Physical and mathematical evidences for a negative-rank tensor

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We propose and study the properties of a new potential demanded by the self-consistency of the duality scheme in electromagnetic-like field theories of totally anti-symmetric tensors in diverse dimensions. Physical implications of this new potential is manifest under the presence of scalar condensates in the Julia-Toulouse mechanism for the nucleation of topological defects with consequences for the confinement phenomenon.

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Physics is an experimental science and theoretical advances are generally driven by careful and delicate observations coming from lab measurements aimed to gain further insights in the overall structure of the theories which describes physical phenomena. Most of the time experimental and theoretical efforts feed each other back to advance in the general goal of physics - the description and interpretation of Nature. There are, however, occasions when adjusts in the theoretical framework become necessary to preserve the structure of the basic postulates under new observations. Sometimes these purely theoretical works may come to deeply modify the way we understand the natural phenomena. Probably the most striking examples are (i) Einstein’s Special Relativity, inspired by the existence of a dramatic coincidence in the Faraday’s law [1] and (ii) Dirac’s charge-monopole quantization [2] suggested by an asymmetry in Maxwell’s equations, opening the field of topology to theoretical physics. In this Letter we want to report on a lack of consistency of the duality symmetry in the generalized electromagnetic framework of theories described by massless anti-symmetric tensors of arbitrary ranks in diverse dimensions [3]. We have found that a new and unexpected object becomes necessary in order to maintain the tensorial structure of the duality symmetry. It is important to mention that our present study is not just a theoretical curiosity. We have in mind physical implications for the condensation of topological defects and duality to various field theories in which the notion of confinement is of fundamental phenomenological and experimental significance [4].

The confinement of the fundamental constituents of matter is probably one of the fundamental and long-standing issues of theoretical physics whose solution has evaded complete comprehension despite intense theoretical and experimental effort. In particular, field theories that yield a linear potential are very important to particle physics, since those theories may be used to describe the confinement of quarks and gluons and be considered as effective theories of quantum chromodynamics. In the past, many authors have shown that the condensation of topological defects such as vortices and monopoles may lead to quark confinement [5]. Due to intense interest mainly in string related topics, these studies have been extended out of the four dimensional domain to theories of antisymmetric tensors of arbitrary ranks in arbitrary space-time dimensions that appear as low-energy effective field theories of strings. This has also helped us to gain insights over the mechanisms of confinement in different contexts [6].

Duality, on the other hand, is a concept of paramount and ever-increasing importance playing a fundamental role in distinct areas of nowadays physics [7]. The electromagnetic duality is probably the main paradigm of this notion displaying a beautiful symmetry between the electric and magnetic sectors of the Maxwell’s equations [8]. Because of the major role played by the extended electromagnetic theories of anti-symmetric tensors in QCD, Super-Gravity and String Theory, duality has been intensely examined in those scenarios in recent years, including couplings with extended sources (branes) of proper dimensions.

Certainly the best way to describe tensors in this extended electromagnetism is by use of differential forms. However, in order to allow access to a larger audience we shall avoid the use of such a language or, in occasions, use it in a more relaxed and intuitive form [23]. Let us consider then theories described by a rank-\( p \) totally anti-symmetric potential \( A_p = A_{\mu_1 \cdots \mu_p} \) whose field-strength reads, as usual

\[
F_{p+1}(A_p) \equiv \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]} ,
\]

being proportional to the exterior derivative of \( A_p \), with Greek indices taking values over the space-time dimensions.
μ = 0, 1, ⋯ , D − 1 [24]. The action controlling the classical dynamics is given by the standard textbook form

$$S_p = N_p \int F_{p+1} F^{p+1},$$  \hspace{1cm} (2)

where $N_p$ is a proper normalization constant. This is also the starting point to the canonical quantization program. It is noticeable that in the free action the potential tensor $A_p$ itself is not present, only its exterior derivative.

For $p = 0$, the potential $A_0$ is just a scalar field whose field equations are described by the massless Klein-Gordon operator. The $p = 1$ case describes the usual Maxwell case with the vector potential $A_1 \equiv A_\mu$ giving origin to the electric/magnetic fields described by the rank-two field tensor $F_2(A_1) \equiv F_{\mu\nu}(A_\mu)$ as,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$  \hspace{1cm} (3)

whose action and field equations are invariant under the usual gauge symmetry,

$$\delta A_\mu = \partial_\mu \xi.$$  \hspace{1cm} (4)

Another interesting example is the $p = 2$ case described by the Kalb-Ramond field $A_2 \equiv A_{\mu\nu}$ whose action is invariant under the rank-two gauge transformation $\delta A_{\mu\nu} = \partial_{[\mu} \xi_{\nu]}$. In general, the rank-$p$ theory is invariant under an extended gauge transformation as,

$$\delta A_p = \partial_{[\alpha} \xi_{p-1]} \equiv F_p (\xi_{p-1}),$$  \hspace{1cm} (5)

involving a gauge function $\xi_{p-1}$ of one degree lesser.

A canonical analysis, taking into account the gauge constraints of these theories reveals that the model described by the tensor potential $A_p$ possesses $C_{D-2}^p$ degrees of freedom, where

$$C_{D-2}^p = \frac{(D-2)!}{p!(D-p-2)!},$$  \hspace{1cm} (6)

is the binomial coefficient. For vector potentials in the Maxwell case, the number of degrees of freedom is just $(D-2)$ giving the two well known polarization of the four-dimensional photon, just a single degree of freedom in the three-dimensional case and no transverse propagating degrees of freedom for a theory defined on the line, as it should be expected. The Pascal triangle symmetry rule demands that

$$C_{D-2}^p = C_{q}^{D-2},$$  \hspace{1cm} (7)

which is valid if

$$p + q + 2 = D.$$  \hspace{1cm} (8)

There should exist therefore, a dual tensor potential $\tilde{A}_q$, to be properly defined below, that although being in general of distinct rank, will propagate exactly the same number of transverse degrees of freedom as the original $A_p$ potential which suggests that they are indeed related by duality. In fact, one can show through the duality mapping, a Legendre transformation followed by the solution of a simple constraint, that this assumption is indeed correct. We shall refer to the simple rule (8) as the massless duality condition. An important instance is the $p = 1$ case where the dual tensor is also a vector for $D = 4$ but it is a scalar for $D = 3$. It is fundamental to notice that apparently the vector Maxwell theory on the line has no dual-partner. This difficulty is generic for the whole class of extended electromagnetism and it is the main point of discussion in this paper.

Let us next consider the inclusion of sources, the extended objects known as branes. They may appear as topological defects coming from the low-energy sector of theories having broken symmetries, where the defects are solitonic solutions. Current conservation demands the branes to be closed. Closed $(p - 1)$-branes are characterized by their charge, say $e$, and the Chern-Kernel $\Lambda_{q+1}$ [9], giving the localization of the brane, such that their (conserved) $p$-current may be written as [25],

$$J^p(\Lambda) = e^D \partial_1 \Lambda_{q+1}.$$  \hspace{1cm} (9)

This form automatically displays the closeness of the source and shows an invariance under

$$\delta \Lambda_{q+1} = \partial_{[1} X_{q]}.$$

$$\delta \Lambda_{q+1} = \partial_{[1} X_{q]}.$$  \hspace{1cm} (10)
meaning that its $p$-dimensional Dirac-brane, whose Chern-Kernel is $\chi_q$ is physically unobservable. The “smallest” brane is the instanton whose dimension is $(-1)$. The usual point-charge is a zero-brane and a string is an one-dimensional brane. The “biggest” brane fitting in a $D$-dimensional space-time is the $(D-2)$-brane whose Chern-Kernel is $\Lambda_0$ and its $d$-current is $J^d = e^{d,1} \partial_1 \Lambda_0$.

A $(p-1)$-brane is an extended charge which, upon time-evolution, describes a $p$-dimensional object, its world-volume current. These objects may therefore couple transversally to a rank-$p$ potential and longitudinally to a rank-$q$ potential, with $p$ and $q$ keeping a relation as in (8). In the usual electromagnetic jargon, the $A_p$ potential “sees” the $(p-1)$-brane as an electric brane while its dual tensor, $\tilde{A}_q$ experiences it as magnetic. Therefore, the potential $A_p$ may couple electrically and magnetically to two distinct branes of dimensions $(p-1)$ and $(q-1)$, respectively, according to the following diagram,

$$\begin{array}{ccc}
(q-1) : g ; \Omega_{p+1} & \xrightarrow{MC} & A_p \\
\xrightarrow{EC} & \xrightarrow{e} & (p-1) : e ; \Lambda_{q+1}
\end{array}$$

Here, the $e$-charged $(p-1)$-brane is described by its Chern-Kernel $\Lambda_{q+1}$ while the $g$-charged $(q-1)$-brane is described by $\Omega_{p+1}$. The classical action for this system is [2, 8]

$$S = \int \left\{ \left[ F_{p+1}(A_p) - g \Omega_{p+1} \right]^2 + e A_p e^D \partial_1 \Lambda_{q+1} \right\},$$

whose equations of motion read

$$\begin{align*}
\partial_1 F^{p+1} &= e J^p(\Lambda) \\
\partial_1 \ast F^{q+1} &= g J^q(\Omega),
\end{align*}$$

where the Hodge dual tensor is defined as

$$\ast F^{q+1} = e^{q+1,p+1} F_{p+1}. \quad (14)$$

These equations display a pair of gauge symmetries [10, 11]. The usual (electric) gauge transformation (5) leaves invariant both terms in (12) thanks to the conservation of the electric current (or the closeness of the brane). There is also a “magnetic” like gauge transformation combining $\delta \Omega_{p+1} = \partial_1 \chi_q$ with $\delta A_p = g \chi_p$ which leaves invariant the first term in (12). The invariance of the second term under the magnetic gauge transformation is only obtained if the charges of the branes are quantized according to Dirac’s condition [12],

$$e g = 2\pi n ; \quad n \in \mathbb{Z}. \quad (15)$$

Duality maps the $A_p$ potential into its dual $\tilde{A}_q$ and reverts the couplings, as shown in the dual action

$$\tilde{S} = \int \left\{ \left[ \tilde{F}_{q+1}(\tilde{A}_q) - e \Lambda_{q+1} \right]^2 + g \tilde{A}_p e^D \partial_1 \Omega_{p+1} \right\},$$

and, thanks to Dirac’s quantization rule (15), an interacting brane-brane term is not induced into the dual action. In a certain sense, the duality map and the charge quantization constraint displayed by these electromagnetic-like systems are restricted by the condition (8). The massless duality rule (8) contains indeed quite interesting information. It shows, for instance, that for $D = 4$ the usual vector potential is dual to another vector potential, a well known fact. Since both dual tensors are of the same rank we call this situation as self-duality. However, even in four-dimensions, there is another, much less known, possibility for duality, between a scalar field and a rank-2 Kalb-Ramond potential [13].

Now we make an important remark. Notice that for every dimension there is a highest-rank potential, $A_{D-1}$, whose field strength must be proportional to the totally anti-symmetric Levi-Civita symbol,

$$F_D = f e_D \quad (17)$$

These fields carry no propagating degrees of freedom but they seem to be of importance for the gauge representation of the cosmological constant with important consequences. For instance, in [14] the rank-3 tensor of a four-dimensional space-time was suggested as a possible candidate for dark matter or dark energy. This maximum rank potential couples transversally (electrically) to the maximum-dimension brane. However, there is no longitudinal coupling for this maximum rank potential because for that it would be necessary a brane of dimension $(-2)$, see Eq.(11).

What seems peculiar here is that according to the massless duality rule (8), the tensor dual to the maximum-rank potential $A_{D-1}$ should have negative rank. In fact it should be a $(-1)$-rank potential, say $A_{-1}$. If such an object is
expected to exist, as demanded by duality, it would couple longitudinally (magnetically) with the maximum-dimension brane but the transverse coupling (electric) would demand a \((-2)\)-brane. The existence of a negative rank tensor (NRT) would therefore complete the duality scheme in electromagnetic-like theories. This situation is illustrated in the diagram below for the special case of \(D = 4\), where the maximum-dimension brane is the membrane and the maximum rank potential is the 3-tensor \(A_3\),

\[
\begin{array}{cccc}
\text{Membrane} & \text{String} & \text{Charge} & \text{Instanton} \\
A_{-1} & A_0 & A_1 & A_2 \\
\text{MC} & \text{MC} & \text{MC} & \text{MC} \\
\text{EC} & \text{EC} & \text{EC} & \text{EC} \\
\text{Instanton} & \text{Charge} & \text{String} & \text{Membrane}
\end{array}
\]

(18)

The important point to be observed at this juncture is that according to standard classification of differential geometry the scalar field, say \(A_0\), is the one expected to be the minimum-rank tensor so that there is no room for the negative rank tensor [15]. So, at this point we have to face the dilemma of either break the duality scheme (8), keeping it asymmetrical, or take a rather pragmatic point of view and extend the well established differential geometry classification by introducing a new and unusual object. We decided to take chance with the latter case.

In fact, the differential in favor of the NRT seems to be the absence of a transverse coupling for it. The existence of such a coupling would demand the presence of the tensor itself while the longitudinal coupling only demands the presence of its (exterior) derivative and not the tensor itself. This fact suggests that adopting as the definition for the NRT just that its exterior derivative leads to a zero-rank tensor,

\[
A_0 = d A_{-1}
\]

(19)

suffices to obtain its proper action both free or longitudinally coupled to the maximum-dimension brane. The NRT per se is never needed. Even when coupled to the maximum-brane its action would only contain the scalar field without any derivative. Therefore it will not propagate any degree of freedom, as expected from duality arguments. Incidentally, it is interesting to observe that such definitions are not new in Physics. Let us recall the example of the Dirac “delta function” which is only properly defined “inside an integral operator”. Similarly, the NRT as well is only meaningfully defined “upon application of the exterior derivative operator.”

Next, let us discuss a more physically motivated aspect of the subject. To this end we consider the nucleation of topological defects in the phenomenological framework developed by Quevedo and Trugenberger (QT) that considers condensation of topological defects induced by quantum fluctuations [16]. Upon nucleation, the condensate of the \(q\)-charged \((q - 1)\)-dimensional branes will absorb the degrees of freedom of its longitudinally coupled tensor \(A_p\), see Eq.(12). The resulting condensate becomes massive while its Chern-Kernel is elevated to the status of propagating field describing the long-wavelength hydrodynamical fluctuations of the condensate. This phenomenon has been christened as Julia-Toulouse mechanism (JTM) [17] in (QT). Its main characteristic, that distinguishes it from its dual, the Higgs mechanism, is the occurrence of rank-jump - the massive condensate has the rank given by the Chern-Kernel which is one degree higher than the original electromagnetic potential while in the Higgs mechanism the massive condensate inherits the degree of the tensor.

The JTM has drastic and important consequences for the electric objects living inside the condensate in the form of the confinement phenomenon known as dual superconductivity. To see this consider the \(q\)-charged brane which, before condensation, was coupled transversally to a rank-\(q\) tensor, \(B_q\) and longitudinally to a tensor \(A_p\), whose action reads,

\[
S = \int \left\{ [F_{p+1} (A_p) - g \Lambda_{p+1}]^2 + g B_q \epsilon^D \partial_1 \Lambda_{p+1} + F_{q+1}^2 (B_q) \right\}.
\]

(20)

After condensation, the action involving the condensate and the \(B_q\) tensor should read [18]

\[
S_{\text{cond}} = \int \left[ F_{p+2}^2 (\Lambda_{p+1}) + m^2 \Lambda_{p+1}^2 + g B_q \epsilon^D \partial_1 \Lambda_{p+1} + F_{q+1}^2 (B_q) \right]
\]

(21)

displaying two phenomenological parameters \(m\) and \(g\) with mass dimension and a \(B \wedge F(\Lambda)\) coupling, with important consequences. To see that after condensation electrically charged objects living inside the condensate get irremediably confined we compute the effective theory for the \(B_q\) tensor by integrating out the condensate degrees of freedom \(\Lambda_{p+1}\),
to obtain [18],
\[
\mathcal{L} = \frac{1}{2} F_{q+1}(B_q) \left[ 1 + \frac{g^2}{\Box + m^2} \right] F_{q+1}(B_q).
\] (22)

This allows us to illustrate the discussion by computing the interaction energy between static \( q \)-charged point-like sources a distance \( L \) apart, for the theory under consideration. As is well known, the interaction between two static charges is generally investigated in terms of Wilson loops. We can, however, more directly recover the interaction potential by using the gauge-invariant variables formalism along the lines of Ref. [18]. In such a case, the corresponding static potential is given by
\[
V(L) = \cdots + \sigma(m)L,
\] (23)
where the “string” tension \( \sigma(m,q) \) is expected to depend both on the mass \( m \) of the condensate and on the charge of the probes. The ellipses represent the screening part that is very much dimensional dependent and must be computed case by case. An specific example will be given below. This confinement property is independent of the degree of the condensate that, in its turn, only depends of the dimension of the condensing brane. This simple explanation for the confinement phenomenon as the result of quantum fluctuations impart important physical significance to the QT phenomenology of the JTM. In [18] the static potential above has been proposed as a way to measure the phenomenological parameter \( m \), the mass of the condensate, in the theory put forward in [16].

One can expect therefore to obtain condensates of distinct degrees for each spacetime dimension. The degree of the highest condensate is \( (D-1) \) coming from the condensation of magnetic instantons. Clearly a \( D \)-degree condensate cannot be induced since that requires a brane with dimension \( (-2) \). An interesting instance in \( D=4 \) is given by the condensation of a magnetic string, described by a degree-one Chern-Kernel that couples longitudinally with a scalar field and transversally with a Kalb-Ramond tensor. After the nucleation, the Chern-Kernel of the magnetic string becomes a vector field that, according to the JTM, absorbs the single degree of freedom of the scalar to become a vectorial massive condensate.

What is however quite peculiar in this case is the fact that, according to the QT scheme, the condensate of lowest degree has vectorial character. There is no room for a scalar condensate unless the NRT is included, as demanded by the duality scheme. Indeed the existence of a scalar condensate would necessarily come from the condensation of the biggest brane, a \( (D-2) \)-dimensional object, whose zero-degree Chern-Kernel would then lead to the proper condensate. The presence of a NRT would allow for the application of the JTM, as proposed by QT, to the biggest brane. The existence of such a scalar condensate is demanded by consistency of the duality mapping. By this we mean that the condensation of this very same brane when examined from the dual point of view, the corresponding Higgs mechanism, is indeed possible.

A well known instance of a scalar condensate interacting with electric charges is given by investigations in \( D=2 \), the long-time used laboratory for quantum field theory, known as massive Schwinger model [19]. Let us examine how the phenomenology of this model is reobtained from the QT-JTM point of view. An electromagnetic-like field theory can be defined on the line with three tensors \( (A_{-1}, A_0, A_1) \) and has only enough “room” for instantons and charges, higher dimensional branes not being allowed. By considering the above general scheme for condensation of magnetic monopoles in \( D=2 \), we obtain the following JT action,
\[
\mathcal{S} = \int \left[ (F_0(A_{-1}) - g \Lambda_0)^2 + g B_1 e^D \partial_1 \Lambda_0 + F_2^2(B_1) \right] \text{JT} \int \left[ F_2^2(\Lambda_0) + m^2 \Lambda_0^2 + g B_1 e^D \partial_1 \Lambda_0 + F_2^2(B_1) \right]
\] (24)
which we recognize as the action for the massive Schwinger model [26]. In the QT/JTM scheme, the (bosonized) scalar field which is induced by quantum fluctuations of the magnetic monopoles, describes the degrees of freedom of the condensate. The effective theory for electric charges exchanging (vectorial) photons inside such a condensate is given by Eqs. (20-22) with \( p = -q = -1 \). The static potential is obtained studying,
\[
\mathcal{L} = \frac{1}{2} F_2(B_1) \left[ 1 + \frac{g^2}{\Box + m^2} \right] F_2(B_1) + B_1 J^1
\] (25)
where \( J_1 \) describes two static heavy point charges. As in the previous example, the interaction energy between pointlike sources clearly displays the confinement phenomenon [20]:
\[
V(L) = \frac{g^2}{2A} \left( 1 + \frac{m^2}{\Lambda^2} \right) (1 - e^{-\lambda L}) + \frac{g^2}{2} \left( 1 - \frac{g^2}{\Lambda^2} \right) L,
\] (26)
with \( \lambda^2 = g^2 + m^2 \) which displays both the screening and the confining part of this interaction. Of course, if we let the system returns to the dilute phase, i.e., \( m \to 0 \), the static potential above shows that confinement disappears, as expected.
It is interesting, at this juncture, to make connection between the monopole condensation described above involving the NRT and the the fermion condensate described by bosonization of the massive Thirring model (MTM) [21]. In fact one can understand a mass term in the MTM model in terms of perturbed Conformal Field Theory. Recall that a cosine potential in the sine-Gordon model is a perturbation from a massless free scalar theory. A mass term in the action of the MTM model corresponds to a cosine potential through the abelian bosonization, and we can identify a mass term in the MTM model as a perturbation from a massless Thirring model. Correspondingly, the monopole effect realized by a cosine potential in the sine-Gordon model is described by a mass term in the MTM model. Such a correspondence leads us to expect the relation between the monopole condensate in compact QED and the fermion condensate in the MTM model [22].

In summary, we have shown that consistency of the duality picture in the generalized electromagnetism of totally anti-symmetric tensors in arbitrary dimensions requires the existence of a new and unusual potential, the negative-rank tensor. In this work we have shown that this object is allowed in the duality scheme because only its (exterior) derivative is required on the action and field equations, not the tensor itself. This feature allows us to incorporate this object into the well defined scheme of differential geometry without destroying the well established classification scheme: it suffices that the exterior derivative of the NRT maps it into a zero-form. A physical motivation for the existence of this object is given in the context of the Julia-Toulouse mechanism that describes the condensation of topological defects. It was observed that without the presence of the NRT it is unable to generate condensates of scalar nature, therefore breaking duality with the Higgs mechanism. Theories that present confinement due to existence of a scalar condensate, such as the $D = 2$ sine-Gordon model, would not otherwise fit into the JT scheme without the presence of such a tensor.

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[23] We shall follow the simplified notation introduced in [4] which basically do not distinguish between a form and its components.
[24] In differential form language the field tensor is given by the exterior derivative of the potential as $F_{p+1} = d A_p$.
[25] According to the simplified notation introduced in [4], we have $J_{\mu_1 \cdots \mu_p}^{\mu_0} = \epsilon^{\mu_1 \cdots \mu_p \alpha_1 \cdots \alpha_{q+1}} \partial_{\alpha_1} \Lambda_{\alpha_1 \cdots \alpha_{q+1}}$.
[26] There is however a subtlety here. The QT approach is just a phenomenological way to incorporate quantum fluctuations of topological defects into the theory. Because of the strong demands they made on gauge invariance, Lorentz symmetry and linearity only part of this contribution is obtained, although with enough physical content to produce confinement. For the case at hand the complete contribution would give the sine-Gordon potential whose first non-trivial contribution gives the mass term.