Similarity Measures of Spherical Fuzzy Sets Based on Cosine Function and Their Applications

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ABSTRACT In this paper, we defined ten similarity measures between spherical fuzzy sets (SFSs) based on the cosine function by considering the degree of positive-membership, degree of neutral-membership, degree of non-membership and degree of refusal-membership in SFSs. Then, we applied these similarity measures and weighted similarity measures between SFSs to pattern recognition and medical diagnosis. Finally, two illustrative examples are designed to show the efficiency of the similarity measures for pattern recognition and medical diagnosis.

INDEX TERMS Spherical fuzzy sets (SFSs), cosine function, cosine similarity measure, pattern recognition, medical diagnosis.

I. INTRODUCTION

The similarity measures are very important and useful tools for determining the degree of similarity between two objects. Measures of similarity between fuzzy sets have gained attention from researchers for their wide applications in various fields, such as pattern recognition, machine learning, decision making and image processing, many measures of similarity between fuzzy sets have been investigated in recent years [1]–[4]. Fuzzy set theory, introduced by Zadeh [5], has been widely employed to model uncertainty in real-world applications. Atanassov [6], [7] extended fuzzy sets to intuitionistic fuzzy sets (IFSs), many different similarity measures between IFSs have been investigated [8]. Li and Cheng [9] proposed a suitable similarity measure between IFSs for pattern recognition problems. Furthermore, Mitchell [10] modified Li and Cheng’s measures [9]. Based on Hamming distance on fuzzy sets, Szmidt and Kacprzyk [11], [12] developed a similarity measure between IFSs along with the Hamming distance. Hung and Yang [13] computed the distance between IFSs along with the Hausdorff distance. Liu [14] developed some novel similarity measures between IFSs. Hung and Yang [15] proposed the similarity measures between IFSs along with the Lp metric. Xu and Xia [16] defined the geometric distance and similarity measures of IFSs for GDM. Ye [17] defined the cosine similarity measure between IFSs. Hung [18] explored the likelihood-based measurement of IFSs for the medical diagnosis and bacteria classification problems. Shi and Ye [19] further provided the cosine similarity measure of IFSs. Tian [20] explored the cotangent similarity measure between IFSs for medical diagnosis issues. Rajarajeswari and Uma [21] provided the cotangent similarity measure which considering membership, nonmembership and hesitation degrees in IFSs. Szmidt [22] discussed distances between IFSs and introduced a family of similarity measures in IFSs. Ye [23] provided two novel cosine similarity measures and weighted cosine similarity measures. Son and Phong [24] gave the intuitionistic vector similarity measures for medical diagnosis. Smarandache [25] defined the concept of neutrosophic set to depict the uncertain information.

More recently, Pythagorean fuzzy set (PFS) [26] has appeared as an useful tool for depicting uncertainty in MADM. The PFS is also succinctly depicted by the membership degree and the non-membership degree, whose sum of squares is less than or equal to 1, the PFS is more general than the IFS. In some cases, the PFS can solve the problems that the IFS can’t, for example, if a DM thinks the membership degree and the non-membership degree as 0.8 and 0.6, respectively, then it is only valid for the PFS. In other words, all the degrees of IFSs are a part of degrees of PFSs, which declared that the PFS is more powerful to tackle the uncertain...
problems. Zhang and Xu [27] explored a Pythagorean fuzzy TOPSIS for handling the MCDM issue. Peng and Yang [28] proposed the division and subtraction operations for PFNs. Afterwards, Beliakov and James [29] centered on “averaging” in the case of PFNs. Reformat and Yager [30] applied the PFNs in solving the collaborative-based recommender system. Gou et al. [31] listed the Properties of Continuous Pythagorean fuzzy information. Ren et al. [32] proposed the Pythagorean fuzzy TODIM approach to MADM. Garg [33] presented the weighted correlation coefficient to measure the accuracy function under interval-valued Pythagorean fuzzified Dice similarity measures for PFSs. Garg [35] presented the new generalized Pythagorean fuzzy operators by using Einstein Operations. Wang et al. [34] developed generalized Dice similarity measures for PFSs. Garg [35] presented the weighted correlation coefficient to measure the relationship between two PFSs. Tang et al. [37] proposed the Pythagorean fuzzy Muirhead mean operators in MADM. Wei [38] explored the Pythagorean fuzzy Hamacher power operators in MADM. Li et al. [39] proposed Pythagorean fuzzy Hamy mean operators in MADM. Gao [40] developed the Pythagorean fuzzy Hamacher prioritized p operators. Wei et al. [41] proposed Maclaurin symmetric mean operators for IVPFN. Li et al. [42] defined Hamy mean operators for IVPFN. Tang et al. [43] gave some Muirhead Mean operators with IVPFN. Some works [44], [45] are combined PFSs with 2-tuple linguistic information. Some papers [46]–[50] connected PFSs with dual hesitant fuzzy sets.

Although, Atanassov’s IFSs theory [6], [7] has been successfully employed in diverse areas, but there are situations in real life which can’t be represented by IFSs [51]–[54]. Picture fuzzy sets are extension of IFSs [55]–[60]. The PFS is characterized by the positive-membership degree, neutral-membership degree and the non-membership degree, whose sum of them is less than or equal to 1, the PFS is more general than the IFS. As a generalization of PFS, the spherical fuzzy sets (SFSs) are also depicted by the positive-membership degree, neutral-membership degree and the non-membership degree, whose sum of squares is less than or equal to 1. In some cases, the SFS can solve the problems that the IFS and PFS can’t, for example, if a DM gives positive-membership degree, neutral-membership degree and the non-membership degree as 0.6, 0.4 and 0.1, respectively, then it is only valid for the PFS. In other words, all the picture fuzzy degrees are a part of the Spherical fuzzy degrees, which indicates that the SFS is more powerful to handle the uncertain problems. Therefore in order to tackle these types of situations, in this paper we introduce the concept of similarity measures for Spherical fuzzy sets (SFSs) based on the cosine functions, which is a new extension of the similarity measure of IFSs based on the cosine functions. In order to do so, the remainder of this paper is set out as follows. In the next section, we review some basic concepts related to IFSs and some similarity measure between IFSs. In Section 3, we propose some similarity measure and some weighted similarity measure between SFSs along with the cosine function. In Section 4, the similarity measures for SFSs are applied to pattern recognition and medical diagnosis. Section 5 concludes the paper with some meaningful remarks.

II. PRELIMINARIES

In the following, we review some basic concepts related to IFSs and some similarity measure between IFSs.

Definition 1 ([6], [7]): An IFS A in X is given by

\[ A = \{ (x, \mu_A(x), \nu_A(x)) | x \in X \} \]  

where \( \mu_A : X \rightarrow [0, 1] \) and \( \nu_A : X \rightarrow [0, 1] \), where, \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X \). The number \( \mu_A(x) \) and \( \nu_A(x) \) represents, respectively, the membership degree and non-membership degree of the element x to the set A.

Definition 2 ([6], [7]): For each IFS A in X, if

\[ \pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \quad \forall x \in X. \]  

Then \( \pi_A(x) \) is named as the degree of indeterminacy of x to A.

Suppose that there are two IFSs A = \{ (x_j, \mu_A(x_j), \nu_A(x_j)) | x_j \in X \} and B = \{ (x_j, \mu_B(x_j), \nu_B(x_j)) | x_j \in X \} in the universe of discourse X = \{ x_1, x_2, \ldots, x_n \}.

Ye [17] explored the cosine similarity measure between IFSs A and B:

\[ IFC^1(A, B) = \frac{1}{n} \sum_{j=1}^{n} \frac{\mu_A(x_j) \mu_B(x_j) + \nu_A(x_j) \nu_B(x_j)}{\sqrt{\mu_A^2(x_j) + \nu_A^2(x_j) \sqrt{\mu_B^2(x_j) + \nu_B^2(x_j)}}} \]  

Shi and Ye [19] further presented the cosine similarity measure by considering membership degree, non-membership degree and hesitancy degree in IFSs in (4), as shown at the bottom of the next page.

Based on cosine function, Ye [23] proposed two cosine similarity measures between IFSs A and B.

\[ IFC^1_A(B) = \frac{1}{n} \sum_{j=1}^{n} \cos \left[ \frac{\pi}{2} \left( \frac{\left| \mu_A(x_j) - \mu_B(x_j) \right|}{\sqrt{\nu_A(x_j) - \nu_B(x_j)}} \right) \right] \]  

\[ IFC^2_A(B) = \frac{1}{n} \sum_{j=1}^{n} \cos \left[ \frac{\pi}{4} \left( \frac{\left| \mu_A(x_j) - \mu_B(x_j) \right|}{\left| \nu_A(x_j) - \nu_B(x_j) \right|} \right) \right] \]  

On the other hand, Tian [20] proposed a cotangent similarity measure between IFSs A and B as following (7), as shown at the bottom of the next page.

where the symbol “\( \vee \)” is the maximum operation.

When membership degree, non-membership degree and hesitancy degree are considered in IFSs, Rajarajeswari and Uma [21] explored the cotangent similarity measure of IFSs (8), as shown at the bottom of the next page.

In the following, we reviewed the weighted cosine and cotangent similarity measures between IFSs A and B, respectively [17], [19]–[23] (9)–(14), as shown at the bottom of the next page, where \( \omega_j (j = 1, 2, \ldots, n) \) is the weight of an
element $x_j, \omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$ and the symbol “$\lor$” is the maximum operation.

### III. SOME SIMILARITY MEASURE BASED ON COSINE FUNCTION FOR SFSs

In such section, we propose some similarity measure and some weighted similarity measure between SFSs along with the concept of the cosine function.

**Definition 3** ([25], [26]): Let $X$ be a fix set. A SFS is an object having the form

$$A = \{(x, (P_A (x), I_A (x), N_A (x))) | x \in X\}$$

(15)

where the function $P_A : X \rightarrow [0, 1]$ defines the degree of positive-membership, the function $I_A : X \rightarrow [0, 1]$ defines the degree of neutral-membership and $N_A : X \rightarrow [0, 1]$ defines the degree of non-membership of the element $x \in X$ to $A$, respectively, and, for every $x \in X$, it holds that

$$(P_A (x))^2 + (I_A (x))^2 + (N_A (x))^2 \leq 1.$$  

(16)

### A. COSINE SIMILARITY MEASURE FOR SFSs

Let $A$ be an SFS in an universe of discourse $X = \{x\}$, the SFS is characterized by the degree of positive-membership $P_A (x)$, the degree of neutral-membership $I_A (x)$, the degree of non-membership $N_A (x)$ and the degree of refusal-membership degree

$$R_A (x) \left( R_A (x) = \sqrt{1 - [(P_A (x))^2 + (I_A (x))^2 + (N_A (x))^2]} \right)$$

Therefore, a cosine similarity measure and a weighted cosine similarity measure for SFSs are proposed in an analogous manner to the cosine similarity measure based on Bhat- tacharya’s distance [61], [62] and cosine similarity measure for IFSs [17].

$$IFC^2 (A, B) = \frac{1}{n} \sum_{j=1}^{n} \frac{\mu_A (x_j) \mu_B (x_j) + \nu_A (x_j) \nu_B (x_j) + \pi_A (x_j) \pi_B (x_j)}{\sqrt{\mu_A^2 (x_j) + \nu_A^2 (x_j) + \pi_A^2 (x_j)} \sqrt{\mu_B^2 (x_j) + \nu_B^2 (x_j) + \pi_B^2 (x_j)}}$$

(4)

$$IFCT^1 (A, B) = \frac{1}{n} \sum_{j=1}^{n} \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \left( |\mu_A (x_j) - \mu_B (x_j)| \lor |\nu_A (x_j) - \nu_B (x_j)| \right) \right]$$

(7)

$$IFCT^2 (A, B) = \frac{1}{n} \sum_{j=1}^{n} \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \left( |\mu_A (x_j) - \mu_B (x_j)| \lor |\nu_A (x_j) - \nu_B (x_j)| \right) \right]$$

(8)

$$WIFC^1 (A, B) = \sum_{j=1}^{n} \omega_j \frac{\mu_A (x_j) \mu_B (x_j) + \nu_A (x_j) \nu_B (x_j)}{\sqrt{\mu_A^2 (x_j) + \nu_A^2 (x_j)}} \sqrt{\mu_B^2 (x_j) + \nu_B^2 (x_j)}}$$

(9)

$$WIFC^2 (A, B) = \sum_{j=1}^{n} \omega_j \frac{\mu_A (x_j) \mu_B (x_j) + \nu_A (x_j) \nu_B (x_j) + \pi_A (x_j) \pi_B (x_j)}{\sqrt{\mu_A^2 (x_j) + \nu_A^2 (x_j) + \pi_A^2 (x_j)}} \sqrt{\mu_B^2 (x_j) + \nu_B^2 (x_j) + \pi_B^2 (x_j)}}$$

(10)

$$WIFCS^1 (A, B) = \sum_{j=1}^{n} \omega_j \cos \left[ \frac{\pi}{2} \left( |\mu_A (x_j) - \mu_B (x_j)| \lor |\nu_A (x_j) - \nu_B (x_j)| \lor |\pi_A (x_j) - \pi_B (x_j)| \right) \right]$$

(11)

$$WIFCS^2 (A, B) = \sum_{j=1}^{n} \omega_j \cos \left[ \frac{\pi}{4} \left( |\mu_A (x_j) - \mu_B (x_j)| \lor |\nu_A (x_j) - \nu_B (x_j)| \lor |\pi_A (x_j) - \pi_B (x_j)| \right) \right]$$

(12)

$$WIFCT^1 (A, B) = \sum_{j=1}^{n} \omega_j \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \left( |\mu_A (x_j) - \mu_B (x_j)| \lor |\nu_A (x_j) - \nu_B (x_j)| \lor |\pi_A (x_j) - \pi_B (x_j)| \right) \right]$$

(13)

$$WIFCT^2 (A, B) = \sum_{j=1}^{n} \omega_j \cot \left[ \frac{\pi}{4} + \frac{\pi}{4} \left( |\mu_A (x_j) - \mu_B (x_j)| \lor |\nu_A (x_j) - \nu_B (x_j)| \lor |\pi_A (x_j) - \pi_B (x_j)| \right) \right]$$

(14)
Suppose that there are two SFSs $A = \{x_j, P_A (x_j)\}, I_A (x_j), N_A (x_j) \{x_j \in X \}$ and $B = \{x_j, P_B (x_j), I_B (x_j), N_B (x_j) \} \{x_j \in X \}$ in the universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$.

A cosine similarity measure between SFSs $A$ and $B$ is proposed as follows (17), as shown at the bottom of this page. If we take $n = 1$, then the cosine similarity measure between SFSs $A$ and $B$ becomes the correlation coefficient between SFSs $A$ and $B$, i.e. $C_{SFS} (A, B) = K_{SFS} (A, B)$. Therefore, the cosine similarity measure between SFSs $A$ and $B$ also satisfies the following properties:

1. $0 \leq SFC^1 (A, B) \leq 1$;
2. $SFC^1 (A, B) = SFC^1 (B, A)$;
3. $SFC^1 (A, B) = 1$, if $A = B$.

Proof: (1) It is obvious that the proposition is true according to the cosine value.

(2) It is obvious that the proposition is true.

When $A = B$, there are $P_A (x_j) = P_B (x_j), I_A (x_j) = I_B (x_j)$ and $N_A (x_j) = N_B (x_j)$ for $j = 1, 2, \cdots, n$. So, there is $C^1_{SFS} (A, B) = 1$. Therefore, we have finished this proofs.

Then, we investigate the distance measure of the angle as follows (17), as shown at the bottom of this page.

\[
SFC^1 (A, B) = \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{P^2_A (x_j) P^2_B (x_j) + I^2_A (x_j) I^2_B (x_j) + N^2_A (x_j) N^2_B (x_j)}{P^2_A (x_j) + I^2_A (x_j) + N^2_A (x_j) \sqrt{P^4_B (x_j) + I^4_B (x_j) + N^4_B (x_j)}} \right]
\]  \hspace{1cm} (17)

\[
SFC^1_j (A (x_j), B (x_j)) = \frac{P^2_A (x_j) P^2_B (x_j) + I^2_A (x_j) I^2_B (x_j) + N^2_A (x_j) N^2_B (x_j)}{\sqrt{P^4_A (x_j) + I^4_A (x_j) + N^4_A (x_j) \sqrt{P^4_B (x_j) + I^4_B (x_j) + N^4_B (x_j)}}}
\]

\[
SFC^1 (B (x_j), C (x_j)) = \frac{P^2_B (x_j) P^2_C (x_j) + I^2_B (x_j) I^2_C (x_j) + N^2_B (x_j) N^2_C (x_j)}{\sqrt{P^4_B (x_j) + I^4_B (x_j) + N^4_B (x_j) \sqrt{P^4_C (x_j) + I^4_C (x_j) + N^4_C (x_j)}}}
\]

\[
SFC^1 (A (x_j), C (x_j)) = \frac{P^2_A (x_j) P^2_C (x_j) + I^2_A (x_j) I^2_C (x_j) + N^2_A (x_j) N^2_C (x_j)}{\sqrt{P^4_A (x_j) + I^4_A (x_j) + N^4_A (x_j) \sqrt{P^4_C (x_j) + I^4_C (x_j) + N^4_C (x_j)}}}
\]
\[ j = 1, 2, \ldots, n, \sum_{j=1}^{n} \omega_j = 1. \text{ In particular, if } \omega = (1/n, 1/n, \ldots, 1/n)^T, \text{ then the weighted cosine similarity measure always reduces to cosine similarity measure. That’s to say, if we suppose } \omega_j = \frac{1}{n}, j = 1, 2, \ldots, n, \text{ then there is } \text{WSFC}^1(A, B) = \text{SFC}^1(A, B). \]

Obviously, the weighted cosine similarity measure of two SFSs \( A \) and \( B \) also satisfies the following properties:

1. \( 0 \leq \text{WSFC}^1(A, B) \leq 1, \)
2. \( \text{WSFC}^1(A, B) = \text{WSFC}^1(B, A), \)
3. \( \text{WSFC}^1(A, B) = 1, \text{ if } A = B, i = 1, 2, \ldots, n. \)

Similar to the previous proof, we can prove the above three properties.

### B. SIMILARITY MEASURES OF SFSs BASED ON COSINE FUNCTION

Based on the cosine function \cite{Wei2019}, we propose four cosine similarity measures between SFSs and analyze their properties.

**Definition 4:** Suppose that there are two sets of SFSs \( A = \{\{x_j, (P_A(x_j), I_A(x_j), N_A(x_j))\} | x_j \in x\} \) and \( B = \{\{x_j, (P_B(x_j), I_B(x_j), N_B(x_j))\} | x_j \in x\} \) in \( X = \{x_1, x_2, \ldots, x_n\}. \) Then, we define two cosine similarity measures between SFSs \( A \) and \( B, \) respectively, as follows:

\[
\text{SFC}^1(A, B) = \frac{1}{n} \sum_{j=1}^{n} \cos \left( \frac{1}{2} \left( \frac{|P_A(x_j) - P_B(x_j)|}{|I_A(x_j) - I_B(x_j)|} \right) \right)
\]

\[
\text{SFC}^2(A, B) = \frac{1}{n} \sum_{j=1}^{n} \cos \left( \frac{1}{4} \left( \frac{|P_A(x_j) - P_B(x_j)|}{|I_A(x_j) - I_B(x_j)|} + \frac{|N_A(x_j) - N_B(x_j)|}{|I_A(x_j) - I_B(x_j)|} \right) \right)
\]

where the symbol \( \lor \) is the maximum operation.

**Proposition 1:** For two SFSs \( A \) and \( B \) in \( X = \{x_1, x_2, \ldots, x_n\}, \) the cosine similarity measures \( \text{SFC}^k(A, B) (k = 1, 2) \) should satisfy the following properties (1)-(4):

1. \( 0 \leq \text{SFC}^k(A, B) \leq 1; \)
2. \( \text{SFC}^k(A, B) = 1 \text{ if and only if } A = B; \)
3. \( \text{SFC}^k(A, B) = \text{SFC}^k(B, A); \)
4. If \( C \) is a SFS in \( X \) and \( A \subseteq B \subseteq C, \) then \( \text{SFC}^k(A, C) \leq \text{SFC}^k(B, C) \) and \( \text{SFC}^k(A, C) \leq \text{SFC}^k(B, C). \)

**Proof:**

1. Since the value of the cosine function is within \([0, 1], \) the similarity measure based on the cosine function is also within \([0, 1]. \) Thus, there is \( 0 \leq \text{SFC}^k(A, B) \leq 1. \)

2. For two SFSs \( A \) and \( B \) in \( X = \{x_1, x_2, \ldots, x_n\}, \) if \( A = B, \) then \( P_A^2(x_j) = P_B^2(x_j), I_A^2(x_j) = I_B^2(x_j), N_A^2(x_j) = N_B^2(x_j), R_A^2(x_j) = R_B^2(x_j), \) for \( j = 1, 2, \ldots, n. \) Hence, \( \text{SFC}^k(A, B) = 1, k = 1, 2. \)

3. If \( \text{SFC}^k(A, B) = 0, k = 1, 2, \) this implies \( P_A^2(x_j) = P_B^2(x_j), I_A^2(x_j) = I_B^2(x_j), N_A^2(x_j) = N_B^2(x_j), R_A^2(x_j) = R_B^2(x_j), \) for \( j = 1, 2, \ldots, n. \)

4. If \( A \subseteq B \subseteq C, \) then \( P_A(x_j) \leq P_B(x_j) \leq P_C(x_j), I_A(x_j) \geq I_B(x_j) \geq I_C(x_j), N_A(x_j) \geq N_B(x_j) \geq N_C(x_j), \) for \( j = 1, 2, \ldots, n. \) Thus, we have

\[
|P_A^2(x_j) - P_B^2(x_j)| \leq |P_A^2(x_j) - P_C^2(x_j)|.
\]
\[ \left| P_B^2(x_j) - P_C^2(x_j) \right| \leq \left| P_A^2(x_j) - P_C^2(x_j) \right|, \]
\[ \left| I_A^2(x_j) - I_B^2(x_j) \right| \leq \left| I_A^2(x_j) - I_C^2(x_j) \right|, \]
\[ \left| I_B^2(x_j) - I_C^2(x_j) \right| \leq \left| I_A^2(x_j) - I_C^2(x_j) \right|, \]
\[ \left| N_A^2(x_j) - N_B^2(x_j) \right| \leq \left| N_A^2(x_j) - N_C^2(x_j) \right|, \]
\[ \left| N_B^2(x_j) - N_C^2(x_j) \right| \leq \left| N_A^2(x_j) - N_C^2(x_j) \right|, \]

Hence, \( \text{SFCS}^k(A, C) \leq \text{SFCS}^k(A, B) \) and \( \text{SFCS}^k(A, C) \leq \text{SFCS}^k(B, C) \) for \( k = 1, 2 \), as the cosine function is a decreasing function within interval \([-1, 1]\).

Thus, the proofs of these properties are completed.

When the four terms like degree of positive-membership, degree of neutral-membership, degree of non-membership and degree of refusal-membership are considered in SFSs, suppose that there are two SFSs \( A = \{x_j, P_A(x_j), I_A(x_j), N_A(x_j), R_A(x_j) \} \) and \( B = \{x_j, P_B(x_j), I_B(x_j), N_B(x_j), R_B(x_j) \} \) in the universe of discourse \( X = \{x_1, x_2, \ldots, x_n\} \), we further propose two cosine similarity measures between SFSs as follows:

\[ \text{SFCS}^3(A, B) = \frac{1}{n} \sum_{j=1}^{n} \cos \left[ \frac{\pi}{2} \left( \left| P_A^2(x_j) - P_B^2(x_j) \right| + \left| I_A^2(x_j) - I_B^2(x_j) \right| + \left| N_A^2(x_j) - N_B^2(x_j) \right| + \left| R_A^2(x_j) - R_B^2(x_j) \right| \right) \right] \]

\[ \text{SFCS}^4(A, B) = \frac{1}{n} \sum_{j=1}^{n} \cos \left( \frac{\pi}{4} \left( \left| P_A^2(x_j) - P_B^2(x_j) \right| + \left| I_A^2(x_j) - I_B^2(x_j) \right| + \left| N_A^2(x_j) - N_B^2(x_j) \right| + \left| R_A^2(x_j) - R_B^2(x_j) \right| \right) \right) \]

In many cases, the weight of the elements \( x_j \in X \) should be taken into account. For example, in MADM, the considered attributes usually have different importance, and thus need to be assigned different weights [64]–[69]. As a result, two weighted cosine similarity measure between SFSs \( A \) and \( B \) is also proposed as follows:

\[ \text{WSFCS}^1(A, B) = \sum_{j=1}^{n} \omega_j \cos \left( \frac{\pi}{2} \left( \left| P_A^2(x_j) - P_B^2(x_j) \right| + \left| I_A^2(x_j) - I_B^2(x_j) \right| + \left| N_A^2(x_j) - N_B^2(x_j) \right| + \left| R_A^2(x_j) - R_B^2(x_j) \right| \right) \right) \]

\[ \text{WSFCS}^2(A, B) = \sum_{j=1}^{n} \omega_j \cos \left( \frac{\pi}{4} \left( \left| P_A^2(x_j) - P_B^2(x_j) \right| + \left| I_A^2(x_j) - I_B^2(x_j) \right| + \left| N_A^2(x_j) - N_B^2(x_j) \right| + \left| R_A^2(x_j) - R_B^2(x_j) \right| \right) \right) \]

\[ \text{WSFCS}^3(A, B) = \sum_{j=1}^{n} \omega_j \cos \left( \frac{\pi}{2} \left( \left| P_A^2(x_j) - P_B^2(x_j) \right| + \left| I_A^2(x_j) - I_B^2(x_j) \right| + \left| N_A^2(x_j) - N_B^2(x_j) \right| + \left| R_A^2(x_j) - R_B^2(x_j) \right| \right) \right) \]

\[ \text{WSFCS}^4(A, B) = \sum_{j=1}^{n} \omega_j \cos \left( \frac{\pi}{4} \left( \left| P_A^2(x_j) - P_B^2(x_j) \right| + \left| I_A^2(x_j) - I_B^2(x_j) \right| + \left| N_A^2(x_j) - N_B^2(x_j) \right| + \left| R_A^2(x_j) - R_B^2(x_j) \right| \right) \right) \]

\[ \text{WSFCS}^4(A, B) = \sum_{j=1}^{n} \omega_j \cos \left( \frac{\pi}{4} \left( \left| P_A^2(x_j) - P_B^2(x_j) \right| + \left| I_A^2(x_j) - I_B^2(x_j) \right| + \left| N_A^2(x_j) - N_B^2(x_j) \right| + \left| R_A^2(x_j) - R_B^2(x_j) \right| \right) \right) \]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is the weight vector of \( x_j (j = 1, 2, \ldots, n) \), along with \( \omega_j \in [0, 1], j = 1, 2, \ldots, n \).

Obviously, the weighted cosine similarity measures also satisfy the axiomatic requirements of similarity measures in Proposition 2.

**Proposition 2:** For two SFSs \( A \) and \( B \) in \( X = \{x_1, x_2, \ldots, x_n\} \), \( \text{SFCS}^k(A, B) \) \( (k = 1, 2, 3, 4) \) should satisfy the following properties (1)-(4):

(1) \( 0 \leq \text{WSFCS}^k(A, B) \leq 1; \)
(2) \( \text{WSFCS}^k(A, A) = 1 \) if and only if \( A = B; \)
(3) \( \text{WSFCS}^k(A, B) = \text{WSFCS}^k(B, A); \)
(4) If \( C \) is a SFS in \( X \) and \( A \subseteq B \subseteq C \), then \( \text{WSFCS}^k(A, C) \leq \text{WSFCS}^k(A, B) \) and \( \text{WSFCS}^k(A, C) \leq \text{WSFCS}^k(B, C). \)

By using similar proof in Proposition 1, we can give the proofs of these properties (1)-(4).

**C. SIMILARITY MEASURES OF SFSs BASED ON COTANGETANT FUNCTION**

In such section, we propose two cotangent similarity measures between SFSs.

**Definition 5:** Let \( A = \{x_j, (P_A(x_j), I_A(x_j), N_A(x_j), R_A(x_j)) \} \) \( \forall x_j \in X \) and \( B = \{x_j, (P_B(x_j), I_B(x_j), N_B(x_j), R_B(x_j)) \} \) \( \forall x_j \in X \) be any two SFSs in \( X = \{x_1, x_2, \ldots, x_n\} \). Then, we explore two cotangent similarity measures between SFSs \( A \) and \( B \), respectively:

\[ \text{SFCT}^1(A, B) = \frac{1}{n} \sum_{j=1}^{n} \cot \left( \frac{\pi}{4} \left( \left| P_A(x_j) - P_B(x_j) \right| + \left| I_A(x_j) - I_B(x_j) \right| + \left| N_A(x_j) - N_B(x_j) \right| + \left| R_A(x_j) - R_B(x_j) \right| \right) \right) \]

\[ \text{SFCT}^2(A, B) = \frac{1}{n} \sum_{j=1}^{n} \cot \left( \frac{\pi}{8} \left( \left| P_A(x_j) - P_B(x_j) \right| + \left| I_A(x_j) - I_B(x_j) \right| + \left| N_A(x_j) - N_B(x_j) \right| + \left| R_A(x_j) - R_B(x_j) \right| \right) \right) \]

where the symbol “\(^\bot\) is the maximum operation.

When the four terms like degree of positive-membership, degree of neutral-membership, degree of non-membership and degree of refusal-membership are considered in SFSs, suppose that there are two SFSs \( A = \{x_j, P_A(x_j), I_A(x_j), N_A(x_j), R_A(x_j) \} \) \( \forall x_j \in X \) and \( B = \{x_j, P_B(x_j), I_B(x_j), N_B(x_j), R_B(x_j) \} \) \( \forall x_j \in X \).
In the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, we explore two cosine similarity measures between SFSs:

**$SFCT^3 (A, B)$**

$$\frac{1}{n} \sum_{j=1}^{n} \cot \left( \frac{\pi}{4} + \frac{\pi}{4} \left( \frac{|p_A^2(x_j) - p_B^2(x_j)|}{|p_A^2(x_j) - p_B^2(x_j)| \lor \ |I_A^2(x_j) - I_B^2(x_j)| \lor \ |N_A^2(x_j) - N_B^2(x_j)| \lor \ |R_A^2(x_j) - R_B^2(x_j)|} \right) \right)$$

**$SFCT^4 (A, B)$**

$$\frac{1}{n} \sum_{j=1}^{n} \cot \left( \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{|p_A^2(x_j) - p_B^2(x_j)|}{|p_A^2(x_j) - p_B^2(x_j)| \lor \ |I_A^2(x_j) - I_B^2(x_j)| \lor \ |N_A^2(x_j) - N_B^2(x_j)| \lor \ |R_A^2(x_j) - R_B^2(x_j)|} \right) \right)$$

where the symbol “$\lor$” is the maximum operation.

In many cases, the weight of the elements $x_j \in X$ should be taken into account. For example, in MADM, the considered attributes usually have different importance, and thus need to be assigned different weights. As a result, four weighted cotangent similarity measure between SFSs $A$ and $B$ is explored:

**$WSFCT^1 (A, B)$**

$$\sum_{j=1}^{n} \omega_j \cot \left( \frac{\pi}{4} + \frac{\pi}{4} \left( \frac{|p_A^2(x_j) - p_B^2(x_j)|}{|p_A^2(x_j) - p_B^2(x_j)| \lor \ |I_A^2(x_j) - I_B^2(x_j)| \lor \ |N_A^2(x_j) - N_B^2(x_j)| \lor \ |R_A^2(x_j) - R_B^2(x_j)|} \right) \right)$$

**$WSFCT^2 (A, B)$**

$$\sum_{j=1}^{n} \omega_j \cot \left( \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{|p_A^2(x_j) - p_B^2(x_j)|}{|p_A^2(x_j) - p_B^2(x_j)| \lor \ |I_A^2(x_j) - I_B^2(x_j)| \lor \ |N_A^2(x_j) - N_B^2(x_j)| \lor \ |R_A^2(x_j) - R_B^2(x_j)|} \right) \right)$$

**$WSFCT^3 (A, B)$**

$$\sum_{j=1}^{n} \omega_j \cot \left( \frac{\pi}{4} + \frac{\pi}{4} \left( \frac{|p_A^2(x_j) - p_B^2(x_j)|}{|p_A^2(x_j) - p_B^2(x_j)| \lor \ |I_A^2(x_j) - I_B^2(x_j)| \lor \ |N_A^2(x_j) - N_B^2(x_j)| \lor \ |R_A^2(x_j) - R_B^2(x_j)|} \right) \right)$$

**$WSFCT^4 (A, B)$**

$$\sum_{j=1}^{n} \omega_j \cot \left( \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{|p_A^2(x_j) - p_B^2(x_j)|}{|p_A^2(x_j) - p_B^2(x_j)| \lor \ |I_A^2(x_j) - I_B^2(x_j)| \lor \ |N_A^2(x_j) - N_B^2(x_j)| \lor \ |R_A^2(x_j) - R_B^2(x_j)|} \right) \right)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $x_j (j = 1, 2, \ldots, n)$, with $\omega_j \in [0, 1]$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} \omega_j = 1$ and the symbol “$\lor$” is the maximum operation. In particular, if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the weighted cotangent similarity measure reduces to cotangent similarity measure.

**IV. APPLICATIONS**

In such section, the cosine similarity measures for SFSs are applied to pattern recognition and medical diagnosis. Now, we utilized Garg’s two numerical examples [36] to illustrate the feasibility of the proposed methods and deliver a comparative analysis.

**A. EXAMPLE1-PATTERN RECOGNITION**

Let us consider a three known patterns $A_i (i = 1, 2, 3)$ which are depicted by the SFSs $A_j (i = 1, 2, 3)$ in the feature space $X = \{x_1, x_2, x_3\}$ as $A_1, A_2, A_3$, shown at the bottom of this page.

Consider an unknown pattern $A \in SFSs(X)$ that will be recognized, where

$$A = \{(x_1, 0.7, 0.5, 0.3), (x_2, 0.8, 0.4, 0.2), (x_3, 0.9, 0.2, 0.3)\}$$

The main purpose of this problem is classify the pattern $A$ in one of classes $A_1, A_2$ and $A_3$. For it, the proposed similarities degrees have been used to compute between $A$ to $A_i (i = 1, 2, 3)$ and are listed in Table 1.

**Table 1. The similarity measures between $A_i (i = 1, 2, 3)$ and $A$.**

| similarity measures | $(A_i, A)$ |
|---------------------|------------|
| $SFC^1 (A_i, A)$    | 0.9389     |
| $SFC^2 (A_i, A)$    | 0.8504     |
| $SFC^3 (A_i, A)$    | 0.9005     |
| $SFC^4 (A_i, A)$    | 0.8641     |
| $WSFCT^1 (A_i, A)$  | 0.6750     |
| $WSFCT^2 (A_i, A)$  | 0.6941     |
| $WSFCT^3 (A_i, A)$  | 0.6625     |
| $WSFCT^4 (A_i, A)$  | 0.5922     |
| $SFCS^1 (A_i, A)$   | 0.9344     |
| $SFCS^2 (A_i, A)$   | 0.9464     |
| $SFCS^3 (A_i, A)$   | 0.9235     |
| $SFCS^4 (A_i, A)$   | 0.9051     |
| $SFCT^1 (A_i, A)$   | 0.6239     |
| $SFCT^2 (A_i, A)$   | 0.7095     |
| $SFCT^3 (A_i, A)$   | 0.6781     |
| $SFCT^4 (A_i, A)$   | 0.6490     |

**A1 = \{(x_1, 0.8, 0.3, 0.4), (x_2, 0.7, 0.1, 0.4), (x_3, 0.6, 0.4, 0.3)\}**

**A2 = \{(x_1, 0.5, 0.4, 0.6), (x_2, 0.6, 0.4, 0.2), (x_3, 0.7, 0.3, 0.2)\}**

**A3 = \{(x_1, 0.8, 0.1, 0.2), (x_2, 0.9, 0.3, 0.1), (x_3, 0.7, 0.1, 0.5)\}**
From the Table 1, we know that the degree of similarity between \( A_3 \) and \( A \) is the largest one. That is, all the ten similarity measures assign the unknown class \( A \) to the known class \( A_3 \) according to the principle of the maximum degree of similarity between SFSs. Compared with Garg’s correlation coefficients method [36], we can derive same result.

If the weight of \( x_i (i = 1, 2, 3) \) are 0.2, 0.3 and 0.5, respectively, the proposed weighted similarities measures have been used to derive between \( A \) to \( A_i (i = 1, 2, 3) \) and are expressed in Table 2.

From Table 2, except for the \( WSFCS^2 (A_i, B) \) and \( WSFCT^2 (A_i, B) (i = 1, 2, 3, 4, 5) \), we know that the similarity measures between \( A_3 \) and \( A \) is the largest one as derived by ten similarity measures. That is, the eight similarity measures assign the unknown class \( A \) to the known class \( A_3 \) according to the principle of the maximum degree of similarity between SFSs except for the \( WSFCS^2 (A_i, B) \) and \( WSFCT^2 (A_i, B) (i = 1, 2, 3, 4, 5) \).

### B. EXAMPLE 2-MEDICAL DIAGNOSIS

Let us consider a set of diagnoses \( D = \{D_1 \text{ (Viral fever)}, D_2 \text{ (Malaria)}, D_3 \text{ (Typhoid)}, D_4 \text{ (Stomach Problem)}, D_5 \text{ (Chest Problem)} \} \) and a set of symptoms \( S = \{s_1 \text{ (Temperature)}, s_2 \text{ (Headache)}, s_3 \text{ (Stomach Pain)}, s_4 \text{ (Cough)}, s_5 \text{ (Chest Pain)} \} \). Suppose that a patient, with respect to all symptoms, can be depicted by the following SFS \( P \) (Patient), as shown at the bottom of this page.

And then each diagnoses \( D_i (i = 1, 2, 3, 4, 5) \) can viewed as SFSs with respect to all the symptoms as follows in \( D_1 \text{ (Viral fever)}, D_2 \text{ (Malaria)}, D_3 \text{ (Typhoid)}, D_4 \text{ (Stomach Problem)}, D_5 \text{ (Chest Problem)} \), as shown at the bottom of the next page.

Our main purpose is to classify the pattern \( P \) in one of the classes \( D_i (i = 1, 2, 3, 4, 5) \). For this, the proposed similarities measures have been used to compute between \( P \) to \( D_i (i = 1, 2, 3, 4, 5) \) and are given in Table 3.

\[
P (\text{Patiment}) = \begin{cases} (s_1, 0.7, 0.2, 0.4), (s_2, 0.6, 0.5, 0.3), (s_3, 0.3, 0.4, 0.1), \\ (s_4, 0.5, 0.3, 0.2), (s_5, 0.4, 0.5, 0.6) \end{cases}
\]
TABLE 3. The similarity measures between $D_i$ ($i = 1, 2, 3, 4, 5$) and $P$.

| similarity measures | $(D_1, P)$ | $(D_2, P)$ | $(D_3, P)$ | $(D_4, P)$ | $(D_5, P)$ |
|---------------------|------------|------------|------------|------------|------------|
| $SFC^1 (D_i, P)$    | 0.8189     | 0.8059     | 0.9480     | 0.7016     | 0.7799     |
| $SFC^2 (D_i, P)$    | 0.8263     | 0.8518     | 0.9331     | 0.7101     | 0.8214     |
| $SFCS^1 (D_i, P)$   | 0.9087     | 0.9494     | 0.9843     | 0.8777     | 0.9197     |
| $SFCS^2 (D_i, P)$   | 0.9365     | 0.9516     | 0.9814     | 0.9172     | 0.9481     |
| $SFCS^3 (D_i, P)$   | 0.9013     | 0.9105     | 0.9643     | 0.8123     | 0.8846     |
| $SFCS^4 (D_i, P)$   | 0.8786     | 0.8863     | 0.9515     | 0.7795     | 0.8729     |
| $SFCT^1 (D_i, P)$   | 0.6470     | 0.7335     | 0.8412     | 0.5993     | 0.6646     |
| $SFCT^2 (D_i, P)$   | 0.7003     | 0.7381     | 0.8264     | 0.6601     | 0.7326     |
| $SFCT^3 (D_i, P)$   | 0.6316     | 0.6491     | 0.7874     | 0.5329     | 0.6073     |
| $SFCT^4 (D_i, P)$   | 0.5950     | 0.6094     | 0.7412     | 0.4971     | 0.5930     |

From the Table 3, we know that the similarity measures between $D_3$ and $P$ is the largest one as derived by ten similarity measures. That is, the ten similarity measures assign the unknown class $P$ to the known class $D_3$. Compared with Garg’s correlation coefficients method [36], we can get same result that the ten similarity measures assign the unknown class $P$ to the known class $D_3$ according to the principle of the maximum degree of similarity between SFSs.

C. ADVANTAGES OF THE PROPOSED SIMILARITY MEASURES

Although, IFSs theory has been successfully applied in diverse areas, but there are situations in real life which can’t be represented by IFSs. Picture fuzzy sets are extension of IFSs. The PFS is characterized by the positive-membership degree, neutral-membership degree and the non-membership degree, whose sum of them is less than or equal to 1, the PFS is more general than the IFS. As a generalization of PFS, the spherical fuzzy sets (SFSs) are also expressed by the positive-membership degree, neutral-membership degree and the non-membership degree, whose sum of squares is less than or equal to 1. In some cases, the SFS can solve the problems that the IFS and PFS cannot, for example, if a DM gives positive-membership degree, neutral-membership degree and
the non-membership degree as 0.6, 0.4 and 0.1, respectively, then it is only valid for the SFS. In other words, all the picture fuzzy degrees are a part of the Spherical fuzzy degrees, which indicates that the SFS is more powerful to tackle the uncertain problems. Therefore, the MADM with SFSs is more suitable for real scientific and engineering applications.

Also it has been observed from the existing studies [17], [19], [20], [21], [23] that the various researchers proposed some algorithms by using similarity measures for IFSs. As mentioned above, there are some situations that can’t be depicted by IFSs, so their corresponding algorithm may not give appropriate results.

The similarity measures for IFSs are special case of the similarity measures of SFSs. Therefore, the proposed similarity measures are more general and suitable to tackle the real-life problem more accurately than the existing ones.

V. CONCLUSION

In this paper, we presented ten similarity measures between SFSs based on the cosine function by considering the degree of positive-membership, degree of neutral-membership membership, degree of non-membership membership and degree of refusal-membership in SFSs. Then, we applied these similarity measures and weighted similarity measures between SFSs to pattern recognition and medical diagnosis. Finally, two illustrative examples are supplied to validate the efficiency of the similarity measures. In the future, the application of the proposed cosine similarity measures of SFSs needs to be explored in complex decision making [70]–[74], risk analysis [75], [76] and many other fields under uncertain environments [52], [77]–[86]. And SFSs are particular cases of Hyperspherical Neutrosophic numbers, thus, we shall extend our work to Hyperspherical Neutrosophic numbers [87].

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