Higher Dimensional Kerr–Schild Spacetimes with (A)dS Background

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Abstract. Geometric properties of higher dimensional Kerr–Schild (KS) spacetimes with (A)dS background are summarised. For Einstein spaces algebraic types of the Weyl tensor compatible with the KS ansatz are established, optical properties of the KS vector are given. Important examples of exact solutions belonging to this class, such as Kerr–de Sitter metrics in arbitrary dimension are also briefly discussed.

1. Introduction

Four-dimensional Kerr–Schild ansatz with Minkowski background was first used in [2]. It led to the discovery of the Kerr rotating black hole. Moreover, many other physically important exact vacuum solutions belong to this class, for instance Kerr–Newmann black hole and Weyl type N Kundt spacetimes including pp-waves. The generalization of KS ansatz to higher dimensions led to the discovery of Myers–Perry solutions [3]. General properties of higher dimensional Ricci-flat KS metric were studied recently in [4].

Rotating black hole solutions with a cosmological constant in four, five and higher dimensions [5–7] can be cast to a generalized Kerr–Schild (GKS) form, where the flat background metric is replaced by a de Sitter or an anti-de Sitter spacetime. We will study general properties of GKS spacetimes and generalize the results for the Ricci-flat case [4] to the Einstein spaces.

Generalized Kerr–Schild ansatz is a metric of the form

\[ g_{ab} = \bar{g}_{ab} - 2\mathcal{H}k_ak_b, \]

where the background metric \( \bar{g}_{ab} \) is (A)dS and \( \mathcal{H} \) is a scalar function. The KS vector \( k \) is null vector with respect to the full metric \( g_{ab} \). Therefore \( k \) is also null vector with respect to the background metric \( \bar{g}_{ab} \) and the inverse metric has the simple form \( g^{ab} = \bar{g}^{ab} + 2\mathcal{H}k^ak_b \). In the following we consider the background metric in the canonical form \( \bar{g}_{ab} = \Omega (-dt + dx_1 + \ldots + dx_{n-1}) \). The conformal factor \( \Omega \) in the de Sitter and anti-de Sitter background metric is given by

\[ \Omega_{dS} = \frac{(n-2)(n-1)}{2\Lambda^2}, \quad \Omega_{AdS} = -\frac{(n-2)(n-1)}{2\Lambda^2}. \]
2. General properties

Let us start with the simplest Ricci frame component of the highest boost weight \( R_{00} = R_{ab} k^a k^b \) since many terms in this contraction vanish. Indeed, we obtain

\[
R_{00} = 2 \mathcal{H} k_{a;b} k^a k^b .
\]  

(3)

From Einstein’s equations it immediately follows

**Proposition 1** The null vector \( k \) in the generalized Kerr–Schild metric (1) is geodetic if and only if the component of the energy-momentum tensor \( T_{00} = T_{ab} k^a k^b \) vanishes.

Note that \( T_{00} \) vanishes not only in Einstein spaces but also in spacetimes with aligned matter content, such as aligned Maxwell field \( F_{ab} k^a \propto k_b \) or aligned pure radiation \( T_{ab} \propto k_a k_b \).

Various quantities constructed from the full GKS metric can be easily used for derivation of the corresponding quantities constructed from the background metric by setting \( \mathcal{H} \) to zero. This allows us to compare geodecticity of KS congruence in both metrics

\[
k_{a;b} k^a = k_{a;b} k^b , \quad k^a_{;b} k^b = k^a_{;b} k^b ,
\]  

(4)

where \( k_{a;b} \) denotes a covariant derivative with respect to the background metric. Thus the KS vector \( k \) is geodetic in the full metric if and only if it is geodetic in the background metric.

From now on we assume \( k \) being geodetic, which significantly simplifies our calculations, e.g. boost weight 1 and 0 components of the Ricci tensor vanish.

One may introduce a null frame in the full metric consisting of two null vectors \( \ell, n \) and \( n - 2 \) spacelike vectors \( m^{(i)} \) such that \( g_{ab} = 2 \ell_{(a} n_{b)} + \delta_{ij} m^{(i)}_{a} m^{(j)}_{b} \). Similarly, we can define a null frame \( \ell, \bar{n}, m^{(i)} \) in the background metric putting \( \bar{n}_a = n_a + \mathcal{H} k_a \). If we identify the frame vector \( \ell \) with the KS vector \( k \) optical matrices \( \bar{L}_{ij} \) and \( L_{ij} \) in the full and background spacetimes obey

\[
L_{ij} \equiv k_{a;b} m^{(i)a} m^{(j)b} = k_{a;b} m^{(i)a} m^{(j)b} \equiv \bar{L}_{ij} .
\]  

(5)

In other words the KS vector \( k \) has the same optical properties in both metrics.

It can be shown [1] that for geodetic KS vector \( k \) all positive boost weight frame components of the Weyl tensor vanish

**Proposition 2** Generalized Kerr–Schild spacetime with geodetic vector \( k \) is algebraically special with \( k \) being the multiple WAND.

From now on, let us consider only Einstein spacetimes. We employ Einstein’s field equations in order to analyze constraints imposed on the parameters of the GKS ansatz. One of the equations [1] can be written as

\[
(n - 2) \partial (D \log \mathcal{H}) = \sigma^2 + \omega^2 - (n - 2)(n - 3) \theta^2 .
\]  

(6)

The function \( \mathcal{H} \) appears in (6) only if \( \theta \neq 0 \) and, consequently, we will solve the non-expanding case \( \theta = 0 \) and expanding case \( \theta \neq 0 \) separately.

3. Non-expanding solutions

Vanishing expansion \( \theta = 0 \) in (6) implies that the KS congruence \( k \) is non-shearing, \( \sigma = 0 \), and non-twisting, \( \omega = 0 \), and thus the optical matrix vanishes, \( L_{ij} = 0 \). It can be shown [1] that consequently all boost weight 0 and -1 components of the Weyl tensor vanish. We can conclude

**Proposition 3** Einstein generalized Kerr–Schild spacetimes with non-expanding KS congruence \( k \) are of type N with \( k \) being the multiple WAND. Twist and shear of the KS congruence \( k \) necessarily vanish and these solutions thus belong to the class of Einstein type N Kundt spacetimes.

Note that in the Ricci-flat case [4] the class of non-expanding KS spacetimes is in fact equivalent with the class of type N Kundt spacetimes.
4. Expanding solutions

In the expanding case Einstein’s equations lead to the “optical constraint”

\[ L_{ik}L_{jk} = \frac{L_{ik}L_{jk}}{(n-2)\theta} S_{ij} \]  

(7)

and it follows that \( L \) is a normal matrix, \( LL^T - L^T L = 0 \). Therefore, in an appropriate frame the optical matrix \( L_{ij} \) has a block-diagonal form consisting of \( 2 \times 2 \) blocks with vanishing symmetric trace-free part (i.e. there is no shear in the corresponding two dimensional spacelike planes).

The block-diagonal form of \( L_{ij} \) allows us to solve Sachs equation (\( DL_{ij} = -L_{ik}L_{kj} \)) and thus explicitly determine \( r \)-dependence (with \( r \) being an affine parameter along \( k \)) of the optical matrix \( L_{ij} \) which consists of \( p \) blocks \( \mathcal{L}(\mu) \) and a diagonal matrix \( \tilde{\mathcal{L}} \)

\[
\mathcal{L}(\mu) = \begin{pmatrix}
    s_{(2\mu)} & A_{2\mu,2\mu+1} \\
    -A_{2\mu,2\mu+1} & s_{(2\mu)}
\end{pmatrix}, \\
\tilde{\mathcal{L}} = r^{-1} \text{diag}(1,\ldots,1,0,\ldots,0),
\]

(8)

with \( 0 \leq 2p \leq m \leq n-2 \) and \( m \) denoting the rank of \( L_{ij} \). Using this result and integrating equation (6) we get

\[ \mathcal{H} = \frac{\mathcal{H}_0}{r^{m-2p-1}} \prod_{\mu=1}^{p} \frac{1}{r^2 + (a_{(2\mu)}^0)^2} . \]

(9)

The KS function \( \mathcal{H} \) may introduce a singularity to the full GKS metric whenever \( \mathcal{H} \) diverges. Obviously, \( \mathcal{H} \) blows up at \( r = 0 \) when \( 2p \neq m \) or \( 2p \neq m-1 \) and in the special case \( 2p = m \) (\( m \) even) or \( 2p = m-1 \) (\( m \) odd) if some of \( a_{(2\mu)}^0 \) vanish at some points of spacetime. In both cases the Kretschmann scalar shows presence of a curvature singularity.

It can be also shown [1] that expanding Einstein GKS spacetimes are not compatible with Weyl types III and N and thus

**Proposition 4** Einstein generalized Kerr–Schild spacetimes with expanding KS congruence \( k \) are of Weyl types II or D or conformally flat.

4.1. Example: 5D Kerr–(A)dS

Higher dimensional Kerr–(A)dS metric is written in the GKS form in [7]. In five dimensions, the background metric \( \bar{g} \), KS vector \( k \) and KS function \( \mathcal{H} \) read

\[
\begin{align*}
\bar{g} &= -\frac{(1 - \lambda r^2)\Delta}{(1 + \lambda^2)(1 + \lambda b^2)} \text{d}t^2 + \frac{r^2 \rho^2}{(1 - \lambda r^2)(r^2 + a^2)(r^2 + b^2)} \text{d}r^2 + \frac{\rho^2}{\Delta} \text{d}\theta^2 \\
&\quad + \frac{(r^2 + a^2) \sin^2 \theta}{1 + \lambda a^2} \text{d}\phi^2 + \frac{(r^2 + b^2) \cos^2 \theta}{1 + \lambda b^2} \text{d}\psi^2 , \quad \mathcal{H} = \frac{M}{\rho^2} , \quad \Delta = 1 + \lambda a^2 \cos^2 \theta + \lambda b^2 \sin^2 \theta.
\end{align*}
\]

(10)

\[ k = \frac{\Delta}{(1 + \lambda a^2)(1 + \lambda b^2)} \text{d}t + \frac{r^2 \rho^2}{(1 - \lambda r^2)(r^2 + a^2)(r^2 + b^2)} \text{d}r - \frac{a \sin^2 \theta}{1 + \lambda a^2} \text{d}\phi - \frac{b \cos^2 \theta}{1 + \lambda b^2} \text{d}\psi , \]

where \( \rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta \) and \( \Delta = 1 + \lambda a^2 \cos^2 \theta + \lambda b^2 \sin^2 \theta \). Using the null frame given in [1] we can express \( L_{ij} \) in the block-diagonal form with \( n = 5, m = 3, p = 1 \). The optical matrix and function \( \mathcal{H} \) of 5D Kerr–(A)dS and GKS spacetime are compared in the following table.
Let us briefly discuss presence of a curvature singularity. In the fist case $a \neq 0$, $b \neq 0$ although $2p = m - 1$ corresponds to the special case, $a_{(2)}^0$ is sum of squares and hence there is no singularity. If we set one of the spins to zero, e.g. $b = 0$, then $a_{(2)}^0$ vanishes for $\theta = \frac{\pi}{2}$ and there is a ring shaped singularity known from the four-dimensional Kerr solution.

Putting $a = b = 0$ leads to (Anti-)de Sitter–Schwarzschild–Tangherlini limit where now $p = 0$ and neither $2p = m - 1$ nor $2p = m$ and therefore a singularity is present at $r = 0$.

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