Abductive forgetting

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Abstract

Abductive forgetting is removing variables from a logical formula while maintaining its abductive explanations. It is defined in either of two ways, depending on its intended application. Both differ from the usual forgetting, which maintains consequences rather than explanations. Differently from that, abductive forgetting from a propositional formula may not be expressed by any propositional formula. A necessary and sufficient condition tells when it is. Checking this condition is \( \Pi^3_p \)-complete, and is in \( \Pi^4_p \) if minimality of explanations is required. A way to guarantee expressibility of abductive forgetting is to switch from propositional to default logic. Another is to introduce new variables.

1 Introduction

Logical forgetting is restricting a logical formula on a subset of its constituent elements, such as its variables [Boo54, Del17, EKI19]. It has been extensively studied in many settings [Lin01, LLM03, Moi07, LR94, BKL+17, Lei17, GKL21, KWW09, WWWZ15, BB22]. It is also known as variable elimination in the context of automated reasoning [DP60, DR94, SP04] and by its dual concept of uniform interpolation in modal and description logic [KWW09].

The three key ingredients of logical forgetting come from its very definition. Propositional forgetting is “given a propositional formula (Point 1), reduce its variables (Point 2) while maintaining its consequences (Point 3) on the remaining ones.” The three points are the three key ingredients. Changing them gives different versions of forgetting.

- rather than a propositional formula, forget from a first-order formula [LR94], a modal logic formula [KWW09], from circumscription [WWWZ15], from an ontology [BKL+17], a logic program [Lei17, GKL21], an argumentation framework [BDR20].

- rather than the variables, restrict the literals [LLM03], the objects [Del17], the subformulae [FWL+18];

- rather than the consequences, maintain something else.

The last point is not vague by mistake. It lacks examples because it is the least explored direction of research. Logical forgetting in Answer Set Programming (ASP) recently introduced strong persistence: maintaining the consequences is not enough, maintaining the

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semantics of the formula after adding new parts is also required. This is one “something else” that may be required to be maintained after forgetting: the consequences after enlarging the formula. The implicit premise is that a logical formula is a representation of knowledge, and knowledge is not static. It may increase. Forgetting something from a formula may not by itself be prepared for the following changes. Changing the reduced version of a formula may not match changing the formula unless specifically designed. It needs a switch from “maintain the consequences” to “maintain the result of enlarging”. Not only respect inference, also respect additions.

Inference and additions are not the only two applications of logic. Even just in propositional logic, checking what follows from a formula and enlarging it do not exhaust its uses. What is possible is another. What explains something is another. What changes after a removal or revision is another. Each use gives a version of forgetting.

**what is possible:** reduce the alphabet of a propositional formula while maintaining the formulae on the reduced alphabet that are consistent with it;

**what explains something:** reduce the alphabet of a propositional formula while maintaining its abductive explanations [EG95] on the reduced alphabet;

**what holds after removal or revision:** reduce the alphabet of a propositional formula while maintaining its consequences on the reduced alphabet after a contraction or revision [RGR11];

This article is about the second extension: maintain the abductive explanations. A formula represents knowledge about a certain domain; abduction infers information such as “this, this and this explain this”. Some facts explain another. They constitute a possible reason for it.

Forgetting removes some of these facts from consideration. An explanation is “A, B and C explain D and E”. Forget about C and E. Result: “A and B explain D”. An explanation remains an explanation, just approximated by the removal of the causes and effects that are not of interest.

This example makes abductive forgetting look obvious. Forgetting a side effect like E does not make an explanation less valid. This is exactly the point of forgetting, removing information of secondary interest. A side effect, indeed. In the same way, disregarding C does not mean that C is false; that would invalidate the explanation “A, B and C explain D”. It means that C is not of interest. Listing among the causes is a distraction.

In spite of its seeming triviality, this example hides an important principle. Maintaining the consequences, rather than the explanations, of a formula “A, B and C implies D and E” empties it. It does not just remove the forgotten variables. It removes everything. The same would be incorrect in abduction. It amounts to a bizarre dialog like:

“Why is D true?”
“It may be because A, B and C explain D”.
“Yes, fine. But I only care about A and B. Forget about C”.
“Oh. Then, D is unexplainable”.

It is not unexplainable. It is an effect of A and B; and also of something not of interest, but that is not of interest. In consequential forgetting, the removal of a consequence like “A, B and C” is not a problem because its role is taken over by the other consequence “A and
B”. In abductive forgetting, nothing would take over “A, B and C explain D” if this is the only explanation of D and is removed just because it contains a part to disregard.

This is about forgetting hypotheses. Forgetting effects is a different story.

Explaining something not of interest is not of interest. Something like “A explains E” is not of interest when E is not of interest. That “A” explains nothing is implicit, unless it explains something.

The same goes when the manifestations to explain do not only include E. Or not, depending on the context.

No doubt “A explains E” ceases to be of interest altogether when E is not of interest. If something is uninteresting, its possible causes are as well. What happens if something is partly interesting? Are its causes uninteresting?

The difference is not clear-cut as it may look.

Two examples motivate two different choices when forgetting about E in an explanation “A explains D and E”.

A doctor is searching for the diagnosis of a patient. The currently available test results only tell that D is the case. Since A causes D and E, a test for E is prescribed, but not yet done. Forgetting about E does not rule out A as an explanation of D. “A explains D” remains.

A professor is writing some teaching material on the same topic. The first chapter is about the basics, which include A and D, but not E. A further chapter will cover E. In order not to give incomplete information “A explains D” just to be later corrected as “A explains D and E”, this fact is delayed to the later chapter.

The difference comes from the aim of forgetting. A doctor wants to concentrate on symptoms that are known; an explanation that includes other symptoms is still a relevant explanation. A professor wants to concentrate on a specific topic in order not to confuse students; the explanations that require knowledge of other topics will be covered later. None of the two solutions is right, none is wrong. They serve different purposes: curing and teaching; more generally, forgetting what is out of reach or what is not of interest.

Both may be inexpressible as abduction.

This is common in logics other than the propositional calculus. For example, forgetting a predicate in first-order logic may not be first-order definable [LR94]. Forgetting symbols from the description logics ALC may not give an ALC formula [WWT+09]. Strongly persistent forgetting from a logic program may be impossible [GKL16b]. Some modal logics do not have uniform interpolants; therefore, they do not always allow forgetting [FLvD19]. This never happens for consequential forgetting in propositional logics: forgetting variables from a propositional formula when maintaining its consequences always gives another propositional formula. Not so when shifting from maintaining consequences to maintaining explanations.

The starting point of abductive forgetting is a propositional formula. Abduction tells the explanations it supports. Forgetting changes them; at a minimum, the forgotten variables are removed; depending on the application, explanations may be deleted altogether. Either way, the surviving explanations may not be the abductive explanations of any propositional formula.

This is the expressibility problem: forgetting something from a formula in a certain
language may not be representable within the same language. It may happen in first-order logic [LR94]. It may happen in description logics [WWT+09]. It may happen in answer set programming [GKL16a]. It may happen in modal logics [FLvd19]. It may happen in abductive forgetting.

It may happen. It may not. Forgetting some variables from a specific formula may not give a propositional formula. Forgetting some variables from another may. A first question is how the difference relates to the definition of forgetting and the definition of abduction.

What to do when a specific formula is given? An algorithm that computes forgetting as a formula if any exists is given. It is used to derive a necessary and sufficient condition for the feasibility of abductive forgetting within propositional logic. It is employed to prove the complexity of the problem.

If forgetting does not stay within propositional logic, does it stay in some more expressive logics? It does in default logic. It does in propositional logic with new variables.

The article is organized as follows. Section 2 fixes the formalism and gives the two definitions of forgetting; Section 3 shows different cases where forgetting can be done or not; Section 4 presents an algorithm for forgetting when possible; Section 5 shows a necessary and sufficient condition for the representability of forgetting within propositional logic; Section 6 is about the complexity of the problem; Section 7 extends the analysis to minimal explanations; Section 8 shows that default rules always allow forgetting; introducing new variables also does, as shown in Section 9; Section 10 tells what changes when causes and effects are not disjoint.

2 The two definitions of abductive forgetting

The starting point of abduction is a propositional formula build over an alphabet that includes two subsets: a set of hypotheses and a set of manifestations. In this article, they are assumed disjoint [EG95, MB06]; how this assumption affects the given result is discussed in Section 10.

The propositional formula represents all available knowledge about the relationships between hypotheses and manifestations. A given subset of hypotheses $E$ may or may not explain a nonempty subset of manifestations $M$. It does if it is consistent with the formula and entails these manifestations. An explanation like this is written $E \Rightarrow M$. It is supported by a formula $F$ if $F \cup E$ is consistent and entails $M$. This condition is written $F \models E \Rightarrow M$.

Which abductive explanations are supported depends not only on $F$ but also on the set of all possible hypotheses and manifestations, since these sets cap $E$ and $M$.

Definition 1 An explanation over hypotheses $I$ and manifestations $C$ is a pair of propositional sets of variables $E \subseteq I$ and $M \subseteq C$ with $M \neq \emptyset$, written $E \Rightarrow M$.

The hypotheses $I$ and manifestations $C$ are two sets of variables. They are assumed disjoint unless stated otherwise: $I \cap C = \emptyset$.

Formally, abduction is finding explanations out of a propositional formula given the sets of all hypotheses and manifestations.

Definition 2 An abduction frame is a triple $\langle F, I, C \rangle$ where $F$ is a propositional formula and $I$ and $C$ are two disjoint sets of variables.

4
The formal definition of support of an explanation follows.

**Definition 3** An abduction frame \( \langle F, I, C \rangle \) supports an explanation \( E \Rightarrow M \) over \( I \) and \( C \) if \( F \cup E \) is consistent and entails \( M \). This condition is written \( \langle F, I, C \rangle \models E \Rightarrow M \). The set of explanations supported by an abduction frame \( \langle F, I, C \rangle \) is denoted \( \text{abduct}(\langle F, I, C \rangle) \).

The set of all hypotheses \( I \) and all manifestations \( C \) are implicit in this article: they are given for the formula before forgetting; they are \( I \cap R \) and \( C \cap R \) after forgetting, where \( R \) is the set of variables to remember. This makes \( I \) and \( C \) fixed in both cases. They are omitted for simplicity: \( \langle F, I, C \rangle \) supports an explanation over \( I \) and \( C \) is shortened to \( F \) supports an explanation. The sets \( I \) and \( C \) are implicit.

To simplify notation, the explanation \( \{a, b, c\} \Rightarrow \{d, e\} \) is written \( abc \Rightarrow de \).

The set of explanations supported by a formula may include both \( a \Rightarrow c \) and \( ab \Rightarrow c \). While the second is still a valid explanation, it may be considered less relevant as it includes the unnecessary hypotheses \( b \). In the same way, \( a \Rightarrow e \) may be preferred to \( bcd \Rightarrow e \) because it requires fewer hypotheses: one instead of three.

Two definitions of minimality are considered: by set containment or by cardinality [Pau93, EG95]. They only select the supported explanations \( E \Rightarrow M \) such that \( E' \Rightarrow M \) is not supported by any \( E' \) less than \( E \), where “less” is either \( E' \subset E \) or \( E' \) comprising fewer hypotheses than \( E \). An arbitrary order \( < \) can be used instead of these two. This article assumes that \( E \subset E' \) implies \( E < E' \). This is the case for the two specific orderings above.

**Summary of the assumptions:**

- \( M \) is not empty: the explanation \( E \Rightarrow \emptyset \) is invalid;
- hypotheses and effects are disjoint; what happens when lifting this assumption is shown in Section 10;
- \( E \subset E' \) implies \( E < E' \).

The explanation \( abc \Rightarrow de \) tells that \( a, b, \) and \( c \) causes \( d \) and \( e \) according to the formula. The available knowledge tells that if \( d \) and \( e \) are the case, a possible reason is that \( a, b \) and \( c \) are as well.

What if the hypothesis \( c \) and the manifestation \( e \) are not of interest?

As outlined in the introduction, forgetting a hypothesis like \( c \) and forgetting a manifestation like \( e \) are two separate issues. The first has a unique treatment; the second does not.

**Forgetting a hypothesis.** The explanation \( abc \Rightarrow de \) turns into \( ab \Rightarrow de \) when \( c \) is not of interest for whichever reason. For example, reading “not of interest” as “a detail to neglect”, neglecting that \( c \) is a part of the cause of \( d \) and \( e \) means that is left out. Reading “not of interest” as “an information that is not of any use anyway”, \( c \) is removed from the explanation as useless. Regardless of the use of forgetting, \( abc \Rightarrow de \) turns into \( ab \Rightarrow de \) when forgetting \( c \).

The alternative would be to remove \( abc \Rightarrow de \) altogether as an explanation. This leads to paradoxical cases such as \( d \) becoming unexplainable, which happens if \( abc \) is its
only explanation. Even if \( d \) has another cause \( fgh \), removing \( abc \Rightarrow de \) hides \( ab \) as an alternative. Overconfidence is dangerous. Both \( fgh \) and \( ab \) cause the symptoms \( d \) and \( e \); course of action: require further analysis. Only \( fgh \) causes \( d \); course of action: cure \( fgh \). Even if it is dangerous in presence of \( a \).

**Forgetting a manifestation.** A tempting solution is to follow the same principle, turning \( abc \Rightarrow de \) into \( abc \Rightarrow d \) when forgetting \( e \). This way of forgetting has some ground. A medical manual may say that \( a, b, \) and \( c \) cause \( d \) and \( e \). Since \( e \) cannot be established for now, it is better left out. Forgotten. Still, \( a, b \) and \( c \) explain \( d \). Forgetting does not mean that \( e \) is false. It means that it is not known. It may still be the case.

\[
\begin{align*}
abc & \Rightarrow de \\
f & \Rightarrow d \\
fg & \Rightarrow de \\
\end{align*}
\]

An example highlights the difference. If \( e \) is false, \( f \) is the rationally cheapest explanation of \( d \). If \( e \) is true, \( f \) alone does not suffice to explain it; also \( g \) and \( h \) are required. This complete explanation is as large as the alternative \( a, b \) and \( c \).

That \( e \) is not of interest does not mean that it is false. It means it is unimportant for now. It may be false, but it also may be true. If it is true, it is still true even if it is not under consideration. The explanation \( f \) is still insufficient; also \( g \) and \( h \) are necessary. The alternative \( a, b \) and \( c \) remains valid.

This argument hinges on “forgetting is not falsity”. Forgetting \( e \) does not mean that \( e \) is false. Still better, it is not assumed false just because it is not of interest.

Yet, sometimes it is.

The same doctor that searches for a diagnosis for \( d \) and \( f \) may the same day teach students about the illness \( d \). If it occurs alone, it is most likely caused by \( f \). Only the complication \( e \) would make \( a, b, \) and \( c \) a reasonable alternative, while it would be too complicated a reason for \( d \) alone. For a quick overview, the alternative is left out. It would be too long to tell everything right out of the bat. Only when going into the details of \( d \), the alternative will be explained.

While forgetting a hypothesis always removes it from the explanations, forgetting a manifestation may remove it from the explanations or remove explanations altogether. It depends on the application of forgetting. A manifestation may have thousands of explanations, but only in presence of other manifestations. The alternatives are still possible, just less relevant. They may still be relevant when forgetting the other manifestations or not depending on the context. Disregarding an alternative explanation of some symptoms may be tragic; delaying its exposition when teaching may be necessary.

The two ways of forgetting are called focusing (like when blowing up a part of a photograph with a magnifying glass) and summarizing (like when giving an overview on a specific topic).
Definition 4 Focusing an abduction frame $\langle F, I, C \rangle$ on a set of variables $R$ is the following set of explanations.

\[
\text{focus}(\langle F, I, C \rangle, R) = \\
\{E \Rightarrow M \mid E \subseteq I \cap R, M \subseteq C \cap R, M \neq \emptyset, \\
\exists E' \subseteq I \setminus R, \exists M' \subseteq C \setminus R, EE' \Rightarrow MM' \in \text{abduct}(\langle F, I, C \rangle)\}
\]

Definition 5 Summarizing an abduction frame $\langle F, I, C \rangle$ on a set of variables $R$ is the following set of explanations.

\[
\text{summarize}(\langle F, I, C \rangle, R) = \\
\{E \Rightarrow M \mid E \subseteq I \cap R, M \subseteq C \cap R, \exists E' \subseteq I \setminus R, EE' \Rightarrow M \in \text{abduct}(\langle F, I, C \rangle)\}
\]

Consequential forgetting does not distinguish between looking at a detail (focusing) and giving a quick overview of a particular (summarizing). Both are given as applications of forgetting with no distinction. For example, Eiter and Kern-Isberner [EKI19] wrote: “not all information can be kept and treated in the same way. […] forgetting […] helps us to deal with information overload and to put a focus of attention”. This is an example of summarizing, as a mean to omit details that are not important. Botoeva et al. [BKL+17] wrote “As an example, consider Snomed CT, which contains a vocabulary for a multitude of domains related to health care, including clinical findings, symptoms, diagnoses, procedures, body structures, organisms, pharmaceuticals, and devices. In a concrete application such as storing electronic patient records, only a small part of this vocabulary is going to be used”. This is an example of focusing, as a mean to limit information to what is necessary to a specific task. Such applications of forgetting are commonly seen in the literature without any distinction made. It is unnecessary since it would make no difference.

Defining focusing and summarizing on the variables to remember $R$ instead of those to forget simplifies the technical treatment, but the concept is the same: focusing or summarizing on some variables is forgetting the others. Both remove hypotheses from the explanations. Focusing also removes manifestations, summarizing removes whole explanations. In the other way around, focusing restricts each explanation $E \Rightarrow M$ to $E \cap R \Rightarrow M \cap R$; summarizing restricts $E \Rightarrow M$ into $E \cap R \Rightarrow M$ if $M \subseteq R$, and removes it altogether otherwise.

The sets $I$ and $C$ in the abductive frame are considered fixed. Consequently, their version after forgetting are always fixed too, to $I \cap R$ and $C \cap R$. They can therefore be left implicit, leading to $\text{focus}(F, R)$ and $\text{summarize}(F, R)$.

Both focusing and summarizing are forms of forgetting. When their difference is not important, such as when they coincide because all manifestations are remembered, the generic term forgetting is used in place of focusing or summarizing.

Theorem 1 If $C \subseteq R$ then $\text{focus}(\langle F, I, C \rangle, R) = \text{summarize}(\langle F, I, C \rangle, R)$.

Proof. Focusing $\text{focus}(\langle F, I, C \rangle, R)$ is defined as follows.

\[
\{E \Rightarrow M \mid E \subseteq I \cap R, M \subseteq C \cap R, M \neq \emptyset, \exists E' \subseteq I \setminus R, \exists M' \subseteq C \setminus R, \langle F, I, C \rangle \models EE' \Rightarrow MM'\}
\]
Since $C$ is contained in $R$, none of its elements is outside $R$. In terms of sets, $C \setminus R$ is empty. The only subset $M'$ of an empty set is the empty set: $M' = \emptyset$. The definition of focus($F, R$) can therefore be rewritten as:

$$\{E \Rightarrow M \mid E \subseteq I \cap R, M \subseteq C \cap R, M \neq \emptyset, \exists E' \subseteq I \setminus R, \langle F, I, C \rangle \models EE' \Rightarrow M\}$$

The definition of $\langle F, I, C \rangle \models EE' \Rightarrow M$ includes $M \neq \emptyset$, which can be therefore be removed from the definition of the set.

$$\{E \Rightarrow M \mid E \subseteq I \cap R, M \subseteq C \cap R, \exists E' \subseteq I \setminus R, \langle F, I, C \rangle \models EE' \Rightarrow M\}$$

This is the definition of summarize($\langle F, I, C \rangle, R$).

Neither focusing nor summarizing produce an explanation like $E \Rightarrow \emptyset$ as a result of forgetting. It is a consequence in the second definition, but is explicitly stated in the first. Without this constraint, forgetting $m$ would turn $ab \Rightarrow m$ into $ab \Rightarrow \emptyset$. An explanation of something to be completely forgotten is remembered. Still worse, it is remembered as an explanation of nothing: $ab \Rightarrow \emptyset$ means that $a$ and $b$ explain why nothing is observed.

Both focusing and summarizing gives a set of explanations. Why not a formula? Consequential forgetting in propositional logic used to be defined as $F[\text{true}/x] \lor F[\text{false}/x]$, the disjunction of two copies of the formula where the variable to forget is respectively replaced by true and false. This definition is valid, but hides the point of consequential forgetting: restricting to what is the case and only involves things to remember. Such consequences happen to be the consequences of $F[\text{true}/x] \lor F[\text{false}/x]$ in propositional logics. None of the more powerful logics appear to be this simple. They either require more complex formulae to represent the consequences of forgetting [ACF+19, ZSW+20] or do not allow such a representation at all [FLvD19]. Abductive forgetting is among them, as shown in the next section.

An alternative definition of forgetting is a formula the supports exactly focus($F, R$) or summarize($F, R$), with the caveat that forgetting may or may not exist. This would however be conceptually wrong, as it links the feasibility of forgetting to a specific representation of the result, by a propositional formula. As will be seen in a following section, a set of explanations can also be represented for example by a default theory.

### 3 Expressing forgetting

In both its forms, focusing and summarizing, forgetting produces a set of explanations. This set may coincide with the set of explanations supported by some formula. If it does, the formula can be viewed as a representation of forgetting. This is analogous to consequential forgetting, which is commonly defined as a formula having the same consequences of the original on the restricted alphabet.

A formula $G$ such that its abductive explanations abduct($G$) coincide with focus($F, R$) or summarize($F, F$) may exist or not. The same may happen for minimal abductive explanations. If such a $G$ exists, this condition is shortened to “forgetting is supported by a formula”.

The following list contains results about forgetting being supported. The ones not specifying it as focusing and forgetting are counterexamples with $C \subseteq R$, a case where the two definitions of forgetting coincide by Theorem 1.
1. forgetting may not be supported by any formula;

2. forgetting may be supported by a formula when restricting to size-minimal explanations but not for containment-minimal or all explanations;

3. forgetting may be supported by a formula when restricting to size-minimal and cardinality-minimal explanations but not for all explanations;

4. a formula supporting forgetting from minimal explanations may not support forgetting from all explanations, even if another supporting that exists;

5. focusing and summarizing coincide if selecting all explanations, they do not if selecting only the minimal explanations;

6. a formula supporting summarizing may exist when none does for focusing;

7. abductive forgetting may not coincide with consequential forgetting.

The first result is that forgetting may not be supported by any formula: forgetting produces a set of explanations that is not the set of explanations supported by any formula.

The formula is $F = \{ab \rightarrow x, ac \rightarrow y, bc \rightarrow \perp\}$, the hypotheses $I = \{a, b, c\}$, the manifestations $C = \{x, y\}$ and the variables to forget $b$ and $c$. Focusing and summarizing coincide since $C \subseteq R$. The explanations supported by $F$ are $ab \Rightarrow x$ and $ac \Rightarrow y$. The explanation $abc \Rightarrow xy$ is not supported since the conjunction of $b$ and $c$ is inconsistent with $F$.

Forgetting $b$ and $c$ turns these explanations into $a \Rightarrow x$ and $a \Rightarrow y$ only. It does not include $a \Rightarrow xy$. A formula supporting the first two also supports the third. The conclusion is that no formula supports $\{a \Rightarrow x, a \Rightarrow y\}$; no formula supports exactly these explanations.

The second result is that forgetting may be supported by a formula when restricting to minimal explanations only, but it is not without this restriction.

This is shown by a small change in the previous instance, where the same hypotheses explain $x$ alone, explain $y$ alone, but they do not explain $x$ and $y$ at the same time. The change is that these hypotheses are not minimal in size. Therefore, they are only relevant when restricting to cardinality-minimal explanation.

The original formula is $F = \{abc \rightarrow x, abd \rightarrow y, cd \rightarrow \perp ef \rightarrow x, ef \rightarrow y\}$, the hypotheses $I = \{a, b, c, a'\}$, the manifestations $C = \{x, y\}$, the variables to forget $b$, $c$ and $f$. The problematic explanations are $ab \Rightarrow x$ and $ab \Rightarrow y$. They are not minimal since $a'$ is smaller than both. Therefore, they are no relevant when restricting to the smallest explanations.

The third result is that forgetting may be supported when restricting to size-minimal and cardinality-minimal explanations but not for all explanations. Only a small change is required: $ab$ and $ac$ are made not minimal by set containment, in addition to cardinality.

The original formula is $F = \{abc \rightarrow x, abd \rightarrow y, cd \rightarrow \perp ace \rightarrow x, ace \rightarrow y\}$, the hypotheses $I = \{a, b, c, a'\}$, the manifestations $C = \{x, y\}$, the variables to forget $b$ and $c$. Again, the problem is on $ab \Rightarrow x$ and $ab \Rightarrow y$. Their hypothesis $ab$ strictly contain the alternative explanation $a$. They are not relevant when minimizing by set containment.
The fourth result is that a formula may support forgetting from minimal explanation, but it does not support forgetting for all explanations, even if one doing that exists. This is proved by $F = \{a \rightarrow x, ab \rightarrow \bot, cd \rightarrow y\}$ where the hypotheses are $I = \{a, b, c, d\}$, the manifestations $C = \{x, y\}$ and $c$ is forgotten. The resulting explanations are $a \Rightarrow x$ and $c \Rightarrow y$, all minimal. The formula $\{a \Rightarrow x, c \Rightarrow y\}$ does not support exactly these explanations because it also supports $ab \Rightarrow x$. Yet, it minimally supports these explanations.

The fifth result is that focusing and summarizing coincide when selecting all explanation, but may not when selecting only the minimal explanations.

Focusing and summarizing change or remove explanations. They both remove the variables to forget from all explanations, but they remove an explanation only if its manifestations contains some manifestations to be remembered and some to be forgotten: $EE' \Rightarrow MM'$ when forgetting $E' \cup M'$ and neither $M$ nor $M'$ is empty. Focusing turns it into $E \Rightarrow M$, summarizing removes it. The original $EE' \Rightarrow MM'$ is supported by the formula $F$. Therefore, $F \cup E \cup E'$ is consistent and entails $M \cup M'$. As a result, it entails $M$ alone. The same formula $F$ also supports $EE' \Rightarrow M$. Since $M$ is not empty and comprises only variables to remember, both forms of forgetting turn it into $E \Rightarrow M$.

Focusing and summarizing differ when selecting only minimal explanations. An example is $F = \{a \rightarrow x, ab \rightarrow y\}$, where the hypotheses are $I = \{a, b\}$, the manifestations $C = \{x, y\}$ and $y$ is forgotten. The minimal explanations supported by $F$ are $a \Rightarrow x$, $ab \Rightarrow y$ and $ab \Rightarrow xy$. The explanation $ab \Rightarrow x$ is supported, but is not minimal. Focusing removes the second explanation and turn the others into $a \Rightarrow x$ and $ab \Rightarrow x$. Summarizing turns the first into $a \Rightarrow x$ and removes the other two.

The previous example also proves the sixth result: focusing may not be supported by any formula when summarizing is. This is the case when selecting the minimal explanations; otherwise, focusing and summarizing coincide. Focusing gives $a \Rightarrow x$ and $ab \Rightarrow x$. The second is not minimal. Therefore, no formula supports them minimally. Summarizing gives $a \Rightarrow x$ only. A formula supporting it minimally is $\{a \rightarrow x\}$.

The seventh and final result is that abductive forgetting may not coincide with consequential forgetting. An example is $F = \{ab \rightarrow x\}$, where the hypotheses are $I = \{a, b\}$, the manifestations $C = \{x\}$ and $b$ is forgotten. Both definitions of forgetting produce $a \Rightarrow x$, which is also minimal. This explanation is supported by the formula $\{a \rightarrow x\}$. Consequential forgetting gives an empty formula instead. Replacing $b$ with false turns $F = \{ab \rightarrow x\}$ into true. As a result, the disjunction $F[\text{true}/b] \lor F[\text{false}/b]$ is true as well.

4 Algorithmic generation of forgetting

Both focusing and summarizing produce a set of explanations $S = \{E \Rightarrow M\}$. This set may comprise the explanations supported by a formula or not. If it does, the formula can be viewed as the result of forgetting. Forgetting takes a formula and gives a formula. More precisely, abductive forgetting turns abduction from the original formula into abduction from the generated formula. Many logics, such as propositional logic, possess this language
invariance [LLM03]. Others, such as description logics, do not [LR94, WWT+09, GKL16b].

This section and the following generalize the question to an arbitrary set: given a set of explanations \( S \), is there a formula \( G \) such that \( \text{abduct}(G) = S \)? When \( S \) is either focus(\( F, R \)) or summarize(\( F, R \)), this is the question of forgetting being supported by a formula.

This section defines an algorithm for generating such a formula \( G \) from \( S \) if any exists. If \( S \) is the result of focusing or summarizing, it finds a formula that represents the result of forgetting: its abductive explanations are exactly those resulting from forgetting. The next section uses the algorithm for proving a necessary and sufficient condition to the existence of such a formula \( G \). Again, this is proved for an arbitrary set of explanations \( S \). When this set is the result of either focusing or summarizing, the condition expresses the existence of a formula supporting forgetting. The complexity results rely on this condition.

The algorithm synthesizes a formula that supports a given set of explanations \( S = \{ E ⇒ M \} \) if one such formula exists. In its current form it is meant to be used in proofs. It is not intended to be used in practice since it is exhaustive on the explanations in \( S \).

**Definition 6** The tentative-supporting formula of a set of explanations \( S \) over disjoint hypotheses \( I \) and manifestations \( C \) is the formula \( G(S) \) that comprises:

1. a clause \( E \rightarrow m \) for each \( E ⇒ m \in S \);
2. a clause \( E \rightarrow \bot \) for each \( E ⇒ m \notin S \) such that \( E' ⇒ m \in S \) for some \( E' \subset E \).

No optimization is attempted, such as neglecting an explanation \( E ⇒ m \) of \( S \) when \( E' ⇒ m \) is also in \( S \) with \( E' \subset E \). This would produce a smaller formula, but complicates the proofs where the algorithm is employed.

The root result about the tentative-supporting formula is that it is sort of closed with regard to entailment: it entails a clause if and only if contains that clause or one entailing it.

**Lemma 1** If \( I \cap C = \emptyset \) and \( F \) only contains clauses \( E \rightarrow m \) and \( E \rightarrow \bot \) with \( E \subseteq I \) and \( m \in C \), then \( F \) entails a clause if and only if it contains a clause with the same head if any and a subset of its body.

**Proof.** If \( F \) contains a clause \( E' ⇒ m \) with \( E' \subseteq E \), it entails \( E ⇒ m \) because this clause is a superset of \( E' ⇒ m \). The same for \( E' ⇒ \bot \).

In the other direction, if \( F \models E \rightarrow h \) then \( F \cup E \models h \). This is the same as \( h \) being generated by propagation from \( F \cup E \). Let \( E' ⇒ m \) be the last clause used in this derivation. By assumption, \( E' \) is a subset of \( I \). Since no clause of \( F \) contains any variable of \( I \) in the head, no variable in \( E' \) follows from propagation. Therefore, they are all in \( E \). A similar argument proves the claim for the clauses \( E ⇒ \bot \).

This lemma relies on the assumption that hypotheses and manifestations are disjoint. The effects of lifting this constraint are considered in Section 10. Also crucial is that no head is in a body, but this is guaranteed by the definition of the tentative-supporting formula \( G(S) \).

The main result about the tentative-supporting formula is that it supports the set of explanations if and only if such a formula exists.
**Theorem 2** A formula $F$ supports exactly the explanations $S$ over disjoint hypotheses and manifestations if and only if $G(S)$ does.

Proof. The assumption is that every $E \Rightarrow M \in S$ satisfies $E \subseteq I$ and $M \subseteq C$. The claim is that $\text{abduct}((G(S), I, C)) = S$ holds if and only if a formula $F$ such that $\text{abduct}((F, I, C)) = S$ exists, where every $E \Rightarrow M \in S$ satisfies $E \subseteq I$ and $M \subseteq C$.

If $G(S)$ supports $S$ then a formula supporting $S$ exists because it is $G(S)$ itself.

The rest of the proof shows the converse: if a formula $F$ supports $S$, then $G(S)$ does as well. The assumption is that $F$ supports $S$.

A first preliminary result is that $S$ contains an explanation $E \Rightarrow M$ if and only if it contains $E \Rightarrow m$ for every $m \in M$.

Since $F$ supports $S$, this set contains $E \Rightarrow M$ if and only if $F \cup E$ is consistent and entails $M$. This is the same as $F \cup E$ being consistent and entailing every $m \in M$. By definition, it is also the same as $F$ supporting $E \Rightarrow m$ for every $m \in M$. Thanks to the assumption that $F$ supports $S$, this is also the same as $S$ containing $E \Rightarrow m$ for every $m \in M$.

A second preliminary result is that $F$ entails $G(S)$.

This is proved by showing that $F$ entails every clause of $G(S)$. A clause $E \rightarrow m$ is in $G(S)$ if and only if $S$ contains $E \Rightarrow m$. Since $F$ supports $S$, it supports its member $E \Rightarrow m$.

The definition of support includes $M \cup E$ for every $m \in M$ such that $M \subseteq C$.

A clause $E \rightarrow \bot$ is in $G(S)$ if $E \Rightarrow m$ is not in $S$, but $E' \Rightarrow m$ is for some $E' \subseteq E$. Since $F$ supports exactly $S$, it supports $E' \Rightarrow m$ but not $E \Rightarrow m$. As a result, $F \cup E'$ is consistent and entails $m$, and either $F \cup E$ is inconsistent or does not entail $m$. The second possibility, $F \cup E \not\models m$, is excluded since $F \cup E$ is a superset of $F \cup E'$, which entails $m$. As a result, the first possibility is the case: $F \cup E$ is inconsistent. This is the same as $F \models E \rightarrow \bot$.

The claim that $G(S)$ supports $S$ is now proved by contradiction. The converse is that either $G(S)$ does not support an explanation of $S$ or that it supports an explanation that is not in $S$. Both cases are proved contradictory.

- $G(S)$ does not support $E \Rightarrow M \in S$;

By the first preliminary result, $E \Rightarrow M \in S$ is the same as $E \Rightarrow m \in S$ for every $m \in M$. By construction, $E \rightarrow m$ is in $G(S)$ for every $m \in M$. A consequence is that $G(S)$ entails $E \rightarrow M$.

By assumption, $G(S)$ does not support $E \Rightarrow M$. Yet, it entails $E \rightarrow M$. Therefore, $G(S) \cup E$ is inconsistent. This is the same as $G(S) \models E \rightarrow \bot$. Since $F$ entails $G(S)$ by the second preliminary result, it also entails $E \rightarrow \bot$. As a result, $F \cup E$ is inconsistent. Therefore, $F$ does not support $E \Rightarrow M$, contradicting the assumption that $E \Rightarrow M$ is in $S$.

- $G(S)$ supports an explanation $E \Rightarrow M \not\in S$;

The definition of support is that $G(S) \cup E$ is consistent and entails $E \rightarrow M$. The latter is the same as $G(S) \models E \rightarrow m$ for every $m \in M$. By Lemma 1, $G(S)$ contains a clause $E' \rightarrow m$ with $E' \subseteq E$ for every $m \in M$. By the definition of $G(S)$, this is possible only if $S$ contains an explanation $E' \Rightarrow m$ for every $m \in M$.
Two possibilities are explored: either \( E \Rightarrow m \) is in \( S \) for all \( m \in M \), or it is not for some. In the second case, \( S \) contains \( E' \Rightarrow m \) but not \( E \Rightarrow m \) with \( E' \subseteq E \). The case \( E' = E \) is not possible since otherwise \( E \Rightarrow m \) and \( E' \Rightarrow m \) would be the same, while the latter is in \( S \) and the former is not. Therefore, \( E' \) is strictly contained in \( E \). By construction, \( G(S) \) contains \( E \rightarrow \perp \). This contradicts the assumption that \( G(S) \) supports \( E \Rightarrow M \). The conclusion is that \( E \Rightarrow m \) is in \( S \) for every \( m \in M \). By the first preliminary result above, \( S \) contains \( E \Rightarrow M \), contrary to assumption.

The assumption that the explanations of \( G(S) \) are not \( S \) leads to contradiction. The consequence is that \( G(S) \) supports \( S \).

The tentative-supporting formula may be exponentially larger than other formulae supporting the same explanations. This is not a problem because its motivation is theoretical: it is employed in some of the following proofs.

5  Necessary and sufficient conditions

A necessary and sufficient condition is proved for a set of explanations being supported by a formula. As in the previous section, the set of explanations may be the result of forgetting, either focusing or summarizing, but not necessarily. If it is, the condition specializes on whether forgetting is expressed by a formula.

The first part of the condition is that manifestations are jointly explained if and only if each is explained.

Definition 7 (Conjunctive condition) A set of explanations \( S = \{ E \Rightarrow M \} \) satisfies the conjunctive condition if it contains both \( E \Rightarrow M_1 \) and \( E \Rightarrow M_2 \) if and only if it contains \( E \Rightarrow M_1 \cup M_2 \).

If \( S \) is the set of explanations supported by a formula, it satisfies this condition.

Lemma 2  The set of explanations supported by an arbitrary formula satisfies the conjunctive condition.

Proof. The definition of a formula \( F \) supporting an explanation \( E \Rightarrow M_1 M_2 \) is that \( F \cup E \) is consistent and entails \( M_1 \cup M_2 \). The latter is the same as \( F \cup E \) entailing both \( M_1 \) and \( M_2 \). These conditions are also the same as \( F \) supporting both \( E \Rightarrow M_1 \) and \( E \Rightarrow M_2 \) since \( F \cup E \) is consistent.

Forgetting may not satisfy the conjunctive condition. A counterexample is \( F = \{ ab \rightarrow m, ac \rightarrow m', \neg b \lor \neg c \} \) where the hypotheses are \( I = \{ a, b, c \} \) and the manifestations \( C = \{ m, m' \} \). The explanations supported by \( F \) are \( ab \Rightarrow m \) and \( ac \Rightarrow m' \). The explanation \( abc \Rightarrow mm' \) is not supported since \( b \) and \( c \) are not together consistent with \( F \). Forgetting \( b \) and \( c \) turns these explanations into \( a \Rightarrow m \) and \( a \Rightarrow m' \), while still not producing \( a \Rightarrow mm' \).

A set of explanations \( S \) satisfying the conjunctive condition is completely defined by the explanation \( E \Rightarrow m \) with a single manifestation. The others are derived by the rule that \( E \Rightarrow M \) is in \( S \) if and only if \( E \Rightarrow m \) is in \( m \) for all \( m \in M \).

While the conjunctive condition is necessary to \( S \) being supported by a formula, it is not sufficient. An additional condition is required.
Definition 8 (Overreaching monotony condition) A set of explanations \( S = \{ E \Rightarrow M \} \) satisfies overreaching monotony if \( E \Rightarrow m \in S \) and \( E'' \Rightarrow m' \in S \) imply \( E' \Rightarrow m \in S \) when \( E \subseteq E' \subseteq E'' \).

This is a sort of “converging monotony” or “bilateral monotony”: \( E \Rightarrow m \) implies \( E' \Rightarrow m \) while \( E'' \Rightarrow m' \) implies the consistency of \( E' \) with the formula. A consequence is \( E' \Rightarrow m \). This is always the case for the abductive explanations of a formula.

Lemma 3 The set of explanations supported by an arbitrary formula satisfies the overreaching monotony condition.

Proof. The assumption is that a formula \( F \) supports a set of explanations \( S \). The claim is that \( S \) satisfies overreaching monotony: \( E \Rightarrow m \in S \), \( E'' \Rightarrow m' \in S \) and \( E \subseteq E' \subseteq E'' \) imply \( E \Rightarrow m \in S \).

A consequence of \( E \Rightarrow m \in S \) is that \( E \Rightarrow m \) is supported by \( F \). By definition, \( F \cup E \) entails \( m \). Since \( F \cup E' \) is a superset of \( F \cup E \), it entails \( m \) as well.

For the same reason, \( E'' \Rightarrow m' \in S \) implies that \( E'' \Rightarrow m' \) is supported by \( F \). This implies the consistency of \( F \cup E'' \). Since \( F \cup E' \) is a subset of \( F \cup E'' \), it is consistent as well.

The conclusions are that \( F \cup E' \) is consistent and entails \( m \). This defines \( F \) supporting \( E' \Rightarrow m \). Since \( S \) is the set of explanations of \( F \), it contains \( E' \Rightarrow m \).

Overreaching monotony is defined on single manifestations only because it only matters when the conjunctive property is satisfied.

Overreaching monotony is always met by the explanations supported by a formula, but not always by forgetting. A counterexample is \( F = \{ ab \Rightarrow m, \neg b \lor \neg c, ac \Rightarrow m' \} \) where the hypotheses are \( I = \{ a, b, c, d \} \) and the manifestations \( C = \{ m, m' \} \). Forgetting \( b \) turns \( ab \Rightarrow m \) into \( a \Rightarrow m \) and leaves \( ac \Rightarrow m' \) unaffected. Since \( F \) supports neither \( ac \Rightarrow m \) nor \( abc \Rightarrow m \), forgetting does not produce \( ac \Rightarrow m \).

The converse of the two lemmas is the case: the two conditions are not only necessary for a set of explanations to be supported by some formula, they are also sufficient.

Theorem 3 A set of explanations over disjoint hypotheses and manifestations is supported by a formula if and only if it satisfies the conjunctive and overreaching monotony conditions.

Proof. Lemma 2 and 3 tell that the set of explanations of a formula satisfies both conditions. In the other way around, if a set of explanations does not satisfy either condition, it is not the set of explanations of any formula.

The claim is proved by showing the opposite direction: if a set of explanations is not the set of explanations supported by any formula, it violates either condition. This is proved by showing that satisfying the conjunctive condition implies violating the overreaching monotony condition.

If \( S \) is not the set of explanations of any formula, it is not the set of explanations of \( G(S) \) since this is a formula. Two cases are possible: either \( G(S) \) supports an explanation that is not in \( S \), or it does not support an explanation in \( S \).

- \( G(S) \) supports \( E \Rightarrow m \notin S \);
The definition of \(G(S)\) supporting \(E \Rightarrow m\) is that \(G(S) \cup E\) is consistent and entails \(m\). The latter is the same as \(G(S) \models E \rightarrow m\). By Lemma 1, \(G(S)\) contains a clause \(E' \rightarrow m\) with \(E' \subseteq E\).

The containment of \(E'\) in \(E\) may be strict or not. The latter case \(E' = E\) implies \(E \rightarrow m \in G(S)\), which by construction implies \(E \Rightarrow m \in S\), which is not the case by assumption. The conclusion is that \(G(S)\) contains \(E' \rightarrow m\) with \(E' \subset E\). By construction, \(S\) contains \(E' \Rightarrow m\).

Since \(S\) contains \(E' \Rightarrow m\) and does not contain \(E \Rightarrow m\) where \(E' \subset E\), by construction \(G(S)\) contains \(E \rightarrow \bot\). This clause contradicts the consistency of \(G(S) \cup E\).

- \(G(S)\) does not support \(E \Rightarrow m \in S\);

Since \(E \Rightarrow m\) is in \(S\), by construction \(G(S)\) contains \(E \rightarrow m\). Since \(G(S)\) does not support \(E \Rightarrow m\), either \(G(S) \cup E\) is inconsistent, or it does not entail \(m\). The second case is not possible because \(G(S)\) contains \(E \rightarrow m\). As a result, \(G(S) \cup E\) is inconsistent.

This is the same as \(G(S) \models E \rightarrow \bot\). By Lemma 1, \(G(S)\) contains a clause \(E' \rightarrow \bot\) with \(E' \subseteq E\). This clause is in \(G(S)\) only if \(S\) does not contain \(E' \Rightarrow m'\), but it contains \(E'' \Rightarrow m'\) for some \(E'' \subseteq E'\) and some manifestation \(m'\). At the same time, \(S\) contains \(E \Rightarrow m\) by assumption. Also \(E' \subseteq E\) is the case.

In summary, \(S\) contains \(E'' \Rightarrow m'\) and \(E \Rightarrow m\); it does not contain \(E' \Rightarrow m'\); also \(E'' \subseteq E' \subseteq E\) holds. By swapping the names \(E\) and \(E''\) and the names \(m\) and \(m'\), these conditions become: \(S\) contains \(E \Rightarrow m\) and \(E'' \Rightarrow m'\) and does not contain \(E' \Rightarrow m\) where \(E \subseteq E' \subseteq E''\). Overreaching monotony would imply \(E' \Rightarrow m \in S\), and is therefore violated.

The conditions equate the existence of a formula supporting a given set of explanations. If this set is the result of forgetting variables from a formula, it is the existence of a formula supporting forgetting.

### 6 Complexity

The problem of establishing whether abductively forgetting variables from a propositional formula is supported by a formula is \(\Pi_p^3\)-complete. While it is harder than the building blocks of abduction, propositional satisfiability and entailment, it is still within the polynomial hierarchy. It cannot probably be solved in a polynomial amount of time, but requires only a polynomial amount of memory. Furthermore, its complexity is similar to that of other problems in abduction [EG95].

The actual proof is made of two parts: membership to \(\Pi_p^3\) and \(\Pi_p^3\)-hardness. Both rely on the necessary and sufficient condition proved in the previous section. The second also proves the conjunctive condition \(\Pi_p^3\)-hard. A separate proof shows the same for overreaching monotony. Checking each condition alone is \(\Pi_p^3\)-hard.

15
6.1 Membership

An arbitrary set of explanations $S = \{E \Rightarrow M\}$ is supported by a formula if and only if it meets both the conjunctive and overreaching monotony conditions. If this set of explanations is the result of forgetting, this is the problem of whether forgetting is expressed by a propositional formula.

A specific explanation $E \Rightarrow M$ results from forgetting if and only if $E' \subseteq I \cap R$ and $M' \subseteq C \cap R$ exists such that $EE' \Rightarrow MM'$ is supported. Depending on the definition of forgetting, $M \neq \emptyset$ or $M' = \emptyset$ is also required.

Which of the two definitions of forgetting is used does not matter. The result of forgetting changes, but the existence of a formula supporting it does not when selecting all explanations and not only the minimal ones. The proofs are for the second definition because it is slightly simpler, as it does not require a set $M'$ at all. An explanation $E \Rightarrow M$ is supported by forgetting if $ED \Rightarrow M$ where $D$ is an arbitrary subset of $I \cap R$.

Lemma 4 Checking whether $\text{summarize}(F, R)$ satisfies the conjunctive property is in $\Pi^p_3$.

Proof. The set of explanations $\text{summarize}(F, R)$ violates the conjunctive condition if either of the two following conditions is the case:

$$\exists E, M_1, M_2. \quad E \Rightarrow M_1 \notin \text{summarize}(F, R)$$

$$E \Rightarrow M_1 M_2 \in \text{summarize}(F, R)$$

$$\exists E, M_1, M_2. \quad E \Rightarrow M_1 \in \text{summarize}(F, R)$$

$$E \Rightarrow M_2 \in \text{summarize}(F, R)$$

$$E \Rightarrow M_1 M_2 \notin \text{summarize}(F, R)$$

The definition of $E \Rightarrow M \in \text{summarize}(F, R)$ is $\exists D \subseteq I \setminus R. ED \Rightarrow M \in \text{abduct}(F)$.

The first of the two conditions is never the case. Its second part $E \Rightarrow M_1 M_2 \in \text{summarize}(F, R)$ is the same as $\exists D \subseteq I \setminus R. ED \Rightarrow M_1 M_2 \in \text{abduct}(F)$, which is the same as the existence of a subset $D$ of $I \setminus R$ such that $F \cup E \cup D \models \perp$ and $F \cup E \cup D \models M_1 M_2$. Since $F \cup E \cup D \models M_1 M_2$ implies $F \cup E \cup D \models M_1$, this implies $\exists D. F \cup E \cup D \models \perp$ and $F \cup E \cup D \models M_1$, which define $E \Rightarrow M_1 \in \text{summarize}(F, R)$.

The second of the two conditions can be rewritten as follows.

$$\exists E, M_1, M_2. \quad \exists D \subseteq I \setminus R. ED \Rightarrow M_1 \in \text{abduct}(F)$$

$$\exists D \subseteq I \setminus R. ED \Rightarrow M_2 \in \text{abduct}(F)$$

$$\forall D \subseteq I \setminus R. ED \Rightarrow M_1 M_2 \notin \text{abduct}(F)$$

Making $\text{abduct}(F)$ explicit:

$$\exists E, M_1, M_2. \quad \exists D \subseteq I \setminus R. F \cup E \cup D \models \perp \text{ and } F \cup E \cup D \models M_1$$

$$\exists D \subseteq I \setminus R. F \cup E \cup D \models \perp \text{ and } F \cup E \cup D \models M_2$$

$$\forall D \subseteq I \setminus R. F \cup E \cup D \models \perp \text{ or } F \cup E \cup D \models \perp 
\text{ and } M_1 M_2 \models \perp$$
The first two subconditions seem to negate the third, but they do not since the subset $D$ may differ from each other. An example is a first subcondition satisfied only by a subset $D$ not entailing $M_2$, and a second subcondition satisfied only by another subset $D$ not entailing $M_1$.

Checking entailment requires a universal quantifier, checking consistency requires an existential one. The most alternations are in the third subcondition: $\exists E, \ldots \forall D. F \cup E \cup D \not\models M_1M_2$. Three quantifiers, first existential equals $\Sigma_3^n$.

This is the complexity class of checking whether the conjunctive condition is violated. The class of checking whether it is met is its complement, $\Pi_3^n$.

Checking overreaching monotony has the same complexity: is in $\Pi_3^n$.

**Lemma 5** Checking whether $\text{summarize}(F, R)$ satisfies overreaching monotony is in $\Pi_3^n$.

**Proof.** Overreaching monotony is violated by $\text{summarize}(F, E)$ if:

\[
\exists E, E', E'', m, m'. \quad E \subseteq E' \subseteq E'' \\
E \Rightarrow m \in \text{summarize}(F, R) \\
E' \Rightarrow m \not\in \text{summarize}(F, R) \\
E'' \Rightarrow m' \in \text{summarize}(F, R)
\]

The definition of $E \Rightarrow m \in \text{summarize}(F, R)$ is $\exists D \subseteq I \setminus R. E \Rightarrow m \in \text{abduct}(F)$.

\[
\exists E, E', E'', m, m'. \quad E \subseteq E' \subseteq E'' \\
\exists D \subseteq I \setminus R. E \Rightarrow m \in \text{abduct}(F) \\
\forall D \subseteq I \setminus R. E' \Rightarrow m \not\in \text{abduct}(F) \\
\exists D \subseteq I \setminus R. E'' \Rightarrow m' \in \text{abduct}(F)
\]

In turn, $ED \Rightarrow m \in \text{abduct}(F)$ is defined as $F \cup E \cup D \not\models \bot$ and $F \cup E \cup D \models m$.

\[
\exists E, E', E'', m, m'. \quad E \subseteq E' \subseteq E'' \\
\exists D \subseteq I \setminus R. F \cup E \cup D \not\models \bot \text{ and } F \cup E \cup D \models m \\
\forall D \subseteq I \setminus R. F \cup E' \cup D \models \bot \text{ or } F \cup E' \cup D \not\models m \\
\exists D \subseteq I \setminus R. F \cup E'' \cup D \not\models \bot \text{ and } F \cup E'' \cup D \models m'
\]

Checking entailment requires a universal quantifier, checking consistency requires an existential one. The most alternations are in the second part: $\exists E, \ldots \forall D. F \cup E' \cup D \not\models m$. Three quantifiers, first existential equals $\Sigma_3^n$.

This is the complexity of checking whether overreaching monotony is violated. The checking whether it is met is its complement, $\Pi_3^n$. \qed

The two lemmas gives an upper bound to the complexity of checking whether forgetting is supported by some formula.

**Lemma 6** Checking whether $\text{summarize}(F, R)$ is supported by a formula is in $\Pi_3^n$. 

\[17\]
Proof. The problem amounts to checking the conjunctive and the overreaching monotony conditions in parallel. Each is in $\Pi^p_3$ by Lemma 4 and Lemma 5. The problem is therefore in $\Pi^p_3$.

6.2 Hardness

The hardness of checking the existence of a formula supporting forgetting is shown to be $\Pi^p_3$-hard. The following lemma proves this claim by showing that the conjunctive condition is $\Pi^p_3$-hard.

Lemma 7 The following two problems are $\Pi^p_3$-hard: checking whether $\text{summarize}(F, R)$ satisfies the conjunctive property; checking whether $\text{summarize}(F, R)$ is supported by some formula.

Proof. The hardness of the conjunctive condition is shown first. The reduction is then shown to also prove the hardness of checking whether forgetting is supported by some formula. Both results are proved by showing the reverse problem $\Sigma^p_3$-hard.

The reduction for the conjunctive property is from the validity of a quantified Boolean formula $\exists X \forall Y \exists Z. F$. Its variables are $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_n\}$. The corresponding formula $G$ is the following, where all variables not in $X \cup Y \cup Z$ are new.

\[
G = XPS \cup XNS \cup XPN \cup XSS \cup XPX \cup XNX \cup YPY \cup YNY \cup \{FM\}
\]

\[
XPS = \{x^p_i x^s_i \rightarrow m_i \mid 1 \leq i \leq n\}
\]

\[
XNS = \{x^n_i x^s_i \rightarrow m_i \mid 1 \leq i \leq n\}
\]

\[
XPN = \{x^p_i x^n_i \rightarrow \bot \mid 1 \leq i \leq n\}
\]

\[
XSS = \{x^s_i \land m_1 \land \cdots \land m_n \rightarrow \bot \mid 1 \leq i \leq n\}
\]

\[
XPX = \{x^p_i \rightarrow x_1 \mid 1 \leq i \leq n\}
\]

\[
XNX = \{x^n_i \rightarrow \neg x_1 \mid 1 \leq i \leq n\}
\]

\[
YPY = \{y^p_i \rightarrow y_1 \mid 1 \leq i \leq n\}
\]

\[
YNY = \{y^n_i \rightarrow \neg y_1 \mid 1 \leq i \leq n\}
\]

\[
FM = F \lor (\neg x^p_1 \land \neg x^n_1) \lor \cdots \lor (\neg x^p_n \land \neg x^n_n) \lor (\neg y^p_1 \land \neg y^n_1) \lor \cdots \lor (\neg y^p_n \land \neg y^n_n) \lor (m_1 \land \cdots \land m_n)
\]

The hypotheses $I$, manifestations $C$ and variables to remember $R$ are:

\[
I = \{x^p_i, x^n_i, x^s_i, y^p_i, y^n_i \mid 1 \leq i \leq n\}
\]

\[
C = \{m_i \mid 1 \leq i \leq n\}
\]

\[
R = \{x^p_i, x^n_i, m_i \mid 1 \leq i \leq n\}
\]

The explanations $\text{summarize}(G, R)$ violate the conjunctive condition when the quantified Boolean formula $\exists X \forall Y \exists Z. F$ is true.
What does the parts of the formula do?
The first part $XPS \cup XNS \cup XPN$ comprises three clauses for each index $i$:

\[
\begin{align*}
  x_i^p & \rightarrow m_i \\
  x_i^n & \rightarrow m_i \\
  x_i^p & \rightarrow \bot 
\end{align*}
\]

Each manifestation $m_i$ is explained by $\{x_i^p, x_i^n\}$ and by $\{x_i^n, x_i^p\}$ but not by their union. When forgetting $x_i^p$, these explanations turn into $x_i^p \Rightarrow m_i$ and $x_i^n \Rightarrow m_i$, while $\{x_i^p, x_i^n\}$ and its supersets do not explain $m_i$.

This extends from single manifestations to sets of manifestations, with an exception. For example, $\{m_1, m_2\}$ is explained by any combination of one hypothesis among $x_1^p$ and $x_1^n$ and one among $x_2^p$ or $x_2^n$. The exception is the set of all manifestations $\{m_1, \ldots, m_n\}$. The clauses of $XSS$ rule out these explanations.

\[x_i^p \land m_1 \land \cdots \land m_n \rightarrow \bot\]

Regardless of how $m_1, \ldots, m_n$ are entailed, they contradict all variables $x_i^p$. No explanation of $\{m_1, \ldots, m_n\}$ contains any $x_i^p$. The arbitrary combinations of $x_i^p$ and $x_i^n$ no longer explain these manifestations.

The conclusion is that all sets of manifestations are explained by an arbitrary combination of $x_i^p$ and $x_i^n$, except the set of all manifestations $\{m_1, \ldots, m_n\}$.

This is a violation of the conjunctive condition unless all such combinations of $x_i^p$ and $x_i^n$ explain $\{m_1, \ldots, m_n\}$ through other clauses.

The other hypotheses to forget are $y_i^p$ and $y_i^n$. The clauses allowing them to entail $m_1, \ldots, m_n$ are $XPX \cup XNX \cup YPY \cup YNY$:

\[
\begin{align*}
  x_i^p & \rightarrow x_i \\
  x_i^n & \rightarrow \neg x_i \\
  y_i^p & \rightarrow y_i \\
  y_i^n & \rightarrow \neg y_i
\end{align*}
\]

The only variables to remember among these are $x_i^p$ and $x_i^n$. A complete combination of them, supplemented by a complete combination of $y_i^p$ and $y_i^n$, forces a complete evaluation over the variables $X \cup Y$. Only such evaluations falsify the central part of the subformula $FPN$:

\[
F \lor (\neg x_1^p \land \neg x_1^n) \lor \cdots \lor (\neg x_n^p \land \neg x_n^n) \lor (\neg y_1^p \land \neg y_1^n) \lor \cdots \lor (\neg y_n^p \land \neg y_n^n) \lor (m_1 \land \cdots \land m_n)
\]

Only when the central part of this formula is false, and $F$ is falsified by the evaluation over $X$ and $Y$ regardless of $Z$, the final part $m_1 \land \cdots \land m_n$ is forced to be true. In the other way around, when an evaluation over $X$ exists such that for all values of $Y$ the formula $F$ is satisfiable, $\{m_1, \ldots, m_n\}$ is not explained this way.
This proves that $\exists X \forall Y \exists Z. F$ equals \{m_1, \ldots, m_n\} not being explained by a combination of $x^p_i$ and $x^n_i$ for all $1 \leq i \leq n$, in violation of the conjunctive condition.

The conclusion is that the conjunctive condition is falsified exactly when the QBF is true. This proves that the conjunctive condition is $\Pi^P_3$-hard.

The correspondence between the validity of the QBF and the violation of the conjunctive condition can be rewritten as follows: if the conjunctive condition is false, the QBF is true; if the QBF is true, the conjunctive condition is false. The latter proves that forgetting is not supported by any formula if the QBF is true. The only missing part is that overreaching monotony is true when the QBF is false. Rather than proving that, a formula that supports

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The correspondence between the validity of the QBF and the violation of the conjunctive condition can be rewritten as follows: if the conjunctive condition is false, the QBF is true; if the QBF is true, the conjunctive condition is false. The latter proves that overreaching monotony is true when the QBF is false. Rather than proving that, a formula that supports

When the QBF is false, each $m_i$ is explained by $x^p_i$, $x^n_i$ and every superset that does not contain two variables of the same index. These are the explanations supported by the formula \{ $x^p_i \rightarrow m_i$, $x^n_i \rightarrow m_i$, $x^p_i x^n_i \rightarrow \bot \mid 1 \leq i \leq n$ \}. This concludes the proof. □

### 6.3 Hardness of overreaching monotony

The previous lemma shows that checking the conjunctive condition is $\Pi^P_3$-hard. This may suggest that this is the difficult part of the problem. This is not the case: overreaching monotony is equally hard.

**Lemma 8** Checking whether summarize($F, R$) satisfies overreaching monotony is $\Pi^P_3$-hard.

**Proof.** Reduction is from the validity of a quantified Boolean formula $\exists X \forall Y \exists Z. F$ to the violation of overreaching monotony. The three sets of variables are $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_n\}$. The corresponding formula $G$ is the following, where all variables not in $X \cup Y \cup Z$ are new.

\[
G = \{AC\} \cup XPX \cup XNX \cup \{XC\} \cup YPY \cup YNY \cup \{FM\} \cup \{AB\}
\]

\[
AC = ac \rightarrow m
\]

\[
XPX = \{x_i^p \rightarrow x_i \mid 1 \leq i \leq n\}
\]

\[
XNX = \{x_i^n \rightarrow \neg x_i \mid 1 \leq i \leq n\}
\]

\[
XC = a(x_1^p \lor x_1^n) \ldots (x_n^p \lor x_n^n) \rightarrow \neg c
\]

\[
YPY = \{y_i^p \rightarrow y_i \mid 1 \leq i \leq n\}
\]

\[
YNY = \{y_i^n \rightarrow \neg y_i \mid 1 \leq i \leq n\}
\]

\[
FM = F \lor \neg a \lor c \lor (\neg y_1^p \land \neg y_1^n) \lor \cdots \lor (\neg y_n^p \land \neg y_n^n) \lor m
\]

\[
AB = ab \rightarrow m
\]

The hypotheses $I$, manifestations $C$ and variables to remember $R$ are:

\[
I = \{a, b, c\} \cup \{x_i^p, x_i^n, y_i^p, y_i^n \mid 1 \leq i \leq n\}
\]

\[
C = \{m\}
\]

\[
R = \{a, b\} \cup \{x_i^p, x_i^n \mid 1 \leq i \leq n\} \cup \{m\}
\]
The claim is that $\exists X \forall Y \exists Z. F$ is true if and only if overreaching monotony is violated by summarize($G, R$). It is proved by linking each evaluation over $X$ that satisfies $\forall Y \exists Z. F$ with a pair of sets $E', E''$ that falsifies overreaching monotony for $m, m'$ and some $E$.

Such a violation occurs when summarize($G, R$) contains $E \Rightarrow m$ and $E'' \Rightarrow m'$, but not $E' \Rightarrow m$ for some $E \subseteq E' \subseteq E''$. This is proved with $m' = m, E = \{a\}, E'$ equal to $E$ with the addition of either $x_i^p$ or $x_i^n$ depending on the value of $x_i$ for each $i$ between 1 and $n$, and $E''$ equal to $E'$ with the addition of $b$.

The explanation $E \Rightarrow m$ is in summarize($G, R$) because $G$ contains $ac \rightarrow m$ and $G \cup \{a, c\}$ is consistent; for example, it is satisfied by setting $a, c$ and $m$ to true and all other variables to false.

The explanation $E'' \Rightarrow m$ is also in summarize($G, R$). The clause $ab \rightarrow m$ allows entailing $m$ since $E''$ contains both $a$ and $b$. The consistency of $G \cup E''$ is proved by showing a model that satisfies it. This model sets $x_i$ to true if $E''$ contains $x_i^p$ and to false if $E''$ contains $x_i^n$. It also sets $a, b$ and $m$ to true and all other variables to false.

Since forgetting contains both $E \Rightarrow m$ and $E'' \Rightarrow m$ for every evaluation of $X$, overreaching monotony requires $E' \Rightarrow m$ as well for every evaluation of $X$. This is proved to be the case if the QBF is false. This way, overreaching monotony is violated when the QBF is true.

Two clauses of $G \cup E'$ may entail $m$: $ac \rightarrow m$ and $FM = F \lor \neg a \lor c \lor \neg (y_1^p \land y_1^n) \lor \ldots \lor \neg (\neg y_n^p \land \neg y_n^n) \lor m$. The clause $ab \rightarrow m$ does not entail $m$ since one of its preconditions is $b$, which occurs positive neither in $G$ nor in $E'$.

The QBF is false if $\exists Y \forall Z. \neg F$ holds for all evaluations of $X$. Each such evaluation corresponds to a set of hypotheses $E'$ containing $x_i^p$ if the evaluation sets $x_i$ to true and $x_i^n$ otherwise. This way, either $x_i^p$ or $x_i^n$ is in $E'$ for every index $i$. Since $E'$ also contains $a$, the clause $XC = a(x_i^p \lor x_i^n) \ldots (x_n^p \lor x_n^n) \rightarrow \neg c$ makes $G \cup E'$ entail $\neg c$. Therefore, adding $c$ to $E'$ violates consistency.

The clauses $XPX = \{x_i^p \rightarrow x_i \mid 1 \leq i \leq n\}$ and $XNX = \{x_i^n \rightarrow \neg x_i \mid 1 \leq i \leq n\}$ force the values of $x_i$ as in the evaluation. The same goes for $Y$: for each of its evaluations, the corresponding hypotheses $y_i^p$ and $y_i^n$ make $G$ entail the value of $y_i$. Adding these hypotheses to $E'$ makes it entail $\neg F$ since $\forall Z. \neg F$ holds for these evaluations of $X$ and $Y$. As a result, $FM = F \lor \neg a \lor c \lor \neg (y_1^p \land y_1^n) \lor \ldots \lor \neg (\neg y_n^p \land \neg y_n^n) \lor m$ and $E'$ entail $m$. This proves that if the QBF is false, then every such $E'$ explains $m$, satisfying overreaching monotony. The converse is also the case. If the QBF is true, the preconditions of this clause is falsified by a value of $Z$. The head $m$ is not entailed. Overreaching monotony is violated.

This proves that overreaching monotony is violated if the QBF is true. The proof is completed by showing that if the QBF is false, summarize($G, R$) is supported by a formula. Lemma 3 then proves that summarize($G, R$) satisfies overreaching monotony.

The set of explanations summarize($G, R$) always comprises $a \Rightarrow m, ab \Rightarrow m$, and $abE \Rightarrow m$ for all sets $E$ including some $x_i^p$ and $x_i^n$ but not both for the same index. It also comprises $aE \Rightarrow m$ for all $E$ including some $x_i^p$ and $x_i^n$ but not both for the same index, and not for all indexes $i$. Since the QBF is false, it also includes the explanations $aE \Rightarrow m$ where $E$ contains either $x_i^p$ or $x_i^n$ but not both for all indexes $i$. Such explanations are supported by the following formula.

$$\{a \rightarrow m\} \cup \{x_i^a x_i^b \rightarrow \bot \mid 1 \leq i \leq n\}$$

This provides an alternative proof of the hardness of the existence of a formula supporting summarize($G, R$).
6.4 Complexity characterization

The following theorem sums up what proved about the complexity of the problem of the existence of a formula supporting forgetting.

**Theorem 4** The following problems are $\Pi^p_3$-complete:

- checking whether $\text{summarize}(F, R)$ satisfies the conjunctive property;
- checking whether $\text{summarize}(F, R)$ satisfies overreaching monotony;
- checking whether $\text{summarize}(F, R)$ is supported by some formula.

*Proof.* Consequence of Lemma 4, Lemma 5, Lemma 7 and Lemma 8.

7 Minimal explanations

While all explanations are possible causes of the manifestations, some contain unjustified elements. An extreme example is $b$ in the explanation $ab$ of the manifestation $c$ according to $\{a \rightarrow c\}$. Not only $b$ is unnecessary, is not even mentioned by the formula. It is not justified by the need of explaining $c$. Such redundant parts of the explanations can be excluded by selecting only minimal explanations.

Minimality may be according to set containment, or to cardinality, or to some complex criteria [PPU03, DSTW04]. An ordering $\leq$ is assumed given; only explanations not greater than others are accepted.

The ordering may be $\subseteq$, it may be cardinality comparison, it may be something else [EG95, PPU03, DSTW04]. It is assumed to refine set containment: if $E' \subseteq E$, then $E' < E$.

The analysis of forgetting for minimal explanations replicates that for all: first, the tentative-supporting formula is defined and proved to minimally support the given set of explanations if any does; second, a necessary and sufficient condition to a set of explanations being minimally supported by some formula is given; third, complexity results are proved.

7.1 The tentative supporting formula

The tentative-supporting formula of a set of explanations is a formula that minimally supports it if one exists. The set of explanations is arbitrary; it could be the result of forgetting, but not necessarily. The tentative-supporting formula is built from clauses that are mandatory to support the explanations.

The same principle applies to minimal explanations. The difference is that instead of the necessary clauses to support the explanations, it comprises the necessary clauses to minimally support them.

The set of explanations is still arbitrary at this early stage. It may for example contain both $E \Rightarrow m$ and $E' \Rightarrow m$ with $E < E'$, which are of course impossible to minimally support at the same time. As said upfront, supporting a set of explanations may not be possible anyway.
The first rule is that $E \Rightarrow M$ requires the formula to entail $E \rightarrow M$. Since the formula entails the implication, it may as well contain it; semantically, there will be no difference. To support $E \Rightarrow M$, all clauses $E \rightarrow m$ for $m \in M$ are necessary.

The second rule is for clauses $E \rightarrow \bot$. A formula supporting a set of explanations entails such a clause whenever it entails $E \rightarrow M$ for other reasons, but $E \Rightarrow M$ is not in the set of explanations to support. For minimal explanations, the clause is not required when the set contains $E' \Rightarrow M$ with $E' < E$.

**Definition 9** The tentative-supporting minimal abduction formula of a set of explanations $S$ is the formula $G(S)$ comprising:

1. a clause $E \rightarrow m$ for each $E \Rightarrow M \in S$ and $m \in M$;
2. a clause $E \rightarrow \bot$ for each $E \Rightarrow M \notin S$ such that
   
   (a) for all $m \in M$ there exist $E'$ and $M'$ such that $E' \subseteq E$, $m \in M'$ and $E' \Rightarrow M' \in S$, and
   
   (b) for all $E' < E$, it holds $E' \Rightarrow M \notin S$.

Condition 1 cannot be simplified into "$E \Rightarrow m \in S$". When minimizing by cardinality, $E$ may minimally explain $M$ but none of its elements because each of them is explained by a set $E'$ smaller than $E$. For the same reason, Condition 2a cannot be simplified into "$E' \Rightarrow m \in S$ for every $m \in M$".

The tentative-supporting formula $G(S)$ is proved to support $S$ if and only if such a formula exists. Necessity is obvious: if $G(S)$ supports $S$, a formula supporting $S$ exists because it is $G(S)$ itself. The bulk of the proof is the other direction: if a formula supports $S$, then $G(S)$ does as well.

The claim is proved in two steps: first, if a formula $F$ supports $S$ but $G(S)$ does not, then $G(S)$ contains a clause $E \rightarrow \bot$ that $F$ does not entail; second, every formula supporting $S$ entails every clause of $G(S)$.

The first part is stated in a slightly different way to be also useful in a following proof.

It assumes, like all following results, that $E' \subseteq E$ implies $E' < E$. Also assumed is the separation of hypotheses and manifestations: $I \cap C = \emptyset$, which is required by Lemma 1. The further requirement of the lemma, that no head of the formula is in a body, is guaranteed by the definition of the tentative-supporting formula $G(S)$ and the separation of hypotheses and manifestations.

**Lemma 9** If $G(S)$ does not minimally support a set of explanations $S$ over disjoint hypotheses and manifestations then either:

1. $S$ contains both $E \Rightarrow M$ and $E' \Rightarrow M$ for some $E' < E$;
2. for some $E$, $E' \subseteq E$ and $M$, the following two conditions hold:
   
   (a) $S$ contains $E \Rightarrow M$;
   
   (b) $E' \rightarrow \bot \in G(S)$.
Proof. The assumption is that \( G(S) \) does not minimally support \( S \). This may happen for two reasons: either \( G(S) \) does not support an explanation of \( S \), or it supports an explanation not in \( S \). This condition is summarized as “\( G(S) \) and \( S \) differ on \( E \Rightarrow M \)”.

The claim is proved by induction. Both the base and the induction steps rely on the same fact: if \( G(S) \) and \( S \) differ on an explanation \( E \Rightarrow M \), then either:

1. \( G(S) \) and \( S \) also differ on another explanation \( E' \Rightarrow M' \) with \( E' < E \);
2. \( S \) contains both \( E \Rightarrow M \) and \( E' \Rightarrow M \) for some \( E' < E \); or
3. for some \( E, E' \subseteq E \) and \( M \), the following two conditions hold:
   (a) \( S \) contains \( E \Rightarrow M \);
   (b) \( E' \rightarrow \bot \in G(S) \).

This is the induction claim; the base case is when \( E \) is minimal: since no \( E' < E \) exists, either the second or the third alternative is true, as the claim of the lemma demands.

If \( G(S) \) and \( S \) differ on some explanation, then either \( G(S) \) does not support an explanation in \( S \) or it supports an explanation not in \( S \).

- **\( G(S) \) does not support \( E \Rightarrow M \in S \)**

  Since \( E \Rightarrow M \) is in \( S \), by construction \( G(S) \) contains \( E \rightarrow m \) for every \( m \in M \). As a result, \( G(S) \cup E \models M \). Since \( G(S) \) does not support \( E \Rightarrow M \), either it is inconsistent with \( E \) or it supports another explanation \( E' \Rightarrow M \) with \( E' < E \).

  The first case is \( G(S) \models E \Rightarrow \bot \). By Lemma 1, \( G(S) \) contains a clause \( E' \rightarrow \bot \) with \( E' \subseteq E \). With the assumption \( E \Rightarrow M \in S \), this is the third of the three alternatives of the claim.

  The second case is that that \( G(S) \) supports \( E' \Rightarrow M \) with \( E' < E \). This explanation \( E' \Rightarrow M \) may be in \( S \) or not.

  - If \( S \) contains \( E' \Rightarrow M \), it satisfies the second of the three alternatives of the claim, since it contains both \( E \Rightarrow M \) by assumption and \( E' \Rightarrow M \) with \( E' < E \).
  - If \( S \) does not contain \( E' \Rightarrow M \), it differs from \( G(S) \) on this explanation since \( G(S) \) supports \( E' \Rightarrow M \) instead. Since \( E' < E \), this is the first of the three alternatives of the claim.

- **\( G(S) \) supports \( E \Rightarrow M \notin S \)**

  Since \( G(S) \) supports \( E \Rightarrow M \), it entails \( E \rightarrow M \) and \( G(S) \cup E \) is consistent.

  The entailment \( G(S) \models E \rightarrow M \) can be rewritten as \( G(S) \models E \rightarrow m \) for every \( m \in M \). Lemma 1 tells that each such entailment implies the presence of some clause \( E' \rightarrow m \) in \( G(S) \) with \( E' \subseteq E \). Such a clause is in \( G(S) \) only if \( S \) contains an explanation \( E' \Rightarrow M' \) with \( m \in M \). In summary, for each \( m \in M \) there exist \( E' \) and \( M' \) such that \( E' \subseteq E, m \in M \), and \( E' \rightarrow M' \in S \).

  This is the second premise of the presence of \( E \rightarrow \bot \) in \( G(S) \). The first is \( E \Rightarrow M \notin S \), which is an assumption. The conclusion \( E \rightarrow \bot \in G(S) \) conflicts with the consistency of \( G(S) \cup E \). As a result, the third premise is false.
The third premise is that $S$ does not contain any explanation $E' \Rightarrow M$ with $E' < E$. Its converse is that $S$ contains $E' \Rightarrow M$ with $E' < E$. By construction, $G(S)$ contains $E' \rightarrow m$ for each $m \in M$; therefore, it entails $E' \rightarrow M$.

Either $G(S) \cup E'$ is consistent, or it is not.

- $G(S) \cup E'$ is consistent
  Since $G(S)$ entails $E' \rightarrow M$, it supports $E' \Rightarrow M$. This contradicts the assumption that $G(S)$ minimally supports $E \Rightarrow M$ since $E' < E$.

- $G(S) \cup E'$ is inconsistent
  A consequence is that $G(S)$ does not support $E' \Rightarrow M$. This explanation is in $S$. As a result, $G(S)$ and $S$ differ on an explanation $E' \rightarrow M$ with $E' < E$. This is the first of the three alternatives of the claim.

The next step is that $G(S)$ could fail at supporting $S$ only because it contains a clause that is not necessary.

**Lemma 10** If $G(S)$ does not minimally support a set of explanations $S$ over disjoint hypotheses and manifestations, then it contains a clause $E \Rightarrow \bot$ that is not entailed by any formula $F$ supporting $S$.

**Proof.** The assumptions are that $F$ supports $S$ and that $G(S)$ does not support $S$.

Since $G(S)$ does not support $S$, Lemma 9 applies: either the first or the second condition is true.

The definition of minimal support forbids $F$ to support both $E \Rightarrow M$ and $E' \Rightarrow M$ with $E' < E$. Since $S$ is the set of minimal explanations of $F$, it does not contain both explanations. The first alternative claim of Lemma 9 is therefore false. The second is therefore true.

The second alternative claim is that $G(S)$ contains $E' \rightarrow \bot$ and $S$ contains $E \Rightarrow M$ for some $E$, $E' \subseteq E$ and $M$. Since $F$ supports $S$, it supports $E \Rightarrow M$. If it entailed $E' \rightarrow \bot$, it would entail $E \rightarrow \bot$ as well since $E' \subseteq E$. It does not entail $E \rightarrow \bot$ since it supports $E \Rightarrow M$. The conclusion is that $F$ does not entail $E' \rightarrow \bot$, which is in $G(S)$.

The final step is that the clauses $E \rightarrow \bot$ of $G(S)$ are all necessarily entailed.

**Lemma 11** If $F$ minimally supports a set of explanations $S$, it entails every clause $E \rightarrow \bot$ of $G(S)$.

**Proof.** The assumptions are that $F$ supports $S$ and that $G(S)$ contains a clause $E \rightarrow \bot$. The claim is that $F \models E \rightarrow \bot$.

Since $E \rightarrow \bot$ is in $G(S)$, the second condition of Definition 9 is true: there exists $M$ such that $E \rightarrow M \notin S$, for all $m \in M$ there exist $E'$ and $M'$ such that $E' \subseteq E$, $m \in M'$ and $E' \Rightarrow M' \in S$, and for all $E' < E$, it holds $E' \Rightarrow M \notin S$.

Since $F$ supports $S$, these conditions translate to $F$:

- $F$ does not support $E \Rightarrow M$;
• for all \( m \in M \), there exist \( E' \) and \( M' \) such that \( E' \subseteq E \), \( m \in M' \) and \( F \) supports \( E' \Rightarrow M' \); and

• \( F \) does not support \( E' \Rightarrow M \) for any \( E' < E \).

From the second point, \( F \) entails \( E' \Rightarrow M' \). Since \( m \in M' \), it entails \( E \Rightarrow m \). Since \( E' \subseteq E \), it entails \( E \Rightarrow m \). This is the case for all \( m \in M \). Therefore, \( F \) entails \( E \Rightarrow M \).

The union \( F \cup E \) may be consistent or not. If it is consistent, then \( F \) supports \( E \Rightarrow M \). This is a minimal explanation of \( M \) since by the third point \( F \) does not support any explanation \( E' \Rightarrow M \) with \( E' < E \). The conclusion is that \( F \) minimally supports \( E \Rightarrow M \), contradicting the first point above. The conclusion is that the assumed consistency of \( F \cup E \) does not hold.

Since \( F \cup E \) is inconsistent, \( F \) entails \( E \Rightarrow \bot \). This is the claim.

Theorem 5 A formula minimally supporting a set of explanations \( S \) over disjoint hypotheses and manifestations exists if and only if \( G(S) \) does.

Proof. If \( G(S) \) supports \( S \) then a formula supporting \( S \) exists: \( G(S) \).

The other direction is proved by contradiction: \( F \) supports \( S \) and \( G(S) \) does not are shown impossible at the same time. These are the premises of Lemma 11 and Lemma 10. The conclusions respectively are that \( F \) entails every clause \( E \Rightarrow \bot \) of \( G(S) \), and that \( G(S) \) contains a clause \( E \Rightarrow \bot \) not entailed by \( F \). This is a contradiction.

7.2 The equivalent conditions

A necessary and sufficient condition tells whether a set of explanations is minimally supported by a formula. Like the analogous condition for non-minimal support, it still applies to an arbitrary set of explanations, which may or may not be the minimal explanations supported by a formula. It however radically differs from that. Overreaching monotony is not only unnecessary, it is always false in the current assumption that \( E \subset E' \) implies \( E < E' \): the support of an explanation \( E \Rightarrow m \) forbids that of \( E' \Rightarrow m \) if \( E' < E \).

The conjunctive condition holds only in one direction: \( E \Rightarrow M_1 \) and \( E \Rightarrow M_2 \) imply \( E \Rightarrow M_1 M_2 \), but not the other way around. A counterexample is \( F = \{ab \rightarrow x, ac \rightarrow y\} \) where the hypotheses are \( I = \{a, b, c\} \), and the manifestations are \( C = \{x, y\} \). The explanation \( abc \Rightarrow xy \) is minimally supported, but \( abc \Rightarrow x \) is not because of \( ab \Rightarrow x \).

While the conjunctive and overreaching monotony conditions are not even sufficient for minimal explanations, two other conditions replace them.

The first condition is that two explanations of the same manifestations cannot be one less than the other.

Definition 10 A set of explanations \( S \) satisfies the minority condition if it does not contain both \( E \Rightarrow M \) and \( E' \Rightarrow M \) with \( E' < E \).

The second condition is not that simple. It is in essence the converse of the second rule of construction of the tentative-supporting formula \( G(S) \): if a set of hypotheses is inconsistent because an unsupported explanation would otherwise be supported, it does not explain any other manifestation.
Definition 11 A set of explanations $S$ satisfies the inconsistency condition if no $E$, $E' \subseteq E$, $M$ and $M'$ are such that:

1. $E \Rightarrow M \in S$;
2. $E' \Rightarrow M' \notin S$;
3. $\forall m \in M'. \exists E'', M''. E'' \subseteq E', m \in M'', E'' \Rightarrow M'' \in S$;
4. $\forall E'' < E. E'' \Rightarrow M' \notin S$.

In words: a set of hypotheses $E'$ is inconsistent with any supporting formula because otherwise $E' \Rightarrow M' \in S$; consequently, no superset of $E'$ explain any other manifestation.

These two conditions are equivalent to the existence of a formula supporting $S$. The first step of the proof of this claim is that they are both true when $S$ is the set of explanations supported by a formula.

Lemma 12 The set of explanations minimally supported by an arbitrary formula satisfies the minority condition (Definition 10).

Proof. The negation of the condition is that the formula $F$ supports both $E \Rightarrow M$ and $E' \Rightarrow M'$ with $E' < E$. By definition, $E \Rightarrow M$ is minimally supported if $F \cup E$ is consistent, it entails $M$, and $F$ does not support $E' \Rightarrow M'$ with $E' < E$. The latter contradicts the assumption that $E' \Rightarrow M'$ is supported.

The inconsistency condition is also the case.

Lemma 13 The set of explanations minimally supported by an arbitrary formula satisfies the inconsistency condition (Definition 11).

Proof. The contrary of the claim is proved to be contradictory: the following is the case for some $E$, $E' \subseteq E$, $M$ and $M'$, where $F$ is a formula supporting $S$:

1. $F \models E \Rightarrow M$;
2. $F \not\models E' \Rightarrow M'$;
3. $\forall m \in M'. \exists E'', M''. E'' \subseteq E', m \in M'', F \models E'' \Rightarrow M''$;
4. $\forall E'' < E. F \not\models E'' \Rightarrow M'$.

By the third point, $F$ supports $E'' \Rightarrow M''$ for some $E''$ and $M''$. As a result, it entails $E'' \Rightarrow m$ since $m \in M''$. Since $E'' \subseteq E'$, it also entails $E' \Rightarrow m$. Since this is the case for all $m \in M'$, it entails $E' \Rightarrow M'$.

If $F \cup E'$ is inconsistent, then $F \cup E$ is inconsistent as well since $E' \subseteq E$. Therefore, $F$ does not explain $E \Rightarrow M$. This contradicts the first assumption $F \models E \Rightarrow M$. As a result, $F \cup E'$ is consistent.

Since $F$ is consistent with $E'$ and entails $M'$, it supports $E' \Rightarrow M'$. It also supports it minimally because of the last point above: no set of explanations $E''$ that is strictly less than
$E'$ explains $M'$. The conclusion is that $F$ minimally supports $E' \Rightarrow M'$. This is the opposite of the second assumption. □

The set of explanations of every formula satisfies both conditions. The converse is also the case: if a set of explanations satisfies both condition, it is supported by some formula. This is proved in reverse: if $S$ is supported by no formula, it violates either condition.

**Lemma 14** If no formula minimally supports a set of explanations $S$ over disjoint hypotheses and manifestations, then $S$ violates either the minority condition (Definition 10) or the inconsistency condition (Definition 11).

Proof. The assumption is that no formula supports $S$. The claim is that either of the following conditions is true:

1. $E \Rightarrow M$ and $E' \Rightarrow M$ are both in $S$ where $E' < E$; or

2. there exist $E$, $E' \subseteq E$, $M$ and $M'$ such that:
   
   (a) $E \Rightarrow M \in S$;
   
   (b) $E' \Rightarrow M' \notin S$;
   
   (c) for all $m \in M'$ there exist $E''$ and $M''$ such that $E'' \subseteq E'$, $m \in M''$ and $E'' \Rightarrow M'' \in S$;
   
   (d) no $E'' \Rightarrow M'$ is in $S$ for $E'' < E'$.

Since no formula supports $S$ and $G(S)$ is a formula, $G(S)$ does not support $F$. By Lemma 9, either:

1. $S$ contains both $E \Rightarrow M$ and $E' \Rightarrow M$ for some $E' < E$; or

2. for some $E$, $E' \subseteq E$ and $M$, the following two conditions hold:
   
   (a) $S$ contains $E \Rightarrow M$;
   
   (b) $E' \rightarrow \bot \in G(S)$.

The first condition is the same as the first alternative claim of the lemma.

The second condition is shown to imply the second alternative claim of the lemma. The premise is that $S$ contains $E \Rightarrow M$ and that $G(S)$ contains $E' \rightarrow \bot \in G(S)$. The conclusion is the second alternative claim of the lemma: there exist $E$, $E' \subseteq E$, etc.

The first premise $E \Rightarrow M \in S$ is the same as the first part of the claim.

The second premise is that $G(S)$ contains $E' \rightarrow \bot$. Therefore, $S$ satisfies the second condition of Definition 9 for $E' \rightarrow \bot$. Rewriting that part of the definition with $E'$ in place of $E$, $M'$ in place of $M$ and so on, this condition becomes: there exists $M'$ such that:

1. $E' \Rightarrow M' \notin S$;

2. for all $m \in M'$ there exist $E''$ and $M''$ such that $E'' \subseteq E'$, $m \in M''$ and $E'' \Rightarrow M'' \in S$; and

28
3. for all $E'' < E'$, it holds $E'' \Rightarrow M' \not\in S$.

These are the second, third and fourth parts of the claim.

The three lemmas together prove that the minority and inconsistency conditions are necessary and sufficient for a set of explanations to be expressed by some formula.

**Theorem 6** A set of explanations over disjoint hypotheses and manifestations is supported by a formula if and only if it satisfies the minority condition (Definition 10) and the inconsistency condition (Definition 11).

**Proof.** Lemma 12 and Lemma 13 proves that a set of explanations supported by a formula satisfies the two conditions. Lemma 14 proves that a set of explanations not supported by any formula violates either the first or the second conditions.

### 7.3 Complexity

A set of explanations is minimally supported by a formula if and only if the minority and inconsistency conditions are the case. The set of explanations is arbitrary. It can be the result of forgetting.

The definitions of forgetting are the same as for all explanations with support replaced by minimal support. The one of focusing is as follows. The one of summarizing is analogous.

$$\text{focus}(\langle F, R \rangle) = \{ E \Rightarrow M \mid E \subseteq I \cap R, M \subseteq C \cap R, M \neq \emptyset, \exists E' \subseteq I \setminus R, \exists M' \subseteq C \setminus R, EE' \Rightarrow MM' \in \text{minabduct}(F)\}$$

Minimal support is formalized as follows.

$$\text{minabduct}(F) = \{ E \Rightarrow M \in \text{abduct}(F) \mid \nexists E' \Rightarrow M \in \text{abduct}(F) . E' < E\}$$

The most quantifier alternations in the two conditions may seem in the third point of the inconsistency condition, but they are not. The quantifier $m \in M$ can be replaced by a conjunction since the number of possible manifestations is bounded by the size of the input.

**Theorem 7** The problem of checking the existence of a formula such that its minimal abductive explanations are exactly the result of minimally forgetting variables from another formula is in $\Pi^p_4$, if two explanations can be compared in polynomial time.

**Proof.** By Theorem 14, the following conditions are equivalent to the non-existence of a formula supporting the explanations in a set $S$.

1. $S$ contains both $E \Rightarrow M$ and $E' \Rightarrow M$ with $E' < E$; or
2. there exist $E, E' \subseteq E$, $M$ and $M'$ such that:
   (a) $E \Rightarrow M \in S$;
   (b) $E' \Rightarrow M' \not\in S$;
(c) \( \forall m \in M' \cdot \exists E'', M'' \cdot E'' \subseteq E', \ m \in M'', \ E'' \Rightarrow M'' \in S \); and

(d) \( \forall E'' < E. \ E'' \Rightarrow M' \notin S \).

The quantifier \( \forall m \in M' \) equates to the conjunction \( \bigwedge_{m \in M'} \) since the cardinality of \( M' \) is bounded by the size of the set of manifestations, which is part of the input data.

1. \( \exists E, E' < E, M \cdot E \Rightarrow M \in S \) and \( E' \Rightarrow M \in S \); or

2. \( \exists E, E' \subseteq E, M, M' \).
   
   (a) \( E \Rightarrow M \in S \);  
   (b) \( E' \Rightarrow M' \notin S \);  
   (c) \( \bigwedge_{m \in M'} \exists E'', M'' \cdot (E'' \subseteq E', \ m \in M'', \ E'' \Rightarrow M'' \in S) \); and  
   (d) \( \forall E''. \ E'' < E \) implies \( E'' \Rightarrow M' \notin S \).

When \( S \) is the set of explanations minimally supported by focusing a formula \( F \) on some variables, the condition \( E \Rightarrow M \in S \) equates the existence of a set of forgotten hypotheses \( E' \) and a set of forgotten manifestations \( M' \) such that \( E \cup E' \Rightarrow M \cup M' \) is minimally supported by the original formula. The same applies to summarizing with the addition of \( M' = \emptyset \).

Minimal support of \( E \cup E' \Rightarrow M \cup M' \) from a formula \( F \) is defined as \( E \cup E' \Rightarrow M \cup M' \) being supported by \( F \) while \( E'' \Rightarrow M \) is not for every \( E'' < E \cup E' \).

\[
E \Rightarrow M \in S \text{ iff } \exists E', M'.
\]
\[
F \cup E \cup E' \not\models \bot
\]
\[
F \cup E \cup E' \models M \cup M'
\]
\[
\forall E'' < E \cup E' \cdot F \cup E'' \not\models \bot \text{ or } F \cup E'' \not\models M
\]

The maximal number of quantifier alternations in this subcondition is three, starting from an existential one: \( \exists E' \forall E'' \ldots F \cup E'' \not\models M \), since the negation of entailment is the same as the existence of a model satisfying \( F \cup E' \) and falsifying \( M \). As a result, every \( E \Rightarrow M \in S \) in the condition above is a \( \exists \forall \exists \) alternation and every \( E' \Rightarrow M' \notin S \) is \( \forall \exists \forall \).

1. \( \exists E, E' < E, M \cdot \exists \forall \exists \) and \( \forall \exists \); or

2. \( \exists E, E' \subseteq E, M, M' \).
   
   (a) \( \exists \forall \exists \);  
   (b) \( \forall \exists \);  
   (c) \( \bigwedge_{m \in M'} \exists E'', M'' \cdot (E'' \subseteq E', \ m \in M'', \ \exists \forall \exists) \); and  
   (d) \( \forall E''. \ E'' < E \) implies \( \forall \exists \forall \).

The maximum number of quantifier alternations is four, starting from an existential one: \( \forall \exists \forall \). The problem of non-existence of a formula supporting forgetting is in \( \Sigma^p_4 \). The converse problem of existence is in \( \Pi^p_4 \). □

30
8 Default logic

Forgetting may not be supported by any propositional formula. The keys of this sentence are “supported” and “propositional”. Forgetting always exists: it is a set of explanations. It could be just stored as such, if not for its sheer size: a tiny formula may support a myriad of explanations. This is a primary reason for finding a propositional formula supporting it, because that formula may have a reasonable size. Another is that a formula may provide insight of what these explanations collectively indicate.

For some logics, forgetting can always be expressed in the logic itself [LLM03]. For some others, it may not [GKL16a, FLvD19, ZSW+20]. In such cases, a solution is to switch to a more powerful logic [GKL16a, ZSW+20]. For example, strongly persistent forgetting is not always possible in logic programming [GKL16b, GKLW20], but it is extending the language with forks [ACF+19].

For propositional logic, the first choice are other logics made from simple propositions, no objects or functions. This rules out first-order and description logics, for example. Obvious candidates are modal logic and nonmonotonic logics.

Default logic is an example. An explanation \( E \Rightarrow M \) is supported if the default theory \( \langle D, W \cup E \rangle \) is consistent and entails \( M \) [EGL97, Tom03]. The question is: given a set of explanations \( S \), is there any default theory \( \langle D, W \rangle \) that supports them, and them only? More specifically, if \( S \) is the result of forgetting, is there a theory supporting its explanations and no other?

The definition of forgetting from a default theory requires some additional specifications. The same default theory may have multiple extensions. Some may entail \( M \) and some may not. In such cases, \( M \) may be considered entailed or not. Accepting \( M \) as a consequence only if entailed by all extensions satisfies the conjunctive condition: if \( a \) and \( b \) are consequences of all extensions, so is \( a \land b \). No default theory supports any set of explanations violating the conjunctive condition, which forgetting may produce. Accepting \( M \) as a consequence if entailed by some extensions does not suffer from this limitation: \( a \) may be entailed only by extensions that do not entail \( b \) and vice versa; none entail \( a \land b \).

The definition of support in default logic is: \( E \Rightarrow M \) is supported by \( \langle D, W \rangle \) if \( \langle D, W \cup E \rangle \) has at least a consistent extension where \( M \) holds.

A simple case demonstrates that default logic supports sets of explanations that propositional logic does not. The default theory \( \langle D, \emptyset \rangle \) that follows supports \( \{a \Rightarrow x, a \Rightarrow y\} \), which violates the conjunctive condition because it does not contain \( a \Rightarrow x \land y \).

\[
D = \left\{ \frac{a : x \land \neg y}{x \land \neg y}, \frac{a : \neg x \land y}{\neg x \land \neg y} \right\}
\]

\[ W = \emptyset \]

The two extensions of \( \langle D, \emptyset \cup \{a\} \rangle \) respectively entail \( x \land \neg y \) and \( \neg x \land y \). One entails \( x \), one entails \( y \), none entail \( x \land y \). The only supported explanations are \( a \Rightarrow x \) and \( a \Rightarrow y \). No propositional theory supports them without also supporting \( a \Rightarrow x \land y \). The conclusion is that default logic support some sets of explanations that propositional logics does not.

The question is whether forgetting from a propositional formula is always supported by a default theory.

The answer is: yes, in theory.
The principle is demonstrated by the example: every explanation $E \Rightarrow M$ turns into a default, which requires $E$ and produces $M$ but none of its supersets.

The resulting theory supports exactly the explanations that result from forgetting. This claim is proved in three steps:

- a condition on sets of explanations is defined;
- it is proved to be equivalent to the set being the result of forgetting from a propositional formula;
- it is proved to be equivalent to the set being supported by a default theory.

The conclusion is that a set of explanations is the result of forgetting from a propositional theory if and only if it supported by a default theory. In short, forgetting from a propositional theory results in a default theory.

**Definition 12 (Consequential monotony)** A set of explanations $S$ satisfies consequential monotony if it contains $E \Rightarrow M'$ whenever it contains $E \Rightarrow M$ with $M' \subseteq M$.

Consequential monotony characterizes forgetting. A set of explanations is the result of forgetting some variables from some formulae if and only if it satisfies consequential monotony. Since no ordering is involved, forgetting and summarizing coincide $\text{focus}(F, R) = \text{summarize}(F, R)$. The first step of the proof is that forgetting satisfies consequential monotony.

**Lemma 15** For every formula $F$ and set of variables $R$, consequential monotony is satisfied by focus$(F, R)$.

Proof. The premises are $E \Rightarrow M' \in \text{focus}(F, R)$ and $M \subseteq M'$. The conclusion is $E \Rightarrow M \in \text{focus}(F, R)$. The first premise $E \Rightarrow M' \in \text{focus}(F, R)$ is defined as the existence of $E'$ and $M''$ such that $F \models EE' \Rightarrow M'M''$. This is in turn defined as $F \cup E \cup E'$ being consistent and entailing $M' \cup M''$. The latter $F \cup E \cup E' \models M' \cup M''$ implies $F \cup E \cup E' \models F \cup M \cup M''$ since $M \subseteq M'$. With the consistency of $F \cup E \cup E'$, this entailment defines $F \models EE' \Rightarrow M'M''$. This is the case for some $E'$ and $M''$. This is the definition of $E \Rightarrow M \in \text{focus}(F, R)$.

The second step of the proof is that if a set of explanations satisfies consequential monotony, it is a result of forgetting.

**Lemma 16** If a set of explanations $S$ over disjoint hypotheses $I$ and manifestations $C$ satisfies consequential monotony then there exist a formula $F$ such that $S = \text{focus}(F, I \cup C)$.

Proof. A new hypothesis is created for each explanation $E \Rightarrow M$ in $S$. To this aim, $S$ is assumed enumerated: $S = \{E_i \Rightarrow M_i \mid 1 \leq i \leq m\}$. The new hypotheses are $a_1, \ldots, a_m$. The formula $F$ comprises the following clauses for each pair of indices $i$ and $j \neq i$:

$$E_i a_i \rightarrow M_i$$

$$a_i a_j \rightarrow \bot$$

$$\{a_i e \rightarrow \bot \mid e \in I \setminus E_i\}$$

32
These clauses make $F$ support $E_i \cup \{a_i\} \Rightarrow M_i$ for every explanation $E_i \Rightarrow M_i$ of $S$. Forgetting $a_i$ turns it into $E_i \Rightarrow M_i$, as required. This proves that every explanation of $S$ is in focus($F, I \cup C$).

The rest of the proof shows that the explanations that are not in $S$ are not supported.

This is proved by contradiction: some explanation $E \Rightarrow M$ not in $S$ is assumed to be in focus($F, I \cup C$). By definition, this is only possible if $F \cup E \cup D$ is consistent and entails $M$ for some $D \subseteq \{a_1, \ldots, a_m\}$.

If $D$ is empty, $F \cup E \cup D$ is satisfied by the model that sets all variables $a_i$ and $C$ to false, since all clauses of $F$ contain a negative occurrence of a variable $a_i$. This model falsifies $M$, contradicting the assumption that $F \cup E \cup D$ entails $M$.

If $D$ contains two variables $a_i$ and $a_j$, it falsifies the clause $a_i a_j \rightarrow \bot$, contradicting the assumed consistency of $F \cup E \cup D$.

The conclusion is that $D$ contains exactly one variable $a_i$. Therefore, $F \cup E \cup D$ is $F \cup E \cup \{a_i\}$.

The unions $F \cup E \cup \{a_i\}$ and $F \cup E \cup \{a_i\} \cup \neg M$ contain only negative occurrences of the variables $a_j$ with $j \neq i$. Removing the clauses containing them does not affect satisfiability. The remaining clauses of $F$ are $E_i a_i \rightarrow M_i$ and $a_i e \rightarrow \bot$ for every $e \in I \setminus E_i$. That $F \cup E \cup \{a_i\}$ is consistent and entails $M$ simplify as follows.

\[
\{E_i a_i \rightarrow M_i\} \cup \{a_i e \rightarrow \bot \mid e \in I \setminus E_i\} \cup E \cup \{a_i\} \models \bot
\]

If a variable $e$ of $E$ is not in $E_i$ then it is in $I \setminus E_i$. As a result, the premise of the first entailment contains $a_i$, $e$ and $a_i e \rightarrow \bot$, contradicting its consistency. The conclusion is that all variables of $E$ are in $E_i$, which is the same as $E \subseteq E_i$.

If a variable $e$ of $E_i$ is not in $E$, the second entailment is contradicted by the model that sets all variables of $\{a_i\} \cup E$ to true and all others to false, including $e$. This model satisfies $E_i a_i \rightarrow M_i$ because it sets $e$ to false. It satisfies every clause $a_i e \rightarrow \bot$ because every $e$ in $I \setminus E_i$ is in $I \setminus E$ since $E \subseteq E_i$, and is therefore set to false by the model. The same model falsifies $M$, contradicting the second entailment above.

The conclusion is that $E = E_i$. It turns the second entailment above into the following one.

\[
\{E_i a_i \rightarrow M_i\} \cup \{a_i e \rightarrow \bot \mid e \in I \setminus E_i\} \cup E_i \cup \{a_i\} \models M
\]

Since the premise contains $a_i$ and every $e \in E_i$, all negative occurrences of these variables can be removed from the clauses where they occur.

\[
M_i \cup \{\neg e \mid e \in I \setminus E_i\} \cup E_i \cup \{a_i\} \models M
\]

Because of the separation of the variables, this is the same as $M_i \models M$, which is the same as $M \subseteq M_i$. Since $E_i \Rightarrow M_i$ is in $S$, by consequential monotony also $E_i \Rightarrow M$ is in $S$. This implies that $E \Rightarrow M$ in $S$ since $E = E_i$. Contradiction with the assumption that $E \Rightarrow M$ is not in $S$ is reached.

The two lemmas prove that consequential monotony characterizes forgetting. 

\[\square\]
Theorem 8 A set of explanations $S$ over disjoint hypotheses $I$ and manifestations $C$ satisfies consequential monotony if and only if there exist a formula $F$ such that $S = \text{focus}(F, I \cup C)$.

Proof. Lemma 15 states that $\text{focus}(F, R)$ satisfies consequential monotony for every set of variables $R$, including $I \cup C$ for whichever disjoint sets $I$ and $C$. Lemma 16 proves the other direction.

Consequential monotony also equates the existence of a default theory supporting the set of explanations.

Theorem 9 A set of explanations $S$ over disjoint hypotheses and manifestations satisfies consequential monotony if and only if it supported by a default theory.

Proof. The claim comprises two parts: first, if $S$ does not satisfy consequential monotony, no default theory supports it; second, if $S$ satisfies consequential monotony, a default theory supports it.

The first part holds because the explanations supported by an arbitrary default theory satisfy consequential monotony. The premise is that an explanation $E \Rightarrow M$ is supported by a default theory $\langle D, W \rangle$. This is defined as the existence of a consistent extension of $\langle D, W \cup E \rangle$ that entails $M$. An extension is just a propositional formula. Since it entails $M$, it also entails every subset $M' \subseteq M$. As a result, the default theory supports $E \Rightarrow M'$, as required.

The second part of the claim is that every set of explanations $S$ over disjoint hypotheses and manifestations satisfying consequential monotony is supported by some default theory. This default theory is $\langle D, \emptyset \rangle$, comprising the following defaults.

$$D = \left\{ E : E \land \neg (I \setminus E) \land M \land \neg (C \setminus M) \mid E \Rightarrow M \in S \right\}$$

The claim is that $E \Rightarrow M$ is in $S$ if and only if some consistent extension of $\langle D, \emptyset \cup E \rangle$ entails $M$.

The inclusion of $E$ in the justification and consequent of the default is redundant, but facilitates some parts of the proof.

Since the defaults of $D$ are normal and the background theory $\emptyset \cup E$ is consistent since $E$ is a set of positive literals, $\langle D, \emptyset \cup E \rangle$ always has at least an extension, and all its extensions are consistent.

A preliminary result is that no two defaults can be applied together. Since $S$ is a set and not a multiset, its explanations $E \Rightarrow M$ differ from each other. Every two of them differ either on $E$ or on $M$: if $E \Rightarrow M$ and $E' \Rightarrow M'$ are both in $S$, then either $E \neq E'$ or $M \neq M'$. Four cases are possible:

- $E$ contains a hypothesis not in $E'$;
- $E'$ contains a hypothesis not in $E$;
- $M$ contains a manifestation not in $M'$;
- $M'$ contains a manifestation not in $M$. 

34
Only the first case is considered, the other three are similar due to the symmetry of the defaults. Let \( e \in E \backslash E' \). This variable belongs to \( E \), and therefore occurs positive in the justification and consequent of the default of \( E \Rightarrow M \); since it belongs to \( E \backslash E' \), it belongs to its superset \( I \backslash E' \); as a result, it occurs negative in the justification and consequent of the default of \( E' \Rightarrow M' \). Applying the default of \( E \Rightarrow M \) results in the generation of \( e \), which blocks the application of the default of \( E' \Rightarrow M' \). Applying the latter results in the generation of \( \neg e \), which blocks the application of the former.

This proves that no two defaults can be applied together.

The main claim can now be proved: \( E \Rightarrow M \) is in \( S \) if and only if a consistent extension of \( \langle D, \emptyset \cup E \rangle \) entails \( M \). Two cases are considered: either \( E \Rightarrow M \) is in \( S \), or it is not. The claim is that an extension of \( \langle D, \emptyset \cup E \rangle \) entails \( M \) in the first case and no extension entails \( M \) in the second.

\( E \Rightarrow M \in S \) By construction, \( D \) contains the default of \( E \Rightarrow M \). Its premise is \( E \), which holds in \( \langle D, \emptyset \cup E \rangle \). Its consequent includes \( E \) itself and \( M \); this part is consistent with \( \emptyset \cup E \) because all these sets comprise positive literals only; it also includes \( \neg(I \backslash E) \) and \( \neg(C \backslash M) \); these two sets comprise negative literals, but they are disjoint from the positive literals \( E \cup M \) because of the separation between hypotheses and manifestations \( I \cap C = \emptyset \). A consequence of this consistency is that the default is applicable. Its application blocks all other defaults. An extension is generated, and this extension includes \( M \).

\( E \Rightarrow M \notin S \) The claim that no extension of \( \langle D, \emptyset \cup E \rangle \) entails \( M \) is proved by contradiction: an extension entailing \( M \) is assumed to exist.

As proved above, every extension is generated by the application of zero or one default. Applying zero defaults to \( \langle D, \emptyset \cup E \rangle \) adds nothing to the background theory \( \emptyset \cup E \). This set does not entail \( M \) because of the assumptions \( M \neq \emptyset \) and \( I \cap C = \emptyset \). This contradicts the assumption that the extension entails \( M \).

The other possibility is that the extension results from applying exactly one default. By construction, every default comes from an explanation \( E' \Rightarrow M' \) of \( S \). Since this explanation belongs to \( S \), it is not the same as \( E \Rightarrow M \), which does not. Either \( E' \neq E \) or \( M' \neq M \).

Since the default of \( E' \Rightarrow M' \) is applied to the background theory \( \emptyset \cup E \), its precondition is entailed and its justification is consistent. The entailment \( \emptyset \cup E \models E' \) implies \( E' \subseteq E \). The consistency of \( \emptyset \cup E \cup \{E' \land \neg(I \backslash E') \land M' \land \neg(C \backslash M')\} \) implies that \( E \) does not contain any hypothesis in \( I \backslash E' \). This condition \( E \cap (I \backslash E') = \emptyset \) translates into \( E \subseteq E' \).

Since the converse also holds, the containment is actually an equality: \( E = E' \).

The extension generated by the application of this default is the deductive closure of its consequent and the background theory. It was assumed to entail \( M \). This entailment is the same as \( \emptyset \cup E \cup \{E' \land \neg(I \backslash E') \land M' \land \neg(C \backslash M')\} \models M \). Because of the separation of hypotheses and manifestations, this is the same as \( M' \land \neg(C \backslash M') \models M \), which is also the same as \( M' \models M \), or \( M \subseteq M' \).

What proved so far is \( E = E' \) and \( M \subseteq M' \) for some explanation \( E' \Rightarrow M' \) of \( S \). A consequence is that \( S \) contains \( E \Rightarrow M' \) with \( M \subseteq M' \). By consequential monotony, \( S \) also contains \( E \Rightarrow M \), contrary to the assumption.
This proves that abductive forgetting in propositional logics is supported by default logic abduction.

In theory.

In practice, the default theory that supports a set of explanations is nothing more than the explanations themselves, each turned into a default. It gives no intuition other than that, which is a problem if the aim of forgetting is to provide a summary of knowledge. Even if it is not, it is a computational drawback. Forgetting may generate many explanations even from a small formula. A default theory that always contains a default for every explanation is as large as the set of explanations itself. This size increase may be unavoidable in the worst case, but should be avoided if possible.

Theory proves that every consequential monotonic set of explanations is supported by a certain default theory, but that default theory may be too large for practical purposes. At the same time, theory does not forbid smaller default theories to support the same set.

A smaller default theory may exploit the background theory \( W \). Instead of \( \emptyset \), the consequential forgetting of \( F \) could be used instead. Consequential forgetting retains all and only the implications \( E \rightarrow M \) that do not contain hypotheses to forget. The default theory \( \langle \emptyset, W \rangle \) supports all explanations \( E \Rightarrow M \) of this kind. It is however incomplete, as forgetting also contains explanations \( E \Rightarrow M \) that are not supported by the original formula. An example is \( a \Rightarrow x \) where the original formula supports \( ab \Rightarrow x \) instead and \( b \) is forgotten. Consequential forgetting does not turn \( ab \rightarrow x \) into \( a \rightarrow b \). Therefore, it has to be enhanced to support \( a \Rightarrow x \). A way to do this could be by a default rule.

\[
\frac{\begin{array}{c} E : M \land \neg E_1 \land \cdots \land \neg E_m \\ \end{array}}{M}
\]

The justification and the consequent are no longer the same. This default is not normal. It could not, because adding \( \neg E_i \) to the background theory might have unwanted consequences. For example, it could entail the negation of a manifestation in \( M \). For this reason, \( \neg E_i \) is only in the justification, to block the application of this default without producing consequences.

9 New variables

Abductive forgetting from a propositional formula may produce a set of explanations that is not supported by a propositional formula. It is supported by a default theory, which is an extension of a propositional formula. Is there any other solution, one that does not require an extended logic?

An analysis of what makes propositional logic fail at abductive forgetting suggests it.

Abductive forgetting mainly turns explanations like \( ab \Rightarrow m \) into \( a \Rightarrow m \). It also removes the explanations of the manifestations that comprise or include manifestations to forget, depending on the definition, but this is less of a problem. What makes forgetting difficult is the removal of hypotheses from explanations.

The meaning of \( ab \Rightarrow m \) is that that \( m \) is explained by \( ab \). Similarly, \( a \Rightarrow m \) means that \( m \) is explained by \( a \). At the level of English sentences, “\( a \) and \( b \) explain \( m \)” and “forget about \( b \)” result in “\( a \) explains \( m \)”. If the only allowed verb is “explain”, this is the best that can be said: “\( a \) explains \( m \)”. As a matter of fact, “\( a \) might explain \( m \)” would be better. In certain
conditions, $a$ explains $m$. The certain conditions are $b$. These conditions are neglected. They should, because this is what forgetting is supposed to do: neglect the conditions to forget.

Yet, these conditions may interact with each other and may interact with the other hypotheses.

- an example of interaction between neglected conditions is $F = \{ab \rightarrow m, ac \rightarrow m', abc \rightarrow \bot\}$: forgetting $b$ and $c$ produces $a \Rightarrow m$ and $a \Rightarrow m'$; in certain conditions ($b$), an explanation of $m$ is $a$; in certain other conditions ($c$), an explanation of $m'$ is $a$. Yet, these two certain conditions never materialize together because of $abc \rightarrow \bot$;

- an example of interaction between neglected conditions and hypotheses is $F = \{ab \rightarrow m, abc \rightarrow \bot\}$: forgetting $b$ only turns $ab \Rightarrow m$ into $a \Rightarrow m$; in certain conditions ($b$), an explanation of $m$ is $a$; yet, these conditions $b$ prevent the hypothesis $c$ to be the case, excluding $ac$ as a further explanation of $m$.

Forgetting may be supported by specifying the “certain conditions” that do not materialize together, and the ones that prevent other hypotheses to materialize.

A way to formalize this is by attacks like in argumentation theory [Dun95]: $a \Rightarrow m$ and $a \Rightarrow m'$ attack each other, meaning that the conditions that make $a$ an explanation of $m$ conflict with the ones that makes it an explanation of $m'$; in the same way, $a \Rightarrow m$ attacks $c$, meaning that conditions that make $a$ an explanation of $m$ conflict with $c$.

A simpler solution is to use introduce new hypotheses.

This is always possible because the new hypotheses can just be the forgotten ones. Yet, they might not. The “certain conditions” may be complicated but what matters might be only that two of them conflict with each other. For example, the original theory may entail $aC \rightarrow m$ and $aC' \rightarrow m'$, where $C$ and $C'$ are complicated formulae that are not consistent with each other. Forgetting their variables result in $a \Rightarrow m$ and $a \Rightarrow m'$ only, without $a \Rightarrow mm'$. The same is the result of forgetting $b$ and $c$ from $\{ab \rightarrow m, ac \rightarrow m', bc \rightarrow \bot\}$. Complicated conditions $C$ and $C'$ are turned into two simple hypotheses $b$ and $c$. A simple clause $bc \rightarrow \bot$ forbids them to be both true at the same time.

Forgetting is supported by forgetting, of course. But may not only be supported by forgetting the same variables from the same formula. It may be supported by forgetting variables from a simpler formula. This is economy of concepts: hypotheses are introduced only when they are necessary to support the explanations resulting from forgetting. If a forgotten hypothesis does not conflict with any other, it can be just removed without the need of introducing any new one. In other cases, multiple forgotten hypotheses involved in complex subformulae can be summarized with two new hypotheses, like in the example $b$ and $c$ take over $C$ and $C'$.

An extreme example shows that sometimes a single new hypothesis can replace arbitrarily many forgotten ones.

\[ F = \{ a_i x_i \rightarrow m_i, a_i b_i x_i \rightarrow \bot, a_i b_i \rightarrow m_i' \mid 1 \leq i \leq m \} \cup \{ a_i a_j \rightarrow \bot, a_i b_j \rightarrow \bot, a_i x_j \rightarrow \bot \mid 1 \leq i, 1 \leq j \leq m, i \neq j \} \]

The explanations supported by this formula are $a_i x_i \Rightarrow m_i$ and $a_i b_i \Rightarrow m_i'$ for every index $i$ between $1$ and $m$. Forgetting $x_i$ turn them into $a_i \Rightarrow m_i$ and $a_i b_i \Rightarrow m_i'$, which are not
supported by any formula because they violate overreaching monotony. The variables $x_i$ need not be all different to produce them. A single variable $x$ suffices.

$$F' = \{a_ix \rightarrow m_i, a_ib_i \rightarrow \bot, a_ib_i \rightarrow m'_i \mid 1 \leq i \leq m\} \cup \{a_ia_j \rightarrow \bot, a_ib_j \rightarrow \bot \mid 1 \leq i \leq m, 1 \leq j \leq m, i \neq j\}$$

This is the case for an arbitrary large $m$: forgetting a single variable may support forgetting arbitrarily many hypotheses, even when that violates overreaching monotony and is therefore not supported by plain abduction without forgetting.

Forgetting a single variable may support forgetting multiple ones. It may, but it also may not. It depends on the formula and on the variables.

This is not merely a matter of numbers, of how many variables are forgotten; what matters is the kind of supported explanations. Forgetting a single variable may violate overreaching monotony, but never violates the conjunctive condition. This is at the same time a blessing and a curse.

- When forgetting is originally done on a single variable, the result is at least guaranteed to satisfy one of the conditions for being supported by a propositional formula.
- When trying to express the result of forgetting, a single additional variable only helps when the conjunctive condition is satisfied.

Forgetting may in general violate the conjunctive condition. It does only when forgetting at least two hypotheses.

**Theorem 10** Forgetting a single hypothesis from a formula satisfies the conjunctive condition.

**Proof.** The claim is shown in reverse: if forgetting violates the conjunctive condition, the forgotten hypotheses are at least two.

Forgetting violates the conjunctive condition when it contains $E \Rightarrow M_1$ and $E \Rightarrow M_2$ and not $E \Rightarrow M_1M_2$. This is the case when the formula $F$ supports $EA \Rightarrow M_1$ and $EB \Rightarrow M_2$ for some sets of hypotheses to forget $A$ and $B$, and it does not support $EC \Rightarrow M_1M_2$ for any set of hypotheses to forget $C$. This includes $C = A \cup B$: it does not support $EAB \Rightarrow M_1M_2$.

Since $F$ supports $EA \Rightarrow M_1$ and $EB \Rightarrow M_2$, it entails $EA \rightarrow M_1$ and $EB \rightarrow M_2$. As a result, it entails $EAB \rightarrow M_1M_2$. If $F \cup E \cup A \cup B$ were consistent, then $F$ would support $EAB \Rightarrow M_1M_2$; it does not; therefore, $F \cup E \cup A \cup B$ is inconsistent.

Since $F \cup E \cup A \cup B$ is inconsistent while its subset $F \cup E \cup A$ is not, their difference $B$ contains at least a hypothesis that is not in their intersection $F \cup E \cup A$, and is therefore not in $A$. For the same reason, $A$ contains a hypothesis that is not in $B$. Since $A$ and $B$ are sets of hypotheses to forget, these are at least two.

This result allows for a different tentative-supporting formula of a set of explanations. The implication $E \rightarrow m$ is generated from $E \Rightarrow m$ only if this explanation is not involved in a violation of overreaching monotony. The other case is the absence of $E' \Rightarrow m$ and the presence of $E'' \Rightarrow m'$ for some other sets of hypotheses $E'$ and $E''$ such that $E \subseteq E' \subseteq E''$ and some other manifestation $m'$. The algorithm generates $Ex \rightarrow m$ and $Ex \rightarrow \neg(E' \backslash E)$.
then. These two clauses support $E \Rightarrow m$ and block $E' \Rightarrow m$ while allowing $E'' \Rightarrow m$ when forgetting $x$.

If forgetting violates overreaching monotony only in one case, the resulting formula supports it. Otherwise, it may support it or not. If all violations are somehow independent on each other, for example they are all about different sets $E$, it works. Otherwise, it may not. It is only an attempt at supporting the given set of explanations anyway, hence the name tentative-supporting formula.

10 Intersecting hypotheses and manifestations

Hypotheses and manifestations are assumed disjoint in the previous sections: $I \cap C = \emptyset$. Concretely, an explanation is never itself explained by something else. Technically, explanations do not chain: $a \Rightarrow b$ and $b \Rightarrow c$ exclude each other; they are not only unsupported, they violate the assumption as $b$ cannot be at the same time a manifestation like in the first explanation and a hypothesis like in the second.

What happens lifting this constraint?

When building the tentative-supporting formula, every explanation $a \Rightarrow b$ turns into a clause $a \rightarrow b$. In the other way around, every clause $a \rightarrow b$ comes from an explanation $a \Rightarrow b$. This forbids the clause $b \rightarrow c$. It should come from an explanation $b \Rightarrow c$, which implies that $b$ is a hypothesis, in addition to being a manifestation because of $a \Rightarrow b$.

Since $a \rightarrow b$ and $b \rightarrow c$ exclude each other, transitivity never applies: $a \Rightarrow c$ is only entailed if the formula contains $a \rightarrow c$ itself or its subset $\emptyset \rightarrow c$.

If $a \Rightarrow c$ is supported, it is entailed. If it is entailed, the formula contains either it or its subset $\emptyset \rightarrow c$. This is Lemma 1, which is relied upon by most other proofs.

This direct link between explanations and clauses breaks when hypotheses and manifestations are not disjoint. A formula that supports $a \Rightarrow b$ and $b \Rightarrow c$ entails $a \rightarrow c$. It supports $a \Rightarrow c$ without containing $a \rightarrow c$. Support no longer implies containment.

This may not be a problem when selecting all explanations or the subset-minimal ones, but it is when selecting the cardinality-minimal ones. The following set exemplifies the problem.

$$S = \{ab \Rightarrow c, c \Rightarrow d, af \Rightarrow h\}$$

Support of $ab \Rightarrow c$ and $c \Rightarrow d$ implies entailment of $ab \rightarrow c$ and $c \rightarrow d$, which entail $ab \rightarrow d$. Every formula supporting $S$ entails $ab \rightarrow d$. Such a formula also supports $ab \Rightarrow c$, and is therefore consistent with $ab$. It entails $ab \rightarrow d$ and is consistent with $ab$; it supports $ab \Rightarrow d$. This explanation is not in $S$. The conclusion is that $S$ is not supported by any formula. The tentative-supporting formula still meet its specification, that requires it to support $S$ only if some formula does.

This is only the case when selecting all explanations or the subset-minimal ones. Things change when selecting only the cardinality-minimal explanations. Even though $ab \Rightarrow d$ is supported, it is not minimally so because of $c \Rightarrow d$. A formula may still minimally support $S$.

The tentative-supporting formula also contains $af \rightarrow h$ to support $af \Rightarrow h$. It therefore entails $afb \rightarrow dh$. The only way to block the explanation $afb \Rightarrow dh$ is the clause $afb \rightarrow \bot$. 

39
Such a clause would be included in the tentative-supporting formula if \( S \) contained \( ab \Rightarrow d \), but it does not. The tentative-supporting formula does not support \( S \).

Yet, a formula supporting \( S \) exists: \( \{ ab \rightarrow c, c \rightarrow d, af \rightarrow h, afb \rightarrow \bot, afc \rightarrow \bot \} \). The clause \( afb \rightarrow \bot \) blocks the explanation \( afb \Rightarrow dh \).

The problem is the disconnection between supported explanations and contained clauses. While both \( ab \Rightarrow c \) and \( c \Rightarrow d \) are supported, \( ab \Rightarrow d \) is not. Since \( ab \Rightarrow d \) is not supported, excluding \( afb \Rightarrow dh \) explicitly looks unnecessary. It is necessary because \( ab \Rightarrow c \) and \( c \Rightarrow d \) cause \( ab \rightarrow d \) to be entailed.

The set of explanations is supported by a formula but not by the tentative-supporting formula when selecting the cardinality-minimal explanations. It is still supported by the tentative-supporting formula when selecting all or the subset-minimal explanations. Whether these formulae work in general or suffer from a similar counterexample is an open question.

A possible solution may be to first produce the definite clauses \( E \rightarrow m \) of \( G(S) \) and then use them in place of the explanations \( E \Rightarrow M \) to generate the negative clauses \( E \rightarrow \bot \). The second condition for adding \( E \rightarrow \bot \) would turn into: the definite clauses of \( G(S) \) entail \( E \rightarrow M \).

Another possible solution may be to change the aim of \( E \rightarrow \bot \). The tentative-supporting formula contains it only when strictly necessary, when an explanation \( E \Rightarrow M \) would be supported while it should not. Sometimes, \( E \rightarrow \bot \) may not be strictly necessary, but not harmful either. When all explanations \( E' \Rightarrow M \) are unsupported for every \( E \subseteq E' \), the clause \( E \rightarrow \bot \) is harmless. It may not be necessary to block an unsupported explanation, but it does not block a supported one either.

Both solutions are to be explored in the future.

11 Conclusions

The common form of forgetting is consequential: “AB is true, A is true, B is true” plus “forget about A” becomes “B is true”. Forgetting about something expresses a disinterest in it—a disinterest in whether it is true or not; a disinterest in what regards it. What regards it may however not be the logical consequences. It may not be logical entailment. It may be what explain what. Abductive forgetting is expressing disinterest about how something is involved in the explanations.

Contrary to consequential forgetting, abductive forgetting is existential rather than universal.

Consequentially forgetting A removes not only “A is true”, but also “AB is true”, leaving only “B is true”. Since B only is of interest, only that it is true is of interest, not that it is true together with some uninteresting else.

Abductive forgetting the same way is incorrect. It removes the explanation “AB explain C” when forgetting about A. If C has no explanation other than AB, it becomes unexplainable, rather than explained by B and something not of interest.

Minimality of explanations generates a further difference from consequential forgetting. While AB may minimally explain neither C nor D, it may minimally explain CD. What does forgetting about D mean? C is still of interest. Its explanations are as well. Depending on the intended application of forgetting, “AB explain CD” is either forgotten or simplified into “AB explain C”. When making a summary by topic, like writing a chapter of textbook,
such a complex situation is better postponed. When making a diagnosis, it is crucial that it remains, even if described in reduced form: AB still explains C if D is unknown.

Abductive forgetting can therefore be defined in two ways.

In both, the result could be defined as a propositional formula. This is common for consequential forgetting: the result of forgetting variables from a formula is another formula. In abductive forgetting, such a formula may not exist. For some formulae and variables to forget, no formula supports exactly the set of explanations after forgetting. This situation is commonly described as “forgetting does not exist”. Yet, it is not forgetting that does not exist, only a formula representing the explanations that result from forgetting. The explanations themselves always exist. This is why forgetting is defined as a set of explanations rather than a formula. This set may or may not be the set of explanations supported by a formula.

Whether forgetting produces the set of explanations supported by a formula is analyzed in this article. It depends on the definition of abduction and on the definition of forgetting, this is the first result about it. An algorithm for building a formula from a set of explanations if any exists is given. A necessary and sufficient condition to this existence is provided. It is made of two parts, the conjunctive condition and the overreaching monotony conditions. The complexity of checking each part separately is located at the third level of the polynomial hierarchy, being $\Pi^p_3$ complete. This is also the complexity of checking them together, which is the same as the existence of a formula supporting the result of forgetting. A similar analysis is done for minimal explanations. When the result of forgetting is not supported in propositional logic, a solution is to allow default rules. The set of explanations generated by forgetting is always supported by a default theory in the credulous semantics. Another solution is to introduce new variables to forget. All previous results assume that hypotheses and manifestations do not intersect. What changes when lifting this constraint is analyzed.

What does emerge from this?

Two definitions of abductive forgetting make sense. None is better than the other. They differ in motivation, in the reason to forget. If the reason is to concentrate on a specific topic, forgetting is focusing, the first definition. If the reason is to provide an overview of a specific topic, forgetting is summarizing, the second definition. The existence of multiple definitions, based on different requisites, is common in forgetting in formalisms other than propositional logics [GKL21, EKI19].

Whether abductive forgetting is supported by a propositional formula is not only a matter of representation. It tells something about the scenario the formula represents. Forgetting A turns “AB explain C” into “B explain C”, meaning: “B may explain C” or better “in certain conditions not of interest, B explain C”. Forgetting being supported by a formula means that the conditions for different explanations do not interact in a relevant way. For example, they do not conflict. They can really be neglected, because they do not influence what forgetting maintains. For a given formula, this may happen when forgetting certain variables but not others. The former are “fully forgettable” because neglecting them does not introduce any complication. The latter are not fully forgettable, as the result of forgetting is still affected by them. A similar observation emerged in Answer Set Programming; Aguado et al. [ACF+19] wrote: “In practice, this means that auxiliary atoms in ASP are more than ‘just’ auxiliary, as they allow one to represent problems that cannot be captured without them.”

The problem of existence of a propositional formula supporting forgetting is similar in complexity level to various other problems of abduction, which mostly range between the
second and the third level of the polynomial hierarchy [EG95].

Many questions are open.

The complexity of the problem is still open for the case of minimal explanations.

The algorithm that produces a formula that supports forgetting if any is exploited in a theoretical context, for proving the necessary and sufficient condition. Its practical application is limited to small formulae, since it is exhaustive: it reads the set of all explanations of forgetting, which is in general exponentially larger than the original formula. For large formulae, its large running time prevents its use. A better choice would be to start by consequentially forgetting, adding clauses only when necessary. The first step is correct because forgetting comprises necessary clauses only: every entailed clauses $E \rightarrow M$ and $E \rightarrow \bot$ made of variables to remember only is needed to ensure the survival or removal of the explanation $E \Rightarrow M$. While this first step is correct, it is not complete. Forgetting may produce other explanations. The clauses needed to ensure need to be added in a second step.

A related question is whether a formula supporting forgetting not only exists, but is also of reasonable size. Forgetting always exists since it is defined as a set of explanations, which may however comprise many explanations. If the formula supporting it is similarly sized, it does not offer any benefit.

The tentative-supporting formula $G(S)$ looks unique in the way it supports a given set of explanations. It is not syntactically minimal, as it may for example contain both $E \rightarrow m$ and $E' \rightarrow m$ with $E \subset E'$. It is however minimal in the sense that it only contains clauses that are strictly necessary to support the given set of explanations $S$. Semantically, it contains as many models as possible, among the formulae that have $S$ as their supported explanations.

Other formulae supporting the same explanations may contain other clauses $E \rightarrow \bot$. If no subset of $E$ explain anything, such a clause is not mandatory. If no superset of $E$ explain anything, it is permitted. When both are the case, the clause can be entailed or not. This looks like the only way formulae supporting the same explanations may significantly differ on the hypotheses and manifestations.

Default logic always expresses forgetting, but it is not the only way of doing that. Other extensions of propositional logic may do that. Yet, many logics have the conjunctive property: they entail a conjunction if and only if they entail each of its parts. Forgetting sometimes requires this not to be the case. The cases where forgetting is not expressed by a formula suggests one: the problem is with the interaction between the forgotten conditions of an explanation and the other maintained explanations; that something invalidates something else is expressed by arguments [Dun95].

Introducing new variables to support forgetting makes sense only when they are as few as possible. The principle is simplicity: the goal of forgetting is to concentrate only on a set of variables; adding new ones may be a necessity, but they are unwanted. Adding many complicates the result, when the aim of forgetting is to simplify. Reducing the new variables as much as possible is an open question.

Explanations and manifestations are always sets of variables in this article. They are positive literals. In general, they can be formulae [LU97]. The two considered orderings about them do not exhaust the possible choices. Extending the results to explanations and manifestations that are formulae and employing complex preference orderings [PPU03, DSTW04] is a further direction of study.

A comparison with related work follows.

Many authors used forgetting as a way to find the explanation of a specific manifesta-
tion. The underlying principle is that an explanation can be found by negating the manifestations, conjoining the theory, forgetting everything but the hypotheses and negating the result [Lin01]. This mechanism is applied to propositional logic [Lin01], description logics [PS15, DPS19, DS19, Koo20], logic programming [Wer13] and modal logics [FWC18].

Lobo and Uzcátegui [LU97] characterized the logical inference relation deriving from abduction. A formula is a consequence of another if it is a consequence of all its explanations. Like $\Rightarrow$ in the present article, this relation is based on the abductive explanations of a fixed formula. Contrary to that, it satisfies the conjunctive condition by construction, being based on logical inference. Expressing such a relation with a cumulative model is akin to supporting $\Rightarrow$ by a formula: a binary relation derived from abduction is expressed as something else. Yet, the binary relations differ, as do their alternative expressions.

Pino-Perez and Uzcátegui [PPU03] also define a binary relation based on abduction, but theirs is almost identical to $\Rightarrow$ apart from the order of their arguments. The only semantical difference is that its second argument is always an explanation of the first. This prevents encoding the explanations of forgetting, which may not be the explanations of a formula. While in the present article $\Rightarrow$ can or cannot not be encoded as a formula to abduce from, their relation can or cannot be defined in terms of an ordering among formulae. While the two relations are mostly the same, the way they are expressed differ.

References

[ACF+19] F. Aguado, P. Cabalar, J. Fandinno, D. Pearce, G. Pérez, and C. Vidal. Forgetting auxiliary atoms in forks. *Artificial Intelligence*, 275:575–601, 2019.

[BB22] R. Baumann and M. Berthold. Limits and possibilities of forgetting in abstract argumentation. In *Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence (IJCAI 2022)*, pages 2539–2545, 2022.

[BDR20] R. Baumann, Gabbay D.M., and O. Rodrigues. Forgetting an argument. In *Proceedings of the Thirty-Fourth AAAI Conference on Artificial Intelligence (AAAI 2020)*, pages 2750–2757. AAAI Press/The MIT Press, 2020.

[BKL+17] E. Botoeva, B. Konev, C. Lutz, V. Ryzhikov, F. Wolter, and M. Zakharyaschev. *Inseparability and Conservative Extensions of Description Logic Ontologies: A Survey*, pages 27–89. Springer, 2017.

[Boo54] G. Boole. *Investigation of The Laws of Thought, On Which Are Founded the Mathematical Theories of Logic and Probabilities*. Walton and Maberly, 1854.

[Del17] J.P. Delgrande. A knowledge level account of forgetting. *Journal of Artificial Intelligence Research*, 60:1165–1213, 2017.

[DP60] M. Davis and H. Putnam. A computing procedure for quantification theory. *Journal of the ACM*, 7:201–215, 1960.

[DPS19] W. Del-Pinto and R.A. Schmidt. Abox abduction via forgetting in ALC. In *Proceedings of the Thirty-Third AAAI Conference on Artificial Intelligence*
usa forget per generare le spiegazioni abduttive.

[DR94] R. Dechter and I. Rish. Directional resolution: The Davis-Putnam procedure, revisited. In *Proceedings of the Fourth International Conference on the Principles of Knowledge Representation and Reasoning (KR’94)*, pages 134–145, 1994.

[DS19] W. Del-Pinto and R.A. Schmidt. Extending forgetting-based abduction using nominals. In *Frontiers of Combining Systems - Proceedings of the twelfth International Symposium, FroCoS 2019*, pages 185–202, 2019.

[DSTW04] J.P. Delgrande, T. Schaub, H. Tompits, and K. Wang. A classification and survey of preference handling approaches in nonmonotonic reasoning. *Computational Intelligence*, 20(2):308–334, 2004.

[Dun95] P.M. Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence*, 77(2):321–358, 1995.

[EG95] T. Eiter and G. Gottlob. The complexity of logic-based abduction. *Journal of the ACM*, 42(1):3–42, 1995.

[EGL97] T. Eiter, G. Gottlob, and N. Leone. Semantics and complexity of abduction from default theories. *Artificial Intelligence*, 90(1-2):177–223, 1997.

[EKI19] T. Eiter and G. Kern-Isberner. A brief survey on forgetting from a knowledge representation and perspective. *KI — Kuenstliche Intelligenz*, 33(1):9–33, 2019.

[FLvD19] L. Fang, Y. Liu, and H. van Ditmarsch. Forgetting in multi-agent modal logics. *Artificial Intelligence*, 266:51–80, 2019.

[FWC18] R. Feng, Y. Wang, and P. Chen. Strongest necessary and weakest sufficient conditions in S5. In *Data Science and Knowledge Engineering for Sensing Decision Support: Proceedings of the 13th International FLINS Conference (FLINS 2018)*, pages 832–839, 2018.

[FWL+18] L. Fang, H. Wan, X. Liu, B. Fang, and Z.-R. Lai. Dependence in propositional logic: Formula-formula dependence and formula forgetting - Application to belief update and conservative extension. In *Proceedings of the Thirty-Second AAAI Conference on Artificial Intelligence (AAAI 2018)*, pages 1835–1844, 2018.

[GKL16a] R. Gonçalves, M. Knorr, and J. Leite. The ultimate guide to forgetting in answer set programming. In *Proceedings of the Fifteenth International Conference on Principles of Knowledge Representation and Reasoning (KR 2016)*, pages 135–144. AAAI Press/The MIT Press, 2016.
[GKL16b] R. Gonçalves, M. Knorr, and J. Leite. You can’t always forget what you want: On the limits of forgetting in answer set programming. In *Proceedings of the Twenty-Second European Conference on Artificial Intelligence (ECAI 2016)*, volume 285, pages 957–965. IOS Press, 2016.

[GKL21] R. Gonçalves, M. Knorr, and J. Leite. Forgetting in answer set programming - A survey. *Theory and Practice of Logic Programming*, 2021. To appear.

[GKLW20] R. Gonçalves, M. Knorr, J. Leite, and S. Woltran. On the limits of forgetting in Answer Set Programming. *Artificial Intelligence*, 2020.

[Koo20] P. Koopmann. LETHE: forgetting and uniform interpolation for expressive description logics. *KI — Künstliche Intelligenz*, 34(3):381–387, 2020.

[KWW09] B. Konev, D. Walther, and F. Wolter. Forgetting and uniform interpolation in extensions of the description logic EL. In *Proceedings of the 22nd International Workshop on Description Logics (DL 2009)*, volume 9, 2009.

[Lei17] J. Leite. A bird’s-eye view of forgetting in answer-set programming. In *Proceedings of the fourteenth Logic Programming and Nonmonotonic Reasoning International Conference, LPNMR 2017*, pages 10–22. Springer, 2017.

[Lin01] F. Lin. On strongest necessary and weakest sufficient conditions. *Artificial Intelligence*, 128(1-2):143–159, 2001.

[LLM03] J. Lang, P. Liberatore, and P. Marquis. Propositional independence — formula-variable independence and forgetting. *Journal of Artificial Intelligence Research*, 18:391–443, 2003.

[LR94] F. Lin and R. Reiter. Forget it! In *Proceedings of the AAAI Fall Symposium on Relevance*, pages 154–159, 1994.

[LU97] J. Lobo and C. Uzcátegui. Abductive consequence relations. *Artificial Intelligence*, 89(1-2):149–171, 1997.

[MB06] J. Meheus and D. Batens. A formal logic for abductive reasoning. *Journal of the Interest Group in Pure and Applied Logic*, 14(2):221–236, 2006.

[Moi07] Y. Moinard. Forgetting literals with varying propositional symbols. *Journal of Logic and Computation*, 17(5):955–982, 2007.

[Pau93] G. Paul. Approaches to abductive reasoning: an overview. *Artificial Intelligence Review*, 7(2):109–152, 1993.

[PPU03] R. Pino Pérez and C. Uzcátegui. Preferences and explanations. *Artificial Intelligence*, 149(1):1–30, 2003.

[PS15] Koopmann P. and R.A. Schmidt. LETHE: saturation-based reasoning for non-standard reasoning tasks. In *Informal Proceedings of the 4th International Workshop on OWL Reasoner Evaluation (ORE-2015)*, volume 1387 of *CEUR Workshop Proceedings*, pages 23–30. CEUR-WS.org, 2015.
O. Rodrigues, D. Gabbay, and A. Russo. Belief revision. In *Handbook of philosophical logic*, pages 1–114. Springer, 2011.

S. Subbarayan and D.K. Pradhan. NiVER: Non-increasing variable elimination resolution for preprocessing SAT instances. In *International conference on theory and applications of satisfiability testing*, pages 276–291. Springer, 2004.

H. Tompits. Expressing default abduction problems as quantified boolean formulas. *AI Communications*, 16(2):89–105, 2003.

C. Wernhard. Abduction in logic programming as second-order quantifier elimination. In *Frontiers of Combining Systems - Ninth International Symposium, FroCoS 2013*, pages 103–119, 2013.

K. Wang, Z. Wang, R.W. Topor, J.Z. Pan, and G. Antoniou. Concept and role forgetting in $\mathcal{ALC}$ ontologies. In *Proceedings of the eighth 8th International Semantic Web Conference, ISWC 2009 25-29, 2009.*, volume 5823 of *Lecture Notes in Computer Science*, pages 666–681. Springer, 2009.

Y. Wang, K. Wang, Z. Wang, and Z. Zhuang. Knowledge forgetting in circumscription: A preliminary report. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence (AAAI 2015)*, pages 1649–1655. AAAI Press/The MIT Press, 2015.

Y. Zhao, R.A. Schmidt, Y. Wang, X. Zhang, and H. Feng. A practical approach to forgetting in description logics with nominals. In *Proceedings of the Thirty-Fourth AAAI Conference on Artificial Intelligence (AAAI 2020)*, pages 3073–3079, 2020.