Shape Theory Via SV Decomposition II

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Abstract
The non isotropic and non central elliptical shape distributions via the Le and Kendall SVD decomposition approach are derived in this paper in the context of invariant polynomials and zonal polynomials. The so termed cone and disk densities here obtained generalise some results of the literature. Finally, some particular densities are applied in a classical data of Biology, and the inference is performed after choosing the best model by using a modified BIC criterion.

1 Introduction and the main principle.

The multivariate statistical shape theory has been developed in the last two decades around the classical works based on normality and isotropy, see Goodall and Mardia (1993) and Dryden and Mardia (1998) and the references there in. Recent works extended this results to elliptical models and partial non isotropy, see for example Caro-Lopera et al (2009) and Díaz-García et al (2003), however some important problems remain, the study of the shape theory without any restriction of the covariance matrix in the elliptical model.

The problem arises from the point of view of applications, the isotropic assumption \( \Theta = I_K \) for an elliptical shape model of the form

\[
X \sim \mathcal{E}_{N \times K}(\mu_X, \Sigma_X, \Theta, h),
\]

restricts substantially the correlations of the landmarks in the figure. Then, we expect the non isotropic model, with any positive definite matrix \( \Theta \), as the best model for considering all the possible correlations among the anatomical (geometrical or mathematical) points.

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This problem can be solved by considering the following procedure: Let be

\[ X \sim E_{N \times K}(\mu_X, \Sigma_X, \Theta, h), \]

if \( \Theta^{1/2} \) is the positive definite square root of the matrix \( \Theta \), i.e., \( \Theta = (\Theta^{1/2})^2 \), with \( K \times K \), Let and Kendall (1993), p. 11, and noting that

\[ X\Theta^{-1}X' = X(\Theta^{-1/2}\Theta^{-1/2})^{-1}X' = X\Theta^{-1/2}(X\Theta^{-1/2})' = ZZ', \]

where

\[ Z = X\Theta^{-1/2}, \]

then

\[ Z \sim E_{N \times K}(\mu_Z, \Sigma_Z, I_K, h) \]

with \( \mu_Z = \mu_X \Theta^{-1/2} \), (see Let and Kendall (1993) p. 20).

And we arrive at the classical starting point in shape theory where the original landmark matrix is replaced by \( Z = X\Theta^{-1/2} \). Then we can proceed as usual, removing from \( Z \), translation, scale, rotation and/or reflection in order to obtain the shape of \( Z \) (or \( X \)) via the QR, SVD, polar decompositions. The shape theory associated with the SVD decomposition can be study from two different approaches, one due to Goodall (1991) and another proposed by Let and Kendall (1993).

We study in this paper the statistical approach of Let and Kendall under a generalised elliptical model. First, recall some facts of this technique (Le and Kendall (1993)). It is known that the shape of an object is all geometrical information that remains after filtering out translation, rotation and scale information of an original figure (represented by a matrix \( X \)) comprised in \( N \) landmarks in \( K \) dimensions. Hence, we say that two figures, \( X_1 : N \times K \) and \( X_2 : N \times K \) have the same shape if they are related with a special similarity transformation \( X_2 = \beta X_1 H + I_N \gamma' \), where \( H : K \times K \in SO(K) \) (the rotation), \( \gamma : K \times 1 \) (the translation), \( I_N : N \times 1 \), \( I_N = (1,1,\ldots,1)' \), and \( \beta > 0 \) (the scale).

Thus, in this context, the shape of a matrix \( X \) is all the geometrical information about \( X \) that is invariant under Euclidean similarity transformations. Then, the shape space is the set of all possible shapes, it is the orbit space of the non-coincident \( N \) landmarks in \( \mathbb{R}^K \) under the action of the Euclidean similarity transformations.

The dimension of this space is \( NK - K - 1 - K(K-1)/2 \), it is, the original dimension \( NK \) is reduced by \( K \) for location, by 1 for uniform scale and by \( K(K-1)/2 \) for rotation. In other words, the shape of \( X \) is the set \( \{ \Gamma : \Gamma \in SO(K) \} \) where \( \Gamma \) is the so termed pre-shape of \( X \) defined as \( \Gamma = LX/\|LX\| \) (\( L \) is Helmert submatrix, for example) which is invariant under translation and scaling of \( X \).

The rotated \( P \) on the pre-shape sphere is termed a fibre of the pre-shape space \( \Sigma_K^N \), these fibres do not overlap and corresponds one to one with shapes in the shape space \( \Sigma_K^N \), it is, the pre-shape space is partitioned into fibres by the rotation group \( SO(K) \) and the fibre is the orbit of \( P \) under the action of \( SO(K) \). Thus, \( \Sigma_K^N \) is the quotient space of \( \Sigma_K^N \) under the action of \( SO(K) \), in notation \( \Sigma_K^N = \Sigma_K^N / SO(K) \), which means that the shape of \( X \) is an equivalent class under the action of the group of similarity transformations.

Now, the statistical theory of shape associated to this approach studies the effect of randomness and assume a probabilistic model for the original matrix in order to obtain the density of the pre-shape (cone) and shape (disk). The complete procedure for obtaining the shape of an original \( X \) can be summarised in the following steps:

\[ LX\Theta^{-1/2} = LZ = Y = V'DH = rV'WH = rV'W(u)H, \]

where the matrix \( L : (N-1) \times N \) has orthonormal rows to \( 1 = (1,\ldots,1)' \). \( L \) can be a submatrix of the Helmert matrix, for example. Here \( Y = V'DH \) is the SVD of matrix.
\( \mathbf{Y} \), with \( \mathbf{V} : n \times (N - 1) \) and \( \mathbf{H} : n \times K \) semiorthogonal matrices and \( \mathbf{D} : n \times n \), \( \mathbf{D} = \text{diag}(D_1, \ldots, D_n) \): \( \mathbf{W} = \mathbf{D}/r \), \( r = \|\mathbf{D}\| = (\sum_{i=1}^n D_i^2)^{1/2} = \|\mathbf{Y}\| \).

Then the standard problem considers a model for \( \mathbf{X} \) and finds the so termed cone and disk densities, which are the densities of \( \mathbf{D} \) and \( \mathbf{W}(\mathbf{u}) \), respectively. A number of published works have studied the classical problem which assumes a non central Gaussian model for \( \mathbf{X} \) very restricted by the isotropy assumption \( \Theta = \mathbf{I}_K \) and \( \Sigma_{\mathbf{X}} = \sigma^2 \mathbf{I}_N \).

These restrictions facilitates the integration in terms of the zonal polynomials and the asymptotic densities can be derived and applied. This procedure based on normality and isotropy is very common in literature of shape under QR, SVD and affine decompositions too, see [Goodall and Mardia, 1993] and [Dryden and Mardia, 1998], and the references there in.

However, it is clear that the gaussian case do not support all the applications and the statistical theory of shape could enriched if complete families of cone and disk densities are available for a particular experiment and the researcher can model the situation by applying a model selection criteria (see for example [Rissanen, 1978], [Kass and Raftery, 1995], [Raftery, 1995] and [Yang and Yang, 2007], among many others).

Therefore, in this paper we propose the statistical theory of shape of Le and Kendall’s approach under any non central elliptical model without any restriction of the covariance matrix.

Explicitly, consider a full covariance elliptical model, indexed by the generator function \( h \),

\[
\mathbf{X} \sim \mathcal{E}_{N \times K}(\mu_{\mathbf{X}}, \Sigma_{\mathbf{X}}, \Theta, h),
\]

so, by the main principle given above, at the beginning of this section, we have that

\[
\mathbf{Z} \sim \mathcal{E}_{N - 1 \times K}(\mu_{\mathbf{Z}}, \Sigma_{\mathbf{X}}, \mathbf{I}_K, h),
\]

with \( \mathbf{Z} = \mathbf{X} \Theta^{-1/2}, \mu_{\mathbf{Z}} = \mu_{\mathbf{X}} \Theta^{-1/2} \).

Then the (SVD) Le and Kendall’s shape coordinates \( \mathbf{u} \) of \( \mathbf{X} \) are constructed in several steps summarised in the expression

\[
L \mathbf{X} \Theta^{-1/2} = LZ = \mathbf{Y} = \mathbf{V}' \mathbf{D} \mathbf{H} = r \mathbf{V}' \mathbf{W} \mathbf{H} = r \mathbf{V}' \mathbf{W}(\mathbf{u}) \mathbf{H}. \tag{1}
\]

Denote \( \mu = \mathbf{L} \mu_{\mathbf{X}} \), so \( \mathbf{Y} : (N - 1) \times K \) is invariant to translations of the figure \( \mathbf{Z} \), and

\[
\mathbf{Y} \sim \mathcal{E}_{N - 1 \times K}(\mu \Theta^{-1/2}, \Sigma \otimes \mathbf{I}_K, h),
\]

where \( \Sigma = \mathbf{L} \Sigma_{\mathbf{X}} \mathbf{L}' \), meanwhile the matrix \( \mathbf{W} \), the shape of \( \mathbf{X} \), is invariant under (translations), rotations and scaling of the landmark data matrix \( \mathbf{X} \).

By considering the above main principle, this paper studies Le and Kendall’s approach for the shape theory based on the SVD decomposition and any non isotropic non central elliptical model. Section 2 obtains the general densities, the so termed cone and disk densities, with some corollaries. Then, the central case and its invariance is studied in Section 3. At the end inference on small and large mouse vertebra data is performed with the classical Gaussian model and two non normal Kotz models, then the best model is chosen by a modified BIC criterion and the corresponding test for equality in mean disk shape is obtained.

\section{Shape Theory via SVD Le and Kendall’s approach}

We start with the jacobian of the corresponding decomposition.
Lemma 2.1. Let be $\mathbf{Y} : N - 1 \times K$, then there exist $\mathbf{V} \in \mathcal{V}_{n,K}$ and $\mathbf{D} : n \times n$, $\mathbf{D} = \text{diag}(D_1, \ldots, D_n)$, $n = \min([N - 1], K)$; $D_1 \geq D_2 \geq \cdots \geq D_n \geq 0$, such that $\mathbf{Y} = \mathbf{V}'\mathbf{D}\mathbf{H}$: This factorisation is termed non-singular part of the SVD. Then

$$(d\mathbf{Y}) = 2^{-n}|\mathbf{D}|^{N-1+K-2n} \prod_{i<j}^n (D_i^2 - D_j^2)^i(d\mathbf{D})(\mathbf{V}d\mathbf{V}')(\mathbf{HdH}').$$

Proof. See Díaz-García et al. (1997).}

In order to obtain the joint density function of $(V, D)$ we need the following generalisation of James (1964, eq. (22)).

Lemma 2.2. Let $\mathbf{X} : K \times n$, $\mathbf{Y} : K \times K$ and $\mathbf{H} \in \mathcal{V}_{n,K}$. Then

1. $\int_{\mathbf{H} \in \mathcal{V}_{n,K}} |\text{tr} (\mathbf{Y} + \mathbf{XH})|^p (\mathbf{HdH}') = \frac{2^n\pi^{Kn/2}}{\Gamma_n[K]} \sum_{f=0}^\infty \sum_{\lambda} (p)_2f(\text{tr} \mathbf{Y})^{p-2f} \frac{C_{\lambda}(\frac{1}{4}\mathbf{XX}')}{f!}$, where $|\text{tr} \mathbf{Y}|^{-1} \text{tr} \mathbf{XH} < 1$ and $\text{tr} \mathbf{Y} \neq 0$.

2. $\int_{\mathbf{H} \in \mathcal{V}_{n,K}} \text{tr} (\mathbf{Y} + \mathbf{XH}) \text{etr}(r(\mathbf{Y} + \mathbf{XH})) (\mathbf{HdH}') = \frac{2^n\pi^{Kn/2}}{\Gamma_n[K]} \text{etr}(\mathbf{Y}) \left\{ \text{tr} \mathbf{Y}_0F_1\left(\frac{1}{2} K; \frac{r^2}{4} \mathbf{XX}'\right) + \sum_{f=0}^\infty \sum_{\lambda} \frac{(f + \frac{1}{2}) C_{\lambda}(\frac{1}{4}\mathbf{XX}')}{(\frac{1}{2} K)_\lambda f!} \right\}$, where $p \in \mathbb{R}$, $r \in \mathbb{R}$, $C_{\lambda}(\mathbf{B})$ are the zonal polynomials of $\mathbf{B}$ corresponding to the partition $\kappa = (f_1, \ldots, f_s)$ of $f$, with $\sum_{i=1}^s f_i = f$; and $(a)_\kappa = \prod_{i=1}^s (a - (j - 1)/2)_{f_i}$, $(a)_f = a(a + 1) \cdots (a + f - 1)$, are the generalised hypergeometric coefficients and $\mathbf{0}F_1$ is the Bessel function, James (1964).

Proof. 1. From Lemma 9.5.3 Muirhead (1982, Lemma 9.5.3, p. 397) we have

$$\int_{\mathbf{H} \in \mathcal{V}_{n,K}} [\text{tr} (\mathbf{Y} + \mathbf{XH})]^p (\mathbf{HdH}') = \frac{2^n\pi^{Kn/2}}{\Gamma_n[K]} \int_{O(K)} [\text{tr} (\mathbf{Y} + \mathbf{XH})]^p (d\mathbf{H}).$$

Furthermore, for $\text{tr} \mathbf{Y} \neq 0$ and $|\text{tr} \mathbf{Y}|^{-1} \text{tr} \mathbf{XH} < 1$

$$[\text{tr} (\mathbf{Y} + \mathbf{XH})]^p = (\text{tr} \mathbf{Y})^p \sum_{f=0}^\infty \frac{(p)_f}{f!} (\text{tr} \mathbf{Y})^{-f} (\text{tr} \mathbf{XH})^f.$$

Now from James (1964, eqs. (46) and (22)) it follows that

$$\int_{\mathbf{H} \in \mathcal{V}_{n,K}} [\text{tr} (\mathbf{Y} + \mathbf{XH})]^p (\mathbf{HdH}') = \frac{2^n\pi^{Kn/2}}{\Gamma_n[K]} \sum_{f=0}^\infty \sum_{\lambda} \frac{(p)_2f(\text{tr} \mathbf{Y})^{-2f}}{(2f)!} \frac{(\frac{1}{2})_f}{(\frac{1}{2} K)_\lambda} C_{\lambda}(\mathbf{XX}'),$$

the result follows, noting that $(\frac{1}{2})_f/(2f)! = 1/(4^f f!)$ and that $C_{\lambda}(\mathbf{aXX}') = a^f C_{\lambda}(\mathbf{XX}')$.  

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2. This follows by expanding the exponentials in series of powers and by applying (22) and (27) from James [1964].

Thus, we can obtain:

**Theorem 2.1.** The joint density of \((V, D)\) is

\[
f_{V,D}(V, D) = \frac{\pi^{\frac{n^2}{2}} |D|^{N-1+K-2n} \prod_{i<j}(D_i^2 - D_j^2)}{\Gamma_n \left[\frac{1}{2}K\right] |\Sigma|^{\frac{1}{2}}} \times \sum_{\kappa} \sum_{t=0}^{\infty} \frac{h^{(2t)}(\sum_{i<j} \text{tr}(\Sigma^{-1}V'D^2V + \Omega))}{t!(\frac{1}{2}K)_{\kappa}} C_{\kappa} (\Omega \Sigma^{-1}V'D^2V).
\]

**Proof.** Let be \(\Omega = \Sigma^{-1} \mu \Theta^{-1} \mu'\), then the density of \(Y\) is given by

\[
f_Y(Y) = \frac{1}{|\Sigma|^{\frac{1}{2}}} h \left[ \text{tr}(\Sigma^{-1}YY' + \Omega) - 2 \text{tr} \mu' \Sigma^{-1}Y \right].
\]

Now, make the change of variables \(Y = V'DH\), so, by Lemma 2.1, the joint density function of \(V, D, H\) is

\[
dF_{V,D,H}(V, D, H) = \frac{2^{-n} |D|^{N-1+K-2n} \prod_{i<j}(D_i^2 - D_j^2)}{|\Sigma|^{\frac{1}{2}}} (VdV')(dD) \times h \left[ \text{tr}(\Sigma^{-1}V'D^2V + \Omega) - 2 \text{tr} \mu' \Sigma^{-1}V'DH \right] (HdH').
\]

Expanding in power series

\[
dF_{V,D,H}(V, D, H) = \frac{2^{-n} |D|^{N-1+K-2n} \prod_{i<j}(D_i^2 - D_j^2)}{|\Sigma|^{\frac{1}{2}}} (VdV')(dD) \times \sum_{t=0}^{\infty} \frac{1}{t!} h^{(t)} \left[ \text{tr}(\Sigma^{-1}V'D^2V + \Omega) \right] \left[ \text{tr} (-2 \mu' \Sigma^{-1}V'DH) \right]^t (HdH').
\]

From Lemma 2.2

\[
\int_{H \in \mathcal{V}_V,K} (-2 \mu' \Sigma^{-1}V'DH)^{2t} (HdH') = \frac{2^n \pi^{\frac{n^2}{2}}}{\Gamma_n \left[\frac{1}{2}K\right]} \sum_{\kappa} \left(\frac{1}{2}\right)_t \frac{1}{t!} C_{\kappa} (\Omega \Sigma^{-1}V'D^2V).
\]

Observing that \(\left(\frac{1}{2}\right)_t = \frac{1}{t!}\), the marginal joint density of \(V, D\) is given by

\[
dF_{V,D}(V, D) = \frac{\pi^{\frac{n^2}{2}} |D|^{N-1+K-2n} \prod_{i<j}(D_i^2 - D_j^2)}{\Gamma_n \left[\frac{1}{2}K\right] |\Sigma|^{\frac{1}{2}}} \times \sum_{t=0}^{\infty} \frac{h^{(2t)}(\sum_{i<j} \text{tr}(\Sigma^{-1}V'D^2V + \Omega))}{t!(\frac{1}{2}K)_{\kappa}} C_{\kappa} (\Omega \Sigma^{-1}V'D^2V) (VdV')(dD).
\]

Now, note that \(D\) contains \(n\) coordinates for which, under this method, its corresponding joint density is termed cone density (or size-and-shape density). Then we have the first main result of this section.
Theorem 2.2. The cone density is given by

\[
  f_D(D) = \frac{2^n \pi^{n(N+K)} |D|^{N-1+K-2n} \prod_{i<j} (D_i^2 - D_j^2)}{\Gamma_n \left[ \frac{1}{2} K \right] \Gamma_n \left[ \frac{1}{2} (N-1) \right] |\Sigma|^{\frac{1}{2}}} \\
  \times \sum_{\theta, \kappa} \sum_{\phi \in \Theta, \kappa} h^{(2l+1)}(\text{tr} \Omega) \Delta^{\theta, \kappa}_\phi C_\phi(D^2) C^{\theta, \kappa}_\phi (\Sigma^{-1}, \Omega \Sigma^{-1}) \frac{1}{t! l! \left( \frac{1}{2} K \right)_\kappa C_\phi(\mathbf{I}_{N-1})}, \tag{2}
\]

where the notation of the sum operators, \( C^{\theta, \kappa}_\phi \) and \( \Delta^{\theta, \kappa}_\phi \) are given in [Davis (1980)], in particular \( \Delta^{\theta, \kappa}_\phi = \frac{C^{\theta, \kappa}_\phi (\mathbf{I}, \mathbf{I})}{C_\phi(\mathbf{I})} \).

Proof. The joint density of \( V, D \) is

\[
  dF_{V,D}(V, D) = \frac{\pi^{\frac{1}{2} K} |D|^{N-1+K-2n} \prod_{i<j} (D_i^2 - D_j^2)}{\Gamma_n \left[ \frac{1}{2} K \right] \Gamma_n \left[ \frac{1}{2} (N-1) \right] |\Sigma|^{\frac{1}{2}}} (dD)(VdV') \\
  \times \sum_{t=0}^{\infty} \frac{h^{(2t)}(\text{tr} \Sigma^{-1} V'D^2 V + \text{tr} \Omega)}{t! \left( \frac{1}{2} K \right)_\kappa} C_\kappa(\Sigma^{-1} V'D^2 V) \\
  \times \prod_{l=1}^{\infty} \phi \in \Theta, \kappa \sum_{\phi \in \Theta, \kappa} C_\phi(D^2) C^{\theta, \kappa}_\phi(\mathbf{I}_{N-1})
\]

Assuming that \( h^{(2l)}(\cdot) \) can be expanded in power series,

\[
  h^{(2l)}(\text{tr} \Sigma^{-1} V'D^2 V + \text{tr} \Omega) = \sum_{l=0}^{\infty} \frac{h^{(2l+1)}(\text{tr} \Omega)}{l!} \left( \text{tr} \Sigma^{-1} V'D^2 V \right)^l = \sum_{l=0}^{\infty} \sum_{\theta} \frac{h^{(2l+1)}(\text{tr} \Omega)}{l!} C_\theta(\Sigma^{-1} V'D^2 V),
\]

where \( C_\theta(A) \) is the zonal polynomial corresponding to the partition \( \theta = (l_1, \ldots, l_\alpha) \), with \( \sum_{i=1}^\alpha l_i = l \).

From [Davis (1980)], eq. (4.13), the integration of \( dF_{V,D}(V, D) \) with respect to \( V \in \mathcal{V}_{n,N-1} \) results

\[
  \int_{V \in \mathcal{V}_{n,N-1}} C_\kappa(\Sigma^{-1} V'D^2 V) C_\theta(\Sigma^{-1} V'D^2 V) (VdV') \\
  = \frac{2^n \pi^{n(N-1)}}{\Gamma_n \left[ \frac{1}{2} (N-1) \right]} \sum_{\phi \in \Theta, \kappa} C^{\theta, \kappa}_\phi (\Sigma^{-1}, \Omega \Sigma^{-1}) C^{\theta, \kappa}_\phi (D^2, D^2) \frac{1}{C_\phi(\mathbf{I}_{N-1})}.
\]

And by [Davis (1980)], eq. (5.1) we have that

\[
  f_D(D) = \frac{2^n \pi^{n(N+K)} |D|^{N-1+K-2n} \prod_{i<j} (D_i^2 - D_j^2)}{\Gamma_n \left[ \frac{1}{2} K \right] \Gamma_n \left[ \frac{1}{2} (N-1) \right] |\Sigma|^{\frac{1}{2}}} \\
  \times \sum_{\theta, \kappa} \sum_{\phi \in \Theta, \kappa} h^{(2l)}(\text{tr} \Omega) \Delta^{\theta, \kappa}_\phi C_\phi(D^2) C^{\theta, \kappa}_\phi (\Sigma^{-1}, \Omega \Sigma^{-1}) \frac{1}{t! l! \left( \frac{1}{2} K \right)_\kappa C_\phi(\mathbf{I}_{N-1})}. \quad \square
\]

Now, we can derive the isotropic version of the cone density
Corollary 2.1. Let be $\Sigma = \sigma^2 I$, then

$$f_D(D) = \frac{2^n \pi^{n(N-1)+1}(N-1)!}{\Gamma_n \frac{1}{2} \Gamma_n \frac{1}{2} (N-1)! \sigma^{N-1} K} \left( \sum_{i<j} D_i D_j \right)^{N-1} \prod_{i<j} \left( D_i^2 - D_j^2 \right) \times \sum_{l=0}^{\infty} \sum_{\kappa} \frac{h^{(2t)}(\text{tr} \Omega + \frac{1}{2\sigma^2} \text{tr} D^2)\kappa (\frac{1}{4\sigma^2} D^2)\kappa (\Omega)}{t! \left( \frac{1}{2} K \right)_\kappa C_k(I)}.$$  \hspace{1cm} (3)

Proof. From Theorem 2.2

1. $\Omega = \Sigma^{-1} \mu \Theta^{-1} \mu' = \frac{1}{\sigma^2} \mu \Theta^{-1} \mu$.

2. From Davis (1980), eq. (5.7),

$$C^{\theta,\kappa}_\phi (\Sigma^{-1}, \Omega \Sigma^{-1}) = C^{\theta,\kappa}_\phi \left( \frac{1}{\sigma^2} I_{N-1}, \frac{1}{\sigma^2} \Omega \right),$$

$$= \left( \frac{1}{\sigma^2} \right)^{2t+1} \sum_{\kappa} \left( \Delta^{\theta,\kappa}_\phi \right)^2 C^{\theta,\kappa}_\phi (I_{N-1}) \left( \Omega \right),$$

Therefore the second line of (2), denoted by $J$, it is simplified as follows:

$$J = \sum_{\theta, \kappa} \sum_{\phi \in \theta, \kappa} \frac{h^{(2t+\ell)}(\text{tr} \Omega)\Delta^{\theta,\kappa}_\phi (\Sigma^{-1}, \Omega) \left( \sum_{\theta, \kappa} C^{\theta,\kappa}_\phi (\Omega) \right)}{t! \left( \frac{1}{2} K \right)_\kappa C_k(I_{N-1})}.$$ 

Note that $\sum_{\phi \in \theta, \kappa} \left( \Delta^{\theta,\kappa}_\phi \right)^2 C^{\theta}_\phi (D^2) = C^{\theta}_\phi (D^2) C^{\theta}_\phi (D^2)$, see Davis (1980), eq. (5.10). Thus

$$J = \sum_{\theta, \kappa} \frac{h^{(2t+\ell)}(\text{tr} \Omega)\sum_{\phi \in \theta, \kappa} \left( \Delta^{\theta,\kappa}_\phi \right)^2 C^{\theta,\kappa}_\phi (\Omega)\kappa (\Omega) \left( \Omega \right)}{t! \left( \frac{1}{2} K \right)_\kappa C_k(I_{N-1})} \left( \text{tr} D^2 \right)^l.$$ 

Now, observe that $h(v) = \sum_{l=0}^{\infty} \frac{h^{(l)}(v)}{l!} (v - a)^l$, with $a = \text{tr} \Omega$, $v = \text{tr} \Omega + \frac{1}{\sigma^2} \text{tr} D^2$, and $h(v) = h^{(2t)}(v)$. Thus

$$J = \sum_{\kappa} \frac{h^{(2t)}(\text{tr} \Omega + \frac{1}{\sigma^2} \text{tr} D^2)\kappa (\Omega) \left( \Omega \right) \kappa (\Omega)}{t! \left( \frac{1}{2} K \right)_\kappa C_k(I_{N-1})} \left( \text{tr} D^2 \right)^l. \hspace{1cm} \Box$$
Now let be \( \mathbf{W} = \mathbf{D}/r, r = \|\mathbf{D}\| = \|\mathbf{V}'\mathbf{D}\mathbf{H}\| = \|\mathbf{Y}\| \) and noting that if \( \mathbf{D} = \text{diag}(D_1, \ldots, D_n) \) we define \( \text{vecp}(\mathbf{D}) = (D_1, \ldots, D_n) \), then
\[
\text{vecp}(\mathbf{D}) = \begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix},
\]
implies that \( \text{vecp}(\mathbf{W}) = \begin{pmatrix} D_1/r \\ \vdots \\ D_n/r \end{pmatrix} = \text{vecp} \frac{\mathbf{D}}{r}, \)
thus
\[
(d\mathbf{W}(\mathbf{u})) = r^m \prod_{i=1}^{m} \sin^{m-i} \theta_i (d\mathbf{u}) \wedge dr = r^m J(\mathbf{u})(d\mathbf{u}) \wedge dr,
\]
with \( m = n - 1, \mathbf{u} = (\theta_1, \ldots, \theta_m)' \).

The shape density under Le and Kendall’s approach it is known as disk density.

**Theorem 2.3.** The disk density is given by
\[
f_{\mathbf{W}}(\mathbf{W}) = \frac{\pi^{\frac{n}{2}} \prod_{i=1}^{n} l_i^{N-1+K-2n} \prod_{i<j} (l_i^2 - l_j^2) J(\mathbf{u})}{\Gamma_n \left[ \frac{1}{2} K \right] |\Sigma|^{\frac{1}{2}}} \times \sum_{t=0}^{\infty} \sum_{\kappa} \frac{1}{t! \left( \frac{1}{2} K \right)_\kappa} \int_{V \in \mathcal{V}_{n,N-1}} C_\kappa \left( \Omega \Sigma^{-1} \mathbf{V}' \mathbf{W}^2 \mathbf{V} \right) \mathbf{V} \mathbf{W}'
\]
\[
\times \int_{0}^{\infty} s^{n(N+K-n-1)+2t-1} h(2t) \left( s^2 + \text{tr} \Omega \right) (ds),
\]
where the number of landmarks \( N \) are selected in such way that \( \frac{n(N+K-n-1)}{2} + t \) is a positive integer, then \( \theta = (l_1, \ldots, l_n) \) is a partition of \( \frac{n(N+K-n-1)}{2} + t \), and \( \sum_{i=1}^{n} l_i = \frac{n(N+K-n-1)}{2} + t \).

**Proof.** From Theorem 2.1
\[
dF_{\mathbf{V}, \mathbf{D}}(\mathbf{V}, \mathbf{D}) = \frac{\pi^{\frac{d}{2}} |\mathbf{D}|^{N-1+K-2n} \prod_{i<j} (D_i^2 - D_j^2)}{\Gamma_n \left[ \frac{1}{2} K \right] |\Sigma|^{\frac{1}{2}}} (d\mathbf{D})(\mathbf{V} \mathbf{D}'')
\]
\[
\times \sum_{t=0}^{\infty} \sum_{\kappa} h(2t) \left[ \text{tr} \left( \Sigma^{-1} \mathbf{V}' \mathbf{D}^2 \mathbf{V} \right) \right] C_\kappa \left( \Omega \Sigma^{-1} \mathbf{V}' \mathbf{D}^2 \mathbf{V} \right)
\]
\[
\times \int_{0}^{\infty} s^{n(N+K-n-1)+2t-1} h(2t) \left( s^2 + \text{tr} \Omega \right) (ds).
\]
Let \( \mathbf{W} = \text{diag}(l_1^*, \ldots, l_n^*), l_i^* = \frac{D_i}{r}, r = \|\mathbf{D}\| = \|\mathbf{Y}\|, \) then
\[
dF_{\mathbf{V}, \mathbf{W}}(\mathbf{V}, \mathbf{W}) = \frac{\pi^{\frac{d}{2}} |r\mathbf{W}|^{N-1+K-2n} \prod_{i<j} (l_i^2 - l_j^2)}{\Gamma_n \left[ \frac{1}{2} K \right] |\Sigma|^{\frac{1}{2}}} (d\mathbf{W})(\mathbf{V} \mathbf{W}'')
\]
\[
\times \sum_{t=0}^{\infty} \sum_{\kappa} h(2t) \left[ r^2 \text{tr} \left( \Sigma^{-1} \mathbf{V}' \mathbf{W}^2 \mathbf{V} \right) \right] C_\kappa \left( r^2 \Omega \Sigma^{-1} \mathbf{V}' \mathbf{W}^2 \mathbf{V} \right)
\]
\[
\times \int_{0}^{\infty} s^{n(N+K-n-1)+2t-1} h(2t) \left( s^2 + \text{tr} \Omega \right) (ds).
\]
Note that
1. \( |r\mathbf{W}|^{N-1+K-2n} = r^n(N-1+K-2n). \)
2. \( \prod_{i<j} r_i^2 (l_i^2 - l_j^2) = (r^2)^{\frac{n(n-1)}{2}} \prod_{i<j} (l_i^2 - l_j^2) \).

3. \( C_\kappa \left( r^2 \Omega^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V} \right) = r^{2t} C_\kappa \left( \Omega \Sigma^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V} \right) \).

Collection powers of \( r \) and defining \( r = \frac{s}{(\text{tr} \: \Omega^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V})^\frac{1}{2}} \), with \( dr = \frac{ds}{(\text{tr} \: \Omega^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V})^\frac{1}{2}} \) and
\[
\int_0^\infty r^{n(N+K-n-1)+2t-1} h^{(2t)} (r^2 \text{tr} \: \Omega^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V}) \, dr
\]
\[
= \int_0^\infty \left( \frac{s}{(\text{tr} \: \Omega^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V})^\frac{1}{2}} \right)^{n(N+K-n-1)+2t-1} h^{(2t)} (s^2) \frac{ds}{(\text{tr} \: \Omega^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V})^\frac{1}{2}}
\]

Thus the marginal density of \( dF_\mathbf{W}(\mathbf{W}) \) is given by
\[
\pi \frac{\sigma^2}{2} \mid \mathbf{W} \mid^{N-1+K-2n} J(u) \prod_{i<j} (l_i^2 - l_j^2) \]
\[
\times \int_{\mathbf{V} \in \mathbb{V}_{n,N-1}} C_\kappa \left( \Omega \Sigma^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V} \right) (\text{tr} \: \Omega^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V})^{\frac{n(N+K-n-1)}{2} + t} (\mathbf{V} d\mathbf{V}')
\]
\[
\times \int_0^\infty s^{n(N+K-n-1)+2t-1} h^{(2t)} (s^2) \, ds.
\]

Now, let be
\[
J = \int_{\mathbf{V} \in \mathbb{V}_{n,N-1}} \frac{C_\kappa \left( \Omega \Sigma^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V} \right) (\mathbf{V} d\mathbf{V}')}{(\text{tr} \: \Omega^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V})^{\frac{n(N+K-n-1)}{2} + t}},
\]
\[
= \int_{\mathbf{V} \in \mathbb{V}_{n,N-1}} \frac{C_\kappa \left( \Omega \Sigma^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V} \right) (\mathbf{V} d\mathbf{V}')}{\sum_{\theta} C_{\theta} \left( \Sigma^{-1} \mathbf{V} \mathbf{W}^2 \mathbf{V} \right)},
\]

where the number of landmarks \( N \) are selected in such way that \( \frac{n(N+K-n-1)}{2} + t \) is a positive integer, then \( \theta = (l_1, \ldots, l_n) \) is a partition of \( \frac{n(N+K-n-1)}{2} + t \), and \( \sum_{i=1}^n l_i = \frac{n(N+K-n-1)}{2} + t \).

Then we obtain the desired result.

The isotropic case of the disk distribution follows

**Theorem 2.4.** The isotropic disk density is given by
\[
f_\mathbf{W}(\mathbf{W}) = \frac{2^n \pi^{\frac{n(N+K-1)}{2}} \prod_{i=1}^n l_i^N \prod_{i<j} (l_i^2 - l_j^2) J(u)}{\Gamma_n \left[ \frac{2K}{N} \right] \Gamma_n \left[ \frac{N-1}{2} \right] \left( \frac{\sigma^2}{2} \right)^{\frac{N-K}{2}}} \times \frac{\sum_{\kappa} C_\kappa \left( \frac{2}{\sigma^2} \mathbf{W}^2 \right) C_\kappa (\Omega)}{C_\kappa (\mathrm{I}_{N-1})} \times \int_0^\infty r^{n(N+K-n-1)+2t-1} h^{(2t)} \left( \text{tr} \: \Omega + \frac{r^2}{\sigma^2} \right) dr.
\]
Proof. The result is obtained from (1) taking $\Sigma = \sigma^2 I$ and observing that $|\Sigma| = |\sigma^2 I| = (\sigma^2)^{(N-1)}$;

$$\text{tr} \Sigma^{-1} V' W^2 V = \frac{1}{\sigma^2} \text{tr} V' W^2 V = \frac{1}{\sigma^2} \text{tr} W^2 V' = \frac{1}{\sigma^2} \text{tr} W^2 = \frac{1}{\sigma^2},$$

recalling that $||W|| = 1$. Hence

$$J = \int_{V \in V_{n-1}} \frac{C_\kappa (\Omega \Sigma^{-1} V' W^2 V) (V dV')}{(\text{tr} \Sigma^{-1} V' W^2 V)^{(N+n-1)}/2 + t}$$

$$= (\sigma^2)^{n(N-K-n-1)+t} \int_{V \in V_{n-1}} C_\kappa \left( \frac{1}{\sigma^2} \Omega V' W^2 V \right) (V dV')$$

$$= \frac{(\sigma^2)^{n(N-K-n-1)+t} 2^n n! (N-1)/2}{\Gamma_n \left[ \frac{1}{2} (N-1) \right]} \frac{C_\kappa \left( \frac{1}{\sigma^2} \Omega \right) C_\kappa \left( W^2 \right)}{C_\kappa \left( I_{N-1} \right)}.$$

Finally, (5) is obtained making the change of variable $s = r/\sigma$ with $ds = dr/\sigma$ in (4) and observing that $C_\kappa \left( \frac{1}{\sigma^2} \Omega \right) C_\kappa \left( W^2 \right) = C_\kappa \left( \Omega \right) C_\kappa \left( \frac{1}{\sigma^2} W^2 \right)$.

Alternatively, let $l_i^* = D_i/r$, $\text{tr} D^2 = \sum_{i=1}^n D_i^2 = r^2$ in (3), therefore

$$f_{W}(W) = \frac{2^n n! \sigma^{N+K-1} \rho_m J(u) \prod_{i=1}^n (r l_i^*)^{N-1+K-2n} \prod_{i<j} r^2 (l_i^* - l_j^*^2)}{\Gamma_n \left[ \frac{1}{2} K \right] \Gamma_n \left[ \frac{1}{2} (N-1) \right] \sigma^2 \left( \frac{N-1}{2} \right)^k} \times \sum_{t=0}^\infty \sum_{\kappa} \frac{h^{(2t)}}{t! \left( \frac{1}{2} K \right)_\kappa} C_\kappa \left( \frac{r^2}{\sigma^2} W^2 \right) C_\kappa \left( \Omega \right) C_\kappa \left( I_{N-1} \right).$$

Observe that

1. $\prod_{i=1}^n (r l_i^*)^{N-1+K-2n} = r^{n(N-1+K-2n)} \prod_{i=1}^n l_i^*^{N-1+K-2n}.$

2. $\prod_{i<j} r^2 (l_i^* - l_j^*^2) = r^{n(n-1)} \prod_{i<j} (l_i^* - l_j^*^2).$

3. $C_\kappa \left( \frac{r^2}{\sigma^2} W^2 \right) = r^{2t} C_\kappa \left( \frac{1}{\sigma^2} W^2 \right).$

Collecting powers of $r$ we have

$$\int_0^\infty r^{n(N-K-n-1)+2t-1} h^{(2t)} \left( \text{tr} \Omega + \frac{r^2}{\sigma^2} \right) dr. \quad \square$$

3 Central Case

We obtain the central cases of the cone and the disk densities.

**Corollary 3.1.** The central cone density is given by

$$f_{D}(D) = \frac{2^n |D|^{N-1+K-2n} \prod_{i<j} (D_i^2 - D_j^2)}{\pi n! \sigma^{N+K-1} \Gamma_n \left[ \frac{1}{2} K \right] \Gamma_n \left[ \frac{1}{2} (N-1) \right] \sigma^2} \sum_{t=0}^\infty \frac{h^{(t)}(0) C_\theta (\Sigma^{-1}) C_\theta (D^2)}{t! C_\kappa \left( I_{N-1} \right)}.$$
Then the required density is given by
\[ dF_Y(Y) = \frac{1}{|\Sigma|^\frac{n}{2}} h \left[ \text{tr} \Sigma^{-1} \mathbf{Y} \mathbf{Y}' \right] (d\mathbf{Y}). \]

The joint density of \( \mathbf{V}, \mathbf{D}, \mathbf{H} \) is
\[ dF_{\mathbf{V}, \mathbf{D}, \mathbf{H}}(\mathbf{V}, \mathbf{D}, \mathbf{H}) = \frac{\prod_{i<j} (D_i^2 - D_j^2)}{2^n |\mathbf{D}|^{-n(K+2n)} |\Sigma|^\frac{n}{2}} h \left[ \text{tr} \Sigma^{-1} \mathbf{V}' \mathbf{D}^2 \mathbf{V} \right] (\mathbf{H} d\mathbf{H}) (d\mathbf{V}). \]

Recalling that
\[ \int_{\mathbf{H} \in V_{n,K}} (\mathbf{H} d\mathbf{H}) = \frac{2^n \pi^\frac{n}{2}}{\Gamma_n \left[\frac{n}{2} K \right]}, \]
we have that
\[ \pi^\frac{n}{2} |\mathbf{D}|^{N-1+K-2n} \prod_{i<j} (D_i^2 - D_j^2) \]
\[ dF_{\mathbf{V}, \mathbf{D}}(\mathbf{V}, \mathbf{D}) = \frac{2^n \pi^\frac{n}{2}}{\Gamma_n \left[\frac{n}{2} K \right] |\Sigma|^\frac{n}{2}} h \left[ \text{tr} \Sigma^{-1} \mathbf{V}' \mathbf{D}^2 \mathbf{V} \right] (d\mathbf{V}). \]

Then the required density is given by
\[ f_{\mathbf{D}}(\mathbf{D}) = \frac{|\mathbf{D}|^{N-1+K-2n} \prod_{i<j} (D_i^2 - D_j^2)}{\Gamma_n \left[\frac{n}{2} K \right] |\Sigma|^\frac{n}{2}} \int_{\mathbf{V} \in V_{n,N-1}} h \left[ \text{tr} \Sigma^{-1} \mathbf{V}' \mathbf{D}^2 \mathbf{V} \right] (d\mathbf{V}). \]

Integrating with respect to \( \mathbf{V} \)
\[ \int_{V_{n,N-1}} h \left[ \text{tr} \Sigma^{-1} \mathbf{V}' \mathbf{D}^2 \mathbf{V} \right] (d\mathbf{V}) \]
\[ = \int_{V_{n,N-1}} \sum_{l=0}^{\infty} \frac{h^{(l)}(0)}{l!} (\text{tr} \Sigma^{-1} \mathbf{V}' \mathbf{D}^2 \mathbf{V})^l (d\mathbf{V}) \]
\[ = \sum_{l=0}^{\infty} \sum_{\theta} \frac{h^{(l)}(0)}{l!} \int_{V_{n,N-1}} C_\theta (\Sigma^{-1} \mathbf{V}' \mathbf{D}^2 \mathbf{V}) (d\mathbf{V}) \]
\[ = \sum_{l=0}^{\infty} \sum_{\theta} \frac{h^{(l)}(0)}{l!} \frac{2^n \pi^\frac{n(N-1)}{2}}{\Gamma_n \left[\frac{n}{2} (N-1) \right]} C_\theta (\Sigma^{-1}) C_\theta (\mathbf{D}^2) \]
\[ = \frac{2^n \pi^\frac{n(N+K-1)}{2}}{\Gamma_n \left[\frac{n}{2} K \right] \Gamma_n \left[\frac{n}{2} (N-1) \right] |\Sigma|^\frac{n}{2}} \sum_{l=0}^{\infty} \sum_{\theta} \frac{h^{(l)}(0)}{l!} C_\theta (\Sigma^{-1}) C_\theta (\mathbf{D}^2). \]

Then,
\[ f_{\mathbf{D}}(\mathbf{D}) = \frac{2^n \pi^\frac{n(N+K-1)}{2}}{\Gamma_n \left[\frac{n}{2} K \right] \Gamma_n \left[\frac{n}{2} (N-1) \right] |\Sigma|^\frac{n}{2}} \sum_{l=0}^{\infty} \sum_{\theta} \frac{h^{(l)}(0)}{l!} C_\theta (\Sigma^{-1}) C_\theta (\mathbf{D}^2). \]

Alternatively, from Theorem 2.2 if we take
1. \( h^{(2l+1)} (\text{tr} \mathbf{\Omega}) = h^{(l)}(0), \)
2. $\Delta_\theta,\kappa = \Delta_\theta,\varphi = C_\phi(I) = 1$

3. $C_\phi(D^2) = C_\phi(D^2)$, $C_\phi(I) = C_\phi(I)$,

4. $C_\phi(\varphi^{-1}, \Omega\varphi^{-1}) = C_\phi(\varphi^{-1}, 0) = C_\phi(\varphi^{-1})$,

the required result follows.

And finally, note that the central disk density is invariant under the elliptical distributions.

**Corollary 3.2.** The central disk density is given by

$$f_W(W) = \frac{\Gamma[\frac{1}{2}(n(N + K - n - 1))] \prod_{i=1}^{n} I_i^{1-n+K-2n} \prod_{i<j} (t_i^2 - t_j^2) J(u)}{2 \pi^{n(N + K - n - 1)} \Gamma_n \left[\frac{1}{2} K\right] \left[\frac{1}{2} (N - 1)\right] (\omega^2)^{(N - 1)K/2}}$$

$$\times \int_{\eta \in \mathbb{V}_{n,N-1}} \sum_\theta C_\theta \left(\Sigma^{-1} V W W V'\right).$$

(7)

where the number of landmarks $N$ are selected in such way that $n(N + K - n - 1)$ is a positive integer, then $\theta = (i_1, \ldots, l_\alpha)$ is a partition of $\frac{n(N + K - n - 1)}{2}$, and $\sum_{i=1}^{\alpha} l_i = \frac{n(N + K - n - 1)}{2}$

where $\theta = (i_1, \ldots, l_n)$ is a partition of the positive integer $n(N + K - n - 1)/2$, $\sum_{i=1}^{n} l_i = n(N + K - n - 1)/2$.

**Proof.** From Theorem 3.3 taking $t = 0$, $\Omega = 0$, $h^{(0)}(\cdot) \equiv h(\cdot)$ and recalling that

$$\int_{0}^{\infty} s^{n(N + K - n - 1) - 1} h(s^2) (ds) = \frac{\Gamma[\frac{1}{2}(n(N + K - n - 1))] (\omega^2)^{(N - 1)K/2}}{\pi^{n(N + K - n - 1)}}$$

the result is obtained.

**4 Example: Mouse Vertebra**

As the reader can check the general cone and disk densities are given in terms of invariant polynomials, so at this time no inference can be performed, except if the series are truncated in the first few terms. However, there is a way to work with an exact density, it is, when we assume an isotropic model.

Consider the isotropic elliptical disk density of theorem 3.3

$$f_W(W) = \frac{2^n \pi^{n(N + K - n - 1)/2} \prod_{i=1}^{n} I_i^{1-n+K-2n} \prod_{i<j} (t_i^2 - t_j^2)}{\Gamma_n \left[\frac{1}{2} K\right] \Gamma_n \left[\frac{1}{2} (N - 1)\right] (\omega^2)^{(N - 1)K/2}}$$

$$\times \sum_{\kappa} \frac{C_{\kappa} \left(\frac{1}{2} W^2\right)}{\kappa!} C_{\kappa} \left(\Omega\right)$$

$$\times \int_{0}^{\infty} s^{n(N + K - n - 1) + 2l - 1} h(2t) \left(\text{tr} \Omega + \frac{r^2}{\sigma^2}\right) dr,$$
and the generator for the subfamily Kotz

\[ h(y) = \frac{R^{T-1+\frac{K(N-1)}{2}}}{\pi^{K(N-1)/2} \Gamma} \left( \frac{K(N-1)}{2} \right)^{T-1} e^{-Ry}, \]

with derivative

\[ \frac{d^k}{dy^k} y^{T-1} e^{-Ry} = (-R)^k y^{T-1} e^{-Ry} \left\{ 1 + \sum_{m=1}^{k} \binom{k}{m} \left[ \prod_{i=0}^{m-1} (T - 1 - i) \right] (-R)^{m} \right\}, \]

see Caro-Lopera et al. (2009) for other families (Pearson VII, Bessel, general Kotz, Jensen Logistic) and their derivatives.

Now, we contrast three models next, the classical Gaussian \((T = 1, R = \frac{1}{2})\) and two non normal \((T = 2, R = \frac{1}{2})\) and \(T = 3, R = \frac{1}{3}\) via the modified BIC criterion, they will be applied to the data of two groups (small and large) of mouse vertebra, and experiment very detailed in Dryden and Mardia (1998).

The isotropic Gaussian disk density is given by

\[ f_W(W) = \frac{2^{\frac{1}{2}(M+n+N-k-2+nN)}}{\pi^M \sigma^{M-n-k-(n+N)}} \prod_{i=1}^{n} (t_{i}^2 - l_{i}^2) J(\mathbf{u}) \]

\[ \times \text{etr} \left( \frac{-\mu' \mu}{2 \sigma^2} \right) \sum_{t=0}^{\infty} \frac{t!}{t!} \left[ \frac{1}{2}(n-1+K-n+N) \right] + t \sum_{k} C_k \left( \mathbf{W}^2 \right) C_k \left( \frac{\mu' \mu}{2 \sigma^2} \right) \]

where \( M = K(N-1) \) and \( n = \min \{(N-1), K\} \).

The Kotz disk density when \( T = 2 \) and \( R = \frac{1}{2} \) follows after some tedious simplification, and it is given by

\[ f_W(W) = \frac{2^{\frac{1}{2}(M+n+N-k-2+nN)}}{\pi^M \sigma^{M-n-k-(n+N)}} \prod_{i=1}^{n} (t_{i}^2 - l_{i}^2) J(\mathbf{u}) \]

\[ \times \text{etr} \left( \frac{-\mu' \mu}{2 \sigma^2} \right) \sum_{t=0}^{\infty} \frac{t!}{t!} \left[ \text{tr} \left( \frac{\mu' \mu}{2 \sigma^2} \right) - 2t \right] \Gamma \left[ \frac{1}{2}(n-1+K-n+N) + t \right] \]

\[ + \Gamma \left[ \frac{1}{2}(n-1+K-n+N) + t + 1 \right] \sum_{k} C_k \left( \mathbf{W}^2 \right) C_k \left( \frac{\mu' \mu}{2 \sigma^2} \right). \]
Finally, the corresponding density for the Kotz model $T = 3$, is obtained as:

$$f(W) = \frac{2^\frac{1}{2}(-2-M+n+Kn^2+nN)}{\pi^\frac{1}{2}(M+n-K-nN)\sqrt{M-n(-1+K-n+N)}\Gamma_n\left[\frac{1}{2}K\right]} \prod_{i=1}^{n} l_i^{N-1} \prod_{i<j} (l_i^2 - l_j^2) \times \frac{J(u)}{M(M+2)} \text{etr} \left( -\frac{\mu^2}{2\sigma^2} \right) \sum_{t=0}^{\infty} \frac{1}{t!} \left( \left[ -8t + 16t^2 - 16t \text{tr} \left( \frac{\mu^2}{2\sigma^2} \right) + 4 \text{tr}^2 \left( \frac{\mu^2}{2\sigma^2} \right) \right] + 4 \left( \text{tr} \left( \frac{\mu^2}{2\sigma^2} \right) - 4t \right) \Gamma \left[ \frac{1}{2}(n(-1+K-n+N)) + t + 1 \right] \right)$$

$$+ 4 \Gamma \left[ \frac{1}{2}(n(-1+K-n+N)) + t + 2 \right] \sum_{\kappa} C_{K} \left( W^2 \right) C_{K} \left( \frac{\mu^2}{2\sigma^2} \right) \frac{1}{\left( \frac{1}{2}K \right)_{K}} C_{K} \left( I_{N-1} \right),$$

The likelihood based on exact densities require the computation of the above series, a carefully comparison with the known hypergeometric of two matrix argument indicates that these distributions can be obtained by a suitable modification of the algorithms of Koev and Edelman (2006).

In order to decide which the elliptical model is the best one, different criteria have been employed for the model selection. We shall consider a modification of the BIC statistic as discussed in Yang and Yang (2007), and which was first achieved by Rissanen (1978) in a coding theory framework. The modified BIC is given by:

$$BIC^* = -2L(\tilde{\mu}, \tilde{\sigma}^2, h) + n_p \log(n + 2) - \log 24,$$

where $L(\tilde{\mu}, \tilde{\sigma}^2, h)$ is the maximum of the log-likelihood function, $n$ is the sample size and $n_p$ is the number of parameters to be estimated for each particular shape density.

As proposed by Kass and Raftery (1995) and Raftery (1995), the following selection criteria have been employed for the model selection.

| BIC* difference | Evidence   |
|-----------------|------------|
| 0–2             | Weak       |
| 2–6             | Positive   |
| 6–10            | Strong     |
| > 10            | Very strong|

The maximum likelihood estimators for location parameters associated with the small and large groups are summarised in the following table:

According to the modified BIC criterion, the Kotz model with parameters $T = 2$, $R = \frac{1}{2}$ and $s = 1$ is the most appropriate, among the three elliptical densities selected, for modeling the data. There is a very strong difference between the non normal and the classical Gaussian model in this experiment.

Let $\mu_1$ and $\mu_2$ be the mean disk of the small and large groups, respectively. We test equal mean shape under the best model, and the likelihood ratio (based on $-2 \log \Lambda \approx \chi^2_{10}$)
for the test $H_0 : \mathbf{\mu}_1 = \mathbf{\mu}_2$ vs $H_a : \mathbf{\mu}_1 \neq \mathbf{\mu}_2$, provides the p-value 0.92, which means that there extremely evidence that the mean shapes of the two groups are equal.

It is important to note that the general densities derived here apply to any elliptical model; some classical elliptical densities as Kotz, Pearson II and VII, Bessel, Jensen-logistic, can be obtained explicitly and applied, however they demand the computation of the the $k$-th derivative of the generator elliptical function $h(\cdot)$. This is not a trivial fact, but for the above mentioned families, the required formulae are available in [Caro-Lopera et al. (2009)]. However, isotropic densities are more tractable because they are expanded in terms of zonal polynomials, instead of non isotropic distributions which require some additional conditions on the number of landmarks in order to obtain a know integral expanded in terms of invariant polynomials. The general densities expanded in terms of invariant polynomials, seem non computable at this date for large degrees.

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| Group | $BIC^{a}$ | $\hat{\mu}_{11}$ | $\hat{\mu}_{12}$ | $\hat{\mu}_{21}$ | $\hat{\mu}_{22}$ | $\hat{\mu}_{31}$ | $\hat{\mu}_{32}$ |
|-------|-----------|------------------|------------------|------------------|------------------|------------------|------------------|
| Small | $-536.1662$ | $-16.3534$ | $-75.7142$ | $40.2639$ | $-3.9264$ | $39.6012$ | $-15.9524$ |
| Large | $-536.9652$ | $-19.3968$ | $-69.2566$ | $41.9566$ | $-1.7093$ | $32.1251$ | $-20.2930$ |

Table 2: Maximum likelihood estimators.
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