ANALYTICAL PROPERTIES OF AN OSTROVSKY-WHITHAM TYPE DYNAMICAL SYSTEM FOR A RELAXING MEDIUM WITH SPATIAL MEMORY AND ITS INTEGRABLE REGULARIZATION

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Abstract

Short-wave perturbations in a relaxing medium, governed by a special reduction of the Ostrovsky evolution equation, and later derived by Whitham, are studied using the gradient-holonomic integrability algorithm. The bi-Hamiltonicity and complete integrability of the corresponding dynamical system is stated and an infinite hierarchy of commuting to each other conservation laws of dispersive type are found. The two- and four-dimensional invariant reductions are studied in detail. The well defined regularization of the model is constructed and its Lax type integrability is discussed.

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1. Introduction

Many important problems of propagating waves in nonlinear media with distributed parameters can be described by means of evolution differential equations of special type. In particular, if the nonlinear medium is endowed still with spatial memory properties, the propagation of the corresponding waves can be modeled by means of the so called generalized Ostrovsky evolution equations [1]. It is also well known [5, 7, 6] that shortwave perturbations in a relaxing one dimensional medium can be described by means of some reduction of the Ostrovsky equations, coinciding with the Whitham type evolution equation

\begin{equation}
\frac{du}{dt} = 2uu_x + \int_{\mathbb{R}} K(x, s)u_s ds,
\end{equation}

discussed first in [5]. Here the kernel \( K : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) depends on the medium elasticity properties with spatial memory and can, in general, be a function of the pressure gradient \( u_x \in C^\infty(\mathbb{R}; \mathbb{R}) \), evolving in respect to equation (1.1). In particular, if \( K(x, s) = \frac{1}{2} |x - s| \), \( x, s \in \mathbb{R} \), then equation (1.1) can be reduced to

\begin{equation}
\frac{du}{dt} = 2uu_x + \partial^{-1}u,
\end{equation}

which was, in particular, studied before in [3, 2, 6, 7, 5].

Since some media possess elasticity properties depending strongly on the spatial pressure gradient \( u_x, \ x \in \mathbb{R} \), the corresponding Whitham kernel looks like

\begin{equation}
K(x, s) := -\theta(x - s)u_s
\end{equation}

for \( x, s \in \mathbb{R} \), naturally modeling the relaxing spatial memory effects. The resulting equation (1.1) with the kernel (1.3) becomes

\begin{equation}
\frac{du}{dt} = 2uu_x - \partial^{-1}u^2 := K[u],
\end{equation}

which appears to possess very interesting mathematical properties. The latter will be the main topic of the next sections below.

2. Lagrangian analysis

A more mathematically correct form of equation (1.4) looks like

\begin{equation}
\frac{du}{dt} = 2(uu_x)_x - u^2_x,
\end{equation}

being a nonlinear hyperbolic flow on the axis \( \mathbb{R} \). Concerning the preceding form of (1.4) it is necessary to define the operation \( \partial^{-1} : C^\infty(\mathbb{R}; \mathbb{R}) \to C^\infty(\mathbb{R}; \mathbb{R}) \), which is not an easy problem. Since equation (2.1) is well defined in the space of \( 2\pi \)-periodic functions \( C^\infty_{2\pi}(\mathbb{R}; \mathbb{R}) \), one can determine upon its subspace \( \tilde{C}^\infty_{2\pi}(\mathbb{R}; \mathbb{R}) \subset C^\infty_{2\pi}(\mathbb{R}; \mathbb{R}) \) of functions under the condition \( \int_0^{2\pi} f(s)ds = 0 \) for any \( f \in \tilde{C}^\infty_{2\pi}(\mathbb{R}; \mathbb{R}) \) the inverse operation

\begin{equation}
\partial^{-1}(\cdot) := \frac{1}{2} \left[ \int_0^\cdot (\cdot)ds - \int_\cdot^{2\pi} (\cdot)ds \right],
\end{equation}

for \( x = \frac{1}{2} \left[ \int_0^{2\pi} (\cdot)ds - \int_0^\cdot (\cdot)ds \right] \).
being definite for all $x \in \mathbb{R}$ and satisfying the natural property $\partial \cdot \partial^{-1} = 1$. Thereby, for convenience, we will consider the flow (1.4) as that in the smooth functional submanifold $M := \bar{C}_0^\infty(\mathbb{R}; \mathbb{R})$. The corresponding vector field $K : M \to T(M)$ defines on $M$ a dynamical system, which appears to possess both Lagrangian and Hamiltonian properties.

To demonstrate them in detail, consider the partial differential equation (2.1) and prove that it is of Lagrangian form, that is

$$(2.3) \quad u_{xt} = \frac{\delta H_\vartheta}{\delta u} := \xi[u],$$

where $H_\vartheta : M \to \mathbb{R}$ is some Fréchet smooth Lagrangian function. To prove (2.3), following the scheme in [8, 14, 12], it is enough to state that only the Volterrian identity $\xi = \xi^*$ holds, that is

$$(2.4) \quad [2(uu_x)_x - u^2_x]' = [2(uu_x)_x - u^2_x]^*,$$

where the sign ”’” means the Fréchet derivative with respect to the variable $u \in M$ and ” * ” means the corresponding conjugation with respect to the natural scalar product on the tangent space $T(M) \simeq T^*(M)$. As a result, there exists a Lagrangian function $H_\vartheta : M \to \mathbb{R}$ in the following explicit form:

$$(2.5) \quad H_\vartheta := \int_0^{2\pi} \mathcal{H}_\vartheta dx = \int_0^{2\pi} uu_x^2 dx,$$

where we used the standard [8, 14] homotopy formula $H_\vartheta = \int_0^1 d\lambda (\text{grad} H_\vartheta[u\lambda], u)$. Thus, expression (2.3) can be presented as the Euler equation

$$(2.6) \quad \delta \mathcal{L}/\delta u = 0,$$

where, by definition,

$$(2.7) \quad \mathcal{L} := \int_0^t \int_0^{2\pi} \left( \frac{1}{2} uu_x u_\tau - \mathcal{H}_\vartheta \right) dx d\tau.$$

Recall now, that owing to the standard results [8, 12, 14, 19, 20], any Lagrangian system in the form (2.6) is Hamiltonian. To show this, rewrite the action functional (2.7) as

$$(2.8) \quad \mathcal{L} = \int_0^t \int_0^{2\pi} [(\varphi, u_\tau) - H_\vartheta] d\tau,$$

where $\varphi := (1/2)u_x \in T^*(M)$. Then the condition (2.6) gives rise to the equality

$$(2.9) \quad u_\tau = -\vartheta \text{ grad } H_\vartheta[u] = K[u],$$

where, by definition,

$$(2.10) \quad \vartheta^{-1} := \varphi' - \varphi^* = \partial/\partial x.$$

As it is easy to see, the operator $\vartheta := \partial^{-1} : T^*(M) \to T(M)$ is necessary implectic [8, 10, 12] and with respect to the flow (2.9) also Noetherian. Thus we have stated the following [16] theorem.

**Theorem 2.1.** The partial differential equation (2.1) is equivalent on the functional manifold $M$ to the Hamiltonian flow (2.9) with the Hamiltonian function (2.5) and co-implectic structure (2.10).
This result means that our flow (1.4) on \( M \), being Hamiltonian, is conservative, thereby one can expect it possesses also an additional hidden infinite hierarchy of conservation laws, which is very important [14, 8, 10, 18, 11] for its integrability analysis. This assumption, as we shall show below, appears to hold really.

3. Gradient-holonomic analysis

Since any conservation law \( \gamma \in D(M) \) satisfies the linear Lax equation

\[
d\psi/dt + K^*\psi = 0,
\]

where \( \psi = \text{grad} \gamma \in T^*(M) \), under the condition of its existence in the form of a local functional on \( M \), it can be found for instance, by means of the asymptotic small parameter method [8]. In particular, one easily gets that expressions

\[
\psi_\vartheta = u_{xx}, \quad \psi_{\eta^{-1}} = \frac{1}{2}(u_x^2 - (u_{xx})^2)
\]
satisfy the Lax equation (3.1) and are the gradients of the corresponding functionals on \( M \), that is

\[
\psi_\vartheta = \text{grad} \gamma_\vartheta, \quad \psi_{\eta^{-1}} = \text{grad} \gamma_{\eta^{-1}},
\]

where

\[
\gamma_\vartheta = \frac{1}{2} \int_0^{2\pi} u_x^2 dx, \quad \gamma_{\eta^{-1}} = \frac{1}{2} \int_0^{2\pi} uu_x^2 dx.
\]

Thus, we have stated that our dynamical system (1.4) allows additional invariants (conservation laws), which can be used within the gradient-holonomic algorithm [8, 15, 18] for finding new associated nontrivial implectic structures on the manifold \( M \). Namely, let us represent conservation laws (3.2) in the scalar product form on \( M \) as

\[
\gamma_\vartheta = (\varphi_\vartheta, u_x) \quad \gamma_{\eta^{-1}} = (\varphi_{\eta^{-1}}, u_x),
\]

where

\[
\varphi_\vartheta = \frac{1}{2} u_x, \quad \varphi_{\eta^{-1}} = -\frac{1}{2} \partial^{-1} u_x^2 \in T^*(M).
\]

Then operators

\[
\vartheta^{-1} = \varphi_\vartheta - \varphi_{\eta^{-1}}^* = \frac{1}{2} \partial - (-\frac{1}{2} \partial) = \partial,
\]

\[
\eta^{-1} = \varphi_{\eta^{-1}} - \varphi_{\eta^{-1}}^* = \partial^{-1} u_{xx} + u_{xx} \partial^{-1}
\]

will be co-implectic [8, 12, 14] on \( M \), and, as it is easy to check, also Noetherian with respect to our dynamical system (1.4). Moreover, via direct calculations one can show that the corresponding implectic operators \( \vartheta, \eta^{-1} : T^*(M) \to T(M) \) are compatible on \( M \), that is for any \( \lambda \in \mathbb{R} \) the expression \( \vartheta + \lambda \eta^{-1} \) is implectic too on \( M \) [8, 9, 12]. Really, it is enough to show [12, 8] that the operator \( \vartheta^{-1} \eta^{-1} \vartheta^{-1} : T^*(M) \to T(M) \) is symplectic on \( M \), that is the differential two-form
(2) \[ \Omega := \int_0^{2\pi} dx (du \wedge \vartheta^{-1} \eta^{-1} du) \in \Lambda^2(M) \text{ is closed, or } d\Omega = 0. \] The latter equality is easily checked by direct calculations. This means, in particular, that all operators of the form

(3.8) \[ \eta_n = \vartheta(\eta^{-1} \vartheta)^n \]

for \( n \in \mathbb{Z} \) will also be implectic on \( M \). Another consequence from this fact is the existence of an infinite hierarchy of invariants \( \gamma_n \in D(M), \ n \in \mathbb{Z}, \) satisfying the expressions

(3.9) \[ K[u] = -\eta_n \, \text{grad} \, \gamma_n. \]

As a particular case one can define an implectic operator \( \eta : T^*(M) \to T(M) \) in the form

(3.10) \[ \eta = \vartheta \eta^{-1} \vartheta = \vartheta^{-2} u_{xx} \vartheta^{-1} + \vartheta^{-1} u_{xx} \vartheta^{-2}. \]

Whence from (2.9) we obtain that

(3.11) \[ u_t = K[u] = -\vartheta \text{grad} \, H = -\eta \text{grad} \, H, \]

where

\[ H_\vartheta = \int_0^{2\pi} u u_x^2 dx, \quad H_\eta = \int_0^{2\pi} u_x^2 dx. \]

The set of expressions (3.8) can be equivalently rewritten in another useful form as

(3.12) \[ \lambda \vartheta \text{grad} \, \gamma(\lambda) = \eta \text{grad} \, \gamma(\lambda), \]

being in some sense equivalent [8, 14, 12] together with equation (3.1) to the adjoint Lax type representation

(3.13) \[ \frac{d\Lambda}{dt} = [\Lambda, K^*] \]

for the dynamical system (1.4), where \( \Lambda := \vartheta^{-1} \eta : T^*(M) \to T^*(M) \) is a so called [14, 12, 8, 12] recursion operator and \( \gamma(\lambda) \in D(M), \ \lambda \in \mathbb{C}, \) is a generating function of the infinite hierarchy of conservation laws (1.4). In particular, as \( |\lambda| \to \infty \) the asymptotic expansion

(3.14) \[ \text{grad} \, \gamma(\lambda)_{|\lambda| \to \infty} \approx \sum_{j \in \mathbb{Z}_+} \lambda^{-j} \text{grad} \, \gamma_j \]

holds, where

(3.15) \[ \text{grad} \, \gamma_n = \Lambda^n \, \text{grad} \, \gamma_0, \quad \gamma_0 := H_\eta, \]

for all \( n \in \mathbb{Z}_+. \) Concerning this infinite hierarchy of conservation laws one can easily check, that all of them are dispersionless. The result obtained above can be formulated as the next theorem.

**Theorem 3.1.** The dynamical system (1.4) on the functional manifold \( M \) is a compatible bi-Hamiltonian flow, possessing an infinite hierarchy of commuting functionally independent dispersionless conservation laws, satisfying the fundamental gradient identity (3.12). The latter is equivalent together with the relationship (3.1) to the adjoint Lax type representation (3.13).

As was mentioned above, the hierarchy of commuting flows \( K_n := -\vartheta \text{grad} \, \gamma_n, \ n \in \mathbb{Z}_+, \) shows an interesting property of their dispersionless. In particular, this entails that they can not be
treated effectively by means of the gradient-holonomic algorithm \[8, 15, 18\]. In particular, the corresponding asymptotic solutions to the Lax equations

\[ \frac{d\varphi}{d\tau_n} + K_n' \varphi = 0, \quad \varphi' \neq \varphi^{+}, \]

where \(|\lambda| \to \infty \) and \(du/d\tau_n = K_n[u], \) \(\tau_n \in \mathbb{R}, \) \(n \in \mathbb{Z}, \) do not give rise to explicit functional expressions, defining a new associated hierarchy of conservation laws for the dynamical system (1.4). Nonetheless, the corresponding hierarchy of dispersive commuting flows on \(M\) does exist for (1.4), being simply associated with the trivial flow \(du/d\tau_0 := 0 \) on \(M\). Namely, let \(H_0 \in D(M)\) be a conservation law of (1.4), satisfying the kernel condition for the operator \(\eta: T^*(M) \to T(M),\)

\[ du/dt_0 = 0 := \eta \, \text{grad} \, H_0. \]

It is easy to find from (3.17) and (3.10) that

\[ \text{grad} \, H_0 = \left[ 2(u_{xx} - 1/2)_{xx} \right] \in T^*(M), \text{ where} \]

\( u \in \mathbb{R}, \) \(H_0 \in \mathbb{C},\)

the nontrivial functionals \(\gamma_{j-1} := \int_0^{2\pi} \sigma_{j-1}[u] dx, \quad j \in \mathbb{Z}_+,\)

are, obviously, functionally independent and commuting conservation laws both of the dynamical system (3.19) and of our dynamical system (1.4). As a result of some simple but slightly tedious calculations one finds that

\[ \sigma_{-1} = \sqrt{u_{xx}}, \quad \sigma_0 = \frac{1}{2}u_{xx}^{-1}u_{xxx}, \quad \sigma_1 = \frac{1}{8}(u_{xx})^{-5/2}u_{xxx}^2, \ldots, \]

and the corresponding hierarchy of already dispersive invariants is given as

\[ \gamma_{-1} = \int_0^{2\pi} \sqrt{u_{xx}} \, dx, \quad \gamma_0 = 0, \]

\[ \gamma_1 = \int_0^{2\pi} \frac{1}{8}u_{xx}^{-5/2}u_{xxx}^2 \, dx, \quad \gamma_2 = 0, \ldots, \]

and so on. Then, owing to conditions (3.19) and (3.20) the generating functional \(\gamma(\lambda) := \int_0^{2\pi} \sigma(x; \lambda) dx, \) \(\lambda \in \mathbb{C},\)

satisfies \([8, 12, 15]\) the following gradient relationship

\[ \lambda^2 \partial \text{grad} \, \gamma(\lambda) = \eta \, \text{grad} \, \gamma(\lambda), \]

suitably modifying the relationship (3.12).
The obtained results are very important for further analytical studying Lax type integrability of the dynamical system (1.4) and finding, in particular, a wide class of its special soliton like and quasi-periodic solutions by means of analytical quadratures. Some of these aspects of the integrability problem are presented in the section below.

4. LAX TYPE REPRESENTATION AND FINITE DIMENSIONAL REDUCTIONS

Since the functional solution (3.21) satisfies the Lax type equation (3.20), it can be considered [15, 18, 10] as a Bloch type eigenfunction of the adjoint Lax type representation (3.13), that is

\[ \Lambda \varphi(x; \lambda) = \lambda^2 \varphi(x; \lambda) \]

for all \( \lambda \in \mathbb{C} \) and \( x \in \mathbb{R} \). This gives rise, following the gradient-holonomic algorithm [15, 18], to the existence of a standard Lax type representation for the associated dynamical system (3.19) and, thereby, for our Whitham type dynamical system (1.4). Omitting here the related calculations, we find surprisingly that this adjoint to (4.1) Lax type spectral problem for the flow (1.4) is equal to

\[ Lf := \begin{pmatrix} -i\lambda & \lambda(u_{xx} - 1) \\ -\lambda & i\lambda \end{pmatrix} f, \]

where an eigenfunction \( f \in L_\infty(\mathbb{R}; \mathbb{C}^2) \) and \( \lambda \in \mathbb{C} \) is a time independent spectral parameter. The result (4.2) can be effectively used for solving our nonlinear equation (1.4), making use either of the inverse spectral transform method [18, 17, 15, 11, 10] or of the dual Bogoyavlensky-Novikov method [18, 8] of finite dimensional reductions. For the latter case we need to construct an finite dimensional invariant symplectic functional submanifolds \( M^{2N} \subset M, N \in \mathbb{Z}_+ \), and to represent the main vector fields \( d/dx \) and \( d/dt \) on them as the corresponding commuting to each other Hamiltonian flows. Moreover, since these flows on \( M^{2N} \) appear to be Liouville-Arnold integrable, we obtain both the complete integrability of our dynamical system (1.4) in quadratures and their exact solutions, expressed, in general, by means of Riemannian theta-functions [18, 17, 15, 10] on some specially constructed algebraic Riemannian surfaces.

Below we consider, for simplicity, the following invariant two- and four-dimensional functional submanifolds:

\[ i) \quad M^2 := \{ u \in M : \text{grad} \mathcal{L}_2[u] = 0 \}, \]

where \( \mathcal{L}_2 := H_\phi + c_\eta H_\eta \in \mathcal{D}(M) \), and

\[ ii) \quad M^4 := \{ u \in M : \text{grad} \mathcal{L}_4[u] = 0 \}, \]

where \( \mathcal{L}_4 := \gamma_{-1} + c_\phi H_\phi + c_\eta H_\eta \in \mathcal{D}(M) \).

**Case i).** We have, therefore [8, 13, 18, 17], on the invariant manifold \( M^2 \) commuting Hamiltonian vector fields \( d/dx \) and \( d/dt \) with respect to the canonical symplectic structure

\[ \omega^{(2)} := d\alpha^{(1)}, \]
where 1-form $\alpha^{(1)} \in \Lambda^1(M)$ is determined by the Gelfand-Dickey [13, 8] relationship

$$dL_2[u] = \text{grad}L_2[u]du + d\alpha^{(1)}/dx,$$

holding on $M$. One now easily finds, that for all $u \in M^2 \subset M$

$$\text{grad}L_2[u] = u_x^2 - 2(u_x)u_x - 2c_\eta u_{xx} = 0,$$

where we have put $p := 2(u + c_\eta)u_x$ and $q := u$. The corresponding Hamiltonian functions $h(x)$ and $h(t) \in D(M^2)$ for Hamiltonian flows

$$\frac{dq}{dx} = \frac{\partial h(x)}{\partial p}, \quad \frac{dp}{dx} = -\frac{\partial h(x)}{\partial q},$$

are found [8], respectively, from the determining relationships

$$\text{grad}L_2[u] u_x := -\frac{\partial h(x)}{\partial x}, \quad \text{grad}L_2[u] u_t := -\frac{\partial h(t)}{\partial x},$$

whence we get that

$$h(x) = (u + c_\eta)u_x = \frac{p^2}{4(q + c_\eta)}, \quad h(t) = -2c_\eta(u + c_\eta)u_x = \frac{-c_\eta p^2}{2(q + c_\eta)}.$$

One sees easily that two flows $d/dx$ and $d/dt$ on the two-dimensional invariant submanifold $M^2 \subset M$ of infinite period are proportional, confirming the classical fact [20, 19]: upon two-dimensional symplectic manifold there exist the only functionally-independent invariant commuting to each other.

The set of Hamiltonian equations (4.9) for the flows $d/dx$ and $d/dt$ has the simple form

$$\frac{dq}{dx} = \frac{p}{2(q + c_\eta)}, \quad \frac{dp}{dx} = \frac{p^2}{4(q + c_\eta)^2},$$

$$\frac{dq}{dt} = -\frac{c_\eta p}{(q + c_\eta)}, \quad \frac{dp}{dt} = -\frac{c_\eta p^2}{2(q + c_\eta)^2},$$

whose solution, given by the exact formula

$$q(x, t) = -c + \left[\frac{3}{2}\sqrt{\bar{h}(x)}(x - 2c_\eta t) + \bar{k}\right]^{2/3} \Rightarrow u(x, t),$$

with $\bar{k} \in \mathbb{R}$ being some real constant, supplies us, evidently, with an exact partial one-parametric solution to our Whitham type nonlinear equation (1.4).

**Case ii.)** Similarly as above, we find the quantities

$$\text{grad}L_4[u] = \left(\frac{1}{2\sqrt{u_{xx}}}\right)_{xx} + c_\eta \left[u_x^2 - 2(uu_x)_x\right] - 2c_\eta u_{xx} = 0,$$

$$\alpha^{(1)} = [2(c_\eta u + c_\eta^2)u_x - \left(\frac{1}{2\sqrt{u_{xx}}}\right)_x]du + \frac{1}{2\sqrt{u_{xx}}}du_x,$$
whence the symplectic structure is given as
\[
\omega^{(2)} = d[2(c_\varphi u + c_\eta)u_x - (\frac{1}{2\sqrt{u_{xx}}})_x] \wedge du + d(\frac{1}{2\sqrt{u_{xx}}}) \wedge du_x
\]
\[
= dp_1 \wedge dq_1 + dp_2 \wedge dq_2,
\]
where we put, by definition, \( q_1 := u, q_2 := u_x, p_1 := 2(c_\varphi u + c_\eta)u_x - (\frac{1}{2\sqrt{u_{xx}}})_x, p_2 := \frac{1}{2\sqrt{u_{xx}}} \) \in M^4, which are canonical symplectic coordinates on the invariant functional submanifold \( M^4 \subset M \).

The commuting to each other Hamiltonian functions, related with flows \( d/dx \) and \( d/dt \), are equal to the next algebraic expressions
\[
h(x) = u_x^2(c_\varphi u + c_\eta) - (\frac{1}{2\sqrt{u_{xx}}})_x u_x - \sqrt{u_{xx}}
\]
\[
= 3q_2^2(c_\varphi q_1 + c_\eta) - q_2p_1 - 1/(2p_2)
\]
and
\[
h(t) = q_1/(2p_2) - 2q_1q_2[p_1 - q_2(c_\varphi q_1 + c_\eta)].
\]
As a result, we have reduced our Whitham type dynamical system (1.4) upon the constructed four-dimensional invariant submanifold \( M^4 \subset M \), on which it is exactly equivalent to two commuting canonical Hamiltonian flows
\[
dq_j/dx = \partial h(x)/\partial p_j, \quad dp_j/dx = -\partial h(x)/\partial q_j,
\]
\[
dq_j/dt = \partial h(t)/\partial p_j, \quad dp_j/dt = -\partial h(t)/\partial q_j
\]
for \( j = 1,2 \), where the corresponding Poisson bracket \( \{h(x),h(t)\} = 0 \) on \( M^4 \). Thereby, owing to the classical Liouville-Arnold theorem [20, 19, 8] our Whitham type dynamical system (1.4), reduced invariantly upon the four-dimensional invariant submanifold \( M^4 \subset M \), is completely integrable by quadratures. This result we will formulate as a final theorem.

**Theorem 4.1.** The Whitham type dynamical system (1.4), reduced upon the invariant two-parametric four-dimensional functional submanifold \( M^4 \subset M \) is exactly equivalent to the set of two commuting to each other canonical Hamiltonian flows (4.17), which are completely Liouville-Arnold integrable by quadratures systems. The corresponding Hamiltonian functions are given by expressions (4.15) and (4.16).

The results obtained above make it possible to construct a wide class of exact two-parametric solutions of the Whitham type nonlinear equation (1.4) by means of quadratures. This very interesting and important problem we plan to investigate in detail in another paper.

5. **Regularization scheme and the integrability problem**

Define a smooth periodic function \( v \in C^\infty_{2\pi}(\mathbb{R};\mathbb{R}) \), such that
\[
v := \partial^{-1}u_x^2
\]
for any \( x, t \in \mathbb{R} \), where the function \( u \in C^\infty_{2\pi}(\mathbb{R};\mathbb{R}) \) solves equation (1.1). Then it is easy to state
that the following nonlinear dynamical system
\begin{align*}
    u_t &= 2uu_x - v \\
    v_t &= 2uv_x \\
\end{align*}
\{ := K[u,v]

of hydrodynamic type, being already well defined on the extended $2\pi$-periodic functional space $\mathcal{M} := \mathcal{C}_2^2(\mathbb{R};\mathbb{R})$, is completely equivalent to that given by expression (1.1). Thereby, mapping (5.1) regularizes the previously not completely determined expression (1.1), making it possible to pose a new integrability problem for dispersionless dynamical system (5.2) of hydrodynamic type in the functional space $\mathcal{M}$. To proceed, one can analyze this problem by means of the gradient-holonomic method [8, 15], which is similar to the approach applied above for studying dynamical system (1.1). Based on preliminary performed calculations and obtained analytical properties of dynamical systems (5.2) we can state that it is also an integrable flow in the functional space $\mathcal{M}$, possessing a suitable Lax type representation. We plan to investigate this problem in detail in another work under preparation.

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