BEYOND QUARTIC HAMILTONIANS

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ABSTRACT. We investigate the Hamiltonian system generated by a homogeneous binary polynomial $U$ of order greater than four. In particular, we study the circumstances under which the associated Hessian is a Weierstrass $\wp$ function and find that the vanishing of two covariants of $U$ is involved.

INTRODUCTION

In [5] we showed that along the Hamilton curves generated by a homogeneous cubic or quartic polynomial in the plane, a suitable rescaling of the associated Hessian satisfies the first-order differential equation

$$F^2 = 4F^3 - g_2F - g_3$$

characteristic of a Weierstrass $\wp$ function, where $g_2$ and $g_3$ are constants of the motion. In the present paper we consider as Hamiltonian a homogeneous binary polynomial of higher order and find that the situation is a little more complicated. Along each Hamilton curve, the associated Hessian again satisfies a differential equation of the same form, but now $g_2$ and $g_3$ need not be constants of the motion: their derivatives are given by constant multiples of two covariants of the homogeneous polynomial; a syzygy relating these covariants to the Hamiltonian and its Hessian then shows that (unless the Hessian itself is constant) constancy of $g_2$ and $g_3$ forces $g_2$ to vanish, as in the case of a homogeneous cubic.

HAMILTONIANS AND HESSIANS

We begin by fixing our conventions with regard to Hamiltonians. As a setting for our study, we take the symplectic plane with linear symplectic coordinates $(p, q)$. Given two functions $W$ and $Z$ in the plane, their classical Poisson bracket $\{W, Z\}$ is their Jacobian $(W, Z)$ with sign reversed: thus

$$\{W, Z\} := \frac{\partial W}{\partial q}\frac{\partial Z}{\partial p} - \frac{\partial W}{\partial p}\frac{\partial Z}{\partial q}$$

coinsides with

$$(Z, W) := \begin{vmatrix} \frac{\partial Z}{\partial p} & \frac{\partial Z}{\partial q} \\ \frac{\partial W}{\partial p} & \frac{\partial W}{\partial q} \end{vmatrix}.$$ 

When the function $U$ is taken as Hamiltonian, the classical Hamilton equations of motion read

$$\dot{q} = \frac{\partial U}{\partial p}, \quad \dot{p} = -\frac{\partial U}{\partial q},$$

by a Hamilton curve we shall mean a solution curve to this system. The derivative of any function $V$ along such a Hamilton curve $\gamma$ is given by

$$(V \circ \gamma)^\circ = \{V, U\} \circ \gamma = (U, V) \circ \gamma.$$ 

We shall often write more simply

$$\dot{V} = \{V, U\} = (U, V) = U_p V_q - U_q V_p$$

where the derivative $^\circ$ is taken along a Hamilton curve, which has been suppressed from the notation.
Throughout, our planar Hamiltonian will be a homogeneous polynomial: a (binary) quantic in classical terminology. The general theory of covariants and invariants for quantics was developed largely by Cayley and Sylvester in the nineteenth century; in particular, Cayley produced a sequence of ten memoirs on the topic between 1854 and 1878. Salmon [6] offers an almost contemporaneous account of this classical theory; Elliott [2] is a slightly more recent elaboration, still in the classical spirit, as is Grace and Young [3]. As we shall encounter several covariants of our quantic, we prepare the analysis by fixing our conventions regarding them.

Thus, let \( U \) be the (binary) quantic of order \( N \) given by

\[
U = a_0 p^N + N a_1 p^{N-1} q + \frac{1}{2} N(N - 1) a_2 p^{N-2} q^2 + \cdots + a_N q^N
\]

where the inclusion of binomial coefficients is both traditional and simplifying. In short,

\[
U = \sum_{n=0}^{N} \binom{N}{n} a_n p^{N-n} q^n
\]

or

\[
U = (a_0, \ldots, a_N)(p, q)^N
\]

in notation introduced by Cayley. Although much of what follows applies quite generally, we shall suppose that \( N > 4 \) unless otherwise stated.

Perhaps the most important covariant associated to \( U \) is its Hessian: the determinant of its matrix of second partial derivatives, thus

\[
\begin{vmatrix}
U_{pp} & U_{pq} \\
U_{qp} & U_{qq}
\end{vmatrix}
\]

Each differentiation brings down the order; we clear factors to normalize the Hessian and write \( H \) for its normalized form: thus

\[
\begin{vmatrix}
U_{pp} & U_{pq} \\
U_{qp} & U_{qq}
\end{vmatrix} = N^2(N - 1)^2 H
\]

where

\[
H = (a_0 a_2 - a_1^2)p^{2N-4} + (N - 2)(a_0 a_3 - a_1 a_2)p^{2N-5} q + \ldots .
\]

The Jacobian \((U, H)\) of \( U \) and \( H \) is a second important covariant of \( U \). Differentiation again introduces factors, which we remove to define the normalized covariant \( G \): thus

\[
\begin{vmatrix}
U_p & U_q \\
H_p & H_q
\end{vmatrix} = N(N - 2) G
\]

where

\[
G = (a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3) p^{3N-6} + \ldots .
\]

Two further covariants associated to \( U \) arise from its quartic emanant; briefly, the details are as follows. Introduce auxiliary variables \((P, Q)\): the fourth emanant \((P \partial_p + Q \partial_q)^4 U\) of \( U \) is a quartic polynomial

\[
AP^4 + 4BP^3Q + 6CP^2Q^2 + 4DPQ^3 + EQ^4
\]

in \((P, Q)\) with coefficients that are polynomials in \((p, q)\). This quartic has familiar invariants

\[
AE - 4BD + 3C^2
\]

and

\[
ACE + 2BCD - AD^2 - B^2E - C^3
\]

which are then covariants of \( U \). Again we normalize, defining the covariants \( S \) and \( T \) of \( U \) by

\[
AE - 4BD + 3C^2 = [N(N - 1)(N - 2)(N - 3)]^2 S
\]

and

\[
ACE + 2BCD - AD^2 - B^2E - C^3 = [N(N - 1)(N - 2)(N - 3)]^3 T
\]
where
\[ S = (a_0a_4 - 4a_1a_3 + 3a_2^2)p^{2N-8} + \ldots \]
and
\[ T = (a_0a_2a_4 + 2a_1a_2a_3 - a_0a_2^2 - a_1^2a_4 - a_2^3)p^{3N-12} + \ldots . \]

We remark that each covariant of \( U \) may be recovered from its source or leader, according to a theorem of Roberts \([4]\): thus, \( H \) may be recovered from its leading coefficient \( a_0a_2 - a_1^2 \) while \( G \) may be recovered from its leading coefficient \( a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3 \); likewise for \( S \) and \( T \). Discussions of this point and its consequences may be found in \([2]\) and \([6]\).

These (normalized) covariants \( H, G, S \) and \( T \) are related to \( U \) by the following syzygy.

**Theorem 1.** \( G^2 + 4H^3 + U^3T = U^2SH \).

**Proof.** Classical: the aforementioned theorem of Roberts \([4]\) implies that we need only check correct behaviour of sources; this check is entirely straightforward. \( \Box \)

The differential equation characteristic of Weierstrass \( \wp \) functions is now on the verge of manifestation. To be explicit, by a Weierstrass equation we shall mean a first-order ordinary differential equation of the form
\[ \Phi^2 = 4\Phi^3 - g_2\Phi - g_3 \]
wherein \( g_2 \) and \( g_3 \) are functions; we shall call this Weierstrass equation proper when \( g_2 \) and \( g_3 \) are constant. In the proper case, introduce the discriminant \( \Delta = g_2^3 - 27g_3^2 \). When \( \Delta \) is zero, \[ \Phi^2 = 4\Phi^3 - g_2\Phi - g_3 \]
has elementary solutions, the cubic on its right side having a repeated root. When \( \Delta \) is nonzero, the nonconstant solutions of \[ \Phi^2 = 4\Phi^3 - g_2\Phi - g_3 \]
are Weierstrass \( \wp \) functions.

Now, a Weierstrass equation emerges from a simple rescaling of the Hessian.

**Theorem 2.** The scalar multiple
\[ \Phi = -[N(N-2)]^2H \]
of the Hessian satisfies the first-order differential equation
\[ \Phi^2 = 4\Phi^3 - N^4(N-2)^4U^2S\Phi - N^6(N-2)^6U^3T \]
along each Hamilton curve of \( U \).

**Proof.** As usual, we suppress notation for the Hamilton curve, differentiation of \( \Phi \) along which results in
\[ \dot{\Phi} = -[N(N-2)]^2\dot{H} = -[N(N-2)]^2(U, H) = -[N(N-2)]^3G \]
whence we deduce that
\[ \bar{\Phi}^2 = [N(N-2)]^6G^2 \]
Substitution from Theorem \([1]\) and rearrangement conclude the argument. \( \Box \)

This differential equation has the Weierstrass form \[ \Phi^2 = 4\Phi^3 - g_2\Phi - g_3 \]
where
\[ g_2 = [N^2(N-2)^2U]^2S \]
and
\[ g_3 = [N^2(N-2)^2U]^3T. \]
Here, the covariants \( \Phi, g_2 \) and \( g_3 \) are evaluated along the Hamilton curve; in general, \( g_2 \) and \( g_3 \) need not be constant.
Our primary concern is with those situations in which this Weierstrass equation is proper: those situations in which \( g_2 \) and \( g_3 \) are constant along the Hamilton curve. Of course, \( U \) itself is constant along each Hamilton curve: indeed,
\[
\circ \quad U = (U,U) = 0.
\]
If this constant value of \( U \) is zero, then \( g_2 = g_3 = 0 \) and the differential equation simplifies, its nonzero solutions being inverse squares. Discounting this case in which \( U \) is zero, we are interested in a Hamilton curve along which the covariants are constant along each Hamilton curve: indeed,\( \circ \)
\[
\text{interested in a Hamilton curve along which the covariants } S \text{ and } T \text{ are constant.}
\]

Naturally, we detect constancy along the Hamilton curve by differentiation. Entirely routine computations reveal that along each Hamilton curve,
\[
\circ \quad S = (U,S) = N(N - 4)[S_0p^{3N-10} + \ldots]
\]
and
\[
\circ \quad T = (U,T) = N(N - 4)[T_0p^{4N-14} + \ldots]
\]
where
\[
S_0 = a_0^2a_5 - 5a_0a_1a_4 + 2a_0a_2a_3 + 8a_1^2a_3 - 6a_1a_2^2
\]
and
\[
T_0 = a_0^2a_2a_5 - a_0^2a_3a_4 - a_0a_2a_5 - 2a_0a_2a_4 + 4a_0a_1a_3 - a_0a_2a_3 + 3a_1^3a_4 - 6a_1a_2a_3 + 3a_1a_4^2.
\]
Accordingly, we focus our attention on the Jacobian covariants \( (U,S) \) and \( (U,T) \).

A useful alternative expression for \( (U,T) \) is as follows.

**Theorem 3.** \( 2(N - 2)(U,T) = N(H,S) \).

*Proof.* By direct calculation of the Jacobian determinants; once again, the work is simplified by the circumstance that only sources need be checked. \( \square \)

Any three (binary) quantics are related by an elementary syzygy that is now perhaps less well known than it should be.

**Theorem 4.** If \( X, Y, Z \) are quantics of orders \( \ell, m, n \) then
\[
\ell X(Y,Z) + m Y(Z,X) + n Z(X,Y) = 0.
\]

*Proof.* By virtue of the Euler theorem on homogeneous functions, \( pZ_p + qZ_q = nZ \) with similar equations for \( X \) and \( Y \). It follows that
\[
nZ(X,Y) = (pZ_p + qZ_q)(X_pY_q - X_qY_p)
\]
with similar expressions for \( \ell X(Y,Z) \) and \( m Y(Z,X) \). Summing,
\[
\ell X(Y,Z) + m Y(Z,X) + n Z(X,Y) = \lambda p + \mu q
\]
where
\[
\lambda = \begin{vmatrix} X_p & X_p & X_q \\ Y_p & Y_p & Y_q \\ Z_p & Z_p & Z_q \end{vmatrix} = 0
\]
and
\[
\mu = \begin{vmatrix} X_q & X_p & X_q \\ Y_q & Y_p & Y_q \\ Z_q & Z_p & Z_q \end{vmatrix} = 0.
\]
\( \square \)

We note here that Grace and Young [3] normalize the Jacobian as a transvectant: in Section 77, their Jacobian of \( X \) and \( Y \) is ours divided by \( \ell m \); with their normalization, Theorem 3 simply asserts that
\[
X(Y,Z) + Y(Z,X) + Z(X,Y) = 0.
\]
A special consequence of this quite general result is the following syzygy between covariants of $U$.

**Theorem 5.** \((N - 4) (U, H) S = (N - 2) [(U, S) H - (U, T) U]\).

*Proof.* Invoke Theorem 4 in case \(X = H\) (of order \(2N - 4\)), \(Y = S\) (of order \(2N - 8\)), \(Z = U\) (of order \(N\)); apply Theorem 3 to replace \(N(H, S)\) by \(2(N - 2)(U, T)\) and rearrange. \(\square\)

Note that if these covariants are considered along a Hamilton curve \(\gamma\) then
\[
(\gamma) (N - 4) H S = (N - 2)[(\gamma) S H - (\gamma) T U]
\]
on account of the circumstance that differentiation along \(\gamma\) is given by \(\dot{V} = (U, V)\).

We may now return to Theorem 2 and the Weierstrass equation for the rescaled Hessian \(\Phi\): thus,
\[
\Phi^2 = 4\Phi^3 - g_2 \Phi - g_3
\]
where \(g_2 = [N^2(N - 2)^2U]^2 S\) and \(g_3 = [N^2(N - 2)^2U]^3 T\). Recall that the differential equation is run along a Hamilton curve \(\gamma\) and that we suppose the constant value of \(U\) to be nonzero. This Weierstrass equation is proper precisely when \(g_2\) and \(g_3\) are constant: thus, precisely when \(S\) and \(T\) are constant; so, precisely when their derivatives \((U, S)\) and \((U, T)\) are constantly zero. Here, constancy is of course along the Hamilton curve \(\gamma\).

At this point, the syzygy of Theorem 5 exerts its influence. As \(N > 4\), the simultaneous vanishing of \((U, S)\) and \((U, T)\) forces the product \((U, H) S\) to vanish along \(\gamma\). If the constant value of \(S\) is nonzero, then the derivative \((U, H)\) vanishes along \(\gamma\) so that \(\Phi\) is constant, its value \(\Phi_0\) satisfying the cubic \(4\Phi_0^3 = g_2 \Phi_0 + g_3\). Consequently, if the (proper) Weierstrass equation considered here has a nonconstant solution then the constant value of \(S\) (and hence of \(g_2\)) is zero. In this connexion, it is interesting to note that \(g_2\) is zero for cubic Hamiltonians but can be nonzero for quartic Hamiltonians; see [5] for this.

These findings may be summarized as follows. The Weierstrass equation of Theorem 2 is proper along the Hamiltonian curve \(\gamma\) precisely when the covariants \((U, S)\) and \((U, T)\) vanish along \(\gamma\). In such a case, if the (rescaled) Hessian \(\Phi\) is nonconstant along \(\gamma\) then the constant value of \(S\) (hence of \(g_2\)) is zero; moreover, a nonconstant \(\Phi\) will be a Weierstrass \(\wp\) function when the constant value of \(T\) is nonzero.

As a special case, if we insist that the covariants \((U, S)\) and \((U, T)\) themselves vanish as quantics, then the Weierstrass equation of Theorem 2 is proper along each Hamilton curve; in this special case, only along those Hamilton curves that originate (and hence remain) in the zero-set of \(S\) can \(\Phi\) be a Weierstrass \(\wp\) function.

In general, given the quantic \(U\) as Hamiltonian, a Hamilton curve \(\gamma\) is determined by its initial point \(\gamma_0\); the question whether or not the covariants \((U, S)\) and \((U, T)\) are zero along \(\gamma\) is therefore decided by \(\gamma_0\). The task of elucidating this matter is reserved for a future publication.

The syzygy in Theorem 5 is only one of many involving the covariants associated to a quantic; the influence of other syzygies is also reserved for a subsequent study.

We should comment on the origin of the Hessian \(H\) within the Hamiltonian theory for the quantic \(U\). Along a Hamilton curve,
\[
\dot{q} = \frac{\partial U}{\partial p}, \quad \dot{p} = -\frac{\partial U}{\partial q}
\]
whence a further differentiation produces
\[
\ddot{p} = U_q U_{qp} - U_{qq} U_p, \quad \ddot{q} = U_p U_{pq} - U_{pp} U_q.
\]
The Euler equation applied to the homogeneous functions $U_p$ and $U_q$ yields
\[ pU_{pp} + qU_{pq} = (N - 1)U_p, \quad pU_{qp} + qU_{qq} = (N - 1)U_q. \]
Isolation of $p$ and $q$ separately results in
\[ (N - 1)[U_qU_{qp} - U_{qq}U_p] = -[U_{pp}U_{qq} - U_{pq}U_{qp}]p \]
and
\[ (N - 1)[U_pU_{pq} - U_{pp}U_q] = -[U_{pp}U_{qq} - U_{pq}U_{qp}]q. \]
Expressing our conclusion in terms of the normalized Hessian,
\[ \overset{\circ}{p} = -[N^2(N - 1)H]p \]
and
\[ \overset{\circ}{q} = -[N^2(N - 1)H]q. \]
Stripping the coordinates,
\[ \overset{\circ}{\gamma} = -[N^2(N - 1)H]\gamma. \]
In terms of the rescaled Hessian $\Phi$ this differential equation assumes the form
\[ \overset{\circ}{\gamma} = \left[ -\frac{N - 1}{(N - 2)^2} \Phi \right]\gamma. \]
In particular, when $\Phi$ is a Weierstrass $\wp$ function, this is a Lamé equation
\[ \overset{\circ}{\gamma} = n(n + 1)\wp\gamma \]
with parameter $n = 1/(N - 2)$ that is not integral (unless $N = 3$).

We should perhaps also comment on the origin of this paper. It began as a continuation of [5] devoted solely to the case of homogeneous quintic polynomials, with only [6] as a reference. In this context, we found the syzygy of Theorem 1 by hand, not by consulting [6]; indeed, this syzygy for quintics does not find its way explicitly into [6]. We then turned to the work of Cayley: this syzygy appears in his eighth memoir and is listed in Table 89 of his ninth memoir [1] as $A^3D - A^2BC + 4C^3 + F^2 = 0$; in fact, it appeared previously on page 173 of [4] but with a sign change in the Hessian. The syzygy of Theorem 5 for quintics already occurs in the second memoir of Cayley; it is listed in Table 89 of [1] as $AI + BF - CE = 0$. Our proof of this syzygy follows a suggestion in Section 194 of [6].

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