Classical dynamics of quantum fluctuations

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It is shown that the vacuum state of weakly interacting quantum field theories can be described, in the Heisenberg picture, as a linear combination of randomly distributed incoherent paths that obey classical equations of motion with constrained initial conditions. We call such paths “pseudoclassical” paths and use them to define the dynamics of quantum fluctuations. Every physical observable is assigned a time-dependent value on each path in a way that respects the uncertainty principle, but in consequence, some of the standard algebraic relations between quantum observables are not necessarily fulfilled by their time-dependent values on paths. We define “collective observables” which depend on a large number of independent degrees of freedom, and show that the dynamics of their quantum fluctuations can be described in terms of unconstrained classical stochastic processes without reference to any additional external system or to an environment. Our analysis can be generalized to states other than the vacuum. Finally, we compare our formalism to the formalism of coherent states, and highlight their differences.

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I. INTRODUCTION

It is well known that some effects of quantum fluctuations of commuting observables in a stationary state of a system can be understood by considering classical stochastic fluctuations with well defined probability distribution functions, for example, photon shot noise in optical systems, the Casimir force between plates, and the Lamb shift in atoms. However, it is not clear in this context to what extent it is possible to describe additional aspects of the dynamics of such quantum fluctuations using classical stochastic concepts.

The dynamics of quantum fluctuations can be manifest when they couple (weakly) to another external system. For example, when considering atoms which interact with vacuum
fluctuations of the electromagnetic field and spontaneously emit photons, a process that can be described with high precision by QED. Similarly, quantum fluctuations can affect spacetime curvature. In this context, the possible gravitational collapse of strong quantum fluctuations towards black holes was considered recently [1], as well as issues concerning classicality and decoherence of quantum fluctuations [2, 3] during and after inflation, and their time evolution [4, 5]. Additionally, the possible dissipative effects [1] of vacuum fluctuations on arbitrary motions in the vacuum were considered.

We propose here a formalism to describe the dynamics of quantum fluctuations in terms of classical stochastic processes which can complement the standard representation in terms of time-dependent quantum correlation functions. This formalism may enable better understanding of the onset of classicality in quantum systems, and in some cases may become a useful tool for performing calculations, in particular numerical simulations of quantum systems.

We represent the vacuum of weakly interacting quantum field theories in the Heisenberg picture as a linear combination of randomly distributed pseudoclassical paths (PCP’s) that do not interfere between themselves, and obey classical equations of motion with constrained initial conditions. Every physical observable is assigned a time-dependent value on each PCP in a way that respects the uncertainty principle. In consequence, some of the standard algebraic relations between quantum observables are not necessarily fulfilled by their pseudoclassical values on paths. We define collective observables which depend on a large number of independent degrees of freedom, and show that their values on PCP’s approximately satisfy standard algebraic relations between classical observables. We use the PCP’s formalism to represent the dynamics of quantum fluctuations of collective observables in terms of classical stochastic processes with unconstrained initial conditions.

Our formalism can be applied to closed systems without explicit reference to observers, measurement, or an environment. We focus in this paper on fluctuations in the vacuum for simplicity and concreteness, but our formalism can be generalized in a straightforward way to describe quantum fluctuations in other states of the system, as we outline in section V.

Our formalism is somewhat reminiscent of the Feynmann path integral formulation (FPI) [7] and the consistent histories formalism of Gell-Mann and Hartle [8] in that it involves paths as basic elements. However, in the FPI formalism, the vacuum is considered as a superposition of coherent paths that do not necessarily fulfill classical equations of motion...
and produce interference patterns in Green's functions. In the consistent histories approach all such coherent paths are grouped into equivalence classes or non-detailed histories that are approximately incoherent. In our formalism the paths obey classical equations of motion and do not interfere at all, but standard algebraic relations are not satisfied on them.

The notion that we propose for the emergence of classical behaviour in the specific context that we discuss is quite different than the case which is often discussed in the literature. In the standard case which relies on the notion of coherent states, for example, in cosmology, or in finite temperature field theory, the emergence of classicality is a consequence of the system being in a highly occupied state. If the number of quanta in a given state is small, as in the vacuum state of a quantum field theory in Minkowski space, there is no notion of classicality. We show that some aspects of classical behaviour can emerge also for states that are not highly populated. A major difference between our formalism and the coherent states formalism is that we identify a classical behaviour associated to a certain class of observables of the system, whereas in the formalism of coherent states classicality is associated with a certain class of quantum states.

The paper is organized as follows, in section II we define the concept of PseudoClassical Paths (PCP’s), in section III we investigate the properties of PCP’s in quantum mechanics and free quantum field theories, and determine the connection between quantum expectation values and ensemble averages over PCP’s. In section IV we define collective observables and show that their quantum fluctuations can be described classically in terms of statistical averages over PCP’s. In section V we outline the extension of our formalism to weakly interacting quantum field theories in states other than the vacuum. In Section VI we compare our formalism to the formalism of coherent states. Section VII contains summary of results.

II. PSEUDOCLASSICAL PATHS

Consider a quantum system with a given complete set \( \{Q_i, i = 1, \ldots, n\} \) of \( n \) commuting observables. Each path corresponds to a set \( \{|q_i, i = 1, \ldots, n\} \) of possible expectation values in the vacuum for operators in \{\(Q\)\}. The sets \( \{|q_i\} \) span the configuration space of the quantum system in its ground state \(|0\rangle\). The probability of any specific point \(|q_i\rangle\) in configuration space to be realized is given by \(|\psi(q)|^2\), \(\psi(q)\) being the wavefunction of the vacuum in the \(Q\)-representation.
We assign at any point $|q_i\rangle$ the “pseudoclassical” value $(Q_i)_{cl} = q_i$ to each operator in $\{Q\}$, and thus define on configuration space $n$ independent random variables $q_i$ whose statistics is determined by $|\psi(q)|^2$. Furthermore, as we will show, the action on the vacuum of a large class of observables $O$ (which may or may not commute with observables in $\{Q\}$) can be expressed in a unique way as the action of a polynomial function $P_o$ of observables in $\{Q\}$, $O|0\rangle = P_o(Q_i)|0\rangle$. Consequently we may assign to each of such operators a random variable which is the same polynomial function of the independent random variables $q_i$ associated with operators in $\{Q\}$. At every point $|q\rangle$ in configuration space we assign the “pseudoclassical” value $O_{cl}(|q\rangle) = P_o(q_i)$ to the observable $O$. In addition, in the Heisenberg picture, time dependence of $O$ is given by the hermitian operator $O(t) = e^{+iHt}Oe^{-iHt}$, where $H$ is the hamiltonian of the system. The observable $O(t)$ can be associated following the same algorithm with a new random variable $P_o(q_i, t)$ defined on configuration space. Thus, each point in configuration space $|q_i\rangle$ is assigned a time-dependent “pseudoclassical” value for each observable $O$ of the system, that is a path whose realization probability is given by $|\psi(q)|^2$. We will show that time evolution of the values assigned on each of these paths to the observables in $\{Q\}$ and their conjugate momenta obey classical equations of motion with constrained initial conditions.

We call the paths “pseudoclassical paths” (PCP’s). They are not classical paths, because the uncertainty principle prevents the values assigned to observables which do not commute with operators in $\{Q\}$ from satisfying standard algebraic relations between classical observables: it turns out that for such observables $(O(t)_{cl})^*O(t)_{cl}$ is not necessarily equal to $(O(t)^*O(t))_{cl}$.

However, as we shall show, when $O$ depends on a large number of degrees of freedom the differences $(O(t))_{cl}^*O(t)_{cl} - (O(t)^*O(t))_{cl}$ are distributed on configuration space with zero mean and negligible dispersion. In other words, classical algebraic relations are approximately recovered on almost all paths for such observables. We call them “collective observables”. We note that quantum fluctuations of collective observables behave classically, and we may borrow classical concepts to define their typical lifetimes. According to the description we have outlined, we may attribute classical aspects to quantum fluctuations of collective observables, in particular, decoherence.
III. PSEUDOCLASSICAL PATHS IN QUANTUM MECHANICS AND IN QUANTUM FIELD THEORY

To discuss our proposal in a concrete and tractable setup we consider a free complex relativistic scalar field $\Phi(x)$ in Minkowski space-time whose lagrangian density is given by

$$L = \partial_{\mu} \Phi \partial^{\mu} \Phi - M^2 \Phi^* \Phi. \quad (1)$$

The vacuum state of this free theory provides the simplest example for which the ideas we have introduced above can be developed and, in addition, serves as a zero'th order perturbative approximation to the actual vacuum of any weakly interacting theory. Later on in section V we will outline how to extend our description to higher orders of perturbation theory, and to states other than the vacuum.

The hamiltonian of the system

$$H = \int d^3 \vec{x} \left( \Pi^*(\vec{x}) \Pi(\vec{x}) + \partial_j \Phi^*(\vec{x}) \partial_j \Phi(\vec{x}) + M^2 \Phi^* \Phi \right), \quad (2)$$

can be expanded in terms of Fourier modes of the field $\Phi(\vec{x})$ and its conjugate momentum $\Pi^*(\vec{x})$:

$$\Phi(\vec{x}) = \sum_{n_1,n_2,n_3=-N}^{+N} \frac{1}{X^{3/2}} q_{n_1,n_2,n_3} e^{i \sum_{j=1}^{3} n_j x_j / X} \quad (3)$$

$$\Pi^*(\vec{x}) = \sum_{n_1,n_2,n_3=-N}^{+N} \frac{1}{X^{3/2}} p_{n_1,n_2,n_3}^* e^{-i \sum_{j=1}^{3} n_j x_j / X}, \quad (4)$$

in order to express $H$ in terms of decoupled harmonic oscillators,

$$H = \sum_{\vec{n}} \left( p_{\vec{n}}^* p_{\vec{n}} + \kappa_{\vec{n}}^2 q_{\vec{n}}^* q_{\vec{n}} \right), \quad (5)$$

where $\kappa_{\vec{n}} = \left[ \left( \frac{2\pi}{X} \right)^2 |\vec{n}|^2 + M^2 \right]^{1/2}$, and $x_j$ are cartesian coordinates. We impose periodic boundary conditions on the three dimensional box $[0,X] \times [0,X] \times [0,X]$. The size of the box $X$ serves as an infrared cutoff, and $x_{\text{min}} = X/N$ serves as an ultraviolet cutoff.

We introduce normalized canonical operators $(Q,P)_{\vec{n}}$, $Q_{\vec{n}} = \sqrt{\kappa_{\vec{n}}} q_{\vec{n}}$, and $P_{\vec{n}} = \frac{1}{\sqrt{\kappa_{\vec{n}}}} p_{\vec{n}}$, which obey canonical commutation relations: $[Q_{\vec{n}}, P_{\vec{m}}^\dagger] = [Q_{\vec{n}}^\dagger, P_{\vec{m}}] = i \delta_{\vec{n},\vec{m}}$ while any other commutators between them vanish. The hamiltonian can be expressed in terms of the $P$’s and $Q$’s, $H = \sum_{\vec{n}} \kappa_{\vec{n}} \left( P_{\vec{n}}^\dagger P_{\vec{n}} + Q_{\vec{n}}^\dagger Q_{\vec{n}} \right)$. 
Let us focus on the Hilbert space of states of one of the modes, labeled by some generic \( \vec{n} \). The hamiltonian for this specific mode is \( H = \kappa (P^\dagger P + Q^\dagger Q) \) (where \( \kappa = \kappa_{\vec{n}} \)), and we denote its vacuum state by \( |0\rangle \). We now show that, for operators \( O \) that can be expressed as a polynomial in \( Q, P \) and their hermitian conjugates \( O = \mathcal{P}(Q^\dagger, Q, P^\dagger, P) \), it is possible to express \( O|0\rangle \) only in terms of a polynomial \( \mathcal{P}_o \) of \( Q \) and its hermitian conjugate \( Q^\dagger \), \( O|0\rangle = \mathcal{P}_o(Q^\dagger, Q)|0\rangle \).

First we would like to determine how to express the action of \( P \) and \( P^\dagger \) on the vacuum. We note that the operators \( (Q,P) \) can be written in terms of hermitian components \( Q = \frac{1}{\sqrt{2}}(Q_R + iQ_I) \) and \( P = \frac{1}{\sqrt{2}}(P_R + iP_I) \). The hermitian operators \( Q_R, P_R, Q_I, P_I \) obey canonical commutation relations \( [Q_R, P_R] = [Q_I, P_I] = i \), and any other commutators between them vanish. Therefore, \( a_R|0\rangle = \frac{1}{\sqrt{2}}(Q_R + iP_R)|0\rangle = 0 \) and \( a_I|0\rangle = \frac{1}{\sqrt{2}}(Q_I + iP_I)|0\rangle = 0 \), so that \( Q_R|0\rangle = -iP_R|0\rangle \), and \( Q_I|0\rangle = -iP_I|0\rangle \). We now substitute into these two equalities the relations \( Q_R = \frac{1}{\sqrt{2}}(Q + Q^\dagger) \), \( P_R = \frac{1}{\sqrt{2}}(P + P^\dagger) \), \( Q_I = -i\frac{1}{\sqrt{2}}(Q - Q^\dagger) \), and \( P_I = -i\frac{1}{\sqrt{2}}(P - P^\dagger) \). Simple algebra leads to the key relations,

\[
P|0\rangle = iQ|0\rangle \tag{6}
\]

\[
P^\dagger|0\rangle = iQ^\dagger|0\rangle. \tag{7}
\]

Note that eq.\((6)\) is not the conjugate of eq.\((7)\).

To proceed with the proof of our claim we look at terms of the form \( PQ \). From the commutation relations of \( P \) and \( Q \) we obtain

\[
P^\dagger Q|0\rangle = (QP^\dagger - i)|0\rangle \tag{8}
\]

\[
PQ|0\rangle = QP|0\rangle, \tag{9}
\]

and the hermitian conjugates of both relations. From \((7)\) we conclude that

\[
QP^\dagger|0\rangle = iQ^\dagger Q|0\rangle, \tag{10}
\]

which upon substitution into eq.\((8)\) yields

\[
P^\dagger Q|0\rangle = i(Q^\dagger Q - 1)|0\rangle. \tag{11}
\]

Substituting eq.\((8)\) into eq.\((9)\) we obtain

\[
PQ|0\rangle = iQ^2|0\rangle. \tag{12}
\]
Additionally, since \( H|0\rangle = \kappa|0\rangle \), \( P^\dagger P + Q^\dagger Q = 1 \), so we conclude that

\[
(P^\dagger P)|0\rangle = (1 - Q^\dagger Q)|0\rangle,
\]

which is, of course, the same result as we would have obtained by using eq. (11) and eq. (12) in sequence.

Using eqs. (3-13) it is possible to express the action on the vacuum of any monomial (and therefore any polynomial) with arbitrary powers and order of \( Q, Q^\dagger, P \) and \( P^\dagger \) as a polynomial in \( Q, Q^\dagger \). For example, commute all the \( P \)'s to the right most position so they act directly on \( |0\rangle \) and use eq. (13) to express them in terms of \( Q \). Then commute all the \( P^\dagger \)'s to the right most position and use eq. (7) to express them in terms of \( Q^\dagger \).

This procedure is guaranteed to be well defined, that is, produce a unique result for all monomials that are related to each other by permuting commuting operators. To see this, assume that two such monomials lead to the same polynomial in \( Q, Q^\dagger \). Then they should be identical when acting on the vacuum, so, their difference (which can also be expressed as a monomial in \( Q, Q^\dagger, P \) and \( P^\dagger \)) vanishes when acting on the vacuum. Conversely, assuming that the same monomial leads to two different polynomials, and using the same logic we conclude that the difference between the two polynomials (which can also be expressed a polynomial in \( Q^\dagger, Q \)) must be the polynomial zero, which means that the resulting polynomials are identical.

The operators \( Q^\dagger, Q \) form a complete representation on the Hilbert space of states that we are considering. In this representation each point \( |q\rangle \) in configuration space is labeled by a single complex number \( q \) and the wavefunction of its vacuum state is

\[
\psi(q) = \frac{\sqrt{2}}{\sqrt{\pi}} e^{-|q|^2},
\]

normalized such that

\[
\langle 0|0\rangle = \int dq|\psi(q)|^2 = \frac{2}{\pi} \int dq e^{-2|q|^2} = 1.
\]

Quantum fluctuations of \( Q^\dagger, Q \) can be thought of as random events that can be described by complex conjugates random variables \( q^*, q \) defined on configuration space, whose distribution function is the gaussian \( |\psi(q)|^2 \). Note that we have chosen the same notation for a point \( |q\rangle \) in configuration space and for the random variable \( q \) whose value at each \( |q\rangle \) is \( Q_{cl} = q \). Since \( O|0\rangle = P_o(Q^\dagger, Q)|0\rangle \), we assign to each point \( |q\rangle \) the value \( O_{cl} = P_o(q^*, q) \equiv o(q^*, q) \) for the observable \( O \). For example, according to eq. (5) we assign the variable \( p = iq \) to the
observable $P$, according to eq.(7) we assign $p^\dagger = iq^*$ to $P^\dagger$, $q2 = q^*q$ to $Q^{\dagger}Q$, and according to eq.(13) $p2 = (1 - q^*q)$ to the observable $P^\dagger P$ and $h = \kappa$ to the hamiltonian $H$.

As we have pointed out previously, in the Heisenberg picture the observable $O$ at time $t$ is given by $O(t) = e^{+iHt}Oe^{-iHt}$ which gets associated according to the rules outlined above with a new random variable $o(t)$. For example, $Q(t)|0\rangle = e^{iqt}Q|0\rangle$ and $P(t) = ie^{iqt}Q|0\rangle$, and therefore for each $|q\rangle$ we assign to these operators the values $q(t) = e^{iqt}$ and $p(t) = ie^{iqt} = iq(t)$. Of course, $H(t) = H$ and therefore $h(t) = h$ at any given time. In this sense each point $|q\rangle$ in configuration space is associated in the Heisenberg picture with a path, since any physical observable can be assigned a time-dependent “pseudoclassical” value on each of them. The paths are random since their realization is determined statistically, but when realized they are in some sense deterministic and obey classical equations, as we will show later. The vacuum state $|0\rangle = \int dq\psi(q)|q\rangle$ can then be thought of as a linear combination of such paths. The probability of each path to be realized due to quantum fluctuations is given at any time by the gaussian distribution $|\psi(q)|^2$.

As advertised, the “pseudoclassical” values assigned to $Q(t)$, $Q_{cl}(t) = q(t)$, and its conjugate momentum $P(t)$ obey the appropriate classical equations of motion for the harmonic oscillator [12], $dp(t)/dt = -\partial H/\partial q^*$ and $dq(t)/dt = \partial H/\partial p^*$, with constrained initial conditions which have to be consistent with the uncertainty principle, and in particular with eqs.(3), (7). The solutions are periodic functions whose period is $T = \frac{2\pi}{\kappa}$, and we may identify the characteristic time scale of these paths with $T$. When considering more general operators that are generic polynomials of $q(t)$ it is possible to construct solutions that have different characteristic time scales, in complete analogy with classical Fourier analysis relating time- and frequency-dependent functions.

We now proceed to show that the quantum vacuum expectation value of an operator $O$ is equal to the ensemble average of its corresponding random variable $o$ over the ensemble of PCP’s, $\langle 0|O|0\rangle = \langle o \rangle_{PCP} \equiv \langle O_{cl} \rangle_{PCP}$. By our construction we can express the vacuum expectation value of a general polynomial only in terms of a polynomial in $Q$ and $Q^\dagger$, $\langle 0|O|0\rangle = \langle 0|P_o(Q^\dagger, Q)|0\rangle$, which is given by

$$\langle 0|P_o(Q^\dagger, Q)|0\rangle = \int dq\frac{2}{\pi}P_o(q^*, q)e^{-2|q|^2}. \hspace{1cm} (16)$$

By definition $\int dq\frac{2}{\pi}P_o(q^*, q)e^{-2|q|^2} = \langle o(q^*, q) \rangle_{PCP} \equiv \langle O_{cl} \rangle_{PCP}$. The PCP’s are incoherent: they do not interfere when we compute “pseudoclassical” average values.
Similarly, expressions can be obtained for the second moments of random variables. First, denote \( \langle o^2 \rangle_{PCP} \equiv \langle (O^\dagger O)_{cl} \rangle_{PCP} \). By construction \( O|0\rangle = P_o(Q^\dagger, Q)|0\rangle \), which implies that
\[
\langle 0|O^\dagger O|0\rangle = \langle 0|[(P_o(Q^\dagger, Q))^\dagger P_o(Q^\dagger, Q)]|0\rangle.
\]
Now notice that \( \langle 0|P_o(Q^\dagger, Q)^\dagger P_o(Q^\dagger, Q)|0\rangle = \langle (O_{cl}^*)O_{cl} \rangle_{PCP} \) which by definition satisfies \( \langle (O_{cl}^*)O_{cl} \rangle_{PCP} = \langle o^*o \rangle \). Thus we have shown that
\[
\langle o^*o \rangle_{PCP} - \langle o^2 \rangle_{PCP} = \langle (O_{cl}^*)O_{cl} - (O^\dagger O)_{cl} \rangle = 0.
\]

Equation (17) holds on average over the ensemble of PCP’s, but it does not necessarily hold on each one of the paths nor even on any of them. The “pseudoclassical” values assigned on a path to physical observables which do not commute with \( Q^\dagger, Q \) do not obey standard algebraic relations between classical observables as a consequence of the uncertainty principle, that is, for such operators \( (O_{cl}^*)O_{cl} - (O^\dagger O)_{cl} \) does not necessarily vanish on each path. This is why we call such paths pseudoclassical, even though they obey classical equations of motion. As a simple example let us consider the operator \( P \). We know that
\[
(P_{cl}^*)^*(P_{cl}) = (iQ_{cl})^*(iQ_{cl}) = (Q^\dagger Q)_{cl} = 1 - (P^\dagger P)_{cl},
\]
so \( (P_{cl}^*)^*(P_{cl}) - (P^\dagger P)_{cl} = 1 - 2(P^\dagger P)_{cl} \) which clearly does not vanish identically on a given PCP, but only on the average because \( \langle 0|P^\dagger P|0\rangle = 1/2 \).

**IV. COLLECTIVE OBSERVABLES AND CLASSICAL DYNAMICS OF THEIR QUANTUM FLUCTUATIONS**

Let us now return to the full field theoretic setup and consider generic local Lorentz-covariant field operators of the type \( O(\vec{x}) = \mathcal{P}(\Phi^\dagger(\vec{x}), \Phi(\vec{x}), \Pi^\dagger(\vec{x}), \Pi(\vec{x})) \), which involve “collective” dynamics of all (or many) of the single harmonic modes in the box. Recall that we have regularized the theory in the UV and in the IR, so the number of modes is finite. The free field theoretic vacuum state is therefore a finite tensor product of the vacuum states of each one of the single decoupled oscillators. It can be expressed as a linear combination of pseudoclassical field configurations (PCFC’s) which are tensor products of the PCP’s of single harmonic oscillator modes, and therefore evolve in time according to classical field equations, since each of the PCP’s evolve in time according to its own classical equations of motion. Every field theoretic observable can be expressed as a function of single mode observables and thus assigned a time-dependent “pseudoclassical” value on each field configuration.
Some field theoretic observables have the property that \((O_d)^*O_d - (O_d^\dagger O_d) \sim 0\), not only in the average sense of eq. (17) but in the stronger sense that the dispersion over the ensemble of PCFC’s of this variable is much smaller than the mean of the variable \((O_d)^*(O_d)\):

\[
\sigma_O = \sqrt{\langle |(O_d)^*(O_d) - (O_d^\dagger O_d)|^2 \rangle_{PCP}} \ll \sqrt{\langle |O_d|^2 \rangle_{PCP}}.
\]

That is, standard classical algebraic relations are approximately recovered on most of the PCFC’s for such observables. Condition (18) is naturally satisfied for observables \(O\) which involve many independent single mode degrees of freedom. We call them “collective observables”. According to the central limit theorem, in this case the probability distribution of the variable \(o^*o - o^2\) is gaussian and therefore its higher statistical moments are also suppressed. Condition (18) effectively removes the constraints on the allowed initial conditions for the classical values of collective operators, and therefore on the initial conditions of the corresponding classical stochastic processes.

Condition (18) does not necessarily hold for observables defined on the Hilbert space of states of a single harmonic mode. For example, let us reexamine the operator \((P_d)^*(P_d) - (P_d^\dagger P_d)\) = \((iQ_d)^*(iQ_d) - (1 - (Q_d^\dagger Q_d))\) = \(-1 + 2(Q_d)^*Q_d\). The mean value of this random variable on configuration space, and \(\sigma_P\), can be computed directly from the distribution function (14) of variables \(Q_d\), \((Q_d)^*\). The mean value vanishes according to (17), but \(\sigma_P = 1 = 2 < |Q_d|^2 \gg 2 < |P_d|^2 >\) is not negligible.

For collective observables that can be expressed as a direct sum of many independent single mode operators,

\[
O(\vec{x}) = \sum_{n_1,n_2,n_3=-N}^{N} \frac{1}{X^{3/2}} \sqrt{\kappa_{\vec{n}} P_{\vec{n}}} e^{\exp \left(-2\pi i (\sum_{j=1}^{3} n_j x_j)/X\right)},
\]

such that \(O_{\vec{n}}O_{\vec{m}}|0\rangle = O_{\vec{n}}|0\rangle \otimes O_{\vec{m}}|0\rangle\) if \(\vec{n} \neq \vec{m}\), we find that the random variable \(o^*o - o^2\) is a quadratic form of \(N\) independent random variables, \(N\) being the effective number of contributing single modes, that is the number of coefficients \(O_{n_1,n_2,n_3}\) which are substantially different from zero. Thus, the dispersion \(\sigma_O\) is suppressed by a factor \(1/\sqrt{N}\) relative to the mean squared value of the variable \(o^*o\): \(\sigma_O \sim \frac{1}{\sqrt{N}} \langle |O_d|^2 \rangle_{PCP}\). We may attribute classical properties to the dynamics of quantum fluctuations of such collective observables.

Let consider, for example, the operator \(\Pi(\vec{x})\) introduced in eq. (11)

\[
\Pi(\vec{x}) = \sum_{n_1,n_2,n_3=-N}^{N} \frac{1}{X^{3/2}} \sqrt{\kappa_{\vec{n}} P_{\vec{n}}} e^{\exp \left(-2\pi i (\sum_{j=1}^{3} n_j x_j)/X\right)}.
\]
Following our rules we assign to $\Pi(\vec{x})$ the random variable

$$\pi(\vec{x}) = \sum_{n_1,n_2,n_3=-N}^{+N} \frac{1}{X^{3/2}} \sqrt{\kappa_{\vec{n}}(i q_{\vec{n}})} \exp \left( -2\pi i \left( \sum_{j=1}^{3} n_j x_j \right) / X \right),$$ (21)

defined in terms of $N^3$ independent random variables $q_{\vec{n}}$. Therefore,

$$(\pi(\vec{x}))^* \pi(\vec{x}) = \sum_{\vec{n},\vec{m}} \frac{1}{X^3} \sqrt{\kappa_{\vec{n}} \kappa_{\vec{m}}} \left( p_{\vec{n}}^* p_{\vec{m}} \right) \exp \left( 2\pi i (\vec{n} - \vec{m}) \cdot \vec{x} / X \right).$$ (22)

On the other hand,

$$(\Pi(\vec{x}))^\dagger \Pi(\vec{x}) = \sum_{\vec{n},\vec{m}} \frac{1}{X^3} \sqrt{\kappa_{\vec{n}} \kappa_{\vec{m}}} (p_{\vec{n}}^\dagger p_{\vec{m}}) \exp \left( 2\pi i (\vec{n} - \vec{m}) \cdot \vec{x} / X \right),$$ (23)

so that, using $(p_{\vec{n}}^\dagger p_{\vec{m}})_cl = \delta_{\vec{n}\vec{m}} - (Q_{\vec{n}}^\dagger Q_{\vec{m}})_cl$, we find that the corresponding random variable is

$$\pi^2(\vec{x}) = \sum_{\vec{n},\vec{m}} \frac{1}{X^3} \sqrt{\kappa_{\vec{n}} \kappa_{\vec{m}}} \left( \delta_{\vec{n}\vec{m}} - q_{\vec{n}}^* q_{\vec{m}} \right) \exp \left( 2\pi i (\vec{n} - \vec{m}) \cdot \vec{x} / X \right),$$ (24)

and therefore

$$(\pi(\vec{x}))^* \pi(\vec{x}) - \pi^2(\vec{x}) = - \sum_{\vec{n}} \frac{1}{X^3} \kappa_{\vec{n}} + 2 \sum_{\vec{n},\vec{m}} \frac{1}{X^3} \sqrt{\kappa_{\vec{n}} \kappa_{\vec{m}}} (p_{\vec{n}}^* p_{\vec{m}}) \exp \left( 2\pi i (\vec{n} - \vec{m}) \cdot \vec{x} / X \right),$$ (25)

which, as anticipated, is a quadratic form in many independent gaussian random variables. Its mean is zero as expected because $\langle (p_{\vec{n}}^* p_{\vec{m}})_{PCP} \rangle = \frac{1}{2} \delta_{\vec{n}\vec{m}}$ from (14). The dominant contribution to the variables appearing in eq.(23) comes from modes whose momentum is about the ultraviolet cutoff, so that effectively $\mathcal{N} \sim N^2$. The dispersion $\sigma_\Pi$ is of order $\sqrt{\mathcal{N}} \sim N$. On the other hand the average value of $(\pi(\vec{x}))^* \pi(\vec{x})$ is of order $\mathcal{N} \sim N^2$. We see explicitly that $\Pi(\vec{x})$ is a collective observable.

As another concrete illustration of our formalism and for a more detailed discussion of lifetimes of quantum fluctuations of collective observables let us consider the collective operator (introduced in [1]),

$$\mathcal{H}_V = \frac{1}{2X^3} \sum_{\vec{n},\vec{m} \neq 0} \frac{1}{\sqrt{\kappa_{\vec{n}} \kappa_{\vec{m}}}} F_V(\vec{n} - \vec{m}, X) \left[ \kappa_{\vec{n}} \kappa_{\vec{m}} P_{\vec{n}}^\dagger P_{\vec{m}} + \kappa_{-\vec{n}} \kappa_{-\vec{m}} Q_{\vec{n}}^\dagger Q_{\vec{m}} \right] + h.c.,$$ (26)

which describes the energy contained in a finite volume $V$ enclosed in the 3D box $[0,X]^3$. On the right hand side of the expression we have introduced the notation $\kappa_{\vec{n},\vec{m}} = \frac{(2\pi)^2}{X} (\vec{n} \cdot \vec{m}) + M^2$, where the $M$ is the mass of the scalar field. The function $F_V(\vec{k}, X) = \int_V d^3 \vec{x} e^{-2\pi i \vec{k} \cdot \vec{x}}$ can be evaluated analytically for simple geometries, for example,
a box with dimensions $L_1, L_2, L_3 < X$. In this particular case: $F_V(\vec{k}, X) = V$, if $\vec{k} = 0$; and $F_V(\vec{k}, X) = \prod_{j=1,2,3} \frac{X}{L_j} \left( e^{2\pi i k_j L_j / X} - 1 \right)$, if $\vec{k} \neq 0$.

As noticed in [1], if $V$ is the whole box then the vacuum $|0\rangle$ of the system is an eigenstate of $H_V$, so it does not fluctuate. For partial volumes whose typical size $V^{1/3}$ is parametrically larger than the ultraviolet cutoff $X/N$, the vacuum $|0\rangle$ is no longer an eigenstate of $H_V$, and therefore $H_V$ fluctuates quantum mechanically.

Using our rules it is straightforward to obtain the random variable $h_V$ assigned to $H_V$,

$$h_V(t) = h_V^0 + \frac{1}{X^3} \sum_{\vec{n}, \vec{m} \neq 0} \text{Re} \left[ \frac{F_V(\vec{n} - \vec{m}, X)}{\sqrt{\kappa_{\vec{n}} \kappa_{\vec{m}}}} (\kappa_{\vec{n}} \kappa_{\vec{m}}) q_{\vec{n}}^* q_{\vec{m}} \right] \exp(i(\kappa_{\vec{n}} + \kappa_{\vec{m}})t), \quad (27)$$

where $h_V^0 = \frac{V}{X^3} \sum_{\vec{n}} \kappa_{\vec{n}}$ is the average value of quantum energy fluctuations in the volume $V$ and $q_{\vec{n}}, q_{\vec{m}}$ are independent gaussian random variables.

We follow [9] to define the lifetime of classical stochastic fluctuations: first, we construct the time-dependent correlation function $f(t_1 - t_2) = \text{Re} \left( (O_{cl}(t_1))^* O_{cl}(t_2) \right)$; then, we define the lifetime of the fluctuation as the inverse of the width in frequency $\omega$ of the Fourier transform $F(\omega)$. The ensemble average $\langle f(t_1 - t_2) \rangle_{PCP} = \langle O_{cl}(t_1) O(t_2) + O(t_2) O_{cl}(t_1) \rangle$ coincides with the standard definition of lifetime of quantum fluctuations. Our definition of the ’lifetime’ of the fluctuations is the definition that is usually used to describe the time scale of a virtual excitation and is not related to the more common definition of transition time from a quantum state to other quantum state induced by a coupling term in the hamiltonian. We believe that our definition is the appropriate definition to describe fluctuations of a quantum system without any reference to an environment or an external perturbing system.

Notice that $\omega$ is a scalar, for example, in the expansion (27) above $\omega = \kappa_{\vec{n}} + \kappa_{\vec{m}}$, so that all the different modes labeled by vectors $\vec{n}$ and $\vec{m}$ which give the same $\omega$ contribute to a unique Fourier mode. The integration over the angles of these vectors contributes to the coefficient of this mode, but not to the width which goes into the definition of the lifetime. Modes that have the same frequency oscillate coherently, and according to the above definition, their lifetime would be infinite. Finite lifetime result from the superposition of many modes with different frequencies. The width in $\omega$ gives the range over which these frequencies are distributed.

The main contribution to $h_V(t)$ in eq.(27) comes from modes whose momentum is about the ultraviolet cutoff $|\vec{n}| \sim |\vec{m}| \sim N$. Its Fourier transform is peaked around $\omega_{UV}$, and the dispersion in $\omega$, $\Delta \omega$, is of order $\omega_{UV}$. Therefore, we find that lifetime of dominant
contributions to the energy fluctuation \( t \simeq \frac{2\pi}{\Delta \omega} \) is inversely proportional to the ultraviolet cutoff of the theory. Subdominant contributions from modes \(|\tilde{n}| \sim |\tilde{n}| < N\) have a longer lifetime, as noticed in [4].

\[ V. \text{ EXTENSIONS} \]

A. Extension to states other than the vacuum

The description of the free vacuum in terms of PCFC’s that we have presented in the previous sections can be generalized to describe other states of the free theory, stationary or non-stationary. In this subsection we show how to carry out this extension. In the next subsection we will show how the description in terms of PCFC’s can be further generalized to quantum states in weakly interacting theories.

Since any state of a free quantum field theory can be represented in a basis of tensor products of single mode stationary states, we will first show how to extend the formalism of PCP’s to stationary excited states of a single harmonic mode. Our immediate goal is to express the action of each single mode operator \( O \) on a generic single mode stationary state \(|\psi\rangle\) in terms of some specific function \( F^\psi_o \) (which is not necessarily a polynomial) of \( Q \) and \( Q^\dagger \) on \(|\psi\rangle\): 

\[ O|\psi\rangle = F^\psi_o(Q^\dagger, Q)|\psi\rangle. \]

Recall that \( Q \) can be written in terms of two hermitian components \( Q = \frac{1}{\sqrt{2}}(Q_R + iQ_I) \) and, therefore, the stationary states of the single harmonic mode that we are considering are labeled by two natural numbers \(|J_1, J_2\rangle = \frac{1}{\sqrt{J_1! J_2!}}(a^\dagger_R)^{J_1}(a^\dagger_I)^{J_2}|0\rangle, \ J_1, J_2 = 0, 1, 2, ..., \) where we are using the notation introduced in section III. For simplicity we will restrict this discussion to the sector of the Hilbert space generated by \( a^\dagger_R \): \(|J_1\rangle = \frac{1}{\sqrt{J_1!}}(a^\dagger_R)^{J_1}|0\rangle. \)

The state \(|J_1\rangle\) can be expressed in terms of the Hermite polynomial \( H_{J_1}(Q_R) \) acting upon the vacuum,

\[ |J_1\rangle = \sqrt{\frac{1}{\pi 2^{J_1} J_1!}} H_{J_1}(Q_R)|0\rangle. \]

The wavefunction of \(|J_1\rangle\) vanishes at the zeroes of the Hermite polynomial and therefore it is possible to invert eq.(28)

\[ |0\rangle = \sqrt{\pi 2^{J_1} J_1!} (H_{J_1}(Q_R))^{-1} |J_1\rangle, \]

which allows us to express the vacuum as a function of the operator \( Q_R \) acting upon the excited state \(|J_1\rangle\).
We use eq. (28) to express \( O|J\rangle = \sqrt{\frac{1}{\sqrt{2\pi}J_1}} O H J_1(Q_R)|0\rangle \) as an operator acting on the free vacuum. Then, we use the rules we stated in section III to determine the action of this operator on the vacuum as a polynomial in \( Q_R: O H J_1(Q_R)|0\rangle = P_{OHJ_1}(Q_R)|0\rangle \). Finally, we use eq. (29) to express the free vacuum as a function of \( Q_R \) acting on \( |J\rangle \). This completes the algorithm that gives the required expression \( O|J\rangle = F^*_{OHJ_1}(Q_R)|J\rangle \). As each single step in the algorithm is uniquely defined the resulting function is also uniquely defined.

As an example, let us apply the proposed algorithm to the particular case of the momentum operator \( P_R \) acting upon the first excited harmonic mode \( |1\rangle \). First, express the excited state by the first Hermite polynomial acting on the free vacuum:

\[
|1\rangle = \sqrt{\frac{1}{\sqrt{\pi}2}} 2Q_R|0\rangle, \tag{30}
\]

to get

\[
P_R|1\rangle = \sqrt{\frac{1}{\sqrt{\pi}2}} 2P_R Q_R|0\rangle. \tag{31}
\]

Then, follow the standard rules stated in section III \( P_R Q_R|0\rangle = i(Q_R^2 - 1)|0\rangle \), so that \( P_R|1\rangle = \sqrt{\frac{1}{\sqrt{\pi}2}} 2i(Q_R^2 - 1)|0\rangle \). Finally, invert eq. (30)

\[
P_R|1\rangle = i \frac{Q_R^2 - 1}{Q_R} |1\rangle. \tag{32}
\]

The algorithm can be applied in similar ways to time dependent operators \( O(t) \), so that we can apply the formalism of pseudoclassical paths. Each PCP corresponds to a point \( |q_r\rangle \) in configuration space. The probability of such path to be realized is now given by the modulus squared of the wavefunction of the first excited state \( |\psi_1(q_r)|^2 \) at such point. Let us point out again that the operator \( 1/Q_R \) is well defined on \( |1\rangle \) because \( \psi_1(q_r) \) vanishes at \( q_r = 0 \).

It is also possible generalize our formalism to non-stationary quantum states of single harmonic modes. Any state \( |\psi\rangle \) can be expanded in a basis of stationary states \( |\psi\rangle = \sum c_{J1}|J1\rangle \).

All finite linear combinations of basis states can also be expressed as a polynomial in \( Q_R \) acting upon the free vacuum

\[
|\psi\rangle = \sum c_{J1}|J1\rangle = \sum c_{J1} \sqrt{\frac{1}{\sqrt{2\pi}J_1}} H J_1(Q_R)|0\rangle = H(Q_R)|0\rangle.
\]

The subsequent steps of the algorithm are identical to those defined for the case of a single stationary state, in particular inversion is possible because \( H(Q_R) \) is a polynomial and the wavefunction of the state vanishes at points where \( H(Q_R) \) vanishes.

We would like to emphasize that the essential reason that pseudoclassical values \( p(t) \) and \( q(t) \) assigned to observable \( P(t) \) and \( Q(t) \) obey classical equations of motion on each PCP
for general quantum states is the operator equations that \( Q_R(t) \) and \( P_R(t) \) obey,

\[
\frac{dP_R(t)}{dt} = -Q_R(t) \quad \frac{dQ_R(t)}{dt} = P_R(t).
\] (33)

The operator equations (33) assure that the classical equations of motion are valid when the operators act on any quantum state

\[
\frac{dP_R(t)}{dt}|\psi\rangle = \frac{dF_{P_R}^{|\psi\rangle}(Q_R, t)}{dt}|\psi\rangle = -Q_R(t)|\psi\rangle
\]

\[
\frac{dQ_R(t)}{dt}|\psi\rangle = P_R(t)|\psi\rangle = F_{P_R}^{|\psi\rangle}(Q_R, t)|\psi\rangle.
\] (34)

However, the particular relation between the classical initial value of the generalized coordinate and its conjugate momentum are different for each different quantum state, as can be realized for example by comparing equations (6) and (7) with equation (32). In other words, different sets of initial conditions correspond to PCP’s of different quantum states.

The description in terms of PCFC’s can now be extended to quantum field theoretic states other than the vacuum. States that can be obtained as a tensor product of single mode states can be described in terms of PCFC’s which are tensor products of the PCP’s of single harmonic oscillator modes. Other quantum field theoretic states that can be obtained as a linear combination of tensor products can be written in terms of a polynomial in \( Q_{\vec{n}} \) and \( Q_{\vec{m}}^\dagger \) acting upon the free field vacuum \(|\psi\rangle = H(Q_{\vec{n}}, Q_{\vec{m}}^\dagger)|0\rangle\). The rest of the discussion is identical to that for a single mode state.

**B. Extension to Interacting Quantum Field Theories**

In this subsection we outline how to extend the formalism of PCFC’s to the case in which the hamiltonian contains weak interaction terms. This is not intended to be an exhaustive study, rather we wish to show that there are no obvious problems in the implementation of our formalism to interacting theories, and thus show that it is not based on some particular properties of free theories.

As a representative example we consider a theory with a polynomial potential \( \lambda V(\phi^\dagger \phi) \). If this interaction term can be treated perturbatively, the Hilbert space of states of the theory can be still expressed as a tensor product of the Hilbert spaces of single harmonic modes,
as in the free field theory case. Therefore, the same rules we have used to assign classical values to each field operator in the free theory can still be applied. The only difference is that in this case the Hamiltonian that appears in the definition of time dependent operators contains the interaction term.

Hence, the only issue to be addressed here is whether PCP’s obey the appropriately modified classical equations of motion. In order to settle this we consider the operator equations that the field operator $\Phi(\vec{x}, t)$ and its conjugate momentum $\Pi(\vec{x}, t)$ obey

$$\frac{d\Pi(\vec{x}, t)}{dt} = -\Phi(\vec{x}, t) - \frac{\delta V}{\delta \Phi^*(\Phi\Phi)}$$

$$\frac{d\Phi(\vec{x}, t)}{dt} = \Pi(\vec{x}, t).$$

(35)

These operator equations guarantee that classical values $(\Pi(\vec{x}, t))_{cl}$ and $(\Phi(\vec{x}, t))_{cl}$ assigned in the vacuum or other quantum states (as explained in the previous subsection) obey the same classical equations of motion on each PCFC.

VI. PSEUDOCLASSICAL PATHS VS. QUANTUM COHERENT STATES

Quantum coherent states are considered as the best description of a quantum system whose dynamics approximately follows a classical trajectory, because the quantum uncertainty is minimal on such states as their wave function is not dispersed during its evolution and, in addition, the average values of the position and conjugate momentum operators evolve according to classical equations of motion. In this sense, each coherent state can be identified, within the precision bounds imposed by the uncertainty principle, with a certain classical trajectory (see, for example, the references given in [1]).

In this section we describe the quantum coherent states of the harmonic oscillator using the formalism of PCP’s that we have developed. This exercise will give us the opportunity to compare the newly introduced concept of pseudoclassical paths to the established concept of coherent quantum states. We will show that a coherent state cannot be identified with a single PCP, and that a single PCP cannot describes any (coherent or not) quantum state. A whole set of PCP’s with their corresponding probability distribution is the only formal object that can describe in our formalism the dynamics of any given quantum state.

Let start with a brief review of the coherent quantum states of a single harmonic oscillator. As we did in section V, we consider the Hilbert space generated by $\hat{a}_R^\dagger$: $|J1\rangle = \frac{1}{\sqrt{J!}}(\hat{a}_R^\dagger)^J|0\rangle$, 
where $J_1 = 0, 1, ...$ is a natural number. For the sake of simplicity we can take $\kappa = 1$, and then the hamiltonian of the system is $H_R = \frac{1}{2} (Q_R^2 + P_R^2)$.

Coherent states are the eigenstates of the annihilation operator $a_R = \frac{1}{\sqrt{2}} (Q_R + iP_R)$:

$$a_R |z\rangle = z |z\rangle,$$  \hspace{1cm} (36)

where $z$ is a complex number. The spectrum of the operator $a_R$ is the whole complex plane.

The normalized eigenstate can be expressed in the basis of stationary states:

$$|z\rangle = e^{-|z|^2/2} \sum_{J_1=0}^{\infty} \frac{z^{J_1}}{\sqrt{J_1!}} |J_1\rangle,$$ \hspace{1cm} (37)

and its wavefunction in the $\{ |q_R\rangle \}$ representation is given by the expression,

$$\Psi_z(q_r) = \frac{1}{\sqrt{\pi^{1/4}}} e^{\frac{-1}{2} (|z|^2 - z^2)} e^{\frac{1}{2} (q_R - \sqrt{2}z)^2}, \hspace{1cm} q_R \in R. \hspace{1cm} (38)$$

In particular, the vacuum state of the harmonic oscillator, $|0\rangle$, is the coherent state that corresponds to the eigenvalue $z = 0$ of the annihilation operator: $a_R |0\rangle = 0$.

Although coherent states are not eigenstates of the hamiltonian $H_R$, their time evolution closely resembles that of stationary states. In the Schroedinger picture:

$$|z, t\rangle = \exp(-iH_Rt) |z\rangle = e^{-it/2} |ze^{-it}\rangle,$$ \hspace{1cm} (39)

where $|ze^{-it}\rangle$ is the coherent state that corresponds to the eigenvalue $z' = ze^{-it}$ of the annihilation operator: $a_R |ze^{-it}\rangle = ze^{-it} (|ze^{-it}\rangle)$. Simple algebra shows that

$$|\Psi_z(q_r; t)|^2 = \frac{1}{\sqrt{\pi}} \exp \left( \left[ q_r - \langle z, t | Q_R | z, t \rangle \right]^2 \right),$$ \hspace{1cm} (40)

the probability density of the wavepacket evolves coherently, centered at $\langle z, t | Q_R | z, t \rangle$ without dispersion, and the quantum uncertainty is minimal: $(\Delta Q_R) \cdot (\Delta P_R) = \frac{1}{2}$.

As the average values of the position and momentum operators obey classical equations of motion:

$$\frac{d \langle z, t | Q_R | z, t \rangle}{dt} = \langle z, t | P_R | z, t \rangle$$ \hspace{1cm} (41)

$$\frac{d \langle z, t | P_R | z, t \rangle}{dt} = -\langle z, t | Q_R | z, t \rangle,$$ \hspace{1cm} (42)

with initial conditions

$$\langle z | Q_R | z \rangle = \sqrt{2} \text{Re}(z),$$ \hspace{1cm} (43)

$$\langle z | P_R | z \rangle = \sqrt{2} \text{Im}(z),$$ \hspace{1cm} (44)
the quantum state is considered as the closest description of a quantum state that approximately follows a classical motion.

We would like to point out that in spite of this suggestive interpretation the similarity between the dynamics of the coherent quantum state and that of a classical trajectory is limited. For example, the average value of the Hamiltonian in the quantum state $|z\rangle$,

$$
\langle z|H_R|z\rangle = \frac{1}{2}(1 + \langle z|Q_R|z\rangle^2 + \langle z|P_R|z\rangle^2) = \frac{1}{2}(1 + 2|z|^2),
$$

(45)
does not match the energy of the corresponding classical trajectory,

$$
E_{cl} = \frac{1}{2}(\langle z|Q_R|z\rangle^2 + \langle z|P_R|z\rangle^2) = |z|^2,
$$

(46)
although the agreement does become relatively better for a large $|z|$.

Let us now examine the coherent state $|z\rangle$ using the formalism of PCP’s. The quantum state $|z\rangle$, as any other quantum state, corresponds to an infinite set of PCP’s, each labeled by a real number $q_R$ and whose probability to occur at any given time is given by the squared modulus of the wavefunction (38). As we know from eq.(33) the time-dependent values $q_R(t)$ and $p_R(t)$ assigned to the operators $Q_R$ and $P_R$ on each of these PCP’s obey classical equations of motion. In addition, eq.(36) implies that

$$
P_R|z\rangle = i(Q_R - \sqrt{2}z)|z\rangle,
$$

(47)
which fixes the initial conditions on each PCP:

$$
q_R(t = 0) = q_R \\
p_R(t = 0) = i(q_R - \sqrt{2}z).
$$

(48)

As an additional example of how the formalism works we compute the value of the Hamiltonian operator on each of these PCP’s. First, we express the operator $H_R$ acting on $|z\rangle$ in terms of some function of the operator $Q_R$ acting on $|z\rangle$:

$$
H_R|z\rangle = \frac{1}{2}(2\sqrt{2}zQ_R - 2z^2 + 1)|z\rangle.
$$

(49)
Then, using this expression we assign to the PCP labeled by $q_R$ the classical value for the energy

$$
h(q_R) = \frac{1}{2}(2\sqrt{2}zq_R - 2z^2 + 1).
$$

(50)
As we have already noticed before, the weighted average value of $h(q_R)$ on the whole set of PCP’s is equal to $\langle z| H_R | z \rangle$. Let us recall that classical values assigned to operators $O$ that commute with the Hamiltonian (in particular $H_R$ itself) do not depend on time since $O(t) = e^{iH_R t}Oe^{-iH_R t} = O$.

In the context of this discussion we want to emphasize that coherent states, as other quantum states, are described by a whole set of PCP’s and, therefore, cannot be identified with a single PCP. Furthermore, a single PCP cannot be identified with a particular quantum state. Let us consider, for example, a pair of different coherent states $|z_1\rangle$ and $|z_2\rangle$, and consider now the PCP labeled by a certain $q_R$ in each of these two quantum states: we immediately see from eq. (48), or eq. (50) that the two paths do not describe the same classical trajectory.

The formalism of PCP’s should be understood as a formal construction that allows to describe the dynamics of quantum states, coherent as well as non-coherent, in terms of pseudoclassical stochastic events as the paths do not interfere at all between themselves. In addition, when the formalism is extended to describe quantum field states in terms of stochastic classical field configurations, the usual algebraic relations between classical observables are approximately recovered for collective quantum operators.

Such a description is not, in general, possible in terms of coherent states, although any quantum state of the harmonic oscillator can be written as a linear superposition of such coherent states, $|\Psi\rangle = \int dz \Psi(z)|z\rangle$. After having identified each coherent state $|z\rangle$ with a certain classical trajectory we could be tempted to describe the generic state $|\Psi\rangle$ as a superposition of classical trajectories as we did with the PCP’s. Nevertheless, coherent states, unlike PCP’s, are actually quantum states that do interfere between themselves and, therefore, the average value of a generic operator $O$ in the state $|\Psi\rangle$, $\langle \Psi | O | \Psi \rangle$, cannot be obtained as the average of its values on each of the coherent trajectories, $\int dz |\Psi(z)|^2 \langle z | O | z \rangle$.

Furthermore, even in the limit when coherent trajectories can be approximately considered as non-interfering trajectories, each with a given probability $\sim |\Psi(z)|^2$ to be realized, the usual algebraic relations between classical observables are not necessarily restored as we will show.

In fact, coherent states are not orthogonal to each other:

$$\langle z | z' \rangle = e^{-(|z|^2+|z'|^2)/2} \sum_{J_1=0}^{\infty} \frac{(z^* z')^{J_1}}{J_1!} = e^{-(|z|^2+|z'|^2)/2} e^{z^* z'}.$$  (51)
Therefore, given a certain operator $O$ expressed in normal ordered form $O(a_R^+, a_R)$ it is immediate to obtain, using eq. (36), that

$$\langle z | O(a_R^+, a_R) | z' \rangle = O(z^*, z') \langle z | z' \rangle = e^{-(|z|^2 + |z'|^2)/2} e^{z^* z'},$$  \hspace{1cm} (52)

and then,

$$| \langle z | O(a_R^+, a_R) | z' \rangle | = |O(z^*, z')| e^{-\frac{1}{2} \sum_n \left( (\text{Re}(z^r_n) - \text{Re}(z'^r_n))^2 + (\text{Im}(z^i_n) - \text{Im}(z'^i_n))^2 \right)}.$$

This last equation means that for a general polynomial operator $O$ only coherent states corresponding to eigenvalues $z, z'$ whose real and imaginary parts are significantly different can be approximately considered as non-interfering.

When we consider the Fock space of more than one harmonic mode, as in quantum field theory, the multi-mode coherent states are obtained as tensor products of single-mode coherent states. Each coherent state is then labeled by a sequence $\{z_n\}$ of complex numbers, each corresponding to the eigenvalue of the annihilation operator of a particular single harmonic mode in the coherent state. This set of states spans the multi-mode Fock space although the coherent states are not orthogonal to each other. From the generalization of eq. (53) to multi-mode coherent states:

$$| \langle \{z_n\} | O(a_{R,n}^+, a_{R,n}) | \{z'_n\} \rangle | = |O(z^*, z')| e^{-\frac{1}{2} \sum_n \left( (\text{Re}(z^r_n) - \text{Re}(z'^r_n))^2 + (\text{Im}(z^i_n) - \text{Im}(z'^i_n))^2 \right)}.$$

it is clear that the exponential suppression factor is stronger the larger the number of harmonic modes involved. In this limit, the classical trajectories associated with each coherent state do not interfere much between themselves.

Once we have identified this set of slightly interfering coherent states, each labeled by a sequence of complex numbers $\{z_n\}$, every observable $O(t)$ can be assigned on each of them the time-dependent value $\langle \{z_n\} | O(t) | \{z_n\} \rangle$, as we did in eqs. (41), (42) for the position and conjugate momentum operators. Then we could try to use this basis to interpret any other quantum state in terms of a superposition of classical trajectories.

Let us consider, in particular, the operator $\Pi(\vec{x})$ introduced in eq. (4). As we have noted, the operators $P_{\vec{n}}$ can be expressed in terms of two hermitian components and, therefore, for each given field theoretic coherent state the index $\vec{n}$ indeed labels a pair $(z^r_{r,\vec{n}}, z^i_{i,\vec{n}})$ of complex numbers. The “classical” value of $\Pi(\vec{x})$ on a coherent state trajectory is given by

$$\langle \{z_{\vec{n}}\} | \Pi(\vec{x}) | \{z_{\vec{n}}\} \rangle = \sum_{\vec{n}} \frac{K_{\vec{n}}^{1/2}}{X^{3/2}} (\text{Im}(z^r_{r,\vec{n}}) + i \text{Im}(z^i_{i,\vec{n}})) e^{2\pi i \vec{x} \cdot \vec{n}/X},$$

(55)
which, in particular, vanishes for the free vacuum \( \langle 0 | \Pi(\vec{x}) | 0 \rangle = 0 \). On the other hand, the value for the operator \( \langle \{ z_{\vec{n}} \} | \Pi^\dagger(\vec{x}) \Pi(\vec{x}) | \{ z_{\vec{n}} \} \rangle \) can be computed,

\[
\langle \{ z_{\vec{n}} \} | \Pi^\dagger(\vec{x}) \Pi(\vec{x}) | \{ z_{\vec{n}} \} \rangle = \sum_{\vec{n}} \frac{\kappa_{\vec{n}}}{X^3},
\]

for any coherent state. We can see that for the vacuum or other coherent states with low \( z \), conditions (17) and (18) are not satisfied, even in the case of collective observables. These conditions, which express the recovery in the quantum formalism of the usual algebraic relations between classical observables, are satisfied in the formalism of coherent states only for states with large \( z \), which correspond to highly populated states.

Therefore, the formalism of coherent states does not permit an adequate description of the dynamics of quantum fluctuations of physical observables in scarcely populated states in terms of classical trajectories. For example, the free vacuum state is itself the coherent state corresponding to the set of eigenvalues \( z_{\vec{k}} = 0 \), or in other words, \( |0\rangle = \int dz \delta(z) |z\rangle \). Such a state would correspond in the description of coherent states to a single classical harmonic trajectory with initial conditions \( \langle Q \rangle = \langle P \rangle = 0 \), which corresponds to a static trajectory. In contrast, in the formalism of PCP’s the free vacuum is represented as a linear superposition of an infinite number of randomly distributed paths, which we have used to describe the dynamics of the quantum fluctuations of physical observables, and conditions (17) and (18) are recovered on each ”pseudoclassical” trajectory for collective observables, as we have proved.

Let us recall that in the PCP’s formalism each classical path is labeled by a set of eigenvalues of a complete representation of commuting observables, for example, the \( \{ Q \} \) representation, but none of these paths can be understood as a quantum state. In the formalism of coherent states, on the other hand, each classical trajectory corresponds to a coherent quantum state which is labeled by the eigenvalues of a set of annihilation operators, which are not hermitian operators and, therefore, do not correspond to physical observables.

Finally, an additional comment on the concept of classicality that we have introduced together with the concept of PCP’s (or PCFC’s for quantum field theoretic states) is pertinent. We have found that the pseudoclassical values on PCP’s assigned to certain class of observables, which we have identified as collective observables, obey approximately the standard algebraic relations between classical observables. The dynamics of the quantum system can then be approximately described, as far as these observables are concerned, as
a classical statistical system. It should be stressed, though, that the indeterminacy of the system has a quantum character. In the formalism of PCP’s we identify classical dynamics associated with a certain class of observables, whereas in the formalism of coherent states classicality is associated with a certain class of quantum states.

VII. SUMMARY

We have developed a formalism in which the quantum states of weakly interacting systems can be described as a linear combination of incoherent paths which obey classical equations of motion with constrained initial conditions and have definite probabilities to be realized due to quantum fluctuations. In this formalism we may represent the dynamics of quantum fluctuations of collective operators which depend on many degrees of freedom in terms of unconstrained classical stochastic processes and the concept of classical dynamics of quantum fluctuations can be defined. Classicality and decoherence of quantum fluctuations are viewed as collective effects.

Although we have focused on a complex relativistic scalar field the whole formalism, and in particular the expression for the random fluctuations of the energy contained in a finite volume, can be generalized with minor changes to describe quantum fluctuations of gauge bosons, fermions or non-relativistic fields with the condition that the independent degrees of freedom are only weakly coupled.

The formalism is based on the following basic properties of weakly coupled quantum field theories:

1) the Hilbert space of weakly interacting theories is the tensor product of the spaces of single modes states

2) Lorentz covariant collective observables can be expressed as a direct sum of independent single-mode operators that are necessarily causally disconnected at some initial time $t = 0$ whatever the actual quantum state of the theory is;

3) the canonical operators $Q, P$ in the Heisenberg picture obey classical equations of motion.
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[11] we use units in which $\hbar = 1, c = 1$.
[12] Similarly, we could have started from relations $\langle 0|a_R^\dagger = 0$ and $\langle 0|a_I^\dagger = 0$ to obtain $\langle 0|P^\dagger = -i\langle 0|Q^\dagger$ and $\langle 0|P = -i\langle 0|Q$ instead of eqs. [3],[7]. Following the same rules we have stated above we obtain the other set of possible solutions to the classical equations $q(t) = exp(-i\epsilon t)q$, $p(t) = -i\epsilon exp(-i\epsilon t)q$. 
