Diffusion on Fractal Cesàro Curve

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Abstract

In this paper, we apply $F^\alpha$-calculus on fractal Koch and Cesàro curves with different dimensions. Generalized Newton’s second law on the fractal Koch and Cesàro curves is suggested. Density of moving particles which absorbed on fractal Cesàro are derived. More, illustrative examples are given to present the details of $F^\alpha$-integrals and $F^\alpha$-derivatives.

Keywords: $F^\alpha$-calculus; Fractal Koch curve; Staircase function; Fractal Cesàro curve

1 Introduction

Fractional calculus which include derivatives and integrals with arbitrary orders is applied in science, engineering and etc. [1, 2, 3]. The fractional derivatives are used to model non-conservative systems, processes with the memory effect and anomalous diffusion [4, 5, 6]. The fractional derivatives are non-local but the most measurements in physics are local [7]. As a result, in view of the fractional local derivatives and Chapman-Kolmogorov condition new Fokker-Planck equation is given [8]. The local fractional derivatives lead to new measure on fractal sets [9]. Fractal geometry which is the generalized Euclidean geometry has important role in science, engineering and medical science. Fractals are the shapes which have self-similar properties and fractional dimensions. Many methods have been used to build analysis on fractal sets and processes [10, 11, 12, 13, 14, 15, 16].

Recently, $F^\alpha$-calculus is suggested in the seminal paper as a framework by A. Parvate and A. D. Gangal which is the generalized standard calculus. $F^\alpha$-calculus is calculus on fractals with the algorithmic property [17, 18, 19]. Researchers have explored this area giving new insight into $F^\alpha$-calculus [20, 21, 22, 23]. The fractal Cantor sets are considered as grating in the diffraction phenomena [24]. Regarding the above mentioned research we apply the $F^\alpha$-calculus on fractal curves in the case of fractal Koch curves. Differential equation corresponding for a motion of the particle on fractal curves is purposed.

Outline of the paper is as follows:
Figure 1: We have plotted fractal Koch curve and fractal Cesáro curve in blue and corresponding staircase functions in red.

In Section 2 we summarize $F^\alpha$-calculus on fractal Koch and Cesáro curves without proofs. In Section 3 we give our new result in this paper which includes equation of motion of the particles. Section 3 contains our conclusion.

2 Preliminaries

In this section, we summarize $F^\alpha$-calculus on parameterize fractal curves and use in the case of fractal Koch and Cesáro curves (see for review Refs. [19] [25].

Calculus on fractal Koch and Cesáro curves:

Let us consider fractal Koch and Cesáro curves which are denoted by $F \subset \mathbb{R}^3$ and define corresponding staircase function. Fractal Koch and Cesáro curves are called continuously parameterizable if there exists a function $w(t) : [a_0, b_0] \rightarrow F$, $a_0, b_0 \in \mathbb{R}$ which is continuous one to one and onto $F$ [19] [25].

Definition: For the fractal curves $F$ and a subdivision $P_{[a,b]}$, $a < b$, $[a, b] \subset [a_0, b_0]$ mass function is defined [19]

$$\gamma^\alpha(F, a, b) = \lim_{\delta \rightarrow 0} \inf_{P : |P| \leq \delta} \sum_{i=0}^{n-1} \frac{|w(t_{i+1}) - w(t_i)|^\alpha}{\Gamma(\alpha + 1)},$$  \hspace{1cm} (1)$$

where $|.|$ indicates the Euclidean norm on $R^3$ and $P_{[a,b]} = \{a = t_0, ..., t_n = b\}$.

Definition: The staircase functions for fractal Koch and Cesáro curves are defined

$$S_F^\alpha(t) = \begin{cases} 
\gamma^\alpha(F, p_0, t) & t \geq p_0, \\
-\gamma^\alpha(F, t, p_0) & t < p_0,
\end{cases}$$  \hspace{1cm} (2)$$

where $p_0 \in [a_0, b_0]$ is arbitrary point.

In Figure 1 we have sketched fractal Koch and Cesáro curves and $S_F^\alpha(t)$ setting $\alpha = 1.26$ and $\alpha = 1.78$.

Definition: The $\gamma$-dimension of fractal Koch and Cesáro curves $(F)$ are
defined

\[ \dim_\gamma(F) = \inf\{\alpha : \gamma^\alpha(F, a, b) = 0\} = \sup\{\alpha : \gamma^\alpha(F, a, b) = \infty\}. \] (3)

**Definition:** \(F^\alpha\)-derivative of function \(f\) at \(\theta \in F\) is defined

\[ D^\alpha_F f(\theta) = F - \lim_{\theta' \to \theta} \frac{f(\theta') - f(\theta)}{J(\theta') - J(\theta)}, \] (4)

where \(J(\theta) = S^\alpha_F(w^{-1}(\theta)), \theta \in F\) and if the limit exists \[19, 25\].

**Definition:** A number \(l\) is \(F\)-limit of the function \(f\) if we have

\[ \theta' \in F \quad \text{and} \quad |\theta' - \theta| < \delta \Rightarrow |f(\theta') - l| < \epsilon. \] (5)

If such a number exists \[19\]. It is indicate by

\[ l = F - \lim_{\theta' \to \theta} f(\theta'). \] (6)

A segment \(C(t_1, t_2)\) of fractal Koch and Cesàro curve is define as

\[ C(t_1, t_2) = \{w(t') : t' \in [t_1, t_2]\}, \] (7)

and \(M, m\) are defined as follows \[19, 25\]

\[ M[f, C(t_1, t_2)] = \sup_{\theta \in C(t_1, t_2)} f(\theta), \] (8)

\[ m[f, C(t_1, t_2)] = \inf_{\theta \in C(t_1, t_2)} f(\theta). \] (9)

**Definition:** The upper and the lower \(F^\alpha\)-sum for the function \(f\) over the subdivision \(P\) are defined

\[ U^\alpha[f, F, P] = \sum_{i=0}^{n-1} M[f, C(t_i, t_{i+1})][S^\alpha_F(t_{i+1}) - S^\alpha_F(t_i)], \] (10)

\[ L^\alpha[f, F, P] = \sum_{i=0}^{n-1} m[f, C(t_i, t_{i+1})][S^\alpha_F(t_{i+1}) - S^\alpha_F(t_i)]. \] (11)

**Definition:** \(F^\alpha\)-integral of the function \(f\) is defined

\[ \int_{C(a,b)} f(\theta)d^\alpha_F \theta = \int_{C(a,b)} f(\theta)d^\alpha_F \theta = \sup_{P[a,b]} L^\alpha[f, F, P] \]

\[ = \int_{C(a,b)} f(\theta)d^\alpha_F \theta = \inf_{P[a,b]} U^\alpha[f, F, P]. \] (12)

**Fundamental theorems of \(F^\alpha\)-calculus:**

**First Part:** If \(f : F \to R\) is \(F^\alpha\)-differentiable function and \(h : F \to R\) is \(F\)-continuous such that \(h(\theta) = D^\alpha_F f(\theta)\), then we have

\[ \int_{C(a,b)} h(\theta)d^\alpha_F \theta = f(w(b)) - f(w(a)). \] (13)
Second part: If $f$ is bounded, $F$-continuous on $C(a,b)$ and $g : F \to R$ then we have

$$g(w(t)) = \int_{C(a,t)} f(\theta) d_F^\alpha \theta, \quad t \in [a, b],$$

where we have

$$D_F^\alpha g(\theta) = f(\theta).$$

For the proofs we refer the reader to [19].

Some of the properties:

1) If $f(\theta) = k \in R$ then we have $D_F^\alpha f = 0$.
2) If $f$ is a $F$-continuous and $D_F^\alpha f = 0$ then $f = k$.
3) Generalized Taylor Series on fractal Koch curves is

$$h(\theta) = \sum_{n=0}^{\infty} \frac{(J(\theta) - J(\theta'))^n}{n!}(D_F^\alpha)^n h(\theta'), \quad \theta \in F.$$ (16)

$$\int_{C(a,b)} f(\theta) d_F^\alpha \theta = \int_{C(a,b)} 1 d_F^\alpha \theta$$

$$= S_F^\alpha(b) - S_F^\alpha(a) = J((w(b)) - J((w(a))),$$ (17)

where $f(\theta) = 1$ is constant function [19, 25].

Note: $F^\alpha$-derivative and $F^\alpha$-integral on fractal Koch curves are linear operators.

Example 1. Consider $f(t) : F \to R$ on the fractal Koch curves as

$$f(t) = (S_F^\alpha(t))^2.$$ (18)

The $F^\alpha$-derivative and the $F^\alpha$-integral of $f$ are

$$\int_{C(a,t)} f(t) d_F^\alpha t = \frac{(S_F^\alpha(t))^3}{3} + k,$$

and

$$D_F^\alpha f(t) = 2 S_F^\alpha(t),$$

where $k$ is constant. Figure [2] shows the graphs of $f$, $F^\alpha$-integral of $f$, and $F^\alpha$-derivative of $f$. 
3 Equation of motion on fractal curves

Generalized Newton’s second law on fractal Koch and Cesáro curves is suggested

\[ m(D_F^\alpha)^2 r_F^\alpha(t) = f_F^\alpha, \]  \hspace{1cm} (19)

where \( r_F^\alpha : F \rightarrow \mathbb{R} \), \( v_F^\alpha(t) = D_F^\alpha r_F^\alpha \) and \( a_F^\alpha(t) = (D_F^\alpha)^2 r_F^\alpha \) are called generalized position, generalized velocity and generalized acceleration on fractal Koch and Cesáro curves, respectively.

**Example 2.** Consider a force \( f_F^\alpha = k(\hat{i} + \hat{j}) \) [ \( ML^\alpha T^2 \) ] such that \( f_F^\alpha : F \rightarrow \mathbb{R} \) apply on a particle with mass \( m \) on fractal Koch curves. One sees immediately that generalized acceleration, velocity and position are

\[
\begin{align*}
    a_F^\alpha(t) &= \frac{k}{m}, \\
    v_F^\alpha(t) &= \frac{k}{m} S_F^\alpha(t) + v_F^\alpha(0), \\
    r_F^\alpha(t) &= \frac{k}{2m} S_F^\alpha(t)^2 + v_F^\alpha(0) S_F^\alpha(t) + r_F^\alpha(0).
\end{align*}
\]  \hspace{1cm} (20)
We present the graph of the Eq. (20) in Figure 3

**Example 3.** Consider particles moving along the fractal Cesàro curve which absorb the particles. The mathematical model for this phenomenon is given by

\[ D^\beta_F \zeta(t) = -k\zeta(t), \quad \beta = 1.78. \]  

(21)

where \( \zeta \) is the density of particles on fractal Cesàro curve. Using \( F^\alpha \)-integral, it is easy to obtain the solution

\[ \zeta(t) = \zeta(0)e^{-kS^\beta_F(t)}. \]  

(22)

Figure 4: We plot the density of particles for the flux of particles on fractal Cesàro curve with absorption

Figure 4 shows the graph of \( \zeta(t) \) on fractal Cesàro curve.

### 4 Conclusion

In this paper, \( F^\alpha \)-calculus is the generalization of the standard calculus on the fractals with fractional dimension and self-similar properties. In the sense of the standard calculus the fractal Koch and Cesàro curves are not differentiable and integrable. The \( F^\alpha \)-calculus is used to define \( F^\alpha \)-integral and \( F^\alpha \)-derivative on fractal Koch and Cesàro curves. Some illustrative examples are given for presenting the details. Finally, generalized differential equation corresponding to the motions on the fractal Koch and Cesàro curves are suggested and solved.
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