THE PLANE FIXED POINT PROBLEM

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Dedicated to Harold Bell

Abstract. In this paper we present proofs of basic results, including those developed so far by H. Bell, for the plane fixed point problem. Some of these results had been announced much earlier by Bell but without accessible proofs. We define the concept of the variation of a map on a simple closed curve and relate it to the index of the map on that curve: Index = Variation + 1. We develop a prime end theory through hyperbolic chords in maximal round balls contained in the complement of a non-separating plane continuum $X$. We define the concept of an outchannel for a fixed point free map which carries the boundary of $X$ minimally into itself and prove that such a map has a unique outchannel, and that outchannel must have variation $= -1$. We also extend Bell’s linchpin theorem for a foliation of a simply connected domain, by closed convex subsets, to arbitrary domains in the sphere.

We introduce the notion of an oriented map of the plane. We show that the perfect oriented maps of the plane coincide with confluent (that is composition of monotone and open) perfect maps of the plane. We obtain a fixed point theorem for positively oriented, perfect maps of the plane. This generalizes results announced by Bell in 1982 (see also [1]). It follows that if $X$ is invariant under an oriented map $f$, then $f$ has a point of period at most two in $X$.

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We denote the plane by $\mathbb{C}$, the Riemann sphere by $\mathbb{C}^\infty = \mathbb{C} \cup \{\infty\}$, the real line by $\mathbb{R}$ and the unit circle by $S^1 = \mathbb{R}/\mathbb{Z}$. Let $X$ be a plane continuum. Since $\mathbb{C}$ is locally connected and $X$ is closed, complementary domains of $X$ are open. By $T(X)$ we denote the topological hull of $X$ consisting of $X$ union all of its bounded complementary domains. Thus, $\mathcal{U}^\infty = \mathbb{C}^\infty \setminus T(X)$ is a simply-connected open domain containing $\infty$. The following is a long-standing question in topology.

**Fixed Point Question:** “Does a continuous function taking a non-separating plane continuum into itself always have a fixed point?”

It is easy to see that a map of a plane continuum to itself can be extended to a perfect map of the plane. We study the slightly more general question, “Is there a plane continuum $Z$ and a perfect continuous function $f : \mathbb{C} \to \mathbb{C}$ taking $Z$ into $T(Z)$ without fixed points in $T(Z)$?” A Zorn’s Lemma argument shows that if one assumes the answer is “yes,” then there is a subcontinuum $X \subset Z$ minimal with respect to these properties. It will follow from Theorem 6.5 that for such a minimal continuum, $f(X) = X = \partial T(X)$ (though it may not be the case that $f(T(X)) \subset T(X)$). Here $\partial T(X)$ denotes the boundary of $T(X)$. We recover Bell’s result [2] (see also Sieklucki [23], and Iliadis [14]) that the boundary of $X$ is indecomposable (with a dense channel, explained later).
In this paper we use tools first developed by Bell to elucidate the action of a fixed point free map (should one exist). We are indebted to Bell for sharing his insights with us. Many of the results of this paper were first obtained by him. Unfortunately, many of the proofs were not accessible. We believe that they deserve to be developed in order to be useful to the mathematical community. The results of this paper are also crucial to several recent results regarding the extension of isotopies of plane continua [20], the existence of fixed points for branched covering maps of the plane [7], fixed points in non-invariant plane continua [5], the existence of locally connected models for all connected Julia sets of complex polynomials [6] and an estimate on the number of attracting and neutral periodic orbits of complex polynomials [8].

We have stated many of these results using existing notions such as prime ends. We introduce Bell’s notion of variation and prove his theorem that index equals variation +1; Theorem 2.13. We also extended Bell’s linchpin Theorem 4.5 for simply connected domains to arbitrary domains in the sphere and given a proof using an elegant argument due to Kulkarni and Pinkall [15]. Our version of this theorem (Theorem 3.5) is essential for the results later in the paper. Theorem 7.1 (Unique Outchannel) is a new result due to Bell. Complete proofs of Theorems 2.13, 2.14, 4.5 and 7.1 appear in print for the first time.

The classical fixed point question asks whether each map of a non-separating plane continuum into itself must have a fixed point. Cartwright and Littlewood [10] showed that the answer is yes if the map can be extended to an orientation-preserving homeomorphism of the plane. It was 25 years before Bell [4] extended this to the class of all homeomorphisms of the plane. Bell announced in 1984 (see also Akis [1]) that the Cartwright-Littlewood Theorem can be extended to the class of all holomorphic maps of the plane. These maps behave like orientation-preserving homeomorphisms in the sense that they preserve local orientation. Compositions of open, perfect and of monotone, perfect surjections of the plane are confluent and naturally decompose into two classes, one of which preserves and the other of which reverses local orientation. We show that any confluent map of the plane is itself a composition of a monotone and a light-open map of the plane. We also show that an oriented map of the plane induces a map to the circle of prime ends of an acyclic continuum from the circle of prime ends of a component of its pre-image. Finally we will show that each invariant non-separating plane continuum, under a positively-oriented map of the plane, must contain a fixed point. It follows that any confluent map of the plane has a point of period at most two in any non-separating invariant sub-continuum.
For the convenience of the reader we have included an index at the end of the paper.

2. Tools

Let \( p : \mathbb{R} \to S^1 \) denote the covering map \( p(x) = e^{2\pi ix} \). Let \( g : S^1 \to S^1 \) be a map. By the degree of the map \( g \), denoted by \( \text{degree}(g) \), we mean the number \( \hat{g}(1) - \hat{g}(0) \), where \( \hat{g} : \mathbb{R} \to \mathbb{R} \) is a lift of the map \( g \) to the universal covering space \( \mathbb{R} \) of \( S^1 \) (i.e., \( p \circ \hat{g} = g \circ p \)). It is well-known that \( \text{degree}(g) \) is independent of the choice of the lift.

2.1. Index. Let \( g : S^1 \to \mathbb{C} \) be a map and \( f : g(S^1) \to \mathbb{C} \) a fixed point free map. Define the map \( v : S^1 \to S^1 \) by

\[
v(t) = \frac{f(g(t)) - g(t)}{|f(g(t)) - g(t)|}.
\]

Then the map \( v : S^1 \to S^1 \) lifts to a map \( \hat{v} : \mathbb{R} \to \mathbb{R} \). Define the index of \( f \) with respect to \( g \), denoted \( \text{ind}(f,g) \), by

\[
\text{ind}(f,g) = \hat{v}(1) - \hat{v}(0) = \text{degree}(v).
\]

Note that \( \text{ind}(f,g) \) measures the net number of revolutions of the vector \( f(g(t)) - g(t) \) as \( t \) travels through the unit circle one revolution in the positive direction.

**Remark 2.1.** (a) If \( g : S^1 \to \mathbb{C} \) is a constant map with \( g(S^1) = c \) and \( f(c) \neq c \), then \( \text{ind}(f,g) = 0 \).

(b) If \( f \) is a constant map and \( f(\mathbb{C}) = w \) with \( w \notin g(S^1) \), then \( \text{ind}(f,g) = \text{wind}(g,S^1,w) \), the winding number of \( g \) about \( w \). In particular, if \( f : S^1 \to T(S^1) \setminus S^1 \) is a constant map, then \( \text{ind}(f, \text{id}|_{S^1}) = 1 \), where \( \text{id}|_{S^1} \) is the identity map on \( S^1 \).

Suppose \( S \subset \mathbb{C} \) is a simple closed curve and \( A \subset S \) is a subarc of \( S \) with endpoints \( a \) and \( b \). Then we write \( A = [a,b] \) if \( A \) is the arc obtained by traveling in the counter-clockwise direction from the point \( a \) to the point \( b \) along \( S \). In this case we denote by \( < \) the linear order on the arc \( A \) such that \( a < b \). We will call the order \( < \) the counterclockwise order on \( A \). Note that \( [a,b] \neq [b,a] \).

More generally, for any arc \( A = [a,b] \subset S^1 \), with \( a < b \) in the counterclockwise order, define the fractional index \( \text{ind}[9] \) of \( f \) on the sub-path \( g|_{[a,b]} \) by

\[
\text{ind}(f, g|_{[a,b]}) = \hat{v}(b) - \hat{v}(a).
\]

While, necessarily, the index of \( f \) with respect to \( g \) is an integer, the fractional index of \( f \) on \( g|_{[a,b]} \) need not be. We shall have occasion to use fractional index in the proof of Theorem 2.13.
Proposition 2.2. Let \( g : S^1 \to \mathbb{C} \) be a map with \( g(S^1) = S \), and suppose \( f : S \to \mathbb{C} \) has no fixed points on \( S \). Let \( a \neq b \in S^1 \) with \([a, b] \) denoting the counterclockwise subarc on \( S^1 \) from \( a \) to \( b \) (so \( S^1 = [a, b] \cup [b, a] \)). Then \( \text{ind}(f, g) = \text{ind}(f, g|_{[a, b]}) + \text{ind}(f, g|_{[b, a]}) \).

2.2. Stability of Index. The following standard theorems and observations about the stability of index under a fixed point free homotopy are consequences of the fact that index is continuous and integer-valued.

Theorem 2.3. Let \( h_t : S^1 \to \mathbb{C} \) be a homotopy. If \( f : \cup_{t \in [0, 1]} h_t(S^1) \to \mathbb{C} \) is fixed point free, then \( \text{ind}(f, h_0) = \text{ind}(f, h_1) \).

An embedding \( g : S^1 \to S \subset \mathbb{C} \) is orientation preserving if \( g \) is isotopic to the identity map \( id|_{S^1} \). It follows from Theorem 2.3 that if \( g_1, g_2 : S^1 \to S \) are orientation preserving homeomorphisms and \( f : S \to \mathbb{C} \) is a fixed point free map, then \( \text{ind}(f, g_1) = \text{ind}(f, g_2) \). Hence we can denote \( \text{ind}(f, g) \) by \( \text{ind}(f, S) \) and if \([a, b] \) is a positively oriented subarc of \( S^1 \) we denote \( \text{ind}(f, g_1|_{[a, b]}) \) by \( \text{ind}(f, g_1([a, b])) \), by some abuse of notation when the extension of \( g_1 \) over \( S^1 \) is understood.

Theorem 2.4. Suppose \( g : S^1 \to \mathbb{C} \) is a map with \( g(S^1) = S \), and \( f_1, f_2 : S \to \mathbb{C} \) are homotopic maps such that each level of the homotopy is fixed point free on \( S \). Then \( \text{ind}(f_1, g) = \text{ind}(f_2, g) \).

In particular, if \( S \) is a simple closed curve and \( f_1, f_2 : S \to \mathbb{C} \) are maps such that there is a homotopy \( h_t : S \to \mathbb{C} \) from \( f_1 \) to \( f_2 \) with \( h_0 \) fixed point free on \( S \) for each \( t \in [0, 1] \), then \( \text{ind}(f_1, S) = \text{ind}(f_2, S) \).

Corollary 2.5. Suppose \( g : S^1 \to \mathbb{C} \) is an orientation preserving embedding with \( g(S^1) = S \), and \( f : S \to T(S) \) is a fixed point free map. Then \( \text{ind}(f, g) = \text{ind}(f, S) = 1 \).

Proof. Since \( f(S) \subset T(S) \) which is a disk with boundary \( S \) and \( f \) has no fixed point on \( S \), there is a fixed point free homotopy of \( f|_S \) to a constant map \( c : S \to \mathbb{C} \) taking \( S \) to a point in \( T(S) \setminus S \). By Theorem 2.4 \( \text{ind}(f, g) = \text{ind}(c, g) \). Since \( g \) is orientation preserving it follows from Remark 2.1 (b) that \( \text{ind}(c, g) = 1 \). \( \square \)

Theorem 2.6. Suppose \( g : S^1 \to \mathbb{C} \) is a map with \( g(S^1) = S \), and \( f : T(S) \to \mathbb{C} \) is a map such that \( \text{ind}(f, g) \neq 0 \), then \( f \) has a fixed point in \( T(S) \).

Proof. Notice that \( T(S) \) is a locally connected, non-separating, plane continuum and, hence, contractible. Suppose \( f \) has no fixed point in \( T(S) \). Choose point \( g \in T(S) \). Let \( c : S^1 \to \mathbb{C} \) be the constant map \( c(S^1) = \{g\} \). Let \( H \) be a homotopy from \( g \) to \( c \) with image in \( T(S) \). Since \( H \) misses the fixed point set of \( f \), Theorem 2.3 and Remark 2.1 (a) imply \( \text{ind}(f, g) = \text{ind}(f, c) = 0 \). \( \square \)
2.3. Variation. In this subsection we introduce the notion of variation of a map on an arc and relate it to winding number.

**Definition 2.7** (Junctions). The standard junction \( J_O \) is the union of the three rays \( J_O^+ = \{ z \in \mathbb{C} \mid z = re^{i\pi/2}, \ r \in [0,\infty) \}, \ J_O^- = \{ z \in \mathbb{C} \mid z = r, \ r \in [0,\infty) \} \), having the origin \( O \) in common. A junction \( J_v \) is the image of \( J_O \) under any orientation-preserving homeomorphism \( h: \mathbb{C} \to \mathbb{C} \) where \( v = h(O) \). We will often suppress \( h \) and refer to \( h(J_O) \) as \( J_v \), and similarly for the remaining rays in \( J_v \). Moreover, we require that for each neighborhood \( W \) of \( v \), \( d(J_v^+ \setminus W, J_v^- \setminus W) > 0 \).

**Definition 2.8** (Variation on an arc). Let \( S \subset \mathbb{C} \) be a simple closed curve, \( f: S \to \mathbb{C} \) a map and \( A = [a,b] \) a subarc of \( S \) such that \( f(a), f(b) \in T(S) \) and \( f(A) \cap A = \emptyset \). We define the variation of \( f \) on \( A \) with respect to \( S \), denoted \( \text{var}(f,A,S) \), by the following algorithm:

1. Let \( v \in A \) and let \( J_v \) be a junction with \( J_v \cap S = \{v\} \).
2. Counting crossings: Consider the set \( M = f^{-1}(J_v) \cap [a,b] \). Each time a point of \( f^{-1}(J_v^+) \cap [a,b] \) is immediately followed in \( M \), in the counterclockwise order < on \( [a,b] \subset S \), by a point of \( f^{-1}(J_v^-) \) count +1 and each time a point of \( f^{-1}(J_v^-) \cap [a,b] \) is immediately followed in \( M \) by a point of \( f^{-1}(J_v^+) \) count −1. Count no other crossings.
3. The sum of the crossings found above is the variation \( \text{var}(f,A,S) \).

Note that \( f^{-1}(J_v^+) \cap [a,b] \) and \( f^{-1}(J_v^-) \cap [a,b] \) are disjoint closed sets in \([a,b]\). Hence, in (2) in the above definition, we count only a finite number of crossings and \( \text{var}(f,A,S) \) is an integer. Of course, if \( f(A) \) does not meet both \( J_v^+ \) and \( J_v^- \), then \( \text{var}(f,A,S) = 0 \).

If \( \alpha: S \to \mathbb{C} \) is any map such that \( \alpha|_A = f|_A \) and \( \alpha(S \setminus (a,b)) \cap J_v = \emptyset \), then \( \text{var}(f,A,S) = \text{win}(\alpha,S,v) \). In particular, this condition is satisfied if \( \alpha(S \setminus (a,b)) \subset T(S) \setminus \{v\} \). The invariance of winding number under suitable homotopies implies that the variation \( \text{var}(f,A,S) \) also remains invariant under such homotopies. That is, even though the specific crossings in (2) in the algorithm may change, the sum remains invariant. We will state the required results about variation below without proof. Proofs can also be obtained directly by using the fact that \( \text{var}(f,A,S) \) is integer-valued and continuous under suitable homotopies.

**Proposition 2.9** (Junction Straightening). Let \( S \subset \mathbb{C} \) be a simple closed curve, \( f: S \to \mathbb{C} \) a map and \( A = [a,b] \) a subarc of \( S \) such that \( f(a), f(b) \in T(S) \) and \( f(A) \cap A = \emptyset \). Any two junctions \( J_v \) and \( J_u \)
with \( u, v \in A \) and \( J_w \cap S = \{ w \} \) for \( w \in \{ u, v \} \) give the same value for \( \text{var}(f, A, S) \). Hence \( \text{var}(f, A, S) \) is independent of the particular junction used in Definition 2.8.

The computation of \( \text{var}(f, A, S) \) depends only upon the crossings of the junction \( J_v \) coming from a proper compact subarc of the open arc \((a, b)\). Consequently, \( \text{var}(f, A, S) \) remains invariant under homotopies \( h_t \) of \( f|\[a,b\] \) in the complement of \( \{ v \} \) such that \( h_t(a), h_t(b) \notin J_v \) for all \( t \). Moreover, the computation is stable under an isotopy \( h_t \) of the plane that moves the entire junction \( J_v \) (even off \( A \)), provided in the isotopy \( h_t(v) \notin f(A) \) and \( f(a), f(b) \notin h_t(J_v) \) for all \( t \).

In case \( A \) is an open arc \((a, b) \subseteq S \) such that \( \text{var}(f, A, S) \) is defined, it will be convenient to denote \( \text{var}(f, A, S) \) by \( \text{var}(f, A, S) \).

The following Lemma follows immediately from the definition.

**Lemma 2.10.** Let \( S \subset \mathbb{C} \) be a simple closed curve. Suppose that \( a < c < b \) are three points in \( S \) such that \( \{ f(a), f(b), f(c) \} \subset T(S) \) and \( f([a, b]) \cap [a, b] = \emptyset \). Then \( \text{var}(f, [a, b], S) = \text{var}(f, [a, c], S) + \text{var}(f, [c, b], S) \).

**Definition 2.11** (Variation on a finite union of arcs). Let \( S \subset \mathbb{C} \) be a simple closed curve and \( A = [a, b] \) a subcontinuum of \( S \) with partition a finite set \( F = \{ a = a_0 < a_1 < \cdots < a_n = b \} \). For each \( i \) let \( A_i = [a_i, a_{i+1}] \). Suppose that \( f \) satisfies \( f(a_i) \in T(S) \) and \( f(A_i) \cap A_i = \emptyset \) for each \( i \). We define the variation of \( f \) on \( A \) with respect to \( S \), denoted \( \text{var}(f, A, S) \), by

\[
\text{var}(f, A, S) = \sum_{i=0}^{n-1} \text{var}(f, [a_i, a_{i+1}], S).
\]

In particular, we include the possibility that \( a_n = a_0 \) in which case \( A = S \).

By considering a common refinement of two partitions \( F_1 \) and \( F_2 \) of an arc \( A \subset S \) such that \( f(F_1) \cup f(F_2) \subset T(S) \) and satisfying the conditions in Definition 2.11, it follows from Lemma 2.10 that we get the same value for \( \text{var}(f, A, S) \) whether we use the partition \( F_1 \) or the partition \( F_2 \). Hence, \( \text{var}(f, A, S) \) is well-defined. If \( A = S \) we denote \( \text{var}(f, S, S) \) simply by \( \text{var}(f, S) \).

2.4. **Index and variation for finite partitions.** What links Theorem 2.6 with variation is Theorem 2.13 below, first announced by Bell in the mid 1980’s (see also Akis [1]). Our proof is a modification of Bell’s unpublished proof. We first need a variant of Proposition 2.9.
Let \( r : \mathbb{C} \to T(S^1) \) be radial retraction: \( r(z) = \frac{z}{|z|} \) when \( |z| \geq 1 \) and \( r|_{T(S^1)} = \text{id}|_{T(S^1)} \).

**Lemma 2.12** (Curve Straightening). Suppose \( f : S^1 \to \mathbb{C} \) is a map with no fixed points on \( S^1 \). If \([a, b] \subset S^1\) is a proper subarc with \( f([a, b]) \cap [a, b] = \emptyset\), \( f((a, b)) \subset \mathbb{C} \setminus T(S^1)\) and \( f\{a, b\} \subset S^1\), then there exists a map \( \tilde{f} : S^1 \to \mathbb{C} \) such that \( \tilde{f}|_{S^1 \setminus (a, b)} = f|_{S^1 \setminus (a, b)}\), \( \tilde{f}|_{[a, b]} : [a, b] \to (\mathbb{C} \setminus T(S^1)) \cup \{f(a), f(b)\}\) and \( \tilde{f}|_{[a, b]}\) is homotopic to \( f|_{[a, b]}\) in \( \{a, b\} \cup \mathbb{C} \setminus T(S)\) relative to \( \{a, b\}\), so that \( r|_{\tilde{f}([a, b])}\) is locally one-to-one. Moreover, \( \text{var}(f, [a, b], S^1) = \text{var}(\tilde{f}, [a, b], S^1)\).

Note that if \( \text{var}(f, [a, b], S^1) = 0\), then \( r\) carries \( \tilde{f}([a, b])\) one-to-one onto the arc (or point) in \( S^1 \setminus (a, b)\) from \( f(a)\) to \( f(b)\). If the \( \text{var}(f, [a, b], S^1) = m > 0\), then \( r \circ \tilde{f}\) wraps the arc \([a, b]\) counterclockwise about \( S^1\) so that \( \tilde{f}([a, b])\) meets each ray in \( J_m\) \( m\) times. A similar statement holds for negative variation. Note also that it is possible for index to be defined yet variation not to be defined on a simple closed curve \( S\). For example, consider the map \( z \to 2z\) with \( S\) the unit circle since there is no partition of \( S\) satisfying the conditions in Definition 2.8.

**Theorem 2.13** (Index = Variation + 1, Bell). Suppose \( g : S^1 \to \mathbb{C} \) is an orientation preserving embedding onto a simple closed curve \( S\) and \( f : S \to \mathbb{C} \) is a fixed point free map. If \( F = \{a_0 < a_1 < \cdots < a_n\}\) is a partition of \( S\) and \( A_i = [a_i, a_{i+1}]\) for \( i = 0, 1, \ldots, n\) with \( a_{n+1} = a_0\) such that \( f(F) \subset T(S)\) and \( f(A_i) \cap A_i = \emptyset\) for each \( i\), then

\[
\text{ind}(f, S) = \text{ind}(f, g) = \sum_{i=0}^{n} \text{var}(f, A_i, S) + 1 = \text{var}(f, S) + 1.
\]

**Proof.** By an appropriate conjugation of \( f\) and \( g\), we may assume without loss of generality that \( S = S^1\) and \( g = \text{id}\). Let \( F\) and \( A_i = [a_i, a_{i+1}]\) be as in the hypothesis. Consider the collection of arcs \( \mathcal{K} = \{K \subset S \mid K \text{ is the closure of a component of } S \cap f^{-1}(f(S) \setminus T(S))\}\).

For each \( K \in \mathcal{K}\), there is an \( i\) such that \( K \subset A_i\). Since \( f(A_i) \cap A_i = \emptyset\), it follows from the remark after Definition 2.8 that \( \text{var}(f, A_i, S) = \sum_{K \subset A_i, K \in \mathcal{K}} \text{var}(f, K, S)\). By the remark following Proposition 2.9, we can compute \( \text{var}(f, K, S)\) using one fixed junction for \( A_i\). It is now clear that there are at most finitely many \( K \in \mathcal{K}\) with \( \text{var}(f, K, S) \neq 0\). Moreover, the images of the endpoints of each \( K\) lie on \( S\).

Let \( m\) be the cardinality of the set \( \mathcal{K}_f = \{K \in \mathcal{K} \mid \text{var}(f, K, S) \neq 0\}\). By the above remarks, \( m < \infty\) and \( \mathcal{K}_f\) is independent of the partition \( F\). We prove the theorem by induction on \( m\).
Suppose for a given $f$ we have $m = 0$. Observe that from the definition of variation and the fact that the computation of variation is independent of the choice of an appropriate partition, it follows that,

$$\text{var}(f, S) = \sum_{K \in \mathcal{K}} \text{var}(f, K, S) = 0.$$ 

We claim that there is a map $f_1 : S \to \mathbb{C}$ with $f_1(S) \subset T(S)$ and a homotopy $H$ from $f|_S$ to $f_1$ such that each level $H_t$ of the homotopy is fixed point free and $\text{ind}(f_1, \text{id}|_S) = 1$.

To see the claim, first apply the Curve Straightening Lemma 2.12 to each $K \in \mathcal{K}$ (if there are infinitely many, they form a null sequence) to obtain a fixed point free homotopy of $f|_S$ to a map $\tilde{f} : S \to \mathbb{C}$ such that $r|_{\tilde{f}(K)}$ is locally one-to-one on each $K \in \mathcal{K}$, where $r$ is radial retraction of $\mathbb{C}$ to $T(S)$, and $\text{var}(\tilde{f}, K, S) = 0$ for each $K \in \mathcal{K}$. Let $K$ be in $\mathcal{K}$ with endpoints $x, y$. Since $\tilde{f}(K) \cap K = \emptyset$ and $\text{var}(\tilde{f}, K, S) = 0$, $r|_{\tilde{f}(K)}$ is one-to-one, and $r \circ \tilde{f}(K) \cap K = 0$. Define $f_1|_K = r \circ \tilde{f}|_K$. Then $f_1|_K$ is fixed point free homotopic to $f|_K$ (with endpoints of $K$ fixed). Hence, if $K \in \mathcal{K}$ has endpoints $x$ and $y$, then $f_1$ maps $K$ to the subarc of $S$ with endpoints $f(x)$ and $f(y)$ such that $K \cap f_1(K) = \emptyset$. Since $\mathcal{K}$ is a null family, we can do this for each $K \in \mathcal{K}$ and set $f_1|_{S^1 \cup \mathcal{K}} = f|_{S^1 \cup \mathcal{K}}$ so that we obtain the desired $f_1 : S \to \mathbb{C}$ as the end map of a fixed point free homotopy from $f$ to $f_1$. Since $f_1$ carries $S$ into $T(S)$, Corollary 2.5 implies $\text{ind}(f_1, \text{id}|_S) = 1$.

Since the homotopy $f \simeq f_1$ is fixed point free, it follows from Theorem 2.4 that $\text{ind}(f, \text{id}|_S) = 1$. Hence, the theorem holds if $m = 0$ for any $f$ and any appropriate partition $F$.

By way of contradiction suppose the collection $\mathcal{F}$ of all maps $f$ on $S^1$ which satisfy the hypotheses of the theorem, but not the conclusion is non-empty. By the above $0 < |\mathcal{K}_f| < \infty$ for each. Let $f \in \mathcal{F}$ be a counterexample for which $m = |\mathcal{K}_f|$ is minimal. By modifying $f$, we will show there exists $f_1 \in \mathcal{F}$ with $|\mathcal{K}_{f_1}| < m$, a contradiction.

Choose $K \in \mathcal{K}$ such that $\text{var}(f, K, S) \neq 0$. Then $K = [x, y] \subset A_i = [a_i, a_{i+1}]$ for some $i$. By the Curve Straightening Lemma 2.12 and Theorem 2.4, we may suppose $r|_{f(K)}$ is locally one-to-one on $K$. Define a new map $f_1 : S \to \mathbb{C}$ by setting $f_1|_{S \setminus K} = f|_{S \setminus K}$ and setting $f_1|_K$ equal to the linear map taking $[x, y]$ to the subarc $f(x)$ to $f(y)$ on $S$ missing $[x, y]$. Figure 1 (left) shows an example of a (straightened) $f$ restricted to $K$ and the corresponding $f_1$ restricted to $K$ for a case where $\text{var}(f, K, S) = 1$, while Figure 1 (right) shows a case where $\text{var}(f, K, S) = -2$. 


Since on $S \setminus K$, $f$ and $f_1$ are the same map, we have
\[
\text{var}(f, S \setminus K, S) = \text{var}(f_1, S \setminus K, S).
\]
Likewise for the fractional index,
\[
\text{ind}(f, S \setminus K) = \text{ind}(f_1, S \setminus K).
\]
By definition (refer to the observation we made in the case $m = 0$),
\[
\text{var}(f, S) = \text{var}(f, S \setminus K, S) + \text{var}(f, K, S)
\]
\[
\text{var}(f_1, S) = \text{var}(f_1, S \setminus K, S) + \text{var}(f_1, K, S)
\]
and by Proposition 2.2,
\[
\text{ind}(f, S) = \text{ind}(f, S \setminus K) + \text{ind}(f, K)
\]
\[
\text{ind}(f_1, S) = \text{ind}(f_1, S \setminus K) + \text{ind}(f_1, K).
\]
Consequently,
\[
\text{var}(f, S) - \text{var}(f_1, S) = \text{var}(f, K, S) - \text{var}(f_1, K, S)
\]
and
\[
\text{ind}(f, S) - \text{ind}(f_1, S) = \text{ind}(f, K) - \text{ind}(f_1, K).
\]
We will now show that the changes in index and variation, going from $f$ to $f_1$ are the same (i.e., we will show that $\text{var}(f, K, S) - \text{var}(f_1, K, S) = \text{ind}(f, K) - \text{ind}(f_1, K)$). We suppose first that $\text{ind}(f, K) = n + \alpha$ for some nonnegative $n \in \mathbb{N}$ and $0 \leq \alpha < 1$. That is, the vector $f(z) - z$ turns through $n$ full revolutions counterclockwise and $\alpha$ part of a revolution counterclockwise as $z$ goes from $x$ to $y$ counterclockwise.
along $S$. (See Figure 1 (left) for a case $n = 0$ and $\alpha$ about 0.8.) Then as $z$ goes from $x$ to $y$ counterclockwise along $S$, $f_1(z)$ goes along $S$ from $f(x)$ to $f(y)$ in the clockwise direction, so $f_1(z) - z$ turns through $-(1 - \alpha) = \alpha - 1$ part of a revolution. Hence, $\text{ind}(f_1, K) = \alpha - 1$. It is easy to see that $\text{var}(f, K, S) = n + 1$ and $\text{var}(f_1, K, S) = 0$. Consequently,

$$\text{var}(f, K, S) - \text{var}(f_1, K, S) = n + 1 - 0 = n + 1$$

and

$$\text{ind}(f, K) - \text{ind}(f_1, K) = n + \alpha - (\alpha - 1) = n + 1.$$

In Figure 1 on the left we assumed that $f(x) < x < y < f(y)$. The cases where $f(y) < x < y < f(x)$ and $f(x) = f(y)$ are treated similarly. In this case $f_1$ still wraps around in the positive direction, but the computations are slightly different: $\text{var}(f, K) = 1$, $\text{ind}(f, K) = 1 + \alpha$, $\text{var}(f_1, K) = 0$ and $\text{ind}(f_1, K) = \alpha$.

Thus when $n \geq 0$, in going from $f$ to $f_1$, the change in variation and the change in index are the same. However, in obtaining $f_1$ we have removed one $K \in K_f$, reducing the minimal $m = |K_f|$ for $f$ by one, producing a counterexample $f_1$ with $|K_{f_1}| = m - 1$, a contradiction.

The cases where $\text{ind}(f, K) = n + \alpha$ for negative $n$ and $0 < \alpha < 1$ are handled similarly, and illustrated for $n = -2$ and $\alpha$ about 0.4 in Figure 1 (right).

2.5. Locating arcs of negative variation. The principal tool in proving Theorem 7.1 (unique outchannel) is the following theorem first obtained by Bell (unpublished). It provides a method for locating arcs of negative variation on a curve of index zero.

**Theorem 2.14** (Lollipop Lemma, Bell). Let $S \subset \mathbb{C}$ be a simple closed curve and $f : T(S) \to \mathbb{C}$ a fixed point free map. Suppose $F = \{a_0 < \cdots < a_n < a_{n+1} < \cdots < a_m\}$ is a partition of $S$, $a_{m+1} = a_0$ and $A_i = [a_i, a_{i+1}]$ such that $f(F) \subset T(S)$ and $f(A_i) \cap A_i = \emptyset$ for $i = 0, \ldots, m$. Suppose $I$ is an arc in $T(S)$ meeting $S$ only at its endpoints $a_0$ and $a_{n+1}$. Let $J_{a_0}$ be a junction in $(\mathbb{C}\setminus T(S)) \cup \{a_0\}$ and suppose that $f(I) \cap (I \cup J_{a_0}) = \emptyset$. Let $R = T([a_0, a_{n+1}] \cup I)$ and $L = T([a_{n+1}, a_{m+1}] \cup I)$. Then one of the following holds:

1. If $f(a_{n+1}) \in R$, then

$$\sum_{i \leq n} \text{var}(f, A_i, S) + 1 = \text{ind}(f, I \cup [a_0, a_{n+1}]).$$
(2) If $f(a_{n+1}) \in L$, then
\[
\sum_{i > n} \operatorname{var}(f, A_i, S) + 1 = \operatorname{ind}(f, I \cup [a_{n+1}, a_{m+1}]).
\]

(2) Prove. Without loss of generality, suppose $f(a_{n+1}) \in L$. Let $C = [a_{n+1}, a_{m+1}] \cup I$ (so $T(C) = L$). We want to construct a map $f' : C \to \mathbb{C}$, fixed point free homotopic to $f|_C$, that does not change variation on any arc $A_i$ in $C$ and has the properties listed below.

(1) $f'(a_i) \in L$ for all $n + 1 \leq i \leq m + 1$. Hence $\operatorname{var}(f', A_i, C)$ is defined for each $i > n$.

(2) $\operatorname{var}(f', A_i, C) = \operatorname{var}(f, A_i, S)$ for all $n + 1 \leq i \leq m$.

(3) $\operatorname{var}(f', I, C) = \operatorname{var}(f, I, S) = 0$.

(4) $\operatorname{ind}(f', C) = \operatorname{ind}(f, C)$.

Having such a map, it then follows from Theorem 2.13 that
\[
\operatorname{ind}(f', C) = \sum_{i = n+1}^{m} \operatorname{var}(f', A_i, C) + \operatorname{var}(f', I, C) + 1.
\]

By Theorem 2.4, $\operatorname{ind}(f', C) = \operatorname{ind}(f, C)$. By (2) and (3), $\sum_{i > n} \operatorname{var}(f', A_i, C) + \operatorname{var}(f', I, C) = \sum_{i > n} \operatorname{var}(f, A_i, S)$ and the Theorem would follow.

It remains to define the map $f' : C \to \mathbb{C}$ with the above properties. For each $i$ such that $n + 1 \leq i \leq m + 1$, chose an arc $I_i$ joining $f(a_i)$ to $L$ as follows:

(a) If $f(a_i) \in L$, let $I_i$ be the degenerate arc $\{f(a_i)\}$.

(b) If $f(a_i) \in R$ and $n + 1 < i < m + 1$, let $I_i$ be an arc in $R \setminus \{a_0, a_{n+1}\}$ joining $f(a_i)$ to $L$.

(c) If $f(a_0) \in R$, let $I_0$ be an arc joining $f(a_0)$ to $L$ such that $I_0 \cap (L \cup J_{a_0}) \subset A_{n+1} \setminus \{a_{n+1}\}$.

Let $x_{n+1} = y_{n+1} = a_{n+1}$, $y_0 = y_{m+1} = 1 \setminus \{a_0, a_{n+1}\}$ and $x_0 = x_{m+1} = A_m \setminus \{a_0, a_{n+1}\}$. For $n + 1 < i < m + 1$, let $x_i \in A_{i-1}$ and $y_i \in A_i$ such that $y_{i-1} < x_i < a_i < y_i < x_{i+1}$. For $n + 1 < i < m + 1$ let $f'(a_i)$ be the endpoint of $I_i$ in $L$, $f'(x_i) = f'(y_i) = f(a_i)$ and extend $f'$ continuously from $x_i, a_i \cup [a_i, y_i]$ onto $I_i$ and define $f'$ from $[y_i, x_{i+1}] \subset A_i$ onto $f(A_i)$ by $f'|_{[y_i, x_{i+1}]} = f \circ h_i$, where $h_i : [y_i, x_{i+1}] \to A_i$ is a homeomorphism such that $h_i(y_i) = a_i$ and $h_i(x_{i+1}) = a_{i+1}$. Similarly, define $f'$ on $[y_0, a_{n+1}] \subset I$ to $f(I)$ by $f|[y_0, a_{n+1}] = f \circ h_0$, where $h_0 : [y_0, a_{n+1}] \to I$ is an onto homeomorphism such that $h(a_{n+1}) = a_{n+1}$ and extend $f'$ from
Figure 2. Bell's Lollipop.

$[x_{m+1}, a_0] \subset A_m$ and $[a_0, y_0] \subset I$ onto $I_0$ such that $f'(x_{m+1}) = f'(y_0) = f(a_0)$ and $f'(a_0)$ is the endpoint of $I_0$ in $L$.

Note that $f'(A_i) \cap A_i = \emptyset$ for $i = n+1, \ldots, m$ and $f'(I) \cap [I \cup J_{a_0}] = \emptyset$.

To compute the variation of $f'$ on each of $A_m$ and $I$ we can use the junction $J_{a_0}$. Hence $\text{var}(f', I, C) = 0$ and, by the definition of $f'$ on $A_m$, $\text{var}(f', A_m, C) = \text{var}(f, A_m, S)$. For $i = n+1, \ldots, m-1$ we can use the same junction $J_v$ to compute $\text{var}(f', A_i, C)$ as we did to compute $\text{var}(f, A_i, S)$. Since $I_i \cup I_{i+1} \subset T(S) \setminus A_i$ we have that $f'([a_i, y_i]) \cup f'([x_{i+1}, a_{i+1}]) \subset I_i \cup I_{i+1}$ misses that junction and, hence, make no contribution to variation $\text{var}(f', A_i, C)$. Since $f^{-1}(J_v_i) \cap [y_i, x_{i+1}]$ is
isomorphic to $f^{-1}(J_{v_i}) \cap A_i$, $\var(f', A_i, C) = \var(f, A_i, S)$ for $i = n + 1, \ldots, m$.

To see that $f'$ is fixed point free homotopic to $f|_C$, note that we can pull the image of $A_i$ back along the arcs $I_i$ and $I_{i+1}$ in $R$ without fixing a point of $A_i$ at any level of the homotopy. Since $f'$ and $f|_C$ are fixed point free homotopic and $f$ has no fixed points in $T(S)$, it follows from Theorems 2.4 and 2.6 that $\ind(f', C) = \ind(f, C) = 0$.  

Note that if $f$ is fixed point free on $T(S)$, then $\ind(f, S) = 0$ and the next Corollary follows.

**Corollary 2.15.** Assume the hypotheses of Theorem 2.14. Suppose, in addition, $f$ is fixed point free on $T(S)$. Then if $f(a_{n+1}) \in R$ there exists $i \leq n$ such that $\var(f, A_i, S) < 0$. If $f(a_{n+1}) \in L$ there exists $i > n$ such that $\var(f, A_i, S) < 0$.

### 2.6. Crosscuts and bumping arcs.

For the remainder of Section 2, our Standing Hypotheses are that $f : C \to C$ takes continuum $X$ into $T(X)$ with no fixed points in $T(X)$, and $X$ is minimal with respect to these properties.

**Definition 2.16 (Bumping Simple Closed Curve).** A simple closed curve $S$ in $C$ which has the property that $S \cap X$ is nondegenerate and $T(X) \subset T(S)$ is said to be a bumping simple closed curve for $X$. A subarc $A$ of a bumping simple closed curve, whose endpoints lie in $X$, is said to be a bumping (sub)arc for $X$. Moreover, if $S'$ is any bumping simple closed curve for $X$ which contains $A$, then $S'$ is said to complete $A$.

A crosscut of $U^\infty = \mathbb{C}^\infty \setminus T(X)$ is an open arc $Q$ lying in $U^\infty$ such that $\overline{Q}$ is an arc with endpoints $a \neq b \in T(X)$. In this case we will often write $Q = (a, b)$. (As seems to be traditional, we use “crosscut of $T(X)$” interchangeably with “crosscut of $U^\infty$.”) If $S$ is a bumping simple closed curve so that $X \cap S$ is nondegenerate, then each component of $S \setminus X$ is a crosscut of $T(X)$. A similar statement holds for a bumping arc $A$. Given a non-separating continuum $T(X)$, let $A \subset C$ be a crosscut of $U^\infty = \mathbb{C}^\infty \setminus T(X)$. Given a crosscut $A$ of $U^\infty$ denote by $\text{Sh}(A)$, the shadow of $A$, the bounded component of $\mathbb{C} \setminus [T(X) \cup A]$.

Since $f$ has no fixed points in $T(X)$ and $X$ is compact, we can choose a bumping simple closed curve $S$ in a small neighborhood of $T(X)$ such that all crosscuts in $S \setminus X$ are small, have positive distance to their image and so that $f$ has no fixed points in $T(S)$. Thus, we obtain the following corollary to Theorem 2.6.
Corollary 2.17. There is a bumping simple closed curve \( S \) for \( X \) such that \( f|_{T(S)} \) is fixed point free; hence, by \([2.6]\), \( \text{ind}(f, S) = 0 \). Moreover, any bumping simple closed curve \( S' \) for \( X \) such that \( S' \subset T(S) \) has \( \text{ind}(f, S') = 0 \). Furthermore, any crosscut \( Q \) of \( T(X) \) for which \( f \) has no fixed points in \( T(X \cup Q) \) can be completed to a bumping simple closed curve \( S \) for \( X \) for which \( \text{ind}(f, S) = 0 \).

Proposition 2.18. Suppose \( A \) is a bumping subarc for \( X \). If \( \text{var}(f, A, S) \) is defined for some bumping simple closed curve \( S \) completing \( A \), then for any bumping simple closed curve \( S' \) completing \( A \), \( \text{var}(f, A, S) = \text{var}(f, A, S') \).

Proof. Since \( \text{var}(f, A, S) \) is defined, \( A = \bigcup_{i=1}^{n} A_i \), where each \( A_i \) is a bumping arc with \( A_i \cap f(A_i) = \emptyset \) and \( |A_i \cap A_j| \leq 1 \) if \( i \neq j \). By the remark following Definition \([2.11]\), it suffices to assume that \( A \cap f(A) = \emptyset \). Let \( S \) and \( S' \) be two bumping simple closed curves completing \( A \) for which variation is defined. Let \( J_a \) and \( J_{a'} \) be junctions whereby \( \text{var}(f, A, S) \) and \( \text{var}(f, A, S') \) are respectively computed. Suppose first that both junctions lie (except for \( \{a, a'\} \)) in \( C \setminus (T(S) \cup T(S')) \). By the Junction Straightening Proposition \([2.9]\), either junction can be used to compute either variation on \( A \), so the result follows. Otherwise, at least one junction is not in \( C \setminus (T(S) \cup T(S')) \). But both junctions are in \( C \setminus T(X \cup A) \). Hence, we can find another bumping simple closed curve \( S'' \) such that \( S'' \) completes \( A \), and both junctions lie in \( (C \setminus T(S'')) \cup \{a, a'\} \). Then by the Propositions \([2.9]\) and the definition of variation, \( \text{var}(f, A, S) = \text{var}(f, A, S'') = \text{var}(f, A, S') \). \( \square \)

It follows from Proposition \([2.18]\) that variation on a crosscut \( Q \), with \( Q \cap f(Q) = \emptyset \), of \( T(X) \) is independent of the bumping simple closed curve \( S \) for \( T(X) \) of which \( Q \) is a subarc and is such that \( \text{var}(f, S) \) is defined. Hence, given a bumping arc \( A \) of \( X \), we can denote \( \text{var}(f, A, S) \) simply by \( \text{var}(f, A) \) when \( X \) is understood.

The following proposition follows from Corollary \([2.17]\), Proposition \([2.18]\) and Theorem \([2.13]\).

Proposition 2.19. Suppose \( Q \) is a crosscut of \( T(X) \) such that \( f \) is fixed point free on \( T(X \cup Q) \) and \( f(Q) \cap Q = \emptyset \). Suppose \( Q \) is replaced by a bumping subarc \( A \) with the same endpoints such that \( Q \cup T(X) \) separates \( A \setminus X \) from \( \infty \) and each component \( Q_i \) of \( A \setminus X \) is a crosscut such that \( f(Q_i) \cap Q_i = \emptyset \). Then

\[ \text{var}(f, Q, X) = \sum_i \text{var}(f, Q_i, X) = \text{var}(f, A, X) \]
2.7. Index and Variation for Carathéodory Loops. We extend the definitions of index and variation to Carathéodory loops.

**Definition 2.20** (Carathéodory Loop). Let \( g : S^1 \to \mathbb{C} \) such that \( g \) is continuous and has a continuous extension \( \psi : \mathbb{C}^\infty \setminus T(S^1) \to \mathbb{C}^\infty \setminus T(g(S^1)) \) such that \( \psi|_{\mathbb{C} \setminus T(S^1)} \) is an orientation preserving homeomorphism from \( \mathbb{C} \setminus T(S^1) \) onto \( \mathbb{C} \setminus T(g(S^1)) \). We call \( g \) (and loosely, \( S = g(S^1) \)), a Carathéodory loop.

In particular, if \( g : S^1 \to g(S^1) = S \) is a continuous extension of a Riemann map \( \psi : \Delta^\infty \to \mathbb{C}^\infty \setminus T(g(S^1)) \), then \( g \) is a Carathéodory loop, where \( \Delta^\infty = \{ z \in \mathbb{C}^\infty \mid |z| > 1 \} \) is the “unit disk” about \( \infty \).

Let \( g : S^1 \to \mathbb{C} \) be a Carathéodory loop and let \( f : g(S^1) \to \mathbb{C} \) be a fixed point free map. In order to define variation of \( f \) on \( g(S^1) \), we do the partitioning in \( S^1 \) and transport it to the Carathéodory loop \( S = g(S^1) \). An allowable partition of \( S^1 \) is a set \( \{ a_0 < a_1 < \cdots < a_n \} \) in \( S^1 \) ordered counterclockwise, where \( a_0 = a_n \) and \( A_i \) denotes the counterclockwise interval \([a_i, a_{i+1}]\), such that for each \( i, f(g(a_i)) \in T(g(S^1)) \) and \( f(g(A_i)) \cap g(A_i) = \emptyset \). Variation \( \text{var}(f, A_i, g(S^1)) = \text{var}(f, A_i) \) on each path \( g(A_i) \) is then defined exactly as in Definition 2.8 except that the junction (see Definition 2.7) is chosen so that the vertex \( v \in g(A_i) \) and \( J_v \cap T(g(S^1)) \subset \{ v \} \), and the crossings of the junction \( J_v \) by \( f(g(A_i)) \) are counted (see Definition 2.8). Variation on the whole loop, or an allowable subarc thereof, is defined just as in Definition 2.11 by adding the variations on the partition elements. At this point in the development, variation is defined only relative to the given allowable partition \( F \) of \( S^1 \) and the parameterization \( g \) of \( S \): \( \text{var}(f, F, g(S^1)) \).

Index on a Carathéodory loop \( S \) is defined exactly as in Section 2.1 with \( S = g(S^1) \) providing the parameterization of \( S \). Likewise, the definition of fractional index and Proposition 2.2 apply to Carathéodory loops.

Theorems 2.3, 2.4, Corollary 2.5, and Theorem 2.6 (if \( f \) is also defined on \( T(S) \)) apply to Carathéodory loops. It follows that index on a Carathéodory loop \( S \) is independent of the choice of parameterization \( g \). The Carathéodory loop \( S \) is approximated, under small homotopies, by simple closed curves \( S_i \). Allowable partitions of \( S \) can be made to correspond to allowable partitions of \( S_i \) under small homotopies. Since variation and index are invariant under suitable homotopies (see the comments after Proposition 2.9) we have the following theorem.

**Theorem 2.21.** Suppose \( S = g(S^1) \) is a parameterized Carathéodory loop in \( \mathbb{C} \) and \( f : S \to \mathbb{C} \) is a fixed point free map. Suppose variation of \( f \) on \( S^1 = A_0 \cup \cdots \cup A_n \) with respect to \( g \) is defined for some partition
$A_0 \cup \cdots \cup A_n$ of $S^1$. Then

$$\text{ind}(f, g) = \sum_{i=0}^{n} \text{var}(f, A_i, g(S^1)) + 1.$$ 

2.8. **Prime Ends.** Prime ends provide a way of studying the approaches to the boundary of a simply-connected plane domain with non-degenerate boundary. See [11] or [18] for an analytic summary of the topic and [25] for a more topological approach. We will be interested in the prime ends of $U^\infty = \mathbb{C}^\infty \setminus T(X)$. Recall that $\Delta^\infty = \{ z \in \mathbb{C}^\infty \mid |z| > 1 \}$ is the “unit disk about $\infty$.” The Riemann Mapping Theorem guarantees the existence of a conformal map $\phi : \Delta^\infty \to U^\infty$ taking $\infty \to \infty$, unique up to the argument of the derivative at $\infty$. Fix such a map $\phi$. We identify $S^1 = \partial \Delta^\infty$ with $\mathbb{R}/\mathbb{Z}$ and identify points $e^{2\pi it}$ in $\partial \Delta^\infty$ by their argument $t \pmod{1}$. Crosscut and shadow were defined in Section 2.6.

**Definition 2.22 (Prime End).** A chain of crosscuts is a sequence $\{Q_i\}_{i=1}^{\infty}$ of crosscuts of $U^\infty$ such that for $i \neq j$, $Q_i \cap Q_j = \emptyset$, $\text{diam}(Q_i) \to 0$, and for all $j > i$, $Q_i$ separates $Q_j$ from $\infty$ in $U^\infty$. Hence, for all $j > i$, $Q_j \subset \text{Sh}(Q_i)$. Two chains of crosscuts are said to be equivalent iff it is possible to form a sequence of crosscuts by selecting alternately a crosscut from each chain so that the resulting sequence of crosscuts is again a chain. A prime end $E$ is an equivalence class of chains of crosscuts.

If $\{Q_i\}$ and $\{Q'_i\}$ are equivalent chains of crosscuts of $U^\infty$, it can be shown that $\{\phi^{-1}(Q_i)\}$ and $\{\phi^{-1}(Q'_i)\}$ are equivalent chains of crosscuts of $\Delta^\infty$ each of which converges to the same unique point $e^{2\pi it} \in S^1 = \partial \Delta^\infty$, $t \in [0, 1)$, independent of the representative chain. Hence, we denote by $E_t$ the prime end of $U^\infty$ defined by $\{Q_i\}$.

**Definition 2.23 (Impression and Principal Continuum).** Let $E_t$ be a prime end of $U^\infty$ with defining chain of crosscuts $\{Q_i\}$. The set

$$\text{Im}(E_t) = \bigcap_{i=1}^{\infty} \text{Sh}(Q_i)$$

is a subcontinuum of $\partial U^\infty$ called the impression of $E_t$. The set

$$\text{Pr}(E_t) = \{ z \in \partial U^\infty \mid \text{for some chain } \{Q'_i\} \text{ defining } E_t, Q'_i \to z \}$$

is a continuum called the principal continuum of $E_t$.

For a prime end $E_t$, $\text{Pr}(E_t) \subset \text{Im}(E_t)$, possibly properly. We will be interested in the existence of prime ends $E_t$ for which $\text{Pr}(E_t) = \text{Im}(E_t) = \partial U^\infty$. 

Definition 2.24 (External Rays). Let \( t \in [0, 1) \) and define
\[
R_t = \{ z \in \mathbb{C} \mid z = \phi(re^{2\pi it}), 1 < r < \infty \}.
\]
We call \( R_t \) the external ray (with argument \( t \)). If \( x \in R_t \) then the \((X,x)\)-end of \( R_t \) is the bounded component \( K_x \) of \( R_t \setminus \{ x \} \).

The external rays \( R_t \) foliate \( U^\infty \).

Definition 2.25 (Essential crossing). An external ray \( R_t \) is said to cross a crosscut \( Q \) essentially if and only if there exists \( x \in R_t \) such that the \((T(X),x)\)-end of \( R_t \) is contained in the bounded complementary domain of \( T(X) \cup Q \). In this case we will also say that \( Q \) crosses \( R_t \) essentially.

The results listed below are known.

Proposition 2.26 ([11]). Let \( E_t \) be a prime end of \( U^\infty \). Then \( \operatorname{Pr}(E_t) = R_t \setminus R_t \). Moreover, for each \( 1 < r < \infty \) there is a crosscut \( Q_r \) of \( U^\infty \) with \( \{ \phi(re^{2\pi it}) \} = R_t \cap Q_r \) and \( \operatorname{diam}(Q_r) \to 0 \) as \( r \to 1 \) and such that \( R_t \) crosses \( Q_r \) essentially.

Definition 2.27 (Landing Points and Accessible Points). If \( \operatorname{Pr}(E_t) = \{ x \} \), then we say \( R_t \) lands on \( x \in T(X) \) and \( x \) is the landing point of \( R_t \). A point \( x \in \partial T(X) \) is said to be accessible (from \( U^\infty \)) iff there is an arc in \( U^\infty \cup \{ x \} \) with \( x \) as one of its endpoints.

Proposition 2.28. A point \( x \in \partial T(X) \) is accessible iff \( x \) is the landing point of some external ray \( R_t \).

Definition 2.29 (Channels). A prime end \( E_t \) of \( U^\infty \) for which \( \operatorname{Pr}(E_t) \) is nondegenerate is said to be a channel in \( \partial U^\infty \) (or in \( T(X) \)). If moreover \( \operatorname{Pr}(E_t) = \partial U^\infty = \partial T(X) \), we say \( E_t \) is a dense channel. A crosscut \( Q \) of \( U^\infty \) is said to cross the channel \( E_t \) iff \( R_t \) crosses \( Q \) essentially.

When \( X \) is locally connected, there are no channels, as the following classical theorem proves. In this case, every prime end has degenerate principal set and degenerate impression.

Theorem 2.30 (Carathéodory). \( X \) is locally connected iff the Riemann map \( \phi : \Delta^\infty \to U^\infty = \mathbb{C}^\infty \setminus T(X) \) taking \( \infty \to \infty \) extends continuously to \( S^1 = \partial \Delta^\infty \).

3. Kulkarni-Pinkall Partitions

Throughout this section let \( K \) be a compact subset of the plane whose complement \( U = \mathbb{C} \setminus K \) is connected. In the interest of completeness we define the Kulkarni-Pinkall partition of \( U \) and prove the
basic properties of this partition that are essential for our work in Section 4. Kulani-Pinkall [15] worked in closed \( n \)-manifolds. We will follow their approach and adapt it to our situation in the plane.

We think of \( K \) as a closed subset of the Riemann sphere \( \mathbb{C}^\infty \), with the spherical metric and set \( U^\infty = \mathbb{C}^\infty \setminus K = U \cup \{ \infty \} \). Let \( \mathcal{B}^\infty \) be the family of closed, round balls \( B \) in \( \mathbb{C}^\infty \) such that \( \text{Int}(B) \subset U^\infty \) and \( |\partial B \cap K| \geq 2 \). Then \( \mathcal{B}^\infty \) is in one-to-one correspondence with the family \( \mathcal{U} \) of closed subsets \( B \) of \( \mathbb{C} \) which are the closure of a complementary component of a straight line or a round circle in \( \mathbb{C} \) such that \( \text{Int}(B) \subset U \) and \( |\partial B \cap K| \geq 2 \).

**Proposition 3.1.** If \( B_1 \) and \( B_2 \) are two closed round balls in \( \mathbb{C} \) such that \( B_1 \cap B_2 \neq \emptyset \) but does not contain a diameter of either \( B_1 \) or \( B_2 \), then \( B_1 \cap B_2 \) is contained in a ball of diameter strictly less than the diameters of both \( B_1 \) and \( B_2 \).

**Proof.** Let \( \partial B_1 \cap \partial B_2 = \{ s_1, s_2 \} \). Then the closed ball with center \( (s_1 + s_2)/2 \) and radius \( |s_1 - s_2|/2 \) contains \( B_1 \cap B_2 \). \( \square \)

If \( B \) is the closed ball of minimum diameter that contains \( K \), then we say that \( B \) is the *smallest ball* containing \( K \). It is unique by Proposition 3.1. It exists, since any sequence of balls of decreasing diameters that contain \( K \) has a convergent subsequence.

We denote the *Euclidean convex hull* of \( K \) by \( \text{conv}_E(K) \). It is the intersection of all closed half-planes (a closed half-plane is the closure of a component of the complement of a straight line) which contain \( K \). Hence \( p \in \text{conv}_E(K) \) if \( p \) cannot be separated from \( K \) by a straight line.

Given a closed ball \( B \in \mathcal{B}^\infty \), \( \text{Int}(B) \) is conformally equivalent to the unit disk in \( \mathbb{C} \). Hence its interior can be naturally equipped with the hyperbolic metric. Geodesics \( g \) in this metric are intersections of \( \text{Int}(B) \) with round circles \( C \subset \mathbb{C}^\infty \) which perpendicularly cross the boundary \( \partial B \). For every hyperbolic geodesic \( g \), \( B \setminus g \) has exactly two components. We call the closure of such components *hyperbolic half-planes* of \( B \). Given \( B \in \mathcal{B}^\infty \), the *hyperbolic convex hull* of \( K \) in \( B \) is the intersection of all (closed) hyperbolic half-planes of \( B \) which contain \( K \cap B \) and we denote it by \( \text{conv}_H(B \cap K) \).

**Lemma 3.2.** Suppose that \( B \in \mathcal{B} \) is the smallest ball containing \( K \subset \mathbb{C} \) and let \( c \in B \) be its center. Then \( c \in \text{conv}_H(K \cap \partial B) \).

**Proof.** By contradiction. Suppose that there exists a circle that separates the center \( c \) from \( K \cap \partial B \) and crosses \( \partial B \) perpendicularly. Then there exists a line \( \ell \) through \( c \) such that a half-plane bounded by \( \ell \)
contains $K \cap \partial B$ in its interior. Let $B' = B + v$ be a translation of $B$ by a vector $v$ that is orthogonal to $\ell$ and directed into this halfplane. If $v$ is sufficiently small, then $B'$ contains $K$ in its interior. Hence, it can be shrunk to a strictly smaller ball that also contains $K$, contradicting that $B$ has smallest diameter. \hfill \Box

Lemma 3.3. Suppose that $B_1, B_2 \in \mathcal{B}^\infty$ with $B_1 \neq B_2$. Then
$$\text{conv}_H(B_1 \cap \partial U) \cap \text{conv}_H(B_2 \cap \partial U) \subset \partial U.$$ In particular, $\text{conv}_H(B_1 \cap \partial U) \cap \text{conv}_H(B_2 \cap \partial U)$ contains at most two points.

Proof. A picture easily explains this, see Figure 3. Note that $\partial U \cap [B_1 \cup B_2] \subset \partial(B_1 \cup B_1)$. Therefore $B_1 \cap \partial U$ and $B_2 \cap \partial U$ share at most two points. The open hyperbolic chords between these points in the respective balls are disjoint. \hfill \Box

It follows that any point in $U^\infty$ can be contained in at most one hyperbolic convex hull. In the next lemma we see that each point of $U^\infty$ is indeed contained in $\text{conv}_H(B \cap K)$ for some $B \in \mathcal{B}^\infty$. So $\{U^\infty \cap \text{conv}_H(B \cap K) \mid B \in \mathcal{B}^\infty\}$ is a partition of $U^\infty$.

Since hyperbolic convex hulls are preserved by Möbius transformations, they are more easy to manipulate than the Euclidean convex hulls used by Bell (which are preserved only by Möbius transformations that fix $\infty$). This is illustrated by the proof of the following lemma.

Lemma 3.4 (Kulkarni-Pinkall inversion lemma). For any $p \in \mathbb{C}^\infty \setminus K$ there exists $B \in \mathcal{B}^\infty$ such that $p \in \text{conv}_H(B \cap K)$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Fig3.png}
\caption{Maximal balls have disjoint hulls.}
\end{figure}
Proof. We prove first that there exists $B^* \in \mathcal{B}_\infty$ such that no line or circle which crosses $\partial B^*$ perpendicularly separates $K \cap \partial B^*$ from $\infty$.

Let $B'$ be the smallest round ball which contains $K$ and let $B = \mathbb{C} \setminus B'$. Then $B^* = B \cup \{\infty\} \in \mathcal{B}_\infty$. If $L$ is a circle which crosses $\partial B^* = \partial B'$ perpendicularly and separates $K \cap \partial B'$ from $\infty$, then it also separates $K \cap \partial B'$ from the center $c'$ of $B'$, contrary to Lemma 3.2.

[To see this note that if a hyperbolic geodesic $g$ of $B^*$ separates $K \cap B^*$ from $\infty$, then $g$ is contained in a round circle $C$ and $C \cap B'$ separates $c'$ from $B' \cap K$, a contradiction.] Hence, $\infty \in \text{conv}_H(B^* \cap K)$.

Now let $p \in C_\infty \setminus K$. Let $M : C_\infty \to C_\infty$ be a Möbius transformation such that $M(p) = \infty$. By the above argument there exists a ball $B^* \in \mathcal{B}_\infty$ such that $\infty \in \text{conv}_H(B^* \cap M(K))$. Then $B = M^{-1}(B^*) \in \mathcal{B}_\infty$ and, since $M$ preserves perpendicular circles, $p \in \text{conv}_H(B \cap K)$ as desired. $\Box$

From Lemmas 3.3 and 3.4, we obtain the following Theorem which is a special case of a Theorem of Kulkarni and Pinkall [15].

**Theorem 3.5.** Suppose that $K \subset \mathbb{C}$ is a nondegenerate compact set such that its complement $U_\infty$ in the Riemann sphere is non-empty and connected. Then $U_\infty$ is partitioned by the family

$$
\mathcal{KPP} = \{U_\infty \cap \text{conv}_H(B \cap K) : B \in \mathcal{B}_\infty\}.
$$

Theorem 3.5 is the linchpin of the theory of geometric crosscuts. An analogue of it was known to Harold Bell and used by him implicitly since the early 1970’s. Bell considered non-separating plane continua $K$ and he used the equivalent notion of Euclidean convex hull of the sets $B \cap \partial U$ for all maximal balls $B \in \mathcal{B}$ (see the comment following Theorem 4.5).

We will denote the partition $\{U_\infty \cap \text{conv}_H(B \cap K) : B \in \mathcal{B}_\infty\}$ of $U_\infty$ by $\mathcal{KPP}$. Let $B \in \mathcal{B}_\infty$. If $B \cap \partial U_\infty$ consists of two points $a$ and $b$, then its (hyperbolic) hull is an open circular segment $g$ with endpoints $a$ and $b$ and perpendicular to $\partial B$. We will call the crosscut $g$ a $\mathcal{K}P$ crosscut or simply a $\mathcal{K}P$ chord. If $B \cap \partial U$ contains three or more points, then we say that the hull $\text{conv}_H(B \cap \partial U)$ is a gap. A gap has nonempty interior. Its boundary in $\text{Int}(B)$ is a union of open circular segments (with endpoints in $K$), which we also call $\mathcal{K}P$ crosscuts or $\mathcal{K}P$ chords. We denote by $\mathcal{K}P$ the collection of all open chords obtained as above using all $B \in \mathcal{B}_\infty$.

The following example may serve to illustrate Theorem 3.5.

**Example.** Let $K$ be the unit square $\{x + yi : -1 \leq x, y \leq 1\}$. There
are five obvious members of $\mathcal{B}$. These are the sets

$$\text{Im} z \geq 1, \text{Im} z \leq -1, \text{Re} z \geq 1, \text{Re} z \leq -1, |z| \geq \sqrt{2},$$

four of which are half-planes. These are the only members of $\mathcal{B}$ whose hyperbolic convex hulls have non-empty interiors. However, for this example the family $\mathcal{B}$ defined in the introduction of Section 3 is infinite. The hyperbolic hull of the half-plane $\text{Im} z \geq 1$ is the semi-disk $\{z \mid |z - i| \leq 1, \text{Im} z > 1\}$. The hyperbolic hulls of the other three half-planes given above are also semi-disks. The hyperbolic hull of $|z| \geq \sqrt{2}$ is the unbounded region whose boundary consists of the four semi-circles lying (except for their endpoints) outside $K$ and contained in the circles of radius $\sqrt{2}$ and having centers at $-2, 2, -2i$ and $2i$, respectively. These hulls do not cover $U$ as there are spaces between the hulls of the half-planes and the hull of $|z| \geq \sqrt{2}$.

If $C$ is a circle that circumscribes $K$ and contains exactly two of its vertices, such as $1 \pm i$, then the exterior ball $B$ bounded by $C$ is maximal. Now $\text{conv}_H(B \cap K)$ is a single chord and the union of all such chords foliates the remaining spaces in $C \setminus K$.

**Lemma 3.6.** If $g_i$ is a sequence of $\mathcal{KP}$ chords with endpoints $a_i$ and $b_i$, and $\lim a_i = a \neq b = \lim b_i$, then $\overline{g_i}$ is convergent and $\lim \overline{g_i} = C$, where $g = C \setminus \{a, b\} \in \mathcal{KP}$ is also a $\mathcal{KP}$ chord.

**Proof.** For each $i$ let $B_i \in \mathcal{B}^\infty$ such that $g_i \subset \text{conv}_H(B_i \cap K)$. Then $B_i$ converges to some $B \in \mathcal{B}^\infty$ and $\overline{g_i}$ converges to a closed circular arc $C$ in $B$ with endpoints $a$ and $b$, and $C$ is perpendicular to $\partial B$. Hence $g = C \setminus \{a, b\} \subset \text{conv}_H(B \cap K)$. So $g \in \mathcal{KP}$. \qed

By Lemma 3.6, the family $\mathcal{KP}$ of chords has continuity properties similar to a foliation.

**Lemma 3.7.** For $a, b \in K \cap \partial U^\infty$, define $C(a, b)$ as the union of all $\mathcal{KP}$ chords with endpoints $a$ and $b$. Then if $C(a, b) \neq \emptyset$, $C(a, b)$ is either a single chord, or $C(a, b) \cup \{a, b\}$ is a closed disk whose boundary consists of two $\mathcal{KP}$ chords contained in $C(a, b)$ together with $\{a, b\}$.

**Proof.** Suppose $g$ and $h$ are two distinct $\mathcal{KP}$ chords between $a$ and $b$. Then $S = g \cup h \cup \{a, b\}$ is a simple closed curve. Choose a point $z$ in the complementary domain $V$ of $S$ contained in $U^\infty$. Since the hyperbolic hulls partition $U^\infty$, there exists $B \in \mathcal{B}^\infty$ such that $z \in \text{conv}_H(B \cap K)$ and $\text{conv}_H(B \cap K)$ can only intersect $S \cap K$ in $\{a, b\}$. So $\text{conv}_H(B \cap K) \cap K = \{a, b\}$ and it follows that $V$ is contained in $C(a, b)$.

The rest of the Lemma follows from 3.6. \qed
4. Hyperbolic foliation of simply connected domains

In this section we will apply the results from Section 3 to the case that $K$ is a non-separating plane continuum (or, equivalently, that $U^\infty = C^\infty \setminus K$ is simply connected). The results in this section are essential to [20] but are not used in this paper. The reader who is only interested in the fixed point question can skip this section.

Let $D$ be the open unit disk in the plane. In this section we let $\phi : D \to C^\infty \setminus K = U^\infty$ be a Riemann map onto $U^\infty$. We endow $D$ with the hyperbolic metric, which is carried to $U^\infty$ by the Riemann map. We use $\phi$ and the Kulkarni-Pinkall hulls to induce a closed collection $\Gamma$ of chords in $D$ that is a hyperbolic geodesic lamination in $D$ (see [24]).

Let $g \in \mathcal{KP}$ be a chord with endpoints $a$ and $b$. Then $a$ and $b$ are accessible points in $K$ and $\phi^{-1}(g)$ is an arc in $D$ with endpoints $z, w \in \partial D$. Let $G$ be the hyperbolic geodesic in $D$ joining $z$ and $w$. Let $\Gamma$ be the collection of all $G$ such that $g \in \mathcal{KP}$. We will prove that $\Gamma$ inherits the properties of the family $\mathcal{KP}$ as described in Theorem 3.5 and Lemma 3.6 (see Lemma 4.3, Theorem 4.5 and the remark following 4.5).

Since members of $\mathcal{KP}$ do not intersect (though their closures are arcs which may have common endpoints) the same is true for distinct members of $\Gamma$. We will refer to the members of $\Gamma$ (and their images under $\phi$) as hyperbolic chords or hyperbolic geodesics. Given $g \in \mathcal{KP}$ we denote the corresponding element of $\Gamma$ by $G$ and its image under $\phi$ in $U^\infty$ by $g$. Note that $\Gamma$ is a lamination of $D$ in the sense of Thurston [24]. By a gap of $\Gamma$ (or of $\phi(\Gamma)$), we mean the closure of a component of $D \setminus \bigcup \Gamma$ in $D$ (or its image under $\phi$ in $U^\infty$, respectively).

**Lemma 4.1** (Jørgensen [21, p.91]). Let $B$ be a closed round ball such that its interior is in $U^\infty$. Let $\gamma \subset D$ be a hyperbolic geodesic. Then $\phi(\gamma) \cap B$ is connected. In particular, if $R_t$ is an external ray in $U^\infty$ and $B \in \mathfrak{B}^\infty$, then $R_t \cap B$ is connected.

If $a, b \in \partial U^\infty$, recall that $C(a, b)$ is the union of all $\mathcal{KP}$ chords with endpoints $a$ and $b$. From the viewpoint of prime ends, all chords in $C(a, b)$ are the same. That is why all the chords in $C(a, b)$ are replaced by a single hyperbolic chord $g \in \phi(\Gamma)$. The following lemma follows.

**Lemma 4.2.** Suppose $g \in \mathcal{KP}$ and $g \subset \text{conv}_{\mathcal{H}}(B \cap \partial U^\infty)$ joins the points $a, b \in \partial U^\infty$ for some $B \in \mathfrak{B}^\infty$. We may assume that the Riemann map $\phi : D \to U^\infty$ is extended over all points $x \in S^1$ so that $\phi(x)$ is an accessible point of $U^\infty$. Let $\phi^{-1}(a) = \bar{a}$, $\phi^{-1}(b) = \bar{b}$ and $\phi^{-1}(B) = \bar{B}$, and let $G$ be the hyperbolic geodesic joining the points $\bar{a}$ and $\bar{b}$ in $D$. Then $g = \phi(G) \subset B$. 

Proof. Suppose, by way of contradiction, that \( x \in G \setminus \tilde{B} \). Let \( C \) be the component of \( \overline{D \setminus \phi^{-1}(g)} \) which does not contain \( x \). Choose \( a_i \to \tilde{a} \) and \( b_i \to \tilde{b} \) in \( S^1 \cap C \) and let \( H_i \) be the hyperbolic geodesic in \( \mathbb{D} \) joining the points \( a_i \) and \( b_i \). Then \( \lim H_i = G \) and \( H_i \cap \tilde{B} \) is not connected. This contradiction with Lemma 4.1 completes the proof. \( \square \)

Lemma 4.3. Suppose that \( \{G_i\} \) is a sequence of hyperbolic chords in \( \Gamma \) and suppose that \( x_i \in G_i \) such that \( \{x_i\} \) converges to \( x \in \mathbb{D} \). Then there is a unique hyperbolic chord \( G \in \Gamma \) that contains \( x \). Furthermore, \( \lim G_i = G \).

Proof. We may suppose that the sequence \( \{G_i\} \) converges to a hyperbolic chord \( G \) which contains \( x \). Let \( g_i \in KP \) so that \( \phi^{-1}(g_i) \) is an open arc which joins the endpoints of \( G_i \). By Lemma 3.6, \( \lim g_i = g \in KP \). It follows that \( G \) is the hyperbolic chord joining the endpoints of \( \phi^{-1}(g) \). Hence \( G \in \Gamma \). \( \square \)

So we have used the family of \( KP \) chords in \( U^\infty \) to stratify \( \mathbb{D} \) to the family \( \Gamma \) of hyperbolic chords. By Lemma 4.2, for each \( KP \) chord \( g \subset \text{conv}_H(B \cap \partial U^\infty) \) its associated hyperbolic chord \( g = \phi(G) \subset B \). Hence, there is a deformation of \( U^\infty \) that maps \( \bigcup KP \) onto \( \bigcup \phi(\Gamma) \), which suggests that components of \( U^\infty \setminus \bigcup \phi(\Gamma) \) naturally correspond to the interiors of the gaps of the Kulkarni-Pinkall partition. That this is indeed the case is the substance of the next lemma.

Lemma 4.4. There is a \( 1 \to 1 \) correspondence between complementary domains \( Z \subset \mathbb{D} \setminus \bigcup \Gamma \) and the interiors of Kulkarni-Pinkall gaps \( \text{conv}_H(B \cap K) \). Moreover, for each gap \( Z \) of \( \Gamma \) there exists a unique \( B \in \mathfrak{B}^\infty \) such that \( Z \) corresponds to the interior of the \( KP \) gap \( \text{conv}_H(B \cap K) \cap U^\infty \) in that \( \partial Z \cap \mathbb{D} = \bigcup \{ G \in \Gamma \mid g \in KP \quad \text{and} \quad g \subset \text{conv}_H(B \cap K) \} \) and \( \phi(Z) \subset B \).

Proof. Let \( g \) and \( h \) be two distinct \( KP \) chords in the boundary of the gap \( \text{conv}_H(B \cap K) \) for some \( B \in \mathfrak{B}^\infty \). Let \( \{a, b\} \) and \( \{c, d\} \) be the endpoints of \( \phi^{-1}(g) \) and \( \phi^{-1}(h) \), respectively. Then \( G \) has endpoints \( \{a, b\} \) and \( H \) has endpoints \( \{c, d\} \). There exist disjoint irreducible arcs \( A \) and \( C \) in \( \partial \mathbb{D} \) between the sets \( \{a, b\} \) and \( \{c, d\} \). Since \( g \) and \( h \) are contained in the same gap, no hyperbolic leaf of \( \Gamma \) has one endpoint in \( A \) and the other endpoint in \( C \). Hence there exists a gap \( Z \) of \( \Gamma \) whose boundary includes the hyperbolic chords \( G \) and \( H \). It now follows easily that for any \( g' \in KP \) which is contained in the boundary of the same gap \( \text{conv}_H(B \cap K) \), \( G' \) is contained in the boundary of \( Z \). Hence the \( KP \) gap \( \text{conv}_H(B \cap K) \) corresponds to the gap \( Z \) of \( \Gamma \). Conversely, if \( Z \) is a gap of \( \Gamma \) in \( \mathbb{D} \) then a similar argument, together with Lemmas 4.1 and
 implies that $Z$ corresponds to a unique gap $\text{conv}_H(B \cap K)$ for some $B \in \mathcal{B}^\infty$. The rest of the Lemma now follows from Lemma 4.2. □

So if $U^\infty = \mathbb{C}^\infty \setminus K$ is endowed with the hyperbolic metric induced by $\phi$, then there exists a family of geodesic chords that share the same endpoints as elements of $\mathcal{KP}$. The complementary domains of $U^\infty \setminus \bigcup \{g \mid g \in \mathcal{KP}\}$ corresponds to the Kulkarni-Pinkall gaps. We summarize the results:

**Theorem 4.5.** Suppose that $K \subset \mathbb{C}$ is a non-separating continuum and let $U^\infty$ be its complementary domain in the Riemann sphere. There exists a family $\phi(\Gamma)$ of hyperbolic chords in the hyperbolic metric on $U^\infty$ such that for each $g \in \phi(\Gamma)$ there exists $B \in \mathcal{B}^\infty$ and $g \subset \text{conv}_H(B \cap \partial U^\infty)$ so that $g$ and $g$ have the same endpoints and $g \subset B$. Each domain $Z$ of $U^\infty \setminus \phi(\Gamma)$ naturally corresponds to a Kulkarni-Pinkall gap $\text{conv}_H(B \cap \partial U^\infty)$ The bounding hyperbolic chords of $Z$ in $U^\infty$ correspond to the $\mathcal{KP}$ chords (i.e., chords in $\mathcal{KP}$) of $\text{conv}_H(B \cap \partial U^\infty)$.

In order to obtain Bell’s Euclidean foliation \cite{3} we could have modified the $\mathcal{KP}$ family as follows. Suppose that $B \in \mathcal{B}$. Instead of replacing a $\mathcal{KP}$ chord $g \in \text{conv}_H(B \cap K)$ by a geodesic in the hyperbolic metric on $U^\infty$, we could have replaced it by a straight line segment; i.e., the geodesic in the Euclidean metric. Then we would have obtained a family of open straight line segments. In so doing we would have replaced the gaps $\text{conv}_H(B \cap \partial U^\infty)$ by $\text{conv}_E(B \cap \partial U^\infty)$, which is the way in which Bell originally foliated $\text{conv}_E(K) \setminus K$. We hope that the above argument provides a more transparent proof of Bell’s result. Note that both in the hyperbolic and Euclidean case the elements of the foliation are not necessarily disjoint (hence we use the word “foliate” rather then “partition”). However, in both cases every point of $U^\infty$ is contained in either a unique chord or in the interior of a unique gap.

5. $\mathcal{KP}$ CHORDS AND PRIME ENDS

We will follow the notation from Section 3 in the case that $K = T(X)$ where $X$ is a plane continuum. Here we assume, as in the introduction to this paper, that $f : \mathbb{C} \to \mathbb{C}$ takes continuum $X$ into $T(X)$ with no fixed points in $T(X)$, and $X$ is minimal with respect to these properties. We apply the Kulkarni-Pinkall partition to $U^\infty = \mathbb{C}^\infty \setminus T(X)$. Recall that $\mathcal{KP} = \{\text{conv}_H(B \cap K) \cap U^\infty \mid B \in \mathcal{B}^\infty\}$ is the Kulkarni Pinkall partition of $U^\infty$ as given by Theorem 3.5.

Let $B^\infty \in \mathcal{B}^\infty$ be the maximal ball such that $\infty \in \text{conv}_H(B^\infty \cap K)$. As before we use balls on the sphere. In particular, straight lines in the plane correspond to circles on the sphere containing the point at
infinity. The subfamily of $\mathcal{KP}$ whose elements are of diameter $\leq \delta$ in the spherical metric is denoted by $\mathcal{KP}_\delta$. The subfamily of chords in $\mathcal{KP}$ of diameter $\leq \delta$ is denoted by $\mathcal{KP}_\delta$.

By Lemma 3.6 we know that the families $\mathcal{KP}$ and $\mathcal{KPP}$ have nice continuity properties. However, $\mathcal{KP}$ and $\mathcal{KPP}$ are not closed in the hyperspace of compact subsets of $\mathbb{C}^\infty$: a sequence of chords or hulls may converge to a point in the boundary of $U^\infty$ (in which case it must be a null sequence).

**Proposition 5.1 (Closedness).** Let $\{g_i\}$ be a convergent sequence of distinct elements in $K_{\mathcal{P}_\delta}$, then either $g_i$ converges to a chord $g$ in $K_{\mathcal{P}_\delta}$ or $g_i$ converges to a point of $X$. In the first case, for large $i$ and $\delta$ sufficiently small, $\text{var}(f, g, T(X)) = \text{var}(f, g_i, T(X))$.

**Proof.** By Lemma 3.6 we know that the first conclusion holds if $g = \lim g_i$ contains a point of $U^\infty$. Hence we only need to consider the case when $\lim g_i = g \subset \partial U^\infty \subset T(X)$. If the diameter of $g_i$ converged to zero, then $g$ is a point as desired. Assume that this is not the case and let $B_i$ be the maximal ball that contains $g_i$. Under our assumption, the diameters of $\{B_i\}$ do not decay to zero. Then $\lim B_i = B \in B^\infty$ and it follows $\lim g_i$ is a piece of a round circle which crosses $\partial B$ perpendicularly. Hence $\lim g_i \cap \text{Int}(B) \neq \emptyset$, contradicting the fact that $g \subset \partial U^\infty \subset T(X)$. Note that for $\delta$ sufficiently small, $\mathcal{F} \cap f(B_i) = \emptyset$. Hence, $\text{var}(f, g_i, T(X))$ and $\text{var}(f, g_i, T(X))$ are defined for all $i$ sufficiently large. Then last statement in the Lemma follows from stability of variation (see Section 2.3). □

**Corollary 5.2.** For each $\varepsilon > 0$, there exist $\delta > 0$ such that for all $g \in \mathcal{KP}$ with $g \subset B(T(X), \delta)$, $\text{diam}(g) < \varepsilon$.

**Proof.** Suppose not, then there exist $\varepsilon > 0$ and a sequence $g_i$ in $\mathcal{KP}$ such that $\lim g_i \subset X$ and $\text{diam}(g_i) \geq \varepsilon$ a contradiction to Proposition 5.1. □

The proof of the following well-known proposition is omitted.

**Proposition 5.3.** For each $\varepsilon > 0$ there exists $\delta > 0$ such that for each open arc $A$ with distinct endpoints $a, b$ such that $A \cap T(X) = \{a, b\}$ and $\text{diam}(A) < \delta$, $T(T(X) \cup A) \subset B(T(X), \varepsilon)$.

**Proposition 5.4.** Let $\varepsilon, \delta$ be as in Proposition 5.3 above with $\delta < \varepsilon/2$ and let $B \in B^\infty$. Let $A$ be a crosscut of $T(X)$ such that $\text{diam}(A) < \delta$. If $x \in T(A \cup T(X)) \cap \text{conv}_H(B \cap T(X)) \setminus T(X)$ and $d(x, A) \geq \varepsilon$, then the radius of $B$ is less than $\varepsilon$. Hence, $\text{diam}(\text{conv}_H(B \cap T(X))) < 2\varepsilon$. 
Proof. Let $z$ be the center of $B$. If $d(z,T(X)) < \varepsilon$ then $\mathrm{diam}(B) < 2\varepsilon$ and we are done. Hence, we may assume that $d(z,T(X)) \geq \varepsilon$. We will show that this leads to a contradiction. By Proposition 5.3 and our choice of $\delta, z \in \mathbb{C}^\infty \setminus T(A \cup X)$. The straight line segment $\ell$ from $x$ to $z$ must cross $T(X) \cup A$ at some point $w$. Since the segment $\ell$ is in the interior of the maximal ball $B$, it is disjoint from $T(X)$, so $w \in A$. Hence $d(x,w) \geq \varepsilon$ and, since $x \in B, B(w, \varepsilon) \subset B$. This is a contradiction since $A \subset B(w, \delta)$ and $\delta < \varepsilon/2$ so $A$ would be contained in the interior of $B$ which is impossible since $A$ is a crosscut of $T(X)$. \[\square\]

**Proposition 5.5.** Let $C$ be a crosscut of $T(X)$ and let $A$ and $B$ be disjoint closed sets in $T(X)$ such that $\overline{C} \cap A \neq \emptyset \neq \overline{C} \cap B$. For each $x \in C$, let $F_x \in \mathcal{KPP}$ so that $x \in F_x$. If each $F_x$ intersects $A \cup B$, then there exists an $F_\infty \in \mathcal{KPP}$ such that $F_\infty$ intersects $A$, $B$ and $\overline{C}$.

Proof. Let $a \in A, b \in B$ be the endpoints of $\overline{C}$. Let $C_a, C_b \subset C$ be the set of points $x \in C$ such that $F_x$ intersects $A$ or $B$, respectively. Then $C_a$ and $C_b$ are closed subsets by Proposition 5.1. Note that $d(A,B) > 0$. If $C_a = \emptyset$, choose $x_i \in C$ converging to $a \in A \cap \overline{C}$. Then $\overline{F_{x_i}} \cap B_\infty \neq \emptyset$ and $\lim F_{x_i} = F_\infty \subset \overline{\gamma}(B_\infty \cap K) \in \mathcal{KPP}$, $\lim B_i = B_\infty$ and where $B_\infty, B_i \in \mathcal{B}^\infty$ such that $F_{x_i} \subset \overline{\gamma}(B_i \cap T(X))$ by Lemma 3.6. Then $F_\infty \cap B \neq \emptyset$ and $a \in A \cap \overline{C} \cap F_\infty$. Suppose now $C_a \neq \emptyset \neq C_b$. Then $C_a$ and $C_b$ are closed and, since $C$ is connected, $C_a \cap C_b \neq \emptyset$. Let $y \in C_a \cap C_b$. Then $F_y \cap A \neq \emptyset \neq F_y \cap B$ and $y \in F_y \cap C$. \[\square\]

Proposition 5.5 allows us to replace small crosscuts which essentially cross a prime end $\mathcal{E}_t$ with non-trivial principal continuum by small nearby $\mathcal{KPP}$ chords which also essentially cross $\mathcal{E}_t$. For if $C$ is a small crosscut of $T(X)$ with endpoints $a$ and $b$ which crosses the external ray $R_t$ essentially, let $A$ and $B$ be the closures of the sets in $T(X)$ accessible from $a$ and $b$, respectively by small arcs missing $R_t$. If the $F_\infty$ of proposition 5.5 is a gap $\overline{\gamma}(B \cap T(X))$, then a $\mathcal{KPP}$ chord in its boundary crosses $R_t$ essentially.

Fix a Riemann map $\varphi : \Delta^\infty \to U^\infty = \mathbb{C}^\infty \setminus T(X)$ with $\varphi(\infty) = \infty$. Recall that an external ray $R_t$ is the image of the radial line segment with argument $2\pi t$ under the map $\varphi$.

**Proposition 5.6.** Suppose the external ray $R_t$ lands on $x \in T(X)$, and $\{g_i\}_{i=1}^\infty$ is a sequence of crosscuts of $T(X)$ converging to $x$ such that there exists a null sequence of arcs $A_i \subset \mathbb{C} \setminus T(X)$ joining $g_i$ to $R_t$. Then for sufficiently large $i$, $\var{f, g_i, T(X)} = 0$.

Proof. Since $f$ is fixed point free on $T(X)$ and $f(x) \in T(X)$, we may choose a connected neighborhood $W$ of $x$ such that $f(W) \cap (W \cup R_t) = \emptyset$. Then for sufficiently large $i$, $\var{f, g_i, T(X)} = 0$. \[\square\]
For sufficiently large \( i \), \( A_i \cup g_i \subset W \). Then for each such \( i \) there exists a junction \( J \) starting from a point in \( g_i \), staying in \( W \) close to \( A_i \) until it reaches \( R_t \), then following \( R_t \) to \( \infty \). By our choice of \( W \), \( \var(f, g_i, T(X)) = 0 \).

**Proposition 5.7.** Suppose that for an external ray \( R_t \) we have \( R_t \cap \text{Int}(\text{conv}_E(T(X))) \neq \emptyset \). Then there exists \( x \in R_t \) such that the \((T(X), x)\)-end of \( R_t \) is contained in \( \text{conv}_E(T(X)) \). In particular there exists a chord \( g \in KP \) such that \( R_t \) crosses \( g \) essentially.

**Proof.** External rays in \( U^\infty \) correspond to geodesic half-lines starting at infinity in the hyperbolic metric on \( \mathbb{C}^\infty \setminus T(X) \). Half-planes are conformally equivalent to disks. Therefore, Jørgensen’s lemma applies: the intersection of \( R_t \) with a half-plane is connected, so it is a half-line. Since the Euclidean convex hull of \( T(X) \) is the intersection of all half-planes containing \( T(X) \), \( R_t \cap \text{conv}_E(T(X)) \) is connected. \( \square \)

**Lemma 5.8.** Let \( E_i \) be a channel (that is, a prime end such that \( \text{Pr}(E_i) \) is non-degenerate) in \( T(X) \). Then for each \( x \in \text{Pr}(E_i) \), for every \( \delta > 0 \), there is a chain \( \{g_i\}_{i=1}^\infty \) of chords defining \( E_i \) selected from \( KP_\delta \) with \( g_i \to x \in \partial T(X) \).

**Proof.** Let \( x \in \text{Pr}(E_i) \) and let \( \{C_i\} \) be a defining chain of crosscuts for \( \text{Pr}(E_i) \) with \( \{x\} = \lim C_i \). By Proposition 5.5 in particular by the remark following the proof of that proposition, there is a sequence \( \{g_i\} \) of \( KP \) chords such that \( d(g_i, C_i) \to 0 \) and \( \text{Pr}(E_i) \) crosses each \( g_i \) essentially. By Proposition 5.4 the sequence \( g_i \) converges to \( \{x\} \). \( \square \)

**Lemma 5.9.** Suppose an external ray \( R_t \) lands on \( a \in T(X) \) with \( \{a\} = \text{Pr}(E_i) \neq \text{Im}(E_i) \). Suppose \( \{x_i\}_{i=1}^\infty \) is a collection of points in \( U^\infty \) with \( x_i \to x \in \text{Im}(E_i) \setminus \{a\} \) and \( \varphi^{-1}(x_i) \to t \). Then there is a sequence of \( KP \) chords \( \{g_i\}_{i=1}^\infty \) such that \( g_i \) separates \( x_i \) from \( \infty \), \( g_i \to a \) and \( \varphi^{-1}(g_i) \to t \) (for sufficiently large \( i \)).

**Proof.** The existence of the chords \( g_i \) again follows from the remark following Proposition 5.5. It is easy to see that \( \lim \varphi^{-1}(g_i) = t \). \( \square \)

5.1. **Auxiliary Continua.** We use \( KP \) chords to form Carathéodory loops around the continuum \( T(X) \).

**Definition 5.10.** Fix \( \delta > 0 \). Define the following collections of chords:

\[
\begin{align*}
KP^+_\delta &= \{g \in KP_\delta \mid \var(f, g, T(X)) \geq 0\} \\
KP^-_\delta &= \{g \in KP_\delta \mid \var(f, g, T(X)) \leq 0\}
\end{align*}
\]
To each collection of chords above, there corresponds an auxiliary continuum defined as follows:

\[ T(X)_\delta = T(X \cup (\bigcup \mathcal{KP}_\delta)) \]

\[ T(X)^+_\delta = T(X \cup (\bigcup \mathcal{KP}^+_\delta)) \]

\[ T(X)^-_\delta = T(X \cup (\bigcup \mathcal{KP}^-_\delta)) \]

**Proposition 5.11.** Let \( Z \in \{ T(X)_\delta, T(X)^+_\delta, T(X)^-_\delta \} \), and correspondingly \( W \in \{ \mathcal{KP}_\delta, \mathcal{KP}^+_\delta, \mathcal{KP}^-_\delta \} \). Then the following hold:

1. \( Z \) is a nonseparating plane continuum.
2. \( \partial Z \subset T(X) \cup (\bigcup W) \).
3. Every accessible point \( y \) in \( \partial Z \) is either a point of \( T(X) \) or a point interior to a chord \( g \in W \).
4. If \( y \in \partial Z \cap g \) with \( g \in W \), then \( y \) is accessible, \( g \subset \partial Z \) and \( \partial Z \) is locally connected at each point of \( g \). Hence, if \( \varphi : \Delta^\infty \to \mathbb{C}^\infty \setminus Z \) is the Riemann map and \( R_t \) is an external ray landing at \( y \), then \( \varphi \) extends continuously to an open interval in \( S_1 \) containing \( t \).

**Proof.** By Proposition 5.11, \( T(X) \cup (\cup W) \) is compact. Moreover, \( T(X) \cup (\cup W) \) is connected since each crosscut \( A \in W \) has endpoints in \( T(X) \). Hence, the topological hull \( T(T(X) \cup (\cup W)) \) is a nonseparating plane continuum, establishing (1).

Since \( Z \) is the topological hull of \( T(X) \cup (\cup W) \), no boundary points can be in complementary domains of \( T(X) \cup (\cup W) \). Hence, \( \partial Z \subset T(X) \cup (\cup W) \), establishing (2). Conclusion (3) follows immediately.

Suppose \( y \in \partial Z \cap g \) with \( g \in W \). Then \( \text{Sh}(g) \subset Z \) and there exists \( y_i \in \mathbb{C} \setminus Z \) such that \( \lim y_i = y \). We may assume that all the points \( y_i \) are on the “same side” of the arc \( g \) (i.e., \( y_i \in \mathbb{C} \setminus \text{Sh}(g) \)). This side of \( g \) is either (1) a limit of \( \mathcal{KP} \) chords \( g_j \), or (2) there exists a gap \( \text{conv}_\mathcal{H}(B \cap X) \) on this side with \( g \) in its boundary. In case (1), \( g \subset \text{Sh}(g_j) \) and, since \( y_i \in \mathbb{C} \setminus Z \) for all \( i \), \( g_j \not\in W \). Hence each \( g_j \subset \mathbb{C} \setminus Z \) for all \( j \). It follows that every point of \( g \) is accessible, \( g \subset \partial Z \) and \( \partial Z \) is locally connected at each point of \( g \). In case (2) there exists a chord \( g' \neq g \) in the boundary of \( \text{conv}_\mathcal{H}(B \cap X) \) which separates \( g \) from infinity. Then \( g' \not\in W \) and the interior of \( \text{conv}_\mathcal{H}(B \cap X) \subset \mathbb{C} \setminus Z \). Hence the same conclusion follows.

The last part of (4) follows from the proof of Carathéodory’s theorem (see [21]).

**Proposition 5.12.** \( T(X)_\delta \) is locally connected; hence, \( \partial T(X)_\delta \) is a Carathéodory loop.
Proof. Suppose that \( T(X)_\delta \) is not locally connected. Then \( T(X)_\delta \) has a non-trivial impression and there exist \( 0 < \varepsilon < \delta/2 \) and a chain \( A_i \) of crosscuts of \( T(X)_\delta \) such that diam(Sh\((A_i)\)) > \( \varepsilon \) for all \( i \). We may assume that \( \lim A_i = y \in T(X)_\delta \).

By Proposition 5.11 (4) we may assume \( y \in X \). Choose \( z_i \in \text{Sh}(A_i) \) such that \( d(z_i, y) > \varepsilon \). We can enlarge the crosscut \( A_i \) of \( T(X)_\delta \) to a crosscut \( C_i \) of \( T(X) \) as follows. Suppose that \( A_i \) joins the points \( a_i^+ \) and \( a_i^- \) in \( T(X)_\delta \). If \( a_i^+ \in T(X) \), put \( y_i^+ = a_i^+ \). Otherwise \( a_i^+ \) is contained in a chord \( g_i^+ \in \mathcal{KP}_\delta \), with endpoints in \( T(X) \), which is contained in \( T(X)_\delta \). Since \( \lim A_i = y \), we can select one of these endpoints and call it \( y_i^- \) such that \( d(y_i^+ , a_i^-) \rightarrow 0 \). Define \( g_i^- \) and \( y_i^- \) similarly. Then \( g_i^+ \cup A_i \cup g_i^- \) contains a crosscut \( C_i \) of \( T(X) \) joining the points \( y_i^+ \) and \( y_i^- \) such that \( \lim C_i = y \). We claim that \( z_i \in \text{Sh}(C_i) \). To see this note that, since \( z_i \in \text{Sh}(A_i) \), there exists a halfray \( R_i \subset \mathbb{C} \setminus T(X)_\delta \) joining \( z_i \) to infinity such that \( |R_i \cap A_i| \) is an odd number and each intersection is transverse. Since \( R_i \cap C_i = R_i \cap A_i \) it follows that \( z_i \in \text{Sh}(C_i) \). Let \( \text{conv}_H(B_i \cap X) \) be the unique hull of the Kulkarni-Pinkall partition \( \mathcal{KP} \) which contains \( z_i \). Since diam\((C_i) \rightarrow 0 \) and \( d(z_i, y) > \varepsilon \), it follows from Proposition 5.11 that diam\((\text{conv}_H(B_i \cap X)) < 2\varepsilon < \delta \). This contradicts the fact that \( z_i \in \mathbb{C} \setminus T(X)_\delta \) and completes the proof.

\[ \square \]

6. Outchannels

Suppose that \( f : \mathbb{C} \rightarrow \mathbb{C} \) is a perfect map, \( X \) is a continuum, \( f \) has no fixed point in \( T(X) \) and \( X \) which is minimal with respect to \( f(X) \subset T(X) \). Fix \( \eta > 0 \) such that for each \( \mathcal{KP} \) chord \( g \subset T(X)_\eta \), \( \overline{g} \cap f(\overline{g}) = \emptyset \) and \( f \) is fixed point free on \( T(X)_\eta \). In this case we will say that \( \eta \) defines variation near \( X \) and that the triple \((f, X, \eta)\) satisfy the standing hypothesis. In this section we will show that \( X \) has at least one negative outchannel, which is defined as follows. Note that for each \( \mathcal{KP} \) chord \( g \) in \( T(X)_\eta \), var\((f, g, T(X)) = \text{var}(f, g) \) is defined.

**Definition 6.1 (Outchannel).** Suppose that \((X, f, \eta)\) satisfy the standing hypothesis. An outchannel of the nonseparating plane continuum \( T(X) \) is a prime end \( \mathcal{E}_i \) of \( \mathbb{C}^\infty \setminus T(X) \) such that for some chain \( \{g_i\} \) of crosscuts defining \( \mathcal{E}_i \), var\((f, g_i, T(X)) \neq 0 \) for every \( i \). We call an outchannel \( \mathcal{E}_i \) of \( T(X) \) a geometric outchannel iff for sufficiently small \( \delta \), every chord in \( \mathcal{KP}_\delta \), which crosses \( \mathcal{E}_i \) essentially, has nonzero variation. We call a geometric outchannel negative (respectively, positive) (starting at \( g \in \mathcal{KP} \)) iff every \( \mathcal{KP} \) chord \( h \subset T(X)_\eta \cap \text{Sh}(\overline{g}) \), which crosses \( \mathcal{E}_i \) essentially, has negative (respectively, positive) variation.
Lemma 6.2. Suppose that \((f, X, \eta)\) satisfy the standing hypothesis and \(\delta \leq \eta\). Let \(Z \in \{T(X)_{-\delta}^+, T(X)_{-\delta}^-\}\). Fix a Riemann map \(\varphi : \Delta^\infty \rightarrow \mathbb{C}^\infty \setminus Z\) such that \(\varphi(\infty) = \infty\). Suppose \(R_t\) lands at \(x \in \partial Z\). Then there is an open interval \(M \subset \partial \Delta^\infty\) containing \(t\) such that \(\varphi\) can be extended continuously over \(M\).

Proof. Suppose that \(Z = T(X)_{-\delta}^-\) and \(R_t\) lands on \(x \in \partial Z\). By proposition [5.11] we may assume that \(x \in X\). Note first that the family of chords in \(KP_{-\delta}\) form a closed subset of the hyperspace of \(\mathbb{C} \setminus X\), by Proposition 5.1. By symmetry, it suffices to show that we can extend \(\psi\) over an interval \([t', t] \subset S^1\) for \(t' < t\).

Let \(\phi : \Delta^\infty \rightarrow \mathbb{C} \setminus T(X)\) be the Riemann map for \(T(X)\). Then there exists \(s \in S^1\) so that the external ray \(R_s\) of \(\mathbb{C} \setminus T(X)\) lands at \(x\). Suppose first that there exists a chord \(g \in KP_{-\delta}\) such that \(G = \varphi^{-1}(g)\) has endpoints \(s'\) and \(s\) with \(s' < s\). Since \(KP_{-\delta}\) is closed, there exists a minimal \(s'' \leq s' < s\) such that there exists a chord \(h \in KP_{-\delta}\) so that \(H = \varphi^{-1}(h)\) has endpoints \(s''\) and \(s\). Then \(h \subset \partial Z\) and \(\phi\) can be extended over an interval \([t', t]\) for some \(t' < t\), by Proposition 5.11 (4).

Suppose next that no such chord \(g\) exists. Choose a junction \(J_x\) for \(T(X)_{-\delta}^-\) and a neighborhood \(W\) of \(x\) such that \(f(W) \cap [W \cup J_x] = \emptyset\). We will first show that there exists \(\nu \leq \delta\) such that \(x \in \partial T(X)_{\nu}\). We may assume that \(\nu\) is so small that any chord of \(KP_{\nu}\) with endpoint \(x\) is contained in \(W\). For suppose that this is not the case. Then there exists a sequence \(g_i \in KP\) of chords such that \(x \in Sh(g_{i+1}) \subset Sh(g_i)\), \(\lim g_i = x\) and \(\var(f, g_i) > 0\) for all \(i\). This contradicts Proposition 5.6. Hence \(x \in \partial T(X)_{\nu}\) for some \(\nu > 0\).

By Proposition 5.12 the boundary of \(T(X)_{\nu}\) is a simple closed curve \(S\) which must contain \(x\). If there exists a chord \(h \in KP_{\nu}\) with endpoint \(x\) such that \(H\) has endpoints \(s'\) and \(s\) with \(s' < s\) then, since \(h \subset W\), \(f(h) \cap J_x = \emptyset\), \(\var(f, h) = 0\) and \(h \in KP_{-\delta}\), a contradiction. Hence, chords \(h\) close to \(x\) in \(S\) so that \(H\) has endpoints less than \(s\) are contained in \(W\) and \(\var(f, h) = 0\) by Proposition 5.6. Hence a small interval \([x', x] \subset S\), in the counterclockwise order on \(S\) is contained in \(T(X)_{\nu}\). It now follows easily that a similar arc exists in the boundary of \(T(X)_{-\delta}\) and the desired result follows.

By a narrow strip we mean the image of an embedding \(h : \{(x, y) \in \mathbb{C} \mid x \geq 0 \text{ and } -1 \leq y \leq 1\} \rightarrow \mathbb{C}\) such that \(\lim_{x \to \infty} \text{diam}(h([x] \times [-1, 1])) = 0\).

Lemma 6.3. Suppose that \((f, X, \eta)\) satisfy the standing hypothesis. If there is a chord \(g\) of \(T(X)\) of negative (respectively, positive) variation,
such that there is no fixed point in $T(T(X) \cup g)$, then there is a negative (respectively, positive) geometric outchannel $E_t$ of $T(X)$ starting at $g$.

Moreover, if $E_t$ is a positive (negative) geometric outchannel starting at the $\mathcal{KP}$ chord $g$, and $\mathcal{G} = \bigcup \{ \text{conv}_H(B \cap T(X)) \mid \text{conv}_H(B \cap T(X)) \subset T(X) \cap \text{Sh}(g) \}$ and a chord in $\text{conv}_H(B \cap T(X))$ crosses $R_t$ essentially. Then $\mathcal{G}$ is an infinite narrow strip in the plane whose remainder is contained in $T(X)$ and which is bordered by a $\mathcal{KP}$ chord and two halflines $H_1$ and $H_2$ (see figure 4).

**Proof.** Without loss of generality, assume $\text{var}(f, g, T(X)) = \text{var}(f, g) < 0$. If $g \subset T(X)_\eta$, put $g' = g$. Otherwise consider the boundary of $T(X)_\eta$ which is locally connected by Proposition 5.12 and, hence, a Carathéodory loop. Then a continuous extension $g : S^1 \to \partial T(X)_\eta$ of the Riemann map $\phi : \Delta^\infty \to \mathbb{C}^\infty \setminus T(X)_\eta$ exists. Whence the boundary of $T(X)_\eta$ contains a sub-path $A = g([a, b])$, which is contained in $\text{Sh}(g)$, whose endpoints coincide with the endpoints of $g$. Note that for each component $C$ of $A \setminus X$, $\text{var}(f, C)$ is defined. Then it follows from Proposition 2.19 applied to a Carathéodory path, that there exists a
component $C = g'$ such that $\text{var}(f, g') < 0$. Note that $g'$ is a $\mathcal{KP}$ chord contained in the boundary of $T(X)_\eta$.

To see that a geometric outchannel, starting with $g$ exists, note that for any chord $g'' \subset T(X)_\delta$ with $\text{var}(f, g'', X) < 0$, if $g'' = \lim g_i$, then there exists $i$ such that for any chord $h$ which separates $g_i$ and $g''$ in $U^\infty$, $\text{var}(f, h, X) < 0$ (see Proposition 5.6). Also by the argument above, if in addition $g'' \subset \text{conv}_H(B \cap T(X))$, where $\text{conv}_H(B \cap T(X))$ is a gap, such that $g''$ separates $\text{conv}_H(B \cap T(X)) \setminus g''$ from infinity in $U^\infty$, then there exists $h \neq g''$ in $\text{conv}_H(B \cap T(X))$ such that $g''$ separates $h$ from infinity in $U^\infty$ and $\text{var}(f, h, X) < 0$. The remaining conclusions of the Lemma follow from these two facts. □

6.1. Invariant Channel in $X$. We are now in a position to prove Bell’s principal result on any possible counter-example to the fixed point property, under our standing hypothesis.

**Lemma 6.4.** Suppose $\mathcal{E}_t$ is a geometric outchannel of $T(X)$ under $f$. Then the principal continuum $\text{Pr}(\mathcal{E}_t)$ of $\mathcal{E}_t$ is invariant under $f$. So $\text{Pr}(\mathcal{E}_t) = X$.

**Proof.** Let $x \in \text{Pr}(\mathcal{E}_t)$. Then for some chain $\{g_i\}_{i=1}^\infty$ of crosscuts defining $\mathcal{E}_t$ selected from $\mathcal{KP}_\delta$, we may suppose $g_i \to x \in \partial T(X)$ (by Lemma 5.8) and $\text{var}(f, g_i, X) \neq 0$ for each $i$. The external ray $R_t$ meets all $g_i$ and there is, for each $i$, a junction from $g_i$ which “parallels” $R_t$. Since $\text{var}(f, g_i, X) \neq 0$, each $f(g_i)$ intersects $R_t$. Since $\text{diam}(f(g_i)) \to 0$, we have $f(g_i) \to f(x)$ and $f(x) \in \text{Pr}(\mathcal{E}_t)$. We conclude that $\text{Pr}(\mathcal{E}_t)$ is invariant. □

**Theorem 6.5** (Dense channel, Bell). Suppose that $(X, f, \eta)$ satisfy our standing hypothesis. Then $T(X)$ contains a negative geometric outchannel; hence, $\partial U^\infty = \partial T(X) = X = f(X)$ is an indecomposable continuum.

**Proof.** By Lemma 5.12 $\partial T(X)_\eta$ is a Carathéodory loop. Since $f$ is fixed point free on $T(X)_\eta$, $\text{ind}(f, \partial T(X)_\eta) = 0$. Consequently, by Theorem 2.13 for Carathéodory loops, $\text{var}(f, \partial T(X)_\eta) = -1$. By the summability of variation on $\partial T(X)_\eta$, it follows that on some chord $g \subset \partial T(X)_\eta$, $\text{var}(f, g, T(X)) < 0$. By Lemma 6.3 there is a negative geometric outchannel $\mathcal{E}_t$ starting at $g$.

Since $\text{Pr}(\mathcal{E}_t)$ is invariant under $f$ by Lemma 6.4 it follows that $\text{Pr}(\mathcal{E}_t)$ is an invariant subcontinuum of $\partial U^\infty \subset \partial T(X) \subset X$. So by the minimality condition in our Standing Hypothesis, $\text{Pr}(\mathcal{E}_t)$ is dense in $\partial U^\infty$. Hence, $\partial U^\infty = \partial T(X) = X$ and $\text{Pr}(\mathcal{E}_t)$ is dense in $X$. It then follows from a theorem of Rutt 22 that $X$ is an indecomposable continuum. □
Theorem 6.6. Suppose that \((X, f, \eta)\) satisfy our standing Hypothesis and \(\delta \leq \eta\). Then the boundary of \(T(X)_\delta\) is a simple closed curve. The set of accessible points in the boundary of each of \(T(X)^+_{\delta}\) and \(T(X)^-_{\delta}\) is an at most countable union of continuous one-to-one images of \(\mathbb{R}\).

Proof. By Theorem 6.5, \(X\) is indecomposable, so it has no cut points. By Proposition 5.12, \(\partial T(X)_\delta\) is a Carathéodory loop. Since \(X\) has no cut points, neither does \(T(X)_\delta\). A Carathéodory loop without cut points is a simple closed curve.

Let \(Z \in \{T(X)^{+}_{\delta}, T(X)^{-}_{\delta}\}\) with \(\delta \leq \eta\). Fix a Riemann map \(\phi : \Delta^{\infty} \to \mathbb{C}^{\infty} \setminus Z\) such that \(\phi(\infty) = \infty\). Corresponding to the choice of \(Z\), let \(W \in \{KP^{+}_{\delta}, KP^{-}_{\delta}\}\). Apply Lemma 6.2 and find the maximal collection \(J\) of disjoint open subarcs of \(\partial \Delta^{\infty}\) over which \(\phi\) can be extended continuously. The collection \(J\) is countable. Since \(X\) has no cutpoints the extension is one-to-one over \(\bigcup J\). Since angles that correspond to accessible points are dense in \(\partial \Delta^{\infty}\), so is \(\bigcup J\). If \(Z = T(X)^{+}_{\delta}\), then it is possible that \(\bigcup J\) is all of \(\partial \Delta^{\infty}\) except one point, but it cannot be all of \(\partial \Delta^{\infty}\) since there is at least one negative geometric outchannel by Theorem 6.5. □

Theorem 6.6 still leaves open the possibility that \(Z \in \{T(X)^{+}_{\delta}, T(X)^{-}_{\delta}\}\) has a very complicated boundary. The set \(C = \partial \Delta^{\infty} \setminus \bigcup J\) is compact and zero-dimensional. Note that \(\phi\) is discontinuous at points in \(C\). We may call \(C\) the set of outchannels of \(Z\). In principle, there could be an uncountable set of outchannels, each dense in \(X\). The one-to-one continuous images of half lines in \(\mathbb{R}\) lying in \(\partial Z\) are the “sides” of the outchannels. If two elements \(J_1\) and \(J_2\) of the collection \(J\) happen to share a common endpoint \(t\), then the prime end \(E_t\) is an outchannel in \(Z\), dense in \(X\), with images of half lines \(\phi(J_1)\) and \(\phi(J_2)\) as its sides. It seems possible that an endpoint \(t\) of \(J \in J\) might have a sequence of elements \(J_i\) from \(J\) converging to it. Then the outchannel \(E_t\) would have only one (continuous) “side.” Such exotic possibilities are eliminated in the next section.

In the lemma below we summarize several of the results in this section and show that an arc component \(K\) of the set of accessible points of the boundary of \(T(X)^{-}_{\delta}\) is efficient in connecting close points in \(K\).

Proposition 6.7. Suppose that \((X, f, \eta)\) satisfy our standing Hypothesis, that the boundary of \(T(X)^{-}_{\delta}\) is not a simple closed curve, \(\delta \leq \eta\) and that \(K\) is an arc component of the boundary of \(T(X)^{-}_{\delta}\) so that \(K\) contains an accessible point. Let \(\varphi : \Delta^{\infty} \to \mathbb{C}^{\infty} \setminus T(X)^{-}_{\delta}\) be a conformal map such that \(\varphi(\infty) = \infty\). Then:
The same conclusion holds for $T(X)$ since its boundary cannot be a simple closed curve.

Proof. By Proposition 5.11 and Theorem 6.3 and its proof, $\varphi$ extends continuously and injectively to a map $\tilde{\varphi} : \tilde{\Delta}^\infty \to \tilde{U}^\infty$, where $\tilde{\Delta}^\infty \setminus \Delta^\infty$ is a dense and open subset of $S^1$ which contains $K$ in its image. Let $\tilde{\varphi}^{-1}(K) = (t',t) \subset S^1$ with $t' < t$ in the counterclockwise order on $S^1$. Hence $\tilde{\varphi}$ induces an order $< \text{ on } K$. If $x < y \in K$, we denote by $< x, y >$ the subarc of $K$ from $x$ to $y$ and by $< x, \infty > = \bigcup_{y>x} < x, y >$.

(2) $E_t$ and $E_{t'}$ are positive geometric outchannels of $T(X)$.

(3) Let $R_t$ be the external ray of $T(X)_{\delta^+}$ with argument $t$. There exists $s \in R_t$, $B \in \mathcal{B}^\infty$ and $g \in \mathcal{K}P$ such that $s \in g \subset \text{conv}_H(B \cap X)$ and $s$ is the last point of $R_t$ in $\text{conv}_H(B \cap X)$ (from $\infty$), $g$ crosses $R_t$ essentially and for each $B' \in \mathcal{B}$ with $\text{conv}_H(B' \cap X) \setminus X \subset \text{Sh}(g)$, diam$(B') < \delta$.

(4) There exists $\hat{x} \in K$ such that if $B' \in \mathcal{B}^\infty$ with $\text{Int}(B') \subset \text{Sh}(g)$, then $\text{conv}_H(B' \cap X) \cap < \hat{x}, \infty >$ is a compact ordered subset of $K$ so that if $C$ is a component of $< x, \infty > \setminus \text{conv}_H(B' \cap X)$ with two endpoints in $\text{conv}_H(B' \cap X)$, $C \in \mathcal{K}P_{\delta^+}$.

(5) Let $\mathcal{B}_t^\infty \subset \mathcal{B}^\infty$ be the collection of all $B \in \mathcal{B}^\infty$ such that $R_t$ crosses a chord in the boundary of $\text{conv}_H(B \cap X)$ essentially and $\text{Int}(B) \subset \text{Sh}(g)$. Then $\mathcal{G} = \bigcup_{B \in \mathcal{B}^\infty} \text{conv}_H(B \cap X)$ is a narrow strip in the plane, bordered by two halflines $H_1$ and $H_2$, which compactifies on $X$ and one of $H_1$ or $H_2$ contains the set $< \hat{x}', \infty >$ for some $\hat{x}' \in K$.

In particular, if $\max(\hat{x}, \hat{x}') < p < q$ and diam$(< p, q >) > 2\delta$, then there exists a chord $g \in \mathcal{K}P$ such that one endpoint of $g$ is in $< p, q >$ and $g$ crosses $R_t$ essentially.
Let $B' \in \mathcal{B}^\infty$ with $\text{Int}(B') \subset \text{Sh}(g)$, $\hat{x} \notin B'$ and $< \hat{x}, \infty > \cap \text{conv}_\mathcal{H}(B' \cap X)$ not connected. Let $h \in \mathcal{K}\mathcal{P}$ with $h \subset \text{conv}_\mathcal{H}(B' \cap X)$ which separates $\text{conv}_\mathcal{H}(B' \cap X) \setminus h$ from $\infty$ in $U^\infty$. Let $a$ and $b$ be the endpoints of $h$. Then $\{a, b\} \subset < \hat{x}, \infty >$. If there is a chord $h' \in \mathcal{K}\mathcal{P}$ with endpoints $a$ and $b$ and $h' \subset < \hat{x}, \infty >$ we are done. If such a $\mathcal{K}\mathcal{P}$ chord does not exist, then $\text{var}(f, h) > 0$. By Lemma 6.3, there is a geometric outchannel $\mathcal{E}_t^r$ starting at $h$. This outchannel disconnects the arc $< a, b >$ between $a$ and $b$, a contradiction. Hence (4) holds.

Next choose $\hat{x}' \in K$ such that each point of $< \hat{x}', \infty >$ is accessible from $\text{Sh}(g)$. Then each subarc $< p, q >$ of $< \hat{x}, \infty >$ of diameter bigger than $2\delta$ cannot be contained in a single element of the $\mathcal{K}\mathcal{P}$ partition. Hence there exists a $\mathcal{K}\mathcal{P}$ chord $g$ which crosses $R_t$ essentially and has one endpoint in $< p, q >$.

Note that for each chord $h \subset \text{Sh}(g)$ which crosses $R_t$ essentially, $\text{var}(f, h) > 0$. By Lemma 6.3, $\bigcup_{B \in \mathcal{B}^\infty} \text{conv}_\mathcal{H}(B \cap X)$ is a strip in the plane, bordered by two halflines $H_1, H_2$, which compactify on $X$. These two halflines, consist of chords in $\mathcal{K}\mathcal{P}_\delta$ and points in $X$, one of which, say $H_1$ meets $< \hat{x}', \infty >$. If $< \hat{x}', \infty >$ is not contained in $H_1$ then, as in the proof of (4), there exists a chord $h \subset P_1$ with $\text{var}(f, h) > 0$ joining two points of $x, y \in < \hat{x}, \infty >$ with all the points of $< x, y >$ accessible from $\text{Sh}(h)$. As above this leads to the contradiction that $< x, y >$ contains a chord of positive variation and the proof is complete. □

7. Uniqueness of the Outchannel

Theorem 6.5 asserts the existence of at least one negative geometric outchannel which is dense in $X$. We show below that there is exactly one geometric outchannel, and that its variation is $-1$. Of course, $X$ could have other dense channels, but they are “neutral” as far as variation is concerned.

**Theorem 7.1** (Unique Outchannel). Suppose that $(X, f, \eta)$ satisfy the standing hypothesis. Then there exists a unique geometric outchannel $\mathcal{E}_t$ for $X$, which is dense in $X = \partial T(X)$. Moreover, for any sufficiently small chord $g$ in any chain defining $\mathcal{E}_t$, $\text{var}(f, g, X) = -1$, and for any sufficiently small chord $g'$ not crossing $R_t$ essentially, $\text{var}(f, g', X) = 0$.

**Proof.** Suppose by way of contradiction that $X$ has a positive outchannel. Let $0 < \delta \leq \eta$ such that if $M \subset T(B(T(X), 2\delta))$ with $\text{diam}(M) < 2\delta$, then $f(M) \cap M = \emptyset$. Since $X$ has a positive outchannel, $\partial T(X)^\mathcal{H}$ is not a simple closed curve. By Theorem 6.6, $\partial T(X)^\mathcal{H}$ contains an arc component $K$ which is the one-to-one continuous image of $\mathbb{R}$. Note that each point of $K$ is accessible.
Let $\varphi : \Delta^\infty \to V^\infty = \mathbb{C} \setminus T(X)_\delta^-$ a conformal map. By Proposition 6.7, $\varphi$ extends continuously and injectively to a map $\tilde{\varphi} : \tilde{\Delta}^\infty \to \tilde{V}^\infty$, where $\tilde{\Delta}^\infty \setminus \Delta^\infty$ is a dense and open subset of $S^1$ which contains $K$ in its image. Then $\tilde{\varphi}^{-1}(K) = (t', t) \subset S^1$ is an open arc with $t' < t$ in the counterclockwise order on $S^1$ (it could be that $(t', t) = S^1 \setminus \{t\}$ and $t = t'$). By abuse of notation, let $<$ denote the order in $K$ induced by $\tilde{\varphi}$ and for $x < y$ in $K$, denote the arc in $K$ with endpoints $x$ and $y$ by $\langle x, y \rangle$. For $x \in K$, let $\langle x, \infty \rangle = \cup_{y > x} \langle x, y \rangle$.
Let $\mathcal{E}_t$ be the prime-end corresponding to $t$. By Proposition 6.7, $\Pr(\mathcal{E}_t)$ is a positive geometric outhannel and, hence, by Lemma 6.4 $\Pr(\mathcal{E}_t) = X$. Let $R_t = \varphi(re^{it})$, $r > 1$, be the external conformal ray corresponding to the prime-end $\mathcal{E}_t$ of $T(X)_s$. Since $\overline{R_t \setminus R_t} = X$ and the small chords $g_x$ which define $\Pr(\mathcal{E}_t)$ have at least one endpoint in $K$, cross $R_t$ essentially at $x$ and have diameter going to zero as $x$ approaches $X$ along $R_t$ (by Proposition 6.4 and Lemma 5.8). Hence, $X = \overline{R_t \setminus R_t} = R_t$. By Proposition 6.7 there is $s \in R_t$ such that if $B \in \mathfrak{B}^\infty$ such that $\text{conv}_H(B \cap X) \cap (X, s)$-end of $R_t \setminus \emptyset$, then $\text{diam}(B) < \delta/2$. Let $\mathfrak{B}^\infty_s = \{B \in \mathfrak{B}^\infty \mid \text{conv}_H(B \cap X) \cap (X, s)\text{-end of } R_t \setminus \emptyset\}$. By Proposition 6.7 there exists $\hat{x} \in K \cap X$ such that for each arc $A \subset \hat{x}, \infty >$ with diameter $> 2\delta$, there is a $\mathcal{KP}$ chord $g$ which contains a point of $A$ as an endpoint and crosses $R_t$ essentially.

Let $a_0 \in K \cap X$ so that $a_0 > \hat{x}$ and $J_{a_0}$ is a junction of $T(X)_s$. Let $W$ be an open disk, with simple closed curve boundary, about $a_0$ such that $\text{diam}(W) < \delta/4$ and $f(W) \cap W \cup J_{a_0} = \emptyset$. Let $a < a_0 < b$ in $K \cap \partial W$ such that $< a, b >$ is the component of $K \cap W$ which contains $a_0$. We may suppose that $< b, \infty > \cap W$ is contained in one component of $W \setminus < a, b >$ since one side of $K$ is accessible from $C \setminus T(X)_s$ and $a_0 \in X$. If $a \in X$, let $p = a$. If not, then there exists a $\mathcal{KP}$ chord $h$ such that $a \in h$. Then $h \subset K$ by Proposition 5.11. Let $p$ be the endpoint of $h$ such that $p < a$.

Since $X \subset < x, \infty >$ there are components of $< b, \infty > \cap W$ which are arbitrarily close to $a_0$. Choose $b < c < d$ in $K$ so that the $< c, d >$ is the closure of a component of $W \cap < b, \infty >$ such that:

1. $a$ and $d$ lie in the same component of $\partial W \setminus \{b, c\}$.
2. There exists $z < c, d > \cap X \cap W$ and an arc $I \subset \{a_0, z\} \cup [W \setminus < p, d >] \cup [W \setminus < p, d >]$ joining $a_0$ to $z$.
3. There is a $\mathcal{KP}$ chord $g \subset W$ with $z$ and $y$ as endpoints which crosses $R_t$ essentially. Hence, $\text{var}(f, g) > 0$.
4. $\text{diam}(f(g)) < d(J_{a_0} \setminus W, J_{a_0} \setminus W)$.

Conditions (1) and (2) follow because $J_{a_0}$ is a connected and closed set from $a_0$ to $\infty$ in $\{a_0\} \cup [C \setminus T(X)_s]$ and the ray $< b, \infty >$ approaches both $a_0$ and $p$. Conditions (3) and (4) follow from Proposition 6.7. If $d \in X$, put $q = d$. Otherwise, let $q < < d, \infty >$ such that there is a $\mathcal{KP}$ chord $h \subset K$ containing $d$ with endpoint $q$ and such that $d < q$.

By a simple extension of Corollary 2.17, there exists a bumping arc $A'$ of $T(X)$ from $p$ to $q$ such that variation is defined on each component
of $A' \setminus X$, $S' = A' \cup <p,q>$ is a simple closed curve with $T(X) \subset T(S')$ and $f$ is fixed point free on $T(S')$. Since $\overline{g} \cap X = \{z,y\}$, we may assume that $A' \cap \overline{g} = \{y\}$. Let $C$ be the arc in $\partial W$ from $a$ to $d$ disjoint from $b$. The arc $A'$ may enter $W$ and intersect $I$ several times. However, in this case $A'$ must enter $W$ through $C$. Since we want to apply the Lollipop lemma, we will modify the arc $A'$ to a new arc $A$ which is disjoint from $I$.

Let $A$ be the set of points in $A' \cup C$ accessible from $\infty$ in $\mathbb{C} \setminus [S' \cup C]$. Then $A$ is a bumping arc from $p$ to $q$, $A \cap I = \emptyset$, $\text{var}(f,A)$ is defined, $S = A \cup <p,q>$ is a simple closed curve with $T(X) \subset T(S)$ and $f$ is fixed point free on $T(S)$. Note that $y \in A$. Then the Lollipop lemma applies to $S$ with $R = T(<a_0,z> \cup I)$ and $R = T(I \cup <z,q> \cup A \cup <p,a_0>)$.

Claim: $f(z) \in R$. Hence by Corollary $2.15$, $<a_0,z>$ contains a chord $g_1$ with $\text{var}(f,g_1) < 0$.

Proof of Claim. Note that the positive direction along $g$ is from $z$ to $y$. Since $z,y \in X$, $\{f(z),f(y)\} \subset X \subset T(S) = T(R) \cup T(L)$. Choose a junction $J_y$ such that $J_{a_0} \setminus W \subset J_y$ and $J_y$ runs close to $R_t$ on its way to $g$. Since $g$ crosses $R_t$ essentially, $\text{var}(f,g) > 0$. Since variation is invariant under suitable homotopies, we may assume that for each $* \in \{+,i,-\}$, $J_y^* \setminus W$ consists of exactly two components one of which is contained in $J_{a_0}$. Let $C_y^*$ be the component of $J_y^* \setminus W$ which is disjoint from $J_{a_0}$. Then $C_y^*$ separates $R \cup C_y^+$ from $L \cup C_y^-$ in $\mathbb{C} \setminus W$ (see figure 5).

Since $f(g) \cap J_{a_0} = \emptyset$, if $f(z) \not\in T(R)$, $\text{var}(f,g) \leq 0$, a contradiction. Hence $f(z) \in T(R)$ (and, in fact, $f(y) \in T(L)$) as desired.

Since $f(z) \in R$, $<a_0,z>$ contains a chord $g$ with $\text{var}(f,g) < 0$. Repeating the same argument, replacing $a_0$ by $y$ and $J_0$ by $J_y$ we obtain a second chord $g_2$ contained in $<y,\infty>$ such that $\text{var}(f,g_2) < 0$.

We will now show that the existence of two distinct chords $g_1$ and $g_2$ in $K$ with variation $<0$ on each leads to a contradiction. Recall that $a_0 \in <b,\infty>$. Hence we can find $y' \in <b,\infty>$ with $y' \in X$ such that $g_1 \cup g_2 \subset <a_0,y'>$ and there exists a small arc $I' \subset W'$ such that $I' \cap <a_0,y'> = \{a_0,y'\}$. Since $f(I') \cap J_{a_0} = \emptyset$, $\text{var}(f,I') = 0$.

We may also assume that $f$ is fixed point free on $T(S')$, where $S' = I' \cup <a_0,y'>$. Since $<a_0,y'>$ contains both $g_1$, $g_2$ and no chords of positive variation, $\text{var}(f,<a_0,y'>) \leq -2$ and $\text{var}(f,S') \leq -2$. Then $\text{ind}(f,S') = \text{var}(f,S') + 1 \leq -1$ a contradiction with Theorem $2.6$.

Hence $X$ has no positive geometric outchannel.

By Theorems $6.5$ and $2.13$, $X$ has exactly one negative outchannel and its variation is $-1$. □
Note that the following Theorem follows from Lemma 6.7 and Theorem 7.1.

**Theorem 7.2.** Suppose that $X$ is a minimal counter example to the plane fixed point problem. Then there exists $\delta > 0$ such that the continuum $Y = T(X)_{\delta}^+$ is a non-separating continuum, $f$ is fixed point free on $Y$ and all accessible points of $Y$ are contained in one arc component $K$ of the boundary of $Y$. In other words, $Y$ is homeomorphic to a disk with exactly one channel removed which corresponds to the unique geometric outchannel of variation $-1$ of $X$. This channel compactifies on $X$. The sides of this channel are halflines consisting entirely of chords of zero variation and points in $X$. There exist arbitrarily small homeomorphisms of the tails of these halflines to the tails of $R_t$ which is the external ray corresponding to this channel.

8. Oriented maps

A *perfect map* is a closed continuous function each of whose point inverses is compact. *We will assume in the remaining sections that all maps of the plane are perfect.* Let $X$ and $Y$ be spaces. A map $f : X \to Y$ is *monotone* provided for each continuum $K \subset Y$, $f^{-1}(K)$ is connected. A map $f : X \to Y$ is *confluent* provided for each continuum $K \subset Y$ and each component $C$ of $f^{-1}(K)$, $f(C) = K$ and $f$ is *light* provided for each point $y \in Y$, $f^{-1}(y)$ is totally disconnected.

It is well known that each homeomorphism of the plane is either orientation-preserving or orientation-reversing. In this section we will establish an appropriate extension of this result for confluent perfect mappings of the plane (Theorem 8.7) by showing that such maps either preserve or reverse local orientation. As a consequence it follows that all perfect and confluent maps of the plane satisfy the Maximum Modulus Theorem. We will call such maps positively- or negatively-oriented maps, respectively. For perfect mappings of the plane, Lelek and Read have shown that confluent is equivalent to the composition of open and monotone maps [17]. Holomorphic maps are prototypes of positively-oriented maps but positively-oriented maps, unlike holomorphic maps, do not have to be light. A non-separating plane continuum is said to be *acyclic*.

**Definition 8.1 (Degree of $f_p$).** Let $f : U \to \mathbb{C}$ be a map from a simply connected domain $U \subset \mathbb{C}$ into the plane. Let $S \subset \mathbb{C}$ be a positively oriented simple closed curve in $U$, and $p \in U \setminus f^{-1}(f(S))$ a point.
Define $f_p : S \to \mathbb{S}^1$ by

$$f_p(x) = \frac{f(x) - f(p)}{|f(x) - f(p)|}.$$  

Then $f_p$ has a well-defined degree, denoted $\text{degree}(f_p)$. Note that $\text{degree}(f_p)$ is the winding number $\text{win}(f, S, f(p))$ of $f|_S$ about $f(p)$.

**Definition 8.2.** A map $f : U \to \mathbb{C}$ from a simply connected domain $U$ is positively-oriented (respectively, negatively-oriented) provided for each simple closed curve $S$ in $U$ and each point $p \in T(S) \setminus f^{-1}(f(S))$, $\text{degree}(f_p) > 0$ (respectively, $\text{degree}(f_p) < 0$).

**Definition 8.3.** A perfect surjection $f : \mathbb{C} \to \mathbb{C}$ is oriented provided for each simple closed curve $S$ and each $x \in T(S)$, $f(x) \in T(f(S))$.

Clearly every positively oriented and each negatively oriented map is oriented. It will follow that all oriented maps satisfy the Maximum Modulus Theorem [8.7].

It is well-known that both open maps and monotone maps (and hence compositions of such maps) of continua are confluent. It will follow (Lemma [8.6]) from a result of Lelek and Read [17] that each perfect, oriented surjection of the plane is the composition of a monotone map and a light open map. The following Lemmas are in preparation for the proof of Theorem [8.7].

**Lemma 8.4.** Suppose $f : \mathbb{C} \to \mathbb{C}$ is a perfect surjection. Then $f$ is confluent if and only if $f$ is oriented.

**Proof.** Suppose that $f$ is oriented. Let $A$ be an arc in $\mathbb{C}$ and let $C$ be a component of $f^{-1}(A)$. Suppose that $f(C) \neq A$. Let $a \in A \setminus f(C)$. Since $f(C)$ does not separate $a$ from infinity, we can choose a simple closed curve $S$ with $C \subset T(S)$, $S \cap f^{-1}(A) = \emptyset$ and $f(S)$ so close to $f(C)$ that $f(S)$ does not separate $a$ from $\infty$. Then $a \notin T(f(S))$. Hence there exists a $y \in A \cap f(S)$. This contradicts the fact that $A \cap f(S) = \emptyset$. Thus $f(C) = A$.

Now suppose that $K$ is an arbitrary continuum in $\mathbb{C}$ and let $L$ be a component of $f^{-1}(K)$. Let $x \in L$ and let $A_i$ be a sequence of arcs in $\mathbb{C}$ such that $\lim A_i = K$ and $f(x) \in A_i$ for each $i$. Let $M_i$ be the component of $f^{-1}(A_i)$ containing the point $x$. By the previous paragraph $f(M_i) = A_i$. Since $f$ is perfect, $M = \limsup M_i \subset L$ is a continuum and $f(M) = K$. Hence $f$ is confluent.

Suppose next that $f : \mathbb{C} \to \mathbb{C}$ is not oriented. Then there exists a simple closed curve $S$ in $\mathbb{C}$ and $p \in T(S) \setminus f^{-1}(f(S))$ such that $f(p) \notin T(f(S))$. Let $L$ be a half-line with endpoint $f(p)$ running to
infinity in \(C \setminus f(S)\). Let \(L^*\) be an arc in \(L\) with endpoint \(f(p)\) and diameter greater than the diameter of the continuum \(f(T(S))\). Let \(K\) be the component of \(f^{-1}(L^*)\) which contains \(p\). Then \(K \subset T(S)\), since \(p \in T(S)\) and \(L \cap f(S) = \emptyset\). Hence, \(f(K) \neq L^*\), and so \(f\) is not confluent. \(\square\)

**Lemma 8.5.** Let \(f : \mathbb{C} \to \mathbb{C}\) be a light, open, perfect surjection. Then there exists an integer \(k\) and a finite subset \(B \subset \mathbb{C}\) such that \(f\) is a local homeomorphism at each point of \(\mathbb{C} \setminus B\), and for each point \(y \in \mathbb{C} \setminus f(B)\), \(|f^{-1}(y)| = k\).

**Proof.** Let \(\mathbb{C}^\infty\) be the one point compactification of \(\mathbb{C}\). Since \(f\) is perfect, we can extend \(f\) to a map of \(\mathbb{C}^\infty\) onto \(\mathbb{C}^\infty\) so that \(f^{-1}(\infty) = \infty\). By abuse of notation we also denote the extended map by \(f\). Then \(f\) is a light open mapping of the compact 2-manifold \(\mathbb{C}^\infty\). The result now follows from a theorem of Whyburn [26, X.6.3]. \(\square\)

The following is a special case, for oriented perfect maps, of the monotone-light factorization theorem.

**Lemma 8.6.** Suppose that \(f : \mathbb{C} \to \mathbb{C}\) is an oriented, perfect map. It follows that \(f = g \circ h\), where \(h : \mathbb{C} \to X\) is a monotone perfect surjection with acyclic fibers and \(g : X \to \mathbb{C}\) is a light, open perfect surjection.

**Proof.** By the monotone-light factorization theorem [19, Theorem 13.3], \(f = g \circ h\), where \(h : \mathbb{C} \to X\) is monotone, \(g : X \to \mathbb{C}\) is light, and \(X\) is the quotient space obtained from \(\mathbb{C}\) by identifying each component of \(f^{-1}(y)\) to a point for each \(y \in \mathbb{C}\). Let \(y \in \mathbb{C}\) and let \(C\) be a component of \(f^{-1}(y)\). If \(C\) were to separate \(\mathbb{C}\), then \(f(C) = y\) would be a point while \(f(T(C))\) would be a non-degenerate continuum. Choose an arc \(A \subset \mathbb{C} \setminus \{y\}\) which meets both \(f(T(C))\) and its complement and let \(x \in T(C) \setminus C\) such that \(f(x) \in A\). If \(K\) is the component of \(f^{-1}(A)\) which contains \(x\), then \(K \subset T(C)\). Hence \(f(K)\) cannot map onto \(A\) contradicting the fact that \(f\) is confluent. Thus, for each \(y \in \mathbb{C}\), each component of \(f^{-1}(y)\) is acyclic.

By Moore’s Plane Decomposition Theorem [12], \(X\) is homeomorphic to \(\mathbb{C}\). Since \(f\) is confluent, it is easy to see that \(g\) is confluent. By a theorem of Lelek and Read [17] \(g\) is open since it is confluent and light (also see [19, Theorem 13.26]). Since \(h\) and \(g\) factor the perfect map \(f\) through a Hausdorff space \(\mathbb{C}\), both \(h\) and \(g\) are perfect [13, 3.7.5]. \(\square\)

**Theorem 8.7** (Maximum Modulus Theorem). Suppose that \(f : \mathbb{C} \to \mathbb{C}\) is a perfect surjection. Then the following are equivalent:

1. \(f\) is either positively or negatively oriented.
2. \(f\) is oriented.
(3) $f$ is confluent.

Moreover, if $f$ is oriented, then for any non-separating continuum $X$, $\partial(f(X)) \subset f(\partial(X))$.

**Proof.** It is clear that (1) implies (2). By Lemma 8.4 every oriented map is confluent. Hence suppose that $f : \mathbb{C} \to \mathbb{C}$ is a perfect confluent map. By Lemma 8.6, $f = g \circ h$, where $h : \mathbb{C} \to \mathbb{C}$ is a monotone perfect surjection with acyclic fibers and $g : \mathbb{C} \to \mathbb{C}$ is a light, open perfect surjection. By Stoilow’s Theorem [27] there exists a homeomorphism $j : \mathbb{C} \to \mathbb{C}$ such that $g \circ j$ is an analytic surjection. Then $f = g \circ h = (g \circ j) \circ (j^{-1} \circ h)$. Since $k = j^{-1} \circ h$ is a monotone surjection of $\mathbb{C}$ with acyclic fibers, it is a near homeomorphism [12, Theorem 25.1]. That is, there exists a sequence $k_i$ of homeomorphisms of $\mathbb{C}$ such that $\lim k_i = k$. We may assume that all of the $k_i$ have the same orientation.

Let $f_i = (g \circ j) \circ k_i$. $S$ a simple closed curve in the domain of $f$ and $p \in T(S) \setminus f^{-1}(f(S))$. Note that $\lim f_i^{-1}(f(S)) \subset f^{-1}(f(S))$. Hence $p \in T(S) \setminus f_i^{-1}(f(S))$ for $i$ sufficiently large. Moreover, since $f_i$ converges to $f$, $f_i|S$ is homotopic to $f|S$ in the complement of $f(p)$ for $i$ large. Thus for large $i$, $\text{degree}((f_i)_p) = \text{degree}(f_p)$, where

$$\left( f_i(p) = \frac{f_i(x) - f_i(p)}{f_i(x) - f_i(p)} \right) \text{ and } f_p(x) = \frac{f(x) - f(p)}{f(x) - f(p)}.$$

Since $g \circ j$ is an analytic map, it is positively oriented and $\text{degree}((f_i)_p) = \text{degree}(f_p) > 0$ if $k_i$ is orientation preserving and $\text{degree}((f_i)_p) = \text{degree}(f_p) < 0$ if $k_i$ is orientation reversing. Thus, $f$ is positively-oriented if each $k_i$ is orientation-preserving and $f$ is negatively-oriented if each $k_i$ is orientation-reversing.

Suppose that $X$ is a non-separating continuum and $f$ is oriented. Let $y \in \partial(f(X))$. Choose $y_i \in \partial(f(X))$ and rays $R_i$, joining $y_i$ to $\infty$ such that $R_i \cap f(X) = \{y_i\}$ and $\lim y_i = y$. Choose $x_i \in X$ such that $f(x_i) = y_i$. Since $f$ is confluent, there exists closed and connected sets $C_i$, joining $x_i$ to $\infty$ such that $C_i \cap X \subset f^{-1}(y_i)$. Hence there exist $x'_i \in f^{-1}(y_i) \cap \partial(X)$. We may assume that $\lim x'_i = x_\infty \in \partial(X)$ and $f(x_\infty) = y$ as desired. \qed

We shall need the following three results in the next section.

**Lemma 8.8.** Let $X$ be a plane continuum and $f : \mathbb{C} \to \mathbb{C}$ a perfect surjective map such that $f^{-1}(T(X)) = T(X)$ and $f|_{\mathbb{C} \setminus f^{-1}(T(X))}$ is confluent. Then for each $y \in \mathbb{C} \setminus T(X)$, each component of $f^{-1}(y)$ is acyclic.

**Proof.** Suppose there exists $y \in \mathbb{C} \setminus T(X)$ such that some component $C$ of $f^{-1}(y)$ is not acyclic. Then there exists $z \in T(C) \setminus f^{-1}(y) \cup T(X)$. 


By unicoherence of \( C \), \( T(X) \cup \{ y \} \) does not separate \( f(z) \) from infinity in \( C \). Let \( L \) be a ray in \( C \setminus \{ T(X) \cup \{ y \} \} \) from \( f(z) \) to infinity. Then \( L = \cup L_i \), where each \( L_i \subset L \) is an arc with endpoint \( f(z) \). For each \( i \) the component \( M_i \) of \( f^{-1}(L_i) \) containing \( z \) maps onto \( L_i \). Then \( M = \cup M_i \) is a connected closed subset in \( C \setminus f^{-1}(y) \) from \( z \) to infinity. This is a contradiction since \( z \) is contained in a bounded complementary component of \( f^{-1}(y) \).

\[ \square \]

**Theorem 8.9.** Let \( X \) a plane continuum and \( f : C \rightarrow C \) a perfect, surjective map such that \( f^{-1}(T(X)) = T(X) \) and \( f|_{C\setminus f^{-1}(T(X))} \) is confluent. If \( A \) and \( B \) are crosscuts of \( T(X) \) such that \( B \cup X \) separates \( A \) from \( \infty \) in \( C \), then \( f(B) \cup T(X) \) separates \( f(A) \setminus f(B) \) from \( \infty \).

**Proof.** Suppose not. Then there exists a half-line \( L \) joining \( f(A) \) to \( \infty \) in \( C \setminus (f(B) \cup T(X)) \). As in the proof of Lemma 8.8 there exists a closed and connected set \( M \subset C \setminus (B \cup X) \) joining \( A \) to \( \infty \), a contradiction. \[ \square \]

**Proposition 8.10.** Under the conditions of Theorem 8.9, if \( L \) is a ray irreducible from \( T(X) \) to \( \infty \), then each component of \( f^{-1}(L) \) is closed in \( C \setminus X \) and is a connected set from \( X \) to \( \infty \).

### 9. Induced maps of prime ends

Suppose that \( f : C \rightarrow C \) is an oriented perfect surjection and \( f^{-1}(Y) = X \), where \( X \) and \( Y \) are acyclic continua. We will show that in this case the map \( f \) induces a confluent map \( F \) of the circle of prime ends of \( X \) to the circle of prime ends of \( Y \). This result was announced by Mayer in the early 1980’s but never appeared in print. It was also used (for homeomorphisms) by Cartwright and Littlewood in [10]. There are easy counterexamples that show if \( f \) is not confluent then it may not induce a continuous function between the circles of prime ends.

**Theorem 9.1.** Let \( X \) and \( Y \) be non-degenerate acyclic plane continua and \( f : C \rightarrow C \) a perfect map such that:

1. \( Y \) has no cut point,
2. \( f^{-1}(Y) = X \) and
3. \( f|_{C\setminus X} \) is confluent.

Let \( \varphi : \Delta^\infty \rightarrow C^\infty \setminus X \) and \( \psi : \Delta^\infty \rightarrow C^\infty \setminus Y \) be conformal mappings. Define \( \hat{f} : \Delta^\infty \rightarrow \Delta^\infty \) by \( \hat{f} = \psi^{-1} \circ f \circ \varphi \).

Then \( \hat{f} \) extends to a map \( \tilde{f} : \Delta^\infty \rightarrow \Delta^\infty \). Moreover, \( \tilde{f}^{-1}(S^1) = S^1 \) and \( F = \tilde{f}|_{S^1} \) is a confluent map.
Proof. Note that $f$ takes accessible points of $X$ to accessible points of $Y$. For if $P$ is a path in $[C \setminus X] \cup \{p\}$ with endpoint $p \in X$, then by (2), $f(P)$ is a path in $[C \setminus Y] \cup \{f(p)\}$ with endpoint $f(p) \in Y$.

Let $A$ be a crosscut of $X$ such that the diameter of $f(A)$ is less than half of the diameter of $Y$ and let $U$ be the bounded component of $C \setminus (X \cup A)$. Let the endpoints of $A$ be $x, y \in X$ and suppose that $f(x) = f(y)$. If $x$ and $y$ lie in the same component of $f^{-1}(f(x))$ then each crosscut $B \subset U$ of $X$ is mapped to a generalized return cut of $Y$ based at $f(x)$ (i.e., the endpoints of $B$ map to $f(x)$) by (1). Note that in this case by (1), $\partial f(U) \subset f(A) \cup \{f(x)\}$.

Now suppose that $f(x) = f(y)$ and $x$ and $y$ lie in distinct components of $f^{-1}(f(x))$. Then by unicoherence of $C$, $\partial U \subset A \cup X$ is a connected set and $\partial U \not\subset A \cup f^{-1}(f(x))$. Now $\partial U \setminus (A \cup f^{-1}(f(x))) = \partial U \setminus f^{-1}(f(A))$ is an open non-empty set in $\partial U$ by (2). Thus there is a crosscut $B \subset U \setminus f^{-1}(f(A))$ of $X$ with $B \setminus B \subset \partial U \setminus f^{-1}(f(A))$. Now $f(B)$ is contained in a bounded component of $C \setminus (Y \cup f(A)) = C \setminus (Y \cup f(A))$ by Theorem 8.9 Since $Y \cap f(A) = \{f(x)\}$ is connected and $Y$ does not separate $C$, it follows by unicoherence that $f(B)$ lies in a bounded component of $C \setminus f(A)$. Since $Y \setminus \{f(x)\}$ meets $f(B)$ and misses $f(A)$ and $Y \setminus \{f(x)\}$ is connected, $Y \setminus \{f(x)\}$ lies in a bounded complementary component of $f(A)$. This is impossible as the diameter of $f(A)$ is smaller than the diameter of $Y$. It follows that there exists a $\delta > 0$ such that if the diameter of $A$ is less than $\delta$ and $f(x) = f(y)$, then $x$ and $y$ must lie in the same component of $f^{-1}(f(x))$.

In order to define the extension $\tilde{f}$ of $f$ over the boundary $S^1$ of $\Delta^\infty$, let $C_i$ be a chain of crosscuts of $\Delta^\infty$ which converge to a point $p \in S^1$ such that $A_i = \varphi(C_i)$ is a null chain of crosscuts or return cuts of $X$ with endpoints $a_i$ and $b_i$ which converge to a point $x \in X$. There are three cases to consider:

Case 1. $f$ identifies the endpoints of $A_i$ for some $A_i$ with diameter less than $\delta$. In this case the chain of crosscuts is mapped by $f$ to a chain of generalized return cuts based at $f(a_i) = f(b_i)$. Hence $f(a_i)$ is an accessible point of $Y$ which corresponds (under $\psi^{-1}$) to a unique point $q \in S^1$ (since $Y$ has no cutpoints). Define $\tilde{f}(p) = q$.

Case 2. Case 1 does not apply and there exists an infinite subsequence $A_{i_j}$ of crosscuts such that $f(A_{i_j}) \cap f(A_{i_k}) = \emptyset$ for $j \neq k$. In this case $f(A_{i_j})$ is a chain of generalized crosscuts which converges to a point $f(x) \in Y$. This chain corresponds to a unique point $q \in S^1$ since $Y$ has no cut points. Define $\tilde{f}(p) = q$.

Case 3. Cases 1 and 2 do not apply. Without loss of generality suppose there exists an $i$ such that for $j > i$ $f(A_i) \cap f(A_j)$ contains...
$f(a_i)$. In this case $f(A_j)$ is a chain of generalized crosscuts based at the accessible point $f(a_i)$ which corresponds to a unique point $q$ on $S^1$ as above. Define $\tilde{f}(p) = q$.

It remains to be shown that $\tilde{f}$ is a continuous extension of $\hat{f}$ and $F$ is confluent. For continuity it suffices to show continuity at $S^1$. Let $p \in S^1$ and let $C$ be a small crosscut of $\Delta^\infty$ whose endpoints are on opposite sides of $p$ in $S^1$ such that $A = \varphi(C)$ has diameter less than $\delta$ 18 and such that the endpoints of $A$ are two accessible points of $X$. Since $f$ is uniformly continuous near $X$, the diameter of $f(A)$ is small and since $\psi^{-1}$ is uniformly continuous with respect to connected sets in the complement of $Y$ (25), the diameter of $B = \psi^{-1} \circ f \circ \varphi$ is small. Also $B$ is either a generalized crosscut or generalized return cut. Since $\hat{f}$ preserves separation of crosscuts, it follows that the image of the domain $U$ bounded by $C$ which does not contain $\infty$ is small. This implies continuity of $\tilde{f}$ at $p$.

To see that $F$ is confluent let $K \subset S^1$ be a subcontinuum and let $H$ be a component of $f^{-1}(K)$. Choose a chain of crosscuts $C_i$ such that $\varphi(C_i) = A_i$ is a crosscut of $X$ meeting $X$ in two accessible points $a_i$ and $b_i$, $C_i \cap f^{-1}(K) = \emptyset$ and $\lim C_i = H$. It follows from the preservation of crosscuts (see Theorem 8.9) that $\hat{f}(C_i)$ separates $K$ from $\infty$. Hence $\hat{f}(C_i)$ must meet $S^1$ on both sides of $K$ and $\hat{f}(C_i) = K$. Hence $F(H) = \lim \hat{f}(C_i) = K$ as required.

**Corollary 9.2.** Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a perfect, oriented map of the plane, $X \subset \mathbb{C}$ is a subcontinuum without cut points and $f(X) = X$. Let $\hat{X}$ be the component of $f^{-1}(f(X))$ containing $X$. Let $\varphi : \Delta^\infty \to \mathbb{C}^\infty \setminus T(\hat{X})$ and $\psi : \Delta^\infty \to \mathbb{C}^\infty \setminus T(X)$ be conformal mappings. Define $\hat{f} : \Delta^\infty \setminus \varphi^{-1}(f^{-1}(X)) \to \Delta^\infty$ by $\hat{f} = \psi^{-1} \circ f \circ \varphi$. Put $S^1 = \partial \Delta^\infty$.

Then $\hat{f}$ extends over $S^1$ to a map $\tilde{f} : \Delta^\infty \to \Delta^\infty$. Moreover $\tilde{f}^{-1}(S^1) = S^1$ and $F = \tilde{f}|_{S^1}$ is a confluent map.

**Proof.** By Lemma 8.6 $f = g \circ m$ where $m$ is a monotone perfect and onto mapping of the plane with acyclic point inverses, and $g$ is an open and perfect surjection of the plane to itself. By Lemma 8.3 $f^{-1}(X)$ has finitely many components. It follows that there exist a simply connected open set $V$, containing $T(X)$, such that if $U$ is the component of $f^{-1}(V)$ containing $\hat{X}$, then $U$ contains no other components of $f^{-1}(X)$. It is easy to see that $f(U) = V$ and that $U$ is simply connected. Hence $U$ and $V$ are homeomorphic to $\mathbb{C}$. Then $f|_U : U \to V$ is a confluent map. The result now follows from Theorem 9.1 applied to $f$ restricted to $U$. \qed
10. Fixed points for positively oriented maps

In this section we will consider a positively oriented map of the plane. As we shall see below, a straightforward application of the tools developed above will give us the desired fixed point result. We will assume, by way of contradiction, that $f : \mathbb{C} \to \mathbb{C}$ is a positively oriented map, $X$ is a plane continuum such that $f(X) \subset T(X)$ and $T(X)$ contains no fixed points of $f$.

**Lemma 10.1.** Let $f : \mathbb{C} \to \mathbb{C}$ be a map and $X \subset \mathbb{C}$ a continuum such that $f(X) \subset T(X)$. Suppose $C = (a,(b) is a crosscut of the continuum $T(X)$. Let $v \in (a,b)$ be a point and $J_v$ be a junction such that $J_v \cap (X \cup C) = \{v\}$. Then there exists an arc $I$ such that $S = I \cup C$ is a simple closed curve, $T(X) \subset T(S)$ and $f(I) \cap J_v = \emptyset$.

**Proof.** Since $f(X) \subset T(X)$ and $J_v \cap X = \emptyset$, it is clear that there exists an arc $I$ with endpoints $a$ and $b$ sufficiently close to $T(X)$ such that $I \cup C$ is a simple closed curve, $T(X) \subset T(I \cup C)$ and $f(I) \cap J_v = \emptyset$. This completes the proof.

**Corollary 10.2.** Suppose $X \subset \mathbb{C}$ is a continuum, $f : \mathbb{C} \to \mathbb{C}$ a positively oriented map such that $f(X) \subset T(X)$. Then for each crosscut $C$ of $T(X)$ such that $f(C) \cap C = \emptyset$, $\var(f, C) \geq 0$.

**Proof.** Suppose that $C = (a, b)$ is a crosscut of $T(X)$ such that $f(C) \cap \overline{C} = \emptyset$ and $\var(f, C) \neq 0$. Choose a junction $J_v$ such that $J_v \cap (X \cup C) = \{v\}$ and $v \in C \setminus X$. By Lemma 10.1 there exists an arc $I$ such that $S = I \cup C$ is a simple closed curve and $f(I) \cap J_v = \emptyset$. Moreover, by choosing $I$ sufficiently close to $X$, we may assume that $v \in \mathbb{C} \setminus f(S)$. Hence $\var(f, C) = \text{Win}(f, S, v) \geq 0$ by the remark following Definition 2.8.

**Theorem 10.3.** Suppose $f : \mathbb{C} \to \mathbb{C}$ is a positively oriented map and $X$ is a continuum such that $f(X) \subset T(X)$. Then there exists a point $x_0 \in T(X)$ such that $f(x_0) = x_0$.

**Proof.** Suppose we are given a continuum $X$ and $f : \mathbb{C} \to \mathbb{C}$ a positively oriented map such that $f(X) \subset T(X)$. Assume that $f|_{T(X)}$ is fixed point free. Choose a simple closed curve $S$ such that $X \subset T(S)$ and points $a_0 < a_1 < \ldots < a_n$ in $S \cap X$ such that for each $i$ $C_i = (a_i, a_{i+1})$ is a sufficiently small crosscut of $X$, $f(C_i) \cap \overline{C_i} = \emptyset$ and $f|_{T(S)}$ is fixed point free. By Corollary 10.2, $\var(f, C_i) \geq 0$ for each $i$. Hence by Theorem 2.13, $\text{ind}(f, S) = \sum \var(f, C_i) + 1 \geq 1$. This contradiction with Theorem 2.6 completes the proof.
Corollary 10.4. Suppose \( f : \mathbb{C} \rightarrow \mathbb{C} \) is a perfect, oriented map and \( X \) is a continuum such that \( f(X) \subset T(X) \). Then there exists a point \( x_0 \in T(X) \) of period at most 2.

Proof. By Theorem 8.7, \( f \) is either positively or negatively oriented. In either case, the second iterate \( f^2 \) is positively oriented and must have a fixed point in \( T(X) \) by Theorem 10.3. \( \square \)

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