Research Article

Global Existence and Decay for a System of Two Singular Nonlinear Viscoelastic Equations with General Source and Localized Frictional Damping Terms

Salah Mahmoud Boulaaras,1,2 Rafik Guefaifia,3 Nadia Mezouar,4 and Ahmad Mohammed Alghamdi5

1Department of Mathematics, College of Sciences and Arts, Al-Rass, Qassim University, Saudi Arabia
2Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Benbella, Algeria
3Department of Mathematics, College of Exact Sciences, University of Tebessa, Tebessa 12002, Algeria
4Mascara University, Faculty of Economics, Mascara, Algeria
5Department of Mathematical Sciences, College of Applied Science, Umm Al-Qura University, Saudi Arabia

Correspondence should be addressed to Salah Mahmoud Boulaaras; s.boularas@qu.edu.sa

Received 19 July 2020; Revised 21 August 2020; Accepted 29 August 2020; Published 9 September 2020

Academic Editor: Fanglei Wang

Copyright © 2020 Salah Mahmoud Boulaaras et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The current paper deals with the proof of a global solution of a viscoelasticity singular one-dimensional system with localized frictional damping and general source terms, taking into consideration nonlocal boundary condition. Moreover, similar to that in Boulaaras’ recent studies by constructing a Lyapunov functional and use it together with the perturbed energy method in order to prove a general decay result.

1. Introduction

The evolution problem with integral conditions is related with many branches of sciences ([1–6]). Cause of this, interest in it occurs naturally in inflation cosmology, nuclear physics, supersymmetric field theories, and quantum mechanics (see for example [2, 7]). Later, by the motivation of this work, some authors gave necessary and sufficient conditions for the hyperbolic equation with source term (see, e.g., [8–10]).

This manuscript is devoted to the study of the global existence and decay for a system of two singular one-dimensional nonlinear viscoelastic equations with general source terms.
where \( L < \infty, T < \infty, \mu \in C^1((0, \alpha)), g_1(\cdot), g_2(\cdot): \mathbb{R}^+ \rightarrow \mathbb{R}^+ \)
and \( f_1(\cdot), f_2(\cdot): \mathbb{R}^2 \rightarrow \mathbb{R} \) are given functions, which will be specified later.

The problems related with localized frictional damping have extensively studied by many teams as [11], where the authors obtained an exponential rate of decay for the solution of the viscoelastic nonlinear wave equation:

\[
\left\{ \begin{array}{l}
    u_{tt} - \frac{1}{X}(xu_x)_x + \int_{0}^{t} g_1(t-s) \frac{1}{X}(xu_x(x,s))_x ds + \mu(x)u_t = f_1(u,v), \text{ in } (0, L) \times (0, T), \\
v_{tt} - \frac{1}{X}(xv_x)_x + \int_{0}^{t} g_2(t-s) \frac{1}{X}(xv_x(x,s))_x ds + \mu(x)v_t = f_2(u,v), \text{ in } (0, L) \times (0, T), \\
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in (0, L), \\
v(x, 0) = v_0(x), v_t(x, 0) = v_1(x), x \in (0, L), \\
u(L, t) = v(L, t) = 0, \int_{0}^{L} xu(x, t) dx = \int_{0}^{L} xv(x, t) dx = 0,
\end{array} \right.
\]

(1)

Very recently, in ([9]), we study the following singular one-dimensional nonlinear equations that arise in generalized viscoelasticity with long-term memory:

\[
\left\{ \begin{array}{l}
    u_{tt} - \frac{1}{X}(xu_x)_x + \int_{0}^{t} g_1(t-s) \frac{1}{X}(xu_x(x,s))_x ds + \alpha (x)u_t = |v|^{q_1} |u|^{p_1} u, \text{ in } (0, L) \times (0, T), \\
v_{tt} - \frac{1}{X}(xv_x)_x + \int_{0}^{t} g_2(t-s) \frac{1}{X}(xv_x(x,s))_x ds + \alpha (x)v_t = |v|^{q_1} |u|^{p_1} v, \text{ in } (0, L) \times (0, T), \\
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in (0, \alpha), \\
v(x, 0) = v_0(x), v_t(x, 0) = v_1(x), x \in (0, \alpha), \\
u(\alpha, t) = v(\alpha, t) = 0, \int_{0}^{\alpha} xu(x, t) dx = \int_{0}^{\alpha} xv(x, t) dx = 0.
\end{array} \right.
\]

(3)

for damping term \( \alpha (x)u_t \) may be null for some part of the domain.

We used the techniques in [11]; we have proven in [8] the existence of a global solution using the potential well theory for the following viscoelastic system with nonlocal boundary condition and localized frictional damping:

\[
\left\{ \begin{array}{l}
    u_{tt} - \Delta u + f(x, t, u) + \int_{0}^{T} g_1(t-s) \Delta u(s) ds + a(x)u_t = 0, \text{ in } (0, L) \times (0, T), \\
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in (0, L), \\
v(x, 0) = v_0(x), v_t(x, 0) = v_1(x), x \in (0, L), \\
u(L, t) = v(L, t) = 0, \int_{0}^{L} xu(x, t) dx = \int_{0}^{L} xv(x, t) dx = 0.
\end{array} \right.
\]

(4)
In view of the articles mentioned above in ([8, 9, 11]), much less effort has been devoted to the existence of a global solution to the system of two singular nonlinear equations which arise in generalized viscoelasticity with localized frictional damping terms using the potential-well theory. Moreover, we prove a general decay result by constructing a Lyapunov functional and use it together with the perturbed energy method.

The structure of the work is as follows: To facilitate the description, firstly in Section 2, we give the fundamental definitions and theorems on function spaces that will be needed in the body of the paper and state the local existence theorem. In Section 3, the energy function $E(t)$ is defined and proved to be a nonincreasing function of time. Finally, the main result is obtained, which gives the general decay conditions:

$$g_i'(t) \leq -\xi(t)g_i(t), \quad i = 1, 2. \tag{5}$$

### 2. Preliminaries

Let $L^p=L^p((0,L))$ be the weighted Banach space equipped with the norm

$$\|u\|_{L^p} = \left( \int_0^L x |u|^pdx \right)^{1/p}, \tag{6}$$

when $p = 2$, we get a Hilbert space, and we denote by $H = L_2^2$, it provided with the finite norm

$$\|u\|_H = \left( \int_0^L x u^2dx \right)^{1/2}. \tag{7}$$

$V = V_{2,1}((0,L))$ be the Hilbert space equipped with the norm

$$\|u\|_V = (\|u\|_H^2 + \|u_x\|_H^2)^{1/2}. \tag{8}$$

We get the following lemma by combining the Poincare inequality and (see [8]).

**Lemma 1.** Let $V_0$ space defined as follows

$$V_0 = \{ u \in V \text{ such that } u(L) = 0 \}. \tag{9}$$

Then, for $2 \leq p \leq 4$, we have

$$\int_0^L x |v|^pdx \leq C_* \|v_x\|_H^p, \quad \forall u \in V_0. \tag{10}$$

where $C_*$ is a constant depending on $L$ and $p$ only and for $p = 2$, $C_* = C_p$ is the Poincare constant.

**Remark 2.** It is clear that $\|u\|_{V_0} = \|u_x\|_H$ defines an equivalent norm on $V_0$.

The next theorem confirms that our problem has a local solution under some condition on $p$ and the relaxation functions $g_i$, the proof can be established by following the argument of [12].

**Theorem 3.** We take $(u_0, v_0) \in V_0^2$ and $(v_1, v_2) \in H^2$. If $p < 3$ and

$$g_i(0) > 0, \quad \left( 1 - \int_0^\infty g_i(s)ds \right) = l > 0, \quad \text{for } i = 1, 2, \tag{11}$$

then, there exists $t_* > 0$ small enough such that the problem (1) has a unique local solution

$$u \in C(0, t_* ; V_0) \cap C^1(0, t_* ; H). \tag{12}$$

**Remark 4.** The condition on $p$ is needed so that the embedding of $V_0$ in $L^2_2$ is Lipschitz.

We need the following assumptions to get our results.

($G_1$) For $i = 1, 2, g_i(t): \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nonincreasing $C_2$ function such that

$$\begin{cases}
    g_i(s) \geq 0, \quad g_i'(s) \leq 0 \text{ and,} \\
    g_i(0) \geq 0, \quad 1 - \int_0^\infty g_i(s)ds = l_1 > 0,
\end{cases} \tag{13}$$

and

($G_2$)

$$g_i'(t) \leq -\xi(t)g_i^a(t), \quad t \geq 0, \quad 1 \leq \sigma < \frac{3}{2}, \tag{14}$$

where $\xi(t)$ is a positive differentiable function. It satisfies for some positive constant $l$

$$\frac{\xi'(t)}{\xi(t)} \leq l, \quad \xi'(t) \leq 0, \quad \int_0^\infty \xi(s)ds = +\infty, \forall t > 0. \tag{15}$$

Furthermore, for any $t_0 > 0$ and $1 < \sigma < 3/2$, there exists a positive constant $C_\sigma$ such that

$$\frac{t}{\left(1 + \int_0^t \xi(s)ds\right)^{1/2(\sigma - 1)}} \leq C_\sigma \forall t \geq t_0. \tag{16}$$

($G_3$) We take

$$\begin{align*}
f_1(u, v) &= a|u + v|^{r+1} (u + v) + b|u|^{2} |v|^{r+2}, \\
f_2(u, v) &= a|u + v|^{r+1} (u + v) + b|v|^{2} |u|^{r+2},
\end{align*} \tag{17}$$

where $a, b > 0$ are constants and $r > -1$.

One can easily verify that

$$uf_1(u, v) + vf_2(u, v) = 2(r + 2)F(u, v), \quad V(u, v) \in \mathbb{R}^2. \tag{18}$$
where

$$F(u, v) = \frac{1}{2(r + 2)} \left[ |u| + |v|^{2(r+2)} + 2b|uv|^{r+2} \right].$$  \hfill (19)

$$(G_{k}) \mu \geq 0, \mu > 0 \text{ in } (L_0, L), \text{ where } 0 \leq L_0 \leq L.$$  

**Lemma 5.** For $r > -1$, there exist $\eta > 0$ such that for any $u, v \in V \cap V_0(0, L)$, we have

$$\|u + v\|_{L^2(x, t)}^{2(r+2)} + 2\|uv\|_{L^2(x, t)}^{(r+2)} \leq \eta \left( I_1 \|u_x\|_{L^2} + I_2 \|v_x\|_{L^2} \right)^{r+2}. $$  \hfill (20)

**Proof.** From Minkowski inequality, we have

$$\|u + v\|_{L^2(x, t)}^{2(r+2)} \leq 2 \left( \|u\|_{L^2(x, t)}^{2(r+2)} + \|v\|_{L^2(x, t)}^{2(r+2)} \right). $$  \hfill (21)

We apply successively Holder’s and Young’s inequalities we obtain

$$\|uv\|_{L^2(x, t)}^{(r+2)} \leq \|u\|_{L^2(x, t)} \|v\|_{L^2(x, t)} = C \left( I_1 \|u_x\|_{L^2} + I_2 \|v_x\|_{L^2} \right)^2. $$  \hfill (22)

We combine the two previous inequalities and the embedding $V \cap V_0(0, L) \hookrightarrow L^2(x, t)$, we get (20).

**Lemma 6.** There exist two positive constants $\Lambda_1$ and $\Lambda_2$ such that

$$\int_{0}^{L} x|f_1(u, v)|^2 \, dx \leq \Lambda_1 \left( I_1 \int_{0}^{L} xu_x^2 \, dx + I_2 \int_{0}^{L} xv_x^2 \, dx \right)^{2(r+3)}, \forall x \in (0, L), i = 1, 2.$$  \hfill (23)

**Proof.** It is clear that

$$|f_1(u, v)| \leq C \left( |u| + |v|^{r+2} + |u|^{r+1} |v|^{r+2} \right) \leq C \left( |u|^{2(r+3)} + |u|^{r+3} + |v|^{r+3} \right).$$  \hfill (24)

Applying Young’s inequality with exponents $q = (2r + 3)/(r + 1), q' = (2r + 3)/(r + 2)$, in the last term in the above inequality, we get

$$|f_1(u, v)| \leq C \left( |u|^{2(r+3)} + |v|^{2(r+3)} \right).$$  \hfill (25)

Consequently, by using Poincaré’s inequality and (20), we obtain

$$\int_{0}^{L} x|f_1(u, v)|^2 \, dx \leq C \left( \|u_x\|_{L^2}^{2(r+3)} + \|v_x\|_{L^2}^{(2r+3)} \right) \leq \Lambda_1 \left( I_1 \|u_x\|_{L^2} + I_2 \|u_x\|_{L^2} \right)^{2(r+3)}. $$  \hfill (26)

Similarly, we get the inequality for $f_2$. The proof is completed.

We define the energy function as

$$E(t) = \frac{1}{2} \int_{0}^{L} xu_x^2 \, dx + \frac{1}{2} \int_{0}^{L} xv_x^2 \, dx + \frac{1}{2} \left( 1 - \int_{0}^{t} g_1(s) \, ds \right) \int_{0}^{L} xu_x^2 \, dx + \frac{1}{2} \left( 1 - \int_{0}^{t} g_1(s) \, ds \right) \int_{0}^{L} xv_x^2 \, dx - \int_{0}^{L} F(u, v) \, dx + \frac{1}{2} \left( g_2 \circ u_x \right)(t) + \frac{1}{2} \left( g_2 \circ u_x \right)(t).$$  \hfill (27)

where

$$\left( g_2 \circ u_x \right)(t) = \int_{0}^{t} xg(t-s) |u_x(x, t) - u_x(x, s)|^2 \, dx.$$  \hfill (28)

**Lemma 7.** Let $(u, v)$ be the solution of system (1) then for all $t \geq 0$

$$\frac{d}{dt} [E(t)] = - \int_{0}^{L} x\mu(x)u_x^2 \, dx - \int_{0}^{L} x\nu(x)v_x^2 \, dx + \frac{1}{2} \left( g' \circ u_x \right)(t) - \frac{1}{2} g_x(t) \int_{0}^{L} xu_x^2 \, dx + \frac{1}{2} \left( g' \circ v_x \right)(t) - \frac{1}{2} g_x \int_{0}^{L} xv_x^2 \, dx.$$  \hfill (29)

Hence, $E(t)$ is a nonincreasing function.

**Proof.** Multiplying the first and the second equations in (1) by $xu_t$ and $xv_t$, respectively, integrating over $(0, L)$, summing up, we obtain (30)

$$\int_{0}^{L} xu_t u_t \, dx - \int_{0}^{L} x(u_x)_s u_t \, dx + \int_{0}^{L} g_x(t-s) (xu_x(x, s))_s \, ds \, dx$$

$$\int_{0}^{L} xv_t v_t \, dx - \int_{0}^{L} x(v_x)_s v_t \, dx + \int_{0}^{L} g_x(t-s) (xv_x(x, s))_s \, ds \, dx$$

$$= - \int_{0}^{L} x\mu(x)u_x^2 \, dx - \int_{0}^{L} x\nu(x)v_x^2 \, dx$$

$$+ \int_{0}^{L} \left[ a|u + v|^{2(r+1)} + b|u|^{r+1} |v|^{r+2} \right] xu_x \, dx$$

$$+ \int_{0}^{L} \left[ a|u + v|^{2(r+1)} + b|v|^{r+1} |u|^{r+2} \right] xv_x \, dx.$$  \hfill (30)

By integration by parts, we obtain

$$\int_{0}^{L} xu_t u_t \, dx = \frac{1}{2} \frac{d}{dt} \left[ \int_{0}^{L} xu_x^2 \, dx \right],$$  \hfill (31)

$$\int_{0}^{L} xv_t v_t \, dx = \frac{1}{2} \frac{d}{dt} \left[ \int_{0}^{L} xv_x^2 \, dx \right],$$  \hfill (32)

$$- \int_{0}^{L} (xu_x)_s u_t \, dx = \frac{1}{2} \frac{d}{dt} \left[ \int_{0}^{L} xu_x^2 \, dx \right],$$  \hfill (33)
\[ -\frac{d}{dt} \int_0^L (xv_x) v_x dx = \frac{1}{2(r + 1)} \int_0^L x v_x^2 dx, \tag{34} \]

\[-\frac{1}{2(r + 2)} \int_0^L x f_1(u,v) uu_x dx + \frac{1}{2(r + 1)} \int_0^L x f_2(u,v) vv_x dx = \frac{1}{2(r + 2)} \frac{d}{dt} \int_0^L \int_{\mathbb{R}^2} |u + v|^{2(r + 2)} + 2b|uv|^{r+1} dx, \tag{35} \]

\[ \int_0^t \int_0^L g_1(t - s) (xu_x(s))_x ds u_x(t) dx = \frac{1}{2} \frac{d}{dt} \left[ (g_1 \ast u_x)(t) - \int_0^t g_1(s) ds \int_0^L x u_x^2 dx \right] - \frac{1}{2} \left( (g_1 \ast u_x)(t) + \int_0^L x u_x^2 dx, \right) \tag{36} \]

\[ \int_0^t \int_0^L g_2(t - s) (xv_x(s))_x dv_x(t) dx = \frac{1}{2} \frac{d}{dt} \left[ (g_2 \ast v_x)(t) - \int_0^t g_2(s) ds \int_0^L x v_x^2 dx \right] - \frac{1}{2} \left( (g_2 \ast v_x)(t) + \int_0^L x v_x^2 dx, \right) \tag{37} \]

Combining (32)–(2.22) in (31), we get (30).

3. Global Existence

In order to state and prove the global existence, we set the following notation

\[ I(t) = I(u(t), v(t)) = \left( 1 - \int_0^t g_1(s) ds \right) \int_0^L x u_x^2 dx + \left( 1 - \int_0^t g_2(s) ds \right) \int_0^L x v_x^2 dx + (g_1 \ast u_x)(t) + (g_2 \ast v_x)(t), \tag{38} \]

\[ J(t) = J(u(t), v(t)) = \frac{1}{2} \left( 1 - \int_0^t g_1(s) ds \right) \int_0^L x u_x^2 dx + \frac{1}{2} \left( 1 - \int_0^t g_2(s) ds \right) \int_0^L x v_x^2 dx + \frac{1}{2} \left( g_1 \ast u_x \right)(t) + \frac{1}{2} \left( g_2 \ast v_x \right)(t), \tag{39} \]

we remark that

\[ E(t) = J(t) + \frac{1}{2} \int_0^L x u_x^2 dx + \int_0^L x v_x^2 dx. \tag{40} \]

Lemma 8. Assume that \((G_1), (G_2), \) and (20) hold also for any \((u_0, v_0) \in V_0^2 \) and \((u_1, v_1) \in H_2 \) satisfying

\[ I(0) > 0, \beta = \eta \left( \frac{2(r + 2)}{r + 1} E(0) \right)^{r+1} < 1, \tag{41} \]

where

\[ E(0) = J(0) + \frac{1}{2} \int_0^L x u_x^2 dx + \frac{1}{2} \int_0^L x v_x^2 dx. \tag{42} \]

Then, there exists \( t_+ > 0 \) such that

\[ I(t) > 0, \forall t \in [0, t_+). \tag{43} \]

Proof. Since \( I(0) > 0 \), then from the continuity of \( I(t) \), there exist \( t_m \leq t_+ \) such that \( I(t) \geq 0 \) for all \( t \in [0, t_m) \); this implies that we have a maximum time value noting \( T_m \) such that

\[ \{ I(T_m) = 0 \text{ and } I(t) > 0, \text{ for all } 0 \leq t < T_m \}. \tag{44} \]

From formulas of \( J(t) \) and \( I(t) \) together with \((G_1)\), we have

\[ J(t) = \frac{r + 1}{2(r + 2)} \left[ \left( 1 - \int_0^t g_1(s) ds \right) \int_0^L x u_x^2 ds + \left( 1 - \int_0^t g_2(s) ds \right) \int_0^L x v_x^2 ds \right] + \left( 1 - \int_0^t g_2(s) ds \right) \int_0^L x u_x^2 ds + \left( 1 - \int_0^t g_2(s) ds \right) \int_0^L x v_x^2 ds \]

\[ + \frac{1}{2(r + 2)} \left[ (g_1 \ast u_x)(t) + \frac{1}{2} \left( g_2 \ast v_x \right)(t) \right] \]

\[ + \frac{1}{2(r + 2)} \left[ (g_1 \ast u_x)(t) + \frac{1}{2} \left( g_2 \ast v_x \right)(t) \right], \tag{45} \]

hence,

\[ l_1 \int_0^L x u_x^2 dx + l_2 \int_0^L x v_x^2 dx \]

\[ \leq \frac{2(r + 2)}{r + 1} J(t) \leq \frac{2(r + 2)}{r + 1} E(t) \]

\[ \leq \frac{2(r + 2)}{r + 1} E(0), \forall t \in [0, T_m). \tag{46} \]
Recalling Lemma 5 and (41), we get

\[
2(r + 2) \int_0^t F(u(T_m), v(T_m)) dx \\
\leq \eta \left( l_1 \int_0^t x u_x^2 dx + l_2 \int_0^t x v_x^2 dx \right) ^{\gamma+2} \\
\leq \eta \left( \frac{2(r + 2)}{r + 1} E(0) \right) ^{\gamma+1} \left( l_1 \int_0^t x u_x^2 dx + l_2 \int_0^t x v_x^2 dx \right) \\
= \beta \left( l_1 \int_0^t x u_x^2 dx + l_2 \int_0^t x v_x^2 dx \right) \\
< \left( 1 - \int_0^t g_1(s) ds \right) \left( l_1 \int_0^t x u_x^2 dx \right) \\
+ \left( 1 - \int_0^t g_2(s) ds \right) \left( l_2 \int_0^t x v_x^2 dx \right) \\
+ (g_1 \circ u_x)(t) + (g_2 \circ v_x)(t),
\]

consequently

\[
\left( 1 - \int_0^t g_1(s) ds \right) \left( l_1 \int_0^t x u_x^2 dx \right) \\
+ \left( 1 - \int_0^t g_2(s) ds \right) \left( l_2 \int_0^t x v_x^2 dx \right) \\
+ (g_1 \circ u_x)(t) - 2(r + 2) \int_0^t x F(u, v) dx > 0,
\]

we deduce that \( I(t) > 0, \forall t \in [0, t_m). \) By repeating the procedure, \( T_m \) is extended to \( t_*. \)

**Theorem 9.** Suppose that \((G_1), (G_2),\) and (20) hold. Then, for any \((u_0, v_0) \in V_0^2\) and \((u, v) \in H^2\) satisfying (41), the solution of system (1) is a bounded and globally in time.

**Proof.** To achieve the proof of this theorem, it suffices to show that \( \|u_x\|_{H^2}^2 + \|v_x\|_{H^2}^2 + \|u_t\|_{H^2}^2 + \|v_t\|_{H^2}^2 \) is bounded independently of \( t. \) As \( E(t) \) is a nonincreasing function, we have

\[
E(0) \geq E(t),
\]

in the other hand and for the definition of \( I(t), \) we have

\[
x \left[ a|u|^2(r+2) + 2b|uv|^{r+2} \right] dx \\
= I(t) - \left( 1 - \int_0^t g_1(s) ds \right) \int_0^t x u_x^2 dx \\
- \left( 1 - \int_0^t g_2(s) ds \right) \int_0^t x v_x^2 dx \\
- (g_1 \circ u_x)(t) \\
- (g_2 \circ v_x)(t),
\]

we introduce (49) into (50), we get

\[
E(0) \geq E(t) = \frac{1}{2} \int_0^t x u_t^2 dx + \frac{1}{2} \int_0^t x v_t^2 dx \\
+ \frac{1}{2} \left( 1 - \int_0^t g_1(s) ds \right) \int_0^t x u_x^2 dx \\
+ \frac{1}{2} \left( 1 - \int_0^t g_2(s) ds \right) \int_0^t x v_x^2 dx \\
+ \frac{1}{2} (g_1 \circ u_x)(t) + \frac{1}{2} (g_2 \circ v_x)(t) + I(t),
\]

by using (14), (15), and (41), (51) yields

\[
E(0) \geq E(t) \geq \frac{1}{2} \int_0^t x u_t^2 dx + \frac{1}{2} \int_0^t x v_t^2 dx \\
+ \frac{2}{(r + 2)} \left( l_1 \int_0^t x u_x^2 dx + l_2 \int_0^t x v_x^2 dx \right) \\
\geq \mu_0 \left( \int_0^t x u_t^2 dx + \int_0^t x v_t^2 dx + \int_0^t x u_x^2 dx + \int_0^t x v_x^2 dx \right),
\]

So

\[
\|u_x\|_{H^2}^2 + \|v_x\|_{H^2}^2 + \|u_t\|_{H^2}^2 + \|v_t\|_{H^2}^2 \leq \tau E(0),
\]

where

\[
\tau = \max \left\{ 2, \frac{2(r + 2)}{(r + 1)l_1}, \frac{2(r + 2)}{(r + 1)l_2} \right\}.
\]

The proof is completed.

### 4. Decay of Solutions

Throughout this section, we will study the asymptotic behavior of solutions’ decay by constructing a suitable Lyapunov function; to do so, for \( N, \varepsilon_1, \) and \( \varepsilon_2, \) are positive constants, we define the following function as

\[
F(t) = E(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \chi(t) + \psi(t),
\]

where

\[
\Phi(t) = \xi(t) \int_0^t x u_t u dx + \xi(t) \int_0^t x v_t v dx \\
+ \frac{\xi(t)}{2} \int_0^t x \mu(u^2 + v^2) dx,
\]

\[
x(t) = -\xi(t) \int_0^t g_1(t-s)(u(t) - u(s)) ds dx \\
- \xi(t) \int_0^t x v_t \int_0^t g_2(t-s)(v(t) - v(s)) ds dx,
\]
\[
\psi(t) = \xi(t) \int_0^t x \mu(t) u(x) \, dx + \xi(t) \int_0^t x \nu(t) \, dx,
\]
(58)

with \( h \in C^1([0, L]) \), \( h(0) = h(L) = 0 \), \( (xh(x))' \leq x \).

In the first step, we prove the equivalence between \( F(t) \) and \( E(t) \) given in the following lemma.

**Lemma 10.** For a choice of \( \varepsilon_1 \) and \( \varepsilon_2 \) small enough, we find two positive constants \( \alpha_1 \) and \( \alpha_2 \) such that

\[
\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t).
\]
(59)

**Proof.** By using Young inequality, follow by recalling Lemma 1 and the fact that \( 0 < \xi(t) \leq \xi(0) \), we get

\[
|\varepsilon_1 \xi(t) \int_0^t x \mu(t) \, dx| \leq \frac{\varepsilon_1}{2} \xi(0) \left( \int_0^t x^2 \, dx + C_p \int_0^t x \, dx \right),
\]
(60)

\[
|\varepsilon_1 \xi(t) \int_0^t x \nu(t) \, dx| \leq \frac{\varepsilon_1}{2} \xi(0) \left( \int_0^t x^2 \, dx + C_p \int_0^t x \, dx \right),
\]
(61)

\[
|\varepsilon_1 \xi(t) \int_0^t g_1(t-s)(u(t)-u(s)) \, ds| \leq \frac{\varepsilon_1}{2} \xi(0) \left( \int_0^t x^2 \, dx + C_p \int_0^t x \, dx \right),
\]
(62)

\[
|\varepsilon_1 \xi(t) \int_0^t g_2(t-s)(v(t)-v(s)) \, ds| \leq \frac{\varepsilon_1}{2} \xi(0) \left( \int_0^t x^2 \, dx + C_p \int_0^t x \, dx \right),
\]
(63)

\[
|\xi(t) \int_0^t x \mu(t) \, dx| \leq \frac{\xi(0)}{2} \|h\|_{\infty} \left( \int_0^t x^2 \, dx + \int_0^t x \, dx \right),
\]
(64)

\[
|\xi(t) \int_0^t x \nu(t) \, dx| \leq \frac{\xi(0)}{2} \|h\|_{\infty} \left( \int_0^t x^2 \, dx + \int_0^t x \, dx \right),
\]
(65)

\[
|\xi(t) \int_0^t x \mu(t) \, dx| \leq \frac{\xi(0)}{2} \|h\|_{\infty} \left( \int_0^t x^2 \, dx + \int_0^t x \, dx \right),
\]
(66)

\[
|\xi(t) \int_0^t x \nu(t) \, dx| \leq \frac{\xi(0)}{2} \|h\|_{\infty} \left( \int_0^t x^2 \, dx + \int_0^t x \, dx \right),
\]
(67)

If we choose \( \varepsilon_1, \varepsilon_2 \) small enough, and \( N \) large enough we find \( \alpha_1, \alpha_2 > 0 \) such that (59) holds true.

Now, we state a Lemma corresponding to the boundness of \( (gw_x)_x(t) \). It will be used in the calculus.

**Lemma 11.** Let \( w \in L^{\infty}((0, T) ; H) \) be such that \( w_x \in L^{\infty}((0, T) ; H) \) and \( g \) be a continuous function on \([0, T]\) and suppose that, then there exists a constant \( C \) such that

\[
(gw_x)_x(t) \leq C \left( \sup_{0 < s < t} \|w_x(s)\|_{H} \right)^{\frac{p}{p-1}} \left( \int_0^t g^\theta(s) \, ds \right)^{\frac{p}{p-1}}
\]
(68)

\[
\times \left( \int_0^t g^\rho(t-s) \|w_x(s)\|_{H}^{2} \, ds \right)^{\frac{1}{2}}
\]
(69)

\[\forall 0 < \theta < 1 \text{ and } p > 1.\]

**Proof.**

(1) For any \( \sigma > 1 \), we have

\[
(gw_x)_x(t) = \int_0^t \left( g(t-s) \right)^{(1-\theta)\sigma} \|w_x(s)\|_{H}^{2} \left( g(t-s) \right)^{(1-\sigma)\sigma} \|w_x(s)\|_{H}^{2} \, ds.
\]
(70)
Applying Holder’s inequality with exponents $\sigma$ and $\sigma/\sigma - 1$, we get
\[
(gw_x(t)) \leq \left( \int_0^t \left( g^{1-\theta}(t-s) \|w_x(\cdot,t)-w_x(\cdot,s)\|^2\right) ds \right)^{1/\sigma} \cdot \left( \int_0^t \left( g^\sigma \theta(1-\sigma)(t-s) \|w_x(\cdot,t)-w_x(\cdot,s)\|^2\right) ds \right)^{1-1/\sigma},
\]
(71)

We set $\sigma = (\rho - 1 + \theta)/(\rho - 1)$, (71) yields
\[
(gw_x(t)) \leq \left( \int_0^t \left( g^{1-\theta}(t-s) \|w_x(\cdot,t)-w_x(\cdot,s)\|^2\right) ds \right)^{1/\rho \pm 1+\theta} \cdot \left( \int_0^t \left( g^\sigma \theta(1-\sigma)(t-s) \|w_x(\cdot,t)-w_x(\cdot,s)\|^2\right) ds \right)^{\theta/\rho \pm 1+\theta},
\]
(72)

It is easy to see that
\[
\int_0^t \left( g^{1-\theta}(t-s) \|w_x(\cdot,t)-w_x(\cdot,s)\|^2\right) ds \leq C \sup_{t \in [0,\infty)} \|w_x(\cdot,s)\|_{H^r} \int_0^t \left( g^{1-\theta}(s)\right) ds.
\]
(73)

We obtain (68) by combining (72) and (73).

(2) We set $\theta = 1$ in (72) and it suffices to note that
\[
\int_0^t \|w_x(\cdot,t)-w_x(\cdot,s)\|^2_{H^r} ds \leq 2t \|w_x(\cdot,t)\|^2_{H^r} + 2 \int_0^t \|w_x(\cdot,s)\|^2_{H^r} ds,
\]
(74)
to arrive at (69).

In the next, we present three lemmas in which we give an upper bound of each derivative’s functions in $F(t)$.

Lemma 12. Suppose that $r > -1$ and (39) hold. Then
\[
\Phi'(t) \leq \left( 1 + \frac{l}{2b} \right) \xi(t) \int_0^t x u_x^2 dx + \left( 1 + \frac{l}{2b} \right) \xi(t) \int_0^t x v_x^2 dx - \xi(t) \left( \frac{1}{2} - \frac{C_p l}{2} \right) \int_0^t x u_x^2 dx - \xi(t) \left( \frac{1}{2} - \frac{C_p l}{2} \right) \int_0^t x v_x^2 dx - \frac{\xi(t)}{2} \left( \int_0^t g_{r}^{2-\sigma}(s) ds \right) \left( g_{r}^\theta \circ u_x \right)(t)
\]
(75)

For any $\delta > 0$.

Proof. After derivation of (56), we recall the differential equations in (1), we get
\[
\Phi'(t) = \xi'(t) \int_0^t x u_x u_x dx + \xi(t) \int_0^t x v_x^2 dx + \xi'(t) \int_0^t x v_x v_x dx + \xi(t) \int_0^t x u_x^2 dx + \xi(t) \int_0^t x u_x v_x dx
\]
(76)

By Young’s inequality and from, (G1), (G2), and Lemma 1, we arrive at
\[
\xi(t) \int_0^t x u_x(t) \left( \int_0^t g_{r}^{2-\sigma}(s) ds \right) \left( g_{r}^\theta \circ u_x \right)(t) dx
\]

In the next, we present three lemmas in which we give an upper bound of each derivative’s functions in $F(t)$.

Lemma 12. Suppose that $r > -1$ and (39) hold. Then
\[
\Phi'(t) \leq \left( 1 + \frac{l}{2b} \right) \xi(t) \int_0^t x u_x^2 dx + \left( 1 + \frac{l}{2b} \right) \xi(t) \int_0^t x v_x^2 dx - \xi(t) \left( \frac{1}{2} - \frac{C_p l}{2} \right) \int_0^t x u_x^2 dx - \xi(t) \left( \frac{1}{2} - \frac{C_p l}{2} \right) \int_0^t x v_x^2 dx - \frac{\xi(t)}{2} \left( \int_0^t g_{r}^{2-\sigma}(s) ds \right) \left( g_{r}^\theta \circ u_x \right)(t)
\]
(75)

For any $\delta > 0$.

Proof. After derivation of (56), we recall the differential equations in (1), we get
\[
\Phi'(t) = \xi'(t) \int_0^t x u_x u_x dx + \xi(t) \int_0^t x v_x^2 dx + \xi'(t) \int_0^t x v_x v_x dx + \xi(t) \int_0^t x u_x^2 dx + \xi(t) \int_0^t x u_x v_x dx
\]
(76)

By Young’s inequality and from, (G1), (G2), and Lemma 1, we arrive at
\[
\xi(t) \int_0^t x u_x(t) \left( \int_0^t g_{r}^{2-\sigma}(s) ds \right) \left( g_{r}^\theta \circ u_x \right)(t) dx
\]
\[
\xi(t) = \frac{1}{2} (1 + \frac{1}{n_1}) (1 - l_1)^2 \left( \int_0^t xu_s^2 \, dx \right)
\]
\[
+ \frac{\xi(t)}{2} \left( 1 + \frac{1}{n_2} \right) \left( \int_0^t g_1^{2-\sigma}(s) \, ds \right) (g_1^1 \circ u_x)(t) + \frac{\xi(t)}{2} \left( 1 + \frac{1}{n_1} \right) \left( \int_0^t g_1^{2-\sigma}(s) \, ds \right) (g_1^2 \circ v_x)(t)
\]
\[
+ \frac{\xi(t)}{2} \left( 1 + \frac{1}{n_2} \right) \left( \int_0^t \left[a|u + v|^{2(r+2)} + 2b|uv|^{r+2} \right] \, dx, \right.
\]
\[
\text{we choose } n_1 = l_1/1 - l_1 \text{ and } n_2 = l_2/1 - l_2, \text{ hence (75) is established.}
\]

**Lemma 13.** Suppose that \( r > -1, (G_1), (G_2), \text{ and (41) hold. Then}
\[
\chi'(t) \leq \xi(t) \left[ 2 + c_1 + c_2 + 2(1 - l_1)^2 \right] \left( \int_0^t xu_s^2 \, dx \right)
\]
\[
+ \xi(t) \left[ 2 + c_2 + 2(1 - l_1)^2 \right] \left( \int_0^t xu_v \, dx \right)
\]
\[
+ \frac{\xi(t)}{\theta} \left[ \theta - \left( \int_0^t g_1(s) \, ds \right) + \theta \right] \left( \int_0^t xu_u \, dx \right)
\]
\[
+ \frac{1}{2\theta} + 2\theta + \frac{C_p(2 + l)}{\theta} \right] \times \xi(t) \left( \int_0^t g_1^{2-\sigma}(s) \, ds \right) (g_1^2 \circ v_x)(t)
\]
\[
- \frac{C_p}{\theta} \xi(t) \left( \int_0^t g_1(t) \, ds \right) (g_1^1 \circ u_x)(t)
\]
\[
- \frac{C_p}{\theta} \xi(t) \left( \int_0^t g_1(t) \, ds \right) (g_1^2 \circ v_x)(t),
\]
\[
\text{for any } \theta > 0.
\]

**Proof.** A derivation of (57) gives
\[
\chi'(t) = -\xi'(t) \left( \int_0^t \frac{g_1(t-s)}{\eta} \, ds \right) \left( \int_0^t xu_u \, dx \right)
\]
\[
- \xi(t) \left( \int_0^t xu_u \, dx \right) \frac{d}{dt} \left( \int_0^t g_1(t-s) \, ds \right) \left( \int_0^t u(t) - u(s) \, ds \right)
\]
\[
- \xi(t) \left( \int_0^t xu_u \, dx \right) \frac{d}{dt} \left( \int_0^t g_1(t-s) \, ds \right) \left( \int_0^t v(t) - v(s) \, ds \right)
\]
\[
- \xi(t) \left( \int_0^t xu_u \, dx \right) \frac{d}{dt} \left( \int_0^t g_2(t-s) \, ds \right) \left( \int_0^t v(t) - v(s) \, ds \right)
\]
\[
- \xi(t) \left( \int_0^t xu_u \, dx \right) \frac{d}{dt} \left( \int_0^t g_2(t-s) \, ds \right) \left( \int_0^t v(t) - v(s) \, ds \right)
\]
\[
\text{(84)}
\]
by using Liebniz’s formula, we get

\begin{align}
\chi'(t) &= -\xi'(t)\int_0^t xu(t)\int_0^s g_1(t-s)(u(t)-u(s))\,ds\,dx \\
&- \xi(t)\int_0^t xu(t)\int_0^s g_1(t-s)(u(t)-u(s))\,ds\,dx \\
&- \xi(t)\int_0^t Xu(t)\int_0^s g_1(t-s)(u(t)-u(s))\,ds\,dx \\
&- \xi(t)\left(\int_0^t g_1(s)\,ds\right)\int_0^t xu^2\,dx \\
&- \xi(t)\int_0^t xu(t)\int_0^s g_2(t-s)(v(t)-v(s))\,ds\,dx \\
&- \xi(t)\int_0^t xu(t)\int_0^s g_2(t-s)(v(t)-v(s))\,ds\,dx \\
&- \xi(t)\left(\int_0^t g_2(s)\,ds\right)\int_0^t xv^2\,dx.
\end{align}

(85)

Recollecting the differentials equation in (1), we get

\begin{align}
\chi'(t) &= -\xi'(t)\int_0^t xu(t)\left(\int_0^s g_1(t-s)(u(t)-u(s))\,ds\right)\,dx \\
&+ \xi(t)\int_0^t xu(t)\left(\int_0^s g_1(t-s)(u(t)-u(s))\,ds\right)\,dx \\
&+ \xi(t)\int_0^t xu(t)\left(\int_0^s g_1(t-s)(u(t)-u(s))\,ds\right)\,dx \\
&- \xi(t)\int_0^t \left(\int_0^s g_1(t-s)(u(t)-u(s))\,ds\right)\,dx \\
&- \xi(t)\int_0^t \left(\int_0^s g_1(t-s)(u(t)-u(s))\,ds\right)\,dx \\
&- \xi(t)\int_0^t \left(\int_0^s g_1(t-s)(u(t)-u(s))\,ds\right)\,dx \\
&+ \xi(t)\int_0^t xu(t)\left(\int_0^s g_2(t-s)(v(t)-v(s))\,ds\right)\,dx \\
&+ \xi(t)\int_0^t xu(t)\left(\int_0^s g_2(t-s)(v(t)-v(s))\,ds\right)\,dx \\
&- \xi(t)\int_0^t \left(\int_0^s g_2(t-s)(v(t)-v(s))\,ds\right)\,dx \\
&- \xi(t)\int_0^t \left(\int_0^s g_2(t-s)(v(t)-v(s))\,ds\right)\,dx \\
&- \xi(t)\int_0^t \left(\int_0^s g_2(t-s)(v(t)-v(s))\,ds\right)\,dx \\
&- \xi(t)\int_0^t \left(\int_0^s g_2(t-s)(v(t)-v(s))\,ds\right)\,dx.
\end{align}

(86)

We will estimate all term in (86) by Young’s inequality, Lemma 1, \((G_1)\), and \((G_2)\).

\begin{align}
-\xi'(t)\int_0^t xu(t)\left(\int_0^s g_1(t-s)(u(t)-u(s))\,ds\right)\,dx \\
&\leq \xi(t)\left|\xi'(t)\right|\left[\theta\int_0^t xu^2\,dx + \frac{C_1}{40}\left(\int_0^t g_1^2(s)\,ds\right)(g_1' \circ u_0)(t)\right] \\
&\leq \theta\xi(t)\int_0^t xu^2\,dx + \frac{C_1}{40}\xi(t)\left(\int_0^t g_1^2(s)\,ds\right)(g_1' \circ u_0)(t),
\end{align}

(87)

\begin{align}
-\xi(t)\int_0^t xu(t)\left(\int_0^s g_1(t-s)(u(t)-u(s))\,ds\right)\,dx \\
&\leq \xi(t)\left|\xi'(t)\right|\left[\theta\int_0^t xu^2\,dx + \frac{C_1}{40}\left(\int_0^t g_1^2(s)\,ds\right)(g_1' \circ u_0)(t)\right] \\
&\leq \xi(t)\left|\xi'(t)\right|\left[\theta\int_0^t xu^2\,dx + \frac{C_1}{40}\left(\int_0^t g_1^2(s)\,ds\right)(g_1' \circ u_0)(t)\right],
\end{align}

(88)

and

\begin{align}
-\xi'(t)\int_0^t xu(t)\left(\int_0^s g_2(t-s)(v(t)-v(s))\,ds\right)\,dx \\
&\leq \xi(t)\left|\xi'(t)\right|\left[\theta\int_0^t xu^2\,dx + \frac{C_1}{40}\xi(t)\left(\int_0^t g_2^2(s)\,ds\right)(g_2' \circ u_0)(t)\right] \\
&\leq 2\theta(1-l_1)\xi(t)\int_0^t xu^2\,dx + \left(2\theta + \frac{1}{40}\right)\xi(t) \\
&- \theta\xi(t)\int_0^t xu^2\,dx - \frac{1}{40}C_1\xi(t)\left(\int_0^t g_2^2(s)\,ds\right)(g_2' \circ u_0)(t),
\end{align}

(89)

and

\begin{align}
-\xi'(t)\int_0^t xu(t)\left(\int_0^s g_2(t-s)(v(t)-v(s))\,ds\right)\,dx \\
&\leq \xi(t)\left|\xi'(t)\right|\left[\theta\int_0^t xu^2\,dx + \frac{C_1}{40}\xi(t)\left(\int_0^t g_2^2(s)\,ds\right)(g_2' \circ u_0)(t)\right] \\
&\leq \frac{C_1}{40}\xi(t)\left(\int_0^t g_2^2(s)\,ds\right)(g_2' \circ u_0)(t) \\
&+ c_1C_1\xi(t)\int_0^t xu^2\,dx + c_2\theta\xi(t)\int_0^t xv^2\,dx,
\end{align}

(90)
\[
\begin{aligned}
\{ c_1 = \Lambda_1 \left( \frac{2(r+2)}{r+1} \right) E(0)^{2^{(r+1)}} \}, \\
\{ c_2 = \Lambda_2 \left( \frac{2(r+2)}{r+1} \right) E(0)^{2^{(r+1)}} \}
\end{aligned}
\] (91)

By the same technique, we obtain estimations on integrals corresponding to \(v, g_2,\) and \(f_2\)

\[
-\xi'(t) \int_0^L x v(t) \left( \int_0^t g_2(t-s)(v(t)-v(s)) ds \right) dx \\
\leq \theta \xi(t) \int_0^L x v(t) dx + \frac{C_p}{4} \xi(t) \left( \int_0^L g_2(t-s)(v(t)-v(s)) ds \right) (g_2^\circ v_x)(t),
\]

\[
-\xi(t) \int_0^L x \left( \int_0^t g_2(t-s)v_x(t) ds \right) dx \\
\leq \theta \xi(t) \int_0^L x v(t) \left( \int_0^t g_2(t-s)v_x(t) ds \right) dx \\
\leq 2 \theta (1-l_2)^2 \xi(t) \int_0^L x v(t) dx + \left( \int_0^L g_2(t-s)(v(t)-v(s)) ds \right) (g_2^\circ v_x)(t),
\]

\[
-\xi(t) \int_0^L x v(t) \left( \int_0^t g_2(t-s)(v(t)-v(s)) ds \right) dx \\
\leq \theta \xi(t) \int_0^L x v(t) dx - \frac{\theta}{4} \left( \frac{2r_0}{C_0} \right) (g_2^\circ v_x)(t),
\] (92)

and

\[
-\xi(t) \int_0^L x f_2(t, u, v) \left( \int_0^t g_2(t-s)(v(t)-v(s)) ds \right) dx \\
\leq \frac{C_p}{4} \xi(t) \left( \int_0^L g_2(t-s)(v(t)-v(s)) ds \right) (g_2^\circ v_x)(t) \\
+ c_2 \theta \xi(t) \int_0^L x v(t) dx + c_2 \theta \xi(t) \int_0^L x v(t) dx,
\] (93)

A combination of (87)–(93) into (86) yields (83).

Lemma 14. Suppose that \(r > -1, (G_j), (G_x), (G_u),\) and (41) hold. Then

\[
\psi'(t) \leq \xi(t) \left[ 1 + ||h||_\infty \theta(2 + ||\mu||_\infty) \right] \left( \int_0^L x v(t) dx \right) \\
+ \xi(t) \left[ 1 + ||h||_\infty \theta \right] \left( \int_0^L x v(t) dx \right) \\
+ \xi(t) \left[ 1 + ||h||_\infty \theta \right] \left( \int_0^L x v(t) dx \right) \\
+ \xi(t) \left[ 1 + ||h||_\infty \theta \right] \left( \int_0^L x v(t) dx \right) \\
+ \xi(t) \left[ 1 + ||h||_\infty \theta \right] \left( \int_0^L x v(t) dx \right) \\
+ \xi(t) \left[ 1 + ||h||_\infty \theta \right] \left( \int_0^L x v(t) dx \right),
\] (95)

For any \(\theta > 0.\)

Proof. We derivate (58) and integrate by part and we finish by using the differential equations in (1), we get

\[
\psi'(t) = \xi'(t) \int_0^L u_x x h(t) u_x dx + \xi'(t) \int_0^L v_x x h(t) v_x dx \\
+ \xi(t) \int_0^L u_x x h(t) u_x dx + (q+1) \xi'(t) \int_0^L v_x x h(t) v_x dx \\
+ \xi(t) \int_0^L u_x x h(t) u_x dx + (q+1) \xi'(t) \int_0^L v_x x h(t) v_x dx \\
= \xi'(t) \int_0^L u_x x h(t) u_x dx + \xi'(t) \int_0^L v_x x h(t) v_x dx \\
- \frac{1}{2} \xi(t) \int_0^L (x h(t))' u_x dx - \frac{1}{2} \xi(t) \int_0^L (x h(t))' v_x dx \\
- \frac{1}{2} \xi(t) \int_0^L (x h(t))' u_x dx - \frac{1}{2} \xi(t) \int_0^L (x h(t))' v_x dx \\
- \xi(t) \int_0^L x u(t) h(t) u_x dx - \xi(t) \int_0^L x u(t) h(t) v_x dx
\]
Applying Young's and Poincare's inequalities, we get

\[ \xi'(t) \int_0^L u_x x h(x) u_x dx \leq \xi(t) \|h\|_{\infty} \left( \theta \int_0^L x u_x^2 dx + \frac{1}{4\theta} \int_0^L x u_x^2 dx \right), \]

(97)

\[ \int_0^L x \mu(x) h(x) u_x u_x dx \leq \|h\|_{\infty} \left( \theta \int_0^L x \mu(x) u_x^2 + \frac{1}{4\theta} \int_0^L x \mu(x) u_x^2 dx \right), \]

(98)

finally

\[ \int_0^L x h(x) u_x \left( \int_0^L g_1(t-s)(u(t)-u(s)) ds \right) dx \leq \|h\|_{\infty} \left( \theta \int_0^L x u_x^2 dx + \frac{1}{4\theta} \int_0^L x u_x^2 dx \right), \]

(99)

similarly

\[ \xi'(t) \int_0^L v_x x h(x) v_x dx \leq \xi(t) \|h\|_{\infty} \left( \theta \int_0^L x v_x^2 dx + \frac{1}{4\theta} \int_0^L x v_x^2 dx \right), \]

(100)

\[ \int_0^L x \mu(x) h(x) v_x v_x dx \leq \|h\|_{\infty} \left( \theta \int_0^L x \mu(x) v_x^2 + \frac{1}{4\theta} \int_0^L x \mu(x) v_x^2 dx \right), \]

and

\[ \int_0^L x h(x) v_x \left( \int_0^L g_1(t-s)(v(t)-v(s)) ds \right) dx \leq \|h\|_{\infty} \left( \theta \int_0^L x v_x^2 dx + \frac{1}{4\theta} \int_0^L x v_x^2 dx \right), \]

(101)

Using (97)–(101) and \( (x h(x))' \leq x \), we obtain (95).

Next Theorem show that solutions decreases exponentially with respect to \( \xi(t) \) and \( \sigma \).

**Theorem 15.** Suppose that \( r > -1 \), \((G_1)\), and \((G_2)\) hold and taking \( u_{i0}, v_{i0} \in V_{00}^r \) and \( (u_i, v_i) \in H^2 \) such that (41) hold true. Then, for each \( t_0 > 0 \), there exist positive constants \( K \) and \( k \) such that

\[ E(t) \leq \left\{ \begin{array}{ll} Ke^{-k \int_{t_0}^t \xi(s) ds}, & \sigma = 1, \\ K \left( 1 + \int_{t_0}^t \xi(s) ds \right)^{-1/\sigma - 1}, & 1 < \sigma < \frac{3}{2}, \forall t \geq t_0. \end{array} \right. \]

(102)

**Proof.** Since \( g_1 \) and \( g_2 \) are continuous and \( g_1(0) > 0, g_2(0) > 0 \) then for any \( t_0 > 0 \), we have

\[ \left\{ \int_0^t g_i(s) ds \geq \int_{t_0}^t g_i(s) ds = g_{10, 0} > 0, \quad \forall t \geq t_0, i = 1, 2. \right. \]

(103)

As \( \xi \) is a positive decreasing function hence \( 1 < \xi(t) / \xi(0) \) and by recalling Lemmas 7, 12, 13, 14, and (101), we get

\[ F'(t) = E'(t) + \xi, \Phi'(t) + \varepsilon_2 \chi' + \psi'(t) \]

\[ \leq -\int_0^t \left[ N - \frac{\|h\|_{\infty}}{4\theta} \right] \mu(x) - (1 + \frac{\|h\|_{\infty}}{4\theta}) \]

\[ - \xi_1 \left( 1 + \frac{1}{2\theta} \right) + \varepsilon_2 \left( g_{1, 0} - \theta - \xi(0) \right) \int_0^t x h(t) dx \]

\[ + \frac{1}{2} - \xi_2 \left( g_{2, 1} \right) \int_0^t x h(t) dx \]

\[ - \frac{1}{2} \xi_2 \left( 1 + \frac{\|h\|_{\infty}}{4\theta} \right) \]

\[ + \varepsilon_2 \left( 1 + \frac{1}{2\theta} \right) + \varepsilon_2 \left( g_{2, 0} - \theta - \xi(0) \right) \int_0^t x h(t) dx \]

\[ + (2\varepsilon_1 + 1) \xi(t) \int_0^t x [u \varepsilon + v^2 + 2b |v|^{\omega + r + 2}] dx \]

\[ + \frac{1}{2} - \xi_2 \left( 1 + \frac{\|h\|_{\infty}}{4\theta} \right) \]

\[ + \frac{1}{2} \xi_2 \left( g_{2, 0} \right) \int_0^t x h(t) dx \]

\[ + \varepsilon_2 \left( 1 + \frac{1}{2\theta} \right) + \varepsilon_2 \left( g_{2, 0} - \theta - \xi(0) \right) \int_0^t x h(t) dx \]

\[ + \varepsilon_2 \left( 1 + \frac{1}{2\theta} \right) + \varepsilon_2 \left( g_{2, 0} - \theta - \xi(0) \right) \int_0^t x h(t) dx \]

\[ + \varepsilon_2 \left( 1 + \frac{1}{2\theta} \right) + \varepsilon_2 \left( g_{2, 0} - \theta - \xi(0) \right) \int_0^t x h(t) dx \]

\[ \times \xi(t) \left[ \int_0^t x h(t) dx \right] + \frac{1}{2} \xi_2 \left( 1 + \frac{1}{2\theta} \right) + \varepsilon_2 \left( 1 + \frac{1}{2\theta} \right) \]

(104)
We take $\delta < (1/2C_p)\min \{l_1, l_2\}$, hence

$$
\begin{cases}
(l_1 - \delta C_p) > \frac{l_1}{2}, \\
(l_2 - \delta C_p) > \frac{l_2}{2},
\end{cases}
$$

(105)

Also, we take $\Theta > \max \{g_{1,0}/2(1 + l), g_{2,0}/2(1 + l)\}$, hence

$$
\begin{cases}
(g_{1,0} - (1 + l)\Theta) < \frac{1}{2} g_{1,0}, \\
(g_{2,0} - (1 + l)\Theta) < \frac{1}{2} g_{2,0}.
\end{cases}
$$

(106)

Now, we choose $\epsilon_2$ small enough such that

$$
k_1 = \left(1 - \frac{\epsilon_2}{4\Theta} C_p g_1(0)\right) > 0,
k_2 = \left(1 - \frac{\epsilon_2}{4\Theta} C_p g_2(0)\right) > 0,
$$

(107)

As far as $\delta, \Theta$, and $\epsilon_2$ are fixed, we then pick $\epsilon_1$ so small that

$$
k_3 = \frac{\epsilon_1}{4} \left(1 - 2C_p\mu||\epsilon\|_{\infty}\right) - \epsilon_2 \Theta \left(1 + c_1 + c_1' + 2(1 - l_1)^2\right) - 1 + ||h||_{\infty}\Theta(2 + ||\mu||_{\infty}) > 0,
k_4 = \frac{\epsilon_1}{4} \left(1 - 2C_p\mu||\epsilon\|_{\infty}\right) - \epsilon_2 \Theta \left(1 + c_2 + c_2' + 2(1 - l_2)^2\right) - 1 + ||h||_{\infty}\Theta(2 + ||\mu||_{\infty}) > 0,
k_5 = \left[-\epsilon_1 \left(1 + \frac{1}{2\delta}\right) + \epsilon_2 \left(g_{1,0} - (1 + l)\Theta\right)\right] > 0,
k_6 = \left[-\epsilon_1 \left(1 + \frac{1}{2\delta}\right) + \epsilon_2 \left(g_{2,0} - (1 + l)\Theta\right)\right] > 0,
$$

(108)

$$
= \left\{\frac{\epsilon_1}{2l_1} + \epsilon_2 \left(1 + \frac{1}{2\delta} + 2\Theta + \frac{2 + l + 2C_p||h||_{\infty} + 4\Theta}{4\Theta}\right)\right\} \left(\int_0^t g_2^{2-\sigma}(s)\,ds\right) > 0,
$$

and

$$
= \left\{\frac{\epsilon_1}{2l_3} + \epsilon_2 \left(1 + \frac{1}{2\delta} + \frac{2 + l + \frac{2}{\Theta} + 4\Theta}{4\Theta}\right)\right\} \cdot \left(\int_0^t g_2^{2-\sigma}(s)\,ds\right) > 0.
$$

(109)

Finally we choose $N > ||h||_{\infty}/4\Theta$ large enough such that

$$
\left(N - \frac{||h||_{\infty}}{4\Theta}\right) \mu(x) - \left(1 + \frac{||h||_{\infty}}{4\Theta}\right) > 0.
$$

(110)

After fixed all this choices then from (14) and (27), we obtain for some $\sigma > 0$,

$$
F'(t) \leq -\sigma \xi(t) \left(\int_0^t g_1^{1-\sigma}(s)\,ds + \int_0^t g_2^{1-\sigma}(s)\,ds\right) + \int_0^t \left[a|x| + b|\nabla x|^2 + 2b|\nabla u|^2\right] dx
$$

$$
- \int_0^t x \left[a|x| + b|\nabla x|^2 + 2b|\nabla u|^2\right] dx
+ \int_0^t x u_t^2 \,dt + \int_0^t \left[\frac{2}{\alpha} \left(\frac{2}{\alpha} \right) + \frac{1}{\alpha} \right] \left[\left(\frac{2}{\alpha} \right) + \frac{1}{\alpha} \right] \left(\frac{2}{\alpha} \right) + \frac{1}{\alpha} \right] \left(\frac{2}{\alpha} \right) + \frac{1}{\alpha} \right]
$$

(111)

$$
\leq -\gamma \xi(t) E(t) \forall t \geq t_0.
$$

We distinguish two cases according $\sigma$

Case 16. $\sigma = 1$

We recall lemma 10, and the estimate (111) gives

$$
F'(t) \leq -\gamma a_1 \xi(t) F(t), \forall t \geq t_0.
$$

(112)

A simple integration of the above inequality over $(t_0, t)$ leads to

$$
F'(t) \leq F(t_0) e^{-\gamma a_1} \int_{t_0}^t \xi(s)\,ds, \forall t \geq t_0.
$$

(113)

Hence, (102) is established.

Case 17. $1 < \sigma < 3/2$

By using (16), we get

$$
g_i(t)^{1-\sigma} \geq (\sigma - 1) \left(\int_0^t \xi(s)\,ds\right) + g_i(t_0)^{1-\sigma}, \quad i = 1, 2.
$$

(114)

So, for $\forall 0 < \tau < 2 - \sigma < 1$, (hence $(1 - \tau/\sigma - 1) > 1$), we have

$$
\int_0^t g_i^{1-\tau}(s)\,ds \leq \int_0^\infty \frac{1}{\Gamma(\sigma - 1)} \left(\int_0^t \xi(s)\,ds\right) + g_i(t_0)^{1-\sigma} \left(\sigma - 1\right)\right)^{1-\tau/\sigma - 1}\,ds.
$$

(115)
In the other hand, by using the fact that \( \int_0^\infty \xi(s) ds = +\infty \), we obtain
\[
\int_0^\infty g_i^{1-\tau}(s) ds < \infty, \forall 0 < \tau < 2 - \sigma, \text{ for } i = 1, 2. \quad (116)
\]

So form (i) of Lemma 11 with \( \theta = \tau \) and \( \rho = \sigma \) and (41) yield
\[
(g_i \circ w_x)(t) \leq C_i \left(E(0) \int_0^\infty g_i^{1-\tau}(s) ds \right)^{\sigma-1/\sigma-1+\tau}
\]
\[
\cdot \left(\left(g_i^\sigma \circ w_x\right)(t)\right)^{\tau/\sigma-1+\tau},
\]
\[
\leq C_i^\prime \left(\left(g_i^\sigma \circ w_x\right)(t)\right)^{\tau/\sigma-1+\tau},
\]
for \( i = 1, 2 \) and \( w = u, v \), respectively, with \( C_i^\prime \) are positive constants.

Therefore, for any \( \sigma_1 > 1 \), we arrive at
\[
E^{\sigma_1}(t) \leq C'' E^{\sigma_1-1}(0) \left(\int_0^t xu_x^2 dx + \int_0^t xv_x^2 dx + \int_0^t xu_x^2 dx \right)
\]
\[
+ \int_0^t xv_x^2 dx - \int_0^t x [a|u| + v|2(r+2)] + 2b|uv|^{r+2} dx\right)
\]
\[
+ C'' \left(\left(g_i \circ u_x\right)(t)\right)^{\sigma_1} + C'' \left(\left(g_i \circ v_x\right)(t)\right)^{\sigma_1}
\]
\[
\leq C'' E^{\sigma_1-1}(0) \left(\int_0^t xu_x^2 dx + \int_0^t xv_x^2 dx + \int_0^t xu_x^2 dx \right)
\]
\[
+ \int_0^t xv_x^2 dx - \int_0^t x [a|u| + v|2(r+2)] + 2b|uv|^{r+2} dx\right)
\]
\[
+ C'' \left(\left(g_i \circ u_x\right)(t)\right)^{\tau/\sigma-1+\tau}
\]
\[
+ C'' \left(\left(g_i \circ v_x\right)(t)\right)^{\tau/\sigma-1+\tau}.
\]
\[
(118)
\]

By setting \( \tau = 1/2 \) and \( \sigma_1 = 2\sigma - 1 \) which give \( \tau/\sigma-1+\tau = 1 \), the estimate (119) becomes,
\[
E^{\sigma_1}(t) \leq T \left[\int_0^t xu_x^2 dx + \int_0^t xv_x^2 dx + \int_0^t xu_x^2 dx + \int_0^t xv_x^2 dx \right]
\]
\[
- \int_0^t x [a|u| + v|2(r+2)] + 2b|uv|^{r+2} dx\right]
\]
\[
+ \left(\left(g_i \circ u_x\right)(t)\right) + \left(\left(g_i \circ v_x\right)(t)\right),
\]
\[
(119)
\]

for some \( T > 0 \). Form the equivalence between \( F \) and \( E \) and by combining the above inequality with (111), we obtain
\[
F^\prime(t) \leq -\frac{\sigma}{T} \xi(t) E^{\sigma_1}(t) \leq -\frac{\sigma}{T} \alpha_1^\prime F^{\sigma_1}(t), \quad \forall t \geq t_0.
\]
\[
(120)
\]

By twice integration over \((t_0, t)\) and \((t_0, \infty)\) of (119) successively leads to
\[
F(t) \leq C_1^\prime \left(1 + \int_{t_0}^t \xi(s) ds \right)^{-1/\sigma-1}, \quad \forall t \geq t_0,
\]
\[
(121)
\]
\[
\int_{t_0}^\infty F(t) dt \leq C_1^\prime \int_{t_0}^\infty \left(1 + \int_{t_0}^t \xi(s) ds \right)^{-1/\sigma-1} dt.
\]
\[
(122)
\]

Since \( 1/\sigma-1 > 0 \) and \( 1 + \int_{t_0}^t \xi(s) ds \rightarrow +\infty \) as \( t \rightarrow +\infty \), we get
\[
\int_{t_0}^\infty F(t) dt \leq \infty.
\]
\[
(123)
\]

Again from (121), we have
\[
T F(t) \leq \frac{C_1^\prime t}{\left(1 + \int_{t_0}^t \xi(s) ds \right)^{-1/\sigma-1}} \leq C_0, \quad \forall t \geq t_0,
\]
\[
(124)
\]

which implies that
\[
\sup_{t \geq t_0} t F(t) < \infty.
\]
\[
(125)
\]

Summing (123) and (125), we get
\[
\int_{t_0}^\infty F(t) dt + \sup_{t \geq t_0} (t F(t)) < \infty.
\]
\[
(126)
\]

By recalling (ii) of Lemma 11 with \( \rho = \sigma \), Lemma 10, and by using (27) and the above bounded, we have
\[
(g_i \circ u_x)(t) \leq C_1^\prime t \|u_x(x, t)\|_{L^2}^2 + \int_0^t \|w_x(x, s)\|_{L^2}^2 ds \right)^{1/\sigma}
\]
\[
\cdot \left(\left(\left(g_i^\sigma \circ u_x\right)(t)\right)\right)^{1/\sigma}
\]
\[
\leq C_1^\prime \left(t F(t) + \int_{t_0}^t \left(\left(\left(g_i^\sigma \circ u_x\right)(t)\right)\right)^{1/\sigma}
\]
\[
(127)
\]

Hence,
\[
(g_i^\sigma \circ u_x)(t) \geq C_1^\sigma \left(\left(\left(g_i^\sigma \circ u_x\right)(t)\right)\right)^{1/\sigma},
\]
\[
(128)
\]

for \( i = 1, 2, w = u, v \), respectively, and some positive constants \( C_1^\sigma \).
Consequently, introduce (128) in (111) and in, (118), we find

\[
F'(t) \leq -C_4 \xi(t) \left[ \int_0^t x u_x^2 dx + \int_0^t x v_x^2 dx + \int_0^t x u_x^2 dx + \int_0^t x v_x^2 dx \right.
\]
\[
+ \int_0^t x v_x^2 dx - x \left[ a[u + \alpha^2(r + 1)] + 2b|\nabla u|^2 \right] dx + \left. \right]_0^t \left[ (g_1 \circ u_x(t))^\sigma + ((g_2 \circ v_x(t))^\sigma \right],
\]

(129)

and

\[
E'(t) \leq C_5 \xi(t) \left[ \int_0^t x u_x^2 dx + \int_0^t x v_x^2 dx + \int_0^t x u_x^2 dx + \int_0^t x v_x^2 dx \right. \]
\[
- \int_0^t x \left[ a[u + \alpha^2(r + 1)] + 2b|\nabla u|^2 \right] dx + \left. \right]_0^t \left[ (g_1 \circ u_x(t))^\sigma + ((g_2 \circ v_x(t))^\sigma \right],
\]

(130)

for all \( t \geq 0 \) and some positive constant \( C_4, C_5 \).

By combining the last two inequalities and along the equivalence between \( F \) and \( E \), we obtain

\[
F'(t) \leq -C_6 \xi(t) F(t), \quad \forall t \geq t_0,
\]

(131)

for some constant \( C_6 > 0 \).

A simple integration of (131) over \((t_0, t)\) gives

\[
F(t) \leq C_0 \left( 1 + \int_{t_0}^t \xi(s) ds \right)^{-1/\sigma - 1}, \quad \forall t \geq t_0.
\]

(132)

The proof is completed by using (59).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there is no conflict of interests regarding the publication of this manuscript. The authors declare that they have no competing interests.

**Authors’ Contributions**

The authors contributed equally in this article. They have all read and approved the final manuscript.

**References**

[1] B. Cahlon, D. M. Kulkarni, and P. Shi, “Stepwise stability for the heat equation with a nonlocal constraint,” *SIAM Journal on Numerical Analysis*, vol. 32, no. 2, pp. 571–593, 1995.

[2] Y. S. Choi and K. Y. Chan, “A parabolic equation with nonlocal boundary conditions arising from electrochemistry,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 18, no. 4, pp. 317–331, 1992.

[3] A. V. Kartynnik, “Three-point boundary value problem with an integral space-variable condition for a second order parabolic equation,” *Differential Equations*, vol. 26, pp. 1160–1162, 1990.

[4] L. S. Pulkina, “A nonlocal problem with integral conditions for hyperbolic equations,” *Electronic Journal of Differential Equations*, vol. 45, pp. 1–6, 1999.

[5] L. S. Pulkina, “The 2 solvability of a nonlocal problem with integral conditions for a hyperbolic equation,” *Differential Equations*, vol. 36, no. 2, pp. 316–318, 2000.

[6] N. I. Yurchuk, “Mixed problem with an integral condition for certain parabolic equations,” *Differential Equations*, vol. 22, pp. 1457–1463, 1986.

[7] P. Shi and M. Shillor, “On design of contact patterns in one dimensional thermoelasticity,” in *Theoretical aspects of industrial design*, SIAM, Philadelphia, 1992.

[8] S. Boulaaras and N. Mezouar, “Global existence and decay of solutions of a singular nonlocal viscoelastic system with a nonlinear source term, nonlocal boundary condition, and localized damping term,” *Mathematical Methods in the Applied Sciences*, vol. 43, no. 10, pp. 6140–6164, 2020.

[9] S. Boulaaras, R. Gueftaifia, and N. Mezouar, “Global existence and decay for a system of two singular one-dimensional nonlinear viscoelastic equations with general source terms,” *Applicable Analysis*, pp. 1–25, 2020.

[10] W. Liu, Y. Sun, and G. Li, “On decay and blow-up of solutions for a singular nonlinear viscoelastic problem with a nonlinear source term.” *Topol. Methods Nonlinear Analysis*, vol. 49, pp. 299–323, 2017.

[11] M. M. Cavalcanti, N. Valeria, D. Cavalcanti, and J. A. Soriano, “Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping,” *Electronic Journal of Differential Equations*, vol. 2002, no. 44, pp. 1–14, 2002.

[12] P. Shi, “Weak solution to an evolution problem with a nonlocal constraint,” *SIAM Journal on Mathematical Analysis*, vol. 24, no. 1, pp. 46–58, 1993.