RIEMANNIAN METRICS ON THE SPHERE WITH ZOLL FAMILIES OF MINIMAL HYPERSURFACES

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Abstract. In this paper we construct smooth Riemannian metrics on the sphere which admit smooth Zoll families of minimal hypersurfaces. This generalizes a theorem of Guillemin for the case of geodesics. The proof uses the Nash-Moser Inverse Function Theorem in the tame maps setting of Hamilton. This answers a question of Yau on perturbations of minimal hypersurfaces in positive Ricci curvature. We also consider the case of the projective space and characterize those metrics on the sphere with minimal equators.

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1. Introduction

In this paper we construct smooth Riemannian metrics on the sphere which admit $n$-parameter families of $(n-1)$-dimensional minimal hypersurfaces as the canonical family of equators. These are the intersections of $n$-dimensional linear hyperplanes of $\mathbb{R}^{n+1}$ with $S^n$ (the unit sphere) and are totally geodesic for the canonical metric $\text{can}$. They form a family of embedded $(n-1)$-dimensional spheres smoothly parametrized by the projective space $\mathbb{RP}^n$. In addition there is a unique

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equator tangent to any \((n - 1)\)-dimensional space in the tangent space of any point. We say that a Riemannian metric \(g\) on \(S^n\) is in \(\mathcal{Z}\) if there exists a smooth family \(\{\Sigma_\sigma\}_{\sigma \in \mathbb{RP}^n}\) of smoothly embedded \((n - 1)\)-dimensional minimal spheres in \((S^n, g)\) such that for each \(x \in S^n\) and each \((n - 1)\)-dimensional space \(\pi \subset T_x S^n\) there exists a unique \(\sigma \in \mathbb{RP}^n\) such that \(T_x \Sigma_\sigma = \pi\) (\(\{\Sigma_\sigma\}_\sigma\) is called a Zoll family), and that \(g\) is in \(\mathcal{Z}'\) if moreover \(\text{area}(\Sigma_\sigma, g) = \text{area}(S^{n-1}, \text{can})\) for all \(\sigma \in \mathbb{RP}^n\).

**Theorem A.** Let \(\dot{\rho}\) be a smooth odd function on the sphere \(S^n, n \geq 3\). Then there exists a smooth one-parameter family of smooth functions \(\rho(t)\) on \(S^n\), \(-\delta < t < \delta\), with \(\rho(0) = 0\) and \(\rho'(0) = \dot{\rho}\) such that \(e^{2\rho(t)}\text{can} \in \mathcal{Z}'\) for every \(t\). In fact there exists a neighborhood \(W\) of the origin in the space of smooth odd functions and a smooth map \(\lambda : W \to \mathcal{Z}'\) with \(\lambda(0) = \text{can}\) and \(D\lambda(0) \cdot \psi = 2\psi\text{can}\).

The existence of metrics in the space \(\mathcal{Z}\) (other than the constant curvature metrics) was proven by Zoll [21] in the case of the two-sphere. These examples are rotationally symmetric. Riemannian metrics with all geodesics closed and of the same length are called Zoll metrics ([2]).

Our Theorem A was proven by Guillemin [10] for the two-sphere. Funk [6] knew that the function \(\dot{\rho}\) must be odd for the deformation to exist. Funk used his transform that sends a function \(\psi\) on the sphere to the function

\[\gamma \mapsto \int_\gamma \psi,\]

where \(\gamma\) denotes a closed geodesic in the round metric. A generalized version of this map will be key in our argument.

The proof of Theorem A will use the Nash-Moser Inverse Function Theorem stated by Hamilton [12]. We will make use of a variational interpretation of the problem, differing from the dynamical approach for geodesics of Guillemin. Our proof also recovers Guillemin’s result.

The methods used in the proof of Theorem A can be applied to the case of general metric variations (see Theorem E).

We also prove that metrics like in Theorem A do not exist on \(\mathbb{RP}^3\):

**Theorem B.** The canonical metric in \(S^3\) has an open neighborhood \(W\) in the smooth topology so that if \(\rho\) is an even function with \(e^{2\rho}\text{can} \in W \cap \mathcal{Z}\), then \(\rho\) is a constant function.

In the discussion of three-manifolds with positive Ricci curvature in [20] (page 128) Yau posed the question of whether there can be continuous families of minimal surfaces when the ambient has no symmetry. The next theorem proves that this is actually possible.
Theorem C. For every $n \geq 3$, there exist metrics $e^{2\rho} \text{can} \in \mathcal{Z}$ on $S^n$ which are arbitrarily close to the canonical metric and have trivial isometry group.

Green showed that Zoll metrics on $\mathbb{RP}^2$ have constant curvature \cite{9}. The next theorem is a consequence of the classification of metrics on $S^n$ with minimal equators (Section 9.4). This uses the method of Hangan \cite{13} who classified metrics on $\mathbb{R}^n$ with minimal hyperplanes.

Theorem D. There exist metrics on $\mathbb{RP}^3$ with minimal equators and discrete isometry group.

We also obtain metrics on $\mathbb{RP}^n$, for each $n \geq 3$, with minimal equators, that do not have constant sectional curvature.

Theorem A can be generalized to include nonconformal variations:

Theorem E. Let $h$ be a smooth symmetric two-tensor on the sphere $S^n$, $n \geq 2$, of the form

$$h = f \text{can} + \mathcal{L}_X(\text{can}) + \bar{h},$$

where $f$ is a smooth odd function on $S^n$, $\mathcal{L}_X$ is the Lie derivative by a smooth vector field $X$ on $S^n$, and $\bar{h}$ is a transverse-traceless symmetric two-tensor on $(S^n, \text{can})$. Then there exists a smooth one-parameter family of Riemannian metrics $g(t)$ on $S^n$, $-\delta < t < \delta$, with $g(0) = \text{can}$ and $g'(0) = h$, such that $g(t) \in \mathcal{Z}'$ for every $t$. In fact there exists a neighborhood $W$ of the origin in the space of smooth tensors $h$ as in (1) and a smooth map

$$\lambda : W \rightarrow \mathcal{Z}'$$

with $\lambda(0) = \text{can}$ and $D\lambda(0) \cdot h = h$.

For the case of the three-sphere, Theorem E (or Theorem A) provides examples of nonhomogeneous metrics for which the uniqueness theorem of Gálvez and Mira \cite{7} applies: for such metrics any minimal sphere belongs to the Zoll family.

For $n \geq 3$, Riemannian metrics with Zoll families of totally geodesic hypersurfaces have constant sectional curvature.

2. Graphical perturbations of equators

We will start by describing the families of spheres modeled after the family of equators. Let $S^n$, $\mathbb{RP}^n$ be the sphere and projective space respectively. Given $v \in S^n$, let

$$\Sigma_v = \{ p \in S^n : \langle p, v \rangle = 0 \}$$
denote the equator that is orthogonal to \( v \). As sets, \( \Sigma_{-v} = \Sigma_v \). The assignment
\[
\sigma = [v] \in \mathbb{RP}^n \mapsto \Sigma_\sigma = \Sigma_v \subset S^n
\]
is a bijection between \( \mathbb{RP}^n \) and the set of equators of \( S^n \).

The standard metric on \( S^n \) is denoted by \( \text{can} \). The unit tangent bundle of \( S^n \) is
\[
T_1 S^n = \{ (p, v) \in S^n \times S^n : \langle p, v \rangle = 0 \}.
\]
Consider the space \( C^\infty_{*, \text{odd}}(T_1 S^n) \) of smooth functions \( \Phi \) defined on \( T_1 S^n \) that are odd with respect to the second variable:
\[
\Phi(p, -v) = -\Phi(p, v).
\]
Given \( \Phi \in C^\infty_{*, \text{odd}}(T_1 S^n) \), for each \( v \in S^n \) we define the set
\[
\Sigma_v(\Phi) = \{ \cos(\Phi(x,v))x + \sin(\Phi(x,v))v \in S^n : x \in \Sigma_v \}. \tag{2}
\]
Clearly, \( \Sigma_v(0) = \Sigma_v \). We will also use \( \Sigma_v(\Phi) \) to denote the map
\[
\Sigma_v(\Phi) : x \in \Sigma_v \mapsto \cos(\Phi(x,v))x + \sin(\Phi(x,v))v \in S^n.
\]
We think of \( \Sigma_v(\Phi) \) as the normal graph over the equator \( \Sigma_v \) of the function
\[
\Phi_v = \Phi(-, v) \in C^\infty(\Sigma_v).
\]
Notice that \( \Sigma_{-v}(\Phi) = \Sigma_v(\Phi) \) both as sets and as maps. Hence we can assign without ambiguity to each point \( \sigma = [v] \in \mathbb{RP}^n \) the hypersurface \( \Sigma_\sigma(\Phi) = \Sigma_v(\Phi) \) of \( S^n \).

Assume that \( \Phi \) has sufficiently small \( C^1 \) norm so that all maps \( \Sigma_v(\Phi) \) are smooth embeddings of the \((n - 1)\)-dimensional sphere into \( S^n \). We will require more conditions on \( \Phi \) later as needed.

Lemma 2.1. The following formulas hold:

i) The tangent space of \( \Sigma_v(\Phi) \) at the point \( y = \Sigma_v(\Phi)(x) \) consists of the vectors
\[
\cos(\Phi(x,v))u + D\Phi(x,v) \cdot (u, 0) \Sigma_v^\perp(\Phi)(x), \quad u \in T_x \Sigma_v,
\]
where
\[
\Sigma_v^\perp(\Phi)(x) = -\sin(\Phi(x,v))x + \cos(\Phi(x,v))v.
\]

ii) The unit normal of \( \Sigma_v(\Phi) \) at the point \( y = \Sigma_v(\Phi)(x) \) that points towards \( v \) is the vector
\[
N_v(\Phi)(x) = \frac{\cos(\Phi(x,v))\Sigma_v^\perp(\Phi)(x) - \nabla^{\Sigma_v} \Phi_v(x)}{\sqrt{\cos(\Phi(x,v))^2 + |\nabla^{\Sigma_v} \Phi_v(x)|^2(x)}},
\]
where \( \nabla^{\Sigma_v} \Phi_v \) is the gradient of \( \Phi_v \) in \( (\Sigma_v, \text{can}) \). In particular,
\[
N_{-v}(\Phi)(x) = -N_v(\Phi)(x) \quad \text{for all} \quad x \in \Sigma_v.
\]
iii) The Jacobian determinant of the map \( \Sigma_v(\Phi) \) at \( x \in \Sigma_v \) is
\[
|J\text{ac} \Sigma_v(\Phi)|(x) = \cos^{n-2}(\Phi(x,v))\sqrt{\cos(\Phi(x,v))^2 + |\nabla^{\Sigma_v} \Phi_v|^2(x)}.
\]

Proof. Let \( u \in T_x\Sigma_v \). Then
\[
D\Sigma_v(\Phi)_x \cdot u = \cos(\Phi(x,v))u + D\Phi_{(x,v)} \cdot (u,0)(-\sin(\Phi(x,v))x + \cos(\Phi(x,v))v),
\]
which proves \( i) \).

Note that \( \langle \Sigma_v^\perp(\Phi)(x), u \rangle = 0 \) for every \( u \in T_x\Sigma_v \) and \( |\Sigma_v^\perp(\Phi)(x)|^2 = 1 \). Hence
\[
|\cos(\Phi(x,v))\Sigma_v^\perp(\Phi)(x) - \nabla^{\Sigma_v} \Phi_v(x)|^2 = \cos(\Phi(x,v))^2 + |\nabla^{\Sigma_v} \Phi_v(x)|^2(x).
\]

Now
\[
\langle \cos(\Phi(x,v))\Sigma_v^\perp(\Phi)(x) - \nabla^{\Sigma_v} \Phi_v(x), \Sigma_v(\Phi)(x) \rangle
\]
\[= \cos(\Phi(x,v))\langle \Sigma_v^\perp(\Phi)(x), \cos(\Phi(x,v))x + \sin(\Phi(x,v))v \rangle.
\]

For \( u \in T_x\Sigma_v \),
\[
\langle \cos(\Phi(x,v))\Sigma_v^\perp(\Phi)(x) - \nabla^{\Sigma_v} \Phi_v(x), \cos(\Phi(x,v))u + D\Phi_{(x,v)} \cdot (u,0)\Sigma_v^\perp(\Phi)(x) \rangle
\]
\[= \cos(\Phi(x,v))D\Phi_{(x,v)} \cdot (u,0) - \cos(\Phi(x,v))\langle \nabla^{\Sigma_v} \Phi_v(x), u \rangle = 0,
\]
finishing the proof of \( ii) \).

Let \( \{u_i\} \subset T_x\Sigma \) be an orthonormal basis. Then
\[
g_{ij} = \langle \cos(\Phi(x,v))u_i + D\Phi_{(x,v)}(u_i,0)\Sigma_v^\perp(\Phi)(x), \cos(\Phi(x,v))u_j + D\Phi_{(x,v)}(u_j,0)\Sigma_v^\perp(\Phi)(x) \rangle
\]
\[= \cos(\Phi(x,v))^2\delta_{ij} + \langle \nabla^{\Sigma_v} \Phi_v(x), u_i \rangle \langle \nabla^{\Sigma_v} \Phi_v(x), u_j \rangle.
\]

Hence
\[
\det g_{ij} = \cos(\Phi(x,v))^{2(n-2)}\left( \cos(\Phi(x,v))^2 + |\nabla^{\Sigma_v} \Phi_v(x)|^2 \right),
\]
proving \( iii) \).

\( \square \)

2.1. Incidence sets. Given the family of hypersurfaces \( \{\Sigma_\sigma(\Phi)\} \), define its incidence set by
\[
[F(\Phi)] = \{(y,\sigma) \in S^n \times \mathbb{RP}^n : y \in \Sigma_\sigma(\Phi)\}
\]
and its orientation-inducing incidence set by
\[
F(\Phi) = \{(y,v) \in S^n \times S^n : y \in \Sigma_v(\Phi)\}.
\]
The projection of \( S^n \times S^n \) onto \( S^n \times \mathbb{RP}^n \) induces a two-to-one map \( F(\Phi) \to [F(\Phi)] \). Clearly, \( F(0) = T_1S^n \).
The set $F(\Phi)$ is the image of the smooth map

$$(x, v) \in T_1S^n \mapsto (\Sigma_v(\Phi)(x), v) \in S^n \times S^n.$$  \hspace{1cm} (3)

Since this is a $C^1$-perturbation of the inclusion of $T_1S^n$ into $S^n \times S^n$, we obtain

**Proposition 2.2.** The sets $F(\Phi)$ and $[F(\Phi)]$ are smooth, embedded, compact hypersurfaces of $S^n \times S^n$ and $S^n \times \mathbb{RP}^n$ respectively.

and

**Proposition 2.3.** The projections

$\pi_1 : (p, v) \in F(\Phi) \mapsto p \in S^n, \quad \pi_2 : (p, v) \in F(\Phi) \mapsto v \in S^n$

and

$\pi_1 : (p, \sigma) \in [F(\Phi)] \mapsto p \in S^n, \quad \pi_2 : (p, \sigma) \in [F(\Phi)] \mapsto \sigma \in \mathbb{RP}^n$

are smooth submersions.

For $p \in S^n$, we define the dual hypersurface

$\Sigma_p^*(\Phi) = \{ \sigma \in \mathbb{RP}^n : (p, \sigma) \in [F(\Phi)] \}$.

If $\Phi = 0$ these sets are the linear projective hyperplanes in $\mathbb{RP}^n$:

$\Sigma_p^* = \Sigma_p^*(0) = \{ [v] \in \mathbb{RP}^n : \langle v, p \rangle = 0 \}$,

parametrized by points $p \in S^n$.

Notice that

$p \in \Sigma_\sigma(\Phi) \iff \sigma \in \Sigma_p^*(\Phi)$.

We have $\Sigma_\sigma(\Phi) = \pi_1(\pi_2^{-1}(\sigma)) \subset S^n$ and $\Sigma_p^*(\Phi) := \pi_2(\pi_1^{-1}(p)) \subset \mathbb{RP}^n$.

2.2. **Generalized Gauss map.** It will be useful to consider $\Sigma(\Phi)$ as the map

$\Sigma(\Phi) : (x, v) \in T_1S^n \mapsto \Sigma_v(\Phi)(x) \in S^n$, 

and similarly the unit normal (with respect to $can$) as the map

$N(\Phi) : (x, v) \in T_1S^n \mapsto N_v(\Phi)(x) \in S^n$.

The generalized Gauss map of the family $\{\Sigma_\sigma(\Phi)\}$ is the map

$G(\Phi) : (x, v) \in T_1S^n \mapsto (\Sigma(\Phi)(x, v), N(\Phi)(x, v)) \in T_1S^n$.

The map $G(0)$ is the identity.

Assume that $\Phi$ has sufficiently small $C^2$ norm. Then Lemma 2.1 implies:

**Proposition 2.4.** The generalized Gauss map $G(\Phi)$ is a smooth diffeomorphism of $T_1S^n$.

**Proof.** $G(\Phi)$ is $C^1$ close to the identity map. \qed
3. THE AREA FUNCTIONAL

We denote by \( C_{\text{even}}^\infty(S^n) \) the set of smooth even functions \( h \) on \( S^n \) \((h(-v) = h(v))\) and by \( C_{\text{odd}}^\infty(S^n) \) the set of smooth odd functions \( h \) on \( S^n \) \((h(-v) = -h(v))\).

Let \( \rho \) be a smooth function on \( S^n \), and \( \Phi \in C_{\text{odd}}^\infty(T_1S^n) \). Let \( A(\rho, \Phi) \) denote the map that assigns to each \( \sigma \in \mathbb{RP}^n \) the area of the surface \( \Sigma_\sigma(\Phi) \) computed with respect to the conformal metric \( e^{2\rho} \text{can} \). Explicitly, for all \( \sigma = [v] \in \mathbb{RP}^n \),

\[
A(\rho, \Phi)(\sigma) = \text{area}(\Sigma_\sigma(\Phi), e^{2\rho} \text{can}) = \int_{\Sigma_v} e^{(n-1)\rho(\Sigma_v(\Phi)(x))} |\text{Jac}(\Sigma_v(\Phi))(x)| dA_{\text{can}}(x). \tag{4}
\]

Hence \( A(\rho, \Phi) \in C^\infty(\mathbb{RP}^n) \). It will be convenient to think of \( A(\rho, \Phi) \) as a function in \( C_{\text{even}}^\infty(S^n) \) as well, and the notation will change accordingly.

3.1. The first variation and the Euler-Lagrange operator. From the identity (4) we have

\[
A(\rho, \Phi)(v) = \int_{\Sigma_v} A(\rho)((x, v), \Phi(x, v), \nabla^{\Sigma_v} \Phi_v(x)) dA_{\text{can}}(x)
\]

for the smooth function \( A(\rho) : T_1S^n \times \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R} \) given explicitly by

\[
A(\rho)((x, v), U, L) = e^{(n-1)\rho(U + \cos(U)v)} \cos^{n-2}(U) \sqrt{\cos^2(U) + |L|^2}.
\]

Hence, for every \( \phi \in C_{\text{odd}}^\infty(T_1S^n) \) and \( v \in S^n \), the identity

\[
\frac{d}{dt}_{|t=0} A(\rho, \Phi + t\phi)(v) = \int_{\Sigma_v} \mathcal{H}(\rho, \Phi)(x, v) \phi(x, v) dA_{\text{can}}(x) \tag{5}
\]

holds for the map

\[
\mathcal{H}(\rho, \Phi)(x, v) = -\text{div}(\Sigma_v, \text{can}) \left( D_3 A(\rho)((x, v), \Phi(x, v), \nabla^{\Sigma_v} \Phi_v(x)) \right)
+ D_2 A(\rho)((x, v), \Phi(x, v), \nabla^{\Sigma_v} \Phi_v(x)), \tag{6}
\]

which we call the Euler-Lagrange operator. (In the first term on the right-hand side, \( v \) is frozen and the divergence is the divergence, at the point \( x \), of a vector field that is tangent to \( \Sigma_v \)).

In view of (6), \( \mathcal{H}(\rho, \Phi) \) is a smooth function on \( T_1S^n \) depending on up to one derivative of \( \rho \) and up to two derivatives of \( \Phi \). Note that \( \mathcal{H}(\rho, \Phi) \in C_{\text{odd}}^\infty(T_1S^n) \).
The first variation formula for the area says that
\[
\frac{d}{dt}_{|t=0} A(\rho, \Phi + t\phi)(v) = \int_{\Sigma_v(\Phi)} \mathcal{H}(\rho, \Phi)(x, v)
\cdot e^{2\rho(y)} \left( \frac{d}{dt}_{|t=0} \Sigma_v(\Phi + t\phi)(x), e^{-\rho(y)} N_v(\Phi)(x) \right) dA_{e^2\rho\text{can}}(y)
\]
\[
= \int_{\Sigma_v} \mathcal{H}(\rho, \Phi)(x, v) e^{n\rho(\Sigma_v(\Phi)(x))} \cos^{n-1}(\Phi(x, v)) \phi(x, v) dA_{\text{can}}(x),
\]
where \( \mathcal{H}(\rho, \Phi)(x, v) \) is the mean curvature of \( \Sigma_v(\Phi) \) in \( (S^n, e^{2\rho \text{can}}) \) at \( y = \Sigma_v(\Phi)(x) \). Hence
\[
\mathcal{H}(\rho, \Phi)(x, v) = \mathcal{H}(\rho, \Phi)(x, v) e^{n\rho(\Sigma_v(\Phi)(x))} \cos^{n-1}(\Phi(x, v)).
\]
The geometric meaning of the equation
\[
\mathcal{H}(\rho, \Phi) = 0
\]
is therefore that it holds if and only if all hypersurfaces \( \Sigma_v(\Phi) \) are critical points of the \( (n-1) \)-dimensional area functional of \( (S^n, e^{2\rho \text{can}}) \). In other words, \( \mathcal{H}(\rho, \Phi) = 0 \) if and only if the family \( \{ \Sigma_v(\Phi) \}_{v \in \mathbb{R}^n} \) consists of minimal hypersurfaces of \( (S^n, e^{2\rho \text{can}}) \).

3.2. The linearization of \( \mathcal{H}(\rho, \Phi) \).

**Proposition 3.1.** The following formulas hold:

i) For every \( f \in C^\infty(S^n) \),
\[
\frac{d}{dt}_{|t=0} \mathcal{H}(\rho + tf, \Phi) = (n - 1)(f \circ \Sigma(\Phi)) \mathcal{H}(\rho, \Phi)
\]
\[
+ (n - 1)(\cos \circ \Phi)^{n-1} \left( \nabla f \circ \Sigma(\Phi), N(\Phi) \right) e^{(n-1)\rho \Sigma(\Phi)}.
\]

ii) For every \( \phi \in C^\infty_{*, \text{odd}}(T_1 S^n) \) and \( v \in S^n \),
\[
\left( \frac{d}{dt}_{|t=0} \mathcal{H}(\rho, \Phi + t\phi) \right)_v = \mathcal{J}_v(\rho, \Phi)(\phi_v)
\]
for some symmetric second-order linear partial differential operator \( \mathcal{J}_v(\rho, \Phi) : C^\infty(\Sigma_v) \to C^\infty(\Sigma_v) \). Moreover, \( \mathcal{J}_{-v}(\rho, \Phi) = \mathcal{J}_v(\rho, \Phi) \).

**Proof.** From the formula of the conformal change of the mean curvature we have
\[
\mathcal{H}(\rho + tf, \Phi) = e^{-tf \circ \Sigma(\Phi)} \left( \mathcal{H}(\rho, \Phi) + (n - 1)e^{-\rho \Sigma(\Phi)} N(\Phi)(tf) \right).
\]
Hence
\[
\frac{d}{dt}|_{t=0} \hat{H}(\rho + tf, \Phi) = -(f \circ \Sigma(\Phi))\hat{H}(\rho, \Phi) + (n - 1)e^{-\rho(\Sigma(\Phi))}N(\Phi)(f).
\]
Since
\[
\mathcal{H}(\rho, \Phi)(x, v) = \hat{H}(\rho, \Phi)(x, v)e^{n\rho(\Sigma_v(\Phi)(x))}\cos^{n-1}(\Phi(x, v)),
\]
we have
\[
\frac{d}{dt}|_{t=0} \mathcal{H}(\rho + tf, \Phi) = \frac{d}{dt}|_{t=0} \hat{H}(\rho + tf, \Phi)e^{n\rho(\Sigma_v(\Phi)(x))}\cos^{n-1}(\Phi(x, v))
\]
\[
+ nf(\Sigma_v(\Phi)(x))\hat{H}(\rho, \Phi)(x, v)e^{n\rho(\Sigma_v(\Phi)(x))}\cos^{n-1}(\Phi(x, v)).
\]
Therefore
\[
\frac{d}{dt}|_{t=0} \mathcal{H}(\rho + tf, \Phi) = (n - 1)(f \circ \Sigma(\Phi))\mathcal{H}(\rho, \Phi)
\]
\[
+ (n - 1)N(\Phi)(f)e^{(n-1)\rho(\Sigma_v(\Phi)(x))}\cos^{n-1}(\Phi(x, v)).
\]

The fact that \( J_v(\rho, \Phi) \) is a second-order linear differential operator follows from equation (6). Symmetry of \( J_v(\rho, \Phi) \) follows from equation (5), since
\[
\frac{\partial}{\partial t}|_{t=0} \frac{\partial}{\partial s}|_{s=0} A(\rho, \Phi + t\phi + s\psi)(v) = \int_{\Sigma_v} \psi_v(x) J_v(\rho, \Phi) \cdot \phi_v(x)dA_{can}(x)
\]
and partial derivatives commute. (See also [19], Proposition 1.1). The identity \( J(\rho, \Phi)_{-v} = J(\rho, \Phi)_{v} \) follows from
\[
A(\rho, \Phi + t\phi + s\psi)(-v) = A(\rho, \Phi + t\phi + s\psi)(v).
\]

Remark 3.2. When \( (\rho, \Phi) = (0, 0) \), the operator \( J_v(0, 0) \) coincides with the Jacobi operator of the equator \( \Sigma_v \) as a minimal hypersurface of \( (S^n, can) \), that is
\[
J_v(0, 0) = -\Delta_{(\Sigma_v, can)} - (n - 1).
\]
In fact the graphical perturbation \( t \mapsto \Sigma_v(t\phi) \) of \( \Sigma_v(0) \) has normal speed \( \phi_v \) at \( t = 0 \). It is well-known that the kernel of the operator \( J(0, 0)_v \) consists precisely of the linear functions
\[
x \in \Sigma_v \mapsto (x, u) \in \mathbb{R} \text{ for all } u \in T_v S^n.
\]
By Fredholm alternative and regularity theory, for every
\[
\psi \in \ker(J_v(0, 0))^{\perp} \cap C^\infty(\Sigma_v)
\]
(i.e. \( L^2 \)-orthogonal to the linear functions) there exists a unique \( \phi \in \ker(J_v(0, 0))^{\perp} \cap C^\infty(\Sigma_v) \) that solves the equation \( J_v(0, 0)\phi = \psi \).
Denote by $P_v : L^2(\Sigma_v, \text{can}) \to \ker(\mathcal{J}_v(0, 0))^\perp$ the $L^2$-orthogonal projection. The adjoint $P^*_v$ is the inclusion map. We denote by $C^0_{0, \text{odd}}(T_1 S^n)$ the space of $\phi \in C^\infty_{*, \text{odd}}(T_1 S^n)$ such that $\phi_v \in \ker(\mathcal{J}_v(0, 0))^\perp$ for every $v \in S^n$.

Assume the $C^3$ norms of $\rho$ and $\Phi$ are sufficiently small.

**Proposition 3.3.** The linear operator $\mathcal{J}_v(\rho, \Phi)$ is elliptic Fredholm of index zero, and

$$\mathcal{P}_v(\rho, \Phi) : \ker(\mathcal{J}_v(0, 0))^\perp \cap C^\infty(\Sigma_v) \to \ker(\mathcal{J}_v(0, 0))^\perp \cap C^\infty(\Sigma_v)$$

given by $\mathcal{P}_v(\rho, \Phi) = P_v \circ \mathcal{J}_v(\rho, \Phi) \circ P^*_v$ is bijective. Moreover, for any fixed $\alpha \in (0, 1)$ there is a positive constant $c = c(n, \alpha)$ so that

$$\|\mathcal{P}_v(\rho, \Phi)\psi\|_{C^{0, \alpha}(\Sigma_v)} \geq c \|\psi\|_{C^{2, \alpha}(\Sigma_v)}$$

(7)

for every $\psi \in C^\infty_{0, \text{odd}}(T_1 S^n)$ and every $v \in S^n$.

**Proof.** The coefficients of the second-order part of $\mathcal{J}_v(\rho, \Phi)$ are close in $C^0$ norm to the coefficients of $\mathcal{J}_v(0, 0)$ uniformly in $v \in S^n$. This implies $\mathcal{J}_v(\rho, \Phi)$ is elliptic Fredholm and by symmetry it has index zero.

When $(\rho, \Phi) = (0, 0)$, by Fredholm alternative and regularity theory,

$$\mathcal{J}_v(0, 0) : \ker(\mathcal{J}_v(0, 0))^\perp \cap C^{2, \alpha}(\Sigma_v) \to \ker(\mathcal{J}_v(0, 0))^\perp \cap C^{0, \alpha}(\Sigma_v)$$

is a continuous bijection. There is a constant $c > 0$ depending only on $n$ and $\alpha$ such that

$$c \|\psi\|_{C^{2, \alpha}(\Sigma_v)} \leq \|\mathcal{J}_v(0, 0)\psi\|_{C^{0, \alpha}(\Sigma_v)}$$

for every $\psi \in \ker(\mathcal{J}_v(0, 0))^\perp \cap C^{2, \alpha}(\Sigma_v)$. We have

$$c \|\psi\|_{C^{2, \alpha}(\Sigma_v)} \leq \|\mathcal{J}_v(0, 0)\psi\|_{C^{0, \alpha}(\Sigma_v)} \leq \|\mathcal{P}_v(\rho, \Phi)\psi\|_{C^{0, \alpha}(\Sigma_v)} + \|\mathcal{P}_v(\rho, \Phi) - \mathcal{P}_v(0, 0)\| \|\psi\|_{C^{2, \alpha}(\Sigma_v)}.$$ 

The assumptions on $\rho$ and $\Phi$ allow us to absorb the last term into the left-hand side and prove (7) with a different constant. Elliptic regularity theory implies that

$$\mathcal{P}_v(\rho, \Phi) : \ker(\mathcal{J}_v(0, 0))^\perp \cap C^\infty(\Sigma_v) \to \ker(\mathcal{J}_v(0, 0))^\perp \cap C^\infty(\Sigma_v)$$

is a bijection.

\[\square\]

3.3. The center and solution maps. Let $\Omega^1_{\text{even}}(S^n)$ denote the set of smooth differential one-forms $\omega$ on $S^n$ ($\omega \in \Omega^1(S^n)$) that are even with respect to the antipodal map $A$, i.e. $A^*\omega = \omega$. The following definition is motivated by Remark 3.2.
Definition 3.4. The center map is the linear map

\[ C : C_{*,\text{odd}}^\infty(T_1S^n) \to \Omega_{\text{even}}^1(S^n) \]

that assigns to each \( \Psi \) the one-form

\[ C(\Psi)(v)(u) = \int_{\Sigma_v} \Psi(x,v)(x,u) dA_{\text{can}}(x) \quad \text{for all} \quad u \in T_vS^n. \]

We call \( C(\Psi) \) the center of \( \Psi \).

We have that

\[ \text{Ker} \, C = \{ \Psi \in C_{*,\text{odd}}^\infty(T_1S^n) : C(\Psi) = 0 \} = C_{0,\text{odd}}^\infty(T_1S^n). \]

The space \( C_{0,\text{odd}}^\infty(T_1S^n) \) is complemented in \( C_{*,\text{odd}}^\infty(T_1S^n) \) by a set of functions that have a simple description.

Proposition 3.5. There is a direct sum decomposition

\[ C_{*,\text{odd}}^\infty(T_1S^n) = C_{0,\text{odd}}^\infty(T_1S^n) \oplus j\Omega_{\text{even}}^1(S^n), \]

where \( j : \Omega_{\text{even}}^1(S^n) \to C_{*,\text{odd}}^\infty(T_1S^n) \) is a right-inverse of the center map that satisfies the following property: for every \( \phi \in C_{0,\text{odd}}^\infty(T_1S^n) \) and every \( \omega \in \Omega_{\text{even}}^1(S^n) \),

\[ \int_{\Sigma_v} \phi_v(x)(j\omega)_v(x)dA_{\text{can}}(x) = 0 \quad \text{for all} \quad v \in S^n. \]

Proof. Let

\[ j : \Omega_{\text{even}}^1(S^n) \to C_{*,\text{odd}}^\infty(T_1S^n) \]

denote the linear map that assigns to each \( \omega \in \Omega_{\text{even}}^1(S^n) \) the smooth function \( j\omega \) on \( T_1S^n \) defined by

\[ j\omega(x,v) = \alpha_n \omega_v(x) \quad \text{for every} \quad (x,v) \in T_1S^n, \]

where \( \alpha_n \) is a positive dimensional constant defined so that

\[ \langle X,Y \rangle = \alpha_n \int_{\Sigma_v} \langle X,x \rangle \langle x,Y \rangle dA_{\text{can}}(x) \quad \text{for all} \quad X,Y \in T_vS^n. \]

We have

\[ j\omega(x,-v) = \alpha_n \omega_{-v}(x) = -\alpha_n \omega_{-v}(-x) = -\alpha_n (A^\ast \omega)_v(x), \]

hence \( j\omega \) is odd in the second variable because \( \omega \) is even.

By duality, a one-form \( \omega \) on \( S^n \) corresponds to a tangent vector field \( X \) on \( S^n \) such that \( \omega_v(u) = \langle X(v), u \rangle \) for all \( u \in T_vS^n \). For every \( (x,v) \in T_1S^n \), we have \( (j\omega)(x,v) = \alpha_n \omega_v(x) = \alpha_n \langle X(v), x \rangle \). Hence, for every \( u \in T_vS^n \),

\[ C(j\omega)_v(u) = \alpha_n \int_{\Sigma_v} \langle X(v), x \rangle \langle x,u \rangle dA_{\text{can}}(x) = \langle X(v), u \rangle = \omega_v(u), \]
that is

\[ C(j\omega) = \omega \quad \text{for every } \omega \in \Omega_{\text{even}}^1(S^n). \]

Moreover, for every \( \phi \in C_0^{\infty}(T_1S^n) \) and every \( \omega \in \Omega_{\text{even}}^1(S^n) \),

\[
\int_{\Sigma_v} \phi_v(x)(j\omega)_v(x) dA_{\text{can}}(x) = \alpha_n C(\phi)_v(X(v)) = 0 \quad \text{for all } v \in S^n.
\]

Since \( j \) is a right-inverse of the centre map \( C \), we can write

\[
\phi = (\phi - jC(\phi)) + jC(\phi)
\]

for \( \phi \in C_0^{\infty}(T_1S^n) \) and \( C(\phi - jC(\phi)) = 0 \), which induces a decomposition of \( C_0^{\infty}(T_1S^n) \) with the required properties. \( \square \)

Notice that \( \psi \in C_0^{\infty}(T_1S^n) \) if and only if \( \psi_v \in C^{\infty}(\Sigma_v) \) lies in \( \text{Ker}(J_v(0,0)) \perp \) for all \( v \in S^n \). Hence, by Remark 3.2, there exists a unique \( \phi \in C_0^{\infty}(T_1S^n) \) so that \( J_v(0,0)(\phi_v) = \psi_v \) for all \( v \in S^n \). The smoothness of \( \phi \) in the combined variable \( (x,v) \) follows by elliptic theory and differentiation of the equation.

More generally, we consider the map

\[
\mathcal{P}(\rho, \Phi) : C_0^{\infty}(T_1S^n) \to C_0^{\infty}(T_1S^n)
\]
given by

\[
\mathcal{P}(\rho, \Phi)(\phi) = \frac{d}{dt}_{|t=0} (H(\rho, \Phi + t\phi) - jC(H(\rho, \Phi + t\phi))). \tag{8}
\]

Proposition 3.3 implies

\[
\mathcal{P}(\rho, \Phi)(\phi)(x, v) = \mathcal{P}_v(\rho, \Phi)(\phi_v)(x).
\]

The inverse of \( \mathcal{P}(\rho, \Phi) \) will be denoted by

\[
\mathcal{S}(\rho, \Phi) : C_0^{\infty}(T_1S^n) \to C_0^{\infty}(T_1S^n), \tag{9}
\]

and is called the solution map. A priori we obtain that \( \mathcal{S}(\rho, \Phi)(\psi) \) is smooth in \( x \) and continuous in \( v \). Again by implicit differentiation, using the uniqueness of the solution and elliptic theory, we get that \( \mathcal{S}(\rho, \Phi)(\psi) \in C_0^{\infty}(T_1S^n) \).

3.4. The variational constraint. Given \( (x,v) \in T_1S^n, u \in T_vS^n \), define

\[
\eta(\Phi)(x,v,u) = -\langle x,u \rangle + D\Phi_{(x,v)} \cdot (-\langle x,u \rangle v,u) - \tan \Phi(x,v) \langle \nabla^{\Sigma_v} \Phi_v(x), u \rangle. \tag{10}
\]

Note that \( \eta(\Phi)(x,v,u) \) depends linearly on \( u \) and

\[
\eta(\Phi)(x,-v,u) = \eta(\Phi)(x,v,u).
\]
Definition 3.6. The constraint map 
\[ \mathcal{K} : C_{*,\text{odd}}(T_1S^n) \times C_{*,\text{odd}}(T_1S^n) \to \Omega^1_{\text{even}}(S^n) \]
is defined by 
\[ \mathcal{K}(\Phi, \Psi)_v(u) = \int_{\Sigma_v} \Psi(x, v) \eta(\Phi)(x, v, u) dA_{\text{can}}(x) \text{ for all } u \in T_vS^n. \]

The map \( \mathcal{K} \) generalizes the center map. In fact by (10),
\[ C(\Psi) = -\mathcal{K}(0, \Psi). \]

Theorem 3.7 (Variational constraint). We have
\[ \mathcal{K}(\Phi, \mathcal{H}(\rho, \Phi)) = dA(\rho, \Phi). \]

Proof. To use identity (13), we will write \( \Sigma_v(t)(\Phi) \) (where \( v(t) \) is a variation of \( v \)) as a graph over \( \Sigma_v \) of the type \( x \mapsto \cos \Phi(\tilde{x}(t), v(t)) \tilde{x}(t) + \sin \Phi(\tilde{x}(t), v(t)) v(t) \in \Sigma_v(\Phi), \)
with \( \langle x(t), v \rangle = 0 \), so \( \Sigma_v(t)(\Phi) \) is the normal graph over \( \Sigma_v \) of \( \Gamma(t, \cdot) \).
Hence \( \Gamma(0, x) = \Phi(x, v). \)

We want to compute \( \eta = \frac{\partial}{\partial t}|_{t=0} \Gamma(0, \cdot). \) Since
\[ \sin \Gamma(t, x(t)) = -\cos \Phi(\tilde{x}(t), v(t)) \frac{t \langle x, u \rangle}{\sqrt{1 + t^2} \langle x, u \rangle^2} + \sin \Phi(\tilde{x}(t), v(t)) \frac{1}{\sqrt{1 + t^2}}, \]
it follows by differentiating that
\[
\cos \Phi(x,v)\langle x, v \rangle = -\cos \Phi(x,v)x(0) + \cos \Phi(x,v)D\Phi_{(x,v)} \cdot (-\langle x, u \rangle v, u).
\]
Hence
\[
\eta(x) + D_2\Gamma(0,x) \cdot x'(0) = -\langle x, u \rangle + D\Phi_{(x,v)} \cdot (-\langle x, u \rangle v, u).
\]
Since \( \Gamma(0, x) = \Phi(x,v) \), we have \( D_2\Gamma(0,x) \cdot x'(0) = D\Phi_{(x,v)}(x'(0),0) \).

Now we differentiate
\[
\cos \Gamma(t, x(t))x(t) = \cos \Phi(\tilde{x}(t), v(t)) \frac{1}{\sqrt{1 + t^2\langle x, u \rangle^2}}x \\
+ \sin \Phi(\tilde{x}(t), v(t)) \frac{t}{\sqrt{1 + t^2}u
\]
at \( t = 0 \). Hence
\[
-\sin \Phi(x,v)\langle x, v \rangle \eta(x) + D_2\Gamma(0,x) \cdot x'(0) \rangle x + \cos \Phi(x,v)x'(0) \\
= -\sin \Phi(x,v)D\Phi_{(x,v)} \cdot (-\langle x, u \rangle v, u)x + \sin \Phi(x,v)u.
\]
Therefore
\[
\sin \Phi(x,v)(\langle x, u \rangle x - u) + \cos \Phi(x,v)x'(0) = 0,
\]
so
\[
x'(0) = \tan \Phi(x,v)\langle x, u \rangle x + u).
\]
We conclude that
\[
\eta(x) = -\langle x, u \rangle + D\Phi_{(x,v)} \cdot (-\langle x, u \rangle v, u) \\
- \tan \Phi(x,v)D\Phi_{(x,v)}(-\langle x, u \rangle x + u, 0) \\
= \eta(x,v,u).
\]
The result follows from identity (5), since \( \Sigma_{\nu(t)}(\Phi) \) is the normal graph over \( \Sigma_{\nu} \) of \( \Gamma(t,\cdot) \) and \( \frac{\partial}{\partial t} \Gamma(0,\cdot) = \eta(\Phi)(\cdot, v, u) \):
\[
dA(\rho, \Phi)_{v}(u) = \int_{\Sigma_{\nu}} \mathcal{H}(\rho, \Phi)(x,v)\eta(\Phi)(x,v,u)dA_{can}(x).
\]

We have:

**Proposition 3.8.** The following assertions are equivalent:

i) The hypersurfaces \( \Sigma_\sigma(\Phi) \) are minimal in \((S^n, e^{2\rho}can)\), i.e.
\[
\mathcal{H}(\rho, \Phi) = 0.
\]
ii) The hypersurfaces $\Sigma_\sigma(\Phi)$ have the same area in $(S^n, e^{2\rho \text{can}})$ and there exists $\omega \in \Omega^1_{\text{even}}(S^n)$ such that

$$\mathcal{H}(\rho, \Phi) = j\omega.$$ 

**Proof.** If the hypersurfaces $\Sigma_\sigma(\Phi)$ are minimal in $(S^n, e^{2\rho \text{can}})$, Theorem 3.7 implies $A(\rho, \Phi)$ is constant. Then i) implies ii) with $\omega = 0$.

Suppose ii). By the Variational Constraint Theorem 3.7,

$$\mathcal{K}(\Phi, j\omega) = \mathcal{K}(\Phi, \mathcal{H}(\rho, \Phi)) = dA(\rho, \Phi) = 0.$$ 

Hence, writing $j\omega(x,v) = \alpha_n \langle X(v), x \rangle$,

$$\int_{\Sigma_v} -\alpha_n \langle X(v), x \rangle \langle x, u \rangle dA_{\text{can}}(x) = \int_{\Sigma_v} j\omega(x,v)\eta(\Phi)(x,v,u)dA_{\text{can}}(x) - \int_{\Sigma_v} \alpha_n \langle X(v), x \rangle (\eta(\Phi)(x,v,u) + \langle x, u \rangle) dA_{\text{can}}(x).$$

Choosing $u = X(v)$, we have

$$\langle X(v), X(v) \rangle = \int_{\Sigma_v} \alpha_n \langle X(v), x \rangle (\eta(\Phi)(x,v,X(v)) + \langle x, X(v) \rangle) dA_{\text{can}}(x).$$

Hence, by (10), we have $|X(v)|^2 \leq C \|\Phi\|_{C^1} |X(v)|^2$. If $\|\Phi\|_{C^1}$ is sufficiently small, $X = 0$ and hence $\mathcal{H}(\rho, \Phi) = j\omega = 0$. 

\[\square\]

4. The Funk transform

The Funk transform (or Funk-Radon transform) associated to the family $\{\Sigma_\sigma(\Phi)\}$ in $(S^n, e^{2\rho \text{can}})$ is the linear map that sends each smooth function $f$ on $S^n$ to the function $F(\rho, \Phi)(f)$ on $\mathbb{R}P^n$, defined by

$$F(\rho, \Phi)(f)(\sigma) = \int_{\Sigma_\sigma(\Phi)} f(y) dA_{e^{2\rho \text{can}}}(y) = \int_{\Sigma_v} f(\Sigma_v(\Phi)(x)) e^{(n-1)\rho(\Sigma_v(\Phi)(x))} |\text{Jac}(\Sigma_v(\Phi))|(x) dA_{\text{can}}(x)$$

for every $\sigma = [v] \in \mathbb{R}P^n$. The Funk transform is well-defined as a linear map

$$F(\rho, \Phi) : C^\infty(S^n) \mapsto C^\infty(\mathbb{R}P^n).$$

The transform $F = F(0,0)$ maps a smooth function $f$ on $S^n$ to the function

$$\sigma \in \mathbb{R}P^n \mapsto \int_{\Sigma_\sigma} f(x) dA_{\text{can}}(x)$$
that computes the integral of the restriction of \( f \) to equators of \( S^n \). This functional was used by Funk \[6\] and Guillemin \[10\] in their works on deformations of Zoll metrics for \( n = 2 \).

Then:

**Proposition 4.1.**

\[
F(\rho, \Phi)(f)(\sigma) = \frac{1}{n-1} \frac{d}{dt}|_{t=0} A(\rho + tf, \Phi)(\sigma) \quad \text{for all} \quad \sigma \in \mathbb{RP}^n.
\]

*Proof.* This follows from identity (4) by differentiating. \( \square \)

5. The Implicit Function Theorem

In this paper we will use the Nash-Moser Inverse Function Theorem as stated by Hamilton \[12\]. The following theorem is a modification of the Implicit Function Theorem with quadratic error (\[12\], Part III, Theorem 3.3.1). We refer the reader to that article for the definitions of the terms that appear in the statement. The proof of the next theorem also uses \[11\].

**Theorem 5.1.** Let \( F \) and \( H \) be tame Fréchet spaces and let \( \Lambda \) be a smooth tame map defined on an open set containing the origin \( U \subset F \),

\[
\Lambda : U \subset F \to H,
\]

with \( \Lambda(0) = 0 \). Suppose there is a smooth tame map \( V(f)h \) linear in \( h \)

\[
V : U \times H \to F,
\]

and a smooth tame map \( Q(f)\{h,k\} \) bilinear in \( h \) and \( k \),

\[
Q : U \times H \times H \to H,
\]

such that for all \( f \in U \) and all \( h \in H \) we have

\[
D\Lambda(f)V(f)h = h + Q(f)\{\Lambda(f), h\}.
\]

(Notice that \( V(f) \) is a right-inverse for \( D\Lambda(f) \) for all \( f \in \Lambda^{-1}(0) \)). Then there exists a neighborhood \( W \subset U \) with \( 0 \in W \) and a smooth tame map

\[
\Gamma : \text{Ker}D\Lambda(0) \cap W \to \Lambda^{-1}(0)
\]

such that

\[
\Gamma(0) = 0 \quad \text{and} \quad D\Gamma(0)v = v \quad \text{for every} \quad v \in \text{Ker}(D\Lambda(0)).
\]

*Proof.* Similarly to the proof of the aforementioned Theorem 3.3.1 in \[12\], consider the modified map

\[
G : (U \subset F) \times H \to F \times H
\]
given by
\[ G(f, h) = (f - V(f)\Lambda(f), h - \Lambda(f)) \]
([12], Part III, Lemma 3.3.2). By construction, the map \( G \) is a smooth tame map such that \( G(f, h) = (f, h) \) if and only if \( \Lambda(f) = 0 \). In other words,
\[ \Lambda^{-1}(0) \cap U = \pi_F(\text{Fix}(G)), \]
where \( \pi_F : F \times H \to F \) is the standard projection and \( \text{Fix}(G) \) denotes the set of fixed points of \( G \).

We compute
\[
DG(f, h) \cdot (g, k) = (g - V(f)[D\Lambda(f) \cdot g] - [DV(f) \cdot g]\Lambda(f),
\]
\[ k - D\Lambda(f) \cdot g). \]

Hence
\[
DG(f, h) \cdot (G(f, h) - (f, h)) = DG(f, h) \cdot (-V(f)\Lambda(f), -\Lambda(f))
\]
\[ = (-V(f)\Lambda(f) + V(f)[D\Lambda(f) \cdot V(f)\Lambda(f)]
\]
\[ + [DV(f) \cdot V(f)\Lambda(f)]\Lambda(f), -\Lambda(f) + D\Lambda(f) \cdot V(f)\Lambda(f))
\]
\[ = (-V(f)\Lambda(f) + V(f)[\Lambda(f) + Q(f)\{\Lambda(f), \Lambda(f)\}]
\]
\[ + [DV(f) \cdot V(f)\Lambda(f)]\Lambda(f), -\Lambda(f) + \Lambda(f) + Q(f)\{\Lambda(f), \Lambda(f)\})
\]
\[ = (V(f)Q(f)\{\Lambda(f), \Lambda(f)\} + [DV(f) \cdot V(f)\Lambda(f)]\Lambda(f),
\]
\[ Q(f)\{\Lambda(f), \Lambda(f)\}). \]

Therefore, for every \( x = (f, g) \in (U \subset F) \times H \),
\[
DG(x) \cdot (G(x) - x) + \tilde{Q}(x) \cdot \{G(x) - x, G(x) - x\} = 0
\]
where
\[
\tilde{Q} : (U \subset F) \times H \times (F \times H) \times (F \times H) \to F \times H
\]
is the map given by
\[
\tilde{Q}(f, h) \cdot \{(\tilde{a}, \tilde{c}), (a, c)\} =
\]
\[- ([DV(f) \cdot \{\tilde{a}\}] \cdot \{c\} + V(f)Q(f) \cdot \{\tilde{c}, c\}, Q(f) \cdot \{\tilde{c}, c\}). \]

The map \( \tilde{Q} \) is a smooth tame map that is bilinear in \((\tilde{a}, \tilde{c})\) and \((a, c)\).

Therefore the map \( G \) is a near-projection in the sense of Hamilton ([11], Section 2). As a consequence ([11]), there exists a smooth tame map
\[
P : \tilde{W} \subset (F \times H) \to F \times H
\]
deﬁned on an open neighborhood of the origin \( \tilde{W} \subset U \times H \) that is a projection:
\[
P \circ P = P,
\]
and that moreover has the same fixed point set as $G$ in $\tilde{W}$: $\text{Fix}(P) = \text{Fix}(G) \cap \tilde{W}$.

The value of the map $P$ at a point $x$ is defined explicitly by an inductive algorithm ([11]), from which we can extract some information. For example, $P(0) = 0$ follows as a consequence of the fact that $G(0) = 0$ ([11], page 26). Furthermore, whenever $DG(0) \cdot u = u$ (so that the pair $(0, u)$ is a fixed point of the tangent map $TG$) we have $DP(0) \cdot u = u$ as well ([11], Section 2.5, page 38).

Let $W = \{ f : (f, 0) \in \tilde{W} \} \subset F$ and

$$\Gamma : \text{Ker}(DA(0)) \cap W \rightarrow F$$

be defined by

$$\Gamma(f) = \pi_F(P(f, 0)).$$

(Notice that $\text{Ker}(DA(0))$ is a tame direct summand of $F$, and therefore a tame Fréchet space itself by [12], Part II, Definition 1.3.1 and Corollary 1.3.3. Take $L : \text{Ker}(DA(0)) \rightarrow F$ given by $L(f) = f$, $M : F \rightarrow \text{Ker}(DA(0))$ given by $M(f) = f - V(0)DA(0)f$ and check that $M \circ L = \text{Id}$.)

The map $\Gamma$ is a smooth tame map (as a composition of smooth tame maps, [12], Part II, Theorem 2.1.6) and its image lies in $\Lambda^{-1}(0)$, because $P(f, 0) \in \text{Fix}(P) = \text{Fix}(G) \cap \tilde{W}$. Since by definition of the near-projection $G$ we have $G(0, 0) = (0, 0)$ and $DG(0, 0) \cdot (v, 0) = (v, 0)$ for every $v \in \text{Ker}(DA(0))$, by construction of $P$ we have $P(0, 0) = (0, 0)$ and $DP(0, 0) \cdot (v, 0) = (v, 0)$. Therefore $\Gamma(0) = 0$ and

$$D\Gamma(0) \cdot v = \pi_F(v, 0) = v \quad \text{for all} \quad v \in \text{Ker}(DA(0)),$$

as we wanted to prove. \qed

The following corollary shows how Theorem 5.1 can be useful to prove Theorem A.

**Corollary 5.2.** For every $v \in \text{ker}(DA(0))$ there exists a smooth one-parameter family $u_t$, $t \in (-\delta, \delta)$, such that $u_0 = 0$, $\dot{u}_0 = v$ and

$$u_t \in \Lambda^{-1}(0) \quad \text{for all} \quad t \in (-\delta, \delta).$$

**Proof.** Take $u_t = \Gamma(tv)$, choosing $\delta > 0$ such that $tv \in W$ for all $t \in (-\delta, \delta)$. \qed

5.1. **The suitable map.** Let

$$F = C^\infty(S^n) \times C^\infty_{0, \text{odd}}(T_1S^n) \quad \text{and} \quad H = C^\infty_0(\mathbb{R}P^n) \times C^\infty_{0, \text{odd}}(T_1S^n),$$

where $C^\infty_0(\mathbb{R}P^n)$ is the set of zero average smooth functions on $\mathbb{R}P^n$. 
Assume $(\rho, \Phi) \in U$, where $U$ is a neighborhood of the origin so that a right-inverse $R(\rho, \Phi)$ of the Funk transform can be constructed. We will show later that $U$ can be taken to be a $C^{3n+4}$ neighborhood of the origin.

Define
\[ \Lambda = (\Lambda_1, \Lambda_2) : (U \subset F) \rightarrow H \]
by setting
\[ \Lambda_1(\rho, \Phi) = A(\rho, \Phi) - \int_{\mathbb{R}P^n} A(\rho, \Phi)(\sigma) dA_{\text{can}}(\sigma) \]  
(12)
and
\[ \Lambda_2(\rho, \Phi) = H(\rho, \Phi) - jC(H(\rho, \Phi)). \]  
(13)

As a consequence of Proposition 3.8,
\[ \Lambda(\rho, \Phi) = 0 \iff H(\rho, \Phi) = 0. \]
Moreover, $\Lambda(0,0) = (0,0)$ since the equators are minimal hypersurfaces in $(S^n, \text{can})$.

5.2. The approximate right-inverse and the quadratic error.
To apply Theorem 5.1 to the map $\Lambda$ we will find a right-inverse for $D\Lambda$ modulo a quadratic error.

From (12) and Proposition 4.1,
\[ D_1 \Lambda_1(\rho, \Phi) \cdot f = (n-1) \left( F(\rho, \Phi)(f) - \int_{\mathbb{R}P^n} F(\rho, \Phi)(f)(\sigma) dV_{\text{can}}(\sigma) \right), \]
\[ D_2 \Lambda_1(\rho, \Phi) \cdot \phi = D_2 A(\rho, \Phi) \cdot \phi - \int_{\mathbb{R}P^n} \left( D_2 A(\rho, \Phi) \cdot \phi \right)(\sigma) dV_{\text{can}}(\sigma), \]  
(14)
where, by the definition of the Euler-Lagrange operator $H(\rho, \Phi),$
\[ (D_2 A(\rho, \Phi) \cdot \phi)([v]) = \int_{\Sigma_v} H(\rho, \Phi)(x,v) \phi(x,v) dA_{\text{can}}(x). \]
From the second part of Proposition 3.5, for all $\phi \in C_{0,\text{odd}}^\infty(T_1 S^n)$ and $v \in S^n$ we have
\[ (D_2 A(\rho, \Phi) \cdot \phi)([v]) = \int_{\Sigma_v} \Lambda_2(\rho, \Phi)(x,v) \phi(x,v) dA_{\text{can}}(x). \]
Likewise, from (13) and (8) we have
\[ D_1 \Lambda_2(\rho, \Phi) \cdot f = D_1 H(\rho, \Phi) \cdot f - jC(D_1 H(\rho, \Phi) \cdot f) \]  
(15)
and

\[ D_2 \Lambda_2(\rho, \Phi) \cdot \phi = \mathcal{P}(\rho, \Phi)(\phi). \tag{16} \]

Let \( S(\rho, \Phi) \) be the solution map as in (9). We define

\[ V = (V_1, V_2) : (U \subset F) \times H \to F \]

by

\[ V_1(\rho, \Phi) \cdot (b, \psi) = \frac{1}{n-1} \mathcal{R}(\rho, \Phi)(b) \]

and, setting \( f = V_1(\rho, \Phi) \cdot (b, \psi), \)

\[ V_2(\rho, \Phi) \cdot (b, \psi) = S(\rho, \Phi)(\psi - (D_1 \mathcal{H}(\rho, \Phi) \cdot f - jC(D_1 \mathcal{H}(\rho, \Phi) \cdot f)). \]

By construction, \( V(\rho, \Phi) \) is a linear map for all \((\rho, \Phi) \in U\).

We now show that \( V \) is a right-inverse for \( \mathcal{D} \Lambda \) modulo a quadratic error. Consider the map

\[ Q = (Q_1, Q_2) : (U \subset F) \times H \times H \to H, \]

bilinear in \((\tilde{b}, \tilde{\psi}), (b, \psi)\), defined by

\[ Q_1(\rho, \Phi) \cdot \{(\tilde{b}, \tilde{\psi}), (b, \psi)\}([v]) = \int_{\Sigma_v} \tilde{\psi}(x, v)(V_2(\rho, \Phi) \cdot (b, \psi))(x, v)dA_{can}(x) \]

\[ - \int_{\mathbb{R}^n} \left( \int_{\Sigma_v} \tilde{\psi}(x, v)(V_2(\rho, \Phi) \cdot (b, \psi))(x, v)dA_{can}(x) \right) dV_{can}(\sigma) \]

and

\[ Q_2(\rho, \Phi) \cdot \{(\tilde{b}, \tilde{\psi}), (b, \psi)\} = 0. \]

We have, with \( V_i = V_i(\rho, \Phi)(b, \psi), \)

\[ D\Lambda_1(\rho, \Phi) \cdot V(\rho, \Phi)(b, \psi) = D_1\Lambda_1(\rho, \Phi) \cdot V_1 + D_2\Lambda_1(\rho, \Phi) \cdot V_2 \]

\[ = (n - 1) \left( \mathcal{F}(\rho, \Phi)(V_1) - \int_{\mathbb{R}^n} \mathcal{F}(\rho, \Phi)(V_1)(\sigma)dV_{can}(\sigma) \right) \]

\[ + D_2\mathcal{A}(\rho, \Phi) \cdot V_2 - \int_{\mathbb{R}^n} (D_2\mathcal{A}(\rho, \Phi) \cdot V_2)(\sigma)dV_{can}(\sigma) \]

\[ = b \]

\[ + \int_{\Sigma_v} \Lambda_2(\rho, \Phi)(x, v)V_2(x, v)dA_{can}(x) \]

\[ - \int_{\mathbb{R}^n} \left( \int_{\Sigma_v} \Lambda_2(\rho, \Phi)(x, v)V_2(x, v)dA_{can}(x) \right) dV_{can}(\sigma) \]

\[ = b + Q_1(\rho, \Phi)(\Lambda(\rho, \Phi), (b, \psi)). \]
And
\[ D\Lambda (\rho, \Phi) \cdot V (\rho, \Phi)(b, \psi) = D_1 \Lambda_2 (\rho, \Phi) \cdot V_1 + D_2 \Lambda_2 (\rho, \Phi) \cdot V_2 \]
\[ = D_1 \mathcal{H}(\rho, \Phi) \cdot V_1 - jC(D_1 \mathcal{H}(\rho, \Phi) \cdot V_1) + \mathcal{P}(\rho, \Phi)(V_2) \]
\[ + \psi - (D_1 \mathcal{H}(\rho, \Phi) \cdot V_1 - jC(D_1 \mathcal{H}(\rho, \Phi) \cdot V_1)) \]
\[ = \psi \]
\[ = \psi + Q_2(\rho, \Phi)(\Lambda(\rho, \Phi), (b, \psi)). \]

Therefore we deduce that
\[ D\Lambda (\rho, \Phi) \cdot V (\rho, \Phi)(b, \psi) = (b, \psi) + Q(\rho, \Phi) \cdot \{\Lambda(\rho, \Phi), (b, \psi)\} \]
holds for all functions \((\rho, \Phi) \in U\) and \((b, \psi) \in H\), which is what we wanted.

The tameness and smoothness properties of all maps and spaces involved will be proven later.

6. ON SOME INTEGRAL OPERATORS

Let \( T_1 \mathbb{R}P^n \) denote the unit tangent bundle of \((\mathbb{R}P^n, can)\). Set
\[ \Omega = T_1 \mathbb{R}P^n \times [-\pi/4, \pi/4]/\sim \quad \text{where } ((\sigma, \theta), t) \sim ((\sigma, -\theta), -t) \]
and let
\[ \Omega_0 = \{[((\sigma, \theta), 0)] \in \Omega : (\sigma, \theta) \in T_1 \mathbb{R}P^n\} \subset \Omega. \]

If \( \Delta \) is the diagonal of \( \mathbb{R}P^n \times \mathbb{R}P^n \), we glue \( \Omega \) to \( \mathbb{R}P^n \times \mathbb{R}P^n \setminus \Delta \) by identifying \( \{((\sigma, \theta), t)\} \) with \( (\sigma, \exp_\sigma(t\theta)) \). The resulting smooth compact manifold is denoted by \( B(\mathbb{R}P^n \times \mathbb{R}P^n) \).

Since \( \mathbb{R}P^n \times \mathbb{R}P^n \setminus \Delta \) is an open and dense set in \( B(\mathbb{R}P^n \times \mathbb{R}P^n) \), smooth functions on \( B(\mathbb{R}P^n \times \mathbb{R}P^n) \) are uniquely determined by their restrictions to \( \mathbb{R}P^n \times \mathbb{R}P^n \setminus \Delta \).

The distance in \( S^n \) or \( \mathbb{R}P^n \) computed with respect to the canonical metric is denoted by \( d \). Associated to any \( k \in C^\infty(B(\mathbb{R}P^n \times \mathbb{R}P^n)) \) there exists a uniquely defined kernel
\[ K \in C^\infty(\mathbb{R}P^n \times \mathbb{R}P^n \setminus \Delta), \quad K(\sigma, \tau) = \frac{k(\sigma, \tau)}{\eta(d(\sigma, \tau))}, \]
where \( \eta : \mathbb{R} \to \mathbb{R} \) is a nondecreasing smooth function with \( \eta(t) = t \) for \( t \leq \frac{\pi}{5} \) and \( \eta(t) = \frac{\pi}{4} \) for \( t \geq \frac{2\pi}{5} \). We will consider the operator that associates to every \( f \in C^\infty(\mathbb{R}P^n) \) the function
\[ L(k)(f)(\sigma) = \int_{\mathbb{R}P^n} K(\sigma, \tau)f(\tau)dV_{\text{can}}(\tau), \quad \sigma \in \mathbb{R}P^n. \]
Since for each $\sigma \in \mathbb{RP}^n$ the function $\tau \mapsto K(\sigma, \tau)$ is in $L^1(\mathbb{RP}^n)$, the function $L(k)(f)$ is well-defined and bounded.

Given a compact manifold $M$ and $s \in \mathbb{R}$, $H^s(M)$ denotes the Sobolev space (see Section 1.3 of [8]) with norm denoted by $||f||_s$.

**Theorem 6.1.** For every $k \in C^\infty(B(\mathbb{RP}^n \times \mathbb{RP}^n))$, $L(k)$ is a pseudo-differential operator of order $1 - n$. Thus

$$L(k) : C^\infty(\mathbb{RP}^n) \to C^\infty(\mathbb{RP}^n)$$

and there are bounded extensions to Sobolev spaces

$$L(k) : H^s(\mathbb{RP}^n) \to H^{s+n-1}(\mathbb{RP}^n), \quad s \in \mathbb{R}.$$

Moreover, there is some positive constant $c_n$ so that

$$||L(k)(f)||_{n-1} \leq c_n||k||_{C^{3n}}||f||_0 \quad \text{for all } f \in H^0(\mathbb{RP}^n). \quad (17)$$

Similarly, for $q \in \mathbb{N}$ there is some positive constant $c_{n,q}$ such that

$$||L(k)(f)||_{n-1+q} \leq c_{n,q}||k||_{C^{3n+q}}||f||_q \quad \text{for all } f \in H^q(\mathbb{RP}^n). \quad (18)$$

Finally, suppose that for some $k_0 \in C^\infty(B(\mathbb{RP}^n \times \mathbb{RP}^n))$ $L(k_0)$ is elliptic and $L(k_0) : H^0(\mathbb{RP}^n) \to H^{n-1}(\mathbb{RP}^n)$ is invertible. Then there exist constants $c > 0$ and $\eta_0 > 0$ so that for all $k \in C^\infty(B(\mathbb{RP}^n \times \mathbb{RP}^n))$ with $||k - k_0||_{3n} < \eta_0$,

$L(k)$ is elliptic, $L(k) : H^0(\mathbb{RP}^n) \to H^{n-1}(\mathbb{RP}^n)$ is invertible, and

$$||L(k)^{-1}(g)||_0 \leq c||g||_{n-1} \quad \text{for all } g \in H^{n-1}(\mathbb{RP}^n). \quad (19)$$

Also $L(k) : H^q(\mathbb{RP}^n) \to H^{n-1+q}(\mathbb{RP}^n)$ is invertible for $q \geq 0$, and there exist positive constants $\eta_0,q, c'_{n,q}, c''_{n,q}$, such that if $||k - k_0||_{3n+q} < \eta_{0,q}$, then

$$||L(k)^{-1}(g)||_q \leq c'_{n,q}||g||_{n-1+q} \quad \text{for all } g \in H^{n-1+q}(\mathbb{RP}^n). \quad (20)$$

If $||k' - k_0||_{3n+q} < \eta_{0,q}$, then

$$||(L(k)^{-1} - L(k')^{-1})(g)||_q \leq c''_{n,q}||k - k'||_{C^{3n+4}}||g||_{n-1+q} \quad (21)$$

for all $g \in H^{n-1+q}(\mathbb{RP}^n)$.

**Proof.** Let $\chi$ be a smooth cut-off function that vanishes outside $[0, \pi/4)$ and is equal to one in $[0, \pi/5)$. Write the kernel $K$ as the sum of

$$K_1(\sigma, \tau) = \chi(d(\sigma, \tau))K(\sigma, \tau) \quad \text{and} \quad K_2(\sigma, \tau) = (1-\chi(d(\sigma, \tau)))K(\sigma, \tau).$$
Since $K_2$ vanishes near $\Delta$ and is therefore smooth on all of $\mathbb{R}P^n \times \mathbb{R}P^n$, the integral operator associated to $K_2$ is a smoothing operator. By differentiating we get that the operator

$$L_2(f)(\sigma) = \int_{\mathbb{R}P^n} K_2(\sigma, \tau)f(\tau)dV_{\text{can}}(\tau)$$

satisfies

$$||L_2(f)||_{n-1} \leq c_n'||k||_{C^{n-1}}||f||_0$$

for $f \in H^0(\mathbb{R}P^n)$. Thus, in order to show that $L(k)$ is a pseudodifferential operator satisfying (17), it suffices to consider the operator $L_1$ defined by

$$L_1(f)(\sigma) = \int_{\mathbb{R}P^n} K_1(\sigma, \tau)f(\tau)dV_{\text{can}}(\tau).$$

Let $\{\eta_i\}_i$ be a partition of unity subordinated to a finite covering of $\mathbb{R}P^n$ by geodesic balls of some radius so that each pair of balls is either disjoint or contained in a ball of radius $\pi/25$. We have $L_1 = \sum_{i,j} \eta_i L_1 \eta_j$.

For those $i,j$ corresponding to disjoint balls we have that

$$(\eta_i L_1 \eta_j)(f)(\sigma) = \int_{\mathbb{R}P^n} \eta_i(\sigma)K_1(\sigma, \tau)\eta_j(\tau)f(\tau)dV_{\text{can}}(\tau)$$

is a smoothing operator with kernel $(\sigma, \tau) \mapsto \eta_i(\sigma)K_1(\sigma, \tau)\eta_j(\tau)$. As in the case of $L_2$ one has

$$||(\eta_i L_1 \eta_j)(f)||_{n-1} \leq c''_n||k||_{C^{n-1}}||f||_0$$

for $f \in H^0(\mathbb{R}P^n)$ for some positive $c''_n$.

For those $i,j$ corresponding to intersecting balls we have that there is a geodesic ball of radius $\pi/25$ containing the supports of $\eta_i, \eta_j$. Hence we consider the following situation.

Let $\phi : \mathbb{R}^n \to \mathbb{R}P^n$ be the exponential map based on a chosen but otherwise arbitrary point $\tilde{\sigma}$ and set $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ to be, respectively, the open balls of radius $\pi/14$ and $\pi/25$ centered at the origin. The map $\phi$ restricted to $U$ is a diffeomorphism onto its image. Given $\eta, \eta' \in C^\infty_c(\phi(V))$, a calculation in polar coordinates in this coordinate system shows that for every $h \in C^\infty_c(V)$ we have:

$$(\eta' L_1 \eta)(h \circ \phi^{-1})(\sigma) = \eta'(\sigma)\int_{\mathbb{R}^n} K_1(\sigma, \phi(y))\frac{\sin(|y|^{n-1})}{|y|^{n-1}}\eta(\phi(y))h(y)dy.$$  

Claim. The function $u : U \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ given by

$$u(x, w) = |w|\eta(\phi(x - w))K_1(\phi(x), \phi(x - w)) \left(\frac{\sin|x - w|}{|x - w|}\right)^{n-1}$$
if $|w| < 2\pi/5$ and otherwise vanishing is smooth. Moreover, the map

$$ (x, \theta, t) \in U \times S^{n-1} \times (0, +\infty) \mapsto u(x, t\theta) \in \mathbb{R} $$

has a smooth extension to $U \times S^{n-1} \times [0, +\infty)$.

Suppose $|w| < 2\pi/5$. Then $|x - w| < 2\pi/5 + \pi/14 < \pi/2$. Since \( \phi \) is injective in the open ball of radius \( \pi/2 \) centered at the origin, \( \phi(x) = \phi(x - w) \) if and only if \( w = 0 \). Hence

$$ (x, w) \mapsto K_1(\phi(x), \phi(x - w)) $$

is smooth on $U \times (B_{2\pi/5}(0) \setminus \{0\})$. To prove smoothness on $U \times \mathbb{R}^n \setminus \{0\}$, since $\sin(t)/t = F(t^2)$ for some smooth function $F$, it is enough to show that the expression defining $u(x, w)$ for $|w| < 2\pi/5$ vanishes in a neighborhood of $|w| = 2\pi/5$. If $|w| > 2\pi/5 - \delta$, then $|x - w| \geq |w| - |x| \geq 2\pi/5 - \delta - \pi/14 > \pi/25$ if $\delta$ is sufficiently small. Hence $\eta(\phi(x - w)) = 0$ so $u(x, w) = 0$.

Let $(x, w) \in U \times B_{2\pi/5}(0)$. Then $d(\phi(x), \phi(x - w)) < |w| < \pi/2$, and we can consider the vector $w'(x, w) \in T_{\phi(x)}\mathbb{RP}^n$ uniquely defined by the identities

$$ \phi(x - w) = \exp_{\phi(x)}(w'(x, w)) \text{ and } |w'(x, w)| = d(\phi(x), \phi(x - w)). $$

The function

$$ (x, \theta, t) \in U \times S^{n-1} \times (-2\pi/5, 2\pi/5) \mapsto w'(x, \theta, t) = w'(x, t\theta) $$

is smooth and satisfies $w'(x, \theta, 0) = 0$. Hence there exists a smooth map $(x, \theta, t) \mapsto \tilde{w}(x, \theta, t)$ such that $w'(x, \theta, t) = t\tilde{w}(x, \theta, t)$. Since $d\phi_x$ is an isomorphism, we have that $w(x, \theta, 0) \neq 0$ for every $(x, \theta) \in U \times S^{n-1}$. For $t > 0$, $|w'(x, \theta, t)| = t|\tilde{w}(x, \theta, t)|$. Hence the map

$$ (x, \theta, t) \in U \times S^{n-1} \times (0, 2\pi/5) \mapsto t\tilde{w}(x, \theta, t) = |w'(x, \theta, t)| $$

extends smoothly to $U \times S^{n-1} \times [0, 2\pi/5)$. Since

$$ u(x, t\theta) = tK(\phi(x), \phi(x - t\theta))\eta(\phi(x - t\theta)) (F(|x - t\theta|^2))^{n-1} $$

$$ = \frac{k(\phi(x), \phi(x - t\theta)) \eta(\phi(x - t\theta)) (F(|x - t\theta|^2))^{n-1}}{\tilde{w}(x, \theta, t)} $$

$$ = \frac{k(\phi(x), \phi(x - t\theta)) \eta(\phi(x - t\theta)) (F(|x - t\theta|^2))^{n-1}}{|\tilde{w}(x, \theta, t)|} $$

for $0 < t < \pi/5$, we can extend smoothly to $t \in [0, \pi/5)$. This finishes the proof of the claim (see also proof of Lemma 2.1 of [14]).
It follows from the claim that we can apply Lemma 2.3 of [14] to the function $u$ above and conclude that the function

$$a(x, \xi) = \chi(4|x|) \int_{\mathbb{R}^n} \frac{u(x, w)}{|w|} e^{-i\langle \xi, w \rangle} dw \quad (22)$$

satisfies the inequalities

$$|\partial^\alpha_x \partial^\beta_x a(x, \xi)| \leq C||k||_{C^{n+|\alpha|+|\beta|}}(1 + |\xi|)^{1-n-|\alpha|} \quad (23)$$

for every multi-index $\alpha, \beta \in \mathbb{N}^n$ and every $(x, \xi) \in U \times \mathbb{R}^n \setminus \{0\}$, where $C$ depends only on $|\alpha|, |\beta|$, and $n$ (and the chosen $\chi$). The specific bound $C||k||_{C^{n+|\alpha|+|\beta|}}$ in (23) follows from inspecting the proof.

Hence $a$ is a symbol of order $1-n$ (see [8, Section 1.2] for definition) with $x$ support in $U$.

The Fourier transform (see [8]) of a function $f$ in the Schwartz space is defined as

$$F(f)(\xi) = c \cdot \int_{\mathbb{R}^n} f(x) e^{-i\langle \xi, x \rangle} dx,$$

for some positive dimensional constant $c$ (which might depend on the line).

For $\sigma = \phi(x), x \in V$, we have

$$(\eta' L_1 \eta)(h \circ \phi^{-1})(\phi(x))$$

$$= \eta'(\phi(x)) \int_{\mathbb{R}^n} K_1(\phi(x), \phi(y)) \frac{\sin(|y|)^n}{|y|^{n-1}} \eta(\phi(y)) h(y) dy$$

$$= \eta'(\phi(x)) \int_{\mathbb{R}^n} \frac{u(x, x-y)}{|x-y|} h(y) dy.$$

By using Fourier’s Inversion Formula and Fubini we have that for all $h \in C^\infty_c(V)$,

$$(\eta' L_1 \eta)(h \circ \phi^{-1})(\phi(x))$$

$$= c \cdot \eta'(\phi(x)) \int_{\mathbb{R}^n} \frac{u(x, x-y)}{|x-y|} \left( \int_{\mathbb{R}^n} e^{i\langle \xi, y \rangle} F(h)(\xi) d\xi \right) dy$$

$$= c \cdot \int_{\mathbb{R}^n} \eta'(\phi(x)) a(x, \xi) e^{i\langle \xi, x \rangle} F(h)(\xi) d\xi.$$

Hence $\eta' L_1 \eta$ is a pseudodifferential operator of order $1-n$ localized in $\phi(V)$, and hence a pseudodifferential operator of order $1-n$ in $\mathbb{R}^n$. Moreover, an inspection of the proof of Lemma 1.2.1 of [8] shows that, if for some constant $C$ we have

$$|\partial^\beta_x a(x, \xi)| \leq C(1 + |\xi|)^{1-n}$$
for all \((x, \xi)\) and all multi-index \(\beta\) with \(|\beta| \leq 2n\), then there is a dimensional constant \(d_n\) so that

\[
\|\eta' L_1 \eta(f)\|_{n-1} \leq d_n C \|f\|_0 \quad \text{for all } f \in H^0(\mathbb{R}^n).
\]

Thus we deduce (17) from the estimate above combined with (23). Similarly for (18).

This shows that \(L(k)\) is a pseudo-differential operator of order \(1 - n\) on \(\mathbb{R}^n\) satisfying (17). Hence \(L(k)\) maps smooth functions into smooth functions. The operator \(L(k)\) admits bounded extensions from the Sobolev spaces \(H^s(\mathbb{R}^n)\) to \(H^{s+n-1}(\mathbb{R}^n)\) for all \(s \in \mathbb{R}\), as proven in Lemma 1.3.4 of [8].

Let us now suppose that \(L(k_0)\) is elliptic and \(L(k_0) : H^0(\mathbb{R}^n) \to H^{n-1}(\mathbb{R}^n)\) is invertible. Then \(L_1(k_0)\) is elliptic because \(L_2(k_0)\) is smoothing. Let \(\eta_l(\phi(x))k_0^{(i)}(x, \xi)\) denote the symbol of \(\eta_lL_1(k_0)\eta_l\) which is computed as in (22). The estimate (23) applied to \(L(k - k_0)\) implies that for all \((x, \xi) \in U \times \mathbb{R}^n \setminus \{0\}\)

\[
|a^{(i)}(x, \xi) - a_0^{(i)}(x, \xi)| \leq C \|k - k_0\|_{C^0}(1 + |\xi|)^{1-n}.
\]

Hence, assuming that \(\|k - k_0\|_{C^0} < \delta\) for some small \(\delta\), the ellipticity of \(L_1(k_0)\) implies the ellipticity of \(L_1(k)\) and hence the ellipticity of \(L(k)\).

The set of invertible operators is open in the operator norm, hence \(L(k) : H^0(\mathbb{R}^n) \to H^{n-1}(\mathbb{R}^n)\) is invertible if \(\delta\) is sufficiently small. Since \(L(k_0)\) is invertible, there is a positive constant \(c\) such that

\[
\|L(k_0)(f)\|_{n-1} \geq 2c \|f\|_0 \quad \text{for all } f \in H^0(\mathbb{R}^n).
\]

By (17), if \(\|k - k_0\|_{C^0} < c/c_n\) then

\[
2c \|f\|_0 \leq \|L(k_0)(f)\|_{n-1} \leq \|L(k - k_0)(f)\|_{n-1} + \|L(k)(f)\|_{n-1} \leq c \|f\|_0 + \|L(k)(f)\|_{n-1}.
\]

This shows (19).

Since \(L(k) : H^0(\mathbb{R}^n) \to H^{n-1}(\mathbb{R}^n)\) is invertible, its index as a Fredholm operator vanishes. But the index of \(L(k) : H^q(\mathbb{R}^n) \to H^{n-1+q}(\mathbb{R}^n)\) does not depend on \(q\) (Lemma 1.4.5 of [8]). The fact that \(L(k)\) is injective on \(H^q(\mathbb{R}^n)\), \(q \geq 0\), implies the invertibility of \(L(k) : H^q(\mathbb{R}^n) \to H^{n-1+q}(\mathbb{R}^n)\). The inequality (20) follows as \(19\). The inequality (21) follows by using the identity

\[
L(k)^{-1} - L(k')^{-1} = L(k)^{-1} \circ L(k' - k) \circ L(k')^{-1},
\]

finishing the proof of the theorem.
7. The dual of the Funk transform

7.1. The dual family. The generalized Gauss map $G(\Phi)$ defined in Section 2.2 is a diffeomorphism and so we can consider

$$G^{-1}(\Phi) : (q, w) \in T_1 S^n \mapsto (\Upsilon_q(\Phi)(w), \Xi_q(\Phi)(w)) \in T_1 S^n.$$ 

Since $N_{-v}(\Phi)(p) = -N_v(\Phi)(p)$ for every $(p, v) \in T_1 S^n$,

$$\Upsilon_q(\Phi)(-w) = \Upsilon_q(\Phi)(w) \quad \text{and} \quad \Xi_q(\Phi)(-w) = -\Xi_q(\Phi)(w) \quad (24)$$

for every $(q, w) \in T_1 S^n$.

Clearly, $[\Xi_q(0)(w)] = [w] \in \mathbb{RP}^n$ for every $w \in \Sigma_q$, or equivalently $[w] \in \Sigma^*_q$. More generally:

**Proposition 7.1.** For $q \in S^n$, the map

$$[\Xi_q(\Phi)] : [w] \in \Sigma^*_q \mapsto [\Xi_q(\Phi)(w)] \in \mathbb{RP}^n$$

is a well-defined smooth embedding whose image is the dual hypersurface $\Sigma^*_q(\Phi)$.

**Proof.** The map is well-defined as a consequence of (24). Let $[w] \in \Sigma^*_q$. Define $(p, v) = G^{-1}(\Phi)(q, w) = (\Upsilon_q(\Phi)(w), \Xi_q(\Phi)(w))$. Then

$$(q, w) = G(\Phi)(p, v) = (\Sigma(\Phi)(p, v), N(\Phi)(p, v)).$$

Therefore $q \in \Sigma_v(\Phi)$, and hence $[v] \in \Sigma^*_v(\Phi)$. This shows $[\Xi_q(\Phi)(w)] \in \Sigma^*_q(\Phi)$. Similarly one can prove that any element of $\Sigma_q^*(\Phi)$ is of the form $[\Xi_q(\Phi)(w)]$ for some $[w] \in \Sigma_q^*$.

Since $G(\Phi)$ is $C^1$ close to the identity map, we have that $G^{-1}(\Phi)$ is also $C^1$ close to the identity. Hence $[\Xi_q(\Phi)]$ is a $C^1$ perturbation of the inclusion map and therefore it is a smooth embedding.

□

Similarly the map

$$\Xi_q(\Phi) : w \in \Sigma_q \mapsto \Xi_q(\Phi)(w) \in S^n$$

is an embedding of the $(n - 1)$-dimensional sphere $\Sigma_q$ into $S^n$. Let

$$N^*_q(\Phi) : w \in \Sigma_q \mapsto N^*_q(\Phi)(w) \in T_{\Xi_q(\Phi)(w)} S^n$$

be the unique unit normal vector field along $\Xi_q(\Phi)$ with $\langle N^*_q(\Phi)(w), q \rangle > 0$ for all $(q, w) \in T_1 S^n$. The map $N^*(\Phi)(q, w) = N^*_q(\Phi)(w)$ is a smooth function of $(q, w) \in T_1 S^n$ satisfying $N^*(\Phi)(q, -w) = N^*(\Phi)(q, w)$. And $N^*_q(0)(w) = q$ for all $w \in \Sigma_q$. 

7.2. Intersections of hypersurfaces. Since the map $G(\Phi)$ is a diffeomorphism, any two hypersurfaces $\Sigma_\sigma(\Phi)$ and $\Sigma_\tau(\Phi)$, $\sigma \neq \tau$, intersect transversely. For the purposes of proving tame estimates we need to have explicit parametrizations of the intersections $\Sigma_\sigma(\Phi) \cap \Sigma_\tau(\Phi)$.

For each $v \in S^n$ we first construct $I_v(\Phi) \in C^\infty(S^n)$ so that $I_v(\Phi)$ coincides with the sine of the signed distance function to $\Sigma_v(\Phi)$ in its neighborhood and such that $I_v(\Phi)^{-1}(0) = \Sigma_v(\Phi)$. The construction needs to depend smoothly on $\Phi$, and some care is required.

Fix $\zeta \in C^\infty_c(\mathbb{R})$ an even cutoff function that is one near zero and zero outside the interval $[-\pi/5, \pi/5]$. Consider the smooth map

$$D(\Phi) : S^n \times S^n \to S^n$$

so that if $v \in S^n$, $x \in \Sigma_v$, and $|t| \leq 2\pi/5$ then

$$D(\Phi)(\exp_x(tv), v) = \cos(t)\Sigma_v(\zeta(t)\Phi)(x) + \sin(t)N_v(\zeta(t)\Phi)(x).$$

Notice that $D(\Phi)(\cdot, v)$ is the identity map outside a neighborhood of $\Sigma_v$ (if $|t| \geq \pi/5$) and that $D(0)(p, v) = p$ for all $(p, v) \in S^n \times S^n$. We have $D(\Phi)(x, v) = \Sigma_v(\Phi)(x)$ if $x \in \Sigma_v$ (case $t = 0$). Also

$$D(\Phi)(\exp_x(tv), -v) = D(\Phi)(\exp_x((-t)(-v)), -v) = D(\Phi)(\exp_x(tv), v)$$

for $|t| \leq 2\pi/5$. Hence $D(\Phi)(p, -v) = D(\Phi)(p, v)$ for all $(p, v) \in S^n \times S^n$.

The map $(p, v) \mapsto (D(\Phi)(p, v), v)$ is a diffeomorphism of $S^n \times S^n$ since it is $C^1$ close to the identity map. Hence for every $v \in S^n$, the map $p \mapsto D(\Phi)(p, v)$ is a diffeomorphism of $S^n$. Denote its inverse by $p \mapsto D^{-1}(\Phi)(p, v)$. We have $D^{-1}(\Phi)(p, -v) = D^{-1}(\Phi)(p, v)$.

If $\Phi$ is $C^l$ close to $\Phi'$, then $D(\Phi)$ is $C^{l-1}$ close to $D(\Phi')$. Consequently, if $\Phi$ is $C^l$ close to $\Phi'$, $l \geq 2$, then $D^{-1}(\Phi)$ is $C^{l-1}$ close to $D^{-1}(\Phi')$.

The function $I_v(\Phi) \in C^\infty(S^n)$ is then defined as

$$I_v(\Phi)(p) = \langle D^{-1}(\Phi)(p, v), v \rangle.$$ (25)

The map satisfies $I_v(\Phi) = -I_{-v}(\Phi)$ for all $v \in S^n$. When $\Phi = 0$, we simply have $I_v(0)(p) = \langle p, v \rangle$. If $\Phi$ is $C^l$ close to $\Phi'$, then $I(\Phi)$ is $C^{l-1}$ close to $I(\Phi')$.

Then $I_v(\Phi)(p) = 0 \iff D^{-1}(\Phi)(p, v) \in \Sigma_v \iff p \in \Sigma_v(\Phi)$. Suppose $y = \cos(t)\Sigma_v(\Phi)(x) + \sin(t)N_v(\Phi)(x)$ with sufficiently small $t$. Then $y = D(\Phi)(\exp_x(tv), v)$. Therefore

$$I_v(\Phi)(y) = \langle \exp_x(tv), v \rangle = \sin(t).$$

We have proved that $I_v(\Phi)$ has the desired properties.
The \((n-1)\)-dimensional spheres \(\Sigma_\sigma\) and \(\Sigma_\tau\) intersect transversely for all \(\sigma \neq \tau \in \mathbb{RP}^n\) and, for all \((v, \theta) \in T_1S^n\), \(0 < |t| < \pi/2\),
\[
\Sigma[v] \cap \Sigma[\exp_v(t \theta)] = \{ x \in S^n : \langle x, v \rangle = \langle x, \theta \rangle = 0 \}.
\]
Hence the family
\[
S_q = \Sigma_\sigma \cap \Sigma_\tau\quad \text{whenever} \quad q = (\sigma, \tau) \in \mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta,
\]
extends smoothly to a family \(\{S_q\}_{q \in B(\mathbb{RP}^n \times \mathbb{RP}^n)}\) of \((n-2)\)-dimensional spheres of \(S^n\).

**Lemma 7.2.** There exists a smooth map
\[
J(\Phi) : B(\mathbb{RP}^n \times \mathbb{RP}^n) \times S^n \to S^n
\]
that is \(C^1\) close to \(q, x \mapsto x\), with \(J(0)(q, x) = x\) and such that
\begin{enumerate}
  \item \(x \mapsto J(\Phi)(q)(x)\) is a diffeomorphism for all \(q \in B(\mathbb{RP}^n \times \mathbb{RP}^n)\);
  \item for all \(q = (\sigma, \tau) \in \mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta\) we have
    \[
    J(\Phi)(q)(S_q) = \Sigma_\sigma(\Phi) \cap \Sigma_\tau(\Phi).
    \]
\end{enumerate}

If \(\Phi\) is \(C^l\) close to \(\Phi', l \geq 3\), \(J(\Phi)\) is \(C^{l-2}\) close to \(J(\Phi')\).

**Proof.** For \(u \in S^n\) with \(u \neq \pm v\), set
\[
\theta(v, u) = \frac{u - \langle u, v \rangle v}{\sqrt{1 - \langle u, v \rangle^2}} \in S^n.
\]
The vector is uniquely determined by the property that \(\langle \theta(v, u), v \rangle = 0\) and \(\exp_v(d(u, v)\theta(u, v)) = u\). Consider the map
\[
Z(\Phi) : (\mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta) \times S^n \to \mathbb{R}^{n+1}
\]
where
\[
Z(\Phi)([v], [u])(x) = x - \langle v, x \rangle v - \langle \theta(v, u), x \rangle \theta(v, u)
+ I_v(\Phi)(x)v + \frac{(I_u(\Phi)(x) - \langle u, v \rangle I_v(\Phi)(x))}{\sqrt{1 - \langle u, v \rangle^2}} \theta(v, u).
\]
The map is well-defined because
\[
I_{-v}(\Phi) = -I_v(\Phi), \quad \theta(v, -u) = -\theta(v, u), \quad \text{and} \quad \theta(-v, u) = \theta(v, u).
\]

We now argue that \(Z(\Phi)\) can be extended to a smooth map defined on \(B(\mathbb{RP}^n \times \mathbb{RP}^n) \times S^n\). Choose a sufficiently small \(\delta > 0\). The smooth map on \(T_1S^n \times [-\delta, \delta] \times S^n\) given by
\[
(v, \theta, t, x) \mapsto I_{\exp_v(t \theta)}(\Phi)(x) - \cos(t)I_v(\Phi)(x)
\]
vanishes on \(T_1S^n \times \{0\} \times S^n\). Therefore there exists a smooth map
\[
Q(\Phi) : T_1S^n \times [-\delta, \delta] \times S^n \to \mathbb{R}
\]
such that

\[ I_{\exp,v}(\Phi)(x) = \cos(t)I_v(\Phi)(x) + \sin(t)Q(\Phi)(v, \theta, t, x). \]

This identity and the fact that

\[ \theta(v, \exp_v(t\theta)) = \frac{t}{|t|} \theta \quad \text{for all} \quad (v, \theta) \in T_1S^n, \quad 0 < |t| < \delta, \]

imply

\[ Z(\Phi)([v], [\exp_v(t\theta)])(x) = x - \langle v, x \rangle v - \langle \theta, x \rangle \theta + I_v(\Phi)(x)v + Q(\Phi)(v, \theta, t, x)\theta. \quad (29) \]

From this expression we see that \( Z(\Phi) \) can be extended smoothly to \( B(\mathbb{RP}^n \times \mathbb{RP}^n) \times S^n \). If \( \Phi \) is \( C^l \) close to \( \Phi' \), then \( Q(\Phi) \) is \( C^{l-2} \) close to \( Q(\Phi') \). Hence \( Z(\Phi) \) is \( C^{l-2} \) close to \( Z(\Phi') \).

When \( \Phi = 0 \) we have \( Z(0)(q)(x) = x \) for all \( q, x \in B(\mathbb{RP}^n \times \mathbb{RP}^n) \times S^n \). Hence we obtain that \( Z(\Phi)(q)(x) \neq 0 \) for all \( q, x \in B(\mathbb{RP}^n \times \mathbb{RP}^n) \times S^n \). As a result, the map

\[ \tilde{Z}(\Phi) : B(\mathbb{RP}^n \times \mathbb{RP}^n) \times S^n \to S^n, \quad (q, x) \mapsto Z(\Phi)(q)(x)/|Z(\Phi)(q)(x)| \]

is well-defined. Since \( \tilde{Z}(\Phi) \) is \( C^1 \) close to \( \tilde{Z}(0) \), the map \( x \mapsto \tilde{Z}(\Phi)(q)(x) \)

is a diffeomorphism of \( S^n \) for every \( q \in B(\mathbb{RP}^n \times \mathbb{RP}^n) \). Thus we obtain a smooth map

\[ J(\Phi) : B(\mathbb{RP}^n \times \mathbb{RP}^n) \times S^n \to S^n, \quad (q, y) \mapsto \tilde{Z}(\Phi)(q)^{-1}(y). \]

If \( \Phi \) is \( C^l \) close to \( \Phi' \), \( l \geq 3 \), \( J(\Phi) \) is \( C^{l-2} \) close to \( J(\Phi') \).

We are left to prove (27). If \( q = ([v], [u]) \in \mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta \), we have that

\[ y \in J(\Phi)(q)(\Sigma_{[v]} \cap \Sigma_{[u]}) \]

if and only if \( Z(\Phi)(q)(y) \in \Sigma_{[v]} \cap \Sigma_{[u]} \), which is equivalent to

\[ \langle Z(\Phi)(q)(y), v \rangle = 0 \quad \text{and} \quad \langle Z(\Phi)(q)(y), \theta(v, u) \rangle = 0. \]

From (28) we see that the first identity occurs if and only if \( I_v(\Phi)(y) = 0 \) and the second identity occurs if and only if \( I_v(\Phi)(y) = \langle u, v \rangle I_v(\Phi)(y) \).

Hence this is equivalent to \( y \in \Sigma_{[v]}(\Phi) \cap \Sigma_{[u]}(\Phi) \).

\[ \square \]

7.3. **The dual of the Funk transform.** Recall the incidence set defined in Section 2.1

\[ [F(\Phi)] = \{(p, \sigma) \in S^n \times \mathbb{RP}^n : p \in \Sigma_\sigma(\Phi)\}, \]

and the projections \( \pi_1 \) and \( \pi_2 \) of points in \([F(\Phi)]\) onto the first and second coordinates respectively. We endow \([F(\Phi)]\) with the induced metric can from the product \((S^n \times \mathbb{RP}^n, can \times can)\). Thus \( \pi_1 \) maps
For every \( p \) maps \( \pi_1^{-1}(p) \) isometrically into \( \Sigma_\sigma(\Phi) \), and \( \pi_2 \) maps \( \pi_2^{-1}(p) \) isometrically into \( \Sigma_p^*(\Phi) \). Similarly for the orientation-inducing set \( F(\Phi) \).

For \((p, v) \in F(\Phi)\), we denote by \( N^*(\Phi)(p, v) \) the unit normal to \( \pi_1^{-1}(p) \subset S^n \) at \( v \in S^n \) with \( \langle N^*(\Phi)(p, v), p \rangle > 0 \). Writing \( (p, v) = (q, \Xi_q(\Phi)(w)) \), we have that \( N^*(\Phi)(p, v) = N_q^*(\Phi)(w) \) for the \( N^* \) defined earlier. It satisfies \( N^*(\Phi)(p, -v) = N^*(\Phi)(p, v) \) and constitutes a smooth map \( N^*(\Phi) : F(\Phi) \to S^n \).

The dual of the Funk transform is the operator \( F^*(\rho, \Phi) \) such that

\[
\int_{\mathbb{R}P^n} F(\rho, \Phi)(f)(\sigma)g(\sigma)\,dV_{can}(\sigma) = \int_{S^n} f(x)F^*(\rho, \Phi)(g)(x)\,dV_{can}(x)
\]

for all \( f \in C^\infty(S^n) \), \( g \in C^\infty(\mathbb{R}P^n) \).

**Proposition 7.3.** For every \( g \in C^\infty(\mathbb{R}P^n) \) and every \( p \in S^n \),

\[
F^*(\rho, \Phi)(g)(p) = e^{(n-1)\rho(p)}\int_{\Sigma_p^*(\Phi)} g(\tau)U(\Phi)(p, \tau)\,dA_{can}(\tau),
\]

where \( U(\Phi) \in C^\infty([F(\Phi)]) \) is

\[
U(\Phi)(p, [v]) = \frac{\sqrt{\cos(\Phi(x, v))^2 + |\nabla N^*_v\Phi|^2(x)}}{|\eta(\Phi)(x, v, N^*(\Phi)(p, v))| \cos(\Phi(x, v))}
\]

for \( p = \Sigma_V(\Phi)(x) \).

**Proof.** According to the co-area formula for Riemannian submersions ([4], Chapter III),

\[
\int_{[F(\Phi)]} F(x, \tau)\,dV_{can} = \int_{\mathbb{R}P^n} \left( \int_{\Sigma_{\sigma}(\Phi)} \frac{F(x, \tau)}{|Jac(\pi_2)|(x, \tau)}\,dA_{can}(x) \right)\,dV_{can}(\tau)
\]

\[
= \int_{S^n} \left( \int_{\Sigma^*_p(\Phi)} \frac{F(x, \tau)}{|Jac(\pi_1)|(x, \tau)}\,dA_{can}(\tau) \right)\,dV_{can}(x)
\]

for every \( F \in C^\infty([F(\Phi)]) \). Here, \( |Jac(\pi_i)| = \sqrt{\det[D\pi_i \circ (D\pi_i)^*]} \).

Thus, for every \( f \in C^\infty(S^n) \) and every \( g \in C^\infty(\mathbb{R}P^n) \),

\[
\int_{\mathbb{R}P^n} \left( \int_{\Sigma_{\sigma}(\Phi)} f(x)e^{(n-1)\rho(x)}\,dA_{can}(x) \right) g(\tau)\,dV_{can}(\tau)
\]

\[
= \int_{[F(\Phi)]} e^{(n-1)\rho(x)}f(x)g(\tau)|Jac(\pi_2)|(x, \tau)\,dV_{can}(x, \tau)
\]

\[
= \int_{S^n} f(x) \left( e^{(n-1)\rho(x)} \int_{\Sigma^*_p(\Phi)} g(\tau)\frac{|Jac(\pi_2)|(x, \tau)}{|Jac(\pi_1)|(x, \tau)}\,dA_{can}(\tau) \right)\,dV_{can}(x).
\]
By the definition of $\mathcal{F}^*(\rho, \Phi)(g)$, this formula proves (30) with
\[
U(\Phi)(p, \sigma) = \frac{|\text{Jac}(\pi_2)|(p, \sigma)}{|\text{Jac}(\pi_1)|(p, \sigma)} \quad \text{for all } (p, \sigma) \in [F(\Phi)].
\]

The tangent space of the $(2n-1)$-dimensional manifold $[F(\Phi)]$ at the point $(p, \sigma)$ admits an orthogonal decomposition
\[
T_{(p,\sigma)}\pi_2^{-1}(\sigma) \oplus T_{(p,\sigma)}\pi_1^{-1}(p) \oplus \text{span}\{V(p, \sigma)\}
\]
where both components of $V(p, \sigma) = (V_1(p, \sigma), V_2(p, \sigma))$ are non-zero, $T_{(p,\sigma)}\pi_2^{-1}(\sigma) = T_p\Sigma_\sigma(\Phi) \times \{0\}$ and $T_{(p,\sigma)}\pi_1^{-1}(p) = \{0\} \times T_\sigma\Sigma_\rho^*(\Phi)$.

By definition of $|\text{Jac}(\pi_i)|$, we compute
\[
|\text{Jac}(\pi_i)|(p, \sigma) = \frac{|V_i(p, \sigma)|}{\sqrt{|V_1(p, \sigma)|^2 + |V_2(p, \sigma)|^2}}
\]
so that
\[
\frac{|\text{Jac}(\pi_2)|(p, \sigma)}{|\text{Jac}(\pi_1)|(p, \sigma)} = \frac{|V_2(p, \sigma)|}{|V_1(p, \sigma)|}.
\]

It remains to compute $V_1$ and $V_2$. Let $\sigma = [v]$ and $p = y = \Sigma_v(\Phi)(x)$. Using the notation of Section 3.4, we choose $u \in T_y\mathbb{S}^n$, $|u| = 1$, and define $v(t), y(t)$. We can choose $u = \mathbf{N}_*(\Phi)(p, v)$. Notice that the projection $[u]$ of $u$ on $T_1\mathbb{R}^\mathbb{P}^n$ is orthogonal to $T_1\Sigma^*(\Phi)$. Since $y(t) \in \Sigma_v(t)(\Phi)$, we have that $(y(t), [v(t)]) \in [F(\Phi)]$ and $(y(0), [v(0)]) = (p, \sigma)$. We have $y'(0) = u$ and
\[
y'(0) = \cos \Phi(x, v)x'(0) - \sin \Phi(x, v)(\eta(x) + D_2\Psi(0, x) \cdot x'(0))x
+ \cos \Phi(x, v)(\eta(x) + D_2\Psi(0, x) \cdot x'(0))v
= \sin \Phi(x, v)u - \sin \Phi(x, v)(D\Phi(x, v)(-\langle x, u \rangle v, u))x
+ \cos \Phi(x, v)(-\langle x, u \rangle + D\Phi(x, v)(-\langle x, u \rangle v, u))v.
\]

We have
\[
\langle y'(0), N_v(\Phi)(x) \rangle = Z(y'(0), \cos \Phi(-\sin \Phi x + \cos \Phi v) - \nabla^{\Sigma_v} \Phi_v(x))
= Z(-\cos \Phi \langle u, x \rangle - \sin \Phi \langle u, \nabla^{\Sigma_v} \Phi_v(x) \rangle + \cos \Phi D\Phi)
= Z \cos \Phi \eta(\Phi)(x, v, u)
\]
by (111), where $\Phi = \Phi(x, v)$, $Z = (\cos(\Phi))^2 + \left|\nabla^{\Sigma_v} \Phi_v(x)\right|^2 \right)^{-\frac{1}{2}}$ and $D\Phi = D\Phi(x, v)(-\langle x, u \rangle v, u)$.

Let $\{e_i\}$ be an orthonormal basis of $T_y\Sigma_v(\Phi) = T_{(p,\sigma)}\pi_2^{-1}(\sigma)$. We choose $k_i$ such that $(y'(0) - \sum k_i e_i, [u])$ is orthogonal to $T_y\Sigma_v(\Phi) \oplus T_{[v]}\Sigma_\rho^*(\Phi)$, or equivalently such that $y'(0) - \sum k_i e_i$ is orthogonal to $T_y\Sigma_v(\Phi)$. Then $V_1 = y'(0) - \sum k_i e_i$ and $V_2 = [u]$. 
We have
\[ |V_1| = |\langle V_1, N_v(\Phi)(x) \rangle| = |\langle y'(0), N_v(\Phi)(x) \rangle| \]
and \(|V_2| = 1|\).

Hence
\[
U(\Phi)(p, [v]) = \frac{1}{|V_1|} \frac{\sqrt{\cos((x, v))^2 + |\nabla v^{\Phi}||^2(x)}}{|\eta(\Phi)(x, v, N^*(\Phi)(p, v))| \cos(\Phi(x, v))}.
\]

\[ \square \]

7.4. The operator \(\mathcal{F} \circ \mathcal{F}^*\) as an integral operator. For \((y, v) \in F(\Phi)\), writing \((y, v) = (\Sigma_v(\Phi)(x), v)\), we define \(N(\Phi)(y, v) = N_v(\Phi)(x)\). This defines a smooth map \(N(\Phi) : F(\Phi) \to S^n\). Hence the generalized Gauss map can be seen as a map \(G(\Phi) : (y, v) \in F(\Phi) \mapsto (y, N(\Phi)(y, v)) \in \mathcal{T}_1S^n\).

Notice that \(G(\Phi) = G(\Phi) \circ T(\Phi)\), where \(T(\Phi)\) is the map in (3). Thus the map \(G(\Phi)\) must be a diffeomorphism as well.

Consider the smooth function \(K(\rho, \Phi)\) defined on \(\mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta\) by
\[
K(\rho, \Phi)((v), [u]) = \int_{\Sigma_v(\Phi) \cap \Sigma_u(\Phi)} e^{2(n-1)(u)} \sqrt{1 - |N(\Phi)(y, v), N(\Phi)(y, v)|}^2 dA_{\text{can}}(y). \tag{31}
\]
The function is well-defined and smooth on \(\mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta\) because \(G(\Phi)\) is a diffeomorphism.

**Proposition 7.4.** The function
\[
k(\rho, \Phi)(\sigma, \tau) = \eta(d(\sigma, \tau))K(\rho, \Phi)(\sigma, \tau)
\]
extends to a smooth function \(k(\rho, \Phi) \in C^\infty(B(\mathbb{RP}^n \times \mathbb{RP}^n))\), where \(\eta\) is as in Section 6. If \((\rho, \Phi)\) is \(C^l\) close to \((\rho', \Phi')\), for \(l \geq 3\), then \(k(\rho, \Phi)\) is \(C^{l-3}\) close to \(k(\rho', \Phi')\).

**Proof.** We will refer to the families of spheres \(\{S_q\}_{q \in B(\mathbb{RP}^n \times \mathbb{RP}^n)}\) defined in (26) and to the map \(J(\Phi)\) defined in Section 7.2. We take as representatives of \([(v), [u]) \in \mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta\) a pair of points \(v, u\) in \(S^n\) such that \(d(u, v) \leq \pi/2\).

Using (27) we have that, for all \((\sigma, \tau) \in \mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta\),
\[
k(\rho, \Phi)((\sigma, \tau)) = \eta(d(\sigma, \tau))K(\rho, \Phi)(\sigma, \tau)
\]
\[ = \int_{S_{(\sigma, r)}} V(\Phi)(\sigma, \tau)(x)e^{2(n-1)(\sigma, \tau)(x)} dA_{\text{can}}(x), \tag{32}\]
where \(\rho(\sigma, \tau) = \rho \circ J(\Phi)(\sigma, \tau)\) and \(V(\Phi)(\sigma, \tau)(x)\) is given by
\[
\frac{\eta(d(u, v))}{|N(\Phi)(y, u) - N(\Phi)(y, v)|} \sqrt{4 - |N(\Phi)(y, v) - N(\Phi)(y, u)|^2} \tag{33}\]
with \( \sigma = [v] \), \( \tau = [u] \), and \( y = J(\Phi)(\sigma, \tau)(x) \). The domain of \( V(\Phi) \) is the open set in

\[
B = \{(x, q) \in S^n \times B(\mathbb{R}P^n \times \mathbb{R}P^n) : x \in S_q\}
\]

consisting of those elements \((x, q)\) where \( q \in \mathbb{R}P^n \times \mathbb{R}P^n \setminus \Delta \).

We wish to extend \( V(\Phi) \) to a smooth function on \( B \), because if that is the case we have from (32) that indeed \( k(\rho, \Phi) \) extends to a smooth function on \( B(\mathbb{R}P^n \times \mathbb{R}P^n) \).

We have that

\[
I_v(\Phi)(y) = \sin d_{\Sigma_v(\Phi)}(y) = \lambda_n(y) = \frac{1}{2} \left| \lambda_n(x) \right| \left| \langle x, y \rangle \right|
\]

for \( y \) in some \( \delta \)-neighborhood of \( \Sigma_v(\Phi) \), where \( d_{\Sigma_v(\Phi)} \) is the signed distance to the hypersurface \( \Sigma_v(\Phi) \). Then

\[
\nabla_{S^n} I_v(\Phi)(y) = \left( \cos d_{\Sigma_v(\Phi)}(y) \right) \nabla_{S^n} d_{\Sigma_v(\Phi)}(y).
\]

Hence, if \( y \in \Sigma_v(\Phi) \) (equivalently, if \((y, v) \in F(\Phi)\)) then

\[
N(\Phi)(y, v) = \nabla_{S^n} I_v(\Phi)(y).
\]

Therefore, choosing \((\sigma, \tau) = ([v], \exp_v(t\theta)) = q\),

\[
2 |\text{Jac} J(\Phi)(\sigma, \tau)|_{\mathcal{S}_q(x)} = \frac{2 |\text{Jac} J(\Phi)(q)|_{\mathcal{S}_q(x)}}{\sqrt{4 - |N(\Phi)(y, v) - N(\Phi)(y, u)|^2}}
\]

with \( y = J(\Phi)(q)(x) \) and \( t > 0 \) extends smoothly to \( B \).

Now define

\[
h(\Phi)(v, \theta, t, y) = \nabla_{S^n} I_v(\Phi)(y) - \nabla_{S^n} I_{\exp_v(t\theta)}(\Phi)(y)
\]

for \((v, \theta, t, y) \in T_1 S^n \times \mathbb{R} \times S^n\). Notice that

\[
h(\Phi)(v, \theta, 0, y) = 0,
\]

hence there exists a smooth function \( \lambda(\Phi)(v, \theta, t, y) \) such that

\[
h(\Phi)(v, \theta, t, y) = t\lambda(\Phi)(v, \theta, t, y).
\]

Therefore, setting \((\sigma, \tau) = ([v], \exp_v(t\theta)) = q\), for sufficiently small \( t > 0 \)

\[
\eta(d(u, v)) = \frac{t}{|N(\Phi)(y, u) - N(\Phi)(y, v)|} = \frac{1}{|h(\Phi)(v, \theta, t, y)|} \frac{1}{|\lambda(\Phi)(v, \theta, t, y)|}
\]

with \( y = J(\Phi)(q)(x) \) extends smoothly to \( B \) provided

\[
\lambda(\Phi)(v, \theta, 0, y) \neq 0
\]

for \( x \in S_q \), \( q = ([v, \theta]), 0 \).
We will compute \( \lambda(0)(v, \theta, t, y) \). Since we have \( I_v(0)(y) = \langle y, v \rangle \), \( \nabla_{S^n} I_v(0)(y) = v - \langle v, y \rangle y \) and \( \exp_v(t\theta) = (\cos t)v + (\sin t)\theta \), we obtain
\[
\begin{align*}
\lambda(0)(v, \theta, t, y) & = v - \langle v, y \rangle y \\
& - ((\cos t)v + (\sin t)\theta - ((\cos t)v + (\sin t)\theta, y)y \\
& = (1 - \cos t)v - (\sin t)\theta - ((1 - \cos t)v - (\sin t)\theta, y)y.
\end{align*}
\]
Therefore \( \lambda(0)(v, \theta, 0, y) = -\theta + \langle \theta, y \rangle y \). If \( y = J(0)(q)(x) = x \in S_q \) with \( q = (\langle v, \theta \rangle, 0) \), then \( \langle y, v \rangle = \langle y, \theta \rangle = 0 \) so
\[
\lambda(0)(v, \theta, 0, y) = -\theta.
\]
Then we can assume \( |\lambda(\Phi)(v, \theta, 0, y)| > \frac{1}{2} \) for \( y = J(\Phi)(q)(x) \) and \( x \in S_q \). This proves the existence of the smooth extension. The statement about closeness in \( C^l \) norms follows from the proof.

\[
\square
\]

Together with Proposition 7.4, the next proposition identifies the operators \( \mathcal{F}(\rho, \Phi) \circ \mathcal{F}^*(\rho, \Phi) \) as integral operators associated to kernels \( k(\rho, \Phi) \) like those considered in Section 6.

**Proposition 7.5.** For every \( f \in C^\infty(\mathbb{RP}^n) \), we have
\[
\langle \mathcal{F}(\rho, \Phi) \circ \mathcal{F}^*(\rho, \Phi) \rangle (f)(\sigma) = \int_{\mathbb{RP}^n} K(\rho, \Phi)(\sigma, \tau) f(\tau) \, dV_{\text{can}}(\tau).
\]

**Proof.** By (11) and (30),
\[
\begin{align*}
\mathcal{F}(\rho, \Phi)[\mathcal{F}^*(\rho, \Phi)(f)](\sigma) & = \int_{\Sigma^*(\Phi)} e^{(n-1)\rho(y)} \mathcal{F}^*(\rho, \Phi)(f)(y) \, dA_{\text{can}}(y) \\
& = \int_{\Sigma^*(\Phi)} \left( \int_{\Sigma^*(\Phi)} e^{2(n-1)\rho(y)} f(\tau) U(\Phi)(y, \tau) \, dA_{\text{can}}(\tau) \right) \, dA_{\text{can}}(y)
\end{align*}
\]
where \( U(\Phi) \in C^\infty([F(\Phi)]) \) is as in Proposition 7.3.

Fix a point \( \sigma \in \mathbb{RP}^n \). Consider the set \([B_\sigma(\Phi)] \subset [F(\Phi)] \) defined by
\[
[B_\sigma(\Phi)] = \{(y, \tau) \in S^n \times \mathbb{RP}^n : \tau \in \Sigma^*_y(\Phi), y \in \Sigma^*_\sigma(\Phi)\}.
\]
Equivalently,
\[
[B_\sigma(\Phi)] = \{(y, \tau) \in S^n \times \mathbb{RP}^n : y \in \Sigma^*_\tau(\Phi) \cap \Sigma^*_\sigma(\Phi)\}.
\]

Let \( I_v(\Phi) \) denote the smooth function on \( S^n \) that was defined in (25) with \( \sigma = [v] \). Then
\[
[B_\sigma(\Phi)] = \{(y, \tau) \in [F(\Phi)] : I_v(\Phi)(y) = 0\}.
\]

For \((y, \tau) \in [B_\sigma(\Phi)]\), write \( y = \Sigma^*_\sigma(\Phi)(x) \). Since \( \pi_1 : [F(\Phi)] \to S^n \) is a submersion, there exists \( X \in T_{(y, \tau)}[F(\Phi)] \) with \((d\pi_1)_{(y, \tau)}(X) = \)
$N_v(\Phi)(x)$. Then $d(I_v(\Phi) \circ \pi_1)(y,\tau)(X) = 1$. Hence 0 is a regular value of the map $(y, \tau) \in [F(\Phi)] \mapsto I_v(\Phi)(y) \in \mathbb{R}$. Thus $[B_\sigma(\Phi)]$ is a smooth embedded hypersurface of $[F(\Phi)]$, hence of dimension $2n - 2$.

From (36), we see that the projection $Q_1 = (\pi_1)|_{[B_\sigma(\Phi)]} : [B_\sigma(\Phi)] \to \Sigma_\sigma(\Phi)$ of $[B_\sigma(\Phi)]$ onto the first factor is a surjective submersion since $Q_1(y,\sigma) = y$ for $y \in \Sigma_\sigma(\Phi)$. By the co-area formula

$$
\int_{\Sigma_\nu(\Phi)} \int_{\Sigma_\nu(\Phi)} e^{2(n-1)\rho(y)} f(\tau)U(\Phi)(y, \tau)dA_{can}(\tau)dV_{can}(y) = \\
= \int_{[B_\sigma(\Phi)]} e^{2(n-1)\rho(y)} f(\tau)U(\Phi)(y, \tau)|Jac(Q_1)|(\sigma, y, \tau)dV_{can}(y, \tau),
$$

where we write the Jacobian of $Q_1$ with an extra entry so to remind us that this map depends on $\sigma$ (which is fixed in the argument).

Let $(y, \tau) \in [B_\sigma(\Phi)]$, $y = \Sigma_\nu(\Phi)(x)$. Then

$$
T_{(y,\tau)}[B_\sigma(\Phi)] = \{X \in T_{(y,\tau)}[F(\Phi)] : dI_v(\Phi)d\pi_1X = 0\} = \{X \in T_{(y,\tau)}[F(\Phi)] : d\pi_1X \in T_y\Sigma_\nu(\Phi)\}.
$$

As in Section 7.3 it admits an orthogonal decomposition

$$
((T_y\Sigma_\tau(\Phi) \cap T_y\Sigma_\sigma(\Phi)) \times \{0\}) \oplus \{0\} \times T_{\tau}\Sigma_\gamma_\nu(\Phi)) \oplus \text{span}\{W(y, \tau)\}
$$

for $\tau \neq \sigma$, with $W(y, \tau) = (W_1(y, \tau), W_2(y, \tau))$.

Let $\{e_i\}_i$ be an orthonormal basis of $T_y\Sigma_\tau(\Phi)$ with $\langle e_i, N(\Phi)(y, v) \rangle = 0$ for $i \geq 2$. Let $\{f_j\}_j$ be an orthonormal basis of $T_{\tau}\Sigma_\gamma_\nu(\Phi)$. We write

$$
W(y, \tau) = V(y, \tau) - \sum_i a_ie_i - \sum_j b_jf_j.
$$

Recall that

$$
V_1(y, \tau) = Z \cos \Phi\eta(\Phi)(\bar{x}, w, N^*(\Phi)(y, w))N(\Phi)(\bar{x}, w)
$$

and

$$
V_2(y, \tau) = [N^*(\Phi)(y, w)],
$$

where $\tau = [w]$, $y = \Sigma_\nu(\Phi)(\bar{x})$, $\Phi = \Phi(\bar{x}, w)$, and $Z = (\cos(\Phi))^2 + |\nabla\Sigma_\nu(\Phi)(\bar{x})|^{-\frac{1}{2}}$.

Then $W(y, \tau) = V(y, \tau) - a_1e_1$ by orthogonality, and

$$
e_1 = \frac{N(\Phi)(y, v) - \langle N(\Phi)(y, v), N(\Phi)(y, w) \rangle N(\Phi)(y, w)}{|N(\Phi)(y, v) - \langle N(\Phi)(y, v), N(\Phi)(y, w) \rangle N(\Phi)(y, w)|}.
$$

Hence $W_2(y, \tau) = V_2(y, \tau)$.

We need $\langle W_1(y, \tau), N(\Phi)(y, v) \rangle = 0$, hence

$$
\langle V_1(y, \tau) - a_1e_1, N(\Phi)(y, v) \rangle = 0.
$$
Therefore
\[
a_1 = \lambda \frac{Z \cos \Phi \eta(\Phi)(\bar{x}, w, N^*(\Phi)(y, w))\langle N(\Phi)(y, w), N(\Phi)(y, v)\rangle}{1 - \langle N(\Phi)(y, v), N(\Phi)(y, w)\rangle^2},
\]
where \(\lambda = |N(\Phi)(y, v) - \langle N(\Phi)(y, w), N(\Phi)(y, v)\rangle N(\Phi)(y, w)|\). Hence
\[
a_1 = \frac{Z \cos \Phi \eta(\Phi)(\bar{x}, w, N^*(\Phi)(y, w))\langle N(\Phi)(y, w), N(\Phi)(y, v)\rangle}{\sqrt{1 - \langle N(\Phi)(y, v), N(\Phi)(y, w)\rangle^2}}.
\]

To compute the Jacobian of \(Q_1\), we choose the orthonormal basis 
\[
\{e_i\}_{i \geq 2}, K = \frac{N(\Phi)(y, w) - \langle N(\Phi)(y, w), N(\Phi)(y, v)\rangle N(\Phi)(y, v)}{\sqrt{1 - \langle N(\Phi)(y, v), N(\Phi)(y, w)\rangle^2}}
\]
of \(T_y \Sigma_v(\Phi)\). Then
\[
|Jac(Q_1)|(\sigma, y, \tau) = \frac{|\langle W_1, K \rangle|}{|W|}.
\]
We want to compute also the Jacobian of \(Q_2 = (\pi_2|_{B_\sigma(\Phi)}): [B_\sigma(\Phi)] \to \mathbb{R}^m\). We decompose \(T_y \Sigma_v(\Phi) = T_y \Sigma_v^*(\Phi) \oplus \text{span}\{[N^*(\Phi)(y, w)]\}\). Hence
\[
Jac(Q_2)(\sigma, y, \tau) = \left| \frac{W_2}{|W|}, [N^*(\Phi)(y, w)] \right| = \frac{1}{|W|}.
\]
Hence
\[
\frac{|Jac(Q_1)|(\sigma, y, \tau)}{|Jac(Q_2)|(\sigma, y, \tau)} = |\langle W_1, K \rangle| = |\langle V_1 - a_1 e_1, K \rangle| = \frac{|Z \cos \Phi \eta(\Phi)(\bar{x}, w, N^*(\Phi)(y, w))|}{\sqrt{1 - \langle N(\Phi)(y, v), N(\Phi)(y, w)\rangle^2}}.
\]
Since the integrand is a smooth function, and \(Q_2^{-1}(\sigma) = \Sigma_{\sigma}(\Phi) \times \{\sigma\}\) has dimension \(n - 1\), we have that
\[
\int_{[B_\sigma(\Phi)]} e^{2(n-1)\rho(y)} f(\tau)U(\Phi)(y, \tau)|Jac(Q_1)|(\sigma, y, \tau)dV_{\text{can}}(y, \tau)
\]
\[
= \lim_{\delta \to 0} \int_{[B_\sigma(\Phi)] \setminus Q_2^{-1}(B_\delta(\sigma))} e^{2(n-1)\rho(y)} f(\tau)U(\Phi)(y, \tau) \cdot |Jac(Q_1)|(\sigma, y, \tau)dV_{\text{can}}(y, \tau).
\]
Now by the co-area formula
\[ \int_{B_\sigma(Q_1)(y,\tau)} e^{2(n-1)\rho(y)} f(\tau) U(\Phi)(y,\tau) \cdot |\text{Jac}(Q_1)| \sigma(y,\tau) d\text{can}(y,\tau) \]
\[ = \int_{\mathbb{R}^n \setminus B_\delta(\sigma)} \left( \int_{\Sigma_y(\Phi) \cap \Sigma_{\tau}(\Phi)} e^{2(n-1)\rho(y)} f(\tau) \frac{dA_{\text{can}}(y)}{\sqrt{1 - \langle N(\Phi)(y,w), N(\Phi)(y,v) \rangle^2}} \right) dV_{\text{can}}(\tau) \]
\[ = \int_{\mathbb{R}^n \setminus B_\delta(\sigma)} f(\tau) K(\rho,\Phi)(\sigma,\tau) dV_{\text{can}}(\tau). \]

Since \(|K(\rho,\Phi)(\sigma,\tau)| \leq Cd(\sigma,\tau)^{-1}\), \(\tau \mapsto f(\tau)K(\rho,\Phi)(\sigma,\tau)\) is integrable and

\[ \lim_{\delta \to 0} \int_{\mathbb{R}^n \setminus B_\delta(\sigma)} f(\tau) K(\rho,\Phi)(\sigma,\tau) dV_{\text{can}}(\tau) \]
\[ = \int_{\mathbb{R}^n} f(\tau) K(\rho,\Phi)(\sigma,\tau) dV_{\text{can}}(\tau), \]

which proves the proposition. \(\square\)

7.5. **The operator \(F \circ F^*\) as a pseudo-differential operator.** Consider the kernel \(k(\rho,\Phi)\) given by (31) and Proposition 7.4. From Proposition 7.5 we have that

\[ F(\rho,\Phi) \circ F^*(\rho,\Phi)(f) = L(k(\rho,\Phi))(f) \quad \text{for all } f \in C^\infty(\mathbb{R}^n), \]

where \(L(k)\) is as in Section 6 and so we obtain from Theorem 6.1 that the operator

\[ F(\rho,\Phi) \circ F^*(\rho,\Phi) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n) \]

is a pseudo-differential operator of order \(1 - n\).

The standard Funk transform \(F = F(0,0)\) is such that

\[ L(k(0,0)) = F \circ F^* \]

is an invertible elliptic pseudo-differential operator of order \(1 - n\), see Corollary 9.5. Let \(\eta_0\) be the constant determined by Theorem 6.1 with \(k_0 = k(0,0)\).

Assume that \((\rho,\Phi)\) has sufficiently small \(C^{3n+3}\) norm. Hence we can suppose by Proposition 7.4 that

\[ ||k(\rho,\Phi) - k(0,0)||_{C^{3n}} < \eta_0/2. \]
Therefore $F(\rho, \Phi) \circ F^*(\rho, \Phi) : H^0(\mathbb{RP}^n) \to H^{n-1}(\mathbb{RP}^n)$ is an invertible elliptic pseudo-differential operator of order $1 - n$. Thus with these assumptions we have proved:

**Theorem 7.6.** The operator

$$F(\rho, \Phi) \circ F^*(\rho, \Phi) : H^0(\mathbb{RP}^n) \to H^{n-1}(\mathbb{RP}^n)$$

is an invertible elliptic pseudo-differential operator of order $1 - n$. The operator

$$R(\rho, \Phi) = F^*(\rho, \Phi) \circ (F(\rho, \Phi) \circ F^*(\rho, \Phi))^{-1} : C^\infty(\mathbb{RP}^n) \to C^\infty(S^n)$$

is well-defined, and it is a right-inverse for the Funk transform $F(\rho, \Phi)$:

$$F(\rho, \Phi) \circ R(\rho, \Phi) = Id.$$

### 8. Checking tameness

This section is based on [12] and we refer the reader to that paper for the appropriate definitions.

**8.1. Tame estimates for $L(k)$**. We start deriving the estimates for the integral operators $L(k)$ defined in Section 6.

Let $V(\mathbb{RP}^n)$ be the space of Killing vector fields of $(\mathbb{RP}^n, can)$. Given $X \in V(\mathbb{RP}^n)$ and $k \in C^\infty(B(\mathbb{RP}^n \times \mathbb{RP}^n))$ there exists $X^B(k) \in C^\infty(B(\mathbb{RP}^n \times \mathbb{RP}^n))$ such that for all $(\sigma, \tau) \in \mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta$

$$X^B(k)(\sigma, \tau) = \langle X(\tau), \nabla_\tau k(\sigma, \tau) \rangle + \langle X(\sigma), \nabla_\sigma k(\sigma, \tau) \rangle. \quad (37)$$

Indeed, if $X$ is generated by isometries $\{R_s\}_s \in PSO(n + 1)$ with $R_0 = Id$ we define $\phi_s : B(\mathbb{RP}^n \times \mathbb{RP}^n) \to B(\mathbb{RP}^n \times \mathbb{RP}^n)$ so that

$$\phi_s(\sigma, \tau) = (R_s(\sigma), R_s(\tau)) \quad \text{if} \quad (\sigma, \tau) \in \mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta$$

and

$$\phi_s[(\sigma, \theta), t] = [(R_s(\sigma), dR_s|_{\sigma}(\theta)), t] \quad \text{if} \quad [(\sigma, \theta), t] \in \Omega.$$

The maps $\{\phi_s\}_s$ are well-defined smooth diffeomorphisms and hence generate a smooth vector field $X^B$ of $B(\mathbb{RP}^n \times \mathbb{RP}^n)$ so that the derivative $X^B(k) \in C^\infty(B(\mathbb{RP}^n \times \mathbb{RP}^n))$ satisfies (37).
By differentiating, we have that

\[
X(L(k)(f))(\sigma) = \frac{d}{ds}_{s=0} L(k)(f)(R_s\sigma)
= \frac{d}{ds}_{s=0} \int_{\mathbb{R}^n} K(\rho, \Phi)(R_s\sigma, \tau)f(\tau)dV_{\text{can}}(\tau)
= \frac{d}{ds}_{s=0} \int_{\mathbb{R}^n} k(\rho, \Phi)(R_s\sigma, R_s\tau)\eta(d(\sigma, \tau))f(R_s\tau)dV_{\text{can}}(\tau)
= \int_{\mathbb{R}^n} X^B(k)(\sigma, \tau)f(\tau) + k(\sigma, \tau)X(f)(\tau)\eta(d(\sigma, \tau))dV_{\text{can}}(\tau).
\]

Thus

\[
X(L(k)(f)) = L(X^B(k))(f) + L(k)(X(f)). \tag{38}
\]

We are going to use that

\[
||Df||_q \leq d_{n,q} \sum_{l=1}^{N} ||X_l(f)||_q \tag{39}
\]

for all integers \(q \geq 0\), where \(\{X_1, \ldots, X_N\}\) is a chosen basis of \(V(\mathbb{R}^n)\) and \(d_{n,q} > 0\).

The proofs of Section II.2.2 \cite{12} give that for all \(f \in C^\infty(\mathbb{R}^n)\), \(k \in C^\infty(B(\mathbb{R}^n \times \mathbb{R}^n))\), and \(p, q, j, l \in \mathbb{N}_0\) with \(p \geq j, q \geq l\), we have

\[
||k||_{C^p} ||f||_q \leq c_{n,p,q,j,l}(||k||_{C^{p-j}} ||f||_{q+j} + ||k||_{C^{p+l}} ||f||_{q-l}) \tag{40}
\]

where \(c_{n,p,q,j,l}\) is a positive constant depending only on \(n, p, q, j, l \in \mathbb{N}_0\).

**Proposition 8.1.** For all \(k \in C^\infty(B(\mathbb{R}^n \times \mathbb{R}^n))\), \(f \in C^\infty(\mathbb{R}^n)\), and \(q \in \mathbb{N}_0\) we have

\[
||L(k)(f)||_{n-1+q} \leq c_{n,q}(||k||_{C^{3n}} ||f||_q + ||k||_{C^{3n+q}} ||f||_0),
\]

where \(c_{n,q}\) is a positive constant depending only on \(n\) and \(q\).

**Proof.** We argue by induction on \(q \in \mathbb{N}_0\). The case \(q = 0\) corresponds to inequality \((17)\).

For each \(X \in V(\mathbb{R}^n)\), using \((38)\) and the induction hypothesis (applied to \(X^B(k)\) and \(X(f)\)) we obtain

\[
||X(L(k)(f))||_{n-1+q} \leq ||L(X^B(k))(f))||_{n-1+q} + ||L(k)(X(f))||_{n-1+q}
\]

\[
\leq c_{n,q}(||X^B(k)||_{C^{3n}} ||f||_q + ||X^B(k)||_{C^{3n+q}} ||f||_0
\]

\[
+ ||k||_{C^{3n}} ||X(f)||_q + ||k||_{C^{3n+q}} ||X(f)||_0)
\]

\[
\leq c(||k||_{C^{3n+1}} ||f||_q + ||k||_{C^{3n+q+1}} ||f||_0
\]

\[
+ ||k||_{C^{3n}} ||f||_{q+1} + ||k||_{C^{3n+q}} ||f||_1).
\]
By the interpolation inequalities (40)
\[ |k|C^{3n+1}||f||_q + ||k||C^{3n+q}||f||_1 \leq c(||k||C^{3n}||f||_{q+1} + ||k||C^{3n+q+1}||f||_0). \]
Hence
\[ ||X(L(k)(f))||_{n+1} \leq c(||k||C^{3n}||f||_{q+1} + ||k||C^{3n+q+1}||f||_0). \]
The desired result follows now from (39).

**Proposition 8.2.** Suppose \( k_0 \in C^\infty(B(\mathbb{RP}^n \times \mathbb{RP}^n)) \) is such that \( L(k_0) \) is elliptic and \( L(k_0) : H^0(\mathbb{RP}^n) \rightarrow H^{n-1}(\mathbb{RP}^n) \) is invertible.

There exists \( \eta_1 > 0 \) such that for all functions \( f \in C^\infty(\mathbb{RP}^n) \) and \( k \in C^\infty(B(\mathbb{RP}^n \times \mathbb{RP}^n)) \) with \( ||k - k_0||C^{3n+1} < \eta_1 \) we have
\[ ||f||_q \leq c_q(k_0)(||L(k)(f)||_{n-1} + ||k||C^{3n+q}||L(k)(f)||_n), \]
where \( c_q(k_0) \) is a positive constant depending only on \( q \) and \( k_0 \).

**Proof.** We argue by induction. The case \( q = 0 \) was proven in (19).

For each \( X \in V(\mathbb{RP}^n) \), the induction hypothesis (applied to \( X(f) \)) and (38) give
\[ ||X(f)||_q \leq c_q(k_0)(||L(k)(X(f))||_{n-1} + ||k||C^{3n+q}||L(k)(X(f))||_n) \]
\[ 
\leq c(||L(k)(f)||_{n+1} + ||k||C^{3n+q}||L(k)(f)||_n)
\]
\[ + c_q(k_0)(||L(X^B(k))(f)||_{n-1}) \]
\[ + ||k||C^{3n+q}||L(X^B(k))(f)||_n. \]

By assumption, \( ||k||C^{3n+1} \) is bounded by a constant depending only on \( k_0 \). Thus, from Proposition 8.1, the induction hypothesis, and (19), we can estimate
\[ ||L(X^B(k))(f)||_{n+1} \leq c_{n,q}(||X^B(k)||C^{3n}||f||_q + ||X^B(k)||C^{3n+q}||f||_0) \]
\[ \leq c(||k||C^{3n+1}||f||_q + ||k||C^{3n+q+1}||f||_0) \]
\[ \leq c(||f||_q + ||k||C^{3n+q+1}||f||_0) \]
\[ \leq c(||L(k)(f)||_{n-1} + ||k||C^{3n+q+1}||L(k)(f)||_n) \]
and, likewise,
\[ ||L(X^B(k))(f)||_{n-1} \leq c||L(k)(f)||_{n-1}. \]
Hence
\[ ||X(f)||_q \leq c(||L(k)(f)||_{n+1} + ||k||C^{3n+q}||L(k)(f)||_n) \]
\[ + ||k||C^{3n+q+1}||L(k)(f)||_n. \]

By the interpolation inequality and the uniform bound on \( ||k||C^{3n} \),
\[ ||k||C^{3n+q}||L(k)(f)||_n \leq c(||L(k)(f)||_{n+1} + ||k||C^{3n+q+1}||L(k)(f)||_n). \]
Therefore,
\[ \|X(f)\|_q \leq c(\|L(k)(f)\|_{n+q} + \|k\|_{C^{3n+q+1}}\|L(k)(f)\|_{n-1}). \]
The desired result follows at once from (39).

8.2. Tame estimates for \( \Lambda \) and its right-inverse. The spaces of smooth functions and smooth one-forms, or more generally smooth sections of a vector bundle, on compact manifolds are tame Fréchet spaces with their standard gradings, by [12], Part II, Theorem 1.3.6 and Corollary 1.3.9. We will denote by \( C^\infty(X,Y) \) the space of smooth maps from \( X \) to \( Y \).

Lemma 8.3. The spaces \( C^\infty_{s,\text{odd}}(T_1S^n) \subset C^\infty(T_1S^n) \) and \( \Omega^1_{\text{even}}(S^n) \subset \Omega^1(S^n) \) are tame Fréchet spaces with the gradings induced by the inclusions. The maps
\[ C : C^\infty_{s,\text{odd}}(T_1S^n) \to \Omega^1_{\text{even}}(S^n) \]
and
\[ j : \Omega^1_{\text{even}}(S^n) \to C^\infty_{s,\text{odd}}(T_1S^n) \]
are tame linear maps.

Proof. The space \( \Omega^1_{\text{even}}(S^n) \) is a tame direct summand of \( \Omega^1(S^n) \) as the map
\[ \omega \in \Omega^1(S^n) \mapsto \frac{\omega + A^*\omega}{2} \in \Omega^1_{\text{even}}(S^n) \]
(with \( A \) the antipodal map) is a tame linear map that is a left-inverse of the inclusion \( \Omega^1_{\text{even}}(S^n) \to \Omega^1(S^n) \). Similarly, \( C^\infty_{s,\text{odd}}(T_1S^n) \) is a tame direct summand of \( C^\infty(T_1S^n) \). Therefore these spaces are also tame by [12], Part II, Lemma 1.3.3.

It follows from the definition that both \( C \) and \( j \) satisfy a tame estimate of degree and base equal to zero, so they are tame linear maps.

Lemma 8.4. The spaces
\[ F = C^\infty(S^n) \times C^\infty_{0,\text{odd}}(T_1S^n) \quad \text{and} \quad H = C^\infty_0(\mathbb{R}P^n) \times C^\infty_{0,\text{odd}}(T_1S^n) \]
are tame Fréchet spaces with their standard gradings. Let \( U \subset F \) be a \( C^{3n+4} \)-neighborhood of the origin. The maps
\[ A : (\rho, \Phi) \in U \subset F \mapsto A(\rho, \Phi) \in C^\infty(\mathbb{R}P^n) \quad \text{and} \]
\[ H : (\rho, \Phi) \in U \subset F \mapsto H(\rho, \Phi) \in C^\infty_{s,\text{odd}}(T_1S^n) \]
are smooth tame maps. Thus the map
\[ \Lambda : (U \subset F) \to H \]
defined in Section 5.1 is a smooth tame map.
Proof. Since $F$ and $H$ are Cartesian products, by \cite{12}, Part II, Lemma 1.3.4, it is enough to argue that their factors are tame Fréchet spaces.

The maps $L : C^\infty(\mathbb{R}P^n) \to C^\infty(\mathbb{R}P^n)$ and $M : C^\infty(\mathbb{R}P^n) \to C^\infty(\mathbb{R}P^n)$ given by $L(f) = f$ and $M(f) = f - \int_{\mathbb{R}P^n} f(\sigma) dV_{can}(\sigma)$ are tame linear maps such that $ML$ is the identity. Hence $C^\infty_0(\mathbb{R}P^n)$ is a tame direct summand of $C^\infty(\mathbb{R}P^n)$ and hence is tame.

Since $C : C_{s,\text{odd}}(T_1S^n) \to \Omega^1_{\text{even}}(S^n)$ and $j : \Omega^1_{\text{even}}(S^n) \to C_{s,\text{odd}}(T_1S^n)$ are tame linear maps, the projection

$$\Psi \in C_{s,\text{odd}}(T_1S^n) \mapsto \Psi - jC(\Psi) \in C_{0,\text{odd}}(T_1S^n)$$

is a tame linear map that is a left-inverse for the inclusion $C_{0,\text{odd}}(T_1S^n) \to C_{s,\text{odd}}(T_1S^n)$. Therefore $C_{0,\text{odd}}(T_1S^n)$ is a tame direct summand of $C_{s,\text{odd}}(T_1S^n)$, and hence is tame.

Notice that (using the identification $C^\infty(\mathbb{R}P^n) \simeq C^\infty_{\text{even}}(S^n)$)

$$A(\rho, \Phi)(v) = \int_{\Sigma_v} \tilde{A}(\rho, \Phi)(x, v) dA_{\text{can}}(x) \quad \text{for all } v \in S^n$$

where

$$\tilde{A} : U \subset C^\infty(S^n) \times C_{s,\text{odd}}(S^n) \to C^\infty(T_1S^n)$$

is given in \cite{4}. Hence $A$ is the composition of $\tilde{A}$ and a tame linear map. Explicitly,

$$\tilde{A}(\rho, \Phi)(x, v) = e^{(n-1)\rho(\cos(\Phi(x,v))x + \sin(\Phi(x,v)))v} \cos^{n-2}(\Phi(x,v))$$

$$\cdot \frac{\cos^2(\Phi(x,v))}{\sqrt{\cos^2(\Phi(x,v)) + |\nabla_{\Sigma_v} \Phi_v(x)|^2}}.$$

The map that sends $\Phi$ to

$$(x, v) \mapsto \cos^{n-2}(\Phi(x,v))\sqrt{\cos^2(\Phi(x,v)) + |\nabla_{\Sigma_v} \Phi_v(x)|^2}$$

is a nonlinear differential operator of degree 1, hence smooth tame by \cite{12}, Part II, 2.2.7. The map that sends $\Phi$ to

$$(x, v) \mapsto \cos(\Phi(x,v))x + \sin(\Phi(x,v))v$$

in $C^\infty(T_1S^n, S^n) \subset C^\infty(T_1S^n, \mathbb{R}^{n+1})$ is also smooth tame (by \cite{12}, Part II, Corollary 2.3.2, $C^\infty(T_1S^n, S^n)$ is a tame manifold). The composition map

$$\mathcal{C} : C^\infty(S^n) \times C^\infty(T_1S^n, S^n) \to C^\infty(T_1S^n),$$

$\mathcal{C}(h, \Gamma) = h \circ \Gamma$, is smooth tame by \cite{12}, Part II, 2.3.3. Therefore the map that sends $(\rho, \Phi)$ to

$$(x, v) \mapsto e^{(n-1)\rho(\cos(\Phi(x,v))x + \sin(\Phi(x,v)))v}$$

is smooth tame. This proves that $\tilde{A}$ is smooth tame which implies $A$ is smooth tame.
The fact $\mathcal{H}$ is a smooth tame map follows similarly as for $\tilde{A}$ using (6). Hence Lemma 8.3 implies $\Lambda$ is smooth tame.

\[\square\]

**Lemma 8.5.** The map $\mathcal{F}^* : ((\rho, \Phi), g) \in (U \subset F) \times C^\infty(\mathbb{R}P^n) \mapsto \mathcal{F}^*(\rho, \Phi)(g) \in C^\infty(S^n)$ is a smooth tame map. Similarly, $\mathcal{F} : ((\rho, \Phi), f) \in (U \subset F) \times C^\infty(S^n) \mapsto \mathcal{F}(\rho, \Phi)(f) \in C^\infty(\mathbb{R}P^n)$ is smooth tame.

**Proof.** For every $g \in C^\infty(\mathbb{R}P^n)$ and every $q \in S^n$,

$$\mathcal{F}^*(\rho, \Phi)(g)(q) = e^{(n-1)\rho(q)} \int_{\Sigma^*_q(\Phi)} g(\tau)U(\Phi)(q, \tau)dA_{can}(\tau),$$

where $U(\Phi) \in C^\infty([F(\Phi)])$ is given by

$$U(\Phi)(q, [v]) = \frac{\sqrt{\cos(\Phi(x, v))^2 + |\nabla^2\Phi_v|^2(x)}}{\eta(\Phi)(x, v, \mathbf{N}^*(\Phi)(q, v))\cos(\Phi(x, v))},$$

for $q = \Sigma_v(\Phi)(x)$.

The map $\tilde{g}$ that sends $\Phi$ to

$$(x, v) \mapsto (\Sigma(\Phi)(x, v), N(\Phi)(x, v))$$

in $C^\infty(T_1S^n, \mathbb{R}^{2n+2})$ is a nonlinear differential operator of degree 1. Hence the generalized Gauss map defined in Section 2.2 regarded as a map

$$\mathcal{G} : (\rho, \Phi) \in U \subset F \mapsto \mathcal{G}(\Phi) \in \text{Diff}(T_1S^n),$$

is smooth tame. The inverse map

$$\mathcal{G}^{-1} : (\rho, \Phi) \in U \subset F \mapsto \mathcal{G}^{-1}(\Phi) \in \text{Diff}(T_1S^n),$$

is also smooth tame ([12], Part II, Theorem 2.3.5).

Since

$$\mathcal{G}^{-1}(\Phi) : (q, w) \in T_1S^n \mapsto (\Upsilon_q(\Phi)(w), \Xi_q(\Phi)(w)) \in T_1S^n,$$

the map that sends $\Phi$ to

$$(q, w) \in T_1S^n \mapsto \Xi_q(\Phi)(w) \in S^n$$

in $C^\infty(T_1S^n, S^n)$ is smooth tame. The map $\mathcal{W} \subset C^\infty(T_1S^n, \mathbb{R}^{n+1}) \to C^\infty(T_1S^n)$ that sends $\Xi$ to

$$(q, w) \in T_1S^n \mapsto \text{Jac}(\Xi_q|_{\Sigma_q})(w),$$

where $\mathcal{W}$ is a sufficiently small neighborhood of the map $(q, w) \to w$, is a nonlinear differential operator of degree 1. Hence it is smooth tame.
The map $\mathcal{W} \cap C^\infty(T_1S^n, S^n) \to C^\infty(T_1S^n, \mathbb{R}^{n+1})$ that sends $\Xi$ to
\[(q, w) \mapsto n(\Xi)(q, w) \in T_{\Xi(q, w)}S^n,\]
where $n(\Xi)(q, w)$ is the unit normal to $\Xi(q, \Sigma_q)$ at $\Xi(q, w)$ satisfying $\langle n(\Xi)(q, w), q \rangle > 0$, is the composition of the inclusion map
\[
\mathcal{W} \cap C^\infty(T_1S^n, S^n) \to \mathcal{W} \cap C^\infty(T_1S^n, \mathbb{R}^{n+1})
\]
and a nonlinear differential operator of degree 1
\[
\mathcal{W} \cap C^\infty(T_1S^n, \mathbb{R}^{n+1}) \to C^\infty(T_1S^n, \mathbb{R}^{n+1}).
\]
Hence it is smooth tame.

We have that $T_1S^n$ is a compact hypersurface of $S^n \times S^n$. For $\delta > 0$ sufficiently small, consider $V_\delta(T_1S^n)$ the tubular neighborhood of $T_1S^n$ of radius $\delta$ and $\bar{P} : V_\delta(T_1S^n) \to T_1S^n$ the nearest point projection. If $\Phi \in C^\infty_{s, odd}(T_1S^n)$ is in some sufficiently small neighborhood of the origin then $F(\Phi) \subset V_\delta(T_1S^n)$ and $\bar{P}|_{F(\Phi)} : F(\Phi) \to T_1S^n$ is a diffeomorphism. The map that sends $\Phi$ to
\[
\left( (x, v) \mapsto \bar{P}(\Sigma_v(\Phi)(x), v) \right) \in \text{Diff}(T_1S^n)
\]
is smooth tame. The same is true for the map
\[
\left( (q, w) \mapsto \bar{P}(\Xi_q(\Phi)(w)) \right) \in \text{Diff}(T_1S^n).
\]
This implies that the map that sends such a $\Phi$ to
\[
X(\Phi) : \left( (q, w) \mapsto (x = x(\Phi)(q, w), v = v(\Phi)(q, w)) \right) \in \text{Diff}(T_1S^n)
\]
with
\[
(\Sigma_v(\Phi)(x), v)) = (q, \Xi_q(\Phi)(w))
\]
is smooth tame. Similarly for $\Phi \mapsto X(\Phi)^{-1}$.

We have that for $v = \Xi_q(\Phi)(w)$
\[
N^*(\Phi)(q, v) = n(\Xi(\Phi))(q, w).
\]
Hence the map that sends $\Phi$ to
\[
(x, v) \in T_1S^n \mapsto N^*(\Phi)(\Sigma_v(\Phi)(x), v) = n(\Xi(\Phi)) \circ X(\Phi)^{-1}(x, v) \in \mathbb{R}^{n+1}
\]
is smooth tame.

Let $V$ be the pullback of the vector bundle $T^vS^n$ under the map $(x, v) \in T_1S^n \mapsto v \in S^n$. Given $(x, v) \in T_1S^n$, $u \in T_vS^n$, recall
\[
\eta(\Phi)(x, v, u) = -\langle x, u \rangle + D\Phi(x, v) \cdot (-\langle x, u \rangle v, u) - \tan \Phi(x, v) \langle \nabla^\Sigma_v, \Phi_v(x), u \rangle.
\]
Then the map that sends $\Phi$ to $\eta(\Phi) \in C^\infty(V)$ is smooth tame. Therefore the map that sends $\Phi$ to

$$(x, v) \mapsto \eta(\Phi)(x, v, N^*(\Phi)(\Sigma_v(\Phi)(x), v))$$

in $C^\infty(T_1S^n)$ is smooth tame.

This proves that the map that sends $\Phi$ to $U'(\Phi) : (x, v) \mapsto U(\Phi)(\Sigma_v(\Phi)(x), v)$ in $C^\infty(T_1S^n)$ is smooth tame.

In conclusion, the map that sends $\Phi, g$ to the function in $C^\infty(T_1S^n)$

$$\Gamma(q, w) = g([\Xi_q(\Phi)(w)])U(\Phi)(q, [\Xi_q(\Phi)(w)])Jac(\Xi_q(\Phi)(|\Sigma_q))(w),$$

satisfying $\Gamma(q, w) = \Gamma(q, -w)$, is smooth tame.

Since $(\rho, \Phi)$ has sufficiently small $C^{3n+4}$ norm, we can apply Proposition 8.2 to $k(\rho, \Phi)$ with $k_0 = k(0, 0)$.

Lemma 8.6. The maps

$$L : ((\rho, \Phi), g) \in (U \subset F) \times C^\infty(\mathbb{R}P^n)$$

$$\mapsto (\mathcal{F}(\rho, \Phi) \circ \mathcal{F}^*(\rho, \Phi))(g) \in C^\infty(\mathbb{R}P^n)$$

and

$$V : ((\rho, \Phi), g) \in (U \subset F) \times C^\infty(\mathbb{R}P^n)$$

$$\mapsto (\mathcal{F}(\rho, \Phi) \circ \mathcal{F}^*(\rho, \Phi))^{-1}(g) \in C^\infty(\mathbb{R}P^n)$$

are smooth tame.

Proof. The map $((\rho, \Phi), g) \mapsto \mathcal{F}(\rho, \Phi)(\mathcal{F}^*(\rho, \Phi)(g))$ is the composition of the map $((\rho, \Phi), g) \mapsto ((\rho, \Phi), \mathcal{F}^*(\rho, \Phi)(g))$ and the map $\mathcal{F}$ of Lemma 8.5. Hence $L$ is smooth tame.
Recall that \((F(\rho, \Phi) \circ F^*(\rho, \Phi))^{-1}(g) = L(k(\rho, \Phi))^{-1}(g)\) for \(g \in C^\infty(\mathbb{R}^n)\). Choose \((\rho', \Phi')\), \(g'\) and \(q \in \mathbb{N}\). Then we write, using Theorem 6.1,

\[
\|L(k(\rho, \Phi))^{-1}(g) - L(k(\rho', \Phi'))^{-1}(g')\|_q \\
\leq \|L(k(\rho, \Phi))^{-1}(g) - L(k(\rho, \Phi))^{-1}(g')\|_q \\
+ \|L(k(\rho, \Phi))^{-1}(g') - L(k(\rho', \Phi'))^{-1}(g')\|_q \\
\leq c_q(\|g - g'\|_{n-1+q} + \|k(\rho, \Phi) - k(\rho', \Phi')\|_{C^{3n+q}}\|g'\|_{n-1+q})
\]

if \(\|k - k'\|_{3n+q}\) is sufficiently small. Hence, if \((\rho, \Phi)\) is \(C^{3n+3+q}\) close to \((\rho', \Phi')\) and \(g \in H^{p+n-1}\)-close to \(g'\) then \(\mathcal{V}((\rho, \Phi), g)\) is \(H^q\) close to \(\mathcal{V}((\rho', \Phi'), g')\). By the Sobolev embedding theorems, since \(q\) is arbitrary, we have that the map \(\mathcal{V}\) is continuous.

We will argue that the map

\[k : (\rho, \Phi) \in U \subset F \mapsto k(\rho, \Phi) \in C^\infty(B(\mathbb{R}^n \times \mathbb{R}^n))\]

given by Proposition 7.4 is smooth tame. Since \(D(\Phi)(y, v)\) (as in Section 7.2) is a nonlinear function of \((y, v), \Phi(x, v), \nabla^2V \Phi_v(x)\) where \(y = \exp_x(tv), x \in \Sigma_u, \) for \(|t| \leq 2\pi/5\), and \(D(\Phi)(y, v) = y\) for \(|t| \geq \pi/5\), we have that the map

\[D : \Phi \in C^\infty_{*, \text{odd}}(T_1S^n) \rightarrow D(\Phi) \in C^\infty(S^n \times S^n, S^n)\]

is smooth tame. Since \((y, v) \mapsto (D(\Phi)(y, v), v)\) is a diffeomorphism of \(S^n \times S^n\) with inverse \((p, v) \mapsto (D^{-1}(\Phi)(p, v), v)\), the map \(\Phi \mapsto D^{-1}(\Phi)\) is smooth tame and hence the map

\[I : \Phi \in C^\infty_{*, \text{odd}}(T_1S^n) \rightarrow I(\Phi) \in C^\infty(S^n \times S^n)\]

is smooth tame.

Recall the proof of Lemma 7.2. If

\[\lambda(\Phi)(v, \theta, t, x) = I_{\exp_v(\theta)}(\Phi)(x) - \cos(t)I_v(\Phi)(x),\]

then

\[\lambda(\Phi)(v, \theta, t, x) = t \frac{\sin(t)}{t} Q(\Phi)(v, \theta, t, x).\]

Hence

\[Q(\Phi)(v, \theta, t, x) = \frac{t}{\sin(t)} \int_0^1 \frac{\partial}{\partial t} \lambda(\Phi)(v, \theta, th, x) dh.\]

This implies that the map

\[Z : \Phi \in C^\infty_{*, \text{odd}}(T_1S^n) \rightarrow Z(\Phi) \in C^\infty(B(\mathbb{R}^n \times \mathbb{R}^n) \times S^n, \mathbb{R}^{n+1})\]

is smooth tame, by using formulas (28) and (29). This implies \(Z\) is smooth tame. Since we have that \((q, x) \mapsto (q, Z(\Phi)(q)(x))\) is a diffeomorphism of \(B(\mathbb{R}^n \times \mathbb{R}^n) \times S^n\) with inverse \((q, y) \mapsto (q, Z(\Phi)^{-1}(q)(y))\),
the map $\overline{Z}^{-1}$ and

$$J : \Phi \in C^\infty_{*, \text{odd}}(T_1 S^n) \to J(\Phi) \in C^\infty(B(\mathbb{R}^n \times \mathbb{R}^{mp}) \times S^n, S^n)$$

are smooth tame.

Using formulas \((32), (33), (34), (35)\) and an argument similar to what we used for the map $Z$, we conclude that the map $k$ is smooth tame.

By Proposition \((8.2)\) with $k_0 = k(0, 0)$, we have

$$\|f\|_q \leq c_q(||L(k(\rho, \Phi))(f)||_{n-1} + ||k(\rho, \Phi)||_{C^{n+q}} ||L(k(\rho, \Phi))(f)||_{n-1})$$

for every $f \in C^\infty(\mathbb{R}^m)$. Hence setting $f = \mathcal{V}(\rho, \Phi, g)$,

$$\|\mathcal{V}(\rho, \Phi, g)\|_q \leq c_q(||g||_{n-1} + ||k(\rho, \Phi)||_{C^{n+q}} ||g||_{n-1})$$

for every $g \in C^\infty(\mathbb{R}^m)$. If $\|g\|_{n-1} \leq C$, we get

$$\|\mathcal{V}(\rho, \Phi, g)\|_q \leq c_q(||g||_{n-1} + C ||k(\rho, \Phi)||_{C^{n+q}}$$

for every $q \geq 0$. Hence the tameness of $\mathcal{V}$ follows from the tameness of $k$. By \((12)\), Part II, Theorem 3.1.1, the map $\mathcal{V}$ is smooth tame.

\[\square\]

**Lemma 8.7.** The right-inverse of the Funk transform

$$\mathcal{R} : ((\rho, \Phi), g) \in (U \subset F) \times C^\infty(\mathbb{R}^m)$$

$$\mapsto \mathcal{F}^*(\rho, \Phi)((\mathcal{F}(\rho, \Phi) \circ \mathcal{F}^*(\rho, \Phi))^{-1}(g)) \in C^\infty(S^n)$$

is smooth tame.

**Proof.** Follows from Lemma \((8.5)\) and Lemma \((8.6)\) \[\square\]

**Lemma 8.8.** The solution map

$$\mathcal{S} : ((\rho, \Phi), \psi) \in (U \subset F) \times C^\infty_{0, \text{odd}}(T_1 S^n) \mapsto \mathcal{S}(\rho, \Phi)(\psi) \in C^\infty_{0, \text{odd}}(T_1 S^n),$$

of \((9)\) is smooth tame.

**Proof.** We can identify with a diffeomorphism a neighborhood of $\Sigma_v \times \{v\} \subset T_1 S^n$ with $S^{n-1} \times D^n$, where $D^n$ is an $n$-dimensional disk. In that way a function $\phi \in C^\infty(T_1 S^n)$ can be locally represented by a function $\phi'$ in $C^\infty(S^{n-1} \times D^n)$. We can use a diffeomorphism of the form $(x, w) \in S^{n-1} \times D^n \mapsto (R_w^{-1}(x), w) \in T_1 S^n$ where $R : D^n \to SO(n + 1)$ is a smooth map satisfying $R_w(w) = v$ (and $R_v = Id$), thinking of $D$ as a neighborhood of $v \in S^n$ and identifying $S^{n-1}$ with $\Sigma_v$. Hence the function $\phi'$ is linear in $x$ if and only if the function $\phi_w$ is linear for every $w \in D$. And

$$\int_{S^{n-1}} \phi'(x, w) \eta'(x, w) dA_{\text{can}}(x) = \int_{\Sigma_w} \phi(y, w) \eta(y, w) dA_{\text{can}}(y).$$
We will use [12], Part II, Section 3.3 on elliptic equations. It follows from (6) that the map \( \mathcal{L} \) sending \((\rho, \Phi)\) to the linear operator
\[
\phi \mapsto \frac{d}{dt|_{t=0}} \mathcal{H}(\rho, \Phi + t\phi)(x, v)
\]
seen, using the diffeomorphism, as a map taking values in
\[
C^\infty(S^{n-1} \times D^n, \pi_1^*(D^2(S^{n-1})))
\]
is smooth tame. Here \( \pi_1(x, w) = x \) and \( D^2(S^{n-1}) \) is the bundle of coefficients of linear differential operators of degree 2 on \( S^{n-1} \).

We take \( W \) to be the space of linear functions on \( \mathbb{R}^n \), \( j: C^\infty(S^{n-1}) \to W \) to be the orthogonal projection and \( i: W \to C^\infty(S^{n-1}) \) to be the inclusion. For some \( C^1 \) neighborhood \( \tilde{U} \) of \( \Delta_{S^{n-1}} + (n - 1) \) in \( C^\infty(S^{n-1}, D^2(S^{n-1})) \), the map
\[
\tilde{L}: (\tilde{U} \subset C^\infty(S^{n-1}, D^2(S^{n-1}))) \times C^\infty(S^{n-1}) \times W \to C^\infty(S^{n-1}) \times W
\]
given by \( \tilde{L}(f, h, p) = (k = L(f)h + ip, q = jh) \) is an isomorphism. Here \( L(f) \) denotes the linear operator associated to \( f \). By [12], Part II, Theorem 3.3.3, the solution
\[
\tilde{S}: (\tilde{U} \subset C^\infty(S^{n-1}, D^2(S^{n-1}))) \times C^\infty(S^{n-1}) \times W \to C^\infty(S^{n-1}) \times W,
\]
\( \tilde{S}(f, k, q) = (h, p) \), is smooth tame. Notice that if \( k \) is orthogonal to the linear functions and if \( \tilde{S}(f, k, 0) = (h, p) \), then \( h \) is orthogonal to the linear functions and
\[
(L(f)h)^\perp = k.
\]
In our case we are dealing with functions of an extra parameter (in \( C^\infty(S^{n-1} \times D^n) \)). Let \( \tilde{U}_D \subset C^\infty(S^{n-1} \times D^n, \pi_1^*D^2(S^{n-1})) \) be a \( C^1 \) neighborhood of \( \Delta_{S^{n-1}} + (n - 1) \) such that if \( f \in \tilde{U}_D \) then \( f_w \in \tilde{U} \) for every \( w \in D \). We can assume that \( \mathcal{L}(\rho, \Phi) \in \tilde{U}_D \) for all \( (\rho, \Phi) \in U \).

We denote by \( C^\infty_\perp(S^{n-1} \times D^n) \) the tame space of \( \psi' \in C^\infty(S^{n-1} \times D^n) \) such that \( x \mapsto \psi'(x, w) \) is orthogonal to the linear functions for every \( w \in D^n \). The same inductive scheme of Theorem II.3.3.1 and Theorem II.3.3.3 in [12] based on differentiating the equation can be used to show that the map defined by
\[
\tilde{S}_D(f, \psi')_w = \pi_2(\tilde{S}(f_w, \psi'_w, 0))
\]
for every \( w \in D^n \), where \( (f, \psi') \in \tilde{U}_D \times C^\infty_\perp(S^{n-1} \times D^n) \), takes values in \( C^\infty_\perp(S^{n-1} \times D^n) \) and is smooth tame. Hence the map
\[
\mathcal{S}': ((\rho, \Phi), \psi') \in (U \subset F) \times C^\infty_\perp(S^{n-1} \times D^n) \to C^\infty_\perp(S^{n-1} \times D^n)
\]
defined by
\[
\mathcal{S}'((\rho, \Phi), \psi') = \tilde{S}_D(\mathcal{L}((\rho, \Phi)), \psi')
\]
is smooth tame which proves $S$ is smooth tame.

\[\square\]

**Corollary 8.9.** The maps

\[V : U \subset F \times H \to H \quad \text{and} \quad Q : (U \subset F) \times H \times H \to H\]

of Section 5.1 are smooth tame.

**Proof.** By definition, $V = (V_1, V_2)$ where

\[V_1(\rho, \Phi)(b, \psi) = \frac{1}{n-1} \mathcal{R}(\rho, \Phi)(b)\]

and

\[V_2(\rho, \Phi)(b, \psi) = S(\rho, \Phi)(\psi - (D_1 \mathcal{H}(\rho, \Phi) \cdot f - jC(D_1 \mathcal{H}(\rho, \Phi) \cdot f))\]

with $f = V_1(\rho, \Phi) \cdot (b, \psi)$. By Lemmas 8.3, 8.4, 8.7 and 8.8, we conclude that $V_1$, $V_2$ and therefore $V$ are all smooth tame.

The map $Q$ can be written as the map

\[Q(\rho, \Phi) \cdot \{(\tilde{b}, \tilde{\psi}), (b, \psi)\} = (Q_1(\rho, \Phi) \cdot \{(\tilde{b}, \tilde{\psi}), (b, \psi)\}, 0)\]

where

\[Q_1(\rho, \Phi) \cdot \{(\tilde{b}, \tilde{\psi}), (b, \psi)\}(\sigma)\]

\[= \int_{\Sigma_n} \tilde{\psi}(x, v)(V_2(\rho, \Phi) \cdot (b, \psi))(x, v)dA_{can}(x)\]

\[\quad - \int_{\mathbb{R}^n} \left( \int_{\Sigma_n} \tilde{\psi}(x, v)(V_2(\rho, \Phi) \cdot (b, \psi))(x, v)dA_{can}(x) \right) dA_{can}(\sigma)\]

The tameness of $Q$ follows from the tameness of $V_2$.

\[\square\]

9. **Proofs of the Main Theorems**

We prove the theorems stated in the Introduction.

9.1. **Proof of Theorem A.** The Fréchet spaces appearing in the definitions of the maps $\Lambda$, $V$ and $Q$ are tame Fréchet spaces with their standard gradings by Lemma 8.4. The map $\Lambda$ defined in Section 5.1 is smooth tame by Lemma 8.4 and so are the maps $V$ and $Q$ defined in Section 5.2 due to Corollary 8.9. Hence, we can apply Theorem 5.1 and conclude the existence of an open subset $W \subset U \subset F$ containing the origin and a smooth tame map

\[\Gamma : \text{Ker}(DA(0, 0)) \cap W \to \Lambda^{-1}(0)\]

such that $\Gamma(0) = 0$ and $D\Gamma(0)v = v$ for all $v \in \text{Ker}(DA(0, 0))$. 


Proposition 9.1. The kernel of $D\Lambda(0,0)$ consists of all the pairs $(f, \phi)$ in $C^\infty(S^n) \times C^\infty_{0,\text{odd}}(T_1S^n)$ such that

i) $f$ is the sum of a constant function and an odd function;

ii) $\phi$ is such that

$$\Delta_{(\Sigma_v,\text{can})} \phi_v + (n-1)\phi_v = (n-1)\langle \nabla f, v \rangle \quad \text{on} \quad \Sigma_v$$

for every $v \in S^n$.

Moreover, every $f$ as in i) uniquely determines $\phi$ as in ii).

Proof. From (14) we have

$$D_1\Lambda_1(0,0) \cdot f = (n-1)\left( F(0,0)(f) - \int_{\mathbb{R}P^n} F(0,0)(f)(\sigma)dA_{\text{can}}(\sigma) \right).$$

Also $D_2\Lambda_1(0,0) \cdot \phi = 0$ since $\mathcal{H}(0,0) = 0$. Hence, we see from Lemma A.1 that $D\Lambda_1(0,0) \cdot (f, \phi) = 0$ is equivalent to $f$ being the sum of a constant function and an odd function.

From Proposition 3.1, item i),

$$(D_1\mathcal{H}(0,0) \cdot f)(x,v) = (n-1)\langle \nabla f(x), v \rangle$$

for all $(x,v)$ in $T_1S^n$. In particular $D_1\mathcal{H}(0,0) \cdot f \in C^\infty_{0,\text{odd}}(T_1S^n)$, because for each $v \in S^n$ the function $x \mapsto \langle \nabla f(x), v \rangle$ is an even function on $\Sigma_v$.

From (15) and (16) we see that $D\Lambda_2(0,0) \cdot (f, \phi) = 0$ is equivalent to

$$(n-1)\langle \nabla f, v \rangle + P_v(0,0)(\phi_v) = 0$$

for all $v \in S^n$. The result follows from Remark 3.2 and the fact that $\phi \in C^\infty_{0,\text{odd}}(T_1S^n)$. \hfill \Box

Given an arbitrary $f \in C^\infty_{\text{odd}}(S^n)$, there is a unique function $\phi \in C^\infty_{0,\text{odd}}(T_1S^n)$ such that

$$(f, \phi) \in \ker(D\Lambda(0,0))$$

by Proposition 9.1. Then, there exists $\delta > 0$ such that

$$(\rho_t, \Phi_t) = \Gamma(tf, t\phi) \in \Lambda^{-1}(0), \quad t \in (-\delta, \delta),$$

(as in Corollary 5.2) defines a path of conformal metrics $e^{2\rho_t\text{can}}$ on the sphere $S^n$ and families $\{\Sigma_\sigma(\Phi_t)\}_{\sigma \in \mathbb{R}P^n}$ of $e^{2\rho_t\text{can}}$-minimal spheres. Notice that $\rho_0 = 0$ and $\rho_0 = f$. A map $\lambda : W \rightarrow \mathcal{Z}$ can be defined as

$$\lambda(f) = e^{2\Gamma_1(f, -S(0,0)(D_1\mathcal{H}(0,0)\cdot f))\text{can}}.$$

Since $\frac{d}{dt}|_{t=0}\text{area}(\Sigma_\sigma(\Phi_t), e^{2\rho_t\text{can}}) = 0$, appropriate deformations in $\mathcal{Z}'$ can be obtained by scaling. This finishes the proof of Theorem A.
9.2. Proof of Theorem B. Let
\[ \mathcal{L} = \{ \Sigma_p : \sigma = [p] \in \mathbb{RP}^3 \} \]
be the space of totally geodesic spheres in \( S^3 \).

**Proposition 9.2.** For every neighborhood \( U \) of \( \mathcal{L} \) in the space of smoothly embedded spheres in \( S^3 \), there is a neighborhood \( V \) in the smooth topology of the canonical metric such that if \( g \in V \), every minimal embedded sphere of \( (S^3, g) \) lies in \( U \).

**Proof.** This follows by the compactness result of [5] applied to the space of minimal spheres and the fact that every minimal embedded sphere in \( (S^3, \text{can}) \) is totally geodesic \([1]\). □

We can choose \( V \) so that the Perturbation Theorem 3.2 of [19] gives a map
\[ \bar{\psi} : V \times \mathbb{RP}^3 \to U \]
so that \( \sigma \mapsto \bar{\psi}(g, \sigma) \) is an embedding for all \( g \in V \), \( \bar{\psi}(\text{can}, [p]) = \Sigma_p \), and \( \bar{\psi}(g, \mathbb{RP}^3) \) contains every minimal embedded sphere with respect to the metric \( g \) that lies in \( U \). Hence by the previous lemma \( \bar{\psi}(g, \mathbb{RP}^3) \) contains every minimal embedded sphere with respect to \( g \in V \).

Suppose \( g = e^{2\rho} \text{can} \in V \) is conformal to the standard metric. For \( [p] \in \mathbb{RP}^3 \), \( \bar{\psi}(g, [p]) \) can be written as a graph over \( \Sigma_p \) of a function \( \psi(\rho) \) as in (2). This defines a function \( \psi(\rho) \) on \( T_1 S^3 \) which according to Perturbation Theorem 3.2 of [19] will be of class \( C^k \) (where \( k \) can be chosen a priori) and \( C^k \) close to the origin (by adjusting the neighborhood \( V \)).

We now claim the existence of \( \alpha > 0 \) and smooth functions
\[ G_i' : T_1S^3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad i = 1, \ldots, 4, \]
such that for all \( v \in S^3 \), and \( i = 1, \ldots, 4, \)
\[ \int_{\Sigma_v} \psi(\rho)(v) x_i dA_{\text{can}} = \int_{\Sigma_v} G_i'(x, v, \psi(\rho)(v), |\nabla^{\Sigma_v} \psi(\rho)(v)|^2) dA_{\text{can}}(x) \quad (41) \]
and, for all \( \omega \in T_1S^3 \), \( s, t \in \mathbb{R} \), and \( i = 1, \ldots, 4, \) the following conditions are met:
\[ |G_i'(\omega, s, t)| + |D_\omega G_i'(\omega, s, t)| \leq \alpha(|s|^2 + |t|), \quad |D_s G_i'(\omega, s, t)| \leq \alpha(|s| + |t|), \quad \text{and} \quad |D_t G_i'(\omega, s, t)| \leq \alpha. \quad (42) \]

With \( M(4 \times 4) = \mathbb{R}^{16} \) being the space of \( 4 \times 4 \) matrices we choose
\[ \Omega : U \to M(4 \times 4), \quad \Omega(\Sigma) = \left( \int_{\Sigma} x_i x_j dA_{\text{can}} \right)_{i,j} . \]
The map \( \Omega \) is an embedding of \( \mathcal{L} \) and let \( L = \Omega(\mathcal{L}) \). Let \( P \) be the smooth map which projects a tubular neighborhood of \( L \) onto \( L \). The
map $\bar{\psi}$ (as constructed in [19, Theorem 3.2]) has the property that for all $\sigma \in \mathbb{RP}^3$

$$P \circ \Omega(\bar{\psi}(g, \sigma)) = P \circ \Omega(\bar{\psi}(\text{can}, \sigma))$$

and so

$$(\Omega(\bar{\psi}(g, \sigma)) - \Omega(\Sigma_\sigma)) \perp T_{\Omega(\Sigma_\sigma)}L.$$ \hfill (44)

Notice that

$$\Omega(\bar{\psi}(g, [v]))_{ij} - \Omega(\Sigma[v])_{ij} = \int_{\Sigma_v(\psi(\rho))} x_i x_j dA_{\text{can}} - \int_{\Sigma_v} x_i x_j dA_{\text{can}}$$

$$= \int_{\Sigma_v} \psi(\rho)_v(x_i \langle v, e_j \rangle + x_j \langle v, e_i \rangle) dA_{\text{can}}$$

$$+ \int_{\Sigma_v} G_{i,j}((x, v), \psi(\rho)_v |\nabla^{\Sigma_v} \psi(\rho)|^2) dA_{\text{can}}(x)$$

where $G_{i,j} : T_3 S^3 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function satisfying similar conditions to (42) and (43) for some constant $\alpha$. Set

$$G_i((x, v), s, t) = -\sum_{j=1}^4 \langle v, e_j \rangle G_{i,j}((x, v), s, t)$$

and

$$G((x, v), s, t) = \sum_{i,j=1}^4 \langle v, e_i \rangle \langle v, e_j \rangle G_{i,j}((x, v), s, t).$$

We obtain for $i = 1, \ldots, 4$,

$$w_i = \sum_{j=1}^4 \left( \Omega(\bar{\psi}(g, [v]))_{ij} - \Omega(\Sigma[v])_{ij} \right) \langle v, e_j \rangle$$

$$= \int_{\Sigma_v} \psi(\rho)_v x_i dA_{\text{can}} - \int_{\Sigma_v} G_i((x, v), \psi(\rho)_v |\nabla^{\Sigma_v} \psi(\rho)|^2) dA_{\text{can}}(x).$$

For each $Z$ orthogonal to $v$, we have that the vector $(Z_i v_j + Z_j v_i)_{i,j}$ is in $T_{\Omega(\Sigma[v])}L$. Hence condition (44) implies that

$$\sum_{i,j} (\Omega(\bar{\psi}(g, [v]))_{ij} - \Omega(\Sigma[v])_{ij}) (Z_i v_j + Z_j v_i)_{i,j} = 0$$

for such $Z$. Hence $\sum_i w_i Z_i = 0$. This implies that vector $w = \sum_{i=1}^4 w_i e_i \in \mathbb{R}^4$ is parallel to $v$ and thus $w = \langle w, v \rangle v$.

On the other hand

$$\langle w, v \rangle = \int_{\Sigma_v} G((x, v), \psi(\rho)_v |\nabla^{\Sigma_v} \psi(\rho)|^2) dA_{\text{can}}(x)$$
and so
\[
\int_{\Sigma_v} \psi(\rho)_v x_i dA_{can} = \langle v, e_i \rangle \int_{\Sigma_v} G((x, v), \psi(\rho)_v, |\nabla_{\Sigma_v} \psi(\rho)_v|^2) dA_{can}(x) \\
+ \int_{\Sigma_v} G_i((x, v), \psi(\rho)_v, |\nabla_{\Sigma_v} \psi(\rho)_v|^2) dA_{can}(x).
\]
This proves the desired claim.

Given \( \phi \in C^1(T_1 S^3) \) and \( u \in T_v S^3 \), we consider \( \partial_u \phi \in C^0(\Sigma_v) \) defined as
\[
\partial_u \phi(x) = d\phi(x, v)(-\langle x, u \rangle v, u).
\]
We will need the following pointwise estimate
\[
|\partial_u |\nabla_{\Sigma_v} \psi(\rho)_v|^2| \leq \alpha_1 (|\nabla_{\Sigma_v} \partial_u \psi(\rho)|^2 + |\nabla_{\Sigma_v} \psi(\rho)_v|^2),
\]
where \( \alpha_1 \) is some positive constant.

Recall that \( \Sigma(\psi(\rho))(x, v) = \cos(\psi(\rho)(x, v)) x + \sin(\psi(\rho)(x, v)) v \). We define \( \beta(\rho) = \rho \circ \Sigma(\psi(\rho)) \).

From (4) and Lemma 2.1 (iii) we see that there is a smooth function \( A : T_1 S^3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) such that
\[
A(\rho, \psi(\rho))(v) = 4\pi + 2 \int_{\Sigma_v} \beta(\rho) dA_{can} \\
+ \int_{\Sigma_v} A(\beta(\rho), \psi(\rho)_v, |\nabla_{\Sigma_v} \psi(\rho)_v|^2) dA_{can}(x),
\]
where \( A \) satisfies for some other positive constant \( \alpha \),
\[
|A(l, s, t)| \leq \alpha (|l|^2 + |s|^2 + |t|),
\]
and
\[
|D_l A|(\omega, l, s, t) \leq \alpha (|l| + |s|^2 + |t|), |D_s A|(\omega, l, s, t) \leq \alpha |s|, |D_t A|(\omega, s, t) \leq \alpha.
\]

We can suppose that the \( C^{3,\alpha} \)-norm of \( \psi(\rho) \) and \( \rho \) are uniformly bounded by some positive constant \( C_0 \) that is independent of \( \rho \in V \).

We use the notation \( f \lesssim g \) to mean an inequality of the form \( f \leq C g \) where \( C \) is a constant that depends only on \( C_0 \) and \( \alpha \).

The \( k \)-derivatives of \( \phi \in C^k(T_1 S^3) \) are denoted by \( D^k \phi \). We define
\[
\nabla^k \beta(\rho) = \nabla^k \rho \circ \Sigma(\psi(\rho)), \quad k = 1, 2.
\]
With this notation we have for all \( (x, v) \in T_1 S^3 \)
\[
|D \beta(\rho)|(x, v) \lesssim |\nabla \beta(\rho)|(x, v)
\]
and
\[ |D^2\beta(\rho)|(x, v) \lesssim |\nabla^2\beta(\rho)|(x, v) + |\nabla\beta(\rho)|(x, v). \]
We use \( ||\phi||_{H^k(S_v)} \) to denote \( ||\phi_v||_{H^k(S_v)} \). We write \( ||\nabla^k\beta(\rho)||_{L^2(S_v)} \) to denote the \( L^2 \)-norm of \( |\nabla^k\rho| \circ \Sigma(\psi(\rho)) \) restricted to \( S_v \).

**Lemma 9.3.** We have (by adjusting \( V \) if necessary) that for all metrics \( e^{2\rho} \in V \cap \mathcal{Z} \):
\[
||\psi(\rho)||_{H^2(S_v)} \lesssim ||\nabla\beta(\rho)||_{L^2(S_v)}
\]
and
\[
||\partial_u\psi(\rho)||_{H^2(S_v)} \lesssim ||\nabla\beta(\rho)||_{L^2(S_v)} + ||\nabla^2\beta(\rho)||_{L^2(S_v)}
\]
for all \( v \in S^3 \) and unit vectors \( u \in T_vS^3 \).

*Proof.* Since the generalized Gauss map \( G(\psi(\rho)) \) is a diffeomorphism, for every tangent plane at some point of \( S^3 \) there is a unique surface among the \( \{ \Sigma_\sigma(\psi(\rho)) \}_{\sigma \in \mathcal{S}_n} \) tangent to it. Therefore if \( g = e^{2\rho} \) can \( \in \mathcal{Z} \) it follows that all \( \Sigma_\sigma(\psi(\rho)) \) are minimal in \( (S^3, g) \).

Using (6) and the fact that \( \mathcal{H}(\rho, \psi(\rho)) = 0 \) we see that \( \mathcal{H}(0, \psi(\rho)) \) can be expressed in terms of \( \psi(\rho), D\psi(\rho) \) and \( \nabla\beta(\rho) \). For our purposes the important properties are that, for all \( v \in S^3 \),
\[
\int_{\Sigma_v} |\mathcal{H}(0, \psi(\rho))|^2(x, v) dA_{can}(x) \lesssim ||\nabla\beta(\rho)||_{L^2(S_v)}^2
\]
and, for every unit vector \( u \in T_vS^3 \),
\[
\int_{\Sigma_v} |\partial_u \mathcal{H}(0, \psi(\rho))|^2 dA_{can} \lesssim ||\nabla\beta(\rho)||_{L^2(S_v)}^2 + ||\nabla^2\beta(\rho)||_{L^2(S_v)}^2.
\]

From the expression for \( \mathcal{H}(0, \psi(\rho)) \) given by (6) we see that
\[
E(\rho) = \mathcal{J}_v(0, 0)(\psi(\rho)) - \mathcal{H}(0, \psi(\rho))
\]
is a quadratic term:
\[
|E(\rho)|(x, v) \lesssim \psi(\rho)^2(x, v) + |\nabla^{\Sigma_v}\psi(\rho)|^2(x, v).
\]
Differentiating both sides of the expression for \( E(\rho) \) we have that for all \( u \in T_vS^3 \),
\[
E_u(\rho) = \mathcal{J}_v(0, 0)(\partial_u\psi(\rho)) - \partial_u \mathcal{H}(0, \psi(\rho))
\]
is also a quadratic term:
\[
|E_u(\rho)|(x, v) \lesssim \psi(\rho)^2(x, v) + |\nabla^{\Sigma_v}\psi(\rho)|^2(x, v) + |(\nabla^{\Sigma_v})^2\psi(\rho)|^2(x, v).
\]
Denote by $\phi^T_v$ the projection of $\phi_v$ onto the kernel of $J_v(0,0)$. Standard energy estimates, (49), and (51) show that
\[
||\psi(\rho)||_{H^2(\Sigma_v)} \lesssim ||H(0, \psi(\rho))||_{L^2(\Sigma_v)} + ||E(\rho)||_{L^2(\Sigma_v)} + ||\psi(\rho)^T||_{L^2(\Sigma_v)} \\
\lesssim ||\nabla \beta(\rho)||_{L^2(\Sigma_v)} + ||\psi(\rho)||^2_{W^{1,4}(\Sigma_v)} + ||\psi(\rho)^T||_{L^2(\Sigma_v)}.
\]
Likewise, energy estimates, (50), and (52), show that
\[
||\partial_u \psi(\rho)||_{H^2(\Sigma_v)} \lesssim ||\partial_u H(0, \psi(\rho))||_{L^2(\Sigma_v)} + ||E_u(\rho)||_{L^2(\Sigma_v)} + ||\partial_u \psi(\rho)^T||_{L^2(\Sigma_v)} \\
\lesssim ||\nabla \beta(\rho)||_{L^2(\Sigma_v)} + ||\nabla^2 \beta(\rho)||_{L^2(\Sigma_v)} + ||\psi(\rho)||^2_{W^{1,4}(\Sigma_v)} + ||\partial_u \psi(\rho)^T||_{L^2(\Sigma_v)}.
\]
From (41) and (42):
\[
||\psi(\rho)^T||_{L^2(\Sigma_v)} \lesssim ||\psi(\rho)||^2_{H^1(\Sigma_v)}
\] (54)
and so, using the previous estimate for $||\psi(\rho)||_{H^2(\Sigma_v)}$ we obtain
\[
||\psi(\rho)||_{H^2(\Sigma_v)} \lesssim ||\nabla \beta(\rho)||_{L^2(\Sigma_v)} + ||\psi(\rho)||^2_{H^1(\Sigma_v)} + ||\psi(\rho)||^2_{W^{1,4}(\Sigma_v)}.
\] (55)
The continuity of the map $\psi$ implies that if we reduce the neighborhood $V$ further we can absorb the last term of the right hand side and conclude
\[
||\psi(\rho)||_{H^2(\Sigma_v)} \lesssim ||\nabla \beta(\rho)||_{L^2(\Sigma_v)}.
\]
We now estimate $||\partial_u \psi(\rho)^T||_{L^2(\Sigma_v)}$. Differentiating the left-hand side of (41) with respect to $u \in T_v S^3$ we obtain
\[
\int_{\Sigma_v} \partial_u \psi(\rho)_v x_idA_{can} - \langle v, e_i \rangle \int_{\Sigma_v} \psi(\rho)_v \langle x, u \rangle dA_{can}.
\]
Differentiating the right-hand side of (41) with respect to the unit vector $u \in T_v S^3$ we obtain (with simplified notation)
\[
\int_{\Sigma_v} D\omega G'_i(-\langle x, u \rangle v, u) dA_{can}(x)
\]
\[
+ \int_{\Sigma_v} D_s G'_i D_u \psi(\rho)_v dA_{can} + \int_{\Sigma_v} D_t G'_i \partial_u |\nabla \psi(\rho)_v|^2 dA_{can}.
\]
Therefore, using the pointwise estimate (15), we deduce from (42), and (43), that
\[
||\partial_u \psi(\rho)^T||_{L^2(\Sigma_v)} \lesssim ||\psi(\rho)^T||_{L^2(\Sigma_v)} + ||\psi(\rho)||_{H^1(\Sigma_v)} + ||\partial_u \psi(\rho)||^2_{H^1(\Sigma_v)}.
\]
Using (54) this simplifies to
\[
||\partial_u \psi(\rho)^T||_{L^2(\Sigma_v)} \lesssim ||\psi(\rho)||_{H^1(\Sigma_v)} + ||\partial_u \psi(\rho)||^2_{H^1(\Sigma_v)}.
\]
Inserting this inequality in (53) and using (55) we have

\[ ||\partial_u \psi(\rho)||_{H^2(\Sigma_v)} \lesssim ||\nabla \beta(\rho)||_{L^2(\Sigma_v)} + ||\nabla^2 \beta(\rho)||_{L^2(\Sigma_v)} \]

\[ + ||\nabla \beta(\rho)||_{L^2(\Sigma_v)}^2 + ||\partial_u \psi(\rho)||_{H^1(\Sigma_v)}^2. \]

Reducing the neighborhood \( V \subset C^7(S^3) \) if necessary, we can absorb
the last terms on the right-hand side and conclude

\[ ||\partial_u \psi(\rho)||_{H^2(\Sigma_v)} \lesssim ||\nabla \beta(\rho)||_{L^2(\Sigma_v)} + ||\nabla^2 \beta(\rho)||_{L^2(\Sigma_v)}. \]

Set \( c(\rho) = 2\pi - A(\rho, \psi(\rho))/2. \)

**Lemma 9.4.** We have (by adjusting \( V \) if necessary) that for all metrics \( e^{2\rho} \text{can} \in V \cap \mathcal{Z}, \)

\[ |\mathcal{F}(\rho) + c(\rho)| \lesssim ||\beta(\rho)||_{L^2(\Sigma_v)}^2 + ||\nabla \beta(\rho)||_{L^2(\Sigma_v)}^2 + ||\nabla \rho||_{L^2(\Sigma_v)}^2 \]

and

\[ |\nabla \mathcal{F}(\rho)(v)| \lesssim ||\nabla \rho||_{L^2(\Sigma_v)}^2 + ||\nabla^2 \rho||_{L^2(\Sigma_v)}^2 \]

\[ + ||\beta(\rho)||_{L^2(\Sigma_v)}^2 + ||\nabla \beta(\rho)||_{L^2(\Sigma_v)}^2 + ||\nabla^2 \beta(\rho)||_{L^2(\Sigma_v)}^2. \]

**Proof.** From (46) and (47) we have that for all \( v \in S^3 \)

\[ 2 \int_{\Sigma_v} \beta(\rho) dA_{\text{can}} + 4\pi - A(\rho, \psi(\rho)) \lesssim ||\beta(\rho)||_{L^2(\Sigma_v)}^2 + ||\psi(\rho)||_{H^1(\Sigma_v)}^2. \]

We have that for all \( (x, v) \in T_v S^3 \)

\[ |\beta(\rho)(x, v) - \rho(x)| \lesssim |\nabla \rho(x)|^2 + |\psi(\rho)(x, v)|^2. \]

Thus

\[ |\mathcal{F}(\rho) + c(\rho)| \lesssim ||\beta(\rho)||_{L^2(\Sigma_v)}^2 + ||\psi(\rho)||_{H^1(\Sigma_v)}^2 + ||\nabla \rho||_{L^2(\Sigma_v)}^2, \]

which when combined with Lemma 9.3 implies

\[ |\mathcal{F}(\rho) + c(\rho)| \lesssim ||\beta(\rho)||_{L^2(\Sigma_v)}^2 + ||\nabla \beta(\rho)||_{L^2(\Sigma_v)}^2 + ||\nabla^2 \beta(\rho)||_{L^2(\Sigma_v)}^2. \]

This proves the first estimate.

To prove the second estimate we use the fact that \( c(\rho) \) is constant to
differentiate (46) in the direction \( u \in T_v S^3 \) (with \( u \) being a unit vector) to obtain, in light of (47), (48), and the pointwise estimate (45),

\[ \left| \int_{\Sigma_v} \partial_u \beta(\rho) dA_{\text{can}} \right| \lesssim ||\beta(\rho)||_{L^2(\Sigma_v)}^2 + ||\partial_u \beta(\rho)||_{L^2(\Sigma_v)}^2 \]

\[ + ||\partial_u \psi(\rho)||_{H^1(\Sigma_v)}^2 + ||\psi(\rho)||_{H^1(\Sigma_v)}^2. \quad (56) \]
Set $\Sigma(t)(x, v) := \cos(t\psi(\rho))x + \sin(t\psi(\rho))v$ and
\[ f(t) := \partial_u(\rho \circ \Sigma(t)) \in C^{2,\alpha}(T_1S^3). \]
Notice that $f(1) = \partial_u\beta(\rho), f(0)(x, v) = -\langle x, u \rangle \langle \nabla \rho(x), v \rangle$, and
\[ \langle \nabla F(\rho)(v), u \rangle = \int_{\Sigma_v} f(0) dA_{can}(x). \]

An explicit computation shows
\[ f'(0) = \partial_u(\psi(\rho) \langle \nabla \rho(x), v \rangle) \]
\[ = \langle \nabla \rho, v \rangle \partial_u \psi(\rho) - \langle x, u \rangle \nabla^2 \rho(v, v) \psi(\rho) + \langle \nabla \rho, u \rangle \psi(\rho) \]
and
\[ |f''(t)(x, v)| \lesssim |\partial_\alpha \psi(\rho)(x, v)|^2 + |\psi(\rho)(x, v)|^2 \]
for all $0 \leq t \leq 1, (x, v) \in T_1S^3$. Hence
\[ \left| \langle \nabla F(\rho)(v), u \rangle - \int_{\Sigma_v} \partial_u \beta(\rho) dA_{can} \right| \leq \int_{\Sigma_v} |f(0) - f(1)| dA_{can} \]
\[ \lesssim ||\partial_u \psi(\rho)||^2_{L^2(\Sigma_v)} + ||\psi(\rho)||^2_{L^2(\Sigma_v)} + ||\nabla \rho||^2_{L^2(\Sigma_v)} + ||\nabla^2 \rho||^2_{L^2(\Sigma_v)}. \]
Combining with Lemma 9.3 and (56) we deduce
\[ |\langle \nabla F(\rho)(v), u \rangle| \lesssim ||\nabla \rho||^2_{L^2(\Sigma_v)} + ||\nabla^2 \rho||^2_{L^2(\Sigma_v)} \]
\[ + ||\beta(\rho)||^2_{L^2(\Sigma_v)} + ||\nabla \beta(\rho)||^2_{L^2(\Sigma_v)} + ||\nabla^2 \beta(\rho)||^2_{L^2(\Sigma_v)}. \]
The arbitrariness of the unit vector $u \in T_vS^3$ implies the result.

We need the following lemma:

**Lemma 9.5.** If $\Phi \in C^2_{*,\text{odd}}(T_1S^n)$ has sufficiently small $C^2$ norm, then the following holds for some positive dimensional constant $c_n$. Given $f \in C^0(S^n)$, set $f(\Phi) = f \circ \Sigma(\Phi) \in C^0(T_1S^n)$. Then
\[ \int_{S^n} ||f(\Phi)||^2_{L^2(\Sigma_v)} dV_{can}(v) \leq c_n ||f||^2_{L^2(S^n)} \]
and
\[ \int_{S^n} ||f(\Phi)||^4_{L^2(\Sigma_v)} dV_{can}(v) \leq c_n ||f||^4_{L^4(S^n)}. \]

**Proof.** Let $\pi_1$ and $\pi_2$ denote the projections of points in $F(\Phi)$ onto the first and second coordinates. If $\Phi = 0, F(0) = T_1S^n = \{(p, v) \in S^n \times S^n : \langle p, v \rangle = 0\}$. We have that $|\text{Jac}(\pi_1)| = \sqrt{\det |D\pi_1 \circ (D\pi_1)^*|} = \frac{1}{\sqrt{2}}$ and similarly $|\text{Jac}(\pi_2)| = \sqrt{\det |D\pi_2 \circ (D\pi_2)^*|} = \frac{1}{\sqrt{2}}$.

For general $\Phi$, we can assume that both Jacobians are in $(1/2, 1)$, and similarly that $|\text{Jac}(\Sigma_v(\Phi))| \geq 1/2$ for all $v \in S^n$. Also we have
that $\Sigma_p^*(\Phi)$ is $C^1$ close to $\Sigma'_p(0)$ for all $p \in S^n$. Thus we can assume that $\text{area}(\Sigma_p^*(\Phi), \text{can}) \leq \omega_{n-1}$ for all $p \in S^n$, where $\omega_{n-1} = \text{area}(S^{n-1}, \text{can})$. (The area of $\Sigma'_p(0)$ is $\omega_{n-1}/2$).

We have that
\[ \int_{S^n} ||f(\Phi)||^2_{L^2(\Sigma^c)} dV_{\text{can}}(v) = \int_{S^n} \int_{\Sigma_v} |f \circ \Sigma(\Phi)|^2(x, v) dA_{\text{can}}(x) dV_{\text{can}}(v), \]
hence
\[ \int_{S^n} ||f(\Phi)||^2_{L^2(\Sigma^c)} dV_{\text{can}}(v) \leq 2 \int_{S^n} \int_{\Sigma_v(\Phi)} |f|^2(p) dA_{\text{can}}(p) dV_{\text{can}}(v) \]
\[ = 2 \int_{S^n} \int_{(\pi_2)^{-1}(\pi_1(v))} |f|^2 \circ \pi_1 dA_{\text{can}} dV_{\text{can}}(v). \]

Using the co-area formula [4, Chapter III] for $\pi_2 : F(\Phi) \rightarrow S^n$ we deduce
\[ \int_{S^n} ||f(\Phi)||^2_{L^2(\Sigma^c)} dV_{\text{can}}(v) \leq 2 \int_{F(\Phi)} |f|^2(p) |\text{Jac}(\pi_2)|(p, v) dV_{\text{can}}(p, v) \]
\[ = 2 \int_{F(\Phi)} |f|^2(p) \frac{|\text{Jac}(\pi_2)|(p, v)}{|\text{Jac}(\pi_1)|(p, v)} |\text{Jac}(\pi_1)|(p, v) dV_{\text{can}}(p, v). \]

Applying the co-area formula again for $\pi_1 : F(\Phi) \rightarrow S^n$
\[ \int_{S^n} ||f(\Phi)||^2_{L^2(\Sigma^c)} dV_{\text{can}}(v) \]
\[ \leq 2 \int_{S^n} |f|^2(p) \int_{(\pi_1)^{-1}(\pi_2)(p)} \frac{|\text{Jac}(\pi_2)|(p, v)}{|\text{Jac}(\pi_1)|(p, v)} dA_{\text{can}}(v) dV_{\text{can}}(p) \]
\[ \leq 8 \omega_{n-1} \int_{S^n} |f|^2(p) dV_{\text{can}}(p). \]

The second inequality follows from the first inequality because we have $\||f(\Phi)||_{L^2(\Sigma^c)}^2 \leq c_n ||f(\Phi)||_{L^2(\Sigma^c)}^2$ and $||f^2||_{L^2(S^n)}^2 = ||f||_{L^4(S^n)}^4$. □

We have from Lemma 9.5 that
\[ \int_{S^3} ||\beta(\rho)||_{L^2(\Sigma^c)}^4 dV_{\text{can}} \lesssim ||\rho||_{L^4(S^3)}^4 \]
\[ \int_{S^3} ||\nabla \beta(\rho)||_{L^2(\Sigma^c)}^4 dV_{\text{can}} \lesssim ||\nabla \rho||_{L^4(S^3)}^4 \]
\[ \int_{S^3} ||\nabla^2 \beta(\rho)||_{L^2(\Sigma^c)}^4 dV_{\text{can}} \lesssim ||\nabla^2 \rho||_{L^4(S^3)}^4. \]
Similar inequalities hold for \( ||\nabla \rho||_{L^2(S^3)}^2 \) and \( ||\nabla^2 \rho||_{L^2(S^3)}^2 \). Using these inequalities in Lemma 9.4 we have

\[
\int_{S^3} |F(\rho) + c(\rho)|^2 + |\nabla F(\rho)|^2 dV_{\text{can}} \\
\lesssim ||\rho||_{L^4(S^3)}^4 + ||\nabla \rho||_{L^4(S^3)}^4 + ||\nabla^2 \rho||_{L^4(S^3)}^4. \tag{57}
\]

**Lemma 9.6.** For a metric \( e^{2\rho_{\text{can}}} \in V \cap \mathcal{Z} \) with \( \int_{S^3} \rho dV_{\text{can}} = 0 \), we have

\[
|c(\rho)| \lesssim ||\rho||_{L^2(S^3)}^2 + ||\nabla \rho||_{L^2(S^3)}^2.
\]

**Proof.** Integrating \( F(\rho) \) over \( S^3 \) and using the co-area formula we have the identity

\[
\int_{S^3} F(\rho)(v) dV_{\text{can}} = 4\pi \int_{S^3} \rho(x) dV_{\text{can}}(x) = 0.
\]

Thus from Lemma 9.4 we deduce

\[
2\pi^2 |c(\rho)| = \left| \int_{S^3} F(\rho)(v) dV_{\text{can}} + 2\pi^2 c(\rho) \right| = \left| \int_{S^3} (F(\rho) + c(\rho)) dV_{\text{can}} \right| \\
\lesssim \int_{S^3} ||\beta(\rho)||_{L^2(S^3)}^2 + ||\nabla \beta(\rho)||_{L^2(S^3)}^2 + ||\nabla \rho||_{L^2(S^3)}^2 dV_{\text{can}} \\
\lesssim ||\rho||_{L^2(S^3)}^2 + ||\nabla \rho||_{L^2(S^3)}^2,
\]

where in the last inequality we used Lemma 9.5. \( \square \)

To prove Theorem B, we can suppose that \( \rho \) is even and \( \int_{S^3} \rho dV_{\text{can}} = 0 \).

The previous lemma and (57) show that

\[
||F(\rho)||_{H^1(S^3)}^2 = \int_{S^3} |F(\rho)|^2 + |\nabla F(\rho)|^2 dV_{\text{can}} \\
\lesssim ||\rho||_{L^4(S^3)}^4 + ||\nabla \rho||_{L^4(S^3)}^4 + ||\nabla^2 \rho||_{L^4(S^3)}^4 = ||\rho||_{W^{2,4}(S^3)}^4.
\]

Because we are assuming that \( \rho \) is even, we know from Lemma A.1 (ii) that \( ||\rho||_{L^2(S^3)} \lesssim ||F(\rho)||_{H^1(S^3)} \). Hence we obtain \( ||\rho||_{L^2(S^3)} \lesssim ||\rho||_{W^{2,4}(S^3)}^2 \).

From Gagliardo–Nirenberg interpolation inequality we have that

\[
||\rho||_{W^{2,4}(S^3)}^2 \lesssim ||\rho||_{C^4(S^3)} ||\rho||_{L^2(S^3)}.
\]

Hence

\[
||\rho||_{L^2(S^3)} \lesssim ||\rho||_{C^4(S^3)} ||\rho||_{L^2(S^3)}.
\]

Thus by adjusting \( V \) we can absorb the right-hand side on the left-hand side. This proves that \( \rho = 0 \), and hence \( e^{2\rho_{\text{can}}} = \text{can} \) proving the Theorem.
9.3. Proof of Theorem C. Let $W \subset C^\infty_{odd}(S^n)$ denote the space of linear functions. Consider the set $W^\perp \cap C^\infty_{odd}(S^n)$ of functions that are $L^2$-orthogonal to $W$. Let $X$ be the set of all $r \in W^\perp \cap C^\infty_{odd}(S^n)$ that are Morse with critical points $\{x_1, \ldots, x_k\} \subset S^n$, depending on $r$, satisfying
\[ r(x_i) \neq r(x_j) \]
for $i \neq j$, such that $\nabla^2 r(x_i)$ has distinct eigenvalues (with respect to the round metric) for each $i$, and such that $\nabla^3 r(x_i) \cdot (w, w, w) \neq 0$ for each $i$ and $w$ eigenvector of $\nabla^2 r(x_i)$. Since $S^n$ can be embedded in some Euclidean space by spherical harmonics of degree three, a classic result says that any smooth function on $S^n$ can be perturbed by one of these to become Morse. This implies the set of Morse functions is open and dense in $W^\perp \cap C^\infty_{odd}(S^n)$, from which one can see $X$ is also open and dense.

Let $r \in X$ and $f \in C^\infty(S^n)$ be the unique solution in $W^\perp \subset C^\infty(S^n)$ of
\[ \Delta f + nf = -\frac{1}{2(n-1)}r. \]
Uniqueness implies that $f \in C^\infty_{odd}(S^n)$.

We apply Theorem A to $f$ to get $\rho_t$. Let $g(t) = e^{2\rho_t}c$. The scalar curvature $R_{g(t)}$ of $g(t)$ can be computed as
\[ R_{g(t)} = e^{-2\rho_t} \left( n(n-1) - 2(n-1)\Delta \rho_t - (n-1)(n-2)|\nabla \rho_t|^2 \right). \]
Hence
\[ \frac{d}{dt} \bigg|_{t=0} R_{g(t)} = -2(n-1)(\Delta f + nf) = r. \]
This implies
\[ \frac{R_{g(t)} - n(n-1)}{t} = r + O(t). \]
Since the set of Morse functions is open in $C^2(S^n)$, for sufficiently small $t$ we have that $R_{g(t)}$ is Morse and has as many critical points as $r$. If $I_t \in Isom(S^n, g(t))$, we have that $R_{g(t)}(I_t(x)) = R_{g(t)}(x)$ for every $x \in S^n$. Then $I_t$ leaves invariant the set of critical points of $R_{g(t)}$, which in turn converges to the set of critical points of $r$ as $t \to 0$.

If $\{x_1, \ldots, x_k\}$ is the set of critical points of $r$, let $\{x_1(t), \ldots, x_k(t)\}$ be the set of critical points of $R_{g(t)}$ with $x_i(t) \to x_i$ as $t \to 0$. We have
\[ R_{g(t)}(x_i(t)) = n(n-1) + tr(x_i(t)) + O(t^2) \]
\[ = n(n-1) + tr(x_i) + o(t). \]
The conditions satisfied by $r$ then imply that $I_t(x_i(t)) = x_i(t)$ for each $i$. Hence $I_t$ is determined by any of the linear maps $DI_t(x_i(t)) : T_{x_i(t)}S^n \to T_{x_i(t)}S^n$. 
Now \( \frac{d}{dt}|_{t=0} \nabla^2_{g(t)} R_{g(t)} = \nabla^2_{can} r \), since \( R_{can} \) is constant. Therefore

\[
\frac{\nabla^2_{g(t)} R_{g(t)}(x_i(t))}{t} = \nabla^2_{can} r(x_i(t)) + O(t),
\]

which implies that for each \( i \) the bilinear form \( \nabla^2_{g(t)} R_{g(t)}(x_i(t)) \) has distinct eigenvalues with respect to \( g(t) \). Since \( I_t \) is an isometry of \( (S^n, g(t)) \) that fixes the critical points of \( R_{g(t)} \), we have that

\[
\nabla^2_{g(t)} R_{g(t)}(x_i(t))(DI_t(x_i(t)) \cdot v, DI_t(x_i(t)) \cdot v) = \nabla^2_{g(t)} R_{g(t)}(x_i(t))(v, v)
\]

for each \( i \) and \( v \in T_{x_i(t)} S^n \). Then \( DI_t(x_i(t)) \) will send an eigenvector of \( \nabla^2_{g(t)} R_{g(t)}(x_i(t)) \) into an eigenvector of \( \nabla^2_{g(t)} R_{g(t)}(x_i(t)) \) with the same eigenvalue. Since the eigenvalues are distinct, \( DI_t(x_i(t)) \cdot w = \pm w \) for each eigenvector \( w \).

Suppose there is a sequence of positive numbers \( t_j \to 0 \) such that \( (S^n, g(t_j)) \) admits an isometry \( I_{t_j} \) that is not the identity map. Then \( I_{t_j} \to I \in O(n + 1) \) (after maybe passing to a subsequence). For each \( x_i \in S^n \) there exists \( w_i \in T_{x_i} S^n \) eigenvector of \( \nabla^2 r(x_i) \) such that \( DI(x_i) \cdot w_i = -w_i \). Since \( r(I(x)) = r(x) \) for every \( x \in S^n \), we get \( \nabla^3 r(x_i)(w_i, w_i, w_i) = 0 \). This is a contradiction and hence for sufficiently small \( t > 0 \) the metric \( (S^n, e^{2t}can) \) has no nontrivial isometry.

If \( f \in W \), the same arguments apply to \( f + l \) instead of \( f \) since we also have \( \Delta(f + l) + n(f + l) = -\frac{1}{2(n-1)} r \). The space of all such functions \( f + l \) is open and dense in \( C^\infty_{odd}(S^n) \) as desired. (An alternative proof of the result can be obtained by generalizing [2], 4.71.)

### 9.4. Proof of Theorem D

We will start by establishing a correspondence between the set of Riemannian metrics on \( S^n \) with minimal equators and the set of positive definite Killing symmetric tensors on \( (S^n, can) \). The explicit description given here is based on the method of Hangan for the Euclidean space [13]. It can also be obtained from [13] by use of the gnomonic projection.

Let \((M^n, g)\) be a Riemannian manifold. A symmetric \( p \)-tensor \( k \) on \( M \) is called Killing when the symmetrization of its covariant derivative \( \nabla^g k \) vanishes. The vector space consisting of Killing symmetric \( p \)-tensors will be denoted by \( K_p(M^n, g) \). For instance, \( k \in K_2(M^n, g) \) if and only if

\[
\nabla^g k(X, Y, Z) + \nabla^g k(Y, Z, X) + \nabla^g k(Z, X, Y) = 0
\]

for all vector fields \( X, Y \) and \( Z \) on \( M^n \).

Given Killing vector fields \( K_1, \ldots, K_p \) on \((M^n, g)\), the symmetric
product \( K_1 \odot \ldots \odot K_p \), defined by

\[
(K_1 \odot \ldots \odot K_p)(X_1, \ldots, X_p) = \sum_{\sigma} g(K_{\sigma(1)}, X_1) \cdot \ldots \cdot g(K_{\sigma(p)}, X_p)
\]  

(58)

is an element of \( K_p(M^n, g) \). In the above formula, the summation is over the set of all permutations of elements of the set \( \{1, \ldots, p\} \). It was proven in \([17]\) that all symmetric Killing \( p \)-tensors in \((S^n, \text{can})\) are linear combinations of symmetric products of \( p \) Killing vector fields as in (58). The dimension of the space of Killing \( p \)-tensors of \((S^n, \text{can})\) has been computed \([18]\): for every integer \( p \geq 1 \),

\[
\dim K_p(S^n, \text{can}) = \frac{1}{n} \binom{n+p}{p+1} \binom{n+p-1}{p}.
\]

Given a three-tensor \( t \) on \( S^n \), we denote by \( t^S \) its cyclic symmetrization given by

\[
t^S(X,Y,Z) = t(X,Y,Z) + t(Y,Z,X) + t(Z,X,Y).
\]

**Proposition 9.7.** Let \( g \) be a Riemannian metric on an open subset \( W \) of the sphere \( S^n \). The following statements are equivalent

i) All \((n-1)\)-equators intersecting \( W \) are minimal in \((S^n, g)\).

ii) The metric \( g \) and the smooth positive function \( \psi \) on \( W \) uniquely defined by \( dV_g = \psi dV_{\text{can}} \) are such that

\[
\left( \nabla^\text{can} g - \frac{4}{n+1} d \log(\psi) \otimes g \right)^S = 0 \quad \text{on} \quad W.
\]

**Proof.** A point \( p \) in \( W \) belongs to the \((n-1)\)-equator \( \Sigma_v, v \neq 0 \), if and only if \( v \) is orthogonal to \( p \), that is if and only if \( v \) belongs to \( T_p S^n \). Let \( v \in \mathbb{R}^{n+1} \setminus \{0\} \). Define \( \hat{V} : S^n \to \mathbb{R} \) by \( \hat{V}(p) = \langle p, v \rangle \). The formula \( N^g = \frac{\nabla^g V}{|\nabla^g V|^g} \) defines a unit normal vector field on \( \Sigma_v \) in \((S^n, g)\). The second fundamental form of \( \Sigma_v \) in \((S^n, g)\) at a point \( p \) in \( \Sigma_v \) is then given by

\[
A(X,Y) = g(\nabla^g_X N^g, Y) = \frac{1}{|\nabla^g V|^g} \text{Hess}_g \hat{V}(X,Y) \quad \text{for all} \quad X,Y \in T_p \Sigma_v.
\]

Notice that \( \text{Hess}_{\text{can}} \hat{V} = -\hat{V} \text{can} \), hence \( \text{Hess}_{\text{can}} \hat{V} = 0 \) on \( \Sigma_v \). Therefore

\[
\text{Hess}_g \hat{V}(X,Y) = XY \hat{V} - (\nabla^g_X Y) \hat{V} = (\nabla^\text{can} X Y - \nabla^g_X Y) \hat{V}
\]

for every \( p \in \Sigma_v, X,Y \in T_p S^n \). We define the tensor

\[
T_g(X,Y,Z) = g(\nabla^g_X Y - \nabla^\text{can}_X Y, Z).
\]
Since the linear map \( v \in T_pS^n \mapsto \nabla^g \hat{V}(p) \in T_pS^n \) is an isomorphism, the condition that all equators passing by \( p \) are minimal at \( p \) is equivalent to

\[
\text{tr}^{12}_g \mathcal{T}_g(Z) = \frac{\mathcal{T}_g(Z, Z, Z)}{|Z|_g^2}
\]

for every \( Z \in T_pS^n \). This is equivalent to requiring that \( \mathcal{S}_g(Z, Z, Z) = 0 \) for every \( Z \), with \( \mathcal{S}_g = \mathcal{T}_g - g \otimes \text{tr}^{12}_g \mathcal{T}_g \). Since \( \mathcal{S}_g \) is symmetric in the first two variables, this is equivalent to vanishing of the cyclic symmetrization of \( \mathcal{S}_g \).

We have

\[
Xg(Y, Z) = \nabla^\text{can} g(Y, Z, X) + g(\nabla^\text{can}_X Y, Z) + g(\nabla^\text{can}_Z X, Y)
\]

and

\[
Xg(Y, Z) = g(\nabla^g_X Y, Z) + g(\nabla^g_Z Y, X).
\]

By subtracting,

\[
\nabla^\text{can} g(Y, Z, X) = \mathcal{T}_g(X, Y, Z) + \mathcal{T}_g(X, Z, Y) = \mathcal{T}_g(X, Y, Z) + \mathcal{T}_g(Z, X, Y).
\]

Hence \((\mathcal{T}_g)^S = \frac{1}{2}(\nabla^\text{can} g)^S\).

Notice that \( \text{tr}^{12}_g \mathcal{S}_g = (1 - n)\text{tr}^{12}_g \mathcal{T}_g \), so

\[
\text{tr}^{12}_g(\mathcal{S}_g^S) = (1 - n)\text{tr}^{12}_g \mathcal{T}_g + 2\text{tr}^{13}_g(\mathcal{S}_g) = 2\text{tr}^{13}_g(\mathcal{T}_g) - (n + 1)\text{tr}^{12}_g(\mathcal{T}_g).
\]

If \( \{e_i\} \) is a local \( g \)-orthonormal positive frame, we have

\[
\psi^{-1} = \text{vol}_\text{can}(e_1, \ldots, e_n).
\]

Thus \(-\psi^{-2}X\psi(p) = -\sum_i \mathcal{T}_g(X, e_i, e_i)\psi^{-1} \), which implies \( d\log \psi = \text{tr}^{13}_g(\mathcal{T}_g) \). Hence the proposition follows from the identity

\[
\mathcal{S}_g^S = \frac{1}{2} \left( \nabla^\text{can} g - \frac{4}{n + 1} g \otimes d \log \psi + \frac{2}{n + 1} g \otimes \text{tr}^{12}_g(\mathcal{S}_g^S) \right)^S.
\]

We have all the elements to describe the correspondence:

**Theorem 9.8.** Let \( W \) be an open subset of the sphere \( S^n \), \( n \geq 2 \). Let \( g \) be a Riemannian metric on \( W \) and denote by \( F_g \) the positive function on \( W \) so that \( dV_g = F_g^{(n+1)/4}dV_\text{can} \). The map

\[
g \mapsto k_g = \frac{1}{F_g} g
\]

is a bijection between the set of metrics \( g \) with minimal \((n-1)\)-equators and the set of positive definite Killing symmetric two-tensors \( k \) of \((W,\text{can})\). The metric \( g_k \) corresponding to \( k \) is given by \( g_k = \frac{1}{D_k} k \) where \( dV_k = D_k^{(n-1)/4}dV_\text{can} \).
Proof. Using the notation of Proposition 9.7, we have $\psi = F_g^{(n+1)/4}$. Hence, the symmetric two-tensor $k_g = (1/F_g)g$, which is clearly positive definite, satisfies

$$\nabla^{can}k_g = \frac{1}{F_g}\nabla^{can}g - \frac{1}{F_g^2}(g \otimes dF_g) = \frac{1}{F_g}\left(\nabla^{can}g - \frac{4}{n+1}g \otimes d\log(\psi)\right).$$

By Proposition 9.7, $g$ has minimal $(n - 1)$-equators if and only if $k_g$ is Killing symmetric for the canonical metric. The result follows by checking that $k_{g_k} = k$. □

We observe that the aforementioned classification of Killing symmetric two-tensors on $(S^n, can)$ in terms of Killing vector fields implies, using Theorem 9.8, that all metrics on $S^n$ with minimal $(n - 1)$-equators are invariant under the antipodal map.

We will now discuss some examples. We justify that not all metrics on $S^n$ with minimal equators have constant sectional curvature if $n \geq 3$. Let $K_1, K_2$ be nontrivial Killing vector fields on $(S^n, can)$ that are orthogonal to each other everywhere. This choice is possible if $n \geq 3$. Let $k(t) = can + tK_1 \otimes K_2$. Hence $k(t)$ is a symmetric Killing two-tensor on $(S^n, can)$ which is positive definite for sufficiently small $t$. Then $k(0) = can$ and $k'(0) = K_1 \otimes K_2$. It is not difficult to check that $tr_{can}k'(0) = 2can(K_1, K_2) = 0$ and that $\text{div}_{can}k'(0) = -\frac{1}{2}d(tr_{can}k'(0)) = 0$. If $g(t) = g_{k(t)}$ is as in Theorem 9.8, then $g(0) = can$ and $g'(0) = k'(0)$. Since $g'(0) = k'(0)$ is divergence-free, it is $L^2$-orthogonal in the canonical metric to the Lie derivative $L_Xcan$ for any vector field $X$ on $S^n$. Hence, for sufficiently small positive $t$, $g_{k(t)}$ has minimal equators and does not have constant sectional curvature.

In dimension $n = 3$, using the structure of $S^3$ as the Lie group of unit quaternions, we exhibit positive definite symmetric Killing two-tensors $k$ that correspond to Riemannian metrics $g_k$ on $S^3$ with minimal equators that are moreover arbitrarily close to $can$ in the smooth topology and whose isometry group is discrete. These metrics are invariant by the antipodal map hence this would prove Theorem D.

Let $i, j, k$ be the basic quaternions. Using quaternionic multiplication, define the left-invariant vector fields

$$X_i(p) = p \cdot i, \quad X_j(p) = p \cdot j, \quad X_k(p) = p \cdot k \quad \text{for all} \quad p \in S^3,$$

and the right-invariant vector fields

$$Y_i(p) = i \cdot p, \quad Y_j(p) = j \cdot p, \quad Y_k(p) = k \cdot p \quad \text{for all} \quad p \in S^3.$$
Note that these are Killing vector fields of \((S^3, \text{can})\), and also
\[
\text{can} = \frac{1}{2}(X_i \odot X_i + X_j \odot X_j + X_k \odot X_k) = \frac{1}{2}(Y_i \odot Y_i + Y_j \odot Y_j + Y_k \odot Y_k).
\]
(59)

The vector fields \(X_i, X_j, X_k, Y_i, Y_j, Y_k\) form a basis of the space of Killing vector fields of \((S^3, \text{can})\).

Choose real numbers \(\alpha_1 > \alpha_2 > \alpha_3 > \beta_1 > \beta_2 > \beta_3 > 0\) with \(\alpha_1, \alpha_2, \alpha_3\) sufficiently close to 1 and \(\beta_1, \beta_2, \beta_3\) sufficiently close to 0 to guarantee that
\[
\bar{k} = \frac{1}{2} \sum_{i=1}^{3} \alpha_i X_i \odot X_i + \frac{1}{2} \sum_{i=1}^{3} \beta_i Y_i \odot Y_i
\]
(60)
is positive definite by virtue of being sufficiently close to \(\text{can}\). Apply Theorem 9.8 so to obtain an antipodally-invariant Riemannian metric \(g_k\) on \(S^3\) with minimal equators. We claim that the isometry group of \(g_k\) is discrete.

Proposition 9.9. Let \(G\) be the group consisting of diffeomorphisms of the sphere \(S^n\) that map \((n-1)\)-equators into \((n-1)\)-equators. Every element of \(G\) is of the form \(\phi(T)(x) = Tx/|Tx|\) for some \(T \in GL(n+1, \mathbb{R})\).

Proof. Since the intersection of distinct \(k\)-equators is a \((k-1)\)-equator, a diffeomorphism \(\phi\) in \(G\) maps \(k\)-equators into \(k\)-equators for every \(k = 0, \ldots, n-1\). Therefore the map that assigns to each proper vector subspace \(V\) of \(\mathbb{R}^{n+1}\) the proper vector subspace generated by the \((\dim(V) - 1)\)-equator \(\phi(V \cap S^n)\) is a collineation, in the sense that it permutes proper vector subspaces of \(\mathbb{R}^{n+1}\) while preserving the partial order induced by inclusion. By the Fundamental Theorem of Projective Geometry, there exists \(T \in GL(n+1, \mathbb{R})\) such that, for every proper vector subspace \(V\) of \(\mathbb{R}^{n+1}\), the subspace \(T(V)\) is precisely the subspace generated by \(\phi(V \cap S^n)\). We conclude that for every \(x\) in \(S^n\) the point \(\phi(x)\) in \(S^n\) must be either equal to \(T(x)/|T(x)|\) or to \(-T(x)/|T(x)|\). Since \(T\) is a linear isomorphism, either \(\phi = \phi(T)\) or \(\phi = \phi(-T)\). \(\square\)

As in the proof of Theorem B, every minimal sphere of \((S^3, g_k)\) is an equator (we could also have used the uniqueness result of [7]). An isometry \(\psi\) of \((S^3, g_k)\) will send a minimal surface into a minimal surface, hence it permutes equators. Therefore by Proposition 9.9, every isometry of \((S^3, g_k)\) is of the form \(\phi(T)\) for some \(T \in GL(4, \mathbb{R})\).

The correspondence from Theorem 9.8 implies that \(\phi(T)^* g_k = g_k\) if and only if \(k \cdot T = k\), where the action of \(T \in GL(n+1, \mathbb{R})\) on Killing
symmetric two-tensors of $(S^n, can)$ is given by

$$(k \cdot T)_x = \frac{|T_x|^4}{(\det T)^{\frac{n+1}{2}}}(\phi(T)^*k)_x.$$ 

Notice that $(K_1 \odot K_2) \cdot T = (\det T)^{-\frac{n+1}{2}}(T^*K_1T) \odot (T^*K_2T)$.

If $\{T_i\} \subset GL(n+1, \mathbb{R})$ is such that $\phi(T_i)$ converges to the identity map as diffeomorphisms of $S^n$, we have $\phi(T_i)^*dV_{can} \to dV_{can}$ which implies $|T_i|^{-(n+1)} \det T_i \to 1$ for every $x \in S^n$. Hence $\det T_i > 0$ and $(\det T_i)^{-\frac{1}{n+1}}T_i \to Id$. Since we can suppose by scaling that $\det T_i = 1$, we have that the group of isometries of $(S^n, g_k)$ of the form $\phi(T)$ is discrete if and only if the equation

$$k \cdot t = 0,$$

$t \in \mathfrak{sl}(4, \mathbb{R})$ (or equivalently $tr t = 0$), admits only the trivial solution $t = 0$. Here $k \to k \cdot t$ is the linearization of the action $k \to k \cdot T$, $T \in SL(n+1, \mathbb{R})$, at the identity. We are going to check that this property holds with $k = \overline{t}$.

Notice that $(K_1 \odot K_2) \cdot t = (K_1 \cdot t) \odot K_2 + K_1 \odot (K_2 \cdot t)$, where $K \cdot t = t^*K + Kt$. If we set $W_1, \ldots, W_6$ equal to $X_1, X_2, X_3, Y_1, Y_2, Y_3$ in that order, as matrices the basis $\{\frac{1}{2}W_i\}$ of $\mathfrak{so}(4, \mathbb{R})$ is orthonormal for the scalar product $\langle A, B \rangle = tr(A^*B)$. For every $t \in \mathfrak{sl}(4, \mathbb{R})$ we have

$$W_p \cdot t = \frac{1}{4} \sum_{q=1}^{6} tr((W_p \cdot t)^*W_q)W_q = -\frac{1}{4} \sum_{q=1}^{6} tr(t^*W_pW_q + W_ptW_q)W_q$$

$$= -\frac{1}{4} \sum_{q=1}^{6} tr(W_q^*W_p t + W_qW_pt)W_q = -\frac{1}{2} \sum_{q=1}^{6} tr(W_qW_p t)W_q.$$

Hence, for every $t \in \mathfrak{sl}(4, \mathbb{R})$ and every $\overline{t}$ as in (60), by setting $w_1, \ldots, w_6$ equal to $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ in that order,

$$\overline{t} \cdot t = \sum_{p=1}^{6} w_p(W_p \cdot t) \odot W_p = -\frac{1}{2} \sum_{p,q=1}^{6} w_p tr(W_qW_p t)W_q \odot W_p$$

$$= -\frac{1}{2} \sum_{1 \leq p < q \leq 6} (w_p tr(W_qW_p t) + w_q tr(W_pW_q t))W_p \odot W_q,$$

where we used the symmetry $K_1 \odot K_2 = K_2 \odot K_1$ and the fact that $W_pW_p = -Id$ for every $p$.

Since $\dim K_2(S^3) = 20$ and $K_2(S^3)$ is generated by products of Killing vector fields, the symmetry $K_1 \odot K_2 = K_2 \odot K_1$ and the identity
(59) imply that \( \{ W_p \circ W_q : 1 \leq p < q \leq 6 \} \) is a linearly independent set. Hence 
\[
\bar{k} \cdot t = 0 \iff w_p tr(W_q W_p t) + w_q tr(W_p W_q t) = 0 \text{ for all } 1 \leq p < q \leq 6.
\]
The quaternionic algebra gives that if \( 1 \leq p < q \leq 3 \) or \( 4 \leq p < q \leq 6 \) we have \( W_p W_q = -W_q W_p \), and if \( 1 \leq p \leq 3, 4 \leq q \leq 6 \), we have \( W_p W_q = W_q W_p \). In any case by the choice of \( \{ w_p \} \) we have 
\[
tr(W_q W_p t) = 0 \text{ for every } 1 \leq p < q \leq 6.
\]
Hence \( t \) is orthogonal to \( X_i, Y_j \) and \( X_i Y_j \) for every \( 1 \leq i, j \leq 3 \). Since the set \( \{ X_i Y_j \}_{1 \leq i,j \leq 3} \) is a basis of the space of symmetric matrices with zero trace, it follows that \( t = 0 \). This finishes the proof.

9.5. Proof of Theorem E. Let \( \text{Sym}_2(S^n) \) denote the space of smooth symmetric two-tensors on \( S^n \). We replace \( A(\rho, \Phi) \) and \( H(\rho, \Phi) \) by \( A(g, \Phi) \) and \( H(g, \Phi) \) to indicate the dependence on a general Riemannian metric, rather than on a conformal factor. Similarly as before we have the identity 
\[
\frac{d}{dt}_{|t=0} A(g, \Phi + t\phi)(v) = \int_{\Sigma_v} H(g, \Phi)(x, v)\phi(x, v) dA_{\text{can}}(x),
\]
for all \( \phi \in C^{\infty}_{s, \text{odd}}(T_1 S^n) \) (see [19], Theorem 1.1).

The derivative \( D_2 H(g, \Phi) \) induces ([19], Theorem 1.1) symmetric second order elliptic operators on each equator \( \Sigma_v \) as before, with 
\[
D_2 H(\text{can}, 0)(\phi)_v = -\Delta_{\text{can}} \phi_v - (n-1)\phi_v
\]
for every \( \phi \in C^{\infty}_{s, \text{odd}}(T_1 S^n) \). Therefore we also have
\[
\mathcal{P}(g, \Phi) : C^{\infty}_{0, \text{odd}}(T_1 S^n) \to C^{\infty}_{0, \text{odd}}(T_1 S^n)
\]
as in (8), and a solution map 
\[
\mathcal{S}(g, \Phi) : C^{\infty}_{0, \text{odd}}(T_1 S^n) \to C^{\infty}_{0, \text{odd}}(T_1 S^n)
\]
as in (9).

The constraint equation still holds true:
\[
\mathcal{K}(\Phi, H(g, \Phi)) = dA(g, \Phi),
\]
and \( H(g, \Phi) = 0 \) if and only if
\[
dA(g, \Phi) = 0 \quad \text{and} \quad H(g, \Phi) = j\omega \quad \text{for some } \omega \in \Omega^1_{\text{even}}(S^n).
\]

Thus we can define a map \( \Lambda \) as in Section 5.1. We let \( \Lambda = (\Lambda_1, \Lambda_2) \) where
\[
\Lambda_1(g, \Phi) = A(g, \Phi) - \int_{\mathbb{RP}^n} A(g, \Phi)(\sigma) dA_{\text{can}}(\sigma)
\]
\[ \Lambda_2(g, \Phi) = H(g, \Phi) - jC(H(g, \Phi)). \]

Hence
\[ \Lambda(g, \Phi) = 0 \iff H(g, \Phi) = 0. \]

Moreover, \( \Lambda(\text{can}, 0) = (0, 0) \) since the equators are minimal hypersurfaces in \((S^n, \text{can})\).

We need to find an approximate right-inverse for \( D\Lambda \) to apply Theorem 5.1. We have
\[ D_1\Lambda_1(g, \Phi) \cdot h = F(g, \Phi)(h) - \int_{\mathbb{RP}^n} F(g, \Phi)(h)(\sigma)dV_{\text{can}}(\sigma), \]
where \( F(g, \Phi) \) is the generalized Funk transform
\[ F(g, \Phi)(h)(\sigma) = \frac{1}{2} \int_{\Sigma_x(\Phi)} \text{tr}(\Sigma_x(\Phi), g) h(x)dA_2(x) \]
for a smooth symmetric two-tensor \( h \) on \( S^n \). Notice that
\[ F(g, \Phi) : \text{Sym}_2(S^n) \to C^\infty(\mathbb{RP}^n). \]

Then
\[ D_2\Lambda_1(g, \Phi) \cdot \phi = D_2A(g, \Phi) \cdot \phi - \int_{\mathbb{RP}^n} (D_2A(g, \Phi) \cdot \phi)(\sigma)dV_{\text{can}}(\sigma), \]
where, by the definition of the Euler-Lagrange operator \( H(g, \Phi) \),
\[ (D_2A(g, \Phi) \cdot \phi)([v]) = \int_{\Sigma_v} H(g, \Phi)(x, v)\phi(x, v)dA_{\text{can}}(x). \]

For all \( \phi \in C^\infty_{0, \text{odd}}(T_1S^n) \) and \( v \in S^n \) we have
\[ (D_2A(g, \Phi) \cdot \phi)([v]) = \int_{\Sigma_v} \Lambda_2(g, \Phi)(x, v)\phi(x, v)dA_{\text{can}}(x). \]

Likewise, we have
\[ D_1\Lambda_2(g, \Phi) \cdot h = D_1H(g, \Phi) \cdot h - jC(D_1H(g, \Phi) \cdot h) \]
and
\[ D_2\Lambda_2(g, \Phi) \cdot \phi = P(g, \Phi)(\phi). \]

If we can construct a right-inverse
\[ \mathcal{R}(g, \Phi) : C^\infty(\mathbb{RP}^n) \to \text{Sym}_2(S^n) \]
for the generalized Funk transform \( F(g, \Phi) \), we can use it to define \( V \) and \( Q \) as in Section 5.2.

Notice that for \( \zeta \in C^\infty(S^n) \), we have
\[ F(g, \Phi)(\zeta g)(\sigma) = \frac{(n-1)}{2} \int_{\Sigma_x(\Phi)} \zeta(x)dA_2(x). \]
We define

\[ \mathcal{F}^\circ(g, \Phi) : C^\infty(\mathbb{RP}^n) \to \text{Sym}_2(S^n) \]

by setting

\[ \mathcal{F}^\circ(g, \Phi)(f)_x = \frac{2}{n-1} \left( \int_{\Sigma^*_x(\Phi)} f(\tau) U(g, \Phi)(x, \tau) dA_{\text{can}}(\tau) \right) g_x \]

for all \( x \in S^n \), where

\[ U(g, \Phi) = \frac{|\text{Jac}(\pi_2)|}{|\text{Jac}(\pi_1)|} \in C^\infty([F(\Phi)]) \]

is the quotient of the Jacobians of the projections of \([F(\Phi)]\) onto the factors \((S^n, g)\) and \((\mathbb{RP}^n, \text{can})\).

Hence

\[ \mathcal{F}(g, \Phi)(\mathcal{F}^\circ(g, \Phi)(f))(\sigma) \]

\[ = \int_{\Sigma^*_x(\Phi)} \left( \int_{\Sigma^*_x(\Phi)} f(\tau) U(g, \Phi)(x, \tau) dA_{\text{can}}(\tau) \right) dA_g(x) \]

\[ = \int_{[B_\sigma(\Phi)]} f(\tau) U(g, \Phi)(x, \tau) |\text{Jac}(Q_1)|(|\sigma, x, \tau|) dV_{g\times\text{can}}(x, \tau) \]

by the co-area formula, where \( Q_1 : [B_\sigma(\Phi)] \to \Sigma_\sigma(\Phi) \) is the projection onto the first coordinate as in the proof of Proposition 7.5. Using the co-area formula again with \( Q_2 : [B_\sigma(\Phi)] \to \mathbb{RP}^n \) we get

\[ \mathcal{F}(g, \Phi)(\mathcal{F}^\circ(g, \Phi)(f))(\sigma) \]

\[ = \int_{\mathbb{RP}^n} \left( \int_{\Sigma^*_x(\Phi) \cap \Sigma^*_\tau(\Phi)} f(\tau) U(g, \Phi)(x, \tau) \times \frac{|\text{Jac}(Q_1)|(|\sigma, x, \tau|)}{|\text{Jac}(Q_2)|(|\sigma, x, \tau|) dA_g(x)} \right) dV_{\text{can}}(\tau). \]

The proofs of Propositions 7.3 and 7.5 give that

\[ \frac{|\text{Jac}(\pi_2)|}{|\text{Jac}(\pi_1)|} \frac{|\text{Jac}(Q_1)|}{|\text{Jac}(Q_2)|} = \frac{1}{\sqrt{1 - g(N(g, \Phi)(x, v), N(g, \Phi)(x, w))^2}} \]

for \([v] = \sigma, [w] = \tau\), where \( N(g, \Phi)(x, v), N(g, \Phi)(x, w) \) denote unit normals in the \( g \)-metric to \( \Sigma_\sigma(\Phi), \Sigma_\tau(\Phi) \), respectively, at \( x \).

Therefore

\[ \mathcal{F}(g, \Phi)(\mathcal{F}^\circ(g, \Phi)(f))(\sigma) = \int_{\mathbb{RP}^n} K(g, \Phi)(\sigma, \tau) f(\tau) dV_{\text{can}}(\tau), \]
where
\[ K(g, \Phi)(\sigma, \tau) = \int_{\Sigma_\sigma(\Phi) \cap \Sigma_\tau(\Phi)} \frac{1}{\sqrt{1 - g(N(g, \Phi)(x, v), N(g, \Phi)(x, w))}} dA_g(x). \]

The analysis of this integral operator is the same as before: the kernel is of the form \( K(g, \Phi) = k(g, \Phi)/(\eta \circ d_{can}) \) where the map \( k(g, \Phi) \) extends smoothly to the blow-up \( B(\mathbb{RP}^n \times \mathbb{RP}^n) \). The operator \( F(g, \Phi) \circ R(g, \Phi) = L(k(g, \Phi)) : C^\infty(\mathbb{RP}^n) \to C^\infty(\mathbb{RP}^n) \) is a pseudodifferential operator of order \( 1 - n \) that is elliptic and invertible if \( (g, \Phi) \) is sufficiently close to \( (can, 0) \) in the smooth topology. Hence we can define \( R(g, \Phi) = F^\circ(g, \Phi) \circ (F(g, \Phi) \circ F^\circ(g, \Phi))^{-1} \) so
\[ F(g, \Phi) \circ R(g, \Phi) = \text{id}. \]

As before we can check that the maps \( \Lambda, V \) and \( Q \) are smooth tame and hence Theorem 5.1 can be applied. This gives a smooth tame map
\[ \Gamma : W \subset \text{Ker}D\Lambda(can, 0) \to \Lambda^{-1}(0, 0), \]
defined on an open subset \( W \) of \( \text{Ker}D\Lambda(can, 0) \), with \( \Gamma(0) = 0 \) and \( D\Gamma(0) \cdot v = v \) for every \( v \in \text{Ker}D\Lambda(can, 0) \). Theorem E follows as Theorem A once we establish that for each smooth symmetric two-tensor \( h \) as in (I) we can find \( \phi \in C^\infty_{0, \text{odd}}(T_1S^n) \) such that \( (h, \phi) \in \text{Ker}D\Lambda(can, 0) \), and then set \( (g, \Phi) = \Gamma(h, t\phi) \).

This is a consequence of the next proposition.

**Proposition 9.10.** The kernel of \( D\Lambda(can, 0) \) consists of all the pairs \( (h, \phi) \) in \( \text{Sym}_2(S^n) \times C^\infty_{0, \text{odd}}(T_1S^n) \) such that:

i) There exists a constant \( c \), a smooth odd function \( f \) on \( S^n \), a smooth vector field \( X \) on \( S^n \) and a symmetric transverse-traceless two-tensor tensor \( \overline{h} \) on \( (S^n, can) \) such that
\[ h = (c + f)can + L_X can + \overline{h}; \]

and

ii) The function \( \phi \) is such that
\[ \Delta_{(\Sigma_v, can)} \phi_v + (n - 1)\phi_v = (D_1H(can, 0) \cdot h)_v \quad \text{on} \quad \Sigma_v \]
for every \( v \in S^n \).

Moreover, every \( h \) as in i) uniquely determines \( \phi \) as in ii).
A calculation yields

\[ D_1 \mathcal{H}(\text{can}, 0)(h)(x, v) = -(\text{div}_\text{can} h)_x(v) + \frac{1}{2} d\tr_{\text{can}}(h)_x(v) + \frac{1}{2} \nabla^v h_x(v, v) \]

for all \((x, v) \in T_1 S^n\) and \(h \in \text{Sym}_2(S^n)\).

**Proof.** By differentiating the constraint equation, we have

\[
d(\mathcal{F}(\text{can}, 0)(h)) = d(D_1 A(\text{can}, 0) \cdot h) = K(0, D_1 \mathcal{H}(\text{can}, 0) \cdot h) = -C(D_1 \mathcal{H}(\text{can}, 0) \cdot h).
\]

Recall also that

\[
D_2 \mathcal{H}(\text{can}, 0)(\phi)_v = -\Delta_{\text{can}} \phi_v - (n - 1) \phi_v
\]

is \(L^2\)-orthogonal to the linear functions on \((\Sigma_v, \text{can})\). This implies that 
\(C(D_2 \mathcal{H}(\text{can}, 0)(\phi)) = 0\).

From the explicit formulas for \(DA\), we deduce that

\[
(h, \phi) \in \ker DA(\text{can}, 0)
\]

if and only if \(\mathcal{F}(\text{can}, 0)(h)\) is a constant function on \(S^n\) and \(\phi_v\) is the unique solution of

\[
\Delta_{(\Sigma_v, \text{can})} \phi_v + (n - 1) \phi_v = (D_1 \mathcal{H}(\text{can}, 0) \cdot h)_v
\]

that is \(L^2(\Sigma_v, \text{can})\)-orthogonal to the linear functions. This solution exists because in this case we have 
\(C(D_1 \mathcal{H}(\text{can}, 0) \cdot h) = 0\).

It remains to determine for which symmetric two-tensors \(h\) on \(S^n\) the function \(\mathcal{F}(\text{can}, 0)(h)\) is constant. Recall that any \(h \in \text{Sym}_2(S^n)\) can be decomposed as a sum

\[
h = (f \text{can} + \mathcal{L}_X \text{can}) + \mathcal{T},
\]

where \(f \in C^\infty(S^n)\), \(\mathcal{L}_X\) denotes the Lie derivative in the direction of the smooth vector field \(X\) on \(S^n\), and \(\mathcal{T}\) is transverse-traceless tensor on \((S^n, \text{can})\), that is, a tensor such that \(\text{div}_\text{can} \mathcal{T} = \tr_{\text{can}} \mathcal{T} = 0\) ([3], Lemma 4.57).

**Claim 1:** The Lie derivatives of \(\text{can}\) are in the kernel of \(\mathcal{F}(\text{can}, 0)\).

If \(\psi_t\) denotes the one-parameter family of diffeomorphisms generated by \(X \in \mathcal{X}(S^n)\), then \(\frac{d}{dt}|_{t=0}(\psi_t)^*\text{can} = \mathcal{L}_X \text{can}\) and, for every \(\sigma \in \mathbb{RP}^n\),

\[
\mathcal{F}(\text{can}, 0)(\mathcal{L}_X \text{can})(\sigma) = D_1 A(\text{can}, 0)(\mathcal{L}_X \text{can})(\sigma)
\]

\[
= \frac{d}{dt}|_{t=0} \text{area}(\Sigma_{\sigma}, (\psi_t)^*\text{can}) = \frac{d}{dt}|_{t=0} \text{area}(\psi_t(\Sigma_{\sigma}), (\psi_t)^*\text{can}) = 0,
\]
where we used the minimality of $\Sigma_\sigma$ in $(S^n, can)$.

Claim 2: Any transverse-traceless symmetric two-tensor $\overline{h}$ is in the kernel of $F(can, 0)$.

For $v \in S^n$, the function $f(x) = \langle x, v \rangle$ satisfies $\text{Hess} f = -f \text{can}$. If $\omega = \overline{h}(\nabla \text{can} f, \cdot)$, we get

\[
\text{div} \text{can} \omega = (\text{div} \text{can} \overline{h})(\nabla \text{can} f) + \langle \overline{h}, \text{Hess} f \rangle = 0
\]

since $\overline{h}$ is divergence-free and traceless. If $\Omega_v$ denotes the hemisphere containing $v$ with $\partial \Omega_v = \Sigma_v$, we have

\[
0 = \int_{\Omega_v} (\text{div} \text{can} \omega) dV \text{can} = -\int_{\Sigma_v} \omega_x(v) dA \text{can}(x) = -\int_{\Sigma_v} \overline{h}_x(v, v) dA \text{can}(x).
\]

But, with $\sigma = [v]$,

\[
2F(can, 0)(\overline{h})(\sigma) = \int_{\Sigma_v} tr_{\Sigma_v, \text{can}} \overline{h}(x) dA \text{can}(x)
\]

\[
= -\int_{\Sigma_v} \overline{h}_x(v, v) dA \text{can}(x) = 0
\]

since $tr_{\Sigma_v} \overline{h} = 0$.

Claim 3: If $f \in C^\infty(S^n)$, then $F(can, 0)(f \text{can})$ is constant if and only if $f$ is the sum of a constant function and an odd function.

In fact, since

\[
\frac{2}{n-1}F(can, 0)(f \text{can}) = \int_{\Sigma_v} f(x) dA \text{can}(x)
\]

is the standard Funk transform on $(S^n, can)$, the result follows from Lemma A.1.

\[\square\]

Appendix. The standard Funk transform

The following lemma summarizes the most relevant properties of the standard Funk transform $F$. Its content and method of proof are well-known (see [10], Appendix A and [16], §4, Lemma 4.3).

Lemma A.1.

i) The kernel of $F$ is the set of odd functions:

\[
C^\infty_{\text{odd}}(S^n) = \{ f \in C^\infty(S^n) : F(f) = 0 \}.
\]
ii) For every non-negative integer $k$, $\mathcal{F}$ restricted to $\mathcal{C}^\infty_{even}(S^n)$ extends to a linear isomorphism

$$\mathcal{F}: H^k_{even}(S^n) \mapsto H^{k+\frac{n-1}{2}}_{even}(S^n).$$

and for a positive constant $c$,

$$\frac{1}{c} \|f\|_k \leq \|\mathcal{F}(f)\|_{k+\frac{n-1}{2}} \leq c \|f\|_k$$

for all $f \in H^k_{even}(S^n)$.

**Corollary A.2.** The standard Funk transform $\mathcal{F}$ of $(S^n, can)$ is such that $\mathcal{F} \circ \mathcal{F}^*$ is elliptic,

$$\mathcal{F} \circ \mathcal{F}^*: H^k(\mathbb{RP}^n) \to H^{k+n-1}(\mathbb{RP}^n)$$

is invertible for all $k \in \mathbb{N}$ and there is a positive constant $c$ such that

$$\frac{1}{c} \|g\|_k \leq \|\mathcal{F} \circ \mathcal{F}^*(g)\|_{k+n-1} \leq c \|g\|_k$$

for all $g \in H^k(\mathbb{RP}^n)$.

**Proof.** Since $\mathcal{F} \circ \mathcal{F}^* = L(k(0,0))$, it follows from (31) that

$$(\mathcal{F} \circ \mathcal{F}^*)(f)(\sigma) = \omega_{n-2} \int_{\mathbb{RP}^n} \frac{1}{\sin d(\sigma, \tau)} f(\tau) dV_{can}(\tau),$$

where $\omega_{n-2} = area(S^{n-2})$. The ellipticity follows as in [15] (Lemma 6.2). Seen as a map from $\mathcal{C}^\infty_{even}(S^n)$ to $\mathcal{C}^\infty_{even}(S^n)$ we have $\mathcal{F}^* = \frac{1}{2} \mathcal{F}$. Hence the invertibility of $\mathcal{F} \circ \mathcal{F}^*$ follows from Lemma A.1. \qed

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