EXISTENCE, UNIQUENESS AND EXPONENTIAL ERGODICITY UNDER
LYAPUNOV CONDITIONS FOR MCKEAN-VLASOV SDES WITH
MARKOVIAN SWITCHING

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Abstract. The paper is dedicated to studying the problem of existence and uniqueness of solutions as well as existence of and exponential convergence to invariant measures for McKean-Vlasov stochastic differential equations with Markovian switching. Since the coefficients are only locally Lipschitz, we need to truncate them both in space and distribution variables simultaneously to get the global existence of solutions under the Lyapunov condition. Furthermore, if the Lyapunov condition is strengthened, we establish the exponential convergence of solutions’ distributions to the unique invariant measure in Wasserstein quasi-distance and total variation distance, respectively. Finally, we give two applications to illustrate our theoretical results.

1. Introduction

Owing to the increasing demands on practical financial markets, ecological systems and social systems, much attention has been drawn to those processes which satisfy the McKean-Vlasov stochastic differential equation (MVSDE) with Markovian switching:

\begin{equation}
\begin{aligned}
dX_t &= b(t, X_t, \mathcal{L}_{X_t}, \alpha_t)dt + \sigma(t, X_t, \mathcal{L}_{X_t}, \alpha_t)dW_t \\
P(\alpha_{t+\Delta t} = j|\alpha_t = i, (X_s, \alpha_s), s \leq t) &= q_{ij}(X_t)\Delta t + o(\Delta t)
\end{aligned}
\end{equation}

for \( i \neq j \), where \( \mathcal{L}_{X_t} \) denotes the law of \( X_t \). A salient feature of such processes is the inclusion of the microcosmic site, the macrocosmic distribution of particles and the discrete event. For instance, the change rate of prices in a financial market may depend on the macrocosmic distribution, and may be very different for different time slots.

When the coefficients are Lipschitz and satisfy the linear growth condition, there are some related works on the stochastic system (1.1) with both McKean-Vlasov property (i.e. coefficients depending on the distribution) and Markovian switching property as follows. In [32], Zhang et al proved the existence and uniqueness of Markov regime switching mean-field type stochastic control systems with state-independent switching in a finite state space. Nguyen et al [18] showed that the limit of SDEs with mean-field interactions and Markovian switching is characterized as the stochastic McKean-Vlasov differential equation with Markovian switching in which the distribution term is actually the conditional distribution (given the history of the switching); the diffusion coefficient is assumed to be bounded. Nguyen et al [19] obtained existence and uniqueness for conditional-distribution dependent stochastic control systems with state-independent switching in a finite state space.

If \( b \) and \( \sigma \) do not depend on Markovian switching \( \alpha_t \), the equation (1.1) is called a McKean-Vlasov SDE or mean-field SDE. Such SDEs are used to study the interacting particle systems and
mean-field games. It was first studied by Kac [12] in the framework of the Boltzmann equation for the particle density in diluted monatomic gases, as well as in the stochastic toy model for the Vlasov kinetic equation for plasma. In [16], McKean studied the propagation of chaos in physical systems of \( N \)-interacting particles related to the Boltzmann equation for the statistical mechanics of rarefied gases. In [24, 25], Sznitman showed the propagation of chaos and the limit equation in a different framework. The limit equation can be described as an evolution equation known as the aforementioned MVSDE. The solution of a MVSDE is a “nonlinear” Markov process, whose transition function may not only depends on the current state but also on the current distribution. Due to its importance and reality, the MVSDE is studied extensively.

Larsy, Lions [13, 14, 15] and Huang, Malhame and Caines [9, 10] independently introduced mean-field games in order to study large population deterministic and stochastic differential games. Veretennikov [26] obtained the existence and uniqueness of invariant measures and weak convergence to invariant measures for McKean-Vlasov SDEs with additive noise. Butkovsky [5] considered ergodic properties of nonlinear Markov chains and McKean-Vlasov equations with additive noise. Buckdahn et al [3] established the relationship between the functionals of the form \( Ef(t, X_t, \mathcal{L}_{X_t}) \) and the associated second-order PDE, involving derivatives with respect to (w.r.t. in short) the law. In [28], Wang showed the well-posedness, existence and uniqueness of invariant measures under monotone conditions. Bogachev et al [4] obtained convergence in variation of probability measure solutions to stationary measures of nonlinear Fokker-Planck equations. Mishura and Veretennikov [17] established weak and strong existence/uniqueness results for solutions of multi-dimensional MVSDEs under relaxed regularity conditions. Barbu and Röckner [2] got the existence of weak solutions to MVSDEs using the superposition principle. Song [23] studied exponential ergodicity for MVSDEs with jumps. In [20], Ren et al proved the existence and uniqueness of solutions in infinite dimension under a Lyapunov condition (different from ours in the present paper).

Along another line, if \( b \) and \( \sigma \) do not depend on the distribution \( \mathcal{L}_{X_t} \), then (1.1) reduces to the so-called switching diffusion system, also known as hybrid switching system, which has gained increasing popularity because of its ability to handle numerous real-world applications in which continuous and discrete dynamics coexist and interact. The behavior of a diffusion process in different environments may be very different. Thus, it can provide more opportunity for realistic models. For instance, the work [1] of Barone-Adesi and Whaley is one of the early efforts using switching processes for financial applications, and in [30] an optimization problem leads to switching diffusion limits under suitable conditions. Yin and his cooperators have systematically studied switching diffusions, such as regularity, Feller property, recurrence, ergodicity and numerical approximation, see e.g. [31, 33]. In [31], they established the existence and uniqueness of solutions under the Lipschitz condition and Lyapunov condition respectively, and ergodicity using cycles and induced Markov chains which is similar to the classical situation. Cloez and Hairer [6] proved the ergodicity with state-dependent switching in a finite state space, using the weak form of Harris’ Theorem (Hairer et al [8]). In [21] Shao obtained the ergodicity with state-independent switching in both finite and infinite state spaces, and in [22] he got the existence and uniqueness of strong solutions with switching in an infinite state space.

The main purpose of our paper is to investigate the existence and uniqueness of solutions as well as exponential ergodicity for the equation (1.1), which we derive under Lyapunov type conditions in a unified way. Since the coefficients depend on the distribution of solutions which is a global property, the classical truncation in the space variable does not work in this situation. Following Ren et al [20], we need to truncate the equation in both space and distribution variables to overcome this difficulty. For the existence of and convergence to invariant measures, we do not appeal to the streamlined method of Hairer and Mattingly [7] which works for the
convergence in Wasserstein distance as well as total variation distance and is now widely adopted such as in Bogachev et al [4], Wang [29]. Instead, we use Lyapunov function itself to achieve the same goal, which we think is simple and interesting in its own right, and also consistent with our Lyapunov function method throughout the paper. Our method works also for the convergence in total variation distance, but the price we pay is that the convergence only works in Wasserstein quasi-distance instead of Wasserstein distance in [7]; see the comment following (H5) in Section 4 for details.

The rest of this paper is arranged as follows. In Section 2, we collect a number of preliminary results concerning switching, transition semigroup and optimal transportation cost. Section 3 presents existence and uniqueness under the Lyapunov condition. Section 4 establishes exponential convergence to invariant measures under the condition of integrable Lyapunov function, both in Wasserstein quasi-distance and weighted total variation distance. In Section 5, we provide two examples to illustrate our theoretical results.

2. Preliminary

Throughout the paper, let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a filtered complete probability space. We assume that the filtration \(\{\mathcal{F}_t\}\) satisfies the usual condition, i.e. it is right continuous and \(\mathcal{F}_0\) contains all \(P\)-null sets. Let \(W\) be an \(n\)-dimensional Brownian motion defined in \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\). We denote by \(B^\top\) the transpose of matrix \(B \in \mathbb{R}^{n \times n_2}\) with \(n_1, n_2 \geq 1\), \(tr(B)\) the trace of \(B\) and \(|B| := \sqrt{tr(B^\top B)}\) the norm of \(B\). Suppose that \(\alpha\) is a stochastic process with right-continuous sample paths, taking values in a finite set \(\mathcal{M} = \{1, 2, \cdots, m\}\), and having \(x\)-dependent generator \(Q = (q_{i,j}) : \mathbb{R}^d \to \mathbb{R}^{m \times m}\) such that for a suitable function \(V(\cdot, \cdot)\),

\[
Q(x)V(x, \cdot)(i) = \sum_{j \in \mathcal{M}} q_{i,j}(x)V(x, j) = \sum_{j \in \mathcal{M}, j \neq i} q_{i,j}(x)(V(x, j) - V(x, i)), \ x \in \mathbb{R}^d, i \in \mathcal{M}.
\]

We say \(Q\) satisfies the \(q\)-property, if \(q_{i,j}(\cdot)\) is Borel measurable, uniformly bounded, \(q_{i,j}(x) \geq 0\) for \(j \neq i\) and \(q_{i,i}(x) = -\sum_{j \neq i} q_{i,j}(x)\) for all \(i, j \in \mathcal{M}\) and \(x \in \mathbb{R}^d\). Assume that \((X_t, \mathcal{L}_t, \alpha_t)_{t \geq 0}\) is a triplet such that \(X_t\) is a continuous component taking values in \(\mathbb{R}^d\), \(\mathcal{L}_t\) denotes the distribution of \(X_t\) taking values in \(\mathcal{P}(\mathbb{R}^d)\) and \(\alpha_t\) is a jump component taking values in \(\mathcal{M}\), where \(\mathcal{P}(\mathbb{R}^d)\) is the space of probability measures on \(\mathbb{R}^d\). The process \((X_t, \alpha_t)\) can be described by the following MVSDE with switching:

\[
\begin{cases}
    dX_t = b(t, X_t, \mathcal{L}_t, \alpha_t)dt + \sigma(t, X_t, \mathcal{L}_t, \alpha_t)dW_t \\
    X_0 = \xi, \alpha_0 = \zeta,
\end{cases}
\]

and for \(i \neq j\),

\[
P(\alpha_{t+\Delta t} = j | \alpha_t = i, (X_s, \alpha_s), s \leq t) = q_{i,j}(X_t)\Delta t + o(\Delta t),
\]

where \(b : [0, \infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}^d\), \(\sigma : [0, \infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}^{d \times n}\), and \(\xi\) is \(\mathcal{F}_0\)-measurable and satisfies some integrable condition to be specified below. The MVSDE has a generator \(L\) given as follows. For each \(i \in \mathcal{M}\) and any twice continuously differentiable function \(V(\cdot, \cdot)\),

\[
LV(x, i) = \frac{1}{2} tr(\sigma \sigma^\top \nabla^2 V(x, i)) + b(t, x, \mu, i)\nabla V(x, i) + Q(x)V(x, \cdot)(i)
\]

where \(x \in \mathbb{R}^d\), \(\nabla^2 V(\cdot, i)\) and \(\nabla V(\cdot, i)\) denote the Hessian and gradient of \(V(\cdot, i)\) respectively.

Note that the evolution of the discrete component \(\alpha\) can be represented as a stochastic integral with respect to a Poisson random measure. Indeed, for any \(x \in \mathbb{R}^d\) and \(i, j \in \mathcal{M}\) with \(i \neq j\),
let $\Delta_{ij}(x)$ be consecutive (w.r.t. the lexicographic ordering on $\mathcal{M} \times \mathcal{M}$), left closed, right open interval of the real line, each having length $q_{ij}(x)$. Define a function $h: \mathbb{R}^d \times \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ by

$$h(x,i,z) := \begin{cases} j - i, & z \in \Delta_{ij}(x), \\ 0, & \text{else.} \end{cases}$$

Then it is equivalent to

$$d\alpha_t = \int_{\mathbb{R}} h(X_t, \alpha_t, z)p(dt,dz),$$

where $p(dt,dz)$ is a Poisson random measure with intensity $dt \times \tilde{m}(dz)$; here $\tilde{m}$ is the Lebesgue measure on $\mathbb{R}$. The poisson random measure $p(\cdot,\cdot)$ is independent of the Brownian motion $W$.

The generalized Itô’s formula reads

$$V(X_t,\alpha_t) - V(X_0,\alpha_0) = \int_0^t LV(X_s,\alpha_s)ds + M_1(t) + M_2(t),$$

where

$$M_1(t) = \int_0^t \langle \nabla V(X_s,\alpha_s), \sigma(s,X_s,\mathcal{L}_{X_s,\alpha_s})dW_s \rangle,$$

$$M_2(t) = \int_0^t \int_{\mathbb{R}} (V(X_s,\alpha_0 + h(X_s,\alpha_{s-},z)) - V(X_s,\alpha_s))\mu(ds,dz),$$

and

$$\mu(ds,dz) = p(ds,dz) - ds \times \tilde{m}(dz).$$

When strong existence and uniqueness of solutions holds for (2.1)-(2.2), the solution $(X_t,\alpha_t)_{t \geq s}$ is a Markov process which is determined by solving the equation from $s$ with initial value $(X_s,\alpha_s)$. More precisely, denote by $\{X_{s,t}^\alpha(\xi)\}_{t \geq s}$ the solution of the equation from $s$ with initial value $X_{s,s} = \xi, \alpha_s = \alpha$, then the uniqueness implies

(2.3)  
$$X_{s,t}^\alpha(\xi) = X_{r,t}^\alpha(X_{s,r}^\alpha(\xi)), \quad 0 \leq s \leq r \leq t.$$  

However, in general, the solution is not strong Markovian because we do not have $\mathcal{L}_{X_s} = \mathcal{L}_{X_t}$ on the set $\{\tau = t\}$ for a stopping time $\tau$ and $t \geq 0$. Moreover, the associated Markov operator $P_t$ given by

$$P_tf(x,\alpha) := Ef(X_t^\alpha(x),\alpha(t)), \quad \alpha \in \mathcal{M}, x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d \times \mathcal{M})$$

is not a semigroup, where $\mathcal{B}_b(\mathbb{R}^d \times \mathcal{M})$ denotes the space of bounded measurable functions on $\mathbb{R}^d \times \mathcal{M}$.

We will consider solutions of (2.1)-(2.2) with some integrable conditions, so let us introduce some basic notations as follows. Let $\mathcal{P}(\mathbb{R}^d \times \mathcal{M})$ be the space of probability measures on $\mathbb{R}^d \times \mathcal{M}$, and $\rho: (\mathbb{R}^d \times \mathcal{M}) \times (\mathbb{R}^d \times \mathcal{M}) \to \mathbb{R}^+$ be a distance-like function satisfying $\rho((x,i),(y,j)) = 0$ if and only if $x = y$, $i = j$. Denote $\mathcal{P}_\rho := \{\mu \in \mathcal{P}(\mathbb{R}^d \times \mathcal{M}) : \int_{\mathbb{R}^d \times \mathcal{M}} \rho((x,i),(0,1))\mu(dx \times \{i\}) < \infty\}$.

If the weak uniqueness holds for (2.1)-(2.2) in $\mathcal{P}_\rho$, we may define a semigroup $P^*_{s,t}$ on $\mathcal{P}_\rho$ by letting $P^*_{s,t}\mu := \mathcal{L}_{X_{s,t},\alpha_{s,t}}(\mu)$ for $\mathcal{L}_{X_{s,t},\alpha_{s,t}}(\mu)$. Then we have

$$P^*_{s,t} \mu = P^*_{r,t} P^*_{s,r} \mu \quad \text{for } 0 \leq s \leq r \leq t.$$  

Note that the semigroup $P^*_{s,t}$ is nonlinear, i.e.

$$P^*_{s,t}\mu \neq \int_{\mathbb{R}^d \times \mathcal{M}} (P^*_{s,t}\delta_{x,i})\mu(dx \times \{i\}), \quad 0 \leq s \leq t.$$  

In the time homogeneous case, i.e. $b$ and $\sigma$ do not depend on $t$, we have $P^*_{s,t} = P^*_{t-s}$ for $0 \leq s \leq t$. A measure $\mu \in \mathcal{P}_\rho$ is said to be invariant measure of $P^*_t$ if $P^*_t\mu = \mu$ for all $t \geq 0$, and the equation
is said to be ergodic if there exists $\mu \in \mathcal{P}_\rho$ such that $\lim_{t \to \infty} P_t^\nu \mu = \mu$ weakly for any $\nu \in \mathcal{P}_\rho$. It is obvious that ergodicity implies uniqueness of invariant measures.

We now introduce the Wasserstein quasi-distance based on $\rho$. For any $\mu, \nu \in \mathcal{P}_\rho$, let

\[
W_\rho(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathcal{M} \times (\mathbb{R}^d \times \mathcal{M})} \rho((x, i), (y, j)) \pi(dx \times \{i\}, dy \times \{j\}) = \inf \mathbb{E}\rho(X, Y),
\]

where $\mathcal{C}(\mu, \nu)$ is the set of couplings between $\mu$ and $\nu$, and the second infimum is taken over all random variables $X, Y$ on $\mathbb{R}^d \times \mathcal{M}$ whose laws are $\mu, \nu$ respectively. In general, $W_\rho$ is not a distance because the triangle inequality may not hold. But it is complete in the sense that any $W_\rho$–Cauchy sequence in $\mathcal{P}_\rho$ is convergent, i.e. for any Cauchy sequence $\{\mu_n\} \subset \mathcal{P}_\rho$, there exists a measure $\mu \in \mathcal{P}_\rho$ such that $W_\rho(\mu_n, \mu) \to 0$ as $n \to \infty$. When $\rho$ is a distance on $\mathbb{R}^d \times \mathcal{M}$, $W_\rho$ satisfies the triangle inequality and is hence a distance on $\mathcal{P}_\rho$.

We also use the usual Wasserstein distance $W_p$ on $\mathcal{P}(\mathbb{R}^d)$ with $p = 1, 2$ in what follows, i.e. $\mathcal{P}_p(\mathbb{R}^d) := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^p \mu(dx) < \infty\}$ and $W_p(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dx, dy)^{1/p}$ for $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$. This should not cause confusion with $W_\rho$ and $\mathcal{P}_\rho$ introduced above. As usual, we also denote $\mu(f) := \int_{\mathbb{R}^d} f(x) \mu(dx)$ in what follows for any function $f$ defined on $\mathbb{R}^d$ and $\mu \in \mathcal{P}(\mathbb{R}^d)$.

3. Existence and uniqueness of solutions

In this section, we consider the existence and uniqueness of the equation (2.1)–(2.2) under the Lyapunov function condition. Firstly, we consider the existence and uniqueness under Lipschitz and linear growth conditions.

**Theorem 3.1.** Suppose that $b : [0, \infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}^d$, and $\sigma : [0, \infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}^{d \times n}$ are measurable, and satisfy the following conditions: for each $t \in [0, \infty), \alpha \in \mathcal{M}, x, y \in \mathbb{R}^d, \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, there exist constants $K, L > 0$ such that

\[
\begin{align*}
|b(t, x, \mu, \alpha) - b(t, y, \nu, \alpha)| &\leq L |x - y| + W_2(\mu, \nu), \\
|\sigma(t, x, \mu, \alpha) - \sigma(t, y, \nu, \alpha)| &\leq L |x - y| + W_2(\mu, \nu), \\
|b(t, x, \mu, \alpha)| + |\sigma(t, x, \mu, \alpha)| &\leq K \left(1 + |x| + (\mu(\cdot, \cdot, \cdot)^2)^{1/2}\right).
\end{align*}
\]

The generator $Q = (q_{i,j}) : \mathbb{R}^d \to \mathbb{R}^{m \times m}$ is a bounded, continuous function and satisfy the q-property. Then for any $T > 0$, $\alpha \in \mathcal{M}$ and $X_0 \in L^2(\Omega, \mathcal{F}_0, P)$, (2.1) has a unique solution $(X_t, \alpha_t)$ with the given initial data in which the evolution of the jump process $\alpha_t$ is specified by (2.2) and $X_t$ satisfies

\[
E \sup_{0 \leq t \leq T} |X_t|^2 < \infty.
\]

**Proof.** This result is known well. So for brevity we only outline main steps.

1. Uniqueness. Suppose $(X_t, \mathcal{L}_X, \alpha_t)$ and $(Y_t, \mathcal{L}_Y, \tilde{\alpha}_t)$ are solutions. If $\alpha_t = \tilde{\alpha}_t$ a.s., the uniqueness follows from Itô’s formula and Gronwall’s inequality since the coefficients are Lipschitz. Otherwise, define $\tau := \inf\{t \geq 0 : \alpha_t = \tilde{\alpha}_t\}$. We can prove $\tau = \infty$ a.s. This proof is similar to the forthcoming Theorem 3.3 so we omit it.

2. Existence. Let $X_t^0 = X_0$, $\nu_t^0 = \mathcal{L} X_0$. For any $n \geq 1$, let $X_t^n$ solve the SDE with Markovian switching

\[
\begin{cases}
X_t^n = b(t, X_t^n, \mu_t^{n-1}, \alpha_t^n)dt + \sigma(t, X_t^n, \mu_t^{n-1}, \alpha_t^n)dW_t \\
X_0^n = X_0, \alpha_0^n = \alpha
\end{cases}
\]
and for \( i \neq j \),

\[
P(\alpha^n_{i+\Delta t} = j | \alpha^n_i = i, (X^n_s, \alpha^n_s), s \leq t) = q_{ij}(X^n_t) \Delta t + o(\Delta t).
\]

As the coefficients are Lipschitz and satisfy the linear growth condition, we can prove that

\[ E \sup_{0 \leq t \leq T} |X^n_t|^2 < \infty \]

and \( \{X^n_t\} \) is a Cauchy sequence, and hence has a limit \( X_t \) in the space \( C([0, T]) \) as \( n \to \infty \), which is a solution.

Now we introduce some assumptions for the equation \((2.1)-(2.2)\).

(H1) For any \( N \geq 1 \), \( \alpha \in M \), there exists a constant \( C_N \geq 0 \) such that for any \( |x|, |y| \leq N \) and \( \sup \mu, \sup \nu \in B(0, N) \) we have

\[
|b(t, x, \mu, \alpha)| + |\sigma(t, x, \mu, \alpha)| \leq C_N,
\]

\[
|b(t, x, \mu, \alpha) - b(t, y, \nu, \alpha)| + |\sigma(t, x, \mu, \alpha) - \sigma(t, y, \nu, \alpha)| \leq C_N(|x - y| + W_2(\mu, \nu)).
\]

Here \( B(0, N) \) denotes the closed ball in \( \mathbb{R}^d \) centered at the origin with radius \( N \).

(H2) (Lyapunov function) There exists a function \( V : \mathbb{R}^d \times M \to \mathbb{R}^+ \) that is twice continuously differentiable with respect to \( x \in \mathbb{R}^d \) for each \( i \in M \) such that there exist constants \( \lambda_1, \lambda_2 \in \mathbb{R} \) satisfying for all \( (t, x, \mu, i) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times M \)

\[
(LV)(t, x, \mu, i) \leq \lambda_1 V(x, i) + \lambda_2 \int_{\mathbb{R}^d} \varphi(x)\mu(dx),
\]

\[
V_R := \inf_{|x| \geq R, i \in M} V(x, i) \to \infty \text{ as } R \to \infty,
\]

where function \( \varphi : \mathbb{R}^d \to \mathbb{R}^+ \) satisfying \( \varphi(x) \leq V(x, i) \) for all \( x \in \mathbb{R}^d, i \in M \).

(H3) (Continuity) For any \( \alpha \in M \) and bounded sequences \( \{x_n, \mu_n\} \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \) with \( x_n \to x \) and \( \mu_n \to \mu \) weakly in \( \mathcal{P}(\mathbb{R}^d) \) as \( n \to \infty \), we have

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} |b(t, x_n, \mu_n, \alpha) - b(t, x, \mu, \alpha)| + |\sigma(t, x_n, \mu_n, \alpha) - \sigma(t, x, \mu, \alpha)| = 0.
\]

where \( \mathcal{P}(\mathbb{R}^d) := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} V(x, i)\mu(dx) < \infty, \forall i \in M\} \).

(H4) There exist constants \( K, \epsilon > 0 \) and increasing unbounded function \( L : \mathbb{N} \to (0, \infty) \) such that for any \( \alpha \in M, \ N \geq 1, |x| \vee |y| \leq N \) and \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \) satisfying

\[
|b(t, x, \mu, \alpha) - b(t, y, \nu, \alpha)| + |\sigma(t, x, \mu, \alpha) - \sigma(t, y, \nu, \alpha)| \leq L_N(|x - y| + W_2(\mu, \nu) + K e^{-\epsilon L_N(1 \wedge W_2(\mu, \nu))}),
\]

where

\[
W^2_{2,N}(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |\phi_N(x) - \phi_N(y)|^2 \pi(dx, dy), \phi_N(x) := \frac{N x}{N \vee |x|}.
\]

Remark 3.2. If the function \( L : \mathbb{N} \to (0, \infty) \) in (H4) is bounded, i.e. \( b \) and \( \sigma \) are globally Lipschitz, then \( K \) should be 0. This then reduces to the the case of Theorem 3.1 so we assume that the function \( L \) is unbounded in (H4).

Theorem 3.3. Assume (H1)-(H3). Then for any \( T > 0, X_0 \in L^2(\Omega, \mathcal{F}_0, P) \) and \( \alpha_0 \in M, \)

\[ EV(X_t, \alpha_t) \leq e^{(\lambda_1 + \lambda_2)t} EV(X_0, \alpha_0), \quad \text{for } t \geq 0. \]

Moreover, if (H4) holds, then the solution is unique.
Proof. (i) Existence.

1. In order to construct a solution using Theorem 3.1 we take a sequence of truncations of \( b \) and \( \sigma \) as follows. For any \( n \geq 1, t \in [0, T], x \in \mathbb{R}^d, \mu \in \mathcal{P}(\mathbb{R}^d), \alpha \in \mathcal{M}, \) define
\[
b^n(t, x, \mu, \alpha) := b(t, \phi_n(x), \mu \circ \phi_n^{-1}, \alpha),
\]
\[
\sigma^n(t, x, \mu, \alpha) := \sigma(t, \phi_n(x), \mu \circ \phi_n^{-1}, \alpha).
\]
For each \( n \geq 1, \) \( b^n \) and \( \sigma^n \) are Lipschitz and satisfy the linear growth condition. Therefore, by Theorem 3.1 the equation
\[
(3.1) \quad \begin{cases}
X^n_t = b^n(t, X^n_t, \mathcal{L}_{X^n_t}, \alpha^n_t)dt + \sigma^n(t, X^n_t, \mathcal{L}_{X^n_t}, \alpha^n_t)dW_t \\
X^n_0 = X_0, \alpha^n_0 = \alpha_0
\end{cases}
\]
and for \( i \neq j, \)
\[
(3.2) \quad P(\alpha^n_{t+\Delta t} = j|\alpha^n_t = i, (X^n_s, \alpha^n_s), s \leq t) = q_{ij}(X^n_t)\Delta t + o(\Delta t)
\]
has a unique solution \((X^n_t, \alpha^n_t)\). Define \( \tau^n := \inf\{t \geq 0 : |X^n_t| \geq n\}. \) By the definition of \( \phi_n, \) we have
\[
\phi_n(X^n_t) = \frac{X^n_t \cdot n}{|X^n_t| \lor n} = X^n_{t \lor \tau^n}.
\]
Moreover, for any measurable set \( A \subset \mathbb{R}^d, \) we obtain
\[
(\mathcal{L}_{X^n_t}) \circ \phi_n^{-1}(A) = P(X^n_t \in \phi_n^{-1}(A)) = P(\phi_n(X^n_t) \in A) = \mathcal{L}_{\phi_n}(X^n_t)(A) = \mathcal{L}_{X^n_{t \lor \tau^n}}(A).
\]
So the equation \((3.1)-(3.2)\) becomes
\[
(3.3) \quad \begin{cases}
X^n_t = b(t, X^n_{t \lor \tau^n}, \mathcal{L}_{X^n_{t \lor \tau^n}}, \alpha^n_t)dt + \sigma(t, X^n_{t \lor \tau^n}, \mathcal{L}_{X^n_{t \lor \tau^n}}, \alpha^n_t)dW_t \\
X^n_0 = X_0, \alpha^n_0 = \alpha
\end{cases}
\]
and for \( i \neq j, \)
\[
(3.4) \quad P(\alpha^n_{t+\Delta t} = j|\alpha^n_t = i, (X^n_s, \alpha^n_s), s \leq t) = q_{ij}(X^n_t)\Delta t + o(\Delta t).
\]
2. Applying Itô’s formula to \( V(X^n_t, \alpha^n_t) \), we have
\[
V(X^n_t, \alpha^n_t) - V(X^n_0, \alpha^n_0)
= \int_0^t L^n V(X^n_s, \alpha^n_s)ds + \int_0^t \nabla V \sigma^n(s, X^n_s, \mathcal{L}_{X^n_s}, \alpha^n_s)dW_s
+ \int_0^t \int_\mathbb{R} V(X^n_s, \alpha_0 + h(X^n_s, \alpha^n_s, z)) - V(X^n_s, \alpha^n_s)\mu(ds, dz),
\]
where \( L^n \) represents the generator corresponding to the \( n \)-th equation for \((X^n, \alpha^n)\). Thus, taking expectation on both sides, we get
\[
EV(X^n_{t \lor \tau^n}, \alpha^n_{t \lor \tau^n}) - EV(X^n_0, \alpha^n_0)
= E \int_0^{t \lor \tau^n} L^n V(X^n_s, \alpha^n_s)ds
= E \int_0^t 1_{[0,\tau^n]}(s)L^n V(X^n_s, \alpha^n_s)ds
= E \int_0^t 1_{[0,\tau^n]}(s)\nabla V(X^n_{s \lor \tau^n}, \alpha^n_{s \lor \tau^n})b^n(s, X^n_{s \lor \tau^n}, \mathcal{L}_{X^n_{s \lor \tau^n}}, \alpha^n_{s \lor \tau^n})ds
+ \frac{1}{2} E \int_0^t 1_{[0,\tau^n]}(s)\nabla^2 V(X^n_{s \lor \tau^n}, \alpha^n_{s \lor \tau^n})A^n(s, X^n_{s \lor \tau^n}, \mathcal{L}_{X^n_{s \lor \tau^n}}, \alpha^n_{s \lor \tau^n})ds.
\]
\[ + E \int_0^t 1_{[0,\tau^n]}(s)Q(X^n_{s\wedge\tau^n})V(X^n_{s\wedge\tau^n},::)(\alpha^n_{s\wedge\tau^n})ds \]
\[ = E \int_0^t 1_{[0,\tau^n]}(s)\nabla V(X^n_{s\wedge\tau^n},\alpha^n_{s\wedge\tau^n})b(s, X^n_{s\wedge\tau^n}, \mathcal{L}X^n_{s\wedge\tau^n}, \alpha^n_{s\wedge\tau^n})ds \]
\[ + \frac{1}{2} E \int_0^t 1_{[0,\tau^n]}(s)\nabla^2 V(X^n_{s\wedge\tau^n},\alpha^n_{s\wedge\tau^n})A(s, X^n_{s\wedge\tau^n}, \mathcal{L}X^n_{s\wedge\tau^n}, \alpha^n_{s\wedge\tau^n})ds \]
\[ + E \int_0^t 1_{[0,\tau^n]}(s)Q(X^n_{s\wedge\tau^n})V(X^n_{s\wedge\tau^n},::)(\alpha^n_{s\wedge\tau^n})ds \]
\[ = E \int_0^t 1_{[0,\tau^n]}(s)LV(X^n_{s\wedge\tau^n},\alpha^n_{s\wedge\tau^n})ds \]
\[ \leq E \int_0^t 1_{[0,\tau^n]}(s)[\lambda_1 V(X^n_{s\wedge\tau^n},\alpha^n_{s\wedge\tau^n}) + \lambda_2 EV(X^n_{s\wedge\tau^n})]ds \]
\[ \leq \int_0^t 1_{[0,\tau^n]}(s)(\lambda_1 + \lambda_2)EV(X^n_{s\wedge\tau^n},\alpha^n_{s\wedge\tau^n})ds, \]
where
\[ A^n(s, x, \mu, \alpha) := \sigma^n(s, x, \mu, \alpha)\sigma^n(s, x, \mu, \alpha)^{\top}, \]
\[ A(s, x, \mu, \alpha) := \sigma(s, x, \mu, \alpha)\sigma(s, x, \mu, \alpha)^{\top}. \]

Applying Gronwall’s inequality, we get
\[ EV(X^n_{t\wedge\tau^n},\alpha^n_{t\wedge\tau^n}) \leq e^{(\lambda_1 + \lambda_2)t} EV(X^n_0,\alpha^n_0) \]
\[ \leq e^{(\lambda_1 + \lambda_2)T} EV(X^n_0,\alpha^n_0) := \delta. \]

Denote \( \tau^n_N := \inf\{t \geq 0 : |X^n_t| \geq N\} \), \( n \geq N \geq 1 \) and let \( t = T \wedge \tau^n_N \), we have
\[ EV(X^n_{T\wedge\tau^n_N},\alpha^n_{T\wedge\tau^n_N}) \leq e^{(\lambda_1 + \lambda_2)T} EV(X^n_0,\alpha^n_0) := \delta. \]

Consequently, \( \tau^n_N \) satisfies
\[ P(\tau^n_N < T) \leq \frac{\delta}{V(N, \alpha^n_{T\wedge\tau^n_N})}. \]

3. Let \( l \geq 1 \) to be determined. By (H1) and BDG’s inequality, there exists a constant \( C(N, l) > 0 \) such that for any \( n \geq N \) we have
\[ E\left( \sup_{t \in [s,(s+\epsilon)\wedge T]} |X^n_{t\wedge\tau^n_N} - X^n_{s\wedge\tau^n_N}|^2 \right) \]
\[ = E\left( \sup_{t \in [s,(s+\epsilon)\wedge T]} \left( \int_{s\wedge\tau^n_N}^{t\wedge\tau^n_N} b^n(r, X^n_r, \mathcal{L}X^n_r, \alpha^n_r)dr + \int_{s\wedge\tau^n_N}^{t\wedge\tau^n_N} \sigma^n(r, X^n_r, \mathcal{L}X^n_r, \alpha^n_r)dW_r \right)^2 \right) \]
\[ \leq C(l) E\left[ \sup_{t \in [s,(s+\epsilon)\wedge T]} C_N c_2^l + \sup_{t \in [s,(s+\epsilon)\wedge T]} \left( \int_{s\wedge\tau^n_N}^{t\wedge\tau^n_N} |\sigma^n(r, X^n_r, \mathcal{L}X^n_r, \alpha^n_r)|^2 dr \right)^l \right] \]
\[ \leq C(N, l)c^l. \]

Let \( k = \lceil \frac{L}{\epsilon} \rceil + 1 \) where \( \lfloor a \rfloor \) denotes the integer part of \( a \in \mathbb{R} \). Then we obtain
\[ E\left( \sup_{s,t \in [0,T],|t-s| \leq \epsilon} |X^n_{t\wedge\tau^n_N} - X^n_{s\wedge\tau^n_N}|^2 \right) \]
\[ \leq C(l) \sum_{j=1}^{k} E( \sup_{t \in [j-1] \cup \{j\} \cup [1, T]} |X_{t \wedge \tau_N}^n - X_{s \wedge \tau_N}^n|^2 ) \]

\[ \leq C(N, l) (T + \epsilon) \epsilon^{l-1}. \]

Therefore, by Hölder's inequality we have

\[ E( \sup_{s, t \in [0, T], |t-s| \leq \epsilon} |X_{t \wedge \tau_N}^n - X_{s \wedge \tau_N}^n|) \leq (C(N, l) (T + \epsilon))^{\frac{1}{2}} \epsilon^{\frac{1}{2} - \frac{1}{2} l}. \]

Taking \( l = 2 \), we get

\[ (3.5) \quad E( \sup_{s, t \in [0, T], |t-s| \leq \epsilon} |X_{t \wedge \tau_N}^n - X_{s \wedge \tau_N}^n|) \leq (C(N) (T + \epsilon))^{\frac{1}{2}} \epsilon^{\frac{1}{2}}. \]

When \( n = N, \tau_N^n = \tau_N^N = \tau_N \). By Arzela-Ascoli type theorem for measures, the sequence \( \{\mu^n := L_{X_{t \wedge \tau_N}^n}\} \) is tight in \( \mathcal{P}(C[0, T]) \). Therefore, by the Prokhorov theorem, there exists a subsequence, still denoted \( \{\mu^n\} \), such that \( \mu^n \rightarrow \mu \) weakly in \( \mathcal{P}(C[0, T]) \) as \( n \rightarrow \infty \).

4. Define \( \Upsilon_{N, m} := \tau_N \wedge \tau_N^m \). Then for any \( m \geq n \geq N \)

\[ (3.6) \quad \phi_N(X_{t \wedge \Upsilon_{N, m}}^j) = X_{t \wedge \Upsilon_{N, m}}^j, \quad j \in \{n, m\} \]

and

\[ (3.7) \quad \lim_{n \rightarrow \infty} \sup_{m \geq n} \mu^n \circ \phi^{-1}_m = \mu \text{ weakly in } \mathcal{P}(\mathbb{R}^d). \]

By Cauchy-Schwarz inequality and BDG's inequality, we arrive

\[ E( \sup_{0 \leq s \leq t} |X_{s \wedge \tau_N}^n - X_{s \wedge \tau_N^m}|^2 ) \]

\[ \leq 2E \left( \sup_{0 \leq s \leq t} \left| \int_0^{s \wedge \tau_N^n} (b^n(r, X^m_r, \mathcal{L}_{X^m_r}, \alpha^m_r) - b^m(r, X^m_r, \mathcal{L}_{X^m_r}, \alpha^m_r))dr \right|^2 \right) \]

\[ + 2E \left( \sup_{0 \leq s \leq t} \left| \int_0^{s \wedge \tau_N^m} (\sigma^n(r, X^m_r, \mathcal{L}_{X^m_r}, \alpha^m_r) - \sigma^m(r, X^m_r, \mathcal{L}_{X^m_r}, \alpha^m_r))dW_r \right|^2 \right) \]

\[ \leq 2E \int_0^{t \wedge \tau_N^n} |b^n(r, X^m_r, \mathcal{L}_{X^m_r}, \alpha^m_r) - b^m(r, X^m_r, \mathcal{L}_{X^m_r}, \alpha^m_r)|^2 dr \]

\[ + CE \int_0^{t \wedge \tau_N^m} |\sigma^n(r, X^m_r, \mathcal{L}_{X^m_r}, \alpha^m_r) - \sigma^m(r, X^m_r, \mathcal{L}_{X^m_r}, \alpha^m_r)|^2 dr \]

\[ = 2E \int_0^{t \wedge \tau_N^n} |b(r, X^m_r, \mu^m_r, \alpha^m_r) - b(r, X^m_r, \mu^m_r, \alpha^m_r)|^2 dr \]

\[ + CE \int_0^{t \wedge \tau_N^m} |\sigma(r, X^m_r, \mu^m_r, \alpha^m_r) - \sigma(r, X^m_r, \mu^m_r, \alpha^m_r)|^2 dr \]

\[ \leq 4E \int_0^{t \wedge \tau_N^n} |b(r, X^m_r, \mu^m_r, \alpha^m_r) - b(r, X^m_r, \mu^m_r, \alpha^m_r)|^2 dr \]

\[ + 4E \int_0^{t \wedge \tau_N^m} |b(r, X^m_r, \mu^m_r, \alpha^m_r) - b(r, X^m_r, \mu^m_r, \alpha^m_r)|^2 dr \]

\[ + CE \int_0^{t \wedge \tau_N^m} |\sigma(r, X^m_r, \mu^m_r, \alpha^m_r) - \sigma(r, X^m_r, \mu^m_r, \alpha^m_r)|^2 dr \]
By (3.6), (3.7) and (H1), there exists a family of constants \( \{ \epsilon_{n,m} : m \geq n \geq 1 \} \) with \( \epsilon_{n,m} \to 0 \) as \( n \to \infty \) such that

\[
|b(t, X_{t \wedge T}^{n,m}, \mu_{t}, \alpha_{t}^{n}) - b(t, X_{t \wedge T}^{m,m}, \mu_{t}^{m}, \alpha_{t}^{m})| \\
\leq |b(t, X_{t \wedge T}^{n,m}, \mu_{t}^{n} \circ \phi_{n}^{-1}, \alpha_{t}^{n}) - b(t, X_{t \wedge T}^{m,m}, \mu_{t}^{n} \circ \phi_{n}^{-1}, \alpha_{t}^{n})| \\
+ |b(t, X_{t \wedge T}^{m,m}, \mu_{t}^{n} \circ \phi_{n}^{-1}, \alpha_{t}^{n}) - b(t, X_{t \wedge T}^{m,m}, \mu_{t}^{m}, \alpha_{t}^{m})| \\
\leq C_{N} \cdot |X_{t \wedge T}^{n,m} - X_{t \wedge T}^{m,m}| + C_{N} \cdot \epsilon_{n,m}
\]

when \( n \geq N \). Next, we treat the term with different switching. Partition the interval \([0, T]\) by \( \epsilon_{n,m} \) (for short \( \epsilon \)). We obtain

\[
\begin{align*}
E & \int_{0}^{T} |b(r, X_{r \wedge T}^{m,m}, \mu_{r}^{m}, \alpha_{r}^{m}) - b(r, X_{r \wedge T}^{m,m}, \mu_{r}, \alpha_{r}^{m})|^2 \, dr \\
& \leq E \sum_{k=0}^{[T/\epsilon]} \int_{k \epsilon}^{(k+1) \epsilon} 1_{[0, \tau_{N}^{n,m}]}(r) |b(r, X_{r \wedge T}^{m,m}, \mu_{r}^{m}, \alpha_{r}^{m}) - b(r, X_{r \wedge T}^{m,m}, \mu_{r}, \alpha_{r}^{m})|^2 \, dr \\
& \quad + E \sum_{k=0}^{[T/\epsilon]} \int_{k \epsilon}^{(k+1) \epsilon} 1_{[0, \tau_{N}^{n,m}]}(r) |b(r, X_{r \wedge T}^{m,m}, \mu_{r}^{m}, \alpha_{r}^{m}) - b(r, X_{r \wedge T}^{m,m}, \mu_{r}, \alpha_{r}^{m})|^2 \, dr \\
& \quad + E \sum_{k=0}^{[T/\epsilon]} \int_{k \epsilon}^{(k+1) \epsilon} 1_{[0, \tau_{N}^{n,m}]}(r) |b(r, X_{r \wedge T}^{m,m}, \mu_{r}^{m}, \alpha_{r}^{m}) - b(r, X_{r \wedge T}^{m,m}, \mu_{r}, \alpha_{r}^{m})|^2 \, dr.
\end{align*}
\]

For the first term of the right-hand side of (3.8), by the local Lipschitz continuity of coefficient \( b \) we have

\[
\sum_{k=0}^{[T/\epsilon]} \int_{k \epsilon}^{(k+1) \epsilon} 1_{[0, \tau_{N}^{n,m}]}(r) |b(r, X_{r \wedge T}^{m,m}, \mu_{r}^{m}, \alpha_{r}^{m}) - b(r, X_{r \wedge T}^{m,m}, \mu_{r}, \alpha_{r}^{m})|^2 \, dr \\
\leq \sum_{k=0}^{[T/\epsilon]} \int_{k \epsilon}^{(k+1) \epsilon} 1_{[0, \tau_{N}^{n,m}]}(r) C_{N}^{2} |X_{r \wedge T}^{m,m} - X_{r \wedge T}^{m,m}|^2 \, dr \\
\leq \sum_{k=0}^{[T/\epsilon]} C(N) \epsilon^{2} \leq C(N, T) \epsilon.
\]

In the same way, for the last term of (3.8) we get

\[
E \sum_{k=0}^{[T/\epsilon]} \int_{k \epsilon}^{(k+1) \epsilon} 1_{[0, \tau_{N}^{n,m}]}(r) |b(r, X_{r \wedge T}^{m,m}, \mu_{r}^{m}, \alpha_{r}^{m}) - b(r, X_{r \wedge T}^{m,m}, \mu_{r}, \alpha_{r}^{m})|^2 \, dr \leq C(N, T) \epsilon.
\]

As for the second term of the right-hand side of (3.8), we have for each \( k = 0, 1, ..., [T/\epsilon] \)

\[
E \int_{k \epsilon}^{(k+1) \epsilon} 1_{[0, \tau_{N}^{n,m}]}(r) |b(r, X_{r \wedge T}^{m,m}, \mu_{r}^{m}, \alpha_{r}^{m}) - b(r, X_{r \wedge T}^{m,m}, \mu_{r}, \alpha_{r}^{m})|^2 \, dr
\]
\begin{align}
(3.9) \quad \leq& 2E \int_{k\epsilon}^{(k+1)\epsilon} 1_{[0,\tau_n^{-}\eta]}(r) |b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k}) - b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k})|^2 dr \\
& + 2E \int_{k\epsilon}^{(k+1)\epsilon} 1_{[0,\tau_n^{-}\eta]}(r) |b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k}) - b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k})|^2 dr.
\end{align}

For the second term of the right-hand side of (3.9), we get

\begin{align}
E \int_{k\epsilon}^{(k+1)\epsilon} 1_{[0,\tau_n^{-}\eta]}(r) |b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k}) - b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k})|^2 dr \\
= E \sum_{j \neq i, i \in \mathcal{M}} \int_{k\epsilon}^{(k+1)\epsilon} 1_{[0,\tau_n^{-}\eta]}(r) |b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k}) - b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k})|^2 dr \\
\leq 4C_N^2 E \sum_{i \in \mathcal{M}} \int_{k\epsilon}^{(k+1)\epsilon} 1_{\alpha_k^{m} = i} \left( \sum_{j \neq i} qij(X_{k\epsilon}^{m})(r - k\epsilon) + o(r - k\epsilon) \right) dr \\
\leq C(N, M) \epsilon^2,
\end{align}

where $M$ denotes the bound of $Q$. To treat the first term of the right-hand side of (3.9), we use the technique of basic coupling of Markov processes. Denote by $\hat{Q}(x_1, x_2) := (\hat{q}(i, j, k)(x_1, x_2))$ the basic coupling of $Q(x_1)$ and $Q(x_2)$, which satisfies

\begin{align}
\hat{Q}(x_1, x_2) f(k, l) &= \sum_{(j, i) \in \mathcal{M} \times \mathcal{M}} \hat{q}(i, j, k)(x_1, x_2) (f(j, i) - f(k, l)) \\
&= \sum_{j \in \mathcal{M}} (q_{kj}(x_1) - q_{ij}(x_2))^+ (f(j, l) - f(k, l)) \\
&\quad + \sum_{j \in \mathcal{M}} (q_{ij}(x_2) - q_{kj}(x_1))^+ (f(k, j) - f(k, l)) \\
&\quad + \sum_{j \in \mathcal{M}} (q_{kj}(x_1) \land q_{ij}(x_2))(f(j, j) - f(k, l))
\end{align}

for any function $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$. Consequently, let $(\alpha_k^n, \alpha_k^m)$ be a stochastic process on a finite state space $\mathcal{M} \times \mathcal{M}$ with generator $\hat{Q}(x_1, x_2)$. Then for any $i_1, i_2, j \in \mathcal{M}$ with $j \neq i_2$, $r \in [k\epsilon, k\epsilon + \epsilon)$ we have

\begin{align}
E(1_{\alpha_k^n = j}) |\alpha_k^m = i_1, \alpha_k^m = i_2, X_{k\epsilon}^{n} = x_1, X_{k\epsilon}^{m} = x_2) \\
= \sum_{l \in \mathcal{M}} E(1_{\alpha_k^n = j}) 1_{\alpha_k^m = l}) |\alpha_k^n = i_1, \alpha_k^m = i_2, X_{k\epsilon}^{n} = x_1, X_{k\epsilon}^{m} = x_2) \\
= \sum_{l \in \mathcal{M}} \hat{q}(i_1, i_2, l, j)(x_1, x_2)(r - k\epsilon) + o(r - k\epsilon) \leq m\hat{M} \epsilon,
\end{align}

where $\hat{M}$ denotes the bound of $\hat{Q}$. Thus, for the first term of (3.9) we obtain

\begin{align}
E \int_{k\epsilon}^{(k+1)\epsilon} 1_{[0,\tau_n^{-}\eta]}(r) |b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k}) - b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k})|^2 dr \\
= E \sum_{j \neq i, i \in \mathcal{M}} \int_{k\epsilon}^{(k+1)\epsilon} 1_{[0,\tau_n^{-}\eta]}(r) |b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k}) - b(r, X_{k\epsilon}^{m, \mu_r, \alpha_k})|^2 1_{\alpha_k^n = j} 1_{\alpha_k^m = i} dr
\end{align}
Then it follows that

\[ \lim_{n \to \infty} X^n = x \]

Therefore, for any \( \epsilon > 0 \)

Now we arrive

Similarly, we have for any \( m \geq n \geq N \)

So we obtain

Replacing this with (3.10), for any \( N \geq 1, \epsilon > 0 \) we have

Letting \( N \to \infty \), we get that \( X^n \) converges to a process \( X \), in probability uniformly in \([0, T]\). Therefore, there exists a subsequence, still denoted \( \{X_n\} \), such that \( P\text{-a.s.} \)

\[ \lim_{n \to \infty} \sup_{0 \leq t \leq T} |X^n_t - X_t| = 0. \]
Especially, $\mathcal{L}_{X_n} \to \mathcal{L}_{X_1}$ weakly in $\mathcal{P}(C[0, T])$. By the uniqueness of limit, we have $\mathcal{L}_{X_1} = \mu_\tau$, $t \in [0, T]$. Therefore, combining this with (H1) and (H3), we let $n \to \infty$ in (3.1)–(3.2) to conclude that $X$ satisfies

$$X_t = X_0 + \int_0^t b(s, X_s, \mathcal{L}_{X_s}, \alpha_s)ds + \int_0^t \sigma(s, X_s, \mathcal{L}_{X_s}, \alpha_s)dW_s,$$

and for $i \neq j$,

$$P(\alpha_{t+\Delta t} = j|\alpha_t = i, (X_s, \alpha_s), s \leq t) = q_{ij}(X_t)\Delta t + o(\Delta t).$$

5. By Itô’s formula and (H2), we have

$$E[V(X_t, \alpha_t) - V(X_0, \alpha_0)] = E \int_0^t \mathcal{L}(X_s, \alpha_s)ds \leq (\lambda_1 + \lambda_2)E \int_0^t V(X_s, \alpha_s)ds.$$

The estimate mentioned in the theorem now follows from Gronwall’s inequality.

(ii). Uniqueness.

Assume that $(X_t, \alpha_t)$ and $(Y_t, \bar{\alpha}_t)$ are two solutions with the same initial value.

1. If $\alpha_t = \bar{\alpha}_t$, $t \geq 0, a.s.$

We first prove the pathwise uniqueness up to a time $t_0 \in [0, T]$. Define the stopping time

$$\tau_n := \tau_n^X \wedge \tau_n^Y = \inf\{t \geq 0 : |X(t)| \vee |Y(t)| \geq n\}, \quad n \geq 1.$$

Then by (H4) and BDG’s inequality we have

$$E|X(t \wedge \tau_n) - Y(t \wedge \tau_n)|^2 \leq 2TE \int_0^{t \wedge \tau_n} |b(s, X_s, \mathcal{L}_{X_s}, \alpha_s) - b(s, Y_s, \mathcal{L}_{Y_s}, \alpha_s)|^2ds + C\int_0^{t \wedge \tau_n} |\sigma(s, X_s, \mathcal{L}_{X_s}, \alpha_s) - \sigma(s, Y_s, \mathcal{L}_{Y_s}, \alpha_s)|^2ds$$

$$\leq (2L_nT + C \cdot L_n)E \int_0^{t \wedge \tau_n} |X_s - Y_s|^2 + W_{2n}(\mathcal{L}_{X_s}, \mathcal{L}_{Y_s})^2 + Ke^{-L_n\epsilon}ds$$

$$\leq 2(L_nT + C \cdot L_n)E \int_0^{t \wedge \tau_n} |X_{s \wedge \tau_n} - Y_{s \wedge \tau_n}|^2 + Ke^{-L_n\epsilon}ds.$$

Applying Gronwall’s inequality, we get

$$E|X(t \wedge \tau_n) - Y(t \wedge \tau_n)|^2 \leq e^{2L_nT + C \cdot L_n}2(L_nT + C \cdot L_n)TKe^{-L_n\epsilon}$$

$$= 2(L_nT + L_n)TKe^{-L_n(\epsilon - 2(T+1)t)},$$

Therefore, letting $n \to \infty$ and using Fatou’s lemma, we get the uniqueness up to the time

$$t_0 := \frac{\epsilon}{2(T+1)} \wedge T.$$

If $t_0 = T$, the proof is finished. Otherwise, because of $X_{t_0} = Y_{t_0}$, we can use the same method to prove that the uniqueness holds up to the time $2t_0 \wedge T$. Repeating this procedure, we can prove the uniqueness up to the time $T$.

2. If $\alpha_t$ and $\bar{\alpha}_t$ are not equal almost surely. Define $\tau := \inf\{t \geq 0 : \alpha_t \neq \bar{\alpha}_t\}$, we want to prove $\tau = \infty$ a.s. Obviously, this is equivalent to $\tau \wedge N = N$ for any $N > 0$. Let $\eta := \tau \wedge N$ and $E := \{\omega : \eta(\omega) < N\}$.

Claim: $P(E) = 0$.

Indeed, if $P(E) > 0$, then for a.s. $\omega \in E$ we have

$$X_s(\omega) = Y_s(\omega), \alpha_s(\omega) = \bar{\alpha}_s(\omega), \forall s \leq \tau(\omega) < N.$$
Let $\eta_\alpha = \inf \{ s > \eta : \alpha_s \neq \alpha_\eta \}$, $\eta_\tilde{\alpha} = \inf \{ s > \eta : \tilde{\alpha}_s \neq \tilde{\alpha}_\eta \}$. By the definition of $\eta_\alpha, \eta_\tilde{\alpha}$ and $\eta$, we have $\eta_\alpha \geq \eta, \eta_\tilde{\alpha} \geq \eta$ and there exists $\delta > 0$ such that
\[
\inf_{x \in \mathbb{R}^d, i \in \mathcal{M}} P(\eta_\alpha > \eta + \delta : \alpha_\eta = i, X_\eta = x) \geq 1 - \frac{1}{4} P(B),
\]
\[
\inf_{y \in \mathbb{R}^d, i \in \mathcal{M}} P(\eta_\tilde{\alpha} > \eta + \delta : \tilde{\alpha}_\eta = i, Y_\eta = y) \geq 1 - \frac{1}{4} P(B).
\]
Therefore, we get
\[
P(\eta_\alpha > \eta + \delta) = \int_{\mathbb{R}^d \times \mathcal{M}} P(\eta_\alpha > \eta + \delta : \alpha_\eta = i, X_\eta = x) P((X_\eta, \alpha_\eta) \in (dx, di)) \geq 1 - \frac{1}{4} P(B).
\]
In the same way, we have
\[
P(\eta_\tilde{\alpha} > \eta + \delta) \geq 1 - \frac{1}{4} P(B).
\]
Thus, we arrive
\[
P(\{\eta_\tilde{\alpha} > \eta + \delta \} \cap B) \geq P(\eta_\tilde{\alpha} > \eta + \delta) - P(B^c) \geq \frac{3}{4} P(B) > 0.
\]
Moreover, we obtain
\[
P(\{\eta_\alpha > \eta + \delta \} \cap \{\eta_\tilde{\alpha} > \eta + \delta \} \cap B) \geq 1 - \frac{1}{4} P(B) - (1 - \frac{3}{4}) P(B) = \frac{1}{2} P(B) > 0.
\]
Define $\tilde{\eta} := \min \{\eta_\alpha, \eta_\tilde{\alpha}\}$ and $\tilde{\tau} := \tilde{\eta} 1_{\tau \leq M} + \zeta 1_{\tau > M}$. Then we get
\[
P(\{\tilde{\tau} > \tau \} \cap B) \geq P(\{\tilde{\eta} > \eta + \delta \} \cap B) > 0,
\]
and if $\tau \leq M$, we have $\tilde{\tau} = \tilde{\eta} \geq \tau = \gamma = \tau \wedge M$. Therefore, there exists a subset $A$ of $B$ such that $\tau < \tilde{\tau}$ and $\alpha_t = \tilde{\alpha}_t$ for any $t \leq \tilde{\tau}$. This contradicts the definition of $\tau$. The proof is complete.

Remark 3.4. (i) When the Lyapunov function $V$ is independent of switching, we can choose $\varphi = V$ in (H2).

(ii) When $\mathcal{M} = \{1\}$, i.e. there is no switching in the equation (2.1)–(2.2). Comparing with Ren at al [20], it seems that our Lyapunov function condition is simpler; note also that their Lyapunov function cannot grow faster than $|x|^2$, while our condition has no this kind of restriction. By taking $|x|^2$ as the Lyapunov function in our Theorem 3.3, our result reduces to that of Hu [11, Theorem 2.1].

4. Invariant measures and exponential convergence

In this section, we investigate long time behaviors of solutions to (2.1)–(2.2), i.e. the existence and uniqueness of invariant measures and exponential convergence to them. We divide this section into two parts: $\mathcal{M} = \{1\}$ and $\mathcal{M} = \{1, 2, ..., m\}$. 


4.1. The MVSDE case. We first consider the special case $\mathcal{M} = \{1\}$, i.e. MVSDEs.

(H5) (Integrable Lyapunov condition) There exists a function $\tilde{V} : \mathbb{R}^d \to \mathbb{R}^+$ which is twice continuously differentiable w.r.t. $x \in \mathbb{R}^d$ and satisfies $\tilde{V}(x) = 0$ if and only if $x = 0$, such that there is a constant $\gamma > 0$ satisfying for each $\pi \in \mathcal{C}(\mu, \nu),$

\begin{equation}
\int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{V}(x-y)\pi(dx,dy) \leq -\gamma \int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{V}(x-y)\pi(dx,dy),
\end{equation}

where $\tilde{L}$ is defined by

$$\tilde{L}V(x-y) := (b(t,x,\mu,1) - b(t,y,\nu,1))\nabla \tilde{V}(x-y) + \frac{1}{2} tr(\nabla^2 \tilde{V}(x-y)A(t,x,y,\mu,\nu,1)),$$

with $A(t,x,y,\mu,\nu,1) = (\sigma(t,x,\mu,1) - \sigma(t,y,\nu,1))(\sigma(t,x,\mu,1) - \sigma(t,y,\nu,1))^\top$.

The function $\tilde{V}$ induces naturally a Wasserstein quasi-distance which is given by

$$W_{\tilde{V}}(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{V}(x-y)\pi(dx,dy) \quad \text{for } \mu, \nu \in \mathcal{P}_{\tilde{V}},$$

where $\mathcal{P}_{\tilde{V}} := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \mu(\tilde{V}) < \infty\}$. In general, $W_{\tilde{V}}$ is not a distance because the triangle inequality may not hold. But it is complete in the sense that any $W_{\tilde{V}}$-Cauchy sequence in $\mathcal{P}_{\tilde{V}}$ is convergent. When $d(x,y) := \tilde{V}(x-y)$ is a distance on $\mathbb{R}^d$, $W_{\tilde{V}}$ satisfies the triangle inequality and is hence a distance on $\mathcal{P}_{\tilde{V}}$. In what follows, we will study the exponential ergodicity under this distance, which is simple and different from that of Hairer and Mattingly [7]. They used a Lyapunov function to construct a family of distances on both the state space and the probability measure space to conclude the exponential ergodicity in total variation distance, which is now extensively adopted.

We have the following result on invariant measures and exponential convergence for MVSDEs.

**Theorem 4.1.** Assume that (H1)-(H5) hold and $\mathcal{M} = \{1\}$.

(i) For any initial measures $\mu_0, \nu_0 \in \mathcal{P}_{\tilde{V}}$, we have for $t \geq 0$

$$W_{\tilde{V}}(P^*_t \mu_0, P^*_t \nu_0) \leq e^{-\gamma t} W_{\tilde{V}}(\mu_0, \nu_0),$$

(ii) If the coefficients $b$ and $\sigma$ are independent of $t$ and there exists $\nu_0 \in \mathcal{P}_{\tilde{V}}$ such that

\begin{equation}
\sup_{t \geq 0} W_{\tilde{V}}(P^*_t \nu_0, \nu_0) < \infty,
\end{equation}

then there exists a unique invariant measure $\mu_{\tilde{I}} \in \mathcal{P}_{\tilde{V}}$ such that

$$W_{\tilde{V}}(P^*_t \mu_0, \mu_{\tilde{I}}) \leq e^{-\gamma t} W_{\tilde{V}}(\mu_0, \mu_{\tilde{I}}) \quad \text{for } t \geq 0, \quad \mu_0 \in \mathcal{P}_{\tilde{V}}.$$

**Proof.** (i). For any initial measures $\mu_0, \nu_0 \in \mathcal{P}_{\tilde{V}}(\mathbb{R}^d)$. Let $X_t$ and $Y_t$ be two solutions such that $\mathcal{L}_{X_0} = \mu_0, \mathcal{L}_{Y_0} = \nu_0$, and

$$W_{\tilde{V}}(\mu_0, \nu_0) = E\tilde{V}(X_0 - Y_0).$$

Denote $\mu_t = \mathcal{L}_{X_t}, \nu_t = \mathcal{L}_{Y_t}$. By Itô’s formula and (H5), we have

$$E\tilde{V}(X_t - Y_t) = E\tilde{V}(X_0 - Y_0) + E \int_0^t \tilde{L}\tilde{V}(X_s - Y_s)ds \leq E\tilde{V}(X_0 - Y_0) - \gamma E \int_0^t \tilde{V}(X_s - Y_s)ds.$$

Applying Gronwall’s inequality, we get

$$E\tilde{V}(X_t - Y_t) \leq e^{-\gamma t} E\tilde{V}(X_0 - Y_0).$$
Thus,
\[ W_{\tilde{V}}(\mu, \nu) \leq e^{-\gamma t} W_{\tilde{V}}(\mu_0, \nu_0). \]

(ii). We first prove that \( \{P_t^s\nu_0\} \) is a \( W_{\tilde{V}} \)-Cauchy sequence. Indeed, from (i) we know that
\[ W_{\tilde{V}}(P_t^s \nu_0, P_{t+s}^s \nu_0) \leq e^{-\gamma t} W_{\tilde{V}}(\nu_0, P_s^s \nu_0) . \]
Thus, by (4.2) we obtain
\[ \lim_{t \to \infty} \sup_{s \geq 0} W_{\tilde{V}}(P_t^s \nu_0, P_{t+s}^s \nu_0) = 0. \]

Since \( P_{\tilde{V}} \) is complete w.r.t. \( W_{\tilde{V}} \), there exists a measure \( \mu_\Xi \in P_{\tilde{V}} \) such that
\[ \lim_{t \to \infty} W_{\tilde{V}}(P_t^s \nu_0, \mu_\Xi) = 0. \]

Consequently, by Lemma 4.2 in Villani [27], we have for any \( t \geq 0 \)
\[ W_{\tilde{V}}(P_t^s \mu_\Xi, \mu_\Xi) \leq \lim_{s \to \infty} W_{\tilde{V}}(P_t^s P_s^s \nu_0, \mu_\Xi) = 0. \]
That is, \( \mu_\Xi \) is an invariant measure. Therefore, by (i) for any \( \mu_0 \in P_{\tilde{V}} \), we have
\[ W_{\tilde{V}}(P_t^s \mu_0, \mu_\Xi) \leq e^{-\gamma t} W_{\tilde{V}}(\mu_0, \mu_\Xi). \]

The proof is complete. \( \square \)

**Remark 4.2.** (i) The condition (4.2) means that there is a “bounded orbit” in \( P_{\tilde{V}} \), which is necessary and natural because the system cannot have an invariant measure if any orbit is unbounded. Note by Theorem 4.1 (i) that existence of one “bounded orbit” implies the boundedness of all the orbits in \( P_{\tilde{V}} \).

(ii) If inequality (4.1) in (H5) is replaced by
\[ \tilde{L}_{\tilde{V}}(x - y) \leq -\gamma_1 \tilde{V}(x - y) + \gamma_2 W_{\tilde{V}}(\mu, \nu) \]
with \( \gamma_1 > \gamma_2 > 0 \), it is immediate to see that the results of Theorem 4.1 are still valid.

(iii) By taking \( \tilde{V}(\cdot) = |\cdot|^2 \) in (H5), our result Theorem 4.1 reduces to that of Hu [11, Theorem 4.1], which in turn is a type of generalization of Wang [28, Theorem 3.1]. Wang’s results considered the exponential ergodicity under the Lyapunov and monotone conditions; note that the diffusion coefficient in [29] requires to be non-degenerate and independent of the distribution, while our results do not need these assumptions.

As a direct consequence of Theorem 4.1, we have

**Corollary 4.3.** Under the conditions of Theorem 4.1, for any measures \( \mu_0, \nu_0 \in P_{\tilde{V}} \) we have
\[ \|P_t^s \mu_0 - P_t^s \nu_0\|_{\text{Var}, \tilde{V}} \to 0, \text{ as } t \to \infty. \]
And there exists a unique invariant measure \( \mu_\Xi \in P_{\tilde{V}} \) such that for any measure \( \mu_0 \in P_{\tilde{V}} \),
\[ \|P_t^s \mu_0 - \mu_\Xi\|_{\text{Var}, \tilde{V}} \to 0, \text{ as } t \to \infty. \]

Here,
\[ \|\mu - \nu\|_{\text{Var}, \tilde{V}} := \sup_{|f| \leq \tilde{V}, f \in C_b} |\mu(f) - \nu(f)| \text{ for } \mu, \nu \in P_{\tilde{V}}. \]

**Proof.** According to the Kantorovich duality (see e.g. [27]), we have
\[ \|\mu - \nu\|_{\text{Var}, \tilde{V}} \leq \sup_{\phi, \psi \in C_b} (\nu(\phi) - \mu(\psi)) = W_{\tilde{V}}(\mu, \nu) \]
for any \( \mu, \nu \in P_{\tilde{V}} \). Combining this with Theorem 4.1, the result immediately follows. \( \square \)
4.2. The case of MVSDEs with switching. Next, we consider MVSDEs with Markovian switching, i.e. \( \mathcal{M} = \{1, 2, \ldots, m\} \). For each fixed environment \( i \in \mathcal{M} \), the corresponding diffusion process \( X_t^{(i)} \) is defined by
\[
dX_t^{(i)} = b(t, X_t^{(i)}, \mathcal{L}_{X_t}, i)dt + \sigma(t, X_t^{(i)}, \mathcal{L}_{X_t}, i)dW_t.
\]
Note that it should be \( \mathcal{L}_{X_t} \) instead of \( \mathcal{L}_{X_t^{(i)}} \) in above equation. Let \( Y_t^{(i)} \) be defined the same as \( X_t^{(i)} \) and denote by \( \tilde{L}^{(i)} \) the infinitesimal generator of \( X_t^{(i)} - Y_t^{(i)} \), i.e. for any twice continuously differentiable function \( f : \mathbb{R}^d \to \mathbb{R}^{+} \)
\[
\tilde{L}^{(i)} f(x-y) := (b(t, x, \mu, i) - b(t, y, \nu, i))\nabla f(x-y) + \frac{1}{2} tr(\nabla^2 f(x-y)A(t, x, y, \mu, i))
\]
with \( A(t, x, y, \mu, i) = (\sigma(t, x, \mu, i) - \sigma(t, y, \nu, i)) (\sigma(t, x, \mu, i) - \sigma(t, y, \nu, i))^\top \).

(H6) (Integrable Lyapunov condition) There exists a function \( \hat{V} : \mathbb{R}^d \to \mathbb{R}^+ \), which is twice continuously differentiable with respect to \( x \in \mathbb{R}^d \), \( \hat{V}(x) = 0 \) iff \( x = 0 \) and \( \hat{V}(x-y) \leq K \max\{\hat{V}(x), \hat{V}(y)\} \) for some constant \( K \) and all \( x, y \in \mathbb{R}^d \), such that there is a constant \( \theta > 0 \) satisfying for each \( \pi \in \mathcal{C}(\mu, \nu), i \in \mathcal{M} \),
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{L}^{(i)} \hat{V}(x-y)\pi(dx, dy) \leq -\theta \int_{\mathbb{R}^d \times \mathbb{R}^d} \hat{V}(x-y)\pi(dx, dy).
\]
Let
\[
d((x, i), (y, j)) := \sqrt{1 + \hat{V}(x-y)}, \quad x, y \in \mathbb{R}^d, \quad i, j \in \mathcal{M},
\]
\[
\mathcal{P}_d := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d \times \mathcal{M}) : \int_{\mathbb{R}^d \times \mathcal{M}} d((x, i), (0, 1)) \mu(dx \times \{i\}) < \infty \right\}.
\]
Define a Wasserstein quasi-distance on \( \mathcal{P}_d \) by
\[
W_d(\mu, \nu) := \inf \text{Ed}(X, Y) \quad \text{for } \mu, \nu \in \mathcal{P}_d,
\]
where the infimum is taken over all random variables \( X, Y \) on \( \mathbb{R}^d \times \mathcal{M} \) whose laws are \( \mu, \nu \) respectively. It is complete in the space \( \mathcal{P}_d \), i.e. any \( W_d \)-Cauchy sequence in \( \mathcal{P}_d \) converges w.r.t. \( W_d \). Note that \( W_d \) is a distance on \( \mathcal{P}_d \) when \( d \) is a distance on \( \mathbb{R}^d \times \mathcal{M} \). In particular, when the mapping \( (x, y) \mapsto \hat{V}(x-y) \) is a distance on \( \mathbb{R}^d \), \( d \) is a distance on \( \mathbb{R}^d \times \mathcal{M} \) and \( \hat{V}(x-y) \leq 2 \max\{\hat{V}(x), \hat{V}(y)\} \).

Denote
\[
\mathcal{P}_V := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d \times \mathcal{M}) : \int_{\mathbb{R}^d \times \mathcal{M}} \hat{V}(x) \mu(dx \times \{i\}) < \infty \right\}.
\]

Theorem 4.4. Assume that (H1)–(H4) and (H6) hold, \( Q(x) \equiv Q \) and for any \( \nu \in \mathcal{P}_V \) we have
\[
\sup_{t \geq 0} (P_t^\nu(V)) < \infty.
\]
Then there exists a constant \( \bar{\theta} > 0 \) such that for any initial measures \( \mu_0, \nu_0 \in \mathcal{P}_V \) we have
\[
W_d(P_t^\nu \mu_0, P_t^\nu \nu_0) \leq Ce^{-\bar{\theta}t}, \quad t \geq 0
\]
for some constant \( C = C(\mu_0, \nu_0) \). In particular, if the coefficients \( b \) and \( \sigma \) are independent of \( t \), then there exists a unique invariant measure \( \mu_\pi \in \mathcal{P}_V \) such that for any \( \mu_0 \in \mathcal{P}_\pi \) we have
\[
W_d(P_t^\pi \mu_0, \mu_\pi) \leq Ce^{-\bar{\theta}t}, \quad t \geq 0
\]
with \( C = C(\mu_0) \).
Proof. (i). Suppose that \((X_t, \alpha_t)\) and \((Y_t, \tilde{\alpha}_t)\) are solutions whose initial distributions are \(\mu_0, \nu_0\) respectively. Denote \(\mu_t = \mathcal{L}(X_t, \alpha_t), \nu_t = \mathcal{L}(Y_t, \tilde{\alpha}_t)\).

1. We first consider the special case \(\alpha_0 = \tilde{\alpha}_0\) a.s. Then \(\alpha_t = \tilde{\alpha}_t\) a.s. In this case, by (H6) and Itô’s formula we get

\[
E\hat{V}(X_t - Y_t) = E\hat{V}(X_0 - Y_0) + E \int_0^t \hat{L}^{(\alpha)} \hat{V}(X_s - Y_s) ds
\]

\[
\leq E\hat{V}(X_0 - Y_0) - \theta E \int_0^t \hat{V}(X_s - Y_s) ds.
\]

Applying Gronwall’s inequality, we obtain

\[
(4.4) \quad E\hat{V}(X_t - Y_t) \leq E\hat{V}(X_0 - Y_0)e^{-\theta t}.
\]

Thus, by Jensen’s inequality we have

\[
W_d(\mu_t, \nu_t) \leq Ed((X_t, \alpha_t), (Y_t, \tilde{\alpha}_t)) = E\sqrt{\hat{V}(X_t - Y_t)}
\]

\[
\leq \sqrt{E\hat{V}(X_t - Y_t)} \leq \sqrt{E\hat{V}(X_0 - Y_0)e^{-\theta t}}
\]

\[
\leq e^{-\frac{\theta t}{4}} KEV(X_0) + KE\hat{V}(Y_0).
\]

2. If \(\alpha_0 = \tilde{\alpha}_0\) a.s. does not hold, define \(\tau := \inf\{t \geq 0 : \alpha_t = \tilde{\alpha}_t\}\). Recall that if \(\alpha_t\) and \(\tilde{\alpha}_t\) are two independent finite-state Markov chains with generator \(Q\), then there exist constants \(C_c, \theta_c > 0\) such that

\[
P(\tau > t) \leq C_c e^{-\theta_c t}, \quad \forall t \geq 0.
\]

Thus, by Hölder’s inequality, Jensen’s inequality and (4.4) there exists a constant \(C > 0\) such that

\[
Ed((X_t, \alpha_t), (Y_t, \tilde{\alpha}_t)) = E \left( \sqrt{1_{\tau > \frac{t}{2}}} + \hat{V}(X_t - Y_t) \cdot 1_{\tau > \frac{t}{2}} \right) + E \left( \sqrt{\hat{V}(X_t - Y_t)} \cdot 1_{\tau \leq \frac{t}{2}} \right)
\]

\[
\leq \sqrt{P(\tau > \frac{t}{2})} \cdot \sqrt{E(1 + \hat{V}(X_t - Y_t)) + E(\hat{V}(X_t - Y_t)1_{\tau \leq \frac{t}{2}})}
\]

\[
\leq \sqrt{C_c e^{-\frac{\theta_c t}{4}}} \sqrt{1 + K \max\{E\hat{V}(X_t), E\hat{V}(Y_t)\} + \sqrt{E(\hat{V}(X_t - Y_t)|\mathcal{F}_\tau)1_{\tau \leq \frac{t}{2}}} + \sqrt{E(\hat{V}(X_t - Y_t))1_{\tau \leq \frac{t}{2}}} + \sqrt{E(\hat{V}(X_t - Y_t))e^{-\frac{\theta t}{2}}} \}
\]

\[
\leq C e^{-\hat{\theta} t},
\]

where \(\hat{\theta} = \frac{1}{4}(\theta \wedge \theta_c)\).

(ii). The proof is completely similar to that of Theorem 4.1 (ii), so we omit it.

\[
\Box
\]

Remark 4.5. We have the following comments on Theorem 4.4

(i) If inequality (4.3) in (H6) is replaced by

\[
\hat{L}^{(i)} \hat{V}(x - y) \leq -\theta_1 \hat{V}(x - y) + \theta_2 W(\mu, \nu)
\]

with \(\theta_1 > \theta_2 > 0\), the results are still valid.

(ii) The condition \(\sup_{t \geq 0}(P_t^\nu(\hat{V})) < \infty\) for any \(\nu \in \mathcal{P}_\hat{V}\) means all the orbits in \(\mathcal{P}_\hat{V}\) are bounded, which is natural and necessary to guarantee that the system is ergodic in \(\mathcal{P}_\hat{V}\).
(iii) We assume \( \mu_0 \in \mathcal{P}_V \) instead of \( \mu_0 \in \mathcal{P}_d \) since \( \mu_0 \in \mathcal{P}_d \) does not guarantee \( \mu_0(\hat{V}) < \infty \).

(iv) In [31], Yin and Zhu showed the ergodicity for SDE with Markovian switching using the classical Khasminskii’s method. But in the present paper, the solution of MVSDE with Markovian switching is not strong Markovian, so the classical Khasminskii’s method does not apply; on the other hand, the corresponding Fokker-Planck equation is nonlinear, so the classical Krylov-Bogolyubov argument for the existence of invariant measures is invalid, either.

As a direct consequence of Theorem 4.4, we have the following corollary whose proof is omitted since it is similar to that of Corollary 4.3.

**Corollary 4.6.** Under the conditions of Theorem 4.4, for any measures \( \mu_0, \nu_0 \in \mathcal{P}_V \), we have
\[
\|P_t^\mu \mu_0 - P_t^\nu \nu_0\|_{\text{Var},d} \to 0, \quad \text{as } t \to \infty.
\]
And there exist a unique invariant measure \( \mu_2 \in \mathcal{P}_V \) such that for any measure \( \mu_0 \in \mathcal{P}_V \),
\[
\|P_t^\mu \mu_0 - \mu\|_{\text{Var},d} \to 0, \quad \text{as } t \to \infty.
\]
Here \( \|\mu - \nu\|_{\text{Var},d} := \sup_{|f| \leq d, f \in C_0} |\mu(f) - \nu(f)| \) for \( \mu, \nu \in \mathcal{P}_d \).

5. Applications

In this section, we provide two examples to illustrate our results.

**Example 5.1.** For each \( x \in \mathbb{R}, \mu \in \mathcal{P}(\mathbb{R}) \) and \( i \in \{1, 2\} \), consider
\[
b(x, \mu, 1) = -x^3 - 2 \int_\mathbb{R} (x + \beta y)\mu(dy), \quad b(x, \mu, 2) = -2x,
\]
\[
\sigma(x, \mu, 1) = \int_\mathbb{R} (x + \beta y)\mu(dy), \quad \sigma(x, \mu, 2) = x,
\]
where \( \beta \in \mathbb{R} \). Then the following results hold: (i) the SDE
\[
dX_t = b(X_t, \mathcal{L}_X, \alpha_t)dt + \sigma(X_t, \mathcal{L}_X, \alpha_t)dW_t
\]
has a unique solution for any \( \beta \in \mathbb{R} \) and when \( E|X_0|^2 < \infty \) we have
\[
E|X_t|^2 \leq e^{(-2 + 2\beta^2)t}E|X_0|^2, \quad \text{for } t \geq 0.
\]
(ii) If there is no switching and \( \beta \in (-1, 1) \), there exists a unique invariant measure to which the solutions’ distributions are exponentially convergent under \( W_2 \) and \( \|\cdot\|_{\text{Var},d} \). Moreover, if the switching’s generator is state-independent and \( \beta \in (-1, 1) \), there exists a unique invariant measure to which the solutions’ distributions are exponentially convergent under \( W_d \) and \( \|\cdot\|_{\text{Var},d} \), where \( d((x, i)(y, j)) = \sqrt{1_{i \neq j} + |x - y|^2} \) for \( x, y \in \mathbb{R}, i, j \in \{1, 2\} \).

**Proof.** (i) It is immediate to see that (H1), (H3) and (H4) hold. We now check the assumptions (H2), (H5) and (H6). Let \( V(x, i) = x^2 \) for \( x \in \mathbb{R} \) and \( i = 1, 2 \). By Itô’s formula and Cauchy-Schwarz inequality, we have
\[
LV(x, 1) = \sigma^2(x, \mu, 1) + b(x, \mu, 1) \cdot 2x
\]
\[
\leq \left( \int_\mathbb{R} (x + \beta y)\mu(dy) \right)^2 - 4x \left( \int_\mathbb{R} (x + \beta y)\mu(dy) \right)
\]
\[
= -3x^2 - 2\beta x \int_\mathbb{R} y\mu(dy) + \beta^2 \left( \int_\mathbb{R} y\mu(dy) \right)^2
\]
\[
\leq -2x^2 + 2\beta^2 \int_\mathbb{R} x^2\mu(dx),
\]
And in the same way we get
\[ LV(x, 2) = -3x^2. \]
Thus, in this example, \( \varphi(x) = V(x) = x^2 \) for \( x \in \mathbb{R} \), \( \lambda_1 = -2 \), \( \lambda_2 = 2\beta^2 \), i.e. (H2) holds. Therefore, by Theorem 3.3 there exists a unique solution \((X_t, \alpha_t)\) and we have
\[ E|X_t|^2 \leq e^{(-2+2\beta^2)t} E|X_0|^2, \quad \text{for} \ t \geq 0. \]

(ii) By Itô’s formula and Cauchy-Schwarz inequality, we have
\[
\int_{\mathbb{R} \times \mathbb{R}} \tilde{L}^{(1)} V(x-y) \pi(dx, dy) \leq -2 \int_{\mathbb{R} \times \mathbb{R}} |x-y|^2 \pi(dx, dy) + 2\beta^2 \int_{\mathbb{R}} x\mu(dx) - \int_{\mathbb{R}} y\nu(dy)^2 \\
\leq (-2 + 2\beta^2) \int_{\mathbb{R} \times \mathbb{R}} |x-y|^2 \pi(dx, dy),
\]
\[
\tilde{L}^{(2)} V(x-y) = 2\langle -2x + 2y, x-y \rangle + |x-y|^2 \\
= -3|x-y|^2.
\]

Thus, \( \tilde{V}(x) = \hat{V}(x) = x^2 \) for \( x \in \mathbb{R} \) and \( \gamma = \theta = 2 - 2\alpha^2 \), i.e. (H5) and (H6) hold. Therefore, by Theorem 4.1, Corollary 4.3, Theorem 4.4 and Corollary 4.6 we get the desired results. \( \square \)

**Example 5.2.** Assume that for each \( x \in \mathbb{R} \), \( \mu \in \mathcal{P}(\mathbb{R}) \) and \( i \in \{1, 2\} \),
\[
b(x, \mu, 1) = -x^3 - x, \quad b(x, \mu, 2) = -\frac{1}{2}x,
\]
\[
\sigma(x, \mu, 1) = \int_\mathbb{R} x\mu(dx), \quad \sigma(x, \mu, 2) = x + 2 \int_\mathbb{R} x\mu(dx).
\]
\( \alpha(t) \) is a two-state random jump process with \( x \)-dependent generator
\[
\begin{pmatrix}
-\frac{1}{3} - \frac{1}{4} \cos x & \frac{1}{3} + \frac{1}{4} \cos x \\
\frac{1}{3} + \frac{1}{2} \sin x & -\frac{1}{3} - \frac{1}{2} \sin x
\end{pmatrix}.
\]

Then the following results hold: (i) there exists a unique solution \((X_t, \alpha_t)\) and when \( E|X_0| < \infty \) we have
\[ E|X_t| \leq e^{-\frac{\theta}{12}t} E|X_0|, \quad \text{for} \ t \geq 0. \]

(ii) When there is no switching, there exists a unique invariant measure to which the solutions’ distributions are exponentially convergent under \( W_1 \) and \( ||\cdot||_{\text{Var}.,t} \). When the generator of switching is state-independent, we obtain a unique invariant measure to which the solutions’ distributions are exponentially convergent under \( W_d \) and \( ||\cdot||_{\text{Var},d} \) where \( d((x, i)(y, j)) = \sqrt{1 + |x-y|} \) for \( x, y \in \mathbb{R}, \ i, j \in \{1, 2\} \).

**Proof.** (i) The coefficients \( b \) and \( \sigma \) clearly satisfy (H1), (H3) and (H4). Consider the Lyapunov function
\[ V(x, 1) = |x|, \quad V(x, 2) = 2|x| \]
for \( x \in \mathbb{R} \). Then we have
\[
LV(x, 1) = \text{sign} x \cdot (-x^3 - x) + \left( \frac{1}{3} + \frac{1}{4} \cos x \right)(2 - 1)|x| \\
\leq -|x| + \frac{7}{12}|x| = -\frac{5}{12}|x| = -\frac{5}{12} V(x, 1), \\
LV(x, 2) = 2\text{sign} x \times (-\frac{1}{2}x) + \left( \frac{7}{3} + \frac{1}{2} \sin x \right)(1 - 2)|x|
\]
\[ \leq -|x| - \frac{11}{6}|x| = -\frac{17}{12}V(x, 2). \]

Thus, in this example, \( \varphi(x) = |x| \) for \( x \in \mathbb{R} \), \( \lambda_1 = -\frac{5}{12}, \lambda_2 = 0 \), i.e. (H2) holds. Therefore, by Theorem 3.3 there exists a unique solution \((X_t, \alpha_t)\) and we have

\[ E|X_t| \leq e^{-\frac{5}{12}t}E|X_0|, \quad \text{for} \quad t \geq 0. \]

(ii) Let \( \tilde{V}(x) = \hat{V}(x) = |x| \) for \( x \in \mathbb{R} \). In the same way, we obtain

\[ \tilde{L}(1)\tilde{V}(x - y) = \text{sign}(x - y) \cdot (-x^3 - x + y^3 + y) \leq -|x - y| = -\tilde{V}(x - y), \]

\[ \tilde{L}(2)\tilde{V}(x - y) = \text{sign}(x - y) \left(-\frac{1}{2}x + \frac{1}{2}y\right) = -\frac{1}{2}|x - y| = -\frac{1}{2}\tilde{V}(x - y), \]

where \( \pi \in \mathcal{C}(\mu, \nu) \). Thus, \( \gamma = 1, \theta = \frac{1}{2}, \) i.e. (H5) and (H6) hold. Therefore, by Theorem 4.1, Corollary 4.3, Theorem 4.4 and Corollary 4.6, we get the desired results. \( \square \)

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