THE CANONICAL SYMMETRY AND
HAMILTONIAN FORMALISM.
I. CONSERVATION LAWS

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Abstract
The properties of the canonical symmetry of the nonlinear Schrödinger equation are investigated. The densities of the local conservation laws for this system are shown to change under the action of the canonical symmetry by total space derivatives.

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1 Introduction

The canonical symmetry of an integrable system is a special discrete symmetry transformation having a number of remarkable properties [1]–[4]. In particular, as we have shown in Ref. [4], this symmetry, considered as a transformation of the phase space of the system, was a canonical transformation. In the same paper, considering the case of the nonlinear Schrödinger equation, we have established that the densities of the lowest conservation laws change under the action of the canonical symmetry by total space derivatives. Here we show that in this case all the conservation laws possess this property. From this it follows, in particular, that the whole hierarchy of the nonlinear Schrödinger equations is invariant with respect to the action of the canonical symmetry.

Since, from our point of view, discrete transformations of the nonlinear differential equations did not attract early the proper attention, we found it useful to give a short review of the basic facts on such transformations and their properties. This is done in sections 2 and 3. Trying to make the paper self-contained but not too long, we decided to give here the proofs only for those facts that could not be found in the standard reference on the symmetry properties of the nonlinear differential equations [5].

In Ref. [4] to show that the canonical symmetry is a canonical transformation we used the method of generating functions. At the beginning of section 4 for the case of the nonlinear Schrödinger equation we give another proof of this fact based on the representation of the Poisson bracket with the help of the corresponding matrix differential operator [5]. The rest of section 4 is devoted to the proof of our main result concerning the behaviour of the densities of the conservation laws under the canonical symmetry.

The summation over repeated indexes is implied.

2 Hamiltonian Approach to Evolution Equations

Here we recall the main facts on the Hamiltonian formulation of the nonlinear evolution equations [5]. We restrict ourselves to the case of two independent variables $t$, $x$ and $A$ dependent variables $u^a$, $a = 1, \ldots, A$. An evolution equation is a system of equations having the form

$$u_t^a = K^a(x, u, u_1^a, \ldots, u_K^a),$$

(2.1)

where for any $k$ $(k)^a$ denote a set formed by the $k$–th derivatives $(k)^a$ of the dependent variables $u^a$ over $x$. It is convenient to put $(0)^a = u^a$.

Consider the space $\mathcal{A}$ of the functions of the variables $x$, $u^a$, and the derivatives of $u^a$ over $x$ up to some finite order. For a general function of such type we use the following notation

$$f[u] = f(x, u, u_1^a, \ldots, u_K^a).$$

(2.2)

The operator of the total derivative over $x$ is defined as

$$D \equiv \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} \sum_{(k)^a} \frac{\partial}{\partial (k)^a}.$$

(2.3)
Actually, when the operator $D$ acts on an element of $A$ we have only a finite number of members in series (2.3). For any two functions $f$ and $g$ we have

$$D^n(fg) = \sum_{k=0}^{m} \binom{m}{k} D^{m-k}(f)D^k(g).$$  

(2.4)

The operator

$$a[u] = \sum_{k=0}^{N(a)} a_k[u]D^k,$$

(2.5)

where $a_k \in A$, is called a differential operator of the order $N(a)\text{[2]}$. Using Eq. (2.4), we can in an obvious way define the multiplication of differential operators. It can be shown that the multiplication of differential operators is an associative operation.

The transposed (adjoint) operator of the differential operator $a$, given by Eq. (2.5), is a differential operator $a^T$, defined as

$$a^T \equiv \sum_{k=0}^{N(a)} (-1)^k D^k a_k = \sum_{k=0}^{N(a)} \sum_{m=k}^{N(a)} (-1)^m \binom{m}{n} D^{m-k}(a_m)D^k.$$  

(2.6)

Here and below we use the following convention: if the operator $D$, or some degree of the operator $D$, precedes some expression surrounded by brackets, it acts only on the expression in the brackets.

A differential operator $a$ is called symmetric (self-adjoint) if $a^T = a$, and it is called skew-symmetric (skew-adjoint) if $a^T = -a$.

A matrix with the matrix elements being differential operators is called a matrix differential operator. Any matrix differential operator $A$ can be written as

$$A = \sum_{k=0}^{N(A)} A_k D^k,$$

(2.7)

where $A_k$ are matrices with the matrix elements being elements of $A$. The transposed operator $A^T$ of the matrix differential operator $A$, given by Eq. (2.7), is defined as

$$A^T = \sum_{k=0}^{N(A)} (-1)^k D^k A_k^T,$$

(2.8)

where $A_k^T$ is the transposed matrix of the matrix $A_k$.

Introduce on $\mathcal{A}$ the operators of the variational derivatives over $u^a$, defined as

$$\frac{\delta}{\delta u^a} \equiv \sum_{k=0}^{\infty} (-1)^k D^k \frac{\partial}{\partial (k) u^a}.$$  

(2.9)

It can be shown that

$$\frac{\delta f}{\delta u^a} = 0, \quad a = 1, \ldots, A,$$

(2.10)

\[\text{The operators, involving negative degrees of } D, \text{ can also be defined, but we will not use such operators in this paper.}\]
if and only if there exists a function \( g \), such that
\[
f = Dg. \tag{2.11}
\]

Hence, defining
\[
\text{Ker } \frac{\delta}{\delta u} \equiv \{ f \in A \mid \delta f / \delta u^a = 0, \ a = 1, \ldots, A \}, \tag{2.12}
\]
we can write
\[
\text{Ker } \frac{\delta}{\delta u} = DA. \tag{2.13}
\]

Two functions \( f \) and \( g \) from \( A \) are said to be equivalent if there exists a function \( h \) such that
\[
f = g + Dh. \tag{2.14}
\]

It is easy to prove that we have defined an equivalence relation. The corresponding equivalence classes are called functionals. We denote the equivalence class containing a function \( f \) by \( F \). In accordance with the established tradition we write in this case \( F = \int f \, dx \).

The space of all functionals is denoted by \( \mathcal{F} \). If a functional \( F \) is given we denote by \( f \) an arbitrary representative from the corresponding equivalence class and call \( f \) the density of the functional \( F \).

A bilinear map \( \{ \cdot, \cdot \} : \mathcal{F} \times \mathcal{F} \to \mathcal{F} \), is said to be a Poisson bracket on \( \mathcal{F} \) if it satisfies the relations
\[
\{ F, G \} = -\{ G, F \}, \tag{2.15}
\]
\[
\{ F, \{ G, H \} \} + \{ G, \{ H, F \} \} + \{ H, \{ F, G \} \} = 0 \tag{2.16}
\]
for any functionals \( F, G \) and \( H \). Equality (2.16) is called the Jacobi identity.

A matrix differential operator \( J \) with the matrix elements \( J_{ab}, a, b = 1, \ldots, A \), is said to be a Hamiltonian operator if the equality
\[
\{ F, G \} = \int \frac{\delta f}{\delta u^a} J_{ab} \frac{\delta g}{\delta u^b} \, dx \tag{2.17}
\]
defines a Poisson bracket on \( \mathcal{F} \). It is clear that the right–hand side of equality (2.17) does not depend of the choice of \( f \) and \( g \). The Poisson bracket (2.16) can be shown to satisfy Eq. (2.15) if and only if the operator \( J \) is skew–symmetric. The Jacobi identity imposes more complicated conditions on the operator \( J \). We do not discuss them here, and only note that if \( J \) is a constant matrix then the Jacobi identity is valid.

We say that evolution equation (2.1) can be written in the Hamiltonian form if there exists a Hamiltonian operator \( J \) and a function \( h \in A \) such that
\[
K^a = J_{ab} \frac{\delta h}{\delta u^b}. \tag{2.18}
\]

If this is the case, we write the equations of motion in the form
\[
u^a_i = J_{ab} [u] \frac{\delta h[u]}{\delta u^b} \tag{2.19}
\]
and call equations (2.19) the Hamilton equations. Actually Hamilton equations (2.19) are defined not by the function \( h \) but by the corresponding functional \( H \), called the Hamiltonian of the system.

A conservation law of evolution equation (2.1) is a relation of the form
\[
\frac{\partial P}{\partial t} + \{P, H\} = 0, 
\]
that should be valid in virtue of equation (2.1). In Eq. (2.20) \( p \) and \( f \) are elements of \( \mathcal{A} \), which are called the density and the flow of the conservation law, respectively. If we have conservation law (2.20) we say that the functional \( P \) is a conserved quantity, or an integral of motion. For the case of Hamilton equations (2.19), the functional \( P \), satisfying the equality
\[
\frac{\partial P}{\partial t} + \{P, H\} = 0, 
\]
is a conserved quantity.

3 Differential Transformations

Consider a transformation of the space described by \( x \), dependent variables \( u^a \), and their \( x \)-derivatives, that is determined by the relations
\[
\tilde{u}^a = \varphi^a[u] = \varphi^a(x, u, \dot{u}, \ldots, \ddot{u}). 
\]
We call such a transformation a differential transformation. If there exists a transformation
\[
u^a = \tilde{\varphi}^a[\tilde{u}],
\]
such that
\[
\tilde{\varphi}^a[\varphi[u]] = u^a,
\]
then we have an invertible transformation. The transformation law for the variables \( \dot{u}^a \) is defined by
\[
\dot{\tilde{u}}^a = (\dot{\varphi})^a[u] \equiv D^m \tilde{\varphi}^a[u].
\]

Define the operator \( \varphi^* \) that acts on a function \( f[u] = f(x, u, \dot{u}, \ldots, \ddot{u}) \) according to the rule
\[
\varphi^* f[u] \equiv f[\varphi[u]] = f(x, \varphi[u], \dot{\varphi}[u], \ldots, \ddot{\varphi}[u]).
\]
It is not difficult to get convinced that for any \( f \in \mathcal{A} \) we have
\[
D \varphi^* f = \varphi^* D f.
\]

Let \( \chi \in \mathcal{A}^I \), i. e. \( \chi \) is a set of functions \( \chi^i \in \mathcal{A}, \) \( i = 1, \ldots, I \). The Fréchet derivative of \( \chi \) is the \( I \times A \)-matrix differential operator \( \chi' \), defined as
\[
\chi^i_a[u] v^a[u] \equiv \left. \frac{d}{d\epsilon} \chi^i[u + \epsilon v[u]] \right|_{\epsilon=0}.
\]

\(^3\)We allow \( P \) to depend explicitly on \( t \).
for any \( v^a \in \mathcal{A} \), \( a = 1, \ldots, A \). From this definition we get

\[
\chi^{\eta}_a = \sum_{k=0}^{\infty} \frac{\partial \chi^i}{\partial \left( \frac{\partial}{\partial u^a} \right)^k} D^k.
\]

(3.8)

In particular, for \( f \in \mathcal{A} \) we have

\[
f'_a = \sum_{k=0}^{\infty} \frac{\partial f}{\partial \left( \frac{\partial}{\partial u^a} \right)^k} D^k.
\]

(3.9)

It follows from this equality that

\[
\frac{\delta f}{\delta u^a} = f'^T_a (1).
\]

(3.10)

Let \( \chi \in \mathcal{A}^I \), denote the set of functions \( \phi^* \chi^i \), \( i = 1, \ldots, I \), by \( \phi^* \chi \). Let us show that

\[
(\phi^* \chi)'[u] = \chi'[\phi[u]] \phi'[u].
\]

(3.11)

Using Eq. (3.8) we get

\[
(\phi^* \chi)^i_a = \sum_{n,m=0}^{\infty} \phi^* \left( \frac{\partial \chi^i}{\partial \left( \frac{\partial}{\partial u^b} \right)^m} \right) \frac{\partial \phi^b}{\partial \left( \frac{\partial}{\partial u^a} \right)^n} D^n.
\]

(3.12)

It is not difficult to show that

\[
\frac{\partial}{\partial u^a} D^m = \sum_{k=0}^{m} \binom{m}{k} D^{m-k} \frac{\partial}{\partial \left( \frac{\partial}{\partial u^a} \right)^{n-k}}, \quad n > m,
\]

(3.13)

\[
\frac{\partial}{\partial \left( \frac{\partial}{\partial u^a} \right)^n} D^m = \sum_{k=0}^{n} \binom{m}{k} D^{m-k} \frac{\partial}{\partial \left( \frac{\partial}{\partial u^a} \right)^{n-k}}, \quad n \leq m.
\]

(3.14)

From this equalities and Eq. (3.4) it follows that

\[
(\phi^* \chi)^i_a = \sum_{m=0}^{\infty} \phi^* \left( \frac{\partial \chi^i}{\partial \left( \frac{\partial}{\partial u^b} \right)^m} \right) \sum_{n=0}^{m} \sum_{n=m+1}^{\infty} \sum_{k=0}^{m} \binom{m}{k} D^{m-k} \frac{\partial \phi^b}{\partial \left( \frac{\partial}{\partial u^a} \right)^n} D^n.
\]

(3.15)

Now using the identity

\[
\sum_{n=0}^{m} \sum_{k=0}^{n} = \sum_{k=0}^{m} \sum_{n=k}^{m},
\]

(3.16)

we can reduce Eq. (3.15) to

\[
(\phi^* \chi)^i_a = \sum_{m=0}^{\infty} \phi^* \left( \frac{\partial \chi^i}{\partial \left( \frac{\partial}{\partial u^b} \right)^m} \right) \sum_{k=0}^{m} \binom{m}{k} D^{m-k} \frac{\partial \phi^b}{\partial \left( \frac{\partial}{\partial u^a} \right)^n} D^n.
\]

(3.17)

That was to be proved.

If differential transformation (3.1) is invertible, then using Eqs. (3.3) and (3.11) we get

\[
\tilde{\phi}^a_c [\phi[u]] \phi^c_b [u] = \delta^a_b.
\]

(3.18)
From Eq. (3.11) for any \( f \in A \) we also have
\[
(\varphi^* f)'_a[u] = f'_b[\varphi[u]]\varphi'^b_a[u].
\] (3.19)

Taking into account Eq. (3.10), we get
\[
\delta \varphi^* f = \varphi'^T a^b \varphi^* \delta f / \delta u^b.
\] (3.20)

Consider now the behaviour of Hamiltonian equations (2.19) under differential transformations. Note first that from Eq. (2.19) it follows that
\[
\big(n\big) u^a_t = D^n J^{ab}[u] \frac{\delta h^c[u]}{\delta u^c}.
\] (3.21)

For the transformed solution \( \tilde{u}^a \) we have
\[
\tilde{u}^a_t = \sum_{n=0}^{\infty} \frac{\partial \varphi^a[u]}{\partial u^b} (\big) u^b_t = \varphi'^a_b[u] J^{bc}[u] \frac{\delta h^c[u]}{\delta u^c}.
\] (3.22)

Let us suppose that we deal with an invertible transformation, then we can define the differential operator \( \tilde{J} \) and the function \( \tilde{h} \) by
\[
\tilde{J}[\varphi[u]] = \varphi'[u] J[u] \varphi'^T[u],
\]
\[
\tilde{h}[\varphi[u]] = h[u].
\] (3.23, 3.24)

In this case from Eq. (3.20) we get
\[
\frac{\delta h^c[u]}{\delta u^a} = \varphi'^b a^a[u] \frac{\delta h^c[u]}{\delta u^b}.
\] (3.25)

Hence, equality (3.22) can be written as
\[
\tilde{u}^a_t = \tilde{J}^{ab}[\tilde{u}] \frac{\delta \tilde{h}[\tilde{u}]}{\delta u^b}.
\] (3.26)

Thus, if we have a solution of equations (2.19), then after transformation (3.1) we get a solution of equations (3.26).

In the case when \( \tilde{J} = J \), i.e. when
\[
\varphi'[u] J[u] \varphi'^T[u] = J[\varphi[u]],
\] (3.27)
we call transformation (3.1) a canonical transformation. If, in addition, \( \tilde{h} - h \in \text{Ker}\, \delta / \delta u \), i.e.
\[
h[\varphi[u]] - h[u] \in \text{Ker}\, \delta / \delta u,
\] (3.28)
then we call transformation (3.1) a differential symmetry transformation. In this case equations (3.26) have the same form as equations (2.19).
4 Nonlinear Schrödinger Equation

In fact, here and Ref. [4] we consider the following complex extension of the nonlinear Schrödinger equation:

\begin{align*}
    i\dot{q} + q'' - 2\epsilon rq^2 &= 0, \\
    i\dot{r} - r'' + 2\epsilon r^2 q &= 0,
\end{align*}

(4.1)

where \( q \) and \( r \) are arbitrary complex functions of the variables \( x \) and \( t \), \( \epsilon \) is the coupling constant. In Eq. (4.1) and below dot and prime mean the partial derivative over \( t \) and \( x \), respectively. The canonical symmetry for this system has the form \[1, 3, 6\]

\begin{align*}
    \tilde{q} &= \frac{1}{\epsilon r}, \\
    \tilde{r} &= \epsilon r^2 q - r'' + \frac{r'^2}{r},
\end{align*}

(4.2)

The inverse transformation is

\begin{align*}
    q &= \epsilon \tilde{q}^2 \tilde{r} - \tilde{q}'' + \frac{\tilde{q}'^2}{\tilde{q}}, \\
    r &= \frac{1}{\epsilon \tilde{q}}.
\end{align*}

(4.3)

Denoting \( u^1 \equiv q \), \( u^2 \equiv r \), we can write equations (4.1) in Hamiltonian form (2.19), where

\begin{align*}
    J &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
    h[q, r] &= r'q' + \epsilon r^2 q^2.
\end{align*}

(4.4)

(4.5)

The differential operator \( \varphi' \), corresponding to transformation (4.2), can be written as

\begin{equation}
    \varphi' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},
\end{equation}

(4.6)

where

\begin{align*}
    \alpha &= 0, \\
    \beta &= -\frac{1}{\epsilon r^2}, \\
    \gamma &= \epsilon r^2, \\
    \delta &= -D^2 + 2\frac{r''}{r}D + 2\epsilon rq - \frac{r'^2}{r^2}.
\end{align*}

(4.7)

(4.8)

It is easy to show that in this case equality (3.27) is valid, hence transformation (4.2) is a canonical transformation.

In Ref. [4] we have shown that for \( h \), given by Eq. (4.6) we have

\begin{equation}
    h[\tilde{q}, \tilde{r}] - h[q, r] = D \left( irq - \frac{i r'^2}{2\epsilon r^2} \right).
\end{equation}

(4.9)

Thus, transformation (4.2) is a symmetry transformation.

The densities of the conservation laws for the system, described by Eq. (4.1), are given by the formula \[4, 8\]

\begin{equation}
    p_n[q, r] = rw_n[q, r], \quad n = 1, 2, \ldots,
\end{equation}

(4.10)
where
\[ w_1[q, r] = q, \] (4.11)
and the quantities \( w_n \) for \( n > 1 \) can be found from the recursive relation
\[ w_{n+1} = -iD(w_n) + \epsilon r \sum_{m=1}^{n-1} w_m w_{n-m}. \] (4.12)

Multiplying Eq. (4.12) by \( r \) and using Eq. (4.10), we get the following recursive relation for the densities \( p_n \):
\[ p_{n+1} = -iD(p_n) + \frac{ir'}{r} p_n + \epsilon \sum_{m=1}^{n-1} m p_m p_{n-m}, \] (4.13)
with the initial condition
\[ p_1[q, r] = rq. \] (4.14)

It is convenient for our purposes to introduce a fictitious conservation law
\[ p_0[q, r] = \frac{ir'}{cr} \] (4.15)
and write Eq. (4.13) as
\[ p_{n+1} = -iD(p_n) + \epsilon \sum_{m=0}^{n-1} m p_m p_{n-m}. \] (4.16)

Consider now the behaviour of the densities \( p_n \) under transformation (4.2). Denote by \( f_n \) the difference between the transformed density and the initial one, i.e.
\[ f_n[q, r] = p_n[q, r] - p_n[q, r]. \] (4.17)
For \( n = 0, 1 \) we have
\[ f_0 = \frac{i}{\epsilon} \left( \frac{iD^2(p_0) + D(p_1)}{iD(p_0) + p_1} \right) = \frac{i}{\epsilon} D(\ln(iD(p_0) + p_1)), \quad f_1 = iD(p_0). \] (4.18)
From Eq. (4.16) we get for \( n > 1 \) the following recursive relation
\[ f_{n+1} = -iD(f_n) + \sum_{m=0}^{n-1} (f_m p_{n-m} + p_m f_{n-m} + f_m f_{n-m}). \] (4.19)
Using this relation, we find subsequently
\[ f_2 = iD \left( p_1 + \frac{\epsilon}{2} p_0^2 \right), \] (4.20)
\[ f_3 = iD \left( p_2 + \epsilon p_0 p_1 + \frac{\epsilon^2}{3} p_0^3 \right), \] (4.21)
\[ f_4 = iD \left( p_3 + \epsilon p_0 p_2 + \frac{\epsilon^2}{2} p_1^2 + \frac{\epsilon^2}{3} p_0^2 p_1 + \frac{\epsilon^3}{4} p_0^3 \right). \] (4.22)
Thus, it is natural to suppose that for $n \geq 1$

$$f_n = D(d_n),$$

(4.23)

where the quantities $d_n$ are given by

$$d_n = i \sum_{m=1}^{n} \left( \frac{\epsilon^{m-1}}{m} \sum_{i_1+\cdots+i_m+m=n} p_{i_1} \cdots p_{i_m} \right).$$

(4.24)

Let us prove this statement.

From Eqs. (4.23) and (4.24) it follows that

$$f_n = i \sum_{m=1}^{n} \left( \frac{\epsilon^{m-1}}{m} \sum_{i_1+\cdots+i_m+m=n} D(p_{i_1})p_{i_2} \cdots p_{i_m} \right).$$

(4.25)

It is not difficult to show that

$$f_{n+1} = i D(p_n) + \epsilon \sum_{m=0}^{n-1} p_m f_{n-m}.$$

Hence, it is enough to show that the equality

$$i D(p_n + f_n) = \epsilon \sum_{m=0}^{n-1} f_m (p_{n-m} + f_{n-m})$$

(4.27)

is true. From Eq. (4.24), using repeatedly Eq. (4.16), we get

$$p_n + f_n = (i D(p_0) + p_1)c_{n-1},$$

(4.28)

where $c_0 = 1$, and for $n \geq 1$

$$c_n = \sum_{m=1}^{n} \left( \frac{\epsilon^m}{m} \sum_{i_1+\cdots+i_m+m=n} p_{i_1} \cdots p_{i_m} \right).$$

(4.29)

Taking into account Eq. (4.29) and the explicit expression for $f_0$, we reduce Eq. (4.27) to the relation

$$D(c_{n-1}) = -i \epsilon \sum_{m=1}^{n-1} f_m c_{n-1-m}.$$ 

(4.30)

The validity of Eq. (4.30) follows directly from Eqs. (1.23) and (1.29).

Thus, we have shown that in the case of the nonlinear Schrödinger equation the densities of the local conservation laws change under the action of the canonical symmetry by total space derivatives. We call such densities quasi–invariants of the differential transformation under consideration. A direct consequence of the fact, we have proved, is the invariance of the higher nonlinear Schrödinger equations under the action of the transformations, given by Eq. (1.2).
5 Conclusion

It is likely that only the densities of the local conservation laws of the nonlinear Schrödinger equation are quasi-invariant under the action of transformations (4.2). If it is the case, then it is interesting to find the conditions under which the invariants of some discrete differential transformation, being a canonical transformation, are involutive with respect to the Poisson bracket.

We see, that the condition of the quasi–invariance under the canonical symmetry can be considered as a characteristic of the densities of the local conservation laws. Note that this fact does not give us a constructive method to find such densities. We can use for this the recursion operator, which can be constructed with the help of a relevant Hamiltonian pair [5], corresponding to the considered system of nonlinear equations. In the next paper we show, that the notion of the canonical symmetry is useful in looking for such operators.

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