EXPRESSING FINITE-INFINITE MATRICES INTO PRODUCTS OF COMMUTATORS OF FINITE ORDER ELEMENTS

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Abstract. Let $R$ be an associative ring with identity 1 and consider $k \in \mathbb{N}$ such that $1 + 1 + \ldots + 1 = k$ is invertible. Denote by $\omega$ an arbitrary $k$th root of unity in $R$ and let $UT_{\infty}(R)$ be the group of upper triangular infinite matrices whose entries lying on the main diagonal are such that solves the equation $x^k = 1$. We show that every element of the group $UT_{\infty}(R)$ can be expressed as a product of $4k - 6$ commutators all depending on powers of elements in $UT_{\infty}(R)$ of order $k$. In the case that $R$ is a complex field or the real number field we prove that, in $SL_n(R)$ and in the subgroup $SL_{VK}(\infty, R)$ of the Vershik-Kerov group over $R$, each element in these groups can be decomposed into a product of at most $4k - 6$ commutators of elements of order $k$.

1. Introduction

Expressing matrices as a product of involutions was studied by several authors a few years ago. For example Halmos in [1] proved that every square matrix over a field, with determinant $\pm 1$, is the product of not more than four involutions and Solwik in [2] proved that for any field, every element of group of upper triangular infinite matrices whose entries lying on the main diagonal are equal to either 1 or $-1$ can be expressed as a product of at most five involutions.

Following the same direction, there are works to express matrices as the product of commutators of matrices. For example Zheng in [3] proved that every matrix $A$ in $SL_n(F)$ is a product of at most two commutators of involutions, where $F$ is the complex number field or the real number field and Hou in [4] proved that the group of upper triangular infinite matrices whose entries lying on the main diagonal are equal to 1 can be expressed as a product of at most two commutators of involutions.

Recently Slowik in [5] and Grunenfelder in [6] study when a matrix can be expressed as a product of fixed order matrices.

In this paper, the authors generalize the work done by Hou in [4] about the group $UT_n^{(k)}(R)$ and $UT_{\infty}(R)$ of matrices whose elements in $T_n(R)$ and

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\end{itemize}
$T_\infty(R)$ respectively have entries in the diagonal of order $k$, obtaining results of when a matrix is the product of commutators of matrix of fixed order.

The main result of this paper is stated as follows:

**Theorem 1.1.** Assume that $R$ is an associative ring with identity 1 and that $1+1+\cdots+1 = k$ is an invertible element of $R$. Then every element of the group $UT_\infty(R)$ and $UT_n(R)$ ($n \in \mathbb{N}$) can be expressed as a product of at most $4k-6$ commutators of elements of order $k$ in $UT_\infty^{(k)}(R)$ and $UT_n^{(k)}(R)$, respectively.

In the section 3, consider the case $R = \mathbb{K}$ a complex field or the real field and the group $SL_n(\mathbb{K})$ we have the following result:

**Theorem 1.2.** All element in $SL_n(\mathbb{K})$ can be written as a product of at most $4k-6$ commutators of elements of order $k$ in $GL_n(\mathbb{K})$.

And, if we consider $GL_{VK}(\infty, \mathbb{K})$ the Vershik-Kerov group, we have:

**Theorem 1.3.** Assume that $\mathbb{K}$ is a complex field or the real number field. Then every element of the group $SL_{VK}(\infty, \mathbb{K})$ can be expressed as a product of at most $4k-6$ commutators of elements of order $k$ in $GL_{VK}(\infty, \mathbb{K})$.

2. Preliminaries

Fix $k \in \mathbb{N}$, $k \geq 2$ and let $R$ be an associative ring with identity and denote by $\omega$ an arbitrary $k$th root of unity in $R$. Denote by $T_n(R)$ and $T_\infty(R)$ the group of upper triangular matrices over a ring $R$, of dimension $n$ and infinite respectively. Anagolously, denote by $UT_n(R)$ and $UT_\infty(R)$ the group of upper unitriangular matrices. We also put

\[
UT_n^{(k)}(R) = \{g \in T_n(R); \ g_{ii}^k = 1\},
\]

\[
UT_\infty^{(k)}(R) = \{g \in T_\infty(R); \ g_{ii}^k = 1\}.
\]

\[
D_n^{(k)}(R) = \{g \in UT_n^{(k)}(R); \ g_{ij} = 0, \text{ if } i \neq j\}.
\]

\[
D_\infty^{(k)}(R) = \{g \in UT_\infty^{(k)}(R); \ g_{ij} = 0, \text{ if } i \neq j\}.
\]

The following remark is immediate from Remark 2.1 of Slowik [2]:

**Remark 1.** Let $G$ a group

1. If $g \in G$ is an element of order $k$, i.e. $g^k = 1$, then for every $h \in G$ the conjugation $g^h = hgh^{-1}$ is an elment of order $k$.

2. If $g \in G$ is a product of $r$ element of order $k$, then for every $h \in G$ the conjugation $g^h$ is a product of $r$ elements of order $k$.

Define the commutator $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. Denote by $E_{ij}$ the finite or infinite matrix with an unique nonzero entry equal to 1 in the position $(i, j)$. Then, if $A \in UT_n^{(k)}(R)$ or $UT_\infty^{(k)}(R)$ we can write

\[
A = \sum_{i,j} a_{i,j}E_{i,j}.
\]
Denote by $J_{\infty}(R)$ the set of all matrices with entries in $R$ in which all entries outside of the first superdiagonal equal 0. Let $A \in UT_{\infty}(R)$ and denote by $J(A)$ the matrix of $J_{\infty}(R)$ that has the same entries on the first super diagonal as $A$. Denote by $Z$ the center of the ring $R$, and by $D_{\infty}(Z)$ the subring of all diagonal infinite matrices with entries in $Z$. We say that $A$ is coherent when there is a sequence $(D_k)_{k \leq 1}$ of elements of $D_{\infty}(Z)$ such that

$$A = \sum_{k=0}^{\infty} D_k J(A)^k.$$ 

In such a sequence we can always require $D_0 = \text{diag}(A)$ and $D_1$ to equal $I_{\infty}$, in which case we say that the sequence $(D_k)_{k \leq 1}$ is normalized.

### 3. Expressing Matrices into Products of Commutators

Assume that 2 is an invertible element of $R$. The following results are of Hou in [4]:

**Lemma 3.1.** If $A \in UT_{\infty}(R)$ is coherent, then $A^2$ is coherent.

**Lemma 3.2.** Assume that $R$ is an associative ring with identity 1 and that 2 is an invertible element of $R$. Let $J \in J_{\infty}(R)$. Then, there exists a coherent matrix $A \in UT_{\infty}(R)$ such that $J(A) = J$ and $A$ is the commutator of two involutions in $T_{\infty}(R)$.

and

**Lemma 3.3.** Assume that $R$ is an associative ring with identity 1 and that 2 is an invertible element of $R$. Let $A, B$ be coherent matrices of $UT_{\infty}(R)$ such that $J(A) = J(B)$. Then, $A$ and $B$ are conjugated in the group $UT_{\infty}(R)$.

The following result generalized the Lemma 3.1

**Lemma 3.4.** If $A \in UT_{\infty}(R)$ is coherent, then $A^k$ is coherent, for all $k \in \mathbb{N}$ such that $k$ is invertible in $R$.

**Proof.** The case for $k = 2$ has been demonstrated by Hou in [4], here shown that if $A = \sum_{i=0}^{\infty} D_i J(A)^i$ with $D_0 = D_1 = I_{\infty}$ and $D_2, D_3, \cdots \in D_{\infty}(Z)$, then

$$A^2 = \sum_{i=0}^{\infty} \left( \sum_{j=1}^{i} D_j S^j(D_{i-j}) \right) J(A)^i,$$

where, if $D \in D_{\infty}(Z)$ have diagonal entries $d_1, d_2, \cdots, d_k, \cdots$, then define $S(D) \in D_{\infty}(Z)$ as the matrix with diagonal entries $d_2, d_3, \cdots, d_{k+1}, \cdots$. Denote by $D^{(2)}_i = \sum_{j=1}^{i} D_j S^j(D_{i-j})$ we can write

$$A^2 = \sum_{i=0}^{\infty} D^{(2)}_i J(A)^i,$$
this follows that

\[ A^3 = \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} D_j S^j(D_{i-j}^{(2)}) \right) J(A)^i, \]

and, in general if

\[ A^i = \sum_{i=0}^{\infty} D_i J(A)^i, \]

where \( D_0^{(l)} = D_1^{(l)} = I_\infty \) and \( D_2^{(l)}, D_3^{(l)}, \cdots \in D_\infty(R) \), then

\[ A^{i+1} = \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} D_j S^j(D_{i-j}^{(l)}) \right) J(A)^i. \]

Note that \( J(A)^3 = (D_0 D_1^{(2)} + D_1 S(D_0^{(2)})) J(A) = (2 + 1) J(A) = 3 J(A) \), in general \( J(A)^l = l J(A) \) for \( l \geq 2 \) and using the invertibility of \( k \) we conclude that \( A^k \) is coherent.

\[ \square \]

In our case, we adapted the lemma 3.2.

**Lemma 3.5.** Assume that \( R \) is an associative ring with identity 1 and that \( 1 + 1 + \cdots + 1 = k \) is an invertible element of \( R \). Let \( J \in J_\infty(R) \). Then, there exists a coherent matrix \( A \in U T_\infty(R) \) such that \( J(A) = J \) and \( A \) is the product of \( 2k - 3 \) commutators all depending of two matrices \( B, C \) in \( U T_\infty(k)(R) \) of order \( k \).

First we proof the following:

**Lemma 3.6.** Suppose that \( k > 1 \) and that \( B, C \) are matrices of order \( k \). Then \((BC)^k \) can be represented as products of \( 2k - 3 \) commutators all depending of exponents of \( B \) and \( C \).

**Proof.** Define by recurrence:

\[ F_2(B, C) = [B, C], \]

\[ F_3(B, C) = F_2(B, C)[C, B^2][B^2, C^2] \]

and

\[ F_{i+1}(B, C) = F_i(B, C)[C^{(i-1)}, B^{-(k-i)}][B^{-(k-i)}, C^i], \]

the we have for example:

\[ F_2(B, C) = [B, C], \]

\[ F_3(B, C) = [B, C][C, B^2][B^2, C^2] \]

\[ F_4(B, C) = F_3(B, C)[C^2, B^3][B^3, C^3] = [B, C][C, B^2][B^2, C^2][C^2, B^3][B^3, C^3] \]

\[ F_5(B, C) = F_4(B, C)[C^3, B^4][B^4, C^4] = [B, C][C, B^2][B^2, C^2][C^2, B^3][B^3, C^3][C^3, B^4][B^4, C^4], \]
then we affirmate that

\[(1) \quad (BC)^i = F_i(B, C)C^{i-1}B^{-(k-i)}C.\]

Its clear that this is true for \(i = 2\). Suppose that this is true for \(i = h\), then, multiplying by \((BC)\) in both sides of \([1]\) we have:

\[
(BC)^{h+1} = (BC)^h(BC) = F_h(B, C)C^{h-1}B^{-(k-h)}C(BC)
= F_h(B, C)C^{h-1}B^{-(k-h)}(C^{-(h-1)}B^{(k-h)}B^{-(k-h)}C^{(h-1)})CBC
= F_h(B, C)[C^{h-1}, B^{-(k-h)}]B^{-(k-h)}C^hBC
= F_{h+1}(B, C)C^{h-1}B^{-(k-h)}[B^{-(k-h)}, C^h]C^hB^{-(k-h)}BC
= F_{h+1}(B, C)C^{h}B^{-(k-(h+1))}C.
\]

This proof the our affirmation \([1]\). In the case that \(i = k - 1\) then we have that

\[
(BC)^{(k-1)} = F_{k-1}(B, C)C^{k-2}B^{-(k-(k-1))}C
\]
or

\[
(BC)^{(k-1)} = F_{k-1}(B, C)C^{k-2}B^{-1}C,
\]
multiplying by \((BC)\) in both sides and remember that \(B\) and \(C\) has order \(k\), we have then

\[
(BC)^k = (F_{k-1}(B, C)C^{k-2}B^{-1}C)(BC)
= F_{k-1}(B, C)[C^{k-2}, B^{-1}]B^{-1}C^{k-2}CBC
= F_{k-1}(B, C)[C^{k-2}, B^{-1}]B^{-1}C^{k-1}BC
= F_{k-1}(B, C)[C^{k-2}, B^{-1}]B^{-1}C^{k-1}C
= F_{k}(B, C),
\]
this finalizing the proof.

\[\square\]

**Proof of Lemma 3.3** Let \(J = \sum_{i=1}^{\infty} a_{i,i+1}E_{i,i+1} \in J_\infty(R)\) and define

\[
B = \begin{pmatrix}
1 & 0 & \frac{1}{w}a_{23} & \cdots \\
0 & w & 1 & \frac{1}{k}a_{43} \\
& 0 & w & \frac{1}{k}a_{45} \\
& & \ddots & \ddots \\
& & & \ddots & \ddots
\end{pmatrix} = \text{diag}\left(1, \left(\begin{array}{ccc}
w & \frac{1}{w}a_{23} \\
0 & 1
\end{array}\right), \left(\begin{array}{ccc}
w & \frac{1}{k}a_{45} \\
0 & 1
\end{array}\right), \cdots\right),
\]

\[
= \sum_{i=1}^\infty \left(E_{2i-1,2i-1} + wE_{2i,2i} + \frac{1}{k}a_{2i,2i+1}E_{2i,2i+1}\right)
\]
\[ C = \begin{pmatrix}
1 & \frac{1}{k}a_{12} & \cdots \\
\frac{1}{k}a_{34} & \ddots & \ddots \\
\end{pmatrix}
\]

\[ = \text{diag} \left( \begin{pmatrix}
1 & \frac{1}{k}a_{12} \\
0 & \frac{1}{k}a_{34} \\
\end{pmatrix}, \ldots \right), \]

Observe that \( B \) and \( C \) are two matrices of order \( k \). Using the Lemma 3.6, denote by \( A = (BC)^k = F_k(B, C) \) then by lemma 3.4 \( J(A) = J((BC)^k) = kJ(BC) \). Observe that

\[ BC = I_\infty + \frac{1}{k} \sum_{i=1}^\infty a_{i,i+1}E_{i,i+1} + \frac{1}{k^2} \sum_{i=1}^\infty a_{2i,2i+1}a_{2i+1,2i+2}E_{2i,2i+2} \]

And \( BC \) is coherent with \( D_0 = D_1 = I_\infty \), and \( D_2 = \sum_{i=1}^\infty E_{2i,2i} \). This conclude the lemma.

From the lemmas 3.3, 3.4 and 3.5, we obtain the following corollary

**Corollary 3.7.** Assume that \( R \) is an associative ring with identity 1 and that \( 1 + 1 + \cdots + 1 = k \) is an invertible element of \( R \). Every matrix in \( UT_\infty(R) \) and \( UT_n(R) \) (\( n \in \mathbb{N} \)), whose entries except in the main diagonal and the first super diagonal are all equal to zero, is a product of \( 2k - 3 \) commutators of powers of two elements of order \( k \) in \( UT_\infty(R) \).

**Proof.** Let \( A \) be in \( UT_\infty(R) \), then by the Lemma 3.5 for \( J(A) \) there is an element \( B \in UT_\infty(R) \) such that \( J(B) = J(A) \) and \( B \) is the product of \( 2k - 3 \) commutators of powers of two elements of order \( k \) in \( UT_\infty(R) \) and by Lemma 3.3 we conclude that \( A \) and \( B \) are conjugated.

We enunciated the followings results adapted from [4]:

**Lemma 3.8.** Assume that \( R \) is an associative ring with identity 1 and let \( n \in \mathbb{N} \):

1. If \( A, B \) are elements of \( UT_n^{(k)}(R) \) such that \( a_{i,i+1} = b_{i,i+1} = 1 \) for all \( 1 \leq i \leq n - 1 \), then \( A \) and \( B \) are conjugated in \( UT_n^{(k)}(R) \).
2. If \( A, B \) are elements of \( UT_\infty^{(k)}(R) \) such that \( a_{i,i+1} = b_{i,i+1} = 1 \) for all \( 1 \leq i \) then \( A \) and \( B \) are conjugated in \( UT_\infty^{(k)}(R) \).

**Proof.** Consider \( A = (a_{ij}) \) and

\[ J = \begin{pmatrix}
a_{11} & 1 & \cdots \\
a_{22} & 1 & \ddots \\
a_{33} & 1 & \ddots \\
\end{pmatrix} \]
All blank entries are equal to 0. We need only to prove that $A$ is conjugated to the matrix $J$, for this we can constructed a matrix $X = (x_{ij}) \in UT^{(k)}_\infty (R)$ such that $X^{-1}AX = J$ or $AX = XJ$. For the first super diagonal entries of $X$ we choose

a.) In the case that $a_{ii} = a_{i+1,i+1}$ then $x_{i,i+1}$ can be choose any element in $R$.

b.) In the case that $a_{ii} \neq a_{i+1,i+1}$ then $x_{i,i+1} = 1$.

When we have first for the second super diagonal entries we can choose

a.) If $a_{ii} = a_{jj}$ then choose $x_{ii} = x_{jj}$ and $x_{ij}$ can be choose any element in $R$.

b.) If $a_{ii} \neq a_{jj}$ then choose $x_{ii} \neq x_{jj}$ and

$$x_{ij} = [a_{ii} - a_{jj}]^{-1}\{x_{i,i-j} - \sum_{k=2}^{j-i} a_{i,i+k}x_{i+k,j}\}$$

Thus the Lemma 3.8 is proved. □

Also:

Corollary 3.9. Assume that $R$ is an associative ring with identity 1 and that $1 + 1 + \cdots + 1 = k$ is an invertible element of $R$. Every matrix in $UT_\infty (R)$ and $UT_n (R)$ $(n \in \mathbb{N})$, whose entries in the first super diagonal are all equal to the identity 1, is a product of $2k - 3$ commutators of matrices of order $k$ in $UT_n^{(k)} (R)$ or $UT_\infty^{(k)} (R)$.

Proof. Consider $A \in UT_\infty (R)$, then by the Lemma 3.8 $A$ is conjugated to matrix $J = J(A)$, and by the Corollary 3.7, $J$ is product of $2k - 3$ commutators of matrices of order $k$. □

Then, with this results, we show the Theorem 1.1.

Proof of Theorem 1.1. Let $A$ be an arbitrary matrix of $UT_\infty (R)$. We can write

$$A = I_\infty + \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} a_{i,j}E_{i,j} \in UT^{(k)}_\infty (R),$$

and consider the following matrix

$$B = I_\infty + \sum_{i=1}^{\infty} (a_{i,i+1} - 1)E_{i,i+1} \in UT_\infty (R).$$
By Corollary 3.7, $B$ is a product of $2k - 3$ commutators of two matrices of order $k$ in $UT^{(k)}_{\infty}(R)$. Note that

$$C = B^{-1}A \in UT_{\infty}(R),$$

is a matrix whose entries in the main diagonal and the first super diagonal are all equal to $1$. By the Corollary 3.9, $C$ is also a product of $2k - 3$ commutators of powers of elements of order $k$ in $UT^{(k)}_{\infty}(R)$. Finally, $A = BC$ is a product of $4k - 6$ commutators of powers of four elements of order $k$ in $UT^{(k)}_{\infty}(R)$.

\[\square\]

4. Case $R = \mathbb{K}$ a complex field or the real field

Consider $R = \mathbb{K}$ be a complex field or the real number field, then we show the Theorem 1.2.

**Proof of Theorem 1.2.** The case $k = 2$ is proved in [4]. Suppose that $k \geq 3$. Consider $A \in SL_n(\mathbb{K})$ not a scalar matrix, then by [10], Theorem 1, we can find a lower-triangular matrix $L$ and an upper-triangular matrix $U$ such that $A$ is similar to $LU$, and both $L$ and $U$ are unipotent. By the corollary 3.9 it follows that each of the matrices $L$ and $U$ is a product of $2k - 3$ commutators of powers of two elements of order $k$ from $SL_n(\mathbb{K})$. The scalar case $A = \alpha I$ with $det(A) = 1$ it suffices to consider the case when $n$ is exactly the order of $\alpha$ (see [4]). Assume that $n = 2m + 1$, then consider $\beta = \alpha^{\frac{n+1}{2}}$ we have that $\beta^n = 1$ and $\beta^2 = \alpha$. First observe the follows decomposition

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 0 & -\beta \end{bmatrix} \cdot \begin{bmatrix} \beta & 1 \\ 0 & -\beta \end{bmatrix}.$$

Then, the matrix $\alpha I$ is a product of two matrices both in the form

$$B = \begin{bmatrix} \beta & 1 \\ 0 & -\beta \end{bmatrix} \oplus \begin{bmatrix} \beta & 1 \\ 0 & -\beta \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \beta & 1 \\ 0 & -\beta \end{bmatrix} \oplus [\beta],$$

and $B \in UT^{(n)}(R)$. By the Corollary 3.7 we conclude that $B$ is product of $2k - 3$ commutators of matrices of order $k$. Finally, $A = \alpha I = B \cdot B$ and thus the Theorem is proved.

For the case $n = 2m$ suppose that $\mathbb{F} = \mathbb{R}$, then the relation $\alpha^n = 1$ obtain $\alpha = \pm 1$. If $\alpha = 1$ then $A = I = [BC]^n$ with $B = C = I \in UT^{(n)}(\mathbb{R})$ and its obviously product of commutators. In the case that $\alpha = -1$ then we observe that

$$\alpha I_n = (-I_2) \oplus (-I_2) \oplus \cdots \oplus (-I_2),$$

and

$$-I_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

In $GL_n(\mathbb{R})$ both matrices are similar to $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ respectively and by the Corollary 3.9 each one is product of $2k - 3$ commutators.
Finally, suppose that $K = \mathbb{C}$. Assume that $n \geq 4$. Decompose $\alpha I_n$ as

$$\alpha I_n = \text{diag}(1, 1, \alpha^2, \alpha^2, \ldots, \alpha^{(n-2)}, \alpha^{(n-2)})$$

$$\times \text{diag}(\alpha^{(n+1)}, \alpha^{(n+1)}, \alpha^{(n-1)}, \alpha^{(n-1)}, \alpha^3, \alpha^3),$$

observe that each sub matrix included in this decomposition can be written in the form

$$\begin{bmatrix} \alpha^s & 0 \\ 0 & \alpha^s \end{bmatrix} = \left( \begin{bmatrix} \alpha^{s/n} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \alpha^{s/n} \end{bmatrix} \right)^n$$

and by the Corollary 3.9 this matrix is a product of $2n - 3$ commutators for each $s = 0, 1, \ldots, n + 1$. Therefore, by the decomposition, we conclude that $\alpha I_n$ is a product of $4n - 6$ commutators. The conclusion of the Theorem has been verified.

Consider the $GL_{V,K}(\infty, K)$ the Vershik-Kerov group consisting of all infinite matrices of the form

$$\begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix}$$

where $M_1 \in GL(n, K)$ for some $n \in \mathbb{N}$ and $M_3 \in T_\infty(K)$.

And we can use the following lemma (see [4]):

**Lemma 4.1.** Assume that $K$ is a complex field or the real number field. Let $A \in GL_n(K)$ of which 1 is no eigenvalue, and let $T$ be an infinite unitriangular matrix. In the Vershik-Kerov group, any matrix of the form

$$\begin{pmatrix} A & B \\ 0 & T \end{pmatrix}$$

is conjugated to

$$\begin{pmatrix} A & 0 \\ 0 & T \end{pmatrix}$$

Then, we shown the Theorem 1.3.

**Proof of Theorem 1.3.** Consider $M \in SL_{V,K}(K)$ in the form $M = \begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix}$, with $M_1 \in SL_n(K)$ and $M_3 \in UT_\infty(K)$. From the proof of Theorem 1.3 in [4], $M$ is conjugated to an infinite matrix of the form $\begin{pmatrix} A & 0 \\ 0 & T \end{pmatrix}$, with $A \in SL_n(K)$ for which 1 is no eigenvalue and $T \in UT_\infty(K)$. By the Theorem 1.1 and the Theorem 1.2 both are products of $4k - 6$ commutators of elements of order $k$ and we know that the direct sum of $A$ and $T$ is also a product of $4k - 6$ commutators of elements of order $k$ then this shows that $M$ is also a product of $4k - 6$ commutators of elements of order $k$. \qed
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