Characteristic Polynomial of a Triangular and Diagonal Strictly Double $\mathbb{R}$-astic Matrices over Interval Max-Plus Algebra

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Abstract. Max-plus algebra is the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{ \varepsilon \}$ where $\mathbb{R}$ is a set of all real numbers and $\varepsilon = -\infty$ which is endowed with max ($\oplus$) and plus ($\otimes$) operations. A matrix in which its components are the element of $\mathbb{R}_\varepsilon$ is called matrix over max-plus algebra. From matrix, we can define characteristic polynomial over max-plus algebra. Max-plus algebra has been generalized into interval max-plus algebra. Interval max-plus algebra is the set of interval over $\mathbb{R}_\varepsilon$, denoted by $I(\mathbb{R})_\varepsilon$, which is endowed with $\oplus$ and $\otimes$ operations. A matrix in which its components are the element of $I(\mathbb{R})_\varepsilon$ is called interval matrix. Interval matrix has some unique forms, two of which are triangular and diagonal strictly double $\mathbb{R}$-astic matrices. From interval matrix, we can define characteristic polynomial over interval max-plus algebra which is used to determine eigenvalues. In this research will discussed about the characteristic polynomial of a triangular and diagonal strictly double $\mathbb{R}$-astic matrices over interval max-plus algebra that will also be used to determine eigenvalues.

1. Introduction

Max-plus algebra is the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{ \varepsilon \}$ where $\mathbb{R}$ is a set of all real numbers and $\varepsilon = -\infty$ which is endowed with max ($\oplus$) and plus ($\otimes$) operations. A matrix in which its components are the element of $\mathbb{R}_\varepsilon$ is called matrix over max-plus algebra. Let $A = (a_{ij}) \in \mathbb{R}_\varepsilon^{n \times n}$, then its permanent is $\text{maper}(A) = \bigoplus_{\pi \in P_n} \bigotimes_{i \in N} a_{i, \pi(i)}$ and its characteristic polynomial is
\( \chi_A(x) = \text{maper}(A \oplus x \otimes I) \), where \( I \) is identity matrix (Butkovic [1]). Eigenvalue of \( A \) is \( \lambda \in \mathbb{R} \) that satisfied the equation \( A \otimes x = \lambda \otimes x \), where \( x \in \mathbb{R}^n \) (Tam [5]). The greatest corner of a max-polynomial is the principal eigenvalue (Butkovic [1]).

In 2011, Rudhito et al. ([3]) did a research about interval max-plus algebra. Interval max-plus algebra is the set of interval over \( \mathbb{R}_\varepsilon \), denoted by \( I(\mathbb{R})_\varepsilon \), which is endowed with \( \oplus \) and \( \otimes \) operations. A matrix in which its components are the element of \( I(\mathbb{R})_\varepsilon \) is called interval matrix. In 2012, Jafari and Hosseinyazdi ([2]) did a research about characteristic max-polynomial of a triangular and certain strictly double \( \mathbb{R} \)-astic matrices. According to Jafari and Hosseinyazdi’s research and interval max-plus algebra concept, this research will discussed about the characteristic polynomial of a triangular and diagonal strictly double \( \mathbb{R} \)-astic matrices over interval max-plus algebra also its principal eigenvalue.

First, we introduce some preliminaries about triangular and certain strictly double \( \mathbb{R} \)-astic matrices over max-plus algebra.

**Definition 1.1.** ([2]) A matrix is called triangular matrix if all entries of above or below diagonal are equal to \( \varepsilon \).

**Theorem 1.2.** ([2]) The permanent of triangular matrix is product of its diagonal entries.

**Theorem 1.3.** ([2]) Characteristic polynomial of a triangular matrix \( A, A \in \mathbb{R}^n_{\geq} \), is

\[
\chi_A(x) = (x \oplus a_{11}) \otimes (x \oplus a_{22}) \ldots \otimes (x \oplus a_{nn}).
\]

**Theorem 1.4.** ([1]) The greatest corner of triangular matrix is \( \delta_1 = \bigoplus_{i \in \mathbb{N}} a_{ii} \).

**Definition 1.5.** ([2]) A square matrix \( A \) is called strictly double \( \mathbb{R} \)-astic, if it satisfies following conditions :

(i) \( a_{ij} < +\infty \) and 

(ii) On each row and each column of \( A \), there is one and only one finite element.

**Theorem 1.6.** ([2]) The permanent of each strictly double \( \mathbb{R} \)-astic is equal to product of finite elements.

**Theorem 1.7.** ([2]) If \( A \) is a strictly double \( \mathbb{R} \)-astic whose diagonal entries are finite, then characteristic polynomial of \( A \) is

\[
\chi_A(x) = (x \oplus a_{11}) \otimes (x \oplus a_{22}) \ldots \otimes (x \oplus a_{nn}).
\]

**Theorem 1.8.** ([1]) The greatest corner of diagonal strictly double \( \mathbb{R} \)-astic matrix is \( \delta_1 = \bigoplus_{i \in \mathbb{N}} a_{ii} \).
Theorem 1.9. ([1]) The greatest corner of a max-polynomial is the principal eigenvalue.

Also, we introduce some preliminaries about interval max-plus algebra.

Definition 1.10. ([3]) Set of interval max-plus algebra is

\[ I(\mathbb{R})_\varepsilon = \{ x = [x, x] \mid x, x \in \mathbb{R}, \varepsilon \leq x \leq \bar{x} \} \cup \{ [\varepsilon, \varepsilon] \}, \]

and set of matrix over interval max-plus algebra is

\[ I(\mathbb{R})_{\varepsilon}^{m \times n} = \{ A = (a_{ij}) \mid (a_{ij}) \in I(\mathbb{R})_\varepsilon, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \}. \]

Definition 1.11. ([3]) For every interval matrix \( A \in I(\mathbb{R})_{\varepsilon}^{m \times n} \), defined a matrix interval of \( A \) that is

\[ [\underline{A}, \overline{A}] = \{ A \in \mathbb{R}_{\varepsilon}^{m \times n} \mid \underline{A} \leq A \leq \overline{A} \}, \]

where \( \underline{A} = (\underline{a}_{ij}) \in \mathbb{R}_{\varepsilon}^{m \times n} \) and \( \overline{A} = (\overline{a}_{ij}) \in \mathbb{R}_{\varepsilon}^{m \times n} \). Then, set of matrix interval \( A \) denoted by \( I(\mathbb{R}_{\max})_{\varepsilon}^{m \times n} \).

Definition 1.12. ([4]) The characteristic polynomial of \( A \), \( A \in I(\mathbb{R})_{\varepsilon}^{n \times n} \) is

\[ \chi_A(x) \approx [\chi_{\underline{A}}(x), \chi_{\overline{A}}(x)], \]

where \( \chi_{\underline{A}}(x) = \text{maper}(\underline{A} \oplus x \otimes I) \) and \( \chi_{\overline{A}}(x) = \text{maper}(\overline{A} \oplus x \otimes I) \).

2. Main Results

This section will discuss the results of this research about characteristic polynomial of triangular and diagonal strictly double \( \mathbb{R} \)-astic matrices over interval max-plus algebra.

2.1. The Triangular Matrix

Definition 2.1. A matrix \( A \in I(\mathbb{R})_{\varepsilon}^{n \times n} \) is called triangular interval matrix if all entries of above or below diagonal are equal to \( [\varepsilon, \varepsilon] \).

Theorem 2.2. The permanent of triangular interval matrix \( A \in I(\mathbb{R})_{\varepsilon}^{n \times n} \) is product of its diagonal entries.

Proof. Let \( A \) be an upper triangular interval matrix with diagonal entries are \([\underline{a}_{11}, \overline{a}_{11}], [\underline{a}_{22}, \overline{a}_{22}], \ldots, [\underline{a}_{nn}, \overline{a}_{nn}]\),

\[
A = \begin{pmatrix}
[\underline{a}_{11}, \overline{a}_{11}] & [\underline{a}_{12}, \overline{a}_{12}] & \cdots & [\underline{a}_{1n}, \overline{a}_{1n}] \\
[\varepsilon, \varepsilon] & [\underline{a}_{22}, \overline{a}_{22}] & \cdots & [\underline{a}_{2n}, \overline{a}_{2n}] \\
\vdots & \vdots & \ddots & \vdots \\
[\varepsilon, \varepsilon] & [\varepsilon, \varepsilon] & \cdots & [\underline{a}_{nn}, \overline{a}_{nn}]
\end{pmatrix}.
\]
Then we can obtain lower bound matrix and upper bound matrix of matrix $A$, that is

$$A = \begin{pmatrix}
\bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\
\varepsilon & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon & \varepsilon & \cdots & \bar{a}_{nn}
\end{pmatrix} \quad \text{and} \quad \overline{A} = \begin{pmatrix}
\overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\
\varepsilon & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon & \varepsilon & \cdots & \overline{a}_{nn}
\end{pmatrix}$$

such that $A \approx [A, \overline{A}]$. So that

$$\text{maper}(A) = \text{maper} \begin{pmatrix}
\bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\
\varepsilon & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon & \varepsilon & \cdots & \bar{a}_{nn}
\end{pmatrix} = \bar{a}_{11} \otimes \bar{a}_{22} \otimes \cdots \otimes \bar{a}_{nn},$$

$$\text{maper} (\overline{A}) = \text{maper} \begin{pmatrix}
\overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\
\varepsilon & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon & \varepsilon & \cdots & \overline{a}_{nn}
\end{pmatrix} = \overline{a}_{11} \otimes \overline{a}_{22} \otimes \cdots \otimes \overline{a}_{nn}.$$ 

Thus

$$\text{maper}(A) = [\text{maper}(\bar{A}), \text{maper}(\overline{A})] = [\bar{a}_{11} \otimes \bar{a}_{22} \otimes \cdots \otimes \bar{a}_{nn}, \overline{a}_{11} \otimes \overline{a}_{22} \otimes \cdots \otimes \overline{a}_{nn}] = [\bar{a}_{11}, \overline{a}_{11}] \otimes [\bar{a}_{22}, \overline{a}_{22}] \otimes \cdots \otimes [\bar{a}_{nn}, \overline{a}_{nn}].$$

With the same method we can prove the theorem for a lower triangular interval matrix. \hfill \Box

**Definition 2.3.** *The characteristic polynomial of triangular interval matrix $A \in I(\mathbb{R})^{n \times n}$ is

$$\chi_A(x) \approx [\chi_{\bar{A}}(x), \chi_{\overline{A}}(x)],$$

where $\chi_{\bar{A}}(x) = \text{maper}(\bar{A} \oplus \bar{x} \otimes I)$ and $\chi_{\overline{A}}(x) = \text{maper}(\overline{A} \oplus \overline{x} \otimes I).$*

**Theorem 2.4.** *The greatest corner of triangular interval matrix $A \in I(\mathbb{R})^{n \times n}$ is

$$\delta_1 \approx [\delta_{11}, \overline{\delta}_1],$$

where $\delta_{11} = \bigoplus_{i \in N} \overline{a}_{ii}$ and $\overline{\delta}_1 = \bigoplus_{i \in N} \overline{a}_{ii}.$*
Proof. Let A be a triangular interval matrix, where $A \approx [A, \overline{A}]$. Then, based on Definition 2.3, characteristic polynomial of A is

$$\chi_A(x) \approx [\chi_A(x), \chi_{\overline{A}}(x)],$$

where $\chi_A(x) = \text{maper}(A \oplus x \otimes I)$ and $\chi_{\overline{A}}(x) = \text{maper}(A \oplus x \otimes I)$. Based on Theorem 1.8, we can obtain that the greatest corner of characteristic polynomial $\chi_A(x) = \text{maper}(A \oplus x \otimes I)$ is

$$\delta_1 = \bigoplus_{i \in N} a_{ii},$$

also the greatest corner of characteristic polynomial $\chi_{\overline{A}}(x) = \text{maper}(A \oplus x \otimes I)$ is

$$\delta_1 = \bigoplus_{i \in N} \overline{a}_{ii}.$$ 

Then we can conclude that the greatest corner of characteristic polynomial $\chi_A(x)$ where $\chi_A(x) \approx [\chi_A(x), \chi_{\overline{A}}(x)]$ is $\delta_1 \approx [\delta_1, \delta_1]$. $\square$

**Theorem 2.5.** The principal eigenvalue of triangular interval matrix is its greatest corner.

Proof. Let A be a triangular interval matrix, where $A \approx [A, \overline{A}]$. Then, $\delta_1$ is the greatest corner of polynomial $\chi_A(x)$ and $\delta_1$ is the greatest corner of polynomial $\chi_{\overline{A}}(x)$. Based on Theorem 1.9, we can obtain that $\delta_1$ is the principal eigenvalue of matrix $A$ and $\delta_1$ is the principal eigenvalue of matrix $\overline{A}$. So, we can conclude that the principal eigenvalue of triangular matrix A, $A \approx [A, \overline{A}]$ is $\delta_1 \approx [\delta_1, \delta_1]$, where $\delta_1$ is the greatest corner of A. $\square$

### 2.2. The Diagonal Strictly Double $\mathbb{R}$-astic Matrix

**Definition 2.6.** A matrix $A \in I(\mathbb{R})^{n \times n}$, $A \approx [A, \overline{A}]$, is called strictly double $\mathbb{R}$-astic matrix if both $A$ and $\overline{A}$ are strictly double $\mathbb{R}$-astic.

**Definition 2.7.** An element $a_{ij} = [a_{ij}, \overline{a}_{ij}]$ of matrix A is called finite if both $a_{ij}$ and $\overline{a}_{ij}$ are finite.

**Theorem 2.8.** The permanent of strictly double $\mathbb{R}$-astic matrix $A \in I(\mathbb{R})^{n \times n}$ is equal to product of finite elements.

Proof. Without loss of generality, let A be a strictly double $\mathbb{R}$-astic matrix whose diagonal entries are finite,

$$A = \begin{pmatrix}
[a_{11}, \overline{a}_{11}] & [\varepsilon, \varepsilon] & \cdots & [\varepsilon, \varepsilon] \\
[\varepsilon, \varepsilon] & [a_{22}, \overline{a}_{22}] & \cdots & [\varepsilon, \varepsilon] \\
\vdots & \vdots & \ddots & \vdots \\
[\varepsilon, \varepsilon] & [\varepsilon, \varepsilon] & \cdots & [\overline{a}_{nn}, \overline{a}_{nn}]
\end{pmatrix}.$$
Then we can obtain lower bound matrix and upper bound matrix of matrix $A$, that is

$$A = \begin{pmatrix}
\tilde{a}_{11} & \varepsilon & \cdots & \varepsilon \\
\varepsilon & \tilde{a}_{22} & \cdots & \varepsilon \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon & \varepsilon & \cdots & \tilde{a}_{nn}
\end{pmatrix} \quad \text{and} \quad \bar{A} = \begin{pmatrix}
\bar{a}_{11} & \varepsilon & \cdots & \varepsilon \\
\varepsilon & \bar{a}_{22} & \cdots & \varepsilon \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon & \varepsilon & \cdots & \bar{a}_{nn}
\end{pmatrix}$$

such that $A \approx [A, \bar{A}]$. So that

$$maper(\bar{A}) = maper\left(\begin{pmatrix}
\tilde{a}_{11} & \varepsilon & \cdots & \varepsilon \\
\varepsilon & \tilde{a}_{22} & \cdots & \varepsilon \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon & \varepsilon & \cdots & \tilde{a}_{nn}
\end{pmatrix}\right) = \tilde{a}_{11} \otimes \tilde{a}_{22} \otimes \cdots \otimes \tilde{a}_{nn},$$

$$maper(\bar{A}) = maper\left(\begin{pmatrix}
\bar{a}_{11} & \varepsilon & \cdots & \varepsilon \\
\varepsilon & \bar{a}_{22} & \cdots & \varepsilon \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon & \varepsilon & \cdots & \bar{a}_{nn}
\end{pmatrix}\right) = \bar{a}_{11} \otimes \bar{a}_{22} \otimes \cdots \otimes \bar{a}_{nn}.$$

Thus

$$maper(A) = [maper(\bar{A}), maper(\bar{A})] = [\tilde{a}_{11} \otimes \tilde{a}_{22} \otimes \cdots \otimes \tilde{a}_{nn}, \bar{a}_{11} \otimes \bar{a}_{22} \otimes \cdots \otimes \bar{a}_{nn}] = [\tilde{a}_{11}, \bar{a}_{11}] \otimes [\tilde{a}_{22}, \bar{a}_{22}] \otimes \cdots \otimes [\tilde{a}_{nn}, \bar{a}_{nn}].$$

\[ \square \]

**Definition 2.9.** A strictly double $\mathbb{R}$-astic matrix whose diagonal entries are finite is called diagonal strictly double $\mathbb{R}$-astic matrix.

**Definition 2.10.** The characteristic polynomial of diagonal strictly double $\mathbb{R}$-astic matrix $A \in I(\mathbb{R})_{n \times n}$ is

$$\chi_A(x) \approx [\chi_A(\tilde{x}), \chi_A(\bar{x})],$$

where $\chi_A(\tilde{x}) = maper(\tilde{A} \oplus x \otimes I)$ and $\chi_A(\bar{x}) = maper(\bar{A} \oplus x \otimes I)$.

**Theorem 2.11.** The greatest corner of diagonal strictly double $\mathbb{R}$-astic matrix $A \in I(\mathbb{R})_{n \times n}$ is

$$\delta_1 \approx [\delta_1, \bar{\delta}_1],$$

where $\delta_1 = \bigoplus_{i \in N} \tilde{a}_{i1}$ and $\bar{\delta}_1 = \bigoplus_{i \in N} \bar{a}_{i1}$. 
Proof. Let $A$ be a diagonal strictly double $\mathbb{R}$-astic matrix, where $A \approx [\underline{A}, \overline{A}]$. Then, based on Definition 2.3, characteristic polynomial of $A$ is

$$
\chi_A(x) \approx [\chi_{\underline{A}}(x), \chi_{\overline{A}}(x)],
$$

where $\chi_{\underline{A}}(x) = \text{maper}(\underline{A} \oplus x \otimes I)$ and $\chi_{\overline{A}}(x) = \text{maper}(\overline{A} \oplus x \otimes I)$.

Based on Theorem 1.8, we can obtain that the greatest corner of characteristic polynomial $\chi_{\underline{A}}(x) = \text{maper}(\underline{A} \oplus x \otimes I)$ is

$$
\delta_1 = \bigoplus_{i \in \mathbb{N}} a_{ii},
$$

also the greatest corner of characteristic polynomial $\chi_{\overline{A}}(x) = \text{maper}(\overline{A} \oplus x \otimes I)$ is

$$
\overline{\delta}_1 = \bigoplus_{i \in \mathbb{N}} \pi_{ii}.
$$

Then we can conclude that the greatest corner of characteristic polynomial $\chi_A(x)$ where $\chi_A(x) \approx [\chi_{\underline{A}}(x), \chi_{\overline{A}}(x)]$ is $\delta_1 \approx [\underline{\delta}_1, \overline{\delta}_1]$. $\square$

**Theorem 2.12.** The principal eigenvalue of diagonal strictly double $\mathbb{R}$-astic matrix is its greatest corner.

Proof. Let $A$ be a diagonal strictly double $\mathbb{R}$-astic matrix, where $A \approx [\underline{A}, \overline{A}]$. Then, $\underline{\delta}_1$ is the greatest corner of polynomial $\chi_{\underline{A}}(x)$ and $\overline{\delta}_1$ is the greatest corner of polynomial $\chi_{\overline{A}}(x)$. Based on Theorem 1.9, we can obtain that $\underline{\delta}_1$ is the principal eigenvalue of matrix $\underline{A}$ and $\overline{\delta}_1$ is the principal eigenvalue of matrix $\overline{A}$. So, we can conclude that the principal eigenvalue of triangular matrix $A$, $A \approx [\underline{A}, \overline{A}]$ is $\delta_1 \approx [\underline{\delta}_1, \overline{\delta}_1]$, where $\delta_1$ is the greatest corner of $A$. $\square$

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