Proving Unsolvability of Set Agreement Task with Epistemic $\mu$-Calculus

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Two methods for task unsolvability

 отметить

❖ Topological method
  * Model: Simplicial complexes.
  * Strategy: Find a breach in topological invariant.
  * Method: Tools from combinatorial topology.

❖ Logical method [Goubault-Ledent-Rajsbaum2021]
  * Model: (Simplicial) Kripke models.
  * Strategy: Find a logic formula (logical obstruction) that is inconsistent between the models.
  * Method: Epistemic logic reasoning
Unsolvability of 1-set agreement (Topology)

$I$

$P = IS$ (standard chromatic subdivision)

$T$

simplicial map $\mu$

immediate snapshot
Unsolvability of 1-set agreement (Topology)

General case argues higher dimensional connectivity, resorting to tools from combinatorial topology (Sperner’s lemma).

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Two methods for task unsolvability

- **Topological method**
  - Model: Simplicial complexes.
  - Strategy: Find a breach in topological invariant.
  - Method: Tools from combinatorial topology.

- **Logical method** [Goubault-Ledent-Rajsbaum2021]
  - Model: (Simplicial) Kripke models.
  - Strategy: Find a logic formula (called logical obstruction) that is inconsistent between the models.
  - Method: Epistemic logic reasoning
Task solvability in simplicial Kripke model

Every map \( f : I \rightarrow O \) over simplicial complexes induces a product update model \( I[O] \), a binary relation encoding of \( f \).

Every product update model \( I[O] \) is a simplicial complex, which induces a simplicial Kripke model for epistemic reasoning.
If there exists a positive epistemic formula $\varphi$ and facet $X \in I[P]$ such that, for any $\delta$: $I[P] \rightarrow I[T]$, $I[P], X \not\models \varphi$ but $I[T], \delta(X) \models \varphi$, then the task is not solvable (i.e., there is no $\delta$).
Pros and cons of logical method

😊 Just find a logical obstruction $\varphi$ to show unsolvability.

😊 $\varphi$ accounts for the reason of unsolvability in the formal language of epistemic logic.

😢 Limited instances of logical obstructions known to date.

* 1-set agreement & approximate agreement
  [Goubault-Ledent-Rajsbaum2021]

* k-set agreement ($k>1$) [Nishida2020] (w/ distributed knowledge),
  later generalized for adversary model [Yagi-Nishimura2020]
  ✬ This works only for single-round protocol.

* General logical obstruction in an extended simplicial model
  [vanDitramsche-Goubault-Lazic-Ledent-Rajsbaum2021]
  ✬ The general formula involves no epistemic contents and
  provides no hints for the reason of unsolvability.
Goal of this talk

- Find an epistemic formula $\Phi$ such that
  - $\Phi$ is a logical obstruction to \textit{k-set agreement}.
  - $\Phi$ contains epistemic contents that \textit{account for the reason of unsolvability}.
  - $\Phi$ works for \textit{multi-round protocols} (where processes are allowed to communicate arbitrarily many times).
Our strategy

❖ To find inconsistency between simplicial Kripke models,
  ✶ Rework on “Sperner’s lemma” to rephrase it as a statement on higher dimensional connectivity.

❖ To express the inconsistency in the language of logic,
  ✶ Use **epistemic μ-calculus**, which extends epistemic logic with:
    ✶ **Distributed knowledge**, a modal operator for higher-dimensional connectivity, and
    ✶ **Propositional greatest fixpoint** for transitive closure.
Sperner’s lemma as connectivity
Sperner’s lemma. Any subdivision of a simplex with Sperner coloring has odd number of fully-colored facets (maximal simplexes).
Proof of Sperner’s lemma (induction on dim.)

dim. = 2
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Proof of Sperner’s lemma (induction on dim.)

- Each graph node other than special node is of degree 1 or 2.
- A graph node is of odd degree iff it is a fully-colored or a special node (I.H.)

\[
\text{dim.}=2 \quad \text{even}
\]

\[
\text{(\# of fully-colored nodes)} = \text{(\# of nodes of odd degree)} - (1 \text{ special node}) = \text{odd}
\]
Proof of Sperner’s lemma (all dimensions)

dim.=2

dim.=1

dim.=0
Sperner’s lemma in a single unified graph

Traversing from the initial node of dimension 0, we eventually reach a fully-colored facet.
If there were no fully-colored facet, there would be a cycle-free, ever-lasting path in the graph.
Logical obstruction in epistemic μ-calculus
Epistemic logic for DC

Epistemic logic = Propositional modal logic for knowledge higher dimensional connectivity

* $K_a \phi$ — Process $a$ knows $\phi$.
* $D_A \phi$ — The collection $A$ of processes know $\phi$.

+ $M, X \models D_A \phi$ iff $\forall Y \in W.(X \sim_A Y \Rightarrow M, Y \models \phi)$

where $X \sim_A Y$ iff $X \sim_a Y$ for every $a \in A$.

\[ M, X \models D_{\{\text{●, ●}\}} p \]
Epistemic logic for DC

Epistemic logic = Propositional modal logic for knowledge
higher dimensional connectivity

* $K_a \varphi$ — Process $a$ knows $\varphi$.

* $D_A \varphi$ — The collection $A$ of processes know $\varphi$.

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  where $X \sim_A Y$ iff $X \sim_a Y$ for every $a \in A$.

$p\neg p$

$M, X \models D_{\{p\}} p$
Epistemic μ-calculus for DC

\[ \varphi ::= p \mid \neg p \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid D_A \varphi \mid \nu Z. \varphi \]

* Distributed knowledge \( D_A \varphi \) for higher dimensional connectivity.

* Greatest fixpoint \( \nu Z. \varphi \) for transitive closure of connectivity
  * greatest solution for \( Z = \varphi \)  (i.e., \( \nu Z. \varphi \Leftrightarrow \varphi[\nu Z. \varphi/Z] \))

* Formulas are positive.
Logical obstruction in extended simplicial model

- Extension with atomic propositions on output values.

\([\text{vanDitramsch-Goubault-Lazic-Ledent-Rajsbaum2021}]\)

Process \(a\) has input \(i\)  
Process \(a\) decides output \(i\)

\[ p ::= \text{input}^i_a \mid \text{decide}^i_a \quad (a \in \Pi, i \in \text{Value}) \]

If there exists a positive epistemic formula \(\varphi\) and a facet 
\(X \in I[IS^m]\) such that, for any \(\delta: I[SA_k] \rightarrow I[IS^m]\),

\[ I[SA_k], \delta(X) \models \varphi \quad \text{but} \quad I[IS^m]_\delta, X \not\models \varphi, \]

then \(k\)-set agreement task is not solvable by \(m\)-round protocol.

Logical obstruction

extended models

\(k\)-set agreement

\(m\) rounds
The logical obstruction to k-set agreement

Single output per each process

\[ \Phi_k = \nu Z. \left[ \text{OFUN} \land \text{VALID} \land \bigwedge_{\emptyset \subset A \subseteq \Pi} (\text{DEC}_A \Rightarrow D_A (\text{KNOW} \land \text{AGREE}_k \land Z)) \right] \]

Validity of agreement

A pair of facets agree on the output of processes that they share.

Collection \( A \) of processes decide outputs from the values \( \{0, \ldots, |A|-1\} \).

k-set agreement

\text{OFUN} = \bigwedge_{a \in \Pi} \left( \bigwedge_{d,e \in \Pi, d \neq e} \neg (\text{decide}_a^d \land \text{decide}_a^e) \land \bigvee_{d \in \Pi} \text{decide}_a^d \right)

\text{VALID} = \bigwedge_{a \in \Pi} \bigwedge_{d \in \Pi} (\text{decide}_a^d \Rightarrow \bigvee_{b \in \Pi} \text{input}_b^d)

\text{AGREE}_k = \bigvee_{A \subseteq \Pi, 0 < |A| \leq k} \bigwedge_{a \in \Pi} \bigvee_{d \in A} \text{decide}_a^d

\text{KNOW} = \bigwedge_{A \subseteq \Pi} \bigwedge_{a \in A} \bigwedge_{d \in \Pi} (\text{decide}_a^d \Rightarrow D_A \text{decide}_a^d)

\text{DEC}_A = \bigwedge_{d=0}^{|A|-1} \bigvee_{a \in A} \text{decide}_a^d
The logical obstruction to k-set agreement

- $I[SA_k], \delta(X) \models \Phi_k$
  - Obviously holds because OFUN, VALID, etc. are all valid.

- $I[IS^m], X \not\models \Phi_k$
  - $I[IS^m], X \models \Phi_k$ implies a cycle-free ever-lasting path such as:
Combinatorial presentation of facets

- Facet in $I[IS]$ (1\textsuperscript{st} round) = ordered set partition
  [Kozlov2012]

- Facet in $I[IS^m]$ (m-th round) = sequence of $m$ ordered set partitions

---

1st round

2nd round

\[
\begin{align*}
\langle 2,0,1 \rangle & : (2,0,1) \\
\langle 2,1,0 \rangle & : (2,1,0) \\
\langle 0,2,1 \rangle & : (0,2,1) \\
\langle 0,1,2 \rangle & : (0,1,2) \\
\langle 1,2,0 \rangle & : (1,2,0) \\
\langle 1,1,2 \rangle & : (1,1,2) \\
\langle 0,1,2 \rangle & : (0,1,2) \\
\langle 0,0,2 \rangle & : (0,0,2) \\
\langle 1,0,2 \rangle & : (1,0,2) \\
\langle 1,1,2 \rangle & : (1,1,2)
\end{align*}
\]
Unsolvability for k-concurrency submodel
A 2-round immediate snapshot ($IS^2$) where simultaneous execution is restricted up to $k$ processes.

- 2-concurrency in 3-process system

**Theorem** [Gafni-He-Kuznetsov-Rieutord2016] $\ell$-set agreement task is solvable by $k$-concurrency model iff $\ell \geq k$. 

White facets are excluded because of high congestion.
Take $\Phi_\ell$ as the logical obstruction for $\ell$-set agreement.

E.g., in 2-concurrency model, $\Phi_1$ is a logical obstruction to 1-set agreement, because the model includes all the facets relevant to the proof.
Summary and Future Topics
Summary

Unsolvability of k-set agreement task in logical method:

- Formula of epistemic μ-calculus as an account for the reason of unsolvability.

- Sperner’s lemma as a statement for higher-dimensional connectivity.

- Greatest fixpoint for expressing long-range, higher-dimensional connectivity.
Future topics

◉ More instances!

◉ From topology to logic
  * Sperner’s lemma
    → higher-dimensional connectivity as a greatest fixpoint in epistemic μ-calculus

  * Others?? (Index lemma, Nerve lemma, ...)

Thank you for listening.

Manuscript on arXiv:
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E. Gafni, Y. He, P. Kuznetsov and T. Rieutord, “Read-write memory and k-set consensus as an affine task”, OPODIS 2016.

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Epistemic logic

Epistemic logic = Propositional logic with modality $K_a \varphi$

* $K_a \varphi$  Process $a$ knows $\varphi$.

Kripke model semantics $M = (W, \sim, L)$

* $W$ is the set of epistemic states (possible worlds).
* $L(X)$ gives the set of true propositions in $X \in W$.
* $\sim_a$ (for each $a \in \Pi$) is an equivalence relation over $W$.

\[ M, X \models K_a \varphi \iff \forall Y \in W. (X \sim_a Y \Rightarrow M, Y \models \varphi) \]

Every complex $C$ gives rise to a simplicial Kripke model:

* $W$ is the set of facets in $C$.
* $X \sim_a Y$ iff $X \sim_a Y$ share a common vertex of color $a$. 
Simplicial Kripke model semantics

- Simplicial Kripke model $M = (W, \sim, L)$
  - $W$ is the set of facets (maximal simplexes) in a chromatic simplicial complex.
  - $L(X)$ gives the set of true props. in $X \in W$.
  - $\sim_a (a \in \Pi)$ is an equivalence relation over $W$ defined by:
    \[ X \sim_a Y \iff X \text{ and } Y \text{ are simplexes sharing a common vertex of color } a. \]

- Semantics of knowledge modality $K_a \varphi$
  - $M, X \models K_a \varphi$ iff $\forall Y \in W. (X \sim_a Y \Rightarrow M, Y \models \varphi)$
There exists no $\delta$ that makes the following diagram commute (hence the task is not solvable),

If there exists a *positive* epistemic formula $\varphi$ and facet $X \in C$ such that $I[T], \delta(X) \Vdash \varphi$ but $I[P], X \nvdash \varphi.z$

**Knowledge gain theorem.** Suppose $C \xrightarrow{\delta} D$, $X \in C$, and $\varphi$ is a *positive* epistemic formula. Then, $D, \delta(X) \Vdash \varphi$ implies $C, X \vdash \varphi$. 

 logical obstruction
If there exists a positive epistemic formula $\varphi$ and facet $X \in I[IS^m]$ such that $I[SA_k], \delta(X) \models \varphi$ but $I[IS^m]_\delta, X \not\models \varphi$, then $k$-set agreement task is not solvable by $m$-round protocol.
k-set agreement task

**Input**  Each of \((n+1)\) processes has its private input value.

**Output**  Each process decides an output value satisfying:

* **Validity.** Each process decides a value out of \((n+1)\) inputs.
* **Agreement.** Processes decide at most \(k\) different values.

A 2-set agreement:
**k-set agreement task**

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A 2-set agreement:

Fact. k-set agreement task is *not solvable* by a (wait-free, asynchronous) system of \(n+1\) processes, unless \(k \geq n+1\).
Topological model for DC

- Chromatic simplex of dimension $n$ = system state of $(n+1)$ processes
- Chromatic simplicial complex = nondeterministic set of states
- Task solvability (topological)

Vertex = color×value

$\exists \mu \in T$ for every simplex $X \in P$.

Color-preserving vertex-to-vertex mapping.
Common knowledge as fixpoint

\[ C_A P \]

\[ \iff \nu Z. \left( P \land \bigwedge_{a \in A} K_a Z \right) \]

\[ \iff P \land \bigwedge_{a \in A} K_a \left( \nu X. \left( P \land \bigwedge_{a \in A} K_a X \right) \right) \]

\[ \iff P \land \bigwedge_{a \in A} K_a \left( P \land \bigwedge_{a \in A} K_a \left( P \land \bigwedge_{a \in A} K_a \cdots \right) \right) \]