Lorentz and “apparent” transformations of the electric and magnetic fields

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It is recently discovered that the usual transformations of the three-dimensional (3D) vectors of the electric and magnetic fields differ from the Lorentz transformations (LT) (boosts) of the corresponding 4D quantities that represent the electric and magnetic fields. In this paper, using geometric algebra formalism, this fundamental difference is examined representing the electric and magnetic fields by bivectors.

I. INTRODUCTION

Recently\textsuperscript{1,2,3} it is proved that, contrary to the general belief, the usual transformations of the three-dimensional (3D) vectors of the electric and magnetic fields, see, e.g., Ref. 4, Eqs. (11.148) and (11.149), differ from the Lorentz transformations (LT) (boosts) of the corresponding 4D quantities that represent the electric and magnetic fields. (The usual transformations will be called the “apparent” transformations (AT) and the name will be explained in Sec. IV.) Comparisons with experiments, the motional emf,\textsuperscript{2} the Faraday disk\textsuperscript{3} and the Trouton-Noble experiment,\textsuperscript{5,6} show that this new approach\textsuperscript{1−3,5,6} with 4D geometric quantities always agrees with the principle of relativity and with experiments, while it it is not the case with the usual approach in which the electric and magnetic fields are represented by the 3D vectors $\mathbf{E}$ and $\mathbf{B}$ that transform according to the AT. The mentioned agreement with experiments is independent of the chosen reference frame and of the chosen system of coordinates in it. The main point in the geometric approach\textsuperscript{1−3,5,6} is that the physical meaning, both theoretically and experimentally, is attributed to 4D geometric quantities, and not, as usual, to 3D quantities.

In this paper I shall present a simplified version of the proof of the difference between the LT and the AT that is already given in Secs. 3.3 and 4 in Ref. 2. For all mathematical details for the used geometric algebra formalism readers can consult Refs. 7, 8.

As shown in Refs. 2, 3 the electric and magnetic fields can be represented by different algebraic objects: 1-vectors, bivectors or their combination. The representation with 1-vectors $\mathbf{E}$ and $\mathbf{B}$ is simpler than others and also closer to the usual expressions with the 3D vectors $\mathbf{E}$ and $\mathbf{B}$, but here we shall only deal with bivectors. The reason is that the representation with bivectors, as in our Eq. (2), is always employed in Refs. 7, 8 and we want to make comparison with their results. In Sec. II A a new Lorentz invariant representation, $E_v$ and $B_v$, is presented that is introduced in Refs. 2 and 3. In Sec. II B we simply
II. ELECTRIC AND MAGNETIC FIELDS AS BIVECTORS

In this geometric approach physical quantities will be represented by 4D geometric quantities, multivectors, that are defined without reference frames, or, when some basis has been introduced, these quantities are represented as 4D geometric quantities comprising both components and a basis. For simplicity and for easier understanding, only the standard basis \{\gamma_\mu; 0, 1, 2, 3\} of orthonormal 1-vectors, with timelike vector \gamma_0 in the forward light cone, will be used in the Minkowski spacetime \(M^4\), but remembering that the approach with 4D geometric quantities holds for any choice of basis in \(M^4\). The basis vectors \gamma_\mu generate by multiplication a complete basis for the spacetime algebra: \(1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \gamma_\nu, \gamma_5 \) (16 independent elements). \gamma_5 is the right-handed unit pseudoscalar, \(\gamma_5 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3\). Any multivector can be expressed as a linear combination of these 16 basis elements of the spacetime algebra. It is worth noting that the standard basis \{\gamma_\mu\} corresponds, in fact, to the specific system of coordinates,
i.e., to Einstein’s system of coordinates. In the Einstein system of coordinates
the Einstein synchronization\(^9\) of distant clocks and Cartesian space coordinates
\(x^i\) are used in the chosen inertial frame of reference. However different systems
of coordinates of an inertial frame of reference are allowed and they are all
equivalent in the description of physical phenomena.

A. Lorentz invariant electric and magnetic fields

The electromagnetic field is represented by a bivector-valued function \(F = F(x)\) on
the spacetime. As shown in Refs. 2, 3 the observer independent \(F\) can be
decomposed into two bivectors \(E_v\) and \(B_v\) representing the electric and
magnetic fields and the unit time-like 1-vector \(v/c\) as

\[
F = E_v + cIB_v, \quad E_v = (1/c^2)(F \cdot v) \wedge v = (1/2c^2)(F - vFv),
\]

\[
IB_v = (1/c^3)(F \wedge v) \cdot v = (1/2c^3)(F + vFv),
\]

where \(I\) is the unit pseudoscalar and \(v\) is the velocity (1-vector) of a family of
observers who measures \(E_v\) and \(B_v\) fields. Observe that \(E_v\) and \(B_v\) depend
not only on \(F\) but on \(v\) as well. All quantities \(F, E_v, B_v, I\) and \(v\) are defined
without reference frames. (\(I\) is defined algebraically without introducing any
reference frame, as in Ref. 10 Sec. 1.2.) Such 4D geometric quantities will be
called the absolute quantities (AQs), while their representations in some basis
will be called coordinate-based geometric quantities (CBGQs). For example,
in the \(\{\gamma_\mu\}\) basis the AQ \(E_v\) from (1) is represented by the following CBGQ
\(E_v = (1/c^2)F^{\mu\nu}v_\mu v_\nu \wedge \gamma_\beta \gamma_\mu \wedge \gamma_\beta\)

B. Electric and magnetic fields in the \(\gamma_0\) - frame

For comparison with the usual treatments\(^7,8\) let us choose the frame in which
the observers who measure \(E_v\) and \(B_v\) are at rest. For them \(v = c\gamma_0\). This frame
will be called the frame of “fiducial” observers or the \(\gamma_0\) - frame. In that frame
\(E_v\) and \(B_v\) from (1) become the observer dependent (\(\gamma_0\) - dependent) \(E_H\) and
\(B_H\) and instead of Eq. (1) we have

\[
F = E_H + c\gamma_0 B_H, \quad E_H = (F \cdot \gamma_0)\gamma_0 = (1/2)(F - \gamma_0 F\gamma_0),
\]

\[
\gamma_0 B_H = (1/c)(F \wedge \gamma_0)\gamma_0 = (1/2c)(F + \gamma_0 F\gamma_0).
\]

(The subscript \(H\) is for “Hestenes.”) \(E_v\) and \(B_v\) in the \(\gamma_0\) - frame are denoted
as \(E_H\) and \(B_H\) since they are identical to 4D quantities used by Hestenes\(^7\) and
the Cambridge group\(^8\) for the representation of the electric and magnetic fields.
We note that such procedure is never used by Hestenes\(^7\) and the Cambridge group\(^8\) since they deal from the outset only with \(\gamma_0\) and thus with a space-time
split in the \(\gamma_0\) - frame, i.e., with the relations (2). This shows that the space-
time split and the corresponding observer dependent form for the electric and
magnetic fields, (2), which is always used in Refs. 7, 8, is simply obtained in
our approach going to the frame of the "fiducial" observers, i.e., replacing some general velocity \( v \) in \(<1>\) by \( c \gamma_0 \).

\( E_H \) and \( B_H \) from \(<2>\) can be written as CBGQs in the standard basis \( \{ \gamma_{\mu} \} \). They are

\[
E_H = F^{i0} \gamma_i \wedge \gamma_0, \quad B_H = (1/2c)\varepsilon^{kli} F_{kl} \gamma_i \wedge \gamma_0. \tag{3}
\]

It follows from \(<8>\) that the components of \( E_H, B_H \) in the \( \{ \gamma_{\mu} \} \) basis (i.e., in the Einstein system of coordinates) give rise to the tensor (components) \((E_H)^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot E_H) = (\gamma^\nu \wedge \gamma^\mu) \cdot E_H, \) (and the same for \((B_H)^{\mu\nu}\) which, written out as a matrix, have entries

\[
(E_H)^{i0} = F^{i0} = E^i, \quad (E_H)^{ij} = 0,
(E_H)^{(i0} = (1/2c)\varepsilon^{kli} F_{kl} = B^i, \quad (B_H)^{ij} = 0. \tag{4}
\]

\((E_H)^{\mu\nu}\) is antisymmetric, i.e., \((E_H)^{\mu\nu} = -(E_H)^{\nu\mu}\), and the same holds for \((B_H)^{\mu\nu}\). \((E_H)^{\mu\nu}\) from Eq. \(<8>\) can be written in a matrix form as

\[
(E_H)^{\mu\nu} = \begin{bmatrix}
0 & -E^1 & -E^2 & -E^3 \\
E^1 & F^{10} & 0 & 0 \\
E^2 & F^{20} & 0 & 0 \\
E^3 & F^{30} & 0 & 0
\end{bmatrix}, \tag{5}
\]

and readers can check that the same matrix form is obtained for \((B_H)^{\mu\nu}\). ((\(B_H)^{(i0} = (1/c)F^{32} = B^1\).)

Thus we see from \(<3>\), and \(<4>\) or \(<5>\), that

(i) both bivectors \( E_H \) and \( B_H \) are parallel to \( \gamma_0 \), \( E_H \wedge \gamma_0 = B_H \wedge \gamma_0 = 0 \), and consequently all space-space components of \((E_H)^{\mu\nu}\) and \((B_H)^{\mu\nu}\) are zero, \((E_H)^{ij} = (B_H)^{ij} = 0.\)

In the usual covariant approaches\(^4\) the components of the 3D \( E \) and \( B \) are identified with six independent components of \( F^{\mu\nu} \) according to the relations

\[
E_i = F^{i0}, \quad B_i = (-1/2c)\varepsilon_{ikl} F_{kl}. \tag{6}
\]

In \(<7>\) and hereafter the components of the 3D \( E \) and \( B \) are written with lowered (generic) subscripts, since they are not the spatial components of the 4D quantities. This refers to the third-rank antisymmetric \( \varepsilon \) tensor too. The super- and subscripts are used only on the components of the 4D quantities.

Comparing \(<1>\) and \(<8>\) we see that they similarly identify the components of the electric and magnetic fields with six independent components of \( F^{\mu\nu} \). However there are important differences between the relations \(<3>\), \(<4>\) or \(<5>\), and \(<6>\). In the usual covariant approaches, e.g., Ref. 4, the 3D \( E \) and \( B \), as \textit{geometric quantities in the 3D space}, are constructed from these six independent components of \( F^{\mu\nu} \) and the \textit{unit 3D vectors} \( i, j, k \), e.g., \( E = F^{10}i + F^{20}j + F^{30}k. \) Observe that the mapping, i.e., the simple identification, Eq. \(<8>\), of the components \( E_i \) and \( B_i \) with some components of \( F^{\mu\nu} \) (defined on the 4D spacetime) is not a permissible tensor operation, i.e., it is not a mathematically correct procedure. The same holds for the construction of the 3D \textit{vectors} \( E \) and
in which the components of the 4D quantity $F^\mu\nu$ are multiplied with the unit 3D vectors, see Ref. 3 for the more detailed discussion. On the other hand, as seen from Eqs. (3), (4) or (5), $E_H$ and $B_H$ and their components $(E_H)^{\mu\nu}$ and $(B_H)^{\mu\nu}$ are obtained by a correct mathematical procedure from the geometric 4D quantities $F$ and $\gamma^\mu$. The components $(E_H)^{\mu\nu}$ and $(B_H)^{\mu\nu}$ are multiplied by the unit bivectors $\gamma_i \wedge \gamma_0$ (4D quantities) to form the geometric 4D quantities $E_H$ and $B_H$. In such a treatment the unit 3D vectors $i, j, k$, (geometric quantities in the 3D space) do not appear at any point.

Furthermore it is worth noting that $F^\mu\nu$ are only components (numbers) that are (implicitly) determined in Einstein’s system of coordinates. Components are frame-dependent numbers (frame-dependent because the basis refers to a specific frame). Components tell only part of the story, while the basis contains the rest of the information about the considered physical quantity. These facts are completely overlooked in all usual covariant approaches and in the above identifications (6) of $E_i$ and $B_i$ with some components of $F^\mu\nu$.

III. LT OF ELECTRIC AND MAGNETIC FIELDS AS BIVECTORS

Let us now apply the active LT (only boosts are considered) to $E_H$ and $B_H$ from Eq. (3). In the usual geometric algebra formalism\(^7,8\) the LT are considered as active transformations; the components of, e.g., some 1-vector relative to a given inertial frame of reference (with the standard basis \(\{\gamma_\mu\}\)) are transformed into the components of a new 1-vector relative to the same frame (the basis \(\{\gamma_\mu\}\) is not changed). Furthermore the LT are described with rotors $R, R^\dagger R = 1$, in the usual way as $p \rightarrow p' = Rp^\dagger R = p_\mu' \gamma^\mu$. Remember that the reverse $R$ is defined by the operation of reversion according to which $\tilde{AB} = \tilde{B}\tilde{A}, \tilde{a} = a$, for any vector $a$, and it reverses the order of vectors in any given expression. Every rotor in spacetime can be written in terms of a bivector as $R = e^{\theta/2}$. For boosts in arbitrary direction the rotor $R$ is

$$R = e^{\theta/2} = (1 + \gamma + \gamma\beta\gamma_0 n)/(2(1 + \gamma))^{1/2},$$  \hspace{1cm} (7)

$\theta = \alpha\gamma_0 n, \beta$ is the scalar velocity in units of $c, \gamma = (1 - \beta^2)^{-1/2}, or in terms of an ‘angle’ $\alpha$ we have $\tanh \alpha = \beta, \cosh \alpha = \gamma, \sinh \alpha = \beta\gamma,$ and $n$ is not the basis vector but any unit space-like vector orthogonal to $\gamma_0; e^{\theta} = \cosh \alpha + \gamma_0 n \sinh \alpha$. One can also express the relationship between two relatively moving frames $S$ and $S'$ in terms of rotor as $\gamma_{\mu}' = R\gamma_{\mu}R$. For boosts in the direction $\gamma_1$ the rotor $R$ is given by the relation (4) with $\gamma_1$ replacing $n$ (all in the standard basis $\{\gamma_\mu\}$). For simplicity we shall only consider boosts in the direction $\gamma_1$.

As said in Sec. IV in Hestenes’ paper\(^7\) in AJP Lorentz rotations preserve the geometric product. This implies that any multivector $M$ transforms by the active LT in the same way as mentioned above for the 1-vector $p$, i.e.,

$$M \rightarrow M' = RM\tilde{R},$$  \hspace{1cm} (8)
see, e.g., Eq. (69) in Hestenes’ paper\(^7\) in AJP. It is not important is \(M\) a simple blade or a Clifford aggregate, is it a function of some other multivectors or it is not.

Hence, according to [3], under the active LT \(E_H\) from [2] must transform in the following way

\[
E'_H = R[(1/2)(F - \gamma_0 F \gamma_0)] R = (1/2)[F' - \gamma_0' F' \gamma_0'] = (F' \cdot \gamma_0') \gamma_0', \quad (9)
\]

where \(F' = R F \bar{R}\) and \(\gamma_0' = R \gamma_0 \bar{R}\). However, as will be shown in Sec. IV, it is surprising that neither Hestenes\(^7\) nor the Cambridge group\(^8\) transform \(E_H\) in the way in which all other multivectors are transformed, i.e., according to [3] and [9].

When the active LT are applied to \(E_H\) from [3], thus when \(E_H\) is written as a CBGQ, then \(E'_H\) becomes

\[
E'_H = R[E' \gamma_1 \wedge \gamma_0] \bar{R} = E^1 \gamma_1 \wedge \gamma_0 + \gamma (E^2 \gamma_2 \wedge \gamma_0 + E^3 \gamma_3 \wedge \gamma_0) - \beta \gamma (E^2 \gamma_2 \wedge \gamma_1 + E^3 \gamma_3 \wedge \gamma_1). \quad (10)
\]

(We denoted, as in Eq. [4], \(E^i = F^{i0}\).) The components \(\mu\nu (E'_H)^{\mu\nu} = -(E_H)^{\mu\nu}\) can be written in a matrix form as

\[
(E'_H)^{\mu\nu} = \begin{pmatrix}
0 & -E^1 & -\gamma E^2 & -\gamma E^3 \\
E^1 & 0 & \beta \gamma E^2 & \beta \gamma E^3 \\
\gamma E^2 & -\beta \gamma E^2 & 0 & 0 \\
E^3 & -\beta \gamma E^3 & 0 & 0
\end{pmatrix}, \quad (11)
\]

The same form can be easily find for \(B'_H\) and its components \((B'_H)^{\mu\nu}\). (This is left for readers.) Eq. (10) is the familiar form for the active LT of a bivector, here \(E_H\), but written as a CBGQ.

(For some general bivector \(N\) the components transform by the LT as the components of a second-rank tensor

\[
N^{i23} = N^{23}, \quad N^{i31} = \gamma (N^{31} - \beta N^{30}), \quad N^{i12} = \gamma (N^{12} + \beta N^{20}), \\
N^{i10} = N^{10}, \quad N^{i20} = \gamma (N^{20} + \beta N^{12}), \quad N^{i30} = \gamma (N^{30} + \beta N^{13}). \quad (12)
\]

From (12) one easily find \((E'_H)^{\mu\nu}\) taking into account that the components \((E_H)^{\mu\nu}\) are determined by Eq. (5).)

It is important to note that

(i') \(E'_H\) and \(B'_H\), in contrast to \(E_H\) and \(B_H\), are not parallel to \(\gamma_0\), i.e., both \(E'_H \wedge \gamma_0 \neq 0\) and \(B'_H \wedge \gamma_0 \neq 0\), and thus there are the space-space components, \((E'_H)^{ij} \neq 0\) and \((B_H)^{ij} \neq 0\). Furthermore,

(ii') the components \((E_H)^{\mu\nu}\) \((B_H)^{\mu\nu}\) transform upon the active LT again to the components \((E'_H)^{\mu\nu}\) \((B'_H)^{\mu\nu}\); there is no mixing of components. Under the active LT \(E_H\) transforms to \(E'_H\) and \(B_H\) to \(B'_H\). Actually, as already said, this is the way in which every bivector transforms under the active LT. Instead of using the active LT we can deal with the passive LT. The essential difference relative to the usual covariant picture is the presence of the
basis in a CBGQ. The existence of the basis causes that every 4D CBGQ is invariant under the passive LT; the components transform by the LT and the basis by the inverse LT leaving the whole 4D CBGQ unchanged. This means that such CBGQ represents the same physical quantity for relatively moving 4D observers. For some general bivector \( N \) the components transform according to (12), whereas the basis \( \gamma'_\mu \wedge \gamma'_\nu \) transform by the inverse LT giving that the whole \( N \) is unchanged

\[
N = (1/2)N^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = (1/2)N'^{\mu\nu}\gamma'_\mu \wedge \gamma'_\nu,
\]

where all primed quantities are the Lorentz transforms of the unprimed ones. It can be checked by the use of (5) and (11) that (13) holds for \( E_H \), i.e., that

\[
E_H = (1/2)(E_H)^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = (1/2)(E'_H)^{\mu\nu}\gamma'_\mu \wedge \gamma'_\nu,
\]

and the same for \( B_H \).

In addition, let us see how one can find the expression for \( E_v \) from (1) as a CBGQ in the \( S' \) frame and in the \( \{\gamma'_\mu\} \) basis. In the \( S' \) frame the “fiducial” observers (that are in the \( S \) frame) are moving with velocity \( v \) whose components are \( v'^\mu = (\gamma c, -\gamma \beta c, 0, 0) \). Of course, for the whole CBGQ \( v \) it holds that \( v = v'^\mu \gamma'_\mu = v^\mu \gamma_\mu \), where the components \( v^\mu \) from \( S \) are \( v^\mu = (c, 0, 0, 0) \). Then \( E_v \) becomes \( E_v = \gamma^0 (\gamma^1 \wedge \gamma_0 + \gamma^2 (F^{20} + \beta F^{21}) \gamma^4 \wedge \gamma_0 + \gamma^2 (F^{00} + \beta F^{03}) \gamma'^1 \wedge \gamma'_0 - \beta \gamma^2 (F^{20} + \beta F^{21}) \gamma'^2 \wedge \gamma'_0 - \beta \gamma^2 (F^{00} + \beta F^{03}) \gamma'^4 \wedge \gamma'_0) \). If the components \( F'^{\mu\nu} \) are expressed in terms of \( F^{\mu\nu} \) from \( S \) using (12) then the same components are obtained as in (14).

**IV. APPARENT TRANSFORMATIONS OF ELECTRIC AND MAGNETIC FIELDS AS BIVECTORS**

In contrast to the LT of \( E_H \) (and \( B_H \)), Eqs. (9) and (10), it is accepted in the usual geometric algebra formalism that \( E_H \) (and \( B_H \)) do not transform as all other multivectors transform, but that they transform as

\[
E'_{H,at} = (1/2)[F' - \gamma_0 F' \gamma_0] = (F' \cdot \gamma_0) \gamma_0,
\]

where \( F' = RF\tilde{R} \). (The subscript “at” is for AT.) It is seen from (14) that only \( F \) is transformed while \( \gamma_0 \) is not transformed. The transformation (14) is nothing else than the usual transformation of the electric field that is given in Ref. 7, Space-Time Algebra, Eq. (18.22), New Foundations for Classical Mechanics, Ch. 9, Eqs. (3.51a,b) and Ref. 8, Sec. 7.1.2, Eq. (7.33).

When (14) is written with CBGQs then instead of the LT (10) we find the AT

\[
E'_{H,at} = F'^{\mu0} \gamma_\mu \wedge \gamma_0 = E^1 \gamma_1 \wedge \gamma_0 + \gamma (E^2 - \beta cB^3) \gamma_2 \wedge \gamma_0 + \gamma (E^3 + \beta cB^2) \gamma_3 \wedge \gamma_0,
\]
In (16) \( E^i = F^{i0} \) and \( B^i = (1/2c)\varepsilon^{kl0}F_{kl} \), as in (4). When the components
\[(E'_{H,at})^{\mu\nu} = \gamma^\nu (\gamma^\mu \cdot (E'_{H,at})) \]
from (10) are written in a matrix form they are
\[
(E'_{H,at})^{\mu\nu} = \begin{bmatrix}
0 & -E'_1^0 & -E'_2^0 & -E'_3^0 \\
E'_1^0 & F_{10} & 0 & 0 \\
E'_2^0 & 0 & F_{20} & 0 \\
E'_3^0 & 0 & 0 & F_{30}
\end{bmatrix}, \tag{17}
\]
where
\[
E'_1^0 = E^1, \quad E'_2^0 = \gamma(E^2 - \beta c B^3), \quad E'_3^0 = \gamma(E^3 + \beta c B^2). \tag{18}
\]
The same matrix form can be obtained for \((B'_{H,at})^{\mu\nu}\) with
\[
B'_1^0 = B^1, \quad B'_2^0 = \gamma(B^2 + \beta E^3/c), \quad B'_3^0 = \gamma(B^3 - \beta E^2/c). \tag{19}
\]
Observe that the transformations (18) and (19) are exactly the familiar expressions for the usual transformations of the components of the 3D \( E \) and \( B \), Ref. 4, Eq. (11.148), which are quoted in every textbook and paper on relativistic electrodynamics from the time of Lorentz, Poincaré and Einstein.

We see from (15), (16), (17), (18) and (19) that
\[
(i') \quad (E'_{H,at})^{\mu\nu} \quad \text{and} \quad (B'_{H,at})^{\mu\nu}, \quad \text{in the same way as} \quad E_H \quad \text{and} \quad B_H, \quad \text{are parallel to} \quad \gamma_0, \quad \text{i.e.,} \quad (E'_{H,at})^{\mu\nu} \wedge \gamma_0 = (B'_{H,at})^{\mu\nu} \wedge \gamma_0 = 0, \quad \text{whence it again holds that the space-space components are zero,} \quad (E'_{H,at})^{ij} = (B'_{H,at})^{ij} = 0. \quad \text{Furthermore, it is seen from} \quad \text{the relations} \quad (15), \quad (18) \quad \text{and} \quad (19) \quad \text{that}
\[
(ii') \quad \text{in contrast to the LT of} \quad E_H \quad \text{and} \quad B_H, \quad \text{Eq. (10), the components} \quad E''_1^0 \quad \text{of} \quad (E'_{H,at})^{\mu\nu} \quad \text{are expressed by the mixture of} \quad E^i, \quad \text{and} \quad B^i, \quad \text{and} \quad \text{the same holds for} \quad (B'_{H,at}).
\]

In all geometric algebra formalisms, e.g., Refs. 7, 8, the AT (16) for \( E'_{H,at} \) (and similarly for \( B'_{H,at} \)) are considered to be the LT of \( E_H \) (\( B_H \)). However, contrary to the generally accepted opinion, the transformations (15), (16), (17), (18) and (19) are not the LT. The LT cannot transform the matrix (3) with \((E_H)^{ij} = 0 \) to the matrix (17) with \((E'_{H,at})^{ij} = 0 \). Furthermore Eq. (13) is not fulfilled,
\[
(1/2)(E'_{H,at})^{\mu\nu} \gamma'_\mu \wedge \gamma'_\nu \neq (1/2)(E_H)^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \tag{20}
\]
which means that these two quantities are not connected by the LT, and consequently they do not refer to the same 4D quantity for relatively moving observers. As far as relativity is concerned these quantities are not related to one another. The fact that they are measured by two observers (\( \gamma_0 \) - and \( \gamma'_0 \) - observers) does not mean that relativity has something to do with the problem. The reason is that observers in the \( \gamma_0 \) - frame and in the \( \gamma'_0 \) - frame are not looking at the same physical quantity but at two different quantities. Every observer makes measurement on its own quantity and such measurements are not related by the LT. The LT of \( E_H \) are correctly given by Eqs. (15), (16) and (17). Therefore we call the transformations (15) and (16) for geometric quantities, and (18) and (19) for components, the “apparent” transformations, the AT. The same name is introduced by Rohrlich\(^{11}\) for the Lorentz contraction;
the Lorentz contracted length and the rest length are not connected by the LT
and consequently they do not refer to the same 4D quantity.

In the usual covariant approaches the components of the 3D \( E' \) and \( B' \) are
identified, in the same way as in (6), with six independent components of \( F'_{\mu\nu} \),
\( E'_i = F'^{\mu 0}_i \), \( B'_i = (1/2c)\varepsilon_{ijk}F'^{\mu}_{jk} \). This then leads to the AT (18) and (19). The 3D \( E' \) and \( B' \) as geometric quantities in the 3D space, are constructed multiplying
the components \( E'_i \) and \( B'_i \) by the unit 3D vectors \( i', j', k' \). The important objections to such usual construction of \( E' \) and \( B' \) are the following: First, the components \( E'_i \) and \( B'_i \) are determined by the AT (18) and (19) and not by the LT. Second, there is no transformation which transforms the unit 3D vectors \( i, j, k \) into the unit 3D vectors \( i', j', k' \). Hence it is not true that, e.g., the 3D vector \( E' = E'_i i' + E'_j j' + E'_k k' \) is obtained by the LT from the 3D vector \( E = E_1 i + E_2 j + E_3 k \). Consequently the 3D vector \( E' \) and \( E \) are not the same quantity for relatively moving inertial observers, \( E' \neq E \). Thus, although it is possible to identify the components of the 3D \( E \) and \( B \) with the components of \( F \) (according to Eq. (6)) in an arbitrary chosen \( \gamma_0 \)-frame with the \( \{\gamma_{\mu}\} \) basis such an identification is meaningless for the Lorentz transformed \( F' \).

V. CONCLUSIONS

The main conclusion that can be drawn from this paper, and Refs. 1-3, is
that the usual transformations of the electric and magnetic fields are not the
LT. It is believed by the whole physics community that the LT of the matrix
of components \( (E_H)^{\mu\nu} \), Eq. (6), for which the space-space components \( (E_H)^{ij} \) are zero and \( (E_H)^{0\theta} = E^\theta \), transform that matrix to the matrix \( (E'_{H,at})^{\mu\nu} \), Eq. (17), in which again the space-space components \( (E'_{H,at})^{ij} \) are zero and the time-space components \( (E'_{H,at})^{0\theta} = E'_{\theta i} \) are given by the usual transformations for the components of the 3D vector \( E \), Eq. (18); the transformed components \( E'_{\theta i} \) are expressed by the mixture of \( E' \) and \( B' \) components. (This statement is equivalent to saying that the transformations (18) and (19) are the LT of the components of the 3D \( E \) and \( B \).) However, according to the correct mathematical procedure, the LT of the matrix of components \( (E_H)^{\mu\nu} \), Eq. (5), transform that matrix to the matrix \( (E'_{H})^{\mu\nu} \), Eq. (11), with \( (E'_{H})^{ij} \neq 0 \). As seen from (11) all transformed components \( (E'_{H})^{\mu\nu} \) of the electric field are determined only by three components \( E' \) of the electric field; there is no mixture with three components \( B' \) of the magnetic field.

It is worth noting that the whole consideration is much clearer when using
1-vectors \( E \) and \( B \), as in Refs. 2, 3, for the representation of the electric and magnetic fields. Then, e.g., \( E = (1/c)F \cdot v \). In the frame of “fiducial” observers it becomes \( E = F \cdot \gamma_0 \), \( E = E' \gamma_3 = F'^{00} \gamma_3 \). By the active LT the electric field \( E \) transforms again to the electric field (according to (3)) \( E' = R(F \cdot \gamma_0)\hat{R} = F' \cdot \gamma_0' \), i.e., \( E' = E'^{\mu} \gamma_\mu = -\beta E' \gamma_0 \gamma_0 + \gamma E' \gamma_3 \gamma_1 + \gamma E' \gamma_2 \gamma_1 + \gamma E' \gamma_1 \gamma_2 \), which now contains the temporal component \( E'^{00} = -\beta E' \gamma_0 \). This is the way in which a 1-vector transforms. (Generally, for components, \( E'^{00} = \gamma(E'^{00} - \beta E' \gamma_0) \), \( E'^{01} = \gamma(E'^{01} - \beta E' \gamma_1) \), \( E'^{2,3} = E'^{2,3} \).) For the passive LT it holds that \( E = E'^{\mu} \gamma_\mu = E'^{\mu} \gamma_\mu' \); \( E \) is the
same quantity for relatively moving observers. On the other hand the AT for components are obtained taking that \( E'_{at} = E^\gamma_\gamma \), only \( F \) is transformed but not \( \gamma_0 \), i.e., \( E'_{at} = 0 \gamma_0 + E^{\alpha}_\alpha \gamma_\alpha \),

\[
E'_{at} = E^1 \gamma_1 + \gamma(E^2 - \beta \epsilon B^3) \gamma_2 + \gamma(E^3 + \beta \epsilon B^2) \gamma_3 ,
\]

and obviously \( E \) and \( E' \) are not the same quantity for relatively moving observers, \( E^\gamma_\gamma \neq E^{\alpha}_\alpha \gamma_\alpha \). All the same as for bivectors \( E_H \) and \( B_H \) but much simpler and closer to the usual formulation with the 3D \( E \) and \( B \). However there is already extensive literature, e.g., Refs. 7, 8, in which the bivectors \( E_H \) and \( B_H \) are employed. Therefore, in this paper, the elaboration of the fundamental difference between the AT and the LT is given using bivectors and not 1-vectors.

These results will be very surprising for all physicists since we are all, and always, taught that the transformations (18) and (19) are the LT of the components of the 3D \( E \) and \( B \). But, the common belief is one thing and clear mathematical facts are quite different thing. The true agreement of these new results with electrodynamic experiments, as shown in Refs. 2, 3 and Refs. 5, 6, substantially support the validity of the results from Refs. 1 - 3 and Refs. 5, 6. Ultimately, these new results say that the Lorentz invariant 4D geometric quantities are physical ones, and not, as usually accepted, the 3D geometric quantities.

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