Presentation of the subgroups of Mathieu Group using Groupoid

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Abstract. Mathieu groups are one type of the sporadic simple groups, they turn out not to be isomorphic to any member of the infinite families of finite simple groups. Study these groups is interesting since their orders are very high. Groupoid can be used to find the presentation of the subgroups of the Mathieu groups. The idea is creating a groupoid by acting the Mathieu group on a subset of this group and then calculating the presentation of the vertex group of the groupoid which represents the presentation of the subgroup as the vertex groups are isomorphic.

1. Introduction
There was an attempt to complete the classification of the finite groups in particular those which are non-abelian. There are several families under the umbrella of the big family of the finite simple group. One of these families called the sporadic simple groups, it includes 26 groups, all of them are finite. Émile Léonard Mathieu (1861, 1873) introduced a special type of groups, they are multiply transitive permutation groups on \( n \) objects (\( n \in \{11, 12, 22, 23, 24\} \)). The Mathieu groups are sporadic simple groups, they are the first five in the family of sporadic groups. They are denoted by \( M_{11}, M_{12}, M_{22}, M_{23} \) and \( M_{24} \) [7].

Groups that act on sets of nine, ten, twenty and twenty one points, respectively are denoted by \( M_9, M_{10}, M_{20} \) and \( M_{21} \). These groups are subgroups of a group with large size and they are not sporadic simple groups but they can be used to construct the larger ones. This sequence can extended up to obtain \( M_{13} \), the Mathieu groupoid acting on thirteen points. Also \( M_{21} \) which is not a sporadic simple group and it becomes isomorphic to projective special linear group PSL(3,4) [5]. Table 1 is showing the orders of the Mathieu group.

| Mathieu Group | Order  |
|---------------|--------|
| 11            | 7920   |
| 12            | 95040  |
| 22            | 443520 |
| 23            | 10200960 |
| 24            | 244823040 |
$M_{12}$ has simple subgroup of a maximal order 660 which is isomorphic to the projective special linear group $PSL2(F_{11})$ over the field of 11 elements. The group $M_{11}$ is the stabilizer of a point in the group $M_{12}$. While the group $M_{10}$ is not sporadic and it is the stabilizer of 2 points, it is an almost simple group whose commutator subgroup is the alternating group $A_6$. The stabilizer of three points is $PSU(3,2^2)$ “the projective special unitary group”. The stabilizer of four points is $Q_8$ “the quaternion group”. Also, $M_{24}$ has a subgroup of order 6072 (simple group) which is a maximal subgroup and it is isomorphic to $PSL2(F_{23})$. The stabilizers of one and two points, $M_{23}$ and $M_{22}$ also becomes simple sporadic groups. The stabilizer of three points is simple and isomorphic to $PSL3(4)$ “the projective special linear group” [4, 8].

We will try to find a presentation of a subgroup of Mathieu group by construction first a finitely presented groupoid by acting the Mathieu group on the set that generate the subgroup of the Mathieu group and then finding the presentation of the vertex group of the groupoid.

The groupoid is an algebraic structure which is a generalization of the group. It is a category in which all arrows are isomorphisms. So a group is a groupoid with one object and arrows the elements of the group.

In the context of topology, the best example of groupoid is the fundamental groupoid of a topological space in which the objects set is a set of point taken from the space and an arrow from point $a$ to point $b$ to be equivalence classes of paths from $a$ to $b$ [3]. This is generalisation of the idea of the fundamental group.

In this paper, we construct a groupoid whose objects set is the left cosets

$$gH = \{ mh \mid h \text{ an element } H \}$$

and $m$ is an element in $M$ (Mathieu group) and $H$ is a subgroup of $M$. The morphism of the groupoid is induced by the group action, more details later.

2. Groupoids and vertex group

2.1. Groupoids, free groupoids and finitely presented groupoids

A groupoid is a special type of category which is a generalization of a group.

Definition 2.1. [6] A groupoid $G$ is a category in which for each morphism (arrow) $f: A \to B$ there is a morphism (arrow) $f^{-1}: B \to A$ such that $f \circ f^{-1} = 1_B, f^{-1} \circ f = 1_A$. The morphism $f^{-1}$ is called the inverse of $f$.

A groupoid $G$ is connected if for each pair of objects $A$ and $B \in Obj(G)$ there is at least one arrow $w \in Arr(G)$ with the property $source(w) = A$ and $target(w) = B$.

The notion “free groupoid” is the corner stone of this work. Since for any free groupoid there is an underlying graph (directed graph). So let us recall the definition and required mathematical fact that help to construct such free groupoid.

Definition 2.2. A directed graph $\Gamma = (V, E, s, t)$ consists of a set $V$ called the set of vertices, a set $E$ called the set of edges of $\Gamma$ and two functions $s, t: E \to V$. The vertex $s(e)$ is the source of an edge $e \in E$. The vertex $t(e)$ is the target of an edge $e \in E$.

A map of directed graphs $(V, E, s, t) \mapsto (V', E', s', t')$ consists of functions $f_1: V \to V'$, $f_2: E \to E'$ such that $s(f_2(e)) = f_1(s(e))$ and $t(f_2(e)) = f_1(t(e))$ for all $e \in E$.

Definition 2.3. The disjoint union $\Gamma = \Gamma_1 \sqcup \Gamma_2$ of directed graphs $\Gamma_1$ and $\Gamma_2$ with disjoint vertex sets $V(\Gamma_1)$ and $V(\Gamma_2)$ and edge sets $E(\Gamma_1)$ and $E(\Gamma_2)$ is the directed graph with $V(\Gamma) = V(\Gamma_1) \sqcup V(\Gamma_2)$ and $E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2)$.

Definition 2.4. A maximal tree $T$ of a directed graph $\Gamma$ is a subgraph which includes every vertex of $\Gamma$ and contains no cycle.
Let $\text{Graphs}$ denote the category whose objects are directed graphs and whose morphisms are maps of directed graphs. Let $\text{Groupoids}$ denote the category whose objects are groupoids and whose morphisms are functors between groupoids. There is a functor

$$U : \text{Groupoids} \to \text{Graphs}$$

which simply forgets the partial composition on a groupoid. If $\mathcal{G}$ is a groupoid, then the vertices of $U(\mathcal{G})$ are precisely the objects of $\mathcal{G}$. The directed edges of $U(\mathcal{G})$ are the arrows of $\mathcal{G}$.

There is a functor

$$F : \text{Graphs} \to \text{Groupoids}$$

where for a directed graph $\Gamma$, the groupoid $F(\Gamma)$ is characterized, up to isomorphism, by the following universal property.

Universal property of a free groupoid on $\Gamma$. There is a map of directed graphs $\iota : \Gamma \to U(F(\Gamma))$. For any groupoid $\mathcal{G}$ and any map of directed graphs $f : \Gamma \to U(\mathcal{G})$ there exists a unique groupoid morphism $\bar{f} : F(\Gamma) \to \mathcal{G}$ for which the following diagram commutes in the category of directed graphs.

$$\begin{array}{ccc}
\Gamma & \xrightarrow{\iota} & U(F(\Gamma)) \\
\downarrow f & & \downarrow U(\bar{f}) \\
U(\mathcal{G}) & & U(\mathcal{G})
\end{array}$$

We call $F(\Gamma)$ the free groupoid on $\Gamma$. The existence of $F(\Gamma)$ is established by an explicit construction in terms of words $x_1^\epsilon x_2^\epsilon ... x_n^\epsilon$ where $\epsilon = \pm 1$, $x_i \in E(\Gamma)$, and $s(x_i^\epsilon) = t(x_{i+1}^\epsilon)$. When the directed graph $\Gamma$ has just a single vertex we say that $F(\Gamma)$ is the free group on the set $E(\Gamma)$.

**Proposition 2.1.** $F(\Gamma)$ is unique up to isomorphism of groupoids.

**Proof.** For simplicity we denote $U(\mathcal{G})$ by $\mathcal{G}$ for any groupoid $\mathcal{G}$.

Let $\Gamma$ be a directed graph, and let $F(\Gamma)$ and $F'(\Gamma)$ be free groupoids on $\Gamma$. Let $\iota : \Gamma \to F(\Gamma)$ be a map, and another map $\iota' : \Gamma \to F'(\Gamma)$. By the universal property of free groupoid there is a unique groupoid morphism $\bar{\iota} : F(\Gamma) \to \mathcal{G}$ for which the following diagram commutes.

$$\begin{array}{ccc}
\Gamma & \xrightarrow{\iota} & F(\Gamma) \\
\downarrow \iota' & & \downarrow 1_F(\Gamma) \\
F'(\Gamma) & & F(\Gamma)
\end{array}$$

commutes. Now we obtain

$$\begin{array}{ccc}
\Gamma & \xrightarrow{\iota} & F(\Gamma) \\
\downarrow \iota' & \quad & \downarrow 1_F(\Gamma) \\
\mathcal{G} & & \mathcal{G}
\end{array}$$

By uniqueness, $\iota' \circ \bar{\iota} = 1_{F(\Gamma)}$. Similarly, $\bar{\iota} \circ \iota' = 1_{F'(\Gamma)}$. Therefore, $F(\Gamma)$ is isomorphic to $F'(\Gamma)$. \qed
Let $\mathcal{G}$ be a groupoid with object set $\text{Obj}(\mathcal{G}) = V$. Let $\mathcal{N}$ be a discrete subgroupoid of $\mathcal{G}$ with the same object set $\text{Obj}(\mathcal{N}) = V$. Thus every arrow of $\mathcal{N}$ is an arrow of $\mathcal{G}$ and $\mathcal{N}$ is closed under groupoid composition. The collection of groups $\{\mathcal{G}(v,v) \mid v \in V\}$ is an example of a discrete subgroupoid of $\mathcal{G}$. We say that a discrete subgroupoid $\mathcal{N}$ is normal in $\mathcal{G}$ if $\mathcal{N}(v,v)$ is a normal subgroup of $\mathcal{G}(v,v)$ for each $v \in V$. Given a discrete normal subgroupoid $\mathcal{N}$ in $\mathcal{G}$ we can form the quotient groupoid $\mathcal{G}/\mathcal{N}$ which is characterized up to groupoid isomorphism by the following universal property.

**Universal property of a quotient groupoid.** There is a morphism of groupoids $\phi : \mathcal{G} \to \mathcal{G}/\mathcal{N}$. For any groupoid $\mathcal{Q}$ with object set $\text{Obj}(\mathcal{Q}) = V$, and for any morphism $\psi : \mathcal{G} \to \mathcal{Q}$ that is the identity on $V$ and that sends each element of $\mathcal{N}$ to an identity element, there exists a unique morphism of groupoids $\psi' : \mathcal{G}/\mathcal{N} \to \mathcal{Q}$ such that the following diagram in the category of groupoids commutes.

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\phi} & \mathcal{G}/\mathcal{N} \\
\downarrow{\psi} & & \downarrow{\psi'} \\
\mathcal{Q} & & 
\end{array}
\]

**Proposition 2.2.** For discrete $\mathcal{N}$, $\mathcal{G}/\mathcal{N}$ is unique up to isomorphism of groupoids.

*Proof.* Similar to the proof of the proposition 2.1.

**Definition 2.5.** We say that a set $r$ of arrows in a discrete subgroupoid $\mathcal{N}$ normally generates $\mathcal{N}$ if any normal discrete subgroupoid of $\mathcal{G}$ containing $r$ also contains the subgroupoid $\mathcal{N}$.

Let $\mathcal{G}$ be a groupoid with vertex set $V = \text{Obj}(\mathcal{G})$, and let $\mathcal{F}(\Gamma)$ be a free groupoid on a directed graph $\Gamma = (V, x, s, t)$, and suppose that there is a morphism of groupoids

$$
\phi : \mathcal{F}(\Gamma) \to \mathcal{G}
$$

(3)

that is the identity on objects and that is surjective on arrows. By $\text{ker} \phi$ we mean the groupoid with vertex set $V$ and with arrows those elements $r$ in $\mathcal{F}(\Gamma)$ mapping to an identity arrow $1_{s(r)}$ in $\mathcal{G}$. The groupoid $\text{ker} \phi$ is a discrete normal subgroupoid and $\mathcal{F}(\mathcal{G})/\text{ker} \phi$ is isomorphic to $\mathcal{G}$. Let $r$ be a set of elements in $\text{ker} \phi$ that normally generates $\text{ker} \phi$. The data $\langle x \mid r \rangle$ is called a free presentation of the groupoid $\mathcal{G}$.

### 2.2. Vertex group

Let $\mathcal{G}$ be a groupoid with object set $\text{Obj}(\mathcal{G}) = V$. For each object (vertex) $v \in V$ we let $\mathcal{G}(v,v)$ denote the group of arrows with source and target equal to $v$. We refer to $\mathcal{G}(v,v)$ as the vertex group or isotropy group or object group at $v$. The vertex group $\mathcal{G}(v,v)$ actually is a subgroupoid consisting of one object $v$ and all arrows of the form $v \to v$.

Let $\mathcal{G}$ be a connected groupoid, we can define a homomorphism

$$
\theta : \mathcal{G} \to \mathcal{G}(v,v)
$$

(4)

in the following sense.

Let $\Gamma$ be the generating graph of $\mathcal{G}$, (i.e. $\mathcal{F}(\Gamma) = \mathcal{G}$), and let $T$ be a maximal tree in $\Gamma$. The tree $T$ generates a subgroupoid $\mathcal{H}$ of $\mathcal{G}$, which called a tree of groupoid. The map $\theta$ is defined as

$$
\begin{align*}
\theta(a) &= v & a & \in \text{Obj}(\mathcal{G}) \\
\theta(w) &= xwy, & w & \in \text{Arr}(\mathcal{G}), x, y & \in \mathcal{H}
\end{align*}
$$

(5)

such that $t(y) = s(w), s(x) = t(w)$ and $s(y) = t(x) = v$.

For $c, d \in \mathcal{H}$ (such that $s(c) = t(d) = u$ and $t(c) = s(d) = v$), the product $dc = 1_u$. Its obvious that the map $\theta$ maps the whole $\mathcal{H}$ onto $1_v$. 


Proposition 2.3. The vertex groups of a connected groupoid are all isomorphic.

Proof. Let $G$ be a groupoid with $\text{Obj}(G) = V$. Let $v \in V$ and $G(v, v)$ is the vertex group on $v$. To prove that all vertex groups are isomorphic to $G(v, v)$, let us choose any object $w \in V$, and any arrow $x$ such that $s(x) = v$ and $t(x) = w$. The map $h \mapsto xhx^{-1}$ is an isomorphism from the vertex group at $G(v, v)$ to the vertex group at $G(w, w)$. \hfill $\Box$

Theorem 2.1. Let $G = \langle x \mid r \rangle$ be a finitely presented connected groupoid, If $G(v, v)$ is the vertex group at $v \in \text{Obj}(G)$, then $G(v, v) = \langle x' \mid r' \cup t \rangle$, where $x' = \{\theta(x) : x \in x\}$ and $r' = \{\theta(r) : r \in r\}$ with expressing $\theta(r)$ as a word $x_1^\epsilon x_2^\epsilon \ldots x_k^\epsilon$, $x_i \in x'$, $\epsilon_i \in \pm 1$ and $t = \{t : t$ edge in a maximal tree of $G\}$.

Proof. Let $x = (V, E, s, t)$ be a connected directed graph. Let $F(x)$ denote the free groupoid on $x$. An arrow $r \in \text{Arr}(F(x))$ is said to be a loop if $s(r) = t(r)$. Let $r$ denote a set of loops in the groupoid $F(x)$. Let $R$ denote the normal subgroupoid of $F(x)$ generated by $r$.

The data $\langle x \mid r \rangle$ is a presentation for the quotient groupoid

$$G = F(x)/R.$$  

Let $t$ denote a maximal tree in the graph $x$. Fix some vertex $v \in V$. Then each vertex $w \in V$ determines a unique simple path $p(w)$ in the tree $t$ with $s(p(w)) = w$ and $t(p(w)) = v$. In other words, $p(w)$ is a path in $t$ from $w$ to $v$.

For each arrow $a$ in the groupoid $F(x)$ let us set

$$\theta(a) = p(s(a))^{-1} * a * p(t(a)).$$

Thus $\theta(a)$ is a loop in the groupoid $F(x)$ with source and target equal to $v$.

Now define

$$x' = \{\theta(a) : a$ is a directed edge in $x$ and $a \notin t\},$$

$$r' = \{\theta(a) : a$ is an arrow in $r\}.$$

Note that $x'$ is a free generating set for the free group $F(x)$ and $r'$ is a subset of $F(x, v)$. Let $R(v, v)$ denote the normal subgroupoid of $F(v, v)$ normally generated by $r'$.

We can now regard $\langle x' \mid r' \rangle$ as a free presentation for the finitely presented group

$$F(v,v)/R(v,v).$$

To prove the theorem we need to see that $F(v,v)/R(v,v)$ is isomorphic to the vertex group $G(v, v)$ in $G$.

There is a canonical set theoretic function $\lambda' : x \to G$. This function induces a group homomorphism

$$\lambda : F(v, v) \to G(v, v)$$

The kernel of $\lambda$, by definition, consists of all loops in $F(x)$ at $v$ that can be written as a product of conjugates of loops in $r$. So clearly the kernel of $\lambda$ is normally generated by $r'$ and the proof is complete. \hfill $\Box$

The theorem and propositions above are implemented in GAP as part of the package FpGd [2] available in GitHub website [1].
3. Group actions produce a groupoid

**Proposition 3.1.** Suppose that a group $G$ acts on a set $S$ and that $x$ is a set of generators for $G$. Then the groupoid $\mathcal{Gpd}(G, S)$ is generated by the collection of arrows $x \times S = \{(x, s) : x \in x, s \in S\}$.

**Proof.** An arbitrary arrow $(g, s)$ in $\mathcal{Gpd}(G, S)$ can be expressed as

$$(x_1^{e_1}, x_2^{e_2}, \ldots, x_n^{e_n}) (x_2^{e_2}, x_3^{e_3}, \ldots, x_n^{e_n}) \ldots (x_{n-1}^{e_{n-1}}, x_n^{e_n}) (x_n^{e_n}, s)$$

where $x_i \in x, e_i = \pm 1$. If $e_i = -1$ then

$$(x_i^{-1}, s) = (x_i, x_i^{-1} s)^{-1}.$$ 

Each arrow in $\mathcal{Gpd}(G, S)$ is a sequence of arrows in $x \times S$. □

Let $G$ be a group with subgroup $U$. Let $G/U = \{gU : g \in G\}$ denote the collection of left cosets $gU = \{u : u \in U\}$. There is an action of $G$ on the set $X = G/U$ given by $(g, hU) \rightarrow ghU$ for $g, h \in G$. This action gives rise to a groupoid $\mathcal{Gpd}(G, U)$.

**Proposition 3.2.** For a group $G$ and subgroup $U$ the groupoid $\mathcal{Gpd}(G, U)$ is connected and all vertex groups are isomorphic to $U$.

**Proof.** The object set of the groupoid $\mathcal{Gpd}(G, U)$ is

$$\{U, U_1, \ldots, U_n\}, \text{ where } n = \text{Index}(U) - 1.$$ 

Since any coset $U_i = y_i U$ for some $y_i \in G$, the groupoid is connected.

To prove that all vertex groups are isomorphic to $U$, let us choose any object $U_i$ and any element $x = x_1^{e_1} \ldots x_n^{e_n}$ such that $x_2^{e_2} \ldots x_n^{e_n} U = U_i$. The map $h \mapsto x^{-1} h x$ is an isomorphism between the vertex group at $U$ to the vertex group at $U_i$. □

**Proposition 3.3.** Let $G = \langle x \mid \mathbf{r} \rangle$ be a finitely presented group with finite index subgroup $U$. Then the groupoid $\mathcal{G} = \mathcal{Gpd}(G, U)$ is finitely presented as follows. The objects of $\mathcal{G}$ are the left cosets $gU$. The generators of $\mathcal{G}$ are the arrows $(x, gU)$ for $x \in x$. Each relator $r = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n} \in \mathbf{r}$ and coset $gU$ give rise to a word

$$(r, gU) = (x_1^{e_1}, x_2^{e_2}, \ldots, x_n^{e_n} gU) \ldots (x_{n-1}^{e_{n-1}}, x_n^{e_n} gU)(x_n^{e_n} gU)$$

in the groupoid generators. These words $(r, gU)$ are the relators for the groupoid.

**Proof.** Let $F(x)$ be the free group on $x$. Let $R$ denote the normal subgroup of $F$ normally generated by $\mathbf{r}$. It yields

$$F/R \cong G = \langle x \mid \mathbf{r} \rangle.$$ 

Let $U$ be a subgroup of the group $G$ and let $G/U$ be the set of left cosets of $U$ in $G$.

Let $\mathcal{G}$ denote the finitely presented groupoid $\mathcal{Gpd}(G, U)$. By definition $\mathcal{G}$ is generated by the set

$$x' = \{(x, gU) : x \in x, gU \in G/U\}.$$ 

Let $\mathfrak{G}$ be the free groupoid generated by $x'$ (i.e. $\mathfrak{G} = \mathcal{Gpd}(F, U)$). So each arrow $a \in \mathfrak{G}$ can be expressed as

$$a = (g_i, S_j),$$
where $S_j \in F/U$ and 
\[ g_i = x_1^{e_{i1}} x_2^{e_{i2}} \ldots x_k^{e_{ik}} \in F \]

There is a groupoid homomorphism $\phi : \mathfrak{F} \to \mathfrak{G}$ such that the kernel of $\phi$ consists of all arrows of the form $(g_i, gU)$ for which the source and target is $gU$. That means $\phi(g_i) = 1_{gU}$ and that yields $g_i \in R$. It is readily seen that $\langle r \rangle = R$.

4. Algorithms and implementation

In order to get the presentation of a subgroup $H$ of a finite fp group $G$, we need to create an fp groupoid induced by the group action of $G$ on $H$. We then evaluate the vertex group on the subgroup under consideration. This is one of the applications of the groupoid techniques. We implement Propositions 3.1 and 3.3. This implementation follows the Algorithm 1.

**Algorithm 1:** Fp groupoid induced by group action

**Result:** Fp groupoid

**procedure**

\[
\begin{align*}
\text{obj}(\mathfrak{G}) &= G/H; \\
\text{gens}(\mathfrak{G}) &= [ ]; \\
\text{for } x \text{ in GeneratorsOfGroup}(G) \text{ do} & \quad \text{for } c \text{ in obj}(\mathfrak{G}) \text{ do} \\
& \quad \quad \text{add(gens}(\mathfrak{G}), x^c); \\
\text{end} \\
\text{end} \\
\text{rels}(\mathfrak{G}) &= [ ]; \\
\text{for } r \text{ in RelatorsOfFpGroup}(G) \text{ do} & \quad \text{for } c \text{ in obj}(\mathfrak{G}) \text{ do} \\
& \quad \quad \text{add(rels}(\mathfrak{G}), r^c); \\
\text{end} \\
\text{end} \\
\text{return } \text{FpGroupoid}(\text{obj}(\mathfrak{G}), \text{gens}(\mathfrak{G}), \text{rels}(\mathfrak{G})); \\
\end{align*}
\]

**Example 4.1.** Consider the Mathieu group $M_{11}$ which is generated by two generators, say $a$ and $b$. Let $L = [a^{-1}ba, (ab)^{-1}b]$ is a set of some members of $M_{11}$. The presentation for the subgroup $S = M_{11}/L$ can be calculated using our algorithm which is implemented in GAP as a function $\text{FpGroupoid}$, the input Mathieu group $M$ and a subgroup $S \leq M$ and it returns a presentation for the groupoid $\mathfrak{G}(M, S)$ and finally calculate the presentation for the vertex group using our GAP function $\text{VertexGroup}$ which serves as the presentation for the subgroup $S$.

\[ S = \langle x, y \mid (yx)^2, x^4, y^4, yx^2yx^{-1}y \rangle \]

The calculation is shown in the following GAP session:

```
gap> M:=MathieuGroup(11);;
gap> H:=Image(IsomorphismFpGroup(M));;
gap> h:=GeneratorsOfGroup(H);;
gap> L:=[h[1]^-1*h[2]*h[1],h[2]^-1*h[1]^-1*h[2]];
[ a^-1*b*a, b^-1*a^-1*b ]
gap> U:=Subgroup(H,L);
gap> Group([ a^-1*b*a, b^-1*a^-1*b ])
```

7
gap> G:=FpGroupoid(H,L);;
gap> v:=Source(GeneratorsOfGroupoid(G)[1]);;
gap> S:=VertexGroup(G,v);; S:=SimplifiedFpGroup(S);
<fp group on the generators [ f1, f2 ]>
gap> RelatorsOfFpGroup(S);
[ (f2*f1)^2, f1^4, f2^4, f2*f1^2*f2*f1^-1*f2 ]

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