Polysymmetric functions and motivic measures of configuration spaces

ASVIN G AND ANDREW O’DESKY

Abstract. We introduce a generalization of symmetric functions and apply the resulting theory to compute the class in the Grothendieck ring of varieties of the space of geometrically irreducible hypersurfaces of a fixed degree in projective space.

1. Introduction

Over the last few decades, there have been many fruitful connections between combinatorics and algebraic geometry. Some recent highlights of applying algebraic geometry to combinatorics are [1], [6] and [12]. A different strand of research, applying combinatorics to algebraic geometry, begins with Grothendieck and Atiyah’s definition of a lambda ring, axiomatizing the symmetric and alternating tensor power operations on the Grothendieck ring of vector bundles. In this paper, we encounter a new application of combinatorics to algebraic geometry extending this strand of research.

Let Λ denote the ring of symmetric functions, and let Λ(d) be a copy of Λ whose grading has been multiplicatively scaled by d. The aim of this paper is to introduce and give some applications of the graded λ-ring with one free generator in each positive degree,

\[ PΛ = Λ(1) \otimes Λ(2) \otimes Λ(3) \otimes \cdots, \]

which we call the ring of polysymmetric functions. This ring is closely connected with the combinatorics of splitting types and extends much of the theory of symmetric functions.

1.1. Arrangement numbers. The ring PΛ has two distinguished bases which serve a foundational role in what follows. Let \( x^{\ast \ast} = \{x_{d,j}\}_{d,j=1}^{\infty} \) be a set of indeterminates where \( x_{d,j} \) has degree \( d \) and identify \( Λ(d) \) with symmetric functions in the variables \( x_{d,\ast} \). We say the monomial \( x_{d_{1,1}}^{m_1} \cdots x_{d_{r,r}}^{m_r} \) with \( m_1, \ldots, m_r \) positive and \( r \) minimal has type \( \tau = \bar{d}\bar{m} = (d_1^{m_1} d_2^{m_2} \cdots d_r^{m_r}) \) (regarded as an unordered multiset of tuples \( (d_k, m_k) \)). The monomial polysymmetric function \( M_\tau \) of type \( \tau \) is the sum over all monomials of type \( \tau \). Next we set

\[ H_d = \sum_{\text{monomials } f \text{ of degree } d} f. \]

The complete homogeneous polysymmetric function \( H_\tau \) of type \( \tau \) is

\[ H_\tau = \prod_{d^m \in \tau} H_d(x^{m}_{\ast \ast}). \]
We will see that \( P\Lambda = \bigoplus \tau \mathbb{Z}M\tau \), \( P\Lambda \otimes \mathbb{Q} = \bigoplus \tau \mathbb{Q}H\tau \) and construct a partial order on types such that
\[
H\lambda = \sum \tau a_{\tau\lambda}M\tau = \sum \tau \leq \lambda a_{\tau\lambda}M\tau \quad (a_{\tau\lambda} \in \mathbb{Z}_{\geq 0}).
\]
The integer \( a_{\tau\lambda} \) for splitting types \( \tau = \vec{d} \vec{m} \) and \( \lambda = \vec{c} \vec{n} \) has a simple combinatorial interpretation as the number of \( \mathbb{Z}_{\geq 0} \)-matrices \( A \) satisfying
\[
A\vec{n} = \vec{m} \quad \text{and} \quad A^T \vec{d} = \vec{c}.
\]
We prove the following identity satisfied by arrangement numbers
\[
a_{\tau\lambda} = a_{\lambda'\tau'}
\]
where \( \tau' = (d \vec{m})^t = m \vec{d} \). We show in §4.1 that this involution on types clarifies some classical identities for transition matrices between bases of symmetric functions.

1.2. Coefficients of the plethystic logarithm. The \( H \) and \( M \) bases of \( P\Lambda \) arise naturally as the coefficients of the plethystic exponential and logarithm. Following [26], a plethystic exponential on a commutative ring \( R \) is a group isomorphism \( \text{Exp} : \left( tR[[t]], + \right) \to \left( 1 + tR[[t]], \times \right) \) satisfying the following properties

1. \( \text{Exp}(t) = (1 - t)^{-1} \),
2. \( \text{Exp}(at) = 1 + at + O(t^2) \),
3. \( \text{Exp}(f(t^n)) = \text{Exp}(f(t))|_{t \mapsto t^n} \),
4. for every \( k \geq 0 \) there exists \( n \geq 0 \) such that the \( k \)-jet of \( \text{Exp}(f) \) is determined by the \( n \)-jet of \( f \).

In order to study plethystic exponentials in a universal setting one may proceed as follows. To compute \( \text{Exp}(ut) \) for some element \( u \) of a \( \lambda \)-ring \( R \), it suffices to compute it once and for all by taking \( u = h_1 \) in the \( \lambda \)-ring \( \Lambda \). The ring \( \Lambda \) is the free \( \lambda \)-ring on one generator \( h_1 \), meaning that for any element \( u \) of any special \( \lambda \)-ring \( R \) there is a unique \( \lambda \)-ring homomorphism
\[
I_u : \Lambda \to R
\]
sending \( h_1 \) to \( u \). The universal expression for \( \text{Exp}(ut) \) is given by
\[
\text{Exp}(h_1t) = 1 + h_1t + h_2t^2 + h_3t^3 + \cdots
\]
where \( h_1, h_2, \ldots \) are the complete homogeneous symmetric functions, in the sense that \( \text{Exp}(ut) \) can be evaluated by applying \( I_u \) to the right-hand side. It is natural to ask for an analogous identity for a general element \( u_1t + u_2t^2 + \cdots \in tR[[t]] \) in place of \( ut \). In order to be universal, each \( u_d \) should generate an independent copy of \( \Lambda \) in some larger universal \( \lambda \)-ring. The minimal such \( \lambda \)-ring is \( P\Lambda \) which is free on the generators \( M_d = h_1 \in \Lambda_{(d)} \subset P\Lambda \), and we may regard the expansion of
\[
\text{Exp}(M_1t + M_2t^2 + M_3t^3 + \cdots) \in P\Lambda[[t]]
\]
as the universal plethystic exponential in the same sense that
\[
\exp(t) = 1 + t + \frac{1}{2}t^2 + \cdots \in \mathbb{Q}[t]
\]
is the universal exponential function. We will show that the following identity holds:

$$\text{Exp}(M_1 t + M_2 t^2 + M_3 t^3 + \cdots) = 1 + H_1 t + H_2 t^2 + H_3 t^3 + \cdots.$$ 

In other words, the $d$th coefficient of the universal plethystic exponential is the $d$th complete homogeneous polysymmetric function $H_d$, and the $d$th coefficient of the universal plethystic logarithm is the $d$th monomial polysymmetric function $M_d$.

We use this to give a combinatorial interpretation of a formula of Getzler–Kapranov. When $R$ is a special $\lambda$-ring, Getzler and Kapranov prove a formula [15, Prop. 8.6] for the plethystic logarithm

$$\text{Log} = \text{Exp}^{-1} : (1 + tR[t], \times) \to (tR[t], +)$$

$$1 + x_1 t + x_2 t^2 + \cdots \mapsto u_1 t + u_2 t^2 + \cdots.$$ 

For a partition $\lambda = (1^{n_1} \cdots m^{n_m}) \vdash m$ set $x^\lambda = x_1^{n_1} \cdots x_m^{n_m}$ and $\ell = n_1 + \cdots + n_m$. Their formula shows that the $d$th coefficient of $\text{Log}$ is

$$u_d = \sum_{d=k m, \lambda=m} (-1)^{\ell-1} \frac{\mu(k)}{k!} \binom{\ell}{n_1, \ldots, n_m} \psi_k x^\lambda$$

(1)

where $\psi_k$ is the $k$th Adams operation. We prove that these coefficients are transition matrix coefficients for the change of basis $M \to H$ in $\Lambda$. Let us call a type $\tau$ $k$-pure if $b^c \in \tau \implies c = k$, and mixed otherwise. There is a bijection between $k$-pure types of degree $d$ and partitions of degree $m = d/k$ given by $\tau = \lambda^k = (b_1^k \cdots b_r^k) \leftrightarrow \lambda : b_1 \geq \cdots \geq b_r$.

**Theorem 1.1.** Let $a^{-1}$ be the inverse of the arrangement numbers $a$ in the incidence algebra on types of degree $d$. Then

1. $a^{-1}_{rd}$ vanishes if $\tau$ is mixed.
2. If $\tau = \lambda^k$ is $k$-pure, then

$$a^{-1}_{rd} = (-1)^{\ell-1} \frac{\mu(k)}{k!} \binom{\ell}{n_1, \ldots, n_m}$$

where $\lambda = (1^{n_1} \cdots m^{n_m}) \vdash m = d/k$.

In the context of configuration spaces, $H_\tau$ (resp. $M_\tau$) corresponds to the closed $\tau$ stratum (resp. open $\tau$ stratum). The identity (1) is limited to $H_d$ and $M_d$, corresponding to the maximal type $\tau = d = d^1$ of degree $d$. This theorem incorporates (1) into the framework of inclusion-exclusion with respect to the poset of types, making it possible to compute motivic measures of general strata.

**Remark 1** (A construction of Specht). Let $G$ be a finite group and let $R(G_n)$ be the representation ring of $G_n = G^\ell \Sigma_n$ where $\Sigma_n$ is the symmetric group on $n$ letters. Induction along $G_m \times G_n \subset G_{m+n}$ makes $R(G) = \oplus_n R(G_n)$ into a commutative ring where $uv = \text{ind}_{G_{m+n}} G_m \times G_n (u \times v)$, and Specht constructed a ring isomorphism [25, Ch. I, Appendix B]

$$R(G) \cong \bigotimes_{c \in \text{Conj}(G)} \Lambda.$$ 

For polysymmetric functions, there is an isomorphism $R(SL_2) \cong \mathfrak{sl}_2 \Lambda$. 

**POLYSYMMETRIC FUNCTIONS AND CONFIGURATION SPACES**

3
1.3. Cohomology of the space of irreducible hypersurfaces. A motivating example for the geometric applications of the ring $P\Lambda$ is the following problem. Let $\text{Irr}_d = \text{Irr}_{d,n}$ denote the variety of geometrically irreducible hypersurfaces of degree $d$ in projective $n$-space. Recent work of Chen [11] shows that the cohomology of $\text{Irr}_d$ stabilizes as $d$ or $n$ goes to infinity. Chen has shown that for any cohomological degree $k$ in the stable range $2(\dim \text{Irr}_{d,n} - \dim \text{Irr}_{d,n-1} + n) \leq k \leq 2 \dim \text{Irr}_{d,n}$, the $k$th compact cohomology group is

$$H^k_c(\text{Irr}_d(\mathbb{C}), \mathbb{Q}) = \begin{cases} \mathbb{Q} & k \text{ even}, \\ 0 & k \text{ odd}. \end{cases}$$

Unfortunately this only accounts for a vanishing proportion of cohomological degrees as $d \to \infty$.

In a different direction, the compact Euler characteristic of the related space $\text{Irr}_d^R$ of $\mathbb{R}$-irreducible degree $d$ real hypersurfaces was computed by Hyde [20]. Hyde showed that

$$\chi(\text{Irr}_d^R) = \begin{cases} b_k & \text{if } d = 2^k, \\ 0 & \text{otherwise}. \end{cases}$$

where $b_k \in \{0,1,-1\}$ is defined by the following rule: any positive integer $n$ may be uniquely expressed as an alternating sum of descending powers of 2,

$$n = 2^{k_{2m}} - 2^{k_{2m-1}} + 2^{k_{2m-2}} - \cdots + 2^{k_2} - 2^{k_1},$$

for integers $k_{2m} > \cdots > k_1 \geq 0$ and $b_k$ is the coefficient of $2^k$ in this expression if $k \in \{k_1, \ldots, k_{2m}\}$ or $b_k = 0$ otherwise. This formula suggests the cohomology of $\text{Irr}_d$ does not have a simple description.

Following Hyde and Chen, our strategy is to use the stratification by splitting types on the projective space of all hypersurfaces. Where Hyde used generating function methods and Chen used the spectral sequence associated to this stratification, we will use plethysm and polysymmetric functions to compute the motivic class of $\text{Irr}_d$ in the Grothendieck ring of varieties $K_0(\text{Var})$ over any base field. Our formula determines the contribution from unstable cohomological degrees to the number of points of $\text{Irr}_d$ over finite fields. For the stratum of hypersurfaces with an arbitrary splitting type we give a weaker result, namely a formula for its class in a certain quotient ring of $K_0(\text{Var})$.

**Theorem 1.2.** The class of $\text{Irr}_d$ in the Grothendieck ring of varieties $K_0(\text{Var})$ is

$$[\text{Irr}_d] = \sum_{d=km} \sum_{\lambda\vdash m} (-1)^{\ell-1} \frac{\mu(k)}{k\ell} \left( \begin{array}{c} \ell \\ n_1, \ldots, n_m \end{array} \right) \prod_{j=1}^m \left( 1 + [A^1]^k + \cdots + [A^1]^{k\left(\binom{n+j}{j}-1\right)} \right)^{n_j}.$$

Let $C(\text{Var})$ be the largest $\lambda$-quotient of $K_0(\text{Var})$ that is a binomial ring. Let $\text{Irr}_\tau$ be the space of hypersurfaces in projective $n$-space with geometric splitting type $\tau$ and let $[\text{Irr}_\tau]$ be its class in $C(\text{Var}_k)$. The binomial class $[\text{Irr}_\tau]$ is zero unless $d^m \in \tau$ implies $d = 1$ in which case

$$[\text{Irr}_\tau] = \left( [A^1]^m(n-1) + 1 \right)^{\binom{1^\tau}{1^1} \cdots \binom{1^\tau}{1^d}}$$

where $\tau[1^j]$ is the number of occurrences of $1^j$ in $\tau$. 
\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
\(d,n\) & \([\text{Irr}_d] \text{ as a polynomial in } q = [A^d] \) & \\
\hline
\((4,2)\) & \(q^{14} + q^{13} + q^{12} - 2q^{10} - 2q^9 - q^8 + q^7 + q^6\) & \\
\hline
\((5,2)\) & \(q^{20} + q^{19} + q^{18} + q^{17} - q^{15} - 3q^{14} - 3q^{13} - q^{12} + 2q^{11} + 3q^{10} + q^9 - q^8 - q^7\) & \\
\hline
\((3,3)\) & \(q^{19} + q^{18} + q^{17} + q^{16} + q^{15} + q^{14} + q^{13} - q^{11} - 2q^{10} - 3q^9 - 2q^8 - q^7 + q^5 + q^4\) & \\
\hline
\end{tabular}
\end{table}

**Figure 1.** Some cases of the theorem. Powers of \(q\) in Chen’s stable range are in bold.

**Remark 2.** Along the way we prove a general formula for the binomial class of a generalized configuration space (Theorem 6.3) with the help of a useful recurrence (Proposition 6.2) satisfied by their motivic classes.

**Remark 3.** By using the \(\lambda\)-ring \(\mathbb{Z}[q]\) in place of \(K_0(\text{Var})\), our method also computes the number \(M_{d,n}(q)\) of arithmetically irreducible degree \(d\) hypersurfaces in projective \(n\)-space over the finite field \(\mathbb{F}_q\), a problem studied by several authors [8], [9], [19], [10], [4], [5]:

\[
M_{d,n}(q) = \sum_{d=km} \sum_{\lambda=m} (-1)^{\ell-1} \frac{\mu(k)}{k\ell} \left( \ell \atop n_1, \ldots, n_m \right) \prod_{j=1}^{m} \left( 1 + q + \cdots + q^{(n+j)-1} \right)^{n_j}.
\]

When \(n = 1\) this recovers the classical formula of Gauss

\[
\frac{1}{d} \sum_{d=km} \mu(k) q^m
\]

for the number\(^1\) of monic irreducible degree \(d\) polynomials in \(\mathbb{F}_q[t]\).

**Acknowledgements.** The authors would like to thank Suki Dasher, Jordan Ellenberg, Darij Grinberg, Trevor Hyde, Laurențiu Maxim, Victor Reiner, Harry Richman for helpful discussions and comments on earlier versions of this paper. We would also like to thank Steven Sam for pointing out to us the connection with Specht’s dissertation, as well as an anonymous referee for bringing [15, Prop. 8.6] to our attention.

## 2. Polysymmetric functions

Let \(x^* = \{x_{ij}\}_{i,j=1}^\infty\) be a set of indeterminates where \(x_{ij}\) has degree \(i\). Let \(\mathbb{Z}[x^*]\) denote the ring of all formal infinite sums of monomials in \(x^*\) with integer coefficients and bounded degree. Let \(\Sigma\) denote the subgroup of permutations of \(\mathbb{Z}_{\geq 0}^2\) fixing the first coordinate. This group acts on \(\mathbb{Z}[x^*]\) by \((\sigma \cdot f)(x_{ij}) = f(x_{\sigma(ij)})\). The **ring of polysymmetric functions** is the subring

\[
\mathcal{P}\Lambda = \mathbb{Z}[x^*]^\Sigma
\]

of \(\Sigma\)-invariants in \(\mathbb{Z}[x^*]\). We set \(\mathcal{P}\Lambda_R = \mathcal{P}\Lambda \otimes_{\mathbb{Z}} R\) for a ring \(R\) and \(\mathcal{P}\Lambda = \mathcal{P}\Lambda_{\mathbb{Q}}\) (note that in the introduction we wrote \(\mathcal{P}\Lambda\) for \(\mathcal{P}\Lambda_{\mathbb{Z}}\)).

\(^1\)Strictly speaking, in the degenerate case \(d = 1\) our formula counts the point at infinity while Gauss’s does not.
Definition 2.1. A type $\tau$ of degree $d$ (written $\tau \vdash d$) is a finite multiset
$$\tau = \{(p_1, m_1), \ldots, (p_r, m_r)\}$$
of ordered pairs of positive integers satisfying $p_1m_1 + \cdots + p rm_r = d$. We also employ
exponential notation, writing $\tau = p_1^{m_1}$ or $(p_1^{m_1}, p_2^{m_2}, \ldots, p_r^{m_r})$. The $p_j$'s and $m_j$'s are called the
degrees and multiplicities of $\tau$. We let $\tau[p^m]$ denote the number of occurrences of $p^m$ in $\tau$.

Remark 4. As a general rule, types serve the same role for polysymmetric functions as par-
titions do for symmetric functions. For instance, the Hilbert series of $P\Lambda$ is the generating
function for types. It satisfies $\sum_{d=0}^{\infty} \dim \mathcal{P}_d = \sum_{d=0}^{\infty} \#{\{\tau \vdash d\}} t^d = \prod_{k=1}^{\infty} (1 - t^k)^{-\sigma_0(k)}$
where $\sigma_0(k)$ is the number of divisors of $k$.

Definition 2.2. The monomial $x_{i_1j_1}^{m_1} \cdots x_{i_rj_r}^{m_r}$ with $m_1, \ldots, m_r$ positive and $r$ minimal has
type $(i_1^{m_1} i_2^{m_2} \cdots i_r^{m_r})$. The monomial polysymmetric function $M_\tau$ of type $\tau$ is the sum over all
monomials of type $\tau$.

Remark 5. If $\tau_w$ is the multiset of integers $k$ such that $(w, k) \in \tau$, then $M_\tau$ equals
$m_{\tau_1}(x_{1*}) \cdots m_{\tau_r}(x_{d*})$ where $m_\lambda$ is the monomial symmetric function of shape $\lambda$.

Example 1.
$$M_{121213} = \sum_{1 \leq i < j < k < \infty} x_{1i}^2 x_{1j}^2 x_{1k}^3 \quad \text{and} \quad M_{121221} = \sum_{1 \leq i < j < k < \infty, i \neq j} x_{1i}^2 x_{1j}^2 x_{2k}^3.$$

Note that $M_{\tau \cup \tau'} \neq M_\tau M_\tau'$ (e.g. $M_{11} \neq M_1^2 = 2M_{11} + M_{12}$).

For a graded ring $A = \bigoplus_d A_d$ and positive integer $n$, let $A_{(n)}$ be the graded ring supported
only in degrees that are multiples of $n$ and satisfying $(A_{(n)})_{dn} = A_d$ for all $d$. There is a
natural injection $A_{\mathbb{Z},(n)} \to P\Lambda_{\mathbb{Z}}$ induced by mapping the indeterminate $y_j$ to $x_{dj}$. These
combine into a homomorphism
$$\bigotimes_{n=1}^{\infty} A_{\mathbb{Z},(n)} \cong \mathcal{P}\Lambda_{\mathbb{Z}}$$
which is an isomorphism of graded rings. In particular, the set $\{M_\tau\}_{\tau \text{ type}}$ is a linear basis
for $P\Lambda_{\mathbb{Z}}$ as a free $\mathbb{Z}$-module, and the rank of $P\Lambda_{\mathbb{Z},d}$ is the number of types of degree $d$.

Remark 6. The tensor product description of $P\Lambda$ implies that any linear basis of $\Lambda$ induces
a linear basis of $P\Lambda$ consisting of pure tensors. Among the five polysymmetric bases
constructed in this paper — $HEE \cdot PM$ — only the monomial polysymmetric basis consists
of pure tensors with respect to one of the classical bases of symmetric functions. The other
polysymmetric bases are non-classical in this sense.

2.1. Coefficients of the plethystic logarithm. Let $R$ be a commutative ring with unity.

2.1.1. $\lambda$-rings. A $\lambda$-ring structure for $R$ is a group homomorphism $\lambda : R \to 1 + tR[t]$ satisfying
$\lambda(r) = 1 + rt + O(t^2)$ for all $r$. Let $\lambda_k(r)$ denote the $k$th coefficient of $\lambda(r)$. A homomorphism
of $\lambda$-rings $\phi : R \to S$ is a ring homomorphism satisfying $\phi(\lambda_k(r)) = \lambda_k(\phi(r))$ for
all $r$. A $\lambda$-ring is special if $\lambda : R \to 1 + tR[t]$ is a $\lambda$-ring homomorphism when $1 + tR[t]$ is
equipped with its canonical $\lambda$-structure (Example 3).

Remark 7. If $\lambda = \lambda_1$ is a $\lambda$-structure for $R$, then $\sigma_1 = \lambda_1^{-1}$ is another $\lambda$-structure for $R$,
called the associated $\sigma$-structure or the opposite $\lambda$-structure.
For any integral symmetric function $f \in \Lambda_{\mathbb{Z}} = \mathbb{Z}[e_1, e_2, \ldots]$ expressed as a polynomial in elementary symmetric functions, let

$$f \circ -: R \to R$$

$$r \mapsto f \circ r = f(\lambda_1(r), \lambda_2(r), \ldots).$$

The next proposition is well-known.

**Proposition 2.3.** For any $r \in R$, the function $- \circ r: \Lambda_{\mathbb{Z}} \to R$ is a ring homomorphism, which commutes with $\lambda$-operations if and only if $(g \circ f) \circ r = g \circ (f \circ r)$ for all $f, g \in \Lambda_{\mathbb{Z}}$. If $R$ is special then $(g \circ f) \circ r = g \circ (f \circ r)$ for any $f, g \in \Lambda_{\mathbb{Z}}, r \in R$.

**Example 2** ([18, Lemma 4.1]). Let $(A, \otimes)$ be a $\mathbb{Q}$-linear tensor category and let $p_{\varepsilon, n} \in \mathbb{Q}[\Sigma_n] (\varepsilon \in \{\pm 1\})$ denote the element of the group algebra of the symmetric group given by $p_{\varepsilon, n} = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \varepsilon^\sigma \sigma$. Then $\lambda(a) = \sum_{n=0}^\infty p_{-1, n}(a^{\otimes n}) t^n$ defines a special $\lambda$-structure on the Grothendieck ring $K_0(A, \otimes)$. Its opposite $\lambda$-structure has operations $\lambda_n(a) = p_{1, n}(a^{\otimes n})$.

**Example 3.** Let $\Lambda(R) = 1 + tR[t]$. There is a ring structure on $\Lambda(R)$ whose addition is ordinary multiplication of power series and whose multiplication $*$ is determined by the formula $(1 - at)^{-1} * (1 - bt)^{-1} = (1 - abt)^{-1}$ for $a, b \in R$ and extended to general power series by linearity and $t$-adic continuity. The ring $\Lambda(R)$ (ring of big Witt vectors of $R$ [23]) has a canonical $\lambda$-structure for which it is a special $\lambda$-ring [29, §2].

2.1.2. **Adams operations.** The $n$th Adams operation $\psi_n: R \to R$ is the function defined by $\psi_n = p_n \circ -$ where $p_n$ is the $n$th power symmetric function. It is well-known that Adams operations are additive (see e.g. [2] for the case of special $\lambda$-rings). If the $\lambda$-structure is special, then $\psi_n$ is a $\lambda$-endomorphism and $\psi_{\ell n} = \psi_n \psi_{\ell}$ (function composition).

**Lemma 2.4.** Let $f: R \to S$ be a ring homomorphism between $\lambda$-rings and assume $S$ has no additive torsion. If $f$ commutes with Adams operations then $f$ commutes with $\lambda$-operations.

**Proof.** This follows from Newton’s identities and an induction argument, see [29, Corollary 3.16]. The same proof there given for special $\lambda$-rings works for general $\lambda$-rings.

2.1.3. **Coefficients of the plethystic logarithm.** Let $R$ be a not necessarily special $\lambda$-ring without additive torsion and write $\text{Log}(1 + x_1 t + x_2 t^2 + \cdots) = u_1 t + u_2 t^2 + \cdots$. Equivalently, $x_1, x_2, \ldots$ are elements in $R$ satisfying

$$1 + \sum_{k=1}^\infty x_k t^k = \prod_{d=1}^\infty \exp \left( \sum_{r=1}^\infty \psi_r u_d \frac{t^r}{r} \right).$$

(2)

The formula for $u_d$ proven in [15, Prop. 8.6] assumed that $R$ was special. This assumption is too strong in practice (we will later take $R = K_0(\text{Var}_k)$ the Grothendieck ring of varieties which is not special) so here we give a simple proof of the same formula under minimal assumptions.

**Theorem 2.5.** Suppose that $x_1, \ldots, x_d$ are contained in a $\lambda$-subring of $R$ on which Adams operations separate, i.e. $\psi_{mn} = \psi_m \psi_n$. Then

$$u_d = \sum_{d=km \lambda=m} (-1)^{\ell - 1} \frac{\mu(k)}{k\ell} \binom{\ell}{n_1, \ldots, n_m} \psi_k x^\lambda.$$
Remark 8. Getzler–Kapranov [15] attribute this formula to Cadogan [7]. In the special case when $R = \mathbb{Q}[t]$ this was rediscovered in [14, Lemma 6.3].

Lemma 2.6. Suppose a ring $R$ is equipped with additive functions $\psi_k: R \to R$. If $(x_k)_{k=1}^d$ and $(u_k)_{k=1}^d$ are elements in $R$ such that $\psi_1 u_d = u_d$ and $\psi_{ab} u_c = \psi_a \psi_b u_c$ for any integers $a, b, c$ with $abc = d$, and satisfying

$$x_d = \sum_{k|d} \psi_{d/k} u_k,$$

then

$$u_d = \sum_{k|d} \mu(d/k) \psi_{d/k} x_k.$$

Proof. \[\sum_{k|d} \mu(d/k) \psi_{d/k} x_k = \sum_{k|d} \mu(d/k) \psi_{d/k} \sum_{\ell|k} \psi_{k/\ell} u_\ell \]

\[= \sum_{\ell|d} \mu(d/k) \psi_{d/\ell} u_\ell \]

\[= \sum_{\ell|d} \psi_{d/\ell} u_\ell \sum_{n|d/\ell} \mu(n) = u_d. \]

Call a subset separable if it satisfies the hypothesis of the theorem.

Lemma 2.7. \(\{x_1, \ldots, x_d\}\) is separable if and only if \(\{u_1, 2u_2, \ldots, du_d\}\) is separable.

Proof. First we prove an identity. Taking the logarithm of (2) obtains

$$\log \left(1 + \sum_{n=1}^\infty x_n t^n\right) = \sum_{d,k=1}^\infty \psi_k u_d \frac{t^d}{k} = \sum_{m=1}^\infty \frac{t^m}{m} \sum_{k|m} \psi_k u_{m/k} \frac{m}{k}.$$

The coefficient of $\frac{t^d}{k}$ on the left-hand side is $M_k(x_1, \ldots, x_k)$ where $M_k(h_1, \ldots, h_k)$ is defined by\(2\)

$$1 + h_1 t + h_2 t^2 + \cdots = \exp \left( \sum_{k=1}^\infty p_k \frac{t^k}{k} \right) = \exp \left( \sum_{k=1}^\infty M_k(h_1, \ldots, h_k) \frac{t^k}{k} \right).$$

Then

$$M_d(x_1, \ldots, x_d) = \sum_{k|d} \psi_{d/k} u_k k. \quad (3)$$

With this identity in hand, now suppose \(\{x_1, \ldots, x_d\}\) is separable. As $x_1 = u_1$, the singleton set \(\{u_1\}\) is separable. Assume we have shown \(\{u_1, \ldots, (d-1)u_{d-1}\}\) is separable. By (3),

$$u_d d = M_d(x_1, \ldots, x_d) - \sum_{k|d, k \neq d} \psi_{d/k} u_k k.$$

The complete homogeneous symmetric functions are an integral basis for Λ which implies that $M_d(x_1, \ldots, x_d)$ is contained in any ring generated by the $x_1, \ldots, x_d$. This shows the

\(\text{2}^\text{The } m\text{th Faber polynomial } F_m \text{ is defined by } \sum_{m=0}^\infty F_m(a_1, \ldots, a_m; t) z^{-m-1} = \frac{\partial}{\partial t} \log(f(z) - t) \text{ where } f(z) = z + a_1 + a_2 z^{-1} + a_3 z^{-2} + \cdots. \text{ They satisfy } F_m(h_1, \ldots, h_m; 0) = -M_m(h_1, \ldots, h_m).\)
right-hand side is contained in a separable λ-subring of \( R \) by hypothesis, which shows that \( u_d \delta \) is as well.

Conversely, if \( \{u_1, 2u_2, \ldots, du_d\} \) is separable, then it is clear from (2) that each \( x_k \) is contained in a separable λ-subring of \( R \).

**Proof of Theorem 2.5.** By Lemma 2.7 the Adams operations separate over \( \{u_1, \ldots, du_d\} \), so we may apply Möbius inversion with Adams operations to (3) to obtain

\[
\begin{align*}
u_d \delta &= \sum_{m|d} \mu(d/m) \psi_{d/m} M_m(x_1, \ldots, x_m) \\
&= \sum_{m|d} \mu(d/m) \psi_{d/m}(\sum_{n_1+2n_2+\cdots+m_n=m} \frac{m(-1)^{1+\sum_k n_k} (\sum_k n_k)!}{n_1! \cdots n_m!} \prod_{k=1}^m x_k^{n_k}).
\end{align*}
\]

where the expression for \( M_m \) comes from (7). Note the right-hand side is actually an integral expression in powers of \( x_1, \ldots, x_d \) and Adams operations in view of the first equality. \( \square \)

2.1.4. Binomial rings. A ring \( B \) satisfying any of the equivalent conditions of the following proposition is a binomial ring.

**Proposition 2.8** ([29, §5]). The following statements are equivalent:

1. \( B \) is \( \mathbb{Z} \)-torsion-free and \( (\binom{b}{k}) = \frac{b! (b-k+1)!}{k!} \in B \subset B \otimes \mathbb{Q} \) for all \( b \in B \) and \( k \geq 1 \),
2. \( B \) is \( \mathbb{Z} \)-torsion-free and \( a^p \equiv a \pmod{pB} \) for any \( a \in B \) and prime \( p \),
3. \( B \) is a special λ-ring with trivial Adams operations.

The λ-operations on a binomial ring are given by \( \lambda_k(b) = \binom{b}{k} \).

**Remark 9.** Any λ-ring \( R \) has a canonical quotient binomial ring \( R_B \), namely the quotient of \( R \) by the smallest ideal closed under λ-operations containing the additive torsion of \( R \) and \( \psi_m(r) - r \) for every \( r \in R \) and \( m \geq 1 \).

The following should be well-known though we do not know a reference. We leave the proof to the reader.

**Proposition 2.9.** Let \( \lambda = (1^{n_1} 2^{n_2} \cdots d^{n_d}) \) be a partition of degree \( d \). If \( B \) is a binomial ring, then for any \( x \in B \) we have

\[
m_\lambda \circ x = \left( \begin{array}{c} x \\ n_1, \ldots, n_d \end{array} \right).
\]

2.2. Higher plethysm. Let \( R \) be a λ-ring. Classically plethysm is defined as a function \( f \circ \cdot : R \to R \) for \( f \in \lambda \). Here we extend it to a function \( F \circ \cdot : tR[t] \to R \) for \( F \in \mathcal{P}_\lambda \).

(Classical plethysm is recovered by identifying \( R \xrightarrow{\sim} tR \subset tR[t] \).) The idea is that the higher copies \( \Lambda_{(1)}, \Lambda_{(2)}, \ldots \subset \mathcal{P}_\lambda \) “witness” the higher order terms of \( r_1 t + r_2 t^2 + \cdots \in tR[t] \).

Any polysymmetric function \( F \in \mathcal{P}_\lambda \mathbb{Z} = \bigotimes_d \Lambda_{\mathbb{Z},(d)} \) can be expressed as a polynomial

\[
F(f_1, \ldots, f_k)
\]

where \( f_j \in \Lambda_{\mathbb{Z},(j)} \). Let

\[
F \circ \cdot : tR[t] \to R
\]
be the function defined by
\[ F \circ (r_1 t + r_2 t^2 + \cdots) = F(f_1 \circ r_1, \ldots, f_k \circ r_k). \]

We state some basic facts about higher plethysm, omitting the proofs as they are analogous to the ordinary case.

**Proposition 2.10.** Let \( r = r_1 t + r_2 t^2 + \cdots \in tR[t] \).

1. \( - \circ r \) is a ring homomorphism \( \mathcal{P}A_Z \to R \) and it commutes with \( \lambda \)-operations if and only if \( (g \circ f) \circ r = g \circ (f \circ r) \) for all \( f, g \in \mathcal{P}A_Z \). If \( R = \bigoplus_d R_d \) is a graded \( \lambda \)-ring\(^3\) and \( r_d \in R_d \) for all \( d \), then \( - \circ r : \mathcal{P}A_Z \to R \) is a graded ring homomorphism.

2. There is a natural bijection \( \text{Hom}_{\text{GrLaRing}}(\mathcal{P}A_Z, R) \overset{\sim}{\to} \prod_{d=1}^\infty R_d \phi \mapsto (\phi(M_d))_d \).

### 3. Families of polysymmetric functions

#### 3.1. Definitions and identities.

The length \( \text{len} f \) of a monomial \( f \) is defined by \( y^{\text{len} f} f(x_{**}) = f(yx_{**}) \). Let \( d \) be a positive integer.

- The \( d \)th complete homogeneous polysymmetric function is
  \[ H_d = \sum_{\text{monomials } f \text{ of degree } d} f. \]
- The \( d \)th elementary polysymmetric function is
  \[ E_d = \sum_{\text{sq.free monomials } f \text{ of degree } d} (-1)^{\text{len} f} f. \]
- The \( d \)th unsigned elementary polysymmetric function is
  \[ E_d^+ = \sum_{\text{sq.free monomials } f \text{ of degree } d} f. \]
- The \( d \)th power polysymmetric function is
  \[ P_d = \sum_{k|d} k \sum_{\text{monomials } f \text{ of type } k^{d/k}} f = \sum_{k|d} k \sum_{j=1}^\infty x_{k,j}^{d/k}. \]

For a type \( \tau \) we define
\[ H_\tau = \prod_{d^n \in \tau} \psi_m(H_d) = \prod_{d^n \in \tau} H_d(x_{**}^m) \]
where \( \psi_m \) is the \( m \)th Adams operation on \( \mathcal{P}A \). The polysymmetric functions \( E_\tau, E_\tau^+ \) and \( P_\tau \) are defined analogously.\(^4\)

\(^3\)A graded \( \lambda \)-ring is a graded ring with \( \lambda \)-operations satisfying \( \lambda_n(R_d) \subset R_{nd} \) for all \( n \) and \( d \).

\(^4\)\( H_2 \) always means \( \psi_2 H_2 = H_2(x_{**}^2) \) and never \( H_4 \).
Remark 10. One can also define $H_{dm}$, $E_{dm}$, and $E_{dm}^+$ with generating functions: $\sum_d H_{dm} t^{md} = \prod_{i,j} (1 - x_{ij} t^{mi})^{-1}$, $\sum_d E_{dm}^+ t^{md} = \prod_{i,j} (1 + x_{ij} t^{mi})$, and $\sum_d E_{dm} t^{md} = \prod_{i,j} (1 - x_{ij} t^{mi})$.

These bases behave similarly to their symmetric siblings. For instance, the polysymmetric analogue of

$$\sum_{d=0}^{\infty} h_d t^d = \left( \sum_{d=0}^{\infty} (-1)^d e_d t^d \right)^{-1} = \sum_{\lambda \text{ partition}} m_{\lambda} t^{\|\lambda\|} = \exp \left( \sum_{k=1}^{\infty} \frac{t^k}{k} \right)$$

is

$$\sum_{d=0}^{\infty} H_d t^d = \left( \sum_{d=0}^{\infty} E_d t^d \right)^{-1} = \left( \sum_{d=0}^{\infty} E_{d}^+(-x_{ss}) t^d \right)^{-1} = \sum_{\tau \text{ type}} M_{\tau} t^{\tau} = \exp \left( \sum_{k=1}^{\infty} P_k \frac{t^k}{k} \right).$$

We record a list of further identities for future reference:

$$\sum_{d=0}^{\infty} H_d t^d = \prod_{n=1}^{\infty} \sigma_n(M_n), \quad \sum_{d=0}^{\infty} H_d t^d = \prod_{n=1}^{\infty} \sum_{k=0}^{\infty} h_{k,(n)} t^{kn},$$

$$\sum_{d=0}^{\infty} H_d t^d = \sum_{\tau \text{ type}} M_{\tau} t^{\tau}, \quad \sum_{d=0}^{\infty} E_d t^d = \sum_{\tau \text{ squarefree}} M_{\tau} t^{\tau},$$

$$\sum_{k=0}^{d} H_{k^2} E_{d-k}^+ = H_d, \quad \sum_{k=0}^{d} H_k E_{d-k} = \delta_{d,0},$$

$$P_d = \sum_{k|d} k \psi_{d/k} M_k, \quad M_d = \frac{1}{d} \sum_{k|d} \mu(\frac{d}{k}) \psi_{d/k} P_k.$$  

Remark 11. In the language of polysymmetric functions, Gorsky’s formula [16, Theorem 1] expresses $M_d$ in terms of the power polysymmetric basis.

3.2. Generating properties.

Theorem 3.1. Each of the sets $\{E_{\tau}^+\}_{\tau \text{ type}}$, $\{H_{\tau}\}_{\tau \text{ type}}$ is a linear basis of $\mathcal{P}\Lambda$. Each of the sets $\{E_{dm}^+\}_{d,m=1}^{\infty}$, $\{H_{dm}\}_{d,m=1}^{\infty}$ is an algebraic basis of $\mathcal{P}\Lambda$.

Proof. The matrix coefficients in the monomial polysymmetric basis of the linear mapping of $\mathcal{P}\Lambda_d$ defined by $M_\tau \mapsto H_\tau$ are given by $a$. This matrix is invertible as $a$ is invertible in the incidence algebra $I_d$ of types over $\mathbb{Q}$ (see Proposition 4.5 below). Thus, the set $\{H_{\tau}\}_{\tau \mapsto d}$ is a $\mathbb{Q}$-basis of $\mathcal{P}\Lambda_d$. The set $\{H_{\tau}\}_{\tau \text{ type}}$ is precisely the set of monomials in $\{H_{dm}\}_{d,m=1}^{\infty}$, which shows that $\mathcal{P}\Lambda$ is free as a $\mathbb{Q}$-algebra on the set $\{H_{dm}\}_{d,m=1}^{\infty}$. The same argument works with $E_{\tau}^+$ in place of $H_{\tau}$ and $e$ in place of $a$. \(\square\)

The above theorem is useful for constructing maps out of $\mathcal{P}\Lambda$.

Proposition 3.2. The ring endomorphism $\Omega: \mathcal{P}\Lambda \to \mathcal{P}\Lambda$ defined by $\Omega(H_{dm}) = E_{dm}$ is an involution.

Proof. Extend $\Omega$ to an endomorphism of $\mathcal{P}\Lambda[t]$ by acting trivially on $t$. Define $\mathcal{H} := \sum_{d=0}^{\infty} H_d t^d$ and $\mathcal{E} := \sum_{d=0}^{\infty} E_d t^d$. We have the factorizations $\mathcal{H} = \prod_{i,j=1}^{\infty} (1 - x_{ij} t^i)$ and $\mathcal{E} = \prod_{i,j=1}^{\infty} (1 - x_{ij} t^i)$, and so $1 = \Omega(\mathcal{H})\Omega(\mathcal{E}) = \mathcal{E}\Omega(\mathcal{E})$ which implies $\Omega(\mathcal{E}) = \mathcal{E}^{-1} = \mathcal{H}$. We
conclude $\Omega(E_d) = H_d$ for all $d$. Running the same argument for $\psi_m(\mathcal{E})$ and $\psi_m(\mathcal{H})$ shows that $\Omega(E_{dm}) = H_{dm}$.

**Corollary 3.3.** $\{E_{dm}\}_{d,m=1}^\infty$ is an algebraic basis of $\mathcal{P}\Lambda$. $\{E_\tau\}_\tau$ type is a linear basis of $\mathcal{P}\Lambda$.

### 3.3. A particular specialization.

Let $R$ denote the ring $\Lambda_Q$ equipped with the unique $\lambda$-structure for which it has trivial Adams operations (this differs from its standard $\lambda$-structure). There is a unique $\lambda$-homomorphism satisfying

$$\Phi: \mathcal{P}\Lambda \to R$$

$$M_d \mapsto q^d$$

where $q_1, q_2, q_3, \ldots \in \Lambda_Q$ are defined by

$$\sum_{d=0}^\infty h_d t^d = \prod_{d=1}^\infty (1 - t^d)^{-q_d}.$$

**Proposition 3.4.** $\Phi(H_{km}) = h_k$ and $\Phi(P_{km}) = p_k$ for any $k, m \geq 1$.

**Proof.** As $R$ has trivial Adams operations and $\Phi$ is a $\lambda$-homomorphism (thus commuting with Adams operations) it suffices to consider the case when $m = 1$. Applying $\Phi$ to

$$\sum_{d=0}^\infty H_d t^d = \prod_{d=1}^\infty \sigma_d(M_d) = \prod_{d=1}^\infty \exp \left( \sum_{k=1}^\infty \psi_k M_d \frac{t^{kd}}{k} \right)$$

results in

$$\sum_{d=0}^\infty \Phi(H_d)t^d = \prod_{d=1}^\infty \exp \left( \sum_{k=1}^\infty q_d \frac{t^{kd}}{k} \right) = \prod_{d=1}^\infty (1 - t^d)^{-q_d}$$

which proves $\Phi(H_d) = h_d$. Now applying $\Phi$ to

$$1 + H_1 t + H_2 t^2 + H_3 t^3 + \cdots = \exp(P_1 + P_2 \frac{t^2}{2} + P_3 \frac{t^3}{3} + \cdots)$$

results in

$$1 + h_1 t + h_2 t^2 + h_3 t^3 + \cdots = \exp(\Phi(P_1) + \Phi(P_2) \frac{t^2}{2} + \Phi(P_3) \frac{t^3}{3} + \cdots),$$

which shows that $\Phi(P_d) = p_d$ as the left-hand side is $\exp(p_1 + p_2 \frac{t^2}{2} + p_3 \frac{t^3}{3} + \cdots)$. □

### 4. Transition matrices

Now we turn to the relationships between polysymmetric bases. It is well-known that the matrix coefficients between the classical bases of symmetric functions admit combinatorial interpretations. These generalize naturally to the polysymmetric setting.

#### 4.0.1. Duality.

There is a non-classical duality on types:

$$\vec{b}^{\vec{a}} \leftrightarrow \vec{m}^{\vec{b}}.$$

Although this duality has no analogue for partitions, this turns out to be responsible for some classical facts about symmetric functions (see (1), (4) and (5) below and their generalizations (i), (iv), (v) proven using the duality of types).

---

\(^5\)One can show that $\{q_d\}_{d=1}^\infty$ is an algebraic basis of $\Lambda_Q$. These symmetric functions seem to have received little attention from the literature.
4.1. **Polysymmetric extensions of some classical constructions.** Let λ and µ denote partitions of degree d. Let us recall a few classical constructions:

- the bilinear form $\langle \cdot, \cdot \rangle: \Lambda \otimes \Lambda \to \mathbb{Z}$ defined by
  $$\langle m_\lambda, h_\mu \rangle_\Lambda = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise}, \end{cases}$$

- the numbers $M_{\lambda\mu}$ defined by the transition $e \to m$
  $$e_\lambda = \sum_{\mu \vdash d} M_{\lambda\mu} m_\mu,$$

- the numbers $N_{\lambda\mu}$ defined by the transition $h \to m$
  $$h_\lambda = \sum_{\mu \vdash d} N_{\lambda\mu} m_\mu.$$

Let $\vec{\lambda}$ denote the list of parts of $\lambda$ in weakly descending order. These constructions satisfy the following properties [28, §7]:

1. the bilinear form $\langle \cdot, \cdot \rangle_\Lambda$ is symmetric,
2. $M_{\lambda\mu}$ equals the number of $\{0, 1\}$-matrices $A$ satisfying $A\vec{\lambda} = \vec{\lambda}$ and $A^T\vec{\mu} = \vec{\mu}$,
3. $N_{\lambda\mu}$ equals the number of $\mathbb{Z}_{\geq 0}$-matrices $A$ satisfying $A\vec{\lambda} = \vec{\lambda}$ and $A^T\vec{\mu} = \vec{\mu}$,
4. $M_{\lambda\mu} = M_{\mu\lambda}$,
5. $N_{\lambda\mu} = N_{\mu\lambda}$.

Now we turn to the polysymmetric setting. Define a bilinear form by declaring

$$\langle M_\lambda, H_\tau \rangle = \begin{cases} 1 & \text{if } \tau^T = \lambda, \\ 0 & \text{otherwise}. \end{cases}$$

For any type $\lambda \vdash d$ we have

$$E^+_\lambda = \sum_{\tau \vdash d} e(\tau, \lambda) M_\tau$$

for uniquely determined integers $e(\tau, \lambda)$. Similarly,

$$H_\lambda = \sum_{\tau \vdash d} a(\tau, \lambda) M_\tau$$

for uniquely determined integers $a(\tau, \lambda)$. These transition matrices $a$ and $e$ extend $N$ and $M$, in the sense that for partitions $\lambda$ and $\mu$,

$$e(\vec{\lambda}^\top, \vec{\mu}^\top) = M_{\lambda\mu} \quad \text{and} \quad a(\vec{\lambda}^\top, \vec{\mu}^\top) = N_{\lambda\mu}.$$ 

The bilinear form $\langle \cdot, \cdot \rangle_\Lambda$ is recovered by restricting the polysymmetric bilinear form $\langle \cdot, \cdot \rangle$ to the two copies $\Lambda_1$ and $\Lambda_2$ in $PA$ given by

$$\Lambda_1 \hookrightarrow PA : m_\lambda \mapsto M_\lambda \quad \text{and} \quad \Lambda_2 \hookrightarrow PA : m_\lambda \mapsto M_\lambda^\top.$$ 

**Theorem 4.1.** Let $\tau = \vec{b}\vec{m}$ and $\lambda = \vec{c}\vec{n}$ be types of degree $d$.

(i) The bilinear form $\langle \cdot, \cdot \rangle$ is symmetric,

(ii) $e(\tau, \lambda)$ equals the number of $\{0, 1\}$-matrices $A$ satisfying

$$A\vec{m} = \vec{m} \quad \text{and} \quad A^T\vec{b} = \vec{c},$$
(iii) $a(\tau, \lambda)$ equals the number of $\mathbb{Z}_{\geq 0}$-matrices $A$ satisfying (4),
(iv) $e(\tau, \lambda) = e(\lambda^T, \tau^T)$, and
(v) $a(\tau, \lambda) = a(\lambda^T, \tau^T)$.
These generalize properties (1)-(5), respectively.

Motivated by this, we define an arrangement of $\vec{b}^\vec{n}$ into $\vec{c}^\vec{m}$ to be a $\mathbb{Z}_{\geq 0}$-matrix $A$ satisfying $A\vec{n} = \vec{m}$ and $A^T\vec{b} = \vec{c}$. An arrangement is squarefree if its entries are in $\{0, 1\}$.

**Remark 12.** Transition matrices between the $HE^+ EP$ bases and pure tensor polysymmetric bases formed from symmetric bases have been studied in [21].

4.2. **Arrangements record factorizations.** To prove Theorem 4.1 we show that arrangements record factorizations of monomials in weighted variables. Let $f_\tau(x_{rs})$ be a monomial of type $\tau$. If $m_1, \ldots, m_r$ are positive integers and $f_{1}^{m_1} \cdots f_{r}^{m_r} = f_\tau$ then we say that $(f_1, \ldots, f_r; m_1, \ldots, m_r)$ is an factorization of $f_\tau$ of type $\lambda$ where $\lambda$ is the type $\{(\deg f_1, m_1), \ldots, (\deg f_r, m_r)\}$.

Assertions (ii) and (iii) of Theorem 4.1 follow from the next two propositions.

**Proposition 4.2.** Let $\tau, \lambda \vdash d$ and let $f_\tau$ be a monomial of type $\tau$. The number of factorizations of $f_\tau$ of type $\lambda$ is equal to $a(\tau, \lambda)$. The number of factorizations of $f_\tau$ of type $\lambda$ using only squarefree monomials $f_1, \ldots, f_r$ is equal to $e(\tau, \lambda)$. In particular, both quantities are independent of the chosen monomial $f_\tau$.

**Proof.** Write $\lambda = \vec{d}^{\vec{n}}$. In the expansion of the product

$$H_\lambda = \left( \sum_{\text{monomials } f_1 \text{ of degree } d_1} f_1^{m_1} \right) \cdots \left( \sum_{\text{monomials } f_r \text{ of degree } d_r} f_r^{m_r} \right),$$

a monomial $f_\tau$ of type $\tau$ will appear each time $(f_1, \ldots, f_r; m_1, \ldots, m_r)$ is a factorization of $f_\tau$ of type $\lambda$. Collecting monomials according to their splitting type expresses $H_\lambda$ in the monomial polysymmetric basis. This proves that the number of factorizations of $f_\tau$ of type $\lambda$ equals $a(\tau, \lambda)$. Similarly, the coefficient of $M_\tau$ in the expression

$$E_\lambda^+ = \left( \sum_{\text{sq.free monomials } f_1 \text{ of degree } d_1} f_1^{m_1} \right) \cdots \left( \sum_{\text{sq.free monomials } f_r \text{ of degree } d_r} f_r^{m_r} \right)$$

is the number of factorizations of $f_\tau$ of type $\lambda$ using only squarefree monomials. \hfill $\Box$

**Proposition 4.3.** Let $\tau = \{(d_1, m_1), \ldots, (d_r, m_r)\}$ and $\lambda = \{(e_1, n_1), \ldots, (e_s, n_s)\}$. Let $f_\tau = x_{d_1,k_1}^{m_1} \cdots x_{d_r,k_r}^{m_r}$ be a monomial of type $\tau$. If $(f_1, \ldots, f_s; n_1, \ldots, n_s)$ is a factorization of $f_\tau$ of type $\lambda$, then the matrix $A$ defined by

$$f_j = x_{d_1,k_1}^{A_{1,j}} \cdots x_{d_r,k_r}^{A_{r,j}}$$

for all $j \in \{1, \ldots, s\}$ is an arrangement of $\tau$ into $\lambda$. This defines a bijection between the following sets:

1. factorizations of $f_\tau$ of type $\lambda$,
2. arrangements of $\tau$ into $\lambda$. 

This restricts to a bijection between the following subsets:

(1') factorizations of \( f_\tau \) of type \( \lambda \) using only squarefree monomials,
(2') arrangements of \( \tau \) into \( \lambda \) with entries in \{0,1\}.

Proof. Let \((f_1, \ldots, f_s; n_1, \ldots, n_s)\) be a factorization of \( f_\tau \) of type \( \lambda \). For each \( j \in \{1, \ldots, s\} \) there are uniquely determined non-negative integers \( A_{ij}, \ldots, A_{rj} \) satisfying

\[
f_j = x_{d_1 k_1}^{A_{1j}} \cdots x_{d_r k_r}^{A_{rj}}
\]

which shows that \( A \) is well-defined. The equality \( f_1^{n_1} \cdots f_s^{n_s} = f_\tau \) implies that

\[
\sum_{j=1}^s A_{ij} n_j = m_i
\]

for each \( i \in \{1, \ldots, r\} \). Since \( \deg f_j = e_j \), we also have

\[
\sum_{i=1}^r A_{ij} d_i = e_j
\]

for each \( j \in \{1, \ldots, s\} \). Conversely, it is clear that any arrangement \( A \) of \( \tau \) into \( \lambda \) determines monomials \( f_1, \ldots, f_s \) according to (5) which give a factorization \((f_1, \ldots, f_s; n_1, \ldots, n_s)\) of \( f_\tau \) of type \( \lambda \).

Restricting the factorization \((f_1, \ldots, f_s; n_1, \ldots, n_s)\) defined by \( A \) to use squarefree monomials is the same as restricting each exponent in (5) to be 0 or 1. \( \square \)

Now we finish the proof of Theorem 4.1.

Proof of (i), (iv), and (v). The equalities in (iv) and (v) are realized by a bijection defined by matrix transposition. More precisely, \( A \mapsto A^T \) gives a bijection from the set of arrangements of \( \tau \) into \( \lambda \) to the set of arrangements of \( \lambda^t \) into \( \tau^t \). This proves (iv) and (v) in view of (ii) and (iii). Since the set \( \{H_\lambda\}_\lambda \) is a linear basis of \( \mathcal{P} \Lambda \) (Theorem 3.1), the symmetry of \( \langle \cdot, \cdot \rangle \) is equivalent to (v). \( \square \)

Remark 13. Arrangements may be regarded as morphisms between types where composition is defined by matrix composition. This gives a refinement of the partial order as \( \tau \leq \lambda \iff a_{\tau\lambda} > 0 \).

4.2.1. Self-arrangements.

**Lemma 4.4.** Let \( \tau = (d_1^{m_1} d_2^{m_2} \cdots, d_r^{m_r}) \) be a type. Let \( \alpha \) be a self-arrangement of \( \tau \). Let \( f_\tau = x_{d_1 j_1}^{m_1} \cdots x_{d_r j_r}^{m_r} \) be a monomial of type \( \tau \). Then

\[
a_\alpha = (x_{d_1 j_{\alpha(1)}}, x_{d_2 j_{\alpha(2)}}, \ldots, x_{d_r j_{\alpha(r)}}, m_1, \ldots, m_r)
\]

is a factorization of \( f_\tau \) of type \( \tau \). Conversely, every factorization of \( f_\tau \) of type \( \tau \) is of the form \( a_\alpha \) for a unique self-arrangement \( \alpha \) of \( \tau \).

Proof. It is immediate to see that \( a_\alpha \) is a factorization of \( f_\tau \) of type \( \tau \). Conversely, let \( a \) be a factorization of \( f_\tau \) of type \( \tau \), and let \( A \) be the arrangement of \( \tau \) into \( \tau \) corresponding to \( a \) by Proposition 4.3. Then \( A \) is an \( r \times r \) matrix with non-negative integer entries, and \( d \) and \( m \) are positive vectors such that \( m = Am \) and \( d = A^T d \). We claim that this forces \( A \) to be a permutation matrix. Multiplying \((A - 1)m = 0\) on the left by the row vector \((1,1,\ldots,1)\) shows that

\[
m_1(A_{1,1} + \cdots + A_{r,1} - 1) + \cdots + m_r(A_{1,r} + \cdots + A_{r,r} - 1) = 0.
\]

By
positivity of \( m \), each column of \( A \) must sum to 1. Similarly, \((A^T - 1)d = 0\) shows each row of \( A \) sums to 1, so \( A \) is a permutation matrix. It is easy to see that \( A \) preserves the degrees and multiplicities of \( \tau \).

4.3. The ‘merge and forget’ partial order. We introduce a partial order on the set of types. First we define two operations on types.

- Suppose that two elements of a type \( \tau = \{(d_1, m_1), \ldots, (d_r, m_r)\} \), say \((d_1, m_1)\) and \((d_2, m_2)\), have the same multiplicity, i.e. \( m_1 = m_2 \). We say that the type
  \[ M(\tau) = \{(d_1 + d_2, m_1), (d_3, m_3), \ldots, (d_r, m_r)\} \]
  is obtained from \( \tau \) by an \textit{elementary merge} \( M \).
- Let an element of \( \tau \) be chosen, say \((d_1, m_1)\). Let \( a \) be a positive integer which is strictly less than \( m_1 \). We say that the type
  \[ F(\tau) = \{(d_1 - a, m_1), (d_1, a), (d_2, m_2), \ldots, (d_r, m_r)\} \]
  is obtained from \( \tau \) by an \textit{elementary forget} \( F \).

A composition of elementary merges (resp. forgets) is called a merge (resp. forget). These operations are dual under \( \tau \mapsto \tau^t \). Let \( T_d \) denote the set of types of degree \( d \) equipped with the partial order generated by merges and forgets, and let \( I_d \) denote its incidence algebra over \( \mathbb{Q} \).

**Proposition 4.5.**

1. Elementary merges \( M : \tau \rightarrow \lambda = M(\tau) \) correspond to squarefree arrangements \( A : \tau \rightarrow \lambda \) with exactly one 1 in each column.
2. Elementary forgets \( F : \tau \rightarrow \lambda = F(\tau) \) correspond to squarefree arrangements \( A : \tau \rightarrow \lambda \) with exactly one 1 in each row.
3. Any arrangement is equal to a forget followed by a merge.
4. The partial order on \( T_d \) is equivalent to the partial order obtained by declaring \( \tau \leq \lambda \iff a(\tau, \lambda) > 0 \). In particular, the transition matrices \( a \) and \( e \) are in the incidence algebra \( I_d \).
5. The transition matrices \( a \) and \( e \) are invertible in the incidence algebra \( I_d \).

For the proof we apply merge and forget operations to factorizations. If \( \alpha \) is a factorization of type \( \tilde{v}^m \), then any merge or forget which is admissible for \( \tilde{v}^m \) may be applied to \( \alpha \) in the obvious manner (for instance, we may merge the first two parts of \( \alpha = (f_1, f_2, f_3; m, m, n) \) to get \( M(\alpha) = (f_1 f_2, f_3; m, n) \).

**Proof.** The first two assertions are immediate from the definitions. The fourth is a consequence of the third assertion. For the third assertion, let \( f = x_{d_1 k_1}^{m_1} \cdots x_{d_r k_r}^{m_r} \) be a monomial of type \( \tau \) and let \( \beta = (f_1, \ldots, f_s; n_1, \ldots, n_s) \) be a factorization of \( f \) of type \( \lambda \). Let \( \alpha_0 \) denote the factorization \((x_{d_1 k_1}, \ldots, x_{d_r k_r}; m_1, \ldots, m_r)\) of \( f \) of type \( \tau \). If \( n_1 \) is less than the multiplicities of every part in \( \alpha_0 \) corresponding to a variable appearing in the monomial \( f_1 \), then one can apply elementary forgets to \( \alpha_0 \) until each variable in \( f_1 \) corresponds to some part in the new factorization \( \alpha_1 \) with multiplicity \( n_1 \). Replacing \( \alpha_0 \) with \( \alpha_1 \), we repeat the above process for each \( i \in \{2, \ldots, r\} \). In the final factorization \( \alpha_r \) each variable appearing in each \( f_i \) corresponds to some part in \( \alpha_r \) with exact multiplicity \( n_i \), and one can now merge parts of \( \alpha_r \) to obtain \( \beta \).
(1) \( a_{\alpha}^{-1} \) vanishes if \( \alpha \) is mixed.
(2) If \( \alpha = (b_1^m \cdots b_r^m) \) is \( m \)-pure, then
\[
a_{\alpha}^{-1} = \frac{\mu(m)}{m} \frac{(-1)^{r-1}}{r!} \prod_{k=1}^r \frac{\tau[k^m]}{\tau[1^m]! \cdots \tau[(d/m)^m]!}.
\]

Proof. We begin with the first assertion. Recall that \( \mathcal{P}_Q \) has the linear basis \( \{H_{\alpha}\}_{\alpha} \).
Define the linear projection operator \( \Theta : \mathcal{P}_Q \to \mathcal{P}_Q \) given by
\[
\Theta(H_{\alpha}) = \begin{cases} 
H_{\alpha} & \text{if } \alpha \text{ is pure of some multiplicity,} \\
0 & \text{otherwise.}
\end{cases}
\]
Recall that \( 1 + H_1 t + H_2 t^2 + H_3 t^3 + \cdots = \exp(P_1 + P_2 \frac{t^2}{2} + P_3 \frac{t^3}{3} + \cdots) \). Applying the \( m \)th Adams operation acting trivially on \( t \) and taking a logarithm results in
\[
\log(1 + H_1 t^m + H_2 t^2 + H_3 t^3 + \cdots) = P_1 t^m + P_2 \frac{t^2}{2} + P_3 \frac{t^3}{3} + \cdots.
\]
The coefficient of any power of $t$ on the left-hand side is of the form $H_\tau$ for a pure type $\tau$ so both sides are fixed by $\Theta$. We conclude that $\Theta(P_{km}) = P_{km}$ for all $k$ and $m$. Therefore

$$\Theta(M_d) = \Theta \left( \frac{1}{d} \sum_{k|d} \mu \left( \frac{d}{k} \right) P_{d/k} \right) = \frac{1}{d} \sum_{k|d} \mu \left( \frac{d}{k} \right) P_{d/k} = M_d$$

and then

$$\sum_{\tau \leq d} a_{\tau d}^{-1} H_\tau = M_d = \Theta(M_d) = \sum_{\tau \leq d} a_{\tau d}^{-1} H_\tau.$$

The set $\{H_\tau\}_\tau$ is linearly independent, so this shows that $a_{\tau d}^{-1} = 0$ if $\tau$ is mixed.

Now we prove the second assertion. Let $R$ denote the $\lambda$-ring $\Lambda_Q$, equipped with the unique $\lambda$-structure for which it has trivial Adams operations. Since $P\Lambda$ is the universal graded $\lambda$-ring, there is a unique $\lambda$-homomorphism satisfying

$$\Phi : P\Lambda \to R \quad M_d \mapsto q_d$$

where $q_1, q_2, q_3, \ldots \in \Lambda_Q$ are defined by

$$\sum_{d=0}^{\infty} h_d t^d = \prod_{d=1}^{\infty} (1 - t^d)^{-q_d}.$$

We have shown earlier that $\Phi(H_{km}) = h_k$ and $\Phi(P_{km}) = p_k$ for all $k, m \geq 1$ (Proposition 3.4). Applying $\Phi$ to $M_d = \frac{1}{d} \sum_{k|d} \mu \left( \frac{d}{k} \right) \psi_{d/k} P_k$ results in

$$\Phi(M_d) = q_d = \frac{1}{d} \sum_{k|d} \mu \left( \frac{d}{k} \right) p_k = \sum_{\tau \leq d} a_{\tau d}^{-1} \prod_{b^m \in \tau} h_b = \sum_{m|d} \sum_{\tau \leq d} \prod_{b^m \in \tau} a_{\tau d}^{-1} \prod_{b^m \in \tau} h_b.$$

This equation is inhomogeneous; we take the degree $d/m$ piece to see that

$$\Phi(M_d) = q_d = \frac{1}{d} \sum_{k|d} \mu \left( \frac{d}{k} \right) p_k = \sum_{\tau \leq d} \prod_{b^m \in \tau} a_{\tau d}^{-1} h_b.$$

Now we compare with the expansion of power symmetric functions in terms of the basis $\{h_\lambda\}_\lambda$. Expanding the Taylor series for the logarithm in the relation

$$\log(1 + h_1 t + h_2 t^2 + h_3 t^3 + \cdots) = p_1 + p_2 t^2 + p_3 t^3 + \cdots$$

shows that

$$p_m = \sum_{n_1 + 2n_2 + \cdots + mn_m = m} \frac{m(-1)^{1+\sum_k n_k}}{\prod_k n_k!} \prod_{k=1}^{m} h_k^{n_k}.$$

We observe two consequences of (6):

- if $m$ is square-free and $\tau$ is $m$-pure, then $\mu(m) a_{\tau d}^{-1}$ is the coefficient of $\prod b^m \in \tau h_b$ in the linear expansion of $p_{d/m}$ in the basis $\{h_\lambda\}_\lambda$,
- if $m$ is not square-free and $\tau$ is $m$-pure, then $a_{\tau d}^{-1}$ is zero.
In either event, if $\tau$ is $m$-pure then $a_{\tau d}^{-1}$ is equal to $\mu(m)/d$ times the coefficient of $\prod_{h^m \in \tau} h_b$ in the linear expansion of $p_{d/m}$, i.e.

$$a_{\tau d}^{-1} = \frac{\mu(m) (d/m)(-1)^{1+\sum_k n_k} (\sum_k n_k)!}{\sum_k n_k} \frac{\mu(m)(-1)^{r-1}}{m} \frac{r!}{\tau[1^m]! \cdots \tau[(d/m)^m]!},$$

where $n_k = \tau[k^m]$ denotes the number of occurrences of $k^m$ in $\tau$.

\[\square\]

4.4.1. An observation on the Möbius function of types. Call a type unramified if all of its multiplicities are one and ramified otherwise.

**Proposition 4.7.** $\mu_{\tau d} = 0$ if $\tau$ is ramified.

**Proof.** Recall that $\mu$ is defined to satisfy $\delta_{\tau \lambda} = \sum_{\tau \leq \kappa \leq \lambda} \mu_{\kappa \lambda}$. Any type $\tau$ has a unique unramified type $\tau'$ defined by applying all possible maximal forget operations to $\tau$ (e.g. if $\tau = (3^22^21)$ then $\tau' = (332222)$; the type $\tau'$ has the defining property that it is minimal among all unramified types that are greater than or equal to $\tau$ (this follows from Proposition 4.5).

Choose any function $j : T_d \to \mathbb{Z}_{\geq 0}$ satisfying $\tau < \lambda \implies j(\tau) < j(\lambda)$. Define the depth of $\tau$ to be $j(d) - j(\tau)$. We proceed by induction on depth. The claim is vacuously true if $\tau = d$ (depth zero). For positive depth, first observe that

$$\delta_{\tau d} = \sum_{\tau \leq \lambda, \lambda \text{ramified}} \mu_{\lambda d} + \sum_{\tau' \leq \lambda} \mu_{\lambda d}.$$ 

If $\tau$ is unramified then there is nothing to show, while if $\tau$ is ramified then by the induction hypothesis this is equal to $\mu_{\tau d} + \sum_{\tau' \leq \lambda} \mu_{\lambda d} = \mu_{\tau d} + \delta_{\tau' d} = \mu_{\tau d}$. We conclude that $\mu_{\tau d} = \delta_{\tau d} = 0$. \[\square\]

4.5. Grids and tilings. We sketch a pictorial interpretation for types and arrangements. Types will be associated with diagrams of boxes (“grids”). For example, the types $\mu = (13^21^22)$ and $\lambda = (1^21^22^12^1)$ are associated with the two grids, respectively:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{grid1}\end{array} \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{grid2}\end{array}
\]

**Definition 4.8.** A tiling of $\lambda$ by $\mu$ is a bijection between the boxes of $\mu$ and the boxes of $\lambda$ satisfying two properties:

1. horizontally adjacent boxes in $\mu$ map to horizontally adjacent boxes in $\lambda$,
2. vertically adjacent boxes in $\lambda$ come from vertically adjacent boxes in $\mu$.

Two tilings of $\lambda$ by $\mu$ are considered equivalent if they differ by a permutation of the boxes of $\lambda$ which stabilizes every block and permutes the set of columns of a given block.

For example, the following two tilings of $\lambda = (1^21^22^12^1)$ by $\mu = (1^31^22)$ are inequivalent:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{tile1}\end{array} \quad \rightarrow \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{tile2}\end{array}\]
\]
The following three tilings of $\lambda = (3)$ by $\mu = (1^21)$ are equivalent:

\[\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{tiling1}}
\end{array} \rightarrow \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{tiling2}}
\end{array} \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{tiling3}}
\end{array} \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{tiling4}}
\end{array}\]

The following assertions are easily verified: types of degree $d$ are in bijection with equivalence classes of grids of area $d$; arrangements are in bijection with equivalence classes of tilings; the duality of types corresponds to rotating the blocks of a grid by 90 degrees; any tiling of $\lambda$ by $\mu$ is sent to a tiling of $\mu^t$ by $\lambda^t$ under the rotation of grids.

5. Applications

In this section we survey some geometric and combinatorial applications. We will focus on the top stratum since general strata are studied in the next section.

5.1. The motivic class of $\text{Irr}_{d,n}$. The Grothendieck ring of varieties $K_0(\text{Var}_k)$ over a field $k$ is additively generated by isomorphism classes $[X]$ of varieties $X$ over $k$ modulo the relations $[X] - ([X \setminus Z] + [Z])$ for $Z \subset X$ a closed subvariety. Its ring multiplication is defined by $[X][Y] = [X \times Y]$. The ring $K_0(\text{Var}_k)$ has a $\sigma$-structure given by

$$
\sigma_t(X) := Z_X(t) = \sum_{n \geq 0} [S^n X] \cdot t^n
$$

where $S^n X$ is the unordered configuration space of degree $n$ on $X$. Note that $\lambda_n(X) \neq \text{Conf}_n(X)$ in general.

**Remark 15.** It is known that $K_0(\text{Var}_k)$ is not special. Indeed Larsen–Lunts [22] have shown that $Z_{C_1 \times C_2}(t)$ is irrational for curves $C_i$ of positive genus, while $Z_{C_1} \ast Z_{C_2}$ (Witt product) is rational since $Z_{C_i}$ is rational.

The coefficients of $\text{Exp}([U_1] t)$ are known to have a geometric interpretation by [17, Theorem 1]. The next proposition extends this to any effective element of $tK_0(\text{Var}_k)[t]$.

**Proposition 5.1.** Let $U_1, U_2, \ldots$ be a collection of varieties. For each nonnegative integer $k$ let $X_k$ be the unordered configuration space of degree $k$ on $\bigsqcup \mathcal{U}_d$ where points of $\mathcal{U}_d$ have weight $d$. Then in $K_0(\text{Var}_k)[t]$ we have the identity

$$
\text{Exp} ([U_1] t + [U_2] t^2 + \cdots) = \sum_{k=0}^{\infty} [X_k] t^k.
$$

**Proof.** By the first two identities of the list in §3.1,

$$
\sum_{n=0}^{\infty} H_n t^n = \prod_{d=1}^{\infty} \sigma_d(M_d) = \prod_{d=1}^{\infty} \sum_{k=0}^{\infty} (\sigma_k M_d) t^{kd} = \prod_{d=1}^{\infty} \sum_{k=0}^{\infty} h_{k,(d)} t^{kd}.
$$

Let $S^k \mathcal{U}_d$ denote the $k$th symmetric product of $\mathcal{U}_d$. Let $\mathcal{U} = [U_1] t + [U_2] t^2 + \cdots$. Applying the higher plethysm $- \circ \mathcal{U}$ obtains

$$
\sum_{n=0}^{\infty} (H_n \circ \mathcal{U}) t^n = \prod_{d=1}^{\infty} \sum_{k=0}^{\infty} (h_{k,(d)} \circ \mathcal{U}) t^{kd} = \prod_{d=1}^{\infty} \sum_{k=0}^{\infty} (h_k \circ \mathcal{U}_d) t^{kd} = \prod_{d=1}^{\infty} \sum_{k=0}^{\infty} [S^k \mathcal{U}_d] t^{kd}.
$$



*Many authors equip $K_0(\text{Var}_k)$ with the opposite $\lambda$-structure, however our convention is in better analogy with the standard identification of $\lambda$-operations with exterior powers on vector bundles.*
The coefficient of $t^d$ is equal to
\[
\bigcup_{n_1+2n_2+\ldots+dn_d\geq 0} \prod_{k=1}^{d} [S^{n_k} U_k] = [X_d],
\]
which proves the claim. □

Next we combine the proposition with Theorem 2.5. Let
\[\xi: K_0(\text{Var}_k) \to R\]
be a $\lambda$-ring homomorphism to a $\lambda$-ring $R$ without additive torsion ("motivic measure").

**Corollary 5.2.** If $\xi(X_1), \ldots, \xi(X_d)$ are contained in a $\lambda$-subring of $R$ on which Adams operations separate (e.g. $R$ special), then
\[
\xi(U_d) = \sum_{d=k \cdot m} \sum_{\lambda \cdot m} (-1)^{\ell-1} \frac{\mu(k)}{k\ell} \binom{\ell}{n_1, \ldots, n_m} \psi_k \prod_{j=1}^{m} \xi(X_j)^{n_j}. \tag{8}
\]

**Remark 16.** Every denominator in (8) is a divisor of $d$ (this follows from (7) and the fact that the $h_k$ are an integral basis of $\Lambda$).

**Proof of Theorem 1.2, top stratum.** The special case of Theorem 1.2 is
\[
X_d = \mathbb{P}_k^{(n+d)-1} = \{\text{degree } d \text{ hypersurfaces in } \mathbb{P}_k\},
\]
\[
U_d = \{\text{geometrically irreducible hypersurfaces}\}.
\]
The motivic measure $\xi$ is the canonical map from $K_0(\text{Var}_k)$ to its quotient modulo additive torsion. The Adams operations separate over the polynomial subring generated by $[A_k^{\lambda}]$, and this subring contains $\xi([X_1]), \ldots, \xi([X_d])$. This proves the formula for $[\text{Irr}_{d,n}]$ in $K_0(\text{Var}_k)$ up to additive torsion. Let $X_\tau = U_\tau$ be the space of hypersurfaces of splitting type $\leq \tau$. We have that $[X_\tau] = \sum_{\lambda \leq \tau} [U_\lambda]$ so by M"obius inversion $[\text{Irr}_{d,n}] = [U_d] = \sum_{\tau \leq d} \mu_{\tau d} [X_\tau]$. Since each $X_\tau$ is a projective space, this shows that $[\text{Irr}_{d,n}]$ is contained in the polynomial subring of $K_0(\text{Var}_k)$ generated by $[A_k^{\lambda}]$. This subring contains no additive torsion, and the formula for $[\text{Irr}_{d,n}]$ is proven. □

We give two examples of motivic measures $\xi$ to which the theorem may be applied.

**5.1.1. Example 1: $E$-polynomials.** The polynomial ring $\mathbb{Z}[u, v]$ has a $\sigma$-structure defined by $\sigma_n(u^p v^q) = u^{np} v^{nq}$ which is special. It is well-known that there is a unique $\lambda$-ring homomorphism $E: K_0(\text{Var}_C) \to \mathbb{Z}[u, v]: x \mapsto E_x$ such that for any variety $X$ one has
\[
E_X = \sum_{p, q, k \geq 0} (-1)^k h^{p, q}(H^k_c(X, \mathbb{Q})) u^p v^q
\]
where $h^{p, q}(H) = \dim \text{Gr}_F^p \text{Gr}_W^{p+q} H_C$ for a mixed Hodge structure $(H, F, W)$. 
5.1.2. Example 2: Zeta function over a finite field. Let \( k = \mathbb{F}_q \). Recall (Example 3) that \( \Lambda(\mathbb{Q}) \) is a special \( \lambda \)-ring. Its Adams operations satisfy

\[
\psi_m \exp \left( \sum_{d=1}^{\infty} \frac{w_d t^d}{d} \right) = \exp \left( \sum_{d=1}^{\infty} \frac{w_{dm} t^d}{d} \right).
\]

The function sending a variety \( X/k \) to its zeta function \( \zeta_X(t) \) uniquely extends to a ring homomorphism \( \zeta: K_0(\text{Var}_{\mathbb{F}_q}) \to \Lambda(\mathbb{Q}) \). We compute Adams operations using (9) which shows this projection sends \( \zeta \).

**Proposition 5.3.** The homomorphism \( \zeta: K_0(\text{Var}_{\mathbb{F}_q}) \to \Lambda(\mathbb{Q}) \) commutes with \( \lambda \)-operations.

**Proof.** It is well-known that the motivic zeta function \( Z_X(t) \) of \( X \) equals \( \exp \left( \sum_r \psi_r(X) \frac{F^n}{r^n} \right) \) and specializes to \( \zeta_X = \exp \left( \sum_r |X(\mathbb{F}_q^n)| \frac{F^n}{r^n} \right) \) under the point-counting ring homomorphism \( \xi_q: K_0(\text{Var}_{\mathbb{F}_q}) \to \mathbb{Z} \). Let \( \xi_{q,q^d}: K_0(\text{Var}_{\mathbb{F}_q}) \to \mathbb{Z} \) be \( |X(\mathbb{F}_q^n)| \). More generally, since the formation of symmetric products commutes with flat base change, \( \xi_{q,q^d}Z_X(t) = \sum_n |S^nX(\mathbb{F}_q^n)|t^n = \sum_n |(S^nX)(\mathbb{F}_q^n)|t^n = \sum_n |(S^nX)(\mathbb{F}_q^n)|t^n \), so \( \zeta_{q,q^d} \) equals the image of \( Z_X(t) \) under \( \xi_{q,q^d} \). This shows that \( \xi_{q,q^d}(\psi_r(X)) = |\psi_r(X)(\mathbb{F}_q^n)| = |X(\mathbb{F}_q^n)| \).

Since Adams operations are additive this shows that \( \zeta \) commutes with Adams operations on all of \( K_0(\text{Var}_{\mathbb{F}_q}) \). As \( \Lambda(\mathbb{Q}) \) is free of additive torsion, commuting with Adams operations implies commuting with \( \lambda \)-operations (Lemma 2.4).

**Corollary 5.4.**

\[
|\mathcal{U}_d(\mathbb{F}_q)| = \sum_{d=km} \sum_{\lambda \vdash m} (-1)^{\ell-1} \frac{\mu(k)}{k\ell} \binom{\ell}{n_1, \ldots, n_m} \prod_{j=1}^m |\mathcal{X}_j(\mathbb{F}_q^n)|^{n_j}.
\]

**Proof.** Apply (8) to the motivic measure \( \zeta \) to obtain

\[
\zeta_{d} = \sum_{d=km} \sum_{\lambda \vdash m} (-1)^{\ell-1} \frac{\mu(k)}{k\ell} \binom{\ell}{n_1, \ldots, n_m} \prod_{j=1}^m (\psi_{\ell} \mathcal{X}_j)^{n_j}.
\]

Identify \( \Lambda(\mathbb{Q}) \) with the ring of big Witt vectors valued in \( \mathbb{Q} \). The corollary follows from applying the projection to the first ghost coordinate:

\[
\frac{d}{dt} \circ \log \mid_{t=0} : \Lambda(\mathbb{Q}) \to \mathbb{Q} : f(t) \mapsto \frac{f'(0)}{f(0)}.
\]

We compute Adams operations using (9) which shows this projection sends \( \psi_r(\zeta_X) \) to \( X(\mathbb{F}_q^n) \).

**5.2. Inverse Pólya enumeration.** Consider a collection of objects whose weights are positive integers and assume that the number \( u_d \) of objects of any given weight \( d \) is finite. Consider the problem of determining the integers \( u_1, u_2, \ldots \) given the integers \( x_1, x_2, \ldots \) where \( x_d \) is the total number of unordered combinations of objects of weight \( d \). This setting
corresponds to the $\lambda$-ring $R = \mathbb{Z}$ with trivial Adams operations, and the coefficients of the plethystic logarithm for $\mathbb{Z}$ are the following universal polynomials:

$$u_d = \sum_{d=km} \sum_{\lambda \vdash m} (-1)^{\ell-1} \sum_{k} \left( \frac{\ell}{k} \right) x^\lambda \in \mathbb{Q}[x_1, \ldots, x_d].$$

For example, $u_2 = x_2 - \frac{1}{2}(x_1^2 + x_1)$ and $u_3 = x_3 - x_2x_1 + \frac{1}{3}(x_1^3 - x_1)$. The problem of determining $x_1, x_2, \ldots$ given $u_1, u_2, \ldots$ is a form of Pólya enumeration, whence the name.

6. Motivic measures of general strata

In this section we investigate the motivic measure of a general stratum $U_\tau$ in $X_d$ of points with exactly splitting type $\tau$. We work in the same abstract setting as in Proposition 5.1. That is to say, $U_1, U_2, \ldots$ denotes an arbitrary collection of varieties and $X_\lambda$ is the unordered configuration space of degree $k$ on $\sqcup_d U_d$ where points of $U_d$ have weight $d$.

6.0.1. The type stratification. For any integers $d, e$ we have the morphism $X_d \times X_e \to X_{d+e}$ induced by multiplication. Any arrangement $A$ of $\lambda$ into $\tau$ determines a morphism

$$\pi_A : \prod_{b^m \in \lambda} X_b \to \prod_{c^n \in \tau} X_c$$

$$(c_1, \ldots, c_r) \mapsto (c_1^{A_{11}} \cdots c_r^{A_{r1}}, \ldots, c_1^{A_{1s}} \cdots c_r^{A_{rs}}).$$

We write $\pi$ for the distinguished morphism $\prod_{b^m \in \lambda} X_b \to X_d$ corresponding to the unique arrangement of $\lambda$ into the maximal type $d$. The $\tau$-stratum $X_\tau \subset X_d$ is the Zariski closure of $\pi(\prod_{c^n \in \tau} X_c)$. The subspace $U_\tau \subset X_d$ of elements of (exact) type $\tau$ is the locally closed subscheme $X_\lambda \cap \bigcup_{\lambda \subset \tau} X_\lambda$.

6.1. Generalized configuration spaces. Let $X$ be a variety over a field $k$. Let $\tilde{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geq 0}$ and let $\Delta \subset X^{n_1+\cdots+n_d}$ be the “big diagonal” consisting of points where at least two of the coordinates are equal. Let

$$\text{Conf}_{\tilde{n}}(X) = (X^{n_1+\cdots+n_d} - \Delta) / \Sigma_{n_1} \times \cdots \times \Sigma_{n_d}.$$

For a type $\tau \equiv d$ with associated partitions $\tau_p$ corresponding to degree $p$, let $\tilde{n}_p = (\tau[p^1], \ldots, \tau[p^d]) \in \mathbb{Z}^d_{\geq 0}$. We leave the proof of the following easy lemma to the reader.

Lemma 6.1. Combining configurations gives a canonical isomorphism between any open stratum $U_\tau$ with a product of generalized configuration spaces:

$$U_\tau = \prod_{p=1}^d \text{Conf}_{\tilde{n}_p}(U_\tau).$$

Generalized configuration spaces satisfy the following recurrence relation in $K_0(\text{Var}_k)$.

Let $m \geq 1$ be an integer, $\tilde{w} = (n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geq 0}$ and $\tilde{v} = (n_1, \ldots, n_d, m)$. We denote $|\tilde{v}| = \sum_i n_i$. For vectors $\tilde{u}_1, \tilde{u}_2$ we write $\tilde{u}_1 \leq \tilde{u}_2$ if each coordinate satisfies the inequality.

Proposition 6.2. For any $\tilde{v}, \tilde{w}$ as above, we have the following recurrence in $K_0(\text{Var}_k)$,

$$\text{Conf}_{\tilde{v}}(X) = \text{Conf}_{\tilde{w}}(X) \times \text{Conf}_m(X) - \sum_{\tilde{u} \leq \tilde{w} \in \mathbb{Z}^d_{\geq 0}} \text{Conf}(\tilde{w} - \tilde{u}, m - |\tilde{u}|, \tilde{u}) (X)$$

(10)

where $(\tilde{w} - \tilde{u}, m - |\tilde{u}|, \tilde{u})$ denotes the vector obtained by concatenation.
Proof. The open immersion

\[ \text{Conf}_X(X) \to \text{Conf}_X(X) \times \text{Conf}_X(X) \]

\[ (D_1, \ldots, D_d, E) \mapsto ((D_1, \ldots, D_d), E) \]

identifies \( \text{Conf}_X(X) \) with an open subspace of \( \text{Conf}_X(X) \times \text{Conf}_X(X) \). The complement consists of points \((D_1, \ldots, D_d, E)\) where the support of \(E\) is not disjoint from the supports of the \(D_i\). Such points are stratified by vectors \(\vec{u} = (u_1, \ldots, u_d)\) where \(u_i\) equals the degree of the intersection of \(E\) with \(D_i\). The morphism

\[ \text{Conf}_{(\vec{w} - \vec{u}, m - |\vec{u}|, \vec{u})}(X) \to \text{Conf}_X(X) \times \text{Conf}_X(X) \]

\[ (D_1, \ldots, D_d, E, F_1, \ldots, F_d) \mapsto (D_1 + F_1, \ldots, D_d + F_d, E + \sum_i F_i) \]

identifies the open stratum corresponding to \(\vec{u}\) with \(\text{Conf}_{(\vec{w} - \vec{u}, m - |\vec{u}|, \vec{u})}(X)\). \(\square\)

Remark 17. Define the generating series

\[ C_d(t_1, \ldots, t_d)(X) = \sum_{\vec{w} = (n_1, \ldots, n_d) \geq \vec{0}} [\text{Conf}_X(X)]^{n_1} \cdots t_d^{n_d} \in K_0(\text{Var}_k)[[t_1, \ldots, t_d]]. \]

Then Proposition 6.2 is equivalent to the identity

\[ C_{2d+1}(t_1, \ldots, t_d, s, t_1s, \ldots, t_ds)(X) = C_d(t_1, \ldots, t_d)(X)C_1(s)(X). \]

Remark 18. Configuration spaces with restricted multiplicities have been studied in [13], [3]. The polysymmetric analogue of these spaces is \(E^d_n \in \mathcal{P}_A\mathbb{Z}\), the sum of all products \(f_1 \cdots f_r\) where \(f_1, \ldots, f_r\) are monomials in \(\mathbb{Z}[x_i]\) with respective degrees \(d_1, \ldots, d_r\) whose greatest common divisor is \(n\)-powerfree. Let \(\vec{t} = (t_1, \ldots, t_r)\) be a vector of indeterminates, and for \(\vec{d} = (d_1, \ldots, d_r)\) write \(\vec{t}^\vec{d} = t_1^{d_1} \cdots t_r^{d_r}\) and \(|\vec{d}| = d_1 + \cdots + d_r\). One can prove the following identity in \(\mathcal{P}_A[\vec{t}]\).

\[ \left( \sum_{d \geq 0} E^d_n t^\vec{d} \right) \left( \sum_{d = 0}^\infty (\psi_n H^d)(t_1 \cdots t_r)^{nd} \right) = \prod_{i=1}^r \sum_{d=0}^\infty H^d t_i^d. \]

This is a polysymmetric analogue of [3, Example 4.1.1-(3)].

6.2. The ring of characteristic cycles. The ring \(C(\text{Var}_k)\) of characteristic cycles over \(k\) is defined to be \(K_0(\text{Var}_k)_B\), the quotient binomial ring of \(K_0(\text{Var}_k)\) (cf. Remark 9). Let \([\mathcal{U}]\) denote the class of a variety in \(C(\text{Var}_k)\). The ring \(C(\text{Var}_k)\) avoids certain pathologies exhibited by \(K_0(\text{Var}_k)\). One such pathology is that in general \(M_\tau \circ (\mathcal{U}_1 t + \mathcal{U}_2 t^2 + \cdots)\) cannot be identified with the stratum \(\mathcal{U}_r\) except when \(\tau = d\). Even if a motivic measure \(\xi\) is valued in a special \(\Lambda\)-ring we may not have \(\xi(M_\tau \circ (\mathcal{U}_1 t + \mathcal{U}_2 t^2 + \cdots)) = \xi(\mathcal{U}_r)\). Nonetheless we will show that

\[ [M_\tau \circ (\mathcal{U}_1 t + \mathcal{U}_2 t^2 + \cdots)] = [\mathcal{U}_r]. \]

Another such pathology is that in view of the formula \([S^n X] = \sigma_n[X]\) and the analogy with the standard \(\lambda\)-operations on vector bundles, one might expect that \([\text{Conf}_n(X)] = \lambda_n[X]\) however this generally fails. Nonetheless we have the following result of Macdonald [24]

\[ \chi(\text{Conf}_n(X)) = \binom{\chi(X)}{n} = \lambda_n(\chi(X)) \]
so even if \([\text{Conf}_n(X)] - \lambda_n([X])\) fails to vanish it is at least in the kernel of \(\chi\). In fact, Macdonald’s identity holds for any binomial measure; equivalently, the expected formula

\[
[\text{Conf}_n(X)] = \binom{[X]}{n} = \lambda_n[X]
\]

holds in \(C(\text{Var}_k)\). The next result generalizes this to a wider class of varieties, namely generalized configuration spaces.

**Theorem 6.3.** Let \(\vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_{\geq 0}^d\) and let \(X\) be a variety over \(k\). Then

\[
[\text{Conf}_{\vec{n}}(X)] = \binom{[X]}{n_1, \ldots, n_d} = \frac{[X]([X] - 1) \cdots ([X] - N + 1)}{n_1! \cdots n_d!}
\]

where \(N = \sum_i n_i\).

**Proof.** First suppose \(d = 1\). Then by [3, Example 4.1.1-(3)] we have that

\[
\sum_{m=0}^{\infty} [\text{Conf}_m(X)] t^m = \frac{Z_X(t)}{Z_X(t^2)}.
\]

Let \([Z_X(t)]\) denote the image of \(Z_X(t)\) in \(C(\text{Var}_k)[t]\). Then

\[
[Z_X(t)] = \exp \left( \sum_{r=1}^{\infty} \psi_r([X]) \frac{t^r}{r} \right) = (1 - t)^{-[X]}
\]

so

\[
\sum_{m=0}^{\infty} [\text{Conf}_m(X)] t^m = \left( \frac{1 - t^2}{1 - t} \right)^{[X]} = \sum_{m=0}^{\infty} \binom{[X]}{m} t^m.
\]

For the general case we proceed by induction on \(|\vec{n}|\). Suppose \(d \geq 2\) and set \(\vec{n}' = (n_1, \ldots, n_{d-1})\). Applying \([-\cdot\] to (10) and using the induction hypothesis\(^7\) obtains

\[
[\text{Conf}_{\vec{n}}(X)] = \binom{[X]}{n_1, \ldots, n_{d-1}} \binom{[X]}{n_d}
\]

\[
- \sum_{\vec{u} \leq \vec{n}' < \vec{n}} \binom{[X]}{n_1 - u_1, \ldots, n_{d-1} - u_{d-1}, n_d - \vec{u}, u_1, \ldots, u_{d-1}}.
\]

For any binomial ring \(R\) and element \(x \in R\), the formal series

\[
B_d(x; t_1, \ldots, t_d) := (1 + t_1 + \cdots + t_d)^x = \sum_{(n_1, \ldots, n_d) \geq \vec{u}} \binom{x}{n_1, \ldots, n_d} t_1^{n_1} \cdots t_d^{n_d} \in R[t_1, \ldots, t_d]
\]

satisfies the identity from Remark 17, i.e.

\[
B_{2d+1}(x; t_1, \ldots, t_d, s, t_1 s, \ldots, t_d s) = (1 + t_1 + \cdots + t_d + s + t_1 s + \cdots + t_d s)^x = (1 + t_1 + \cdots + t_d)^x (1 + s)^x = B_d(x; t_1, \ldots, t_d) B_1(x; s).
\]

\(^7\)Note that \(|\vec{n}' - \vec{u}| + (n_d - \vec{u}) + |\vec{u}| < |\vec{n}|\).
Therefore the following identity holds (as polynomials in $x$),

\[
\begin{pmatrix} x \\ n_1, \ldots, n_d \end{pmatrix} = \begin{pmatrix} x \\ n_1, \ldots, n_{d-1} \end{pmatrix} \begin{pmatrix} x \\ n_d \end{pmatrix} - \sum_{\vec{u} \leq \vec{n} \in \mathbb{Z}^d_{\geq 0}} \begin{pmatrix} x \\ n_1 - u_1, \ldots, n_{d-1} - u_{d-1}, n_d - |\vec{u}|, u_1, \ldots, u_{d-1} \end{pmatrix},
\]

Taking $x = [X]$ proves the theorem. \hfill \Box

**Corollary 6.4.** For any type $\tau \models d$ we have

\[
[U_\tau] = \prod_{p=1}^d \left[ \tau[p^1], \ldots, \tau[p^d] \right].
\]

**Remark 19 (rationality of the motivic zeta function).** The ring of characteristic cycles over $k$ is a special $\lambda$-ring, and therefore receives a homomorphism from the specialization $K_0(\text{Var}_k)_{sp}$ of the Grothendieck ring of varieties (cf. [22, Lemma 4.12]). Larsen and Lunts have asked whether the motivic zeta function of a variety in $K_0(\text{Var}_k)_{sp}[t]$ is rational [22, Question 8.8]. Is the image of the motivic zeta function of a variety in $C(\text{Var}_k)[t]$ rational?

**Theorem 6.5.** For any type $\tau \models d$ we have

\[
[U_\tau] = M_\tau \circ \[U\] \quad \text{and} \quad [X_\tau] = H_\tau \circ \[U\].
\]

In particular, if $\xi$ is a motivic measure valued in a binomial ring then for any type $\lambda$,

\[
\xi(U_\lambda) = \sum_{\tau \leq \lambda} a_{\tau \lambda}^{-1} \prod_{b^\alpha \in \tau} \xi(X_b) \quad \text{and} \quad \xi(X_\lambda) = \sum_{\tau \leq \lambda} a_{\tau \lambda} \prod_{b^\alpha \in \tau} \xi(U_b).
\]

For a type $\tau$ let $\tau_p$ denote the multiset of $k$ satisfying $p^k \in \tau$, regarded as a partition. We write each partition $\tau_p$ as $(1^{n_1} \cdots d^{n_d})$ for non-negative integers $\vec{n}_p = (n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geq 0}$ (so $\vec{n}_p = (\tau[p^1], \ldots, \tau[p^d])$).

**Proof.** Recall that $M_\tau = m_{\tau_1}(x_1) \cdots m_{\tau_d}(x_d)$ (Remark 5). Let $[U]$ denote the element $[U_1]t + [U_2]t^2 + \cdots$ of $tC(\text{Var}_k)[t]$. By definition of higher plethysm,

\[
M_\tau \circ [U] = \prod_{p=1}^d m_{\tau_p} \circ [U_p].
\]

On the other hand by Lemma 6.1,

\[
[U_\tau] = \prod_{p=1}^d \left[ \text{Conf}_{\vec{n}_p}(U_p) \right].
\]

We conclude by Theorem 6.3 and Proposition 2.9 that $[U_\tau] = M_\tau \circ [U]$. Now observe that $[X_\tau] = \sum_{\lambda \leq \tau} [U_\lambda] = \sum_{\lambda \leq \tau} (M_\lambda \circ [U]) = (\sum_{\lambda \leq \tau} M_\lambda) \circ [U] = H_\tau \circ [U].$ \hfill \Box

**Lemma 6.6.** $[A_1]_k$ is an idempotent in $C(\text{Var}_k)$ and it is not equal to 0 or 1.
Proof. Sym^n(A^1) = A^n by the fundamental theorem of symmetric functions. Thus
\[ \sum_{r \geq 1} \psi_r([A^1_k]) \frac{t^r}{r} = \log \sum_{n \geq 0} \sigma_n([A^1_k]) t^n = \log \left( \frac{1}{1 - [A^1_k] t} \right) \]
implies that \( \psi_r([A^1_k]) = [A^1_k]^r \in K_0(\text{Var}_k) \). In the binomial quotient, \([A^1_k]^2 = \psi_2([A^1_k]) = [A^1_k]\). Moreover, one easily finds ring homomorphisms mapping \([A^1_k]\) to either 0 or 1. □

The next theorem contains the rest of Theorem 1.2.

**Theorem 6.7.** The binomial class \([\text{Irr}_\tau]\) is zero unless \(d^m \in \tau \implies d = 1\) in which case
\[ [\text{Irr}_\tau] = \left( [A^1_k](n - 1) + 1 \right) \left( \tau[1^1], \ldots, \tau[1^d] \right). \]

Proof. By Corollary 6.4 it suffices to prove the case \( \tau = d \). As \([A^1_k]\) is an idempotent (Lemma 6.6) we see that \([P^N_k] = \sum_{i=0}^N [A^1_k]^N = (N + 1)[A^1_k] + (1 - [A^1_k])\). Therefore, by Proposition 5.1 we have
\[ [Z_{\text{Irr}_d}(t)] = \prod_{d=1}^\infty (1 - t^d)^{-[\text{Irr}_d]} = \sum_{d=0}^\infty [\mathcal{Y}_d] t^d \]
\[ = [A^1_k] \sum_{d=0}^\infty \binom{n + d}{d} t^d + (1 - [A^1_k]) \sum_{d=0}^\infty t^d \]
\[ = \frac{[A^1_k]}{(1 - t)^n} + \frac{1 - [A^1_k]}{1 - t}. \]

Since \([A^1_k]\) is an idempotent, we can prove an identity in \(C(\text{Var}_k)\) by verifying it after applying the two projections \(q_1, q_0\) specializing \([A^1_k] = 1\) or \([A^1_k] = 0\). In the first case, we have
\[ \prod_{d=1}^\infty (1 - t^d)^{-q_1([\text{Irr}_d])} = (1 - t)^{-n} \]
while in the second case, we have
\[ \prod_{d=1}^\infty (1 - t^d)^{-q_0([\text{Irr}_d])} = (1 - t)^{-1}. \]
Comparing the exponents on both sides and appealing to the uniqueness of such a factorization obtains the claimed identity. □
### Appendix A. Arrangement numbers $a_{\lambda\tau}$ in degrees $\leq 5$

| $\lambda\tau$ | (1) | (1^2) | (1^3) | (1^21) | (1^211) | (1^311) | (2^1) | (2^11) | (3^1) | (3^2) | (3^21) | (4^1) | (5) |
|----------------|------|-------|-------|--------|--------|--------|------|-------|-------|------|--------|------|-----|
| (1)            | 1    | 1     | 1     | 1      | 1      | 1      | 1    | 1     | 1     | 1    | 1      | 1    | 1   |
| (1^2)          |      |       |       |        |        |        |      |       |       |      |        |      |     |
| (1^3)          | 1    | 1     | 1     | 1      | 1      | 1      | 1    | 1     | 1     | 1    | 1      | 1    | 1   |
| (1^4)          |      |       |       |        |        |        |      |       |       |      |        |      |     |
| (1^5)          |      |       |       |        |        |        |      |       |       |      |        |      |     |
| (1^21)         | 0    | 1     | 0     | 1      | 1      | 2      | 4    | 10    | 3     | 7    | 2      | 4    | 1   |
| (1^211)        | 0    | 0     | 1     | 2      | 1      | 1      | 3    | 5     | 2     | 4    | 1      | 3    | 2   |
| (1^311)        | 0    | 0     | 0     | 2      | 1      | 0      | 6    | 20    | 3     | 13   | 1      | 7    | 0   |
| (1^31)         |      |       |       |        |        |        |      |       |       |      |        |      |     |
| (1^41)         |      |       |       |        |        |        |      |       |       |      |        |      |     |
| (2^1)          |      |       |       |        |        |        |      |       |       |      |        |      |     |
| (2^11)         |      |       |       |        |        |        |      |       |       |      |        |      |     |
| (3^1)          |      |       |       |        |        |        |      |       |       |      |        |      |     |
| (3^2)          |      |       |       |        |        |        |      |       |       |      |        |      |     |
| (4^1)          |      |       |       |        |        |        |      |       |       |      |        |      |     |
| (5)            |      |       |       |        |        |        |      |       |       |      |        |      |     |
### Appendix B. Inverse arrangement numbers $a_{r}^{-1}$ in degrees $\leq 5$

|   | (1) | (11) | (12) | (14) | (121) | (1212) | (1111) | (212) | (211) | (31) | (22) | (4) |
|---|-----|------|------|------|-------|--------|--------|-------|-------|------|------|----|
| (1) | 1  | −1   | 0    | −1   | 1     | −1/2   | 1/4    | 0     | 0     | 0    | −1/2 | 1/4 | 0  |
| (11) | 0  | 1/2  | −1   | 0    | 1/2   | −1/2   | 1/2    | 0     | 0     | −1/2 | 0    | 1/2 | 0  |
| (12) | 0  | 0    | 1    | 0    | 0     | −1    | −1/2   | 0     | 0     | 0    | −1/2 | 1/2 | 0  |

|   | (1^3) | (12) | (111) | (21) | (3) | (1^4) | (131) | (121) | (112) | (1111) | (212) | (211) | (31) | (22) | (4) |
|---|-------|------|-------|------|----|-------|-------|-------|-------|-------|-------|-------|----|------|----|
| (1^3) | 1    | −1   | 0    | −1   | 0  | 1/2   | −1   | 0    | 0    | −1/2   | 0    | 0    | −1   | 0   | 1/2   | 0  |
| (1^21) | 0    | 1    | −1/2 | −1/2 | 0  | 1/6   | −1/2 | 1/3   | 0    | 0    | 0    | 1     | 0   | 0   | −1/2 | 0  |
| (111) | 0    | 0    | 0    | 0    | 1/2 | −1/2 | 0    | 0    | 0    | −1/2   | 0    | 0    | 1/2 | 0   | −1/2 | 0  |
| (21) | 0    | 0    | 0    | 1    | −1   | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1   | −1/2 | −1/2 | 0  |
| (3) | 0    | 0    | 0    | 0    | 1    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0   | 0   | 1   | 0  |

|   | (1^5) | (1^312) | (1^41) | (1^311) | (1^211) | (1^2111) | (2111) | (311) | (221) | (32) | (41) | (5) |
|---|-------|---------|--------|---------|---------|---------|--------|-------|-------|------|------|----|
| (1^5) | 1    | −1    | −1    | 0    | 1    | −1   | 1/2    | 0    | 0    | 0    | 0    | 0    | 0   | 1/6   | 0  |
| (1^312) | 0    | 1    | 0    | −1/2 | −1   | 0    | 1/6    | −1/2 | 1/2   | −1   | 0    | 0    | 0    | 0    | 1/6   | 0  |
| (1^41) | 0    | 0    | 1    | −1   | 0    | −1/2 | 1     | 0    | 0    | 0    | −1/2 | 1/2 | 0   | 0    | −1/2 | 0  |
| (1^311) | 0    | 0    | 0    | 0    | 1/2 | −1/2 | 0    | −1/2 | 0    | 1/3   | 0    | 0    | 1/3 | 0   | −1/2 | 0  |
| (21^3) | 0    | 0    | 0    | 0    | 1/2 | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1/2 | 0   | −1/2 | 0  |
| (1^211) | 0    | 0    | 0    | 0    | 0    | −1/2 | 0    | 0    | 0    | 0    | 0    | 0    | −1/2 | 0   | 1/2   | 0  |
| (1^2111) | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | −1/2 | 0   | 0    | 0  |
| (11111) | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1/2 | 0   | −1/2 | 0  |
| (21^2) | 0    | 0    | 0    | 0    | 0    | 0    | 1    | −1/2 | 0    | 0    | −1/2 | 0   | 1/2 | 0   | −1/2 | 0  |
| (2111) | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1/2 | −1    | 0  |
| (31^2) | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1  |
| (311) | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1  |
| (2^21) | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1  |
| (221) | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1  |
| (32) | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1  |
| (41) | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1  |
| (5) | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1  |
### Appendix C. Möbius function of types in degrees ≤ 5

| Type | 1^2 | 1^3 | 1^4 | 1^5 |
|------|-----|-----|-----|-----|
| (1^2) | 1 | -1 | 0 | 1 |
| (1^3) | 0 | 1 | -1 | 1 |
| (1^4) | 0 | 0 | 1 | 0 |
| (1^5) | 0 | 0 | 0 | 0 |

| Type | 2^1 | 2^2 | 2^3 | 2^4 | 2^5 |
|------|-----|-----|-----|-----|-----|
| (1^2) | 1 | -1 | 1 | 0 | 0 |
| (1^3) | 0 | 1 | -1 | 0 | 0 |
| (1^4) | 0 | 0 | 1 | 0 | 0 |
| (1^5) | 0 | 0 | 0 | 0 | 0 |

| Type | 3^1 | 3^2 | 3^3 | 3^4 | 3^5 |
|------|-----|-----|-----|-----|-----|
| (1^2) | 1 | -1 | 1 | 0 | 0 |
| (1^3) | 0 | 1 | -1 | 0 | 0 |
| (1^4) | 0 | 0 | 1 | 0 | 0 |
| (1^5) | 0 | 0 | 0 | 0 | 0 |

| Type | 4^1 | 4^2 | 4^3 | 4^4 |
|------|-----|-----|-----|-----|
| (1^2) | 1 | -1 | 1 | 0 |
| (1^3) | 0 | 1 | -1 | 0 |
| (1^4) | 0 | 0 | 1 | 0 |
| (1^5) | 0 | 0 | 0 | 0 |

| Type | 5 | |
|------|-----|-----|
| (1^2) | 1 | -1 |
| (1^3) | 0 | 1 |
| (1^4) | 0 | 0 |
| (1^5) | 0 | 0 |

The table continues with more rows and columns, detailing the Möbius function for different types and degrees.
References

[1] K. Adiprasito, J. Huh, and E. Katz. Hodge theory for combinatorial geometries. *Annals of Mathematics*, 188(2):381–452, 2018.
[2] M. F. Atiyah and D. O. Tall. Group representations, \(\lambda\)-rings and the \(J\)-homomorphism. *Topology*, 8:253–297, 1969.
[3] M. Bilu, R. Das, and S. Howe. Zeta statistics and Hadamard functions. *Adv. Math.*, 407:68, 2022.
[4] A. Bodin. Number of irreducible polynomials in several variables over finite fields. *The American Mathematical Monthly*, 115(7):653–660, 2008.
[5] A. Bodin. Generating series for irreducible polynomials over finite fields. *Finite Fields Appl.*, 16(2):116–125, 2010.
[6] T. Braden, J. Huh, J. P. Matherne, N. Proudfoot, and B. Wang. Singular Hodge theory for combinatorial geometries. *arXiv preprint arXiv:2010.06088*, 2020.
[7] C. C. Cadogan. The Möbius function and connected graphs. *J. Comb. Theory, Ser. B*, 11:193–200, 1971.
[8] L. Carlitz. The distribution of irreducible polynomials in several indeterminates. *Illinois J. Math.*, 7:371–375, 1963.
[9] L. Carlitz. The distribution of irreducible polynomials in several indeterminates. II. *Canadian J. Math.*, 17:261–266, 1965.
[10] E. Cesaratto, J. von zur Gathen, and G. Matera. The number of reducible space curves over a finite field. *J. Number Theory*, 133(4):1409–1434, 2013.
[11] W. Chen. Stability of the cohomology of the space of complex irreducible polynomials in several variables. *Int. Math. Res. Not. IMRN*, (22):17256–17276, 2021.
[12] B. Elias, S. Makisumi, U. Thiel, G. Williamson, B. Elias, S. Makisumi, U. Thiel, and G. Williamson. The Hodge theory of Soergel bimodules. *Introduction to Soergel Bimodules*, pages 347–367, 2020.
[13] B. Farb, J. Wolfson, and M. M. Wood. Coincidences between homological densities, predicted by arithmetic. *Advances in Mathematics*, 352:670–716, 2019.
[14] C. Florentino, A. Nozad, and A. Zamora. Serre polynomials of \(\mathrm{SL}_n\)- and \(\mathrm{PGL}_n\)-character varieties of free groups. *Journal of Geometry and Physics*, 161:104008, 2021.
[15] E. Getzler and M. M. Kapranov. Modular operads. *Compos. Math.*, 110(1):65–126, 1998.
[16] E. Gorsky. Adams operations and power structures. *Mosc. Math. J.*, 9(2):305–323, 2009.
[17] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández. A power structure over the Grothendieck ring of varieties. *Math. Res. Lett.*, 11(1):49–57, 2004.
[18] F. Heinloth. A note on functional equations for zeta functions with values in Chow motives. *Ann. Inst. Fourier (Grenoble)*, 57(6):1927–1945, 2007.
[19] X.-d. Hou and G. L. Mullen. Number of irreducible polynomials and pairs of relatively prime polynomials in several variables over finite fields. *Finite Fields Appl.*, 15(3):304–331, 2009.
[20] T. Hyde. Euler characteristic of the space of real multivariate irreducible polynomials. *Proc. Am. Math. Soc.*, 150(6):2331–2343, 2022.
[21] A. Kanna and N. A. Loehr. Transition matrices and Pieri-type rules for polysymmetric functions, 2024.
[22] M. Larsen and V. A. Lunts. Rationality criteria for motivic zeta functions. *Compos. Math.*, 140(6):1537–1560, 2004.
[23] H. W. Lenstra. Construction of the ring of Witt vectors. *Eur. J. Math.*, 5(4):1234–1241, 2019.
[24] I. G. Macdonald. The Poincaré polynomial of a symmetric product. *Proc. Cambridge Philos. Soc.*, 58:563–568, 1962.
[25] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, second edition, 2015.
[26] S. Mozgovoy. Motivic classes of quot-schemes on surfaces, 2019.
[27] N. Ramachandran. Zeta functions, Grothendieck groups, and the Witt ring. *Bulletin des Sciences Mathématiques*, 139(6):599–627, 2015.
[28] R. P. Stanley and S. Fomin. *Enumerative Combinatorics*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.
[29] D. Yau. *Lambda-rings*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.

Department of Mathematics  
University of Wisconsin–Madison

*Email address:* gasvinseeker94@gmail.com

Department of Mathematics  
Princeton University

*Email address:* andy.odesky@gmail.com