Decoding Algebraic Geometry codes by a key equation

J. I. Farrán*

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Abstract

A new effective decoding algorithm is presented for arbitrary algebraic-geometric codes on the basis of solving a generalized key equation with the majority coset scheme of Duursma. It is an improvement of Ehrhard’s algorithm, since the method corrects up to the half of the Goppa distance with complexity order $O(n^{2.81})$, and with no further assumption on the degree of the divisor $G$.

Key words – AG codes, Ehrhard’s key equation, majority coset decoding.

1 Introduction

Decoding algebraic-geometric codes (AG codes in short) in an effective way can be done by means of solving a key equation, generalizing the ideas of the Berlekamp-Massey algorithm for BCH codes or the Euclidean algorithm for classical Goppa codes (see [1]). In the original version of Porter, Shen and Pellikaan (see [12]), only one-point codes with further assumptions on the curve were decoded, but the main ideas of the method can be extended for arbitrary curves and AG codes with Ehrhard’s version of the key equation. Nevertheless, this algorithm does not correct up to the Goppa distance, but the complexity is only $O(n^3)$ (more details in [4]). Our aim is to include in this method a majority scheme which generalizes the ideas of Feng and Rao for one point codes (see [6]), together with giving an improvement of the complexity by using the new methods given in [14] to solve linear equations. Thus, the algorithm that we propose improves both the decoding capacity and the complexity without losing the generality of its application to arbitrary AG codes. It uses the majority coset decoding scheme, which was introduced by Duursma, with the only further assumption that there is an extra rational point in the curve which is not used

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in the construction of the codes (more details in [2]). This hypothesis is actually a weakening of the assumptions required by Porter’s method.

In section 2 we rewrite Ehrhard’s key equation in a way that is closer to the original ideas of Porter, Shen and Pellikaan, in order to show the explicit connection between both works. Afterwards, we summarize in section 3 the main ideas of Duursma’s majority coset scheme, in order to give in section 4 an algorithm which includes the above majority scheme in the key equation, so that one can increase the error capacity without the assumption deg $G \geq 6g - 2\tau - 2$, where $\tau$ is the gonality of the curve, which is required in Ehrhard’s algorithm given in [5] (see also [3] for further details). In the paper, we fix a non-singular absolutely irreducible projective algebraic curve $\chi$ defined over $\mathbb{F}_q$ and rational points $P_1, \ldots, P_n$ of $\chi$.

2 Key equation and decoding

Let $G$ be a rational divisor whose support is disjoint to $D = P_1 + \ldots + P_n$. Assume that $2g - 2 < \text{deg } G < n + g$, and consider the code $C = C_{\Omega}(D, G)$, that is the image of the linear injective map

$$\text{res}_D : \Omega(G - D) \to \mathbb{F}_q^n$$

$$\eta \mapsto (\text{res}_{P_1}(\eta), \ldots, \text{res}_{P_n}(\eta))$$

with dimension $k \geq n - \text{deg } G + g - 1$ and minimum distance $d \geq d^* = \text{deg } G + 2 - 2g$, where $g$ is the genus of the curve. In the sequel, we fix a divisor $G^*$ with $\ell(G^*) = 0$ and $G \geq G^*$. In order to decode $C$, we will give a result for preparation.

Lemma 1 There exists a vector space $V$ of differential forms such that $\Omega(G - D) \subseteq V$ and $\text{res}_D : V \to \mathbb{F}_q^n$ is an isomorphism.

Proof:

Since $\Omega(G - D) \subseteq \Omega(G^* - D)$, it suffices to prove that $\text{res}_D$ is surjective on $\Omega(G^* - D)$, because it is injective on $\Omega(G - D)$. But the kernel of $\text{res}_D$ considered on $\Omega(G^* - D)$ is $\Omega(G^*)$; hence the rank is $i(G^* - D) - i(G^*) = \text{deg } G^* - \text{deg } (G^* - D) = n$, because of the Riemann-Roch formula.

□

Remark 1 In the sequel we fix an arbitrary differential form $\eta \neq 0$ and write $K = (\eta)$. Then for any rational divisor $H$ consider the isomorphism

$$\mathcal{L}(K - H) \to \Omega(H)$$
given by
\[ f \mapsto f \eta \]

This map is compatible with inclusions and restrictions, and so the inclusions \( \Omega(G - D) \subseteq V \subseteq \Omega(G^* - D) \) give the corresponding \( \mathcal{L}(K + D - G) \subseteq U \subseteq \mathcal{L}(K + D - G^*) \), where the map \( f \mapsto \text{res}_D(f \eta) \) is an isomorphism from \( U \) onto \( \mathbb{F}_q^n \). Denote the inverse of this last map by \( y \mapsto h_y \), i.e. \( h_y \) is the unique element in \( U \) such that \( \text{res}_D(h_y \eta) = y \).

Because of the bijection \( C \cong \mathcal{L}(K + D - G) \) given by \( y \leftrightarrow h_y \), the decoding problem can be obviously described as follows:

\[ (*) \quad \text{Given } y \in \mathbb{F}_q^n, \text{ find a function } h_c \in \mathcal{L}(K + D - G) \text{ such that } h_c \eta \text{ has a minimal number of poles in sup } (D), \text{ where } h_c = h_y - h_e. \]

This problem will be solved by the following definition and results.

**Definition 1** Given an arbitrary divisor \( F \), a solution of the key equation for the received word \( y \) (related to \( F \)) is a triple \((f, q, r) \in (\mathcal{L}(F) \setminus \{0\}) \times \mathcal{L}(K + F + D - G) \times \mathcal{L}(K + F - G^*)\) such that \( f h_y = q + r \).

Notice that this definition means that \( h_y = \frac{q}{f} + \frac{r}{f} \) and \( h_e = \frac{q}{f} + \frac{r}{f} \in \mathcal{L}(K + D - G) \). Thus, what we need to solve the decoding problem is giving conditions so that \( h_e = \frac{r}{f} \) has few poles in \( \text{sup } (D) \). This is done by the following theorem.

**Theorem 1 (Decoding theorem)** Let \( y = c + e \), where \( c \in C \). Then:

1. If \( \mathcal{L}(F - D_e) \neq 0 \), then there exists a solution of the key equation.
2. If \( \deg F + \text{wt}(e) < d^* \), then any solution \((f, q, r)\) of the key equation satisfies

\[ \text{res}_D \left( \frac{q \eta}{f} \right) = c \quad \text{and} \quad \text{res}_D \left( \frac{r \eta}{f} \right) = e \]

**Proof:**

1. Take a non-zero function \( f \in \mathcal{L}(F - D_e) \subseteq \mathcal{L}(F) \). Then \((f h_e) \geq G - D - F - K, (f h_e) \geq -F + D_e + G^* - D_e - K = G^* - F - K \) and \( f h_y = f h_e + f h_e \); hence the triple \((f, f h_e, f h_e)\) is a solution of the key equation.

2. Denote by \( D_e \) the divisor of poles of \( h_e \eta \) in the support of \( D \). Let \((f, q, r)\) be a solution of the key equation and set \( \varphi \equiv r - f h_e = f h_e - q \). One can estimate the following divisors:

\[ K + (r - f h_e) \geq \min\{G^* - F, G^* - F - D_e\} = G^* - F - D_e \]
and

\[ K + (f h_e - q) \geq G - F - D \]

what means that \( \varphi \in L(K + F + D_e - G^*) \cap L(K + F + D - G) = L(K + F + D_e - G) = 0 \), since by assumption \( \text{deg}(K + F + D_e - G) = 2g - 2 + \text{deg}(F) + \text{wt}(e) - \text{deg}(G) < 0 \). Hence \( \varphi = r - f h_e = f h_e - q = 0 \), what yields the theorem.

\[ \square \]

Assume from now on that \( L(F - D_e) \neq 0 \) and \( \text{deg} F + \text{wt}(e) < d^* \) (notice that both assumptions are satisfied if \( \text{wt}(e) \leq \nu \) and \( \text{deg}(F) = \nu + g \), where \( \nu = \left\lfloor \frac{d^* - g - 1}{2} \right\rfloor \); that is, when there are few errors and \( F \) is small). Thus, for a fixed \( y \in \mathbb{F}_q^* \) define the linear map

\[ \epsilon_y : L(F) \rightarrow L(K + F + D - G^*) \]

\[ f \mapsto f h_y \]

Since \( \text{deg}(G - F) > \text{deg} G - d^* = 2g - 2 \), one has \( L(K + F + D - G) \cap L(K + F - G^*) = L(K + F - G) = 0 \), and hence there exists a vector space \( W \) such that

\[ L(K + F + D - G^*) = L(K + F + D - G) \oplus L(K + F - G^*) \oplus W \]

Denoting by \( \pi_W \) and \( \pi^* \) the natural projections onto \( W \) and \( L(K + F - G^*) \) respectively, notice that the key equation means that \( \epsilon_y(f) \) has a null projection onto \( W \). Therefore, if there exists a codeword \( c \) satisfying \( \text{wt}(y - c) \leq t \), where \( 0 < t \leq \left\lfloor \frac{d^* - g - 1}{2} \right\rfloor \) is fixed, one can compute the error vector with the following algorithm, where a suitable basis for every above function space is assumed to be previously calculated. Such bases can be computed by means of Brill-Noether algorithm (see [3]).

**Algorithm 1** \((K_G(F))\)

1. Compute a matrix for the linear map \( \epsilon_y \).
2. Find a non-zero function \( f \in \ker(\pi_W \circ \epsilon_y) \).
3. Compute \( r = \pi^*(\epsilon_y(f)) \).
4. Compute \( e = \text{res}_D \left( \text{res}_D^{m_h} \right) \), checking that \( y - e \in C \) and \( \text{wt}(e) \leq \left\lfloor \frac{d^* - 1}{2} \right\rfloor \).
Notice that most of the calculations in this algorithm are concentrated in the first two steps, and thus its complexity is that of solving linear equations (see \([4]\)). Also notice that the algorithm may fail in the second or forth steps if the number of committed errors is greater than the bound \(\left\lfloor \frac{d^* - g - 1}{2} \right\rfloor\), and hence it cannot correct in general up to the half of the Goppa distance. In order to do it, we can use a majority voting scheme, what will be explained in the next sections.

Remark 2 We show now how the above results generalize those of Porter, Shen and Pellikaan, and why they are stronger. Following the notations from \([12]\), the original algorithm works with the codes \(C = C_Ω(D, E - \mu P)\), where \(E\) is the divisor of zeros of a function \(h \in K_∞(P)\) without zeros in \(\text{sup}(D)\), \(K_∞(P)\) being the ring of those functions having poles only at \(P\), \(P\) being a rational point distinct from \(\text{sup}(D)\), and where \(\mu\) is a positive integer. In this case, we can obviously take \(G^* = -\mu P\). For the sake of simplicity, assume that there exists a differential form \(\eta\) such that \((\eta) = (2g - 2)P\).

Firstly, from the isomorphism given by lemma 1 we obtain a basis \(ε_1, \ldots, ε_n\) of \(V\) such that \(\text{res}_D(ε_1), \ldots, \text{res}_D(ε_n)\) is the canonical basis of \(\mathbb{F}_q^n\). Then, Porter defines a “syndrome function” by

\[ S_y \cdot η = \sum_{j=1}^{n} y_j \left( 1 - \frac{h}{h(P_j)} \right) ε_j \]

Notice that \(S_y \in K_∞(P)\), \(S_y \equiv h_y \pmod{h}\) and \(-v_P(S_y) \leq m + 2g - 1\), where \(m = -v_P(h)\). On the other hand, Porter’s result to decode \(C\) can be rewritten as follows (see \([3]\) for further details):

If there is an integer \(t\) such that \(t + \text{wt}(e) < d^*\) and functions \(f, q, r \in K_∞(P)\) satisfying \(-v_P(r) \leq t + 2g - 2 + \mu, -v_P(f) \leq t\) and the “polynomial key equation”

\[ f S_y = gh + r \]

then \(h_e = \frac{r}{f}\).

Such triples \((f, g, r)\) are called “valid solutions” in \([12]\). Thus, by taking \(K = (\eta) = (2g - 2)P\) and \(F = tP\), one has \(f \in L(F)\) and \(r \in L(K + F - G^*)\), and hence this is a particular case of our method \([1]\). Moreover, one obtains \(e = \text{res}_D \frac{r}{f}\) where \(f\) has few zeros for \(F\) “small” (because of \((f) + F \geq 0\) and thus, for a suitable choice of \(t\) and \(\text{wt}(e)\), \(\frac{r}{f}\) has a minimal number of poles

\[^1\text{In particular, the condition of being minimal for a valid solution can be dismissed from the results of Porter.}\]
in \( \sup(D) \), according to the formulation (*) of the decoding problem. This is actually the underlying idea of Porter, which was carried out by a "row reduction process" at a certain resultant matrix, but of course it can also be done by simple techniques of linear algebra, as we have explained above.

Thus, the results of our paper are stronger than the originals, since they work with an arbitrary divisor \( G \) and we do not require any special differential form \( \eta \) or rational function \( h \), what is actually a very strong restriction. Moreover, one obtains a quite similar formula to compute the error just from \( h_y \), without the need of the syndrome \( S_y \).

3 Majority coset decoding

This section is abstracted from \([2]\). Assume that there exists a rational point \( P_\infty \notin \sup(D) \), and let \( H_1 \) be a rational divisor whose support is disjoint to \( \sup(D) \). Set \( H_0 = H_1 - P_\infty \) and \( H_2 = H_1 + P_\infty \). For \( i = 0, 1, 2 \), let \( C_i = C_0(D, H_i) \) and \( d_i^* = \deg(H_i) + 2 - 2g \). One obviously has \( C_0 \supseteq C_1 \supseteq C_2 \).

For an error vector \( e \) such that \( \wt(e) \leq (d_1^* - 1)/2 \) we want to solve the following problem:

Given \( y_1 \) with \( y_1 - e \in C_1 \), finding \( y_2 \) such that \( y_2 - e \in C_2 \).

Such a problem is called \( \text{coset decoding procedure related to the extension } C_1 \supseteq C_2 \), where we obviously can assume that \( C_1 \neq C_2 \).

Thus, for a given \( y \in \mathbb{F}_q^n \) and for any rational function \( h \) without poles in \( \sup(D) \), one defines the syndrome \( S_y(h) \) by the expression

\[
S_y(h) = \sum_{j=1}^n y_j h(P_j) \in \mathbb{F}_q
\]

which is linear with respect to both \( y \) and \( h \).

It is very easy to prove that the syndrome is a \( \text{coset invariant} \), i.e. \( S_y(h) = S_y(h) \) for all \( h \in L(H_i) \) if and only if \( y - e \in C_i \), for \( i = 0, 1, 2 \). Hence, \( y \in C_i \) if and only if \( S_y(h) = 0 \) for all \( h \in L(H_i) \).

On the other hand, for an arbitrary divisor \( F \) defined over \( \mathbb{F}_q \) and \( i = 0, 1, 2 \), one defines the kernels \( K_i(F) \) associated to the error vector \( e \) by

\[
K_i(F) = \{ f \in L(F) \mid S_e(f \cdot g) = 0, \forall g \in L(H_i - F) \}
\]

All the vector spaces \( K_1(F + P_\infty)/K_0(F), K_0(F)/K_1(F), L(H_1 - F)/L(H_1 - F - P_\infty), K_1(F + P_\infty)/K_2(F + P_\infty) \) and \( K_2(F + P_\infty)/K_1(F) \) have dimension at most one. Thus, we are interested in the following conditions:

\[
\begin{align*}
(A1) \ K_1(F + P_\infty) & \neq K_0(F) \\
(A2) \ K_0(F) & = K_1(F) \\
(A3) \ L(H_1 - F) & \neq L(H_1 - F - P_\infty) \\
(B1) \ K_1(F + P_\infty) & = K_2(F + P_\infty) \\
(B2) \ K_2(F + P_\infty) & \neq K_1(F)
\end{align*}
\]
Define the conditions \((A) \iff (A1) \land (A2) \land (A3)\) and \((B) \iff (B1) \land (B2)\). Since one has \((A1) \land (B1) \iff (A2) \land (B2)\), the conditions \((A)\) and \((B)\) are equivalent to \((A1), (A3)\) and \((B1)\).

It follows from [2] (sections II and III) that if \((A)\) and \((B)\) are satisfied, then the coset decoding procedure can be implemented by the following algorithm, where \(D\) and \(P_\infty\) are fixed.

**Algorithm 2 \((C_{H_1}(F))\)**

**Input**: \(y_1\).

If \(C_1 = C_2\) then \(y_2 = y_1\) else:

- Find \(c_0 \in C_1 \setminus C_2\).
- Find \(f \in K_1(F + P_\infty) \setminus K_0(F)\).
- Find \(g \in \mathcal{L}(H_1 - F) \setminus \mathcal{L}(H_1 - F - P_\infty)\).
- Compute \(\lambda = S_{y_1}(fg)/S_{c_0}(fg)\).
- Set \(y_2 = y_1 - \lambda c_0\).

**Output**: \(y_2\).

Unfortunately we are not able in practice to check the condition \((B)\), since \(K_2(F + P_\infty)\) is not known from the received word \(y\). This problem can be solved by means of a majority voting, on the basis of the following result due to Duursma (see [3] for further details).

**Theorem 2 (Main theorem)** Let \(C_0 \supseteq C_1 \supseteq C_2\) be the extension of codes given by \(C_i \equiv C_{\Omega}(D, H_i)\), where \(H_1\) has disjoint support with \(D\), \(H_0 \equiv H_1 - P_\infty\) and \(H_2 \equiv H_1 + P_\infty\). Assume that the genus is \(g \geq 1\), and take numbers \(t, r \geq 0\) such that \(2t + r + 1 \leq d^*_1 = \deg H_1 + 2 - 2g\). Take an arbitrary divisor \(F_0\) of degree \(t\), and define \(F_i \equiv F_0 + iP_\infty\) for \(i = 1, \ldots, 2g - 1\). For an error vector \(e\) with weight \(\text{wt}(e) \leq t\), define:

\[
I \doteq \{r, r+1, \ldots, 2g-2\}
\]

\[
T \doteq \{i \in I \mid (A) \land (B) \text{ hold for } F = F_i\}
\]

\[
F \doteq \{i \in I \mid (A) \land \neg(B) \text{ hold for } F = F_i\}
\]

Then at least one of the following conditions holds:

\[
\begin{align*}
(i) & \quad \mathcal{L}(H_1 - F_{2g-1} - D_e - rP_\infty) \neq 0 \\
(ii) & \quad \mathcal{L}(F_r - D_e) \neq 0 \\
(iii) & \quad \sharp T > \sharp F
\end{align*}
\]

In the last section we will see how to apply this majority scheme in order to improve the correction capacity of the decoding algorithm by solving the Ehrhard’s key equation up to the half of the Goppa distance. The so obtained procedure is thus the best possible one by solving a key equation, looking at the generality and the capacity of the algorithm.
4 Decoding by a key equation with majority voting

Let $C = C_Ω(D, G)$ be a strongly algebraic-geometric code, i.e. such that $2g - 2 < \deg(G) < n$. For our purpose, we can assume that $g > 0$, since otherwise the key equation corrects $C$ up to the half of the Goppa distance and we do not need any majority voting.

Consider successive divisors $G_r = G + rP_∞$, for $r = 0, 1, \ldots, g$. Notice that for any such divisor $G_r$ one has $2g - 2 < \deg(G_r) < n + g$, and thus all these divisors are in the situation of the first paragraph in section 2. On the other hand, take $t = \lfloor d^* - 1/2 \rfloor$, where $d^* \equiv \deg(G) + 2 - 2g$, and assume $t > 0$. Take then a divisor $F_0$ with degree $t$ and set $F_i = F_0 + iP_∞$ for $i = 1, \ldots, 2g - 1$.

Thus we can consider the following algorithm, which brings together the methods of Ehrhard and Duursma. In the algorithm, the main idea is that the conditions (i) and (ii) given by theorem 2 allows us to get the error vector by means of a key equation for some suitable $G$ and $F$, and otherwise the condition (iii) provides us with a majority test to solve the coset decoding problem and decrease the size of the code. We assume that bases for the involved function and differential spaces are previously calculated together with the spaces $U, V, W$ as in section 2, for all of the possible cases when algorithm 1 is applied.

Algorithm 3 ($D_G(F_0)$)

$Input := y \in \mathbb{F}_q^n$.

Set $y_1 = y$.

From $r = 0$ to $r = g$ do:

- Set $H_1 = G + rP_∞$.
- If $K_G(G - F_{2g-1})$ gets the error vector from $y_1$, then return $e$ and STOP.
- Otherwise, if $K_{H_1}(F_r)$ gets the error vector from $y_1$, then return $e$ and STOP.
- Otherwise, compute $I_A \doteq \{ i = r, r+1, \ldots, 2g-2 \mid (A) \text{ holds for } F = F_i \}$, apply the coset decoding procedure $C_{H_i}(F_i)$ for $i \in I_A$ with input $y_1$ and get a vector $y_2$ whose coset with respect to $C_2 \doteq C_Ω(D, H_1 + P_∞)$ occurs most of the times.
  
  Set $y_1 = y_2$ and NEXT $r$.

Notice that algorithm 1 is always applied to one of the divisors $G_r$. Thus, if we take a divisor $G^*$ such that $\ell(G^*) = 0$ and $G^* \leq G \leq G_r$, we can use the same divisor $G^*$ for all the involved key equations.
Finally, since every functional code can be expressed as a differential code and vice versa, we can prove the following new result, which incorporates the Duursma’s version of the majority voting scheme into the Ehrard’s version of the key equation.

**Theorem 3** Let $\chi$ be a non-singular absolutely irreducible projective algebraic curve defined over the finite field $\mathbb{F}_q$ with at least $n + 1$ rational points. Let $C = C_\Omega(D, G)$ be an algebraic-geometric code with length $n$ such that $2g - 2 < \deg(G) < n$. Let $F_0$ be any divisor with degree $t \geq \left\lceil \frac{d^* - 1}{2} \right\rceil$, where $d^* \equiv \deg(G) + 2 - 2g$ is the Goppa distance of $C$. Then the algorithm $D_G(F_0)$ decodes $C$ up to $t$ errors with complexity $O(n^{2.81})$.

Proof:

First of all, the condition $2t + r + 1 \leq d_1^* = \deg(H_1) + 2 - 2g$ is satisfied by every divisor $H_1 = G_r$ from $r = 0$ to $r = g$, and for $t \geq \left\lceil \frac{(d^* - 1)/2} \right\rceil$; thus we can apply theorem 2 in every step of the algorithm, provided $\text{wt}(\mathbf{e}) \leq t$.

For a fixed $H_1 = G_r$, if the condition (i) $\mathcal{L}(H_1 - F_{2g-1} - D_e - rP_\infty) = \mathcal{L}(G - F_{2g-1} - D_e) \neq 0$ holds together with $\text{wt}(\mathbf{e}) \leq t$, then the key equation $\mathcal{K}(F)$ obtains the error vector for $F = G - F_{2g-1}$, since $\deg F + \text{wt}(\mathbf{e}) < d^*$ and $\mathcal{L}(F - D_e) \neq 0$, and theorem 1 can be applied.

In the same way, if the condition (ii) $\mathcal{L}(F_r - D_e) \neq 0$ holds together with $\text{wt}(\mathbf{e}) \leq t$, then the key equation $\mathcal{K}(F_r)$ obtains the error vector, since $\deg F_r + \text{wt}(\mathbf{e}) < \deg(G) + 2 - 2g$ and $\mathcal{L}(F_r - D_e) \neq 0$, and theorem 1 can also be applied.

Otherwise, the condition (iii) implies that the algorithm $\mathcal{C}(F_r)$ is correct for most of the “candidates” $i \in I_A$, and we can carry on with the next step. Finally, for $r = g$ the condition $\mathcal{L}(F_r - D_e) \neq 0$ is always true and the algorithm stops at most in $g + 1$ steps, if not too many errors occur.

Notice that still the complexity of this algorithm is equivalent to solve a linear system of size $n$, since most of the computations come from either applications of the algorithm $\mathcal{K}(F)$ or finding a function in $K_1(F + P_\infty) \setminus K_0(F)$ (more details in $\text{(2)}$). Thus, the complexity is actually $O(n^{2.81})$ since solving linear equations can be done faster than Gaussian elimination (see for instance $\text{(4)}$).

\[ \square \]

**Remark 3** Notice that the complexity $O(n^{2.81})$ is even better than the complexity of Sakata’s algorithm $O(n^{3.81})$ if the curve $\chi$ is embedded in an affine $r$-space with $r > 10$ (what happens in the constructions of asymptotically good codes given in $\text{(3)}$). Thus, general decoding methods which are based on solving linear equations are not so far from “fast decoding” as they are supposed to (see $\text{(7)}$ for a survey on decoding).

\[ ^2 \text{Nowadays there are even some improvements of this complexity.} \]
Example 1 Consider the Klein quartic $X^3Y + Y^3Z + Z^3X = 0$ over $\mathbb{F}_8$. This curve has genus $g = 3$ and 24 rational points, namely, $Q_0 = (1 : 0 : 0)$, $Q_1 = (0 : 1 : 0)$ and $Q_2 = (0 : 0 : 1)$ on the coordinate lines, and all the others are in the affine plane, namely, $P_1, \ldots, P_{21}$ (see [10] for details). Set $H_1 = G = 4(Q_0 + Q_1 + Q_2), D = P_1 + \ldots + P_{21}$ and define the code $C_1 = C = C_{11}(D, G)$, with parameters $[21, 11, \geq 8]$. Consider the vector $y_1 = (1, 0, 1, \alpha, 0, \ldots, 0)$ as a received word, where $\alpha \in \mathbb{F}_8$ satisfies $\alpha^3 + \alpha + 1 = 0$, and take the divisor $F_0 = 3P_\infty$, where $P_\infty = Q_2$. Notice that the correction capacity of our algorithm is $t = 3$, whereas the key equation only corrects two errors.

Thus, in the step $r = 0$ one easily checks that the conditions (i) and (iii) from theorem 2 are not satisfied, and hence the key equation cannot correct this error. Then, one computes the set $I_A = \{3\}$ and applies $C_G(F)$ to the only candidate $F = F_3$:

- Take $c = (\alpha, \alpha^5, \alpha^3, 0, \alpha^4, \alpha^6, 1, 0, 1, 0, \ldots, 0) \in C_1 \setminus C_2$.
- Take $f = \alpha^3 + \frac{Z^3}{X^3Y} \in K_1(F_3 + P_\infty) \setminus K_0(F_3)$.
- Take $g = \frac{X}{Y} \in L(G - F_3) \setminus L(G - F_3 - P_\infty)$.
- Compute $\lambda = S_{y_1}(fg)/S_c(fg) = \alpha^3$.
- Return $y_2 = y_1 - \lambda c = (\alpha^5, \alpha, \alpha^2, \alpha, 1, \alpha^5, \alpha^2, \alpha^3, 0, \alpha^3, \alpha^3, 0, \ldots, 0)$.

In this case we have no voting since there is an only candidate, and the above solution is the new $y_1$ for the next step of the algorithm, which works in a smaller code, and go on until the key equation gets the error vector.

Example 2 Consider now the Hermite curve $Y^4Z + YZ^4 + X^5 = 0$ over $\mathbb{F}_{16}$. It has 64 affine rational points and only one point $P_\infty$ at infinity. Let $D = P_1 + \ldots + P_{64}, G_1 = 23P_\infty$ and define the code $C = C_{11}(D, G_1)$, which is of type $[64, 46, \geq 13]$. Consider then $y_1 = (\alpha^{12}, \alpha^4, \alpha^7, \alpha^8, \alpha^9, 0, \ldots, 0)$ as a received word, where $\alpha \in \mathbb{F}_{16}$ satisfies $\alpha^4 + \alpha + 1 = 0$, and take the divisor $F_0 = 6P_\infty$.

Now for $r = 0$ again (i) and (ii) do not hold, and one computes $I_A = \{1, 2, 3, 5, 7, 8, 9\}$. In this case, voting actually occurs and the procedure is equivalent to the algorithm of Feng and Rao (see [3]).

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