Scalar field fluctuations in Schwarzschild–de Sitter spacetime

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Abstract
We calculate quantum fluctuations of a free scalar field in the Schwarzschild–
de Sitter spacetime, adopting the planar coordinates that are pertinent to
the presence of a black hole in an inflationary Universe. In a perturbation
approach, expanding in powers of a small black hole event horizon compared
to the de Sitter cosmological horizon, we obtain time evolution of the quantum
fluctuations and then derive the scalar power spectrum.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The cosmic microwave background that we observe today is almost homogeneous and
isotropic. The background temperature in our sky is about 2.7 K with a tiny fluctuation
at a level of about $10^{-5}$ K. This is consistent with measurements of the matter content of
the Universe, altogether prevailing in a spatially flat Universe. The inflation scenario can explain
the homogeneity, the isotropy and the flatness of the present Universe [1]. Moreover, quantum
fluctuations of the inflaton field during inflation can give rise to primordial density fluctuations
with a nearly scale-invariant power spectrum which is consistent with the recent WMAP
data on cosmic microwave background anisotropies [2]. Therefore, a general assumption
usually made in most cosmological models is that the background metric is homogeneous and
isotropic. For example, a de Sitter metric is used in the inflationary era and a flat Friedmann–
Robertson–Walker metric is used in the subsequent hot big bang. An interesting notion is
the recent discovery of a dominant component in the matter content, dubbed the dark energy,
which exerts a negative pressure to drive an accelerated expansion of the Universe [3]. If dark
energy is a form of vacuum energy, our Universe will coast to the de Sitter spacetime or the
inflating phase again in the future.
Apparently our present Universe is not so homogeneous and isotropic because we observe local nonlinear structures such as stars, galaxies, clusters of galaxies and very massive black holes. An appropriate spacetime, for example, for a massive black hole sitting in the accelerating Universe, would be described by the Schwarzschild metric in the vicinity of the black hole and by the de Sitter metric at places far from the black hole. Presently, these local structures are decoupled from the Hubble flow, so it suffices to use the Friedmann–Robertson–Walker metric to study the large scale structures of the Universe. However, in the early Universe the gravitational effect of a black hole to the background metric may be important and should be addressed. For example, the existence of a black hole or a distribution of black holes at the onset of inflation can invalidate the use of a homogeneous and isotropic background metric for the calculation of de Sitter quantum fluctuations. This also applies to the situation when we are very near to one of these black holes that still exist today or not.

The cosmic no hair conjecture infers that the inflationary Universe approaches asymptotically the de Sitter spacetime till the end of inflation [4]. Nevertheless, the effects of matter and spacetime inhomogeneities to inflation should be considered. Several authors have studied the onset of inflation under inhomogeneous initial conditions to determine whether large inhomogeneity during the very early Universe can prevent the Universe from entering an inflationary era [5]. It was found that in some cases a large initial inhomogeneity may suppress the onset of inflation [6]. If the inflaton field is sufficiently inhomogeneous, the wormhole can form from collapsing vacuum energy density peaks before the inhomogeneity is damped by the exponential expansion [7]. In the case of inhomogeneities in a dust era before inflation, some inhomogeneities can collapse into a black-hole spacetime [8]. Furthermore, for the inhomogeneities of the spacetime itself, energies in the form of gravitational waves can also form a black-hole spacetime [9]. As a consequence, at the onset of inflation, the distortion of the metric by these inhomogeneities should be taken into account.

With these considerations in mind, in this work we will investigate the quantum fluctuations of a free massless scalar field in the Schwarzschild–de Sitter (SdS) spacetime. In the static coordinate system, the line element of the SdS spacetime is given by

$$ds^2 = - \left(1 - \frac{2GM}{r} - H^2 r^2\right) dt^2 + \left(1 - \frac{2GM}{r} - H^2 r^2\right)^{-1} dr^2 + r^2 d\Omega^2,$$

(1)

where $G = M^{2}_{\text{Pl}}$, $M$ is the mass of the black hole and $H$ is the Hubble parameter for inflation. Here we use the convention with $c = h = 1$. As is well known, the SdS metric has a black hole horizon and a cosmological horizon. The casual structure of the SdS spacetime is depicted in the Penrose diagram given in, for example, [10]. In the static coordinates (1) an observer can only receive a signal inside or just right on the cosmological horizon. This static metric is insufficient for our purpose because in the cosmological setting we aim at studying the temporal evolution of a Fourier mode of the scalar quantum fluctuations that crosses the cosmological horizon during inflation. Therefore, we will instead use the planar coordinates for the SdS metric [10], which is given by

$$ds^2 = - f(r, \tau) \, dt^2 + h(r, \tau)(dr^2 + r^2 d\Omega^2),$$

(2)

where $dt = a^{-1}(t) \, dr$ is the conformal time and $a(t) = e^{Ht}$. In equation (2), for simplicity we have used the same notations, $t$ and $r$, actually referring to different local coordinates than those in equation (1). The $f$ and $h$ functions are given by

$$f(r, \tau) = a^2(\tau) \left[1 - \frac{GM}{2a(\tau) r}\right]^2 \left[1 + \frac{GM}{2a(\tau) r}\right]^{-2}, \quad h(r, \tau) = a^2(\tau) \left[1 + \frac{GM}{2a(\tau) r}\right]^2,$$

(3)

with the cosmic scale factor $a(\tau) = -1/(H \tau)$. In these coordinates, the black hole horizon corresponds to $r = GM/(2a)$ and the cosmological horizon is given by $r = a/H$. For our
purpose, we will restrict the range of validity of $\tau$ and $r$ to $-1/H < \tau < 0$ and $GM/(2a) < r$. Note that at late times (i.e. $\tau \to 0^-$) the planar coordinates behave like a de Sitter expansion.

It is well known that a black hole evaporates into the Hawking radiation which leads to a mass loss of the black hole and to its eventual disappearance [11]. For a Schwarzschild black hole with mass $M$, the evaporation time is given by

$$t_{ev} = \frac{5120\pi G^2 M^3}{a^2}. \quad (4)$$

In general the evaporation time scale of a SdS black hole is different from a Schwarzschild one. However, for small SdS black holes with the black hole temperature much higher than the de Sitter temperature, the evaporation time scale should be of the same order as the Schwarzschild case. Therefore, our present consideration requires the condition that the evaporation time scale of the black hole is longer than the time scale of inflation, i.e. $Ht_{ev} > 1$. This gives the lower bound on the mass of the black hole for a given inflation scale:

$$\frac{M}{M_{Pl}} > 3.96 \times 10^{-2} \left( \frac{M_{Pl}}{H} \right)^{1/3}. \quad (5)$$

In the next section, we will review the scalar quantum fluctuations in the de Sitter metric. Then, we will introduce a perturbation method to expand the planar metric in powers of the black hole mass $M$ to find an approximate solution of the scalar equation. Section 3 contains the numerical results of the first-order solutions. Section 4 is our conclusion.

2. Perturbation approach to the Schwarzschild–de Sitter spacetime

2.1. Classical solution

Consider a massless scalar field $\phi$ which satisfies the Klein–Gordon equation in the SdS spacetime:

$$\partial_{\mu}(\sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi) = 0. \quad (6)$$

Since the space is spherically symmetric, one can expand $\phi$ as

$$\phi(x) = \sum_{lm} \psi_l(r, \tau) Y_{lm}(\theta, \phi). \quad (7)$$

If one further writes $\psi_l(r, \tau)$ in a spectral form in terms of spherical Bessel functions $j_l$ [12],

$$\psi_l(r, \tau) = \int_0^\infty dk k^2 j_l(kr) \varphi_{kl}(\tau), \quad (8)$$

then we will have

$$\phi(x) = \int_0^\infty dk \sum_{lm} \varphi_{klm}(x), \quad \varphi_{klm}(x) = k^2 j_l(kr) \varphi_{kl}(\tau) Y_{lm}(\theta, \phi). \quad (9)$$

We define the spectral function of the fluctuations of the field $\phi(x)$ as

$$S_{kl}(r, \tau) = (2l + 1) j_l^2(kr) P_{kl}(r, \tau), \quad P_{kl}(r, \tau) = \frac{k^5}{4\pi} |\varphi_{kl}(\tau)|^2. \quad (10)$$

The power spectrum $P_{kl}$ which gives the power of the fluctuations in a logarithmic interval of $k$ is useful for comparing theoretical predictions with observations.

Our next task is to calculate the function $\varphi_{kl}(\tau)$ in equation (10) in the SdS planar metric (2). The Klein–Gordon equation (6) becomes

$$-\frac{1}{\sqrt{f} \mathcal{F}} \partial_r \left( \frac{\mathcal{F}}{f} \partial_r \varphi_l \right) + \frac{1}{r^2 \sqrt{f} \mathcal{F}} \partial_r \left( r^2 \sqrt{f} \mathcal{F} \partial_r \varphi_l \right) - \frac{l(l+1)}{r^2} \varphi_l = 0. \quad (11)$$
where we have separated out the angular part of $\phi(x)$ in equation (7). There is no exact solution to this equation. We therefore adopt a perturbative approach assuming that the quantity

$$\epsilon \equiv GMH$$

is a small parameter. But condition (5) implies that

$$\epsilon > 3.96 \times 10^{-2} \left( \frac{H}{M_{Pl}} \right)^{\frac{1}{2}}.$$  

However, this shows that the smallness of $\epsilon$ can be easily satisfied for any reasonable inflation scale. Let us rewrite equation (11) as

$$\partial_\tau^2 \varphi_l - \frac{2}{\tau} \partial_\tau \varphi_l - \frac{2}{r} \partial_r \varphi_l + \frac{l(l+1)}{r^2} \varphi_l = \left( 1 - \frac{h}{f} \right) \partial_\tau^2 \varphi_l - \frac{2}{\tau} \left( 1 - \frac{h}{f} \right) \partial_\tau \varphi_l$$

$$- \frac{1}{\tau^2 \sqrt{f h}} \partial_\tau \left( \sqrt{\frac{h^3}{f}} \right) \partial_r \varphi_l + \frac{1}{\sqrt{f h}} \partial_r (\sqrt{f h}) \partial_r \varphi_l.$$  

(14)

We then expand the functions $f$ and $h$ in powers of $\epsilon$ as defined in equation (12) and write

$$\varphi_l = \varphi_l^{(0)} + \varphi_l^{(1)} + \varphi_l^{(2)} + \cdots$$

as an expansion in orders of $\epsilon$. It is straightforward to show that the right-hand side of equation (14) becomes

$$8 \left( \frac{\epsilon \tau}{2r} \right) \left( \partial_\tau^2 \varphi_l - \frac{1}{\tau} \partial_\tau \varphi_l \right) - 30 \left( \frac{\epsilon \tau}{2r} \right)^2 \left( \partial_\tau^2 \varphi_l - \frac{1}{15 \tau} \partial_\tau \varphi_l - \frac{1}{15 r} \partial_r \varphi_l \right) + \cdots.$$  

(16)

Note that this is essentially expanded in terms of $\epsilon \tau / (2r) = GM/(2ar)$. As we adopt a perturbative approach, we cannot really take $ar$ close to the black hole horizon $2GM$ and should put a lower cutoff on $r$ of the order of $2GM/a$ in the calculation. However, we find that taking the cutoff to zero will not affect the results that we will obtain below.

2.1.1. Zeroth order. From equations (14)–(16), the zeroth order corresponds to the de Sitter case with

$$\partial_\tau^2 \varphi_l^{(0)} - \frac{2}{\tau} \partial_\tau \varphi_l^{(0)} - \frac{2}{r} \partial_r \varphi_l^{(0)} + \frac{l(l+1)}{r^2} \varphi_l^{(0)} = 0.$$  

(17)

To solve this equation, we take the Bessel transform

$$\varphi_l^{(0)}(r, \tau) = \int_{0}^{\infty} dk k^2 j_l(kr) \varphi_l^{(0)}(\tau).$$  

(18)

Then we have

$$\partial_\tau^2 \varphi_l^{(0)}(\tau) - \frac{2}{\tau} \partial_\tau \varphi_l^{(0)}(\tau) + k^2 \varphi_l^{(0)}(\tau) = 0,$$

(19)

and the solution is

$$\varphi_l^{(0)}(\tau) = C_1(-k\tau)^{\frac{1}{2}} H_{\frac{l}{2}}^{(1)}(-k\tau) + C_2(-k\tau)^{\frac{1}{2}} H_{\frac{l}{2}}^{(2)}(-k\tau).$$  

(20)
where $H_2^{(1)}$ and $H_2^{(2)}$ are the Hankel functions of order 3/2 [12]. If we take the boundary conditions

$$C_1 = -\frac{H}{k^2 \sqrt{2k}} \quad \text{and} \quad C_2 = 0,$$

(21)

then we will have

$$\psi_{kl}^{(0)}(\tau) = -\frac{H\tau}{k\sqrt{\pi k}} \left(1 - \frac{i}{k\tau}\right) e^{-i\tau},$$

(22)

and

$$P_{kl}^{(0)}(\tau) = \frac{k^5}{4\pi} \left|\psi_{kl}^{(0)}(\tau)\right|^2 = \frac{H^2}{4\pi^2} (1 + k^2 \tau^2).$$

(23)

As $\tau \to 0$, $P_{kl}^{(0)}(\tau) \to H^2/(4\pi^2)$. This result gives rise to a scale-invariant power spectrum that is preferred by observational data [2]. Also it matches the well-known scale-invariant power spectrum of de Sitter quantum fluctuations [13], which presumably undergo decoherence to become classical fluctuations.

2.1.2. First order. With the perturbative expansion for $\psi_l$ in equation (15), the Klein–Gordon equation in equation (14) can be solved perturbatively. The first order $\psi_{l}^{(1)}(r, \tau)$ then satisfies

$$\partial_\tau^2 \psi_{l}^{(1)} - 2\partial_\tau \psi_{l}^{(1)} - 2\partial_r \psi_{l}^{(1)} + \frac{l(l + 1)}{r^2} \psi_{l}^{(1)} = J_1,$$

(24)

where the source term is given by

$$J_1(r, \tau) = \frac{4\epsilon \tau}{r} \left(\partial_\tau^2 \psi_{l}^{(0)} - \frac{1}{\tau} \partial_\tau \psi_{l}^{(0)}\right)$$

$$= \frac{4\epsilon H\tau^2}{\sqrt{\pi r}} \int_0^\infty dk k^2 j_l(kr) j_l(k\tau) e^{-i\tau}.$$

(25)

To solve this inhomogeneous equation, we use the Green’s function $G(r, \tau; r', \tau')$ which satisfies the equation

$$\partial_\tau^2 G - 2\partial_\tau G - \partial_r^2 G - \frac{2}{r} \partial_r G + \frac{l(l + 1)}{r^2} G = \frac{\delta(r - r')\delta(\tau - \tau')}{r^2}.$$

(26)

Using the completeness property of the spherical Bessel functions,

$$\int_0^\infty dk k^2 \left[\sqrt{\frac{2}{\pi}} r j_l(kr)\right]\left[\sqrt{\frac{2}{\pi}} r' j_l(kr')\right] = \delta(r - r'),$$

(27)

and taking

$$G_1(r, \tau; r', \tau') = \int_0^\infty dk k^2 g_k(\tau, \tau') j_l(kr) j_l(kr'),$$

(28)

equation (26) becomes

$$\partial_\tau^2 g_k - 2\partial_\tau g_k + k^2 g_k = \frac{2}{\pi} \delta(\tau - \tau').$$

(29)

For the retarded Green’s function, $g_k = 0$ for $\tau' > \tau > \tau_i$, where $\tau_i$ denotes an initial time when the source begins to operate. For $0 > \tau > \tau'$,

$$g_k(\tau, \tau') = \frac{i}{2\tau'^2 k^2} \left[(-k\tau)^2 H_2^{(1)}(-k\tau)(-k\tau')^2 H_2^{(2)}(-k\tau') \right.\right.$$

$$- (k\tau')^2 H_2^{(1)}(-k\tau)(-k\tau)^2 H_2^{(2)}(-k\tau)\left.\right].$$

(30)
With this retarded Green’s function, the first order \( \varphi^{(1)}_i (r, \tau) \) can be expressed as

\[
\varphi^{(1)}_i (r, \tau) = \int_0^\infty dk' k^2 j_i (k r) \varphi^{(1)}_{kl} (\tau) = \int_0^\infty dr' r^2 \int_0^\infty dr' G(r, \tau; r', \tau') J_i (r', \tau').
\]  
(31)

Hence, we find that

\[
\varphi^{(1)}_{kl} (\tau) = \frac{2i e H}{\sqrt{\pi} k} \int_0^\infty dk' k^{3/2} (k')^2 \int_0^\infty dr' j_i (k r') j_i (k' r') \int_0^\tau dr' e^{-ik'r'}
\]
\[
\times \left[ (-k \tau)^2 H^{(1)}_{\frac{3}{2}} (-k \tau) (-k' \tau)^2 H^{(2)}_{\frac{3}{2}} (-k' \tau) \right] - (-k' \tau)^2 H^{(1)}_{\frac{3}{2}} (-k' \tau) (-k \tau)^2 H^{(2)}_{\frac{3}{2}} (-k \tau) .
\]  
(32)

It is useful to rewrite \( \varphi^{(1)}_{kl} (\tau) \) as

\[
\varphi^{(1)}_{kl} (\tau) = \epsilon \alpha_{kl} (\tau) \varphi^{(0)}_{kl} (\tau) + \beta_{kl} (\tau) \varphi^{(0)*}_{kl} (\tau).
\]  
(33)

The integral over \( r' \) can be performed:

\[
\int_0^\infty dr' j_i (k r') j_i (k' r') = \frac{\pi}{2 \sqrt{k - k'}} \left( \frac{1}{k} \right) \left( \frac{k'}{k} \right)^{1/2} \frac{\Gamma(l + 1)}{\Gamma(l + \frac{3}{2}) \Gamma(\frac{1}{2})} F \left( l + 1, \frac{1}{2}; l + \frac{3}{2}, \frac{k^2}{k'} \right),
\]  
(34)

where \( \Gamma \) is the Gamma function, \( F \) is the hypergeometric function and \( k_+ (k_-) \) represents the smaller (bigger) one of \( k \) and \( k' \) [12]. After some rescalings the coefficients \( \alpha_{kl} \) and \( \beta_{kl} \) can be simplified to

\[
\alpha_{kl} (\tau) = \frac{-2i \Gamma(l + 1)}{\Gamma(l + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_1^\infty dk' k^{l+\frac{3}{2}} F \left( l + 1, \frac{1}{2}; l + \frac{3}{2}, k' \right) \int_{k_+}^{k_{\tau}} dr' e^{-ik'r'} e^{ir'} (\tau + i)
\]
\[
+ \int_{k_+}^{\infty} dk' k^{l+\frac{3}{2}} F \left( l + 1, \frac{1}{2}; l + \frac{3}{2}, k' \right) \int_{k_+}^{k_{\tau}} dr' e^{-ik'r'} e^{ir'} (\tau + i)
\]  
(35)

and

\[
\beta_{kl} (\tau) = \frac{2i \Gamma(l + 1)}{\Gamma(l + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_1^\infty dk' k^{l+\frac{3}{2}} F \left( l + 1, \frac{1}{2}; l + \frac{3}{2}, k' \right) \int_{k_+}^{k_{\tau}} dr' e^{-ik'r'} e^{-ir'} (\tau - i)
\]
\[
+ \int_{k_+}^{\infty} dk' k^{l+\frac{3}{2}} F \left( l + 1, \frac{1}{2}; l + \frac{3}{2}, k' \right) \int_{k_+}^{k_{\tau}} dr' e^{-ik'r'} e^{-ir'} (\tau - i).
\]  
(36)

The power spectrum in this order is then given by

\[
P_{kl} (\tau) = P^{(0)}_{kl} (\tau) \left[ 1 + \epsilon \Delta^{(1)}_{kl} (\tau) \right],
\]  
(37)

where using \( \varphi^{(1)}_{kl} \) in equation (32) (noting that it has an \( \epsilon \) inside), we have introduced the dimensionless quantity

\[
\Delta^{(1)}_{kl} (\tau) = 2 \epsilon^{-1} \left| \varphi^{(0)}_{kl} (\tau) \right|^2 \text{Re} \left[ \varphi^{(0)*}_{kl} (\tau) \varphi^{(1)}_{kl} (\tau) \right] = 2 \text{Re} \left[ \alpha_{kl} (\tau) \right] + 2 \left| \varphi^{(0)}_{kl} (\tau) \right|^2 \text{Re} \left[ \beta^{(0)*}_{kl} (\tau) \varphi^{(0)}_{kl} (\tau) \right].
\]  
(38)

As \( \tau \to 0 \), from equation (22),

\[
\Delta^{(1)}_{kl} (0) = 2 \text{Re} \left[ \alpha_{kl} (0) - \beta_{kl} (0) \right].
\]  
(39)
From equations (35) and (36), we find that
\[
\Delta_1^{(1)}(0) = \frac{8G(l + 1)}{\Gamma(l + 1)} \left[ \int_0^1 dk' k'^{l+1} F \left( l + 1, \frac{1}{2}; l + \frac{3}{2}; k'/k \right) 
+ \int_1^{\infty} dk' k'^{l+1} F \left( l + 1, \frac{1}{2}; l + \frac{3}{2}; k'/k \right) \right] 
 \times \frac{1}{(k^2 - 1)^2} \left[ \cos(k\tau_i) [2 \sin(k'\tau_i) - k'(k^2 - 1)k\tau_i \cos(k'k\tau_i)] 
+ \sin(k\tau_i) [k'(k^2 - 3) \cos(kk\tau_i) - (k^2 - 1)k\tau_i \sin(k'k\tau_i)] \right].
\] (40)

2.2. Quantization

A unique mode function \( \varphi_{klm}(x) \) can be obtained once an appropriate vacuum is chosen. By using this mode function the scalar field can be quantized in the standard manner
\[
\hat{\phi}(x) = \int_0^{\infty} dk \sum_{lm} \left[ \hat{a}_{klm} \varphi_{klm}(x) + \hat{a}_{klm}^\dagger \varphi_{klm}^*(x) \right],
\] (41)
with the commutation relations
\[
[\hat{a}_{klm}, \hat{a}_{k'l'm'}^\dagger] = [\hat{a}_{k'l'm'}^\dagger, \hat{a}_{klm}^\dagger] = 0,
\] (42)
\[
[\hat{a}_{klm}, \hat{a}_{k'l'm'}] = \delta(k - k') \delta_{ll'} \delta_{mm'}.
\]
We have the delta function in \( k \) because there is no coupling between different \( k \)-modes. We will show this explicitly later by a perturbation approach. The delta functions in \( l \) and \( m \) stem from the rotational invariance about the central black hole. The vacuum state is defined as
\[
\hat{a}_{klm}|0\rangle = 0.
\] (43)

Now \( \varphi_{klm}(x) \) is the mode function, so it should also satisfy the Klein–Gordon equation. Since the space is spherically symmetric, one can write
\[
\varphi_{klm}(x) = \varphi_{kl}(r, \tau) Y_{lm}(\theta, \phi).
\] (44)

But \( \varphi_{kl}(r, \tau) \) cannot in general be separated as a product of functions with only one variable like \( k^2 j_l(kr) \phi_l(\tau) \) as we have done in the classical solution. It is because \( \varphi_{klm}(x) \) is required to satisfy the Klein–Gordon equation for each \( k, l \) and \( m \), while for the classical wave only \( \phi(x) \) itself is required to do so. However, \( \varphi_{kl}(r, \tau) \) can still be obtained perturbatively. The Klein–Gordon equation for the mode function is the same as equation (11), given by
\[
-\frac{1}{\sqrt{f\hbar}} \frac{\partial}{\partial \tau} \left( \frac{\hbar^3}{f} \frac{\partial}{\partial \tau} \varphi_{kl}(r, \tau) \right) + \frac{1}{r^2 \sqrt{f\hbar}} \frac{\partial}{\partial \tau} (r^2 \sqrt{f\hbar} \partial_r \varphi_{kl}(r, \tau)) - \frac{l(l + 1)}{r^2} \varphi_{kl}(r, \tau) = 0.
\] (45)

The two-point correlation function is then given by
\[
\langle 0 | \hat{\phi}(x) \hat{\phi}(x') | 0 \rangle = \int_0^{\infty} dk \sum_{lm} \varphi_{klm}(x) \varphi_{klm}^*(x')
= \int_0^{\infty} dk \sum_l \frac{2l + 1}{4\pi} \varphi_{kl}(r, \tau) \varphi_{kl}(r', \tau) P_l(\cos \gamma),
\] (46)
where \( \gamma \) is the separation angle between the two points. As \( x' \to x \), we have
\[
\langle 0 | \hat{\phi}^2(x) | 0 \rangle = \int_0^{\infty} \frac{dk}{k} \sum_l \frac{2l + 1}{4\pi} k \varphi_{kl}(r, \tau)^2.
\] (47)
In terms of $\psi_{kl}(r, \tau)$, the spectral function of the fluctuations of the quantum field $\phi(x)$ can be defined as

$$S_{kl}(r, \tau) = (2l + 1) j_{l}^{2}(kr) P_{kl}(r, \tau), \quad P_{kl}(r, \tau) = \frac{k}{4\pi j_{l}^{2}(kr)} |\psi_{kl}(r, \tau)|^{2}. \quad (48)$$

Perturbatively,

$$P_{kl}(r, \tau) = \frac{k}{4\pi j_{l}^{2}(kr)} \left|\psi_{kl}^{(0)}(r, \tau) + \psi_{kl}^{(1)}(r, \tau) + \cdots\right|^{2}, \quad (49)$$

where we have defined

$$P_{kl}^{(0)}(r, \tau) = \frac{k}{4\pi j_{l}^{2}(kr)} \left|\psi_{kl}^{(0)}(r, \tau)\right|^{2}, \quad (50)$$

$$\Delta_{kl}^{(1)}(r, \tau) = 2\epsilon^{-1} |\psi_{kl}^{(0)}(r, \tau)|^{-2} \text{Re}[\psi_{kl}^{(0)}(r, \tau)\psi_{kl}^{(1)*}(r, \tau)]. \quad (51)$$

2.2.1. Zeroth order. Following the same steps in equations (14)–(16), to the zeroth order we have as before

$$\frac{\partial^{2}}{\partial \tau^{2}} \psi_{kl}^{(0)}(r, \tau) - \frac{2}{\tau} \frac{\partial}{\partial \tau} \psi_{kl}^{(0)}(r, \tau) - \frac{\partial_{r}^{2}}{r} \psi_{kl}^{(0)}(r, \tau) = \frac{2}{r} \frac{\partial}{\partial r} \psi_{kl}^{(0)}(r, \tau) + \frac{l(l + 1)}{r^{2}} \psi_{kl}^{(0)}(r, \tau) = 0, \quad (52)$$

and the general solution is found to be

$$\psi_{kl}^{(0)}(r, \tau) = k^{2} j_{l}(kr)\psi_{kl}^{(0)}(\tau), \quad (53)$$

where $\psi_{kl}^{(0)}(\tau)$ is given by equation (20). Boundary conditions (21) indeed correspond to the choice of the Bunch–Davies vacuum that selects the mode function

$$\psi_{kl}^{(0)}(\tau) = -\frac{H \tau}{k\sqrt{\pi k}} \left(1 - \frac{1}{k^{2}}\right) e^{-i k \tau}, \quad (54)$$

and hence the zeroth order power spectrum in equation (50) is

$$P_{kl}^{(0)}(r, \tau) = \frac{k}{4\pi j_{l}^{2}(kr)} \left|\psi_{kl}^{(0)}(r, \tau)\right|^{2} = \frac{H^{2}}{4\pi^{2}}(1 + k^{2} \tau^{2}). \quad (55)$$

2.2.2. First order. To the next order, we have

$$\frac{\partial^{2}}{\partial \tau^{2}} \psi_{kl}^{(1)}(r, \tau) - \frac{2}{\tau} \frac{\partial}{\partial \tau} \psi_{kl}^{(1)}(r, \tau) - \frac{\partial_{r}^{2}}{r} \psi_{kl}^{(1)}(r, \tau) - \frac{2}{r} \frac{\partial}{\partial r} \psi_{kl}^{(1)}(r, \tau) + \frac{l(l + 1)}{r^{2}} \psi_{kl}^{(1)}(r, \tau) = J_{1}, \quad (56)$$

where

$$J_{1}(r, \tau) = \frac{4\epsilon \tau}{r} \left[\frac{\partial_{r}^{2}}{\partial r^{2}} \psi_{kl}^{(0)}(r, \tau) + \frac{1}{\tau} \frac{1}{\partial \tau^{2}} \psi_{kl}^{(0)}(r, \tau)\right]$$

$$= \frac{4\epsilon H \tau^{2}}{\sqrt{\pi} r} \left[k^{2} j_{l}(kr)\psi_{kl}^{(1)}(\tau)\right]. \quad (57)$$

The retarded Green’s function necessary to solve this equation is the same as before. Then,

$$\psi_{kl}^{(1)}(r, \tau) = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} dr' r'^{2} G(r, \tau; r', \tau') J_{1}(r', \tau')$$

$$= \frac{2\epsilon H k^{2}}{\sqrt{\pi}} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{\tau_{1}}^{\tau} \int_{\tau_{1}}^{\tau} d\tau' r' j_{l}(kr') j_{l}(k' r') \left[(-k' \tau')(k' r')^{2} H_{2}^{(1)}(-k' \tau')\right]$$

$$- (k' \tau')(k' r')^{2} H_{2}^{(1)}(-k' \tau')$$

$$- (-k' \tau')(k' r')^{2} H_{2}^{(2)}(-k' \tau') \quad (58)$$
This is in the form of
\[
\psi_{kl}^{(1)}(r, \tau) = \epsilon \int_0^\infty \frac{dk'}{k'} \left[ \alpha_{kk'}(\tau) \psi_{kl}^{(0)}(r, \tau) + \beta_{kk'}(\tau) \psi_{kl}^{(0)*}(r, \tau) \right],
\]
where
\[
\alpha_{kk'}(\tau) = -\left( \frac{4i}{\pi} \right) k^2 k'^\frac{1}{2} \int_0^\infty dr' r' j_j(kr') j_l(k'r') \int_\tau^\infty d\tau' (k' \tau' + i)e^{-ik-k'}\tau'.
\]
\[
\beta_{kk'}(\tau) = \left( \frac{4i}{\pi} \right) k^2 k'^\frac{1}{2} \int_0^\infty dr' r' j_j(kr') j_l(k'r') \int_\tau^\infty d\tau' (k' \tau' - i)e^{-ik+k'}\tau'.
\]
As the perturbative formalism in quantum mechanics, the eigenfunctions in the first order consist of zeroth order eigenfunctions of different energies.

Now we calculate the leading correction (51) to the power spectrum. Substituting equation (60) in equation (59), we have
\[
\Delta_{kl}^{(1)}(r, \tau) = \int_0^\infty \frac{dk'}{k'} \frac{k^2}{k'^2} \frac{j_j(k'r')}{j_l(kr)} \left| \psi_{kl}^{(0)}(\tau) \right|^2 \times 2 \text{Re} \left[ \alpha_{kk'}^{(0)}(\tau) \psi_{kl}^{(0)*}(\tau) + \beta_{kk'}^{(0)}(\tau) \psi_{kl}^{(0)}(\tau) \right].
\]
As \( \tau \to 0 \), this becomes
\[
\Delta_{kl}^{(1)}(r, 0) = \int_0^\infty \frac{dk'}{k'} \left( \frac{k}{k'} \right)^{\frac{1}{2}} \frac{j_j(k'r')}{j_l(kr)} \left[ \alpha_{kk'}^{(0)}(0) - \beta_{kk'}^{(0)}(0) \right].
\]
Here we obtain from equations (60) and (61) that
\[
2 \text{Re} \left[ \alpha_{kk'}^{(0)}(0) - \beta_{kk'}^{(0)}(0) \right] = \frac{16}{\pi} \left( \frac{k}{k'} \right)^{\frac{1}{2}} \frac{k^2}{k'^2} \int_0^\infty dr' r' j_j(kr') j_l(k'r') \times \left( \frac{k^2}{k'^2} - 1 \right)^{-2} \left[ \cos(k' \tau) \left[ 2 \sin(k \tau) - \left( \frac{k^2}{k'^2} - 1 \right) k \tau \cos(k \tau) \right] + \sin(k' \tau) \left[ \frac{k}{k'} \left( \frac{k^2}{k'^2} - 3 \right) \cos(k \tau) - \left( \frac{k^2}{k'^2} - 1 \right) k' \tau \sin(k \tau) \right] \right],
\]
where the remaining integral over \( r' \) can be evaluated using formula (34). This quantity diverges as \( k \to k' \). However, we find that the integral in equation (63) is finite and weakly dependent on \( r \). In the limit of \( r \to 0 \), it can be approximated by
\[
\Delta_{kl}^{(1)}(r, 0) \simeq \int_0^\infty \frac{dk'}{k'} \left( \frac{k'}{k} \right)^{1-\frac{1}{2}} \left[ \alpha_{kk'}^{(0)}(0) - \beta_{kk'}^{(0)}(0) \right].
\]

3. Numerical results

Equations (38) and (62) are the main results of this paper. However, they are complicated integrals and are not illuminating. Therefore, we perform numerical calculations of the black-hole corrections to the power spectrum. Assume that inflation begins at the initial time \( t = 0 \).
Figure 1. Time evolution of the first-order normalized power spectra $\Delta_{kl}^{(1)}(\tau)$ in equation (38) for $l = 2, 22, 40$. For each case, we have selected some $k$-modes. The time is expressed in terms of the e-folding number $H_1$. 
and \( a(t = 0) = 1 \). Then, the initial conformal time is \( \tau_i = -1/H \) and the final conformal time is \( \tau = -e^{-Ht}/H \), where \( Ht \) is the e-folding number of inflation. It is useful to note that the Fourier mode with momentum \( k \) crosses out the cosmological horizon at time \( \tau = -1/k \), or equivalently, when the e-folding number is \( Ht = \ln(k/H) \).

In figure 1, we show the time evolution of the first-order contribution to the scalar fluctuations from equation (37). Actually, it is more convenient to plot the normalized power spectrum \( \Delta_{kl}^{(1)}(\tau) \) in equation (38) against the e-folding number \( Ht \). We have chosen the angular momentum \( l = 2, 22, 40 \) as examples. For each \( l \), different \( k \) modes are shown. From all the plots, we can see that for a \( k \)-mode \( \Delta_{kl}^{(1)}(\tau) \) oscillates when the mode is still sub-horizon. Once the mode crosses out the horizon, \( \Delta_{kl}^{(1)}(\tau) \) stops oscillating and gradually approaches a constant value. This behavior can be easily explained by equation (25), where the source term dies off as \( \varphi_{kl}^{(0)} \) goes super-horizon and then gets frozen. In figure 2, we plot the asymptotic values \( \Delta_{kl}^{(1)}(0) \) in equation (39) for \( 1 < l < 50 \) and \( 1 < k/H < 50 \). The figure shows that \( \Delta_{kl}^{(1)}(0) \) is suppressed in low-\( l \) and low-\( k \) regions. This can be explained taking into account that in this limit one considers fluctuations on a large scale, where the effects of the black hole should be negligible. Also the magnitude of \( |\Delta_{kl}^{(1)}(0)| \) is of order 1, so the first-order contribution, compared to the zero-order de Sitter power spectrum, is roughly downsized by the expansion parameter \( \epsilon \).

For the quantum case, we plot the asymptotic values \( \Delta_{kl}^{(1)}(r, 0) \) in equation (63) for \( Hr = 0.01, 0.1 \) against \( l \) and \( k/H \) for \( 1 < l < 50 \) and \( 1 < k/H < 50 \). Note that if inflation lasts for about 60 e-folds, \( r = H^{-1} \) will be about the size of the present Universe and \( Hr < 1 \) corresponds to sub-horizon length scales. As shown in figure 3, the general trend is similar to the classical case in figure 2, except that there are some differences at both low \( l \) and low \( k \). This is indeed an explicit example that quantum fluctuations generated during inflation behave like classical waves. We have also calculated \( \Delta_{kl}^{(1)}(r, 0) \) for larger values of \( Hr \), which becomes fluctuating but in general the value of the amplitude is getting smaller. It is expected because the farther away the black hole is, the lesser is its effect to the perturbation.
4. Conclusions

We have presented a perturbation method to compute the effect of the presence of a black hole in the de Sitter space to the quantum fluctuations of a free massless scalar field. The method is valid as long as the expansion parameter $\epsilon \equiv GMH \ll 1$, i.e. the size of the black hole event horizon is smaller than that of the de Sitter cosmological horizon. The calculation can be easily applied to a vector field or a gravitational wave. Here the first-order contribution is computed and the results are given in the assumption that the black hole is located at the origin of the coordinates. Higher-order corrections can be worked out perturbatively though complicated. It would be interesting to consider the effect due to the distribution of black holes in the de Sitter space. In fact, the perturbation can be in the form of cosmological defects such as monopoles, cosmic strings or domain walls.
Let us briefly discuss some cosmological implications of the results that we have obtained in this work. Assume that inflation lasts for about 60 e-folds. Then, the wavelength of the Fourier mode with $k/H = 1$ is about the size of the present Universe. If the location of the black hole that exists during inflation is near the Earth, the suppressed power of the first-order correction to the de Sitter inflaton fluctuations in low $l$ and low $k$ regions may result in a blue-tilted density power spectrum on large angular scales. This in turn gives rise to a suppression of the large-scale cosmic microwave background anisotropy that may be relevant to the observed low quadrupole in the WMAP cosmic microwave background anisotropy data [14]. A detailed calculation of the effect of the cosmic microwave background anisotropy is underway, including the case in which the black hole is located somewhere else in the Universe.

If inflation lasts for a longer time, then the wavelength of the Fourier mode that corresponds to the size of the present Universe will be given by a larger value of $k/H$. In figure 4, we plot the asymptotic values $\Delta_{kl}^{(1)}(0)$ against $l$ for $k/H = 1, 10, 100$ and 250, which correspond to the inflation with 60, 62.3, 64.6 and 65.5 e-folds, respectively. As expected, the longer the inflation duration, the lesser pronounced are the effects of the black hole to large-scale or low-$l$ observations. This can also be seen in the quantum case with $\Delta_{kl}^{(1)}(r, 0)$ in figure 3.

It is worth noting that this work gives a realization of the general discussions in [15] about the potentially observable effects of a small violation of translational invariance during inflation, as characterized by the presence of a preferred point, line or plane. This violation may induce derivations from pure statistical isotropy of cosmological perturbations, thus leaving anomalous imprints on the cosmic microwave background anisotropy [15].

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