Divisor Labeling of Some Special Graphs

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Abstract.
An assignment of labels (mostly, integers) to the vertices of a graph \( G(V, E) \), or simply \( G \), on \( n \) vertices subject to certain constraints is called a vertex labeling of \( G \). A graph \( G \) is a divisor graph if the vertices of \( G \) can be labeled with integers (necessarily distinct) in such a way that either \( x \) divides \( y \) or \( y \) divides \( x \) for any two adjacent vertices \( x \) and \( y \). In this paper, we encounter the divisor labeling of certain classes of graphs.

Keywords: Graph Labeling, Divisor Labeling, Divisor Graph

1. Introduction
We consider only non-trivial, simple, finite, undirected, and connected graphs throughout this article. Let \( Z \) and \( N \) be the set of integers and natural numbers, respectively. An assignment of integers to the vertices of \( G(V(G), E(G)) \), or simply \( G \), subject to some constraints is called a vertex labeling of \( G \). There are hundreds of variants of graph labeling techniques which are introduced for wide range of applications in real life. For a detailed survey on graph labeling we refer to [1]. Santhosh et al. [6] called \( G \) a divisor graph if \( V(G) = Z \) & \( xy \in E(G) \) iff \( x \) divides \( y \) or vice versa. In other words, \( G \) is called a "divisor graph" if there is a finite set of positive integers \( \{c_1, c_2, \ldots, c_n\} \) such that \( V(G) = \{c_1, c_2, \ldots, c_n\} \) and \( c_i c_j \in E(G) \) if \( c_i/c_j \) or \( c_j/c_i \). It is shown that "no divisor graph contains an induced odd cycle of length greater than 3" and "every induced subgraph of a divisor graph is a divisor graph". It is also proved that complete, bipartite, complete multipartite, and joins of divisor graphs are divisor graphs. Seoud et al. [7] have discussed some "necessary and sufficient conditions" for a graph to be a divisor graph. For a detailed study on divisor graphs one can see [1]. In this paper we study a different version of divisor labeling.

Specifically, we say that a graph \( G \) is a divisor graph if there exists a one-to-one labeling of its vertices \( d : V(G) \rightarrow N \) such that for any two adjacent vertices \( x \) and \( y \), either \( d(x) \) divides \( d(y) \) or vice versa. We also define \( d(xy) = \frac{d(x)}{d(y)} \) or \( \frac{d(y)}{d(x)} \). We call \( d \) a divisor labeling of \( G \), so \( G \) is a divisor graph if and only if there exists a divisor labeling of \( G \). Note that in a divisor labeling,
the labels on the vertices of $G$ must be distinct, but the labels on the edges need not be. Also note that by our definition, the label $d(k)$ of a vertex $k \in G$ may still divide the label $d(l)$ of a vertex $l \in G$ if $kl$ is not an edge of $G$. In this paper, we establish the divisor labeling of some special graphs such as lotus inside a circle, butterfly graph, Mobius ladder etc.

**Note:** A divisor labeling of a graph $G$ is not unique.

2. Main Results

In this section we derive the divisor labeling of certain classes of graphs.

2.1. Divisor Labeling of Certain Graphs

This section is devoted to obtain the divisor labeling of some graphs such as lotus inside a circle, line graph of sunlet graph, and the generalized butterfly graph. First we recall the definitions of these graphs for the sake of completeness.

**Definition 1.** [9] A graph $G$ is said to be a bipartite graph if it does not contain any cycle of odd length.

**Lemma 1.** [1] Every bipartite graph is a divisor graph.

**Definition 2.** [9] A vertex of degree one is known as a pendant vertex. An edge of a graph is said to be pendant if one of its end vertices is a pendant vertex.

**Definition 3.** [9] A cycle $C_n$ is a closed path whose initial and terminal vertices are the same.

**Definition 4.** [5] A “lotus inside a circle, denoted $LC_n$, is a graph obtained from the cycle $C_n : u_1, u_2, \ldots, u_n, u_1$ and a star $K_{1,n}$ (with the central vertex $v_0$ and the pendant vertices $v_1, v_2, \ldots, v_n$) by joining each $v_i$ to $u_i$ and $u_{i+1} \ (mod \ n)$”.

**Theorem 1.** The lotus inside a circle graph $LC_n$ is a divisor graph for $n \geq 3$.

![Figure 1. A divisor labeling of $LC_6$.](image)

**Proof.** Let $LC_n, n \geq 3$ be the given lotus inside a circle graph. Let $v_0$ be the central vertex. Label the vertices of $LC_n$ as $v_1, v_2, \ldots, v_n$. Clearly $|LC_n| = 2n + 1$. Define an injective function $d : V(LC_n) \rightarrow N$ as follows: without loss of generality, let $d(v_i) = 2^i$ for $1 \leq i \leq 2n$ and $d(v_0) = 2^{2n+1}$. A simple check clearly shows that $d$ is the required divisor labeling of $LC_n$. 

\[\square\]
Definition 5. [3] The “Mobius ladder, denoted by $M_n$, is a graph with even number $n$ of vertices, formed from $C_n$ by adding edges connecting opposite pairs of vertices in $C_n$.”

Lemma 2. [3] The Mobius ladder $M_n$ is bipartite when $n \equiv 2(\text{mod } 4)$ and not bipartite when $n \equiv 0(\text{mod } 4)$.

Theorem 2. The Mobius ladder $M_n$ admits a divisor labeling for $n \geq 4$.

Proof. Let $M_n$ be the given Mobius ladder on $n$ vertices (where $n$ is an even number). We label the vertice of $M_n$ as $v_1, v_2, ..., v_n, v_{n+1}, ..., v_{2n}$ as given in Figure 2. Then there are two cases.

Case 1: When $n \equiv 2(\text{mod } 4)$

The proof is direct from Lemma 3 and Lemma 1.

Case 2: When $n \equiv 0(\text{mod } 4)$

Define an one-to-one labeling $d : V(M_n) \to NS$ as follows: without loss of generality, let $d(v_i) = 2^i$ for $1 \leq i \leq 2n$. Then it is clear to see that $d$ is the desired divisor labeling of $M_n$. □
Definition 6. A “shell graph, denoted by $C(n, n-3)$, is a graph defined as a cycle $C_n$ with $(n-3)$ chords sharing a common end point called the apex”.

Theorem 3. The shell graph $C(n, n-3)$ admits a divisor labeling for $n \geq 3$.

Proof. Let $C(n, n-3)$ be the given shell graph on $n \geq 3$ vertices. Define an injective labeling $d : V(C(n,n-3)) \to \mathbb{N}$ as follows: without loss of generality, let $d(v_0) = 1$. Then $d(v_i) = 2d(v_{i-1})$ for $1 \leq i \leq n-1$. Clearly $d$ is the required divisor labeling of $C(n, n-3)$.

Definition 7. [8] The “sunlet graph, denoted $S_n$, is a graph on $2n$ vertices obtained by attaching $n$-pendant edges to the cycle $C_n$”.

Definition 8. [8] The “line graph of a graph $G$, denoted by $L(G)$, is a graph whose vertices are the edges of $G$, and if $uv \in E(G)$ then $uv \in E(L(G))$ if $u$ and $v$ share a vertex in $G$”.

Theorem 4. The line graph of a sunlet graph $L(S_n)$ is a divisor graph for $n \geq 3$.

Proof. Let $S_n, n \geq 3$ be the given sunlet graph on $2n$ vertices. Obtain the line graph of a sunlet graph $L(S_n)$ whose vertex set $V(L(S_n))$ is defined as $\{v_1, v_2, ..., v_{2n}\}$. Now define an injective labeling $d : V(L(S_n)) \to \mathbb{N}$ as follows: without loss of generality, let $d(v_i) = 2^i$ for $1 \leq i \leq 2n$. One can see that $d$ is the required divisor labeling of $L(S_n)$ as $f(v_1)$ and $f(v_2)$ divide $f(v_{2n})$.

Figure 4. A divisor labeling of $L(S_8)$

Definition 9. [2] The “generalized butterfly graph, denoted $BF_n$, is a graph obtained by inserting vertices to every wing with the assumption that sum of inserting vertices to every wing are same”.

Theorem 5. The generalized butterfly graph $BF_n$ is a divisor graph for $n \geq 3$. 
Define an injective labeling $d$.

**Case 2:**

The proof is direct from Lemma 1.

Let $BF_n$, $n \geq 3$ be the given generalized butterfly graph. One can note that $|V(BF_n)| = 2n + 1$ and $|E(BF_n)| = 4n - 2$. Let $v_0$ be the apex (central vertex). Let $V(BF_n)$ be

$\{v_i : i = 1, 2, ..., 2n\}$

and edge set $E(BF_n)$ be

$\{(v_i, v_{i+1}) : i = 1, 2, ..., n - 1, n + 1, ..., 2n - 1\} \cup \{(v_0, v_i) : i = 1, 2, ..., 2n\}$.

We name the vertices on right wing as $\{v_1, v_2, v_3, ..., v_{n-1}, v_n\}$ and the vertices on left wing as $\{v_{n+1}, v_{n+2}, v_{n+3}, ..., v_{2n-1}, v_{2n}\}$. Define an one-to-one function $d : V(BF_n) \rightarrow N$ as follows: without loss of generality, let $d(v_0) = 2$. Then $d(v_i) = 2d(v_{i-1})$ for $1 \leq i \leq 2n$. One can easily see that $d$ gives the divisor labeling of $BF_n$. The proof is complete.

**Definition 10.** [4] A graph $B_{n,m}$ (where $n$ and $m$ are any arbitrary positive integers) is said to be a butterfly graph if "two cycles of same order sharing a common vertex with an arbitrary number of pendant edges attached at the common vertex".

**Theorem 6.** A butterfly graph $B_{n,m}$ permits a divisor labeling for any $n, m \in N$.

Let $B_{n,m}$ be the butterfly graph with $n \geq 3$ and $m \in N$. Define the vertex set of $B_{n,m}$ as $V(B_{n,m}) = \{v_1, v_2, ..., v_n\} \cup \{u_1, u_2, ..., u_m\} \cup \{w_1, w_2, ..., w_m\}$. Clearly $|V(B_{n,m})| = 2n + m$ as $u_1 = v_1$. We consider the following two cases.

**Case 1:** $C_n$, when $n$ is even

The proof is direct from Lemma 1.

**Case 2:** $C_m$, when $n$ is odd

Define an injective labeling $d : V(B_{n,m}) \rightarrow N$ as follows: without loss of generality, let $d(v_i) = 5^i$ for $1 \leq i \leq n$. Next let $d(u_2) = 5d(v_n)$ and $d(u_i) = 5d(u_{i-1})$ for $3 \leq i \leq n$. Finally let $d(w_1) = 5d(u_n)$ and $d(w_i) = 5d(w_{i-1})$ for $2 \leq i \leq m$. A simple check shows that the vertex labels are distinct and $d$ induces a divisor labeling of $B_{n,m}$.

**Definition 11.** [5] The join of two graphs $H_1 \cup H_2$, $H_1 + H_2$, is a graph formed with vertex set $V(H_1 + H_2) = V(H_1) \cup V(H_2) \cup E(H_1 + H_2) = E(H_1) \cup E(H_2) \cup \{uv : u \in V(H_1), v \in V(H_2)\}$.

**Definition 12.** [9] A graph in which any two vertices are adjacent is called a complete graph and is denoted by $K_n$. 

![Figure 5. A divisor labeling of $BF_{11}$](image)
Theorem 7. The graph $K_2 + mK_1$ admits a divisor labeling for all $m \in \mathbb{N}$.

Proof. Let $K_1$ and $K_2$ be the given complete graphs on 1 and 2 vertices, respectively. We label the vertices of $K_2$ as $v_1$ and $v_2$. We also label the vertices of $mK_1$ as $u_1, u_2, ..., u_m$. Obtain $K_2 + mK_1$ by using Definition 11. Now an one-one function $d : V(K_2 + mK_1) \to \mathbb{N}$ as follows: without loss of generality, let $d(v_1) = 1$ and $d(v_2) = 2$. Then $d(u_i) = 4m$ for $1 \leq i \leq m$. One can see that $d$ produces the required divisor labeling of $K_2 + mK_1$.

CONCLUSION

The divisor labeling of certain graphs have been investigated. Obtaining the divisor labeling of other families of graphs is still open and this is for future work. We also believe that the concept of divisor labeling may find applications in graph-based cryptography and network security.

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