FI\(_W\)-modules and stability criteria for representations of the classical Weyl groups

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Abstract

In this paper we develop machinery for studying sequences of representations of any of the three families of classical Weyl groups, extending work of Church, Ellenberg, Farb, and Nagpal [CEF12], [CEFN12] on the symmetric groups \(S_n\) to the signed permutation groups \(B_n\) and the even-signed permutation groups \(D_n\). For each family \(\mathcal{W}_n\), we present an algebraic framework where a sequence \(V_n\) of \(\mathcal{W}_n\)-representations is encoded into a single object we call an FI\(_W\)-module. We prove that if an FI\(_W\)-module \(V\) satisfies a simple finite generation condition then the structure of the sequence is highly constrained. Two consequences are:

1. The pattern of irreducible representations in the decomposition of each \(V_n\) eventually stabilizes in a precise sense.
2. The characters of \(V_n\) are, for \(n\) large, given by a character polynomial in signed-cycle-counting class functions, independent of \(n\).

We apply this theory to obtain new results about a number of sequences associated to the classical Weyl groups:

(a) the cohomology of hyperplane complements,
(b) the cohomology of the pure string motion groups,
(c) the cohomology of generalized flag varieties, and more generally the \(r\)-diagonal coinvariant algebras.

We analyze the algebraic structure of the category of FI\(_W\)-modules, and introduce restriction and induction operations that enable us to study interactions between the three families of groups. We use this theory to prove analogues of Murnaghan’s 1938 stability theorem for Kronecker coefficients for the families \(B_n\) and \(D_n\). The theory of FI\(_W\)-modules gives a conceptual framework for stability results such as these.
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1 Introduction

Let $W_n$ denote any of the one-parameter families of Weyl groups: the symmetric groups $S_n$, the hyperoctahedral groups (signed permutation groups) $B_n$, or the even-signed permutation group $D_n$. In this paper we develop theory to study sequences $\{V_n\}$ of $W_n$-representations. These Weyl groups’ connections to Lie theory and realizations as finite reflection groups make such sequences prevalent in a broad range of mathematical subject areas.

We prove that if a sequence of $W_n$-representations has the structure of what we call a finitely generated $FI_{W}$-module (Section 1.1), there are strong constraints on the growth of the representations $V_n$, the form of the characters, and the pattern of irreducible $W_n$-representations in the decomposition of $V_n$. Our work builds on the theory of $FI$-modules developed by Church, Ellenberg, Farb, and Nagpal to study sequences of $S_n$-representations [CEF12], [CEFN12].

To establish this finitely generated $FI_{W}$-module structure, it is enough to verify certain elementary compatibility and finiteness conditions on $\{V_n\}$. These conditions are often easily checked in practice, and hold for a wealth of examples of sequences in geometry, algebraic topology, algebra, and combinatorics.

We give applications to the following sequences of representations:

I. the cohomology $H^m(\Sigma_n, \mathbb{Q})$ of the pure string motion group $\Sigma_n$ as representations of $B_n$ (Section 7.1),

II. the diagonal coinvariant algebras $C_r(n)$ associated to $S_n$, $B_n$, and $D_n$; for $r = 1$ these are the cohomology algebras of the associated generalized flag varieties (Section 7.2),

III. the cohomology $H^m(X_n, \mathbb{Q})$ of the hyperplane complements associated to $S_n$, $B_n$, and $D_n$ (Section 7.3).

Our work implies the following results about these sequences, which are new in many cases. We will define the terminology more precisely below.

**Theorem 1.1.** Let $\{V_n\}$ be any of the sequences I, II, or III as above.
1. The dimension of each sequence $V_n$ is eventually polynomial in $n$.

2. The characters of each sequence of $W_n$-representations are, for $n$ large, equal to a character polynomial, a polynomial in the signed cycle counting functions, which is independent of $n$.

3. Each sequence of $W_n$-representations is uniformly representation stable. In particular, the multiplicity of each irreducible $W_n$-representation $V(\lambda)_n$ in $V_n$ is eventually independent of $n$.

These results for sequences II and III in type A recover work of Church–Ellenberg–Farb [CEF12, Theorems 3.4 and 4.7], and the proof of representation stability for sequence III in type B/C recovers work of Church–Farb [CF13, Theorems 4.6].

The set of $FI_W$–modules has a rich algebraic structure. $FI_W$–modules in many ways resemble modules over a ring: there are natural notions of $FI_W$–module maps with quotients, kernels, and cokernels. We prove in Section 4.3 that $FI_W$–modules are Noetherian in the sense that sub-$FI_W$–modules of finitely generated $FI_W$–modules are themselves finitely generated. There are direct sum and tensor product operations on $FI_W$–modules, which we analyze in Section 6. In Sections 3.5 and 3.6 we develop restriction and induction operations between sequences of the different families of Weyl groups, using the category-theoretic concept of a Kan extension. This algebraic structure provides a conceptual framework and many powerful tools for analyzing sequences of $W_n$–representations.

Results of this $FI_W$–modules theory include an analogue of Murnaghan’s 1938 stability theorem for Kronecker coefficients [Mur38] for the hyperoctahedral group $B_n$ and even-signed permutation group $D_n$, which we prove in Section 6. These are stated here using notation for rational irreducible $B_n$ and $D_n$–representations defined in Section 2.2.

**Theorem 6.4. (Murnaghan’s stability theorem for $B_n$).** For any pair of double partitions $\lambda = (\lambda^+, \lambda^-)$ and $\mu = (\mu^+, \mu^-)$, there exist nonnegative integers $g_{\lambda,\mu}^\nu$, independent of $n$, such that for all $n$ sufficiently large:

$$V(\lambda)_n \otimes V(\mu)_n = \bigoplus_{\nu} g_{\lambda,\mu}^\nu V(\nu)_n.$$  \hspace{2cm} (7)

The coefficients $g_{\lambda,\mu}^\nu$ are nonzero for only finitely many double partitions $\nu$.

Theorem 6.4 implies the following:
1.1 \( \text{FI}_W \)-modules and finite generation

**Corollary 6.5.** (Murnaghan’s stability theorem for \( D_n \)). With double partitions \( \lambda = (\lambda^+, \lambda^-) \) and \( \mu = (\mu^+, \mu^-) \) as above, for all \( n \) sufficiently large the tensor product of the \( D_n \)-representations \( V(\lambda)_n \otimes V(\mu)_n \) has a stable decomposition:

\[
V(\lambda)_n \otimes V(\mu)_n = \bigoplus_{\nu} g^\nu_{\lambda, \mu} V(\nu)_n
\]

where \( g^\nu_{\lambda, \mu} \) are the structure constants of Equation (7).

In the context of \( \text{FI}_W \)-module theory, these stability results follow easily from a structural property of \( \text{FI}_W \)-modules: tensor products of finitely generated \( \text{FI}_W \)-modules are themselves finitely generated \( \text{FI}_W \)-modules.

Many aspects of the theory of \( \text{FI}_W \)-modules parallels the work [CEF12] and [CEFN12]. We encounter numerous additional challenges, however, particularly in type D. Section 1.7 summarizes the relation to recent work and new phenomena in this paper.

1.1 \( \text{FI}_W \)-modules and finite generation

We will now define our central concepts, \( \text{FI}_W \)-modules and finite generation.

**Definition 1.2.** (The Category \( \text{FI}_W \)). Let \( W_n \) denote the Weyl group in type \( A_{n-1}, B_n/C_n, \) or \( D_n \), and accordingly let \( \text{FI}_W \) denote the category \( \text{FI}_{A_n}, \text{FI}_{BC_n}, \) or \( \text{FI}_{D_n} \), as shown in the table below.

| Category | Objects | Morphisms |
|----------|---------|-----------|
| \( \text{FI}_{BC_n} \) | \( n = \{\pm1, \pm2, \ldots, \pm n\} \) | \{ injections \( f : m \to n \mid f(-a) = -f(a) \ \forall a \in m \} \ |
| \( 0 = \emptyset \) | \( \text{End}(n) \cong B_n \) |
| \( \text{FI}_{D_n} \) | \( n = \{\pm1, \pm2, \ldots, \pm n\} \) | \{ injections \( f : m \to n \mid f(-a) = -f(a) \ \forall a \in m \}; \ |
| \( 0 = \emptyset \) | \( \text{isomorphisms must reverse an even number of signs} \} \ |
| \( \text{End}(n) \cong D_n \) |
| \( \text{FI}_{A_n} \) | \( n = \{\pm1, \pm2, \ldots, \pm n\} \) | \{ injections \( f : m \to n \mid f(-a) = -f(a) \ \forall a \in m \); \ |
| \( 0 = \emptyset \) | \( f \) preserves signs \} \ |
| \( \text{End}(n) \cong S_n \) |
In each case, the objects of $\text{FI}_W$ are indexed by the natural numbers $\mathbb{Z}_{\geq 0}$; we will write these objects in boldface throughout the paper. The endomorphisms $\text{End}(n)$ are isomorphic to the group $W_n$, and the morphisms are generated by $\text{End}(n)$ and the natural inclusions $I_n : n \to (n + 1)$. The category $\text{FI}_A$ is equivalent to the category $\text{FI}$ defined by Church–Ellenberg–Farb [CEF12] as the category of Finite sets and Injective maps. There are inclusions of categories $\text{FI}_A \hookrightarrow \text{FI}_D \hookrightarrow \text{FI}_{BC}$.

**Definition 1.3. (FI$_W$–module).** Let $\text{FI}_W$ denote $\text{FI}_A$, $\text{FI}_{BC}$, or $\text{FI}_D$, and accordingly let $W_n$ denote $S_n$, $B_n$, or $D_n$. We define an $\text{FI}_W$–module $V$ over a ring $k$ to be a (covariant) functor from $\text{FI}_W$ to the category of $k$–modules. We will assume $k$ is commutative and with unit. The image of an $\text{FI}_W$–module is a sequence of $W_n$–representations $V_n := V(n)$ equipped with an array of maps $V_m \to V_n$ compatible with the $W_n$–action. For $f \in \text{Hom}_{\text{FI}_W}(m, n)$, we write $f_*$ (or simply $f$) to denote the linear map $V(f) : V_m \to V_n$.

This definition of an $\text{FI}_A$–module is equivalent to that of an $\text{FI}$–module given by [CEF12]. A schematic of an $\text{FI}_W$–module is shown in Figure 1.

**Definition 1.4. (Finite generation, Degree of generation).** We say an $\text{FI}_W$–module $V$ is finitely generated if there is a finite set of elements of $\bigsqcup_{n=0}^{\infty} V_n$ that are not contained in any proper sub–$\text{FI}_W$–module. The images of these elements under the $\text{FI}_W$ morphisms span each $k[W_n]$–module $V_n$. We say $V$ is finitely generated in degree $\leq d$ if it has a finite generating set $\{v_i\}$ with $v_i \in V_{m_i}$, $m_i \leq d$ for each $i$.
Example 1.5. (Some finitely and infinitely generated FI-W–modules). For a basic example to illustrate Definition 1.4, let $V_n := k[x_1, \ldots, x_n]$ be the polynomial ring on $n$ variables $x_i$ with the obvious inclusions $V_{n-1} \hookrightarrow V_n$. The group $W_n$ acts on $V_n$ by permuting and (for $D_n$ or $B_n$) negating the variables. The FI-W–module formed by the spaces $V_n$ is infinitely generated, but for each integer $d \geq 0$ the subspaces of homogeneous degree-$d$ polynomials $k[x_1, \ldots, x_n]_{(d)}$ form a sub–FI-W–module finitely generated in degree $\leq d$. Figure 2 shows a finite generating set for the FI-W–module of homogenous degree-2 polynomials.

![Figure 2: The finitely generated FI-W–module $k[x_1, \ldots, x_n]_{(2)}$](image)

The property of being finitely generated is easy to verify in many applications, but has strong implications for the structure of the underlying sequence of $W_n$–representations.

1.2 Character polynomials in type B/C and D

Let $k$ be a field of characteristic zero. One of our main results is that the sequence of characters of a finitely generated FI-W–module over $k$ is, for $n$ large, equal to a character polynomial which does not depend on $n$. This was proven for symmetric groups in [CEF12, Theorem 2.67], and here we extend these results to the groups $D_n$ and $B_n$.

Character polynomials for the symmetric groups date back to Murnaghan [Mur51] and Specht [Spe60]; they are described in Macdonald [Mac79, I.7.14]. In Section 5 we introduce character polynomials for the groups $B_n$ and $D_n$, in two families of signed variables. We use the classical results for $S_n$ to derive formulas for the character polynomials of irreducible $B_n$–representations (Theorem 5.10), and use these formulas to study these character polynomials in type
B/C and D.

Conjugacy classes of the hyperoctahedral group are classified by signed cycle type, see Section 2.1.2 for a description. We define the functions $X_r, Y_r$ on $\bigoplus_{n=0}^\infty B_n$ such that

$$X_r(\omega) \text{ is the number of positive } r-\text{cycles in } \omega,$$

$$Y_r(\omega) \text{ is the number of negative } r-\text{cycles in } \omega.$$

The functions $X_r, Y_r$ are algebraically independent as class functions on $\bigoplus_{n=0}^\infty B_n$, and so they form a polynomial ring

$$k[X_1, Y_1, X_2, Y_2, \ldots]$$

whose elements span the class functions on $B_n$ for each $n \geq 0$.

We prove that the sequence of characters of $\{V_n\}$ associated to any finitely generated $\text{FI}_{BC}$–module or $\text{FI}_D$–module $V$ over a field of characteristic zero are equal to a unique element of $k[X_1, Y_1, X_2, Y_2, \ldots]$ for all $n$ sufficiently large.

**Example 1.6. (Signed permutation matrices: A first example of a character polynomial).** As an elementary example of a sequence of $B_n$–representations described by a character polynomial, consider the canonical action of the hyperoctahedral groups $B_n$ on the vector space $\mathbb{Q}^n$ by signed permutation matrices, that is, generalized permutation matrices with nonzero entries $\pm 1$. The trace of a signed permutation matrix $\sigma$ is

$$\text{Tr}(\sigma) = \# \{1\text{’s on the diagonal of } \sigma\} - \# \{(-1)\text{’s on the diagonal of } \sigma\}$$

$$= \# \{ \text{ positive one cycles of } \sigma \} - \# \{ \text{ negative one cycles of } \sigma \}$$

$$= X_1(\sigma) - Y_1(\sigma)$$

and so the characters $\chi_n$ of this sequence are given by the function

$$\chi_n = X_1 - Y_1 \quad \text{for all values of } n.$$

The group $D_n$ is canonically realized as the subgroup of this signed permutation matrix group comprising those matrices with an even number of entries equal to $(-1)$. The character of this representation is the restriction of the character $\chi_n$ to the subgroup $D_n \subseteq B_n$, and so again this sequence of characters is equal to the character polynomial $\chi_n = X_1 - Y_1$ for all values of $n$. 

8
Conjugacy classes of the groups $D_n \subseteq B_n$ are not fully classified by their signed cycle type, due to the existence of certain 'split' classes when $n$ is even; see Section 2.1.3 for details. The functions $\{X_r, Y_r\}$ therefore do not span the space of class functions on any group $D_n$ with $n$ even. We prove, however, that when a sequence of representations $\{V_n\}$ of $D_n$ has the structure of a finitely generated $\text{FI}_D$–module, for $n$ large the characters depend only on the signed cycle type of the classes. Remarkably, the characters associated to $\{V_n\}$ are, for $n$ large, equal to a character polynomial independent of $n$.

**Theorem 5.15. (Characters of finitely generated $\text{FI}_W$–modules are eventually polynomial).** Let $k$ be a field of characteristic zero. Suppose that $V$ is a finitely generated $\text{FI}_{BC}$–module with weight $\leq d$ and stability degree $\leq s$, or, alternatively, suppose that $V$ is a finitely generated $\text{FI}_D$–module with weight $\leq d$ such that $\text{Ind}_{D}^{BC} V$ has stability degree $\leq s$. In either case, there is a unique polynomial

$$F_V \in k[X_1, Y_1, X_2, Y_2, \ldots]$$

such that the character of $W_n$ on $V_n$ is given by $F_V$ for all $n \geq s + d$. The polynomial $F_V$ has degree $\leq d$, with $\deg(X_i) = \deg(Y_i) = i$.

Weight and stability degree are defined in Sections 4.1 and 4.2; these quantities are always finite for finitely generated $\text{FI}_W$–modules and associated induced $\text{FI}_W$–modules.

Theorem 5.15 generalizes the result of Church–Ellenberg–Farb [CEF12, Theorem 2.67] that the characters of finitely generated $\text{FI}_A$–module are, for $n$ sufficiently large, given by a character polynomial in the class functions $X_r$ on $\bigsqcup_{n=0}^{\infty} S_n$ that takes a permutation $\sigma$ and returns the number of $r$–cycles in its cycle type.

In our applications, it remains an open problem to compute the character polynomials in all but a few small degrees. Since we can often establish explicit upper bounds on the degrees and stable ranges of these polynomials, the problem is much more tractable: to find the character polynomials – and so determine the characters for all values of $n$ – it is enough to compute the characters for finitely many specific values of $n$.

**Eventually polynomial dimensions.** Suppose that $V$ is a finitely generated $\text{FI}_W$–module with character polynomial $F_V$. For each $n$ in the stable range, the
1.3 Connection to representation stability

Prior to their work with Ellenberg on FI–modules, Church and Farb defined and developed the theory of representation stability for families of groups $G_n$ including $S_n$ and $B_n$ in [CF13]. For a sequence $V_n$ of rational $G_n$–representations to be representation stable, the multiplicities of the irreducible constituents $V(\lambda)_n$ of $V_n$ must eventually be constant in $n$; a key to this definition is the appropriate classification of irreducible $G_n$–representations $V(\lambda)_n$ as functions of $n$. We describe these definitions in more detail in Section 2.2, where we also introduce a definition of representation stability for sequences of $D_n$–representations.

It is shown in [CEF12, Theorem 1.14] that, for sequences of $S_n$–representations with the structure of an FI–module, finite generation is equivalent to uniform representation stability. We prove this phenomenon holds more generally:

**Theorem 5.19. (Polynomial growth of dimension over arbitrary fields).** Let $k$ be any field, and let $V$ be a finitely generated FI$_W$–module over $k$. Then there exists an integer-valued polynomial $P(T) \in \mathbb{Q}[T]$ such that

$$
\dim_k(V_n) = P(n) \quad \text{for all } n \text{ sufficiently large.}
$$
Theorems 4.28 and 4.29. (FI$_W$–modules are uniformly representation stable iff they are finitely generated). Suppose that $k$ is a field of characteristic zero, and $W_n$ is $S_n$, $D_n$, or $B_n$. Let $V$ be a finitely generated FI$_W$–module. Take $d$ to be an upper bound on the weight of $V$, $g$ an upper bound on its degree of generation, and $r$ an upper bound on its relation degree; when $W_n$ is $D_n$, take $r$ to be an upper bound on the relation degree of Ind$^{BC}_D V$. Then $\{V_n\}$ is uniformly representation stable with respect to the maps induced by the natural inclusions $I_n : n \to (n + 1)$, stabilizing once $n \geq \max(g, r) + d$; when $W_n$ is $D_n$ and $d = 0$ we need the additional condition that $n \geq g + 1$.

Suppose conversely that $V$ is an FI$_W$–module, and that $\{V_n, (I_n)_*\}$ is uniformly representation stable for $n \geq N$. Then $V$ is finitely generated in degree $\leq N$.

The classification of rational irreducible $B_n$ and $D_n$–representations are described in Sections 2.1.2 and 2.1.3, and the precise definition of $V(\lambda)_n$ and criteria for representation stability are given in Section 2.2.

1.4 FI$_W$♯–modules

In Section 4.6 we described a certain class of FI$^{BC}$–modules called FI$^{BC}$♯–modules, analogues of the FI♯–modules (“FI sharp modules”) defined by Church–Ellenberg–Farb. An FI$^{BC}$♯–module is a sequence of $B_n$–representations that simultaneously admits a functor from FI$^{BC}$ and a functor from the dual category FI$^{BC}$ in some compatible sense; see Definition 4.35.

A finitely generated FI$^{BC}$♯–module structure places even stronger constraints on the structure of a sequence of $B_n$–representations. For example, we show in Section 5.5 that a FI$^{BC}$♯–module finitely generated in degree $\leq d$ has characters equal to a unique character polynomial of degree at most $d$ for all values of $n$, and dimensions given by a polynomial in $n$ of degree at most $d$ for all $n$.

We prove in Theorem 4.42 that FI$^{BC}$♯–modules are direct sums of sequences of the form

$$
\left\{ \bigoplus_{m=0}^{d} \text{Ind}_{B_{m} \times B_{n-m}}^{B_n} U_m \boxtimes k \right\}_n
$$

Here, $k$ denotes the trivial $B_{n-m}$–representation, and $U_m$ is a $B_m$–representation, possibly 0. The external tensor product

$$(U_m \boxtimes k)$$

is the $k$–module $(U_m \otimes_k k)$ as a $(B_m \times B_{n-m})$–representation. Theorem 4.42
extends [CEF12, Theorem 2.24], which is the analogous statement in type A.

1.5 Some applications

The theory of $\text{FI}_W$–modules developed in this paper gives new, concrete results about a variety of known objects in geometry and combinatorics. In Section 7 we give applications to the pure string motion group $P\Sigma_n$, diagonal coinvariant algebras associated to the reflection groups $W_n$, and hyperplane complements associated to the reflection groups $W_n$.

**Application: the pure string motion group.** Let $P\Sigma_n$ be the group of pure string motions. This motion group is a generalization of the pure braid group, and can be realized as the group of pure symmetric automorphisms of the free group $F_n$; see Section 7.1 for a definition.

**Theorem 7.3.** Let $k$ be $\mathbb{Z}$ or $\mathbb{Q}$. The cohomology rings $H^*(P\Sigma_n, k)$ form an $\text{FI}_{BC}^{\sharp}$–module, and a graded $\text{FI}_{BC}$–algebra of finite type, with $H^m(P\Sigma_n, k)$ finitely generated in degree $\leq 2m$. In particular the $\text{FI}_{BC}$–algebra $H^*(P\Sigma_n, \mathbb{Q})$ has slope $\leq 2$.

We recover (with considerably less effort) the main result of our previous paper [Wil12], which stated that for each $m$, the sequence of $B_n$–representations

$$\{H^m(P\Sigma_n, \mathbb{Q})\}_n$$

is uniformly representation stable.

**Corollary 7.4.** For each $m$, the sequence $\{H^m(P\Sigma_n; \mathbb{Q})\}_n$ of representations of $B_n$ (or $S_n$) is uniformly representation stable, stabilizing once $n \geq 4m$.

A consequence of uniform representation stability, which follows from stability for the trivial representation and a transfer argument, is rational homological stability for the the string motion group $\Sigma_n$. This recovers the rational case of a result of Hatcher and Wahl [HW10, Corollary 1.2]. More details are given in Section 7 of [Wil12].

Another consequence of Theorem 7.3 is the existence of character polynomials. Because these cohomology groups are $\text{FI}_{BC}$–modules, their characters are equal to the character polynomial for all values of $n$, and not just $n$ sufficiently large.
Corollary 7.6. Let $k$ be $\mathbb{Z}$ or $\mathbb{Q}$. Fix an integer $m \geq 0$. The characters of the sequence of $B_n$–representations $\{H^m(P\Sigma_n; k)\}_n$ are given, for all values of $n$, by a unique character polynomial of degree $\leq 2m$.

We compute these character polynomials explicitly in degree 1 and 2:

$$
\chi_{H^1(P\Sigma_n; \mathbb{Z})} = X_1^2 - X_1 - Y_1^2 + Y_1
$$

$$
\chi_{H^2(P\Sigma_n; \mathbb{Z})} = 2X_2 + Y_2^2 + 2Y_2^2 - X_1^2Y_1^2 - \frac{3}{2}Y_1^3 + \frac{1}{2}X_1^4 + X_1^2 - 2X_2^2 - \frac{3}{2}X_1^3
\quad + \frac{1}{2}X_1^4 + \frac{1}{2}X_1Y_1^2 - X_1Y_2 - X_2Y_1 - Y_1Y_2 + \frac{1}{2}X_2^2Y_1 - X_1X_2 - 2Y_2
$$

It is an open problem to compute these polynomials for larger values of $m$.

**Application: diagonal coinvariant algebras.** Let $k$ be a field, and consider the canonical action of $W_n$ on

$$
V_n := k^n \cong \text{Span}_k \langle x_1, \ldots, x_n \rangle.
$$

The group $S_n$ acts by permutation matrices, $B_n$ by signed permutation matrices, and $D_n$ by signed permutation matrices with an even number of entries equal to $-1$.

There is an induced diagonal action of $W_n$ on $V_n^{\oplus r}$, and so an induced action on the symmetric algebra $\text{Sym}(V_n^{\oplus r})$, isomorphic to the polynomial algebra

$$
k[x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}].
$$

The $r$-diagonal coinvariant algebra $C^{(r)}(n)$ is the quotient of this algebra by the ideal $I_n$ of constant-term-free $W_n$–invariant polynomials. The algebra $C^{(r)}(n)$ has a natural multigrading by $r$-tuples $J = (j_1, \ldots, j_r)$, where $j_\ell$ specifies the total degree of a monomial in the variables $x_1^\ell, \ldots, x_n^\ell$.

The coinvariant algebras $C^{(1)}(n)$ were studied classically for their connections to representation theory of Lie groups. The $r$-diagonal coinvariant algebras have been studied since the 1990s, with major contributors including Garsia, Haiman, Hagland, Gordon, Bergeron, and Biagioli; see Section 7.2 for more history. Haiman [Hai02a] and Bergeron [Ber09] offer in-depth background on coinvariant algebras and their many connections to other areas of algebraic combinatorics.

In Section 7.2 we prove that each multigraded component $C^{(r)}_J(n)$ of $C^{(r)}(n)$
is a finitely generated co–FI\(_W\)–module. Understanding the characters of the multigraded components of \(C^{(r)}(n)\) is a well-known open problem; little is known except for very small values of \(r\) and \(n\). The following result, inspired by the work of [CEF12] and [CEFN12] on diagonal coinvariant algebras in type A, reveals underlying structure and patterns in these sequences of representations.

**Theorem 7.8.** Let \(k\) be a field, and let \(V_n \cong k^n\) be the canonical representation of \(W_n\) by (signed) permutation matrices. Given \(r \in \mathbb{Z}_{>0}\), the sequence of coinvariant algebras

\[
C^{(r)} := k[V^{\oplus r}] / \mathcal{I}
\]

is a graded co–FI\(_W\)–algebra of finite type. When \(k\) has characteristic zero, the weight of the multigraded component \(C_J^{(r)}\) is \(\leq |J|\).

**Corollary 7.9.** Let \(k\) be a field of characteristic zero. For \(n\) sufficiently large (depending on the \(r\)-tuple \(J\)), the sequence \(C_J^{(r)}(n)\) is uniformly multiplicity stable.

**Corollary 7.10.** Let \(k\) be a field of characteristic zero. For \(n\) sufficiently large (depending on the \(r\)-tuple \(J\)), the characters of \(C_J^{(r)}(n)\) are given by a character polynomial \(F_J\) of degree \(\leq |J|\). In particular the dimension of \(C_J^{(r)}(n)\) is given by the degree \(|J|\) polynomial

\[
\dim_k C_J^{(r)}(n) = F_J(n, 0, 0, 0 \ldots)
\]

for all \(n\) in the stable range.

**Corollary 7.11.** Let \(k\) be an arbitrary field. Then for each \(r\)-tuple \(J\), there exists a polynomial \(P_J \in \mathbb{Q}[T]\) (depending on \(k\)) so that

\[
\dim_k C_J^{(r)}(n) = P_J(n)
\]

for all \(n\) sufficiently large (depending on \(k\) and \(J\)).

Theorem 7.8 and its corollaries were proven in type A over characteristic zero by Church–Ellenberg–Farb [CEF12, Theorem 3.4]. In later work with Nagpal these authors extend their work to fields of arbitrary characteristic [CEFN12, Proposition 4.2], and in particular they prove Corollary 7.11 in type A [CEFN12, Theorem 1.9].

In the special case \(r = 1\), the algebras \(C^{(1)}(n)\) are isomorphic to the cohomology rings of the generalized flag varieties associated to the Lie groups in
Some applications

1.5 Some applications

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type \( W \); see Section 7.2 for details. A corollary of Theorem 7.8 is representation stability and the existence of character polynomials for these cohomology groups.

In Section 7.2 we state the character polynomials in type B/C for \(| J | \leq 3 \); in general, computing these character polynomials is an open problem.

Application: hyperplane complements. Each family of groups \( W_n \) has a canonical action on \( \mathbb{R}^n \) by signed permutation matrices; we denote by \( \mathcal{M}_W(n) \) the set of complexified hyperplanes fixed by reflections in \( W_n \), and

\[
\mathcal{M}_W(n) := \mathbb{C}^n \backslash \bigcup_{H \in \mathcal{A}(n)} H
\]

the associated hyperplane complement. See Section 7.3 for explicit descriptions of these spaces, and a brief survey of results on the structure of their cohomology rings. In type A, the space \( \mathcal{M}_A(n) \) is precisely the ordered \( n \)-point configuration space of \( \mathbb{C} \), and Church–Ellenberg–Farb show its cohomology groups are finitely generated \( \text{FI}_A \)–modules [CEF12, Theorem 4.7]. Using a presentation for \( H^*(\mathcal{M}_W(n); \mathbb{C}) \) computed by Brieskorn [Bri73] and Orlik–Solomon [OS80], we generalize the results of [CEF12] to all three families of classical Weyl groups.

**Theorem 7.14.** Let \( \mathcal{M}_W \) be the complex hyperplane complement associated with the Weyl group \( W \) in type \( A_{n-1}, B_n/C_n, \) or \( D_n \). In each degree \( m \), the groups \( H^m(\mathcal{M}_A(\bullet), \mathbb{C}) \) form an \( \text{FI}_A \)–module finitely generated in degree \( \leq 2m \), and both \( H^m(\mathcal{M}_{BC}(\bullet), \mathbb{C}) \) and \( H^m(\mathcal{M}_D(\bullet), \mathbb{C}) \) are \( \text{FI}_{BC} \)–modules finitely generated in degree \( \leq 2m \).

**Corollary 7.15.** In each degree \( m \), the sequence of cohomology groups \( \{ H^m(\mathcal{M}_W(n), \mathbb{C}) \} \) is uniformly representation stable in degree \( \leq 4m \).

**Corollary 7.16.** In each degree \( m \), the sequence of characters of the \( W_n \)–representations \( H^m(\mathcal{M}_W(n), \mathbb{C}) \) are given by a unique character polynomial of degree \( \leq 2m \) for all \( n \).

We emphasize that, because these sequences are \( \text{FI}_{W} \)–modules, their characters are equal to the character polynomial for every value of \( n \).

Corollary 7.15 recovers the work of Church–Farb [CF13, Theorem 4.1 and 4.6] in types A and B/C. In type A, Theorem 7.14 recovers the work of Church–Ellenberg–Farb [CEF12] on the cohomology of the ordered configuration space of the plane.
Character polynomials and stable decompositions for $H^m(\mathcal{M}_A(\bullet), \mathbb{C})$ are computed in [CEF12] for some small values of $m$. In Type B/C and D, we can also compute the character polynomials by hand in small degree:

\[
\chi_{H^1(\mathcal{M}_{BD}(\bullet), \mathbb{C})} = 2 \left( \frac{X_1}{2} \right)^2 + 2 \left( \frac{Y_1}{2} \right)^2 + 2X_2 \\
\chi_{H^1(\mathcal{M}_{BC}(\bullet), \mathbb{C})} = 2 \left( \frac{X_1}{2} \right)^2 + 2 \left( \frac{Y_1}{2} \right)^2 + 2X_2 + X_1 - Y_1
\]

See Section 7.3 for the character polynomials and stable decompositions in when the degree $m$ is 1 and 2.

### 1.6 Remarks on the general theory

We briefly highlight some key tools and results of the theory of FI$_W$-modules.

**The structure of finitely generated FI$_W$-modules.** A crucial fact about finitely generated FI$_W$-modules is that they can be realized as quotients of sequences of the form

\[
\left\{ \bigoplus_{m=0}^{g} \text{Ind}_{W_{n-m}}^{W_n} k \right\}_n,
\]

where $k$ denotes the trivial $W_{n-m}$-representation, as shown in Proposition 3.17. Over fields of characteristic zero, the combinatorics of these induced representations is governed by the branching rules for each family $W_n$-rules that are well understood for $S_n$ and $B_n$, though more complex for $D_n$ (see, for example, Geck-Pfeiffer [GP00]).

We prove in Theorem 4.22 that sub–FI$_A$-modules of finitely generated FI$_W$-modules are themselves finitely generated. This Noetherian property was proven for FI$_A$-modules by Church–Ellenberg–Farb [CEF12, Theorem 2.6] over Noetherian rings containing the rationals, and later proven by Church–Ellenberg–Farb–Nagpal [CEFN12, Theorem 1.1] over arbitrary Noetherian rings; our proof uses their results. These properties of finite generation are used extensively throughout this paper.

**Restriction and Induction of FI$_W$-Modules** Given the inclusions of categories $\text{FI}_A \hookrightarrow \text{FI}_D \hookrightarrow \text{FI}_{BC}$, there is a natural restriction operation of FI$_{BC}$ and FI$_D$-modules down to FI$_D$ or FI$_A$-modules, and we show in Proposition 3.24 that the restriction of functors between these categories preserves the property...
1.7 Relationship to earlier work

1.7.1 Recent work

Representation stability. In 2010, Church–Farb [CF13] introduced the concept of representation stability for sequences of rational representations of several families of groups: $S_n$, $B_n$, and the linear groups $SL_n(Q)$, $GL_n(Q)$, and $Sp_{2n}(Q)$. For each family they formulated stability criteria in terms of the pattern of irreducible subrepresentations, patterns which they show appear in ubiquitous examples throughout mathematics. They give a host of applications to classical representation theory, the cohomology of groups arising in geometric group theory, Lie algebras and their homology, the (equivariant) cohomology of flag and Schubert varieties, and algebraic combinatorics.

$FI$–modules. Two years later Church–Ellenberg–Farb [CEF12] significantly refined the theory for sequences of $S_n$–representations by introducing $FI$–modules. This new work accomplished several things: They proved criteria for representation stability that are simple and easily established – a finitely generated $FI$–modules structure. They strengthened their results with the observation that the characters of a representation stable sequence have an associated character polynomial, and gave a number of consequences including polynomial growth of dimension. They gave a framework for studying sequences of $S_n$–representations over arbitrary coefficient rings, which does not depend on the combinatorial particulars of the classification of irreducible rational representations. The category $FI$, and the concept of finite generation, are natural and elementary constructs. Their theory provides a structured, unified context and a vocabulary to describe patterns and stability phenomenon that could not be captured otherwise.
Using the theory of FI-modules, Church–Ellenberg–Farb prove new results about a number of fundamental sequences $V_n$ of $S_n$–representations. These include the cohomology of the $n$-point configuration space of a manifold, the cohomology of the moduli space of $n$-puncture surfaces, certain subalgebras of the cohomology of the genus $n$ Torelli group, and the diagonal $S_n$-coinvariant algebras on $r$ sets of $n$ variables.

Central stability. Putman [Put12] independently developed a theory that extends representation stability to positive characteristic. He established stability results for level $q$ congruence subgroups of $GL_n(R)$ for a large class of rings $R$ with ideals $q$. His main definition, central stability, is closely related to the notion of a finitely generated FI–module; see for example Remark 3.36. Putman proved that central stability implies representation stability and polynomial dimension growth. He integrated his theory of central stability with the classical homological stability machinery developed by Quillen. This representation-theoretic homological stability apparatus applies to a variety of geometric and algebraic applications over numerous coefficient systems.

FI–modules over Noetherian rings. Shortly after the appearance of Putman’s work, the work on FI–modules [CEF12] were strengthened further by Church–Ellenberg–Farb–Nagpal [CEFN12]. These authors extended several results to broader classes of coefficients: they prove polynomial growth of dimension over fields of positive characteristic, and the Noetherian property over arbitrary Noetherian rings. They generalize their results for several of the above applications to coefficients in the integers or positive characteristic.

Twisted commutative algebras. In 2010, Snowden [Sno13] independently proved, using different language, several fundamental properties of FI–modules. His work centres on modules over a class of objects called twisted commutative algebras; FI–modules are an example. His results include the Noetherian and polynomial growth properties for finitely generated complex FI–modules, results which he uses to study syzygies of Segre embeddings. See Sam–Snowden [SS12b] for an accessible introduction to the theory of twisted commutative algebras. Following the work of Church–Ellenberg–Farb, Sam–Snowden [SS12a] performed a deeper analysis of the category of FI–modules over a field of characteristic zero and proved a number of algebraic and homological finiteness properties.
We would be interested to better understand how the work of Snowden and Sam–Snowden relates to the theory of FI$_W$–modules developed here.

1.7.2 New obstacles and new phenomena

Much of the theory of FI$_W$–modules parallels the work of Church–Ellenberg–Farb [CEF12], and frequently their methods of proof adapt to our more general context. Some additional hurdles and some new phenomena do emerge, however, for the Weyl groups $B_n$ and $D_n$. These include:

- **Character polynomials in type B/C.** The existence of character polynomials for finitely generated FI$_A$–modules follows immediately from representation stability and classical results in algebraic combinatorics: the formula for the character polynomial of the irreducible $S_n$–representation $V(\lambda)_n$ appear in texts such as MacDonald [Mac79]. The achievement of [CEF12] here was uncovering this (regrettably little-known) formula and recognizing its implications for the study of FI$_A$–modules. The analogous formulas for the irreducible $B_n$–representations are not so readily available, however, and we compute these in Section 5.2. These signed character polynomials now involve two sets of variables $X_r$ and $Y_r$, corresponding to the positive and negative cycles for these groups.

- **Restriction, induction, and coinduction.** The restriction and induction operations between the three categories FI$_A$, FI$_D$, and FI$_{BC}$ give FI$_W$–modules a new level of structure. In Sections 3.5 and 3.6 we define and study these operations from a category-theoretic perspective.

- **Branching rules in type D.** The combinatorics of the branching rules for the hyperoctahedral groups, like the symmetric groups, are well understood. With these formulas, many of the methods of proof used by Church–Ellenberg–Farb [CEF12] for FI$_A$–modules adapt beautifully to FI$_{BC}$–modules, including the proof that finite generation is equivalent to uniform representation stability. In contrast, the branching rules for the groups $D_n$ are more subtle, and it is not clear that the methods in [CEF12] adapt to type D. We proceed instead by analyzing the restriction and inductions operations between FI$_D$ and FI$_{BC}$. To recover the main results in type D, we relate a finitely generated FI$_D$–module $V$ to the FI$_D$–module

$$\text{Res}^{BC}_D \text{Ind}^{BC}_D V,$$
defined in Section 3.6, and appeal to our results for finitely generated $\text{FI}_{BC}$–modules.

- **Representation stability in type D.** It was initially unclear how *representation stability* ought to be defined for representations of the even-signed permutation groups $D_n$. The classification of irreducible $D_n$–representations (Section 2.1.3), which involves *unordered* pairs of partitions and ‘split’ representations in even degree, did not suggest any deterministic growth rules of the form defined by Church–Farb [CF13] for $S_n$ and $B_n$ (Section 2.2). More to the point, it was not clear that we could expect any specific constraints on the patterns of irreducible representations in a class of sequences as broad and commonly occurring as the finitely generated $\text{FI}_D$–modules. The definition of representation stability in type D was ultimately written late in the course of this project, after the discovery of an unanticipatedly strong result: If $V$ is a finitely generated $\text{FI}_D$–module, then, for $n$ sufficiently large, $V_n$ is the restriction of a $B_n$–representation.

- **Character polynomials in type D.** Given the classification of conjugacy classes in type D (Section 2.1.3), and the existence of ‘split’ classes that could not be characterized by signed cycle type, we had not expected an analogue of character polynomials to exist for sequences of $D_n$–representations, except in exceptional cases. A finitely generated $\text{FI}_D$–module *does* have characters equal, for large $n$, to a character polynomial. We again establish this existence result by realizing the tail of a finitely generated $\text{FI}_D$–module $V$ as the restriction of an $\text{FI}_{BC}$–module, using properties of categorical induction $\text{Ind}_{D}^{BC}$.

- **A category $\text{FI}_{D^?}$?** There does not appear to be a suitable analogue of $\text{FI}^?_D$ for the category $\text{FI}_{D^?}$; see Remark 4.36. Fortunately, and perhaps not by coincidence, the applications in type D where we have expected this extra structure, such as the cohomology groups of the hyperplane complements $\mathcal{M}_D(n)$, turned out to be restrictions of $\text{FI}_{BC^?}$–modules to $\text{FI}_D \subseteq \text{FI}_{BC^?}$.

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2 Background

2.1 The Weyl groups of classical type

The classical Weyl groups comprise three one-parameter families of finite reflection groups. The symmetric group \( S_n \) is the Weyl group of type \( A_{n-1} \); the hyperoctahedral group (or signed permutation group) \( B_n \) is the Weyl group of the (dual) root systems of types \( B_n \) and \( C_n \), and its subgroup the even-signed permutation group \( D_n \) is the Weyl group of type \( D_n \). We briefly review the representation theory of these groups.

We note that the finite dimensional complex representations of \( S_n \), \( B_n \), and \( D_n \) are defined over the rational numbers [GP00, Theorem 5.4.5, Theorem 5.5.6, Corollary 5.6.4].

2.1.1 The symmetric group \( S_n \)

The rational representation theory of the symmetric group \( S_n \) is well understood; a standard reference is Fulton–Harris [FH04]. The irreducible representations of \( S_n \) are in natural bijection with the set of partitions \( \lambda \) of \( n \), which we denote

\[
\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r), \quad \text{with } \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_r \text{ and } \lambda_0 + \lambda_1 + \cdots + \lambda_r = n.
\]

Each integer \( \lambda_i \) is a part or addend of the partition. We write

\[
\lambda \vdash n \quad \text{or} \quad |\lambda| = n
\]
to indicate the size of the partition. The length $\ell(\lambda)$ of $\lambda$ is the number of parts.

We write $V_\lambda$ to denote the $S_n$–representation associated to $\lambda$.

### 2.1.2 The hyperoctahedral group $B_n$

The hyperoctahedral group $B_n$ is the wreath product

$$B_n = \mathbb{Z}/2\mathbb{Z} \wr S_n := (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n,$$

where $S_n$ acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permuting the coordinates. It is the symmetry group of the $n$–hypercube, dually, the $n$–hyperoctahedron. There is a canonical representation of $B_n$ as the group of signed permutation matrices, that is, $n \times n$ generalized permutation matrices with nonzero entries $\pm 1$. We can also characterize the hyperoctahedral group as the symmetry group of the set

$$\{-1,1\}, \{-2,2\}, \ldots, \{-n,n\},$$

where the $k^{th}$ factor of $(\mathbb{Z}/2\mathbb{Z})^n$ transposes the elements in the block $\{-k,k\}$, and $S_n$ permutes the $n$ blocks. As such, $B_n$ is also called the signed permutation group.

It is often convenient to consider $B_n$ as a subgroup of the symmetric group $S_{2n}$ that acts on the $2n$ letters

$$\Omega = \{-1,1,-2,2,\ldots,-n,n\}.$$

We frequently write elements of $B_n$ in the cycle notation of permutations of $\Omega$.

**The rational representation theory of $B_n$.** The representation theory of the hyperoctahedral group was developed Young in the 1920s [You30], and further refined by authors including Mayer [May75]; Geissinger and Kinch [GK78]; al-Aamily, Morris, and Peel [aAMP]; and Naruse [Nar85]. It is described in [GP00].

The rational irreducible representations of $B_n$ can be built up from those of the symmetric group $S_n$. These irreducible $B_n$–representations are classified by double partitions of $n$, that is, ordered pairs of partitions

$$(\lambda, \nu) \quad \text{with} \quad |\lambda| + |\nu| = n.$$
2.1 The Weyl groups of classical type

For \( \lambda \vdash n \), define \( V(\lambda, \emptyset) \) to be the \( B_n \)-representation pulled back from \( S_n \)-representation \( V_\lambda \) under the surjection \( \pi : B_n \to S_n \). Let \( \mathbb{Q}^\varepsilon \) denote the one-dimensional “sign” representation associated to the character

\[
\varepsilon : B_n \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \to \{\pm 1\}
\]

where the canonical generators of \( (\mathbb{Z}/2\mathbb{Z})^n \) act by \((-1)\), and elements of \( S_n \) act trivially. Define

\[
V(\emptyset, \nu) := V(\nu, \emptyset) \otimes \mathbb{Q}^\varepsilon.
\]

Then, generally, for \( \lambda \vdash m \) and \( \nu \vdash (n - m) \), we define

\[
V(\lambda, \nu) := \text{Ind}_{B_m \times B_{n-m}}^{B_n} V(\lambda, \emptyset) \boxtimes V(\emptyset, \nu),
\]

where \( \boxtimes \) denotes the external tensor product of the \( B_m \)-representation \( V(\lambda, \emptyset) \) with the \( B_{n-m} \)-representation \( V(\emptyset, \nu) \). Each double partition \((\lambda, \nu)\) yields a distinct irreducible representation of \( B_n \), and every irreducible representation has this form.

The conjugacy classes of \( B_n \). The conjugacy classes of \( B_n \) were described by Young [You30]. More modern exposition can be found in, for example, Carter [Car72], or Naruse [Nar85]. A similar classification is found in [GP00, Chapter 3].

An element of \( B_n \), viewed as a permutation on \( \Omega = \{\pm 1, \ldots, \pm n\} \), can be decomposed into cycles. Define the negation of a cycle \( \beta = (s_1 s_2 \cdots s_r) \) as

\[
-\beta := (-b_1 - b_2 \cdots - b_r).
\]

The cycles in a signed permutation come in two flavours:

Definition 2.1. (Positive and negative cycles in \( B_n \)).

1. Cycles \( \sigma = (s_1 s_2 \cdots s_r - s_1 - s_2 \cdots - s_r) \). These cycles satisfy \(-\sigma = \sigma\).

   In the natural surjection \( B_n \to S_n \), the cycle \( \sigma \) is mapped to the \( r \)-cycle

   \[
   (|s_1| |s_2| \cdots |s_r|).
   \]

   The power \( \sigma^r \) is the product of \( r \) involutions

   \[
   (s_1 - s_1) (s_2 - s_2) \cdots (s_r - s_r).
   \]
For this reason, Carter calls the element $\sigma$ a negative cycle of length $r$. We note that these elements reverse the sign of an odd number of digits $\{1, \ldots, n\}$.

2. Cycles $\alpha = (a_1 \ a_2 \ \cdots \ a_r)$ with $|a_i| \neq |a_j|$ if $i \neq j$. These cycles satisfy $-\alpha \neq \alpha$.

For any signed permutation $\omega \in B_n$, these cycles occur in pairs $\alpha(-\alpha)$. The surjection $B_n \rightarrow S_n$ maps $\alpha(-\alpha)$ to the $r$–cycle

$$\left( |a_1| \ |a_2| \ \cdots \ |a_r| \right),$$

and $(\alpha(-\alpha))^r$ is the identity element. Accordingly, Carter calls the product $\alpha(-\alpha)$ a positive cycle of length $r$. We note that these elements reverse the sign of an even number of digits $\{1, \ldots, n\}$.

For example,

$$(1 \ 2)(-1 \ -2) \quad \text{and} \quad (1 \ -2)(-1 \ 2)$$

are both examples of positive two-cycles in $B_n$;

$$(1 \ 2 \ -1 \ -2) \quad \text{and} \quad (1 \ -2 \ -1 \ 2)$$

are both negative two-cycles.

The cycle structure of an element $\omega \in B_n$ is encoded by double partitions $(\lambda, \nu)$ of $n$, where $\lambda$ designates the lengths of the positive cycles, and $\mu$ designates the lengths of the negative cycles. The double partition $(\lambda, \nu)$ is called the signed cycle type of the element $\omega$.

For example, the identity element has cycle type $((1^n), \emptyset)$. The element

$$w_o = (-1 \ 1)(-2 \ 2) \cdots (-n \ n)$$

has cycle type $((\emptyset, (1^n))$. The element

$$x = (1 \ -1)(2 \ -3 \ 7 \ -2 \ 3 \ -7)(4 \ 5)(-4 \ -5)(-6) \in B_7$$

has cycle type $((2, 1), (3, 1))$.

The following result dates back to Young [You30]. See also [Car72, Proposition 24] and [GP00, Proposition 3.4.7].
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Proposition 2.2. (Classification of conjugacy classes of \( B_n \)). Two elements \( x, y \in B_n \) are conjugate in \( B_n \) if and only if they have the same signed cycle type. Thus the conjugacy classes of \( B_n \) are classified by double partitions \((\lambda, \nu)\) of \( n \).

Branching rules and Pieri’s formula for \( B_n \). The branching rules for \( B_n \) are a main tool in our development of the theory of \( \text{FI}_{BC} \)-modules over fields of characteristic zero. These rules are described in, for example, Geck–Pfeiffer [GP00, Lemma 6.1.3]:

\[
\text{Ind}_{B_a \times B_n - a}^{B_n} V(\lambda^+,\lambda^-) \boxtimes V(\mu^+,-\mu^-) = \bigoplus_{(\nu^+,\nu^-)} C^{\nu^+}_{\lambda^+,(n-a)} C^{\nu^-}_{\lambda^-,\mu^-} V(\nu^+,\nu^-)
\]

where \( C^{\nu^+}_{\lambda^+,(n-a)} \) denotes the Littlewood–Richardson coefficient. We will use in particular Pieri’s formula, the case where \( V(\mu^+,-\mu^-) \) is the trivial representation \( k = V((n-a),\emptyset) \).

\[
\text{Ind}_{B_a \times B_n - a}^{B_n} V(\lambda^+,\lambda^-) \boxtimes k = \bigoplus_{(\nu^+,\nu^-)} C^{\nu^+}_{\lambda^+,(n-a)} C^{\nu^-}_{\lambda^-,\emptyset} V(\nu^+,\nu^-)
\]

\[
= \bigoplus_{\nu^+} C^{\nu^+}_{\lambda^+,(n-a)} V(\nu^+,-\lambda^-)
\]

\[
= \bigoplus_{\nu^+} V(\nu^+,-\lambda^-)
\]

where the final sum is taken over all partitions \( \nu^+ \) that can be constructed by adding \((n-a)\) boxes to \( \lambda^+ \), with no two boxes added to the same column. Some small cases are shown in Figure 3.

Figure 3: Illustrating the branching rules for \( B_n \).

\[
\begin{align*}
\text{Ind}_{B_7 \times B_2}^{B_7} V(\square,\infty) \boxtimes k &= V(\square,\infty) \oplus V(\square,\square) \oplus V(\square,\infty) \oplus V(\square,\infty) \\
\text{Ind}_{B_5 \times B_2}^{B_5} V(\square,\infty) \boxtimes k &= V(\square,\infty) \oplus V(\square,\square) \oplus V(\square,\emptyset) \oplus V(\square,\infty) \oplus V(\square,\emptyset)
\end{align*}
\]

By Frobenius reciprocity, the multiplicity of \( V(\lambda^+,\nu^-) \boxtimes k \) in the restriction
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\[ \text{Res}_{B_n \times B_{n-a}}^{B_n \times B_{n-a}} V_{(\nu^+, \nu^-)} \] is

\[ \begin{cases} 
1 & \text{if } \nu^+ \text{ can be constructed by removing } (n-a) \text{ boxes from distinct columns of } \lambda^+, \\
0 & \text{otherwise.} 
\end{cases} \]  

(3)

The decomposition of induced representations \( \text{Ind}_{B_n \times B_{n-1}}^{B_n} V(\lambda^+, \lambda^-) \) are described by Geck–Pfeiffer [GP00, Lemma 6.1.9]:

\[ \text{Ind}_{B_n \times B_{n-1}}^{B_n} V(\lambda^+, \lambda^-) = \bigoplus_{\lambda^+} V(\lambda^+, \lambda^-) + \bigoplus_{\lambda^-} V(\lambda^+, \lambda^-) \]  

(4)

summed over all \( \lambda^+ \) that can be constructed by adding a single box to \( \lambda^+ \), and all \( \lambda^- \) that can be constructed by adding a single box to \( \lambda^- \). By iteratively applying this law to the trivial \( B_{n-m} \)-module \( k \), we find:

\[ \text{Ind}_{B_{n-m}}^{B_n} k = \text{Ind}_{B_{n-m-1}}^{B_{n-m}} \cdots \text{Ind}_{B_{n-m}}^{B_{n-1}} V((n-m), \emptyset) \]

\[ = \bigoplus_{\lambda^+, \lambda^-} V(\lambda^+, \lambda^-) \]  

(5)

summed over \( V(\lambda^+, \lambda^-) \) with multiplicity equal to the number of ways that the double partition \( (\lambda^+, \lambda^-) \) can be built up from \( ((n-m), \emptyset) \) by adding one box at a time to either partition. There are no restrictions on columns, though the addition of each box must form a valid double partition.

Restriction from \( B_n \) to \( S_n \).

The restriction of a \( B_n \)-representation \( V(\lambda^+, \lambda^-) \) to \( S_n \subseteq B_n \) is

\[ \text{Res}_{S_n}^{B_n} V(\lambda^+, \lambda^-) = \bigoplus_{\lambda} C_{\lambda^+, \lambda^-}^{\lambda} V_{\lambda} \]  

(6)

where \( C_{\lambda^+, \lambda^-}^{\lambda} \) again is the Littlewood–Richardson coefficient. See Geck–Pfeiffer [GP00, Lemma 6.1.4].

2.1.3 The even-signed permutation group \( D_n \)

We described a representation of the hyperoctahedral group \( \varepsilon : B_n \to \mathbb{Z}/2\mathbb{Z} \) that counts the number of \(-1\)'s (mod 2) appearing in a signed permutation matrix \( w \). The kernel of this map is the index–2 normal subgroup \( D_n \) of \( B_n \), the
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even-signed permutation group. If we classify elements of $B_n$ by cycle type as in Definition 2.1, the subgroup $D_n$ comprises exactly those elements of $B_n$ with an even number of negative cycles.

The rational representation theory of $D_n$. The representation theory of $D_n$ is given, for example, in [GP00, Chapter 5.6]. The irreducible representations derive from those of $B_n$. Given an irreducible representation $V_{(\lambda, \nu)}$ of $B_n$, the restriction to the action $D_n$ decomposes as either one or two distinct irreducible representations. When $\lambda \neq \nu$, the two irreducible $B_n$–representations $V_{(\lambda, \nu)}$ and $V_{(\nu, \lambda)}$ restrict to the same representation of $D_n$; each distinct set of nonequal partitions $\{\lambda, \nu\}$ gives a different irreducible representation $V_{(\lambda, \nu)}$ of $D_n$. When $n$ is even, for any partition $\lambda \vdash \frac{n}{2}$, the irreducible $B_n$–representation $V_{(\lambda, \lambda)}$ restricts to a sum of two nonisomorphic irreducible $D_n$–representations of equal dimension. Thus, the irreducible representations of $D_n$ are classified by the set

$$\{\{\lambda, \nu\} | \lambda \neq \nu, |\lambda| + |\nu| = n\} \bigcup \{ (\lambda, \pm) | |\lambda| = \frac{n}{2}\},$$

with the ‘split’ irreducible representations $V_{(\lambda, +)}$ and $V_{(\lambda, -)}$ only occurring for even $n$.

The conjugacy classes of $D_n$. The structure of the conjugacy classes of $D_n$ was described by Young [You30], and more recently by Carter [Car72, Proposition 25] and [GP00, Proposition 3.4.12]. As with $B_n$, the conjugacy classes of $D_n$ are classified by signed cycle type, with one exception. When $n$ is even, the elements for which all cycles are positive and have even length are now split between two conjugacy classes, as follows:

Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition of $n$ with all parts $\lambda_i$ of even length. Then the elements

$$\alpha^+ := (1 \ 2 \ \cdots \ \lambda_1) (-1 \ -2 \ \cdots \ -\lambda_1) (1+\lambda_1 \ 2+\lambda_1 \cdots \lambda_\ell+\lambda_1)(-1-\lambda_1 \ -2-\lambda_1 \cdots -\lambda_2-\lambda_1) \cdots$$

and $\alpha^- := (1 \ -1)\alpha^+(1 \ -1)$

are representatives of the two conjugacy classes of elements with signed cycle type $(\lambda, \emptyset)$, which we will denote $(\lambda, +)$ and $(\lambda, -)$, respectively. In summary:

Proposition 2.3. (Classification of conjugacy classes of $D_n$) The conjugacy classes
of $D_n$, are classified by the set

$$\{(\lambda, \nu) \mid |\lambda| + |\nu| = n, \ \nu \text{ has an even number of parts};$$

$$\text{if } \nu = \emptyset \text{ then not all parts of } \lambda \text{ are even } \}$$

$$\coprod \{(\lambda, \pm) \mid |\lambda| = n, \ \text{all parts of } \lambda \text{ are even } \}$$

with the ‘split’ conjugacy classes $(\lambda, \pm)$ only occurring when $n$ is even.

2.2 Representation stability

In a precursor to their work on FI–modules, Church and Farb [CF13] define a form of stability for a sequence $\{V_n\}$ of $G_n$–representations, for various families of groups $G_n$ with inclusions $G_n \hookrightarrow G_{n+1}$, including the symmetric and hyperoctahedral groups. We recall their definitions, and additionally introduce a notion of stability for the even-signed permutation groups $D_n$.

For the symmetric groups $S_n$, in order to compare representations for different values of $n$, Church–Farb identify those irreducible representations associated to partitions of $n$ that differ only in their largest parts – that is, two irreducible representations are considered ‘the same’ if the Young diagram for one can be constructed by adding boxes to the top row of the Young diagram for the other.

Accordingly, for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of $m$, we write $V(\lambda)_n$ to denote the irreducible $S_n$–representation associated to

$$\lambda[n] := ((n-m), \lambda_1, \lambda_2, \ldots, \lambda_t)$$

whenever it is defined, that is,

$$V(\lambda)_n := \begin{cases} V_{\lambda[n]} & (n-m) \geq \lambda_1, \\ 0 & \text{otherwise.} \end{cases}$$

We call $\lambda[n] \vdash n$ the padded partition associated to $\lambda$.

Similarly, for the hyperoctahedral groups $B_n$, two double partitions are identified if they differ only in the largest part of the first partition. For a double partition $\lambda = (\lambda^+, \lambda^-)$ with $\lambda^+ \vdash \ell$ and $\lambda^- \vdash m$, we define

$$\lambda[n] := (\lambda^+[n-m], \lambda^-)$$
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to be the padded double partition associated to $\lambda = (\lambda^+, \lambda^-)$, and we write $V(\lambda)_n$ or $V(\lambda^+, \lambda^-)_n$ to denote the irreducible $B_n$–representation

$$V(\lambda)_n := \begin{cases} V_{\lambda[n]} & (n - m) \geq \lambda_1^+, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we introduce a stable notation for certain representations of the even-signed permutation groups $D_n$. Let $\lambda = (\lambda^+, \lambda^-)$ be a double partition with $\lambda^+ \vdash \ell$ and $\lambda^- \vdash m$. Then we write $V(\lambda)_n$ to denote the $D_n$–representation

$$V(\lambda)_n := \text{Res}^{B_n}_{D_n} V(\lambda)_n.$$

Explicitly, $V(\lambda)_n$ is the $D_n$–representation

$$V(\lambda)_n = \begin{cases} V(\lambda^+[n-m],\lambda^-) & (n - m) \geq \lambda_1^+ \text{ and } \lambda^+[n-m] \neq \lambda^-, \\ V(\lambda^-,+) \oplus V(\lambda^-,\lnot) & (n - m) \geq \lambda_1^+ \text{ and } \lambda^+[n-m] = \lambda^-, \\ 0 & \text{otherwise.} \end{cases}$$

We note that $V(\lambda)_n$ is an irreducible $D_n$–representation for all but at most one value of $n$.

**Definition 2.4. (Consistent sequence).** Let $\{V_n\}$ be a sequence of $G_n$–representations with maps $\phi_n : V_n \rightarrow V_{n+1}$. The sequence $\{V_n, \phi_n\}$ is consistent if $\phi_n$ is equivariant with respect to the $G_n$–action on $V_n$ and the $G_{n+1}$–action on $V_{n+1}$ under restriction to the subgroup $G_n \hookrightarrow G_{n+1}$.

**Definition 2.5. (Representation stability).** A consistent sequence $\{V_n, \phi_n\}$ of finite dimensional $G_n$–representations is representation stable if it satisfies three properties:

I. **Injectivity.** The maps $\phi_n : V_n \rightarrow V_{n+1}$ are injective, for all $n$ sufficiently large.

II. **Surjectivity.** The image $\phi_n(V_n)$ generates $V_{n+1}$ as a $k[G_{n+1}]$–module, for all $n$ sufficiently large.

III. **Multiplicities.** Decompose $V_n$ into irreducible $G_n$–representations:

$$V_n = \bigoplus_{\lambda} c_{\lambda,n} V(\lambda)_n.$$
For each \( \lambda \), the multiplicity \( c_{\lambda,n} \) of \( V(\lambda)_n \) is eventually independent of \( n \).

**Definition 2.6. (Uniform representation stability).** Let \( \{V_n, \phi_n\} \) be a representation stable sequence with the multiplicity \( c_{\lambda,n} \) constant for all \( n \geq N_\lambda \). The sequence \( \{V_n, \phi_n\} \) is uniformly representation stable if \( N = N_\lambda \) can be chosen independently of \( \lambda \).

### 3 \( \text{FI}_W \)-modules and related constructions

#### 3.1 The category \( \text{FI}_W \)

In this section we recall the main definitions and establish some notation. Let \( W_n \) denote the Weyl group \( S_n, D_n, \) or \( B_n \). In Definition 1.2 we defined the category \( \text{FI}_W \) with objects indexed by the natural numbers \( \mathbb{Z}_{\geq 0} \), and morphisms generated by its endomorphisms

\[
\text{End}(n) \cong W_n
\]

and the canonical inclusions

\[
I_n : n \mapsto (n + 1).
\]

Throughout this paper, we will let \( I_n \) denote this natural inclusion of sets, and write \( I_{m,n} : m \to n \) to denote the composite

\[
I_{m,n} := I_{n-1} \circ \ldots \circ I_m.
\]

Any \( \text{FI}_W \) morphism \( f : m \to n \) factors as the composite of \( I_{m,n} \) with some (signed) permutation \( \sigma \in W_n \).

**The stabilizer of \( I_{m,n} \).** The group \( W_n \) acts transitively on the set of morphisms \( \text{Hom}_{\text{FI}_W}(m, n) \) by postcomposition. We denote by \( H_{m,n} = H_{m,n}^W \) the stabilizer of \( I_{m,n} \) in \( W_n \). As depicted in Figure 4, the group \( H_{m,n} \) is the copy of \( W_{n-m} \subseteq W_n \) that pointwise fixes the image \( I_{m,n}(m) \subseteq n \).

**Remark 3.1.** (\( \text{Hom}_{\text{FI}_D}(m, n) = \text{Hom}_{\text{FI}_{BC}}(m, n) \) for \( m \neq n \)). When \( m < n \), the \( \text{FI}_D \) morphisms \( \text{Hom}_{\text{FI}_D}(m, n) \) may reverse an even or odd number of signs. These morphisms are by definition generated by \( I_{m,n} \) and \( \text{End}_{\text{FI}_D}(n) \cong D_n \).
For example, although
\[(1 - 1)(n - n) \in D_n,\]
the involution \((n - n)\) is in the stabilizer of \(I_{m,n}\). Thus
\[(1 - 1)(n - n) \circ I_{m,n} = (1 - 1) \circ I_{m,n} \in \text{Hom}_{FI_W}(m, n)\]
negates only \(\pm 1\). When \(m \neq n\), the set of \(\text{FI}_D\) morphisms \(f : m \to n\) coincides exactly with the set of \(\text{FI}_{BC}\) morphisms \(f : m \to n\).

### 3.2 \(\text{FI}_W\)–modules

Recall from Definition 1.3 that an \(\text{FI}_W\)–module over a ring \(k\) is a functor

\[V : \text{FI}_W \to k\text{–Mod}.\]

For a fixed family of Weyl groups \(W_n\) and ring \(k\), the set of all \(\text{FI}_W\)–modules over \(k\) form a category. A map of \(\text{FI}_W\)–modules \(F : V \to V'\) is a natural transformation, that is, it is a sequence of maps

\[F_n : V_n \to V'_n\]

that commute with the \(\text{FI}_W\) morphisms in the sense that

\[F_n \circ V(f) = V'(f) \circ F_m \quad \text{for every } f \in \text{Hom}_{\text{FI}_W}(m, n).\]

**Example 3.2.** The spaces \(V_n = \mathbb{Q}^n\) form an \(\text{FI}_W\)–module with the canonical action of \(W_n\) by (signed) permutation matrices, and the standard inclusions

\[(I_n)_* : \mathbb{Q}^n \hookrightarrow \mathbb{Q}^{n+1}.\]

**Example 3.3.** Church–Ellenberg–Farb showed in [CEF12, Proposition 2.56] that,
for any partition $\lambda$ of $n$, the sequence of $S_n$–representations $V_n = V(\lambda)_n$ admits and FI$_A$–module structure. We will show in Definition 4.32 and Proposition 4.33 that, analogously, for any double partition $\lambda = (\lambda^+, \lambda^-)$ of $n$, the sequence of $B_n$–representations $V_n = V(\lambda)_n$ admits a FI$_{BC}$–module structure. Restriction of this FI$_{BC}$–module to FI$_D$ gives the sequence of $D_n$–representations $V_n = V(\lambda)_n$ an FI$_D$–module structure.

**Recognizing FI$_W$–modules.** An FI$_W$–module gives a consistent sequence of $W_n$–representations in the sense of Definition 2.4, as the images of the natural inclusions $I_n$ give maps

$$\phi_n = (I_n)_*: V_n \to V_{n+1}$$

compatible with the action of $W_n = \text{End}(\mathfrak{n})$. Not all consistent sequences arise from FI$_W$–modules, however. The following lemma gives necessary and sufficient conditions for a consistent sequence $\{V_n, \phi_n\}$ of $W_n$–representations to have the structure of an FI$_W$–module.

**Lemma 3.4.** (FI$_W$–modules vs. consistent sequences). A consistent sequence $\{V_n, \phi_n\}$ of $W_n$–representations can be promoted to an FI$_W$–module with $\phi_n = (I_n)_*$ if and only if, for all $m, n$, the stabilizer

$$H_{m,n} := \text{Stab}(I_{m,n}) \cong W_{n-m}$$

acts trivially on the image of $I_{m,n}(V_m) \subseteq V_n$.

**Proof.** An element of $\tau \in H_{m,n}$ by definition satisfies $\tau \circ I_{m,n} = I_{m,n}$, so given any FI$_W$–module $V$ these elements necessarily act trivially on the image

$$(I_{m,n})_*(V_m) \subseteq V_n.$$

Conversely, consider a consistent sequence $\{V_n, \phi_n\}$ of $W_n$–representations. Define

$$\phi_{m,n} := \phi_{n-1} \circ \cdots \circ \phi_m.$$

Given any $f \in \text{Hom}_{FI_W}(m, n)$, we can factor

$$f = \sigma \circ I_{m,n} \quad \text{for some } \sigma \in W_n.$$
We can realize \{V_n, \phi_n\} as an FI\_W–module by assigning

\[ V : f \mapsto f_* := \sigma \circ \phi_{m,n}; \]

the condition of the lemma is precisely the condition needed to ensure that this assignment is well-defined, independent of choice of factorization of \( f \). It is straightforward to check that the consistency of the sequence \{V_n, \phi_n\} ensures that the assignment \( f \mapsto f_* \) respects composition.

The result of Lemma 3.4 was proven for FI\_A–modules by Church–Ellenberg–Farb [CEF12, Lemma 2.1]; they show that a consistent sequence \{V_n, \phi_n\} of \( S_n \)–representations can be promoted to an FI\_A–module if and only if for all \( m \leq n \), \( \sigma, \sigma' \in S_n \), and \( v \in V_n \) lying in the image of \( V_m \),

\[ \sigma|_{\{1,2,\ldots,m\}} = \sigma'|_{\{1,2,\ldots,m\}} \implies \sigma(v) = \sigma'(v). \]

**Example 3.5. (The regular representations do not form an FI\_W–module).** When \( W_n \) is any of \( S_n \), \( B_n \), or \( D_n \), the sequence of regular representations

\[ V_n := k[W_n] \]

is a consistent sequence that is not an FI\_W–module. In each case, for example, the permutation that transposes \( n \) and \( (n-1) \) acts nontrivially on the image \( I_{(n-2),n}(V_{n-2}) \), violating the conditions of Lemma 3.4.

**Example 3.6. (Alternating and sign representations do not form FI\_W–modules).** A second example: The sequence of alternating representations \( V_n \cong k \) of the symmetric groups \( S_n \), or its pullbacks to \( B_n \) or \( D_n \), give a consistent sequence with no FI\_W–module structure. Again, the 2-cycle that transposes \( n \) and \( (n-1) \) acts nontrivially on the image \( I_{(n-2),n}(V_{n-2}) \). For similar reasons, the sign representations \( \varepsilon \) defined in Section 2.1.2 provide a consistent sequence of \( B_n \)–representations with no FI\_BC–module structure.

In summary: to verify that a sequence \{V_n, \phi_n\} has FI\_W–module structure, we must check two conditions. The sequence must be consistent in the sense of Definition 2.4, and it must satisfy the condition on stabilizers described in Lemma 3.4.

**Some additional definitions.** Following the model of [CEF12], we define:
Definition 3.7. (co–FI\(_W\)–modules). A co–FI\(_W\)–module over a ring \(k\) is a functor from the dual category \(\text{FI}^\text{op}_W\) to \(k\)–Modules.

Definition 3.8. (FI\(_W\)–space; co–FI\(_W\)–spaces). An FI\(_W\)–space (respectively co–FI\(_W\)–space) is a functor \(X\) from FI\(_W\) (respectively FI\(_W^\text{op}\)) to the category Top of topological spaces. We similarly define (co–)FI\(_W\)–spaces up to homotopy as functors to \(\text{hTop}\), the homotopy category of topological spaces.

For fixed integer \(i > 0\) and ring \(k\), composing the above functors \(X\) with the homology or cohomology functors \(H_i(\quad; k)\) or \(H^i(\quad; k)\) realizes the sequence of \(i\)th (co)homology groups of spaces \(X(n)\) as an FI\(_W\)– or co–FI\(_W\)–module.

### 3.3 The FI\(_W\)–modules \(M_W(m)\) and \(M_W(U)\)

In analogy to [CEF12, Definition 2.5], we define the FI\(_W\)–modules \(M_W(m)\). These are in a sense the ‘free’ finitely generated FI\(_W\)–modules; we will see in Proposition 3.17 that every finitely generated FI\(_W\)–module is a quotient of a sum of FI\(_W\)–modules of this form. This property will be critical to our development of the theory of FI\(_W\)–modules.

Definition 3.9. (The FI\(_W\)–module \(M_W(m)\)). Define \(M_W(m)\) to be the FI\(_W\)–module such that \(M_W(m)_n\) is the \(k\)-module with basis \(\text{Hom}_{\text{FI}_W}(m, n)\) and an action of \(W_n\) by post-composition.

Since \(W_n\) acts transitively on \(\text{Hom}_{\text{FI}_W}(m, n)\), we can identify the \(W_n\)–set \(\text{Hom}_{\text{FI}_W}(m, n)\) with the cosets of the stabilizer

\[ H_{m,n} := \text{Stab}(I_{m,n}) \cong W_{n-m} \subseteq W_n. \]

This gives an isomorphism of \(W_n\)–representations

\[ M_W(m)_n \cong \text{Ind}_{W_{n-m}}^{W_n} k \]

where \(k\) has a trivial \(W_n\) action. Over a field of characteristic zero, the decomposition of these representations are described in Pieri’s rules; see Equation (5) for the hyperoctahedral formula.

Observe

\[ M_W(m)_n = 0 \quad \text{when} \ n < m. \]
The first nonzero degree \( n = m \) is the regular representation

\[
M_W(m)_m \cong k[W_m].
\]

In general, \( M_W(m)_n \) can be considered the permutation representation of \( \mathcal{W}_n \) on the set of \( m \)-tuples

\[
\left( f(1), f(2), \ldots, f(m) \right) \subseteq n
\]

that designate the images of the FI\(_W\) morphisms \( f : m \to n \).

**Example 3.10.** (\( M_W(0) \) and \( M_W(1) \)) The FI\(_W\)-module \( M_W(0) \) is the sequence of trivial representations

\[
M_W(0)_n \cong k.
\]

The FI\(_A\)-module \( M_A(1) \) is the sequence of canonical \( S_n \)-representations as permutation matrices. Over characteristic zero, in the notation of Section 2.2, we get the following decomposition:

\[
M_A(1)_n \cong V(\emptyset)_n \oplus V(\emptyset) \quad \text{for all } n.
\]

The FI\(_{BC}\)-module \( M_{BC}(1) \) is the sequence of \((2n)\)-dimensional representations of \( B_n \) permuting a basis \( \{e_1, e_{-1}, \ldots, e_n, e_{-n}\} \). Over characteristic zero, \( M_{BC}(1)_n \) decomposes as follows.

\[
M_{BC}(1)_n = V(\emptyset, \emptyset)_n \oplus V(\emptyset, \emptyset)_n \oplus V(\emptyset, \emptyset)_n \quad \text{for all } n.
\]

Here,

\[
V(\emptyset, \emptyset)_n = \langle (e_1 - e_{-1}), \ldots, (e_n - e_{-n}) \rangle
\]

is the canonical \( B_n \)-representation by signed permutation representations, and

\[
V(\emptyset, \emptyset)_n \oplus V(\emptyset, \emptyset)_n = \langle (e_1 + e_{-1}), \ldots, (e_n + e_{-n}) \rangle
\]

is the pullback of the canonical \( S_n \) permutation representation. It is an exercise to verify that these decompositions are consistent with Pieri’s rule, Equation (5).

The representation \( M_D(1)_1 \) is trivial, but for \( n > 1 \) the \( D_n \)-representation \( M_D(1)_n \) is the restriction of the \( B_n \)-representation \( M_{BC}(1)_n \) described above.
Remark 3.11. Recall from Remark 3.1 that
\[ \text{Hom}_{\text{FI}}(m, n) = \text{Hom}_{\text{FI}_{BC}}(m, n) \quad \text{whenever } m \neq n. \]
There is therefore an isomorphism if \( D_n \)-representations
\[ M_D(m)_n \cong \text{Res}_{D_n}^B M_{BC}(m)_n \quad \text{whenever } m \neq n. \]
These isomorphisms will be crucial to our study of induction of \( \text{FI}_D \)-modules in Section 3.6.

3.3.1 An adjunction

Definition 3.12. Let \( \mathcal{W}_m \)-Rep denote the category of \( \mathcal{W}_m \)-representations over a ring \( k \). For each fixed integer \( m \geq 0 \), analogous to the definition of \( \pi_m \) given by Church–Ellenberg–Farb [CEF12], we define the forgetful functor
\[ \pi_m : \text{FI}_W\text{-Mod} \to \mathcal{W}_m\text{-Rep} \]
\[ V \mapsto V(m). \]
and, for each integer \( m \geq 0 \), we define the functor
\[ \mu_m : \mathcal{W}_m\text{-Rep} \to \text{FI}_W\text{-Mod} \]
\[ U \mapsto M_W(m) \otimes_{k[W_m]} U. \]
As in [CEF12, Proposition 2.6], we note that since
\[ M_W(m)_n \cong \text{Ind}_{W_{n-m}}^{W_n} k \cong k[W_n/W_{n-m}], \]
we can equivalently describe \( \mu_m \) by the formula:
\[ \mu_m : \mathcal{W}_m\text{-Rep} \to \text{FI}_W\text{-Mod} \]
\[ (\mu_m(U))_n = \begin{cases} 
0 & n < m, \\
\text{Ind}_{W_m \times W_{n-m}}^{W_n} U \boxtimes k & n \geq m.
\end{cases} \]
where \( \boxtimes \) denotes the external tensor product, and \( k \) denotes the trivial \( W_{n-m} \)-representation.
Proposition 3.13. The functor

\[ \mu_m : \mathcal{W}_m\text{-Rep} \to \mathcal{FI}_W\text{-Mod} \]

is the left adjoint to

\[ \pi_m : \mathcal{FI}_W\text{-Mod} \to \mathcal{W}_m\text{-Rep}. \]

The proof of the adjunction follows from the same argument given for [CEF12, Proposition 2.6], by considering any Weyl group \( \mathcal{W}_m \) in place of the symmetric group \( S_m \).

We remark that

\[ \mu_m(k[\mathcal{W}_m]) = M_{\mathcal{W}}(m) \otimes_{k[\mathcal{W}_m]} k[\mathcal{W}_m] \cong M_{\mathcal{W}}(m). \]

More generally, if \( U \) is a finite-dimensional \( \mathcal{W}_m \)-representation, we denote \( \mu_m(U) \) by \( M_{\mathcal{W}}(U) \). Following [CEF12, Definition 2.7], we extend the functor \( M_{\mathcal{W}} \) to \( \bigoplus_m \mathcal{W}_m\text{-Rep} \).

Definition 3.14. Define \( M_{\mathcal{W}} \) to be the map

\[ M_{\mathcal{W}} : \bigoplus_m \mathcal{W}_m\text{-Rep} \to \mathcal{FI}_W\text{-Mod} \]

\[ U_m \mapsto \mu_m(U_m) \]

3.4 Generation of \( \mathcal{FI}_W\text{-modules} \)

Church–Ellenberg–Farb defined notions of span, finite generation, and degree of generation for \( \mathcal{FI} \)-modules, which apply equally in the more general context of \( \mathcal{FI}_W \)-modules. These definitions are summarized below.

Definition 3.15. (Span; Generating set). If \( V \) is an \( \mathcal{FI}_W \)-module, and \( S \) is a subset of the disjoint union \( \bigsqcup V_n \), then the span of \( S \), denoted \( \text{span}_V(S) \), is the minimal \( \mathcal{FI}_W \)-submodule of \( V \) containing the elements of \( S \). We call \( \text{span}_V(S) \) the sub-\( \mathcal{FI} \)-module generated by \( S \).

Recall from Definition 1.4 that an \( \mathcal{FI}_W \)-module \( V \) is finitely generated if there is a finite set of elements

\[ S = \{v_1, \ldots, v_l\} \subseteq \bigsqcup V_n \]
such that \( \text{span}_V(S) = V \). Moreover \( V \) is generated in degree \( \leq m \) if

\[
V = \text{span}_V(\prod_{i=0}^{m} V_i).
\]

We call the minimum such \( m \) the degree of generation of \( V \), if it exists.

**Example 3.16.** The \( \text{FI}_W \)-module \( M_W(m) \) is generated in degree \( m \) by the identity map

\[
\text{id}_m \in \text{Hom}_{\text{FI}_W}(m, m), \quad \text{the basis for } M_W(m)_m.
\]

More generally, given a nonzero \( W_m \)-representation \( U \), the \( \text{FI}_W \)-module

\[
M_W(U) := M_W(m) \otimes_{k[W_m]} U
\]

is generated in degree \( m \) by \( M_W(U)_m = U \).

Just as in [CEF12, Remark 2.13, Proposition 2.16], the finitely generated \( \text{FI}_W \)-modules are precisely those which admit a surjection by an \( \text{FI}_W \)-module of the form \( \oplus_i M_W(m_i) \).

**Proposition 3.17.** An \( \text{FI}_W \)-module is finitely generated in degree \( \leq m \) if and only if it admits a surjection \( \oplus_i M_W(m_i) \rightarrow V \) for some finite sequence of integers \( \{m_i\} \), with \( m_i \leq m \) for each \( i \).

**Proof.** Given any finitely generated \( \text{FI}_W \)-module \( V_n \), with generators \( v_1, \ldots, v_\ell \), with \( v_i \in V_{m_i} \), the map

\[
\bigoplus_{i=1}^\ell M_W(m_i) \rightarrow V
\]

\[
f \mapsto f_*(v_i) \quad f \in \text{Hom}_W(m_i, n), \quad \text{the basis for } M_W(m_i)_n
\]

is the desired surjection of \( \text{FI}_W \)-modules.

Conversely, the image of an \( \text{FI}_W \)-module

\[
\bigoplus_{i=1}^\ell M_W(m_i)
\]

under an \( \text{FI}_W \)-module map is generated by the images of the identity morphisms \( \{\text{id}_{m_i}\}_{i=1}^\ell \).
3.4 Generation of FI\(_W\)–modules

Given an FI\(_W\)–module \(V\), any \(n\), and any \(v \in V_n\), then we have a surjective map of FI\(_W\)–modules

\[ M_W(n) \twoheadrightarrow \text{Span}_V(\{v\}) \subseteq V \quad \text{given by } f \mapsto f_* (v). \]

Moreover, any map \(M_W(n) \rightarrow V\) can be described in this way by taking \(v\) to be the image of \(\text{id}_n \in M_W(n)_n\). This observation is a form of Yoneda lemma for the category of FI\(_W\)–modules.

**Remark 3.18.** (\(M_W(U) \rightarrow \text{Span}(U)\)). Given an FI\(_W\)–module \(V\), and \(W_m\) subrepresentation \(U\) of \(V_m\), then by an argument as in Proposition 3.17, the FI\(_W\)–module

\[ M_W(U) := U \otimes_{k[W_m]} M_W(m) \]

surjects onto the span of \(U\) in \(V\).

In [CEF12, Proposition 2.17], Church–Ellenberg–Farb describe the compatibility of degree of generation, and finite generation, with short exact sequences of FI–modules. Their results hold for FI\(_W\)–modules:

**Proposition 3.19.** Let \(0 \rightarrow U \rightarrow V \rightarrow Q \rightarrow 0\) be a short exact sequence of FI\(_W\)–modules. If \(V\) is generated in degree \(\leq m\) (resp. finitely generated), then \(Q\) is generated in degree \(\leq m\) (resp. finitely generated). Conversely, if both \(U\) and \(Q\) are generated in degree \(\leq m\) (resp. finitely generated), then \(V\) is generated in degree \(\leq m\) (resp. finitely generated).

These statements can be shown by considering images or lifts of an appropriate generating set.

**Definition 3.20.** (Finite Presentation). A finitely generated FI\(_W\)–module \(V\) is **finitely presented** with generator degree \(g\) and relation degree \(r\) if there is a surjection

\[ \bigoplus_{i=1}^g M_W(m_i)^{\oplus b_i} \twoheadrightarrow V \]

with a kernel finitely generated in degree at most \(r\).

The Noetherian property, proved in Section 4.3 below, implies that all finitely generated FI\(_W\)–modules are in fact finitely presented.
3.4 Generation of $\text{FI}_W$–modules

3.4.1 The functor $H_0$

In analogy with [CEF12, Definition 2.18], we define a functor

$$H_0 : \text{FI}_W\text{-Mod} \to \bigoplus_m W_m\text{-Rep}$$

with the property that

$$H_0(M_W(U))_m = U_m,$$

that is, $H_0$ is a left inverse to $M_W$. As in [CEF12], we will see in Section 4.6 that additionally

$$M_W(H_0(V)) = V$$

when $V$ has the additional structure of an $\text{FI}_W\sharp$–module.

**Definition 3.21. (The Functor $H_0$).** Given an $\text{FI}_W$–module $V$, we define the functor $H_0$ by

$$H_0 : \text{FI}_W\text{-Mod} \to \bigoplus_m W_m\text{-Rep}$$

$$(H_0(V))_n = V_n / \left( \text{span}_k \left( \bigoplus_{k < n} V_k \right) \right)$$

The spaces $(H_0(V))_n$ are a minimal set of $W_n$–representations generating the $\text{FI}_W$–module $V$. As noted in [CEF12], these representations vanish for $n > m$ if and only if $V$ is generated in degree $\leq m$, and moreover $V$ is finitely generated if and only if $H_0(V_n)$ is a finitely generated $k$–module.

We can put an $\text{FI}_W$–module structure on the $W_n$–representations $(H_0(V))_n$ by letting $I_n$ act by 0 for all $n$. We denote this $\text{FI}_W$–module by $H_0(V)^{\text{FI}_W}$.

There is a natural surjection

$$V \twoheadrightarrow H_0(V)^{\text{FI}_W}.$$ 

Note that we could equivalently characterize the $\text{FI}_W$–module $H_0(V)^{\text{FI}_W}$ as the largest quotient of $V$ with the property that all $\text{FI}_W$ morphisms

$$f : m \to n \quad \text{with} \quad m \neq n$$

act by 0: in any such quotient, all images $f_*(V_m) \subseteq V_n$ must necessarily be 0.
Remark 3.22. \( (M_W(H_0(V)) \twoheadrightarrow V) \). Let \( V \) be an \( \text{FI}_W \)-module over characteristic zero. As suggested by Remark 3.18, there is a (noncanonical) surjection

\[ M_W(H_0(V)) \twoheadrightarrow V. \]

The proof given in [CEF12, Proposition 2.43] for \( \text{FI}_A \)-modules applies directly to the cases of \( \text{FI}_{BC} \) and \( \text{FI}_D \).

### 3.5 Restriction of \( \text{FI}_W \)-modules

The natural embeddings \( S_n \hookrightarrow D_n \hookrightarrow B_n \) give inclusions of categories

\[ \text{FI}_A \hookrightarrow \text{FI}_D \hookrightarrow \text{FI}_{BC}, \]

which define restriction operations on the corresponding \( \text{FI}_W \)-modules. These operations, together with the \textit{induction} functors that we will define in Section 3.6, will be our main tools for studying the interactions of the three families of Weyl groups.

Notably, we will show in Proposition 3.24 that restriction of \( \text{FI}_W \)-modules preserves the property of finite generation. We will use this result to establish the Noetherian property for \( \text{FI}_D \) and \( \text{FI}_{BC} \)-modules, Theorem 4.22. We use Proposition 3.24 again to prove Theorem 5.19, which states that the dimensions of finitely generated \( \text{FI}_D \) and \( \text{FI}_{BC} \)-modules over arbitrary fields are eventually polynomial. In both cases, Proposition 3.24 reduces the proofs to the type A case, which are established by Church–Ellenberg–Farb–Nagpal [CEF12].

Definition 3.23. (Restriction). Given a family of inclusions \( \mathcal{W}_n \hookrightarrow \mathcal{W} \), any \( \text{FI}_{\mathcal{W}} \)-module \( V \) inherits the structure of an \( \text{FI}_W \)-module by restricting the functor \( V \) to the subcategory \( \text{FI}_W \) in \( \text{FI}_{\mathcal{W}} \). We call this construction \( \text{Res}_W^V \), the \textit{restriction} of \( V \) to \( \text{FI}_W \).

Proposition 3.24. (Restriction preserves finite generation). For each family of Weyl groups \( \mathcal{W} \subseteq \mathcal{W}_n \), the restriction \( \text{Res}_W^V \) of a finitely generated \( \text{FI}_W \)-module \( V \) is finitely generated as an \( \text{FI}_W \)-module. Specifically,

1. Given an \( \text{FI}_{BC} \)-module \( V \) finitely generated in degree \( \leq m \), \( \text{Res}^B_C V \) is finitely generated as an \( \text{FI}_A \)-module in degree \( \leq m \).

2. Given an \( \text{FI}_{BC} \)-module \( V \) finitely generated in degree \( \leq m \), \( \text{Res}^D_C V \) is finitely generated as an \( \text{FI}_D \)-module in degree \( \leq m \).
3.5 Restriction of $\text{Fl}_W$-modules

3. Given an $\text{Fl}_D$-module $V$ finitely generated in degree $\leq m$, $\text{Res}^D_A V$ is finitely generated as an $\text{Fl}_A$-module in degree $\leq (m + 1)$.

Proof of Proposition 3.24(1). The key to the proof is the fact that for each $m, n$ with $m \leq n$, the actions of $S_n$ on the right and $B_m$ on the left are together transitive on the cosets $B_n/B_{n-m} \cong \text{Hom}_{\text{Fl}_{BC}}(m, n)$.

We first prove the claim for the $\text{Fl}_{BC}$-module $M_{BC}(m)$ for fixed $m$. Recall that $M_{BC}(m)_n = \text{Span}_k \{ e_f | f \in \text{Hom}_{\text{Fl}_{BC}}(m, n) \}$; we identify $\text{Hom}_{\text{Fl}_{BC}}(m, n)$ with the set of inclusions

$$f : \{ \pm 1, \pm 2, \ldots, \pm m \} \to \{ \pm 1, \pm 2, \ldots, \pm n \}$$

satisfying $f(-c) = -f(c)$ for all $c = \pm 1, \ldots, \pm m$.

Take as generating set the basis

$$S = \{ e_w | w \in \text{Hom}_{\text{Fl}_{BC}}(m, m) \cong B_m \}$$

for $M_{BC}(m)_m$, and take any inclusion $f \in \text{Hom}_{\text{Fl}_{BC}}(m, n)$; we will show $e_f$ is in the $\text{Fl}_A$ span of $S$. There is some $\sigma^{-1} \in S_n$ so that the postcomposite $\sigma^{-1} \circ f$ has image

$$\{ \pm 1, \pm 2, \ldots, \pm m \} \subseteq \{ \pm 1, \pm 2, \ldots, \pm n \}.$$

Additionally, there is some $w^{-1} \in B_m$ so that the precomposite $\sigma^{-1} \circ f \circ w^{-1}$ is the natural inclusion $I_{m,n}$. Thus $f$ factors as $f = \sigma \circ I_{m,n} \circ w$, and so

$$e_f = (\sigma_* \circ (I_{m,n})_*)(e_w)$$

is in the $\text{Fl}_A$-span of $S$.

It follows that the restriction of $M_{BC}(m)$ is finitely generated as an $\text{Fl}_A$-module by degree--$m$ generators.

Now, let $V$ be any finitely generated $\text{Fl}_{BC}$-module. By Proposition 3.17, there is an $\text{Fl}_{BC}$-module map

$$\bigoplus_{a=0}^m M_{BC}(a)^{\otimes b_a} \to V$$

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which consists of a sequence of surjections of the underlying $k$–modules. Considered as a map of $\text{FI}_A$–modules, this same map is a surjection

$$\text{Res}^{BC}_A\left(\bigoplus_{a=0}^m M_{BC}(\alpha)^{\oplus b_a}\right) = \bigoplus_{a=0}^m (\text{Res}^{BC}_A M_{BC}(\alpha))^{\oplus b_a} \twoheadrightarrow \text{Res}^{BC}_A V.$$ 

It follows that $\text{Res}^{BC}_A V$ is finitely generated over $\text{FI}_A$ by generators of degree $\leq m$. \hfill \Box

**Proof of Proposition 3.24(2).** This follows from Proposition 3.24(1), which implies that $\text{Res}^{BC}_D V$ is finitely generated in degree $\leq m$ by the action of $\text{FI}_A \subseteq \text{FI}_D$. \hfill \Box

**Proof of Proposition 3.24(3).** The proof of Proposition 3.24(3) is similar to that of Proposition 3.24(1). However, $B_m$ acts transitively by precomposition on the subset of maps in $\text{Hom}_{\text{FI}_D}(m, n)$ with a given image, whereas when $n > m$ there are two orbits of maps in $\text{Hom}_{\text{FI}_D}(m, n)$ with a given image under the action of $D_m$ – the orbit of those maps which reverse an even number of signs, and the orbit of those maps which reverse an odd number. For this reason, $\text{Res}^{D}_D M_D(m)$ is not generated in degree $\leq m$.

We again begin with the $\text{FI}_D$–module $M_D(m)$. We have

$$M_D(m)_n = \text{Span}_k \{ e_f \mid f \in \text{Hom}_{\text{FI}_D}(m, n) \};$$

where each $f$ is an inclusion

$$f : \{\pm1,\pm2,\ldots,\pm m\} \hookrightarrow \{\pm1,\pm2,\ldots,\pm n\} \quad \text{satisfying } f(-c) = -f(c)$$

for all $c = \pm1,\ldots,\pm m$.

If $m = n$, then $f$ must reverse an even number of signs; if $n < m$, then $f$ can
reverse an even or odd number of signs.

Take as generating set the bases for $M_D(m)_m$ and $M_D(m)_{m+1}$,

$$S = \{ e_w \mid w \in \text{Hom}_{R_D}(m, m) \text{ or } \text{Hom}_{R_D}(m, (m + 1)) \}.$$  

Suppose $n > m$, and let $f \in \text{Hom}_{R_D}(m, n)$. Take $\sigma^{-1} \in S_n$ so that $\sigma^{-1} \circ f$ has image

$$\{ \pm 1, \pm 2, \ldots, \pm m \} \subseteq \{ \pm 1, \pm 2, \ldots, \pm n \}.$$  

Then there is some $g \in \text{Hom}_{R_D}(m, m+1)$ so that $\sigma^{-1} \circ f = I_{m+1,n} \circ g$, and so

$$e_f = \sigma_{\ast} \circ (I_{m+1,n})_{\ast}(e_g).$$  

Thus $M_D(m)$ is generated by the generators $S$ in degrees $m$ and $(m + 1)$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (Dm) at (0,0) {$\{ \pm 1, \pm 2, \pm 3 \}$};
\node (Dm1) at (3,0) {$\{ \pm 1, \pm 2, \pm 3, \pm 4 \}$};
\node (S) at (6,0) {$\{ \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6 \}$};
\node (Immn1) at (1.5,-1.5) {$I_{m,m+1}$};
\node (Imm1n) at (4.5,-1.5) {$I_{m+1,n}$};
\node (g) at (1.5,0) {$g$};
\node (sigma) at (4.5,0) {$\sigma \in S_n$};
\node (f) at (0,0) {$f = \sigma \circ I_{m+1,n} \circ g$};
\draw[->] (Dm) to (g);
\draw[->] (g) to (Dm1);
\draw[->] (Dm1) to (sigma);
\draw[->] (sigma) to (S);
\draw[->] (g) to (f);
\end{tikzpicture}
\caption{Hom$_{R_D}(m, n) = S_n \cdot I_{m+1,n} \cdot \text{Hom}_{R_D}(m, (m + 1))$}
\end{figure}

Again, any finitely generated FI$_D$–module $V$ admits a surjection by some FI$_D$–module of the form

$$\bigoplus_{a=0}^m M_D(a)^{b_a}.$$  

It follows that $\text{Res}_A^D V$ is generated by the images of generating sets for $\text{Res}_A^D M_D(m_i)$ for each $i$, each in degree $\leq (m + 1)$.  

\begin{remark}
(Res$^D_{S_n}$ does not preserve ‘surjectivity’ of consistent sequences).
We note the FI$_{BC}$–module structure in Proposition 3.24(1) is necessary. Consider, for example, the sequence of regular representations $k[B_n]$ and inclusions

$$k[B_{n-1}] \hookrightarrow k[B_n].$$  

This sequence does not have an FI$_{BC}$–module structure, but does form a consistent sequence in the sense of Definition 2.4. It ‘surjects’ in the sense of Def-
3.6 Induction of FI\(_W\)-modules

In Section 3.5 we analyzed the restriction functor on FI\(_W\)-modules. Just as with group representations, restriction has a left adjoint, a procedure for inducing FI\(_A\) and FI\(_D\)-modules up to functors from FI\(_D\) or FI\(_BC\). This construction, which uses the theory of Kan extensions, was described to us by Peter May. In this section we will define induction of FI\(_W\)-modules and establish some properties of this operation.

For present purposes, we are particularly interested in studying induction from FI\(_D\) to FI\(_BC\). This will enable us to use our theory of FI\(_BC\)-modules to recover results for finitely generated FI\(_D\)-modules, including representation stability (Section 4.4) and existence of character polynomials (Section 5.3). The results for FI\(_BC\) make extensive use of the branching rules for the hyperoctahedral group, but the D\(_n\) analogues of these rules are more troublesome. The properties of induction established here make our main results accessible in type D.

Remark 3.27. (The naive definition of induction). We note that the naive “pointwise” definition of induction of FI\(_W\)-modules is not well defined: If we were to define Ind\(_W^n V\) so that in degree \(n\) it were the representation

\[
\text{Ind}_W^n V_n,
\]

then the resulting sequence would not in general have the structure of an FI\(_W^n\)-module.

Consider, for example, the sequence of trivial D\(_n\)-representations, with
$V_n = k$ for all $n$, and all $\text{FI}_D$ maps acting as isomorphisms. Then

$$\text{Ind}^{B_n}_{D_n} k \cong k \oplus k^\varepsilon$$

is a sum of the trivial representation $k$ and the one-dimensional sign representation $k^\varepsilon$ associated to the character

$$\varepsilon : B_n \to B_n/D_n \cong \{\pm 1\}.$$

This cannot be a $\text{FI}_{BC}$-module since, for example, the signed permutation $(-n\ n) \in B_n$ acts by multiplication by $-1$ on a summand of the image $I_{m,n}(V_m) \subseteq V_n$ for any $m < n$, in violation of Lemma 3.4.

There is, however, a natural way to define induction of $\text{FI}_{W}$-modules, using a standard category-theoretic universal construct: the left Kan extension. General constructions and properties of Kan extensions are given in Mac Lane [ML98, Chapter 10] (see also notes by Riehl [Rie09]), which we briefly outline. Then in Definition 3.29 below we will define induction of $\text{FI}_{W}$-modules using a concrete description of these constructions as they apply to the categories $\text{FI}_{W}$.

Given a subcategory $\text{FI}_{W} \subseteq \text{FI}_{W}$, and an $\text{FI}_{W}$-module $V$, we denote by $\text{Ind}_{W}^{W} V$ the left Kan extension of $V$ along the inclusion of categories. This is an $\text{FI}_{W}$-module

$$\text{Ind}_{W}^{W} V : \text{FI}_{W} \to k\text{-Mod}$$

The induction map

$$\text{Ind}_{W}^{W} : \text{FI}_{W}\text{-Mod} \to \text{FI}_{W}\text{-Mod}$$

is functorial on the functor category of $\text{FI}_{W}$-modules. In particular, given two $\text{FI}_{W}$-modules $V$ and $W$ and a map of $\text{FI}_{W}$-modules $F : V \to W$, there is a corresponding map of $\text{FI}_{W}$-modules

$$\text{Ind}_{W}^{W} F : \text{Ind}_{W}^{W} V \to \text{Ind}_{W}^{W} W;$$
assigned in a manner that respects composition of $\text{FI}_W$-module maps.

The functor $\text{Ind}_W^W$ is the left adjoint to $\text{Res}_W^W$, and satisfies the associated properties recognizable from the familiar adjunction for induction and restriction of group representations. For any $\text{FI}_W$-module $V$, there is a canonical map of $\text{FI}_W$-modules

$$\eta_V : V \rightarrow \text{Res}_W^W(\text{Ind}_W^W V)$$

defined by the unit map associated to $\text{Ind}_W^W$ and $\text{Res}_W^W$, the natural transformation

$$\eta : id \rightarrow (\text{Res}_W^W \text{Ind}_W^W).$$

Given any $\text{FI}_{pp}$-module $U$ and $\text{FI}_W$-module map $V \rightarrow \text{Res}_W^W U$, there exists a unique map of $\text{FI}_W$-modules

$$\alpha : \text{Ind}_W^W V \rightarrow U$$

such that the following diagram commutes.

\[
\begin{array}{ccc}
\text{Res}_W^W (\text{Ind}_W^W V) & \xrightarrow{\eta} & \text{Res}_W^W U \\
V & \xrightarrow{\text{Res}_W^W \alpha} & \text{Res}_W^W U
\end{array}
\]

This correspondence defines a bijection

$$\left\{ \text{FI}_W \text{-Module Maps} \right\} \quad \longleftrightarrow \quad \left\{ \text{FI}_{pp} \text{-Module Maps} \right\}$$

which is natural in the inputs $V$ and $U$.

We can describe the induced functor explicitly, as in Mac Lane [ML98, Chapter 10]. Before giving any further details, we will motivate this construction with a somewhat nonstandard characterization of induction of group representations.

**Remark 3.28.** (Induction as a coequalizer). Given a group $G$, a subgroup $H \subseteq G$, and a $H$-representation $V$, we could define the usual induced representation
3.6 Induction of $\text{FI}_W$–modules

J C H Wilson

Induction of $\text{FI}_W$–modules

Induction $\text{Ind}_W^G V$ as the coequalizer

$k[G] \otimes_k k[H] \otimes_k V \xrightarrow{\phi} k[G] \otimes_k V \xrightarrow{\psi} \text{Ind}_W^G V$

$f \otimes g \otimes v \xrightarrow{\phi} f \otimes g(v)$

$f \otimes g \otimes v \xrightarrow{\psi} f \circ g \otimes v$

Our formula for the induced functor $\text{Ind}_W^G V$ generalizes this construction from $k$–modules to the categorical setting.

Following Mac Lane [ML98, Chapter 10.4], we define the $W_n$–representation $(\text{Ind}_W^G V)^n$ as a certain coend, the coequalizer of two maps $\phi$ and $\psi$.

$\bigoplus_{p \leq q \leq n} M_W(q)_n \otimes_k M_W(p)_q \otimes_k V_p \xrightarrow{\phi} \bigoplus_{r \leq n} M_W(r)_n \otimes_k V_r \xrightarrow{\psi} (\text{Ind}_W^G V)^n$

$f \otimes g \otimes v \xrightarrow{\phi} f \otimes g(v)$

$f \otimes g \otimes v \xrightarrow{\psi} f \circ g \otimes v$

In parallel with the $k$–modules $\text{Ind}_W^G V := k[G] \otimes_k k[H] V$, the induced functor $\text{Ind}_W^G V$ is sometimes called a tensor product of functors over a category and written $\text{FI}_W \otimes_{\text{FI}_W} V$. We summarize its construction in the following definition.

**Definition 3.29. (Induction).** Given an $\text{FI}_W$–module $V$, and an inclusion of categories $\text{FI}_W \hookrightarrow \text{FI}_W$, we define the induced $\text{FI}_W$–module $\text{Ind}_W^G V$ by

$$(\text{Ind}_W^G V)^n = \bigoplus_{r \leq n} M_W(r)_n \otimes_k V_r \bigg/ \langle \ f \otimes g_\star(v) = (f \circ g) \otimes v \quad | \quad g \text{ is an } \text{FI}_W \text{ morphism} \rangle.$$

with the action of $h \in \text{Hom}_{\text{FI}_W}(m, n)$ by

$$h_\star : g \otimes v \mapsto (h \circ g) \otimes v.$$
We emphasize that induction is left adjoint to restriction, and satisfies the naturality properties described above.

We observe that $(\text{Ind}_W^W V)_n$ is, in fact, a quotient of the $W_n$-representation $\text{Ind}_W^W (V_n)$. Given a pure tensor

$$g \otimes v \quad \text{with } g : r \to n \text{ and } v \in V_r,$$

we can factor $g = \tilde{g} \circ I_{r,n}$ for some $\tilde{g} \in W_n$, and so

$$g \otimes v = \tilde{g} \otimes I_{r,n}(v) \in M_{W_n}(n) \otimes V_n.$$

Hence $(\text{Ind}_W^W V)_n$ is a quotient of the induced representation

$$\text{Ind}_W^W (V_n) \cong M_{W_n}(n) \otimes V_n \bigg/ \langle f \otimes g(v) = f \circ g \otimes v \mid g \in W_n \rangle,$$

modulo additional relations which require the stabilizer $H_{\ell,n} = \text{Stab}(I_{\ell,n})$ to act trivially on the image of $(\text{Ind}_W^W V)_\ell$ in $(\text{Ind}_W^W V)_n$, and so overcome the obstructions described in Remark 3.27.

Proposition 3.30. (Induction respects finite generation). If $V$ is an $F^W_n$-module finitely generated in degree $\leq m$, then the induced module $\text{Ind}_W^W V$ is a finitely generated $F^W_n$-module in degree $\leq m$.

Proof of Proposition 3.30. If $V$ is generated by a finite set of elements $v_m \in V_m$, then it is easily seen that $\text{Ind}_W^W V$ is generated by the images of the elements

$$\text{id}_m \otimes v_m \in M_{W_n}(m) \otimes_k V_m$$

in $(\text{Ind}_W^W V)_m$.

Proposition 3.31. $(\text{Ind}_W^W M_W(m) \cong M_{W_n}(m))$. Given categories $F^W \subseteq F^W_n$ and any integer $m$, there is an isomorphism of $F^W_n$-modules

$$\text{Ind}_W^W M_W(m) \cong M_{W_n}(m).$$

In other words, the functor $\text{Ind}_W^W$ preserves represented functors.
3.6 Induction of FI\(_W\)–modules

**Proof of Proposition 3.31.** It is straightforward to verify that the map

\[
\bigoplus_{m \leq r \leq n} M_{\mathbb{P}}(r)_n \otimes_k M_{\mathbb{W}}(m)_r \to M_{\mathbb{P}}(m)_n
\]

\[g \otimes f \mapsto g \circ f \quad \text{with} \quad f \in \text{Hom}_{\mathbb{W}}(m, r), \quad g \in \text{Hom}_{\mathbb{P}}(r, n)\]

factors through an isomorphism of FI\(_W\)–modules

\[(\text{Ind}_{\mathbb{W}}^W M_{\mathbb{W}}(m))_n \xrightarrow{\cong} M_{\mathbb{P}}(m)_n. \quad \Box\]

**Corollary 3.32.** Given an FI\(_W\)–module finitely generated in degree \(\leq m\), the natural surjection of FI\(_W\)–modules of Proposition 3.17

\[S : \bigoplus_a M_{\mathbb{W}}(a)^{ba} \to V\]

can be promoted to a surjection of FI\(_\mathbb{P}\)–modules

\[(\text{Ind}_{\mathbb{W}}^W S) : \bigoplus_a M_{\mathbb{P}}(a)^{ba} \to \text{Ind}_{\mathbb{P}}^W V.\]

**Proposition 3.33.** \((V \hookrightarrow (\text{Res}_{\mathbb{W}}^W \text{Ind}_{\mathbb{W}}^W V))\). Given an FI\(_W\)–module \(V\) and an inclusion of categories FI\(_W\) \hookrightarrow FI\(_\mathbb{P}\), the natural map of FI\(_W\)–modules

\[V \to \text{Res}_{\mathbb{W}}^W \text{Ind}_{\mathbb{W}}^W V\]

\[v \in V_n \mapsto \text{id}_n \otimes v \in M_{\mathbb{P}}(n)_n \otimes V_n\]

is injective.

It should not be surprising that the map

\[V \to (\text{Res}_{\mathbb{W}}^W \text{Ind}_{\mathbb{W}}^W V)\]

is injective since, heuristically, the relations defining the quotient \((\text{Ind}_{\mathbb{W}}^W V)_n\) come from relations already imposed on \(V\) by its FI\(_W\)–module structure.

It seems that there ought to be a proof of Proposition 3.33 – injectivity of the unit map \(\eta_V\) – using formal properties of the adjunction, and the fact that injectivity holds for the represented functors \(M_{\mathbb{W}}(m)\). We have found this formal proof to be elusive, however. A proof specific to the three categories FI\(_W\)
is given below.

**Proof of Proposition 3.33.** We first address the case where \( W_n \) is the symmetric group \( S_n \), and \( B_n \) or \( D_n \). To prove the proposition, it suffices to show that the underlying maps of \( k \)-modules

\[
V_n \rightarrow \left( \text{Res}^W_A \text{Ind}^W_A V \right)_n
\]

are injective; to do so we will construct left inverses \( \tilde{L}_n \) of the \( k \)-module maps.

Fix \( n \). We define a map of \( k \)-modules

\[
L : \bigoplus_{r \leq n} M_W(r)_n \otimes_k V_r \rightarrow V_n
\]

as follows: For any pure tensor of the form

\[
g \otimes v \in M_W(r)_n \otimes_k V_r \quad \text{with FI}_W \text{ morphism } g : r \rightarrow n,
\]

we can factor \( g \) as \( g = \sigma \circ \tilde{g} \), where

\[
\sigma \in (\mathbb{Z}/2\mathbb{Z})^n \subset B_n \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n,
\]

and \( \tilde{g} \) is a (uniquely determined) FI\(_A\)-morphism. We assign

\[
L : \bigoplus_{r \leq n} M_W(r)_n \otimes_k V_r \rightarrow V_n
\]

\[
g \otimes v \mapsto \tilde{g}(v)
\]

Because the assignment \( g \mapsto \tilde{g} \) respects composition,

\[
L(f \circ h \otimes v) = L(f \otimes h(v)) \quad \text{for all FI}_A \text{ morphisms } h.
\]

Hence, \( L \) factors through the quotient \( (\text{Ind}^W_A V)_n \)

\[
L : \bigoplus_{r \leq n} M_W(r)_n \otimes_k V_r \rightarrow (\text{Ind}^W_A V)_n \tilde{L}_n \rightarrow V_n.
\]

The composite map of \( k \)-modules
3.6 Induction of $\text{FI}_W$–modules

\[
V_n \longrightarrow (\text{Ind}_A^W V)_n = (\text{Res}_A^W \text{Ind}_A^W V)_n \xrightarrow{\tilde{L}} V_n
\]

\[
v \mapsto id_n \otimes v \Longrightarrow id_n(v) = v
\]

is the identity, which implies that the natural map of $\text{FI}_A$–modules

\[
V_n \longrightarrow (\text{Res}_A^W \text{Ind}_A^W V)_n
\]

is injective.

Next we address the induction of $\text{FI}_D$–modules up to $\text{FI}_{BC}$–modules. We will use the same outline as in the first case, but there are additional subtleties: unlike with $S_n$, the group $D_n$ is not a quotient of $B_n$, and there is no way to associate an $\text{FI}_D$ morphism $\tilde{g}$ to each $\text{FI}_{BC}$ morphism $g$ in a manner that respects composition.

We will, however, still define a left inverse $\tilde{L}$ as above. Again, for each fixed $n$, we define a map of $k$–modules

\[
L : \bigoplus_{r \leq n} M_{BC}(r)_n \otimes_k V_r \longrightarrow V_n
\]

on the pure tensors

\[
g \otimes v \quad \text{with} \quad g : r \to n \text{ and } v \in V_r
\]

as follows. If $r \neq n$, then

\[
\text{Hom}_{\text{FI}_{BC}}(r, n) \cong \text{Hom}_{\text{FI}_{D}}(r, n)
\]

by Remark 3.1, and so $g(v)$ is a well-defined element of $V_n$. In this case we define

\[
L : g \otimes v \mapsto g(v) \in V_n.
\]

Similarly, suppose $g \in \text{End}_{\text{FI}_{BC}}(n)$ but $v$ is in the image of $V_r$ for some $r < n$, say, $v = f(u)$ for some $\text{FI}_D$–morphism $f : r \to n$. Then

\[
g \circ f \in \text{Hom}_{\text{FI}_{BC}}(r, n) = \text{Hom}_{\text{FI}_D}(r, n),
\]
and \((g \circ f)(u)\) is a well-defined element of \(V_n\). In this case we define

\[
L : g \otimes f(u) \mapsto (g \circ f)(u) \in V_n.
\]

Both assignments satisfy

\[
L(g \circ h \otimes v) = L(g \otimes h(v)) \quad \text{for all FI}_D\text{-morphisms } h.
\]

Finally, suppose \(g \otimes v\) is a pure tensor with

\[
g \in B_n \cong \text{End}_{\text{FI}_B}(n) \quad \text{and} \quad v \in V_n \text{ such that } v \notin \text{Span}_V(V_{n-1}).
\]

Since \(D_n \subseteq B_n\) has index two, either

\[
g \in D_n, \quad \text{or} \quad (-n) \circ g \in D_n.
\]

We define

\[
L(g \otimes v) = \begin{cases} 
  g(v) & \text{if } g \in D_n, \\
  (-n) \circ g)(v) & \text{if } g \notin D_n.
\end{cases}
\]

In this case, too,

\[
L(g \otimes h(u)) = L(g \circ h \otimes u) \quad \text{for all FI}_D\text{-morphisms } h:
\]

since, by assumption on \(v\), we can write \(v = h(u)\) only if \(h\) is an element of \(\text{End}_{\text{FI}_D}(n) \cong D_n\), and so \(g \in D_n\) if and only if \(g \circ h \in D_n\).

Once again, \(L\) will factor through the quotient \((\text{Ind}_D^{BC} V)_n\), and gives the desired left inverse. The map

\[
V \rightarrow \text{Res}_D^{BC} \text{ Ind}_D^{BC} V
\]

is injective, as desired.

Having established Proposition 3.33, we can now prove a critical fact about finitely generated \(\text{FI}_D\)-modules.

**Proposition 3.34.** \((V_n \cong (\text{Res}_D^{BC} \text{ Ind}_D^{BC} V)_n \text{ for } n \text{ large})\). Suppose \(V\) is an \(\text{FI}_D\)-module finitely generated in degree \(\leq m\). Then

\[
V_n \cong (\text{Res}_D^{BC} \text{ Ind}_D^{BC} V)_n
\]
is an isomorphism of $D_n$–representations for all $n > m$. In particular, every finitely generated $FI_D$–module $V$ is, for $n$ greater than its degree of generation, the restriction of an $FI_{BC}$–module.

**Proof of Proposition 3.34.** The map

$$V_n \rightarrow (\text{Res}_D^{BC} \text{Ind}_D^{BC} V)_n$$

is injective by Proposition 3.33, and so it suffices to show that this map is surjective for $n > m$. Since $V$ is finitely generated in degree $\leq m$, by Proposition 3.17 we have a surjection of $FI_D$–modules

$$S : \bigoplus_{a=0}^{m} M_{D}(a)^{\oplus b_a} \twoheadrightarrow V.$$

Inducing both sides up to $FI_{BC}$ gives a surjective map

$$\bigoplus_{a=0}^{m} M_{BC}(a)^{\oplus b_a} = \text{Ind}_D^{BC} \left( \bigoplus_{a=0}^{m} M_{D}(a)^{\oplus b_a} \right) \xrightarrow{\text{Ind}_D^{BC} S} \text{Ind}_D^{BC} V$$

where the first equality follows from Proposition 3.31. By naturality of the unit map $\eta$, these maps fit together into a commutative diagram

$$\bigoplus_{a=0}^{m} M_{D}(a)^{\oplus b_a} \xrightarrow{S} V$$

$$\bigoplus_{a=0}^{m} \text{Res}_D^{BC} M_{BC}(a)^{\oplus b_a} \xrightarrow{\text{Res}_D^{BC} \text{Ind}_D^{BC} S} \text{Res}_D^{BC} \text{Ind}_D^{BC} V$$

By Remark 3.11, the left vertical arrow

$$\bigoplus_{a=0}^{m} M_{D}(a)^{\oplus b_a} \rightarrow \bigoplus_{a=0}^{m} \text{Res}_D^{BC} M_{BC}(a)^{\oplus b_a}$$

is an isomorphism of $D_n$–representations for $n > m$, and so the composite

$$\bigoplus_{a=0}^{m} M_{D}(a)^{\oplus b_a} \cong \bigoplus_{a=0}^{m} \text{Res}_D^{BC} M_{BC}(a)^{\oplus b_a} \rightarrow (\text{Res}_D^{BC} \text{Ind}_D^{BC} V)_n$$

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is surjective for $n > m$. By commutativity, the right vertical arrow

$$V_n \rightarrow (\text{Res}^B_D \text{Ind}^B_D V)_n$$

must also surject for these values of $n$, which proves the claim. \hfill \Box

**Remark 3.35.** Let $V$ be an $\text{FI}_D$–module. In Proposition 3.34 we proved the isomorphism of $D_n$–representations

$$V_n \simeq (\text{Res}^B_D \text{Ind}^B_D V)_n \quad \text{for } n > m.$$

We note that the analogous statements about $\text{Ind}^D_A$ and $\text{Ind}^B_A$ are false. This is apparent from the $\text{FI}_A$–modules $M_A(m)$ with $m > 0$. We have

$$\text{rank}_k M_A(m)_n = [S_n : S_{n-m}] = \frac{n!}{(n-m)!}.$$ 

In contrast, by Proposition 3.31, we have

$$\text{Ind}^D_A M_A(m) = M_D(m) \quad \text{and} \quad \text{Ind}^B_A M_A(m) = M_{BC}(m),$$

with

$$\text{rank}_k M_D(n)_m = [D_n : D_{n-m}] = \begin{cases} 2^{m-1}m! & n = m \\ 2^m n! & n > m \\ (n-m)! \end{cases}$$

$$\text{rank}_k M_{BC}(n)_m = [B_n : B_{n-m}] = \frac{2^m n!}{(n-m)!}.$$ 

We see that $M_A(m)_n$ is a proper sub–$k[S_n]$–module of $\text{Res}^D_A \text{Ind}^D_A M_A(m)_n$ and $\text{Res}^B_A \text{Ind}^B_A M_A(m)_n$ for all $n$.

**Remark 3.36.** We remark that the concept of central stabilization defined by Putman [Put12] can be understood in terms of categorical induction. Let $\mathcal{N}_2$ denote the full subcategory of $\text{FI}_A$ on the objects $(N-1)$ and $N$. Let $\mathcal{N}_3$ denote the full subcategory on $(N-1)$, $N$, and $(N+1)$. Any $S_{N-1}$–equivariant map $\phi_{N-1}$ between an $S_{N-1}$–representation $V_{N-1}$ and an $S_N$–representation $V_N$ can be realized as the map

$$\phi_{N-1} = (I_{N-1,N})_*$$
of a functor

\[ V : \mathcal{N}_2 \to k\text{-Mod}. \]

Then Putman’s central stabilization of \( \phi_{N-1} \) is precisely the \( S_{N+1} \)-representation obtained from the induced functor \( \text{Ind}_{N_2}^{N_3} V \),

\[ \mathcal{C}(V_{N-1} \xrightarrow{\phi_{N-1}} V_N) \cong \left( \text{Ind}_{N_2}^{N_3} V \right)_{N+1} \]

and the associated central stabilization sequence is the induced \( \text{FI}_A \)-module \( \text{Ind}_{N_2}^{N_3} V \).

### 3.6.1 Coinduction of \( \text{FI}_W \)-modules

Given an inclusion of categories \( \text{FI}_W \subseteq \text{FI}_{\overline{W}} \) and an \( \text{FI}_W \)-module \( V : \text{FI}_W \to k\text{-Mod} \), we have defined the induced \( \text{FI}_{\overline{W}} \)-module \( \text{Ind}_{\overline{W}}^W V \). There is also a dual construction, the coinduced \( \text{FI}_{\overline{W}} \)-module \( \text{Coind}_{\overline{W}}^W V \), described to us by Peter May. Although we will not use this construction in this paper, we include a brief discussion for theoretical interest.

**The right adjoint to restriction.** The functor \( \text{Coind}_{\overline{W}}^W \) : \( \text{FI}_W \to k\text{-Mod} \) is the right Kan extension of \( V \) along the inclusion \( \text{FI}_W \to \text{FI}_{\overline{W}} \).

\[
\begin{array}{ccc}
\text{FI}_W & \xrightarrow{V} & k\text{-Mod} \\
\downarrow & & \downarrow \\
\text{FI}_{\overline{W}} & \xrightarrow{\text{Coind}_{\overline{W}}^W} & \text{FI}_{\overline{W}} \text{-Mod}
\end{array}
\]

The map \( V \mapsto \text{Coind}_{\overline{W}}^W V \) is functorial on the category of \( \text{FI}_W \)-modules

\[ \text{Coind}_{\overline{W}}^W : \text{FI}_W\text{-Mod} \to \text{FI}_{\overline{W}}\text{-Mod}. \]

As in Mac Lane [ML98], we construct \( \text{Coind}_{\overline{W}}^W V \) as an end, the equalizer of maps \( \phi \) and \( \psi \):

\[ (\text{Coind}_{\overline{W}}^W V)_n \longrightarrow \prod_r \text{Hom}_k(M_{\overline{W}}(n)_r, V_r) \xrightarrow{\phi} \prod_{p,q} \text{Hom}_k(M_W(q)_p \otimes k M_{\overline{W}}(n)_q, V_p) \]

\[ \xrightarrow{\psi} \prod_{r,s} \text{Hom}_k(M_{\overline{W}}(s)_r \otimes k M_W(n)_s, V_s) \]
3.6 Induction of $\mathcal{F}_W$--modules

$$\phi : \left\{ \begin{array}{c}
F_r : M_{\mathcal{F}_W}(n)_r \to V_r \\
f \mapsto F_r(f)
\end{array} \right\} \mapsto \left\{ \phi(F_r) : M_{\mathcal{F}_W}(q)_r \otimes_k M_{\mathcal{F}_W}(n)_q \to V_r \\
g \otimes h \mapsto F_r(g \circ h)
\right\}$$

$$\psi : \left\{ \begin{array}{c}
F_r : M_{\mathcal{F}_W}(n)_r \to V_r \\
f \mapsto F_r(f)
\end{array} \right\} \mapsto \left\{ \psi(F_r) : M_{\mathcal{F}_W}(r)_p \otimes_k M_{\mathcal{F}_W}(n)_r \to V_p \\
g \otimes h \mapsto g \ast (F_r(h))
\right\}$$

Concretely,

$$(\text{Coind}_{\mathcal{F}_W}^W V)_n = \text{Span}_k \left\{ F = (F_r) \in \prod_{r \geq n} \text{Hom}_k(M_{\mathcal{F}_W}(n)_r, V_r) \right\}
| F_q(g \circ f) = g(F_p(f)) \text{ for all } \mathcal{F}_W--\text{morphisms } g : p \to q \right\},$$

that is, $(\text{Coind}_{\mathcal{F}_W}^W V)_n$ is the $k$--span of the natural transformations of $\mathcal{F}_W$--modules

$$F : \text{Res}_{\mathcal{F}_W}^W M_{\mathcal{F}_W}(n) \to V.$$

$\text{Coind}_{\mathcal{F}_W}^W V$ inherits its (covariant) $\mathcal{F}_W$--module structure from the (contravariant) action of an $\mathcal{F}_W$--morphism $f : m \to n$ by precomposition on $M_{\mathcal{F}_W}(-)_r$, combined with the (contravariant) functor $\text{Hom}_k(-, V_r)$.

$$f_* : (\text{Coind}_{\mathcal{F}_W}^W V)_m \to (\text{Coind}_{\mathcal{F}_W}^W V)_n
\left\{ \begin{array}{c}
F_r : M_{\mathcal{F}_W}(m)_r \to V_r \\
g \mapsto F_r(g)
\end{array} \right\} \mapsto \left\{ f_* (F_r) : M_{\mathcal{F}_W}(n)_r \to V_r \\
h \mapsto F_r(h \circ f)
\right\}.$$ 

The induction functor $\text{Ind}_{\mathcal{F}_W}^W$ is the left adjoint to the restriction functor $\text{Res}_{\mathcal{F}_W}^W$, and $\text{Coind}_{\mathcal{F}_W}^W$ is the right adjoint to $\text{Res}_{\mathcal{F}_W}^W$. The functor $\text{Coind}_{\mathcal{F}_W}^W$ comes with a family of $\mathcal{F}_W$--module maps

$$\eta : \text{Res}_{\mathcal{F}_W}^W \text{Coind}_{\mathcal{F}_W}^W V \to V$$

that define a natural bijection

$$\left\{ \begin{array}{c}
\mathcal{F}_W--\text{Module Maps} \\
\text{Res}_{\mathcal{F}_W}^W U \to V
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{c}
\mathcal{F}_W--\text{Module Maps} \\
U \to \text{Coind}_{\mathcal{F}_W}^W V
\end{array} \right\}.$$
Constraints on finitely generated $\text{FI}_W$–modules  J C H Wilson

Relationship to induction; Examples. For representations of finite groups, induction and coinduction are equivalent. This is not the case for induction and coinduction of $\text{FI}_W$–modules. For example, consider the trivial $\text{FI}_W$–module $M_W(0)$. We claim

$$\text{Coind}^{BC}_D M_D(0) = M_{BC}(0).$$

In contrast, $\text{Coind}^{BC}_A M_A(0)$ is the sequence of $B_n$–representations

$$\left(\text{Coind}^{BC}_A M_A(0)\right)_n = k\left[(\mathbb{Z}/2\mathbb{Z})^n\right].$$

where $B_n$ acts on its normal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ by conjugation. Not only is $\text{Coind}^{BC}_A M_A(0)$ not the trivial $\text{FI}_{BC}$–module, it is infinitely generated, with generators in every dimension.

Questions. These examples raise a number of questions: How can we interpret $\text{Coind}^{W}_W V$? For which $\text{FI}_W$–modules $V$ will $\text{Coind}^{W}_W V$ be finitely generated? For which $V$ are $\text{Ind}^{W}_W V$ and $\text{Coind}^{W}_W V$ isomorphic as $\text{FI}_W$–modules? What structure does coinduction reveal regarding the relationships between the categories of $\text{FI}_W$–modules for the three families of groups?

4 Constraints on finitely generated $\text{FI}_W$–modules

Church–Ellenberg–Farb [CEF12] relate finite generation of an $\text{FI}_A$–module to certain constraints on the shape of the Young diagrams in the irreducible representations of each representation $V_n$. We develop analogous results for $\text{FI}_D$ and $\text{FI}_{BC}$.

4.1 The weight of an $\text{FI}_W$–module

Definition 4.1. (Weight). Let $k$ be a field of characteristic zero. Church–Ellenberg–Farb [CEF12, Definition 2.50] define the weight of an $\text{FI}_A$–module to be $\leq d$ if for every $n \geq 0$ and every irreducible constituent $V(\lambda)_n$ of $V_n$ has

$$|\lambda| \leq d$$

(in the notation described in Section 2.2). Similarly, we define the weight of a $B_n$–representation $V_n$ to be $\leq d$ if every irreducible representation $V(\lambda)_n =$
4.1 The weight of an \(\text{FI}_W\)-module

\[ V(\lambda^+, \lambda^-)_n \text{ in } V_n \text{ satisfies} \]
\[ |\lambda^+| + |\lambda^-| \leq d. \]

We define the weight of an \(\text{FI}_{BC}\)-module \(V\) to be \(\leq d\) if \(V_n\) has weight \(\leq d\) for each \(n\). We define the weight of an \(\text{FI}_D\)-module \(V\) as the weight of the \(\text{FI}_{BC}\)-module \(\text{Ind}^{\text{BC}}_{\text{D}} V\).

An \(\text{FI}_W\)-module \(V\) has finite weight if it is of weight \(\leq d\) for some \(d \geq 0\), and we call the minimum such \(d\) the weight of \(V\), \(weight(V)\). We say that the weight of a Young diagram \(\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_\ell)\) is

\[ \lambda_1 + \cdots + \lambda_\ell = |\lambda| - \lambda_0. \]

Over characteristic zero, the weight of submodules and quotients of \(V\) are at most \(\text{Weight}(V)\).

**Proposition 4.2.** Let \(k\) be a field of characteristic zero. Then (in the notation of Section 2.2), a \(\text{W}_n\)-representation \(V(\lambda)_n\) is contained in \(M_{\text{W}}(m)\) if and only if \(|\lambda| \leq m\). In type \(D\), \(M_{\text{D}}(m)\) decomposes completely into representations of the form \(V(\lambda)_n\).

In type \(A\) and \(B/C\), it is immediate that all representations over characteristic zero decompose into irreducible representations of the form \(V(\lambda)_n\). In type \(D\), this is precisely the statement that all ‘split’ irreducible representations occur in pairs \(V\{\mu, +\} \oplus V\{\mu, -\}\).

**Proof of Proposition 4.2.** The branching rules for the symmetric groups implies that

\[ V(\lambda)_n \text{ occurs in } M_{\text{A}}(m)_n \cong \text{Ind}^{\text{S}_n}_{\text{S}_{n-m}} k \]

if and only if \(\lambda[n]\) can be built from the partition \((n - m)\) by adding one box at a time; these are exactly those diagrams \(\lambda[n]\) with largest part \(n - |\lambda| \geq (n - m)\), equivalently, with \(|\lambda| \leq m\).

Similarly, by the branching rules for the hyperoctahedral group (Equation (5), Section 2.1.2),

\[ V(\lambda)_n = V(\lambda^+, \lambda^-)_n \text{ appears in } M_{\text{BC}}(m)_n \]

precisely when \((\lambda^+, \lambda^-)_n\) contains the double partition \(((n - m), \emptyset)\), that is, when the largest part of \(\lambda^+, n - (|\lambda^+| + |\lambda^-|)\), is at least \((n - m)\). We conclude that \(V(\lambda)_n\) is contained in \(M_{\text{BC}}(m)\) if and only if \(|\lambda^+| + |\lambda^-| \leq m\).
Finally, in type D, by Remark 3.11 we have

\[ M_D(m)_n = \begin{cases} k[D_m] & \text{if } n = m, \\
\text{Res}_D^{BC} M_{BC}(m)_n & \text{if } n > m, \end{cases} \]

When \( n = m \), all \( D_n \)-representations \( V(\lambda)_m \) necessarily satisfy \( |\lambda| < m \), and conversely every irreducible subrepresentation appears in the regular representation \( k[D_m] \) with multiplicity equal to its dimension. The split representations \( V(\lambda^-) \) and \( V(\lambda^+) \), being of equal dimension, occur in pairs. For \( n > m \), the result follows immediately from the identification \( M_D(m)_n \cong \text{Res}_D^{BC} M_{BC}(m)_n \) and the result in type B/C.

Theorem 4.2 (with Proposition 3.31 in Type D) imply:

**Corollary 4.3.** The FI\(_{W} \)-module \( M_W(m) \) has weight \( m \).

**Theorem 4.4. (Degree of generation bounds weight).** Suppose that \( V \) is an FI\(_{W} \)-module over a field of characteristic zero. If \( V \) is finitely generated in degree \( \leq m \), then \( \text{weight}(V) \leq m \).

**Proof of Theorem 4.4.** By Proposition 3.17, any FI\(_{W} \)-module \( V \) finitely generated in degree \( \leq m \) is a quotient of some FI\(_{W} \)-module of the form \( \bigoplus_{a=0}^{m} M_W(n)^{k_a} \). Therefore, we conclude Theorem 4.4 from Proposition 4.2 and (for FI\(_D \)-modules) Corollary 3.32.

Theorem 4.4 is proven for FI\(_A \)-modules in [CEF12, Proposition 2.51].

Theorem 4.4 strongly constrains which irreducible representations can occur in \( V_n \) once \( n \) is large relative to the degree of generation of \( V \). The following corollary gives some examples of irreducible components which are excluded.

**Corollary 4.5.** Suppose that \( V \) is an FI\(_{W} \)-module over a field of characteristic zero, generated in degree \( \leq m \).

- If \( W_n \) is \( S_n \), then for all \( n > (m + 1) \) the \( S_n \)-representation \( V_n \) cannot contain the alternating representation.
- If \( W_n \) is \( D_n \) or \( B_n \), then for all \( n > (m + 1) \) the \( W_n \)-representation \( V_n \) cannot contain the pullback of the alternating representation.
- If \( W_n \) is \( B_n \), then for all \( n > m \) the \( B_n \)-representation \( V_n \) cannot contain the ‘sign’ representation associated to the character \( \varepsilon : B_n \rightarrow B_n/D_n \cong \{ \pm 1 \} \).
4.1 The weight of an FI\(_W\)–module

Proof of Corollary 4.5. If \(V\) is an FI\(_W\)–module generated in degree \(\leq m\) as above, then \(\text{weight}(V) \leq m\) by Theorem 4.4. The alternating \(S_n\)–representation

\[ V(1,1,\ldots,1) \]

has weight \((n - 1)\), as does its pullback to \(B_n\)

\[ V((1,1,\ldots,1),\emptyset) \]

so neither representation can occur in \(V_n\) once \(n > (m + 1)\). The hyperoctahedral sign representation

\[ V(\emptyset,(n)) \]

has weight \(n\), so it can occur in \(V_n\) only when \(n \leq m\).

The alternating \(S_n\)–representation pulls back to the \(D_n\)–representation

\[ V\{(1,1,\ldots,1),\emptyset\}. \]

Suppose there existed a FI\(_D\)–module \(V\) wherein this pullback occurred in \(V_n\) for some \(n > (m + 1)\). By the classification of \(D_n\)–representations described in Section 2.1.3, this pullback \(D_n\)–representation is the restriction of either the \(B_n\)–representation \(V((1,1,\ldots,1),\emptyset)\) of weight \((n - 1)\) or the \(B_n\)–representation \(V(\emptyset,(1,1,\ldots,1))\) of weight \(n\); it does not occur in the restriction of any other \(B_n\)–representation. By Proposition 3.34, the FI\(_{BC}\)–module \(\text{Ind}_{BC}^D V\) must contain one of these two representations in degree \(n\), contradicting Theorem 4.4.

\[ \square \]

Remark 4.6. (Weight and Restriction from \(B_n\) to \(S_n\)). Consider a \(B_n\)–representation \(V(\lambda^+,\lambda^-)_n\) such that \(|\lambda^+| + |\lambda^-| = d\). By Formula (6), Section 2.1.2, its restriction to \(S_n\) is

\[ \text{Res}^{B_n}_{S_n} V(\lambda^+,\lambda^-) = \sum_{\lambda [n]} C_{\lambda^+,\lambda^-}^{\lambda} V(\lambda)_n. \]

The Littlewood–Richardson coefficient \(C_{\lambda^+,\lambda^-}^{\lambda}\) is nonzero only if and \(\lambda^+ \subseteq \lambda\), and so \(\lambda^+_0 \subseteq \lambda_0\). This implies that, for a FI\(_{BC}\)–module \(V\),

\[ \text{weight}(\text{Res}^{B_n}_{S_n} V) \leq \text{weight}(V). \]

Conversely, \(C_{\lambda^+,\lambda^-}^{\lambda} = 1\) for \(\lambda\) such that \(\lambda = \lambda^+_i + \lambda^-_i\). Thus, if the restriction of some FI\(_{BC}\)–module \(V\) yields an FI\(_A\)–module of weight \(\leq d\), then necessarily
weight(λ⁺) + weight(λ⁻) ≤ d for all irreducible summands V(λ⁺, λ⁻) in Vₙ for each n. These facts about Littlewood–Richardson coefficients can be found, for example, in Fulton [Ful97, Chapter 5].

**Proposition 4.7.** (Split representations do not occur in finitely generated FI₉-modules for n > 2m). Let k be a field of characteristic zero, and suppose that V is an FI₉-module over k finitely generated in degree ≤ m. Then for any n > 2m, the Dₙ-representation Vₙ does not contain any 'split' irreducible representations, that is, all of its irreducible components are of the form V(λ, µ) for λ ≠ µ.

**Proof of Proposition 4.7.** By Proposition 3.17, there is a surjection of FI₉-modules

\[ \bigoplus_{a=0}^{m} M_D(a)^{λ_a} \rightarrow V; \]

every irreducible component of Vₙ must appear in M_D(a)_n for some a ≤ m. Moreover, by Remark 3.11 we have an isomorphism of Dₙ-representations

\[ M_D(a)_n = \text{Res}_{D_n}^{B_n} M_{BC}(a)_n, \]

and so every irreducible component of Vₙ must appear in Res_{D_n}^{B_n} M_{BC}(a)_n for some a ≤ m.

The branching rules (Equation (5), Section 2.1.2) imply that the irreducible representation V(λ, µ) ⊆ M_{BC}(a)_n only if

\[ ((n - m), \emptyset) \subseteq ((n - a), \emptyset) \subseteq (λ, µ), \]

and so in particular

\[ |λ| \geq (n - m) > m \geq |µ| \quad \text{for all } n > 2m. \]

Thus, V(λ, µ) ⊆ M_{BC}(a)_n only if |λ| ≠ |µ|, and so by restriction to Dₙ we conclude that when n > 2m, Vₙ only contains irreducible components of the form V(λ, µ) for λ ≠ µ. □

### 4.2 Coinvariants and stability degree

**Shifted FI₉-modules.** The category FI₉ contains isomorphic copies of itself as proper subcategories. We use these inclusions to define the shifting operation on FI₉-modules. As in [CEF12, Section 2.4], by shifting and passing
4.2 Coinvariants and stability degree

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to coinvariants, we define the stability degree of an \( \text{FI}_W \)–module, and, in Lemma 4.20, we find a lower bound on the stability degree of a finitely presented \( \text{FI}_W \)–module. In Section 4.4, we will use this concept to prove the equivalence of finite generation of an \( \text{FI}_{BC} \)–module with representation stability in the sense of Church–Farb [CF13].

**Definition 4.8. (Shifts \( \Pi_{[-a]} : \text{FI}_W \to \text{FI}_W \))** For each \( a \geq 0 \), there are maps

\[
\begin{align*}
\{ \pm 1, \ldots, \pm n \} &\hookrightarrow \{ \pm 1, \ldots, \pm (n + a) \} \\
 d &
\mapsto (d + a)
\end{align*}
\]

These maps define functors

\[
\Pi_{[-a]} : \text{FI}_W \to \text{FI}_W
\]

\[
\begin{array}{c}
n \mapsto (n + a) \\
\{ f : m \to n \} \mapsto \{ \Pi_{[-a]}(f) : (m + a) \to (n + a) \}
\end{array}
\]

where

\[
\Pi_{[-a]}(f) \text{ maps } \begin{cases} d \mapsto d & \text{if } d \leq a, \\
(d + a) \mapsto (f(d) + a). \end{cases}
\]

Figure 7 gives a schematic of the functor \( \Pi_{[-2]} \).

**Figure 7:** The functor \( \Pi_{[-a]}(f) : \text{FI}_W \to \text{FI}_W \)

**Definition 4.9. (Shifted \( \text{FI}_W \)–modules).** Given an \( \text{FI}_W \)–module \( V : \text{FI}_W \to k \)–
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Mod, we define the shifted FI\(_W\)-module \(S_{+a}V\) by

\[S_{+a}V = V \circ \Pi^{-a}.\]

The \(W_n\)-representation \((S_{+a}V)_n\) is the restriction of \(V_{n+a}\) to the copy of \(W_n\) acting on \(\{\pm (1 + a), \ldots \pm (n + a)\} \subseteq (n + a)\).

**Definition 4.10. (Coinvariants functor \(\tau\)).** We define

\[\tau : \text{FI}_W\text{-Mod} \to \text{FI}_W\text{-Mod}\]

be the coinvariants functor, as follows: for an FI\(_W\)-module \(V\), let \(\tau V\) be the FI\(_W\)-module with

\[(\tau V)_n = (V_n)_{W_n} := k \otimes_k [W_n] V_n\]

That is, \((\tau V)_n\) is the largest quotient of \(V_n\) on which \(W_n\) acts trivially. When \(k\) is a field of characteristic zero, the map \(V_n \rightarrow (V_n)_{W_n}\) is the projection onto the invariant subspace \((V_n)_{W_n}\).

**Definition 4.11. (The graded \(k[T]\)-module \(\Phi_a(V)\)).** Fix an integer \(a \geq 0\). We define

\[\Phi_a : \text{FI}_W\text{-Mod} \rightarrow k[T]\text{-Mod}

\[V \mapsto \bigoplus_{n \geq 0} (\tau \circ S_{+a}V)_n = \bigoplus_{n \geq 0} (V_{n+a})_{W_n}\]

The action of \(T\) is by the maps \((V_{n+a})_{W_n} \rightarrow (V_{n+1+a})_{W_{n+1}}\) induced by the maps \((I_{n+a})_* : V_{n+a} \rightarrow V_{n+1+a}\).

**Remark 4.12.** Note that each graded piece \(\Phi_a(V)_n = (V_{n+a})_{W_n}\) has the structure of an \(W_a\)-module, and \(T\) acts \(W_a\)-equivariantly. Over characteristic zero, the multiplicity of a \(W_a\)-representation \(U\) in \((V_{n+a})_{W_n}\) is equal to the multiplicity of \(U \otimes k\) in the restriction \(\text{Res}_{W_a \times W_n}^{W_{n+a}} V_{n+a}\) given by the branching rules (Equation (3), Section 2.1.2.)

**Definition 4.13. (Injectivity degree; Surjectivity degree).** An FI\(_W\)-module has injectivity degree \(\leq s\) (respectively, surjectivity degree \(\leq s\)) if for every \(a \geq 0\) and for all \(n \geq s\), the map \(\Phi_a(V)_n \rightarrow \Phi_a(V)_{n+1}\) induced by \(T\) is injective (respectively, surjective). The minimum such \(s\) is called the injectivity degree (respectively, the surjectivity degree).
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Definition 4.14. (Stability degree). An \( \text{FI}_W \)-module has stability degree \( \leq s \) if for every \( a \geq 0 \) and \( n \geq s \), the map

\[
\Phi_a(V)_n \to \Phi_a(V)_{n+1}
\]

induced by \( T \) is an isomorphism of vector spaces; equivalently, an isomorphism of \( \mathcal{W}_a \)-representations. The stability degree is the minimum such \( s \); it is the maximum of injectivity and surjectivity degree.

Explicitly, \( V \) has stability degree \( \leq s \) if

\[
(V_{n+a})_{\mathcal{W}_n} \cong (V_{n+1+a})_{\mathcal{W}_{n+1}} \quad \text{for every } a \geq 0 \text{ and } n \geq s.
\]

Figure 8 shows an \( \text{FI}_W \)-module \( V \) with stability degree 3.

Remark 4.15. Let \( V \) be an \( \text{FI}_W \)-module with surjectivity degree \( \leq s \) and injectivity degree \( \leq t \). Since \( \Phi_a \) is right exact, quotients of \( V \) also have surjectivity degree \( \leq s \). Furthermore, when \( k \) contains \( \mathbb{Q} \), \( \Phi_a \) is left exact, and any submodule of \( V \) has injectivity degree \( \leq t \).

Remark 4.15 generalizes [CEF12, Remark 2.36]. The following proposition

![Figure 8: Stability degree 3](image-url)
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is proven in [CEF12, Proposition 2.37] for Fl\textsubscript{A}. We adapt their proof to Fl\textsubscript{BC} and Fl\textsubscript{D}.

**Proposition 4.16.** For \( m \geq 0 \), the Fl\textsubscript{W}–module \( M\text{W}(m) \) has injectivity degree 0 and surjectivity degree \( m \).

**Proof of Proposition 4.16.** Fix a \( a \geq 0 \). By definition, \((S+aM\text{W}(m))_n\) is the free vector space over \( \text{Hom}_{\text{FlW}}(m,(n+a)) \), where \( W_n \) acts by postcomposition, permuting \( \{ \pm(1+a), \ldots, \pm(n+a) \} \subseteq (n+a) \).

In type BC, two maps in \( \text{Hom}_{\text{FlBC}}(m,(n+a)) \) are in the same orbit if they restrict to the same function on their inverse images of \( \{ \pm 1, \ldots, \pm a \} \). Thus we can identify \( \Phi_a(M\text{BC}(m)) \) with the vector spaces with bases:

\[
n \neq 0, \quad B^{BC}_{\leq n} = \{ f : \{ \pm 1, \ldots, \pm m \} \to \{ \pm 1, \ldots, \pm a, \star \} |\quad \text{Away from } \star, \text{ } f \text{ injects and } f(-d) = -f(d); \quad |f^{-1}(\star)| \leq n \}
\]

\[
n = 0, \quad B^{BC}_{\leq 0} = \text{Hom}_{\text{FlBC}}(m,a).
\]

Type D, however, is slightly more subtle. When \( n > (m-a) \), then the orbit of a map \( g \in \text{Hom}_{\text{FlD}}(m,(n+a)) \) is no longer determined by the restriction of \( g \) to \( g^{-1}(\{ \pm 1, \ldots, \pm a \}) \). Since \( g \) can reverse an even or odd number of signs, but all elements of \( D_n \) only reverse an even number, the orbit of \( g \) will also depend on whether \( g \) reverses an even or odd number of signs of numerals in \( g^{-1}(\{ \pm(a+1), \ldots, \pm(n+a) \}) \).

Thus \( \Phi_a(M\text{D}(m)) \) are the vector spaces freely spanned by:

\[
n \neq 0, n > (m-a), \quad B^{D}_{\leq n} \text{ is two copies of } \\
\{ f : \{ \pm 1, \ldots, \pm m \} \to \{ \pm 1, \ldots, \pm a, \star \} |\quad \text{Away from } \star, \text{ } f \text{ injects and } f(-d) = -f(d); \quad |f^{-1}(\star)| \leq n \}
\]

\[
n \neq 0, n = m-a, \quad B^{D}_{\leq (m-a)} \text{ is one copy of } \\
\{ f : \{ \pm 1, \ldots, \pm m \} \to \{ \pm 1, \ldots, \pm a, \star \} |\quad \text{Away from } \star, \text{ } f \text{ injects and } f(-d) = -f(d); \quad |f^{-1}(\star)| \leq n \}
\]

\[
n = 0, \quad B^{D}_{\leq 0} = \text{Hom}_{\text{FlD}}(m,a)
\]
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In all cases, however, we have inclusions $B_{\leq n}^W \hookrightarrow B_{\leq (n+1)}^W$, and so $M_W^m$ has injectivity degree $0$. Moreover, once $n \geq m$, all maps $f$ automatically satisfy the condition on $|f^{-1}(\star)|$, and so $B_{\leq n}^W = B_{\leq (n+1)}^W$ in this range. We conclude that $M_W^m$ has surjectivity degree $m$. \hfill \Box

**Proposition 4.17.** If an $\text{FI}_W$–module $V$ is generated in degree $\leq m$, then $V$ has surjectivity degree $\leq m$.

Church–Ellenberg–Farb prove this result for $\text{FI}_A$ in [CEF12, Proposition 2.39].

**Proof of Proposition 4.17.** By Lemma 3.17, any $\text{FI}_W$–module $V$ generated in degree $\leq m$ admits a surjection from some $\text{FI}_W$–module $\bigoplus_{a=0}^{m} M_W(a)^{\otimes h_a}$, which has surjectivity degree $m$ by Proposition 4.16. The result follows from Remark 4.15. \hfill \Box

**Proposition 4.18.** Given a nonzero $W_m$–representation $U$, the $\text{FI}_W$–module $M_W(U)$ has injectivity degree $0$ and surjectivity degree $\leq m$.

**Proof of 4.18.** Since $M_W(U)$ is generated in degree $m$, it has surjectivity degree $\leq m$ by Proposition 4.17. Church–Ellenberg–Farb show that $M_A(U)$ has injectivity degree $0$ [CEF12, Proposition 2.38] by noting that (in the notation of the proof of Proposition 4.16), $k[B_{\leq n}^A]$ embeds as a $k[S_m]$–equivariant summand of $k[B_{\leq (n+1)}^A]$ and so the maps

$$\Phi_a(M_A(U))_n = \Phi_a(M_A(m))_n \otimes_{k[S_m]} U \rightarrow \Phi_a(M_A(m))_{n+1} \otimes_{k[S_m]} U = \Phi_a(M_A(U))_{n+1}$$

are injective for all $a, n$. Applying the same argument to $B_{\leq n}^{BC}$ and $B_{\leq n}^D$, we conclude that $M_{BC}(U)$ and $M_D(U)$ have injectivity degree $0$. \hfill \Box

**Proposition 4.19.** Let $f : V \rightarrow U$ be a morphism of $\text{FI}_W$–modules, and assume that $k$ contains $\mathbb{Q}$. Suppose that $V$ has injectivity degree $\leq B$ and surjectivity degree $\leq C$, and that $U$ has injectivity degree $\leq D$ and surjectivity degree $\leq E$. Then $\ker(f)$ has injectivity degree $\leq B$ and surjectivity degree $\leq \max(C, D)$, and $\text{coker}(f)$ has injectivity degree $\leq \max(C, D)$ and surjectivity degree $\leq E$.

**Proof of Proposition 4.19.** The proof given for $\text{FI}_A$ in [CEF12, Proposition 2.44] carries through directly. The idea is to use exactness of the functor $\Phi_a$, and
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perform diagram chases on the following diagrams:

\[
\begin{array}{ccc}
0 & \longrightarrow & \Phi_a(\ker f) \\
& & \downarrow \\
0 & \longrightarrow & \Phi_a(V) \\
& & \downarrow \\
& & \Phi_a(U) \\
& & \downarrow \\
& & \Phi_a(coker f) \\
& & \longrightarrow 0
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
0 & \longrightarrow & \Phi_a(\ker f) \\
& & \downarrow \\
0 & \longrightarrow & \Phi_a(V) \\
& & \downarrow \\
& & \Phi_a(U) \\
& & \downarrow \\
& & \Phi_a(coker f) \\
& & \longrightarrow 0
\end{array}
\]

Lemma 4.20. Suppose \( k \) contains \( \mathbb{Q} \). Let \( V \) be a finitely presented \( F_{I_W} \)-module with generator degree \( g \) and relation degree \( r \). Then \( V \) has stability degree \( \leq \max(r, g) \).

We mimic the proof of [CEF12, Proposition 2.47].

**Proof of Lemma 4.20.** By assumption, there is an exact sequence

\[
0 \longrightarrow K \longrightarrow \bigoplus_{a=0}^g M_{I_W}(a)^{\oplus b_a} \longrightarrow V \longrightarrow 0
\]

with kernel \( K \) generated in degree \( \leq r \). By Proposition 4.16, \( \bigoplus_{a=0}^g M_{I_W}(m)^{\oplus b_i} \) has injectivity degree 0 and surjectivity degree \( g \). By Proposition 4.17, \( K \) has surjectivity degree \( \leq r \). The result then follows from Proposition 4.19.

Lemma 4.21. (\( \lambda_1^+ \leq \text{stability degree} \)). Suppose that \( V \) is an \( F_{I_{BC}} \)-module over a field \( k \) of characteristic zero, and suppose \( V \) has stability degree \( s \). For any \( n \geq 0 \), and (in the notation of Section 2.2) any \( V(\lambda^+, \lambda^-)_n \) in \( V_n \), the largest part \( \lambda_1^+ \) of \( \lambda^+ \) satisfies \( \lambda_1^+ \leq s \).

Lemma 4.21 parallels [CEF12, Proposition 2.42], and we adapt this argument. In Theorem 4.27, we will use Lemma 4.21 to relate the stability degree of an \( F_{I_{BC}} \)-module \( V \) to the representation stability of \( \{V_n\} \).

**Proof of Lemma 4.21.** Let \( \lambda \) and \( \mu \) denote the double partitions \( \lambda = (\lambda^+, \lambda^-) \) and \( \mu = (\mu^+, \mu^-) \), and let \( \lambda_1^+ \) and \( \mu_1^+ \) denote the largest parts of \( \lambda^+ \) and \( \mu^+ \), respectively. Let \( m = |\mu| \), and denote \( \nu = \lambda[n] \). First we note that every irreducible representation \( V(\lambda)_n = V_\nu \) in \( M_{BC}(\mu)_n \) must satisfy \( \lambda_1^+ \leq \mu_1^+ \). By the branching rules for \( B_n \), Equation (2),

\[
V(\lambda)_n = V_\nu \quad \subseteq \quad M_{BC}(\mu)_n = \text{Ind}_{B_m \times B_{n-m}}^{B_n} V_\mu \boxtimes k
\]

\( \iff \)

\( \nu^- = \mu^- \) and \( \nu^+ \) can be built from \( \mu^+ \) by adding \( (n-m) \) boxes in distinct columns.
A box can be added to the end of the second row $\mu_2^+$ only if $\mu_1^+ > \mu_2^+$. This implies that $\lambda_1^+$, the second row in $\nu^+ = \lambda^+[n - |\lambda^-|]$, must be no larger than $\mu_1^+$, the first row of $\mu^+$. Some small cases are shown in Figure 9.

\[
M_{BC}(\begin{array}{c} \ast \ast \\ \ast \ast \end{array}, \begin{array}{c} \ast \ast \\ \ast \ast \end{array})_{11} = V(\begin{array}{c} \ast \ast \\ \ast \ast \end{array}) \oplus V(\begin{array}{c} \ast \ast \| \\ \ast \ast \end{array}) \oplus V(\begin{array}{c} \ast \ast \\ \ast \ast \| \end{array}) \oplus V(\begin{array}{c} \ast \ast \| \\ \ast \ast \end{array}) \\
\oplus V(\begin{array}{c} \ast \ast \| \\ \ast \ast \| \end{array}) \oplus V(\begin{array}{c} \ast \ast \| \\ \ast \ast \end{array}) \oplus V(\begin{array}{c} \ast \ast \| \\ \ast \ast \end{array})
\]

\[
M_{BC}(\begin{array}{c} \ast \ast \| \\ \ast \ast \| \\ \ast \ast \end{array}, \begin{array}{c} \ast \ast \| \\ \ast \ast \| \\ \ast \ast \end{array})_{10} = V(\begin{array}{c} \ast \ast \| \\ \ast \ast \| \\ \ast \ast \end{array}) \oplus V(\begin{array}{c} \ast \ast \| \\ \ast \ast \| \| \end{array}) \oplus V(\begin{array}{c} \ast \ast \| \\ \ast \ast \| \end{array}) \oplus V(\begin{array}{c} \ast \ast \| \\ \ast \ast \| \end{array}) \oplus V(\begin{array}{c} \ast \ast \| \\ \ast \ast \| \end{array}) \oplus V(\begin{array}{c} \ast \ast \| \\ \ast \ast \| \end{array}) \oplus V(\begin{array}{c} \ast \ast \| \\ \ast \ast \| \end{array})
\]

Figure 9: Illustrating the inequality $\lambda_1^+ \leq \mu_1^+$. The double partitions $\mu$ are bolded, and $\lambda$ are shaded in.

We next claim that if $V$ has stability degree $s$, then every irreducible sub-representation $V_\mu$ of $H_0(V)_m$ must satisfy $\mu_1 \leq s$. Recall from Definition 3.21 that $H_0(V)^{FL_{BC}}$ denotes the $FL_{BC}$–module structure on $H_0(V)$ wherein all morphisms $I_n$ act by zero. The surjection $V \to H_0(V)^{FL_{BC}}$ implies by Remark 4.15 that $H_0(V)^{FL_{BC}}$ has surjectivity degree $\leq s$. Suppose that $V_\mu \subseteq H_0(V)_m$, and let $\eta$ be the double partition such that $\mu = \eta[m]$. Remark 4.12 and the branching rules (Equation (3), Section 2.1.2) imply that $(V_\mu)|_{\mu_1^+} = V_\eta$. It follows that when $a = m - \mu_1^+$,

\[
\Phi_a(H_0(V)^{FL_{BC}})_{\mu_1^+} = (H_0(V)^{FL_{BC}})_{B_{\mu_1^+}} \quad \text{contains} \quad V_\eta,
\]

and so in particular the coinvariants $\Phi_a(H_0(V)^{FL_{BC}})_{\mu_1^+}$ are nonzero. Since $T$ acts by zero, the surjectivity degree of $H_0(V)^{FL_{BC}}$ must be greater than $\mu_1^+$, and we conclude that $\mu_1^+ \leq s$.

Now, suppose that $V$ is an $FL_{BC}$–module over characteristic zero with stability degree $s$. We’ve shown that every irreducible component $V_\mu$ of $H_0(V)$ satisfies $\mu_1^+ \leq s$, and so by the first paragraph above, for each $n$, every $V(\lambda)_n$ in $M_{BC}(H_0(V))_n$ satisfies $\lambda_1^+ \leq \mu_1^+ \leq s$. Since by Remark 3.22 there is a surjection $M_{BC}(H_0(V)) \to V$, we conclude that for each $n$, every irreducible constituent $V(\lambda)_n$ of $V_n$ satisfies $\lambda_1^+ \leq s$. 

\[\square\]
4.3 The Noetherian property

In [CEF12, Theorem 2.60], Church–Ellenberg–Farb prove a Noetherian property for FI\(_A\)–modules over Noetherian rings containing \(\mathbb{Q}\). This theorem is later generalized by Church–Ellenberg–Farb–Nagpal in [CEFN12, Theorem 1.1] to arbitrary Noetherian rings. We will show that the same is true for modules over all three categories FI\(_W\).

**Theorem 4.22. (FI\(_W\)–modules are Noetherian).** Let \(k\) be a Noetherian ring. Then any sub–FI\(_W\)–module of a finitely generated FI\(_W\)–module over \(k\) is itself finitely generated.

**Proof.** Suppose that \(V\) is a finitely generated FI\(_W\)–module, and \(U\) is any sub–FI\(_W\)–module. When \(W\) is \(S_n\), the result is [CEFN12, Theorem 1.1]. If \(W_n\) is \(B_n\) or \(D_n\), then by Propositions 3.24(1) and (3), the restriction of \(V\) to \(FIA\) is finitely generated as an \(FIA\)–module. Thus by [CEFN12, Theorem 1.1], \(U\) is finitely generated over \(FIA \subseteq FIW\), and therefore it is finitely generated over \(FIW\).

4.4 Finite generation and representation stability

Representation stability is a concept introduced by Church and Farb in [CF13]; the definition is given in Section 2.2. In [CEF12, Section 2.7], Church–Ellenberg–Farb prove that for an FI\(_A\)–module \(V\) over a field \(k\) of characteristic zero, finite generation is equivalent to uniform representation stability for the sequence of \(Sn\)–representations \(\{V_n\}\). We will show, in Theorems 4.28 and 4.29, the analogous results for FI\(_BC\). In summary:

**Theorem 4.23.** Let \(k\) be a field of characteristic zero. An FI\(_W\)–module \(V\) is finitely generated if and only if \(\{V_n\}\) is uniformly representation stable with respect to the maps induced by the natural inclusions \(I_n : n \leftrightarrow (n + 1)\).

We first show that, in the notation of Section 2.2, sequences of \(B_n\)–representations of the form \(\oplus_{\lambda} c_{\lambda} V(\lambda)_n\) over characteristic zero are determined up to isomorphism by their coinvariants. We will use this result to relate finite generation with representation stability.

**Lemma 4.24. (The \(B_n\)–representations \((V(\lambda)_n)_{B_{n-a}}\) stabilize for \(n \geq a + \lambda_1^+\))**

Suppose \(k\) is a field of characteristic zero. Given a double partition \(\lambda = (\lambda^+, \lambda^-)\), the \(B_n\)–representations \((V(\lambda)_n)_{B_{n-a}}\) are independent of \(n\) for \(n \geq a + \lambda_1^+\), where \(\lambda_1^+\) denotes the largest part of \(\lambda^+\).
4.4 Finite generation and representation stability

Proof of Lemma 4.24. By Remark 4.12 and the branching rules (Equation 3),

\[(V(\lambda^+, \lambda^-)_n)_{B_{n-a}} = V(\lambda^+[n - |\lambda^-|], \lambda^-)_{B_{n-a}} = \bigoplus_{\mu^+} V(\mu^+, \lambda^-)
\]

summed over all partitions \(\mu^+\) that can be constructed from \(\lambda^+[n - |\lambda^-|]\) by removing \((n - a)\) boxes from distinct columns. Once \((n - a) > \lambda^+_1\), at least one box must be removed from the top row of \(\lambda^+[n - |\lambda^-|]\). Removing one box from the top row of the padded partition \(\lambda^+[n - |\lambda^-|]\) associated to \(n\) yields the padded partition \(\lambda^+[(n-1) - |\lambda^-|]\) associated to \((n-1)\), and removing the remaining \(((n-1)-a)\) boxes from distinct columns gives the decomposition of \((V(\lambda^+, \lambda^-)_{n-1})_{B_{(n-1)-a}}\). Thus, as \(B_a\)-representations,

\[(V(\lambda^+, \lambda^-)_{n-1})_{B_{(n-1)-a}} \cong (V(\lambda^+, \lambda^-)_n)_{B_{n-a}} \quad \text{for all } n > a + \lambda^+_1,
\]

and the lemma follows. \(\square\)

Lemma 4.25. (\(B_n\)-representations are determined by their coinvariants). Assume that \(k\) is a field of characteristic zero. Let \(\Lambda\) be a set of double partitions \(\lambda = (\lambda^+, \lambda^-)\) of size at most \(d\), and set \(M = \max_{\lambda} \lambda^+_1\), where \(\lambda^+_1\) denotes the largest part of \(\lambda^+\). Let \(n \geq m \geq (d + M)\) be nonnegative integers. Suppose that for some \(B_m\)-representation \(V_m\) and \(B_n\)-representation \(V_n\),

\[V_m \cong \bigoplus_{\lambda \in \Lambda} b_{\lambda} V(\lambda)_m \quad \text{and} \quad V_n \cong \bigoplus_{\lambda \in \Lambda} c_{\lambda} V(\lambda)_n,
\]

If for each \(0 \leq a \leq d\), the coinvariants

\[(V_m)_{B_{m-a}} \cong (V_n)_{B_{n-a}}\]

are isomorphic as \(B_a\)-representations, then \(c_{\lambda} = b_{\lambda}\) for all \(\lambda \in \Lambda\).

Corollary 4.26. With \(n\) and \(d\) as above, the coinvariants \((V_n)_{B_{n-a}} = 0\) for all \(0 \leq a \leq d\) if and only if \(V_n = 0\).

Church–Ellenberg–Farb prove an analogous result to Lemma 4.25 for the symmetric group in [CEF12, Lemma 2.40 and Proposition 2.58]. We adapt their methods in the following proof.

Proof of Lemma 4.25. We will prove that \(c_{\lambda} = b_{\lambda}\) for all \(|\lambda| \leq p\) for each \(p\) with \(0 \leq p \leq d\), proceeding by induction on \(p\).
If $p = 0$, then the only double partition $\lambda$ of size at most $p$ is the double partition $\lambda = (\emptyset, \emptyset)$ associated to the trivial representation. Taking $a = 0$, we see

$$c_\lambda = \dim((V_n)_{B_n}) = \dim((V_m)_{B_m}) = b_\lambda.$$ 

The conclusion follows for $p = 0$.

Consider some double partition $\lambda = (\lambda^+, \lambda^-)$. By Remark 4.12 and the branching rules (Equation (3), Section 2.1.2), the multiplicity of $V(\nu^+, \nu^-)$ in $(V(\lambda)_n)_{B_n-a}$ is:

$$\begin{cases} 
1 & \text{if } \nu^+ \text{ can be constructed by removing } (n-a) \text{ boxes from } \lambda^+[n-|\lambda^-|], \\
0 & \text{otherwise.}
\end{cases}$$

Since the largest part of $\lambda^+[n-|\lambda^-|]$ is $(n-|\lambda|)$, the coinvariants must vanish when $|\lambda| > a$, and when $|\lambda| = a$, $(V(\lambda)_n)_{B_n-a}$ is a single copy of the $B_a$–representation $V_\lambda$.

Now, suppose (as inductive hypothesis) that $c_\lambda = b_\lambda$ for all $|\lambda| < p$, and consider the coinvariants corresponding to $a = p$. We have:

$$(V_m)_{B_m-p} = \bigoplus_{|\lambda|=p} c_\lambda V_\lambda \bigoplus_{|\lambda|<p} c_\lambda (V(\lambda)_m)_{B_m-p}$$

$$(V_n)_{B_n-p} = \bigoplus_{|\lambda|=p} b_\lambda V_\lambda \bigoplus_{|\lambda|<p} b_\lambda (V(\lambda)_n)_{B_n-p}$$

By the inductive hypothesis, the subrepresentations of $V_m$ and $V_n$ of weight $< p$ are isomorphic. Since by assumption $p + \max_\lambda \lambda^+_1 \leq d + M \leq m, n$, Lemma 4.24 implies that the coinvariants of these subrepresentations are isomorphic. Thus $(V_m)_{B_m-p} \cong (V_n)_{B_n-p}$ only if $c_\lambda = b_\lambda$ for all $\lambda$ with $|\lambda| = p$. The lemma follows by induction.

**Theorem 4.27.** Suppose that $k$ is a characteristic zero field, and that $V$ is a FI$_{BC}$–module with weight $\leq d$, and stability degree $N$. Then, $\{V_n\}$ is uniformly representation stable with respect to the maps $\phi_n : V_n \to V_{n+1}$ induced by the natural inclusions $I_n : n \mapsto (n+1)$. The sequences stabilizes for $n \geq N + d$.

The arguments used in [CEF12, Theorem 2.58] carry through to type B/C; we briefly give these arguments here.

**Proof of Theorem 4.27:** We note that, by Lemma 4.21, for all $n$ and all irreducible
components $V(\lambda^+, \lambda^-)_n$ in $V_n$, the largest part $\lambda^+_1$ of $\lambda^+$ is less than $N$. We can therefore apply Lemma 4.25 and Corollary 4.26 to the representations $V_n$ for any $n \geq N + d \geq \max \lambda^+_1 + d$.

I. Injectivity Let $K_n$ denote the kernel of $\phi_n$. By assumption that $V$ has stability degree $N$, the composite

$$(V_n)_{B_n-d} \to (V_{n+1})_{B_{n-d}} \to (V_{n+1})_{B_{n+1-d}}$$

is an isomorphism for $n \geq N + d$, which implies that the first map is injective. The operation of taking coinvariants is exact in characteristic zero, and so it follows that its kernel is isomorphic to $(K_n)_{B_{n-d}}$. Thus $(K_n)_{B_{n-d}} = 0$, and so $K_n = 0$ by Corollary 4.26. This proves injectivity of $\phi_n$ for $n \geq N + d$.

II. Surjectivity To prove that $\phi_n(V_n)$ generates $V_{n+1}$ as a $k[B_{n+1}]$-module, it suffices to show that the induced map

$$\text{Ind}(\phi_n) : \text{Ind}^{B_{n+1}}V_n \to V_{n+1}$$

is surjective. Let $C_{n+1}$ denote the cokernel of this map. The composition

$$(V_n)_{B_{n-d}} \to (\text{Ind}^{B_{n+1}}V_n)_{B_{n+1-d}} \to (V_{n+1})_{B_{n+1-d}}$$

is an isomorphism for $n \geq N + d$ by assumption. Thus $(C_{n+1})_{B_{n-d}}$ vanishes, and so $C_{n+1}$ vanishes by Corollary 4.26, and $I_n$ surject for $n \geq N + d$.

III. Multiplicity Stability By assumption,

$$(V_n)_{B_{n-a}} \cong (V_{n+1})_{B_{n+1-a}} \quad \text{for all } a \geq 0 \text{ and } n \geq N + a.$$

Thus for $n \geq N + d$, Lemma 4.25 implies that the multiplicity of each irreducible $V(\lambda)_n$ in $V_n$ is constant. This completes the proof.

\[\square\]

**Theorem 4.28.** (Finitely generated $\mathcal{F}_{\lambda W}$-modules are uniformly representation stable). Suppose that $k$ is a field of characteristic zero, and $\mathcal{W}_n$ is $S_n$, $D_n$ or $B_n$. Let $V$ be a finitely generated $\mathcal{F}_{\lambda W}$-module. Take $d$ to be an upper bound on the weight of $V$, $g$ an upper bound on its degree of generation, and $r$ an upper bound on its relation degree; when $\mathcal{W}_n$ is $D_n$, take $r$ to be an upper bound on the relation degree of $\text{Ind}^{B_{n+1}}_{D^n} V$. 

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Then, \( \{ V_n \} \) is uniformly representation stable with respect to the maps induced by the natural inclusions \( I_n : n \to (n + 1) \), stabilizing once \( n \geq \max(g, r) + d \); when \( W_n \) is \( D_n \) and \( d = 0 \) we need the additional condition that \( n \geq g + 1 \).

**Proof of Theorem 4.28.** Suppose first that \( W_n \) is \( S_n \) or \( B_n \). By Lemma 4.20, \( V \) has stability degree \( \max(g, r) \). The conclusion follows from [CEF12, Proposition 2.58] in type A and Theorem 4.27 in Type B/C.

Next suppose \( W_n \) is \( D_n \), and consider the FI\(_{BC}\)-module \( \text{Ind}_{D}^{BC} V \). \( \text{Ind}_{D}^{BC} V \) has weight \( d \) by the definition of weight for FI\(_D\)-modules. By Lemma 3.30, \( \text{Ind}_{D}^{BC} V \) will also have generator degree \( g \), and it has relation degree \( r \) by assumption. Hence \( \text{Ind}_{D}^{BC} V \) is uniformly representation stable with respect to the \( B_n \) action for \( n \geq \max(g, r) + d \). However, by Proposition 3.34,

\[
V_n \cong (\text{Res}_{D}^{BC} \text{Ind}_{D}^{BC} V)_n \quad \text{as } D_n\text{-representations}, \quad \text{for } n \geq g + 1
\]

and so \( V \) is a uniformly representation stable sequence of \( D_n\)-representations in this range.

Note that, since an FI\(_{BC}\)-module \( V \) must have weight \( \leq g \) by Theorem 4.4, it is uniformly representation stable for \( n \geq \max(2g, g + r) \).

**Theorem 4.29.** (Uniformly representation stable FI\(_W\)-modules are finitely generated). Suppose conversely that \( V \) is an FI\(_W\)-module, and that \( \{ V_n, (I_n)_* \} \) is uniformly representation stable for \( n \geq N \). Then \( V \) is finitely generated in degree \( \leq N \).

**Proof.** For \( n \geq N \), the “surjectivity” criterion for representation stability implies that \( (I_n)_*(V_n) \) generates \( V_{n+1} \) as a \( k\{W_{n+1}\}\)-module. Since each vector space \( V_n \) is finite dimensional by assumption, we can take bases for \( \{V_m\}_{m \leq N} \) to be our finite generating set.

**Remark 4.30.** (FI\(_W\)-modules cannot be non-uniformly representation stable). We note that the assumption of uniformity of representation stability was not needed in the proof of Theorem 4.29. It follows that, over characteristic zero, any sequences of either \( S_n \) or \( B_n \)-representations that is non-uniformly representation stable cannot admit an FI\(_W\)-module structure. If such a sequence did, representation stability would imply finite generation, which would imply uniform representation stability, a contradiction. The alternating representations of \( S_n \) and sign representation of \( B_n \) are examples of
4.5 The functor $\tau_{\geq d}$ and the $\FI_W$–module $V(\lambda)$

Given an $\FI_A$ or $\FI_{BC}$–module $V$, we will define a filtration $\tau_{\geq d}V$ of $V$ into sub-$\FI_W$–modules by weight. We will use this filtration to construct an $\FI_{BC}$–module structure on the sequence of $B_n$–representations $\{V(\lambda^+,\lambda^-)\}_{n\in\mathbb{N}}$, and an $\FI_D$–module structure on the sequence of $D_n$–representations $\{V(\lambda^+,\lambda^-)\}_{n\in\mathbb{N}}$, associated to any double partition $(\lambda^+ ,\lambda^-)$.

The FI$_{BC}$–module $V(\lambda)$ will feature later in the proof of Theorem 6.4, the type B/C analogues of Murnaghan’s stability theorem for tensor products of $S_n$–representations.

**Proposition 4.31.** (The functor $\tau_{\geq d}$). Suppose that $W_n$ is $S_n$ or $B_n$, and that $k$ is a field of characteristic zero. Fix $d \geq 0$. Any $\FI_W$–module $V$ over $k$ contains a sub-$\FI_W$–module, which we denote $\tau_{\geq d}V$, defined by

$$(\tau_{\geq d}V)_n \text{ is the sum of all components } V(\lambda)_n \text{ of } V_n \text{ with } |\lambda| \geq d.$$ 

**Proof.** For $W_n = S_n$, this is proven by Church–Ellenberg–Farb [CEF12, Proposition 2.54]. Their arguments also adapt to type B/C:

The spaces $(\tau_{\geq d}V)_n$ are by construction $B_n$–invariant, and so it suffices to show that the image of $(I_{m,n})_* : (\tau_{\geq d}V)_m \to V_n$ lies in $(\tau_{\geq d}V)_n$ for all $m,n$. The branching rules for the hyperoctahedral group (deduced from Equation (4) and Frobenius reciprocity) assert that a $B_m$–representation $V(\mu)_m = V(\mu^+,\mu^-)_m$ is contained in the restriction of a $B_n$–representation $V(\lambda)_n = V(\lambda^+,\lambda^-)_n$ only if $\mu[m] \subseteq \lambda[n]$. In particular, if $|\mu| \geq d$ then necessarily $|\lambda| \geq d$. Since $(I_{m,n})_*$ is $B_m$–equivariant with respect to the restriction of $V_n$ to $B_m$, the branching rules imply that $(I_{m,n})_* ((\tau_{\geq d}V)_m) \subseteq (\tau_{\geq d}V)_n$, as required. 

The decomposition of the filtration

$$M_A \left( \begin{array}{c} \mathbb{P} \\ \mathbb{P} \end{array} \right) = \tau_{\geq 1}M_A \left( \begin{array}{c} \mathbb{P} \\ \mathbb{P} \end{array} \right) \supseteq \tau_{\geq 2}M_A \left( \begin{array}{c} \mathbb{P} \\ \mathbb{P} \end{array} \right) \supseteq \tau_{\geq 3}M_A \left( \begin{array}{c} \mathbb{P} \\ \mathbb{P} \end{array} \right) \supseteq \tau_{\geq 4}M_A \left( \begin{array}{c} \mathbb{P} \\ \mathbb{P} \end{array} \right) = 0.$$ 

is shown in Figure 10.

Suppose $k$ is a characteristic zero field. Let $\lambda$ be a partition with largest part $\lambda_1$. In [CEF12, Proposition 2.56], Church–Ellenberg–Farb define the FI$_A$–
4.5 The functor $\tau_{\geq d}$ and the $FI_W$–module $V(\lambda)$

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module

$$V(\lambda) := \tau_{\geq |\lambda|} M_A(\lambda).$$

They show that $V(\lambda)$ is finitely generated in degree $|\lambda| + \lambda_1$, and satisfies

$$V(\lambda)_n = \begin{cases} V_{\lambda[n]} & \text{if } n \geq |\lambda| + \lambda_1, \\ 0 & \text{otherwise}. \end{cases}$$

We give the analogous construction for $FI_{BC}$.

**Definition 4.32.** (The $FI_{BC}$–module $V(\lambda) = V(\lambda^+, \lambda^-)$) Let $k$ be a field of characteristic zero. Let $\lambda = (\lambda^+, \lambda^-)$ be a double partition. Define the $FI_{BC}$–module $V(\lambda)$ by

$$V(\lambda) := \tau_{\geq |\lambda|} M_{BC}(\lambda).$$

This is consistent with the notation given in Section 2.2.

**Proposition 4.33.** Let $\lambda^+_1$ denote the largest part of $\lambda^+$. The $FI_{BC}$–module $V(\lambda) = V(\lambda^+, \lambda^-)$ satisfies

$$V(\lambda)_n = \begin{cases} V_{\lambda[n]} & \text{if } n \geq |\lambda^+| + |\lambda^-| + \lambda^+_1, \\ 0 & \text{otherwise}. \end{cases}$$

and $V(\lambda)$ is finitely generated in degree $|\lambda^+| + |\lambda^-| + \lambda^+_1$.

**Proof.** Let $a = |\lambda|$. By definition,

$$M_{BC}(\lambda) = \begin{cases} 0 & n < a, \\ \text{Ind}_{B_n^+ \times B_n^-} V_{\lambda} \otimes k & n \geq a. \end{cases}$$

Figure 10: Filtration of $M_A(2, 1)$ by weight.
and so the branching rule (Equation (2)) implies that, for \( n \geq |\lambda| \),

\[
M_{BC}(\lambda)_n = \bigoplus_{\mu^+} V(\mu^+, \lambda^-)
\]

summed over all partitions \( \mu^+ \) constructed by adding \( (n - |\lambda^+| - |\lambda^-|) \) boxes in distinct columns of \( |\lambda^+| \). An irreducible component \( V(\mu^+, \lambda^-) \) can appear in \( (\tau_{\geq |\lambda|} M_{BC}(\lambda))_n \) only if \( \lambda_1^+ \) boxes are added to each column of \( \lambda^+ \) below the top row. This happens only once \( n \geq |\lambda^+| + |\lambda^-| + \lambda_1^+ \), and gives the single irreducible \( V(\lambda^+, \lambda^-)_n \).

Since \( V(\lambda) \) consists of a single irreducible \( B_n \)-representation for all \( n \geq |\lambda^+| + |\lambda^-| + \lambda_1^+ \), to prove finite generation in degree \( |\lambda^+| + |\lambda^-| + \lambda_1^+ \) it suffices to show that the maps \( V(\lambda)_n \to V(\lambda)_{n+1} \) are nonzero in this range. We can consider \( V(\lambda) \) as a sub-\( \text{FI}_{BC} \)-module of

\[
M_{BC}(\lambda) = M_{BC}(\alpha) \otimes_{k[B_n]} V_\lambda.
\]

By their definition the maps

\[
M_{BC}(\alpha)_n \to M_{BC}(\alpha)_{n+1}
\]

are injective, and since \( V_\lambda \) is a flat \( k[B_n] \)-module in characteristic zero, the maps

\[
M_{BC}(\lambda)_n \to M_{BC}(\lambda)_{n+1}
\]

are injective, and the conclusion follows.

By restricting the \( \text{FI}_{BC} \)-module \( V(\lambda^+, \lambda^-) \) to the subcategory \( \text{FI}_D \), we construct an \( \text{FI}_D \)-module with the following properties.

**Corollary 4.34.** Given any ordered pair of partitions \( \lambda = (\lambda^+, \lambda^-) \) (with \( \lambda_1^+ \) the largest part of \( \lambda^+ \)), there is an \( \text{FI}_D \)-module \( V(\lambda) \) such that

\[
V(\lambda)_n = \begin{cases} 
V_{(\lambda^+ [n-|\lambda^-|], \lambda^-)} & \text{if } n \geq |\lambda^+| + |\lambda^-| + \lambda_1^+, \text{ and } \lambda^+[n-|\lambda^-|] \neq \lambda^- \\
V_{(\lambda^-, +)} \oplus V_{(\lambda^-, -)} & \text{if } n \geq |\lambda^+| + |\lambda^-| + \lambda_1^+, \text{ and } \lambda^+[n-|\lambda^-|] = \lambda^- \\
0 & \text{otherwise.}
\end{cases}
\]

As examples, the \( \text{FI}_{BC} \)-module \( V(\square, \square) \) and its restriction to \( \text{FI}_D \) are shown in Figure 11.
4.6 \textit{FI}_{W^\sharp}–modules

Church–Ellenberg–Farb [CEF12, Definition 2.19] define \textit{FI}^\sharp–modules, a class of sequences of $S_n$–representations which carry compatible \textit{FI}_A and co–\textit{FI}_A–module structures. We will give several characterizations of the analogous constructions in type B/C. An \textit{FI}_{BC}^\sharp–module structure imposes strong constraints on a sequence of $B_n$–representations; just as for \textit{FI}_A^\sharp–modules [CEF12, Theorem 2.24]: the underlying \textit{FI}_{BC}–module structure on these sequences must be of the form $\bigoplus_r M_{BC}(U_r)$ for some set of $B_r$–representations $U_r$.

\textbf{Definition 4.35.} For $n \geq 0 \in \mathbb{Z}$, let

$$n_0 := \{0, \pm 1, \pm 2, \ldots, \pm n\}.$$ 

We think of the digit 0 as a basepoint. Define \textit{FI}_{BC}^\sharp to be the category with objects $n_0$ for $n \geq 0 \in \mathbb{Z}$, and morphisms

$$f : n_0 \to n_0 \quad \text{such that} \quad f(-a) = -f(a) \text{ for all } a \in n_0$$

and

$$|f^{-1}(b)| \leq 1 \text{ for } 1 \leq |b| \leq n.$$ 

These morphisms are “injective away from zero”. Note that the conditions imply $f(0) = 0$.

We define \textit{FI}_A^\sharp to be the subcategory with the same objects and morphisms preserving signs. In both cases, an \textit{FI}_{W^\sharp}–module over a ring $k$ is a functor from
4.6 \( FI_W \)-modules

\( FI_W \) to the category of \( k \)-modules.

In both types A and B/C, the injective maps in \( FI_W \) are precisely the \( FI_W \) morphisms. We call \( f \in \text{Hom}_{FI_W}(m_0, n_0) \) a projection if \( |f^{-1}(\pm b)| = 1 \) for \( 1 \leq b \leq n \); these projections are left inverses to the \( FI_W \) morphisms.

For a morphism \( f : m_0 \to n_0 \), we call \( \left| f^{-1}(\{\pm 1, \ldots, \pm n\}) \right| \) the rank of \( f \).

This description of \( FI_W \)-modules was suggested to us by Peter May. The category \( FI_A \) appears independently in work by May and Merling studying the Segal equivariant infinite loop space machine [MM12], where the category is denoted II.

**Remark 4.36. (A Category \( FI_D \)?)**. Unlike with \( FI_A \) and \( FI_{BC} \), we cannot introduce partial inverses to the morphisms in the category \( FI_D \) without also introducing additional automorphisms – and, in fact, generating the entire category \( FI_{BC} \). It is not clear that we can create any satisfactory analogue of \( FI \)-module theory in type D, since critical properties fail: the \( FI_D \)-module structure on \( M_D(m) \) does not extend to an \( FI_{BC} \)-module structure.

**4.6.1 Alternate Characterizations of the Categories \( FI_W \)**

In this section we give two alternate descriptions of the categories \( FI_W \). The first relates these categories back to the definition of \( FI \) given by Church–Ellenberg–Farb [CEF12, Definition 2.19]. The second frames \( FI_W \) as the category of spans on \( FI_W \). This description will be convenient for proving Proposition 4.49, which characterizes \( FI_W \)-modules as simultaneous \( FI_W \)-modules and co–\( FI_W \)-modules.

**Remark 4.37.** Church–Ellenberg–Farb [CEF12, Definition 2.19] defined \( FI_A \) to be the category whose objects are finite sets, in which \( \text{Hom}_{FI_A}(S, T) \) is the set of triples \((A, B, \phi)\) with \( A \) a subset of \( S \), \( B \) a subset of \( T \) and \( \phi : A \to B \) an isomorphism. The composition of two morphisms is given by composition of functions, where the domain is the largest set on which the composition is defined, and the codomain is its bijective image.

We can generalize this definition. We call a subset \( A \subseteq n \) symmetric if

\[
a \in A \iff -a \in A.
\]

Then we define \( FI_{BC} \) to be the category whose objects are the finite sets \( n = \{\pm 1, \pm 2, \ldots, \pm n\} \), and whose morphisms \( \text{Hom}(m, n) \) are triples \((A, B, \phi)\) such
that $A$ is a symmetric subset of $\mathfrak{m}$, $B$ is a symmetric subset of $\mathfrak{n}$, and $\phi : A \to B$ is an injective map satisfying

$$\phi(-a) = -\phi(a) \quad \text{for every } a \in A.$$ 

$\text{FI}_{A^\#}$ is the subcategory in which all morphisms preserve signs; this coincides with the definition of $\text{FI}_{A^\#}$ given in [CEF12, Definition 2.19].

These definitions of the categories $\text{FI}_{\mathfrak{m}^\#}$ and $\text{FI}_{\mathfrak{n}^\#}$ are equivalent to Definition 4.35. We identify the morphism $(A, B, \phi) \in \text{Hom}_{\text{FI}_{\mathfrak{m}^\#}}(\mathfrak{m}, \mathfrak{n})$ with the map $f : \mathfrak{m}_0 \to \mathfrak{n}_0$ defined by

$$f : \mathfrak{m}_0 \to \mathfrak{n}_0$$

$$j \mapsto \begin{cases} 
\phi(j) & j \in A, \\
0 & j \notin A.
\end{cases}$$

Conversely, we can identify $f : \mathfrak{m}_0 \to \mathfrak{n}_0$ with a triple $(A, B, \phi)$ by taking $A = f^{-1}(\{\pm 1, \ldots, \pm n\})$, $B = f(A)$, and $\phi = f|_A$. One can check that these identifications of morphisms are consistent with the composition rules, and give an isomorphism of categories.

**Remark 4.38.** The description of the categories $\text{FI}_{\mathfrak{m}^\#}$ in Remark 4.37 suggests a third characterization: as the category of ‘spans’ of $\text{FI}_{\mathfrak{m}^\#}$, an instance of a general classical construction to introduce partial inverses of the morphisms in a category. A morphism $(A, B, \phi) \in \text{Hom}_{\text{FI}_{\mathfrak{m}^\#}}(\mathfrak{m}_1, \mathfrak{m}_2)$ of rank $m$ can be instead described as an equivalence class of pairs of maps

$$\{ (f, g) \in \text{Hom}_{\text{FI}_{\mathfrak{m}^\#}}(\mathfrak{m}, \mathfrak{m}_1) \times \text{Hom}_{\text{FI}_{\mathfrak{m}^\#}}(\mathfrak{m}, \mathfrak{m}_2) | f(\mathfrak{m}) = A, g(\mathfrak{m}) = B, \text{ and } \phi = g \circ f^{-1}|_A : A \to B \}.$$ 

In other words, the morphisms $\text{Hom}_{\text{FI}_{\mathfrak{m}^\#}}(\mathfrak{m}_1, \mathfrak{m}_2)$ of rank $m$ are all pairs

$$(f, g) \in \text{Hom}_{\text{FI}_{\mathfrak{m}^\#}}(\mathfrak{m}, \mathfrak{m}_1) \times \text{Hom}_{\text{FI}_{\mathfrak{m}^\#}}(\mathfrak{m}, \mathfrak{m}_2),$$

considered up to precomposition with an isomorphism of $\mathfrak{m}$. The composition of morphisms $(f, g) \in \text{Hom}_{\text{FI}_{\mathfrak{m}^\#}}(\mathfrak{m}_1, \mathfrak{m}_2)$ and $(h, i) \in \text{Hom}_{\text{FI}_{\mathfrak{m}^\#}}(\mathfrak{m}_2, \mathfrak{m}_3)$ is given by taking the fibre product of $g$ and $h$ (which is well-defined up to isomorphism):
4.6 \( \text{FI}_{\mathcal{W}} \sharp \)-modules

Then \((h,i) \circ (f,g) = (f \circ p, i \circ q) \in \text{Hom}_{\text{FI}_{\mathcal{W}}} (m_1, m_3)\).

4.6.2 Examples of \( \text{FI}_{\mathcal{W}} \sharp \)-modules

We prove in Proposition 4.39 and Corollary 4.40 that \( \text{FI}_{\mathcal{B}} \)-modules of the form \( \text{M}_{\mathcal{B}}(m) \) or \( \text{M}_{\mathcal{B}}(U) \) have \( \text{FI}_{\mathcal{B}} \sharp \)-module structures.

**Proposition 4.39.** (\( \text{M}_{\mathcal{W}}(a) \) is an \( \text{FI}_{\mathcal{W}} \sharp \)-module). Let \( \mathcal{W}_n \) be \( S_n \) or \( B_n \). For \( a \geq 0 \), the \( \text{FI}_{\mathcal{W}} \)-module structure on \( \text{M}_{\mathcal{W}}(a) \) extends to an \( \text{FI}_{\mathcal{W}} \sharp \)-structure.

**Proof of Proposition 4.39.** By Definition 3.9,

\[
\text{M}_{\mathcal{W}}(a)_m = \text{Span}_k \{ e_s \mid s \in \text{Hom}_{\text{FI}_{\mathcal{W}}}(a, m) \}.
\]

Take any \( \text{FI}_{\mathcal{W}} \sharp \)-morphism \( f : m_0 \rightarrow n_0 \). We define

\[
f \cdot e_s = \begin{cases} 
    e_{f \circ s} & \text{if } 0 \notin f(s(a)) \\
    0 & \text{otherwise}
\end{cases}
\]

The condition \( 0 \notin f(s(a)) \) is the statement that \( f \circ s \) is an injective map \( a \rightarrow n \). Given \( g : n_0 \rightarrow p_0 \), we note that

\[
0 \in (g \circ f)(s(a)) \iff 0 \in g((f \circ s)(a));
\]

this implies that the action \((g \circ f) \cdot e_s = g \cdot (f \cdot e_s)\) is functorial. \( \square \)

**Corollary 4.40.** (\( \text{M}_{\mathcal{W}}(U) \) is an \( \text{FI}_{\mathcal{W}} \sharp \)-module). Let \( \mathcal{W}_n \) be \( S_n \) or \( B_n \). Given a \( \mathcal{W}_n \)-representation \( U \), the \( \text{FI}_{\mathcal{W}} \)-module

\[
\text{M}_{\mathcal{W}}(U) := \text{M}_{\mathcal{W}}(a) \otimes_{k[\mathcal{W}_n]} U
\]

has the structure of an \( \text{FI}_{\mathcal{W}} \sharp \)-module.
Remark 4.41. We note the proof of Proposition 4.39 does not work in type $D$, as the space $M_D(m)_m \subseteq M_{BC}(m)_m$ is not closed under the action of action of the $\text{FI}_{BC}^{\#}$-morphisms. Choose any $s \in D_m$ and $f$ such that $f \in B_m, f \not\in D_m$. Then there is a basis element $e_s \in M_D(m)_m$, but its designated image $e_{fs}$ under the $\text{FI}_{BC}^{\#}$ morphism $f : m_0 \rightarrow m_0$ is not an element of $M_D(m)_m$.

4.6.3 Classification of $\text{FI}_{BC}^{\#}$-modules

The structure of an $\text{FI}_{BC}^{\#}$-module is highly constrained. In Corollary 4.40 we saw that $M_{BC}(U)$ is an $\text{FI}_{BC}^{\#}$-module. Just as Church–Ellenberg–Farb proved with $\text{FI}_A^{\#}$-modules [CEF12, Theorem 2.24], we will now find that all $\text{FI}_{BC}^{\#}$-modules are sums of $\text{FI}_{BC}^{\#}$-modules of this form.

Theorem 4.42. ($\text{FI}_{W}^{\#}$-modules take the form $\bigoplus_{a=0}^{\infty} M_W(U_a)$). Let $W_n$ be $S_n$ or $B_n$. Every $\text{FI}_{W}^{\#}$-module $V$ is of the form

$$V = \bigoplus_{a=0}^{\infty} M_W(U_a), \quad U_a \text{ a representation of } W_a \text{ (possibly } U_a = 0),$$

and moreover that the maps

$$M_W(-) : \bigoplus_a W_a-\text{Rep} \longrightarrow \text{FI}_W-\text{Mod} \quad \text{and} \quad H_0(-) : \text{FI}_W-\text{Mod} \longrightarrow \bigoplus_a W_a-\text{Rep}$$

are inverses, defining an equivalence of categories.

This theorem is proved for $\text{FI}_A$ in [CEF12, Theorem 2.24], and their proof adapts readily to the general case.

Church–Ellenberg–Farb proceed by induction. Assume $V$ is an $\text{FI}_A^{\#}$-module, and fix $n$ such that $V_m = 0$ for all $m < n$ (possibly $n = 1$). They define a particular idempotent endomorphism of $\text{FI}_A^{\#}$-modules $E : V \rightarrow V$, and show that the resultant decomposition

$$V = EV \oplus \ker(E) \cong M_A(V_a) \oplus \ker(E).$$

Their same proof carries through exactly in the case $\text{FI}_{BC}^{\#}$-modules if we
redefine the endomorphism $E$ as follows, for $m \geq n,$

$$E_m : V_m \to V_m$$

$$E_m = \sum_{\substack{S \subseteq m, \ |S| = n \\ S \text{ symmetric}}} I_S$$

where $I_S(j) = \begin{cases} j & \text{if } j \in S, \\ 0 & \text{otherwise} \end{cases}$

$$\in \text{Hom}_{\text{FI}^{\sharp}}(m_0, m_0).$$

In the notation of Remark 4.37, $I_S = (S, S, \text{identity}).$ Again we conclude

$$V \cong M_{BC}(V_n) \oplus \ker(E)$$

with $\ker(E)$ vanishing in degree $n,$ and the desired decomposition follows by induction on $n.$

Church–Ellenberg–Farb further argue that, since maps $F : V \to V'$ of $\text{FI}_A\#$–modules commute with $E$ and preserves this decomposition, the map $M_A(V_n) \to M_A(V'_n)$ must be induced by some map of $S_n$–representations $V_n \to V'_n.$ Their arguments hold for $\text{FI}_{BC}$–modules, and imply the equivalence of the categories $\bigoplus_a B_a\text{-Rep}$ and $\text{FI}_{BC}\text{-Mod}.$

**Corollary 4.43.** Let $W_n$ be $S_n$ or $B_n.$ With $V$ an $\text{FI}_W\#$–module as above, any sub–$\text{FI}_W\#$–module of $V$ is of the form $\bigoplus_{a=0}^{\infty} M_W(U'_a)$ for some $W_a$–representation $U'_a \subseteq U_a.$

**Corollary 4.44.** Let $W_n$ be $S_n$ or $B_n.$ If $V$ is an $\text{FI}_W$–module generated in degree $\leq d,$ then any sub–$\text{FI}_W\#$–module of $V$ is also generated in degree $\leq d.$

Example 3.16 and Proposition 4.18 imply:

**Corollary 4.45.** Let $W_n$ be $S_n$ or $B_n.$ An $\text{FI}_W\#$–module $V$ has injectivity degree 0. If $V$ is generated in degree $d,$ then $V$ has stability degree $\leq d,$ as do its sub–$\text{FI}_W$–submodules.

**Corollary 4.46.** If $V$ is an $\text{FI}_{BC}\#$–module over characteristic zero, generated in degree $d.$ Then $\{V_n\}$ is uniformly representation stable in degree $\leq 2d.$

**Proof of Corollary 4.46.** Any such $\text{FI}_{BC}\#$–module has weight $\leq d$ by Theorem 4.4, and stability degree $\leq d$ by Corollary 4.45. The conclusion follows from Theorem 4.27. \[\square\]
**Corollary 4.47.** Let $W_n$ be $S_n$ or $B_n$. An FI$W_n$–module $V$ is completely determined by the sequence of $W_n$–representations $\{V_n\}$, equivalently (over characteristic zero) by the sequence of characters $\{\chi_n\}$.

**Proof of Corollary 4.47.** We can construct $H_0(V)$ inductively from the sequence $\{V_n\}$ of $W_n$–representations:

$$H_0(V)_0 = V_0 \quad \text{and} \quad H_0(V)_n = V_n / \text{span} \bigoplus_{k<n} M_W(H_0(V)_k)_n$$

The FI$W_n$–module structure on $V$ is determined by the identification

$$V \cong M_W(H_0(V)). \quad \square$$

**Corollary 4.48.** If $k$ is a field, and $V$ an FI$B/C_n^\#$–module $k$. Then

$V$ is finitely generated in degree $\leq d$

$\iff \dim_k(V_n) = O(n^d)$

$\iff \dim_k(V_n) = P(n)$ for some polynomial $P \in \mathbb{Q}[T]$ of degree at most $d$

If $k$ is a commutative ring, then an FI$B/C_n^\#$–module $V$ over $k$ is finitely generated in degree $\leq d$ if and only if $V_n$ is generated by $O(n^d)$ elements.

**Proof of Corollary 4.48.** The statements follow from Theorem 4.42 and the same argument used to prove [CEF12, Corollary 2.27]. In type B/C, the polynomial $P$ is determined by the formula

$$\dim_k M_{BC}(U)_n = \binom{n}{m} \dim_k U \quad \text{for a } B_m \text{–representation } U. \quad \square$$

### 4.6.4 Criteria for an FI$W_n^\#$–module structure

To verify that a given sequence has the structure of an FI$W_n^\#$–module it is convenient to define an FI$W_n^\#$–module structure in terms of compatible FI$W$–module and co–FI$W$–module structures.

**Proposition 4.49.** Let $W_n$ be $S_n$ or $B_n$. Suppose that $V$ is an FI$W$–module and a co–FI$W$–module, satisfying the following Condition (•):

For every pair of morphisms $f : m \to n$ and $g : m' \to n$, the map $g^* \circ f_*$ factors
through any pullback \((m \times_n m', p, q)\) in the sense that \(g^* \circ f_* = q_\ast \circ p^\ast\).

\[
\begin{array}{ccc}
 m \times_n m' & \xrightarrow{p} & m \\
 & & \downarrow f \\
 m & \xrightarrow{g} & m' \\
 & & \downarrow q \\
 & & n
\end{array}
\]

Then \(V\) is an \(FI_{W^\sharp}\)-module.

Conversely, any \(FI_{W^\sharp}\)-module structure on \(V\) makes \(V\) an \(FI_W\)-module and a co-\(FI_W\)-module satisfying this condition.

Note that \(m \times_n m' = d\), where \(d = |f(m) \cap g(m')|\), and that the maps \(p\) and \(q\) are inclusions

\[
p : d \to f^{-1}(f(m) \cap g(m')) \subseteq m \quad \text{and} \quad q : d \to g^{-1}(f(m) \cap g(m')) \subseteq m'
\]

making the diagram commute. This choice of \(FI_W\) morphisms \(p, q\) is unique up to the action of \(W_d\) on \(m \times_n m'\).

**Remark 4.50.** We note that if we take \(f\) and \(g\) to be the identity maps \(id_n\), and \(p, q\) to be any (signed) permutation \(\sigma \in W_n\), Condition \((*)\) implies \(\sigma^* = (\sigma_\ast)^{-1}\).

More generally, if we take a pullback of \(f = g : m \to n\), we conclude \(f^* \circ f_* = id_m\). In contrast, \(f_* \circ f^*\) need not be the identity on \(V_n\).

\[
\begin{array}{ccc}
 n & \xleftarrow{id_n} & n \\
 & \swarrow \sigma & \searrow \sigma \\
 n & \xrightarrow{id_n} & n
\end{array}
\]

**Proof of Proposition 4.49.** Suppose that \(V\) is an \(FI_W\)-module and a co-\(FI_W\)-module satisfying Condition \((*)\). We can extend these to an \(FI_{W^\sharp}\)-module structure on \(V\) as follows:

Given an \(FI_{W^\sharp}\) morphism \(f \in \text{Hom}_{FI_{W^\sharp}}(m, n)\) of rank \(d\), we can factor \(f\) through \(d_0\)

\[
m_0 \xrightarrow{p} d_0 \xrightarrow{i} n_0
\]

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where $p$ is a projection and $i$ is an injection. The maps $p$ and $i$ are well-defined up to the action of $\mathcal{W}$ on $d_0$. The projection $p$ has a right inverse, an injective map $\bar{p} : d_0 \to m_0$. The restriction of $i$ and $\bar{p}$ to $d \subseteq d_0$ are FI$_W$ morphisms, which (by abuse of notation) we also denote $i$ and $\bar{p}$. We define the action of $f$ on $V$ by $f_* := i_* \circ \bar{p}^*$.

\[
\begin{array}{ccc}
V_m & \overset{\bar{p}^*}{\longrightarrow} & V_d \\
\downarrow & & \downarrow \\
\overset{i_*}{\longrightarrow} & & \overset{i_*}{\longrightarrow} \\
V_n & \overset{i_*}{\longrightarrow} & V_n
\end{array}
\]

This action is well-defined, since, given another factorization of $f$

\[
m_0 \overset{\sigma^{-1} \circ p}{\longrightarrow} d_0 \overset{i \circ \sigma}{\longrightarrow} n_0 \quad \text{for } \sigma \in \mathcal{W}_d
\]

we would find that $\bar{p} \circ \sigma$ is the right inverse for $\sigma^{-1} \circ p$, and so

\[
f_* = (i \circ \sigma)_* \circ (\bar{p} \circ \sigma)^* = i_* \circ \sigma_* \circ \sigma^* \circ \bar{p}^* = i_* \circ \bar{p}^*.
\]

We can check that this assignment respects composition in FI$_W^\sharp$. Suppose that $f_1 : m_0^1 \to m_0^2$ and $f_2 : m_0^2 \to m_0^3$ are FI$_W^\sharp$ morphisms of rank $d_1$ and $d_2$. We can factor

\[
f_1 = i_1 \circ p_1 \quad \text{and} \quad f_2 = i_2 \circ p_2
\]

into composites of projections and injections.

By Condition (*), the map $(\bar{p}_2)^* \circ (i_1)_*$ factors through the pullback

\[
(\bar{p}_2)^* \circ (i_1)_* = q_* \circ p^*.
\]
This concludes the proof that $V$ is an FI$_W^*$–module.

Conversely, suppose $V$ is an FI$_W^*$–module. For any FI$_W$ morphism $f : m \to n$, we can extend $f$ to an FI$_W^*$ morphism $m_0 \to n_0$ by taking $f(0) = 0$. Then by assigning $f_* : V_m \to V_n$ to act as this FI$_W^*$ morphism, and $f^* : V_n \to V_m$ to act as its left inverse, we endow $V$ with an FI$_W$–module and co–FI$_W$–module structure. Using the functoriality of $V$ as an FI$_W^*$–module, it is straightforward to check that these assignments satisfy Condition ($*$). □

5 The character polynomials

5.1 Character polynomials for the symmetric groups

A major result of Church–Ellenberg–Farb is that, given a finitely generated FI$_A$–module over a field of characteristic zero, the characters of the $S_n$–representations $V_n$ have a particularly nice form. They are, for $n$ sufficiently large, given by a character polynomial (independent of $n$), as we now define.

Definition 5.1. (Character Polynomials for $S_n$). Let $k$ be a characteristic zero field. For $r \geq 1$ and $n \geq 0$, let $X_r$ be the class function on $S_n$ defined by

$$X_r(s) := \text{the number of } r\text{–cycles in the cycle type of } s.$$
the disjoint union $\bigsqcup_{n=0}^{\infty} S_n$, however, the functions $X_r$ are algebraically independent, and define a polynomial ring $k[X_1, X_2, \ldots]$. We call elements of this ring the character polynomials of the symmetric groups, and define the total degree of a character polynomial by assigning $\deg(X_r) = r$.

**Theorem 5.2.** [CEF12, Theorem 2.67] (Polynomiality of characters for $S_n$). Let $k$ be a field of characteristic zero, and let $V$ be a finitely generated $\text{FI}_A$–module with weight $\leq d$ and stability degree $\leq s$. There exists a unique polynomial $P_V \in k[X_1, X_2, \ldots]$ such that $\chi_{V_n}(\sigma) = P_V(\sigma)$ for all $n \geq s + d$ and all $\sigma \in S_n$.

The polynomial $P_V$ has degree at most $d$. By setting $F_V(n) = P_V(n, 0, \ldots, 0)$ we have:

$$\dim_k(V_n) = \chi_{V_n}(\text{id}) = F_V(n) \quad \text{for all } n \geq s + d.$$ 

If $V$ is an $\text{FI}_A^\sharp$–module then the above equalities hold for all $n \geq 0$.

**Background and formulas for character polynomials of the symmetric groups.**

Church–Ellenberg–Farb [CEF12] prove these theorems using the classical result that the character of the irreducible representations $V(\lambda)_n$, written here in the notation defined in Section 2.2, is given by a character polynomial $P^\lambda$ independent of $n$. These character polynomials were described by Murnaghan in 1951 [Mur51] and by Specht in 1960 [Spe60]. This independence of the characters from $n$ was known to Murnaghan in 1937 [Mur37].

Formulas for the character polynomial $P^\lambda$ associated to the irreducible representations $V(\lambda)_n$ are given in Macdonald’s book [Mac79]. In 2009, new formulas were published by Garsia and Goupil [GG09], which they used to study the combinatorics of Kronecker coefficients. To state these formulas, we use the following notation:

Let $\lambda$ be a partition of $n$. We define the length of $\lambda$,

$$\ell(\lambda) = \text{the number of parts of } \lambda.$$ 

For $r \geq 1$, we write

$$n_r(\lambda) = \text{the number of parts of } \lambda \text{ of length } r.$$ 

We further define the integer $z_\lambda$ so that $\frac{n!}{z_\lambda}$ is the number of elements in $S_n$. 

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5.1 Character polynomials for the symmetric groups

of cycle type \( \lambda \). Explicitly,

\[
z_\lambda = \prod_r n_r(\lambda) n_r(\lambda)!
\]

We write \( \chi^\lambda \) to mean the character of the irreducible \( S_n \)-representation \( V_\lambda \).

We define \( \chi^\emptyset := 1 \).

If \( \chi \) is any class function on \( S_n \), and \( \rho \) a partition of \( n \), then we write \( \chi^\rho \) to mean the value of \( \chi \) on elements of cycle type \( \rho \).

**Definition 5.3. (Generalized Binomial Coefficients).** Let \( \rho \) be a partition of \( m \). Following Macdonald [Mac79, I.7.13(a)], we define generalized binomial coefficients:

\[
\binom{X}{\rho} := \prod_r \binom{X_r}{n_r(\rho)} = \prod_r X_r(X_r - 1) \cdots (X_r - n_r(\rho) + 1) / n_r(\rho)!,
\]

For example,

\[
\binom{X}{(3, 2, 2, 1)} := \binom{X_3}{1} \binom{X_2}{2} \binom{X_1}{1} = X_3 X_2(X_2 - 1)/2 X_1 = 1/2 X_3 X_2 X_1 - 1/2 X_3 X_2 X_1
\]

**Remark 5.4. (Indicator Functions for the Conjugacy Classes of \( S_m \)).** Given a partition \( \lambda \) of \( m \), and \( s \in S_m \), note that the generalized binomial coefficient

\[
\binom{X}{\lambda}(s) = \begin{cases} 1 & \text{if } s \text{ has cycle type } \lambda, \\ 0 & \text{otherwise}. \end{cases}
\]

This polynomial’s restriction to \( S_m \subseteq \bigsqcup_{n \geq 0} S_n \) is an indicator function for the conjugacy class of cycle type \( \lambda \). Polynomials of this form give a convenient basis for \( k[\{X_1, X_2, \ldots\}] \).

Since the binomial coefficient in an indeterminate \( X \)

\[
\binom{X}{m} = X(X - 1)(X - 2) \cdots (X + 1)/m!
\]

is a polynomial in \( X \) of degree \( m \), the generalized binomial coefficient \( \binom{X}{\lambda} \) is a polynomial of total degree \( \sum r \cdot n_r(\lambda) = m \) in \( k[\{X_1, X_2, \ldots\}] \).

**Proposition 5.5. ([Mac79, I.7.14])**

For \( \lambda \vdash m \), a formula for the character \( P^{\lambda} \) of the irreducible \( S_n \)-representation
5.2 Character polynomials in type B/C and D

$V(\lambda)_n$ is given as follows:

$$P^\lambda = \sum_{\text{Partitions } \rho, \sigma \text{ such that } |\rho| + |\sigma| = |\lambda|} \frac{(-1)^{f(\sigma)}}{z_\sigma} \chi^\lambda_{(\rho,\sigma)} \left( X_\rho \right).$$

By Remark 5.4, this is a character polynomial of degree $|\lambda| = m$.

5.2 Character polynomials in type B/C and D

We can analogously define character polynomials for the hyperoctahedral group $B_n$ and the even-signed permutation groups $D_n$. We will first develop the theory in type B/C, and from there we can use our methods of inducing FI$_D$–modules to FI$_{BC}$ to recover results in type D.

Recall from Section 2.1.2 that the conjugacy classes for $B_n$ are classified by double partitions $(\alpha, \beta)$ of $n$, designating the signed cycle type of each element. Given a character (or class function) $\chi$ of a $B_n$–representation, we will write $\chi_{(\alpha,\beta)}$ to denote the value of $\chi$ on elements of signed cycle type $(\alpha, \beta)$.

**Definition 5.6. (Character Polynomials for $B_n$).** For $r \geq 1$ and $n \geq 0$, let $X_r$ and $Y_r$ be the class functions on $B_n$ defined by

$$X_r(w) = \text{the number of positive } r\text{-cycles in the cycle type of } w.$$ $Y_r(w) = \text{the number of negative } r\text{-cycles in the cycle type of } w.$$ Again, these functions form a polynomial ring $k[X_1, Y_1, X_2, Y_2, \ldots]$ where we designate $\text{deg}(X_r) = \text{deg}(Y_r) = r$.

**Example 5.7.** Consider $V_n = V((n-1), (1)) \cong k^n$, the canonical $B_n$–representation by signed permutation matrices. Recall from Example 1.6 that a signed permutation matrix has a 1 appearing on its diagonal for each positive one-cycle $(i)(-i)$, and a $-1$ appearing on its diagonal for every negative one-cycle $(-i \ i)$. The characters of $V_n$ are

$$\chi^V = X_1 - Y_1 \quad \text{for all } n.$$
Similarly, one can compute that the characters of $V_n = \bigwedge^2 V((n-1), (1))$ are
\[
\chi^{\bigwedge^2 V} = \frac{1}{2}X_1(X_1 - 1) + \frac{1}{2}Y_1(Y_1 - 1) - X_1Y_1 - X_2 + Y_2 \quad \text{for all } n,
\]
and that the characters of $V_n = \text{Sym}^2 V((n-1), (1))$ are
\[
\chi^{\text{Sym}^2 V} = \frac{1}{2}X_1(X_1 + 1) + \frac{1}{2}Y_1(Y_1 + 1) - X_1Y_1 + X_2 - Y_2 \quad \text{for all } n.
\]

**Remark 5.8. (Indicator Functions for the Conjugacy Classes of $B_m$)** Given a double partition $(\lambda, \nu)$ of $m$, and $w \in B_m$, note that the degree $m$ character polynomial
\[
\left( \begin{array}{c} X \\ \lambda \\ \end{array} \right) \left( \begin{array}{c} Y \\ \nu \\ \end{array} \right)(w) = \begin{cases} 1 & \text{if } w \text{ has signed cycle type } (\lambda, \nu), \\ 0 & \text{otherwise}. \end{cases}
\]
Again $\left( \begin{array}{c} X \\ \lambda \\ \end{array} \right) \left( \begin{array}{c} Y \\ \nu \\ \end{array} \right)$ is a polynomial of degree $\sum r \cdot n_r(\lambda) + \sum r \cdot n_r(\nu) = m$ that is an indicator function on $B_m$ of the signed conjugacy class $(\lambda, \nu)$.

**Remark 5.9. (Restricting Characters to $S_n \subseteq B_n$).** The symmetric group $S_n$ forms the subgroup of $B_n$ generated by the (necessarily positive) cycles that preserve signs. Thus, if $V$ is a $B_n$-representation with character $\chi^V$ given by some character polynomial $P_V \in k[X_1, Y_1, X_2, Y_2, \ldots]$, the character for $\text{Res}_{B_n}^{S_n} V$ is given by the character polynomial in $k[X_1, X_2, \ldots]$ obtained by evaluating each variable $Y_r$ in $P_V$ at 0.

**5.2.1 The character of $V(\lambda, \mu)_n$ is independent of $n$**

Recall from Section 2.2 that, given a double partition $(\lambda, \nu)$ of $d$ with $\nu \vdash m$, then $V(\lambda, \nu)_n$ denotes the irreducible $B_n$-representation associated to the double partition $(\lambda|m-n|, \nu)$.

**Theorem 5.10. (The character of $V(\lambda, \mu)_n$ is independent of $n$).** If $(\lambda, \nu)$ is a double partition of $d$, then there is a character polynomial $P^{(\lambda, \nu)}$ of degree at most $d$ equal to the character of the irreducible $B_n$-representations $V(\lambda, \nu)_n$ for all $n$.

Explicitly, $P^{(\lambda, \nu)}$ is given as follows. Let $m = |\nu|$, and define $\mu$ so that $\nu = \mu|m|$. 

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for $\nu = \emptyset$ take $\mu = \emptyset$. Then

$$P(\lambda, \nu) = \sum_{(\alpha, \beta)} \sum_{\text{Partitions } \rho, \sigma} (-1)^{\ell(\beta)} \sum_{\text{Partitions } \xi, \eta} (-1)^{\ell(\eta)} \chi^\lambda_{(\rho \cup \sigma)} \left( \left( X_r - n_r(\alpha) + n_r(\beta) \right) \left( X_r \right) \left( \frac{X_r}{n_r(\alpha)} \right) \left( \frac{X_r}{n_r(\beta)} \right) \right).$$

For example,

$$P(\emptyset, \emptyset) = (X_1 - Y_1)(X_1 + Y_1 - 2)$$

$$P(\emptyset, \emptyset, \emptyset) = \left( \frac{X_1}{2} \right) + \left( \frac{Y_1}{2} \right) - X_2 - X_1Y_1 + Y_2$$

We will prove Theorem 5.10 in four steps. Our first step, Lemma 5.11, is to prove the result for representations of the form $V(\lambda, \emptyset)_n$. In the second step, Lemma 5.12, we produce a formula for characters of representations $V(\emptyset, \lambda)_n$. Our third step, Lemma 5.14, is to compute the character of an induced representation of the form $\text{Ind}_{B_n \times B_{n-m}} U \boxtimes U'$, and the final step will be to derive the formula in Theorem 5.10.

**Lemma 5.11. (Step 1: The character of $V(\lambda, \emptyset)_n$).** Let $\lambda$ be a partition of $m$. Then, for each $n$, the character of $B_n$-representation $V(\lambda, \emptyset)_n$ is given by the character polynomial $P(\lambda, \emptyset)$.

$$P(\lambda, \emptyset) = \sum_{\text{Partitions } \rho, \sigma} \sum_{\text{Partitions } \xi, \eta} (-1)^{\ell(\sigma)} \chi^\lambda_{(\rho \cup \sigma)} \left( \left( X_r + Y_r \right) \left( \frac{X_r + Y_r}{n_r(\rho)} \right) \right).$$

**Proof of Lemma 5.11.** As described in Section 2.1.2, the $B_n$-representations $V(\lambda, \emptyset)_n$ are by definition the pullback of the $S_n$-representation $V(\lambda)_n$ under the natural surjection $B_n \twoheadrightarrow S_n$. This map takes positive and negative $r$-cycles in $B_n$ to $r$-cycles in $S_n$; a signed permutation of signed cycle type $(\mu, \nu)$ is mapped to
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a permutation of type \( \mu \cup \nu \). It follows that a hyperoctahedral character polynomial for \( V(\lambda, \varnothing)_n \) can be obtained from the symmetric character polynomial for \( V(\lambda)_n \) by replacing each \( X_r \) with the sum \( X_r + Y_r \). The formula therefore follows from Macdonald’s formula, Proposition 5.5.

**Lemma 5.12. (Step 2: The character of \( V(\varnothing, \lambda[n]) \)).** Let \( n \) be fixed, and consider a partition \( \lambda[n] \) of \( n \). Then the character \( \chi(\varnothing, \lambda[n]) \) of the \( B_n \)-representation \( V(\varnothing, \lambda[n]) \) takes the following value on \( B_n \) elements of signed cycle type \( (\alpha, \beta) \):

\[
\chi(\varnothing, \lambda[n], (\alpha, \beta)) = (-1)^{\ell(\beta)} P(\lambda, 0)(\alpha, \beta).
\]

**Remark 5.13.** We note that this formula for the character \( V(\varnothing, \lambda[n]) \) is not a \( B_n \) character polynomial, since the coefficient \((-1)^{\ell(\beta)}\) depends on the cycle type \((\alpha, \beta)\).

**Proof of Lemma 5.12.** Recall from Section 2.1.2 that

\[ \varepsilon : B_n \to B_n / D_n \cong \{ \pm 1 \} \]

is the character mapping an element \( w \in B_n \) to \(-1\) precisely when \( w \) reverses an odd number of signs. Since positive cycles reverse an even number of signs, and negative cycles reverse an odd number, the character \( \varepsilon \) takes the value \((-1)^{\ell(\beta)}\) on elements of signed cycle type \((\alpha, \beta)\).

By definition,

\[ V(\varnothing, \lambda[n]) = V(\lambda[n], \varnothing) \otimes \varepsilon = V(\lambda, \varnothing)_n \otimes \varepsilon \]

and so the formula follows from Lemma 5.12. \( \square \)

**Lemma 5.14. (Step 3: The character of \( \text{Ind}_{B_m \times B_{n-m}} B_n \otimes U' \)).** Suppose that \( U \) is a \( B_m \)-representation with character \( \chi_U \), and that \( U' \) is a \( B_{n-m} \)-representation, with character \( \chi_{U'} \). Then the character \( \chi_{(U \otimes U')/B_n} \) of the induced \( B_n \)-representation \( \text{Ind}_{B_m \times B_{n-m}} B_n \otimes U' \) is given by:

\[
\chi_{(\rho, \sigma)}^{(U \otimes U')} = \sum_{(\alpha, \beta), |\alpha| + |\beta| = m, \delta, \gamma} \chi_{(\alpha, \beta)}^{U} \chi_{(\delta, \gamma)}^{U'} \left( \frac{X}{\alpha} \right) \left( \frac{Y}{\beta} \right) \left( \rho, \sigma \right)
\]

where \( (\delta, \gamma) \) is the double partition of \((n-m)\) such that \((\rho, \sigma) = (\alpha \cup \delta, \beta \cup \gamma)\). It is well-defined, since \( \left( \frac{X}{\alpha} \right) \left( \frac{Y}{\beta} \right) \left( \rho, \sigma \right) \) will vanish unless such a decomposition of

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$(\rho, \sigma)$ exists.

We note that Lemma 5.14 holds when $k$ is $\mathbb{Z}$ or any field.

**Proof of Lemma 5.14.** Let $w \in (B_m \times B_{n-m})$, and let $p_m$ and $p_{n-m}$ denote the projections of $w$ onto $B_m$ and $B_{n-m}$, respectively. The character of the $(B_m \times B_{n-m})$–representation $U \boxtimes U'$ is

$$\chi^{U \boxtimes U'} = \chi^U(p_m(w)) \cdot \chi^{U'}(p_{n-m}(w)).$$

The character of the induced representation $\text{Ind}_{B_m \times B_{n-m}}^{B_n} U \boxtimes U'$ is

$$\chi^{(U,U')} (w) = \sum_{\{\text{cosets } C \mid w.C = C\} \forall s \in C} \chi^{U \boxtimes U'}(s^{-1}ws)$$

$$= \sum_{\{\text{cosets } C \mid w.C = C\} \forall s \in C} \chi^U(p_m(s^{-1}ws)) \cdot \chi^{U'}(p_{n-m}(s^{-1}ws))$$

summed over all cosets $C$ in $B_n/(B_m \times B_{n-m})$ that are stabilized by $w$, equivalently, those cosets $C$ such that $s^{-1}ws \in (B_m \times B_{n-m})$ for any $s \in C$.

The cosets $B_n/(B_m \times B_{n-m})$ correspond to the orbit of the sets

$$\{-1,1\}, \ldots, \{-m,m\} \quad \text{and} \quad \{-(m+1),(m+1)\}, \ldots, \{-n,n\}$$

under the action of $B_n$; they are indexed by all partitions of

$$\{-1,1\}, \{-2,2\}, \ldots, \{-n,n\}$$

into a set of $m$ blocks and a set of $(n-m)$ blocks. An element $w \in B_n$ can be conjugated into $(B_m \times B_{n-m})$ precisely when its positive and negative cycles can be partitioned into a set of cycles of total length $m$, and a set of cycles of total length $(n-m)$. If we fix a double partition $(\alpha, \beta)$ of $m$, then the cycles of $w$ can be factored into an element $w_m$ of cycle type $(\alpha, \beta)$ and its complement $w_{n-m}$ in the following number of ways (possibly 0):

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}(w) := \left( \begin{array}{c} X_1(w) \\ n_1(\alpha) \end{array} \right) \left( \begin{array}{c} X_2(w) \\ n_2(\alpha) \end{array} \right) \cdots \left( \begin{array}{c} X_m(w) \\ n_m(\alpha) \end{array} \right) \left( \begin{array}{c} Y_1(w) \\ n_1(\beta) \end{array} \right) \left( \begin{array}{c} Y_2(w) \\ n_2(\beta) \end{array} \right) \cdots \left( \begin{array}{c} Y_m(w) \\ n_m(\beta) \end{array} \right).$$

Each such factorization of $w$ corresponds to a coset $C \in B_n/(B_m \times B_{n-m})$ that is stabilized by $w$. For any representative $s \in C$, $p_m(s^{-1}ws)$ has signed cycle
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Thus, if we denote the signed cycle type of $w_{n-m}$ by $(\delta, \gamma)$, we conclude

$$\chi^{(U, U')} (w) = \left( \sum_{(\alpha, \beta) \atop |\alpha| + |\beta| = m} \chi^{(\alpha, \beta)} (\delta, \gamma) \chi^{(\alpha)} (X) \chi^{(\beta)} (Y) \right) (w). \quad \square$$

**Proof of Theorem 5.10. (Step 4: The Character of $V(\lambda, \nu)_n$).** Let $(\lambda, \nu)$ be a double partition of $d$, with $|\nu| = m$ and $|\lambda| = (d - m)$. From the construction of the irreducible representations of $B_n$ described in Section 2.1.2,

$$V(\lambda, \nu)_n = \text{Ind}_{B_{n-m} \times B_m}^{B_n} V(\lambda, \emptyset)_{n-m} \boxtimes V(\emptyset, \nu).$$

We wish to compute a character polynomial $P^{(\lambda, \nu)}$ which gives the character for $V(\lambda, \nu)_n$ for each $n$.

By Lemma 5.14,

$$\chi^{(\lambda[n], \nu)} (w) = \left( \sum_{(\alpha, \beta) \atop |\alpha| + |\beta| = |\nu|} \chi^{(\alpha, \beta)} (\lambda[n-m], \emptyset) \chi^{(\alpha)} (X) \chi^{(\beta)} (Y) \right) (w)$$

with $(\delta, \gamma)$ the double partition of $(n - m)$ such that $(\alpha \cup \delta, \beta \cup \gamma)$ is the signed cycle type of $w$.

We write $\nu = \mu[m]$, where $\mu$ is the partition obtained from $\nu$ by discarding the largest part; thus, by Lemmas 5.12 and 5.11,

$$\chi^{(\emptyset, \mu[m])} (\alpha, \beta) = (-1)^{\ell(\beta)} p^{(\mu, 0)} (\alpha, \beta)$$

$$= (-1)^{\ell(\beta)} \sum_{\text{Partitions } \rho, \sigma \atop |\rho| + |\sigma| = |\mu|} \left( \chi^{(\rho \cup \sigma)} (\mu) \chi^{(\rho \cup \sigma)} (X) \chi^{(\rho \cup \sigma)} (Y) \right) (\alpha, \beta)$$

Moreover, since for each $r$ we have

$$n_r(\delta) = X_r(w) - n_r(\alpha) \quad \text{and} \quad n_r(\gamma) = Y_r(w) - n_r(\beta),$$

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we can use Lemma 5.11 to compute:

\[
\chi_{(\delta, \gamma)}^{(\lambda[n-m], \emptyset)} = P^{(\lambda, 0)}(\delta, \gamma) = \sum_{\text{Partitions } \xi, \eta \mid |\xi| + |\eta| = |\lambda|} \frac{(-1)^\ell(\eta) \chi^\lambda(\xi \cup \eta)}{z_\eta} \prod_r \left( X_r + Y_r - n_r(\xi) \right)(\delta, \gamma) = \\
= \sum_{\text{Partitions } \xi, \eta \mid |\xi| + |\eta| = |\lambda|} \frac{(-1)^\ell(\eta) \chi^\lambda(\xi \cup \eta)}{z_\eta} \prod_r \left( X_r - n_r(\alpha) + Y_r - n_r(\beta) \right)(w)
\]

Putting these together,

\[
\chi_{(\delta, \gamma)}^{(\lambda[n-m], \nu)}(w) = \sum_{(\alpha, \beta) \mid |\alpha| + |\beta| = |\nu|} \chi_{(\alpha, \beta)}^{(\lambda[n-m], \emptyset)} \left( \begin{array}{c}
X \\
\alpha
\end{array} \right) \left( \begin{array}{c}
Y \\
\beta
\end{array} \right) (w) = \\
= \left( \sum_{(\alpha, \beta) \mid |\alpha| + |\beta| = |\nu|} \chi_{(\alpha, \beta)}^{(\lambda[n-m], \emptyset)} \left( \begin{array}{c}
X \\
\alpha
\end{array} \right) \left( \begin{array}{c}
Y \\
\beta
\end{array} \right) (w) \right) (-1)^\ell(\beta) \left( \sum_{\text{Partitions } \rho, \sigma \mid |\rho| + |\sigma| = |\mu|} \frac{(-1)^\ell(\sigma) \chi^\mu(\rho \cup \sigma)}{z_\sigma} \prod_r \left( n_r(\rho) \right) \right) \\
= \left( \sum_{\text{Partitions } \xi, \eta \mid |\xi| + |\eta| = |\lambda|} \frac{(-1)^\ell(\eta) \chi^\lambda(\xi \cup \eta)}{z_\eta} \prod_r \left( X_r - n_r(\alpha) + Y_r - n_r(\beta) \right) \right)(w)
\]

which gives the desired formula.

Note that the degree of \( P^{(\lambda, \nu)} \)

\[
\deg(P^{(\lambda, \nu)}) \leq \left( |\alpha| + |\beta| + \max_{\text{Partitions } \xi, \eta \mid |\xi| + |\eta| = |\lambda|} |\xi| \right) \\
= (|\nu| + |\lambda|) \\
= d
\]

so \( \deg(P^{(\lambda, \nu)}) \) is at most the size of the double partition \((\lambda, \nu)\), as claimed. This concludes the proof. \(\square\)

### 5.3 Finite generation and character polynomials

We can now use Theorem 5.10 to prove the existence of character polynomials for finitely generated \( \text{FI}_{\mathcal{B}C} \)-modules in Theorem 5.15. As a consequence of Theorems 5.2 and 5.15, we can determine a number of constraints on the
structure of finitely generated FI$W$–modules.

**Theorem 5.15. (Characters of finitely generated FI$W$–modules are eventually polynomial).** Let $k$ be a field of characteristic zero. Suppose that $V$ is a finitely generated FI$BC$–module with weight $\leq d$ and stability degree $\leq s$, or, alternatively, suppose that $V$ is a finitely generated FI$D$–module with weight $\leq d$ such that $\text{Ind}_{D}^{BC} V$ has stability degree $\leq s$. In either case, there is a unique polynomial $F_{V} \in k[X_{1}, Y_{1}, X_{2}, Y_{2}, \ldots]$ such that the character of $W_{n}$ on $V_{n}$ is given by $F_{V}$ for all $n \geq s + d$. The polynomial $F_{V}$ has degree $\leq d$, with $\deg(X_{i}) = \deg(Y_{i}) = i$.

We remark that, by Theorem 4.4, $d$ is at most the degree of generation of $V$.

**Proof of Theorem 5.15.** Assume first that $V$ is a finitely generated FI$BC$–module. By Theorem 4.27, for $n \geq s + d$, $V_{n}$ has a decomposition

$$V_{n} = \bigoplus_{\lambda} c_{\lambda} V(\lambda)_{n}$$

where by assumption $c_{\lambda}$ is only nonzero for $|\lambda| \leq d$. Thus for $n \geq s + d$ the characters $V_{n}$ are given by a character polynomial of degree $\leq d$ by Theorem 5.10.

We will now use this result to prove the theorem for type $D$. That $V$ is an FI$D$–module of weight $\leq d$ means by definition that $\text{Ind}_{D}^{BC} V$ is an FI$BC$–module of weight $\leq d$, and $\text{Ind}_{D}^{BC} V$ moreover has stability degree $\leq s$ by assumption. Hence the $B_{n}$–representations $(\text{Ind}_{D}^{BC} V)_{n}$ are given by a unique character polynomial $F_{V}$ for all $n \geq s + d$. Moreover, if $V$ is generated in degree $\leq m$, then

$$V_{n} \cong (\text{Res}_{D}^{BC} \text{Ind}_{D}^{BC} V)_{n} \quad \text{for all } n \geq m + 1$$

by Proposition 3.34, and so the character of $V_{n}$ is given by the restriction of $F_{V}$ to $D_{n}$ in this range. The theorem follows.

**Corollary 5.16. (Polynomial growth of dimension for finitely generated FI$W$–modules).** Given a finitely generated FI$W$–module $V$ over a field of characteristic zero with associated character polynomial $F_{V}$, the dimension $\text{dim}(V_{n})$ of $V_{n}$ is given by $F_{V}(n, 0, 0, 0, \ldots)$ in the stable range. In particular, if $V$ is finitely generated in degree $\leq d$, then $\text{dim}(V_{n})$ is eventually a polynomial in $n$ of degree at most $d$. 

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5.4 Polynomial dimension over positive characteristic

Corollary 5.17. (Characters only depend on short cycles). Suppose that $k$ is a field of characteristic zero, and let $V$ be a finitely generated $\text{FI}_W$–module. Let $\chi_n$ denote the character of the $B_n$–representation $V_n$. Then there exists some positive integer $d \leq \text{weight}(V)$, independent of $n$, such that for every $w \in W_n$, the value $\chi_n(w)$ depends only on cycles in $w$ of length at most $d$.

Remark 5.18. (Character polynomials of co–$\text{FI}_W$–modules). Suppose that $V$ is a co–$\text{FI}_W$–module over a field of characteristic 0. We define its dual $V^*$ to be the $\text{FI}_W$–module with $(V^*)_n = (V_n)^*$. Suppose $V^*$ is a finitely generated $\text{FI}_W$–module of weight $\leq d$ and stability degree $\leq s$, and that $F_V$ is the associated character polynomial. Since $(V_n)^* \cong (V_n)$ (see Geck–Pfeiffer [GP00, Corollary 3.2.14]), the characters of $\chi_{V_n} = F_V$ in the range $n \geq s + d$.

5.4 Polynomial dimension over positive characteristic

Church–Ellenberg–Farb–Nagpal proved that the dimensions of finitely generated $\text{FI}_A$–modules over a field $k$ are eventually polynomial even when $k$ has positive characteristic [CEFN12, Theorem 1.2]. We use their result to prove the same for all $\text{FI}_W$–modules.

Theorem 5.19. (Polynomial growth of dimension over arbitrary fields). Let $k$ be any field, and let $V$ be a finitely generated $\text{FI}_W$–module over $k$. Then there exists an integer-valued polynomial $P(T) \in \mathbb{Q}[T]$ such that

$$\dim_k(V_n) = P(n) \quad \text{for all } n \text{ sufficiently large}.$$ 

We note that, in contrast to the result over characteristic zero, Theorem 5.19 does not come with bounds on the degree of $P(T)$ or the range of $n$-values for which the equality holds.

Proof of Theorem 5.19. When $V$ is a finitely generated $\text{FI}_A$–module, the result follows from [CEFN12, Theorem 1.2]. If $V$ is a finitely generated $\text{FI}_{BC}$ or $\text{FI}_D$–module, then by Proposition 3.24 its restriction to $\text{FI}_A$ is a finitely generated $\text{FI}_A$–module, and the result again follows from [CEFN12, Theorem 1.2].

5.5 The character polynomials of $\text{FI}_W$–modules

In this section we compute the character polynomials of the $\text{FI}_{BC}$–modules $M_{BC}(U)$, Proposition 5.20. We conclude that the character polynomial of an
5.5 The character polynomials of \( F_{W} \)-modules

\( F_{BC} \)-module \( V \) must equal \( \chi_{V} \) for all values of \( n \). The formula given in Proposition 5.20 is moreover useful for computing character polynomials of \( F_{BC} \)-modules, such as in our applications in Sections 7.1 and 7.3. We end this section with Proposition 5.22, the character polynomials of the \( F_{W} \)-modules \( M_{W}(m) \) for each family \( W_{n} \).

**Proposition 5.20. (The Character of \( M_{BC}(U)_{n} \)).** Let \( k \) be a field of characteristic zero. Let \( U \) be a representation of \( B_{m} \) with character \( \chi^{U} \). Then the character \( \chi^{M_{BC}(U)_{n}} \) is, for each \( n \), given by the character polynomial \( P^{U} \):

\[
P^{U}(w) = \left( \sum_{|\alpha| + |\beta| = m} \chi^{U}_{(\alpha, \beta)} \binom{X}{\alpha} \binom{Y}{\beta}(w) \right)
:= \sum_{|\alpha| + |\beta| = m} \chi^{U}_{(\alpha, \beta)} \binom{X_{1}(w)}{n_{1}(\alpha)} \binom{X_{2}(w)}{n_{2}(\alpha)} \cdots \binom{X_{m}(w)}{n_{m}(\alpha)} \binom{Y_{1}(w)}{n_{1}(\beta)} \binom{Y_{2}(w)}{n_{2}(\beta)} \cdots \binom{Y_{m}(w)}{n_{m}(\beta)}
\]

**Proof of Proposition 5.20.** Since \( M_{BC}(U)_{n} = \text{Ind}_{B_{n} \times B_{n-m}}^{B_{m}} U \otimes k \), with \( k \) the trivial \( B_{n-m} \)-representation, the result follows from Lemma 5.14.

**Corollary 5.21.** Let \( V \) be an \( F_{BC} \)-module \( V \) over a field of characteristic zero. Then if \( V \) is finitely generated in degree \( \leq d \), the characters of \( V_{n} \) are equal to a unique character polynomial \( F_{V} \in k[X_{1}, Y_{1}, X_{2}, Y_{2}, \ldots] \) of degree at most \( d \), with equality for every value of \( n \geq 0 \). The dimensions of \( V_{n} \) are given by a polynomial of degree at most \( d \)

\[
dim_{k}(V_{n}) = F_{V}(n, 0, 0, \ldots) \quad \text{for every value of } n.
\]

We can find explicit formulas for the \( F_{W} \)-modules \( M_{W}(m) \).

**Proposition 5.22.** Let \( k \) be \( \mathbb{Z} \) or a field of characteristic zero. When \( W_{n} \) is \( S_{n} \), the character polynomial of \( M_{A}(m) \) is

\[
\chi^{M_{A}(m)} = m! \binom{X_{1}}{m}.
\]

When \( W_{n} \) is \( B_{n} \), the character polynomial of \( M_{BC}(m) \) is

\[
\chi^{M_{BC}(m)} = 2^{m} m! \binom{X_{1}}{m}.
\]
When $W_n$ is $D_n$, $M_D(m)$ is also given by a character polynomial for $n > m$:

$$\chi^{M_D(m)} = 2^m m! \binom{X_1}{m} \quad \text{when } n > m.$$  

When $n = m$, the character of $M_D(m)_m$ take the value $2^{m-1} m!$ on the identity and vanishes otherwise.

**Proof of Proposition 5.22.** Take as basis for $M_W(m)_n$ the set

$$S = \{e_f \mid f \in \text{Hom}_{FI\mathcal{W}}(m, n)\}.$$  

An element $w \in W_n$ will permute these basis elements; the trace of $w$ is the size of its fixed set in $S$. A basis element $e_f$ is fixed by $w$ only if $w$ fixes its image $f(m) \subseteq n$ pointwise; conversely for every choice of $m$ (positive) 1-cycles

$$(a_1)(-a_1), (a_2)(-a_2), \ldots, (a_m)(-a_m)$$

in $w$, $w$ will fix all basis elements $e_f$ for which the image of $f$ is

$$f(m) = \{\pm a_1, \ldots, \pm a_m\} \subseteq n.$$  

When $W_n$ is $S_n$, there are $m!$ such maps. When $W_n$ is $B_n$, there are $2^m m!$ such maps. When $W_n$ is $D_n$, there are $2^m m!$ such maps whenever $n > m$; when $n = m$ there are only $2^{m-1} m!$, since in this case each endomorphism $f$ must reverse an even number of signs. The formulas follow. \(\square\)

### 6 Tensor products and FI$_W$–algebras

In this section we define the tensor product of FI$_W$–modules, and show that it respects weight and degree of generation. As a consequence we derive Theorem 6.4, the hyperoctahedral analogue of Murnaghan’s theorem on the stability of Kronecker coefficients. We define graded FI$_W$–modules and FI$_W$–algebras, and study some finiteness properties (finite type and slope) of these objects.
6.1 Tensor products and Murnaghan’s theorem for $B_n$ and $D_n$

Definition 6.1. (Tensor product of FI$_W$–modules). Given FI$_W$–modules $V$ and $W$, the tensor product $V \otimes W$ is the FI$_W$–module such that

$$(V \otimes W)_n = V_n \otimes W_n$$

and the FI$_W$–morphisms act diagonally.

Proposition 6.2. (Tensor products respect finite generation). If $V$ and $W$ are finitely generated FI$_W$–modules, then so is $V \otimes W$. If $V$ is generated in degree $\leq m$ and $W$ in degree $\leq m'$, then $V \otimes W$ is generated in degree $\leq m + m'$. If $k$ is a field of characteristic zero, then weight($V \otimes W$) $\leq$ weight($V$) + weight($W$).

Proof of Proposition 6.2. To prove finite generation, we follow the arguments of [CEF12, Proposition 2.61]. By Proposition 3.17, the FI$_W$–modules $V$ and $W$ are quotients of FI$_W$–modules of the form $\oplus_{a=0}^m M_W(a)^{b_a}$ and $\oplus_{a=0}^{m'} M_W(a)^{b'_{a}}$, respectively. It is therefore enough to show that the FI$_W$–module

$$X := M_W(m) \otimes M_W(m')$$

is finitely generated in degree $\leq (m + m')$. The $W_n$–representation $X_n$ is, by definition,

$$X_n = \text{Span}_k \{ (f, f') \in \text{Hom}_{FI_W}(m, n) \times \text{Hom}_{FI_W}(m', n) \}.$$  

When $n \geq m + m'$, for given $(f, f') \in X_n$ there exists some $(g, g') \in X_{m+m'}$ and some $h \in \text{Hom}_{FI_W}(m + m', n)$ so that

$$h_* (g, g') := (h \circ g, h \circ g') = (f, f').$$

We conclude that $X$ is finitely generated in degree $\leq (m + m')$.

To prove subadditivity of weights, it suffices to show that, in the notation of Section 2.2, any $W_n$–representation $V(\nu)_n$ occurring in the product $V(\mu)_n \otimes V(\lambda)_n$ must satisfy $|\nu| \leq (|\mu| + |\lambda|)$.

Fix $n$. By Proposition 4.2, $V(\mu)_n$ and $V(\lambda)_n$ occur in $M_W(|\mu|)_n$ and $M_W(|\lambda|)_n$, respectively, and so $V(\mu)_n \otimes V(\lambda)_n$ is a $W_n$–subrepresentation of $M_W(|\mu|)_n \otimes M_W(|\lambda|)_n$. By the previous paragraph, $M_W(|\mu|) \otimes M_W(|\lambda|)$ is generated in degree $\leq (|\mu| + |\lambda|)$, and so by Theorem 4.4 it has weight $\leq (|\mu| + |\lambda|)$. $\square$
6.1 Tensor products and Murnaghan’s theorem for $B_n$ and $D_n$

Using the formulas for the characters of $M_A(m)$ and $M_{BC}(m)$ in Proposition 5.22, and the identity relating binomial and multinomial coefficients:

\[
\binom{z}{m} \binom{z}{p} = \sum_{d=0}^{m} \binom{m+p-d}{d,m-d,p-d} \binom{z}{m+p-d}
\]

we conclude that

\[
\chi_{M_A(m) \otimes M_A(p)} = m! \binom{X_1}{m} p! \binom{X_1}{p} = m! p! \sum_{d=0}^{m} \binom{m+p-d}{d,m-d,p-d} \binom{X_1}{m+p-d}
\]

\[
\chi_{M_{BC}(m) \otimes M_{BC}(p)} = 2^m m! \binom{X_1}{m} 2^p p! \binom{X_1}{p} = 2^m+p m! p! \sum_{d=0}^{m} \binom{m+p-d}{d,m-d,p-d} \binom{X_1}{m+p-d}
\]

By Corollary 4.47, these characters completely determine the FI$_W$--structure.

**Proposition 6.3.** The following tensor product decompositions hold when $W_n$ is $S_n$ or $B_n$:

\[
M_A(m) \otimes M_A(p) = \bigoplus_{d=0}^{m} \frac{m! p!}{(m+p-d)!} \binom{m+p-d}{d,m-d,p-d} M_A(m+p-d)
\]

\[
M_{BC}(m) \otimes M_{BC}(p) = \bigoplus_{d=0}^{m} \frac{2^d m! p!}{(m+p-d)!} \binom{m+p-d}{d,m-d,p-d} M_{BC}(m+p-d)
\]

**Theorem 6.4. (Murnaghan’s stability theorem for $B_n$).** For any pair of double partitions $\lambda = (\lambda^+, \lambda^-)$ and $\mu = (\mu^+, \mu^-)$, there exist nonnegative integers $g_{\lambda,\mu}^\nu$, independent of $n$, such that for all $n$ sufficiently large:

\[
V(\lambda)_n \otimes V(\mu)_n = \bigoplus_{\nu} g_{\lambda,\mu}^\nu V(\nu)_n.
\]

The coefficients $g_{\lambda,\mu}^\nu$ are nonzero for only finitely many double partitions $\nu$.

**Proof of Theorem 6.4.** The FI$_{BC}$--modules $V(\lambda)$ and $V(\mu)$ are finitely generated by Proposition 4.33, and so by Proposition 6.2 their product $V(\lambda) \otimes V(\mu)$ is finitely generated, and therefore is uniformly representation stable by Theorem 4.28.

By restricting both sides of Equation (7) to action of $D_n$, we conclude

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Corollary 6.5. (Murnaghan’s stability theorem for $D_n$). With double partitions $\lambda = (\lambda^+, \lambda^-)$ and $\mu = (\mu^+, \mu^-)$ as above, for all $n$ sufficiently large the tensor product of the $D_n$–representations $V(\lambda)_n \otimes V(\mu)_n$ has a stable decomposition:

$$V(\lambda)_n \otimes V(\mu)_n = \bigoplus_{\nu} g_{\lambda,\mu}^\nu V(\nu)_n$$

where $g_{\lambda,\mu}^\nu$ are the structure constants of Equation (7).

The analogous stability result for the Kronecker coefficients of the symmetric group is a classical result of Murnaghan [Mur38]. The observation that Murnaghan’s theorem follows from the theory of finitely generated $\mathcal{F}_A$–modules is given by [CEF12, Theorem 2.65]. Theorem 6.4 is a natural counterpart to Murnaghan’s theorem, however, we have consulted with a number of experts and have not been able to find the result in the literature.

6.2 Graded $\mathcal{F}_{W}$–modules and graded $\mathcal{F}_{W}$–algebras

In analogy to Church–Ellenberg–Farb [CEF12, Section 2.10], we define graded $\mathcal{F}_{W}$–modules, $\mathcal{F}_{W}$–algebras, $\mathcal{F}_{W}$–ideals, and the dual notions for each. We give define the finiteness criteria finite type and slope.

Definition 6.6. (Graded $\mathcal{F}_{W}$–modules; Finite type; Slope). A graded $\mathcal{F}_{W}$–module $V = \bigoplus_i V^i$ is a functor from $\mathcal{F}_{W}$ to the category of graded $k$–modules. Each graded piece $V^i$ is an $\mathcal{F}_{W}$–module; we say $V$ has finite type if $V^i$ is a finitely generated for all $i$.

Suppose $k$ is a field of characteristic zero, and let $V$ be a graded $\mathcal{F}_{W}$–module supported in nonnegative degrees. We say that the slope of $V$ is $\leq m$ if $V^i$ has weight $\leq m \cdot i$ for all $i$.

Example 6.7. The polynomial algebras $V_n = k[x_1, \ldots, x_n]$ from Example 1.5 form a graded $\mathcal{F}_{W}$–module of finite type, graded by total degree. The graded piece $V^d_n := k[x_1, \ldots, x_n]|_{(d)}$ is finitely generated in degree $\leq d$, and so when $k$ is a field of characteristic zero $V$ has slope $\leq 1$ by Theorem 4.4.

The tensor product of graded $\mathcal{F}_{W}$–modules $U = \bigoplus_i U^i$ and $W = \bigoplus_j W^j$ is the graded $\mathcal{F}_{W}$–module

$$U \otimes W = \bigoplus_\ell \left( \bigoplus_{i+j=\ell} (U^i \otimes W^j) \right).$$
By applying Proposition 6.2 to each summand \((U^i \otimes W^j)\), we conclude that the induced grading on the tensor product of graded \(\text{FI}_W\)-modules respects weight and finite generation properties, in the following sense.

**Proposition 6.8. (Tensor product preserves finite type and slope).** Let \(U\) and \(W\) be graded \(\text{FI}_W\)-modules of finite type, supported in nonnegative grades, with \(U_0 \cong W_0 \cong M_W(0)\). Then the tensor product \(U \otimes W\) is a graded \(\text{FI}_W\)-module of finite type. When \(k\) is a characteristic zero field, \(U \otimes W\) will have slope \(\leq m\) whenever \(U\) and \(V\) have slopes \(\leq m\).

Church–Ellenberg–Farb prove this result in type A [CEF12, Proposition 2.70].

**Definition 6.9. (\(\text{FI}_W\)-algebras).** A (graded) \(\text{FI}_W\)-algebra \(A = \bigoplus A^i\) is a functor from \(\text{FI}_W\) to the category of (graded) \(k\)-algebras. A sub-\(\text{FI}_W\)-module \(V\) generates \(A\) as an \(\text{FI}_W\)-algebra if \(V_n\) generates \(A_n\) as a \(k\)-algebra for all \(n\).

**Definition 6.10. (Free associative \(\text{FI}_W\)-algebras).** Given a graded \(\text{FI}_W\)-module \(V\), we define the free associative algebra on \(V\) as the graded \(\text{FI}_W\)-algebra

\[
k\langle V \rangle := \bigoplus_{j=0}^{\infty} V^j.
\]

Any \(\text{FI}_W\)-algebra \(A\) generated by \(V\) admits a surjection of \(\text{FI}_W\)-algebras

\[k\langle V \rangle \twoheadrightarrow A.\]

Proposition 6.8 implies that \(k\langle - \rangle\) respects the weight and finite generation properties of the gradings of a graded \(\text{FI}_W\)-module \(V\), and consequently so does any \(\text{FI}_W\)-algebra that \(V\) generates. Propositions 6.11 and 6.12 are proven in type A by Church–Ellenberg–Farb [CEF12, Proposition 2.73 and Theorem 2.74].

**Proposition 6.11. (The functor \(k\langle - \rangle\) preserves finite type and slope).** Let \(V\) be a graded \(\text{FI}_W\)-module supported in nonnegative grades, with \(V_0 \cong M_W(0)\). If \(V\) has finite type, then \(k\langle V \rangle\) has finite type. If \(V\) is a graded \(\text{FI}_W\)-module over characteristic zero with slope \(\leq m\), then \(k\langle V \rangle\) as slope \(\leq m\).

If \(A\) is an \(\text{FI}_W\)-algebra generated by an \(\text{FI}_W\)-module \(V\), we can deduce Propositions 6.12 and 6.13 from the surjection of graded \(\text{FI}_W\)-algebras \(k\langle V \rangle \twoheadrightarrow A\).
6.2 Graded $\text{FI}_W$–modules and graded $\text{FI}_W$–algebras

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Proposition 6.12. (Finite type $\text{FI}_W$–modules generate $\text{FI}_W$–algebras of finite type and slope). Suppose that $A$ is an $\text{FI}_W$–algebra generated by an graded $\text{FI}_W$–module $V$ of finite type, supported in nonnegative grades. Then if $V$ has finite type, so does $A$. For $k$ a field of characteristic zero, if $V$ has slope $\leq m$ then $A$ has slope $\leq m$.

Proposition 6.13. Let $A$ be an $\text{FI}_W$–algebra generated by a graded $\text{FI}_W$–module $V$ concentrated in grade $d$. If $V$ is finitely generated in degree $\leq m$, then the $i$th graded piece $A^i$ is finitely generated in degree $\leq \left(\frac{i}{d}\right)m$, and moreover if $k$ is a characteristic zero field then weight $(A^i) \leq \left(\frac{i}{d}\right)$ weight $(V)$.

Definition 6.14. (FI$W$–ideals). Given a graded FI$W$–algebra $A$, an FI$W$–ideal $I$ of $A$ is a graded sub–FI$W$–algebra of $A$ such that $I_n$ is a homogeneous ideal in $A_n$ for each $n$.

Definition 6.15. (Co–FI$W$–modules, Co–FI$W$–algebras, finite type). A graded co–FI$W$–module is functor from the dual category $\text{FI}_W^{op}$ to the category of graded $k$–modules, and similarly a graded co–FI$W$–module is a functor to the graded $k$–algebras. When $k$ is a field, then we say that a graded co–FI$W$–module $V$ has finite type if its dual $V^*$, defined by $V^*_n = \text{Hom}_k(V_n, k)$, has finite type. Similarly, $V$ has slope $\leq m$ if $V^*$ does.

Proposition 6.16. (Finite type co–FI$W$–modules generate co–FI$W$–algebras of finite type). Let $k$ be a Noetherian commutative ring. Suppose that $A$ is a graded co–FI$W$–algebra containing a graded co–FI$W$–module $V$ supported in positive grades. If $V$ has finite type, then the subalgebra $B$ of $A$ generated by $V$ is a graded co–FI$W$–algebra of finite type. When $k$ is a field of characteristic zero and $V$ is a graded co–FI$W$–module of slope $\leq m$, then $B$ has slope $\leq m$.

Proof of Proposition 6.16. The proposition follows just as in the proof of [CEF12, Proposition 2.77], by considering the dual space $B^*$ as a graded sub–FI$W$–algebra of $k(V)^*$. Theorem 4.22, the Noetherian property for FI$W$–modules over Noetherian rings, implies that each graded piece of $B^*$ is finitely generated. Moreover, over a characteristic zero field, the weights of the graded pieces of $k(V)^*$ give an upper bound of on the weights of those of $B^*$, and the result follows.

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7 Some applications

Fl\(W\)-modules arise naturally in numerous areas of mathematics. In this section we give some applications of the theory developed in this paper to the cohomology of the group of pure string motions, the generalized \(r\)-diagonal coinvariant algebras, and the cohomology of the complements of the Weyl groups' reflecting hyperplanes.

7.1 The cohomology of the group of pure string motions

In [Wil12], we proved that the cohomology pure string motion group \(P\Sigma_n\) is uniformly representation stable with respect to a natural action of the hyperoctahedral group.

The group \(\Sigma_n\) of string motions is a generalization of the braid group. It is defined as the group of motions of \(n\) smoothly embedded, oriented, unlinked, unknotted circles in \(\mathbb{R}^3\); see for example Brownstein–Lee [BL93] for a complete definition. The work of Dahm (see Dahm [Dah62] or Goldsmith [Gol81]) identifies \(\Sigma_n\) with the symmetric automorphism group of the free group \(F_n\) on \(n\) letters \(x_1, \ldots, x_n\), the subgroup of automorphisms generated by the following elements:

\[
\alpha_{i,j} = \begin{cases} 
  x_i \mapsto x_j x_i x_j^{-1} \\
  x_\ell \mapsto x_\ell \quad (\ell \neq i)
\end{cases}
\]

\[
\tau_i = \begin{cases} 
  x_i \mapsto x_{i+1} \\
  x_{i+1} \mapsto x_i \\
  x_\ell \mapsto x_\ell \quad (\ell \neq i, i+1)
\end{cases}
\]

\[
\rho_i = \begin{cases} 
  x_i \mapsto x_i^{-1} \\
  x_\ell \mapsto x_\ell \quad (\ell \neq i)
\end{cases}
\]

The subgroup \(P\Sigma_n = \langle \alpha_{i,j} \rangle \subseteq \Sigma_n\) is the group of pure symmetric automorphisms (or pure string motions), the analogue of the pure braid group.

The central theorem of [Wil12]:

**Theorem 7.1.** [Wil12, Theorem 6.1] For each fixed \(m \geq 0\), the sequence of \(B_n\)-representations

\[\{H^m(P\Sigma_n; \mathbb{Q})\}_{n \in \mathbb{N}}\]

is uniformly representation stable with respect to the maps

\[\phi_n : H^m(P\Sigma_n; \mathbb{Q}) \to H^m(P\Sigma_{n+1}; \mathbb{Q})\]

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induced by the ‘forgetful’ map $P\Sigma_{n+1} \to P\Sigma_n$. The sequence stabilizes once $n \geq 4m$.

The theory of FI$_{BC}$–modules developed here allows for a significantly simplified proof of this result, and new perspective on the structure of these cohomology groups.

The integral homology group $H_1(P\Sigma_n; \mathbb{Z}) = P\Sigma_n/[P\Sigma_n, P\Sigma_n]$ is the free abelian group $\mathbb{Z}[\alpha_{i,j} \mid i \neq j]$, and the cohomology ring is generated by the dual elements $\alpha^*_{i,j}$. A presentation for the integral cohomology was conjectured by Brownstein and Lee [BL93, Conjecture 4.6] and proven by Jensen, McCammond, and Meier [JMM06, Theorem 6.7] (see also Griffin [Gri13b, Section 4]). Jensen–McCammond–Meier study the action of $P\Sigma_n/\text{Inn}(F_n)$ on the MacCullough–Miller complex [MM96] to obtain Theorem 7.2.

**Theorem 7.2.** [JMM06, Theorem 6.7]. The cohomology ring $H^*(P\Sigma_n; \mathbb{Z})$ is the exterior algebra generated by the degree-one classes $\alpha^*_{i,j}$, with $i, j \in [n], i \neq j$, modulo the relations

\begin{align*}
(1) \quad & \alpha^*_{i,j} \wedge \alpha^*_{j,i} = 0 \\
(2) \quad & \alpha^*_{i,j} \wedge \alpha^*_{j,i} - \alpha^*_{i,j} \wedge \alpha^*_{i,j} + \alpha^*_{i,j} \wedge \alpha^*_{i,i} = 0
\end{align*}

In [Wil12], to prove that the sequence $H^m(P\Sigma_n; \mathbb{Q})$ is uniformly representation stable, we use a combinatorial description of the cohomology groups given by Jensen–McCammond–Meier [JMM06] and an orbit–stabilizer argument to decompose each group into a sum of induced representation of a particular form. We then use a result of Church–Farb [CF13, Theorem 4.6] (inspired by the work of Hemmer [Hem10, Theorem 2.4]), to deduce from the combinatorics of the branching rules that these induced representations are uniformly representation stable.

Here, we can recover uniform representation stability for $H^m(P\Sigma_n; \mathbb{Q})$ as a $B_n$–representation almost immediately by demonstrating that it is finitely generated as an FI$_{BC}$–$\mathbb{Z}$–module, as follows.

**Theorem 7.3.** Let $k$ be $\mathbb{Z}$ or $\mathbb{Q}$. The cohomology rings $H^*(P\Sigma_n, k)$ form an FI$_{BC}$–module, and a graded FI$_{BC}$–algebra of finite type, with $H^m(P\Sigma_n, k)$ finitely generated in degree $\leq 2m$. In particular the FI$_{BC}$–algebra $H^*(P\Sigma_n, \mathbb{Q})$ has slope $\leq 2$.

**Proof of Theorem 7.3.** The map induced by an FI$_{BC}$–morphism $f : m_0 \to n_0$ on
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\[ H^1(P\Sigma \star, \mathbb{Z}) \]
is:

\[ f^* : H^1(P\Sigma_m; k) \rightarrow H^1(P\Sigma_n; k) \]

\[ \alpha_{i,j}^* \mapsto \begin{cases} 
\alpha_{[f(i)],[f(j)]}^* & \text{if } f(i) \neq 0, f(j) > 0 \\
-\alpha_{[f(i)],[f(j)]}^* & \text{if } f(i) \neq 0, f(j) < 0 \\
0, & \text{if } f(i) = 0 \text{ or } f(j) = 0 
\end{cases} \]

It is straightforward to verify that \( f^* \) extends to an algebra map on \( H^*(P\Sigma \star, k) \), and that this action is functorial.

The \( \text{Fl}_{BC} \)-module \( H^1(P\Sigma \star; k) \) is generated in degree 2 by \( \alpha_{1,2} \), and \( V^* = H^*(P\Sigma \star; k) \) is generated as an \( \text{Fl}_{BC} \)-algebra by \( H^1(P\Sigma \star; k) \). We conclude from Proposition 6.13 that \( V^m \) is finitely generated in degree \( \leq 2m \), and that \( H^*(P\Sigma \star; \mathbb{Q}) \) is a graded \( \text{Fl}_{BC} \)-algebra of slope \( \leq 2 \).

Corollary 7.4. For each \( m \), the sequence \( \{ H^m(P\Sigma_n; \mathbb{Q}) \}_n \) of representations of \( B_n \) (or \( S_n \)) is uniformly representation stable, stabilizing once \( n \geq 4m \).

Remark 7.5. We only defined representation stability for \( \text{Fl}_W \)-modules over fields of characteristic zero (Definition 2.5). However, the presentation for the groups \( H^m(P\Sigma_n; \mathbb{Z}) \) given by Jensen-McCammond–Meier shows that these groups are free abelian ([JMM06, Theorem 6.7], see Theorem 7.2), and we can identify \( H^m(P\Sigma_n; \mathbb{Z}) \) with the integer span of the basis \( \alpha_{i_1,j_1}^* \wedge \cdots \wedge \alpha_{i_m,j_m}^* \) for \( H^m(P\Sigma_n; \mathbb{Q}) \). Hence we get a version of representation stability for the integral groups \( H^m(P\Sigma_n; \mathbb{Z}) \) by redefining the subrepresentation \( V(\lambda)_n \) as the integral Specht module associated to the partition \( \lambda[n] \).

Theorem 5.15 implies that the characters of the sequence \( \{ H^m(P\Sigma_n; \mathbb{Q}) \}_n \) are given by a character polynomial of degree \( \leq 2m \). As in the above remark, since the integral cohomology is free abelian, these same character polynomials give characters for the integral cohomology.
Corollary 7.6. Let $k$ be $\mathbb{Z}$ or $\mathbb{Q}$. Fix an integer $m \geq 0$. The characters of the sequence of $B_n$–representations $\{H^m(P\Sigma_n;k)\}_n$ are given, for all values of $n$, by a unique character polynomial of degree $\leq 2m$.

This concludes a simpler proof of [Wil12, Theorems 6.1 and 6.4]. We have moreover extended the results of [Wil12] to integer coefficients, and obtained polynomiality results on the characters.

For small values of $m$, we can compute the FI$_{BC}$– and FI$_{A^\sharp}$–module structures and character polynomials of $H^m(P\Sigma_n;\mathbb{Z})$ by computing traces on an explicit basis for $H^m(P\Sigma_n;\mathbb{Z})$, $n = 1, \ldots, 2m$, and using Proposition 5.20. The result is, as an FI$_{BC}$–module,

$$H^1(P\Sigma_n;\mathbb{Z}) = M_{BC}(\emptyset,\emptyset)$$

$$\chi_{H^1(P\Sigma_n;\mathbb{Z})} = 2 \binom{X_1}{2} - 2 \binom{Y_1}{2} = X_1(X_1 - 1) - Y_1(Y_1 - 1).$$

In degree 2:

$$H^2(P\Sigma_n;\mathbb{Z}) = M_{BC}(\emptyset,\emptyset) \oplus M_{BC}(\emptyset,\emptyset) \oplus M_{BC}(\emptyset,\emptyset) \oplus M_{BC}(\emptyset,\emptyset)$$

$$\chi_{H^2(P\Sigma_n;\mathbb{Z})} = 12 \binom{X_1}{4} + 12 \binom{Y_1}{4} + 9 \binom{X_1}{3} + 9 \binom{Y_1}{3} - 4 \binom{X_2}{2} + 4 \binom{Y_2}{2}$$

$$- 4 \binom{X_1}{2} \binom{Y_1}{2} - X_1X_2 - X_1Y_2 - X_2Y_1 - Y_1Y_2 - \binom{X_1}{2}Y_1 - X_1 \binom{Y_1}{2}$$

$$= 2X_2 + Y_1^2 + 2Y_2^2 - X_1^2Y_1^2 - \frac{3}{2}Y_1^3 + \frac{1}{2}Y_1^4 + X_1^2 - 2X_2^2 - \frac{3}{2}X_1^3 + \frac{1}{2}X_1^4$$

$$+ \frac{1}{2}X_1Y_1^2 - X_1Y_2 - X_2Y_1 - Y_1Y_2 + \frac{1}{2}X_1^2Y_1 - X_1X_2 - 2Y_2$$

By restricting to the action of the symmetric groups we find, as an FI$_{A^\sharp}$–
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module,

\[ H^1(P\Sigma_\bullet; \mathbb{Z}) = M_A(\square) \oplus M_A(\square) \]

\[ \chi_{H^1(P\Sigma_\bullet; \mathbb{Z})} = 2 \left( \frac{X_1}{2} \right) = X_1(X_1 - 1) \]

\[ H^2(P\Sigma_\bullet; \mathbb{Z}) = M_A(\square)^2 \oplus M_A(\square) \oplus M_A(\square)^2 \oplus M_A(\square)^3 \oplus M_A(\square)^2 \oplus M_A(\square)^2 \]

\[ \chi_{H^2(P\Sigma_\bullet; \mathbb{Z})} = 12 \left( \frac{X_1}{4} \right) + 9 \left( \frac{X_1}{3} \right) - X_1X_2 - 4 \left( \frac{X_2}{2} \right) \]

\[ = \frac{1}{2}X_1^4 - \frac{3}{2}X_1^3 + X_1^2 - X_1X_2 - 2X_2^2 + 2X_2 \]

Problem 7.7. For each \( m \), compute the \( B_n \) character polynomial of \( H^m(P\Sigma_\bullet; \mathbb{Z}) \), and compute its decomposition as an \( \text{FI}_{BC}^\# \)-module into a sum of induced representations \( M_{BC}(U) \).

7.1.1 Generalizations

There are several families of groups that naturally generalize the (pure) braid groups and (pure) symmetric automorphism groups, which we outline below. With each family, there are open questions concerning whether the cohomology rings admit the structure of a finite type \( \text{FI}_{W} \) or \( \text{FI}_{W}^\# \)-algebra, and how this structure reflects the structure of the groups.

Partially symmetric automorphisms. The group \( \Sigma^k_n \) of partially symmetric automorphisms of the free group \( F_n = \langle x_1, \ldots, x_n \rangle \) are those automorphisms that send each of the first \( k \) generators \( x_1, \ldots, x_k \) to a conjugate of one of the elements \( x_1, x_1^{-1}, \ldots, x_k, x_k^{-1} \). We impose no restrictions on the images of \( x_{k+1}, \ldots, x_n \). The pure partially symmetric automorphism group \( P\Sigma^k_n \) is the subgroup of \( \Sigma^k_n \) of automorphisms that send each generator \( x_j \) with \( 1 \leq j \leq k \) to a conjugate of itself. We note that

\[ \Sigma^n_n = \Sigma_n, \quad P\Sigma^n_n = P\Sigma_n, \quad \text{and} \quad P\Sigma^0_n = \Sigma^0_n = \text{Aut}(F_n) \]
these groups interpolate between the (pure) symmetric automorphism group and the full automorphism group of $F_n$.

The groups $P\Sigma_n^k$ were studied by Jensen–Wahl [JW04] for their relationships to mapping class groups. Jensen–Wahl have computed a presentation and established certain homological properties of the groups. Bux–Charney–Vogtmann [BCV09] determined that the image of the group $P\Sigma_n^k$ in $\text{Out}(F_n)$ has virtual cohomological dimension $2n - k - 2$ when $k \neq 0$. They exhibit a proper action of these outer automorphism groups on a $(2n - k - 2)$-dimensional deformation retract of a certain contractible subcomplex of the spine of Culler–Vogtmann’s Outer space; see Charney–Vogtmann [CV09] for details.

Zaremsky [Zar12] proved that both families $P\Sigma_n^k$ and $\Sigma_n^k$ are, for fixed $k$, rationally homologically stable in $n$. He proved moreover that for fixed $n$, the groups $\Sigma_{n+k}^k$ are rationally homologically stable in $k$. Zaremsky obtains these results by studying the groups’ actions on subcomplexes of the spine of Auter space. He uses methods from discrete Morse theory to prove that the filtered pieces of certain subcomplexes are highly connected, extending techniques of McEwen–Zaremsky [MZ09].

Given these results, it would be interesting to determine whether there is a $\text{FI}_{BC}$ or $\text{FI}_{BC}^\sharp$–module structure on the rational cohomology groups of $P\Sigma_n^k$ as a sequence in $k$, and, if so, to determine the associated stable decompositions and character polynomials.

**Symmetric automorphisms of free products.** Given a group $G$, let $G^{*n}$ denote its $n$-fold free product

$$G^{*n} := \underbrace{G \ast G \ast \cdots \ast G}_{n \text{ copies}}.$$  

The automorphism group $\text{Aut}(G^{*n})$ contains a copy of the symmetric group $S_n$ which permutes the $n$ free factors. These permutations normalize the following subgroups of automorphisms; see for example Griffin [Gri13a, Gri13b] for details.

- The *Fouxe-Rabinovitch group* $\text{FR}(G^{*n}) \subseteq \text{Aut}(G^{*n})$ generated by *partial conjugations* of $G^{*n}$. A partial conjugation is an automorphism that conjugates the $i^{th}$ free factor $G$ by some $g$ in the $j^{th}$ factor $G$ with $i \neq j$. All factors other than the $i^{th}$ are fixed.
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- The inner automorphisms of each factor $\prod_n \text{Inn}(G)$
- All automorphisms of each factor $\prod_n \text{Aut}(G)$
- The Whitehead automorphism group $\text{Wh}(G^n) := \text{FR}(G^n) \rtimes \prod_n \text{Inn}(G)$
- The pure automorphism group $\text{PAut}(G^n) := \text{FR}(G^n) \rtimes \prod_n \text{Aut}(G)$

The symmetric automorphism group of $G^n$ is the group

$$\Sigma\text{Aut}(G^n) := \left(\text{PAut}(G^n) \rtimes S_n\right).$$

We note that

$$P\Sigma_n \cong \text{FR}(\mathbb{Z}^n) \cong \text{Wh}(\mathbb{Z}^n) \quad \text{and} \quad \Sigma_n \cong \Sigma\text{Aut}(\mathbb{Z}^n).$$

Griffin constructs a classifying space for $\text{FR}(G^n)$, which he defines as a moduli space of cactus products, and alternatively characterizes combinatorially in terms of diagonal complexes comprised of forest posets. Using this classifying space he computes the homology of the groups $\text{FR}(G^n)$, $\text{PAut}(G^n)$, and $\Sigma\text{Aut}(G^n)$.

Collinet–Djament–Griffin [CDG12] have proven that if $G$ does not contain $\mathbb{Z}$ as a free factor, the sequences $\text{Aut}(G^n)$ and $\Sigma\text{Aut}(G^n)$ are (integrally) homologically stable, stabilizing in degree $i$ once $n \geq 2i + 2$. Their work complements the results of Hatcher [Hat95] for $G \cong \mathbb{Z}$ and extends results of Hatcher–Wahl [HW10] for several important classes of groups $G$ coming from low-dimensional topology. Collinet–Djament–Griffin prove their results using the theory of functor homology, and an analysis of the action of $\text{FR}(G^n)$ on a variation of the MacCullough–Miller complex [MM96] due to Chen–Glover–Jensen [CGJ05].

We would be interested to better understand the relationship between the work done on the groups $\text{FR}(G^n)$, $\text{Wh}(G^n)$, and $\text{PAut}(G^n)$ and the theory of $\text{FI}_A$–modules.

**Virtual and flat braid groups.** The (pure) virtual braid group and the (pure) flat braid group are generalizations of the (pure) braid group that allow virtual or flat crossings of strands. This additional structure was introduced by
Kauffman [Kau99], motivated by the study of knots in thickened higher-genus surfaces and the combinatorial theory of Gauss codes. Virtual and flat crossings are distinct from the under- and over-crossings in familiar knot and braid diagrams, and each have their own admissible Redemeister moves. For details see for example Kauffman [Kau99, Kau00], Vershinin [Ver01], Kauffman–Lambropoulou [KL04], Bardakov [Bar04], and Bartholdi–Enriquez–Etingof–Rains [BEER06].

In [Lee13], Peter Lee analyzes the cohomology of the pure virtual braid groups and the pure flat braid groups as representations of the symmetric groups. He proves that, for both families, the rational cohomology groups are uniformly representation stable [Lee13, Corollaries 1 and 5]. His work raises the questions of whether these cohomology sequences are in fact $\text{FI}_A$ or $\text{FI}_A^\#-algebras$, the structure of the associated character polynomials, and whether these results extends to integral cohomology.

### 7.2 Diagonal coinvariant algebras

Let $W_n$ be a finite reflection group acting on an $n$–dimensional vector space $V$ over a field $k$. Let $x_1, x_2, \ldots, x_n$ denote a basis for $V$. Then

$$k[X^{(r)}(n)] := k[x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}]$$

is a polynomial ring isomorphic to the symmetric algebra on $V^\oplus r$; the algebra $k[X^{(r)}(n)]$ has an action of $W_n$ induced by the diagonal action of $W_n$ on $V^\oplus r$. This ring has a natural grading by $r$–tuples

$$J = (j_1, \ldots, j_r) \in \mathbb{Z}_{\geq 0}^r,$$

where $j_i$ designates the total degree in variables $x_1^{(i)}, \ldots, x_n^{(i)}$.

Let $I_n$ be the ideal generated by the constant-term-free $W_n$–invariant polynomials. The $r$-diagonal coinvariant algebra is the $k$–algebra

$$C^{(r)}(n) := k[X^{(r)}(n)]/I_n.$$ 

Since $I_n$ is homogeneous with respect to the multigrading on $k[X^{(r)}(n)]$, the
The structure of $C^{(r)}(n)$ as a $W_n$–representation over characteristic zero has been the subject of extensive study. The coinvariant algebra $C^{(1)}(n)$ appeared in classical representation theory and Lie theory; Borel [Bor53] proved that the algebra $C^{(1)}(n)$ is the cohomology of a generalized flag manifold, which we will define below. The diagonal coinvariant algebras $C^{(2)}(n)$ were first investigated in type A by Garsia and Haiman [GH93] for their relationship to MacDonald polynomials, but these algebras were subsequently found to have rich connections to numerous objects in algebraic combinatorics; see Haiman [Hai02a] for a survey.

In 2002 Haiman established a formula for the characters of the $S_n$–representations $C^{(2)}(n)$ in terms of MacDonald polynomials, and deduces a number of combinatorial consequences for the spaces $C^{(2)}(n)$ [Hai02b]. A refinement of the formulas for these characters was conjectured by Haglund–Haiman–Loehr–Remmel–Ulyanov [HHL+05].

In 2003 Gordon [Gor03] studied coinvariant algebras associated to a Coxeter group $W_n$. He resolved a conjecture of Haiman [Hai94] by computing the Hilbert series of a quotient ring closely related to $C^{(2)}(n)$. Bergeron and Biagioli computed the trivial and alternating component of $C^{(2)}(n)$ in type B [BB06]. In 2011, Bergeron analyzed the algebras $C^{(r)}(n)$ associated to a general complex reflection group $W = G(m,p,n)$ [Ber11]. Bergeron shows, for fixed group $W$, the multigraded Hilbert polynomial associated to $C^{(r)}(n)$ can be described in terms of Schur polynomials in a form independent of $r$, and Bergeron computes these series in special cases. In general, the structure (or even dimension) of $C^{(r)}(n)$ is not known for $n > 3$. Additional background on coinvariant algebras can be found in Bergeron’s book [Ber09].

Church–Ellenberg–Farb [CEF12, Theorem 3.4] proved that when $W_n$ is $S_n$ acting on the representation $V_n = M_A(1)_n$ over a field $k$ of characteristic zero, the resultant coinvariant algebra

$$C^{(r)} := k[M_A(1)^{\oplus r}] / I$$

is a graded co–FI$_A$–module of finite type, and that moreover the graded pieces
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$(C^{(r)}_J)^*$ of the dual FI$_A$–module have weight at most $|J|$. Together with Nagpal, these authors showed that even over positive characteristic, the dimensions of the graded pieces are eventually polynomial [CEFN12, Theorem 1.9]. We can extend their results as follows.

**Theorem 7.8.** (Diagonal coinvariant algebras are finite type). Let $k$ be a field, and let $V_n \cong k^n$ be the canonical representation of $W_n$ by (signed) permutation matrices. Given $r \in \mathbb{Z}_{>0}$, the sequence of coinvariant algebras

$$C^{(r)} := k[V^\oplus r]/I$$

is a graded co–FI$_W$–algebra of finite type. When $k$ has characteristic zero, the weight of the multigraded component $C^{(r)}_J$ is $\leq |J|$.

**Proof of Theorem 7.8.** Let $W_n$ be the Weyl group $S_n$, $D_n$, or $B_n$, then let $V$ be the FI$_W$–module associated to the canonical $n$–dimensional $W_n$–representations $V_n \cong k^n$ by (signed) permutation matrices. Then $V$ is $M_A(1)$ in type A, $M_{BC}(\varnothing, \square)$ in type B/C, and $\text{Res}^{BC}_D M_{BC}(\varnothing, \square)$ in type D. In each type, the sequence $\{V_n\}$ has a co–FI$_W$–module structure by Proposition 4.39 and Corollary 4.40. The ideals $\mathcal{I}_n$ form a co–FI$_W$–module, determined by the $W_n$–action and the maps

$$(I_n)^* : \mathcal{I}_{n+1} \rightarrow \mathcal{I}_n$$

$$x_i \rightarrow \begin{cases} x_i & i \leq n, \\ 0 & i = n + 1. \end{cases}$$

Since $C^{(r)}(n)^*$ is generated as an algebra by its degree 1 part, the co–FI$_W$–algebra $C^{(r)}$ has finite type by Proposition 6.16.

Over characteristic zero, the FI$_W$–module $(V^\oplus r)^*$ has weight 1 by Theorem 4.4. The graded piece $C^{(r)}_J(n)$ is a subquotient of the degree $|J|$ tensor product on $(V_n)^\oplus r$, and weight is additive under tensor products by Proposition 6.2.

**Corollary 7.9.** Let $k$ be a field of characteristic zero. For $n$ sufficiently large (depending on the $r$-tuple $J$), the sequence $C^{(r)}_J(n)$ is uniformly multiplicity stable.

Since representations of $W_n$ are self-dual (a consequence of [GP00, Corollary 3.2.14]), the characters of $C^{(r)}_J(n)$ are given by the character polynomial for its dual, with degree bounded by Theorems 5.2 and 5.15.
Corollary 7.10. Let \( k \) be a field of characteristic zero. For \( n \) sufficiently large (depending on the \( r \)-tuple \( J \)), the characters of \( \mathcal{C}_J^{(r)}(n) \) are given by a character polynomial \( F_J \) of degree \( \leq |J| \). In particular the dimension of \( \mathcal{C}_J^{(r)}(n) \) is given by a polynomial

\[
\dim_k \mathcal{C}_J^{(r)}(n) = F_J(n, 0, 0, \ldots)
\]

for all \( n \) in the stable range.

Theorem 5.19 implies that over fields of any characteristic, the dimensions of the graded pieces of \( \mathcal{C}^{(r)} \) are eventually polynomial.

Corollary 7.11. Let \( k \) be an arbitrary field. Then for each \( r \)-tuple \( J \), there exists a polynomial \( P_J \in \mathbb{Q}[T] \) (depending on \( k \)) so that

\[
\dim_k \mathcal{C}_J^{(r)}(n) = P_J(n)
\]

for all \( n \) sufficiently large (depending on \( k \) and \( J \)).

The cohomology of generalized flag manifolds. Take \( k \) to be the complex numbers \( \mathbb{C} \). Let \( G^W_n \) be a semisimple complex Lie group with Weyl group \( W_n \), and let \( B^W_n \) be a Borel subgroup of \( G^W_n \). Borel proved that the complex coinvariant algebra \( \mathcal{C}^{(1)}(n) \) is isomorphic as a graded \( k[W_n] \)-algebra to the cohomology \( H^*(G^W_n/B^W_n; \mathbb{C}) \) of the generalized flag manifold \( G^W_n/B^W_n \) [Bor53]; the isomorphism multiplies the grading by 2. Specifically, we have

Type A_{n-1}:
\[ G^A_n = \text{SL}_n(\mathbb{C}) \]
\[ G^A_n/B^A_n = \{ 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \mid \dim \mathbb{C} V_m = m \} \]
The complete flag variety

Type B_n:
\[ G^B_n = \text{SO}_{2n+1}(\mathbb{C}) \] (Quadratic form \( Q \))
\[ G^B_n/B^B_n = \{ 0 \subseteq V_1 \subseteq \cdots \subseteq V_{2n+1} = \mathbb{C}^{2n+1} \mid \dim \mathbb{C} V_m = m, \ Q(V_i, V_{2n+1-i}) = 0 \} \]
The variety of complete flags equal to their orthogonal complements

Type C_n:
\[ G^C_n = \text{Sp}_{2n}(\mathbb{C}) \] (Symplectic form \( L \))
\[ G^C_n/B^C_n = \{ 0 \subseteq V_1 \subseteq \cdots \subseteq V_{2n} = \mathbb{C}^{2n} \mid \dim \mathbb{C} V_m = m, \ L(V_i, V_{2n-i}) = 0 \} \]
The variety of complete flags equal to their symplectic complements

Type D_n:
\[ G^D_n = \text{SO}_{2n}(\mathbb{C}) \] (Quadratic form \( Q \))
\[ G^D_n/B^D_n = \{ 0 \subseteq V_1 \subseteq \cdots \subseteq V_{2n} = \mathbb{C}^{2n} \mid \dim \mathbb{C} V_m = m, \ Q(V_i, V_{2n-i}) = 0 \} \]
The variety of complete flags equal to their orthogonal complements
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See (for example) Fulton–Harris [FH04] for more details. Theorem 7.8 therefore implies:

**Corollary 7.12.** Let \( W \) denote type A, B, C, or D. The cohomology rings \( H^*(G^W_n/B^W_n; \mathbb{C}) \) are graded co–FI\_\( W \)--algebras of finite type, that is, for each \( m \), \( H^m(G^W_n/B^W_n; \mathbb{C}) \) are co–FI\_\( W \)--modules of weight \( \leq \frac{m}{2} \). In particular, for each \( m \), the sequence of \( W \)--representations \( H^m(G^W_n/B^W_n; \mathbb{C}) \) is uniformly representation stable, and the associated sequence of characters are eventually equal to a character polynomial of degree at most \( \frac{m}{2} \).

We can compute the character polynomials for the \( r \)--diagonal coinvariant algebras \( \mathcal{C}^{(r)} \) for small values of \( r \) by hand, by computing the trace of the action of \( W_n \) at each point in a resolution for \( \mathcal{C}^{(r)}(n) \) by \( k[\mathcal{W}_n] \)--modules. When \( W_n \) is \( B_n \), we find the following characters \( \chi^{(r)}_J(n) \) of \( \mathcal{C}^{(r)}(n) \).

\[
\begin{align*}
\chi^{(1)}_{(1)} &= X_1 - Y_1 \quad (n \geq 1) \\
\chi^{(1)}_{(2)} &= X_1 + Y_1 + \left( \frac{X_1}{2} \right) + \left( \frac{Y_1}{2} \right) + X_2 - Y_2 - X_1 Y_1 - 1 \quad (n \geq 2) \\
\chi^{(1)}_{(3)} &= 2 \left( \frac{X_1}{2} \right) - 2 \left( \frac{Y_1}{2} \right) + \left( \frac{X_1}{3} \right) + X_1 Y_1 \left( \frac{X_1}{2} \right) - Y_1 \left( \frac{X_1}{3} \right) \\
&\quad + X_3 - Y_3 + X_1 X_2 - Y_1 X_2 - X_1 Y_2 + Y_1 Y_2 \quad (n \geq 3) \\
\chi^{(2)}_{(1,1)} &= X_1 + Y_1 + 2 \left( \frac{X_1}{2} \right) + 2 \left( \frac{Y_1}{2} \right) - 2 X_1 Y_1 - 1 \quad (n \geq 2) \\
\chi^{(2)}_{(2,1)} &= Y_1 - X_1 + 4 \left( \frac{X_1}{2} \right) - 4 \left( \frac{Y_1}{2} \right) + X_2 X_1 - X_2 Y_1 - X_1 Y_2 + Y_1 Y_2 \\
&\quad + 3 \left( \frac{X_1}{3} \right) - 3 \left( \frac{Y_1}{3} \right) + 3 X_1 \left( \frac{Y_1}{2} \right) - 3 Y_1 \left( \frac{X_1}{3} \right) \quad (n \geq 3) \\
\chi^{(3)}_{(1,1,1)} &= -2 X_1 + 2 Y_1 + 6 \left( \frac{X_1}{2} \right) - 6 \left( \frac{Y_1}{2} \right) + 6 \left( \frac{X_1}{3} \right) - 6 \left( \frac{Y_1}{3} \right) \\
&\quad + 6 X_1 \left( \frac{Y_1}{2} \right) - 6 Y_1 \left( \frac{X_1}{2} \right) \quad (n \geq 3)
\end{align*}
\]

We note that the character of \( \chi^{(r)}_{(j_1,\ldots,j_r)} = \chi^{(r+1)}_{(j_1,\ldots,j_r,0)} \), and moreover the characters \( \chi^{(r)}_{(j_1,\ldots,j_r)} \) are fixed under permutations of the ordered \( r \)--tuple \( J \). It follows that the above character polynomials determine all characters \( \chi^{(r)}_J \) for \( |J| \leq 3 \).
Problem 7.13. For each graded piece $c^{(r)}_j$, compute the associated character polynomial and the stable decomposition into irreducible representations. Determine the stable ranges of each.

7.3 The cohomology of hyperplane complements

Let $W_n$ be the Weyl group in type $A_{n-1}$, $B_n/C_n$, or $D_n$, and consider the canonical action of $W_n$ on $\mathbb{C}^n$ by (signed) permutation matrices. Let $\mathcal{A}(n)$ be the set of hyperplanes fixed by the (complexified) reflections of $W_n$, and let $M_W = M_W(n)$ be their complement

$$M_W(n) := \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}(n)} H.$$ 

The group $W_n$ permutes the set of hyperplanes, and acts on $M_W$. For each family $\{W_n\}$, the hyperplane complements can be described explicitly:

- $M_A(n) = \{(v_1, \ldots, v_n) \in \mathbb{C}^n \mid v_i \neq v_j \text{ for } i \neq j\}$
- $M_D(n) = \{(v_1, \ldots, v_n) \in \mathbb{C}^n \mid v_i \neq \pm v_j \text{ for } i \neq j\}$
- $M_{BC}(n) = \{(v_1, \ldots, v_n) \in \mathbb{C}^n \mid v_i \neq \pm v_j \text{ for } i \neq j; v_i \neq 0 \text{ for all } i\}$

We note that $M_{BC}(n) \subseteq M_D(n) \subseteq M_A(n)$.

The hyperplane complement $M_A(n)$ is the ordered $n$–point configuration space of the plane $\mathbb{C}$; it is an Eilenberg–Mac Lane space with fundamental group the pure braid group on $n$ strands. Arnol’d computed its integral cohomology in 1969 [Arn69]. Its quotient $M_A(n)/S_n$ is an Eilenberg–Mac Lane space with fundamental group the braid group on $n$ strands. Brieskorn showed that $M_{BC}(n)$ and $M_D(n)$ and their quotients $M_{BC}(n)/B_n$ and $M_D(n)/S_n$ are also Eilenberg–Mac Lane spaces [Bri73, Proposition 2]; their fundamental groups are sometimes called generalized (pure) braid groups.

Brieskorn [Bri73] and Orlik–Solomon [OS80] studied the cohomology of the complement $M$ of a general arrangement of complex hyperplanes containing the origin. Define a set of hyperplanes $H_1, \ldots, H_p$ to be dependent if

$$\text{codim}(H_1 \cap \cdots \cap H_p) < p.$$
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Let $E(\mathcal{A})$ be the complex exterior algebra

$$E(\mathcal{A}) := \bigwedge \langle e_H \mid H \in \mathcal{A} \rangle$$

and let $I(\mathcal{A}) \subseteq E(\mathcal{A})$ be the ideal

$$I(\mathcal{A}) := \langle \sum_{\ell=1}^{p} (-1)^{\ell} e_{H_1} \cdots \hat{e}_{H_\ell} \cdots e_{H_p} \mid H_1, \ldots, H_p \text{ dependent} \rangle$$

Orlik–Solomon proved that $H^*(\mathcal{M}_W, \mathbb{C})$ is isomorphic to

$$A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A})$$

as a graded algebra [OS80, Theorem 5.2]. Their work implies that

$$H^*(\mathcal{M}_A(n), \mathbb{C}) \cong A(\mathcal{A})$$

as a graded $\mathbb{C}[W_n]$–module under the $W_n$–action $w \cdot e_H = e_{wH}$. The structure of $H^*(\mathcal{M}_A(n), \mathbb{C})$ as an $S_n$–representation is described by Lehrer–Solomon [LS86], and the structure of the $B_n$–representations $H^*(\mathcal{M}_{BC}(n), \mathbb{C})$ is described by Douglass [Dou92]. Lehrer–Solomon and Douglass give decompositions of the $W_n$–representations $H^*(\mathcal{M}_W(n), \mathbb{C})$ in type $A$ and $B/C$, respectively, as sums of certain explicitly described induced representations. Lehrer–Solomon conjectured that, as they prove in type $A$, the cohomology groups $H^m(\mathcal{M}_W, \mathbb{C})$ decompose into a sum of induced one-dimensional representations of centralizers, summed over the set of $W_n$ conjugacy classes [LS86, Conjecture 1.6]. Recent progress has been made on this conjecture; see Douglass–Pfeiffer–Röhrle [DPR12].

Church and Farb prove that, for each degree $m$, the sequence $H^m(\mathcal{M}_A(n), \mathbb{Q})$ is a uniformly representation stable sequence of $S_n$–representations [CF13, Theorem 4.1]. Church–Ellenberg–Farb further prove that $H^m(\mathcal{M}_A, \mathbb{Q})$ is a graded FI$^+$–algebra of finite type; this is a special case of their much more general results on the ordered configuration space of manifolds [CEF12, Theorem 4.7; see also Theorems 4.1 and 4.2]. In [CF13, Theorem 4.6], Church–Farb analyze the stability behaviour of the sequence $H^m(\mathcal{M}_{BC}, \mathbb{C})$ of $B_n$–representations.

The following result recovers [CF13, Theorem 4.1 and 4.6] in types $A_{n-1}$ and $B_n/C_n$. It recovers the work of Church–Ellenberg–Farb on the cohomology of the ordered configuration space of $\mathbb{C}$. 

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Theorem 7.14. Let \( \mathcal{M}_W \) be the complex hyperplane complement associated with the Weyl group \( W_n \) in type \( A_{n-1}, B_n/C_n \), or \( D_n \), as described above. In each degree \( m \), the cohomology groups \( H^m(\mathcal{M}_A(\bullet), \mathbb{C}) \) form an \( F_{IA} \)-module finitely generated in degree \( \leq 2m \), and both \( H^m(\mathcal{M}_{BC}(\bullet), \mathbb{C}) \) and \( H^m(\mathcal{M}_D(\bullet), \mathbb{C}) \) are \( F_{IB} \)-modules finitely generated in degree \( \leq 2m \). For each \( W \), the cohomology \( H^*(\mathcal{M}_W(\bullet), \mathbb{C}) \) of the hyperplane complements is a graded \( F_{IW} \)-module of slope 2.

Proof of Theorem 7.14. For each \( W \), the projection map

\[
\mathcal{M}_W(n+1) \longrightarrow \mathcal{M}_W(n)
\]

\[
P : (v_1, \ldots, v_n, v_{n+1}) \longmapsto (v_1, \ldots, v_n)
\]

has a section

\[
S : \mathcal{M}_W(n) \longrightarrow \mathcal{M}_W(n+1)
\]

\[
(v_1, \ldots, v_n) \longmapsto (v_1, \ldots, v_n, 1 + \sum_{i=1}^{n} |v_i|)
\]

and so induces an injective map on cohomology, as follows. We associate each hyperplane \( H \subseteq \mathbb{C}^n \) to its orthogonal complement, the span of the vectors

\[
\pm (e_i - e_j), \; \pm (e_i + e_j), \; \text{or} \; \pm e_i \quad \text{for } i, j = 1, \ldots, n.
\]

The inclusion of these normal vectors \( \mathbb{C}^n \to \mathbb{C}^{n+1} \) gives an identification of the hyperplane \( H \subseteq \mathbb{C}^n \) with a hyperplane \( H \subseteq \mathbb{C}^{n+1} \), which define the induced map \( P^* \).

\[
P^* : H^*(\mathcal{M}_W(n); \mathbb{C}) \longrightarrow H^*(\mathcal{M}_W(n+1), \mathbb{C})
\]

\[
e_H \longmapsto e_H
\]

These inclusions are \( W_n \)-equivariant maps, and give \( H^*(\mathcal{M}_W(\bullet); \mathbb{C}) \) the structure of a graded \( F_{IW} \)-module.

The \( F_{IA} \)-module \( H^1(\mathcal{M}_A(\bullet); \mathbb{C}) \) is finitely generated in degree \( \leq 2 \) by element \( e_{(e_i - e_j)}^\pm \), and the \( F_{IB} \)-module \( H^1(\mathcal{M}_{BC}(\bullet); \mathbb{C}) \) is finitely generated in degree \( \leq 2 \) by elements \( e_{(e_i - e_j)}^\pm, e_{(e_i + e_j)}^\pm \), and \( e_{(e_i)}^\pm \). It follows from Proposition 6.13 that \( H^m(\mathcal{M}_W(\bullet); \mathbb{C}) \) is finitely generated in degree \( \leq 2m \) in types A and B/C. The bound on the slope of the \( F_{IW} \)-algebra \( H^*(\mathcal{M}_W(\bullet); \mathbb{C}) \) follows from Theorem 4.4.
The section \( S \) induces a map

\[
S^* : H^*(\mathcal{M}_W(n+1); \mathbb{C}) \rightarrow H^*(\mathcal{M}_W(n), \mathbb{C});
\]

when \( W_n \) is \( S_n \) or \( B_n \) these sections give \( H^*(\mathcal{M}_W(\bullet), \mathbb{C}) \) the structure of an FI\(_W\)-module, just as in the proof of [CEF12, Theorem 4.6]. We can describe this structure explicitly: an FI\(_{BC}\)-morphism \( f : m_0 \rightarrow n_0 \) acts on the generators \( e_H \) as follows.

\[
e_{(e_i - e_j)^\perp} \mapsto \begin{cases} 
e_{(e_{f(i)} - e_{f(j)})^\perp}, & \text{if } f(i), f(j) \neq 0 \\ 0, & \text{if } f(i) = 0 \text{ or } f(j) = 0 \end{cases}
\]

\[
e_{(e_i + e_j)^\perp} \mapsto \begin{cases} 
e_{(e_{f(i)} + e_{f(j)})^\perp}, & \text{if } f(i), f(j) \neq 0 \\ 0, & \text{if } f(i) = 0 \text{ or } f(j) = 0 \end{cases}
\]

\[
e_{(e_i)^\perp} \mapsto \begin{cases} 
e_{(e_{f(i)})^\perp}, & \text{if } f(i) \neq 0 \\ 0, & \text{if } f(i) = 0 \end{cases}
\]

Here, we use the convention that \( e_{-i} := -e_i \). It is straightforward to verify that these maps are functorial.

In type \( A \), this action restricts to an FI\(_A\)-module structure on the ring \( H^*(\mathcal{M}_A(n), \mathbb{C}) \) generated by the elements \( e_{(e_i - e_j)^\perp} \). For type \( D \), observe that the inclusion of hyperplane complements

\[
\mathcal{M}_{BC}(n) \hookrightarrow \mathcal{M}_D(n)
\]

induces an inclusion of cohomology groups

\[
H^*(\mathcal{M}_D(n); \mathbb{C}) \hookrightarrow H^*(\mathcal{M}_{BC}(n), \mathbb{C}).
\]

The subspaces \( H^*(\mathcal{M}_D(n); \mathbb{C}) \subseteq H^*(\mathcal{M}_{BC}(n), \mathbb{C}) \) form the \( B_n \)-invariant subring generated by the elements \( e_{(e_i - e_j)^\perp} \) and \( e_{(e_i + e_j)^\perp}, i \neq j \). These inclusions realize \( H^*(\mathcal{M}_D(n); \mathbb{C}) \) as a sub–FI\(_{BC}\)-module of \( H^*(\mathcal{M}_{BC}(n), \mathbb{C}) \) generated as an FI\(_{BC}\)-algebra by the FI\(_{BC}\)-module \( H^1(\mathcal{M}_D(n); \mathbb{C}) \). Since \( H^1(\mathcal{M}_D(n); \mathbb{C}) \) is finitely generated in degree \( \leq 2 \), it follows again that \( H^*(\mathcal{M}_D(n); \mathbb{C}) \) is an FI\(_{BC}\)-algebra of slope 2 with \( H^{m}(\mathcal{M}_D(n); \mathbb{C}) \) finitely generated in degree \( \leq 2m \). \( \square \)
7.3 The cohomology of hyperplane complements

Theorem 7.14 has the following consequences.

**Corollary 7.15.** In each degree \( m \), the sequence of cohomology groups \( \{ H^m(\mathcal{M}_W(n), \mathbb{C}) \} \) of the associated hyperplane complement is uniformly representation stable in degree \( \leq 4m \).

In types A and B/C, Corollary 7.15 recovers [CF13, Theorem 4.1 and 4.6].

**Corollary 7.16.** In each degree \( m \), the sequence of characters of the \( \mathcal{W}_n \)-representations \( H^m(\mathcal{M}_W(n), \mathbb{C}) \) are given by a unique character polynomial of degree \( \leq 2m \) for all \( n \).

**Proof of Corollary 7.16.** The statement follows for \( S_n \) from [CEF12, Theorem 2.67], and in type \( B_n \) from Proposition 5.15. Since the \( D_n \) characters are the restriction of the characters of \( B_n \) on the \( B_n \)-subrepresentations \( H^* (\mathcal{M}_D(n); \mathbb{C}) \subseteq H^* (\mathcal{M}_BC(n), \mathbb{C}) \), these \( D_n \) characters are given by the character polynomial for \( B_n \) on this sub-FI\( _{BC} \)-module of \( H^* (\mathcal{M}_BC(\bullet), \mathbb{C}) \).

The character polynomials for \( H^m(\mathcal{M}_A(\bullet), \mathbb{C}) \) are computed in [CEF12] for some low values of \( m \). The decompositions for \( H^1(\mathcal{M}_D(\bullet), \mathbb{C}) \) and \( H^1(\mathcal{M}_BC(\bullet), \mathbb{C}) \) are:

\[
H^1(\mathcal{M}_D(\bullet), \mathbb{C}) = 2M_D(\{ \square, \emptyset \}) \\
\chi_{H^1(\mathcal{M}_D(\bullet), \mathbb{C})} = 2 \left( \frac{X_1}{2} \right) + 2 \left( \frac{Y_1}{2} \right) + 2X_2
\]

\[
H^1(\mathcal{M}_BC(\bullet), \mathbb{C}) = M_{BC}(\square, \emptyset) \oplus M_{BC}(\square, \emptyset) \oplus M_{BC}(\emptyset, \square) \\
\chi_{H^1(\mathcal{M}_BC(\bullet), \mathbb{C})} = 2 \left( \frac{X_1}{2} \right) + 2 \left( \frac{Y_1}{2} \right) + 2X_2 + X_1 - Y_1
\]

The decompositions for \( H^2(\mathcal{M}_D(\bullet), \mathbb{C}) \) and \( H^2(\mathcal{M}_BC(\bullet), \mathbb{C}) \) are:

\[
H^2(\mathcal{M}_D(\bullet), \mathbb{C}) = M_D(\{ \square, \emptyset \}) \oplus M_D(\{ \square, \emptyset \}) \oplus M_D(\{ \square, \emptyset \}) \oplus M_D(\{ \square, \emptyset \}) \oplus M_D(\{ \square, \emptyset \}) \oplus M_D(\{ \square, \emptyset \}) \oplus M_D(\{ \square, \emptyset \}) \oplus M_D(\{ \square, \emptyset \})
\]

\[
H^2(\mathcal{M}_BC(\bullet), \mathbb{C}) = M_{BC}(\square, \emptyset) \oplus M_{BC}(\square, \emptyset) \oplus M_{BC}(\square, \emptyset) \oplus M_{BC}(\square, \emptyset) \oplus M_{BC}(\square, \emptyset) \oplus M_{BC}(\square, \emptyset) \oplus M_{BC}(\square, \emptyset) \oplus M_{BC}(\square, \emptyset)
\]
Problem 7.17. For each $m$, compute the character polynomial of the $\mathbb{F}_{A^2}$-module $H^m(\mathcal{M}_A(\bullet), \mathbb{C})$, and compute its decomposition into induced representations $M_A(U)$. Compute the character polynomials of the $\mathbb{F}_{BC^2}$-modules $H^m(\mathcal{M}_{BC}(\bullet), \mathbb{C})$ and $H^m(\mathcal{M}_D(\bullet), \mathbb{C})$ for each $m$, and compute the decomposition into induced representations $M_{BC}(U)$.

References

[aAMP] E. al Aamily, A. O. Morris, and M. H. Peel. The representations of the Weyl groups of type $B_n$. 

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| Reference | Citation |
|-----------|----------|
| [Arn69]   | V. I. Arnol’d. The cohomology ring of the colored braid group. *Mathematical Notes*, 5(2):138–140, 1969. |
| [Bar04]   | V. G. Bardakov. The virtual and universal braids. *arXiv preprint math/0407400*, 2004. |
| [BB06]    | F. Bergeron and R. Biagioli. Tensorial square of the hyperoctahedral group coinvariant space. *the electronic journal of combinatorics*, 13(R38):1, 2006. |
| [BCV09]   | K.-U. Bux, R. Charney, and K. Vogtmann. Automorphisms of two-dimensional RAAGS and partially symmetric automorphisms of free groups. *Groups, Geometry, and Dynamics*, 3(4):541–554, 2009. |
| [BEER06]  | L. Bartholdi, B. Enriquez, P. Etingof, and E. Rains. Groups and lie algebras corresponding to the Yang–Baxter equations. *Journal of Algebra*, 305(2):742–764, 2006. |
| [Ber09]   | F. Bergeron. *Algebraic combinatorics and coinvariant spaces*. AK Peters, 2009. |
| [Ber11]   | F. Bergeron. Multivariate diagonal coinvariant spaces for complex reflection groups. *arXiv preprint arXiv:1105.4358*, 2011. |
| [BL93]    | A. Brownstein and R. Lee. Cohomology of the group of motions of n strings in 3–space. In *Mapping class groups and moduli spaces of Riemann surfaces: proceedings of workshops held June 24–28, 1991, in Göttingen, Germany, and August 6-10, 1991, in Seattle, Washington...*, volume 160, page 51. Amer Mathematical Society, 1993. |
| [Bor53]   | A. Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953. |
| [Bri73]   | E. Brieskorn. Sur les groupes de tresses [d’apres V. I. Arnol’d]. *Séminaire Bourbaki vol. 1971/72 Exposés 400–417*, pages 21–44, 1973. |
| [Car72]   | R. W. Carter. Conjugacy classes in the Weyl group. *Compositio Math*, 25(1):1–59, 1972. |
| [CDG12]   | G. Collinet, A. Djament, and J. T. Griffin. Stabilité homologique pour les groupes d’automorphismes des produits libres. *International Mathematics Research Notices*, 2012. |
| [CEF12]   | T. Church, J. Ellenberg, and B. Farb. FI-modules: A new approach to stability for $S_n$–representations. *arXiv preprint arXiv:1204.4533*, 2012. |
| [CEFN12]  | T. Church, J.S. Ellenberg, B. Farb, and R. Nagpal. FI–modules over Noetherian rings. *arXiv preprint arXiv:1210.1854*, 2012. |
| [CF13]    | T. Church and B. Farb. Representation theory and homological stability. *Arxiv preprint arXiv:1008.1368* (2010), *to appear in Advances in Mathematics*, 245:250–314, 2013. |
[CGJ05] Y. Chen, H. H. Glover, and C. A. Jensen. Proper actions of automorphism groups of free products of finite groups. *International Journal of Algebra and Computation*, 15(02):255–272, 2005.

[CV09] R. Charney and K. Vogtmann. Finiteness properties of automorphism groups of right-angled artin groups. *Bulletin of the London Mathematical Society*, 41(1):94–102, 2009.

[Dah62] D. M. Dahn. A generalization of braid theory. *Princeton Univ. Ph. D. thesis*, 1962.

[Dou92] J. M. Douglass. On the cohomology of an arrangement of type $B_l$. *J. Algebra*, 146:265–282, 1992.

[DPR12] J. M. Douglass, G. Pfeiffer, and G. Röhrle. An inductive approach to coxeter arrangements and solomons descent algebra. *Journal of Algebraic Combinatorics*, 35(2):215–235, 2012.

[FH04] W. Fulton and J. Harris. *Representation Theory: A First Course*. Springer, 2004.

[Ful97] W. Fulton. *Young tableaux*. Cambridge Univ. Press, 1997.

[GG09] A. M. Garsia and A. Goupil. Character polynomials, their $q$-analogs and the Kronecker product. *Work*, 1:1, 2009.

[GH93] A. M. Garsia and M. Haiman. A graded representation model for Macdonald’s polynomials. *Proceedings of the National Academy of Sciences*, 90(8):3607–3610, 1993.

[GK78] L. Geissinger and D. Kinch. Representations of the hyperoctahedral groups. *Journal of Algebra*, 53(1):1–20, 1978.

[Gol81] D. L. Goldsmith. The theory of motion groups. *The Michigan Mathematical Journal*, 28(1):3–17, 1981.

[Gor03] I. Gordon. On the quotient ring by diagonal invariants. *Inventiones Mathematicae*, 153(3):503–518, 2003.

[GP00] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori-Hecke algebras*. Oxford University Press, USA, 2000.

[Gri13a] J. T. Griffin. Diagonal complexes and the integral homology of the automorphism group of a free product. *Proceedings of the London Mathematical Society*, 106(5):1087–1120, 2013.

[Gri13b] James Thomas Griffin. Automorphisms of free products of groups. 2013.

[Hai94] M. Haiman. Conjectures on the quotient ring by diagonal invariants. *Journal of Algebraic Combinatorics*, 3(1):17–76, 1994.

[Hai02a] M. Haiman. Combinatorics, symmetric functions, and Hilbert schemes. *Current developments in mathematics*, 2002:39–111, 2002.
REFERENCES

[Hai02b] M. Haiman. Vanishing theorems and character formulas for the hilbert scheme of points in the plane. Inventiones Mathematicae, 149(2):371–407, 2002.

[Hat95] A. Hatcher. Homological stability for automorphism groups of free groups. Commentarii Mathematici Helvetici, 70(1):39–62, 1995.

[Hem10] D. J. Hemmer. Stable decompositions for some symmetric group characters arising in braid group cohomology. Journal of Combinatorial Theory, Series A, 2010.

[HHL+05] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. A combinatorial formula for the character of the diagonal coinvariants. Duke Mathematical Journal, 126(2):195–232, 2005.

[HW10] A. Hatcher and N. Wahl. Stabilization for mapping class groups of 3–manifolds. Duke Mathematical Journal, 155(2):205–269, 2010.

[JMM06] C. A. Jensen, J. McCammond, and J. Meier. The integral cohomology of the group of loops. Geometry & Topology, 10:759–784, 2006.

[JW04] C. A. Jensen and N. Wahl. Automorphisms of free groups with boundaries. Algebr. Geom. Topol, 4:543–569, 2004.

[Kau99] L. H. Kauffman. Virtual knot theory. European Journal of Combinatorics, 20(7):663–690, 1999.

[Kau00] L. H. Kauffman. A survey of virtual knot theory. Knots in Hellas, 98:143–202, 2000.

[KL04] L. H. Kauffman and S. Lambropoulou. Virtual braids. arXiv preprint math.GT/0407349, 2004.

[Lee13] P. Lee. On the action of the symmetric group on the cohomology of groups related to (virtual) braids. arXiv preprint arXiv:1304.4645, 2013.

[LS86] G. I. Lehrer and L. Solomon. On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes. Journal of Algebra, 104(2):410–424, 1986.

[Mac79] I. G. Macdonald. Symmetric functions and Hall polynomials. Clarendon Press Oxford, 1979.

[May75] S. J. Mayer. On the characters of the Weyl group of type C. Journal of Algebra, 33(1):59–67, 1975.

[ML98] S. Mac Lane. Categories for the working mathematician, volume 5. Springer verlag, 1998.

[MM96] D. MacCullough and A. Miller. Symmetric automorphisms of free products, volume 582. American Mathematical Society, 1996.
REFERENCES

[MM12] J. P. May and M. Merling. *The Segal equivariant infinite loop space machine*. 2012. In preparation.

[Mur37] F. D. Murnaghan. The characters of the symmetric group. *American Journal of Mathematics*, 59(4):739–753, 1937.

[Mur38] F. D. Murnaghan. The analysis of the Kronecker product of irreducible representations of the symmetric group. *American journal of mathematics*, 60(3):761–784, 1938.

[Mur51] F. D. Murnaghan. The characters of the symmetric group. *Proc. Nat. Acad. Sci. U.S.A.*, 37:55–58, 1951.

[MZ09] R. McEwen and M. C. B. Zaremsky. A combinatorial proof of the Degree Theorem in Auter space. *arXiv preprint arXiv:0907.4642*, 2009.

[Nar85] H. Naruse. Representation theory of Weyl group of type $C_n$. *Tokyo Journal of Mathematics*, 8(1):177–190, 1985.

[OS80] P. Orlik and L. Solomon. Combinatorics and topology of complements of hyperplanes. *Inventiones Mathematicae*, 56(2):167–189, 1980.

[Put12] A. Putman. Stability in the homology of congruence subgroups. *preprint*, 2012.

[Rie09] E. Riehl. Homotopy (limits and) colimits. 2009. Notes.

[Sno13] A. Snowden. Syzygies of Segre embeddings and $\Delta$-modules. *Duke Mathematical Journal*, 162(2):225–277, 2013.

[Spe60] W. Specht. Die charaktere der symmetrischen gruppe. *Mathematische Zeitschrift*, 73(4):312–329, 1960.

[SS12a] S. V. Sam and A. Snowden. GL–equivariant modules over polynomial rings in infinitely many variables. *arXiv preprint arXiv:1206.2233*, 2012.

[SS12b] S. V. Sam and A. Snowden. Introduction to twisted commutative algebras. *arXiv preprint arXiv:1209.5122*, 2012.

[Ver01] V. V. Vershinin. On homology of virtual braids and Burau representation. *Journal of Knot Theory and Its Ramifications*, 10(05):795–812, 2001.

[Wil12] J. C. H. Wilson. Representation stability for the cohomology of the pure string motion groups. *Algebr. Geom. Topol*, 12(2):909–931, 2012.

[You30] A. Young. On quantitative substitutional analysis (fifth paper). *Proceedings of the London Mathematical Society*, 2(1):273, 1930.

[Zar12] M. C. B. Zaremsky. Rational homological stability for groups of partially symmetric automorphisms of free groups. *arXiv preprint arXiv:1203.4845*, 2012.
REFERENCES

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