Characteristics of conservations laws of chiral-type systems

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Abstract

In this note a new way to construct the characteristics of conservations laws of integrable chiral-type systems is proposed.

Key words: chiral-type systems, Lax representation, characteristics of conservation laws, Killing fields.

1. Introduction

Chiral-type systems (see, for example, [1]) are the systems of partial differential equations of the form

$$U_{xy} + G_{\beta\gamma}^\alpha U_x U_y + Q^\alpha = 0.$$  \hspace{0.5cm} (1)

Here, the Greek indices $\alpha, \beta, \gamma$ range from 1 to $n$ and the subscripts denote partial derivatives with respect to the independent variables $x$ and $y$. The coefficients $G_{\beta\gamma}^\alpha, Q^\alpha$ are assumed to be smooth functions of the variables $U_1, U_2, \ldots, U^n$. The summation rule over the repeated indices is also assumed.

If the system (1) is a system of Euler-Lagrange equations, then it is called a nonlinear generalized sigma model.

Following [2], recall that the characteristic of the conservation law $L = (L_1, L_2)$ of the system (1) is a set of functions $R = \{R_\alpha\}$ such that

$$\text{Div } L = D_x L_1 + D_y L_2 = R_\alpha \Delta^\alpha,$$  \hspace{0.5cm} (2)

where $\Delta^\alpha$ denotes the l.h.s. of Eq. (1).

We understand integrable systems as the systems admitting a Lax representation. Suppose that the matrix $g$-valued Lax representation is of the form:

$$D_y \tilde{A} - D_x \tilde{B} + [\tilde{A}, \tilde{B}] = S_\alpha \Delta^\alpha,$$  \hspace{0.5cm} (2)

where
\[ \tilde{A} = A_\alpha U_\alpha^x + \lambda M, \quad \tilde{B} = B_\alpha U_\alpha^y + \frac{1}{\lambda} N, \]  

(3)

\[ A, B, M, N \] are smooth functions of the variables \( U^1, U^2, ..., U^n \), taking values in a matrix Lie algebra \( \mathfrak{g} \), and \( S_\alpha = S_\alpha(U^1, U^2, ..., U^n) \).

**Remark 1.** Note that the set of such matrices as \( S_\alpha \) in the general case was investigated by M. Marvan [3], and named a characteristic element of a Lax representation.

Let \( f : \mathfrak{g} \times \mathfrak{g} \times ... \mathfrak{g} \mapsto R \) be a symmetric \( p \)-linear ad-invariant form on \( \mathfrak{g} \), i.e., for all \( x_1, x_2, ..., x_p, y \in \mathfrak{g} \) the following identity is valid:

\[ f([y, x_1], x_2, ..., x_p) + f(x_1, [y, x_2], x_3, ..., x_p) + ... + f(x_1, x_2, ..., [y, x_p]) = 0. \]  

(4)

For \( p = 2 \) one can take the Killing metric of the Lie algebra \( \mathfrak{g} \).

The purpose of this note is to prove the following main theorem which states that the sets \( R_\alpha \) and \( \tilde{R}_\alpha \) defined by the expressions

\[ R_\alpha = f(S_\alpha, \frac{\partial \tilde{A}}{\partial \lambda}, \frac{\partial \tilde{A}}{\partial \lambda}, ..., \frac{\partial \tilde{A}}{\partial \lambda}), \quad \tilde{R}_\alpha = f(S_\alpha, \frac{\partial \tilde{B}}{\partial \lambda}, \frac{\partial \tilde{B}}{\partial \lambda}, ..., \frac{\partial \tilde{B}}{\partial \lambda}) \]  

(5)

are characteristics of conservation laws of the system \( (1) \).

2. **Main theorem and examples**

Denote the \( \alpha \)-th Euler operator by

\[ E_\alpha = \sum J(-D)_J(\frac{\partial}{\partial U_j^\alpha}), \]

where the sum extending over all multi-indices \( J = (j_1, j_2) \).

For the sequel, we need the following technical lemma 1.

**Lemma 1** Let

\[ \tilde{A} = A_\alpha U_\alpha^x + M, \quad \tilde{B} = B_\alpha U_\alpha^y + N, \]  

(6)

where \( A, B, M, N \) are smooth functions of the variables \( U^1, U^2, ..., U^n \), taking values in a matrix Lie algebra \( \mathfrak{g} \), and such that the following conditions are satisfied:

\[ M_\alpha = [B_\alpha, M], \]  

(7)

\[ N_\alpha = [A_\alpha, N]. \]  

(8)
Here and further, the comma denotes the partial derivatives, that is, \( P,\alpha = \frac{\partial P}{\partial U^\alpha} \).

Assume that \( f : \underbrace{\mathfrak{g} \times \mathfrak{g} \times \ldots \mathfrak{g}}_p \mapsto R \) is a symmetric \( p \)-linear ad-invariant form on \( \mathfrak{g} \).

Then the following identities are valid:

\[
E_\alpha (f(D_y \tilde{A} - D_x \tilde{B} + [\tilde{A}, \tilde{B}]), M_\alpha, ..., M_\alpha) = 0,
\]

\[
E_\alpha (f(D_y \tilde{A} - D_x \tilde{B} + [\tilde{A}, \tilde{B}]), N_\alpha, ..., N_\alpha) = 0.
\]

We will prove this lemma in the appendix.

**Theorem 1** Let the chiral-type system \((1)\) admit the Lax representation of the form \((2),(3)\) in a matrix Lie algebra \( \mathfrak{g} \). Then for each ad-invariant symmetric \( p \)-form \( f \) of the Lie algebra \( \mathfrak{g} \) the sets of functions \((5)\) are characteristics of conservation laws of the system \((1)\).

**Proof.**

In the case under consideration, up to a constant factor, we evidently have:

\[
R_\alpha = f(S_\alpha, M_\alpha, ..., M_\alpha),
\]

\[
\tilde{R}_\alpha = f(S_\alpha, N_\alpha, ..., N_\alpha).
\]

Further, consider the case of Eq. \((11)\). The case of Eq. \((12)\) can be proved in a similar way. One can easily verify that the coefficients by first derivatives \( U_\alpha^x, U_\alpha^y \) in l.h.s. of Eq. \((2)\) vanish whenever Eqs. \((7),(8)\) are fulfilled.

Substituting \((2)\) in the polynomial \( f \) as the first argument and \( M \) as the remaining arguments, we obtain the relations

\[
f(D_y \tilde{A} - D_x \tilde{B} + [\tilde{A}, \tilde{B}], M_\alpha, ..., M_\alpha) = R_\alpha [U_\alpha^x U_\alpha^y + G_\alpha^r U_\alpha^x U_\alpha^y + Q^\alpha].
\]

The use of the lemma 1 completes the proof.

Next, we notice an invariant properties of the characteristics of conservation laws.
One can readily verify that under arbitrary non-degenerate transformations of the variables $U^1, U^2, ..., U^n$ the functions $G_{\beta\gamma}^\alpha$ in (1) are transformed as the coefficients of an affine connection. Therefore, we assume $G_{\beta\gamma}^\alpha$ to be the coefficients of an affine connection in the local coordinate system $U^1, U^2, ..., U^n$ of a space $V^n$. We say that the connection is associated to the system (1) and denote the covariant derivatives w.r.t. this connection by $\nabla_{\alpha}$.

Further on, we consider characteristics of conservation laws only of the form $R_{\alpha} = R_{\alpha}(U^1, U^2, ..., U^n)$.

**Theorem 2** A set $R_{\alpha}$ is the characteristic of the conservation law of system (1) iff the following conditions are satisfied:

1) $R_{\alpha}$ is a Killing covector field, i.e.,

$$\nabla_{(\alpha}R_{\beta)} = 0;$$

(13)

2) $R_{\alpha}Q^\alpha = \text{const};$

(14)

3) the form $\nabla_{\alpha}R_{\beta}dU^\alpha \wedge dU^\beta$ is closed, i.e.,

$$d(\nabla_{\alpha}R_{\beta}dU^\alpha \wedge dU^\beta) = 0.$$  

(15)

**Proof.**

As it is well known (see for example [2]), a function $R_{\alpha}\Delta^\alpha$ is a divergence iff $E_{\beta}(R_{\alpha}\Delta^\alpha) = 0$.

One can readily verify that

$$E_{\alpha}(R_{\beta}\Delta^\beta) = K_{\alpha\beta}U_x^\beta + \tilde{L}_{\alpha\beta\gamma}U_x^\beta U_y^\gamma + L_{\alpha},$$

where

$$K_{\alpha\beta} = \nabla_{(\alpha}R_{\beta)};$$

(16)

$$\tilde{L}_{\alpha\beta\gamma} = R_{\alpha\beta\gamma} - (R_{\delta}G_{\alpha\gamma}^\delta)_{,\beta} - (R_{\delta}G_{\beta\gamma}^\delta)_{,\alpha} + (R_{\delta}G_{\beta\alpha}^\delta)_{,\gamma};$$

(17)

$$L_{\alpha} = (R_{\delta}Q^\delta)_{,\alpha}.$$  

(18)

Now, one can see that conditions (13) and (14) are fulfilled. Taking into account condition (13), Eq. (17) results in

$$\tilde{L}_{\alpha\beta\gamma} = (\nabla_{\gamma}R_{\alpha})_{,\beta} + (\nabla_{\beta}R_{\gamma})_{,\alpha} + (\nabla_{\alpha}R_{\beta})_{,\gamma}.$$  

This completes the proof.
Corollary 1 Let the chiral-type system (1) admit the Lax representation of the form (2), (3) in a compact semisimple Lie algebra $\mathfrak{g}$, where:

1) not all of the coefficients $Q^\alpha$ vanish;
2) the number $n$ of components of the system (1) and the dimension of the Lie algebra $\mathfrak{g}$ satisfy the condition:

$$n \leq \dim \mathfrak{g} \leq n + 1; \quad (19)$$

3) the matrices $S_\alpha$ are linear independent.

Then the system (1) admits at least one non-trivial conservation law whose characteristic $R_\alpha$ depends on $U^1, U^2, ..., U^n$ and satisfy the conditions (13), (15), and

$$R_\alpha Q^\alpha = 0.$$

Proof.

First, using the Lax representation (2), we find that $2S_\alpha Q^\alpha = [M, N]$. In view of linear independence of $S_\alpha$ we conclude that $M$ and $N$ are linear independent too. Let the 2-form $f$ be the Killing metric on the Lie algebra $\mathfrak{g}$. Taking into account non-degeneracy and positivity of the form $f$ and consideration of the dimensionality, we conclude that at least one of the fields $R_\alpha = f(S_\alpha, M), \tilde{R}_\alpha = f(S_\alpha, N)$ is non-vanishing. Now, the use of the theorem 2 and observation that $R_\alpha Q^\alpha = f(Q^\alpha S_\alpha, M) = f(\frac{[M, N]}{2}, M) = 0$ complete the proof.

The tensor fields $R_\alpha, \tilde{R}_\alpha$ admit the following generalization.

Theorem 3 Let the chiral-type system (1) admit the Lax representation taking value in a Lie algebra $\mathfrak{g}$. Define for each ad-invariant symmetric $p$-form $f$ of the Lie algebra $\mathfrak{g}$ the tensor fields:

$$R_{\alpha_1 \alpha_2 ... \alpha_k} = f(S_{\alpha_1}, S_{\alpha_2}, ..., S_{\alpha_k}, M, ..., M), \quad (20)$$

$$\tilde{R}_{\alpha_1 \alpha_2 ... \alpha_k} = f(S_{\alpha_1}, S_{\alpha_2}, ..., S_{\alpha_k}, N, ..., N). \quad (21)$$

Then these fields are the Killing fields.
Example 1 Consider Pohlmeier-Lund-Regge system [5] which has the form:

\[ \Delta^1 = U^1_{xy} + \frac{1}{\sin U^2} (U^1_x U^2_x + U^1_y U^2_y) = 0, \]

\[ \Delta^2 = U^2_{xy} - \frac{\sin U^2}{(1 + \cos U^2)^2} U^1_x U^1_y - p \sin U^2 = 0, \]

where \( p \) is an arbitrary constant.

It will be convenient to write the Lax representation of PLR system in the form (2),(3), where

\[ \tilde{A} = \begin{pmatrix} 0 & \lambda p - \frac{\cos U^2 U^1_x}{2 \cos^2 U^2} & -\frac{\tan U^2}{2} U^1_x \\ -\left(\frac{p}{\lambda} - \frac{\cos U^2 U^1_x}{2 \cos^2 U^2}\right) & 0 & U^1_x \\ \frac{\tan U^2}{2} U^1_x & -U^2_x & 0 \end{pmatrix}, \]

\[ \tilde{B} = \begin{pmatrix} 0 & -\left(\frac{\cos U^2}{\lambda} + \frac{U^1_y}{2 \cos^2 U^2}\right) & -\tan U^2 \\ \left(\frac{\cos U^2}{\lambda} + \frac{U^1_y}{2 \cos^2 U^2}\right) & 0 & 0 \\ -\tan U^2 & 0 & 0 \end{pmatrix}. \]

Choose the basis \( B \) of the Lie algebra \( so(3) \) as

\[ \vec{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \]

then we have

\[ D_y \tilde{A} - D_x \tilde{B} + [\tilde{A}, \tilde{B}] = S_\alpha \Delta^\alpha = \begin{pmatrix} 0 & \frac{\tan U^2}{2} & -\frac{\tan U^2}{2} \\ -\frac{\tan U^2}{2} & 0 & 0 \\ \frac{\tan U^2}{2} & 0 & 0 \end{pmatrix} \Delta^1 \]

\[ + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \Delta^2. \]

That is, w. r. t. the basis \( B \)

\[ S_1 = (0, \frac{tg U^2}{2}, \frac{tg^2 U^2}{2}), S_2 = (1, 0, 0). \]

\[ M = (0, 0, p), N = (0, \sin U^2, -\cos U^2). \]
Assume that the 2-form \( f \) is the Killing metric of the Lie algebra \( \mathfrak{so}(3) \), which w. r. t. the basis \( B \) is \( \delta_{ij} \) up to a constant factor.

Next, we obtain \( R = (f(S_1, M), f(S_2, M)) = (ptg^2 \frac{U^2}{2}, 0), \tilde{R} = (f(S_1, N), f(S_2, N)) = (tg^2 \frac{U^2}{2}, 0) \). Indeed, one can verify that the 1-form \( \phi = tg^2 \frac{U^2}{2}(U^3_x dx - U^3_y dy) \) is the conservation law of PLR system with the characteristic \( R = p\tilde{R} \).

**Remark 2** It can be proved that PLR-system admits only one conservation law whose characteristic depends on \( U^1, U^2 \). It would be interesting to investigate in which cases of integrable chiral-type systems all of the characteristics depending on \( U^1, U^2, ..., U^n \) can be found by using expression (5).

**Example 2** Consider the 3-component system

\[
\begin{align*}
U^1_{xy} + U^3 U^1_y \cot U^3 - \frac{1}{\sin U^3} U^3_y U^2_x &= 0, \\
U^2_{xy} + U^3 U^2_y \cot U^3 - \frac{1}{\sin U^3} U^3_x U^1_y &= 0, \\
U^3_{xy} + U^1_y U^2_x \sin U^3 - p \sin U^3 &= 0,
\end{align*}
\]

where \( p \) is an arbitrary constant.

This system admits the Lax representation of the form (2), (3), where [6]:

\[
\begin{align*}
\tilde{A} &= \begin{pmatrix} 0 & i\lambda M^3 & -i\lambda M^2 \\
-i\lambda M^3 & 0 & i\lambda M^1 \\
i\lambda M^2 & -i\lambda M^1 & 0 \end{pmatrix}, \\
\tilde{B} &= \begin{pmatrix} 0 & \frac{1}{\chi} (\cos U^3 U^1_y + U^2_y) & -b_{31} \\
-b_{31} + (\cos U^3 U^1_y + U^2_y) & 0 & b_{23} \\
-b_{23} & 0 & 0 \end{pmatrix}, \\
M^1 &= p \sin U^3 \sin U^2, M^2 = -p \sin U^3 \cos U^2, M^3 = p \cos U^3, \\
b_{31} &= \sin U^3 \cos U^2 U^1_y - \sin U^2 U^3_y, b_{23} = -\cos U^2 U^3_y - \sin U^2 \sin U^3 U^1_y.
\end{align*}
\]

Consider the same basis \( B \) and the same 2-form \( f \) as in example 1. Then, we find \( S_1 = (\sin U^2 \sin U^3, -\cos U^2 \sin U^3, \cos U^3), S_2 = (0, 0, 1), S_3 = (\cos U^2, \sin U^2, 0) \), and \( R = \{f(S_\alpha, M)\} = \{p, p \cos U^3, 0\}, \tilde{R} = \{f(S_\alpha, M)\} = \{\cos U^3, 1, 0\} \). Now, one can see that the set \( R \) up to factor \( p \) is the characteristic of the conservation law \( \phi_1 = (\cos U^3 U^2_x + U^1_x) dx \), and the set \( \tilde{R} \) is the characteristic of the conservation law \( \phi_2 = (\cos U^3 U^1_y + U^2_y) dy \).
Remark 3. It turns out that the expressions (5) for characteristics are still valid not only for the case of chiral-type systems, but in some cases of evolution equations.

Example 3. Korteweg–de Vries equation. Write the Lax representation of KdV in the form [7]:

\[D_t \tilde{A} - D_x \tilde{B} + [\tilde{A}, \tilde{B}] = i(U_y - 6UU_x + U_{xxx}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\]

where

\[
\tilde{A} = i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i \begin{pmatrix} 0 & U \\ 1 & 0 \end{pmatrix},
\]

\[
\tilde{B} = -4\lambda^2 \tilde{A} - 2i\lambda \begin{pmatrix} -U & -iU_x \\ 0 & U \end{pmatrix} + \begin{pmatrix} U_x & iU_{xx} + 2iU^2 \\ 2iU & -U_x \end{pmatrix}.
\]

Then, we find

\[S = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g = \mathfrak{sl}(2) \text{ and, assuming } f \text{ to be the Killing metric, we obtain}
\]

\[f(S_1, \frac{\partial \tilde{A}}{\partial \lambda}) = -2V, \quad f(S_2, \frac{\partial \tilde{A}}{\partial \lambda}) = -2U \]

and

\[-2V \Delta^1 - 2U \Delta^2 = -D_t(2UV) - D_x(UV_x - VU_x).\]
3. Conclusion

In this note we prove that expressions (5) are characteristics of conservation laws for the chiral-type systems admitting a Lax representation. It would be interesting to investigate for which systems of evolution equations the same is valid.

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Appendix

Here, we give proof of lemma 1.

Let us first prove Eq. (9). The case of Eq. (10) can be proved in a similar way. Denote by $R_{1\alpha} = E_\alpha(f(D_y\bar{A}, M, ..., M))$, $R_{2\alpha} = E_\alpha(D_x\bar{B}, M, ..., M))$, $R_{3\alpha} = E_\alpha([\bar{A}, \bar{B}], M, ..., M))$. It is readily verified that collecting the terms at $U^\beta_{xy}, U^\beta_x U^\gamma_y, U^\beta_x$ in $R_{i\alpha}$, $i = 1, 2, 3$, taking into account Eqs. (7), (8), and ad-invariancy of the form $f$, we obtain the relations

$$R_{1\alpha} = Z_{i\alpha\beta} U^\beta_{xy} + T_{i\alpha\beta\gamma} U^\beta_x U^\gamma_y + W_{i\alpha\beta} U^\beta_x,$$

where

$$Z_{1\alpha\beta} = 2(p-1)f(A(\alpha, M, M, M, ..., M),$$

$$Z_{2\alpha\beta} = 2(p-1)f(B(\alpha, M, M, ..., M),$$

$$= 2(p-1)f(B(\alpha, [B, \beta], M, ..., M), M) - 2(p-1)f([A(\alpha, B)], M, ..., M),$$

$$Z_{3\alpha\beta} = -2(p-1)f([A(\alpha, B)], M, ..., M),$$

$$T_{1\alpha\beta\gamma} = (p-1)\{2f(A(\alpha, \beta, M, ..., M), f(A(\alpha, \beta, M, ..., M))$$

$$+ f(A, M, \gamma, M, ..., M) + (p-2)f(A, M, \gamma, M, \beta, M, ..., M))\},$$

$$T_{2\alpha\beta\gamma} = (p-1)\{2f(B(\alpha, \gamma, M, ..., M), f(B(\alpha, \gamma, M, ..., M))$$

$$+ f(B, M, \gamma, M, ..., M) + (p-2)f(B, M, \gamma, M, \beta, M, ..., M))\},$$

$$T_{3\alpha\beta\gamma} = f([A, B], M, ..., M) - f([A, B], \beta, M, ..., M)$$

$$- f([A, B], M, ..., M) + (p-1)f([A, B], M, M, ..., M)$$

$$- f([A, B], M, ..., M) - f([A, B], M, \gamma, M, ..., M))\},$$

$$W_{1\alpha\beta} = 0, W_{2\alpha\beta} = (p-1)(f(N, M, \alpha, M, ..., M)$$

$$- f(N, \alpha, M, \beta, M, ..., M)),$$

$$W_{3\alpha\beta} = f([A, B], M, ..., M) - f([A, B], N, M, ..., M)$$

$$+ (p-1)(f([A, B], N, M, M, ..., M)) - f([A, B], N, \beta, M, ..., M))\}.$$

We claim that all of the coefficients of the expression $R_{1\alpha} - R_{2\alpha} + R_{3\alpha}$ vanish.
In view of relations (23)-(25) and (7), we obtain the condition
\[ Z_{1\alpha\beta} - Z_{2\alpha\beta} + Z_{3\alpha\beta} = 0. \]

Further, using Eqs. (26)-(28) and the identities of the form
\[ f([A_\beta, B_\gamma], \alpha, M, ..., M) = f(A_\beta, B_\gamma, \alpha, M, ..., M) + (p - 1)f(A_\beta, M_\gamma, ..., M), \]
we obtain the equations:
\[ T_{1\alpha\beta\gamma} - T_{2\alpha\beta\gamma} + T_{3\alpha\beta\gamma} = (p - 1)f(A_\alpha, M_\gamma, M_\beta, M, ..., M) \]
\[ - (p - 2)f(A_\alpha, M_\gamma, M_\beta, M, ..., M), \]
\[ - f(B_\gamma, M_\alpha, M, ..., M) - f(B_\alpha, M_\gamma, M, ..., M) \]
\[ - (p - 2)f(B_\alpha, M_\gamma, M_\beta, M, ..., M) + f([A_\beta, B_\gamma, M_\alpha, M, ..., M]) \]
\[ + f([A_\beta, B_\gamma], M_\alpha, M, ..., M) - f([A_\beta, B_\gamma, M_\beta, M, ..., M]) - f([A_\beta, B_\alpha, M_\gamma, M, ..., M]). \]

Denote the sum of the underlined terms by \( D \). We claim that \( D = 0 \).

Indeed, using the identities
\[ f([A_\alpha, B_\gamma], M_\beta, M, ..., M) = (p - 1)f(A_\alpha, [B_\gamma, M_\beta, M, ..., M]), \]
\[ f([A_\alpha, B_\gamma], M_\beta, M, ..., M) = f(A_\alpha, [B_\gamma, M_\beta, M, ..., M]) \]
\[ + (p - 2)f(A_\alpha, M_\gamma, M_\beta, M, ..., M), \]
reduce \( D \) in the form
\[ D = (p - 1)f(A_\alpha, M_\gamma, M, ..., M) + (p - 1)f(A_\alpha, M_\gamma, M_\beta, M, ..., M) \]
\[ - f(A_\alpha, [B_\gamma, M_\beta, M, ..., M]) - (p - 2)f(A_\alpha, M_\gamma, M_\beta, M, ..., M) \]
\[ - f(A_\alpha, [B_\gamma, M_\beta, M, ..., M]). \] Thus we conclude that \( D = 0 \).

Define \( D_1 \) as follows:
\[ D_1 = (p - 1)f([A_\beta, B_\gamma], M_\alpha, M, ..., M) - f([A_\beta, B_\gamma], M_\gamma, M, ..., M) \]
\[ + f([A_\beta, B_\alpha], M, ..., M) - f([A_\beta, B_\alpha], M_\gamma, M, ..., M). \]

Using Eq. (31), one can verify that \( D_1 \) results in \( D_1 = (p - 1)f(A_\beta, M_\gamma - M_\alpha, M, ..., M), \) that is \( D_1 = 0. \)
Denote by $D_2 = T_{1\alpha\beta\gamma} - T_{2\alpha\beta\gamma} + T_{3\alpha\beta\gamma}$. Now taking into account the previous computations, $D_2$ yields that

$$D_2 = -(p - 1)\{f(B_{\gamma,\beta}, M, \ldots, M) - f(B_{\gamma,\alpha}, M, \ldots, M) + f(B_{\gamma,\beta}, M, \ldots, M) + f(B_{\alpha, M, \beta}, M, \ldots, M) + (p - 2)f(B_{\alpha, M, \beta}, M, \ldots, M)\}.$$

Further, using the following identities:

$$f(B_{\alpha, M, \beta, \gamma}, M, \ldots, M) = f(B_{\alpha, [B_{\beta, \gamma}], M}, M, \ldots, M) + f(B_{\alpha, [B_{\beta, \gamma}], M}, M, \ldots, M) = - f([B_{\beta, \gamma}, B_{\alpha}], M, \ldots, M) - (p - 2)f(B_{\alpha, M, \beta, \gamma}, M, \ldots, M),$$

$D_2$ results in

$$D_2 = -(p - 1)\{2f([B_{\gamma, \beta}, M, \ldots, M], M, \ldots, M) + 2f([B_{\alpha, \gamma}, M, \ldots, M], M, \ldots, M) + f([B_{\alpha, M, \beta}, M, \ldots, M], M, \ldots, M)\}.$$

Taking into account the identities:

$$2[B_{[\alpha, \beta], M}] = -[B_{[\alpha, M, \beta]}, M] + [B_{\beta, M, \alpha}] = -[B_{[\alpha, B_{\beta}], M}],$$

$$2f(B_{[\gamma, \beta], M, \alpha, M, \ldots, M}) = 2f(B_{[\gamma, \beta], [B_{\alpha}, M], M, \ldots, M}) = -f([B_{[\alpha, M, \beta]}, B_{[\beta, M, \alpha]}], M, \ldots, M),$$

transform $D_2$ to the form:

$$D_2 = (p - 1)\{f([B_{\gamma, \beta}, M, \alpha, M, \ldots, M], M, \ldots, M) + f([B_{\alpha, \gamma}, M, \beta, M, \ldots, M], M, \ldots, M) + f([B_{\beta, B_{\alpha}], M, \gamma, M, \ldots, M], M, \ldots, M)\}.$$

In view of relations

$$f([B_{\alpha}[B_{\gamma}, B_{\beta}], M, \ldots, M], M) = -(p - 1)f([B_{\gamma}, B_{\beta}], M, \alpha, M, \ldots, M)$$

and Jacobi’s identity, $D_2$ vanishes.
Finally, denote by

$$\triangle_1 = W_{1\alpha\beta} - W_{2\alpha\beta} + W_{3\alpha\beta}. \quad (32)$$

Substituting Eq. (29) and (30) in (32), we obtain the relations

$$\triangle_1 = f([A_{\beta}, N_{\alpha}], M, ..., M) - f([A_{\alpha}, N_{\beta}], M, ..., M).$$

These equations result in

$$\triangle_1 = f(N_{,\alpha\beta} - N_{,\beta\alpha}, M, ..., M) = 0.$$

Therefore we have proved the lemma.

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