THE CRITICAL BEHAVIOUR OF POTTS MODELS WITH SYMMETRY BREAKING FIELDS

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Abstract

The $Q$-state Potts model in two dimensions in the presence of external magnetic fields is studied. For general $Q \geq 3$ special choices of these magnetic fields produce effective models with smaller $Z(Q')$ symmetry ($Q' < Q$). The phase diagram of these models and their critical behaviour are explored by conventional finite-size scaling and conformal invariance. The possibility of multicritical behavior, for finite values of the symmetry breaking fields, in the cases where $Q > 4$ is also analysed. Our results indicate that for effective models with $Z(Q')$ symmetry ($Q' \leq 4$) the multicritical point occurs at zero field. This last result is also corroborated by Monte Carlo simulations.

1 Introduction

The ferromagnetic $Q$-state Potts model in two dimensions is among the most studied models of statistical mechanics (see [1] for a review). In the absence of external fields the model has a global $Z(Q)$ invariance [2] which, for low
temperatures, is spontaneously broken giving arise to phase transitions of second order for $Q \leq 4$ and first order for $Q > 4$ [3]. The critical fluctuations for $Q = 2, 3$ and 4 are governed by conformal field theories with central charges $c = \frac{1}{2}, \frac{4}{5}$ and 1, respectively, and the whole operator content of these models with several boundary conditions is known [4].

In this paper we study the critical behaviour of these models on the square lattice, in the presence of symmetry breaking magnetic fields. In general these fields will break completely the $Z(Q)$ symmetry, but for some special choices of these magnetic fields the resulting effective model will have a residual symmetry $Z(Q')$, with $Q' < Q$, and a domain wall structure at low temperatures, similar to these in the $Q'$-state Potts model.

On general grounds [5], we do expect that in order-disorder phase transitions, of discrete symmetry models, the critical behavior is dictated mainly by the number of ground-states (zero temperature configurations) and the relative surface energy of the infinite domain walls connecting these ground states. This reasoning induces us to expect, for arbitrary values of the symmetry breaking fields, producing a $Z(Q')$ model, an effective model in the same universality class as the $Q'$-state Potts model. However this analysis is not valid in general since some critical and multicritical models, like the Ising model and the tricritical Ising model [6], although having the same number of ground-states and domain wall structure at low temperatures, exhibit distinct critical properties.

An earlier mean field analysis on these models [7] indicate that for $Q \leq 4$ the $Z(Q') (Q' < Q)$ model, produced by the breaking fields, are in the same universality class of the $Q'$-state Potts model, for nonzero values of the fields, while for $Q > 4$, with $Q' < 4$, there is a multicritical point for finite values of the critical field, where the phase transition changes from first to second order.

Our study will be done numerically by using standard finite-size scaling [8] to obtain the phase diagram of the models, and the machinery arising from conformal invariance [4, 10] to distinguish the several possible critical behaviours. In the location of multicritical points for $Q > 4$ we also perform Monte Carlo some simulations by calculating the fourth-order cumulant of the magnetization.
2 The model

Defining at each lattice site \( \vec{r} = (i, j) \) of a square lattice an integer variable \( n_{\vec{r}} = 0, 1, \ldots, Q - 1 \), the Hamiltonian of the \( Q \)-state Potts model with \( n_h \) (\( n_h = 0, 1, \ldots, Q - 1 \)) symmetry breaking fields \( \{ \tilde{h}_m \} \) \( (m = 0, 1, \ldots, n_h - 1) \) is given by

\[
H_Q(\epsilon, \{ \tilde{h}_m \}) = -\epsilon \sum_{\vec{r}, \vec{r}'} \delta_{n_{\vec{r}}, n_{\vec{r}'}} - \sum_{m=0}^{n_h-1} \sum_{\vec{r}} \tilde{h}_m \delta_{n_{\vec{r}}, m}.
\]

(1)

where \( \epsilon > 0 \) is the ferromagnetic coupling and the first sum runs over nearest-neighbour sites. In the absence of external fields the model has a \( Z(Q) \) symmetry, since the configuration \( \{ n_{\vec{r}} \} \) and \( \{ n_{\vec{r}+l, \mod. Q} \} \) \( (l = 1, 2, \ldots, Q) \) has the same energy. The fields \( \{ \tilde{h}_m \} \), depending on their relative values, break this symmetry totally or partially. The interesting cases where the symmetry is partially broken are those where \( \tilde{h}_m = \tilde{h} > 0 \) \( (m = 0, 1, \ldots, n_h - 1) \) and the remaining symmetry is \( Z(n_h) \otimes Z(Q - n_h) \). This symmetry corresponds to \( Z(n_h) \) rotations among the variables pointing in the field directions and \( Z(Q - n_h) \) rotations among the other variables. At zero temperature we have \( n_h \) ground states and we do expect that the \( Z(n_h) \) symmetry is spontaneously broken.

Rather than working with the above Euclidean version of the model it is convenient to consider its quantum Hamiltonian version in order to simplify our numerical analysis. The row-to-row transfer matrix as well the associated \( \tau \)-continum quantum Hamiltonian [11] can be derived by a standard procedure (see [12] for example). The associated one-dimensional quantum Hamiltonian in a \( L \)-site chain is given by

\[
\hat{H} = -\frac{L^2}{g} \sum_{l=1}^{L} \sum_{\alpha=0}^{g-1} \left( \hat{S}_l \hat{S}^\dagger_{l+1} \right)^\alpha + \lambda \hat{R}^\alpha_l + \sum_{m=0}^{n_h-1} h_m \left( \hat{S}_l e^{-\frac{2\pi i m}{Q}} \right)^\alpha,
\]

(2)

where \( \lambda \) plays the role of temperature and the magnetic fields \( \{ h_m \} \) \( (m = 0, 1, \ldots, n_h - 1) \) are related with the fields \( \{ \tilde{h}_m \} \) in [1]. In (2) \( \hat{S}_l \) and \( \hat{R}_l \) are \( L^Q \otimes L^Q \) matrices satisfying the \( Z(Q) \) algebra

\[
[\hat{R}_l, \hat{S}_k] = (\theta - 1) \delta_{k,l} \hat{S}_k \hat{R}_l \quad \hat{R}^q_l = \hat{S}^q_l = 1.
\]

where \( \theta = \exp(\frac{2\pi i}{Q}) \), and in the basis where \( \hat{S}_l \) is diagonal they are given by

\[
\hat{S}_l = 1 \otimes 1 \otimes \cdots 1 \otimes S \otimes 1 \cdots \otimes 1,
\]

\[\hat{R}_l = 1 \otimes \cdots 1 \otimes R_l \otimes 1 \cdots \otimes 1,\]
\[ \hat{R}_l = 1 \otimes 1 \otimes \cdots 1 \otimes R \otimes 1 \cdots \otimes 1 \]

where the matrices \( S \) and \( R \) are in the \( l \)th position in the product and are given by

\[ S = \sum_{i=0}^{Q-1} \theta^i |i><i|, \quad R = \sum_{i=0}^{Q-1} |i><[i+1]_Q|, \]

and the symbol \([x + y]_Q\) means the addition \((x + y)\), modulo \(Q\).

In the absence of magnetic fields \((h_m = 0; m = 0, 1, \ldots, n_h - 1)\) the \(Z(Q)\) symmetry of (1) is reflected in (2) by its commutation with the \(Z(Q)\)-charge operator

\[ \hat{P} = \prod_{l=1}^{L} \hat{R}_l, \quad \hat{P}^Q = 1. \quad (3) \]

The Hilbert space associated with (2) can therefore be separated into disjoint sectors labelled by the eigenvalues \(\exp(\frac{i \pi q}{Q})\), \((q = 0, 1, \ldots, Q - 1)\) of (3). The interesting cases, which we will concentrate on in this paper, are obtained by choosing in (2) equal values for the magnetic fields

\[ h_1 = h_2 = \ldots = h_{n_h} = h > 0. \quad (4) \]

In this case the symmetry \(Z(n_h) \otimes Z(Q - n_h)\) of (3) is reflected by the simultaneous commutation of (2) with the "parity" operators

\[ \hat{V} = \prod_{l=1}^{L} \hat{V}_l, \quad \hat{V}^{n_h} = 1, \quad (5) \]

\[ \hat{W} = \prod_{l=1}^{L} \hat{W}_l, \quad \hat{W}^{Q-n_h} = 1, \quad (6) \]

with

\[ \hat{V}_l = 1 \otimes 1 \otimes \cdots 1 \otimes V \otimes 1 \cdots \otimes 1 \]

\[ \hat{W}_l = 1 \otimes 1 \otimes \cdots 1 \otimes W \otimes 1 \cdots \otimes 1 \]

and \(V\) and \(W\), located at the \( l \)th position in the product, are \(Q \times Q\) matrices given by

\[ V = \sum_{i=0}^{n_h-1} |i><[i+1]_{n_h}| + \sum_{i=n_h}^{Q-1} |i><i| \]

\[ W = \sum_{i=0}^{n_h-1} |i><i| + \sum_{i=n_h}^{Q-1} |i><[i+1]_{Q-n_h}|. \]
The Hilbert space associated to (2) is now separated into \( n_h (Q - n_h) \) disjoint sectors labelled by the eigenvalues \( \exp(i \frac{2\pi v}{n_h}) \) and \( \exp(i \frac{2\pi w Q - n_h}{Q - n_h}) \) (\( v = 0, 1, \ldots, n_h - 1 \), \( w = 0, 1, \ldots, Q - n_h - 1 \)) of the operators \( \hat{\mathcal{V}} \) and \( \hat{\mathcal{W}} \), respectively.

In the numerical diagonalization of (2), with periodic boundaries, all the above symmetries, together with the translational invariance, enables us to handle large lattices with modest computer time and memory. We use the Lanczos method to diagonalize (2) up to \( L = 10, 11 \) and 13, for \( Q = 5, 4 \) and 3, respectively.

## 3 Results

We considered in our study only the interesting cases where \( Q > n_h \geq 2 \) and \( h_1 = h_2 = \ldots = h_{n_h} = h \geq 0 \), since in these cases always a remaining symmetry \( Z(Q') \) (\( Q' = n_h \geq 2 \)) still remains for \( h \neq 0 \).

When \( h = 0 \) the model is self-dual with a phase transition at \( \lambda_c(0) = 1 \), which has a second order or first order nature depending if \( Q \leq 4 \) or \( Q > 4 \), respectively. In the limit \( h \to \infty \) the eigenvectors of \( \hat{\mathcal{S}}_i \) in (2) with eigenvalues \( \theta^l, l = n_h, n_h + 1, \ldots, Q - 1 \), are forbidden and we have an effective \( n_h \)-state Potts model at zero field. Analysing the effect of (2) in the remaining \( n_L \)-dimensional Hilbert space it is not difficult to see that the phase transition happens at

\[
\lambda_c(h \to \infty) = \frac{Q}{n_h} \tag{7}
\]

Between those two extremum values of \( h \) we estimate the phase transition curve \( \lambda_c(h) \) by using standard finite-size scaling. The curve is evaluated by extrapolations to the bulk limit (\( L \to \infty \)) of sequences \( \lambda_c(h, L) \) obtained by solving

\[
\Gamma_L(\lambda_c) L = \Gamma_{L+1}(\lambda_c)(L + 1), \quad L = 2, 3, \ldots, \tag{8}
\]

where \( \Gamma_L(\lambda_c) \) is the mass gap of the Hamiltonian (2) with \( L \) sites.

Once the transition curve is estimated, in the region of continuous phase transitions we expect the model is conformally invariant. This symmetry allows us to infer the critical properties from the finite-size corrections to the eigenspectrum at \( \lambda_c \). The conformal anomaly \( c \) can be calculated from the large \( L \) behaviour of the ground-state energy \( E_0(L) \). For periodic chains
$E_0(L)$ behaves as
\[
\frac{E_0(L)}{L} = \epsilon_\infty - \frac{\pi c v_s}{6L^2} + o(L^{-2}),
\]

where $\epsilon_\infty$ is the ground-state energy, per site, in the bulk limit and $v_s$ is the sound velocity. The scaling dimensions of operators governing the critical fluctuations (related to critical exponents) are evaluated from the finite-$L$ corrections of the excited states. For each primary operator, with dimension $x_\phi$ and spin $s_\phi$, in the operator algebra of the system, there exists an infinite tower of eigenstates of the quantum Hamiltonian, whose energy $E_{m,m'}^\phi$ and momentum $P_{m,m'}^\phi$, in a periodic chain are given by
\[
E_{m,m'}^\phi(L) = E_0 + \frac{2\pi v_s}{L}(x_\phi + m + m') + o(L^{-1})
\]
\[
P_{m,m'}^\phi = (s_\phi + m - m')\frac{2\pi}{L}
\]

where $m, m' = 0, 1, \ldots$.

We present our results separately for the cases $Q \leq 4$ and $Q > 4$ in the next sections.

### 3.1 Models with $Q \leq 4$

There exist three interesting cases, namely, the 3-state Potts model with two fields ($Q = 3, n_h = 2$) and the 4-state Potts model with three and two fields ($Q = 4, n_h = 3, 2$).

The critical curves were obtained by solving (8). As an example, in Fig.1 we show the extrapolated curve for the case of $Q = 3$ and $n_h = 2$. We also show in this figure the curve obtained by solving (8) for $L = 5$. We clearly see an agreement with the limiting values $\lambda_c(0) = 1$ and $\lambda_c(h \rightarrow \infty) = \frac{3}{2}$, predict by (7). Similar curves are obtained in the other cases.

The conformal anomaly and anomalous dimensions are obtained using relations (9,10), for several values of $h$. In table 1,2 and 3 we show for some values of $h$ the extrapolated results obtained for the cases ($Q = 3, n_h = 2$) ($Q = 4, n_h = 3$) and ($Q = 4, n_h = 2$), respectively. Our personal estimative of errors are in the last digit. In these tables we also present our conjectured values. The dimensions $x_n(k, v)$ and $x_n(k, v, w)$ appearing in these tables are obtained by using in (10) the $n$th ($n = 1, 2, 3, \ldots$) eigenenergy in the sector with momentum $\frac{2\pi k}{L}$ ($k = 0, 1, \ldots$) and eigenvalues $\exp(i\frac{2\pi v}{Q-n_h})$ and
\[ \exp(i \frac{2\pi w}{Q-n_h})(v = 0, 1, \ldots, n_h - 1, w = 0, 1, \ldots, Q - n_h - 1) \] of the operators $\hat{V}$ and $\hat{W}$ defined in (3) and (4), respectively.

We see in table 1 that for all values of $h$ the conformal anomaly is $c = \frac{1}{2}$, indicating that the model share the same universality class as the $Z(2)$ Ising model. The dimensions $1$ and $\frac{1}{8}$ correspond to the dimensions of the energy and magnetic operator in the Ising model and the dimension $2 = 1 + 1$ and $\frac{9}{8} = 1 + \frac{1}{8}$ are the next dimensions in the tower of these operators (see Eq. (10)).

In table 2 we clearly see that the conformal anomaly, for all values of $h$ is $c = \frac{4}{5}$. There exists two modular invariant universality classes of conformal theories with $c = \frac{4}{5}$ [14]. One of these can be represent by the restricted solid-on-solid (RSOS) model [15, 16] and the other by the 3-state Potts model with no magnetic fields. These two models, although having the same conformal anomaly $c = \frac{4}{5}$, have distinct operator content. The dimension $x = 0, \frac{4}{5}, \frac{14}{5}, \frac{2}{3}$ in table 2 and the degeneracy of the sectors where the operator $\hat{V}$ in (3) has eigenvalues $\exp(i \frac{2\pi}{3})$ and $\exp(i \frac{4\pi}{3})$ indicate that the model belongs to the same universality class as the 3-state Potts model.

In table 3, as in table 1, the conformal anomaly is $c = 1/2$ and the dimensions are those of the Ising model indicating that both models are in the same universality class.

Beyond the values of $h$ presented in tables 1-3 we also performed a careful analysis for small values of $(h \sim 0.01)$ finding similar results as those presented in those tables. This indicate that for $Q \leq 4$, where the model has a second-order phase transition in the absence of external fields, the introduction of $n_h$ ($Q > n_h \geq 2$) fields, of arbitrary strength, brings the model into the universality class of a Potts model with $n_h$ states.

### 3.2 Models with $Q > 4$

In this case, while in the absence of external fields the models exhibit a first-order phase transition the introduction of $n_h$ magnetic fields ($4 \geq n_h \geq 2$) of infinite and equal strength ($h \rightarrow \infty$) render them to an effective $n_h$-state Potts model, which has a second order phase transition. This brings the interesting possibility of a multicritical behaviour for a finite value of $h$, when the transition curve changes from second to first order as we decreases $h$ from the infinite value. In fact this is the mean field prediction [7].
Since the Hilbert space grows exponentially with $Q$, the simplest case where the above critical point may occur is the 5-state Potts model in the presence of $n_h = 2$ magnetic fields. In table 4 we present, for some values of $h$, our results for the finite-size sequences of the conformal anomaly of the model. These sequences are obtained from (10) and (9)

$$c_{L,L+1} = \frac{12[(L + 1)E_0(L) - LE_0(L + 1)]}{[(L + 1)^2 - L^2][E_2(L + 1) - E_1(L + 1)]}$$

(11)

where $E_0(L)$ is the ground-state energy for the chain of length $L$ and $E_2(L)$, $E_1(L)$ are the lowest eigenenergies with momentum 0 and $\frac{2\pi}{L}$, respectively, in the sector where the operators $\hat{V}$ and $\hat{W}$, defined in (3) and (4) has eigenvalues (-1) and (1). We see in table 4 that for $h > \sim 0.05$ we still have an Ising-like behaviour with $c = \frac{1}{2}$. For $h < 0.05$, and for the lattice sizes we were able to handle, it is not possible to obtain reliable results using (11).

An heuristic method, which was proved to be effective in obtaining multicritical points in earlier works [17] is to simultaneously solve Eq. (8) for three different lattice sizes

$$\Gamma_L(\lambda_c) = \Gamma_{L+1}(\lambda_c)(L + 1) = \Gamma_{L+2}(\lambda_c)(L + 2).$$

(12)

We tried to solve these equations for $0.5 > \lambda > 0$ ($L = 5$) and we found no consistent solutions, which indicates the absence of a tricritical point for a finite value of $h$.

Another method, also used to locate multicritical points [14], is obtained from the simultaneous crossing of two different gaps on a given pair of lattices (instead of three lattices as in (12)). Trying several different gaps we also did not find, within this method, a multicritical point for $h \neq 0$.

Since these methods are heuristic and the lattice sizes we are considering may not be enough to obtain the bulk limit ($L \rightarrow \infty$) in the region $h \sim 0$ we decide to supplement our results by Monte Carlo simulations. These simulations will enable us to distinguish the order of the phase transition as we change the magnetic field strength. We simulate the systems by the heatbath algorithm and analyse the fourth-order cumulant of the magnetization as a function of the magnetic field. The simulations were done directly in the classical version of the model (1). Since the evidence of a multicritical point, for non-zero values of $h$, would be large for higher values of $Q$, we choose $Q = 7$ and $n_h = 2$ ($\tilde{h}_0 - \tilde{h}_1 = h$) for extensive calculations.
The fourth-order cumulant of the magnetization is defined by

$$U_L = 1 - \frac{< m^4 >_L}{3 < m^2 >^2_L} \quad (13)$$

where the averages are done on an $L \times L$ lattice. The magnetization in (13), for a given configuration $\{n_\vec{r}\}$ of classical variables, is defined by

$$m = \frac{1}{L^2} \sum_\vec{r} (\delta_{n_\vec{r},0} - \delta_{n_\vec{r},1})$$

Following Binder [18], the cumulant (13) will be zero for $T > T_c$ and $U_L = \frac{2}{3}$ for $T < T_c$. At the transition temperature $T_c$ (13) will be zero for continuous phase transitions and negative for first-order phase transitions.

In Fig. 2 we show the values of $U_L$ for lattice sizes $50 \times 50, 80 \times 80$ and $170 \times 170$. In the simulations we choose $\epsilon = 1$, $\bar{h}=0.01$ and each point was obtained by averaging $5.10^4$ iterations, after thermalization. We see in this figure that while for $L = 50$ the phase transition appears to be first order, as $L$ grows the numerical results indicates the phase transition to be continuous. The result for the smaller lattice $L = 50$ is clearly due to the finite size of the lattice. By repeating these simulations for even smaller values of $\bar{h}$ we should expect that these finite-size effects will be apparent for even larger lattices, and our simulations are in favour of a multicritical point only at $\bar{h}=0$.

4 Conclusion

We have calculated the phase transition diagram and critical properties of the $Q$-state Potts model in the presence of $n_h$ ($Q > n_h > 1$) external magnetic fields of equal strength $h > 0$. In the case where $Q > n_h \geq 2$ the original symmetry, at $h = 0$, breaks into a $Z(n_h) \otimes Z(Q-n_h)$ symmetry. The $Z(n_h)$ part of the above symmetry relates the configurations of the $n_h$ distinct ground-state configurations at zero temperature, and by standard arguments, should be spontaneously broken at low temperatures.

Our results, based on conformal invariance and supplemented by Monte Carlo simulation indicate that, for arbitrary values of $h$, the order-disorder phase transition associated with the global $Z(n_h)$ symmetry is in the same universality class of the $n_h$-state Potts model. Moreover, for $Q > 4$, contrary
to the mean field prediction, we do not see any evidence of a multicritical point for non-zero values of $h$.

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Figure Captions

Fig. 1 - Estimates for the critical curve of the 3-state Hamiltonian (2) with $n_h = 2$ fields. The curve in the largest scale, for $0 < h < 5$, interpolates the points obtained by extrapolating the solutions of (8) for $L = 2 - 13$ (circles). The inserted curve for $0 < h < 50$ was obtained by solving (8) for $L = 5$.

Fig. 2 - Fourth-order cumulant of the magnetization (13) as a function of $\beta$ for the 7-state Potts model in presence of $n_h = 2$ external fields $\tilde{h}_0 = \tilde{h}_1 = 0.01$ (see Eq. (11)). The lattice sizes are $L = 50 \times 50$, $L = 80 \times 80$ and $L = 170 \times 170$.
Table Caption

Table 1 - Extrapolated and conjectured results for the conformal anomaly $c$ and anomalous dimensions $x_n(k, v)$ of the 3-states Potts chain $(2)$ with $n_h = 2$ magnetic fields ($h_0 = h_1 = h$) (see the text). The conjectured values, in parenthesis, are the corresponding ones in the critical Ising model.

Table 2 - Extrapolated and conjectured results for the conformal anomaly $c$ and anomalous dimensions $x_n(k, v)$ of the 4-states Potts chain $(2)$ with $n_h = 3$ magnetic fields ($h_0 = h_1 = h_2 = h$) (see the text). The conjectured values, in parenthesis, are the corresponding ones appearing in the 3-state Potts model.

Table 3 - Extrapolated and conjectured results for the conformal anomaly $c$ and anomalous dimensions $x_n(k, v, w)$ of the 4-states Potts chain $(2)$ with $n_h = 2$ magnetic fields ($h_0 = h_1 = h$) (see the text). The conjectured values, in parenthesis, are the corresponding ones in the critical Ising model.

Table 4 - Finite-size sequences $c_{L,L+1}$, defined in $(3)$ for the 5-state Potts model Hamiltonian $(2)$ with $n_h = 2$ ($h_0 = h_1 = h$) external magnetic fields. The last line gives the extrapolated results.
Table 1

| h   | c      | $x_2(0,0)$ | $x_1(1,0)$ | $x_1(0,1)$ | $x_1(1,1)$ |
|-----|--------|------------|------------|------------|------------|
| 0.5 | 0.50087 (0.5) | 1.00073 (1) | 2.001 (2) | 0.1251 (0.125) | 1.1251 (1.125) |
| 1.0 | 0.50000 (0.5) | 0.99996 (1) | 1.9995 (2) | 0.12497 (0.125) | 1.12497 (1.125) |
| 1.5 | 0.50003 (0.5) | 1.00000 (1) | 1.99997 (2) | 0.1249 (0.125) | 1.1249 (1.125) |
| 2.0 | 0.50008 (0.5) | 1.00000 (1) | 2.00000 (2) | 0.12500 (0.125) | 1.12500 (1.125) |

Table 2

| h   | c      | $x_2(0,0)$ | $x_3(0,0)$ | $x_2(0,1)$ | $x_3(0,1)$ | $x_1(1,1)$ | $x_2(1,1)$ |
|-----|--------|------------|------------|------------|------------|------------|------------|
| 0.5 | 0.800 (0.8) | 0.8001 (0.8) | 2.79 (2.8) | 0.1333 (0.133...) | 1.332 (1.33...) | 1.1333 (1.133...) | 2.332 (2.33...) |
| 1.0 | 0.7991 (0.8) | 0.7997 (0.8) | 2.8 (2.8) | 0.1329 (0.133...) | 1.330 (1.33...) | 1.1329 (1.133...) | 2.331 (2.33...) |
| 1.5 | 0.79 (0.8) | 0.799 (0.8) | 2.83 (2.8) | 0.1334 (0.133...) | 1.330 (1.33...) | 1.1334 (1.133...) | 2.3319 (2.33...) |
| 2.0 | 0.7991 (0.8) | 0.7999 (0.8) | 2.84 (2) | 0.13345 (0.133...) | 1.330 (1.33...) | 1.13345 (1.133...) | 2.3319 (2.33...) |
Table 3

| h   | c   | $x_2(0, 0, 0)$ | $x_1(1, 0, 0)$ | $x_1(0, 1, 0)$ | $x_1(1, 1, 0)$ |
|-----|-----|----------|-------------|-------------|-------------|
| 0.5 | 0.5002 (0.5) | 0.997 (1) | 1.944 (2) | 0.1248 (0.125) | 1.1248 (1.125) |
| 1.0 | 0.505 (0.5) | 1.0000 (1) | 1.999 (2) | 0.12509 (0.125) | 1.12509 (1.125) |
| 1.5 | 0.5000 (0.5) | 0.9999 (1) | 1.9996 (2) | 0.1249 (0.125) | 1.1249 (1.125) |
| 2.0 | 0.5000 (0.5) | 0.9999 (1) | 1.9997 (2) | 0.12499 (0.125) | 1.12499 (1.125) |

Table 4

| $N \backslash h$ | 0.05 | 0.1 | 0.5 | 1.0 |
|-----------------|------|-----|-----|-----|
| 4               | 1.494578 | 1.293239 | 0.822347 | 0.747102 |
| 5               | 1.150201 | 0.947010 | 0.665147 | 0.632826 |
| 6               | 0.957555 | 0.784083 | 0.603697 | 0.583638 |
| 7               | 0.834717 | 0.696948 | 0.572906 | 0.558058 |
| 8               | 0.753559 | 0.646537 | 0.554626 | 0.542796 |
| 9               | 0.698775 | 0.615352 | 0.542598 | 0.532888 |
| 10              | 0.660931 | 0.594802 | 0.534149 | 0.526071 |
| $\infty$       | 0.5693   | 0.5448   | 0.4898   | 0.500    |