Some cosmological aspects of Hořava-Lifshitz gravity: integrable and nonintegrable models

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Abstract

In this work, some new integrable and nonintegrable cosmological models of the Hořava-Lifshitz gravity are proposed. For some of them, exact solutions are presented. Then these results extend for the F(R) Hořava-Lifshitz gravity theory case. In particular, several integrable cosmological models of this modified gravity theory were constructed in the explicit form.

1 Introduction

More than one year ago Hořava proposed a theory, the so-called Hořava-Lifshitz quantum gravity, which is a power-counting renormalizable theory with consistent ultra-violet behavior [1]. In this theory, the scaling of the gravitational system at short distances exhibits a strong anisotropy between space and time

\[ x^i \rightarrow bx^i, \quad t \rightarrow b^z t. \]  

(1.1)

This important relation between space and time coordinates can be realized with the Arnowitt-Deser-Misner decomposition of the metric of the form

\[ ds^2 = -L^2 dt^2 + g_{ij}(dx^i + L_i dt)(dx^j + L_j dt), \]  

(1.2)

where \( g_{ij} \) is the spatial metric (roman letters indicate spatial indices), \( L \) and \( L_i \) are the lapse and shift functions, respectively. Recently the F(R) Hořava-Lifshitz quantum gravity has been proposed [2] (see also [3]-[8]). In this work, we consider integrable aspects of the usual and F(R) modified Hořava-Lifshitz gravity theories (see e.g. [10]-[11]).

The paper is organized as follows. In section 2, we study some integrable and nonintegrable FRW cosmological models of Hořava-Lifshitz gravity. In the next section 3, we extend some of these results for the modified F(R) Hořava-Lifshitz quantum gravity case and present its some exact integrable models. Exact solutions of some integrable models of the usual Hořava-Lifshitz gravity were considered in the section 4. Section 5 is devoted to the conclusion.

2 Cosmological models of Hořava-Lifshitz gravity

In this work, we restrict ourselves to the detailed balance case. In this case, the action of Hořava-Lifshitz gravity takes the form (\( z = 3 \)):

\[ S = \int dt d^3x \sqrt{\mathcal{N}} \left( \frac{\kappa^2}{2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2}{2w^4} C_{ij} C^{ij} - \frac{\kappa^2 \mu \epsilon^{ijk}}{2w^2 \sqrt{3}} R_{il} \nabla_j R^l_k \right), \]

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\[
\frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} - \frac{\kappa^2 \mu^2}{8(3\lambda - 1)} (1 - \frac{4\lambda}{3}) R^2 + \Lambda R - 3\Lambda^2), \tag{2.1}
\]

where \( \nabla_i \) are the covariant derivatives defined with respect to the spatial metric \( g_{ij} \), \( \epsilon^{ijk} \) is the totally antisymmetric unit tensor and \( \kappa, \lambda, \mu, w = \text{consts} \). Here \( K_{ij} \) and \( C^{ij} \) are the extrinsic curvature and the Cotton tensor, respectively, which are given by

\[
K_{ij} = 0.5 L^{-1} (g_{ij} - \nabla_i L_j - \nabla_j L_i), \quad C^{ij} = g^{-0.5} \epsilon^{ijk} \nabla_k (R^j_i - 0.25 R \delta^j_i). \tag{2.2}
\]

As our main interest in this work is the cosmological aspects of HL gravity, we impose the projectability condition. Consider FRW spacetime with the scale factor \( a(t) \) and metric

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \tag{2.3}
\]

that is

\[
L = 1, \quad g_{ij} = a^2(t) \gamma_{ij}, \quad L^i = 0, \tag{2.4}
\]

where \( k \) can take any value but it is related to \((-0, +) \) curvatures according to sign. In this case, the Friedmann equations we can write in the \( H \)-form

\[
p = -\alpha(2\dot{H} + 3H^2) + f_1(a), \tag{2.5}
\]

\[
\rho = 3\alpha H^2 + f_2(a), \tag{2.6}
\]

\[
\dot{\rho} = -3H(\rho + p) \tag{2.7}
\]

or in the \( N \)-form

\[
p = -\alpha(2\dot{N} + 3\dot{N}^2) + f_1(a), \tag{2.8}
\]

\[
\rho = 3\alpha \dot{N}^2 + f_2(a), \tag{2.9}
\]

\[
\dot{\rho} = -3\dot{N}(\rho + p). \tag{2.10}
\]

Here \( \dot{N} = \ln a \), \( H = \dot{N} \) and

\[
f_1 = \beta_0 + \beta_2 a^{-2} + \beta_4 a^{-4}, \quad f_2 = \eta_0 + \eta_2 a^{-2} + \eta_4 a^{-4}, \tag{2.11}
\]

where \( \alpha = 2\kappa^{-2}(3\lambda - 1) \) and

\[
\beta_0 = 0.75 \mu^2 \Lambda^2 \alpha^{-1}, \quad \beta_2 = -0.5k \mu^2 \Lambda \alpha^{-1}, \quad \beta_4 = -0.25 \mu^2 k^2 \alpha^{-1}, \tag{2.12}
\]

\[
\eta_0 = -0.75 \mu^2 \Lambda^2 \alpha^{-1}, \quad \eta_2 = 1.5k \mu^2 \Lambda \alpha^{-1} = -3\beta_2, \quad \eta_4 = 0.75 \mu^2 k^2 \alpha^{-1} = -3\beta_4. \tag{2.13}
\]

In the next sections we will study some integrable aspects of the Friedmann equations in the \( H \)-form (2.5)-(2.7) or in the \( N \)-form (2.8)-(2.10) and try solve some of them.

### 2.1 Integrable models

Let us start from the presentation of some examples of integrable HL cosmological models. We assume that \( N, a, H \) satisfy one of Painlevé equations so that we obtain 30 new integrable HL models. Consider examples [below \( \alpha, \beta, \gamma, \delta, \kappa \) and \( \mu \) are arbitrary constants] (see also [10]-[11]).

1) \( P_T \) - models. In our cosmological case, this model as and other \( P \)-type models have 5 particular submodels that means 5 type cosmological models. For this case we have:

i) \( P_{IA} \) - model. The \( P_{IA} \) - model we write in the following closed form

\[
p = -\alpha(2\dot{H} + 3H^2) + f_1(a), \tag{2.14}
\]

\[
\rho = 3\alpha H^2 + f_2(a), \tag{2.15}
\]

\[
\dot{N} = 6\dot{N}^2 + t, \tag{2.16}
\]

\[
\dot{\rho} = -3H(\rho + p). \tag{2.17}
\]
or
\[
\begin{align*}
p &= -\alpha(2\ddot{N} + 3\dot{N}^2) + f_1(a), \\
\dot{\rho} &= 3\dot{N}^2 + f_2(a), \\
\ddot{N} &= 6\dot{N}^2 + t, \\
\dot{p} &= -3\ddot{N}(\rho + p).
\end{align*}
\] (2.18)

We believe that the system (2.14)-(2.17) [or its equivalent (2.18)-(2.21)] is integrable. Note that the case \(\alpha = 1, \ f_i = 0\) corresponds to General Relativity (GR). Keeping in mind that the PIA-model has the form (2.14)-(2.17) [or equivalently (2.18)-(2.21)], for short, we here write it as
\[
\ddot{N} = 6\dot{N}^2 + t. \tag{2.22}
\]

We also note that in the systems (2.14)-(2.17) or (2.18)-(2.21), the equation (2.22) plays the role of the EoS \(p = p(\rho)\). Finally we would like to note that similarly the other models we below write in the same short form as (2.22).

2) PII - models.

i) PIIA - model:
\[
\ddot{N} = 2\dot{N}^3 + tN + \nu. \tag{2.27}
\]

ii) PIIIB - model:
\[
\ddot{a} = 2a^3 + ta + \nu. \tag{2.28}
\]

iii) PIIIC - model:
\[
\ddot{H} = 2H^3 + tH + \nu. \tag{2.29}
\]

iv) PIIID - model:
\[
H_{aa} = 2H^3 + aH + \nu. \tag{2.30}
\]

v) PIIIE - model:
\[
H_{NN} = 2H^3 + NH + \nu. \tag{2.31}
\]

3) PIII - models.

i) PIIIA - model:
\[
\ddot{N} = \frac{1}{N}\ddot{N}^2 - \frac{1}{t}(\ddot{N} - \alpha N^2 - \beta) + \gamma N^3 + \frac{\delta}{N}. \tag{2.32}
\]

ii) PIIIB - model:
\[
\ddot{a} = \frac{1}{a}\ddot{a}^2 - \frac{1}{t}(\ddot{a} - \alpha a^2 - \beta) + \gamma a^3 + \frac{\delta}{a}. \tag{2.33}
\]

iii) PIIIC - model:
\[
\ddot{H} = \frac{1}{H}\ddot{H}^2 - \frac{1}{t}(\ddot{H} - \alpha H^2 - \beta) + \gamma H^3 + \frac{\delta}{H}. \tag{2.34}
\]

iv) PIIID - model:
\[
H_{aa} = \frac{1}{H}H_{aa}^2 - \frac{1}{a}(H_{aa} - \alpha H^2 - \beta) + \gamma H^3 + \frac{\delta}{H}. \tag{2.35}
\]

v) PIIIE - model:
\[
H_{NN} = \frac{1}{H}H_{NN}^2 - \frac{1}{N}(H_{NN} - \alpha H^2 - \beta) + \gamma H^3 + \frac{\delta}{H}. \tag{2.36}
\]
4) $P_{IV}$ - models.
i) $P_{IVA}$ - model:
\[ \dot{N} = \frac{1}{2N} \dot{N}^2 + 1.5N^3 + 4tN^2 + 2(t^2 - \alpha)N + \frac{\delta N}{N} \] (2.37)

ii) $P_{IVB}$ - model:
\[ \ddot{a} = \frac{1}{2a} \dot{a}^2 + 1.5a^3 + 4ta^2 + 2(t^2 - \alpha)a + \frac{\delta a}{a} \] (2.38)

iii) $P_{IVC}$ - model:
\[ \ddot{H} = \frac{1}{2H} \dot{H}^2 + 1.5H^3 + 4tH^2 + 2(t^2 - \alpha)H + \frac{\delta H}{H} \] (2.39)

iv) $P_{IVD}$ - model:
\[ H_{aa} = \frac{1}{2H} H_a^2 + 1.5H^3 + 4aH^2 + 2(a^2 - \alpha)H + \frac{\delta \alpha}{\alpha} \] (2.40)

v) $P_{IVE}$ - model:
\[ H_{NN} = \frac{1}{2H} H_N^2 + 1.5H^3 + 4N^2H^2 + 2(N^2 - \alpha)H + \frac{\delta \alpha}{\alpha} \] (2.41)

5) $P_{V}$ - models.
i) $P_{VA}$ - model:
\[ \ddot{N} = \left( \frac{1}{2N} + \frac{1}{N - 1} \right) \dot{N}^2 - \frac{1}{t} (\dot{N} - \gamma N) + t^{-2} (N - 1)^2 (\alpha N + \beta N^{-1}) + \frac{\delta N(N + 1)}{N - 1} \] (2.42)

ii) $P_{VB}$ - model:
\[ \ddot{a} = \left( \frac{1}{2a} + \frac{1}{a - 1} \right) \dot{a}^2 - \frac{1}{t} (\dot{a} - \gamma a) + t^{-2} (a - 1)^2 (\alpha a + \beta a^{-1}) + \frac{\delta a(a + 1)}{a - 1} \] (2.43)

iii) $P_{VC}$ - model:
\[ \ddot{H} = \left( \frac{1}{2H} + \frac{1}{H - 1} \right) \dot{H}^2 - \frac{1}{t} (\dot{H} - \gamma H) + t^{-2} (H - 1)^2 (\alpha H + \beta H^{-1}) + \frac{\delta H(H + 1)}{H - 1} \] (2.44)

iv) $P_{VD}$ - model:
\[ H_{aa} = \left( \frac{1}{2H} + \frac{1}{H - 1} \right) H_a^2 - \frac{1}{a} (H_a - \gamma H) + a^{-2} (H - 1)^2 (\alpha H + \beta H^{-1}) + \frac{\delta H(H + 1)}{H - 1} \] (2.45)

v) $P_{VE}$ - model:
\[ H_{NN} = \left( \frac{1}{2H} + \frac{1}{H - 1} \right) H_N^2 - \frac{1}{N} (H_N - \gamma H) + N^{-2} (H - 1)^2 (\alpha H + \beta H^{-1}) + \frac{\delta H(H + 1)}{H - 1} \] (2.46)

6) $P_{VI}$ - models.
i) $P_{VIA}$ - model:
\[ \ddot{N} = 0.5 \left( \frac{1}{N} + \frac{1}{N - 1} + \frac{1}{N - t} \right) \dot{N}^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{N - t} \right) \dot{N} \]
\[ + t^{-2} (t - 1)^{-2} N(N - 1)(N - t) \left[ \alpha + \beta tN^{-2} + \gamma(t - 1)(N - 1)^{-2} + \delta t(t - 1)(N - t)^{-2} \right] . \] (2.47)

ii) $P_{VIB}$ - model:
\[ \ddot{a} = 0.5 \left( \frac{1}{a} + \frac{1}{a - 1} + \frac{1}{a - t} \right) \dot{a}^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{a - t} \right) \dot{a} \]
\[ + t^{-2} (t - 1)^{-2} a(a - 1)(a - t) \left[ \alpha + \beta a^{-2} + \gamma(t - 1)(a - 1)^{-2} + \delta t(t - 1)(a - t)^{-2} \right] . \] (2.48)

iii) $P_{VIC}$ - model:
\[ \ddot{H} = 0.5 \left( \frac{1}{H} + \frac{1}{H - 1} + \frac{1}{H - t} \right) \dot{H}^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{H - t} \right) \dot{H} \]
+ t^{-2}(t - 1)^{-2}H(H - 1)(H - t) \left[ \alpha + \beta tH^{-2} + \gamma(t - 1)(H - 1)^{-2} + \delta t(t - 1)(H - t)^{-2} \right]. \; (2.49)

iv) P_{VID} - model:
\[ H_{aa} = 0.5 \left( \frac{1}{H} + \frac{1}{H - 1} + \frac{1}{H^- a} \right) H_a^2 - \left( \frac{1}{a} + \frac{1}{a - 1} + \frac{1}{H^- a} \right) H_a \]
+ a^{-2}(a - 1)^{-2}H(H - 1)(H - a) \left[ \alpha + \beta aH^{-2} + \gamma(a - 1)(H - 1)^{-2} + \delta a(a - 1)(H - a)^{-2} \right]. \; (2.50)

v) P_{VIE} - model:
\[ H_{NN} = 0.5 \left( \frac{1}{H} + \frac{1}{H - 1} + \frac{1}{H^- N} \right) H_N^2 - \left( \frac{1}{N} + \frac{1}{N - 1} + \frac{1}{H^- N} \right) H_N \]
+ N^{-2}(N - 1)^{-2}H(H - 1)(H - N) \left[ \alpha + \beta NH^{-2} + \gamma(N - 1)(H - 1)^{-2} + \delta N(N - 1)(H - N)^{-2} \right]. \; (2.51)

Note that all Painlevé equations can be represented as Hamiltonian systems that is as (see e.g. \cite{10} and references therein)
\[
\dot{q} = \frac{\partial E}{\partial r}, \quad (2.52)
\]
\[
\dot{r} = -\frac{\partial E}{\partial q}, \quad (2.53)
\]

where \( E(q, r, t) \) is the (non-autonomous) Hamiltonian function. Consider some examples (see e.g. \cite{10} and references therein).

1) \( P_I \)-models. In this case, \( q, r, F \) read as
\[
\dot{q} = r, \quad (2.54)
\]
\[
\dot{r} = 6q^2 + t, \quad (2.55)
\]
\[
E = 0.5r^2 - 2q^3 - tq. \quad (2.56)
\]

2) \( P_{II} \)-models. In this case we have
\[
\dot{q} = r - q^2 - 0.5t, \quad (2.57)
\]
\[
\dot{r} = 2qr + \alpha + 0.5, \quad (2.58)
\]
\[
E = 0.5r^2 - (q^2 + 0.5t)r - (\alpha + 0.5)q. \quad (2.59)
\]

3) \( P_{III} \)-models. In this case we have
\[
t\dot{q} = 2q^2r - k_2tq^2 - (2\theta_1 + 1)q + k_1t, \quad (2.60)
\]
\[
t\dot{r} = -2qr^2 + 2k_2tqr + (2\theta_1 + 1)r - k_2((\theta_1 + \theta_2)t, \quad (2.61)
\]
\[
tE = q^2r^2 - [k_2tq^2 + (2\theta_1 + 1)q - k_1t]r + k_2(\theta_1 + \theta_2)tq. \quad (2.62)
\]

So in this subsection we presented 24 new HL cosmological models. Note that as integrable systems, these models admit n-soliton solutions, infinite number commuting integrals of motion, Lax representations etc.

### 2.2 Nonintegrable models

Here we consider some known and new HL models induced by some ODEs. These ODEs are nonintegrable so that the corresponding HL cosmological models are nonintegrable (see also \cite{11}).

1) \( \Lambda_{CDM} \) cosmology. We start with the \( \Lambda_{CDM} \) cosmology. Here we present 5 submodels [We remark that in fact only the \( \Lambda_1 \) - model corresponds to \( \Lambda_{CDM} \) cosmology],

i) \( \Lambda_1 \) - model:
\[
\dot{N} = 0.5\Lambda - 1.5N^2. \quad (2.63)
\]

ii) \( \Lambda_2 \) - model:
\[
\ddot{a} = 0.5\Lambda - 1.5\dot{a}^2. \quad (2.64)
\]
iii) $\Lambda_3$ - model: $$\ddot{H} = 0.5\Lambda - 1.5\dot{H}^2. \quad (2.65)$$

iv) $\Lambda_4$ - model: $$H_{aa} = 0.5\Lambda - 1.5H_a^2. \quad (2.66)$$

v) $\Lambda_5$ - model: $$H_{NN} = 0.5\Lambda - 1.5H_N^2. \quad (2.67)$$

2) **Pinney cosmology.** It induced by the Pinney equation. Let us present 5 submodels.

i) Pinney$_1$ - model: $$\ddot{N} = \xi(t)N + \frac{k}{N^3}, \quad (2.68)$$

where $\xi = \xi(t), \quad k = const.$

ii) Pinney$_2$ - model: $$\ddot{a} = \xi(t)a + \frac{k}{a^3}. \quad (2.69)$$

iii) Pinney$_3$ - model: $$\ddot{H} = \xi(t)H + \frac{k}{H^3}. \quad (2.70)$$

iv) Pinney$_4$ - model: $$H_{aa} = \xi(a)H + \frac{k}{H^3}. \quad (2.71)$$

v) Pinney$_5$ - model: $$H_{NN} = \xi(N)H + \frac{k}{H^3}. \quad (2.72)$$

3) **Schrödinger cosmology.** For this model also write 5 submodels.

i) Schrödinger$_1$ - model: $$\ddot{N} = uN + kN, \quad (2.73)$$

where $u = u(t), \quad k = const.$

ii) Schrödinger$_2$ - model: $$\ddot{a} = ua + ka. \quad (2.74)$$

iii) Schrödinger$_3$ - model: $$\ddot{H} = uH + kH. \quad (2.75)$$

iv) Schrödinger$_4$ - model: $$H_{aa} = uH + kH. \quad (2.76)$$

v) Schrödinger$_5$ - model: $$H_{NN} = uH + kH. \quad (2.77)$$

4) **Hypergeometric cosmology.** Let us we present 5 submodels.

i) $H_1$ - model: $$\ddot{N} = t^{-1}(1-t)^{-1} \{(\alpha + \beta + 1)t - \gamma\} \dot{N} + \alpha\beta N. \quad (2.78)$$

ii) $H_2$ - model: $$\ddot{a} = t^{-1}(1-t)^{-1} \{(\alpha + \beta + 1)t - \gamma\} \dot{a} + \alpha\beta a. \quad (2.79)$$

iii) $H_3$ - model: $$\ddot{H} = t^{-1}(1-t)^{-1} \{(\alpha + \beta + 1)t - \gamma\} \dot{H} + \alpha\beta H. \quad (2.80)$$

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iv) \( H_4 \) - model:

\[
H_{aa} = a^{-1}(1 - a)^{-1}\{[(\alpha + \beta + 1)a - \gamma]H_a + \alpha \beta H}\]. \hspace{1cm} (2.81)

v) \( H_5 \) - model:

\[
H_{NN} = N^{-1}(1 - N)^{-1}\{[(\alpha + \beta + 1)N - \gamma]H_N + \alpha \beta H}\]. \hspace{1cm} (2.82)

Finally we would like to note that the above presented HL models, in particular, have some solutions which describe the accelerated expansion of the universe.

3 Cosmological models of \( F(R) \) Horava-Lifshitz gravity

Let us now consider the \( F(R) \) Horava-Lifshitz gravity. Following [2], its action we write as

\[
S = 0.5\kappa^{-2} \int dt d^3x \sqrt{g} \mathcal{F}(\tilde{R}),
\]

where \( \kappa^2 = 16\pi G \) and

\[
\tilde{R} = K_{ij}K^{ij} - \lambda K^2 + R + 2\mu \nabla_\mu (n^\nu n_\nu - n^\nu \nabla_\nu n^\mu) - L(3)g_{ij}.
\]

The case \( \mathcal{F}(\tilde{R}) = \tilde{R} \) corresponds to the original Horava-Lifshitz gravity. In this section, we construct integrable and nonintegrable \( F(R) \) Horava-Lifshitz gravity models induced by some ODSs.

For the metric

\[
ds^2 = -L^2 dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),
\]

the scalar \( \tilde{R} \) takes the form

\[
\tilde{R} = 3(1 - 3\lambda + 6\mu)H^2L^{-2} + 6\mu H^{-1} \ln(HL^{-1})t.
\]

In the FRW case and as \( L = 1 \), the equations of motion read as \([2]\)

\[
0 = F - 2(1 - 3\lambda + 3\mu)(\dot{H} + 3H^2)F' - 2(1 - 3\lambda)\tilde{R}F'' + 2\mu \tilde{R}^2 F''' + \kappa^2 p_m, \hspace{1cm} (3.5)
\]

\[
0 = F - 6[(1 - 3\lambda + 3\mu)H^2 + \mu \dot{H}]F' + 6\mu H \tilde{R}F'' - \kappa^2 \rho_m,
\]

where \( p_m \) and \( \rho_m \) are the pressure and energy density of a perfect fluid that fills the Universe.

3.1 Integrable models

Some classes integrable models can be constructed with the help of known integrable ODEs. It is the main idea of the work (see also [11]). One of famous representatives of such integrable ODEs are Painlevé equations. It is the case that we are going to use to get integrable Horava-Lifshitz gravity models. Let us demonstrate it. The lazy way to do it is the assumption that the function \( \mathcal{F}(\tilde{R}) \) or its sister \( f(\tilde{R}) = \mathcal{F}(\tilde{R}) - \tilde{R} \) satisfy some integrable ODEs, in our case, one of Painlevé equations. Note that there are 6 integrable Painlevé equations with each of them can be relate 4 integrable \( F(\tilde{R}) \) Horava-Lifshitz models so that totally we have 30 models. Now we are in the position to present these equations (or models) \([F' \equiv dF/d\tilde{R}, \ F' \equiv dF/dt \text{ etc}].\)

1) \( F_I(\tilde{R}) \) - models.

a) \( F_{IA}(\tilde{R}) \) - model:

\[
f'' = 6f^2 + \tilde{R}
\]

or

\[
F'' = 6F^2 + \tilde{R}.
\]

b) \( F_{IB}(\tilde{R}) \) - model:

\[
f = 6f^2 + t
\]

or

\[
\bar{F} = 6F^2 + t.
\]
c) $F_{IC}(R)$ - model:

\[ f_{aa} = 6f^2 + a \]  \hspace{1cm} (3.11)

or

\[ F_{aa} = 6F^2 + a. \]  \hspace{1cm} (3.12)

d) $F_{ID}(R)$ - model:

\[ f_{NN} = 6f^2 + N \]  \hspace{1cm} (3.13)

or

\[ F_{NN} = 6F^2 + N. \]  \hspace{1cm} (3.14)

e) $F_{IE}(R)$ - model:

\[ f_{HH} = 6f^2 + H \]  \hspace{1cm} (3.15)

or

\[ F_{HH} = 6F^2 + H. \]  \hspace{1cm} (3.16)

2) $F_{II}(R)$ - model.

a) $F_{IIA}(R)$ - model:

\[ f'' = 2f^3 + \tilde{R}f + \alpha \]  \hspace{1cm} (3.17)

or

\[ F'' = 2F^3 + \tilde{R}F + \alpha. \]  \hspace{1cm} (3.18)

b) $F_{IIB}(R)$ - models:

\[ \ddot{f} = 2f^3 + tf + \alpha \]  \hspace{1cm} (3.19)

or

\[ \ddot{F} = 2F^3 + tF + \alpha. \]  \hspace{1cm} (3.20)

c) $F_{IIC}(R)$ - models:

\[ f_{aa} = 2f^3 + af + \alpha \]  \hspace{1cm} (3.21)

or

\[ F_{NN} = 2F^3 + NF + \alpha. \]  \hspace{1cm} (3.22)

d) $F_{IID}(R)$ - models:

\[ f_{NN} = 2f^3 + NF + \alpha \]  \hspace{1cm} (3.23)

or

\[ F_{NN} = 2F^3 + NF + \alpha. \]  \hspace{1cm} (3.24)

e) $F_{IIE}(R)$ - models:

\[ f_{HH} = 2f^3 + HF + \alpha \]  \hspace{1cm} (3.25)

or

\[ F_{HH} = 2F^3 + HF + \alpha. \]  \hspace{1cm} (3.26)

3) $F_{III}(R)$ - models.

a) $F_{IIIA}(R)$ - model:

\[ f'' = \frac{1}{f} f'^2 - \frac{1}{R} (f' - \alpha f^2 - \beta) + \gamma f^3 + \frac{\delta}{f} \]  \hspace{1cm} (3.27)

or

\[ F'' = \frac{1}{F} F' f^2 - \frac{1}{R} (F' - \alpha F^2 - \beta) + \gamma F^3 + \frac{\delta}{F}. \]  \hspace{1cm} (3.28)
b) $F_{III B} (R)$ - model:
\[ \ddot{f} = \frac{1}{f} f^2 - \frac{1}{f} (\dot{f} - \alpha f^2) + \frac{1}{f} (\ddot{f} - \dot{f}^2 - \beta) + \frac{1}{f} (f^3 + \delta) \] (3.29)
or
\[ \ddot{F} = \frac{1}{F} F^2 - \frac{1}{F} (\dot{F} - \alpha F^2) + \frac{1}{F} (\ddot{F} - \dot{F}^2 - \beta) + \frac{1}{F} (F^3 + \delta) \] (3.30)

c) $F_{III C} (R)$ - model:
\[ \ddot{f}_{aa} = \frac{1}{f} f^2 - \frac{1}{f} (f_{aa} - \alpha f^2 - \beta) + \frac{1}{f} (f^3 + \delta) \] (3.31)
or
\[ \ddot{F}_{aa} = \frac{1}{F} F^2 - \frac{1}{F} (F_{aa} - \alpha F^2 - \beta) + \frac{1}{F} (F^3 + \delta) \] (3.32)

d) $F_{III D} (R)$ - model:
\[ \ddot{f} = \frac{1}{f} f^2 - \frac{1}{f} (\dot{f} - \alpha f^2) + \frac{1}{f} (\ddot{f} - \dot{f}^2 - \beta) + \frac{1}{f} (f^3 + \delta) \] (3.33)
or
\[ \ddot{F} = \frac{1}{F} F^2 - \frac{1}{F} (\dot{F} - \alpha F^2) + \frac{1}{F} (\ddot{F} - \dot{F}^2 - \beta) + \frac{1}{F} (F^3 + \delta) \] (3.34)

e) $F_{III E} (R)$ - model:
\[ \ddot{f} = \frac{1}{f} f^2 - \frac{1}{f} (\dot{f} - \alpha f^2) + \frac{1}{f} (\ddot{f} - \dot{f}^2 - \beta) + \frac{1}{f} (f^3 + \delta) \] (3.35)
or
\[ \ddot{F} = \frac{1}{F} F^2 - \frac{1}{F} (\dot{F} - \alpha F^2) + \frac{1}{F} (\ddot{F} - \dot{F}^2 - \beta) + \frac{1}{F} (F^3 + \delta) \] (3.36)

4) $F_{IV} (R)$ - models.

a) $F_{IVA} (R)$ - model:
\[ \dddot{f} = \frac{1}{f} f^2 + 1.5 f^3 + 2 \dot{f} - \alpha f^2 + 2 (\dot{f} - \alpha)^2 f + \frac{1}{f} \delta \] (3.37)
or
\[ \dddot{F} = \frac{1}{F} F^2 + 1.5 F^3 + 2 \dot{F} - \alpha F^2 + 2 (\dot{F} - \alpha)^2 F + \frac{1}{F} \delta \] (3.38)

b) $F_{IVB} (R)$ - model:
\[ \dddot{f} = \frac{1}{f} f^2 + 1.5 f^3 + 4 \dot{f} - \alpha f^2 + 2 (\dot{f} - \alpha)^2 f + \frac{1}{f} \delta \] (3.39)
or
\[ \dddot{F} = \frac{1}{F} F^2 + 1.5 F^3 + 4 \dot{F} - \alpha F^2 + 2 (\dot{F} - \alpha)^2 F + \frac{1}{F} \delta \] (3.40)

c) $F_{IVC} (R)$ - model:
\[ \dddot{f}_{aa} = \frac{1}{f} f^2 + 1.5 f^3 + 4 a \dot{f} - \alpha f^2 + 2 (a^2 - \alpha)^2 f + \frac{1}{f} \delta \] (3.41)
or
\[ \dddot{F}_{aa} = \frac{1}{F} F^2 + 1.5 F^3 + 4 a \dot{F} - \alpha F^2 + 2 (a^2 - \alpha)^2 F + \frac{1}{F} \delta \] (3.42)

d) $F_{IVD} (R)$ - model:
\[ \dddot{f}_{NN} = \frac{1}{f} f^2 + 1.5 f^3 + 4 N \dot{f} - \alpha f^2 + 2 (N^2 - \alpha)^2 f + \frac{1}{f} \delta \] (3.43)
or
\[ \dddot{F}_{NN} = \frac{1}{F} F^2 + 1.5 F^3 + 4 N \dot{F} - \alpha F^2 + 2 (N^2 - \alpha)^2 F + \frac{1}{F} \delta \] (3.44)

e) $F_{IV E} (R)$ - model:
\[ \dddot{f}_{HH} = \frac{1}{f} f^2 + 1.5 f^3 + 4 H \dot{f} - \alpha f^2 + 2 (H^2 - \alpha)^2 f + \frac{1}{f} \delta \] (3.45)
or

\[ F_{HH} = \frac{1}{2F} F'^2_h + 1.5 F^3 + 4H F^2 + 2(H^2 - \alpha) F + \delta F. \]  

(3.46)

5) \( F_V(R) - \) models.

a) \( F_{VA}(R) \) - model:

\[ f'' = (\frac{1}{2f} + \frac{1}{f-1}) f'^2 - \frac{1}{R} (f' - \gamma f) + \tilde{R}^{-2}(f-1)^2(\alpha f + \beta f^{-1}) + \frac{\delta f(f+1)}{f-1}. \]

(3.47)

\[ F'' = (\frac{1}{2F} + \frac{1}{F-1}) F'^2 - \frac{1}{R} (f' - \gamma F) + \tilde{R}^{-2}(F-1)^2(\alpha F + \beta F^{-1}) + \frac{\delta F(F+1)}{F-1}. \]

(3.48)

b) \( F_{VB}(R) \) - model:

\[ \tilde{f} = (\frac{1}{2f} + \frac{1}{f-1}) \tilde{f}'^2 - \frac{1}{t} (\tilde{f}' - \gamma f) + t^{-2}(f-1)^2(\alpha f + \beta f^{-1}) + \frac{\delta f(f+1)}{f-1}. \]

(3.49)

\[ \tilde{F} = (\frac{1}{2F} + \frac{1}{F-1}) \tilde{F}'^2 - \frac{1}{t} (\tilde{F}' - \gamma F) + t^{-2}(F-1)^2(\alpha F + \beta F^{-1}) + \frac{\delta F(F+1)}{F-1}. \]

(3.50)

c) \( F_{VC}(R) \) - model:

\[ f_{aa} = (\frac{1}{2f} + \frac{1}{f-1}) f_{aa}'^2 - \frac{1}{a} (f_a - \gamma f) + a^{-2}(f-1)^2(\alpha f + \beta f^{-1}) + \frac{\delta f(f+1)}{f-1}. \]

(3.51)

\[ F_{aa} = (\frac{1}{2F} + \frac{1}{F-1}) F_{aa}'^2 - \frac{1}{a} (F_a - \gamma F) + a^{-2}(F-1)^2(\alpha F + \beta F^{-1}) + \frac{\delta F(F+1)}{F-1}. \]

(3.52)

d) \( F_{VD}(R) \) - model:

\[ f_{NN} = (\frac{1}{2f} + \frac{1}{f-1}) f_{NN}'^2 - \frac{1}{N} (f_N - \gamma f) + a^{-2}(f-1)^2(\alpha f + \beta f^{-1}) + \frac{\delta f(f+1)}{f-1}. \]

(3.53)

\[ F_{NN} = (\frac{1}{2F} + \frac{1}{F-1}) F_{NN}'^2 - \frac{1}{N} (F_N - \gamma F) + N^{-2}(F-1)^2(\alpha F + \beta F^{-1}) + \frac{\delta F(F+1)}{F-1}. \]

(3.54)

e) \( F_{VE}(R) \) - model:

\[ f_{HH} = (\frac{1}{2f} + \frac{1}{f-1}) f_{HH}'^2 - \frac{1}{H} (f_H - \gamma f) + H^{-2}(f-1)^2(\alpha f + \beta f^{-1}) + \frac{\delta f(f+1)}{f-1}. \]

(3.55)

\[ F_{HH} = (\frac{1}{2F} + \frac{1}{F-1}) F_{HH}'^2 - \frac{1}{H} (H_H - \gamma H) + H^{-2}(H-1)^2(\alpha H + \beta H^{-1}) + \frac{\delta F(F+1)}{F-1}. \]

(3.56)

6) \( F_{V1}(R) \) - models.

a) \( F_{V1A}(R) \) - model:

\[ f'' = 0.5 \left( \frac{1}{f} + \frac{1}{f-1} + \frac{1}{f-R} \right) f'^2 - \left( \frac{1}{R} + \frac{1}{R-1} + \frac{1}{f-R} \right) f' + \tilde{R}^{-2}(\tilde{R}-1)^{-2} f(f-1)(f-\tilde{R}) \left[ \alpha + \beta \tilde{R} f^{-2} + \gamma (\tilde{R}-1)(f-1)^{-2} + \delta \tilde{R} (\tilde{R}-1)(f-\tilde{R})^{-2} \right]. \]

(3.57)

Instead of this equation we can consider the following one

\[ F'' = 0.5 \left( \frac{1}{F} + \frac{1}{F-1} + \frac{1}{F-R} \right) F'^2 - \left( \frac{1}{R} + \frac{1}{R-1} + \frac{1}{F-R} \right) F'. \]
\[ + \tilde{R}^{-2}(\tilde{R} - 1)^{-2}F(F - 1)(F - \tilde{R}) \left[ \alpha + \beta \tilde{R}F^{-2} + \gamma(\tilde{R} - 1)(F - 1)^{-2} + \delta \tilde{R}(\tilde{R} - 1)(F - \tilde{R})^{-2} \right]. \quad (3.58) \]

d) \( F_{V1D}(R) \) - model:
\[
\tilde{f} = 0.5 \left( \frac{1}{f} + \frac{1}{F - 1} + \frac{1}{F - t} \right) f^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{F - t} \right) \dot{f}
+ t^{-2}(t - 1)^{-2}f(f(t) - f - t) \left[ \alpha + \beta tf^{-2} + \gamma(f - 1) + \beta(t - 1)(f - t)^{-2} \right]. \quad (3.59)
\]
and
\[
\tilde{F} = 0.5 \left( \frac{1}{F} + \frac{1}{F - 1} + \frac{1}{F - t} \right) \tilde{F}^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{F - t} \right) \tilde{F}
+ t^{-2}(t - 1)^{-2}F(F - 1)(F - t) \left[ \alpha + \beta tF^{-2} + \gamma(t - 1)(F - 1)^{-2} + \delta(t - 1)(F - t)^{-2} \right]. \quad (3.60)
\]

c) \( F_{V1C}(R) \) - model:
\[
f_{aa} = 0.5 \left( \frac{1}{f} + \frac{1}{f - 1} + \frac{1}{f - a} \right) f^2 - \left( \frac{1}{a} + \frac{1}{a - 1} + \frac{1}{f - a} \right) f_a
+ a^{-2}(a - 1)^{-2}f(f - 1)(f - a) \left[ \alpha + \beta a f^{-2} + \gamma(a - 1)(f - 1)^{-2} + \delta a(a - 1)(f - a)^{-2} \right]. \quad (3.61)
\]
and
\[
F_{aa} = 0.5 \left( \frac{1}{F} + \frac{1}{F - 1} + \frac{1}{F - a} \right) F^2 - \left( \frac{1}{a} + \frac{1}{a - 1} + \frac{1}{F - a} \right) F_a
+ a^{-2}(a - 1)^{-2}F(F - 1)(F - a) \left[ \alpha + \beta a F^{-2} + \gamma(a - 1)(F - 1)^{-2} + \delta a(a - 1)(F - a)^{-2} \right]. \quad (3.62)
\]

d) \( F_{V1D}(R) \) - model:
\[
f_{NN} = 0.5 \left( \frac{1}{f} + \frac{1}{f - 1} + \frac{1}{f - N} \right) f^2 - \left( \frac{1}{N} + \frac{1}{N - 1} + \frac{1}{f - N} \right) f_N
+ N^{-2}(N - 1)^{-2}f(f - 1)(f - N) \left[ \alpha + \beta N f^{-2} + \gamma(N - 1)(f - 1)^{-2} + \delta N(N - 1)(f - N)^{-2} \right]. \quad (3.63)
\]
and
\[
F_{NN} = 0.5 \left( \frac{1}{F} + \frac{1}{F - 1} + \frac{1}{F - N} \right) F^2 - \left( \frac{1}{N} + \frac{1}{N - 1} + \frac{1}{F - N} \right) F_N
+ N^{-2}(N - 1)^{-2}F(F - 1)(F - N) \left[ \alpha + \beta N F^{-2} + \gamma(N - 1)(F - 1)^{-2} + \delta N(N - 1)(F - N)^{-2} \right]. \quad (3.64)
\]

e) \( F_{V1E}(R) \) - model:
\[
f_{HH} = 0.5 \left( \frac{1}{f} + \frac{1}{f - 1} + \frac{1}{f - H} \right) f^2 - \left( \frac{1}{H} + \frac{1}{H - 1} + \frac{1}{f - H} \right) f_H
+ H^{-2}(H - 1)^{-2}f(f - 1)(f - H) \left[ \alpha + \beta H f^{-2} + \gamma(H - 1)(f - 1)^{-2} + \delta H(H - 1)(f - H)^{-2} \right]. \quad (3.65)
\]
and
\[
F_{HH} = 0.5 \left( \frac{1}{F} + \frac{1}{F - 1} + \frac{1}{F - H} \right) F^2 - \left( \frac{1}{H} + \frac{1}{H - 1} + \frac{1}{F - H} \right) F_H
+ H^{-2}(H - 1)^{-2}F(F - 1)(F - H) \left[ \alpha + \beta H F^{-2} + \gamma(H - 1)(F - 1)^{-2} + \delta H(H - 1)(F - H)^{-2} \right]. \quad (3.66)
\]

So we presented new 30 HL cosmological \( F_j(R) \) models \( (J = I, II, III, IV, V, VI) \) which are integrable due to integrability of Painlevé equations.

**Exact solutions of integrable HL models.** All above constructed integrable HL models admit (may be infinity number) exact solutions. Let us here present some of them [for simplicity, we give just some particular solutions for the sister function \( f(\tilde{R}) = F(\tilde{R}) - \tilde{R} \) and only for some models] (see e.g. [10]).

1) The \( F_{IIA}(R) \) - model.
i) As our first example, we consider the $F_{IIA}(R)$ - model (3.17). Let us present its some solutions. For example, this model has the following particular solutions:

\[
\begin{align*}
\psi(R) &\equiv \psi(R;1.5) = \psi - (2\psi^2 + R)^{-1}, \\
\phi(R) &\equiv \phi(R;1) = -\frac{1}{R}, \\
\psi(R) &\equiv \psi(R;2) = \frac{1}{R} - \frac{3\hat{R}^2}{R^3 + 4}, \\
\phi(R) &\equiv \phi(R;3) = \frac{3\hat{R}^2}{R^3 + 4} - \frac{6\hat{R}^2(R^3 + 10)}{R^6 + 20R^3 - 80}, \\
\psi(R) &\equiv \psi(R;4) = \frac{1}{R} + \frac{6\hat{R}^2(R^3 + 10)}{R^6 + 20R^3 - 80} - \frac{9\hat{R}^4(R^3 + 40)}{R^9 + 60R^6 + 11200}, \\
\phi(R) &\equiv \phi(R;0.5\epsilon) = -\epsilon\psi.
\end{align*}
\]

and so on. Here

\[
\psi = (\ln \phi)_R, \quad \phi(R) = C_1Ai(-2^{-1/3}R) + C_2Bi(-2^{-1/3}R),
\]

$C_i = \text{consts}$ and $Ai(x)$, $Bi(x)$ are Airy functions.

ii) Similarly, for the $F_{IIB}(R)$ - model (3.19) we have the following particular solutions

\[
\begin{align*}
\psi(R) &\equiv \psi(t;1.5) = \psi - (2\psi^2 + t)^{-1}, \\
\phi(R) &\equiv \phi(t;1) = -\frac{1}{t}, \\
\psi(R) &\equiv \psi(t;2) = \frac{1}{R} - \frac{3t^2}{t^3 + 4}, \\
\phi(R) &\equiv \phi(t;3) = \frac{3t^2}{t^3 + 4} - \frac{6t^2(t^3 + 10)}{t^6 + 20t^3 - 80}, \\
\psi(R) &\equiv \psi(t;4) = \frac{1}{t} + \frac{6t^2(t^3 + 10)}{t^6 + 20t^3 - 80} - \frac{9t^4(t^3 + 40)}{t^9 + 60t^6 + 11200}, \\
\phi(R) &\equiv \phi(t;0.5\epsilon) = -\epsilon\psi.
\end{align*}
\]

and so on. Here

\[
\psi = (\ln \phi)_t, \quad \phi(t) = C_1Ai(-2^{-1/3}t) + C_2Bi(-2^{-1/3}t).
\]

2) The $F_{IIA}(R)$ - model.

i) Our next example is the $F_{IIA}(R)$ - model (3.27). It has the following particular solutions:

\[
\begin{align*}
\psi(R) &\equiv \psi(R;\nu_1,0,0,-\nu_1\nu_2^2) = \nu_2\sqrt{\hat{R}}, \\
\phi(R) &\equiv \phi(R;0,-2\nu_1,0,4\nu_1\nu_2 - \nu_2^2) = \hat{R}[\ln(\hat{R}^{\nu_1})^2 + \ln(\hat{R}^{\nu_2}e^{\nu_2})], \\
\psi(R) &\equiv \psi(R;\nu_1^2,\nu_2,0,\nu_2^2(\nu_2^2 - \nu_3\nu_4),0) = \frac{\hat{R}^{\nu_1 - 1}}{\nu_3\hat{R}^{2\nu_1} + \nu_2\hat{R}^{2\nu_1} + \nu_4}, \\
\phi(R) &\equiv \phi(R;2\nu_1 + 3,-2\nu_1 + 1,1,-1) = \frac{\hat{R} + \nu_1}{\hat{R} + \nu_1 + 1}, \\
\psi(R) &\equiv \psi(R;\nu_1,-\nu_1\nu_2^2,\nu_2,-\nu_2\nu_4^2) = \nu_2, \\
\phi(R) &\equiv \phi(R;\nu_1,\nu_2,\nu_3,\nu_4) = -\epsilon_1(\ln \phi)_R
\end{align*}
\]

and so on. Here

\[
\varphi(R) = \hat{R}^{\nu}[C_1J_{\nu}(\sqrt{\epsilon_1\epsilon_2\hat{R}}) + C_2Y_{\nu}(\sqrt{\epsilon_1\epsilon_2\hat{R}})], \quad (\epsilon_1, C_1 = \text{consts}),
\]

and $J_{\nu}(x), Y_{\nu}(x)$ are Bessel functions.
IIIB

Let us here present some known and new HL models induced by some ODEs. These ODEs are nonintegrable so that the corresponding HL cosmological models are nonintegrable. Consider examples. 

i) We start from models induced by the hypergeometric differential equation. We write here 5 versions of this model.

R-version:

\[ \ddot{R}(1 - \dot{R})f'' + [c - (\alpha + b + 1)\dot{R}]f' - \alpha bf = 0. \]  

(3.85)

It has the solution \( f(\dot{R}) = 2F(\alpha, b; c; \dot{R}) \) which is the hypergeometric function.

t-version:

\[ t(1 - t)\ddot{f} + [c - (\alpha + b + 1)t]f' - \alpha bf = 0 \]  

(3.86)

with the solution \( f(t) = 2F(\alpha, b; c; t) \).

a-version:

\[ a(1 - a)f_{aa} + [c - (\alpha + b + 1)t]f_a - \alpha bf = 0 \]  

(3.87)

with the solution \( f(a) = 2F(\alpha, b; c; a) \).

N-version:

\[ N(1 - N)f_{NN} + [c - (\alpha + b + 1)t]f_N - \alpha bf = 0 \]  

(3.88)

with the solution \( f(N) = 2F(\alpha, b; c; N) \).

H-version:

\[ H(1 - H)f_{HH} + [c - (\alpha + b + 1)t]f_H - \alpha bf = 0 \]  

(3.89)

with the solution \( f(H) = 2F(\alpha, b; c; H) \).

\[ ii) \text{ Another example is the case when } f(\dot{R}) \text{ satisfies the Pinney equation} \]

\[ f'' + \xi_1(\dot{R})f + \frac{\xi_2(\dot{R})}{f^3} = 0. \]  

(3.90)

If \( \xi_1 = 1, \quad \xi_2 = \kappa = \text{const} \), this equation has the following solution (see e.g. [11])

\[ f(\dot{R}) = \cos^2 \dot{R} + \kappa^2 \sin^2 \dot{R}. \]  

(3.91)

Its "t-form" is \( f(t) = \cos^2 t + \kappa^2 \sin^2 t \) which is the solution of the Pinney equation

\[ \ddot{f} + \xi_1(t)f + \frac{\xi_2(t)}{f^3} = 0 \]  

(3.92)

as \( \xi_1 = 1, \quad \xi_2 = \kappa = \text{const} \).

\[ iii) \text{ Let us we present one more example. Let the function } f \text{ satisfies the equation} \]

\[ f'' = 6f^2 - 0.5g_2, \quad (g_2 = \text{const}) \]  

(3.93)
These equations admit the following solutions

\[ f(\hat{R}) = \varphi(\hat{R}) \]  

(3.105)

and

\[ f(\hat{R}) \equiv f(t) = \varphi(t), \]  

(3.106)

where \( \varphi(\hat{R}) \) and \( \varphi(t) \) are the Weierstrass elliptic functions.

Finally we would like to note that the above presented HL models, in particular, have some solutions which describe the accelerated expansion of the universe.

### 4 Cosmological solutions

It is important find exact solutions of HL gravity models (see e.g. [4]-[9]). In our case, all above presented HL models admit exact solutions. It is important that some of these solutions describe accelerated expansion of the universe. Let us present some cosmological solutions of some above presented HL models. As an example, consider the PII accelerated expansion of the universe. Let us present some cosmological solutions of some above presented HL models. As an example, consider the PII accelerated expansion of the universe. Let us present some cosmological solutions of some above presented HL models. As an example, consider the PII accelerated expansion of the universe. Let us present some cosmological solutions of some above presented HL models. As an example, consider the PII accelerated expansion of the universe. Let us present some cosmological solutions of some above presented HL models. As an example, consider the PII accelerated expansion of the universe.

i) **\( P_{IIA} \) - model (2.27):**

\[
N(t) \equiv N(t; \nu_1, 0, 0, -\nu_1 \nu_2^2) = \nu_2 \sqrt{t},
\]

(4.1)

\[
N(t) \equiv N(t; 0, -2\nu_1, 0, 4\nu_1 \nu_2 - \nu_3^2) = t[(\ln t \nu_1)² + \ln (t \nu_2 \nu_3)],
\]

(4.2)

\[
N(t) \equiv N(t; -\nu_1^2 \nu_2, 0, \nu_1^2 (\nu_2 - \nu_3 \nu_4), 0) = \frac{\nu_2 t^{2\nu_1} + \nu_2 \nu_1 + \nu_4}{\nu_3 t^{2\nu_1} + \nu_2 \nu_1 + \nu_4},
\]

(4.3)

\[
N(t) \equiv N(t; 2\nu_1 + 3, -2\nu_1 + 1, 1, -1) = \frac{t + \nu_1}{t + \nu_1 + 1},
\]

(4.4)

\[
N(t) \equiv N(t; \nu_1, -\nu_1 \nu_2^2, \nu_3, -\nu_3 \nu_4^2) = \nu_2,
\]

(4.5)

\[
N(t) \equiv N = -\epsilon_1 (\ln \varphi) t,
\]

(4.6)

Hence we get the corresponding expressions for the scale factor \( a(t) \). We have

\[
a(t) \equiv a(t; \nu_1, 0, 0, -\nu_1 \nu_2^2) = e^{\nu_2 \sqrt{t}},
\]

(4.7)

\[
a(t) \equiv a(t; 0, -2\nu_1, 0, 4\nu_1 \nu_2 - \nu_3^2) = e^{\nu_2 \ln (t \nu_1)² + \ln (t \nu_2 \nu_3)},
\]

(4.8)

\[
a(t) \equiv a(t; -\nu_1^2 \nu_2, 0, \nu_1^2 (\nu_2 - \nu_3 \nu_4), 0) = e^{\nu_2 \ln t^{2\nu_1} + \ln t^{2\nu_1} + \ln \nu_4},
\]

(4.9)

\[
a(t) \equiv a(t; 2\nu_1 + 3, -2\nu_1 + 1, 1, -1) = e^{\frac{\nu_1 + 1}{t + \nu_1 + 1}},
\]

(4.10)

\[
a(t) \equiv a(t; \nu_1, -\nu_1 \nu_2^2, \nu_3, -\nu_3 \nu_4^2) = e^{\nu_2},
\]

(4.11)

\[
a(t) \equiv a = e^{-\epsilon_1 (\ln \varphi) t},
\]

(4.12)

ii) **\( P_{IIB} \) - model (2.28):**

\[
a(t) \equiv a(t; \nu_1, 0, 0, -\nu_1 \nu_2^2) = \nu_2 \sqrt{t},
\]

(4.13)

\[
a(t) \equiv a(t; 0, -2\nu_1, 0, 4\nu_1 \nu_2 - \nu_3^2) = t[\ln (t \nu_1)² + \ln (t \nu_2 \nu_3)],
\]

(4.14)

\[
a(t) \equiv a(t; -\nu_1^2 \nu_2, 0, \nu_1^2 (\nu_2 - \nu_3 \nu_4), 0) = \frac{\nu_2 t^{2\nu_1} + \nu_2 \nu_1 + \nu_4}{\nu_3 t^{2\nu_1} + \nu_2 \nu_1 + \nu_4},
\]

(4.15)

\[
a(t) \equiv a(t; 2\nu_1 + 3, -2\nu_1 + 1, 1, -1) = \frac{t + \nu_1}{t + \nu_1 + 1},
\]

(4.16)

\[
a(t) \equiv a(t; \nu_1, -\nu_1 \nu_2^2, \nu_3, -\nu_3 \nu_4^2) = \nu_2,
\]

(4.17)

\[
a(t) \equiv a = -\epsilon_1 (\ln \varphi) t,
\]

(4.18)
iii) P_{IIC} - model (2.29):

\[
H(t) \equiv H(t; \nu_1, 0, 0, -\nu_1 \nu_2^2) = \nu_2 \sqrt{t},
\]

(4.19)

\[
H(t) \equiv H(t; 0, -2\nu_1, 0, 4\nu_1 \nu_2 - \nu_3^2) = t [\ln (t \sqrt{\nu_1})^2 + \ln (\nu_3 e^{\nu_2})],
\]

(4.20)

\[
H(t) \equiv H(t; -\nu_1^2 \nu_2, 0, \nu_1^2 (\nu_2^4 - \nu_3^4)), 0) = \frac{t^{\nu_1^{-1}}}{\nu_3 t^{2\nu_1} + \nu_2 t^{\nu_1} + \nu_4},
\]

(4.21)

\[
H(t) \equiv H(t; 2\nu_1 + 3, -2\nu_1 + 1, 1, -1) = \frac{t + \nu_1}{t + \nu_1 + 1},
\]

(4.22)

\[
H(t) \equiv H(t; \nu_1, -\nu_1 \nu_2^2, \nu_3, -\nu_3 \nu_4^2) = \nu_2,
\]

(4.23)

\[
H(t) \equiv H = -\epsilon (\ln \varphi) t
\]

(4.24)

and so on. These exact solutions correspond to the different cosmologies. Some of them describe the accelerated and decelerated phases of the Universe. Finally we note that similarly we can present exact solutions of the other HL models constructed in the previous sections.

5 Conclusion

In this work, a new class integrable and nonintegrable cosmological models of the Hořava-Lifshitz gravity were proposed. For some of them, exact solutions are presented. To construct integrable models we use the well-known integrable systems, namely, Painleve equations. Then these results extend for the F(R) Hořava-Lifshitz gravity theory case. In particular, several integrable cosmological models of this modified gravity theory were constructed in the explicit form.

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