Cosmic Acceleration and the notion of ‘Alternative Vacuum’ in Nash Theory

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We discuss the cosmological implications in the Nash theory of gravity, where the field equations are derived from a lagrangian quadratic in Ricci invariants. An exact cosmological solution is found which is compared with the observational expectations, using a Markov chain Monte Carlo simulation of JLA + OHD + BAO data sets. Departures from a standard ΛCDM cosmology are discussed. We propose that the Nash vacuum dynamics can be imagined as the effective dynamics of a growing vacuum in General Relativity, written as a time-evolving scalar field. The resulting mild evolution of the masses of fundamental particles is found to be within the observed scale of variation measured through the molecular absorption spectra of a series of Quasars.

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I. INTRODUCTION

Einstein’s General Theory of Relativity (GR) allows us to put space-time, matter and gravity within one bracket; a single mathematical framework on the macroscopic level. Not only the theory stands out for its elegance, it keeps on passing experimental tests over the years; the most recent on the list being the detection of gravitational waves [1] and the image of black-hole shadows captured by Event Horizon Telescope [2]. There are albeit a few remaining riddles within the theory related to patches of observational inconsistencies [3–8] and also, some limitations in probing the theory in higher curvature regime. The latter problem is rooted in the incompatibility of GR and quantum field theory, however, higher curvature corrections are often considered as a possible solution. Both of these issues leave open the scope of working out viable modified theories of gravity. The particular theory of gravity proposed by John. Nash Jr. [9] (Nash Theory from here onwards) is one of a kind in this class.

The signature of Nash theory is a specific choice of gravitational action, quadratic in the Ricci scalar and Ricci tensor. The action has some resemblance with string inspired theories of gravity and also a few characteristic departures [10]. For instance, it has no terms linear in Ricci scalar in the action and the comparative weighting of the two quadratic terms $R^{\alpha \beta} R_{\alpha \beta}$ and $R^2$ are different from a Lovelock invariant [11]. This indicates higher order field equations. This is probably one of the reasons for which the theory is relatively less explored (apart from a few attempts [12–14]), despite accommodating some remarkable features. For instance, the theory is a special case of the renormalizable Stelle Gravity [17], and therefore a possible candidate for quantum gravity theories. Moreover, the scalar equation of Nash Theory in a four dimensional vacuum is a wave equation and it can enable a wider variety of gravitational waves compared to GR. After all, it is not wrong to expect new solutions governing the nature of space-time geometry from a set of higher order field equations. On this note, we focus on vacuum cosmological solutions of the Nash field equations. The manner in which an energy-momentum distribution can enter the Nash theory is a different question altogether and we avoid that question in this work. We find an exact solution of the spatially flat cosmological equations in vacuum. We also show that a Nash vacuum may be rendered as a combination of two parts : vacuum Einstein field equations plus an exotic self-interacting scalar field. This vacuum field has a symmetry breaking self-interaction (like Higgs) and allows a mildly varying vacuum expectation value (vev). This leads to variations of quark masses and written in a quantified manner through a variation of proton-to-electron mass ratio $\mu$. We show that these variations are on a desired scale as predicted by the molecular absorption spectra of a series of Quasars.

In section II we briefly review the action and the field equations of Nash theory. We introduce the spatially flat FRW equations in the same section and solve them in

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section III. Evolution of the Hubble parameter and kinematic quantities governing the late-time acceleration of the universe are discussed in the same section. In section IV we discuss the structure of an envisaged vacuum scalar field and the scale of variation of different fundamental couplings. We conclude the manuscript in section V.

II. NASH EQUATIONS FOR VACUUM

The Nash action is written as

\[ S = \int d^4x \sqrt{-g} \left[ 2R^{\alpha\beta} R_{\alpha\beta} - R^2 \right]. \]

We can see straightaway that there is no term linear in Ricci scalar in the action, i.e., the standard Einstein-Hilbert part is switched off. The field equations are derived by taking a metric variation and written as

\[ N^{\mu\nu} \equiv \Box G^{\mu\nu} + G^{\alpha\beta} \left( 2R^{\mu\nu}_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} \right) = 0, \quad (1) \]

which will henceforth be regarded as the Nash equations. The \( \Box \) is the d’Alembertian and \( G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \) is the standard Einstein tensor. One peculiarity of gravitational theories with higher curvature terms is that it yields equations of motion of higher-order in derivatives of the metric. Due to Ostrogradsky’s theorem it is a well-known fact that non-degenerate Lagrangians containing second (or higher) derivatives of a field give rise to some classical instabilities. A further analysis in [13] has shown specifically that the presence of the term \( R^{\mu\nu} R_{\mu\nu} \) gives rise to two unconstrained instabilities per space point. But then again one can argue that Ostrogradsky instability is only a classical instability and it does not necessarily mean a quantum instability as argued in [12]. Moreover, it has been suggested by these authors that these type of instabilities can be removed by quantum physics. So keeping this in mind we can venture further and see what the Nash theory predicts. Finding motivations from the writings of Nash [4], we take it forward that any solution of vacuum Einstein equations is also a solution of these equations. However, at the same time they can provide for a wider variety of solutions due to their nature, describing motions in a gravitational field. We take a spatially flat FRW spacetime as

\[ ds^2 = -dt^2 + a(t)^2 \left( dr^2 + r^2 d\Omega^2 \right). \quad (2) \]

The components of the Nash tensor take the form

\[ N^{tt} = -3 \left( 2\dot{H} H + 6H^2 - \dot{H}^2 \right), \quad (3) \]

and

\[ N^{rr} = 2\ddot{H} + 12\dot{H} H + 9\dot{H}^2 + 18\dddot{H} H^2. \quad (4) \]

All the components are written as a function of \( H = \frac{\dot{a}}{a} \), the Hubble function. We can then write the trace of the Nash tensor \( (N = g_{\mu\nu} N^{\mu\nu}) \) as

\[ N = tr(N) = 6 \left( \dddot{H} + 7\dot{H} H + 4\dot{H}^2 + 12\dddot{H} H^2 \right). \quad (5) \]

We can further simplify the equations by introducing the variable \( y = H^{1/2} \),

\[ y = H^{1/2}, \quad (6) \]

which leads to the equations

\[ N^{tt} = -12y^3 \left( \dddot{y} + 3g^2 \dot{y} \right) = 0, \quad (7) \]

\[ N^{rr} = 4 \left( \dot{y} y + 6\dddot{y} y^3 + 3g\dot{y} + 9g^5 \dot{y} + 15g^2 y^2 \right) = 0. \quad (8) \]

With a simple manipulation we rewrite these as

\[ \frac{d}{dt} (\dot{y} + y^3) = 0, \quad (9) \]

\[ y^2 \frac{d^2}{dt^2} (\dot{y} + y^3) - 3 \frac{d}{dt} \left( (\dot{y} + y^3)^2 \right) = 0. \quad (10) \]

It is curious to note, that only one independent component \( \frac{d}{dt} (\dot{y} + y^3) \), dictates the structure of the field equations for a Nash vacuum. From the field Eq. 3, this component is identified as

\[ -3 \left( 2\ddot{H} H + 6\dot{H}^2 - \dot{H}^2 \right) = 0. \quad (11) \]

This feature of one independent component Eq. 11 dictating dynamics in gravitational field is a unique feature of Nash gravity. Such an equation can not be found in standard GR. It is already known that Nash equations accommodate two solutions of standard GR, the Milne and the de-Sitter metric [12], however, these do not support a late-time acceleration of the universe. In the next section we focus on solving the spatially flat Nash vacuum equations for a solution that can describe the present acceleration of the universe, preceded by a deceleration.

III. EXACT SOLUTION AND COSMOLOGICAL ANALYSIS

We find that the independent Nash Eq. 11 can be written in terms of purely kinematic parameters involved in cosmology, in particular, the deceleration \( (q) \) and the jerk parameter \( (j) \). The standard parameters, Hubble, deceleration, jerk and statefinder are defined respectively as
For all \( q < 0 \), the statefinder is negative. This indicates that a Nash vacuum allows the statefinder parameter to attain only negative values during an epoch of acceleration. Only for \( q > \frac{1}{2} \), the statefinder can be positive. This restriction is not found in standard GR.

3. At any ‘critical point’ where the universe goes through a transition from deceleration into acceleration or vice versa, \( q = 0 \), which indicates \( j = \frac{3}{2} \) and \( s = -\frac{1}{2} \). Therefore, the point of transition(s) is fixed on the parameter space at the outset.

4. By definition, at \( q = \frac{1}{2} \), there is a divergence of the statefinder parameter (See Eq. (15)). Therefore, the deceleration can take a value either in the \( q < \frac{1}{2} \) domain or in the \( q > \frac{1}{2} \) domain. The former one seems more plausible, since the deceleration is supposed to have a smooth transition from positive into negative domain and vice versa during the evolution of the universe, without any discontinuity.

Altogether, a Nash vacuum cosmology seems to provide a constrained cosmological dynamics, with specific allowed values of kinematic parameters in different phases of the universe. For an exact solution we only need to solve Eq. (11). Most of the available alternative models of late-time cosmology closely follow and reiterate the dynamics of standard ΛCDM cosmology. However, it is an interesting case when a simple modified theory of gravity with a specific restriction on the kinematic parameters as in Eq. (16) can produce an accelerating solution even without any matter. First, we write Eq. (11) as an equation whose arguments are redshift rather than cosmic time

\[
\frac{H''}{H} + \frac{1}{2} \frac{H'^2}{H^2} - \frac{2H'}{Hx} = 0. \tag{18}
\]

Primes are derivatives with respect to \((1 + z) = x\). Solving this we write an exact analytical form for Hubble as

\[
H(z) = H_0 \left[ C_0 \left\{ (1 + z)^3 - 1 \right\} + C_0(1 + 2C_1) \right]^\frac{2}{3}, \tag{19}
\]

where \( H_0 \) is the value of Hubble parameter at the present time, i.e., \( z = 0 \). \( C_0 \) and \( C_1 \) are constants of integration.

This is an exact solution and although we do not expect this to match with observational data extremely well, we can indeed establish some basic requirements in view of a consistent late-time cosmology. We need to ensure that, we have (i) a consistent present value of the Hubble and deceleration parameter and (ii) evolution curves of the Hubble function and the deceleration fitting in with observations without any discontinuity. An estimation of the model parameters are done to meet these requirements, for which we use the following set of observations

- Joint Light Curve Analysis from SDSS – II and SNLS collaborations, resulting in the the Supernova distance modulus data [18],

We write Eq. (11) as a linear combination of \( j \) and \( q \) as follows

\[
-2(j-1) + (q+1)^2 = 0. \tag{16}
\]

This is already a new result that puts significant constraints on the allowed evolution of the universe. We note below a few non-trivial features and resulting speculations from Eq. (16), even before solving for \( H \).

1. It is not possible for the jerk parameter to have a negative value, anywhere during the expansion of the Universe described by a Nash vacuum. This restriction is not found in standard GR.

2. The statefinder parameter becomes

\[
s = \frac{(q + 1)^2}{6(q - \frac{1}{2})}. \tag{17}
\]
TABLE I: Best Fit Parameter values of three parameters: (i) Dimensionless Hubble $h_0$, (ii) $C_0$ and (iii) $C_1$.

| $H_0$  | $C_0$  | $C_1$  |
|--------|--------|--------|
| 73.2$^{+1.2}_{-1.2}$ | 0.103$^{+0.005}_{-0.005}$ | 4.5$^{+0.2}_{-0.2}$ |

$OHD + JLA + BAO$

FIG. 2: Parameter Space Confidence Contours showing estimation of the uncertainty, the best fit and the likelihood analysis of parameters (combination of data from $OHD+JLA+BAO$).

- estimated measurement of Hubble parameter value in the present epoch ($OHD$) [19] and
- the Baryon Acoustic Oscillation (BAO) data from the BOSS collaborations and 6dF Galaxy Survey [20].

We have used a numerical code Markov Chain Monte Carlo simulation (MCMC) written in python. The analysis is statistical and the results are shown in the form of confidence contours on the parameter space, as in Fig. 2. The contours point out to the best fit values of three parameters: (i) Dimensionless Hubble $h_0$ ($\sim H_0/100\, \text{km\,Mpc}^{-1}\, \text{sec}^{-1}$), (ii) $C_0$ and (iii) $C_1$. The best possible values of the three parameters and $1\sigma$ error estimations are written in Table I. $H_0$, the current value of Hubble parameter is quite consistent with the observations [19]. The evolution of $H(z)$ as a function of $z$ is shown in Fig. 3. Observational data points are fitted alongwith the curve which shows sufficient match during the late-times. However, for $z > 1$, the departure from standard $ΛCDM$ is apparent.

The curves depicting deceleration and the jerk parameter evolutions are shown in Fig. 4 and they suggest a departure from standard cosmology. Around $z \sim 0$, i.e., the present time, the deceleration is close to $-0.8$ which indicates a universe expanding at a slightly faster rate than the anticipated acceleration (for standard $ΛCDM$ cosmology this value is $\sim 0.65$). The redshift of transition from deceleration into the acceleration is $z_t < 1$ which goes well with observations. For a higher range of redshift, the deceleration becomes positive, indicating a decelerated expansion immediately prior to the present epoch. The jerk parameter is close to 2 around $z \sim 0$ which too, is different from standard $ΛCDM$, for which jerk is equal to 1. The effective equation of state $ω_{eff}$ as shown in Fig. 5 shows a dark energy dominated acceleration but for $z > 1$ suggests a radiation dominated deceleration. The departures are minor and we believe a proper description of the source of energy-momentum distribution in Nash cosmology can come in as a correction on the level of field equations and provide a lot more observational equivalence.
IV. NASH CURVATURE CORRECTION AS A SCALAR FIELD

The exact solution portrays a cosmic acceleration and tells us that Nash theory accommodates more solutions compared to GR, even in vacuum. This is due to a geometric source of energy which finds its origin in the higher curvature terms. From what we have seen from the parameter estimation in the last section, this geometric source may not exactly replicate a desired dark energy evolution during the present epoch in Nash theory. However, the fact that it produces an acceleration even without any matter is a notion worthy of further discussions. A few parallels of standard cosmic acceleration can be found in other relevant gravitational systems carrying quadratic curvature terms. For example, it was proved that theories with only a quadratic action can admit vacua of arbitrary constant curvature \cite{21}, and a mixed Einstein-quadratic curvature action can generate the effect of a unique cosmological constant term, sometimes termed a unique vacua \cite{22}. Given the fact that a Nash action has no Einstein-Hilbert term, we compare the acceleration with a residual vacuum dynamics generated from a unique, self-interacting scalar field. We write the Nash Eqs. 3 and 4 in the following manner,

\[3H^2 - 3H^2 - 3\beta \left(2\dot{H}H + 6\dot{H}H^2 - \dot{H}^2\right) = 0,\]  
\[-2\dot{H} - 3H^2 + (2\dot{H} + 3H^2) + \beta \left(2\ddot{H} + 12\dot{H}^2 + 9\ddot{H} + 18\dot{H}H^2\right) = 0.\]  
\[\rho_{GR} = 3H^2 = \rho_{vac} = 3\beta \left(H^2 + 2\ddot{H}H + 6\dot{H}H^2 - \dot{H}^2\right),\]
\[p_{GR} = -2\dot{H} - 3H^2 = p_{vac} = -\beta \left(2\ddot{H} + 12\dot{H}H + 9\ddot{H} + 18\dot{H}H^2\right),\]
\[\frac{\dot{\phi}^2}{2} + V(\phi) = 3\beta \left(\frac{H^2}{\beta} + 2\ddot{H}H + 6\dot{H}H^2 - \dot{H}^2\right),\]
\[\frac{\dot{\phi}^2}{2} - V(\phi) = -\beta \left(2\ddot{H} + 12\dot{H}H + 9H^2 + 18\dot{H}H^2 + \frac{2H + 3H^2}{\beta}\right).\]

Let us suppose that the scalar self-interaction is given by the $\phi^4$ potential used in spontaneous symmetry breaking. Then it is easy to see that for mathematical consistency the mass term must be time-dependent. We can then write $V(\phi)$ as

\[V(\phi) = V_0 - M(t)^2\phi^2 + \frac{\lambda}{4}\phi^4.\]
The scalar field and $M(t)$ both have mass dimension one while $\lambda$ is dimensionless. The vacuum expectation value ($vev$) $v$ can be derived from this Higgs-type potential as

$$\frac{\partial V}{\partial \phi} \bigg|_v = 0, \quad v = \sqrt{\frac{2M(t)^2}{\lambda}}. \quad (25)$$

If we further carry forward the analogy with the Higgs field, this time variation leads to a variation of standard couplings. The forebear of any such ideas allowing variations of fundamental couplings is the Large Numbers Hypothesis of Dirac [23], which received many theoretical treatments over time. Phenomenological analysis of these theories are popular and have produced some consequences of considerable interest [24]. For example, variation of a fundamental coupling such as the fine structure constant generates non-trivial variations in the proton-to-electron mass ratio through the QCD scale $\Lambda_{QCD}$ [25]. There is also a constant (alongwith the Yukawa coupling) $\lambda_{e,q}$, which received many theoretical treatments over time. Phenomenological analyses of these data-sets one can also infer that the $\mu$ variation is directly proportional with a fine structure constant variation and $\nu$-variation are connected through the equation (for more discussion see [31]).

$$\Delta v/v = \frac{\Delta m_p - \Delta m_e}{m_p - m_e} = -91 \frac{\Delta v}{100 v}. \quad (30)$$

If $v_0$ and $v_z$ are the values of Higgs vev at the present epoch (the present value of Higgs vev is $v_0 = 246 GeV$) and at some redshift $z$, then $\Delta v/v$ is equivalent to $(v_z - v_0)/v_0$. This variation can be measured from the molecular absorption spectra of a series of quasars. From the observational data-sets one can also infer that the $\mu$ variation is directly proportional with a fine structure constant variation, with the proportionality constant being measured as $R \sim 50$ [30], and related to high-energy scales in theories of unification. The theoretical $\mu$ variation and $\nu$-variation are connected through the equation (for more discussion see [31]).

$$\Delta v/v = \frac{(v_z - v_0)}{v_0}. \quad (31)$$

We compare the theoretically derived $\Delta \mu/\mu$ with the observational bound found from the data analysis of Cesium Atomic Clock spectroscopy [32].

$$\frac{\Delta \mu}{\mu} = (-0.5 \pm 1.6) \times 10^{-16} \text{ year}^{-1}. \quad (32)$$

It is more practical to write the variation in comparison with the Hubble parameter whose present value is $H_0 \approx 7 \times 10^{-11} \text{ year}^{-1}$. Thus, the variation is in a scale of $\frac{\Delta \mu}{\mu} \approx 10^{-6} H_0$. The observed variations in different redshifts are given in Table. III as weighted average analyses of Hydrogen molecular spectra from different quasars [33–39].

While the data in Table III clearly shows a variation of the coupling during cosmic expansion, one cannot claim
TABLE III: The data-table for the variation of $\Delta \alpha/\alpha$. Unit of variation is parts per million (ppm).

| Source          | Redshift | $\Delta \alpha/\alpha$ (ppm) | Spectrograph. |
|-----------------|----------|------------------------------|---------------|
| J0026−2857      | 1.02     | $3.5 \pm 8.9$               | UVES          |
| J0058+0041      | 1.07     | $-1.4 \pm 7.2$              | HIRES         |
| 3 sources       | 1.08     | $4.3 \pm 3.4$               | HIRES         |
| HS1549+1919     | 1.14     | $-7.5 \pm 5.5$              | UVES/HIRES/HDS|
| HE0515−4414     | 1.15     | $-1.4 \pm 0.9$              | UVES          |
| J1237+0106      | 1.31     | $-4.5 \pm 8.7$              | HIRES         |
| HS1549+1919     | 1.34     | $-0.7 \pm 6.6$              | UVES/HIRES/HDS|
| J0841+0312      | 1.34     | $3.0 \pm 4.0$               | HIRES         |
| J0841+0312      | 1.34     | $5.7 \pm 4.7$               | UVES          |
| J0108−0037      | 1.37     | $-8.4 \pm 7.3$              | UVES          |
| HE0001−2340     | 1.58     | $-1.5 \pm 2.6$              | UVES          |
| J1029+1039      | 1.66     | $-4.7 \pm 5.3$              | HIRES         |
| HE1104−1805     | 1.69     | $1.3 \pm 2.6$               | UVES          |
| HS1946+7658     | 1.74     | $-7.9 \pm 6.2$              | HIRES         |
| HS1549+1919     | 1.80     | $-6.4 \pm 7.2$              | UVES/HIRES/HDS|
| Q1103−2645      | 1.84     | $3.5 \pm 2.5$               | UVES          |
| Q2206−1958      | 1.92     | $-4.6 \pm 6.4$              | UVES          |
| Q1755+57        | 1.97     | $4.7 \pm 4.7$               | HIRES         |
| PHL957          | 2.31     | $-0.7 \pm 6.8$              | HIRES         |
| PHL957          | 2.31     | $-0.2 \pm 12.9$             | UVES          |

The parameter $M_0$ is of mass dimension one in natural units. The dimensionless function of redshift $u(z)$ holds the key for any variation of $\mu$. Next, using Eqs. (23) and (24) we write

$$M(t)^2 = M_0^2 u(z).$$  \hspace{1cm} (33)

Using the solution for the scalar field we solve for $u(z)$, or $M(t)^2$, and estimate the Higgs vev $v(z)$. Then using $v(z)$, we find $\Delta \mu(z)$ and plot it in the top panel of Fig. 7. The data points from Table. III are also fitted with the curve. Although the curve does not exactly move through all the data points, at a scale of variation given by $\simeq 10^{-6} H_0$ it is not expected to do so either. The curve falls within the overall range of variation taken from observations of Molecular absorption spectra, only if $0.001 < \beta < 0.01$. The plot in Fig.
is for $\beta = 0.005$. In our interpretation it is the best fit parameter value of $\beta$, although no proper estimation technique is involved here apart from a classic trial and error method. This mild cosmic variation and its comparison with molecular absorption lines of Quasar spectra also provides us motivation to look for variations in fine structure constant $\alpha$. Once again, we refer to the data set from a combined analysis of constraints that uses molecular absorption spectroscopy of different Quasars [40–44].

The variations are quite mild in the scale of $10^{-6}H_0$ or parts per million (ppm) and as expected, more mild compared to the $\mu$-variation. The specific set of measurements of $(\alpha_z - \alpha_0)/\alpha_0 = \Delta\alpha/\alpha$ taken here are from the HIRES and UVES spectrographs [45–48], operated at the Keck and VLT telescopes. The measurements from a variety of source Quasars are written in the form of Table. III. We fit these measurements with the derived evolution of $\Delta\alpha/\alpha$ and give the combined curve in the bottom panel of Fig. 7. The scale of variation is shown for $\beta = 0.005$. The parameter $\beta$ therefore seems to be a necessary coupling of the theory. It can be called a weight factor, that serves our requirement of observational validity. Looking into Eq. (22), it is evident that $\beta$ should have a dimension $-2$ in natural units ($M_\odot^{-2}$). At this point, we make a speculation that $\beta \sim H_0^{-2}$ and leave this particular issue to future discussions. We also note that the derived variation of $\Delta\mu/\mu$ and $\Delta\alpha/\alpha$ seems like one half of a sinusoid and inspires speculations regarding a probable oscillatory behavior of the variation, within proper scale. Similar variations have recently been reported in the context of generalized Brans-Dicke type theories [31] and models of running vacuum [49].

\section*{V. CONCLUSIONS}

Quite a few quadratic actions of gravity have been considered in gravitational physics and their implications in different issues, related to renormalizability or cosmic expansion history are more or less well documented. A Nash theory appears to be a specific choice within a larger class of theories [17], that promote field equations of higher order and remains relatively unexplored. Due to the specific comparative weighting of the term quadratic in the scalar curvature and that quadratic in the Ricci tensor in the action, the field equations of Nash theory provides a few unique properties. The primary motivation is to look into these properties from a cosmological perspective, even though the original theory was just conceived out of a mathematical curiosity: to write ‘an interesting equation’ for a supposedly Yukawa-like gravitational field that can relate to Klein-Gordon equations in a non-relativistic context. The additional purpose of this manuscript is to reaffirm that a Nash equation, the equation with a Yukawa-like aspect can indeed be portrayed as an alternative vacuum equation [5].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7.png}
\caption{Top Panel: Evolution of the conjectured $\Delta\mu/\mu$ as a function of redshift along with the fitted observational points from quasar absorption spectra, following Table III. The associated error bars are also shown in the graph. Bottom Panel: Evolution of the conjectured $\Delta\alpha/\alpha$ as a function of redshift along with the fitted observational points from quasar absorption spectra, following Table III. The associated error bars are also shown in the graph.}
\end{figure}

We work with the spatially flat cosmological equations in the Nash theory. No source of matter, fluid, exotic fields or a cosmological constant is taken. The field equations, although carrying fourth order derivatives of the scale factor, due their characteristic symmetry, is governed by a single independent component in the form of Eq. (11). The component leads to a simple yet non-trivial relation between the kinematic parameters jerk and deceleration in the form of Eq. (10) which is a distinct property of Nash cosmology. It ensures that the cosmological evolution is constrained at the outset such that one may not choose these parameters at will. We solve the field equations directly to write an exact solution for the Hubble parameter. The viability of this solution is discussed in comparison with a wide array of observational data from the present cosmological epoch. The Hubble evolution is satisfactory compared to the luminosity distance modulus measurements and the evolution of deceleration parameter suggests an epoch of deceleration prior to the present acceleration. The redshift of transition is found to be quite satisfactory. There are, however, a few departures. The present value of deceleration does not match exactly with standard $\Lambda$CDM cosmology; the effective EOS of the system suggests a radiation dominated era of deceleration rather than a matter-dominated one. These issues
can indeed be addressed, gradually, once we figure out the best possible way to include a energy momentum tensor within the theory to engineer some fine-tunings in this otherwise crude yet promising cosmological behavior.

To portray the Nash equations as a modified vacuum equation, we incorporate a Higgs-like scalar field. We manipulate the vacuum equations such that they seem equivalent to a standard GR vacuum plus the Higgs scalar and its self-interaction. We find that the vacuum expectation value of this so-called Higgs scalar field should be a function of coordinates (time alone in this case) in order for this construction to be consistent. This opens up an interesting discussion if we identify this scalar as the Standard Model Higgs field. In such a case, this variation leads to non-trivial variations in proton-electron mass ratio and the fine structure constant which needs to be within a certain range to minimize the variation of fundamental couplings. One may imagine this parameter to be a different scalar field of the dimension $H^2$, and formulate a Higgs-Nash theory. We keep in mind that any solution of an Einstein vacuum is already a solution of Nash vacuum, however, this never excludes the possibility of finding a larger class of new solutions. Therefore, another way to proceed might be to look for new static solutions in the theory and check for viability of standard principles that are viable in GR, such as the Birkhoff Theorem. These will be addressed in near future by the authors.

We conclude with a positive intent and curiosity, by commenting that it is possible for the Nash theory to emerge as a good theory and provide more of such interesting phenomenology. A simple vacuum solution of the theory can challenge our usual understandings of the nature, for instance, the endurance of a fundamental constant approach or the viability of the Equivalence principles. We need to formulate a stronger version of the theory that can fill in different gaps. One possible way of doing that is perhaps already hinted in this manuscript, through the weight parameter $\beta$ which needs to be within a certain range to minimize the variation of fundamental couplings. One may imagine this parameter to be a different scalar field of the dimension $H^2$ and formulate a Scalar-Nash theory. We keep in mind that any solution of an Einstein vacuum is already a solution of Nash vacuum, however, this never excludes the possibility of finding a larger class of new solutions. Therefore, another way to proceed might be to look for new static solutions in the theory and check for viability of standard principles that are viable in GR, such as the Birkhoff Theorem. These will be addressed in near future by the authors.

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