A differential analog of a theorem of Chevalley

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Abstract

In this note a proof of a differential analog of Chevalley’s theorem [C] on homomorphism extensions is given. An immediate corollary is a condition of finiteness of extensions of differential algebras and several equivalent definitions of a differentially closed field, including Kolchin’s Nullstellensatz.

In this note I give a proof of the following differential analog of Chevalley’s theorem [C] on homomorphism extensions.

Theorem 1. Let S be a differential algebra over $\mathbb{Q}$ with no zero divisors and let $b$ be a non-zero element of S. Let $R$ be a differential subalgebra of $S$ over which $S$ is differentially finitely generated. Let $\mathcal{F}$ be a differentially closed field of characteristic 0. Then there exists a non-zero element $a$ of $R$ such that any homomorphism $\phi : R \to \mathcal{F}$ which does not annihilate $a$ extends to a homomorphism $\psi : S \to \mathcal{F}$ which does not annihilate $b$.

An almost immediate consequence of the proof of Theorem 1 is

Theorem 2. Let $\mathcal{F}$ be a differentially closed field of characteristic 0 and let $S \supset R$ be differentially finitely generated differential algebra and subalgebra over $\mathcal{F}$. Suppose that there exists a non-zero element $b$ of $S$ such that any homomorphism $\phi : R \to \mathcal{F}$ has only finitely many extensions $\psi : S \to \mathcal{F}$ satisfying $\psi(b) \neq 0$. Then the field extension Fract $S \supset$ Fract $R$ is finite.

In particular, if any homomorphism $\phi : R \to \mathcal{F}$ has at most $d$ extensions $\psi : S \to \mathcal{F}$ with $\psi(b) \neq 0$, then the degree of Fract $S$ over Fract $R$ is at most $d$.

An immediate corollary of Theorem 1 is Kolchin’s Nullstellensatz [K] and its earlier weaker versions by Ritt [R], Cohn [Cohn] and Seidenberg [S].

As far as I can understand it, Theorem 1 is closely related to Blum’s elimination of quantifiers theorem [Blum], [M] in the model theory of differentially closed fields.

In Section 1 I explain the necessary background on Differential Algebra, in Sections 2 and 3 give proofs of Theorems 1 and 2 and in Section 4 give about a dozen of equivalent definitions of

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a differentially closed field of characteristic 0 (Theorem 1 and Kolchin’s Nullstellensatz being among them).

This note is an offshoot of a course in Differential Algebra that I gave at M.I.T. in the fall of 2000. The motivation for teaching this course came from close connections of the theory of conformal algebras to differential algebras [BDR]. Namely, each Lie conformal algebra defines a functor from the category of differential algebras to the category of Lie algebras (very much like a group scheme defines a functor from the category of algebras to the category of groups). I am grateful to my students for their enthusiasm, especially to A. De Sole for several corrections and improvements. I would like to thank A. Buium for very useful and enlightening correspondence and J. Young for bringing Marker’s paper [M] to my attention and explaining some parts of it.

1. Here I recall some terminology and facts from Differential Algebra. All of this can be easily found in two excellent books [R] and [K], the primary source of both being Ritt’s foundational book [R].

By an algebra we always mean a commutative associative unital algebra. A differential algebra $R$ is an algebra over a field with a fixed derivation $\delta$. By a homomorphism of differential algebras we always mean a homomorphism commuting with derivations. Likewise subalgebras and ideals are assumed to be $\delta$-invariant, though to emphasize this we shall often call them differential subalgebras and ideals.

The first important observation of Ritt is that in a differential algebra over $\mathbb{Q}$ the radical of a differential ideal is a differential ideal (see [R], [Ra], [K] Lemma 1.8). (A counterexample in characteristic $p$ is the zero ideal in $R = \mathcal{F}[x]/(x^p)$, $\delta = d/dx$.) Another important fact is the differential Krull theorem: any differential radical ideal is an intersection of differential prime ideals (see [Ra], [K] Theorem 2.1). These facts imply, in particular, that maximal (among) differential ideals are prime.

If a differential algebra $R$ has no zero divisors, we can form the field of fractions $\text{Fract} R$ and extend the derivation $\delta$ to this field.

One says that a differential algebra $S$ is differentially generated by elements $x_1, \ldots, x_n$ over a differential subalgebra $R$, and writes $S = R\{x_1, \ldots, x_n\}$, if the algebra $S$ is generated by all elements from $R$ and all derivatives $x_i^{(k)}$, $i = 1, \ldots, n$, $k \in \mathbb{Z}_+$ = $\{0, 1, 2, \ldots\}$, of the elements $x_i$.

Let $S = R\{x\}$ be a differential algebra with no zero divisors. One says that $x$ is differentially transcendental over $R$ if all elements $x^{(k)}$, $k \in \mathbb{Z}_+$, are algebraically independent over $\text{Fract} R$; otherwise $x$ is called differentially algebraic over $R$.

The algebra of differential polynomials over a differential algebra $R$ in the differential indeterminates $y_1, \ldots, y_n$ is the algebra of polynomials $R[y_i^{(k)}, i = 1, \ldots, n, k \in \mathbb{Z}_+]$ with the derivation $\delta$ extended from $R$ by the rule $\delta(y_i^{(k)}) = y_i^{(k+1)}$. This algebra is denoted by $R\{y_1, \ldots, y_n\}$.

Consider the differential algebra $R\{y\}$ of differential polynomials over $R$ in one differential indeterminate $y$. Let $A(y) \in R\{y\}\backslash R$, be a “non-constant” differential polynomial. The largest $r$ for which $y^{(r)}$ is present in $A(y)$ is called the order of $A(y)$ and is denoted by ord $A(y)$. One can write in a unique way:

$$A(y) = I_A(y)y^{(r)d} + I_1(y)y^{(r)d-1} + \cdots + I_d(y),$$

where ord $I_A(y) < r$, ord $I_j(y) < r$ and $I_A(y) \neq 0$. Then $d$ is called the degree of $A(y)$ and is denoted by deg $A(y)$, and $I_A(y)$ is called the initial of $A(y)$. The differential polynomial $S_A(y) = \cdots + I_d(y)$, where
\( \frac{\partial A(y)}{\partial y^{(r)}} \) is called the separant of \( A(y) \). The important property of the characteristic 0 case is that \( S_A(y) \neq 0 \) if \( A \notin R \).

For \( A, B \in R\{y\} \backslash R \) we write \( A < B \) if either \( \text{ord} A < \text{ord} B \), or \( \text{ord} A = \text{ord} B \) and \( \deg A < \deg B \). We also write \( A < B \) if \( A \in R, B \notin R \). Note that \( I_A < A \) and \( S_A < A \).

The basic result of Ritt’s theory (see \[\text{R}] \text{Lemma 7.3 or } [\text{R}] \text{(2.3)}) is the following division algorithm:

Given a non-constant differential polynomial \( A(y) \), for any \( F(y) \in R\{y\} \) there exists \( G(y) \in R\{y\} \) with \( G < A \) and \( m, n \in \mathbb{Z}_+ \) such that:

\[
I_A(y)^m S_A(y)^n F(y) \equiv G(y) \mod [A(y)],
\]

where \([A(y)]\) is the ideal of \( R\{y\} \) generated by \( A(y)^{(k)} \), \( k \in \mathbb{Z}_+ \). Furthermore, one can take \( m = 0 \) if only \( \text{ord} G \leq \text{ord} A \) is required, which is called the weak division algorithm.

The important notion of a differentially closed field was introduced by Robinson \[\text{Rob}\]. His axioms have been considerably simplified by Blum \[\text{Blum}\], and it is her definition, given below, that is commonly used. A differential field \( \mathcal{F} \) is called differentially closed if for any two differential polynomials \( A(y), B(y) \in \mathcal{F}\{y\} \) with \( B \neq 0 \) and \( \text{ord} B < \text{ord} A \) there exists \( \alpha \in \mathcal{F} \) such that \( A(\alpha) = 0 \) and \( B(\alpha) \neq 0 \).

Any differentially closed field is algebraically closed (i.e., has no non-trivial finite extensions), but it always has differentially algebraic extensions. Nevertheless, it turned out to be the right substitute for Differential Algebra of the notion of an algebraically closed field.

The existence of a differentially closed field containing a given differential field of characteristic 0 is easy to establish in the framework of Model Theory (see e.g. \[\text{M}\]). An elementary proof (i.e., without reference to model theory) may be found in \[\text{B}] \text{(5.2)}.

2. Proof of Theorem 1. The general scheme of the proof is the same as Chevalley’s \[\text{C}\]. We have:

\[
(1) \quad S = R\{x_1, \ldots, x_{n-1}\}\{x_n\} \supset R\{x_1, \ldots, x_{n-1}\} \supset R.
\]

By induction on \( n \) we reduce the proof to the case \( n = 1 \). Indeed, from being true for \( n = 1 \), we conclude that there exists a non-zero \( b_1 \in R\{x_1, \ldots, x_{n-1}\} \) such that any homomorphism \( \psi_1 : R\{x_1, \ldots, x_{n-1}\} \to \mathcal{F} \) with \( \psi_1(b_1) \neq 0 \) extends to \( \psi : R\{x_1, \ldots, x_n\} \to \mathcal{F} \) with \( \psi(b) \neq 0 \), and by the inductive assumption, there exists a non-zero \( \alpha \in R \) such that any homomorphism \( \varphi : R \to \mathcal{F} \) with \( \varphi(\alpha) \neq 0 \) extends to \( \psi_1 : R\{x_1, \ldots, x_{n-1}\} \to \mathcal{F} \) with \( \psi_1(b_1) \neq 0 \).

Thus, we may assume that \( S = R\{x\} \). Given a homomorphism \( \varphi : R \to \mathcal{F} \), we may extend it to the algebras of differential polynomials \( A(y) \mapsto A^\varphi(y) \) by applying \( \varphi \) to coefficients. Let \( B(y) \in R\{y\} \) be such that \( B(x) = b \). We consider separately two cases.

Case 1: \( x \) is differentially transcendental over \( R \). Let \( a \in R \) be any non-zero coefficient of \( B(y) \). If \( \varphi(a) \neq 0 \), then \( B^\varphi(y) \in \mathcal{F}\{y\} \) is a non-zero differential polynomial, hence there exists \( \alpha \in \mathcal{F} \) which is not a root of \( B^\varphi(y) \) (we take \( B = B^\varphi(y) \) and \( A \) with \( \text{ord} A > \text{ord} B \) in the definition of a differentially closed field). Then \( \varphi(Q(x)) = Q^\varphi(\alpha) \) is a well defined homomorphism \( S \to \mathcal{F} \) with \( \psi(b) \neq 0 \).

Case 2: \( x \) is differentially algebraic over \( R \). Let \( A(y) \in R\{y\} \) be a minimal in the partial ordering \( < \) irreducible (in the usual sense) over \( \text{Fract} R \) differential polynomial such that \( A(x) = 0 \). If \( F(y) \in R\{y\} \) is such that \( F(x) = 0 \), apply the division algorithm:

\[
S_A(y)^m I_A(y)^n F(y) \equiv G(y) \mod [A(y)],
\]
where $G < A$. Since $F(x) = 0$ and $A(x) = A'(x) = \cdots = 0$, we see that $G(x) = 0$, hence, due to minimality of $A$, $G(y) = 0$, and we have

\[(2) \quad S_A(y)^m I_A(y)^n F(y) \in [A(y)].\]

Let $a_1 \in R$ be a non-zero coefficient of $I_A(y)$. Let $D(y) \in R\{y\}$ be the discriminant of $A(y)$ viewed as a polynomial in $y^{(\text{ord } A)}$. Note that $D(y) \neq 0$ since $A(y)$ is an irreducible polynomial, and that ord $D(y) < \text{ord } A(y)$. Let $a_2 \in R$ be a non-zero coefficient of $D(y)$.

Suppose that $\varphi(a_1a_2) \neq 0$. Then ord $A^\varphi(y) = \text{ord } A(y) > \text{ord } I_A^\varphi(y)D^\varphi(y)$ and $I_A^\varphi(y)D^\varphi(y) \neq 0$. Since $F$ is differentially closed, there exists $\alpha \in F$ which is a root of $A^\varphi(y)$ but not a root of $I_A^\varphi(y)D^\varphi(y)$. Since $D^\varphi(\alpha) \neq 0$, $A^\varphi(y)$ and $S_A^\varphi(y)$ have no common roots, hence $S_A^\varphi(\alpha) \neq 0$. Since also $I_A^\varphi(\alpha) \neq 0$, but $A_A^\varphi(\alpha) = 0$, we conclude from (2) that $F^\varphi(\alpha) = 0$.

Therefore $\psi(Q(x)) = Q^\varphi(\alpha)$ is a well defined homomorphism $S \to F$ which extends $\varphi : R \to F$.

It remains to take care of the condition $\psi(b) \neq 0$ by an appropriate choice of $\alpha$ (satisfying the above conditions as well). By the weak division algorithm we have:

\[(3) \quad S_A^n(y)B(y) = B_1(y) \mod [A(y)],\]

where $n \in \mathbb{Z}_+$ and ord $B_1(y) \leq \text{ord } A(y)$. Letting $y = x$ in (3) we get

\[(4) \quad B_1(x) = S_A^n(x)b.\]

Since $S_A^\varphi(\alpha) \neq 0$, we conclude that $S_A(x) \neq 0$, hence $B_1(x) \neq 0$, due to (3). Let $r(y) \in R\{y\}$ be the resultant of the polynomials $B_1(y)$ and $A(y)$ viewed as polynomials in $y^{(\text{ord } A)}$. Since $A(x) = 0$ and $B_1(x) \neq 0$, we conclude that $r(y) \neq 0$ (otherwise $A(y)$ and $B_1(y)$ would have a common root in Fract $R$ and therefore $B_1(y)$ would be divisible by $A(y)$ due to its irreducibility). Note that ord $r(y) < \text{ord } A(y)$. Let $a_3 \in R$ be a non-zero coefficient of $r(y)$.

We let $a = a_1a_2a_3$ and suppose that $\varphi(\alpha) \neq 0$. Choose $\alpha \in F$ such that, as before, $A^\varphi(\alpha) \neq 0$, $I_A^\varphi(\alpha)D^\varphi(\alpha) \neq 0$, and, in addition, $r^\varphi(\alpha) \neq 0$. As before, define $\psi(Q(x)) = Q^\varphi(\alpha)$. As before, this is a well defined homomorphism $S \to F$, and, by (4): $S_A^\varphi(\alpha)\psi(b) = B_1^\varphi(\alpha)$. But $B_1^\varphi(\alpha) \neq 0$, since $r^\varphi(\alpha) \neq 0$ and therefore $A^\varphi(y)$ and $B_1^\varphi(y)$ have no common roots. Hence $\psi(b) \neq 0$.

**Corollary 1.** If $S$ is a differentially finitely generated differential algebra over $F$, then for any non-zero $b \in S$ there exists an $F$-algebra homomorphism $\psi : S \to F$ such that $\psi(b) \neq 0$.

3. In order to prove Theorem 2, we need the following lemma.

**Lemma 1.** Let $F$ be a differentially closed filed. Then

(a) For any non-zero $A(y) \in F\{y\}$ there exists infinitely many $\alpha \in F$ such that $A(\alpha) \neq 0$.

(b) If $A(y) \in F\{y\}\setminus F$ and there exists $B(y) \in F\{y\}$ such that $B(y) \neq 0$, and $B(y) < \text{ord } A(y)$ and only finitely many roots of $A(y)$ are not roots of $B(y)$, then ord $A(y) = 0$.

**Proof.** (a) Take a sequence $A_j(y) \in F\{y\}$, $j \in \mathbb{Z}_+$, of increasing order, such that $A_0(y) = A(y)$. For each $j = 1, 2, \ldots$ there exists $\alpha_j \in F$ which is a root of $A_j$, but not of $A_0 \cdots A_{j-1}$. Hence all $\alpha_1, \alpha_2, \ldots$ are not roots of $A(y)$.
(b) Let \( \alpha_1, \ldots, \alpha_n \in F \) be all roots of \( A(y) \) which are not roots of \( B(y) \). Note that \( n \geq 1 \) since \( F \) is differentially closed. Let \( B_1(y) = \prod_{i=1}^{n} (y - \alpha_i)B(y) \) and suppose that \( \text{ord } A(y) > 0 \). Then \( \text{ord } B_1(y) < \text{ord } A(y) \), hence there exists \( \beta \in F \) which is a root of \( A(y) \), but not of \( B_1(y) \). But then \( \beta \not= \alpha_i \) for all \( i \), a contradiction. \( \square \)

Proof of Theorem 2. Again, using (b), we reduce the proof to the case \( S = R\{x\} \). Certainly any homomorphism \( R\{x_1, \ldots, x_{n-1}\} \to F \) extends in only finitely many ways to \( S \to F \) that does not annihilate \( b \). Hence from \( n = 1 \) case we conclude that Fract \( S \) is finite over Fract \( R\{x_1, \ldots, x_{n-1}\} \). By Theorem 1 there exists \( b' \neq 0 \) in \( R\{x_1, \ldots, x_{n-1}\} \) such that any homomorphism \( \varphi \) to \( F \) with \( \varphi(b') \neq 0 \) extends to \( \psi: S \to F \) with \( \psi(b) \neq 0 \). Hence there exists only finitely many homomorphisms \( \psi: R\{x_1, \ldots, x_{n-1}\} \to F \) which extend \( \varphi: R \to F \) such that \( \psi(b') \neq 0 \). Hence, by the inductive assumption, Fract \( R\{x_1, \ldots, x_{n-1}\} \) is finite over Fract \( R \) and Fract \( S \) is finite over Fract \( R \).

Since \( R \) is differentially finitely generated over \( F \), by Corollary 1, for any non-zero \( a \in R \) we can find a homomorphism \( \varphi: R \to F \) with \( \varphi(a) \neq 0 \). Again we consider separately two cases of \( S = R\{x\} \), keeping notations of the proof of Theorem 1.

Case 1: \( x \) is differentially transcendental over \( R \). Let \( a \in R \) be a non-zero coefficient of \( B(y) \), so that \( B^\varphi(y) \neq 0 \). By Lemma 1, \( B^\varphi(y) \) has infinitely many non-roots \( \alpha \), and for each of them the homomorphism \( \psi(B(x)) = B^\varphi(\alpha) \) extends \( \varphi \) such that \( \psi(b) \neq 0 \). Thus, this case is impossible.

Case 2: \( x \) is differentially algebraic over \( R \). Take \( a = a_1a_2a_2 \) from the proof of Theorem 1 and \( \varphi: R \to F \) with \( \varphi(a) \neq 0 \). From the proof of Theorem 1 we know that \( \varphi \) extends to \( \psi: S \to F \) with \( \psi(b) \neq 0 \) by letting \( \psi(Q(x)) = Q^\varphi(\alpha) \), where \( \alpha \) is a root of \( A^\varphi(y) \) and is not a root of \( I_{A}^\varphi(y)D^\varphi(y)r^\varphi(y) \). Since, by conditions of the theorem, there exists only finitely many such \( \alpha \), we conclude by Lemma 1 that \( \text{ord } A^\varphi(y) = 0 = \text{ord } A(y) \). In other words, \( A(y) \) is an ordinary (non-zero) polynomial over \( R \) such that \( A(x) = 0 \). Hence Fract \( S \) is finite over Fract \( R \).

\( \square \)

4. Theorem 3. The following properties of a differential filed \( F \) of characteristic 0 are equivalent:

(a) \( F \) is differentially closed.

(b) If \( A, B \in F\{y\} \) are such that \( A \) is irreducible, \( B \) is not divisible by \( A \) and \( \text{ord } A \geq \text{ord } B \), then there exists \( \alpha \in F \) such that \( A(\alpha) = 0, B(\alpha) \neq 0 \) and \( S_A(\alpha) \neq 0 \).

(c) If \( J \) is a prime differential ideal of \( F\{y\} \) and \( B \in F\{y\}\setminus J \), then there exists \( \alpha \in F \) such that \( f(\alpha) = 0 \) for all \( f \in J \) and \( B(\alpha) \neq 0 \).

(d) The same as (c), where \( J \) is a radical differential ideal.

(e) The same as (c) (resp. d) where \( F\{y\} \) is replaced by \( F\{y_1, \ldots, y_n\} \), \( n \geq 1 \).

(f) If \( J \) is a proper prime differential ideal of \( F\{y_1, \ldots, y_n\} \), \( n \geq 1 \) arbitrary, then there exists \( \alpha \in F \) such that \( f(\alpha) = 0 \) for all \( f \in J \).

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(g) The same as (f), where $J$ is a proper differential ideal (resp. radical differential ideal).

(h) The same as (f), where $J$ is a proper maximal differential ideal.

(i) Given any $S, R$ and $b$ as in Theorem, the conclusion of Theorem holds for $F$.

Proof. (a) implies (i) by Theorem. (i) implies (c) by taking $S = F\{y\}/J$, $b =$ image of $B$ in $S$ and applying Corollary (and the differential Krull theorem). (e) trivially implies (c) and (d). A standard argument of adding extra indeterminates shows that (f) is equivalent to (e). By the differential Krull theorem, (g) is equivalent to (f). Since every prime differential ideal can be included in a maximal one, which is prime too, (h) is equivalent to (f). (b) trivially implies (a) (by taking in (a) an irreducible factor of $A$ of the same order). Finally, the implication (c) $\Rightarrow$ (b) is proved in the same way as 3) $\Rightarrow$ 1) in Theorem 5.1 of [B]. Indeed, $\{A\} : S_A$ is a prime ideal by Ritt’s structure theorem Theorem 2.5 or Lemma 7.9, and this ideal does not contain $S_AB$ by Ritt’s divisibility theorem Theorem 2.4 or Lemma 7.8.

Remark. All proofs can be extended without difficulty to the case of several commuting derivations $\delta_1, \ldots, \delta_m$.

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