Abstract

Hilbert space representations of the cross product ∗-algebras of the Hopf ∗-algebra $U_q(su_2)$ and its module ∗-algebras $O(S^2_{qr})$ of Podleś spheres are investigated and classified by describing the action of generators. The representations are analyzed within two approaches. It is shown that the Hopf ∗-algebra $O(SU_q(2))$ of the quantum group $SU_q(2)$ decomposes into an orthogonal sum of projective Hopf modules corresponding to irreducible integrable ∗-representations of the cross product algebras and that each irreducible integrable ∗-representation appears with multiplicity one. The projections of these projective modules are computed. The decompositions of tensor products of irreducible integrable ∗-representations with spin $l$ representations of $U_q(su_2)$ are given. The invariant state $h$ on $O(S^2_{qr})$ is studied in detail. By passing to function algebras over the quantum spheres $S^2_{qr}$, we give chart descriptions of quantum line bundles and describe the representations from the first approach by means of the second approach.

Keywords: Quantum groups, unbounded representations
Mathematics Subject Classifications (2000): 17B37, 81R50, 46L87

0 Introduction

Podleś quantum spheres [13] (see [7, Section 4.5] for a short treatment) are a one-parameter family $S^2_{qr}$, $r \in [0, \infty]$, of mutually non-isomorphic quantum homogeneous spaces of the quantum group $SU_q(2)$, where $0 < q < 1$. Each of
These spaces can be considered as a quantum analogue of the classical 2-sphere. Their coordinate algebras \( \mathcal{O}(S^2_{qr}) \) are right coideal \(*\)-subalgebras of the coordinate Hopf \(*\)-algebra \( \mathcal{O}(SU_q(2)) \) of the quantum group \( SU_q(2) \) and hence left module \(*\)-algebras of the Hopf \(*\)-algebra \( \mathcal{U}_q(su_{12}) \). Therefore, the left cross product \(*\)-algebra \( \mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(su_{12}) \) is defined. The subject of this paper are Hilbert space representations of the \(*\)-algebra \( \mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(su_{12}) \).

There is now an extensive literature on Podleś spheres (see e.g. [1–3, 5, 6, 12]). Let us restate some algebraic results from these papers that are relevant for our investigations. Let \( \mathcal{C} \) denote the coalgebra \( \mathcal{O}(SU_q(2))/\mathcal{O}(S^2_{qr})^+\mathcal{O}(SU_q(2)) \) with quotient map \( \rho : \mathcal{O}(SU_q(2)) \rightarrow \mathcal{C} \), where \( \mathcal{O}(S^2_{qr})^+ = \{ x \in \mathcal{O}(S^2_{qr}) ; \varepsilon(x) = 0 \} \). The algebra \( \mathcal{O}^*(S^2_{qr}) \) is then the algebra of left \( \mathcal{C} \)-covariant elements of \( \mathcal{O}(SU_q(2)) \).

Note that only in the case \( r = 0 \) the coalgebra \( \mathcal{C} \) is a Hopf algebra and the quantum sphere \( S^2_{qr} \) is the quotient by a quantum subgroup. M. Dijkhuizen and T. Koornwinder [3] have found a skew-primitive element \( X_r \in \mathcal{U}_q(su_{12}) \) such that \( \mathcal{O}(S^2_{qr}) \) is the subalgebra of right \( X_r \)-invariant elements of \( \mathcal{O}(SU_q(2)) \). A major step have been the results of E. F. Müller and H.-J. Schneider [12]. They showed that \( \mathcal{O}(SU_q(2)) \) is faithfully flat as a left (and right) \( \mathcal{O}(S^2_{qr}) \)-module and that \( \mathcal{C} \) is spanned by group-like elements. As a consequence, \( \mathcal{C} \) is the direct sum of simple subalgebras \( \mathcal{C}_j \). Then

\[
M_j = \{ x \in \mathcal{O}(SU_q(2)) ; \rho(x_{(1)}) \otimes x_{(2)} \in \mathcal{C}_j \otimes \mathcal{O}(SU_q(2)) \}
\]

is a finitely generated projective relative \( \mathcal{O}(SU_q(2)), \mathcal{O}(S^2_{qr}) \)-Hopf module and \( \mathcal{O}(SU_q(2)) \) is the direct sum of these Hopf modules \( M_j \) [12, p. 186]. In the subgroup case \( r = 0 \), the corresponding projections and their Chern numbers have been computed in [6] and [5], respectively. A family of group-like elements spanning the coalgebra \( \mathcal{C} \) was determined in [2].

In the present paper, we reconsider and extend these algebraic results in the Hilbert space setting. We prove that \( \mathcal{O}(SU_q(2)) \) is the orthogonal direct sum of the Hopf modules \( M_j \) and we give an explicit description of this decomposition. Here the skew-primitive element \( X_r \) plays a crucial role. Moreover, we determine the projections of the projective modules \( M_j \). All this is carried out in Section 3. Each Hopf module \( M_j \) corresponds to an irreducible \(*\)-representation \( \pi_j \) of the cross product \(*\)-algebra \( \mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(su_{12}) \) such that the restriction of \( \pi_j \) to \( \mathcal{U}_q(su_{12}) \) is a direct sum of spin \( l \) representations \( T_l, l \in \frac{1}{2}\mathbb{N}_0 \). Let us call a \(*\)-representation of \( \mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(su_{12}) \) integrable if it has the latter property. In Section 4 we classify integrable \(*\)-representations of \( \mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(su_{12}) \) and prove that each irreducible integrable \(*\)-representation of \( \mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(su_{12}) \) is unitarily equivalent to one of the
representations \( \pi_j \). In the course of this classification, we describe the structure of these \(*\)-representations \( \pi_j \) by explicit formulas for the actions of generators of \( \mathcal{O}(S^2_{qr}) \) on an orthonormal basis of weight vectors for the representation \( T_i \) of \( \mathcal{U}_q(\mathfrak{su}_2) \). In the terminology of our previous paper [15], this is the first approach to representations of the cross product algebra \( \mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2) \). We also derive a formula for the decomposition of the tensor product representation \( \pi_j \otimes T_i \) into a direct sum of representations \( \pi_j \).

The corresponding second approach [15] is developed in Section 5. Here we begin with a representation of the \(*\)-algebra \( \mathcal{O}(S^2_{qr}) \) given in a canonical form and we extend it to a representation of \( \mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2) \). The main technical tool for this is to “decouple” the cross relations of the cross product algebra by finding an auxiliary \(*\)-subalgebra \( \mathcal{Y}_r \) of \( \mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2) \) which commutes with the \(*\)-algebra \( \mathcal{O}(S^2_{qr}) \). Such decouplings have been found and studied in [4].

Section 6 starts by defining algebras of functions which extend the coordinate algebras \( \mathcal{O}(S^2_{qr}) \) and by describing invariant functionals on such function algebras. The algebras of functions together with the invariant functionals will be used to give another description of irreducible integrable \(*\)-representations. It should be emphasized that though all irreducible integrable \( \pi_j \) of \( \mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2) \) involve unbounded operators, their restrictions to the \(*\)-subalgebra \( \mathcal{O}(S^2_{qr}) \otimes \mathcal{Y}_r \) are given by bounded operators only. In Subsection 6.2, we show that the restriction of an irreducible integrable \(*\)-representation to \( \mathcal{O}(S^2_{qr}) \otimes \mathcal{Y}_r \) decomposes into a direct sum of two representations which can be realized on algebras of functions with support in the positive and the negative spectrum of a certain self-adjoint operator. This self-adjoint operator represents a coordinate function of the quantum sphere and the two algebras of functions can be considered as “charts” of the projective module \( M_j \). The representation of \( \mathcal{O}(S^2_{qr}) \otimes \mathcal{Y}_r \) on each chart leads again to an irreducible \(*\)-representation of the cross product algebra which is not integrable and can be described by the formulas from the second approach. To round off this circle of investigations, we recover in Section 7 the irreducible integrable representation by taking the direct sum of both charts and passing to another domain.

In Section 1, we briefly mention the correspondence between relative Hopf modules and modules of cross product algebras and we characterize direct sums of Heisenberg representations for cross product algebras \( A \rtimes \mathcal{U} \) of Hopf \(*\)-algebras \( A \) of compact quantum groups. Section 2 collects a number of definitions and basic facts on Podleś spheres and on the cross product algebras \( \mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2) \) which are needed in what follows.

All facts and notions on quantum groups used in this paper can be found, for
instance, in [7]. The algebra $U_q(\mathfrak{sl}_2)$ is due to P. P. Kulish and N. Y. Reshetikhin [8]. The quantum group $SU_q(2)$ was invented in [19] and [17] and the quantum spheres $S^2_{qr}$ were discovered in [13].

Let us introduce some notation. Throughout this paper, $q$ is a real number of the open interval $(0, 1)$. We abbreviate
\[ \lambda := q - q^{-1}, \quad \lambda_n := (1 - q^{2n})^{1/2}, \quad [n] := (q^n - q^{-n})/(q - q^{-1}), \]
where $n \in \mathbb{N}_0$. For an algebra $\mathcal{A}$, the notation $\mathcal{A}^n$ stands for the direct sum of $n$ copies of $\mathcal{A}$ and $M_n(\mathcal{A})$ denotes the set of $n \times n$-matrices with entries from $\mathcal{A}$. Let $I$ be an at most countable index set, $V$ a linear space and $D = \oplus_{i \in I} V_i$, where $V_i = V$ for all $i \in I$. We denote by $\eta_i$ the vector of $D$ which has the element $\eta \in V$ as its $i$-th component and zero otherwise. It is understood that $\eta_i = 0$ whenever $i \notin I$. If $V$ is a Hilbert space, then $\overline{\oplus}_{i \in I} V_i$ denotes the closed linear span of $\{\eta_i; \eta \in V, \ i \in I\}$. Given a dense linear subspace $D$ of a Hilbert space,
\[ \mathcal{L}^+(D) := \{ y \in \text{End}(D); \ D \subset D(y^*), \ y^*D \subset D \} \]
is a unital *-algebra of closeable operators with involution $y \mapsto y^* |D$. As customary, $D(x)$ denotes the domain of an operator $x$. By a *-representation of a unital *-algebra $\mathcal{A}$ on the domain $D$, we mean a unit preserving *-homomorphism $\pi$ from $\mathcal{A}$ into $\mathcal{L}^+(D)$ (see e.g. [14]). When no confusion can arise, we omit the letter which denotes the representation and write $x$ instead of $\pi(x)$.

1 Relative Hopf modules and representations of the cross product algebras

Let $\mathcal{U}$ be a Hopf *-algebra and let $\mathcal{X}$ be a left $\mathcal{U}$-module *-algebra, that is, $\mathcal{X}$ is a unital *-algebra with left $\mathcal{U}$-action $\triangleright$ satisfying
\[ f \triangleright xy = (f(1) \triangleright x)(f(2) \triangleright y), \quad f \triangleright 1 = \varepsilon(f)1, \quad (f \triangleright x)^* = S(f)^* \triangleright x^* \quad (1) \]
for $x, y \in \mathcal{X}$ and $f \in \mathcal{U}$. Here $\Delta(f) = f(1) \otimes f(2)$ is the Sweedler notation for the comultiplication $\Delta(f)$ of $f \in \mathcal{U}$. Then the left cross product *-algebra $\mathcal{X} \rtimes \mathcal{U}$ is the *-algebra generated by the two *-subalgebras $\mathcal{X}$ and $\mathcal{U}$ with respect to the cross commutation relations
\[ fx = (f(1) \triangleright x)f(2), \quad x \in \mathcal{X}, \ f \in \mathcal{U}. \quad (2) \]
Let $A$ be a Hopf algebra and $X$ a right $A$-coideal subalgebra of $A$. A relative Hopf module in $X \mathcal{M}^A$ (see e.g. [11]) is a right $A$-comodule $M$ which is also a left $X$-module such that the right coaction of $A$ is $X$-linear, that is,

$$
(xm)(1) \otimes (xm)(2) = x(1)m(1) \otimes x(2)m(2), \quad x \in X, \ m \in M. \tag{3}
$$

Here we write simply $xm$ for the left module action of $x$ at $m$ and we use the Sweedler notation for the coaction.

Let $\langle \cdot, \cdot \rangle$ be a dual pairing of Hopf algebras $U$ and $A$. Then any right $A$-comodule $M$ determines a left $U$-module by

$$
f \triangleleft m := \langle f, m(2) \rangle m(1), \quad f \in U, \ m \in M, \tag{4}
$$

and the right $A$-comodule algebra $X$ becomes a left $U$-module algebra. Hence the cross product algebra $X \rtimes U$ is defined. The cross relations (2) are given by

$$
f x = \langle f(1), x(2) \rangle x(1) f(2), \quad x \in X, \ f \in U. \tag{5}
$$

The following simple well-known lemma is crucial in what follows.

**Lemma 1.1** Let $M$ be a left $X$-module and a right $A$-comodule.

(i) If $M \in X \mathcal{M}^A$, then $M$ is a left $X \rtimes U$-module.

(ii) Suppose that $U$ separates the points of $A$, that is, $\langle f, a \rangle = 0$ for all $f \in U$ implies $a = 0$. If $M$ is a left $X \rtimes U$-module, then $M \in X \mathcal{M}^A$.

**Proof.** As $M$ is a right $A$-comodule, it is a left $U$-module. From (4), we obtain

$$
f \triangleleft (xm) = \langle f, (xm)(2) \rangle (xm)(1), \tag{6}
$$

$$
\langle f(1), x(2) \rangle x(1) (f(2) \triangleright m) = \langle f, x(2) m(2) \rangle x(1) m(1). \tag{7}
$$

If $M \in X \mathcal{M}^A$, then (3) holds and hence the right hand sides of (6) and (7) coincide. That is, the cross relations (5) of $X \rtimes U$ are satisfied, so we have a well defined left $X \rtimes U$-module. Conversely, if the right hand sides of (6) and (7) are equal, then (3) follows by using the assumption that $U$ separates $A$. \hfill \Box

Suppose now that $\langle \cdot, \cdot \rangle$ is a dual pairing of Hopf $\ast$-algebras $U$ and $A$ and that $X$ is a $\ast$-invariant right $A$-coideal. Suppose that there exists a positive linear functional $h$ on $X$ (i.e. $h(x^\ast x) \geq 0$ for $x \in X$) which is $U$-invariant (i.e. $h(f \triangleleft x) = \varepsilon(f) h(x)$ for $x \in X$ and $f \in U$). Then there is a unique $\ast$-representation $\pi_h$,
called the Heisenberg representation of the $\star$-algebra $\mathcal{A} \rtimes \mathcal{U}$, with domain $D_h$ such that $\pi_h[\mathcal{A}]$ is the GNS-representation of $\mathcal{A}$ with cyclic vector $\varphi_h = \pi_h(1)$, $D_h = \pi(\mathcal{A})\varphi_h$ and $\pi(f)\varphi_h = \varepsilon(f)\varphi_h$ for $f \in \mathcal{U}$. (Note that we have taken in [15] the closure of $\pi_h$ as the Heisenberg representation obtained from $h$.)

Recall that a Hopf $\star$-algebra $\mathcal{A}$ is called a CQG-algebra (compact quantum group algebra) if $\mathcal{A}$ is the linear span of matrix elements of finite dimensional unitary corepresentations of $\mathcal{A}$ [3], [7, Subsection 11.3.1]. A CQG-algebra has a unique Haar state $h$.

The next proposition is a Hilbert space version of the well-known algebraic fact that Hopf modules in $\mathcal{A}M^A$ are trivial.

**Proposition 1.2** Suppose that $\langle \cdot, \cdot \rangle$ is a dual pairing of a Hopf $\star$-algebra $\mathcal{U}$ and a CQG Hopf $\star$-algebra $\mathcal{A}$ such that $\mathcal{U}$ separates the points of $\mathcal{A}$. Let $\pi$ be a $\star$-representation of the cross product $\star$-algebra $\mathcal{A} \rtimes \mathcal{U}$ on a domain $D$. Then $\pi$ is unitarily equivalent to a direct sum of Heisenberg representations $\pi_h$ of $\mathcal{A} \rtimes \mathcal{U}$ if and only if $D$ is a right $\mathcal{A}$-comodule such that $\pi(f)\varphi = f^\varphi$ for $f \in \mathcal{U}$ and $\varphi \in D$.

**Proof.** To prove the necessity, it suffices to check the above condition for the Heisenberg representation $\pi_h$. Because the Haar state of $\mathcal{A}$ is faithful, there is a well defined right coaction $\phi$ of $\mathcal{A}$ on $D$ given by $\phi(\varphi) = \pi_h(a_{(1)})\varphi_h \otimes a_{(2)}$ for $\varphi = \pi_h(a)\varphi_h$, $a \in \mathcal{A}$. Let $f \in \mathcal{U}$. Using the cross relation (2) and $\mathcal{U}$-invariance of the cyclic vector $\varphi_h$, we obtain

$$\pi_h(f)\varphi = \pi_h(fa)\varphi = \langle f_{(1)}, a_{(2)} \rangle \pi_h(a_{(1)}f_{(2)}) \varphi_h = \langle f, a_{(2)} \rangle \pi_h(a_{(1)}) \varphi_h = f^\varphi.$$

Now we prove the sufficiency. Suppose $D$ is a right $\mathcal{A}$-comodule such that $\pi(f)\varphi = f^\varphi \equiv \langle f, \varphi(2) \rangle \varphi(1)$ for $\varphi \in D$ and $f \in \mathcal{U}$. Let $D_{inv}$ be the subspace of vectors $\omega(\varphi) := \pi(S^{-1}(\varphi(3)))\varphi(1)$, $\varphi \in D$. Let $f \in \mathcal{U}$ and $\varphi \in D$. Using the fact that $\pi$ is a representation and relation (2), we compute

$$\pi(f)\omega(\varphi) = \pi(fS^{-1}(\varphi(2)))\varphi(1) = \langle f_{(1)}, S^{-1}(\varphi(3)) \rangle \pi(S^{-1}(\varphi(3))) \pi(f_{(2)}) \varphi(1) = \langle f_{(1)}S^{-1}(\varphi(3)) \rangle \pi(S^{-1}(\varphi(4))) \langle f_{(2)}, \varphi(2) \rangle \varphi(1) = \langle f, S^{-1}(\varphi(3)) \varphi(2) \rangle \pi(S^{-1}(\varphi(4))) \varphi(1) = \varepsilon(f)\omega(\varphi).$$

Hence the functional $\langle \pi(\cdot)\omega(\varphi), \omega(\varphi) \rangle$ on $\mathcal{A}$ is $\mathcal{U}$-invariant. Since $\mathcal{U}$ separates the points of $\mathcal{A}$, it is also $\mathcal{A}$-invariant and so a multiple of the Haar state $h$. Thus, we have

$$\langle \pi(x)\omega(\varphi), \omega(\varphi) \rangle = \|\omega(\varphi)\|^2 h(x), \quad x \in \mathcal{A}. \quad (8)$$
Next we show that $\omega(\varphi) \perp \omega(\varphi')$ for $\varphi, \varphi' \in \mathcal{D}$ implies

$$
\pi(A \rtimes \mathcal{U})\omega(\varphi) \perp \pi(A \rtimes \mathcal{U})\omega(\varphi').
$$

(9)

Let $u \in \mathbb{C}$, $|u| = 1$, and let $x \in A$ be hermitian. Clearly, $\omega(\varphi) + u\omega(\varphi') \in \mathcal{D}_{\text{inv}}$. By (8), we obtain

$$
\langle \pi(x)\omega(\varphi), \omega(\varphi') \rangle = \langle \pi(x)\omega(\varphi) + u\omega(\varphi'), \omega(\varphi') \rangle + 2\text{Re} \, u \langle \pi(x)\omega(\varphi), \omega(\varphi') \rangle
$$

Using Zorn’s lemma, we choose a maximal set $\{\omega(\varphi_i) ; i \in I\}$ of orthonormal vectors of the vector space $\mathcal{D}_{\text{inv}}$. Let $\mathcal{D}_i = \pi(A \rtimes \mathcal{U})\omega(\varphi_i)$ and let $\pi_i$ be the restriction of the representation $\pi$ to $\mathcal{D}_i$. Since $\langle \pi_i(x)\omega(\varphi_i), \omega(\varphi_i) \rangle = h(x) \quad \text{for} \quad x \in A$ by (8), $\pi_i$ is unitarily equivalent to the Heisenberg representation $\pi$ of $A \rtimes \mathcal{U}$. By (7), $\mathcal{D}_i \perp \mathcal{D}_j$ for $i \neq j$. Since $\varphi = \pi(\varphi_2)\omega(\varphi_{(1)})$ for $\varphi \in \mathcal{D}$, the domain $\mathcal{D}$ is the linear span of subspaces $\mathcal{D}_i, i \in I$. Putting the preceding together, we have shown that $\pi$ is unitarily equivalent to a direct sum of Heisenberg representations.

Let $A$ be the $CQG$-algebra $\mathcal{O}(SU_q(2))$ and let $\mathcal{U} = U_q(su_2)$. Then the hypothesis in Proposition [1.2] is satisfied if and only if $\pi$ is integrable, that is, its restriction to $U_q(su_2)$ is a direct sum of spin $l$ representations $T_l, l \in \frac{1}{2}\mathbb{N}_0$. Therefore, by Proposition [1.2], a $*$-representation $\pi$ of the $*$-algebra $\mathcal{O}(SU_q(2)) \rtimes U_q(su_2)$ is a direct sum of Heisenberg representations $\pi_h$ if and only $\pi$ is integrable. This was proved in [15] by another method. A similar result holds for compact forms of standard quantum groups.

2 The cross product algebra $\mathcal{O}(S^2_{qr}) \rtimes U_q(su_2)$

The Hopf $*$-algebra $U_q(su_2)$ is generated by elements $E, F, K, K^{-1}$ with relations

$$
KK^{-1} = K^{-1}K = 1, \quad KE = qEK, \quad FK = qKF, \quad EF - FE = \lambda^{-1}(K^2 - K^{-2}), \quad (10)
$$

involution $E^* = F, \quad K^* = K$, comultiplication

$$
\Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K,
$$

7
counit $\varepsilon(E) = \varepsilon(F) = \varepsilon(1 - K) = 0$ and antipode $S(K) = K^{-1}$, $S(E) = -qE$, $S(F) = -q^{-1}F$. There is a dual pairing $\langle \cdot, \cdot \rangle$ of the Hopf $*$-algebras $\mathcal{U}_q(\mathfrak{su}_2)$ and $\mathcal{O}(\mathfrak{su}_2(2))$ given on generators by

$$\langle K^{+1}, d \rangle = \langle K^{+1}, a \rangle = q^{+1/2}, \quad \langle E, c \rangle = \langle F, b \rangle = 1$$

and zero otherwise, where $a, b, c, d$ are the usual generators of $\mathcal{O}(\mathfrak{su}_2(2))$ (see e.g. [7, Chapter 4]).

We shall use the definition of the coordinate algebras $\mathcal{O}(S^2_{q,r})$, $r \in [0, \infty]$, of Podleś spheres as given in [13]. For $r \in [0, \infty)$, $\mathcal{O}(S^2_{q,r})$ is the $*$-algebra with generators $A = A^*, B, B^*$ and defining relations

$$AB = q^{-2}BA, \ AB^* = q^2B^*A, \ B^*B = A - A^2 + r, \ BB^* = q^2A^2 - q^{-1}A^2 + r. \quad (11)$$

For $r = \infty$, the defining relations of $\mathcal{O}(S^2_{q,\infty})$ are

$$AB = q^{-2}BA, \ AB^* = q^2B^*A, \ B^*B = -A^2 + 1, \ BB^* = -q^{-2}A^2 + 1. \quad (12)$$

Let $r < \infty$. As shown in [13], $\mathcal{O}(S^2_{q,r})$ is a right $\mathcal{O}(\mathfrak{su}_2(2))$-comodule $*$-algebra such that

$$x_{-1} := q^{-1}(1 + q^2)^{1/2}B, \ x_1 := -(1 + q^2)^{1/2}B^*, \ x_0 := 1 - (1 + q^2)A. \quad (13)$$

transform by the spin 1 matrix corepresentation $(t_{ij}^1)$ of $\mathfrak{su}_2(2)$. Hence $\mathcal{O}(S^2_{q,r})$ is a left $\mathcal{U}_q(\mathfrak{su}_2)$-module $*$-algebra with left action given by $f \cdot x_j = \sum_i x_i \langle f, t_{ij} \rangle$ for $f \in \mathcal{U}_q(\mathfrak{su}_2), j = -1, 0, 1$. Inserting the form of the matrix $(t_{ij}^1)$ (see [13] or [7, Subsection 4.5.1]) and the Hopf algebra pairing $\langle \cdot, \cdot \rangle$ into (5), we derive the following cross relations for the cross product algebra $\mathcal{O}(S^2_{q,r}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$:

$$KA = AK, \ EA = AE + q^{-1/2}B^*K, \ FA = AF - q^{-3/2}BK,$$

$$KB = q^{-1}BK, \ EB = qBE - q^{1/2}(1 + q^2)AK + q^{1/2}K, \ FB = qBF,$$

$$KB^* = qB^*K, \ EB^* = q^{-1}B^*E, \ FB^* = q^{-1}B^*F + q^{-1/2}(1 + q^2)AK - q^{-1/2}K.$$

That is, $\mathcal{O}(S^2_{q,r}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$ is the $*$-algebra with generators $A, B, B^*, E, F, K, K^{-1}$, with defining relations (10), (11) and the preceding set of cross relations.

For $r = \infty$, we set

$$x_{-1} := q^{-1}(1 + q^2)^{1/2}B, \ x_1 := -(1 + q^2)^{1/2}B^*, \ x_0 := -(1 + q^2)A. \quad (14)$$
Then the cross relations for $\mathcal{O}(S^2_{q\infty}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$ can be written as

$$KA = AK, \quad EA = AE + q^{-1/2}B^*K, \quad FA = AF - q^{-3/2}BK,$$
$$KB = q^{-1}BK, \quad EB = qBE - q^{1/2}(1 + q^2)AK, \quad FB = qBF,$$
$$KB^* = qB^*K, \quad EB^* = q^{-1}B^*E, \quad FB^* = q^{-1}B^*F + q^{-1/2}(1 + q^2)AK.$$  

There is an infinitesimal description of Podleś quantum spheres which was discovered in [3]. For $r \in [0, \infty]$, define an element $X_r \in \mathcal{U}_q(\mathfrak{su}_2)$ by

$$X_r = q^{1/2}(q^{-1} - q)^{-1}r^{-1/2}(1 - K^2) + EK + qFK, \quad r \in (0, \infty),$$
$$X_0 = 1 - K^2, \quad X_{\infty} = EK + qFK.$$  \hspace{1cm} (15)

Then $\Delta(X_r) = 1 \otimes X_r + X_r \otimes K^2$. As shown in [3], the right coideal $*$-subalgebra $\mathcal{X} := \{ x \in \mathcal{O}(\mathrm{SU}_q(2)) : \langle X_r, x_{(1)} \rangle x_{(2)} = 0 \}$ of infinitesimal invariants with respect to $X_r$ can be identified with the coordinate $*$-algebra $\mathcal{O}(S^2_{q_r})$ of Podleś quantum 2-spheres. The generators $x_{-1}, x_0, x_1$ of $\mathcal{O}(S^2_{q_r})$ are then identified with the elements

$$x_{-1} = (1 + q^{-2})^{1/2}(r^{1/2}a^2 + ac - qr^{1/2}c^2),$$
$$x_0 = (1 + q^{-2})^{1/2}(r^{1/2}ab + 1 + (q + q^{-1})bc - (1 + q^2)r^{1/2}dc),$$
$$x_1 = (1 + q^{-2})^{1/2}(r^{1/2}b^2 + bd - qr^{1/2}d^2) \quad \text{for } r < \infty;$$
$$x_{-1} = (1 + q^{-2})^{1/2}(a^2 - qc^2),$$
$$x_0 = (1 + q^{-2})(ab - q^2dc),$$
$$x_1 = (1 + q^{-2})^{1/2}(b^2 - qd^2) \quad \text{for } r = \infty.$$

3 Decomposition of $\mathcal{O}(\mathrm{SU}_q(2))$

Throughout this section, we denote by $\mathcal{X}$ the coordinate algebras $\mathcal{O}(S^2_{q_r})$ and by $\mathcal{A}$ and $\mathcal{U}$ the Hopf $*$-algebras $\mathcal{O}(\mathrm{SU}_q(2))$ and $\mathcal{U}_q(\mathfrak{su}_2)$, respectively.

First we recall a few crucial algebraic results from [12] needed in what follows. Let $\mathcal{C}$ denote the coalgebra $\mathcal{A}/\mathcal{X}^+\mathcal{A}$ with quotient map $\rho : \mathcal{A} \to \mathcal{C}$, where $\mathcal{X}^+ = \{ x \in \mathcal{X} : \varepsilon(x) = 0 \}$. By [12], $\mathcal{C}$ is spanned by group-like elements and, as a consequence, it is the direct sum of simple subcoalgebras $\mathcal{C}_j$, $j \in \frac{1}{2}\mathbb{Z}$. Set

$$M_j = \{ x \in \mathcal{A} : \rho(x_{(1)}) \otimes x_{(2)} \in \mathcal{C}_j \otimes \mathcal{A} \}.$$
Then $M_j$ is a finitely generated projective Hopf module in $\chi \mathcal{M}^A$ and $A$ is the direct sum of all $M_j$ (cf. [12, p. 163]). Since the dual pairing of $U$ and $A$ is non-degenerate [7, Section 2.4], it follows from Lemma 1.1 that Hopf modules in $\chi \mathcal{M}^A$ are in one-to-one correspondence with left $\chi \times U$-modules.

In this section, we reconsider these algebraic results in the Hilbert space setting. Since the Haar state on $A$ is faithful, we can equip $A$ with the inner product

$$\langle a, b \rangle := h(b^* a), \quad a, b \in A.$$  \hspace{1cm} (16)

The Heisenberg representation of $A$ is just the left $\chi \times U$-module $A$ endowed with the inner product (16). Hence each left $\chi \times U$-module $M_j$ corresponds to a $*$-representation, denoted by $\hat{\pi}_j$, of the cross product $*$-algebra $O(S^2_{qr}) \times U_q(su_2)$. We will return to this representation in Proposition 4.6 below.

In Theorem 3.1, we give a direct proof that $A = O(SU_q(2))$ is the orthogonal direct sum of Hopf modules $M_j$ and describe this decomposition explicitly. Moreover, the projections of the projective $\chi$-modules $M_j$ are computed.

Let us first introduce some more notation. We abbreviate

$$\lambda_\pm := 1/2 \pm (r + 1/4)^{1/2} \text{ for } r < \infty, \quad \lambda_\pm := \pm 1 \text{ for } r = \infty,$$

$$s := 0 \text{ for } r = 0, \quad s := -r^{-1/2} \lambda_- \text{ for } r \in (0, \infty), \quad s := 1 \text{ for } r = \infty.$$  

Note that $s = r^{1/2} \lambda_+^{-1}$ when $r \in [0, \infty)$. For $j \in \frac{1}{2} \mathbb{N}$, define

$$u_j := (d + q^{-1} sb)(d + q^{-2} sb) \ldots (d + q^{-2j} sb), \hspace{1cm} (17)$$

$$w_j := (a - qsc)(a - q^2 sc) \ldots (a - q^{2j} sc), \hspace{1cm} (18)$$

$$u_{-j} := E^{2j} w_j, \hspace{1cm} (19)$$

and set $u_0 = w_0 = 1$. From [2], we conclude that $\rho_L(u_j), \rho_L(w_j), j \in \frac{1}{2} \mathbb{N}_0$, are group like elements that span the coalgebra $A/\chi^+ A$, where $\rho_L : A \rightarrow A/\chi^+ A$ is the canonical mapping. (In order to apply the results in [2], one has to interchange the right $A$-comodule algebra $\chi$ with the left $A$-comodule algebra $\theta(\chi)$ using the $*$-algebra automorphism and coalgebra anti-homomorphism $\theta : A \rightarrow A$ determined by $\theta(a) = a, \theta(d) = d, \theta(b) = -qc, \theta(c) = -q^{-1}b$.) In particular, the simple coalgebras $C_j$ are given by $C_j = \mathbb{C}\rho_L(u_j)$ and $C_{-j} = \mathbb{C}\rho_L(w_j), j \in \frac{1}{2} \mathbb{N}_0$.

As a consequence, $u_j \in M_j$ for $j \in \frac{1}{2} \mathbb{Z}$. A crucial role will play the elements

$$v_{kj}^l := N_{kj}^l E^{l-k} (x_1^{l-|j|} u_j), \quad l \in \frac{1}{2} \mathbb{N}_0, \quad j, k = -l, -l + 1, \ldots, l.$$  \hspace{1cm} (20)
of \(A\), where \(N_{kj}^l = \|F^{l-k}(x_j\l h|u_j)\|^{-1}\). To describe the projection of the projective module \(M_j\), we define a \((2|j|+1)\times(2|j|+1)\)-matrix \(P_j\) with entries from \(A\) by

\[
P_j = \left( q^{-(n+m)[2|j|+1]} v_{nj}^{[j]} v_{mj}^{[j]} \right)_{n,m=-|j|}.
\]

**Theorem 3.1** The decomposition \(A = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} M_j\) is an orthogonal direct sum with respect to the inner product given by \([15]\). The set \(\{v_{kj}^l; l \in \mathbb{Z} \setminus 0, k, j = -l, -l+1, \ldots, l\}\) is an orthonormal basis of \(A\) and

\[
M_j = \text{Lin}\{v_{kj}^{l+n}; n \in \mathbb{N}_0, k = -(|j|+n), -(|j|+n) + 1, \ldots, |j|+n\}.
\]

As a left \(\mathcal{X}\)-module, \(M_j\) is generated by \(\{v_{kj}^l; k = -|j|, -|j|+1, \ldots, |j|\}\), the matrices \(P_j\) are orthogonal projections in \(M_{2|j|+1}(\mathcal{X})\), and \(M_j\) is isomorphic to \(\mathcal{X}^{2|j|+1}P_j\). Each vector \(v_{kj}^l\) is cyclic for the \(*\)-representation \(\hat{\pi}_j\) of \(\mathcal{X} \rtimes \mathcal{U}\) on \(M_j\).

**Proof.** Obviously, \(v_{kj}^l\) is a highest weight vector of weight \(l\) and the linear space \(V_j^l := \text{Lin}\{v_{kj}^l; k = -l, -l+1, \ldots, l\}\) is an irreducible \(\mathcal{U}_q(\mathfrak{su}_2)\)-module of spin \(l\). Hence \(v_{kj}^l\) and \(v_{kj'}^l\) are orthogonal whenever \(l \neq l'\) or \(k \neq k'\) because then they belong to representations of different spin or are vectors of different weights.

It remains to prove that \(v_{kj}^l\) and \(v_{kj'}^l\) are orthogonal when \(j \neq j'\). The idea of the proof [12] is to show that \(v_{kj}^l\) and \(v_{kj'}^l\) are eigenvectors of different eigenvalues of a hermitian operator \(\hat{X}_r\) acting on \(A\). Let \(\hat{X}_r\) be defined by

\[
\hat{X}_r(a) := a \circ X_r = (X_r, a_{(1)})a_{(2)}, \quad a \in A,
\]

where \(X_r \in \mathcal{U}_q(\mathfrak{su}_2)\) is given by \([15]\). The relation \(S(X_r)^* = S(X_r)\) implies that the operator \(\hat{X}_r\) is hermitian. Indeed, for \(a, b \in A\), we have

\[
\langle a, b \circ X_r \rangle = h((b \circ X_r)^*a) = h((b^* \circ S(X_r)^*)a) = h((b^* \circ S(X_r))a) = \varepsilon((X_r(2))h((b^* \circ S(X_r(1)))a) = h((b^* \circ S(X_r(1))X_r(2))(a \circ X_r(3))) = h(b^*(a \circ X_r)) = \langle a \circ X_r, b \rangle.
\]

For \(j \in \frac{1}{2}\mathbb{Z}\), set

\[
\mu_j := 1 - q^{2j} \quad \text{for} \quad r = 0,
\]

\[
\mu_j := q^{1/2}(q^{-1} - q)^{-1/2}(1 - q^{-2j} \lambda_+ - q^{2j} \lambda_-) \quad \text{for} \quad r \in (0, \infty),
\]

\[
\mu_j := q^{1/2}(q^{-1} - q)^{-1}(q^{-2j} - q^{2j}) \quad \text{for} \quad r = \infty.
\]
We claim that $u_j \circ X_r = \mu_j u_j$, where the vectors $u_j$, $j \in \frac{1}{2}\mathbb{Z}$, are defined by (17)–(19). Let $r \in (0, \infty)$. For $j = 0$, we have $u_0 \circ X_r = \varepsilon(X_r) = 0$. Assume that the assertion holds for $j \in \frac{1}{2}\mathbb{N}_0$. Using $\Delta(X_r) = 1 \otimes X_r + X_r \otimes K^2$ and $s = -r^{-1/2} \lambda_- = r^{-1/2} \lambda^+_1$, we compute

$$
u_{j+\frac{1}{2}} X_r = u_j \left((d + q^{-2}) \circ X_r\right) + u_j \circ X_r \left((d + q^{-2}) \circ K^2\right)$$

$$= u_j \left((q^{1/2} r^{-1/2} (q^2 - 1) q^{-1} (1 - q) - q^{1/2} r^{-1/2} q^{2j} \lambda_+ + q \mu_j) d\right)$$

$$+ (q^{1/2} r^{-1/2} (q^2 - 1) q^{-1} (1 - q) + q^{1/2} r^{-1/2} q^{2j} \lambda_+ + q^{-1} \mu_j) q^{-2} \circ X_r = u_{j+1/2} \left((d + q^{-2}) \circ X_r\right) = \mu_{j+1/2} u_{j+1/2}.$$ 

By induction, the claim follows for $j \in \frac{1}{2}\mathbb{N}_0$. Similarly, one proves that $w_j \circ X_r = \mu_{-j} w_j$ for $j \in \frac{1}{2}\mathbb{N}$. Since $(E^{2j} \circ w_j) \circ X_r = E^{2j} \circ (w_j \circ X_r)$, we obtain $u_j \circ X_r = \mu_j u_j$ for all $j \in \frac{1}{2}\mathbb{Z}$. Analogous, but simpler, computations show that the claim also holds for $r = 0$ and $r = \infty$. Using $x_1 \circ X_r = 0$, we obtain

$$\left(N_{k_j}^{j} \circ \left(x_1^{-1/|j|} u_j\right)\right) \circ X_r = N_{k_j}^{j} (x_1^{-1/|j|} u_j \circ X_r) = \mu_j N_{k_j}^{j} \circ (x_1^{-1/|j|} u_j),$$

so that $\tilde{X}_r(v_{k,j}^l) = \mu_j v_{k,j}^l$. Since $\tilde{X}_r$ is hermitian and $\mu_j \neq \mu_{j'}$, we conclude that $v_{k,j}^l$ and $v_{k,j'}^l$ are orthogonal whenever $j \neq j'$.

The decomposition of $\mathcal{A}$ into an orthogonal direct sum is a consequence of above results. Let $t_{k,j}^l \in \mathcal{A}$, $l \in \frac{1}{2}\mathbb{N}_0$, $k, j = -l - 1, \ldots, l$, denote the matrix elements from the Peter-Weyl decomposition of $\mathcal{A}$ [7, Section 4.2]. As $v_{k,j}^l$ is a weight vector with weight $k$ of a spin $l$ representation of $U_q(\mathfrak{s}u_2)$, we know that $v_{k,j}^l \in \text{Lin}\{t_{k,i}^l : i = -l - 1, \ldots, l\}$. A simple dimension argument shows that $\text{Lin}\{v_{k,i}^l : i = -l - 1, \ldots, l\} = \text{Lin}\{t_{k,i}^l : i = -l - 1, \ldots, l\}$. Since the elements $t_{k,j}^l$ span $\mathcal{A}$, we conclude that $\{v_{k,j}^l : l \in \frac{1}{2}\mathbb{N}_0, k, j = -l - 1, \ldots, l\}$ is an orthonormal basis of $\mathcal{A}$. Recall that $v_{|j|,j}^{|j|} = N_{|j|,j}^{|j|} \circ u_j \in M_j$. As $M_j \in \mathcal{A} M^4$, it follows by the definition of $v_{k,j}^l$ that $v_{k,j}^l \in M_j$ for all $l = |j|, |j| + 1, \ldots$ and $k = -l, -l + 1, \ldots$. Since $\mathcal{A}$ is the direct sum of $M_j$, we conclude that $M_j = \text{Lin}\{v_{k,j}^l : l = |j|, |j| + 1, \ldots, k = -l, -l + 1, \ldots\}$ and the decomposition $\mathcal{A} \oplus_{j \in \frac{1}{2}\mathbb{Z}} M_j$ is an orthogonal sum.

Writing

$$v_{k,j}^l = N_{k,j}^l \circ (F^{l-k} \circ x_1^{-1/|j|}) \circ (F^{l-k} \circ u_j)$$

$$= N_{k,j}^l \circ (F^{l-k} \circ x_1^{-1/|j|}) \circ (F^{l-k} \circ u_j) \circ v_{|j|,j}^{|j|}$$

12
and keeping in mind that \((F^l - k)(2^j v_{|j,j|}) \in \text{Lin}\{v_{|j,j|}; k = -|j|, -|j| + 1, \ldots, |j|\}\), it is clear that \(M_j\) is generated by \(\{v_{|j,j|}; k = -|j|, -|j| + 1, \ldots, |j|\}\) as a left \(\mathcal{X}\)-module and that \(v_{|j,j|}\) is cyclic for the \(*\)-representation \(\hat{\pi}_j\) of \(\mathcal{X} \rtimes \mathcal{U}\) on \(M_j\).

We turn now to the projections of the projective modules \(M_j\). Defining

\[ v_j := [2|j| + 1]^{-1/2}(q_{|j|}v_{-|j,j|}, \ldots, q_{|j|}v_{|j,j|})^t, \]

we can write \(P_j = v_j v_j^*\). This immediately implies that \(P_j^* = P_j\). In order to prove \(P_j^2 = P_j\), it is sufficient to show that \(v_j^* v_j = 1\). Recall that \(K v_{|j,j|} = q^j v_{|j,j|}\) and \(F v_{|j,j|} = [l+k]^{1/2}[l-k+1]^{1/2} v_{k-1,j}\) (see also Equation (23) below). By the third equation in (1), \(K v_{|j,j|} = q^{-k} v_{|j,j|}\) and \(F v_{|j,j|} = -q^{-l-k-2} v_{k,j}\). From this, we conclude that \(v_j^* v_j = [2|j| + 1]^{-1}(q_{|j|} v_{-|j,j|}^* v_{|j,j|} + \ldots + q_{-|j|} v_{|j,j|}^* v_{|j,j|})\) is a linear combination of vectors of weight 0. We show that \(v_j^* v_j\) belongs to a spin 0 representation. Since \(K v_{|j,j|} = v_j^* v_j\), this is equivalent to \(F v_{|j,j|} = 0\).

Inserting the expressions for \(v_j\) gives

\[ F v_{|j,j|} = \sum_{k=-|j|}^{+|j|} q^{-k} [2|j| + 1]^{-1} \left((F v_{|j,j|}) (K v_{|j,j|}) + (K^{-1} v_{|j,j|}) (F v_{|j,j|})\right) \]

\[ = [2|j| + 1]^{-1} \sum_{k=-|j|}^{+|j|} \left( -q^{-k} [2|j| + 1]^{1/2} v_{k+1,j}^* v_{|j,j|} + q^{-k} [2|j| + 1]^{1/2} v_{k+1,j}^* v_{|j,j|}\right) \]

which telescopes to zero. Since \(v_j^* v_j\) belongs to a spin 0 representation and \(h(v_{|j,j|} v_{|j,j|}) = ||v_{|j,j|}||^2 = 1\), we have \(v_j^* v_j = h(v_j^* v_j) = \sum_{k=-|j|}^{+|j|} q^{-2k} [2|j| + 1]^{-1} = 1\) by (21) as desired. Hence \(P_j^2 = P_j\) is an orthogonal projection.

Next we verify that \(v_{|j,j|} v_{|j,j|}^*\) belongs to \(\mathcal{X}\). In order to do so, we use the fact that \(\mathcal{X}\) is the set of elements \(x \in \mathcal{A}\) such that \(x \circ X_r = 0\). Since \(v_{|j,j|} X_r = \hat{X}_r (v_{|j,j|}) = \mu_j v_{|j,j|}^*\) and \(v_{|j,j|}^* = N_{|j,j|} S(F^{|j| - m}) \circ u_{|j,j|}^*\), we get

\[ (v_{|j,j|} v_{|j,j|}^*) \circ X_r = N_{|j,j|} v_{|j,j|}^* (S(F^{|j| - m}) \circ u_{|j,j|}^*) \circ X_r + (v_{|j,j|} X_r) (S(F^{|j| - m}) \circ u_{|j,j|}^*) \circ K^2 \]

\[ = N_{|j,j|} v_{|j,j|}^* S(F^{|j| - m}) \circ u_{|j,j|}^* \circ X_r + \mu_j u_{|j,j|}^* \circ K^2. \]

Hence it suffices to show that \(u_{|j,j|}^* \circ X_r + \mu_j u_{|j,j|}^* \circ K^2 = 0\) for all \(j \in \frac{1}{2} \mathbb{Z}\). This can be done by induction. Let \(r \in (0, \infty)\). For \(j = 0\), the assertion is true since
This completes the proof. \( X \) -module, so \( \Psi \) 

Hence \( u^* \) 

Recall that \( u \) 

Finally we prove that \( u^* \circ X_r = 0 \) and \( \mu_0 = 0 \). Assume that \( u^*_j \circ X_r + \mu_j u^*_j \circ K^2 = 0 \) holds for \( j \in \frac{1}{2}\mathbb{N}_0 \). Then \( u^*_j \circ X_r = -\mu_j (u^*_j \circ K^2) \). As \( u^*_{j+1/2} = (a - q^{-2j}sc)u^*_j \), we compute

\[
\begin{align*}
&u^*_{j+1/2} \circ X_r + \mu_{j+1/2} (u^*_{j+1/2} \circ K^2) = (a - q^{-2j}sc)(u^*_j \circ X_r) \\
&+ (a - q^{-2j}sc) \circ X_r (u^*_j \circ K^2) + \mu_{j+1/2} (a - q^{-2j}sc) \circ K^2 (u^*_j \circ K^2) \\
&= \left[ -\mu_j (a - q^{-2j}sc) + (q^{1/2}r^{-1/2}(q^{-1} - q)^{-1}(1 - q^{-1}) - q^{-1/2}q^{-2j}s)a \\
&- (q^{1/2}r^{-1/2}(q^{-1} - q)^{-1}(1 - q) - q^{2j+3/2}s^{-1}) q^{-2j}sc \\
&+ \mu_{j+1/2} (q^{-1}a - q^{-2j+1}sc) \right] (u^*_j \circ K^2).
\end{align*}
\]

Inserting the expressions for \( s, \mu_j \) and \( \mu_{j+1/2} \), the preceding equation yields zero. This proves that \( u^*_j \circ X_r + \mu_j u^*_j \circ K^2 = 0 \) and hence \( (v^{|j|}_{nl}v^{|j|}_{mj}) \circ X_r = 0 \) for \( j \in \frac{1}{2}\mathbb{N}_0 \). In the same way, one can show that \( u^*_j \circ X_r + \mu_{j+1/2} u^*_j \circ K^2 = 0 \) for \( j \in \frac{1}{2}\mathbb{N}_0 \).

Since \( u^*_j \circ X_r + \mu_j u^*_j \circ K^2 = S(E^{2j})^* [w^*_j \circ X_r + \mu_{j+1/2} w^*_j \circ K^2] \), we conclude that \( u^*_j \circ X_r + \mu_j u^*_j \circ K^2 = 0 \) holds for all \( j \in \frac{1}{2}\mathbb{Z} \). For \( r = 0 \) and \( r = \infty \), the proof is similar.

Finally we prove that \( M_j \) is isomorphic to \( \mathcal{X}^{2|j|+1}P_j \) as a left \( \mathcal{X} \)-module. Define a mapping \( \Psi_j : \mathcal{X}^{2|j|+1}P_j \to M_j \) by

\[
\Psi_j((y_{-|j|}, \ldots, y_{|j|})P_j) := [2|j|+1]^{-1/2} \sum_{k=-|j|}^{|j|} q^{-k}y_kv^{|j|}_{kj}.
\]

Recall that \( P_j = v_jv^*_j \) and \( v^*_jv_j = 1 \). Suppose we are given \( y_{-|j|}, \ldots, y_{|j|} \in \mathcal{X} \) such that \( (y_{-|j|}, \ldots, y_{|j|})P_j = 0 \). Multiplying by \( v_j \) from the right yields

\[
0 = (y_{-|j|}, \ldots, y_{|j|})P_jv_j = (y_{-|j|}, \ldots, y_{|j|})v_j(v^*_jv_j) = [2|j|+1]^{-1/2} \sum_{k=-|j|}^{|j|} q^{-k}y_kv^{|j|}_{kj}.
\]

Hence \( \Psi_j \) is well defined. Furthermore, \( [2|j|+1]^{-1/2} \sum_{k=-|j|}^{|j|} q^{-k}y_kv^{|j|}_{kj} = 0 \) implies

\[
0 = ([2|j|+1]^{-1/2} \sum_{k=-|j|}^{|j|} q^{-k}y_kv^{|j|}_{kj})v^*_j = (y_{-|j|}, \ldots, y_{|j|})P_j,
\]

so \( \Psi_j \) is injective. Since \( M_j \) is generated by \( \{v^{|j|}_{kj} : k = -|j|, \ldots, |j| \} \) as a left \( \mathcal{X} \)-module, \( \Psi_j \) is also surjective. Whence \( \Psi_j \) realizes the desired isomorphism. This completes the proof. \( \square \)
4 Integrable ∗-representations of the cross product algebras

4.1 Classification of integrable ∗-representations of the cross product algebra $\mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$

For $l \in \frac{1}{2} \mathbb{N}_0$, let $T_l$ denote the type 1 spin $l$ representations of $\mathcal{U}_q(\mathfrak{su}_2)$. Recall that $T_l$ is an irreducible ∗-representation of the ∗-algebra $\mathcal{U}_q(\mathfrak{su}_2)$ acting on a $(2l+1)$-dimensional Hilbert space with orthonormal basis $\{v_j^l : j = -l, -l+1, \ldots, l\}$ by the formulas (see, for instance, [7, Subsection 3.2.1])

$$Kv_j^l = q^j v_j^l, \quad Ev_j^l = [l-j+1]^{1/2}v_{j+1}^l, \quad Fv_j^l = [l+j+1]^{1/2}v_{j-1}^l.$$ (23)

As mentioned in the introduction, we call a ∗-representation of $\mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$ integrable if it has the following property:

The restriction to $\mathcal{U}_q(\mathfrak{su}_2)$ is the direct sum of representations $T_l$, $l \in \frac{1}{2} \mathbb{N}_0$.

The aim of this subsection is to classify all integrable ∗-representations of the cross product ∗-algebra $\mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$. We shall use the generators $x_j$ defined by (13) (resp. (14)) rather than $A, B, B^*$.

Suppose we have such a representation acting on the domain $\mathcal{H}$. Then $\mathcal{H}$ can be written as $\mathcal{H} = \bigoplus_{l \in \frac{1}{2} \mathbb{N}_0} V_l$ such that $V_l = \bigoplus_{j=-l}^l V_j^l$, where each $V_j^l$ is the same Hilbert space $V_j^1$, say, and the generators of $\mathcal{U}_q(\mathfrak{su}_2)$ act on $V_j^l$ by (23). We claim that there exist operators $\alpha^\pm(l, j) : V_j^l \to V_j^{l+1}, \alpha^0(l, j) : V_j^l \to V_j^{l+1}, \beta^+(l, j) : V_j^l \to V_j^{l+1}$, and self-adjoint operators $\beta^0(l, j) : V_j^l \to V_j^l$ such that

$$x_1 v_j^l = \alpha^+(l, j) v_j^l + \alpha^0(l, j) v_j^l + \alpha^-(l, j) v_j^l,$$ (24)

$$x_0 v_j^l = \beta^+(l, j) v_j^l + \beta^0(l, j) v_j^l + \beta^-(l, j) v_j^l,$$ (25)

$$x_{-1} v_j^l = -q^{-1} \left( \alpha^-(l+1, j-1)^* v_j^l + \alpha^0(l, j-1)^* v_j^l + \alpha^+(l-1, j-1)^* v_j^l \right).$$ (26)

for $v_j^l \in V_j^l$. Indeed, let $v_j^l \in V_j^l$. Since $Kx_j = qx_j K$, $x_1 v_j^l$ is a weight vector with weight $j+1$, and since $E^{l-j+1} x_1 v_j^l = q^{-l+j-1} x_1 E^{l-j+1} v_j^l = 0$, $x_1 v_j^l$ is in the linear span of vectors $w_j^{r+1} \in V_j^{r+1}$, where $r \leq l+1$. Similarly, replacing $x_1$ by $x_{-1}$ and $E$ by $F$, we conclude that $x_{-1} v_j^l$ belongs to the span of vectors $w_j^{r-1} \in V_j^{r-1}$, $r \leq l+1$. Therefore, since $x_{-1} = -q^{-1} x_1^*$, we have $x_{\pm 1} v_j^l \in V_j^{l-1} \oplus V_j^{l+1} \oplus V_j^{l+1}$, so $x_{\pm 1} v_j^l$ is of the form (24) (resp. (26)). From the last two relations of (11) (resp. (12)) and from $x_0 v_j^l = x_0$, it follows that $x_0 v_j^l$ is of the form (25). Note that all
operators $\alpha^\pm(l,j)$, $\alpha^0(l,j)$, $\beta^+(l,j)$, $\beta^0(l,j)$ are bounded because the operators $x_{-1}$, $x_0$, $x_1$ are bounded for any *-representation of the *-algebra $\mathcal{O}(S^2_{qr})$.

Inserting (23) and (24) into the equation $Ex_1v_j^l = q^{-1}x_1Ev_j^l$, we get

$$[l-j]^{1/2}[l+j+3]^{1/2}\alpha^+(l,j) = q^{-1}[l-j]^{1/2}[l+j+1]^{1/2}\alpha^+(l,j+1),$$
$$[l-j-1]^{1/2}[l+j+2]^{1/2}\alpha^0(l,j) = q^{-1}[l-j]^{1/2}[l+j+1]^{1/2}\alpha^0(l,j+1),$$
$$[l-j-2]^{1/2}[l+j+1]^{1/2}\alpha^-(l,j) = q^{-1}[l-j]^{1/2}[l+j+1]^{1/2}\alpha^-(l,j+1).$$

The solutions of these recurrence relations are given by

$$\alpha^+(l,j) = q^{-l+j}[l+j+1]^{1/2}[l+j+2]^{1/2}[2l+1]^{-1/2}[2l+2]^{-1/2}\alpha^+(l,l),$$
$$\alpha^0(l,j) = q^{-l+j+1}[l-j]^{1/2}[l+j+1]^{1/2}[2l]^{-1/2}\alpha^0(l,l-1),$$
$$\alpha^-(l,j) = q^{-l+j+2}[l-j-1]^{1/2}[l-j]^{1/2}[2l]^{-1/2}\alpha^-(l,l-2).$$

Similarly, from the equation $Ex_0v_j^l = (x_0E + [2]^{1/2}x_1K)v_j^l$, we obtain

$$[l-j+1]^{1/2}[l+j+2]^{1/2}\beta^+(l,j) = [l-j]^{1/2}[l+j+1]^{1/2}\beta^+(l,j+1) + [2]^{1/2}q^l\alpha^+(l,j),$$
$$[l-j]^{1/2}[l+j+1]^{1/2}\beta^0(l,j) = [l-j]^{1/2}[l+j+1]^{1/2}\beta^0(l,j+1) + [2]^{1/2}q^l\alpha^0(l,j).$$

The equation $Ex_0v_j^l = (x_0E + [2]^{1/2}x_1K)v_j^l$ yields in addition

$$\beta^+(l,l) = q^{l}[2]^{1/2}[2l+2]^{-1/2}\alpha^+(l,l).$$

(27)

Further, the equation $0 = \langle v_j^l, (x_1F + q[2]^{1/2}x_0K - qFx_1)v_j^l \rangle$ implies that

$$\alpha^0(l,l-1) = -[2]^{1/2}[2l]^{-1/2}q^{l+1}\beta^0(l,l).$$

(28)

As a consequence, $\alpha^0(l,l-1)$ is self-adjoint. Using (27) and (28), it follows that the above recurrence relations for $\beta^+(l,j)$ and $\beta^0(l,j)$ have the following solutions:

$$\beta^+(l,j) = q^j[l-j+1]^{1/2}[l+j+1]^{1/2}[2]^{1/2}[2l+1]^{-1/2}[2l+2]^{-1/2}\alpha^+(l,l),$$

(29)

$$\beta^0(l,j) = (1 - q^{l+j+1}[l-j][2][2l]^{-1})\beta^0(l,l).$$

(30)

From $0 = \langle v_{j-1}^l, (Ex_{-1} - qx_{-1}E - [2]^{1/2}x_0K)v_{j-1}^l \rangle$, we derive $\beta^+(l-1,l-1) = -q^{-l}[2l-1]^{1/2}\alpha^-(l,l-2)$. Combining this equation with (27) gives

$$\alpha^-(l,l-2) = -q^{2l-1}[2]^{1/2}[2l-1]^{-1/2}[2l]^{-1/2}\alpha^+(l-1,l-1).$$

(31)
From (27)–(31), it follows that the representation is completely described if the operators $\alpha^+(l, l)$ and $\beta^0(l, l)$ for $l \in \frac{1}{2}\mathbb{N}_0$ are known. Our next aim is to determine these operators. This will be done by induction on $l$, where consecutive steps increase $l$ by 1.

We begin by analyzing the relations between $\beta^0(l, l)$ and $\alpha^+(l, l)$. Let $r < \infty$ and abbreviate $\rho := 1 + [2]^2 r$. On $V_l^\dagger$, the two equations $BB^* = r + q^2 A - q^4 A^2$ and $B^* B = r + A - A^2$ lead to the operator relations

\[
|\alpha^+(l-1, l-1)|^2 + \alpha^0(l, l-1)^2 + |\alpha^-(l+1, l-1)|^2 = (1 + q^2)^{-1} \left( q^2 \rho + (1 - q^2) \beta^0(l, l) - (\beta^0(l, l)^2 + |\beta^+(l, l)|^2) \right),
\]

\[
|\alpha^+(l, l)|^2 = (1 + q^2)^{-1} \left( q^2 \rho - (1 - q^2) \beta^0(l, l) - q^4(\beta^0(l, l)^2 + |\beta^+(l, l)|^2) \right).
\]

Here it is understood that $\alpha^0(0, -1) = 0$ and $\alpha^+(l-1, l-1) = 0$ for $l = 0, 1/2$. Inserting (27), (28) and (31), these two relations give after some calculations

\[
|\alpha^+(l-1, l-1)|^2 + q^{2l+1}[2l+1]^{-1}[2l+2]^{-1}[2l+3]^{-1} \alpha^+(l, l)^2 = [2]^{-1} (q \rho + (q^{-1} - q) \beta^0(l, l) - q^2[2l+2]^{-1}(2l+3)^{-1} \beta^0(l, l)^2),
\]

\[
|\alpha^+(l, l)|^2 = [2]^{-1} (\rho - (1 - q^2) \beta^0(l, l) - q^2 \beta^0(l, l)^2).
\]

Eliminating $\alpha^+(l, l)$ from these equations yields

\[
[2][2l+1] |\alpha^+(l-1, l-1)|^2 = [2l] \rho + (1 - q^2) [2l+2] \beta^0(l, l) - [2l]^{-1} [2l+2] q^2 \beta^0(l, l)^2.
\]

(34)

Now we start with the induction procedure. Our first aim is to show that $\beta^0(0, 0) = 0$. A computation similar to the above shows that

\[
q[2]^{-1} [3] |\alpha^+(0, 0)|^2 = (1 + q^2)^{-1} (q^2 \rho - (1 - q^2) q^2 \beta^0(0, 0) - q^4 \beta^0(0, 0)^2),
\]

\[
q[2]^{-1} [3] |\alpha^+(0, 0)|^2 = (1 + q^2)^{-1} (q^2 \rho + (1 - q^2) \beta^0(0, 0) - \beta^0(0, 0)^2).
\]

Eliminating $|\alpha^+(0, 0)|^2$ gives $0 = \beta^0(0, 0) - \beta^0(0, 0)^2$. Since $\beta^0(0, 0)$ is self-adjoint, it is an orthogonal projection. Assume to the contrary that $\beta^0(0, 0) \neq 0$. Then there is $v_0^0 \in V_0^0$ such that $\beta^0(0, 0) v_0^0 = v_0^0$ and $\|v_0^0\| = 1$. Note that $h(\cdot) := \langle \cdot, v_0^0 \rangle$ is the unique $U_q(\mathfrak{su}_2)$-invariant state on $\mathcal{O}(S^2_{q^2})$. From this, we conclude $1 = \langle \beta^0(0, 0) v_0^0, v_0^0 \rangle = \langle x_0 v_0^0, v_0^0 \rangle = h(x_0) = 0$, which is a contradiction. Thus $\beta^0(0, 0) = 0$.

For $l = 1/2$, Equation (34) becomes

\[
0 = (q[3] \beta^0(1/2, 1/2))^2 - (1 - q^2) [3] \beta^0(1/2, 1/2) - \rho.
\]
This operator identity can only be satisfied if the self-adjoint operator \( \beta^0(1/2, 1/2) \) has purely discrete spectrum with eigenvalues

\[
\beta^0(1/2, 1/2)_{1/2}^\pm := [3]^{-1}(q^{-2}\lambda_\pm - \lambda_\mp).
\]

Clearly, \( \beta^0(1/2, 1/2)_{1/2}^+ \neq \beta^0(1/2, 1/2)_{1/2}^- \). Denoting the corresponding eigenspaces by \( \mathcal{K}_{-1/2} \) and \( \mathcal{K}_{1/2} \), we can write \( V_{1/2}^1 = \mathcal{K}_{-1/2} \oplus \mathcal{K}_{1/2} \), and \( \beta^0(1/2, 1/2) \) acts on \( V_{1/2}^1 \) by

\[
\beta^0(1/2, 1/2)w_{-1/2} = \beta^0(1/2, 1/2)_{-1/2}w_{-1/2}, \quad w_{-1/2} \in \mathcal{K}_{1/2}, \\
\beta^0(1/2, 1/2)w_{1/2} = \beta^0(1/2, 1/2)_{1/2}^+w_{1/2}, \quad w_{1/2} \in \mathcal{K}_{1/2}.
\]

Here we do not exclude the cases \( \mathcal{K}_{-1/2} = \{0\} \) and \( \mathcal{K}_{1/2} = \{0\} \).

Next let \( l \in \frac{1}{2}\mathbb{N}_0 \). Assume that there exist (possibly zero) Hilbert spaces \( \mathcal{K}_{l-1}, \mathcal{K}_{l-1} \oplus \ldots \oplus \mathcal{K}_{l} \) such that \( V_l^1 = \mathcal{K}_{l-1} \oplus \ldots \oplus \mathcal{K}_{l} \). For \( j \in \frac{1}{2}\mathbb{N}_0, j \leq l \), set

\[
\beta^0(l, l)_{j}^\pm = [2l+2]^{-1/2}([2j][q^{-2}\lambda_\pm - \lambda_\mp] - (1 - q^{-2})[l-j][l+j+1]),
\]

\[
\alpha^+(l, l)_{j}^\pm = \sqrt{2}[l+3]^{-1/2}[2l+2]^{-1/2}([2l+2]^2(c+1/4) - (q^{-1} - q)[l-j][l+j+1]/2 \pm [2j](c+1/4)^2)^{1/2}.
\]

where we already inserted (35) into (33). Assume that \( \beta^0(l, l) \) acts on \( V_l^1 \) by

\[
\beta^0(l, l)w_{-j} = \beta^0(l, l)_{j}^-w_{-j}, \quad w_{-j} \in \mathcal{K}_{-j}, \quad \beta^0(l, l)w_{j} = \beta^0(l, l)_{j}^+w_{j}, \quad w_{j} \in \mathcal{K}_{j}.
\]

We show that there exist Hilbert spaces \( \mathcal{K}_{l+1}^{l+1} \) and \( \mathcal{K}_{l+1} \) such that, up to unitary equivalence, \( V_{l+1}^{l+1} = \mathcal{K}_{l+1}^{l+1} \oplus \ldots \oplus \mathcal{K}_{l} \oplus \mathcal{K}_{l+1} \), the operator \( \beta^0(l+1, l+1) \) acts on \( V_{l+1}^{l+1} \) by the formulas (37), and \( \alpha^+(l, l) : V_l^1 \to V_{l+1}^{l+1} \) is given by

\[
\alpha^+(l, l)w_{-j} = \alpha^+(l, l)_{j}^-w_{-j}, \quad w_{-j} \in \mathcal{K}_{-j}, \quad \alpha^+(l, l)w_{j} = \alpha^+(l, l)_{j}^+w_{j}, \quad w_{j} \in \mathcal{K}_{j},
\]

where \( j = -l, -l+1, \ldots, l \).

Observe that \( \ker B = \{0\} \) (cf. Subsection 5.3 below). Hence \( \ker \alpha^+(l, l) = \{0\} \). Let \( \alpha^+(l, l) = U[\alpha^+(l, l)] \) denote the polar decomposition of \( \alpha^+(l, l) \). Then \( U \) is an isometry from \( V_l^1 \) onto \( \overline{\alpha^+(l, l)V_l^1} \). On the other hand, we have the decomposition \( V_{l+1}^{l+1} = \alpha^+(l, l)V_l^1 \oplus \ker \alpha^+(l, l)^* \). After applying a unitary transformation, we can assume that \( V_{l+1}^{l+1} = V_l^1 \oplus \ker \alpha^+(l, l)^* = \mathcal{K}_{l-1} \oplus \ldots \oplus \mathcal{K}_l \oplus \ker \alpha^+(l, l)^* \) and \( \alpha^+(l, l)v_l^1 = |\alpha^+(l, l)|v_l^1 \) for all \( v_l^1 \in V_l^1 \). From (33) and (37), it follows that the
action of $\alpha^+(l,l)$ on $V_l^l$ is determined by Equation (38) as asserted. We proceed by describing the action of $\beta^0(l+1,l+1)$. By (34), $|\alpha^+(l,l)^*|^2$ commutes with $\beta^0(l+1,l+1)$. Hence $\beta^0(l+1,l+1)$ leaves the subspaces $V_l^l$ and $\ker \alpha^+(l,l)^*$ of $V_{l+1}^{l+1}$ invariant. Let $\tilde{\beta}^0(l+1,l+1)$ and $\beta^0(l+1,l+1)$ denote the restrictions of $\beta^0(l+1,l+1)$ to $V_l^l$ and $\ker \alpha^+(l,l)^*$, respectively. Evaluating the relation $0 = ((x_0x_1 - q^2x_1x_0 - (1 - q^2)x_1)v^l_l, v^l_{l+1})$ for $v^l_l \in V_l^l$ and $v^l_{l+1} \in V_{l+1}^{l+1}$ yields

$$\beta^0(l+1,l+1)\alpha^+(l,l) - q^2\alpha^+(l+1,l+1)\beta^0(l,l) - (1 - q^2)\alpha^+(l,l) = 0.$$ 

Inserting (27) and observing that $\alpha^+(l,l)\beta^0(l,l) = \beta^0(l,l)\alpha^+(l,l)$ by (37) and (38), we deduce the following operator equation on $V_l^l$

$$\{(1 + q^24^2)(2[2l+2]^{-1})\beta^0(l+1,l+1) - q^2(\alpha^+(l,l) - (1 - q^2))\alpha^+(l,l) = 0.$$

Since $V_l^l = \alpha^+(l,l)V_l^l$, the operator in braces must be zero. This implies that

$$\tilde{\beta}^0(l+1,l+1) = [2l+4]^{-1}[2l+2](\beta^0(l,l) + q^{-2} - 1)$$

Hence the operator $\beta^0(l+1,l+1)$ acts on $V_l^l = \mathcal{K}_{-l} \oplus \ldots \oplus \mathcal{K}_l$ by

$$\beta^0(l+1,l+1)w_j = [2l+4]^{-1}[2l+2](\beta^0(l,l)_j + q^{-2} - 1)w_j$$

$$= [2l+4]^{-1}[2l+2](j)(q^{-2}\lambda_\epsilon - \lambda_{-\epsilon} + (q^{-2} - 1)[l - j + 1][l + j + 2])w_j,$$

where $\epsilon = \text{sign}(j)$. The last equation is obtained by straightforward computations. On $\ker \alpha^+(l,l)^*$, Equation (34) reads

$$0 = \rho + (1 - q^2)(2l+2)^{-1}[2l+4]\tilde{\beta}^0(l+1,l+1) - ([2l+2]^{-1}[2l+4]q\tilde{\beta}^0(l+1,l+1))_j.$$

Since the solution of the quadratic equation $0 = -\rho - (q^{-1} - q)t + t^2$ is given by $t^{\pm} = q^{-1}\lambda_\pm - q\lambda_\mp$, we conclude that $\beta(l+1,l+1)$ has purely discrete spectrum consisting of the two distinct eigenvalues

$$\beta^0(l+1,l+1)_j^\pm = [2l+4]^{-1}[2l+2](q^{-2}\lambda_\pm - \lambda_{\mp}),$$

and $\ker \alpha^+(l,l)^*$ splits into the direct sum $\ker \alpha^+(l,l)^* = \mathcal{K}_{-(l+1)} \oplus \mathcal{K}_{l+1}$, where $\mathcal{K}_{-(l+1)}$ and $\mathcal{K}_{l+1}$ denote the eigenspaces of $\tilde{\beta}^0(l+1,l+1)$ corresponding to the eigenvalues $\beta^0(l+1,l+1)_l^-$ and $\beta^0(l+1,l+1)_l^+$, respectively. Accordingly, the operator $\beta^0(l+1,l+1)$ acts on $\ker \alpha^+(l,l)^*$ by

$$\beta^0(l+1,l+1)w_{-(l+1)} = \beta^0(l+1,l+1)_l^-w_{-(l+1)}, \quad w_{-(l+1)} \in \mathcal{K}_{-(l+1)},$$

$$\beta^0(l+1,l+1)w_{l+1} = \beta^0(l+1,l+1)_l^+w_{l+1}, \quad w_{l+1} \in \mathcal{K}_{l+1}.$$
This shows that $\beta^0(l+1, l+1)$ is of the same form as $\beta^0(l, l)$. By induction, the operators $\beta^0(l, l)$ and $\alpha^+(l, l)$ are now completely determined.

In the case $r = \infty$, there are only minor changes in the preceding reasoning. Set $\rho = (q + q^{-1})^2$. Then, Equations (33)–(34) remain valid if one omits the first order term of $\beta^0(l, l)$. Equations (35) and (36) become

$$\beta^0(l, l)_j^\pm = \pm q^{-1}[2][2j][2l + 2]^{-1},$$

$$\alpha^+(l, l)_j^\pm = [2]^{1/2}[2(l + j + 1)]^{1/2}[2(l - j + 1)]^{1/2}[2l + 2]^{-1/2}[2l + 3]^{-1/2},$$

and $\alpha^+(l, l)$ is again given by (38).

Note that whenever a Hilbert space $K_j$ is non-zero, it appears as a direct summand in each $V_l^k$, where $l = |j|$, $|j| + 1, \ldots$ and $k = -l, -l + 1, \ldots, l$. Moreover, the generators of $\mathcal{O}(S^2_{qr}) \times U_q(\mathfrak{su}_2)$ leave this decomposition invariant. Hence the representation of $\mathcal{O}(S^2_{qr}) \times U_q(\mathfrak{su}_2)$ on $\bigoplus_{l \in \mathbb{Z}_0} \bigoplus_{k = -l}^l V_l^k$ splits into a direct sum of representations on $\bigoplus_{n \in \mathbb{Z}_0} \bigoplus_{k = -(|j| + n)}^{|j| + n} K_j^{j + n}$, where each $K_j^{j + n}$ is the same Hilbert space $K_j$ and is considered as a direct summand of $V_l^k$.

It still remains to prove that Equations (23)–(26) define a representation of $\mathcal{O}(S^2_{qr}) \times U_q(\mathfrak{su}_2)$ when we insert the expressions for the operators obtained in the previous discussion. This can be done by showing that the defining relations of $\mathcal{O}(S^2_{qr})$ and the cross relations of $\mathcal{O}(S^2_{qr}) \times U_q(\mathfrak{su}_2)$ are satisfied. We have checked this; the details of these lengthy and tedious computations are omitted.

We summarize the outcome of above considerations in the next theorem.

**Theorem 4.1** Each integrable $*$-representation of the cross product $*$-algebra $\mathcal{O}(S^2_{qr}) \times U_q(\mathfrak{su}_2)$ is, up to unitary equivalence, a direct sum of representations $\pi_j^\pm$, $j \in \frac{1}{2}\mathbb{N}_0$, of the following form:

The domain is the direct sum $\bigoplus_{l \in \mathbb{Z}_0} \bigoplus_{k = -(|j| + 1)}^{|j| + 1} K_j^{j + l}$, where each $K_j^{j + l}$ is the same Hilbert space $K$. The generators $E$, $F$, $K$ of $U_q(\mathfrak{su}_2)$ act on $\bigoplus_{l \in \mathbb{Z}_0} \bigoplus_{k = -(|j| + 1)}^{|j| + 1} K_j^{j + l}$ by (23). The actions of the generators $x_1$, $x_0$, $x_{-1}$ of $\mathcal{O}(S^2_{qr})$ are determined by

$$x_1 v_k^l = q^{-l-k}[l+k+1]^{1/2}[l+k+2]^{1/2}[2l+1]^{-1/2}[2l+2]^{-1/2} \alpha^0(l, l)_{j}^\pm v_{k+1}^l \pm 1,$$

$$- q^{l+2}[l-k]^{1/2}[l+k+1]^{1/2}[2l]^{1/2}[2l+1]^{1/2} \beta^0(l, l)_{j}^\pm v_{k+1}^l,$$

$$- q^{l+k+1}[l-k-1]^{1/2}[l-k]^{1/2}[2l-1]^{-1/2}[2l]^{-1/2} \alpha^0(l-1, l-1)_{j}^\pm v_{k+1}^l,$$

$$x_0 v_k^l = q^{l-k+1}[l+k+1]^{1/2}[l+k+2]^{1/2}[2l+1]^{-1/2}[2l+2]^{-1/2} \alpha^0(l, l)_{j}^\pm v_{k}^l \pm 1,$$

$$+ (1 - q^{l+k+1}[l-k][2][2l]^{-1}) \beta^0(l, l)_{j}^\pm v_{k}^l,$$

$$+ q^{k}[l-k]^{1/2}[l+k]^{1/2}[2l-1]^{-1/2}[2l]^{-1/2} \alpha^0(l-1, l-1)_{j}^\pm v_{k}^l.$$
\[ x_1 v_k = q^{l+k}[l-k+1]^{1/2}[l-k+2]^{1/2}[2l+1]^{-1/2}[2l+2]^{-1/2} \alpha^+(l, l)_{j}^+ v_{k-1}^{l+1} \\
+ q^k[l-k+1]^{1/2}[l+k]^{1/2}[2l]^{-1/2} \beta^0(l, l)_{j}^+ v_{k-1}^{l} \\
- q^{-l+k-1}[l+k-1]^{1/2}[l+k]^{1/2}[2l-1]^{-1/2}[2l-2]^{-1/2} \alpha^+(l-1, l-1)_{j}^+ v_{k-1}^{l}, \]

where, for \( r < \infty \), the real numbers \( \beta^0(l, l)_{j}^+ \) and \( \alpha^+(l, l)_{j}^+ \) are given by (35) and (36), respectively, and, for \( r = \infty \), by (39) and (40), respectively.

Representations corresponding to different pairs of labels \((j, \pm)\) are not unitarily equivalent (with only one obvious exception: \( \pi_0^+ = \pi_0^- \)). A representation of this list is irreducible if and only if \( \mathcal{K} = \mathbb{C} \).

An immediate consequence of Theorem 4.1 is the following

**Corollary 4.2** (i) Each integrable \(*\)-representation of \( \mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(su_2) \) is a direct sum of integrable irreducible \(*\)-representations.

(ii) Each integrable irreducible \(*\)-representation of \( \mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(su_2) \) is unitarily equivalent to a \(*\)-representations \( \pi_j^\pm, j \in \frac{1}{2} \mathbb{N}_0 \), with \( \mathcal{K} = \mathbb{C} \).

It will be convenient to introduce the following notation. If \( \pi_j^\pm, j \in \frac{1}{2} \mathbb{N}_0 \),

\[ \pi_j := \pi_{-j} \text{ for } j < 0, \quad \pi_j := \pi_j^+ \text{ for } j \geq 0, \quad j \in \frac{1}{2} \mathbb{Z}. \]

### 4.2 Decomposition of tensor products of irreducible integrable representations with spin \( l \) representations

**Lemma 4.3** Suppose that \( \mathcal{X} \) is a left module \(*\)-algebra of a Hopf \(*\)-algebra \( \mathcal{U} \).

Let \( \pi \) and \( T \) be \(*\)-representations of the \(*\)-algebras \( \mathcal{X} \times \mathcal{U} \) and \( \mathcal{U} \) on domains \( D \) and \( V \), respectively. Then there is a \(*\)-representation, denoted by \( \pi \otimes T \), of the \(*\)-algebra \( \mathcal{X} \times \mathcal{U} \) on the domain \( D \otimes V \) such that \( (\pi \otimes T)(x) = \pi(x) \otimes T(1) \) for \( x \in \mathcal{X} \) and \( (\pi \otimes T)(f) = \pi(f_{(1)}) \otimes T(f_{(2)}) \) for \( f \in \mathcal{U} \).

**Proof.** It suffices to check that \( \pi \otimes T \) respects the cross relation (2), that is,

\[ (\pi \otimes T)(f)(\pi \otimes T)(x) = (\pi \otimes T)(f_{(1)} \otimes x)(\pi \otimes T)(f_{(2)}) \]

for \( x \in \mathcal{X} \) and \( f \in \mathcal{U} \). The details of this easy verification are left to the reader. \( \square \)

Clearly, the tensor product \( \pi \otimes T \) of an integrable \(*\)-representation \( \pi \) of the cross product algebra \( \mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(su_2) \) and a spin \( l \) representation \( T \) of \( \mathcal{U}_q(su_2) \) is
again integrable, so Corollary 4.2 applies. The decomposition of $\pi_j \otimes T_l$ into a direct sum of irreducible integrable representations of $\mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(su_2)$ is described in the next proposition.

**Proposition 4.4** For $j \in \frac{1}{2} \mathbb{Z}$ and $l \in \frac{1}{2} \mathbb{N}_0$, let $\pi_j$ be an irreducible integrable $\ast$-representation of $\mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(su_2)$ and $T_l$ a spin $l$ representation of $\mathcal{U}_q(su_2)$. Then, up to unitary equivalence, $\pi_j \otimes T_l = \pi_{j-l} \oplus \pi_{j-l+1} \oplus \ldots \oplus \pi_{j+l}$.

**Proof.** For $l = 0$, there is nothing to prove. Let $l = 1/2$. From Theorem 4.1, it follows that the restriction of $\pi_j$ to $\mathcal{U}_q(su_2)$ is of the form $\oplus_{n \in \mathbb{N}_0} T_{|j|+n}$. Assume first that $j \neq 0$. By the Clebsch–Gordon decomposition, we have

$$\pi_j \otimes T_{1/2} = T_{|j|-1/2} \oplus (\oplus_{n \in \mathbb{N}_0} 2T_{|j|+n+1/2})$$

(41)

as representations of $\mathcal{U}_q(su_2)$. By Corollary 4.2, $\pi_j \otimes T_{1/2}$ decomposes into a direct sum of irreducible integrable $\ast$-representations described in Theorem 4.1. In (41), there occur no spin $k$ representations for $k < |j| - 1/2$, exactly one spin $|j| - 1/2$ representation, and two spin $k$ representations for $k = |j| + 1/2, |j| + 3/2, \ldots$. Thus we must have $\pi_j \otimes T_{1/2} = \pi_{|j|-1/2} \oplus \pi_{|j|+1/2}^2$, where $\epsilon_1, \epsilon_2 \in \{+, -\}$ are to be determined. Moreover, $\pi_{|j|-1/2}^1$ and $\pi_{|j|+1/2}^2$ are irreducible. Again by the Clebsch–Gordon decomposition,

$$v_{|j|-1/2}^{[j]} = [2|j|+1]^{-1/2} \left(q^{1/2}[2|j|]^{1/2} v_{|j|}^{[j]} \otimes v_{-1/2}^{1/2} - q^{-|j|} v_{|j|-1}^{[j]} \otimes v_{1/2}^{1/2}\right)$$

(42)

is the (unique) highest weight vector of the spin $|j| - 1/2$ representation in (41). If $|j| = 1/2$, then $v_0^{[j]}$ belongs to a spin 0 representation and $\pi_{1/2}^1 = \pi_0$ is the Heisenberg representation. Let $|j| > 1/2$ and $\epsilon = \text{sign}(j)$. Then a straightforward computation gives

$$\langle v_{|j|-1/2}^{[j]}, \pi_j \otimes T_{1/2}(x_0) v_{|j|-1/2}^{[j]} \rangle = [2|j|+2][2|j|-1][2|j|+1]^{-1}[2|j|]^{-1} \beta(0(|j|, |j|))_{|j|}.$$ 

Hence $\epsilon_1 = \text{sign}(\langle v_{|j|-1/2}^{[j]}, \pi_j \otimes T_{1/2}(x_0) v_{|j|-1/2}^{[j]} \rangle) = \text{sign}(\beta(0(|j|, |j|))_{|j|}) = \text{sign}(j)$. The linear space of highest weight vectors belonging to spin $|j| + 1/2$ representations in (41) is 2-dimensional and spanned by the orthonormal vectors

$$u_{|j|+1/2}^{[j]} = [2|j|+3]^{-1/2} \left(q^{1/2}[2|j|+2]^{1/2} v_{|j|+1}^{[j]} \otimes v_{-1/2}^{1/2} - q^{-|j|-1} v_{|j|}^{[j]} \otimes v_{1/2}^{1/2}\right)$$

$$u_{|j|+1/2}^{[j]} = v_{|j|}^{[j]} \otimes v_{1/2}^{1/2}.$$ 

22
The vector
\[ v := [2|j|+3]^{1/2}[2|j|+2]^{-1/2}\alpha^+([j], [j])\|u_{|j|+1/2} - q[2]^{1/2}[2|j|]^{-1/2}\beta^0([j], [j])\|u_{|j|+1/2} \]
is orthogonal to \( x_1 v_{|j| - 1/2} \). Hence \( \|v\|^{-1}v \) is the (unique) highest weight vector of the spin \(|j| + 1/2\) representation belonging to the decomposition of \( \pi^+_{|j|+1/2} \). Our goal is to determine \( \epsilon_2 = \text{sign}(\|v\|^{-1}v, x_0\|v\|^{-1}v) = \text{sign}(\langle v, x_0v \rangle) \). Using (55), (56) and the formulas in Theorem 4.1, we obtain
\[
\langle v, x_0v \rangle = \{[2|j|+3][2|j| - 1][2|j|]^{-1}[2|j|+1]^{-1}(\alpha^+([j], [j])\|\beta^0([|j|], [|j|])\|)\}^j + [2][2|j|]^{-1}[2|j|+2]^{-1}\}^j \beta^0([|j|], [|j|])\|j].
\]
As the expression in braces is positive, we deduce \( \epsilon_2 = \text{sign}(\beta^0([|j|], [|j|])\|j]) = \text{sign}(j) \).

Next, assume that \( j = 0 \). By similar arguments as above, we conclude that \( \pi_0 \otimes T_{1/2} = \pi_0^{1/2} \oplus \pi_1^{1/2} \). Let \( u_1 \) and \( u_2 \) denote the highest weight vectors of the spin 1/2 representations belonging to \( \pi_0^{1/2} \) and \( \pi_1^{1/2} \), respectively. Then \( \langle u_1, x_0u_1 \rangle = \beta^0(1/2, 1/2)_{1/2}^{1/2} \) and \( \langle u_2, x_0u_2 \rangle = \beta^0(1/2, 1/2)_{1/2}^{1/2} \). The vector \( \|w_1^{1/2}, x_0w_1^{1/2} \) belongs to the span of the orthonormal vectors \( u_1 \) and \( u_2 \). Since \( \langle w_1^{1/2}, x_0w_1^{1/2} \rangle = 0 \), we get \( \text{sign}(\beta^0(1/2, 1/2)_{1/2}^{1/2}) \neq \text{sign}(\beta^0(1/2, 1/2)_{1/2}^{1/2}) \), whence \( \epsilon_1 \neq \epsilon_2 \). Summarizing the preceding results, we conclude that, for all \( j \in \frac{1}{2}\mathbb{Z} \), the representation \( \pi_j \otimes T_{1/2} \) decomposes into \( \pi_j \otimes T_{1/2} = \pi_j \otimes T_{1/2} = \pi_{j-1/2} \oplus \pi_{j+1/2} \).

The proposition can now be proved by induction. Let \( k \in \frac{1}{2}\mathbb{N} \). Assume that Proposition 4.4 holds for all \( l = 0, 1/2, \ldots, k \). By Corollary 4.2, \( \pi_j \otimes T_k \otimes T_{1/2} = (\pi_j \otimes T_k) \otimes T_{1/2} = \pi_j \otimes (T_k \otimes T_{1/2}) \) decomposes into irreducible integrable \( * \)-representations. By our induction hypothesis and (41), we have
\[
(\pi_j \otimes T_k) \otimes T_{1/2} = \pi_{j-k-1/2} \oplus (\bigoplus_{n=0}^{2k-1} \pi_{j-k+n+1/2}) \oplus \pi_{j+k+1/2}.
\]
On the other hand,
\[
\pi_j \otimes (T_k \otimes T_{1/2}) = (\bigoplus_{n=0}^{2k-1} \pi_{j-k+n+1/2}) \oplus (\pi_j \otimes T_{k+1/2}).
\]
Comparing both results shows that
\[
\pi_j \otimes T_{k+1/2} = \pi_{j-k-1/2} \oplus \pi_{j-k+1/2} \oplus \ldots \oplus \pi_{j+k+1/2}.
\]
4.3 Realization of irreducible integrable \(*\)-representations of $$\mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(\mathfrak{su}_2)$$ on $$\mathcal{O}(SU_q(2))$$

In this subsection, we relate the irreducible representation $$\pi_j$$ from Theorem 4.1 to the representation $$\hat{\pi}_j$$ corresponding to the Hopf module $$M_j$$ from Section 3.

**Lemma 4.5** Let $$l \in \frac{1}{2}\mathbb{N}$$. Define $$e^+_{l/2} = d + q^{-2l-1}sb$$ and $$e^-_{l/2} = c - q^{2l+1}sa$$. Set $$e^+_{-1/2} = F^e_{1/2}$$ and $$e^-_{1/2} = E^e_{-1/2}$$. Then

$$q^{1/2}[l]^{1/2}v^l_{-1/2}e^+_{-1/2} - q^{-1}v^l_{-1/2}e^-_{-1/2} = q^{1/2}[l]^{1/2}v^l_{-l,-l}e^-_{1/2} - q^l v^l_{l+1,-l}e^-_{-1/2} = 0.$$  \hspace{1cm} (43)

**Proof.** Recall that $$v^l_{1/2} = N^l_{1/2}(d + q^{-1}sb) \cdot \cdots \cdot (d + q^{-2l}sb)$$ and that $$v^l_{-1/2} = [l]^{-1/2}F^e_{1/2}v^l_{1/2} = [l]^{-1/2}N^l_{1/2}(F^e_{1/2})$$. A straightforward induction argument shows that $$F^e_{1/2}v^l = q^{-1/2}[l]u_{-l/2}(c + q^{-2l}sa)$$. Now a direct calculation gives

$$u_t e^+_{-1/2} - q^{-1}d^{-1/2}[l]^{-1}(F^e_{1/2})e^+_{1/2} = u_{-l/2}(d + q^{-2l}sb)(c + q^{-2l-1}sa) - q^{-1}u_{-l/2}(c + q^{-2l}sa)(d + q^{-2l-1}sb) = 0$$

which implies the first equality in (43). The second equality is proved in the same way by using $$v^l_{-l,-l} = \|w_t\|^{-1}w_t$$. \hspace{1cm} \Box

**Proposition 4.6** Let $$j \in \frac{1}{2}\mathbb{Z}$$. The irreducible integrable \(*\)-representation $$\pi_j$$ of $$\mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(\mathfrak{su}_2)$$ from Theorem 4.1 is unitarily equivalent to the \(*\)-representation $$\hat{\pi}_j$$ on the left $$\mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(\mathfrak{su}_2)$$-module $$M_j$$ from Theorem 3.1.

**Proof.** By Theorem 3.1, the restriction of the representation $$\pi_j$$ to $$\mathcal{U}_q(\mathfrak{su}_2)$$ is of the form $$\oplus_{n \in \mathbb{N}_0} T_{j|n}$$. Likewise, by Theorem 4.1, the restriction of the representation $$\pi_j$$ to $$\mathcal{U}_q(\mathfrak{su}_2)$$ is $$\oplus_{n \in \mathbb{N}_0} T_{j|n}$$. Therefore, by Corollary 4.2(i), $$\hat{\pi}_j$$ is unitarily equivalent either to $$\pi_{\hat{j}|\hat{n}}$$ or to $$\pi^*_{\hat{j}|\hat{n}}$$. That is, it only remains to specify the label + or -. Recall from Theorem 3.1 that $$M_j$$ is the linear span of vectors $$v^{|j|+n}_{kj}$$, where $$n \in \mathbb{N}_0$$ and $$k = -(|j| + n), \ldots, |j| + n$$.

For $$j = 0$$, there is only the unique Heisenberg representation. A direct calculation yields $$\langle v^{|1/2,-1/2|}_{1,2}, x_0 v^{|1/2|}_{1,2,-1/2,2} \rangle = \beta^0(1/2, 1/2)_{1/2}$$ and $$\langle v^{|1/2|}_{1/2,1,2}, x_0 v^{|1/2|}_{1,2,1,2} \rangle = \beta^0(1/2, 1/2)_{1/2}$$ so that $$\hat{\pi}_{-1/2} = \pi_{-1/2}$$ and $$\hat{\pi}_{1/2} = \pi_{1/2}$$.

We proceed by induction. Let $$l \in \frac{1}{2}\mathbb{N}$$. Assume that $$\pi_l$$ is unitarily equivalent to $$\hat{\pi}_l$$. Recall that $$T_{1/2}$$ is the spin 1/2 representation on $$V_{1/2} = \text{Lin}\{v^{|1/2|}_{-1/2,1/2}, v^{|1/2|}_{1/2,-1/2}\}$$.
where \( v_{l \pm 1/2} \) are the weight vectors. By Proposition \ref{4.4}, the tensor product representation \( \pi_l \otimes T_{1/2} \cong \pi_l \otimes T_{1/2} \) decomposes into the direct sum \( \pi_{l-1/2} \oplus \pi_{l+1/2} \) on \( M_l \otimes V_{1/2} = W_{l-1/2} \oplus W_{l+1/2} \). Set \( e_{1/2} := \|d + q^{-2l-1}sb\|^{-1}(d + q^{-2l-1}sb) \) and \( e_{-1/2} := Fv_{1/2} \). Consider the linear mapping \( \Phi : M_l \otimes V_{1/2} \rightarrow \mathcal{O}(SU_q(2)) \) defined by \( \Phi(v \otimes v_{1/2}^{l \pm 1/2}) = ve_{1/2} \). Clearly, \( \Phi \) intertwines the actions of \( \mathcal{O}(S^2_{qr}) \times U_q(su_2) \) on \( M_l \otimes V_{1/2} \) and \( \mathcal{O}(SU_q(2)) \). From (42) and the first equality in (43), we conclude that \( \Phi(v_{l-1/2}^{l-1/2}) = 0 \). Since the highest weight vector \( v_{l-1/2}^{l-1/2} \) is cyclic for the representation \( \pi_{l-1/2} \), the latter implies that \( \Phi(W_{l-1/2}) = \{0\} \). On the other hand, it follows from (17) and (20) that there is a non-zero constant \( \gamma_l \) such that \( \Phi(v_{l+1/2}^{l+1/2}) = \gamma_l v_{l+1/2}^{l+1/2} \). Therefore, \( \Phi \) maps \( W_{l+1/2} \) into \( M_{l+1/2} \), so \( \Phi \) is a non-trivial intertwiner of the irreducible integrable representations \( \pi_{l+1/2} \) on \( W_{l+1/2} \) and \( \pi_{l+1/2} \) on \( M_{l+1/2} \). By a standard application of Schur’s lemma, \( \pi_{l+1/2} \) and \( \pi_{l+1/2} \) are unitarily equivalent. This proves the assertion for \( l + 1/2 \).

The proof for negative integers is similar. One replaces \( v_{l+1/2} \) by \( v_{l-1/2} \) and uses \( e_{-1/2} := \|a - q^{2l+1}sc\|^{-1}(a - q^{2l+1}sc) \) and the second equality in (43).

By Proposition \ref{4.4} and Theorem \ref{3.1}, each irreducible integrable \(*\)-representation of the cross product algebra \( \mathcal{O}(S^2_{qr}) \times U_q(su_2) \) can be realized on \( \mathcal{O}(SU_q(2)) \) and \( \mathcal{O}(SU_q(2)) \) decomposes into the orthogonal direct sum of all irreducible integrable \(*\)-representation of \( \mathcal{O}(S^2_{qr}) \times U_q(su_2) \), each with multiplicity one.

## 5 Representations of the cross product \(*\)-algebras: Second approach

### 5.1 “Decoupling” of the cross product algebras

Let us first suppose that \( r \in [0, \infty) \). From the relations \( AB = q^{-2}BA \) and \( AB^* = q^2 B^*A \), it is clear that \( S = \{A^n; n \in \mathbb{N}_0\} \) is a left and right Ore set of the algebra \( \mathcal{O}(S^2_{qr}) \). Moreover the algebra \( \mathcal{O}(S^2_{qr}) \) has no zero divisors. Hence the localization algebra, denoted by \( \hat{\mathcal{O}}(S^2_{qr}) \), of \( \mathcal{O}(S^2_{qr}) \) at \( S \) exists. The \(*\)-algebra \( \hat{\mathcal{O}}(S^2_{qr}) \) is then a \(*\)-subalgebra of \( \hat{\mathcal{O}}(S^2_{qr}) \) and all elements \( A^n, n \in \mathbb{N}_0 \), are invertible in \( \hat{\mathcal{O}}(S^2_{qr}) \). From [9, Theorem 3.4.1], we conclude that \( \hat{\mathcal{O}}(S^2_{qr}) \) is a left \( U_q(su_2) \)-module \(*\)-algebra such that \( \mathcal{O}(S^2_{qr}) \) is a left \( U_q(su_2) \)-module \(*\)-subalgebra. The
left action of the generators $E, F, K$ on $A^{-1}$ is given by

$$E \circ A^{-1} = -q^{-5/2} B^* A^{-2}, \quad F \circ A^{-1} = q^{1/2} B A^{-2}, \quad K \circ A^{-1} = A^{-1}. $$

Hence the left cross product algebra $\hat{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$ is a well defined $*$-algebra containing $\mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$ as a $*$-subalgebra.

Let $\mathcal{Y}_r$ denote the $*$-subalgebra of $\mathcal{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$ generated by

\begin{align}
X &:= q^{3/2} \lambda FK^{-1} A + qB = q^{3/2} \lambda AF K^{-1} + q^{-1} B, \quad (44) \\
X^* &:= q^{3/2} \lambda AK^{-1} E + qB^* = q^{3/2} \lambda K^{-1} EA + q^{-1} B^*, \quad (45) \\
Y &:= qK^{-2} A \quad (46)
\end{align}

and let $\hat{\mathcal{Y}}_r$ be the subalgebra of $\hat{O}(S^2_{qr})$ generated by $\mathcal{Y}_r$ and $Y^{-1}$. Note that $Y = Y^*$. It is straightforward to check that the elements $X, X^*, Y, Y^{-1}$ commute with the generators $A$, $B$, $B^*$ of $\mathcal{O}(S^2_{qr})$. Hence the algebras $\hat{\mathcal{Y}}_r$ and $\hat{O}(S^2_{qr})$ commute inside the cross product algebra $\hat{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$. Moreover, the generators of $\mathcal{Y}_r$ satisfy the commutation relations

$$ Y X = q^2 XY, \quad Y X^* = q^{-2} X^* Y, \quad X^* X - q^2 XX^* = (1 - q^2)(Y^2 + r). \quad (47) $$

We denote by $\hat{\mathcal{U}}_q(\mathfrak{su}_2)$ the Hopf $*$-subalgebra of $\mathcal{U}_q(\mathfrak{su}_2)$ generated by $e := EK$, $f := K^{-1} F$ and $k := K^2$. As an algebra, $\mathcal{U}_q(\mathfrak{su}_2)$ has generators $e, f, k, k^{-1}$ with defining relations

$$ kk^{-1} = k^{-1} k = 1, \quad ke = q^2 ek, \quad kf = q^{-2} fk, \quad ef - fe = \lambda^{-1}(k - k^{-1}). \quad (48) $$

From (44)–(46), we obtain

$$ f = q^{-1/2} \lambda^{-1} (X - qB) A^{-1}, \quad e = q^{1/2} \lambda^{-1} (X^* - q^{-1} B^*) Y^{-1}, \quad k = qY^{-1} A. \quad (49) $$

Hence the two commuting algebras $\hat{\mathcal{Y}}_r$ and $\hat{O}(S^2_{qr})$ generate the $*$-subalgebra $\hat{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$ of $\hat{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$. In this sense, $\hat{\mathcal{Y}}_r$ and $\hat{O}(S^2_{qr})$ “decouple” the cross product algebra $\hat{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$. By choosing a PBW-basis of $\hat{O}(S^2_{qr})$ and $\hat{\mathcal{Y}}_r$, it is easy to show that the subalgebra generated by $\hat{O}(S^2_{qr})$ and $\hat{\mathcal{Y}}_r$ is isomorphic to the tensor product algebra $\hat{O}(S^2_{qr}) \otimes \hat{\mathcal{Y}}_r$. Therefore we can consider $\hat{O}(S^2_{qr}) \otimes \hat{\mathcal{Y}}_r$ as a $*$-subalgebra of $\hat{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$ by identifying $x \otimes y$ with $xy$, $x \in \hat{O}(S^2_{qr})$, $y \in \hat{\mathcal{Y}}_r$. Similarly, $\hat{O}(S^2_{qr}) \otimes \hat{\mathcal{Y}}_r$ becomes a $*$-subalgebra of $\hat{O}(S^2_{qr}) \rtimes \mathcal{U}_q(\mathfrak{su}_2)$.  

26
There is an alternative way to define the $*$-algebra $\hat{O}(S^2_{qr}) \times \hat{U}_q(su_2)$ by taking the two sets $A = A^*, A^{-1}, B, B^*$ and $X, X^*, Y = Y^*, Y^{-1}$ of pairwise commuting generators with defining relations (11), (47) and the obvious relations

$$AA^{-1} = A^{-1}A = 1, \quad YY^{-1} = Y^{-1}Y = 1. \quad (50)$$

Indeed, if we define $e, f, k$ by (49), then the relations (48) of $\hat{U}_q(su_2)$ and the cross relations of $\hat{O}(S^2_{qr}) \times \hat{U}_q(su_2)$ can be derived from this set of defining relations.

The larger cross product algebra $\hat{O}(S^2_{qr}) \times \hat{U}_q(su_2)$ can be redefined in a similar manner if we replace the generator $Y$ by $\hat{K}$. That is, $\hat{O}(S^2_{qr}) \times \hat{U}_q(su_2)$ is the $*$-algebra with the generators $A = A^*, A^{-1}, B, B^*, X, X^*, \hat{K} = K^*, K^{-1}$ satisfying the defining relations (11), (50).

$$KA = AK, \quad BK = qKB, \quad KB^* = qB^*K, \quad X^*X - q^2XX^* = (1 - q^2)(q^2K^{-4}A^2 + r), \quad (51)$$

and $A, B, B^*$ commute with $X$ and $X^*$. The generators $F$ and $E$ are then given by

$$F = q^{-3/2}^{\lambda^{-1}}(X - qB)KA^{-1}, \quad E = q^{-3/2}^{\lambda^{-1}}A^{-1}K(X^* - qB^*). \quad (52)$$

The preceding considerations and facts carry over almost verbatim to the case $r = \infty$. The only difference is that in the case $r = \infty$ one has to set $r = 1$ in the third equations of (47) and (51).

### 5.2 Operator representations of the $*$-algebra $\mathcal{Y}_r$

For the study of representations of the $*$-algebra $\mathcal{Y}_r$, we need two auxiliary lemmas. The first one restates the Wold decomposition of an isometry (see [18, Theorem 1.1]), while the second is Lemma 4.2(ii) in [15].

**Lemma 5.1** Each isometry $v$ on a Hilbert space $\mathcal{K}$ is up to unitary equivalence of the following form: There exist Hilbert subspaces $\mathcal{K}^u$ and $\mathcal{K}^s_0$ of $\mathcal{K}$ and a unitary operator $v_u$ on $\mathcal{K}^u$ such that $v = v_u \oplus v_s$ on $\mathcal{K} = \mathcal{K}^u \oplus \mathcal{K}^s$ and $v_s$ acts on $\mathcal{K}^s = \bigoplus_{n=0}^{\infty} \mathcal{K}^s_n$ by $v_s \zeta_n = \zeta_{n+1}$, where each $\mathcal{K}^s_n$ is $\mathcal{K}^0_0$ and $\zeta_n \in \mathcal{K}^s_n$. Moreover, $\mathcal{K}^u = \bigcap_{n=0}^{\infty} v^n \mathcal{K}$.

**Lemma 5.2** Let $v$ be the operator $v_s$ on $\mathcal{K}^s = \bigoplus_{n=0}^{\infty} \mathcal{K}^s_n$, $\mathcal{K}^s_n = \mathcal{K}^0_0$, from Lemma 5.1, and let $Y$ be a self-adjoint operator on $\mathcal{K}^s$ such that $q^2vY \subseteq Yv$. Then there is a self-adjoint operator $Y_0$ on the Hilbert space $\mathcal{K}^0_0$ such that $Y_0 = q^{2n}Y_0\zeta_n$ and $v\zeta_n = \zeta_{n+1}$ for $\zeta_n \in \mathcal{K}^s_n$ and $n \in \mathbb{N}_0$.
It suffices to treat the case \( r \in [0, \infty) \) because the algebra \( \mathcal{V}_\infty \) is isomorphic to \( \mathcal{V}_1 \). Suppose that we have a representation of relations (47) by closed operators \( X, X^* \) and a self-adjoint operator \( Y \) acting on a Hilbert space \( \mathcal{K} \). We assume that \( Y \) has trivial kernel.

Let \( X = v|X| \) be the polar decomposition of the operator \( X \). Since \( q \in (0, 1) \), \( r \geq 0 \) and \( \ker Y = \{0\} \), we have \( \ker X = \{0\} \) by the third equation of (47). Hence \( v \) is an isometry on \( \mathcal{K} \), that is, \( v^*v = I \).

By (47), \( YX^*X = X^*XY \). We assume that the self-adjoint operators \( Y \) and \( X^*X \) strongly commute. Then \( Y \) and \( |X| = (X^*X)^{1/2} \) also strongly commute. Again by (47), \( Yv|X| = YX = q^2XY = q^2v|X|Y = q^2vY|X| \). Since \( \ker |X| = \ker X = \{0\} \), we conclude that

\[
q^2vY = Yv. \tag{53}
\]

Since \( X^* = |X|v^* \), the third equation of (47) rewrites as

\[
|X|^2 = q^2v|X|^2v^* + (1-q^2)(Y^2 + r). \tag{54}
\]

Multiplying by \( v \) gives \( |X|^2v = v(q^2|X|^2 + (1-q^2)(q^4Y^2 + r)) \) since \( v^*v = I \) and, by (53), \( Y^2v = q^4vY^2 \). Proceeding by induction, we derive

\[
|X|^2v^n = v^n(q^{2n}|X|^2 + (1-q^{2n})(q^{2n+2}Y^2 + r)). \tag{55}
\]

Now we use the Wold decomposition \( v = v_u \oplus v_s \) on \( \mathcal{K} = \mathcal{K}^u \oplus \mathcal{K}^s \) of the isometry \( v \) by Lemma 5.1. Since \( \mathcal{K}^u = \bigcap_{n=0}^{\infty} v^n \mathcal{K} \) and \( Yv = q^2vY \), the Hilbert subspace \( \mathcal{K}^u \) reduces the self-adjoint operator \( Y \). From (55), it follows that \( |X|^2 \) leaves a dense subspace of the space \( \mathcal{K}^u \) invariant. We assume that \( \mathcal{K}^u \) reduces \( |X|^2 \). Hence we can consider all operators occurring in relation (55) on the subspace \( \mathcal{K}^u \). For the unitary part \( v_u \) of \( v \), we have \( v_u v_u^* = I \) on \( \mathcal{K}^u \). Multiplying (55) by \((v_u^*)^n\) and using again Equation (53), we derive on \( \mathcal{K}^u \) the relation

\[
0 \leq q^{2n}v_u^n|X|^2(v_u^*)^n = |X|^2 - (1-q^{2n})r + q^2Y^2 - q^{-2n+2}Y^2
\]

for all \( n \in \mathbb{N} \). Letting \( n \to \infty \) and remembering that \( \ker Y = \{0\} \) and \( q \in (0, 1) \), we conclude that the latter is only possible when \( \mathcal{K}^u = \{0\} \). That is, the isometry \( v \) is (unitarily equivalent to) the unilateral shift operator \( v_s \).

Since \( v = v_s \), Lemma 5.2 applies to the relation \( q^2vY = Yv \). Hence there exists a self-adjoint operator \( Y_0 \) on \( \mathcal{K}^u_0 \) such that \( Y\zeta_n = q^{2n}Y_0\zeta_n \) on \( \mathcal{K}^s = \bigoplus_{n=0}^{\infty} \mathcal{K}^u_0 \),
\( \mathcal{K}_n^s = \mathcal{K}_n^s \). Since \( \nu^* \zeta_0 = 0 \), (54) yields \( |X|^2 \zeta_0 = (1 - q^2)(Y_0^2 + r)\zeta_0 \). Using this equation and (54), we compute

\[
|X|^2 \zeta_n = |X|^2 v^n \zeta_0 = v^n (q^{2n}|X|^2 + (1-q^{2n})(q^{2n+2}Y_0^2 + r)) \zeta_0 \\
= v^n (q^{2n}(1-q^2)(Y_0^2 + r)\zeta_0 + (1-q^{2n})(q^{2n+2}Y_0^2 + r)\zeta_0) \\
= \lambda_n^2 (q^{2n}Y_0^2 + r)\zeta_n
\]

for \( \zeta_n \in \mathcal{K}_n^s \). We assumed above that the pointwise commuting self-adjoint operator \( |X|^2 \) and \( Y \) strongly commute. Hence their reduced parts on each subspace \( \mathcal{K}_n^s \) strongly commute. Therefore, (56) implies \( |X|\zeta_n = \lambda_{n+1}(q^{2n}Y_0^2 + r)^{1/2} \zeta_n \).

Recall that \( X = \nu|X| \) and \( X^* = |X|\nu^* \). Renaming \( \mathcal{K}_n^s \) by \( \mathcal{K}_n \) and summarizing the preceding, we obtain the following form of the operators \( X, X^*, Y \):

\[
X \zeta_n = \lambda_{n+1}(q^{2n}Y_0^2 + r)^{1/2} \zeta_{n+1}, \quad X^* \zeta_n = \lambda_n(q^{2n-2}Y_0^2 + r)^{1/2} \zeta_{n-1}, \\
Y \zeta_n = q^{2n}Y_0 \zeta_n \quad \text{on} \quad \mathcal{K} = \oplus_{n=0}^\infty \mathcal{K}_n, \quad \mathcal{K}_n = \mathcal{K}_0,
\]

where \( Y_0 \) is self-adjoint operator with trivial kernel on the Hilbert space \( \mathcal{K}_0 \). Conversely, it is easy to check that these operators \( X, X^*, Y \) satisfy the relations (47), so they define indeed a *-representation of the *-algebra \( \mathcal{Y}_r \). The representation is irreducible if and only if \( \mathcal{K}_0 = \mathbb{C} \). In this case, \( Y_0 \) is a non-zero real number.

### 5.3 Representations of the cross product *-algebras

First let us review the representations of the *-algebra \( \mathcal{O}(S^2_{qr}) \) from [13]. Recall the definition of \( \lambda_{\pm} \) from Section 3. For \( r < \infty \), we set

\[
c_{\pm}(n) := (r + \lambda_{\pm} q^{2n} - (\lambda_{\pm} q^{2n})^2)^{1/2}.
\]

Let \( \mathcal{H}_0, \mathcal{H}_+^r \) and \( \mathcal{H}_0^r \) be Hilbert spaces and let \( u \) be a unitary operator on \( \mathcal{H}_0 \). Let \( \mathcal{H}_\pm^r = \oplus_{n=0}^\infty \mathcal{H}_n^\pm \), where \( \mathcal{H}_n^\pm = \mathcal{H}_0^\pm \), and let \( \mathcal{H} = \mathcal{H}_0^r \oplus \mathcal{H}_+^r \oplus \mathcal{H}_-^r \) for \( r \in (0, \infty) \) and \( \mathcal{H} = \mathcal{H}_0^r \oplus \mathcal{H}_+^r \) for \( r = 0 \). The generators of \( \mathcal{O}(S^2_{qr}) \) act on the Hilbert space \( \mathcal{H} \) by the following formulas:

\[
r \in [0, \infty) : \quad A = 0, \quad B = r^{1/2}u, \quad B^* = r^{1/2}u^* \quad \text{on} \quad \mathcal{H}_0^r, \\
a_n = \lambda_{\pm} q^{2n}a_n, \quad b_n = c_{\pm}(n)b_{n-1}, \quad b^* a_n = c_{\pm}(n+1)a_{n+1} \quad \text{on} \quad \mathcal{H}_\pm^r,
\]

\[
r = \infty : \quad A = 0, \quad B = u, \quad B^* = u^* \quad \text{on} \quad \mathcal{H}_0^r, \\
a_n = \pm q^{2n}a_n, \quad b_n = (1-q^{4n})^{1/2}b_{n-1}, \quad b^* a_n = (1-q^{4(n+1)})^{1/2}a_{n+1} \quad \text{on} \quad \mathcal{H}_\pm^r.
\]
Recall that, for \( r = 0 \), there is no Hilbert space \( \mathcal{H}^- \). From [13, Proposition 4], it follows that, up to unitary equivalence, each \(*\)-representation of \( \mathcal{O}(S^2_{qr}) \) is of the above form. Note that all operators are bounded and \( \mathcal{H}^0 = \ker A, A > 0 \) on \( \mathcal{H}^+ \) and \( A < 0 \) on \( \mathcal{H}^- \).

Next we show that, for each \(*\)-representation of the algebras \( \mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(\mathfrak{su}_2) \) and \( \mathcal{O}(S^2_{qr}) \times \hat{\mathcal{U}}_q(\mathfrak{su}_2) \), the space \( \mathcal{H}^0 \) is \( \{0\} \), so the operator \( A \) is invertible. We carry out the reasoning for \( \mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(\mathfrak{su}_2) \). Assuming that the commuting self-adjoint operators \( A \) and \( K \) strongly commute, it follows that \( K \) leaves \( \mathcal{H}^0 = \ker A \) invariant. By (11) (resp. (12)), \( B\mathcal{H}^0 \subseteq \mathcal{H}^0 \). Let \( \xi \in \mathcal{H}^0 \). Using the relation \( FA = AF - q^{-3/2}BK \), we see that

\[
q^{-3}\|BK\xi\|^2 = \|AF\xi\|^2 = \langle q^{-3/2}BK\xi, AF\xi \rangle = 0.
\]

That is, \( BK\xi = 0 \) and \( AF\xi = 0 \), so that \( F\xi \in \mathcal{H}^0 \). For \( r \in (0, \infty) \), \( BK \) is invertible on \( \mathcal{H}^0 \) and hence \( \xi = 0 \). For \( r = 0 \), we have \( B^*\xi = B^*F\xi = 0 \). From the cross relation \( F B^* = q^{-1}B^*F + q^{-1/2}(1 + q^2)AK - q^{-1/2}K \), we get \( K\xi = 0 \) and so \( \xi = 0 \). Thus, \( \mathcal{H}^0 = \{0\} \).

Consider now a \(*\)-representation of the \(*\)-algebra \( \hat{\mathcal{O}}(S^2_{qr}) \times \hat{\mathcal{U}}_q(\mathfrak{su}_2) \). Its restriction to \( \mathcal{O}(S^2_{qr}) \) is a \(*\)-representation of the form described above with \( \mathcal{H}^0 = \{0\} \). All operators of the \(*\)-subalgebra \( \mathcal{Y}_r \) commute with \( A \) and \( B \). Let us assume that the spectral projections of the self-adjoint operator \( A \) commute also with all operators of \( \mathcal{Y}_r \). (Note that, for a \(*\)-representation by bounded operators on a Hilbert space, the latter fact follows. For unbounded \(*\)-representation, it does not and we restrict ourselves to the class of well behaved representations which satisfy this assumption.) Since \( \mathcal{H}_n^\pm \) is the eigenspace of \( A \) with eigenvalue \( \lambda_{\pm}q^{2n} \) and these eigenvalues are pairwise distinct, the operators of \( \mathcal{Y}_r \) leave \( \mathcal{H}_n^\pm \) invariant. That is, we have a \(*\)-representation of \( \mathcal{Y}_r \) on \( \mathcal{H}_n^\pm \). But \( \mathcal{Y}_r \) commutes also with \( B \). Since \( B \) is a weighted shift operators with weights \( c_{\pm}(n) \neq 0 \) for \( n \in \mathbb{N} \), it follows that the representations of \( \mathcal{Y}_r \) on \( \mathcal{H}_n^\pm = \mathcal{H}_0^\pm \) are the same for all \( n \in \mathbb{N}_0 \). Using the structure of the representation of the \(*\)-algebra \( \mathcal{Y}_r \) on the Hilbert space \( \mathcal{K} := \mathcal{H}_0^\pm \) derived in Subsection 5.2 and inserting the formulas for \( X, X^*, Y \) and \( A, B, B^* \) into (49), one obtains the action of the generators \( e, f, k, k^{-1} \) of \( \hat{\mathcal{U}}_q(\mathfrak{su}_2) \). We do not list these formulas, but we will do so below for the generators of \( \mathcal{U}_q(\mathfrak{su}_2) \).

Let us turn to a \(*\)-representation of the larger \(*\)-algebra \( \hat{\mathcal{O}}(S^2_{qr}) \times \mathcal{U}_q(\mathfrak{su}_2) \). Then, in addition to the considerations of the preceding paragraph, we have to deal with the generator \( K \). We assumed above that \( K \) and \( A \) are strongly commuting self-adjoint operators. Therefore, \( K \) commutes with the spectral projections of \( A \). Hence each space \( \mathcal{H}_n^\pm \) is reducing for \( K \). The relation \( BK = qKB \) implies that
there is an invertible self-adjoint operator $K_0$ on $\mathcal{H}_0^\pm$ such that $K\eta_n = q^nK_0\eta_n$ for $\eta_n \in \mathcal{H}_n^\pm$. Recall that we have $XK = qKX$ and $YK = KY$ in the algebra $\mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(su_2)$. Inserting for $X$ and $Y$ the corresponding operators on $K \equiv \mathcal{H}_0^\pm = \oplus_{m=0}^\infty K_m$ from Subsection 5.2 we conclude that there exists an invertible self-adjoint operator $H$ on $\mathcal{K}$ such that $K_0\zeta_m = q^{-m}H\zeta_m$ for $\zeta_m \in \mathcal{K}_m$. Further, since $Y = qK^{-2}A$, we have $Y\zeta_0 = Y_0\zeta_0 = qH^{-2}\lambda_\pm\zeta_0$ for $\zeta_0 \in \mathcal{K}_0$. Inserting the preceding facts and the results from Subsection 5.2 into (52) and renaming by $\mathcal{G}$, we obtain the following $*$-representations (satisfying the assumptions made above) of the cross product $*$-algebra $\mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(su_2)$ for $r \in (0, \infty)$:

\[(I)_{+,H}: \ A\eta_{nm} = \lambda_+ q^{2n}\eta_{nm}, \ B\eta_{nm} = c_\pm(n)\eta_{n-1,m}, \ B^*\eta_{nm} = c_\pm(n+1)\eta_{n+1,m}, \ E\eta_{nm} = q^{-1/2}\lambda^{-1} [q^{-n}\lambda_m (\lambda_\pm^2 q^{-2m} + H^{-4})^{1/2} H\eta_{n,m-1} - q^{-m} (\lambda_\pm^2 q^{-2m-2} + \lambda_\pm^{-1} - q^{2n+2})^{1/2} H\eta_{n+1,m}], \ F\eta_{nm} = q^{-1/2}\lambda^{-1} [q^{-n}\lambda_{m+1} (\lambda_\pm^2 q^{-2m-2} + H^{-4})^{1/2} H\eta_{n,m+1} - q^{-m} (\lambda_\pm^2 q^{-2m} + \lambda_\pm^{-1} - q^{2n})^{1/2} H\eta_{n-1,m}], \ K\eta_{nm} = q^n H\eta_{nm},\]

where $H$ is an invertible self-adjoint operator on a Hilbert space $\mathcal{G}$. The domain is the direct sum $\mathcal{H} = \oplus_{n=m=0}^\infty \mathcal{H}_{nm}$, where $\mathcal{H}_{nm} = \mathcal{G}$. In the case $r = 0$, there is only the representation $(I)_{+,H}$. The case $r = \infty$ has already been treated in [15].

6 Algebras of functions and representations of the cross product algebras

6.1 Algebras of functions and the invariant state on $S^2_{qr}$

Let $\mathcal{F}(\sigma(A))$ be the $*$-algebra of all complex Borel functions on the set $\sigma(A) := \{q^{2n}\lambda_\pm, q^{2n}\lambda_\pm; n \in \mathbb{N}_0\}$, let $\mathcal{F}_b(\sigma(A))$ be the $*$-algebra of all bounded complex Borel functions on $\sigma(A)$, and let $\mathcal{F}^\infty(\sigma(A))$ be the set of all $f \in \mathcal{F}(\sigma(A))$ for which there exist an $\varepsilon > 0$ and a function $\hat{f} \in C^\infty((-\varepsilon, \varepsilon))$ such that $f = \hat{f}$ on $\sigma(A) \cap (-\varepsilon, \varepsilon)$. For $f \in \mathcal{F}^\infty(\sigma(A))$, we can assign unambiguously the value $f(0)$ at 0 by taking the limit $f(0) = \lim_{t \to 0} f(t)$. In order to be in accordance with our previous notation, we write $A$ for the function $id(t) = t$ and also for the argument of functions $f \in \mathcal{F}(\sigma(A))$. We denote by $\mathcal{F}(S^2_{qr})$, $r \in [0, \infty)$, the unital $*$-algebra generated by the $*$-algebra $\mathcal{F}(\sigma(A))$ and two generators $B, B^*$ with defining
relations

\[
B f(A) = f(q^2A)B, \quad f(A)B^* = B^* f(q^2A), \quad f \in \mathcal{F}(\sigma(A)); \quad (57)
\]
\[
B^* B = A - A^2 + r, \quad BB^* = q^2A - q^4A^2 + r, \quad \text{for } r \in [0, \infty), \quad (58)
\]
\[
B^* B = -A^2 + 1, \quad BB^* = -q^4A^2 + 1 \quad \text{for } r = \infty. \quad (59)
\]

For \( k \in \mathbb{Z} \), we set \( B^{\#k} = B^k \) if \( k \geq 0 \) and \( B^{\#k} = B^{-k} \) if \( k < 0 \). As a vector space, \( \mathcal{F}(S^2_{qr}) \) is spanned by \( \{ B^n f_1(A), f_2(A)B^{\#k}; f_1, f_2 \in \mathcal{F}(\sigma(A)), \ n, k \in \mathbb{N}_0 \} = \{ f(A)B^{\#k}; f \in \mathcal{F}(\sigma(A)), \ k \in \mathbb{Z} \} \). We denote by \( \mathcal{F}_b(S^2_{qr}) \) and \( \mathcal{F}^\infty(S^2_{qr}) \) the \(*\)-subalgebras of \( \mathcal{F}(S^2_{qr}) \) generated by \( \mathcal{F}_b(\sigma(A)) \) and \( \mathcal{F}^\infty(\sigma(A)) \), respectively, and \( B \) and \( B^* \). We introduce a left action \( \circ \) of the Hopf algebra \( \mathcal{U}_q(\mathfrak{su}_2) \) on \( \mathcal{F}^\infty(S^2_{qr}) \) by setting

\[
E \circ p(B)f(A) = q^{1/2} \left[ \frac{p(q^{-1}B) - p(qB)}{(q^{-1} - q)B} - (1 + q^2) \frac{p(q^{-3}B) - p(qB)}{(q^{-3} - q)B} \right] f(A) + q^{-1/2}p(qB)B^* \frac{f(A) - f(q^2A)}{(1 - q^2)A},
\]
\[
E \circ f(A)p(B^*) = q^{-1/2} \frac{f(q^{-2}A) - f(A)}{(q^{-2} - 1)A} B^* p(qB^*),
\]
\[
F \circ p(B)f(A) = -q^{-3/2}Bp(qB) \frac{f(q^{-2}A) - f(A)}{(q^{-2} - 1)A},
\]
\[
F \circ f(A)p(B^*) = -q^{-3/2} \frac{f(A) - f(q^2A)}{(1 - q^2)A} Bp(qB^*)
\]
\[
- q^{-1/2} f(A) \left[ \frac{p(q^{-1}B^*) - p(qB^*)}{(q^{-1} - q)B^*} - (1 + q^2) \frac{p(q^{-3}B^*) - p(qB^*)}{(q^{-3} - q)B^*} \right],
\]
\[
K \circ p(B)f(A) = p(q^{-1}B)f(A), \quad K \circ f(A)p(B^*) = f(A)p(qB^*),
\]

where \( f \) is a function in \( \mathcal{F}^\infty(\sigma(A)) \) and \( p \) is a polynomial. It can be shown that these formulas define indeed an action of the Hopf \(*\)-algebra \( \mathcal{U}_q(\mathfrak{su}_2) \) such that \( \mathcal{F}^\infty(S^2_{qr}) \) is a left \( \mathcal{U}_q(\mathfrak{su}_2) \)-module \(*\)-algebra. We omit the details of these lengthy, but straightforward computations.

From the defining relations, it is clear that the coordinate \(*\)-algebra \( \mathcal{O}(S^2_{qr}) \) of the quantum sphere is a \(*\)-subalgebra of \( \mathcal{F}^\infty(S^2_{qr}) \). On \( B, A, B^* \) considered as
elements of $\mathcal{F}^\infty(S_{qr}^2)$, the preceding formulas coincide with corresponding formulas for the actions on the generators $B$, $A$, $B^*$ of $O(S_{qr}^2)$. Hence $O(S_{qr}^2)$ is a left $\mathcal{U}_q(\mathfrak{su}_2)$-module $*$-subalgebra of $\mathcal{F}^\infty(S_{qr}^2)$.

Now we turn to the construction of a $\mathcal{U}_q(\mathfrak{su}_2)$-invariant linear functional $h$. For $f \in \mathcal{F}_b(\sigma(A))$, we put

$$h_0(f(A)) := \gamma_+ \sum_{n=0}^{\infty} f(\lambda_+ q^{2n}) q^{2n} + \gamma_- \sum_{n=0}^{\infty} f(\lambda_- q^{2n}) q^{2n},$$

where $\gamma_+ := (1-q^2)\lambda_+(\lambda_+ - \lambda_-)^{-1}$ and $\gamma_- := (1-q^2)\lambda_-(\lambda_- - \lambda_+)^{-1}$. Note that $\gamma_\pm := (1-q^2)(1/2 \pm (r+4)^{-1/2})$. (When $r = 0$, the above equation simplifies to $h_0(f(A)) = (1-q^2)\sum_{n=0}^{\infty} f(q^{2n}) q^{2n}$ since in this case $\gamma_+ = \lambda_+ = 1$ and $\gamma_- = 0$.) Define a functional $h$ on $\mathcal{F}_b(S_{qr}^2)$ by

$$h(p(B)f(A)) = p(0)h_0(f(A)), \quad h(f(A)p(B^*)) = p(0)h_0(f(A)). \quad (60)$$

For $g \in \mathcal{F}_b(\sigma(A))$, we have

$$h(g(A)) = q^2 h(g(q^2A)) + \gamma_+ g(\lambda_+) + \gamma_- g(\lambda_-). \quad (61)$$

First we show that $h$ is a faithful state on the $*$-algebra $\mathcal{F}_b(S_{qr}^2)$. We restrict ourselves to the case $r \in (0, \infty]$. Let $x = \sum_k (B^k f_k(A) + g_k(A) B^{*k}) \in \mathcal{F}_b(S_{qr}^2)$ with $f_k, g_k \in \mathcal{F}_b(\sigma(A))$. From $(57)$–$(59)$, we obtain $B^k B^* = \Pi_{i=0}^{k-1} p_r(q^{-2i}A)$ and $B^k B^{*k} = \Pi_{i=1}^{k} p_r(q^{2i}A)$, where $p_r(A) := (\lambda_+ - A)(A - \lambda_-)$. By $(60)$, we have

$$h(x^*x) = \sum_k h(B^k g_k B^{*k} + \overline{B^{*k}} B^k f_k)$$

$$= \sum_k h_0(|g_k(q^{2k}A)|^2 \prod_{i=1}^{k} p_r(q^{2i}A)) + h_0(|f_k(A)|^2 \prod_{i=0}^{k-1} p_r(q^{-2i}A)) \quad (62)$$

Note that $p_r(\lambda_\pm) = 0$ and $p_r(q^{2j}\lambda_\pm) > 0$ for $j \in \mathbb{N}$. Hence $h(x^*x) \geq 0$ by $(62)$. Assume that $h(x^*x) = 0$. From $(62)$ and the definition of $h_0$, it follows that $g_k(q^{2n}\lambda_\pm) = 0$ for $n \in \mathbb{N}_0$ and $f_k(q^{2n}\lambda_\pm) = 0$ for $n \geq k$. The latter implies $g_k(A) = 0$ and $B^k f_k(A) = 0$, so $x = 0$. That is, $h$ is a faithful state.

Now we prove that $h$ is $\mathcal{U}_q(\mathfrak{su}_2)$-invariant on the $*$-subalgebra $\mathcal{F}^\infty(S_{qr}^2)$, that is, $h(y^*x) = \varepsilon(y)h(x)$ for $y \in \mathcal{U}_q(\mathfrak{su}_2)$ and $x \in \mathcal{F}^\infty(S_{qr}^2)$. Clearly, it suffices to verify the latter condition for the generators $y = E, F, K$. For $y = K$, the assertion is obvious. Since the functional $h$ is hermitian and $\mathcal{F}^\infty(S_{qr}^2)$ is a $\mathcal{U}_q(\mathfrak{su}_2)$-module $*$-algebra, it is sufficient to check the invariance with respect to $E$. By
the definitions of the action of $E$ and the functional $h$, it is enough to show that $h(Ebf(A)) = 0$ for all $f(A) \in \mathcal{F}^\infty(S_{qr}^2)$. We carry out the proof for $r \in (0, \infty)$. Since $f \in \mathcal{F}^\infty(S_{qr}^2)$, the function $g(A) := f(A) - Af(A) + r(f(A) - f(0))/A$ is bounded on $\sigma(A)$. Hence $g \in \mathcal{F}_b(\sigma(A))$ and (61) applies to this function. Observe that $1 - \lambda_+ + r\lambda_-^1 = 0$ and $\gamma_+\lambda_+^{-1} + \gamma_-\lambda_-^{-1} = 0$ by the definitions of these constants. Using these facts and the equation $BB^* = q^2A - q^4A^2 + r$, we compute

$$q^{-1/2}Ebf(A) = f(A) - (1 + q^2)Af(A) + BB^* \frac{f(A) - f(q^2A)}{(1 - q^2)A}$$

and hence

$$q^{-1/2}(1 - q^2)h(Ebf(A)) = \gamma_+g(\lambda_+) + \gamma_-g(\lambda_-) = \gamma_+f(\lambda_+)(1 - \lambda_+ + r\lambda_-^1) + \gamma_-f(\lambda_-)(1 - \lambda_- + r\lambda_-^1) - f(0)(\gamma_+\lambda_+^{-1} + \gamma_-\lambda_-^{-1}) = 0.$$ 

Thus, $h$ is a $\mathcal{U}_q(\mathfrak{su}_2)$-invariant state on $\mathcal{F}^\infty(S_{qr}^2)$.

For the coordinate algebra $O(S_{qr}^2)$, the preceding description of the invariant functional was obtained in [10]. However for our consideration below it is crucial to have the invariant state on the larger $\ast$-algebra $\mathcal{F}^\infty(S_{qr}^2)$.

The preceding proof shows that, with the action of $E$ defined by the above formula, $h(Ebf(A)) = 0$ for $f \in \mathcal{F}_b(\sigma(A))$ if $f(0) := \lim_{t \to 0} f(t)$ exists and the function $(f(A) - f(0))/A$ is bounded on $\sigma(A)$. For instance, $h(EbfA\chi_+/(A)) = 0$, where $\chi_+$ is the characteristic function of $[0, \infty)$. But we get $h(EbfA\chi_+/(A)) = q^{1/2}(1 - q^2)^{-1}\gamma_+\lambda_- \neq 0$ for $r \in (0, \infty]$, so the $\mathcal{U}_q(\mathfrak{su}_2)$-invariance of the functional $h$ does not hold on the larger $\ast$-algebra $\mathcal{F}_b(S_{qr}^2)$.

Now we develop a second operator-theoretic approach to the $\mathcal{U}_q(\mathfrak{su}_2)$-module structure of $\mathcal{F}^\infty(S_{qr}^2)$ and to the invariant state $h$. Suppose that $\pi$ is a $\ast$-representation of the $\ast$-algebra $\mathcal{F}_b(S_{qr}^2)$ on a Hilbert space $\mathcal{H}$ such that $\ker \pi(A) = \{0\}$. Then all operators $\pi(x), x \in \mathcal{F}_b(S_{qr}^2)$, are bounded and leave the dense domain $\mathcal{D} := \cap_{n=1}^\infty \mathcal{D}(\pi(A)^{-n})$ of $\mathcal{H}$ invariant. For notational simplicity, we write $x$ for the operators $\pi(x)$ and $\pi(x)/\mathcal{D}$. From the form of the representations of $O(S_{qr}^2)$ described in Subsection 5.3 it is clear that $\text{sign}A := A|A|^{-1}$ commutes with all representation operators. For $T \in \mathcal{L}^+(\mathcal{D})$, we define

$$EbfT = -q^{1/2}\lambda_1^{-1}A^{-1}[B^*, |A|^{1/2}T|A|^{-1/2}]$$

$$= -q^{-1/2}\lambda_1^{-1}\text{sign}A|A|^{-1/2}[B^*, T]|A|^{-1/2}, \tag{63}$$

34
representations on \( \in \mathbb{R} \) orthonormal basis of eigenvectors with eigenvalues take the representation on \( \text{Tr} \) multiplicity one, defines a state on the \( \) is a \( f \) for \( x \) similar manner. Since \( \) is of trace class. Therefore, \( \) acts on \( \) for \( x \)\( \) has an \( \) is of \( \) and \( \) of \( \) listed above.

In order to define the invariant state \( h \), we specialize the representation \( \pi \). For \( r \in (0, \infty) \), let \( \pi \) be the direct sum representation on \( \mathcal{H} := \mathcal{H}^+ \oplus \mathcal{H}^- \), where the representations on \( \mathcal{H}^\pm \) are given in Subsection 5.3 with \( \mathcal{H}^+_0 = \mathbb{C} \). For \( r = 0 \), we take the representation on \( \mathcal{H} := \mathcal{H}^+ \) with \( \mathcal{H}^+_0 = \mathbb{C} \). Since the operator \( A \) has an orthonormal basis of eigenvectors with eigenvalues \( q^{2n} \lambda_\pm \) and each eigenvalue has multiplicity one, \( A \) is of trace class and so is \( |A|x \) for all \( x \in \mathcal{F}_b(S^2_{qr}) \). Obviously, \( \text{Tr}_{\mathcal{H}}|A| = (1 - q^2)^{-1}(\lambda_+ - \lambda_-) \). Therefore,

\[
h(x) = (1 - q^2)(\lambda_+ - \lambda_-)^{-1}\text{Tr}_{\mathcal{H}}|A|x, \quad x \in \mathcal{F}_b(S^2_{qr}),
\]

defines a state on the \( \ast \)-algebra \( \mathcal{F}_b(S^2_{qr}) \).

Next we show that \( h \) is \( \mathcal{U}_q(su_2) \)-invariant on \( \mathcal{F}^\infty(S^2_{qr}) \). We carry out this verification in the case \( r \in (0, \infty) \) for the generator \( E \) and for an element \( x = B^{\#n}f(A) \) of \( \mathcal{F}^\infty(S^2_{qr}) \), where \( n \in \mathbb{Z} \) and \( f \in \mathcal{F}^\infty(\sigma(A)) \). The other cases are treated in a similar manner. Since \( f \in \mathcal{F}^\infty(\sigma(A)) \), there is a bounded function \( g \) on \( \sigma(A) \) such that \( f(A) - f(0) = Ag(A) \). Write \( x = x_1 + x_2 \), where \( x_1 = B^{\#n}g(A)A \) and \( x_2 = f(0)B^{\#n} \). By (63) and (66), we have

\[
h(E_b y) = \text{const} \text{Tr}_{\mathcal{H}}|A|^{-1}[B^*, y] = \text{const} \left( \text{Tr}_{\mathcal{H}^+}[B^*, y] - \text{Tr}_{\mathcal{H}^-}[B^*, y] \right)
\]

for \( y \in \mathcal{F}^\infty(S^2_{qr}) \). Since \( g(A) \) and \( B^{\#n} \) are bounded operators and \( A \) is of trace class, \( x_1 \) is of trace class. Therefore, \( \text{Tr}_{\mathcal{H}^+}[B^*, x_1] = 0 \), so that \( h(E_b x_1) = 0 \) by (67). Since \( [B^*, B^{\#n}] \in \mathcal{A}\mathcal{O}(S^2_{qr}) \) by Lemma 6.2 below, \( [B^*, B^{\#n}] \) is of trace class. If \( n \neq 1 \), then \( \text{Tr}_{\mathcal{H}}|A|^{-1}[B^*, B^{\#n}] = 0 \) because in this case \( |A|^{-1}[B^*, B^{\#n}] \) acts on \( \mathcal{H} \) as a weighted shift operator. Hence \( h(E_b x_2) = 0 \). Suppose that \( n = 1 \). Since \( [B^*, B] = (1 - q^2)(A - (1 + q^2)A^2) \) and

\[
(1 - q^2)\text{Tr}_{\mathcal{H}^\pm}(A - (1 + q^2)A^2) = \lambda_\pm - \lambda_\pm^2 = \lambda_\pm + \lambda_-,
\]

it follows from (67) that \( h(E_b x_2) = 0 \). Thus, \( h(E_b x) = 0 \). This completes the proof of the \( \mathcal{U}_q(su_2) \)-invariance of \( h \) on \( \mathcal{F}^\infty(S^2_{qr}) \).

In both approaches, we have proved the following theorem.
Theorem 6.1 With the foregoing definitions, $F_{\infty}(S^2_{qr})$ is a left $U_q(su_2)$-module $*$-algebra which contains $O(S^2_{qr})$ as a $U_q(su_2)$-module $*$-subalgebra. The functional $h$ is a faithful $U_q(su_2)$-invariant state on the $U_q(su_2)$-module $*$-algebra $F_{\infty}(S^2_{qr})$.

In the next subsection, we shall need the following lemma. It is the algebraic version of the operator-theoretic formulas (63)–(65) stated above.

Lemma 6.2 For any $x \in O(S^2_{qr})$, we have
\[
Ax = (K^{2v}x)A, \quad xA = A(K^{-2v}x),
\]
\[
[B^*, x] = -q^{1/2}A(K^{-1}E^v x), \quad [B, x] = -q^{3/2}A(K^{-1}F^v x).
\]

In particular, the commutators $[B, x]$ and $[B^*, x]$ are in $A \cdot O(S^2_{qr})$.

It suffices to prove the lemma for elements $x$ of the vector space basis $\{A^n B^{\#k}; n \in \mathbb{N}_0, k \in \mathbb{Z}\}$ of $O(S^2_{qr})$. This can be done by a straightforward induction argument. We omit the details.

6.2 Description of quantum line bundles by charts

Throughout this subsection, we suppose that $j \in \frac{1}{2}\mathbb{Z}$.

From Proposition 4.6 and Theorems 3.1 and 4.1, it follows that each irreducible integrable $*$-representation of $O(S^2_{qr}) \rtimes U_q(su_2)$ can be realized as a projective module $M_j \cong O(S^2_{qr})^{2|j|+1} P_j$. It is known that the projective modules $M_j$ can be considered as line bundles over the quantum spheres $S^2_{qr}$ [2, 6]. In this section, we describe the quantum line bundles $M_j$ by two “charts”. The charts will be realized by algebras of functions on the positive and the negative part of the spectrum of the self-adjoint operator $A$. These function algebras lead to $*$-representations of the $*$-algebra $F_{b}(S^2_{qr}) \otimes Y_r$ by left and right multiplications. It will be shown that each chart is related to a tensor product of an irreducible $*$-representation of $O(S^2_{qr})$ from Subsection 5.3 and an irreducible $*$-representation of the $*$-algebra $Y_r$ from Subsection 5.2.

Recall that the isomorphism $\Psi_j$ realizing $M_j \cong O(S^2_{qr})^{2|j|+1} P_j$ is given by Equation (22). In what follows, we consider $O(S^2_{qr})^{2|j|+1} P_j$ as a subspace of $F_{b}(S^2_{qr})^{2|j|+1}$. Our next aim is to equip $F_{b}(S^2_{qr})^{2|j|+1}$ with an inner product such that $\Psi_j$ becomes an isometry. We begin by proving an auxiliary lemma.

Lemma 6.3 Let $x \in O(S^2_{qr})$ and $k, l \in \{-|j|, -|j| + 1, \ldots, -|j|\}$. Then
\[
h(v^{|j|}_{kj} x v^{|j|}_{lj}) = c_j q^{2|j|-2k} h(x^{|j|}_{lj} v^{|j|}_{kj}^*),
\]
where \( c_j = h(v_{j,j}^{[j]})^{-1} \). For \( y \in F_0(S^2_{q^r}) \) and \( g \in F_0(\sigma(A)) \), we have

\[
h(yg(A)) = h(g(A)y), \quad h(yB) = q^2 h(By), \quad h(yB^*) = q^{-2} h(B^*y).
\] (69)

**Proof.** We first show that it suffices to prove (68) for \( k = |j| \). Indeed, from (20), it follows that there is a non-zero real constant \( \gamma_k^j \) such that \( v_{k,j}^{[j]} = \gamma_k^j F^{[j]}_{\lambda^k} v_{[j]}^{[j]} \). Recall that \( h((f\circ y)^\ast z) = h(y^\ast f\circ z) \) and \( (f\circ y)^\ast = S(f)^\ast y^\ast \) for \( f \in U_q(\mathfrak{su}_2) \), \( y, z \in \mathcal{O}(SU_q(2)) \). Assuming that (68) holds for \( \kappa = |j| \), we get

\[
h(v_{k,j}^{[j]} x v_{l,j}^{[l]}) = \gamma_k^j h((F^{[j]}_{\lambda^k} v_{[j]}^{[j]})^\ast x v_{l,j}^{[l]}) = \gamma_k^j h(v_{[j]}^{[j]} E^{[j]}_{\lambda^k} (x v_{l,j}^{[l]}))
\]

\[
= c_j \gamma_k^j h((S(E^{[j]}_{\lambda^k}) v_{l,j}^{[l]} v_{l,j}^{[l]})^\ast v_{[j]}^{[j]}) = c_j \gamma_k^j h(x v_{l,j}^{[l]} S(E^{[j]}_{\lambda^k}) v_{l,j}^{[l]} v_{l,j}^{[l]})
\]

\[
= c_j \gamma_k^j h(x v_{l,j}^{[l]} (S^2(E^{[j]}_{\lambda^k}) v_{l,j}^{[l]} v_{l,j}^{[l]}) v_{l,j}^{[l]}) = c_j q^{2(|j|-k)} h(x v_{l,j}^{[l]} v_{[j]}^{[j]}).
\]

Next we note that it suffices to prove that

\[
h(v_{l,j}^{[j]} v_{l,j}^{[j]+n}) = c_j h(v_{l,j}^{[j]+n} v_{l,j}^{[j]*})
\] (70)

for all \( n \in \mathbb{N}_0 \) since \( x v_{l,j}^{[j]} \in M_j \) and the elements \( v_{l,j}^{[j]+n} \) span \( M_j \) by Theorem 3.1. As the elements \( v_{l,j}^{[j]+n} \) form an orthonormal set in \( \mathcal{O}(SU_q(2)) \) with inner product defined by (16), we only have to show that the right-hand side of (70) vanishes whenever \( l \neq |j| \) or \( n > 0 \). (Observe that Equation (70) is trivially satisfied for \( l = |j| \) and \( n = 0 \).) If \( l \neq |j| \), then it follows from \( K v_{l,j}^{[j]+n} v_{l,j}^{[j]*} = q^{2(|j|-l)} v_{l,j}^{[j]+n} v_{l,j}^{[j]*} \) and the \( U_q(\mathfrak{su}_2) \)-invariance of \( h \) that \( h(v_{l,j}^{[j]+n} v_{l,j}^{[j]*}) = 0 \). If \( n > 0 \), then \( v_{l,j}^{[j]+n} = \kappa_l^{[j]+n} F v_{l,j}^{[j]+n} \) with a non-zero real constant \( \kappa_l^{[j]+n} \). Hence

\[
h(v_{l,j}^{[j]+n} v_{l,j}^{[j]*}) = \kappa_l^{[j]+n} h((S(F)^* S v_{l,j+1,j}^{[j]+n}) v_{l,j+1,j}^{[j]+n}) = \kappa_l^{[j]+n} h(v_{l,j+1,j}^{[j]+n} S(F) v_{l,j+1,j}^{[j]+n})
\]

\[
= \kappa_l^{[j]+n} q^{-2} h(v_{l,j+1,j}^{[j]} (E v_{l,j}^{[j]} v_{l,j}^{[j]*}) v_{l,j}^{[j]*}) = 0
\]

since \( v_{l,j}^{[j]} \) is a highest weight vector.

To prove (69), we can assume that \( y = f(A) B^{*k} \), where \( f \in F_0(\sigma(A)) \) and \( k \in \mathbb{Z} \), because these elements span \( F_0(S^2_{q^r}) \). Since \( h(yg(A)) = h(g(A)y) = 0 \) for \( k \neq 0 \), the first equality of (69) is obvious. The third equality of (69) follows from the second one because the state \( h \) is hermitian. As \( h(yB) = h(By) = 0 \) for \( k \neq -1 \), it remains to treat the case \( k = -1 \). Using the relations \( B^* B = \)
\((A - \lambda_-)(\lambda_+ - A), BB^* = (q^2A - \lambda_-)(\lambda_+ - q^2A)\) and Equation \((61)\), we get
\[
\begin{align*}
h(yB) &= h(f(A)B^*B) = h(f(A)(A - \lambda_-)(\lambda_+ - A)) \\
&= h(q^2f(q^2A)(q^2A - \lambda_-)(\lambda_+ - q^2A)) = h(q^2f(q^2A)BB^*) \\
&= h(q^2Bf(A)B^*) = h(q^2By).
\end{align*}
\]
\(\square\)

**Proposition 6.4** Let \(\mathcal{L}_2(S^2_{qr})^{2|j|+1}\) be the Hilbert space completion of \(\mathcal{F}_h(S^2_{qr})^{2|j|+1}\) with respect to the inner product given by
\[
\langle(y_{-|j|}, \ldots, y_{|j|}), (z_{-|j|}, \ldots, z_{|j|})\rangle = c_jq^{2|j|}\sum_{k=-|j|}^{|j|} q^{-2k}h(z_ky_k), \quad (71)
\]
where \(c_j = h(v^{|j|}_{-|j|}v^{|j|}_{|j|})^{-1}\). Then the right multiplication by \(P_j\) on \(\mathcal{F}_h(S^2_{qr})^{2|j|+1}\) defines an orthogonal projection on the Hilbert space \(\mathcal{L}_2(S^2_{qr})^{2|j|+1}\). The isomorphism \(\Psi_j : \mathcal{O}(S^2_{qr})^{2|j|+1}P_j \xrightarrow{\cong} M_j\) from Equation \((22)\) is an isometry.

**Proof.** As \(P^2_j = P_j\), we only have to show that \(P_j\) is self-adjoint with respect to the inner product \((71)\). Since \(K^*v^{|j|}_{-|j|}v^{|j|}_{|j|} = q^{-|k|}v^{|j|}_{-|j|}v^{|j|}_{|j|}\), there exists a polynomial \(p^{|j|}_{|k|}(A)\) such that \(v^{|j|}_{-|j|}v^{|j|}_{|j|} = B^{2|k|}q^{-|k|}p^{|j|}_{|k|}(A)\). Hence, by \((69)\),
\[
h(xv^{|j|}_{-|j|}v^{|j|}_{|j|}) = h(q^{2(k-|j|)}v^{|j|}_{-|j|}v^{|j|}_{|j|}x) \quad (72)
\]
for \(x \in \mathcal{F}_h(S^2_{qr})\). This gives
\[
\begin{align*}
\langle(y_{-|j|}, \ldots, y_{|j|})P_j, (z_{-|j|}, \ldots, z_{|j|})\rangle &= c_jq^{2|j|}\sum_{k=-|j|}^{|j|} q^{-2k}h([2|j|+1]^{-1}q^{-(l+k)}z_ky_lv^{|j|}_{-|j|}v^{|j|}_{|j|}) \\
&= c_jq^{2|j|}\sum_{k=-|j|}^{|j|} q^{-2l}h([2|j|+1]^{-1}q^{-(l+k)}y_lv^{|j|}_{-|j|}v^{|j|}_{|j|}z_ky_l) \\
&= \langle(y_{-|j|}, \ldots, y_{|j|}), (z_{-|j|}, \ldots, z_{|j|})P_j\rangle,
\end{align*}
\]
which proves the first assertion of the proposition.

Let \((y_{-|j|}, \ldots, y_{|j|}) = (y_{-|j|}, \ldots, y_{|j|})P_j \in \mathcal{O}(S^2_{qr})^{2|j|+1}P_j\) and \((z_{-|j|}, \ldots, z_{|j|}) = (z_{-|j|}, \ldots, z_{|j|})P_j \in \mathcal{O}(S^2_{qr})^{2|j|+1}P_j\). From the definition of \(P_j\), it follows that
Similarly, we have the matrix

This shows that since

we compute

\[ \langle \Psi_j(y_{-|j|}, \ldots, y_{|j|}), \Psi_j(z_{-|j|}, \ldots, z_{|j|}) \rangle = [2|j|+1]^{-1} \sum_{n=-|j|}^{|j|} q^{-n} z_n v_{nj}^{|j|} \]

which shows the second assertion of the proposition.

Since \( \Psi_j \) is an isometric isomorphism, the \(*\)-representation \( \hat{\pi}_j \) of the crossed product algebra \( \mathcal{O}(S^2_{qr}) \ast \mathcal{U}_q(su_{2}) \) on \( M_j \) is unitarily equivalent to the \(*\)-representation \( \hat{\pi}_j := \Psi_j^{-1} \circ \pi_j \circ \Psi_j \) on \( \mathcal{O}(S^2_{qr})^{2|j|+1} P_j \). The restriction of \( \hat{\pi}_j \) to \( \mathcal{O}(S^2_{qr}) \otimes \mathcal{Y}_r \) can be described by left and right multiplications by matrices with entries from \( \mathcal{O}(S^2_{qr}) \). Clearly, for \( z \in \mathcal{O}(S^2_{qr}) \), \( \hat{\pi}_j(z)(y_{-|j|}, \ldots, y_{|j|}) P_j = z(y_{-|j|}, \ldots, y_{|j|}) P_j \). Since \( X = q^{3/2} \lambda AF K^{-1} + q^{-1} B \) commutes with \( y_k \in \mathcal{O}(S^2_{qr}) \), we obtain

\[ [2|j|+1]^{1/2} \hat{\pi}_j(X)(y_{-|j|}, \ldots, y_{|j|}) = \Psi_j^{-1}( \sum_{k=-|j|}^{|j|} q^{-k} y_k X v_{kj}^{|j|} ) \]

This shows that \( \hat{\pi}_j(X)((y_{-|j|}, \ldots, y_{|j|}) P_j) = ((y_{-|j|}, \ldots, y_{|j|}) \mathbf{m}_j) P_j \), where the matrix \( \mathbf{m}_j = (m^j_{kj}^l)_{k,l=-|j|} \in M_{2|j|+1}(\mathcal{O}(S^2_{qr})) \) has the entries

\[
m^j_{kk} = q^{-1} B, \quad m^j_{k,k-1} = q^{1/2-k} \lambda[|j|-k+1]^{1/2}[|j|+k]^{1/2} A, \quad m^j_{kl} = 0, \quad l \neq k, k-1.
\]

Similarly, we have \( \hat{\pi}_j(X^*)(y_{-|j|}, \ldots, y_{|j|}) P_j = ((y_{-|j|}, \ldots, y_{|j|}) \mathbf{m}^*_j) P_j \) with matrix \( \mathbf{m}^*_j = (m^{\dagger j}_{kj}^l)_{k,l=-|j|} \) given by

\[
m^{\dagger j}_{kk} = qB^*, \quad m^{\dagger j}_{k-1,k} = q^{5/2-k} \lambda[|j|-k+1]^{1/2}[|j|+k]^{1/2} A, \quad m^{\dagger j}_{kl} = 0, \quad l \neq k, k-1.
\]
and \( \hat{\pi}_j(Y)((y_{-|j|}, \ldots, y_{|j|}) P_j) = ((y_{-|j|}, \ldots, y_{|j|}) n_j) P_j \) with \( n_j = (n^{ij}_{kk})_{k,l = -|j|} \) given by
\[
n^{ij}_{kk} = q^{-2k+1}A, \quad n^{ij}_{kl} = 0, \quad l \neq k.
\]

It is easy to check that
\[
n_j m_j = q^2 m_j n_j, \quad n_j m_j = q^{-2} m_j n_j, \quad m_j m_j - q^2 m_j^\dagger m_j = (1 - q^2)(n_j^2 + r).
\]

Note that all operators are bounded. Hence the restriction of \( \hat{\pi}_j \) to \( \mathcal{O}(S^2_{qr}) \otimes \mathcal{Y}_r \) yields a bounded \( * \)-representation \( \hat{\pi}^b_j \) on the Hilbert space \( \mathcal{L}_2(S^2_{qr})^{2|j|+1} P_j \). This representation can be extended to a bounded \( * \)-representation, denoted again by \( \hat{\pi}^b_j \), of \( \mathcal{F}_b(S^2_{qr}) \otimes \mathcal{Y}_r \) on \( \mathcal{L}_2(S^2_{qr})^{2|j|+1} P_j \) such that \( \hat{\pi}^b_j(f), f \in \mathcal{F}_b(S^2_{qr}), \) acts on \( \mathcal{F}_b(S^2_{qr})^{2|j|+1} P_j \) by left multiplication.

Let \( \chi_- \) and \( \chi_+ \) denote the characteristic functions of the intervals \((-\infty, 0]\) and \([0, \infty)\), respectively. Set \( \mathcal{F}_b(S^2_{qr})^+ := \mathcal{F}_b(S^2_{qr}) \chi_+(A) \) and, for \( r > 0 \), \( \mathcal{F}_b(S^2_{qr})^- := \mathcal{F}_b(S^2_{qr}) \chi_-(A) \). Since \( \chi_\pm(A) \) commutes with all elements of \( \mathcal{F}_b(S^2_{qr}), \mathcal{F}_b(S^2_{qr}) \) is a unital \( * \)-algebra with unit element \( \chi_\pm(A) \) and an ideal of \( \mathcal{F}_b(S^2_{qr}) \). Our next aim is to describe the \textit{“charts”} \( \mathcal{F}_b(S^2_{qr})^{2|j|+1} P_j \) of the quantum line bundle \( \mathcal{F}_b(S^2_{qr})^{2|j|+1} P_j \) by the function algebras \( \mathcal{F}_b(S^2_{qr})^- \) and \( \mathcal{F}_b(S^2_{qr})^+ \) themselves.

**Lemma 6.5** Set \( \xi(s) := -sab + (s^2 - 1)qbc + sq^2dc \), where the parameter \( s \) is defined as in Section 3. Then

(i) \( (a - qsc)(d + sb) = 1 - \xi(s), \quad (d + q^{-1}sb)(a - sc) = 1 - q^{-2}\xi(s), \)
\( (b - qsd)(-qc - qsa) = q^2s^2 + \xi(s), \quad (qc + sa)(-b + sd) = s^2 + \xi(s). \)

(ii) \( (a - qsc)\xi(gs) = q^2\xi(s)(a - qsc), \quad (d + q^{-1}sb)\xi(q^{-1}s) = q^{-2}\xi(s)(d + q^{-1}sb), \)
\( (b - qsd)\xi(gs) = \xi(s)(b - qsd), \quad (c + q^{-1}sa)\xi(q^{-1}s) = \xi(s)(c + q^{-1}sa). \)

(iii) \( v^j_{jj} v^{-j*}_{jj} = \gamma^+_j((\lambda_+ - A)(\lambda_+ - q^{-2}A) \ldots (\lambda_+ - q^{-4j+2}A), \quad j > 0, \)
\( v^{-j*}_{jj} v^j_{jj} = \gamma^-_j((q^2A - \lambda_-)(q^4A - \lambda_-) \ldots (q^{4j}A - \lambda_-), \quad j > 0, \)
\( v^j_{lj} v^{-j*}_{lj} = \gamma^+_j((\lambda_+ - q^2A)(\lambda_+ - q^4A) \ldots (\lambda_+ - q^{4j}A), \quad j < 0, \)
\( v^{-j*}_{lj} v^j_{lj} = \gamma^-_j((\lambda_+ - q^2A)(\lambda_+ - q^4A) \ldots (\lambda_+ - q^{4j}A - \lambda_-), \quad j < 0, \)
with non-zero constants \( \gamma^\pm_j \in \mathbb{R}. \)

(iv) The function \( v^{|j|}_{jj} v^{-|j|*}_{jj} \chi_-(A) \) is invertible in \( \mathcal{F}_b(S^2_{qr})^- \) for \( r > 0. \)

The function \( v^{-|j|}_{-jj} v^{|j|*}_{-jj} \chi_+(A) \) is invertible in \( \mathcal{F}_b(S^2_{qr})^+. \)
Lemma 6.6

With the inner product on \( F_b(S_{qr}^2) \) defined by \( \langle f, g \rangle := h(g^*f) \), there is an isometric isomorphism \( \hat{\Psi}_{j,\pm} \) from \( F_b(S_{qr}^2) \) onto \( F_b(S_{qr}^2)^{\pm} \) given by

\[
\hat{\Psi}_{j,-}(f) = q^{2j}[2j+1]^{1/2}(0, \ldots, 0, f(v_j^j v_j^j - (A))^{-1/2}) P_j, \quad j > 0,
\]

\[
\hat{\Psi}_{j,-}(f) = q^{2j}|2j+1|^{1/2}(f(v_j^j v_j^j - (A))^{-1/2}, 0, \ldots, 0) P_j, \quad j < 0,
\]

\[
\hat{\Psi}_{j,+}(g) = q^{-2j}[2j+1]^{1/2}(g(v_j^j v_j^j - (A))^{-1/2}, 0, \ldots, 0) P_j, \quad j > 0,
\]

\[
\hat{\Psi}_{j,+}(g) = q^{-2j}[2j+1]^{1/2}(g(v_j^j v_j^j - (A))^{-1/2}, 0, \ldots, 0) P_j, \quad j < 0,
\]

where \( f \in F_b(S_{qr}^2) \) and \( g \in F_b(S_{qr}^2) \).

Proof. (i) and (ii) follow by straightforward computations.

(iii) Let \( j > 0 \). Then \( \hat{\Psi}_{j,\pm} = \hat{\Psi}_{j,\pm} \) where \( \hat{\Psi}_{j,\pm} \) is defined by Equation (17). From (i) and (ii), it follows that

\[
(d + q^{-1}sb)(d + q^{-2j}sb)(a - q^{-2j+1}sc)(a - sc) = (1 - q^{-2}x(s))(1 - q^{-4j}x(s)).
\]

Inserting \( x(s) = \pi^{-1/2} A = \lambda^{-1} q^2 A \) gives the result. The other cases are treated analogously.

(iv) As \( A < 0 \) on the interval \((-\infty, 0]\) and \( \lambda > 0 \), each factor \( (\lambda - q^{-2}A) \), \( k \in \mathbb{Z}, \) is invertible in \( F_b(S_{qr}^2)_- \). Likewise, for \( r > 0 \), each factor \( (q^{2k}A - \lambda_-) \), \( k \in \mathbb{Z}, \) is invertible in \( F_b(S_{qr}^2)_+ \) since \( \lambda_- < 0 \).

(v) Let \( j > 0 \). In the proof of Proposition 6.4, we argued that \( v_j^j v_j^j = B_j^j k_j^j (A) \). Note that \( B_j^j = q_2^2 \xi_2^j B \). This together with (iii) gives

\[
v_j^j v_j^j (v_j^j v_j^j - (A))^{-1} v_j^j v_j^j = (N_{jj}^j(1 - q^{2(j-k)} s) \xi_2^j)(1 - q^{2(j+k)} s)(1 - q^{2(j-k)} s) v_j^j v_j^j.
\]

On the other hand, \( v_j^j v_j^j = N_{jj}^j(1 - \xi_2^j) \xi_2^j(1 - q^{2j} s)(1 - q^{2j-2} s)(1 - q^{2j-4} s) \xi_2^j \) by (i) and (ii).

The vector \( v_j^j_k \) is a linear combination of products consisting of factors \( d + q^{-n}sb \) and \( (c + q^{-m}sa) \), \( 1 \leq n, m \leq 2j \), where the terms \( d + q^{-n}sb \) and \( (c + q^{-m}sa) \) appear \( j + k \)-times and \( j - k \)-times, respectively. Applying (i) and (ii), we see that \( v_j^j v_j^j v_j^j = N_{jj}^j(1 - q^{2j} s)(1 - q^{2j-2} s)(1 - q^{2j-4} s) \xi_2^j v_j^j \). Inserting this identity into above equation gives the result. The other cases are handled similarly. \( \square \)
Proof. We carry out the proof for $\tilde{\Psi}_{j,-}$ and $j > 0$. The other cases are treated in the same manner. By Lemma 6.5, $v_{j,j}^j, v_{j,j}^j*(A)$ is invertible. Hence $\tilde{\Psi}_{j,-}$ is well defined. Fix $(y_{-j}, \ldots, y_j) = (y_{-j}, \ldots, y_j)P_j \in \mathcal{F}_b(S^2_{qr})_{-}^{2j+1} P_j$. Let $z = \sum_{k=-j}^j q^{2j-k} y_k v_{k,j}^k v_{j,j}^j*(v_{j,j}^j v_{j,j}^j* (A))^{-1}$ and $(z_{-j}, \ldots, z_j) = (0, \ldots, 0, z)P_j$. By Lemma 6.5

$$z_i = [2j+1]^{-1} \sum_{k=-j}^j q^{-(k+l)} y_k v_{k,j}^k v_{j,j}^j*(v_{j,j}^j v_{j,j}^j* (A))^{-1} v_{j,j}^j v_{j,j}^j$$

$$= [2j+1]^{-1} \sum_{k=-j}^j q^{-(k+l)} y_k v_{k,j}^k v_{j,j}^j* (A) = y_i.$$  

Thus $\tilde{\Psi}_{j,-}(q^{-2j}[2j+1]^{-1/2} z(v_{j,j}^j v_{j,j}^j* (A))^{1/2}) = (y_{-j}, \ldots, y_j)$, so $\tilde{\Psi}_{j,-}$ is surjective.

Next we verify that $\tilde{\Psi}_{j,-}$ is isometric. Let $f \in \mathcal{F}_b(S^2_{qr})_{-}$. Since $P_j$ is a projection, we have $[2j+1]^{-2} \sum_{k=-j}^j q^{-2(j+k)} v_{j,j}^j v_{j,j}^j* v_{j,j}^j = [2j+1]^{-1} q^{-2j} v_{j,j}^j v_{j,j}^j$. Using Equations (69) and (72), we conclude that

$$h(x(v_{j,j}^j v_{j,j}^j* (A))^{-1/2} v_{j,j}^j v_{j,j}^j) = q^{2(k-j)} h((v_{j,j}^j v_{j,j}^j* (A))^{-1/2} v_{j,j}^j v_{j,j}^j).$$

From these relations, it follows that, for $f \in \mathcal{F}_b(S^2_{qr})_{-}$,

$$\|\tilde{\Psi}_{j,-}(f)\|^2 = [2j+1]^{-1} q^{4j} \sum_{k=-j}^j q^{-2k}$$

$$\times h(q^{-2(j+k)} v_{j,j}^j v_{j,j}^j* (v_{j,j}^j v_{j,j}^j* (A))^{-1/2} f* f((v_{j,j}^j v_{j,j}^j* (A))^{-1/2} v_{j,j}^j v_{j,j}^j))$$

$$= h(f* f) = \|f\|^2.$$  

As multiplication by elements of $\mathcal{F}_b(S^2_{qr})$ leaves the decomposition $\mathcal{F}_b(S^2_{qr}) = \mathcal{F}_b(S^2_{qr})_{-} \oplus \mathcal{F}_b(S^2_{qr})_{+}$ invariant, we have $\mathcal{F}_b(S^2_{qr})^{2j+1} P_j = \mathcal{F}_b(S^2_{qr})_{-}^{2j+1} P_j \oplus \mathcal{F}_b(S^2_{qr})_{+}^{2j+1} P_j$ and the $*$-representation $\tilde{\pi}^b_j$ of $\mathcal{F}_b(S^2_{qr}) \otimes \mathcal{Y}$ decomposes into a direct sum $\tilde{\pi}^b_j = \tilde{\pi}^b_{j,+} \oplus \tilde{\pi}^b_{j,-}$ of $*$-representations $\tilde{\pi}^b_j$, on $\mathcal{F}_b(S^2_{qr})_{-} \otimes \mathcal{Y}$ on $\mathcal{F}_b(S^2_{qr})_{+}$. Using the isometric isomorphism $\tilde{\Psi}_{j,+}$, the $*$-representation $\tilde{\pi}^b_{j,+}$ is unitarily equivalent to a $*$-representation $\rho_{j,+} := (\tilde{\Psi}_{j,+})^{-1} \circ \tilde{\pi}^b_{j,+} \circ \tilde{\Psi}_{j,+}$ of $\mathcal{F}_b(S^2_{qr})_{-} \otimes \mathcal{Y}$ on $\mathcal{F}_b(S^2_{qr})_{+}$.

Theorem 6.7 The $*$-representation $\rho_{j,+} := (\tilde{\Psi}_{j,+})^{-1} \circ \tilde{\pi}^b_{j,+} \circ \tilde{\Psi}_{j,+}$ of $\mathcal{F}_b(S^2_{qr}) \otimes \mathcal{Y}$ on $\mathcal{F}_b(S^2_{qr})_{+}$ is given by

$$\rho_{j,-}(X)(f) = q^{-1} f B(\lambda_+ - q^{-2j} A)^{1/2}(\lambda_+ - A)^{-1/2}, \quad \rho_{j,-}(Y)(f) = q^{2j+1} f A, \quad \rho_{j,-}(X^*)(f) = qf(\lambda_+ - q^{-2j} A)^{1/2}(\lambda_+ - A)^{-1/2} B^*,$$

42
\[ \rho_{j,+}(X)(f) = q^{-1} f B(q^{4j} A - \lambda_-)^{1/2}(A - \lambda_-)^{-1/2}, \quad \rho_{j,+}(Y)(f) = q^{2j+1} f A, \]
\[ \rho_{j,+}(X^*)(f) = q f (q^{4j} A - \lambda_-)^{1/2}(A - \lambda_-)^{-1/2} B^*, \]

and \( \rho_{j,\pm}(x)(f) = xf \) for \( x \in F_b(S_{qr}^2), \ f \in F_b(S_{qr}^2)_\pm \).

In particular, all representation operators \( \rho_{j,\pm}(y), \ y \in F_b(S_{qr}^2) \otimes \mathbb{V}_r \), are bounded and \( \rho_{j,\pm} \) extends to a \(*\)-representation, denoted also by \( \rho_{j,\pm} \), on the Hilbert space completion \( L_2(S_{qr}^2)_\pm \) of \( F_b(S_{qr}^2)_\pm \). The restriction of this \(*\)-representation \( \rho_{j,\pm} \) to \( \mathcal{O}(S_{qr}^2) \otimes \mathbb{V}_r \) is unitarily equivalent to a tensor product representation \( \sigma^\pm \otimes \sigma_j^\pm \) on \( \mathcal{H}^\pm \otimes \mathcal{K}^\pm \), where \( \sigma^\pm \) denotes the irreducible \(*\)-representation of \( \mathcal{O}(S_{qr}^2) \) on \( \mathcal{H}^\pm \) from Subsection 5.2 and \( \sigma_j^\pm \) denotes the irreducible \(*\)-representation of \( \mathcal{V}_r \) on \( \mathcal{K}^\pm \) from Subsection 5.2 with \( Y_0 = q^{j+1}\lambda_\pm \). The restriction of the \(*\)-representation \( \pi_j \cong \bar{\pi}_j \cong \pi_j \) of \( \mathcal{O}(S_{qr}^2) \rtimes U_q(\mathfrak{su}_2) \) to \( \mathcal{O}(S_{qr}^2) \otimes \mathcal{V}_r \) is unitarily equivalent to the direct sum representation \( \rho_{j,-} \oplus \rho_{j,+} \).

**Proof.** Clearly, for \( x \in F_b(S_{qr}^2) \), \( \rho_{j,-}(x) \) acts by left multiplication, that is, \( \rho_{j,-}(x)(g) = x g \), \( g \in F_b(S_{qr}^2)_- \). Our next aim is to compute the action of the generators of \( \mathcal{V}_r \). Let \( g \in F_b(S_{qr}^2)_- \). By the definition the matrix \( n_j \),
\[ \rho_{j,-}(Y)(g) = (\hat{\Psi}_{j,-})^{-1}(q^{2j}[2j+1]^{1/2}(0, ..., 0, f(v^j_{jj} v^j_{jj}^* \chi_-(A))^{-1/2}) n_j P_j \]
\[ = (\hat{\Psi}_{j,-})^{-1}(q^{2j}[2j+1]^{1/2}(0, ..., 0, q^{-2j+1} f A(v^j_{jj} v^j_{jj}^* \chi_-(A))^{-1/2}) P_j) = q^{-2j+1} f A. \]

From Lemma 6.5 (iii), it follows that
\[ (v^j_{jj} v^j_{jj}^* \chi_-(A))^{-1/2} B^*(v^j_{jj} v^j_{jj}^* \chi_-(A))^{1/2} = \chi_-(A)(\lambda_+ - q^{-4j} A)^{-1/2}(\lambda_- - A)^{-1/2} B^*. \]

Using this identity and the explicit form of the matrix \( m_j^\pm \), we see that
\[ \rho_{j,-}(X^*)(g) = (\hat{\Psi}_{j,-})^{-1}(q^{2j}[2j+1]^{1/2}(0, ..., 0, f(v^j_{jj} v^j_{jj}^* \chi_-(A))^{-1/2}) m_j^\pm P_j) \]
\[ = q f(\lambda_+ - q^{-4j} A)^{1/2}(\lambda_- - A)^{-1/2} B^*. \]

The operator \( \rho_{j,-}(X) \) is determined by the relation \( \rho_{j,-}(X) = \rho_{j,-}(X^*)^* \). Since \( h(g^* f \varphi(A) B^*) = h(q^{-2} g B \varphi(A)) f \) for all \( f, g, \varphi(A) \in F_b(S^2_{qr}) \), we obtain
\[ \rho_{j,-}(X)(g) = q^{-1} g B(\lambda_+ - q^{-4j} A)^{1/2}(\lambda_- - A)^{-1/2}. \]

This proves the formulas of the theorem for \( \rho_{j,-}, j > 0 \). The other cases are treated in the same way. From the preceding formulas it is clear that all representation operators \( \rho_{j,\pm}(y), y \in F_b(S_{qr}^2) \otimes \mathbb{V}_r \), are bounded.
For \( n \in \mathbb{N}_0 \), let \( \chi_{n, \pm} \) denote the characteristic function of the point \( q^{2n}\lambda_{\pm} \).

Define
\[
\vartheta_{nl}^\pm := c_{nl}^\pm \chi_{n, \pm}(A)B^nl, \quad n \in \mathbb{N}_0, \quad l \in \mathbb{Z}, \quad l \geq -n, 
\]
where, with \( \gamma_{\pm} \) defined in Subsection 6.1,
\[
c_{n,0}^\pm = \gamma_{\pm}^{-1/2} q^{-n}, \\
c_{nl}^\pm = \gamma_{\pm}^{-1/2} q^{-n-l} \left( \prod_{m=0}^{l-1} (q^{2(n-m)} \lambda_{\pm} - \lambda_-)(\lambda_+ - q^{2(n-m)} \lambda_{\pm}) \right)^{-1/2}, \quad l < 0, \\
c_{nl}^\pm = \gamma_{\pm}^{-1/2} q^{-n-l} \left( \prod_{m=1}^{l} (q^{2(n+m)} \lambda_{\pm} - \lambda_-)(\lambda_+ - q^{2(n+m)} \lambda_{\pm}) \right)^{-1/2}, \quad l > 0.
\]

For \( r = 0 \), only \( \vartheta_{nl}^+ \) is considered. The set \( \{ \vartheta_{nl}^\pm : n \in \mathbb{N}_0, l \in \mathbb{Z}, l \geq -n, \} \) is an orthonormal basis of \( \mathcal{L}_2(S^2_{q^r}) \). Note that \( c_{n}(n) = \lambda_{n}(q^{2n}\lambda_{2}^{2} + r)^{1/2} = ((q^{2n}\lambda_{\pm} - \lambda_-)(\lambda_+ - q^{2n}\lambda_{\pm}))^{1/2} \). Using the commutation rules in \( \mathcal{F}_b(S^2_{q^r}) \) and applying \( f(A)\chi_{n, \pm}(A) = f(q^{2n}\lambda_{\pm})\chi_{n, \pm}(A) \) for all \( f \in \mathcal{F}_b(\sigma(A)) \), we get
\[
\rho_{j, \pm}(A)\vartheta_{nl}^\pm = \lambda_{\pm} q^{2n}\vartheta_{nl}^\pm, \\
\rho_{j, \pm}(B)\vartheta_{nl}^\pm = c_{\pm}(n)\vartheta_{n-1,l+1}^\pm, \quad \rho_{j, \pm}(B^*)\vartheta_{nl}^\pm = c_{\pm}(n+1)\vartheta_{n+1,l-1}^\pm, \\
\rho_{j, \pm}(Y)\vartheta_{nl}^\pm = \lambda_{\pm} q^{\pm 2j+1} q^{2(n+l)}\vartheta_{nl}^\pm, \\
\rho_{j, \pm}(X)\vartheta_{nl}^\pm = \lambda_{n+l+1}(q^{2(n+l)}(\lambda_{\pm} q^{\pm 2j+1})^2 + r)^{1/2} \vartheta_{n,l+1}^\pm, \\
\rho_{j, \pm}(X^*)\vartheta_{nl}^\pm = \lambda_{n+l}(q^{2(n+l-1)}(\lambda_{\pm} q^{\pm 2j+1})^2 + r)^{1/2} \vartheta_{n,l-1}^\pm.
\]

Renaming \( \zeta_{nk}^\pm := \vartheta_{nk}^\pm, k, n \in \mathbb{N}_0 \), we obtain
\[
\rho_{j, \pm}(A)\zeta_{nk}^\pm = \lambda_{\pm} q^{2n}\zeta_{nk}^\pm, \quad \rho_{j, \pm}(B)\zeta_{nk}^\pm = c_{\pm}(n)\zeta_{n-1,k}^\pm, \\
\rho_{j, \pm}(B^*)\zeta_{nk}^\pm = c_{\pm}(n+1)\zeta_{n+1,k}^\pm, \\
\rho_{j, \pm}(Y)\zeta_{nk}^\pm = q^{2k} q^{\pm 2j+1} \lambda_{\pm} \zeta_{nk}^\pm, \quad \rho_{j, \pm}(X)\zeta_{nk}^\pm = \lambda_{n+1}(q^{2k}(q^{\pm 2j+1} \lambda_\pm)^2 + r)^{1/2} \zeta_{n,k+1}^\pm, \\
\rho_{j, \pm}(X^*)\zeta_{nk}^\pm = \lambda_{k}(q^{2(k-1)}(q^{\pm 2j+1} \lambda_\pm)^2 + r)^{1/2} \zeta_{n,k-1}^\pm.
\]

Let \( \mathcal{H}^\pm \) and \( \mathcal{K}^\pm \) be the closed linear spans of orthonormal systems \( \{ \eta_{n}^\pm ; n \in \mathbb{N}_0 \} \) and \( \{ \xi_{k}^\pm ; k \in \mathbb{N}_0 \} \), respectively. Setting \( \zeta_{nk}^\pm := \eta_{n}^\pm \otimes \xi_{k}^\pm \), we see that the restriction of \( \rho_{j, \pm} \) to \( \mathcal{O}(S^2_{q^r}) \otimes \mathcal{Y}_r \) is unitarily equivalent to the tensor product representation \( \sigma^\pm \otimes \sigma_j^\pm \). The last assertion of the theorem follows immediately from the preceding since \( \mathcal{O}(S^2_{q^r}) \otimes \mathcal{Y}_r \subset \mathcal{F}_b(S^2_{q^r}) \otimes \mathcal{Y}_r \). \( \square \)
7 Description of the irreducible integrable representations by the second approach

As we have seen in Subsection 6.2 the irreducible integrable representation \( \overline{\pi}_j \) of \( \mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(su_2) \) leads to bounded \(*\)-representations \( \rho_{j,\pm} \) of \( \mathcal{O}(S^2_{qr}) \otimes \mathcal{Y}_r \) on the charts \( \mathcal{F}_b(S^2_{qr})_{\pm} \). In this section, we recover the irreducible integrable representations \( \overline{\pi}_j \) of \( \mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(su_2) \) by taking the orthogonal sum of both charts and passing to another domain. Because this construction is based on \(*\)-representations of \( \mathcal{O}(S^2_{qr}) \otimes \mathcal{Y}_r \), we say that we have described the irreducible integrable representation \( \overline{\pi}_j \) by the second approach.

We begin by showing that the bounded \(*\)-representation \( \rho_{j,\pm} \) of \( \mathcal{O}(S^2_{qr}) \otimes \mathcal{Y}_r \) on the Hilbert space completion \( \mathcal{L}_2(S^2_{qr})_{\pm} \) of \( \mathcal{F}_b(S^2_{qr})_{\pm} \) leads to a \(*\)-representation of the \(*\)-algebra \( \hat{\mathcal{O}}(S^2_{qr}) \times \mathcal{U}_q(su_2) \). By a slight abuse of notation, we use the same symbol \( \rho_{j,\pm} \) to denote the representation of the cross product algebra. It is a \(*\)-representation by unbounded operators acting on the invariant dense domain

\[
\mathcal{D}_{j,\pm} := \cap_{n,m=0}^{\infty} \mathcal{D}(\rho_{j,\pm}(A)^{-n}\rho_{j,\pm}(Y)^{-m}) \subset \mathcal{L}_2(S^2_{qr})_{\pm}
\]

Note that \( \pi(A) \) and \( \pi(Y) \) are commuting bounded self-adjoint operators but their inverses are unbounded. For \( \varphi \in \mathcal{D}_{j,\pm} \), define

\[
\rho_{j,\pm}(K)\varphi := q^{1/2}\rho_{j,\pm}(Y)|^{-1/2}\rho_{j,\pm}(A)^{1/2}\varphi, \quad \rho_{j,\pm}(A^{-1})\varphi := \rho_{j,\pm}(A)^{-1}\varphi
\]

(75)

and \( \rho_{j,\pm}(K^{-1}) := \rho_{j,\pm}(K)^{-1} \). When \( \varphi \) belongs to \( \mathcal{F}_b(S^2_{qr})_{\pm} \cap \mathcal{D}_{j,\pm} \), we can write \( \rho_{j,\pm}(K)\varphi = q^{1/2}|A|^{1/2}\varphi |A|^{-1/2} \) and \( \rho_{j,\pm}(A^{-1})\varphi := A^{-1}\varphi \). From the commutation rules in the algebra \( \hat{\mathcal{O}}(S^2_{qr}) \otimes \mathcal{Y}_r \), it follows easily that (75) defines indeed a \(*\)-representation of the larger \(*\)-algebra \( \hat{\mathcal{O}}(S^2_{qr}) \times \mathcal{U}_q(su_2) \) on \( \mathcal{D}_{j,\pm} \).

Using the isometric isomorphism \( \bar{\psi}_{j,\pm} \), we obtain an irreducible \(*\)-representation \( \bar{\rho}_{j,\pm} := \bar{\psi}_{j,\pm} \circ \rho_{j,\pm} \circ (\bar{\psi}_{j,\pm})^{-1} \) of \( \hat{\mathcal{O}}(S^2_{qr}) \times \mathcal{U}_q(su_2) \) on

\[
\mathcal{D}(\bar{\rho}_{j,\pm}) := \cap_{n,m=0}^{\infty} \mathcal{D}(\bar{\rho}_{j,\pm}(A)^{-n}\bar{\rho}_{j,\pm}(Y)^{-m}) \subset \mathcal{L}_2(S^2_{qr})_{\pm}^{2|j|+1} P_{j},
\]

The restriction of the direct sum \( \bar{\rho}_{j} := \bar{\rho}_{j,\pm} \ominus \bar{\rho}_{j,\pm} \) to \( \mathcal{O}(S^2_{qr}) \otimes \mathcal{Y}_r \) yields a bounded representation which can be extended to a representation on the Hilbert space \( \mathcal{L}_2(S^2_{qr})^{2|j|+1} P_{j} = \mathcal{L}_2(S^2_{qr})^{-2|j|+1} P_{j} \oplus \mathcal{L}_2(S^2_{qr})^{2|j|+1} P_{j} \). By the definitions of \( \rho_{j,\pm} \) and \( \bar{\rho}_{j,\pm} \), it is obvious that this representation coincides with \( \overline{\pi}_j \) and so it coincides with the restriction of \( \overline{\pi}_j \) to \( \mathcal{O}(S^2_{qr}) \otimes \mathcal{Y}_r \) on its common domain. However, the restriction of \( \bar{\rho}_{j} \) to \( \mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(su_2) \) is not unitarily equivalent to \( \bar{\pi}_j \) since the
latter is irreducible while the former is not. Theorem \textbf{7.2} below shows that we can reconstruct the irreducible integrable representations \( \tilde{\pi}_j \cong \tilde{\pi}_j \cong \pi_j \) from \( \tilde{\rho}_j \). The proof of Theorem \textbf{7.2} is based on the following lemma.

\textbf{Lemma 7.1} Let \( \eta \in \mathcal{O}(S^2_{q,r})^{2|j|+1} P_j \). Then

\[
\tilde{\pi}_j^{b}(A)\tilde{\pi}_j(K^{-1}E)\eta = q^{-3/2} \gamma^{-1} \tilde{\pi}_j^{b}(X^* - qB^*)\eta, \tag{76}
\]

\[
\tilde{\pi}_j^{b}(A)\tilde{\pi}_j(FK^{-1})\eta = q^{-3/2} \gamma^{-1} \tilde{\pi}_j^{b}(X - q^{-1}B)\eta, \tag{77}
\]

\[
|\tilde{\pi}_j^{b}(Y)|^{1/2} \tilde{\pi}_j(K)\eta = q^{1/2}|\tilde{\pi}_j^{b}(A)|^{1/2}\eta, \quad |\tilde{\pi}_j^{b}(A)|^{1/2} \tilde{\pi}_j(K^{-1})\eta = q^{-1/2}|\tilde{\pi}_j^{b}(Y)|^{1/2}\eta. \tag{78}
\]

\textbf{Proof.} Let \( \eta = (y_{-j}, \ldots, y_{j}) P_j \in \mathcal{O}(S^2_{q,r})^{2|j|+1} P_j \). From the uniqueness of the square root of a positive operator, it follows that

\[
|\tilde{\pi}_j^{b}(A)|^{1/2} (y_{-j}, \ldots, y_{j}) P_j = (|A|^{1/2} y_{-j}, \ldots, |A|^{1/2} y_{j}) P_j, \\
|\tilde{\pi}_j^{b}(Y)|^{1/2} (y_{-j}, \ldots, y_{j}) P_j = q^{1/2} (q^{1/2} y_{-j}|A|^{1/2}, \ldots, q^{-1/2} y_{j}|A|^{1/2}) P_j.
\]

Further, the commutation rules in \( \mathcal{F}_b(S^2_{q,r}) \) imply for all \( y \in \mathcal{O}(S^2_{q,r}) \)

\[
|A|^{1/2} y = (K^s y)|A|^{1/2}, \quad y|A|^{1/2} = |A|^{1/2} (K^{-1} y).
\]

On the other hand, since \( \tilde{\pi}_j := \Psi_j^{-1} \circ \tilde{\pi}_j \circ \Psi_j \), we have

\[
\tilde{\pi}_j(K^{\pm 1}) (y_{-j}, \ldots, y_{j}) P_j = [2|j|+1]^{-1/2} \Psi_j^{-1}(K^{\pm 1} \gamma \sum_{k=|j|}^{1} q^{-k} y_{k}y_{|k|}) = (q^{\mp |j|} K^{\pm 1} \gamma y_{-j}, \ldots, q^{\pm |j|} K^{\pm 1} \gamma y_{j}) P_j.
\]

Combining the preceding equations proves (78).

Computing the action of \( \tilde{\pi}_j(FK^{-1}) = \Psi_j^{-1} \circ \tilde{\pi}_j \circ \Psi_j(FK^{-1}) \) on the element \( \eta = (y_{-j}, \ldots, y_{j}) P_j \) gives

\[
\tilde{\pi}_j(FK^{-1}) (y_{-j}, \ldots, y_{j}) P_j = [2|j|+1]^{-1/2} \Psi_j^{-1} \sum_{k=-|j|}^{1} q^{-k} ((FK^{-1} \gamma y_{k}) y_{|k|}) \\
+ q^{-k}[|j| - k + 1]^{1/2}[|j| + k]^{1/2} (K^{-2} \gamma y_{k}) y_{|k-1|,1})
\]

so that, by using Lemma 6.2,

\[
\tilde{\pi}_j^{b}(A)\tilde{\pi}_j(FK^{-1}) (y_{-j}, \ldots, y_{j}) P_j = (A(FK^{-1} \gamma y_{-j}), \ldots, A(FK^{-1} \gamma y_{j}))) P_j \\
+ q^{-2}(q^{1/2}[|j|][1])^{1/2} A(K^{-2} \gamma y_{-j+1}), \ldots, q^{-|j|+1}([1][2|j|])^{1/2} A(K^{-2} \gamma y_{j}), 0) P_j \\
= -q^{-5/2} \gamma^{-1} ([B, y_{-j}], \ldots, [B, y_{j}]) P_j \\
+ q^{-2}(q^{1/2}[|j|][1])^{1/2} y_{-j+1}, \ldots, q^{-|j|+1}([1][2|j|])^{1/2} y_{j} A, 0) P_j.
\]

46
Comparing the last identity with the action of $\hat{\pi}_j^b(X)$ and $\hat{\pi}_j^b(B)$ on $\mathcal{O}(S^2_q)\otimes \mathcal{O}(\mathfrak{su}_2)$ from Subsection 6.2 (see the discussion preceding Proposition 6.4) shows (77). Equation (76) is proved similarly.

Let us recall the notion of the adjoint of a $*$-representation $\pi$ of a $*$-algebra $\mathcal{X}$ (see e.g. [14, Section 8.1]). It is a representation $\pi^*$ of $\mathcal{X}$ acting on the domain $\mathcal{D}(\pi^*) := \cap_{x \in \mathcal{X}} \mathcal{D}(\pi(x)^*)$ by $\pi^*(x) = \pi(x^*)^* x$, where $x \in \mathcal{D}(\pi^*)$. In general, $\pi^*$ is not a $*$-representation.

**Theorem 7.2** The irreducible integrable $*$-representation $\hat{\pi}_j$ of $\mathcal{O}(S^2_q) \times \mathcal{U}_q(\mathfrak{su}_2)$ is the restriction of the adjoint $\hat{\pi}_j^*$ of the $*$-representation $\hat{\rho}_j = \hat{\rho}_{j,-} \oplus \hat{\rho}_{j,+}$ of $\hat{\mathcal{O}}(S^2_q) \times \mathcal{U}_q(\mathfrak{su}_2)$ to the domain $\mathcal{O}(S^2_q)\otimes \mathcal{O}(\mathfrak{su}_2)$ with $\mathcal{D}(\pi_j^*)$ determined by the formulas (1) from Subsection 5.3.

**Proof.** First note that, by definition, $\hat{\rho}_j(Z) = \hat{\pi}_j^b(Z)$ for all $Z \in \mathcal{O}(S^2_q) \otimes \mathcal{Y}_r$ and $\varphi \in \mathcal{D}(\hat{\rho}_{j,+})$. In particular, $\langle \hat{\rho}_j(A^{-1}) \varphi, \hat{\pi}_j^b(A) \eta \rangle = \langle \varphi, \eta \rangle$ for all $\eta$ from the Hilbert space $L_2(S^2_q)^{2|j|+1}$. Now let $\eta \in \mathcal{O}(S^2_q)^{2|j|+1}$ and $\varphi \in \mathcal{D}(\hat{\rho}_{j,+})$. Then, by Lemma 7.1,

$$\langle \hat{\rho}_j(K^{-1} E) \varphi, \eta \rangle = q^{-3/2} \lambda^{-1} \langle \hat{\rho}_j(X^* - q^{-1} B^*) \hat{\rho}_j(A^{-1}) \varphi, \eta \rangle = q^{-3/2} \lambda^{-1} \langle \hat{\rho}_j(A^{-1}) \varphi, \hat{\pi}_j^b(X - q^{-1} B) \eta \rangle = \langle \hat{\rho}_j(A^{-1}) \varphi, \hat{\pi}_j^b(A) \eta \rangle = \langle \varphi, \hat{\pi}_j(\mathcal{F}K^{-1}) \eta \rangle.$$

Similarly one shows $\langle \hat{\rho}_j(FK^{-1}) \varphi, \eta \rangle = \langle \varphi, \hat{\pi}_j(K^{-1}E) \eta \rangle$. As a above, we have $\langle |\hat{\rho}_j(Y)|^{-1/2} \varphi, \hat{\pi}_j^b(Y) \rangle = \langle \varphi, \eta \rangle$ and thus, again by Lemma 7.1,

$$\langle \hat{\rho}_j(K) \varphi, \eta \rangle = q^{1/2} \langle |\hat{\rho}_j(A)|^{1/2} |\hat{\rho}_j(Y)|^{-1/2} \varphi, \eta \rangle = q^{1/2} \langle |\hat{\rho}_j(Y)|^{-1/2} \varphi, |\hat{\pi}_j^b(A)|^{1/2} \eta \rangle = \langle |\hat{\rho}_j(Y)|^{-1/2} \varphi, |\hat{\pi}_j^b(Y)|^{1/2} \hat{\pi}_j(K) \eta \rangle = \langle \varphi, \hat{\pi}_j(K) \eta \rangle.$$

Likewise, $\langle \hat{\rho}_j(K^{-1}) \varphi, \eta \rangle = \langle \varphi, \hat{\pi}_j(K^{-1}) \eta \rangle$. Clearly, $\langle \hat{\rho}_j(x^*) \varphi, \eta \rangle = \langle \varphi, \hat{\pi}_j(x) \eta \rangle$ for all $x \in \mathcal{O}(S^2_q)$. As the elements $K^\pm 1, K^{-1} E, FK^{-1}$ and $x \in \mathcal{O}(S^2_q)$ generate the algebra $\mathcal{O}(S^2_q) \times \mathcal{U}_q(\mathfrak{su}_2)$, it follows that $\mathcal{O}(S^2_q)\otimes \mathcal{D}(\hat{\rho}_j^*)$ and $\hat{\rho}_j^*(Z) = \hat{\pi}_j^b(Z)$ for all $Z \in \mathcal{O}(S^2_q) \times \mathcal{U}_q(\mathfrak{su}_2)$ and $\eta \in \mathcal{O}(S^2_q)^{2|j|+1}$.

Computing the action of $\rho_{j,-}(K), \rho_{j,-}(E) = q^{-3/2} \lambda^{-1} \rho_{j,-}(A^{-1} K(X^* - qB^*))$, $\rho_{j,-}(F) = \rho_{j,-}(E)^*$ on the basis vectors $\zeta_{nm}^\pm$, from the proof of Theorem 6.7 shows that $\rho_{j,+}$ is determined by the formulas (1) from Subsection 5.3. Setting $\eta_{nm}^\pm = \hat{\psi}_{j,\pm}(\zeta_{nm}^\pm)$ establishes the second assertion of the theorem. \qed
Let us make the case \( j = 0 \) more explicit. Then \( P_0 = 1, M_0 \cong \mathcal{O}(S^2_{qr}) \), and \( \pi_0 \cong \tilde{\pi}_0 \) is just the Heisenberg representation of \( \mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(\mathfrak{su}_2) \). Accordingly, \( \tilde{\pi}_0(f)\eta = f \cdot \eta \) for \( f \in \mathcal{U}_q(\mathfrak{su}_2), \eta \in \mathcal{O}(S^2_{qr}) \). The \( 1 \times 1 \)-matrices \( m_0, m_0^\dagger \) and \( n_0 \) from Subsection 6.2 have the entries \( q^{-1}B, qB^* \) and \( qA \), respectively. Hence we obtain for \( z, \phi \in \mathcal{F}_b(S^2_{qr}) \)
\[
\begin{align*}
\tilde{\pi}_0^b(z)\phi &= z\phi, & \tilde{\pi}_0^b(X)\phi &= q^{-1}\phi B, & \tilde{\pi}_0^b(X^*)\phi &= q\phi B^*, & \tilde{\pi}_0^b(Y)\phi &= q\phi A.
\end{align*}
\]
Inserting these formulas into (76)–(78), we recover Equations (63)–(65) which we used to define a \( \mathcal{U}_q(\mathfrak{su}_2) \)-action on the operator algebra \( \mathcal{L}^+(\mathcal{D}) \). In particular, Equations (63)–(65) and (79) give a new description of the Heisenberg representation of \( \mathcal{O}(S^2_{qr}) \times \mathcal{U}_q(\mathfrak{su}_2) \) on \( \mathcal{O}(S^2_{qr}) \subset \mathcal{F}_b(S^2_{qr}) \) by left and right multiplications.

References

[1] Brzeziński, T., Quantum homogeneous spaces as quantum quotient spaces. J. Math. Phys. 37 (1996), 2388–2399.

[2] Brzeziński, T. and S. Majid, Quantum geometry of algebra factorisations and coalgebra bundles. Commun. Math. Phys. 213 (2000), 491–521.

[3] Dijkhuizen, M. and T. K. Koornwinder, Quantum homogeneous spaces, quantum duality and quantum 2-spheres. Geometriae Dedicata 52 (1994), 291–315.

[4] Fiore, G., On the decoupling of the homogeneous and inhomogeneous parts in inhomogeneous quantum groups. J. Phys. A 35 (2002), 657–678.

[5] Hajac, P. M., Bundles over quantum sphere and noncommutative index theorem. K-Theory 21 (1996), 141–150.

[6] Hajac, P. M. and S. Majid, Projective module description of the q-monopole. Commun. Math. Phys. 206 (1999), 247–264.

[7] Klimyk, K. A. and K. Schmüdgen, Quantum Groups and Their Representations. Springer, Heidelberg, 1997.

[8] Kulish, P. P. and N. Yu. Reshetikhin, Quantum linear problem for the sine-Gordon equation and higher representations. Zap. Nauchn. Sem. LOMI 101 (1981), 101–110.
[9] Lunts, V. A. and A. L. Rosenberg, Differential operators on noncommutative rings. Sel. math. 3 (1997), 335–359.

[10] Mimachi, K. and M. Noumi, Quantum 2-spheres and big q-Jacobi polynomials. Commun. Math. Phys. 128 (1990), 521–531.

[11] Montgomery, S., Hopf algebras and their actions on rings. Amer. Math. Soc., Providence, R.I., 1993.

[12] Müller, E. F. and H.-J. Schneider, Quantum homogeneous spaces with faithfully flat module structures. Israel J. Math. 111 (1999), 157–190.

[13] Podleś, P., Quantum spheres. Lett. Math. Phys. 14 (1987), 193–202.

[14] Schmüdgen, K., Unbounded Operator Algebras and Representation Theory. Birkhäuser, Basel, 1990.

[15] Schmüdgen, K. and E. Wagner, Hilbert space representations of cross product algebras. J. Funct. Anal. 200 (2003), 451–493.

[16] Schneider, H.-J., Principal homogeneous spaces for arbitrary Hopf algebras. Israel J. Math. 72 (1990), 167–195.

[17] Soibelman, Ya. S. and L. L. Vaksman, Algebra of functions on the quantum SU(2). Funct. Anal. Appl. 22 (1988), 170–181.

[18] Sz.-Nagy, B. and C. Foias, Analyse harmonique des operateurs de l’espace de Hilbert. Academiai Kiado, Budapest, 1979.

[19] Woronowicz, S. L., Twisted SU(2) group. An example of a non-commutative differential calculus. Publ. RIMS Kyoto Univ. 23 (1987), 117–181.