Properties of multitype subcritical branching processes in random environment

Vladimir Vatutin, Elena Dyakonova

Abstract

We study properties of a \( p \)-type subcritical branching process in random environment initiated at moment zero by a vector \( z = (z_1, \ldots, z_p) \) of particles of different types. Assuming that the process belongs to the class of the so-called strongly subcritical processes we show that its survival probability to moment \( n \) behaves for large \( n \) as \( C(z) \lambda^n \) where \( \lambda \) is the upper Lyapunov exponent for the product of mean matrices of the process and \( C(z) \) is an explicitly given constant. We also demonstrate that the limiting conditional distribution of the number of particles given the survival of the process for a long time does not depend on the vector \( z \) of the number of particles initiated the process.

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1 Introduction and statement of main results

We investigate asymptotic properties of a class of subcritical multitype branching processes in random environment (MBPRE’s). To formulate the results of the paper we introduce some notation.

For \( p \)-dimensional vectors \( x = (x_1, \ldots, x_p) \) and \( y = (y_1, \ldots, y_p) \) we set

\[
(x, y) := \sum_{i=1}^{p} x_i y_i, \quad |x| := \sum_{i=1}^{p} |x_i| \quad \text{and} \quad x^y := \prod_{i=1}^{p} (x_i)^{y_i}.
\]

The standard basis vectors will be denoted by \( e_i, \ i = 1, 2, \ldots, p \). We set \( 0 := (0, 0, \ldots, 0) \) and \( 1 := (1, 1, \ldots, 1) \), denote by \( \mathbb{N}_0^p \) the set of all \( p \)-dimensional vectors with non-negative integer-valued components and introduce the notation \( \mathbb{N}_0^p := \mathbb{N}_0^p \setminus \{0\} \).

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†Novosibirsk State University, Novosibirsk, Russia, e-mail: vatutin@mi-ras.ru

‡Novosibirsk State University, Novosibirsk, Russia, e-mail: elena@mi-ras.ru
For every \( p \)-tuple \((\mu_1, \mu_2, \ldots, \mu_p)\) of probability measures on \( \mathbb{N}_0^p \), we define its multidimensional multivariate generating function \( f = (f^1, f^2, \ldots, f^p) \) by the relations

\[
f^i(s) := \sum_{z \in \mathbb{N}_0^p} \mu^i(z) s^z, \quad s = (s_1, \ldots, s_p) \in [0, 1]^p, \ i = 1, 2, \ldots, p.
\]

Any sequence \( \{f_n = (f^1_n, f^2_n, \ldots, f^n_p), n \geq 1\} \) of multidimensional multivariate generating functions is called a varying environment. The corresponding measure \( \mu^i_n \) describes the offspring law of type \( i \) particles in generation \( n - 1 \).

The model we are planning to investigate is a \( p \)-type branching process \( Z_n = (Z^1_n, Z^2_n, \ldots, Z^n_p), n \geq 0 \), where \( Z^i_n \) is the number of type \( i \) particles in the process at moment \( n \). This process has a (may be random) starting point \( Z_0 \) and the population sizes of the subsequent generations of the process are specified by the recursion

\[
Z^i_n = \sum_{j=1}^{p} \sum_{k=1}^{Z^j_{n-1}} X^i_{n,j,k}, \quad i = 1, 2, \ldots, p, \ n \geq 1, \quad (1)
\]

where \( X_{n,j,k} = (X^1_{n,j,k}, X^2_{n,j,k}, \ldots, X^p_{n,j,k}), k \geq 1 \), are independent random vectors distributed according to \( \mu^j_n \). Here and in what follows we denote random objects by upper case symbols using the respective bold symbols if the objects are vectors or matrices.

Equipping the set of all tuples \((\mu^1, \mu^2, \ldots, \mu^p)\) with the total variation distance we obtain a metric space \( \mathcal{P}_p(\mathbb{N}_0^p) \) which, due to the one-to-one correspondence \( (\mu^1, \mu^2, \ldots, \mu^p) \leftrightarrow f = (f^1, f^2, \ldots, f^p) \), we identify with the space of all multidimensional multivariate probability generating functions. Thus, we are able to specify a probability measure \( \mathbb{P} \) on the space of such generating functions. This agreement allows us to introduce a sequence of independent, identically distributed random \( p \)-dimensional generating functions \( \mathcal{E} := \{F_n = (F^1_n, F^2_n, \ldots, F^n_p), n \geq 1\} \) which will be called a random environment. We say that \( Z = \{Z_n, n \geq 0\} \) is a \( p \)-type BPRE, if for every fixed realisation of the environmental sequence its conditional distribution is determined by \((1)\).

BPRE’s with one type of particles have been intensively investigated during the last two decades and many their properties are well understood. The reader may find a unified presentation of the corresponding results in [16]. The multi-dimensional case is much less studied. For instance, only recently estimates for the survival probability of the critical and some classes of subcritical multitype BPRE’s were found under relatively general conditions, see [9],[10],[11],[12],[17],[18],[19],[20].

It known that asymptotic properties of MBPRE’s are often described in terms of properties of the (random) mean matrices

\[
M_n = (M^i_{ij}(n))_{i,j=1}^p := \left( \frac{\partial F^i_n}{\partial s_j}(1) \right)_{i,j=1}^p, \quad n \geq 1.
\]
If $F_n$ are independent and distributed as a generating function $F = (F^1, \ldots, F^p)$ then $M_n$ are independent probabilistic copies of the random matrix

$$M = (M_{ij})_{i,j=1}^p := \left( \frac{\partial F^i}{\partial s_j} (1) \right)_{i,j=1}^p.$$

We assume that the distribution of $M$ satisfies the following assumptions:

**Condition H1.** The set $\Theta^+ := \{ \theta > 0 : \mathbb{E} \left[ \|M\|^\theta \right] < \infty \}$ is nonempty, where $\|M\|$ is the operator norm of $M$.

**Condition H2.** The support of the distribution of $M$ acts strongly irreducibly on the semi-group of matrices with non-negative entries, i.e. no proper finite union of subspaces of $\mathbb{R}^p$ is invariant with respect to all elements of the multiplicative semi-group $S^+$ of $p \times p$ matrices generated by the support of $M$.

**Condition H3.** There exists a positive number $\gamma > 1$ such that

$$1 \leq \frac{\max_{i,j} M_{ij}}{\min_{i,j} M_{ij}} \leq \gamma.$$ 

To introduce one more condition we define the cone

$$\mathcal{C} = \{ x = (x_1, \ldots, x_p) \in \mathbb{R}^p : x_i \geq 0 \text{ for any } i = 1, \ldots, p \},$$

and the space

$$\mathcal{X} = \mathcal{C} \cap \{ x : x \in \mathbb{R}^p, |x| = 1 \}.$$

**Condition H4.** There exists $\delta > 0$ such that

$$\mathbb{P} \left( h \in S^+ : \text{ for any } x \in \mathcal{X}, \log |xh| \geq \delta \right) > 0.$$ 

To formulate the next assumption we define matrices

$$B(k) := \left( \frac{\partial^2 F^k}{\partial s_i \partial s_j} (1) \right)_{i,j=1}^p$$

and

$$B_n(k) := \left( \frac{\partial^2 F^k_n}{\partial s_i \partial s_j} (1) \right)_{i,j=1}^p$$

and random variables

$$\mathcal{T} := \frac{1}{\|M\|^2} \sum_{k=1}^p \|B(k)\| \text{ and } \mathcal{T}_n := \frac{1}{\|M_n\|^2} \sum_{k=1}^p \|B_n(k)\|, n = 1, 2, \ldots.$$ 

Thus, $\mathcal{T}_n$ are independent probabilistic copies of $\mathcal{T}$.

**Condition H5.** There exists $\varepsilon > 0$ such that

$$\mathbb{E} \left[ \|M\| \log \mathcal{T} \right]^{1+\varepsilon} < \infty.$$ 

Conditions H1 – H5 impose restrictions on the integral characteristics of the MPBRE. Our next assumption directly concerns the reproduction laws of particles and forces particles of all types to die without children with positive probability.

**Condition H6.** $\mathbb{P} (F^k[0] > 0, k = 1, \ldots, p) = 1$. 

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It is well known (see, for instance, point 2.1.2 in [5]) that the limit
\[
\lambda(\theta) := \lim_{n \to \infty} \left( \mathbb{E} \left[ \|M_n \cdots M_1\|^{\theta} \right] \right)^{1/n} < \infty
\]
exists for every \( \theta \in \Theta := \{0\} \cup \Theta_+ \). Set
\[
\Lambda(\theta) := \log \lambda(\theta), \quad \theta \in \Theta.
\]
Along with the random mean matrix \( M \) we consider its expectation
\[
m := \mathbb{E} M = (\mathbb{E} M_{ij})_{i,j=1}^{p} = (m_{ij})_{i,j=1}^{p}.
\]
Clearly, the elements of \( m \) are positive. Let \( \lambda \) be the Perron root of \( m \) and let \( V = (V_1, \ldots, V_p) \) and \( U = (U_1, \ldots, U_p) \) be the strictly positive left and right eigenvectors of \( m \) corresponding to the eigen-value \( \lambda \)
\[
V \cdot m = \lambda V, \quad m \cdot U = \lambda U \quad \text{and scaled as} \quad (V, 1) = 1, \quad (V, U) = 1. \tag{2}
\]
In view of (2) and positivity of the components of \( U \)
\[
\lim_{n \to \infty} |m^n|^{1/n} = \lim_{n \to \infty} |m^n U|^{1/n} = \lambda.
\]
Observe further that
\[
\frac{1}{p} |M_n \cdots M_1 U| \leq \|M_n \cdots M_1\| \leq \sup_{|x|=1} |M_n \cdots M_1 x| \leq \|m\| \leq \frac{1}{\min_{1 \leq i \leq p} U_i} |M_n \cdots M_1 U|.
\]
Thus, by independency of random matrices and the Perron-Frobenius theorem
\[
\lambda(1) = \lim_{n \to \infty} (\mathbb{E} [\|M_n \cdots M_1\|])^{1/n} = \lim_{n \to \infty} (\mathbb{E} [m^n \cdots m])^{1/n} = \lim_{n \to \infty} |m^n U|^{1/n} = \lambda.
\]
Therefore, \( \lambda(1) \) is the Perron root of \( m \). For this reason we rather often write below \( \lambda \) for \( \lambda(1) \).

Now we are ready to formulate the main results of the paper describing properties of a MBPRE which is initiated by a deterministic vector
\[
Z_0 = (Z_0^1, Z_0^2, \ldots, Z_0^p) = z = (z_1, z_2, \ldots, z_p) \in \mathbb{N}_+^p
\]
of particles at time 0. Introduce the event
\[
D_n(z) := \{Z_n \neq 0 \mid Z_0 = z\} \tag{3}
\]
and denote by int\( \Theta \) the interior of the set \( \Theta \).
Theorem 1. Let Conditions $\textbf{H1} - \textbf{H5}$ be valid, $1 \in \text{int} \Theta$ and $\Lambda'(1) < 0$. Then for all $z \in \mathbb{N}_+^p$, as $n \to \infty$,

$$\mathbb{P}(\mathcal{D}_n(z)) \sim \frac{(z, U)}{E[(T_\infty, U)]} \lambda^n(1)$$  \quad (4)

and, for any $s \in [0, 1]^p$ and all $z \in \mathbb{N}_+^p$

$$\lim_{n \to \infty} E[s^{Z_n}|Z_n \neq 0; Z_0 = z] = \sum_{x \in \mathbb{N}_+^p} t_x s^x := T(s) = E[s^{T_\infty}]$$  \quad (5)

exists, is independent of $z$, specifies a proper distribution on $\mathbb{N}_+^p$ and solves the equation

$$E[T(F(s))] = \lambda(1)T(s) + 1 - \lambda(1).$$  \quad (6)

Remark 1. The case $|z| = 1$ was considered in [19], where, for all $i = 1, ..., p$ it was shown that

$$\mathbb{P}(\mathcal{D}_n(e_i)) \sim K(e_i) \lambda^n(1)$$  \quad (7)

as $n \to \infty$ and that

$$\lim_{n \to \infty} E[s^{Z_n}|Z_n \neq 0; Z_0 = e_i] = T_i(s)$$  \quad (8)

exists. However, no explicit expressions for the constants $K(e_i)$ in (7) were given and independence of the limit in (8) on the initial value $z = e_i, i = 1, ..., p$ was not established. Our Theorem 1 fills this gap. The proofs of (4) and (5) are based on a combination of the methods from [19] used in Section 2 and the change of measure of a new type applied in Section 5.

Remark 2. If $\Lambda'(1) < 0$ then $\Lambda'(0) < 0$ by convexity of $\lambda(\theta)$. In particular, if $p = 1$ then the inequalities $\Lambda'(0) < 0$ and $\Lambda'(1) < 0$ are reduced to $E[M_{11}] < 0$ and $E[M_{11} \log M_{11}] < 0$. Such single-type BPRE’s (called strongly subcritical) were investigated, for instance, in [1], [2], [3], [14] and [15].

Multitype subcritical branching processes were considered in [7] and [11] under the additional restriction that all mean matrices of the reproduction laws have a common deterministic left or right eigenvector corresponding to the Perron roots of these matrices. We do not require the validity of such condition.

The following two corollaries provide an additional information about the limiting distributions of the process $\{Z_n, n \geq 1\}$ under two types of conditioning.

Corollary 2. Let the conditions of Theorem 1 be valid. Then

(i) for each fixed $m \geq 1$ and $z, j_0, j_1, ..., j_m \in \mathbb{N}_+^p$

$$\lim_{n \to \infty} \mathbb{P}(Z_{n-m} = j_0, Z_{n-m+1} = j_1, ..., Z_m = j_m|Z_n \neq 0; Z_0 = z) = \frac{1}{\lambda^m(1)} \mathbb{P}(Z_1 = j_1, ..., Z_m = j_m|Z_n \neq 0; Z_0 = j_0) \mathbb{P}(T_\infty = j_m);$$  \quad (9)

(ii) if the tuple $0 = n_0 < n_1 < ... < n_r = n$ is such that

$$n_* := \min_{0 \leq k \leq r-1} (n_{k+1} - n_k) \to \infty,$$
then
\[
\mathbb{P}(Z_{n_1} = j_1, ..., Z_{n_r} = j_r | Z_n \neq 0; Z_0 = z) \\
\rightarrow \left( \prod_{i=1}^{r-1} (j_i, K) \mathbb{P}(T_\infty = j_i) \right) \mathbb{P}(T_\infty = j_r), \quad (10)
\]

where
\[
K = \frac{U}{\mathbb{E}[(T_\infty, U)]}. \quad (11)
\]

Thus, given survival of the strongly subcritical MBPRE to a distant moment \(n\), the vectors of the number of particles at moments being far from each other are asymptotically independent. Such phenomenon for single-type strongly subcritical BPRE’s was described in [1].

We now introduce the so-called \(Q\)-process \(Y := \{\tilde{Y}_n, n \geq 1\}\) which may be nonrigorously considered as the process \(\{Z_n, n \geq 0\}\) conditioned to survive in the distant future and whose multi-dimensional distributions are specified by the formulas
\[
\mathbb{P} \left( \tilde{Y}_1 = j_1, \tilde{Y}_2 = j_2, ..., \tilde{Y}_n = j_n | \tilde{Y}_0 = y \right)
= \lim_{n \to \infty} \mathbb{P} (Z_1 = j_1, Z_2 = j_2, ..., Z_n = j_n | Z_{n+m} \neq 0; Z_0 = y).
\]

The next statement gives a more explicit representation for the distributions of \(\mathcal{Y}\).

**Corollary 3** Under the conditions of Theorem 1 for each fixed \(n \geq 1\)
\[
\mathbb{P} \left( \tilde{Y}_1 = j_1, \tilde{Y}_2 = j_2, ..., \tilde{Y}_n = j_n | \tilde{Y}_0 = y \right)
= \frac{1}{\lambda^n(1)} \frac{(j_n, K)}{(y, K)} \mathbb{P} (Z_1 = j_1, Z_2 = j_2, ..., Z_n = j_n | Z_0 = y)
\]

and for any \(j, y \in \mathbb{N}_+^p\)
\[
\lim_{n \to \infty} \mathbb{P} \left( \tilde{Y}_n = j | \tilde{Y}_0 = y \right) = \frac{(j, U)}{\mathbb{E}[(T_\infty, U)]} \mathbb{P}(T_\infty = j).
\]

Thus, the \(Q\)-process \(\mathcal{Y}\) has a kind of size-biased distribution in the limit.

The rest of the paper is organized as follows. To prove the desired statements we need to perform two changes of measure. The first change of measure is introduced in Section 2. Using the new measure we obtain in Section 3 a rough representation for the survival probability of a class of subcritical MBPRE’s and prove in Section 4 a Yaglom-type limit theorem for the distribution of the number of particles of different types given the process is initiated by several particles. In Section 5 we make the second change of measure and complete the proof of Theorem 1 by justifying asymptotic representation (4). A description of properties of \(Q\)-processes is given in Section 6.
2 First change of measure

We agree to denote $p \times p$ deterministic matrices with non-negative entries by bold lower case symbols and, as a rule, use the same notation for row and column vectors. For instance, if $\mathbf{x}$ is a $p$-dimensional vector and $\mathbf{h}$ is a $p \times p$ deterministic matrix then, the $\mathbf{x}$ will be treated as a row vector in the product $\mathbf{xh}$ and as a column vector in the product $\mathbf{hx}$. It will be clear from the context which form is used.

For every $\mathbf{x} \in \mathbb{X}$ and a $p \times p$ matrix $\mathbf{h} \in \mathbb{S}^+$ we specify the column vector

$$\mathbf{h} \circ \mathbf{x} := \frac{\mathbf{hx}}{|\mathbf{hx}|} \in \mathbb{X} \quad (12)$$

if $\mathbf{hx} \neq 0$ and the row vector

$$\mathbf{x} \circ \mathbf{h} := \frac{\mathbf{hx}}{|\mathbf{hx}|} \in \mathbb{X} \quad (13)$$

if $\mathbf{xh} \neq 0$.

Denote by $\mathcal{C}_b(\mathbb{X})$ the set of all bounded continuous functions on $\mathbb{X}$. For $\theta \in \Theta$, $g \in \mathcal{C}_b(\mathbb{X})$, and $\mathbf{x} \in \mathbb{X}$ define the transition operators

$$P_\theta g(\mathbf{x}) := \mathbb{E} \left[ |\mathbf{Mx}|^\theta g(\mathbf{M} \circ \mathbf{x}) \right]$$

and

$$P^*_\theta g(\mathbf{x}) := \mathbb{E} \left[ |\mathbf{M}^T \mathbf{x}|^\theta g(\mathbf{M}^T \circ \mathbf{x}) \right].$$

If the assumptions $\textbf{H1} - \textbf{H3}$ hold, then, according to Proposition 3.1 in [5], $\lambda(\theta)$ is the spectral radius of $P_\theta$ and $P^*_\theta$ and there exist a unique strictly positive function $r_\theta \in \mathcal{C}_b(\mathbb{X})$ and a unique probability measure $l_\theta$ such that

$$\int_{\mathbb{X}} r_\theta(\mathbf{x}) dl_\theta(\mathbf{x}) = 1$$

and

$$l_\theta P_\theta = \lambda(\theta) l_\theta, \quad P_\theta r_\theta = \lambda(\theta) r_\theta. \quad (14)$$

Here $l_\theta P_\theta$ is a unique probability measure satisfying the equality

$$\int_{\mathbb{X}} g(\mathbf{x})(l_\theta P_\theta)(d\mathbf{x}) = \int_{\mathbb{X}} (P_\theta g(\mathbf{x}))l_\theta(d\mathbf{x})$$

for each $g \in \mathcal{C}_b(\mathbb{X})$ (see [6], page 2070) and $P_\theta r_\theta = \lambda(\theta) r_\theta$ means that

$$\mathbb{E} \left[ |\mathbf{Mx}|^\theta r_\theta(\mathbf{M} \circ \mathbf{x}) \right] = \lambda(\theta) r_\theta(\mathbf{x}) \quad (15)$$

for all $\mathbf{x} \in \mathbb{X}$.

Similarly, there exists a pair $(r^*_\theta, l^*_\theta)$ possessing the same properties relative to $P^*_\theta$ as $(r_\theta, l_\theta)$ relative to $P_\theta$. Moreover,

$$r_\theta(\mathbf{x}) = c \int_{\mathbb{X}} (\mathbf{x}, \mathbf{y})^\theta l^*_\theta(d\mathbf{y})$$
Therefore, by definition of the process and set \( Z \)

\[ \text{constant } K \]

\[ S \]

probability distribution on measures \( \lambda^n(\theta) \). The precise expression for \( M_h \) on our original probability space \( (\Omega, F) \) is not difficult to check that, for every \( n \in \mathbb{N} \), it is not difficult to check that, for every \( n \geq 1 \), every \( x \in X \) and every matrix \( h \in S^+ \),

\[ E_p^n(\theta)(x, h) := \frac{|hx|^\theta}{\lambda^n(\theta)} \frac{r_\theta(h \circ x)}{r_\theta(x)}, \quad x \in X, n \geq 1. \]  

(16)

Using (15) it is not difficult to check that, for every \( n \geq 1 \), every \( x \in X \) and every matrix \( h \in S^+ \),

\[ E_p^n(\theta)(x, h) = p^n_{\theta-1}(x, h). \]  

(17)

In particular,

\[ E_p^n(\theta)(x, L_{n,1}) = 1, \]  

(18)

where \( L_{n,k} := M_n \cdots M_k \) if \( 1 \leq k \leq n \) and \( L_{n,n+1} \) is the unit \( p \times p \) matrix.

Let \( \mathbb{I}_A \) be the indicator of the event \( A \). It follows from (18) that

\[ \mathbb{P}_n(\theta)(A) = \mathbb{P}_n(\theta)(A) := E_p^n(\theta)(x, L_{n,1}) \mathbb{I}_A, \quad A \in F_n, \]  

(19)

is a probability measure on \( F_n \). Furthermore, (17) implies that the sequence of measures \( \{\mathbb{P}_n(\theta), n \geq 1\} \) is consistent and can be extended to a probability measure \( \mathbb{P}(\theta) \) on our original probability space \( (\Omega, F) \).

### 3 Asymptotic of the survival probability

We prove in this section that \( \mathbb{P}(D_n(z)) \sim K(z)\lambda^n(1) \) as \( n \to \infty \) for some constant \( K(z) \). The precise expression for \( K(z) \) will be found in Section 5.

For every environment \( \mathcal{E} = \{F_n, n \geq 1\} \) and \( 0 \leq k < n \) we define iterations

\[ F_{n,k}(s) := F_{k+1}(F_{k+2}(\ldots F_{n-1}(s) \ldots)), \quad F_{k,n}(s) := F_{k+1}(F_{k+2}(\ldots F_n(s) \ldots)), \]  

and set

\[ F_{n,n}(s) := s. \]

By definition of the process \( X_n \)

\[ E[s^Z_n|Z_0 = e_1, F_2, \ldots, F_n] = F_{0,n}^i(s). \]

Therefore,

\[ \mathbb{P}(Z_n \neq 0|Z_0 = e_1, F_2, \ldots, F_n) = 1 - F_{0,n}^i(0). \]  

(20)
This equality and definition (3) imply
\[ P(\mathcal{D}_n(z)) = E \left[ 1 - \prod_{i=1}^{p} (F_{0,n}(0))^{z_i} \right]. \quad (21) \]

Let \( \bar{\theta} := (\theta_1, \ldots, \theta_p) \) be a vector with nonnegative real components \( \theta_i \in \{0\} \cup [1, \infty), i = 1, \ldots, p \), such that \( \theta = \theta_1 + \ldots + \theta_p \geq 1 \). Along with (21) we consider the quantity
\[ Y(n, \bar{\theta}) := E \left[ \prod_{i=1}^{p} (1 - F_{0,n}^i(0))^{\theta_i} \right] = E \left[ \prod_{i=1}^{p} (1 - F_{n,0}^i(0))^{\theta_i} \right]. \quad (22) \]

This function has the following probabilistic meaning. Let \( \mathcal{A}_n(z) := \bigcap_{i=1}^{p} \bigcap_{j=1}^{z_i} \{ Z_{ij}^i \neq 0 \mid Z_{ij}^i = e_1 \} \)
\[ A_n(z) := \bigcap_{i=1}^{p} \bigcap_{j=1}^{z_i} \{ Z_{ij}^i \neq 0 \mid Z_{ij}^i = e_1 \} \]
is the event that each initial particle has a nonempty number of descendants at moment \( n \), then
\[ Y(n, z) = P(\mathcal{A}_n(z)) = E \left[ \prod_{i=1}^{p} (1 - F_{0,n}^i(0))^{z_i} \right]. \quad (24) \]

Since
\[ 1 - F_{n,0}^i(s) = (e_i, 1 - F_{n,0}(s)) \leq (e_i, M_n(1 - F_{n-1,0}(s))), \quad (25) \]
it follows that
\[ 1 - F_{n,0}^i(0) \leq e_i L_{n,1} 1 \leq |L_{n,1}|. \]
Using this representation we get by (22)
\[ Y(n, \bar{\theta}) \leq E \left[ \prod_{i=1}^{p} |L_{n,1} 1|^{\theta_i} \right] = E \left[ |L_{n,1} 1|^{\theta} \right], \quad (26) \]
where \( \theta = \theta_1 + \ldots + \theta_p \).

**Lemma 4** Let \( \bar{\theta} := (\theta_1, \ldots, \theta_p) \) be a vector with nonnegative components \( \theta_i \in \{0\} \cup [1, \infty), i = 1, \ldots, p \), such that \( \theta = |\bar{\theta}| = \theta_1 + \ldots + \theta_p \geq 1 \). Assume that Conditions H1–H3 are valid and \( \theta \in \Theta \). Then there exists a constant \( C = C(\theta) \) such that
\[ Y(n, \bar{\theta}) \leq C \lambda^n(\theta) \]
for all \( n \geq 1 \).
Proof. We make the change of measure at the right-hand side of (26) by means of the density $p_n^{(\theta)}(x,h)$ in (16) with $\theta = |\theta|$ and $x = p^{-1}1$. As a result we get

$$
Y(n, \theta) \leq \lambda^n(\theta) r_\theta(x) \mathbb{E} \left[ \frac{1}{r_\theta(L_{n,1} \circ x)} \frac{1}{|L_{n,1}1|^\theta} \right]
$$

$$
= p^\theta \lambda^n(\theta) r_\theta(p^{-1}1) \mathbb{E}^{(\theta)} \left[ \frac{1}{r_\theta(L_{n,1} \circ (p^{-1}1))} \right].
$$

(27)

Since $r_\theta(x)$ is a continuous positive function on the compact $X$, there are constants $c_1$ and $c_2$ such that

$$
0 < c_1 \leq r_\theta(x) \leq c_2 < \infty
$$

for all $x \in X$. Hence it follows that

$$
Y(n, \theta) \leq p^\theta c_2 \lambda^n(\theta) =: C \lambda^n(\theta).
$$

Lemma 4 is proved.

Having this result in hands we check the validity of the following statement.

**Lemma 5** Let Conditions H1 – H5 be valid, $1 \in \text{int}\theta$ and $\Lambda'(0) < 0$. Then for any $z \in \mathbb{N}^p_+$, as $n \to \infty$

$$
\mathbb{P}(D_n(z)) \sim \sum_{i=1}^{p} z_i \mathbb{P}(D_n(e_i)).
$$

(29)

**Remark 3.** One may think that equivalence (29) is evident, since $\mathbb{P}(D_n(e_i)) \to 0$ as $n \to \infty$. Indeed, this is always the case for the ordinary multitype Galton-Watson processes. However, such equivalence is, in general, not true for BPRE's.

For instance, Bansaye [4] has shown, analyzing a single type subcritical BPRE $\{Z_n, n \geq 0\}$, that if the process is weakly subcritical (i.e., satisfies the conditions $\Lambda'(0) < 0, \Lambda'(1) > 0$ in our notation) then there exists $\beta \in (0, 1)$ such that the quantity

$$
\alpha_z := \lim_{n \to \infty} \frac{\mathbb{P}(Z_n > 0 | Z_0 = z)}{\mathbb{P}(Z_n > 0 | Z_0 = 1)}
$$

is of order $z^\beta \log z$ as $z \to \infty$.

**Proof of Lemma 5.** We need the following inequalities

$$
0 \leq \sum_{k=1}^{p} z_k (b_k - a_k) - \left( \prod_{k=1}^{p} b_k^{z_k} - \prod_{k=1}^{p} a_k^{z_k} \right) \leq \sum_{k,l=1}^{p} z_k z_l (b_k - a_k) (b_l - a_l)
$$

(30)

valid for $z_k \in \mathbb{N}_0$ and real numbers $0 \leq a_k \leq b_k \leq 1, k = 1, 2, ..., p$. Hence, using (21) we obtain that, for any $\theta_0 \in (1, 2)$

$$
0 \leq \sum_{i=1}^{p} z_i \mathbb{P}(D_n(e_i)) - \mathbb{P}(D_n(z)) \leq \sum_{i,j=1}^{p} z_i z_j \mathbb{E} \left[ (1 - F_{0,n}^i(0)) \left( 1 - F_{0,n}^j(0) \right) \right]
$$

$$
\leq 2 |z| \sum_{i,j=1}^{p} z_i \mathbb{E} \left[ (1 - F_{0,n}^i(0))^2 \right] \leq 2 |z| \sum_{i,j=1}^{p} z_i \mathbb{E} \left[ (1 - F_{0,n}^i(0))^{\theta_0} \right].
$$

(31)

10
Since $\Lambda'(1) < 0$ and $1 \in \text{int}\Theta$, there exists $\theta_0 > 1$ such that $\theta_0 \in \text{int}\Theta$ and $\Lambda'(\theta_0) < 0$. By Lemma 4

$$
\mathbb{E} \left[ (1 - F_{0,n}(0))^{\theta_0} \right] \leq C \lambda^n(\theta_0) \quad (32)
$$
as $n \to \infty$. Since $\lambda(\theta)$ is a convex function for $\theta \in \Theta$, it follows that $\lambda(1) > \lambda(\theta_0) > 0$. Combining (31) and (7) gives, as $n \to \infty$

$$
P(D_n(z)) \sim \sum_{i=1}^p z_i \mathbb{P}(D_n(e_i)) \sim \sum_{i=1}^p z_i \lambda^n(1) \lambda(1) \quad (33)
$$
with still not specified explicitly positive constants $K(e_i)$.

Lemma 5 is proved.

4 Yaglom limit theorem

Now we extend statement (8) to the case of processes initiated by several particles.

Theorem 6 Assume that Conditions H1 – H5 are valid, $1 \in \text{int}\Theta$ and $\Lambda'(1) < 0$. Then

$$
\lim_{n \to \infty} \mathbb{E} \left[ s^{Z_n|Z_n \neq 0, Z_0 = z} \right] = \sum_{j \in \mathbb{N}_+^p} t_j(z)s^j =: T(z, s) \quad (34)
$$
exists for all $z \in \mathbb{N}_+^p$, specifies a proper distribution on $\mathbb{N}_+^p$ and solves the equation

$$
\mathbb{E} \left[ T(z, F(s)) \right] = \lambda(1)T(z, s) + 1 - \lambda(1). \quad (35)
$$

Proof. We write

$$
\mathbb{E} \left[ s^{Z_n|Z_n \neq 0, Z_0 = z} \right] = \frac{\mathbb{E} \left[ s^{Z_n|Z_n \neq 0, Z_0 = z} \right]}{\mathbb{P}(Z_n \neq 0|Z_0 = z)} = 1 - \frac{\mathbb{E} \left[ 1 - s^{Z_n|Z_0 = z} \right]}{\mathbb{P}(Z_n \neq 0|Z_0 = z)} = 1 - \frac{\mathbb{E} \left[ 1 - \prod_{i=1}^p (F_{0,n}(s_i))^{z_i} \right]}{\mathbb{P}(D_n(z))}.
$$

By (30), (31), and (32) we see that

$$
0 \leq \sum_{i=1}^p z_i \mathbb{E} \left[ 1 - F_{0,n}(s) \right] - \mathbb{E} \left[ 1 - \prod_{i=1}^p (F_{0,n}(s_i))^{z_i} \right] \leq \sum_{i,j=1}^p z_i z_j \mathbb{E} \left[ (1 - F_{0,n}(s)) (1 - F_{0,n}(s)) \right] \leq 2 |z| \sum_{i,j=1}^p z_i \mathbb{E} \left[ (1 - F_{0,n}(0))^{\theta_0} \right] = o(\lambda^n(1)) \quad (36)
$$
as $n \to \infty$. As shown in Theorem 1 of [19]

$$\lim_{n \to \infty} \frac{E\left[1 - F_{0,n}^i(s)\right]}{P(D_{n}(e_i))} =: \phi_i(s)$$

exists for any $i \in \{1, \ldots, p\}$, where $\phi_i(s), s \in [0, 1]^p$, is a nondegenerate multivariate function with $\phi_i(1) = 0$.

Using (33) and setting

$$K^* = (K(e_1), \ldots, K(e_p)),$$

where $K(e_i), i = 1, \ldots, p$, are the same as in (7), we see that

$$\lim_{n \to \infty} \frac{P(D_n(e_i))}{P(D_n(z))} = \frac{K(e_i)}{(z, K^*)}.$$

Thus, as $n \to \infty$

$$\frac{1}{P(D_n(z))} \sum_{i=1}^{p} z_i E\left[1 - F_{0,n}^i(s)\right] = \sum_{i=1}^{p} z_i \frac{P(D_n(e_i))}{P(D_n(z))} \frac{E\left[1 - F_{0,n}^i(s)\right]}{P(D_n(e_i))} \sim \frac{1}{(z, K^*)} \sum_{i=1}^{p} z_i K(e_i) \phi_i(s).$$

Setting $\Phi_i(s) = 1 - \phi_i(s)$ we obtain

$$T(z, s) := \lim_{n \to \infty} E\left[s^{Z_n} | Z_n \neq 0; Z_0 = z\right]$$

$$= \sum_{i=1}^{p} z_i K(e_i) \Phi_i(s) \frac{1}{(z, K^*)} = \sum_{j \in \mathbb{N}_1} t_j(z)s^j,$$

(38)

where $T(z, s)$ is a nondegenerate multivariate function in $s$ and $T(z, 1) = 1$. Since the generating function $F_{n+1}$ is independent of the tuple $F_1, \ldots, F_n$ we get by (37) and the dominated convergence theorem that, as $n \to \infty$

$$\frac{E\left[1 - F_{0,n+1}^i(s)\right]}{P(D_{n+1}(z))} = \frac{P(D_n(e_i))}{P(D_{n+1}(z))} \frac{E\left[1 - F_{0,n}^i(F_{n+1}(s))\right]}{P(D_n(e_i))}$$

$$= \frac{P(D_n(e_i))}{P(D_{n+1}(z))} \int_{y \in [0, 1]^p} \frac{E\left[1 - F_{0,n}^i(y)\right]}{P(D_n(e_i))} P(F_{n+1}(s) \in dy)$$

$$\to \frac{1}{\lambda(1)} \frac{K(e_i)}{(z, K^*)} E[\phi_i(F_{n+1}(s))] = \frac{1}{\lambda(1)} \frac{K(e_i)}{(z, K^*)} E[\phi_i(F(s))].$$

Therefore,

$$T(z, s) = \lim_{n \to \infty} E\left[s^{Z_{n+1}} | Z_{n+1} \neq 0; Z_0 = z\right]$$

$$= 1 - \lim_{n \to \infty} \sum_{i=1}^{p} z_i E\left[1 - F_{0,n+1}^i(s)\right]$$

$$= 1 - \frac{1}{\lambda(1)} \sum_{i=1}^{p} z_i K(e_i) \frac{1}{(z, K^*)} E[\phi_i(F(s))] = 1 - \frac{1}{\lambda(1)} E[1 - T(z, F(s))].$$
Thus,
\[
E[T(z, F(s))] = \lambda(1) T(z, s) + 1 - \lambda(1).
\] (39)

Theorem 6 is proved.

5 Second change of measure

To get more information about the limiting function \( T(z, s) \) we need one more measure \( P^* \) on the \( \sigma \)-field generated by \( Z_1, Z_2, \ldots; F_1, F_2, \ldots \). To this aim we fix \( Z_0 = z \) and, for every non-negative measurable functional \( \varphi \) on \( \mathbb{N}_+^p \times \mathcal{P}_k^p(\mathbb{N}_0^p) \), \( k \geq 1 \), set
\[
E^*[\varphi(Z_1, \ldots, Z_k; F_1, \ldots, F_k)] := \frac{E[\varphi(Z_1, \ldots, Z_k; F_1, \ldots, F_k)(Z_k, U)]}{(z, U)\lambda^k}. \] (40)

Lemma 7 Relation (40) defines a probability measure \( P^* \) on the \( \sigma \)-algebra generated by \( Z_1, Z_2, \ldots; F_1, F_2, \ldots \).

Proof. Set \( R_{1,k} = M_1 M_2 \cdots M_k \). Clearly,
\[
E[Z_k|Z_0 = z] = E[E[Z_k | Z_1, \ldots, Z_{k-1}; \mathcal{E}]] = E[Z_{k-1} M_k|Z_0 = z] = E[z M_1 \cdots M_k] = z E[R_{1,k}] = zm^k.
\]

Hence, using (2) we see that
\[
E^*[1] : = E[(Z_k, U)|Z_0 = z] = \frac{(E[Z_k|Z_0 = z], U)}{(z, U)\lambda^k} = \frac{(zm^k, U)}{(z, U)\lambda^k} = \frac{(z, m^k U)}{(z, U)\lambda^k} = 1.
\]

Thus, \( P^* \) is, indeed, a probability measure for each \( k \). Furthermore, the following consistency condition holds: If \( Z_0 = z \in \mathbb{N}_+^p \) and functions \( \varphi_k \) and \( \varphi_{k+1} \) satisfy
\[
\varphi_{k+1}(z_1, \ldots, z_{k+1}; f_1, \ldots, f_{k+1}) = \varphi_k(z_1, \ldots, z_k; f_1, \ldots, f_k)
\]
for all \( z_i \in \mathbb{N}_0^p \) and \( f_i \in \mathcal{P}_p(\mathbb{N}_0^p) \), \( 1 \leq i \leq k + 1 \), then

\[
\mathbb{E}^*[\varphi_{k+1}(Z_1, \ldots, Z_{k+1}; F_1, \ldots, F_{k+1})] = \mathbb{E}[\varphi_{k+1}(Z_1, \ldots, Z_{k+1}; F_1, \ldots, F_{k+1})]^{(z, U)k+1} = \mathbb{E}[\varphi_k(Z_1, \ldots, Z_k; F_1, \ldots, F_k)]^{(z, U)k+1} = \mathbb{E}[\varphi_k(Z_1, \ldots, Z_k; F_1, \ldots, F_k)\mathbb{E}[Z_{k+1} | Z_1, \ldots, Z_k; \mathcal{E}]]^{(z, U)k+1} = \mathbb{E}[\varphi_k(Z_1, \ldots, Z_k; F_1, \ldots, F_k)]^{(z, U)k+1} = \mathbb{E}^*[\varphi_k(Z_1, \ldots, Z_k; F_1, \ldots, F_k)],
\]

where for the last three equalities we have used the relation

\[
\mathbb{E}[Z_{k+1} | Z_k = x, \mathcal{E} = (f_1, f_2, \ldots)] = xM_{k+1},
\]

and the independency of \( M_{k+1} \) and the random vector \((Z_1, \ldots, Z_k, F_1, \ldots, F_k)\).

The lemma is proved.

**Remark 4.** Note that the measure \( \mathbb{P}^* \) has essentially different nature in comparison with the measure \( \mathbb{P}^{(1)} \) defined by formula (19). For instance, under \( \mathbb{P}^{(1)} \) the MBPRE \( Z_n \) may take value \( 0 \) with probability

\[
\mathbb{P}_n^{(1)}(Z_n = 0) = \mathbb{E} \left[ p_n^{(1)}(x, L_{n, 1}) \mathbb{I}_{\{Z_n = 0\}} \right] > 0,
\]

while under \( \mathbb{P}^* \) the probability of the event \( \{Z_n = 0\} \) always is zero.

**Remark 5.** If \( \varphi_k(z_1, \ldots, z_k; f_1, \ldots, f_k) = \varphi_k(f_1, \ldots, f_k) \), i.e. depends only on generating functions, then

\[
\mathbb{E}^*[\varphi(F_1, \ldots, F_k)] = \mathbb{E}[\varphi(F_1, \ldots, F_k)]^{(z, U)k+1} = \mathbb{E}[\varphi(F_1, \ldots, F_k)]^{(z, U)k+1} = \mathbb{E}[\varphi(F_1, \ldots, F_k)]^{(z, U)k+1} = \mathbb{E}[\varphi(F_1, \ldots, F_k)]^{(z, U)k+1}
\]

for each \( k \in \mathbb{N} \).
We recall that according to the definition of MBPRE’s the transition probabilities \( P_{xy} \) of the Markov chain \( \{Z_n, n \geq 0\} \) are

\[
P_{xy} := \mathbb{E}[(F^x)[y]], \quad x, y \in \mathbb{N}_0^p.
\]

(42)

where \((F^x)[y]\) is the weight assigned by \( F^x \) to the point \( y \in \mathbb{N}_0^p \). Using this definition we can identify the distribution of the process \( \{Z_n, n \geq 0\} \) under the measure \( P^* \) as the law of a certain Markov chain. Note that (40) implies

\[
P^*(Z_1 = z_1, \ldots, Z_k = z_k | Z_0 = z_0) := \frac{(z_k, U) \prod (z_i | z_{i-1})}{(z_0, U) \lambda^k}
\]

(43)

for every \( z_j \in \mathbb{N}_+^p \), \( 0 \leq j \leq k \), where

\[
P^*_{xy} := \frac{(y, U) P_{xy}}{(x, U) \lambda}, \quad x, y \in \mathbb{N}_+^p.
\]

(44)

By linearity of expectation

\[
\sum_{y \in \mathbb{N}_0^p} P^*_{xy} = \frac{1}{(x, U) \lambda} \sum_{y \in \mathbb{N}_0^p} (y, U) P_{xy} = \frac{1}{(x, U) \lambda} \mathbb{E}[(Z_1, U) | Z_0 = x] = \frac{1}{(x, U) \lambda} \mathbb{E}[(M_1, U) | x, mU) = 1
\]

for every \( x \in \mathbb{N}_+^p \). Thus, \( P^* \) indeed, specifies a Markov chain on \( \mathbb{N}_+^p \).

There is an evident connection between the \( k \)-step transition probabilities of the \( P \)-chain and the \( k \)-step transition probabilities of the \( P^* \)-chain which we express by the following formula:

\[
P^*(Z_{j+k} = y | Z_j = x) = P^*_{xy} = \frac{(y, U) P_{xy}}{(x, U) \lambda}
\]

(45)

for every \( x, y \in \mathbb{N}_+^p \) and \( j, k \geq 0 \). Here \( P^*_{xy}, P^k \) denote the \( k \)th power of the transition matrices \( P^* \) and \( P \).

For \( x = (x_1, \ldots, x_p) \in \mathbb{N}_+^p \) we write the representation

\[
\mathbb{E}[(F(s))^x] = \mathbb{E}\left( \prod_{i=1}^{p} (F^i(s))^{x_i} \right) = \sum_{y \in \mathbb{N}_0^p} P_{xy} y^x = \sum_{y \in \mathbb{N}_0^p} P (Z_1 = y | Z_0 = x) s^y.
\]

(46)
Thus, for each $y \neq 0$
\[
\sum_{x \in \mathbb{N}^p_+} t_x(z) P_{xy} = \lambda \sum_{y \in \mathbb{N}^p_+} t_y(z) s^y + 1 - \lambda.
\]
Thus, if $y \neq 0$ then
\[
\sum_{x \in \mathbb{N}^p_+} t_x(z) P_{xy} = \lambda t_y(z),
\]
and if $y = 0$ then
\[
\sum_{x \in \mathbb{N}^p_+} t_x(z) P_{x0} = 1 - \lambda.
\]

Note that by Fatou's lemma
\[
W(z) := \sum_{y \in \mathbb{N}^p_+} (y, U)t_y(z) = \sum_{y \in \mathbb{N}^p_+} (y, U) \lim_{n \to \infty} \mathbb{P}(Z_n = y | Z_0 \neq 0; Z_0 = z)
\]
\[
\leq \liminf_{n \to \infty} \mathbb{E}[(Z_n, U) | Z_0 \neq 0; Z_0 = z]
\]
\[
= \liminf_{n \to \infty} \frac{\mathbb{E}[(Z_n, U) | Z_0 = z]}{\mathbb{P}(Z_0 = z)}
\]
\[
= \lim_{n \to \infty} \frac{\mathbb{P}(Z_n = z, U)}{\mathbb{P}(Z_0 = z)} = \lim_{n \to \infty} \frac{\lambda^n (z, U)}{(z, K^*) \lambda^n} = \frac{(z, U)}{(z, K^*)} < \infty.
\]

Let
\[
t^*_x(z) = \frac{(x, U) t_x(z)}{W(z)}, \quad x \in \mathbb{N}^p_+.
\]
Clearly, for each $z \in \mathbb{N}^p_+$
\[
\sum_{x \in \mathbb{N}^p_+} t^*_x(z) = 1.
\]
Using (44) we obtain for $y \in \mathbb{N}^p_+$
\[
\sum_{x \in \mathbb{N}^p_+} t^*_x(z) P^*_x = \sum_{x \in \mathbb{N}^p_+} \frac{(x, U) t_x(z)}{W(z)} \times \frac{(y, U) P_{xy}}{(x, U) \lambda}
\]
\[
= \frac{(y, U)}{W(z) \lambda} \sum_{x \in \mathbb{N}^p_+} t_x(z) P_{xy} = \frac{(y, U)}{W(z) \lambda} t_y(z) = t^*_y(z).
\]
As a result,
\[
\sum_{x \in \mathbb{N}_+^p} t^*_x(z) P^*_x y = t^*_y(z), \quad y \in \mathbb{N}_+^p.
\] (52)

Lemma 8 Let Conditions $H1 - H6$ be valid, $1 \in \text{int} \Theta$ and $\Lambda'(1) < 0$. Then
(i) The probability measure $t^*$ from (50) is an invariant distribution for the
Markov chain generated by transition probabilities $P^*_x y$.
(ii) The chain has a single recurrent class $R = \text{supp} t^*$. The class $R$ is
positive recurrent and aperiodic.

Proof. (i) By (51), the measure $t^*$ has total mass 1. The invariance follows
from (52).

(ii) We first show that there are states which can be reached from any other
state of the chain in two steps. Recall that $m_{ij} > 0$ for all $i, j \in \{1, ..., p\}$. Therefore, for any $x \in \mathbb{N}_+^p$ and $i \in \{1, ..., p\}$
\[
\sum_{w=(w_1, ..., w_p) \in \mathbb{N}_+^p : w_i \geq 1} P_{xw} > 0.
\]
Further, in view of Condition $H6$, there exist $i \in \{1, ..., p\}$ and $y = (y_1, ..., y_p) \in \mathbb{N}_+^p$ with $y_i \geq 1$ such that
\[
\mathbb{P}(F^i[y] \neq 0 \text{ and } F^k[0] > 0, k = 1, ..., p) > 0,
\]
i.e., type $i$ individuals of the same generation of the original branching process
may have both 0 or $y \neq 0$ children with a positive probability. For such $y$ and any $x \in \mathbb{N}_+^p$
\[
P^2_{xy} \geq \sum_{w \in \mathbb{N}_+^p : w_i \geq 1} P_{xw} P_{wy} \geq \sum_{w \in \mathbb{N}_+^p : w_i \geq 1} P_{xw} \mathbb{E}[F^i[y](F^0[0])^{w-e_i}] > 0
\]
and, therefore,
\[
P^2_{xy} = \frac{(y, U) P^2_{xy}}{(x, U) \lambda^2} > 0.
\] (53)
Besides, for the $y$
\[
P^*_y y = \lambda^{-1} P^*_y y \geq \lambda^{-1} \mathbb{E}[F^i[y](F^0[0])^{y-e_i}] > 0.
\] (54)

The second assertion of the lemma now follows from standard results from
Markov chain theory: Since any invariant probability distribution is supported
by positive recurrent states (see, e.g. the criterion in Section XV.7 of [13]), part
(i) of the proposition shows that the chain has at least one such class. In view of
(53) there can be at most one recurrent class. Clearly, this class $R$, say, contains
all $y$ which satisfy (54). Since $P^*_y y > 0$ for such $y$, the class is aperiodic. The
fact that $R = \text{supp} t^*$ again follows from part (i), because the equilibrium weight $t^*_x(z)$ is the reciprocal of the expected return time to $y$ (see, e.g., Theorem 1 in
Section XV.7 of [13]) and, therefore, is unique for all $z \in \mathbb{N}_+^p$.

The lemma is proved.
Lemma 9 If the conditions of Theorem 6 are valid then the function \( T(z, s) \) in (35) is one and the same for all \( z \in \mathbb{N}_+^p \).

**Proof.** Since \( P^* \) has a unique recurrent class and does not depend on a particular \( z \) it follows that the quantities \( t^*_y(z), y \in \mathbb{N}_+^p \), also do not depend on \( z \). We write them as \( t^*_y \). Recalling (50) we see that, for each \( y \) the ratio

\[
\frac{t^*_y}{(y, U)} = \frac{t_y(z)}{W(z)}
\]

is one and the same for all \( z \in \mathbb{N}_+^p \). This, in turn, means that the left-hand side of the equality

\[
\sum_{y \in \mathbb{N}_+^p} \frac{t_y(z)}{W(z)} P_{y0} = 1 - \lambda \frac{1}{W(z)}
\]

is one and the same for all \( z \in \mathbb{N}_+^p \). Therefore, \( W(z) = W \) is also independent of \( z \). In view of (55) the last implies independence of \( t_y(z) = t_y \) of \( z \in \mathbb{N}_+^p \). Therefore, \( T(z, s) \) is one and the same for all \( z \in \mathbb{N}_+^p \), and we write

\[
T(s) = \sum_{y \in \mathbb{N}_+^p} t_y s^y = \mathbb{E} \left[ s^{T^\infty} \right], \quad T(1) = 1,
\]

where, in view of (49)

\[
W = \sum_{y \in \mathbb{N}_+^p} (y, U) t_y = \mathbb{E} \left[ (T^\infty, U) \right] < \infty,
\]

and

\[
t^*_y = \frac{(y, U) t_y}{W}, y \in \mathbb{N}_+^p.
\]

This completes the proof of Lemma 9.

Now we would like to find an explicit form for the vector \( K^* \) in (4).

Lemma 10 Under the conditions of Theorem 1

\[
K^* = \frac{U}{\mathbb{E} \left[ (T^\infty, U) \right]}.
\]

**Proof.** It follows from (5) and (4) that for any \( z \in \mathbb{N}_+^p \)

\[
\lim_{n \to \infty} \mathbb{P}(Z_n = y | Z_0 = z) = \lim_{n \to \infty} \frac{P_{zy}^n}{(z, K^*)^n \lambda^n} = t_y.
\]

Given \( t_y > 0 \) we get

\[
t^*_y = \frac{(y, U)}{(z, U)} t_y = \lim_{n \to \infty} \frac{(y, U) P_{zy}^n}{(z, U) \lambda^n} = \frac{(y, U)(z, K^*)}{(z, U)} t_y = W(z, K^*) \frac{t_y}{(z, U)}
\]

This completes the proof of Lemma 10.
Thus,

\[
\frac{(z, K^*)}{(z, U^*)} = \frac{1}{W^*}
\]

and, therefore, the left-hand side is one and the same for all \(z \in \mathbb{N}^p\). Selecting sequentially \(z = e_i, i = 1, \ldots, p\), gives \(K^* = W^{-1}U = K\), where \(K\) is the same vector as in (11).

Lemma 10 is proved.

**Proof of Theorem 1.** Combining Theorem 6 and Lemmas 9 and 10 gives the proof of Theorem 1.

6 Q-processes

**Proof of Corollary 2.** Relation (9) easily follows from Theorem 1.

Thus, we prove (10). For \(0 = n_0 < n_1 < \ldots < n_r = n\) we write

\[
P(Z_{n_1} = j_1, Z_{n_2} = j_2, \ldots, Z_{n_r} = j_r | Z_{n_r} \neq 0; Z_0 = j_0) = \frac{\lambda^n}{\lambda^n} \prod_{i=1}^{r} \mathbb{P}(Z_{n_i} = j_i | Z_{n_{i-1}} = j_{i-1})
\]

We know by Theorem 1 that

\[
\lim_{n \to \infty} \mathbb{P}(Z_n = y | Z_0 = x) = \frac{\lambda^n}{\lambda^n} \mathbb{P}(Z_n = y | Z_0 = x) = (y, K) \mathbb{P}(T_\infty = y).
\]

Using (57) and (4) we deduce by (56) that

\[
\lim_{n \to \infty} \mathbb{P}(Z_{n_1} = j_1, Z_{n_2} = j_2, \ldots, Z_{n_r} = j_r | Z_{n_r} \neq 0; Z_0 = j_0) = \frac{1}{(j_r, K)} \left( \prod_{i=1}^{r} \mathbb{P}(Z_{n_i} = j_i | T_\infty = j_i) \right)
\]

as desired.

**Proof of Corollary 3.** By definition

\[
\mathbb{P}(\hat{Y}_1 = j_1, \hat{Y}_2 = j_2, \ldots, \hat{Y}_n = j_n | \hat{Y}_0 = y) = \lim_{m \to \infty} \mathbb{P}(Z_1 = j_1, Z_2 = j_2, \ldots, Z_n = j_n | Z_{n+m} \neq 0; Z_0 = y).
\]
In view of (4) and Theorem 1

\[ \mathbb{P} \left( \hat{Y}_1 = j_1, \hat{Y}_2 = j_2, ..., \hat{Y}_n = j_n | \hat{Y}_0 = y \right) = \mathbb{P} \left( Z_1 = j_1, Z_2 = j_2, ..., Z_n = j_n | Z_0 = y \right) \lim_{m \to \infty} \frac{\mathbb{P} \left( Z_m \neq 0 | Z_0 = j_n \right)}{\mathbb{P} \left( Z_{m+n} \neq 0 | Z_0 = y \right)} \]

\[ = \frac{1}{\lambda^n (y, K)} \mathbb{P} \left( Z_1 = j_1, Z_2 = j_2, ..., Z_n = j_n | Z_0 = y \right). \]

Hence, recalling (57) and (4) we get

\[ \lim_{n \to \infty} \mathbb{P} \left( \hat{Y}_n = j | \hat{Y}_0 = y \right) = \lim_{n \to \infty} \frac{1}{\lambda^n (y, K)} \mathbb{P} \left( Z_n = j, Z_n \neq 0 | Z_0 = y \right) \]

\[ = (j, K) \lim_{n \to \infty} \frac{\mathbb{P} \left( Z_n \neq 0 | Z_0 = y \right)}{\mathbb{P} \left( Z_n = j | Z_n \neq 0, Z_0 = y \right)} = (j, K) \mathbb{P} \left( T_\infty = j \right) = \frac{(j, U)}{E \left( \mathbb{E}(T_\infty, U) \right)} \mathbb{P} \left( T_\infty = j \right). \]

Corollary 2 is proved.

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