Universal graded Hopf algebra and classical mechanics

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Abstract. The concept of the tangent and cotangent bundles plays a central role in describing the dynamics of mechanical systems with singular configuration space. There is no any unified approach to the description of these objects at the present time. The reason for considering a particular construction is that the constructed model gives these bundles in the classical case. One example of such a generalization is the algebra of cosymbols of differential operators of the smooth functions algebra on a singular manifold in the category of geometric modules. This algebra has the natural structure of the Hopf algebra and the algebra dual to it in the classical case coincides with the cotangent bundle of a smooth manifold. Generalizing this example, we introduce the notion of a universal graded algebra for which we can define the structure of the Hopf algebra and the Poisson bracket is defined in a natural way on the dual algebra. This allows us to determine the evolution equation of the system.

1. Introduction
The article describes one of the possible approaches to the conceptual definition of the concepts of tangent and cotangent bundles. In classical (smooth) differential geometry, these objects are described geometrically and are naturally related by mutual duality in the algebraic sense, but they use the concept of smoothness in the sense of analysis. However, the need to consider, for example, manifolds with singularities requires the ”algebraization” of concepts. Such an algebraization is completely natural for the tangent vector if we consider it as differentiation at the point of some algebra. This implies the definition of the tangent bundle. However, various possible algebraizations of the concept of a cotangent bundle lead to objects that are not isomorphic to each other. The question arises of finding a conceptual definition that allows either to make a choice or simply to determine the limits of the applicability of concepts. The tangent space is the spectrum of some universal object in any differentially closed category. The reason for this situation is the fact that the graded algebra of cosymbols of differential operators is multiplicatively generated by 1-cosymbol (for any rings, algebras, and categories), in contrast to the algebra of symbols. In particular, Hopf algebras naturally appear as universal objects in commutative geometry.

2. Universal graded Hopf algebra
Let $K$ be unitary commutative ring, $A$ — commutative associative and unitary $K$-algebra, $C_i$, $i \geq 1$, are $A$-modules, $d : A \to C_1$ — $K$-homomorphism. Let’s consider a graded commutative $A$-algebra $C = \oplus_{i=0}^{\infty}C_i$, $C_0 = A$, $C_k \cdot C_s \subset C_{k+s}$.
we should verify three relations

1° the module \( C_1 \) is generated by the image of homomorphism \( d \) over \( A \), i.e. \( C_1 = < d(A)>_A; \)

2° every element of \( C_k \), \( k \geq 2 \), is \( A \)-linear combination of elements \( c_1 \cdot \ldots \cdot c_k \), \( c_i \in C_1 \), \( C_k = < C_1^k >_A. \)

**Theorem 1.1.** The universal algebra \( A \) is the involutive Hopf algebra.

**Proof.** Consider the tensor product

\[
C \bigotimes_A C = \bigoplus_{k=0}^{\infty} \left( C_1 \bigotimes_A C_{k-1} \right)
\]

with componentwise multiplication \((\theta_1 \otimes \theta_2) \cdot (\theta_1' \otimes \theta_2') = (\theta_1 \cdot \theta_2) \otimes (\theta_1' \cdot \theta_2')\). This algebra is graded commutative algebra with unit \( 1 \otimes 1 \). The \( C \)-multiplication \( \mu : C \otimes_A C \to C \) is the unitary \( A \)-homomorphism of algebras.

Let us introduce the map on the generating elements \( c_{\kappa^k} = c_1 \cdot \ldots \cdot c_k \), where \( \kappa^k = \{1, \ldots, k\} \) is multi-index, \( \Delta : C \to C \otimes_A C \), by the formula

\[
\Delta(c_{\kappa^k}) = \sum_{i=0}^{k} \sum_{|\tau|=i} c_{\kappa^i} \otimes c_{\kappa^k-i}.
\]

By the definition \( c_{\kappa^0} = 1_A \), \( \Delta(1) = 1 \otimes 1 \), \( \Delta(c) = 1 \otimes c + c \otimes 1 \), where \( c \in C_1 \).

Introduced multiplication is \( A \)-homomorphism of algebras. Indeed, \( A \)-linearity of \( \Delta \) follows directly from the definition. Let us check the multiplicativity on the basic elements. Fix some set of elements \( \{c_1, \ldots, c_{k+s}\} \) and multi-index \( \kappa \subset \kappa^{k+s} \), \( |\kappa| = k \). In the first place

\[
\Delta(c_{\kappa^i} \cdot c_{\kappa^j}) = \Delta(c_{\kappa^{i+j}}) = \sum_{i=0}^{k+s} \sum_{|\mu|=i} c_{\mu} \otimes c_{\kappa^j} = \Delta(c_{\kappa^{i+j}}) = \sum_{i=0}^{k+s} \sum_{|\mu|=i} c_{\mu} \otimes c_{\kappa^j}.
\]

On the other hand

\[
\Delta(c_{\kappa^i}) = \sum_{i=0}^{k} \sum_{|\tau|=i} c_{\tau} \otimes c_{\kappa^k-i}, \quad \tau \cup \tau = \kappa,
\]

\[
\Delta(c_{\kappa^j}) = \sum_{j=0}^{s} \sum_{|\tau|=j} c_{\tau} \otimes c_{\kappa^j}, \quad \sigma \cup \sigma = \kappa.
\]

Write the production of these elements

\[
\Delta(c_{\kappa^i}) \cdot \Delta(c_{\kappa^j}) = \sum_{i=0}^{k} \sum_{j=0}^{s} \sum_{|\tau|=i} \sum_{|\sigma|=j} c_{\tau \cup \sigma} \otimes c_{\tau \cup \sigma}.
\]

Since any multi-index \( \nu \) can be uniquely represented by the unity of multi-indices \((\nu \cap \kappa) \cup (\nu \cap \kappa') = \tau \cup \sigma\), the both expressions (1) and (2) coincide. The multiplicivity is proved.

Consider the projection on the first summand of grade \( \epsilon : C \to A \), which is obviously the \( A \)-homomorphism of algebras. The triple \((C, \Delta, \epsilon)\) is a commutative coalgebra. To proof this we should verify three relations

\[
(\Delta \otimes \text{id}_C) \circ \Delta = (\text{id}_C \otimes \Delta) \circ \Delta,
\]

\[
(\epsilon \otimes \text{id}_C) \circ \Delta = (\text{id}_C \otimes \epsilon) \circ \Delta = \text{id}_C,
\]

\[
\tau \circ \Delta = \Delta.
\]
Here the factor permutation $\tau : C \otimes_A C \rightarrow C \otimes_A C$ is a unitary $A$-homomorphism. The commutativity (3) follows directly from the definition of comultiplication $\Delta$. To prove (4) note, that the counit $\varepsilon$ maps all 0-graded elements to zero, hence on the basic elements holds

$$(\varepsilon \otimes \text{id}_C)(\Delta(\theta)) = (\varepsilon \otimes \text{id}_C)(1 \otimes \theta + \ldots) = \theta.$$ 

Left multiplication can be checked in the same way. The relation (5) is obvious.

As the comultiplication and counit are homomorphisms, the described coalgebra together with multiplication and unit $\eta : A \rightarrow C$ is bealgebra. To prove that the algebra is involutive Hopf algebra we should find an antipod $S \in \text{Hom}_A(A,C)$, with relation

$$\mu \circ (\text{id}_C, S) \circ \Delta = \mu \circ (S, \text{id}_C) \circ \Delta = \eta \circ \varepsilon.$$ 

Show that the following homomorphism is the antipod

$$S = \bigoplus_{i=0}^{\infty} (-1)^i \text{Id}_{C_i(A)}.$$ 

Consider a basic element $a_{\pi^k}$. For $k = 0$

$$\mu((\text{id}_C, S)(\Delta(a))) = \mu((\text{id}_C, S)(a \otimes 1)) = \mu(a \otimes 1) = a = \eta(\varepsilon(a)).$$ 

For $k > 0$ the comultiplication of basic element is the sum of $2^k$ tensor products and we can calculate explicitly

$$\mu((\text{id}_C, S)(\Delta(c_{\pi^k}))) = \mu \left( \text{id}_C, S \left( \sum_{i=0}^{k} \sum_{|\pi^i|} c_{\pi^i} \otimes c_{\pi^i} \right) \right) =$$

$$\mu \left( \sum_{i=0}^{k} \sum_{|\pi^i|} (-1)^{k-i} c_{\pi^i} \otimes c_{\pi^i} \right) = \sum_{i=0}^{k} \sum_{|\pi^i|} (-1)^{k-i} c_{\pi^i} \cdot c_{\pi^i} =$$

$$\sum_{i=0}^{k} \sum_{|\pi^i|} (-1)^{k-i} c_{\pi^i} = \left( \sum_{i=0}^{k} \sum_{|\pi^i|} (-1)^{k-i} \right) c_{R^k} = 0,$$

on the other hand $\eta(\varepsilon(c_{R^k})) = \eta(0) = 0$. $\square$

3. Dual algebra and Poisson structure

For the universal graded algebra $C$ consider the dual graded $A$-module

$$C^* = \bigoplus_{i=0}^{\infty} \text{Hom}_A(C_i, A).$$

Define on $C^*$ multiplication of elements of the grade $f \in C^*_k = \text{Hom}_A(C_k, A)$, $g \in C^*_s = \text{Hom}_A(C_s, A)$

$$(f \cdot g)(\theta) := \sum_i f(\theta(i)) \cdot g(\theta(i)).$$
where $\Delta(\theta) = \sum_i \theta(i) \otimes \tilde{\theta}(i)$ is the result of comultiplication of basic element. We get the element of module $\text{Hom}_A(C_{k+s}, A)$. Obviously, such a multiplication is commutative. Calculate the product with counit
\[
(f \cdot \varepsilon)(\theta) := \sum_i f(\theta(i)) \cdot \varepsilon(\tilde{\theta}(i)) = f(\theta) \cdot \varepsilon(1) = f(\theta).
\]
Hence defined algebra is unitary.

Now define the Poisson bracket
\[
\{f, g\}(\theta) = \sum_i \left[ f \left( d(g(\theta(i))) \cdot \tilde{\theta}(i) \right) - g \left( d(f(\theta(i))) \cdot \tilde{\theta}(i) \right) \right],
\]
where elements of the dual algebra $f, g$ are considered as $A$-homomorphisms and dot-multiplication denotes those in the Hopf algebra. The linearity and skew commutativity of the bracket can be easily checked. The straightforward calculation shows that the Jacoby identity also holds. Hence, we get the Poisson structure on the dual Hopf algebra.

It is well known that the cotangent bundle is the spectrum of the algebra of the differential operator symbols over the smooth functions algebra [1]. This fact allowed giving pure algebraic conceptual formulation of classical Hamiltonian formalism over smooth manifolds [2]. This in turn gave the possibility to generalize the concept of Hamiltonian mechanics on objects more general than smooth manifolds. However, the straightforward reformulation of several constructions met some principal difficulties. For instance, for object more general than smooth manifold there are several natural algebraic generalizations of concept of cotangent bundle. One can generalize the concept of covector and set the collection of them over points of algebra or consider just the spectrum of algebra. These objects do not coincide in principal. This gave the impact to search another approach to generalization of Hamiltonian formalism. We have based our considerations on the fundamental remark of A.M. Vinogradov, that the symbol algebra is the dual object to the algebra of cosymbols. As it turned out the representative objects of natural functors of differential calculus over commutative unitary algebra compose the filtered algebra, so called infinity jet algebra [3]. The jet algebra in standard way generates the graded algebra of cosymbols of differential operators. This universal representative object of differential calculus seems to be more natural object to formulate Hamiltonian formalism. Moreover, it was noted that this algebra equipped with the natural comultiplication, due clear structure of the cosymbol modules. All standard dual constructions generate the known structures of the differential operator symbols algebra. Due the universal differentiation, i.e. exterior differential, which is defined for universal representative objects in differentially closed categories, it is possible to describe Poisson bracket on dual object [4] in the way described in this article.

References

[1] Mishra D and Mishra L N 2015 *Facta Universitatis (NIS)* Ser. Math. Inform. 30(5) 1–12

[2] Vinogradov A, Kalnitsky V and Sorokina M 2011 *Tangent Space of a Commutative Algebra Presentation at ESF Exploratory Workshop on Current Problems in Differential Calculus over Commutative Algebras, Secondary Calculus, and Solution Singularities of Nonlinear PDEs, Vietri sul Mare, Italy* (http://sites.google.com/site/levicivitainstitute/Activities/workshops/ESF-EW-2011/Kalnitsky.pdf)

[3] Krasil’shchik I S, Lychagin V V and Vinogradov A M 1986 *Geometry of jet spaces and nonlinear partial differential equations*, v. 1 (New York: Gordon and Breach Science Publishers)

[4] Kalnitsky V 2012 *Poisson structure on the dual Hopf algebra Presentation at 6th European Congress of Mathematics, Krakow, Poland* (http://www.6ecm.pl/docs/Kalnitsky_Vyacheslav_703.pdf)