String Threshold corrections from Field Theory

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Abstract

A field theory approach to summing threshold effects to the gauge couplings in a two-torus compactification is presented and the link with the (heterotic) string calculation is carefully investigated. We analyse whether the complete UV behaviour of the theory may be described on pure field theory grounds, as due to momentum modes only, and address the role of the winding modes, not included by the field theory approach. “Non-decoupling” effects in the low energy limit, due to a small dimension are discussed. The role of modular invariance in ensuring a finite (heterotic) string result is addressed.

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1 Introduction

There has recently been growing interest in the physics of (large) extra dimensions from a field theory perspective. The phenomenological implications of the associated Kaluza-Klein states (whether charged or not under the gauge group) in such models have been extensively addressed. Many of these phenomenological studies are performed in an effective field theory approach, and the link between such Kaluza-Klein models and a full string theory (either heterotic or type I), where such effects can be consistently investigated, is not always clear.

For the purpose of the present work, we would like to address this link for the particular problem of threshold effects to the gauge couplings due to Kaluza-Klein (and, for closed string theory, winding modes) excitations associated with the compact dimensions, in a generic orbifold compactification to 4 dimensions. This is considered to be a $\mathcal{N} = 1$ orbifold, with a (two dimensional) $\mathcal{N} = 2$ sub-sector which survives compactification. The massive momentum and winding excitations associated with the latter fall into $\mathcal{N} = 2$ multiplets and give significant corrections to the gauge couplings, while the massless states fall into $\mathcal{N} = 1$ multiplets associated with the light spectrum and bring in a logarithmic contribution to the RG flow of the gauge couplings.

A field theory calculation of the threshold effects due (only) to Kaluza-Klein states corresponds to a zero slope limit $\alpha' \rightarrow 0$ at string level. In this limit string physics, such as all ultra-heavy winding modes, decouple. Of course this limit is singular, and string thresholds present a divergent behaviour. Therefore it is not surprising that the field theory result is also divergent. By comparing the two results we would like to investigate, whether this divergence is entirely due to Kaluza-Klein modes, or whether winding modes can bring additional UV effects. There are differences in the structure of the string thresholds in the type I case relative to the heterotic case: in the former the quadratic divergence in the zero slope limit is absent, while the latter and field theory are sensitivity to a high cut-off scale. We briefly comment the origin of such behaviour.

Unlike the heterotic string case, the effective field theory approach is a priori unable to account for the additional effects of the winding states which do play a role in the compactification. Even decoupled, winding modes may in principle still provide some additional (UV finite) contribution because their decoupling is, in fact, only asymptotic [7]. Effects which may vanish in the limit $\alpha' \rightarrow 0$ may not be visible in a field theory description. We identify the link between these effects and the mixing between momentum and winding modes. For the type I string case there are no winding modes contributions to the gauge couplings, thus one may expect the finite part be similar to that of regularised effective field theory. Our purpose is thus to clarify for the heterotic case which contributions of the (string) threshold correction to the gauge couplings cannot be computed on field theory grounds. We analyse the correctness of the naive expectation that their origin be due to winding modes alone. We also comment on the link of type I result with a field theory approach.

The plan of the work is as follows: in the next section we quote the heterotic and type I string corrections to the gauge couplings, due to momentum and winding modes associated with the extra compact dimensions of the string. This will set up the notation and stage the results from the string to which we report our findings from a field theory approach. In Section 2 we use Coleman-Weinberg formula to compute the effects of the same Kaluza-Klein spectrum to the gauge couplings. We take into account the effects of the shape of the manifold (two torus), so far not considered at field theory level. Section 4 clarifies the link field theory - string theory and the differences in the results obtained. Appendix A, B detail calculations of section 3. Appendix C - the link among the regularisation schemes used in the field theory approach to computing string thresholds, while a detailed exposition of Poisson re-summation applied to Gaussian sums like the string partition.
2 Thresholds results from string theory

String threshold effects have been computed in the heterotic string case in \[2, 3, 4, 5, 6\] and in type I case in \[8\]. Before proceeding with a field theory investigation of these effects, we quote these well known results for later comparison and for establishing our notations/conventions.

Kaluza-Klein states and in the case of the heterotic string also winding states affect significantly the value of the gauge couplings at a scale of the order of the inverse radii of the manifold of the extra compact dimensions. In the case an \(\mathcal{N} = 2\) sector “survives” compactification, the effective gauge couplings \(\alpha_i = g_i^2/(4\pi)\) receive the corrections

\[
\alpha_i^{-1}(Q) = k_i\alpha_u^{-1} + \frac{b_i}{2\pi} \ln \frac{M_s}{Q} + \Delta_i + \cdots,
\]

at scale \(Q\), with \(k_i\) is the Kac-Moody level, \(\Delta_i\) is the (one-loop) gauge group dependent string threshold correction (a function of the moduli fields) induced by \(\mathcal{N} = 2\) multiplets. \(b_i\) is the one-loop beta function of the light modes (\(\mathcal{N} = 1\) sector). In the case of the heterotic string a universal (gauge group independent) correction \(\sigma\) exists, but in \(\mathcal{N} = 2\) it is absorbed \(\mathcal{N} = 1\) into the re-defined coupling \(\alpha_u\) which should then be identified with the effective coupling at field theory level \(\mathcal{N} = 1\).

Heterotic string results

In this case six of the dimensions are compactified on an orbifold, \(T^6/G\), and we consider the generic case when the spectrum splits into \(\mathcal{N} = 1\), \(\mathcal{N} = 2\) and \(\mathcal{N} = 4\) sectors, with the latter two associated with a \(T^2 \times T^4\) split of the \(T^6\) torus. Due to the supersymmetric non-renormalisation theorem, the \(\mathcal{N} = 4\) sector does not contribute to the holomorphic coupling, nor does it affect the effective coupling \(\alpha_u\) in the one-loop case. The \(\mathcal{N} = 1\) sector brings in a logarithmic running associated with light states, eq. (1). The moduli dependence brought in by \(\Delta_i\) comes with a coefficient proportional to the beta function of the \(\mathcal{N} = 2\) sector. All heterotic states are closed string states and at one loop the string world sheet has the topology of the torus \(T^2\). For the case of a six-dimensional supersymmetric string vacuum compactified on a two torus \(T^2\) the string correction \(2, 3\) (see also \(3, 6\)) is the result of two double-sums over the Kaluza-Klein and winding modes, respectively, associated with \(T^2\), and takes the form

\[
\Delta_i^H = -\frac{\beta_i}{4\pi} \ln \left\{ \frac{8\pi e^{1-\gamma_E}}{3\sqrt{3}} U_2 |\eta(U)|^4 T_2 |\eta(T)|^4 \right\}, \quad \eta(T) = e^{\frac{T}{2} T} \prod_{k=1}^{\infty} \left( 1 - e^{2\pi ikT} \right).
\]

Here \(\beta_i\) is the beta coefficient associated with the \(\mathcal{N} = 2\) sub-sector, and the Dedekind function \(\eta(T)\) is used. The complex moduli parameter \(T = T_1 + iT_2\) and \(U = U_1 + iU_2\) can be expressed in function
of the metric $G_{ij}$ (and its determinant $G$) and anti-symmetric tensor $B_{ij} = B\epsilon_{ij}$ background as

$$T = 2\left[B + i\sqrt{G}\right] = 2\left[B + i\frac{R_1 R_2 \sin \theta}{2\alpha'}\right], \quad U = \frac{1}{G_{11}} \left[G_{12} + i\sqrt{G}\right] = \frac{R_2}{R_1} e^{i\theta},$$

(3)

with the string parameter $\alpha'$ related to the string scale $M_s$ by $M_s = 2 e^{(1-\gamma_E)/2} 3^{-3/4} \sqrt{2\pi \alpha'}$ (in $\overline{DR}$ scheme, see [2]). The volume of compactification is characterized by $T_2$ ($\alpha'$ dependent), while $U$ describes the shape of the torus ($\alpha'$ independent). In particular, for an orthogonal torus $\theta = \pi/2$ the shape parameter $U$ is purely imaginary. (Generalisations of the above result for the case of Wilson line moduli exist [6], but for the purpose of this work we restrict ourselves to the case when these are not present.)

**Type I string results**

The threshold effects in four dimensional $\mathbb{Z}_N$ orientifold models of type I string with $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetry have a structure closely related to that of the heterotic string. There exists a contribution from the twisted moduli: their VEVs remove the orbifold singularities and allow a transition to smooth (Calabi-Yau) manifolds. When their VEVs are set to zero by the vanishing of the D-terms of anomalous $U(1)$'s the corrections to the gauge couplings, eq. (1) are [8]

$$\Delta_i^I = -\frac{b_i}{4\pi} \ln \left\{ 4\pi e^{-\gamma_E} U_2 |\eta(U)|^4 T_2 \right\}.$$  

(4)

The definition of the complex structure moduli $U$ is similar to the heterotic case, while $T$ is now expressed in type I string units [8]. An additional summation (not written explicitly) is present if each of the three complex dimensions comes with a $\mathcal{N} = 2$ sector. The constant under the logarithm depends on the regularisation used at string level [8]. The essential difference in (4) from the heterotic case (2) is the absence of the contribution due to Dedekind eta function $\ln |\eta(T)|^4$, dominated by a power-like (with the scale) term, with an additional milder (exponentially suppressed) contribution.

**3 Field theory calculation of the thresholds**

At field theory level the effects on the gauge couplings of a finite/infinite tower of Kaluza Klein states (on a two torus) can be considered as well. We perform this calculation and compare the field theory result $\Omega_i$ to the full string result for the gauge dependent part of the thresholds ($\Delta_i^H$ or $\Delta_i^I$). We will thus identify the contribution of Kaluza Klein states only, and should this be possible, analyse the UV role (if any) of pure winding modes’ contributions by comparing to the full (heterotic) string result.

We start with the general Coleman Weinberg formula for the threshold effects [2] to the gauge couplings. We have

$$\Omega_i = \frac{1}{4\pi} \sum_i T(R_i) \sum_{m_{1,2}} \int_{-\infty}^{\infty} dt e^{-\pi t M_{m_{1,2}}^2/\mu^2}.$$  

(5)

This constant is here chosen in agreement with the regularisation we used below for the field theory case.
The result of summing a finite or infinite tower of Kaluza-Klein states to the gauge couplings is divergent, thus the need for a regularisation. We introduced a UV dimensionless (proper time) regulator $\xi \to 0$ as the lower limit of the integral above. This scheme will be used in the following. Other regularisations can be used, with formal equations relating them presented in Appendix \ref{app:other-reg}. An alternative method to computing (\ref{eq:Omega-def}) is given in Appendix \ref{app:other-reg} using dimensional reduction or zeta function regularisation scheme. This scheme is explicitly related to the cut-off regularisation that we use in this section. The method used in Appendix \ref{app:other-reg} clarifies the link between these regularisations, to relate results obtained in such different schemes. This method is general and may be applied to other calculations as well (e.g. the scalar potential in Kaluza Klein models). Finally, we mention that in all regularisations employed in this work, the Clifford algebra is worked out first, and only after is the regularisation of the scalar sum-integrals of type (\ref{eq:Omega-def}) performed. Therefore, all supersymmetric cancellations that rely on spinor properties are taken into account, before the regularisation of $\Omega_i$.

In (\ref{eq:Omega-def}) $\mu$ is an arbitrary (finite) mass scale, which we introduced to render the above equation dimensionless. A “prime” on the sum in eq. (\ref{eq:Omega-def}) stands for the absence of the $(m_1, m_2) = (0, 0)$ Kaluza Klein massless mode which at this stage is subtracted out. This is because this ($N = 1$) massless mode is accounted for by the logarithm of the cut-off already present in eq. (\ref{eq:T-def}). In (\ref{eq:Omega-def}) $M^2_{m_1, m_2}$ is the mass expression for Kaluza-Klein states for a two dimensional case (torus), with “shape effects” of the torus characterised by the ratio of radii and angle $\theta$. These together define the quantity $U$ (with definition as in string case eq. (\ref{eq:U-def}) since it is scale independent) and the volume $T$

$$T(\mu) \equiv i T_2(\mu) = i \mu^2 R_1 R_2 \sin \theta .$$

(6)

Definition (\ref{eq:U-def}) for $T_2$ will only be used in the field theory case, instead of its string version (\ref{eq:U-def}). Thus the mass of Kaluza-Klein states on a two-torus can be written in terms of $U$ and $T$ as (for details see e.g. \ref{app:other-reg})

$$M^2_{m_1, m_2} = \frac{1}{\sin^2 \theta} \left[ \frac{m_1^2}{R_1^2} + \frac{m_2^2}{R_2^2} - \frac{2m_1 m_2 \cos \theta}{R_1 R_2} \right] = \frac{|m_2 - Um_1|^2}{(\mu^2 T_2) U_2} .$$

(7)

In the limiting case $\theta = \pi/2$, $M^2_{m_1, m_2} \to m_1^2/R_1^2 + m_2^2/R_2^2$ which is valid for an orthogonal torus, more commonly encountered in the literature than the general case of arbitrary $\theta$ that we consider here. Eq. (\ref{eq:masses2}) shows that even for large compactification radii $R_{1,2}$, the Kaluza-Klein states may have very large mass and thus be undetectable if $\theta \ll 1$ as may be seen [9] by expanding (7) for small $\theta$.

The integrand of (\ref{eq:Omega-def}) may be written as

$$I \equiv \sum_{m_1, m_2 \in \mathbb{Z}} e^{-\frac{\pi i}{\tau_1 \tau_2} |U m_1 - m_2|^2} = \sum_{m_2 \in \mathbb{Z}} e^{-\frac{\pi i}{\tau_1 \tau_2} m_2^2} + \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} e^{-\frac{\pi i}{\tau_1 \tau_2} |U m_1 - m_2|^2}$$

(8)

$$= \sum_{m_2 \in \mathbb{Z}} e^{-\frac{\pi i}{\tau_2 \tau_2} m_2^2} + \left[ \frac{T_2 U_2}{t} \right]^{\frac{1}{2}} \sum_{m_1 \in \mathbb{Z}} e^{-\pi t \frac{U_1}{U_2} m_1^2} + \left[ \frac{T_2 U_2}{t} \right]^{\frac{1}{2}} \sum_{m_1 \in \mathbb{Z}} \sum_{p \in \mathbb{Z}} e^{-\frac{\pi i}{\tau_2 \tau_2} m_1^2 - \frac{\pi i}{\tau_2 \tau_2} T_2 U_2 p^2 - 2 \pi i m_1 p U_2} ,$$

(9)

with $m_2$ replaced in (\ref{eq:Omega-def}) by the Poisson re-summed index $p$, using eq. (D-10). Each of the sums above can be integrated over $t \in [\xi, \infty)$ separately. From (\ref{eq:Omega-def}), (\ref{eq:U-def}) we then have

$$\Omega_i = \frac{\tau_i}{4\pi} \left( \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 \right) ,$$

(10)
with \( \delta_i = -2T(G) + 2 \sum_r T_i(r) \) the \( N = 2 \) beta function and with (see Appendix [A-1]) and (A-12)

\[
J_1 \equiv \int_\xi^\infty \frac{dt}{t} \sum_{m_2 \in \mathbb{Z}} e^{-\frac{\pi T_2 m_2^2}{T_2}} = -\ln \left[ 4\pi e^{-\gamma_e} \frac{T_2}{\xi} \right] + 2 \left[ \frac{T_2 U_2}{\xi} \right],
\]

\[
J_2 \equiv \int_\xi^\infty \frac{dt}{t} \left[ \frac{T_2 U_2}{t} \right]^{\frac{1}{2}} \sum_{m_1 \in \mathbb{Z}} e^{-\frac{\pi T_2 m_1^2}{T_2}} = \frac{\pi}{3} U_2 + \frac{T_2}{\xi} - 2 \left[ \frac{T_2 U_2}{\xi} \right],
\]

\[
J_3 \equiv \int_\xi^\infty \frac{dt}{t} \left[ \frac{T_2 U_2}{t} \right]^{\frac{1}{2}} \sum_{m_1 \in \mathbb{Z}} \sum_{p \in \mathbb{Z}} e^{-\frac{\pi T_2 m_1^2}{T_2} - \frac{\pi T_2 U_2 p^2}{T_2}} e^{-2\pi i m_1 p U_2}
\]

\[= 4 U_2^{\frac{1}{2}} \sum_{m_1 \geq 1} \sum_{p \geq 1} \left[ \frac{m_1}{p} \right]^{\frac{1}{2}} (e^{-2\pi i m_1 p U_2} + c.c.) K_{-\frac{1}{2}}(2\pi U_2 m_1 p) = -\ln \prod_{m_1 \geq 1} \left| 1 - e^{2\pi i m_1 U} \right|^4.
\]

In (13) the regulator \( \xi \) was removed since the integral was well defined in both UV and IR. In fact, all expressions of \( J_i \) (i=1,2,3) after the (second) equal sign are only true in the limit \( \xi \to 0 \), and we will comment on the size of the errors induced for \( \xi \neq 0 \) by using the results of Appendix [A-1]. In (A-12) we used the integral representation of the modified Bessel function \( K_\nu \) of index \( \nu = \frac{1}{2} \).

We identified the sources of all (UV) divergences in \( \Omega_i \) as coming from \( J_1 \) and \( J_2 \). These are logarithmic in scale, \( \ln(T_2/\xi) \), due to “original” momentum “zero modes” (see eqs. (A-4) to (A-7)) and quadratic in scale \( (T_2/\xi) \) which are due to Poisson re-summed momentum “zero modes” (see eqs. (A-10) to (A-18)). The linear divergence \( (T_2/\xi)^{1/2} \) in (11) and (12) cancels out. The above identification will be useful for comparing to the heterotic case, where the same states will be integrated over a different region (the fundamental domain) and this will avoid the divergent behaviour of field theory.

From (13), (14), (11), (12), (13) we find

\[
\Omega_i = -\frac{\delta_i}{4\pi} \ln \left[ 4\pi e^{-\gamma_e} e^{-T_2} T_2 U_2 |\eta(U)|^4 \right],
\]

where we introduced the notation

\[
T_2^* \equiv \left. \frac{T_2}{\xi} \right|_{\xi \to 0} = \Lambda^2 R_1 R_2 \sin \theta, \quad \Lambda^2 \equiv \frac{\mu^2}{\xi}.
\]

Eq. (15) is the main result of this work. It describes the effects of the tower of Kaluza Klein states alone, and it includes shape moduli effects (\( U \) dependence) not considered by previous field theory approaches [13]. They have important effects as we discuss in the next section.

The above field theory result holds true if the (dimensionless) cut-off \( \xi \to 0 \) or equivalently if the (dimensionful) cut-off scale \( \Lambda \) introduced in (11) satisfies \( \Lambda \to \infty \). Indeed, \( J_i \) (i=1,2,3) of (11), (12) and (13) have corrections \( \delta_i \), evaluated in Appendix [A-1], eqs. (A-20), (A-21), (A-22). The sum of these corrections vanishes only if \( \xi \to 0 \). This requirement may actually be “relaxed” and the following sufficient condition for the field theory cut-off (or radii) exists, derived in eqs. (A-31),

\[\text{Note that the non-holomorphic contribution } \ln(T_2 U_2) \text{ in eq. (11), which propagates into the final result for } \Omega_i, \text{ is essentially due to a massless state } (m_1, m_2) = (0, 0). \text{ This may be observed by a careful examination of eqs. (8), (9), (10) and also (A-4) to (A-7) showing the origin of the emergent } \ln(T_2 U_2). \text{ For additional details on the non-holomorphic contribution at string level see [13].}\]
If this condition is not satisfied, corrections $\delta_i$ to the field theory result (15) may become significant. Using $T^*_2$ and $U_2$ definitions, this translates into

$$\Lambda R_2 \sin \theta > \Lambda R_1 \gg 1, \quad (U_2 > 1),$$

(18)

$$\Lambda R_1 > \Lambda R_2 \sin \theta \gg (\Lambda R_1)^{\frac{1}{2}}, \quad (U_2 < 1),$$

(19)

which lead to a large compactification volume $T^*_2 \equiv \Lambda^2 R_1 R_2 \sin \theta \gg 1$. While the constraint for the radius be larger than $1/\Lambda$ is not surprising, one notices the role that the angle $\theta$ plays in the validity of the field theory result, eq. (13). The constraints are important for very small angle $\theta$, and have implications for the validity of phenomenological studies with more than one dimension.

The field theory result (13) is very close to the heterotic string result of eq. (2) and it shows quadratic and logarithmic divergences. A comparison with the (heterotic) string result (2) shows that the field theory cut-off is replaced at string level by $\Lambda^2 \equiv \mu^2/\xi \rightarrow 1/\alpha'$. The condition of removing the regulator, $\xi \rightarrow 0$ in (2) corresponds to an infinite string scale limit $\alpha' \rightarrow 0$. String effects which vanish in this limit may not necessarily be recovered by a field theory approach and these will be identified later on. When $\alpha' \rightarrow 0$ the string result (2) also shows quadratic and logarithmic divergences, similar to (13). This is a good check of our calculation and shows the two results are indeed very similar in this limit. One notices however different coefficients for the quadratic divergence terms, $T^*_2$ in $\Omega_i$ and $(\pi/3) T_2$ in $\Delta^H$. A suitable re-definition of the regulator $\xi, \xi \rightarrow (3/\pi) \xi$ may avoid this difference. This may be regarded as the correct (re)definition for a cut-off of the field theory, such that the UV behaviour is identical to that from string theory. The conclusion is that in the absence of a full string calculation to compare with, one cannot claim that the whole UV behaviour of a Kaluza Klein model is that found on pure field theory grounds. While the nature of the divergence (quadratic) is correctly reproduced, its coefficient is not that of string theory. For this a full string calculation is necessary. We comment on this issue in Section 4.1.1.

3.1 “Non-decoupling” effects of a small dimension.

In this section we discuss the contribution of “shape effects” ($U$ dependence) in (13) versus “volume” effects ($T$ dependence).

Consider first the case $U_2 \gg 1$. Since $U_2 \equiv R_2/R_1 \sin \theta$, this condition may be respected if for example $\theta = \pi/2$ and $R_2 \gg R_1$. In such case, using the definition of the Dedekind function (2) and its asymptotic behaviour, we find that the leading contribution to $\Omega_i$ due to shape effects is significant, $\Omega_i \sim (\pi/3) U_2 \gg 1$. We compare this contribution to “volume” effects. The leading contribution in $T$ to $\Omega_i$ is $\Omega_i \sim T^*_2$ which is larger than $(\pi/3) U_2$ because $T^*_2 \gg U_2$ for $U_2 > 1$, if eq. (17) is respected. Therefore “volume” effects are more important than “shape” effects. However, the latter are still significant and do not necessarily decouple, although the manifold is effectively one-dimensional, $R_2 \gg R_1$. This can have important phenomenological implications.

5We do not address here the implications of large moduli from string theory point of view. In particular large $T_2$ may imply (for the heterotic string) a large 10D string coupling and the reaching of its non-perturbative limit.
Consider now the case $U_2 \ll 1$. This is possible if for example $R_1 = R_2$ and $\theta \ll 1$. In this case one finds that the $U$ dependence of the thresholds is $\Omega_i \sim \pi/(3\theta) + \ln \theta$ which is again large (but still smaller than volume effects. This is because the leading “volume” part is again $\pi/(3\theta)$ because $T_2^* \gg 1/U_2 \approx 1/\theta$ for small $\theta$, eq. (17).

One reason why shape effects do not “decouple” when $U_2 \gg R_1$ is that $U$ is scale independent. A second reason is that leading $U_2$ dependence of $\Omega_i$ similarly to its $T_2$ dependence arises from (12) as an effect of “mixing” of states associated each with one of the two dimensions. Indeed $J_2$ is the result of one tower of momentum states (sum over $m_1$) due to one dimension, combined with that of the second dimension, the Poisson re-summed “zero” state ($p = 0$) in $J_2$ (see (3) from which $J_2$ arises). A final reason for the non-decoupling of $U$ dependence is that it is due to contributions of infinitely many states associated with the small(er) dimension $R_1$, therefore their overall effect does not necessarily decouple. As a further illustration of this point consider the following case with $U_2 > 1$

$$R_2 \gg R_1 \equiv \frac{k}{\Lambda \sin \theta} \gg \frac{1}{\Lambda}, \quad k = O(1).$$

(20)

The last inequality is due to (18) and requires small $\theta, \theta \ll 1$ which together with the assumption $U_2 > 1$ gives $R_2 \gg R_1$ of eq. (20). Given this relation between the radii, the case is “effectively” one dimensional. The leading behaviour of $\Omega_i$ of (18) is then

$$\Omega_i = -\ln \frac{T_i}{4\pi} \left[ 4\pi e^{-\gamma_E} e^{-k\Lambda R_2 (\Lambda R_2 \sin \theta)^2} \left| \eta(R_2/R_1 e^{i\theta}) \right|^4 \right] \sim \frac{T_i}{4\pi} (k\Lambda R_2)$$

(21)

and a linear divergence emerges, as expected for an effectively one-dimensional case, $R_2 \gg R_1$. This result should be compared to that for one extra dimension ($R_2$) only. The contribution to the gauge thresholds in the one dimensional case (A-1) is

$$\Omega_i^{(1)} = -\ln \frac{b_i}{4\pi} \left[ 4\pi e^{-\gamma_E} e^{-2\Lambda R} (\Lambda R)^2 \right] \sim \frac{b_i}{4\pi} (2\Lambda R_2).$$

(22)

Comparing (22) and (21) we conclude that the coefficient of the leading (linear) ultraviolet divergence is in general different in the two cases, unless a very specific choice is made, ($k = 2$). Therefore, in addition to the (sub-dominant) shape effects which do not decouple in the limit $R_2 \gg R_1$, we see the same is true for the more important UV contribution $\Lambda R_2$, which has a different coefficient. The difference is essentially due to additional (infinitely many) states of large mass $\sim 1/R_1 \gg 1/R_2$, c.f. (21), included in (21) but not present in (22), and which control the value of this UV (linear) divergence coefficient.

In the light of this discussion, applied to the two-torus compactification of the previous section, we can say (from a field theory point of view) that the coefficient (equal to 1) of the quadratic divergence in (13) is different from that equal to $\pi/3$ of the (heterotic) string case (2) due to additional states of large mass. Such states exist in the string case and may contribute, and were not considered in eq. (14) (winding modes).

4 The link with string theory

Since the field theory result (13) is close to both the heterotic and type I string results, eqs. (2), (4), a comparison of these results is performed in the following. To do this we first review the (heterotic) string calculation. We then point out the similarities and differences with our field theory calculation.

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6This argument applies for $U_2 \gg 1$. For $U_2 \ll 1$ see $J_i$ of eq. (14).
4.1 Heterotic String Case

In this subsection we review the standard calculation of the heterotic string leading to \( (2) \). We consider a six-dimensional supersymmetric string vacuum compactified on a two torus \( T^2 \); the remaining dimensions are associated with higher amount of supersymmetry and do not contribute to the holomorphic coupling. The string correction is

\[
\Delta_i = \frac{b_i}{4\pi} \int_{\Gamma} \frac{d\tau_1 d\tau_2}{\tau_2} (Z_{\text{torus}} - 1),
\]

where \(-1\) subtracts out the contribution of massless modes included separately in \( (1) \). The fundamental domain is defined by \( \Gamma = \{ \tau_2 > 0, |\tau_1| < 1/2, |\tau| > 1 \} \) and \( \tau = \tau_1 + i\tau_2 \) is the modulus of the world sheet torus. The usual partition function in terms of winding modes \((n_{1,2})\) and Kaluza-Klein modes \((m_{1,2})\) is (for an additional discussion see Appendix \( D \))

\[
Z_{\text{torus}} = \sum_{n_{1,2},m_{1,2} \in \mathbb{Z}} \exp \left[ 2\pi i \tau (m_1 n_1 + m_2 n_2) \right] \exp \left[ -\frac{\pi \tau_2}{T U_2} (T U_2 + T n_1 - U m_1 + m_2) \right] \]  
\[
= \frac{T_2}{\tau_2} \sum_A e^{-2\pi i T \det A} \exp \left[ -\frac{\pi T_2}{\tau_2 U_2} \left( (1 \quad U) \left( \begin{array}{c} \tau_1 \\ 1 \end{array} \right) \right)^2 \right], \quad A = \begin{pmatrix} n_1 & p_1 \\ n_2 & p_2 \end{pmatrix},
\]

with the sum over \( 2 \times 2 \) matrices \( A \) with integer elements. In the last step a Poisson re-summation over the original Kaluza-Klein modes \( m_{1,2} \) was performed, \( m_{1,2} \rightarrow p_{1,2} \), to give two double sums over \( p_{1,2} \) and original winding modes \( n_{1,2} \).

Following eqs. \( (23) \) and \( (24) \), the heterotic string result can be written as a sum over the orbits of the modular group \( SL(2, \mathbb{Z}) \) in the space of \( GL(2, \mathbb{Z}) \) to give

\[
\Delta_i = \frac{T_i}{4\pi} \left[ \mathcal{J}(A=0) + \mathcal{J}(\det A=0) + \mathcal{J}(\det A \neq 0) + \int_{\Gamma} \frac{d\tau_1 d\tau_2}{\tau_2} (-1) \right].
\]

The first contribution (zero orbit) is due to (Poisson re-summed) Kaluza-Klein modes only with \( p_{1,2} = 0 \) (no winding modes, \( n_{1,2} = 0 \)), which may be due to infinitely many original Kaluza-Klein modes \( m_{1,2} \). The second contribution \( \mathcal{J}(\det A=0) \) (degenerate orbit) and the third \( \mathcal{J}(\det A \neq 0) \) (non-degenerate orbit) mix both windings \( (n_{1,2}) \) and original momentum \( m_{1,2} \) (or equivalently \( p_{1,2} \)) modes. Finally the integral of the last term accounts for the (field theory) massless states \( (m_{1,2} = n_{1,2} = 0) \) contribution. The first contribution obtained after integrating over the fundamental domain \( \Gamma \) is finite

\[
\mathcal{J}(A=0) = \int_{\Gamma} \frac{d\tau_1 d\tau_2}{\tau_2} T_2 = \frac{\pi}{3} T_2.
\]

The domain of integration \( \Gamma \) ensures the heterotic string gives a finite result.

The degenerate orbit contribution mixes both winding and Kaluza-Klein modes. An “unfolding” procedure of the fundamental domain \( [11] \) is then used to re-write the four sums over (Poisson re-summed, \( p_{1,2} \)) Kaluza-Klein and winding \( (n_{1,2}) \) modes as two sums over a mixture of these modes,

\footnote{Throughout this work we make a distinction between original momentum modes \( m_{1,2} \) and the Poisson re-summed “modes”, \( p_{1,2} \). In particular \( p_{1,2} = 0 \) may correspond to a large number of original Kaluza-Klein states.}
while the remaining two sums “conspire” with the fundamental domain of integration to give instead
an integral over the half strip only, \((\tau_2 > 0, -1/2 < \tau_1 < 1/2)\). The integral over \(\tau_1\) is then trivial to
give
\[
J^{(\det A=0)} = T_2 \int_0^\infty \frac{d\tau_2}{\tau_2^2} \sum_{j,p \in \mathbb{Z}} e^{-\frac{\pi T_2}{2} (j + U p)^2} \left[ 1 - e^{-\frac{N}{\tau_2}} \right] = \ln N - \ln \left[ 4\pi e^{-2\gamma E T_2} U_2 |\eta(U)|^4 \right],
\]
(28)
Note that \(j, p\) above are not Kaluza-Klein levels, but a mixture of both original winding and (Poisson
re-summed, \(p_{1,2}\)) momentum modes. In the decoupling limit of winding modes \(\alpha' \to 0, j,p\) are
mainly (Poisson re-summed) Kaluza-Klein “levels”. To evaluate (28) an infrared (IR) regulator
\((1 - e^{(-N/\tau_2)})\) is introduced \((N \to \infty)\) since \(J^{(\det A=0)}\) has a divergence coming from the IR integration
region \((\tau_2 \to \infty)\). This IR divergence cancels out upon subtracting out the (regularised) massless
\((m_{1,2} = n_{1,2} = 0)\) contribution, see (23)
\[
\int_{\Gamma} \frac{d\tau_1 d\tau_2}{\tau_2} (-1) \left[ 1 - e^{-\frac{N}{\tau_2}} \right] = -\ln N - \left[ \gamma E + 1 - \ln 3\sqrt{3}/2 \right]
\]
(29)
There remains the contribution \(J^{(\det A\neq 0)}\) whose calculation leads to
\[
J^{(\det A\neq 0)} = -\ln \prod_{n_1 > 0} \left| 1 - e^{2\pi i n_1 T} \right|^4,
\]
(30)
with \(T\) expressed as usual in \(\alpha'\) units. \(J^{(\det A\neq 0)}\) vanishes in the regime of Kaluza-Klein modes only,
of \(\alpha' \to 0\), see eqs. (23) with \(T_2 \propto 1/\alpha'\). This contribution is in essence due to the phase factor in (24)
and is zero if \(n_{1,2} = 0\) (no winding modes). More exactly, (30) is the result of a mixed contribution
(re-summed) momentum modes - winding modes, see definition of \(A\), eq. (25). Adding together
(27), (28), (29) and (30) gives the string result, eq. (2). This ends our review of the heterotic string
calculation.

4.1.1 Analysis - Heterotic string versus field theory calculation
We can now address the similarities and differences between the (heterotic) string and field theory
results.

The term \((\pi/3) T_2\) of eq. (27) (zero orbit) gives in the regime of Kaluza Klein modes \(\alpha' \to 0\)
\((T_2\text{ large})\) when winding modes are asymptotically decoupled, the leading contribution to the string
thresholds. This term corresponds at field theory level to the quadratic divergence of eq. (14), (15)
as a result of integrating there from \(\xi \to 0\). Thus the domain of integration \(\Gamma\) leading to (27) ensures
a finite (heterotic) string result, while at field theory level a regularisation is required. Further,
the coefficient \(\pi/3\) in (27) is itself a consequence of integrating over \(\Gamma\). Since \(\Gamma\) has no field theory
correspondent, a \(\pi/3\) coefficient for its quadratic divergence is not recoverable on pure field theory
grounds. In field theory this coefficient is regularisation scheme dependent, but may be chosen
to be equal to that in heterotic string, by a suitable regulator re-definition in eq. (15), \(\xi \to (3/\pi) \xi\).
Note that the leading (quadratic) UV behaviour is both in string and field theory due to Poisson
re-summed Kaluza-Klein zero modes, \(p_{1,2} = 0\), eqs. (27) and (12).

The above arguments do not imply that winding modes do not have any role in the UV. They
only mean that in the regime of \(\alpha' \to 0\) of field theory with a suitable choice or redefinition of
the regulator, the UV behaviour of the couplings can be described (in the heterotic case) on pure
field theory grounds only, eq. (15). Winding modes enable the symmetries of the string (modular invariance) and require that the \( \tau \) integration in (27) take place over the fundamental domain \( \Gamma \). They thus (indirectly) “control” the value of the resulting coefficient \( \pi/3 \) of (27), not recovered on pure field theory grounds.

In the (field theory) limit of vanishing \( \alpha' \) the result (28) (degenerate orbits) is due to Kaluza-Klein modes only, having in this limit an UV logarithmic divergence, \( \ln T_2 \) with field theory correspondent in the UV logarithmic divergence \( \ln \xi \) of (11), (15). The remaining finite part is similar in both cases. For example the whole dependence on \( U \) moduli (\( \ln U_2 |\eta(U)|^4 \)) is re-obtained by the field theory calculation. This may be explained by the fact that \( U \) is (scale) \( \alpha' \) independent. This enables one to take account at field theory level of the \( U \) dependence, as we did in eq. (7) for the mass spectrum of Kaluza Klein states. This spectrum displays the symmetry \( U \rightarrow 1/U \) (or equivalently \( R_1 \rightarrow R_2 \), \( \theta \rightarrow -\theta \)), and is not surprising that the \( U \) dependent part of the final result (15) has (unlike the \( T \) dependent part) this symmetry as well, just like the string. Also note that ignoring “zero modes” contributions, eq. (28) is equal after double Poisson re-summation (D-9) to the field theory integrand, see eqs. (5). Finally, the result (28) has some winding modes’ effects included, recovered on field theory grounds simply because the integration limit in (15), (\( \xi, \infty \)) recovers for \( \xi \rightarrow 0 \) the integration range \( (0, \infty) \) of (28). In (28) this is the result of an “unfolding” effect of (additional) summing over winding modes. Taking \( \xi \rightarrow 0 \) thus “recovers” some winding modes effects on field theory grounds, but then the field theory calculation requires an UV regularisation.

In string theory winding modes are needed in deriving (28) to unfold the fundamental domain to the integration range \( \tau_2 > 0 \) and they ensure a finite result in the UV (\( \tau_2 \rightarrow 0 \)). The result (28) has an infrared IR divergence, \( \ln N \), well known in string theory, coming from integration in the region \( \tau_2 \rightarrow \infty \) in eq. (28). Thus the sum over all modular orbits, given by the first three terms in (26) is IR divergent as well. It is also invariant under modular transformations for \( \tau \). Therefore the string corrections to the gauge coupling, although respect this symmetry, have a divergence essentially due to massless (field theory) modes contribution. Further, this divergence (\( \ln N \)) is subtracted out (see eq. (29)) so that \( \Delta H \) computes only (massive modes’) corrections to the field theory contribution, with the latter introduced separately in (1), the logarithmic term. (This last contribution is not modular invariant). Modular invariance does not “forbid” an IR divergence at string level which needs to be regularised. It is not clear to us to what extent the fact that the regulators used (for degenerate orbits integral) in string theory, are not invariant under this symmetry, should be a matter for concern, given that this symmetry is used at an earlier stage in “unfolding” and computing the same integral(s).

Finally we observe that at field theory level one only required an UV regularisation for the contribution of the massive KK modes, while at string level an IR regularisation was instead needed. The regularisation constant in field theory case (\( -\ln(4\pi e^{-\gamma}) \)) of (13) emerged from (11), (A-8), (A-3) where zeta function/DR regularisation was used. This constant is equal (see Appendix A in [14]) to that emerging at heterotic string from eq. (28), provided the same regularisation (DR) is performed in (28) in the IR regime. This implies a possible relationship between the UV and IR regimes of the field theory (which at string level is enabled by winding modes).

The contribution from the non-degenerate orbits (31) is the only part missed by the field theory result, eq. (15) and is related to the presence in string case of the phase factors in (24), (25) not present at field theory level. Indeed the Coleman Weinberg formula eq. (1) is similar to the string partition function only if \( n_{1,2} = 0 \). The dependence on \( T_1 = \text{Re} T = 2B \) in (30) where \( B \) is related to the (world sheet) antisymmetric tensor field does not have a (direct) field theory explanation either. In fact (31) is a non-perturbative correction, of topological nature (world sheet instantons), as it...
corresponds to a mixed contribution (re-summed) Kaluza-Klein - winding modes, thus unrecoverable on field theory grounds. It is interesting to note that a term of structure similar to \( \frac{\pi}{3} \) but with \( T \rightarrow U \) could however be computed by field theory, see eq. (13). Finally, the term (30) brings only a finite renormalisation of the couplings and does not affect their leading (UV) behaviour as found on field theory grounds.

4.1.2 Heterotic string regularisation of the field theory result

Here we provide a physical (string) regularisation of the effects of Kaluza-Klein states. We noticed that the coefficient \( \pi/3 \) in [27] arises as a result of integrating over \( \Gamma \) leading to a finite (power-like with the scale) result, unlike the case of field theory where a quadratic divergence appears. For a physical understanding of this result we analyse what string gives without performing the Poisson re-summation (eq. (24)), since it is not clear how to interpret (in field theory) a Poisson re-summed Kaluza-Klein state of a given level. Thus we do not necessarily require the presence of an infinite tower of Kaluza-Klein states. From eq. (23) and (24) setting \( n_{1,2} = 0 \) gives that

\[
\Delta_i(n_{1,2} = 0) \equiv \Delta^K_i = \frac{\bar{b}_i}{4\pi} \int_{-1/2}^{1/2} d\tau_1 \int_0^\infty d\tau_2 \frac{1}{T_2 U_2} \left\{ \sum_{m_1, m_2 \in \mathbb{Z}} \exp \left[ -\frac{\pi \tau_2}{T_2 U_2} |m_2 - U m_1|^2 \right] - 1 \right\}.
\]

(31)

Note the similarity of the Coleman-Weinberg formula in field theory, eq. (5), with that of (31) integrated over \( \Gamma \) instead of \( (0, \infty) \). We restrict the sums over \( m_1, m_2 \) to a finite number of terms of mass below some field theory cut-off \( \Lambda \) to find, from (31) and from the relation between \( \alpha' \) and the string scale \( M_s \) (see below eq. (3)), that

\[
\Delta^K_i = \frac{\bar{b}_i}{2\pi} \sum_{(m_1, m_2) \neq (0, 0)} \ln \frac{M_s}{M_{m_1, m_2}} - \frac{\bar{b}_i}{4\pi} \sum_{(m_1, m_2) \neq (0, 0)} \int_{-1/2}^{1/2} d\tau_1 \int_0^1 \frac{\kappa_m \sqrt{1-\tau_1^2}}{t} (e^{-t} - 1),
\]

(32)

where \( M_{m_1, m_2} \) and \( \kappa_m \) are given by

\[
M_{m_1, m_2}^2 = \frac{|m_2 - U m_1|^2}{(\alpha' T_2) U_2}, \quad (\alpha' T_2) = R_1 R_2 \sin \theta, \quad \kappa_m \equiv \pi \alpha' M_{m_1, m_2}^2.
\]

We denote by \( \tilde{\Delta}_i \) the second term in (32) and restrict both sums in (32) to terms with \( \kappa_m \ll 1 \), thus

\[
M_{m_1, m_2} \ll M_s
\]

(34)

A necessary (but not sufficient) condition for this to hold true is (see [3])

\[
M_s R_1 \sin \theta \gg m_1^2, \quad M_s R_2 \sin \theta \gg m_2^2
\]

(35)

to be compared to (17). Eqs. (34) and (33) impose restrictions not only on the radii but also on the shape parameter \( \theta \) as well. Indeed, even if the radii are large, a small angle \( \theta \) can render the mass of Kaluza-Klein states very large and then (34) is not respected. If inequality (34) holds, then

\[
\Delta^K_i(\kappa_m \leq 1) = \frac{\bar{b}_i}{2\pi} \sum_{(m_1, m_2) \neq (0, 0)}^{\kappa_m \leq 1} \ln \frac{M_s}{M_{m_1, m_2}} - \tilde{\Delta}_i(\kappa_m \leq 1),
\]

(36)

\[\text{Here we used that, for } z \ll 1 \text{ we have } \int_z^\infty \frac{dt}{t} e^{-t} = -\gamma_E - \ln z - \int_0^z \frac{dt}{t} (e^{-t} - 1) \approx -\gamma_E - \ln z.\]
which generalises a previous result of [13] with $U$ dependence (hidden in $M_{m_1,m_2}$) taken into account. The term $\Delta_i$ is negligible in the limit [14] and we are only left with (a finite number of) logarithmic terms. These are just those expected in an effective field theory approach. Eq. (36) is the effective field theory limit of the string thresholds, valid when the energy scale is below/of the order of the string scale, so that only a finite number of modes of mass small relative to this scale contribute with the usual logarithmic corrections. Comparing (36) to (15) confirms the power-like (in scale) regime of the couplings only exists in the limit of infinite number of Kaluza-Klein states.

The above overlap of the results in effective field theory to those of string theory in the case of including only a truncated tower of Kaluza-Klein modes shows how the string regularise the otherwise (quadratically) divergent behaviour of the effective field theory, by restricting to the case where the physical cut-off of the effective theory satisfies $\Lambda \leq M_s$. Similar mechanism applies at the level of vacuum energy when only a truncated tower of Kaluza-Klein states (below the string scale) is included. In this procedure modular invariance of the (heterotic) string - essentially the lower limit of integration $\sqrt{1-\tau_1^2}$ of (31) - played an important, “regularisation” role to ensure a finite result. In a sense winding modes are present in (31) by setting the lower limit of integration of the fundamental domain. Finally, note that (31) (which has $n_{1,2} = 0$) and subsequent equations correspond to the orbits with $\det A = 0$ or $A = 0$, so $\Delta_{i}^{KK}$ recovers the (leading) part due to (27) and (28). However, $\Delta_{i}^{KK}$ misses the part due to summing over winding modes $n_{1,2}$ in (31) which essentially would bring an integration limit $\tau_2 \geq 0$ rather than from $\tau_2 \geq \sqrt{1-\tau_1^2}$ as in (31) for cases with $\det A = 0$. This part was however included by the result (15) of summing an infinite tower of Kaluza-Klein states.

4.2 Type I String case

Comparing the field theory result eq. (15) with the type I string result eq. (4) we notice the following. Firstly, there is a logarithmic contribution $\ln T_2$ present in both the field and string theory case, and is traced to the infinite sum of the effects of Kaluza-Klein states. Second, there is no winding mode contribution in type I string case, this explains the absence of terms of type (30) (manifest in the heterotic case), and this again agrees with the field theory calculation, where it is absent too, eq. (15).

The similarity field theory - type I results is not surprising since the type I string calculation is in fact just a summation over a tower of momentum modes just like in field theory, but over a geometry (the annulus $A$ and Möbius $M$ strip) different from that considered in deriving (15) and this has implications for the comparison with field theory. Indeed, according to [8] one $\mathcal{N} = 2$ sub-sector gives

$$\Delta_i^I = \frac{\bar{b}_i}{4\pi} \int_0^{\infty} dt \sum_{m_1,m_2 \in \mathbb{Z}} e^{-\frac{\pi t}{T_2^2} |m_1 + Um_2|^2} \bigg|_{reg}. \quad (37)$$

The notable difference to field theory calculation is that after a Poisson re-summation, the coefficient of $T_2$ in $\Delta_i^I$ is just zero. This also differs from (27) of the heterotic case (where the fundamental domain of the torus gave a $\pi/3$ coefficient to $T_2$). Thus the would-be UV regularised quadratic divergence of the Möbius strip and annulus just cancel each other in the closed string channel, and this is marked by the symbol ”reg” in (37) to stress the need for a regularisation even at string level, for both $A$ and $M$. The coefficients these two contributions come with are controlled by the tadpole cancellation condition [8], with no clear equivalent at field theory level. (This situation is similar to the heterotic case where the coefficient $\pi/3$ of eq. (27) of the leading contribution could not be
recovered on field theory grounds). Therefore the two approaches differ by the power-like term which is present in field theory. For the same reason there is no physical regularisation as in section 4.1.2 of the heterotic case, as a smooth transition from a field theory to a type I string regime.

5 Conclusions

Using a field theory approach in a two torus compactification we computed the effects of an infinite tower of Kaluza-Klein states to the gauge couplings, and included “shape” effects not considered before. These may give significant effects even in the case $R_2 \gg R_1$, with UV behaviour different from the pure one dimensional case and with possible phenomenological implications.

The field theory result obtained using cut-off regularisation and a summation over the tower of Kaluza Klein states only is very close to that of the heterotic string. A comparison of these two results shows that the logarithmic divergence is reproduced precisely as in string theory in the limit $\alpha' \to 0$. Further, a quadratic divergence is also obtained in field theory, in agreement with the string result in this limit. The full UV behaviour of the gauge couplings may be described in a field theory approach, as due to Kaluza-Klein states, only if a suitable (re)definition of the regulator is made. This re-definition is necessary since the coefficient of the UV quadratic divergence of the string in the limit $\alpha' \to 0$ cannot be recovered on pure field theory grounds. This is because the exact value of this coefficient is controlled by the symmetries of the string not manifest at field theory level (i.e. modular invariance and thus winding modes’ presence).

Within a field theory framework one cannot say that the UV finite part of the result is identical to that in string theory because the field theory calculation corresponds at (heterotic) string level to the limit $\alpha' \to 0$ when additional string effects may vanish. We identified this part as due to topological excitations (combined effect winding - Poisson re-summed momentum modes) associated with the extra dimensions considered (world-sheet instantons). This part is unrecoverable on field theory grounds. Within a physical regularisation provided by the heterotic string, we showed how the string regularises an effective field theory result of including a finite number of momentum states, each of them bringing the usual logarithmic contribution.

The role of modular invariance in ensuring an UV finite result was discussed. It remains unclear to us the implications (if any) of using in string case an infrared regulator which does not respect world-sheet symmetries used in computing the same string integrals. We also stress that while the string result requires an IR regularisation of the (degenerate orbits) string correction, the field theory approach of summing massive KK modes contribution requires an UV regularisation only. This may actually point out a relationship between the UV and IR regimes of the theory (enabled at string level by winding modes).

In type I case there are no winding modes’ effects, and the finite part of the field theory result is just that of the string calculation; however, the issue of a physical regularisation in this case remains an open question as it seems to be no smooth transition from the field theory regime of including some (light) Kaluza-Klein states below the string scale to the full string regime.

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Appendix

A Field theory calculation of the thresholds effects. (I).

In this section we compute $J_1$ and $J_2$ of eqs. (11), (12). To compute $J_1$ we introduce the integral

$$S(\xi, \rho) = \int_\xi^\infty \frac{dt}{t} \sum_{m \in \mathbb{Z}} e^{-\pi t m^2 \rho},$$

(A-1)

and compute it in the limit $\xi \to 0$. Since

$$S(\xi, \rho) = S(\xi \rho, 1),$$

(A-2)

we will only need to evaluate explicitly $S(\xi, 1)$. With the above definition $J_1$ is given by

$$J_1 = S(\xi, 1/(T_2 U_2)),$$

(A-3)

To evaluate $S(\xi, 1)$ we proceed as follows (a prime on a sum stands for the absence of $m \neq 0$ state):

$$S(\xi, 1) = \int_1^\xi \frac{dt}{t} \sum_{m \in \mathbb{Z}} e^{-\pi t m^2} + \int_1^\infty \frac{dt}{t} \sum_{m \in \mathbb{Z}} e^{-\pi t m^2}$$

(A-4)

$$= \int_1^\xi \frac{dt}{t} \left[-1 + t^{-1/2} + t^{-1/2} \sum_{p \in \mathbb{Z}} e^{-\pi t p^2}\right] + \int_1^\infty \frac{dt}{t} \sum_{m \in \mathbb{Z}} e^{-\pi t m^2}$$

(A-5)

$$= \int_1^\xi \frac{dt}{t} \left[-1 + \frac{1}{\sqrt{t}}\right] + \int_1^{1/\xi} \frac{dt}{\sqrt{t}} \sum_{p \in \mathbb{Z}} e^{-\pi t p^2} + \int_1^\infty \frac{dt}{t} \sum_{m \in \mathbb{Z}} e^{-\pi t m^2}$$

(A-6)

$$= \left[\frac{2}{\sqrt{\xi}} + \ln \xi - 2\right] + \int_1^{1/\xi} \frac{dt}{\sqrt{t}} \left[t^{-1/2} + t^{-1}\right] \sum_{m \in \mathbb{Z}} e^{-\pi t m^2}, \quad \xi \ll 1$$

(A-7)

Under the first integral we first added/subtracted “-1” which enabled us further perform a one-dimensional Poisson re-summation eq. (D-10) and a subsequent change of variable, $t \to 1/t$, and removed the regulator in the second integral of (A-6). The integral in (A-7) is equal to the limit $\mathcal{K}(\alpha \to 0)$ where we introduced

$$\mathcal{K}(\alpha) = \int_1^\infty dt \left[t^{-1/2-\alpha} + t^{\alpha-1}\right] \sum_{m \in \mathbb{Z}} e^{-\pi t m^2} = 2 e^{-\alpha} \Gamma(\alpha) \zeta(2\alpha) - \frac{1}{\alpha(2\alpha-1)}.$$  

(A-8)

This is just the Riemman integral representation of $\zeta$ function, see e.g. (9.513), valid for all complex/real $\alpha$. The limit $\mathcal{K}(\alpha \to 0)$ is well defined and equals $\mathcal{K}(\alpha \to 0) = -\ln (4\pi e^{-2-\gamma})$. From (A-1), (A-3), (A-7) we find $S(\xi, 1)$ and the results used in (11), (22)

$$S(\xi, 1) = \ln \xi + \frac{2}{\sqrt{\xi}} - \ln (4\pi e^{-\gamma}), \quad \xi \ll 1,$$

(A-9)

$$J_1 = S(\xi/(T_2 U_2), 1) = -\ln \left[4\pi e^{-2\gamma} U_2^2 T_2^2 \xi\right] + 2 \left[\frac{T_2 U_2}{\xi}\right]^{1/2}, \quad \xi/(T_2 U_2) \ll 1,$$

(A-10)

$$S(1/\Lambda^2, 1/R^2) = -\ln \left[4\pi e^{-2\gamma}(\Lambda R)^2 e^{-2\Lambda R}\right], \quad 1/\Lambda \ll R,$$

(A-11)
To compute \( J_2 \) of eq. (12) the analysis proceeds identically. We introduce

\[
\mathcal{R}(\xi, \rho) = \int_{\xi}^{\infty} \frac{dt}{t^{3/2}} \sum_{m \in \mathbb{Z}} e^{-\pi tm^2},
\]

which we compute in the limit \( \xi \to 0 \). Since

\[
\mathcal{R}(\xi, \rho) = \sqrt{\rho} \mathcal{R}(\xi \rho, 1),
\]

we only have to compute \( \mathcal{R}(\xi, 1) \). With the above definitions, \( J_2 \) is given by

\[
J_2 = \sqrt{T_2 U_2} \mathcal{R}(\xi, U_2/T_2).
\]

We have

\[
\mathcal{R}(\xi, 1) = \int_{\xi}^{1} \frac{dt}{t^{3/2}} \sum_{m \in \mathbb{Z}} e^{-\pi tm^2} + \int_{1}^{\infty} \frac{dt}{t^{3/2}} \sum_{m \in \mathbb{Z}} e^{-\pi tm^2}
\]

\[
= \int_{\xi}^{1} \frac{dt}{t^{3/2}} \left[ -1 + t^{-1/2} + t^{-1/2} \sum_{p \in \mathbb{Z}} e^{-\pi tp^2} \right] + \int_{1}^{\infty} \frac{dt}{t^{3/2}} \sum_{m \in \mathbb{Z}} e^{-\pi tm^2}
\]

\[
= \int_{\xi}^{1} \frac{dt}{t^{3/2}} \left[ -1 + \frac{1}{\sqrt{\pi}} \right] + \int_{1}^{1/\xi} \frac{dt}{t^{3/2}} \sum_{p \in \mathbb{Z}} e^{-\pi tp^2} + \int_{1/\xi}^{\infty} \frac{dt}{t^{3/2}} \sum_{m \in \mathbb{Z}} e^{-\pi tm^2}
\]

\[
= 1 + \frac{1}{\xi} - \frac{2}{\sqrt{\pi}} + \int_{1/\xi}^{\infty} dt \left[ 1 + t^{-2/3} \right] \sum_{m \in \mathbb{Z}} e^{-\pi tm^2} = \frac{1}{\xi} - \frac{2}{\sqrt{\pi}} + \frac{\pi}{3}, \quad \xi \ll 1,
\]

where the last integral is just \( K(-1/2) = -1 + \pi/3 \), see definition (A-8). From (A-12), (A-14), (A-18) we find

\[
J_2 = \frac{T_2}{\xi} - 2 \left[ \frac{T_2 U_2}{\xi} \right]^{1/2} + \frac{\pi}{3} U_2, \quad \xi U_2/T_2 \ll 1
\]

used in eq. (12). For a generalisation of integrals (A-1) and (A-12) to cases with arbitrary powers of \( t \) in the denominators of the integrands, see Appendix C in [17].

A.1 (Vanishing) errors in the (regularised) field theory result.

While computing \( J_1, J_2, J_3 \) we introduced finite errors \( \delta_i \) (i=1,2,3) for each \( J_i \), respectively. Their overall sum should vanish in the limit \( \xi \to 0 \) for (A-1), (A-3), (A-4) to hold true. The errors arise from (A-6) for \( J_1 \) (see also (A-3), (A-2)), from (A-17) for \( J_2 \) (see also (A-14), (A-13)), and from (13) for \( J_3 \). Here we evaluate upper bounds on each of them. The errors are

\[
\delta_1 = \int_{1/\xi}^{1/\xi} \frac{dt}{t^{1/2}} \sum_{m \in \mathbb{Z}} e^{-\pi tm^2}, \quad \xi \equiv \frac{\xi}{T_2 U_2},
\]

\[
\delta_2 = U_2 \int_{1/\xi}^{0} \frac{dt}{t^{1/2}} \sum_{m \in \mathbb{Z}} e^{-\pi m^2/t}, \quad \xi \equiv \frac{\xi U_2}{T_2},
\]

\[
\delta_3 = \sqrt{T_2 U_2} \int_{1/\xi}^{0} \frac{dt}{t^{3/2}} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} e^{-\pi tm^2/T_2 - \pi T_2 U_2 n^2 / t - 2\pi i mn U_2}.
\]
To obtain bounds on these errors we use that for $a > 0$

$$\sum_{m \geq 1} e^{-am^2} \leq \sum_{m \geq 1} e^{-am} = \frac{e^{-a}}{1 - e^{-a}}, \quad (A-23)$$

With this we find bounds on $\delta_i$

$$|\delta_1| \leq \frac{2}{1 - e^{-\pi/\xi}} \int_{1/\xi}^{\infty} dt e^{-\pi t}, \quad \xi \equiv \frac{T_2 U_2}{T}, \quad (A-24)$$

$$|\delta_2| \leq \frac{2 U_2}{1 - e^{-\pi/\xi}} \int_{1/\xi}^{\infty} dt e^{-\pi t}, \quad \tilde{\xi} \equiv \frac{\xi U_2}{T}, \quad (A-25)$$

$$|\delta_3| \leq \frac{4 \sqrt{T_2 U_2}}{1 - e^{-\pi(T_2 U_2)/\xi}} \int_{1/\xi}^{\xi} \frac{dt}{t^{3/2}} \frac{e^{-\pi t U_2/T_2}}{1 - e^{-\pi t U_2/T_2}} e^{-\pi T_2 U_2/t}, \quad (A-26)$$

where for each $\delta_i$ we further “relaxed” the inequalities by replacing one factor of their integrands originating from (A-23) by the (larger) contribution in front of the integrals. For $\delta_3$ we use that (for arbitrary $T_2$, $U_2$, $t$)

$$\frac{e^{-\pi t U_2/T_2}}{1 - e^{-\pi t U_2/T_2}} \leq \frac{T_2}{\pi t U_2} \quad (A-27)$$

The integrals for $\delta_1$ and $\delta_3$ are then further approximated by the maximal value of their integrands times the integration range. We find

$$|\delta_1| \leq \frac{2}{1 - e^{-\pi(T_2 U_2)/\xi}} \left[ \frac{3}{2 \pi e} \right]^{\frac{3}{2}} \frac{\xi}{T_2 U_2}, \quad (A-28)$$

$$|\delta_2| \leq \frac{2}{1 - e^{-\pi T_2/(\xi U_2)}} \frac{U_2}{\pi} e^{-\pi T_2/(\xi U_2)}, \quad (A-29)$$

$$|\delta_3| \leq \frac{4}{1 - e^{-\pi(T_2 U_2)/\xi}} \left[ \frac{5}{2 \pi e} \right]^{\frac{5}{2}} \frac{\xi}{T_2 U_2} \frac{1}{\pi U_2^2}. \quad (A-30)$$

Sufficient, but not necessary conditions for the absolute value of the overall sum of the errors to vanish are found from the requirement each of them vanish separately (the first two are just those of (A-10) and (A-19)).

$$|\delta_1| \ll 1 : \quad \frac{T_2 U_2}{\xi} \gg 1, \quad (A-31)$$

$$|\delta_2| \ll 1 : \quad \frac{T_2 U_2}{\xi} \gg U_2^2, \quad (A-32)$$

$$|\delta_3| \ll 1 : \quad \frac{T_2 U_2}{\xi} \gg \frac{1}{U_2}. \quad (A-33)$$

Overall, a sufficient condition is

$$\frac{T_2 U_2}{\xi} \gg \max \left\{ U_2^2, \frac{1}{U_2^2} \right\}. \quad (A-34)$$
B Field theory calculation of the thresholds effects. (II).

Here we give a different evaluation of the thresholds, eq. (5) by explicitly relating proper time regularisation with dimensional or zeta function regularisation. The procedure is general, may be applied to other calculations (e.g. scalar potential in Kaluza Klein models) and is important for relating results in different regularisation schemes. The gauge independent part Ω of (5) defined by Ω_i ≡ b_i/(4π) Ω, may be split in two integrals, over (ξ, 1) and (1, ∞) to give (a “prime” on a double sum stands for the absence of (m_1, m_2) = (0, 0) state)

\[ Ω = \int_ξ^∞ \frac{dt}{t} \sum_{m_{1,2} \in \mathbb{Z}} \epsilon^{-\frac{π t}{T_2 U_2}}|U_{m_1-m_2}|^2 \] (B-1)

\[ = \int_ξ^1 \frac{dt}{t} \sum_{m_{1,2} \in \mathbb{Z}} \epsilon^{-\frac{π t}{T_2 U_2}}|U_{m_1-m_2}|^2 + \int_1^∞ \frac{dt}{t} \sum_{m_{1,2} \in \mathbb{Z}} \epsilon^{-\frac{π t}{T_2 U_2}}|U_{m_1-m_2}|^2 \] (B-2)

\[ = \int_ξ^1 \frac{dt}{t} \left[T_2 \sum_{m_{1,2} \in \mathbb{Z}} \epsilon^{-\frac{π t}{T_2 U_2}}|U_{m_1-m_2}|^2 + T_2 - 1 \right] + \int_1^∞ \frac{dt}{t} \sum_{m_{1,2} \in \mathbb{Z}} \epsilon^{-\frac{π t}{T_2 U_2}}|U_{m_1-m_2}|^2 \] (B-3)

where we used double Poisson re-summation, eq. (D-9). The second integral in (B-3) which we denote by \( L \), is well defined both in the UV and IR and is finite. All possible divergences in Ω come from the first integral in (B-3). They are due to (Poisson re-summed) “zero momentum modes” term \( T_2/t^2 \) (quadratic in scale), while \(-1/t \) (logarithmic) corresponds to original Kaluza Klein massless modes.

To compute the second integral of (B-3) (i.e. \( L \)) we consider first the quantity \( L_ε \), introduced below, eq. (B-4), which is well defined in the limit \( ε \to 0 \). In this limit \( L_ε \) equals \( L \). We have

\[ L_ε \equiv \int_1^∞ \frac{dt}{t^{1+ε}} T_2 \sum_{m_{1,2} \in \mathbb{Z}} \epsilon^{-\frac{π t}{T_2 U_2}}|U_{m_1-m_2}|^2 + \int_1^∞ \frac{dt}{t^{1+ε}} \sum_{m_{1,2} \in \mathbb{Z}} \epsilon^{-\frac{π t}{T_2 U_2}}|U_{m_1-m_2}|^2 \] (B-4)

\[ = \int_1^∞ \frac{dt}{t^{1+ε}} \left[1 - T_2 + \frac{1}{t} \sum_{m_{1,2} \in \mathbb{Z}} \epsilon^{-\frac{π t}{T_2 U_2}}|U_{m_1-m_2}|^2 \right] + \int_1^∞ \frac{dt}{t^{1+ε}} \sum_{m_{1,2} \in \mathbb{Z}} \epsilon^{-\frac{π t}{T_2 U_2}}|U_{m_1-m_2}|^2 \] (B-5)

\[ = \int_0^1 \frac{dt}{t^{1+ε}} \left[1 - \frac{T_2}{t} \right] + \int_0^∞ \frac{dt}{t^{1+ε}} \sum_{m_{1,2} \in \mathbb{Z}} \epsilon^{-\frac{π t}{T_2 U_2}}|U_{m_1-m_2}|^2 . \] (B-6)

In (B-5), we added/subtracted “\(-T_2\)” under the first integral in (B-3) which enabled us to perform a “double” Poisson re-summation (D-9), and isolated its zero mode, \( 1/t \). For the same integral a change of variable is then performed, \( t \to 1/t \) while \( ε \) presence allows us to split it into two contributions to reach eq. (B-6).

The second integral in (B-6) which we denote by \( M \) is very close to a dimensionally regularised version of Kaluza Klein states’ contributions to gauge couplings, see eq. (5). We thus related the two regularisation schemes via computing the finite quantity (\( L \)). We have

\[ M \equiv \sum_{m_{1,2} \in \mathbb{Z}} \int_0^∞ \frac{dt}{t^{1+ε}} e^{-\frac{π t}{T_2 U_2}}|U_{m_1+m_2}|^2 = \Gamma(-ε) \frac{π}{T_2 U_2} \sum_{m_{1,2} \in \mathbb{Z}} \frac{1}{|U_{m_1+m_2}|^2} , \] (B-7)
We then use that (for derivation see \cite{12})

\[
\sum_{m,n \in \mathbb{Z}} \frac{1}{|Um + n|^{2s}} = 2\zeta(2s) + \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{\Gamma(s)} |U_2|^{1 - 2s} 2\zeta(2s - 1) \\
+ \frac{8\pi s}{\Gamma(s)} |U_2|^{\frac{1}{2} - s} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \left( \frac{p}{m} \right)^{s - \frac{1}{2}} \cos(2\pi pmU_1) K_{s - \frac{1}{2}} (2\pi pm|U_2|),
\]  

with \( K \) the modified Bessel function and \( U = U_1 + iU_2 \). Notice that the source of possible divergences \( \Gamma(-\epsilon) \) in eq. (B-7) are canceled by terms of eq. (B-8). Therefore, in the remaining two terms we may put \( s = -\epsilon = 0 \). The sums over the modified Bessel function \( K_{-\frac{1}{2}} \) can be rewritten in terms of the Dedekind eta-function as

\[
\sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \sqrt{\frac{m}{p}} \cos(2\pi pmU_1) K_{-\frac{1}{2}} (2\pi pm|U_2|) = -\frac{1}{4\sqrt{U_2}} \left( \ln|\eta(U)|^2 + \frac{\pi}{6} U_2 \right).
\]  

We thus find

\[
\mathcal{M} = 2 \left[ \frac{\pi}{T_2 U_2} \right]^\epsilon \Gamma(-\epsilon)\zeta(-2\epsilon) - \ln \left[ |\eta(U)|^4 \right].
\]  

Using

\[
\Gamma(-\epsilon) = -\frac{1}{\epsilon} - \gamma_E, \quad x^\epsilon = 1 + \epsilon \ln |x|, \quad \zeta(-2\epsilon) = -\frac{1}{2} + \epsilon \ln(2\pi),
\]  

we also find the alternative form

\[
\mathcal{M} = \frac{1}{\epsilon} - \ln \left[ 4\pi e^{-\gamma_E} T_2 U_2 |\eta(U)|^4 \right].
\]  

From eqs. (B-6) and (B-12) we find the (finite) result for \( \mathcal{L} \)

\[
\mathcal{L} = -\ln \left[ 4\pi e^{-\gamma_E} e^{-T_2} T_2 U_2 |\eta(U)|^4 \right].
\]  

leading to

\[
\Omega = -\ln \left[ 4\pi e^{-\gamma_E} e^{-\frac{T_2}{\xi}} T_2 \xi U_2 |\eta(U)|^4 \right].
\]  

in agreement with the result of eq. (15) in the text.

To obtain the type of divergence (quadratic and logarithmic in scale) due to terms \( T_2/\xi \) in \( \Omega \) we used the cut-off regularization. With the zeta-function or dimensional regularisation procedure this behaviour can also be recovered (in (B-12)) if an IR regulator is introduced in addition. However, the calculations then become much more involved. (This situation is somewhat similar to the regularization of the integral \( \int d^4p \frac{1}{p^2} \), which is quadratically divergent in the cut-off scheme, but vanishes with dimensional regularisation. This is not in contradiction with (C-5) that relates these two schemes, since that integral is not well-defined. By using an IR regulator mass, the dimensional regularised result is divergent and is quadratically sensitive to this IR regulator mass.)
C The link among various regularisation schemes

In this section relations among various regularisation schemes and in particular their link with the Coleman-Weinberg integral used in the text, eq. (5), are presented. Each of them can in principle be used for the threshold calculations discussed in this work. Let $\Delta$ be the Laplacian on the compact two torus with the spectrum $M_{m_1,m_2}$ with $m_1, m_2 \in \mathbb{Z}$. All relations given in this appendix do not rely on the precise spectrum of the Laplacian $\Delta$. Let $\Box$ be the d’Alembertian of the non-compact 4 dimensional space combined with the compact torus. The basic quantity that we compute is the trace $\text{Tr}$ of the d’Alembertian

$$I = \text{Tr} \Box^{-\alpha} = \text{tr} \int \frac{d^4p_4}{(2\pi)^4} (p_4^2 + \Delta)^{-\alpha}. \quad (C-1)$$

The $\text{tr}$ is over the spectrum of the Laplacian $\Delta$ only. Such traces naturally appear when computing loop corrections. However it may require regularisation for to be well-defined.

In $\zeta$-function regularisation a complex number $\delta$ is introduced to define the operator trace $I$

$$I_\zeta(\delta) = \text{Tr} \Box^{-\alpha-\delta} = \frac{\Gamma(\alpha + \delta - 2)}{16\pi^2 \Gamma(\alpha)} \text{tr} \Delta^{2-\alpha-\delta}, \quad (C-2)$$

where after the second equal sign we have performed the integration over the non-compact four dimensions. Heat-kernel regularisation $I_H(s) = \text{Tr} \Box^{-\alpha} e^{-s\Box}$ is related by a Mellin transform to $\zeta$-function regularisation

$$\frac{1}{\Gamma(\delta)} \int_0^\infty ds s^{-1+\delta} I_H(s) = I_\zeta(\delta). \quad (C-3)$$

This result and the Coleman-Weinberg integral representation over Schwinger proper time

$$I_{CW}(\delta) = \frac{1}{16\pi^2 \Gamma(\alpha)} \text{tr} \int_0^\infty dt t^{3-\alpha-\delta} e^{-t\Delta} = I_\zeta(\delta) \quad (C-4)$$

follow directly from rescalings of the definition of the $\Gamma$-function. The Coleman-Weinberg integral is also often regulated using a cut-off $\xi$

$$I_{CO}(\xi) = \frac{1}{16\pi^2 \Gamma(\alpha)} \text{tr} \int_\xi^\infty dt t^{3-\alpha} e^{-t\Delta}, \quad \int_0^\infty d\xi \frac{\delta}{\xi^{1-\delta}} I_{CO}(\xi) = I_\zeta(\delta). \quad (C-5)$$

Another option is to use dimensional regularisation on the non-compact 4 dimensions. By integrating over the $D_4 = 4 - 2\epsilon_4$ dimensions this almost becomes the $\zeta$-function regularised result

$$I_D(\epsilon_4) = \text{tr} \int \frac{d^{D_4}p_4}{(2\pi)^{D_4}} (p_4^2 + \Delta)^{-\alpha} = (4\pi)^{\epsilon_4} I_\zeta(\epsilon_4). \quad (C-6)$$

Finally using dimensional regularisation for both compact and non-compact dimensions \[18\], the regularised expression is given by

$$I_D(\epsilon_4, \epsilon_5) = \int \frac{d^{D_4}p_4}{(2\pi)^{D_4}} \int_\Box \frac{d^{D_5}p_5}{2\pi^5} (p_4^2 + p_5^2)^{-\alpha} \text{tr} \frac{1}{p_5 - \Delta}, \quad (C-7)$$

with $D_5 = 1 - 2\epsilon_5$. This can be expressed as a trace over the Laplacian $\Delta$ and hence can be rewritten in terms of the $\zeta$-function regularised expression.
D Poisson re-summation of the string partition function

String theory on a n-dimensional torus $T^n$ is described by the partition function $Z$ obtained by summing over the combined lattice points $\Lambda_{m,n}$ spanned by the winding modes (the lattice $\Lambda_n = \Lambda_{n,0}$) and by the momentum modes (on the dual lattice $\Lambda_\alpha = \Lambda_{0,n}$). Details on string partition functions and dualities can be found in [14]. In general the lattice obtained by combining a lattice and a dual lattice of dimensions $m$ and $n$, respectively is denoted by $\Lambda_{m,n}$. Using the Poisson re-summation formulae

$$\frac{1}{V_\Lambda} \sum_{p \in \Lambda} e^{2\pi i p^T x} = \sum_{w \in \Lambda} \delta(x - w), \quad \frac{1}{V_\Lambda} \sum_{w \in \Lambda} e^{-2\pi i q^T w} = \sum_{p \in \Lambda} \delta(q - p), \quad (D-1)$$

the lattice $\Lambda_{m,n}$ can be turned into the (“winding-like”) lattice $\Lambda_{m+n}$ or into the dual (“momentum-like”) lattice $\Lambda_{m+n}$, respectively. The purpose of this section is then to show that either of these latter two lattices/descriptions are equivalent, but this does not remove effects due to mixing between winding and momentum modes (Poisson re-summed or not). In (D-1) the volume of the unit (dual) lattice cell is denoted by $V_\Lambda$, $(V_\Lambda)$. In the following we choose to apply (D-1) to the following function

$$G = \sum_{v \in \Lambda_{m,n}} e^{-\pi (v^T \alpha v + 2\beta^T v + c)}, \quad a = \begin{pmatrix} \alpha & \beta \\ \beta^T & \delta \end{pmatrix}, \quad b = \begin{pmatrix} \mu \\ \nu \end{pmatrix}, \quad (D-2)$$

$G$ is a function of a symmetric matrix $a = a^T$, a vector $b$ and a scalar $c$. Using the properties of Gaussian integrals, one obtains the following two representations of the function $G = \tilde{G} = \tilde{G}$ using the properties of Gaussian integrals

$$G = \frac{1}{\sqrt{\det \delta V_\Lambda}} \sum_{w \in \Lambda_{m+n}} e^{-\pi (\overline{w}^T \alpha \overline{w} + 2\beta^T \overline{w} + c)}, \quad \tilde{G} = \frac{1}{\sqrt{\det a V_\Lambda}} \sum_{v \in \Lambda_{m+n}} e^{-\pi (v^T \bar{a} v + 2\bar{b}^T v + \bar{c})}, \quad (D-3)$$

where we assumed that both $\alpha$ and $\delta$ are invertible, with

$$\overline{a} = \begin{pmatrix} \alpha - \beta \delta^{-1} \beta^T & -i\beta \delta^{-1} \\ -i\delta^{-1} \beta^T & \delta^{-1} \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} \mu - \beta \delta^{-1} \nu \\ -i\delta^{-1} \nu \end{pmatrix}, \quad \bar{c} = c - \nu^T \delta^{-1} \nu, \quad \overline{a} = \begin{pmatrix} \alpha^{-1} \\ \alpha^{-1} \beta^T \alpha^{-1} \end{pmatrix}, \quad \bar{a} = \begin{pmatrix} i\alpha^{-1} \mu \\ \nu + \beta^T \alpha^{-1} \mu \end{pmatrix}, \quad \bar{c} = c - \mu^T \alpha^{-1} \mu. \quad (D-4)$$

In obtaining $G$ the momentum modes were “Poisson re-summed” to leave a sum over the “winding-like” lattice ($\Lambda_{m+n}$) while in $\tilde{G}$ the winding modes were “Poisson re-summed” to leave a sum over the “momentum-like” lattice ($\Lambda_{m+n}$). Note that the off diagonal entries in $a$ “mix” original winding modes with Poisson re-summed momentum modes, while the off diagonal entries in $\bar{a}$ mix original momentum modes with Poisson re-summed winding modes. Also note that the matrix $\bar{a}$ becomes $\bar{a}$ if we perform the following substitutions $\alpha \to (\alpha - \beta \delta^{-1} \beta^T)^{-1}$, $\beta \to -\alpha^{-1} \beta (\delta - \beta^T \alpha^{-1} \beta)^{-1}$ and $\delta \to (\delta - \beta^T \alpha^{-1} \beta)^{-1}$.

We can now apply the above formulae in the context of string theory. The Hamiltonians for the left and right movers on the string on a $n$ dimensional torus with metric $g$ and anti-symmetric tensor $b$ is

$$H_{\pm} = \frac{1}{2} P_{\pm} T g P_{\pm} = \frac{1}{2} T^{\pm} G_{\pm} v, \quad \text{with} \quad G_{\pm} = \left( \begin{array}{cc} g - bg^{-1} b & \frac{1}{2}(\pm 1 + bg^{-1}) \\ \frac{1}{2}(\pm 1 - bg^{-1}) & \frac{1}{4} g^{-1} \end{array} \right), \quad (D-5)$$
derived from the momenta $P_\pm = \pm w + \frac{i}{2} g^{-1} p - g^{-1} bw$. The resulting string partition function reads

$$Z = \sum_{(w, p)^T \in \Lambda_{n,n}} e^{2\pi i (\tau H_+ - \bar{\tau} H_-)} = \sum_{v \in \Lambda_{n,n}} e^{-\pi v^T (2\tau_2 G_- - i\tau \Sigma) v}, \quad \Sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  \hfill (D-6)

Since $Z$ is of the standard form of eq. (D-2), we can apply the Poisson re-summation formulae (D-3) with $b = 0$ and $c = 0$. Thus only $a$ and $\bar{a}$ have to be computed in this case using (D-4). With the notation $g = 2g$ and $\bar{g} = \frac{1}{2} (g - bg^{-1} b)^{-1}$, we find for $a$ and $\bar{a}$

$$a = \frac{1}{\tau_2} \begin{pmatrix} |\tau|^2 g & -(\tau_1 + i\tau_2 g^{-1} b) \bar{g} \\ -g(\tau_1 - i\tau_2 bg^{-1}) & g \end{pmatrix}, \quad \bar{a} = \frac{1}{\tau_2} \begin{pmatrix} \bar{g} & \bar{g}(\tau_1 + i\tau_2 bg^{-1}) \\ (\tau_1 - i\tau_2 g^{-1} b)\bar{g} & |\tau|^2 \bar{g} \end{pmatrix}.$$  \hfill (D-7)

This shows that, provided that the appropriate metric is taken into account, the two descriptions below with either Poisson re-summed momentum modes or Poisson re-summed winding modes respectively, are equivalent:

$$Z = \frac{1}{\sqrt{\text{det} g}} \sum_{v \in \Lambda_n} e^{-\pi v^T \text{det} g v} = \frac{1}{\sqrt{\text{det} \bar{g}} \sum_{\bar{v} \in \bar{\Lambda}_n} e^{-\pi \bar{v}^T \text{det} \bar{g} \bar{v}}}.$$  \hfill (D-8)

This equivalence justifies an expectation for some similarity between regularised field theory (with infinite towers of momentum states only) and (heterotic) string results. As already noted, the off diagonal entries in $a$ “mix” original winding modes with Poisson re-summed Kaluza-Klein modes, while the off diagonal entries in $\bar{a}$ mix original Kaluza-Klein modes with Poisson re-summed winding modes. The off-diagonal term with $b$ does not depend on $\tau_2$ (thus unlikely to bring UV effects).

Because of the anti-symmetry of $b$ we can divide the Poisson re-summed and original Kaluza-Klein or winding modes into three classes: 1. the zero orbit, which has all Poisson re-summed Kaluza-Klein (winding) and original winding (Kaluza-Klein) modes set to zero; 2. the degenerate orbit, where the re-summed Kaluza-Klein (winding) and original winding (Kaluza-Klein) modes are linearly dependent; 3. the non-degenerate case.

By applying the formalism developed above to the case of a two torus with generic shape (as considered in the main text) one finds the “double” Poisson re-summation formula

$$\sum_{m_1, 2 \in \mathbb{Z}} e^{-\pi \frac{1}{\tau_2 |U m_1 - m_2|^2}} = T_2 \sum_{\tilde{m}_1, 2 \in \mathbb{Z}} e^{-\pi |\tilde{m}_1 + U \tilde{m}_2|^2}$$  \hfill (D-9)

where as usual $U_1$ and $U_2$ denote the real and imaginary parts of the complex shape parameter $U$, respectively. Also

$$\sum_{n \in \mathbb{Z}} e^{-\pi A(n - \sigma)^2} = \frac{1}{\sqrt{A}} \sum_{\tilde{n} \in \mathbb{Z}} e^{-\pi A^{-1} \tilde{n}^2 - 2i\pi \tilde{n} \sigma}.$$  \hfill (D-10)
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