Growth rates of groups associated with face 2-coloured triangulations and directed Eulerian digraphs on the sphere

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Abstract
Let $G$ be a properly face 2-coloured (say black and white) piecewise-linear triangulation of the sphere with vertex set $V$. Consider the abelian group $A_W$ generated by the set $V$, with relations $r + c + s = 0$ for all white triangles with vertices $r$, $c$ and $s$. The group $A_B$ can be defined similarly, using black triangles. These groups are related in the following manner $A_W \cong A_B \cong \mathbb{Z} \oplus \mathbb{Z} \oplus C$ where $C$ is a finite abelian group.

The finite torsion subgroup $C$ is referred to as the canonical group of the triangulation. Let $m_t$ be the maximal order of $C$ over all properly face two-coloured spherical triangulations with $t$ triangles of each colour. By relating properly face two-coloured spherical triangulations to directed Eulerian spherical embeddings of digraphs whose abelian sand-pile groups are isomorphic to $C$ we provide improved upper and lower bounds for $\limsup_{t \to \infty} (m_t)^{1/t}$.

1 Introduction

Let $G$ be a graph. We will denote the vertex set of $G$ by $V(G)$ and the edge set of $G$ by $E(G)$. Suppose that there exists a face 2-coloured, black and

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white say, triangulation of the sphere, i.e. a spherical triangulation, $\mathcal{G}$ of $G$. Denote the set of white faces by $W$ and the set of black faces by $B$. As the faces are properly face 2-coloured $G$ is Eulerian and, by a well known result of Heawood \[14\], regardless of whether or not $G$ is simple, $G$ has a proper vertex 3-colouring. If $G$ is simple, then the rotation at every vertex is a cycle, i.e. the triangulation is piecewise-linear. See Figure 1 for an illustration of a face 2-coloured spherical triangulation where the graph is simple.

Figure 1: A face 2-coloured spherical triangulation. A vertex, $r_0$, has been placed at infinity.

Define $\mathcal{A}_W$ to be the abelian group with generating set $V(G)$, subject to the relations $\{r + c + s = 0 : r, c, s$ are the vertices of a white face of $\mathcal{G}\}$. Define $\mathcal{A}_B$ similarly but using the black faces. In [1] Blackburn and the current author proved that $\mathcal{A}_W \cong \mathcal{A}_B \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathcal{C}$

where $\mathcal{C}$ is a finite abelian group. In the same paper the question of the growth rate of the maximal order of $\mathcal{C}$, in the terminology established in [13] the canonical group of the face 2-coloured spherical triangulation, was raised. More precisely:

**Question** (Blackburn & McCourt, [1]). Let $m_t$ be the maximal order of the canonical group over all properly face two-coloured spherical triangulations.
of simple graphs with \( t \) faces of each colour. What is the value of

\[
\limsup_{t \to \infty} (m_t)^{1/t}?
\]

In [1] a lower bound of 1.201 was obtained. Earlier work of Cavenagh and Wanless [7] provided an upper bound of \( 6^{1/3} < 1.818 \) and of Drápal and Kepka of \( e^{1/e} < 1.445 \). More recently Grubman and Wanless [13] improved the lower bound to \( 5123^{1/30} > 1.329 \). In Section 3 we will provide an improved upper bound of \( \exp\left(\frac{\ln(3) + \ln(2)}{5}\right) < 1.431 \) and in Section 4 an improved lower bound of \( (27/2)^{1/8} > 1.384 \).

In order to establish these new bounds we will make use of a connection between canonical groups of face 2-coloured spherical triangulation and abelian sand-pile groups of directed Eulerian spherical digraphs. In Section 2 we will discuss the background for both of these groups as well as further motivation for addressing the above question.

\section{Background and motivation}

\subsection{Spherical latin bitrades}

Let \( G \) be a properly vertex 3-coloured simple graph with \( v \) vertices. If the edges of \( G \) can be partitioned into copies of \( K_3 \), then such a partition is called a \textit{partial latin square}. The set of triples of vertices of each of the copies of \( K_3 \) completely describes such a partial latin square and we will use the two descriptions interchangeably. We will refer to the graph \( G \) as the \textit{support graph} of the partial latin square.

Let \( P \) be a partial latin square, then the three vertices in each triple of \( P \) are each from a different vertex colour classes, say \( R \) (the \textit{rows}), \( C \) (the \textit{columns}) and \( S \) (the \textit{symbols}), in the support graph. Suppose that \( \max\{|R|, |C|, |S|\} = n \), then any triple of the partial latin square is of the form \( \{r_i, c_j, s_k\} \), where \( r_i \in R \), \( c_j \in C \) and \( s_k \in S \), and such a triple can be thought of as the symbol \( k \) occurring in row \( i \), column \( j \) of a \( n \times n \) array.

Two partial latin squares are said to be \textit{isotopic} if they are equal up to a relabelling of their sets of rows, columns and symbols. A partial latin square \( P \) is said to \textit{embed} in an abelian group \( A \) if and only if it is isotopic to a partial latin subsquare contained in the Cayley table of \( A \). An abelian group \( A \) is said to be a \textit{minimal abelian representation} for the partial latin square.
Define $\mathcal{A}_P$ to be the abelian group with generating set $V(G)$, subject to the relations $\{r + c + s = 0 : \{r, c, s\} \in P\}$. The motivation for this definition is that if $P$ embeds in an abelian group, then it embeds in $\mathcal{A}_P$ and, in particular, any minimal abelian representation $A$ of $P$ is a quotient of the finite torsion subgroup of $\mathcal{A}_P$, see [1] and [11] for details.

A latin bitrade is an ordered pair $(W, B)$ of non-empty partial latin squares such that for each triple $\{r_i, c_j, s_k\} \in W$ (respectively $B$) there exist unique $r_{i'} \neq r_i, c_{j'} \neq c_j$ and $s_{k'} \neq s_k$ such that

$$\{\{r_{i''}, c_j, s_k\}, \{r_i, c_{j'}, s_{k'}\}, \{r_i, c_j, s_{k''}\}\} \subseteq B \text{ (respectively } W)$$

That is, they are disjoint decompositions of the edge set of the same simple support graph. The arrays in Figure 2 correspond to a pair of partial latin squares which form a latin bitrade $(W, B)$. Note that the two partial latin squares, $W$ and $B$, are not isotopic.

![Figure 2: A pair of partial latin squares that together form a latin bitrade.](image)

Suppose that $\mathcal{G}$ is a face 2-coloured spherical triangulation of a simple graph $G$ with face colour classes $W$ and $B$ and a proper vertex 3-colouring given by $R, C$ and $S$. Then the faces of $W$ (respectively $B$) form a partial latin square. As $W$ and $B$ are decompositions of the same simple graph and, provided $|W| > 1$, no face occurs in both $W$ and $B$, the pair $(W, B)$ is a latin bitrade. For example, the face 2-coloured spherical triangulation illustrated in Figure 1 corresponds to the latin bitrade $(W, B)$ in Figure 2, the white faces corresponding to the entries in $W$ and the grey faces the entries in $B$.

In general the partial latin squares forming a bitrade do not necessarily embed in an abelian group, see [7]. However, the partial latin squares forming a bitrade $(W, B)$ arising from a face 2-coloured spherical triangulation both embed in abelian groups, and hence $W$ embeds in $\mathcal{A}_W$ and $B$ embeds in $\mathcal{A}_B$. If $P$ embeds in $A$ and, for all embeddings of $P$ in $A$, the isotopic copy of $P$ in the Cayley table of $A$ generates $A$. 
A_B, [7] [10], answering a question from [9]. In [7] Cavenagh and Wanless conjectured that \( A_W \cong A_B \); this was proved in a more general setting in [1] as discussed in Section 1.

2.2 Directed Eulerian spherical digraphs and abelian sand-pile groups

Let \( G \) be a graph; we will denote the degree of a vertex \( v \in V(G) \) by \( \deg_G(v) \) and the maximum degree over all vertices of \( G \) by \( \Delta(G) \). Let \( G \) be an embedding of \( G \) in a sphere. We arbitrarily fix an orientation for the vertices, and denote the rotation at a vertex \( v \in V(G) \) by \( \rho(v) \). Suppose \( \rho(v) = (u_1, u_2, \ldots, u_{\deg_G(v)}) \) for some \( v \in V(G) \); if \( G \) is a triangulation and \( G \) is a simple graph, then the set of vertices \( \{u_0, u_1, \ldots, u_{\deg_G(v)} - 1\} \) induces a cycle in \( G \) where, interpreting \( u_{\deg_G(v)} \) as \( u_0 \), the edges are between \( u_i \) and \( u_{i+1} \). In a slight abuse of notation we will denote this cycle as \( \rho(v) \).

Let \( D \) be a (not necessarily simple) digraph. Label the vertices of \( D \) as \( v_1, v_2, \ldots, v_n \). The adjacency matrix \( A = [a_{ij}] \) of \( D \) is the \( n \times n \) matrix where the entry \( a_{ij} \) equals the number of arcs from vertex \( v_i \) to vertex \( v_j \). The asymmetric Laplacian of \( D \) is the \( n \times n \) matrix \( L(D) = B - A \) where \( B \) is the diagonal matrix whose entry \( b_{ii} \) is the out-degree of \( v_i \). A sink in a digraph is a vertex with 0 out-degree.

A digraph \( D \) is Eulerian if the out-degree at each vertex of \( D \) equals the in-degree of each vertex of \( D \). In this case, for each \( v \in V(D) \) we will refer to out-degree and in-degree of \( v \) simply as the degree of \( v \) and denote it by \( \deg_D(v) \). Suppose \( D \) is an Eulerian digraph with vertex set \( V(D) = \{v_1, v_2, \ldots, v_n\} \). Let \( u \in V(D) \), then the digraph obtained by deleting all the arcs leaving vertex \( u \) from \( D \) is called the Eulerian digraph \( D \) with sink \( u \).

Let \( D \) be a strongly connected Eulerian digraph where \( V(D) = \{v_1, v_2, \ldots, v_n\} \); fix an \( i \), where \( 1 \leq i \leq n \). A reduced asymmetric Laplacian, \( L'(D) \), for \( D \) is obtained by removing row \( i \) and column \( i \) from \( L(D) \). The abelian sand-pile group of the Eulerian digraph \( D \) with sink \( v_i \), denoted \( S(D) \) is \( \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L'(D) \). The group obtained is invariant of the choice of vertex for the sink, see [15, Lemma 4.12]. As such, without loss of generality, we can define \( S(D) = \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L'(D) \) where \( L'(D) \) is the asymmetric Laplacian obtained by removing row and column \( n \) of \( L(D) \). (An equivalent definition of \( S(D) \) is the finite torsion subgroup of \( \mathbb{Z}^n/\mathbb{Z}^nL(D) \).)

Let \( D \) be a digraph and let \( v \in V(D) \). An arborescence diverging from \( v \)
is a directed sub-tree of $D$ in which all the arcs are directed away from $v$. If $D$ is Eulerian, and hence strongly connected, then the number of spanning arborescences diverging from a vertex $v$ does not depend on $v$, see [20, Theorem VI.23]; this number is known as the tree number of $D$ and we will denote it by $T(D)$. By the Matrix-Tree Theorem, [20, Theorem VI.28], $T(D)$ is the determinant of $L'(D)$; which in turn is the order of the abelian sand-pile group $S(D)$, see [15, Lemma 2.8]. A recent and comprehensive survey of results on abelian sand-pile groups of digraphs is given by [15].

In [17] Ribó Mor uses a probabilistic argument via Suen’s Inequality, [19], to establish an upper bound on the order of the abelian sand-pile group in an undirected planar graph in terms of the number of vertices. In the same thesis Ribó Mor establishes a tighter bound using non-probabilistic techniques. This bound has subsequently been improved on in [4].

Consider an embedding of an Eulerian digraph. If each face of the embedding is a directed cycle, equivalently the arc rotation at each vertex alternates between incoming and outgoing arcs, the embedding is called a directed Eulerian digraph embedding, see [2] and [3]. If the embedding is on the sphere we call it a directed Eulerian spherical digraph. Eulerian digraph embeddings in surfaces of arbitrary genus have been studied in [2] and [8], and in [3] Bonnington et al. provide Kuratowski type theorems for directed Eulerian spherical digraphs.

In the following Subsection we will discuss a connection between the canonical groups of face 2-coloured spherical triangulations and the abelian sand-pile groups of directed Eulerian spherical digraphs.

### 2.3 Canonical groups and abelian sand-pile groups

Let $G$ be a face 2-coloured spherical triangulation with a proper vertex 3-colouring where the vertex colour classes are $R$, $C$ and $S$. Let $I \in \{R, C, S\}$; we will construct a directed Eulerian spherical digraph $D_I(G)$ (or simply $D_I$) with vertex set $I$. The digraph will potentially have, for any pair of vertices $u$ and $v$, multiple arcs from $u$ to $v$. Let $\{I_0, I_1, I_2\} = \{R, C, S\}$. Consider a vertex $i \in I_0$, then the rotation at $i$ is

$$
\rho(i) = (u_1, v_1, u_2, v_2, \ldots, u_{\frac{1}{2}\deg_G(i)}, v_{\frac{1}{2}\deg_G(i)}),
$$

where, without loss of generality, $u_j \in I_1$ and $v_j \in I_2$ for all $1 \leq j \leq \frac{1}{2}\deg_G(i)$ and the edge $e_j$ between $u_j$ and $v_j$ in the rotation is contained in a black
face. Then in $D_I$ there are $\frac{1}{2}\deg_G(i)$ outgoing arcs $a_j$ with initial vertex $i$, one for each black face, and the terminal vertex for arc $a_j$ is the vertex in $I$ contained in the white face containing edge $e_j$. Clearly, the graph $D_I$ inherits a spherical embedding from $G$ in which the arc rotation at each vertex alternates between incoming and outgoing arcs. Hence $G$ is a directed Eulerian spherical embedding. Figure 3 illustrates the graph $D_R$ (the arcs of which are shown as dashed) obtained from a face 2-coloured spherical triangulation.

![Diagram](image)

Figure 3: A face 2-coloured spherical triangulation together with corresponding digraph $D_R$. The vertex colour classes are $R = \{r_0, r_1, r_2, r_3, r_4\}$, where vertex $r_0$ has been placed at infinity; $C = \{c_0, c_1, c_2, c_3\}$; and $S = \{s_0, s_1, s_2, s_3, s_4\}$.

**Lemma 1.** Given a strongly connected directed Eulerian spherical digraph $D$, there exists a face 2-coloured spherical triangulation $\mathcal{G}$ with a vertex 3-colouring given by the vertex sets $R, C$ and $S$, such that for some $I \in \{R, C, S\}$,

$$D_I(\mathcal{G}) \cong D.$$
Proof. In short, we reverse the construction above.

Denote the faces of $D$ as $f_1, f_2, \ldots, f_k$. Insert a new vertex $z_i$ into each face $f_i$ for all $1 \leq i \leq k$. Consider an arc of $D$, a say that has $x$ as its initial vertex and $y$ as its terminal vertex. Then on one side of $a$ there is a new vertex $u$ and on the other a new vertex $w$. Replace $a$ with two triangular faces; a black face with vertex set $\{x, u, w\}$ and a white face with vertex set $\{y, u, w\}$. As $D$ is strongly connected this results in a triangulation of the sphere and as $D$ is a directed Eulerian digraph the resulting triangulation is properly face 2-coloured. \qed

We now list some observations on Lemma 1 and the above construction.

Observation 1. Let $\mathcal{G}$ be a face 2-coloured spherical triangulation of a graph $G$ with a proper vertex 3-colouring where the colour classes are $R$, $C$ and $S$. Let $I \in \{R, C, S\}$.

(i) If $v \in I$, then $\deg_{D_I}(v) = \frac{1}{2} \deg_G(v)$.

(ii) A face $f$ of size $k$ in $D_I$ corresponds to a vertex in $G$ with degree $2k$.

(iii) Let $\{I, J, K\} = \{R, C, S\}$. A face $f$ of size $d$ in $D_I$ corresponds to a face of size $d$ in, without loss of generality, $D_J$ and a vertex of (out-)degree $d$ in $D_K$. While a vertex of (out-)degree $d$ in $D_I$ corresponds to a face of size $d$ in $D_J$ and a face of size $d$ in $D_K$.

The following lemma is implicit in [1].

Lemma 2. Let $\mathcal{G}$ be a face 2-coloured spherical triangulation with a proper vertex 3-colouring where the vertex colour classes are $R$, $C$ and $S$. Let $I \in \{R, C, S\}$, then $D_I$ is strongly connected and $\mathcal{S}(D_R) \cong \mathcal{S}(D_C) \cong \mathcal{S}(D_S) \cong \mathcal{C}$, where $\mathcal{C}$ is the canonical group of $\mathcal{G}$.

In the following sections we will focus on bounding the number of spanning arborescences in the directed graph $D_I$, where $I \in \{R, C, S\}$, obtained from a face 2-coloured spherical triangulation $\mathcal{G}$ of a simple graph $G$. In Section 3 considering all such $\mathcal{G}$ with a fixed number of faces in each colour class yields the improved upper bound. In Section 4 we provide a construction for face 2-coloured spherical triangulations for which the associated graphs $D_I$, where $I \in \{R, C, S\}$, have many spanning arborescences, obtaining a lower bound. Before doing so we will discuss the construction of face 2-coloured spherical triangulations that yield specific canonical groups.
2.4 Constructing abelian groups

Proposition 1. Let \( m \geq 1 \). There exists a face 2-coloured spherical triangulation of a simple graph with canonical group \( C \cong \mathbb{Z}_m \).

Proof. Let \( D \) be the strongly connected directed Eulerian spherical digraph with two vertices, \( v_0 \) and \( v_1 \) say, and \( 2m \) arcs, \( m \) from \( v_0 \) to \( v_1 \) and \( m \) from \( v_1 \) to \( v_0 \) where the edge rotation at each vertex alternates between incoming and outgoing arcs. Then \( L(D) = \begin{bmatrix} m & -m \\ -m & m \end{bmatrix} \) and \( L'(D) = [m] \), so \( S \cong \mathbb{Z}_m \).

By Lemma 1, there exists a face 2-coloured spherical triangulation, with a vertex 3-colouring given by the sets \( R, C \) and \( S \) where \( D = D_I \) for some \( I \in \{ R, C, S \} \). It is easy to see that in this case the triangulation is of a simple graph. \(\square\)

We will use recursive applications of the following elementary lemma to prove Proposition 2.

Lemma 3. Given two Eulerian digraphs \( D_1 \) and \( D_2 \) with disjoint vertex sets the graph \( D \) obtained by identifying a vertex in \( D_1 \) with a vertex in \( D_2 \) has an abelian sand-pile group isomorphic to \( S(D_1) \oplus S(D_2) \).

Proof. Let \( v_1 \in V(D) \) and \( v_2 \in V(D_2) \) be the vertices identified to form \( D \) and denote the identified vertex as \( v \). As \( D_1 \) and \( D_2 \) are strongly connected and Eulerian, \( D \) is also strongly connected and Eulerian. Let \( L'(D_1) \) (respectively \( L'(D_2), L'(D) \)) be the reduced asymmetric Laplacians with sink \( v_1 \) in \( D_1 \) (respectively \( v_2 \) in \( D_2 \) and \( v \) in \( D \) ). Then applying, possibly trivial, row and column permutations to \( L'(D) \) yields

\[
\begin{bmatrix}
L'(D_1) & 0 \\
0 & L'(D_2)
\end{bmatrix}.
\]

\(\square\)

Proposition 2. Consider an arbitrary finite abelian group \( \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k} \). Then there exists a face 2-coloured spherical triangulation with canonical group isomorphic to \( \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k} \).

Proof. Using Proposition 1 construct graphs \( D_i \) for \( 1 \leq i \leq k \) where \( S(D_i) = \mathbb{Z}_{m_i} \). Take any spherical embedding of a tree with \( k \) edges, labelled \( e_1, \ldots, e_k \), and replace each edge with \( D_i \). It is easy to see that this can be done so
that the resulting embedded digraph, $D$, is a strongly connected directed Eulerian spherical digraph. The resulting graph can also be obtained by recursive applications of Lemma 3 and hence has an abelian sand-pile group isomorphic to $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$. Therefore, by Lemma 1, there exists a face 2-coloured triangulation that has $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$ as its canonical group.

Figure 4 illustrates the construction used in the proof of Proposition 2 in the case where the canonical group of the face 2-coloured triangulation is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{A face 2-coloured spherical triangulation whose canonical group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. A vertex has been placed at infinity and the digraph $D_R$, where $R = \{r_0, r_1, r_2\}$, is shown with dashed arcs.}
\end{figure}

Note that the construction used in the proof of Proposition 2 yields triangulations of graphs that are not simple, i.e. they do not correspond to latin bitrades.

Let $A = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_{k-1}}$ where, without loss of generality, $m_i > 1$ for $1 \leq i \leq k - 1$. The construction in the proof of Proposition 2 yields a triangulation $G$ with a proper vertex three colouring given by $R$, $C$ and $S$ such that $D \cong D_R(G)$ and neither $D_C(G)$ nor $D_S(G)$ contain any cut vertices. Hence a set $T$ of nonisomorphic trees on $k$ vertices yields $|T|$ nonisomorphic face 2-coloured triangulations all of which have canonical groups isomorphic to $A$. (Otter [18] showed that the number of nonisomorphic trees on $k$ vertices is asymptotically $0.4399237(2.95576)^{k3/2}$.)
3 Improving the upper bound

In this section all the face 2-coloured spherical triangulations will be of simple graphs. Moreover, as we are concerned with the behaviour of $m_t$ as $t \to \infty$, in the following discussion, we take $t \geq 4$. Hence every vertex in any triangulation considered is contained in at least four faces.

Similarly to the approach taken by Ribó Mor in [17], the improved upper bound for $\lim \sup_{t \to \infty} (m_t)^{1/t}$ is obtained using a probabilistic argument based on Suen’s Inequality. However, the results in [17] are concerned with the growth of the number of spanning trees in terms of the number of vertices in the graph, rather than the number of arcs. A vital component of Ribó Mor’s argument is the addition of edges to a planar graph to obtain a triangulation. Thus, although the beginning of our argument follows that of [17] (in setting up the use of a refinement of Suen’s Inequality), as we are interested in the growth rate as the number of arcs increases, the remainder necessarily follows a different approach.

Let $\{I_i\}_{i \in I}$ be a finite family of Bernoulli random variables each with success probability $p_i$; i.e. $\mathbb{P}(I_i) = p_i$. A simple graph $\Gamma$ where $V(\Gamma) = I$ is called a dependency graph for $\{I_i\}_{i \in I}$ if when two disjoint subsets of $I$, $A$ and $B$ say, are mutually independent, there is no edge between any vertex in $A$ and any vertex in $B$. In particular two distinct variables $I_i$ and $I_j$ are independent unless there is an edge between $i$ and $j$. For ease of notation, when discussing a dependency graph, if there exists an edge between vertices $i$ and $j$, we write $i \sim j$.

We will make use of the following refinement to Suen’s Inequality (note that in our case both Suen’s Inequality and that presented in Theorem 1 yield the same bound).

**Theorem 1** (Janson [16]). Let $I_i$, where $i \in I$ be a finite family of Bernoulli random variables with success probability $p_i$, having a dependency graph $\Gamma$. Let $S = \sum_{i \in I} I_i; \mu = \mathbb{E}(S) = \sum_{i \in I} p_i; \Delta = \frac{1}{2} \sum_{i \in I} \sum_{j \in I, i \sim j} \mathbb{E}(I_iI_j);$ and $\delta = \max_{i \in I} \sum_{k \sim i} p_k$. Then

$$\mathbb{P}(S = 0) \leq \exp \left( -\mu + \Delta e^{2\delta} \right).$$

Let $G$ be a properly face 2-coloured triangulation of a simple graph $G$ with $t$ faces of each colour such that the order of its canonical is maximum over all such triangulations. Fix a vertex $i_0$ of $D_I(G)$ (for the remainder of this section we will write $D_I$ for $D_I(G)$). Let $\mathcal{R}$ be a random selection of incoming
arcs, one for each vertex of \(V(D_I) - i_0\). Then, denoting the subgraph of \(D_I\) induced by the arcs of \(R\) as \(D_I[R]\), we have

\[
T(D_I) = \mathbb{P}\left( D_I[R] \text{ is a spanning arborescence rooted at } i_0 \right) \prod_{i \in V(D_I) - i_0} \deg_{D_I}(i).
\]

Equivalently

\[
T(D_I) = \mathbb{P}(D_I[R] \text{ contains a directed cycle}) \prod_{i \in V(D_I) - i_0} \deg_{D_I}(i).
\]

We can now use Theorem 1 to provide an upper bound for the probability that \(D_I[R]\) does not contain a directed cycle (as \(R\) contains exactly one incoming arc for each vertex not equal to \(i_0\) if the underlying graph contains a cycle, it must be directed).

Let \(D_{I-i_0}\) denote the set of all directed cycles in \(D_I\) that do not contain the vertex \(i_0\). For each \(\gamma \in D_{I-i_0}\), define:

\[
I_\gamma = \begin{cases} 1 & \gamma \text{ is a subgraph of } D_I[R]; \\
0 & \text{otherwise}. \end{cases}
\]

From the definition of \(R\) the arcs of \(\gamma\) are independent events and an arc from a vertex \(u\) to a vertex \(v\), where there is an arc from \(u\) to \(v\) in \(\gamma\) occurs in \(R\) with probability \(1/(\deg_{D_I}(u))\). Hence, \(I_\gamma\) is a Bernoulli random variable taking the value 1 with probability

\[
p_\gamma = \frac{1}{\prod_{u \in V(\gamma)} \deg_{D_I}(u)}.
\]

Therefore \(S = \sum_{\gamma \in D_I} I_\gamma\) counts the number of cycles in \(D_I[R]\), and \(\mathbb{P}(S = 0)\) measures the probability that no cycle exists in \(D_I[R]\).

Define a graph \(\Gamma\) on the vertex set \(D_I\), with an edge between vertex \(\alpha\) and \(\beta\) if and only if \(\alpha\) and \(\beta\) share a vertex in \(D_I\). (Note that two cycles in \(D_I[R]\) can never share a vertex.) Thus, \(\alpha \sim \beta\) implies that \(\mathbb{E}(I_\alpha I_\beta) = 0\), and hence, the value of \(\Delta\) from Theorem 1 is zero.

Applying Theorem 1 with \(\mu = \sum_{\gamma \in D_{I-i_0}} p_\gamma\), we have:

\[
T(D_I) \leq \exp\left(-\mu\right) \prod_{i \in V(D_I) - i_0} \deg_{D_I}(i).
\]
Let $D_I$ denote the set of all directed cycles in $D_I$ and $D_{i_0}$ denote the set of all directed cycles in $D_I$ that contain $i_0$. Then, by Inequality 1 we have:

$$T(D_I) \leq \exp \left[ \left( \sum_{i \in I} \ln(\deg_{D_I}(i)) \right) - \ln(\deg_{D_I}(i_0)) - \left( \sum_{\gamma \in D_I} p_{\gamma} \right) \right].$$

**Lemma 4.**

$$T(D_I) \leq \exp \left[ \left( \sum_{i \in I} \ln(\deg_{D_I}(i)) \right) - \left( \sum_{\gamma \in D_I} p_{\gamma} \right) \right].$$

**Proof.** We show that $-\ln(\deg_{D_I}(i_0)) + \left( \sum_{\gamma \in D_{i_0}} p_{\gamma} \right) < 0$.

As $D_I$ is Eulerian there are $\deg_{D_I}(i_0)$ cycles passing through $i_0$. Hence,

$$-\ln(\deg_{D_I}(i_0)) + \left( \sum_{\gamma \in D_{i_0}} p_{\gamma} \right) \leq -\ln(\deg_{D_I}(i_0)) + \deg_{D_I}(i_0) \max_{\gamma \in D_I} \{p_{\gamma}\}.$$

Now $i_0$ is necessarily a vertex in each cycle passing through it and the minimum (out-)degree of any vertex in $D_I$ is 2, so

$$\max_{\gamma \in D_I} \{p_{\gamma}\} = \max_{\gamma \in D_I} \left\{ \frac{1}{\prod_{u \in V(\gamma)} \deg_{D_I}(u)} \right\} \leq \frac{1}{2 \deg_{D_I}(i_0)}.$$

Therefore,

$$-\ln(\deg_{D_I}(i_0)) + \left( \sum_{\gamma \in D_{i_0}} p_{\gamma} \right) \leq -\ln(2) + \frac{\deg_{D_I}(i_0)}{2 \deg_{D_I}(i_0)} < 0.$$

By Lemma 2, $T(D_R) = T(D_C) = T(D_S)$, so, by Lemma 4 we have:

$$T(D_I)^3 \leq T(D_R)T(D_C)T(D_S) \leq \exp \left[ \left( \sum_{r \in R} \ln(\deg_{D_R}(r)) + \sum_{c \in C} \ln(\deg_{D_C}(c)) + \sum_{s \in S} \ln(\deg_{D_S}(s)) \right) - \left( \sum_{\gamma \in D_R} p_{\gamma} + \sum_{\gamma \in D_C} p_{\gamma} + \sum_{\gamma \in D_S} p_{\gamma} \right) \right].$$
Denote the set of faces in $D_I$, for $I \in \{R, C, S\}$, by $F_I$; and, in a slight abuse of notation, the set of vertices on a face $f$ by $V(f)$. As the triangulation is of a simple graph it is piecewise linear. Hence, the facial walk of any face in $F_I$ is a cycle and so $F_I \subseteq D_I$. As $\deg_{D_I}(i) = \frac{1}{2} \deg_G(i)$, we have

$$3 \ln(T(D_I)) \leq \left( \sum_{v \in V} \ln \left( \frac{1}{2} \deg_G(v) \right) \right) - \left( \sum_{f \in F_R} \frac{1}{\prod_{r \in V(f)} \deg_{D_R}(r)} \right) - \left( \sum_{f \in F_C} \frac{1}{\prod_{c \in V(f)} \deg_{D_C}(c)} \right) - \left( \sum_{f \in F_S} \frac{1}{\prod_{s \in V(f)} \deg_{D_S}(s)} \right).$$

Let $\{I_0, I_1, I_2\} = \{R, C, S\}$. Consider a vertex $i \in I_0$, then the rotation at $i$ is

$$\rho(i) = (u_1, v_1, u_2, v_2, \ldots, u_{\frac{1}{2} \deg_G(i)}, v_{\frac{1}{2} \deg_G(i)}),$$

where, without loss of generality, $u_j \in I_1$ and $v_j \in I_2$ for all $1 \leq j \leq \frac{1}{2} \deg_G(i)$. Note that $i$ corresponds to the face with facial walk $(u_1, u_2, \ldots, u_{\frac{1}{2} \deg_G(i)})$ in $G_{I_1}$ and the face with facial walk $(v_1, v_2, \ldots, v_{\frac{1}{2} \deg_G(i)})$ in $G_{I_2}$. Hence, defining $\rho_1(i) = \{u_1, u_2, \ldots, u_{\frac{1}{2} \deg_G(i)}\}$ and $\rho_2(i) = \{v_1, v_2, \ldots, v_{\frac{1}{2} \deg_G(i)}\}$ we have the following upper bound for $3 \ln(T(D_I))$.

$$\left( \sum_{v \in V} \ln \left( \frac{1}{2} \deg_G(v) \right) \right) - \left( \sum_{v \in V} \frac{1}{\prod_{j \in \rho_1(v)} \frac{1}{2} \deg_G(j)} \right) - \left( \sum_{v \in V} \frac{1}{\prod_{j \in \rho_2(v)} \frac{1}{2} \deg_G(j)} \right).$$

Let $n_k$ denote the number of degree $k$ vertices in $G$. Then arguing from Inequality 2 we prove the following theorem.

**Theorem 2.** Let $m_t$ be the maximal order of the canonical group of all properly face two-coloured spherical triangulations of simple graphs with $t$ faces of each colour. Then

$$\limsup_{t \to \infty} (m_t)^{1/t} \leq \exp \left( \frac{\ln(3) + \ln(2)}{5} \right) < 1.431.$$  

**Proof.** Define a function $g : V \to \mathbb{Z}$ by

$$g : v \mapsto \begin{cases} 
2, & \text{if all the neighbours of } v \text{ have degree } \leq 6; \\
1, & \text{if all the neighbours of } v \text{ in precisely one of the two colour classes in } \rho(v) \text{ have degree } \leq 6; \\
0, & \text{otherwise.}
\end{cases}$$
Let $N_4 = \{ v \in V(G) : \deg(v) = 4 \}$ and $N_6 = \{ v \in V(G) : \deg(v) = 6 \}$. Further let $0 \leq \alpha \leq 2$ and $0 \leq \beta \leq 2$ be such that $\alpha n_4 = \sum_{v \in N_4} g(v)$ and $\beta n_6 = \sum_{v \in N_6} g(v)$.

Rewriting the upper bound (2) in terms of the $n_k$'s, and bounding the second summation in terms of $\alpha$ and $\beta$ we have

$$3 \ln(T(D_I)) \leq \left( \sum_{i=2}^{\Delta(G)/2} \ln(i) n_{2i} \right) - \frac{\alpha n_4}{3^2} - \frac{\beta n_6}{3^3}.$$ 

As the average degree of a vertex in $G$ is $6 - 12/n$, for each vertex of degree $2i > 6$ we can associate $(2i - 6)/2$ degree four vertices. Hence we have that $n_4 = 6 + \sum_{i=4}^{\Delta(G)/2} (i - 3)n_{2i}$. Thus

$$3 \ln(T(D_I)) < \ln(3) n_6 + \left( \sum_{i=4}^{\Delta(G)/2} (\ln(i) + (i - 3) \ln(2)) n_{2i} \right) - \frac{\alpha n_4}{9} - \frac{\beta n_6}{27}.$$ 

Consider a vertex $v$ in $N_j$, where $j \in \{4, 6\}$. Recall that if $\rho_1(v)$ contains a vertex with degree greater than 6 but $\rho_2(v)$ does not, then $g(v) = 1$; if both $\rho_1(v)$ and $\rho_2(v)$ do not contain vertices of degree larger than 6, then $g(v) = 2$; and otherwise $g(v) = 0$. Hence, $3\alpha n_4 + \beta n_6 \geq 3 \left( 2n_4 - \sum_{i=4}^{\Delta(G)/2} 2(i - 3)n_{2i} \right) + \left( 2n_6 - \sum_{i=4}^{\Delta(G)/2} 2(i - (i - 3))n_{2i} \right) = 3(12) + 2n_6 - 6 \sum_{i=4}^{\Delta(G)/2} n_{2i} > 2n_6 - 6(n - n_6 - n_4)$. Therefore,

$$3 \ln(T(D_I)) < \ln(3) n_6 + \left( \sum_{i=4}^{\Delta(G)/2} (\ln(i) + (i - 3) \ln(2)) n_{2i} \right) - \frac{A}{27},$$

where

$$A = \begin{cases} 8n_6 + 6n_4 - 6n, & \text{if } 8n_6 + 6n_4 > 6n; \\ 0, & \text{otherwise}. \end{cases}$$

In $\sum_{i=4}^{\Delta(G)/2} (\ln(i) + (i - 3) \ln(2)) n_{2i}$, the coefficient $\ln(i) + (i - 3) \ln(2)$ corresponds to the average contribution of $i - 2$ vertices (one of degree $2i$ and $i - 3$ of degree four). Hence the sum corresponds to the contribution of all the vertices of degree not equal to 6. As $3 \ln(2)/2 \geq (\ln(i) + (i - 3) \ln(2))/(i - 2)$ for all $i \geq 4$ we have that

$$3 \ln(T(D_I)) < \ln(3)n_6 + \frac{3 \ln(2)}{2} (n - n_6) = \left( \ln(3) - \frac{3 \ln(2)}{2} \right) n_6 + \frac{3 \ln(2)n_6}{2},$$

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regardless of whether or not $A = 0$.

Now, suppose that $8n_6 + 6n_4 > 6n$. Then $A \neq 0$ and

$$3 \ln(T(D_I)) < \ln(3)n_6 + \sum_{i=4}^{\Delta(G)/2} \left( \ln(i) + (i - 3) \ln(2) \right) n_{2i} - \frac{8n_6 + 6n_4 - 6n}{27}.$$

As $n_4 = 6 + \sum_{i=4}^{\Delta(G)/2} (i - 3) n_{2i}$ we have that

$$3 \ln(T(D_I)) < \frac{6n}{27} + \left( \ln(3) - \frac{8}{27} \right) n_6 + \sum_{i=4}^{\Delta(G)/2} \left( \ln(i) + \left( \ln(2) - \frac{6}{27} \right) (i - 3) n_{2i} \right).$$

As $\frac{3 \ln(2)}{2} - \frac{6}{54} \geq \frac{1}{i-2} \left( \ln(i) + (i - 3) \left( \ln(2) - \frac{6}{27} \right) \right)$ for all $i \geq 4$ we have that

$$3 \ln(T(D_I)) < \frac{6n}{27} + \left( \ln(3) - \frac{8}{27} \right) n_6 + \left( \frac{3 \ln(2)}{2} - \frac{6}{54} \right) (n - n_6).$$

$$= \left( \frac{3 \ln(2)}{2} + \frac{6}{54} \right) n + \left( \ln(3) - \frac{3 \ln(2)}{2} - \frac{10}{54} \right) n_6.$$

So we have overall upper bound for $3 \ln(T(D_I))$ when

$$\left( \ln(3) - \frac{3 \ln(2)}{2} \right) n_6 + \frac{3 \ln(2)}{2} n = \left( \frac{3 \ln(2)}{2} + \frac{6}{54} \right) n + \left( \ln(3) - \frac{3 \ln(2)}{2} - \frac{10}{54} \right) n_6;$$

i.e., when $n_6/n = 3/5$. Hence,

$$\limsup_{t \to \infty} (m_t^{1/t}) \leq \limsup_{t \to \infty} (m_t^{1/t}) \left( \exp \left( \frac{t(\ln(3) + \ln(2))}{5} \right) \right)^{1/t}$$

$$= \exp \left( \frac{\ln(3) + \ln(2)}{5} \right).$$

\[\square\]

A family of face 2-coloured spherical triangulations that has attracted recent interest, see [5] and [6], are triangulations that contain precisely six degree 4 vertices and all the other vertices have degree 6, i.e. near-homogeneous
face 2-coloured spherical triangulations. Part of the motivation for their study comes from their connection to a solved case of Barnette’s Conjecture [12]. When restricting ourselves to the near-homogeneous case we can significantly improve the upper bound.

**Theorem 3.** Let \( h_t \) be the maximal order of the canonical group of all near-homogeneous properly face two-coloured spherical triangulations. Then

\[
\limsup_{t \to \infty} \left( h_t \right)^{1/t} < \left( \exp \left( \ln(3) - \frac{2}{27} \right) \right)^{1/3} < 1.4071.
\]

**Proof.** As a near-homogeneous spherical triangulation has exactly six degree four vertices and every other vertex has degree six, the upper bound reduces to \( 6 \ln(2) + \left( \ln(3) - \frac{2}{27} \right) (n - 6) \), and the result follows. \( \square \)

## 4 Improving the lower bound

In [13], Grubman and Wanless analyse the effect, to order of the canonical group of face 2-coloured spherical triangulations, of applying several recursive constructions. They obtain a lower bound on the growth rate of \( 5123^{1/30} \) by using a construction that identifies a black triangle in one face 2-coloured spherical triangulation, \( G_1 \), with a white triangle in a second face 2-coloured spherical triangulation, \( G_2 \). When viewed as a recursive construction applied to edges of the related digraphs \( D_I(G_1) \) and \( D_I(G_2) \) this equates to removing an arc from a vertex \( u \) to a vertex \( u' \) in \( D_I(G_1) \) and an arc from a vertex \( w \) to a vertex \( w' \) in \( D_I(G_2) \), then identifying \( u \) and \( w' \) and adding an arc from \( w \) to \( u' \). Denote the resulting strongly connected digraph as \( D \), then, by considering the spanning arborescences rooted at \( u = w' \), it follows that \( \mathcal{T}(D) = \mathcal{T}(D_I(G_1))\mathcal{T}(D_I(G_2)) \). Note that the construction can be applied so that \( D \) has a directed embedding in the sphere.

By considering recursive constructions applied to faces, rather than the arcs, of \( D_R, D_C \) and \( D_S \), taking care to ensure the resulting related undirected triangulations are still simple, we will provide an improved lower bound.

**Lemma 5.** Let \( G \) be a face 2-coloured spherical triangulation of a simple graph \( G \), with a proper vertex 3-colouring given by the colour classes \( R, C \) and \( S \), and with canonical group \( C \). Suppose that \( G \) has \( t \) faces of each colour class. Further suppose that \( D_I(G) \) for some \( I \in \{R,C,S\} \) where \( |I| > k \) contains a face, \( f \) say, of size \( k \) the vertices of which all have (out-)degree 2.
Then there exists a face 2-coloured spherical triangulation $G'$ of a simple graph with $t + 2k$ faces with a proper vertex 3-colouring given by the colour classes $R', C'$ and $S'$ with canonical group $C'$ such that: there exists a $I \in \{R', C', S'\}$ where $D_I(G')$ contains a face of size $k$ in which all the vertices have (out-)degree 2; and

$$|C'| \geq \left( \sum_{j=0}^{k-1} \frac{1}{2j-2} \binom{k-1}{j} \right) |C|.$$ 

Proof. Denote the vertices of the face $f$ by $v_1, v_2, \ldots, v_k$ so that the arcs on the boundary of the face are from $v_i$ to $v_{i+1}$, where subscripts are taken modulo $k$. Insert a new vertex into the interior of $f$, call this vertex $u$, also add an arc from $u$ to $v_j$ and an arc from $v_j$ to $u$, for all $1 \leq j \leq k$, maintaining a directed Eulerian spherical embedding, $D'$ say. (We have replaced a face of size $k$ with $k$ triangular faces and $k$ digons.)

We next calculate a lower bound for the number of spanning arborescences in $D'$. Let $A$ be the set of all spanning arborescences in $D$ rooted at $x \notin \{v_1, v_2, \ldots, v_k\}$. Choose a vertex $v \in \{v_1, v_2, \ldots, v_k\}$. Let $1 \leq j \leq k - 1$ and select $j$ distinct vertices from $\{v_1, v_2, \ldots, v_k\} \setminus \{v\}$, denote them $v_1', \ldots, v_j'$. For each arborescence in $A$, remove the ingoing arc with end vertex $v_i'$, for all $1 \leq i \leq j$. As $\deg_D(v_i') = 2$ this yields at least $\frac{1}{2} |A|$ different subgraphs. Now, to each of these subgraphs, add the arc from $v$ to $u$ and the arcs from $u$ to $v_i'$ for all $1 \leq i \leq j$. This results in $\frac{1}{2} |A|$ spanning arborescences of $D'$ rooted at $x$. There were $k$ choices for $v$ and $\binom{k-1}{j}$ choices for the other $j$ vertices. Hence we have at least

$$\left( \sum_{j=0}^{k-1} \frac{1}{2j-2} \binom{k-1}{j} \right) |A|$$

spanning arborescences rooted at $x$ in $D'$.

To complete the proof we need to show that $D'$ corresponds to a face 2-coloured spherical embedding of a simple graph $G'$ with a vertex 3-colouring with colour classes $R', C'$ and $S'$ and that there exists a $I \in \{R', C', S'\}$ such that $D_I(G')$ has a face of size $k$ in which all the vertices have (out-)degree 2.

By Lemma 1 $D'$ corresponds to a face 2-coloured spherical triangulation $G'$. To see that this new triangulation is also of a simple graph note the following. The triangulation $G'$ can be obtained from $G$ by first deleting the vertex of degree $2k$ that corresponds to $f$ in $D_I$ and all the faces and
edges incident to it and replacing them with a single face of size $2k$. Denote the vertices of this new face by $w_0, w_1, \ldots, w_{2k-1}$ so that the edges on the boundary of the face are from $w_i$ to $w_{i+1}$ where subscripts are taken modulo $2k$. Next insert $2k + 1$ new vertices, $z, z_0, \ldots, z_{2k-1}$ and edges into the new face so that the rotations at the new vertices are:

$$\rho(z) = (z_0, z_1, \ldots, z_{2k-1})$$
$$\rho(z_i) = (z, z_{i-1}, w_i, z_{i+1}),$$

where $0 \leq i \leq 2k - 1$ and subscripts are taken modulo $2k$. Hence $G'$ is also a triangulation of a simple graph.

By Observation 1, $G'$ contains a vertex (z in the previous paragraph) with degree $2k$ whose neighbours are all contained in precisely four faces (two white and two black). Hence there exists a $D_I(G')$ with a face of size $k$ in which all the vertices have (out-)degree 2.

**Theorem 4.** Let $m_t$ be the maximal order of the canonical group of all properly face two-coloured spherical triangulations of simple graphs with $t$ faces of each colour. Then

$$\limsup_{t \to \infty} (m_t)^{1/t} \geq \left( \frac{27}{2} \right)^{1/8} > 1.384.$$ 

**Proof.** Consider the face 2-coloured spherical triangulation of a simple graph illustrated in Figure 3. The vertex set $R = \{r_0, r_1, r_2, r_3, r_4\}$ forms a colour class of a vertex three colouring (the other classes being $\{c_1, c_2, c_3, c_4\}$ and $\{s_0, s_1, s_2, s_3, s_4\}$). The digraph $D_R$ contains a face of size 4 in which all the vertices have (out-)degree 2. Repeated application of Lemma 5 obtains the result.

Similar base triangulations for Lemma 5 to be recursively applied to can easily be obtained for face sizes other than 4, but the resulting families have smaller growth rates.

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