SOLUTION OF THE INVERSE SPECTRAL PROBLEM
FOR A CONVOLUTION INTEGRO-DIFFERENTIAL OPERATOR
WITH ROBIN BOUNDARY CONDITIONS

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Abstract. The operator of double differentiation on a finite interval with Robin boundary conditions perturbed by the composition of a Volterra convolution operator and the differentiation one is considered. We study the inverse problem of recovering the convolution kernel along with a coefficient of the boundary conditions from the spectrum. We prove the uniqueness theorem and that the standard asymptotics is a necessary and sufficient condition for an arbitrary sequence of complex numbers to be the spectrum of such an operator. A constructive procedure for solving the inverse problem is given.

Key words: integro-differential operator, convolution, Robin boundary conditions, inverse spectral problem, nonlinear integral equation

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1. INTRODUCTION

Consider the boundary value problem $L = L(M, h, H)$ of the form

\[ \ell y := -y'' + \int_0^x M(x-t)y'(t) \, dt = \lambda y, \quad 0 < x < \pi, \]

\[ U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(\pi) + Hy(\pi) = 0, \]

where $\lambda$ is the spectral parameter, $M(x)$ is a complex-valued function, $(\pi-x)M(x) \in L^2(0, \pi)$ and $h, H \in \mathbb{C}$.

We study an inverse spectral problem for $L$. Inverse problems of spectral analysis consist in recovering operators from given their spectral characteristics. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences and engineering. The greatest success in the inverse problem theory has been achieved for the Sturm–Liouville operator $\ell_1 y := -y'' + q(x)y$ (see, e.g., [1–4]) and afterwards for higher-order differential operators [5–7]. For example, it is known that the potential $q(x)$ can be uniquely determined by specifying the spectra of two boundary value problems for equation $\ell_1 y = \lambda y$ with one common boundary condition.

For integro-differential and other classes of nonlocal operators inverse problems are more difficult for investigation, and the classical methods either are not applicable to them or require essential modifications (see [4, 8–19] and the references therein). In [9] a perturbation of the Sturm–Liouville operator with Dirichlet boundary conditions by the Volterra convolution operator was considered. It was proven that the specification of only the spectrum uniquely determines the convolution component. Moreover, developing the idea of Borg’s method a constructive procedure for solving this inverse problem was obtained along with the local solvability and stability. In [15] the global solvability was proved by reducing this inverse problem to solving the so-called main nonlinear integral equation, which was solved globally. Earlier by a particular case of this approach the analogous results were obtained for the

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operator (1) with Dirichlet boundary conditions [12]. Here we study the case of Robin boundary conditions. A short version of this preprint is to appear in [20].

Let \( \varphi(x, \lambda) \) be a solution of equation (1) satisfying the initial conditions

\[
\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h. \tag{3}
\]

Clearly, the eigenvalues of the problem \( L \) with account of multiplicity coincide with the zeros of the function

\[
\Delta(\lambda) := V(\varphi(x, \lambda)), \tag{4}
\]

which is called the characteristic function of \( L \). By the wellknown method (see, e.g., [4]) involving Rouché’s theorem one can prove that the spectrum of \( L \) consists of infinitely many eigenvalues \( \lambda_k, \ k \geq 0 \). Moreover, the following theorem holds.

**Theorem 1.** The spectrum \( \{\lambda_k\}_{k \geq 0} \) has the form

\[
\lambda_k = (k + \kappa_k)^2, \quad \{\kappa_k\} \in l_2. \tag{5}
\]

As compared with the Dirichlet boundary conditions the Robin ones (2) bring additional difficulties in studying the inverse problem for \( L \). First, we consider the following problem.

**Inverse Problem 1.** Given \( \{\lambda_k\}_{k \geq 0} \) and \( h, H \), find \( M(x) \).

For this inverse problem we prove the following uniqueness theorem.

**Theorem 2.** The specification of the spectrum \( \{\lambda_k\}_{k \geq 0} \) uniquely determines the function \( M(x) \), provided that the coefficients \( h, H \) are known a priori.

We note that Inverse Problem 1 is overdetermined. Indeed, the spectrum possesses also some information on the coefficients of the boundary conditions. In particular, we prove that if \( h = 0 \), then along with \( M(x) \) the coefficient \( H \) is also determined, i.e. a uniqueness theorem holds for the following problem.

**Inverse Problem 2.** Given \( \{\lambda_k\}_{k \geq 0} \) and \( h = 0 \), find \( H \) and \( M(x) \).

Moreover, the following theorem holds.

**Theorem 3.** For arbitrary complex numbers \( \lambda_k, \ k \geq 0 \), of the form (5) there exists a unique (up to values on a set of measure zero) function \( M(x) \), \( (\pi - x)M(x) \in L_2(0, \pi) \), and a unique number \( H \in \mathbb{C} \), such that \( \{\lambda_k\}_{k \geq 0} \) is the spectrum of the problem \( L(M, 0, H) \).

Thus, the asymptotics (5) is a necessary and sufficient condition for the solvability of Inverse Problem 2. The importance of the assumption \( h = 0 \) is explained in Remark 1 (see Section 4). We leave open whether the analogous criterium can be obtained assuming only that \( h \) is known a priori but \( h \neq 0 \), or symmetrically: \( H \) is given while \( h \) is unknown.

In the next section we derive the main nonlinear integral equation of the inverse problem and prove the global solvability of this nonlinear equation. In Section 3 we prove some auxiliary assertions along with Theorem 1. In Section 4 we give the proof of Theorems 2 and 3, which is constructive, and provide algorithms for solving the inverse problems (Algorithms 1 and 2).

**2. MAIN NONLINEAR INTEGRAL EQUATION**

Let the functions \( C(x, \lambda), S(x, \lambda) \) be solutions of equation (1) satisfying the initial conditions

\[
C(0, \lambda) = S'(0, \lambda) = 1, \quad C'(0, \lambda) = S(0, \lambda) = 0. \]
Thus, according to (3), (4) we have $\varphi(x, \lambda) = C(x, \lambda) + h S(x, \lambda)$ and

$$\Delta(\lambda) = C'(\pi, \lambda) + h S'(\pi, \lambda) + H C(\pi, \lambda) + h H S(\pi, \lambda).$$

Let $\rho^2 = \lambda$. The following representation is wellknown:

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x P(x, t) \frac{\sin(\rho(x - t))}{\rho} dt, \quad 0 \leq x \leq \pi,$$

where $P(x, t)$ is the kernel of the transformation operator. In [12] it was shown that

$$P(x, t) = \sum_{\nu=1}^\infty \frac{(x-t)^\nu}{\nu!} N^{*\nu}(t),$$

where

$$N^{*1}(x) = N(x), \quad N^{*(\nu+1)}(x) = N * N^{*\nu}(x) = \int_0^x N(x-t) N^{*\nu}(t) dt, \quad \nu \geq 1,$$

and the function $N(x), (\pi - x)N(x) \in L_2(0, \pi)$, is connected with $M(x)$ by the relation

$$M(x) = 2N(x) - \int_0^x dt \int_0^t N(t-\tau) N(\tau) d\tau, \quad 0 < x < \pi.$$

The following lemma gives further representations, which we use in the sequel.

**Lemma 1.** The following representations hold:

$$S(x, \lambda) = K(x, x) - \rho \int_0^x K(x, t) \sin \rho(x - t) dt,$$

$$S'(x, \lambda) = R(x, x) - \rho \int_0^x R(x, t) \sin \rho(x - t) dt,$$

$$C(x, \lambda) = 1 - \rho \int_0^x Q(x, t) \sin \rho(x - t) dt,$$

$$C'(x, \lambda) = -\rho \sin \rho x - \rho \int_0^x P(x, t) \sin \rho(x - t) dt,$$

where

$$K(x, t) = t + \int_0^t (t-\tau) P(x, \tau) d\tau,$$

$$R(x, t) = 1 + \int_0^t P(x, \tau) d\tau + \int_0^t (t-\tau) P_x(x, \tau) d\tau,$$

$$Q(x, t) = 1 + \int_0^t P(x - t + \tau, \tau) d\tau.$$

**Proof.** The integration by parts in (7) gives

$$S(x, \lambda) = \int_0^x \left(1 + \int_0^t P(x, \tau) d\tau \right) \cos \rho(x - t) dt,$$
Integrating (17) by parts we arrive at (10) and (14). Then differentiating (10) with respect to $x$ we get

$$S'(x, \lambda) = \frac{d}{dx}K(x, x) - \rho \int_0^x K_x(x, t) \sin \rho (x - t) \, dt - \lambda \int_0^x K(x, t) \cos \rho (x - t) \, dt.$$

Integrating by parts in the last term we arrive at

$$S'(x, \lambda) = \frac{d}{dx}K(x, x) - \rho \int_0^x (K_x(x, t) + K_t(x, t)) \sin \rho (x - t) \, dt.$$

Taking (14) and (15) into account we get (11).

Further, we have

$$C(x, \lambda) = 1 - \lambda \int_0^x S(t, \lambda) \, dt. \quad (18)$$

Indeed, put

$$v(x) := S'(x, \lambda) - \int_0^x M(x - t)S(t, \lambda) \, dt, \quad u(x) := 1 - \lambda \int_0^x S(t, \lambda) \, dt.$$

Since $v(0) = u(0)$ and $v'(x) = u'(x)$, we have $v(x) = u(x)$. On the other hand, since

$$u'(x) = -\lambda S(x, \lambda), \quad u''(x) = -\lambda S'(x, \lambda),$$

we get

$$\ell u = \lambda \left( S'(x, \lambda) - \int_0^x M(x - t)S(t, \lambda) \, dt \right) = \lambda v(x) = \lambda u(x).$$

Taking into account $u(0) = C(0, \lambda)$ and $u'(0) = C'(0, \lambda)$ we arrive at (18).

Differentiating (18) and substituting (7) therein we get (13).

Finally, substituting (7) into (18) we get

$$C(x, \lambda) = 1 - \rho \int_0^x \sin \rho t \, dt - \rho \int_0^x dt \int_0^t P(t, t - \tau) \sin \rho \tau \, d\tau.$$

Changing the order of integration yields

$$C(x, \lambda) = 1 - \rho \int_0^x \left( 1 + \int_{x-t}^x P(\tau, \tau - x + t) \, d\tau \right) \sin \rho (x - t) \, dt,$$

which gives (12) and (16). \qed

The next lemma is a direct corollary of formulae (6), (10)–(13).

**Lemma 2.** The characteristic function has the form

$$\Delta(\lambda) = -\rho \sin \rho \pi + \alpha + \rho \int_0^\pi w(x) \sin \rho x \, dx, \quad w(x) \in L^2(0,\pi). \quad (19)$$

Here

$$\alpha = hR(\pi, \pi; N) + H + hHK(\pi, \pi; N), \quad (20)$$

$$-w(\pi - x) = P(\pi, x; N) + hR(\pi, x; N) + HQ(\pi, x; N) + hHK(\pi, x; N), \quad (21)$$

where we add the argument ”N” in order to indicate the dependence on $N(x)$. 
The relation (21) can be considered as a nonlinear equation with respect to \(N(x)\). We call it main nonlinear integral equation of the inverse problem. The main equation (21) can be rewritten in the explicit form. Indeed, by virtue of (8), (14)–(16) we have

\[
P(\pi, x; M) = \sum_{\nu=1}^{\infty} \frac{(\pi - x)^{\nu}}{\nu!} N^{*\nu}(x),
\]

where \(\nu\) is a nonnegative integer. Hence (26) has a unique solution for \(f \in L^2(0, \pi)\), where \(f = \Psi(x, t)\). Indeed, by virtue of (8), (14)–(16) we have

\[
R(\pi, x; M) = 1 + \frac{1}{\nu!} \int_{\pi/2}^{\pi} (\pi - t)^{\nu} N^{*\nu}(t) dt + \int_{\pi/2}^{\pi} \nu(x - t)(\pi - t)^{\nu-1} N^{*\nu}(t) dt,
\]

by virtue of Theorem 4 in [10], equation (26) has a unique solution for \(f \in L^2(0, \pi)\).

Substituting (22)–(25) into (21) we get

\[
f(x) = \sum_{\nu=1}^{\infty} \left( \psi_\nu(x) N^{*\nu}(x) + \int_{\pi/2}^{\pi} \Psi_\nu(x, t) N^{*\nu}(t) dt \right),
\]

where \(f(x) = -w(x) - h - H - hHx\), \(\psi_\nu(x) = (\pi - x)^{\nu}/\nu!\) and

\[
\Psi_\nu(x, t) = \frac{1}{\nu!} \left( H(\pi - x)^{\nu} + h(\pi - t)^{\nu-1}(\pi - t + (x - t)(\nu + H(\pi - t))) \right).
\]

**Theorem 4.** For each function \(f(x) \in L^2(0, \pi)\) and any complex numbers \(h, H\) equation (26) has a unique solution \(N(x)\), which belongs to \(L^2(0, T)\) for each \(T \in (0, \pi)\). Following [10] we represent this solution in the form

\[
N(x) = N_1(x) + N_2(x),
\]

where \(N_1(x) \in L^2(0, \pi)\) and \(N_2(x) = 0\) on \((0, \pi/2)\). Then we have

\[
N^{*\nu}(x) = N_1^{*\nu}(x) + \nu N_1^{*(\nu-1)} N_2(x), \quad \nu \geq 2.
\]

Substituting this into (26) we arrive at

\[
f(x) - \mu_1(x) = (\pi - x)N_2(x) + \int_{\pi/2}^{\pi} A(x, t) (\pi - t)N_2(t) dt, \quad \frac{\pi}{2} < x < \pi,
\]

where

\[
\mu_1(x) = \sum_{\nu=1}^{\infty} \left( \psi_\nu(x) N_1^{*\nu}(x) + \int_{\pi/2}^{\pi} \Psi_\nu(x, t) N_1^{*\nu}(t) dt \right),
\]

\[
A(x, t) = \frac{1}{\pi - t} \left( \Psi_1(x, t) + \sum_{\nu=2}^{\infty} \nu \left( \psi_\nu(x) N_1^{*(\nu-1)}(x - t) + \int_{t}^{\pi} \Psi_\nu(x, \tau) N_1^{*(\nu-1)}(\tau - t) d\tau \right) \right)
\]

are square-integrable functions. Hence \((\pi - x)N_2(x) \in L^2(0, \pi)\).  \(\square\)
3. AUXILIARY ASSERTIONS

In this section for convenience of the reader we prove Theorem 1 and further auxiliary assertions, which we use in Section 4 for solving the inverse problems. We note that Theorem 1 is a direct corollary of the following assertion.

Lemma 3. Any function of the form (19) has infinitely many zeros \( \lambda_k, k \geq 0 \), having the asymptotics (5).

Proof. For \( \rho \in G_\delta := \{ \rho : |\rho - k| \geq \delta, k \in \mathbb{Z} \}, \delta > 0 \), we have \( |\sin \rho \pi| \geq C_\delta \exp(|\text{Im} \rho| \pi), C_\delta > 0 \). Therefore, for sufficiently large \( |\rho|, \rho \in G_\delta \), the following inequality holds

\[
|\rho \sin \rho \pi| > |\alpha + \rho \int_0^\pi w(x) \sin \rho x \, dx|.
\]

According to Rouche’s theorem there are exactly \( N + 1 \) zeros \( \lambda_k, k = 0, N \), of the function \( \Delta(\lambda) \) lying inside the contour \( \Gamma_N = \{ \lambda : |\lambda| = (N + 1/2)^2 \} \) for sufficiently large \( N \). Moreover, for each \( \delta > 0 \) there exists \( k_\delta \) such that for \( |k| > k_\delta \) there is exactly one zero \( \rho_k \) of the function \( \Delta(\rho^2) \) inside the contour \( \gamma_k(\delta) = \{ \rho : |\rho - k| = \delta \} \). Thus, \( \rho_k = \sqrt{\lambda_k} = k + \kappa_k \), where \( \kappa_k = o(1) \). Substituting this into (19) we get \( \{ \kappa_k \} \in l_2 \) and (5) is proved. \( \square \)

Analogously to Theorem 1.1.4 in [4] using Hadamard’s factorization theorem one can obtain the following assertion.

Lemma 4. The function \( \Delta(\lambda) \) is uniquely determined by its zeros by the formula

\[
\Delta(\lambda) = \pi(\lambda_0 - \lambda) \prod_{k=1}^\infty \frac{\lambda_k - \lambda}{k^2}.
\] (27)

Proof. It follows from (19) that \( \Delta(\lambda) \) is entire in \( \lambda \) of order 1/2, and consequently by Hadamard’s factorization theorem, \( \Delta(\lambda) \) is uniquely determined up to a multiplicative constant by its zeros:

\[
\Delta(\lambda) = C \lambda^s \prod_{\lambda_k \neq 0} \left( 1 - \frac{\lambda}{\lambda_k} \right),
\] (28)

where \( s \geq 0 \) is the multiplicity of the eigenvalue \( \lambda = 0 \). Consider the function

\[
\Delta_0(\lambda) := -\rho \sin \rho \pi = -\pi \prod_{k=1}^\infty \left( 1 - \frac{\lambda}{k^2} \right).
\]

Then

\[
\frac{\Delta(\lambda)}{\Delta_0(\lambda)} = C \frac{\lambda^s}{\pi} \prod_{\lambda_k=0} \frac{n_k}{k^2 - \lambda} \prod_{\lambda_k \neq 0} \frac{n_k}{\lambda_k} \prod_{\lambda_k \neq 0} \left( 1 + \frac{\lambda_k - k^2}{k^2 - \lambda} \right), \quad \text{where} \quad n_k = \begin{cases} k^2, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}
\]

On the other hand, taking (5) and (19) into account we calculate

\[
\lim_{\lambda \to -\infty} \frac{\Delta(\lambda)}{\Delta_0(\lambda)} = 1, \quad \lim_{\lambda \to -\infty} \prod_k \left( 1 + \frac{\lambda_k - k^2}{k^2 - \lambda} \right) = 1,
\]

and hence

\[
C = (-1)^s \pi \prod_{\lambda_k=0} \frac{1}{n_k} \prod_{\lambda_k \neq 0} \frac{\lambda_k}{n_k}.
\]
Substituting this into (28) we arrive at (27).

Thus, according to Lemmas 3 and 4 any function of the form (19) is uniquely determined by formula (27) from its zeros, which, in turn, have the asymptotics (5). By the standard approach (see, e.g., Lemma 3.3 in [12]) one can prove the following inverse assertion.

**Lemma 5.** (i) Let arbitrary complex numbers \( \lambda_k, k \geq 0 \), of the form (5) be given. Then the function \( \Delta(\lambda) \) determined by (27) has the form (19) with certain \( \alpha \in \mathbb{C} \) and complex-valued function \( w(x) \in L_2(0, \pi) \).

**Proof.** Once again using the function \( \Delta_0(\lambda) := -\rho \sin \rho \pi = -\pi \lambda \prod_{k=1}^{\infty} \frac{k^2 - \lambda}{k^2} \), from (27) we get

\[
\Delta(\lambda) = \Delta_0(\lambda) F(\lambda), \quad F(\lambda) = \frac{\lambda - \lambda_0}{\lambda} \prod_{k=1}^{\infty} \frac{\lambda_k - \lambda}{k^2 - \lambda}.
\]

Put

\[
\beta_k := \Delta(k^2), \quad \theta_k := \frac{\beta_k - \alpha}{k}, \quad k \geq 1.
\]

By virtue of (29), (30), we have

\[
\beta_k = (-1)^{k+1} \frac{\pi}{2k^2} (\lambda_0 - k^2)(\lambda_k - k^2) d_k, \quad d_k = \prod_{\nu=1, \nu \neq k}^{\infty} \frac{\lambda_\nu - k^2}{\nu^2 - k^2}, \quad k \geq 1.
\]

Hence, according to (5), (31), (32) we get

\[
\theta_k = (-1)^{k+1} \frac{\pi}{2k^3} (\lambda_0 - k^2)(2\kappa_k + k^2) d_k - \frac{\alpha}{k}.
\]

Since the sequence \( \{d_k\} \) is bounded, we have \( \{\theta_k\} \in l_2 \). Determine the function \( w(x) \in L_2(0, \pi) \) such that

\[
\theta_k = \int_0^\pi w(x) \sin kx \, dx, \quad k \geq 1,
\]

and denote

\[
\theta(\rho) = \int_0^\pi w(x) \sin \rho x \, dx.
\]

Consider the function

\[
S(\rho) := \frac{\theta(\rho) - \Delta(\rho^2) - \alpha}{\sin \rho \pi} = \frac{\theta(\rho) + F(\rho^2) + \alpha}{\rho \sin \rho \pi} - 1,
\]

which is, by (31) and (33) after removing the singularities, entire in \( \rho \). Let us show that \( |F(\rho^2)| < C_\delta \) for \( \rho \in G_\delta = \{ \rho : |\rho - k| \geq \delta, k \in \mathbb{Z} \}, \ \delta > 0 \). Denote \( \kappa_{-k} := -\kappa_k, k \in \mathbb{N} \). Then

\[
F(\rho^2) = \frac{\rho^2 - \lambda_0}{\rho^2} \prod_{k \in \mathbb{N}} \left( 1 + \frac{\kappa_k}{k - \rho} \right).
\]

For a fixed \( \delta > 0 \) choose \( N \) such that \( |\kappa_k| \leq \delta/2 \) for \( k \geq N \). We have for \( \rho \in G_\delta \)

\[
F(\rho^2) = \exp(H_N(\rho)) \frac{\rho^2 - \lambda_0}{\rho^2} \prod_{0 < |k| < N} \left( 1 + \frac{\kappa_k}{k - \rho} \right),
\]

(34)
where
\[ H_N(\rho) = \sum_{|k| \geq N} \ln \left( 1 + \frac{\kappa_k}{k - \rho} \right) = \sum_{|k| \geq N} \frac{\kappa_k}{k - \rho} \sum_{\nu=0}^\infty \frac{(-1)^\nu}{\nu + 1} \left( \frac{\kappa_k}{k - \rho} \right)^\nu. \]

For \( \rho \in G_\delta \) using the Cauchy-Bunyakovsky inequality we arrive at
\[ |H_N(\rho)| \leq C \left( \sum_{|k| \geq N} \frac{1}{|k-\rho|^2} \right)^{1/2} \]
and consequently \( H_N(\rho) \) is bounded in \( G_\delta \). Thus, from (34) it follows that \( |F(\rho^2)| < C_\delta \) for \( \rho \in G_\delta \). Moreover, it is easy to see that \( F(\rho^2) \to 1 \) as \( |\text{Im}\rho| \to \infty \). Using the maximum modulus principle we conclude that \( S(\rho) \) is bounded and consequently, according to Liouville’s theorem, it is constant. Since \( S(\rho) \to 0 \) for \( |\text{Im}\rho| \to \infty \), \( S(\rho) \equiv 0 \) and formula (19) holds. □

4. SOLUTION OF THE INVERSE PROBLEM

By virtue of (19), (27), the number \( \alpha \) can be found by the formula
\[ \alpha = \pi \lambda_0 \prod_{k=1}^\infty \frac{\lambda_k}{k^2}. \]  
(35)

According to (19) the function \( w(x) \) can be reconstructed as the Fourier series
\[ w(x) = \frac{2}{\pi} \sum_{k=1}^\infty \frac{\Delta(k)}{k} \sin kx - \alpha \frac{\pi - x}{\pi}. \]  
(36)

Proof of Theorem 2. According to Lemmas 2 and 4 the specification of the spectrum uniquely determines the function \( w(x) \). Since \( h \) and \( H \) are given, by virtue of Theorem 4 the function \( N(x) \) is a unique solution of the main equation (26). Hence, the function \( M(x) \) is determined uniquely and can be reconstructed by formula (9). □

Let the spectrum \( \{\lambda_k\}_{k \geq 0} \) of a problem \( L(M,0,H) \) along with the numbers \( h, H \) be given. Then the function \( M(x) \) can be constructed by the following algorithm.

Algorithm 1. (i) Construct the function \( w(x) \) by formulae (27), (35) and (36);
(ii) find \( N(x) \) by solving the main equation (26);
(iii) construct \( M(x) \) by formula (9).

Proof of Theorem 3. Using the given numbers \( \lambda_k \) we construct the function \( \Delta(\lambda) \) by formula (27). According to Lemma 5 it has the representation (19) with certain number \( \alpha \) and function \( w(x) \in L_2(0,\pi) \). Let \( N(x) \) be the solution of the main equation (26) with this function \( w(x) \) and \( h = 0, H = \alpha \). Determine the function \( M(x), (\pi - x)M(x) \in L_2(0,\pi) \), by formula (9) and consider the corresponding boundary value problem \( L = L(M,0,H) \). It is easy to see that the constructed function \( \Delta(\lambda) \) is the characteristic function of this problem \( L \). Thus, the spectrum of the latter coincides with \( \{\lambda_k\}_{k \geq 0} \). According to (9), (19) and (27) the uniqueness of \( M(x) \) follows from the uniqueness of the solution of equation (26). □

Let a sequence \( \{\lambda_k\}_{k \geq 0} \) of the form (5) be given. According to Theorem 3 there exists a unique boundary value problem \( L(M,0,H) \) with the spectrum \( \{\lambda_k\}_{k \geq 0} \), which can be constructed by the following algorithm.
Algorithm 2. (i) Having calculated the number $\alpha$ by formula (35), put

$$h := 0, \quad H := \alpha;$$

and construct the function $w(x)$ by formulae (27), (36);
(ii) find $N(x)$ by solving the main equation (26);
(iii) construct $M(x)$ by formula (9).

Remark 1. The importance of the assumption $h = 0$ can be seen from formula (20). Indeed, since $N(x)$ is unknown, the number $\alpha$ being, in turn, uniquely determined by the spectrum determines $H$ only if $h = 0$.

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