Higher solutions of Hitchin’s self-duality equations

LYNN HELLER AND SEBASTIAN HELLER†

Leibniz Universität Hannover, Institut für Differentialgeometrie, Welfengarten 1, 30167 Hannover, Germany

†Corresponding author. Email: seb.heller@gmail.com

Communicated by: Alexander I. Bobenko

[Received on 15 March 2020; accepted on 7 August 2020]

Solutions of Hitchin’s self-duality equations correspond to special real sections of the Deligne–Hitchin moduli space—twistor lines. A question posed by Simpson in 1997 asks whether all real sections give rise to global solutions of the self-duality equations. An affirmative answer would in principle allow for complex analytic procedures to obtain all solutions of the self-duality equations. The purpose of this article is to construct counter examples given by certain (branched) Willmore surfaces in three-space (with monodromy) via the generalized Whitham flow. Though these sections do not give rise to global solutions of the self-duality equations on the whole Riemann surface $M$, they induce solutions on an open and dense subset of it. This suggest a connection between Willmore surfaces, i.e., rank 4 harmonic maps theory, with the rank 2 self-duality theory.

Keywords: self-duality equations and Higgs bundles; Willmore surfaces and harmonic maps; $\lambda$-connections and Deligne–Hitchin moduli spaces.

Introduction

The starting point of our investigations are Hitchin’s self-duality equations on a compact Riemann surface [1]

$$F^V = -[\Phi, \Phi^*]; \quad \bar{\partial}^V \Phi = 0,$$

where $F^V$ is the curvature of a special unitary connection $\nabla$ on a rank 2 hermitian bundle $V$ over the Riemann surface $M$, and $\Phi$ is a $(1, 0)$-form with values in the trace-free endomorphism bundle $\text{End}_0(V)$. These equations are a two-dimensional reduction of the self-dual Yang–Mills equations invariant under the (unitary) gauge group. Though the self-duality equations cannot be explicitly solved so far, the moduli space $\mathcal{M}$ of solutions possesses a very rich geometric structure.

Restricting to irreducible solutions, Hitchin [1] showed that $\mathcal{M}$ is a smooth manifold of dimension $12(g - 1)$ for Riemann surfaces of genus $g \geq 2$. Moreover, these irreducible solutions are uniquely determined by their Higgs pair $(\bar{\partial}^V, \Phi)$ up to unitary gauge transformations. From this perspective, $\Phi$ is a holomorphic $\text{End}_0(V)$-valued 1-form for the holomorphic vector bundle $(V, \bar{\partial}^V)$, and the irreducibility of the self-duality solution translates to the stability of the Higgs pair: $\Phi$-invariant holomorphic line subbundles of $V$ have strictly negative degree. Conversely, Hitchin [1] and Simpson [2] have shown independently that every stable Higgs pair gives rise to an irreducible solution of the self-duality equations. Therefore, there exist a 1:1 correspondence between the moduli spaces of stable Higgs pairs and irreducible
self-duality solutions—the Hitchin–Kobayashi correspondence. By construction, the moduli space of stable Higgs bundles \((\tilde{\omega}, \Phi)\) is a holomorphic symplectic manifold containing the cotangent bundle of the moduli space of stable holomorphic bundles as an open dense subset. Thus through the Hitchin–Kobayashi correspondence (the smooth part of) \(\mathcal{M}\) inherits a complex structure \(I\).

From another point of view it was observed that for solutions \((\nabla, \Phi)\) of the self-duality equations the connection \(\nabla + \Phi + \Phi^*\) is flat. Donaldson [3], using Eells and Sampson’s [4] heat flow construction, showed that every irreducible flat \(SL(2, \mathbb{C})\)-connection uniquely determines a solution of the self-duality equations (up to unitary gauge equivalence). Since the moduli space of irreducible flat \(SL(2, \mathbb{C})\)-connections is again a holomorphic symplectic manifold, \(\mathcal{M}\) naturally inherits a second complex structure \(J\). Composing the two complex structures, a third complex structure \(K\) is obtained rendering \(\mathcal{M}\) into a hyper-Kähler manifold: the three complex structures anti-commute and are Kähler with respect to the same natural \(L^2\)-metric.

The transition between the different pictures and thus the dependence of the different complex structures of \(\mathcal{M}\) on each other is difficult, except in the case where the underlying Riemann surface is a torus and all self-duality solutions are reducible. The construction of the Deligne–Hitchin moduli space \(\mathcal{M}_{\text{DH}} \rightarrow \mathbb{C}P^1\) [5, 6] is an effort to interpolate between these pictures using a parameter \(\lambda \in \mathbb{C}P^1\), where the Higgs pair can be found at \(\lambda = 0\) and the flat connection \(\nabla + \Phi + \Phi^*\) at \(\lambda = 1\). It is constructed such that the so-called associated family of flat connections

\[
\lambda \in \mathbb{C}^* \longmapsto \nabla^\lambda := \nabla + \lambda^{-1}\Phi + \lambda\Phi^*
\]

(0.1)
yields a holomorphic section of \(\mathcal{M}_{\text{DH}} \rightarrow \mathbb{C}P^1\) of a particularly simple form: it satisfies a reality condition (see (1.4)) and gives rise to a so-called twistor line when identifying \(\mathcal{M}_{\text{DH}} \rightarrow \mathbb{C}P^1\) with the twistor space of \(\mathcal{M}\), see [5]. A natural question, due to Simpson [5], is whether all real (holomorphic) sections are twistor lines, i.e., whether they all correspond to solutions of Hitchin’s self-duality equations. As noted by Simpson, an affirmative answer would allow, at least ‘philosophically’, for a complex analytic procedure to obtain all solutions of the self-duality equations.

This article gives a negative answer to this question by constructing counter examples arising from certain Willmore surfaces. Willmore surfaces \(f: M \rightarrow S^3\) are critical points of the Willmore functional

\[
\int_M (|\vec{H}|^2 + 1)dA,
\]

(0.2)
where \(\vec{H}\) denotes the mean curvature of the immersion and \(dA\) is the induced area form. The Willmore functional is invariant under conformal transformations of the 3-sphere. Examples of Willmore surfaces are given by minimal surfaces in the constant curvature subgeometries of the conformal 3-sphere. Willmore surfaces have been studied via integrable systems techniques, see for example in [7, 8]. We adjust the generalized Whitham flow for minimal and constant mean curvature surfaces in the 3-sphere developed in [9] to flow from rotational symmetric Willmore cylinders discovered by Babich and Bobenko [10] to (branched) Willmore surfaces of higher genus. The key observation here is that the Babich–Bobenko examples solve Hitchin’s self-duality equations away from their umbilic lines, i.e., they are solutions to the self-duality equations away from one-dimensional singularity sets on a torus, see [11, Figure 2 (a)].

In order to construct real sections of the Deligne–Hitchin moduli space which do not correspond to twistor lines, we start with Willmore surfaces of Babich–Bobenko type. We show that they correspond to families of flat connections satisfying the reality condition of the self-duality equations. Then, we flow these initial families of flat connections with the generalized Whitham flow introduced in [9] towards
families of flat connections on higher genus surfaces. At small rational times \( \rho \) we obtain the desired counter examples on high genus surfaces. In order to avoid singular points of the moduli space, i.e., reducible flat connections, we drop the extrinsic closing condition of the surfaces and fix the spectral curve \( \Sigma \) of the initial surface instead. When applied to solutions of spectral genus 0, the flow yields global \((\Bbb{Z}_{g+1}\)-symmetric) solutions of the self-duality equations. Therefore, we call these new real sections (corresponding to Willmore tori of spectral genus 1—see Section 2) higher solutions of the self-duality equations. They turn out to solve the self-duality equations on an open and dense subset of the Riemann surface \( M \).

There are in fact two types of real sections covering the antipodal involution \( \lambda \mapsto -\lambda^{-1} \) of \( \Bbb{CP}^1 \) corresponding to the two real subgroups \( \text{SU}(2) \) and \( \text{SU}(1,1) \) of \( \text{SL}(2, \Bbb{C}) \). The \( \text{SU}(1,1) \)-case corresponds to \((\pi_1(M)\)-equivariant) harmonic maps into the space of oriented circles in the \( 2\)-sphere \( \text{SL}(2, \Bbb{C})/\text{SU}(1,1) \), and examples are constructed in [12]. In contrast to the examples constructed in this article, those differ from twistor lines by a \( \Bbb{Z}_2 \)-invariant, see Sections 1.3 and Remark 1.22, and are not related to solutions of the self-duality equations.

The article is organized as follows. In Section 1, we first introduce the notion and the most important properties of the Deligne–Hitchin moduli space \( \mathcal{M}_{\text{DH}} \) of a compact Riemann surface. In Section 2, we describe the families of flat \( \text{SL}(2, \Bbb{C}) \)-connections on tori in terms of (algebro-geometric) spectral data. The spectral data constructed in Theorem 2.1 will serve as initial data for a flow. In Section 3, we recall the construction of a 2:1 covering of the moduli space of flat connections \( \nabla^{\lambda} \) on the 4-punctured sphere with prescribed local monodromies of \( \nabla^{\lambda} \) around the punctures. Thereafter, we use the eigenvalue \( \rho \) of the logarithmic local monodromy as the flow parameter and adapt the generalized Whitham flow techniques of [9] to deform our initial data on a torus in Section 3. In Section 4, we prove the existence of real sections of \( \mathcal{M}_{\text{DH}} \) over the Riemann surface \( M_q \) of genus \( g(\rho) \) for rational \( \rho = -\frac{p}{q} \) which map into the smooth part of the moduli space. Finally, we show in Section 5 that these new real sections give rise to solutions of the self-duality equations on open and dense subsets of the compact Riemann surface \( M_q \).

1. The Deligne–Hitchin moduli space \( \mathcal{M}_{\text{DH}} \)

The Deligne-Hitchin moduli space \( \mathcal{M}_{\text{DH}} = \mathcal{M}_{\text{DH}}(M) \) of a compact Riemann surface \( M \) provides a natural tool to study associated families of flat connections of solutions to the self-duality equations. It was first defined by Deligne (see [5, 6]) as a complex analytic reincarnation of the twistor space (see [13]) associated to the hyper-Kähler moduli space of self-duality solutions.

**Definition 1.1** For \( \lambda \in \Bbb{C} \) fixed, a (integrable) \( \lambda \)-connection on a \( C^\infty \) complex vector bundle \( V \to M \) over a Riemann surface \( M \) is a pair \( (\bar{\partial}, D) \) consisting of a holomorphic structure on \( V \) and a linear first-order differential operator

\[
D : \Gamma(M, V) \to \Omega^{(1,0)}(M, V)
\]

satisfying the \( \lambda \)-Leibniz rule

\[
D(fs) = \lambda \bar{\partial}f \otimes s + fDs
\]

for functions \( f \) and sections \( s \), and the integrability condition

\[
D\bar{\partial} + \bar{\partial}D = 0. \quad (1.1)
\]
Example 1.2 Let $V = M \times \mathbb{C}^n$ be a trivial rank $n$ bundle equipped with the trivial connection $d$. Let $d = d' + d'' = \bar{\partial}^0 + \partial^0$ be its decomposition into its $(1, 0)$ and $(0, 1)$ parts. Then, for $\lambda \in \mathbb{C}$ fixed, the pair 

$$(\bar{\partial}^0, \lambda \partial^0)$$

gives the so-called trivial $\lambda$-connection on $V$. For $\lambda = 0$, the corresponding $\lambda$-connection reduces to the trivial holomorphic structure on $V$.

Remark 1.3 The operators $\bar{\partial}$ and $\partial$ also act on $(0, 1)$-forms and $(1, 0)$-forms respectively. For $\lambda = 0$ the integrability condition (1.1) is equivalent to 

$$D = \Phi \in H^0(M, K\text{End}(V))$$

being a holomorphic endomorphism-valued 1-form, and for $\lambda \neq 0$ we have that 

$$\nabla = \bar{\partial} D + \partial$$

is a flat connection. Here and in the following $K = (T^*M)^{(1,0)}$ is the canonical bundle of the Riemann surface $M$.

Definition 1.4 A $\lambda$-connection $(\bar{\partial}, D)$ for $\lambda = 0$ is called a Higgs pair. In this case, $D = \Phi \in H^0(M, K\text{End}(V))$ will be referred to as a Higgs field.

In this article, we restrict to the subclass of $\lambda$-connections corresponding to the group $G_c = \text{SL}(2, \mathbb{C})$. Note that a $\lambda$-connection on a vector bundle $V$ induces $\lambda$-connections on all associated tensor bundles, e.g., $V^*$ and $\Lambda^n V$. For $\lambda = 0$ and $n = \text{rank}(V)$, the induced $\lambda$-connection of $(\bar{\partial}, D)$ on $\Lambda^n V$ is given by the induced holomorphic structure on the determinant bundle and the trace of $D$.

Definition 1.5 A $\text{SL}(2, \mathbb{C})$ $\lambda$-connection is a $\lambda$-connection on a rank 2 vector bundle $V \to M$ over a compact Riemann surface $M$, such that the induced $\lambda$-connection on $\Lambda^2 V$ is trivial.

For the rest of the section, we consider the case where $M$ is compact and has genus $g \geq 2$. We may assume without loss of generality that 

$$V = M \times \mathbb{C}^2,$$

since every complex vector bundle $V$ over a surface with trivial determinant is trivial as a $C^\infty$-vector bundle.

Definition 1.6 Let $M$ be a compact Riemann surface. A $\text{SL}(2, \mathbb{C})$ $\lambda$-connection $(\bar{\partial}, D)$ is called stable, if every $\bar{\partial}$-holomorphic subbundle $L \subset V = \mathbb{C}^2$ with 

$$D(\Gamma(M, L)) \subset \Omega^{(1,0)}(M, L)$$

satisfies 

$$\deg(L) < 0$$
and semi-stable if

\[ \deg(L) \leq 0. \]

All other \( \lambda \)-connections are called unstable. A \( \text{SL}(2, \mathbb{C}) \) \( \lambda \)-connection is called polystable if it is stable or the direct sum of \( \lambda \)-connections on degree zero line bundles.

For \( \lambda \neq 0 \), every \( \bar{\partial} \)-holomorphic and \( D \)-invariant line subbundle \( L \subset V \) is automatically parallel with respect to the flat connection \( \nabla = \frac{1}{2} D + \bar{\partial} \). Therefore, the degree of \( L \) is 0 and the \( \lambda \)-connection \( (\bar{\partial}, D) \) is semi-stable. Moreover, \( (\bar{\partial}, D) \) is stable if and only if the flat connection \( \nabla = \frac{1}{2} D + \bar{\partial} \) is irreducible. For \( \lambda = 0 \) there exist unstable \( \lambda \)-connections and their gauge orbits are infinitesimally close to the gauge orbits of (certain) stable \( \lambda \)-connections. Moreover, the gauge orbits of certain semi-stable \( \lambda \)-connections are infinitesimally close to each other, see also the notion of \( S \)-equivalence for the case of holomorphic bundles [14]. In order to obtain a well-behaved moduli space, we restrict to polystable \( \lambda \)-connections.

For \( \lambda \in \mathbb{C} \) fixed, let \( \mathcal{A}_2^\lambda \) denote the space of (integrable) \( \text{SL}(2, \mathbb{C}) \) \( \lambda \)-connections, and \( \dot{\mathcal{A}}_2^\lambda \) the subspace of polystable \( \lambda \)-connections. Then there is a natural action of the gauge group

\[ G = \{ g : M \to \text{SL}(2, \mathbb{C}) \mid g \text{ is a } C^\infty \text{ map.} \} \]

on \( \mathcal{A}_2^\lambda \) and for \( \lambda \neq 0 \) the quotient

\[ \dot{\mathcal{A}}_2^\lambda / G \]

is biholomorphic to the moduli space of flat totally reducible \( \text{SL}(2, \mathbb{C}) \)-connections. Recall that a totally reducible connection is by definition a direct sum of irreducible connections. As such it is a complex analytic space which is smooth away from (gauge orbits of) reducible flat connections. For \( \lambda = 0 \), the quotient

\[ \dot{\mathcal{A}}_0^2 / G \]

is the moduli space of polystable Higgs bundles \( \mathcal{M}_{\text{Dol}} \) [1].

**Definition 1.7** Let \( M \) be a compact Riemann surface of genus \( g \geq 2 \). The Hodge moduli space \( \mathcal{M}_{\text{Hod}} = \mathcal{M}_{\text{Hod}}(M) \) is the space of all polystable \( \text{SL}(2, \mathbb{C}) \) \( \lambda \)-connections on \( V = M \times \mathbb{C}^2 \to M \) modulo gauge transformations.

The gauge-equivalence class of a \( \lambda \)-connection \( (\lambda, \bar{\partial}, D) \) is denoted by

\[ [\lambda, \bar{\partial}, D] \in \mathcal{M}_{\text{Hod}} \]

or by

\[ [\lambda, \bar{\partial}, D]_M \in \mathcal{M}_{\text{Hod}}(M) \]

to emphasis its dependence on the Riemann surface.
Remark 1.8 The Hodge moduli space can be equipped with an algebraic structure through the GIT construction [5]. However, we prefer to think of $\mathcal{M}_{\text{Hod}}$ as a complex analytic space (with quotient topology). The gauge orbits of stable $\lambda$-connections are smooth points of $\mathcal{M}_{\text{Hod}}$. These form an open and dense subset $\mathcal{M}^\text{stable}_\text{Hod} \subset \mathcal{M}_{\text{Hod}}$. We restrict to the locus of gauge orbits of stable $\lambda$-connections in this paper. As a stable $\lambda$-connection does not permit non-trivial automorphisms (trivial automorphisms are constant multiples of the identity), the smooth structure can be constructed by standard gauge theoretic methods. The complex dimension of this space is $\dim(\mathcal{M}_{\text{Hod}}) = \dim(M) + 1$, e.g., in the SL(2, $\mathbb{C}$)-case over a surface of genus $g$ the dimension is $6g - 5$.

The Hodge moduli space admits a holomorphic map $f = f_M : \mathcal{M}_{\text{Hod}} \to \mathbb{C}$; $[\lambda, \bar{\partial}, D] \mapsto \lambda$ whose fibre at $\lambda = 0$ is the (polystable) Higgs moduli space $\mathcal{M}_{\text{Dol}}$ of Hitchin [1], see also [12], and at $\lambda = 1$ it is the deRham moduli space of flat (totally reducible) SL(2, $\mathbb{C}$)-connections $\mathcal{M}_{\text{dR}}$, which we consider as complex analytic spaces endowed with their natural complex structures $I$ and $J$, respectively.

1.1 The gluing construction of the Deligne–Hitchin moduli space

Let $M$ be a Riemann surface and $\overline{M}$ be its complex conjugate Riemann surface. As differentiable manifolds we have $M \cong \overline{M}$ and thus their deRham moduli spaces of flat SL(2, $\mathbb{C}$)-connections are naturally isomorphic (as complex analytic spaces, not as algebraic spaces). Through the Deligne gluing [6]

$$\Psi : \mathcal{M}_{\text{Hod}}(M) \setminus f_M^{-1}(0) \to \mathcal{M}_{\text{Hod}}(\overline{M}) \setminus f_{\overline{M}}^{-1}(0); [\lambda, \bar{\partial}, D]_M \mapsto [\lambda, \bar{\partial}, D, \frac{1}{\lambda} \bar{\partial}]_{\overline{M}}$$

we can define the Deligne–Hitchin moduli space to be

$$\mathcal{M}_{\text{DH}} = \mathcal{M}_{\text{Hod}}(M) \cup_{\Psi} \mathcal{M}_{\text{Hod}}(\overline{M}).$$

The Deligne–Hitchin moduli space admits a natural fibration $f : \mathcal{M}_{\text{DH}} \to \mathbb{C}P^1$ whose restriction to $\mathcal{M}_{\text{Hod}}(M)$ is $f_M$ and whose restriction to $\mathcal{M}_{\text{Hod}}(\overline{M})$ is $1/f_{\overline{M}}$.

Remark 1.9 Note that the Deligne-gluing map $\Psi$ maps stable $\lambda$-connections on $M$ to stable $\frac{1}{\lambda}$-connections on $\overline{M}$. Hence, it maps the smooth locus of $\mathcal{M}_{\text{Hod}}(M)$ (consisting of stable $\lambda$-connections) to the smooth locus of $\mathcal{M}_{\text{Hod}}(\overline{M})$, and $\mathcal{M}_{\text{DH}}$ is equipped with a structure of a complex manifold at all of its stable points.

Definition 1.10 A section of $\mathcal{M}_{\text{DH}}$ is a holomorphic map

$$s : \mathbb{C}P^1 \to \mathcal{M}_{\text{DH}}$$

such that $f \circ s = \text{Id}$.

It is well-known (and one of the motivations behind its definition) that

$$f : \mathcal{M}_{\text{DH}} \to \mathbb{C}P^1$$
is holomorphic isomorphic to the twistor fibration $\mathcal{P} \rightarrow \mathbb{C}P^1$ of the hyper-Kähler metric on the moduli space of solutions to Hitchin’s self-duality equations, at least at the smooth points, see [5]. The isomorphism is given as follows. Take a solution $(\nabla, \Phi)$ of the self-duality equation and the twistor line

$$\lambda \mapsto (\lambda, \bar{\partial}^{\nabla}, \Phi)$$

with respect to the $C^\infty$-trivialization $\mathcal{P} \cong M_{Dol} \times \mathbb{C}P^1$. Then, this twistor line is holomorphically isomorphic to the section given by the holomorphic map

$$\lambda \in \mathbb{C} \mapsto [\lambda, \bar{\partial}^{\nabla} + \lambda \Phi^*, \lambda \bar{\partial}^{\nabla} + \Phi]_M \in \mathcal{M}_{Hod}(M) \subset \mathcal{M}_{DH}.$$  \hfill (1.2)

It follows from the work of Hitchin [1] and Donaldson [3] that every stable point in $\mathcal{M}_{DH}$ uniquely determines a twistor line.

**Definition 1.11** A holomorphic section $s$ of $\mathcal{M}_{DH}$ is called stable, if the $\lambda$-connection $s(\lambda)$ is stable for all $\lambda \in \mathbb{C}^*$ and if the Higgs pairs $s(0)$ on $M$ and $s(\infty)$ on $\overline{M}$ are stable.

Note that a twistor line $s$ is already stable if $s(\lambda_0)$ is stable for one $\lambda_0 \in \mathbb{C}$.

**1.2 Automorphisms of the Deligne–Hitchin moduli space**

The Deligne–Hitchin moduli space admits some natural automorphisms which will play important roles in the later sections. First of all, for every $\mu \in \mathbb{C}^*$ the (multiplicative) action of $\mu$ on $\mathbb{C}P^1$ has a natural lift to $\mathcal{M}_{DH}$ by

$$\mu([\lambda, \bar{\partial}, D]) = [\mu \lambda, \mu \bar{\partial}, \mu D].$$

**Definition 1.12** We denote by $N : \mathcal{M}_{DH} \rightarrow \mathcal{M}_{DH}$ the map given by multiplication with $\mu = -1$, namely

$$[\lambda, \bar{\partial}, D] \mapsto [-\lambda, \bar{\partial}, -D].$$

Note that by dropping the special linear condition from Definition 1.5 we may define the Deligne–Hitchin moduli space for the case $GL(n, \mathbb{C})$ analogously to the $SL(2, \mathbb{C})$-case. In the general case (e.g. for $GL(n, \mathbb{C})$ rather than $SL(2, \mathbb{C})$ connections) taking the dual of a flat connection gives rise to an automorphism of the moduli space of flat connections, which extends to the Deligne–Hitchin moduli space we denote by $\sigma$. The automorphism $\sigma : \mathcal{M}_{DH} \rightarrow \mathcal{M}_{DH}$ is given by

$$[\lambda, \bar{\partial}, D] \mapsto [\lambda, \bar{\partial}^*, D^*],$$

where $(\cdot)^*$ denotes the dual operator. Since $SL(2, \mathbb{C})$-connections and $SL(2, \mathbb{C}) \lambda$-connections are self-dual, $\sigma$ is just the identity map in our case.

The reality conditions for sections are related to a natural anti-holomorphic automorphism denoted by $C$. 

Definition 1.13 Let $C: \mathcal{M}_{DH} \longrightarrow \mathcal{M}_{DH}$ be the continuation of the map $\tilde{C}: \mathcal{M}_{Hod}(M) \longrightarrow \mathcal{M}_{Hod}(\overline{M})$ given by

$\tilde{C}([\lambda, \tilde{\partial}, D]_M) \mapsto [\overline{\lambda}, \overline{\tilde{\partial}}, \overline{D}]_{\overline{M}}.$ (1.3)

To be more concrete, for $\overline{\partial} = \overline{\partial}_0 + \eta$ and $D = \lambda(\overline{\partial}_0) + \omega$ where $d = \partial^0 + \overline{\partial}^0$ is the trivial connection, $\eta \in \Omega^{0,1}(M, \mathfrak{sl}(n, \mathbb{C}))$, and $\omega \in \Omega^{1,0}(M, \mathfrak{sl}(n, \mathbb{C}))$, we define the complex conjugate on the trivial $\mathbb{C}^n$-bundle over $\overline{M}$ to be

$\overline{\partial} = \partial^0 + \overline{\eta}$ and $\overline{D} = \overline{\lambda}(\overline{\partial}^0) + \overline{\omega}.$

The map $C$ covers the map

$\lambda \in \mathbb{C}P^1 \mapsto \overline{\lambda}^{-1} \in \mathbb{C}P^1.$

It is important to note that $C$ and $N$ commute. Moreover, both maps are involutive. Thus, their composition

$T = CN$

is an involution as well, covering the fixed-point free involution $\lambda \mapsto -\overline{\lambda}^{-1}$ on $\mathbb{C}P^1$.

Remark 1.14 For Deligne–Hitchin moduli spaces associated to other Lie groups than $\text{SL}(2, \mathbb{C})$ there exist different anti-holomorphic involutions

$\mathcal{M}_{DH} \longrightarrow \mathcal{M}_{DH}$

covering $\lambda \mapsto \overline{\lambda}^{-1}$. For $\text{SL}(n, \mathbb{C})$ and $\text{GL}(n, \mathbb{C})$, another natural choice is given by

$[\lambda, \tilde{\partial}, D]_M \mapsto [\lambda, \tilde{\partial}^*, D^*]_{\overline{M}}$

with respect to a hermitian metric on $V$. For $n = 2$, we have (with respect to the standard hermitian metric on $\mathbb{C}^2 \rightarrow M$)

$\overline{\partial} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\partial}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\overline{D} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$

Thus both definitions coincide.

In the case of rank 1 Deligne–Hitchin moduli spaces, we denote by $C$ the real involution on $\lambda$-connections induced by complex conjugation analogous to (1.3).
1.3 Real sections

In the following, let $M$ be a compact Riemann surface of genus $g \geq 2$. We consider the antiholomorphic involution of the associated Deligne–Hitchin moduli space

$$\mathcal{T} = CN: \mathcal{M}_{\text{DH}} \longrightarrow \mathcal{M}_{\text{DH}}$$

covering the antipodal involution

$$\lambda \mapsto -\bar{\lambda}^{-1}$$

of $\mathbb{CP}^1$. A holomorphic section $s$ of $\mathcal{M}_{\text{DH}}$ is called real with respect to $\mathcal{T}$ if

$$\mathcal{T}(s(-\bar{\lambda}^{-1})) = s(\lambda)$$

(1.4)

holds for all $\lambda \in \mathbb{CP}^1$.

Example 1.15 We claim that twistor lines (1.2) are real holomorphic sections with respect to $\mathcal{T}$ in the case $G_v = \text{SL}(2, \mathbb{C})$. Let $(\nabla, \Phi)$ be a solution of the self-duality equations on $M$ with respect to the standard hermitian metric on $\mathbb{C}^2 \rightarrow M$. Because we are dealing with $\mathfrak{sl}(2, \mathbb{C})$-matrices, the unitary connection $\nabla$ satisfies

$$\nabla = \nabla^* = \bar{\nabla} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is equivalent to

$$\bar{\partial}^\nabla = \bar{\partial} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \partial^\nabla = \partial \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and analogously we have

$$\Phi = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Phi^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \Phi^* = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

compare with Remark 1.14. Therefore, the twistor line (1.2) satisfies

$$\mathcal{T}(s(\lambda)) = \mathcal{T}([\lambda, \bar{\nabla}^\lambda + \lambda \Phi^* + \Phi + \lambda \partial^\nabla]_M)$$

$$= [\bar{\lambda}^{-1}, \bar{\nabla}^{-1} (\Phi + \lambda \partial^\nabla), -\bar{\lambda}^{-1} (\bar{\partial}^\nabla + \lambda \Phi^*)]_M$$

$$= [\bar{\lambda}^{-1}, \bar{\nabla} + \bar{\lambda}^{-1} \Phi, -\Phi^* - \bar{\lambda}^{-1} \bar{\partial}^\nabla]_M$$

$$= [(-\bar{\lambda}^{-1}, \bar{\partial}^\nabla + \bar{\lambda}^{-1} \Phi, -\Phi^* - \bar{\lambda}^{-1} \bar{\partial}^\nabla)]_M$$

$$= [(-\bar{\lambda}^{-1}, \bar{\partial}^\nabla - \bar{\lambda}^{-1} \Phi^*, \Phi - \bar{\lambda}^{-1} \partial^\nabla)]_M = s(-\bar{\lambda}^{-1}).$$

(1.5)
In order to obtain a global lift of a section \( s \) of \( \mathcal{M}_{\text{DH}} \) to the space of flat connections or integrable \( \lambda \)-connections, it is technically necessary to consider \( \lambda \)-connections with connection 1-forms that are only \( C^k \) on \( M \) (rather than \( C^\infty \)), for some \( k \in \mathbb{N}^{\geq 2} \). For every \( \lambda \in \mathbb{C} \), these \( C^k \)-\( \lambda \)-connections are gauge equivalent to \( C^\infty \)-\( \lambda \)-connections, compare with [15, Section 14].

We will make use of the following lemma which is proven analogously to the proof of Theorem 8 in [16].

**Lemma 1.16** Let \( s \) be a holomorphic and stable section of \( \mathcal{M}_{\text{DH}} \rightarrow \mathbb{C}P^1 \). Then there exists a holomorphic lift \( \hat{s}_k \) of \( s \) on \( C \subset \mathbb{C}P^1 \) to the space of \( C^k \)-\( \lambda \)-connections on \( M \) for every \( k \in \mathbb{N}^{\geq 2} \).

We restrict ourselves from now on to stable sections \( s \). Admissible sections are particularly well-behaved sections \( s \) of \( \mathcal{M}_{\text{DH}} \) of the form

\[
s(\lambda) = [\lambda, \tilde{\partial} + \lambda \Psi, \lambda D + \Phi]
\]

for a holomorphic structure \( \tilde{\partial} \), a \( \partial \)-operator \( D \), an endomorphism-valued \((1,0)\)-form \( \Phi \) and an endomorphism-valued \((0,1)\)-form \( \Psi \), such that \((\tilde{\partial}, \Phi)\) and \((D, \Psi)\) are stable Higgs pairs on \( M \) and \( \overline{M} \), respectively. In particular, twistor lines are admissible.

Consider a real, holomorphic and stable section \( s \). By Lemma 1.16, \( s \) admits a \( C^k \)-lift on \( \mathbb{C} \)

\[
\lambda \mapsto (\lambda, \tilde{\partial}^\lambda, D^\lambda).
\]

Let

\[
\lambda \in \mathbb{C}^* \subset \mathbb{C}P^1 \mapsto \nabla^\lambda := \tilde{\partial}^\lambda + \frac{i}{\lambda} D^\lambda
\]

be the corresponding family of flat connections.

Then \( s \) being real translates to the existence of a gauge transformation \( g(\lambda) \), for every \( \lambda \in \mathbb{C}^* \), satisfying

\[
\nabla^\lambda \cdot g(\lambda) = \overline{\nabla^{-\lambda^{-1}}}. \tag{1.6}
\]

We will show in Lemma 1.18 below the existence of a holomorphic \( \mathbb{C}^* \)-family of \( \text{SL}(2, \mathbb{C}) \) gauge transformations

\[
\lambda \in \mathbb{C}^* \mapsto g(\lambda)
\]

satisfying (1.6). We call a family of flat connections \( \nabla^\lambda \) satisfying (1.6) real. Applying equation (1.6) twice, we obtain

\[
\nabla^\lambda \cdot g(\lambda) g(-\frac{1}{\lambda}) = \nabla^\lambda.
\]

Because the section \( s \) is stable, the connections \( \nabla^\lambda \) are irreducible for all \( \lambda \in \mathbb{C}^* \). Therefore \( g(\lambda) g(-\frac{1}{\lambda}) \) is a constant multiple of the identity for every \( \lambda \in \mathbb{C}^* \). Therefore,

\[
\mathcal{H}_\lambda := \{ g \in \mathcal{C}^{k+1}(M, (\mathbb{C}^2)^* \otimes \mathbb{C}^2) \mid g \text{ is parallel with respect to (} \overline{\nabla^{-\lambda^{-1}}} \otimes (\nabla^\lambda) \}\}
is a one-dimensional vector space for every $\lambda \in \mathbb{C}^*$. We consider gauge transformations of class $C^{k+1}$ for $\lambda \in \mathbb{C}^*$, since the gauge action is

$$\omega \mapsto g^{-1}dg + g^{-1}\omega g,$$

for a connection 1-form $\omega$. Therefore, a gauge transformation of class $C^{k+1}$ yields a connection form of class $C^k$. At $\lambda = 0$ the gauge behaves slightly differently. A Higgs pair of class $C^k$ is gauge equivalent to a Higgs pair of class $C^\infty$ by a gauge transformation of Sobolev class $W^{2,k+1}$ by [15]. Nevertheless, by the Sobolev embedding theorem the gauge transformation is then at least $C^{k-1}$.

Because the connections $\nabla^\lambda$ are irreducible, $C^k$ and satisfy (1.6), $H_\lambda$ is one-dimensional for all $\lambda \in \mathbb{C}$. Clearly, if $g(\lambda)$ satisfies (1.6) then $g(\lambda) \in H_\lambda$. Hence, for $g \in H_\lambda \setminus \{0\}$

$$\text{Det}(g) \in \mathbb{C}^*$$

is constant on $M$ (as we only consider $\text{SL}(2, \mathbb{C})$-connections) and non-zero.

Consider the complex vector space of $W^{2,k+1}$-endomorphisms

$$E^{k+1} := \{ g : M \to g(2, \mathbb{C}) \mid g \text{ is of class } W^{2,k+1} \}$$

and let

$$H := \bigcup_{\lambda \in \mathbb{C}^*} H_\lambda \subset (\mathbb{C}^* \times E^{k+1} \to \mathbb{C}^*).$$

We claim that $H$ is a holomorphic line sub-bundle over $\mathbb{C}^*$: The kernel bundle $K$ of the holomorphic structures

$$(\overline{\partial}_{\nabla^\lambda - 1})^* \otimes \overline{\partial}_{\nabla^\lambda}$$

is a holomorphic vector bundle over $\mathbb{C}^*$, see [17, Proposition 3.1]. The bundle extends holomorphically through the points where the dimension of the kernel jumps [17]. By construction $H \subset K$. The family of holomorphic structures $\overline{\partial}_{\nabla^\lambda}$ considered in this article will be generically irreducible. Hence, the fibres of $K$ are one-dimensional, from which we obtain that $H = K$ is a holomorphic line bundle as claimed.

As $\mathbb{C}^*$ is a non-compact Riemann surface and hence Stein there exists a nowhere vanishing holomorphic section

$$\lambda \in \mathbb{C}^* \mapsto g(\lambda) \in H_\lambda \subset E^{k+1}$$

(1.7)

of the bundle $H \to \mathbb{C}^*$. In particular, the map

$$g : \mathbb{C}^* \times M \to \text{End}(\mathbb{C}^2)$$

is of class $C^{k-1}$ by Sobolev embedding theorem, and for every $p \in M$ the map

$$\lambda \in \mathbb{C}^* \mapsto g(\lambda, p) \in \text{GL}(2, \mathbb{C})$$
is holomorphic. Note that for every $\lambda \in \mathbb{C}^*$ the gauge $g(\lambda)$ has constant determinant $d(\lambda)$ on $M$. We claim that there exists a holomorphic section $g$ as in (1.7) such that the holomorphic map

$$\lambda \in \mathbb{C}^* \mapsto d(\lambda) = \text{Det}(g(\lambda,p)),$$

is also constant, where $p \in M$ is arbitrary. We remark that in the case of non-real holomorphic sections of Deligne–Hitchin moduli spaces, the analogous statement does not hold in general, see for example [12, Proposition 2.11].

**Lemma 1.17** Let $\lambda \in \mathbb{C}^* \mapsto \nabla^\lambda$ be a real and holomorphic family of irreducible flat $\text{SL}(2, \mathbb{C})$-connections and $g$ the corresponding holomorphic family of $\text{GL}(2, \mathbb{C})$-gauge transformations in the sense of (1.7). Then, there exists a holomorphic map

$$h : U \to \mathbb{C}^*$$

defined on an open neighbourhood $U$ of the closed unit disc $D_1 := \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$, such that $\tilde{g} := hg$ satisfies

$$\tilde{g}(\lambda)\tilde{g}(-\bar{\lambda}^{-1}) = \pm \text{Id} \quad (1.8)$$

for all $\lambda \in \mathbb{C}^*$. The sign on the right hand does not depend on the choice of $h$.

**Proof.** By irreducibility of $\nabla^\lambda$ we have

$$g(\lambda)g(-\bar{\lambda}^{-1}) = \hat{f}(\lambda)\text{Id}$$

for a holomorphic function $\hat{f}$ without zeros along $S^1$. Moreover, we can compute the index of the curve $\gamma = \hat{f}|_{S^1}$ to be

$$\text{Ind}_0(\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{d\hat{f}}{\hat{f}} = \frac{1}{4\pi i} \int_\gamma \frac{d(\hat{f}^2)}{(\hat{f})^2} = \frac{1}{4\pi i} \int_\gamma \frac{d(\text{det } g(\lambda))}{\text{det } g(\lambda)} + \frac{1}{4\pi i} \int_\gamma \frac{d(\text{det } g(-\bar{\lambda}^{-1}))}{\text{det } g(-\bar{\lambda}^{-1})} = 0.$$ 

Therefore, there exists a well-defined holomorphic function $f$ such that $\hat{f}(\lambda) = \exp(\lambda)$). Consider the Laurent series of $f$ for $\lambda \in S^1$

$$f(\lambda) = \sum_{k \in \mathbb{Z}} f_k \lambda^k.$$

Then $\hat{f}(\lambda) = \hat{f}(-\bar{\lambda}^{-1})$ yields for $k \neq 0$

$$(-1)^k\hat{f}_k = f_{-k}$$

and

$$f_0 = \hat{f}_0 + n2\pi i,$$
for some \( n \in \mathbb{Z} \). Hence \( \tilde{g} = hg \) for the holomorphic function

\[
h(\lambda) = \exp \left( - \sum_{k \in \mathbb{Z} > 0} f_k \lambda^k - \frac{1}{2} \text{Re} f_0 \right)
\]

has the desired properties.

Let \( \hat{h} \) be another holomorphic map such that \( \hat{g} = \hat{h} \tilde{g} \) satisfies (1.8). The map \( \hat{h} \) is holomorphic and therefore we have

\[
\hat{h}(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, \quad \text{and} \quad \overline{\hat{h}(\bar{\lambda}^{-1})} = \sum_{n=0}^{\infty} (-1)^n \overline{a_n} \lambda^{-n}.
\]

(1.9)

From (1.8), we obtain

\[
\hat{h}(\lambda) \overline{\hat{h}(\bar{\lambda}^{-1})} = \pm 1.
\]

By (1.9) \( \hat{h}(\lambda) \) has only non-negative Laurent coefficients and \( \overline{\hat{h}(\bar{\lambda}^{-1})} \) has only non-positive Laurent coefficients. Hence, both maps must be constant. Therefore,

\[
\hat{h}(\lambda) \overline{\hat{h}(\bar{\lambda}^{-1})} = 1
\]

showing the claimed independence of the choice of \( h \) as long as \( h \) is well defined on the closed unit disc.

\[\square\]

Although not every holomorphic section \( s \) of \( \mathcal{M}_{DH} \) admits lifts over

\[
\mathbb{C} \subset \mathbb{C}P^1 \quad \text{and} \quad \mathbb{C}P^1 \setminus \{0\}
\]

which are related by a holomorphic \( \mathbb{C}^* \)-family of \( \text{SL}(2, \mathbb{C}) \)-valued gauge transformations (see Definition 2.8 and Proposition 2.11. in [12]), this property holds for real sections.

**Lemma 1.18** Let \( \lambda \in \mathbb{C}^* \mapsto \nabla^{\lambda} \) be a real and holomorphic family of irreducible flat \( \text{SL}(2, \mathbb{C}) \)-connections. Then there is a holomorphic family of gauge transformations

\[
\lambda \in \mathbb{C}^* \mapsto g(\lambda) \in \mathcal{H}_\lambda \subset \mathcal{E}^{k+1}
\]

with \( \det[g(\lambda)] \equiv 1 \) satisfying (1.6). The family \( g(\lambda) \) is unique up to sign.

**Proof.** Assume that \( g \) cannot be chosen to be \( \text{SL}(2, \mathbb{C}) \)-valued for all \( \lambda \in \mathbb{C}^* \). Then the index of the curve \( \tilde{\gamma} = \text{det}g|_{S^1} \)

\[
\text{Ind}_0(\tilde{\gamma}) = \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{d\text{det}(g)}{\text{det}(g)}
\]
must be odd. Otherwise, the square root of $\det(g)$ would be well defined and $\tilde{g} = \frac{1}{\sqrt{\det(g)}} g$ defines a family of $\text{SL}(2, \mathbb{C})$-gauge transformations satisfying (1.6). By multiplying $g$ with a suitable holomorphic function defined on $\mathbb{C}^*$, the index of $\tilde{\gamma}$ changes by an even integer. Thus, we can assume without loss of generality that $\text{Ind}_0(\tilde{\gamma}) = 1$. Applying Lemma 1.17, we can further assume

$$g(\lambda)g(-\bar{\lambda}^{-1}) = \pm \text{Id}. \quad (1.10)$$

For $p \in M$ fixed, consider the Birkhoff factorization (see [18, Chapter 8] or Section 5 for a short summary) of $g_p(\lambda)$

$$g_p(\lambda) = g_+(\lambda) \begin{pmatrix} \lambda^{l+1} & 0 \\ 0 & \lambda^{-l} \end{pmatrix} g_-(\lambda) \quad (1.11)$$

for some $l \in \mathbb{N}$, where $g_+$ is a holomorphic map into $\text{SL}(2, \mathbb{C})$ that extends holomorphically to $\lambda = 0$ while $g_-$ extends holomorphically to $\lambda = \infty$. The diagonal matrix

$$\begin{pmatrix} \lambda^{l+1} & 0 \\ 0 & \lambda^{-l} \end{pmatrix}$$

accounts for the fact that $\text{Ind}_0(\tilde{\gamma}) = 1$. By the uniqueness part of the Birkhoff factorization, every other pair $(\tilde{g}_+, \tilde{g}_-)$ satisfying (1.11) is given by

$$\tilde{g}_+(\lambda) = g_+(\lambda) \begin{pmatrix} \frac{1}{a} & -\frac{1}{ad}b(\lambda^{-1})\lambda^{2l+1} \\ 0 & \frac{1}{a} \end{pmatrix}$$

and

$$\tilde{g}_-(\lambda) = \begin{pmatrix} a & b(\lambda^{-1}) \\ 0 & d \end{pmatrix} g_-(\lambda),$$

with constants $a, d \in \mathbb{C}^*$ and a polynomial $b$ (in the variable $\lambda^{-1}$) of degree at most $2l + 1$. By (1.10), we can relate the Birkhoff factorizations of $g(\lambda)$ and $g(-\bar{\lambda}^{-1})$:

$$g(\lambda)^{-1} = \pm g_+(-\bar{\lambda}^{-1}) \begin{pmatrix} (-1)^{l+1} & 0 \\ 0 & (-1)^{l} \lambda^l \end{pmatrix} g_-(\bar{\lambda}^{-1})$$

and therefore

$$g(\lambda) = \pm (-1)^{l}g_-(\bar{\lambda}^{-1})^{-1} \begin{pmatrix} -\lambda^{l+1} & 0 \\ 0 & \lambda^{-l} \end{pmatrix} g_+(-\lambda^{-1})^{-1}.\quad$$

Hence, there exist $a, d \in \mathbb{C}^*$ and a polynomial $b(\lambda^{-1})$ such that

$$\pm (-1)^{l}g_-(\bar{\lambda}^{-1})^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = g_+(\lambda) \begin{pmatrix} \frac{1}{a} & -\frac{1}{ad}b(\lambda^{-1})\lambda^{2l+1} \\ 0 & \frac{1}{a} \end{pmatrix}.$$
and
\[
g_+(-\bar{\lambda}^{-1})^{-1} = \begin{pmatrix} a & b(\lambda^{-1}) \\ 0 & \bar{d} \end{pmatrix} g_-(\lambda).
\]

Putting the last two equations together yields that either \(-a\bar{a} = 1\) or \(-\bar{d}d = 1\), depending on the sign of \((-1)^{i}\) and the sign of (1.10), giving a contradiction in either case. The uniqueness of \(g\) up to sign follows from the stability of \(s\). \(\square\)

For every holomorphic \(\mathbb{C}^*\)-lift \(\nabla^\lambda\) of a real holomorphic section \(s\) of \(\mathcal{M}_{\text{DH}}\), the two lemmas above yield the existence of a holomorphic family of \(\text{SL}(2, \mathbb{C})\)-gauge transformations
\[
\lambda \in \mathbb{C}^* \mapsto g(\lambda) \in \mathcal{H}_i \subset \mathcal{E}^{k+1}
\]
(unique up to sign) satisfying (1.6) and
\[
g(\lambda)g(-\bar{\lambda}^{-1}) = \pm \text{Id}. \quad (1.12)
\]

By [12, Lemma 2.15], the sign on the right-hand side is independent of the lift \(\nabla^\lambda\) of \(s\) motivating the following definition.

**Definition 1.19** [12, Definition 2.16] A stable real section \(s\) of \(\mathcal{M}_{\text{DH}}\) is called positive or negative depending on the sign of (1.12).

**Remark 1.20** A real section \(s\) corresponding to a solution of Hitchin’s self-duality equations is negative. In fact, a canonical lift is given by the associated family of flat connections, and for the standard hermitian structure on \(\mathbb{C}^2\) we obtain that
\[
g(\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
is constant in \(\lambda\) and squares to \(-\text{Id}\).

**Question 1.21** Simpson [5, Section 4] raised the question whether every real holomorphic section of the Deligne–Simpson twistor space \(\mathcal{M}_{\text{DH}}\) induces a solution of the self-duality equations.

**Remark 1.22** It is shown in [12, Theorem 3.6] that all stable, admissible, and negative sections of \(\mathcal{M}_{\text{DH}}\) are twistor lines. It is also proved [12] (Theorem 3.4) that there exists admissible positive stable sections of \(\mathcal{M}_{\text{DH}}\). In Simpson’s notation, [19] being admissible is equivalent to being pure as a mixed twistor structure. The new sections of the Deligne–Hitchin moduli space constructed in this article are stable and negative but not admissible.
1.4 The conformal Gauss map

In the definition of positive and negative real holomorphic sections, the stability of the lift is crucial. The conformal Gauss map is a geometric example where dropping the stability condition at $\lambda = 0$ gives two lifts of the section $s$ on $\mathbb{C}^*$ with different signs in (1.12).

Given a real holomorphic stable section $s$ of the Deligne–Hitchin moduli space with a (non-zero) nilpotent Higgs field $\Phi$ at $\lambda = 0$, we consider the kernel bundle

$$L := \ker \Phi$$

and a complementary (smooth) subbundle $\tilde{L}$ of $V = \mathbb{C}^2$. Define the family of gauge transformations

$$\lambda \in \mathbb{C}^* \mapsto h(\lambda) := \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

with respect to

$$V = L \oplus \tilde{L}.$$ 

A direct computation gives (see [20])

$$\tilde{\nabla}^\lambda := \nabla^\lambda h(\lambda) = \lambda^{-1} \tilde{\Phi} + \tilde{\nabla} + ...$$  \hspace{1cm} (1.13)

for a new nilpotent Higgs pair $(\tilde{\partial} \tilde{\nabla}, \tilde{\Phi})$. This Higgs pair $(\tilde{\partial} \tilde{\nabla}, \tilde{\Phi})$ is not stable, since $\tilde{L} = \ker \tilde{\Phi}$ and

$$\deg(\tilde{L}) = - \deg(L) > 0$$

is positive by assumption. Thus $\tilde{\nabla}^\lambda$ is not a lift of $s$ on the whole complex plane $\mathbb{C} \subset \mathbb{C}P^1$.

Lemmas 1.17 and 1.18 give rise to a family of SL(2, $\mathbb{C}$) gauge transformations $g(\lambda)$ satisfying

$$\overline{\nabla}^{-\lambda^{-1}} = \nabla^\lambda g(\lambda).$$

This family is unique up to sign. Moreover, the family of SL(2, $\mathbb{C}$) gauge transformations

$$\tilde{g}(\lambda) := \lambda h^{-1}(\lambda) g(\lambda) h(-\bar{\lambda}^{-1}).$$  \hspace{1cm} (1.14)

satisfies

$$\overline{\nabla}^{-\lambda^{-1}} = \tilde{\nabla}^\lambda \tilde{g}(\lambda)$$

and a direct computation shows

$$\tilde{g}(\lambda) \tilde{g}(-\bar{\lambda}^{-1}) = -g(\lambda) g(-\bar{\lambda}^{-1}).$$

This means that we have been able to change the sign by gauging to an unstable Higgs pair at $\lambda = 0$. 

Remark 1.23 The construction (1.13) is well-known in the theory of immersed surfaces in 3-space. In our case, a twistor line gives rise to an equivariant harmonic map $f$ into the hyperbolic 3-space $SL(2, \mathbb{C})/SU(2)$. The Higgs field being nilpotent corresponds to the harmonic map being conformal, hence $f$ is minimal. Consider the hyperbolic 3-space as a subspace of the round 3-sphere with a round 2-sphere as its boundary at infinity, e.g., the Poincare ball model. Every point $p$ of the surface $f$ uniquely determines the best approximating 2-sphere at that point, i.e., the sphere and the surface touch with the same orientation and the same mean curvature at $p$. In the minimal surface case, these 2-spheres are totally geodesic and intersect the boundary at infinity perpendicularly, see [21] and the references therein, or [20, Section 4] for a short summary. This yields a conformal harmonic map $G$ from $M$ into the space of oriented circles in the 2-sphere (the boundary at infinity). The associated families of flat connections $\tilde{\nabla}^\lambda$ and $\tilde{\nabla}^\lambda$ of $f$ and $G$, respectively, are related via the construction (1.13). The map $G$ is called the conformal Gauss map or central sphere congruence.

Example 1.24 Even if the section $s$ is not admissible, i.e., the gauges $g(\lambda)$ do not admit a global Birkhoff factorization into positive and negative gauges on $M$ (see Section 5 and Definition 5.6), $\tilde{g}(\lambda)$ in (1.14) might. Let

$$\lambda \in \mathbb{C}^* \mapsto \tilde{\nabla}^\lambda = \nabla + \lambda^{-1}\Phi + \lambda\Phi^\#$$

(1.15)

be a family of flat connections, where $\nabla$ is a SU(1, 1)-connection and $\#$ denotes the adjoint for the standard SU(1, 1) structure $(.,.)$ on $\mathbb{C}^2$. We refer to $(\nabla, \Phi, (.,.)$) as a SU(1, 1) self-duality solution.

Assume there exist a curve $\gamma \subset M$ with

$$L_{|\gamma} = \ker(\Phi)_{|\gamma} = \ker(\Phi^\#)_{|\gamma},$$

i.e., $L$ is a null line with respect to $(.,.)$ along $\gamma$. On $M \setminus \gamma$ we can reverse the construction in (1.13) by taking the gauge

$$\tilde{h}(\lambda) = h^{-1}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with respect to

$$\mathbb{C}^2_{|M \setminus \gamma} = \ker(\Phi)_{|M \setminus \gamma} \oplus \ker(\Phi^\#)_{|M \setminus \gamma}.$$
The sign depends on the component of $M \setminus \gamma$ and is chosen such that the unitary structure is positive definite. Note that $h^{-1}$ becomes singular along $\gamma$. The real holomorphic sections constructed in Section 4 have the same behaviour as this example.

2. Spectral genus 1 solutions on a torus

The aim of this section is to provide spectral data of singular solutions of the cosh-Gordon equation on a Riemann surface $M$ of genus 1. Away from the singularities, these give rise to minimal surfaces in the hyperbolic 3-space and correspond therefore to local solutions of the self-duality equations. The spectral data will serve as initial data for the flow constructed in this article.

We denote the rank one Deligne–Hitchin moduli space corresponding to the group $G_C = \mathbb{C}^*$ of a Riemann surface $M$ by $\mathcal{M}_{DH}^1(M) = \mathcal{M}_{DH}^1$. As in the $SL(2, \mathbb{C})$ case, the rank one Deligne–Hitchin moduli space of $M$ admits the automorphisms $N$ and the real involutions $C$ and $T = CN$. It also has an additional holomorphic involution

$$\sigma : \mathcal{M}_{DH}^1(M) \to \mathcal{M}_{DH}^1(M); [\lambda, \bar{\partial}, D] \mapsto [\lambda, \bar{\partial}^*, D^*],$$

taking a $\lambda$-connection to its dual.

**Theorem 2.1** Let $\Gamma = 2\mathbb{Z} + 2\tau\mathbb{Z}$ and $\Lambda = \mathbb{Z} + \tau_{\text{spec}}\mathbb{Z}$ with $\tau \in i\mathbb{R}^{-1}$, $\tau_{\text{spec}} \in i\mathbb{R}^{-2}$ be rectangular lattices and let $M = \mathbb{C}/\Gamma$ and $\Sigma = \mathbb{C}/\Lambda$ be the corresponding Riemann surfaces. Then there exists a holomorphic map

$$D : \Sigma \to \mathcal{M}_{DH}^1(M)$$

satisfying

1. $\lambda := fM \circ D : \Sigma \to \mathbb{C}P^1$ is a double covering branched over $0, \infty, r, -\frac{1}{r}$ for some $0 < r < 1$;
2. there is a holomorphic involution $\sigma : \Sigma \to \Sigma$ such that
   $$D \circ \sigma = \sigma \circ D$$
   and
   $$\lambda \circ \sigma = \lambda;$$
3. there is a real involution $\eta : \Sigma \to \Sigma$ covering the antipodal involution $\lambda \mapsto -\bar{\lambda}^{-1}$ such that
   $$\sigma(D(\eta(\xi))) = T(D(\xi))$$
   for all $\xi \in \Sigma$.

**Proof.** On the Riemann surface, $\Sigma$ consider the real fixed-point free involution

$$\eta : \Sigma \to \Sigma; \ [\xi] \mapsto [\xi + \frac{1}{2}\tau_{\text{spec}}]$$

and the elliptic involution

$$\sigma : [\xi] \mapsto [-\xi].$$
Since $\eta$ commutes with $\sigma$, it induces a real fixed-point free involution on $\Sigma/\sigma \cong \mathbb{C}P^1$ which, after applying a suitable M"obius transformation, is the antipodal map. Thus, there is a 2-fold covering

$$\lambda : \Sigma \longrightarrow \mathbb{C}P^1$$

with $\lambda \circ \sigma = \lambda, \lambda([0]) = 0, \lambda \circ \eta(\xi) = -\lambda(\xi)^{-1}$ for all $\xi \in \Sigma$. Without loss of generality, we can assume

$$r := \lambda([\frac{1}{2}]) \in \mathbb{R} \quad \text{and} \quad R := \lambda([\frac{r_{\text{spec}}}{2}]) = -\frac{1}{r} \in \mathbb{R}.$$ 

The preimage of $\lambda = \infty$ is $\xi = \frac{1 + r_{\text{spec}}}{2}$. It can be shown that for $r_{\text{spec}} \in i\mathbb{R} > 1$ or equivalently $R < -1$.

For fixed $r_{\text{spec}}$ and $\lambda : \Sigma \rightarrow \mathbb{C}P^1$, we choose constants $a, b \in \mathbb{C}$ such that the Weierstrass $\wp$-function on $\Sigma = \mathbb{C}/\Lambda$ satisfies

$$\int_{1} (a\wp + b) d\xi = 2 \quad (2.1)$$

and

$$\int_{r_{\text{spec}}} (a\wp + b) d\xi = 0, \quad (2.2)$$

where we identify $\Lambda \cong H_1(\Sigma, \mathbb{Z})$. In fact, $a, b$ are real and given by

$$a = -\frac{r_{\text{spec}}}{\pi i} \quad \text{and} \quad b = -2\eta_3 \pi i, \quad (2.3)$$

and $\eta_3 = \zeta(\frac{r_{\text{spec}}}{2})$ for the Weierstrass $\zeta$-function, see [9, p. 10].

Let $\chi$ to be the meromorphic function on $\mathbb{C}$ uniquely determined by

$$d\chi = \frac{\pi i}{2\tau} (a\wp(\xi - \frac{1 + r_{\text{spec}}}{2}) + b)d\xi \quad \text{and} \quad \chi(0) = 0. \quad (2.4)$$

By construction, $\chi$ has a first-order pole in $\xi = \frac{1 + r_{\text{spec}}}{2}$. Moreover, let $\alpha$ be the meromorphic function on $\mathbb{C}$ given by

$$\alpha(\xi) = \frac{\chi(\xi - \frac{1 + r_{\text{spec}}}{2})}{2\tau} + \frac{\pi i}{2\tau}. \quad (2.5)$$

It has a first-order pole in $\xi = 0$.

As before, we decompose $d = \partial^0 + \bar{\partial}^0$ into its $(1, 0)$ and $(0, 1)$-parts. Consider the Riemann surface $M = \mathbb{C}/\Gamma$ equipped with its affine coordinate $w$ and the map

$$\mathcal{D} : \Sigma \rightarrow \mathcal{M}_{1, \mathbb{DH}}^1(M); \quad \xi \mapsto \left[\lambda(\xi), \bar{\partial}^0 - \chi(\xi)d\bar{w}, \lambda(\xi)(\partial^0 + \alpha(\xi)dw)\right]_M.$$
We want to show that \( D \) is well defined on \( \Sigma = \mathbb{C}/\Lambda \). The map \( D \) is well defined on \( \mathbb{C} \setminus \left( \Lambda \cup \left( \frac{1 + \tau_{\text{spec}}}{2} + \Lambda \right) \right) \).

It extends holomorphically to \( \xi = 0 \) as \( \alpha \) has a first-order pole and \( \lambda \) has a zero at \( \xi = 0 \). Using the Deligne-gluing, \( D \) extends holomorphically to \( \xi = \frac{1 + \tau_{\text{spec}}}{2} \) as well.

We claim that

\[
D(\xi) = D(\xi + \gamma) \in M_{1 \text{DH}}^1(M)
\]

for all \( \xi \in \mathbb{C} \setminus \left( \Lambda \cup \left( \frac{1 + \tau_{\text{spec}}}{2} + \Lambda \right) \right) \) and \( \gamma \in \Lambda \), showing that \( D \) is well defined on \( \Sigma \). Recall that \( \Lambda \) is generated by 1 and \( \tau_{\text{spec}} \). By (2.2), we get

\[
\chi(\xi + \tau_{\text{spec}}) = \chi(\xi) \quad \text{and} \quad \alpha(\xi + \tau_{\text{spec}}) = \alpha(\xi).
\]

Recall that \( \lambda(\xi) = \lambda(\xi + 1) \). Hence, \( D(\xi) = D(\xi + \tau_{\text{spec}}) \). By (2.1), we get

\[
\chi(\xi + 1) = \chi(\xi) + \frac{\pi i}{\tau} \quad \text{and} \quad \alpha(\xi + 1) = \alpha(\xi) + \frac{\pi i}{\tau}.
\]

Then, by the definition of the gauge group action on \( \lambda \)-connections, the two \( \lambda \)-connections

\[
\left( \lambda(\xi), \bar{\partial}^0 - \chi(\xi)d\bar{w}, \lambda(\xi)(\bar{\partial}^0 + \alpha(\xi)dw) \right)_M
\]

and

\[
\left( \lambda(\xi), \bar{\partial}^0 - \left( \chi(\xi) + \frac{\pi i}{\tau} \right)d\bar{w}, \lambda(\xi)(\bar{\partial}^0 + (\alpha(\xi) + \frac{\pi i}{\tau})dw) \right)_M
\]

are gauge equivalent by the gauge transformation

\[
g : M \to \mathbb{C}^*; \ z \mapsto \exp \left( \frac{\pi i}{\tau} (z - \bar{z}) \right).
\]

Therefore, \( D \) is well defined on \( \Sigma \). The properties (1)–(3) are easy to check. \( \square \)

**Remark 2.2** In the following, we refer to \( \Sigma \) as the spectral curve and to \( (\Sigma, \lambda, \chi, \alpha) \) as spectral data, where \( \chi \) and \( \alpha \) are as in the proof of Theorem 2.1.

**Lemma 2.3** With the notations of Theorem 2.1 and for generic \( \tau \in i\mathbb{R}^{>0} \) the image of the map

\[
\xi \in S := \lambda^{-1}(S^1) \subset \Sigma \mapsto [\bar{\partial}^0 - \chi(\xi)d\bar{w}] \in \text{Jac}(M)
\]

does not contain a spin bundle on \( M \). In particular,

\[
[\bar{\partial}^0 - \chi(\cdot)d\bar{w}] : \lambda^{-1}(D_1) \subset \Sigma \longrightarrow \text{Jac}(M)
\]

only maps \([0], [\frac{1}{2}] \) to the trivial line bundle in \( \text{Jac}(M) \) for large \( \tau \in i\mathbb{R}^{>0} \).
Remark 2.4 A spin bundle $S$ on a Riemann surface $M$ is a holomorphic line bundle for which $S^2 = K$ is the canonical bundle of $M$. In the case of $M$ being a torus, a spin bundle is a square root of the trivial holomorphic line bundle.

Proof. The holomorphic structure

$$\bar{\partial}^0 - \chi(\xi)d\bar{w} \quad \text{on} \quad \mathbb{C}^2 \to M$$

is spin if and only if

$$\frac{4\tau}{\pi i} \chi(\xi) \in \mathbb{Z} \oplus \tau \mathbb{Z}.$$ (2.6)

We first show that $\frac{4\tau}{\pi i} \chi|_{\xi}$ is nowhere real valued, i.e., that its imaginary part never vanishes. Note that for $\tau_{\text{spec}} \in i\mathbb{R} > 1$ one component of $S$ is contained in

$$\{x + iy \mid 0 \leq x \leq 1; 0 < y < \frac{\tau_{\text{spec}}}{2}\}.$$

Due the properties of the Weierstrass $\wp$-function for rectangular lattices, the function $\frac{4\tau}{\pi i} \chi$ has real values along the real line and along the line $\frac{\tau_{\text{spec}}}{2} + \mathbb{R}$. Moreover,

$$\frac{4\tau}{\pi i} \chi(\xi) \in i\mathbb{R}^{>0} \quad \text{for} \quad \xi \in \{iy \mid 0 < y < \frac{\tau_{\text{spec}}}{2}\}$$

and

$$\frac{4\tau}{\pi i} \chi(\xi) \in 2 + i\mathbb{R}^{>0} \quad \text{for} \quad \xi \in \{1 + iy \mid 0 < y < \frac{\tau_{\text{spec}}}{2}\}.$$

Let

$$\mathcal{R} := \{\xi \in \{x + iy \mid 0 < x < 1; 0 < y < \frac{\tau_{\text{spec}}}{2}\} \mid \frac{4\tau}{\pi i} \chi(\xi) \in \mathbb{R}\}.$$

We claim that the zeros of the meromorphic function $a\wp + b$ are simple and contained in the shifted imaginary axis $\frac{1}{2} + i\mathbb{R} \mod \Lambda$. Because $a, b \in \mathbb{R}$ are real, the function $a\wp + b$ takes real values along the axes

$$\mathbb{R} \mod \Lambda \quad \text{and} \quad i\mathbb{R} \mod \Lambda$$

and along the shifted axes

$$\frac{\tau_{\text{spec}}}{2} + i\mathbb{R} \mod \Lambda \quad \text{and} \quad \frac{1}{2} + i\mathbb{R} \mod \Lambda.$$

Moreover, $a\wp + b$ has a double pole at $[0]$ and nowhere else, and therefore has exactly two zeros counted with multiplicity. If there is no zero of the meromorphic function $a\wp + b$ on the set

$$\{\frac{1}{2} + \tau_{\text{spec}}t \mid 0 < t < \frac{1}{2} \text{ or } \frac{1}{2} < t < 1\},$$
the function would be either non-negative or non-positive on the shifted imaginary axis $\frac{1}{2} + i \mathbb{R} \mod \Lambda$ which contradicts (2.2). Hence, there are two simple zeros of the function $a\wp + b$ contained in the set $\{\frac{1}{2} + t \mathbb{Z} \mid 0 < t < \frac{1}{2} \text{ or } \frac{1}{2} < t < 1\}$.

Therefore, the function $\frac{2\pi}{\tau} \chi$ has only simple poles and its critical points (which are all of order one since $a\wp + b$ has only simple zeros) are contained in $\mathbb{Z} + i \mathbb{R}$. Thus we have

$$\mathcal{R} \cap \partial \{x + iy \mid 0 < x < 1; 0 < y < \frac{\text{spec}}{2}\} = \emptyset$$

and $\mathcal{R}$ is a closed submanifold in

$$\{x + iy \mid 0 < x < 1; 0 < y < \frac{\text{spec}}{2}\} \subset \mathbb{C}.$$ If $\mathcal{R}$ would be non-empty, it contains a critical point of $\frac{2\pi}{\tau} \chi$ giving a contradiction.

By (2.4), the map $\frac{2\pi}{\tau} \chi$ does not depend on $\tau$, its real part is real analytic and non-constant on $S$. Hence, there exists $k \in \mathbb{N}$ and $\xi_1, \ldots, \xi_k \in S$ such that

$$\{\xi \in S \mid \Re \left(\frac{4\pi}{\tau} \chi(\xi)\right) \in \mathbb{Z}\} = \{\xi_1, \ldots, \xi_k\}.$$ Since the imaginary part of $\frac{2\pi}{\tau} \chi|_S$ never vanishes, the set of $\tau \in i \mathbb{R}^{>0}$ for which (2.6) holds is given by

$$T_{\text{spin}} = \{\tau \in i \mathbb{R}^{>0} \mid \exists \xi \in S : \frac{4\pi}{\tau} \chi(\xi) \in \mathbb{Z} \oplus \tau \mathbb{Z}\}$$

$$= \{\tau \in i \mathbb{R}^{>0} \mid \exists l \in \mathbb{N}^k : \Im \left(\frac{4\pi}{\tau} \chi(\xi_l)\right) \in \mathbb{Z}\}$$

(where we recall that $\chi$ depends on $\tau$ by definition). Hence, the set $T_{\text{spin}}$ is discrete. Moreover, we obtain that for $\tau_{\text{spec}} \in i \mathbb{R}^{>0}$ fixed, the set

$$\{|\tau| \mid \tau \in T_{\text{spin}}\}$$

is bounded from above but accumulates at 0.

For the second assertion, recall that the holomorphic structure

$$\bar{\partial}^0 - \chi(\xi) d\bar{w} \quad \text{on} \quad \mathbb{C}^2 \rightarrow M$$

is trivial if and only if

$$\frac{2\pi}{\tau} \chi(\xi) \in \mathbb{Z} \oplus \tau \mathbb{Z}. \quad (2.7)$$

Again, the function $\xi \mapsto \frac{2\pi}{\tau} \chi(\xi)$ does not depend on $\tau$. Let

$$m := \sup \{|\Im (\frac{2\pi}{\tau} \chi(\xi))| \mid \xi \in S\} < \infty.$$

Hence, if $\tau$ is large enough, i.e., $\frac{\tau}{\tau} > m$, trivial spin structures $\bar{\partial}^0 - \chi(\xi) d\bar{w}$ for $[\xi] \in \lambda^{-1}(D_i)$ can only occur for real $\xi \mod \Lambda$. Recall that $\frac{2\pi}{\tau} \chi$ is only real along the real axis, and strictly monoton increasing
with period 2. Because \( \frac{2\pi}{\tau} \chi(0) = 0 \) and because \( \frac{2\pi}{\tau} \chi \) is an odd function of \( \xi \), the only points in \( \lambda^{-1}(D_1) \) which are mapped to the trivial class in \( \text{Jac}(M) \) are [0] and \([\frac{1}{2}]\).

**Lemma 2.5** With the notations of Theorem 2.1 and for \( \tau \in i\mathbb{R} > 0 \) large enough, the line bundle connection
\[
d - \chi(\xi)d\bar{w} + \alpha(\xi)dw
\]
is never a non-trivial spin connection on the torus \( M \) for all \( \xi \in \lambda^{-1}(D_1) \).

**Proof.** Analogously to the proof of Lemma 2.3 a spectral value \( \xi_0 \in \lambda^{-1}(D_1) \) such that \( \bar{\partial}^0 - \chi(\xi_0)dw \) is a non-trivial spin structure must be real. Because the function \( \xi \mapsto \frac{2\pi}{\tau} \chi(\xi) \) is monoton increasing with period 2 along the real line and due to 2.6 and 2.7 there is a unique \( \xi_1 = 1 - \xi_0 \in [\frac{1}{2}, 1] \) such that \( \bar{\partial}^0 - \chi(\xi_1)dw \). Due to the symmetries, it remains to study the connection \( d - \chi(\xi_0)d\bar{w} + \alpha(\xi_0)dw \) at \( \xi_0 \).

Restricted to the real axis, the function \( \chi \) is monotonic increasing, and the function \( \alpha \) is monotonic decreasing. To be a spin connection, the connection
\[
d - \chi(\xi)d\bar{w} + \alpha(\xi)dw
\]
(2.8)
must have \( \pm 1 \) monodromy. Recall that \( \chi \) has period \( \frac{2\pi}{\tau} \) along \( 1 \in \Lambda \). Because \( \chi \) is odd, i.e., \( \chi(-\xi) = -\chi(\xi) \), we obtain \( \chi(\frac{1}{2}) = \frac{\pi i}{2\tau} \). Similarly, one can show \( \alpha(\frac{1}{2}) = \frac{\pi i}{2\tau} \), and using the well-defined gauge transformation \( g(z) = \exp(\frac{2\pi i}{2\tau}(z - \bar{z})) \) on \( M = 2\mathbb{Z} + 2\tau\mathbb{Z} \) we see that the connection (2.8) is trivial for \( \xi = [\frac{1}{2}] \), i.e., has trivial monodromy. By the monotonicity of the functions \( \chi \) and \( \alpha \), the connection \( d^{\chi(\xi_0),\alpha(\xi_0)} \) cannot be spin. \( \square \)

### 3. Deformations

In this section, we adjust the generalized Whitham flow in [9] to our spectral data \((\Sigma, \lambda, \chi, \alpha)\). We start with describing useful coordinate systems on the moduli spaces of regular singular connections on a 4-punctured sphere.

#### 3.1 Abelianization of flat connections

We first recall the constructions of [22], see also [9, Section 3.1]. Let \( M = \mathbb{C}/\Gamma \) where \( \Gamma = 2\mathbb{Z} + 2\tau\mathbb{Z} \) with \( \tau \in i\mathbb{R} > 0 \) is a rectangular lattice. Let \( \sigma : M \to \mathbb{C}P^1 \) be the elliptic involution \( [w] \mapsto [-w] \) and let
\[
z : M \to \mathbb{C}P^1
\]
be the induced double covering which has four ramification points
\[
P_1 = [0], P_2 = [1], P_3 = [1 + \tau], P_4 = [\tau] \in M.
\]
After a Moebius transformation, the four branch points \( p_k \in \mathbb{C}P^1 \) of \( z \) can be chosen without loss of generality to be

\[
p_1 = z([0]) = 0, \quad p_2 = z([1]) = 1, \quad p_3 = z([1 + \tau]) = \infty, \quad p_4 = z([\tau]) = m
\]

for some \( m \in \mathbb{C} \setminus \{0, 1\} \). For \( \rho \in \mathbb{C} \), consider the moduli space

\[
\mathcal{M}_\rho^2 = \mathcal{M}_\rho^2(\mathbb{C}P^1 \setminus \{p_1, \ldots, p_4\})
\]

of flat SL(2, \( \mathbb{C} \)) connections on the 4-punctured sphere \( \mathbb{C}P^1 \setminus \{p_1, \ldots, p_4\} \) such that the local monodromies around every puncture \( p_k \) lie in the conjugacy class of

\[
\begin{pmatrix}
\exp(2\pi i \frac{2p+1}{4}) & 0 \\
0 & \exp(-2\pi i \frac{2p+1}{4})
\end{pmatrix}
\]

(3.4)

It is a classical fact that \( \mathcal{M}_\rho^2 \) is two-dimensional, see for example [23] and the literature therein or [22].

For convenience of the reader, we shortly describe how \( \mathcal{M}_\rho^2 \) can be parametrized. Consider the lattice \( \frac{1}{2}\Gamma = \mathbb{Z} + \tau \mathbb{Z} \) and the corresponding theta-function \( \vartheta : \mathbb{C} \rightarrow \mathbb{C} \) of \( \frac{1}{2}\Gamma \) uniquely determined (up to a multiplicative constant) by \( \vartheta(0) = 0 \) and

\[
\vartheta(w + 1) = \vartheta(w), \quad \vartheta(w + \tau) = -\vartheta(w) e^{-2\pi iw}
\]

(3.5)

for all \( w \in \mathbb{C} \). For \( x \in \mathbb{C} \setminus \frac{1}{2}\Gamma \) fixed define

\[
\beta_x(w) = \frac{\vartheta(w - x)}{\vartheta(w)} e^{\frac{2\pi i}{\tau} x(w - \bar{w})}.
\]

(3.6)

The function \( \beta_x \) is doubly periodic in \( w \) with respect to the lattice \( \frac{1}{2}\Gamma \) and satisfies

\[
\left( \bar{\vartheta}_\theta - \frac{2\pi i}{\tau - \bar{\tau}} xd\bar{w} \right) \beta_x = 0.
\]

Thus \( \beta_x \) is a meromorphic section of the trivial bundle \( \mathbb{C} \rightarrow \mathbb{C}/\frac{1}{2}\Gamma \) equipped with the holomorphic structure \( \bar{\vartheta}_\theta - \frac{2\pi i}{\tau - \bar{\tau}} xd\bar{w} \). It has a simple zero at \( w = x \) and a first-order pole at \( w = 0 \). By pull-back, we can also consider \( \beta_x \) on the fourfold cover \( M = \mathbb{C}/\Gamma \rightarrow \mathbb{C}/\frac{1}{2}\Gamma \) as a meromorphic section with simple poles at the half lattice points.

Let \( \rho \in \mathbb{C} \). For a given flat \( \mathbb{C}^* \)-connection

\[
d^{x,a} = d + \alpha dw - \chi d\bar{w}
\]

(3.7)

with \( \chi \in \mathbb{C} \setminus (\frac{\pi i}{\tau - \bar{\tau}} + \frac{\pi i\bar{\tau}}{\tau - \bar{\tau}} \mathbb{Z}) \), and \( \alpha \in \mathbb{C} \) let \( x = \frac{\tau - \bar{\tau}}{2\pi i} \chi \) and define the flat singular connection \( ^\rho \hat{\nabla}^{x,a} \) on the trivial rank 2 bundle \( \mathbb{C}^2 \rightarrow M \):

\[
^\rho \hat{\nabla}^{x,a} = d + \begin{pmatrix}
-\chi d\bar{w} + \alpha dw \\
\rho \frac{\vartheta(0)}{\tau - \bar{\tau}} \beta_{2x}(w) dw \\
\end{pmatrix}.
\]

(3.8)
For \( \rho = 0 \), we obtain smooth and totally reducible \( SL(2, \mathbb{C}) \)-connections on the torus \( M \). The off-diagonal part of the connection 1-form in (3.8) only depends on \( \rho \) and on the holomorphic structure \( \bar{\partial}_0 - \chi d\bar{w} \) and is independent of \( \alpha \). By [22, Section 3] \( \rho \bar{\nabla}_x,\alpha \) is gauge equivalent (via a gauge transformation with singularities at \( P_1, \ldots, P_4 \)) to an invariant connection \( \rho \bar{\nabla}_x,\alpha \) with respect to \( [w] \mapsto [-w] \). In other words, \( \rho \bar{\nabla}_x,\alpha \) is well defined on \( \mathbb{CP}^1 \backslash \{p_1, \ldots, p_4\} \). This determines a map

\[
\Pi: \mathcal{M}'_{dr}(M) \backslash (\pi^1)^{-1}(\Lambda) \to \mathcal{M}^2_{\rho}(\mathbb{CP}^1 \backslash \{p_1, \ldots, p_4\}); \quad [d^x,\alpha] \mapsto [\rho \bar{\nabla}_x,\alpha], \tag{3.9}
\]

where

\[
\pi^1: \mathcal{M}'_{dr}(M) \to \text{Jac}(M)
\]

is the natural projection. Replacing \( d^x,\alpha \) by its dual connection \( d^{-x,-\alpha} \) in (3.8) the corresponding connection lies in the same gauge-equivalence class, i.e.,

\[
\Pi([d^{-x,-\alpha}]) = \Pi([d^x,\alpha]) = [\rho \bar{\nabla}_x,\alpha]
\]

giving the first assertion of the following theorem.

**Theorem 3.1** ([22] Theorems 3.4 and 3.5) Let \( \rho \in [-\frac{1}{2}, \frac{1}{2}] \). Then the map \( \Pi \) in (3.9) is a double covering onto an open and dense subset.

This 2:1 correspondence extends to \( \chi = \gamma \in \Lambda = \frac{\pi i}{\tau - \bar{\tau}} \mathbb{Z} + \frac{\pi i}{\tau - \bar{\tau}} \mathbb{Z} \) in the following sense: Let \( U \subset \mathbb{C} \) be open with \( U \cap \Lambda = \{\gamma\} \). Assume \( \chi \in U \backslash \{\gamma\} \mapsto \alpha(\chi) \) is holomorphic. Then

\[
\chi \in U \mapsto \Pi([d^{x,\alpha}(\chi)])
\]

extends holomorphically if and only if \( \alpha \) has a first-order pole at \( \chi = \gamma \) such that

\[
\alpha(\chi) \sim_\gamma \pm \frac{4\pi i}{\tau - \bar{\tau}} \frac{\rho}{\chi - \gamma} + \tilde{\gamma} + \text{higher order terms in } (\chi - \gamma). \tag{3.10}
\]

**Remark 3.2** The ambiguity of the sign of the residue \( \pm \frac{4\pi i}{\tau - \bar{\tau}} \) in (3.10) is meaningful: If \( \rho > 0 \) the underlying parabolic structure is stable for the '+'-sign and unstable otherwise (see [22, Theorem 3.5] for more details). This is reversed for \( \rho < 0 \).

In [9], the implicit function theorem is used to obtain a family of maps \( \chi \) and \( \alpha \) depending on \( \rho \) satisfying closing conditions. There the most important condition was that the constructed family of flat connections \( \nabla^\lambda \) is unitary along the unit circle. In this article, the considered real involution \( \eta \) has no fixed points and thus we need a 2-point construction to realize the reality condition.

**Lemma 3.3** Let \( \bar{\partial}_0 - \chi_i d\bar{w}, i = 1, 2 \), be two non-spin holomorphic structures on the rectangular torus \( M = \mathbb{C}/\Gamma \). Consider the flat \( SL(2, \mathbb{C}) \)-connections

\[
\rho=0 \hat{\nabla}_{x_1,\bar{x}_2} \quad \text{and} \quad \rho=0 \hat{\nabla}_{x_2,\bar{x}_1}
\]
given by (3.8). Then, there is an open neighbourhood \( U_\rho \subset \mathbb{R} \) of \( \rho = 0 \), and open neighbourhoods \( U_{\bar{\rho}} \subset \mathbb{C} \) of \( \bar{\rho} \), and \( \bar{\chi} \), respectively, with the property that for all \( \rho \in U_\rho \), \( \bar{\chi}_1 \in U_{\chi_1} \) and \( \bar{\chi}_2 \in U_{\chi_2} \) there are unique

\[
\bar{\alpha}_i = \bar{\alpha}_i(\rho, \bar{\chi}_1, \bar{\chi}_2) \in U_{\bar{\chi}_i}
\]

for \( i = 1, 2 \) such that

\[
\rho \hat{\nabla} \bar{\chi}_1 \bar{\alpha}_1 \quad \text{and} \quad \rho \hat{\nabla} \bar{\chi}_2 \bar{\alpha}_2
\]

are gauge equivalent on \( M \setminus \{P_1, \ldots, P_4\} \). Moreover, \( \bar{\alpha}_i = \bar{\alpha}_i(\rho, \bar{\chi}_1, \bar{\chi}_2) \) is real analytic in its parameters.

The proof is a direct application of the implicit function theorem (at \( \rho = 0 \)) to Theorem 3.1, compare also with the proof of [9, Lemma 3.2].

**Remark 3.4** The statement of the lemma also holds for the flat \( \text{SL}(2, \mathbb{C}) \)-connections

\[
\rho = 0 \hat{\nabla} \bar{\chi}_1 \bar{\alpha}_1 \quad \text{and} \quad \rho = 0 \hat{\nabla} \bar{\chi}_2 \bar{\alpha}_2
\]

if \( \gamma \in \Lambda \equiv \frac{\pi i}{4-7} \mathbb{Z} + \frac{\pi i \tau}{4-7} \mathbb{Z} \) is the lattice of \( \text{Jac}(\mathcal{M}) \). Therefore, the lemma can be rephrased in terms of the moduli space of flat line bundle connections on \( \mathcal{M} \).

### 3.2 The generalized Whitham flow

For \( \rho = 0 \) consider spectral data \((\Sigma, \lambda, \chi, \alpha)\) of Theorem 2.1 and the corresponding family of flat connections via (3.8). The generalized Whitham flow is the deformation of spectral data (and of the resulting harmonic map) given by varying \( \rho \) while preserving the reality condition and the translational periods (2.1) and (2.2) of \( \chi \) and \( \alpha \). Hence, for some \( d > 1 \), we are looking for deformations \( \tilde{\chi} \) and \( \tilde{\alpha} \) of \( \chi \) and \( \alpha \) of the form

\[
\tilde{\chi} = \chi + \hat{\chi} \quad \text{and} \quad \tilde{\alpha} = \alpha + \hat{\alpha},
\]

where

\[
\hat{\chi}: \lambda^{-1}(\{\lambda \in \mathbb{C} | \lambda \bar{\lambda} < d\}) \subset \Sigma \rightarrow \mathbb{C}
\]

and

\[
\hat{\alpha}: \lambda^{-1}(\{\lambda \in \mathbb{C} | \frac{1}{d} < \lambda \bar{\lambda} < d\}) \subset \Sigma \rightarrow \mathbb{C}
\]

are odd (with respect to the involution \( \sigma \) on \( \Sigma \)) holomorphic maps. In order to study these maps, we first recall that the spectral curve \( \Sigma \) in the torus case \((\rho = 0, \text{Theorem 2.1})\) is given by the algebraic equation

\[
y^2 = \lambda(\lambda - r)(\lambda + \frac{1}{r})
\]

(3.11)
for some $0 < r < 1$. Then, $y: \Sigma \to \mathbb{C}$ is an odd meromorphic function on $\Sigma$. On the domains

$$\lambda^{-1}(\{\lambda \in \mathbb{C} \mid \lambda \lambda < d\}) \subset \Sigma \quad \text{and} \quad \lambda^{-1}(\{\lambda \in \mathbb{C} \mid \frac{1}{d} < \lambda \lambda < d\}) \subset \Sigma$$

of $\hat{x}$ and of $\hat{a}$, respectively, the function $y$ has no poles. Therefore, $\hat{x}$ and $\hat{a}$ can be written as

$$\hat{x} = yx := yx \circ \lambda \quad \text{and} \quad \hat{a} = ya := ya \circ \lambda,$$

for holomorphic functions

$$x: \{\lambda \in \mathbb{C} \mid \lambda \lambda < d\} \to \mathbb{C}$$

and

$$a: \{\lambda \in \mathbb{C} \mid \frac{1}{d} < \lambda \lambda < d\} \to \mathbb{C}.$$

For $d > 1$ fixed, let $B_d$ be the Banach space of bounded holomorphic functions on $D_d := \{\lambda \in \mathbb{C} \mid \lambda \lambda < d\}$ equipped with the supremum norm. Note that $\chi|_S$ is a well-defined bounded map from the compact set $S = \lambda^{-1}(S^1)$ into $\text{Jac}(M)$. Let

$$\chi_1 = \chi(\xi) \quad \text{and} \quad \chi_2 = \chi\left(\xi + \frac{\text{spec}}{2}\right).$$

Then there exist by Lemma 3.3 and Remark 3.4 a $\delta > 0$ and an open neighbourhood $U_0 \subset B_d$ of the zero function such that for all $\xi \in S$, $\rho \in (-\delta, \delta)$, and all $\hat{x} = yx \circ \lambda$ with $\xi \in U_0$

$$\chi(\xi) + \hat{x}(\xi) \in U_{\chi_1} \quad \text{and} \quad \chi\left(\xi + \frac{\text{spec} + 1}{2}\right) + \hat{x}\left(\xi + \frac{\text{spec} + 1}{2}\right) \in U_{\chi_2}.$$

In particular, there is a real-analytic function

$$\alpha^\rho_\xi: S \to \mathbb{C} \quad (3.12)$$

such that

$$\rho \nabla x(\xi) + \hat{x}(\xi) \alpha^\rho_\xi(\xi) \quad \text{and} \quad \rho \nabla x(\xi + \frac{\text{spec} + 1}{2}) + \hat{x}(\xi + \frac{\text{spec} + 1}{2}) \alpha^\rho_\xi(\xi + \frac{\text{spec} + 1}{2})$$

are gauge equivalent on $M$ for all $\xi \in S$.

It also follows from Lemma 3.3 and Remark 3.4 that for $\xi \in U_0 \subset B_d$ the map $\alpha^\rho_\xi$ has the same translational periods as $\alpha$. Thus, for any $\xi \in U_0$ and any $\rho \in (-\delta, \delta)$ the function

$$\alpha^\rho_\xi - \alpha$$
is real analytic and single-valued on both components \( S_1 \) and \( S_2 \) of \( S \). Hence, \( \alpha^\epsilon_\pm - \alpha \) is a holomorphic function defined on an open neighbourhood of \( S \) and odd by uniqueness in Lemma 3.3. Altogether, up to choosing \( d > 1 \) smaller, \( \alpha^\epsilon_\pm \) determines a holomorphic function

\[
\alpha^\epsilon_\pm : A_d := \{ \lambda \in \mathbb{C} \mid \frac{1}{d} < \lambda \bar{\lambda} < d \} \to \mathbb{C}
\]
satisfying

\[
\alpha^\epsilon_\pm - \alpha = y a^\epsilon_\pm \circ \lambda.
\] (3.13)

for all \( \xi \in S \). In this vein, let \( B_{1/d}^d \) the Banach space of bounded holomorphic functions on the annulus \( A_d \).

**Lemma 3.5** Let \( \tau_{\text{spec}}, \tau \) be as in Lemma 2.3. Then, there exist \( d > 1 \), \( \delta > 0 \) and an open neighbourhood \( U_0 \subset B_d \) of the zero function such that for every \( \rho \in (-\delta, \delta) \) and every \( x \in U_0 \) the function \( a^\rho \) defined by (3.13) is bounded and holomorphic on \( A_d \). Moreover, the map

\[
(\rho, x) \in (-\delta, \delta) \times U_0 \mapsto a^\rho \in B_{1/d}^d
\]
is smooth.

**Proof.** Consider the spectral curve \( \Sigma = \mathbb{C}/\Lambda \) and its holomorphic maps \( \lambda \) and \( y \) to \( \mathbb{C}P^1 \) as in (3.11). By the assumptions of Theorem 2.1, there are no branch points over \( \lambda \in S^1 \subset \mathbb{C}P^1 \). Therefore, the space of holomorphic functions \( B_{1/d}^d \), from a small open annulus \( A_d \) of \( S^1 \) can be identified with the space of odd holomorphic functions \( \hat{B}_{1/d}^d \), from the corresponding neighbourhood \( \lambda^{-1}(U) \) of \( S \subset \Sigma \) by multiplying with \( y \). Hence the map

\[
(\rho, x) \in (-\delta, \delta) \times U_0 \mapsto a^\rho \in B_{1/d}^d
\]
is smooth if the map

\[
(\rho, x) \mapsto y \cdot a^\rho \circ \lambda = \alpha^\rho - \alpha
\]
to the Banach space \( \hat{B}_{1/d}^d \) is smooth. To show this, we construct a holomorphic function \( F \) and a smooth map \( h \) such that

\[
y \cdot a^\rho \circ \lambda = F \circ h(\rho, x).
\]

For \( \tau_{\text{spec}} \in i\mathbb{R}^> \), we decompose \( S = S_1 \cup S_2 \) into its two connected components. Since we are working with odd maps, it suffices to study the behaviour of the involved functions on one connected component \( S_1 \) or \( S_2 \) only. Denote by \( \hat{S}_i \), the preimage of \( S_i \), with respect to \( C \mapsto \mathbb{C}/\Lambda = \Sigma \). Moreover, let

\[
W := \{0\} \times \left\{ (\chi(\xi), \chi(\bar{\xi}), \chi(\bar{\xi} + \frac{\tau_{\text{spec}} + 1}{2}), \chi(\bar{\xi} + \frac{\tau_{\text{spec}} + 1}{2})) \mid \xi \in \hat{S}_1 \right\} \subset \mathbb{C}^5.
\]
By analytic continuation there is a holomorphic function from an open neighbourhood \( \hat{W} \) of \( W \)

\[
F: \hat{W} \to \mathbb{C}
\]

which is uniquely determined by

\[
F(\rho, \chi_1, \bar{\chi}_1, \chi_2, \bar{\chi}_2) = \tilde{\alpha}_1(\rho, \chi_1, \chi_2) - \tilde{\alpha}_1(0, \chi_1, \chi_2)
\]

for all \((\rho, \chi_1, \bar{\chi}_1, \chi_2, \bar{\chi}_2) \in W\), where \(\tilde{\alpha}_1(\rho, \chi_1, \chi_2)\) are given by Lemma 3.3. Let

\[
\eta(\xi) = \bar{\xi} + \frac{1 + \text{spec}}{2}
\]

be the lift of the real structure constructed in Theorem 2.1 and let \(n: V \subset \mathbb{C} \to V\) be the holomorphic map determined by \(n|_{\hat{S}_1} = \eta|_{\hat{S}_1}^{-1}\) for some open neighbourhood \(V\) of \(\hat{S}_1\) via analytic continuation. Note the map \(n\) exists on \(V\) because \(\lambda|_{S_k} : S_k \to S^1\) is bijective for \(k = 1, 2\). Then,

\[
h: \mathbb{C} \times B_d \to \mathbb{C} \times (\hat{S}_d)^4
\]

\[
(\rho, \chi) \mapsto (\rho, \chi + \bar{\chi} \circ \lambda, (\chi + \bar{\chi} \circ \lambda)(\eta \circ n), (\chi + \bar{\chi} \circ \lambda) \circ n, (\chi + \bar{\chi} \circ \lambda) \circ \eta)
\]

is smooth and we have by definition (3.13)

\[
y(\xi) a_\rho^0 \circ \lambda(\xi) = F \circ h(\rho, \chi), \tag{3.15}
\]

proving the statement. \(\square\)

**Lemma 3.6** Using the notations of Lemma 2.3 and \(r = \lambda([\frac{1}{2}])\). Then, there exist \(d > 1, \delta > 0\) and an open set

\[
U = \lambda^{-1}\left(\{\lambda \in \mathbb{C} \mid 0 < \lambda \bar{\lambda} < d; \lambda \neq r\}\right)
\]

such that there are two smooth families of odd holomorphic maps

\[
\rho \in (-\delta, \delta) \mapsto \left((\chi_+^\rho, \alpha_+^\rho) : U \subset \Sigma \to \mathcal{M}_{dR}(M)\right)
\]

and

\[
\rho \in (-\delta, \delta) \mapsto \left((\chi_-^\rho, \alpha_-^\rho) : U \subset \Sigma \to \mathcal{M}_{dR}(M)\right)
\]

into the moduli space of flat line bundle connections \(\mathcal{M}_{dR}(M)\) satisfying

1. for \(\rho = 0\) \((\chi_+^0, \alpha_+^0) = (\chi_-^0, \alpha_-^0)\) are the spectral data in Theorem 2.1;
2. for all \(\rho \in (-\delta, \delta)\) the maps \(\chi_\pm^\rho: U \to \text{Jac}(M)\) extend holomorphically through \([0] = \lambda^{-1}(0)\) and \([\frac{1}{2}] = \lambda^{-1}(r) \in \Sigma\).
(3) \( \alpha \pm \) has first-order poles at \([0, \frac{1}{2}] \in \Sigma\), and the expansion of the pole at \( \lambda = \frac{1}{2} \) satisfies (3.10) with the respective sign;

(4) for all \( \rho \in (-\delta, \delta) \) and for all \( \xi \in \mathcal{S} \) the connections

\[ \rho \hat{\nabla}^{\pm}(\xi) \alpha \pm(\xi) \]

are gauge equivalent on the punctured torus \( M \setminus \{P_1, \ldots, P_4\} \).

**Proof.** Using Lemmas 3.3 and 3.5, the proof is analogous to the proof of Theorem 4.2 in [9]. It requires first to apply the implicit function theorem to guarantee (4) and \( \alpha \pm \) having a first-order pole in \( \lambda = [0] \).

The remaining parameters are used to apply the implicit function theorem again to obtain the asymptotic (3.10) at \( \lambda = \frac{1}{2} \). Since the residue in (3.10) vanishes for \( \rho = 0 \), we have a choice of sign here. The main difference is that in contrast to Theorem 4.2 of [9] no extrinsic closing condition is required. This allows us to fix the branch point \( r \) of the spectral curve for all \( \rho \). \( \square \)

**Remark 3.7** The two connections \( \rho \hat{\nabla}^{\pm}(\xi) \alpha \pm(\xi) \) (3.8) are gauge equivalent by a diagonal gauge on the punctured torus \( M \setminus \{P_1, \ldots, P_4\} \). The uniqueness part of the implicit function theorem in the proof of Lemma 3.6 and the sign change in (3.10) then give

\[ (\chi^{\rho}, \alpha^{\rho}) = (\chi^{-\rho}, \alpha^{-\rho}) \]

for all \( \rho \in (-\delta, \delta) \).

**4. Existence of negative real sections**

In this section, we prove the existence of negative, non-admissible, real holomorphic sections in Deligne–Hitchin moduli spaces of Riemann surfaces of high genus. We show that each set of the initial data (Lemma 3.6) give rise to two \( \mathbb{C}^\ast \)-families of flat connections on the 4-punctured sphere (Proposition 4.2). These families induce real holomorphic sections of the Deligne–Hitchin moduli spaces of certain coverings of the 4-punctured sphere.

**Definition 4.1** Let \( \rho \in \mathbb{R} \). The quadruple \((\Sigma, \lambda, \chi, \alpha)\) is called \( \rho \)-spectral data, if

- \( \Sigma \) is the spectral curve and \( \lambda : \Sigma \to \mathbb{C}P^1 \) as in Lemma 3.6;
- \( \chi : \tilde{\Sigma} := \lambda^{-1}(D_d) \subset \Sigma \to \text{Jac}(M) \) is an odd holomorphic map for some \( d > 1 \);
- \((\chi, \alpha) : \tilde{\Sigma} \subset \Sigma \to \mathcal{M}_{d\mathbb{R}}^{1}(M) \) is an odd meromorphic map such that \((\chi, \alpha)\) satisfies the condition (3.10) for \( \rho \).

Let \( M = \mathbb{C}/\Gamma \) be a rectangular torus given by the algebraic equation

\[ y^2 = \frac{(z-z_0)(z-z_1)}{(z-z_2)(z-z_3)} \]
for pairwise distinct points \( z_0, \ldots, z_3 \in \mathbb{C} \). Without loss of generality, we assume that the branch points of \( z \) are the half-lattice points \([0], [1], [1 + \tau] \) and \([\tau] \). For \( q \in \mathbb{N}^+ \) consider the compact Riemann surface \( M_q \) of genus \( q - 1 \) defined by the algebraic equation

\[
y^q = \frac{(z - z_0)(z - z_1)}{(z - z_2)(z - z_3)}.
\]

(4.1)

Naturally, the surface \( M_q \) is equipped with a \( q \)-fold covering \( z \) to the Riemann sphere. If \( q \) is even, it further admits a \( \frac{q}{2} \)-fold covering to the torus \( M \) given by

\[
\pi_q: M_q \to M; (y, z) \mapsto (y^{\frac{q}{2}}, z)
\]

(4.2)

totally branched at \([0], [1], [1 + \tau], [\tau] \in M \).

**Proposition 4.2** Let \( p, q \in \mathbb{N} \) be coprime integers with \( q \) even and \( \rho = -p/q \in (-\frac{1}{2}, 0) \). Let \((\Sigma, \lambda, \chi, \alpha)\) be \( \rho \)-spectral data provided by Lemma 3.6 and \( \pi_q: M_q \to M \) as in (4.2). Fix \( k \in \mathbb{N}^+ \) large. Then the \( \rho \)-spectral data induce a holomorphic family of flat \( \mathcal{O}_k \)-\( \text{SL}(2, \mathbb{C}) \)-connections

\[
\nabla^\lambda, \quad \lambda \in D_d^q = \{ \lambda | 0 < \lambda \bar{\lambda} < d \}
\]

on \( M_q \) with asymptotic expansion

\[
\nabla^\lambda = \lambda^{-1}\Phi + \nabla + \lambda\Phi_1 + \text{higher order terms in } \lambda
\]

(4.3)

around \( \lambda = 0 \) such that \((\tilde{\theta}^\lambda, \Phi)\) is a stable Higgs pair. Furthermore, under the assumptions of Lemma 2.5, the connections \( \nabla^\lambda \) are irreducible for all \( \lambda \in D_d^q \) for \( \rho = -p/q \sim 0 \) small.

**Proof.** The \( \rho \)-spectral data define a family of connections \( \hat{\nabla}^\chi(\xi, \alpha(\xi)) \) (see (3.8)) on the 4-punctured sphere \( \mathbb{C}P^1 \setminus \{p_1, \ldots, p_4\} \) parametrized by \( \xi \in \tilde{\Sigma} \). Because \( \chi \) and \( \alpha \) are odd maps, we can define

\[
\hat{\nabla}^\lambda := \nabla^\chi(\xi, \alpha(\xi)) \end{equation}

via the hyper elliptic involution \( \lambda \) on \( \tilde{\Sigma} \). The condition that \( \alpha \) satisfies (3.10) ensures that there exists a global \( \lambda \)-dependent gauge \( g \) such that \( \hat{\nabla}^\lambda \cdot g \) extends holomorphically through the points \( \xi \), where \( \chi(\xi) \) is a lattice point. The pull-back of \( \hat{\nabla}^\lambda \cdot g \) to \( M_q \) then has apparent singularities in \( \pi_q^{-1}(\{p_1, \ldots, p_4\}) \) and is therefore gauge equivalent to a holomorphic family of smooth connections \( \nabla^\lambda \) over \( M_q \). As in [9, Section 3.3] the Higgs pair at \( \lambda = 0 \) is stable, see also [22, Section 2.4].

It remains to show irreducibility of \( \nabla^\lambda \) for all \( \lambda \in \mathbb{C}^+ \). For \( \rho \neq 0 \), a connection \( \hat{\nabla}^\lambda \) on the 4-punctured sphere is reducible if and only if the corresponding \( d - \chi(\xi) \cdot d\tilde{w} + \alpha(\xi) \cdot dw \) is a non-trivial spin connection by [22, Section 2.4]. Because \( M_q \to \mathbb{C}P^1 \) is a cyclic covering, the same holds for \( \nabla^\lambda \). At \( \rho = 0 \) and under the assumptions of Lemma 2.5, \( d - \chi(\xi) \cdot d\tilde{w} + \alpha(\xi) \cdot dw \) is never a non-trivial spin connection. By continuity of the flow, this remains true for \( \rho \sim 0 \) giving irreducibility for \( \nabla^\lambda \) for \( \rho \neq 0 \).

**Remark 4.3** In the following we use the family \( \hat{\nabla}^\lambda \) of flat connections on the 4-punctured sphere because these connections are irreducible for all \( \rho \sim 0 \). Therefore, the notion of positive and negative real sections
is well defined and preserved by continuous deformations. An analogous statement as in the proof of Proposition 4.3 also holds for $\rho > 0$: we can construct flat connections $\tilde{\nabla}^\lambda$ for every $\lambda \in \mathbb{C}^*$ on the 4-punctured sphere. If $\rho = p/q$ with $p, q \in \mathbb{N}$, $p < q$, then the connections can be pulled back to $M_g$ and, up to gauge transformations, yield a holomorphic $\mathbb{C}^*$-family of flat connections which give rise to a stable Higgs pair at $\lambda = 0$. Due to the stability properties (Remark 3.2), this implies that the bundle type of $\tilde{\nabla}^\lambda$ extends to $\lambda = 0$ as

$$\mathcal{O}(1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$$

for $\rho > 0$, while we obtain the trivial holomorphic bundle for $\rho < 0$, see [22, Section 2.4]. Note that also the conformal Gauss map (Section 1.4) replaces the underlying trivial holomorphic bundle at $\lambda = 0$ by $\mathcal{O}(1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$.

**Theorem 4.4** There is a $g_0 \in \mathbb{N}$ such that for every $g > g_0$ there exists a Riemann surface of genus $g$ admitting a negative real holomorphic section in its Deligne–Hitchin moduli space which is not a twistor line.

**Proof.** For $\rho < 0$, consider $\rho$-spectral data satisfying Lemma 2.5 and let $\delta > 0$ as in Lemma 3.6. Let $g_0 \in \mathbb{N}$ such that $\frac{1}{g_0 + 1} < \delta$. Then, for $q \geq (g_0 + 1)$ and $\rho = -\frac{1}{q}$, we obtain two holomorphic families of flat connections on the $q$-fold covering $M_q \to \mathbb{C}P^1$ (of genus $g = q - 1$) by Lemma 3.6 and Proposition 4.2. Both families give rise to local (in $\lambda$) sections of the Deligne–Hitchin moduli space on $\{ \lambda \in \mathbb{C} \mid \lambda \bar{\lambda} < 1 + \epsilon \} \subset \mathbb{C}P^1$ by the properties (2) and (3) in Lemma 3.6. By property (4) the sections are real with respect to the involution $T$ and hence extend to global sections

$$s_{\pm} : \mathbb{C}P^1 \to \mathcal{M}_{DH}(M_q).$$

These sections are stable by Proposition 4.2, compare with Remark 4.3.

We claim either $s_+ (\pm \rho)$ or $s_- (\mp \rho)$ is negative. Assume that $s_+ (\rho)$ is positive, we want show that in this case $s_- (-\rho)$ must be negative. Consider the local lift of the section $s_+ (\rho)$ on $\{ \lambda \in \mathbb{C} \mid 0 < \lambda \bar{\lambda} < 1 + \epsilon \}$ to the space of flat connections on the 4-punctured sphere given by

$$\rho \tilde{\nabla}^{\lambda(\xi)} = \rho \tilde{\nabla}_{\chi(\xi), \alpha(\xi)}.$$

For $\rho < 0$, the corresponding $\lambda$-connections extend to $\lambda = 0$ as a stable nilpotent parabolic Higgs field on the trivial holomorphic bundle, see [22, Section 2.4]. For $\rho \sim 0$, the connections on the 4-punctured sphere are irreducible by Lemma 2.5.

Applying the conformal Gauss map construction (Section 1.4), there exist a negative lift $\rho \tilde{\nabla}^\lambda$ of $s_+ (\rho)$ which is irreducible for $\lambda \neq 0$ but has an unstable parabolic Higgs bundle at $\lambda = 0$ with respect to the parabolic weight $\rho < 0$. Moreover, the underlying holomorphic bundle is $\mathcal{O}(1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$.

On the 4-punctured sphere, the connections $\rho \tilde{\nabla}^\lambda$ are irreducible for all $\lambda \in \mathbb{C}^* \setminus \{ \lambda(\xi_0) \}$ if $\rho = 0$. Therefore, the sign in (1.12) does not change within the continuous $\rho$-deformation. Hence, at $-\rho$ the family $-\rho \tilde{\nabla}^\lambda$ is negative. Since $-\rho > 0$, and because the underlying holomorphic bundle is $\mathcal{O}(1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$, the parabolic Higgs pair of $-\rho \tilde{\nabla}^\lambda$ at $\lambda = 0$ is stable, see again [22, Section 2.4]. Moreover, the parabolic structure is unstable at the branch point $\lambda(\xi_0)$ for $-\rho > 0$, see Remark 3.2. By Remark 3.7

$$(\chi_+^{\rho}, \alpha_+^{p}) = (\chi_-^{\rho}, \alpha_-^{p}),$$
showing that \(-\rho \nabla^\lambda\) is a lift of \(s_-(-\rho)\). Therefore, \(s_-(-\rho)\) is a negative section on the 4-punctured sphere. Because negativity and stability is preserved by pull-backs \(s_-(-\rho)\) gives also a negative section on \(M_q\).

It remains to prove that these negative real sections are not twistor lines. Observe from (3.4) and (3.8) that the eigenvalues \(\pm \tilde{\rho}\) of the residues of the local monodromies on the 4-punctured sphere and the eigenvalues \(\pm \rho\) of the residues of the local monodromies on the 4-punctured torus are related by

\[
\rho = 2\tilde{\rho} - \frac{1}{2}.
\]  

Solutions to the self-duality equations on punctured Riemann surfaces have been studied in [24]. For the 4-punctured sphere with all parabolic weights being \(\tilde{\rho} = \frac{1}{4}\) and strictly parabolic Higgs fields (i.e. parabolic Higgs fields with nilpotent residues as in our case) the solutions correspond to smooth and reducible solutions on the torus \(M \xrightarrow{2:1} \mathbb{CP}^1 \setminus \{p_1, \ldots, p_4\}\) thanks to (3.8), compare with [25] and Theorem 7 and Remark 8 in [26]. The smooth dependence of the solutions on the parabolic weight \(\tilde{\rho} = \rho_i\) (see [27]) gives that the solutions of the self-duality equations for \(\rho \sim \frac{1}{4}\) are close to these reducible solutions (with respect to the supremum norm after applying appropriate gauge transformations). But for \(\rho = 0\), the real sections \(s\) considered in this article are the spectral genus 1 solutions on a torus. As all global self-duality solutions are totally reducible, a short computation shows that global solutions on a torus correspond to spectral genus 0 solutions. (A similar computation is done in [28, Section 6] for the case of harmonic maps to the 3-sphere.) Alternatively, as Higgs field is nilpotent in our examples, the solution correspond to conformal harmonic immersions. These are determined by the conformal factor for the metric (and the Hopf differential which is a constant multiple of \((dw)^2\)). For spectral genus 1 solutions, the metric factor does only depend on one real variable and is explicitly determined by the Weierstrass \(\wp\)-function, see [10]. Hence, also for \(\rho \sim 0\) (or equivalently \(\tilde{\rho} \sim \frac{1}{4}\)) the section \(s\) does not solve the self-duality equations globally on \(M\) and can therefore not be twistor lines. \(\square\)

**Remark 4.5** An independent proof of the fact that the sections are not twistor lines is given in [20]: self-duality solutions with nilpotent Higgs fields are branched minimal surfaces in hyperbolic 3-space. In general, they are only defined on the universal covering but they are equivariant with respect to a representation of the fundamental group, for details see Section 5. For twistor lines with nilpotent Higgs field, the Willmore energy is bounded from above. Otherwise the area of the corresponding minimal surface would be negative. In the case of the sections at hand, this bound does not hold, see [20, Theorem 5.2]. Hence, the sections cannot be twistor lines.

**5. Higher solutions of the self-duality equations**

In Theorem 4.4, we constructed negative real sections for the Deligne–Hitchin moduli space which are not twistor lines. In this section, we want to give a geometric interpretation of these sections in terms of Willmore surfaces in 3-space. To do so, we show that the sections given by Theorem 4.4 give solutions to Hitchin’s self-duality equations on an open and dense subset of the Riemann surface \(M\).

It is well known that self-duality solutions for \(G_C = SL(2, \mathbb{C})\) correspond to equivariant harmonic maps into hyperbolic 3-space, see [2, 3]. Given a solution \((\nabla, \Phi, h)\) on a Riemann surface \(M\) for a unitary connection \(\nabla\) with respect to the hermitian metric \(h\) and the Higgs field \(\Phi\) we consider the flat connection

\[
\nabla^1 = \nabla + \Phi + \Phi^*. 
\]  

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Fix a point \( p \in M \) and a unitary frame \( \Psi_p \) at \( p \) of the rank 2 vector space \( V_p = \mathbb{C}^2 \). Through parallel transport we obtain an equivariant frame

\[
\Psi := (e_1, e_2)
\]
on the universal covering \( \tilde{M} \to M \). Then

\[
H : q \in \tilde{M} \mapsto \begin{pmatrix}
h(e_1(q), e_1(q)) & h(e_1(q), e_2(q)) \\
h(e_2(q), e_1(q)) & h(e_2(q), e_2(q))
\end{pmatrix} \in \{ H \in \text{SL}(2, \mathbb{C}) \mid \tilde{H}^T = H \}
\]
is the corresponding equivariant harmonic map into the hyperbolic 3-space

\[
\mathbb{H}^3 = \text{SL}(2, \mathbb{C})/\text{SU}(2) = \{ H \in \text{SL}(2, \mathbb{C}) \mid \tilde{H}^T = H \}^+, 
\]
where \( \{ H \in \text{SL}(2, \mathbb{C}) \mid \tilde{H}^T = H \}^+ \) is the component of \( \{ H \in \text{SL}(2, \mathbb{C}) \mid \tilde{H}^T = H \} \) containing the identity matrix. Without loss of generality, we assume in the following that \( h \) is the standard metric, i.e., \( A^* = \tilde{A}^T \) and the map \( H \) is given by

\[
H = \tilde{\Psi}^T \Psi.
\]

**Remark 5.1** In the following, we will consider \( \{ H \in \text{SL}(2, \mathbb{C}) \mid \tilde{H}^T = H \} \) as the union of two copies of the hyperbolic 3-space, and \( H \) maps into this union. Depending on the component, a harmonic map \( H \) will therefore be given by

\[
H = \pm \tilde{\Psi}^T \Psi.
\]

**Definition 5.2** Let \( M \) be a compact Riemann surface, and \( U \subset M \) be an open dense subset. Let \((\nabla, \Phi, h)\) be a solution of the self-duality equations on \( U \). We say that the solution converges to \( \infty \) for \( p \to \partial U \) if the corresponding equivariant map \( H \) converges to the boundary of \( \mathbb{H}^3 \).

**Remark 5.3** This definition is well defined, since going to the boundary of \( \mathbb{H}^3 \) is invariant under the action of \( \text{SL}(2, \mathbb{C}) \). With respect to the matrix model, the condition just means that the operator norm (with respect to the standard hermitian metric on \( \mathbb{C}^2 \)) of the matrix \( H \) goes to \( \infty \) as \( p \to \partial U \).

There is a further useful description of \( \mathbb{H}^3 \) (see also [10, Section 2]): consider a totally geodesic 2-sphere in the 3-sphere \( S^2 \subset S^3 \). The complement \( S^3 \setminus S^2 \) consists of two three-dimensional hemispheres, and each of them can be equipped with the hyperbolic metric. For explicit computations, we use the stereographic projection of \( S^3 \) determined by a point \( p \in S^2 \subset S^3 \) such that \( S^2 \setminus \{ p \} \) is mapped to \( \mathbb{R}^2 \times \{ 0 \} \). The hyperbolic metric on the two (three-dimensional) half planes is then given by

\[
g_{(x,y,z)} = \frac{1}{z^2}(dx \otimes dx + dy \otimes dy + dz \otimes dz).
\]
The isometry between the matrix and the half-plane model is given by

\[ \{ H \in \text{SL}(2, \mathbb{C}) \mid \tilde{H}^T = H \} \longrightarrow \mathbb{R}^3 \setminus (\mathbb{R}^2 \times \{0\}); \]

\[ \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ x_0 - x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}. \] (5.2)

In this setup, a map intersects the \( \infty \)-boundary of \( \mathbb{H}^3 \) (away from the base point \( p \) of the stereographic projection) in first order, i.e., transversally, if the third component \( z \) in the half-plane model has 0 as a regular value. The action of the isometry group \( \text{PSL}(2, \mathbb{C}) \) of the hyperbolic 3-space extends to an action on \( S^3 \) by conformal transformations. Restricted to the boundary \( S^2 = \mathbb{C}P^1 \) we obtain the action of the Moebius group. In the light cone model, the conformal transformations of \( S^3 \) are given by the (standard) inclusion

\[ \text{SL}(2, \mathbb{C}) \cong \text{SO}(3, 1) \longrightarrow \text{SO}(4, 1) \]

given by the action on \( \{ H \in \text{SL}(2, \mathbb{C}) \mid \tilde{H}^T = H \} \)

\[ (h, g) \mapsto \tilde{g}^T H g. \] (5.3)

We then have the following geometric interpretation of Theorem 4.4.

**Theorem 5.4** Up to choosing \( \rho \sim 0 \) smaller, every real section \( s \) of the Deligne–Hitchin moduli space \( \mathcal{M}_{\text{DH}}(M_q) \) constructed in Theorem 4.4 gives rise to a solution of the self-duality equations on an open dense subset \( U \subset M_q \) which converges to \( \infty \) as \( p \to \partial U \). The boundary \( \partial U \) is given by closed regular curves.

The corresponding equivariant harmonic map \( f : U \to \mathbb{H}^3 \) extends through the boundary \( \partial U \) and gives a smooth map \( \tilde{f} : M_q \to S^3 \) on the universal covering \( \pi : \tilde{M}_q \to M_q \). In particular, \( \tilde{f} \) is a branched Willmore surface equivariant with respect to the monodromy representation of the connection \( \nabla^1 \) acting on \( S^3 \) via (5.3) by conformal transformations.

The proof of the Theorem 5.4 relies on properties of loop group factorizations. We thus briefly recall some basic definitions and concepts for loop groups first, for details see [18, Sections 3 and 8].

**Definition 5.5** A loop is a \( C^\infty \)-map \( \gamma : S^1 \to \text{SL}(2, \mathbb{C}) \). The loop group \( \Lambda \text{SL}(2, \mathbb{C}) \) is the set of all loops \( \gamma \).

\( \Lambda \text{SL}(2, \mathbb{C}) \) has a natural Lie group structure given by point-wise multiplication together with a compatible differentiable structure. A loop is called positive, if it extends to a holomorphic map from the unit disc \( D_1 \subset \mathbb{C} \) to \( \text{SL}(2, \mathbb{C}) \), and negative, if it extends holomorphically to \( \mathbb{C}P^1 \setminus \tilde{D}_1 \). The space of positive, respectively negative, loops is denoted by \( \Lambda^+ \text{SL}(2, \mathbb{C}) \), respectively \( \Lambda^- \text{SL}(2, \mathbb{C}) \).

**Definition 5.6** The big cell of \( \Lambda \text{SL}(2, \mathbb{C}) \) is the open and dense subset of \( \Lambda \text{SL}(2, \mathbb{C}) \) whose elements are given by the product of a positive and a negative loop, i.e.,

\[ \gamma = \gamma_+ \gamma_- \].
The factorization of a given element $\gamma$ in the big cell into its positive and negative loop is called Birkhoff factorization. It extends to the whole loop group by allowing a middle term $J = \text{diag}(\lambda^i, \lambda^{-i})$, where $k$ depends on $\gamma$, i.e., $\gamma = \gamma_+ J \gamma_-$ for every $\gamma \in \text{ASL}(2, \mathbb{C})$. For $\gamma$ lying in the big cell the factorization is uniquely determined by fixing $\gamma_- (\infty) = \text{Id}$. Equivalently, the factorization $\gamma = \gamma_+ \gamma_-$ is unique up to multiplying a constant $B \in \text{SL}(2, \mathbb{C})$, i.e., up to $\tilde{\gamma}_+ = \gamma_+ B$ and $\tilde{\gamma}_- = B^{-1} \gamma_-$. A (real-analytic) loop $\gamma$ can be interpreted as the transition function of a holomorphic rank 2 vector bundle $V_{\gamma}$ trivialized over $U_0 := D_{1+e}$ and $U_\infty := \mathbb{C}P^1 \setminus \frac{1}{1+e}$. Because $\gamma$ maps into $\text{SL}(2, \mathbb{C})$ the bundle $V_{\gamma}$ has trivial determinant. Moreover, if $\gamma = \gamma_+ \gamma_-$ lies in the big cell of $\text{ASL}(2, \mathbb{C})$, then the two holomorphic frames given by $\gamma_+$ on $U_0$ and $\gamma_-^{-1}$ are trivializing frames of $V_{\gamma}$, i.e., $V_{\gamma}$ is the trivial holomorphic $\mathbb{C}^2$-bundle over $\mathbb{C}P^1$. In the context of this article, a real-analytic loop is obtained from the gauge transformations $g(\lambda)$ evaluated at $x \in M$, which we denote by $g_x(\lambda)$. In this case, we have $U_0 = \mathbb{C}$ and $U_\infty = \mathbb{C}P^1 \setminus \{0\}$. Because the initial condition of the generalized Whitham flow is particularly well understood, we can show that the possible bundle types $V_{g_x}$ are very limited for $\rho$ small. The reader might also consult [29] for related work.

Before we start with the technical part, we recall from Section 1 that for a real section $s$ and for every $k \in \mathbb{N}$, we can always find a holomorphic lift into the space of $C^{k+1}$-connections, and a holomorphic family

$$\lambda \in \mathbb{C}^* \mapsto g(\lambda) \in \mathcal{H}_\lambda \subset \mathcal{E}^{k+2}$$

of gauge transformations (note that we have shifted $k \to k+1$ here in order to obtain $g \in C^k$ in the following). By the Sobolev embedding theorem, $g(\lambda)$ is a gauge transformation of class $C^k$ for every $\lambda \in \mathbb{C}^*$.

**Proposition 5.7** Let $x \in M_q$, $\rho \sim 0$ and $\nabla^k$ be a lift of class $C^{k+1}$ of a real section $s$ of the Deligne–Hitchin moduli space over the compact Riemann surface $M_q$ constructed in Theorem 4.4 with gauge transformations of class $C^k g(\lambda)$ satisfying (1.12) with the minus sign. Then the bundle $V_{g_x}$ is either trivial or

$$V_{g_x} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1).$$

The set

$$W := \{x \in M_q | V_{g_x} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)\}$$

is locally given by the zero locus of a real-valued $C^k$-function.

**Proof.** For $\rho = 0$, the initial data are given by spectral genus 1 solutions of the cosh-Gordon equation. The Birkhoff factorization in this case is well understood and has the claimed properties, see [10]. Moreover, the bundles $V_{g_x}$ are trivial at the branch points of $M_q \to \mathbb{C}P^1$.

Since $\lambda \text{SL}(2, \mathbb{C})/\lambda^+ \text{SL}(2, \mathbb{C})$ is stratified by the induced bundle type, see [18, Theorem 8.6.3], the property that $V_{g_x}$ is either trivial or $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ is an open condition and therefore preserved for $\rho \sim 0$. By Lemma 5.12, the set $W$ is the zero set of a real valued $C^k$-function $r$. \hfill $\Box$

**Remark 5.8** We will see below that $W$ is in fact a (compact) submanifold, i.e., the union of closed $C^k$-curves, where $k$ is arbitrary large.
To proceed, we need the following lemma. It is well known for holomorphic maps, but we are not aware of any reference which deals with the case of $C^k$-maps.

**Lemma 5.9** For $k \in \mathbb{N} > 0$ and a $C^\infty$-manifold $M$ let

$$y \in M \mapsto (g_y : \mathbb{C}^* \longrightarrow \text{SL}(2, \mathbb{C}) \text{ holomorphic})$$

be a $C^k$-map into the loop group such that the corresponding rank 2 bundles $V_{g_y}$ over $\mathbb{C}P^1$ are either trivial or $O(1) \oplus O(-1)$. Let $x \in W = \{ y \in M | V_{g_y} \cong O(1) \oplus O(-1) \}$. Then there exists an open neighbourhood $U \subset M$ of $x$, a $C^k$-function $r : U \rightarrow \mathbb{C}$ and $C^k$-maps

$$y \in U \mapsto (h^+_y : \mathbb{C} \longrightarrow \text{SL}(2, \mathbb{C}) \text{ holomorphic})$$

into the positive loop group and

$$y \in U \mapsto (h^-_y : \mathbb{C}P^1 \setminus \{0\} \longrightarrow \text{SL}(2, \mathbb{C}) \text{ holomorphic})$$

into the negative loop group such that for all $y \in U$

$$g_y = h^+_y \begin{pmatrix} \lambda^{-1} & r(y) \\ 0 & \lambda \end{pmatrix} h^-_y. \quad (5.4)$$

In particular, $r(y) \neq 0$ is equivalent to $V_{g_y}$ being trivial.

**Proof.** The last assertion follows from the fact that the Birkhoff splitting of the middle term in (5.4) for $r \neq 0$ is given by

$$\begin{pmatrix} \lambda^{-1} & r \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} r \quad 0 \\ \lambda \quad \lambda^{-1} \end{pmatrix} \begin{pmatrix} \lambda \quad 1 \\ -1 \quad 0 \end{pmatrix}$$

implying $V_{g_y}$ being trivial.

For $y \in M$ consider the $C^k$-family of holomorphic bundles $\tilde{V}_y$ given by the cocycle

$$\lambda \mapsto \lambda g_y(\lambda).$$

In particular, $\tilde{V}_x$ is of type $O \oplus O(2)$ for $x \in W$ and its determinant bundle is $O(2) \rightarrow \mathbb{C}P^1$.

By Riemann Roch, the space of holomorphic sections of $\tilde{V}_y$ is complex four-dimensional for every $y \in M$. Therefore, the space of holomorphic sections of $\tilde{V}_y$ defines a rank four $C^k$-vector bundle $\hat{V}$ over $M$.

Let $x \in W$. Choose two local sections $s_1, s_2$ of $\hat{V}$ in a neighbourhood $U$ of $x$ such that the two holomorphic sections $s_1(x), s_2(x)$ of $\tilde{V}_x$ have determinant

$$\det (s_1(x), s_2(x)) = \lambda(\lambda - 1) \in H^0(\mathbb{C}P^1, O(2)).$$
For \( y \in U \) fixed, \( s_i(y) \) are holomorphic sections of \( \tilde{V}_y \to \mathbb{C}P^1 \). With respect to the local trivialization of the bundle \( \tilde{V}_y \) over \( U_+ = \mathbb{C} \) and \( U_- = \mathbb{C}P^1 \setminus \{0\} \) each \( s_i(y) \) is determined by a pair of \( \mathbb{C}^2 \)-valued holomorphic functions \( f_\pm^i: U_\pm \to \mathbb{C}^2 \). By definition, the frames

\[
F_+(y) = (f_+^1(y), f_+^2(y)) \quad \text{and} \quad F_-(y) = (f_-^1(y), f_-^2(y))
\]

of \( \tilde{V}_y \) over \( U_+ \) and \( U_- \), respectively, satisfy

\[
F_+(y) = \lambda g_y F_-(y)
\]

for all \( y \in U \). We will omit the argument \( y \) of \( F_\pm \) in the following. The \( C^k \)-regularity of the involved maps in \( y \) is not affected by the operations performed. The determinants of \( F_\pm \) satisfy

\[
\det(F_+) = \lambda^2 \det(F_-)
\]

with \( \det(F_+) \) being a polynomial of degree 2 in \( \lambda \). By continuity and by choosing \( U \) small enough \( \det(F_+) \) has two simple zeros (close to \( \lambda = 0 \) and \( \lambda = 1 \)) for all \( y \in U \).

Assume that there exist \( C^k \)-families of \( gl(2, \mathbb{C}) \)-valued polynomials \( P = P_0 + \lambda P_1 + \lambda^2 P_2 \) and \( Q = Q_0 + \lambda^{-1} Q_1 + \lambda^{-2} Q_2 \)

with the property that \( P \) and \( Q \) are invertible for generic \( \lambda \) such that

\[
\tilde{F}_+(y) = F_+(y)P_\star^{-1}(y) \quad \text{is a holomorphic map from } \mathbb{C} \text{ into } SL(2, \mathbb{C})
\] (5.5)

and

\[
\tilde{F}_-(y) = F_-(y)Q_\star^{-1}(y) \quad \text{is a holomorphic map from } \mathbb{C}P^1 \setminus \{0\} \text{ into } SL(2, \mathbb{C})
\] (5.6)

for all \( y \in U \). In particular, we have

\[
\det(P_\star) = \det(F_+(x)) = \lambda(\lambda - 1) \quad \text{and} \quad \det(Q_\star) = \det(F_-(x)) = 1 - \lambda^{-1}. \quad \text{(5.7)}
\]

Then

\[
\frac{1}{\lambda} PQ^{-1} = \tilde{F}_+^{-1} g \tilde{F}_-
\]

is a \( C^k \)-family of holomorphic maps \( \mathbb{C}^* \to SL(2, \mathbb{C}) \).

Since \( \tilde{F}_+^{-1} g \tilde{F}_- \) is \( SL(2, \mathbb{C}) \)-valued and holomorphic in \( \lambda \in \mathbb{C}^* \), the coefficients of the matrix \( \frac{1}{\lambda} PQ^{-1} \in gl(2, \mathbb{C}) \) extend holomorphically through the zeros of \( \det Q \). Writing these coefficients in terms of the coefficients of \( P \) and \( Q \) and using

\[
1 = \det \tilde{F}_+^{-1} g \tilde{F}_- = \det \left( \frac{1}{\lambda} PQ^{-1} \right)
\]

(5.9)
we can compute
\[ \frac{1}{x} PQ^{-1} = \lambda^{-1} r_{-1} + r_0 + \lambda r_1. \] (5.10)
for some matrices \( r_i \in \mathfrak{gl}(2, \mathbb{C}) \).

This gives \( \det r_{-1}(y) = \det r_1(y) = 0 \) for all \( y \in U \). If \( r_1 = 0 \), then \( \frac{1}{x} PQ\tilde{F} \) is a negative loop and 
\( g = \tilde{F}(\frac{1}{x} PQ\tilde{F}) \) is the Birkhoff factorization of \( g \) into positive and negative parts. Therefore,
\[ r_1(x) \neq 0 \quad \text{and} \quad r_{-1}(x) \neq 0 \]
for \( x \in W \). A similar argument gives \( \det(r_0(y)) = 0 \) for all \( y \in U \). Otherwise, \( \frac{1}{x} PQ^{-1} r_0^{-1} \) lies in the big cell.

To obtain (5.4), we write \( r_i \) with respect to a suitable basis of \( \tilde{V}_y \). Firstly, we can find suitable \( \mathcal{C}^k \)-maps
\[ h_1, h_2 : U \rightarrow \text{SL}(2, \mathbb{C}) \]
such that
\[ h_1 r_{-1} h_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]
Equation (5.9) then implies
\[ h_1 r_1 h_2 = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix}. \]
Choosing
\[ h_2 = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} \tilde{h}_2 \]
gives
\[ h_1 r_{-1} h_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad h_1 r_1 h_2 = \begin{pmatrix} 0 & \hat{r} \\ 0 & 1 \end{pmatrix} \]
for some \( \mathcal{C}^k \)-function \( \hat{r} : U \rightarrow \mathbb{C} \). Moreover, due to (5.9),
\[ h_1 r_0 h_2 = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \]
for some \( \mathcal{C}^k \)-function \( r : U \rightarrow \mathbb{C} \). The desired factorization of \( g \) is then given by
\[ g = \tilde{F}_+ h_1^{-1} \begin{pmatrix} 1 & \hat{r}(y) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & r(y) \\ 0 & \lambda \end{pmatrix} h_2^{-1} \tilde{F}^{-1}. \]
i.e.,

\[ h^+_y := \tilde{F}_+h^{-1}_1 \begin{pmatrix} 1 & \tilde{r}(y) \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad h^-_y := h^{-1}_2\tilde{F}^{-1}_- \]

To complete the proof, it remains to show the existence of the polynomials \( P \) and \( Q \) in (5.5) and (5.6). This is done pointwise in \( y \in U \) in the following Lemma 5.10. The \( \mathcal{C}^k \) dependency in \( y \) can be deduced as follows: by (5.7) the determinant of \( F_+ \) has only two simple zeros on an open neighbourhood \( U \) of \( x \). Thus, the two simple zeros \( \lambda_0(y) \) and \( \lambda_1(y) \) are \( \mathcal{C}^k \)-functions in \( y \in U \). Without loss of generality, we can assume that

\[ F_+(x)(\lambda_0(x)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0, \]

because if \( (1 \ 0)^T \) lies in the kernel of \( F_+(x)(\lambda_0(x)) \), then we can choose the vector \( (0 \ 1)^T \) instead, as the zeros of the determinant \( F_+(x)(\lambda_0(x)) \) are simple. Let

\[ F_+ = \begin{pmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{pmatrix} \]

and consider the \( \mathcal{C}^k \)-map

\[ y \in U \mapsto \tilde{P}_0(y) = \begin{pmatrix} 1 & -\frac{f_{1,2}(y)(\lambda_0(y))}{f_{1,1}(y)(\lambda_0(y))} \\ 0 & (\lambda - \lambda_0(y)) \end{pmatrix} \]

into the space of 2 by 2 matrices with polynomial entries in \( \lambda \) of degree less or equal 1. The determinant \( \det \tilde{P}_0^{-1}(y) \) has a simple pole at \( \lambda_0(y) \) and is holomorphic elsewhere.

Therefore, \( F_+(y)(\tilde{P}_0(y))^{-1} \) is holomorphic and non-singular at \( \lambda_0(y) \) by construction of \( \tilde{P}_0(y) \) and has a single simple zero at \( \lambda_1(y) \). Inductively, we can eliminate the zero of \( \det(\tilde{F}_+(y)(\tilde{P}_0(y))^{-1}) \) of

\[ y \in U \mapsto F_+(y)(\tilde{P}_0(y))^{-1} \]

Altogether, we obtain a \( \mathcal{C}^k \)-map

\[ y \in U \mapsto P(y) \]

into the space of a \( \mathfrak{gl}(2, \mathbb{C}) \)-valued polynomials in \( \lambda \) of degree less or equal 2 satisfying (5.5). Analogously, by replacing \( \lambda \) by \( \tilde{\lambda} = \frac{1}{\lambda} \) we obtain the existence of a \( \mathcal{C}^k \)-map

\[ y \in U \mapsto Q(y) \]

into the space of a \( \mathfrak{gl}(2, \mathbb{C}) \)-valued polynomials in \( \lambda^{-1} \) of degree less or equal 2 satisfying (5.6). \( \square \)
Lemma 5.10 Let \( F : \mathbb{C} \to \mathfrak{gl}(2, \mathbb{C}) \) be a holomorphic map such that \( \det(F) \neq 0 \) is a polynomial of degree \( \leq d \). Then there is a polynomial

\[
P : \mathbb{C} \longrightarrow \mathfrak{gl}(2, \mathbb{C})
\]

of degree \( \leq d \) such that \( \det(P) = \det(F) \) and \( FP^{-1} \) extends holomorphically through the zeros of \( \det(F) \).

Proof. We prove the Lemma by induction over the degree \( d \) of the polynomial \( \det(F) \). For \( d = 0 \) we can choose \( P = \text{Id} \) and the assertion holds trivially.

Assume that for every \( F \) such that \( \det(F) \) has degree \( k \leq d \) there exists a polynomial \( P_k \) of degree \( k \) with the desired properties. For \( F \) such that \( \det(F) \) is of degree \( d + 1 \) let \( \lambda_0 \) be a zero of \( \det(F) \). There are two cases to consider.

The first is \( F(\lambda_0) = 0 \). In this case choose \( \tilde{F} : \mathbb{C} \to \mathfrak{gl}(2, \mathbb{C}) \) such that

\[
F = \tilde{F} \cdot (\lambda - \lambda_0)\text{Id}.
\]

Then

\[
\det(F) = (\lambda - \lambda_0)^2 \det(\tilde{F}),
\]

and \( \det(\tilde{F}) \) is of degree \( d - 1 \). By assumption, we can find a polynomial \( P_{d-1} \) of degree \( d - 1 \) with \( \det(\tilde{F}) = \det(P_{d-1}) \) such that \( \tilde{F}P_{d-1}^{-1} \) extends holomorphically through the zeros of \( \det(\tilde{F}) \). Then

\[
P = (\lambda - \lambda_0)\text{Id} \cdot P_{d-1}
\]

satisfies that \( FP^{-1} \) extends holomorphically through the zeros of \( \det(F) \).

In the second case \( F(\lambda_0) \neq 0 \), and \( F(\lambda_0) \) has a one-dimensional kernel \( L \). Decompose \( \mathbb{C}^2 = L \oplus \tilde{L} \) for some complementary line \( \tilde{L} \) and choose \( \tilde{F} \) such that

\[
F = \tilde{F} \cdot \begin{pmatrix} \lambda - \lambda_0 & 0 \\ 0 & 1 \end{pmatrix}
\]

with respect to the splitting. Then \( \det(\tilde{F}) \) is of degree \( d \) and there exists a suitable polynomial \( P_d \) by assumption. By choosing

\[
P(\lambda) = P_d \cdot \begin{pmatrix} \lambda - \lambda_0 & 0 \\ 0 & 1 \end{pmatrix}
\]

with respect to \( \mathbb{C}^2 = L \oplus \tilde{L} \) we therefore obtain that \( FP^{-1} \) extends holomorphically through the zeros of \( \det(F) \) and \( FP^{-1} \) takes values in \( \text{SL}(2, \mathbb{C}) \) by construction. \( \square \)

Definition 5.11 On the loop group \( \Lambda\text{SL}(2, \mathbb{C}) \) we define an involution \( \gamma \mapsto \gamma^* \) by

\[
\gamma^*(\lambda) = \gamma(\bar{\lambda}^{-1}).
\]
**Lemma 5.12** Let \( g \) and \( k \in \mathbb{N} > 0 \) be as in Lemma 5.9 with the additional property that 

\[ gg^* = -\text{Id}. \]

Then, there exists a factorization of the form (5.4) for a real-valued function \( r \) of class \( C^k \).

**Proof.** By Lemma 5.9, there exist a factorization of \( g \) with

\[ g_y = h_+^y \begin{pmatrix} \lambda^{-1} & r(y) \\ 0 & \lambda \end{pmatrix} h_-^y \]

for a complex valued \( C^k \)-function \( r \). We want to find new \( h^\pm \) such that the function \( r \) is real valued. Let \( A := h^-(h^+)^* \). Then \( A \) is a \( C^k \)-map into the (negative) loop group \( \Lambda^- \text{SL}(2, \mathbb{C}) \), and

\[ -\text{Id} = gg^* = h^+ \begin{pmatrix} \lambda^{-1} & r \\ 0 & \lambda \end{pmatrix} h^-(h^+)^* \begin{pmatrix} -\lambda & \bar{r} \\ 0 & -\lambda^{-1} \end{pmatrix} (h^-)^* \]

implies

\[ (A^{-1})^* = \begin{pmatrix} \lambda^{-1} & r \\ 0 & \lambda \end{pmatrix} A \begin{pmatrix} \lambda & -\bar{r} \\ 0 & \lambda^{-1} \end{pmatrix}. \quad (5.11) \]

Comparing the \( \lambda \)-coefficients of both sides of the equation shows that \( A \) is of the form

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \lambda^{-1} \begin{pmatrix} a_1 & 0 \\ c_1 & d_1 \end{pmatrix} + \lambda^{-2} \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix} \]

for some \( C^k \)-functions \( a, \ldots, c_2 : \bar{U} \to \mathbb{C} \) satisfying

\[ c_2 = -\bar{c} \]
\[ c_1 = \bar{c}_1 \]
\[ a_1 = \bar{c}r \]
\[ d_1 = -\bar{c}\bar{r} \]
\[ a + c_1 r = \bar{d} \]
\[ \bar{b} = cr\bar{r} \]
\[ rd = \bar{r}\bar{d}. \]

The fifth and the last equation in (5.13) gives that the \( C^k \)-function

\[ y \mapsto d(y)r(y) \]

is real valued. Moreover, \( b = \bar{c}r\bar{r} \) gives

\[ ad = 1 + bc = 1 + c\bar{c}r\bar{r}, \]
implying that $a$ and $d$ are non-vanishing. Therefore,

$$
\tilde{h}^+ = h^+ \left( \frac{1}{\sqrt{d}} \begin{array}{cc} 0 & 0 \\ \sqrt{d} & 0 \end{array} \right)
$$

$$
\tilde{h}^- = \left( \begin{array}{cc} \sqrt{d} & 0 \\ 0 & \frac{1}{\sqrt{d}} \end{array} \right) h^-
$$

are well defined and satisfy

$$
g_y = \tilde{h}_y^+ \left( \lambda^{-1} \begin{array}{cc} \tilde{r} & 0 \\ 0 & \lambda \end{array} \right) \tilde{h}_y^-
$$

with the real-valued $C^k$-function

$$
\tilde{r} = dr.
$$

We are now in a position to prove Theorem 5.4.

**Proof of Theorem 5.4.** Consider on $M_q$ the family of flat connections $\nabla^\lambda$ and the associated family of $\text{SL}(2, \mathbb{C})$-gauge transformations $g(\lambda)$ satisfying (1.12). The set $U \subset M_q$ of points $y$ where the loop $\lambda \mapsto g_y(\lambda)$ lies in the big cell is by Proposition 5.7 open and dense, see also [29]. In other words, for every $y \in U$ there exists a Birkhoff factorization

$$
g_y(\lambda) = g^+_y(\lambda)g^-_y(\lambda), \quad (5.14)
$$

where $g^+_y(\lambda)$ and $g^-_y(\lambda)$ are positive and negative loops, respectively. We can choose the factorization (5.14) in a way that both factors depend smoothly on $y \in U$, e.g., by imposing the normalization $g^+_y(0) = \text{Id}$.

Recall that without a normalization the Birkhoff factorization $g = g^+ g^-$ is only unique up to

$$
g^+ \mapsto g^+ B^{-1} \quad \text{and} \quad g^- \mapsto B g^-
$$

for some $B \in \text{SL}(2, \mathbb{C})$. Hence, by using (1.12), there exists a smooth map

$$
B : U \to \text{SL}(2, \mathbb{C})
$$

such that

$$
g^-_y(\lambda) = B g^+_y(-\tilde{r}^{-1})^{-1}
$$

and

$$
B \tilde{B} = -\text{Id}.
$$
The last equation implies that

\[ B = \begin{pmatrix} \alpha & \beta \\ \gamma & -\bar{\alpha} \end{pmatrix} \]

for real valued \( \beta, \gamma \), and complex valued \( \alpha \). Because the determinant of \( B \) is 1,

\[ \beta \gamma < 0. \]

Depending on the sign of \( \beta \), \( B \) can be written as

\[ B = \pm C \delta C^{-1} \]

for the smooth map

\[ C = \begin{pmatrix} \sqrt{\pm \beta} & 0 \\ \mp \sqrt{\pm \beta} & 1 \end{pmatrix} : U \rightarrow \text{SL}(2, \mathbb{C}) \]

and

\[ \delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

It can be checked directly that the family of flat connections

\[ \tilde{\nabla}^\lambda = \nabla^\lambda (g^+(\lambda) C) \tag{5.15} \]

gives a solution of the self-duality equations on \( U \) with respect to the standard hermitian metric on \( \mathbb{C}^2 \).

We show that for \( p \rightarrow \partial U \) the matrix norm of the associated harmonic map \( H \) into hyperbolic 3-space goes to \( \infty \), and that \( H \) extends (after gluing the two hyperbolic balls along its \( \infty \)-boundary \( S^2 \)) to a smooth map \( f : \tilde{M}_q \rightarrow S^3 \) on the universal covering \( \tilde{M}_q \rightarrow M_q \).

Recall that on a simply connected subset \( \tilde{U} \subset U \) the harmonic map \( H \) corresponding to the real section \( s \) of \( M_{DH} \) is given by

\[ \pm \tilde{F}^T F : \tilde{U} \rightarrow \mathbb{H}^3 \cup \mathbb{H}^3, \]

where \( F \) is the parallel frame of \( \tilde{\nabla}^1 \) in (5.15). A parallel frame \( \Psi \) of \( \nabla^1 \) is then given by

\[ \Psi = (g^+(1) C) F, \]

with \( g^+(\lambda) \) defined in (5.14). Therefore, we have to analyse the behaviour of

\[ f = \pm \tilde{F}^T F = \pm \tilde{\Psi}^T (\tilde{g}^+(1)^{-1})^T (\tilde{C}^{-1})^T C^{-1} g^+(1)^{-1} \Psi \]

\[ = \tilde{\Psi}^T (\tilde{g}^+(1)^{-1})^T \delta^{-1} (\pm \tilde{C} \delta C^{-1}) g^+(1)^{-1} \Psi = \tilde{\Psi}^T (\tilde{g}^+(1)^{-1})^T \delta^{-1} B g^+(1)^{-1} \Psi \tag{5.16} \]
when $g(\lambda)$ leaves the big cell. Let $x \in W$. By Lemma 5.9, there exists a neighbourhood $\tilde{U} \subset M_q$ of $x$ such that for all $y \in \tilde{U}$ we have the factorization

$$
g_y = h_y^+(\frac{\lambda^{-1}}{r(y)} \begin{pmatrix} r(y) & 0 \\ \frac{1}{r(y)} & \lambda^{-1} \end{pmatrix}) h_y^-
$$

for a $C^k$-function $r: \tilde{U} \to \mathbb{R}$ (Lemma 5.12) with sufficiently large $k \in \mathbb{N}$. Recall that $r \neq 0$ is equivalent to $g_y$ lying in the big cell, since

$$g_y = h^+(\frac{1}{\lambda} \begin{pmatrix} r(y) & 0 \\ \frac{1}{r(y)} & \lambda^{-1} \end{pmatrix}) h^-(y)
$$

with

$$g_y^+ = h^+(y) \begin{pmatrix} r(y) & 0 \\ \lambda & \frac{1}{r(y)} \end{pmatrix}
$$

gives a Birkhoff factorization of $g_y$. For $y \in \tilde{U}$ lying in the big cell (5.16) thus yields

$$f = \tilde{\Psi}^T (\tilde{h}^+(1)^{-1})^T \delta^{-1} \tilde{B} g^+(1)^{-1} \Psi
$$

$$= \tilde{\Psi}^T (h^+(1)^{-1})^T \delta^{-1} \left( \begin{array}{cc} \tilde{r} & 0 \\ 1 & \tilde{r} \end{array} \right) \tilde{B} \left( \begin{array}{cc} \frac{1}{\tilde{r}} & 0 \\ -1 & r \end{array} \right) h^+(1)^{-1} \Psi.
$$

(5.17)

For every $y \in \tilde{U}$ fixed, the map $y \mapsto h^+(1)(y)^{-1} \Psi(y): \tilde{U} \to \text{SL}(2, \mathbb{C})$ is well defined and acts on $\mathbb{H}^3 \cup \mathbb{H}^3$ by isometries via (5.3). Therefore, to analyse the behaviour for $H \to \infty$, it is sufficient to consider the term

$$\delta^{-1} \left( \begin{array}{cc} \tilde{r} & 0 \\ 1 & \tilde{r} \end{array} \right) \tilde{B} \left( \begin{array}{cc} \frac{1}{\tilde{r}} & 0 \\ -1 & r \end{array} \right)
$$

(5.18)

for $r(y) \to 0$.

As in the proof of Lemma 5.12, we consider $A = h^-(h^+)^*$ of the form (5.12) with coefficients satisfying (5.13). Then the constant loop $B$ is given by

$$B = g^-(g^+)^* = \left( \begin{array}{cc} \frac{1}{\lambda} & 0 \\ -1 & 0 \end{array} \right) A \left( \begin{array}{cc} \tilde{r} & 0 \\ -\lambda^{-1} & \frac{1}{\tilde{r}} \end{array} \right) = \left( \begin{array}{cc} c \tilde{r} & d \tilde{r} \\ -a \tilde{r} & -b \tilde{r} \end{array} \right)
$$

(5.19)

and

$$\delta^{-1} \left( \begin{array}{cc} \tilde{r} & 0 \\ 1 & \tilde{r} \end{array} \right) \tilde{B} \left( \begin{array}{cc} \frac{1}{\tilde{r}} & 0 \\ -1 & r \end{array} \right) = \left( \begin{array}{cc} -c + \frac{d}{\tilde{r}} + \frac{d}{r} - \tilde{c} & -\tilde{d} + c r \\ -\tilde{a} + c \tilde{r} & \tilde{b} \end{array} \right)
$$

(5.20)

where we have used $\tilde{b} = cr \tilde{r}$. Therefore, the map in (5.18) takes values in the space of symmetric matrices $\{C \in \text{SL}(2, \mathbb{C}) \mid C \mapsto \tilde{C}^T \}$ by (5.13). Because

$$1 = \det \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = ad - bc = d \tilde{d} - c_1 d r - c \tilde{c} r \tilde{r}
$$
\[ d(y) \bar{a}(y) \to 1 \] for \( r(y) \to 0 \). Therefore, the upper left entry of the right-hand side of (5.20) goes to \( \infty \) with the same order as the vanishing order of \( r \). The other entries remain finite.

It remains to prove that the vanishing order of \( r \) at \( x \in W \) is 1 and that \( \partial U \) is a smooth one-dimensional submanifold for \( \rho \sim 0 \). Let \( \rho = 0 \) and \( x \in W \). In this case, it is well known from [10, Section 6] that the surface \( f \) intersects the boundary at infinity transversely. It follows from (5.20) that the differential of \( r \) at \( x \) does not vanish.

This property is preserved for \( \rho \sim 0 \), because the data depend continuously on \( \rho \). In particular, the upper left entry of the right-hand side of (5.20) goes to \( \infty \) with order 1, while the other entries remain finite valued. Lemma 5.9 shows that \( \partial U \) is given by the vanishing locus of \( r \), thus \( \partial U \) is a smooth one-dimensional submanifold of \( M_q \).

By construction, the Higgs fields of the real sections in Theorem 4.4 are nilpotent. This implies that the corresponding harmonic maps on \( \tilde{M} \) are conformal. Therefore, \( \tilde{f}_{\tilde{U}} \) is a conformally parametrized equivariant minimal surface in the hyperbolic 3-space defined on the preimage \( \tilde{U} = \pi^{-1}(U) \). In particular, \( \tilde{f}_{\tilde{U}} \) is a branched Willmore surface, and as \( \tilde{f} \) is smooth and \( \tilde{U} \subset \tilde{M} \) is an open dense subset, \( \tilde{f} \) satisfies the Willmore Euler–Lagrange equation globally. \( \square \)

**Remark 5.13 (Higher solutions and Willmore surfaces)** The higher solutions of Hitchin’s self-duality equations constructed here are given by isothermic Willmore surfaces that are locally, but not globally, minimal surfaces in a space form. Willmore tori in the 4-sphere are shown to form an integrable system in [30]. They are obtained through an associated family of flat \( \text{SL}(4, \mathbb{C}) \)-connections \( \nabla^\mu \) for \( \mu \in \mathbb{C}^* \) [8]. The associated family is encoded by the spectral curve \( \Sigma \) which is a 4-fold covering of \( \mathbb{C}P^1 \) and possesses an additional involution \( \sigma \) if the target is a three-dimensional space form. The quotient \( \Sigma/\sigma \) is then a hyperelliptic curve. In the case of isothermic Willmore tori this quotient is another \( \mathbb{C}P^1 \) and the family of flat \( \text{SL}(4, \mathbb{C}) \)-connections splits into the direct sum of two (gauge equivalent) rank 2 families of flat connections parametrized by \( \lambda \in \Sigma/\sigma \), see [31]. The double covering of the \( \mu \)-plane by the \( \lambda \)-plane corresponds to taking a square root. Therefore, the rank 2 associated family of flat connections obtained through this construction is invariant under a real involution covering either \( \lambda \mapsto \bar{\lambda}^{-1} \) or \( \lambda \mapsto -\bar{\lambda}^{-1} \) on \( \mathbb{C}P^1 \), i.e., it corresponds to the harmonic maps into the 3-sphere or the self-duality equations case. If \( \Sigma/\sigma \) has genus \( \geq 1 \) then the associated family of flat connections \( \nabla^\mu \) should give rise to a holomorphic map

\[ \Sigma/\sigma \to \mathcal{M}_{DH} \]

which is compatible with the projection \( \mathcal{M}_{DH} \to \mathbb{C}P^1 \) and real with respect to either \( \rho \) or \( \tau \).

**Acknowledgements**

We thank the anonymous referees for providing many suggestions to improve the clarity of the text.

**Funding**

The DFG priority program 2026 ‘Geometry at Infinity’ to L.H.; RTG 1670 *Mathematics inspired by string theory and quantum field theory* funded by the DFG to S.H.

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