On a $C^2$-nonlinear subdivision scheme avoiding Gibbs oscillations

S. Amat *  K. Dadourian†  J. Liandrat‡

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Abstract

This paper is devoted to the presentation and the study of a new nonlinear subdivision scheme eliminating the Gibbs oscillations close to discontinuities. Its convergence, stability and order of approximation are analyzed. It is proved that this scheme converges towards limit functions of Hölder regularity index larger than 1.192. Numerical estimates provide an Hölder regularity index of 2.438. Up to our knowledge, this subdivision scheme is the first one that achieves simultaneously the control of the Gibbs phenomenon and regularity index larger than 1 for its limit functions.

Key Words. Nonlinear subdivision scheme, limit function, regularity, stability, Gibbs phenomenon.

AMS(MOS) subject classifications. 41A05, 41A10, 65D05, 65D17

1 Introduction

Subdivision schemes are useful tools for generating smooth curves and surfaces. For convergent schemes, starting from discrete sets of control points and using basic rules of low complexity, curves or surfaces can be obtained as

*Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena (Spain). Research supported in part by the Spanish grants MTM2007-62945. e-mail:sergio.amat@upct.es
†Ecole Centrale de Marseille. Laboratoire d’Analyse Topologie et Probabilites. e-mail:dadouria@cmi.univ-mrs.fr
‡Ecole Centrale de Marseille. Laboratoire d’Analyse Topologie et Probabilites. e-mail:jliandrat@ec-marseille.fr
limits (called limit functions) of sequences of points generated by recursive applications of the subdivision scheme.

A simple example of subdivision scheme is the family of interpolatory subdivision schemes, based on Lagrange’s interpolation that have been derived and analyzed in [11]. Other example is the family of spline subdivision schemes related to spline spaces [8].

The four-point interpolatory scheme [16], [15] is a convergent linear scheme of the first family, involving four-point stencils at each subdivision, for which the limit function is at least in the space\(^1\) \(C^1\). The Chaikin algorithm [7] is an example of spline subdivision scheme, with lower complexity as the previous example and converging towards \(C^2\) functions\(^2\).

For applications, for instance to computer aided geometric design or image processing, complexity and convergence/regularity are not the only quality criterion. The order of approximation, that characterizes the precision of the scheme, is another important one. Moreover, oscillations that could occur in the limit function at the vicinity of strongly varying data (coming from the sampling of discontinuous functions), called Gibbs oscillation, are really undesirable.

In the last decade, various attempts to improve the properties of linear subdivision schemes, have lead to nonlinear subdivision schemes. For such schemes, the subdivision rules become data dependant; in addition to the previously defined criteria, one should add a stability property that ensures that the nonlinear scheme is linearly affected by perturbations of the data (for linear schemes, the stability is a direct consequence of the convergence).

For nonlinear subdivision schemes, very few results concerning convergence or stability are available, see for instance [5], [9], [12], [21], [10] and [17].

A large family of nonlinear subdivision schemes to which belong the ENO, WENO or PPH schemes [9], [4] is made by the schemes constructed as a perturbation of the four-point linear interpolatory Lagrange scheme based on centered degree 3 polynomial interpolation. These schemes are interpolatory subdivision schemes (i.e. based on interpolation rules) and are constructed to avoid the Gibbs oscillations occurring classically for linear interpolatory schemes (see Figure II). The schemes of this family are unfortunately characterized by a low regularity of the limit functions of type

\(^1\)For \(0 < \alpha < 1\), \(f \in C^\alpha(\mathbb{R})\) iff \(f\) is bounded and \(\exists C > 0\) such that \(\forall x, y \in \mathbb{R}, |f(x) - f(y)| \leq C|x - y|\alpha\). For \(\alpha > 1\), \(f \in C^\alpha(\mathbb{R})\) iff \(f^{(\lfloor \alpha \rfloor)}\) is bounded and \(f^{(\lfloor \alpha \rfloor)} \in C^{(\alpha - \lfloor \alpha \rfloor)}\) where \([\alpha]\) is the integer part of \(\alpha\).

\(^2\)\(C^{\alpha -} = \{f \in C^\alpha, \forall \beta < \alpha\}\)
Moreover, the ENO scheme is unstable.

In [14], a new linear four-point subdivision scheme was presented. Its refinement rule is based on local cubic interpolation followed by a shift of $1/4$ or, in other words, an evaluation at positions $1/4$ and $3/4$ rather than the standard evaluation at $1/2$. This new scheme was shown to be convergent towards a $C^2$ curve.

The aim of this paper is to analyze the scheme obtained using the same trick (shift of $1/4$) for the PPH-type schemes [4] which are derived modifying the classical four-point interpolatory subdivision scheme substituting the harmonic mean to the arithmetic mean. After the definition of the new scheme in section 2, we analyze successively its convergence (section 3), its stability (section 4) and its order of approximation (in section 5). Its behavior in presence of strongly varying data (Gibbs oscillations) is analyzed in section 6. The last section is devoted to concluding remarks.

2 A new nonlinear subdivision scheme

As mentioned above, the starting point of our work is the construction of N. Dyn, M.S. Floater and K. Hormann in [14]. There, a new linear four-point subdivision scheme that generates $C^2$ curves is presented. Its refinement rule is based on the local cubic Lagrange interpolation, followed by evaluation at positions $1/4$ and $3/4$ of the refined interval. For all $f \in \ell^\infty(\mathbb{Z})$, the scheme is then given by

\[
\begin{align*}
(Sf)_{2n} &= -\frac{7}{128}f_{n-1} + \frac{105}{128}f_n + \frac{35}{128}f_{n+1} - \frac{5}{128}f_{n+2}, \\
(Sf)_{2n+1} &= -\frac{5}{128}f_{n-1} + \frac{35}{128}f_n + \frac{105}{128}f_{n+1} - \frac{7}{128}f_{n+2}.
\end{align*}
\] (1)

Following [4] where a nonlinear scheme is derived modifying the classical four-point interpolatory subdivision scheme substituting the harmonic mean to the arithmetic mean, we first obtain two new formulations of the scheme (1).

\[
\begin{align*}
(Sf)_{2n} &= \frac{49}{64}f_n + \frac{14}{64}f_{n+1} + \frac{1}{64}f_{n+2} - \frac{7}{64}\frac{(d^2f_n + d^2f_{n+1})}{2}, \\
(Sf)_{2n+1} &= \frac{15}{64}f_n + \frac{50}{64}f_{n+1} - \frac{1}{64}f_{n+2} - \frac{5}{64}\frac{(d^2f_n + d^2f_{n+1})}{2}.
\end{align*}
\]
where \((d^2 f)\) is defined by
\[ d^2 f_n = f_{n+1} - 2f_n + f_{n-1}. \]

The two formulations differ essentially in the distribution of the points
\(f_n\) contributing to the three first terms of 1 and 2.

Using the same strategy as in [4], we define the new nonlinear subdivision
scheme \(S_{\text{ppha}}\) associated to (1) by

If \(|d^2 f_n| \geq |d^2 f_{n+1}|\),
\[
(S_{\text{ppha}}f)_{2n} = \frac{49}{64} f_n + \frac{14}{64} f_{n+1} + \frac{1}{64} f_{n+2} - \frac{7}{64} \text{pph}(d^2 f_n, d^2 f_{n+1}),
\]
\[
(S_{\text{ppha}}f)_{2n+1} = \frac{15}{64} f_n + \frac{50}{64} f_{n+1} - \frac{1}{64} f_{n+2} - \frac{5}{64} \text{pph}(d^2 f_n, d^2 f_{n+1}).
\]

If \(|d^2 f_n| < |d^2 f_{n+1}|\),
\[
(S_{\text{ppha}}f)_{2n} = -\frac{1}{64} f_{n-1} + \frac{15}{64} f_n + \frac{50}{64} f_{n+1} - \frac{5}{64} \text{pph}(d^2 f_n, d^2 f_{n+1}),
\]
\[
(S_{\text{ppha}}f)_{2n+1} = \frac{1}{64} f_{n-1} + \frac{49}{64} f_n + \frac{14}{64} f_{n+1} - \frac{7}{64} \text{pph}(d^2 f_n, d^2 f_{n+1}),
\]

where \(\text{pph}\) stands for the harmonic mean defined by
\[
(x, y) \in \mathbb{R}^2 \mapsto \text{pph}(x, y) := \frac{xy}{x + y} (\text{sgn}(xy) + 1),
\]
with \(\text{sgn}(x) = 1\) if \(x \geq 0\) and \(\text{sgn}(x) = -1\) if \(x < 0\).

The initial motivation for the substitution of the arithmetic mean by the
harmonic mean is the elimination of oscillations near strong variating data
thanks to the fact that
\[
|\text{pph}(x, y)| \leq 2 \min(|x|, |y|),
\]
substitutes to
\[
\frac{x + y}{2} \leq \max(|x|, |y|).
\]

Before analyzing in details the properties of the new scheme \(S_{\text{ppha}}\)
we summarize the most important properties of the harmonic mean in the
following proposition (see [5] for more details).
Proposition 1 For all \((x, y) \in \mathbb{R}^2\), the harmonic mean \(\text{PPH}(x, y)\) satisfies

1. \(\text{PPH}(x, y) = \text{PPH}(y, x)\).
2. \(\text{PPH}(x, y) = 0\) if \(xy \leq 0\).
3. \(\text{PPH}(-x, -y) = -\text{PPH}(x, y)\).
4. \(\text{PPH}(x, y) = \frac{\text{sign}(x) + \text{sign}(y)}{2} \min(|x|, |y|) \left[ 1 + \left| \frac{x-y}{x+y} \right| \right]\).
5. \(|\text{PPH}(x, y)| \leq \max(|x|, |y|)\).
6. \(|\text{PPH}(x, y)| \leq 2 \min(|x|, |y|)\).
7. For \(x, y > 0\), \(\min(x, y) \leq \text{PPH}(x, y) \leq \frac{x+y}{2}\).
8. If \(x = O(1)\), \(y = O(1)\), \(|y-x| = O(h)\) and \(xy > 0\) then
   \[|\frac{x+y}{2} - \text{PPH}(x, y)| = O(h^2).\]
9. \(|\text{PPH}(x_1, y_1) - \text{PPH}(x_2, y_2)| \leq 2 \max(|x_1 - x_2|, |y_1 - y_2|)\).

3 Convergence and Regularity

We recall the following definition.

Definition 1 A subdivision scheme \(S\) is said to be convergent if

\[\forall f \in l^\infty(\mathbb{Z}), \exists S^\infty f \in C^0(\mathbb{R}) \text{ such that } \lim_{j \to +\infty} \sup_{n \in \mathbb{Z}} |(S^j f)_n - S^\infty f(n2^{-j})| = 0.\]

\[\text{(4)}\]

In order to derive the convergence, we rewrite the nonlinear subdivision scheme \(S_{\text{PPHA}}\) as a perturbation of a classical two-point linear subdivision scheme, \(S_C\), introduced by G. Chaikin in [7] and defined by

\[\text{(5)}\]

\[\begin{align*}
(S_C f)_{2n} &= \frac{3}{4} f_n + \frac{1}{4} f_{n+1}, \\
(S_C f)_{2n+1} &= \frac{1}{4} f_n + \frac{3}{4} f_{n+1}.
\end{align*}\]

The scheme \(S_C\) is known to be convergent with a regularity \(C^{2-}\). Writing
If \(|d^2 f_n| \geq |d^2 f_{n+1}|\),

\[
(S_{\text{PPHA}} f)_{2n} = \frac{3}{4} f_n + \frac{1}{4} f_{n+1} + \frac{1}{64} d^2 f_{n+1} - \frac{7}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}),
\]

\[
(S_{\text{PPHA}} f)_{2n+1} = \frac{1}{4} f_n + \frac{3}{4} f_{n+1} - \frac{1}{64} d^2 f_{n+1} - \frac{5}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}),
\]

If \(|d^2 f_n| < |d^2 f_{n+1}|\),

\[
(S_{\text{PPHA}} f)_{2n} = \frac{3}{4} f_n + \frac{1}{4} f_{n+1} - \frac{1}{64} d^2 f_{n+1} - \frac{5}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}),
\]

\[
(S_{\text{PPHA}} f)_{2n+1} = \frac{1}{4} f_n + \frac{3}{4} f_{n+1} + \frac{1}{64} d^2 f_{n+1} - \frac{7}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}),
\]

we get that \(S_{\text{PPHA}}\) can be expressed as

\[
S_{\text{PPHA}} f = S c f + F(d^2 f),
\]

with

\[
F(d^2 f)_{2n} = \begin{cases} 
\frac{1}{64} d^2 f_{n+1} - \frac{7}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) & \text{if } |d^2 f_n| > |d^2 f_{n+1}|,
\frac{-1}{64} d^2 f_{n+1} - \frac{5}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) & \text{if } |d^2 f_n| < |d^2 f_{n+1}|,
\end{cases}
\]

and

\[
F(d^2 f)_{2n+1} = \begin{cases} 
\frac{-1}{64} d^2 f_{n+1} - \frac{5}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) & \text{if } |d^2 f_n| > |d^2 f_{n+1}|,
\frac{1}{64} d^2 f_{n+1} - \frac{7}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) & \text{if } |d^2 f_n| < |d^2 f_{n+1}|.
\end{cases}
\]

To analyze the convergence of \(S_{\text{PPHA}}\), we use a result proved in [3], [2] that reads:

A sufficient condition for the convergence of a nonlinear subdivision scheme \(S_{NL} : l^\infty(\mathbb{Z}) \rightarrow l^\infty(\mathbb{Z})\) of the form:

\[
\forall f \in l^\infty(\mathbb{Z}), \quad \forall n \in \mathbb{Z} \quad \begin{cases} 
(S_{NL} f)_{2n+1} = (S f)_{2n+1} + F(\delta f)_{2n+1}, \\
(S_{NL} f)_{2n} = f_n,
\end{cases}
\]

where \(F\) is a nonlinear operator defined on \(l^\infty(\mathbb{Z})\), \(\delta\) is a linear and continuous operator on \(l^\infty(\mathbb{Z})\) and \(S\) is a linear and convergent subdivision scheme is:

**Theorem 1** If \(F, S\) and \(\delta\) given in [3] verify:
There exists $M > 0$ such that $\forall d \in l^\infty(\mathbb{Z})$, $\|F(d)\|_\infty \leq M\|d\|_\infty$, (9)

$$
\exists c < 1 \text{ such that } \|\delta S(f) + \delta F(\delta f)\|_\infty \leq c\|\delta f\|_\infty,
$$

(10)

then the subdivision scheme $S_{NL}$ is uniformly convergent. Moreover, if $S$ is $C^\alpha$ convergent then, for all sequence $f \in l^\infty(\mathbb{Z})$, $S_{NL}^\infty(f) \in C^{\beta -}$ with $\beta = \min(\alpha, -\log_2(c))$.

Using theorem [1] we are going to prove the following result.

**Theorem 2** The nonlinear subdivision scheme $S_{PPHA}$ is convergent with a regularity at least $C^{\beta -}$ with $\beta \geq -\log_2(\frac{7}{64}) > 1$.

**Proof**

From the properties of the harmonic mean (Proposition [1]),

$$
|c_1 d^2 f_{n+1} - c_2 PPH(d^2 f_n, d^2 f_{n+1})| \leq \max(c_1, c_2)\|d^2 f\|_\infty.
$$

(11)

For the perturbation $F$ defined in (6) and (7), it is then easy to see that for all $d \in l^\infty(\mathbb{Z})$,

$$
\|F(d)\|_\infty \leq \frac{7}{64}\|d\|_\infty,
$$

(12)

that is hypothesis (9).

We now consider hypothesis (10) related, in this case, to the contraction of the second order differences ($d^2 f$). To simplify the notations we call $f^1 = S_{PPHA}(f)$.

Different cases must be considered:

**Case 1:** $k=2n+1$, study of $(d^2 f^1)_{2n+1} = f^1_{2n+2} - 2f^1_{2n+1} + f^1_{2n}$

- case 1A1: $|d^2 f_n| \geq |d^2 f_{n+1}|$ and $|d^2 f_{n+1}| \geq |d^2 f_{n+2}|$,
- case 1A2: $|d^2 f_n| < |d^2 f_{n+1}|$ and $|d^2 f_{n+1}| < |d^2 f_{n+2}|$,
- case 1B1: $|d^2 f_n| \geq |d^2 f_{n+1}|$ and $|d^2 f_{n+1}| < |d^2 f_{n+2}|$,
- case 1B2: $|d^2 f_n| < |d^2 f_{n+1}|$ and $|d^2 f_{n+1}| \geq |d^2 f_{n+2}|$.

**Case 2:** $k=2n$, study of $(d^2 f^1)_{2n} = f^1_{2n+1} - 2f^1_{2n} + f^1_{2n-1}$

- case 2A1: $|d^2 f_n| \geq |d^2 f_{n+1}|$ and $|d^2 f_{n-1}| \geq |d^2 f_n|$,
- case 2A2: $|d^2 f_n| < |d^2 f_{n+1}|$ and $|d^2 f_{n-1}| < |d^2 f_n|$,
- case 2B1: $|d^2 f_n| \geq |d^2 f_{n+1}|$ and $|d^2 f_{n-1}| < |d^2 f_n|$,
- case 2B2: $|d^2 f_n| < |d^2 f_{n+1}|$ and $|d^2 f_{n-1}| \geq |d^2 f_n|$.
The other cases follow by symmetry.

- Cases 1A: We obtain for the case 1A

\[
(d^2 f^1)_{2n+1} = \frac{1}{4} f_{n+2} - \frac{2}{4} f_{n+1} + \frac{1}{4} f_n + \frac{1}{64} d^2 f_{n+2} + \frac{3}{64} d^2 f_{n+1} - \frac{7}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1})
\]

\[
= \frac{1}{4} d^2 f_{n+1} + \frac{1}{64} d^2 f_{n+2} + \frac{3}{64} d^2 f_n - \frac{7}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1})
\]

\[
= \frac{19}{64} d^2 f_{n+1} + \frac{1}{64} d^2 f_{n+2} - \frac{7}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}).
\]

Using equation (11) for \(d^2 f_{n+1}\) and \(\text{PPH}(d^2 f_{n+1}, d^2 f_{n+2})\) and Proposition 1, we have

\[
|(d^2 f^1)_{2n+1}| \leq \frac{19 + 1 + 3}{64} \|d^2 f\| \|d^2 f\| \leq \frac{23}{64} \|d^2 f\| \|d^2 f\|.
\] (13)

Similarly for the case 1A2, we have

\[
(d^2 f^1)_{2n+1} = \frac{15}{64} d^2 f_{n+1} - \frac{3}{64} d^2 f_n - \frac{5}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{9}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}),
\]

and (13) remains valid.

- Cases 1B: We obtain for the case 1B

\[
(d^2 f^1)_{2n+1} = \frac{1}{4} f_{n+2} - \frac{2}{4} f_{n+1} + \frac{1}{4} f_n - \frac{1}{64} d^2 f_{n+1} + \frac{3}{64} d^2 f_{n+1} - \frac{5}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1})
\]

\[
= \frac{18}{64} d^2 f_{n+1} - \frac{5}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}).
\]
Using equation (11) for $d^2 f_{n+1}$ and $\text{PPH}(d^2 f_{n+1}, d^2 f_{n+2})$, and Proposition 1, we have

$$|(d^2 f^1)_{n+1}| \leq \frac{18 + 3}{64} ||d^2 f||_\infty \leq \frac{21}{64} ||d^2 f||_\infty. \quad (14)$$

Similarly for the case $1B_2$, we have

$$(d^2 f^1)_{2n+1} = \frac{16}{64} d^2 f_{n+1} + \frac{1}{64} d^2 f_{n+2} - \frac{3}{64} d^2 f_n - \frac{7}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{9}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}),$$

and (14) remains valid.

- Cases 2A: We obtain for the case $2A_1$

  $$(d^2 f^1)_{2n} = \frac{1}{4} f_{n+1} - \frac{2}{4} f_n + \frac{1}{4} f_{n-1} - \frac{3}{64} d^2 f_{n+1} - \frac{1}{64} d^2 f_n + \frac{9}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) - \frac{5}{64} \text{PPH}(d^2 f_{n-1}, d^2 f_n)$$

  $$= \frac{1}{4} d^2 f_n - \frac{3}{64} d^2 f_{n+1} - \frac{1}{64} d^2 f_n + \frac{9}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) - \frac{5}{64} \text{PPH}(d^2 f_{n-1}, d^2 f_n)$$

  $$= \frac{15}{64} d^2 f_n - \frac{3}{64} d^2 f_{n+1} + \frac{9}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) - \frac{5}{64} \text{PPH}(d^2 f_{n-1}, d^2 f_n).$$

Using equation (11) for $d^2 f_{n+1}$ and $\text{PPH}(d^2 f_n, d^2 f_{n+1})$, and for $d^2 f_{n-1}$ and $\text{PPH}(d^2 f_{n-1}, d^2 f_n)$, we have

$$|(d^2 f^1)_{2n}| \leq \frac{15 + 9}{64} ||d^2 f||_\infty \leq \frac{28}{64} ||d^2 f||_\infty. \quad (15)$$

Similarly for the case $2A_2$, we have

$$(d^2 f^1)_{2n} = \frac{19}{64} d^2 f_n + \frac{1}{64} d^2 f_{n+1} + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) - \frac{7}{64} \text{PPH}(d^2 f_{n-1}, d^2 f_n),$$

and (15) remains valid.
• Cases 2B: We obtain for the case 2B

\[
(d^2 f^1)_{2n} = \frac{1}{4} f_{n+1} - \frac{2}{4} f_n + \frac{1}{4} f_{n-1} - \frac{3}{64} d^2 f_{n+1} + \frac{1}{64} d^2 f_n + \frac{9}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) - \frac{7}{64} \text{PPH}(d^2 f_{n-1}, d^2 f_n)
\]

\[
= \frac{1}{4} d^2 f_n - \frac{3}{64} d^2 f_{n+1} + \frac{1}{64} d^2 f_{n-1} + \frac{9}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) - \frac{7}{64} \text{PPH}(d^2 f_{n-1}, d^2 f_n).
\]

Using equation (11) for \(d^2 f_{n+1}\) and \(\text{PPH}(d^2 f_n, d^2 f_{n+1})\), and for \(d^2 f_n\) and \(\text{PPH}(d^2 f_{n-1}, d^2 f_n)\), we have

\[
|| (d^2 f^1)_{2n} || \leq \frac{16 + 9 + 1}{64} ||d^2 f||_\infty \leq \frac{28}{64} ||d^2 f||_\infty. \quad (16)
\]

Similarly for the case 2B2, we have

\[
(d^2 f^1)_{2n} = \frac{18}{64} d^2 f_n + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) - \frac{5}{64} \text{PPH}(d^2 f_{n-1}, d^2 f_n),
\]

and (16) remains valid.

From equations (13), (14), (15) and (16), we deduce that for all \(f \in l^\infty(\mathbb{Z})\)

\[
||d^2 S_{\text{PPHA}} f||_\infty \leq \frac{7}{16} ||d^2 f||_\infty. \quad (17)
\]

Therefore \(S_{\text{PPHA}}\) verifies the hypothesis (10) of the Theorem 1. In particular, we obtain the convergence of \(S_{\text{PPHA}}\).

For the regularity, we use again Theorem 1. According to the values \(\alpha = 2\) and \(c = \frac{7}{16}\) we obtain the regularity constant \(\beta = \min (2, -\log_2 \left( \frac{7}{16} \right)) \approx 1.192.

□

**Numerical Regularity**

Following [20], the regularity of a limit function can be evaluated numerically. Using \(S_1\) and \(S_2\) the subdivision schemes for the differences of order 1 and 2 associated to \(S_{\text{PPHA}}\) (that can be derived due to the specific definition of \(S_{\text{PPHA}}\)), the following quantities are estimated for \(k = 1, 2,\)

\[
-\log_2 \left( \frac{2^k ||(S_{k+1}^j f)_{n+1} - (S_{k+1}^j f)_{n}||_\infty}{|| (S_k^j f)_{n+1} - (S_k^j f)_{n}||_\infty} \right).
\]
They provide an estimate for $\beta_1$ and $\beta_2$ such that the limit function belongs to $C^{1+\beta_1-}$ and $C^{2+\beta_2-}$. From table 1, the numerical estimate of the regularity is $C^{2.438-}$. We recall that the corresponding estimate for the linear scheme [14] is $C^{2.67-}$.

| $j$ | 5     | 6     | 7     | 8     | 9     | 10    |
|-----|-------|-------|-------|-------|-------|-------|
| $\beta_1$ | 0.9999 | 0.9999 | 1     | 1     | 1     | 1     |
| $\beta_2$ | 0.4395 | 0.7738 | 1.2615 | 0.6541 | 0.4387 | 0.4388 |

Table 1: Numerical estimates of the limit function regularity $C^{1+\beta_1-}$ and $C^{2+\beta_2-}$ for $S_{\text{PPHA}}$.

4 Stability

For simplicity in notations we call for any initial sequence $f^0$, and any $j \in \mathbb{N}$, $f^{j+1} = S(f^j)$. We recall the following definition.

**Definition 2** A convergent subdivision scheme is stable if

$$\exists C < +\infty \text{ such that } \forall f^0, g^0 \in l^\infty(\mathbb{Z}) \quad ||S^\infty f - S^\infty g||_{L^\infty} \leq C||f^0 - g^0||_{l^\infty}. \quad (18)$$

As for the convergence, to derive the stability of $S_{\text{PPHA}}$ we use the following theorem of [2].

**Theorem 3** If $F, S$ and $\delta$ given in (3) verify: $\exists M > 0, c < 1$ such that $\forall f, g, d_1, d_2$,

$$||F(d_1) - F(d_2)||_{\infty} \leq M ||d_1 - d_2||_{\infty}, \quad (19)$$

$$||\delta(S_{NL}f - S_{NL}g)||_{\infty} \leq c||\delta(f - g)||_{\infty}, \quad (20)$$

then the nonlinear subdivision scheme $S_{NL}$ is stable.

In order to check the hypotheses of Theorem 3 for $S_{NL} = S_{\text{PPHA}}$ we first prove the following lemma.

**Lemma 1** Let be $f, g \in l^\infty(\mathbb{Z})$, if $|d^2f_n| \geq |d^2f_{n+1}|$ and $|d^2g_{n+1}| \geq |d^2g_n|$ then

$$|d^2f_{n+1} + d^2g_{n} - 2\text{PPH}(d^2g_n,d^2g_{n+1})| \leq 3||d^2f - d^2g||_{\infty}. $$

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Proof
We consider different cases.

• If $d^2 g_{n+1}d^2 g_n < 0$, we have
  $$d^2 f_{n+1} + d^2 g_n - 2\text{PPH}(d^2 g_n, d^2 g_{n+1}) = d^2 f_{n+1} + d^2 g_n.$$  
  * if $d^2 f_{n+1}d^2 g_n < 0$, using that $|d^2 f_n| \geq |d^2 f_{n+1}|$
    $$|d^2 f_{n+1} + d^2 g_n| \leq |d^2 f_{n+1} - d^2 g_n| \leq |d^2 f_n - d^2 g_n|.$$  
  * if $d^2 f_{n+1}d^2 g_n \geq 0$, we have $d^2 f_{n+1}d^2 g_{n+1} < 0$ and with $|d^2 g_{n+1}| \geq |d^2 g_n|$, we obtain
    $$|d^2 f_{n+1} + d^2 g_n| \leq |d^2 f_{n+1} - d^2 g_{n+1}|.$$  

• If $d^2 g_{n+1}d^2 g_n \geq 0$, we recall (Proposition 1) that if $x, y > 0$, $\min(x, y) \leq \text{PPH}(x, y) \leq \max(x, y)$.
Without loss of generality, we suppose that $d^2 g_n \geq 0$. We denote
  $$H = d^2 f_{n+1} + d^2 g_n - 2\text{PPH}(d^2 g_n, d^2 g_{n+1}).$$  
  * if $H > 0$,
    $$H \leq d^2 f_{n+1} + d^2 g_n - 2\min(|d^2 g_n|, |d^2 g_{n+1}|)$$  
    $$\leq d^2 f_{n+1} - d^2 g_n$$  
    $$\leq |d^2 f_n - d^2 g_n|.$$  
  * if $H < 0$,
    $$H \geq d^2 f_{n+1} + d^2 g_n - 2\max(|d^2 g_n|, |d^2 g_{n+1}|)$$  
    $$\geq d^2 f_{n+1} + d^2 g_n - 2d^2 g_{n+1}$$  
    $$\geq d^2 f_{n+1} - d^2 g_{n+1} + d^2 g_n - d^2 g_{n+1}.$$  

We have again to consider different cases according to the sign of $d^2 f_n$.
- $d^2 f_n \geq 0$, we have $d^2 f_n - d^2 f_{n+1} \geq 0$
  $$H \geq (d^2 f_{n+1} - d^2 g_{n+1}) + (d^2 g_n - d^2 g_{n+1}) - (d^2 f_n - d^2 f_{n+1})$$  
  $$\geq 2(d^2 f_{n+1} - d^2 g_{n+1}) + (d^2 g_n - d^2 f_n) \geq -3|d^2 f - d^2 g|_\infty.$$
\(-d^2 f_n < 0 \text{ and } d^2 f_{n+1} d^2 f_n \geq 0, \text{ then } d^2 f_{n+1} < 0\)

\[
H \geq (d^2 f_{n+1} - d^2 g_{n+1}) + (d^2 f_{n+1} - d^2 g_{n+1}) \\
\geq 2(d^2 f_{n+1} - d^2 g_{n+1}) \geq -2||d^2 f - d^2 g||_\infty.
\]

\(-d^2 f_n < 0 \text{ and } d^2 f_{n+1} d^2 f_n < 0, \text{ we have from hypothesis that } d^2 f_{n+1} + d^2 f_n < 0,\)

\[
H \geq (d^2 f_{n+1} - d^2 g_{n+1}) + (d^2 f_{n+1} - d^2 g_{n+1}) + d^2 f_n + d^2 g_n \\
\geq 2(d^2 f_{n+1} - d^2 g_{n+1}) + d^2 f_n - d^2 g_n \geq -3||d^2 f - d^2 g||_\infty.
\]

\(\square\)

We are now ready to prove the stability of \(S_{\text{PPHA}}\).

**Theorem 4** The scheme \(S_{\text{PPHA}}\) is stable.

**Proof**

We check the hypotheses of Theorem 3.

Firstly, we start with the hypothesis \((19)\) for \(F\).

Using the expressions of perturbation \(F\), (6) and (7), and Proposition 1, we obtain for all \(d_1, d_2 \in l^\infty(\mathbb{Z})\) that

\[
||F(d_1) - F(d_2)||_\infty \leq \frac{1 + 7 \cdot 2}{64} ||d_1 - d_2||_\infty.
\]

Secondly, we have to verify the contraction hypothesis \((20)\).

For a couple \(f, g \in l^\infty(\mathbb{Z})\), we study \((d^2 f^1 - d^2 g^1)_k\) for \(k = 2n + 1\) (case 1) or \(k = 2n\) (case 2).

We consider different cases, according to the proof of Theorem 2 for \(f\) or \(g\).

For \(k=2n+1\), we have 7 cases to study, see table 2, the others cases being deduced by symmetry.

- Case 1A1 – 1A1: with equation (13) we can obtain directly

\[
|d^2 f_{2n+1}^1 - d^2 g_{2n+1}^1| \leq \frac{19 + 1 + 7 \cdot 2 + 3 \cdot 2}{64} ||d^2 f - d^2 g||_\infty \\
\leq \frac{5}{8} ||d^2 f - d^2 g||_\infty.
\]
Table 2: Cases to consider for $k = 2n + 1$ in the proof of the stability of $S_{\text{PHA}}$.

| Case 1A | Case 1B |
|---------|---------|
| $1A$    | $1B$    |
| $1A_1$  | $1A_2$  |
| $1A_1$  | $1A_1 - 1A_2$ | $1B_1$  | $1A_2$  |
| $1A_1$  | $1A_1 - 1A_2$ | $1B_1$  | $1B_1$  |
| $1B_1$  | $1B_2$  |
| $1B_1$  | $1B_2$  |
| $1B_1 - 1A_2$ | $1B_1 - 1B_1$ |

- Case 1A1 - 1A2: from equations (13) and (14), we obtain
  \[
d^2 f_{2n+1} - d^2 g_{2n+1} = \frac{19}{64} d^2 f_{n+1} - \frac{15}{64} d^2 g_{n+1} + \frac{1}{64} d^2 f_{n+2} + \frac{3}{64} d^2 g_n - \frac{7}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2})
  + \frac{5}{64} \text{PPH}(d^2 g_{n+1}, d^2 g_{n+2}) + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) + \frac{9}{64} \text{PPH}(d^2 g_n, d^2 g_{n+1})
  + \frac{1}{64} (d^2 f_{n+1} - d^2 g_{n+1}) - \frac{7}{64} (\text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) - \text{PPH}(d^2 g_{n+1}, d^2 g_{n+2}))
  + \frac{1}{64} (d^2 f_{n+2} + d^2 g_{n+1} - 2\text{PPH}(d^2 g_{n+1}, d^2 g_{n+2}))
  + \frac{3}{64} (\text{PPH}(d^2 f_n, d^2 f_{n+1}) - \text{PPH}(d^2 g_{n+1}, d^2 g_{n+2}))
  + \frac{3}{64} (d^2 g_n + d^2 f_{n+1} - 2\text{PPH}(d^2 g_n, d^2 g_{n+1})).
\]

Applying Lemma 1 and Proposition 1, we obtain
\[
|d^2 f_{2n+1} - d^2 g_{2n+1}| \leq \frac{16 + 7 \cdot 2 + 2 + 1 + 3 \cdot 2 + 3 \cdot 3}{64} ||d^2 f - d^2 g||_{\infty}
\]
\[
\leq \frac{45}{64} ||d^2 f - d^2 g||_{\infty}.
\]

Similarly
- Case 1A1 - 1B1: from equations (13) and (14), we obtain applying the Lemma 1 and the Proposition 1
  \[
  |d^2 f_{2n+1} - d^2 g_{2n+1}| \leq \frac{21}{32} ||d^2 f - d^2 g||_{\infty}.
  \]

- Case 1A1 - 1B2: from equations (13) and (15), we obtain applying the Lemma 1 and the Proposition 1
  \[
  |d^2 f_{2n+1} - d^2 g_{2n+1}| \leq \frac{23}{32} ||d^2 f - d^2 g||_{\infty}.
  \]
Case 1 \( B_1 - 1A_2 \): from equations (14) and (14), we obtain applying the Lemma 1 and the Proposition 1

\[ |d^2 f_{2n+1}^1 - d^2 g_{2n+1}^1| \leq \frac{5}{8}||d^2 f - d^2 g||_\infty. \]

- Case 1 \( B_1 - B_1 \): from equation (14) we conclude

\[ |d^2 f_{2n+1}^1 - d^2 g_{2n+1}^1| \leq \frac{18 + 5 \cdot 2 + 3 \cdot 2}{64} ||d^2 f - d^2 g||_\infty \leq \frac{17}{32} ||d^2 f - d^2 g||_\infty. \]

- Case 1 \( B_1 - B_2 \): from equations (14) and (15), we obtain applying the Lemma 1 and the Proposition 1

\[ |d^2 f_{2n+1}^1 - d^2 g_{2n+1}^1| \leq \frac{3}{8} ||d^2 f - d^2 g||_\infty. \]

- Case 1 \( B_2 - 1A_2 \): from equations (15) and (14), we obtain applying the Lemma 1 and the Proposition 1

\[ |d^2 f_{2n+1}^1 - d^2 g_{2n+1}^1| \leq \frac{11}{16} ||d^2 f - d^2 g||_\infty. \]

For \( k = 2n \), we have 7 other cases to study, see table 3. The others cases are deduced by symmetry.

We notice that in equations (14) and (15) the same coefficients, but the cases are not completely equivalent.

| Case | f verifies | g verifies | notation | Case | f verifies | g verifies | notation |
|------|-----------|-----------|----------|------|-----------|-----------|----------|
| 2A   | 2A_1      | 2A_1      | 2A_1 - 2A_1 |      | 2B_1      | 2A_2      | 2B_1 - 2A_2 |
|      | 2A_2      | 2A_2      | 2A_1 - 2A_2 |      | 2B_1      | 2B_1      | 2B_1 - 2B_1 |
|      | 2A_1      | 2B_1      | 2A_1 - 2B_1 |      | 2B_1      | 2B_2      | 2B_1 - 2B_2 |
|      | 2A_1      | 2B_2      | 2A_1 - 2B_2 |      | 2B_2      | 2A_2      | 2B_2 - 2A_2 |

Table 3: Cases to consider for \( k = 2n \) in the proof of the stability of \( S_{PPHA} \).
Case 2A1 − 2A1: with equation (15) we can conclude

\[ |d^2 f_{2n}^1 - d^2 g_{2n}^1| \leq \frac{15 + 3 \cdot 2 + 5 \cdot 2}{64}||d^2 f - d^2 g||_\infty \]
\[ \leq \frac{23}{32}||d^2 f - d^2 g||_\infty. \]

Case 2A1 − 2A2: from equations (15) and (16), we obtain

\[
\begin{aligned}
d^2 f_{2n}^1 - d^2 g_{2n}^1 &= \frac{15}{64}d^2 f_n - \frac{19}{64}d^2 g_n - \frac{3}{64}d^2 f_{n+1} + \frac{9}{64}\text{PPH}(d^2 f_n, d^2 f_{n+1}) \\
&- \frac{3}{64}\text{PPH}(d^2 g_n, d^2 g_{n+1}) - \frac{5}{64}\text{PPH}(d^2 f_{n-1}, d^2 f_n) + \frac{7}{64}\text{PPH}(d^2 g_{n-1}, d^2 g_n) \\
&= \frac{16}{64}(d^2 f_n - d^2 g_n) + \frac{9}{64}(\text{PPH}(d^2 f_n, d^2 f_{n+1}) - \text{PPH}(d^2 g_n, d^2 g_{n+1})) \\
&- \frac{3}{64}(d^2 f_{n+1} + d^2 g_n - 2\text{PPH}(d^2 g_n, d^2 g_{n+1})) \\
&- \frac{5}{64}(\text{PPH}(d^2 f_{n-1}, d^2 f_n) - \text{PPH}(d^2 g_{n-1}, d^2 g_n)) \\
&- \frac{1}{64}(d^2 f_n + d^2 g_{n-1} - 2\text{PPH}(d^2 g_{n-1}, d^2 g_n)).
\end{aligned}
\]

Applying the Lemma 1 and the Proposition 1, we obtain

\[ |d^2 f_{2n}^1 - d^2 g_{2n}^1| \leq \frac{16 + 9 \cdot 2 + 3 \cdot 3 + 5 \cdot 2 + 3}{64}||d^2 f - d^2 g||_\infty \]
\[ \leq \frac{28}{32}||d^2 f - d^2 g||_\infty. \]

Similarly,

Case 2A1 − 2B1: from equations (15) and (16), we obtain applying the Lemma 1 and the Proposition 1

\[ |d^2 f_{2n}^1 - d^2 g_{2n}^1| \leq \frac{25}{32}||d^2 f - d^2 g||_\infty. \]

Case 2A1 − 2B2: from equations (15) and (17), we obtain applying the Lemma 1 and the Proposition 1

\[ |d^2 f_{2n}^1 - d^2 g_{2n}^1| \leq \frac{13}{16}||d^2 f - d^2 g||_\infty. \]
• Case 2$B_1 - 2A_2$: from equations (16) and (16), we obtain applying the Lemma 1 and the Proposition 1

\[ |d^2 f_{2n}^1 - d^2 g_{2n}^1| \leq \frac{57}{64} |d^2 f - d^2 g|_\infty. \]

• Case 2$B_1 - 2B_1$: from equation (16) we conclude

\[ |d^2 f_{2n}^1 - d^2 g_{2n}^1| \leq \frac{16 + 3 + 1 + 9 \cdot 2 + 7 \cdot 2}{64} |d^2 f - d^2 g|_\infty \leq \frac{21}{32} |d^2 f - d^2 g|_\infty. \]

• Case 2$B_1 - 2B_2$: from equations (16) and (17), we obtain applying the Lemma 1 and the Proposition 1

\[ |d^2 f_{2n}^1 - d^2 g_{2n}^1| \leq \frac{15}{16} |d^2 f - d^2 g|_\infty. \]

• Case 2$B_2 - 2A_2$: from equations (17) and (16), we obtain applying the lemma 1 and the proposition 1

\[ |d^2 f_{2n}^1 - d^2 g_{2n}^1| \leq \frac{21}{32} |d^2 f - d^2 g|_\infty. \]

Finally, the hypotheses of theorem (20) are verified and stability can be established.

\[ \Box \]

5 Order of approximation

In this section we consider the reproduction of polynomials and the order of approximation of $S_{PPHA}$.

We recall the following definitions.

**Definition 3** A subdivision scheme $S$ is said to reproduce polynomials of degree $k$ if for all polynomial $P$ of degree $k$:

If $\forall n \in \mathbb{Z}, f_n = P(n)$, then $\exists \tilde{P}$ a polynomial of degree $k$ such that $(Sf)_n = \tilde{P}(2^{-1}n)$. 17
**Definition 4** A subdivision scheme \( S \) is said to have an order \( k \) of approximation if for all function \( g \in C^k \) and all \( h > 0 \),

\[
\text{if } f = g(h), \text{ then } |Sf - g(2^{-1}h)| \leq Ch^k.
\]

We then have the following property.

**Proposition 2** \( S_{\text{PPHA}} \) reproduces the polynomials of degree 2 with translation of \( \frac{1}{4} \).

**Proof**

We remark that for any \( P \), polynomial of degree 2, and \( p = (P(n))_{n \in \mathbb{Z}} \), we have

\[
\text{PPH}(d^2p_n, d^2p_{n+1}) = \frac{p_n + p_{n+1}}{2}
\]

Therefore, for the initial sequence \( p = (p_n)_{n \in \mathbb{Z}} \), \( S_{\text{PPHA}}(p) \) coincides with the application to \( p \) of the linear scheme [14]. In particular, the results of N. Dyn, M.S. Floater and K. Hormann [14] can be applied and the property of definition 3 is satisfied with \( \tilde{P}(.) = P(.-1/4) \).

\( \square \)

Concerning the order of approximation the following proposition holds.

**Proposition 3** For all function \( g \in C^4([0,1]) \) and \( h > 0 \), if

\[
f = (g((n - \frac{1}{2})h))_{n \in \mathbb{Z}},
\]

then

\[
|S_{\text{PPHA}}f_n - g(2^{-1}h(n - \frac{1}{2}))|_\infty = O(h^4),
\]

if \( d^2f_n d^2f_{n+1} > 0 \) for all \( n \in \mathbb{Z} \), then

\[
|S_{\text{PPHA}}f_n - g(2^{-1}h(n - \frac{1}{2}))|_\infty = O(h^3).
\]

**Proof**

According to Proposition [1] we have that if \( d^2f_n d^2f_{n+1} > 0 \) for all \( n \in \mathbb{N} \) then

\[
|\text{PPH}(d^2f_n, d^2f_{n+1}) - \frac{d^2f_n + d^2f_{n+1}}{2}| = O(h^4).
\]

Therefore, is \( S \) stands for the linear scheme defined in [14], according to the definition of the \( S_{\text{PPHA}} \),

\[
||S_{\text{PPHA}}f - Sf||_\infty = O(h^4).
\]
Since (see [14]) the scheme $S$ is of order of approximation 4 we get the result when $d^2 f_n d^2 f_{n+1} > 0$. In other case, the reproduction of polynomials leads to

$$|| (S_{PPHA} f)_n - g(2^{-1} h(n - \frac{1}{2})) ||_\infty = O(h^3).$$

\[ \square \]

**Remark 1** Following [20] one can also establish, using the stability of $S_{PPHA}$ that $|| S_{PPHA} f - g ||_\infty = O(h^3)$.

### 6 Elimination of the Gibbs phenomenon

In this section we focus on the behavior of the scheme in presence of strongly varying data. The reference behavior deals with a step function as shown on Figure 1. As it is visible on Figure 1 left, high order linear schemes suffer from an oscillating behavior named as Gibbs phenomenon.

According to D. Gottlieb and C.W. Shu [18], given a punctually discontinuous function $f$ and its sampling $f^h$ defined by $f^h_n = f(nh)$, the Gibbs phenomenon deals with the convergence of $S^\infty(f^h)$ towards $f$. It can be characterized by two features ([18] p. 244):

1. Away from the discontinuity the convergence is rather slow and for any point $x$,
   
   $$|f(x) - S^\infty(f^h)(x)| = O(h).$$

2. There is an overshoot, close to the discontinuity, that does not diminish with reducing $h$; thus

   $$\max |f(x) - S^\infty(f^h)(x)|$$

   does not tend to zero with $h$.

We are now going to prove that the nonlinear schemes $S_{PPHA}$ does not suffer from the Gibbs phenomenon oscillations, as it can be guessed from Figure 1. We have indeed the following

**Proposition 4** Given $0 \leq \xi \leq h$, for any function $f$ defined by:

$$\forall x \leq \xi, f(x) = f_-(x) \quad \text{with} \quad f_- \in C^\infty([ -\infty, \xi]),$$

$$\forall x > \xi, f(x) = f_+(x) \quad \text{with} \quad f_+ \in C^\infty([\xi, +\infty[,$$

and discontinuous in $\xi$, we have:

- if $|x| \geq \frac{9}{2} h$, $|f(x - \frac{1}{2}) - S_{PPHA}^\infty(f^h)(x)| = O(h^3)$,
Figure 1: Comparison of limit functions for the same initial sequence (sampling of function (21)). Left, linear scheme (1), right nonlinear scheme $S_{\text{PPHA}}$

- if $|x| \leq \frac{9}{2}h$, $f_-(0) + O(h) \geq S_{\text{PPHA}}(f^h)(x) \leq f_-(h) + O(h)$.

**Proof**

Without loss of generality, we focus on $[0, +\infty[$ and suppose that $f_-(\xi) > f_+(\xi)$.

We first consider a single application of $S_{\text{PPHA}}$. Using Proposition 3 we get:

- for $n \geq 2$ and $n_1 \in \{2n, 2n+1\}$, $|S_{\text{PPHA}}(f^h)_{n_1} - f_+(2^{-1}h(n_1 - \frac{1}{2})| = O(h^2)$

- for $n = 1$ since $f$ is discontinuous in $\xi$, $d^2 f_n = O(1)$ and $d^2 f_{n+1} = O(h^2)$. Then, from Proposition $\mathbb{I}$ $\text{PPH}(d^2 f_n, d^2 f_{n+1}) = O(h^2)$. Moreover, according to the definition of $S_{\text{PPHA}}$ as a perturbation of the Chaikin scheme $S_C$ we get that $|S_{\text{PPHA}}(f^h)_{n_1} - S_C(f^h)_{n_1}| = O(h^2)$. Since $S_C$ is a second order scheme we get that $|S_{\text{PPHA}}(f^h)_{n_1} - f_+(2^{-1}h(n_1 - \frac{1}{2}))| = O(h^2)$ for $n_1 \in \{2n, 2n+1\}$.

- for $n = 0$, $d^2 f_n d^2 f_{n+1} \leq 0$ and therefore, according to the Proposition $\mathbb{I}$ $\text{PPH}(d^2 f_n, d^2 f_{n+1}) = 0$. It is then easy to check, from the definition of $S_{\text{PPHA}}$ that $f_-(0) \leq S_{\text{PPHA}} f_{2n} \leq S_{\text{PPHA}} f_{2n+1} \leq f_+(h)$. However, writing $S_{\text{PPHA}}$ as a perturbation of $S_C$ we get that $|S_{\text{PPHA}}(f^h)_{n_1} - S_C(f^h)_{n_1}| = O(d^2 f^h)$. 

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Iterating, according to the stability of $S_{\text{PPHA}}$ we get:

- for $x \geq \frac{9}{2}h$, $|S_{\text{PPHA}}^\infty f^h(x) - f_+(x - 1/2)| = O(h^3)$.

- for $0 \leq x < \frac{9}{2}h$, the contraction of the second order differences (equation (17)) and the fact that the Chaikin scheme $S_C$ does not produce Gibbs oscillations allow to conclude.

□

Before concluding this work, we come back to Figure 1 and to the comparison between the limit functions obtained with $S_{\text{PPHA}}$ and the limit function obtained with linear subdivision schemes starting from the sampling $f^h$ of the discontinuous function:

$$f(x) = \begin{cases} 
\sin(\pi x) & \text{for } x \in [0, 0.5] \\
-\sin(\pi x) & \text{for } x \in [0.5, 1] 
\end{cases}$$

(21)

It appears from Figure 1 that the nonlinear scheme $S_{\text{PPHA}}$ exhibits a much better behavior close to the discontinuity than the linear scheme of comparable complexity. From Proposition 3 we know moreover that the scheme $S_{\text{PPHA}}$ is of higher order than the Chaikin scheme.

7 Conclusions

In this paper, a new nonlinear subdivision scheme has been defined. It has many desirable properties. It is convergent with a regularity proved to be at least $C^{1.192-}$ and numerically estimated at $C^{2.438-}$. By construction, it is adapted to the presence of isolated discontinuities and the Gibbs phenomenon is eliminated. The scheme is also stable, that due the nonlinear nature is not a consequence of the convergence. Moreover, its order of convergence is 3. Recalling that it is constructed from a four-point centered stencil, all these properties make of this scheme a very good candidate for various applications.

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