On approximation of maps into real algebraic homogeneous spaces

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Abstract. Let $X$ be a compact (resp. compact and nonsingular) real algebraic variety and let $Y$ be a homogeneous space for some linear real algebraic group. We prove that a continuous (resp. $C^\infty$) map $f: X \to Y$ can be approximated by regular maps in the $C^0$ (resp. $C^\infty$) topology if and only if it is homotopic to a regular map. Taking $Y = S^p$, the unit $p$-dimensional sphere, we obtain solutions of several problems that have been open since the 1980’s and which concern approximation of maps with values in the unit spheres. This has several consequences for approximation of maps between unit spheres. For example, we prove that for every positive integer $n$ every $C^\infty$ map from $S^n$ into $S^n$ can be approximated by regular maps in the $C^\infty$ topology. Up to now such a result has only been known for five special values of $n$, namely, $n = 1, 2, 3, 4$ or 7.

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1 Introduction

In the present paper, we study approximation of continuous or $C^\infty$ maps by real regular maps, that is, real algebraic morphisms. Our results concern maps with values in homogeneous spaces for linear real algebraic groups. Special attention is paid to maps into unit spheres. The main result, Theorem 1.1, and its consequences provide answers to some approximation problems that have been open since the 1980’s.

Throughout this work, by an algebraic variety we always mean a quasiprojective variety. To be precise, a real algebraic variety is a ringed space with structure sheaf of $\mathbb{R}$-algebras of $\mathbb{R}$-valued functions, which is isomorphic to a Zariski locally closed subset of real projective $n$-space $\mathbb{P}^n(\mathbb{R})$, for some $n$, endowed with the Zariski topology and the sheaf of regular functions. This is compatible with [2], which contains a detailed exposition of real algebraic geometry. We use the analogous definition of a complex algebraic variety, replacing $\mathbb{R}$ by $\mathbb{C}$. Recall that each real algebraic variety in the sense used here is actually affine, that is, isomorphic to an algebraic subset of $\mathbb{R}^n$, for some $n$, see [2] Proposition 3.2.10 and Theorem 3.4.4. Morphisms of algebraic varieties are called regular maps (in some of our references they are called entire rational maps, see [3,4,55,56]).

A real algebraic group is a real algebraic variety $G$ endowed with the structure of a group such that the group operations $G \times G \to G$, $(a, b) \mapsto ab$, and $G \to G$, $a \mapsto a^{-1}$ are regular maps. Morphisms of real algebraic groups are regular maps that are group homomorphisms. A real algebraic group is said to be linear if it is isomorphic to a Zariski
closed subgroup of the general linear group $\text{GL}_n(\mathbb{R})$, for some $n$. Obviously, the familiar subgroups of $\text{GL}_n(\mathbb{R})$, the orthogonal group $O(n)$ and special orthogonal group $\text{SO}(n)$, are linear real algebraic groups. A complex algebraic group, linear or not, is defined analogously, replacing $\mathbb{R}$ by $\mathbb{C}$. Clearly, each real or complex algebraic group $G$ is a nonsingular algebraic variety of pure dimension. Moreover, if $G$ is linear, then each Zariski closed subgroup of $G$ is a linear algebraic group.

Any complex algebraic variety $V$ carries the obvious underlying structure $\mathbb{R}V$ of a real algebraic variety, called the realification of $V$ (for example, $\mathbb{R}(\mathbb{C}^n) = \mathbb{R}^{2n}$). If $G$ is a complex algebraic group, then its realification $\mathbb{R}G$ is a real algebraic group with the same group operations as in $G$. If $H$ is a Zariski closed subgroup of $G$, then $\mathbb{R}H$ is a Zariski closed subgroup of $\mathbb{R}G$. Moreover, if the group $G$ is linear, then so is the group $\mathbb{R}G$; indeed, it suffices to note that the real algebraic group $\mathbb{R}\text{GL}_n(\mathbb{C})$ is isomorphic to the image of the real regular embedding

$$\left. \mathbb{R}\text{GL}_n(\mathbb{C}) \to \text{GL}_{2n}(\mathbb{R}), \quad A + \sqrt{-1} B \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \right.$$ 

where $A, B$ are real $n$-by-$n$ matrices. Consequently, each Zariski closed subgroup of $\mathbb{R}\text{GL}_n(\mathbb{C})$ is a linear real algebraic group. In particular, the unitary group $U(n)$ and special unitary group $\text{SU}(n)$ are linear real algebraic groups, being Zariski closed subgroups of $\mathbb{R}\text{GL}_n(\mathbb{C})$.

The general linear group $\text{GL}_n(\mathbb{H})$, where $\mathbb{H}$ is the (skew) field of quaternions, can also be viewed as a linear real algebraic group via the standard embedding into $\text{GL}_{4n}(\mathbb{R})$. Therefore the symplectic subgroup $\text{Sp}(n)$ of $\text{GL}_n(\mathbb{H})$ is a linear real algebraic group.

Let $G$ be a real algebraic group. A $G$-space (or a $G$-variety) is a real algebraic variety $Y$ on which $G$ acts, the action $G \times Y \to Y$, $(a, y) \mapsto a \cdot y$ being a regular map. A homogeneous space for $G$ is a $G$-space on which $G$ acts transitively. Note that each homogeneous space for $G$ is a nonsingular real algebraic variety of pure dimension.

Besides the Zariski topology, every real algebraic variety is endowed with the Euclidean topology determined by the standard metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties will refer to the Euclidean topology.

Given real algebraic varieties $X$ and $Y$, we denote by $\mathcal{R}(X, Y)$ the set of all regular maps from $X$ into $Y$. We regard $\mathcal{R}(X, Y)$ as a subset of the space $C^0(X, Y)$ of all continuous maps endowed with the $C^0$ (that is, compact-open) topology. We say that a continuous map $f: X \to Y$ can be approximated by regular maps in the $C^0$ topology if, for every neighborhood $\mathcal{U}$ of $f$ in $C^0(X, Y)$, there is a regular map $g: X \to Y$ which belongs to $\mathcal{U}$. If both varieties $X$ and $Y$ are nonsingular, then $\mathcal{R}(X, Y)$ is a subset of the space $C^\infty(X, Y)$ of all $C^\infty$ maps endowed with the $C^\infty$ topology (see [33], p. 36] or [64], p. 311] for the definition of this topology and note that in [33] it is called the weak $C^\infty$ topology); therefore the concept of approximation of a $C^\infty$ map $f: X \to Y$ by regular maps in the $C^\infty$ topology is well-defined.

Our main result is the following.

**Theorem 1.1.** Let $X$ be a compact (resp. compact and nonsingular) real algebraic variety and let $Y$ be a homogeneous space for some linear real algebraic group. Then, for a continuous (resp. $C^\infty$) map $f: X \to Y$, the following conditions are equivalent:

(a) $f$ can be approximated by regular maps in the $C^0$ (resp. $C^\infty$) topology.

(b) $f$ is homotopic to a regular map.
The implication \( (a) \Rightarrow (b) \) is obvious because two continuous maps that are sufficiently close in the \( C^0 \) topology are homotopic. The proof of \( (b) \Rightarrow (a) \) is given in Section 4 and depends on the techniques developed in Sections 2 and 3. Some special cases of Theorem 1.1 have been anticipated since the 1980’s, however, the results obtained heretofore have been very incomplete.

As immediate consequences of Theorem 1.1 we get the following two corollaries.

**Corollary 1.2.** Let \( X \) be a compact (resp. compact and nonsingular) real algebraic variety and let \( Y \) be a homogeneous space for some linear real algebraic group. Then every continuous (resp. \( C^\infty \)) null homotopic map from \( X \) to \( Y \) can be approximated by regular maps in the \( C^0 \) (resp. \( C^\infty \)) topology.

**Corollary 1.3.** Let \( Y \) be a homogeneous space for some linear real algebraic group. If \( Y \) is compact, then every continuous (resp. \( C^\infty \)) map \( Y \to Y \) that is homotopic to the identity map can be approximated by regular maps in the \( C^0 \) (resp. \( C^\infty \)) topology.

It should be mentioned that no direct proofs of these corollaries, which do not make use of Theorem 1.1 are available.

In the rest of this section, we discuss the impact of Theorem 1.1 within the framework described in the following example.

**Example 1.4.** Here are some homogeneous spaces of interest.

(i) For every nonnegative integer \( n \) the unit \( n \)-sphere

\[ S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\} \]

is a homogeneous space for the real orthogonal group \( O(n+1) \).

(ii) Let \( F \) stand for one of the fields \( \mathbb{R} \), \( \mathbb{C} \) or \( \mathbb{H} \). Denote by \( G_r(F^n) \) the Grassmannian of \( r \)-dimensional \( F \)-vector subspaces of \( F^n \), regarded as a real algebraic variety (see [2, pp. 72, 73, 352]). Clearly, \( G_r(F^n) \) is a homogeneous space for the group \( GL_n(F) \) viewed as a linear real algebraic group.

(iii) Denote by \( V_r(F^n) \) the Stiefel manifold of all orthonormal \( r \)-frames in \( F^n \), regarded as a real algebraic variety. Clearly, \( V_r(F^n) \) is a homogeneous space for the linear real algebraic group \( O(n) \) if \( F = \mathbb{R} \), \( U(n) \) if \( F = \mathbb{C} \), and \( Sp(n) \) if \( F = \mathbb{H} \).

(iv) Every real algebraic group \( G \) is a homogeneous space for \( G \) under action by left translations.

In view of Theorem 1.1 and Example 1.4(i), we get at once the following result on maps into the unit \( p \)-sphere \( S^p \).

**Corollary 1.5.** Let \( X \) be a compact (resp. compact and nonsingular) real algebraic variety and let \( p \) be a positive integer. Then, for a continuous (resp. \( C^\infty \)) map \( f : X \to S^p \), the following conditions are equivalent:

(a) \( f \) can be approximated by regular maps in the \( C^0 \) (resp. \( C^\infty \)) topology.

(b) \( f \) is homotopic to a regular map.
Until now Corollary 1.5 with dim \( X \geq p \) has only been known for a few special values of \( p \), namely, for \( p = 1, 2 \) or \( 4 \) (see [3]) and \( p = 3 \) or \( 7 \) (see [1]). Of course, if dim \( X < p \), then the set of regular maps \( \mathcal{R}(X, \mathbb{S}^p) \) is dense in the space \( C^0(X, \mathbb{S}^p) \) (resp. \( C^\infty(X, \mathbb{S}^p) \)). Indeed, \( \mathbb{S}^p \) with one point removed is biregularly isomorphic to \( \mathbb{R}^p \) via the stereographic projection. Moreover, for dim \( X < p \), every continuous (resp. \( C^\infty \)) map from \( X \) into \( \mathbb{S}^p \) can be approximated in the space \( C^0(X, \mathbb{S}^p) \) (resp. \( C^\infty(X, \mathbb{S}^p) \)) by maps that are not surjective, and hence the density assertion follows from the Weierstrass approximation theorem.

The question which continuous or \( C^\infty \) maps are homotopic to regular ones has been investigated in numerous works. By applying Theorem 1.1 or Corollary 1.5, some of these results can now be translated into the results on approximation by regular maps. Next we give the first example of this principle.

**Theorem 1.6.** For every positive integer \( n \) the set of regular maps \( \mathcal{R}(\mathbb{S}^n, \mathbb{S}^n) \) is dense in the space of \( C^\infty \) maps \( C^\infty(\mathbb{S}^n, \mathbb{S}^n) \).

**Proof.** Since each \( C^\infty \) map \( \mathbb{S}^n \to \mathbb{S}^n \) is homotopic to a regular one (see [65, Theorem 1] for \( n \) odd, and [4, Corollary 4.2] for \( n \) arbitrary), the conclusion follows at once from Corollary 1.5.

Up to now Theorem 1.6 has only been known for \( n = 1, 2 \) or \( 4 \) (see [3]) and \( n = 3 \) or \( 7 \) (see [1]).

The following conjecture, although plausible, is wide open.

**Conjecture I.** Let \( Y \) be a homogeneous space for some linear real algebraic group. Then, for every positive integer \( n \), the set of regular maps \( \mathcal{R}(\mathbb{S}^n, Y) \) is dense in the space of \( C^\infty \) maps \( C^\infty(\mathbb{S}^n, Y) \). In particular, \( \mathcal{R}(\mathbb{S}^n, \mathbb{S}^p) \) is dense in \( C^\infty(\mathbb{S}^n, \mathbb{S}^p) \) for each pair \((n, p)\) of positive integers.

By [3, Corollary 2.7] (see also [59, Theorems 2.2 and 11.1]), Conjecture I is valid for the homogeneous space \( Y = G_s(\mathbb{F}^d) \) defined in Example 1.4(ii). Recall that \( G_s(\mathbb{F}^d) \) is biregularly isomorphic to the unit sphere \( S^{d-1} \), \( d(\mathbb{F}) = \dim \mathbb{F} \). In particular, Conjecture I holds if \( Y \) is one of the spheres \( S^1, S^2 \) or \( S^4 \). Presently Conjecture I for maps between unit spheres remains open, but it is supported further by Theorem 5.6.

In Section 6 we obtain some results supporting Conjecture I for maps with values in the classical linear real algebraic groups \( O(m), SO(m), U(m), SU(m), Sp(m) \) or in the Stiefel manifold \( V_s(\mathbb{F}^m) \) (see Propositions 6.1 and 6.3 and Theorem 6.2).

In general, the equivalent conditions (a) and (b) in Corollary 1.5 are not satisfied. Indeed, suppose given a compact nonsingular real algebraic variety \( X \) such that every regular map from \( X \) into \( \mathbb{S}^p \) is null homotopic. Then, in view of Corollary 1.5, a \( C^\infty \) map \( f: X \to \mathbb{S}^p \) can be approximated by regular maps if and only if it is null homotopic. Of course, assuming that \( \dim X = p \), there are always \( C^\infty \) maps from \( X \) into \( \mathbb{S}^p \) that are not null homotopic. By [5, Theorem 2.1], for “most” orientable nonsingular algebraic hypersurfaces \( X \) in \( \mathbb{P}^{2n+1}(\mathbb{R}) \), every regular map from \( X \) into \( \mathbb{S}^{2n} \) is null homotopic. Moreover, according to [11, Theorems 1.2 and 7.3], for “most” nonsingular real cubic curves \( C \) in \( \mathbb{P}^2(\mathbb{R}) \), every regular map from \( C^{2n} \) into \( \mathbb{S}^{2n} \) is null homotopic. For real cubic curves one can also substitute other real algebraic curves, see [19, Theorem 1.3] and [12, Corollary 2]. We explicitly state the following.

**Theorem 1.7.** Let \( X \) be a compact connected nonsingular real algebraic variety of odd dimension \( k < 2n \). Then a \( C^\infty \) map \( f: X \times \mathbb{S}^{2n-k} \to \mathbb{S}^{2n} \) can be approximated by regular maps in the \( C^\infty \) topology if and only if it is null homotopic.
Proof. By [5, Theorem 2.4], each regular map from $X \times S^{2n-k}$ into $S^{2n}$ is null homotopic, and hence the conclusion follows from Corollary 1.5.

The behavior of regular maps into odd-dimensional spheres is entirely different.

**Theorem 1.8.** Let $X$ be a compact connected oriented nonsingular real algebraic variety of odd dimension $k$. Then either

(i) the set $R(X, S^k)$ is dense in the space $C^\infty(X, S^k)$, or

(ii) the closure of $R(X, S^k)$ in $C^\infty(X, S^k)$ coincides with the set

$$\{ f \in C^\infty(X, S^k) : \deg(f) \in 2\mathbb{Z} \},$$

where $\deg(f)$ is the topological degree of the map $f$.

Proof. By [4, Corollary 2.5], the set

$$\{ m \in \mathbb{Z} : m = \deg(g), \ g \in R(X, S^k) \}$$

is equal either to $\mathbb{Z}$ or $2\mathbb{Z}$, so the statement follows by combining Hopf’s theorem with Corollary 1.5.

No example of an odd-dimensional variety $X$ for which condition (ii) in Theorem 1.8 holds is known, which raises the possibility that the following conjecture might be true.

**Conjecture II.** Let $X$ be a compact connected nonsingular real algebraic variety and let $k$ be a positive odd integer. If $\dim X = k$, then the set of regular maps $R(X, S^k)$ is dense in the space of $C^\infty$ maps $C^\infty(X, S^k)$.

This conjecture is known to be true for $k = 1$ (see [3, Corollary 1.5]), but remains open for all other odd values of $k$. Example 5.8 shows that the assumption $\dim X = k$ in Conjecture II cannot be replaced by $\dim X \geq k$.

According to the celebrated Nash–Tognoli theorem, every compact $C^\infty$ manifold $M$ is diffeomorphic to a nonsingular real algebraic variety $X$, called an algebraic model of $M$, see [54,61] and [2, Theorem 14.1.10]. Actually, $M$ admits an uncountable family of pairwise nonisomorphic algebraic models, provided $\dim M \geq 1$, see [8]. The behavior of regular maps from $X$ into unit spheres, as $X$ runs through the class of algebraic models of $M$, has been investigated in [4–7,11,41]. The following is a new result in this direction.

**Theorem 1.9.** Let $M$ be a compact connected $C^\infty$ manifold of dimension $m$. Then there exists an algebraic model $X$ of $M$ such that the set of regular maps $R(X, S^m)$ is dense in the space of $C^\infty$ maps $C^\infty(X, S^m)$.

Proof. By [4, Proposition 4.5], there exists an algebraic model $X$ of $M$ such that each $C^\infty$ map from $X$ into $S^m$ is homotopic to a regular map. Hence the conclusion follows from Corollary 1.5.

Recall that a compact oriented $C^\infty$ manifold is said to be an oriented boundary if it is the boundary, with the induced orientation, of a compact oriented $C^\infty$ manifold with boundary.

**Theorem 1.10.** Let $M$ be a compact connected oriented $C^\infty$ manifold of dimension $m$. Assume that either
(i) \( m \equiv 2 \pmod{4} \), or
(ii) \( m \equiv 0 \pmod{4} \) and the disjoint union of two copies of \( M \) is an oriented boundary.

Then there exists an algebraic model \( X \) of \( M \) such that each regular map from \( X \) into \( S^m \) is null homotopic. Moreover, a \( C^\infty \) map \( f : X \to S^m \) can be approximated by regular maps in the \( C^\infty \) topology if and only if it is null homotopic.

**Proof.** The first assertion is proved in [41, Corollary 2.3], and the second follows from Corollary 1.5. \( \Box \)

Let us comment on the boundary assumption in (ii). For each integer \( k \geq 4 \) there exists a compact connected oriented \( C^\infty \) manifold of dimension \( 4k \), which is not an oriented boundary, but the disjoint union of its two copies is; there is no such a manifold in dimensions 4, 8 and 12. This statement follows from the description of the torsion elements in the oriented cobordism ring, see [63, p. 309].

We believe that in Theorem 1.10 (with \( m \equiv 0 \pmod{4} \)) the boundary condition is not only sufficient, but also necessary.

**Conjecture III.** Let \( M \) be a compact connected oriented \( C^\infty \) manifold of dimension \( 4k \). Assume that there exists an algebraic model \( X \) of \( M \) such that each regular map from \( X \) into \( S^{4k} \) is null homotopic. Then the disjoint union of two copies of \( M \) is an oriented boundary.

This conjecture is known to be true if \( \dim M = 4 \), see [5, Theorem 3]. Recall that if \( \dim M = 4 \), then \( M \) is an oriented boundary exactly when its signature \( \sigma(M) \) is zero. If \( \sigma(M) \neq 0 \) and \( X \) is an algebraic model of \( M \), then each \( C^\infty \) map \( f : X \to S^4 \) with \( \deg(f) \) divisible by \( 6\sigma(M) \) can be approximated by regular maps in the \( C^\infty \) topology, see [5, Theorem 5.6].

Along with approximation by regular maps, investigated further in [2, 3, 11, 13, 16, 18, 19, 36, 39, 48, 52, 40, 43, 45, 47, 48, 56, 50], one can also consider approximation by stratified-regular (= regulous = continuous hereditarily rational = continuous rational, the last equality holds assuming nonsingularity of varieties) maps or by piecewise-regular maps [46]. There are remarkable similarities and differences in these three cases.

We have already indicated how the paper is organized. It should be added that the inspiration for Sections 2 and 3, in which we develop the main technical tools, comes from complex geometry, above all Gromov’s paper [32] and the related works of Forstnerič and others, see [28] and the references therein. As an analog for Gromov’s key notion of spray, we introduce the concept of partial spray in the real case.

## 2 Malleable real algebraic varieties

The notions and results of this section will be generalized in the next one. Nevertheless, it seems to be beneficial to discuss first the key special case.

In what follows we work with vector bundles, which are always \( \mathbb{R} \)-vector bundles. Let \( Y \) be a real algebraic variety. Given a vector bundle \( p : E \to Y \) over \( Y \), with total space \( E \) and bundle projection \( p \), we sometimes refer to \( E \) as a vector bundle over \( Y \). If \( y \) is a point in \( Y \), we let \( E_y := p^{-1}(y) \) denote the fiber of \( E \) over \( y \) and write \( 0_y \) for the zero vector in \( E_y \). As usual, we call the set \( Z(E) := \{ 0_y \in E_y : y \in Y \} \) the zero section of \( E \).

The general theory of algebraic vector bundles over real algebraic varieties is discussed in [2, Section 12.1]. For each algebraic vector bundle \( E \) over \( Y \) there exists an algebraic
vector bundle $E'$ over $Y$ such that the direct sum $E \oplus E'$ is algebraically trivial, see [2 Theorem 12.1.7]. Moreover, if $E_1$ is an algebraic vector subbundle of $E$, then there exists an algebraic vector subbundle $E_2$ of $E$ such that $E_1 \oplus E_2 = E$. This is the case since $E$ can be regarded as an algebraic vector subbundle of the product vector bundle $Y \times \mathbb{R}^n$, for some $n$, and hence as $E_2$ one can take the orthogonal complement (with respect to the standard scalar product on $\mathbb{R}^n$) of $E_1$ in $E$.

Assuming that $Y$ is a nonsingular real algebraic variety, we write $TY$ for the tangent bundle to $Y$ and $T_y Y$ for the tangent space to $Y$ at $y \in Y$.

**Definition 2.1.** Let $Y$ be a nonsingular real algebraic variety.

(i) A partial spray for $Y$ is a quadruple $(E, p, E^0, s)$, where $p: E \to Y$ is an algebraic vector bundle over $Y$ and $s: E^0 \to Y$ is a regular map defined on a Zariski open neighborhood $E^0 \subseteq E$ of the zero section $Z(E)$ of $E$ such that $s(0_y) = y$ for all $y \in Y$. (We write “partial” to emphasize that $s$ may be defined on a proper subset $E^0$ of $E$.)

(ii) A partial spray $(E, p, E^0, s)$ for $Y$ is said to be dominating if the derivative

$$d_{0_y} s: T_{0_y} E \to T_y Y$$

maps the subspace $E_y = T_{0_y} E_y$ of $T_{0_y} E$ onto $T_y Y$, that is,

$$d_{0_y} s(E_y) = T_y Y$$

for all $y \in Y$.

(iii) The variety $Y$ is called malleable if it admits a dominating partial spray.

The simplest malleable variety is $Y = \mathbb{R}^n$ which admits a dominating partial spray $(E, p, E^0, s)$, where $p: E := Y \times \mathbb{R}^n \to Y$ is the product vector bundle, $E^0 = E$, and $s: E^0 \to Y$ is defined by $s(y, v) = y + v$ for all $(y, v) \in E^0$. More substantial and useful examples are provided by Examples 2.5 and 2.6 and above all by Proposition 2.8.

In the following two lemmas we describe some basic properties of dominating partial sprays.

**Lemma 2.2.** Any malleable nonsingular real algebraic variety $Y$ admits a dominating partial spray $(E, p, E^0, s)$ such that $p: E \to Y$ is an algebraically trivial vector bundle.

**Proof.** Suppose that $(\tilde{E}, \tilde{p}, \tilde{E}^0, \tilde{s})$ is a dominating partial spray for $Y$. Choose an algebraic vector bundle $\tilde{p}': \tilde{E}' \to Y$ such that the direct sum $p: E := \tilde{E} \oplus \tilde{E}' \to Y$ is an algebraically trivial vector bundle. Setting

$$E^0 := \{(v, v') \in \tilde{E} \oplus \tilde{E}' : v \in \tilde{E}^0\}$$

and defining $s: E^0 \to Y$ by $s(v, v') = \tilde{s}(v)$, we get a dominating partial spray $(E, p, E^0, s)$ for $Y$. \hfill \Box

**Lemma 2.3.** Let $Y$ be a malleable nonsingular real algebraic variety and let $(E, p, E^0, s)$ be a dominating partial spray for $Y$. Then there exists a dominating partial spray $(\hat{E}, \hat{p}, \hat{E}^0, \hat{s})$ for $Y$ such that $\hat{p}: \hat{E} \to Y$ is an algebraic vector subbundle of $p: E \to Y$, $E^0 = \hat{E} \cap E^0$, $\hat{s} = s|_{E^0}$, and the restriction

$$d_{0_y} \hat{s}|_{E^0}: \hat{E}_y \to T_y Y$$

of the derivative $d_{0_y} \hat{s}: T_{0_y} \hat{E} \to T_y Y$ is a linear isomorphism for all $y \in Y$. 

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Proof. Let $\alpha: E \to TY$ be the morphism of algebraic vector bundles over $Y$ defined by
$$\alpha(v) = d_{p(v)}s(v) \quad \text{for all } v \in E.$$ 
Since $\alpha$ is a surjective morphism, its kernel $\mathrm{Ker} \alpha$ is an algebraic vector subbundle of $E$. It follows that $E = \hat{E} \oplus \mathrm{Ker} \alpha$ is the direct sum for some algebraic vector subbundle $\hat{p}: \hat{E} \to Y$ of $p: E \to Y$. This completes the proof. \qed

As a consequence of Lemma 2.3, we obtain the following.

**Proposition 2.4.** Any malleable nonsingular real algebraic variety $Y$ admits a dominating partial spray of the form $(TY, \pi, (TY)^0, \sigma)$, where $\pi: TY \to Y$ is the tangent bundle to $Y$.

**Proof.** With notation as in Lemma 2.3, the vector bundle $\hat{E}$ is isomorphic to the tangent bundle $TY$, which completes the proof. \qed

Next we identify some malleable real algebraic varieties.

**Example 2.5.** Let $G$ denote either the real orthogonal group $\text{O}(n)$ or the real special orthogonal group $\text{SO}(n)$. Then each homogeneous space $Y$ for $G$ is malleable. This assertion can be established by direct computation as follows.

Let $\text{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ be the $\mathbb{R}$-algebra of all $n$-by-$n$ matrices with real entries. Let 0 and 1 stand for the zero matrix and the identity matrix in $\text{Mat}_n(\mathbb{R})$. Using the determinant function $\det$, we obtain a Zariski open neighborhood
$$\Omega := \{ x \in \text{Mat}_n(\mathbb{R}) : \det(1 + x) \neq 0 \}$$
of the zero matrix. We identify the tangent space to $G$ at 1 with the vector subspace $V$ of $\text{Mat}_n(\mathbb{R})$ comprising all skew symmetric matrices; $V \cong \mathbb{R}^d$ where $d = n(n - 1)/2$. Setting $U = V \cap \Omega$ it readily follows that
$$\varphi: U \to G \cap \Omega, \quad v \mapsto (1 - v)(1 + v)^{-1}$$
is a well-defined biregular isomorphism with $\varphi(0) = 1$. We get a dominating partial spray $(E, p, E^0, s)$ for $Y$, where $p: E := Y \times V \to Y$ is the product vector bundle, $E^0 = Y \times U$, and $s: E^0 \to Y$ is defined by $s(y, v) = \varphi(v) \cdot y$ for all $(y, v) \in E^0$.

In particular, for unit spheres we get the following.

**Example 2.6.** For every positive integer $n$ the unit sphere $S^n$ is a malleable real algebraic variety. Indeed, this is the case by Example 2.5 because $S^n$ is a homogeneous space for the real orthogonal group $\text{O}(n + 1)$. Thus, in view of Proposition 2.4 there is a dominating partial spray $(T S^n, \pi, (T S^n)^0, \sigma)$ for $S^n$ in which $\pi: T S^n \to S^n$ is the tangent bundle to $S^n$, however, we do not know how to describe explicitly $(T S^n)^0$ and $\sigma$.

**Question 2.7.** Let $n$ be a positive integer. Is there a dominating partial spray $(E, p, E^0, s)$ for $S^n$ with $E^0 = E$?

For the proof of Theorem 1.1 it is essential to generalize Example 2.5. First we recall some terminology and notation which is compatible with [20, 25, 58]. Let $V$ be an algebraic $\mathbb{R}$-variety, that is, a complex algebraic variety defined over $\mathbb{R}$ (equivalently, $V$ can be viewed as a reduced quasiprojective scheme over $\mathbb{R}$). The set of real points of $V$ will be denoted by $V(\mathbb{R})$. We emphasize the distinction between algebraic $\mathbb{R}$-varieties and real algebraic varieties. An algebraic $\mathbb{R}$-group is an algebraic $\mathbb{R}$-variety $\Gamma$ endowed with the structure
of a group such that the identity element of \( \Gamma \) is in \( \Gamma(\mathbb{R}) \), and the group operations are regular maps defined over \( \mathbb{R} \). An algebraic \( \mathbb{R} \)-group is said to be linear if, for some positive integer \( n \), it is isomorphic over \( \mathbb{R} \) to a Zariski closed subgroup of \( \text{GL}_n(\mathbb{C}) \) defined over \( \mathbb{R} \). If \( \Gamma \) is an algebraic \( \mathbb{R} \)-group, then \( \Gamma(\mathbb{R}) \) is a real algebraic group. Moreover, if \( \Gamma \) is a linear algebraic \( \mathbb{R} \)-group, then \( \Gamma(\mathbb{R}) \) is a linear real algebraic group. Conversely, if each linear real algebraic group \( G \) is up to isomorphism of the form \( \Gamma(\mathbb{R}) \) for some linear algebraic \( \mathbb{R} \)-group \( \Gamma \). Indeed, we may assume that \( G \) is a Zariski closed subgroup of \( \text{GL}_n(\mathbb{R}) \) and take as \( \Gamma \) the Zariski closure of \( G \) in \( \text{GL}_n(\mathbb{C}) \), see \([58, \text{Lemma 2.2.4}]\). In view of this discussion we can make use of Chevalley’s paper \([25]\) in the proof of Proposition 2.8 below.

**Proposition 2.8.** Let \( G \) be a linear real algebraic group. Then each homogeneous space \( Y \) for \( G \) is a malleable variety.

**Proof.** Let \( d = \text{dim} \ G \) and let \( G^0 \) be the irreducible component of the variety \( G \) that contains the identity element 1 of \( G \). By a theorem of Chevalley \([25, \text{Theorem 2}]\) or \([20, \text{Corollary 7.12}]\), \( G^0 \) is a unirational variety, and hence there exist a Zariski open subset \( U \) of \( \mathbb{R}^d \) and a regular map \( \psi: U \to G^0 \) such that the image \( \psi(U) \) is Zariski dense in \( G^0 \). Clearly, at some point \( a \in U \) the derivative \( d_a \psi: \mathbb{R}^d \to T_{\psi(a)}G^0 \) is a linear isomorphism. Using a translation in \( \mathbb{R}^d \) we may assume that \( a = 0 \in U \). The map \( \varphi: U \to G, \ v \mapsto \psi(v)\psi(0)^{-1} \) is regular, \( \varphi(0) = 1 \), and the derivative of \( \varphi \) at 0 is a linear isomorphism. Now we obtain a dominating partial spray \((E,p,E^0,s)\) for \( Y \), where \( p: E := Y \times \mathbb{R}^d \to Y \) is the product vector bundle, \( E^0 = Y \times U \), and \( s: E^0 \to Y \) is defined by \( s(y,v) = \varphi(v) \cdot y \) for all \((y,v) \in E^0\). \( \square \)

# 3 Malleable submersions

This section is devoted to developing techniques used in the proof of Theorem 1.1. Of independent interest are Theorem 3.9 and Corollary 3.10.

**Notation 3.1.** Throughout the present section \( X, Z \) are nonsingular real algebraic varieties, and \( h: Z \to X \) is a regular map which is surjective and submersive. Furthermore, \( V(h) \) denotes the algebraic vector subbundle of the tangent bundle \( TZ \) to \( Z \) defined by

\[
V(h)_z = \text{Ker}(d_z h : T_z Z \to T_{h(z)} X)
\]

for all \( z \in Z \).

Observe that \( V(h)_z \) is the tangent space to the fiber \( h^{-1}(h(z)) \).

Let \( U \) be an open subset of \( X \). A map \( f: U \to Z \) is called a section of \( h: Z \to X \) if \( h(f(x)) = x \) for all \( x \in U \). A section which is a \( \mathbb{R} \)-linear map defined over \( \mathbb{R} \) is called a regular map. A map that is a \( \mathbb{R} \)-linear map is called a regular map. Furthermore, \( V(h) \) is a malleable variety.

**Definition 3.2.** Let \( h: Z \to X \) be the submersion of Notation 3.1.
(i) A partial spray for \( h: Z \to X \) is a quadruple \((E, p, E^0, s)\), where \( p: E \to Z \) is an algebraic vector bundle over \( Z \) and \( s: E^0 \to Z \) is a regular map defined on a Zariski open neighborhood \( E^0 \subseteq E \) of the zero section \( Z(E) \) of \( E \) such that 

\[
s(E_z \cap E^0) \subseteq h^{-1}(h(z)) \quad \text{and} \quad s(0_z) = z \quad \text{for all} \ z \in Z.
\]

(ii) A partial spray \((E, p, E^0, s)\) for \( h: Z \to X \) is said to be dominating if the derivative \( d_0s: T_0_z E \to T_z Z \) maps the subspace \( E_z = T_0_z E_z \) of \( T_0_z E \) onto \( V(h)_z \), that is,

\[
d_0s(E_z) = V(h)_z
\]

for all \( z \in Z \).

(iii) The submersion \( h: Z \to X \) is called malleable if it admits a dominating partial spray.

Significant examples of malleable submersions are provided by combining Proposition 2.8 and Lemma 3.2. Note that if \( X \) is reduced to a point, then Definition 3.2 coincides with Definition 2.1.

The next two lemmas are generalizations of Lemmas 2.2 and 2.3.

**Lemma 3.3.** If the submersion \( h: Z \to X \) is malleable, then it admits a dominating partial spray \((E, p, E^0, s)\) such that \( p: E \to Z \) is an algebraically trivial vector bundle.

**Proof.** Suppose that \((\tilde{E}, \tilde{p}, \tilde{E}^0, \tilde{s})\) is a dominating partial spray for \( h: Z \to X \). Choose an algebraic vector bundle \( \tilde{p}' : \tilde{E}' \to Z \) such that the direct sum \( p : E := \tilde{E} \oplus \tilde{E}' \to Z \) is an algebraically trivial vector bundle. Setting

\[
E^0 := \{(v, v') \in \tilde{E} \oplus \tilde{E}' : v \in \tilde{E}^0\}
\]

and defining \( s : E^0 \to Z \) by \( s(v, v') = \tilde{s}(v) \), we get a dominating partial spray \((E, p, E^0, s)\) for \( h: Z \to X \). \(\square\)

**Lemma 3.4.** Suppose that \((E, p, E^0, s)\) is a dominating partial spray for the submersion \( h: Z \to X \). Then there exists a dominating partial spray \((\tilde{E}, \tilde{p}, \tilde{E}^0, \tilde{s})\) for \( h: Z \to X \) such that \( \tilde{p} : \tilde{E} \to Z \) is an algebraic vector subbundle of \( p : E \to Z \), \( E^0 = \tilde{E} \cap E^0 \), \( \tilde{s} = s|_{E^0} \), and the restriction

\[
d_0s|_{E_z} : \tilde{E}_z \to V(h)_z
\]

of the derivative \( d_0s : T_0_z E \to T_z Z \) is a linear isomorphism for all \( z \in Z \).

**Proof.** Let \( \alpha : E \to V(h) \) be a morphism of algebraic vector bundles over \( Z \) defined by

\[
\alpha(v) = d_{0(p(v))} s(v) \quad \text{for all} \ v \in E.
\]

Since \( \alpha \) is a surjective morphism, its kernel \( \text{Ker} \alpha \) is an algebraic vector subbundle of \( E \). It follows that \( E = \tilde{E} \oplus \text{Ker} \alpha \) is the direct sum for some algebraic vector subbundle \( \tilde{p} : \tilde{E} \to Z \) of \( p : E \to Z \). This completes the proof. \(\square\)

**Notation 3.5.** Suppose that \((E, p, E^0, s)\) is a dominating partial spray for the submersion \( h: Z \to X \). Let \( U \) be an open subset of \( X \) and let \( f : U \to Z \) be a \( C^\infty \) section of \( h: Z \to X \). Denote by \( p_f : E_f \to U \) the pullback of the vector bundle \( p : E \to Z \) under the map \( f : U \to Z \). Recall that \( p_f : E_f \to U \) is a \( C^\infty \) vector bundle over \( U \), where

\[
E_f := \{(x, v) \in U \times E : f(x) = p(v)\}, \quad p_f(x, v) = x.
\]

We define

\[
E_f^0 := \{(x, v) \in E_f : v \in E^0\}
\]

\( s_f : E_f^0 \to Z \), \( s_f(x, v) = s(v) \).

Clearly, \( E_f^0 \) is an open neighborhood of the zero section \( Z(E_f) \) of \( E_f \).
Lemma 3.6. Using Notation 3.3, assume that

\[ d_0s|_{E_z}: E_z \to V(h)_z \]

is a linear isomorphism for all \( z \in Z \). Then \( s_f: E_f^0 \to Z \) maps diffeomorphically an open neighborhood of the zero section \( Z(E_f) \) in \( E_f^0 \) onto an open neighborhood of \( f(U) \) in \( Z \).

**Proof.** Let \( x \in U \). The zero vector in the fiber \((E_f)_x \) is \((x, 0_{f(x)})\), where \( 0_{f(x)} \) is the zero vector in the fiber \( E_{f(x)} \). Since \( s_f(x, 0_{f(x)}) = s(0_{f(x)}) = f(x) \), it follows that \( s_f \) induces a diffeomorphism between \( Z(E_f) \) and \( f(U) \). Moreover, the derivative

\[ d(x, 0_{f(x)})s_f: T(x, 0_{f(x)})E_f \to T_{f(x)}Z \]

is an isomorphism because

\[ d_0s|_{E_z}: E_z \to V(h)_z \]

is an isomorphism for all \( z \in Z \). Consequently, \( s_f \) is a local diffeomorphism at the point \((x, 0_{f(x)})\). Now the lemma follows from \[24\] (12.7).

Lemma 3.7. Suppose that \((E, p, E^0, s)\) is a dominating partial spray for the submersion \( h: Z \to X \). Let \( U \) be an open subset of \( X \) and let \( F: U \times [0, 1] \to Z \) be a homotopy of \( C^\infty \) sections of \( h: Z \to X \). Let \( U_0 \) be an open subset of \( X \) whose closure \( \overline{U_0} \) is compact and contained in \( U \). Let \( t_0 \) be a point in \([0, 1]\). Then there exist a neighborhood \( I_0 \) of \( t_0 \) in \([0, 1]\) and a continuous map \( \xi: U_0 \times I_0 \to E^0 \) such that

\[
\begin{align*}
(3.7.1) & \quad p(\xi(x, t)) = F(x, t_0) \text{ for all } (x, t) \in U_0 \times I_0, \\
(3.7.2) & \quad \xi(x, t_0) = 0_{F(x,t_0)} \text{ for all } x \in U_0, \\
(3.7.3) & \quad s(\xi(x, t)) = F(x, t) \text{ for all } (x, t) \in U_0 \times I_0, \\
(3.7.4) & \quad \text{for every } t \in I_0 \text{ the map } U_0 \to E^0, x \mapsto \xi(x, t) \text{ is of class } C^\infty.
\end{align*}
\]

**Proof.** By Lemma 3.4 we may assume without loss of generality that

\[ d_0s|_{E_z}: E_z \to V(h)_z \]

is an isomorphism for all \( z \in Z \). Defining \( f: U \to Z \) by \( f(x) = F(x, t_0) \), in view of Lemma 3.6 there exist an open neighborhood \( M \subseteq E_f^0 \) of the zero section \( Z(E_f) \) and an open neighborhood \( N \subseteq Z \) of \( f(U) \) such that the restriction \( \sigma: M \to N \) of \( s_f: E_f^0 \to Z \) is a diffeomorphism. Since \( \overline{U_0} \) is a compact subset of \( U \), we can choose an open neighborhood \( I_0 \) of \( t_0 \) in \([0, 1]\) such that \( F_t(U_0) \subseteq N \) for all \( t \in I_0 \). Therefore, for each \( t \in I_0 \), there exists a unique \( C^\infty \) map \( \zeta_t: U_0 \to E_f \) satisfying \( \zeta_t(U_0) \subseteq M \) and \( F_t(x) = \sigma(\zeta_t(x)) \) for all \( x \in U_0 \). Writing \( \zeta_t(x) \) as \( \zeta_t(x) = (\alpha_t(x), \xi_t(x)) \), where \( \alpha_t: U_0 \to U \) and \( \xi_t: U_0 \to E \) are \( C^\infty \) maps, we get

\[ f(\alpha_t(x)) = p(\xi_t(x)) \quad \text{and} \quad s(\xi_t(x)) = F_t(x). \]

By Definition 3.2(4) \( s(\xi_t(x)) \in h^{-1}(h(p(\xi_t(x))))) \), and hence

\[ h(s(\xi_t(x))) = h(f(\alpha_t(x))) = \alpha_t(x). \]

On the other hand,

\[ h(s(\xi_t(x))) = h(F_t(x)) = x. \]
Consequently, \( \alpha_t(x) = x \). It follows that \( \zeta : U_0 \rightarrow E_f \) is a \( C^\infty \) section, over \( U_0 \), of the vector bundle \( p_f : E_f \rightarrow U \). Clearly, \( \zeta_0(U_0) \subseteq Z(E_f) \). Furthermore, the map

\[ \zeta : U_0 \times I_0 \rightarrow E_f, \quad (x, t) \mapsto \zeta_t(x) \]

is continuous. Note that \( \zeta(x, t) = (x, \zeta(x, t)) \), where

\[ \xi : U_0 \times I_0 \rightarrow E^0, \quad (x, t) \mapsto \xi_t(x) \]

is a continuous map with \( p(\xi(x, t)) = f(x) = F(x, t_0) \) for all \((x, t) \in U_0 \times I_0\). By construction, the map \( \xi \) satisfies conditions \[(3.7.1)(3.7.4)\]

**Lemma 3.8.** Assume that the submersion \( h : Z \rightarrow X \) is malleable. Let \( U \) be an open subset of \( X \) and let \( F : U \times [0, 1] \rightarrow Z \) be a homotopy of \( C^\infty \) sections of \( h : Z \rightarrow X \). Let \( U_0 \) be an open subset of \( X \) whose closure \( \overline{U_0} \) is compact and contained in \( U \). Then there exist a dominating partial spray \((E, p, E^0, s)\) for \( h : Z \rightarrow X \) and a continuous map \( \xi : U_0 \times [0, 1] \rightarrow E^0 \) such that \( p : E = Z \times \mathbb{R}^n \rightarrow Z \) is the product vector bundle and \( \xi(x, t) = (F(x, 0), \eta(x, t)) \) for all \((x, t) \in U_0 \times [0, 1]\), where the map \( \eta : U_0 \times [0, 1] \rightarrow \mathbb{R}^n \) satisfies

\[
(3.8.1) \quad \eta(x, 0) = 0 \text{ for all } x \in U_0, \\
(3.8.2) \quad s(F(x, 0), \eta(x, t)) = F(x, t) \text{ for all } (x, t) \in U_0 \times [0, 1], \\
(3.8.3) \quad \text{for every } t \in [0, 1] \text{ the map } U_0 \rightarrow \mathbb{R}^n, x \mapsto \eta(x, t) \text{ is of class } C^\infty.
\]

**Proof.** By Lemma 3.3 the submersion \( h : Z \rightarrow X \) admits a dominating partial spray \((\tilde{E}, \tilde{p}, \tilde{E}^0, \tilde{s})\) such that \( \tilde{p} : \tilde{E} = Z \times \mathbb{R}^m \rightarrow Z \) is the product vector bundle. In view of Lemma 3.7 and the compactness of the interval \([0, 1]\) (see the Lebesgue lemma for compact metric spaces [23, p. 28, Lemma 9.11]), there exists a partition \( 0 = t_0 < t_1 < \cdots < t_k = 1 \) of \([0, 1]\) such that for each \( i = 1, \ldots, k \) there exists a continuous map \( \xi^i : U_0 \times [t_{i-1}, t_i] \rightarrow E^0 \) with the following properties:

- \( \xi^i(x, t) = (F(x, t_i), \eta^i(x, t)) \) for all \((x, t) \in U_0 \times [t_{i-1}, t_i]\),
- \( \eta^i(x, t_{i-1}) = 0 \) for all \( x \in U_0 \),
- \( \tilde{s}(F(x, t_{i-1}), \eta^i(x, t)) = F(x, t) \) for all \((x, t) \in U_0 \times [t_{i-1}, t_i]\),
- \( \text{for every } t \in [t_{i-1}, t_i] \text{ the map } U_0 \rightarrow \mathbb{R}^m, x \mapsto \eta^i(x, t) \text{ is of class } C^\infty. \)

For \( i = 1, \ldots, k \) we define recursively a dominating partial spray \((E(i), p^{(i)}, E(i)^0, s^{(i)})\) for \( h : Z \rightarrow X \) by

\[ (E(i), p^{(i)}, E(i)^0, s^{(i)}) = (\tilde{E}, \tilde{p}, \tilde{E}^0, \tilde{s}) \quad \text{if } i = 1, \]

while for \( i \geq 2 \) we require

\[ p^{(i)} : E(i) = Z \times (\mathbb{R}^m)^i \rightarrow Z \]

to be the product vector bundle and set

\[
E(i)^0 = \{(z, v_1, \ldots, v_i) \in E(i) : (z, v_1, \ldots, v_{i-1}) \in E(i-1)^0, \quad (s^{(i-1)}(z, v_1, \ldots, v_{i-1}), v_i) \in E(1)^0\},
\]

\[
s^{(i)} : E(i)^0 \rightarrow Z, \quad s^{(i)}(z, v_1, \ldots, v_i) = s^{(i)}(s^{(i-1)}(z, v_1, \ldots, v_{i-1}), v_i),
\]
where \( z \in Z \) and \( v_1, \ldots, v_i \in \mathbb{R}^n \).

In particular, \((E, p, E^0, s) := (E(k), p^{(k)}, E(k)^0, s^{(k)})\) is a dominating partial spray for \( h: Z \to X \). Note that \( p: E = Z \times \mathbb{R}^n \to Z \) is the product vector bundle with \( \mathbb{R}^n = (\mathbb{R}^m)^k \). Now, consider a map

\[
\xi: U_0 \times [0,1] \to E^0, \quad \xi(x, t) = (F(x, 0), \eta(x, t)),
\]

where \( \eta: U_0 \times [0,1] \to \mathbb{R}^n = (\mathbb{R}^m)^k \) is defined by

\[
\eta(x, t) = (\eta^1(x, t), 0, \ldots, 0)
\]

for all \((x, t) \in U_0 \times [t_0, t_1]\), and

\[
\eta(x, t) = (\eta^1(x, t_1), \ldots, \eta^{i-1}(x, t_{i-1}), \eta^i(x, t), 0, \ldots, 0)
\]

for all \((x, t) \in U_0 \times [t_{i-1}, t_i]\) with \( i = 2, \ldots, k \). One readily checks that \( \eta \) is a well-defined continuous map satisfying (3.8.1)–(3.8.3).

The main result of this section is the following.

**Theorem 3.9.** Assume that the submersion \( h: Z \to X \) is malleable. Let \( U \) be an open subset of \( X \) and let \( f: U \to Z \) be a \( C^\infty \) section of \( h: Z \to X \) that is homotopic through \( C^\infty \) sections to a regular section. Let \( W \) be an open subset of \( X \) whose closure \( \overline{W} \) is compact and contained in \( U \). Then the restriction \( f|_W: W \to Z \) can be approximated by regular sections of \( h: Z \to X \) in the \( C^\infty \) topology.

**Proof.** Let \( F: U \times [0,1] \to Z \) be a homotopy of \( C^\infty \) sections such that \( F_0 \) is a regular section and \( F_1 = f \). Choose an open subset \( U_0 \) of \( X \) such that its closure \( \overline{U_0} \) is compact and \( \overline{W} \subseteq U_0 \subseteq \overline{U_0} \subseteq U \), and let \((E, p, E^0, s), \xi: U_0 \times [0,1] \to E^0, \xi(x, t) = (F(x, 0), \eta(x, t))\), \( \eta: U_0 \times [0,1] \to \mathbb{R}^n \) be as in Lemma 3.8. In particular, we have

\[
s(F(x, 0), \eta(x, 1)) = F(x, 1) = f(x) \quad \text{for all } x \in U_0.
\]

By the Weierstrass approximation theorem, the \( C^\infty \) map \( \eta_1: U_0 \to \mathbb{R}^n, x \mapsto \eta(x, 1) \) can be approximated by regular maps in the \( C^\infty \) topology. If \( \tilde{\eta}_1: U_0 \to \mathbb{R}^n \) is a regular map sufficiently close to \( \eta_1 \), then \((F(x, 0), \tilde{\eta}_1(x)) \in E^0\) for all \( x \in W \) because \( \overline{W} \) is a compact subset of \( U_0 \). Consequently,

\[
\tilde{f}: W \to Z, \quad x \mapsto s(F(x, 0), \tilde{\eta}_1(x))
\]

is a well-defined regular map close to \( f|_W \) in the \( C^\infty \) topology. Finally, in view of Definition 3.2(1), \( \tilde{f}: W \to Z \) is a section of \( h: Z \to X \), which completes the proof.

It is worthwhile to point out the following special case of Theorem 3.9.

**Corollary 3.10.** Assume that the submersion \( h: Z \to X \) is malleable and the variety \( X \) is compact. Let \( f: X \to Z \) be a \( C^\infty \) section of \( h: Z \to X \) that is homotopic through \( C^\infty \) sections to a regular section. Then \( f \) can be approximated by regular sections of \( h: Z \to X \) in the \( C^\infty \) topology.
4 Proofs of Theorem 1.1 and related results

To begin with we discuss approximation of maps with values in a malleable real algebraic variety. In order to make use of Theorem 3.9 or Corollary 3.10 we need the following observation.

**Lemma 4.1.** Let $X, Y$ be nonsingular real algebraic varieties. Assume that the variety $Y$ is malleable. Then the canonical projection

$$h: X \times Y \to X, \quad (x, y) \mapsto x$$

is a malleable submersion.

**Proof.** Let $(E, p, E^0, s)$ be a dominating partial spray for $Y$. We obtain a dominating partial spray $(\tilde{E}, \tilde{p}, \tilde{E}^0, \tilde{s})$ for $h: X \times Y \to X$ setting

$$\tilde{E} = \{(x, y, v) \in (X \times Y) \times E : y = p(v)\},$$

$$\tilde{p}: \tilde{E} \to X \times Y, \quad ((x, y), v) \mapsto (x, y),$$

$$\tilde{E}^0 = \{(x, y, v) \in \tilde{E} : v \in E^0\},$$

$$\tilde{s}: \tilde{E}^0 \to X \times Y, \quad ((x, y), v) \mapsto (x, s(v)).$$

**Theorem 4.2.** Let $X$ be a compact nonsingular real algebraic variety and let $Y$ be a malleable nonsingular real algebraic variety. Then, for a $C^\infty$ map $f: X \to Y$, the following conditions are equivalent:

(a) $f$ can be approximated by regular maps in the $C^\infty$ topology.

(b) $f$ is homotopic to a regular map.

**Proof.** It suffices to prove (b)⇒(a). To this end let $\Phi: X \times [0, 1] \to Y$ be a homotopy such that $\Phi_0$ is a regular map and $\Phi_1 = f$. We may assume that $\Phi$ is a $C^\infty$ map, see [50, Proposition 10.22]. By Lemma 4.1, the canonical projection $h: X \times Y \to X$ is a malleable submersion. Clearly,

$$F: X \times [0, 1] \to X \times Y, \quad (x, t) \mapsto (x, \Phi(x, t))$$

is a homotopy of $C^\infty$ sections of $h: X \times Y \to X$. Therefore, according to Corollary 3.10 the $C^\infty$ section

$$X \to X \times Y, \quad x \mapsto (x, f(x))$$

can be approximated by regular sections in the $C^\infty$ topology, which implies that (a) holds.

The following variant of Theorem 4.2 can be derived from Lemma 3.8.

**Theorem 4.3.** Let $X$ be a compact (possibly singular) real algebraic variety and let $Y$ be a malleable nonsingular real algebraic variety. Then, for a continuous map $f: X \to Y$, the following conditions are equivalent:

(a) $f$ can be approximated by regular maps in the $C^0$ topology.

(b) $f$ is homotopic to a regular map.
Proof. It suffices to prove (b) $\Rightarrow$ (a). Suppose that (b) holds, and let $g: X \to Y$ be a regular map homotopic to $f$. We may assume that $X, Y$ are Zariski closed subsets of $\mathbb{R}^k, \mathbb{R}^l$, respectively. Then there exists a Zariski open neighborhood $\Omega \subseteq \mathbb{R}^k$ of $X$ and a regular map $\tilde{g}: \Omega \to \mathbb{R}^l$ with $\tilde{g}|_X = g$, see [2, p. 62]. By the Tietze extension theorem, there exists a continuous map $\tilde{f}: \Omega \to \mathbb{R}^l$ with $\tilde{f}|_X = f$. Now, let $\rho: T \to Y$ be a $C^\infty$ tubular neighborhood of $Y$ in $\mathbb{R}^l$, where $T$ is an open neighborhood of $Y$ in $\mathbb{R}^l$ and $\rho$ is a $C^\infty$ retraction. Choose an open neighborhood $U \subseteq \Omega$ of $X$ such that $\tilde{f}(U) \subseteq T$ and $\tilde{g}(U) \subseteq T$. We can also choose a $C^\infty$ map $U \to T$ arbitrarily close to $\tilde{f}|_U: U \to T$ in the $C^\infty$ topology; shrinking $U$, if necessary, such a map is homotopic to $\tilde{f}|_U: U \to T$. Therefore, for the proof of (a) we may assume that the map $\tilde{f}|_U: U \to T$ is of class $C^\infty$. Clearly, the maps $\varphi, \psi: U \to Y$, defined by

$$\varphi(x) = \rho(\tilde{g}(x)), \quad \psi(x) = \rho(\tilde{f}(x)) \quad \text{for all } x \in U,$$

are of class $C^\infty$ and satisfy $\varphi|_X = g$, $\psi|_X = f$. Shrinking $U$ further, the variety $X$ is a continuous retract of $U$ (recall that $X$ is triangulable [2, Theorem 9.2.1]). Consequently, the $C^\infty$ maps $\varphi, \psi$ are homotopic, and hence there exists a $C^\infty$ homotopy $\Psi: U \times [0, 1] \to Y$ with $\Phi_0 = \varphi$ and $\Phi_1 = \psi$. Let $U_0 \subseteq \mathbb{R}^k$ be an open neighborhood of $X$ whose closure $\overline{U}_0$ is compact and contained in $U$.

Consider the canonical projection $h: \mathbb{R}^k \times Y \to \mathbb{R}^k$ and the homotopy of its $C^\infty$ sections

$$F: U \to \mathbb{R}^k \times Y, \quad x \mapsto (x, \Phi(x(t))).$$

According to Lemma 4.1, $h: \mathbb{R}^k \times Y \to \mathbb{R}^k$ is a malleable submersion. Hence, by Lemma 3.8 there exist a dominating partial spray $(E, p, E^0, s)$ for $h: \mathbb{R}^k \times Y \to \mathbb{R}^k$ and a continuous map $\xi: U_0 \times [0, 1] \to E^0$ such that $p: E = (\mathbb{R}^k \times Y) \times \mathbb{R}^n \to \mathbb{R}^k \times Y$ is the product vector bundle and $\xi(x(t)) = (F(x, 0), \eta(x(t)))$, where the continuous map $\eta: U_0 \times [0, 1] \to \mathbb{R}^n$ satisfies

$$s(F(x, 0), \eta(x(t))) = F(x, t) \quad \text{for all } (x, t) \in U_0 \times [0, 1].$$

In particular,

$$s(F(x, 0), \eta(x, 1)) = F(x, 1) = (x, \psi(x)) \quad \text{for all } x \in U_0.$$

By the Weierstrass approximation theorem, the continuous map $X \to \mathbb{R}^n, x \mapsto \eta(x, 1)$ can be approximated by regular maps in the $C^0$ topology. Since $F(x, 0) = (x, \varphi(x))$ and $g = \varphi|_X: X \to Y$ is a regular map, it follows that the continuous map $f = \psi|_X: X \to Y$ can be approximated by regular maps in the $C^0$ topology. Thus (a) holds and the proof is complete.

Now our main theorem follows immediately.

Proof of Theorem 1.1. By Proposition 2.8, the variety $Y$ is malleable. Hence the conclusion follows from Theorems 1.2 and 4.3.

Theorem 1.1 holds, in particular, for maps with values in an arbitrary linear real algebraic group $G$. The assumption that $G$ is linear cannot be omitted. Indeed, each linear real algebraic group is up to isomorphism of the form $\Gamma(\mathbb{R})$ for some linear algebraic $\mathbb{R}$-group $\Gamma$ (see Section 2). In the category of algebraic $\mathbb{R}$-groups the linear ones can be characterized as follows.

**Theorem 4.4.** Let $k$ be a positive integer and let $\Gamma$ be an irreducible algebraic $\mathbb{R}$-group. Then the following conditions are equivalent:
(a) \( \Gamma \) is a linear algebraic \( \mathbb{R} \)-group.

(b) Every continuous null homotopic map from \( S^k \) into \( \Gamma(\mathbb{R}) \) can be approximated by regular maps in the \( C^0 \) topology.

**Proof.** By Theorem [1.1] (a) implies (b). To prove the reversed implication suppose that (a) does not hold. Then, by Chevalley’s theorem [26] (see [27] for a modern treatment), there exists a surjective morphism of algebraic \( \mathbb{R} \)-groups \( \varphi: \Gamma \to A \), where \( A \) is an Abelian \( \mathbb{R} \)-variety, \( \dim A \geq 1 \). Since \( \varphi \) is defined over \( \mathbb{R} \), its restriction \( \varphi(\mathbb{R}): \Gamma(\mathbb{R}) \to A(\mathbb{R}) \) is a regular map of real algebraic varieties. The image \( \varphi(\mathbb{R})(\Gamma(\mathbb{R})) \) is Zariski dense in \( A(\mathbb{R}) \), the map \( \varphi \) being surjective. Hence there exists a point \( a \in \Gamma(\mathbb{R}) \) at which the derivative \( d_a \varphi: T_a \Gamma(\mathbb{R}) \to T_{\varphi(\mathbb{R})(a)}A(\mathbb{R}) \) is surjective. Therefore, in view of the rank theorem for \( C^\infty \) maps, we can find a continuous null homotopic map \( f: S^k \to \Gamma(\mathbb{R}) \) such that the composite map \( \varphi \circ f: S^k \to A(\mathbb{R}) \) is not constant. It follows that \( \varphi(\mathbb{R}) \circ f \) cannot be approximated by regular maps in the \( C^0 \) topology because each regular map from \( S^k \) into \( \Gamma(\mathbb{R}) \) is constant, see [53, Corollary 3.9]. Consequently, (b) does not hold. In other words, (b) implies (a).

5 Further results on regular maps into unit spheres

The first result of this section can be viewed as a generalization of Theorem 1.6.

**Theorem 5.1.** Let \( X \) be a compact connected oriented nonsingular real algebraic variety of dimension \( n \). Then the set of regular maps \( \mathcal{R}(X,S^n) \) is dense in the space of \( C^\infty \) maps \( C^\infty(X,S^n) \) if and only if there exists a regular map from \( X \) into \( S^n \) of topological degree 1.

**Proof.** Suppose that \( f: X \to S^n \) is a regular map with \( \deg(f) = 1 \). By [4, Corollary 4.2], for every integer \( d \) there exists a regular map \( \varphi_d: S^n \to S^n \) with \( \deg(\varphi_d) = d \). Since the composite map \( \varphi_d \circ f \) is regular and \( \deg(\varphi_d \circ f) = d \), it follows from Hopf’s theorem that each \( C^\infty \) map from \( X \) into \( S^n \) is homotopic to a regular map. Therefore the set \( \mathcal{R}(X,S^n) \) is dense in the space \( C^\infty(X,S^n) \) by Corollary 1.5. The converse is obvious.

Here is an illuminating example, which itself is a generalization of Theorem 1.6.

**Example 5.2.** Let \( n, k \) be two positive integers and let \( S^n_{2k} \) be the Fermat \( n \)-sphere of degree 2k,

\[ S^n_{2k} = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}: x_0^{2k} + \cdots + x_n^{2k} = 1\}. \]

Clearly, \( S^n_{2k} \) is a nonsingular real algebraic variety diffeomorphic to the unit \( n \)-sphere \( S^n = S^n_2 \). If \( k \) is odd, then the set of regular maps \( \mathcal{R}(S^n_{2k},S^n) \) is dense in the space of \( C^\infty \) maps \( C^\infty(S^n_{2k},S^n) \). Indeed, the regular map

\[ f: S^n_{2k} \to S^n, \quad (x_0, \ldots, x_n) \mapsto (x_0^k, \ldots, x_n^k) \]

is a homeomorphism, and hence \( \deg(f) = 1 \) if \( S^n_{2k} \) is oriented in the standard way. Therefore the assertion follows from Theorem 5.1.

The following is a variant of Theorem 5.1 for nonorientable varieties.

**Theorem 5.3.** Let \( X \) be a compact connected nonorientable nonsingular real algebraic variety of dimension \( n \). Then either

(i) the set \( \mathcal{R}(X,S^n) \) is dense in the space \( C^\infty(X,S^n) \), or


(ii) the closure of $\mathcal{R}(X, S^n)$ in $C^\infty(X, S^n)$ coincides with the set of all $C^\infty$ null homotopic maps from $X$ into $S^n$.

Proof. By Hopf’s theorem, there are exactly two homotopic classes of $C^\infty$ maps from $X$ into $S^n$. One of them is represented by a constant map (which obviously is a regular map). Therefore the proof is complete in view of Corollary 1.5. \qed

Next we give a relevant example.

Example 5.4. Let $k$ be a positive integer and let $X_k$ be the blowup of the 2-sphere $S^2$ at $k$ points. Clearly, $X_k$ is a compact connected nonsingular real algebraic surface, which is a $C^\infty$ nonorientable surface of genus $k$. According to [3, Theorem 1.7], the set $\mathcal{R}(X_k, S^2)$ is dense in $C^\infty(X_k, S^2)$ for all $k \geq 1$. Furthermore, if $k$ is odd, then for every algebraic model $X$ of $X_k$ the set $\mathcal{R}(X, S^2)$ is dense in $C^\infty(X, S^2)$ by [5, Theorem 2]. However, if $k$ is even, then there exists an algebraic model $Y$ of $X_k$ such that the closure of $\mathcal{R}(Y, S^2)$ in $C^\infty(Y, S^2)$ consists precisely of all $C^\infty$ null homotopic maps from $Y$ into $S^2$, see [5, Theorem 3.3].

It is convenient to bring into play the homotopy groups. Let $n$ be a positive integer. As a base point in the unit $n$-sphere $S^n$ we choose $s_n = (1,0,\ldots,0)$. For any given real algebraic variety $Y$ with base point $y_0 \in Y$, let $\pi_n^{alg}(Y, y_0)$ denote the subset of the $n$th homotopy group $\pi_n(Y, y_0)$ comprising the homotopy classes represented by regular maps from $S^n$ into $Y$ that preserve the base points. We write $\pi_n^{alg}(S^p)$, $\pi_n(S^p)$ instead of $\pi_n^{alg}(S^p, s_p)$, $\pi_n(S^p, s_p)$, respectively. It is an open problem whether $\pi_n^{alg}(S^p)$ is a subgroup of $\pi_n(S^p)$ for all pairs of positive integers $(n, p)$, see [2, Proposition 1.1] for partial results.

Theorem 1.1 implies the following.

Proposition 5.5. Let $Y$ be a homogeneous space for a linear real algebraic group $G$ and let $y_0$ be a point in $Y$. Assume that $\pi_n^{alg}(Y, y_0) = \pi_n(Y, y_0)$ for some positive integer $n$. Then the set of regular maps $\mathcal{R}(S^n, Y)$ is dense in the space of $C^\infty$ maps $C^\infty(S^n, Y)$.

Proof. Let $f: S^n \to Y$ be a $C^\infty$ map. Choose an element $a \in G$ such that $a \cdot f(s_n) = y_0$. Since $\pi_n^{alg}(Y, y_0) = \pi_n(Y, y_0)$, the $C^\infty$ map $g: S^n \to Y$, $x \mapsto a \cdot f(x)$ is homotopic to a regular map, and hence, by Theorem 1.1, it can be approximated by regular maps in the $C^\infty$ topology. Consequently, the map $f$ can be approximated by regular maps in the $C^\infty$ topology because $f(x) = a^{-1} \cdot g(x)$ for all $x \in S^n$. \qed

In particular, for every pair $(n, p)$ of positive integers, the set $\mathcal{R}(S^n, S^p)$ is dense in the space of $C^\infty$ maps $C^\infty(S^n, S^p)$ if and only if $\pi_n^{alg}(S^p) = \pi_n(S^p)$. The following result is another generalization of Theorem 1.6 and provides and additional support for Conjecture 1.1.

Theorem 5.6. Let $(n, p)$ be a pair of positive integers. Then the set of regular maps $\mathcal{R}(S^n, S^p)$ is dense in the space of $C^\infty$ maps $C^\infty(S^n, S^p)$ in each of the following five cases:

(i) $p = 1, 2$ or 4.

(ii) $n - p \leq 3$.

(iii) $4 \leq n - p \leq 5$ with possible exception for the pairs $(9,5), (7,3), (11,6), (10,5)$ and $(8,3)$.

(iv) The homotopy group $\pi_n(S^p)$ is finite cyclic of odd order, and $p$ is odd with $n \leq 2p - 2$.

(v) $n = p + 13$, where $p$ is odd and $p \geq 15$.  

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Proof. \((i)\) This is proved in \([3\) Theorem 1.1\] and has already been mentioned in Section \(I\).

\((ii)\) The case \(n = p\) is contained in Theorem \(1.6\). For \(1 \leq n - p \leq 3\), one has \(\pi_n^{\text{alg}}(S^p) = \pi_n(S^p)\) by \([57\), Corollaries 1.2, 1.3 and 1.4\]. The case \(n < p\) is trivial.

\((iii)\) One has \(\pi_{p+4}(S^p) = 0\) for \(p \geq 6\) and \(\pi_{p+5}(S^p) = 0\) for \(p \geq 7\) (see \([34\) pp. 331, 332\]), which together with \((i)\) completes the proof.

\((iv)\) In this case, \(\pi_n^{\text{alg}}(S^p) = \pi_n(S^p)\) by \([4\) Theorem 2.1 and p. 163\].

\((v)\) One has \(\pi_{p+13}(S^p) = Z/3\) if \(p \geq 15\) (see \([60\) p. 188\]), so the assertion follows from \((iv)\). \(\square\)

Recall a basic notion from algebraic topology. An \(H\)-space is a pointed topological space \(X\) with base point \(e\), together with a continuous map (\(H\)-space multiplication) \(\mu: X \times X \to X\) such that \(\mu(e, e) = e\), and the maps \(X \to X\) defined by

\[x \mapsto \mu(x, e) \quad \text{and} \quad x \mapsto \mu(e, x)\]

are homotopic to the identity map of \(X\) through homotopies that keep the base point \(e\) fixed. For every positive integer \(n\), the group operation in the homotopy group \(\pi_n(X) := \pi_n(X, e)\) is induced by the \(H\)-space multiplication \(\mu\), see \([23\) p. 443, Theorem 4.1\].

For our purpose relevant examples of \(H\)-spaces are real algebraic groups (in particular, the unit sphere \(S^3\) with the multiplication of the quaternions of norm 1) and the unit sphere \(S^7\), the latter with the multiplication of the octonions of norm 1.

**Proposition 5.7.** Let \(Y\) be either a real algebraic group or the unit sphere \(S^7\). Then, for every positive integer \(n\), the subset \(\pi_n^{\text{alg}}(Y)\) is a subgroup of the homotopy group \(\pi_n(Y)\).

**Proof.** In the case under consideration, the \(H\)-space multiplication \(Y \times Y \to Y, (a, b) \mapsto ab\) and the inverse operation \(Y \to Y, a \mapsto a^{-1}\) are regular maps. The assertion follows since the group operation in \(\pi_n(Y)\) is induced by the \(H\)-space multiplication. \(\square\)

The following example has already been alluded to in connection with Conjecture \(II\) in Section \(I\).

**Example 5.8.** Let \((n, p)\) be a pair of integers, \(n > p \geq 1\). Then there exist an algebraic model \(X\) of the \(n\)-dimensional torus \((S^1)^n\) and a \(C^\infty\) map \(f: X \to S^p\) such that \(f\) is not homotopic to any regular map from \(X\) into \(S^p\).

This assertion can be proved as follows. Let \(h: (S^1)^p \to S^p\) be a \(C^\infty\) map of topological degree 1. Then the induced homomorphism in homology \(h_*: H_p((S^1)^p; Z/2) \to H_p(S^p; Z/2)\) is an isomorphism. Let \(w\) be a generator of the cohomology group \(H^p(S^p; Z/2) \cong Z/2\); so \(w\) corresponds via the Poincaré duality to a point in \(S^p\). Clearly, if \([[S^1]^p] \in H_p((S^1)^p; Z/2)\) is the fundamental class of \((S^1)^p\), then the Kronecker index \(\langle w, h_*([[S^1]^p]) \rangle\) is nonzero. Therefore the assertion holds by \([17\) Theorem 2.8\] (with \(K = (S^1)^p\), \(L = (S^1)^{n-p}\), \(Y = S^p\)).

Consequently, the set of regular maps \(R(X, S^p)\) is not dense in the space of \(C^\infty\) maps \(C^\infty(X, S^p)\).

Next we illustrate the behavior of regular maps from the product of spheres \(S^p \times S^q\) into the sphere \(S^{p+q}\).

**Example 5.9.** The results depend strongly on the specific values of \(p\) and \(q\). In what follows by a degree of a map we mean a topological degree.
(i) According to [51] Theorem 12, there exists a regular map \( S^4 \times S^2 \to S^6 \) of degree 1. Hence, in view of [4] Corollary 4.2, for every integer \( d \) there exists a regular map \( S^4 \times S^2 \to S^6 \) of degree \( d \). Now, it follows from Hopf’s theorem that each \( C^\infty \) map \( S^4 \times S^2 \to S^6 \) is homotopic to a regular map. Consequently, by Corollary 1.5 the set of regular maps \( \mathcal{R}(S^4 \times S^2, S^6) \) is dense in the space of \( C^\infty \) maps \( C^\infty(S^4 \times S^2, S^6) \).

(ii) The same argument shows that \( \mathcal{R}(S^4 \times S^1, S^5) \) is dense in \( C^\infty(S^4 \times S^1, S^5) \).

(iii) By [51] Theorem 14 and [4] Corollary 4.2, for every even integer \( d \) there exists a regular map \( S^2 \times S^2 \to S^4 \) of degree \( d \). It is an open problem whether there exists a regular map \( S^2 \times S^2 \to S^4 \) of odd degree; if it does, then \( \mathcal{R}(S^2 \times S^2, S^4) \) is dense in \( C^\infty(S^2 \times S^2, S^4) \).

(iv) Assuming that both \( p \) and \( q \) are odd positive integers, according to Theorem 1.7 a \( C^\infty \) map \( f : S^p \times S^q \to S^{p+q} \) can be approximated by regular maps if and only if it is null homotopic.

6 Regular maps into real algebraic groups

For maps into classical groups or Stiefel manifolds (see Example 1.4(iii) for the notation), we have the following result supporting Conjecture I.

**Proposition 6.1.** Let \((n, m)\) be a pair of positive integers and let \(Y\) be one of the following real algebraic varieties:

(i) \( Y = O(m) \) or \( Y = SO(m) \) with \( n \leq m - 2 \) and \( n = 8k + l, k \in \mathbb{Z}, l = 2, 4, 5 \) or 6;

(ii) \( Y = U(m) \) or \( Y = SU(m) \) with \( n \leq 2m - 1 \) and \( n = 2k, k \in \mathbb{Z} \);

(iii) \( Y = Sp(m) \) with \( n \leq 4m + 1 \) and \( n = 8k + l, k \in \mathbb{Z}, l = 0, 1, 2 \) or 6;

(iv) \( Y = \mathbb{V}_r(\mathbb{F}^m) \) with \( n \leq (m - r + 1)d(\mathbb{F}) - 2 \), where \( d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F} \).

Then the set of regular maps \( \mathcal{R}(S^n, Y) \) is dense in the space of \( C^\infty \) maps \( C^\infty(S^n, Y) \).

**Proof.** According to Bott’s periodicity theorem [22] (see also [35] Chap. 8, Remark 4.2, and 12.2) the \( n \)th homotopy group \( \pi_n(Y) \) is trivial in cases (i) (ii) and (iii). Furthermore, by [35] Chap. 8, Theorem 5.1], the homotopy group \( \pi_n(Y) \) is trivial in case (iv). Therefore the proof is complete in view of Corollary 1.2.

Next we consider regular maps from \( S^n \) into \( U(m) \) or \( SU(m) \), for \( n \) odd with \( n \leq 2m - 1 \). This is harder to handle than Proposition 6.1(ii) since the homotopy groups \( \pi_n(U(m)) \) and \( \pi_n(SU(m)) \) are nontrivial if \( n \) is odd \( (n \neq 1 \) for the latter group). We have the following partial result.

**Theorem 6.2.** Let \((n, m)\) be a pair of positive integers and let \( G(m) \) denote either \( U(m) \) or \( SU(m) \). Assume that \( n = 2k - 1 \) is odd and \( m \geq 2^{k-1} \). Then the set of regular maps \( \mathcal{R}(S^n, G(m)) \) is dense in the space of \( C^\infty \) maps \( C^\infty(S^n, G(m)) \).

As a preparation for the proof of Theorem 6.2 we briefly summarize the discussion contained in [21] Part III B]. Let \( k, p \) be two integers with \( 1 \leq k \leq p \). For any continuous map \( f : S^{2k-1} \to GL_p(\mathbb{C}) \), its degree \( \deg(f) \) is an integer defined as follows. The map \( f \)
is homotopic to the composite of some continuous map \( h: S^{2k-1} \to GL_k(\mathbb{C}) \) and the inclusion map \( GL_k(\mathbb{C}) \hookrightarrow GL_p(\mathbb{C}) \). The first column \( h_1 \) of \( h \) determines a continuous map \( h_1: S^{2k-1} \to \mathbb{C}^k \setminus \{0\} \), and therefore one gets a continuous map

\[
\psi = \frac{h_1}{\|h_1\|}: S^{2k-1} \to S^{2k-1},
\]

where \( S^{2k-1} \) is regarded as a subset of \( \mathbb{C}^k = \mathbb{R}^{2k} \). The topological degree \( \deg(\psi) \) of \( \psi \) is divisible by \( (k-1)! \), and one sets

\[
\deg f = (-1)^{k-1} \frac{\deg(\psi)}{(k-1)!}.
\]

The definition of \( \deg(f) \) does not depend on the choice of \( h \). Furthermore, one has the following variant of Bott’s periodicity theorem: If \( 1 \leq k \leq p \), then

\[
\deg: \pi_{2k-1}(GL_p(\mathbb{C})) \to \mathbb{Z}, \quad [f] \mapsto \deg(f)
\]

is a well-defined group isomorphism.

As usual, for any complex matrix \( A \), let \( A^* \) denote its conjugate transpose. For any positive integer \( r \), denote by \( I_r \) the identity \( r \)-by-\( r \) matrix.

Suppose given two continuous maps \( f: S^{2k-1} \to GL_p(\mathbb{C}) \) and \( g: S^{2l-1} \to GL_q(\mathbb{C}) \), where \( 1 \leq k \leq p \) and \( 1 \leq l \leq q \). The product

\[
f \# g: S^{2(k+l)-1} \to GL_{2pq}(\mathbb{C})
\]

is a continuous map defined by

\[
(x, y) \mapsto \begin{pmatrix} F(x) \otimes I_q & -I_p \otimes G(y)^* \\ I_p \otimes G(y) & F(x)^* \otimes I_q \end{pmatrix},
\]

where \( x \in \mathbb{C}^k, \ y \in \mathbb{C}^l, \ \|(x, y)\| = 1 \), while \( F \) and \( G \) are the homogeneous extensions of \( f \) and \( g \), respectively. To be precise,

\[
F: \mathbb{C}^k \to \text{Mat}_p(\mathbb{C}), \quad F(x) = \begin{cases} \|x\| f\left(\frac{x}{\|x\|}\right) & \text{for } x \in \mathbb{C}^k \setminus \{0\}, \\ 0 & \text{for } x = 0,
\end{cases}
\]

where \( \text{Mat}_p(\mathbb{C}) \) is the space of all complex \( p \)-by-\( p \) matrices, and \( G: \mathbb{C}^l \to \text{Mat}_q(\mathbb{C}) \) is defined analogously. One has the formula

\[
\deg(f \# g) = \deg(f) \deg(g).
\]

Now, starting with the map

\[
a: S^1 \to GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}, \quad z \mapsto z,
\]

we define a sequence of continuous maps

\[
a_k: S^{2k-1} \to GL_{2k-1}(\mathbb{C}), \quad k = 1, 2, \ldots
\]

by a recursive formula: \( a_1 = a \) and \( a_k = a_{k-1} \# a \) for \( k \geq 2 \). Since \( \deg(a) = 1 \), it follows that \( \deg(a_k) = 1 \) for all \( k \geq 1 \). Thus, by the variant of Bott’s periodicity theorem stated
above, for every \( k \geq 1 \) the homotopy group \( \pi_{2k-1}(\text{GL}_{2k-1}(\mathbb{C})) \cong \mathbb{Z} \) is generated by the homotopy class represented by \( a_k \).

The maps \( a_k \) have some other useful properties. Let \( A_k : \mathbb{C}^k \to \text{Mat}_{2k-1}(\mathbb{C}) \) be the homogeneous extension of \( a_k, k \geq 1 \). Clearly, \( A_1(z_1) = z_1 \) and

\[
A_k(z_1, \ldots, z_k) = \left( \begin{array}{c} A_{k-1}(z_1, \ldots, z_{k-1}) \ & -\bar{z}_k I_{2k-2} \\ \bar{z}_k I_{2k-2} \ & A_{k-1}(z_1, \ldots, z_{k-1})^* \end{array} \right)
\]

for \( k \geq 2 \),

where \((z_1, \ldots, z_k) \in \mathbb{C}^k\) and \( \bar{z}_k \) is the conjugate of \( z_k \). It follows that \( A_k \) is an \( \mathbb{R} \)-linear map for \( k \geq 1 \). Furthermore,

\[
A_k(z_1, \ldots, z_k)A_k(z_1, \ldots, z_k)^* = \left( \sum_{j=1}^k z_j \bar{z}_j \right) I_{2k-1}
\]

for \( k \geq 1 \),

\[
\det A_1(z_1) = z_1 \bar{z}_1, \quad \det A_k(z_1, \ldots, z_k) = \left( \sum_{j=1}^k z_j \bar{z}_j \right)^{2k-2}
\]

for \( k \geq 2 \).

In particular, \( a_k \) can be regarded as a regular map \( a_k : S^{2k-1} \to SU(2k-1) \) for \( k \geq 2 \).

**Proof of Theorem 6.2.** By Proposition 5.5, it is sufficient to show that \( \pi_{2k-1}^\text{alg}(G(m)) = \pi_{2k-1}(G(m)) \).

**Case 1.** Suppose that \((k, m) = (1, m)\) with \( m \geq 1 \). It is well known that \( \pi_1(SU(m)) = 0 \), and the inclusion map \( U(1) \hookrightarrow U(m) \) induces an isomorphism \( \pi_1(U(1)) \cong \pi_1(U(m)) \). Case 1 follows since \( U(1) = S^1 \) and \( \pi_1^\text{alg}(S^1) = \pi_1(S^1) \).

**Case 2.** Suppose that \( k \geq 2 \) and \( m \geq 2k-1 \). The inclusion map \( G(2k-1) \hookrightarrow \text{GL}_{2k-1}(\mathbb{C}) \) induces an isomorphism of the homotopy groups \( \pi_{2k-1}(G(2k-1)) \) and \( \pi_{2k-1}(\text{GL}_{2k-1}(\mathbb{C})) \). As explained above, the latter group is generated by the homotopy class represented by the map \( a_k \). Since \( a_k \) is a regular map with values in \( G(2k-1) \), we get \( \pi_{2k-1}^\text{alg}(G(2k-1)) = \pi_{2k-1}(G(2k-1)) \) by Proposition 5.7. Moreover, the inclusion map \( G(2k-1) \hookrightarrow G(m) \) induces an isomorphism of the homotopy groups \( \pi_{2k-1}(G(2k-1)) \) and \( \pi_{2k-1}(G(m)) \) (see [35, Chap. 8, Remark 4.2]), and hence \( \pi_{2k-1}^\text{alg}(G(m)) = \pi_{2k-1}(G(m)) \).

We have one more result on regular maps into the groups \( U(m) \) or \( SU(m) \).

**Proposition 6.3.** Let \((n, m)\) be a pair of positive integers and let \( G(m) \) denote either \( U(m) \) or \( SU(m) \). Assume that \( 1 \leq n \leq 6 \). Then the set of regular maps \( R(S^n, G(m)) \) is dense in the space of \( \mathcal{C}^\infty \) maps \( \mathcal{C}^\infty(S^n, G(m)) \), possibly with the exception of \((n, m) = (5, 3)\) or \((n, m) = (6, 3)\).

**Proof.** Suppose that \((n, m) \neq (5, 3)\) and \((n, m) \neq (6, 3)\). Then the only cases not already covered by Proposition 6.1 and Theorem 6.2 are \((n, m)\) equal to \((4, 2), (5, 2)\) and \((6, 2)\). Hence, by Proposition 5.5, it is sufficient to show that \( \pi_{2n}^\text{alg}(G(2)) = \pi_{2n}(G(2)) \) for \( n = 4, 5 \) and 6. This is proved in [57, Corollaries 1.2, 1.3 and 1.4] for \( G(2) = SU(2) = S^3 \). The case \( G(2) = U(2) \) follows since the inclusion map \( SU(m) \hookrightarrow U(m) \) induces for all \( i \geq 2 \) an isomorphism between the corresponding \( i \)th homotopy groups, see [35, Chap. 8, 12.2].

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