Abstract

Thomas-Fermi model is considered here to make it cogent to capture the Planck-scale effect with the use of a generalization of uncertainty relation. Here generalization contains both linear and quadratic terms of momentum. We first reformulate the Thomas-Fermi model for the non-relativistic case. It has been shown that it can also be reformulated for taking into account the relativistic effect. Dialectic screening for the non-relativistic cases has been studied and the expression of screening length has been found out explicitly.

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I. INTRODUCTION

Over the decades, there has been a huge interest in the study of predicting the probable behavior of the physical system in the vicinity of Planck-scale. Three are different ways of studying the probable behavior of physical systems in the vicinity of Planck-scale through some well-designed formulation which has a very close link with the well-developed theory (string theory) believed to be suitable to the Planck-scale regime. The generalization of uncertainty relation is one of the most important and well-accepted frameworks in this respect. It acquired a great deal of attention when it has been found to be concomitant with the string theory, loop quantum gravity, and non-commutativity of space-time as well, at the conceptual level \[1, 3–5\]. The generalization of uncertainty relation is made in various ways however the generalization associated with minimum length and minimum length along with the maximum momentum are the two basic criteria of the generalization since the one-dimensional object namely string which is supposed to appear at the Plank scale would appear with the length of order of Planck-length.

There are a heap of literature on these two types of generalized uncertainty relation in the different branches of physics. Although generalization of uncertainty generally called \textit{generalized uncertainty principle (GUP)}\(^1\) with quadratic term in momentum initially showed the way to introduce that concept of minimum length, the concept of minimum length along with the maximum momentum was developed later through linear-quadratic GUP and these two are equally potent to incorporate the Planck-scale correction. The articles \[1, 3–10\] offer lucid and elaborate discussion on how the concept of minimum length is correlated to the well-developed theory of Planck-scale (string theory).

The formal development of statistical physics to make it amenable to the Planck-scale is carried out in the articles \[11, 16\]. In condensed matter physics the the generalization of uncertainty relation is used for the same purpose in the articles \[17, 22\]. In black-hole physics also we find the extensive use of different types of deformed uncertainty algebra in the articles \[23, 33\] to incorporate Planck-scale effect. The Quark-Gluon Plasma physics too has been extended with the GUP framework in the article \[34, 37\] for the same purpose.

The recent work of Shebabi \textit{et al.} \[18, 19\] on the Thomas-Fermi(Thomas \[38\], Fermi \[39\]) are the fascinating extension of generalized uncertainty relation having quadratic momentum dependence. Thomas-Fermi (TF) model is a many-particle statistical model that came up as a heuristic semi-classical method to describe the electrostatic potential and the charge densities in large atoms, metals, and in astrophysical objects such as neutron stars \[40, 41\]. Here the electrons are considered to behave like ideal gas that obeys the Fermi-Dirac statistics. In its original formulation, the effect of exchange forces is not taken into account, and the temperature of the system is taken as \(T = 0\). Dirac has extended the theory to include the effects of exchange forces. Relativistic corrections over it were developed by Vallarta

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\(^1\) We will prefer to use generalization of uncertainty relation in place of principle to extend our due respect to the celebrated Heisenberg uncertainty principle that brought a breakthrough in fundamental development of physics.
and Rosen \[42\] by replacing the non-relativistic electron kinetic energy term in the Thomas-Fermi formulation by its relativistic counterpart. On the other hand, the thermal effects on the Thomas-Fermi formulation were considered by Marshak and Bethe \[43\] and Feynman, Metropolis, and Teller \[44\], among others the thermal effects are as important as the relativistic effects if the temperature are of the order of $10^7 \, 0 K$ or more. In its original formulation, the effect of exchange forces was not taken into account, and the temperature of the system is taken as $T = 0$. Dirac has extended the theory to include the effects of exchange forces \[45\]. The extension of Shehabi et al. would be useful to capture the Planck-scale physics not only in the solid-state physics but also in the astrophysics. In the article, the author has studied the TF model with a special type of GUP. So a natural extension of this model is the application of another GUP on it and study the outcome of it. We have already mentioned that there is another class of GUP that is associated with the minimum length along with the maximum momentum. The introduction of linear-quadratic GUP and its interesting extension in different branches of physics are found in the articles \[46–54\]. And in many cases, the extension of the same model in different GUP has come up since the basic foundation of the deformations differs in a distinct manner. So, naturally, the correction followed are also distinctly different.

The TF model is so popular and useful in providing the different aspects of the charged many-particle system that extension of this model with other deformed Heisenberg algebra would be instructive. In this respect, it would be of worth investigation of Thomas-Fermi model with the generalized Heisenberg algebra having linear and quadratic term of momentum. In this article, we, therefore, make an attempt to carry out the formulation of the TF model with the the generalized Heisenberg algebra having both linear and quadratic term of momentum. We should mention here that in the article \[54\], the TF model has been attempted to solve with this type of generalized algebra, however the approach we will follow here is different. In \[54\], the correction is introduced at the operator level within the Hamiltonian and a perturbation technique is used to evaluate the energy of the system. But in our extension correction will be incorporated through the density of state-level as has been found in \[18, 19\] following the formalism \[55\].

The article is organized as follows. In Sec.I, a general discussion over Heisenberg algebra with linear and quadratic term in momentum is given. Sec II is devoted with the formulation of the Thomas-Fermi model with Linear-Quadratic generalization to make it cogent to capture the Planck-scale phenomena. In Sec. III, the screening effect is studied with this reformulated model. Sec. IV is an extension to include a relativistic effect in the present situation. A brief summary and discussion are given in Sec. V.

II. GENERAL DESCRIPTION OF HEISENBERG ALGEBRA WITH THE LINEAR AND QUADRATIC TERM IN MOMENTUM

The celebrated Heisenberg algebra is given by

$$[x, p] = i\hbar,$$

and the uncertainty relation from which the above algebra generates is

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$  \hspace{1cm} (2)

Generalized Heisenberg algebra with quadratic term in momentum reads

$$[x, p] = i\hbar (1 + \beta^2 p^2).$$ \hspace{1cm} (3)

However for generalization of Heisenberg with linear and quadratic term in momentum algebra has the following form:

$$[x, p] = i\hbar (1 - 2\alpha p + 4\alpha^2 p^2).$$ \hspace{1cm} (4)

The uncertainty relation corresponding to the algebra \[12\] is

$$\Delta x, \Delta p \geq \frac{\hbar}{2} \left(1 + \frac{\alpha}{\sqrt{\langle p^2 \rangle}} + 4\alpha^2 \Delta p^2 + 4\alpha^2 \langle p^2 \rangle^2 - 2\alpha \sqrt{\langle p^2 \rangle^2} \right).$$ \hspace{1cm} (5)

Here $\alpha = \frac{m_p c^2}{\sqrt{\langle p^2 \rangle}}$, where $m_p c^2 = 10^{19} \, GeV$, and the Planck length $l_p = 10^{-35} \, m$. Unlike the quadratic generalization, concept of maximum momentum along with the minimum length is admissible here, and here lies the fundamental difference between these two. The minimum length and maximum momentum admissible to this deformed algebra respectively are

$$\delta x \geq \delta x_{\text{min}} \approx \alpha_0 l_p, \delta p \leq \delta p_{\text{min}} \approx \frac{m_p c}{\alpha_0}.$$ \hspace{1cm} (6)
For $D$ dimension the equation (4) gets generalized to the following

\[ [x_i, p_j] = i\hbar[\delta_{ij} - \alpha(\delta_{ij}p + \frac{p_ip_j}{p}) + \alpha^2(\delta_{ij}p^2 + 3p_ip_j)]. \] (7)

Note that the equation (7) is satisfied by the following representation of position and momentum respectively:

\[ x_i = x_{i0}, \quad p_i = p_{i0}(1 - \alpha p + 2\alpha^2 p^2). \] (8)

The invariant phase space volume for this deformation entails the following modification

\[ [D\mu] = \frac{d^3x d^3p}{J}, \] (9)

where $J$ is the jacobian for transformation that reads

\[ J^{-1} = (2\pi)^3[1 - \alpha p + (\frac{2}{D+1} + \frac{1}{2})\alpha^2 p^2]^{D+1} \] (10)

for the deformed uncertainty relation considered here. These are the necessary input for this generalized uncertainty relation having linear and quadratic term of momentum to deal with it. We, are therefore, in a position to apply it to the model we have considered for extension. To be precise we will consider the finite temperature TF model and pursue the non-relativistic case to start with in the framework to which we now turn.

**III. FINITE TEMPERATURE THOMAS-FERMI MODEL WITH A GENERALIZED UNCERTAINTY RELATION**

Thomas-Fermi model model is formulated with the consideration that a many electron system is equivalent to a gas of fermions obeying Fermi-Dirac statistics and these electrons are occupying the phase space uniformly with one spin up and one spin down electron per unit cell having volume $\hbar^3$. Therefore, the density of electron is given

\[ n(r) = \frac{N}{V} = g\frac{2}{\hbar^3} \int f(E)d^3p, \] (11)

when position and momenta satisfy the usual Heisenberg algebra. Here $N$ represents the total number and $V = \int d^3x$. The Fermi-Dirac distribution function is denoted by $f(E)$:

\[ F(E) = \frac{1}{e^{\frac{E-\mu}{kT}} + 1}, \] (12)

and $g$ represents the degeneracy. In this situation $g = 2$. So the total number of electron can be obtained by summing or integrating over the energy $E$ as applicable according to the nature of the problem. According to the Thomas-Fermi model the screened coulomb potential is given by

\[ \nabla^2 \Phi = 4\pi e(\eta - \eta_0) - 4\pi q\delta(r), \] (13)

where $\eta$ and $\eta_0$ are the number of particles in the excited and ground state respectively. We, therefore, have

\[ \nabla^2 \phi = 4\pi e(I - I_0) - 4\pi q\delta(r), \] (14)

where

\[ I = g\frac{2}{\hbar^3} \int \frac{1}{J} \frac{4\pi p^2dp}{exp(\frac{-p^2}{2K}\tilde{\mu} + 1)} \] (15)

here $J$ represents the jacobian for transformation as given in (10), and

\[ I(0) = I(\phi=0). \] (16)

and $\tilde{\mu} = \mu + e\phi$. Here $\phi$ represents electrostatic potential, $\mu$ is the chemical potential, $T$ stands for temperature and $K$ is the Boltzmann constant. It is a general practice to set $K = 1$ without any loss of generality. In the limit of large
volume, it is reasonable to use integral over all phase space keeping jacobian within $J$ since the correction enters through that.

The generalized uncertainty which we are going to use to study the finite temperature Thomas-Fermi model to be potent to acquire Planck-scale effect is

$$[x, p] = i\hbar(1 - \eta p + \eta^2 p^2)$$  \hspace{1cm} (17)$$

It is convenient to use $2\alpha = \eta$ to get rid of factors of 2 or its multiples in different parts of the computation. So the phase space volume in presence of this generalized Heisenberg algebra.

$$D\mu = \frac{d^3x d^3p}{h^3} (1 - \eta p + \eta^2 p^2)^{-4}$$

$$= \frac{d^3x d^3p}{h^3} (1 + 4\eta p + 6\eta^2 p^2 - 20\eta^3 p^3 + 10\eta^4 p^4 + \ldots).$$  \hspace{1cm} (18)

Here $\eta \ll 1$. So the density of states retaining the terms up to order $\eta^2$ is given by

$$D\mu \approx \frac{d^3x d^3p}{h^3} (1 - 4\eta p + 6\eta^2 p^2).$$  \hspace{1cm} (19)

Using the above expression of density states, the particle density per unit volume $n = \frac{N}{\pi^2}$ is obtained which is given by

$$n = \frac{2\pi}{h^3} (2mT)^\frac{7}{2} \int_0^\infty \frac{y^2 dy}{e^{y - \frac{T}{2}} + 1} + 4\eta (2mT)^\frac{7}{2} \int_0^\infty \frac{y dy}{e^{y - \frac{T}{2}} + 1} + 6\eta^2 (2mT)^\frac{7}{2} \int_0^\infty \frac{y^2 dy}{e^{y - \frac{T}{2}} + 1}$$

$$- 20\eta^3 (2mT)^\frac{7}{2} \int_0^\infty \frac{y^2 dy}{e^{y - \frac{T}{2}} + 1} + 10\eta^4 (2mT)^\frac{7}{2} \int_0^\infty \frac{y^2 dy}{e^{y - \frac{T}{2}} + 1}$$  \hspace{1cm} (20)

In the above expression we make the substitution $y = \frac{\phi}{T}$ and $\epsilon = \frac{p^2}{2m}$. The equation (21) contains fermi integral

$$f_\nu(\varsigma) = \frac{1}{\Gamma(\nu + 1)} \int_0^\infty \frac{y^{\nu - 1} dy}{e^{\varsigma - \frac{T}{2}} + 1}$$  \hspace{1cm} (21)

where $\Gamma(\nu)$ is the Euler Gamma function. In weakly non-degenerate case $|\varsigma| \gg 1$. With this limit the equation(21) has the expansion

$$f_\nu(\varsigma) = \varsigma^\nu \Gamma(\nu + 1) \left[ 1 + \nu(\nu + 1)\frac{\pi^2}{6\epsilon^2} + \ldots \right],$$  \hspace{1cm} (22)

So the Thomas-Fermi density of a non-relativistic particle in the presence of the deformed Heisenberg algebra is found out to be

$$n = \frac{\sqrt{2}}{\pi^2 h^3} (me)^{\frac{7}{2}} \left[ \frac{1}{3} + \frac{\pi^2 T^2}{24e^2 (\phi + \frac{\beta}{e})^2} \right] + \eta (2me)^{\frac{7}{2}} \left[ 1 + \frac{\pi^2 T^2}{3e^2 (\phi + \frac{\beta}{e})^2} \right]$$

$$+ \frac{6}{5} \eta^2 (2me)^{\frac{7}{2}} \left( 1 + \frac{5\pi^2 T^2}{8e^2 (\phi + \frac{\beta}{e})^2} \right) - \frac{10}{3} \eta^3 (2me)^{\frac{7}{2}} \left( 1 + \frac{\pi^2 T^2}{e^2 (\phi + \frac{\beta}{e})^2} \right)$$

$$+ \frac{10}{7} \eta^4 (2me)^{\frac{7}{2}} \left( 1 + \frac{35\pi^2 T^2}{24e^2 (\phi + \frac{\beta}{e})^2} \right).$$  \hspace{1cm} (23)

The use Poisson’s equation along with the TF density (23) enables us to get generalized TF equation:

$$\nabla^2 \phi = 4\pi e n \approx 4\pi e \frac{\sqrt{2}}{\pi^2 h^3} (me)^{\frac{7}{2}} \left[ \frac{1}{3} + \frac{\pi^2 T^2}{24e^2 (\phi + \frac{\beta}{e})^2} \right]$$

$$+ \eta (2me)^{\frac{7}{2}} \left( 1 + \frac{\pi^2 T^2}{3e^2 (\phi + \frac{\beta}{e})^2} \right) + \frac{6}{5} \eta^2 (2me)^{\frac{7}{2}} \left( 1 + \frac{5\pi^2 T^2}{8e^2 (\phi + \frac{\beta}{e})^2} \right)$$

$$- \frac{10}{3} \eta^3 (2me)^{\frac{7}{2}} \left( 1 + \frac{\pi^2 T^2}{e^2 (\phi + \frac{\beta}{e})^2} \right) + \frac{10}{7} \eta^4 (2me)^{\frac{7}{2}} \left( 1 + \frac{35\pi^2 T^2}{24e^2 (\phi + \frac{\beta}{e})^2} \right)$$  \hspace{1cm} (24)

The terms containing $\eta$ represent the correction due to the use of deformed Heisenberg algebra. Note that the above equation lands onto the usual Thomas-Fermi equation if we set $\eta = 0$. The equation (23) is the generalized form of the Thomas-Fermi equation with linear-quadratic generalization which will be equally useful to capture the Plank-scale effect.
IV. THE DIELECTRIC SCREENING PROCESS IN PRESENCE GENERALIZED UNCERTAINTY RELATION

As an application we will now turn towards the investigation of the impacts of this alternative deformed Heisenberg algebra on dielectric screening process. To this end we assume a uniform gas of electrons having charge density $-n_0e$ is superimposed on a lattice of shielded nuclei with charge density $n_0e$ and place a positive point charge $Q$ as a test charge in this charged sea with a coulomb potential $\phi$ to study the screening effect in this situation. It is straightforward to obtain the screening potential since the screening potential would be a solution of Poisson’s equation in this particular environment.

$$\nabla^2 \phi = 4\pi e(n - n_0) - 4\pi Q \delta(r). \quad (25)$$

Our objective is to incorporate the Planck-scale effect through the specified generalized uncertainty. So it is to bear in mind that the density that has to be used is the modified TF density to have a solution of the equation (25) with $g = 2$. So what we have is

$$\nabla^2 \phi = \frac{4me^2\sqrt{2}me}{\pi \hbar^3}((\phi + \frac{\mu}{e})^2\left[\frac{1}{3} + \frac{\pi^2 T^2}{24e^2(\phi + \frac{\mu}{e})^2}\right] + \eta(2me)^2\left[1 + \frac{\pi^2 T^2}{3e^2(\phi + \frac{\mu}{e})^2}\right]$$

$$+ \frac{6}{5} \eta^2(2me)(\phi + \frac{\mu}{e})\left[1 + \frac{5\pi^2 T^2}{8e^2(\phi + \frac{\mu}{e})^2}\right] - \frac{10}{3} \eta^3(2me)^2(\phi + \frac{\mu}{e})\left[1 + \frac{25\pi^2 T^2}{8e^2(\phi + \frac{\mu}{e})^2}\right]$$

$$+ \frac{10}{7} \eta^4(2me)^2(\phi + \frac{\mu}{e})^2\left[1 + \frac{35\pi^2 T^2}{24e^2(\phi + \frac{\mu}{e})^2}\right]$$

$$- \frac{10}{7} \eta^4(2me)^2(\phi + \frac{\mu}{e})^2\left[1 + \frac{35\pi^2 T^2}{24e^2(\phi + \frac{\mu}{e})^2}\right] - 4\pi Q \delta(r). \quad (26)$$

If we keep ourselves restricted with the linear response only ignoring all nonlinear effect it will allow us to consider the $|\frac{\eta}{\hbar}| \ll 1$. With this approximation the above equation gets simplified into

$$\nabla^2 \phi = \frac{3}{2} \phi(\frac{4\pi n_0 e^2}{\mu})\left[1 - \frac{\pi^2 T^2}{24\mu^2} + \frac{3m\pi^2 T^2 \eta^2}{2\mu} - \frac{40\sqrt{2} m \pi^2 T^2 \eta^2}{3 \mu^2} + 25(m \pi T \eta)^2\right] - 4\pi Q \delta(r). \quad (27)$$

Note that temperature independent term is not considered here. The equation (27) can be casted in to following simplified form

$$\nabla^2 \phi = \frac{3}{2} \phi \left[\lambda_F^{(\eta)}\right]^{-2} - 4\pi, Q \delta(r) \quad (28)$$

where $\phi$ is given by

$$\phi = \frac{Q}{r} e^{-\sqrt{2}(\frac{s}{\sqrt{\eta}})}. \quad (29)$$

Here $n_0 = \frac{1}{3\pi^2(\frac{2m}{\hbar^2})^{\frac{3}{2}}}$ and the screening length length is found out to be

$$\left[\lambda_F^{(\eta)}\right]^{-2} = \frac{4\pi n_0 e^2}{\mu}(1 - \frac{\rho}{3} + 6\sigma - 40\tilde{\sigma} + 25\zeta) \quad (30)$$

where $\rho = \frac{\pi^2 T^2}{8\mu^2}$, $\sigma = \frac{\pi^2 T^2 m^2}{4\mu^2}$, $\tilde{\sigma} = \frac{\sqrt{2} m \pi^2 T^2 \eta^2}{\sqrt{\mu}}$ and $\zeta = (m \pi T)^2 \eta^4$.

V. RELATIVISTIC THOMAS-FERMI MODEL WITH LINEAR-QUADRATIC GENERALIZATION

We are now going to generalized the TF model with linear-quadratic GUP to incorporate the Planck-scale correction for relativistic domain. The modified phase space volume which follows from equations (10), (11) and (12).

$$D\mu \simeq \frac{d^3 xx^3 \rho}{h^3} (1 + 4\eta \rho + 6\eta^2 \rho^2). \quad (31)$$
 retaining the terms up to 2nd order in $\eta$. So the TF density in the relativistic regime is described by

$$n(r) = \frac{1}{\pi^2 \hbar^3} \int_0^\infty \frac{p^2 dp (1 + 4\eta p + 6\eta^2 p^2)}{1 + \exp\{b(\sqrt{p^2 c^2 + m^2 c^2 - mc^2 - e(\phi - \phi_0)})\}}, \quad (32)$$

where $m$ is the mass of the particle, $b$ is just the inverse temperature since $K = 1$ is already set. However, the exact expression of $b$ ($b \equiv 1/K_T$), and $\mu = -e\phi_0$ is the chemical potential.

It would be useful to introduce a Juttner’s transformation [56] at this stage since it will make the computation tractable a lot. Explicitly it is

$$\frac{p}{mc} = \sinh \theta. \quad (33)$$

With this transformation the TF density reads

$$n(r) = \frac{m^3 c^3}{\pi^2 \hbar^3} \int_0^\infty \frac{\sinh^2 \theta \cosh \theta d\theta (1 + 4\eta mc \sinh \theta + 6\eta^2 m^2 c^2 \sinh^2 \theta)}{1 + \frac{1}{\Lambda} \exp(bmc^2 \cosh \theta)} + \frac{4\eta mc}{1 + \frac{1}{\Lambda} \exp(bmc^2 \cosh \theta)}$$

$$+ \frac{6\eta^2 m^2 c^2}{1 + \frac{1}{\Lambda} \exp(bmc^2 \cosh \theta)} \int_0^\infty \frac{\sinh^3 \theta \cosh \theta d\theta}{1 + \frac{1}{\Lambda} \exp(bmc^2 \cosh \theta)}, \quad (34)$$

where $\Lambda = e^{b(\mu mc^2 + e\phi(r))}$. If we now adopt a new variable $\omega$ in the similar way has been used in [58]

$$\omega = bmc^2 \cosh \theta. \quad (35)$$

the above equation (34) reduces to

$$n(r) = \frac{m^3 c^3}{\pi^2 \hbar^3} \frac{1}{bmc^2} \int_0^\infty \frac{d\xi_1(\omega)}{1 + \frac{1}{\Lambda} e^\omega} d\omega + 4\eta mc \int_0^\infty \frac{d\xi_1(\omega)}{1 + \frac{1}{\Lambda} e^\omega} d\omega$$

$$+ \frac{6\eta^2 m^2 c^2}{1 + \frac{1}{\Lambda} e^\omega} \int_0^\infty \frac{d\xi_2(\omega)}{1 + \frac{1}{\Lambda} e^\omega} d\omega, \quad (36)$$

where

$$\frac{d\xi_1}{d\omega} = \sinh \theta \cosh \theta, \quad (37)$$

$$\frac{d\xi_2}{d\omega} = \sinh^2 \theta \cosh \theta, \quad (38)$$

$$\frac{d\xi_3}{d\omega} = \sinh^3 \theta \cosh \theta. \quad (39)$$

At this stage it would be useful to use the use the Sommerfeld lemma [57] which leads us to have the following simplified from from the equation (36):

$$n(r) = \frac{m^3 c^3}{\pi^2 \hbar^3} \frac{1}{bmc^2} \left[ \xi_1(\omega_0) + \frac{\pi^2}{6} \xi_1''(\omega_0) + \ldots \right]$$

$$+ 6\eta^2 m^2 c^2 \left[ \xi_2(\omega_0) + \frac{\pi^2}{6} \xi_2''(\omega_0) + \ldots \right]$$

$$+ 4\eta mc \left[ \xi_3(\omega_0) + \frac{\pi^2}{6} \xi_3''(\omega_0) + \ldots \right], \quad (40)$$
where $\omega_0$ is function of $\Lambda$ which has the following explicit expression

$$\omega_0 = \ln \Lambda = b[\mu + mc^2 + e\phi(r)] = bmc^2 \cosh \theta_0. \quad (41)$$

To obtain the expression of $n(r)$ in a desired form a new variable $s$ is introduced which has the following definition

$$s = \sinh \theta_0 = \left[ \frac{(\mu + mc^2 + e\phi(r))}{m^2c^4} \right] - 1^{\frac{1}{2}}, \quad (42)$$

the equations (37), (38) and (39) in terms of $s$ look

$$\xi_1(\omega_0) = \frac{bmc^2}{3} s^3, \xi_1''(\omega_0) = \frac{1}{bmc^2} \left( \frac{2s^2 + 1}{s} \right), \quad (43)$$

$$\xi_3(\omega_0) = \frac{bmc^2}{4} s^4, \xi_3''(\omega_0) = \frac{1}{bmc^2} [2(1 + s^2) + s^2], \quad (44)$$

$$\xi_2(\omega_0) = \frac{bmc^2}{5} s^5, \xi_2''(\omega_0) = \frac{1}{bmc^2} [3s(1 + s^2) + s^3]. \quad (45)$$

Substitution of the equations (43), (44) and (45) in the equation (40) leads us to the required result of the TF charge density which is compatible to relativistic regime:

$$n(r) = \frac{m^3c^4}{4\pi^2h^3} \left\{ \frac{bmc^2}{3} s^3 + \frac{\pi^2}{6} \frac{1}{bmc^2} \left( \frac{2s^2 + 1}{s} \right) + \ldots \right\}$$

$$n(r) + 4\eta mc \left\{ \frac{bmc^2}{4} s^4 + \frac{\pi^2}{6} \frac{1}{bmc^2} \left( 2(1 + s^2) + s^2 \right) + \ldots \right\}$$

$$+ 6\eta^2 m^2 c^2 \left\{ \frac{bmc^2}{5} s^5 + \frac{\pi^2}{6} \frac{1}{bmc^2} \left( 3s(1 + s^2) + s^3 \right) + \ldots \right\} + o(\eta^3) \} \quad (46)$$

Poisson’s equation with the use of this TF density leads to obtain the general form of coulomb potential in the relativistic regime where Planck-scale correction has got incorporate with the framework used here.

$$\nabla^2 \phi = 4\pi e\rho(r) \simeq 4\pi e \frac{m^3c^3}{\pi^2h^3} \left( \frac{\mu + mc^2 + e\phi}{m^2c^4} \right)^2 - 1^{\frac{1}{2}} \left\{ 1 + \frac{\pi^2}{2(bmc^2)^2} \left( \frac{\mu + mc^2 + e\phi}{m^2c^4} - 1 \right)^{-2} \right\}$$

$$+ 4\eta mc \left\{ \frac{3}{4} \left( \frac{\mu + mc^2 + e\phi}{m^2c^4} - 1 \right)^{-\frac{1}{2}} \right\}$$

$$+ 6\eta^2 m^2 c^2 \left\{ \frac{3}{5} \left( \frac{\mu + mc^2 + e\phi}{m^2c^4} - 1 \right) \right\} + \frac{\pi^2}{2(bmc^2)^2} \left( \frac{\mu + mc^2 + e\phi}{m^2c^4} - 1 \right)^{-2} \} \quad (47)$$

If we now adopt the new variables

$$\Phi = \frac{\phi + \mu}{Ze/r}, \quad r = ax, \quad a = \frac{1}{2} \frac{9\pi^2}{128Z^2} \frac{h^2}{me^2}, \quad \lambda = \frac{4Z^2}{3\pi} \frac{\mu}{h^2c^2}, \quad (48)$$

we will reach to the final expression of relativistic TF equation in the presence of linear-quadratic generalization:

$$\frac{d^2 \Phi}{dx^2} = \frac{\Phi^2}{\sqrt{x}} [1 + \lambda \Phi] \frac{x^2}{\Phi^2} (1 + \rho \frac{x^2}{\Phi^2}) (1 + \frac{x^2}{\Phi^2})^{-2} + \frac{4}{mc} \sigma_r \left( \frac{3}{4} [1 + \lambda \Phi] \right) \frac{x^2}{\Phi^2} [1 + \lambda \Phi]^{-2}$$

$$+ \frac{6}{m^2 c^2} \sigma_r \left( \frac{3}{5} [1 + \lambda \Phi] \right) + \frac{x^2}{\Phi^2} \right[ (1 + \lambda \Phi)^{-2} \right] \} \quad (49)$$

where $\rho_r, \sigma_r$ and $\sigma_r$ have the following expression respectively

$$\rho_r = \frac{1}{8} \frac{\pi^2 a^2}{s^2 c^2 Z^2}, \quad \sigma_r = \frac{1}{8} \frac{\pi^2 m^2 \eta a}{s^2 c^2 Z^2}, \quad \sigma_r = \frac{1}{8} \frac{\pi^2 m^4 \eta^2 a^2}{s^2 c^2 Z^2} \quad (50)$$

The equation (49) will be equally useful in the vicinity of Planck-scale when relativistic effect will be taken into account. Note that in the limit $\rho \ll mc$, the above equation leads to TF equation in the absence of relativistic effects:

$$\frac{d^2 \Phi}{dx^2} = \frac{\Phi^2}{\sqrt{x}} (1 + \rho \frac{x^2}{\Phi^2}) + \frac{4}{mc} \sigma_r \left( \frac{3}{4} + \frac{x^2}{\Phi^2} \right) + \frac{6}{m^2 c^2} \sigma_r \left( \frac{3}{5} + \frac{x^2}{\Phi^2} \right). \quad (51)$$
In the limit $\eta \to 0$, or in the absence of thermal effects, the relativistic TF equation becomes

$$\frac{d^2 \Phi}{dx^2} = \frac{\Phi^2}{\sqrt{x}} \left[ 1 + \lambda \frac{\Phi}{x} \right]^2$$

(52)

VI. SUMMARY AND DISCUSSION

This present paper is an extension and elaboration of the TF model with linear-quadratic generalization to make it potent to capture the Plank-scale effect. Initially, we consider the non-relativistic case. It is then extended to the relativistic regime. For the non-relativistic case screening process has been studied with the evaluation of screening length. For the relativistic case also the screening length is easily calculable. The generalization done here may be useful for high-density system as a star. How the screening process will be affected in the quark-gluon plasma system by the quantum gravity effect can also be studied using this modified TF equation for quark-gluon plasma at a very high density.

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