Chern Classes of Bundles on Rational Surfaces

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Abstract
Consider the blow up \( \pi : \widetilde{X} \to X \) of a rational surface \( X \) at a point. Let \( \widetilde{V} \) be a holomorphic bundle over \( \widetilde{X} \) whose restriction to the exceptional divisor is \( \mathcal{O}(j) \oplus \mathcal{O}(-j) \) and define \( V = (\pi_* \widetilde{V})^\vee \). Friedman and Morgan gave the following bounds for the second Chern classes
\[ j \leq c_2(\widetilde{V}) - c_2(V) \leq j^2. \]
We show that these bounds are sharp.

1 Introduction.
In this paper our basic setting will be the following. \( X \) will be a rational surface, \( \pi : \widetilde{X} \to X \) the blow-up of \( X \) at point \( x \in X \) and \( \ell \) the exceptional divisor. \( \widetilde{V} \) will be a rank two bundle over the surface \( \widetilde{X} \) satisfying \( \det \widetilde{V} \simeq \mathcal{O}_\widetilde{X} \) and \( \widetilde{V}|_\ell \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j), j \geq 0 \). Let \( V = \pi_* \widetilde{V}^\vee \). Friedman and Morgan \[2\] gave the following estimate for the second Chern classes
\[ j \leq c_2(\widetilde{V}) - c_2(V) \leq j^2. \]
We show that these bounds are sharp by giving an algebraic procedure to calculate \( c_2(\widetilde{V}) - c_2(V) \) directly from the transition matrix defining \( \widetilde{V} \) in a neighborhood of \( \ell \).

Since \( \widetilde{V}|_{\widetilde{X} - \ell} = \pi^* V|_{X - p} \) it is natural to localize our study of \( \widetilde{V} \) to a neighborhood of the exceptional divisor. If \( x \) is the blown-up point, we choose an open set \( U \supset x \) that is biholomorphic to \( \mathbb{C}^2 \). Then \( \pi^*(U) \) gives a neighborhood \( N_\ell \simeq \widetilde{\mathbb{C}^2} \). We remark that since \( \widetilde{X} \) is rational we can only guaranty the existence of a neighborhood of \( x \) that is biholomorphic to \( \mathbb{C}^2 \) minus a finite number of points. However, all our calculations use holomorphic functions, which always extend over these points since we are in dimension 2. Therefore we may from the start assume without loss of generality that \( N(\ell) = \mathbb{C}^2 \).

We use the following results:

**Theorem 1.1**: ([5], Thm. 2.1) Let \( E \) be a holomorphic bundle on \( \mathbb{C}^2 \) with \( E_\ell \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j) \). Then \( E \) has a transition matrix of the form
\[
\begin{pmatrix}
  z^j & p \\
  0 & z^{-j}
\end{pmatrix}
\]
from \( U \) to \( V \), where \( p = \sum_{i=1}^{j-2} \sum_{l=-j+1}^{j-1} p_{il} z^l u^i \).
Corollary 1.2 ([6], Cor. 4.1) Every holomorphic rank two vector bundle over $\tilde{X}$ with vanishing first Chern class is topologically determined by a triple $(V, j, p)$ where $V$ is a rank two holomorphic bundle on $X$ with vanishing first Chern class, $j$ is a non-negative integer, and $p$ is a polynomial.

It follows that for each polynomial $p$ (in 3 variables $z, z^{-1},$ and $u$) there is a canonical construction that assigns a bundle $\tilde{V}$ over $\tilde{X}$ whose topological type is determined by $p$. We remark that N. Buchdahl [1] has shown that these bounds are sharp in a more general setting by entirely different methods.

Following Friedman and Morgan [2, p. 302], we define the sheaf $Q$ by the exact sequence
\[ 0 \to \pi_*\tilde{V} \to V \to Q \to 0. \]
where $Q$ is a sheaf supported only at $x$. From the exact sequence (1) it follows immediately that $c_2(\pi_*\tilde{V}) - c_2(V) = l(Q)$, where $l$ stands for length. An application of Grothendieck-Riemann-Roch (cf [2]) gives that $c_2(\tilde{V}) - c_2(V) = l(Q) + l(R^1\pi_*\tilde{V})$. Since $Q$ and $R^1\pi_*\tilde{V}$ are supported at $x$ we can compute their lengths by looking at $\tilde{V}|_{\mathcal{N}}$ and at $\pi$ as the blow-up map $\pi : \mathbb{C}^2 \to \mathbb{C}^2$. The lengths of $Q$ and $R^1\pi_*\tilde{V}$ can be explicitly computed using the Theorem on Formal Functions of Grothendieck (see[3]).

2 Calculation of Chern classes.

2.1 The upper bound occurs for $p = 0$.

Proposition 2.1 : If the bundle $\tilde{V}$ splits on a neighborhood of the exceptional divisor, then $c_2(\tilde{V}) - c_2(V) = j^2$.

Proof: We show that for $\tilde{V}|_{\mathcal{N}(\ell)} = \mathcal{O}(j) \oplus \mathcal{O}(-j)$, we have $l(R^1\pi_*\tilde{V}) = j(j+1)/2$ and $l(Q) = j(j-1)/2$, which we state as lemmas. It follows that $c_2(\tilde{V}) - c_2(V) = l(Q) + l(R^1\pi_*\tilde{V}) = j(j+1)/2 + j(j-1)/2 = j^2$. □

We now prove the lemmas we just used.

Lemma 2.2 If the bundle $\tilde{V}$ splits on a neighborhood of the exceptional divisor, then $l(Q) = j(j+1)/2$.

Proof: Since the length of $Q$ equals the dimension of $Q^\wedge_x$ as a $k(x)$-vector space, we need to study the map $(\pi_*\tilde{V}^\wedge_x) \to V_x^\wedge$ and compute the dimension
of the cokernel as a $k(x)$-vector space. But as $V^\wedge = (\pi_* \check{V}_x^\wedge)^{\vee\vee}$, we need to compute the $\mathcal{O}_x^\wedge$-module structure on $M = (\pi_* \check{V}_x)^\wedge$ and study the natural map $M \hookrightarrow M^{\vee\vee}$ of $\mathcal{O}_x^\wedge$-modules. By the Formal Functions Theorem

$$M \simeq \lim_{\leftarrow} H^0(\ell_n, \check{V}\mid\ell_n)$$

as $\mathcal{O}_x^\wedge(\simeq \mathbb{C}[[x,y]])$-modules, where $\ell_n \simeq N_\ell \times \mathcal{O}_x/m_\ell^{n+1}$ is the n-th infinitesimal neighborhood of $\ell$. To do this we will have to isolate the action of $\mathcal{O}_x^\wedge$ on $H^0(\ell_n, \check{V}\mid\ell_n)$.

We first write the blow-up of $\mathbb{C}^2$ with two charts $U \sim V \sim \mathbb{C}^2$ with $(z, u) \mapsto (z^{-1}, zu)$ in $U \cap V$. Then the blow-up map $\pi : \check{C}^2 \to \mathbb{C}^2$ is given on the $U$ chart by $(x, y) = \pi(z, u) = (u, zu)$. We give the natural action of $x$ and $y$ on this space; that is, $x$ acts by multiplication by $u$ and $y$ acts by multiplication by $zu$. This yields $M = \mathbb{C}[[x, y]] < \alpha, \beta_0, \beta_1, \ldots, \beta_j >$, where

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 \\ z \end{pmatrix}, \quad \ldots, \quad \beta_j = \begin{pmatrix} 0 \\ z^j \end{pmatrix}.$$ 

With relations:

$$\begin{cases} x\beta_1 - y\beta_0 \\ x\beta_2 - y\beta_1 \\ \vdots \\ x\beta_j - y\beta_{j-1} \end{cases}, \quad \begin{cases} x^2\beta_2 - y^2\beta_0 \\ \vdots \\ x^2\beta_j - y^2\beta_{j-2} \end{cases}.$$ 

All together there are $j(j+1)/2$ relations. Now writing the generators of $M^\wedge$ and $M^{\vee\vee}$ we see that $\text{coker}(M \hookrightarrow M^{\vee\vee})$ is a $j(j+1)/2$ dimensional vector space over $\mathbb{C} \simeq k(x)$, hence $l(Q) = j(j+1)/2$. 

**Lemma 2.3** If we have a split extension on a neighborhood of the exceptional divisor, that is, when $\check{V}\mid_{N(\ell)} \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j)$, then $l(R^1 \pi_* \check{V}) = j(j-1)/2$.

**Proof:** Here again we follow the same method using the Theorem on Formal Functions and compute that

$$h^1(\ell_n, \check{V}\mid_{\ell_n}) = \begin{cases} (j-1) & \text{for } n = 0 \\ (j-1) + (j-2) + \cdots + (j-n) & \text{for } 1 \leq n \leq j-1 \\ j(j-1)/2 & \text{for } n \geq j-1. \end{cases}$$
Also, $\text{Ker} \left( H^1(\ell_n, V|_{\ell_n}) \to H^1(\ell_{n-1}, V|_{\ell_{n-1}}) \right)$ has dimension $(j - n)$ for $2 \leq n \leq j - 1$. Computing the inverse limit (cf. Lang [4]) this implies that

$$l(R^1\pi_*\tilde{V}) = \dim \lim_{\leftarrow n} H^1(\ell_n, V|_{\ell_n}) = 1 + 2 + \cdots + (j - 1) = j(j - 1)/2.$$ 

This can also be verified by invoking the short exact sequence on p.388 of Hartshorne [3].

Remark: We can also proof Proposition 2.1 in a simpler way, by explicitly constructing a generic section of $\tilde{V}$ and counting its zeros.

2.2 The lower bound occurs for $p = u$.

Theorem 2.4 If $\tilde{V}$ is a bundle corresponding to the triple $(V, j, u)$ (according to Cor. 1.2), then $c_2(\tilde{V}) - c_2(V) = j$.

Proof: We show that $l(R^1\pi_*\tilde{V}) = j - 1$ and $l(Q) = 1$, which we state as lemmas. It follows that $c_2(\tilde{V}) - c_2(V) = (j - 1) + 1 = j$.

Lemma 2.5 If $\tilde{V}$ is given by $(V, j, u)$ then $l(Q) = 1$.

Proof: The bundle $\tilde{V}$ is given over $N(\ell)$ (according to Theorem 2.1) by the transition matrix $\begin{pmatrix} z^j & u \\ 0 & z^{-j} \end{pmatrix}$. Here calculations are similar to the ones in the proof of Lemma 2.2. We set $M = (\pi_*\tilde{V})^\wedge$ and study the natural map $\rho : M \hookrightarrow M^{\vee\vee}$ of $O_x^\wedge$-modules. The value of $l(Q)$ is the dimension of cokernel of $\rho$.

The $O_x$-module structure of $M$ is completely determined by the structure of the sections of the bundle $V'$. Writing sections of $V'$ in the form $\begin{pmatrix} a \\ b \end{pmatrix} = \left( \frac{\sum a_{ik}z^k u^i}{\sum b_{ik}z^k u^i} \right)$ it is simple to see which restrictions are imposed in the coefficients $a_{ik}$ and $b_{ik}$. Some terms of the general form for the sections are

$$\begin{pmatrix} a \\ b \end{pmatrix} = b_{00} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b_{01} \begin{pmatrix} 0 \\ z \end{pmatrix} + b_{0j} \begin{pmatrix} -u \\ z^{-j} \end{pmatrix} + \cdots + a_{j0} \begin{pmatrix} u^j \\ 0 \end{pmatrix} + \cdots.$$ 

In fact, one verifies that these terms are enough to generate all the sections. It then follows that $M = < \beta_{00}, \beta_{01}, \beta_{0j}, \alpha_{j0} > / R$ where $\beta_{00} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\beta_{01} =$
\[(0)
\]

\[\beta_{0j} = \begin{pmatrix} -u \\ z^j \end{pmatrix}, \alpha_{j0} = \begin{pmatrix} u^j \\ 0 \end{pmatrix}\] and \(R\) is the set of relations

\begin{align*}
\{ x \beta_{01} - y \beta_{00} = 0 \\
\alpha_{j0} + x^{j-1} \beta_{0j} - y^{j-1} \beta_{01} = 0 \\
\}
\end{align*}

Using the second relation, one eliminates \(\alpha_{j0}\) from the set of generators and gets a simpler presentation \(M \simeq \langle \beta_{00}, \beta_{01}, \beta_{0j} \rangle / R'\) where \(R'\) now has the single relation \(x \beta_{01} - y \beta_{00} = 0\). It is now a matter of simple algebra to find that \(M^\vee = \langle a, b \rangle\) is free on two generators, where \(a = \begin{cases} 
\beta_{00} \mapsto x \\
\beta_{01} \mapsto y \\
\beta_{0j} \mapsto 0
\end{cases}\) and \(b = \begin{cases} 
\beta_{00} \mapsto 0 \\
\beta_{01} \mapsto 0 \\
\beta_{0j} \mapsto 1
\end{cases}\). Then naturally \(M^{\vee\vee} = \langle a^*, b^* \rangle\) is generated by the dual basis, namely \(a^* = \begin{cases} 
a \mapsto 1 \\
b \mapsto 0
\end{cases}\) and \(b^* = \begin{cases} 
a \mapsto 0 \\
b \mapsto 1
\end{cases}\). The map \(\rho\) is given by evaluation and we have \(\text{im} \rho = \langle x a^*, y a^*, b^* \rangle\) and therefore the cokernel is \(\text{coker} \ rho = \langle a^* \rangle\) and \(l(Q) = \text{dim} \text{coker} \rho = 1\).

\textbf{Lemma 2.6} If \(\widetilde{V}\) is given by \((V, j, u)\) then \(l(R^1 \pi_* \widetilde{V}) = j - 1\).

\textbf{Proof:} We claim that \(H^1(\ell_n, \widetilde{V}|_{\ell_n})\) is generated by the 1-cocycles \(\left( \begin{array}{c} z^k \\ 0 \end{array} \right)\) for \(-j \leq k \leq -1\) and that the maps \(H^1(\ell_n, \widetilde{V}|_{\ell_n}) \to H^1(\ell_{n-1}, \widetilde{V}|_{\ell_{n-1}})\) are the identity. Hence \(l(R^1 \pi_* \widetilde{V}) = \text{dim} \lim \text{lim} H^1(\ell_n, \widetilde{V}|_{\ell_n}) = j - 1\). In fact, if \(T\) is the transition matrix for \(\widetilde{V}|_{N(\ell)}\) then the equality

\[B = \sum_{i=0}^{\infty} \sum_{k=-\infty}^{\infty} \left( \begin{array}{c} 0 \\ b_{ik} z^k u^i \end{array} \right) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \left( \begin{array}{c} 0 \\ b_{ik} z^k u^i \end{array} \right) + T^{-1} \sum_{i=0}^{\infty} \sum_{k=-\infty}^{-1} \left( \begin{array}{c} 0 \\ b_{ik} z^k u^{i+1} \end{array} \right)\]

shows that \(B\) is a coboundary, since the first term on the r.h.s. is holomorphic in \(U\) and the last term of the r.h.s. is holomorphic in \(V\). As a consequence every 1-cocycle has a representative of the form \(\alpha = \sum_{i=0}^{\infty} \sum_{k=-\infty}^{\infty} \left( a_{ik} z^k u^i \right)\).

Analogously \(A = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \left( a_{ik} z^k u^i \right) + T^{-1} \sum_{i=0}^{\infty} \sum_{k=-\infty}^{\infty} \left( a_{ik} z^k u^{i+1} \right)\) is a coboundary. Therefore the only terms that give nonzero cohomology classes in \(\alpha\) are the ones with indices \(k\) for \(-j \leq k \leq -1\) and the claim follows. 

5
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