INJECTIVE AND PROJECTIVE MODEL STRUCTURES ON ENRICHED DIAGRAM CATEGORIES

LYNE MOSER

Abstract. We prove the existence of the injective and projective model structures on the category of simplicial functors $D \to sSet_*$, where $D$ is a small simplicial category, and that these model structures are simplicial and proper. Then we generalize this result by replacing $sSet_*$ with other symmetric monoidal categories which are locally presentable bases, and accessible model categories.

1. Introduction

Categories of diagrams, functors from a small category to a model category, might admit an injective or projective model structure. The projective model structure on diagrams in combinatorial model categories was first constructed using cofibrant generation, while the construction of an injective model structure is more delicate, as explained in the introduction of Joyal’s letter to Grothendieck (see [Joy84]). The breakthrough by Hess, Kędziorek, Riehl, and Shipley in [HKRS17] provides new tools which unify the proof of existence of both injective and projective model structures under rather mild assumptions (see [HKRS17, Theorem 3.4.1]).

Categories of enriched diagrams, in particular simplicial functors $sSet^\text{fin}_* \to sSet_*$ from finite pointed simplicial sets to pointed simplicial sets, play a prominent role in Goodwillie calculus. Such categories might also admit an injective or projective model structure. The projective one has been established in [CD09], and is used to develop a model theoretic framework for Goodwillie calculus, where $n$-excisive approximations are seen as fibrant replacements. The injective one is the key of the definition of homotopy nilpotent groups by Biedermann and Dwyer in [BD10]. Quite a few authors rely on their original work to understand what homotopy nilpotent groups are, among others [CS15], [Eld16], [CSV14], and [Bie17]. It is rather difficult to find a reference for the existence of the injective model structure on enriched diagram categories, except Lurie’s proof in [Lur09, Proposition A.3.3.2].

The aim of this article is to provide a short and self-contained proof of the existence of both injective and projective model structures using the methods of [HKRS17].

Theorem 3.6. Let $D$ be a small $sSet_*$-category. The category of simplicial functors $D \to sSet_*$ admits both injective and projective model structures, and these structures are simplicial and proper.
The methods of [HKRS17] consist of inducing model structures from adjunctions. In their article, there is a proof for the existence of the injective and projective model structures on a category $\mathcal{M}^D$ of functors, where $\mathcal{M}$ is an accessible model category and $D$ is a small category. They induce these two model structures from the Kan extension adjunctions,

$$
\begin{array}{ccc}
\mathcal{M}^D & \xrightarrow{i^*} & \mathcal{M}^{\operatorname{Ob}D} \\
\downarrow & & \downarrow \iota^* \\
\mathcal{M} & \xleftarrow{i_*} & \mathcal{M}^{\operatorname{Ob}D}
\end{array}
$$

where $\operatorname{Ob}D$ denotes the category of objects of $D$, and $i: \operatorname{Ob}D \to D$ is the canonical inclusion. The idea here is to adapt this proof to the enriched setting.

The proof presented in this article also works well for other symmetric monoidal categories $\mathcal{V}$, which are locally presentable bases (see Definition 2.1) and accessible model categories. More precisely, the methods of [HKRS17] allow us to prove the existence of injective and projective model structures on the category of $\mathcal{V}$-functors $D \to \mathcal{V}$, for any small $\mathcal{V}$-category $D$. To go further, we would like to prove that it is true for any category of $\mathcal{V}$-functors $D \to A$, where $A$ is a locally presentable $\mathcal{V}$-category whose underlying category is an accessible model category, and $D$ is a small $\mathcal{V}$-category. But we do not know if one of the necessary hypotheses is satisfied in this case, namely that the $\mathcal{V}$-category of $\mathcal{V}$-functors $D \to A$ is locally presentable (see Remark 4.4).

Throughout the whole article, the following notations are used.

**Notation 1.1.** Let $(\mathcal{V}, \otimes, I)$ be a symmetric monoidal category. There is a 2-functor $(-)_0: \mathcal{V}\text{-CAT} \to \text{CAT}$ sending a $\mathcal{V}$-category to its underlying category (see [Rie14, Proposition 3.5.10]). Let $\mathcal{A}$ be a $\mathcal{V}$-category. For $A, B \in \mathcal{A}$, we denote by

- $\operatorname{Hom}_A(A, B)$, the hom-object in $\mathcal{V}$ for $A, B$, and
- $\mathcal{A}(A, B) = \operatorname{Hom}_A(A, B)_0$, the underlying set of morphisms from $A$ to $B$.

If the monoidal category $\mathcal{V}$ is closed, for $X, Y \in \mathcal{V}$, we denote by

- $\mathcal{V}(X, Y)$, the set of morphisms from $X$ to $Y$, and
- $\operatorname{Hom}_\mathcal{V}(X, Y)$, the internal hom for $X, Y$.

In particular, we have that $\operatorname{Hom}_\mathcal{V}(X, Y)_0 = \mathcal{V}(X, Y)$, since the underlying category of the $\mathcal{V}$-category $\mathcal{V}$ is $\mathcal{V}$ itself (see [Rie14, Lemma 3.4.9]). If $D$ is a small $\mathcal{V}$-category, we denote by

- $[D, \mathcal{V}]$, the $\mathcal{V}$-category of $\mathcal{V}$-functors from $D$ to $\mathcal{V}$, and
- $[D, \mathcal{V}]_0$, the ordinary category of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations.

Section 2 explains under which conditions on $\mathcal{V}$ the enriched functor category $[D, \mathcal{V}]_0$ is locally presentable, a necessary condition to apply the methods of [HKRS17]. Section 3 gives a proof of the main result, namely that $[D, \text{sSet}_*]_0$ admits both injective and projective model structures. This result is then generalized to other symmetric monoidal categories in Section 4. Finally, in Section 5, it is shown that, if the model structure on $\mathcal{V}$ is enriched or proper, so are the injective and projective model structures on $[D, \mathcal{V}]_0$. 
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2. Locally presentable enriched diagram categories

The aim is to show that the category \([D, \text{sSet}_*]_0\) of simplicial functors admits both injective and projective model structures, when \(D\) is a small simplicial category. In order to do so, we adapt the proof of [HKRS17, Theorem 3.4.1] to an enriched case. To use their methods, the first step is to show that the category \([D, \text{sSet}_*]_0\) is locally presentable.

In this section, we work in a more general setting. Let \((\mathcal{V}, \otimes, I)\) be a closed symmetric monoidal category, which is a locally presentable base (Definition 2.1). We show that the category \([D, \mathcal{V}]_0\) is locally presentable, for any small \(\mathcal{V}\)-category \(D\). To see this, we use two results coming from [BQR98]. The first result says that the \(\mathcal{V}\)-category \([D, \mathcal{V}]\) of \(\mathcal{V}\)-functors is locally presentable in the enriched sense, and the second says that the underlying category of a locally presentable \(\mathcal{V}\)-category is locally presentable (in the ordinary sense). All definitions of this section come from [BQR98].

Definition 2.1. Let \(\alpha\) be a regular cardinal. A **locally \(\alpha\)-presentable base** is a closed symmetric monoidal category \((\mathcal{V}, \otimes, I)\) which admits a strongly generating family of \(\alpha\)-presentable objects containing the unit \(I\) and closed under tensor products. We say that \(\mathcal{V}\) is a **locally presentable base** if it is a locally \(\alpha\)-presentable base for some regular cardinal \(\alpha\).

Example 2.2. The categories sSet and sSet_* are locally presentable bases (see [BQR98, Example 1.4.c]).

Let \((\mathcal{V}, \otimes, I)\) be a locally presentable base. Here are the definitions of a presentable object and a locally presentable \(\mathcal{V}\)-category in the enriched setting.

Definition 2.3. Let \(\mathcal{A}\) be a \(\mathcal{V}\)-category. An object \(A\) of \(\mathcal{A}\) is **\(\alpha\)-presentable** (in the enriched sense) if the representable \(\mathcal{V}\)-functor \(\text{Hom}_\mathcal{A}(A, -) : \mathcal{A} \to \mathcal{V}\) preserves \(\alpha\)-filtered \(\mathcal{V}\)-colimits.

Definition 2.4. Let \(\alpha\) be a regular cardinal. A \(\mathcal{V}\)-category \(\mathcal{A}\) is **locally \(\alpha\)-presentable** (in the enriched sense) if it is \(\mathcal{V}\)-cocomplete and it admits a strongly \(\mathcal{V}\)-generating family of (enriched) \(\alpha\)-presentable objects. We say that \(\mathcal{A}\) is **locally presentable** if it is locally \(\alpha\)-presentable for some regular cardinal \(\alpha\).

Let \(D\) be a small \(\mathcal{V}\)-category. The \(\mathcal{V}\)-category \([D, \mathcal{V}]\) is \(\mathcal{V}\)-cocomplete with \(\mathcal{V}\)-colimits computed pointwise and, by the enriched Yoneda Lemma, the representable \(\mathcal{V}\)-functors \(\text{Hom}_D(d, -)\) form a family of \(\alpha\)-presentable objects which is a strong \(\mathcal{V}\)-generating family. This implies the following proposition.

Proposition 2.5 ([BQR98 Example 6.2]). The \(\mathcal{V}\)-category \([D, \mathcal{V}]\) of \(\mathcal{V}\)-functors is locally presentable (in the enriched sense).
To show that the category \([D, \mathcal{V}]_0\) is locally presentable, it remains to see that the underlying category of a \(\mathcal{V}\)-category is locally presentable when the \(\mathcal{V}\)-category is locally presentable in the enriched sense.

**Proposition 2.6** ([BQR98 Proposition 6.6]). Let \(\mathcal{A}\) be a \(\mathcal{V}\)-cocomplete \(\mathcal{V}\)-category and denote by \(\mathcal{A}_0\) the underlying category of \(\mathcal{A}\). If \(\mathcal{A}\) is locally \(\alpha\)-presentable in the enriched sense for some regular cardinal \(\alpha\), then \(\mathcal{A}_0\) is locally \(\alpha\)-presentable in the ordinary sense.

**Corollary 2.7.** Let \(D\) be a small \(\mathcal{V}\)-category. The category \([D, \mathcal{V}]_0\) of \(\mathcal{V}\)-functors is locally presentable (in the ordinary sense).

**Proof.** This follows directly from Propositions 2.5 and 2.6. □

3. **Simplicial diagrams in \(\text{sSet}_*\)**

Consider the symmetric monoidal category \((\text{sSet}_*, \wedge, S^0)\) of pointed simplicial sets, and let \(D\) be a small \(\text{sSet}_*\)-category. Since \(\text{sSet}_*\) is a locally presentable base (see Example 2.2), the category of simplicial functors \([D, \text{sSet}_*]_0\) is locally presentable in the ordinary sense, by Corollary 2.7. In this section, it is shown that this simplicial functor category admits both injective and projective model structures, which are left- and right-induced from a category \([\text{Ob} D, \text{sSet}_*]_0\) similar to the one used in the non-enriched case, namely \(\text{sSet}_*\text{Ob} D\). As in the non-enriched case, we want this category to be a category of simplicial functors defined only on objects of \(D\), and in which the simplicial natural transformations \(\alpha: F \Rightarrow G\) correspond to families of maps of pointed simplicial sets \(\{\alpha_d: Fd \to Gd\}_{d \in D}\) with no further conditions. This last statement allows us to define a model structure on \([\text{Ob} D, \text{sSet}_*]_0\) coming from the one of \(\text{sSet}_*\), in which cofibrations, fibrations, and weak equivalences are defined pointwise.

**Definition 3.1.** Let \(D\) be a small \(\text{sSet}_*\)-category. The **simplicial category of objects** of \(D\) is the \(\text{sSet}_*\)-category \(\text{Ob} D\) with the same objects as \(D\), in which the hom-objects are given by

\[
\text{Hom}_{\text{Ob} D}(d, d') = \Delta^0, \quad \text{and} \quad \text{Hom}_{\text{Ob} D}(d, d) = S^0,
\]

for every \(d \neq d' \in D\), and the identity morphisms are given by

\[
\text{id}_d = \text{id}_{S^0} : S^0 \to \text{Hom}_{\text{Ob} D}(d, d) = S^0,
\]

for every \(d \in D\). There is an inclusion \(i: \text{Ob} D \to D\) given by the \(\text{sSet}_*\)-functor which is the identity on objects, and with

\[
i_{d,d'} : \text{Hom}_{\text{Ob} D}(d, d') = \Delta^0 \to \text{Hom}_D(d, d')
\]
corresponding to the inclusion of the basepoint, for every \(d \neq d' \in D\), and

\[
i_{d,d} : \text{Hom}_{\text{Ob} D}(d, d) = S^0 \to \text{Hom}_D(d, d)
\]
corresponding to the identity morphism \(\text{id}_d\) in \(D\), for every \(d \in D\).

It follows from the definition that an \(\text{sSet}_*\)-functor \(F: \text{Ob} D \to \text{sSet}_*\) corresponds to a family of pointed simplicial sets \(\{Fd\}_{d \in D}\). To see this, note that each map of pointed simplicial sets

\[
F_{d,d'} : \text{Hom}_{\text{Ob} D}(d, d') = \Delta^0 \to \text{Hom}_{\text{sSet}_*}(Fd, Fd')
\]
corresponds to the inclusion of the basepoint, for every \( d \neq d' \in D \), and each map

\[
F_{d,d} : \text{Hom}_{\text{Ob}
D}(d, d) = S^0 \to \text{Hom}_{s\text{Set}^*}(Fd, Fd)
\]

is the identity morphism \( \text{id}_{Fd} \) in \( s\text{Set}^* \), since the following diagram commutes,

\[
\begin{array}{ccc}
S^0 & \xrightarrow{\text{id}_{S^0}} & S^0 \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{Ob}
D}(d, d) & \xrightarrow{\text{id}_{Fd}} & \text{Hom}_{s\text{Set}^*}(Fd, Fd)
\end{array}
\]

for every \( d \in D \). Moreover, an \( s\text{Set}^* \)-natural transformation \( \alpha : F \to G \) in \( [\text{Ob}
D, s\text{Set}^*]\) corresponds to a family of maps of pointed simplicial sets \( \{ \alpha_d : Fd \to Gd \}_{d \in D} \). This follows from the fact that a map \( S^0 \to \text{Hom}_{s\text{Set}^*}(Fd, Gd) \) corresponds to a map of pointed simplicial sets \( Fd \to Gd \), for every \( d \in D \), and from the fact that the following diagrams commute,

\[
\begin{array}{cccc}
\text{Hom}_{\text{Ob}
D}(d, d') = \Delta^0 & \xrightarrow{F_{d,d'}} & \text{Hom}_{s\text{Set}^*}(Fd, Fd') & \text{Hom}_{\text{Ob}
D}(d, d) = S^0 \\
\downarrow & (\alpha_d)' \downarrow & \downarrow \alpha_d & \downarrow \alpha_d \\
\text{Hom}_{s\text{Set}^*}(Gd, Gd') & \xrightarrow{(\alpha_d)} & \text{Hom}_{s\text{Set}^*}(Fd, Gd') & \text{Hom}_{s\text{Set}^*}(Gd, Gd)
\end{array}
\]

for every \( d \neq d' \in D \). Hence there is a model structure on \( [\text{Ob}
D, s\text{Set}^*]_0 \), in which cofibrations, fibrations and weak equivalences are defined pointwise.

Since \( s\text{Set}^* \) is complete and cocomplete in the enriched sense, we have the following adjunctions

\[
\begin{array}{ccc}
[D, s\text{Set}^*]_0 & \xrightarrow{i^*} & [\text{Ob}
D, s\text{Set}^*]_0 \\
\downarrow & \swarrow & \downarrow \\
i_* & & i_*
\end{array}
\]

where \( i^* : [D, s\text{Set}^*]_0 \to [\text{Ob}
D, s\text{Set}^*]_0 \) is the precomposition functor, and

\[
i_* , i_* : [\text{Ob}
D, s\text{Set}^*]_0 \to [D, s\text{Set}^*]_0
\]

are the \textit{enriched} left and right Kan extension functors. The aim is to induce model structures on \( [D, s\text{Set}^*]_0 \) from the pointwise model structure on \( [\text{Ob}
D, s\text{Set}^*]_0 \) through these adjunctions. Next theorem gives a criterion to check if the model structure induced from an adjunction exists. The starting model category is supposed to be \textit{accessible}, which means that the category is locally presentable and the weak factorization systems consisting of cofibrations and trivial fibrations, respectively trivial cofibrations and fibrations, are accessible (see [HKRS17, Definition 3.1.6]).
Notation 3.2. Let $\mathcal{C}$ denote a class of morphisms in a category $\mathcal{M}$. The class $\mathcal{C}^\Box$ is the class of morphisms in $\mathcal{M}$ which have the right lifting property with respect to all morphisms in $\mathcal{C}$, and the class $\mathcal{C}^\otimes$ is the class of morphisms in $\mathcal{M}$ which have the left lifting property with respect to all morphisms in $\mathcal{C}$.

Theorem 3.3 ([HKRS17, Corollary 3.3.4]). Let $(\mathcal{M}, F, C, W)$ be an accessible model category, and let $\mathcal{A}$ and $\mathcal{B}$ be two locally presentable categories. Suppose we have the following adjunctions

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{V} & \mathcal{M} & \xleftarrow{U} & \mathcal{B} \\
\xrightarrow{\bot} & & \xleftarrow{\bot} & & \\
\mathcal{R} & & & & \mathcal{L}
\end{array}
\]

(i) The left-induced model structure on $\mathcal{A}$ exists if and only if $V^{-1}\mathcal{C}^\Box \subseteq V^{-1}W$.

(ii) The right-induced model structure on $\mathcal{B}$ exists if and only if $\mathcal{B}(U^{-1}F) \subseteq U^{-1}W$.

In the proof of the existence of left- and right-induced model structures on functor categories, [HKRS17] uses algebraic weak factorization systems (see [BG16, Section 2], for an introduction to algebraic weak factorization systems). An algebraic weak factorization system on a category $\mathcal{M}$ consists of a pair $(L, R)$, with $L = (L, \epsilon, \Delta)$ a comonad on $\mathcal{M}^2$ and $R = (R, \eta, \mu)$ a monad on $\mathcal{M}^2$, together with a distributive law $\delta: LR \Rightarrow RL$, where $\mathcal{M}^2$ is the category of morphisms and commutative squares in $\mathcal{M}$, and $(L, R)$ forms a functorial factorization on $\mathcal{M}$ (see [BG16, Section 2.2]). This notion is useful to describe a model category. One can associate two algebraic weak factorization systems $(L, R)$ to any model category (except a few of them), in which the cofibrations (resp. trivial cofibrations) correspond to the $L$-coalgebras, and the trivial fibrations (resp. fibrations) correspond to the $R$-algebras. Moreover, an accessible model category can equivalently be defined as a locally presentable category equipped with a pair of accessible algebraic weak factorization systems (see [HKRS17, Remark 3.1.8]).

Denote by $\text{Sq}(\mathcal{M})$ the double category of objects, (horizontal) morphisms, (vertical) morphisms, and commutative squares in $\mathcal{M}$, and let $(L, R)$ be an algebraic weak factorization system on $\mathcal{M}$. There are two double forgetful functors $\text{Coalg}_L \rightarrow \text{Sq}(\mathcal{M})$ and $\text{Alg}_R \rightarrow \text{Sq}(\mathcal{M})$. Moreover, a single of these two double functors determines uniquely the algebraic weak factorization system $(L, R)$ to any model category (except a few of them), in which the cofibrations (resp. trivial cofibrations) correspond to the $L$-coalgebras, and the trivial fibrations (resp. fibrations) correspond to the $R$-algebras. Moreover, an accessible model category can equivalently be defined as a locally presentable category equipped with a pair of accessible algebraic weak factorization systems (see [HKRS17] Remark 3.1.8)).

Remark 3.4. In particular, it follows that a double functor $\text{Coalg}_L \rightarrow \text{Coalg}_L'$ over $\text{Sq}(\mathcal{M})$ corresponds to a morphism $(L, R) \rightarrow (L', R')$ of algebraic weak factorization systems, and a double functor $\text{Alg}_R \rightarrow \text{Alg}_R$ over $\text{Sq}(\mathcal{M})$ corresponds to a morphism $(L, R) \rightarrow (L', R')$.
of algebraic weak factorization systems (see [BG16, Section 2.9], for the definition of a morphism of algebraic weak factorization systems).

Next theorem explains how to get an algebraic weak factorization system, whose underlying factorization system is left-induced from the one of another category. The following result is proven in [BG16, Proposition 13], but the statement as presented here comes from [HKRS17, Theorem 3.3.1]. Moreover, a dual version for the right-induced one can be found in [HKRS17, Theorem 3.3.2].

**Theorem 3.5.** Let \( A \) and \( M \) be locally presentable categories, and let \( V : A \to M \) be a cocontinuous functor between them. If \( M \) admits an accessible algebraic weak factorization system \((L, R)\), then the pullback \( A \overset{V}{\to} S_q(M) \) defines an accessible algebraic weak factorization system on \( A \), whose underlying factorization system is left-induced from the one of \( M \).

We now have all the tools to prove the main theorem.

**Theorem 3.6.** Let \( D \) be a small \( sSet_* \)-category. The category \([D, sSet_*]_0\) of simplicial functors admits both injective and projective model structures, and these structures are simplicial and proper.

**Proof.** We show that the injective model structure on \([D, sSet_*]_0\) exists. The proof for the projective one is dual. Let \( \text{Ob } D \) and \( i : \text{Ob } D \to D \) be as in Definition 3.1. We have an adjunction,

\[
[D, sSet_*]_0 \rightleftarrows [\text{Ob } D, sSet_*]_0.
\]

where \( i^* \) is the precomposition functor, and \( i_* \) is the enriched right Kan extension functor. Let \((C, F_t)\) be the accessible algebraic weak factorization system on \( sSet_* \) of cofibrations and trivial fibrations, which exists since \( sSet_* \) is combinatorial (see [HKRS17, Corollary 3.1.7]). It induces pointwise algebraic weak factorization systems \((C^D, F^D_t)\) and \((C^{\text{Ob } D}, F^{\text{Ob } D}_t)\) on \([D, sSet_*]_0\) and \([\text{Ob } D, sSet_*]_0\), respectively, by postcomposing with the comonad \( C \) and the monad \( F_t \). Since colimits in \([D, sSet_*]_0\) are computed pointwise (see [KecS2, Section 3.3]), the functor \( i^* : [D, sSet_*]_0 \to [\text{Ob } D, sSet_*]_0\) is cocontinuous, and, by Corollary [2.7], the category \([D, sSet_*]_0\) is locally presentable. Moreover, like in the non-enriched case, the category \([\text{Ob } D, sSet_*]_0\) is also locally presentable, and the algebraic weak factorization system \((C^{\text{Ob } D}, F^{\text{Ob } D}_t)\) on \([\text{Ob } D, sSet_*]_0\) is accessible. Therefore, we can apply Theorem 3.5 and we obtain a left-induced algebraic weak factorization...
system \((C^{\text{inj}}, F^{\text{inj}})\) on \([D, sSet_*]_0\). Let \(\mathcal{C}\) denote the class of pointwise cofibrations in \([\text{Ob} D, sSet_*]_0\), and \(\mathcal{W}\) the class of pointwise weak equivalences in \([\text{Ob} D, sSet_*]_0\). Then, by Theorem 3.5, the pair \(((i^*)^{-1})^{-1}\mathcal{C}, ((i^*)^{-1})^{-1}\mathcal{P})\) corresponds to the underlying weak factorization system of \((C^{\text{inj}}, F^{\text{inj}})\). By Theorem 3.3, it remains to show that 

\[ (i^*)^{-1}\mathcal{C} \subseteq (i^*)^{-1}\mathcal{W}. \]

In order to see this, it suffices to show that \(F^{\text{inj}}\)-algebras are \(F^{D}\)-algebras, which implies that injective trivial fibrations are in particular pointwise trivial fibrations, and thus pointwise weak equivalences.

We first show that \(C^D\)-coalgebras are \(C^{\text{inj}}\)-coalgebras. Note that a \(C^D\)-coalgebra corresponds to a natural transformation which is pointwise a \(C\) coalgebra and such that its naturality squares are \(C\)-coalgebra morphisms, while a \(C^{\text{inj}}\)-coalgebra is a natural transformation which is pointwise a \(C\)-coalgebra with no further condition. Since the exterior square in the following diagram commutes,

\[
\begin{array}{ccc}
\text{Coalg}_{C^D} & \xrightarrow{\delta} & \text{Coalg}_{C^{\text{inj}}} \\
V \downarrow & & \downarrow U \\
\text{Sq}([D, sSet_*]_0) & \xrightarrow{\text{Sq}(i^*)} & \text{Sq}([\text{Ob} D, sSet_*]_0)
\end{array}
\]

there exists a morphism \(\text{Coalg}_{C^D} \to \text{Coalg}_{C^{\text{inj}}}\) of double categories, which corresponds to the natural inclusion of \(C^D\)-coalgebras into \(C^{\text{inj}}\)-coalgebras. By Remark 3.4, this double functor determines uniquely a morphism \((C^D, F^D) \to (C^{\text{inj}}, F^{\text{inj}})\) of algebraic weak factorization systems, which gives rise to a double functor \(\text{Alg}_{F^{\text{inj}}} \to \text{Alg}_{F^D}\). This shows that an \(F^{\text{inj}}\)-algebra is in particular an \(F^D\)-algebra.

Finally, it follows from Propositions 5.4 and 5.6 below that the injective and projective model structures are simplicial and proper.

**Remark 3.7.** Considering the symmetric monoidal category \((sSet, \times, \Delta^0)\) instead of the symmetric monoidal category \((sSet_*, \wedge, S^0)\), one can do a similar proof to show that \([D, sSet]_0\) admits both injective and projective model structures, for any small simplicial category \(D\), but with \(\text{Ob} D\) defined as the simplicial category with the same objects as \(D\), and with

\[
\text{Hom}_{\text{Ob} D}(d, d') = \emptyset, \quad \text{and} \quad \text{Hom}_{\text{Ob} D}(d, d) = \Delta^0,
\]

for every \(d \neq d' \in D\). There is also a canonical simplicial functor \(i: \text{Ob} D \to D\).

4. **Generalization to other symmetric monoidal categories**

In this section, we generalize Theorem 3.6 to any locally presentable base which is an accessible model category. Suppose \((\mathcal{V}, \otimes, I)\) is such a category. Recall that Corollary 2.7 implies that the category \([D, \mathcal{V}]_0\) is locally presentable, for any small \(\mathcal{V}\)-category \(D\). To adapt the proof of Theorem 3.6 to the general setting, we need
to define a \( \mathcal{V} \)-category \( \text{Ob} \mathcal{D} \), associated to any small \( \mathcal{V} \)-category \( \mathcal{D} \). Note that, since \( \mathcal{V} \) is a model category, it is cocomplete, and hence it admits an initial object \( \emptyset \).

**Definition 4.1.** Let \( \mathcal{D} \) be a small \( \mathcal{V} \)-category. The **enriched category of objects** of \( \mathcal{D} \) is the \( \mathcal{V} \)-category \( \text{Ob} \mathcal{D} \) with the same objects as \( \mathcal{D} \), in which the hom-objects are given by

\[
\text{Hom}_{\text{Ob}\mathcal{D}}(d, d') = \emptyset, \quad \text{and} \quad \text{Hom}_{\text{Ob}\mathcal{D}}(d, d) = I,
\]

for every \( d \neq d' \in \mathcal{D} \), and the identity morphisms are given by

\[
id_d = id_I: I \to \text{Hom}_{\text{Ob}\mathcal{D}}(d, d) = I,
\]

for every \( d \in \mathcal{D} \). There is an inclusion \( i: \text{Ob} \mathcal{D} \to \mathcal{D} \) given by the \( \mathcal{V} \)-functor which is the identity on objects, and with

\[
i_{d,d'}: \text{Hom}_{\text{Ob}\mathcal{D}}(d, d') = \emptyset \to \text{Hom}_\mathcal{D}(d, d')
\]

corresponding to the unique morphism in \( \mathcal{V} \) from the initial object to \( \text{Hom}_\mathcal{D}(d, d') \), for every \( d \neq d' \in \mathcal{D} \), and

\[
i_{d,d}: \text{Hom}_{\text{Ob}\mathcal{D}}(d, d) = I \to \text{Hom}_\mathcal{D}(d, d)
\]

corresponding to the identity morphism \( id_d \) in \( \mathcal{D} \), for every \( d \in \mathcal{D} \).

The following lemma motivates this definition by saying that the \( \mathcal{V} \)-category \( \text{Ob} \mathcal{D} \) in the enriched case plays a role similar to the one of the category of objects in the non-enriched case.

**Lemma 4.2.** Let \( \mathcal{D} \) be a small \( \mathcal{V} \)-category. Then

(i) a \( \mathcal{V} \)-functor \( F: \text{Ob} \mathcal{D} \to \mathcal{V} \) corresponds to a family of objects \( \{Fd\}_{d \in \mathcal{D}} \) in \( \mathcal{V} \), and

(ii) a \( \mathcal{V} \)-natural transformation \( \alpha: F \Rightarrow G \) in \( [\text{Ob} \mathcal{D}, \mathcal{V}]_0 \) corresponds to a family of morphisms \( \{\alpha_d: Fd \to Gd\}_{d \in \mathcal{D}} \) in \( \mathcal{V} \), with no further conditions.

**Proof.** Let \( F: \text{Ob} \mathcal{D} \to \mathcal{V} \) be a \( \mathcal{V} \)-functor. For every \( d \neq d' \in \mathcal{D} \), the morphism

\[
F_{d,d'}: \text{Hom}_{\text{Ob}\mathcal{D}}(d, d') = \emptyset \to \text{Hom}_\mathcal{V}(Fd, Fd')
\]

corresponds to the unique morphism in \( \mathcal{V} \) from the initial object to \( \text{Hom}_\mathcal{V}(Fd, Fd') \) and, for every \( d \in \mathcal{D} \), the morphism

\[
F_{d,d}: \text{Hom}_{\text{Ob}\mathcal{D}}(d, d) = I \to \text{Hom}_\mathcal{V}(Fd, Fd)
\]

is the identity morphism \( id_{Fd} \) in \( \mathcal{V} \), since the following diagram commutes,

\[
\begin{array}{ccc}
\text{Hom}_{\text{Ob}\mathcal{D}}(d, d) = I & \xrightarrow{id_I} & I \\
\downarrow F_{d,d} & & \downarrow F_{d,d} \\
\text{Hom}_\mathcal{V}(Fd, Fd) & \xrightarrow{id_{Fd}} & \text{Hom}_\mathcal{V}(Fd, Fd)
\end{array}
\]

for every \( d \in \mathcal{D} \). Hence the \( \mathcal{V} \)-functor \( F \) corresponds to the family of objects \( \{Fd\}_{d \in \mathcal{D}} \) in \( \mathcal{V} \).

Let \( \alpha: F \Rightarrow G \) be a \( \mathcal{V} \)-natural transformation in \( [\text{Ob} \mathcal{D}, \mathcal{V}]_0 \). Recall that a morphism \( I \to \text{Hom}_\mathcal{V}(Fd, Gd) \) in \( \mathcal{V} \) corresponds to a morphism \( Fd \to Gd \) in \( \mathcal{V} \), for every \( d \in \mathcal{D} \),
since the underlying category of the $\mathcal{V}$-category $\mathcal{V}$ is $\mathcal{V}$ itself. Moreover, the following diagrams trivially commute, for every $d \neq d' \in D$.

\[
\begin{array}{ccc}
\text{Hom}_{\text{Obj} D}(d, d') = \varnothing & \xrightarrow{F_{d,d'}} & \text{Hom}_\mathcal{V}(Fd, Fd') \\
\downarrow G_{d,d'} & & \downarrow (\alpha_{d'})_* \\
\text{Hom}_\mathcal{V}(Gd, Gd') & \xrightarrow{(\alpha_d)_*} & \text{Hom}_\mathcal{V}(Fd, Gd')
\end{array}
\]

\[
\begin{array}{ccc}
\text{Hom}_{\text{Obj} D}(d, d) = I & \xrightarrow{F_{d,d}} & \text{Hom}_\mathcal{V}(Fd, Fd) \\
\downarrow G_{d,d} & & \downarrow \alpha_d \\
\text{Hom}_\mathcal{V}(Gd, Gd) & \xrightarrow{(\alpha_d)_*} & \text{Hom}_\mathcal{V}(Fd, Gd)
\end{array}
\]

This shows that $\alpha: F \Rightarrow G$ corresponds to a family of morphisms $\{\alpha_d: Fd \to Gd\}_{d \in D}$ in $\mathcal{V}$.

Hence there is a model structure on $[\text{Obj} D, \mathcal{V}]_0$ coming from the one of $\mathcal{V}$, in which cofibrations, fibrations and weak equivalences are defined pointwise.

Since $\mathcal{V}$ is $\mathcal{V}$-complete and $\mathcal{V}$-cocomplete, we also have two adjunctions

\[
\begin{array}{ccc}
[D, \mathcal{V}]_0 & \xrightarrow{i_*} & \text{Obj} D, \mathcal{V}]_0 \\
\downarrow & & \downarrow \\
[D, \mathcal{V}]_0 & \xleftarrow{i^*} & \text{Obj} D, \mathcal{V}]_0
\end{array}
\]

where $i^*$ is the precomposition functor, and $i_*, i^*$ are the left and right enriched Kan extension functors. Therefore, we can adapt the proof of Theorem 3.6 to the category $\mathcal{V}$, and we obtain the following more general result.

**Theorem 4.3.** Let $(\mathcal{V}, \otimes, I)$ a locally presentable base. Suppose $\mathcal{V}$ is an accessible model category, and let $D$ be a small $\mathcal{V}$-category. The category $[D, \mathcal{V}]_0$ of enriched functors admits both injective and projective model structures.

**Remark 4.4.** Suppose $\mathcal{A}$ is a locally presentable $\mathcal{V}$-category, and $D$ is a small $\mathcal{V}$-category. If it is true that $[D, \mathcal{A}]$ is also locally presentable (in the enriched sense), we can generalize Theorem 4.3 to a more general statement:

Suppose $\mathcal{V}$ is a locally presentable base. Let $\mathcal{A}$ be a locally presentable $\mathcal{V}$-category such that its underlying category $\mathcal{A}_0$ is an accessible model category, and let $D$ be a small $\mathcal{V}$-category. Then the category $[D, \mathcal{A}]_0$ of $\mathcal{V}$-functors admits both injective and projective model structures.

5. **Enrichment and Properness**

As before, let $(\mathcal{V}, \otimes, I)$ be a locally presentable base, which is an accessible model category. We show that, if the model structure on $\mathcal{V}$ is enriched over $\mathcal{V}$, then so are the injective and projective model structures on $[D, \mathcal{V}]_0$, and similarly for properness. This completes the proof of Theorem 3.6 since $\text{sSet}_*$ is a simplicial, proper model category. We first recall the definition of a $\mathcal{V}$-enriched model category, a notion which generalizes the notion of a simplicial model category (see [Hir03, Definition 9.1.6]).
**Definition 5.1.** A $\mathcal{V}$-category $\mathcal{M}$ is a $\mathcal{V}$-enriched model category if

- [MC1-5]: its underlying category $\mathcal{M}_0$ admits a model structure,
- [MC6]: it is tensored and cotensored over $\mathcal{V}$, and
- [MC7]: if $i: A \to B$ is a cofibration in $\mathcal{M}_0$ and $p: X \to Y$ is a fibration in $\mathcal{M}_0$, the map
  $$\left(i^*, p_*\right): \text{Hom}_\mathcal{M}(B, X) \to \text{Hom}_\mathcal{M}(A, X) \times_{\text{Hom}_\mathcal{M}(A, Y)} \text{Hom}_\mathcal{M}(B, Y)$$
  is a fibration in $\mathcal{V}$, which is trivial if either $i$ or $p$ is a weak equivalence.

We first check that $[D, \mathcal{V}]_0$ satisfies axiom [MC6], for any small $\mathcal{V}$-category $D$.

**Lemma 5.2.** Let $D$ be a small $\mathcal{V}$-category. The category $[D, \mathcal{V}]_0$ is tensored and cotensored over $\mathcal{V}$, with tensor product and power defined pointwise.

**Proof.** Let $F: D \to \mathcal{V}$ be a $\mathcal{V}$-functor, and let $X \in \mathcal{V}$. The aim is to define a tensor product $F(-) \otimes X: D \to \mathcal{V}$ and a power $\text{Hom}_\mathcal{V}(X, F(-)): D \to \mathcal{V}$. Consider the functor $- \otimes X: \mathcal{V} \to \mathcal{V}$ sending an object $Y \in \mathcal{V}$ to $Y \otimes X$ and such that, for $Y, Z \in \mathcal{V}$, the map
  $$( - \otimes X )_{YZ}: \text{Hom}_\mathcal{V}(Y, Z) \to \text{Hom}_\mathcal{V}(Y \otimes X, Z \otimes X)$$
is the adjunct of the evaluation map $\text{ev} \otimes \text{id}_X: \text{Hom}_\mathcal{V}(Y, Z) \otimes Y \otimes X \to Z \otimes X$. Define the tensor product $F(-) \otimes X: D \to \mathcal{V}$ to be the composite
  $$D \xrightarrow{F} \mathcal{V} \xrightarrow{- \otimes X} \mathcal{V}.$$

Then, consider the functor $\text{Hom}_\mathcal{V}(X, -): \mathcal{V} \to \mathcal{V}$ sending an object $Y \in \mathcal{V}$ to $\text{Hom}_\mathcal{V}(X, Y)$, and such that, for $Y, Z \in \mathcal{V}$, the map
  $$\text{Hom}_\mathcal{V}(X, -)_{YZ}: \text{Hom}_\mathcal{V}(Y, Z) \to \text{Hom}_\mathcal{V}(\text{Hom}_\mathcal{V}(X, Y), \text{Hom}_\mathcal{V}(X, Z))$$
is the adjunct of the composition map $\text{Hom}_\mathcal{V}(Y, Z) \otimes \text{Hom}_\mathcal{V}(X, Y) \to \text{Hom}_\mathcal{V}(X, Z)$. Define the power $\text{Hom}_\mathcal{V}(X, F(-)): D \to \mathcal{V}$ to be the composite
  $$D \xrightarrow{F} \mathcal{V} \xrightarrow{\text{Hom}_\mathcal{V}(X, -)} \mathcal{V}.$$

These constructions satisfy the expected adjunction properties. This shows that $[D, \mathcal{V}]_0$ is tensored and cotensored over $\mathcal{V}$, and that both tensor product and power are defined pointwise. \hfill $\Box$

To prove that the injective and projective model structures on $[D, \mathcal{V}]_0$ are $\mathcal{V}$-enriched, it remains to show that $[D, \mathcal{V}]_0$ satisfies axiom [MC7]. Since cofibrations and weak equivalences are defined pointwise in the injective model structure on $[D, \mathcal{V}]_0$, and fibrations and weak equivalences are defined pointwise in the projective one, it is convenient to have equivalent statements to axiom [MC7], where only (trivial) cofibrations or only (trivial) fibrations appear. This allows us to check pointwise that the category $[D, \mathcal{V}]_0$ satisfies this last axiom. Next lemma is an adaptation of [Hir03, Proposition 9.3.7] to the general setting.

**Lemma 5.3.** Let $\mathcal{M}$ be a tensored and cotensored $\mathcal{V}$-category, whose underlying category admits a model structure. Axiom [MC7] of Definition 5.1 is equivalent to each of the following statements.
(i) If \( i: A \to B \) is a cofibration in \( \mathcal{M}_0 \) and \( j: K \to L \) is a cofibration in \( \mathcal{V} \), the map
\[
i \star j: A \otimes L \amalg_{K} B \otimes K \to B \otimes L
\]
is a cofibration in \( \mathcal{M}_0 \), which is trivial if either \( i \) or \( j \) is a weak equivalence.

(ii) If \( p: X \to Y \) is a fibration in \( \mathcal{M}_0 \) and \( j: K \to L \) is a cofibration in \( \mathcal{V} \), the map
\[
(j^*, p_*): X^L \to X^K \times_{Y^L} Y^K
\]
is a fibration in \( \mathcal{M}_0 \), which is trivial if either \( j \) or \( p \) is a weak equivalence.

Proof. This follows immediately from the adjunctions
\[\mathcal{M}(K \otimes A, X) \cong \mathcal{V}(K, \text{Hom}_{\mathcal{M}}(A, X)),\]
where \( A, X \in \mathcal{M} \) and \( K \in \mathcal{V} \).

Proposition 5.4. Let \((\mathcal{V}, \otimes, I)\) a locally presentable base. Suppose \( \mathcal{V} \) is an accessible \( \mathcal{V}\)-enriched model category, and let \( D \) be a small \( \mathcal{V}\)-category. The injective and projective model structures on \([D, \mathcal{V}]_0\) exist, and these structures are \( \mathcal{V}\)-enriched.

Proof. We show that the injective model structure satisfies condition (i) of Lemma 5.3. Let \( \alpha: F \Rightarrow G \) be a cofibration in \([D, \mathcal{V}]_0\), and \( j: K \to L \) be a cofibration in \( \mathcal{V} \). Since the model structure is injective, the morphism \( \alpha_d: F_d \to G_d \) is a cofibration in \( \mathcal{V} \), for every \( d \in D \). The map \( \alpha \star j: F \otimes L \amalg_{K} G \otimes K \to G \otimes L \) has components \( (\alpha \star j)_d = (\alpha_d) \star j \), for \( d \in D \), and every component \( (\alpha_d) \star j \), for \( d \in D \), is a cofibration in \( \mathcal{V} \), since \( \mathcal{V} \) is a \( \mathcal{V}\)-enriched model category. Hence \( \alpha \star j \) is a cofibration in \([D, \mathcal{V}]_0\). Moreover, if either \( \alpha \) or \( j \) is a weak equivalence, the morphism \( (\alpha_d) \star j \) is trivial, for every \( d \in D \), and thus \( \alpha \star j \) is also trivial.

Remark 5.5. Proposition 5.4 generalizes to any \( \mathcal{V}\)-enriched model category \( \mathcal{A} \) such that \([D, \mathcal{A}]_0\) admits injective and projective model structures, by Lemma 5.3. To go further with Remark 4.4, the following statement can be formulated:

Suppose \( \mathcal{V} \) is a locally presentable base and a model category. Let \( \mathcal{A} \) be a locally presentable \( \mathcal{V}\)-enriched accessible model category, and let \( D \) be a small \( \mathcal{V}\)-category. Then the category \([D, \mathcal{A}]_0\) of \( \mathcal{V}\)-functors admits both injective and projective model structures, and these model structures are again \( \mathcal{V}\)-enriched.

This statement would recover Lurie’s theorem (see [Lur09], Proposition A.3.3.2), and even generalize it.

The last result is that, if \( \mathcal{V} \) is a proper model category, the induced injective and projective model structures on \([D, \mathcal{V}]_0\) are also proper, for any small \( \mathcal{V}\)-category \( D \). Recall that a model category is proper if weak equivalences are preserved under pushouts along cofibrations, and under pullbacks along fibrations.

Proposition 5.6. Let \((\mathcal{V}, \otimes, I)\) be a locally presentable base. Suppose \( \mathcal{V} \) is a proper accessible \( \mathcal{V}\)-enriched model category, and let \( D \) be a small \( \mathcal{V}\)-category. The injective and projective model structures on \([D, \mathcal{V}]_0\) exist, and these structures are proper.

Proof. We give a proof for the injective model structure. The one for the projective model structure is dual. Since limits and colimits are defined pointwise in \([D, \mathcal{V}]_0\), and weak equivalences in the injective model structure are defined pointwise, it suffices to
see that injective cofibrations and fibrations in \([D, V]_0\) are pointwise cofibrations and fibrations. It is the case for injective cofibrations by definition. To show that injective fibrations are in particular pointwise fibrations, we adapt the proof of [BHK+15, Lemma 4.6] to the enriched case.

Suppose \(\eta: F \Rightarrow G\) is a fibration in \([D, V]_0\). Let \(i: A \rightarrow B\) be a trivial cofibration in \(V\), and let \(d \in D\). We need to show that \(\eta_d: Fd \rightarrow Gd\) has the right lifting property with respect to \(i: A \rightarrow B\). By the enriched Yoneda lemma, and since \([D, V]\) is enriched and tensored over \(V\) (Lemma 5.2), then

\[
\mathcal{V}(A, Fd) \cong \mathcal{V}(A, \text{Hom}_{[D, V]}(\text{Hom}_D(d, -), F)) \cong [D, V](A \otimes \text{Hom}_D(d, -), F).
\]

Saying that \(\eta_d\) has the right lifting property with respect to \(i\) is then equivalent to saying that \(\eta\) has the right lifting property with respect to \(i \otimes \text{Hom}_D(d, -)\). But each component of \(i \otimes \text{Hom}_D(d, -)\) is a trivial cofibration in \(V\), since the model structure on \(V\) is \(V\)-enriched, and hence \(i \otimes \text{Hom}_D(d, -)\) is an injective trivial cofibration in \([D, V]_0\). □

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UPHESS BMI FSV, ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, STATION 8, CH-1015 LAUSANNE, SWITZERLAND

E-mail address: lyne.moser@epfl.ch