Equivariant extensions of ∗-algebras

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Abstract

A bivariant functor is defined on a category of ∗-algebras and a category of operator ideals, both with actions of a second countable group $G$, into the category of abelian monoids. The element of the bivariant functor will be $G$-equivariant extensions of a ∗-algebra by an operator ideal under a suitable equivalence relation. The functor is related with the ordinary $Ext$-functor for $C^*$-algebras defined by Brown-Douglas-Fillmore. Invertibility in this monoid is studied and characterized in terms of Toeplitz operators with abstract symbol.

Introduction

Extensions of $C^*$-algebras by stable $C^*$-algebras have been thoroughly studied (see [2], [3], [11], [14]) due to their close relation to Toeplitz operators and $KK$-theory (see [11], [14]). The starting point was the article [3] where an abelian monoid $Ext(A)$ was associated to a $C^*$-algebra $A$. This monoid consists of extensions $0 \to K \to E \to A \to 0$ under a certain equivalence relation, here $K$ denotes the ideal of compact operators. The construction can be generalized to a bivariant theory by replacing $K$ with an arbitrary stable $C^*$-algebra $B$ and one obtains an abelian monoid $Ext(A, B)$. In [14] this construction was put into the equivariant setting although only the invertible elements of $Ext_G(A, B)$ were studied. We will study the full extension monoids.

As is shown in [11], and equivariantly in [14], an odd Kasparov $A-B$-module gives an extension of $A$ by $B$ which induces an additive mapping $KK^1_G(A, B) \to Ext_G(A, B)$. It can be shown, as is done in [14] that this is a bijection to the group $Ext_G^{-1}(A, B) \subseteq Ext_G(A, B)$ of invertible elements. A more straightforward approach is the proof in [11] using the Stinespring representation theorem. As a corollary of this proof, if $A$ is nuclear and separable the Choi-Effros lifting theorem implies that $Ext_G(A, B)$ is a group if $G$ is trivial. This is the main motivation of studying extension theory.

The reason for leaving the category of $C^*$-algebras is that most cohomology theories behave badly on $C^*$-algebras and one needs to look at dense subalgebras, see more in [10]. For example, if we use cohomology and Atiyah-Singer index theorem to calculate the index of a Toeplitz operator this is easily done via an explicit integral in terms of the symbol and its derivatives if the symbol is smooth, see more in [7].

With this as motivation we will extend the $Ext_G$-functor to ∗-algebras which embed into separable $C^*$-algebras and actions which extend to $C^*$-automorphisms.
In the first part of this paper we define suitable categories for the first and the second variable of the functor. Then, similarly to the setting with \( C^* \)-algebras, we will construct a bivariant functor \( \mathcal{E}xt_G \) to the category of abelian monoids. In particular there is a natural transformation

\[
\Theta : \mathcal{E}xt_G \rightarrow Ext_G
\]

in the category of abelian monoids. An interesting question to study further is what types of elements are in the kernel of the \( \Theta \)-mapping and if there is some way to make \( \Theta \) surjective?

After that we will move on to study the invertible elements. A rather remarkable result is that the invertible elements are those extensions which arise from a \( G \)-equivariant algebraic \( A \rightarrow \mathcal{J} \)-Kasparov modules. As an example, we will study the case of extensions of the smooth functions on a compact manifold by the Schatten class operators, in this case the \( \Theta \)-mapping turns out to be a surjection. At the end of the paper we describe a certain type of elements in the kernel of the \( \Theta \)-mapping which we will call linear deformations. The linear deformations are analytic in their nature. We end the paper by giving an explicit example of a linear deformation of the ordinary Toeplitz operators on the Hardy space that produces another \( \mathcal{E}xt \)-class but is homotopic to the \( \mathcal{E}xt \)-class defined by the ordinary Toeplitz operators.

## 1 Definitions and basic properties

To begin with we will define the suitable categories. From here on, let \( G \) be a second countable locally compact group. We will say that the group action \( \alpha : \ G \rightarrow Aut(A) \) acts continuously on the \( C^* \)-algebra \( A \) if \( g \mapsto \alpha_g(a) \) is continuous for all \( a \in A \).

**Definition 1.1.** Let \( C^*A_G \) denote the category with objects consisting of pairs \( (A, A) \) where \( A \) is a separable \( C^* \)-algebra with a continuous \( G \)-action and \( A \) is a \( G \)-invariant dense \( \ast \)-subalgebra. A morphism in \( C^*A_G \) between \( (A, A) \) to \( (A', A') \) is a \( G \)-equivariant \( \ast \)-homomorphism \( \varphi : A \rightarrow A' \) bounded in \( C^* \)-norm.

As an abuse of notation we will denote an object \( (A, A) \) in \( C^*A_G \) by \( A \) and its Latin character \( A \) will denote the ambient \( C^* \)-algebra. Observe that a morphism in \( C^*A_G \) is the restriction of an equivariant \( \ast \)-homomorphism \( \tilde{\varphi} : A \rightarrow A' \) uniquely determined by \( \varphi \). This follows from that if \( \varphi : A \rightarrow A' \) is bounded in \( C^* \)-norm it extends to \( \tilde{\varphi} : A \rightarrow A' \) and since \( \varphi \) is equivariant \( \tilde{\varphi} \) will also be equivariant. Conversely, an equivariant \( \ast \)-homomorphism of \( C^* \)-algebras is always \( C^* \)-bounded. When a linear mapping \( T : A \rightarrow A' \), not necessarily equivariant, between two objects is induced by a bounded mapping \( T : A \rightarrow A' \) we will say that \( T \) is \( C^* \)-bounded.

For a \( C^* \)-algebra \( B \) we will denote its multiplier \( C^* \)-algebra by \( M(B) \) and embed \( B \) as an ideal in \( M(B) \). If \( B \) has a \( G \)-action we will equip \( M(B) \) with the induced \( G \)-action.

**Definition 1.2.** If \( (\mathcal{J}, I) \in C^*A_G \) satisfies that the \( C^* \)-algebra \( I \) is equivariantly stable, that is \( I \otimes K \cong I \) where \( K \) has trivial \( G \)-action, and \( \mathcal{J} \) is an ideal in \( M(I) \) the algebra \( \mathcal{J} \) is called a \( C^* \)-stable \( G \)-ideal. Let \( C^*S_I_G \) denote the full subcategory of \( C^*A_G \) consisting of \( C^* \)-stable \( G \)-ideals.
We will call a morphism $\psi : \mathcal{I} \to \mathcal{I}'$ of $C^*$-stable $G$-ideals an embedding of $C^*$-stable $G$-ideals if $\psi : \mathcal{I} \to \mathcal{I}'$ is an isomorphism.

**Proposition 1.3.** For any $C^*$-stable $G$-ideal $\mathcal{I}$ there is an equivariant isomorphism $M_2 \otimes \mathcal{I} \cong \mathcal{I}$ inducing an isomorphism $M_2 \otimes \mathcal{I} \cong \mathcal{I}$. The isomorphism is given by the adjoint action of a $G$-invariant unitary operator $V = V_1 \oplus V_2 : \mathcal{I} \oplus \mathcal{I} \to \mathcal{I}$ between Hilbert modules.

Notice that $V$ being unitary is equivalent to $V_1, V_2 \in \mathcal{M}(\mathcal{I})$ being isometries satisfying

$$V_1V_1^* + V_2V_2^* = 1.$$ 

**Proof.** It is sufficient to construct two $G$-invariant isometries $V_1, V_2 \in \mathcal{M}(\mathcal{I})$ such that $V_1V_1^* + V_2V_2^* = 1$. Then $V := V_1 \oplus V_2$ is a $G$-invariant unitary. Thus $V$ will be an isomorphism of Hilbert modules so $Ad V : M_2 \otimes \mathcal{I} \to \mathcal{I}$ is an isomorphism and since $\mathcal{I}$ is an ideal $Ad V$ induces a isomorphism $M_2 \otimes \mathcal{I} \cong \mathcal{I}$.

Let $K$ denote a separable Hilbert space with trivial $G$-action. Choose a unitary $V' : K \oplus K \to K$. Let $V_1', V_2' \in \mathcal{B}(K)$ be defined by $V'(x_1 \oplus x_2) := V_1'x_1 + V_2'x_2$. We may take the isometries $V_1$ and $V_2$ to be the image of $V_1'$ and $V_2'$ under the equivariant, unital embedding

$$\mathcal{B}(K) = \mathcal{M}(\mathcal{K}) \hookrightarrow \mathcal{M}(\mathcal{I} \otimes \mathcal{K}) \cong \mathcal{M}(\mathcal{I}).$$

One important class of $C^*$-stable $G$-ideals is the class of symmetrically normed operator ideals such as the Schatten class ideals and the Dixmier ideals (see more in [4]) over a separable Hilbert space $H$ with a $G$-action. In order to get equivariant stability we need to stabilize the Hilbert space with another Hilbert space with trivial $G$-action. Let $H'$ denote a separable Hilbert space and define

$$\mathcal{L}_H^p := (\mathcal{L}^p(H \otimes H'), \mathcal{K}(H \otimes H'))$$

and analogously for the Dixmier ideal $\mathcal{L}_H^{n+}$. The $G$-action on the algebras are the one induced from the $G$-action on $H$.

The main study of this paper are equivariant extensions $0 \to \mathcal{I} \to \mathcal{E} \xrightarrow{\phi} \mathcal{A} \to 0$ where $\mathcal{I}$ is a $C^*$-stable $G$-ideal and $\mathcal{A} \in C^*A_G$. In particular we are interested in when such extensions admit $C^*$-bounded splittings of Toeplitz type.

Consider for example the $0$th order pseudodifferential extension $\Psi_0^0(M)$ on a closed Riemannian manifold $M$. This extension is an extension of the smooth functions on the cotangent sphere $S^*M$ by the classical pseudodifferential operators of order $-1$ given by the short exact sequence

$$0 \to \Psi^{-1}(M) \to \Psi^0(M) \to C^\infty(S^*M) \to 0.$$

The algebra $\Psi^{-1}(M)$ is not $C^*$-stable, but $\Psi^{-1}(M)$ is dense in $\mathcal{L}^p(L^2(M))$ for any $p > n$, so the pseudo-differential extension fits in our framework after some modifications. The pseudo-differential extension admits an explicit splitting $T : C^\infty(S^*M) \to \Psi^0(M)$ in terms of Fourier integral operators which is not $C^*$-bounded if $\dim M > 1$. Read more about this in Chapter 18.6 in [3]. In this setting however, the problem can be mended. In [3] a $C^*$-bounded splitting
is constructed for real analytic manifolds \( M \) in terms of Grauert tubes and Toeplitz operators.

We will abuse the notation somewhat by referring both to the object \( \mathcal{E} \) and the extension by \( \mathcal{E} \). Observe that the definition implies that there exists a commutative diagram with equivariant, exact rows

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathcal{E} & \overset{\varphi}{\longrightarrow} & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & E & \overset{\varphi}{\longrightarrow} & A & \longrightarrow & 0
\end{array}
\]

The \(*\)-homomorphism \( \bar{\varphi} : E \to A \) is the extension of \( \varphi \) to \( E \).

**Definition 1.4.** Two \( \mathcal{G} \)-equivariant extensions \( \mathcal{E} \) and \( \mathcal{E}' \) of \( \mathcal{A} \) by \( \mathcal{I} \) are said to be isomorphic if there exists a morphism \( \psi : \mathcal{E} \to \mathcal{E}' \) in \( C^* \mathcal{A} \mathcal{G} \) that fits into a commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathcal{E} & \overset{\varphi}{\longrightarrow} & A & \longrightarrow & 0 \\
\downarrow & & \downarrow \psi & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathcal{E}' & \overset{\varphi'}{\longrightarrow} & A & \longrightarrow & 0
\end{array}
\]

Because of the five lemma, \( \psi \) is an isomorphism.

Choose a linear splitting \( \tau : \mathcal{A} \to \mathcal{E} \) and identify \( \mathcal{I} \) with an ideal in \( \mathcal{E} \). The mapping \( \tau \) being a splitting of an equivariant mapping \( \mathcal{E} \to \mathcal{A} \) implies that

\[
\tau(ab) - \tau(a)\tau(b), \quad \tau(a^*) - \tau(a)^* \in \mathfrak{I} \quad \text{and} \quad \tau(g.a) - g.\tau(a) \in \mathfrak{I} \quad \forall g \in \mathcal{G}. \tag{2}
\]

Given a \( C^* \)-stable \( \mathcal{G} \)-ideal \( \mathfrak{I} \) we define the \( \mathcal{G} \)-\( * \)-algebra \( \mathcal{C}_\mathfrak{I} := \mathcal{M}(\mathfrak{I})/\mathfrak{I} \) and denote by \( q_\mathfrak{I} : \mathcal{M}(\mathfrak{I}) \to \mathcal{C}_\mathfrak{I} \) the canonical surjection. By the equations (2) and (3) the mapping \( q_\mathfrak{I}\tau : \mathcal{A} \to \mathcal{C}_\mathfrak{I} \) is an equivariant \(*\)-homomorphism. We will call the mapping \( \beta_\mathcal{A} := q_\mathfrak{I}\tau \) the Busby mapping for the extensions \( \mathcal{E} \). A Busby mapping that is \( C^* \)-bounded after composing with \( \mathcal{C}_\mathfrak{I} \to \mathcal{M}(\mathfrak{I})/\mathfrak{I} \) is called bounded. A Busby mapping which can be lifted to a \( C^* \)-bounded \( \mathcal{G} \)-equivariant \(*\)-homomorphism of \( \mathcal{A} \) is called trivial.

For an equivariant \(*\)-homomorphism \( \beta : \mathcal{A} \to \mathcal{C}_\mathfrak{I} \) we can define the \(*\)-algebra

\[
\mathcal{E}_\beta := \{ a \oplus x \in \mathcal{A} \oplus \mathcal{M}(\mathfrak{I}) : \beta(a) = q_\mathfrak{I}(x) \}.
\]

The \(*\)-algebra \( \mathcal{E}_\beta \) is closed under the \( \mathcal{G} \)-action on \( \mathcal{A} \oplus \mathcal{M}(\mathfrak{I}) \) so it is a \( \mathcal{G} \)-\( * \)-algebra. Denote the norm closure of \( \mathcal{E}_\beta \) in \( \mathcal{A} \oplus \mathcal{M}(\mathfrak{I}) \) by \( \mathcal{E}_\beta \). We have an injection \( \mathfrak{I} \to \mathcal{E}_\beta \) and a surjection \( \mathcal{E}_\beta \to \mathcal{A} \). The kernel of \( \mathcal{E}_\beta \to \mathcal{A} \) is \( \mathfrak{I} \), so the sequence \( 0 \to \mathfrak{I} \to \mathcal{E}_\beta \to \mathcal{A} \to 0 \) is exact and the arrows are equivariant. The \(*\)-algebra \( \mathcal{E}_\beta \) is a well defined object in \( C^* \mathcal{A} \mathcal{G} \), because Theorem 2.1 of [14] states that the induced \( \mathcal{G} \)-action on \( \mathcal{E}_\beta \) is continuous provided it is continuous on \( \mathfrak{I} \) and on \( \mathcal{A} \).

**Proposition 1.5.** The equivariant \(*\)-homomorphism \( \beta : \mathcal{A} \to \mathcal{C}_\mathfrak{I} \) determines the extension up to a isomorphism, i.e if \( \mathcal{E} \) has Busby mapping \( \beta \), \( \mathcal{E} \) is isomorphic to \( \mathcal{E}_\beta \).
Proof. Suppose that $\beta$ is Busby mapping for $E$. Define $\psi : E \to E_\beta$ as

$$\psi(x) := \varphi(x) \oplus x.$$ 

Since $\varphi$ is equivariant, so is $\psi$. This makes the diagram (1) commutative, thus $\psi$ is an isomorphism of $G$-equivariant extensions.

The most useful class of $G$-equivariant extensions are the ones arising from algebraic $A - \mathcal{I}$-Kasparov modules. This is defined as an algebraic generalization of Kasparov modules for $C^*$-algebras, see more in [11].

**Definition 1.6.** A $G$-equivariant algebraic $A - \mathcal{I}$-Kasparov module is a $C^*$-bounded $G$-equivariant representation $\pi : A \to \mathcal{M}(I)$ and an almost $G$-invariant symmetry $F \in \mathcal{M}(I)$ that is almost commuting with $\pi(A)$, that is:

$$g.F - F \in \mathcal{I} \quad \forall g \in G \quad \text{and} \quad [F, \pi(a)] \in \mathcal{I} \quad \forall a \in A.$$

Since $F$ is a grading we can define the projection $P := (F + 1)/2$. The pair $(\pi, F)$ induces a $\ast$-homomorphism

$$\beta : A \to C_3, \quad a \mapsto q_3(P\pi(a)P).$$

(4)

The requirement $[F, \pi(a)] \in \mathcal{I}$ together with $g.F - F \in \mathcal{I}$ implies that $\beta$ is an equivariant $\ast$-homomorphism.

Let $B_G(A, \mathcal{I})$ denote the set of bounded $G$-equivariant Busby mappings on $A$. This is the correct set to study extensions in. By Proposition 1.5 the set of $G$-equivariant Busby mappings is the same set as the set of isomorphism classes of $G$-equivariant extensions. But we need some useful notion of equivalence of extensions, or by the previous reasoning an equivalence relation on $B_G(A, \mathcal{I})$. For an object $\mathcal{I} \in C^*SI_G$ we define the almost invariant weakly unitaries

$$U^\text{aw}(\mathcal{I}) := q_3^{-1}\{v \in C_3 : g.v = v, \quad v^*v = vv^* = 1\}.$$ 

Let the almost invariant unitaries be defined as $U^\text{a}(\mathcal{I}) := U^\text{aw}(\mathcal{I}) \cap U(\mathcal{M}(\mathcal{I}))$.

**Definition 1.7.** Strong equivalence on $B_G(A, \mathcal{I})$ is the equivalence of Busby mappings by the adjoint $U^{\text{a}}(\mathcal{I})$-action on $C_3$. Weak equivalence on $B_G(A, \mathcal{I})$ is that of the adjoint $U^\text{aw}(\mathcal{I})$-action on $C_3$.

Let $E_G(A, \mathcal{I})$ denote the set of strong equivalence classes of $B_G(A, \mathcal{I})$ and let $E^\text{aw}_G(A, \mathcal{I})$ denote the set of weak equivalence classes. Similarly let $D_G(A, \mathcal{I})$ denote the set of strong equivalence classes of trivial Busby mappings and let $D^\text{aw}_G(A, \mathcal{I})$ denote the set of weak equivalence classes of trivial Busby maps.

The isomorphism $\lambda : M_2 \otimes C_3 \to C_3$ induced by $Ad$ from Proposition 1.3 can be used to define the sum of two $G$-equivariant Busby mappings $\beta_1, \beta_2 \in B_G(A, \mathcal{I})$ as

$$\beta_1 + \beta_2 := \lambda \circ (\beta_1 \oplus \beta_2) : A \to C_3.$$ 

**Proposition 1.8.** The binary operation $+$ on $B_G(A, \mathcal{I})$ induces a well defined abelian semigroup structure on $E_G(A, \mathcal{I})$ independent of the choice of the unitary $V = V_1 \oplus V_2$. The set $D_G(A, \mathcal{I})$ is a subsemigroup.
The proof of the above proposition is the same as the proof of Lemma 3.1 in [14] where the semigroup of equivariant extensions of a $C^*$-algebra is constructed. Two $G$-equivariant Busby mappings $\beta_1, \beta_2 \in B_G(A, \mathfrak{J})$ are said to be stably equivalent if they differ by trivial Busby mappings. That is, if there exist $C^*$-bounded, $G$-equivariant $\ast$-homomorphisms $\pi_1, \pi_2 : A \to M_2 \otimes C_2$ such that

$$\beta_1 \oplus q_3 \pi_1 \equiv \beta_2 \oplus q_3 \pi_2 : A \to M_2 \otimes C_2.$$ 

Stable equivalence induces a well defined equivalence relation on $E_G(A, \mathfrak{J})$ and $E^w_G(A, \mathfrak{J}).$

**Definition 1.9.** We define $\mathcal{E}xt_G(A, \mathfrak{J})$ as the monoid of stable equivalence classes of $E_G(A, \mathfrak{J})$ and $\mathcal{E}xt^w_G(A, \mathfrak{J})$ as the monoid of stable equivalence classes of $E^w_G(A, \mathfrak{J})$. For $G = \{1\}$ we denote the $\mathcal{E}xt$-invariants by $\mathcal{E}xt(A, \mathfrak{J})$ and $\mathcal{E}xt^w(A, \mathfrak{J}).$

The monoids $\mathcal{E}xt_G(A, \mathfrak{J})$ and $\mathcal{E}xt^w_G(A, \mathfrak{J})$ coincide with the semigroup quotients $E_G(A, \mathfrak{J})/D_G(A, \mathfrak{J}),$ respectively $E^w_G(A, \mathfrak{J})/D^w_G(A, \mathfrak{J}).$ It has a zero-element since the class of an element in $D_G(A, \mathfrak{J})$ is zero.

If we are given a $G$-equivariant extension $\mathcal{E}$ of $A$ we will denote the class in $\mathcal{E}xt_G(A, \mathfrak{J})$ of its $G$-equivariant Busby mapping $\beta$ by $[\mathcal{E}]$ or by $[\beta].$

**Proposition 1.10.** If $\mathfrak{J} = I$ there are isomorphisms

$$\mathcal{E}xt^w_G(A, I) \cong \mathcal{E}xt_G(A, I) \cong \mathcal{E}xt_G(A, I) \equiv \mathcal{E}xt_G(A, I) \cong \mathcal{E}xt^w_G(A, I).$$

**Proof.** We will prove the existence of the first and the second isomorphism. The proof of the last isomorphism is a special case of the first isomorphism for $A = A.$

To prove the existence of the first isomorphism it is sufficient to show that weakly equivalent $G$-equivariant Busby mappings are strongly equivalent to stable equivalence. Assume that $\beta_1, \beta_2 \in B_G(A, \mathfrak{J})$ are weakly equivalent via the almost invariant weakly unitary $U \in U^{w\ast}(\mathfrak{J}).$ Then $\beta_1 \oplus 0$ and $\beta_2 \oplus 0$ are weakly equivalent via the almost invariant weakly unitary $U \oplus U^\ast.$ But the operator $U \oplus U^*$ lifts to a unitary $\tilde{U} \in M(M_2 \otimes I)$ since $C_2$ is a $C^*$-algebra. In fact $\tilde{U} \in U^a(M_2 \otimes \mathfrak{J})$ since $U$ is almost invariant. Thus $\beta_1 \oplus 0$ and $\beta_2 \oplus 0$ are strongly equivalent. For the proof that $U \oplus U^\ast$ lifts to a unitary, see Proposition 3.4.1 in [2].

The second isomorphism is given by the mapping $\mathcal{E}xt_G(A, I) \to \mathcal{E}xt_G(A, I),$ $[\mathcal{E}] \mapsto [E].$ In terms of the $G$-equivariant Busby mapping $\beta$ the mapping is given by $[\beta] \mapsto [\beta],$ since $A$ is dense and $\beta$ is bounded by assumption this is a surjection and $\beta$ determines $\beta$ uniquely.

The constructions of $\mathcal{E}xt_G$ and $\mathcal{E}xt^w_G$ are the same as $\mathcal{E}xt_G$ and $\mathcal{E}xt^w_G$ but with $C^*$-algebras. These constructions can be found in [3], [11] and [14]. Proposition [14.10] is a mild generalization of Proposition 15.6.4 in [2]. The proof is the same although $A$ does not need to be a $C^*$-algebra.

Since the two theories are very similar we will focus on $\mathcal{E}xt_G.$ All results stated in this paper are easily verified to also hold for $\mathcal{E}xt^w_G.$

6
2 Functoriality of $\mathcal{E}xt_G$

In this section we will prove that $\mathcal{E}xt_G$ is a functor to the category $\text{Mo}^{ab}$ of abelian monoids. We define this category to have objects of abelian monoids and a morphism is an additive mapping $k : M_1 \to M_2$ such that $k(0) = 0$. We know how $\mathcal{E}xt_G$ acts on the objects of $C^*A_G$ and $C^*SL_G$. What needs to be defined is the action of $\mathcal{E}xt_G$ on the morphisms. We begin by showing that $\mathcal{E}xt_G$ depends covariantly on $\mathcal{J}$.

Let $\psi : \mathcal{J} \to \mathcal{J}'$ be a morphism of $C^*$-stable $G$-ideals. By definition $\psi$ can be extended to an equivariant mapping $\mathcal{M}(I) \to \mathcal{M}(I')$ which induces an equivariant mapping $q_\psi : C_I \to C_{I'}$. Define $\psi_* : E_G(A, \mathcal{J}) \to E_G(A, \mathcal{J}')$ by $\psi_*[\beta] := [q_\psi \circ \beta]$. Clearly, $\psi_*[\beta]$ is independent of the stable equivalence class of $[\beta]$. Hence $\psi$ induces a well defined mapping

$$\psi_* : \mathcal{E}xt_G(A, \mathcal{J}) \to \mathcal{E}xt_G(A, \mathcal{J}').$$

Since $\psi_*$ acting on a trivial extension gives a trivial extension we have a homomorphism of monoids.

Let us move on to proving that $\mathcal{E}xt_G$ depends contravariantly on $\mathcal{A}$. Let $\varphi : A \to A'$ be a morphism in $C^*A_G$. Take a $G$-equivariant Busby mapping $\beta$ of $A'$. Then we can define a $G$-equivariant Busby mapping $\varphi^* \beta := \beta \circ \varphi$ of $A$. This clearly depends on neither strong equivalence class nor stable equivalence class of the $G$-equivariant Busby mapping. If $\beta$ is trivial it follows that $\varphi^* \beta$ is trivial so we have a morphism of monoids

$$\varphi^* : \mathcal{E}xt_G(A', \mathcal{J}) \to \mathcal{E}xt_G(A, \mathcal{J}).$$

We have now proved the following proposition.

**Proposition 2.1.** The functor $\mathcal{E}xt_G : C^*A_G \times C^*SL_G \to \text{Mo}^{ab}$ is a well defined functor. It is covariant in $\mathcal{J}$ and contravariant in $\mathcal{A}$.

As noted above, an extension $E$ of the algebra $A$ by $\mathcal{J}$ gives rise to an extension $E$ of $A$ by $I$. This procedure defines a mapping $E_G(A, \mathcal{J}) \to E_G(A, I)$ which respects stable equivalences.

Let $C^*_G$ denote the category of separable $C^*$-algebras with a continuous $G$-action and $SC^*_G$ the full subcategory of equivariantly stable objects in $C^*_G$. We can define an essentially surjective functor

$$\Gamma_1 : C^*A_G \times C^*SL_G \to C^*_G \times SC^*_G,$$

$$((A, A), (\mathcal{J}, I)) \mapsto (A, I).$$

Its right adjoint is the full and faithful functor

$$\Gamma_2 : C^*_G \times SC^*_G \to C^*A_G \times C^*SL_G$$

$$(A, I) \mapsto ((A, A), (I, I)).$$

Notice that $\Gamma_1 \Gamma_2$ is the identity functor on $C^*_G \times SC^*_G$. Define the functor

$$\mathcal{E}xt_G : C^*_G \times SC^*_G \to \text{Mo}^{ab} \text{ by } \mathcal{E}xt_G := \mathcal{E}xt_G \circ \Gamma_2.$$
Proposition 2.2. The mapping $\Theta$ defines a natural transformation

$$\Theta : \text{Ext}_G \to \text{Ext}_G \circ \Gamma_1.$$ 

Proof. The mapping $\Theta_A$ merely extends Busby mappings to the object’s $C^*$-closure, so $\Theta_A$ commutes with composition of morphisms in $C^*_A \times C^*_S \mathcal{I}_G$ since they are just equivariant $C^*$-bounded $*$-homomorphisms. Thus $\Theta$ is a natural transformation. \qed

3 Invertible extensions

Just as in the case of a $C^*$-algebra one can relate invertibility in the $\text{Ext}_G$-monoid and properties of the splitting. In this section we will study invertibility in $\text{Ext}_G$-monoid in terms of Toeplitz operators.

The main result to be obtained in this section tells us that there is a direct link between algebraic properties in the $\text{Ext}_G$-monoid and analytical properties of the extension. But this tells us nothing about how to construct the inverse or give explicit expressions. We will study this in the case of $G$ being the trivial group and for extensions admitting a $C^*$-bounded, completely positive splitting. Then these explicit constructions are possible in an ideal $\mathcal{J} \supseteq \mathcal{I}$ such that $\mathcal{I}$ is the linear span of $\{a^*a : a \in \mathcal{J}\}$. In this setting an explicit inverse can be given in $\text{Ext}(A, \mathcal{J})$.

Definition 3.1. A $G$-equivariant extension which admits a splitting of the form $a \mapsto P \pi(a) P$, for a $G$-equivariant algebraic $A - \mathcal{J}$-Kasparov module $(\pi, F)$ and $P = (F + 1)/2$, is called a $G$-equivariant Toeplitz extension.

We will sometimes identify the Toeplitz extension with the pair $(P, \pi)$.

Theorem 3.2. An extension $[E] \in \text{Ext}_G(A, \mathcal{J})$ is invertible if and only if $[E]$ can be represented by a $G$-equivariant Toeplitz extension.

For equivariant extensions of $C^*$-algebras this statement is proved in [14] (Lemma 3.2) and the case $G$ trivial is well studied in [11] and [2]. Our proof of Theorem 3.2 is based upon the same ideas adjusted to our setting.

Lemma 3.3. Every strong equivalence class of an invertible $G$-equivariant extension is stably equivalent to a $G$-equivariant Toeplitz extension.

Proof. Assume that $E$ is a $G$-equivariant extension of $A$ by $\mathcal{J}$ with equivariant Busby mapping $\beta_1 : A \to C_2$ which is invertible in $\text{Ext}_G(A, \mathcal{J})$. By definition there is a mapping $\beta_2 : A \to C_3$ and a $U \in U^a(M_2 \otimes \mathcal{J})$ such that

$$U^*(\beta_1 \otimes \beta_2)U : A \to M_2 \otimes C_3$$

can be lifted to an equivariant $C^*$-bounded representation $\pi : A \to M_2 \otimes \mathcal{M}(I)$. Let $P \in M_2 \otimes \mathcal{M}(I)$ denote the almost $G$-invariant projection $U^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U$. Define

$$\beta'(a) := q_2(P\pi(a)P), \quad \beta''(a) := q_3((1 - P)\pi(a)(1 - P)).$$
For $a \in A$, we have
\[
\beta_1(a) = q_3(UPU^*)(\beta_1(a) \oplus \beta_2(a))q_3(UPU^*) = q_3(Uq(P\pi(a)P)q_3(U^*)) = q_3(U)\beta'(a)q_3(U^*),
\]
which implies that up to strong equivalence $\beta$ is the Busby mapping of the extension. By the same reasoning $\beta''$ is strongly equivalent $\beta_2$.

Define $\tau'(a) := P\pi(a)P$ and $\tau''(a) := (1-P)\pi(a)(1-P)$. We express the representation $\pi':=AdU^* \circ \pi$ as follows
\[
\pi'(a) = \begin{pmatrix} U\tau'(a)U^* & \pi_{12}(a) \\ \pi_{21}(a) & U\tau''(a)U^* \end{pmatrix},
\]
Since $q_3\pi' = \beta_1 \oplus \beta_2$, it follows that $\pi_{12}(a), \pi_{21}(a) \in \mathcal{J}$. The calculation
\[
[P, \pi(a)] = U^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pi'(a) U = U^* \begin{pmatrix} 0 & \pi_{12}(a) \\ -\pi_{21}(a) & 0 \end{pmatrix} U \in M_2 \otimes \mathcal{J},
\]
is a consequence of that $M_2 \otimes \mathcal{J}$ is an ideal in $M_2 \otimes I$ and implies that $\tau$ defines a $G$-equivariant Toeplitz extension.

**Proof of Theorem 3.2.** If $[\mathcal{E}]$ is invertible it is given by a Toeplitz extension by Lemma 3.3. Conversely assume that $\mathcal{E}$ is a $G$-equivariant Toeplitz extension $(\pi, P)$ of $A$. We define $P' := 1-P$, $P_2 := P \oplus P'$, $\tau(a) := P\pi(a)P$ and $\tau'(a) := P'\pi(a)P'$. Then the claim from which the theorem will follow is that the Busby mapping $q_3 \circ \tau'$ defines an inverse to $\mathcal{E}$. To prove this, we define the almost $G$-invariant symmetry
\[
U := \begin{pmatrix} P & P' \\ P' & P \end{pmatrix}.
\]
This symmetry satisfies $UP_2U = 1 \oplus 0$. We make the observation that $(\pi \oplus \pi, P_2)$ and $(U\pi \oplus \pi U, P_2)$ defines the same extension because of Proposition 1.5 and that the pair $(\pi, P)$ are $\mathcal{J}$-almost commuting. Since
\[
\pi(a) \oplus 0 = UP_2U(\pi(a) \oplus \pi(a))UP_2U
\]
it follows that
\[
[q_3 \circ \tau] + [q_3 \circ \tau'] = [q_3 \circ (P_2(\pi \oplus \pi)P_2)] = [q_3 \circ (UP_2U^2(\pi \oplus \pi)U^2P_2U)] = [q_3 \circ (UP_2U(\pi \oplus \pi)UP_2U)] = [q_3 \circ \pi \oplus 0] = 0.
\]

Suppose that we are in the situation $G = \{e\}$. In this case we are able to calculate an inverse to extensions admitting positive splitting if we enlarge the ideal somewhat. This should be thought of as passing from $\mathcal{L}^u(H)$ to $\mathcal{L}^{2u}(H)$. First we need an abstract notion of this procedure.

**Proposition 3.4.** Suppose that $\mathcal{J}$ is a $C^*$-stable $G$-ideal. The $*$-algebra
\[
\mathcal{J}_3 := l.s.\{x \in I : x^* x \in \mathcal{J} \text{ and } xx^* \in \mathcal{J}\}.
\]
defines a $C^*$-stable $G$-ideal $(\mathcal{J}_3, I) \in C^* SIG$. We will call $\mathcal{J}_3$ the square root of $\mathcal{J}$.
Proof. Define the two $*$-invariant subsets $J_3^+ := \{ x \in I : x^*x \in \mathcal{J} \}$ and $J_3^- := \{ x \in I : xx^* \in \mathcal{J} \}$. For $x \in J_3^+$ and $a \in \mathcal{M}(I)$, $(xa)^*xa \in \mathcal{J}$ so $xa \in J_3^+$. Since $J_3^+$ is $*$-invariant, $ax \in J_3^+$. Similarly, if $x \in J_3^-$ and $a \in \mathcal{M}(I)$ we have that $ax(ax)^* \in \mathcal{J}$ so $ax \in J_3^-$ and $xa \in J_3^-$. The $*$-algebra $J_3 \equiv \text{l.s.}(J_3^+ \cap J_3^-)$ so $J_3$ is an ideal in $\mathcal{M}(I)$. There is an embedding $\mathcal{J} \subseteq J_3$ because $J$ is a $*$-algebra, so $J_3$ is dense in $I$.

Theorem 3.5. Let $\mathcal{E}$ be an extension of $A$ by $\mathcal{J}$ admitting a $C^*$-bounded splitting $\kappa$ extending to a completely positive contraction $\kappa : A \to \mathcal{M}(I)$. If $i : \mathcal{J} \to J_3$ is the embedding of $\mathcal{J}$ into its square root, $i_*[q_3 \circ \kappa]$ is invertible in $\text{Ext}(A, J_3)$.

Before proving this we need to review the useful construction of the Stinespring representation. This is a standard method for operator algebras and was first introduced by Stinespring in [13].

Theorem 3.6 (Stinespring Representation Theorem). Assume that $A$ is a separable $C^*$-algebra, $I$ is a stable $C^*$-algebra and that $\kappa : A \to \mathcal{M}(I)$ is a completely positive mapping such that $\|\kappa\| \leq 1$. Then there exists a $*$-homomorphism $\pi_\kappa : A \to M_2 \otimes \mathcal{M}(I)$ of $A$ such that

$$
\begin{pmatrix}
\kappa(a) & 0 \\
0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\pi_\kappa(a)
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}.
$$

The $*$-homomorphism $\pi_\kappa$ is called a Stinespring representation of $\kappa$. For proof see [11].

Lemma 3.7. Assume that $\kappa : A \to \mathcal{M}(I)$ is a completely positive contraction. In the notation above

$$
\{ a \in A : \kappa(a^2) - \kappa(a)^2 \in \mathcal{J} \} = \{ a \in A : [P, \pi_\kappa(a)] \in \mathcal{J}_3 \},
$$

where $P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Proof. We express the representation as follows

$$
\pi(a) = \begin{pmatrix} \kappa(a) & \pi_{12}(a) \\ \pi_{21}(a) & \pi_{22}(a) \end{pmatrix},
$$

where $\pi_{12}(a) = P\pi(a)(1 - P)$ and so on. This implies that $\pi_{12}(a)^* = \pi_{21}(a^*)$. Since $\pi$ is a representation

$$
\begin{pmatrix}
\kappa(ab) & * \\
* & *
\end{pmatrix}
= 
\pi(ab)
= 
\pi(a)\pi(b)
= 
\begin{pmatrix}
\kappa(a)\kappa(b) + \pi_{12}(a)\pi_{21}(b) & * \\
* & *
\end{pmatrix},
$$

(5)

So

$$
\kappa(ab) - \kappa(a)\kappa(b) = \pi_{12}(a)\pi_{21}(b).
$$

Thus $\kappa(a^2) - \kappa(a)^2 \in \mathcal{J}$ if and only if $\pi_{12}(a)\pi_{21}(a) \in \mathcal{J}$. After polarization we only need to show that this is equivalent to the statement $[P, \pi_\kappa(a)] \in \mathcal{J}_3$ for self adjoint $a$. But

$$
[P, \pi(a)] = 
\begin{pmatrix}
0 & \pi_{12}(a) \\ -\pi_{21}(a) & 0
\end{pmatrix}.
$$
This proves Theorem 3.2, since this implies that \( \kappa \) defines a Toeplitz extension of \( \mathcal{A} \) by \( \mathcal{J} \) and by Theorem 3.2 the element \( i_\ast [\pi_\circ \kappa] \) is invertible in \( \mathcal{E}xt(\mathcal{A}, \mathcal{J}) \).

To see that the square root of a \( C^\ast \)-stable ideal is needed sometimes, consider the example of the Besov space \( \mathcal{A} = \mathcal{B}^1_{p/2} \) on the circle \( S^1 \). This carries a representation \( \pi : \mathcal{A} \rightarrow \mathcal{B}(L^2(S^1)) \) by multiplication as functions. Let \( P \) be the Hardy projection. By [12], if \( a \in L^\infty(S^1) \) it holds that \( [P, \pi(a)] \in \mathcal{L}^p(L^2(S^1)) \) if and only if \( a \in \mathcal{A} \). Making a similar decomposition of \( \pi \) as in the proof of Lemma 3.7 one can show that the completely positive mapping \( \tau(a) := P\pi(a)P \) is a splitting of an extension of \( \mathcal{A} \) by \( \mathcal{L}^{p/2} \). Since \( \mathcal{A} \equiv \{ a \in L^\infty(S^1) : [P, \pi(a)] \in \mathcal{L}^p(L^2(S^1)) \} \) it follows that \([\xi_{L^{p/2}} \circ \tau] \in \mathcal{E}xt(\mathcal{A}, \mathcal{L}^{p/2})\) is not invertible by Theorem 3.2. But if \( i : \mathcal{L}^{p/2} \rightarrow \mathcal{L}^p \) denotes the inclusion mapping (which coincides with the mapping constructed in Proposition 3.4) the element \( i_\ast [\xi_{L^{p/2}} \circ \tau] \in \mathcal{E}xt(\mathcal{A}, \mathcal{L}^p) \) is invertible by Theorem 3.2.

### 4 Example: Extensions of \( C^\infty(M) \) by Schatten ideals

Commutative \( C^\ast \)-algebras have many good properties such as nuclearity and concrete realizations in geometry. The geometric interpretations of extensions of commutative \( C^\ast \)-algebras over a manifold, such as Toeplitz operators and pseudodifferential operators, are motivating for extension theory and allows for very concrete smooth \( * \)-subalgebras to do calculations in.

For example, the one dimensional case \( M = \mathbb{T} \) can be handled in a fairly straightforward fashion by finding an invertible generator for \( \mathcal{E}xt^{-1}(C^\infty(S^1), \mathcal{L}^p) \) for \( p \geq 2 \) precisely as is done for \( C(S^1) \) in Chapter 7 in [6]. To find a set of generators in the general setting will be difficult. But a more abstract approach together with a topological description of \( K \)-homology of smooth manifolds shows that the \( \Theta \)-mapping in fact is a surjection for \( \mathcal{A} = C^\infty(M) \) and \( \mathcal{J} \) being a Schatten ideal or a Dixmier ideal.

For \( p > n \) define \( i^p : \mathcal{L}^{n^+} \rightarrow \mathcal{L}^p \) to be the embedding of \( C^\ast \)-stable ideals induced by the embedding \( \mathcal{L}^{n^+} \rightarrow \mathcal{L}^p \) of operator ideals.

**Theorem 4.1.** Let \( p > n \). Assume that \( M \) is a compact manifold of dimension \( n \) and \( \mathcal{A} = C^\infty(M) \). Then the mappings

\[
\Theta_{\mathcal{L}^{n^+}}^A : \mathcal{E}xt(\mathcal{A}, \mathcal{L}^{n^+}) \rightarrow \mathcal{E}xt(C(M), \mathcal{K}) = K_1(M) \quad \text{and} \\
\Theta_{\mathcal{L}^p}^A : \mathcal{E}xt(\mathcal{A}, \mathcal{L}^p) \rightarrow \mathcal{E}xt(C(M), \mathcal{K}) \quad \text{are surjective.}
\]

**Proof.** Using the definition of topological \( K \)-homology, see [11], one sees that a class in \( K_1^{op}(M) \equiv K_1(C(M)) \cong \mathcal{E}xt(C(M), \mathcal{K}) \) can be represented as the...
Fredholm module associated to a 0:th order pseudodifferential operator \( F \) over \( M \) and the representation \( \pi \) being pointwise multiplication of functions on \( L^2(M, E) \) for some vector bundle \( E \). Since \( F \) is of order 0 the commutator \([F, \pi(a)]\) is of order \(-1\) for \( a \in A \). Thus \([F, \pi(a)]\) is \( L^{n+}(L^2(M, E)) \) so \((F, \pi)\) is an \( A - L^{n+}\)-Kasparov module. Therefore \( \text{Ext}(A, L^{n+}) \to \text{Ext}(C(M), K) \) is surjective. A similar argument to the above one implies that \( \Theta^A_{L^p} : \text{Ext}(A, L^p) \to \text{Ext}(C(M), K) \) is surjective.

5 Deformations of Toeplitz extensions

To end this paper we will look at a certain part of the set \( \Theta^{-1}([P, \pi]) \) for a Toeplitz extension \((P, \pi)\). The part of \( \Theta^{-1}([P, \pi]) \) we will study are linear perturbations of the projection \( P \). We will give an example of a smooth family of this type of linear deformations which gives a family of extensions \((P, \pi)\) with \( 0 \)-summable extensions \((E, \pi)\) for any \( a \in A \). This example shows that \( \text{Ext} \) is not a homotopy invariant but carries more analytic information than similar bivariant theories.

If \((P, \pi)\) defines an \( \mathcal{J}\)-summable Toeplitz extension we say that \( x \in \text{Ext}(A, \mathcal{J}) \) is a linear deformation of \((P, \pi)\) by \( T \in PIP \) if \( x \) can be represented by an extension with a splitting of the form \( \pi_T : a \mapsto (P + T)\pi(a)(P + T) \). Observe that \( T \in PIP \subseteq I \) implies that \( \Theta(P, \pi) = \Theta(x) \). For \( a, b \in A \) we have that

\[
\tau_T(ab) - \tau_T(a)\tau_T(b) =
\]

\[
= (P + T)\pi(ab)(P + T) - (P + T)\pi(a)(P + T)^2\pi(b)(P + T) =
\]

\[
= \pi(ab)(P + T)^2(P - (P + T)^2) + [P + T, \pi(ab)](P + T)+
\]

\[
+ (P + T)\pi(a)[\pi(b), (P + T)^2](P + T) +
\]

\[
+ [\pi(ab), (P + T)](P + T)^3,
\]

so a sufficient condition for the operator \( T \) to define a linear deformation is that \( T^* - T, T^2 + 2T \in \mathcal{J} \) and \( [T, \pi(a)] \in \mathcal{J} \) for all \( a \in A \).

The main example of a linear deformation is when one considers different representatives of Toeplitz extensions via a pseudo-differential operator on a manifold. Assume that \( D \) is a self-adjoint, elliptic pseudo-differential operator on a smooth, compact manifold \( M \) without boundary and let us take \( P \) as the spectral projection onto the positive spectrum of \( D \). The operator \( P \) is a pseudo-differential operator of order 0 so \([P, a] \in \mathcal{L}^p(L^2(M))\) for any \( a \in C^\infty(M) \) and any \( p > n \). Therefore the linear mapping \( \tau(a) := PaP \) defines an \( \mathcal{L}^p\)-summable Toeplitz extension of \( C^\infty(M) \). Let us take one more self-adjoint, elliptic pseudo-differential operator \( K \) of order \( \varepsilon > n/2p \) and consider the order \(-\varepsilon\) operator

\[
T = P(K(1 + K^2)^{-1/2} - 1)P.
\]

The operator \( T \) satisfies the identity

\[
T^2 + 2T = (T + P)^2 - P = -P(1 + K^2)^{-1}P.
\]

So the operator \( T \) satisfies \( T^2 + 2T \in \mathcal{L}^p \) since we choose \( K \) to have order bigger than \( n/2p \). While \( T \) is of order \(-\varepsilon\), \([T, \pi(a)] \in \mathcal{L}^p(L^2(M))\) and \( T \) is self-adjoint since \( K \) is self-adjoint. Therefore the linear mapping \( \tau_T(a) := (P + T)a(P + T) \) defines an extension which is a linear deformation of \( \tau \).
The model case of the above setting is $K = D$. In this case the operator $P + T$ is given by $PD(1 + D^2)^{-1/2}P$. Up to a finite rank operator, we have that $P = \frac{1}{2}(D|D|^{-1} + 1)$ where the compact operator $|D|^{-1}$ can be defined as the inverse of $\sqrt{D^2}$. Since $D^*D$ and defined to be 0 on the finite-dimensional space $\ker(D^*)$. Define the order 0 pseudo-differential operator

$$\tilde{P}_D := \frac{1}{2}(D(1 + D^2)^{-1/2} + 1).$$

Since $t/|t| - t(1 + t^2)^{-1/2} = O(t^{-2})$ as $t \to \infty$ and the order of $D$ is larger than $n/2p$ we have that

$$PD(1 + D^2)^{-1/2}P - \tilde{P}_D \in L^p(L^2(M)).$$

Therefore the linear deformation of $\tau$ by $PD(1 + D^2)^{-1/2} - 1)P$ coincides in $\mathcal{E}xt(\mathcal{C}^\infty(M), L^p)$ with the extension defined by the linear mapping $a \mapsto \tilde{P}_Da\tilde{P}_D$.

In general, we can not say more of $T$ than $T \in L^{0/\varepsilon}$ since the pseudo-differential operator $K(1 + K^2)^{-1/2} - 1$ is of order $-\varepsilon$. As a consequence, if $\varepsilon < n/p$ one can not expect that the mappings $q_{L^p} \circ \tau$ and $q_{L^p} \circ \tau_T$ coincide. We will by an example show that the two mappings may even lie in different strong equivalence classes.

**Lemma 5.1.** Let $P$ be the Hardy projection on $S^1$ and assume that $T \in \mathcal{K}(H^2(S^1))$ is defined as $Tz^k := \lambda_k z^k$ for some positive sequence $(\lambda_k)_{k \in \mathbb{N}}$ converging to 0. If $a \in \mathcal{C}^\infty(S^1)$ is given by $a(z) := z$ then for any $p \geq 1$ and any unitary $U \in \mathcal{B}(H^2(S^1))$ we have that

$$\|U^*PaPU - (P + T)a(P + T)\|_{L^p(H^2(S^1))} \geq \|T\|_{L^p(H^2(S^1))}.$$

**Proof.** We will use the notation $e_k(z) := z^k$ for $k \geq 0$ and $f_k := Ue_k$. Our first observation is that

$$(P + T)a(P + T)e_k = (1 + \lambda_{k+1} + \lambda_k + \lambda_k \lambda_{k+1})e_{k+1}. \quad (7)$$

If we set $L = U^*PaPU - (P + T)a(P + T)$ we have that

$$L^*L = S_1 + S_2 - S_3 - S_4$$

where

$$S_1 := U^*Pa^*PaPU,$$

$$S_2 := (P + T)a^*(P + T)^2a(P + T),$$

$$S_3 := (P + T)a^*(P + T)U^*PaPU \quad \text{and}$$

$$S_4 := U^*Pa^*PU(P + T)a(P + T).$$

Using (7) we obtain the following equalities:

$$\langle S_1e_k, e_k \rangle = \|Pa f_k\|^2 = 1,$$

$$\langle S_2e_k, e_k \rangle = \|(P + T)a(P + T)e_k\|^2 = (1 + \lambda_{k+1} + \lambda_k + \lambda_k \lambda_{k+1})^2,$$

$$\langle S_3e_k, e_k \rangle = \langle S_4e_k, e_k \rangle = (1 + \lambda_{k+1} + \lambda_k + \lambda_k \lambda_{k+1}) \langle af_k, f_{k+1} \rangle.$$
Using these calculations the fact that \( \lambda_k, \lambda_{k+1} \geq 0 \) together with the elementary estimate \( |\langle af_k, f_{k+1}\rangle| \leq 1 \) implies that

\[
\langle L^*Le_k, e_k\rangle = 1 + (1 + \lambda_{k+1} + \lambda_k + \lambda_k\lambda_{k+1})^2 - 2(1 + \lambda_{k+1} + \lambda_k + \lambda_k\lambda_{k+1})R(af_k, f_{k+1}) = 1 - |\langle af_k, f_{k+1}\rangle|^2 + \vert [1 - \langle af_k, f_{k+1}\rangle + \lambda_{k+1} + \lambda_k + \lambda_k\lambda_{k+1}]^2 \vert \\
\geq (\lambda_{k+1} + \lambda_k + \lambda_k\lambda_{k+1})^2 \geq |\lambda_k|^2.
\]

After reordering the sequence \( \lambda_k \) into a decreasing sequence, we have that the singular values \( \mu_k(L) \in \mathbb{N} \) satisfies that \( \mu_k(L) \geq \| Le_k \| \geq |\lambda_k| \), so by Lidskii’s theorem

\[
\| U^*PaPU - (P + T)a(P + T) \|_{\mathcal{L}^p(H^2(S^1))}^p = \sum_{k\in\mathbb{N}} \mu_k(L)^p \geq \sum_{k\in\mathbb{N}} |\lambda_k|^p.
\]

**Proposition 5.2.** For any \( p > 1 \) there is a smooth family \( (T_\varepsilon)_{\varepsilon \in (1/2p, 2/p)} \subseteq \mathcal{L}^{2p}(H^2(S^1)) \) such that the linear deformations of the Toeplitz extension on the Hardy space by \( T_\varepsilon \) defines a family \( (x_\varepsilon)_{\varepsilon \in (1/2p, 2/p)} \subseteq \mathcal{E}xt(C^\infty(S^1), \mathcal{L}^p) \) where \( x_\varepsilon \neq x_{\varepsilon+1/p} \) for \( \varepsilon \in (1/2p, 1/p) \).

If we would replace the \( \mathcal{E}xt \)-invariant by for instance \( kk \)-theory, see more in [5], one would not be able to separate the elements \( x_\varepsilon \) and \( x_{\varepsilon+1/p} \) since the smooth family \( (T_\varepsilon)_{\varepsilon \in [\varepsilon, \varepsilon+1/p]} \) can be used to construct a homotopy between the classification mappings of the extensions \( x_\varepsilon \) and \( x_{\varepsilon+1/p} \).

**Proof.** Let us start by defining the smooth family \( (T_\varepsilon)_{\varepsilon \in (1/2p, 2/p)} \subseteq \mathcal{L}^{2p}(H^2(S^1)) \). We define \( T_\varepsilon \) for each \( \varepsilon \in (1/2p, 2/p) \) in the same way as in Lemma [5] from the sequence

\[
\lambda_{k,\varepsilon} := 1 - |k|^\varepsilon(1 + |k|^2\varepsilon)^{-1/2}.
\]

This choice of \( \lambda_{k,\varepsilon} \) coincides with that in the example above when \( K = |d/d\theta|^\varepsilon \). Since \( \varepsilon \mapsto \lambda_{k,\varepsilon} \) is smooth, so is \( \varepsilon \mapsto T_\varepsilon \). The sequence \( (\lambda_{k,\varepsilon})_{k \in \mathbb{Z}} \) behaves asymptotically as \( |k|^{-\varepsilon} \) so \( (\lambda_{k,\varepsilon})_{k \in \mathbb{Z}} \in \ell^{2p}(\mathbb{N}) \) since \( \varepsilon > 1/2p \).

When \( \varepsilon \in (1/p, 2/p) \) the sequence \( (\lambda_{k,\varepsilon})_{k \in \mathbb{Z}} \) is \( p \)-summable. Therefore \( (T_\varepsilon)_{\varepsilon \in (1/p, 2/p)} \subseteq \mathcal{L}^{2p}(H^2(S^1)) \) and \( \tau_{T_\varepsilon} \) is isomorphic to the Toeplitz extension on the Hardy space for \( \varepsilon \in (1/p, 2/p) \). However, when \( \varepsilon < 1/p \) we have that \( (\lambda_{k,\varepsilon})_{k \in \mathbb{Z}} \not\in \ell^p(\mathbb{N}) \). The norm estimate of the differences of the Toeplitz extension on the Hardy space and a deformation by \( T_\varepsilon \) in Lemma [5] implies that for any unitary \( U \in \mathcal{B}(H^2(S^1)) \)

\[
U^*PaPU - (P + T_\varepsilon)a(P + T_\varepsilon) \notin \mathcal{L}^p(H^2(S^1)).
\]

Therefore \( \tau \) is not strongly equivalent to \( \tau_{T_\varepsilon} \) for \( \varepsilon \in (1/2p, 1/p) \) and \( x_\varepsilon \neq x_{\varepsilon+1/p} \) for \( \varepsilon \in (1/2p, 1/p) \).
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