SOME PROPERTIES OF GENERALIZED HYPERGEOMETRIC
APPELL POLYNOMIALS

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Abstract. In this paper, we present a new real-valued Appell-type polynomial family $A_n^{(k)}(m, x)$, $n, m \in \mathbb{N}_0$, $k \in \mathbb{N}$, every member of which is expressed by mean of the generalized hypergeometric function

$$\sum_{k=0}^{\infty} \frac{a_1^{(k)} a_2^{(k)} \cdots a_p^{(k)}}{b_1^{(k)} b_2^{(k)} \cdots b_q^{(k)}} \frac{x^k}{k!},$$

where $x^{(n)}$ denotes the Pochhammer symbol (rising factorial) defined by $x^{(n)} = x(x+1)(x+n-1)$ for $n \geq 1$ and $x^{(0)} = 1$, as follows

$$A_n^{(k)}(m, x) = x^n k + p F_q \left[ \frac{a_1, a_2, \ldots, a_p}{b_1, b_2, \ldots, b_q} \right]$$

and is the Appell-type polynomial family simultaneously.

The generating exponential function of this type of polynomials is firstly discovered and the proof that they are of Appell-type ones is given. We present the differential operator formal power series representation as well as an explicit formula over the standard basis, and establish a new identity for the generalized hypergeometric function. Besides, we derive the addition, the multiplication and some other formulas for this polynomial family.

1. Introduction

In [16], P. Appell presented polynomial sequence \{A_n(x)\}, $n = 0, 1, 2, \ldots$, such that $\deg A_n(x) = n$ and satisfying the identity

$$A'_n(x) = n A_{n-1}(x),$$

where $A_0(x) \neq 0$, which is called the Appell polynomials sequence.

An arbitrary Appell polynomial sequence possesses an exponential generating function

$$A(t) e^{xt} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!},$$

here $A(t)$ is a formal power series

$$A(t) = a_0 + a_1 t + a_2 \frac{t^2}{2!} + \cdots + a_n \frac{t^n}{n!} + \cdots , a_0 \neq 0.$$

The Appell-type polynomials $A_n(x)$ are expressed in the terms of \{a_n\} as follows

$$A_n(x) = \sum_{i=0}^{n} \binom{n}{i} a_{n-i} x^i.$$

The simplest example of Appell-type polynomials is the monomial sequence \{x^n\}, $n = 0, 1, \ldots$, other examples are the Bernoulli, the Euler polynomials and the Hermite polynomials. For more examples one can consult [1, 5].
The Appell-type polynomials perform a large variety of features and are widely spread at the different areas of mathematics, namely, at special functions, general algebra, combinatorics and number theory. Recently, the Appell-type polynomials are of big interest. The modern researches give the alternative definitions of Appell-type polynomials and apply new approaches based, for instance, on the determinant method or in Pascal matrix method (see, e.g., [8], [15]). Consequently, many new properties of those polynomials are described and a great deal of identities involving Appell-type polynomials are obtained (see [7, 10, 9]).

Let us recall that the generalized hypergeometric function is defined as follows

$$
{_{p}F_{q}} \left[ \begin{array}{c}
a_1, a_2, \ldots, a_p \\
b_1, b_2, \ldots, b_q
\end{array} \mid z \right] = \sum_{k=0}^{\infty} \frac{a_1^{(k)} \cdots a_p^{(k)}}{b_1^{(k)} \cdots b_q^{(k)}} \frac{z^k}{k!},
$$

where $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q$ are complex parameters and none of $b_i$ equals to a non-positive integer or zero, $x^{(n)}$ denotes the Pochhammer symbol (or rising factorial) defined by $x^{(n)} = x(x+1)(x+2) \cdots (x+n-1)$ for $n \geq 1$ and $x^{(0)} = 1$. Further on, we denote the generalized hypergeometric function by $_pF_q$ for brevity.

We note that the Gauss hypergeometric function $_2F_1$ and the Kummer hypergeometric function $_1F_1$ are the partial cases of (2). Apart from the Appell-type polynomials, there exist some polynomial families which admit representation via the partial cases of the generalized hypergeometric function, i.e., the Jacobi polynomials $P^{(\alpha,\beta)}(z)$

$$
P^{(\alpha,\beta)}_n(z) = \frac{(\alpha + 1)^{(n)}}{n!} {_{2}F_{1}} \left[ \begin{array}{c}
-n, n+\alpha+\beta+1 \\
\alpha+1
\end{array} \mid 1 - z \right].
$$

At the same time, there exists a number of the Appell-type polynomial families which also admit the representation via partial cases of the Gauss hypergeometric function. It is known (II) that the Laguerre polynomials $L_n(x)$ are presented as follows

$$
L_n(x) = {_{1}F_{1}} \left[ \begin{array}{c}
-n \\
1
\end{array} \mid x \right].
$$

Remarkably, the Hermite polynomials $H_n(x)$ are simply expressed in the terms of those functions (II)

$$
H_n(x) = x^n {_{2}F_{0}} \left[ \begin{array}{c}
n \\
\frac{n-1}{2} - \frac{1}{2}
\end{array} \left| - \frac{2}{x^2} \right] = e^{x^2-\frac{1}{2}x^2},
\right. 
G(x,t) = e^{xt-\frac{1}{2}t^2}
$$

The natural way of generalisation of the Hermite polynomials is to expand the array of ratios for another denominators, it was made in [13], the authors obtained the Gould-Hopper polynomials $g_m^n(x,h)$, with $G(x,t) = e^{xt+ht^m}$, which could be also expressed in the terms of the generalized hypergeometric function as follows

$$
g_m^n(x,h) = x^m {_{m}F_{0}} \left[ \begin{array}{c}
n - m - 1 \ldots - n - m + 1 \\
m \ldots m
\end{array} \mid \frac{(-1)^m h^m}{x^m} \right].
$$

The aim of this paper is to find a polynomial family, which would be the Appell-type one and admit the generalized hypergeometric function representation simultaneously. Still, there exist the polynomial families which have the needed representation, e.g., the generalized hypergeometric polynomials $f_n(a_i;b_j;x)$, studied at [12], such that
Some properties of generalized hypergeometric Appell polynomials

\[ f_n(a_i; b_j; x) = \binom{-n, n+1, a_1, a_2, \ldots, a_p}{\frac{1}{2}, b_1, b_2, \ldots, b_q} x^n, \quad n \in \mathbb{N}_0, \]

and the incomplete hypergeometric polynomials associated with generalized incomplete hypergeometric function, studied at [14], but they both are not the Appel-type polynomials.

The difference between all mentioned classes of polynomials, depending if they are of Appell-type or not and if they possess the generalized hypergeometric function representation or do not, has motivated the title of the paper.

Therefore, let us give the following

**Definition 1.** Let \( \Delta(k, -n) \) denote the array of \( k \) ratios

\[ \frac{n}{k}, \frac{n-1}{k}, \ldots, \frac{n-k+1}{k}, \quad n \in \mathbb{N}_0, k \in \mathbb{N}. \]

Then we call the polynomial family

\[ A_n^{(k)}(m, x) = x^n \binom{k}{m} F_q \left[ \frac{a_1, a_2, \ldots, a_p, \Delta(k, -n)}{b_1, b_2, \ldots, b_q} \right], \quad n, m \in \mathbb{N}_0, k \in \mathbb{N} \]

where

\[ k+pF_q = \sum_{i=0}^{[n/k]} \prod_{r=1}^{p} (a_r)^{(i)} \prod_{s=1}^{q} (b_s)^{(i)} \prod_{j=1}^{k} \left( -\frac{n-j+1}{k} \right)^{(i)} \frac{m^i}{i! x^k}, \]

the generalized hypergeometric Appell polynomials.

We note that if \( p = 0, q = 0, k := m, m := (-1)^k h k^k \) the generalized hypergeometric Appell polynomials \( A_n^{(k)}(m, x) \) become the Gould-Hopper polynomials \( g_m^n(x, h) \) and if \( p = 0, q = 0, m = -2, k = 2 \) they become the Hermite polynomials \( H_n(x) \) mentioned above.

The main result of this article is the following basic statement.

**Theorem 1.** The generalized hypergeometric Appell polynomials \( A_n^{(k)}(m, x) \) defined by definition [14] are the Appell type ones.

2. Basic definitions and notation

In addition to the rising factorial we use the falling factorial \( (x)_n = x(x-1)(x-2) \cdots (x-n+1) \) for \( n > 0 \) and \( (x)_0 = 1 \). In these notation, the following relations holds (see [12])

\[ (x)_n = (-1)^n (-x)^{(n)}, \]

and the Gauss product of indexes formula (see [6]) will be written as follows

\[ (-\lambda)^{(mn)} = m^m \prod_{j=1}^{m} \left( -\frac{\lambda - j + 1}{m} \right)^{(n)}, \quad n \in \mathbb{N}_0. \]

We note that in the case when either \( a \) or \( b \) is a non-positive integer, the generalized hypergeometric function reduces to a polynomial:
\[ pFq \left[ \begin{array}{c} -m, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \right| z \right] = \sum_{n=0}^{\infty} (-1)^n \frac{(m)_n}{n!} \prod_{j=2}^{p} a_j^{(n)} \prod_{s=1}^{q} b_s^{(n)} z^n. \]

As far as we deal with the differentiation, the differentiation formula with respect to \( z \) would be useful:

\[ \frac{d}{dx} pFq \left[ \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \right| z \right] = \prod_{j=1}^{p} a_j \prod_{s=1}^{q} b_s \cdot pFq \left[ \begin{array}{c} a_1+1, a_2+1, \ldots, a_p+1 \\ b_1+1, b_2+1, \ldots, b_q+1 \end{array} \right| z \right]. \]

3. Basic properties of the generalized hypergeometric Appell polynomials

3.1. Being of Appell type. Proof of theorem\^[1] To prove the generalized hypergeometric Appell polynomials \( A_n^{(k)}(m, x) \) are the Appell-type polynomials, it is sufficient to show that there exists a formal power series \( A(t) \) such that the following relation holds

\[ A(t)e^{xt} = \sum_{n=0}^{\infty} A_n^{(k)}(m, x) \frac{t^n}{n!}. \]

We set \( (\gamma)^i = \left( \prod_{r=1}^{p} (a_r)^{(i)} \right) / \left( \prod_{s=1}^{q} (b_s)^{(i)} \right) \). Then from definition \( \[2\] \) and relations \( \[5\] \) and \( \[3\] \) it follows that

\[ A_n^{(k)}(m, x) = x^{n+p+k} F_q \left[ \begin{array}{c} a_1, a_2, \ldots, a_p, \Delta(k, -n) \\ b_1, b_2, \ldots, b_q \end{array} \right| \frac{m}{x^k} \right] = x^{n} \sum_{i=0}^{[n/k]} (\gamma)^i (-1)^{ki} (n)_{ki} \frac{m^i}{i!x^{ki}}. \]

We choose

\[ A(t) = pFq \left[ \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \right| (-1)^k \frac{t^k}{m^k} \right]. \]

Using the expansion of \( e^{xt} \) into the power series and changing the product of the series by the double series, we transform the generating function as follows

\[ A(t)e^{xt} = \left( \sum_{n=0}^{\infty} (\gamma)^n \frac{(-1)^k m^k x^n}{n!} \right) \left( \sum_{s=0}^{\infty} \frac{(xt)^s}{s!} \right) = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} (\gamma)^n (-1)^k n^m x^s t^{s+k} \frac{k^{kn}}{k^{kn} s!n!}. \]

Using the infinite sums interchange formula \([2]\)

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p-q,q} \]

and taking into account the multiplicity of \( i \), we have

\[ \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} a_{s,n} = \sum_{n=0}^{\infty} \sum_{i=0}^{[n/k]} a_{n-k,i,i}. \]
then

\[
\sum_{n=0}^{\infty} \left( \sum_{s=0}^{\infty} \left( \gamma \frac{n^{m}(\frac{1}{2})^{kn}}{k^{kn}s!n!} \right) \right) = \sum_{n=0}^{\infty} \left( \sum_{i=0}^{\infty} \frac{m^{i}(\frac{1}{2})^{ki}}{k^{ki}} \frac{x^{n-ki}t^{n}}{(n-ki)!i!} \right) = \sum_{n=0}^{\infty} x^{n} \left( \sum_{i=0}^{\infty} \left( \frac{m^{i}(\frac{1}{2})^{ki}(\gamma)_{i}x^{n-ki}t^{n}}{i!} \right) \right) \frac{t^{n}}{n!}.
\]

The inner sum is precisely equal to the generalized hypergeometric function in the form of (3) and, therefore, the relation (4) holds. This means that the generating function admit the needed representation (3).

It should be noted that there is another way to prove theorem 1, which is to replace \(xt\) by \(t\) and \(m/x^{k}\) by \(x\) in problem 26, p.173 [4].

As a consequence of theorem 1, we derive a new identity for the generalized hypergeometric function.

**Corollary 1.** The following identity holds

\[
n_{x^{n-k-1}}F_{q} \left[ \begin{array}{c} a_{1}, a_{2}, \ldots, a_{p}, \Delta(k, -n + 1) \\ b_{1}, b_{2}, \ldots, b_{q} \end{array} \right] = n_{x^{n-1}}F_{q} \left[ \begin{array}{c} a_{1}, a_{2}, \ldots, a_{p}, \Delta(k, -n) \\ b_{1}, b_{2}, \ldots, b_{q} \end{array} \right] \]

\[
-km\gamma_{1}\Delta(k, -n)x_{n-k-1}F_{q} \left[ \begin{array}{c} a_{1} + 1, a_{2} + 1, \ldots, a_{p} + 1, \Delta(k, -n + k) \\ b_{1} + 1, b_{2} + 1, \ldots, b_{q} + 1 \end{array} \right],
\]

where \(\Delta(k, -n)\) denotes the product

\[
\left( \frac{-n}{k} \right) \left( \frac{-n-1}{k} \right) \ldots \left( \frac{-n-k+1}{k} \right).
\]

**Proof.** The generalized hypergeometric Appell polynomials are the Appell-type ones, hence, the identity

\[
\frac{d}{dx} \left\{ A_{n}^{(k)}(m, x) \right\} = nA_{n-1}^{(k)}(m, x)
\]

fulfils.

Representing the polynomials \(A_{n}^{(k)}(m, x)\) in the terms of the generalized hypergeometric function according to the definition 1 we immediately obtain the left side of the corollary equality.

To obtain its right side we differentiate the hypergeometric representation of the polynomials \(A_{n}^{(k)}(m, x)\) under the Leibnitz rule:

\[
\frac{d}{dx} \left\{ x_{n}F_{q} \left[ \begin{array}{c} a_{1}, a_{2}, \ldots, a_{p}, \Delta(k, -n) \\ b_{1}, b_{2}, \ldots, b_{q} \end{array} \right] \right\} = n_{x^{n-1}}F_{q} \left[ \begin{array}{c} a_{1}, a_{2}, \ldots, a_{p}, \Delta(k, -n) \\ b_{1}, b_{2}, \ldots, b_{q} \end{array} \right] + x_{n}F_{q} \left[ \begin{array}{c} a_{1}, a_{2}, \ldots, a_{p}, \Delta(k, -n) \\ b_{1}, b_{2}, \ldots, b_{q} \end{array} \right].
\]
Performing the derivative of the hypergeometric function, we obtain
\[
x^n \frac{d}{dx} \left\{ p+k \, F_q \left[ \begin{array}{c} a_1, a_2, \ldots, a_p, \Delta(k, -n) \\ b_1, b_2, \ldots, b_q \end{array} \right] \middle| \frac{m}{x^k} \right\} = x^n \left( \frac{(-1)^k(n)_k a_1 \cdot a_p m(-k)}{k^k b_1 \cdots b_q 1!x^{k+1}} ight) + \left( \frac{(-1)^{2k}(n)_{2k} a_1(a_1 + 1) \cdots a_p(a_p + 1) m^2(-2k)}{k^{2k} b_1(b_1 + 1) \cdots b_q(b_q + 1) 2!x^{2k+1}} \right)
\]
\[
+ \left( \frac{(-1)^{3k}(n)_{3k} a_1(a_1 + 1)(a_1 + 2) \cdots a_p(a_p + 1)(a_p + 2) m^3(-3k)}{k^{3k} b_1(b_1 + 1)(b_1 + 2) \cdots b_q(b_q + 1) 3!x^{3k+1}} + \cdots \right)
\]
\]
\[
= x^{n-k-1} m^k \frac{(-1)^{k+1}(n)_k a_1 \cdots a_p}{k^k b_1 \cdots b_q} \left( 1 + \frac{(-1)^{k}(n-k)_k (a_1 + 1) \cdots (a_p + 1) m \cdot 2}{k^{k} (b_1 + 1) \cdots (b_q + 1) 2!x^k} \right)
\]
\[
+ \left( \frac{(-1)^{2k}(n-k)_{2k} (a_1 + 1)(a_1 + 2) \cdots (a_p + 1)(a_p + 2) m^2 \cdot 3}{k^{2k} (b_1 + 1)(b_1 + 2) \cdots (b_q + 1) 3!x^{2k}} + \cdots \right)
\]
\]
\[
= -km\gamma_1 \Delta_1(k, -n) x^{n-k-1} p+k \, F_q \left[ \begin{array}{c} a_1 + 1, a_2 + 1, \ldots, a_p + 1, \Delta(k, -n+k) \\ b_1 + 1, b_2 + 1, \ldots, b_q + 1 \end{array} \right] \middle| \frac{m}{x^k} \right],
\]
that ends the proof. \(\square\)

Since an arbitrary polynomial on one variable \(P_n(x) \in \mathbb{C}[x]\) always permits the formal series representation
\[
P_n(x) = \sum_{i=0}^{n} \alpha_i x^i,
\]
then we are interested in finding those representation for the generalized hypergeometric Appell polynomials.

**Corollary 2.** The generalized hypergeometric Appell polynomials \(A_n^{(k)}(m, x)\) possess

(i) the standard basis \(\{x^i\}_{i=0}^{n}\) representation

\[
A_n^{(k)}(m, x) = \sum_{i=0}^{[n/k]} \frac{n!(-1)^{ki} \gamma^i m^i}{i!k^i(n-k)!} x^{n-ki},
\]

(ii) the differential operator formal power series representation

\[
A_n^{(k)}(m, x) = \left( \sum_{i=0}^{[n/k]} \frac{(-1)^{ki} \gamma^i m^i}{i!k^i} D^k \right) x^n.
\]

**Proof.** (i) We use an approach from [9] which is based on the idea of the connection problem.

Given the two polynomial families of Appell type \(\{P_n(x)\}\) and \(\{Q_n(x)\}\) with generating functions \(A_1(t)\) and \(A_2(t)\) respectively, the solution of its connection problem could be written as follows:
\[
Q_n(x) = \sum_{m=0}^{n} \frac{n!}{m!} \alpha_{n-m} P_m(x),
\]
where

$$\frac{A_2(t)}{A_1(t)} = \sum_{k=0}^{\infty} \alpha_k t^k.$$  

We are searching for the unknown coefficients $\alpha_k$ to decompose the polynomials

$$Q_n(x) = x^n, A_2(t) = 1$$

upon the polynomials $A_n^{(k)}(m, x)$ defined by (3) with generating function $A_1(t)$ defined by (7). Deriving the ratio of generating functions $A_2(t)$ and $A_1(t)$ we have

$$\frac{A_2(t)}{A_1(t)} = \sum_{r=0}^{\infty} \frac{(-1)^r m^r}{k^r r!} (\gamma)^r t^r = \sum_{r=0}^{\infty} \alpha_r t^r,$$

and, constructing the corresponding coefficients $\alpha_{n-m}$, we obtain the needed representation.

(ii) An arbitrary Appell-type polynomial $P_n(x)$ could be also written in the symmetric form

$$P_n(x) = \sum_{i=0}^{n} \binom{n}{i} c_i x^{n-i}.$$  

According to [5], the latter expression is equivalent to the following differential operator representation

$$P_n(x) = \left( \sum_{i=0}^{n} \frac{c_i}{i!} D^i \right) x^n,$$

where $D := d/dx$ is an ordinary differentiation with respect to $x$, consequently,

$$A_n^{(k)}(m, x) = \sum_{i=0}^{[n/k]} \binom{n}{ki} c_i x^{n-ki} = \sum_{i=0}^{[n/k]} \binom{n}{ki} (-1)^{ki} (\gamma)^i m^i (ki)! \frac{i! k^i}{(n-ki)!} x^{n-ki},$$

we deduce a differential operator formal power series representation of the generalized hypergeometric Appell polynomials of the form of (9).

□

Remark. Comparing the power series (1) and operational formula (9) of the generalized hypergeometric Appell polynomials to the corresponding ones of the Gould-Hopper polynomials

$$A(t) = e^{ht}, \quad g_{n}^{m}(x, h) = (e^{hD}) x^n,$$

it is easy to see that the latter have more compact forms.

Symmetry. Substituting the negative value of argument into the formula [8]

$$A_n^{(k)}(m, -x) = \sum_{i=0}^{[n/k]} (-1)^{n-ki} n! (-1)^{ki} (\gamma)^i m^i \frac{i! k^i}{(n-ki)!} x^{n-ki},$$

we conclude that, in the case of even $k$, the generalized hypergeometric Appell polynomials are the even ones themselves while $n$ is an even number, and they are the odd ones themselves while $n$ is an odd number:

$$A_{2n}^{(2k)}(m, -x) = A_{2n}^{(2k)}(m, x), \quad A_{2n+1}^{(2k)}(m, -x) = -A_{2n+1}^{(2k)}(m, x).$$

Otherwise, for any odd $k$ in the case of odd $n$, the summands standing on the even places change their signs into the opposite ones, and the same do the summands standing on the odd places in the case of even $n.$
3.2. Addition and Multiplication Formulas and Other Properties. Here we shall prove the following result.

**Theorem 2.** The following formulas hold for the generalized hypergeometric Appell polynomials

(i) addition formula

\[ A_n^{(k)}(m, x + y) = \sum_{i=0}^{n} \binom{n}{i} y^{n-i} A_i^{(k)}(m, x) = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} A_i^{(k)}(m, y), \]

(ii) multiplication formula

\[ A_n^{(k)}(m, Mx) = \sum_{i=0}^{n} \binom{n}{i} (M - 1)^{n-i} x^{n-i} A_i^{(k)}(m, x), \]

(iii) indexes interchange formula

\[ \sum_{i=0}^{n} \binom{n}{i} A_i^{(k_1)}(m, x) A_{n-i}^{(k_2)}(m, y) = \sum_{i=0}^{n} \binom{n}{i} A_i^{(k_2)}(m, x) A_{n-i}^{(k_1)}(m, y) \]

(iv) convolution type identity

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} A_i^{(k)}(m, x) A_{n-i}^{(k)}(m, x) \]

\[ = \left(\frac{-1}{k^n}\right) \frac{m^{n/k} n!}{k^n} \sum_{i=0}^{[n/k]} a_1^{(i)} \ldots a_p^{(i)} \frac{\Delta(k, -n)}{b_1^{(i)} \ldots b_q^{(i)}} A_{n/k-i}^{(k)} b_1^{(n/k-i)} \ldots b_q^{(n/k-i)}. \]

**Proof.** The addition and the multiplication formulas hold for all Appell-type polynomial families ([5]), consequently, they hold for the generalized hypergeometric Appell polynomials as well. The indexes interchange formulas could be obtained applying methods proposed in [9] and the convolution type identity is obtained by the simple direct calculations at \( x = 0 \). \( \square \)

It is worth stressing, that the polynomials \( A_n^{(k)}(m, Mx) \) lose the property of being of Appell-type. Moreover, the generalized hypergeometric polynomials over the polynomials could be defined in the same manner as the generalized hypergeometric Appell polynomials:

\[ A_n^{(k)}(m, f(x)) = (f(x))^{n+k} \mathcal{F}_q \left[ \begin{array}{c} a_1, a_2, \ldots, a_p, \Delta(k, -n) \\ b_1, b_2, \ldots, b_q \end{array} \right] \frac{m}{(f(x))^k}, \]

where

\[ f(x) = a_0 x^p + a_1 x^{p-1} + \cdots + a_p, \quad a_0 \neq 0, \]

which submit the following differentiation rule

\[ \frac{d}{dx} A_n^{(k)}(m, f(x)) = nf'(x)A_{n-1}^{(k)}(m, f(x)). \]

In particular, in the case when \( p = a_0 = 1 \), we obtain the Appell differentiation.
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