PRIME AMPHICHEIRAL KNOTS WITH FREE PERIOD 2

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Abstract We construct prime amphicheiral knots that have free period 2. This settles an open question raised by the second-named author, who proved that amphicheiral hyperbolic knots cannot admit free periods and that prime amphicheiral knots cannot admit free periods of order $> 2$.

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1. Introduction

A knot $K$ in the 3-sphere is amphicheiral if there exists an orientation-reversing diffeomorphism $\varphi$ of the 3-sphere which leaves the knot invariant. More precisely, the knot is said to be positive-amphicheiral ($+$-amphicheiral for short) if $\varphi$ preserves a fixed orientation of $K$ and negative-amphicheiral ($-$-amphicheiral for short) otherwise. Of course, a knot can be both positive- and negative-amphicheiral. This happens if and only if the knot is both amphicheiral and invertible. Recall that a knot is invertible if there exists a free $\mathbb{Z}/n\mathbb{Z}$-action on the 3-sphere.

A knot $K$ in the 3-sphere is said to have free period $n \geq 2$ if there exists an orientation-preserving periodic diffeomorphism $f$ of the 3-sphere of order $n$ which leaves the knot invariant, such that $f$ generates a free $\mathbb{Z}/n\mathbb{Z}$-action on the 3-sphere.

It was proved by the second-named author that amphicheirality and free periodicity are inconsistent in the following sense (see [24, 25]).

(1) Any amphicheiral prime knot does not have free period $> 2$.

(2) Any amphicheiral hyperbolic knot does not have free period.
On the other hand, it is easy to construct, for any integer \( n \geq 2 \), a composite amphicheiral knot that has free period \( n \). Thus, an open question was raised in [25] whether there is an amphicheiral prime knot that has free period 2.

The main purpose of this paper is to prove the following theorem, which gives an answer to this question.

**Theorem 1.1.**

1. For each \( \epsilon \in \{+,-\} \), there are infinitely many prime knots with free period 2 that are \( \epsilon \)-amphicheiral but not \( -\epsilon \)-amphicheiral; in particular, they are not invertible.
2. There are infinitely many prime knots with free period 2 that are \( \epsilon \)-amphicheiral for each \( \epsilon \in \{+,-\} \); in particular, they are invertible.

For the proof of the theorem, we introduce a specific subgroup \( \mathbb{G} = \langle \gamma_1, \gamma_2 \rangle \) of the orthogonal group \( O(4) \cong \text{Isom}(S^3) \), generated by two commuting orientation-reversing involutions \( \gamma_1 \) and \( \gamma_2 \), where \( \gamma_1 \) is a reflection in a 2-sphere, \( \gamma_2 \) is a reflection in a 0-sphere, and \( f := \gamma_1 \gamma_2 \) is a free involution. Consider a \( \mathbb{G} \)-invariant hyperbolic link \( L = K_0 \cup O_\mu \) in \( S^3 \), consisting of a specific \( \mathbb{G} \)-invariant component \( K_0 \) and a \( \mu \)-component trivial link \( O_\mu \). Then we prove that given such a link \( L \), we can construct a prime amphicheiral knot \( K \) with free period 2, such that the exterior \( E(L) \) of \( L \) is the root \( E_0 \) of the Jaco–Shalen–Johannson decomposition (JSJ decomposition) of the exterior \( E(K) \), i.e. the geometric piece containing the boundary of \( E(K) \), where the boundary torus of the tubular neigbourhood of \( K_0 \) corresponds to \( \partial E(K) \). In fact, we show that each \( \gamma_i \) (to be precise, the restriction of \( \gamma_i \) to \( E(L) \)) extends to an orientation-reversing diffeomorphism of \( S^3 \) preserving the prime knot \( K \), and \( f \) extends to a free involution on \( S^3 \) preserving \( K \). It should be noted that \( \mathbb{G} \) does not necessarily extend to a group action on \( (S^3,K) \) (see Remark 2.3).

The proof of Theorem 1.1 is then reduced to producing examples of links \( L \) fulfilling the above properties. Of course, while it is not difficult to construct links admitting prescribed symmetries, ensuring that they are hyperbolic can be much more delicate. We will construct three links that will provide examples of knots having different properties. We will give theoretic proofs of the fact that they are hyperbolic, although this can also be checked using the computer program SnapPea [28] or SnapPy [8], or the computer-verified program HIKMOT [12].

Moreover, we show that any amphicheiral prime knot with free period 2 is constructed in this way (Theorem 4.1). In other words, if \( K \) is an amphicheiral prime knot with free period 2, then the root \( E_0 \) of the JSJ decomposition of \( E(K) \) is identified with \( E(L) \) for some \( \mathbb{G} \)-invariant link \( L \) with the above property. Furthermore, we prove the following theorem, which provides some insight regarding the root \( E_0 \) with respect to the \( \mathbb{G} \)-action.

**Theorem 1.2.** Let \( K \) be a prime amphicheiral knot with free period 2 and let \( E_0 = E(L) = E(K_0 \cup O_\mu) \) be its root. Then, after an isotopy, \( L \) is invariant by the action of \( \mathbb{G} \) on \( S^3 \), and the following hold.
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(1) \( L \) contains
   • at most two components whose stabilizer is \( G \), one of which must be \( K_0 \);
   • at least one pair of components of \( O_{\mu} \), such that each of the components intersects the 2-sphere \( \text{Fix}(\gamma_1) \) transversely at two points, the stabilizer of each of the components is generated by \( \gamma_1 \) and \( f \) interchanges the two components;
   • no component with stabilizer generated by \( f \).

(2) Assume that \( K \) is positive-amphicheiral. Then \( K_0 \) is contained in \( \text{Fix}(\gamma_1) \).

(3) Assume that \( K \) is negative-amphicheiral. Then \( K_0 \) must contain \( \text{Fix}(\gamma_2) \) and intersect \( \text{Fix}(\gamma_1) \) transversally at two points.

Remark 1.3. In the above theorem, if \( K \) is both positive- and negative-amphicheiral, then \( L \) admits two different positions with respect to the action of \( G \). In other words, there are two subgroups \( G_+ \) and \( G_- \) in \( \text{Diff}(S^3, L) \), such that (i) both \( G_+ \) and \( G_- \) are conjugate to \( G \) in \( \text{Diff}(S^3) \) and (ii) the groups \( G_+ \) and \( G_- \) satisfy conditions (2) and (3), respectively. In fact, \( (S^3, L) \) admits an action of \( (\mathbb{Z}/2\mathbb{Z})^3 \) such that \( G_+ \) and \( G_- \) correspond to \( (\mathbb{Z}/2\mathbb{Z})^2 \oplus 0 \) and \( 0 \oplus (\mathbb{Z}/2\mathbb{Z})^2 \), respectively, where \( 0 \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus 0 \) is generated by \( f \).

For a prime knot \( K \), let \( \mu(K) \) be the number of boundary components of the root \( E_0 \) of the JSJ decomposition of \( E(K) \) that correspond to the tori of the JSJ decomposition, i.e. \( \mu(K) + 1 \) is equal to the number of boundary components of \( E_0 \). Now define \( \mu_+ \) (respectively \( \mu_- \)) to be the minimum of \( \mu(K) \) over all prime knots \( K \) with free period 2 that are positive-amphicheiral (respectively negative-amphicheiral).

The main result of [25] says that \( \mu_+ > 0 \). The following theorem determines both \( \mu_+ \) and \( \mu_- \).

Theorem 1.4. The following hold.

(1) \( \mu_- = 2 \). Moreover, if \( K \) realizes \( \mu_- \), i.e., if \( K \) is a prime negative-amphicheiral knot with free period 2 such that \( E_0 = E(K_0 \cup O_2) \), then \( K_0 \) contains \( \text{Fix}(\gamma_2) \) and the stabilizer of each component of \( O_2 \) is generated by \( \gamma_1 \).

(2) \( \mu_+ = 3 \). Moreover, if \( K \) realizes \( \mu_+ \), i.e., if \( K \) is a prime positive-amphicheiral knot with free period 2 such that \( E_0 = E(K_0 \cup O_3) \), then \( K_0 \) is a (necessarily trivial) knot contained in \( \text{Fix}(\gamma_1) \), and one of the three components of \( O_3 \) is stabilized by \( G \) and thus contains \( \text{Fix}(\gamma_2) \), while the stabilizer of the other two components is generated by \( \gamma_1 \).

We remark that our explicit constructions provide in particular negative-amphicheiral (respectively positive-amphicheiral) knots \( K \) with free period 2 such that \( \mu(K) = 2 \) (respectively \( \mu(K) = 3 \)). We also describe the structure of the subgroup \( \text{Isom}^*(E_0) \) of \( \text{Isom}(E_0) \) consisting of those elements which extend to a diffeomorphism of \( (S^3, K) \) (Proposition 5.10).

The paper is organized as follows. In §2, we show how one can construct prime amphicheiral knots with free period 2 from the exterior of a hyperbolic link with specified
properties. In § 3, we provide examples of such links. In § 4, we show that the requirement on the links are not only sufficient but also necessary (Theorem 4.1). This is used in § 5 to prove Theorems 1.2 and 1.4; Theorem 1.1 is also proved in this section. In § 6, we refine the arguments in § 5 and give more detailed information concerning the root $E_0$. In particular, we present a convenient description of the link $L$ in the case where $K$ is positive-amphicheiral (Remark 6.2). The final §§ 7, 8 and 9 are technical: there we show that the links introduced in § 3 are hyperbolic, completing the proof that they fulfil all the desired requirements.

2. Constructing the knots

In this section, we show that the existence of a hyperbolic link with a specified symmetry and some extra properties is sufficient to ensure the existence of prime amphicheiral knots with free period 2. We start by defining the symmetry we want.

Let $G = \langle \gamma_1, \gamma_2 \rangle$ be the group of isometries of the 3-sphere $S^3 = \{(z_1, z_2) \in C^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ with the standard metric, generated by the following two commuting orientation-reversing isometric involutions $\gamma_1$ and $\gamma_2$:

$$
\gamma_1(z_1, z_2) = (\bar{z}_1, z_2), \quad \gamma_2(z_1, z_2) = (-\bar{z}_1, -z_2).
$$

The involution $\gamma_1$ is a reflection in the 2-sphere $S^3 \cap (\mathbb{R} \times \mathbb{C})$, and the involution $\gamma_2$ is a reflection in the 0-sphere $\{(\pm i, 0)\}$. The composition $f := \gamma_1 \gamma_2$ is an orientation-preserving free involution $(z_1, z_2) \mapsto (-z_1, -z_2)$. If we identify $S^3$ with $\mathbb{R}^3 \cup \{\infty\}$, where $\infty$ corresponds to the point $(i, 0) \in S^3$, then $\gamma_1$ is an inversion in a unit 2-sphere in $\mathbb{R}^3$ and $\gamma_2$ is an antipodal map with respect to the origin. If we identify $S^3$ with $\mathbb{R}^3 \cup \{\infty\}$, where $\infty$ corresponds to the point $(1, 0) \in S^3$, then $\gamma_1$ is a reflection in a 2-plane in $\mathbb{R}^3$ and $\gamma_2$ has a unique fixed point in each of the two half-spaces defined by the 2-plane.

**Definition 2.1.** Let $L = K_0 \cup O_\mu$ be a link of $\mu + 1$ components. We will say that $L$ provides an admissible root if it satisfies the following requirements:

- $L$ is $G$-invariant and $K_0$ is stabilized by the whole group $G$;
- $O_\mu$ is a trivial link with $\mu$ components;
- $L$ is hyperbolic.

A link $L$ providing an admissible root can be used to construct a prime satellite knot in the following way: remove a $G$-invariant open regular neighbourhood of $O_\mu = \cup_{i=1}^\mu O_i$ and glue non-trivial knot exteriors, $E(K_i)$, $i = 1, \ldots, \mu$, along the corresponding boundary components in such a way that the longitude and meridian of $O_i$ are identified with the meridian and longitude of $E(K_i)$, respectively. The image of $K_0$ in the resulting manifold is a prime knot $K$ in $S^3$.

Note that

$$(S^3, K) = (E(O_\mu), K_0) \cup (\cup_{i=1}^\mu (E(K_i), \emptyset)),$$

where $E(O_\mu)$ is the exterior of $O_\mu$, which is $G$-invariant.
We wish to perform the gluing so that the following conditions hold.

(i) Each element of $\mathbb{G}$ ‘extends’ to a diffeomorphism of $(S^3, K)$. To be precise, for each element $g \in \mathbb{G}$, the restriction, $\hat{g}$, of $g$ to $(E(O_\mu), K_0)$ extends to a diffeomorphism, $\hat{g}$, of $(S^3, K)$.

(ii) An ‘extension’ $\hat{f}$ of $f = \gamma_1\gamma_2$ to $(S^3, K)$ is a free involution.

Note that we do not intend to extend the action of $\mathbb{G}$ to $(S^3, K)$. In fact, such an extension does not exist generically (see Remarks 2.3 and 5.9 and Proposition 6.3).

To this end, we need to choose the knot exterior $E(K_i)$ appropriately according to the stabilizer in $\mathbb{G}$ of the component $O_i$ of the link $L$.

**Case 1.** The stabilizer of $O_i$ is trivial. In this case, one can choose $K_i$ to be any non-trivial knot in $S^3$. The same knot exterior must be chosen for the other three components of $O_\mu$ in the same $\mathbb{G}$-orbit as $O_i$, and the gluing must be carried out in a $\mathbb{G}$-equivariant way. Note that this is possible because the elements of $\mathbb{G}$ map meridians (respectively longitudes) of the components of $L$ to meridians (respectively longitudes).

**Case 2.** The stabilizer of $O_i$ is generated by $\gamma_j$ for some $j \in \{1, 2\}$. Let $T_i$ be the boundary component of $E(O_\mu)$ which forms the boundary of a regular neighbourhood of $O_i$, and let $\ell_i$ and $m_i$ be the longitude and the meridian curves in $T_i = \partial E(K_i)$ of the knot $K_i$. Recall that these are the meridian and longitude of $O_i$, respectively. We see that $\gamma_j$ acts on $H_1(T_i; \mathbb{Z})$ as either

\[
(\gamma_j)_* \begin{pmatrix} \ell_i \\ m_i \end{pmatrix} = \begin{pmatrix} \ell_i \\ -m_i \end{pmatrix}
\]

or

\[
(\gamma_j)_* \begin{pmatrix} \ell_i \\ m_i \end{pmatrix} = \begin{pmatrix} -\ell_i \\ m_i \end{pmatrix}.
\]

In the former case, we need to choose $K_i$ to be a positive-amphicheiral knot in $S^3$, while in the latter case we need to choose $E(K_i)$ to be the exterior of a negative-amphicheiral knot. With this choice, the restriction of the involution $\gamma_j$ to $(E(O_\mu), K_0)$ extends to a diffeomorphism of $(E(O_\mu) \cup E(K_i), K_0)$.

This can be seen as follows. Because of the chosen type of chirality of $K_i$, there is an orientation-reversing diffeomorphism, $\nu_i$, of $E(K_i)$ whose induced action on $H_1(T_i; \mathbb{Z})$ is equal to that of $(\gamma_j)_*$. Then the restrictions of $\gamma_j$ and $\nu_i$ to $T_i$ are smoothly isotopic. So, by using a collar neighbourhood of $T_i$, we can glue the diffeomorphisms $\gamma_j$ and $\nu_i$ to obtain the desired diffeomorphism of $(E(O_\mu) \cup E(K_i), K_0)$.

For the other component $O_{i'} = f(O_i)$ of $L$ in the $\mathbb{G}$-orbit of $O_i$, we glue a copy of the same knot exterior $E(K_{i'})$ along $O_{i'}$, so that the free involution extends to a free involution of $(E(O_\mu) \cup E(K_i) \cup E(K_{i'}), K_0)$, which exchanges $E(K_i)$ with $E(K_{i'})$. By the previous argument, $\gamma_j$ also extends to a diffeomorphism of $(E(O_\mu) \cup E(K_i) \cup E(K_{i'}), K_0)$ and so do all elements of $\mathbb{G}$.

**Case 3.** The stabilizer of $O_i$ is generated by $f$. This case cannot arise, as shown by the following lemma.
Lemma 2.2. Let $L$ be a link which is invariant by the action of $G$. Then no component of $L$ can be stabilized precisely by the cyclic subgroup of $G$ generated by $f$.

Proof. Assume that $L_j$ is a component whose stabilizer contains $f$. Since the two balls of $S^3 \setminus \text{Fix}(\gamma_1)$ are exchanged by $f$, $L_j$ cannot be contained in one of them and must thus intersect $\text{Fix}(\gamma_1)$. It follows that $\gamma_1$ belongs to the stabilizer of $L_j$, and hence the stabilizer of $L_j$ is the whole group $G$. \hfill $\square$

Note that for the conclusion of the lemma to be valid, we do not need to assume that $L$ is hyperbolic.

Case 4. The stabilizer of $O_i$ is $G$. In this case we choose $K_i$ to be a non-trivial amphicheiral knot with free period 2; recall that, without appealing to Theorem 1.1, there are composite knots with these properties. In fact, the connected sum of two copies of an $\epsilon$-amphicheiral knot is $\epsilon$-amphicheiral and has free period 2 (cf. [24, Theorem 4]). Now note that the action of $f$ on $H_1(T_i)$ is trivial and therefore the actions of $\gamma_1$ and $\gamma_2$ on $H_1(T_i)$ are identical. Hence we may choose $E(K_i)$ to be the exterior of a positive-or negative-amphicheiral knot accordingly, so that $E(K_i)$ admits an orientation-reversing diffeomorphism, $\nu_i$, such that the action of $\nu_i$ on $H_1(T_i)$ is identical with those of $\gamma_1$ and $\gamma_2$. Thus, both $\gamma_1$ and $\gamma_2$ extend to diffeomorphisms of $(E(O_\mu) \cup E(K_i), K_0)$, as in Case 2.

Although the composition of the extensions of $\gamma_1$ and $\gamma_2$ may not be an involution, we can show that $f = \gamma_1 \gamma_2$ extends to a free involution on $(E(O_\mu) \cup E(K_i), K_0)$. To see this, we use the fact that the strong equivalence class of a free $\mathbb{Z}/n\mathbb{Z}$-action on a torus $T$ is determined by its slope, which is the submodule of $H_1(T;\mathbb{Z}/n\mathbb{Z})$ isomorphic to $\mathbb{Z}/n\mathbb{Z}$, represented by a simple loop on $T$, which is obtained as an orbit of a free circle action in which the $\mathbb{Z}/n\mathbb{Z}$-action embeds (see [24, §2]). Here, two actions are strongly equivalent if they are conjugate by a diffeomorphism which is isotopic to the identity. Now, as in Case 1, let $\ell_i$ and $m_i$ be the longitude and the meridian in $T_i = \partial E(K_i)$ of the knot $K_i$. Then, since $f$ gives a free period 2 of the trivial knot $O_i$, the slope of the action of $f$ on $T_i$ is generated by $\ell_i + m_i$ by [24, Lemma 1.2(3)]. Let $f_i$ be a free involution on $E(K_i)$ which realizes the free periodicity of $K_i$. Then the slope of $f_i$ on $T_i$ is also generated by $\ell_i + m_i$. Thus the restrictions of $f$ and $f_i$ to $T_i$ are strongly equivalent by [24, Lemma 1.1]. Hence, by using a collar neighbourhood of $T_i$, we can glue these free involutions to obtain a free involution on $(E(O_\mu) \cup E(K_i), K_0)$.

The above case-by-case discussion shows that, given a link $L = K_0 \cup O_\mu$ which provides an admissible root, we can construct a knot $(S^3, K)$ by attaching suitable knot exteriors $\{E(K_i)\}_{1 \leq i \leq \mu}$ to $(E(O_\mu), K_0)$, such that all elements of $G$ extend to diffeomorphisms of $(S^3, K)$, and such that an extension of $f$ is a free involution. It is now clear that the resulting knot $K$ is a prime amphicheiral knot and has free period 2.

Remark 2.3. According to Theorem 1.2 (see also Claim 5.2), $L$ must contain a component $O_i$ whose stabilizer is generated by $\gamma_1$, where $\gamma_1$ acts on the boundary torus $T_i$ as a reflection in two meridians of $O_i$. Thus the knot $K_i$ must be positive amphicheiral. The orientation-reversing diffeomorphism $\nu_i$ of $E(K_i)$ realizing the positive amphicheirality of $K_i$ is not necessarily an involution nor a periodic map. Even if $\nu_i$ is an involution, its
restriction to $T_i$ is not strongly equivalent to that of $\gamma_1$. In this case, the square of the diffeomorphism obtained by gluing $\gamma_1$ and $\nu_i$ is a non-trivial Dehn-twist along the JSJ torus $T_i$. As a consequence, the extension of $\gamma_1$ can never be periodic.

3. Some explicit examples of links providing admissible roots

To complete the proof of the existence of prime amphicheiral knots with free period 2 we still need to produce a link satisfying the requirements of Definition 2.1.

In this section we shall define three links, $L_\mu = K_0 \cup O_\mu$ with $\mu = 2, 3, 6$, and show that they provide admissible roots. The three different links will allow us to produce prime amphicheiral knots with different properties.

3.1. The link $L_2$

Consider the link depicted in Figure 1(c). This link was constructed so as to ensure that it admits the desired $G$-action in the following way. Consider a fundamental domain for the $G$-action, define a tangle inside the domain and get a link by symmetrizing the domain and the tangle it contains thanks to the $G$-action.

It is not difficult to see that $G$ has a fundamental domain which consists of ‘half a ball’; indeed, each of the 3-balls bounded by the 2-sphere $\text{Fix}(\gamma_1)$ forms a $\gamma_2$-invariant fundamental domain for $\langle \gamma_1 \rangle$, and the restriction of the antipodal map $\gamma_2$ to each of the 3-ball has half the ball as a fundamental domain. To be even more precise, the half ball can be considered as a cone on a closed disc. The identification induced by the action on its boundary, which consists of gluing the boundary circles of the discs via a rotation of order 2, allows us to obtain the global quotient, which is a cone on a projective plane; the vertex of the cone is a singular point of order two (image of the fixed-points of $\gamma_2$), and the base of the cone is a silvered projective plane (image of the fixed-points of $\gamma_1$).

We define a three-component tangle inside the half ball as shown in Figure 1(a). One component is a ‘trefoil arc’ (i.e. a trefoil knot cut at one point); this component will give rise to $K_0$, which is $G$-invariant and thus contains the vertex of the cone. The other two components are unknotted arcs which are entangled with the first component. The procedure and result of symmetrizing the tangle are shown in Figure 1(b) and (c). Because of its very construction it is now clear that $L_2$ has the required $G$-action where $K_0$ is $G$-invariant. It is also clear from the picture that the components $O_1$ and $O_2$ of $L_2$ form a trivial link.

It remains to show that the link is hyperbolic. The proof of this fact is rather technical and is given in §7.

3.2. The link $L_3$

As in the previous subsection, we build a $G$-symmetric link from a tangle in a fundamental domain for the $G$-action, as illustrated in Figure 2.

The resulting link $L_3$ and its component $K_0$ are $G$-invariant by construction. It is clear from the picture that $O_1 \cup O_2 \cup O_3$ is a trivial link. Hyperbolicity of $L_3$ is again rather technical to establish and will be shown in §8.

We remark that the symmetry group of $L_3$ is larger than $G$. In fact, it contains another reflection $\gamma_3$ in a 2-sphere (see Figure 3). As a consequence, $L_3$ is invariant by the action
Figure 1. (a) The three-string tangle inside the half-ball fundamental domain (top left), (b) the first symmetrization with respect to the antipodal map (top right) and (c) the resulting link $L_2 = K_0 \cup O_1 \cup O_2$ after symmetrizing with respect to the inversion in a sphere.

of another Klein four-group $G' = \langle \gamma_1', \gamma_2' \rangle = r^{-1}Gr$, where $r$ is the $\pi/2$-rotation about $Fix(\gamma_1) \cap Fix(\gamma_3)$, and $\gamma_1' = r^{-1}\gamma_1r = \gamma_3$ and $\gamma_2' = r^{-1}\gamma_2r = \gamma_1\gamma_2\gamma_3$. Notice that the element $h := r^2 = \gamma_1\gamma_3$ is a $\pi$-rotation acting as a strong inversion of both $K_0$ and $O_1$. (Here, a strong inversion of a knot is an orientation-preserving smooth involution of $S^3$ preserving the knot, whose fixed-point set is a circle intersecting the knot transversely at two points.)

3.3. The link $L_6$

In this case, because of the relatively large number of components, the tangle obtained by intersecting the link with a fundamental domain is harder to visualize. Instead of exhibiting the tangle, we give in Figure 4 two pictures showing that the link is symmetric with respect to both $\gamma_1$ and $\gamma_2$. Remark that $L_6$ is a highly symmetric link, namely, it is symmetric with respect to three more reflections in vertical planes perpendicular to $Fix(\gamma_1)$, that is, the plane of projection in Figure 4. Observe that the product of two reflections in these vertical planes results in a $2\pi/3$-rotation $\rho$, while the product of one reflection with $\gamma_1$ is a $\pi$-rotation acting as a strong inversion of $K_0$. 
Figure 2. The link $L_3$ pictured on the left is symmetric with respect to $\gamma_1$, the reflection in the horizontal plane. The tangle obtained as the quotient of $L_3$ by $\gamma_1$ is shown on the right; it is left invariant by the central symmetry of the ball, which lifts to the involution $\gamma_2$ whose fixed point set consists of the two points of $O_1$ marked in red.

Figure 3. An image of $L_3$ displaying the extra symmetry of the link.

Figure 4. Two diagrams of the link $L_6$ showing the presence of different symmetries. On the left, the link is symmetric with respect to the plane of projection; this symmetry corresponds to $\gamma_1$. The pink lines are the axes of the $\pi$-rotations acting as strong inversions on $K_0$; they coincide with the intersection of $\text{Fix}(\gamma_1)$ with the fixed-point sets of the other reflections that leave $L_6$ invariant. On the right, the link is symmetric with respect to a central reflection corresponding to $\gamma_2$. 
Once more, we postpone the proof that $L_6$ is hyperbolic to §9, where we will also see that this example can be generalized to give an infinite family of links providing admissible roots.

4. Structure of prime amphicheiral knots with free period 2

In this section, we show that any prime amphicheiral knot with free period 2 is constructed as in §2.

Theorem 4.1. Let $K$ be a prime amphicheiral knot with free period 2. Then there is a link $L = K_0 \cup O_\mu$ which provides an admissible root and satisfies the following conditions.

1. There are non-trivial knots $K_i (i = 1, \cdots, \mu)$ such that $(S^3, K) = (E(O_\mu), K_0) \cup (\cup_{i=1}^\mu (E(K_i), \emptyset))$.

Here, if a component $O_i$ of $O_\mu$ is stabilized by $\gamma_j$ for some $j = 1$ or 2, then the knot $K_i$ is negative-amphicheiral or positive-amphicheiral according to whether $\gamma_j$ preserves or reverses a fixed orientation of $O_i$.

2. For each element $g$ of $G$, its restriction to $(E(O_\mu), K_0)$ extends to a diffeomorphism of $(S^3, K)$, which we call an extension of $g$. Moreover, some extension of $f = \gamma_1 \gamma_2 \in G$ is a free involution on $S^3$. Furthermore, if $K$ is $\epsilon$-amphicheiral, then an extension of $\gamma_1$ or $\gamma_2$ realizes the $\epsilon$-amphicheirality of $K$.

Let $K$ be a prime amphicheiral knot with free period 2. Then, as already observed, it follows from [25] that the exterior $E(K)$ of $K$ admits a non-trivial JSJ decomposition. Let $E_0$ be the root of the decomposition, i.e. the geometric piece containing the boundary, and let $E_i (1 \leq i \leq \mu)$ be the closure of the components of $E(K) \setminus E_0$. Then $E_0$ is identified with the exterior of a link $L = K_0 \cup O_\mu$, where $O_\mu = \cup_{i=1}^\mu O_i$ is a $\mu$-component trivial link, and $E_i$ is identified with a knot exterior $E(K_i)$ for each $i$ with $1 \leq i \leq \mu$ (see, e.g. [24, Lemma 2.1]).

We show that $L$ provides an admissible root in the sense of Definition 2.1, namely, $E_0$ is hyperbolic and, after an isotopy, both $L$ and $K_0$ are $G$-invariant.

We first prove that the root $E_0$ is hyperbolic. Otherwise, $E_0$ is a Seifert fibred space embedded in $E(K)$ with at least two (incompressible) boundary components and so $E_0$ is a composing space or a cable space (see [14, Lemma VI.3.4] or [13, Lemma IX.22]). Since $K$ is prime, $E_0$ is not a composing space. So $E_0$ is a cable space and hence $K$ is a cable knot. However, a cable knot cannot be amphicheiral, a contradiction. Although this fact should be well known, we could not find a reference, so we include a proof for completeness.

Lemma 4.2. A cable knot is not amphicheiral.

Proof. Let $K$ be a $(p, q)$-cable of some knot, where $p$ is a positive integer greater than 1 and $q$ is an integer relatively prime to $p$. Then the root $E_0$ of the JSJ decomposition of $E(K)$ is the Seifert fibred space with base orbifold an annulus with one cone
point, such that the singular fibre has index \((p,q)\). If \(K\) is amphicheiral, then \(E_0\) admits an orientation-reversing diffeomorphism, \(\gamma\), and we may assume that \(\gamma\) preserves the Seifert fibration (see, e.g. [27, Theorem 3.9]). Hence we have \(q/p \equiv -q/p \in \mathbb{Q}/\mathbb{Z}\), and so \(p = 2\). Thus \(E_0\) is identified with the exterior of the pretzel link \(P(2,-2,q)\). Since \(\gamma\) is a restriction of a diffeomorphism of \((S^3,K)\) to \(E_0\), it extends to a diffeomorphism of \((S^3,P(2,-2,q))\), which reverses the orientation of \(S^3\). This contradicts the fact that \(P(2,-2,q)\) is not amphicheiral. The last fact can be seen, for example, by using \([26, Theorem 4.1]\).

Let Isom\(^*\)(\(E_0\)) be the subgroup of the isometry group of the complete hyperbolic manifold \(E_0\) consisting of those elements \(g\) which extend to diffeomorphisms of \((S^3,K)\). (To be precise, we identify \(E_0\) with the non-cuspidal part of a complete hyperbolic manifold.) Denote by Isom\(^+\)(\(E_0\)) the subgroup of Isom\(^*\)(\(E_0\)) consisting of elements whose extensions to \((S^3,K)\) preserve the orientation of both \(S^3\) and \(K\). Then we have the following lemma, which holds a key to the main result in this section. Thus we include a proof, even though it follows from [24, the last part of the proof of Lemma 2.2].

**Lemma 4.3.**

1. The action of Isom\(^*\)(\(E_0\)) on \(E_0\) extends to a smooth action on \((S^3,L)\).
2. Isom\(^+\)(\(E_0\)) is a finite cyclic group.
3. The extension of Isom\(^+\)(\(E_0\)) to \((S^3,L)\) acts effectively on \(K_0\).

**Proof.**

1. Let \(\ell_i\) and \(m_i\) be the longitude and the meridian, respectively, of the knot \(K_i\) with \(E_i = E(K_i)\) \((1 \leq i \leq \mu)\). Recall that they are the meridian and the longitude of \(O_i\), respectively. Since each element \(g \in \text{Isom}^*(E_0)\) extends to a diffeomorphism of \((S^3,K)\), it follows that if \(g(E_i) = E_j\) then \(g(\ell_i) = \pm\ell_j\), and so \(g\) maps the meridian of \(O_i\) to the meridian (possibly with reversed orientation) of \(O_j\). Hence the action of Isom\(^*\)(\(E_0\)) on \(E_0\) extends to an action on \((S^3,L)\).

2. If Isom\(^+\)(\(E_0\)) is not cyclic, then the restriction of the extended action to \(K_0\) is not effective, i.e. there is a non-trivial element, \(g\), of Isom\(^+\)(\(E_0\)) whose extension, \(\tilde{g}\), to \((S^3,K_0 \cup O_\mu)\) is a non-trivial periodic map with \(\text{Fix}(\tilde{g}) = K_0\). By the positive solution of the Smith conjecture [20], \(\text{Fix}(\tilde{g})\) is a trivial knot, and \(\tilde{g}\) gives a cyclic periodicity of the trivial link \(O_\mu\). By [23, Theorem 1], such periodic maps are ‘standard’, and so there are mutually disjoint discs \(D_i\) \((1 \leq i \leq \mu)\) in \(S^3\) with \(\partial D_i = O_i\) such that, for each \(i \in \{1, \ldots, \mu\}\), either (i) \(\tilde{g}(D_i) = D_i\), so that \(\text{Fix}(\tilde{g})\) intersects \(D_i\) transversely at a single point, or (ii) \(\tilde{g}(D_i) = D_j\) for some \(j \neq i\). If (ii) happens for some \(i\), then \(D_i\) is disjoint from \(\text{Fix}(\tilde{g}) = K_0\) and \(O_\mu - O_i\), so the torus \(T_i\) is compressible in \(E_0\), a contradiction. Hence the link \(K_0 \cup O_\mu\) is as illustrated in Figure 5 and therefore \(E_0\) is a composing space, a contradiction.

3. This follows from the argument in the previous point. \(\square\)
Let $\gamma$ be a diffeomorphism of $(S^3, K)$ which realizes the amphicheirality of $K$, and let $f$ be a free involution of $(S^3, K)$. After an isotopy, $\gamma$ preserves $E_0$ and so determines a self-diffeomorphism of $E_0$. We will denote by $\bar{\gamma}$ the orientation-reversing isometry of $E_0$ isotopic to this diffeomorphism. By $[1, 24]$, we may assume that the involution $f$ restricts to an isometry $\bar{f}$ of $E_0$. We denote by $\bar{\gamma}$ and $\bar{f}$ the periodic diffeomorphisms of $(S^3, L)$ obtained as extensions of $\bar{\gamma}$ and $\bar{f}$, respectively, whose existence is guaranteed by Lemma 4.3.

**Lemma 4.4.** The diffeomorphism $\bar{f}$ is a free involution of $(S^3, L)$.

**Proof.** By construction, $\bar{f}$ is an involution which acts freely on $E_0$. If $\bar{f}$ has a fixed point, it must occur inside a regular neighbourhood of some component $O_i$. Moreover, the fixed-point set must coincide with an $O_i$ which is $\bar{f}$-invariant. It follows that the slope of $\bar{f}$ along $O_i$ is the meridian. Since the meridian of $O_i$ coincides with the longitude of $E_i = E(K_i)$, the action of $f$ on $E(K_i)$ cannot be free by $[24$, Lemma 1.2(3)], against the assumption. □

**Lemma 4.5.** We can choose the diffeomorphism $\gamma$ of $(S^3, K)$ giving amphicheirality of $K$ so that the corresponding isometry $\bar{\gamma}$ of $E_0$ satisfies the condition that the subgroup $\langle \bar{\gamma}, \bar{f} \rangle$ of Isom $(E_0)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Moreover, the new $\gamma$ preserves or reverses the orientation of $K$ according to whether the original $\gamma$ preserves or reverses the orientation of $K$.

**Proof.** Since Isom $(E_0)$ is the isometry group of a hyperbolic manifold with finite volume, it is a finite group. The element $\bar{\gamma}$ reverses the orientation of the manifold, so it must have even order and, up to taking an odd power, we can assume that its order is a power of 2. The cyclic subgroup Isom $^+*(E_0)$ of Isom $^*(E_0)$ is clearly normal. It contains the subgroup generated by $\bar{f}$, which is its only subgroup of order 2. It follows that the subgroup generated by $\bar{f}$ is normalized by $\bar{\gamma}$, and hence $\bar{f}$ and $\bar{\gamma}$ commute. If the order of $\bar{\gamma}$ is 2, we are done. Otherwise, $\bar{\gamma}^2$ belongs to the cyclic group Isom $^+*(E_0)$ and $\bar{f}$ is a power of $\bar{\gamma}^2$. Note that the periodic map $\bar{\gamma}$ of $(S^3, L)$ obtained as the extension of $\bar{\gamma}$ reverses the orientation of $S^3$ and hence it has a non-empty fixed point set. Thus $\bar{\gamma}^2$ also has a non-empty fixed point set, and so does $\bar{f}$, a contradiction. Hence $\langle \bar{\gamma}, \bar{f} \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$. The last assertion is obvious from the construction. □

Consider the subgroup $\langle \bar{\gamma}, \bar{f} \rangle$ of Isom $(E_0)$ in the above lemma, and let $\langle \bar{f}, \gamma \rangle$ be its extension to a group action on $(S^3, L)$. The main result of $[9]$ implies that $\langle \bar{f}, \gamma \rangle$ is conjugate, as a subgroup of Diff$(S^3)$, to a subgroup $\bar{G}$ of the orthogonal group $O(4) \cong Isom(S^3)$. 

![Figure 5. A connected sum of Hopf links.](image)
By using the facts that \( \bar{f} \) is a free involution and that \( \bar{\gamma} \) descends to an orientation-reversing involution on \( S^3/\bar{f} \), we may assume that \( \bar{G} \) is equal to the group \( G = \langle \gamma_1, \gamma_2 \rangle \) (cf. [16]). Thus we may assume that \( L = K_0 \cup O_\mu \) is \( G \)-invariant, and it provides an admissible root in the sense of Definition 2.1. The remaining assertions of Theorem 4.1 follow from the arguments in §2.

5. Proofs of Theorems 1.1, 1.2 and 1.4

Let \( K \) be a prime amphicheiral knot with free period 2. Then, by Theorem 4.1, there is a link \( L = K_0 \cup O_\mu \) which provides an admissible root and non-trivial knots \( K_i \) \((i = 1, \ldots, \mu)\) such that

\[
(S^3, K) = (E(O_\mu), K_0) \cup (\cup_{i=1}^\mu (E(K_i), \emptyset)).
\]

In particular, \( L \) is \( G \)-invariant and the root \( E_0 \) of the JSJ decomposition of \( E(K) \) is identified with \( E(L) = E(K_0 \cup O_\mu) \), where \( \mu = \mu(K) \).

We will start with the proof of Theorem 1.2: each point of the theorem will be proved as a claim in this section.

Claim 5.1. \( L \) contains at most two components whose stabilizer is the whole group \( G \). These components are either a great circle in the 2-sphere \( \text{Fix}(\gamma_1) \) or a trivial or composite knot that meets \( \text{Fix}(\gamma_1) \) and contains the two fixed points of \( \gamma_2 \).

Here, by a great circle we mean an embedded circle in the 2-sphere \( \text{Fix}(\gamma_1) \) which is invariant by the antipodal map induced by \( f \) on \( \text{Fix}(\gamma_1) \).

Proof. Let \( L_i \) be a component which is left invariant by \( G \). If \( L_i \) is contained in \( \text{Fix}(\gamma_1) \) then it must be a trivial knot which is a great circle of \( \text{Fix}(\gamma_1) \). Since the 2-sphere \( \text{Fix}(\gamma_1) \) contains two mutually disjoint great circles, no other component contained in \( \text{Fix}(\gamma_1) \) can have \( G \) as stabilizer. Assume now that \( L_i \) is not contained in \( \text{Fix}(\gamma_1) \). Since its stabilizer is \( G \), \( L_i \) cannot be contained in one of the two balls of \( S^3 \setminus \text{Fix}(\gamma_1) \). As a consequence, it intersects \( \text{Fix}(\gamma_1) \) transversally at two antipodal points, and each of the two balls of \( S^3 \setminus \text{Fix}(\gamma_1) \) in an arc. Since each of these arcs is left invariant by \( \gamma_2 \), it must contain one of the two fixed points of \( \gamma_2 \). Of course, at most one component of \( L \) can contain \( \text{Fix}(\gamma_2) \). Since the two arcs are exchanged by \( \gamma_1 \), it follows that they are both unknotted, in which case \( L_i \) is trivial, or both knotted, in which case \( L_i \) is composite.

Claim 5.2. \( L \) contains at least one pair of components of \( O_\mu \), such that (i) each of the components intersects the 2-sphere \( \text{Fix}(\gamma_1) \) transversely at two points, (ii) the stabilizer of each of the components is generated by \( \gamma_1 \) and (iii) \( f \) interchanges the two components.

Proof. Since \( L \) is hyperbolic, we can assume that \( \gamma_1 \) acts as a hyperbolic isometry on its complement. It follows that each component of \( \text{Fix}(\gamma_1) \setminus L \) is a totally geodesic surface in the hyperbolic manifold \( S^3 - L \). Thus no component of \( \text{Fix}(\gamma_1) \setminus L \) is a sphere, a disc or an annulus. On the other hand, if a component of \( L \) is not disjoint from the 2-sphere \( \text{Fix}(\gamma_1) \), then it is either contained in \( \text{Fix}(\gamma_1) \) or intersects \( \text{Fix}(\gamma_1) \) transversely at two points. (In fact, if a component of \( L \) intersects transversely \( \text{Fix}(\gamma_1) \) at a point, then it
is $\gamma_1$-invariant and so it intersects $\text{Fix}(\gamma_1)$ transversely at precisely two points.) Now the claim follows from the fact that either (a) $\text{Fix}(\gamma_1)$ contains no component of $L$, so that $\text{Fix}(\gamma_1) \setminus L$ is a punctured sphere with at least three (actually four) punctures, or (b) contains some components of $L$, in which case at least two components of $\text{Fix}(\gamma_1) \setminus L$ are punctured discs with at least two punctures. \hfill \Box

\textbf{Claim 5.3.} $L$ contains no component with stabilizer generated by $f$. This was proved in Lemma 2.2.

\textbf{Claim 5.4.}

\begin{enumerate}[(1)]
    \item If $K$ is positive-amphicheiral, then $K_0$ must be contained in $\text{Fix}(\gamma_1)$.
    \item If $K$ is negative-amphicheiral, then $K_0$ must contain $\text{Fix}(\gamma_2)$ and intersect $\text{Fix}(\gamma_1)$ transversally at two points.
\end{enumerate}

\textbf{Proof.} By assumption, $K_0$ is $G$-invariant, so it must be as described in Claim 5.1. Consider the action induced by $G$ on $H_1(K_0; \mathbb{Z})$. Since $f \in G$ is a free involution of $S^3$, the action of $f$ on $H_1(K_0; \mathbb{Z})$ is trivial, so the actions of $\gamma_1$ and $\gamma_2$ on it coincide. It is now easy to see that (i) if $K_0$ is contained in $\text{Fix}(\gamma_1)$ then $\gamma_1$ acts trivially on $H_1(K_0; \mathbb{Z})$, and (ii) if $K_0$ meets $\text{Fix}(\gamma_1)$ transversely then $\gamma_1$ acts on $H_1(K_0; \mathbb{Z})$ as multiplication by $-1$. Since $\epsilon$-amphicheirality of $K$ is realized by an extension of $\gamma_1$ or $\gamma_2$ by Theorem 4.1, we obtain the desired result. \hfill \Box

This ends the proof of Theorem 1.2.

We now explain Remark 1.3. Suppose that the prime knot $K$ with free period 2 is both positive- and negative-amphicheiral. Since $\text{Isom}^*(E_0)$ is a finite cyclic group by Lemma 4.3(2), there is a unique element $f \in \text{Isom}^*(E_0)$ which extends to a smooth involution of $(S^3, K)$ realizing the free period 2. For $\epsilon \in \{+, -\}$, let $\gamma_\epsilon$ be the order 2 element of $\text{Isom}^*(E_0)$ which realizes the $\epsilon$-amphicheirality of $K$, such that $\langle f, \gamma_\epsilon \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$ (cf. Lemma 4.5). Now recall that the finite group $\text{Isom}^*(E_0)$ extends to an action of $(S^3, L)$ by Lemma 4.3, and so we identify it with a finite subgroup of $\text{Diff}(S^3, L)$. Then $\gamma_+ \gamma_-$ preserves the orientation of $S^3$ and reverses the orientation of the component $K_0$, and so it realizes the invertibility of $K_0$. Since $(\gamma_+ \gamma_-)^2$ acts on $K_0$ as the identity map, the periodic map $(\gamma_+ \gamma_-)^2$ must be the identity map according to Lemma 4.3. Thus $(\gamma_+ \gamma_-)^2 = 1$ in $\text{Isom}^*(E_0)$. Hence we see $\langle f, \gamma_+, \gamma_- \rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$. The main result of [9] guarantees that, as a subgroup of $\text{Isom}(S^3)$, this group is smoothly conjugate to a subgroup of $\text{Isom}(S^3)$. Remark 1.3 now follows from this fact.

We need the following lemma in the proof of Theorem 1.4.

\textbf{Lemma 5.5.} Let $L = K_0 \cup O_2$ be a three-component link providing an admissible root. Then $K_0$ cannot be contained in the 2-sphere $\text{Fix}(\gamma_1)$.

\textbf{Proof.} Assume by contradiction that $L$ is a link with three components providing an admissible root and such that $K_0$ is contained in $S := \text{Fix}(\gamma_1)$. Recall that, because of Claim 5.2, each component of $O_2$ must intersect $S$ transversally at two points. Thus $S$ gives a 2-bridge decomposition of $O_2$. (For terminology and standard facts on 2-bridge
links, refer to [17, § 2].) By the uniqueness of the 2-bridge spheres or by the classification of 2-bridge spheres, $S$ is identified with the standard 2-bridge sphere of the 2-bridge link of slope $1/0$. The knot $K_0$ is an essential simple loop on the $4$-times punctured sphere $S \setminus \mathcal{O}_2$, and the isotopy type of any such loop is completely determined by its slope $s \in \mathbb{Q} \cup \{1/0\}$. We can easily check that the involution $\gamma_2$ sends a loop of slope $s$ to a loop of slope $-s$. Hence the slope of $K_0$ is either 0 or $1/0$. According to whether the slope is 0 or $1/0$, the link $L = K_0 \cup \mathcal{O}_2$ is the connected sum of two Hopf links or 3-component trivial link, a contradiction.

We shall now give the proof of Theorem 1.4. Claim 5.2 shows that $\mu_- , \mu_+ \geq 2$. The link $L_2$ defined in § 3, on the other hand, shows that $\mu_- \leq 2$. In fact, both $\gamma_1$ and $\gamma_2$ act on the component $K_0$ by reversing its orientation; they extend to diffeomorphisms of $(S^3, K)$ which give negative-amphicheirality of the knot $K$. The first part of the theorem now follows from Claims 5.2 and 5.4.

For the second part, once again the link $L_3$ defined in § 3 shows that $\mu_+ \leq 3$. Lemma 5.5 ensures that $\mu_+ = 3$. In fact, both $\gamma_1$ and $\gamma_2$ act on the component $K_0$ by preserving its orientation; they extend to diffeomorphisms of $(S^3, K)$ which give positive-amphicheirality of the knot $K$. Now the second part of the theorem follows again from Claims 5.2 and 5.4.

We pass now to the proof of Theorem 1.1. Assuming the hyperbolicity of the links $L_\mu$ with $\mu = 2, 3, 6$, which is proved in §§ 7, 8 and 9, the existence of prime, amphicheiral knots with free period 2 was established in §3. To finish the proof of Theorem 1.1, we only need to check that we can find prime amphicheiral knots admitting free period 2 that are:

1. negative-amphicheiral but not positive-amphicheiral;
2. positive-amphicheiral but not negative-amphicheiral;
3. positive- and negative-amphicheiral at the same time, that is, amphicheiral and invertible.

We show why these statements hold by proving a series of claims.

**Claim 5.6.** The prime amphicheiral knots whose root is the exterior of $L_2$ are negative-amphicheiral and cannot be positive-amphicheiral.

**Proof.** Since $\gamma_1$ acts on $K_0$ by reversing its orientation, the extensions of both $\gamma_1$ and $\gamma_2$ act by inverting $K$, which is thus negative-amphicheiral. Assume now by contradiction that $K$ is also positive-amphicheiral. According to Theorem 1.2, $L_2$ must admit another action $G'$ of the Klein four-group, containing a reflection $\gamma_1'$ in a 2-sphere $\text{Fix}(\gamma_1')$, such that $K_0$ is contained in $\text{Fix}(\gamma_1')$. This is, however, impossible as $K_0$ is not trivial.

**Claim 5.7.** The prime amphicheiral knots whose root is the exterior of $L_3$ are positive-amphicheiral. If the knots $K_i$, $i = 1, 2, 3$, are all positive-amphicheiral but none of them is negative-amphicheiral, then $K$ itself is not negative-amphicheiral.
\textbf{Proof.} The fact that $K$ is positive-amphicheiral can be seen as in Claim \ref{claim:invertible}. Observe that $\gamma_1$ acts on each $O_i$ by reversing its orientation. As a consequence, each $E(K_i)$ must be the exterior of a positive-amphicheiral knot, i.e. each $K_i$ must be positive-amphicheiral. We assume now that each $K_i$ is not negative-amphicheiral. Suppose by contradiction that $K$ is negative-amphicheiral. Then, by Theorem \ref{thm:positive}, $L_3$ admits another action $G' = \langle \gamma_1', \gamma_2' \rangle$ of the Klein four-group such that $K_0'$ contains $\text{Fix}(\gamma_2') \cong S^0$. Since $O_3$ has three components, precisely one of them, say $O_j$, for a $j \in \{1, 2, 3\}$, must be $G'$-invariant and, according to Claim \ref{claim:positive}, it must be contained in the 2-sphere $\text{Fix}(\gamma_1')$. Since $\gamma_1'$ acts trivially on the first integral homology groups of such components, $K_j$ must be negative-amphicheiral against the assumption. \hfill \Box

\textbf{Claim 5.8.} Let $K$ be a prime amphicheiral knot admitting free period 2 whose root is the exterior of $L_\mu$ with $\mu = 3$ or 6, which is constructed as in $\S$ 2 by using the symmetry $G$.

(1) If $\mu = 3$ and $K_1$ is invertible, then $K$ is invertible.

(2) If $\mu = 6$ and all $K_i$ are copies of the same negative-amphicheiral knot, then $K$ is invertible.

\textbf{Proof.}

(1) Consider the link $L_3$. Then, by construction of the knot $K$, $K_1$ is a positive-amphicheiral knot admitting free period 2, and both $K_2$ and $K_3$ are copies of a positive-amphicheiral knot. As noted in $\S$ 3.2, $L_3$ is invariant by the action of the group $\hat{G} = \langle \gamma_1, \gamma_2, \gamma_3 \rangle < \text{Isom}(S^3)$, where the $\gamma_i$ shown in Figure 3 are orientation-reversing involutions. Note that $\hat{G}$ is the direct product of the group $G = \langle \gamma_1, \gamma_2 \rangle$ and the order 2 cyclic group generated by $h := \gamma_1 \gamma_3$. Then $h$ reverses the orientations of $K_0$ and $O_1$, and preserves the orientations of $O_2$ and $O_3$. The stabilizer in $\hat{G}$ of $O_j$ is equal to $\hat{G}$ for $j = 1$ and to $\langle \gamma_1, h \rangle$ for $j = 2, 3$. For $j = 2, 3$, $h$ acts on $H_1(T_j; \mathbb{Z})$ ($j = 2, 3$) trivially and so there is no obstruction in extending $h$ to $E(K_j)$. On the other hand, $h$ acts on $H_1(T_1; \mathbb{Z})$ as $-I$. Thus, if $K_1$ is invertible, then $h$ extends to a diffeomorphism of $(S^3, K)$ that realizes the invertibility of $K$.

(2) Consider the link $L_6$. Then $L_6$ is invariant by the group $\hat{G} = \langle \gamma_1, \gamma_2, \gamma_3 \rangle < \text{Isom}(S^3)$, where $\gamma_3$ is the reflection in the vertical plane intersecting the projection plane in the horizontal pink line shown in Figure 4. If all $K_i$ are copies of the same negative-amphicheiral knot, then each of the $\gamma_3$ and the $2\pi/3$-rotation $\rho$ extends to a diffeomorphism of $(S^3, K)$, and hence every element of $\hat{G}$ extends to a diffeomorphism of $(S^3, K)$. We can observe that any extension of $h := \gamma_1 \gamma_3$ realizes the invertibility of $K$. \hfill \Box

The proof of Theorem 1.1 is now complete.

\textbf{Remark 5.9.}

(1) In the situation described in Claim 5.8, the involution $h = \gamma_1 \gamma_3$ extends to a diffeomorphism $\hat{h}$ of $(S^3, K)$ which realizes the invertibility of the knot. However, any
extension $\hat{h}$ of $h$ (to be precise, of the restriction of $h$ to $E(L_\mu)$) to $(S^3, K)$ cannot be an involution. We explain this in the case where the root of $K$ is the exterior of $L_3$. In this case, $h$ stabilizes all components of $L_3$ and acts as a $\pi$-rotation on each of the components $O_2$ and $O_3$. Thus, for $i = 2, 3$, the restriction of $h$ to $T_i = \partial N(O_i)$ is an orientation-preserving free involution whose slope is the longitude of $O_i$. This means that $h$ acts on $E(K_i)$, $i = 2, 3$ by preserving the meridian of $E(K_i)$. The positive solution to the Smith conjecture implies that $\hat{h}$ cannot have finite order.

The same argument works for the case when the root of $K$ is the exterior of $L_6$.

(2) Consider now the element $\gamma_2\gamma'_1 = fh$. If the root of $K$ is the exterior of $L_6$, then $fh$ extends to a strong inversion of $K$. To see this, it suffices to observe that this element does not stabilize any component of $L_6$ other than $K_0$. In the case where the root of $K$ is the exterior of $L_3$, $fh$ is a strong inversion provided that the knot $K_1$ is strongly invertible, since $O_1$ meets $\text{Fix}(fh)$ and is left invariant by $fh$, while the components $O_2$ and $O_3$ are exchanged.

The following result is a consequence of the discussion in the previous section and of Theorem 1.2.

**Proposition 5.10.** Assume $K$ is a prime amphicheiral knot with free period 2 and let $E_0$ be its root which can be identified with the exterior of a link $L = K_0 \cup O$. Let $2n \geq 2$ be the order of the cyclic group $\text{Isom}^+(E_0)$. Then precisely one of the following situations occurs:

1. $K$ is positive-amphicheiral but not negative-amphicheiral and $\text{Isom}^+(E_0)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$;

2. $K$ is negative-amphicheiral but not positive-amphicheiral and $\text{Isom}^+(E_0)$ is isomorphic to the dihedral group $\mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$;

3. $K$ is invertible and $\text{Isom}^+(E_0)$ is isomorphic to the semi-direct product $\mathbb{Z}/2\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, where one copy of $\mathbb{Z}/2\mathbb{Z}$ acts dihedrally on $\mathbb{Z}/2\mathbb{Z}$ and the other trivially.

**Proof.** Let $\hat{\gamma}$ be an element in $\text{Isom}^+(E_0)$ which reverses the orientation of $E_0$. We saw in Lemma 4.5 that, up to taking an odd power, we can choose $\hat{\gamma}$ to be of order 2. This means that if $K$ is not invertible, the exact sequence

$$1 \longrightarrow \text{Isom}^+(E_0) \longrightarrow \text{Isom}^+(E_0) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

splits and it is enough to understand how $\mathbb{Z}/2\mathbb{Z}$ acts on $\text{Isom}^+(E_0)$. To conclude, it suffices to observe that $\text{Isom}^+(E_0)$ acts on the circle $K_0$ and the action of $\hat{\gamma}$ is effective if and only if it reverses the orientation of the circle and thus acts dihedrally on $\text{Isom}^+(E_0)$.

If $K$ is invertible, the argument is the same provided we can show that the exact sequence

$$1 \longrightarrow \text{Isom}^+(E_0) \longrightarrow \text{Isom}^+(E_0) \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

splits again. This, however, follows easily from Remark 1.3. \qed
6. More information on the root $E_0 = E(L)$

In this section, we give refinements of the arguments in the previous section and present more detailed results on the structure of the root $E_0 = E(L)$.

We first give a characterization of the links $L = K_0 \cup O_\mu$ that provide an admissible root of a positive amphicheiral knot. Recall that, after an isotopy, such a link $L$ is invariant by the action $G$ and $K_0$ is contained in $\text{Fix}(\gamma_1)$. The following proposition provides a more precise description of such links.

**Proposition 6.1.** Let $L = K_0 \cup O_\mu$ be a link providing an admissible root. Assume that $K_0$ is contained in $S = \text{Fix}(\gamma_1)$. Then the following hold.

1. Suppose no component of $O_\mu$ is $f$-invariant. Then there is an $f$-invariant family of pairwise disjoint discs $\{D_i\}_{i=1}^\mu$ such that $\partial D_i = O_i$ and $D_i$ intersects $S$ transversely, precisely in a single arc $(1 \leq i \leq \mu)$.

2. Suppose one component, say $O_1$, of $O_\mu$ is $f$-invariant. Then there is an $f$-invariant family of discs $\{D'_1, D'_2\} \cup \{D_i\}_{i=2}^\mu$ with disjoint interiors, such that $\partial D'_1 = \partial D'_2 = O_1$, $\partial D_i = O_i$ $(2 \leq i \leq \mu)$ and each disc in the family intersects $S$ transversely, precisely in a single arc.

In particular, each component of $O_\mu$ meets $S$ transversally at two points, in both cases.

**Proof.** (1) Suppose no component of $O_\mu$ is $f$-invariant. Then, since $f$ is orientation-preserving and $O_\mu$ is a trivial link, the equivariant Dehn’s lemma [18, Theorem 5] implies that there is an $f$-invariant family of mutually disjoint discs $\{D_i\}_{i=1}^\mu$ such that $\partial D_i = O_i$.

To prove the proposition we will show that we can choose $\{D_i\}_{i=1}^\mu$ so that each $D_i$ intersects $S := \text{Fix}(\gamma_1)$ precisely in one arc.

By small isotopy, we can assume that the family $\{D_i\}_{i=1}^\mu$ intersects the sphere $S$ transversally; in the case where $O_i = \partial D_i$ is contained in $S$, we mean by ‘transversally along $O_i$’ that there is a collar neighbourhood of $\partial D_i$ in $D_i$ that intersects $S$ only along $\partial D_i$.

We want to show that one can eliminate all circle components of $S \cap (\cup_{i=1}^\mu \text{int} D_i)$. We start by observing that for each such circle component $C$ there is a well-defined notion of the inside of $C$ (in the sphere $S$). Indeed, for each $C$ one can consider the circle $f(C)$. These are two disjoint circles in $S$, so they bound disjoint subdiscs of $S$, and the inside of $C$ is defined to be the interior of the subdisc bounded by $C$ that is disjoint from $f(C)$; observe that the notion of the inside is $f$-equivariant, i.e. the inside of $f(C)$ is the image of the inside of $C$ by $f$. Note now that a circle component of $S \cap (\cup_{i=1}^\mu \text{int} D_i)$ can be of two types: either it contains some arc component of $S \cap (\cup_{i=1}^\mu D_i)$ (first type) or it does not (second type).

We can eliminate all circles of the second type in an $f$-equivariant way, as follows. Note that a circle of this type can only contain circles of the same type in its inside. Let $C$ be any such circle which is innermost in $S$ so that $f(C)$ is also innermost in $S$. The circle $C$ (respectively $f(C)$) is also contained in a disc $D_i$ (respectively $f(D_i)$) of our family, where it bounds a subdisc. We now replace by surgery this subdisc in $D_i$ (respectively $f(D_i)$) with a disc parallel to the subdisc of $S$ contained in $C$ (respectively $f(C)$) slightly off $S$,...
chosen appropriately on the side of \( S \) that allows us to eliminate the intersection. Notice that this operation can be carried out even when \( C \) (respectively \( f(C) \)) contains points of \( K_0 \), and it may result in eliminating other circle intersections, since \( C \) (respectively \( f(C) \)) is not necessarily innermost in \( \cup_{i=1}^\mu D_i \).

The above argument shows that we can assume that all circles in \( S \cap (\cup_{i=1}^\mu \text{int} D_i) \) are of the first type, i.e., contain arc components of \( S \cap (\cup_{i=1}^\mu D_i) \). Under this hypothesis, we now show how to eliminate all circles. Let now \( C \) be a circle component of \( S \cap (\cup_{i=1}^\mu \text{int} D_i) \) which is innermost in \( \cup_{i=1}^\mu D_i \); \( f(C) \) is also innermost in \( \cup_{i=1}^\mu D_i \). Let \( \Delta \) be the disc bounded by \( C \) in \( \cup_{i=1}^\mu D_i \); it is entirely contained in one of the two balls bounded by \( S \) that we shall denote by \( B^\pm \). The disc \( f(\Delta) \), bounded by \( f(C) \) is contained in the second ball, denoted by \( B^- \). Consider now \( \gamma_1(\Delta) \in B^- \) and \( \gamma_1(f(\Delta)) = f(\gamma_1(\Delta)) \subset B^+ \). Since \( \mathcal{O}_\mu \) is \( \gamma_1 \)-invariant, the interiors of these discs are disjoint from \( \mathcal{O}_\mu \) and also from \( L \). Up to small isotopy, we can assume that the two discs meet the family \( \cup_{i=1}^\mu D_i \) transversally. If the interior of \( \gamma_1(\Delta) \) is disjoint from \( \cup_{i=1}^\mu D_i \) (and so is the interior of \( f(\gamma_1(\Delta)) \)), then we can use these two discs to remove the circle intersections \( C \) and \( f(C) \). Otherwise, we perform \( f \)-equivariant surgery along the family \( \cup_{i=1}^\mu D_i \) in order to eliminate all intersections in the interior of the two discs \( \gamma_1(\Delta) \) and \( f(\gamma_1(\Delta)) \), as follows. Indeed, let \( C' \) be an innermost circle of intersection in \( \gamma_1(\Delta) \). \( C' \) must be contained in and bound a subdisc of some disc \( D_i \). One can now replace the subdisc of \( D_i \) with a copy of the disc bounded by \( C' \) in \( \gamma_1(\Delta) \) to reduce the intersection. At the same time, one can replace the subdisc bounded by \( f(C') \) in \( f(D_i) \) with the subdisc bounded by \( f(C') \) in \( f(\gamma_1(\Delta)) \); note that these operations take place in disjoint balls. We stress again that such surgery can only diminish the number of components of \( S \cap (\cup_{i=1}^\mu \text{int} D_i) \) because \( C' \) and \( f(C') \) are not necessarily innermost in \( \cup_{i=1}^\mu D_i \).

We continue to denote by \( \{D_i\}_{i=1}^\mu \) the family obtained after surgery. Let \( C \) be the circle chosen at the beginning of the preceding paragraph. If it is no longer contained in \( \cup_{i=1}^\mu D_i \) then there is nothing to do; note that in this case the intersection \( f(C) \) has also been removed. Suppose \( C \) is contained in \( \cup_{i=1}^\mu D_i \). If the interior of \( \gamma_1(\Delta) \) is not disjoint from \( \cup_{i=1}^\mu D_i \), then we repeat the preceding argument to decrease the intersection. If the interior of \( \gamma_1(\Delta) \) is disjoint from \( \cup_{i=1}^\mu D_i \), then we can use \( \gamma_1(\Delta) \) and \( f(\gamma_1(\Delta)) \) to remove \( C \) and \( f(C) \), as in the preceding paragraph.

The above argument shows that the family can be chosen so that \( S \cap (\cup_{i=1}^\mu \text{int} D_i) \) does not contain circle components. This implies immediately that no component of \( \mathcal{O}_\mu \) can be disjoint from the sphere \( S \) or contained in it, as in this case the link \( L \) would be split, contrary to the assumption that it is hyperbolic; indeed, if \( O_i \) is any such component, the interior of the disc \( D_i \) in the family just constructed is disjoint from \( \mathcal{O}_\mu \) and \( S \) and thus does not meet \( K_0 \) either. This completes the proof of the assertion \((2.1)\) of the proposition.

(2) Suppose one component, say \( O_1 \), of \( \mathcal{O}_\mu \) is \( f \)-invariant. By Theorem 1.2(1), the other components of \( \mathcal{O}_\mu \) are not \( f \)-invariant. Then by the equivariant Dehn’s lemma, there is an \( f \)-invariant family of \( \mu + 1 \) discs \( \{D'_1, D''_1\} \cup \{D_i\}_{i=2}^\mu \) with disjoint interiors, such that \( \partial D_i' = \partial D_i'' = O_1 \), \( \partial D_i = O_i \) (\( 2 \leq i \leq \mu \)). We can assume that this family intersects the sphere \( S \) transversally. For each loop component \( C \) of \( \mathcal{I} := S \cap ((\text{int} D'_1 \cup \text{int} D''_1) \cup (\cup_{i=2}^\mu \text{int} D_i)) \), we define its inside and its type as in the proof of \((2.1)\). Note that \( S \cap D'_1 \) and \( S \cap D''_1 \) each contain a single arc component, denoted by \( \alpha'_1 \) and \( \alpha''_1 \), respectively, and the union \( \alpha'_1 \cup \alpha''_1 \) forms a great circle in \( S \). This implies that no loop component of
The arc systems for links \( L_6 \) and \( L_3 \).

\[ \mathcal{I} \] contains \( \alpha'_1 \) or \( \alpha''_1 \) in its inside. (It should also be noted that the loop \( \alpha'_1 \cup \alpha''_1 \) is not contained in \( \mathcal{I} \).) Now, the argument in the proof of (2.1) works verbatim, and we can remove all loop components of \( \mathcal{I} \), completing the proof of (2) of the proposition.

**Remark 6.2.** In the above proposition, the link \( L = K_0 \cup O_\mu \) is recovered from the \( f \)-invariant arc system in \( S \) which is obtained as the intersection of the \( f \)-invariant family of discs with \( S \). To explain this, identify \( S^3 \) with the suspension of \( S \), the space obtained from \( S \times [-1,1] \) by identifying the subspaces \( S \times \{ \pm 1 \} \) to a point. We assume that the \( G \)-action on \( S^3 \) is equivalent to the \( G \)-action on the suspension that is obtained from the natural product action of \( G \) on \( S \times [-1,1] \). Then the following hold.

1. In the first case, set \( \alpha_i = D_i \cap S \) (\( 1 \leq i \leq \mu \)). Then \( L \) is \( G \)-equivariantly homeomorphic to the link in the suspension obtained as the image of

   \[ K_0 \cup (\cup_{i=1}^{\mu} \partial (\alpha_i \times [-1/2,1/2])) \subset S \times [-1,1]. \]

2. In the second case, set \( \alpha'_1 = D'_1 \cap S \) and \( \alpha_i = D_i \cap S \) (\( 2 \leq i \leq \mu \)). Then \( L \) is \( G \)-equivariantly homeomorphic to the link in the suspension obtained as the image of

   \[ K_0 \cup \partial (\alpha'_1 \times [-1,1]) \cup (\cup_{i=2}^{\mu} \partial (\alpha_i \times [-1/2,1/2])) \subset S \times [-1,1]. \]

   It should be noted that the image of \( \partial (\alpha'_1 \times [-1,1]) \) in the suspension of \( S \) is the suspension of \( \partial \alpha'_1 = O_1 \cap S \subset S \). Moreover, if \( \alpha_1 \) is any arc in \( S \) with endpoints \( O_1 \cap S \) such that \( \alpha_1 \cap f(\alpha_1) = \partial \alpha_1 \), then \( L \) is \( \langle \gamma_1 \rangle \)-equivariantly (but not \( G \)-equivariantly) homeomorphic to the link in the suspension obtained as the image of

   \[ K_0 \cup \partial (\alpha_1 \times [-2/3,2/3]) \cup (\cup_{i=2}^{\mu} \partial (\alpha_i \times [-1/2,1/2])) \subset S \times [-1,1]. \]

   For example, the links \( L_6 \) and \( L_3 \), respectively, satisfy conditions (2.1) and (2) of Proposition 6.1, and they are represented by the two arc systems in Figure 6.

Next, we present the following generalization of Remark 5.9(1).
Proposition 6.3. Let $K$ be an invertible amphicheiral knot having free period 2, and let $L = K_0 \cup \mathcal{O}_\mu$ be its root. Let $\{\gamma_1, \gamma_2\}$ and $\{\gamma'_1, \gamma'_2\}$, respectively, be the symmetries of $L$ generating the two $\mathbb{G}$-actions (compare Remark 1.3 and \S 3.2). Then no extension $\hat{h}$ of the element $h = \gamma_1 \gamma'_1$ is ever a strong inversion of $K$.

Proof. Assume by contradiction that $\hat{h}$ is a strong inversion of $K$ extending $h$. Recall that according to Theorem 1.2, $K_0$ is a trivial knot contained in the 2-sphere $S = \text{Fix}(\gamma_1)$; moreover, the 2-sphere $S' = \text{Fix}(\gamma'_1)$ intersects $S$ perpendicularly and meets $K_0$ at two antipodal points. Consider now the sublink $\mathcal{O} = \mathcal{O}_\mu$ of $L$: according to Theorem 1.2 it must contain two components that intersect transversally $S$ and two components that intersect transversally $S'$. We note that, under our hypotheses, a component that meets $S$ (respectively $S'$) transversally cannot intersect $S'$ (respectively $S$) transversally. This follows from the fact that, by the consideration in Remark 5.9, $h$ cannot leave a component invariant and act as a rotation on it, so that either a component intersects $\text{Fix}(h)$ or its linking number with $\text{Fix}(h) = S \cap S'$ must be even, and hence zero. This implies that $\mathcal{O}$ contains (at least two) components that intersect $S'$ transversally and are either contained in $S$ or disjoint from it. This is, however, impossible according to Proposition 6.1. \hfill \Box

Finally, we show that the first assertion of Theorem 1.2 (i.e. Claim 5.1) is the ‘best possible’, in the sense that all situations described in the claim can arise.

Let $L = K_0 \cup \mathcal{O}_\mu$ be a link which provides an admissible root. We first assume that $K_0$ is the unique $\mathbb{G}$-invariant component of $L$. Then $L$ satisfies one of the following conditions.

1. $K_0$ is a trivial knot contained in $\text{Fix}(\gamma_1)$.
2. $K_0$ is a trivial knot meeting $\text{Fix}(\gamma_1)$ at two points.
3. $K_0$ is a composite knot meeting $\text{Fix}(\gamma_1)$ at two points.

The links $L_6$ and $L_2$ in \S 3 provide examples of the first and the third situations, respectively. We show that the second situation can also occur. Consider the configuration of Figure 7, where each small box represents a rational tangle such that (i) the strands of the tangles behave combinatorially as the dotted arcs inside the boxes, and (ii) the tangles are not a sequence of twists.

For each choice of rational tangles as above, the result is a link with three trivial components (note that when forgetting the outer (respectively inner) component, the inner (respectively outer) component and the central one form a Montesinos link). The exterior of the four central small boxes and of the dotted ones is a basic polyhedron (see [15, Chapter 10]), so it is $\pi$-hyperbolic by Andreev’s theorem. The interiors of the dotted boxes have Seifert fibred double covers, regardless of the chosen rational tangles. It follows that for sufficiently large rational tangles inserted into the four central small boxes, the exterior of the dotted boxes is $\pi$-hyperbolic. It follows that for sufficiently large rational tangles, the Bonahon–Siebenmann decomposition [4] of the orbifold associated with the link with branching order 2 consists of three geometric pieces: two Seifert fibred ones and a $\pi$-hyperbolic one which is their complement. Since the Seifert fibred ones are atoroidal (and the $\pi$-hyperbolic one is anannular), the link is hyperbolic. It is now easy to see
Figure 7. A link providing an admissible root with a single trivial $\mathbb{G}$-invariant component containing $\text{Fix}(\gamma_2)$, which forms a trivial knot.

Figure 8. A link providing an admissible root with two $\mathbb{G}$-invariant components consisting of a trivial knot and a composite one.

that the construction can be carried out in a $\mathbb{G}$-equivariant way so that the black central component is $\mathbb{G}$-invariant, providing an admissible root with the desired property.

We next consider the case where $L$ has two $\mathbb{G}$-invariant components. These can both be trivial, as is the case for $L_3$, or one can be trivial and the other composite. An example of the latter type can be built from the same tangle in the half ball used to construct $L_2$, but with a different gluing and an extra component contained in $\text{Fix}(\gamma_1)$ (see Figure 8). The hyperbolicity of this link can be checked using a computer program such as SnapPea, SnapPy or HIKMOT, or can be proved following the same lines as the proof provided for $L_2$ (see §7) by a slight adaptation of Lemma 7.4, since $K_D$ is in a different position; the details are left to the interested reader.

These two observations indicate that the first assertion of Theorem 1.2 (i.e. Claim 5.1) is the ‘best possible’.
7. The link $L_2$ is hyperbolic

Consider the link $L_2 = K_0 \cup O_2$ with $O_2 = O_1 \cup O_2$ in Figure 1(c). Then $L_2$ is the double of the tangle $(B^3, \tau) := (B^3, \tau_0 \cup \tau_1 \cup \tau_2)$ in Figure 1(b). The tangle $(B^3, \tau)$ is regarded as the sum of the tangle $(B^3, t) := (B^3, t_0 \cup t_1 \cup t_2)$ in Figure 1(a) with its mirror image.

**Lemma 7.1.** The tangle $(B^3, t_0 \cup t_1)$ obtained by removing one of the unknotted arcs, as in Figure 9, is hyperbolic with totally geodesic boundary. To be precise, $B^3 \setminus (t_0 \cup t_1)$ admits a complete hyperbolic structure, such that $\partial(B^3 \setminus (t_0 \cup t_1))$ is a totally geodesic surface.

**Proof.** If we take the double of the tangle, we obtain the pretzel link $P(3, 2, -2, -3)$, which is hyperbolic by [5] (see also [3, 11]). The desired result follows from this fact. □

**Lemma 7.2.** The link, $L'$, obtained as the double of $(B^3, t_0 \cup t_1 \cup t_2)$, is prime and unsplittable.

**Proof.** Observe that the link $L'$ is obtained from the link $L'' := P(3, 2, -2, -3)$ by adding a parallel copy of the unknotted component. Thus, the exterior $E(L')$ is obtained from the hyperbolic manifold $E(L'')$ and the Seifert fibred space $P \times S^1$, with $P$ a two-holed disc, by gluing along two incompressible toral boundary components. Since both $E(L'')$ and $P \times S^1$ are irreducible and the torus $\partial P \times S^1$ is incompressible in both $E(L'')$ and $P \times S^1$, we see that $E(L')$ is irreducible. Hence $L'$ is unsplittable. Since $E(L'')$ admits no essential annuli, we see that the only essential annuli in $E(L')$ are saturated annuli contained in $P \times S^1$ with boundary components contained in $\partial(P \times S^1) \setminus \partial E(L'')$. Hence $L'$ is prime. □

**Lemma 7.3.**

(1) Let $\Delta$ be a disc properly embedded in $B^3$ such that $\Delta$ is disjoint from $t$. Then $\Delta$ cuts off a 3-ball in $B^3$ disjoint from $t$.

(2) Let $\Delta$ be a disc properly embedded in $B^3$ such that $\Delta$ intersects $t$ transversely at a single point. Then $\Delta$ cuts off a 3-ball, $B$, in $B^3$ such that $(B, t \cap B)$ is a 1-string trivial tangle.

**Proof.** Suppose that there exists a disc, $\Delta$, properly embedded in $B^3$ such that either $\Delta$ is disjoint from $t$ or $\Delta$ intersects $t$ transversely at a single point, which does not satisfy the desired conditions. Then the double of $\Delta$ gives a 2-sphere in $S^3$ which separates the components of $L'$ or gives a non-trivial decomposition of $L'$. This contradicts Lemma 7.2. □

Let $D$ be the flat disc forming the right side of $\partial B^3$ as illustrated in Figure 1(a), and set $K_D = \partial D$. 

Lemma 7.4. There exists no essential annulus, $A$, in $B^3 \setminus t$ whose boundary is disjoint from $K_D$. To be precise, any incompressible annulus $A$ properly embedded in $B^3 \setminus t$ such that $\partial A \cap K_D = \emptyset$ is parallel in $B^3 \setminus t$ to (i) an annulus in $\partial B^3 \setminus (K_D \cup t)$, (ii) the frontier of a regular neighborhood of $K_D$ or (iii) the frontier of a regular neighborhood of a component $t_i$ of $t$.

**Proof.** Suppose to the contrary that there is an essential annulus, $A$, in $B^3 \setminus t$ such that $\partial A \cap K_D = \emptyset$. Since $A$ is compressible in $B^3$, it bounds a cylinder $B^2 \times I$ with $I = [0, 1]$, where $A = \partial B^2 \times I$ and $B^2 \times \partial I \subset \partial B^3$.

**Case 1.** One component of $\partial A$ is contained in $\text{int} D$ and the other component is contained in $D^c := \partial B^3 \setminus D$. Then we may assume $B^2 \times 0 \subset \text{int} D$ and $B^2 \times 1 \subset D^c$. If $A$ is compressible in $B^3 \setminus (t_0 \cup t_1)$, then the cylinder must contain $t_2$. Since $t_2$ has an endpoint in $\text{int} D$ and the other endpoint in $D^c$, $t_2$ has an endpoint in $B^2 \times 0$ and the other endpoint in $B^2 \times 1$. Since $t_2$ is unknotted, this implies that $t_2$ forms the core of the cylinder and hence $A$ is parallel to the frontier of a regular neighborhood of $t_2$. This contradicts the assumption that $A$ is essential in $B^3 \setminus t$. So we may assume that $A$ is incompressible in $B^3 \setminus (t_0 \cup t_1)$. Thus, Lemma 7.1 implies that $A$ is parallel in $B^3 \setminus (t_0 \cup t_1)$ to (i) the frontier of a regular neighborhood of $t_0$, (ii) the frontier of a regular neighborhood of $t_1$ or (iii) an annulus in $\partial B^3 \setminus (t_0 \cup t_1)$.

**Subcase 1-i.** $A$ is parallel in $B^3 \setminus (t_0 \cup t_1)$ to the frontier of a regular neighborhood $N(t_0)$ of $t_0$. Since $t_0$ is knotted whereas $t_1$ and $t_2$ are unknotted, and since each $t_i$ has an endpoint in $\text{int} D$ and the other endpoint in $D^c$, we see that $t_1$ and $t_2$ are not contained in $N(t_0)$. Hence $A$ is parallel in $B^3 \setminus t$ to the frontier of $N(t_0)$, a contradiction of the assumption that $A$ is essential.

**Subcase 1-ii.** $A$ is parallel in $B^3 \setminus (t_0 \cup t_1)$ to the frontier of a regular neighborhood $N(t_1)$ of $t_1$ in $B^3 \setminus t_0$. If $t_2$ is not contained in $N(t_1)$, then $A$ is parallel in $B^3 \setminus t$ to the frontier of $N(t_1)$. So $t_2$ must be contained in $N(t_1)$. Now, let $L' \cup K_D$ be the link in $S^3$ obtained as the ‘double’ of $(B^3, t \cup K_D)$, and let $T$ be the torus in the
exterior $E(L' \cup K_D)$ obtained as the double of the annulus $A$. Then, by the above observation, $T$ bounds a solid torus, $V$, in $S^3$, such that $V \cap (L' \cup K_D) = O'_1 \cup O'_2$, where $O'_i$ is the double of $t_i$ ($i = 1, 2$). By the proof of Lemma 7.2, we know that the JSJ decomposition of $E(L')$ is given by the torus $T_0 := \partial E(L'') = \partial (P \times S^1)$, and hence $T$ is isotopic to $T_0$ in $E(L')$. Thus, there is a self-diffeomorphism $\psi$ of $(S^3, L')$ pairwise isotopic to the identity, which carries $T_0$ to $T$. Let $A_0$ be the annulus with $\partial A_0 = O'_1 \cup O'_2$ as illustrated in Figure 10. Note that, up to isotopy, $T_0$ is the boundary of a regular neighbourhood of $A_0$. Note that the component $K_D$ intersects $A_0$ transversely at two points, whereas $K_D$ is disjoint from $\psi(T_0) = T \subset E(L' \cup K_D)$ and therefore $K_D$ is disjoint from $\psi(A_0)$. Since $\psi$ is pairwise isotopic to the identity, there is a smooth isotopy $\Psi : K_D \times [0, 1] \to E(L')$ such that $\Psi|_{K_D \times 0} = 1_{K_D}$ and $\Psi(K_D \times 1) = \psi^{-1}(K_D)$.

We may assume that $\Psi$ is transversal to $A_0$ and so $F^{-1}(A_0)$ is a one-dimensional submanifold of $K_D \times [0, 1]$. Since $K_D \cap A_0$ consists of two intersection points and since $\psi^{-1}(K_D) \cap A_0 = \emptyset$, the 1-manifold $\Psi^{-1}(A_0)$ contains precisely one arc component, $\beta$, and it joins the two points $(K_D \cap A_0) \times 0$. Consider the disc, $\delta$, in $K_D \times [0, 1]$ cut off by the arc $\beta$. Since $A_0 \cap E(L')$ is incompressible in $E(L')$ and $E(L')$ is irreducible, we may assume that the interior of $\delta$ is disjoint from $\Psi^{-1}(A_0)$. Thus the loop $\Psi(\partial \delta)$ is null-homotopic in $E(L')$. On the other hand, the loop $\Psi(\partial \delta)$ is the union of one of the two subarcs $\alpha_1$ and $\alpha_2$ of $K_D$ bounded by $K_D \cap A_0 = \partial \beta$ (see Figure 10) and the path $\Psi(\beta)$ in $A_0$ joining the two points $K_D \cap A_0$. However, we can easily observe that $\text{lk}(\alpha_1 \cup \beta', K'_0) = \pm 1$ and $\text{lk}(\alpha_2 \cup \beta', O'_2) = \pm 1$ for any path $\beta'$ in $A_0$ with endpoints $K_D \cap A_0$. This is a contradiction.

**Subcase 1-iii.** $A$ is parallel in $B^3 \setminus (t_0 \cup t_1)$ to an annulus, $A'$, in $\partial B^3 \setminus (t_0 \cup t_1)$. Then $A$ cuts off a solid torus, $W$, from $B^3 \setminus (t_0 \cup t_1)$. Since $A$ is essential, $W$ must contain the component $t_2$. This implies that $O'_2$ is null homotopic in $S^3 \setminus K'_0$, where $K'_0$ is the double of $t_0$. However, we can easily check by studying the knot group of the square knot $K'_0$ that this is not the case.
**Case 2.** Both components of \( \partial A \) are contained in \( \text{int} D \). Since each component of \( t_1 \) has one endpoint in \( \text{int} D \) and the other endpoint in \( D^c \), we see that the components of \( \partial A \) are concentric in \( D \), and that \( B^2 \times 0 \) is contained in \( \text{int} D \) and \( B^2 \times 1 \) contains \( D^c \) in its interior. Since \( t_1 \) joins \( \text{int} D \) with \( D^c \), \( t_1 \) joins \( B^2 \times 0 \) with \( B^2 \times 1 \) in the cylinder \( B^2 \times I \). So the cylinder \( B^2 \times I \) is unknotted in \( B^3 \) and hence the closure of its complement in \( B^3 \) is a solid torus, \( W \). Since \( \partial B^2 \times 0 \) is a meridian of the solid torus \( (S^3 \setminus \text{int} B^3) \cup B^2 \times I \), we see that \( \partial B^2 \times 0 \) is a longitude of \( W \). So \( A \) is parallel to the annulus \( A' := V \cap \partial B^3 \) through \( W \). Since every component of \( t \) joins \( \text{int} D \) with \( D^c \), \( t \) must be contained in the cylinder and hence \( W \) is disjoint from \( t \). This contradicts the assumption that \( A \) is essential.

**Lemma 7.5.** The complement \( B^3 \setminus t \) does not contain an incompressible torus.

**Proof.** This follows from the easily observed fact that the exterior \( E(t) \) is homeomorphic to a genus 3 handlebody. Indeed, the exterior of \( t \) in \( B^3 \) is the trefoil knot exterior with two parallel unknotting tunnels drilled out.

**Lemma 7.6.** The tangle \((B^3, \tau)\) is simple in the following sense.

1. Let \( \Delta \) be a disc properly embedded in \( B^3 \) such that \( \Delta \) is disjoint from \( \tau \). Then \( \Delta \) cuts off a 3-ball in \( B^3 \) disjoint from \( \tau \).

2. Let \( \Delta \) be a disc properly embedded in \( B^3 \) such that \( \Delta \) intersects \( \tau \) transversely at a single point. Then \( \Delta \) cuts off a 3-ball, \( B \), in \( B^3 \) such that \((B, \tau \cap B)\) is a 1-string trivial tangle.

3. There exists no essential annulus, \( A \), in \( B^3 \setminus \tau \). To be precise, any incompressible annulus \( A \) properly embedded in \( B^3 \setminus \tau \) is parallel in \( B^3 \setminus \tau \) to (i) an annulus in \( \partial B^3 \setminus \tau \) or (ii) the frontier of a regular neighbourhood of a component \( \tau_i \) of \( \tau \).

4. \( B^3 \setminus \tau \) does not contain an incompressible torus.

**Proof.** Note that \((B^3, \tau)\) is obtained from \((B^3, t)\) and its mirror image by identifying \( D \) with its copy in the mirror image. Thus the assertion follows from Lemmas 7.2–7.5 by using the standard cut and paste method (see [24, Criterion 6.1]).

Since \((S^3, L_2)\) is the double of the simple tangle \((B^3, \tau)\), we see that \( S^3 \setminus L_2 \) is atoroidal, i.e. it does not contain an essential torus. Moreover, \( S^3 \setminus L_2 \) is not a Seifert fibred space. This can be seen as follows. If it were a Seifert fibred space, then it should be a 2-fold composing space, i.e. homeomorphic to the complement of the connected sum of two Hopf links. Now we use the fact that for a given link \( L \), the greatest common divisor, \( d(L) \), of the linking numbers of the components is an invariant of the link complement. Since \( d(\text{the connected sum of two Hopf links}) = 1 \) and \( d(L_2) = 0 \), \( S^3 \setminus L_2 \) cannot be a Seifert fibred space. Hence, by Thurston’s uniformization theorem for Haken manifolds, \( S^3 \setminus L_2 \) admits a complete hyperbolic structure, i.e. \( L_2 \) is hyperbolic.
8. The link $L_3$ is hyperbolic

In this section, we prove that the link $L_3$ is hyperbolic. We begin with the following lemma.

**Lemma 8.1.** There exists no 2-sphere, $S$, in $S^3$ satisfying the following conditions.

1. $S$ is disjoint from $K_0$.
2. Either $S$ is disjoint from $O_1$ or $S$ intersects $O_1$ transversely at two points.
3. $S$ separates $O_2$ from $O_3$, i.e. $O_2$ and $O_3$ are contained in distinct components of $S^3 \setminus S$.

**Proof.** Suppose to the contrary that there is a 2-sphere $S$ satisfying the conditions. Let $S'$ be the $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-covering of $S^3$ branched over the Hopf link $K_0 \cup O_1$, and let $S$ and $O_i$, respectively, be the inverse image of $S$ and $O_i$ in $S'$. Then $O_2 \cup O_3$ is the link in the 3-sphere $S'$ as illustrated in Figure 11. Observe that any component of $O_2$ and any component of $O_3$ form a Hopf link. On the other hand, the inverse image of $S$ in $S'$ consists of four or two mutually disjoint 2-spheres, each of which separates a component of $O_2$ from each component of $O_3$. This is a contradiction. □

Let $B^3$ be one of the 3-balls in $S^3$ bounded by $S^2$, and set $t_i := O_i \cap B^3$ ($i = 1, 2, 3$). Then the pair $(B^3, t)$ with $t = t_1 \cup t_2 \cup t_3$ is a 3-string tangle, and $K_0$ is a loop in $\partial B^3$. Let $D$ and $D'$ be the 2-discs in $\partial B^3$ bounded by $K_0$ which contain $\partial t_2$ and $\partial t_3$, respectively.

**Lemma 8.2.** The three-punctured disc $D \setminus t$ is incompressible in $B^3 \setminus t$. Namely, there does not exist a disc, $\Delta$, properly embedded in $B^3$ satisfying the following conditions.

1. $\partial \Delta$ is a loop in $\text{int}(D \setminus t)$ which does not bound a disc in $\text{int}(D \setminus t)$.
2. $\Delta$ is disjoint from $t$.

**Proof.** Suppose to the contrary that there exists a disc $\Delta$ satisfying the conditions. Let $D_\Delta$ be the disc in $D$ bounded by $\partial \Delta$, and let $B$ be the 3-ball in $B^3$ bounded by the 2-sphere $\Delta \cup D_\Delta$. Since each of $t_1$ and $t_3$ has an endpoint in $\text{int} D' \subset \partial B^3 \setminus D_\Delta$ and since it is disjoint from $\Delta$, both $t_1$ and $t_3$ are contained in the complement of $B$. So, by the second condition, $t_2$ has an endpoint in $\text{int} D_\Delta$, and therefore $t_2$ is contained in $B$. Hence, by taking the double of $\Delta$, we obtain a 2-sphere in $S^3$ satisfying the conditions in Lemma 8.1, which is a contradiction. □

**Lemma 8.3.** There exists no disc, $\Delta$, properly embedded in $B^3$ satisfying the following conditions.

1. $\partial \Delta$ is a loop in $\text{int}(D \setminus t)$.
2. $\Delta$ intersects $t$ transversely at a single point.
3. $\Delta$ does not cut off a 1-string trivial tangle from $(B^3, t)$. 


Figure 11. The bottom left drawing illustrates the double cover, $M_2(K_0)$, of $S^3$ branched along $K_0$ and the inverse images of $O_3 = O_1 \cup O_2 \cup O_3$. The $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-cover, $M_{2\mathbb{Z}/2}(K_0 \cup O_1)$, of $S^3$ branched along the Hopf link $K_0 \cup O_1$ is obtained as the double cover of $M_2(K_0)$ branched along the inverse image of $O_1$. The drawing on the top illustrates the inverse image of $O_2 \cup O_3$ in $M_{2\mathbb{Z}/2}(K_0 \cup O_1)$, where the left and right sides are glued together.

**Proof.** Suppose to the contrary that there exists a disc $\Delta$ satisfying the conditions. As in the proof of Lemma 8.2, let $D_\Delta$ be the disc in $D$ bounded by $\partial \Delta$, and let $B$ be the 3-ball in $B^3$ bounded by $\Delta \cup D_\Delta$. Since the endpoints of $t_3$ are contained in $\text{int} D' \subset \partial B^3 \setminus D_\Delta$ and since $t_3$ intersects $\Delta$ at one point at most, we see that $t_3$ is contained in the complement of $B$. So, precisely one of $t_1$ and $t_2$ intersects $\Delta$. Suppose that $\Delta$ intersects $t_1$. Then, since $t_1$ is an unknotted arc, the third condition implies that the 3-ball $B$ must contain $t_2$. Hence, by taking the double of $\Delta$, we obtain a 2-sphere in $S^3$. 

satisfying the conditions in Lemma 8.1, which is a contradiction. Hence $t_1$ is disjoint from $\Delta$ and $t_2$ intersects $\Delta$ transversely at a single point. Since both $t_1$ and $t_2$ are disjoint from $B$ and since $t_2$ is an unknotted arc, we see that $(B, t \cap B) = (B, t_2 \cap B)$ is a trivial 1-string tangle. This contradicts the assumption that $\Delta$ satisfies the third condition. □

Let $\delta_2$ and $\delta_3$ be the discs in $B^3$ as illustrated in Figure 12.

**Lemma 8.4.** There exists no disc, $\Delta$, in $B^3$ satisfying the following conditions.

1. $\partial \Delta = \alpha \cup \beta$, where $\alpha$ is an arc properly embedded in $\delta_2 \setminus t$ and $\beta$ is an arc in $\text{int}(D \setminus t)$ such that $\beta \cap \delta_2 = \partial \beta = \partial \alpha$.

2. $\Delta$ is disjoint from $t$.

3. The disc $\Delta$ is non-trivial in the following sense. Let $\delta_{2,\Delta}$ be the subdisc of $\delta_2$ bounded by $\alpha \cup \alpha'$, where $\alpha'$ is the subarc of $\delta_2 \cap D$ bounded by $\partial \alpha$. Then the disc $\Delta \cup \delta_{2,\Delta}$ does not cut off a 3-ball in $B^3$ which is disjoint from $t$.

**Proof.** Suppose to the contrary that there exists a disc $\Delta$ satisfying the conditions. Let $D_\Delta$ be the subdisc of $D$ bounded by $\alpha' \cup \beta$. Since $\Delta$ is disjoint from $t$, the loop $\partial \delta_{2,\Delta}$ is homotopic to the loop $\partial D_\Delta$ in $B^3 \setminus t$. Thus $\delta_{2,\Delta}$ contains the point $t_1 \cap \delta_2$ if and only if $D_\Delta$ contains the point $t_1 \cap D$.

Suppose first that $D_\Delta$ does not contain the point $t_1 \cap D$ and so $\delta_{2,\Delta}$ does not contain the point $t_1 \cap \delta_2$. Then $\Delta \cup \delta_{2,\Delta}$ is a disc properly embedded in $B^3$ disjoint from $t$. Hence the disc $D_\Delta$ in $D$ bounded by $\partial (\Delta \cup \delta_{2,\Delta})$ is disjoint from $t$ by Lemma 8.2. Since $(B^3, t)$ is a 3-strand trivial tangle, $B^3 \setminus t$ is homeomorphic to a genus 3 handlebody (with three annuli on the boundary removed), and so $B^3 \setminus t$ is irreducible. Hence the 2-sphere $\Delta \cup \delta_{2,\Delta} \cup D_\Delta$ bounds a 3-ball in $B^3 \setminus t$. This contradicts the assumption that $\Delta$ satisfies the third condition that $\Delta$ is non-trivial.

Suppose next that $D_\Delta$ contains the point $t_1 \cap D$ and so $\delta_{2,\Delta}$ contains the point $t_1 \cap \delta_2$. Consider an arc in the disc $\delta_{2,\Delta} \cup D_\Delta$ joining the two points $t_1 \cap (\delta_{2,\Delta} \cup D_\Delta)$, and let $\gamma$ be the boundary of its regular neighbourhood in $\delta_{2,\Delta} \cup D_\Delta$. Then $\partial \Delta = \partial (\delta_{2,\Delta} \cup D_\Delta)$ is homotopic to $\gamma$ in $B^3 \setminus t$. Note that $\gamma$ is as illustrated in Figure 13. If we ignore the string $t_2$ and regard $\gamma$ as a loop in $B^3 \setminus (t_1 \cup t_3)$, then the free homotopy class of $\gamma$ up to orientation corresponds to the conjugacy class of the free group $\pi_1 (B^3 \setminus (t_1 \cup t_3))$ represented by $x_1 (x_3 x_3^{-1})$, where $\{x_1, x_3\}$ is the free basis illustrated in Figure 13. This contradicts the fact that $\Delta$ is disjoint from $t$. This completes the proof of Lemma 8.4. □
Remark 8.5. The restriction of the antipodal map $\gamma_2$ to $B^3$ gives a diffeomorphism of $(B^3, K_0 \cup t)$ interchanging $D$ with $D'$ (and $\delta_2$ with $\delta_3$). Hence, we may replace $D$ with $D'$ (and $\delta_2$ with $\delta_3$) in Lemmas 8.2, 8.3 and 8.4.

Lemma 8.6. There exists no essential annulus, $A$, in $B^3 \setminus t$ whose boundary is disjoint from $K_0 = \partial D = \partial D'$. To be precise, any incompressible annulus $A$ properly embedded in $B^3 \setminus t$ such that $\partial A \cap K_0 = \emptyset$ is parallel in $B^3 \setminus t$ to (i) an annulus in $\partial B^3 \setminus (K_0 \cup t)$, (ii) the frontier of a regular neighbourhood of $K_0$ or (iii) the frontier of a regular neighbourhood of a component of $t_i$ of $t$.

Proof. Suppose to the contrary that there is an essential annulus, $A$, in $B^3 \setminus t$ such that $\partial A \cap K_0 = \emptyset$. Since $A$ is compressible in $B^3$, it bounds a cylinder $B^2 \times I$ with $I = [0, 1]$, where $A = \partial B^2 \times I$ and $B^2 \times \partial I \subset \partial B^3$.

Case 1. One component of $\partial A$ is contained in $\text{int}D$ and the other component is contained in $\text{int}D'$. Then we may assume that $B^2 \times 0 \subset \text{int}D$ and $B^2 \times 1 \subset D'$. Since $A$ is incompressible, $B^2 \times 1$ must contain at least one point of $\partial t$. If $B^2 \times 1$ contains exactly one point of $\partial t$, then, pushing $A \cup (B^2 \times 1)$ into the interior of $B^3$, we obtain a properly embedded disc $\Delta$ in $B^3$ which satisfies the first two conditions of Lemma 8.3. The lemma implies that $\Delta$ cuts off a 1-string trivial tangle and so $A$ is parallel to the frontier of the boundary of a regular neighbourhood of a component of $t$, a contradiction. So, $B^2 \times 1$ contains at least two points of $\partial t$. By Remark 8.5, the same argument implies that $B^2 \times 0$ also contains at least two points of $\partial t$. Suppose that $B^2 \times 1$ contains the three points $D' \cap \partial t = D' \cap (t_1 \cup t_3)$. Then the cylinder $B^2 \times I$ contains $t_1 \cup t_3$, because $A$ is disjoint from $t$. Since $B^2 \times 0$ contains at least two points of $\partial t$, it must contain a point of $\partial t_2$. Hence we see that the cylinder $B^2 \times I$ contains the whole $t = t_1 \cup t_2 \cup t_3$. Since the arc $t_1$, which joins a point of $B^2 \times 0$ and a point of $B^2 \times 1$ in the cylinder $B^2 \times I$, is unknotted, the cylinder is unknotted in $B^3$ and hence the closure of its complement is a solid torus. This implies that $A$ is parallel to the frontier of a regular neighbourhood of $K_0$ in $B^3$, a contradiction. Hence both $B^2 \times 0$ and $B^2 \times 1$ contain exactly two points of $\partial t$. Thus we see that $t_2 \cup t_3$ is contained in the cylinder $B^2 \times I$ and $t_1$ is contained in the complement of the cylinder.

Claim 8.7. We may assume that $A$ is disjoint from the discs $\delta_2$ and $\delta_3$. 
Prime amphicheiral knots with free period 2

**Proof.** We may assume that the intersection of $A$ with $\delta_2 \cup \delta_3$ is transversal and the number of components of $A \cap (\delta_2 \cup \delta_3)$ is minimized. Suppose that $A \cap (\delta_2 \cup \delta_3)$ contains loop components. Pick a loop component, $C$, which is innermost in $\delta_2 \cup \delta_3$, and let $\delta_C$ be the subdisc of $\delta_2 \cup \delta_3$ bounded by $C$. Suppose $\delta_C$ contains a point of $t \cap (\delta_2 \cup \delta_3) = t_1 \cap (\delta_2 \cup \delta_3)$. Then $C$ forms a meridian of $t_1$ and hence it is not null-homotopic in $B^3 \setminus t$. So $C$ cannot be null-homotopic in the annulus $A \subset B^3 \setminus t$, and hence it is either incompressible, $A$, or it is not null-homotopic in $B^3 \setminus t_1$, a contradiction. Hence $\delta_C$ is disjoint from $t$. Since $A$ is incompressible, $C$ bounds a disc, $A_C$, in $A$. The union $A_C \cup \delta_C$ is a 2-sphere in the irreducible manifold $B^3 \setminus t$. Thus we can remove the intersection $C$ by using the 3-ball bounded by $A_C \cup \delta_C$. This contradicts the minimality of the intersection $A \cap (\delta_2 \cup \delta_3)$. Thus we have shown that $A \cap (\delta_2 \cup \delta_3)$ has no loop components, and so it is either empty or consists of arcs whose boundaries are contained in $\partial B^3 \cap (\delta_2 \cup \delta_3)$.

Suppose that $A \cap (\delta_2 \cup \delta_3)$ is non-empty, and pick an arc component, $C$, which is ‘outermost’ in $\delta_2 \cup \delta_3$. Let $\delta_C$ be the ‘outermost’ subdisc of $\delta_2 \cup \delta_3$ bounded by $C$ and the subarc of $\partial B^3 \cap (\delta_2 \cup \delta_3)$ bounded by $\partial C$. Since $\partial C$ lies either in $D$ or $D'$, the arc $C$ must be inessential in $A$ and it cuts off a disc, $A_C$, from $A$. If $\partial C \subset D$, then the union $\Delta := A_C \cup \delta_C$ satisfies the first two conditions of Lemma 8.4. Hence the lemma implies that $\Delta$ is ‘trivial’ and so we can remove the intersection $C$. This contradicts the minimality of the intersection $A \cap (\delta_2 \cup \delta_3)$. By Remark 8.5, the same argument works when $\partial C \subset D'$. Thus we obtain the claim. □

By the above claim, $\partial B^2 \times 0$ is isotopic in $B^3 \setminus t$ to the boundary of a regular neighbourhood of $\delta_2 \cap D$ in $D \setminus t_1$, and $\partial B^2 \times 1$ is isotopic in $B^3 \setminus t$ to the boundary of a regular neighbourhood of $\delta_3 \cap D'$ in $D' \setminus t_1$. Hence, the loop $\partial B^2 \times 0$ is null-homotopic in $B^3 \setminus (t_1 \cup t_3)$, and the loop $\partial B^2 \times 1$ represents the conjugacy class of the non-trivial element $((x_1 x_3 x_1^{-1} x_3^{-1}) \pm 1)$ in $\pi_1(\partial B^3 \setminus (t_1 \cup t_3))$ (see Figure 13). This contradicts the fact that these loops are freely homotopic in $B^3 \setminus t$ and hence in $B^3 \setminus (t_1 \cup t_3)$. So, we have shown that Case 1 cannot happen.

**Case 2.** Both components of $\partial A$ are contained in $\text{int} D$, and they bound mutually disjoint discs in $\text{int} D$. Then we have $B^2 \times \partial I \subset \text{int} D$. By the argument in the first step in Case 1, using the incompressibility of $A$ and Lemma 8.3, we can see that both $B^2 \times 0$ and $B^2 \times 1$ contain at least two points of $\partial t$. Then $D$ must contain at least four points of $\partial t$, a contradiction. By Remark 8.5, the same argument works when both components of $\partial A$ are contained in $\text{int} D'$ and bound mutually disjoint discs in $\text{int} D'$.

**Case 3.** Both components of $\partial A$ are contained in $\text{int} D$, and they are concentric in $\text{int} D$. Then we may assume that $B^2 \times 0$ is contained in $\text{int} D$ and that $B^2 \times 1$ contains $D'$ in its interior. Reasoning as in Case 1, we see that $B^2 \times 0$ contains at least two endpoints of $t$; moreover, $B^2 \times 1$ must contain at least three of them since it contains $D'$. As a consequence, the cylinder contains the entire tangle $t$.

**Claim 8.8.** The cylinder $B^2 \times I$ in unknotted in $B^3$, and so the closure of the complement of the cylinder $B^2 \times I$ in $B^3$ is a solid torus, $V$. 
Proof. Assume first that the endpoints of \( t_1 \) in \( D \) belong to \( B^2 \times 0 \). Then, since \( t_1 \) joins \( \text{int} D \) with \( \text{int} D' \), \( t_1 \) joins \( B^2 \times 0 \) with \( B^2 \times 1 \) in the cylinder \( B^2 \times I \). So the cylinder \( B^2 \times I \) is unknotted in \( B^3 \) and hence the closure of its complement in \( B^3 \) is a solid torus \( V \).

Suppose that the above assumption does not hold. Then both endpoints of \( t_1 \) must belong to \( B^2 \times 1 \), while \( B^2 \times 0 \) contains precisely the two endpoints of \( t_2 \). Consider now the disc \( \delta_2 \) introduced before Lemma 8.4. Let \( \eta \) be a small arc contained in \( \delta_2 \) that joins the point \( p \) of \( t_1 \) inside \( \delta_2 \) to a point \( q \) in the interior of \( t_2 \) (see Figure 12). Let \( t'_2 \) be either of the two subarcs of \( t_2 \) going from one of its endpoints to \( q \) and \( t'_1 \) the subarc of \( t_1 \) going from \( p \) to the endpoint of \( t_1 \) contained in \( D' \). We claim that the arc obtained by concatenating \( t'_2, \eta \) and \( t'_1 \) is trivial and contained in the cylinder \( B^2 \times I \). Indeed, this arc cobounds a disc with an arc in \( \partial B^3 \); such a disc is obtained by surgery along the two trivializing discs for \( t_1 \) and \( t_2 \). To see that the arc is contained in the cylinder, it is enough to prove that \( \eta \) is contained inside the cylinder. This follows from the fact, which can be proved as in Claim 8.7, that \( \delta_2 \) does not meet \( A \). Using this trivial arc, we see once more as in the previous case that the cylinder is unknotted and its exterior is a solid torus \( V \). □

By Claim 8.8, \((S^3 \setminus \text{int} B^3) \cup B^2 \times I \) is a solid torus, and \( \partial B^2 \times 0 \) is its meridian. Thus \( \partial B^2 \times 0 \) is a longitude of the solid torus \( V \). So \( A \) is parallel to the annulus \( A' := V \cap \partial B^3 \) through \( V \). Since \( A \) is essential in \( B^3 \setminus t \), \( V \) should contain a component of \( t \). However, this is impossible, since—as observed at the beginning—the entire tangle is contained in the cylinder.

Now the proof of Lemma 8.6 is complete. □

Observe that the exterior of \( t \) in \( B^3 \) is a genus 3 handlebody and so \( B^3 \setminus t \) is atoroidal (i.e. does not contain an essential torus). By using this fact and Lemmas 8.2–8.6, we can see that the link \( S^3 \setminus L_3 \) is atoroidal. Since \( L_3 \) has four components, this implies that \( S^3 \setminus L_3 \) cannot be a Seifert fibred space. Hence \( L_3 \) is hyperbolic by Thurston’s uniformization theorem for Haken manifolds.

9. The link \( L_6 \) is hyperbolic

Consider a fundamental domain for the action of the dihedral group of order 6 acting on \((S^3, L_6)\) and generated by the reflections in three vertical planes, as illustrated in Figure 4. The domain is shown in Figure 14.

This domain can be seen as the intersection of two balls bounded by the fixed-point sets of two reflections, that is, two 2-spheres. The two 2-spheres meet along a circle (a portion of the circle is the pink line of intersection of the two planes in Figure 14). It is not difficult to see that the tangle in this fundamental domain, together with the circle of intersection of the reflecting spheres on the boundary, coincides with the tangle and the equator \( K_0 \) on the right-hand side of Figure 2. Doubling this tangle we then obtain the sublink \( O_3 \) of the hyperbolic link \( L_3 \), while the circle can be identified with \( K_0 \).

Since \( L_3 \) is hyperbolic, it is also \( 2\pi/3 \)-hyperbolic, as it is not the figure-eight knot. Indeed, it follows from Thurston’s orbifold theorem (see [2, 7]) that a hyperbolic link that is not \( 2\pi/3 \)-hyperbolic must be either Euclidean or spherical. Dunbar’s list of geometric
orbifolds with underlying space the 3-sphere \([10]\) shows that the only link with this property is the figure-eight knot.

Consider now the hyperbolic orbifold \((S^3, L_3(2\pi/3))\), which is topologically \(S^3\) with singular set \(L_3\) of cone angle \(2\pi/3\) along every component. The reflection \(\gamma_1\) of \(L_3\) induces a hyperbolic reflection of \((S^3, L_3(2\pi/3))\) along a totally geodesic surface. This implies that the fundamental domain for the dihedral action admits a hyperbolic structure with cone angle \(2\pi/3\) along the components of the tangle and cone angle \(\pi/3\) along the circle cobounding the two totally geodesic three-punctured discs of the silvered boundary.

This cone hyperbolic structure now lifts to a cone hyperbolic structure of \(L_6\). As a consequence, the link \(L_6\) is \(2\pi/3\)-hyperbolic and hence hyperbolic.

**Remark 9.1.** Since the link \(L_3\) is \(2\pi/n\)-hyperbolic for every \(n \geq 3\), this construction provides a whole family of hyperbolic links obtained by gluing together \(2n\) copies of the tangle in the fundamental domain, via reflections in half of their boundaries. The resulting link \(L_{2n}\) will admit a dihedral symmetry of order \(2n\). Moreover, if \(n\) is odd, the link \(L_{2n}\) will also have the required \(G\)-action to provide an admissible root.

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