Separation of Equilibration Time-Scales in the Gradient Expansion

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We study thermalization by applying gradient expansion to the Kadanoff-Baym equations of the 2PI effective action to two-loop in a theory with Dirac fermions coupled to scalars. In addition to those chemical potentials which equilibrate in the on-shell limit, we identify modes which are conserved in this approximation, but which relax when off-shell effects are taken into account. This implies that chemical equilibration does not require higher loop contributions to the effective action and is compatible with the gradient expansion. We explicitly calculate the damping time-scales of both, on- and off-shell, chemical equilibration rates. It is shown that off-shell equilibration is suppressed by the thermal width of the particles in the plasma, which explains the separation of on- and off-shell chemical equilibration time-scales.

I. INTRODUCTION

Kadanoff Baym equations are an effective method for describing the out-of-equilibrium dynamics of quantum fields. For many particular models and initial conditions, thermalization has been demonstrated successfully from numerical solutions to these equations, see e.g. refs. \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\] for some works in the field. While details can depend on the particular settings, it has been revealed that starting from far-from-equilibrium initial conditions, thermalization proceeds typically in several usually overlapping stages:

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• Dephasing/prethermalization \[7, 14\]: In this stage, a constant equation of state is reached and an approximate equipartition between kinetic and potential energy is attained. This dephasing effect does not rely on scattering-driven processes and is a pure quantum phenomenon.

• Kinetic equilibration \[5, 9, 12\]: Within this stage, the quasi-particle distribution functions approach the Bose-Einstein or Fermi-Dirac form, respectively. The interactions which drive the system towards kinetic equilibrium do not necessarily relax chemical potentials through violation of particle number. For example, this is the case for elastic two-by-two scatterings.

• Chemical equilibration \[3, 12\]: Finally, processes that do not conserve particle numbers eventually relax the chemical potentials.

This last stage of chemical equilibration is the subject of the present study. The inclusion of off-shell effects constitutes one qualitative difference between the Kadanoff-Baym and the Boltzmann equations. This has been mentioned as a possible explanation for the discrepancy between numerical solutions within the two approaches \[4\]. Besides, Kadanoff-Baym equations include memory integrals and treat deviations from equilibrium to all order in derivatives, usually referred to as gradients. Boltzmann equations do not incorporate memory integrals and are first order in gradients.

For systems which are close to equilibrium, gradient expansion and neglect of memory integrals is justifiable. On the other hand, due to the possible presence of quantities which only equilibrate when off-shell effects are included, it is desirable to account for these within an analytical approach. For the case of scalar $\lambda \phi^4$ theory in $2 + 1$ dimensions, Boltzmann equations have been generalized to include off-shell processes. Numerical solutions to these equations are found to be in accordance with the solutions to the full Kadanoff-Baym equations, but in disagreement with the on-shell Boltzmann equations \[5, 6, 15\]. The explanation is that while $2 \leftrightarrow 2$ scatterings lead to kinetic equilibration at a time-scale proportional to $\lambda^2$, chemical equilibration is driven by $1 \leftrightarrow 3$ off-shell processes that are suppressed by higher powers in $\lambda$; see also e.g. ref. \[9\] for related discussions and refs. \[8, 10\] for a numerical analysis in $3+1$ dimensions. The case of an interacting system of fermions and scalars is numerically treated in \[12\], where solutions to the Kadanoff-Baym and the Boltzmann equations are compared. It is found that there are chemical potentials which are conserved
by the Boltzmann equations but which are relaxed when Kadanoff-Baym equations are applied. This phenomenon is explained by the fact that the quasi-particle approximation of Boltzmann equations leads to spuriously conserved quantities.

The aim of the current work is to show how to reproduce on- and off-shell chemical equilibration in a semi-classical approach, without resorting to numerical solutions. As a concrete model, we consider a Dirac fermion coupled via a Yukawa coupling to a complex scalar. We expect that this model shares the essential features for the discussion of chemical equilibration in the quark-meson model considered in refs. [3, 12] and the models usually considered for leptogenesis [16] and for chargino mediated electroweak baryogenesis [17]. The classical Boltzmann equations can be derived from the Kadanoff-Baym equations by using several approximations. These are the gradient expansion (to first order), the expansion in coupling constants (typically up to the first nontrivial order that accounts for scattering or decay processes) and the quasi-particle (or on-shell) approximation. We will show that by omitting the last approximation, full chemical equilibration can be described by semi-classical transport equations. In particular, the gradient expansion is valid in this regime and we present analytic expressions for the on- and off-shell relaxation time-scales of the chemical potentials.

The paper is organized as follows. In the next section we set up notation and in section III we introduce the gradient expansion. In the two subsequent sections, the model and the Boltzmann equations in the on-shell limit are discussed. Finally, we calculate the various time-scales of chemical equilibration in section VI, before we conclude in section VII.

II. THE KADANOFF-BAYM EQUATIONS

In this section, we derive the Kadanoff-Baym equations and set up notation. We follow the nomenclature of ref. [18], where a more complete treatment can be found. Statistical systems in equilibrium and non-equilibrium can be described by the Kadanoff-Baym equations which are the Schwinger-Dyson equations in the in-in formalism and in Wigner space [19]. The in-in formalism is designed to describe the time-evolution of density matrix elements rather than of scattering processes to which the usual in-out formalism applies. The in-in formalism is implemented by integrating along the Schwinger-Keldysh [20, 21] closed time path (CTP) from a finite time $t = t_0$ to $t = +\infty$ and back. The Green functions for bosons and fermions
are then defined as the path ordered operators
\[ i\Delta(u,v) = \langle \Omega | T \phi(u)\phi^\dagger(v) | \Omega \rangle, \]
\[ iS_{\alpha\beta}(u,v) = \langle \Omega | T \psi_\alpha(u)\bar{\psi}_\beta(v) | \Omega \rangle. \]

Depending on the locations of the coordinates \( u \) and \( v \) on the branches of the CTP, we obtain the four Green functions
\[ i\Delta'(u,v) = \langle \Omega | T \phi(u)\phi^\dagger(v) | \Omega \rangle, \]
\[ i\Delta^<(u,v) = \langle \Omega | \phi^\dagger(v)\phi(u) | \Omega \rangle, \]
\[ i\Delta^>(u,v) = \langle \Omega | \phi(u)\phi^\dagger(v) | \Omega \rangle, \]
\[ i\Delta^\bar{t}(u,v) = \langle \Omega | T \phi(u)\phi^\dagger(v) | \Omega \rangle, \]

and similar expressions for the fermions. In the following, we focus on the bosonic functions. The fermionic functions fulfill similar corresponding equations (see ref. [18]). From the definitions (3), it is obvious that only two of the four functions are independent while the other two can be expressed as
\[ \Delta'(u,v) = \theta(u_0 - v_0)\Delta^>(u,v) + \theta(v_0 - u_0)\Delta^<(u,v), \]
\[ \Delta^\bar{t}(u,v) = \theta(u_0 - v_0)\Delta^<(u,v) + \theta(v_0 - u_0)\Delta^>(u,v). \]

Besides, the Green functions have the following hermiticity property
\[ (i\Delta^{<,>}(u,v))^\dagger = i\Delta^{<,>}(v,u). \]

The retarded and advanced propagators are defined as
\[ \Delta^r = \Delta' - \Delta^< = \Delta^> - \Delta^\bar{t}, \]
\[ \Delta^a = \Delta' - \Delta^> = \Delta^< - \Delta^\bar{t}, \]

which can be used to define the spectral and hermitian functions
\[ A_\phi = \frac{i}{2}(\Delta^r - \Delta^a) = \frac{i}{2}(\Delta^> - \Delta^<), \]
\[ \Delta_h = \frac{1}{2}(\Delta^r + \Delta^a) = -i \text{sign}(u_0 - v_0)A_\phi. \]

The usefulness of these definitions becomes obvious for the thermal equilibrium Green functions in Wigner space, where one obtains for a free boson of mass \( M \)
\[ A_\phi(p,x) = \int d^4p e^{ip\cdot x}A_\phi(x + r/2, x - r/2) = \pi\delta(h^2 - M^2)\text{sign}(k_0), \]
and
\[ \Delta^< = 2n_B^{eq}(k_0) A_\phi, \quad \Delta^> = 2(n_B^{eq}(k_0) + 1) A_\phi, \quad n_B^{eq}(k_0) = \frac{1}{e^{\beta k_0} - 1}. \]

Using this notation, the Kadanoff-Baym equations for the bosons in Wigner space read [18]
\[ (\Omega_\phi^2 \pm i\Gamma_\phi) \Delta^{r,a} = 1, \quad \Omega_\phi^2 \Delta^{<,>} - \Pi^{<,>} \Delta_h = \frac{1}{2}(\Pi^{>,*} \Delta^< - \Pi^{<,*} \Delta^>). \]

where
\[ A \ast B = A e^{-i\frac{1}{2}(\partial_x \partial_k - \partial_k \partial_x)} B \]
denotes the Moyal star product and the various functions \( \Pi \) are self-energies which have the same time-ordering prescriptions as defined for the Green functions in eqs. [3]. The self-energies can be deduced from the two-particle-irreducible (2PI) effective action. Besides, we have defined
\[ \Omega_\phi^2 = k^2 - M^2 - \Pi_h, \quad \Gamma_\phi = \frac{i}{2}(\Pi^{>,*} - \Pi^{<,*}). \]

In the following, we discuss this system of equations in the gradient expansion.

### III. GRADIENT EXPANSION

We consider spatially homogeneous and isotropic systems, such that all functions depend on the time \( t = x_0 \), but not on the spatial coordinates \( \vec{x} \). At rather late times, when the system is not too far from equilibrium, one expects that the relaxation of the system towards equilibrium is driven by interactions which allow for a gradient expansion in the weak coupling regime.\(^1\) Thus, one can expand the Moyal star product [13]
\[ A \ast B \approx AB - \frac{i}{2}\{A, B\}_{pb} + O(\partial_x^2), \]
where we have defined the Poisson brackets \( \{A, B\}_{pb} = \partial_x A \partial_k B - \partial_k A \partial_x B. \)

At first order in the gradient expansion, eq. [11] reads
\[ (\Omega_\phi^2 \pm i\Gamma_\phi) \Delta^{r,a} - \frac{i}{2}\{\Omega_\phi^2 \pm i\Gamma_\phi, \Delta^{r,a}\}_{pb} = 1, \]

1 This argument can however be jeopardized by oscillation effects in the case of several flavors as discussed in ref. [22, 23].
which is solved by
\[ \Delta^{r,a} = \frac{1}{\Omega^2 + i\Gamma}. \] (17)

Here, we have used the fact that \( \{ A, 1/A \}_{pb} = -\{ A, A \}_{pb}/A^2 = 0 \). This leads to the spectral and hermitian functions
\[ A_\phi = \frac{\Gamma^2}{\Omega^4 + \Gamma^2}, \quad \Delta_h = \frac{\Omega^2}{\Omega^4 + \Gamma^2}. \] (18)

The remaining information is encoded in the anti-hermitian part of eq. (12), which reads
\[- \{ \Omega^2, i\Delta^{<,>} \}_{pb} + \{ i\Pi^{<,>}, i\Delta_h \}_{pb} = \Pi^{>\Delta^{<}} - \Pi^{<\Delta^{>}}. \] (19)

Since we assume that the coupling constant \( y \) is small, the self-energies are at late times effectively of first order in the gradient expansion. We therefore neglect the self-energies in the Poisson brackets (in contrast to the methods used e.g. in ref. [15]), which leads to
\[ 2k_0 \partial_t i\Delta^{<,>} = \Pi^{>\Delta^{<}} - \Pi^{<\Delta^{>}}. \] (20)

Analogously, one finds for a free fermion in equilibrium
\[ iS^< = -2n_{eq}^F(k_0) A_\psi, \quad iS^> = 2(1 - n_{eq}^F(k_0)) A_\psi, \quad n_{eq}^F(k_0) = \frac{1}{e^{\beta k_0} + 1}, \] (21)
and the fermionic Kadanoff-Baym equations
\[ (\Omega_\psi \pm i\Sigma_A) S^{r,a} - \frac{1}{2} \{ \Omega_\psi \pm i\Sigma_A, S^{r,a} \}_{pb} = 1, \] (22)
\[ \gamma_0 \partial_t iS^{<,>} = \frac{1}{2} (\Sigma^> S^< - \Sigma^< S^>) + \text{h.c.}, \] (23)

where \( \Sigma \) denotes the fermionic self-energy and \( \Omega_\psi = \hat{\rho} - \Sigma_h \) and \( \Sigma_A = \frac{i}{2} (\Sigma^> - \Sigma^<) \). This implies that
\[ A_\psi = \frac{\Omega_\psi \Gamma_\psi - \Sigma_A (\Omega_\psi^2 - \Sigma_A^2)}{(\Omega_\psi^2 - \Sigma_A^2)^2 + \Gamma_\psi^2}, \] (24)
\[ S_h = \frac{\Omega_\psi (\Sigma_A^2 - \Sigma_\psi^2) + \Sigma_A \Gamma_\psi}{(\Omega_\psi^2 - \Sigma_A^2)^2 + \Gamma_\psi^2}, \] (25)

where we have defined \( \Gamma_\psi = 2\Omega_\psi \cdot \Sigma_\psi = \{ \Sigma_A, \Omega_\psi \}, \Sigma_A^2 = \Sigma_A \Sigma_A \) and \( \Omega_\psi^2 = \Omega_\psi \Omega_\psi \). In general, the fermionic propagator has a complicated structure and e.g. contains a pole from a collective excitation [24]. However, the corresponding residuum is exponentially small for most particles in the plasma such that one can safely ignore this subtlety.
IV. THE MODEL AND ITS CONSERVED CHARGES

For our purpose of an analytic calculation of the separation of time-scales of on- and off-shell chemical equilibration, we consider a theory with a Lagrange density as in ref. [18]

\[ \mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi + \partial_\mu \phi \partial^\mu \phi - y \bar{\psi}(P_R \phi + P_L \phi^\dagger)\psi, \]  

with a small real coupling constant \( y \). Using the two-loop 2PI effective action, this Lagrangian leads to the following expressions for the self-energies [18]

\[ i\Pi^{<,>}(k,x) = -y^2 \int \frac{d^4 p d^4 q}{(2\pi)^4} \delta(k - p - q) \text{Tr}[P_R i S^{-}\lessgtr (-p,x)P_L i S^{<,>}(q,x)], \]  

and for the fermions (\( \Sigma = \Sigma_L + \Sigma_R \))

\[ i\Sigma_R^{<,>}(p,x) = y^2 \int \frac{d^4 q d^4 k}{(2\pi)^4} \delta(k - p - q) i\Delta^{>\lessgtr}(-k,x)P_L i S^{<,>}(q,x)P_R, \]  

\[ i\Sigma_L^{<,>}(q,x) = y^2 \int \frac{d^4 p d^4 k}{(2\pi)^4} \delta(k - p - q) i\Delta^{<,>}(k,x)P_R i S^{<,>}(-p,x)P_L. \]  

We now define the charges

\[ Q_S = \int \frac{d^4 k}{(2\pi)^4} 2k_0(i\Delta^< + 2\theta(-k_0)A_\phi), \]  

\[ P_\mu^S = \int \frac{d^4 k}{(2\pi)^4} 2k_\mu k_0(i\Delta^< + 2\theta(-k_0)A_\phi), \]  

\[ Q_{L/R} = -\int \frac{d^4 q}{(2\pi)^4} \text{Tr}\gamma_0 P_{L/R}(iS^< + 2\theta(-q_0)A_\psi), \]  

\[ P_{L/R}^\mu = -\int \frac{d^4 q}{(2\pi)^4} \text{Tr}q^\mu \gamma_0 P_{L/R}(iS^< + 2\theta(-q_0)A_\psi), \]

where the terms involving the spectral functions \( \mathcal{A} \) have vanishing time-derivatives and have been introduced to make the expressions finite. The interpretation of these quantities in the on-shell limit is given in the next section. When using the kinetic equations (20) and (23), the form of the collision terms immediately leads to the following conserved quantities:

\[ Q_R + Q_L = \text{const.} \]  

\[ 2Q_S - Q_R + Q_L = \text{const.} \]  

\[ P_S^\mu + P_R^\mu + P_L^\mu = \text{const.} \]

We emphasize that these conservation laws are due to the symmetries of the collision terms and make no assumption about the Green functions and their spectral properties. Of course, this is reflecting that above charges are conserved due to Noether symmetries with respect to rephasings and translations of the Lagrangian [20].
V. THE ON-SHELL LIMIT

The on-shell limit arises when neglecting the influence of interactions on the spectral functions. Notice, that in order to obtain the correct results, one has to take into account that the Breit-Wigner width behaves as \( \Gamma \rightarrow \text{sign}(k_0)\epsilon \) for both bosons and fermions. In this case, we recover the spectral functions of the free theory

\[
A_\phi(k, x) \rightarrow \pi \delta(k^2 - M^2) \text{sign}(k_0),
\]

\[
A_\psi(k, x) \rightarrow \pi \xi \delta(k^2) \text{sign}(k_0),
\]

that fulfill

\[
\int \frac{dk_0}{2\pi} 2k_0 A_\phi = 1,
\]

\[
\int \frac{dk_0}{2\pi} 2\gamma_0 A_\psi = 1.
\]

Next, we employ the ansatz

\[
i\Delta^< = 2n_S(k_0) A_\phi,
\]

\[
i\Delta^> = 2(n_S(k_0) + 1) A_\phi,
\]

\[
iS^< = -2(n_L(k_0) P_L A_\psi P_R + n_R(k_0) P_R A_\psi P_L),
\]

\[
iS^> = 2(A_\psi - n_L(k_0) P_L A_\psi P_R - n_R(k_0) P_R A_\psi P_L),
\]

with the generalized particle distribution functions \( n(k_0) \). This turns the conserved quantities in eq. (29) into

\[
Q_S = \int \frac{d^3k}{(2\pi)^3} (n_S(\omega) - \bar{n}_S(\omega)),
\]

\[
P^0_S = \int \frac{d^3k}{(2\pi)^3} (\omega n_S(\omega) + \omega \bar{n}_S(\omega)),
\]

\[
Q_{L,R} = \int \frac{d^3k}{(2\pi)^3} (n_L,R(k) - \bar{n}_L,R(k)),
\]

\[
P^0_{L,R} = \int \frac{d^3k}{(2\pi)^3} (k n_{L,R}(k) + k \bar{n}_{L,R}(k)),
\]

where we have introduced the bosonic dispersion relation \( \omega^2 = k^2 + M^2 \). Besides, we have defined the anti-particle distribution functions for negative energies

\[
\bar{n}_S(-\omega) = -(n_S(\omega) + 1), \quad \bar{n}_{L,R}(k) = 1 - n_{L,R}(-k_0).
\]
From these definitions, it is clear that the conserved quantities represent total energy and charges. In particular, the knowledge of the initial charges and total energy should suffice to determine the final equilibrium state of the system that is characterized by the temperature and the chemical potentials after equilibration. Assuming kinetic equilibrium, the Kubo-Martin-Schwinger (KMS) relation including chemical potentials are
\[ \Delta^> = e^{\beta(p_0-\mu_S)}\Delta^<, \quad P_L S^> = -e^{\beta(p_0-\mu_L)}P_L S^<, \quad P_R S^> = -e^{\beta(p_0-\mu_R)}P_R S^<. \] (40)

These relations then imply that the distribution functions are of the form
\[ n_{L,R} = 1/(e^{\beta(p_0-\mu_{L,R})} + 1), \quad n_S = 1/(e^{\beta(p_0-\mu_S)} - 1). \] (41)

Under the close-to-equilibrium assumption that \( \mu_{L,R,S}/T \ll 1 \), we can linearly relate the charges to the chemical potentials by introducing statistical factors as
\[ Q_{L,R,S} = \frac{T^2}{6} k_{L,R,S} \mu_{L,R,S}, \] (42)
where
\[ k_{L,R} = 12T^{-3} \int \frac{d^3 k}{(2\pi)^3} n_F^e q(1 - n_F^e) = \frac{6}{\pi^2} \int_0^\infty x^2 dx \frac{e^x}{(e^x + 1)^2} = 1, \] (43)
\[ k_S = 12T^{-3} \int \frac{d^3 k}{(2\pi)^3} n_B^e q(n_B^e + 1) = \frac{6}{\pi^2} \int M/T \sqrt{x^2 - \frac{M^2}{T^2}(e^x - 1)^2}. \] (44)

In the limit \( M/T \to 0 \), the latter integral evaluates to \( k_S = 2 \), and we plot the statistical factors as a function of \( M/T \) in Fig[1].

Next, we note that the self-energies (27) and (28) inherit the KMS property (40) from the Wightman functions,
\[ \Pi^>(k) = \Pi^<(k)e^{\beta(k_0-\mu_L+\mu_R)}, \] (45a)
\[ \Sigma^>_R(k) = -\Sigma^<_R(k)e^{\beta(k_0-\mu_L+\mu_S)}, \] (45b)
\[ \Sigma^>_L(k) = -\Sigma^<_L(k)e^{\beta(k_0-\mu_R-\mu_S)}. \] (45c)

From the Feynman rules in the Schwinger-Keldysh formalism (in particular from energy-momentum conservation) it is possible to derive that these relations even hold to arbitrary order in perturbation theory. Consequently, we find for the collision terms
\[ \Pi^>(k)\Delta^<(k) - \Pi^<(k)\Delta^>(k) = \Pi^>(k)\Delta^<(k)\left(1 - e^{\beta(\mu_L-\mu_R-\mu_S)}\right), \] (46a)
\[ \Sigma^>_L(k)S^<_L(k) - \Sigma^<_L(k)S^>_L(k) = \Sigma^>_L(k)S^<_L(k)\left(1 - e^{\beta(\mu_R-\mu_L+\mu_S)}\right), \] (46b)
\[ \Sigma^>_R(k)S^<_R(k) - \Sigma^<_R(k)S^>_R(k) = \Sigma^>_R(k)S^<_R(k)\left(1 - e^{\beta(\mu_L-\mu_R-\mu_S)}\right), \] (46c)
such that these terms vanish and a static solution of the transport equations in eqs. (20) and (23) is obtained if
\[ \mu_S + \mu_R - \mu_L = 0. \] (47)

Hence, one would expect that the final state is determined by the temperature and the three conserved quantities in eq. (30) only, and that it satisfies the equilibrium condition (47) above.

However, it turns out that in the on-shell approximation, not only the combinations (30) are conserved, but also additional charges in the particle and anti-particle sectors separately [12]. This is a purely kinematic effect. Consider a scalar decaying into a fermion/anti-fermion pair. If all three particles are on-shell, all energies have to have the same sign. This is most easily seen in the rest-frame of the scalar, where \( k_0 = \pm M \). In this frame, the momenta of fermions differ only by a sign and hence their energies are equal and therefore \( q_0 = p_0 = k_0/2 \). This means that the collision integrals in eqs. (20) and (23) have in the on-shell limit only a support in the regions
\[ \text{sign}k_0 = \text{sign}p_0 = \text{sign}q_0. \] (48)

This is sufficient to show that the number of fermions plus twice the number of scalars is separately conserved for particles and for anti-particles, such that the Boltzmann equation cannot reproduce chemical equilibration in the on-shell limit. Formally, we can express these conservation laws by defining
\[ \bar{Q}_{L,R,S} = \int \frac{d^3k}{(2\pi)^3} \left( n_{L,R,S}(\omega) + \bar{n}_{L,R,S}(\omega) \right) \] (49)
and noting
\[ 2Q_S + Q_L + Q_R = \text{const.} \] (50a)
\[ \bar{Q}_L - \bar{Q}_R = \text{const.} \] (50b)

In order to distinguish from the conserved charges defined in eqs. (30), we refer to these combinations as total particle number densities. We introduce an additional set of chemical potentials \( \bar{\mu}_{L,R,S} \) to account for total particle numbers different from their equilibrium values. These are defined through the relations
\[ \Delta^+ = e^{\beta(p_0 - \bar{\mu}_S \text{sign}p_0)} \Delta^-, \] (51)
\[ P_L S^> = -e^{\beta(p_0 - \mu_L \text{sign} p_0)} P_L S^<, \quad P_R S^> = -e^{\beta(p_0 - \mu_R \text{sign} p_0)} P_R S^<. \] (52)

Now by taking combinations of \( \mu_{L,R,S} \) and \( \bar{\mu}_{L,R,S} \), we can adjust the number of particles and anti-particles independently.

For a support of the form as in eq. (48), the collision terms for non-vanishing \( \bar{\mu}_{L,R,S} \) read

\[ \Pi^>(k) \Delta^<(k) - \Pi^<(k) \Delta^>(k) = \Pi^>(k) \Delta^<(k) \left( 1 - e^{\text{sign}(k_0) \beta (\bar{\mu}_L + \bar{\mu}_R - \bar{\mu}_S)} \right), \] (53a)
\[ \Sigma^>_L(k) S^<_L(k) - \Sigma^<_L(k) S^>_L(k) = \Sigma^>_L(k) S^<_L(k) \left( 1 - e^{\text{sign}(k_0) \beta (-\bar{\mu}_R - \bar{\mu}_L + \bar{\mu}_S)} \right), \] (53b)
\[ \Sigma^>_R(k) S^<_R(k) - \Sigma^<_R(k) S^>_R(k) = \Sigma^>_R(k) S^<_R(k) \left( 1 - e^{\text{sign}(k_0) \beta (-\bar{\mu}_L - \bar{\mu}_R + \bar{\mu}_S)} \right), \] (53c)

and these vanish as long as

\[ \bar{\mu}_S - \bar{\mu}_L - \bar{\mu}_R = 0. \] (54)

Notice that this is only true in the two-loop approximation of the effective action.

To summarize, we have defined six chemical charges, \( Q_{L,R,S} \) and \( \bar{Q}_{L,R,S} \). Through on-shell processes, two linear combinations of these charges are not conserved and equilibrate according to the conditions (47) and (54). Therefore, in the on-shell limit there remain four conserved chemical charges as defined by eqs. (30a, 30b) and (50). While the charges (30a, 30b) are conserved by virtue of Lagrangian symmetries, the conservation of the charges (50) is an artifact of the on-shell approximation. Notice that this argument is in principle not based on the gradient expansion, since for these time-independent chemical potentials the collision terms vanish to all orders in gradients as long as the support of the collision integral is of the form as in eq. (18).

**VI. TIME-SCALES**

The considerations of the last section show that the relaxation of some chemical potentials is described by on-shell Boltzmann equations. The rate of these relaxation processes is given by the thermally averaged matrix elements of the on-shell scattering process. Off-shell equilibration on the other hand is additionally suppressed by the Breit-Wigner width of the particles in the plasma, which is why we expect these processes to be slow compared to the on-shell relaxation. In the following, we analytically calculate the on-shell relaxation time-scale \( \tau_{on} \) and the off-shell time-scale \( \tau_{off} \) in linear response theory, and we show that indeed, they exhibit a separation \( \tau_{on} \ll \tau_{off} \). For this purpose, we assume that the
deviations from equilibrium are of the form
\[
\delta \Delta^<(k) = \delta \Delta^>(k) = 2\beta \delta \mu_S(k)n_B^{eq}(k)(n_B^{eq}(k) + 1)A_\phi, \tag{55}
\]
\[
\delta S^<(k) = \delta S^>(k) = -2\beta(P_L\delta \mu_L(k) + P_R\delta \mu_R(k))n_F^{eq}(k)(1 - n_F^{eq}(k))A_\psi. \tag{56}
\]
These expressions follow from the definitions (33) with
\[
n_S(k) = \frac{1}{e^{\beta(k_0 - \delta \mu_S(k))} - 1}, \quad n_{L,R}(k) = \frac{1}{e^{\beta(k_0 - \delta \mu_{L,R}(k))} + 1}, \tag{57}
\]
when expanded for \(\delta \mu_{S,R,L}/T \ll 1\).

A. Charge Equilibration

Let us now calculate the relaxation rate for the charge densities that is damped by on-shell processes. Therefore, we consider the case of chemical potentials without energy dependence, \(\delta \mu_{L,R,S} = \mu_{L,R,S}\).

Using the kinetic equations (20) and (23), one obtains in linear response three equations
\[
\partial_t Q_L = \frac{1}{T}C^{on}(\mu_S + \mu_R - \mu_L), \tag{58a}
\]
\[
\partial_t Q_R = -\frac{1}{T}C^{on}(\mu_S + \mu_R - \mu_L), \tag{58b}
\]
\[
\partial_t Q_S = -\frac{1}{T}C^{on}(\mu_S + \mu_R - \mu_L), \tag{58c}
\]
what is consistent with the conserved combinations in eq. (30). The equation for the non-conserved combination of chemical potentials (47) is
\[
\partial_t (\mu_S + \mu_R - \mu_L) = -\frac{6}{T^3} \left( \frac{2}{k_{L,R}} + \frac{1}{k_S} \right) C^{on}(\mu_S + \mu_R - \mu_L), \tag{59}
\]
where in on-shell approximation
\[
C^{on} = -y^2 \int \frac{d^4p d^4q d^4k}{(2\pi)^8} \delta(k - p - q) \text{Tr}[P_RiS^>(-p)P_LiS^<(q)]i\Delta^>(k) \tag{60}
\]
\[
= 2y^2 \int \frac{d^4p d^4q}{(2\pi)^8} p_\mu q^\nu n_F^{eq}(p_0)n_F^{eq}(q_0)(1 + n_B^{eq}(p_0 + q_0)) \times \text{sign}(q_0)\delta(q^2)\text{sign}(p_0)\delta(p^2)\text{sign}(q_0 + p_0)\delta((q + p)^2 - M^2). \tag{61}
\]
After energy and angular integration, this reads
\[
C^{on} = y^2M^2 \int \frac{dp dq}{16\pi^3} \theta(4qp - M^2)n_F^{eq}(p)n_F^{eq}(q)(1 + n_B^{eq}(p + q)). \tag{62}
\]
FIG. 1: The plots show the coefficients $k_{L,R}$ and $k_S$ as functions of the scalar mass $M$.

The symmetry of this expression becomes more apparent when noting that

$$n_F^\text{eq}(p)n_F^\text{eq}(q)(1 + n_B^\text{eq}(p + q)) = (1 - n_F^\text{eq}(p))(1 - n_F^\text{eq}(q))n_B^\text{eq}(p + q).$$

(63)

The integral (62) has an interpretation in terms of processes that describe the decay of the scalar into a fermion–anti-fermion pair and annihilation of a fermion–anti-fermion pair into a scalar. These processes are suppressed for small scalar masses, since this reduces the phase space of the kinetically allowed final states. We plot $C^\text{on}$ in Fig. 2 from where its features can be verified.

From there the typical time-scale of charge equilibration can be read off:

$$\gamma^Q = \frac{6}{T^3}(k_S^{-1} + 2k_{L,R}^{-1})C^\text{on}.$$  

(64)

The resulting time scale for the choices $y^2 = 1.0$ and $y^2 = 0.1$ is shown in Fig. 3.

For $M/T \gg 1$, it is instructive to evaluate the relevant integrals as

$$k_S \sim \sqrt{18/\pi^3}e^{-\frac{M}{T}} \left(\frac{M}{T}\right)^{\frac{3}{2}}$$

(65)

and

$$C^\text{on} \sim \frac{y^2}{\sqrt{512\pi^6}}M^2T^\frac{3}{4}e^{-\frac{M}{T}},$$

(66)

such that

$$\gamma^Q \sim \frac{M}{16\pi}.$$  

(67)
FIG. 2: The plot shows the collision terms $C_{\text{on}}$ and $C_{\text{off}}$ as functions of the scalar mass $M$. The combination $C_{\text{off}}/y^4$ has a small residual dependence on $y$ and we use $y = 1.0$ in the plot.

FIG. 3: The plots show the time-scales of on-shell chemical equilibration, $\gamma^Q$ given by eq. (64), and the three different time-scales given by the eigenvalues of the system in eq. (69) as a function of the scalar mass $M$. In the left (right) plot a coupling $y^2 = 1.0$ ($y^2 = 0.1$) has been used.

Therefore, chemical equilibration according to (47) is attained faster for larger $M$. We emphasize however, that this does not imply that physical interaction rates become fast. The opposite is true, as can be seen from the exponential decrease of $C_{\text{on}}$. Chemical equilibration can occur, because in the large mass limit, a change in the chemical potential $\mu_S$ only corresponds to an exponentially small change in the physical charge density $Q_S$. The increase of $\gamma^Q$ with larger $M$ has been noted in refs. [25, 26]. There it has been shown that this feature
simplifies considerations of chemical equilibration for electroweak baryogenesis and for the conversion of baryon-minus-lepton number to baryon number, as relevant for leptogenesis.

**B. Total Particle Number Equilibration**

In order to investigate the equilibration of total particle number, consider chemical potentials of the form

$$\delta \mu_{L,R,S}(k) = \bar{\mu}_{L,R,S}\text{sign}(k_0).$$

(68)

Using this ansatz in the kinetic equations (20) and (23) and linearizing in the chemical potentials then leads to

$$\frac{T^3}{6}k_S \partial_t \bar{\mu}_S = -C^{\text{on}}(\bar{\mu}_S - 2\bar{\mu}_+) - C^{\text{off}}\bar{\mu}_S,$$

(69a)

$$\frac{T^3}{6}k_{L,R} \partial_t \bar{\mu}_+ = C^{\text{on}}(\bar{\mu}_S - 2\bar{\mu}_+),$$

(69b)

$$\frac{T^3}{6}k_{L,R} \partial_t \bar{\mu}_- = -C^{\text{off}}\bar{\mu}_-,$$

(69c)

where we have defined the combinations $$\bar{\mu}_\pm = \frac{1}{2}(\bar{\mu}_L \pm \bar{\mu}_R)$$ and the off-shell decay rate

$$C^{\text{off}} = y^2 \int \frac{d^4p d^4q d^4k}{(2\pi)^8}\delta(k - p - q)\text{Tr}[P_R iS^>(-p)P_L iS^<(q)]i\Delta^>(k) \times [\text{sign}(k_0) - (\text{sign}(q_0) + \text{sign}(p_0))/2]^2.$$  

(70)

Notice that without the off-shell contributions, there would be two unsuppressed modes, namely $$\bar{\mu}_-$$ and $$k_S \bar{\mu}_S + k_{L,R} \bar{\mu}_+.$$ On the other hand, including the off-shell effects and in the limit of $$C^{\text{off}}/C^{\text{on}} \ll 1,$$ these are still approximate eigenmodes which are damped at the rates

$$\gamma_1 \approx \frac{6}{T^3}k_{L,R}^{-1}C^{\text{off}}, \quad \gamma_2 \approx \frac{6}{T^3}k_S^{-1}C^{\text{off}}.$$  

(71)

In addition, there is the approximate mode $$\bar{\mu}_S - 2\bar{\mu}_+,$$ which is suppressed by on-shell equilibration at the rate

$$\gamma_3 \approx \frac{6}{T^3}(2k_{L,R}^{-1} + k_S^{-1})C^{\text{on}}.$$  

(72)

Let us now proceed with the evaluation of the decay rate $$C^{\text{off}}$$ close to equilibrium. The main contributions to this integral come from the regions, where two of the particles are on-shell, but the third is off-shell violating the condition in eq. (48). We hence replace two
of the Breit-Wigner spectral functions by delta functions. For simplicity, we present only the contribution where the scalar is off-shell, in which case one obtains the contribution

\[
\mathcal{C}^\text{off}_\phi = y^2 \int \frac{d^4 p d^4 q}{16 \pi^6} p_\mu q_\nu n_F^\text{eq}(p_0) n_F^\text{eq}(q_0) n_B^\text{eq}(-p_0 - q_0) \\
\times \text{sign}(q_0) \delta(q^2) \text{sign}(p_0) \delta(p^2) \frac{\Gamma_\phi(p_0 + q_0, |\vec{p} + \vec{q}|)}{(p^2 + 2p \cdot q + q^2 - M^2)^2 + \Gamma_\phi^2(p_0 + q_0, |\vec{p} + \vec{q}|)} \\
\times [\text{sign}(p_0 + q_0) - (\text{sign}(q_0) + \text{sign}(p_0))/2]^2. \tag{73}
\]

Notice that if the two fermions are on-shell with \(\text{sign}(p_0) \neq \text{sign}(q_0)\), the momentum of the scalar \(k^\mu = (\omega, \vec{k})\) is necessarily space-like, \(\omega^2 - k^2 < 0\). The corresponding Breit-Wigner width is close to equilibrium given by

\[
\Gamma_\phi(\omega, k) = y^2 (\omega^2 - k^2) \int_{(k-\omega)/2}^{(k+\omega)/2} \frac{dp}{8\pi k} n_\psi(p) \\
= y^2 \frac{\omega^2 - k^2}{\beta k} \ln \left( \frac{1 + e^{\frac{k^2}{2}(k-\omega)}}{1 + e^{\frac{k^2}{2}(k+\omega)}} \right). \tag{74}
\]

Integration over the energies yields

\[
\mathcal{C}^\text{off}_\phi = y^2 \int \frac{dp dq \sin \theta d \theta}{8 \pi^4} p^2 q^2 (1 - \cos \theta) \\
\times (n_F^\text{eq}(-p) n_F^\text{eq}(q) n_B^\text{eq}(p-q) - n_F^\text{eq}(p) n_F^\text{eq}(-q) n_B^\text{eq}(q-p)) \\
\times \frac{\Gamma_\phi(|p-q|, \sqrt{p^2 + q^2 - 2pq \cos \theta})}{(2pq(1 - \cos \theta) + M^2)^2 + \Gamma_\phi^2(|p-q|, \sqrt{p^2 + q^2 - 2pq \cos \theta})}. \tag{75}
\]

Here, we have used that \(q_0\) and \(p_0\) need to have opposite signs. Notice that this does not contain the potentially harmful pole in the bosonic distribution function, since the width \(\Gamma\) vanishes in this case. A plot of \(\mathcal{C}_\phi^\text{off}\) is provided in Fig. \[2\].

Since the off-shell effects are proportional to the Breit-Wigner width, they are proportional to \(y^4\) and hence suppressed compared to on-shell effects for small couplings. The resulting equilibration rates given by \(\gamma^Q\) as in eq. \[63\] and by the eigenvalues of the system in eq. \[69\] for the choices \(y^2 = 1.0\) and \(y^2 = 0.1\) are shown in Fig. \[3\].

It is again possible to make an estimate for the \(M/T \gg 1\) limit. The integrand of eq. \[75\] receives only relevant contributions from \(p, q \ll M\), such that we can replace the nominator by \(M^4\). Numerical evaluation then yields

\[
\mathcal{C}_\phi^\text{off} \sim 0.126 y^4 \frac{T^8}{M^4}. \tag{76}
\]
The suppression for large $M$ is not exponential, but given by the Breit-Wigner width. From Fig. 3 it is however evident that when $M/T$ is not too large and there is a sizable number of scalar particles $S$ present in the heat bath, the on-shell equilibration rate is dominating over the off-shell processes. Since the on-shell rate is $\propto y^2$ and the off-shell rate $\propto y^4$, this separation of scales becomes more pronounced for smaller $y$.

**VII. DISCUSSION AND CONCLUSIONS**

We have calculated the various time-scales of chemical equilibration in a theory with a Dirac fermion coupled to a scalar. For that purpose, we have applied a gradient expansion and a linear response ansatz to the Kadanoff-Baym equations. The different time-scales arise hereby from processes that are driven either by on-shell or off-shell effects. While on-shell equilibration is described in a quasi-particle picture with $\delta$-functions accounting for the on-shell conditions, we have shown that by including finite width effects, the calculation can straightforwardly be generalized to determine the rate of off-shell relaxation. We emphasize that already at leading loop order in the 2PI effective action, a self-consistent solution to the spectral and kinetic equations leads to off-shell chemical equilibration. In addition, the gradient expansion is consistent with off-shell chemical equilibration. Parametrically, we find for the time scales of on- and off-shell chemical equilibration

$$\gamma_{\text{on}} \propto y^2, \quad \gamma_{\text{off}} \propto y^4,$$

what explains the separation of time scales. In our model, equilibration of charge results from on-shell processes, while there exist two linear combinations of total particle numbers that only relax through off-shell contributions.

For our discussion of the off-shell relaxation rate, we have focused on the contributions to the collision integral where the scalar particle is off-shell. Contributions of similar size (and same sign, as can easily be checked) are expected from the domains of integration where the fermions are off-shell. The numerical values presented should hence not be taken at face value, but might be larger by a factor $2 - 3$ in the full calculation, when taking the fermionic off-shell effects into account. Since we find that the rate of off-shell effects is of order $y^4$, three-loop contributions to the effective action might also be equally important as the off-shell effects arising from the two-loop effective action. However, the analyses of
Kadanoff-Baym equations in the literature we referred to also take only the two-loop order of the effective action into account. Hence, we neglected higher loop orders in the effective action to facilitate a direct comparison.

We would like to stress that our analysis does not only confirm the common lore that off-shell effects can be important in certain models. In the model at hand, full equilibration requires the violation of eq. (48), which is a stronger condition than deviation from the on-shell limit. As a consequence of eq. (48) in conjunction with the Breit-Wigner form of the spectral function, we see from eq. (76) the decaying behavior of the off-shell equilibration rate for large M. In contrast, if the spectral function would be Gaussian with a width proportional to \( y^2 \), off-shell equilibration would be exponentially suppressed.

The advantage of transport equations using the gradient expansion in Wigner space is that they do not contain memory integrals and therefore are much easier to solve numerically than the full Kadanoff-Baym equations in coordinate space even if no further approximations as e.g. the on-shell approximation or an ansatz in terms of chemical potentials are used. Hence, it is important to understand the range of applicability of transport equations. In summary, our results indicate that the use of the gradient expansion is justified to describe thermalization at rather late times (namely after effects as prethermalization and damping are concluded) if the off-shell effects are properly taken into account and coupling constants are rather small. This is seen explicitly in the presented analytical approach: In the on-shell approximation, the collision terms vanish at all orders in gradients in some cases even though the system is not in its true equilibrium state, while once including off-shell effects, the system achieves full equilibration already at first order in the gradient expansion. Hence, higher orders in the gradient expansion are not decisive to describe full equilibration at late times qualitatively in the present system.

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