A new constraint of the Hamilton cycle algorithm
Heping JIANG
jjhpjhp@gmail.com

Abstract
Grinberg’s theorem is a necessary condition for the planar Hamilton graphs. In this paper, we use cycle bases and removable cycles to survey cycle structures of the Hamiltonian graphs and derive an equation of the interior faces in Grinberg’s Theorem. The result shows that Grinberg’s Theorem is suitable for the connected and simple graphs. Furthermore, by adding a new constraint of solutions to the equation, we find such solutions can be a necessary and sufficient condition for finding Hamiltonian graphs. We use the new constraint to improve the edge pruning technique and obtain a polynomial time algorithm for finding a Hamiltonian cycle in the connected and simple graphs.

Keywords
Hamiltonian graphs  Grinberg’s Theorem  cycle basis  removable cycle
Hamiltonian cycle algorithms

1 Introduction
The graphs considered in the paper are the finite, undirected, connected and simple graphs. A graph \( G = (V, E) \) is a finite nonempty set \( V \) of elements called vertices, together with a set \( E \) of two element subsets of \( V \) called edges. We define a walk in graph \( G \) as a finite alternating sequence of vertices and edges that begins and ends with two distinct vertices in which each edge in the sequence joins the vertex that precedes it in the sequence and following it. No edge appears more than once in a walk. A closed walk in which no vertex (except the beginning and ending vertex) appears more than once is called a cycle (that every vertex in a cycle has degree two). A cycle that contains every vertex of a graph is called a Hamiltonian cycle. A graph is Hamiltonian if it contains a Hamiltonian cycle. The Hamiltonian problem is to find a good characterization [1] as a necessary and sufficient condition of a Hamiltonian cycle in a graph.
In computer science, the main approaches to solve Hamiltonian problems are the backtracking algorithms. In a searching process, by the constraints, the algorithm prunes vertices or edges not satisfied the rules of constructing a Hamiltonian cycle in a graph [2]. It is the key to solve the problems to use the restrictions in a backtracking algorithm. Since no practical conditions have hitherto been found [3], there is not a polynomial time algorithm for finding a Hamiltonian cycle in arbitrary graphs [4].

In this paper, we introduce two notions, cycle basis and removable cycles, into the study of the Hamiltonian problem and re-obtain the same equation associated with the interior faces in Grinberg’s Theorem. This result shows that Grinberg’s Theorem can be used to the connected and simple graphs. In addition, with analyzing the cycle structures, we find a new constraint for the solutions. The constraint solutions can be a necessary and sufficient condition of Hamiltonian graphs. With such solutions, we develop a new edge-pruning technique and obtain a polynomial time algorithm for simple connected graphs.

In 1968, taking a Hamiltonian cycle as a Jordan curve on the plane, E. Grinberg analyzed the relations of the interior faces and outer faces along the curve and derived an equation [5], called Grinberg’s formula or Grinberg’s criterion, and later named Grinberg’s Theorem. The theorem is a necessary condition to determine a planar Hamiltonian graph.

**Theorem 1.1** (Grinberg’s Theorem) Let G be a planar graph with a Hamilton cycle C. Then

\[ \sum_{i=1}^{V} (i - 2)(f'_{i} - f''_{i}) = 0, \]

where \( f'_{i} \) and \( f''_{i} \) are the numbers of faces of degree \( i \) contained in inside \( C \) and outside \( C \), respectively.

Hereafter, Theorem 1.1 was used to survey the Hamiltoncity of the 2-factor graphs, grids, and the Petersen graphs [6, 7, 8] etc., however, no result for the connected and simple graphs has been found. For why Grinberg’s Theorem is used only as a necessary condition, there have no explanations and references from E. Grinberg or others. We, in this paper, obtain the answer in the combinatorial analysis.

Let \( C \) be a cycle in a graph \( G \). It is well known that cycles in \( G \) generate the
cycle space with symmetric difference of $C_1$ and $C_2$ $(C_1 \Theta C_2 = (C_1 \cup C_2) - (C_1 \cap C_2))$ as an addition of cycles $C_1$ and $C_2$. The dimension of the cycle space is equal to $|E| - |V| + 1$. A cycle basis of $G$ is defined as a basis for the cycle space of $G$ which consists entirely of cycles. S. MacLane first used cycle bases to study the combinatorial condition for a graph being planar [9]. Since cycle bases of graphs have a variety of applications in science and engineering, then minimum cycle bases have a special meaning [10, 11]. In 1987 J. D. Horton presented a polynomial time algorithm to find a minimum cycle basis of a graph [12]. From then on the algorithms have been improved [11]. Based on the property of cycle bases and the advancements of the algorithms, we use these two notions in this paper. As matter of fact, there have no differences between a cycle basis and a minimum cycle basis in our research. We use the minimum cycle bases here just for the sake of that it is fundamental to solving the Traveler salesman problem. Therefore, in this paper, a cycle means one in the minimum cycle basis. Note that the given graph considered in Theorem 1.1 is a finite planar graph consisted of elementary cycles partitioned into two kinds, interior and outer faces. Whenever adding a chord into an interior face, it will generate two new faces. If using a cycle to replace a face, we will obtain the same result. It seems that there have no changes for Theorem 1.1 whatever use faces or cycles and actually no references about Theorem 1.1 associated with cycle bases have been found. On the other, the study of removable cycles in connectivity of a graph was initiated by A. M. Hobbs [13]. A cycle C in graph G is called removable if $G - E(C)$ is 2-connected. The following interests concentrated on the study of connectivity associated with removable cycles [14, 15], however, the latest results are still limited to the sufficient conditions [16, 17, 3], which means that it is difficult to determine Hamiltoncity of the graphs even if keeping the 2-connectivity unchangeable. As a result, there have few issues on removable cycles in studying the Hamilton problem and being relevant to Grinberg’s Theorem.

In our research, by substituting faces with cycles, we analysis the combinatorial relations of cycles in a cycle basis of a graph, and derive the same equation associated with interior faces in Theorem 1.1, that is

$$\sum_{i \geq 2} (i f'_i - 2 f''_i) = |V| - 2,$$

called Grinberg Equation in this paper. A set of interior faces $f'_i$ (a set of cycles or a collection of vertex sets), if which satisfied Grinberg Equation, is called solution set, and a set of outer faces $f''_i$ co-solution set, respectively. All the numbers are the
nonnegative integer. Obviously, Grinberg Equation and Hamiltonicity in a graph have the relation as below,

**Theorem 1.2** If a graph $G$ is Hamiltonian, then

$$
\sum_{i \geq 2} (i f_i' - 2 f_i'') = |V| - 2.
$$

Therefore, the following corollary holds,

**Corollary 1.1** If Grinberg Equation of a graph $G$ has no solution, then $G$ is not Hamiltonian.

Theorem 1.2 shows that Grinberg’s Theorem for the planar graphs can be used to the connected and simple graphs. From Theorem 1.2 we know that all the co-solution cycles can be removed and all the left solution cycles will generate a Hamiltonian cycle if the given graph is Hamiltonian. In this paper, we define a removable cycle in a graph $G$ as a cycle such that the remained subgraph $G'(V', E')$ satisfied $V' = V$ and $E' = E - 1$ if remove the cycle from the given cycle basis of $G$.

In the proof of Theorem 1.2 (see the section 2.1), there have two cases satisfying that the cardinality of the union of two combined cycles equals to 2: both cycles combine with a common edge or not. In the case of combining without common edge, we find a new kind of cycles, irremovable co-solution cycles, whose existence implies that having solutions does not mean the given graph is a Hamiltonian graph, vice verse, whose not existence implies that having solutions means the union of such cycle set is a Hamiltonian cycle. We then obtain the following result,

**Theorem 1.3** A graph $G$ is Hamiltonian, if and only if there have solutions for Grinberg Equation of $G$ and the co-solutions set equal to the removable cycles set.

For determining whether or not the co-solutions set equal to the removable cycles set, we improve the backtracking algorithms of Hamiltonian graphs using an approach of edge pruning. It is well known that the key of the edge pruning technique is the constraints based upon the rules of constructing a Hamiltonian cycle (most of the rules are necessary conditions) [18]. The method in our algorithm
is to delete removable cycles that are co-solutions cycles. According to the further analysis on irremovable co-solution cycles (see section 3.1) we obtain a new constraint for the deleting. With adding this constraint to the algorithm, we show the performance for finding a Hamiltonian cycle in an arbitrary connected simple graph takes polynomial time.

Since some notions are used only in the corresponding section, then it is convenient to give those definitions separately. The terminology and notions not defined in this paper can be found in [19], [20] or [21]. The following sections are arranged as follows. Section 2.1 is the proof of Theorem 1.2. Section 2.2 shows the characterizations of irremovable co-solution cycles. Section 2.3 is the proof of Theorem 1.3. Section 3 presents the new constraint, improved algorithm, and analysis of the time complexity of the new algorithm.

2 The main result

2.1 The Proof of Theorem 1.2

Proof  Note that every cycle in a cycle basis of a given graph is corresponding to a subset of vertices or edges. Since a cycle is represented generally by the cardinality of a subset of vertices, called the order of a cycle, then the relations of cycles in a cycle basis is represented by a subset of vertices. Therefore, let \( F \) be a cycle basis of a graph \( G \). \( f_i (\in F, \ i \leq |V|) \) referred to as a cycle with order \( i \) (interior faces in Grinberg theorem). \( |f_i| \) denotes the number of \( f_i \) and \( |F| \) the number of \( F \). Naturally we have \( |f| = |f_3| + |f_4| + \cdots + |f_i|, V_i \in V(f_i) \). Since the union of the whole cycles in a cycle basis is the given graph itself then the number of all vertices of the given graph equals to this union, that is \( |V| = |V_3 \cup V_4 \cup \cdots \cup V_i| \). By inclusion-exclusion principle, we have

\[
|V| = \sum_{a=3}^{i} |V_a| - \sum_{3 \leq a < b \leq i} |V_a \cap V_b| + \sum_{3 \leq a < b < c \leq i} |V_a \cap V_b \cap V_c| - \cdots + (-1)^{i-1}|V_3 \cap V_4 \cap V_5 \cdots \cap V_i| \tag{2.1}
\]

Suppose that \( G \) is a Hamiltonian graph. Since a Hamiltonian cycle in \( G \) can be represented by the symmetric difference of a subset of cycles in a cycle basis of \( G \), then by set operations we can derive this subset such that the union of every pair of
disjoint cycles is null. Let \( V_a \cap V_b \) denote the terms only of that the value of the intersection of every pair of joint cycles is 2. Thus, equation (2.1) can be written as

\[
|V| = \sum_{a=3}^{i} |V_a| - \sum_{3 \leq a < b \leq i} |V_a \cap V_b|.
\]  

(2.2)

Since the number of pair of joint cycles is \(|F| - 1\), and \(|V_a \cap V_b| = 2\), then we have \(\sum_{3 \leq a < b \leq i} |V_a \cap V_b| = 2(|F| - 1)\). When using \(|f| = |f_3| + |f_4| + \cdots + |f_i|, V_i \in V(f_i)\) replace \(|F|\), we obtain

\[
\sum_{3 \leq a < b \leq i} |V_a \cap V_b| = 2(|f_3| + |f_4| + \cdots + |f_i| - 1).
\]  

(2.3)

Furthermore, \(\sum_{a=3}^{i} |V_a|\) is the sum of all subsets of vertices, that is

\[
\sum_{a=3}^{i} |V_a| = |V_3| + |V_4| + \cdots + |V_i|.
\]  

(2.4)

Where \(|V_3| = 3|f_3|\), \(|V_4| = 4|f_4|\), \ldots, \(|V_i| = i|f_i|\), so equation (2.4) can be written as following

\[
\sum_{a=3}^{i} |V_a| = 3|f_3| + 4|f_4| + \cdots + i|f_i|.
\]  

(2.5)

Using equation (2.3) and (2.5) to substitute the corresponding terms in equation (2.2), we derive

\[
\sum_{a=3}^{i} i|f_i| - 2(\sum_{a=3}^{i} |f_i| - 1) = |V|.
\]  

(2.6)

According to the definition of a Hamiltonian cycle, there have \(|V|\) equals to \(|C|\) and it is clear that \(f_i\) can be replaced by interior faces \(f'_i\), then we have

\[
\sum_{a=3}^{i} (i|f'_i| - 2|f'_i|) = |C| - 2.
\]  

(2.7)

Equation (2.7) is the equation associated with the interior faces of Gringberg theorem, called Gringberg equation in this paper. The theorem holds. ■
2.2 A new constraint and some lemmas

2.2.1 Preliminaries

In this section, we partition cycles in a cycle basis into three kinds: removable, irremovable and uncertain. The main application of such partition in our research is to reveal a new kind of cycles in a cycle basis of some graphs so that we can improve the edge pruning technique. For the description of characterizations of that kind of cycles, we here give some notions and definitions that appear in this section. \( R \) refers to as the number of cycles passed an edge in a given graph. An edge of \( R = 1 \) is called a boundary edge. A vertex is called a boundary vertex if it has only two boundary edges on which are two cycles. A vertex, not boundary is called a non-boundary vertex. Especially, a non-boundary vertex is called inner vertex if all edges of this vertex are \( R = 2 \).

It is well known that there have some theorems in [18] (or rules [22]) for constructing a Hamiltonian cycle. In this paper, we utilize both the location of edges of \( R = 1 \) and the number of boundary vertices to depict the characterizations of combinatorial structures of cycles, that replaces some essential rules we need. By the definition of a removable cycle in section 1, we can use a cycle with only one edge of \( R = 1 \) to express a removable cycle. For a cycle in a cycle basis there only have 5 cases that not satisfied the removable condition. Therefore, a cycle \( C \) is irremovable if

1) there are two adjacent edges of \( R = 1 \);

2) there are two edges of \( R = 1 \) not adjacent;

3) the number of non-inner vertices is greater than or equal to 3, though there is only one edge of \( R = 1 \);

4) there have neither edges of \( R = 1 \) nor boundary vertices;

5) there have no edges of \( R = 1 \) but have boundary vertices.
Table 2.2.1-1 5 cases not satisfied the conditions of a removable cycle

In Table 2.2.1-1, for the first three cases, it obviously violates the rules of constructing a Hamiltonian cycle when deleting cycle C. For the fourth case, since it is uncertain whether cycle C is removable or not, we say cycle C is an uncertain cycle. Case 5) can be partitioned into two sub-cases. The first one is that all vertices of C are the boundary. Since cycles circumjacent C are irremovable and no edges of \( R = 1 \), then we say C is an irremovable cycle; the later one is that there are non-boundary vertices on cycle C which could be removable or irremovable (see Lemma 2.2.3 and Theorem 3.1), so it belongs to an uncertain cycle.

In the view of the elements of a graph, we can list six combinatorial structures of irremovable cycles that violate the rules (see Table 2.2.1-2), who have three characterizations on a common vertex that

1) there are two edges of \( R = 1 \) on the same cycle;
2) there have both edges of \( R = 1 \) and edges of \( R \geq 3 \);
3) there have more than and equal to three \( R = 1 \) edges.
Six combinatorial structures of irremovable cycles that violate the rules Table 2.2.1

Clearly, these characterizations imply that not less than 3 edges are forced edges for Hamiltonian cycles. Let \( |P| \) denote the number of such edges on a common vertex of irremovable cycles. Then, it is easy to have

Lemma 2.2.1 Every graph with \( |P| \geq 3 \) is not Hamiltonian.

In this paper, we call \( |P| \geq 3 \) an essential constraint to the algorithm of Hamilton graphs.

2.2.2 A new constraint and some lemmas

In the proof of the theorem 1.2 (see section 2.1), we suppose that the terms which the union of every pair of disjoint cycles is null are zero in the given cycle set. This implies that every pair of joint cycles satisfies that the number of common vertices is 2. We call such cycle set a 2-common-vertex set. In a graph G, there have two kinds of subsets of a 2-common-vertex set, a 2-common-vertex set with common edges or that without common edges. For the case of two joint cycles, if it is a 2-common-vertex set without common edges, then there appears an area between two cycles, see Figure 2.2.2-1. In a cycle basis of a graph, this area is a cycle having boundary vertices but no boundary edges, marked \( C_K \), which has three characterizations, that is
1) irremovable,  
2) irreplaceable,  
3) non-boundary edges.

![a 2-common-vertex set with common edges](image1) ![a 2-common-vertex set without common edges](image2)

Figure 2.2.2-1

In the case of $C_k (|C_k| \neq 0)$, we have three lemmas to depict the relations between the solutions and Hamiltonicity of graphs.

**Lemma 2.2.2-1** Every 2-common-vertex set with $|C_k| = 0$ is the solution set.

**Proof** For a given 2-common-vertex set with $|C_k| = 0$, since every pair of joint cycles is not only a set with common edges but also has two common vertices, that is a 2-common-vertex set with common-edges, then it means the given set satisfies equation (2.7). Hence, the given set is the solution set.  

For the case of common edges in a 2-common-vertex set, we can separate it into two sub-cases, there are 2 cycles on the common vertex (i.e., denoted by X) only and not less than 2 cycles. For example a graph b in Figure 2.2.2-2, there is a vertex X that satisfies $|P| \geq 3$, by Lemma 2.2.1, b is not Hamiltonian. However, it is clear that a 2-common-vertex set without common edges is another cycle basis of a graph B, see the right one in Figure 2.2.2-3 (Actually, there are $E(G) - V(G) + 1$ cycle bases in a graph B). This implies that the non-Hamiltonian property of $|C_k| \neq 0$ in one cycle basis of a graph G can be represented as another one in
which there has a vertex $X$ that satisfies $|P| \geq 3$, see the left one in Figure 2.2.2-3. Since every two bases for a graph $G$ can transform mutually \[12\], then the following lemma holds,

**Lemma 2.2.2-2** In a 2-common-vertex set, if $|C_K| \neq 0$ then $|P| \geq 3$.

**Proof** (Omit)

From Lemma 2.2.2-1 and Lemma 2.2.2-2, a statement yields that it is not sufficient for utilizing the solutions of Grinberg equation to determine whether or not a graph is a Hamiltonian graph. Obviously, with the method in proof of Theorem 1.1, we cannot find there a $C_K$ exists in face set. According to the relation of $C_K$ and $|P|$ in Lemma 2.2.2-2, Lemma2.2.2-3 holds, and we obtain Lemma 2.2.2-4 further.

**Lemma 2.2.2-3** Every 2-common-vertex set with $|C_K| \neq 0$ is not Hamiltonian.

**Proof** (Omit)
Lemma 2.2.2-3 show that it must confirm that there is no $C_k$ in co-solution when deleting a removable cycle from the cycle set. That is the new constraint in this paper: $|C_k| = 0$.

**Lemma 2.2.4** For a 2-common-vertex set, if $|P| \leq 2$ and $|C_k| = 0$, then it is Hamiltonian.

**Proof** In a given 2-common-vertex set of a graph $G$, for pairs of joint cycles there are two types of combinations. If there has $|C_k| = 0$, by Lemma 2.2.2-1, then the given set is the solution set. If there has both $|C_k| = 0$ and $|P| \leq 2$, then the given set is a 2-common-vertex set with common edges (by the three characterizations on a common vertex in section 2.2.1). It means there have and only have two edges of $R = 1$ on every joint vertex, which implies there is a walk passed through every vertex just once. Thus, we obtain a Hamiltonian set.$\Box$

Next lemma will be used to prove Theorem 1.3 in section 2.3.

**Lemma 2.2.5** If there have $|P| \geq 3$ and $|C_k| \neq 0$ in a 2-common-vertex set, then co-solution set $\neq$ removable cycles set.

**Proof** For a given 2-common-vertex set with $|P| \geq 3$ or $|C_k| \neq 0$, in case of $|P| \geq 3$ the given set does not satisfy equation (2.7), so we have co-solution set $\neq$ removable cycles set. In case of $|C_k| \neq 0$, by Lemma 2.2.2-2, it means there has $|P| \geq 3$ in given set, and clearly, we have co-solution set $\neq$ removable cycles set.$\Box$

**2.3 The proof of Theorem 1.3**

Proof According to Lemma 2.2.2-4, we have co-solution set $= \text{removable cycles set}$. In addition, by Lemma 2.2.2-5, we have co-solution set $\neq \text{removable cycles set}$. Hence, the theorem holds.$\Box$

**3 The new algorithm**

Different from the constraint in the existing edge pruning techniques, by adding a new constraint, in this paper, we present an improved edge pruning
technique, in which deleting an edge depends on removing a cycle and removing a cycle depends on the choice of removable cycles in the co-solution set under the conditions of \(|P| \leq 2\) and \(|C_K| = 0\). By Theorem 1.3, the running result is either the co-solution set equals to the removable cycles set or not equals to.

3.1 Preliminaries

In a cycle basis of a graph \(G\), \(K\) is a boundary vertex of degree \(\geq 4\). If there is a cycle \(C\) passed through \(K\), on which there have boundary vertices but no boundary edges, we call \(C\) and all the cycles combining with common vertices and edges \(C\) set. If removing a set of cycles from a \(C\) set results in a Hamiltonian set, we say \(C\) is dismantlable. Let \(S_c\) refer to a solution of a \(C\) set, and \(S'_c\) a co-solution set. We have

**Theorem 3.1** \(C = C_K\) if and only if \(C \in S'_c\) and \(C\) is not dismantleable

**Proof** It is evident that a \(C\) set is simple to determine its Hamiltonicity. Based upon such a fact we only need to discuss the following.

For a \(C\) set, if \(C\) is a co-solution cycle \((C \in S'_c)\) and \(C\) is dismantlable, then the \(C\) set is Hamiltonian by deleting a set of cycles, therefore \(C \neq C_K\). If \(C\) is not dismantlable, then the \(C\) set is not Hamiltonian. It implies that \(C\) has non-boundary vertices and we cannot convert them to the boundary vertices. So \(C = C_K\). ■

3.2 A new algorithm for Hamiltonian graphs and its complexity

By Corollary 1.1, a no-solutions-graph is not Hamiltonian. So, the graphs considered here are the solutions-graphs. Let \(G\) denote a cycle set having solutions. \(\{C_1, C_2, \cdots, C_S\}\) denotes a co-solution set consists of removable cycles. Deleting the elements in \(\{C_1, C_2, \cdots, C_S\}\) consecutively under the conditions of \(|P| \leq 2\) and \(|C_K| = 0\) is called operations under solutions. We say the operations unfinished if \(|P| \geq 3\) or \(C_i = C_K\) when deleting \(C_i\) in step \(i\) \((1 \leq i \leq S)\).

**Theorem 3.2.1** A graph \(G\) is Hamiltonian, if and only if the operations in \(G\) can be finished.
Proof According to the definition, for a given graph $G$, finishing the operations will obtain a Hamiltonian cycle. If not, suppose that there appears either $|P| \geq 3$ or $|C_K| \neq 0$ when deleting $C_i$ at step $i$ ($1 \leq i \leq S$) in the operations. For the case of $|P| \geq 3$, by Lemma 2.2.1, the set when forced to delete $C_i$ is not Hamiltonian, even though the left steps can be finished. For the case of $|C_K| \neq 0$, it means $C_i = C_K$, from Theorem 3.1 we have $C_i$ is not dismantlable, clearly, we obtain the same result of the case of $|P| \geq 3$. ■

Based on Theorem 3.2.1, we present a new algorithm based upon the constraints $|P| \leq 2$ and $|C_K| = 0$. The main steps are:

1) Find a minimum cycle basis of the given graph,
2) Find solutions of Grinberg equation of a minimum cycle basis,
3) Implement the operations under the solutions.

For convenience to analyze the complexity, we decompose the algorithm into four parts that finding a minimum cycle basis, finding solutions of Grinberg equation of a minimum cycle basis, checking whether or not there has a dismantlable $C_K$ if deleting a preselected removable cycle, and checking whether or not there has $|P| \geq 3$ if deleting a preselected removable cycle. The algorithm is presented in Table 3.2.

---

**Input** The adjacent matrix of a graph $G$

**Output** A Hamiltonian cycle of $G$ (if not, $G$ is a non-Hamiltonian graph)

**The main module**

S1 find the minimum cycle basis from the set of cycles; /* sub-module 1*/
S2 classify the cycles by order;
S3 check whether or not there has a case of $|P| \geq 3$ on the vertices of degree $\geq 3$ in $G$;
   if true, then $G$ is not Hamiltonian, and Exit;
   if false, then goto S4;
S4 find the solutions of the set of cycles; /* sub-module 2*/
   if no solutions, then $G$ is not Hamiltonian, and Exit;
   if have solutions, then goto S5;
S5 find a vertex $K$ from the boundary vertices of $G$;
    if a vertex is $K$, then goto S6;
    if not, goto S7;
S6 check the dismantlability of the cycle $C$ /* sub-module 3 */
    if $C$ is not dismantlable, then move $C$ out of the co-solution set of $G$
        and goto S5;
    if $C$ is dismantlable, then move $C$ into the co-solution set of $G$
        and goto S5;
S7 check $C_{(R=1,1)}$; /* where $C_{(R=1,1)}$ refer to a cycle with t edges of $R = i$ */
    if $C_{(R=1,1)} = 0$, then goto S9;
    if $C_{(R=1,1)} \neq 0$, then select a cycle in the set of $C_{(R=1,1)}$ to be a candidate,
        and goto S8;
S8 check the removability of the candidate $C_{(R=1,1)}$ /* sub-module 4 */
    if the candidate $C_{(R=1,1)}$ is removable, then move it into the co-solution set of
        $G$ and delete it from the set of $G$, that is $G - C_{(R=1,1)}$, and goto S5;
    if the candidate $C_{(R=1,1)}$ is not removable, then move it into the solution set of
        $G$ and goto S7;
S9 compare removable cycles set and co-solution set.
    if removable cycles set = co-solution set, then $G$ is Hamiltonian;
    if removable cycles set $\neq$ co-solution set, then $G$ is not Hamiltonian;
Exit

sub-module 1 /* find the minimum cycle basis from the set of cycles */
    Refer to [12]

sub-module 2 /* find the solutions of the set of cycles */
    Refer to [23]

sub-module 3 /* check the dismantlability of the cycle $C$ */
S1 find a cycle $C$ on the vertex $K$ and $C$ set /* Here $C$ is candidate of $C_K$ */
S2 find the unique solution of $C$ set;
    if there is no the unique solution, then $|C_K| = 0$, move $C$ into the co-solution
        set of $G$, and goto S5 of the main module;
if there has the unique solution, then goto S3;
S3 check whether C is an irreplaceable co-solution cycle or not;
  if no, then $|C_K| = 0$, move C into the co-solution set of G, and goto S5 of the main module;
  if yes, then delete all removable cycles from C set, and goto S4;
S4 check the number of edges of $R = 1$ and the order of C set;
  if (the number of edges of $R = 1$) = (the order of C set),
    then C is dismantlable, $|C_K| = 0$, and goto S6 of the main module;
  if (the number of edges of $R = 1$) ≠ (the order of C set),
    then C is not dismantlable, $|C_K| ≠ 0$, and goto S6 of the main module;
sub-module 4 /* check the removability of the candidate $C_{(R=1,1)}$ */
S1 search vertex K being correspondent to vertices on the candidate $C_{(R=1,1)}$ in G;
  If yes, then select $C_K$ and perform sub-module 3;
    When $|C_K| = 0$, goto S2;
    When $|C_K| ≠ 0$,
      If dismantlable, then goto S2;
      If not dismantlable, then move the candidate $C_{(R=1,1)}$ into the irremovable cycles set, and goto S5;
  If no, then goto S2;
S2 check whether there have two or more edges of $R = 1$ on the common vertex of the candidate $C_{(R=1,1)}$ or not;
  If yes, then move the candidate $C_{(R=1,1)}$ into the irremovable cycles set, and goto S5;
  If no, then goto S3;
S3 check whether there have either the edges of $R = 1$ or the edges of $R ≥ 3$ on the common vertex of the candidate $C_{(R=1,1)}$ or not;
  If yes, then move the candidate $C_{(R=1,1)}$ into the irremovable cycles set, and goto S5;
  If no, then goto S4;
S4 check whether there have more than and equal to 3 edges of $R = 1$ on the common vertex of the candidate $C_{(R=1,1)}$ or not;
  If yes, then move the candidate $C_{(R=1,1)}$ into the irremovable cycles set, and goto S5;
If no, then goto S5;
S5 Back to S8 of the main module;

Table 3.2 The program consisted of the main module and 4 sub-modules

Our algorithm consists of the main module and four sub-modules.

The sub-module 1 has a polynomial time algorithm [12]. The sub-module 2 is a procedure for solving a single linear Diophantine equation, which has been shown to have a polynomial time algorithm [23].

The sub-module 3 includes three sequent steps. That is to certain the unique solution for C set, to check whether the preselected cycle is an irreplaceable co-solution cycle or not, and to determine whether a Hamiltonian cycle exists in the C set or not so that we can confirm cycle C on vertex K is dismantlable or not. The time taking in the first step is to resemble quite closely that of the sub-module 2. In the second step, we should compare the co-solution cycles with the preselected cycle which takes the time less than that of E(C) - V(C) + 1. We in the next need to delete all the other co-solution cycles from the C set, which takes the time is same as that in the second step. Hence, the sub-module 3 is polynomial. Sub-module 4 includes three checkings. Before checking, we need to perform the sub-module 3. The time it takes equals to $A \times B \times C$ (where A is the number of co-solution cycles in C set, B is the order of the number of co-solution cycles, and C is the time of performing of sub-module 3), and is polynomial. While the time of other three checking procedures equals to the sum of the whole products of the number of every cycle and its order in C set, that is $\sum \{|C_i| \times V(G_i)\}$, where i is the sequence number. Hence, it is also polynomial.

Since the time taken from S1 to S7 in the main module is a sum of the time that finished procedures needed, then it is polynomial. In S8 of the main program, if the preselected cycle is a removable cycle, then the program will go back to S5 after deleting it. It implies that we will select another one to be as a new preselected cycle from the left of $E(G) - V(G) + 1$ cycles. The time that such an iterative procedure takes equals to the time that the finished main module takes by $(E(G) - V(G) + 1)$ times. Nevertheless, the algorithm in Table 3.2 is still polynomial.
4 Conclusions

Based on the combinatorial analysis of cycles in the cycle basis of the connected simple graphs, we present a new constraint to the algorithm for Hamiltonian graphs. The proofs in this paper show that the algorithm is polynomial.

References

[1] J. Edmonds. (1965). Minimum partition of a matroid into independent subsets. Journal of Research of National Bureau of Standards—B. Mathematical Physics Vol. 69B, Nos. 1 and 2, 67–72.

[2] Basil Vandegriend. (1998). Finding hamiltonian cycles: Algorithms, graphs and Performance. Master Degree's thesis. University of Alberta. 12-15, 33-34, 43-46.

[3] Ronald J. Gould. (2014). Recent advances on the Hamiltonian problem: survey III. Graphs and Combinatorics. 30, 1-46.

[4] Andrew Chalaturnyk. (2008). A fast algorithm for finding Hamilton cycles. Master's thesis, University of Manitoba.

[5] E. Grinberg. (1968). Plane homogeneous graphs of degree 3 without Hamiltonian circuits. Latvian Math. Yearbook. 4, 51-58.

[6] Y. Shimamoto. (1978). On an Extension of the Grinberg Theorem. Journal of Combinatorial Theory, Series B, 24, 169-180.

[7] A. N. M. Salman, E. T. Baskoro, H. J. Broersma. (2003). A Note Concerning Hamilton Cycles in Some Classes of Grid Graphs. PROC. ITB Sains & Tek. Vol. 35 A, No.1. 65-70.

[8] G.L. Chia and Carsten Thomassen. (2011). Grinberg’s Criterion Applied to Some Non-Planar Graphs. Ars Combinatoria, Volume C, July. 3-7.

[9] Saunders MacLane. (1937). A Combinatorial Condition for Planar Graphs. Fundamenta Mathematicae 28. 22-32.

[10] Josef Leydold, Peter F. stadler. (1998). Minimal Cycle bases of Outerplanar graphs. The electronic journal of combinatorics, Volume 5.

[11] K. Mehlhorn, D. Michail. (2009). Minimum Cycle Bases: Faster and Simpler. ACM Transactions on Algorithms, Vol. 6, No.1.

[12] J. D. Horton. A polynomial-time algorithm to find the shortest cycle basis of a
graph. SIAM J. Comput. 16(2)

[13] A. M. Hobbs. personal communication.

[14] Bill Jackson. (1980). Removable cycles in 2-connected graphs of minimum degree at least four. Journal of London Mathematics. Soc. (2), 21, 385-392.

[15] Luis A. Goddyn, Jan van den Heuvel, Sean McGuinness. (1997). Removable cycles in multigraphs. Journal of Combinatorial Theory, Series B, 71, 130-143.

[16] Geng-Hua Fan. (1984). New Sufficient Conditions for Cycles in Graphs, Journal of Combinatorial Theory, Series B, 37, 221-227.

[17] Ronald J. Gould. (2003). Advances on the Hamiltonian problem—a survey, Graphs Combinatorics, 19(1), 7-52.

[18] Basil Vandegriend. (1998). Finding hamiltonian cycles: Algorithms, graphs and Performance. Master Degree's thesis, University of Alberta. 8-9, 39.

[19] C. Berge. (1962). The theory of graphs and its applications. New York, John Wiley & Sons.

[20] Narsingh Deo. (1974). Graph Theory with Applications to Engineering and Computer Science. USA. Prentice-Hall, Inc., Englewood Cliffs, N. J.

[21] R. Gould, Graph Theory. (2012). Dover Publications, Inc.

[22] Jeffrey A. Shufelt and Hans J. Berliner. (1994). Generating Hamiltonian circuits without backtracking from errors, Theoretical Computer Science, 132, 347-375.

[23] Alexander Schrijver. (1999). Theory of linear and integer programming. John Wiley & Sons Ltd., 54.