Pseudoclassical theory of Mayorana–Weyl particle

Grigoryan G.V.*, Grigoryan R.P. **, Tyutin I.V. †

Yerevan Physics Institute, Republic of Armenia

*E-mail:GAGRI@VXC.YERPHY.AM
**E-mail:ROGRI@VXC.YERPHY.AM
† E-mail:TYUTIN@LPI.AC.RU

Abstract

A pseudoclassical theory of Weyl particle in the space–time dimensions $D = 2n$ is constructed. The canonical quantization of that pseudoclassical theory is carried out and it results in the theory of the $D = 2n$ dimensional Weyl particle in the Foldy–Wouthuysen representation. A quantum mechanics of the neutral Weyl particle in even–dimensional space–time is suggested and the connection of this theory with the theory of Mayorana–Weyl particle in QFT is discussed for $D = 10$. 

Yerevan Physics Institute
Yerevan 1995

* P.N.Lebedev Physical Institute, Moscow, Russia
† Partially supported by the grant 211-5291 YPI of the German Bundesministerium für Forschung und Technologie.
1 Introduction

In spite of the bulk of the papers devoted to the theories of point particles and to methods of their quantization, the problem still attracts the attention of the investigators and numerous classical models of particles and superparticles were discussed recently. The renewal of the interest to these theories is primarily due to the problems in the string theory since particles are prototypes of the strings.

As it is known, the theory of RNS string with a GSO projection (see. [1]) is a supersymmetric theory in ten dimensional space–time. The supersymmetry requires that each mass level comprises a supermultiplete. In the massless sector the superpartners to the gauge vector fields, which in $D = 10$ dimensions have eight degrees of freedom, are massless fermions – Mayorana–Weyl bispinors. Hence it is interesting to construct the classical (pseudoclassical) model, which after quantization will bring to the theory of the Mayorana–Weyl bispinor in $D = 10$ dimensions.

The first pseudoclassical description of the relativistic spinning particle was given in paper [2, 3]. It was followed by a great number of papers [4-23] devoted to the different quantization schemes of that theory, to the introduction of the internal symmetries and to the generalization to higher spins.

A pseudoclassical theory of Weyl particle in the space–time $D = 4$ was constructed in [24]. In this paper the method suggested in [24] is generalized to the arbitrary even dimensions $D = 2n$. This is realized in sect.2, where the canonical quantization of that pseudoclassical theory is carried out and it results in the theory of the $D = 2n$ dimensional Weyl particle in the Foldy–Wouthuysen representation. A quantum mechanics of the neutral Weyl particle in even–dimensional space–time is presented in sect.3 and the relation of this theory to the theory of Mayorana–Weyl bispinor in QFT is discussed in sect.4 in $D = 10$ dimensions.
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$D = 2n$–dimensional Weyl particle

Consider a theory with the action given by the expression

$$S = \int d\tau \left\{ \frac{1}{2e} \left( \dot{x}^{\mu} - i \frac{\chi}{2} \xi^{\mu} - \frac{(-i)^{D-2}}{(D-2)!} \varepsilon^{\mu\nu\lambda_1\ldots\lambda_{D-2}} b_{\nu} \xi_{\lambda_1} \cdots \xi_{\lambda_{D-2}} + \tilde{\alpha} b^{\mu} \right)^2 - i \frac{2}{2} \xi^{\mu} \dot{\xi}^{\mu} \right\}, \tag{1}$$

which is a generalization to the space-time dimension $D = 2n$ of the pseudoclassical theory of Weyl particle [24]. Here $x_{\mu}$ are the coordinates of the particle, $\mu = 0, 1, \ldots, D - 1$; $\xi^{\mu}$ are Grassmann variables describing the spin degrees of freedom; $e, \chi, b^{\mu}$ are additional fields, $\tilde{\alpha}$ is a constant; $e, b^{\mu}, \tilde{\alpha}$ being grassmann even, $\chi$- grassmann odd variables; the overdot denotes a differentiation with respect to the parameter $\tau$ along the trajectory.

The action (1) is invariant under three types of gauge transformations: the reparametrization transformations

$$\delta x^{\mu} = u \dot{x}^{\mu}, \quad \delta e = \frac{d}{d\tau}(ue), \quad \delta b^{\mu} = \frac{d}{d\tau}(ub^{\mu}), \quad \delta \xi^{\mu} = u \dot{\xi}^{\mu}, \quad \delta \chi = \frac{d}{d\tau}(u\chi) \tag{2}$$

with the even parameter $u(\tau)$, supergauge transformations

$$\delta x^{\mu} = iv \xi^{\mu}, \quad \delta e = iv \chi, \quad \delta b^{\mu} = 0, \quad \delta \xi^{\mu} = v \frac{z^{\mu}}{e}, \quad \delta \chi = 2 \dot{v},$$

$$z^{\mu} = \dot{x}^{\mu} - \frac{i}{2} \chi \xi^{\mu} - \frac{(-i)^{D-2}}{(D-2)!} \varepsilon^{\mu\nu\lambda_1\ldots\lambda_{D-2}} b_{\nu} \xi_{\lambda_1} \cdots \xi_{\lambda_{D-2}} + \tilde{\alpha} b^{\mu} \tag{3}$$

with the odd parameter $v(\tau)$, and also under transformations [24]

$$\delta x^{\mu} = \frac{(-i)^{D-2}}{(D-2)!} \varepsilon^{\mu\nu\lambda_1\ldots\lambda_{D-2}} \eta_{\nu} \xi_{\lambda_1} \cdots \xi_{\lambda_{D-2}} - \tilde{\alpha} \eta^{\mu},$$

$$\delta \xi^{\mu} = \frac{1}{e} \frac{(-i)^{D-2}}{(D-3)!} \varepsilon^{\mu\nu\delta\lambda_2\ldots\lambda_{D-2}} \eta_{\nu} \varepsilon \xi_{\lambda_2} \cdots \xi_{\lambda_{D-2}},$$

$$\delta b^{\mu} = \frac{d}{d\tau}(\eta^{\mu}), \quad \delta \chi = -2 \eta_{\nu} (p^{\nu} \xi^{\sigma} - p^{\sigma} \xi^{\nu}) b_{\sigma} \delta D_4, \quad \delta e = -2i \eta_{\nu} \xi^{\nu} \xi^{\sigma} b_{\sigma} \delta D_4, \tag{4}$$

with the even parameter $\eta_{\nu}(\tau)$.

Acting in the standard way [25, 26] we obtain the canonical hamiltonian of the theory, which is given by the expression

$$H = \dot{x}^{\mu} p_{\mu} + \dot{\xi}^{\mu} \pi_{\mu} - L = \quad$$
\[
\frac{\epsilon}{2} p^2 + \frac{i}{2} \chi p \xi^\mu - \left( \frac{(-i)^{D-2}}{(D-2)!} \varepsilon^{\mu \lambda_1 \cdots \lambda_{D-2}} p_{\mu} \xi_{\lambda_1} \cdots \xi_{\lambda_{D-2}} + \tilde{\alpha} p^\nu \right) b_\nu,
\]

(5)

primary constraints

\[
\Phi^{(1)}_1 = \pi_e, \quad \Phi^{(1)}_2 = \pi_\chi, \quad \Phi^{(1)}_{3\mu} = \pi_\mu - i \frac{\xi_\mu}{2}, \quad \Phi^{(1)}_{4\mu} = \pi^b_\mu
\]

(6)

and the secondary constraints

\[
\Phi^{(2)}_1 = |p_0| - \omega, \quad \Phi^{(2)}_2 = p_\mu \xi^\mu,
\]

\[
\Phi^{(2)}_{3\mu} = T_\mu = \frac{(-i)^{D-2}}{(D-2)!} \varepsilon_{\mu \nu \lambda_1 \cdots \lambda_{D-2}} p^\nu \xi^{\lambda_1} \cdots \xi^{\lambda_{D-2}} + \tilde{\alpha} p_\mu,
\]

(7)

where \( \omega = |\vec{p}| \), \( \vec{p} = (p_k) \), \( k = 1, \ldots, D - 1 \). One can see now, that the canonical hamiltonian \( H \) is equal to zero on the constraints surface, as it was expected to be.

The constraints \( F \equiv (\Phi^{(1)}_1, \Phi^{(1)}_2, \Phi^{(1)}_{3\mu}, \Phi^{(1)}_{4\mu}) \) are first class. Apart from them there is one more first class constraint \( \varphi \), which is a linear combination of the constraints \( \Phi^{(1)}_{(3\mu)}, \Phi^{(2)}_2 \):

\[
\varphi = p^\mu \Phi^{(1)}_{3\mu} + i \Phi^{(2)}_2 = p_\mu \pi^\mu + i \frac{\xi_\mu}{2}.
\]

(8)

Adhering to the quantization method, when already at the classical level all gauge degrees of freedom are fixed [17], we must introduce into the theory additional constraints, equal in number to that of all first class constraints and conjugated to the latter. However, as it was noted in [24], the constraints \( \Phi^{(2)}_{3\mu} \) for \( D \geq 4 \), are at least quadratic functions of the variables \( \xi^\lambda \), thus complicating the introduction of additional constraints conjugated to \( \Phi^{(2)}_{3\mu} \). For this reason, following [24], the constraints \( \Phi^{(2)}_{3\mu} \) after quantization will be used as conditions on the state vectors. For the remaining first class \( F \) and \( \varphi \) constraints we will introduce equal number of additional constraints \( \Phi^G \) in the form

\[
\Phi^G_1 = x_0 - \kappa \tau, \quad \Phi^G_2 = \chi, \quad \Phi^G_3 = e - \frac{\kappa}{p_0}, \quad \Phi^G_{4\nu} = b_\nu, \quad \Phi^G_5 = \xi_0,
\]

(9)

where \( \kappa = -\text{sign}p_0 \). The constraint \( x_0 - \kappa \tau \approx 0 \) was introduced in [17] as one, conjugated to the constraint \( |\vec{p}| - \omega \approx 0 \); \( \kappa = +1 \) corresponds to the particle sector in the theory, while \( \kappa = -1 \) corresponds to the antiparticle sector. Now the constraints (1), (7) (except for
Φ^{(2)}_{4\mu}) and (9) constitute a system of second class constraints. To use the Dirac method of quantization of the theories with second class constraints, we’ll pass to a set of constraints, which do not depend on time explicitly. To go over to time independent set of constraints we perform a canonical transformation from the variables \( x^\mu, p_\mu \) to variables \( x'^\mu, p'_\mu \), defined by the relations

\[
x'_0 = x_0 - \kappa \tau, \quad x'^i = x^i, \quad p'_\mu = p_\mu
\]  
(10)

( corresponding generating function is given by \( W = x'^\mu p'_\mu - \tau \kappa p_0' \) ). The constraint \( \Phi^G \) in terms of new variables takes the form \( x'_0 \approx 0 \); all other constraints remain unchanged.

The hamiltonian of the system on the constraint surface is given by the expression

\[
H = \omega = [\vec{p}],
\]  
(11)

To find the Dirac brackets for independent variables \( x^i, p_j, \xi^k \) of the theory we’ll make use of the results of the paper [18], where the theory of \( D = 2n \) dimensional relativistic spinning particle was considered. Note, that in comparison with the paper [18], the system of second–class constraints in this theory contains new constraints \( \Phi^{(1)}_{4\mu} = \pi^b_\mu \approx 0 \), \( \Phi^G_{4\mu} = b_\mu \approx 0 \). However they have a special form [26] and they do not affect the final Dirac brackets (the variables \( b_\mu \) and \( \pi^b_\mu \) can be excluded from the theory using the constraints).

This allows to use immediately the results of [18], taken in the massless limit, to obtain the following final Dirac brackets

\[
\{x^i, x^j\}_D = \frac{i}{2\omega^2}[\xi^i, \xi^j], \quad \{x^i, p_j\}_D = \delta^i_j, \quad \{p_i, p_j\}_D = 0,
\]

\[
\{x^i, \xi^j\}_D = \frac{1}{\omega^2}\xi^i p^j, \quad \{\xi^i, \xi^j\}_D = -i \left( \delta^{ij} - \frac{p^i p^j}{\omega^2} \right), \quad \{p_i, \xi^j\}_D = 0.
\]  
(12)

The quantization of the theory is carried out through the realization of the operators \( x^i, \hat{p}_i, \xi^i \) in the form [17, 18]:

\[
\hat{x}^i = \hat{q}^i - \frac{i\hbar}{4\omega^2}\left[\Sigma^i, \hat{p}_k \Sigma^k\right]_-, \quad \hat{\xi}^i = \left( \frac{\hbar}{2} \right)^{1/2} \hat{\kappa} \left[\Sigma^i - \frac{1}{\omega^2}\hat{p}_i \hat{p}_j \Sigma^j\right], \quad \hat{p}_k = -i \frac{\partial}{\partial q^k},
\]  
(13)
where the operators $\hat{q}^i$ (physical coordinate operators) are multiplication operators, the variable $\kappa$ is replaced by the operator $\hat{\kappa}$

$$\hat{\kappa} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \gamma^0 = \tau^3 \otimes I, \quad \hat{\kappa}^2 = 1$$

with eigenvalues $\kappa = \pm 1$, the operators $\Sigma^i = \text{diag}(\sigma^i, \sigma^i)$, $\sigma^i$ are $2^{(D-2)/2} \times 2^{(D-2)/2}$ matrices, which realize the Clifford algebra $[\sigma^i, \sigma^j]_+ = \delta^{ij} I$.

Now using the expressions (13) for $\hat{\xi}^i$, we can find the expressions for the operators $\hat{T}_\mu$, which correspond to the first class constraints $\Phi^{(2)}_\alpha$:

$$\hat{T}_\mu = \left( \frac{\hbar}{2} \right)^{(D-2)/2} \hat{p}_\mu \hat{T}, \quad \hat{T} = \hat{\kappa} \frac{\hat{p}_i \Sigma^i}{\hat{\omega}} - \alpha, \quad \hat{p}_0 = -\hat{\kappa} \hat{\omega}, \quad \alpha = \left( \frac{\hbar}{2} \right)^{-(D-2)/2} \hat{\alpha},$$

To deduce these relations we used the equality $\varepsilon_{01}^{(D-1)} = -\varepsilon_{12}^{(D-1)} = -1$, and also the relation

$$\left( -i \right)^{D-2} (D-2)! \varepsilon^{i j_1 \ldots j_{D-2}} \sigma^{j_1} \ldots \sigma^{j_{D-2}} = \sigma^i$$

from the $\sigma^i$–matrix algebra in $(D - 1)$–dimensional space \[28\].

The canonical generators of the Lorentz transformation

$$J^{\mu\nu} = - \left( x^\mu p^\nu - x^\nu p^\mu + \frac{i}{2} [\xi^\mu, \xi^\nu]_+ \right)$$

after quantization of the theory, in terms of operators of the physical variables, are given by the expressions [18]

$$\hat{j}^{ik} = -\hat{q}^i \hat{p}^k + \hat{q}^k \hat{p}^i - \frac{i\hbar}{4} \left[ \Sigma^i, \Sigma^k \right]_-, \quad \hat{j}^{0k} = -x^0 \hat{p}^k - \frac{1}{2} \hat{\kappa} \left[ \hat{q}^k, \hat{\omega} \right]_+ - \frac{i\hbar}{4\hat{\omega}} \hat{\kappa} \hat{p}^j \left[ \Sigma^k, \Sigma^j \right]_-.$$  

It’s not difficult to check that the $\hat{j}^{\mu\nu}$ operators commute with the operator $\hat{T}$. Furthermore, introducing the projection operators

$$\hat{T}_\pm = -\frac{1}{2} \alpha \hat{T} |_{\alpha = \pm} = \frac{1}{2} \left( 1 \mp \hat{\kappa} \frac{\hat{p}_i \Sigma^i}{\hat{\omega}} \right),$$

which commute with $\hat{j}^{\mu\nu}$, we can represent the $\hat{j}^{\mu\nu}$ operator in the form

$$\hat{j}^{\mu\nu} = \hat{T}_+ \hat{j}^{\mu\nu} + \hat{T}_- \hat{j}^{\mu\nu} = \hat{T}_+ \hat{j}^{\mu\nu} \hat{T}_+ + \hat{T}_- \hat{j}^{\mu\nu} \hat{T}_-,$$
which reflects the property of the reducibility of the representation of the Lorentz group for \( m = 0 \).

As it was already mentioned above, the operators \( \hat{T}_\mu \) will be used to impose conditions on the physical state vectors. For the \( D \)-dimensional space–time the state vector in general has \( 2^{D/2} \) components. Representing the state vector \( f \) in the form

\[
f = \begin{pmatrix} f^1(q) \\ f^2(q) \end{pmatrix}, \quad q = (x^0, q^i),
\]

where \( f^1(q) \) and \( f^2(q) \) are \( 2^{(D-2)/2} \) component spinors, we write down the equations for the state vector in the form:

\[
\hat{T} f = 0.
\]

It is natural to interpret the quantum mechanics constructed above as a theory of Weyl particle in the Foldy–Wouthuysen representation. Indeed, consider the Schrödinger equation \( (i\partial/\partial \tau - \hat{H})f = 0 \), which describes the evolution of the state-vector \( f \) with respect to parameter \( \tau \). Being rewritten in terms of the physical time \( x_0 = \kappa \tau \) it takes the form

\[
(i \frac{\partial}{\partial x^0} - \gamma^0 \hat{\omega})f = 0.
\]

Applying the unitary Foldy–Wouthuysen transformation for the case of massless particle in \( D \) dimensional space–time

\[
f = U \psi, \quad U = \frac{\hat{\omega} + \gamma^i \hat{p}^i}{\hat{\omega} \sqrt{2}},
\]

where \( \psi \) is the wave function in the Dirac representation, \( \gamma^\mu = (\gamma^0, \gamma^i) \) are Dirac \( \gamma \)-matrices in \( D = 2n \)-dimensional space–time, we find [4, 8] that the Schrödinger equation transforms into Dirac equation, the expressions for the Lorentz generators \( \hat{J}^{\mu\nu} \) transform into standard expressions for the Lorentz generators in the Dirac representation. Furthermore, by direct calculation one can prove that the operator

\[
\hat{T} \equiv \hat{T}_{FW} = \gamma^0 \frac{\hat{p}^i \Sigma^i}{\hat{\omega}} - \alpha
\]
transforms into the $\hat{T}_D$ operator

$$\hat{T}_D = U^+ \hat{T}_{FW} U = \gamma^{D+1} - \alpha,$$  \hspace{1cm} (26)

where $\gamma^{D+1}$ is the analogue of Dirac $\gamma^5$–matrix in dimensions $D = 2n$. One can see, that operator $\hat{T}_D$ is proportional to a standard Weyl projector in the Dirac representation.

Thus we see that the quantum mechanical description constructed here after the Foldy–Wouthuysen transformation turns into the Dirac description of the Weyl particle. Hence the above constructed quantum mechanics describes the Weyl particle in the Foldy–Wouthuysen representation.

3 Quantum mechanics of Mayorana–Weyl spinor

Now we will proceed with the construction of the pseudoclassical theory of Mayorana–Weyl particle. Note, that the action (1) is invariant under the transformations

$$\begin{align*}
  x^\mu(\tau) &\to x^\mu(-\tau), & \xi^\mu(\tau) &\to \xi^\mu(-\tau), & \xi_{D+1}(\tau) &\to -\xi_{D+1}(-\tau), \\
  \chi(\tau) &\to \chi(-\tau), & e(\tau) &\to e(-\tau), & b^\mu(\tau) &\to -b^\mu(-\tau), & i &\to -i,
\end{align*}$$ \hspace{1cm} (27)

which correspond to the reparametrization $\tau \to -\tau$. This transformation was not included in the gauge group in the previous section. In that case the model describes the charged particle and in the gauge $x^0 - \kappa \tau \approx 0$ the trajectories with $\kappa = +1$ are interpreted as trajectories of particles and those with $\kappa = -\tau$ as trajectories of antiparticles.

The switching on of the external electromagnetic field confirms this assertion since to the trajectory with a given $\kappa$ corresponds a particle with a charge $\kappa e$ \cite{17} and the action isn’t invariant under the transformation $\tau \to -\tau$. When the action is invariant under the transformation $\tau \to -\tau$, there is a possibility of another interpretation. We can identified the trajectories with $\kappa = +1$ and $\kappa = -1$. This is equivalent to the introduction of the reparametrization $\tau \to -\tau$ in the gauge group \cite{17} and then the theory describes the truly neutral particle.
Thus to describe Mayorana–Weyl spinor we enlarge the symmetry group of the action (1), given by the gauge transformations (2)–(4), by the transformation (27) and will choose for $\Phi_G$ (see. (9)) the constraint $\Phi_G' = x_0 - \tau$.

Formally this means that now we must take all equations of the previous section for the value of $\kappa = +1$. The wave functions described now by a $2^{(D-2)/2}$ column $f = f^1$. The hamiltonian is equal to $H = \hat{\omega}$ and the realization of the operators $\hat{x}^i, \hat{p}_i, \hat{\xi}^i$ are given by (13) with $\kappa = +1$.

Now with $\kappa = +1$ the expression for operators $\hat{T}$ and $\hat{J}^{\mu\nu}$ take the form

$$\hat{T}(\kappa = +1) \equiv \hat{T} = -2\alpha\hat{P}_\alpha, \quad \hat{P}_\alpha = \frac{1}{2} \left( 1 - \frac{\alpha\hat{p}^i\sigma^i}{\hat{\omega}} \right),$$

$$\hat{J}^{ik}(\kappa = +1) \equiv \hat{\mathcal{J}}^{ik} = -\hat{q}^i\hat{p}^k + \hat{q}^k\hat{p}^i - \frac{i\hbar}{4} \left[ \sigma^i, \sigma^k \right]_-, \quad \hat{J}^{0k}(\kappa = +1) \equiv \hat{\mathcal{J}}^{0k} = -x^0\hat{p}^k - \frac{1}{2} \left[ \hat{q}^k, \hat{\omega} \right]_+ - \frac{i\hbar}{4\hat{\omega}}\hat{p}^j \left[ \sigma^k, \sigma^j \right]_-. \quad (28)$$

where we introduced the projectors $\hat{P}_\alpha = (\hat{P}_+, \hat{P}_-)$. As it was pointed above, the operator $\hat{T}$ commutes with generators $\hat{J}^{\mu\nu}$ for all $D$-s, and hence the operator $\hat{P}_\alpha$ commutes with generators $\hat{\mathcal{J}}^{\mu\nu}$. From this follows, that $\hat{\mathcal{J}}^{\mu\nu}$ can be represented in the form

$$\hat{\mathcal{J}}^{\mu\nu} = \hat{P}_+\hat{\mathcal{J}}^{\mu\nu} + \hat{P}_-\hat{\mathcal{J}}^{\mu\nu} = \hat{P}_+\hat{\mathcal{J}}^{\mu\nu}\hat{P}_+ + \hat{P}_-\hat{\mathcal{J}}^{\mu\nu}\hat{P}_-. \quad (30)$$

This again is a reflection of the fact, that the corresponding representation of the Lorentz group is reducible.

Now the condition (22) for the definite value of the $\alpha$ takes the form

$$\hat{P}_\alpha f^1 = 0. \quad (31)$$

Since, as it is known, $\hat{p}^i\sigma^i/\hat{\omega}$ is operator of chirality, the condition (31) extracts from $f^1$ the state with one value of chirality equal to $\alpha$.

Thus we have constructed in any even–dimensional space–time the quantum mechanics of the system, describing a neutral particle (the antiparticle coincides with the particle) which has one value of chirality (equal to $\alpha$). It is natural to call it Mayorana–Weyl particle.
Let us turn now to the clarification of the relation of the description of the Mayorana–Weyl particle constructed here to the Mayorana–Weyl particle in the Dirac picture, which exists only in the dimensions $D = 2(\text{mod } 8)$ \cite{27}). We will consider, for definiteness, the space–time dimension $D = 10$.

Let us write the explicit representation of $\gamma$–matrices for $D = 10$

\[
\Gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \Gamma_{11} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},
\]

where the matrices $\sigma^i$ are given by the expressions

\[
\sigma^1 = i\gamma^2 \otimes \gamma^5, \quad \sigma^2 = i\gamma^1 \otimes I_4, \quad \sigma^3 = \gamma^5 \otimes I_4, \quad \sigma^4 = i\gamma^3 \otimes I_4,
\]

\[
\sigma^5 = i\gamma^2 \otimes \gamma^0, \quad \sigma^6 = \gamma^2 \otimes \gamma^1, \quad \sigma^7 = \gamma^2 \otimes \gamma^2, \quad \sigma^8 = \gamma^2 \otimes \gamma^3, \quad \sigma^9 = \gamma^0 \otimes I_4.
\]

Here $\gamma^5, \gamma^\mu (\mu = 0, 1, 2, 3)$ are usual Dirac $\gamma$–matrices, $I_4$ is the unit $4 \otimes 4$ matrix. The $\sigma^i$ matrices have the properties, which can be easily deduced from the representation (33):

\[
\sigma^{iT} = (-1)^{i+1} \sigma^i, \quad \sigma^i^* = (-1)^{i+1} \sigma^i, \quad \sigma^{i+} = \sigma^i.
\]

It’s convenient to represent $\Gamma$ matrices as a direct product of a matrices of lower dimensions:

\[
\Gamma^0 = \tau^3 \otimes I_4 \otimes I_4, \quad \Gamma^i = i\tau^2 \otimes \sigma^i, \quad i = 1, \ldots, 9, \quad \Gamma_{11} = \tau^1 \otimes I_4 \otimes I_4
\]

where $\tau^1, \tau^2, \tau^3$ are Pauli matrices.

The properties of $\Gamma$ matrices

\[
(\Gamma^0)^{-1} = \Gamma^0, \quad (\Gamma^i)^{-1} = -\Gamma^i, \quad (\Gamma_{11})^{-1} = \Gamma_{11}, \quad (\Gamma^i)^T = (-1)^i \Gamma^i
\]

can be easily deduced from (35) using (34). Write down the expression for the matrix $M \equiv C\Gamma^0$ ($C$ is the charge conjugation matrix), which is the same in both Dirac and Foldy–Wouthuysen representations:

\[
M = \Gamma^2 \Gamma^4 \Gamma^6 \Gamma^8 \Gamma_{11} = \tau^1 \otimes \Lambda, \quad \Lambda = I_2 \otimes \tau^2 \otimes I_2 \otimes \tau^2.
\]
Consider now the equation (31)

\[
\left( \frac{\hat{p}^i \sigma^i}{\omega} - \alpha \right) f^1 = 0
\]  

(38)

the solution of which is a Majorana–Weyl spinor in \( D = 10 \) dimensions in quantum mechanics. Let us write the analogues of equations (38) for the complex conjugated spinor \( f^{1*} \). Taking into account the relation \( \hat{p}^* \mu = -\hat{p} \mu \ (\hat{p}^+ \mu = \hat{p}_\mu) \) we have

\[
\left( \frac{\hat{p}^i \sigma^i}{\omega} - \alpha \right) f^{1*} = 0
\]  

(39)

Using the relation

\[
\Lambda \sigma^i = \sigma^i \Lambda,
\]

(40)

which can be easily checked by direct calculations and commuting \( \Lambda \) with the operators acting on \( f^1 \) in equation (39) we find

\[
\left( \frac{\hat{p}^i \sigma^i}{\omega} + \alpha \right) (\Lambda f^{1*}) = 0
\]  

(41)

The equations (38) and (41) may be written in the equivalent form

\[
\hat{T} f_M = 0,
\]

(42)

where \( \hat{T} \) is given by (15) and the notation

\[
f_M = \begin{pmatrix} f^1 \\ \Lambda f^{1*} \end{pmatrix}
\]

(43)

was introduced. It can be proved, that the existence of a real matrix \( \Lambda \) with the property (40) singles out in QFT the space–time dimensions \( D = 2(\text{mod } 8) \) [27].

The \( f_M \) defined by (43) is just a Mayorana bispinor in the Foldy–Wouthuysen representation. In fact, writing down the condition for the Mayorana bispinor \( f \)

\[
f = f^C = M f^*
\]

(44)

we find, that the spinors, satisfying (44) , have the form of (43).
The equation (42) corresponds to equation (22) which, as it was found in the previous section, are Weyl conditions on the Mayorana spinor in the Foldy–Wouthuysen representation. Furthermore, the Schrödinger equation for $f_M$ has the form of (23). The Lorentz generators for $f_M$ can be constructed through the generators (29) for the $f^1$ using the relation

$$i\hat{J}^{\mu\nu}(\kappa = -1) = -i\Lambda\hat{J}^{\mu\nu*}(\kappa = +1)\Lambda.$$  \hspace{1cm} (45)

Direct calculations of the right hand side of the (45) using (40) shows that the $\hat{J}^{\mu\nu}(\kappa = -1)$ coincides with $\hat{J}^{\mu\nu}(\kappa = -1)$, which ensures the Lorentz covariance of Mayorana bispinor (13).

Thus using the quantum mechanical column $f^1$ we constructed the Mayorana–Weyl bispinor in the Foldy–Wouthuysen representation.

This research was partially supported by the grant 211-5291 YPI of the German Bundesministerium für Forschung und Technologie. I.V.Tyutin was supported in part by grant # M21300 from international Science Foundation and Government of the Russian Federation and by European Community Commission under contract INTAS-94-2317.

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