) = f_0,
\] (4.1)
where the nonlinear operator \(\{f \mapsto \Phi(f)\} : W^{s}_p(\mathbb{R}) \rightarrow \mathcal{L}(W^{s+1}_p(\mathbb{R}), W^{s-2}_p(\mathbb{R}))\), with \(p \in (1, \infty)\) and \(s \in (1 + 1/p, 2)\), is defined as follows. Given \(f \in W^{s}_p(\mathbb{R})\) and \(h \in W^{s+1}_p(\mathbb{R})\), let
\[
\kappa(f)[h] := \frac{h''}{(1 + f^2)^{3/2}}.
\] (4.2)
If \( f \in W^s_p(\mathbb{R}) \), then \( \kappa(f)[f] \) coincides with the curvature \( \kappa(f) \) of the interface \( \{y = f(x)\} \). This operator is smooth, that is
\[
\kappa \in C^\infty(W^s_p(\mathbb{R}), \mathcal{L}(W^{s+1}_p(\mathbb{R}), W^{s-1}_p(\mathbb{R}))), \tag{4.3}
\]
see, e.g. [28, Lemma 3.1]. In virtue of Proposition 3.4 and of \( a_\mu \in (-1, 1) \), there exits, for given functions \( f \in W^s_p(\mathbb{R}) \) and \( h \in W^{s+1}_p(\mathbb{R}) \), a unique solution \( \vartheta := \vartheta(f)[h] \in W^{s-1}_p(\mathbb{R}) \) to the equation
\[
(1 + a_\mu \mathcal{A}(f)[\vartheta]) = b_\mu(\sigma \kappa(f)[h] - \Theta h). \tag{4.4}
\]
A direct consequence of (4.3), of the smoothness of the map which associate to an isomorphism its inverse, of the representation (2.9) of \( \mathcal{A}(f)^* \), and of the property
\[
[f \mapsto B^0_{n,m}(f)] \in C^\infty(W^s_p(\mathbb{R}), \mathcal{L}(W^{s-1}_p(\mathbb{R}))), \tag{4.5}
\]
we obtain that
\[
\vartheta \in C^\infty(W^s_p(\mathbb{R}), \mathcal{L}(W^{s+1}_p(\mathbb{R}), W^{s-1}_p(\mathbb{R}))). \tag{4.6}
\]
The property (4.5) follows from Lemma 2.1 (ii), by arguing as in [33, Appendix C] (where the case \( p = 2 \) is proven in detail). Having introduced these quasilinear operators, we now define the operator \( \Phi \) by setting
\[
\Phi(f)[h] := -(\mathcal{B}(f)[\vartheta(f)[h]])'. \tag{4.7}
\]
In view of the continuity of the operator \( d/dx : W^{s-1}_p(\mathbb{R}) \to W^{s-2}_p(\mathbb{R}) \), we deduce from (2.9), (4.5), and (4.7) that
\[
\Phi \in C^\infty(W^s_p(\mathbb{R}), \mathcal{L}(W^{s+1}_p(\mathbb{R}), W^{s-2}_p(\mathbb{R}))). \tag{4.8}
\]
As a second step, we prove in Theorem 4.1 below that the evolution problem (4.1) is of parabolic type, that is we show that \( \Phi(f) \) is, when viewed as an unbounded operator in \( W^{s-2}_p(\mathbb{R}) \) with definition domain \( W^{s+1}_p(\mathbb{R}) \), the generator of an analytic semigroup in \( \mathcal{L}(W^{s-2}_p(\mathbb{R})) \). This property enables us to use the abstract quasilinear parabolic theory presented in [2, 34] when establishing Theorem 1.1.

**Theorem 4.1.** \textit{Given} \( p \in (1, \infty) \), \( s \in (1 + 1/p, 2) \), \textit{and} \( f \in W^s_p(\mathbb{R}) \), \textit{we have}
\[
-\Phi(f) \in \mathcal{H}(W^{s+1}_p(\mathbb{R}), W^{s-2}_p(\mathbb{R})).
\]

In the proof of Theorem 4.1 we are inspired by the strategy used in the references [12, 18, 20]. The main step in the proof of Theorem 4.1 is provided by the localization result established in Proposition 4.2 below. Before proceeding with this result, we note that the leading order part \( \Phi^\sigma(0) \) of \( \Phi(0) \) is the linear operator
\[
\Phi^\sigma(0) = \sigma b_\mu \frac{d}{dx} H \frac{d^2}{dx^2} = \sigma b_\mu H \frac{d^3}{dx^3}, \tag{4.9}
\]
where \( H \) denotes the Hilbert transform. We point out that \( \Phi^\sigma(0) \) is the Fourier multiplier with symbol \([\xi] \mapsto -\sigma b_\mu |\xi|^3\).

**Proposition 4.2.** \textit{Let} \( p \in (1, \infty) \), \( 1 + 1/p < s' < s < 2 \), \textit{and} \( f \in W^s_p(\mathbb{R}) \), \textit{and} \( \nu > 0 \) \textit{be given}. \textit{Then, there exist} \( \varepsilon \in (0, 1) \), \textit{a} \( \varepsilon \)-localization family \( \{(\pi_j^\varepsilon, x_j^\varepsilon) : -N + 1 \leq j \leq N\} \), \textit{and a constant} \( K = K(\varepsilon) \), \textit{such that}
\[
\|\pi_j^\varepsilon \Phi(\tau f)[h] - \alpha_{\tau,j} \Phi^\sigma(0)[\pi_j^\varepsilon h]\|_{W^{s-2}_p} \leq \nu \|\pi_j^\varepsilon h\|_{W^{s+1}_p} + K \|h\|_{W^{s+1}_{s'+1}}. \tag{4.10}
\]
for all \(-N + 1 \leq j \leq N\), \(\tau \in [0, 1]\), and \(h \in W^{s+1}_p(\mathbb{R})\), where, letting \(\alpha_r := (1 + \tau^2 f'^2)^{-3/2}\), we defined

\[
\alpha_r,N := \lim_{|x| \to \infty} \alpha_r(x) = 1 \quad \text{and} \quad \alpha_r,j := \alpha_r(x_j^\varepsilon), \quad |j| \leq N - 1.
\]

**Proof.** In the following C and \(C_1\), \(0 \leq i \leq 2\), denote constants that do not depend on \(\varepsilon\), while constants denoted by \(K\) may depend on \(\varepsilon\).

Setting \(C_0 := \|d/dx\|_{L(W^{s+1}_p(\mathbb{R}),W^{s-2}_p(\mathbb{R}))}\), for given \(-N + 1 \leq j \leq N\), \(\tau \in [0, 1]\), and \(h \in W^{s+1}_p(\mathbb{R})\) we obtain, in view of (4.7) and of the representation (4.9) of \(\Phi^\varepsilon(0)\), that

\[
\|\pi_j^\varepsilon(\Phi(\tau f))[h] - \alpha_{i,j} \Phi(0)[\pi_j^\varepsilon h]\|_{W^{s-2}_p} = \|\pi_j^\varepsilon(\Phi(\tau f))[h] - \sigma b_\mu \alpha_{i,j}(H[\pi_j^\varepsilon h])'[\pi_j^\varepsilon h]\|_{W^{s-2}_p}
\]

\[
\leq \|\pi_j^\varepsilon \mathcal{B}(\tau f)^{\ast} [\vartheta(\tau f)[h]] + \sigma b_\mu \alpha_{i,j} H[\pi_j^\varepsilon h]''\|_{W^{s-2}_p}
\]

\[
= C_0 \|\pi_j^\varepsilon \mathcal{B}(\tau f)^{\ast} [\vartheta(\tau f)[h]] + \sigma b_\mu \alpha_{i,j} H[\pi_j^\varepsilon h]''\|_{W^{s-1}_p} + K\|h\|_{W^{s+1}_p},
\]

since, by Lemma 2.1 (i) and Theorem 3.1, we have

\[
\|\pi_j^\varepsilon \mathcal{B}(\tau f)^{\ast} [\vartheta(\tau f)[h]]\|_{W^{s-2}_p} \leq K\|\mathcal{B}(\tau f)^{\ast} [\vartheta(\tau f)[h]]\|_{W^{s-2}_p} \leq K\|\vartheta(\tau f)[h]\|_p \leq K\|h\|_{W^{s+1}_p}.\]

It thus remains to estimate the expression \(\pi_j^\varepsilon \mathcal{B}(\tau f)^{\ast} [\vartheta(\tau f)[h]] + \sigma b_\mu \alpha_{i,j} H[\pi_j^\varepsilon h]''\|_{W^{s-1}_p}.\) This is done in several steps.

Step 1. We prove there exists a positive constant \(C_1\) such that

\[
\|\pi_j^\varepsilon \vartheta(\tau f)[h]\|_{W^{s-1}_p} \leq C_1 \|\pi_j^\varepsilon h\|_{W^{s+1}_p} + K\|h\|_{W^{s+1}_p}
\]

for all \(\varepsilon \in (0, 1)\), \(\tau \in [0, 1]\), \(-N + 1 \leq j \leq N\), and \(h \in W^{s-1}_p(\mathbb{R})\). To this end we multiply (4.4) by \(\pi_j^\varepsilon\) to deduce that

\[
(1 + a_\mu A(\tau f)^{\ast} )[\pi_j^\varepsilon \vartheta(\tau f)[h]] = b_\mu \pi_j^\varepsilon (\sigma \kappa) (\tau f)[h] - \Theta h
\]

\[
+ a_\mu (A(\tau f)^{\ast} [\pi_j^\varepsilon \vartheta(\tau f)[h]] - \pi_j^\varepsilon A(\tau f)^{\ast} [\vartheta(\tau f)[h]]).
\]

The first term on the right is estimated as follows

\[
\|\pi_j^\varepsilon (\sigma \kappa) (\tau f)[h] - \Theta h\|_{W^{s-1}_p} \leq C_1 \|\pi_j^\varepsilon h\|_{W^{s+1}_p} + K\|h\|_{W^{s+1}_p}.
\]

Moreover, combining (2.9) and the commutator estimate in Lemma A.1, the arguments used to derive (4.11) yield

\[
\|A(\tau f)^{\ast} [\pi_j^\varepsilon \vartheta(\tau f)[h]] - \pi_j^\varepsilon A(\tau f)^{\ast} [\vartheta(\tau f)[h]]\|_{W^{s-1}_p} \leq K\|\vartheta(\tau f)[h]\|_p \leq K\|h\|_{W^{s+1}_p}.
\]

Proposition 3.4 now ensures that (4.12) indeed holds true.

Step 2. Taking advantage of Lemma A.2 (if \(|j| \leq N - 1\) and Lemma A.3 (if \(|j| = N\), we infer from (2.9) that for each sufficiently small \(\varepsilon \in (0, 1)\) we have

\[
\|\pi_j^\varepsilon \mathcal{B}(\tau f)^{\ast} [\vartheta(\tau f)[h]] + H[\pi_j^\varepsilon \vartheta(\tau f)[h]]\|_{W^{s-1}_p} \leq \frac{\nu}{2C_0 C_1} \|\pi_j^\varepsilon \vartheta(\tau f)[h]\|_{W^{s+1}_p} + K\|\vartheta(\tau f)[h]\|_{W^{s+1}_p}.
\]
for all $\tau \in [0,1], -N + 1 \leq j \leq N$, and $h \in W^{s+1}_p(\mathbb{R})$. Using (4.12) and (4.6) (with $s = s'$), we infer from the latter estimate that
\[
\|\pi_j h\|_{W^{s+1}_p} \leq \frac{\nu}{2C_0} \|\pi_j h\|_{W^{s+1}_p} + K\|h\|_{W^{s'}_{p+1}}
\] 
for all $\tau \in [0,1], -N + 1 \leq j \leq N$, and $h \in W^{s+1}_p(\mathbb{R})$, provided that $\varepsilon$ is sufficiently small.

Step 3. We first note that $\alpha_\tau(f) = (1 + \tau^2 f^2)^{-3/2} \in BUC^{s-1-1/\nu}(\mathbb{R})$, $\tau \in [0,1]$, and $\alpha_\tau(f)(x) \to 1$ for $|x| \to \infty$ (uniformly with respect to $\tau \in [0,1]$). In this step we show that, for each sufficiently small $\varepsilon \in (0,1)$, we have
\[
\|H[\pi_j \partial(\tau f)](h) - \sigma b_\mu \alpha_{\tau,j} H[\pi_j h]|_{W^{s+1}_p} \leq \frac{\nu}{2C_0} \|\pi_j h\|_{W^{s+1}_p} + K\|h\|_{W^{s'}_{p+1}}
\] 
for all $\tau \in [0,1], -N + 1 \leq j \leq N$, and $h \in W^{s+1}_p(\mathbb{R})$.

To proceed, we set $C_2 := \|H\|_{L^1(W^{s+1}_p(\mathbb{R}))}$ and obtain that
\[
\|H[\pi_j \partial(\tau f)](h) - \sigma b_\mu \alpha_{\tau,j} H[\pi_j h]|_{W^{s+1}_p} \leq C_2 \|\pi_j \partial(\tau f)](h) - \sigma b_\mu \alpha_{\tau,j} (\pi_j h)|_{W^{s+1}_p}.
\]
In order to estimate the right side of the latter inequality, we infer from (4.4) that
\[
\pi_j \partial(\tau f)](h) - \sigma b_\mu \alpha_{\tau,j} (\pi_j h)|_{W^{s+1}_p} = -a_\mu \pi_j \hat{A}(\tau, f)^{\star}\{\partial(\tau f)](h) + b_\mu [\pi_j (\sigma \kappa f)](h) - \Theta h) - \sigma \alpha_{\tau,j} (\pi_j h)|_{W^{s+1}_p}.
\]
Using Lemma A.2 (if $|j| \leq N - 1$) and Lemma A.3 (if $j = N$) together with the representation (2.9) of $\hat{A}(\tau, f)^{\star}$, we have
\[
\|\pi_j \hat{A}(\tau, f)^{\star}\{\partial(\tau f)](h)\}|_{W^{s+1}_p} \leq \frac{\nu}{4(1 + |a_\mu|) C_0 C_1 C_2} \|\pi_j \partial(\tau f)](h)\|_{W^{s+1}_p} + K\|\partial(\tau f)](h)\|_{W^{s'}_{p+1}}
\]
and, arguing as in the derivation of (4.13), we conclude that
\[
\|\pi_j \hat{A}(\tau, f)^{\star}\{\partial(\tau f)](h)\}|_{W^{s+1}_p} \leq \frac{\nu}{4(1 + |a_\mu|) C_0 C_2} \|\pi_j h\|_{W^{s+1}_p} + K\|h\|_{W^{s'}_{p+1}}
\]
for all $\tau \in [0,1], -N + 1 \leq j \leq N$, and $h \in W^{s+1}_p(\mathbb{R})$, provided that $\varepsilon$ is sufficiently small. Moreover, for all sufficiently small $\varepsilon \in (0,1)$, we obtain in view of $\chi_j \pi_j = \pi_j$, of the estimate (1.9), and of the Hölder continuity of $\alpha_\tau$ that
\[
\|\pi_j (\sigma \kappa f)](h) - \Theta h) - \sigma \alpha_{\tau,j} (\pi_j h)|_{W^{s+1}_p} \leq \sigma \|\alpha_\tau - \alpha_\tau(x_j)\|_{W^{s+1}_p} + K\|h\|_{W^{s'}_{p+1}}
\]
\[
\leq 2\sigma \|\chi_j (\alpha_\tau - \alpha_\tau(x_j))\|_{W^{s+1}_p} + K\|h\|_{W^{s'}_{p+1}}
\]
\[
\leq \frac{\nu}{4b_\mu C_0 C_2} \|\pi_j h\|_{W^{s+1}_p} + K\|h\|_{W^{s'}_{p+1}}
\]
for all $\tau \in [0,1], |j| \leq N - 1$, and $h \in W^{s+1}_p(\mathbb{R})$. Since $\alpha_\tau - 1$ vanishes at infinity, for $j = N$ we similarly have
\[
\|\pi_N (\sigma \kappa f)](h) - \Theta h) - \sigma \alpha_{\tau,N} (\pi_N h)|_{W^{s+1}_p} \leq \sigma \|\alpha_\tau - 1\|_{W^{s+1}_p} + K\|h\|_{W^{s'}_{p+1}}
\]
\[
\leq 2\sigma \|\chi_N (\alpha_\tau - 1)||_{W^{s+1}_p} + K\|h\|_{W^{s'}_{p+1}}
\]
\[
\leq \frac{\nu}{4b_\mu C_0 C_2} \|\pi_N h\|_{W^{s+1}_p} + K\|h\|_{W^{s'}_{p+1}}.
\]
Gathering all these estimates, we conclude that (4.14) is valid. Combining the estimates (4.11), (4.13), and (4.14), we arrive at (4.10) and the proof is complete. 

\[ \Box \]
We are now in a position to establish Theorem 4.1. The proof relies deeply on Proposition 4.2, the arguments being identical to those in the proof [28, Theorem 3.5].

Proof of Theorem 4.1. Since \( f' \) is a bounded function, there exists a constant \( \kappa_0 \geq 1 \) such that the Fourier multipliers \( \alpha \tau_j \Phi^\tau(0), \tau \in [0, 1] \) and \(-N + 1 \leq j \leq N\), identified in Proposition 3.4 satisfy

\[
\bullet \lambda - \alpha \tau_j \Phi^\tau(0) \in \text{Isom}(W_p^{s+1}(\mathbb{R}), W_p^{s-2}(\mathbb{R})) \quad \text{for all } \text{Re} \lambda \geq 1, \tag{4.15}
\]

\[
\bullet \kappa_0 \| (\lambda - \alpha \tau_j \Phi^\tau(0)) [h] \|_{W_p^{s-2}} \geq |\lambda| \cdot \| h \|_{W_p^{s-2}} + \| h \|_{W_p^{s+1}} \quad \text{for all } h \in W_p^{s+1}(\mathbb{R}), \text{Re} \lambda \geq 1. \tag{4.16}
\]

The properties (4.15)-(4.16) together with the estimate (4.10) in Proposition 4.2 lead now to the desired result, see [28, Theorem 3.5] for full details. \( \square \)

We conclude this section with the proof of our main result which uses to a large extent the abstract quasilinear parabolic theory presented in [2] (see also [34, Theorem 1.1]).

Proof of Theorem 1.1. To start, let

\[
0 < \beta := \frac{2}{3} < \alpha := \frac{s - \frac{3}{2}}{3} < 1.
\]

We further define \( E_1 := W_\beta^{s+1}(\mathbb{R}), E_0 := W_\beta^{-s}(\mathbb{R}), \) and \( E_\eta := (E_0, E_1)_{\eta, p} \) for \( \eta \in (0, 1) \). Recalling the interpolation property (1.8), we have \( E_\alpha = W_\beta^s(\mathbb{R}) \) and \( E_\beta = W_\beta^s(\mathbb{R}) \). We now infer from (4.8) and Theorem 4.1 (both with \( s = \frac{3}{2} \)), that \(-\Phi \in C^\infty(E_\beta, H(E_1, E_0))\). Hence, the assumptions of [34, Theorem 1.1] are satisfied in the context of the Muskat problem (4.1). Consequently, given \( f_0 \in W_\beta^s(\mathbb{R}) \), there exists a unique maximal classical solution \( f = f(\cdot; f_0) \) to (4.1) such that

\[
f \in C([0, T^+), W_\beta^s(\mathbb{R})) \cap C((0, T^+), W_\beta^{s+1}(\mathbb{R})) \cap C^1((0, T^+), W_\beta^{s-2}(\mathbb{R}))
\]

and

\[
f \in C^\infty([0, T^+), W_\beta^s(\mathbb{R})]
\]

where \( T^+ = T^+(f_0) \) is an upper bound for the maximal existence time and the Hölder exponent \( \zeta \in (0, \alpha - \beta] \) can be chosen arbitrary small, cf. [34, Remark 1.2 (ii)]. Moreover, the mapping \([t, f_0] \mapsto f(t; f_0)\) defines a semiflow on \( W_\beta^s(\mathbb{R}) \) which is smooth in the open set

\[
\{(t, f_0) : f_0 \in W_\beta^s(\mathbb{R}), 0 < t < T^+(f_0)\} \subseteq \mathbb{R} \times W_\beta^s(\mathbb{R}).
\]

In remains to establish the parabolic smoothing property (1.5). To this end, we may choose above \( E_0 := L_p(\mathbb{R}) \) and \( E_1 := W_\beta^3(\mathbb{R}) \). Similarly as in the case \( p = 2 \) considered in [29], using Lemma 2.2 and Theorem 3.1, we may deduce that \(-\Phi \in C^\infty(W_\beta^2(\mathbb{R}), H(W_\beta^3(\mathbb{R}), L_p(\mathbb{R})))\).

Applying the quasilinear parabolic theory from [2] in this context and arguing as in the proof of [29, Theorem 1.1], we obtain that, given \( f_0 \in W_\beta^{s+1}(\mathbb{R}) \), there exists a unique solution \( \bar{f} = \bar{f}(\cdot; f_0) \) to (4.1) that satisfies

\[
\bar{f} \in C([0, \bar{T}^+), W_\beta^{s+1}(\mathbb{R})) \cap C((0, \bar{T}^+), W_\beta^3(\mathbb{R})) \cap C^1((0, \bar{T}^+), L_p(\mathbb{R})), \tag{4.17}
\]

with \( \bar{T}^+ = \bar{T}^+(f_0) \in (0, \infty] \) denoting the maximal existence time. Additionally, \([t, f_0] \mapsto \bar{f}(t; f_0)\) defines a semiflow on \( W_\beta^{s+1}(\mathbb{R}) \) and

\[
\bar{f} \in C^\infty((0, \bar{T}^+) \times \mathbb{R}, \mathbb{R}) \cap C^\infty((0, \bar{T}^+), W_\beta^k(\mathbb{R})) \quad \text{for all } k \in \mathbb{N}.
\]
Recalling (4.17), Lemma 2.1 (i) and Theorem 3.1 imply that \( d\tilde{f}/dt \in C([0, \tilde{T}^+), W_p^{-1}(\mathbb{R})) \). This property, (4.17), and the mean value theorem lead now to \( \tilde{f} \in C^S([0, \tilde{T}^+), W_p^{3}(\mathbb{R})) \) for some sufficiently small \( \zeta \in (0, \alpha - \beta) \). Consequently, \( \tilde{T}^+ \leq T^+ \) and \( \tilde{f}(\cdot; f_0) = f(\cdot; f_0) \) on \( [0, \tilde{T}^+] \) for all \( f_0 \in W_p^{\tilde{T}^+}(\mathbb{R}) \).

It actually holds \( \tilde{T}^+ = T^+ \). Indeed, let us assume that \( \tilde{T}^+ < T^+ \). The properties established above enable us to conclude that \( \tilde{f} \in C^S([0, T^+), W_p^2(\mathbb{R})) \) for some sufficiently small \( \zeta' \in (0, 1) \). Since \( f \in C([0, T^+), W_p^{\tilde{T}^+}(\mathbb{R})) \), we may infer from [34, Proposition 2.1] (after possibly choosing a smaller \( \zeta' \)), that there exist positive constants \( \varepsilon > 0 \) and \( \delta > 0 \) such that for all \( |t_0 - \tilde{T}^+| \leq \varepsilon \), the problem

\[
\frac{df}{dt}(t) = \Phi(f(t))[f(t)], \quad t > 0, \quad f(0) = f(t_0),
\]

has a unique solution

\[
\tilde{f} \in C([0, \delta], W_p^{\tilde{T}^+}(\mathbb{R})) \cap C((0, \delta], W_p^3(\mathbb{R})) \cap C^1((0, \delta], L_p(\mathbb{R})) \cap C^\zeta([0, \delta], W_p^2(\mathbb{R})).
\]

Hence, choosing \( t_0 < \tilde{T}^+ \) such that \( t_0 + \delta > \tilde{T}^+ \), we may extend \( \tilde{f} \) to the interval \( [0, t_0 + \delta] \), which contradicts the maximality property of \( \tilde{f} \), hence \( \tilde{T}^+ = T^+ \). This provides the regularity property (1.5) and the proof is complete. \( \square \)

**Appendix A. Localization of the singular integral operators \( B_{n,m}^0(f) \)**

In this section we collect some results that enable us to localize the singular integrals operators \( B_{n,m}^0(f) \) introduced in (2.2). Lemma A.2 and Lemma A.3 can be viewed as generalizations of the method of freezing the coefficients of elliptic differential operators, and, together with the commutator estimate from Lemma A.1, are essential in the proof of Proposition 4.2. Before proceeding, we recall that \( H \) denotes the Hilbert transform.

**Lemma A.1.** Let \( n, m \in \mathbb{N}, p \in (1, \infty), s \in (1 + 1/p, 2), f \in W_p^s(\mathbb{R}), \) and \( \varphi \in \text{BUC}^1(\mathbb{R}) \) be given. Then, there exists a constant \( K \) that depends only on \( n, m, \|\varphi\|_\infty, \) and \( \|f\|_{W_p^s} \) such that

\[
\|\varphi B_{n,m}^0(f)[\theta] - B_{n,m}^0(f)[\varphi \theta]\|_{W_p^1} \leq K\|\theta\|_p
\]

for all \( \vartheta \in L_p(\mathbb{R}) \).

**Proof.** See [1, Lemma 12]. \( \square \)

The next lemmas describe how to localize the operators \( B_{n,m}^0(f) \).

**Lemma A.2.** Let \( n, m \in \mathbb{N}, 1 + 1/p < s' < s < 2, \) and \( \nu \in (0, \infty) \) be given. Let further \( f \in W_p^s(\mathbb{R}) \) and \( a \in \{1\} \cup W_p^{s-1}(\mathbb{R}) \). For any sufficiently small \( \varepsilon \in (0, 1) \), there exists a constant \( K \) that depends only on \( \varepsilon, n, m, \|f\|_{W_p^s}, \) and \( \|a\|_{W_p^{s-1}} \) (if \( a \neq 1 \)) such that

\[
\left\| \pi_j^\varepsilon B_{n,m}^0(f)[a\vartheta] - \frac{a(x_j^\varepsilon)(f'(x_j^\varepsilon))^n}{[1 + (f'(x_j^\varepsilon))^2]^m} H[\pi_j^\varepsilon \vartheta]\right\|_{W_p^{s-1}} \leq \nu\|\pi_j^\varepsilon \vartheta\|_{W_p^{s-1}} + K\|\vartheta\|_{W_p^{s'-1}}
\]

for all \( |j| \leq N - 1 \) and \( \vartheta \in W_p^{s-1}(\mathbb{R}) \).

**Proof.** If \( a = 1 \), see [1, Lemma 13]. If \( a \in W_p^{s-1}(\mathbb{R}) \), this result follows by arguing as in the proof of [33, Lemma D.5] (where the result in the case \( p = 2 \) is established). \( \square \)

Lemma A.3 below describes how to localize the operators \( B_{n,m}^0(f) \) at infinity.


Lemma A.3. Let $n, m \in \mathbb{N}$, $1 + 1/p < s' < s < 2$, and $\nu \in (0, \infty)$ be given. Let further $f \in W^s_p(\mathbb{R})$ and $a \in \{1\} \cup W^{s-1}_p(\mathbb{R})$. For any sufficiently small $\varepsilon \in (0, 1)$, there exists a constant $K$ that depends only on $\varepsilon$, $n$, $m$, $\|f\|_{W^s_p}$, and $\|a\|_{W^{s-1}_p}$ (if $a \neq 1$) such that
\[
\|\pi_0 f B_{0,m}^a(\vartheta)|W^s_p - 1| \leq \nu \|\pi_0 f\|_{W^{s-1}_p} + K \|\vartheta\|_{W^{s'-1}_p}
\]
and
\[
\|\pi_0 f B_{0,m}^a(\vartheta) - H[\pi_0 f]\|_{W^{s-1}_p} \leq \nu \|\pi_0 f\|_{W^{s-1}_p} + K \|\vartheta\|_{W^{s'-1}_p}
\]
for all $\vartheta \in W^{s-1}_p(\mathbb{R})$.

Proof. If $a = 1$, see [1, Lemma 15]. If $a \in W^{s-1}_p(\mathbb{R})$, the desired estimate follows by arguing as in the proof of [33, Lemma D.6] (where the result in the case $p = 2$ is established).

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