Conjugate (nil) clean rings and Köthe’s problem

Jerzy Matczuk*
Institute of Mathematics, Warsaw University
Banacha 2, 02-097 Warsaw, Poland
E-mail: jmatczuk@mimuw.edu.pl

Abstract

Question 3 of [3] asks whether the matrix ring $M_n(R)$ is nil clean, for any nil clean ring $R$. It is shown that positive answer to this question is equivalent to positive solution for Köthe’s problem in the class of algebras over the field $\mathbb{F}_2$. Other equivalent problems are also discussed.

The classes of conjugate clean and conjugate nil clean rings, which lie strictly between uniquely (nil) clean and (nil) clean rings are introduced and investigated.

The notion of clean rings was introduced in 1977 by Nicholson in [9]. Thereafter such rings and their variations were intensively studied by many authors (cf. [12] and references within).

Recall that an element $a$ of a unital ring $R$ is clean if $a = e + u$, where $e$ is an idempotent and $u$ is a unit of $R$. When the above presentation is unique, $a$ is called uniquely clean. The ring $R$ is (uniquely) clean if every element of $R$ is such.

Diesl in [3] undertook to develop a general theory, based on idempotents and decomposition of elements, that would unify some of existing concepts related to cleanness and regularity. In this context a class of (uniquely) nil clean rings, appeared naturally, i.e. rings in which every element can be (uniquely) presented as $e + l$, for some idempotent $e$ and a nilpotent element $l$. It is easy to see that if $a$ is a nil clean element, then $-a$ is clean. Thus nil clean rings are clean. Earlier, uniquely nil clean rings were considered by Chen in [2].

In the paper, we introduce and investigate conjugate clean and conjugate nil clean rings. Those are (nil) clean rings in which idempotents appearing in decompositions of elements described above are unique up to conjugation, i.e. if $a = e + s = f + t$ are such decompositions, then the idempotents $e, f$ are conjugate in $R$. Clearly every uniquely (nil) clean ring is conjugate (nil) clean. In fact, it is not difficult to see that a ring $R$ is uniquely (nil) clean if and only if it is conjugate (nil) clean and all idempotents of $R$ are central.

*This research was supported by the Polish National Center of Science Grant No. DEC-2011/03/B/ST1/04893.
Thus the introduced classes of clean rings seem to be natural extensions of their "unique" counterparts. We offer constructions and characterizations of such rings. In particular, it will become clear that the introduced classes are different and the class of conjugate (nil) clean rings lies strictly between uniquely (nil) clean rings and (nil) clean rings. All this is presented in Section 2.

It is known that the matrix ring $M_n(R)$ over a clean ring $R$ is also clean (cf. Corollary 1 [4]). On the other hand, Wang and Chen [14] constructed a commutative clean ring $R$ such that not every element of $M_n(R)$ can be presented as a sum of an idempotent and a unit that commute with each other. In other words, they proved that a matrix ring over strongly clean ring does not have to be strongly clean.

Let $R$ be a nil clean ring. Then, by the above, $M_n(R)$ is a clean ring. Diesl posed a question (Question 3 [3]) whether $M_n(R)$ is in fact nil clean. This question was the initial motivation for our studies. It remains unsolved, nevertheless we show that positive answer to this question is equivalent to positive solution for Köthe’s problem in the class of algebras over the field $F_2 = \mathbb{Z}/2\mathbb{Z}$. In fact we present in Theorem 3.1 various conditions related to clean rings which are equivalent to Köthe’s problem. It appears that formally weaker statement ”$M_2(R)$ is nil clean for any uniquely nil clean $F_2$-algebra $R$” is, in fact, equivalent to Diesl ’s question. On the other hand, there exist conjugate nil clean rings $R$ such that the matrix ring $M_2(R)$ is not conjugate nil clean.

1 Preliminary results

For a ring $R$, $J(R)$ will denote the Jacobson radical of $R$, $U(R)$ will stand for the group of units of $R$.

The following proposition will be crucial for our considerations.

**Proposition 1.1** (Corollary 11 [5]). Let $e, f \in R$ be idempotents such that $e - f$ is a nilpotent element or $e - f \in J(R)$. Then $e$ and $f$ are conjugate in $R$, i.e. there exists $u \in U(R)$ such that $e = ufu^{-1}$.

Let us present an application of the above proposition. It will be needed later in the text but it is also of independent interest. In the following theorem $T$ is an over ring of a ring $R$ such that $T = R \oplus I$, for some ideal $I$ of $T$. The two-sided annihilator of $I$ in $R$ is define as $\text{ann}_R(I) = \{r \in R \mid rI = Ir = 0\}$. Recall that a ring $R$ is called abelian if all its idempotents are central.

**Theorem 1.2.** Suppose $T = R \oplus I$, where $I$ is an ideal of $T$ such that $J(I) = I$. The following conditions are equivalent:

1. $T$ is an abelian ring;
2. All idempotents of $R$ are central in $T$;

If one of the above equivalent conditions holds then:
(3) All idempotents of $T$ are trivial (i.e. all idempotents of $T$ belong to $R$) and $es = se$, for every idempotent $e$ of $R$ and $s \in I$;

Moreover, when $ann_R(I) = 0$, all the above statements are equivalent to:

(4) All idempotents of $R$ commute with elements of $I$.

Proof. The implications (1) \(\Rightarrow\) (2) and (3) \(\Rightarrow\) (4) are tautologies.

(2) \(\Rightarrow\) (1) and (3). Let $e = e_0 + s$ be an idempotent, where $e_0 \in R$ and $s \in I$. Then $e^2 = e_0^2 + w$, for some $w \in I$. Thus $e_0 \in R$ is an idempotent and the statement (2) implies that $e_0$ is central in $T$. Moreover, as $e - e_0 = s \in I \subseteq J(T)$, we can apply Proposition 1.1 to pick $u \in U(T)$ such that $e = ue_0u^{-1} = e_0 \in R$. This shows that $e = e_0$ belongs to $R$ and it is central in $T$, i.e. statements (1) and (3) hold.

Suppose now that $ann_R(I) = 0$ and the property (4) is satisfied. Let $e$ be an idempotent of $R$, $r \in R$ and $s \in I$. Then, making use of (4), we have $0 = e(rs)(1 - e) = er(1 - e)s$ and $0 = e(sr)(1 - e) = ser(1 - e)$. This shows that $er(1 - e)I = 0 = Ier(1 - e)$, i.e. $er(1 - e) \in ann_R(I) = 0$. Replacing the idempotent $e$ by $(1 - e)$ we also obtain $(1 - e)re = 0$, for any $r \in R$. Hence $e$ is central in $R$ and, by (4), $e$ is central in $T$ i.e. the statement (2) holds. This completes the proof of the theorem.

Notice that implications (1) $\iff$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) in the above theorem hold always without any additional assumptions. Clearly the equivalence of all the conditions do not hold in general. For example, it is easy to construct rings $T = R \oplus I$, such that $ann_R(I)$ contains a noncentral idempotent of $R$ but all idempotents of $R$ commute with elements of $I$ (taking $T = R$ and $I = 0$ we get a trivial example of this kind).

Let $\sigma$ be an endomorphism of a ring $R$ and $R[x; \sigma], R[[x; \sigma]]$ denote skew polynomial and skew power series rings over $R$, respectively.

Corollary 1.3. Let $T$ denote one of the rings $R[[x; \sigma]]$ or $R[x; \sigma]/(x^n)$, where $\sigma$ is an endomorphism of $R$ and $n \geq 2$. The following conditions are equivalent:

(1) $T$ is an abelian ring;

(2) All idempotents of $R$ are central and $\sigma(e) = e$, for every idempotent $e$ of $R$;

(3) All idempotents of $T$ are trivial and $\sigma(e) = e$, for every idempotent $e$ of $R$.

Proof. Notice that if $T = R[[x; \sigma]]$ then $I = Tx$ is the Jacobson radical ideal of $T$ and $T = R \oplus I$.

For $T = R[x; \sigma]/(x^n)$, let $\bar{x}$ denote the canonical image of $x$ in $T$. Then $I = T\bar{x}$ is a nilpotent ideal of $R$ and $T = R \oplus I$.

In both cases $ann_R(I) = 0$ (in fact the left annihilator of $I$ in $R$ is equal to zero). It is also standard to check that an element $a \in R$ commutes with elements of $I$ if and only if $a$ is central in $R$ and $\sigma(a) = a$. Now it is clear that the corollary is a consequence of Theorem 1.2.
We will say that \( a = e + t \) is a clean (nil clean) decomposition of an element \( a \) of \( R \) if \( e = e^2 \) and \( t \) is a unit (a nilpotent element) of \( R \).

**Definition 1.4.** An element \( a \in R \) is called

(i) conjugate clean if it is clean and for any two clean decompositions \( a = e + u = f + v \) of \( a \), the idempotents \( e, f \) are conjugate;

(ii) conjugate nil clean if it is nil clean and for any two nil clean decompositions \( a = e + l = f + m \) of \( a \), the idempotents \( e, f \) are conjugate.

Clearly every uniquely (nil) clean element is conjugate (nil) clean. Let us observe that:

**Remark 1.5.** (1) Proposition 1.1 implies that every idempotent of a nil clean ring \( R \) is conjugate nil clean.

(2) Let \( a \) be a conjugate clean element of \( R \). If either \( a \) or \( a - 1 \) is invertible then \( a \) is uniquely clean. In particular, all nilpotent elements and all units which are conjugate clean are, in fact, uniquely clean.

Let us recall that idempotents lift modulo an ideal \( I \) of \( R \) if, for any \( a \in R \) such that \( a^2 - a \in I \), there exists an idempotent \( e \in R \) such that \( e - a \in I \). If the idempotent \( e \) is uniquely determined by the element \( a \), then we say that idempotents lift uniquely modulo \( I \).

The following lemma is known (cf. [11]). We present its short proof for completeness.

**Lemma 1.6 (Lemma 17 [11]).** Let \( R \) be a clean ring. Then idempotents lift modulo every ideal \( I \) of \( R \).

**Proof.** Let \( a \in R \) be such that \( a^2 - a \in I \). By assumption \( a = e + u \), for some idempotent \( e \) and a unit \( u \) of \( R \). Then \( a - u(1 - e)u^{-1} = e + u(eu^{-1} + u - 1 = (a^2 - a)u^{-1} \in I \). This shows that \( a \) lifts to \( u(1 - e)u^{-1} \).

For an element \( a \in R \) the canonical image of \( a \) in the factor ring \( R/I \) will be denoted by \( \bar{a} \).

**Definition 1.7.** Let \( I \) be an ideal of a ring \( R \). We say that idempotents lift up to conjugation modulo \( I \) if:

(i) idempotents lift modulo \( I \);

(ii) if \( e, f \in R \) are idempotents such that \( \bar{e} = \bar{f} \), then \( e \) and \( f \) are conjugate in \( R \).

Clearly if idempotents lift uniquely modulo \( I \), then they also lift up to conjugation. It is known that if \( I \) is a nil ideal of \( R \), then idempotents lift modulo \( I \). The following lemma, which is a direct consequence of Proposition 1.1, says that in this case idempotents lift up to conjugation.

**Lemma 1.8.** Let \( I \) be an ideal of \( R \) contained in \( J(R) \) such that idempotents lift modulo \( I \). Then idempotents lift up to conjugation modulo \( I \).
Lemma 1.9. Let $I$ be an ideal of $R$ contained in $J(R)$. Then:

1. Let $a \in R$. If $\bar{a} = a + I \in R/I$ is invertible in $R/I$, then $a$ is invertible in $R$. In particular, $a + s$ is invertible, for any $s \in I$.

2. Suppose that idempotents lift modulo $I$. If $e, f$ are idempotents of $R$ such that $\bar{e}, \bar{f}$ are conjugate in $R/I$, then $e, f$ are conjugate in $R$.

Proof. (1) Suppose $\bar{a} \in R/I$ is invertible in $R/I$. Then there exist $b \in R$ and $s, t \in I$ such that $ab = 1 + s, ba = 1 + t$. Since $I \subseteq J(R)$, the elements $1 + s, 1 + t$ are invertible. This yields the thesis.

(2) Let $\bar{u} \in U(R/I)$ be such that $\bar{e} = \bar{u} \bar{f} \bar{u}^{-1}$. Then, by (1), $u$ is invertible in $R$ and $e = uf \bar{u}^{-1} + s$, for some $s \in I \subseteq J(R)$. Now, Proposition 1.1 implies that $e$ and $uf \bar{u}^{-1}$ are conjugate in $R$. Thus $e$ and $f$ are also conjugate.

For any ring $R$, $M_n(R)$ and $U_T n(R)$ will denote the ring of $n$ by $n$ matrices over $R$ and its subring consisting of all upper triangular matrices, respectively. $\mathbb{F}_2$ will stand for the field $\mathbb{Z}/2\mathbb{Z}$.

2 Conjugate (nil) clean rings

We begin with the following definition:

Definition 2.1. A ring $R$ is conjugate (nil) clean if every element of $R$ is conjugate (nil) clean.

Clearly every uniquely (nil) clean ring is conjugate (nil) clean.

It is known (see Lemma 5.5 of [3] and Lemma 4 [11], respectively) that idempotents in uniquely (nil) clean rings are central. Therefore we have:

Remark 2.2. The following conditions are equivalent:

1. $R$ is uniquely (nil) clean ring;
2. $R$ is conjugate (nil) clean, abelian ring.

The above suggests that conjugate (nil) clean rings form a natural extension of the class of uniquely (nil) clean rings.

Theorem 3 of [7] states that a matrix ring $M_n(D)$ over a division ring $D$ is nil clean if and only if $D = \mathbb{F}_2$ (in the case $D$ is a field, this result was obtained earlier in [1]). With the help of this theorem, we get the following characterization:

Theorem 2.3. Let $D$ be a division ring. Then:

1. $M_n(D)$ is conjugate nil clean if and only if $D = \mathbb{F}_2$ and $n \leq 2$,
2. $M_n(D)$ is conjugate clean if and only if $D = \mathbb{F}_2$ and $n = 1$. 
Proof. Clearly a division ring $D$ is nil clean (conjugate clean) if and only if $D = \mathbb{F}_2$. Therefore we can restrict our attention to the case when $n \geq 2$.

(1) Suppose that the ring $M_n(D)$ is conjugate nil clean. Then it is nil clean and Theorem 3 \cite{7} shows that $D = \mathbb{F}_2$. The equation
\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} + \begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0
\end{pmatrix}
\]
implies that $M_n(\mathbb{F}_2)$ is not conjugate nil clean, for any $n \geq 3$. Thus $D = \mathbb{F}_2$ and $n \leq 2$, as required.

By the same theorem, $R = M_2(\mathbb{F}_2)$ is nil clean. Let $a = e + l = f + m$ be two nil clean decompositions of $a \in R$. Then $tr(a) = tr(e) = tr(f)$, where $tr(a)$ denotes the trace of the matrix $a$. If $tr(a) = 1$, then both $e$ and $f$ are conjugate to $\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}$.

Suppose $tr(a) = 0$. Then either $e = 0$ and $a$ is nilpotent or $e = 1$ and $a$ is a unit. This implies that $e = f$, when $tr(a) = 0$. Therefore $M_2(\mathbb{F}_2)$ is conjugate nil clean.

This completes the proof of (1).

(2) The equation
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
-1 & 1 \\
1 & 0
\end{pmatrix}
\]
implies that, for any ring $R$ and $n \geq 2$, the matrix ring $M_n(R)$ is not conjugate clean. This and the remark from the beginning of the proof imply (2). \hfill \Box

The following corollary is a direct consequence of the above theorem and its proof. It shows, in particular, that a conjugate nil clean ring does not have to be conjugate clean. It is worth to mention that every uniquely nil clean ring is uniquely clean (cf. Theorem 5.9 \cite{3}).

**Corollary 2.4.** (1) The ring $M_2(\mathbb{F}_2)$ is conjugate nil clean and it is neither conjugate clean nor uniquely nil clean;

(2) Let $R$ be a ring of characteristic 2 and $n \geq 3$. Then $M_n(R)$ is not conjugate nil clean;

(3) For any ring $R$ and $n \geq 2$, $M_n(R)$ is never conjugate clean.

We will see in Proposition 2.15 that the assumption about the characteristic of $R$ in the above corollary can be removed.

Let us record the following property, its easy proof is left as an exercise.

**Proposition 2.5.** The product $R_1 \times \ldots \times R_n$ is conjugate (nil) clean if and only if all rings $R_i$, $1 \leq i \leq n$, are such.

**Proposition 2.6.** Let $R$ be a Boolean ring. Then:

(1) $M_n(R)$ is nil clean, for any $n \geq 1$;
(2) $M_n(R)$ is conjugate nil clean if and only if $n \leq 2$.

Proof. The statement (1) is exactly Corollary 6 of [1].

For proving (2) we will extend arguments used in [1]. By (1), the ring $M_2(R)$ is nil clean. We claim that it is conjugate nil clean. Let $a = e + l = f + m \in M_2(R)$ be nil clean decompositions of $a$. Let $S$ be a the subring of $R$ generated by all entries of matrices appearing in the above equations. Then $S$ is a finite Boolean ring, so it is isomorphic to a finite direct product of copies of $\mathbb{F}_2$. Hence $M_n(S)$ is isomorphic to a finite product of copies of $M_2(\mathbb{F}_2)$ and Theorem 2.3 and Proposition 2.5 yield that $M_2(S)$ is a conjugate nil clean ring. Therefore $e$ and $f$ are conjugate in $M_2(R)$.

The reverse implication is given by Corollary 2.4(2), as Boolean rings are of characteristic 2.

Proposition 2.7. Let $I$ be an ideal of $R$ contained in $J(R)$ such that idempotents lift modulo $I$. Then $R$ is conjugate clean if and only if $R/I$ is conjugate clean.

Proof. Suppose $R/I$ is conjugate clean. Let $a \in R$ and $\bar{a} = \bar{e} + \bar{v}$ be clean decomposition of $a$ in $R/I$. Since idempotents lift modulo $I$, Lemma 1.9 implies that we may assume that $e$ is an idempotent and $a = e + v + s$, where $v$ is a unit of $R$ and $s \in I \subseteq J(R)$. Then $v + s \in U(R)$ by Lemma 1.9 again, i.e. $a$ is a clean element of $R$. If $a = e + s = f + t$ are two clean presentations of $a$ then, by assumption $\bar{e}, \bar{f}$ are conjugate in $R/I$ and Lemma 1.9 implies that $e, f$ are conjugate in $R$.

Suppose now that $R$ is conjugate clean. Then clearly $R/I$ is clean. By assumptions imposed on $I$, both idempotents and units lift modulo $I$. Using this, it is easy to see that the ring $R/I$ is conjugate clean.

Suppose that $T$ is an over ring of a ring $R$ such that $T = R \oplus I$, for some ideal $I$ of $T$. In this situation it is clear that idempotents lift modulo $I$. The results, which were obtained up to now, give the following corollaries.

Corollary 2.8. Suppose $T = R \oplus I$, where $I$ is an ideal of $T$ such that $J(I) = I$. Then:

(1) $T$ is conjugate clean if and only if the ring $R$ is conjugate clean;

(2) Suppose $ann_R(I) = 0$. Then $T$ is uniquely clean if and only if the ring $R$ is uniquely clean and all its idempotents commute with elements of $I$.

Proof. The first statement is a direct consequence of Proposition 2.7. The second one follows from Theorem 1.2, Remark 2.2 and (1).
(2) Let $\sigma$ be an endomorphism of a ring $R$ and $n \geq 2$. If $T$ denotes one of the rings $R[[x;\sigma]]$, $R[x;\sigma]/(x^n)$, then:

(i) $T$ is conjugate clean if and only if $R$ is conjugate clean;

(ii) $T$ is uniquely clean if and only if $R$ is uniquely clean and $\sigma(e) = e$, for every idempotent $e$ of $R$.

Proof. Corollary 2.8 and Proposition 2.5 give the first statement. The statement (2) is a consequence of Corollaries 2.8 and 1.3.

Using the above, we can easily construct rings which are conjugate clean but are not uniquely clean.

Example 2.10. Let $R$ be a uniquely clean ring. The following rings are conjugate clean but they are not uniquely clean: $UT_n(R)$ and $R[[x;\sigma]]$, $R[x;\sigma]/(x^n)$, where $n \geq 2$ and $\sigma$ is an endomorphism of $R$ such that there exists an idempotent $e$ of $R$ with $\sigma(e) \neq e$.

The following theorem offers characterizations of conjugate clean rings. The statement (3) gives a way of constructing new conjugate clean rings from a given conjugate clean ring. Clearly this construction generalizes the one from Corollary 2.8.

Theorem 2.11. The following conditions are equivalent:

(1) $R$ is conjugate clean;

(2) $R/J(R)$ is conjugate clean and idempotents lift modulo $J(R)$.

(3) There exist a conjugate clean subring $A$ of $R$ and a Jacobson radical ideal $I$ of $R$ such that:

(i) $R = A + I$;

(ii) $U(A) = U(R) \cap A$;

(iii) every idempotent of $R$ is of the form $e + x$, for some $e = e^2 \in A$ and $x \in I$.

Proof. The equivalence (1) $\iff$ (2) is a direct consequence of Proposition 2.7 and Lemma 1.6.

Taking $A = R$ and $I = 0$, one gets (1) $\implies$ (3).

(3) $\implies$ (1) Let $A$ and $I$ be as in (3) and $r \in R$. Then $r = a + x$, for some $a \in A$ and $x \in I$. Since $A$ is clean, there exist $e = e^2 \in A$ and $u \in U(A) \subseteq U(R)$ such that $a = e + u$. By assumption $I \subseteq J(R)$, so $u + x$ is invertible in $R$ and $r = e + (u + x)$ is a clean decomposition of $r$. This shows that $R$ is a clean ring.

Let $r = e + u$ be a clean decomposition of $r \in R$. By (iii), $e = e_0 + x$, for an idempotent $e_0 \in A$ and $x \in I \subseteq J(R)$. Proposition 1.1 shows that $e$ and $e_0$ are conjugate. Moreover $r = e_0 + (u + x)$ is a clean decomposition of $r$. Let us consider two clean decompositions of $r$, say $r = e + u = f + v$. By the above, up to conjugation of idempotents, we may assume that $e, f \in A$, $u, v \in U(R)$. Let $u_0 \in A$ and $x \in I \subseteq J(R)$ be such that $u = u_0 + x$. Then $u - x = u_0 \in U(R) \cap A = U(A)$. Similarly, there exist $v_0 \in U(A)$ and $y \in I$ such that $v = v_0 + y$. Then $r - x = f + v_0 + y - x = e + u_0 \in A$. In particular, we get
Notice that if an element $a \in R$ can be written in a form $a = e + t$, where $e = e^2$ and $t \in J(R)$, then $a = (1 - e) + (2e - 1) + t$. Since $t \in J(R)$ and $2e - 1$ is a unit, the above equation shows that $a$ is a clean element. Therefore rings in which every element can be presented as a sum of an idempotent and an element from $J(R)$ form a natural proper subclass of clean rings. We call such rings $J$-clean rings.

Making use of Lemma 1.6 one can easily check that $R$ is $J$-clean if and only if $R/J(R)$ is a Boolean ring and idempotents lift modulo $J(R)$. Uniquely clean rings were characterized in [11], as rings $R$ such that $R/J(R)$ is Boolean and idempotents lift uniquely modulo $J(R)$. Therefore, the class of uniquely clean rings is contained in the class of $J$-clean rings. The inclusion is strict, since uniquely clean rings are abelian. Notice that, by Theorem 2.11 we get:

**Corollary 2.12.** Every $J$-clean ring is conjugate clean.

The remaining part of this section is focused on properties of conjugate nil clean rings.

Let $I$ be a nil ideal of a ring $R$. Then $h \in R$ is nilpotent if and only if $\bar{h} \in R/I$ is such. In particular, if $h \in R$ is nilpotent, then all elements from the coset $h + I$ are also nilpotent. Using this observation and arguments similar to that of Proposition 2.7 and Theorem 2.11 we can prove the following:

**Proposition 2.13.** Let $I$ be a nil ideal of $R$. The following conditions are equivalent:

1. $R$ is conjugate nil clean;
2. $R/I$ is conjugate nil clean;
3. There exists a conjugate nil clean subring $A$ of $R$ such that:
   i. $R = A + I$;
   ii. every idempotent of $R$ is of the form $e + x$, where $e = e^2 \in A$ and $x \in I$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $R$ is conjugate nil clean. Then clearly $R/I$ is nil clean. Let $\bar{a} = \bar{e} + \bar{l} = \bar{f} + \bar{m}$ be nil clean decompositions of $\bar{a}$. Since $I$ is nil, idempotents lift modulo $I$, so we may assume that $e, f$ are idempotents of $R$. Clearly elements $l, m$ are nilpotent as $\bar{l}, \bar{m}$ are such. Hence, in $R$, we can write $a = e + l + s = f + m + t$ for some suitable $s, t \in I$. Then the elements $l + s$ and $m + t$ are also nilpotent and the fact that $R$ is conjugate clean yields that $e$ and $f$ are conjugate in $R$. Thus $\bar{e}$ and $\bar{f}$ are conjugate in $R/I$, i.e. $R/I$ is conjugate clean.

(2) $\Rightarrow$ (1) Suppose $R/I$ is conjugate nil clean. Let $a \in R$. Then, by assumption, we have nil clean decomposition $\bar{a} = \bar{e} + \bar{l}$ in $R/I$. Since idempotents lift modulo $I$, we may assume that $e$ is an idempotent and $a = e + l + s$, where $l + s$ is nilpotent as $l$ is such and $s \in I$. This shows that $a$ is a nil clean element of $R$. If $a = e + l = f + m$ are two nil clean
presentations of \( a \) then, by assumption, \( \bar{e}, \bar{f} \) are conjugate in \( R/I \) and Lemma 1.9 implies that \( e, f \) are conjugate in \( R \).

(3) \Rightarrow (1) Let \( R = A + I \), where \( A \) is as in (3). Then every \( r \in R \) can be presented in a form \( r = a + x = e + l + x \), where \( a \in A, x \in I \) and \( a = e + l \) is a nil clean decomposition of \( a \) in \( A \). Notice that \( l + x \) is a nilpotent element as \( l \) is nilpotent and \( I \) is a nil ideal. This proves that \( R \) is nil clean.

Let \( r = e + l \) be a nil clean decomposition of \( r \in R \). Using (ii), we may pick an idempotent \( e_0 \in A \) and \( x \in I \) such that \( e = e_0 + x \). Proposition 1.1 shows that \( e \) and \( e_0 \) are conjugate in \( R \). Moreover \( r = e_0 + (l + x) \) is a nil clean decomposition of \( r \). This means that, considering nil clean decompositions \( e + l \) of \( r \in R \) up to conjugation of idempotents, we may assume that \( e \in A \).

Let \( r = e + l = f + m \) be two clean decompositions of \( r \in R \). By the above, we may assume that \( e, f \in A \). Let \( l_0 \in A \) and \( x \in I \) be such that \( l = l_0 + x \). Then \( l_0 = l - x \) is nilpotent, as \( l \) is nilpotent and \( I \) is nil. Similarly, there exist a nilpotent element \( m_0 \in A \) and \( y \in I \) such that \( m = m_0 + y \). Then \( r - x = f + m_0 + y - x = e + l_0 \in A \). In particular, we get \( y - x \in A \cap I \) and \( m_0 + y - x \in A \) is nilpotent. Therefore, as \( A \) is conjugate nil clean and \( f + (m_0 + y - x) = e + l_0 \) are two clean decompositions of \( r - x \) in \( A \), \( e \) and \( f \) are conjugate in \( A \) so in \( R \) as well. This shows that \( R \) is conjugate nil clean.

Taking \( A = R \) one gets (1) \Rightarrow (3). \( \square \)

The first application of the above proposition requires the following observation.

**Lemma 2.14.** Let \( R \) be a nil clean ring. Then \( 2R \) is a nilpotent ideal of \( R \). In particular \( R/2R \) has a structure of \( \mathbb{F}_2 \)-algebra.

**Proof.** Let \( 2 = e + l \) be nil clean decomposition of \( 2 \). Then \( e + el = 2e = e + le \). Hence \( le = e = el \) and, as \( l \) is nilpotent, we obtain \( e = 0 \), i.e. \( 2 = l \) is nilpotent. Thus \( 2R \) is a nilpotent ideal of \( R \) and the ring \( \bar{R} = R/2R \) has a structure of an \( \mathbb{F}_2 \)-algebra, as required. \( \square \)

**Proposition 2.15.** For any ring \( R \) and \( n \geq 3 \), the ring \( M_n(R) \) is not conjugate nil clean.

**Proof.** Let \( n \geq 3 \). Assume that \( R \) is a ring such that \( M_n(R) \) is conjugate nil clean. By Lemma 2.14, \( I = 2M_n(R) = M_n(2R) \) is a nilpotent ideal of \( M_n(R) \). Thus, Proposition 2.13 implies that \( M_n(R)/I \simeq M_n(R/2R) \) is conjugate nil clean. Corollary 2.4(2) shows that this is impossible, as \( R/2R \) is of characteristic 2. Thus such \( R \) can not exist. \( \square \)

It is known (Proposition 3.16 [3]) that if \( R \) is nil clean then \( J(R) \) is nil. In particular \( R \) is nil clean if and only if \( R/J(R) \) is nil clean and \( J(R) \) is nil. Proposition 2.13 gives the following characterization of conjugate nil clean rings.

**Corollary 2.16.** For a ring \( R \), the following conditions are equivalent:

(1) \( R \) is conjugate nil clean;

(2) \( J(R) \) is nil and \( R/J(R) \) is conjugate nil clean;
(3) $J(R)$ is nil and there exist a conjugate nil clean subring $A$ of $R$ such that:

(i) $R = A + J(R)$;

(ii) every idempotent of $R$ is of the form $e + x$, for some $e = e^2 \in A$ and $x \in J(R)$.

Proposition 3.18 states that if $R$ is a nil clean, abelian ring, then $J(R)$ contains all nilpotent elements of $R$. Using Corollary 2.16 and Remark 2.2 one can easily recover (cf. Theorem 5.9) the following characterization of uniquely nil clean rings:

**Corollary 2.17.** For a ring $R$, the following statements are equivalent:

(1) $R$ is uniquely nil clean;

(2) $R/J(R)$ is Boolean, $J(R)$ is nil and idempotents lift uniquely modulo $J(R)$.

We will use the above characterization in the proof of Theorem 3.1.

Applying Proposition 2.13, Remark 2.2 and Theorem 1.2 we obtain the following:

**Corollary 2.18.** Suppose $T = R \oplus I$, where $I$ is a nil ideal of $T$. Then:

(1) $T$ is conjugate nil clean if and only if the ring $R$ is conjugate nil clean;

(2) Suppose $\operatorname{ann}_R(I) = 0$. Then $T$ is uniquely nil clean if and only if the ring $R$ is uniquely nil clean and all idempotents of $R$ commute with elements of $I$.

With the help of the above corollary, similarly as in Corollary 2.9 we get:

**Corollary 2.19.** (1) Let $U_T(R)$ denote the ring of all $n$ by $n$ upper triangular matrices over $R$. Then:

(i) $U_T(R)$ is conjugate nil clean if and only if $R$ is such;

(ii) $U_T(R)$ is not uniquely nil clean when $n \geq 2$.

(2) Let $\sigma$ be an endomorphism of a ring $R$ and $n \geq 2$. Then:

(i) $R[x;\sigma]/(x^n)$ is conjugate nil clean if and only if $R$ is conjugate nil clean;

(ii) $R[x;\sigma]/(x^n)$ is uniquely nil clean if and only if $R$ is uniquely clean and $\sigma(e) = e$, for every idempotent $e$ of $R$;

Let us notice that Corollary 2.17 implies that if $R$ is a uniquely nil clean ring, then the set of all nilpotent elements $N(R)$ of $R$ is equal to $J(R)$. In particular, $N(R)$ is an ideal of $R$ in this case.

**Proposition 2.20.** Let $R$ be a ring such that the set $N(R)$ is an ideal of $R$. The following conditions are equivalent:

(1) $R$ is nil clean;

(2) $R$ is conjugate nil clean;

(3) $R/J(R)$ is a Boolean ring and $J(R)$ is nil.
If one of the above equivalent conditions holds, then \( R \) is conjugate clean.

**Proof.** (1) ⇒ (2) Suppose \( R \) is nil clean. Let \( a = e + l = f + m \) be two nil clean decomposition of \( a \in R \). Then \( e - f = m - l \in N(R) \). In particular \( e - f \) is nilpotent and Proposition 1.1 shows that the idempotents \( e \) and \( f \) are conjugate, i.e. \( R \) is conjugate nil clean.

(2) ⇒ (3) Suppose \( R \) conjugate nil clean. Then, by Corollary 2.16 \( J(R) \) is nil and \( R/J(R) \) is conjugate nil clean. Since \( N(R) \) is an ideal, \( N(R) = J(R) \). This means that \( R/J(R) \) is a reduced nil clean ring, so it is a Boolean ring.

The implication (3) ⇒ (1) is a direct consequence of the fact that idempotents lift modulo nil ideals. The last statement is a consequence of (3) and Theorem 2.11.

In the context of the above proposition, let us recall that the ring \( M_2(\mathbb{F}_2) \) is conjugate nil clean ring but it is not conjugate clean. Thus the proposition does not hold without the assumption made on the set \( N(R) \). Notice also that the power series ring \( \mathbb{F}_2[[x]] \) is a uniquely clean domain and it is not nil clean.

When \( R \) is a commutative ring, then \( N(R) \) is an ideal of \( R \) and \( R \) is conjugate nil clean if and only if \( R \) is uniquely nil clean. Therefore Proposition 2.20 gives the following corollary:

**Corollary 2.21.** For a commutative ring \( R \), the following conditions are equivalent:

(1) \( R \) is nil clean;

(2) \( R \) is uniquely nil clean;

(3) \( R/J(R) \) is a Boolean ring and \( J(R) \) is nil.

The equivalence of (1) and (3) in the above corollary is exactly Corollary 3.20 of [3].

### 3 Nil clean rings and Köthe’s problem

Köthe’s problem was formulated in 1930, it asks whether a ring \( R \) has no nonzero nil one-sided ideals provided \( R \) has no nonzero nil ideals. It is known (see Theorem 6, [8]) that the problem has a positive solution if and only if it has positive solution for algebras over fields. There are many other problems in ring theory which are equivalent or related to it (see [13]). In the theorem below we indicate new ones which are associated with nil clean rings.

Diesl in [3] formulated a few questions on nil clean elements and rings. In particular, he posed a question (Question 3 [3]) whether a matrix ring \( M_n(R) \) over a nil clean ring \( R \) has to be nil clean. We show that positive answer to the above Diesl’s question is equivalent to positive solution for Köthe’s problem in the class of algebras over the field \( \mathbb{F}_2 \).

**Theorem 3.1.** The following conditions are equivalent:

(1) If \( R \) is a nil clean ring, then \( M_n(R) \) is nil clean;
(2) If $R$ is a uniquely nil clean ring, then $M_n(R)$ is nil clean;
(3) If $R$ is a uniquely nil clean ring, then $M_2(R)$ is nil clean;
(4) If $R$ is a uniquely nil clean ring, then $M_2(R)$ is conjugate nil clean;
(5) If $A$ is a nil algebra over $\mathbb{F}_2$, then $M_n(A)$ is nil;
(6) Köthe’s problem has positive solution in the class of $\mathbb{F}_2$-algebras.

Proof. The implication (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is a tautology.

(3) $\Rightarrow$ (4) Let $R$ be a uniquely clean ring. Then, by Corollary 2.17 $\hat{R} = R/J(R)$ is a Boolean ring and Proposition 2.6(2) implies that the matrix ring $M_2(\hat{R})$ is conjugate nil clean.

By assumption, the ring $T = M_2(R)$ is nil clean. In particular, by Proposition 3.16 $[3]$, $J(T)$ is nil. Moreover $T/J(T) = M_2(R)/J(M_2(R)) \simeq M_2(\hat{R})$ is conjugate nil clean. Now, Corollary 2.16 shows that $T = M_2(R)$ is conjugate nil clean, i.e. (4) holds.

(4) $\Rightarrow$ (5) Let $A$ be a nil algebra over the field $\mathbb{F}_2$ and $A^*$ denote the $\mathbb{F}_2$-algebra with unity adjoined with the help of $\mathbb{F}_2$ to $A$. Then $J(A^*) = A$ and $A^*/J(A^*) = \mathbb{F}_2$. Since $A^* = A \cup (1 + A)$, the only idempotents of $A^*$ are 0,1. Thus, by Corollary 2.17, $A^*$ is uniquely nil clean. Therefore, by (4), $M_2(A^*)$ is conjugate nil clean and Corollary 2.16 implies that $J(M_2(A^*)) = M_2(J(A^*)) = M_2(A)$ is nil. Therefore, we have shown that for any nil algebra $A$, the $2 \times 2$ matrix algebra $M_2(A)$ is also nil. It is known and easy that in this case $M_n(A)$ is nil for any $n \geq 2$.

The equivalence of (5) and (6) is known (cf. [3]).

(5) $\Rightarrow$ (1) Let $R$ be uniquely nil clean ring. Thus, by Corollary 2.17, $J(R)$ is nil, $R/J(R)$ is Boolean.

Let $I = 2R$ and $\hat{R} = R/2R$. Then, by Lemma 2.14 $I$ is a nilpotent ideal of $R$ and $\hat{R}$ has a structure of $\mathbb{F}_2$-algebra. Moreover $J(\hat{R}) = J(R)/I$ is nil and $B = \hat{R}/J(\hat{R}) \simeq R/J$ is Boolean. Then, using the statement (5) and Proposition 2.6(1), we obtain that $J = M_n(J(\hat{R}))$ is nil and $M_n(\hat{R})/J = M_n(B)$ is nil clean, respectively. Hence, by Corollary 3.17 $[3]$ we obtain that $M_n(\hat{R})$ is nil clean. Then also $M_n(R)$ is nil clean as $M_n(\hat{R}) \simeq M_n(R)/M_n(I)$ and $M_n(I)$ is a nilpotent ideal of $M_n(R)$.

Let us notice that in the proof of the implication (4) $\Rightarrow$ (5), the property (4) was used only for $\mathbb{F}_2$-algebras. This means that in Theorem 3.16 we can add new equivalent statements replacing rings by $\mathbb{F}_2$-algebras. In particular, Diesl’s question is equivalent to the question whether $M_2(R)$ is conjugate clean for any uniquely clean $\mathbb{F}_2$-algebra. In this context, let us notice that the ring $R = M_2(\mathbb{F}_2)$ is conjugate nil clean however, by Corollary 2.3 $M_2(R) = M_2(\mathbb{F}_2)$ is not conjugate nil clean.

Let us mentioned at the end that relations between Köthe’s problem and properties of clean elements were investigated in [6]. In particular, it was proved that Köthe’s problem has positive solution if and only if the set of clean elements of the polynomial ring $R[x]$ forms a subring, for any clean ring $R$ such that $N(R)$ is an ideal of $R$ (cf. Theorem 2.15 [6]).
Acknowledgement. I would like to thank André Leroy and Jan Okniński for helpful discussions.

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