Soliton stars in the breather limit

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Abstract
This paper presents an asymptotic reduction of the Einstein–Klein–Gordon system with a real scalar field (soliton star problem). A periodic solution of the reduced system, similar to the sine-Gordon breather, is obtained by a variational method. This tallies with numerical computations. As a consequence, a time-periodic redshift for sources close to the center of the star is obtained.

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1. Introduction

The question of whether the matter content of the universe is accounted for by luminous objects has led to the search for mechanisms that would lead to massive objects of a new kind. For instance, one could consider the Einstein field equations with matter terms arising from a real or complex scalar field that solves the (linear) Klein–Gordon equation [7, 14, 15]. Numerical computations [17] suggest that such a configuration, with a real scalar field, should admit long-lived, nearly periodic and strongly localized solutions. They would owe their existence to the nonlinear structure of the Einstein field equations, without any periodic forcing. In this sense, they would be similar to the breather solution of the sine-Gordon equation [1, 10, 12]. For this reason, objects modeled by a solution of the Einstein field equations coupled to a real Klein–Gordon field were called ‘oscillating soliton stars,’ even though they may not emit light, but merely affect light rays in their vicinity. Possible astrophysical applications, in particular to dark matter, are discussed in [17]. If the scalar field decays very fast, it is expected that the metric behaves like a Schwarzschild metric at infinity, with mass related to the energy density of the scalar field. The purpose of this paper is to provide a perturbative construction of such soliton stars, thereby providing a simple set-up to understand such objects analytically. We introduce two assumptions:
(a) the expansion parameter is an amplitude parameter;
(b) the space and time variables are scaled in a way consistent with the dispersion relation of the Klein–Gordon equation.
This procedure leads to a consistent limit, in which the equation for the scalar field reduces to simple harmonic motion

\[ u_{tt} + u = 0; \]

the amplitude depends on space, and is determined by a nonlinear non-secularity condition at second order in perturbation theory [9, 10]. This procedure makes no reference to the complete integrability of the sine-Gordon equation. This ‘breather limit’ is also useful in particle physics [6]. While the main point of [9] was to give a rigorous argument to explain the special role of the sine-Gordon equation in this context, the method of proof also automatically generates approximate solitons, by a procedure of general applicability. This paper gives an analogue of this ‘breather limit’ for the soliton star problem and shows that the Einstein–Klein–Gordon equations admit a consistent limit that is analytically tractable. It is the counterpart of the Newtonian limit of the boson star problem with complex scalar field [16, (2.17)–(2.18)].

After recalling the field equations in section 2, scaled variables are introduced in section 3 leading to the breather limit (theorem 1). At this stage, the metric and scalar field are determined by solving a nonlinear coupled system in two unknowns \( S \) and \( Z \). Intuitively, \( Z \) is the Newtonian potential generated by a mass density proportional to \( S^2 \), and \( S \) solves a Helmholtz equation with potential proportional to \( Z \). A solution for this nonlinear system is obtained in section 4 (theorem 2, proved in section 6), by a variant of the variational method for finding nonlinear ‘ground states’ of nonlinear Klein–Gordon equations [4, 18]. The asymptotic behavior of the metric is given in theorem 3. In particular, the metric component \( g_{00} \) is the sum of a Schwarzschild-like term and a periodic, exponentially decaying correction. Model validation issues, using observations or computations, are discussed in section 5. It follows that light originating in the vicinity of the soliton star should, if this model is valid, exhibit a time-periodic frequency shift.

2. Field equations

Consider a spherically symmetric metric

\[ ds^2 = -N^2 dt^2 + h^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{1} \]

where \( N, h \) and \( \phi \) only depend on \((r, t)\). In the following, subscripts \( r \) and \( t \) indicate derivatives with respect to these variables. We let \((x^0, \ldots, x^3) = (t, r, \theta, \phi)\), so that \( ds^2 = g_{ab} dx^a dx^b \), where Latin indices run from 0 to 3 and the summation convention is used. The scalar field \( \phi \) satisfies the Klein–Gordon equation

\[ -\left( \frac{1}{N^2} \frac{\partial}{\partial t} \right) \left( N^2 \frac{\partial}{\partial t} \phi + \frac{2}{N} \frac{\partial}{\partial r} \phi \right) - m^2 \phi = 0, \]

(2)

We assume that the metric and scalar field are even and periodic with respect to \( t \). We require \( N \to 1, h \to 1 \) and \( \phi \to 0 \) as \( r \to \infty \) for fixed \( t \). The equations for the scalar field and the metric coefficients are as follows [17]:

\[ \frac{1}{N^2} \left[ \phi_{tt} - \frac{N \phi_t}{N} + \frac{h \phi_t}{h} \right] + \frac{1}{h^2} \left[ \phi_{rr} - \frac{h \phi_r}{h} + \frac{N \phi_r}{N} + \frac{2 \phi_r}{r} \right] = m^2 \phi. \tag{3} \]

\[ (N^2)_r = N^2 \left[ (h^2 - 1)/r + 4\pi Gr h^2 \left( \frac{\phi_t^2}{N^2} + \frac{\phi_r^2}{h^2} - m^2 \phi^2 \right) \right], \tag{4} \]

\[ (h^2)_r = h^2 \left[ -(h^2 - 1)/r + 4\pi Gr h^2 \left( \frac{\phi_t^2}{N^2} + \frac{\phi_r^2}{h^2} + m^2 \phi^2 \right) \right], \tag{5} \]

\[ (h^2)_t = 8\pi Gr h^2 \phi_t \phi_t. \tag{6} \]
3. Small-amplitude equations

In this section, we define new variables and scaled unknowns, and prove that the field equations reduce, to leading order, to a system of two equations in two unknowns.

3.1. New variables and unknowns

Define new variables by

\[ \xi = m \varepsilon r, \]
\[ \tau = mt \sqrt{1 - \varepsilon^2}, \]

where \( \varepsilon > 0 \) is a new parameter. The scaling of time variables is motivated by the form of the Klein–Gordon equation\(^1\). Derivatives transform as follows:

\[ \partial_r = \varepsilon m \partial_\xi, \quad \partial_t = m(1 - \varepsilon^2)^{1/2} \partial_\tau. \]

In the following, subscripts \( \xi \) and \( \tau \) denote derivatives with respect to these variables. We let \( \mu = 4\pi G \).

Next, define new unknowns \( u, \tilde{N} \) and \( \tilde{h} \) by

\[ \phi = \varepsilon^2 u(\xi, \tau, \varepsilon), \quad (7a) \]
\[ N^2 = 1 + \varepsilon^2 \tilde{N}(\xi, \tau, \varepsilon), \quad (7b) \]
\[ h^2 = 1 + \varepsilon^2 \tilde{h}(\xi, \tau, \varepsilon). \quad (7c) \]

The assumptions on the metric and scalar field in section 2 lead to the conditions

\[ u, \tilde{N} \text{ and } \tilde{h} \text{ tend to } 0 \text{ as } r \to \infty, \text{ and are } 2\pi \text{-periodic in } \tau. \quad (7d) \]

As a result, the field variables are \( t \)-periodic with period \( 2\pi/\omega \), with \( \omega = m(1 - \varepsilon^2)^{1/2} \).

While, for the linear Klein–Gordon equation, the amplitude of the solution may be chosen independently of the period, this is not true in the nonlinear case, since \( \phi \) is determined by \( \varepsilon \), and therefore by \( \omega \). A general feature of nonlinear oscillators is the dependence of amplitude on period; here, this dependence is reflected in equations (9). In particular, \( \omega \) determines the leading-order amplitude \( \varepsilon^2 S(\xi) \) of the scalar field \( \phi \).

3.2. Field equations at leading order

We prove the following result.

**Theorem 1.** To lowest order in \( \varepsilon \), the soliton star problem (3)–(7) is equivalent to the system

\[ \Delta S - S = SZ, \quad (8a) \]
\[ \Delta Z = \mu S^2, \quad (8b) \]
\[ S \to 0 \text{ and } Z \to 0 \text{ as } \xi \to \infty. \quad (8c) \]

where \( \Delta = \frac{d^2}{d\xi^2} + (2/\xi) \frac{d}{d\xi} \) is the radial Laplacian in three dimensions and \( \mu = 4\pi G \).

The functions \( S \) and \( Z \) determine the scalar field and the metric at order \( \varepsilon^2 \) via

\[ \phi = \varepsilon^2 S(\xi) \cos \tau, \quad (9a) \]

\(^1\) The function \( f = \exp(\varepsilon mr) \cos(\omega t) \) solves \( f_{tt} - f_{rr} + m^2 f = 0 \) if and only if \( \omega^2 = m^2(1 - \varepsilon^2) \).
\[ N^2 = 1 + \epsilon^2 (Z + Y \cos 2\tau), \quad (9b) \]
\[ h^2 = 1 + \epsilon^2 h_0(\xi), \quad (9c) \]

where
\[ h_0(\xi) = \frac{\mu}{\xi} \int_0^\xi y^2 S^2(y) \, dy \quad \text{and} \quad Y(\xi) = \int_\xi^\infty \mu y S^2(y) \, dy. \quad (10) \]

**Remark 1.** The $1/\xi$ decay of $h^2 - 1$ is consistent with the behavior of the Schwarzschild solution. The integral for $Y$ converges because the scalar field decays exponentially fast at infinity, by theorem 3.

**Proof.** Write
\[ u = u_0 + \epsilon^2 u_1 + \cdots, \quad \tilde{N} = N_0 + \epsilon^2 N_1 + \cdots, \quad \tilde{h} = h_0 + \epsilon^2 h_1 + \cdots \]
and insert into equations (3)–(6). In the following, we lump as $O(\epsilon^k)$ various terms which involve a power of $\epsilon$ equal to $k$ or more. Equations (4)–(6) become respectively
\[ \tilde{N}_\xi = \frac{\tilde{h}}{\xi} + \mu \xi (u_\tau^2 - u^2) + O(\epsilon^2), \quad (11a) \]
\[ \tilde{h}_\xi = -\frac{\tilde{h}}{\xi} + \mu \xi (u_\tau^2 + u^2) + O(\epsilon^2), \quad (11b) \]
\[ \tilde{h}_\tau = O(\epsilon^2). \quad (11c) \]

Using relations (7), equation (3) becomes
\[ u = -\frac{(1 - \epsilon^2)}{1 + \epsilon^2 \tilde{N}} \left[ u_{\tau\tau} - \frac{1}{2} \epsilon^2 (\tilde{N}_\tau - \tilde{h}_\tau) u_\tau \right] + \epsilon^2 \frac{u_{\xi \xi}}{1 + \epsilon^2 \tilde{h}} + O(\epsilon^4). \quad (12) \]

Let
\[ \Delta = \frac{\partial^2}{\partial \xi^2} + \frac{2}{\xi} \frac{\partial}{\partial \xi}. \]

Since $(1 - \epsilon^2)/(1 + \epsilon^2 \tilde{N}) = 1 - \epsilon^2(1 + \tilde{N}) + O(\epsilon^4)$, (12) may be simplified to
\[ u_{\tau\tau} + u = \epsilon^2 \left\{ (1 + \tilde{N}) u_{\tau\tau} - \frac{1}{2} (\tilde{N}_\tau - \tilde{h}_\tau) u_\tau + \Delta u \right\} + O(\epsilon^4). \quad (13) \]

At leading order, we therefore obtain the equations
\[ h_{0\tau} = 0, \quad (14a) \]
\[ h_{0\xi} = -\frac{h_0}{\xi} + \mu \xi (u_{\tau\tau}^2 + u_0^2), \quad (14b) \]
\[ N_{0\xi} = \frac{h_0}{\xi} + \mu \xi (u_{\tau\tau}^2 - u_0^2), \quad (14c) \]
\[ u_{0\tau\tau} + u_0 = 0. \quad (14d) \]

Equation (14d) gives, since $u$ is even in time,
\[ u_0 = S(\xi) \cos \tau. \]

This proves (9a). It follows that $u_{0\tau}^2 + u_0^2 = S^2$; equation (14b) now yields
\[ h_{0\xi} = -\frac{h_0}{\xi} + \mu \xi S^2. \quad (15) \]
and \((14a)\) shows that
\[ h_0 = h_0(\xi), \]
hence \((9c)\). Finally, \((14c)\) gives
\[ N_{0\xi} = \frac{h_0(\xi)}{\xi} - \mu \xi S^2 \cos 2\tau. \tag{16} \]
It follows that \(N_0 = Y(\xi) \cos 2\tau + Z(\xi) + N_{00}(\tau)\). Since the metric is asymptotically flat as \(\xi \to \infty\), \(N_{00}\) is constant. Incorporating it into \(Z\), we obtain
\[ N_0 = Z(\xi) + Y(\xi) \cos 2\tau, \tag{17} \]
where \(Y\) and \(Z\) tend to zero as \(\xi \to \infty\). Equation \((9b)\) therefore holds.

Equation \((16)\) now yields
\[ Z_{\xi} = h_0(\xi)/\xi, \tag{18} \]
\[ Y_{\xi} = -\mu \xi S^2. \tag{19} \]

Equations \((15)\) and \((18)\) now yield
\[ \mu \xi^2 S^2 = (\xi h_0)_{\xi} = \xi^2 (Z_{\xi} + \frac{2}{\xi} Z_{\xi}) = \xi (\xi Z)_{\xi}, \]
hence
\[ \Delta Z = \mu S^2. \]
This proves \((8a)\). Finally, equation \((14b)\) yields \((\xi h_0)_{\xi} = \mu \xi^2 S^2\); since \(h_0\) should be regular at the origin, the first part of \((10)\) follows. Equation \((19)\) gives the rest of \((10)\). The convergence of the integral is a consequence of the exponential decay of \(S\) (see theorem 3).

It remains to prove \((8b)\). To this end, consider the terms of order \(\varepsilon^2\) in equation \((13)\):
\[ u_{1\tau\tau} + u_1 = (1 + N_0)u_{0\tau\tau} + \frac{1}{\tau} N_{0\tau} u_{0\tau} + \Delta S \cos \tau. \tag{20} \]
Now,
\[ (1 + N_0)u_{0\tau\tau} + \frac{1}{\tau} N_{0\tau} u_{0\tau} = -(1 + Z + Y \cos 2\tau)S \cos \tau + (Y \sin 2\tau)(S \sin \tau) \]
\[ = -(1 + Z)S \cos \tau - Y S \cos 3\tau. \]
Since \(u_1\) should be \(2\pi\)-periodic in \(\tau\), the right-hand side of \((20)\) should not contain any term proportional to \(\cos \tau\). Therefore,
\[ \Delta S - (1 + Z)S = 0. \]
This completes the proof of the theorem. \(\square\)

Remark 2. The function \(Y\) may also be expressed directly in terms of \(Z\): since \(Y_{\xi} = -\mu \xi S^2\), \((Y + (\xi Z)_{\xi})_{\xi} = 0\) hence, since \(Y\), \(Z\) and \(\xi Z_{\xi} = h_0\) all tend to zero as \(\xi \to \infty\),
\[ Y = -(\xi Z)_{\xi}. \]

4. Solutions of system \((8)\)

We now introduce a variational formulation of system \((8)\) and show that minimizing sequences converge to a solution in which \(S\) decays exponentially and \(Z\) behaves like a Newtonian potential at infinity.
4.1. Variational principle and existence of a solution

Define

\[ E[S, Z] = \frac{1}{2} \int_{0}^{\infty} \left[ S'(\xi)^2 + S(\xi)^2 + Z'(\xi)^2 / (2\mu) \right] \xi^2 \, d\xi, \tag{21} \]

where the prime denotes the derivative with respect to \( \xi \), and

\[ I[S, Z] = \frac{1}{2} \int_{0}^{\infty} S^2 Z(\xi)^2 \, d\xi. \tag{22} \]

These expressions may be written

\[ E = \frac{1}{8\pi} \int_{\mathbb{R}^3} \left[ |\nabla S|^2 + S^2 + \frac{|\nabla Z|^2}{2\mu} \right] \, d^3 x, \]

where \( \nabla \) denotes the (Euclidean) gradient in \( \mathbb{R}^3 \), and

\[ I = \frac{1}{8\pi} \int_{\mathbb{R}^3} S^2 Z \, d^3 x. \]

System (8) is the Euler–Lagrange equation of the Lagrangian \( F[S, Z] := E[S, Z] + I[S, Z] \). This expression is unbounded below \( 2 \). We therefore minimize \( E \) while keeping the value of \( I \) fixed. We may assume \( I = -1 \) without loss of generality: since \( I \) is homogeneous of degree 3, its value may be modified by scaling as long as it is nonzero. We let \((S, Z)\) vary over the space \( H^1_r (\mathbb{R}^3) \times D^1_r (\mathbb{R}^3) \), where \( H^1_r (\mathbb{R}^3) \) is the space of radial functions \( u(\xi) \) on \( \mathbb{R}^3 \) that are square-summable together with their first-order derivatives, while \( D^1_r (\mathbb{R}^3) \) is the closure of the set of compactly supported, smooth radial functions, for the norm \( \| \nabla u \|_{L^2} \); \( D^1_r (\mathbb{R}^3) \) may also be viewed as the space of radial functions in \( L^6 (\mathbb{R}^3) \) that have square-summable first-order derivatives [4]. The space \( H^1_r (\mathbb{R}^3) \times D^1_r (\mathbb{R}^3) \) is a Hilbert space with pairing

\[ \langle S_1, Z_1 | S_2, Z_2 \rangle = \int_{\mathbb{R}^3} \left[ \nabla S_1 \cdot \nabla S_2 + S_1 S_2 + \nabla Z_1 \cdot \nabla Z_2 \right] \, d^3 x \]

and norm

\[ \| S, Z \| = \sqrt{\langle S, Z | S, Z \rangle}. \]

We prove three results:

- The infimum of \( E \) constrained by \( I \) is achieved for some \((S, Z)\).
- \((S, Z)\) generate a solution of (8).
- At infinity, the metric is Schwarzschild-like and \( S \) decays exponentially.

The first two points follow from the following theorem. The third is proved in section 4.2.

**Theorem 2.** The infimum of \( E \) subject to the constraint \( I = -1 \), where \((S, Z)\) varies in \( H^1_r (\mathbb{R}^3) \times D^1_r (\mathbb{R}^3) \), is achieved; after scaling \( Z \), this provides a solution of (8). Furthermore, any solution of (8) in this function space satisfies

\[ M := \mu \int_{0}^{\infty} \xi^2 S(\xi)^2 \, d\xi \geq \frac{3}{8} \pi \sqrt{3}. \tag{23} \]

The somewhat technical proof is deferred to section 6.

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2 Choose \( S \) and \( Z \) so that \( I[S, Z] < 0 \); then \( F[\alpha S, \alpha Z] \to -\infty \) as \( \alpha \to +\infty \).

3 The infimum of \( E \) over the set of functions such that \( I = 0 \) is clearly zero.
Remark 3. The Lagrangian $F$ may be given a geometric interpretation. Consider $\mathcal{L}[N, h, \phi] \sin \theta \, d\theta \, d\phi \, d\psi = (R \sqrt{-g} - 2\mu L) \sin \theta \, d\theta \, d\psi$, where $L$ is the Lagrangian density (2) for the scalar field and $R \sqrt{-g}$ is the Hilbert Lagrangian. Next, perform the change of unknown (7): this leads to the expression

$$\mathcal{M}[S, Z, \varepsilon, \tau] = \mathcal{L}[1 + \varepsilon^2(Z + Y \cos \tau), 1 + \varepsilon^2h_0, \varepsilon^2 S \cos \tau].$$

Expand $\mathcal{M}$ with respect to $\varepsilon$: $\mathcal{M} = \mathcal{M}_0 + \varepsilon^2\mathcal{M}_2 + \varepsilon^4\mathcal{M}_4 + \cdots$. Then, take the average of each term with respect to $\tau$ over one period: $\langle \mathcal{M} \rangle = \langle \mathcal{M}_0 \rangle + \varepsilon^2\langle \mathcal{M}_2 \rangle + \varepsilon^4\langle \mathcal{M}_4 \rangle + \cdots$. After computation, one obtains $\langle \mathcal{M}_0 \rangle = 0$, $\langle \mathcal{M}_2 \rangle = (\xi Z)\xi$ and $\langle \mathcal{M}_4 \rangle = -\frac{1}{2}\varepsilon^2(2\mu)(S^2 + 2S^2(1 + Z) + Z^2) + \Phi_e$, where $\Phi$ is a function of the field variables and their derivatives. Since $\Phi_e$ and $(\xi Z)\xi$ are divergences, they do not contribute to the Euler equation. We are left with $\langle \mathcal{M}_4 \rangle$, which differs from $F$ by a multiplicative constant.

4.2. Decay estimates

We now turn to the decay properties of $S$ and $Z$. We state the result using the original variable $r$ so that the result should be easier to interpret.

**Theorem 3.** For any $\alpha < 1$, $S = O(e^{-r/m})$ and $rZ(\xi) \to -M/(\varepsilon m)$ as $r \to \infty$, where

$$M = \mu \int_0^\infty y^3S^2(y) \, dy.$$  

In addition, $S\sqrt{\mu}$ is independent of $\mu$.

**Proof.** The argument is classical [4, 18]. System (8) may be written

$$(\xi S)'' = \mu S(Z + 1),$$  

$$(\xi Z)'' = \mu\xi S^2.$$  

Therefore, $S = S_1\sqrt{\mu}$, where $S_1$ solves the same system with $\mu = 1$. The decay of higher derivatives follows from interior regularity estimates for the Laplacian; it therefore suffices to prove the decay of $S$ and $Z$.

Let $v(\xi) = \xi S$ and $w(\xi) = \frac{1}{\xi}v^2$. Since $Z \to 0$ at infinity, one may, for any $\alpha \in (0, 1)$, find $R > 0$ such that $1 + Z(\xi) \geq \alpha$ for $\xi \geq R$. Let $\beta = \sqrt{2\alpha}$ and $\psi(\xi) = (w^{\prime} + \beta w)e^{-\beta\xi}$, so that

$$e^{\beta\xi}w'' - \beta^2w = \nu v'' + \nu^2 - \alpha v^2 = [(1 + Z) - \alpha]v^2 + v^2 \geq 0,$$

so that $\psi$ is nondecreasing for $\xi \geq R$. There are now two possibilities.

(a) For every $\xi \geq R$, $\psi(\xi) \leq 0$. In that case, for $\xi \geq R$, we have $(w^{\prime}e^{\beta\xi})' = (w'' + \beta w)e^{\beta\xi} \leq 0$, hence

$$w \leq w(\xi) \leq w(R)e^{-\beta \xi}.$$  

This proves that $w$, hence $S$, decays exponentially.

(b) There is a $\xi_0 > R$ such that $\psi(\xi_0) > 0$. Since $\psi$ is nondecreasing, $w'(\xi) + \beta w(\xi) \geq \psi(\xi_0)e^{\beta\xi}$ for $\xi \geq \xi_0$, and $w' + \beta w$ is not integrable near infinity. However, $\int_0^\infty w' \, d\xi = \int_0^{\xi_0} \frac{1}{\xi^2}S^2 \, d\xi = 8\pi\varepsilon^2 \|S\|^2_{L^2} < \infty$, and $|w'| \leq \frac{1}{\xi}S(\xi S' + S) \leq \frac{1}{\xi}S^2(2S' + S^2) + \xi S^2$ is also integrable near infinity; indeed, $\int_0^\infty \frac{1}{\xi^2}S^2 \, d\xi$ equals $(4\pi)^{-1}\|\nabla S\|^2_{L^2}$, and is therefore finite. This contradiction proves that case (b) cannot occur.

This completes the proof of the exponential decay of $S$.  

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Equation (26b) for $Z$ may now be integrated: since $\xi Z$ vanishes for $\xi = 0$,

$$Z = z_0 + \frac{\mu}{\xi} \int_0^\xi (\xi - y) y S^2(y) \, dy,$$

where $z_0$ is constant. Since $S$ decays exponentially, $Z$ has a limit at infinity. This limit must be zero, since $Z \in L_6$. Therefore,

$$z_0 = -\mu \int_0^\infty y S^2(y) \, dy,$$

and

$$Z(\xi) = -\frac{M}{\xi} + \int_\xi^\infty \left(\frac{y}{\xi} - 1\right) \mu y S^2(y) \, dy.$$

Equation (15) gives the behavior of $h_0$. The behavior of $N$ and $h$ follows. This completes the proof. □

Remark 4. Since Poisson’s equation admits singular solutions, corresponding to point masses, it is natural to ask whether system (8) admits solutions in which $Z$ behaves like $1/\xi$ at the origin. A positive answer may be obtained by the method of reduction: following the strategy described in [11], one can prove that there is a four-parameter family of solutions, defined for small $\xi$, such that

$$S(\xi) = \frac{S_0}{\xi} + S_1 + S_{11} \ln \xi + O(\xi \ln \xi),$$

$$Z(\xi) = \frac{Z_0}{\xi} + Z_1 + Z_{11} \ln \xi + O(\xi \ln \xi),$$

where $S_0, S_1, Z_0, Z_1$ are arbitrary constants, $S_{11} = S_0Z_0$, and $Z_{11} = \mu S_0^2$. The solutions considered so far all have $S_0 = Z_0 = 0$. A consequence of this computation is that it is possible to take $S_0 = 0$ and $Z_0 \neq 0$: a singularity in $Z$ does not necessarily imply a singularity in the scalar field. There is also a reduction with $S$ and $Z$ behaving like $1/\xi^2$, but it involves fewer constants. Note also that system (8) admits a one-parameter family of scaling transformations:

$$(S, Z, \theta) \mapsto (\theta^2 S(\theta \xi), \theta^2 Z(\theta \xi) + \theta^2 - 1).$$

5. Model validation

The information on the asymptotic behavior of $S$ and $Z$ now enables us to relate the parameters $\varepsilon$ and $m$ of the model to observational data. Since we need two parameters, we need two data. For instance, consider the redshift of light originating at two points $B$ and $B'$, of known location $(r = r_B$ and $r_{B'})$ and observed at a point $A$ relatively at rest with respect to $B$ and $B'$ (see [8] for the precise meaning of relative velocity in general relativity for distant objects). We obtain

$$\frac{v_B - v_A}{v_A} = \frac{\langle g_{00}\rangle_B}{\langle g_{00}\rangle_A} - 1 \approx Z(\varepsilon m r_B) + S(\varepsilon m r_B) \cos(mt \sqrt{1 - \varepsilon^2}),$$

where subscripts $A$ and $B$ indicate the points where the frequency ($v$) and metric components ($g_{00}$) are determined. If $r_B$ is large, $S(\xi_B)$ is small relative to $Z(\xi_B) \approx 1 - \frac{\varepsilon m}{m r_B}$, because of the
fast decay of $S$. The redshift for this source should therefore be nearly constant in time. From it, we may estimate $\frac{\epsilon}{mr_B}$, hence the parameter $f = \frac{\epsilon}{m}$.

Next, consider a source $B'$ closer to the center of the putative soliton star. If the present model is correct, one should now observe an oscillatory redshift, with period $T = \frac{2\pi}{m \sqrt{1 - \epsilon^2}}$. Eliminating $m$ gives

$$\frac{2\pi f}{T} = \epsilon \sqrt{1 - \epsilon^2};$$

(27)

since $\epsilon \in (0, 1)$, this expression can never exceed $\frac{1}{2}$. Thus, from the observation of $f$ and $T$, we may check whether $\frac{2\pi f}{T}$ is less than $\frac{1}{2}$ and, if that is the case, compute $\epsilon$ from (27), and deduce the value of $m$ from the relation $m = \frac{\epsilon}{f}$.

Since (27) has in general two roots, the smaller one seems preferable, in view of the assumption of small amplitude.

We may also estimate parameters from the numerical data in [16, 17], where the total mass, defined as

$$\lim_{r \to \infty} r^2 \left(1 - h - 2\right),$$

is equal to 0.52, and the period is $2\pi/\omega$ with $\omega = 0.0196$. In the present notation, this means:

$$\frac{\epsilon M}{m} = 0.52, \quad \frac{\epsilon}{\sqrt{1 - \epsilon^2}} = 0.0196.$$  

Therefore, $\epsilon \lesssim 0.52 \times 0.0196/M$, hence, using estimate (23),

$$\epsilon \lesssim 5 \times 10^{-3}.$$  

6. Proof of theorem 2

6.1. Step 1: convergence of minimizing sequences

Consider a minimizing sequence $(S_n, Z_n)$:

$$E[S_n, Z_n]$$

decreases and tends to $E_0 := \inf_{I(S,Z) = -1} E[S, Z]$ as $n$ tends to infinity. Since $E[S_n, Z_n]$ is in particular bounded, one can prove\(^4\) that for any $p \in (2, 6)$, any bounded sequence in $H^1_0(\mathbb{R}^3)$ admits a subsequence that converges weakly in $H^1_0(\mathbb{R}^3)$, strongly in $L^p(\mathbb{R}^3)$, and pointwise almost everywhere. In other words, there is a sequence, still called $(S_n, Z_n)$ for convenience, and a pair $(S, Z) \in H^1_0(\mathbb{R}^3) \times D^1_0(\mathbb{R}^3)$ such that the following properties hold simultaneously:

$$I[S_n, Z_n] = -1$$

for every $n$ and

$$\lim_{n \to \infty} E[S_n, Z_n] = E_0,$$

(28)

$$\left(\forall (\sigma, \zeta) \in H^1_0 \times D^1_0 \right) \lim_{n \to \infty} \langle S_n, Z_n | \sigma, \zeta \rangle = 0,$$

(29)

$$\lim_{n \to \infty} \| S_n - S \|_{L^p} = 0,$$

(30)

$$S_n \to S$$

almost everywhere,

(31)

$$\left(\forall \xi \in L^{6/5} \right) \lim_{n \to \infty} \int Z_n \xi \, d^3 x = 0.$$  

(32)

\(^4\) For the results on function spaces used here see [18, 4]. In addition, functions $S$ in $H^1_0(\mathbb{R}^3)$ satisfy an estimate of the form

$$|S(\xi)| \leq c_2 \xi^{-1}(\|\nabla S\|_{L^2} + \|S\|_{L^2}).$$

Functions $Z$ in $D^1_0(\mathbb{R}^3)$ satisfy an estimate of the form

$$|Z(\xi)| \leq c_3 \xi^{-1/2} \|\nabla Z\|_{L^2}.$$  

In particular, if $u$ belongs to $D^1_0$, it satisfies $\xi u(\xi) \to 0$ as $\xi \to 0$. For background results on weak convergence, see [5].
6.2. Step 2: \((S, Z)\) is a minimizer

We need to prove that \(E[S, Z] = E_0\) and \(I[S, Z] = -1\). Since the norm in any Hilbert space is weakly lower semi-continuous,

\[
E[S, Z] \leq E_0 = \liminf_{n \to \infty} E[S_n, Z_n],
\]

and since \(E[S, Z]\) cannot be lower than its infimum \(E_0\), we conclude

\[
E[S, Z] = E_0.
\]

Next, let us choose \(p = 12/5\). Since \(\|S_n - S\|_{L^{12/5}} \to 0\), \(S_n\) and \(S_n\) are both bounded in \(L^{12/5}\), and therefore, using Hölder’s inequality, which applies since \(5/12 + 5/12 + 1/6 = 1\),

\[
\left| \int \frac{1}{2} (S_n^2 - S^2) Z_n \, d^3 x \right| \leq \|S_n + S\|_{L^{12/5}} \|S_n - S\|_{L^{12/5}} \|Z_n\|_{L^6} \to 0
\]

as \(n\) tends to infinity. On the other hand, since \(S \in L^{12/5}, S^2 \in L^{6/5}\). Using (32), we obtain

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} S^2 (Z_n - Z) \, d^3 x = 0.
\]

Writing \(S_n^2 Z_n - S^2 Z = (S_n^2 - S^2) Z_n + S^2 (Z_n - Z)\), we obtain

\[
\lim_{n \to \infty} I[S_n, Z_n] = I[S, Z],
\]

hence \(I[S, Z] = -1\).

6.3. Step 3: \(E\) and \(I\) are of class \(C^1\) on \(H^1_0(\mathbb{R}^3) \times D^1_0(\mathbb{R}^3)\)

This is true for \(E\) because it is the square of the norm in a Hilbert space. As for \(I\), its Gâteaux derivative is the map \(dI\) defined by

\[
dI[S, Z] \cdot (\sigma, \zeta) = \int \left( SZ \sigma + \frac{1}{2} S^2 \zeta \right) d^3 x.
\]

A form of the Sobolev embedding theorem [18, 4] shows that there is a constant \(c_3\) such that

\[
\|S\|_{L^{12/5}} + \|Z\|_{L^6} \leq c_3 \|S, Z\|.
\]

Since \(5/12 + 1/6 = 7/12\) and \(S^2\) is bounded in \(L^{6/5}\), Hölder’s inequality yields

\[
\|SZ\|_{L^{12/7}} + \|S^2\|_{L^{6/5}} \leq c_3 \|S, Z\|^2.
\]

Since \(\int (SZ \sigma + \frac{1}{2} S^2 \zeta) \, d^3 x \leq \|SZ\|_{L^{12/7}} \|\sigma\|_{L^{12/5}} + \|S^2\|_{L^{6/5}} \|\zeta\|_{L^6}\),

\[
|dI[S, Z] \cdot (\sigma, \zeta)| \leq c_6 \|S, Z\|^2 \|\sigma, \zeta\|.
\]

This proves that \(dI\) is a continuous linear form on \(H^1_0 \times D^1_0\).

Regarding the continuity of \(dI\) with respect to \((S, Z)\),

\[
|dI[S_1, Z_1] - dI[S_2, Z_2]| \cdot (\sigma, \zeta) |
\]

\[
= \int \left( (S_1 - S_2)Z_1 + S_2(Z_1 - Z_2)\sigma + \frac{1}{2}(S_1 - S_2)(S_1 + S_2)\zeta \right) d^3 x
\]

\[
\leq \left[ \|S_1 - S_2\|_{L^{12/5}} \|Z_1\|_{L^6} + \|S_2\|_{L^{12/5}} \|Z_1 - Z_2\|_{L^6} \right] \|\sigma\|_{L^{12/5}}
\]

\[
+ \frac{1}{2} \left[ \|S_1 - S_2\|_{L^{12/5}} \|S_1 + S_2\|_{L^{12/5}} \|\zeta\|_{L^6} \right]
\]

\[
\leq c_7 \|\sigma, \zeta\| (\|S_1, Z_1\| + \|S_2, Z_2\|) \|S_1 - S_2, Z_1 - Z_2\|.
\]

The continuity of \(dI\) follows.
6.4. Step 4: \((S, Z)\) solves \((8)\)

We first prove that there is a Lagrange multiplier \(\lambda\) such that \(dE = \lambda \, dI\) or, in other words, that
\[
-\Delta S + S = \lambda S Z, \quad -\Delta Z = \lambda \mu S^2.
\] (33)

Because of the constraint, \(S\) is not identically zero; therefore, we may find a positive function \(Z_0\) with compact support such that
\[
\int S^2 Z_0 \, d^3x = 1.
\]

If \(\sigma\) and \(\zeta\) are arbitrary variations of \(S\) and \(Z\), so that \(I[S + \sigma, Z + \zeta]\) may not be equal to \(-1\), one may, if \(\sigma\) is small enough in \(D_1\), define a constant \(\theta\) by
\[
I[S + \sigma, Z + \zeta - \theta Z_0] = -1.
\]

The result is
\[
\theta = \theta(S, Z, \sigma, \zeta) = \int (\nabla^2 S \cdot \nabla \sigma + S (1 - 2\lambda Z) \sigma + \nabla Z \cdot \nabla \zeta - \frac{\lambda}{2} Z S^2 \zeta) \, d^3x = 0,
\]
where \(\lambda = \int \nabla Z \cdot \nabla Z_0 \, d^3x\). After integration by parts, this leads to (33).

If \(\lambda\) were equal to zero, \(Z\) would be a singularity-free solution of Laplace’s equation which tends to zero at infinity. By Liouville’s theorem, \(Z\) would then be identically zero, hence \(I[S, Z] = 0\), violating the constraint. Therefore, \(\lambda \neq 0\). This allows us to consider \((S/\lambda, -Z/\lambda)\), which solves \((8)\). This completes the proof.

**Remark 5.** Multiplying \((8a)\) by \(S\) and integrating by parts (the boundary term which arises in this manner vanishes because \(S\) and its derivatives decay exponentially), we obtain
\[
\int \nabla S \cdot \nabla S + S^2 \, d^3x = -\int S^2 Z \, d^3x = \frac{1}{\mu} \int |\nabla Z|^2 \, d^3x.
\]

Using Hölder’s inequality and interpolation, we obtain
\[
\int S^2 Z \, d^3x \leq \|Z\|_{L^1} \|S\|_{L^3}^2 = \|Z\|_{L^1} \|S\|_{L^6}^2 \leq \|Z\|_{L^1} \|S\|_{L^3}^{3/2} \|S\|_{L^6}^{1/2}.
\]

6.5. Step 5: estimate (23) holds

Since \(M = 4\pi \mu \|S\|_{L^6}^2\), it suffices to estimate the \(L^2\) norm of \(S\). Multiplying \((8a)-(8b)\) by \(S\) and \(Z\) respectively, and integrating, we obtain
\[
\int (|\nabla S|^2 + S^2) \, d^3x = -\int S^2 Z \, d^3x = \frac{1}{\mu} \int |\nabla Z|^2 \, d^3x.
\] (34)

Using Hölder’s inequality and interpolation, we obtain
\[
\left| \int S^2 Z \, d^3x \right| \leq \|Z\|_{L^1} \|S\|_{L^6} \|S\|_{L^3}^2 \leq \|Z\|_{L^1} \|S\|_{L^3}^{3/2} \|S\|_{L^6}^{1/2},
\]

\footnote{This smallness condition guarantees that \(\int (S + \sigma)^2 Z_0 \, d^3x \neq 0\).}
since $\frac{5}{12} = \frac{3}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{6}$. Let $K$ denote the best constant in Sobolev’s inequality:

$$\|u\|_{L^6} \leq K \|\nabla u\|_{L^3}.$$ 

It is known\(^6\) that

$$K = \frac{1}{\sqrt{3}} \left( \frac{4}{\pi^2} \right)^{1/3}.$$

Equation (34) implies

$$K^{-2} \|Z\|_{L^6}^2 \leq \int |\nabla Z|^2 \, d^3x \leq \mu \|Z\|_{L^6} \|S\|_{L^3}^{3/2} \|S\|_{L^6}^{1/2},$$

hence $\|Z\|_{L^6} \leq \mu K^{-2} \|S\|_{L^3}^{3/2} \|S\|_{L^6}^{1/2} $, and

$$K^{-2} \|S\|_{L^6}^2 + \|S\|_{L^6}^2 \leq \int (|\nabla S|^2 + S^2) \, d^3x \leq \mu K^{-2} \|S\|_{L^3}^3 \|S\|_{L^6}.$$

Letting $X = \|S\|_{L^6}$, we conclude that

$$K^{-2} X^2 - \mu K^{-2} \|S\|_{L^6}^3 X + \|S\|_{L^6}^2 \leq 0.$$ 

The discriminant of this quadratic expression in $X$ must therefore be nonnegative. This yields the relation $\mu^2 K^4 \|S\|_{L^6}^6 - 4 K^{-2} \|S\|_{L^6}^8 \geq 0$, or $\mu \|S\|_{L^6}^2 \geq 2 K^{-3}$. It follows that

$$M = \mu \|S\|_{L^6}^2 (4\pi)^{-1} \geq \frac{2}{3} \frac{\pi^2}{4} \frac{3}{2} \frac{3}{2} (4\pi)^{-1} = \frac{3}{8} \frac{\pi}{\sqrt{3}},$$

QED.

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\(^6\) See [2, p 39 sqq.] and [3, 13, 19].