Decays in Quantum Hierarchical Models

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We study the dynamics of a simple model for quantum decay, where a single state is coupled to a set of discrete states, the pseudo continuum, each coupled to a real continuum of states. We find that for constant matrix elements between the single state and the pseudo continuum the decay occurs via one state in a certain region of the parameters, involving the Dicke and quantum Zeno effects. When the matrix elements are random several cases are identified. For a pseudo continuum with small bandwidth there are weakly damped oscillations in the probability to be in the initial single state. For intermediate bandwidth one finds mesoscopic fluctuations in the probability with amplitude inversely proportional to the square root of the volume of the pseudo continuum space. They last for a long time compared to the non-random case.

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The problem of the decay of an excitation into a continuum is a fundamental problem in quantum mechanics [1], appearing in numerous fields of physics. A natural hierarchy of couplings occurs in many physical systems.

For example, a spin of a nucleus may be coupled to the electromagnetic modes of a cavity in which it is situated, and these in turn may be coupled to the modes of a larger box or the vacuum [2]. Hierarchical systems were studied [3, 4, 5, 6, 7, 8, 9, 10] but the time dependence for the model we present was not investigated.

We will first show that in the case of constant couplings \( u_\mu = u \), \( v_\mu = v \) the system decay is dominantly exponential with rate:

\[
\Gamma = \frac{u^2}{\gamma + \frac{d}{\pi}},
\]

where \( \gamma = \pi v^2 \nu_c \), \( \nu_c \) being the RC density of states [12]. As the coupling to the RC becomes stronger, the rate decreases, a phenomenon referred to as the quantum Zeno effect [13, 14]. A diagonalization of Hamiltonian (1) for the case \( \gamma \gg d \) yields a very wide state, similar to the Dicke state [15], and another wide state, which is the most relevant for the decay of the system. The other \( N-1 \) states acquire a small width and are not pertinent for the dynamics of the system until long times of order \( \gamma \log(\gamma) \).

When the couplings to the RC are random we identify four regimes. The boundaries between them are controlled by the typical matrix element \( \bar{u} \sim \frac{\lambda}{\sqrt{N}} \). For \( u_0 < d \) there are decaying Rabi oscillations via one state. For \( u_0 < D \) Fermi’s golden rule (FGR) is obtained, while for \( u_0 \sim D \) novel mesoscopic fluctuations with amplitude
coefficients

Notice that even when we normalize the states linear combination of many states in the PC.

Physical realizations.- The model (11) applies, for example, to a small quantum dot coupled to one or more quantum dots, each of which is coupled to a lead. One can study the time dependence of an injected electron’s probability to remain in the dot, assuming that all relevant levels in the dots are empty. In some cases the matrix elements $u_\mu$ connecting two dots can be taken as constant [14]. For the more generic case of a single dot coupled to a disordered or sufficiently distorted larger quantum dot, $u_\mu$ do not have the same sign, and we take them to be random. These solid state implementations have close analogies when one replaces the quantum dots by atoms in optical cavities [2].

Derivation of the results.- Eliminating the amplitudes in the RC after Laplace transforming the equations of motion, the dynamics of the other $N+1$ amplitudes in the SS and the PC are formulated in terms of a $N+1$ by $N+1$ non-hermitian matrix, which is the matrix describing the original matrix elements between these states, plus a matrix element $-i\gamma_{\mu\nu} = -i\pi\nu v_{\mu}v_{\nu}$ [11, 17]. The reduced Hamiltonian for the system is:

$$H = \sum_{\mu=1}^{N} E_\mu |\mu\rangle\langle \mu | + u_\mu (|0\rangle\langle \mu | + |\mu\rangle\langle 0 |) - \sum_{\mu,\nu} i\gamma_{\mu\nu} |\mu\rangle\langle \nu |.$$

(4)

From now on we shall assume $v_\mu$ to be constant, and therefore $\gamma_{\mu\nu} \equiv \gamma$.

This leads to the (exact) eigenvalue equation:

$$\sum_{\mu} \frac{u_\mu^2}{\lambda - E_\mu} - i\gamma \Sigma_1 \times \frac{u_\mu}{\lambda - E_\mu} = \lambda,$$

(5)

with $\Sigma_1 = \frac{\sum \frac{u_\mu}{\lambda - E_\mu}}{\sum \frac{u_\mu^2}{\lambda - E_\mu} + 1}$.

The generic form for the eigenvectors is:

$$|V_n\rangle = |0\rangle + \sum_{\mu} \frac{u_\mu - i\gamma \Sigma_1}{\lambda_{\mu} - E_\mu} |\mu\rangle,$$

(6)

where $\lambda_n$ is the corresponding eigenvalue.

Although $H$ is non-hermitian, we can still decompose the initial state as a superposition of its eigenvectors [18]:

$$A_0(t) = \sum_{n} C_n e^{-i\lambda_n t} |0\rangle|V_n\rangle.$$

(7)

Notice that even when we normalize the states $|V_n\rangle$, the coefficients $C_n$ are not the usual projections $\langle V_n | 0 \rangle$ [18, 19].

We shall now use this formalism to study the cases of constant and random matrix elements.

A. Constant matrix elements.- First we take the matrix element between the initial SS and the levels of the PC to be a constant $u$ [14].

To analyze the decay, let us write the equations of motion for the amplitudes following from Eq. (4):

$$i\frac{dA_\mu}{dt} = uA_\nu + E_\mu A_\mu.$$

(8)

where $A_\mu$ is the amplitude of state $\mu$

Upon Laplace transforming Eq. [13], we obtain:

$$i\omega A_\mu = i + u \sum_{\nu} A_\nu + E_\mu A_\mu.$$

(9)

$$i\omega A_\mu = uA_\mu - i\gamma \sum_{\nu} A_\nu + E_\mu A_\mu.$$

(10)

For $D \gg \omega \gg d$ and for the SS with energy far enough from the edges of the PC band, we can approximate the sum $\sum_i \frac{1}{\omega - E_i}$ by $-\frac{\Gamma}{\omega}$, which leads to the result:

$$A_0 = \frac{1}{\omega + \Gamma},$$

(11)

with $\Gamma$ given by Eq. [3]. The inverse Laplace transform gives the exponential decay, in a large time window, which for $\gamma \ll d$ is given by $\frac{1}{\omega + \Gamma} < t < \frac{1}{\omega}$.

We shall now analyze the structure of the eigenstates yielding the result of Eq. [3].

For the limit $\gamma = 0$, we have the discrete Wigner-Weisskopf problem [1], and Eq. [3] reduces to FGR. In that case the eigenvalues are real (since the Hamiltonian is hermitian), and it is the superposition of many eigenvectors that gives rise to the decay (for intermediate times).

As $\gamma$ increases and reaches the regime $\gamma \gg d$, the behavior is changed and one state completely dominates the decay of the system. In this regime there is a fast decaying eigenvector approximately of the form $|0\rangle + x \sum_{j} |j\rangle$, with eigenvalue $\lambda \approx -i\gamma N$ and $x \approx \frac{1}{wN}$. This is related to the Dicke effect, where a coherent sum of many states with equal amplitudes is also present [15]. We also find an additional eigenvalue $-\frac{\Gamma}{\omega}$, which gives rise to the decay of the system (for intermediate times).
B. Random matrix elements.- When the disorder is sufficiently large, the matrix elements can be considered random \[10\]. For simplicity, let us consider the case where the level spacing is constant, but the elements \(u_{ij}\) are randomly distributed around 0, with a standard deviation \(\bar{u}\).

To understand the magnitudes of the matrix elements involved in the physical realization of a small quantum dot coupled to a larger disordered one which is coupled to a one channel lead \[11\], it is instructive to look at their site representation. If we denote the sites of the larger dot by \(i\), then the isolated dot eigenstates are \(|\mu\rangle = \frac{1}{\sqrt{N}} \sum_i \phi^\mu_i |i\rangle\), where \(\phi^\mu_i\) are random coefficients of order unity (assuming the disorder is large enough, yet not too large as to make the states localized). We shall assume that out of the \(N\) sites, \(S\) are coupled to \(|0\rangle\), and \(S'\) are coupled to the RC. Changing basis to the set \(|\mu\rangle\), one can verify that the couplings to the RC and \(|0\rangle\) are random. If the matrix elements \(V_{jk}\) (between a site \(j\) and a state \(k\) in the lead) do not depend on \(j\), after the lead elimination the Hamiltonian contains terms \(\sim |\mu\rangle \langle \nu| a_{\mu} a_{\nu}^\dagger\), and by multiplying the states \(|\mu\rangle\) by a phase factor we can obtain a model corresponding to Eq. (1), with \(\gamma_{\mu \nu}\) real and positive. For simplicity we shall assume the magnitude of \(\gamma_{\mu \nu}\) to also be a constant, \(\gamma\) \[21\]. Denoting the typical matrix element connecting a site in the larger dot with a site in the lead by \(v_0\) and the typical matrix element connecting a site in the larger dot with a site in the single state dot \(u_0\), a straightforward calculation shows that the typical tunneling matrix elements in the Hamiltonian \[4\] are given by the relations: \(\gamma = \gamma_0 \sqrt{\frac{\bar{u}}{\bar{u}_0}}\) and \(\bar{u}_0 = \frac{\gamma_0}{\sqrt{\bar{u}}} u_0\) where \(\gamma_0 = \nu_0 |v_0|^2\) and \(\bar{u}\) is the typical matrix element \(u_{\mu\nu}\) in Eq. (4). Notice that \(u_0, v_0, \gamma_0\) are microscopic parameters independent of the system size.

We shall now analyze the dynamics for four different cases: \(u_0 > D, u_0 \sim D, D > u_0 > d, d > u_0\).

Case I.- To understand the behavior for \(u_0 > D\), we study the degenerate case \(D = 0\), for which the Hamiltonian is

\[
H = \sum_{\mu} u_{\mu}(|\mu\rangle \langle \mu| - |\mu\rangle \langle 0| - i\gamma \sum_{\mu, \nu} |\mu\rangle \langle \nu|).
\]

Defining the states \(|W_1\rangle = \frac{1}{\sqrt{N}} \sum_{\mu} |\mu\rangle\) and \(|W_3\rangle = \frac{1}{\sqrt{N}} \sum_{\mu} |\mu\rangle\), the Hamiltonian takes the form:

\[
H = \sqrt{\sum_{\mu} u_{\mu}^2} (|0\rangle \langle W_1| + |W_1\rangle \langle 0|) - iN\gamma |W_3\rangle \langle W_3|. \tag{12}
\]

As \(|W_1\rangle\) and \(|W_3\rangle\) are not orthogonal, it is useful to use a Gram-Schmidt procedure to define \(|W_2\rangle = |W_3\rangle - \langle W_3| \langle W_3| W_1\rangle W_1\rangle\) and then normalize it. Now we can represent the system’s Hamiltonian in the basis formed by \(|0\rangle\), \(|W_2\rangle\) and \(|W_3\rangle\) as a 3x3 matrix:

\[
H = \begin{pmatrix}
0 & U_2 \sqrt{(1 - c^2)} & cU_2 \\
U_2 \sqrt{(1 - c^2)} & 0 & 0 \\
cU_2 & 0 & -i\gamma N
\end{pmatrix}, \tag{13}
\]

where \(U_1 = \sum_j u_j \sim u_0\), \(U_2 = \sqrt{\sum_j u^2_j} \sim u_0\), \(c = \frac{u_0}{U_2 \sqrt{N}} \sim \frac{1}{\sqrt{N}}\).

Diagonalizing \(H\) perturbatively in \(\frac{1}{\sqrt{N}}\) we find (using \(c \sim \frac{\bar{u}}{\bar{u}_0}\)) eigenvectors \(|V_3\rangle = |W_3\rangle + O(\frac{1}{\sqrt{N}})\) with eigenvalue \(-i\gamma_0 + O(\frac{1}{\sqrt{N}})\), and \(|V_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |W_2\rangle)\) with eigenvalues \(\lambda_\pm = \pm U_2 - i\delta\), where \(\delta = -i \frac{U_2^2}{2N\gamma} \sim -i \frac{u_0^2}{2N}\).

Since the projection of the initial state on \(|V_3\rangle\) is negligible, the decay of the initial state is described by a superposition of two exponentially decaying terms, with exponents \(\approx \pm U_2 - i\delta\). This implies Rabi-type oscillations \[1\], vol 1. 447) with a characteristic frequency \(U_2 \sim u_0\), and an envelope decaying exponentially with rate \(\frac{U_2^2}{2N\gamma} \sim \frac{u_0^2}{2N}\) \[22\]. Notice that in the case of finite bandwidth, the above analysis will be approximately correct if \(|\lambda_\pm| \gg D\) and \(D \gg u_0\). The first condition gives the restriction \(D \ll u_0\). Since this relation is \(N\) independent, both cases are physically inaccessible in the limit of large \(N\). The result of a numerical simulation is shown in Fig. (3a).

Notice two peculiarities of the result: First, as in Eq. (3), when \(\gamma\) is increased the decay is slower. Second, the decay is much slower than in the ordered case, by a factor of \(2N\). This can be understood as follows: the sum of all imaginary parts of the eigenvalues exactly equals \(-iN\gamma\). Since the Dicke eigenvector is still of the same form as before, with an eigenvalue approximately given by \(-iN\gamma\), the rest of the eigenvalues have small imaginary parts. Plugging the Dicke eigenvalue into Eq. (5) as \(-iN\gamma + \epsilon\), one finds that \(\epsilon \approx \frac{\gamma_0^2 u_0^2}{2N}\). For constant matrix elements, the part of the decay sum rule not taken up

![Graph showing imaginary part of eigenvalues of Hamiltonian](image-url)
The finite bandwidth case shows a sharp initial decay followed by the Dicke eigenvector, namely \( \text{sinc} \) function.

For the non-degenerate case \( \gamma \sum \) consistently, that \( \Sigma \) \( E \) energies form a band of bandwidth \( D \), we have the interference of \( \Sigma \) elements in Eq. (7), but these are now randomly distributed. Although we have random oscillations, a characteristic frequency seems apparent. This is because typically a few states are, by chance, coupled more than the others. Since in our case \( q_n \sim |\langle V_n | 0 \rangle| \sim \frac{1}{\sqrt{N}} \) the amplitude of the oscillations \( \sim \frac{1}{\sqrt{N}} \) as well.

Case III.- Defining \( \lambda_0 = \pi \bar{a}^2 / d \sim u_0^2 / D \), the FGR rate, we find different behavior for the cases \( D \geq \lambda_0 \). Notice that the condition of the crossover \( D \sim \lambda_0 \) is the same as before, namely \( D \sim u_0 \).

If \( D \gg \lambda_0 (D \gg u_0) \), a coefficient of an eigenvalue \( \lambda \) will have a typical size \( \frac{1}{\mu \bar{a}^2} \). Notice that this fits the exact sum \( \sum C_n^2 = 1 \). Thus, we have a superposition of \( M \sim \pi \bar{a}^2 \) oscillatory signals, with (positive) random coefficients. If furthermore \( \bar{u} \gg d \) (which is equivalent to \( u_0 \sqrt{N} \gg D \)), we will have a large number of random components within a complete Lorentzian, and therefore the FGR exponential decay will be retrieved (but with the Zeno effect suppressed).

Case IV.- For \( u_0 \ll d \), a single state is relevant, and the system will show (decaying) Rabi oscillations.

Conclusions.- We considered the generic problem of a single state coupled to a real continuum via a pseudo continuum. In the ordered case, we found that a single eigenvector characterizes most of the decay of the system, and the decay becomes slower when increasing the coupling to the real continuum. When the bandwidth \( D \) is smaller than the typical matrix element \( u_0 \), adding disorder causes the decay to be much slower, and introduces oscillations in time. When \( D \sim u_0 \), mesoscopic fluctuations in the probability to stay in the single state as a function of time follow, while for a larger bandwidth, FGR exponential decay is retrieved.

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For a single channel lead (as defined by [17]) the matrix element between a state in the PC and in the RC does not depend on the index of the RC state, provided the coupling region is smaller than the typical wavelength [23]. The case of multi channel leads will be treated in future works.

When the matrix elements have alternating sign, the Zeno effect will not appear [24], and FGR holds.

Consider a number of small quantum dots with no tunneling amongst them, at equal distances from the single quantum dot. The matrix elements $u_j$ will be constant.

For a treatment of non-hermitian matrices (including perturbation theory) see: M. M. Sternheim and J. F. Walker, Phys. Rev. C 6, 114 (1972).

To find the coefficients $C_n$, it is helpful to note that since $H$ is symmetric, the scalar product of two different eigenvectors (without taking the complex conjugate of one of them) vanishes. It follows that if we redefine the projection operation as a scalar product, and normalize the eigenvectors accordingly, the usual formulas for the eigenvector decomposition will hold.

Notice that the upper time for which the decay is exponential increases with $\gamma$. For, roughly, $\gamma > \gamma_0$, where $\gamma_0$ is the FGR rate, the coupling to the continuum increases the exponential decay regime.

We have confirmed that fluctuations in the magnitude do not lead to a qualitative difference in the results in most cases, as long as the sign of the matrix elements is constant.

In the tight-binding picture with $S = 1$ the rate has a simple interpretation, being the typical matrix element between the SS and the site it is coupled to in the larger dot.

We have confirmed that fluctuations in the magnitude do not lead to a qualitative difference in the results in most cases, as long as the sign of the matrix elements is constant.