THE ROLE OF THE ROGERS-SHEPHARD INEQUALITY IN THE CHARACTERIZATION OF THE DIFFERENCE BODY

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Abstract. The difference body operator enjoys different characterization results relying on its basic properties such as continuity, SL($n$)-covariance, Minkowski valuation or symmetric image. The Rogers-Shephard and the Brunn-Minkowski inequalities provide upper and lower bounds for the volume of the difference body in terms of the volume of the body itself. In this paper we aim to understand the role of the Rogers-Shephard inequality in characterization results of the difference body and, at the same time, to study the interplay among the different properties. Among others, we prove that the difference body operator is the only continuous and GL($n$)-covariant operator from the space of convex bodies to the origin-symmetric ones which satisfies a Rogers-Shephard type inequality while every continuous and GL($n$)-covariant operator satisfies a Brunn-Minkowski type inequality.

1. Introduction

Let $\mathcal{K}_n$ denote the set of convex bodies (compact and convex sets) in $\mathbb{R}^n$. The support function of a convex body $K \in \mathcal{K}_n$ can be seen as its analytic definition, as $K$ is uniquely determined by means of it

$$h(K, v) = h_K(v) = \max\{\langle v, x \rangle : x \in K\}.$$  

Here $v \in \mathbb{R}^n$ and $\langle v, x \rangle$ stands for the standard inner product of $v$ and $x$ in $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$ is measurable, we write Vol$(A)$ to denote its volume, that is, its $n$-dimensional Lebesgue measure. As usual, we write GL$(n)$ and SL$(n)$ to denote the general linear and special linear groups in $\mathbb{R}^n$. The unit sphere of $\mathbb{R}^n$ will be denoted by $S^{n-1}$.

The difference body $D(K)$ of $K \in \mathcal{K}_n$ is the vector (or Minkowski) sum of $K$ and its reflection in the origin, i.e.,

$$DK := K + (-K).$$

The difference body inequality or Rogers-Shephard inequality (see e.g. [8]) constitutes the fundamental (affine) inequality relating the volume of the difference body $DK$ and the volume of $K$. It is usually introduced together with a lower bound, which is a direct consequence of the Brunn-Minkowski inequality:

Let $K \in \mathcal{K}_n$. Then

$$2^n \text{Vol}(K) \leq \text{Vol}(DK) \leq \binom{2n}{n} \text{Vol}(K).$$

Equality holds on the left-hand side if and only if $K$ is centrally symmetric and on the right hand side precisely if $K$ is a simplex.

We say that an operator $\Diamond : \mathcal{K}_n \to \mathcal{K}_n$ satisfies a Rogers-Shephard type inequality (in short RS) if there exists a constant $C > 0$ such that for all $K \in \mathcal{K}_n$,

$$\text{Vol}(\Diamond K) \leq C \text{Vol}(K).$$

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Analogously, $\hat{\phi}$ satisfies a \textit{Brunn-Minkowski type inequality} (in short BM) if there exists a constant $c > 0$ such that for all $K \in \mathcal{K}^n$,

\begin{equation}
  c \text{Vol}(K) \leq \text{Vol}(\hat{\phi}K).
\end{equation}

Sometimes we will consider other functionals than volume. In this case, we will say that $\hat{\phi} : \mathcal{K}^n \rightarrow \mathcal{K}^n$ satisfies a \textit{Rogers-Shephard type inequality} (respectively, a Brunn-Minkowski type inequality) \textit{for the functional} $\phi : \mathcal{K}^n \rightarrow \mathbb{R}$ if in (4) (resp. (5)) the volume is replaced by $\phi$.

As an operator on convex bodies

\[ D : \mathcal{K}^n \rightarrow \mathcal{K}^n, \]

\[ K \mapsto DK, \]

the difference body enjoys several properties. It is continuous in the Hausdorff metric, SL($n$)-covariant and homogeneous of degree 1. An operator $\hat{\phi} : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is said to be $G$-covariant for a group of transformations $G$ if for any $K \in \mathcal{K}^n$ it holds

\[ \hat{\phi}(gK) = g\hat{\phi}K \text{ for any } g \in G, \]

and it is \textit{homogeneous} of degree $k \in \mathbb{R}$ if for any $K \in \mathcal{K}^n$,

\[ \hat{\phi}(\lambda K) = \lambda^k \hat{\phi}K \text{ for any } \lambda > 0. \]

Further, $K \mapsto DK$ is a translation invariant Minkowski valuation. Here an operator $\hat{\phi}$ is a \textit{Minkowski valuation} if for any $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$,

\[ \hat{\phi}(K \cup L) + \hat{\phi}(K \cap L) = \hat{\phi}(K) + \hat{\phi}(L), \]

where the addition on $\mathcal{K}^n$ is the Minkowski addition. An operator $\hat{\phi}$ is \textit{translation invariant} if

\[ \hat{\phi}(K + t) = \hat{\phi}(K) \text{ for any } t \in \mathbb{R}^n. \]

In fact, in [6] M. Ludwig proved that already continuity, translation invariance, Minkowski valuation and SL($n$)-covariance are enough to classify the difference body operator.

**Theorem A** ([6]). Let $n \geq 2$. An operator $\hat{\phi} : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is a continuous, translation invariant and SL($n$)-covariant Minkowski valuation if and only if there is a $\lambda \geq 0$ such that $\hat{\phi}K = \lambda DK$.

If the image of the operator $\hat{\phi}$ is restricted to centrally symmetric convex bodies, i.e., symmetric with respect to the origin, a characterization in the same direction is provided by R. Gardner, D. Hug and W. Weil in [3]. Following their notation let us introduce the notion of o-symmetrization. Given a class of sets $\mathcal{C}$ and its subclass $\mathcal{C}_s$ of central symmetric (symmetric with respect to the origin) elements of $\mathcal{C}$, an operator $\hat{\phi} : \mathcal{C} \rightarrow \mathcal{C}_s$ is called an \textit{o-symmetrization}.

**Theorem B** ([3]). Let $n \geq 2$. An o-symmetrization $\hat{\phi} : \mathcal{K}^n \rightarrow \mathcal{K}_s^n$ is continuous, translation invariant and GL($n$)-covariant if and only if there is a $\lambda \geq 0$ such that $\hat{\phi}K = \lambda DK$.

We would like to notice that this result is obtained as a by-product of a systematic study of operations between convex sets (see [4] and [7] too).

However, none of the above classifications makes use of the fundamental affine isoperimetric inequalities attached to it, namely, (4). In the spirit of these classifications of $DK$, we derive the following ones in which Rogers-Shephard inequality plays a prominent role.

**Theorem 1.1.** Let $n \geq 2$. An operator $\hat{\phi} : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is continuous, GL($n$)-covariant and satisfies a Rogers-Shephard type inequality if and only if there are $a, b \geq 0$ such that $\hat{\phi}K = aK + b(-K)$.

As a consequence of this we obtain the following corollary.

**Corollary 1.2.** Let $n \geq 2$. An o-symmetrization $\hat{\phi} : \mathcal{K}^n \rightarrow \mathcal{K}_s^n$ is continuous, GL($n$)-covariant and satisfies a Rogers-Shephard type inequality if and only if there is a $\lambda \geq 0$ such that $\hat{\phi}K = \lambda DK$. 

In this paper we also aim to study the interaction of continuity and GL($n$)-covariance with other properties which are usually attached to the difference body. Several results will be obtained in this direction in Section 4. Nevertheless, only when adding the translation invariance property a characterization result for the difference body operator is achieved.

**Theorem 1.3.** Let $n \geq 2$. An operator $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ is continuous, GL($n$)-covariant and translation invariant if and only if there is a $\lambda \geq 0$ such that $\lambda K = DK$.

The role of a Brunn-Minkowski type inequality in classifying the difference body happens not to be relevant accompanied of GL($n$)-covariance and continuity: as we shall prove in Theorem 2.3 satisfying BM is a consequence of the join of both, GL($n$)-covariance and continuity. We provide several examples showing that without GL($n$)-covariance and/or continuity, the difference body is far from being unique.

The paper has the following outline: in Section 2 we prove Theorem 1.3. Further, in Section 3 we study the role of Rogers-Shephard inequality in the already mentioned (classical) classifications of the difference body and provide the proof of Theorem 1.1. In Section 4 we investigate the interplay of continuity and GL($n$)-covariance with other properties, such as monotonicity, additivity or BM. Finally, we provide several examples of operators which share these properties and have no direct relation with the difference body, since they are not continuous or GL($n$)-covariant.

2. **Projection covariance is a powerful tool: proof of Theorem 1.3**

Following the ideas of [3], in this section we will prove that projection covariance happens to be a very powerful assumption in order to classify an operator in the spirit described in the introduction. An operator $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ is projection covariant if for any $E \in \text{Gr}(k,n)$, $1 \leq k \leq n - 1$ and any $K \in \mathcal{K}^n$,

$$\diamond(K|E) = (\diamond K)|E,$$

where $A|E$ is the orthogonal projection of the set $A$ onto $E \in \text{Gr}(k,n)$. As usual, $\text{Gr}(k,n)$ is the Grassmannian of linear $k$-dimensional subspaces of $\mathbb{R}^n$. We shall see that, in our context, projection covariance is equivalent to GL($n$)-covariance and continuity. Moreover, it provides us with an almost explicit description (see Theorem 2.1) of the image of the operators $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ sharing this property in terms of the support function.

Our main results rely strongly on slight variations of [3] Lemma 7.4, Lemma 8.1 and Theorem 8.2. We include most of the details of the proof for completeness.

**Theorem 2.1.** Let $n \geq 2$. The operator $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ is projection covariant if and only if there is a planar convex body $M \subset \mathbb{R}^2$ such that

$$h(\diamond K, x) = h_M(h_K(x), h_{-K}(x)),$$

for all $K \in \mathcal{K}^n$ and all $x \in \mathbb{R}^n$.

If the operator $\diamond$ satisfies the above hypothesis, we will say that the planar convex body $M$ obtained by this result is the associated body to $\diamond$. However, it is known that $M$ needs not to be unique (see e.g. [3] Section 8).

**Proof.** We first show that there is a homogeneous of degree 1 function $f : H \to \mathbb{R}$ such that

$$h(\diamond K, x) = f(h_K(x), h_{-K}(x)), \quad \forall x \in \mathbb{R}^n,$$

with $H = \{(s,t) \in \mathbb{R}^2 : -s \leq t\}$.

Let $u \in S^{n-1}$ and $l_u = \text{lin} u$. Since $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ is projection covariant, we have

$$\langle K \rangle l_u = \langle \diamond K \rangle l_u.$$

If $I$ is a closed interval in $l_u$, then $\diamond I \subset l_u$. Thus we can find functions $f_u : H \to \mathbb{R}$, $g_u : H \to \mathbb{R}$, such that

$$\diamond [-su, tu] = [-f_u(s,t)u, g_u(s,t)u],$$

for all $s,t \in \mathbb{R}$ with $-s \leq t$.

and \(-f_u(s, t) \leq g_u(s, t)\) whenever \((s, t) \in H\).

We let \(0 \leq \alpha \leq 1\) and choose \(v \in S^{n-1}\) such that \(\langle u, v \rangle = \alpha\). Applying (8) with \(K = [-su, tu]\) and \(l_v\) instead of \(l_u\) and (3), we obtain
\[
(\Diamond [-su, tu])| l_v = [-f_u(s, t)u, g_u(s, t)u]| l_v = \alpha[-f_u(s, t)v, g_u(s, t)v]
\]
for all \((s, t) \in H\). Therefore \(f_v(\alpha s, \alpha t) = \alpha f_u(s, t)\) and \(g_v(\alpha s, \alpha t) = \alpha g_u(s, t)\), for all \((s, t) \in H\). We also obtain
\[
f_u(\alpha^2 s, \alpha^2 t) = \alpha^2 f_u(s, t),
\]
for all \((s, t) \in H\).

Taking \(r = \alpha^2\), we have \(f_u(rs, rt) = rf_u(s, t)\), for \(0 \leq r \leq 1\) and \(s, t \in H\). Replacing \(s\) and \(t\) by \(s/r\) and \(t/r\), respectively, it follows that \(f_u\) is homogeneous of degree 1.

Next, we need to prove that for any \(u, v \in S^{n-1}\), \(f_u = f_v\) and \(g_u = g_v\). First we fix \(u \in S^{n-1}\) and let \(v \in S^{n-1}\) be such that \(\langle u, v \rangle > 0\). We put \(\alpha = \langle u, v \rangle\). Using (10) and the homogeneity of \(f_u\), we obtain
\[
\alpha f_u(s, t) = f_u(\alpha s, \alpha t) = \alpha f_u(s, t),
\]
for all \((s, t) \geq 0\). This shows that \(f_v = f_u\) for all such \(v\) and consequently \(f_u = f\) is independent of \(u\). Applying the same arguments to \(g_u\) we obtain \(g_u = g\) and that it is also homogeneous of degree 1.

Joining (3) and (9) and using that \(f_u = f\) and \(g_u = g\), we obtain
\[
(\Diamond K)| l_u = [-h_\Diamond K(-u)u, h_\Diamond K(u)u] = \alpha[-h_{\Diamond K}(-u)u, h_{\Diamond K}(u)u] = [-f(h_{\Diamond K}(u), h_{\Diamond K}(u)) u, g(h_{\Diamond K}(u), h_{\Diamond K}(u)) u],
\]
for all \(u \in S^{n-1}\). Thus,
\[
h_\Diamond K(-u) = f(h_{\Diamond K}(u), h_{\Diamond K}(u)), \quad h_\Diamond K(u) = g(h_{\Diamond K}(u), h_{\Diamond K}(u)),
\]
for all \(u \in S^{n-1}\), which implies \(f(s, t) = g(t, s)\) for \((s, t) \in H\). Now, it is enough to observe that \(f\) and \(g\) determine \(\Diamond K\) for any \(u \in S^{n-1}\) and using the homogeneity of \(f\) we obtain (7) for any \(u \in \mathbb{R}^n\).

Next we prove that \(f : H \rightarrow \mathbb{R}\) is a support function. Notice that with (7) we have obtained, as in [3, Lemma 8.1], a homogeneous of degree 1 function \(f\). Thus, we can use Theorem 8.2 in [3] to obtain the convexity of \(f\).

Indeed, considering the convex body \(K_0 = [-e_2, e_1]\), the segment joining the points \(-e_2\) with \(e_1\), and the projection covariance of \(\Diamond\), one obtains that \(M = \Diamond K_0\) is a feasible body associated to \(\Diamond\), using the properties of \(f\).

The converse of the result follows directly from the properties of support functions. \(\square\)

Centrally symmetric convex bodies play a special role when dealing with the difference body, as they are the only \textit{fixed points} (up to maybe a constant) of the operator. From now on and for the sake of brevity, we will write \textit{o-symmetric convex sets} for origin or centrally symmetric sets.

\textbf{Remark 2.2.} From the above result it immediately follows that if \(K\) is o-symmetric, then \(h(\Diamond K, u) = h_M(1, 1)h(K, u)\) and \(h_M(1, 1) \geq 0\).

In [3] it was proved that an o-symmetrization enjoys continuity and \(\text{GL}(n)\)-covariance if and only if it is projection covariant [3 Lemma 4.3 and Corollary 8.3]. The same proof works for a non-necessarily o-symmetric operator \(\Diamond : K^n \rightarrow K^n\). Hence, we have the following lemma.
Lemma 2.3 (Corollary 8.3 in [3]). The operator $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ is continuous and $\text{GL}(n)$-covariant if and only if it is projection covariant.

As a consequence of this result based on the projection covariance, implied by the $\text{GL}(n)$-covariance and continuity, Theorem 1.3 follows, in the same way as Theorem B was proved in [3]. We include again (most of) the details for completeness.

Proof of Theorem 1.3. For $K \in \mathcal{K}^n$, it has to be shown $h(\diamond K, x) = \lambda h_{DK}(x)$ for any $x \in \mathbb{R}^n$. Since $\diamond$ is invariant under translations, it is enough to find a translation $t$ for which $h(\diamond(K + t), x) = \lambda h_{DK}(x)$. The vector $t \in \mathbb{R}^n$ satisfying $(x, t) = \frac{1}{2}(h(-K, x) - h(K, x))$ fulfills the stated property.

3. ROGERS-SHEPHARD inequality as classifying property

In this section we show that if the condition of translation invariance in Theorem 1.3 is replaced by the condition $\text{RS}$ -of satisfying a Rogers-Shephard type inequality (cf. (1))-, then other operators than the difference body (see Theorem 1.1) appear, but they are not far from it.

If instead, we replace, now in Theorem 1.3 translation invariance by $\text{RS}$ (notice that in this case the assumption of $\alpha$-symmetrization is present) we do actually obtain the difference body (cf. Corollary 1.2).

Now we aim to prove Theorem 1.3 i.e., that a continuous and $\text{GL}(n)$-covariant operator $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ satisfying $\text{RS}$ is necessarily of the form $\diamond K = aK + b(-K)$ for $a, b \geq 0$. We will need the following lemmas for the proof.

Lemma 3.1. Let $n \geq 2$ and $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ be continuous and $\text{GL}(n)$-covariant with $M \in \mathcal{K}^2$ as associated convex body. If $h_M(1, 1) = 0$, then $\diamond K = \{0\}$ for every $K \in \mathcal{K}^n$.

Proof. We show first that $h_M(a, b) \leq 0$ for every $a, b \geq 0$. Let $K$ be a convex body with $h_K(e_1) = h_K(-e_1) \neq 0$, $h_K(e_2) = h_K(-e_2) \neq 0$ but $a := h_K(e_1 + e_2) \neq h_K(-e_1 - e_2) =: b$. Note that for every $a, b > 0$, there exists a convex body satisfying these conditions. Since $\diamond K$ is a convex body, it holds, as claimed,

$$h_M(a, b) = h_M(h_K(e_1 + e_2), h_K(-e_1 - e_2)) \leq h_M(h_K(e_1), h_K(-e_1)) + h_M(h_K(e_2), h_K(-e_2)) = 0.$$

Next we show that $M$ is necessarily a segment of the form $\{(s, -s), (s, -s)\}$, for some $0 \leq s \leq r$. In order to do it, it is enough to prove that $h_M(1, -1) \leq 0$ and $h_M(-1, 1) \leq 0$. Let us consider convex bodies $K, L$ satisfying $h_K(e_1 + e_2) = -1$, $h_K(-e_1 - e_2) = 1$, $h_L(e_1 + e_2) = 1$, $h_L(-e_1 - e_2) = -1$ and $h_K(\pm e_1), h_K(\pm e_2), h_L(\pm e_1), h_L(\pm e_2) > 0$. Since $\diamond K$ and $\diamond L$ are convex we obtain, by using the previous claim, that $h_M(1, -1) \leq 0$ and $h_M(-1, 1) \leq 0$.

Hence, for any convex body $K \in \mathcal{K}^n$, we have $h(\diamond K, u) = h_M(h_K(u), h_K(-u)) = \max\{-s, -r\}(h_K(u) + h_K(-u)) = h(\diamond K, -u)$. Since $h(\diamond K, u) + h(\diamond K, -u) \geq 0$, we obtain $r = s = 0$, and hence, $M = \{(0, 0)\}$.

Remark 3.2. We notice, that from the above proof it follows that

i) the converse of Lemma 3.1 holds. Indeed, the image of a full dimensional $\alpha$-symmetric convex body is the origin if and only if $h_M(1, 1) = 0$ (see Remark 2.2).

ii) if $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous $\text{GL}(n)$-covariant operator and the image of some $\alpha$-symmetric convex body (not 0-dim) is a point, then $\diamond \equiv 0$.

The next lemma will be very useful to prove most of the following results.

We will say that an operator $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ is trivial, if its image consists only of the origin.

Lemma 3.3. Let $n \geq 2$ and $\omega(K, u) = h(K, u) + h(K, -u)$ denote the width of $K \in \mathcal{K}^n$ in the direction $u \in S^{n-1}$.
i) If $\diamond : K^n \to K^n$ is a continuous and GL($n$)-covariant operator with $M \in K^2$ as associated convex body, then
\[
\omega(K, u) h_M(1, 1) \leq \omega(\diamond K, u), \quad \forall u \in S^{n-1}.
\]

ii) If $\diamond : K^n \to K^n$ is a continuous, non-trivial, GL($n$)-covariant operator, then \text{aff} K \subseteq \text{aff} \diamond K$. In particular, $\diamond$ maps convex sets of dimension $n$ into convex sets of dimension $n$.

iii) If $K, L \in K^n$ satisfy $\omega(K, u) \leq \omega(L, u)$ for every $u \in S^{n-1}$, then
\[
\Vol(K) \leq 2^{-n} \left(\frac{2n}{n}\right) \Vol(L).
\]

Proof.

i) It follows directly from the properties of the support function of $M$, that
\[
\omega(\diamond K, u) = h_M(h_K(u), h_K(-u)) + h_M(h_K(-u), h_K(u)) \geq (h_K(u) + h_K(-u)) h_M(1, 1) = \omega(K, u) h_M(1, 1).
\]

ii) It follows directly from the first item and Remark 3.2.

iii) From the hypothesis, it follows that $h(K + (-K), u) \leq h(L + (-L), u)$ for every $u \in S^{n-1}$, thus, $DK \subseteq DL$. Now, using (3) we obtain, as stated,
\[
\Vol(K) \leq 2^{-n} \Vol(DK) \leq 2^{-n} \Vol(DL) \leq 2^{-n} \left(\frac{2n}{n}\right) \Vol(L).
\]

The following remark will be used in the following. We include it for future reference.

Remark 3.4. Let $\diamond$ be a continuous and GL($n$)-covariant operator and $M$ its associated body. If $M$ is a segment on the line $\{(x, y) : x = y\} \subset \mathbb{R}^2$, then $\diamond K = \lambda DK$ for certain $\lambda \geq 0$.

The next result states that if the continuous and GL($n$)-covariant operator $\diamond$ preserves dimensions, then it is not far from being the difference body operator.

Proposition 3.5. Let $n \geq 2$ and $0 \leq k \leq n - 1$. An operator $\diamond : K^n \to K^n$ is continuous, GL($n$)-covariant and satisfies $\dim K = \dim \diamond K$ for any $K \in K^n$ of $\dim K = k$ if and only if there are $a, b \geq 0$ such that $\diamond K = aK + b(-K)$.

Proof. Let $u \in S^{n-1}$ and $K \in K^n$ with $\dim K = k$ be such that $\omega(K, u) = 0$ and $h(K, u) > 0$. Then, since $\dim K = \dim \diamond K$, using Lemma 3.3 we obtain that $\omega(\diamond K, u) = 0$. From $h(K, u) = -h(K, -u)$, it follows
\[
\omega(\diamond K, u) = h_M(h_K(u), h_K(-u)) + h_M(h_K(-u), h_K(u)) = h_K(u) h_M(1, 1) + h_M(-1, 1) = h_K(u) \omega(M, 1, -1)
\]

If $\omega(M, (1, -1)) > 0$, according to the above calculation, $\omega(\diamond K, u) > 0$. Hence, it holds $\omega(M, (1, -1)) = 0$. The latter implies that $M$ is a (possibly degenerated) segment lying in a line parallel to $\{x = y\}$. Let $M = [(a, a), (b, b)] + (p, 0)$ with $a, b, p \in \mathbb{R}$ and $a \leq b$. Then, using Remark 3.4 we obtain
\[
h_{\diamond K}(x) = h_{\lambda h_{K} + (-K)}(x) = h_{[(a, a), (b, b)]}(h_{K}(x), h_{-K}(x)) + ph_{K}(x)
\]
\[
= \lambda h_{K} + pK + h_{(\lambda + p)K} + h_{(-K)}(x).
\]
The converse is clear.

We notice that for the proof it is enough to assume only, that for some $K \in K^n$ with $0 \leq \dim K \leq n - 1$ there is an appropriate translate of it, $K + t$ such that $\dim(K + t) = \dim(\diamond(K + t))$.

We would also like to remark the following consequence:
Corollary 3.6. Let $n \geq 2$. An operator $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ is continuous, $\text{GL}(n)$-covariant and satisfies $\diamond\{p\} = \{0\}$ for some $p \neq 0$ if and only if there is a $\lambda \geq 0$ such that $\diamond K = \lambda DK$.

As a consequence of the above proof we observe the following remark.

Remark 3.7. Let $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous and $\text{GL}(n)$-covariant operator and $M$ the body associated to it. It holds,

i) if $K \in \mathcal{K}^n$ with $0 \leq \dim K \leq n - 1$, then $\omega(\diamond K, u) = h_K(u)\omega(M, (1, -1))$ for every $u \in S^{n-1}$ such that $\omega(K, u) = 0$.

ii) $\omega(M, (1, -1)) \neq 0$ if and only if $\dim K < \dim \diamond K$ for some $K \in \mathcal{K}^n$ of $\dim K \leq n - 1$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. First, we recall that under the assumption of RS, if $K \in \mathcal{K}^n$ with $\dim K \leq n - 1$ then we necessarily have $\dim \diamond K \leq n - 1$. Thus, from Lemma 3.3 and Proposition 3.5 with $k = n - 1$, we can assert that $\omega(M, (1, -1)) = 0$ and we have, as in Proposition 3.5, that $\diamond K = aK + b(-K)$ for some $a, b \geq 0$.

It is clear, that each operator of the form $K \mapsto aK + b(-K)$, $a, b \geq 0$ satisfies a Rogers-Shephard inequality.

The next corollary indicates that asking for RS is as strong as translation invariance if $o$-symmetrization is assumed, as it is the case of Theorem 1.1.

Corollary 3.8. Let $n \geq 2$. An $o$-symmetrization $\diamond : \mathcal{K}^n \to \mathcal{K}_o^n$ is continuous, $\text{GL}(n)$-covariant and satisfies RS if and only if there exists a $\lambda \geq 0$ such that $\diamond K = \lambda DK$.

Proof. From Theorem 1.1 there exist $a, b \geq 0$ such that $\diamond K = aK + b(-K)$ for $K \in \mathcal{K}^n$. We have to prove that $a = b$. In order to do so, it is enough to consider $K \in \mathcal{K}^n$ with $h(K, -u) = 0$ and $h(K, u) > 0$, which an appropriate translation of any full dimensional $(\dim K = n)$ convex body does. Hence, since $K$ satisfies $h(\diamond K, u) = h(aK + b(-K), u) = h(aK + b(-K), -u) = h(\diamond K, -u)$, we easily obtain $a = b$.

4. Interaction of other properties with projection covariance

In the previous sections we have seen that continuity and $\text{GL}(n)$-covariance (or equivalently projection covariance) play a strong classifying role for the difference body, being however, on their own, not enough for it. In this section we gather together some aspects of the interplay of continuity and $\text{GL}(n)$-covariance with further assumptions such as Brunn-Minkowski inequality, monotonicity or additivity. We also provide (further) examples showing that removing either $\text{GL}(n)$-covariance or continuity forces us to assume several additional assumptions in order to get some close to the difference body.

We start proving that continuity and $\text{GL}(n)$-covariance imply the homothety property. We say that a non-trivial operator $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ satisfies the homothety property if there exists some $\lambda > 0$ such that $\diamond K = \lambda K$ for every $K \in \mathcal{K}_o^n$. If $\lambda = 1$ one gets the identity property used in 3.

Proposition 4.1. Let $n \geq 2$. If $\diamond : \mathcal{K}^n \to \mathcal{K}^n$ is a non-trivial, continuous, $\text{GL}(n)$-covariant operator, then it has the homothety property.

Proof. From Theorem 2.1 we know that $h(\diamond K, u) = h_M(h_K(u), h_K(-u))$. Thus, if $K = -K$, we immediately obtain $h(\diamond K, u) = h_M(1, 1)h(K, u)$ from which the result follows.

Remark 4.2. We notice that an operator having the homothety property needs not be either continuous or $\text{GL}(n)$-covariant:

$$\diamond K = \begin{cases} K, & \text{if } K = -K \\
B_n, & \text{otherwise} \end{cases}$$

where $B_n$ is the unit Euclidean ball.
Next we prove that if a non-trivial operator is continuous and GL\((n)\)-covariant, then it satisfies BM. Indeed even more can be proven: such an operator satisfies a Brunn-Minkowski type inequality for every quermassintegral.

We recall that the Steiner formula states that the volume of the Minkowski sum of a convex body \(K\) and a non-negative dilation of \(B_n\) is a polynomial

\[
\text{Vol}(K + \rho B_n) = \sum_{i=0}^{n} \rho^i \binom{n}{i} W_i(K),
\]

whose coefficients \(W_i(K)\), up to normalization, are the so-called quermassintegrals of \(K\). From the above it follows that \(W_0(K) = \text{Vol}(K)\) and \(W_n(K) = \text{Vol}(B_n)\) for all \(K\). Indeed, this is a particular case of a much more general fact (see \cite{11}): for convex bodies \(K_1, \ldots, K_m\) and \(\lambda_1, \ldots, \lambda_m \in \mathbb{R}\) non-negative, the volume of the linear combination \(\lambda_1 K_1 + \cdots + \lambda_m K_m\) is a homogeneous polynomial, whose coefficients are the so-called mixed volumes (of whose quermassintegrals are particular cases), namely

\[
\text{Vol}(\lambda K_1 + \cdots + \lambda K_m) = \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq n} \lambda_{i_1} \cdots \lambda_{i_m} V(K_{i_1}, \ldots, K_{i_m}).
\]

**Theorem 4.3.** Let \(n \geq 2\). If \(\Diamond : \mathcal{K}^n \to \mathcal{K}^n\) is a non-trivial, continuous and GL\((n)\)-covariant operator, then it satisfies a Brunn-Minkowski type inequality for every quermassintegral \(W_i\), \(i = 0, \ldots, n-1\), i.e., there is a constant \(c_{n,i,\Diamond} > 0\) such that

\[
W_i(K) \leq c_{n,i,\Diamond} W_i(\Diamond K).
\]

**Proof.** Let \(M\) be the associated convex body to \(\Diamond\), ensured by Lemma \ref{lem:2.3} and Theorem \ref{thm:2.1}. Since \(\Diamond\) is non-trivial, \(h_M(1,1) > 0\) by Lemma \ref{lem:3.1} and Remark \ref{rem:2.2}.

We will prove first the statement for \(i = 0\), that is, for the volume. Let \(K \in \mathcal{K}^n\). From Lemma \ref{lem:3.3}i, we know that

\[
\omega(h_M(1,1)K, u) \leq \omega(\Diamond K, u),
\]

for all \(u \in S^{n-1}\).

Using now Lemma \ref{lem:3.3}iii, we have

\[
\text{Vol}(K) \leq (2h_M(1,1))^{-n} \binom{2n}{n} \text{Vol}(\Diamond K),
\]

and hence, a Brunn-Minkowski type inequality for \(\Diamond\) and \(W_0 = \text{Vol}\).

We notice that \(M\) is associated to the operator \(\Diamond\) and not to some convex body, hence, the constant is independent of the convex body.

In order to prove the statement for \(i = 1, \ldots, n-1\) we will make use of the so-called Kubota integral recursion formula for quermassintegrals, which states (see e.g. \cite{11} (5.72))

\[
W_{n-i}(\Diamond K) = c(l, n) \int_{E \in \text{Gr}(l, n)} \text{Vol}_l((\Diamond K)|E) dE,
\]

where \(dE\) denotes the probability measure on \(\text{Gr}(l, n)\) and \(c(l, n)\) is a constant depending only on \(n, l\), \(0 \leq l \leq n\).

From Lemmas \ref{lem:3.3}i and \ref{lem:2.3} we have

\[
\omega(h_M(1,1)(K|E), u) \leq \omega(\Diamond (K|E), u) = \omega((\Diamond K)|E, u)
\]

for any \(E \in \text{Gr}(l, n)\) and \(u \in S^{n-1}\).

Lemma \ref{lem:3.3}iii applied on \(E \in \text{Gr}(l, n)\) and the projection covariance yield

\[
\text{Vol}_l(h_M(1,1)(K|E)) \leq 2^{-l} \binom{2l}{l} \text{Vol}_l((\Diamond K)|E).
\]

Using the homogeneity of the volume and plugging the latter inequality in (13) we obtain the result. \(\square\)
Using Proposition 3.4 for an \((n-l-1)\)-dimensional convex body, we obtain that the condition of satisfying a Rogers-Shephard for a quermassintegrals almost characterizes the difference body operator, that is, the following statement holds:

**Theorem 4.4.** Let \( n \geq 2 \). An operator \( \diamond : K^n \to K^n \) is continuous GL\((n)\)-covariant and satisfies a Rogers-Shephard type inequality for some \( l \)-th quermassintegral, \( l \in \{0, 1, \ldots, n-1\} \) if and only if there are \( a, b \geq 0 \) such that \( \diamond K = aK + b(-K) \).

**Proof.** First we note that for \( l = 0 \), the statement coincides with Theorem 1.1.

Let \( M \) be the associated body to the operator \( \diamond \) such that \( W_l(\diamond K) \leq cW_l(K) \) for some \( c > 0 \) and a fixed \( l \in \{1, \ldots, n-1\} \).

Lemma 3.3 yields \( \dim K \leq \dim \diamond K \) for any \( K \in K^n \). Let \( K \in K^n \) be such that \( \dim K = n - l - 1 \), so that \( W_l(K) = 0 \).

Since \( 1 \leq l \leq n - 1 \), we can find \( u \in S^{n-1} \) such that \( \omega(K, u) = 0 \). Then, from Remark 3.7, we have that \( \omega(\diamond K, u) = h(K, u)\omega(M, (1, -1)) \). If \( \omega(K, (1, -1)) > 0 \), just using an appropriate translation of \( K \) instead of \( K \), if necessary, it holds that there exists an \((n - l - 1)\)-dimensional convex \( K \) with \( \dim \diamond K \geq n - l \). Hence, \( \diamond \) does not satisfy a Rogers-Shephard inequality for \( W_l \). Thus, \( \omega(M, (1, -1)) = 0 \), which as in the proof of Proposition 3.4 yields that \( \diamond K = aK + b(-K) \) for some \( a, b \geq 0 \).

We notice, that each operator \( K \mapsto aK + b(-K), a, b \geq 0 \) satisfies a Rogers-Shephard inequality for \( W_l, 1 \leq l \leq n - 1 \), using that \(-K \leq nK \) for \( K \in K^n \) having centroid at the origin. The latter yields the converse. \(\Box\)

Monotonicity happens to be, as BM, a property shared by any continuous and GL\((n)\)-covariant operator. In order to prove it, we will make use of the so called M-sum.

Given a subset \( M \) of \( \mathbb{R}^2 \) and sets \( K, L \in \mathbb{R}^n \), the \( M \)-sum of \( K, L \) is the set

\[
K +_M L = \{ ax + by : x \in K, y \in L, (a, b) \in M \}.
\]

If \( M \in K^2 \) and \( K, L \in K^n \), then it is not difficult to check, that

\[
h(K +_M L, u) = h_M(h_K(u), h_L(u))
\]

for \( u \in S^{n-1} \). For further details we refer to [3]. We notice, that the M-sum was already (implicitly) used in Theorem 2.1.

**Proposition 4.5.** Let \( n \geq 2 \). If \( \diamond : K^n \to K^n \) is a continuous and GL\((n)\)-covariant operator, then it is monotonic.

**Proof.** From Theorem 2.1 we know that

\[
\diamond K = K +_M (-K) = \bigcup_{(a,b)\in M} aK + b(-K).
\]

It follows directly from \( K \subseteq L \), that \( \diamond K \subseteq \diamond L \). \(\Box\)

In the next result we impose additivity to the continuous and GL\((n)\)-covariant operator. We recall that an operator \( \diamond : K^n \to K^n \) is said to be additive if for any \( K, L \in K^n \) it satisfies that \( \diamond (K + L) = \diamond K + \diamond L \).

**Proposition 4.6.** Let \( n \geq 2 \). An operator \( \diamond : K^n \to K^n \) is continuous, GL\((n)\)-covariant and additive if and only if there are \( a, b \geq 0 \) such that \( \diamond K = aK + b(-K) \).

**Proof.** Let \( u \in S^{n-1}, c > 0 \) and \( K, L \in K^n \) such that \( (h_K(u), h_K(-u)) = (c, -c) \) and \( (h_L(u), h_L(-u)) = (-c, c) \). Then,

\[
h(\diamond (K + L), u) = h_M(c - c, -c + c) = 0
\]

and

\[
h(\diamond K + \diamond L, u) = h_M(c, -c) + h_M(-c, c) = c\omega(M, (1, -1)).
\]

Thus, \( \diamond \) is additive if and only if \( \omega(M, (1, -1)) = 0 \), that is, if and only if \( \diamond K = aK + b(-K) \) for some \( a, b \geq 0 \) (cf. Remark 3.3). \(\Box\)
We have seen above, that the assumptions of continuity and GL(n)-covariance, Rogers-Shephard inequality almost characterize the difference body (see Theorem 1).

A similar phenomenon happens when to continuity and GL(n)-covariance is added the hypothesis of Minkowski valuation. In [12], T. Wannerer proved that if we omit the assumption of translation invariance in Ludwig’s classification of the difference body (Theorem A), we are actually not that far from the difference body.

**Theorem C** ([12]). Let \( n \geq 3 \). An operator \( \diamond : \mathcal{K}^n \to \mathcal{K}^n \) is a continous, GL(n)-covariant Minkowski valuation if and only if there are \( a, b, c, d \geq 0 \) such that

\[
\diamond K = aK + b(\neg K) + c \text{conv}\{\{0\} \cup K\} + d \text{conv}\{\{0\} \cup (\neg K)\}.
\]

Let us notice that the continuous, GL(n)-covariant operator \( \diamond K = \text{conv}\{\{0\} \cup K\} \) can be obtained as

\[
h(\diamond K, u) = \max\{0, h(K, u)\} = h_{(0,0), (1,0)}(h_K(u), h_K(-u)),
\]
i.e., a body \( M \), associated to \( \diamond \) (cf. Theorem 2.1) can be chosen to be the segment joining the origin and the point \((1,0)\).

Hence, the body \( M := \{(a, 0)\} + \{(0, b)\} + e(0, 0, (1,0)) + d(0, 0, (0,1)) \) is an associated body to \((14)\). In other words, for any \( u \in S^{n-1} \), \( h(\diamond K, u) \) is given by

\[
h_M(h_K(u), h_K(-u)) = \begin{cases} (a + c)h_K(u) + (b + d)h_K(-u), & \text{if } h_K(u), h_K(-u) \geq 0 \\ (a + c)h_K(u) + bh_K(-u), & \text{if } h_K(u) \geq 0, h_K(-u) < 0 \\ ah_K(u) + (b + d)h_K(-u), & \text{if } h_K(u) < 0, h_K(-u) \geq 0. \end{cases}
\]

Using Theorem 2.1 we prove that Theorem C also holds for \( n = 2 \) and give a new proof of it.

**Theorem 4.7.** Let \( n \geq 2 \). An operator \( \diamond : \mathcal{K}^n \to \mathcal{K}^n \) is a continous, GL(n)-covariant Minkowski valuation if and only if there are \( a, b, c, d \geq 0 \) such that

\[
\diamond K = aK + b(\neg K) + c \text{conv}\{\{0\} \cup K\} + d \text{conv}\{\{0\} \cup (\neg K)\}.
\]

**Proof.** Let \( M \) be an associated convex body to \( \diamond \), \( u \in S^{n-1} \) and \( \alpha, \beta > 0 \). Choose convex bodies \( K, L \) such that

\[
h(K, u) = h(L, -u) = h(K \cap L, u) = h(K \cap L, -u) = 0,
\]

\[
h(L, u) = h(K \cup L, u) = \alpha, \quad h(K, -u) = h(K \cup L, -u) = \beta.
\]

Since \( \diamond : \mathcal{K}^n \to \mathcal{K}^n \) is a Minkowski valuation, it holds

\[
h_M(\alpha, \beta) + h_M(0, 0) = h_M(0, \beta) + h_M(\alpha, 0),
\]

that is,

\[
h_M(\alpha, \beta) = \alpha h_M(1, 0) + \beta h_M(0, 1),
\]

and \( h_M(\alpha, \beta) \) is a linear function when \( \alpha, \beta \geq 0 \).

Next, by choosing \( K' \) and \( L' \) such that

\[
h(K', u) = h(K' \cup L', u) = 0, \quad h(L', -u) = h(K' \cup L', -u) = \beta,
\]

\[
h(K', -u) = h(K' \cap L', -u) = -h(L', u) = -h(K' \cap L', u) = \alpha,
\]

it similarly follows \( h_M(\alpha, \beta) = \alpha(h_M(0, 1) - h_M(-1, 1)) + \beta h_M(0, 1) \).

Finally, by choosing \( K'' \) and \( L'' \) such that

\[
h(K'', u) = h(K'' \cup L'', u) = 0, \quad h(L'', u) = h(K'' \cup L'', u) = \beta,
\]

\[
h(K'', -u) = h(K'' \cap L'', -u) = -h(L'', -u) = -h(K'' \cap L'', -u) = \alpha,
\]

it follows \( h_M(\beta, -\alpha) = \beta h_M(1, 0) + \alpha(h_M(-1, 1) - h_M(1, 0)) \).

The three conditions obtained for the support function of \( h_M \) imply that (15) is satisfied considering \( a = h_M(0, 1) - h_M(-1, 1), b = h_M(-1, 1) - h_M(1, 0), a + c = h_M(1, 0) \) and \( b + d = h_M(0, 1) \). Thus, \( \diamond K \) is given as in (14).
Since each of the summands of \( \partial \) defines a continuous, GL\((n)\)-covariant operator which is a Minkowski valuation, the result follows. \( \square \)

Using Theorem \( \Box \) and Theorem \( \mathcal{L} \), we obtain the following result for \( o\)-symmetrizations:

**Corollary 4.8.** Let \( n \geq 2 \). An \( o\)-symmetrization \( \diamond : \mathcal{K}^n \to \mathcal{K}_s^n \) is a continuous, GL\((n)\)-covariant Minkowski valuation if and only if there are \( a, b \geq 0 \) such that

\[
\diamond K = aDK + bD(\text{conv}(\{0\} \cup K)).
\]

**Proof.** Let \( u \in S^{n-1} \) and \( K \in \mathcal{K}^n \) such that \( h(K, u) > 0 \) and \( h(K, -u) = 0 \). Theorem \( \mathcal{L} \) ensures that

\[
h(\diamond K, u) = ah(K, u) + bh(K, -u) + c \max\{0, h(K, u)\} + d \max\{0, h(K, -u)\}
= (a + c)h(K, u).
\]

On the other hand, using now \(-u\) we have

\[
h(\diamond K, -u) = (b + d)h(K, u).
\]

Since \( \diamond K \) is \( o\)-symmetric, it necessarily holds \( a + c = b + d \). Now choosing a convex body \( K \) such that for \( u \in S^{n-1} \) we have \( h(K, u) > 0 \) and \( h(K, -u) < 0 \), we will have that \( h(\diamond K, u) = h(\diamond K, -u) \) if and only if the following equality holds:

\[
(a + c)h(K, u) + bh(K, -u) = (b + d)h(K, u) + ah(K, -u).
\]

The latter together with the already obtained relation for \( a, b, c, d \) implies directly that \( a = b \) and \( c = d \), which proves the result. \( \square \)

Finally, we would like to remark that under GL\((n)\)-covariance and continuity, an operator is an \( o\)-symmetrization if and only if it is even. Indeed, since \(-\text{Id} \in \text{GL}(n)\), we directly obtain, that \( \diamond (-K) = -\diamond K \).

In the next subsection we will present some examples showing that a random choice of the properties we have been dealing with, does not, in general, get any close to the difference body.

### 4.1. Examples of operators sharing properties with \( DK \), but far from being it.

First we would like to understand the role of RS in Theorems \( \mathcal{A} \) and \( \mathcal{B} \). Since we have Theorem \( \mathcal{A} \) and Theorem \( \mathcal{B} \) it would only make sense to either replace one of the hypothesis in the latter by RS, or weaken any of them and add RS. Let us also recall, that replacing translation invariance by RS in both, Theorem \( \mathcal{A} \) and \( \mathcal{B} \) has already been addressed in Theorem \( \mathcal{L} \) and Corollary \( \mathcal{L} \).

Unfortunately, we are not aware of a non-continuous valuation which is GL\((n)\)-covariant, translation invariant and satisfies a Rogers-Shephard inequality. However, if any of the other assumptions in Theorems \( \mathcal{A} \) and \( \mathcal{B} \) is replaced by Rogers-Shephard inequality, there is, in general, no possibility of getting close to a characterization of the difference body. The following examples illustrate it.

**Example 4.9.** Let \( L \in \mathcal{K}_s^n \) have dimension at most \( n - 1 \). Then, the operator

\[
\diamond : \mathcal{K}^n \to \mathcal{K}_s^n
K \mapsto L
\]

is a continuous, Minkowski valuation which is also an \( o\)-symmetrization and translation invariant. It satisfies RS but it is not GL\((n)\)-covariant. Further, it does not satisfy BM.

**Example 4.10.** Let \( a(K) \) denote either the Steiner point of \( K \) (see e.g. [11] p. 50]) or the center of gravity (centroid) of \( K \) (see e.g. [11] p. 314]). The operator

\[
K \mapsto \text{conv}((K - a(K)) \cup (-K + a(-K)))
\]

satisfies BM and RS \( \mathcal{B} \). Moreover, it is a continuous \( o\)-symmetrization. If \( a(K) \) is the Steiner point, then the operator is also translation invariant.
Example 4.11. Let $p \in \mathbb{R}^n$. The operator
$$
\hat{\Diamond}_p : \mathcal{K}^n \rightarrow \mathcal{K}^n \\
K \mapsto K - p.
$$
is a continuous Minkowski valuation, which clearly satisfies RS and BM. However, $\hat{\Diamond}_p$ is neither an $o$-symmetrization, nor $GL(n)$-covariant or translation invariant.

If $p = \text{st}(K)$, the Steiner point of $K$, then it is further $O(n)$-covariant but not $SL(n)$-covariant.

Example 4.12. The operator
$$
\hat{\Diamond} : \mathcal{K}^n \rightarrow \mathcal{K}^n \\
K \mapsto \text{Vol}(K)DK
$$
is a continuous, translation invariant and $SL(n)$-covariant $o$-symmetrization. This example yields that the $GL(n)$-covariance imposed in Theorems 1.3 and 3 cannot be weakened even to $SL(n)$.

The previous example shows also that the three conditions continuity, Minkowski valuation and RS together, neither characterize the difference body nor imply $GL(n)$-covariance.

Example 4.13. Let $B$ be a symmetric convex body with non-empty interior. We define
$$
\hat{\Diamond} : \mathcal{K}^n \rightarrow \mathcal{K}^n \\
K \mapsto DK \cap B.
$$
The operator $\hat{\Diamond}$ is an $o$-symmetrization, continuous and translation invariant. It also satisfies the Rogers-Shephard inequality.

However, $\hat{\Diamond}$ is neither a Minkowski valuation, nor $GL(n)$-covariant, nor satisfies BM.

The latter shows that the four conditions: $o$-symmetrization, continuity, translation invariance and Rogers-Shephard inequality neither characterize the difference body nor imply $GL(n)$-covariance.

Let us also notice that RS is not directly implied by all of the already treated properties, except for the already proven results dealing with the difference body and its relatives, as the following example shows.

Example 4.14. For $C \in \mathcal{K}_s^2$, the complex difference bodies (see [1]) defined by
$$
h(D_C K, u) = \int_{S^1} h(\alpha K, u)dS(C, \alpha), \quad u \in S^{n-1}, \ K \in \mathcal{K}(C^n)
$$
provide examples of continuous, $o$-symmetrizations, translation invariant Minkowski valuations which satisfy the Brunn-Minkowski inequality but not the Rogers-Shephard inequality. They are neither $GL(n)$-covariant.

The following example deals with the continuity condition.

Example 4.15. Let $L \in \mathcal{K}_s^n$ have dimension at most $n - 1$. Then, the operator
$$
\hat{\Diamond}K = \begin{cases} 
DK, & \text{if } \dim K = n \\
L, & \text{otherwise}
\end{cases}
$$
is an $o$-symmetrization, translation invariant and satisfies both RS and BM. It is however, not continuous.

If $L$ is chosen to be the origin, then it is also $GL(n)$-covariant, monotonic and 1-homogeneous.

Example 4.16. Let $\omega(K)$ denote the mean width of $K$ (see e.g. [11 (1.30)]). Let us consider the operator
$$
\hat{\Diamond} : \mathcal{K}^n \rightarrow \mathcal{K}^n \\
K \mapsto B_{\omega(K)},
$$
where $B_{\omega(K)}$ denotes the ball centered at the origin and of radius $\omega(K).$
The operator ◦ is a continuous, \(o\)-symmetrization, translation invariant and Minkowski valuation which satisfies BM. It is also monotonic and homogeneous of degree 1. However, it neither satisfies RS nor is GL\((n)\)-covariant.

Note that if we change, in the last example, the mean width by any other intrinsic volume, then we would lose the Brunn-Minkowski inequality, since ◦ inherits the homogeneity of the intrinsic volume.

**Remark 4.17.** We notice that homogeneity assumed together with RS and/or BM implies more precise values for the homogeneity degree. Indeed, let \(q\) be the homogeneity degree of ◦. Then using RS and/or BM we can write

\[
c\lambda^n \text{Vol}(K) \leq \text{Vol}(\lambda \diamond K) = \text{Vol}(\lambda^q \diamond K) = \lambda^q \text{Vol}(K) \leq c \text{Vol}(\lambda K) = C \lambda^n \text{Vol}(K).
\]

Thus, \(q \leq 1\) if RS is assumed, \(q \geq 1\) if BM is assumed and, in consequence, \(q = 1\) if BM and RS are assumed. Hence, under RS and BM, we can replace GL\((n)\)-covariance assumption by SL\((n)\)-covariance and homogeneity.

Next examples prove that RS does not imply, in general, further good properties such us homogeneity or translation invariance.

**Example 4.18.** The operator \(\diamond K = K \cap B_n\) satisfies RS. It is also continuous and \(O(n)\)-covariant. It has neither BM nor is translation invariant.

**Example 4.19.** Let, for every \(K \in \mathcal{K}^n\), \(\diamond K = L\) with \(\dim L \leq n - 1\). Then, \(\diamond\) is a Minkowski valuation, homogeneous of degree 0 which satisfies RS.

**Example 4.20.** Let \(\diamond K = \text{Vol}(K)^{1/n} B_n\). It is a continuous \(o\)-symmetrization satisfying BM and RS. Further, it is translation invariant and homogeneous of degree one. It is clearly not a Minkowski valuation.

**Example 4.21.** A remarkable example is given by

\[
K \mapsto L + \text{Vol}(K)S
\]

where \(S\) is a centered segment and \(L\) is an \(o\)-symmetric \((n - 1)\)-dimensional convex body so that \(\dim(S + L) = n\). The operator ◦ is a continuous, translation invariant Minkowski valuation, and also an \(o\)-symmetrization which satisfies a Rogers-Shephard and a Brunn-Minkowski type inequality, since

\[
\text{Vol}(L + \text{Vol}(K)S) = \text{Vol}(K)V(L[n - 1], S).
\]

In a forthcoming work, we shall prove that a continuous, translation invariant Minkowski valuation, which is an \(o\)-symmetrization and satisfies a Rogers-Shephard and a Brunn-Minkowski type inequality is either of the above type or 1-homogeneous.

Finally we would like to observe that in [10] an example of a translation invariant, GL\((n)\)-covariant valuation which is not continuous is provided. If we slightly modify this example, we can obtain a translation invariant, GL\((n)\)-covariant valuation which is also \(o\)-symmetrization, 1-homogeneous and satisfies BM, but is not continuous. Let \(SK\) denote the sum of the segments of the boundary of \(K\), all centered at the origin. Then \(K \mapsto DK + SK\) has the stated properties.

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