Measuring the Hubble constant with black sirens

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We investigate a recently proposed method for measuring the Hubble constant from gravitational wave detections of binary black hole coalescences without electromagnetic counterparts. In the absence of a direct redshift measurement, the missing information on the left-hand side of the Hubble-Lemaître law is provided by the statistical knowledge on the redshift distribution of sources. We assume that source distribution in redshift depends on unknown hyperparameters, modeling our ignorance of the astrophysical binary black hole distribution. With tens of thousands of these “black sirens” – a realistic figure for the third generation detectors Einstein Telescope and Cosmic Explorer – an observational constraint on the value of the Hubble parameter at percent level can be obtained. This method has the advantage of not relying on electromagnetic counterparts, which accompany a very small fraction of gravitational wave detections, nor on often unavailable or incomplete galaxy catalogs.

Keywords: gravitational waves, black hole mergers, cosmological parameters

I. INTRODUCTION

The Hubble constant $H_0$ – the current expansion rate of space – is a fundamental parameter that sets the time and distance scales of the observable Universe. It is then alarming that the local model-independent determination of the Hubble constant via calibrated local Type Ia supernovae [1] is in strong tension with the CMB determination based on the standard ΛCDM model of cosmology [2]. The tension reached $4.5\sigma$ [3] and it could very well signal the need of a new standard model of cosmology [4]. The possibility of physics beyond ΛCDM has been urging the scientific community to measure $H_0$ via the widest range possible of probes and techniques: besides Cepheids, strong lensing time delays, tip of the red giant branch, megamasers, oxygen-rich Miras and surface brightness fluctuations (see [5–8] for details).

Gravitational wave (GW) observations are expected to play an important role in the determination of $H_0$ already in the near future [9], thanks first to the second generation detectors LIGO [10], Virgo [11] and KAGRA [12], and then to the third generation detectors Einstein Telescope [13] and Cosmic Explorer [14]. The reason is twofold. First, GW observations are a new and powerful probe so that an independent and precise measurement of $H_0$ will be obtained. Second, GW observations already with second generation detectors will cover the most interesting redshift range ($0.2 \lesssim z \lesssim 0.7$, Abbott et al. [15]) as far as the Hubble tension is concerned. It is low enough so as to be considered “late Universe” but high enough so that local inhomogeneities are not supposed to have any impact via the so-called cosmic variance on $H_0$ [16]. In other words, GW observations have the potential to shine light in a definitive way on the tension between early- and late-Universe measurements of $H_0$.

So far, different techniques, not mutually exclusive, have been used, all exploiting the fact that compact binary coalescences are standard sirens [17, 18]. If an electromagnetic counterpart is available, then one can break the intrinsic degeneracy between $H_0$ and the coalescence redshift $z$, and precisely determine the Hubble constant with just a few tens of events [19]. The first, and so far unique, of these standard sirens was GW170817 and provided alone a 14% measurement of $H_0$ [20]. On the other hand, most of the observed binary coalescences do not have electromagnetic counterparts and the redshifts of galaxies in the angular position of the coalescence, inferred from galaxy catalogs, can be used to break the $H_0$–$z$ degeneracy (see [17] and the recent [27]). The first of these dark sirens was GW170814 [28]. Although not yet constraining, given the rapidly increasing number of detections, one expects percentage level constraints after 50 events [29], if catalogs are complete enough (see also [30]). Confining oneself to the binary neutron star case, observation of tidal effects can break the gravitational mass-redshift degeneracy, enabling the reconstruction of the Hubble relations without electromagnetic counterparts [31]. Alternatively, one can exploit the spatial clustering scale between galaxies and gravitational wave sources, as

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1 See also [20, 21] for the role of gamma-ray bursts in conjunction with standard sirens, and e.g. [22–25] for measure of cosmic expansion history by using additional probes than standard sirens.
proposed by [32, 33]; this method is expected to produce accurate and precise measurements of the expansion history of the Universe.

Finally, another intriguing method uses the expected gap in the black hole mass function due to the pair-instability supernovae [34]. Features in the mass distribution break indeed the mass-redshift degeneracy intrinsic to GW observations, so that it is possible to measure $H_0$ without electromagnetic counterparts or host galaxy catalogs [35–38].

Here, improving on the idea presented in [39], we propose an alternative method to measure the Hubble constant. This technique uses all observed binary black hole coalescences, which represent the quasi totality of the events: the $H_0$–$z$ degeneracy of these black sirens is broken via the expected (parameter-dependent) redshift distribution of coalescences.\(^2\) As we will argue, instead of using galaxy catalogs, unavailable or incomplete for most events, one can exploit the prior distribution of the coalescence redshift, suitably convolved with the instrument sensitivity of the detectors. In particular, our method is expected to outperform methods that rely on galaxy catalogs in the limit of many observations [O(10^4)] with poor localization at $z \sim 1 – 2$. Therefore, it could be tested with coalescences observed by second generation detectors during their future runs and it should definitely be efficient with third generation detectors.

This paper is organized as follows: Sec. II presents the method, whose limiting cases are treated analytically and discussed in Sec. III. The forecasted results relative to third generation detectors are presented in Sec. IV. We conclude in Sec. V.

## II. METHOD

Throughout this paper we will adopt the standard model of cosmology, according to which the Universe is spatially flat and has an energy content made of vacuum energy (the cosmological constant $\Lambda$) and pressureless matter (mostly cold dark matter, CDM). The low-redshift background evolution of the flat $\Lambda$CDM model is completely specified by the values of the Hubble constant $H_0$ and of the matter density parameter $\Omega_m$. In particular, in our model, the luminosity distance is related to the redshift via:

\[
\begin{align*}
\tilde{d}_L(z) &= \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{E(z')}, \\
E(z) &= \frac{H(z)}{H_0} = \sqrt{\Omega_m (1+z)^3 + 1 - \Omega_m},
\end{align*}
\]

with the comoving distance $d_{\text{com}}(z) = \tilde{d}_L(1+z)$, with the index (t) standing for “theoretical”.

Let us now consider one coalescence event. GW detections measure the luminosity distance $d_L$ so that one can build the posterior distribution $f$ of the cosmological parameters and binary black hole (BBH) redshift as follows:

\[
f(H_0, \Omega_m, z|d_L) = \frac{p_{\text{tot}}(H_0, \Omega_m, z) \mathcal{L}(d_L, |H_0, \Omega_m, z)}{\mathcal{E}},
\]

where here the evidence $\mathcal{E}$ is just a normalization constant. We will now discuss the prior $p_{\text{tot}}$ and the likelihood $\mathcal{L}$.

### A. Prior

Using the product rule, the prior can be written as:

\[
p_{\text{tot}}(H_0, \Omega_m, z) = p(H_0) \, p(\Omega_m) \, p(z|H_0, \Omega_m).
\]

We assumed that $\Omega_m$ and $H_0$ are independent because for the former we use an informative prior from Supernovae Ia, which is independent from $H_0$. We adopt the almost Gaussian prior from the Pantheon dataset [41]:

\[
p(\Omega_m) \propto \exp \left[ -\frac{(\Omega_m - \Omega_m^{(p)})^2}{2\sigma_{m,p}^2} \right],
\]

where $\Omega_m^{(p)} = 0.298$ and $\sigma_{m,p} = 0.022$.

Regarding $p(H_0)$, as we aim at measuring the Hubble constant with black sirens, we adopt a flat broad prior:

\[
p(H_0) \propto \begin{cases} 
\text{const} & \text{if } H_0 \in [20, 140] \text{ km s}^{-1}\text{Mpc}^{-1} \\
0 & \text{otherwise}
\end{cases},
\]

which is the same prior adopted by [28].

The prior on the observed coalescence redshift $p(z|H_0, \Omega_m)$ is the nontrivial piece of information necessary to extract information on $H_0$ from gravitational wave observations. The standard dark-siren approach estimates the redshift prior via a galaxy catalog that covers the sky localization of the event [28, 42, 43]. This approach has the advantage of correlating the coalescence to the actual nearby galaxies and, in particular, to their large-scale structure of voids, filaments and clusters. However, the galaxy catalog may be incomplete or unavailable. The idea at the base of our black-siren method is to estimate $p(z|H_0, \Omega_m)$ theoretically. More precisely, in the present paper we will obtain the redshift prior via an analytical estimation of the star-formation rate, convolved with a suitable star formation to binary coalescence delay, while we leave for future work the use of synthetic galaxy catalogs from state-of-the-art hydrodynamical simulations.

We model the redshift prior via two contributions:

\[
p(z|H_0, \Omega_m) = A(H_0, \Omega_m) \, R_m(z) \, f_C\left(\tilde{d}_L(z)\right),
\]
which we now explain in detail. In the previous equation
A is a normalization constant which may depend on all
the parameters but \( z \).

1. Merger rate

The first contribution \( R_m(z) \) is the rate number (\( N_m \))
density of mergers in the detector frame (number of mergers
per detector time per redshift) which will be expressed via:
\[
R_m^{(\tau)}(z) = \frac{dN_m^{(\tau)}}{dtdz},
\]
where we omit the inconsequential normalization con-
stant and the hyper parameter \( \tau \) is discussed below. Following [44, 45], we model \( R_m \) via the total merger rate per
comoving volume in the source frame \( R_m^{(\tau)} \):
\[
R_m^{(\tau)}(z) = \frac{1}{1 + z} \frac{dV}{dz} R_m^{(\tau)}(z),
\]
where the \( 1 + z \) term in the denominator arises from con-
verting source-frame time \( t_s \) to detector-frame time \( t_d \), and \( dV/dz \) is the cosmology-dependent comoving volume
element per unit redshift interval:
\[
\frac{dV}{dz} = \frac{4\pi c d_L^2(z)}{H(z)(1 + z)^2} = \frac{4\pi}{E(z)} \left( \frac{c}{H_0} \right)^3 \left( \int_0^z \frac{dz'}{E(z')} \right)^2.
\]
Then, we model \( R_m^{(\tau)} \) via a delayed volumetric BBH for-
mation rate \( R_f \). Specifically, we account for the stochastic
delay between star formation and BBH merger via a
Poissonian distribution of characteristic delay \( \tau \):
\[
R_m^{(\tau)}(z) = \frac{1}{\tau} \int_0^\infty dz_f \frac{df}{dz_f} R_f(z_f) \exp \left[ -\frac{t(z_f) - t(z)}{\tau} \right],
\]
where
\[
\tau \equiv \frac{1}{H_0} \int_0^z \frac{dz'}{(1 + z')E(z')}
\]
is the time spent between redshift \( z \) and the present
epoch. Note that \( R_m^{(\tau)} \), apart from the normalization, de-
deps on \( \tau \) only via the dimensionless combination \( H_0\tau \).

Finally, we assume that the BBH volumetric formation
rate is proportional to the star formation rate density
\( \psi(z) \) at the same redshift:
\[
R_f(z_f) = \frac{dN_f}{dV df} \propto \psi(z_f).
\]
In other words we are not considering the time between
star formation and BBH formation, which should be neg-
ligible given the time scale of BBH coalescence. We adopt
the measured star formation rate from [46]:
\[
\psi_{\text{MD14}}(z) = 0.015 \frac{(1 + z)^{2.7}}{1 + (1 + z)^{0.5}} M_\odot \text{ yr}^{-1} \text{Mpc}^{-3},
\]
with \( C = 2.9 \). The merger rate obtained using Eq. (14)
in Eq. (13) may not correspond to the one realized in
nature. We do not account here for the fact that only a
fraction of stars ends up in black holes. Moreover we ne-
glect that both merger rate and time delay distribution
may depend on binary intrinsic properties, like compo-
nent masses and spins. Such dependences can be modeled
by including additional hyperparameters to the proposed
merger rate and eventually marginalizing over them, at
the cost of degrading the precision of the recovery of cos-
mological parameters. However, we will neglect these
details for the moment to show in principle the power of
the method, and in the Appendix we show that the ad-
dition of another hyperparameter can absorb the effect
of our ignorance of the underlying merger rate, and still
produce an unbiased determination of the Hubble con-
stant, at the price of moderately degrading the precision
of parameter estimation. See [47] for a recent applica-
tion of jointly fitting the cosmological parameters and
the source population properties of binary black holes.

As already mentioned, the characteristic delay \( \tau \) is a
hyperparameter of the redshift prior. We adopt a flat
hyperprior:
\[
p(\tau) \propto \begin{cases}
\text{const} & \text{if } \tau \in [100 \text{ Myr}, t_0(H_0, \Omega_m)] \\
0 & \text{otherwise}
\end{cases},
\]
where \( t_0 \) is the age of the Universe (since we observe the
coalescence it must be \( \tau < t_0 \)). One can then consider
the following compound distribution as the coalescence
prior:
\[
R_m(z) = \int_0^\infty d\tau p(\tau) R_m^{(\tau)}(z).
\]
Note that, numerically, it is equivalent to include \( \tau \) as a
nuisance parameter with prior \( p(\tau) \). We will adopt this
point of view when considering a generic number \( n \) of
events.

2. Detector sensitivity

The last piece in Eq. (7), \( f_C(d_C^{(4)}) \), models the LIGO-
Virgo detector sensitivity on the luminosity distance: ob-
viously more distant sources are less likely to be detected
than nearer ones. Indeed, coalescences are observed if
a signal-to-noise ratio (\( SNR \)) larger than 8 is achieved.
The \( SNR \) is computed by comparing the \( f \)-domain wave-
form \( \tilde{h}(f) \) with the detector noise \( S_n(f) \):
\[
SNR = 2 \left[ \int_0^\infty df \frac{\tilde{h}(f)^2}{S_n(f)} \right]^{1/2},
\]
\[
\tilde{h}(f) = F_+ \tilde{h}_+(f) + F_\times \tilde{h}_\times (f),
\]
where the pattern functions \( F_+ \) are function of the two
angles locating the source in the sky \((\alpha, \delta)\) and the pola-
rization angle \( \psi \), and the GW polarizations \( \tilde{h}_+ \) are
which depend on the luminosity distance, the orientation between the binary orbital plane and the observation direction. Fig. 1 shows the square root of the noise spectral density for illustration and are valid only for the inspiral phase of the coalescence. For 3G curve we adopted the noise spectral density “D” from [49].

\[
\tilde{h}_+ = \left( \frac{5}{24} \right)^{1/2} \frac{\pi^{-2/3}}{d_L^{(1)}} \mathcal{M}_{c}^{5/6} f_0^{-7/6} \left( 1 + \cos^2 \theta \right)^2 e^{i\phi(f)},
\]

\[
\tilde{h}_x = \left( \frac{5}{24} \right)^{1/2} \frac{\pi^{-2/3}}{d_L^{(1)}} \mathcal{M}_{c}^{5/6} f_0^{-7/6} \cos \epsilon \ e^{i\phi(f) + \pi/2},
\]

which depend on the luminosity distance, the orientation \( \epsilon \) and the redshifted chirp mass \( \mathcal{M}_c \equiv M_c(1 + z) \). The chirp mass is defined by \( \mathcal{M}_c \equiv \eta^{3/5} M \), where \( M \equiv m_1 + m_2 \), \( \eta \equiv m_1 m_2 / M^2 \), and \( m_i \) are the individual constituent masses. The angle \( \epsilon \) gives the relative orientation between the binary orbital plane and the observation direction. Fig. 1 shows the square root of the noise spectral density \( \sqrt{S_n} \) used to estimate the SNR for second (2G) and third (3G) generation detectors.

To relate the astrophysical to the detected merger rate one needs to take into account selection effect, i.e. to estimate how likely it is to detect a source located at a given distance from the observatory, which is obtained by averaging over the source parameters to get the average distribution of detections as a function of distance. The requirement for detection is that the signal has \( SNR \geq 8 \), and averaging is performed over masses and angles as reported in Table I.

\[\text{Note that the interference term between } \tilde{h}_+ \text{ and } \tilde{h}_x \text{ vanishes in the } SNR \text{ integral. Analytic expressions (19,20) are shown for illustration and are valid only for the inspiral phase of the coalescence.}\]

Table I. Parameter space that is uniformly explored (except for masses) to sample the SNR of Eq. (17). For the individual masses the distribution adopted is a broken power law \( \propto m_i^{-1.5} (m_i^{-3}) \) for \( m_i < 40 M_\odot \) (40 \( < m_i / M_\odot \) < 80) for solar mass black holes and a log prior for intermediate masses 120 \( < M / M_\odot < 10^4 \).

| Parameter          | Quantity | Interval         |
|--------------------|----------|------------------|
| Comoving distance  | \( d_L \) | \([100, 1.2 \times 10^4] \) |
| Individual mass    | \( m_1 / M_\odot \) | \([1.2, 10^4] \) |
| Mass ratio         | \( q = m_2 / m_1 \) | \( > 10^{-3} \) |
| Binary orientation | \( \cos \epsilon \) | \([-1, 1] \) |
| Polarization       | \( \psi \) | \([0, 2\pi] \) |
| Right ascension    | \( \alpha \) | \([0, 2\pi] \) |
| Declination        | \( \delta \) | \([0, \pi] \) |

The astrophysical mass distribution of stellar-mass black holes can be inferred from LIGO/Virgo O1, O2, O3a data as described in [50, 51]. This is relevant for 2G detectors as they are sensitive to binaries with total mass up to \( \sim O(100 M_\odot) \). We can assume that the mass of the heavier binary component is distributed according to a broken power law with exponents \( \alpha_1 = -1.5 \) and \( \alpha_2 = -5 \) for masses between 5 and 60 \( M_\odot \), with the slope change occurring at \( m_{\text{break}} = 40 M_\odot \). The mass ratio \( q \) is assumed to be distributed according to \( p(q) \propto q^{-1} \) with 0.1 \( \leq q \leq 1 \), with a lower cutoff on the lighter mass assuming 1.2\( M_\odot < m_2 \). Third generation detectors will also be sensitive to intermediate-mass black holes with \( m_i \gtrsim 10^2 M_\odot \). As their distribution is completely unknown, we have assumed a mass gap from 80 to 120\( M_\odot \) due to pair-instability supernovae [34] and an uninformative \( \log \) prior up to \( m_i < 10^4 M_\odot \). In the same spirit of the 2G case, that is to use a concrete example to test the method, we assume the distribution of the primary mass to be \( \propto 1 / m_1 \) for 120 \( \leq m_1 / M_\odot \leq 10^4 \) and for the mass ratio in this region the prior \( p(q) \propto q^{1/2} \).

It is important to stress that stellar- and intermediate-mass black hole population properties are not precisely known and that here we wish to use indicative values for the underlying population to test the efficiency of our method in a realistic case. Moreover, the black hole mass function is only used to evaluate the reach of the detector. Besides this detail, its information is not folded into the likelihood to determine cosmological parameters. As we will show, the method proposed here can lead to interesting constraints on \( H_0 \) only for a large number of detections \( \gtrsim O(10^3) \). Hence, we can safely assume that once accumulating so many detections, the population properties of the sources will be known with great accuracy. The use of a different underlying astrophysical mass distribution will impact both the simulated signals and the priors entering the determination of the \( H_0 \) posterior probability distribution, leaving basically unaltered the predictive power of the method.

We use the waveform approximant known as \textit{IMRPhenomD} [52, 53], describing the entire coalescence, for spinless sources generated via LALSuite [48], and noise as
in Fig. 1, representative of second and third generation ground-based GW detectors. After imposing $SNR > 8$ and averaging over all parameters but $d^{(i)}_C$, we obtain the distributions $f(d^{(i)}_C)$ shown in Fig. 2 whose tail in the 2G and 3G cases can be modeled according to:

$$
\begin{align*}
    f_C \left( d^{(i)}_C \right) &\propto \begin{cases} 
    \exp \left[ -\frac{d^{(i)}_C}{d^{(cut2)}_C} \right] & \text{2G} \\
    \exp \left[ -\left( \frac{d^{(i)}_C}{d^{(cut3)}_C} \right)^3 \right] & \text{3G}
    \end{cases}, \\
\end{align*}
$$

\[ (21) \]

where $d^{(cut2)}_C = 320 \text{ Mpc}$ and $d^{(cut3)}_C = 7.9 \text{ Gpc}$.

The decay with the comoving distance is qualitative different in the 2G and 3G cases. In the 2G case, only sources at moderate redshift are visible, as increasing the distance increases the denominator in Eqs. (19,20), thus decreasing the $SNR$.

In the 3G case, signals with $z > 1$ are visible for a wide range of masses, with the result that the $(1 + z)^{5/6}$ dependence at the numerator of Eqs. (19,20) almost cancels the $z$-dependence of $d_C(1 + z)$ at the denominator. As a consequence, the $SNR$ varies with distance approximately according to $d_C^3$ until the redshift pushes the signal to low enough frequencies to fall outside the detector’s band, and this happens around $d_C \simeq 12 \text{ Gpc}$ for a wide range of masses, as that is the value at which $z$ steeply increases for small variation of $d_C$, see Fig. 3.

Note that the $SNR$ depends on the redshifted chirp mass $M_c$ which depends on redshift. To obtain the simulations presented in Fig. 2 the redshift is not varied independently but instead determined from the distance and the fiducial cosmology ($\Lambda$CDM): $z^{(fid)} = z(d^{(i)}_C, H_0^{(fid)}, \Omega_m^{(fid)}) \simeq H_0^{(fid)} d^{(i)}_C/c$, with $H_0^{(fid)} = 69.32 \text{ km s}^{-1}\text{Mpc}^{-1}$ and $\Omega_m^{(fid)} = \Omega_m^{(p)}$.

**B. Likelihood**

In the Gaussian approximation, the likelihood can be written according to:

$$
\mathcal{L}(d_L, | H_0, \Omega_m, z) \propto \exp \left[ -\frac{(d_L - d_L^{(i)}(H_0, \Omega_m, z))^2}{2\sigma_L^2} \right],
$$

\[ (22) \]

where to lighten notation the dependence of the likelihood on the luminosity distance uncertainty $\sigma_L$ has been suppressed.

**C. Posterior for $n$ coalescences**

When combining $n$ coalescences it is convenient to marginalize immediately on the parameters that are specific to a given event so that:

$$
f(H_0, \Omega_m, \tau | d_L) \propto p(H_0) p(\Omega_m) p(\tau)
\times \int dz \, p(z | H_0, \Omega_m, \tau) \mathcal{L}(d_L, | H_0, \Omega_m, z),
$$

\[ (23) \]

where, as discussed earlier, we treated $\tau$ as a nuisance parameter. The expression above can then be generalized to the case of $n$ detections $\{d_{L,i}\}$:

$$
f(H_0, \Omega_m, \tau | \{d_{L,i}\}) \propto p(H_0) p(\Omega_m) p(\tau)
\times \prod_{i=1}^{n} dz_i \, p(z_i | H_0, \Omega_m, \tau) \mathcal{L}(d_{L,i}, | H_0, \Omega_m, z_i).
$$

\[ (24) \]
Numerically, the posterior exploration will be performed on the parameters $H_0, \Omega_m, \tau$. In other words, for each point $\{H_0, \Omega_m, \tau\}$ of the parameter space we will estimate the $n$ 1-dimensional integrals of Eq. (24). We parametrize here the inevitable uncertainty in the knowledge of the underlying merger distribution with only one hyperparameter $\tau$, and we address in the Appendix the issue of the generality of the merger rate function that we adopt in Eq. (16).

### III. LIMITING CASES

To understand analytically the statistical inference on $H_0$ with black sirens it is useful to consider the following limiting cases.

#### A. Low redshift

It is interesting to take the limit $z \to 0$ in Eq. (7). First, one has that $\frac{1}{1+z} \frac{dV}{dz} \sim z^2 \sim d_L^2$. Second, from Eq. (14) it follows that $R_f(z) \sim \text{constant}$ so that, from Eq. (11), one finds that $R_m^{(\tau)}(z) \sim \text{constant}$. One then finds from Eq. (7) that:

$$ p(z|H_0, \Omega_m) \xrightarrow{z \to 0} d_L^2. $$

In other words, the prior cannot break the $H_0$-$z$ degeneracy as it depends just on $d_L$, which is the quantity measured by GW observations. Equivalently, the information that is able to break the $H_0$-$z$ degeneracy comes from a nontrivial $R_f$.

#### B. Negligible luminosity distance error

Next, we can take the limit $\sigma_L/d_L \to 0$ in Eq. (24):

$$ f(H_0, \Omega_m, \tau|\{d_L, i\}) \propto p(H_0) p(\Omega_m) p(\tau) \times \prod_{i=1}^{n} \int dz_i A R_m^{(\tau)}(z_i) e^{-\frac{d_L^2}{2\sigma_L^2}/d_L^2} \delta(d_{L,i} - d_L^2 (H_0, \Omega_m, z_i)) = p(H_0)p(\Omega_m)p(\tau) \prod_{i=1}^{n} \frac{A R_m^{(\tau)}(z_i, H_0, \Omega_m, \tau)}{\left|\frac{\partial d_L^{(\tau)}}{\partial z_i}(H_0, \Omega_m, z_i)\right|}, $$

where we used the properties of the Dirac delta function and $z_i = z^{(t)}(d_{L,i}, H_0, \Omega_m)$ is the theoretical redshift associated with $d_{L,i}$ given $H_0$ and $\Omega_m$, and assuming $f_C = e^{d_L^2/d_L^{(cut)}}$. We see that, in this limit, the detector sensitivity $f_C$ does not contain cosmological information.

#### C. Infinite number of observations

Statistical inference with black sirens suffers from two sources of uncertainties. The first is due to the uncertainty $\sigma_L$ on the measurement of the luminosity distance. The second comes from having a finite sample $n$ of observations. Indeed, we are constraining parameters to recover the actual distribution of coalescence redshifts.

From Eq. (26) it is easy to see how a fiducial model is recovered in the limit of infinite observations. Assuming flat priors on $H_0, \Omega_m$, and $\tau$:

$$ f(H_0, \Omega_m, \tau|\{d_L, i\}) \propto \prod_{i} f(d_{L,i}|H_0, \Omega_m, \tau), $$

where $f(d^{(t)}_L|H_0, \Omega_m, \tau)$ is the theoretical distribution in the luminosity distance given the theoretical model (the Jacobian is absorbed by the change of variable). From the previous equation one sees that in the limit $n \to \infty$ the values of $H_0, \Omega_m$, and $\tau$ that maximize the posterior are the ones that were used to produce the measurements $\{d_{L,i}\}$.

#### D. Toy example

To further simplify the analysis we consider the following redshift prior:

$$ p(z|H_0) = \left(\frac{c}{H_0 d_L^{(cut)}} + \frac{1}{z_f}\right)^3 \frac{z^2}{2} e^{-\frac{z}{z_f}} \exp\left(\frac{-cz}{H_0 d_L^{(cut)}}\right), $$

where we adopted the approximation $d_L^{(cut)} \simeq d_L^{(cut)} \simeq c z/H_0$, so that we can drop the (anyway weak) dependence on $\Omega_m$. Eq. (28) represents a normalized, reasonable toy model where the factor $z^2 e^{-z/z_f}$ intends to
reproduce the astrophysical merger distributions and a detector sensitivity exponentially decaying with redshift has been assumed. Fig. 4 shows this prior for two values of the detector luminosity cut $d_L^{(\text{cut})}$. The vertical lines mark the mean redshifts $z = 3z_fz_c/(z_f + z_c)$, where $z_c(H_0) = H_0d_L^{(\text{cut})}/c$.

Taking again the limit $\sigma_{L,d_L} \to 0$, the posterior becomes:

$$\ln f(H_0|\{d_{L,i}\}) = 3n \ln \left( \frac{H_0d_L^{(\text{cut})}}{cz_f} + 1 \right) - n \frac{H_0d_L}{cz_f} \left( \frac{H_0d_L^{(\text{cut})}}{c} - \frac{H_0d_L^{(\text{cut})}}{cz_f} \right)$$

(29)

where $d_L = \frac{1}{3} \sum_i d_{L,i}$, we omitted additive constants and in the last equation we used:

$$d_L = \frac{z}{H_0^{(\text{fid})}} \frac{c}{z_f} \frac{3z_fz_c(H_0^{(\text{fid})})}{z_f + z_c(H_0^{(\text{fid})})} \frac{c}{H_0^{(\text{fid})}}.$$  

(30)

The posterior maximum (best fit) is found by solving $\partial \ln f/\partial H_0 = 0$, which gives $H_{0,\text{bf}} = H_0^{(\text{fid})}$, that is, the fiducial value of the Hubble constant is recovered in the limit of infinite (infinitely precise) measurements.

Finally, we can compute the Fisher matrix, which in this case, is just a number:

$$F = -\frac{\partial^2 \ln f(H_0|\{d_{L,i}\})}{\partial H_0^2} \bigg|_{H_0^{(\text{fid})}},$$

(31)

so that:

$$\frac{\sigma_{H_0}}{H_0} = F^{-1/2} \frac{1 + z_f/z_c(H_0^{(\text{fid})})}{\sqrt{3n}},$$

(32)

which depends on $z_c = H_0^{(\text{fid})}d_L^{(\text{cut})}/c$.

Fig. 5 shows the forecasted constraints relative to the toy model of Eq. (29) for a second generation (blue line) and third generation (orange line) detector. This result does not take into account the degeneracy of $H_0$ with $\Omega_m$ and $\tau$. In the next Section we will discuss a realistic forecast.

### IV. REALISTIC FORECAST

We now perform the full analysis of Eq. (24). The merger rate of Eq. (9) is represented in Fig. 6 for the fiducial values of $\tau = 5$ Gyr, $H_0 = 69.32$ km s$^{-1}$Mpc$^{-1}$ and $\Omega_m = \Omega_m^{(\text{fid})}$, and for the detector sensitivities of 2G and 3G detectors (see Sec. II A 2). We will now consider the case of the future 3G detectors. Fig. 7 shows the normalized distribution of simulated injections for a 3G detector.

The expected absolute number of binary black hole observations by 3G detectors is poorly constrained because the underlying source distribution is known only to a small extent. By considering very different values of $\tau$ and normalizing the local merger rate density at 50 Gpc$^{-3}$ yr$^{-1}$, one can see that, for instance, 10,000 detections can be accumulated in a time varying between a
Figure 6. Distribution of detections and merger rate (assuming $\tau = 5$ Gyr). The 2G curve is normalized to unity, the 3G curve and merger rate have normalization consistent with the 2G curve.

Figure 7. Normalized distribution of simulated detections for a 3G detector.

Indeed, while one could fix the luminosity distances at their fiducial values, the distribution in redshift of the injections is necessarily stochastic. In other words, here we are considering fully realistic mock datasets.

For the scenario with 10,000 and 20,000 injections, we fully sample the posterior via MCMC using the numerical codes EMCEE [54], through its Bilby implementation [55], and GETDIST [56]. The results are shown in Fig. 8, for the case of a 5% uncertainty in $d_L$, i.e. $\sigma_L/d_L = 0.05$. We can see that already with 10,000 GW observations it is possible to constrain the Hubble parameter at the few % level.

As can be seen, the maximum of the posterior does not coincide exactly with the fiducial value of the parameters (red lines in Fig. 8). This is expected because in the present analysis it is not possible to perform a forecast without fluctuations in the observational quantities.

Figure 8. Marginalized constraints on $H_0$, $\tau$ and $\Omega_m$ for 10,000 and 20,000 simulated injections for the 3G case with 5% relative errors in the measurement of $d_L$. The fiducial values of the parameters are marked with red lines. The prior on $\Omega_m$ is displayed with a green dashed line.

week and few months [44]. Here, we consider the following possible scenarios – 10,000, 20,000 and 40,000 detections – which are realistic given the programmed duration of future 3G observation runs.

For the scenario with 10,000 and 20,000 injections, we analyze the scenario with 40,000 detections via the Fisher matrix approximation, obtained numerically via the NUMDIFFTOOLS library. This is necessary because of the increased computational cost: as shown by Eq. (24) one has $n$ numerical integrals for $n$ injections. As explained earlier, the maximum of the posterior randomly walks around the fiducial value of the parameters and, to obtain a more robust estimate of the Fisher matrix against nonlinearities, we consider several sets of injections and average the corresponding Fisher matrices. The result of this procedure is shown in Fig. 9 (including also the cases that were analyzed via MCMC) and summarized in Table II for the precision and Table III for the average bias in the recovered $H_0$. The results reported in Fig. 8 give 1-$\sigma$ levels for $H_0$ of 5.5% and 3.4% for 10,000 and 20,000 injections respectively, in agreement with the Fisher matrix estimations.

In the previous analysis we assumed that one hyperparameter is enough to model our ignorance on the source distribution. In the Appendix we show that one more hyperparameter can capture a possible bias in the adopted

4 pypi.org/project/numdifftools.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\# injs & $\sigma_{H_0}/H_0$ & \\
& $[\sigma_{dL}/dL = 5\%]$ & $[\sigma_{dL}/dL = 10\%]$ \\
\hline
10,000 & 4.9\% & 12.1\% \\
20,000 & 3.0\% & 7.6\% \\
40,000 & 2.7\% & 6.5\% \\
\hline
\end{tabular}
\caption{Forecasted relative constraints on $H_0$ for third-generation gravitational-wave detectors.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\# injs & $\left\{(H_{0,\text{inj}})/H_{0,\text{sec}} - 1)^2\right\}^{1/2}$ & \\
& $[\sigma_{dL}/dL = 5\%]$ & $[\sigma_{dL}/dL = 10\%]$ \\
\hline
10,000 & 2.7\% & 3.3\% \\
20,000 & 1.0\% & 1.0\% \\
40,000 & 0.5\% & 0.9\% \\
\hline
\end{tabular}
\caption{Forecasted bias on $H_0$ for third-generation gravitational-wave detectors.}
\end{table}

start formation rate model. This prevents the introduction of a bias in the Hubble constant at the price of degrading the precision of its determination, which worsens by a factor $\approx 2$.

V. CONCLUSIONS

Detections of gravitational waves from binary coalescences have opened new ways to investigate cosmology. In particular, while using concurrent observations of redshift and luminosity distance is an obvious way to measure the Hubble constant, data from the first three observation runs of LIGO and Virgo showed that binary black holes, dark sirens without an electromagnetic counterpart, are far more frequent than neutron star binaries with electromagnetic counterparts. Note, however, that forecasts for third generation detectors indicate that one could constrain the Hubble constant to subpercent level by accumulating electromagnetically bright standard sirens over 10 years at a rate of $\sim 30$ bright standard sirens per year [22].

On the other hand, by exploiting the gravitationally measured source location, in the case of a network of at least three detectors, it has been shown that already with $O(200)$ dark siren events one can achieve a few percent measurement of $H_0$ if the galaxy catalogs are at least 25% complete [9]. This can be assumed only for relatively close sources, although galaxy catalogs complete to magnitude 24 are expected to be produced by Euclid [57], allowing to see a Milky Way-type galaxy up to 1 Gpc.

Here, we proposed an independent method, where redshift information comes from our partial knowledge of the source distribution. Marginalizing over the hyperparameter encoding our ignorance of the binary astrophysical distributions we can estimate the Hubble constant with a few percent precision with few tens of thousands black siren detections, without the need of multiple detectors, galaxy catalogs or electromagnetic counterparts to have information about the individual source redshifts. Note that, while the forecasted rate of binary black hole coalescence detections by third generation gravitational wave observatories is subject to large uncertainties, even in the more pessimistic scenarios few $O(10^3)$ detections per month should be made so that our method should be a viable alternative.

There are, however, caveats in our method. First, to take into account detector-related selection effects, we have simulated future detections with a specific black hole mass function. This will be addressed by the time our method will be used. Indeed, 3G detectors will have accumulated tens of thousands of BBH detections so that we expect such mass function to be known accurately. Second, the star formation rate we assumed may not correspond to the one realized in nature and the model we presented in the main text, with only one hyperparameter, may be an oversimplification. To test these assumptions we have performed simulations in which data were injected and analyzed using different star formation rate models. The results reported in the Appendix show that the addition of an another hyperparameter can capture the difference in underlying star formation rate models and prevent the introduction of a bias in the Hubble constant, though degrading the precision of its determination.
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Appendix A: Robustness against unknown star formation rate

Our analysis adopts the star formation rate density $\psi_{MD14}$ of equation (14) from Madau and Dickinson [46]. Here, we investigate the impact of analyzing with $\psi_{MD14}$ data that were produced with the alternative star formation rate density by Robertson and Ellis [58]:

$$\psi_{RE12}(z) = \left[ 0.007 + 0.27 (z/3.7)^{2.5} + 0.003 \right] M_\odot \text{yr}^{-1}\text{Mpc}^{-3},$$

(A1)

to have a proxy of the bias we may introduce in the cosmological parameter estimation by adopting an incorrect underlying star formation and merger distribution. Both functions are plotted in Fig. 10. Fig. 11 shows that despite the two underlying star formation rates are qualitatively different, the resulting merger rates can be made to overlap by adjusting the $C$ parameter of Eq. (14), which we now promote to hyper-parameter (and treat as a nuisance parameter).

We then show in Fig. 12 the results of an analysis in which the probability distributions for $H_0$, $\Omega_m$ and the two nuisance parameters $\tau$ and $C$ are obtained in the case in which the injections are generated assuming the star formation rate (A1) but analyzed with the star formation rate (14). One can see that the hyper-parameter $C$, by taking a value different from the original one of eq. (14), absorbs the effect of a different star formation rate, avoiding a bias in $H_0$. On the other hand, the precision on $H_0$ is degraded to almost 10% percent level, thus requiring several tens of thousand of injections to reach percent level.
Figure 12. Statistical inference for injections generated according to the star formation rate of Eq. (A1) but analyzed using the star formation rate of Eq. (14), for 5% relative errors in the measurement of $d_L$, with 20,000 and 10,000 injections. Here, the model includes the nuisance parameter $C$, which absorbs the effect of a different star formation rate between injection and recovery, with the result of keeping $H_0$ unbiased.