Instanton Effects on the Role of the Low-Energy Theorem for the Scalar Gluonic Correlation Function

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December 3, 2018

Abstract

Instanton contributions to the Laplace sum-rules for correlation functions of scalar gluonic currents are calculated. The role of the constant low-energy theorem term, whose substantial contribution is unique to the leading Laplace sum-rule \( L_{-1} \), is shown to be diminished by instanton contributions, significantly increasing the resulting mass bounds for the ground state of scalar gluonium and improving compatibility with results from higher-weight sum-rules.

1 Introduction

In the chiral limit of \( n_f \) quarks, the low-energy theorem (LET) for scalar gluonic correlation functions is [1]

\[
\Pi(0) = \lim_{Q \to 0} \Pi(Q^2) = \frac{8\pi}{\beta_0} \langle J \rangle ,
\]

where

\[
\Pi(Q^2) = i \int d^4x e^{iq \cdot x} \langle O|T[J(x)J(0)]|O \rangle , \quad Q^2 = -q^2 > 0
\]

\[
J(x) = -\frac{\pi^2}{\alpha \beta_0} \beta(\alpha) G^{a}_{\mu\nu}(x) G^{a}_{\mu\nu}(x)
\]

\[
\beta(\alpha) = \nu^2 \frac{d}{d\nu^2} \left( \frac{\alpha(\nu)}{\pi} \right) = -\beta_0 \left( \frac{\alpha}{\pi} \right)^2 - \beta_1 \left( \frac{\alpha}{\pi} \right)^3 + \ldots
\]

\[
\beta_0 = \frac{11}{4} - \frac{1}{6} n_f , \quad \beta_1 = \frac{51}{8} - \frac{19}{24} n_f
\]

The current \( J(x) \) is renormalization group (RG) invariant for massless quarks [2], and its normalization has been chosen so that to lowest order in \( \alpha \)

\[
J(x) = \alpha G^{a}_{\mu\nu}(x) G^{a}_{\mu\nu}(x) \left[ 1 + \frac{\beta_1}{\beta_0} \frac{\alpha}{\pi} + O(\alpha^2) \right] \equiv \alpha G^2(x) \left[ 1 + \frac{\beta_1}{\beta_0} \frac{\alpha}{\pi} + O(\alpha^2) \right].
\]

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Most applications of dispersion relations in sum-rules are designed to remove dependence on low-energy subtraction constants. However, knowledge of the LET for gluonic correlation functions permits the possibility of sum-rules that contain explicit dependence on the LET subtraction constant \( \Pi(0) \). For example, the dispersion relation appropriate to the asymptotic (perturbative) behaviour of the correlation function (2) is

\[
\Pi(Q^2) = \Pi(0) + Q^2 \Pi'(0) + \frac{1}{2}Q^4 \Pi''(0) - Q^2 \frac{1}{\pi} \int_{t_0}^{\infty} dt \frac{\rho(t)}{t^3(t + Q^2)} .
\]

where \( \rho(t) \) is the hadronic spectral function with physical threshold \( t_0 \) appropriate to the quantum numbers of the current used to construct the correlation function.

Unfortunately, direct application of the dispersion relation (7) is not possible because the theoretical (perturbative) calculation of \( \Pi(Q^2) \) contains a field-theoretical divergence proportional to \( Q^4 \). A related problem is the significant contribution of excited states and the QCD continuum to the integral of \( \rho(t) \) in (7). Enhancement of the lowest-lying resonance contribution in applications to light hadronic systems requires greater high-energy suppression of this integral.

The established technique for dealing with these issues is the Laplace sum-rules (4). A family of Laplace sum-rules can be obtained from the dispersion relation (7) through the Borel transform operator \( B \),

\[
\hat{B} \equiv \lim_{N, Q^2 \rightarrow \infty} \frac{N}{Q^2} \equiv \tau (-1)^k Q^2 k \Pi(0) + (-1)^k \frac{1}{2} Q^4 k + 2 k \Pi'(0) + (-1)^k \frac{1}{2} Q^6 k + 4 k \Pi''(0) \]

\[
- \frac{1}{\pi} \int_{t_0}^{\infty} dt \frac{1}{\tau t^2} \hat{B} \left[ (-1)^k \frac{Q^2 k}{t + Q^2} \right] \rho(t) .
\]

There are some important constraints on \( k \) that will lead to sum-rules with predictive power. Since the perturbative prediction of \( \Pi(Q^2) \) contains divergent constants multiplied by \( Q^4 \), the sum-rules \( L_k(\tau) \) where this contribution is absent require \( k \geq -2 \). However, the low-energy constants \( \Pi'(0) \) and \( \Pi''(0) \) are not determined by the LET \( i.e. \) only the quantity \( \Pi(0) \) appears in (11). Hence the sum-rules \( L_k(\tau) \) which will be independent of \( \Pi'(0) \) and \( \Pi''(0) \) must satisfy \( k \geq -1 \), and only the \( k = -1 \) sum-rule will contain dependence on the LET-determined quantity \( \Pi(0) \):

\[
L_{-1}(\tau) = -\Pi(0) + \frac{1}{\pi} \int_{t_0}^{\infty} dt \frac{1}{t} e^{-\tau t} \rho(t) .
\]

\[
L_k(\tau) = \frac{1}{\pi} \int_{t_0}^{\infty} dt t^k e^{-\tau t} \rho(t) , \quad k > -1
\]
The “resonance(s) plus continuum” model is used to represent the hadronic physics phenomenology contained in \( \rho(t) \) in \(^{13,14}\). In this model, hadronic physics is (locally) dual to the theoretical QCD prediction for energies above the continuum threshold \( s = s_0 \):

\[
\rho(t) \equiv \theta (s_0 - t) \rho^{had}(t) + \theta (t - s_0) \text{Im} \Pi^{QCD}(t)
\]

The contribution of the QCD continuum to the sum-rules is denoted by

\[
c_k (\tau, s_0) = \frac{1}{\pi} \int_{s_0}^{\infty} dt \, t^k e^{-t \tau} \text{Im} \Pi^{QCD}(t)
\]

Since the continuum contribution is determined by QCD, it is usually combined with the theoretical quantity \( \mathcal{L}_k (\tau) \)

\[
\mathcal{S}_k (\tau, s_0) \equiv \mathcal{L}_k (\tau) - c_k (\tau, s_0)
\]

resulting in the following Laplace sum-rules relating QCD to hadronic physics phenomenology:

\[
\mathcal{S}_{-1} (\tau, s_0) = -\Pi(0) + \frac{1}{\pi} \int_{s_0}^{\infty} dt \, \frac{1}{t} e^{-t \tau} \rho^{had}(t)
\]

\[
\mathcal{S}_k (\tau, s_0) = \frac{1}{\pi} \int_{s_0}^{\infty} dt \, t^k e^{-t \tau} \rho^{had}(t) , \quad k > -1
\]

The property

\[
\lim_{s_0 \to \infty} c_k (\tau, s_0) = 0
\]

implies that the sum-rules \(^{13,14}\) are identical in the \( s_0 \to \infty \) limit.

\[
\lim_{s_0 \to \infty} \mathcal{S}_k (\tau, s_0) = \mathcal{L}_k (\tau)
\]

The only appearance of the \( \Pi(0) \) term is in the \( k = -1 \) sum-rule, and as first noted in \(^5\), this LET term comprises a significant contribution in the \( k = -1 \) sum-rule. From the significance of this scale-independent term one can ascertain the important qualitative role of the LET in sum-rule phenomenology. To see this role, we first model the hadronic contributions \( \rho^{had}(t) \) using the narrow resonance approximation

\[
\frac{1}{\pi} \rho^{had}(t) = \sum_{r} F_r^2 m_r^2 \delta (t - m_r^2)
\]

where the sum over \( r \) represents a sum over sub-continuum resonances of mass \( m_r \). The quantity \( F_r \) is the coupling strength of the resonance to the vacuum through the gluonic current \( J(0) \), so the sum-rule for scalar gluonic currents probes scalar gluonium states. In the narrow-width approximation the Laplace sum-rules \(^{18,19}\) become

\[
\mathcal{S}_{-1} (\tau, s_0) + \Pi(0) = \sum_{r} F_r^2 e^{-m_r^2 \tau}
\]

\[
\mathcal{S}_k (\tau, s_0) = \sum_{r} F_r^2 m_r^{2k+2} e^{-m_r^2 \tau} , \quad k > -1
\]

Thus if the (constant) LET term is a significant contribution on the theoretical side of \(^{23}\), then the left-hand side of \(^{23}\) will exhibit reduced \( \tau \) dependence relative to other theoretical contributions. To reproduce this diminished \( \tau \) dependence, the phenomenological (i.e. right-hand) side must contain a light resonance with a coupling larger
than or comparable to the heavier resonances. By contrast, the absence of the \( \Pi(0) \) (constant) term in \( k > -1 \) sum-rules leads to stronger \( \tau \) dependence which is balanced on the phenomenological side by suppression of the lightest resonances via the additional powers of \( m_T^2 \) occurring in \( S_{-1}^f(\tau,s_0) \). Thus if \( \Pi(0) \) is found to dominate \( S_{-1}(\tau,s_0) \), then one would expect qualitatively different results from analysis of the \( k = -1 \) and \( k > -1 \) sum-rules.

Such distinct conclusions drawn from different sum-rules can be legitimate. In the pseudoscalar quark sector, the lowest sum-rule is dominated by the pion, and the low mass of the pion is evident from the minimal \( k \) dependence of the next-to-lowest sum-rule. By contrast, the absence of the \( \Pi(0) \) (constant) term in higher-weight sum-rules and has also corroborated lattice estimates. However, the overall consistency of the

In the absence of instantons \([2]\), explicit sum-rule analyses of scalar gluonium \([3,5,8]\) have been addressed quantitatively.

2 Instanton Effects in the Laplace Sum-Rules

In Section 2, we explicitly calculate the instanton contributions to Laplace sum-rules of scalar gluonic currents. We pay particular attention to the \( k = -1 \) sum-rule and demonstrate that instanton contributions partially cancel against the LET constant \( \Pi(0) \) and serve to appreciably diminish its dominance of this leading order sum-rule. The phenomenological implications of this partial cancellation are investigated in Section 3, and a discussion relating our work to other analyses of instanton effects in the scalar gluonium channel is presented in Section 4.

2 Instanton Effects in the Laplace Sum-Rules

The field-theoretical (QCD) calculation of \( \Pi(Q^2) \) consists of perturbative (logarithmic) corrections known to three-loop order (\( \overline{\text{MS}} \) scheme) in the chiral limit of \( n_f = 3 \) massless quarks \([14]\). QCD vacuum effects of infinite correlation length parameterized by the power-law contributions from the QCD vacuum condensates \([5,10]\), and QCD vacuum effects of finite correlation length devolving from instantons \([16]\)

\[
\Pi(Q^2) = \Pi^{pert}(Q^2) + \Pi^{cond}(Q^2) + \Pi^{inst}(Q^2),
\]

with

1. ... the perturbative contribution (ignoring divergent terms proportional to \( Q^4 \)) given by

\[
\Pi^{pert}(Q^2) = Q^4 \log \left( \frac{Q^2}{\nu^2} \right) \left[ a_0 + a_1 \log \left( \frac{Q^2}{\nu^2} \right) + a_2 \log^2 \left( \frac{Q^2}{\nu^2} \right) \right]
\]

\[
a_0 = -2 \left( \frac{\alpha}{\pi} \right)^2 \left[ 1 + \frac{659}{36} \frac{\alpha}{\pi} + 247.480 \left( \frac{\alpha}{\pi} \right)^2 \right], \quad a_1 = 2 \left( \frac{\alpha}{\pi} \right)^3 \left[ \frac{9}{4} + 65.781 \frac{\alpha}{\pi} \right], \quad a_2 = -10.1250 \left( \frac{\alpha}{\pi} \right)^4
\]

\[\text{The calculation of one-loop contributions proportional to } \langle J \rangle \text{ in } [15] \text{ have been extended non-trivially to } n_f = 3 \text{ from } n_f = 0, \text{ and the operator basis has been changed from } \langle \alpha G^2 \rangle \text{ to } \langle J \rangle.\]
2. . . the condensate contributions given by

$$\Pi^{\text{cond}}(Q^2) = \left[ b_0 + b_1 \log \left( \frac{Q^2}{\nu^2} \right) \right] \langle J \rangle + c_0 \frac{1}{Q^2} \langle O_6 \rangle + d_0 \frac{1}{Q^4} \langle O_8 \rangle$$

$$b_0 = 4\pi \frac{\alpha}{\pi} \left[ 1 + \frac{175 \alpha}{36 \pi} \right] , \quad b_1 = -9\pi \left( \frac{\alpha}{\pi} \right)^2 , \quad c_0 = 8\pi^2 \left( \frac{\alpha}{\pi} \right)^2 , \quad d_0 = 8\pi^3 \frac{\alpha}{\pi}$$

$$\langle O_6 \rangle = \left\langle g f_{abc} G_{\mu \nu}^a G_{\rho \sigma}^b G^c_{\rho \mu} \right\rangle , \quad \langle O_8 \rangle = 14 \left( \langle \alpha f_{abc} G_{\mu \nu}^a G_{\rho \sigma}^b \rangle \right)^2 - \left( \langle \alpha f_{abc} G_{\mu \nu}^a G_{\rho \sigma}^b \rangle \right)^2$$

3. . . and the instanton contribution given by

$$\Pi^{\text{inst}}(Q^2) = 32\pi^2 Q^4 \int \rho^1 \left[ K_2 \left( \rho \sqrt{Q^2} \right) \right]^2 \text{dn}(\rho) \ ,$$

where $K_2(x)$ represents a modified Bessel function [17].

The strong coupling constant $\alpha$ is understood to be the running coupling at the renormalization scale $\nu$, and renormalization group improvement of the Laplace sum-rules implies that $\nu^2 = 1/\tau$ [18]. The instanton contributions represent a calculation with non-interacting instantons of size $\rho$, with subsequent integration over the instanton density distribution $n(\rho)$. The theoretical contributions to the Laplace sum-rules corresponding to [25] are

$$\mathcal{L}_k(\tau) = \mathcal{L}_k^{\text{pert}}(\tau) + \mathcal{L}_k^{\text{cond}}(\tau) + \mathcal{L}_k^{\text{inst}}(\tau) \ .$$

An alternative to the direct calculation of the Laplace sum-rules through the definition of $\hat{B}$ in [8] is obtained through an identity relating the Borel and Laplace transform [19]

$$f(Q^2) = \int_0^{\infty} d\tau F(\tau) e^{-Q^2 \tau} = \mathcal{L}[F(\tau)] \implies \frac{1}{\tau} \hat{B} \left[ f(Q^2) \right] = F(\tau) = \mathcal{L}^{-1} \left[ f(Q^2) \right]$$

$$\mathcal{L}^{-1} \left[ f(Q^2) \right] = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} f(Q^2) e^{Q^2 \tau} dQ^2$$

where the real parameter $b$ in the definition [33] of the inverse Laplace transform must be chosen so that $f(Q^2)$ is analytic to the right of the contour of integration in the complex plane. Using the result [32], the Laplace sum-rules [11] can be obtained from an inverse Laplace transform of the theoretically-determined correlation function:

$$\mathcal{L}_k(\tau) = \mathcal{L}^{-1} \left[ (-1)^k Q^{2k} \Pi(Q^2) \right] \ .$$

In the complex $Q^2$ plane where the inverse Laplace transform [33] is calculated, the QCD expression [25] for the correlation function $\Pi(Q^2)$ is analytic apart from a branch point at $Q^2 = 0$ with a branch cut extending to infinity along the negative-real-$Q^2$ axis. Consequently, analyticity to the right of the contour in [33] implies that $b > 0$. Consider the contour $C(R)$ in Figure [11] $\Pi(Q^2)$ is analytic within and on $C(R)$ and so with $z = Q^2$

$$0 = \frac{1}{2\pi i} \int_{C(R)} (-z)^k e^{z\tau} \Pi(z) dz \ ,$$

which leads to

$$\frac{1}{2\pi i} \int_{b-iR}^{b+iR} (-z)^k e^{z\tau} \Pi(z) dz = -\frac{1}{2\pi i} \int_{\Gamma_1 + \ldots + \Gamma_4} (-z)^k e^{z\tau} \Pi(z) dz - \frac{1}{2\pi i} \int_{\Gamma_5 + \Gamma_6} (-z)^k e^{z\tau} \Pi(z) dz \ .$$

2A factor of 2 to include the sum of instanton and anti-instanton contributions has been included in [26].
Taking the limit as $R \to \infty$, which requires use of the asymptotic behaviour of the modified Bessel function $K_2(z)$,

$$K_2(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad ; \quad |z| \gg 1, \quad |\arg(z)| \leq \frac{\pi}{2}$$  \hfill (37)

the individual integrals over $\Gamma_1, \ldots, \Gamma_4$ are found to vanish, resulting in the following expression for the Laplace sum-rule.

$$\mathcal{L}_k(\tau) = \frac{1}{2\pi i} \int t^k e^{-t\tau} \left[ \Pi(\tau) - \Pi(\tau) \right] dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} (-1)^k \exp(\epsilon e^{i\theta} \tau) e^{k+1} e^{i(k+1)\theta} \Pi(e^{i\theta}) d\theta$$  \hfill (38)

Perturbative and QCD condensate contributions to the Laplace sum-rules are well known [3, 5, 8], and serve as a consistency check for the conventions used to determine the instanton contribution through (38). Keeping in mind the $k \geq -1$ constraint established previously, we see that the perturbative contributions to the $\theta$ integral in (38) are zero in the limit as $\epsilon \to 0$, leaving only the anticipated integral of the discontinuity across the branch cut $i.e. \text{Im}\Pi^{\text{pert}}(t)$ to determine the following perturbative contributions to the Laplace sum-rule.

$$\mathcal{L}_k^{\text{pert}}(\tau) = \frac{1}{2\pi i} \int_{0}^{\infty} t^{k+2} e^{-t\tau} \left[ -a_0 - 2a_1 \log \left( \frac{t}{\nu^2} \right) + \alpha_2 \left( k^2 - 3 \log \frac{t}{\nu^2} \right) \right] dt$$  \hfill (39)

The QCD condensate terms proportional to $b_0, c_0$ and $d_0$ in the correlation function $\Pi(z)$ do not have a branch discontinuity, so their contribution to the Laplace sum-rule arises solely from the contour $\Gamma_\epsilon$ (represented by the term in (38) with the $\theta$ integral), and can be evaluated using the result

$$-\frac{1}{2\pi i} \int_{\Gamma_\epsilon} e^{z\tau} dz = \begin{cases} 0 & n = 0, \quad -1, \quad -2, \ldots \\ \frac{1}{(n-1)!} \gamma_{n-1} & n = 1, \quad 2, \quad 3, \ldots \end{cases}$$  \hfill (40)

The QCD condensate term proportional to $b_1$ requires a more careful treatment. If $\Pi(z)$ is replaced with $\log \left( z/\nu^2 \right)$ in (38) then we find

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} (-z)^k e^{z\tau} \log \left( \frac{z}{\nu^2} \right) dz = -\infty \int_{-\pi}^{\pi} (-1)^k \exp(\epsilon e^{i\theta} \tau) e^{k+1} e^{i(k+1)\theta} \left( \log \left( \frac{\epsilon}{\nu^2} \right) + i\theta \right) d\theta$$  \hfill (41)

The last term in this equation will be zero in the $\epsilon \to 0$ limit except when $k = -1$. Similarly, the $t$ integral is well defined in the $\epsilon \to 0$ limit except when $k = -1$. With $\nu^2 = 1/\tau$, and with evaluation of the $\epsilon \to 0$ limit $i.e. \text{Im}\Pi^{\text{pert}}(t)$, we find

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} (-z)^k e^{z\tau} \log \left( \frac{z}{\nu^2} \right) dz = \begin{cases} -\infty \int_{0}^{\infty} t^k e^{-t\tau} dt & k > -1 \\ -\gamma_{k+1} & k = -1 \end{cases}$$  \hfill (42)

where $\gamma_{k} \approx 0.5772$ is Euler’s constant. It is easily verified that equations (42), (40), and (41) lead to the known results [3, 5, 8] for the non-instanton contributions to the Laplace sum-rules for scalar gluonic currents.

To evaluate the instanton contributions to the Laplace sum-rule, we must calculate the following integral:

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} (-z)^k e^{z\tau} \left[ K_2(\rho \sqrt{z}) \right]^2 dz = -\frac{1}{2\pi i} \int_{-\pi}^{\pi} (-1)^k \exp(\epsilon e^{i\theta} \tau) e^{k+3} e^{i(k+3)\theta} \left[ K_2(\rho e^{i\theta} \sqrt{z}) \right]^2 d\theta$$  \hfill (43)
Simplification of \(43\) requires the following properties of the modified Bessel function \(K_2(z)\) \(17\)

\[
K_2(z) \sim \frac{2}{z^2}, \quad z \to 0
\]

\[
K_2(z) = \begin{cases} 
-\frac{\pi}{4}H_2^{(1)}(ze^{i\pi/2}), & -\pi < \arg(z) \leq \frac{\pi}{2} \\
\frac{i}{2}\pi H_2^{(2)}(ze^{-i\pi/2}), & -\frac{\pi}{2} < \arg(z) \leq \pi 
\end{cases}
\]

where \(H_2^{(1)}(z) = J_2(z) + iY_2(z)\) and \(H_2^{(2)}(z) = J_2(z) - iY_2(z)\). The asymptotic behaviour \(44\) implies that the \(\theta\) integral of \(43\) will be zero in the \(\epsilon \to 0\) limit for \(k > -1\) and the identity \(45\) allows evaluation of the discontinuity in the \(t\) integral of \(43\), leading to the following instanton contribution to the Laplace sum-rules:

\[
L_{-1}^{\text{inst}}(\tau) = -16\pi^3 \int_{0}^{\infty} dn(\rho) \rho^4 \int_{0}^{\infty} t J_2(\rho \sqrt{t}) Y_2(\rho \sqrt{t}) e^{-t\tau} dt - 128\pi^2 \int_{0}^{\infty} dn(\rho) \quad (46)
\]

\[
L_{k}^{\text{inst}}(\tau) = -16\pi^3 \int_{0}^{\infty} dn(\rho) \rho^4 \int_{0}^{\infty} t^{k+2} J_2(\rho \sqrt{t}) Y_2(\rho \sqrt{t}) e^{-t\tau} dt, \quad k > -1
\]

\[
L_{k}^{\text{inst}}(\tau) = \begin{cases} 
128\pi^2 \int_{0}^{\infty} dn(\rho) \frac{a^4 e^{-a}}{\rho^5} [2aK_0(a) + (1 + 2a + a^2)K_1(a)] & , k = 0 \\
256\pi^2 \int_{0}^{\infty} dn(\rho) \frac{a^5 e^{-a}}{\rho^6} [(9 - 4a)K_0(a) + (3 + 7a - 4a^2)K_1(a)] & , k = 1
\end{cases}
\]

where \(a \equiv \rho^2/(2\tau)\) and where \(K_n\) is the modified Bessel function of the first kind of order \(n\) (c.f. \(17\)). Observe the symmetry between \(46\) and \(47\) broken by the term \(-128\pi^2 \int dn(\rho)\) appearing in \(46\) —a term which corresponds to the second integral on the right-hand side of \(43\) and which is nonzero only for \(k = -1\). Conversely, we note that naively substituting \(k = -1\) into \(47\) leads to an incorrect expression for the instanton contribution to the leading order sum-rule. This asymmetric role of the instanton contributions to \(k = -1\) and \(k > -1\) sum-rules is also a property of the LET as illustrated in \(13\) \(14\).

As discussed in Section \(1\) we wish to determine whether the leading order sum-rule \(L_{-1}\) might support the existence of a lowest-lying resonance whose presence is mass-suppressed in subsequent higher order \(k > -1\) sum-rules. Such is indeed the case, for example, for the pion within sum-rules based on a pseudoscalar \(\overline{q}q\) current. Correspondingly, one might anticipate the identification of the lowest-lying scalar gluonium state with a 500–600 MeV \(\sigma\) (i.e. the lower-mass range of the \(f_0(400 - 1200)\) resonance \(14\)) whose contribution to higher order sum-rules is suppressed by additional factors of \(m_{\pi}^2\) [i.e. the additional factors of \(m_{\pi}^2\) in \(21\)], a scenario analogous to the \(m_{\pi}^2\) suppression of pion contributions to the pseudoscalar \(\overline{q}q\) sum-rules \(6\). As already noted in Section \(1\) the LET constant in the absence of instantons supports this scenario for a sub-GeV scalar glueball.

However, the instanton contribution to the \(k = -1\) sum-rule is opposite in sign and comparable in magnitude to the LET subtraction constant \(\Pi(0)\), thereby ameliorating this term’s dominance of the lowest order sum-rule. For example, the contribution of instanton and LET terms to the \(k = -1\) sum rule in the dilute instanton liquid (DIL) model \(11\),

\[
dn(\rho) = n_c \delta(\rho - \rho_c) d\rho \quad ; \quad n_c = 8 \times 10^{-4} \text{ GeV}^4, \quad \rho_c = \frac{1}{600 \text{ MeV}} \quad (48)
\]

renders trivial the remaining integrations in \(46, 47\). If we approximate \(\langle J \rangle\) by \(\langle \alpha G^2 \rangle\) and employ a recently determined value of the gluon condensate \(20\)

\[
\langle \alpha G^2 \rangle = (0.07 \pm 0.01) \text{ GeV}^4 \quad (49)
\]

we obtain via \(11\) and \(15\) an \(n_f = 3\) estimate of the LET subtraction constant:

\[
\Pi(0) \simeq \frac{32\pi}{9} \langle \alpha G^2 \rangle \approx (0.78 \pm 0.11) \text{ GeV}^4 \quad . \quad (50)
\]
In Figure 2, we use (46) and the central value of (50) [with $n_c$ and $\rho_c$ given in (48)] to plot
\[
\Pi(0) + \frac{L_{\rho_{\text{inst}}}(\tau)}{\Pi(0)}
\]
as a function $\tau$. We note that, as anticipated, instanton effects do indeed significantly reduce the impact of the LET on the $k = -1$ sum-rule: anywhere from 20–65% for $\tau$ ranging between 0.6 GeV$^{-2}$ and 1.0 GeV$^{-2}$. Recalling that the dominance of the LET over $\mathcal{S}_{-1}$ is responsible for the discrepancy in gluonium mass scales in the analysis of the $k = -1$ and $k > -1$ sum-rules, we see that suppression of the LET by instanton effects could reconcile this discrepancy, a possibility which is investigated further in the next section.

3 Phenomenological Impact of Instanton Effects in the Laplace Sum-Rules

Ratios of Laplace sum-rules provide a simple technique for extracting the mass of the lightest (narrow) resonance probed by the sum-rules. If only the lightest resonance (of mass $m$) is included in (23) and (24), then for the first few sum-rules we see that
\[
\frac{S_1(\tau, s_0)}{S_0(\tau, s_0)} = m^2
\]
\[
\frac{S_0(\tau, s_0)}{\mathcal{S}_{-1}(\tau, s_0) + \Pi(0)} = m^2.
\]

This method of predicting the mass $m$ requires optimization of $s_0$ to minimize the $\tau$ dependence that can occur in the sum-rule ratios. However, a qualitative analysis which avoids these optimization issues occurs in the $s_0 \to \infty$ limit where bounds on the mass $m$ can also be obtained. These bounds originate from inequalities satisfied on the hadronic physics side of the sum-rule because of the positivity of $\rho^{\text{had}}(t)$. For example,
\[
\frac{1}{\pi} \int_{t_0}^{t_{\rho_{\text{had}}}(t)} dt e^{-t\tau} \rho^{\text{had}}(t) = \frac{1}{\pi} \int_{s_0}^{s_{\rho_{\text{had}}}(t)} dt (t - s_0) e^{-t\tau} \rho^{\text{had}}(t) \leq s_0 \frac{1}{\pi} \int_{t_0}^{t_{\rho_{\text{had}}}(t)} dt e^{-t\tau} \rho^{\text{had}}(t)
\]
\[
\implies S_1(\tau, s_0) \leq s_0 S_0(\tau, s_0).
\]

Furthermore, positivity of $\text{Im}\Pi^{QCD}(t)$ leads to an inequality for the continuum.
\[
\frac{1}{\pi} \int_{s_0}^{s_\infty} dt e^{-t\tau} \text{Im}\Pi^{QCD}(t) = \frac{1}{\pi} \int_{s_0}^{s_\infty} dt (t - s_0) e^{-t\tau} \text{Im}\Pi^{QCD}(t) \leq s_0 \frac{1}{\pi} \int_{s_0}^{s_\infty} dt e^{-t\tau} \text{Im}\Pi^{QCD}(t)
\]
\[
\implies c_1(\tau, s_0) \geq s_0 c_0(\tau, s_0)
\]

These inequalities can be extended to include the $k = -1$ sum-rules and continuum.
\[
S_0(\tau, s_0) \leq s_0 [\mathcal{S}_{-1}(\tau, s_0) + \Pi(0)]
\]
\[
c_0(\tau, s_0) \geq s_0 c_{-1}(\tau, s_0)
\]

We then see that
\[
\frac{L_1(\tau)}{L_0(\tau)} = \frac{S_1(\tau, s_0) + c_1(\tau, s_0)}{S_0(\tau, s_0) + c_0(\tau, s_0)} \geq \frac{S_1(\tau, s_0)}{S_0(\tau, s_0)} = m^2
\]
\[
\frac{L_0(\tau)}{L_{-1}(\tau) + \Pi(0)} = \frac{S_0(\tau, s_0) + c_0(\tau, s_0)}{S_0(\tau, s_0) + c_{-1}(\tau, s_0)} \geq \frac{S_0(\tau, s_0)}{S_{-1}(\tau, s_0) + \Pi(0)} = m^2.
\]
Thus the ratios of the $s_0 \to \infty$ limit of the sum-rules provide bounds on the mass in this single narrow resonance approximation. Extending the analysis to many narrow resonances alters so that the sum-rule ratios are an upper bound on the lightest resonance, upholding the bounds on the mass $m^2$ of the lightest resonance.

\begin{align}
\frac{\mathcal{L}_1(\tau)}{\mathcal{L}_0(\tau)} &\geq m^2 \\
\frac{\mathcal{L}_0(\tau)}{\mathcal{L}_{-1}(\tau) + \Pi(0)} &\geq m^2
\end{align}

The sum-rule bounds in can now be employed to determine the phenomenological impact of the instanton contributions on the sum-rule estimates of the lightest gluonium state, and to assess whether the suppression of the LET contribution by the instanton effects is sufficient to reduce the discrepancy between sum-rule analyses containing or omitting the $k = -1$ sum-rule. Collecting results from equations the first few sum-rules $\mathcal{L}_k(\tau)$ are

\begin{align}
\mathcal{L}_{-1}(\tau) &= \frac{1}{\tau^2} \left[ -a_0 + a_1 (-2 + 2\gamma_E) + a_2 \left( \frac{\pi^2}{2} + 6\gamma_E - 3\gamma_E^2 \right) \right] + \frac{-b_0 + b_1 \gamma_E}{\tau} \langle J \rangle - c_0 \tau \langle \mathcal{O}_6 \rangle - d_0 \frac{\tau^2}{2} \langle \mathcal{O}_8 \rangle \\
&- 64\pi^2 \int d\rho(\rho^a \sum_{n=\pm 1} a(1 + 2 + a + a^2)K_1(a)) \\
\mathcal{L}_0(\tau) &= \frac{1}{\tau^3} \left[ -2a_0 + a_1 (-6 + 4\gamma_E) + a_2 \left( \frac{\pi^2}{2} - 6 + 18\gamma_E - 6\gamma_E^2 \right) \right] - \frac{b_1}{\tau} \langle J \rangle + c_0 \langle \mathcal{O}_6 \rangle + d_0 \tau \langle \mathcal{O}_8 \rangle \\
&+ 128\pi^2 \int d\rho(\rho^a \sum_{n=\pm 1} a(1 + 2 + a + a^2)K_1(a)) \\
\mathcal{L}_1(\tau) &= \frac{1}{\tau^4} \left[ -6a_0 + a_1 (-22 + 12\gamma_E) + a_2 \left( \frac{3\pi^2}{2} - 36 + 66\gamma_E - 18\gamma_E^2 \right) \right] - \frac{b_1}{\tau^2} \langle J \rangle - d_0 \langle \mathcal{O}_8 \rangle \\
&+ 256\pi^2 \int d\rho(\rho^a \sum_{n=\pm 1} a(1 + 2 + a + a^2)K_1(a))
\end{align}

Renormalization-group improvement has been achieved by setting $\nu^2 = 1/\tau$ in the correlation function and in the running coupling $\alpha$:

\begin{align}
\frac{\alpha_s(\nu)}{\pi} &= \frac{1}{\beta_0 L} \left[ -\beta_1 \log L \sum_{n=\pm 1} a \beta_1^2 \left( \log^2 L - \log L - 1 \right) + \bar{\beta}_2 \right] \\
L &= \log \left( \frac{\nu^2}{\Lambda^2} \right), \quad \bar{\beta}_i = \frac{\beta_i}{\beta_0}, \quad \beta_0 = \frac{9}{4}, \quad \beta_1 = 4, \quad \beta_2 = \frac{3863}{384}
\end{align}

with $\Lambda_{\overline{MS}} \approx 300$ MeV for three active flavours, consistent with current estimates of $\alpha_s(M_T)$ and matching conditions through the charm threshold.

The nonperturbative QCD parameters are needed for further analysis of the sum-rules. We employ the DIL model parameters summarized in, as well as vacuum saturation for the dimension-8 gluon condensate.

\begin{align}
\langle \mathcal{O}_8 \rangle &= 14 \left( \langle \alpha f_{ab} G_{\mu\nu} A_{\mu}^a G_{\nu}^b \rangle \right) - \left( \langle \alpha f_{ab} G_{\mu\nu} A_{\mu}^a G_{\nu}^b \rangle \right) = \frac{9}{16} \left( \langle \alpha G^2 \rangle \right) \\
\langle \mathcal{O}_6 \rangle &= \langle g f_{abc} G_{\mu\nu}^a G_{\rho\lambda}^b G_{\nu\lambda}^c \rangle = (0.27 \text{ GeV}^2) \left( \langle \alpha G^2 \rangle \right)
\end{align}

Finally, again using the approximation $\langle J \rangle = \langle \alpha G^2 \rangle$ and the central gluon condensate value [see reference] from reference, we find that the role of instanton contributions to the sum-rules is as illustrated in Figures and. In particular, we see that the instanton contributions diminish $\mathcal{L}_{-1}(\tau) + \Pi(0)$. The LET term $\Pi(0)$, which leads to the asymptotic flattening of $\mathcal{L}_{-1}(\tau) + \Pi(0)$ at a value substantially different from zero when instantons are absent, is
clearly suppressed by instanton effects in the large \( \tau \) region. As noted earlier, such flattening over the \( \tau \leq 1.0 \text{ GeV}^{-2} \) region would be indicative via (23) of a sub-GeV lowest-lying resonance (i.e., \( m^2 \tau \ll 1 \)) over the physically relevant region of \( \tau \leq 1.0 \text{ GeV}^{-2} \). The corresponding effects of instantons on the sum-rule ratios (60–61) is shown in Figures 6 and 7. As expected from the instanton’s impact of lowering \( L_{-1}(\tau) \) and elevating \( L_0(\tau) \), the ratio \( L_0(\tau)/[L_{-1}(\tau) + \Pi(0)] \) is increased substantially by inclusion of instanton effects, increasing the corresponding upper bound on the mass of the lightest gluonium state. Instanton effects also serve to lower the ratio \( L_1(\tau)/L_0(\tau) \), decreasing the corresponding upper bound on the mass of the lightest gluonium state.

Figures 8 and 9 summarize the ratio (mass bound) analysis in the presence and in the absence of instanton effects. It is evident that instanton effects lead to a substantial increase in the mass bound on the lightest gluonium state, but other important features emerge. For example, the instanton suppression of the LET term \( \Pi(0) \) reduces the discrepancy between the ratios including or omitting the \( k = -1 \) sum-rule. Furthermore, a \( \tau \)-minimum stability plateau crucial for establishing a credible upper mass bound is seen to occur at reasonable energy scales \((1/\sqrt{\tau} \leq 1.0 \text{ GeV})\) only when instanton effects are included. The ratios with instanton effects included (see Figure 8) are remarkably flat, suggesting that the mass bounds could be close to the mass prediction that would be obtained from a full sum-rule analysis incorporating the QCD continuum (i.e. \( s_0 < \infty \)) in the phenomenological model.

4 Discussion

We have calculated the instanton contribution to the Laplace sum-rules of scalar gluonium and demonstrated explicitly how, for the lowest order \( k = -1 \) sum-rule, this instanton contribution cancels part of the dominant LET constant.

As noted in the Introduction, a discrepancy between the lowest lying states evident from the lowest and from the next-to-lowest Laplace sum-rules may be indicative of two distinct states. Such is found to be the case, for example, in the pseudoscalar channel in which the pion dominates the leading \((k = 0)\) Laplace sum rule, but the \( \Pi(1300) \) resonance is found to dominate the next-to-leading \((k = 1)\) Laplace sum rule, because of a mass-suppression of the lowest-lying (pion) state in the latter sum rule [6]. Moreover, analyses of the scalar gluonium channel in the absence of explicit instanton contributions seem to exhibit a similar discrepancy between leading \( k = -1 \) and non-leading \( k > -1 \) sum-rules [3, 5, 8].

Prior QCD sum-rule analyses of the instanton contribution to the scalar gluonium channel have focused either on the \( k = -1 \) sum rule exclusively [10] or the LET-insensitive \( k > -1 \) sum-rules [12]. Although these two analyses (which are separated by almost two decades) are both indicative of a lowest lying-resonance mass near or above 1.4 GeV, their input content (parameter values and levels of perturbation theory) are necessarily different, suggesting the need for a single consistent treatment of leading \( k = -1 \) and non-leading \( k > -1 \) Laplace sum-rules in the scalar gluonium channel that is inclusive of instanton effects. We have shown here that careful consideration of the contribution arising from instantons within the \( k = -1 \) sum rule in this channel leads to consistency with higher sum-rules in the estimation of lowest-lying resonance masses in the scalar gluonium channel.\(^3\)

Correspondence with the prior treatment of the \( k = -1 \) sum rule [10] can be obtained by examining the instanton contribution [10] to this sum rule in the high-energy limit of small \( \tau \). This contribution is obtained

\(^3\)Note also that we have incorporated the significant NNLO perturbative corrections, as opposed to LO corrections in [10] and NLO corrections in [12].
through evaluation of the integral in \([46]\) in the large-\(a\) \([\tau = \rho^2/(2\alpha)]\) limit:

\[
\mathcal{L}_{-1}^{\text{inst}}(\tau) = -128\pi^2 \int dn(\rho) - 32\pi^3 \int dn(\rho) \int_0^\infty x^3 J_2(x) Y_2(x) e^{-\frac{x^2}{2\tau}} dx
\]

\[
= -64\pi^2 \int dn(\rho) \left[ \frac{\rho^4}{4\tau^2} \left[ 1 + \frac{\rho^2}{2\tau} \right] K_0 \left( \frac{\rho^2}{2\tau} \right) + \frac{\rho^2}{\tau} \left[ 2 + \frac{\rho^2}{\tau} + \frac{\rho^4}{4\tau^2} \right] K_1 \left( \frac{\rho^2}{2\tau} \right) \right] e^{-\frac{\rho^2}{\tau}} \quad (69)
\]

\[
\rightarrow -16\pi^2 \int dn(\rho) e^{-\frac{\rho^2}{\tau}} \rho^5 \tau^{-\frac{3}{2}} \left[ 1 + O \left( \frac{\tau}{\rho^2} \right) \right], \quad \tau \ll \rho^2 . \quad (70)
\]

In the instanton liquid model \([dn(\rho) = n_c \delta(\rho - \rho_c) d\rho]\), \([70]\) is consistent with eq. (42) of reference \([10]\) for the instanton contribution to the \(k = -1\) sum-rule, which was utilized to anticipate the \((1.6\ \text{GeV})^2\) lattice prediction of the scalar gluonium mass. Figure \([10]\) provides a comparison of this asymptotic form with the exact expression \([69]\) for \(\mathcal{L}_{-1}^{\text{inst}}(\tau)\) under the instanton-liquid assumption. Of particular interest is the difference between the two expressions over the range \(2 \lesssim a \lesssim 3\), corresponding to the \(0.4\ \text{GeV}^{-2} \lesssim \tau \lesssim 0.6\ \text{GeV}^{-2}\) range in Figure \([8]\) for which \(\sqrt{\mathcal{L}_0/\mathcal{L}_{-1} + \Pi(0)}\) is flat. Although both the asymptotic expression of \([10]\) and the exact expression provide negative contributions which mitigate dominance of the positive LET contribution over \(\mathcal{L}_{-1}\), the exact expression is substantially larger in magnitude over the region of phenomenological interest. We speculate that such an increase relative to the analysis of \([10]\) serves to compensate for the larger phenomenological value at present for the gluon condensate within the low-energy theorem term \([10]\), although it may also compensate the non-leading perturbative contributions in \([20]\) not known at the time of \([10]\).

Of course, more sophisticated expressions for the instanton contributions have been utilized to generate consistent scalar gluonium phenomenology both within \([12]\) and complementary to \([25]\) a QCD sum rule framework, as noted above. The key point of the work presented here is the reconciliation of the \(k = -1\) sum rule with higher-\(k\) sum-rules. We reiterate the approximate consistency between scalar-gluonium masses obtained from the flattened regions of the \(\sqrt{\mathcal{L}_0/\mathcal{L}_{-1} + \Pi(0)}\) and \(\sqrt{\mathcal{L}_1/\mathcal{L}_0}\) curves of Figure \([8]\) as well as the drastic reduction of the scalar gluonium mass evident in the curves of Figure \([8]\) when instanton contributions are omitted. Roughly speaking, such contributions account for half the lowest-lying scalar-gluonium mass within a sum-rule context.

**Acknowledgements:** The authors are grateful for research support from the Natural Sciences and Engineering Research Council of Canada (NSERC). We are also grateful to N. Kochelev for helpful correspondence.

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Figure 1: Closed contour $C(R)$ used to obtain the inverse Laplace transform defining the Laplace sum-rules. The inner circular segment $\Gamma_\epsilon$ has a radius of $\epsilon$, and the outer circular segments $\Gamma_2$ and $\Gamma_3$ have a radius $R$. The wavy line on the negative real axis denotes the branch cut of $\Pi(z)$, and the linear segments of the contour above and below the branch cut are denoted by $\Gamma_c$. 
Figure 2: The quantity $[\Pi(0) + \mathcal{L}_{-1}^{\text{inst}}(\tau)] / \Pi(0)$ is plotted as a function of $\tau$, illustrating that instanton effects cancel a significant portion of the LET term in the $k = -1$ sum-rule.
Figure 3: Comparison of the full field-theoretical content of $\mathcal{L}_1(\tau) + \Pi(0)$ with (dashed curve) and without (solid curve) instanton effects.
Figure 4: Comparison of $L_0(\tau)$ with instanton effects included (dashed curve) and instanton effects excluded (solid curve).
Figure 5: Comparison of $L_1(\tau)$ with instanton effects included (dashed curve) and instanton effects excluded (solid curve).
Figure 6: Comparison of the ratio \( \sqrt{\mathcal{L}_0(\tau)} / [\mathcal{L}_{-1}(\tau) + \Pi(0)] \) with instanton effects included (dashed curve) and instanton effects excluded (solid curve).
Figure 7: Comparison of the ratio $\sqrt{\mathcal{L}_1(\tau)/\mathcal{L}_0(\tau)}$ with instanton effects included (dashed curve) and instanton effects excluded (solid curve).
Figure 8: Sum-rule ratios used to obtain scalar gluonium mass bounds with inclusion of instanton effects. The solid curve represents the ratio $\sqrt{\mathcal{L}_0(\tau)/[\mathcal{L}_{-1}(\tau) + \Pi(0)]}$ and the dashed curve represents the ratio $\sqrt{\mathcal{L}_1(\tau)/\mathcal{L}_0(\tau)}$. 
Figure 9: Sum-rule ratios used to obtain scalar gluonium mass bounds with omission of instanton effects. The solid curve represents the ratio $\sqrt{L_0(\tau)/[L_{-1}(\tau) + \Pi(0)]}$ and the dashed curve represents the ratio $\sqrt{L_1(\tau)/L_0(\tau)}$. 
Figure 10: Instanton contributions to $L_{-1}$ in the instanton liquid model as a function of $a = \rho_c^2/(2\tau)$. The solid line represents the complete expression (69) for the instanton contributions, and the dashed line represents the lead term of the large $a$ expansion (70) obtained in [10].