Contextual Bandits with Side-Observations

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Abstract

We investigate contextual bandits in the presence of side-observations across arms in order to design recommendation algorithms for users connected via social networks. Users in social networks respond to their friends’ activity, and hence provide information about each other’s preferences. In our model, when a learning algorithm recommends an article to a user, not only does it observe his/her response (e.g. an ad click), but also the side-observations, i.e., the response of his neighbors if they were presented with the same article. We model these observation dependencies by a graph $G$ in which nodes correspond to users, and edges correspond to social links. We derive a problem-instance-dependent lower-bound on the regret of any consistent algorithm. We propose an optimization (linear programming) based data-driven learning algorithm that utilizes the structure of $G$ in order to make recommendations to users and show that it is asymptotically optimal, in the sense that its regret matches the lower-bound as the number of rounds $T \to \infty$. We show that this asymptotically optimal regret is upper-bounded as $O(|\chi(G)| \log T)$, where $|\chi(G)|$ is the domination number of $G$. In contrast, a naive application of the existing learning algorithms results in $O(N \log T)$ regret, where $N$ is the number of users.

1 Introduction

The contextual multi-armed bandit model is popularly used in order to place ads and make personalized recommendations of news articles to users of web services \[LCLS10, LZ08\]. In this model, both users and contents are represented by sets of features. For example, user features are obtained on the basis of their historical behavior and demographic information; while content feature depends upon its category and descriptive information. A learning algorithm for contextual bandits sequentially recommends articles to users based on contextual information of the articles and preferences of users, while continually adapting its strategy to present articles on the basis of feedback, e.g. ad clicks, downloads, etc., received from users. Its goal is to maximize the cumulative reward, which is equal to the total number of user clicks in the long run.

We consider the problem of making recommendations to users of a social network such as Facebook, Goodreads, LinkedIn. If users’ preferences were known, we could employ an optimal stationary strategy that maps the context of each user to its optimal action (i.e., present her with an article that has the highest click-probability). Since users’ preferences are typically unknown, one could employ
an efficient contextual-bandit learning algorithm as in [LS16] on each user separately. This strategy achieves a regret of $O(N \log T)$, where $N$ is the number of users. However, since the number of users can be very large (e.g. Facebook has 2.5 billion users [Wik20]), this strategy is impractical.

Consider a social network modeled by an undirected graph $G$ in which the nodes correspond to users, and undirected edges correspond to “social links”, i.e., two users are connected by an edge if they are “friends.” Since individual users are connected to a subset of the remaining users, each time the algorithm makes a recommendation to a user, it also obtains feedback from her “neighbors” regarding their potential interest in a similar offer. For example, when a user $i$ is presented with a promotion $x$, his neighbors could be explicitly queried as follows: “Would you be interested in promotion $x$ that was offered to your friend $i$?” The response of user $i$’s friends to this query then constitutes “side observations”. We design learning algorithms that incorporate these side-observations into the decision-making process for making recommendations. We show that the regret of the proposed algorithms scales at most as $O(\left|\chi(G)\right| \log T)$, where $\left|\chi(G)\right|$ is the domination number (see Definition 4)) of graph $G$. Since $\left|\chi(G)\right| \ll N$ for most graphs (see [Wei20] for more details), our algorithms drastically reduce the dependence of the regret on the number of users.

In our setup, choosing an arm in the multi-armed bandit problem corresponds to making a recommendation to a single user in network. We work with a linear contextual bandit models, i.e., the one-step expected reward (e.g. user’s ad-click probability) of an arm is a linear function of the context of the arm. This (unknown) linear function depends upon the user on which this arm is played. For this contextual multi-armed bandit model with side observations, we derive a lower bound on the regret of any consistent algorithm, and show that our algorithms are asymptotically (as $T \to \infty$) optimal, since their asymptotic regret matches this lower bound.

1.1 Related Work

We begin by describing existing works on contextual bandits, and then discuss works that derive learning algorithms for models in which side-observations are present.

**Contextual Bandits with Linear Pay-off Functions:** Contextual bandits with linear pay-off functions have been extensively studied. The efficiency of a learning algorithm is measured by its regret. Upper Confidence Bound (UCB)-based algorithms that use optimism in the face of uncertainty have been developed in works such as [Aue02, LCLS10, CLRS11]. CLRS11 analyzes LinUCB and shows that its minimax/worst-case regret scales as $\tilde{O}(\sqrt{dT})$, where $d$ is the dimension of feature space. CL11 and [AG13] utilize Thompson sampling and prove that its regret scales as $\tilde{O}\left(\frac{d}{\epsilon^2} \sqrt{T^{1+\epsilon}}\right)$, where $0 < \epsilon < 1$. However, we focus on developing algorithms that have provably optimal problem-dependent regret guarantees [BCB12, LS20]. As has been shown in [LS16], performance of learning algorithms based on UCB, or Thompson sampling can be arbitrarily far from optimal in this setting.

Finite-time problem-dependent guarantees for linear bandits have been derived in [RT10, AYPS11], however these are far from optimal. [LS16] studies problem-dependent regret in the asymptotic regime (when $T \to \infty$), and derives algorithm that is asymptotically optimal. We focus exclusively on the problem-dependent setting, and build upon techniques of [LS16].

**Learning with Side observations:** [MS11] introduced the side-observation model in the adversarial multi-armed (non-contextual) bandit setting in which upon choosing an action, the decision maker not only receives reward from the chosen arm, but also gets to observe the rewards of its “neighboring” arms. The observation dependencies are encoded as an undirected graph $G$ in which two nodes $i, j$ are connected by an edge only if pulling an arm also reveals reward of the other arm. [CKLB12, BES14] and [BLES17] extend results of [MS11] to the setup of stochastic multi-armed bandits in which rewards from an arm are i.i.d. across time, and reward distribution depends upon arm. [CKLB12] derives algorithms whose regret scales as $O(\left|\gamma(G)\right| \log T)$, where $\left|\gamma(G)\right|$ is the clique cover number of

\[ \tilde{O}(\cdot) \text{ hides factors that are logarithmic in number of rounds} \, T.\]
Associated with each node \( E \) while undirected edges 

The social network of interest is modeled by a graph \( G \) using which nodes of vertices within which every two vertices are adjacent. Clique cover number is the smallest number of cliques generated by the operational history until round \( t \).

Let \( u_j \) recommend to user \( i \) during round \( t \) also let \( y_{i,j}(t) \) denote the side-observation obtained from \( i \) during round \( t \). Presenting article to a user \( i \) also reveals “side-observations” on its neighboring nodes \( \mathcal{N}_i := \{ j : (i,j) \in \mathcal{E} \} \). These are rewards that would have been obtained if the same article was presented to users in the set \( \mathcal{N}_i \). Let \( r_i(t) \) denote the reward received from recommendation to user \( i \) during round \( t \). Also let \( y_{i,j}(t) \) denote the side-observation obtained from user \( j \) as a result of recommendation to \( i \) during round \( t \).

Let \( \mathcal{F}_t := \sigma \left( \{ r_i(s) \}_{i \in \mathcal{V}}, \{ y_{i,j}(s) : j \in \mathcal{N}_i \}_{i \in \mathcal{V}}, \{ U_i(s) \}_{i \in \mathcal{V}} \right) \) denote the sigma-algebra generated by the operational history until round \( t \). The reward earned from \( i \) is given by

\[
r_i(t) = U_i^T(t) \theta_i^* + \eta_i(t), \quad i \in \mathcal{V},
\]

Our main contributions are as follows:

1. We model the contextual multi-armed bandit problem in the presence of side-observations. We derive an instance dependent lower-bound on the regret of any consistent policy which is domination number of \( G \). The key insight gained from [CKLB12, BES14] is that in the presence of side-observations, not only does an efficient algorithm need to take into account the history of rewards obtained from an arm, but also the location of the arm in the graph \( G \). Thus, for example, it might even be optimal to pull an arm with a low estimate of mean reward, because it is connected to relatively unexplored arms, and the “exploration gains” resulting from side-observations outweigh the (relatively larger) instantaneous regret of this arm. Our work generalizes the side-observations learning model to the case of contextual bandits with linear pay-offs. Our analysis is inspired by results in [LS16].

2. We propose a UCB-type learning algorithm that explores the values of unknown coefficients of users arms using a barycentric spanner of the set of context vectors of all the arms for each user in the network. It maintains confidence balls for the rewards of arms and uses a stopping rule in order to decide when to stop the exploration phase. At the end of exploration phase, it plays those arms that are optimal given the current estimates of coefficients. We analyze its finite time regret, and show that it can be upper-bounded as \( O(|\mathcal{V}| \log T) \) with a prefactor that depends upon the sub-optimality gaps of rewards of arms. This is a simple algorithm with a good regret bound, but does not match the lower bound.

3. To close, the gap mentioned in 2 above, we develop a learning algorithm that is composed of three phases. During the warm-up phase, it samples each user’s coefficient vectors for fixed \( O(\log^{1/2} T) \) number of rounds. Thereafter, in the success phase it uses these samples to estimate the unknown sub-optimality gaps of arms, and inputs these estimate into the LP mentioned above. The solution of this LP then yields the number of times each arm is to be played. The algorithm uses a detector in order to constantly track the quality of estimates obtained at the end of warm-up phase. In the event it detects that these estimates are “bad,” it switches to the UCB-type algorithm described in 2. above. We show that this “data-dependent” algorithm’s regret asymptotically matches the lower-bound.

2 Problem Setting

The social network of interest is modeled by a graph \( G = (\mathcal{V}, \mathcal{E}) \), in which the nodes \( \mathcal{V} \) represent users, while undirected edges \( \mathcal{E} \) represent social connections. We let \( |\mathcal{V}| \) denote the number of users. Associated with each node \( i \in \mathcal{V} \) is a “coefficient vector” \( \theta_i^* \in \mathbb{R}^d \). In each round \( t = 1, 2, \ldots, T \), the decision maker recommends articles to each \( i \in \mathcal{V} \). Let \( U_i(t) \in \mathcal{U} \subset \mathbb{R}^d \) denote the context of article presented to \( i \) during round \( t \). Presenting article to a user \( i \) also reveals “side-observations” on its neighboring nodes \( \mathcal{N}_i := \{ j : (i,j) \in \mathcal{E} \} \). These are rewards that would have been obtained if the same article was presented to users in the set \( \mathcal{N}_i \). Let \( r_i(t) \) denote the reward received from recommendation to user \( i \) during round \( t \). Also let \( y_{i,j}(t) \) denote the side-observation obtained from user \( j \) as a result of recommendation to \( i \) during round \( t \).

Let \( \mathcal{F}_t := \sigma \left( \{ r_i(s) \}_{i \in \mathcal{V}}, \{ y_{i,j}(s) : j \in \mathcal{N}_i \}_{i \in \mathcal{V}}, \{ U_i(s) \}_{i \in \mathcal{V}} \right) \) denote the sigma-algebra generated by the operational history until round \( t \). The reward earned from \( i \) is given by

\[
r_i(t) = U_i^T(t) \theta_i^* + \eta_i(t), \quad i \in \mathcal{V},
\]

\( A \) clique cover of a given undirected graph is a partition of the vertices of the graph into cliques, i.e., subsets of vertices within which every two vertices are adjacent. Clique cover number is the smallest number of cliques using which nodes of \( G \) can be covered.

\( A \) policy whose regret is smaller than \( o(t^p) \), \( \forall p > 0 \) and all possible instances of problem.
where \( \eta_t(t) \sim N(0, 1) \) is Gaussian and independent of \( \mathcal{F}_t \). Side-observations are given by,
\[
y_{t(i,j)}(t) = U_i^T(t) \theta_i^* + \eta_{t(i,j)}(t), \quad \forall (i, j) \in \mathcal{E},
\]
where \( \eta_{t(i,j)}(t) \sim N(0, 1) \) are independent of \( \mathcal{F}_t \), and independent across social links.

**Notation:** Denote \( \theta^* := (\theta_1^*, \theta_2^*, \ldots, \theta_N^*) \in \mathbb{R}^{d \times N} \) the vector consisting of unknown coefficients. An “arm” \( a \) that corresponds to playing context \( u \in \mathcal{U} \) on node (user) \( i \) is denoted by the tuple \( a = (i, u) \). For an arm \( a \), we let \( u_a \) denote its context vector, and \( i_a \) its node. An optimal arm \( b \) for node \( i \) satisfies \( b \in \arg \max_{a \in A_i} \{ u_a^T \theta_i^* \} \). We assume that each node \( i \in \mathcal{V} \) has exactly one optimal arm, which is denoted by \( a_i^* \), and the corresponding optimal context vector is denoted \( u_i^* \).

We let \( A_i \) denote the set of arms that can be played on node \( i \), and let \( A := \cup_{i \in \mathcal{V}} A_i \) denote the set of all arms. \( A_i^{(s)} \) denotes the set of all sub-optimal arms on node \( i \), and \( A^{(s)} := \cup_{i \in \mathcal{V}} A_i^{(s)} \) denotes the set of all sub-optimal arms. We also say that two arms \( a_1 = (i_1, u_1), a_2 = (i_2, u_2) \) are neighboring arms if \( (i_1, i_2) \in \mathcal{E} \). By notational abuse, we let \( N_i \) denote the set of neighboring arms of arm \( a \). When it is clear from the context, we will occasionally use \( a \) to denote \( u_a \). Define
\[
\Delta_a := \max_{u \in \mathcal{U}} u^T \theta_{a^*} - u_a^T \theta_{a^*},
\]
and also,
\[
\Delta_{\min, i} := \min_{a \in A_i^{(s)}} (a_i^*)^T \theta_i^* - a^T \theta_i^*, \quad \Delta_{\max, i} := \max_{a \in A_i^{(s)}} (a_i^*)^T \theta_i^* - a^T \theta_i^*,
\]
\[
\Delta_{\min} := \min_{a \in A} \Delta_a, \quad \Delta_{\max} := \max_{a \in A} \Delta_a.
\]

In what follows, we assume that a node is a neighbor of itself, i.e., \( i \in N_i \), or equivalently \( (i, i) \in \mathcal{E} \). Thus, we let \( y_{t(i,i)}(s) = r_i(s) \). This notation drastically simplifies the exposition.

All vectors are assumed to be column vectors. \( 0_{m \times n} \) denotes an \( m \times n \) matrix comprises of only zeros. For a matrix \( M \), \( \text{tr}(M) \) or \( M^T \) will denote its transpose, while \( \text{trace}(M) \) denotes its trace, and \( \text{col}_k(M) \) denotes its \( k \)-th column. For two vectors \( x, y \in \mathbb{R}^d \), \( \langle x, y \rangle \) denotes the dot product between \( x \) and \( y \). We will use \( < x, y > \) and \( x^T y \) interchangeably for dot product between \( x \) and \( y \). We use \( F_a(t) \) denote the number of times arm \( a \) has been played until round \( t \). For a vector \( x \in \mathbb{R}^d \) we let \( ||x|| \) denote its Euclidean norm, and for a positive-definite matrix \( H \), we let \( ||x||_H^2 := x^T H x \). For two integers \( m, n \) satisfying \( n > m \), we let \( [m, n] := \{ m, m+1, \ldots, n \} \).

**Learning Algorithm and Regret:** A learning algorithm \( \pi : \mathcal{F}_t \rightarrow \otimes_{i \in \mathcal{V}} A_i, t = 1, 2, \ldots, T \) maps the observational history until each round \( t \), to a set of \( |\mathcal{V}| \) arms, one for each node. As discussed earlier, we let \( U_i(t) \) denote the context of arm chosen for node \( i \). The performance of \( \pi \) is measured by its regret \( R_{\pi}^\star(\theta^*, \mathcal{G}, \mathcal{A}) (T) \),
\[
R_{\pi}^\star(\theta^*, \mathcal{G}, \mathcal{A}) (T) := \mathbb{E} \sum_{t=1}^T \left( \sum_{i \in \mathcal{V}} (u_i^*)^T \theta^* - U_i^T(t) \theta_i^* \right),
\]
where the expectation above is taken with respect to the probability measure induced by the algorithm \( \pi \), and randomness of rewards. Our objective is to design a learning algorithm that has a low regret.

**Definition 1. (Consistent Algorithm)** A learning algorithm \( \pi \) is called consistent if for all \( \theta^*, \mathcal{A}, \mathcal{G} \) and \( p > 0 \), it satisfies \( R_{\pi}^\star(\theta^*, \mathcal{G}, \mathcal{A}) (T) = o(T^p) \).

### 2.1 Preliminaries

Now we introduce the concept of a barycentric spanner, and generalize it to the graphical setting, which will be crucial in our algorithm design.

**Definition 2 (Barycentric Spanner of \( \mathcal{U} \) [AK04])**. A set of context vectors \( C \subseteq \mathcal{U} \) is called barycentric spanner of \( \mathcal{U} \) if each \( u \in \mathcal{U} \) can be written as follows,

\[
u = \sum_{w \in C} \alpha_w \ w, \quad \text{where} \ \alpha_w \in [-1, 1].
\]

The following result is Proposition 2.2 and Proposition 2.4 of [AK04].
Lemma 1. There exists a barycentric spanner of \( U \) that has cardinality less than or equal to \( d \). Moreover it can be obtained in time polynomial in \( d \).

Definition 3 (Barycentric Spanner of \((A, G)\)). Let \( C \) be a barycentric spanner of \( U \). Then, the set of arms \( S \),

\[
S := \{(i, u) : i \in V, u \in C\},
\]

is a barycentric spanner of \((A, G)\). In what follows, we let \( S \) be such a barycentric spanner of cardinality \( N_d \).

Definition 4. (Dominating set of a graph) A dominating set of graph \( G = (V, E) \) is a set of nodes such that every node from \( V \) is either in this set, or is a neighbor of some node belonging to this set. Let \( \chi(G) \) be a dominating set with minimum cardinality. \( |\chi(G)| \) is called the domination number of \( G \).

3 Lower Bounds

Define

\[
G_i(t) := \sum_{s=1}^{t} \sum_{j \in N_i} U_j(s)U_j^T(s), \quad \bar{G}_i(t) = \mathbb{E}(G_i(t)), \quad \forall i \in V.
\] (6)

We have the following lower bound on the regret of any consistent learning algorithm. Its proof is deferred to the appendix.

Theorem 1. Under any consistent learning algorithm, we have

\[
\limsup_{T \to \infty} \log(T)\|u_a\|^2_{\bar{G}_a^{-1}(T)} \leq \frac{\Delta_a^2}{2}, \quad \forall a \in A^{(s)}.
\] (7)

Consider the following linear program,

\[
LP : \min_{\alpha(a):a \in A^{(s)}} \sum_{a \in A^{(s)}} \alpha(a)\Delta_a
\] (8)

s.t. \( \|u_a\|^2_{H_a^{-1}(\alpha)} \leq \frac{\Delta_a^2}{2}, \quad \forall a \in A^{(s)}, \)

where \( H_a(\alpha) := \sum_{j \in N_i} \sum_{a:1_a = j} \alpha(a)aa^T, \)

\[
\text{where we have } \alpha = \{\alpha(a)\}_{a \in A}, \text{ and } \alpha(a) \in [0, \infty), \quad \forall a. \text{ Let } c(\theta^*, G, A) \text{ denote its optimal value. We then have}
\]

\[
\limsup_{T \to \infty} \frac{R(T)}{\log T} \geq c(\theta^*, G, A).
\] (11)

Note that solving the \( LP \) requires us to know the values \( \Delta_a \).

4 Stopping Time based Algorithm

We now propose an algorithm for contextual bandits with side-observations. This algorithm is composed of two phases (i) Exploratory Phase, that is followed by (ii) Exploitation phase. The exploratory phase lasts until a stopping criteria is met. More details are as follows.

Exploratory Phase: Only the arms in the barycentric spanner \( S \) are played in a round robin manner. Since \( S \) is composed of \( d \) arms at each node \( i \), this phase is composed of sets of consecutive rounds of the form \([kd + 1, (k + 1)d]\), \( k = 0, 1, \ldots \) such that each arm in \( S \) is played exactly once during each such set. We call each such set an episode. The algorithm maintains the empirical estimates \( \{\hat{\theta}_i(t)\}_{i \in V} \) of the unknown coefficients \( \theta_i^* \), which are obtained as follows,

\[
\hat{\theta}_i(t) := G_i^{-1}(t) \left[ \sum_{s=1}^{t} \sum_{j \in N_i} y_{(j,i)}(s)U_j(s) \right], \quad i \in V,
\] (12)
Algorithm 1 Stopping Time Based Algorithm

**Input:** Arms $\mathcal{A}$, Graph $G$, Confidence parameter $\delta$, Time horizon $T$

**Initialize:** Set $t := 1$, and estimates $\hat{\theta}(t) = (1, 1, \ldots, 1)$ for all $i \in V$

// Exploratory Phase

while $\exists i \in V$ such that $B_{i,1}(t) \cap B_{i,m}(t) \neq \emptyset$ for some $m$ do

Play arms $a \in S$ in round-robin fashion

Update the estimates $\hat{\theta}_i(t)$ using (12)

Update the confidence balls $B_a(t)$ using (13)

end while

Obtain estimates $\hat{\theta}_i(t)$ of the coefficients, and the optimal arms $\hat{a}_i^\star(\tau), i \in V$

// Exploitation Phase

for $t = \tau + 1, \tau + 2, \ldots, T$ do

Play $\{\hat{a}_i^\star(\tau)\}_{i \in V}$ on corresponding nodes

end for

where $G_i(t)$ is as in [1]. Additionally, it also maintains confidence ball $B_a(t)$ around the estimate of mean reward of each arm $a$ as follows,

$$B_a(t) := \left\{ \mu \in \mathbb{R} : |\mu - \hat{\theta}_a(t)| \leq \alpha(t), a \in \mathcal{A} \right\}$$

where

$$\alpha(t) := \sqrt{2 \log \left( \frac{T \sum_{i \in V} |\mathcal{A}_i|/\delta}{t} \right)} d.$$  \hspace{1cm} (14)

It orders the balls $\{B_a(t)\}_{a \in \mathcal{A}_i}$ at each node $i$ in decreasing order of the corresponding values of the estimates of the mean rewards $\{a^T \hat{\theta}_i(t) : a \in \mathcal{A}_i\}$. Let $B_{i,m}(t)$ be the $m$-th such ball at node $i$ during round $t$. Define $\tau_i$ to be the following stopping time,

$$\tau_i := \inf \left\{ t : t = kd \text{ where } k \in \mathbb{N}, B_{i,1}(t) \cap B_{i,m}(t) = \emptyset, \forall m = 2, 3, \ldots, |\mathcal{A}_i| \right\}$$

and $\tau := \max_{i \in V} \tau_i$. \hspace{1cm} (16)

Exploratory phase ends at round $\tau$.

**Exploitation Phase:** Let $\hat{a}_i^\star(t)$ be the optimal arm for node $i$ when the true value of coefficient of $i$ is equal to $\hat{\theta}_i(t)$, i.e., $\hat{a}_i^\star(t) \in \arg \max_{a_i \in A_i} \{a_i^T \hat{\theta}_i(t)\}$, $i \in V$. During rounds $t > \tau$, algorithm plays only the arms $\{\hat{a}_i^\star(\tau), i \in V\}$ at their corresponding nodes. Thus, it uses $\hat{\theta}_i(\tau)$ as a proxy for $\theta^*$, and plays the resulting greedy decisions.

The following result provides upper bound on regret of Algorithm [1]. We defer its proof to the appendix.

**Theorem 2.** The regret $R(T)$ of Algorithm [1] is upper-bounded as

$$R(T) \leq \left( \sum_{i \in V} \Delta_{\text{max},i} \right) \frac{2 \log \left( \frac{T \sum_{i \in V} |\mathcal{A}_i|/\delta}{d} \right)}{(\Delta_{\text{min}}/2)^2} + \delta T \left( \sum_{i \in V} \Delta_{\text{max},i} \right).$$

With $\delta = 1/T$, we obtain the following upper-bound on regret,

$$R(T) \leq \left( \sum_{i \in V} \Delta_{\text{max},i} \right) \frac{2 \log \left( \frac{T^2 \sum_{i \in V} |\mathcal{A}_i|}{d} \right)}{(\Delta_{\text{min}}/2)^2} + \left( \sum_{i \in V} \Delta_{\text{max},i} \right).$$

\footnote{Superscript denotes ordered balls.}
5 \ LP based Optimal Algorithm

Note that the regret of Algorithm 1 scales as \(O(\log T)\); if the parameters \(\Delta_{\text{max}}, \Delta_{\text{min}}\) and the dimension of contexts \(d\) are kept constant, then the regret scales linearly with the number of nodes \(N\).

We now propose an algorithm that uses LP (9)-(10) in order to make decisions. Its regret matches the lower bound of Section 3, and the prefactor can be upper-bounded by domination number \(|\chi(G)|\) of graph \(G\). For graphs \(G\) that satisfy \(|\chi(G)| \ll |V|\), this algorithm can be much more efficient than Algorithm 1.

**Algorithm 2** Optimal Algorithm based on LP

\begin{verbatim}
Input: Arms \(A\), Graph \(G\), Confidence parameter \(\delta\), Time horizon \(T\)

// Warm-up Phase
Play each arm in spanning set \(S\) for \(d \log^{1/2} T\) times

// Success Phase
\(\epsilon_T(t) \leftarrow \max_{a \in A} ||a||_{G^{-1}} \|d \log^{1/2} T\|^{2} g^{1/2}(T)\)
\(\hat{\Delta} \leftarrow \hat{\Delta}(d \log^{1/2} T), \hat{\mu} \leftarrow \hat{\mu}(d \log^{1/2} T)\)
Solve \(LP(\hat{\Delta})\) (20)-(22) to obtain \(\beta^*(\hat{\Delta})\)
\textbf{while} \(t \leq T\) and \(|\hat{\mu}_a - \hat{\mu}_a(t-1)| \leq 2\epsilon_T\) for all \(a \in A\)
\textbf{do}
\quad For each \(i \in V\) play actions in a round robin fashion with \(N_a(t) \leq \beta^*_a(\hat{\Delta})\)
\textbf{end while}

// Recovery Phase
Discard all data and play \(\pi_{ball}\) until \(t = T\)
\end{verbatim}

We begin by introducing some notations. Define,
\[
\begin{align*}
    f(t) :& = 2 \log(t) + cd \log(d \log t) + 2, \\
    g(t) :& = 2 \log(\log t) + \frac{\log(\log t)}{\log 2} + cd \log(d \log t),
\end{align*}
\]
where \(c > 0\) is a constant. Let
\[
\hat{\Delta}_a(t) := \max_{b \in A} (b-a)^T \hat{\theta}_a(t),
\]
denote estimate of sub-optimality gap of arm \(a\) during round \(t\), and \(\Delta(t) := \{\hat{\Delta}_a(t)\}_{a \in A}\).

The proposed algorithm is composed of three phases.

**Warm-up Phase:** Algorithm plays arms in barycentric spanner \(S\) for \(d \log^{1/2} T\) rounds.

**Success Phase:** Denote by \(\hat{\Delta} = \{\hat{\Delta}_a\}_{a \in A}, \hat{\mu} = \{\hat{\mu}_a\}_{a \in A}\) the estimates of sub-optimality gaps and mean values of rewards that are obtained by using the information gained during warm-up phase.

Consider the following linear program obtained from \(LP (9)-(10)\) by replacing the gaps \(\Delta_a\) by their estimates \(\hat{\Delta} = \{\hat{\Delta}_a : a \in A\}\):

\[
LP(\hat{\Delta}) : \min_{\{\hat{\beta}_a\}_{a \in A}} \sum_{a \in A} \beta_a \hat{\Delta}_a
\]
subject to \(f(T)||u_a||_{H^{-1}(\beta)}^2 \leq \frac{\Delta_a^2}{2}, \forall a \in A\),
\[
\text{where } H_i(\beta) := \sum_{j \in N_i, \{a : a = j\}} \beta_a a a^T, \quad i \in V.
\]

Let \(\beta^*(\hat{\Delta}) = \{\beta^*_a(\hat{\Delta}) : a \in A\}\) be a solution of \(LP(\hat{\Delta})\).

The algorithm uses estimates \(\hat{\Delta}\) to solve (20)-(22), and obtains \(\beta^*(\hat{\Delta})\). It then plays each arm \(a\) in a round-robin fashion until it has been played for \(\beta^*_a(\hat{\Delta})\) rounds. Meanwhile, it also continually
Figure 1: Comparison of regret of Algorithm 2 with the algorithms of [LS16], which we denote as Lattimore and Lattimore-N. The plots are obtained after averaging the results of 100 runs. \( N \) and \( K \) denote the number of users, and contexts respectively.

keeps track of the quality of estimates \( \hat{\mu}_a \) of rewards obtained at the end of warm-up phase as follows. Define

\[
\epsilon_T(t) := \max_{a \in A} \|a\|_{G_{t_a}(t)}^{-1} g^{1/2}(T),
\]

where \( g(\cdot) \) is as in (18). If during any round \( t \), it observes that \( |\hat{\mu}_a(t) - \mu_a| > 2\epsilon_T(d \log \frac{1}{2} T) \) for some arm \( a \in A \), then it declares that the estimates \( \hat{\mu} \) are bad, and in this event algorithm enters recovery phase.

Recovery Phase: Algorithm discards all operational history and collected data, and starts playing Algorithm 1.

We next show that this algorithm is asymptotically optimal, i.e., as \( T \to \infty \), its regret matches the lower bound derived in Theorem 1. We defer the proof of following theorem to appendix.

**Theorem 3.** The regret \( R(T) \) of Algorithm 2 satisfies

\[
\limsup_{T \to \infty} \frac{R(T)}{\log T} \leq c(A, \theta^*, \mathcal{G}),
\]

where \( c(A, \theta^*, \mathcal{G}) \) is the optimal value of LP (8)-(10). It then follows from lower bound derived in Theorem 7 that Algorithm 2 is asymptotically optimal as \( T \to \infty \).

**Corollary 1.** Optimal value of LP (8)-(10), \( c(A, \theta^*, \mathcal{G}) \), is less than or equal to \( \frac{\Delta_{\text{max}}}{\Delta_{\text{min}}} |\chi(\mathcal{G})| \). Thus, the regret \( R(T) \) of Algorithm 2 satisfies \( \limsup_{T \to \infty} \frac{R(T)}{\log T} \leq \frac{\Delta_{\text{max}}}{\Delta_{\text{min}}} |\chi(\mathcal{G})| \).

6 Experiments

**Synthetic Data Experiment:** The vector \( \theta = \{\theta_i^*\}_{i \in V} \) that contains the coefficients of the users, and the contexts of the arms, are generated randomly; \( \theta_i \), and the context vectors are drawn from a uniform distribution with support in the set \([0, 1]^d\). The edges in the graph \( \mathcal{G} \) are drawn randomly;
any two nodes $i, j \in V$ are connected with a probability $p$. The noise $\eta_i(t), \eta_{i,j}(t)$ associated with the rewards and the side-observations $1, 2$ are assumed to be Gaussian with standard deviation 0.1. We compare the performance of Algorithm 2 with the algorithm of [LS16], which is denoted Lattimore, and its adaption to the graphical setting which is denoted Lattimore-N. Lattimore-N is a naive adaptation of Lattimore algorithm, it uses side observations to enhance the estimation after warm-up phase, our experimental results show the potential gains from using the side observations alone. However, by leveraging the graph structure, our optimal algorithm shows significant regret reduction, and verifies our theoretical claims. We summarize the results of this evaluation in Figure 1 where we plot the cumulative regret of the algorithms as a function of rounds, averaged over 100 runs.

7 Discussion

In this paper we introduce a framework to incorporate side-observations into the contextual bandits with linear payoff functions. We derive an instance-dependent lower bound on the regret of any learning algorithm, and also an optimal algorithm whose regret matches these lower bounds asymptotically as $T \to \infty$.

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8 Appendix

Appendix A: Proof of Lemma 1

Proof of Lemma 1. Let $S(\{i\})$ be a barycentric spanner for $\{i\}, A_i$, i.e., when only node $i$ is present. Consider the set of arms $S(\{i\}), i \in \chi(G)$. Consider an arm $a$, and let $b$ be an arm such that $i_b \in \chi(G)$ that satisfies $i_b \in N_{i_a}$. Moreover, its context $u_{i_b}$ is equal to $u_i$. Note that such an arm exists because $\chi(G)$ covers all the nodes $V$. We have

$$u_{i_a} = u_{i_b} = \sum_{c \in S(\{i_b\})} \alpha_c u_c,$$

where $|u_c| \leq 1$ since $S(\{i_b\})$ is barycentric spanner of $\{i_b\}, A_{i_b}$. Thus, we have shown that context of an arbitrary arm can be written as a linear combination of contexts of arms belonging to set $S(\{i\}), i \in \chi(G)$, which shows that $\cup_{i \in \chi(G)}S(\{i\})$ is a barycentric spanner of $(A, G)$. This completes the proof.

Appendix B: Proof of Theorem 1

The proof is composed of a sequence of lemmas that culminates in the proof of theorem. The following result is Lemma 5 of [LS16].

Lemma 2. Let $P$ and $P'$ be measures on the same measurable space $(\Omega, F)$. Then, for any event $A \in F$, we have,

$$P(A) + P'(A^c) \geq \frac{1}{2} \exp(-\text{KL}(P, P')),$$

where $\text{KL}(P, P')$ denotes the relative entropy between $P$ and $P'$, which is defined as $+\infty$ if $P$ is not absolutely continuous with respect to $P'$, and is equal to $\int_\Omega dP(\omega) \log \frac{dP'}{dP}(\omega)$ otherwise.

Lemma 3. (Information Processing Lemma) Consider a learning algorithm $\pi$, and let $P, P'$ denote the measures induced by it on the sequence of rewards $\{r_i(s) : i \in I\}_{i=1}^T$, side-observations $\{y(i,j)(s) : (i, j) \in E\}_{i=1}^T$, and actions $\{U_i(s) : i \in V\}_{i=1}^T$, when the graph and arms are equal to $G$ and $A$ in both the cases, while users’ coefficients in the two cases are equal to $\{\theta_{ij}\}_{i \in V}$ and $\{\theta'_{ij}\}_{i \in V}$ respectively. Furthermore, assume that $\theta^*$ and $\theta'$ differ only on a single node $i$, i.e., $\theta^* = \theta'_j, \forall j \neq i$ and $\theta^*_i \neq \theta'_i$. Then we have the following,

$$\text{KL}(P, P') = \frac{1}{2}(\theta^*_i - \theta'_i)\sum_i G_i(T) (\theta^*_i - \theta'_i),$$

where $G_i(T)$ is as in (6), and expectation in above is taken when $\theta^*$ is true parameter.

Constructing Modified Coefficient Vector $\theta'$

Recall that when the coefficient vector is equal to $\theta^*$, arm $a^*_{i}$ is the unique optimal arm for node $i$. We will now construct a coefficient vector $\theta'$ so that the resulting optimal arm for node $i$ will be $b^*$, where $b^* \neq a^*_{i}$. Since we do not modify the coefficient at other nodes, the optimal arms for other nodes $v \in V \setminus \{i\}$ remain unchanged. Let $H > 0$ be a positive-definite matrix that will be specified soon. We let

$$\theta'_v = \begin{cases} \theta^*_v, & \text{if } v \in V \setminus \{i\}, \\ \theta^*_i + \frac{1}{\|b^* - a^*_i\|_H} H(b^* - a^*_i)(\Delta_{b^*} + \epsilon) & \text{if } v = i. \end{cases}$$

Note that under $\theta'$, the mean reward of arm $b^*$ is more than that of $a^*_i$ since,

$$\theta'_v^T (b^* - a^*_i) = \left(\theta^*_v + \frac{1}{\|b^* - a^*_i\|_H} H(b^* - a^*_i)(\Delta_{b^*} + \epsilon)\right)^T (b^* - a^*_i) = -\Delta_{b^*} + \Delta_{b^*} + \epsilon = \epsilon.$$  

Let $R^\pi(\theta^*, G, A)(T)$, $R^\pi(\theta', G, A)(T)$ denote regrets of $\pi$ under $\theta^*$, $\theta'$ respectively. We have the following lower-bound on $R^\pi(\theta^*, G, A)(T) + R^\pi(\theta', G, A)(T)$.
We then have, \[ R^{\pi}_{(\theta^*, G, A)}(T) + R^{\pi'}_{(\theta^*, G, A)}(T) \geq \frac{eT}{2} \left[ P \left( N_{a_1}(T) \leq T/2 \right) + P' \left( N_{a_1}(T) > T/2 \right) \right], \]
where \( N_a(T) \) is the number of plays of arm \( a \) until round \( T \).

**Proof.** Clearly,

\[ R^{\pi}_{(\theta^*, G, A)}(T) \geq \frac{T}{2} \Delta_{\min,i} P \left( N_{a_1}(T) \leq T/2 \right). \]

Similarly, it follows from (25) that

\[ R^{\pi'}_{(\theta^*, G, A)}(T) \geq \frac{T}{2} \epsilon P' \left( N_{a_1}(T) > T/2 \right). \]

The proof then follows by adding the above two inequalities. \[\square\]

**Lemma 5.** Let \( \theta' \) be the coefficient vector constructed as in (24), and \( b^* \) be the optimal arm at node \( i \) under \( \theta' \). Define \[ \delta a_1^* := b^* - a_1^*, \] where \( a_1^* \) is the optimal arm for node \( i \) when its coefficient is equal to \( \theta_1^* \). For \( H > 0 \) define

\[ \rho_i(T; H) := \|\delta a_1^*\|_H^2 \rho_{G,i}(T) \|\delta a_1^*\|_{\tilde{G}_{i-1}(T)}^2 \left( \|\delta a_1^*\|_H^2 \right)^{-1}. \]

We then have,

\[ \frac{(\Delta_{b^*} + \epsilon)^2}{2} \frac{\rho_i(T; H)}{\log T \|\delta a_1^*\|_{\tilde{G}_{i-1}(T)}^2} \geq 1 + \frac{\log \epsilon - \log 2}{\log T} - \frac{\log \left( R^{\pi}_{(\theta^*, G, A)}(T) + R^{\pi'}_{(\theta^*, G, A)}(T) \right)}{\log T}. \]

**Proof.** It follows from Lemma 2 and Lemma 3 that

\[ R^{\pi}_{(\theta^*, G, A)}(T) + R^{\pi'}_{(\theta^*, G, A)}(T) \geq \frac{eT}{2} \exp(-KL(P, P')). \]

Substituting the expression for \( KL(P, P') \) from Lemma 3 into the above inequality, and taking logarithms, we obtain the following.

\[ \frac{1}{2} (b^*_i - \theta'_i)^T G_i(T) (b^*_i - \theta'_i) \geq \log \left( \frac{eT}{2} \right) - \log \left( R^{\pi}_{(\theta^*, G, A)}(T) + R^{\pi'}_{(\theta^*, G, A)}(T) \right). \]

We then substitute value of \( \theta'_i \) from (24) in the above and perform some algebraic manipulations in order to obtain (28). \[\square\]

**Lemma 6.** Let \( \delta a_1^* \) be as in (26), and \( \pi \) be a consistent policy. We then have that,

\[ \liminf_{T \to \infty} \frac{\rho_i(T; H)}{\log T \|\delta a_1^*\|_{\tilde{G}_{i-1}(T)}^2} \geq \frac{2}{\Delta_{b^*}}. \]

**Proof.** Since \( \pi \) is asymptotically consistent, we have \( \limsup_{T \to \infty} \frac{R^{\pi}_{(\theta^*, G, A)}(T) + R^{\pi'}_{(\theta^*, G, A)}(T)}{\log T} \leq 0 \). Substituting this into the bound (28) yields

\[ \frac{(\Delta_{b^*} + \epsilon)^2}{2} \liminf_{T \to \infty} \frac{\rho_i(T; H)}{\log T \|\delta a_1^*\|_{\tilde{G}_{i-1}(T)}^2} \geq 1. \]

The result then follows since the bound holds true for all \( \epsilon > 0 \) and choice of \( b^* \). \[\square\]
Next, define
\[ c := \limsup_{T \to \infty} \log T \|\delta a^*\|^2_{\tilde{G}^{-1}(T)}, \]
and let \( d \in \mathbb{R} \) be such that
\[ d \leq \liminf_{T \to \infty} \frac{\rho_i(T; H)}{\log T \|\delta a^*\|^2_{\tilde{G}^{-1}(T)}}. \]
We then have that
\[ c \leq \frac{\liminf_{T \to \infty} \rho_i(T; H)}{d}, \] (30)
where \( H > 0 \). It follows from Lemma 6 that \( d \) can be taken to be \( 2/\Delta^2 \). We now obtain an upper-bound on \( \liminf_{T \to \infty} \rho_i(T; H) \) which will give us an upper-bound on \( c \).

Lemma 7. Define,
\[ \tilde{H}_i(T) := \frac{G_i^{-1}(T)}{\|G_i^{-1}(T)\|}, \]
and let \( \tilde{H}_i(\infty) \) be a limit point of \( \tilde{H}_i(T) \). We then have that
\[ \lim_{T \to \infty} \rho_i(T; \tilde{H}_i(\infty)) \leq 1. \] (32)

Proof. We have
\[ \rho_i(T; H) = \frac{\|\delta a^*\|^{2}_{\tilde{H}_i(T)} \|\delta a^*\|^{2}_{\tilde{G}^{-1}(T)}}{\|\delta a^*\|^{2}_{\tilde{H}_i(T)}} \]
\[ = \frac{\|\delta a^*\|^{2}_{\tilde{H}_i(T)} \|\delta a^*\|^{2}_{\tilde{H}_i(T)} \|\delta a^*\|^{-1}_{H}}{\|\delta a^*\|^{2}_{\tilde{H}_i(T)}}. \]
The last expression computes to 1 with \( H \) set equal to \( \tilde{H}_i(T) \). It then follows that
\[ \lim_{T \to \infty} \rho_i(T; \tilde{H}_i(\infty)) \leq 1. \]

Lemma 8. Under any consistent policy \( \pi \), we have that
\[ \limsup_{T \to \infty} \log T \|\delta a^*\|^2_{\tilde{G}^{-1}(T)} \leq \frac{\Delta^2}{2}. \]

Proof. Follows by substituting (32) into the inequality (30), and choosing \( d \) to be equal to \( 2/\Delta^2 \).

We are now in a position to prove Theorem 1.

Proof of Theorem 1 Consider a sub-optimal arm \( a \in A_i(\pi) \). We will show that for a consistent algorithm,

We have,
\[ \|a_{a}\|_{\tilde{G}^{-1}(T)} \leq \|a_{a} - a_{a}^*\|_{\tilde{G}^{-1}(T)} + \|a_{a}^*\|_{\tilde{G}^{-1}(T)} \]
\[ \leq \|a_{a} - a_{a}^*\|_{} + \frac{\|a_{a}^*\|}{\sqrt{N_{a_i}(T)}}, \]
where the first inequality follows from the triangle inequality, while the second follows since from (6) we have that \( G_i(T) \geq N_{a_i}(T) a_{a}^* (a_{a}^*)^T \), which yields \( \tilde{G}^{-1}(T) \leq (N_{a_i}(T))^{-1} [a_{a}^* (a_{a}^*)^T]^\dagger \). Multiplying both sides of the above inequality by \( \log^{1/2} T \), we obtain
\[ \limsup_{T \to \infty} \log^{1/2} T \|a_{a}\|_{\tilde{G}^{-1}(T)} \leq \limsup_{T \to \infty} \log^{1/2} T \|a_{a} - a_{a}^*\|_{\tilde{G}^{-1}(T)} + \limsup_{T \to \infty} \frac{\log^{1/2} T}{\sqrt{N_{a_i}(T)}} \|a_{a}^*\|. \]
Under a consistent policy, we have \( \lim_{T \to \infty} \frac{N_a(T)}{T} = 1 \), so that the second term on the r.h.s. vanishes. It follows from Lemma 8 that the first term on the r.h.s. is upper-bounded by \( \Delta_a/\sqrt{T} \). Substituting these into the above inequality yields the proof of (7).

We now prove (11). Let \( \Delta \) vanish. It follows from Lemma 8 that the first term on the r.h.s. is upper-bounded by \( \Delta_a/\sqrt{T} \). Substituting this into the above inequality yields the proof of (7).

Using \( \alpha^T = \alpha^T \) and \( \alpha^T = \alpha^T \), define \( \alpha^T = \{ \alpha^T : \alpha^T \in A \} \). Its regret \( R^\pi(T) \) satisfies

\[
\frac{R^\pi(T)}{\log T} = \sum_{a \in A^T} \alpha^T(a) \Delta_a. \tag{33}
\]

Note that \( \bar{G}_i(T) = (\log T) H_i(\alpha^T) \) or \( \bar{G}_i^{-1}(T) = (\log T)^{-1} H_i^{-1}(\alpha^T) \), where the function \( H_i(\cdot) \) is as defined in (10). Since \( \pi \) is consistent, it then follows from (7) that,

\[
\limsup_{T \to \infty} \|u_a\|_{H_i^{-1}(\alpha^T)}^2 \leq \frac{\Delta_a^2}{2}. \tag{34}
\]

Let \( \alpha^{(\infty)} = \{ \alpha^{(\infty)}(a) : a \in A \} \) be a limit point of \( \alpha^T \). It follows from (34) that the vector \( \alpha^{(\infty)} \) is feasible for the LP (8)-(10). Moreover, it follows from (33) that the regret \( R^\pi(T) \) satisfies

\[
\limsup_{T \to \infty} \frac{R^\pi(T)}{\log T} \geq \sum_{a \in A^T} \alpha^{(\infty)}(a) \Delta_a.
\]

This completes the proof.

**Appendix C: Proof of Theorem 2**

We begin by deriving bounds on the error associated with the estimates \( \hat{\theta}_i(t) \) that are obtained as in (12), i.e., \( \hat{\theta}_i(t) = G_i^{-1}(t) \left[ \sum_{s=1}^{t} \sum_{j \in N_i} j_{(j,i)}(s) U_j(s) \right] \). Substituting the expressions for rewards \( r_i(s) \) and side-observations \( y_{(j,i)}(s) \) from (1) and (2), we obtain the following,

\[
e_i(t) := \hat{\theta}_i(t) - \theta_i^* \\
= G_i^{-1}(t) \sum_{s=1}^{t} \sum_{j \in N_i} j_{(j,i)}(s) U_j(s). \tag{35}
\]

For \( x \in \mathbb{R}^d \), consider:

\[
x^T e_i(t) = \sum_{s=1}^{t} \sum_{j \in N_i} j_{(j,i)}(s) x^T G_i^{-1}(t) U_j(s). \tag{36}
\]

Define the following “error event,”

\[
E_i(x, \alpha, t) := \{ \omega : x^T e_i(t) > \alpha \}, \text{ where } \alpha > 0, i \in \mathcal{V}, t \in [1, T]. \tag{37}
\]

**Lemma 9.** Let decisions \( \{ U_i(t) \}_{i \in \mathcal{V}, t \in [1, T]} \) be deterministic. We then have that,

\[
\mathbb{P}(E_i(x, \alpha, t)) \leq \exp\left(-\frac{\alpha^2}{2\|x\|_{G_i^{-1}(t)}^2}\right). \tag{38}
\]

**Proof.** For \( \lambda > 0 \), it follows from Chebyshev’s inequality that,

\[
\mathbb{P}(x^T e_i(t) > \alpha) \leq \exp(-\lambda \alpha) E \exp(\lambda x^T e_i(t)). \tag{39}
\]

Substituting the expression for \( x^T e_i(t) \) from (36), we obtain,

\[
E \exp(\lambda x^T e_i(t)) = \exp\left(\frac{\lambda^2}{2\|x\|_{G_i^{-1}(t)}^2}\right). \tag{40}
\]
Substituting the above into the inequality (39), we obtain
\[ \mathbb{P} \left( x^T e_i(t) > \alpha \right) \leq \exp \left( -\frac{\lambda^2}{2} \|x\|_{G_i^{-1}(t)}^2 \right). \]
For \( \lambda = \alpha / \|x\|_{G_i^{-1}(t)}^2 \), the above inequality reduces to
\[ \mathbb{P} \left( x^T e_i(t) > \alpha \right) \leq \exp \left( -\frac{\alpha^2}{2 \|x\|_{G_i^{-1}(t)}^2} \right). \] (41)
This completes the proof. ■

Note that the exploration phase is composed of episodes, and each episode lasts for \( d \) rounds. Each arm in \( S \) is played exactly once during an episode. After \( t \) episodes of play of \( S \), the matrices \( G_i(td) \) are as follows
\[ G_i(td) = t \left( \sum_{j \in N_i, a \in S \cap A_j} aa^T \right) = t|N_i| \left( \sum_{u \in C} u u^T \right), \]
so that
\[ G_i^{-1}(td) = \frac{1}{t|N_i|} \left( \sum_{u \in C} u u^T \right)^{-1}. \] (42)

**Lemma 10.** If the decisions \( \{U_i(t) : i \in V\}_{t \in [1, T]} \) are such that only the arms in \( S \) are played in a round-robin manner, then,
\[ \|a\|_{G_i^{-1}(td)}^2 \leq \frac{d}{t|N_i|}, \forall a \in A. \] (43)

**Proof.** Within this proof we let \( i \) denote the node of arm \( a \). Since \( C \) is a barycentric spanner for \( U \) (see Definition 2), we have \( u_a = \sum_{u \in C} \alpha_u u \), where \( \alpha_u \in [-1, 1], \forall u \). Thus,
\[ \|u_a\|_{G_i^{-1}(td)}^2 = \sum_{u \in C} \alpha_u^2 u^T G_i^{-1}(td) u \]
\[ \leq \frac{1}{t|N_i|} \sum_{u \in C} \alpha_u^2 u^T \left( u u^T \right)^T u \]
\[ \leq \frac{1}{t|N_i|} \sum_{u \in C} u^T \left( u u^T \right)^T u, \]
\[ \leq \frac{d}{t|N_i|}, \]
where the first inequality follows from (42), and the second inequality follows since \( |\alpha_u| \leq 1 \). ■

Recall the size of confidence intervals \( \alpha(t) \),
\[ \alpha(t) = \sqrt{\frac{2 \log \left( \frac{T}{\sum_{i \in V} |A_i|/\delta} \right)}{t}} d. \] (44)

**Lemma 11.** Define
\[ \mathcal{E} := \bigcup_{k \in [1, T/d], i \in V, a_i \in A} \mathcal{E}_i(a_i, \alpha(kd), kd), \]
where \( \alpha(t) \) is as in (44). We have the following bound while playing \( S \)
\[ \mathbb{P} \left( \mathcal{E} \right) \leq \delta. \] (46)
Proof. Substituting the bound \((43)\) for \(\|a\|^2_{G_{i,a}^{-1}(kd)}\) into \((58)\), we obtain,

\[
\mathbb{P}(E_i(a_i, \alpha(kd), kd)) \leq \exp\left(-\frac{\alpha^2(kd)|N_i|kd}{2d}\right)
\]

\[
\leq \frac{T}{\sum_{i \in \mathcal{V}} |A_i|},
\]

The proof then follows by using union bound, \(\mathbb{P}(E) \leq \sum_{(k,i,a)} \mathbb{P}(E_i(a_i, \alpha(kd), kd))\).

**Lemma 12.** Consider the set \(E\) as defined in \((45)\). On \(E\), we have \(\hat{a}^*_i(\tau_i) = a^*_i\), i.e., the estimate of optimal arm at node \(i\) is equal to the unique optimal arm for node \(i\). Thus, with a probability greater than \(1 - \delta\), we have \(\hat{a}^*_i(\tau_i) = a^*_i\).

Proof. Note that \(\hat{a}^*_i(\tau_i)\) is the arm that corresponds to \(B_{a,i}^{(s)}(\tau_i)\). On \(E\), we have \(\mu_{a,i} = B_{a,i}^{(s)}(\tau_i)\), and also \(\mu_a \in B_{a,i}^{(s)}(\tau_i)\) for any \(a \in A_i^{(s)}\). This means that on \(E\), the ball \(B_{a,i}^{(s)}(\tau_i)\) is equal to the ball \(B_{a,i}^{(s)}(\tau_i)\), since if this was not the case, then we would have a contradiction that \(\mu_a > \mu_{a,i}\) for some sub-optimal arm \(a\). Hence we conclude that \(\hat{a}^*_i(\tau_i) = a^*_i\) on \(E\). The proof is completed by noting that from \((16)\), we have that \(\mathbb{P}(\mathcal{E}) \geq 1 - \delta\).

Next, we derive an upper-bound on the stopping time \(\tau\) that marks the end of the exploration phase. This yields an upper-bound on the “exploration regret.”

**Lemma 13.** Consider the stopping time \(\tau\) as defined in \((15)-(16)\). The following holds true on the set \(E\),

\[
\tau \leq \frac{2 \log \left( T \sum_{i \in \mathcal{V}} |A_i|/\delta \right) d}{\Delta_{\min}/2}.
\]

Proof. We begin by bounding the time \(\tau_i\). Note that on the set \(E\), since the mean rewards of arms lie within their confidence ball, we have that in order for the ball \(B_{a,i}^{(s)}\) and the ball \(B_a\), corresponding to some sub-optimal \(a \in A_i^{(s)}\), to intersect during round \(t\), we must have

\[
\mu_{a,i} - \alpha(t) \leq \mu_a + \alpha(t), a \in A_i^{(s)},
\]

which gives

\[
\alpha(t) \geq \frac{\Delta_a}{2}, a \in A_i^{(s)}.
\]

Substituting the expression for \(\alpha(t)\) from \((44)\), we obtain

\[
\sqrt{\frac{2 \log \left( T \sum_{i \in \mathcal{V}} |A_i|/\delta \right) d}{\Delta_{\min}/2}} \geq \Delta_a/2, a \in A_i^{(s)},
\]

or

\[
t \leq \frac{2 \log \left( T \sum_{i \in \mathcal{V}} |A_i|/\delta \right) d}{\Delta_{\min}/2}, a \in A_i^{(s)}.
\]

The above implies that on \(E\), the ball \(B_{a,i}^{(s)}(t)\) cannot intersect with \(B_a, a \in A_i^{(s)}\) during rounds \(t > \frac{2 \log(T \sum_{i \in \mathcal{V}} |A_i|/\delta) d}{\Delta_{\min}/2}, a \in A_i^{(s)}\). The proof then follows by noting that \(\tau = \max_i \tau_i\).

Proof. of Theorem 2

It follows from Lemma 12 that on the set \(E\), Algorithm 1 yields 0 regret after round \(\tau\). Thus, we have

\[
\mathbb{1}(E) \sum_{t=1}^{T} \sum_{i \in \mathcal{V}} \Delta U_i(t) \leq \tau \left( \sum_{i \in \mathcal{V}} \Delta_{\max,i} \right).
\]

16
Upon substituting the upper-bound on $\tau$ that was derived in Lemma 13, we obtain the following,

$$
\mathbb{E} \left( \mathbb{1}(\mathcal{E}^c) \sum_{t=1}^{T} \sum_{i \in V} \Delta_{U_i(t)} \right) \leq \frac{2 \log \left( T \sum_{i \in V} |A_i| / \delta \right) d}{(\Delta_{\min}/2)^2} \left( \sum_{i \in V} \Delta_{\max,i} \right). \quad (47)
$$

Similarly, since the regret is bounded by $T$ on any sample path, we have

$$
\mathbb{E} \left( \mathbb{1}(\mathcal{E}) \sum_{t=1}^{T} \sum_{i \in V} \Delta_{U_i(t)} \right) \leq T \left( \sum_{i \in V} \Delta_{\max,i} \right) \mathbb{P}(\mathcal{E}) \leq \delta T \left( \sum_{i \in V} \Delta_{\max,i} \right), \quad (48)
$$

where the last inequality follows from Lemma 11. The proof then follows by combining the inequalities (47) and (48).

**Appendix D: Proof of Theorem 3**

**D.1: Preliminary Results**

We begin by deriving some results that will be useful while analyzing regret of Algorithm 2. Recall that $\hat{\Delta} = \{\hat{\Delta}_a : a \in A\}$ denotes the estimates \(^{(19)}\) of sub-optimality gaps obtained at the end of warm-up phase.

**Lemma 14.** Consider LP($\hat{\Delta}$) \(^{(20)-(22)}\) that takes as input the estimates $\hat{\Delta}$ obtained at the end of warm-up phase. We then have that,

$$
\sum_{a \in A} \beta^*_a(\hat{\Delta}) \leq 2 d^3 f(T) \frac{\hat{\Delta}_{\max}}{\hat{\Delta}_{\min}^3},
$$

where $\beta^*(\hat{\Delta}) = \{\beta^*_a(\hat{\Delta})\}_{a \in A}$ is a solution of \(^{(20)-(22)}\).

**Proof.** Same as proof of Lemma 12 of [LS16].

**Lemma 15.** Define $\delta_T$ as follows,

$$
1 + \delta_T := \max_{a \in A, \Delta_a > 0} \frac{\Delta_a^2}{\Delta_a^2}.
$$

We then have that,

$$
\sum_{a \in A} \beta^*_a(\hat{\Delta}) \hat{\Delta}_a \leq (1 + \delta_T) \sum_{a \in A} \beta^*_a(\Delta) \hat{\Delta}_a. \quad (50)
$$

**Proof.** For an arm $a$, we have

$$
\|a\|^2_{H^{-1}_a((1+\delta_T)\beta^*(\Delta))} = \frac{\|a\|^2_{H^{-1}_a(\beta^*(\Delta))}}{(1 + \delta_T)} \leq \frac{\Delta_a^2}{(1 + \delta_T)f(T)} \leq \frac{\Delta_a^2}{f(T)}, \quad (51)
$$

where $H_a(\cdot)$ is as defined in \(^{(22)}\), the first inequality follows since $\beta^*(\Delta)$ is feasible for LP($\Delta$), and the last inequality follows from definition of $\delta_T$. It follows from inequality \(^{(51)}\) that the vector $\{(1 + \delta_T)\beta^*_a(\Delta) : a \in A\}$ is feasible for LP($\hat{\Delta}$). Hence, the optimal value of LP($\hat{\Delta}$) is upper-bounded by $(1 + \delta_T) \sum_{a \in A} \beta^*_a(\Delta) \hat{\Delta}_a$. This completes the proof.
We analyze the regret on the following sets separately: (i) \( F \).

Thus, for times \( \Delta \) is the number of plays of arms calculated by solving the LP (20)-(22) obtained by substituting \( a \in A \), \( \beta(\Delta) \) is the optimal value of the LP (53). We then have,

\[
\mathbb{P}(F) \leq \frac{1}{\log (T)} \quad \mathbb{P}(F') \leq \frac{1}{T}.
\]

D.2: Regret Analysis of Algorithm 2

We analyze the regret on the following sets separately: (i) \( F^c \), (ii) \( F \cap (F')^c \), and (iii) \( F' \).

**Regret Analysis on \( F^c \)**

**Lemma 18.** Algorithm 2 never enters recovery phase on \( F^c \).

**Proof.** It follows from (52) that on \( F^c \) we have the following,

\[
|\mu_a - \hat{\mu}_a(t)| \leq \|a\|^G_{i_a} g^{1/2}(T) \leq \epsilon_T(t), \quad \forall a \in A.
\]

Thus, for times \( s, t \geq d \log^{1/2} T \), we have

\[
|\hat{\mu}_a(s) - \hat{\mu}_a(t)| \leq \epsilon_T(\min\{s, t\}) \leq \epsilon_T(d \log^{1/2} T).
\]

Since recovery phase occurs only when \( |\hat{\mu}_a(t) - \hat{\mu}_a| > 2 \epsilon_T(d \log^{1/2} T) \), where \( \hat{\mu}_a \) is the estimate of \( \mu_a \) at time \( d \log^{1/2} T \), this shows that the algorithm does not enter the recovery phase on \( F^c \). ■

**Lemma 19.** The cumulative regret of Algorithm 2 during the success phase, on the set \( F^c \), can be bounded as follows,

\[
\limsup_{T \to \infty} \mathbb{E} \left[ \mathbb{I}(F^c) \sum_{t \in T_{\text{success}}} \sum_{i \in V} \Delta u_i(t) \right] \leq c(\theta^*, \mathcal{G}, \mathcal{A}), \tag{54}
\]

where \( c(\theta^*, \mathcal{G}, \mathcal{A}) \) is the optimal value of the LP (5) \( 10 \).

**Proof.** Within this proof, we use \( \hat{\Delta} \) and \( \epsilon_T \) in lieu of \( \Delta(t \log^{1/2} T) \) and \( \epsilon_T(d \log^{1/2} T) \). Recall that \( \beta^*(\hat{\Delta}) \) is the number of plays of arms calculated by solving the LP (20)-(22) obtained by substituting the true value of the sub-optimality gaps \( \Delta_a \), \( \beta^*(\Delta) \) satisfies the following,

\[
\limsup_{T \to \infty} \frac{\sum_{a \in A} \beta^*(\hat{\Delta}) \Delta_a}{\log T} = c(\theta^*, \mathcal{G}, \mathcal{A}). \tag{55}
\]
The regret occurred during the success phase satisfies,
\[
\mathbb{1}(F^c) \sum_{t \in T_{\text{succ}}} \sum_{i \in V} \Delta U_i(t) \leq \sum_{a \in A^{(s)}} \beta_a^*(\hat{\Delta}) \Delta_a
\]
\[
= \sum_{a \in A^{(s)}} \beta_a^*(\hat{\Delta}) \hat{\Delta}_a + \sum_{a \in A^{(s)}} \beta_a^*(\hat{\Delta}) \left[ \Delta_a - \hat{\Delta}_a \right]
\]
\[
\leq (1 + \delta_T) \sum_{a \in A^{(s)}} \beta_a^*(\Delta) \Delta_a + 2c_T \sum_{a \in A^{(s)}} \beta_a^*(\hat{\Delta})
\]
\[
\leq (1 + \delta_T) \sum_{a \in A^{(s)}} \beta_a^*(\Delta) \Delta_a + 2c_T (d \log(1/2)) \left[ (1 + \delta_T) \sum_{a \in A^{(s)}} \beta_a^*(\Delta) + \sum_{a \in A^{(s)}} \beta_a^*(\hat{\Delta}) \right],
\] (56)

where the first inequality follows since under Algorithm 2, the number of plays of an arm \(a\) is almost equal to \(\beta_a^*(\hat{\Delta})\), the second inequality follows from (55) and the fact that on \(F^c\) we have \(|\mu_a - \hat{\mu}_a(t)| \leq \epsilon_T\). We now use the results of Lemma 14 and Lemma 16 in the inequality (56), and also choose \(T\) to be sufficiently large enough so as to satisfy \(2\epsilon_T \leq \delta_{\min}/2\), and obtain the following,
\[
\mathbb{1}(F^c) \sum_{t \in T_{\text{succ}}} \sum_{i \in V} \Delta U_i(t) \leq \left(1 + \frac{16\epsilon_T}{\delta_{\min}}\right) \sum_{a \in A^{(s)}} \beta_a^*(\Delta) \Delta_a + 2c_T \left[ 2 + 2d^2 f(T) \frac{\Delta_{\max}}{\Delta_{\min}^3} \right].
\] (57)

We have
\[
\epsilon_T = O\left( \frac{\log^{1/2}(\log T)}{\log^{1/4} T} \right).
\] (58)

We now divide both sides of the inequality (57) by \(\log T\), and substitute (58) in (57) in order to obtain the following,
\[
\limsup_{T \to \infty} \frac{1}{\log T} \mathbb{1}(F^c) \sum_{t \in T_{\text{succ}}} \sum_{i \in V} \Delta U_i(t) \leq \limsup_{T \to \infty} \frac{1}{\log T} \sum_{a \in A^{(s)}} \beta_a^*(\Delta) \Delta_a \leq c(\theta^*, g, A),
\] (59)

where last inequality follows from (55). The proof then follows from Fatou’s lemma [Fol13].

Regret Analysis on \(F \cap (F')^c\):

**Lemma 20.** We have,
\[
\limsup_{T \to \infty} \mathbb{E} \left[ \frac{\mathbb{1}(F \cap (F')^c) \sum_{t \in T_{\text{succ}}} \sum_{i \in V} \Delta U_i(t)}{\log T} \right] = 0.
\] (60)

**Proof.** We omit the proof since it follows closely the proof of Lemma 13 of [LS16].

**Proof.** of Theorem 3. Throughout this proof, we denote the regret of Algorithm 2 by \(R(T)\). Let \(T_{\text{warm}}, T_{\text{succ}}, T_{\text{rec}}\) denote the rounds spent in the warm-up, success, and recovery phases respectively. Total regret is decomposed as follows
\[
\frac{R(T)}{\log T} = \frac{1}{\log T} \mathbb{E} \left( \sum_{t \in T_{\text{warm}}} \sum_{i \in V} \Delta U_i(t) + \sum_{t \in T_{\text{succ}}} \sum_{i \in V} \Delta U_i(t) + \sum_{t \in T_{\text{rec}}} \sum_{i \in V} \Delta U_i(t) \right).
\] (61)

Since the warm-up phase lasts for \(O(\log^{1/2} T)\) rounds, contribution of the first term is asymptotically 0 as \(T \to \infty\).

Next, we analyze the regret during \(T_{\text{rec}}\). It follows from Lemma 17 and Lemma 18 that \(\mathbb{P}(T_{\text{rec}} \neq 0) \leq 1/\log T\). Also, from Theorem 2 we have that if the algorithm does enter the recovery phase, then its regret is upper-bounded as \(O(\log T)\). Combining these two bounds, we have
We finally analyze the regret in the success phase. We further decompose this term as follows,

\[
\limsup_{T \to \infty} \frac{R(T)}{\log T} = \limsup_{T \to \infty} \frac{1}{\log T} \mathbb{E} \left[ \sum_{t \in T_{\text{succ}}} \sum_{i \in V} \Delta U_i(t) \right]. \tag{62}
\]

We finally analyze the regret in the success phase. We further decompose this term as follows,

\[
\frac{1}{\log T} \mathbb{E} \left[ \sum_{t \in T_{\text{succ}}} \sum_{i \in V} \Delta U_i(t) \right] = \frac{1}{\log T} \mathbb{E} \left[ \mathbb{I}(\mathcal{F}^c) \sum_{t \in T_{\text{succ}}} \sum_{i \in V} \Delta U_i(t) \right] + \frac{1}{\log T} \mathbb{E} \left[ \mathbb{I}(\mathcal{F}) \sum_{t \in T_{\text{succ}}} \sum_{i \in V} \Delta U_i(t) \right]. \tag{63}
\]

It follows from Lemma [20] that the second term on the r.h.s. above asymptotically vanishes. Since from Lemma [17] we have \( \mathbb{P}(\mathcal{F}) \leq 1/T \), and moreover the regret occurred on the set \( \mathcal{F}^c \) can be trivially upper-bounded as \( O(T) \), we conclude that \( \mathbb{E} \left[ \mathbb{I}(\mathcal{F}) \sum_{t \in T_{\text{succ}}} \sum_{i \in V} \Delta U_i(t) \right] \) is upper-bounded by a constant that does not depend upon \( T \). Thus, the last term in the r.h.s. above also vanishes asymptotically. Finally, as shown in Lemma [19] the first term in the r.h.s. is asymptotically upper-bounded by \( c(\theta^*, \mathcal{G}, \mathcal{A}) \). Substituting these three bounds into the relation (63) completes the proof.

**Appendix E: Proof of Corollary [1]**

Consider the following linear program

\[
LP_1 : \min_{\{w_a\}_{a \in \mathcal{A}}} \Delta_{\max} \sum_{a \in \mathcal{A}} w_a \tag{64}
\]

s.t. \( n_i(w) \geq \sqrt{\frac{\mathcal{G}}{\Delta_{\min}}} \), \( \forall i \in \mathcal{V} \), \( \text{where } n_i(w) := \sum_{j \in \mathcal{N}_i} \sum_{a : i = j} w_a, i \in \mathcal{V} \),

\[
w_a = 0 \text{ if } a \notin \mathcal{S}, \text{ and } w_a = w_b \forall a, b \in \mathcal{S} \cap \mathcal{A}_a. \tag{67}
\]

It is easily verified that any vector feasible for \( LP_1 \) is also feasible for \( LP \). Moreover, its objective function is also greater than the objective of \( LP \). Thus, its optimal value, denoted \( c(\mathcal{A}, \theta^*, \mathcal{G}) \), is greater than \( c(\mathcal{A}, \theta^*, \mathcal{G}) \). Consider now a scaled version of \( LP_1 \),

\[
LP_{1,s} : \min_{\{w_a\}_{a \in \mathcal{A}}} \sum_{a \in \mathcal{A}} w_a \tag{68}
\]

s.t. \( n_i(w) \geq 1 \), \( \forall i \in \mathcal{V} \), \( \text{where } n_i(w) := \sum_{j \in \mathcal{N}_i} \sum_{a : i = j} w_a, i \in \mathcal{V} \),

\[
w_a = 0 \text{ if } a \notin \mathcal{S}, \text{ and } w_a = w_b \forall a, b \in \mathcal{S} \cap \mathcal{A}_a. \tag{71}
\]

It is evident that if \( x \) is feasible for \( LP_1 \), then \( x \Delta_{\min}/\sqrt{\mathcal{G}} \) is feasible for \( LP_{1,s} \), and if \( y \) is feasible for \( LP_{1,s} \), then \( y \sqrt{\mathcal{G}}/\Delta_{\min} \) is feasible for \( LP_1 \). Thus, if \( c(\mathcal{A}, \theta^*, \mathcal{G}) \) denotes optimal value of \( LP_{1,s} \), then we have \( c(\mathcal{A}, \theta^*, \mathcal{G}) \) \( c(\mathcal{A}, \theta^*, \mathcal{G}) \) \( \sqrt{\mathcal{G}}/\Delta_{\min} \) \( \Delta_{\max} \). The proof is completed by noting that the optimal value of \( LP_{1,s} \) is a lower bound on \( \chi(\mathcal{G}) \).