A solvable class of renewal processes and its applications

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Abstract

When the distribution of the inter-arrival times of a renewal process is a mixture of geometric laws, we prove that the renewal function of the process is given by the moments of a probability measure which is explicitly related to the mixture distribution. We then observe that this class of renewal processes also provides a solvable family of random polymers. Namely, we obtain an exact representation of the partition function of polymers pinned at sites of the aforementioned renewal processes. In the particular case where the mixture measure is a generalized Arcsine law, the computations can be explicitly handled.

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1 Renewal theory for mixtures of geometric laws

Let $K$ be a positive measure on $\mathbb{N} = \{1, \ldots, \}$ with total mass $\Sigma_K = \sum_{n \geq 1} K(n) \leq 1$. We extend $K$ to a probability measure on $\mathbb{N} \cup \{\infty\}$ by setting $K(\{\infty\}) = 1 - \sum_{n \geq 1} K(n)$. Let $\eta = (\eta_n)_{n \geq 1}$ be an i.i.d. sequence of random variables with law $K$. We consider $\eta$ as inter-arrival times of a renewal process and we define the random variables $\tau_0 = 0$ and for all $n \geq 1$, $\tau_n = \sum_{1 \leq i \leq n} \eta_i$.

Finally, we define the renewal process associated to $K$ as the random set $\tau := \{\tau_i; i \geq 0\}$. We will denote by $P$ the law of $\tau$.

Before stating our main result, let us introduce the Stieltjes transform of a positive measure, which turns out to be the key notion in this framework. Let $C_+ := \{z \in \mathbb{C}, \Im z > 0\}$. Let $\mu$ be a positive measure on $\mathbb{R}$. The Stieltjes transform of $\mu$ is the analytic function $s_\mu$, defined from $C_+$ to $C_+$ by:

$$ s_\mu(z) = \int R \frac{d\mu(x)}{x - z}. $$

Theorem 1. Let $\mu$ be a positive measure on $[0, 1]$ and suppose that, for all $n \geq 1$,

$$ K(n) = \int_0^1 (1 - x)^{n-1} x d\mu(x). $$

Then, there exists a probability measure $\nu$ on $[0, 1]$ such that:

$$ s_\nu(z) s_\mu(1 - z) = \frac{1}{z(1 - z)}. $$

Moreover, for all $N \geq 0$:

$$ P(N \in \tau) = \int_0^1 x^N d\nu(x). $$

Remark 1. The mapping from $\mu$ to $\nu$ is an involution. The family of generalized Arcsine laws with parameters $(1 - v, v)$, $v \in (0, 1)$, defined by the densities

$$ \frac{\sin(\pi v)}{\pi} x^{-v}(1 - x)^{v-1} 1_{x \in [0,1]} dx, $$

are a particular instance of the stated results. In this case, the computations can be explicitly handled.

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are fixed points of this involution. It can be easily checked from the expression of their Stieltjes transforms, which is equal to

\[ \frac{1}{1 - z} \left( \frac{z}{1 - z} \right)^{\nu}. \]

We conjecture that these distributions are actually the only fixed points.

Proof. Let \( G(z) = \sum_{N \geq 0} P(N \in \tau) z^N \) be the generating series associated to the sequence \((P(N \in \tau))_{N \geq 0}\). Then,

\[
G(z) = 1 + E \left[ \sum_{k \geq 1} z^{n_1 + \cdots + n_k} \right] = \frac{1}{1 - E[z^{\tau_1}]}.
\]

Therefore, using (1), we deduce that:

\[ -\frac{1}{z} G \left( \frac{1}{z} \right) = \frac{1}{1 - \int_0^1 \frac{xx}{1 - z - x} d\mu(x)}. \] (5)

The above equality holds for all complex \( z \in \mathbb{C} \) such that \( 1/z \) is inside the disk of convergence of \( G \). It extends analytically to the whole complex upper half-plane \( \mathbb{C}_+ \) thanks to the right-hand side expression. Now, notice that for all \( x \in (0, 1) \), the homographic function \( z \mapsto \frac{zx}{1-z} \) preserves \( \mathbb{C}_+ \) since it is of determinant \( x(1-x) > 0 \). Therefore, the function \( z \mapsto -\frac{1}{z} G \left( \frac{1}{z} \right) \) preserves \( \mathbb{C}_+ \) and is a Nevanlinna’s function. Moreover, \( -\frac{1}{z} G \left( \frac{1}{z} \right) \sim -\frac{1}{z} \) as \( |z| \to +\infty \). Therefore, using the characterization of Nevanlinna functions which are Stieltjes transforms of probability measures \([1]\) (page 93), there exists a probability measure \( \nu \) such that

\[ -\frac{1}{z} G \left( \frac{1}{z} \right) = \int_{\mathbb{R}} \frac{d\nu(x)}{x-z}. \] (6)

Hence using (5) and (6), we get that

\[ \forall z \in \mathbb{C}_+, \quad \left( \int_{\mathbb{R}} \frac{d\nu(x)}{x-z} \right)^{-1} = -z - \int_0^1 \frac{zx}{1-z-x} d\mu(x), \] (7)

which implies Equation (2). Finally, let us notice that since the support of \( \mu \), \( \text{Supp}(\mu) \), is included in \([0, 1]\), its Stieltjes transform \( s_\mu \) is analytic on \( \mathbb{R} \setminus (0, 1) \). In turn, by Equation (2), the Stieltjes transform of \( \nu \) is analytic on \( \mathbb{R} \setminus (0, 1) \), which implies that \( \text{Supp}(\nu) \subset [0, 1] \).

Let us mention that Nagaev [6] obtained a similar moment representation in the particular case where \( \mu \) admits a continuous density with respect to the Lebesgue measure on \([0, 1]\). Let us also mention that his proof involves lengthy analytical computations which do not lead to a clear relation between \( \mu \) and \( \nu \) as in (2).

Let us comment on assumption (1). During the proof of Theorem 1, we crucially rely on it in order to prove that the function

\[ z \mapsto -z E_P \left[ \left( \frac{1}{z} \right)^{\tau_1} \right] \]

preserves the upper half-plane. In the generic case, this property is not satisfied. However, let us mention a slight extension of Theorem 1 which can be proved in the same way.

Remark 2. If \( \mu \) is a measure on \((-1, 1)\) such that \( \mu \) is positive on \((0, 1)\) and negative on \((-1, 0)\), a similar result still holds when \( K(n) = \int_{-1}^1 x^n d\mu(x) \). Namely, there exists a probability measure \( \nu_\beta \) such that

\[ \forall z \in \mathbb{C}_+, \quad \left( \int_{\mathbb{R}} \frac{d\nu_\beta(x)}{x-z} \right)^{-1} = -z - \int_{-1}^1 \frac{zx}{x-z} d\mu(x). \]

And again, for all \( N \geq 0 \), \( P(N \in \tau) = \int_{\mathbb{R}} x^N d\nu_\beta(x) \).
Notice that Theorem 2 presents the advantage of having a direct probabilistic interpretation. Indeed, under its setting, $K$ is a mixture of geometric laws.

Let us finally mention that Theorem 1 allows to easily recover the classical renewal Theorem in our setting.

**Proposition 1** (Basic renewal theorem). Under the setting of Theorem 1,

\[ P(N \in \tau) \xrightarrow{N \to +\infty} \frac{1}{m_K}, \]

where $m_K := \sum_{n \geq 1} nK(n)$.

**Proof.** From Equation (3) and the dominated convergence Theorem,

\[ P(N \in \tau) \xrightarrow{N \to +\infty} \nu(\{1\}). \]

The quantity $\nu(\{1\})$ is given by the residue at $z = 1$ of $s_\nu(z)$. By Equation (2), this is equal to

\[ \frac{1}{s_\mu(0)} = \left( \int_0^1 \frac{d\mu(x)}{x} \right)^{-1} = \left( \int_0^1 \sum_{n \geq 1} n(1-x)^{n-1} x d\mu(x) \right)^{-1} = \frac{1}{m_K}. \]

\[ \square \]

2 Application to polymer pinned on a defect line

2.1 Definition of the model

In this section we introduce a model of polymer and present the main results concerning its study. We adopt the notations of [5] and refer the reader to this book for every details. Let $\beta \in \mathbb{R}$ and $N \geq 1$. The polymer model associated to $K$ with parameter $\beta$ is defined by the following probability measure $P_{N,\beta}$ on subsets of $\{0, \ldots, N\}$ whose density with respect to $P$ is:

\[ \frac{dP_{N,\beta}}{dP}(\tau) := \frac{1}{Z_{N,\beta}} \exp(\beta N(\tau)) \mathbf{1}_{N \in \tau}, \]

where $N(\tau) = |\{1, \ldots, N\} \cap \tau|$ and

\[ Z_{N,\beta} := E_P[\exp(\beta N(\tau)) \mathbf{1}_{N \in \tau}]. \]

The renormalization constant $Z_{N,\beta}$ is called the partition function and captures many information on the model. For example,

\[ \frac{1}{N} \frac{\partial}{\partial \beta} \log Z_{N,\beta} = E_{P_{N,\beta}} \left[ \frac{N(\tau)}{N} \right] \]

is the average time spent at 0 by the polymer. As $N$ tends to infinity, it converges to the derivative of the so-called free energy of the model, which is defined by:

\[ F(\beta) := \lim_{N \to +\infty} \frac{1}{N} \log Z_{N,\beta}. \]

Hence, $F'(\beta)$ corresponds to the asymptotic fraction of time spent at zero by the polymer. Therefore, we speak about a delocalized regime when $F'(\beta) = 0$ and about a localized regime when $F'(\beta) > 0$.

It has been shown that there exists a phase transition for this model. More precisely, let $\beta_c := -\log \Sigma_K$. Then, if $\beta > \beta_c$, the free energy $F(\beta)$ is uniquely determined by

\[ E_K[\exp(-F(\beta)\tau_1)] = \exp(-\beta). \]
Otherwise, if $\beta \leq \beta_c$, the free energy is given by $F(\beta) = 0$. Therefore, the model exhibits a phase transition at $\beta = \beta_c$ from a delocalized regime to a localized regime.

The identification of the free energy is based on a simple rewriting of the partition function that we recall here. First, let us introduce a new family of (sub)-probability measures:

$$\forall n \geq 1, \quad \tilde{K}_\beta(n) := \begin{cases} \exp(\beta) K(n) \exp(-F(\beta)n) & \text{if } \beta \geq \beta_c, \\ \exp(\beta) K(n) & \text{if } \beta < \beta_c. \end{cases}$$

(10)

Notice that from the definition $\beta_c$, the measure $\tilde{K}_\beta$ is a probability measure when $\beta > \beta_c$, whereas it is a sub-probability measure when $\beta < \beta_c$. Let $\tilde{P}_\beta$ be the law of the renewal process associated to $\tilde{K}_\beta$. Then, summing over the inter-arrival times leads to:

$$Z_{N,\beta} = \sum_{n=1}^N \sum_{l_1 + \cdots + l_n = N} \prod_{i=1}^n \exp(\beta) K(l_i)$$

$$= \exp(F(\beta)N) \sum_{n=1}^N \sum_{l_1 + \cdots + l_n = N} \prod_{i=1}^n \tilde{K}_\beta(l_i)$$

$$= \exp(F(\beta)N) \tilde{P}_\beta(N \in \tau).$$

(11)

Under the classical assumption that there exists $\alpha > 0$ and a slowly varying function such that

$$K(n) = \frac{L(n)}{n^{1+\alpha}},$$

(12)

the probability $\tilde{P}_\beta(N \in \tau)$ does not decay exponentially fast. Therefore, the function $F$ defined in (9) is indeed the free energy of the model. Moreover, thanks to the asymptotic theory of renewal processes, Equation (11) also allows to obtain the asymptotic leading term of $Z_{N,\beta}$ as $N \to +\infty$.

### 2.2 Moment representation of the partition function

Under the additional assumption that $\{K(n)\}_{n \geq 1}$ is the sequence of moments of some measure $\mu$, the partition function $Z_{N,\beta}$ is the $N$-th moment of some measure $\nu_\beta$ which corresponds to an explicit transformation of $\mu$. In the following we will denote $C_+ = \{z \in C, \exists z > 0\}$.

**Theorem 2.** Let $\mu$ be a positive measure on $(0,1)$ and suppose that, for all $n \geq 1$,

$$K(n) = \int_0^1 (1-x)^{n-1} x d\mu(x).$$

(13)

Then, for all $\beta \in \mathbb{R}$, there exists a probability measure $\nu_\beta$ such that:

$$s_{\nu_\beta}(z) \left( e^\beta s_{\mu}(1 - z) - \frac{1 - e^\beta}{1 - z} \right) = \frac{1}{z(1 - z)}.$$

(14)

Moreover, for all $N \geq 0$:

$$Z_{N,\beta} = \int_{\mathbb{R}} x^N d\nu_\beta(x).$$

(15)

**Proof.** Let $\beta \in \mathbb{R}$ and define, for all $z \in C_+$, $z_\beta = z \exp(F(\beta))$. Then, thanks to Equation (11), the generating function associated to the sequence $(Z_{N,\beta})_{N \geq 0}$ is equal to:

$$G(z) := \sum_{N \geq 0} Z_{N,\beta} z^N$$

$$= 1 + \sum_{N \geq 1} \tilde{P}_\beta(N \in \tau) z_\beta^N$$

$$= 1 + E_{\tilde{P}_\beta} \left[ \sum_{N \geq 1} 1_{N \in \tau} z_\beta^N \right].$$

(16)
Let \((\eta^{(\beta)}_k)_{k \geq 1}\) be a sequence of i.i.d. random variables with law \(\widetilde{K}_\beta\). Then, Equation (16) becomes:

\[
G(z) = 1 + E_{\widetilde{P}_\beta} \left[ \sum_{k \geq 1} z^{\eta^{(\beta)}_1} \cdots z^{\eta^{(\beta)}_k} \right] = \frac{1}{1 - \exp(\beta) E_{\widetilde{P}_\beta} [z^{\eta^{(\beta)}_1}]}.
\]

Finally, from the definition of \(\widetilde{K}_\beta\) given in (10), we obtain:

\[
G(z) = \frac{1}{1 - \exp(\beta) E_{\widetilde{P}_\beta} [z^{\eta^{(\beta)}_1}]}.
\]

Let \(S(z) = -\frac{1}{z} G \left( \frac{1}{z} \right)\). Then, using Hypothesis (13), it is easy to deduce that:

\[
S(z) = \frac{1}{-z - \exp(\beta) \int_0^1 \frac{x z}{1 - z - x} d\mu(x)}.
\]

Notice that \(S(z) \sim -\frac{1}{z}\) as \(|z| \to +\infty\). Moreover, since for all \(z \in \mathbb{C}_+\) and \(x \in (0,1), \frac{x}{1-x} \in \mathbb{C}_+\), the following inclusion holds: \(S(\mathbb{C}_+) \subset \mathbb{C}_+\). By [1] (page 93), these two properties imply that there exists a probability measure \(\nu_\beta\) such that:

\[
S(z) = \int_{\mathbb{R}} \frac{d\nu_\beta(x)}{x - z},
\]

which ends the proof of Theorem 2.

\[\square\]

**Remark 3.** Note that by (17), the Stieltjes transform of \(\nu_\beta\) is also given by

\[
s_{\nu_\beta}(z) = \frac{1}{-z - \exp(\beta) \int_0^1 \frac{x z}{1 - z - x} d\mu(x)}.
\]

From (15), the free energy \(F(\beta)\) of the model is positive if and only if \(\nu_\beta\) has an atom larger than 1 given by \(\exp(F(\beta))\). This situation corresponds to the existence singularity of \(s_{\nu_\beta}\) which exists if and only if

\[
\int_0^1 \frac{x}{1 - y - x} d\mu(x) = -e^{-\beta}, \quad \text{for some } y > 1.
\]

As a function of \(y\), the left-hand side of the previous equality is increasing on \([1, +\infty)\), is equal to \(-\Sigma_K = \sum_{n \geq 0} K(n)\) at \(y = 1\) and tends to 0 as \(y\) tends to infinity. Therefore, \(\nu_\beta\) has a unique atom larger than 1 if and only if \(\beta > -\log \Sigma_K\). The mass of the atom of \(\nu_\beta\) at \(\exp(F(\beta))\) is given by the residue of \(s_{\nu_\beta}\) at \(\exp(F(\beta))\) which happens to be equal to \(F'(\beta)\), which is the average time spent at zero by the polymer. In particular, as \(N\) tends to infinity,

\[
Z_{N,\beta} \sim F'(\beta) \exp(NF(\beta)).
\]

### 2.3 A family of exactly solvable models

It turns out that some particular choices of probability measure \(\mu\) in Theorem 2 yield explicit computations. More precisely, for all \(v \in (0,1), \) let \(\mu_v\) be the Beta distribution with parameters \((1 - v, v)\), defined by:

\[
\mu_v(dx) := \frac{\sin(\pi v)}{\pi} x^{-v}(1 - x)^{v-1} 1_{x \in [0,1]} dx.
\]

Let \(K_v\) be the measure associated to \(\mu_v\), that is:

\[
\forall n \geq 1, \quad K_v(n) = \int_{\mathbb{R}} (1 - x)^{n-1} x d\mu_v(x).
\]
Notice that since $\sum_{n \geq 1} K_v(n) = \mu_v(0, 1)$, the measure on $Z_+$ defined by the $K_v(n)$'s is a probability measure. Moreover, for all $n \geq 1$:

$$K_v(n) = \frac{\sin(\pi v) \Gamma(n + v - 1) \Gamma(2 - v)}{\Gamma(n + 1)}.$$ 

Therefore, as $n \to +\infty$, $K_v(n) \sim \frac{\sin(\pi v) \Gamma(2 - v)}{\pi} \frac{1}{n^2}$. Hence, the probability measures $K_v$'s satisfy (12).

We will denote by $Z_{N,\beta,v}$ the partition function of the polymer associated to $K_v$. Then, Theorem 2 translates into the following result.

**Theorem 3.** For all $\beta > 0$ and $v \in (0, 1)$, let $\gamma_{v,\beta} = (1 - e^{-\beta})^{1/v}$ and define:

$$f_{v,\beta}(x) = \frac{\sin(\pi v)}{\pi x} \left[ (1 - e^\beta)^2 x^{2(1 - v)} - 2 e^\beta (1 - e^\beta) \cos(\pi v) x^{1 - v} (1 - x)^{1 - v} + e^{2\beta}(1 - x)^{2(1 - v)} \right],$$

$$x_{v,\beta} = \frac{1}{1 - \gamma_{v,\beta}} \quad \text{and} \quad c_{v,\beta} = \frac{\exp(-\beta) \gamma_{v,\beta}}{1 - v} \frac{\gamma_{v,\beta}}{1 - \gamma_{v,\beta}}.$$ 

Let $\nu_{v,\beta}$ be the following probability measure:

$$\nu_{v,\beta}(dx) = f_{v,\beta}(x) \textbf{1}_{x \in (0, 1)} dx + \textbf{1}_{\beta > 0} c_{v,\beta} \delta_{x_{v,\beta}}(dx),$$

Then, for all $N \geq 0$,

$$Z_{N,\beta,v} = \int_{\mathbb{R}} x^N d\nu_{v,\beta}(x).$$

**Remark 4.** Using Remark 3, we deduce that the constant $c_{v,\beta}$ is a positive constant smaller than 1 which corresponds to the asymptotic average time spent at 0 as $N$ tends to infinity.

**Proof.** Let $v \in (0, 1)$ and $\beta \in \mathbb{R}$. Recall that from Equation (14), the Stieltjes transform of $\nu_{v,\beta}$ satisfies:

$$\forall z \in \mathbb{C}_+^+, \quad \frac{1}{s_{v,\beta}(z)} = z(1 - z) \left( e^\beta s_{\mu_v}(1 - z) - \frac{1 - e^\beta}{1 - z} \right).$$

(18)

For all $x \in \mathbb{R}$, let us define $s_{v,\beta}(x) := \lim_{t \to 0^+} s_{v,\beta}(x + it)$ and $s_{\mu_v}(x) := \lim_{t \to 0^+} s_{\mu_v}(x - it)$. Then, Equation (18) becomes:

$$\frac{1}{s_{v,\beta}(x)} = x(1 - x) \left( e^\beta s_{\mu_v}(1 - x) - \frac{1 - e^\beta}{1 - x} \right).$$

(19)

We now use the following identity:

$$s_{\mu_v}(x) = \int_{\mathbb{R}} \frac{d\tilde{\mu}_v(y)}{y - x} - i\pi \frac{d\mu_v}{dx}(x),$$

where the integral in the right-hand side stands for a Cauchy principal value. The latter can be explicitly computed (see [3] page 250). More precisely:

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{y^{-v}(1 - y)^{1 - v}}{y - x} dy = \begin{cases} \frac{1}{\sin(\pi(1 - v))} \frac{1}{1 - x} \left| \frac{x}{1 - x} \right|^{-v} & \text{if } x < 0 \text{ or } x > 1, \\ -x^{-v}(1 - x)^{v - 1} \cos(\pi(1 - v)) & \text{if } 0 < x < 1. \end{cases}$$

Using (20), we deduce that:

$$s_{\mu_v}(1 - x) = \begin{cases} -\frac{1}{x} \left| \frac{x}{1 - x} \right|^{-v} & \text{if } x < 0 \text{ or } x > 1, \\ \cos(\pi v)(1 - x)^{-v} x^{1 - v} - i \sin(\pi v) x^{1 - v}(1 - x)^{v - 1} & \text{if } 0 < x < 1. \end{cases}$$

(21)

From (21) and (19), it is possible to identify the measure $\nu_{v,\beta}$ as explained in the following.
First, the absolutely continuous part of \( \nu_{v,\beta} \) is given by \( \frac{1}{\pi} \sin(\pi v) x^{1-v} (1-x)^{1-v} \). Therefore, it is supported on the interval \((0,1)\) and given by:

\[
\frac{d\nu_{v,\beta}}{dx}(x) = \frac{\sin(\pi v)}{\pi x} \frac{e^{\beta x} x^{1-v} (1-x)^{1-v} - e^{\beta} x^{1-v} (1-x)^{1-v} + e^{2\beta} (1-x)^{2(1-v)}}{(1-e^{\beta})^2 x + 2e^{\beta} (1-e^{\beta}) \cos(\pi v) x^{1-v} (1-x)^{1-v} + e^{2\beta} (1-x)^{2(1-v)}}.
\]

Besides, \( \nu_{v,\beta} \) has an atom at \( x \in \mathbb{R} \) if \( s_{\nu_{v,\beta}}(x) = \infty \). Therefore, the atomic part is contained in \( \mathbb{R} \setminus [0,1] \) and \( x \in \mathbb{R} \setminus [0,1] \) is an atom of \( \nu_{v,\beta} \) if and only if:

\[
1 + e^{\beta} \left( \frac{x}{x-1} \right)^{v-1} - e^{\beta} = 0 \iff x = 1 + \frac{(1-e^{-\beta})^{1/v}}{1 - (1-e^{-\beta})^{1/v}}.
\]

The right-hand side does not belong to \([0,1]\) if and only if \( \beta > 0 \). Therefore, we have the following dichotomy:

- if \( \beta > 0 \), the measure \( \nu_{v,\beta} \) has an atom at \( x_{v,\beta} := 1 + \frac{(1-e^{-\beta})^{1/v}}{1 - (1-e^{-\beta})^{1/v}} > 1 \);
- if \( \beta \leq 0 \), the measure \( \nu_{v,\beta} \) has no atom.

Suppose that \( \beta > 0 \). Then, the atom \( x_{v,\beta} \) coincides with \( \exp(F'(\beta)) \) and by Remark 3 the mass of \( \nu_{v,\beta} \) at \( x_{v,\beta} \) is equal to \( F'(\beta) = \partial_{\beta}(x_{v,\beta})/x_{v,\beta} \), which yields the expression of \( c_{v,\beta} \).

Then, a straightforward consequence of Theorem 3 is the following explicit formula for the free energy of the model, defined in (8).

**Corollary 1.** The following equality holds:

\[
F_v(\beta) = \begin{cases} 0 \quad &\text{if } \beta \leq 0, \\ \log \left( \frac{1}{1-\gamma_{v,\beta}} \right) \quad &\text{if } \beta > 0. \end{cases}
\]

Moreover, when \( \beta > 0 \), as \( N \to +\infty \),

\[
Z_{N,\beta,v} \sim \frac{\exp(-\beta)}{1-v} \frac{\gamma_{v,\beta}^v}{1-\gamma_{v,\beta}} \left( \frac{1}{1-\gamma_{v,\beta}} \right)^N.
\]

### 2.4 The special case of the Arcsine law

When \( v = 1/2 \), \( \mu_v \) is the classical Arcsine law:

\[
\mu_{1/2}(dx) = \frac{1}{\pi} \frac{1}{\sqrt{1-x}x} 1_{x \in (0,1)} dx.
\]

In that case,

\[
\forall n \geq 1, \quad K_{1/2}(n) = \frac{1}{2^{2n}} \frac{1}{2n-1} \binom{2n}{n},
\]

which is also the probability that the first return to 0 of the simple random walk is equal to \( 2n \), see for example [4]. It turns out that in this setting, the phase transition from the delocalized regime to the localized regime for the polymer model corresponds to a famous phase transition in random matrix theory, which we briefly recall in the following.

For all \( n \geq 1 \), let \( X_n \) be a matrix of size \( n \times n \) whose entries are i.i.d. random variables, centered and reduced. Let also \( \Sigma_n = \text{Diag}(2e^{\beta}, 1, \ldots, 1) \), where \( \beta \in \mathbb{R} \). We consider the following random covariance matrix:

\[
S_n = \frac{1}{4n} \Sigma_n^{1/2} X_n X_n^T \Sigma_n^{1/2}.
\]

Denoting \( \lambda_1 \geq \cdots \geq \lambda_n \) the eigenvalues of \( S_n \), it turns out that, in probability, the empirical spectral measure \( \frac{1}{n} \sum_{1 \leq i \leq n} \delta_{\lambda_i} \) weakly converges towards the so-called Marchenko-Pastur law.
with parameter 1, given by the density \((2/\pi)(1 - x)^{1/2}x^{-1/2}1_{0 < x < 1}\)dx. Moreover, the possible existence of an eigenvalue outside the limiting support \((0, 1)\), often called an outlier, is the object of the following phase transition. We will denote by \(\phi_1\) the normalized eigenvector associated to \(\lambda_1\).

**Theorem 4.** Let \(e_1\) be the first vector of the canonical basis. Then, the following convergences hold in probability:

\[
\lambda_1 \xrightarrow{n \to +\infty} \begin{cases} 
1 & \text{if } \beta \leq 0, \\
\frac{e^{\beta}}{2e^{\beta} - 1} \text{otherwise,} 
\end{cases}
\text{and}

|\langle \phi_1, e_1 \rangle|^2 \xrightarrow{n \to +\infty} \begin{cases} 
0 & \text{if } \beta \leq 0, \\
\frac{2e^{\beta} - 2}{e^{\beta} - 1} \text{otherwise.} 
\end{cases}
\]

This result was first proved by Báik, Ben Arous and Péché [2] in a Gaussian setting. Another approach to this problem is to study the spectral measure in direction \(e_1\) – see [7], defined by

\[
\mu(S_n, e_1) := \sum_{i=1}^{n} |\langle \phi_i, e_1 \rangle|^2 \delta_{\lambda_i},
\]

where \(\phi_i\) is the normalized eigenvector associated to \(\lambda_i\). With our notations, it turns out that in probability, \(\mu(S_n, e_1)\) weakly converges to \(\nu_{\frac{1}{2}, \beta}\), which is given by

\[
\nu_{\frac{1}{2}, \beta} = \frac{e^\beta}{\pi x^2 (1 - 2e^{2\beta})} 1_{0 < x < 1} dx + \frac{2e^\beta - 2}{2e^\beta - 1} 1_{\beta > 0} e^{2\beta} \frac{1}{x^{2\beta - 1}} (dx).
\]

In particular, the atomic part of \(\nu_{\frac{1}{2}, \beta}\) allows to retrieve the convergences of Theorem 4. Interestingly, this links the Báik, Ben Arous and Péché phase transition for the largest eigenvalue of deformed random covariance matrices to the phase transition from the delocalized to the localized regime for the polymer model. In the super-critical regimes, the limit of \(\log \lambda_1\) is the free energy of the polymer, and the limit of the square projection of the associated eigenvector is the multiplicative factor in front of the exponential term of the partition function – this can be seen from Equations (22) and (23).

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