Positive Solution to Singular Elliptic Problems with Subcritical nonlinearities

Abstract: In this paper, we study the existence of a non-trivial weak solution to the following singular elliptic equations with subcritical nonlinearities:

\[
\begin{cases}
-\text{div}(\vert x \vert^{-2\beta} \nabla u) - \mu \frac{f(x)u}{\vert x \vert^{2(\beta+1)}} = \frac{\lambda g(x)}{u^\theta} + h(x)u^p & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N \) is an open bounded domain with \( C^1 \) boundary, \( \theta, \lambda > 0, \ 0 < \beta < \frac{N-2}{2}, \ 0 < p < 1, \ 0 < \mu < \left( \frac{N-2(\beta+1)}{2} \right)^2, \ N \geq 3, \ 0 \in \Omega \) and \( 0 \leq f, g, h \in L^\infty(\Omega) \). We show that there exists a solution \( u \in H_0^1(\Omega, \vert x \vert^{-2\beta}) \cap L^\infty(\Omega) \) to this problem.

Keywords: Nonlinear elliptic equation, Hardy potential, positive solution

MSC: Primary 35J05, 35J25; Secondary 46E35

1 Introduction

Let \( \Omega \subset \mathbb{R}^N \) be an open bounded domain with \( C^1 \) boundary, \( 0 \in \Omega \). We consider the following singular elliptic equations with subcritical nonlinearities:

\[
\begin{cases}
-\text{div}(\vert x \vert^{-2\beta} \nabla u) - \mu \frac{f(x)u}{\vert x \vert^{2(\beta+1)}} = \frac{\lambda g(x)}{u^\theta} + h(x)u^p & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

under assumptions \( \theta, \lambda > 0, \ 0 < \beta < \frac{N-2}{2}, \ 0 < p < 1, \ 0 < \mu < \left( \frac{N-2(\beta+1)}{2} \right)^2, \ N \geq 3, \) and \( 0 \leq f, g, h \in L^\infty(\Omega) \). We prove existence of (1.1) in \( H_0^1(\Omega, \vert x \vert^{-2\beta}) \cap L^\infty(\Omega) \). Furthermore, a solution of (1.1) is a function \( u \) in \( W^{1,1}_0(\Omega, \vert x \vert^{-2\beta}) \) with the following:

\[
\forall \ \Omega' \subset \subset \Omega, \ \exists \ c_{\Omega'} : u \geq c_{\Omega'} > 0 \ \text{in } \Omega',
\]

and

\[
\int_{\Omega} \vert x \vert^{-2\beta} \nabla u \nabla \varphi - \mu \int_{\Omega} f(x) \frac{u \varphi}{\vert x \vert^{2(\beta+1)}} = \int_{\Omega} g(x) \frac{\varphi}{u^\theta} + \int_{\Omega} h(x)u^p \varphi, \quad \forall \ \varphi \in C_0^1(\Omega).
\]

We pointed out that the right hand side is well defined by (1.2) since \( \varphi \) has compact support.

*Corresponding Author: Dharmendra Kumar: Indian Institute of Technology Gandhinagar, Gujarat, India - 382355, E-mail: k.dharmendra@iitgn.ac.in, dharamsambey90@gmail.com
The problem of the form (1.1) arise in certain problems in fluid mechanics, chemical heterogenous catalysts, pseudo plastic flow and non-Newtonian fluids (see [3, 9, 13, 14, 17]). Simpler form of the problem (1.1) is of the type

\[ -\text{div}(M(x)\nabla u) = h(x, u), \] (1.4)

where \( M(x) = |x|^{-2\beta} \) and \( h(x, s) \) is singular for \( s = 0 \) and \( x = 0 \). Equation (1.4) with \( M(x) \) is identity matrix, has been widely studied by many authors in the past, see, for instance [2, 5–7, 12, 16] and the reference therein. When \( h(x, s) \) that does not depend on \( x \), Crandall, Rabinowitz and Tartar in [7], showed the existence and continuity properties of the solution of (1.4). In [12], it has been shown that the existence of the solution to (1.4), for \( h(x, u) = \frac{f(x)}{u^n} \), where \( f \) is a continuous function. They showed that the solution \( u \in H^1_0(\Omega) \) if and only if \( \gamma < 3 \) and that, if \( \gamma > 1 \), solution \( u \notin C^1(\Omega) \). For, \( h(x, u) = f(x)g(u) \), Lair and Shaker [10, 11] studied the problem (1.4), and by Zhang and Cheng, see [19], using the first eigenfunction of the Laplacian in \( \Omega \).

The present work is more close to results in [1, 8], although we generalised the results for sublinearities, i.e. for the case \( 0 < p < 1 \) [1, Theorem 2.4]. The aim of this paper is to show the existence to (1.1) for sublinearities.

To obtain a solution to (1.1), we use the similar techniques as used in [1, 2]. The organisation of this paper is as follows. Section 2 deals with some preliminaries, basic facts which are used in the main result. Section 3 is devoted to the proof of the main theorem.

The main result of this paper is the following theorem, which we will prove in the next section.

**Theorem 1.1.** Assume (1.1) with \( 0 < p < 1 \), \( \theta, \Lambda > 0 \), \( 0 < \beta < \frac{N-2}{2} \), \( 0 < \mu < \left( \frac{N-2(\beta+1)}{2} \right)^2 \), and \( 0 \leq f, g, h \in L^\infty(\Omega) \). Then \( \exists \Lambda \in (0, \infty) \) such that for \( \lambda \in (0, \Lambda) \), \( \exists \) a solution \( u \in H^1_0(\Omega, |x|^{-2\beta}) \) to (1.1). Moreover

- If \( 0 < \theta < 1 \), then \( u \in H^1_0(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega) \).

2 Auxiliary results

Let \( n \in \mathbb{N} \), let \( g_n(x) = \min\{g(x), n\} \). To prove the existence results we consider the following approximate problem:

\[
\begin{align*}
-\text{div}(|x|^{-2\beta}\nabla u_n) - \mu &- f(x)u_n \frac{|x|^{2(\beta+1)}}{|x|^{2(\beta+1)} + 1} \frac{1}{n} = \frac{\lambda g_n(x)}{(u_n + \frac{1}{n})^{\theta}} \quad \text{in } \Omega, \\
u_n &> 0 \quad \text{in } \Omega, \\
u_n &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (2.1)

Let us recall the following Caffarelli-Kohn-Nirenberg inequality and a compact embedding theorem (which is an extension of the classical Rellich-Kondrachov compactness theorem).

**Proposition 2.1.** Caffarelli-Kohn-Nirenberg inequality [4]: There is a constant \( C_{\gamma, \beta} > 0 \) such that

\[
\left( \int_{\mathbb{R}^N} |x|^{-2\gamma} |u|^2 \right)^{2/2} \leq C_{\gamma, \beta} \int_{\mathbb{R}^N} |x|^{-2\gamma} |\nabla u|^2,
\] (2.2)

for all \( u \in C^0_0(\mathbb{R}^N) \), where

\[
-\infty < \gamma < \frac{N-2}{2}, \quad \beta \leq \gamma < \beta + 1, \quad 2s = \frac{2N}{N-2d}, \quad d = \gamma + 1 - \beta.
\] (2.3)
Let $H^1_0(\Omega, |x|^{-2\beta})$ be the completion of $C^\infty_0(\mathbb{R}^N)$, with respect to the following weighted norm $\| \cdot \|$ defined by

$$
\|u\| = \left( \int_\Omega |x|^{-2\beta}|\nabla u|^2 \right)^{1/2}.
$$

(2.4)

From the boundedness of $\Omega$ and the standard approximation arguments, it can be shown that (2.2) holds for any $u \in H^1_0(\Omega, |x|^{-2\beta})$ in the sense:

$$
\left( \int_\Omega |x|^{-2\beta q}|u|^q \right)^{2/q} \leq C_\beta \int_\Omega |x|^{-2\gamma}|\nabla u|^2,
$$

(2.5)

for $1 \leq q \leq 2*$.

**Proposition 2.2.** General Hardy-Sobolev inequality: There is a constant $C_{\beta,N,2} > 0$ such that

$$
\left( \int_\Omega |x|^{-2(\beta+1)}|u|^2 \right)^{1/2} \leq C_{\beta,N,2} \int_\Omega |x|^{-2\beta}|\nabla u|^2,
$$

(2.6)

for all $u \in H^1_0(\Omega, |x|^{-2\beta})$, where

$$
-\infty < \beta < \frac{N-2}{2}, \quad C_{\beta,N,2} = \left( \frac{N-2(\beta+1)}{2} \right)^2.
$$

(2.7)

Moreover, $C_{\beta,N,2}^{-1}$ is optimal and it is not achieved.

Let $L^q(\Omega, |x|^{-\beta q})$ be the weighted $L^q$ space with weighted norm defined by

$$
\|u\|_{\beta,q} := \left( \int_\Omega |x|^{-\beta q}|u|^q \right)^{1/q}.
$$

(2.8)

then (2.5) can be written as

$$
\|u\|_{\beta,q} \leq C\|u\|, \quad \forall u \in H^1_0(\Omega, |x|^{-2\beta}),
$$

(2.9)

where $C > 0$ denotes a universal constant, which may change its value from line to line.

**Proposition 2.3.** Compact imbedding theorem[18]: Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with $C^1$ boundary and $0 < \gamma < \frac{N}{2}$, $\beta \leq \gamma < (1 + \beta)$. For $1 \leq q \leq 2^*$, the imbedding $H^1_0(\Omega, |x|^{-2\beta}) \hookrightarrow L^q(\Omega, |x|^{-\gamma q})$ is continuous. For $1 \leq q < 2^*$, the imbedding $H^1_0(\Omega, |x|^{-2\beta}) \hookrightarrow L^q(\Omega, |x|^{-\gamma q})$ is compact.

**Lemma 2.4.** For $C_{\beta,N,2}^{-1}\|f\|_{L^\infty(\Omega)} < \frac{1}{2}$ where $0 < \beta < \frac{N-2}{2}$, $C_{\beta,N,2} = \left( \frac{N-2(\beta+1)}{2} \right)^2$. Then Problem (2.1) has a nonnegative solution in $H^1_0(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega)$.

**Proof.** Let $n \in \mathbb{N}$ be a fixed natural no and $\omega$ be a function in $L^2(\Omega, |x|^{-2\beta})$. By the Lax-Milgram Theorem, the following problem has a unique solution $u$ in $H^1_0(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega)$:

$$
\begin{cases}
-\text{div}(|x|^{-2\beta}\nabla u) - \mu \frac{f(x)u}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{g_n(x)}{(|\omega| + \frac{1}{n})^\beta} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(2.10)
because the operator is coercive due to the assumptions on \( \mu \) and Proposition 2.2.

Now, for any \( u \in L^2(\Omega, |x|^{-2(\beta+1)}) \), we define the mapping

\[
T : L^2(\Omega, |x|^{-2(\beta+1)}) \to H^1_0(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega),
\]

as follows:

\[
T(u) = \omega.
\]

Let us take \( u \) as a test function, we have

\[
\int_{\Omega} |x|^{-2\beta} \nabla u|^2 - \mu \int_{\Omega} \frac{f(x)u^2}{|x|^{2(\beta+1)} + \frac{1}{n}} = \int_{\Omega} \frac{g_n(x)u}{(|\omega| + \frac{1}{n})^\theta},
\]

and so

\[
\int_{\Omega} |x|^{-2\beta} |\nabla u|^2 - \mu \int_{\Omega} \frac{f(x)u^2}{|x|^{2(\beta+1)} + \frac{1}{n}} \leq n^{\theta+1} \int_{\Omega} u,
\]

Since \( g_n(x) = \min \{g(x), n\} \), so \( g_n(x) \leq n \), this implies that

\[
\frac{g_n(x)u}{(|\omega| + \frac{1}{n})^\theta} \leq \frac{nu}{n(|\omega| + 1)} \leq n^{\theta+1} u,
\]

The General Hardy - Sobolev inequality on the left-hand side and Hölder inequality on the right-hand side implies that

\[
\left( C_{\beta,N,2} - \mu \|f\|_{\infty} \right) \int_{\Omega} |x|^{-2(\beta+1)} |u|^2 \leq Cn^{\theta+1} \left( \int_{\Omega} u^2 \right)^{1/2},
\]

for some constant \( C \) independent of \( u \).

This is equivalent to

\[
\left( C_{\beta,N,2} - \mu \|f\|_{\infty} \right) \int_{\Omega} |x|^{-2(\beta+1)} |u|^2 \leq Cn^{\theta+1} |\Omega|^{2(\beta+1)} \left( \int_{\Omega} |x|^{-2(\beta+1)} u^2 \right)^{1/2},
\]

where \( |\Omega| \) denotes the diameter of \( \Omega \).

This implies that

\[
\leq Cn^{\theta+1} |\Omega|^{(\beta+1)} \left( \int_{\Omega} |x|^{-2(\beta+1)} u^2 \right)^{1/2},
\]

i.e. the ball of radius \( Cn^{\theta+1} \) is invariant under \( T \). Using compact imbedding, it can be shown that \( T \) is both continuous and compact on \( L^2(\Omega, |x|^{-2(\beta+1)}) \). So by Schauder’s fixed point theorem, \( \exists \ u_n \in H^1_0(\Omega, |x|^{-2\beta}) \) such that \( u_n = T(u_n) \), i.e., \( u_n \) solves

\[
\begin{cases}
- \text{div}(|x|^{-2\beta} \nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{g_n(x)}{(u_n + \frac{1}{n})^\theta} \quad \text{in } \Omega, \\
u_n > 0 \quad \text{in } \Omega, \\
u_n = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]

Since \( \frac{g_n(x)}{(u_n + \frac{1}{n})^\theta} \geq 0 \) a.e., so the maximum principle implies that \( u_n \geq 0 \) and hence \( u_n \) be a solution of (2.1). By the result of [15], \( u_n \) belongs to \( L^\infty(\Omega) \), because of the right-hand side of (2.10) belongs to \( L^\infty(\Omega) \).
3 Proof of the theorem 1.1

Proof. Consider the following Dirichlet problems

\[ u_n \in H^1_0(\Omega, |x|^{-2\beta}) : -\text{div}(|x|^{-2\beta} \nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{\lambda g_n(x)}{(u_n + \frac{1}{n})^\theta} + h(x)u_n^p \text{ in } \Omega. \quad (3.1) \]

To prove the existence of a solution \( u_n \) of (3.1), we apply Sattinger monotone iteration.

We first obtain a subsolution:

By Lemma (2.4) there exists a weak solution \( u_n \) of the following Dirichlet problem

\[ u_n \in H^1_0(\Omega, |x|^{-2\beta}) : -\text{div}(|x|^{-2\beta} \nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{\lambda g_n(x)}{(u_n + \frac{1}{n})^\theta} \text{ in } \Omega, \quad (3.2) \]

so \( u_n \) solves:

\[ u_n \in H^1_0(\Omega, |x|^{-2\beta}) : -\text{div}(|x|^{-2\beta} \nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{\lambda g_n(x)}{(u_n + \frac{1}{n})^\theta} \text{ in } \Omega. \quad (3.3) \]

Thus

\[ u_n \in H^1_0(\Omega, |x|^{-2\beta}) : -\text{div}(|x|^{-2\beta} \nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} \leq \frac{\lambda g_n(x)}{(u_n + \frac{1}{n})^\theta} + h(x)u_n^p \text{ in } \Omega, \quad (3.4) \]

\( \text{i.e. } u_n \) is a subsolution to the problem (3.1).

Now, we construct a supersolution of (3.1):

Let \( t \geq Y_\lambda \) (yet to determine) and consider the following Dirichlet problems \( \bar{u}_{n,t} : \bar{u}_{n,t} \) is the solution of

\[ \bar{u}_{n,t} \in H^1_0(\Omega, |x|^{-2\beta}) : -\text{div}(|x|^{-2\beta} \nabla \bar{u}_{n,t}) - \mu \frac{f(x)\bar{u}_{n,t}}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{t}{(\bar{u}_{n,t} + \frac{1}{n})^\theta}, \quad (3.5) \]

then (see [2]) \( \exists c_0 > 0 \) such that

\[ ||\bar{u}_{n,t}||_{L^\infty(\Omega)} \leq c_0 t^\frac{1}{\theta}. \quad (3.6) \]

Then the maximum principle implies that \( \bar{u}_{n,t} \geq 0 \) so that \( \bar{u}_{n,t} \) solves

\[ -\text{div}(|x|^{-2\beta} \nabla \bar{u}_{n,t}) - \mu \frac{f(x)\bar{u}_{n,t}}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{t}{(\bar{u}_{n,t} + \frac{1}{n})^\theta}. \quad (3.7) \]

We show that for some \( t > 0 \),

\[ \frac{t}{\left( \frac{1}{n} + \bar{u}_{n,t} \right)^\theta} \geq \frac{\lambda g(x)}{(\frac{1}{n} + \bar{u}_{n,t})^\theta} + h(x)\bar{u}_{n,t}^p, \quad (3.8) \]

\( \text{i.e.} \)

\[ t \geq \lambda g(x) + h(x)\bar{u}_{n,t}^p \left( \frac{1}{n} + \bar{u}_{n,t} \right)^\theta. \quad (3.9) \]

The above inequality is true if \( t \) satisfies the following

\[ t \geq \lambda ||g||_\infty + ||h||_\infty \left[ c_0 t^\frac{1}{\theta} \right]^p \left( \frac{1}{n} + c_0 t^\frac{1}{\theta} \right)^\theta. \quad (3.10) \]

Suppose \( Y_\lambda \) be such that

\[ Y_\lambda = \lambda ||g||_\infty + 2^\theta ||h||_\infty c_0^{p+\theta} Y_\lambda^{p+\theta}. \quad (3.11) \]

Let \( F(Y_\lambda) = Y_\lambda - \lambda ||g||_\infty - 2^\theta ||h||_\infty c_0^{p+\theta} Y_\lambda^{p+\theta}. \)

Then

\[ F(0) = -\lambda ||g||_\infty < 0. \]
Now, let
\[ G(Y_A) = Y_A \frac{\bar{\alpha}}{2} - 2^\theta \|h\|_{\infty} c_0^{\theta+\theta} - \lambda \|g\|_{\infty} Y_A^{\frac{\bar{\alpha}}{2}}. \]

Then let \( Y_A \to \infty \),
\[ \implies G(Y_A) \to \infty, \ i.e.; \]
\[ \exists N \in \mathbb{N} \text{ such that } G(Y_A) > 1, \ \forall Y_A \geq N, \]
\[ \implies G(N) > 1. \]

Now,
\[ F(Y_A) = Y_A^{\frac{\bar{\alpha}}{2}} G(Y_A). \]

Hence
\[ F(N) = N^{\frac{\bar{\alpha}}{2}} G(N) > 1 \cdot N^{\frac{\bar{\alpha}}{2}} > 0. \]

By Mean Value Theorem, \( \exists Y_A \) such that \( 0 < Y_A < N \) and \( F(Y_A) = 0 \). That is there exists \( \Lambda > 0 \) such that for \( 0 < \lambda < \Lambda \), there exists the solution of the equation (3.11). Therefore, we only have to prove that if \( n \) is large enough, we have
\[ \lambda \|g\|_{\infty} + 2^\theta \|h\|_{\infty} c_0^{\theta+\theta} Y_A^{\frac{\bar{\alpha}}{2}} \geq \lambda \|g\|_{\infty} + \|h\|_{\infty} c_0^{\theta+\theta} Y_A^{\frac{\bar{\alpha}}{2}} \left( \frac{1}{n} + c_0 Y_A^{\frac{1}{2}} \right)^{\theta}, \]
\[ (3.12) \]

thus,
\[ 2^\theta c_0^{\theta+\theta} Y_A^{\frac{\bar{\alpha}}{2}} \geq \left( \frac{1}{n} + c_0 Y_A^{\frac{1}{2}} \right)^{\theta}, \]
\[ (3.13) \]

this will hold if \( \frac{1}{n} \leq c_0 Y_A^{\frac{1}{2}} \) and this latter inequality true for \( n \) large.

Since \( Y_A \) independent of \( n \). Thus we proved that for \( Y \geq Y_A \), \( u_{n,t} \) is a super solution to the problem (3.1).

We show that \( u_n \leq u_{n,t} \):

Define
\[ L(v) = -div(|x|^{-2\beta} \nabla v) - \mu \frac{f(x)v}{|x|^{2(\beta+1)} + \frac{1}{\pi}}. \]
\[ (3.14) \]

Next aim to show \( u_n \leq u_{n,t} \). Indeed
\[ L(u_n - u_{n,t}) = \]
\[ (3.15) \]

\[ -div(|x|^{-2\beta} \nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{\pi}} + div(|x|^{-2\beta} \nabla u_{n,t}) + \mu \frac{f(x)u_{n,t}}{|x|^{2(\beta+1)} + \frac{1}{\pi}} \]
\[ = \frac{\lambda g_n(x)}{\left( \frac{1}{n} + |u_n| \right)^{\theta}} - \frac{t}{\left( \frac{1}{n} + |u_{n,t}| \right)^{\theta}}, \]
\[ (3.17) \]

implies
\[ \langle L(u_n - u_{n,t}), (u_n - u_{n,t})^\ast \rangle = \int_\Omega \left[ \frac{\lambda g_n(x)}{\left( \frac{1}{n} + |u_n| \right)^{\theta}} - \frac{t}{\left( \frac{1}{n} + |u_{n,t}| \right)^{\theta}} \right] (u_n - u_{n,t})^\ast. \]
\[ (3.18) \]

Since \( g_n(x) = \min\{n, g(x)\} \leq g(x) \leq \|g(x)\|_{L^\infty(\Omega)}, \) so we have
\[ \langle L(u_n - u_{n,t}), (u_n - u_{n,t})^\ast \rangle \leq \int_\Omega \left[ \frac{\lambda g_n(x)}{\left( \frac{1}{n} + |u_n| \right)^{\theta}} - \frac{t}{\left( \frac{1}{n} + |u_{n,t}| \right)^{\theta}} \right] (u_n - u_{n,t})^\ast. \]

Since \( g_n(x) = \min\{n, g(x)\} \leq g(x) \leq \|g(x)\|_{L^\infty(\Omega)}, \) so we have
\[
\int_{\Omega} \left[ \frac{\lambda||g||_{L^\infty}}{1 + |u_n|^\theta} - \frac{\lambda||g||_{L^\infty}}{1 + |u_{n,t}|^\theta} \right] (u_n - u_{n,t})^+ \\
+ \int_{\Omega} \left[ \frac{t |g|}{1 + |u_n|^\theta} - \frac{t}{1 + |u_{n,t}|^\theta} \right] (u_n - u_{n,t})^+
\]

\[
= \lambda \int_{\{u_n \geq u_{n,t}\}} ||g||_{L^\infty} \left[ \frac{1}{1 + |u_n|^\theta} - \frac{1}{1 + |u_{n,t}|^\theta} \right] (u_n - u_{n,t}) + \int_{\{u_n < u_{n,t}\}} \frac{1}{1 + |u_n|^\theta} \left[ |\lambda||g||_{L^\infty} - t \right] (u_n - u_{n,t}).
\]

(3.19)

Note that, in the last line, the first integral is negative because the real function \( \frac{1}{(1+|t|)^\theta} \) is decreasing if \( l > 0 \) and the second integral is negative because \( t \geq Y_\lambda \) and by the definition (3.11), \( Y_\lambda \geq \lambda||g||_{L^\infty} \).

Next we apply Sattinger Method:

\[
w(s) := \frac{\lambda g(x)}{(\frac{1}{n} + s)} + h(x)s^p + n^{\theta+1}\lambda \delta s, \quad 0 < s < \infty.
\]

(3.20)

It can be shown that function \( w(s) \) is increasing.

Now the classical Amann-Sattinger method will give the existence of \( u_n \) solution of

\[
u_n \in H^1_0(\Omega, |x|^{-2\beta}) : -\text{div}(|x|^{-2\beta}\nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} + n^{\theta+1}\lambda \delta u_n
\]

(3.21)

\[
= \frac{\lambda g(x)}{(\frac{1}{n} + u_n)^\theta} + h(x)(u_n)^p + n^{\theta+1}\lambda \delta u_n,
\]

(3.22)

and such that

\[
u_n \leq u_n \leq \omega_{n,t} \leq c_0 t^{\frac{1}{\theta+1}}.
\]

(3.23)

also using [2], it can be shown that the sequence \( \{u_n\} \) is increasing with respect to \( n \), \( u_n > 0 \) in \( \Omega \) and for every \( \Omega' \subset \subset \Omega \), \( \exists \epsilon_{\Omega'} \) (independent of \( n \)) such that

\[
u_n(x) \geq \omega_1(x) \geq \epsilon_{\Omega'} > 0, \quad \forall x \in \Omega', n \in \mathbb{N}.
\]

(3.24)

We will use \( u_n \) as a test function in the following equation:

\[
u_n \in H^1_0(\Omega, |x|^{-2\beta}) : -\text{div}(|x|^{-2\beta}\nabla u_n) - \mu \frac{f(x)u_n}{|x|^{2(\beta+1)} + \frac{1}{n}} = \frac{\lambda g(x)}{(\frac{1}{n} + u_n)^\theta} + h(x)(u_n)^p.
\]

(3.25)

Indeed,

\[
\int_{\Omega} |x|^{-2\beta} \nabla u_n \cdot \nabla u_n - \frac{\mu}{2} \int_{\Omega} f(x)u_n^2 |x|^{2(\beta+1)} + \frac{1}{n} = \int_{\Omega} \frac{\lambda g(x)u_n}{(\frac{1}{n} + u_n)^\theta} + \int_{\Omega} h(x)(u_n)^p u_n,
\]

(3.26)

\[
\left( 1 - \mu \|f\|_{L^\infty} C_{\beta,N,2}^{-1} \right) \int_{\Omega} |x|^{-2\beta} \nabla u_n \nabla u dx \leq \lambda \|g\|_{L^\infty}(\Omega) \int_{\Omega} u_n^{1-\theta} + ||h||_{L^\infty} \int_{\Omega} |u_n|^{p+1}.
\]

(3.27)

(3.23) implies that \( \{u_n\} \) is bounded in \( H^1_0(\Omega, |x|^{-2\beta}) \). Then \( \exists \) a subsequence \( \{u_n\} \) (still denoted by \( \{u_n\} \)) of sequence \( \{u_n\} \) and \( u \in H^1_0(\Omega, |x|^{-2\beta}) \cap L^\infty(\Omega) \) such that

\[
u_n \rightharpoonup u \quad \text{weakly in } H^1_0(\Omega, |x|^{-2\beta}) \text{ and strongly in } L^2(\Omega, |x|^{-2\beta}),
\]

(3.28)

\[
u_n \rightarrow u \quad \text{a.e. in } \Omega.
\]

(3.29)
\[ \nabla u_n \rightarrow \nabla u \text{ weakly in } L^2(\Omega, |x|^{-2\beta}). \]  

(3.30)

So we have,

\[ \lim_{n \to \infty} \int_{\Omega} |x|^{-2\beta} \nabla u_n \nabla \varphi = \int_{\Omega} |x|^{-2\beta} \nabla u \nabla \varphi, \quad \forall \ \varphi \in H^1_0(|x|^{-2\beta}). \]  

(3.31)

Furthermore, we have, for \( \phi \in H^1_0(\Omega, |x|^{-2\beta}) \),

\[ 0 \leq \frac{\lambda g(x)\phi}{\left(\frac{1}{n} + u_n\right)\theta} \leq \frac{\lambda ||g||_{\infty}\phi}{(c_{\Omega})^\theta}, \]  

(3.32)

since, recalling (3.23) and (3.24), \( u_n \geq u_0 \geq u_1 \geq c_{\Omega} \) and \( 0 < \theta \leq 1 \).

By the Lebesgue dominated convergence theorem, we have

\[ \lim_{n \to \infty} \int_{\Omega} \frac{\lambda g(x)\phi}{\left(\frac{1}{n} + u_n\right)\theta} = \lambda \int_{\Omega} \frac{g(x)\phi}{u^\theta}. \]  

(3.33)

**Remark 3.1.** As in [1], the existence to (1.1) still holds for \( p > 1 \).

**Remark 3.2.** The similar proof as in the Theorem 1.1, will work to treat the following semilinear elliptic problem:

\[
\begin{align*}
& \begin{cases} 
- \operatorname{div}(M(x)\nabla u) - \mu \frac{g(x)u}{|x|^2} = \frac{\lambda f(x)}{|x|} + h(x)u^\theta & \text{in } \Omega, \\
\quad u > 0 & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases} \\
& \gamma < p < 1, \lambda > 0 \text{ small and } M \text{ be bounded elliptic matrix, i.e. } \exists 0 < \alpha < \beta \text{ such that } |\alpha \eta|^2 \leq |M(x)\eta|, |M(x)| \leq \beta.
\end{align*}
\]

(3.34)

As in Theorem 1.1, \( \exists \Lambda \in (0, \infty) \) such that for \( \Lambda \in (0, \Lambda) \) \( \exists \) a solution \( u \), strictly positive in \( \Omega \) of (3.34), in the following sense that

\[ \int_{\Omega} M(x)\nabla u \nabla \varphi - \mu \int_{\Omega} \frac{f(x)u\varphi}{|x|^2} = \lambda \int_{\Omega} \frac{g(x)\varphi}{u^\gamma} + \int_{\Omega} h(x)u^\theta, \quad \forall \varphi \in H^1_0(\Omega), \]  

(3.35)

where \( \Omega' \) is an arbitrary open subset of \( \Omega \), such that \( \Omega' \subset \Omega \).

Moreover if \( 0 < \gamma < 1, \) then \( u \in H^1_0(\Omega) \cap L^\infty(\Omega) \). For the sake of brevity, we omit the proof.

**Remark 3.3.** [8] The similar proof as in the Theorem 1.1, will work to treat the following semilinear elliptic problem:

\[
\begin{align*}
& \begin{cases} 
- \operatorname{div}(M(\xi)\nabla_{\xi} u) - \mu \frac{g(\xi)u}{(|\xi|^4 + t^2)^{1/2}} = \frac{\lambda f(\xi)}{|\xi|} + h(\xi)u^\theta & \text{in } \Omega, \\
\quad u > 0 & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases} \\
& \gamma < p < 1, \lambda > 0 \text{ small and } M \text{ be bounded elliptic matrix, i.e. } \exists 0 < \alpha < \beta \text{ such that } |\alpha \eta|^2 \leq |M(\xi)\eta|, |M(\xi)| \leq \beta.
\end{align*}
\]

(3.36)

As in Theorem 1.1, \( \exists \Lambda \in (0, \infty) \) such that for \( \Lambda \in (0, \Lambda) \) \( \exists \) a solution \( u \), strictly positive in \( \Omega \) of (3.36), in the following sense that

\[ \int_{\Omega} M(\xi)\nabla_{\xi} u \nabla_{\xi} \varphi - \mu \int_{\Omega} \frac{f(\xi)u\varphi}{(|\xi|^4 + t^2)^{1/2}} = \lambda \int_{\Omega} \frac{g(\xi)\varphi}{u^\gamma} + \int_{\Omega} h(\xi)u^\theta, \quad \forall \varphi \in H^1_0(\Omega, \mathbb{H}^n), \]  

(3.37)

where \( \Omega' \) is an arbitrary open subset of \( \Omega \), such that \( \Omega' \subset \Omega \).

Moreover if \( 0 < \gamma < 1, \) then \( u \in H^1_0(\Omega, \mathbb{H}^n) \cap L^\infty(\Omega) \). For the sake of brevity, we omit the proof.
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