Integrability of solutions to mixed stochastic differential equations

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Abstract. We prove that the standard conditions that provide unique solvability of a mixed stochastic differential equations also guarantee that its solution possesses finite moments. We also present conditions supplying existence of exponential moments. For a special equation whose coefficients do not satisfy the linear growth condition, we find conditions for integrability of its solution.

Keywords. Mixed stochastic differential equation, moment of solution, exponential moment of solution

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Introduction

The main object of this article is a stochastic differential equation of the form

\[ X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s + \int_0^t c(s, X_s)dZ_s, \quad t \in [0, T]. \]

The randomness in this equation comes from two processes: a standard Wiener process \( W \) and a process \( Z \) whose paths are Hölder continuous of order greater than \( 1/2 \). In place of the process \( Z \), usually a fractional Brownian motion \( B^H \) with the Hurst parameter \( H > 1/2 \) is taken. Due to such twofold nature of the randomness, equation (1) is called a mixed stochastic differential equation.

Existence and uniqueness of solution to a mixed stochastic differential equation (1) were proved under different conditions in [2, 4, 5, 6, 7]. More generally, mixed equations with jumps were considered in [8] and mixed delay equations, in [9].

The principal aim of this article is to prove existence of moments of a solution to (1). In [7, 8, 9], the existence of moments was proved under an additional assumption of boundedness of the coefficient \( b \). In [10], the exponential integrability of solutions was established under the assumption that all coefficients of (1) are bounded and certain other assumptions. In this paper we will generalize those results. Namely, we will show the existence of moments without any assumptions except those providing the unique solvability and certain exponential integrability of the driver \( Z \). Under additional assumption that the coefficients are bounded we show the exponential integrability of the solution to (1). We also consider an equation with coefficients not satisfying the linear growth condition and prove that all moments of its solution are finite.

The paper is organized as follows. In Section 1, we introduce the main object and provide necessary information on the pathwise (Young) integral. In Section 2 we show the usual and exponential integrability of the solution to (1). In Section 3 we prove existence of moment for a more general equation, whose coefficients do not satisfy the linear growth conditions.
1 Preliminaries

Let \((\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})\) be a complete filtered probability space. We will use the following notation. The symbol \(|\cdot|\) will denote the absolute value of a real number, the Euclidean norm of a vector, and the (Euclidean) operator norm of a matrix. We will use the symbol \(C\) to denote any constant whose value is not important and may vary from one line to another; should this constant depend on certain parameters, we will put them into subscripts. If the value of a constant is important, we will use the symbol \(K\) for it.

Now we proceed to a precise definition of the main object. It is the following stochastic differential equation in \(\mathbb{R}^d\):

\[
X_t = X_0 + \int_0^t a(s, X_s)ds + \sum_{i=1}^m \int_0^t b_i(s, X_s)dW^i_s + \sum_{j=1}^l \int_0^t c_j(s, X_s)dZ^j_s, \quad t \in [0, T],
\]

where the coefficients \(a: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, b_i: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, i = 1, \ldots, m, c_j: [0, T] \times \mathbb{R}^d \to \mathbb{R}^l, j = 1, \ldots, l\), are jointly continuous; \(W = \{W_t = (W^1_t, \ldots, W^m_t), t \in [0, T]\}\) is a standard Wiener process in \(\mathbb{R}^m\), \(Z = \{Z_t (Z^1_t, \ldots, Z^l_t), t \in [0, T]\}\) is an \(\mathbb{F}\)-adapted process in \(\mathbb{R}^l\), whose paths are Hölder continuous of order \(\mu > 1/2\); the initial condition \(X_0\) is non-random. In what follows we will use the short form (1) to write equation (2) and the integrals involved.

In (1), the integral w.r.t. the Wiener process \(W\) is understood as the Itô integral, while that w.r.t. the process \(Z\), as the pathwise Young integral. We will give only basics on it; further information may be found e.g. in [1].

Let functions \(g, h : [a, b] \to \mathbb{R}\) be \(\alpha\)- and \(\beta\)-Hölder continuous correspondingly, with \(\alpha + \beta > 1\). Then the integral \(\int_a^b g(x)dh(x)\) is well defined as a limit of integral sums. Moreover, one has an estimate (the Young–Love inequality)

\[
\left| \int_a^b g(s)dh(s) \right| \leq C_{\alpha, \beta} \|h\|_{a, b, \beta} \left( \|g\|_{a, b, \infty} + \|g\|_{a, b, a} (b - a)^\alpha \right) (b - a)^\beta,
\]

where \(\|f\|_{a, b, \infty} = \sup_{x \in [a, b]} |f(x)|\) and \(\|f\|_{a, b, \gamma} = \sup_{a \leq s < t \leq b} \frac{|f(t) - f(s)|}{|t - s|^{\gamma}}\) are the supremum norm and a \(\gamma\)-Hölder seminorm on \([a, b]\), respectively.

The following assumptions guarantee that equation (1) has a unique solution, see [9]:

A1. For all \(t \in [0, T], x \in \mathbb{R}^d\),

\[
|a(t, x)| + |b(t, x)| + |c(t, x)| \leq C(1 + |x|).
\]

A2. The function \(c\) is differentiable in the second variable, moreover, the derivative is bounded: for all \(t \in [0, T], x \in \mathbb{R}^d\),

\[
|c'(t, x)| \leq C.
\]

A3. For all \(R > 0, t \in [0, T]\) and \(x_1, x_2 \in \mathbb{R}^d\) such that \(|x_1| \leq R, |x_2| \leq R\),

\[
|a(t, x_1) - a(t, x_2)| + |b(t, x_1) - b(t, x_2)| + |c'(t, x_1) - c'(t, x_2)| \leq C_R |x_1 - x_2|.
\]

A4. For some \(\beta \in (1 - \mu, 1/2)\) and any \(s, t \in [0, T], x \in \mathbb{R}^d\)

\[
|c(t, x) - c(s, x)| \leq C |t - s|^\beta (1 + |x|), \quad |c'(s, x) - c'(t, x)| \leq C |s - t|^\beta.
\]

Such formulation of the condition A4 is needed in order to be able to consider linear equations.
2 Integrability of solution

In this section we prove integrability of the solution to equation (1). We use techniques similar to those used in [3, 10].

Theorem 1. Assume that A1–A4 hold and

$$ E \left[ \exp \left\{ c \| Z \|_{0,t,\mu}^{1/\mu} \right\} \right] < \infty. $$

Then for any $p > 0$ the solution $X$ to equation (1) satisfies

$$ E \left[ \| X \|_{p,0,T,\infty}^{p} \right] < \infty. $$

Proof. For $N \geq 1$, $R \geq 1$, denote $\tau_{N,R} = \min \{ t \geq 0 : \| X \|_{0,t,\infty} \geq R \text{ or } \| Z \|_{0,t,\mu} \geq N \}$, $X_{t}^{N,R} = X_{t \wedge \tau_{N,R}}$, $1_{t} = 1_{\{ t \leq \tau_{N,R} \}}$. Put also $I_{t}^{1} = \int_{0}^{t} a(s, X_{s}^{N,R}) 1_{s} d W_{s}$, $I_{t}^{b} = \int_{0}^{t} b(s, X_{s}^{N,R}) 1_{s} d W_{s}$, $I_{t}^{c} = \int_{0}^{t} c(s, X_{s}^{N,R}) 1_{s} d W_{s}$. Fix arbitrary $\theta \in (1 - \mu, \beta]$.

Let $0 \leq s \leq u \leq v \leq t \leq T$. Write

$$ |X_{u}^{N,R} - X_{v}^{N,R}| \leq |I_{u}^{a} - I_{u}^{b}| + |I_{v}^{b} - I_{v}^{c}| + |I_{v}^{c} - I_{v}^{c}|. $$

Estimate first

$$ |I_{u}^{a} - I_{u}^{b}| \leq \int_{u}^{v} \left| a(z, X_{z}^{N,R}) \right| dz \leq C \int_{u}^{v} (1 + \| X_{z}^{N,R} \|) dz \leq C \left( 1 + \| X_{s}^{N,R} \|_{s,t,\infty} \right) (v - u). $$

Further, using (3), we have

$$ |I_{v}^{c} - I_{v}^{c}| \leq C N \left( \left\| c(\cdot, X_{s}^{N,R}) \right\|_{u,v,\infty} + \left\| c(\cdot, X_{s}^{N,R}) \right\|_{u,v,\theta} (v - u) \theta \right) (v - u)^{\mu}. $$

It follows from assumption A1 that

$$ \left\| c(\cdot, X_{s}^{N,R}) \right\|_{u,v,\infty} \leq C \left( 1 + \| X_{s}^{N,R} \|_{u,v,\infty} \right) \leq C \left( 1 + \| X_{s}^{N,R} \|_{s,t,\infty} \right). $$

Since

$$ \left| c(x, X_{x}^{N,R}) - c(y, X_{y}^{N,R}) \right| \leq \left| c(x, X_{x}^{N,R}) - c(y, X_{y}^{N,R}) \right| + \left| c(y, X_{y}^{N,R}) - c(y, X_{y}^{N,R}) \right| \leq C \left( |x - y|^{\beta} (1 + \| X_{x}^{N,R} \|_{s,t,\infty}) + \| X_{x}^{N,R} - X_{y}^{N,R} \|_{s,t,\infty} \right), $$

then

$$ \left\| c(\cdot, X_{s}^{N,R}) \right\|_{u,v,\theta} \leq C \left( (v - u)^{\beta - \theta} (1 + \| X_{s}^{N,R} \|_{u,v,\infty}) + \| X_{s}^{N,R} \|_{u,v,\theta} \right). $$

Therefore,

$$ |I_{u}^{c} - I_{u}^{c}| \leq C N \left( 1 + \| X_{s}^{N,R} \|_{s,t,\infty} + \| X_{s}^{N,R} \|_{s,t,\theta} (v - u)^{\theta} \right) (v - u)^{\mu}. $$

Consequently,

$$ \| X_{s}^{N,R} \|_{s,t,\theta} \leq C \left( 1 + \| X_{s}^{N,R} \|_{s,t,\infty} \right) (t - s)^{1 - \theta} + \| I_{s,t,\theta}^{b} \|_{s,t,\theta} + C N \left( 1 + \| X_{s}^{N,R} \|_{s,t,\infty} (t - s)^{\mu - \theta} + \| X_{s}^{N,R} \|_{s,t,\theta} (t - s)^{\mu} \right) \leq \| I_{s,t,\theta}^{b} \|_{s,t,\theta} + C N \left( 1 + \| X_{s}^{N,R} \|_{s,t,\infty} (t - s)^{\mu - \theta} + \| X_{s}^{N,R} \|_{s,t,\theta} (t - s)^{\mu} \right) $$
with certain non-random constant $K$.

Suppose that $t - s \leq \Delta$ with $\Delta \leq (2KN)^{-1/\mu}$. Then
\[
\|X^{N,R}\|_{s,t,\theta} \leq 2\|I^b\|_{s,t,\theta} + 2KN \left(1 + \|X^{N,R}\|_{s,t,\infty}(t - s)^{\mu - \theta}\right).
\] (4)

Further, from the obvious inequality
\[
\|X^{N,R}\|_{s,t,\infty} \leq |X_s| + \|X^{N,R}\|_{s,t,\theta}(t - s)^{\theta},
\]
using (4), we obtain
\[
\|X^{N,R}\|_{s,t,\infty} \leq \|X^{N,R}\|_{0,s,\infty} + 2\left(\|I^b\|_{s,t,\theta} + KN\right)(t - s)^{\theta} + 2KN \|X^{N,R}\|_{s,t,\infty}(t - s)^{\mu}
\]
\[
\leq \|X^{N,R}\|_{0,s,\infty} + 2\left(\|I^b\|_{s,t,\theta} + KN\right)\Delta^{\theta} + 2KN \|X^{N,R}\|_{s,t,\infty}\Delta^\mu,
\]
whenever $t - s \leq \Delta$. Assuming further that $\Delta \leq (4KN)^{-1/\mu}$, we get
\[
\|X^{N,R}\|_{s,t,\infty} \leq 2\|X^{N,R}\|_{0,s,\infty} + 4\left(\|I^b\|_{s,t,\theta} + KN\right)\Delta^\theta.
\]

Hence we derive for any $p > (1/2 - \theta)^{-1}$ that
\[
E\left[\|X^{N,R}\|^p_{0,t,\infty}\right] \leq C_p \left(E\left[\|X^{N,R}\|^p_{0,s,\infty}\right] + E\left[\|I^b\|^p_{s,t,\theta}\right] \Delta^p + N^p\Delta^p\right). \] (5)

Using the Garsia–Rodemich–Rumsey inequality, we have
\[
E\left[\|I^b\|^p_{s,t,\theta}\right] \leq C_p \int_s^t \int_s^t \frac{E\left[|I^b(x) - I^b(y)|^p\right]}{|x - y|^{p\theta + 2}} dx dy
\]
\[
\leq C_p \int_s^t \int_s^t E\left[\left|\int_x^y b(z, X_{z,R}^N)^2 1_{z \leq t} dz\right|^p\right] |x - y|^{-p\theta - 2} dx dy
\]
\[
\leq C_p \int_s^t \int_s^t \left(1 + E\left[|X^{N,R}|^p\right]\right) \int_s^t \int_x^y |x - y|^{p/2 - p\theta} dx dy
\]
\[
\leq C_p \left(1 + E\left[|X^{N,R}|^p\right]\right) \int_s^t \int_s^t |x - y|^{p/2 - p\theta} dx dy
\]
\[
\leq C_p \left(1 + E\left[|X^{N,R}|^p\right]\right) \Delta^{p/2 - p\theta}.
\]

Plugging this estimate into (5), we arrive at the inequality
\[
E\left[\|X^{N,R}\|^p_{0,t,\infty}\right] \leq K_p \left(E\left[|X^{N,R}|^p\right] + E\left[|X^{N,R}|^p\right] \Delta^{p/2} + N^p\Delta^p\right)
\]
\[
\leq K_p \left(E\left[|X^{N,R}|^p\right] + E\left[|X^{N,R}|^p\right] \Delta^{p/2} + N^p\right)
\]
with certain constant $K_p$. Assuming that $\Delta \leq (2K_p)^{-2/p}$, we get
\[
E\left[\|X^{N,R}\|^p_{0,t,\infty}\right] \leq 2K_p \left(E\left[|X^{N,R}|^p\right] + N^p\right). \] (6)

Finally, put $\Delta = \min\left\{(4KN)^{-1/\mu}, (2K_p)^{-2/p}\right\}$. Splitting the segment $[0, T]$ into $[T/\Delta] + 1$ parts of length at most $\Delta$, we obtain from the estimate (6) that
\[
E\left[\|X^{N,R}\|^p_{0,T,\infty}\right] \leq (2K_p + 1)^{T/\Delta + 1} (|X_0|^p + N^p) \leq C_p \exp\left\{C_p N^{1/\mu}\right\}.
\]
Theorem 1.

Let $E$ be centered Gaussian process $B$. Recall that an $l$-dimensional fractional Brownian motion $B^H = \{B^H_t = (B^{H,1}_t, \ldots, B^{H,l}_t), t \in [0, T]\}$ with the covariance function

$$E\left[ B^H_t B^H_s \right] = \frac{\delta_{ij}}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).$$

It is well known that a fractional Brownian motion has a version which satisfies the Hölder condition with any exponent $\mu < H$. We assume henceforth that this version is taken.

Corollary 3. Let in equation (1) the process $Z = B^H$ be a fractional Brownian motion with a parameter $H \in (1/2, 1)$. Then for any $p > 0$

$$E \left[ \|X\|_{0,T,\mu}^p \right] < \infty.$$

Proof. Let $\mu \in (1/2, H)$. Then the Hölder seminorm

$$\|B^H\|_{0,T,\mu} = \sup_{0 \leq s < t \leq T} \frac{|B^H_t - B^H_s|}{(t-s)^\mu}$$

is an almost surely finite supremum of norms of a centered Gaussian family. Therefore, by Fernique’s theorem, $E \left[ \exp \left\{ a \|B^H\|_{0,T,\mu}^2 \right\} \right] < \infty$ for some $a > 0$. Since $\mu > 1/2$, then $E \left[ \exp \left\{ c \|B^H\|_{0,T,\mu}^{1/\mu} \right\} \right] < \infty$ for all $c > 0$. Thus, the required statement follows from Theorem 1.

The exponential integrability will be proved under a different set of assumptions. Some assumptions are carried forward unchanged, nevertheless we repeat them for convenience.

B1. For all $t \in [0, T], x \in \mathbb{R}^d$, 

$$|a(t,x)| + |b(t,x)| + |c(t,x)| \leq C.$$

B2. For all $t \in [0, T], x \in \mathbb{R}^d$, 

$$|c'_x(t,x)| \leq C.$$
B3. For all \( R > 0, t \in [0, T] \) and \( x_1, x_2 \in \mathbb{R}^d \) such that \( |x_1| \leq R, |x_2| \leq R \),

\[
|a(t, x_1) - a(t, x_2)| + |b(t, x_1) - b(t, x_2)| + |c'_x(t, x_1) - c'_x(t, x_2)| \leq C_R |x_1 - x_2|.
\]

B4. For all \( s, t \in [0, T], x \in \mathbb{R}^d \),

\[
|c(t, x) - c(s, x)| + |c'_x(t, x) - c'_x(s, x)| \leq C|s - t|\beta.
\]

Under these assumption the exponential integrability of the solution to (11) is proved the same way as it is made in (10) for coefficients independent of \( t \). Nevertheless, for completeness we will give principal ideas, omitting unimportant details.

**Theorem 2.** Assume that the assumptions B1–B4 are satisfied and for any \( c > 0, \alpha \in (0, 2) \)

\[
\mathbb{E} \left[ \exp \left\{ c \|Z\|_{0, T, \mu}^\alpha \right\} \right] < \infty.
\]

Then for any \( c > 0, \gamma \in (0, 4\mu/(2\mu + 1)) \) the solution \( X \) to equation (\ref{eq:1}) satisfies

\[
\mathbb{E} \left[ \exp \left\{ c \|X\|_{0, T, \infty}^\gamma \right\} \right] < \infty.
\]

**Proof.** The proof partially repeats that of Theorem (\ref{thm:1}) so some details will be left out.

Denote \( I^a_v = \int_0^v a(s, X_s) ds, I^b_v = \int_0^v b(s, X_s) dW_s, I^c_v = \int_0^v c(s, X_s) dZ_s \) and fix arbitrary \( \theta \in (1 - \mu, \beta], \kappa \in (\theta, 1/2). \)

Let \( 0 \leq s \leq u \leq v \leq t \leq T \). Write

\[
|X_v - X_u| \leq |I^a_v - I^a_u| + |I^b_v - I^b_u| + |I^c_v - I^c_u|.
\]

Estimate

\[
|I^a_v - I^a_u| \leq \int_u^v |a(z, X_z)| dz \leq C(v - u).
\]

Further, using (\ref{eq:3}), we have

\[
|I^b_v - I^b_u| \leq \| I^b \|_{s, t, \kappa} (v - u)^\kappa.
\]

Evidently, \( |I^c_v - I^c_u| \leq \| I^c \|_{s, t, \kappa} (v - u)^\kappa \). The estimates above yield

\[
\|X\|_{s, t, \theta} \leq C(t - s)^{1 - \theta} + \| I^b \|_{s, t, \kappa} (t - s)^{\kappa - \theta} + C \| Z \|_{0, t, \mu} \left( (t - s)^{\mu - \theta} + \| X \|_{s, t, \theta} (t - s)^{\theta} \right)
\]

\[
\leq \| X \|_{s, t, \kappa} (t - s)^{\kappa - \theta} + K(1 + \| Z \|_{0, t, \mu}) \left( (t - s)^{\mu - \theta} + \| X \|_{s, t, \theta} (t - s)^{\theta} \right)
\]

with a positive constant \( K \). Assuming that \( t - s \leq \Delta := (2K(1 + \| Z \|_{0, T, \mu}))^{-1/\mu} \), we have

\[
\|X\|_{s, t, \theta} \leq 2\| I^b \|_{s, t, \kappa} (t - s)^{\kappa - \theta} + 2K(1 + \| Z \|_{0, t, \mu})(t - s)^{\mu - \theta}.
\]

As in the proof of Theorem (\ref{thm:1}) the last estimate implies

\[
\|X\|_{0, t, \infty} \leq \|X\|_{0, s, \infty} + 2\| I^b \|_{s, t, \kappa} (t - s)^\kappa + 2K(1 + \| Z \|_{0, t, \mu})(t - s)^\mu
\]

\[
\leq \|X\|_{0, s, \infty} + 2K\left( \| I^b \|_{s, t, \kappa} \Delta^\kappa + (1 + \| Z \|_{0, t, \mu}) \Delta^\mu \right).
\]
Splitting the segment \([0, T]\) into \([T/\Delta] + 1\) parts of length at most \(\Delta\), we get
\[
\|X\|_{0,T,\infty} \leq |X_0| + 2K(T + 1) \left(\|I^b\|_{0,T,\alpha} \Delta^{\alpha - 1} + 2K(1 + \|Z\|_{0,T,\mu}) \Delta^{\mu - 1}\right)
\]
\[
\leq C \left(1 + \|I^b\|_{0,T,\alpha}(1 + \|Z\|_{0,T,\mu})^{(1-\kappa)/\mu} + (1 + \|Z\|_{0,T,\mu})^{1/\mu}\right)
\]
Now take arbitrary \(\gamma \in (0, 4\mu/(2\mu + 1))\). Since \(2(\gamma - \mu)/\mu < 2/\gamma\), then it is possible to choose \(\kappa\) so that \(1 - \kappa \in (1/2, (2 - \gamma)\mu/\gamma)\), equivalently,
\[
\frac{2(1 - \kappa)}{2 - \gamma} \leq \frac{2}{\gamma}
\]
Now take arbitrary \(\gamma \in (0, 2/(2 - \gamma))\) so that \(\nu := \lambda(1 - \kappa)/\mu < 2/\gamma\), and denote \(\lambda' = \lambda/(1 - \lambda)\) the adjoint exponent for \(\lambda\); from \(\lambda > 2/(2 - \gamma)\) it follows that \(\lambda' < 2/\gamma\). From the Young inequality
\[
\|I^b\|_{0,T,\alpha}(1 + \|Z\|_{0,T,\mu})^{(1-\kappa)/\mu} \leq \frac{1}{\lambda'}\|I^b\|_{0,T,\alpha} + \frac{1}{\lambda}(1 + \|Z\|_{0,T,\mu})^{\nu'}.
\]
Therefore,
\[
\|X\|_{0,T,\infty}^\gamma \leq C \left(1 + \|I^b\|_{0,T,\alpha}^{\lambda'} + (1 + \|Z\|_{0,T,\mu})^{\nu'} + (1 + \|Z\|_{0,T,\mu})^{1/\mu}\right)^\gamma
\]
\[
\leq C \left(1 + \|I^b\|_{0,T,\alpha}^{\lambda'} + (1 + \|Z\|_{0,T,\mu})^{\nu'\gamma} + (1 + \|Z\|_{0,T,\mu})^{\gamma/\mu}\right).
\]
Hence the statement of the theorem follows, because the exponents are less than 2 and for \(\alpha \in (0, 2)\) \(\mathbb{E}\left[\exp\left\{c\|Z\|_{0,T,\mu}^\alpha\right\}\right] < \infty\) by the assumption, \(\mathbb{E}\left[\exp\left\{c\|I^b\|_{0,T,\alpha}^\alpha\right\}\right] < \infty\) by \([10]\) Lemma 1].

**Corollary 4.** Let in \((1)\) the process \(Z\) is a fractional Brownian motion \(B^H\) with the Hurst parameter \(H \in (1/2, 1)\), and the coefficients satisfy the assumptions B1–B4. Then for all \(c > 0\), \(\gamma \in (0, 4H/(2H + 1))\) the solution \(X\) to equation \((1)\) satisfies
\[
\mathbb{E}\left[\exp\left\{c\|X\|_{0,T,\infty}^\gamma\right\}\right] < \infty.
\]

### 5 Integrability of solution for equations without linear growth condition

Consider now equation of a form
\[
Y_t = Y_0 + \int_0^t \tilde{a}(s, X_s, Y_s)ds + \sum_{i=1}^r \int_0^t \tilde{b}_i(s, X_s, Y_s)d\tilde{W}_t^i + \sum_{j=1}^q \int_0^t \tilde{c}_j(s, X_s, Y_s)d\tilde{Z}_t^j, \quad t \in [0, T],
\]
where \(X\) solves \((1)\); the coefficients \(\tilde{a} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^k\), \(\tilde{b}_i : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^k\), \(i = 1, \ldots, r\), \(\tilde{c}_j : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^k\), \(j = 1, \ldots, q\), are jointly continuous; \(\tilde{W} = \{\tilde{W}_t = (\tilde{W}_t^1, \ldots, \tilde{W}_t^r), t \in [0, T]\}\) is a standard Wiener process in \(\mathbb{R}^r\), \(\tilde{Z} = \{\tilde{Z}_t = (\tilde{Z}_t^1, \ldots, \tilde{Z}_t^q), t \in [0, T]\}\) is an \(\mathbb{F}\)-adapted process in \(\mathbb{R}^q\) having \(\mu\)-Hölder continuous paths; the initial condition \(Y_0\) is non-random. Abbreviate this equation as
\[
Y_t = Y_0 + \int_0^t \tilde{a}(s, X_s, Y_s)ds + \int_0^t \tilde{b}(s, X_s, Y_s)d\tilde{W}_t + \int_0^t \tilde{c}(s, X_s, Y_s)d\tilde{Z}_s. \quad (8)
\]
Such equations arise in modeling quite often. For instance, in financial mathematics, a price process in a stochastic volatility model can be driven by an equation

$$S_t = S_0 + \int_0^t \mu u S_u du + \int_0^t \sigma^W S_u dW_u + \int_0^t \sigma^B S_u dB^H_u,$$

(9)

where the stochastic volatility processes $\sigma^W$ and $\sigma^B$ are also solutions to some stochastic differential equations. Another example is the equation satisfied by the Malliavin derivative of the solution to (1):

$$dDX_t = a'(X_t)DX_t dt + b'(X_t)DX_t dW_t + c'(X_t)DX_t dZ_t.$$  

(10)

If we combine equation (9) with volatility equations or equation (10) with (1), then the coefficients of resulting multi-dimensional equation, generally speaking, will not satisfy the linear growth condition. So we need some other techniques to study the integrability.

In our case the role of ‘stochastic volatility’ is played by the solution $X$ to (1). We will assume that the coefficients to (1) satisfy the assumptions B1–B4. We formulate the assumptions on the coefficients (8), using (9) and (10) as model equations. Specifically, we will assume that for some $\rho \in (0, 2/3)$

C1. For all $t \in [0, T], x \in \mathbb{R}^d, y \in \mathbb{R}^k$,

$$|\tilde{a}(t, x, y)| + |\tilde{c}(t, x, y)| \leq C(1 + |x|^\rho)(1 + |y|).$$

C2. For all $t \in [0, T], x \in \mathbb{R}^d, y \in \mathbb{R}^k$

$$|\tilde{b}(t, x, y)| \leq C(1 + |y|).$$

C3. For all $t \in [0, T], x \in \mathbb{R}^d, y \in \mathbb{R}^k$

$$|\tilde{c}'(t, x, y)| \leq C(1 + |x|^\rho).$$

C4. For all $R > 1, t \in [0, T]$ and $x \in \mathbb{R}^d, y_1, y_2 \in \mathbb{R}^k$ such that $|x| \leq R, |y_1| \leq R, |y_2| \leq R$,

$$|\tilde{a}(t, x, y_1) - \tilde{a}(t, x, y_2)| + |\tilde{b}(t, x, y_1) - \tilde{b}(t, x, y_2)| + |\tilde{c}'(t, x, y_1) - \tilde{c}'(t, x, y_2)| \leq C|y_1 - y_2|.$$  

(11)

C5. For all $t \in [0, T], x_1, x_2 \in \mathbb{R}^d, y \in \mathbb{R}^k$,

$$|\tilde{c}(t, x_1, y) - \tilde{c}(t, x_2, y)| \leq C|x_1 - x_2|(1 + |y|).$$

C6. For all $s, t \in [0, T], x \in \mathbb{R}^d, y \in \mathbb{R}^k$,

$$|\tilde{c}(s, x, y) - \tilde{c}(t, x, y)| \leq C|s - t|^\beta (1 + |x|^\rho)(1 + |y|),$$

$$|\tilde{c}'(s, x, y) - \tilde{c}'(t, x, y)| \leq C|s - t|^\beta (1 + |y|).$$

Unfortunately, we were not able to prove existence of all moments under the assumption $|\tilde{b}(t, x, y)| \leq C(1 + |x|^\rho)(1 + |y|)$, so we impose C2.

The proof of the unique solvability for the equation (8) under assumptions C1–C6 is similar to that for equation (11) (see [9]), so we omit it.
Theorem 3. Assume that the coefficients of equation (1) satisfy B1–B4, and the coefficients of (S) satisfy C1–C6 with \( \rho \in (0,2\mu(2\mu-1)/(2\mu+1)) \). Let also for any \( c > 0, \alpha \in (0,2) \)

\[
E \left[ \exp \left\{ c \|Z\|^\alpha_{0,T,\mu} \right\} \right] + E \left[ \exp \left\{ c \|\tilde{Z}\|^\alpha_{0,T,\mu} \right\} \right] < \infty.
\]

Then for any \( p > 0 \) the solution \( Y \) to (S) satisfies

\[
E \left[ \|Y\|^p_{0,T,\infty} \right] < \infty.
\]

Remark 6. The restriction \( \rho < 2\mu(2\mu-1)/(2\mu+1) \) explains why \( \rho < 2/3 \) in C1–C6: the right-hand side of the former inequality increases in \( \mu \) and is equal to 2/3 for \( \mu = 1 \).

Proof. The proof will follow the same scheme as the proofs of Theorem 1 and 2.

Put \( J_1^t = \int_0^t b(s,X_s) dW_s \). Fix arbitrary \( N \geq 1, M \geq 1, R \geq 1, \theta \in (1-\mu,\beta] \), \( \kappa \in (0,1/2) \). Denote

\[
\tau = \min \left\{ t \geq 0 : \|Y\|_{0,t,\infty} \geq R \text{ or } \|Z\|_{0,t,\mu} + \|J^b\|_{0,t,\kappa} \geq N \text{ or } \|X\|^\rho_{0,t,\infty} \geq M \right\},
\]

\( X_t^\tau = X_t \chi_{\tau < \infty}, Y_t^\tau = Y_t \chi_{\tau < \infty}, I_t = 1_{\{t \leq \tau\}} \). Put also \( I_t^\alpha = \int_0^t \tilde{a}(s,X_s^\tau,Y_s^\tau) 1_s ds, I_t^b = \int_0^t \tilde{b}(s,X_s^\tau,Y_s^\tau) 1_s d\tilde{W}_s, I_t^c = \int_0^t \tilde{c}(s,X_s^\tau,Y_s^\tau) 1_s d\tilde{Z}_s \).

Let \( 0 \leq s \leq u \leq v \leq t \leq T \). Write

\[
\|Y_{v}^\tau - Y_{u}^\tau\| \leq |I_v^\alpha - I_u^\alpha| + |I_v^b - I_u^b| + |I_v^c - I_u^c|.
\]

From the condition C1

\[
|I_v^\alpha - I_u^\alpha| \leq \int_u^v |\tilde{a}(z,X_s^\tau,Y_s^\tau)| 1_s dz \leq CM \left( 1 + \|Y^\tau\|_{s,t,\infty} \right) (v-u).
\]

Denoting \( \xi_t = \tilde{c}(t,X_t^\tau,Y_t^\tau) \), we have from (3) that

\[
|I_v^\alpha - I_u^\alpha| \leq CN \left( \|\xi\|_{u,v,\infty} + \|\xi\|_{u,v,\theta} (v-u)^\theta \right) (v-u)^\mu.
\]

The assumption C1 allows to estimate

\[
\|\xi\|_{u,v,\infty} \leq CM \left( 1 + \|Y^\tau\|_{s,t,\infty} \right).
\]

The inequalities

\[
|\tilde{c}(x,X_s^\tau,Y_s^\tau) - \tilde{c}(y,X_s^\tau,Y_s^\tau)| \leq |\tilde{c}(x,X_s^\tau,Y_s^\tau) - \tilde{c}(y,X_s^\tau,Y_s^\tau)|
+ |\tilde{c}(y,X_s^\tau,Y_s^\tau) - \tilde{c}(y,X_s^\tau,Y_s^\tau)|
+ |\tilde{c}(y,X_s^\tau,Y_s^\tau) - \tilde{c}(y,X_s^\tau,Y_s^\tau)|
\leq C \left( |x-y|^\beta (1 + |X_s^\tau|^\rho) (1 + |Y_s^\tau|) + |X_s^\tau - Y_s^\tau| (1 + |Y_s^\tau|) + |Y_s^\tau - Y_s^\tau| (1 + |X_s^\tau|^\rho) \right)
\]

imply that

\[
\|\xi\|_{u,v,\theta} \leq CM \left( (v-u)^{\beta-\theta} (1 + \|Y^\tau\|_{u,v,\infty}) + \|Y^\tau\|_{u,v,\theta} (1 + \|Y^\tau\|_{u,v,\infty}) \right).
\]

From these inequalities, we obtain

\[
\|Y^\tau\|_{s,t,\theta} \leq CM \left( 1 + \|Y^\tau\|_{s,t,\infty} \right) (t-s)^{-\theta} + \|I^b\|_{s,t,\theta} + CMN \left( 1 + \|Y^\tau\|_{s,t,\infty} \right) (t-s)^{-\mu-\theta}
+ CN \left( M (t-s)^{\beta-\theta} (1 + \|Y^\tau\|_{s,t,\infty}) + \|Y^\tau\|_{s,t,\theta} (1 + \|Y^\tau\|_{s,t,\infty}) \right) (t-s)^\mu
\leq \|I^b\|_{s,t,\theta} + CMN \left( 1 + \|Y^\tau\|_{s,t,\infty} \right) (t-s)^{-\theta} + \|Y^\tau\|_{s,t,\theta} (t-s)^\mu
+ CN \left( M (t-s)^{\beta-\theta} (1 + \|Y^\tau\|_{s,t,\infty}) + \|Y^\tau\|_{s,t,\theta} (1 + \|Y^\tau\|_{s,t,\infty}) \right) (t-s)^\mu.
\]
Hence we get, similarly to (7),
\[ \|X^\tau\|_{s,t,\theta} \leq CN(t-s)^{\kappa-\theta} + CN(t-s)^{\mu-\theta} \leq CN(t-s)^{\kappa-\theta}. \]
Consequently,
\[ \|Y^\tau\|_{s,t,\theta} \leq \|I^b\|_{s,t,\theta} + KNM \left( (1 + \|Y^\tau\|_{s,t,\infty}) (t-s)^{\mu-\theta} + \|Y^\tau\|_{s,t,\theta} (t-s)^\mu \right) \]
\[ + KN^2 \left( (1 + \|Y^\tau\|_{s,t,\infty}) (t-s)^{\mu+\kappa-\theta} \right) \]
with some constant \( K \).
Assume that \( t-s \leq \Delta \) with \( \Delta \leq (2KMN)^{-1/\mu} \). Then
\[ \|Y^\tau\|_{s,t,\theta} \leq 2\|I^b\|_{s,t,\theta} + 2KMN \left( (1 + \|Y^\tau\|_{s,t,\infty}) (t-s)^{\mu-\theta} \right) \]
\[ + 2KN^2 \left( (1 + \|Y^\tau\|_{s,t,\infty}) (t-s)^{\mu+\kappa-\theta} \right). \]
Hence, as in Theorems 1 and 2, we have
\[ \|Y^\tau\|_{s,t,\infty} \leq 2\|I^b\|_{s,t,\theta} + 4\|I^b\|_{s,t,\theta}. \]
Therefore, for arbitrary \( p > (1/2 - \theta)^{-1} \)
\[ E \left[ \|Y^\tau\|_{0,t,\infty} \right] \leq C_p \left( E \left[ \|X^\tau\|^p_{0,s,\infty} \right] + E \left[ \|I^b\|_{s,t,\theta} \right] \Delta^{\mu\theta} + N^p \Delta^{\theta} \right). \]
Using the same reasoning as in the proof of Theorem 1 we arrive at the estimate
\[ E \left[ \|Y^\tau\|_{0,t,\infty} \right] \leq K_p \left( E \left[ \|Y^\tau\|^p_{0,s,\infty} \right] + E \left[ \|Y^\tau\|_{s,t,\theta} \right] \Delta^{\mu/2} + N^p \right), \]
with some positive constant \( K_p \). Putting
\[ \Delta = \min \left\{ (8KMN)^{-1/\mu}, (8KN^2)^{-1/(\mu+\kappa)}, (2Kp)^{-2/p} \right\}, \]
we get
\[ E \left[ \|Y^\tau\|_{0,t,\infty} \right] \leq 2K_p \left( E \left[ \|Y^\tau\|^p_{0,s,\infty} \right] + N^p \right), \]
therefore,
\[ E \left[ \|Y^\tau\|_{0,t,\infty} \right] \leq (2Kp + 1)T^{\Delta+1} (|Y_0|^p + N^p) \leq C_p \exp \left\{ C_p \left( N^{1/\mu} M^{1/\mu} + N^{2/(\mu+\kappa)} \right) \right\}. \]
As in the proof of Theorem 1 we derive hence that
\[ \left( E \left[ \|Y^\tau\|_{0,t,\infty} \right] \right)^2 \leq C_p \exp \left\{ C_p \left( \xi^{1/\mu} \eta^{1/\mu} + \xi^{2/(\mu+\kappa)} \right) \right\} \]
where \( \xi = \|Z\|_{0,t,\mu} + \|J_0\|_{0,t,\kappa}, \eta = \|X\|^p_{0,t,\infty} \). Since \( \mu + \kappa > 1 \), then for any \( c > 0 \)
\[ E \left[ \exp \left\{ c \xi^{2/(\mu+\kappa)} \right\} \right] \leq 1 \]. From the restriction on \( \rho \) it follows that \( 2\mu(2\mu - 1)^{-1} < 4\mu^2 \rho^{-1}(2\mu + 1)^{-1} \). Choose arbitrary \( \lambda \in (2\mu(2\mu - 1)^{-1}, 4\mu^2 \rho^{-1}(2\mu + 1)^{-1}) \), denote \( \lambda' = \lambda/(\lambda - 1) \) the exponent adjoint to \( \lambda \) and write by the Young inequality
\[ \xi^{1/\mu} \eta^{1/\mu} \leq C(\xi^{\lambda'/\mu} + \eta^{\lambda'/\mu}). \]
Theorem 2 implies that \( E \left[ \exp \left\{ c \xi^{\lambda'/\mu} \right\} \right] < \infty \) for any \( c > 0 \). It is easy to see that \( \lambda' < 2\mu \), so \( E \left[ \exp \left\{ c \xi^{\lambda'/\mu} \right\} \right] < \infty \) for any \( c > 0 \). Thus, the theorem is proved. \( \square \)
Corollary 7. Let in equations (1) and (8) the processes $Z$ and $\tilde{Z}$ be fractional Brownian motions with the Hurst parameter $H \in (1/2, 1)$, and let the coefficients of the equations satisfy assumptions B1–B4 and C1–C6 with $\rho \in (0, 2H(2H-1)/(2H+1))$, correspondingly. Then for any $p > 0$ the solution $Y$ to equation (8) satisfies

$$\mathbb{E}\left[\|Y\|_{p,T,\infty}^p\right] < \infty.$$ 

Remark 8. The last corollary allows to deduce that the solution to (1) with $Z = B^H$ has an integrable Malliavin derivative provided that the coefficients are differentiable, the derivative of $b$ is bounded, and the derivatives of $a$ and $c$ grow slower than a power function with an exponent less than $(0, 2H(2H-1)/(2H+1))$.

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