THE SPACE OF HARDY-WEIGHTS FOR QUASILINEAR EQUATIONS: MAZ’YA-TYPE CHARACTERIZATION AND SUFFICIENT CONDITIONS FOR EXISTENCE OF MINIMIZERS

UJJAL DAS AND YEHUDA PINCHOVER

Abstract. Let $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^N$ be a domain. Let $A := (a_{ij}) \in L_{\text{loc}}^\infty(\Omega; \mathbb{R}^{N \times N})$ be a symmetric and locally uniformly positive definite matrix. Set $|\xi|^2_A := \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j$, $\xi \in \mathbb{R}^N$, and let $V$ be a given potential in a certain local Morrey space. We assume that the energy functional

$$Q_{p,A,V}(\phi) := \int_{\Omega} [|\nabla \phi|^p_A + V|\phi|^p]d\mu$$

is nonnegative in $W^{1,p}(\Omega) \cap C_c(\Omega)$.

We introduce a generalized notion of $Q_{p,A,V}$-capacity and characterize the space of all Hardy-weights for the functional $Q_{p,A,V}$, extending Maz’ya’s well known characterization of the space of Hardy-weights for the $p$-Laplacian. In addition, we provide various sufficient conditions on the potential $V$ and the Hardy-weight $g$ such that the best constant of the corresponding variational problem is attained in an appropriate Beppo-Levi space.

2000 Mathematics Subject Classification. Primary 49J40; Secondary 31C45, 35B09, 35J62.

Keywords: Beppo-Levi space, capacity, Hardy-type inequality, quasilinear elliptic equation, concentration compactness, criticality theory, positive solutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a domain, and $p \in (1, \infty)$. Let $A := (a_{ij}) \in L_{\text{loc}}^\infty(\Omega; \mathbb{R}^{N \times N})$ be a symmetric and locally uniformly positive definite matrix. Set

$$|\xi|^2_A := \langle A(x)\xi, \xi \rangle = \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j, \quad x \in \Omega, \quad \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N,$$

and let $\Delta_{p,A}u := \text{div} (|\nabla u|^{p-2}_A A(x)\nabla u)$ be the $(p,A)$-Laplacian of $u$ (see [14, 28]).

For a potential $V$ in $\mathcal{M}^q_{\text{loc}}(p; \Omega)$, a certain local Morrey space (see Definition 2.1), we consider the following quasilinear elliptic equation:

$$Q'_{p,A,V}[u] := -\Delta_{p,A}u + V|u|^{p-2}u = 0 \quad \text{in } \Omega, \quad (1.1)$$

together with its energy functional

$$Q_{p,A,V}(\phi) := \int_{\Omega} [|\nabla \phi|^p_A + V|\phi|^p]d\mu \quad \phi \in W^{1,p}(\Omega) \cap C_c(\Omega).$$

Throughout the paper we assume that $Q_{p,A,V} \geq 0$ on $W^{1,p}(\Omega) \cap C_c(\Omega)$. 

arXiv:2202.12324v1 [math.AP] 24 Feb 2022.
We recall that, by the Agmon-Allegretto-Piepenbrink-type theorem [28, Theorem 4.3], $Q_{p,A,V} \geq 0$ on $W^{1,p}(\Omega) \cap C_c(\Omega)$ if and only if (1.1) admits a weak positive solution (or positive supersolution) in $W^{1,p}_{\text{loc}}(\Omega)$.

The first aim of the present paper is to characterize the space of all functions $g \in L^1_{\text{loc}}(\Omega)$ such that the following Hardy-type inequality holds:

$$
\int_{\Omega} |g||\phi|^p dx \leq CQ_{p,A,V}(\phi) \quad \forall \phi \in W^{1,p}(\Omega) \cap C_c(\Omega)
$$

(1.2)

for some $C > 0$. A function $g$ satisfying (1.2) is called a Hardy-weight of $Q_{p,A,V}$ in $\Omega$. We denote the space of all Hardy-weights by

$$
\mathcal{H}_p(\Omega,V) := \{g \in L^1_{\text{loc}}(\Omega) \mid g \text{ satisfies (1.2)}\}.
$$

If $\mathcal{H}_p(\Omega,V) = \{0\}$, then $Q_{p,A,V}$ is said to be critical in $\Omega$, otherwise, $Q_{p,A,V}$ is subcritical in $\Omega$ [28]. If $Q_{p,A,V} \geq 0$ on $W^{1,p}(\Omega) \cap C_c(\Omega)$, then $Q_{p,A,V}$ is said to be supercritical in $\Omega$. Recall that for $p \in (1,N)$, $\Omega = \mathbb{R}^N \setminus \{0\}$, $V = 0$, $A = I_{N \times N}$, and $g(x) = C(p,N)/|x|^p$, inequality (1.2) corresponds to the classical Hardy inequality.

In the context of improving the classical Hardy inequality many examples of Hardy-weights were produced, see [1, 9, 10, 11, 13] and the references therein. For $p = 2$, $A = I_{N \times N}$, and bounded domain $\Omega$, it is well known that $L^2(\Omega) \subset \mathcal{H}_2(\Omega,0)$ with $r > \frac{N}{2}$ [24], $r = \frac{N}{2}$ [3]. Moreover, using the Lorentz-Sobolev embedding, Visciglia [40] showed that the Lorentz space $L^{\frac{N}{r},\infty}(\Omega) \subset \mathcal{H}_p(\Omega,0)$ when $p \in (1,\infty)$, $A = I_{N \times N}$, and $\Omega$ is a general domain. In [26, Theorem 8.5], Maz’ya gave an intrinsic characterization of a Hardy-weight (for the $p$-Laplacian) using the notion of $p$-capacity. In the recent paper [5], using Maz’ya’s characterization, the authors introduced a norm on $\mathcal{H}_p(\Omega,0)$, the space of all Hardy-weights for the $p$-Laplacian, making $\mathcal{H}_p(\Omega,0)$ a Banach function space (see Definition 2.9).

Inspired by [8], we extend the classical definition of $p$-capacity on compact sets in $\Omega$ to the case of the nonnegative functional $Q_{p,A,V}$ (see a detailed discussion in Subsection 2.3).

**Definition 1.1** ($Q_{p,A,V}$-capacity). Let $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ be a positive function. For a compact set $F \subseteq \Omega$, the $Q_{p,A,V}$-capacity of $F$ with respect to $(u,\Omega)$ is defined by

$$
\text{Cap}_u(F,\Omega) := \inf \{Q_{p,A,V}(\phi) \mid \phi \in \mathcal{N}_{F,u}(\Omega)\},
$$

where $\mathcal{N}_{F,u}(\Omega) := \{\phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \mid \phi \geq u \text{ on } F\}$.

Now we are in a position to extend to our setting, the definition of a norm on the space of Hardy-weights. Let $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ be a positive solution of (1.1). Then, we equip $\mathcal{H}_p(\Omega,V)$ with the norm:

$$
\|g\|_{\mathcal{H}_p(\Omega,V)} := \sup \left\{ \frac{\int_F |g||u|^p dx}{\text{Cap}_u(F,\Omega)} \mid F \subseteq \Omega \text{ is compact set s.t. } \text{Cap}_u(F,\Omega) \neq 0 \right\},
$$

which is well defined due to the following extension of Maz’ya’s well known characterization of the space of Hardy-weights for the $p$-Laplacian [26, Theorem 8.5].
Theorem 1.2. Let $p \in (1, \infty)$ and $g \in L^1_{\text{loc}}(\Omega)$. Then $\|g\|_{\mathcal{H}_p(\Omega, V)} < \infty$ if and only if the Hardy-type inequality (1.2) holds. Moreover, let $\mathcal{B}_g(\Omega, V)$ be the best constant in (1.2), then

$$
\|g\|_{\mathcal{H}_p(\Omega, V)} \leq \mathcal{B}_g(\Omega, V) \leq C_H \|g\|_{\mathcal{H}_p(\Omega, V)},
$$

where $C_H$ is independent of $g$. Furthermore, $\|g\|_{\mathcal{H}_p(\Omega, V)} := \mathcal{B}_g(\Omega, V)$ is an equivalent norm on $\mathcal{H}_p(\Omega, V)$. In particular, up to the equivalence relation of norms, the norm $\|\cdot\|_{\mathcal{H}_p(\Omega, V)}$ is independent of the positive solution $u$.

We would like to remark that the main difficulty in proving Theorem 1.2 arises due to the sign-changing behaviour of the potential $V$. We overcome this difficulty by using the $Q_{p,A,V}$-capacity and the simplified energy functional ([29, 30], see Definition 2.6). In view of Theorem 1.2, we identify the space of Hardy-weights as

$$
\mathcal{H}_p(\Omega, V) = \{ g \in L^1_{\text{loc}}(\Omega) \mid \|g\|_{\mathcal{H}_p(\Omega, V)} < \infty \}.
$$

In fact, $\mathcal{H}_p(\Omega, V)$ is a Banach function space (see Definition 2.9), and in particular, $\mathcal{H}_p(\Omega, V)$ is a Banach space. Moreover, under some conditions we show that certain weighted Lebesgue spaces are embedded in $\mathcal{H}_p(\Omega, V)$ (see Theorem 3.5).

Definition 1.3 (Beppo-Levi space). The generalized Beppo-Levi space $\mathcal{D}^{1,p}_{A,V+}(\Omega)$ is the completion of $W^{1,p}(\Omega) \cap C_c(\Omega)$ with respect to the norm

$$
\|\phi\|_{\mathcal{D}^{1,p}_{A,V+}(\Omega)} := \left[\|\nabla \phi\|_{L^p(\Omega)}^p + \|\phi\|_{L^p(\Omega,D^p)}^p\right]^{1/p}.
$$

Recall that for $g \in \mathcal{H}_p(\Omega, V)$, $\mathcal{B}_g(\Omega, V)$ is the best constant for the inequality (1.2), i.e.,

$$
\frac{1}{\mathcal{B}_g(\Omega, V)} = \mathcal{S}_g(\Omega, V) := \inf\{Q_{p,A,V}(\phi) \mid \phi \in W^{1,p}(\Omega) \cap C_c(\Omega), \int_{\Omega} |g||\phi|^p dx = 1\}.
$$

We say that the best constant $\mathcal{B}_g(\Omega, V)$ is attained if $\mathcal{S}_g(\Omega, V)$ is attained in $\mathcal{D}^{1,p}_{A,V+}(\Omega)$.

Remark 1.4. Generally speaking, for $p \neq 2$, the nonnegative functional $Q_{p,A,V}$ does not define a norm on $W^{1,p}(\Omega) \cap C_c(\Omega)$ unless $V \geq 0$ (see the discussion in [31, Section 6]). Therefore, to a subcritical functional $Q_{p,A,V+}$, we associate the Banach space $\mathcal{D}^{1,p}_{A,V+}(\Omega)$ which is a complete, separable and reflexive Banach space of functions (see Remark 4.2). We note that the space $\mathcal{D}^{1,p}_{A,V}(\Omega)$ for $V$ satisfying $\inf V > 0$ appears in the literature (see for example [6] and references therein), in particular for $p = 2$, the space $\mathcal{D}^{1,2}_{A,V}(\Omega)$ when $\inf V > 0$ appears in spectral theory of Schrödinger operators (see for example [7]).

In the present paper, we also investigate the attainment of $\mathcal{S}_g(\Omega, V)$ in $\mathcal{D}^{1,p}_{A,V+}(\Omega)$. In this context, using the standard variational methods, it is easily seen that if $V^- = 0$ and the map

$$
T_g(\phi) := \int_{\Omega} |g||\phi|^p dx
$$

is compact on the Beppo-Levi space $\mathcal{D}^{1,p}_{A,V+}(\Omega)$, then $\mathcal{S}_g(\Omega, V)$ is attained in $\mathcal{D}^{1,p}_{A,V+}(\Omega)$. The second aim of the paper is to characterize the space of all $g \in \mathcal{H}_p(\Omega, V)$ for which $T_g$ is compact on $\mathcal{D}^{1,p}_{A,V+}(\Omega)$. As in [25, Section 2.4.2, pp. 130], [5, Theorem 8], we give a
necessary and sufficient condition on $g$ for which $T_g$ is compact on $\mathcal{D}^{1,p}_{A,V+}(\Omega)$, see Theorem 6.9. Moreover, we identify a subspace of $\mathcal{H}_p(\Omega,V)$ that ensure the compactness of $T_g$. Let

$$\mathcal{H}_{p,0}(\Omega,V) := \mathcal{H}_p(\Omega,V) \cap L^\infty(\Omega)^{1-\|\mathcal{H}_p(\Omega,V)\|_p}.$$

For $A = I_{N \times N}$ and $V = 0$, it has been shown in [5, Theorem 8] that if $g \in \mathcal{H}_{p,0}(\Omega,0)$, then $T_g$ is compact in $\mathcal{D}^{1,p}_{L,0}(\Omega)$. In the present paper, we prove the analogous result for a symmetric, and locally uniformly positive definite matrix $A := (a_{ij}) \in L^\infty_{loc}(\Omega; \mathbb{R}^{N \times N})$, and a potential $V$, see Theorem 4.3 and Remark 4.4-(i). In fact, under some further assumptions on the matrix $A$, the converse is also true for $p \in (1, N)$ (see Remark 4.4-(ii)).

It is worth mentioning that $\mathcal{S}_g(\Omega,V)$ might be achieved in $\mathcal{D}^{1,p}_{A,V+}(\Omega)$ without $T_g$ being compact. This leads to our next aim, which is to provide weaker sufficient conditions on $V$ and $g$ such that $\mathcal{S}_g(\Omega,V)$ is attained by a function in $\mathcal{D}^{1,p}_{A,V+}(\Omega)$. For $A = I_{N \times N}$ and $V = 0$, such problems have been studied in [5, 37, 38] using the well known concentration compactness arguments due to P. L. Lions [21, 22]. In the present article, we give two different sufficient conditions such that $\mathcal{S}_g(\Omega,V)$ is attained by a function in $\mathcal{D}^{1,p}_{A,V+}(\Omega)$ (see theorems 5.1 and 6.1). In each theorem we assume that $Q_{p,A,V}$ admits a (different) spectral gap-type condition, but, while the proof of Theorem 5.1 is based on criticality theory, the proof of Theorem 6.1 uses the standard concentration compactness method.

**Remarks 1.5.** (i) If the operator $Q_{p,A,V-\mathcal{S}_g(\Omega)|g|$ is subcritical in a domain $\Omega$, then the best Hardy constant in (1.2) is not attained [28].

(ii) If $g \in \mathcal{H}_p(\Omega,V)$ is an optimal Hardy-weights (i.e., $Q_{p,A,V-\mathcal{S}_g(\Omega)|g|$ is null-critical with respect to $g$ in $\Omega$) [10, 11], then the best Hardy constant in (1.2) is not attained.

2. Preliminaries

2.1. Notation. Throughout the paper, we use the following notation and conventions:

- For $R > 0$ and $x \in \mathbb{R}^n$, we denote by $B_R(x)$ the open ball of radius $R$ centered at $x$.
- $\chi_S$ denotes the characteristic function of a set $S \subset \mathbb{R}^n$.
- We write $A_1 \subset A_2$ if $\overline{A}_1$ is a compact set, and $\overline{A}_1 \subset A_2$.
- For any subset $A \subset \mathbb{R}^N$, we denote the interior of $A$ by $\overset{\rightharpoonup}{A}$.
- $C$ refers to a positive constant which may vary from line to line.
- Let $g_1, g_2$ be two positive functions defined in $\omega$. We use the notation $g_1 \asymp g_2$ if there exists a positive constant $C$ such that $C^{-1}g_2(x) \leq g_1(x) \leq Cg_2(x)$ for all $x \in \omega$.
- For any real valued measurable function $u$ and $\omega \subset \mathbb{R}^n$, we denote

$$\inf_{\omega} u := \text{ess inf}_{\omega} u, \quad \sup_{\omega} u := \text{ess sup}_{\omega} u, \quad u^+ := \max(0,u), \quad u^- := \max(0,-u).$$

- For a subspace $X(\Omega)$ of measurable functions on $\Omega$, $X_c(\Omega) := \{f \in X(\Omega) | \text{supp } f \subset \Omega\}$.
- Let $(X,\|\cdot\|_X)$ be a normed space, and $Y \subset X$, $\overline{Y}^{\|\cdot\|_X}$ is the closure of $Y$ in $X$.
- For a Banach space $V$ over $\mathbb{R}$, we denote by $V^*$ the space of continuous linear maps from $V$ into $\mathbb{R}$.
- For any $1 \leq p \leq \infty$, $p'$ is the Hölder conjugate exponent of $p$ satisfying $p' = p/(p-1)$.
- For $1 \leq p < n$, $p^* := np/(n-p)$ is the corresponding Sobolev critical exponent.
2.2. Morrey spaces. In this subsection we introduce the local Morrey spaces $\mathcal{M}^q_{\text{loc}}(p; \Omega)$ in which the potential $V$ belongs to, and briefly discuss some of their properties.

**Definition 2.1** (Morrey space). Let $q \in [1, \infty]$ and $\omega \Subset \mathbb{R}^N$ be an open set. For a measurable, real valued function $f$ defined in $\omega$, we set

$$
\|f\|_{\mathcal{M}^q(\omega)} = \sup \left\{ m_q(r) \int_{\omega \cap B_r(y)} |f| \, dx \mid y \in \omega, 0 < r < \text{diam}(\omega) \right\},
$$

where $m_q(r) = r^{-N/q'}$. We write $f \in \mathcal{M}^q_{\text{loc}}(\Omega)$ if for any $\omega \Subset \Omega$, we have $\|f\|_{\mathcal{M}^q(\omega)} < \infty$.

**Remarks 2.2.** (i) Note that $\mathcal{M}^1_{\text{loc}}(\Omega) = L^1_{\text{loc}}(\Omega)$ and $\mathcal{M}^\infty_{\text{loc}}(\Omega) = L^\infty_{\text{loc}}(\Omega)$ (as vector spaces), but $L^q_{\text{loc}}(\Omega) \subsetneq \mathcal{M}^q_{\text{loc}}(\Omega) \subsetneq L^q(\Omega)$ for any $q \in (1, \infty)$.

(ii) For $f \in \mathcal{M}^q_{\text{loc}}(\Omega)$ and $1 \leq q < \infty$, it is easily seen that for $B_\rho(x) \subset \Omega$ we have $\|f\|_{B_\rho(x)} \leq \|f\|_{B_\rho(\rho)}$ for $r \leq \rho$.

Next we define a special local Morrey space $\mathcal{M}^q_{\text{loc}}(p; \Omega)$ which depends on the underlying exponent $1 < p < \infty$.

**Definition 2.3** (Special Morrey space). For $p \neq N$, we define

$$
\mathcal{M}^q_{\text{loc}}(p; \Omega) := \begin{cases} 
\mathcal{M}^q_{\text{loc}}(\Omega) \text{ with } q > \frac{N}{p} & \text{if } p < N, \\
L^1_{\text{loc}}(\Omega) & \text{if } p > N,
\end{cases}
$$

while for $p = N$, the Morrey space $\mathcal{M}^q_{\text{loc}}(N; \Omega)$ consists of all those $f$ such that for some $q > N$ and any $\omega \Subset \Omega$

$$
\|f\|_{\mathcal{M}^q(N;\omega)} = \sup \left\{ m_q(r) \int_{\omega \cap B_r(y)} |f| \, dx \mid y \in \omega, r < \text{diam}(\omega) \right\},
$$

where $m_q(r) = [\log(\frac{\text{diam}(\omega)}{r})]^{q/N'}$ for $0 < r < \text{diam}(\omega)$ [23, Theorem 1.94, and references therein]. For more details on Morrey spaces, see the monograph [23] and references therein.

We recall the Morrey-Adams inequality (see [23, Theorem 4.1] and [33, Theorem 7.4.1]).

**Proposition 2.4** (Morrey-Adams inequality). Let $p \in (1, \infty)$, $\omega \Subset \mathbb{R}^N$ and $f \in \mathcal{M}^q_{\text{loc}}(p; \omega)$. Then there exists a constant $C(N, p, q) > 0$ such that for any $0 < \delta < \delta_0$

$$
\int_\omega |f|^p \, dx \leq \delta \int_\omega |\nabla f|^p \, dx + \frac{C(N, p, q)}{\delta^{pN/q}} \|f\|_{\mathcal{M}^q_{\text{loc}}(p; \omega)} \int_\omega |\phi|^p \, dx \quad \forall \phi \in W_0^{1,p}(\omega). \quad (2.1)
$$

2.3. Generalized Capacity. Let $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ be a positive function. Recall our definition of $\operatorname{Cap}_u(F, \Omega)$, the $Q_{p,A,V}$-capacity of a compact set $F \subset \Omega$ with respect to $(u, \Omega)$ (see Definition 1.1). In this subsection, we briefly discuss some of the properties of $\operatorname{Cap}_u(F, \Omega)$ (cf. [29] and references therein).

**Remarks 2.5.** (i) Let $\Omega_1 \subset \Omega_2$ be domains in $\mathbb{R}^N$. Then $\operatorname{Cap}_u(\cdot, \Omega_2) \leq \operatorname{Cap}_u(\cdot, \Omega_1)$. On the other hand, for two compact sets $F_1 \subset F_2$ in $\Omega$, we have $\operatorname{Cap}_u(F_1, \Omega) \leq \operatorname{Cap}_u(F_2, \Omega)$.

(ii) Notice that, for any compact set $F \subset \Omega$ we have

$$
\operatorname{Cap}_u(F, \Omega) = \inf \{ Q_{p,A,V}(\psi u) \mid \psi u \in W^{1,p}(\Omega) \cap C(\Omega) \text{ with } \psi \geq 1 \text{ on } F \}. \quad (2.1)
$$
(iii) One can easily verify that \( \phi \in N_{F,u}(\Omega) \) implies that \(|\phi| \in N_{F,u}(\Omega) \) and \( Q_{p,A,V}(|\phi|) = Q_{p,A,V}(\phi) \). Hence, in the definition of the \( Q_{p,A,V} \)-capacity, it is enough to consider only nonnegative test functions \( \phi \in N_{F,u}(\Omega) \). Furthermore, for any \( \phi \in N_{F,u}(\Omega) \), define \( \hat{\phi} = \min\{\phi, u\} \). Then, \( \hat{\phi} \in N_{F,u}(\Omega) \) and \( Q_{p,A,V}(\hat{\phi}) \leq Q_{p,A,V}(\phi) \) (cf. the proof of [8, Proposition 4.1] and Lemma 5.5). Thus, it is easy to see that

\[
\text{Cap}_u(F, \Omega) = \inf \left\{ Q_{p,A,V}(\phi) \mid \phi \in \tilde{N}_{F,u}(\Omega) \right\},
\]

where \( \tilde{N}_{F,u}(\Omega) = \{ \phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \mid \phi = u \text{ on } F, 0 \leq \phi \leq u \text{ in } \Omega \} \).

(iv) Choosing \( u = 1 \), we see that our definition of \( Q_{p,A,V} \)-capacity coincides with the classical definitions. For instance, see [14, 26] for the case \( V = 0 \), \( A = I_{N \times N} \), [32] for \( V \neq 0 \), \( A = I_{N \times N} \), and [29] for \( V \neq 0 \), \( A \in L^\infty(\Omega; \mathbb{R}^{N \times N}) \).

(v) The local uniform ellipticity of \( A \in L^\infty(\Omega; \mathbb{R}^{N \times N}) \) ensures that there exists a positive measurable function \( \theta : \Omega \to (0, \infty) \) such that

\[
\frac{1}{\theta(x)}|\xi|_l \leq |\xi|_A \leq \theta(x)|\xi|_l \quad \forall x \in \Omega, \xi \in \mathbb{R}^N,
\]

and \( \theta \) is called a local uniform ellipticity function. Suppose that the local uniform ellipticity function \( \theta \in L^\infty(\Omega) \), then \( 1 \leq \theta(x) \leq \|\theta\|_{L^\infty(\Omega)} \) in \( \Omega \). Therefore, \( A \in L^\infty(\Omega, \mathbb{R}^{N \times N}) \), and \( A \) is uniformly elliptic in \( \Omega \). Furthermore, if \( p \in (1, N) \), then the Gagliardo-Nirenberg-Sobolev inequality implies that for any compact \( F \subset \Omega \) and any \( \phi \in N_{F,1}(F, \Omega) \)

\[
|F|^{\frac{p}{p'}} \leq \left[ \int_{\Omega} |\phi|^{p'} \, dx \right]^{p/p} \leq C \int_{\Omega} |\nabla \phi|_p^p \, dx \leq C \int_{\Omega} |\theta(x)|^p |\nabla \phi|_A^p \, dx \leq C \|\theta\|_{L^\infty(\Omega)}^p \int_{\Omega} |\nabla \phi|_A^p \, dx
\]

for some \( C > 0 \). Thus, if \( V = 0 \), \( p \in (1, N) \), and \( \theta \in L^\infty(\Omega) \), then there exists a constant \( C > 0 \) such that

\[
|F|^{\frac{p}{p'}} \leq C \text{Cap}_1(A, F, \Omega) \quad \forall F \subset \Omega.
\]

(vi) Let \( F \subset \Omega \) be a compact set. We claim that \( \text{Cap}_u(F, \Omega) = 0 \) if and only if \( \text{Cap}_1(F, \Omega) = 0 \). Assume first that \( \text{Cap}_1(F, \Omega) = 0 \). Consider \( u_F = u/\|u\|_{L^\infty} \), where \( u_F \) is the restriction of \( u \) on \( F \). Then, it is clear that \( \text{Cap}_{u_F}(F, \Omega) \leq \text{Cap}_1(F, \Omega) \). This implies that \( \text{Cap}_{u_F}(F, \Omega) = 0 \). Since \( \text{Cap}_{u_F}(F, \Omega) = \|u_F\|_{L^\infty}^p \text{Cap}_u(F, \Omega) \), it follows that \( \text{Cap}_u(F, \Omega) = 0 \). Conversely, suppose that \( \text{Cap}_u(F, \Omega) = 0 \), and consider \( \tilde{u}_F = u/\inf F u \). Then, following a similar argument as above, we conclude that \( \text{Cap}_1(F, \Omega) = 0 \).

2.4. Simplified Energy Functional. Recall our assumption that \( Q_{p,A,V} \geq 0 \) on \( W^{1,p}(\Omega) \cap C_c(\Omega) \), and notice that a priori, the corresponding Lagrangian might take negative values. Fortunately, due to a Picone identity [29] see also (5.3)), this Lagrangian is equal to a nonnegative Lagrangian (depending on a positive solution \( u \) of (1.1)), but the latter Lagrangian contains indefinite terms. Fortunately, \( Q_{p,A,V} \) admits an equivalent functional \( E_u \) depending on \( u \) which, roughly speaking, is easier to handle since it contains only nonnegative terms.

**Definition 2.6 ([29, 30])**. Let \( p \in (1, \infty) \) and \( u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega) \) be a positive solution of (1.1). Then the simplified energy functional \( E_u : W^{1,p}(\Omega) \cap C_c(\Omega) \to \mathbb{R} \) of the functional
$Q_{p,A,V}$ in $\Omega$ is defined as

$$E_u(\phi) := \int_\Omega u^2|\nabla \phi|_A^2(\phi|\nabla u|_A + u|\nabla \phi|_A)^{p-2}dx \quad \phi \in W^{1,p}(\Omega) \cap C_c(\Omega).$$

**Proposition 2.7** ([30, Lemma 2.2] and [29, Lemma 3.4]). Let $p \in (1, \infty)$ and $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ be a positive solution of (1.1). Then there exists $C_1, C_2 > 0$ such that

$$C_1 Q_{p,A,V}(u\phi) \leq E_u(\phi) \leq C_2 Q_{p,A,V}(u\phi) \quad \forall u\phi \in W^{1,p}(\Omega) \cap C_c(\Omega).$$

**Remark 2.8.** If $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ is a positive solution of (1.1), one may equivalently define a $Q_{p,A,V}$-capacity using the simplified energy functional.

### 2.5. Banach function space.

**Definition 2.9.** A normed linear space $(X(\Omega), \|\cdot\|_{X(\Omega)})$ (the value $\|f\|_{X(\Omega)} = \infty$ is admitted) of measurable functions on $\Omega$ is called a Banach function space if the following conditions are satisfied:

1. $\|f\|_{X(\Omega)} = \|f\|_{X(\Omega)}$, for all $f \in X(\Omega)$,
2. if $(f_n)$ is a nonnegative sequence of function in $X(\Omega)$, increasing to $f$, then $\|f_n\|_{X(\Omega)}$ increases to $\|f\|_{X(\Omega)}$.

The norm $\|\cdot\|_{X(\Omega)}$ is called a Banach function space norm on $X(\Omega)$ [41, Section 30, Chapter 6]. Indeed, the restriction of a Banach function space to $\{f \in X(\Omega) \mid \|f\|_{X(\Omega)} < \infty\}$ is complete [41, Theorem 2, Section 30, Chapter 6].

### 3. Characterization of Hardy-weights

The present section is devoted mainly to the proof of Theorem 1.2. As applications of Theorem 1.2, we derive a necessary integral condition as in [16, Theorem 3.1] for $|g|$ being a Hardy-weight, and show that certain weighted Lebesgue spaces are embedded into $H_{p}(\Omega, V)$.

**Proof of Theorem 1.2.** **Necessity:** Let $g \in L^1_{\text{loc}}(\Omega)$ be such that the Hardy-type inequality (1.2) holds. Let $F \subset \Omega$ be a compact set. Then by (1.2) we have for any measurable function $\psi$ such that $\psi u \in W^{1,p}(\Omega) \cap C_c(\Omega)$ with $\psi \geq 1$ on $F$

$$\int_F |g||u|^pdx \leq \int_\Omega |g||\psi u|^pdx \leq B_g(\Omega, V)Q_{p,A,V}(\psi u).$$

By taking infimum over all $\psi u \in W^{1,p}(\Omega) \cap C_c(\Omega)$ with $\psi \geq 1$ on $F$, and using Remark 2.5-(ii) we obtain

$$\int_F |g||u|^pdx \leq B_g(\Omega, V)Cap_u(F, \Omega) \quad \text{for all compact sets } F \text{ in } \Omega.$$

Hence, $\|g\|_{H_{p}(\Omega, V)} \leq B_g(\Omega, V)$.

**Sufficiency:** Assume that $\|g\|_{H_{p}(\Omega, V)} < \infty$, i.e.,

$$\int_F |g||u|^pdx \leq \|g\|_{H_{p}(\Omega, V)}Cap_u(F, \Omega) \quad \text{for all compact sets } F \text{ in } \Omega. \quad (3.1)$$
Consider the measure $\mu = |g||u|^pdx$. Then for any nonnegative $\psi \in W^{1,p}(\Omega) \cap C_c(\Omega)$ (see for example [23, Theorem 1.9]),

$$\int_\Omega |g|\psi|u|^pdx = p \int_0^\infty \mu(\{\psi \geq t\})t^{p-1}dt = p \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \mu(\{\psi \geq t\})t^{p-1}dt$$

$$\leq (2^p - 1) \sum_{j=-\infty}^{\infty} 2^{pj} \mu(\{\psi \geq 2^j\}) \leq ||g||_{L^p(\Omega)}(2^p - 1) \sum_{j=-\infty}^{\infty} 2^{pj} \text{Cap}_u(\{\psi \geq 2^j\}, \Omega).$$  \hspace{1cm} (3.2)

**Case 1.** Let $p \geq 2$. Consider the function $\psi_j \in W^{1,p}(\Omega) \cap C_c(\Omega)$

$$\psi_j(x) := \begin{cases} 0 & \text{if } \psi \leq 2^{j-1}, \\ \frac{\psi}{2^j} - 1 & \text{if } 2^{j-1} \leq \psi \leq 2^j, \\ 1 & \text{if } 2^j \leq \psi. \end{cases}$$

Then $\text{Cap}_u(\{\psi \geq 2^j\}, \Omega) \leq Q_{p,A,V}(\psi_j u)$. Let $A_j := \{x \in \Omega \mid 2^{j-1} \leq \psi(x) \leq 2^j\}$. The simplified energy (Proposition 2.7) implies

$$\text{Cap}_u(\{\psi \geq 2^j\}, \Omega) \leq Q_{p,A,V}(\psi_j u) \leq C \int_{A_j} u^2 |\nabla \psi_j|^2 \left[ |\nabla u|_A + u|\nabla \psi_j|_A \right]^{p-2}dx$$

$$\leq C \left( \frac{C}{2^{(p-1)j}} \right) \int_{A_j} u^2 |\nabla \psi_j|^2 \left[ |\nabla u|_A + u|\nabla \psi_j|_A \right]^{p-2}dx. \hspace{1cm} (3.3)$$

**Case 2.** Let $p < 2$. Consider the function

$$\tilde{\psi}_j(x) := \begin{cases} 0 & \text{if } \psi \leq 2^{j-1}, \\ \left[ \frac{\psi}{2^j} - 1 \right]^{2/p} & \text{if } 2^{j-1} \leq \psi \leq 2^j, \\ 1 & \text{if } 2^j \leq \psi. \end{cases}$$

Then $\tilde{\psi}_j \in W^{1,p}(\Omega) \cap C_c(\Omega)$ and $\text{Cap}_u(\{\psi \geq 2^j\}, \Omega) \leq Q_{p,A,V}(\tilde{\psi}_j u)$, and similarly we obtain

$$\text{Cap}_u(\{\psi \geq 2^j\}, \Omega) \leq Q_{p,A,V}(\tilde{\psi}_j u) \leq C \int_{A_j} u^2 |\nabla \tilde{\psi}_j|^2 \left[ |\nabla u|_A + u|\nabla \tilde{\psi}_j|_A \right]^{p-2}dx$$

$$\leq \left( \frac{4C}{p^2} \right) \int_{A_j} u^2 \left[ \frac{\psi}{2^j} - 1 \right]^{2(2-p)/p} |\nabla \tilde{\psi}_j|^2 \left[ |\nabla u|_A + u\left[ \frac{\psi}{2^j} - 1 \right]^{2(2-p)/p} \right]^{p-2}dx$$

$$\leq \left( \frac{4C}{p^2} \right) \int_{A_j} u^2 \left[ \frac{\psi}{2^j} - 1 \right]^{2(2-p)/p} \left[ |\nabla u|_A + u\left[ \frac{\psi}{2^j} - 1 \right]^{2(2-p)/p} \right]^{p-2}dx$$

$$\leq \left( \frac{16C}{2^p p^2} \right) \int_{A_j} u^2 |\nabla \tilde{\psi}_j|^2 \left[ |\nabla u|_A + u|\nabla \tilde{\psi}_j|_A \right]^{p-2}dx. \hspace{1cm} (3.4)$$
Subsequently, using (3.3) (for \( p \geq 2 \)) and (3.4) (for \( p < 2 \)), we obtain from (3.2) that
\[
\int_{\Omega} |g| |\psi u|^p \, dx \leq C_2 \|g\|_{\mathcal{H}_p(\Omega, V)} \sum_{j=-\infty}^{\infty} 2^{pj} \left( \frac{1}{2^{pj}} \right) \int_{A_j} u^2 |\nabla \psi|^2_A |\psi| |\nabla u| + u |\nabla \psi| A^{p-2} \, dx
\]
\[
= C_2 \|g\|_{\mathcal{H}_p(\Omega, V)} \sum_{j=-\infty}^{\infty} \int_{A_j} u^2 |\nabla \psi|^2_A |\psi| |\nabla u| + u |\nabla \psi| A^{p-2} \, dx
\]
\[
= C_2 \|g\|_{\mathcal{H}_p(\Omega, V)} \int_{\Omega} [u^2 |\nabla \psi|^2_A |\psi| |\nabla u| + u |\nabla \psi| A^{p-2}] \, dx \leq C_H \|g\|_{\mathcal{H}_p(\Omega, V)} Q_{p,A,V}(u \psi). \tag{3.5}
\]
Therefore, for any nonnegative \( \phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \), by taking \( \psi = \phi / u \) in (3.5), we get
\[
\int_{\Omega} |g||\phi|^p \, dx \leq C_H \|g\|_{\mathcal{H}_p(\Omega, V)} Q_{p,A,V}(\phi).
\]
Since \( |\phi| \in W^{1,p}(\Omega) \cap C_c(\Omega) \) for \( \phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \), our claim follows.

Furthermore, \( \|g\|_{B(\Omega, V)} := B_g(\Omega, V) \) also defines a norm on \( \mathcal{H}_p(\Omega) \). From the above proof, it follows that this norm is equivalent to the norm \( \| \cdot \|_{\mathcal{H}_p(\Omega, V)} \). Therefore, up to the equivalence relation of norms, the norm \( \| \cdot \|_{\mathcal{H}_p(\Omega, V)} \) is independent of the positive solution \( u \). \( \square \)

**Remarks 3.1.** (i) Let \( \Omega \) be a domain in \( \mathbb{R}^N \). Theorem 1.2 implies that \( \text{Cap}_u(F, \Omega) = 0 \) for any compact set \( F \Subset \Omega \) if and only if \( Q_{p,A,V} \) is critical in \( \Omega \). In fact, by [29, Theorem 6.8], \( Q_{p,A,V} \) is critical in \( \Omega \) if and only if \( \text{Cap}_1(F, \Omega) = 0 \) for any compact set \( F \Subset \Omega \), and by Remark 2.5-(vi), for any compact set \( F \Subset \Omega \), \( \text{Cap}_1(F, \Omega) = 0 \) if and only if \( \text{Cap}_u(F, \Omega) = 0 \).

(ii) In Theorem 1.2, if we take \( A = I_{N\times N}, V = 0 \), and \( u = 1 \), we obtain Maz’ya’s characterization of Hardy-weights for the \( p \)-Laplacian [26, Theorem 8.5].

(iii) Frequently, it is easier to compute or estimate \( \| \cdot \|_{B(\Omega, V)} \) than \( \| \cdot \|_{\mathcal{H}_p(\Omega, V)} \).

### 3.1. Necessary condition for being a Hardy-weight

In the recent paper [16], the authors proved a necessary condition for \( |g| \) to be a Hardy-weight. Here we show that this condition can be derived directly from Theorem 1.2. First we recall the definition of positive solution of minimal growth at infinity.

**Definition 3.2.** Let \( K_0 \) be a compact subset in \( \Omega \). A positive solution \( u \) of \( \mathcal{Q'}_{p,A,V}[w] = 0 \) in \( \Omega \setminus K_0 \), is called a **positive solution of minimal growth in a neighborhood of infinity in \( \Omega \)** if for any smooth compact subset \( K \) of \( \Omega \) with \( K_0 \Subset K \), any positive supersolution \( \psi \in C(\Omega \setminus K) \) of \( \mathcal{Q'}_{p,A,V}[w] = 0 \) in \( \Omega \setminus K \) such that \( u \leq v \) on \( \partial K \), satisfies \( u \leq v \) in \( \Omega \setminus K \).

**Remark 3.3.** If \( \mathcal{Q}_{p,A,V} \) is subcritical (critical) in a domain \( \Omega \), then respectively, its minimal positive Green function (ground state) is a positive solution of the equation \( \mathcal{Q'}_{p,A,V}[w] = 0 \) of minimal growth in a neighborhood of infinity in \( \Omega \) [28]. For the definition and properties of a ground state see Definition 5.3, Remark 5.4 and [28].

Now we derive the necessary integral condition of [16] in the following proposition.

**Proposition 3.4.** [16, Theorem 3.1] Let \( \Omega \) be a domain and \( K \Subset \Omega \subset \mathbb{R}^N \) with \( K \neq \emptyset \). Let \( \mathcal{Q}^{1,p}_u(\Omega) \cap C(\Omega) \) be a positive solution of \( \mathcal{Q'}_{p,A,V}[w] = 0 \) in \( \Omega \setminus K \) of minimal growth at a neighborhood of infinity in \( \Omega \). If \( g \in \mathcal{H}_p(\Omega, V) \), then \( \int_{\Omega \setminus K_1} |g|u^p \, dx < \infty \), where \( K \Subset K_1 \Subset \Omega \).
Proof. Let \( g \in \mathcal{H}_p(\Omega, V) \), and \( K \Subset \Omega \) such that \( \tilde{K} \neq \emptyset \). By [28, Proposition 4.19], there exists \( 0 \leq V_K \in C_c^\infty(\Omega) \) with support in \( K \) such that \( Q_{p,A,V-V_K} \) is critical in \( \Omega \). Let \( \psi \) be the ground state of \( Q_{p,A,V-V_K} \) in \( \Omega \). Thus, by [15, 29] and Remark 3.1-(i), \( \operatorname{Cap}_p(F, \Omega) = 0 \) for any compact set \( F \Subset \Omega \) with nonempty interior. Let \( \bigcup_n \Omega_n = \Omega \) be a compact smooth exhaustion of \( \Omega \). For each \( n \in \mathbb{N} \), there exists \( 0 \leq \phi_n \in \mathcal{N}_{\Omega_n,\psi}(\Omega) \) such that \( Q_{p,A,V-V_K}(\phi_n) < 1/n \). Hence,

\[
\int_{\Omega_n} |g| \phi_n^p \, dx \leq \int_{\Omega_n} |g| \phi_n^p \, dx \leq C Q_{p,A,V}(\phi_n) = C Q_{p,A,V-V_K}(\phi_n) + C \int_{\Omega} V_K |\phi_n|^p \, dx \\
\leq \frac{C}{n} + C \|V_K\|_{L^\infty(\Omega)} \int_{K} \psi^p \, dx
\]

Using Fatou’s lemma, we conclude that \( \int_{\Omega} |g| \phi_n^p \, dx < \infty \). Hence, it follows from Remark 3.3 that \( \int_{\Omega \setminus K_1} |g| u^p \, dx < \infty \).

\[ \square \]

### 3.2. Weighted Lebesgue space embedded in \( \mathcal{H}_p(\Omega, V) \)

In this subsection, we show that there are certain weighted Lebesgue spaces that are continuously embedded in \( \mathcal{H}_p(\Omega, V) \).

**Theorem 3.5.** Let \( p \geq 2 \) and \( u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega) \) be a positive solution of \( Q_{p,I,V}^r[u] = 0 \) in \( \Omega \). For some \( \alpha \in [1, N/p], \beta \in [1, \infty], \) and \( r > 1 \), suppose that there exists \( C_\alpha > 0 \) such that

\[
|\alpha|^{\frac{1}{\alpha} + \frac{1}{\alpha'} - \frac{1}{p}} \left[ \frac{1}{|A|} \int_A u^{\frac{2p}{\alpha'}} \, dx \right]^{\frac{1}{\alpha'}} \left[ \frac{1}{|A|} \int_A u^{-p'r} \, dx \right]^{\frac{1}{p'}} < C_\alpha,
\]

for any measurable set \( A \) in \( \Omega \). Then, \( L^\alpha(\Omega, u^{\frac{p}{p'}}) \subset \mathcal{H}_p(\Omega, V) \).

**Proof.** By (3.6) and [36, Theorem 1], the following weighted Sobolev inequality holds

\[
\left[ \int_{\Omega} u^{\frac{\alpha p}{p'}} |\phi|^\alpha p \, dx \right]^{\frac{1}{\alpha p}} \leq C \left[ \int_{\Omega} u^p |\nabla \phi|^p \, dx \right]^{\frac{1}{p}} \quad \forall \phi \in W^{1,p}(\Omega) \cap C_c(\Omega)
\]

for some \( C > 0 \). Now, for any \( \phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \), using the simplified energy (Proposition 2.7) and the assumption \( p \geq 2 \), we obtain

\[
\int_{\Omega} |\phi| u^p \, dx \leq \int_{\Omega} |\phi| u^{\frac{p}{p} + (1-\frac{1}{p})} \, dx \leq \int_{\Omega} |\phi| \alpha u^{\frac{p}{p'}} \, dx \leq \left[ \int_{\Omega} |\phi|^\alpha u^{\frac{p}{p'}} \, dx \right]^{\frac{1}{\alpha}} \left[ \int_{\Omega} u^{\frac{p}{p'}} |\nabla \phi|^p \, dx \right] \leq C_1 \left[ \int_{\Omega} |\phi|^\alpha u^{\frac{p}{p'}} \, dx \right]^{\frac{1}{\alpha}} Q_{p,I,V}[u \phi],
\]

for some \( C_1 > 0 \). This proves our claim. \[ \square \]

**Example 3.6.** (i) Let \( 2 \leq p < N, V = 0 \). Then \( u = 1 \) is a positive solution of \( -\Delta_p[u] = 0 \) in \( \mathbb{R}^N \). By taking \( \alpha = \beta = N/p \), the Gagliardo-Nirenberg inequality implies (3.7). Hence, \( L^{N/p}(\mathbb{R}^N) \subset \mathcal{H}_p(\mathbb{R}^N, 0) \). Consequently, \( L^{N/p}(\mathbb{R}^N) \subset \mathcal{H}_p(\mathbb{R}^N, V) \) for \( 2 \leq p < N, \) and \( V \geq 0 \).

(ii) Let \( 2 \leq p < N, V(x) = -\frac{N-p}{p} |x|^{-p} \) and \( \Omega = B_1(0) \). Then \( u = |x|^{\frac{N-p}{p}} \) is a positive solution of \( Q_{p,I,V}^r[u] = 0 \). For \( \alpha = N/p \), and \( \beta = 1 \), (3.7) is satisfied for any \( r > 1 \). Hence, \( L^N(B_1(0), |x|^{-N(p-N)}) \subset \mathcal{H}_p(B_1(0), V) \).

**Proposition 3.7.** Let \( V = 0 \) and \( A \in L^\infty(\Omega; \mathbb{R}^{N \times N}) \). Then for \( p \in (1, N) \) and an open subset \( \Omega \) in \( \mathbb{R}^N, L^{N/p}(\Omega) \) is continuously embedded in \( \mathcal{H}_p(\Omega, 0) \).
Proof. For \( g \in L^{N/p}(\Omega) \), using Remark 2.5 (v), and recalling that \( p^* = pN/(N - p) \), we have

\[
\int_{F} \frac{|g|\, dx}{\text{Cap}_1(F, \Omega)} \leq C \|g\|_{L^{N/p}(\Omega)}
\]

for any compact \( F \subseteq \Omega \). Thus, \( \|g\|_{\mathcal{H}_p(\Omega, 0)} \leq C \|g\|_{L^{N/p}(\Omega)} \). This proves the proposition. \( \square \)

4. Compactness of the functional \( T_g \)

Suppose that \( Q_{p,A,V} \) is subcritical with \( V \geq 0 \), and \( g \in \mathcal{H}_p(\Omega, V) \). In other words, \( g \) satisfies

\[
\int_{\Omega} |g| |\phi|^p \, dx \leq C \int_{\Omega} \left[ |\nabla \phi|_A^p + V |\phi|^p \right] \, dx \quad \forall \phi \in W^{1,p}(\Omega) \cap C_c(\Omega),
\]

for some \( C > 0 \). Then the best constant in (4.1) is achieved in \( D_{A,V}^{1,p}(\Omega) \) if the functional

\[
T_g(\phi) = \int_{\Omega} |g| |\phi|^p \, dx
\]

is compact in \( D_{A,V}^{1,p}(\Omega) \) (for instance see [4, Section 6.2]). We devote this section to discuss the compactness of the map \( T_g \). Let us commence with the following important remarks.

Remarks 4.1. (i) Let \( \Omega \subseteq \mathbb{R}^N \) be a domain. Recall that for a nonnegative \( g \in L_{\text{loc}}^1(\Omega) \), the space \( D_{A,g}^{1,p}(\Omega) \) is the completion of \( W^{1,p}(\Omega) \cap C_c(\Omega) \) with respect to the norm

\[
\|\phi\|_{D_{A,g}^{1,p}} := \left[ \|\nabla \phi|_A\|_{L^p(\Omega)}^p + \|\phi\|_{L^p(\Omega, g \, dx)}^p \right]^{1/p}.
\]

It is known that \( D_{A}^{1,p}(\Omega) := D_{A,0}^{1,p}(\Omega) \) is not a Banach space of functions if and only if \( Q_{p,A,0} \) is critical. However, if \( Q_{p,A,0} \) is subcritical in \( \Omega \), then for \( g \in \mathcal{H}_p(\Omega, 0) \setminus \{0\} \), the norm \( \|\phi\|_{D_{A,0}^{1,p}} \) is equivalent to \( \|\phi\|_{D_{A,0}^{1,p}} \) for \( \phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \), and \( D_{A,0}^{1,p}(\Omega) = D_{A}^{1,p}(\Omega) \). Thus, the Hardy-type inequality (4.1) with Hardy-weight \( g \) holds for all \( \phi \in D_{A}^{1,p}(\Omega) \) as well.

(ii) In view of (i), if \( g \in \mathcal{H}_p(\Omega, 0) \setminus \{0\} \), then it follows that \( D_{A}^{1,p}(\Omega) \hookrightarrow L^p(\Omega, |g| \, dx) \), and hence, \( D_{A}^{1,p}(\Omega) \) is a well defined Banach space of functions. Furthermore, \( D_{A}^{1,p}(\Omega) \) is reflexive and separable. Indeed, consider the space \( L_A^p(\Omega)^N \) with the norm \( \|f\|_{L_A^p(\Omega)} := \left[ \int_{\Omega} |f|^p_A \, dx \right]^{1/p} \).

Consider the map

\[
E : D_{A}^{1,p}(\Omega) \to L_A^p(\Omega)^N \quad \text{given by} \quad E(\phi) := \left( \frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_N} \right).
\]

Clearly, \( E \) is an isometry from \( D_{A}^{1,p}(\Omega) \) onto a closed subspace of \( L_A^p(\Omega)^N \). Thus, it inherits the reflexivity and separability properties of \( L_A^p(\Omega)^N \). Therefore, it is enough to prove that \( (L_A^p(\Omega)^N, \|f\|_{L_A^p(\Omega)^N}) \) is a reflexive Banach space. Recall that the local uniform ellipticity of \( A \) ensures that there exists a positive function \( \theta : \Omega \to \mathbb{R} \) such that

\[
\frac{1}{\theta(x)} |\xi|_I \leq |\xi|_A \leq \theta(x) |\xi|_I, \quad \forall x \in \Omega, \xi \in \mathbb{R}^N.
\]

Consequently,

\[
\|f\|_{L_A^p(\Omega, \theta^{-1} \, dx)} \leq \|f\|_{L_A^p(\Omega)} \leq \|f\|_{L_A^p(\Omega, \theta \, dx)}.
\]
Using standard arguments, one obtains that \((L^p_k(\Omega))^N, \|f\|_{L^p_k(\Omega,\rho^k dx)}\) is a reflexive Banach space for \(k = 1, -1\) and \(1 < p < \infty\). Hence, \((L^p_A(\Omega))^N, \|f\|_{L^p_A(\Omega)}\) is a reflexive and separable Banach space.

(iii) Let \((u_n)\) be a sequence in \(W^{1,p}(\Omega) \cap C_c(\Omega)\) such that \(u_n \to u\) in \(D^{1,p}_A(\Omega)\). Consider any open smooth bounded set \(\tilde{\Omega}\) in \(\Omega\). It is clear that \(u_n \in W^{1,p}(\tilde{\Omega})\) and \((u_n)\) is bounded in \(W^{1,p}(\tilde{\Omega})\) (by Sobolev inequality and the local uniform ellipticity of \(A\)). Thus, (up to a subsequence) \(u_n|_{\tilde{\Omega}} \to u|_{\tilde{\Omega}}\) in \(W^{1,p}(\tilde{\Omega})\). Hence, (up to a subsequence) \(u_n|_{\tilde{\Omega}} \to u|_{\tilde{\Omega}}\) in \(L^p(\tilde{\Omega})\) (by Rellich-Kondrachov compactness theorem).

**Remark 4.2.** Let \(V \in \mathcal{M}_{\text{loc}}^{\phi}(p; \Omega)\), and assume that \(Q_{p,A,V}\) is subcritical in \(\Omega\). Then,

\[
\int_{\Omega} V^{-|\phi|^p} dx \leq \int_{\Omega} [\nabla \phi]^p_A + V^+|\phi|^p dx \quad \forall \phi \in W^{1,p}(\Omega) \cap C_c(\Omega),
\]

(4.2)

Using a standard completion argument as in Remark 4.1-(i), we infer that (4.2) holds for all \(\phi \in D^{1,p}_{A,V^+}(\Omega)\) as well. Now, it is easy to see that, if \(g \in H_p(\Omega, V)\), then

\[
\int_{\Omega} |g||\phi|^p dx \leq S_g(\Omega) \int_{\Omega} [\nabla \phi]^p_A + V|\phi|^p dx \quad \forall \phi \in D^{1,p}_{A,V^+}(\Omega).
\]

(4.3)

Moreover, using a similar argument as in Remark 4.1-(ii), it follows that \(D^{1,p}_{A,V^+}(\Omega)\) is a reflexive, separable Banach space.

For \(A = I_{N \times N}\) and \(V = 0\), there are necessary and sufficient conditions on \(g \in H_p(\Omega, 0)\) for the compactness of \(T_g\) on \(D^{1,p}_A(\Omega)\), for instance, see [25, Section 2.4.2], and [5, Theorem 8]. In Theorem 6.9 we prove an analogous characterization for a general matrix \(A\) and a nonnegative potential \(V\). However, the main aim of this section is to provide a subspace of \(H_p(\Omega, V)\) for which \(T_g\) is compact on \(D^{1,p}_{A,V^+}(\Omega)\). To this end, we define

\[
\mathcal{H}_{p,0}(\Omega, V) := \mathcal{H}_{p}(\Omega, V) \cap L^\infty_c(\Omega)^{1\|H_p(\Omega, V)\|}.
\]

**Theorem 4.3.** Let \(g \in \mathcal{H}_{p,0}(\Omega, V^+).\) Then \(T_g\) is compact in \(D^{1,p}_{A,V^+}(\Omega)\).

**Proof.** Let \(g \in \mathcal{H}_{p,0}(\Omega, V^+),\) and \((\phi_n)\) be a bounded sequence in \(D^{1,p}_{A,V^+}(\Omega)\). By reflexivity, up to a subsequence, \(\phi_n \to \phi\) in \(D^{1,p}_{A,V^+}(\Omega)\). Choose \(\varepsilon > 0\) arbitrary. Then there exists \(g_\varepsilon \in H_p(\Omega, V^+) \cap L^\infty_c(\Omega)\) such that \(\|g - g_\varepsilon\|_{H_p(\Omega, V^+)} < \varepsilon\). Let \(K_\varepsilon\) be the support of \(g_\varepsilon\) and \(K_\varepsilon \subset \Omega_\varepsilon \subset \Omega\), where \(\Omega_\varepsilon\) is a smooth open, bounded set. Notice that

\[
|T_g(\phi_n) - T_g(\phi)| \leq \int_{\Omega} |g - g_\varepsilon|||\phi_n|^p - |\phi|^p| dx + \int_{\Omega} |g_\varepsilon|||\phi_n|^p - |\phi|^p| dx \leq I_1 + I_2.
\]

(4.4)

Using the definition of \(\|\cdot\|_{H_p(\Omega, V^+)},\) the first integral can be estimated as follows:

\[
I_1 \leq C\|g - g_\varepsilon\|_{H_p(\Omega, V^+)} \left(\|\phi_n\|_{D^{1,p}_{A,V^+}(\Omega)}^{p} + \|\phi\|_{D^{1,p}_{A,V^+}(\Omega)}^{p}\right) \leq CM\varepsilon,
\]

(4.5)

for some \(M > 0\) independent of \(n\) (as \((\phi_n)\) is bounded in \(D^{1,p}_{A,V^+}(\Omega)\)).

In order to estimate the second integral, we notice that \(\phi_n|_{\Omega_\varepsilon} \in W^{1,p}(\Omega_\varepsilon)\) for each \(n \in \mathbb{N}\), and \((\phi_n)\) is bounded in \(W^{1,p}(\Omega_\varepsilon)\) (by local uniform ellipticity). Since, \(\Omega_\varepsilon\) is a smooth bounded
domain we obtain by compactness that \( \lim_{n \to \infty} \| \phi_n - \phi \|_{L^p(\Omega)} = 0 \) and \( \phi_n \to \phi \) a.e. (up to a subsequence). Consequently, \( \lim_{n \to \infty} \int_{\Omega} |\phi_n|^p \, dx = \int_{\Omega} |\phi|^p \, dx \). Now, for each \( \varepsilon > 0 \), we have \( |g_\varepsilon| |\phi_n|^p \leq |g_\varepsilon| |\phi|^p \) for all \( n \in \mathbb{N} \), and \( |g_\varepsilon| |\phi_n|^p \to |g_\varepsilon| |\phi|^p \) a.e. Thus, by the generalized dominated convergence theorem [35, Section 4, Theorem 16], we infer that \( \lim_{n \to \infty} \int_{\Omega} |g_\varepsilon| |\phi_n|^p \, dx = \int_{\Omega} |g_\varepsilon| |\phi|^p \, dx \) for each \( \varepsilon > 0 \). Thus, by taking \( n \to \infty \) in (4.4) and using (4.5), we obtain

\[
\limsup_{n \to \infty} |T_g(\phi_n) - T_g(\phi)| \leq CM\varepsilon.
\]

Since \( CM \) is independent of \( \varepsilon \), and \( \varepsilon > 0 \) is arbitrary, we are done. \( \square \)

Remarks 4.4. (i) Clearly, \( \mathcal{H}_p(\Omega, V) \subset \mathcal{H}_p(\Omega, V^+) \), and \( \| \cdot \|_{\mathcal{H}(\Omega, V^+)} \leq \| \cdot \|_{\mathcal{H}(\Omega, V)} \). Thus, \( \mathcal{H}_p(\Omega, V) \cap L^\infty(\Omega) \subset \mathcal{H}_p(\Omega, V^+) \cap L^\infty(\Omega) \). Hence,

\[
\mathcal{H}_p(\Omega, V) \cap L^\infty(\Omega) \subset \mathcal{H}_p(\Omega, V) \cap L^\infty(\Omega) \subset \mathcal{H}_p(\Omega, V^+) \cap L^\infty(\Omega).
\]

Therefore, \( \mathcal{H}_{p,0}(\Omega, V) \subset \mathcal{H}_{p,0}(\Omega, V^+) \). It follows from Theorem 4.3 that if \( g \in \mathcal{H}_{p,0}(\Omega, V) \), then \( T_g \) is compact in \( \mathcal{D}^{1,p}_{A,V^+}(\Omega) \).

(ii) In fact, if \( p \in (1, N) \), \( A = I_{N \times N} \) and \( V = 0 \), then the converse of Theorem 4.3 is also true [5, Theorem 8]. The embedding \( \mathcal{D}^{1,p}_{A}(\Omega) \hookrightarrow L^p(\Omega) \) played a crucial role in their proof. For \( p \in (1, N) \), \( V = 0 \), and a matrix \( A \) with local ellipticity function \( \theta \in L^\infty(\Omega) \), it clearly follows that \( \mathcal{D}^{1,p}_{A}(\Omega) = \mathcal{D}^{1,p}(\Omega) \) as vector spaces having equivalent norms. Indeed, let \( \theta \in L^\infty(\Omega) \). Then

\[
\int_\Omega |\nabla \phi|^p \, dx = \int_\Omega \theta^p \frac{1}{\theta^p} |\nabla \phi|^p \, dx \leq \int_\Omega \theta^p |\nabla \phi|^p \, dx \leq \| \theta \|_{L^\infty(\Omega)}^p \int_\Omega |\nabla \phi|^p \, dx \leq \| \theta \|_{L^\infty(\Omega)}^{2p} \int_\Omega |\nabla \phi|^p \, dx.
\]

Also, the \( Q_{p,A,0} \)-capacity of compact sets is equivalent to the corresponding \( Q_{p,t,0} \)-capacity. Therefore, if \( \theta \in L^\infty(\Omega) \), \( V = 0 \), and \( T_g \) is compact in \( \mathcal{D}^{1,p}_{A}(\Omega) \), then \( g \in \mathcal{H}_{p,0}(\Omega, V) \).

(iii) Let \( 1 < p < N \), \( V = 0 \), and \( \theta \in L^\infty(\Omega) \). Let \( g \in \mathcal{H}_{p}(\Omega, V) \cap L^\infty(\Omega) \) with compact support \( K_g \subset \Omega \). Then, using Remark 2.5-(v), we obtain

\[
\| g \|_{\mathcal{H}_{p}(\Omega, V)} \leq \sup \left\{ \frac{\int_F |g| \, dx}{\text{Cap}_1(F, \Omega)} \mid F \subset \Omega; \text{Cap}_1(F, \Omega) \neq 0 \right\}
\]

\[
\leq \sup \left\{ \frac{\int_F |g| \, dx}{\text{Cap}_1(F, \Omega)} \mid F \text{ compact in } K_g; \text{Cap}_1(F, \Omega) \neq 0 \right\}
\]

\[
\leq \sup \left\{ \frac{\int_F |g|^N \, dx}{\text{Cap}_1(F, \Omega)} \mid F \text{ compact in } K_g; \text{Cap}_1(F, \Omega) \neq 0 \right\} \leq C \| g \|_{L^\infty(\Omega)}^N,
\]

for some \( C > 0 \). Consequently, \( L^N(\Omega) \hookrightarrow \mathcal{H}_{p,0}(\Omega, V) \).
5. Attainment of Best Constant I

In the present section we assume that for \( p \leq N \), the following additional regularity assumption holds in a certain subdomain \( \Omega' \Subset \Omega \):

\[(H0) : A \in C^{0,\gamma}(\Omega' ; \mathbb{R}^{N \times N}) \text{, and } g, V \in M^q_{\text{loc}}(p; \Omega'), \text{ where } 0 < \gamma \leq 1, q > N, \text{ and } g \in \mathcal{H}_p(\Omega, V). \]

Under hypothesis \((H0)\) positive solutions \( v \) of the equation \( Q'_{p, A, V - \lambda g}[u] = 0 \) in \( \Omega' \) \([20,\ \text{Theorem 5.3}]\) are differentiable in \( \Omega' \) and satisfy in any \( \omega \Subset \Omega' \)

\[
\sup_{\omega} |\nabla v| \leq C \sup_{\Omega'} |v|,
\]

for some positive constant \( C \), depending only on \( n, p, \gamma, q, \text{dist}(\omega, \Omega'), \|A\|_{C^{0,\gamma}(\Omega')}, \|\theta^{-1}\|_{L^\infty(\Omega')}, \|g\|_{M^q(\Omega')}, \text{ and } \|V\|_{M^q(\Omega')} \).

Recall that in view of (4.3) the best constant \( B_g(\Omega) \) in (1.2) for \( g \in \mathcal{H}_p(\Omega, V) \) is given by

\[
\frac{1}{B_g(\Omega)} = S_g(\Omega) = \inf \{ Q_{p, A, V}(\phi) \mid \phi \in \mathcal{D}^{1,p}_{A, V+}(\Omega), \int_{\Omega} |g|\|\phi\|^p dx = 1 \}.
\]

In the present and the following section we prove certain sufficient conditions on \( g \) and \( V \) so that the best constant in (1.2) is attained in \( \mathcal{D}^{1,p}_{A, V+}(\Omega) \). Let \( \bigcup_n \Omega_n = \Omega \) be a compact smooth exhaustion of \( \Omega \). We define the following:

\[
S_g^{\infty}(\Omega) := \lim_{i \to \infty} \inf \left\{ Q_{p, A, V}(\phi) \mid \phi \in \mathcal{D}^{1,p}_{A, V+}(\Omega \setminus \Omega_i), \int_{\Omega_i} |g|\|\phi\|^p dx = 1 \right\}.
\]

Clearly, \( S_g(\Omega) \leq S_g^{\infty}(\Omega) \). Using criticality theory we prove below that under some regularity assumptions if the spectral gap condition \( S_g(\Omega) < S_g^{\infty}(\Omega) \) holds, then \( B_g(\Omega) \) is attained.

**Theorem 5.1.** Let \( g \in \mathcal{H}_p(\Omega, V) \cap M^q_{\text{loc}}(p; \Omega) \) be such that \( S_g(\Omega) < S_g^{\infty}(\Omega) \). Let \( K \Subset K_1 \Subset \Omega' \), where \( K \) is a compact set such that \( Q_{p, A, V - \lambda_1 g} \geq 0 \) in \( \Omega \setminus K \) for some \( \lambda_1 \in (S_g(\Omega), \|g\|_{M^q(\Omega)}) \), and for \( p \leq N, \Omega' \Subset \Omega \) is a subdomain in which hypothesis \((H0)\) is satisfied. Let \( G \) be a positive solution the equation \( Q'_{p, A, V - S_g(\Omega) g}[u] = 0 \) in \( \Omega \setminus K \) of minimal growth in a neighborhood of infinity in \( \Omega \). Assume that \( \int_{\Omega \setminus K} V^{-p} dx < \infty \). Then, \( B_g(\Omega) \) is attained in \( \mathcal{D}^{1,p}_{A, V+}(\Omega) \).

We begin with a key lemma, claiming that if the above spectral gap condition is satisfied for a Hardy-weight \( g \), then \( Q_{p, A, V - S_g(\Omega) g} \) is critical. The proof is similar to the proof of \([17, \text{Lemma 2.3}]\).

**Lemma 5.2.** Let \( g \in \mathcal{H}_p(\Omega, V) \cap M^q_{\text{loc}}(p; \Omega) \) be such that \( S_g(\Omega) < S_g^{\infty}(\Omega) \). For \( p \leq N \), assume further that the hypothesis \((H0)\) is satisfied in a subdomain \( \Omega' \), and that there exists a smooth open set \( K_0 \Subset \Omega' \) such that \( Q_{p, A, V - \lambda_1 g} \geq 0 \) in \( \Omega \setminus K_0 \) for some \( \lambda_1 \in (S_g(\Omega), \|g\|_{M^q(\Omega)}) \). Then \( Q_{p, A, V - S_g(\Omega) g} \) is critical in \( \Omega \).

**Proof.** Following \([27, \text{Lemma 4.6}]\) and \([17, \text{Lemma 2.3}]\), we set

\[
S := \{ t \in \mathbb{R} \mid Q_{p, A, V - t g} \geq 0 \text{ in } \Omega \}, \quad S_\infty := \{ t \in \mathbb{R} \mid Q_{p, A, V - t g} \geq 0 \text{ in } \Omega \setminus K \text{ for some } K \Subset \Omega \}.
\]

Clearly, \( S \) and \( S_\infty \) are intervals, and since \( Q_{p, A, V} \) has a spectral gap, it follows that

\[
S = (-\infty, S_g(\Omega)] \subsetneq S_\infty \subset (-\infty, S_g^{\infty}(\Omega)].
\]
For simplicity, we set $\lambda_0 = \mathbb{S}_q(\Omega)$. Let $\lambda_1 \in S_\infty \setminus S$ and $K_0 \in \Omega'$ satisfy the hypothesis of the lemma. Consequently, the equation $Q'_{p,A,V - \lambda_1 |g|}[u] = 0$ in $\Omega \setminus \overline{K}_0$ admits a positive solution.

**Claim** There exists $0 \leq \nu \in M^+_p(\Omega)$ such that $Q_{p,A,V - \lambda_1 |g| + \nu} \geq 0$ in $\Omega$.

Fix a smooth open set $K$ satisfying $K_0 \Subset K \Subset \Omega'$. We first show that there exists a positive solution $v$ of the equation $Q'_{p,A,V - \lambda_1 |g|}[u] = 0$ in $\Omega \setminus \overline{K}$ satisfying $v = 0$ on $\partial K$.

To this end, consider a smooth exhaustion $\{\Omega_i\}_{i \in \mathbb{N}}$ of $\Omega$ by smooth relatively compact subdomains such that $x_0 \in \Omega_1 \setminus \overline{K}$ and such that $K \subset \Omega_{i-1} \Subset \Omega_i$ for all $i > 1$. Let $v_i$ be the unique positive solution of the Dirichlet problem

$$
\begin{align*}
Q'_{p,A,V - \lambda_1 |g|}[u] &= f_i \quad \text{in } \Omega_i \setminus \overline{K}, \\
u &= 0 \quad \text{on } \partial(\Omega_i \setminus \overline{K}),
\end{align*}
$$

where $f_i$ is a nonzero nonnegative function in $C^\infty_c(\Omega_i \setminus \Omega_{i-1})$ normalized in such a way that $v_i(x_0) = 1$. The existence and uniqueness of such a solution is guaranteed by [28, Theorem 3.10] combined with the fact that $Q'_{p,A,V - \lambda_1 |g|}[u] = 0$ admits a positive solution in $\Omega \setminus \overline{K}_0$.

By the Harnack principle and elliptic regularity (see for example, [28]) the sequence $\{v_i\}_{i \in \mathbb{N}}$ admits a subsequence converging locally uniformly to a positive solution $v$ of the equation $Q'_{p,A,V - \lambda_1 |g|}[u] = 0$ in $\Omega \setminus \overline{K}$ satisfying $v = 0$ on $\partial K$. Note that by classical regularity theory we have that $v$ is of class $C^\alpha$ up to $\partial K$. Moreover, for $p \leq N$, hypothesis (H0) implies that $v$ has a locally bounded gradient in $\Omega' \setminus K$.

Let $K_1$ be an open set such that $K \Subset K_1 \Subset \Omega'$ and let $\min_{x \in \partial K_1} v(x) = m > 0$. Fix $\varepsilon > 0$ satisfying $8\varepsilon < m$. Let $F$ be a $C^2$ function from $[0, +\infty]$ to $[0, +\infty]$ such that $F(t) = \varepsilon$ for all $0 \leq t \leq 2\varepsilon$ and $F(t) = t$ for all $t \geq 4\varepsilon$ and such that $F'(t) \neq 0$ for all $t > 2\varepsilon$. Assume also that $|F'(t)|^{p-2}F''(t) \to 0$ as $t \to 2\varepsilon$, hence the function $t \to |F'(t)|^{p-2}F''(t)$ (defined identically equal to zero on $[0, 2\varepsilon]$) is continuous on $[0, +\infty]$ (for this purpose, it is enough for example that $F$ is chosen to be of the type $\varepsilon + (t - 2\varepsilon)^\beta$ for all $t > 2\varepsilon$ sufficiently close to $2\varepsilon$ and $\beta > \max\{p', 2\}$).

We set $\bar{v}(x) := F(v(x))$ for all $x \in K_1 \cap (\Omega \setminus \overline{K})$. By the definition of $\bar{v}$, there exists an open neighborhood $U$ of $\partial K$ such that $\bar{v}(x) = \varepsilon$ for all $x \in U \cap (\Omega \setminus \overline{K})$, and there exists an open neighborhood $U_1$ of $\partial K_1$ such that $\bar{v}(x) = v(x)$ for all $x \in U_1 \cap K_1$. Thus $\bar{v}(x)$ can be extended continuously into the whole of $\Omega$ by setting $\bar{v}(x) = \varepsilon$ for all $x \in \overline{K}$ and $\bar{v}(x) = v(x)$ for all $x \in \Omega \setminus K_1$. By Lemma A.1 in Appendix A, we have in the weak sense that

$$
-\Delta_{p,A}\bar{v} = -|F'(v)|^{p-2}[(p-1)F''(v)|\nabla v|^p_A + F'(v)\Delta_{p,A}v] \quad \text{in } K_1 \setminus \overline{K}.
$$

(5.1)

By our assumptions on $F$ and $v$, it follows that $-\Delta_{p,A}\bar{v} \in L^1_{\text{loc}}(\Omega)$, and

$$
-\Delta_{p,A}\bar{v} = \begin{cases} 
0 & \text{in } K \cup (U \cap (\Omega \setminus \overline{K})), \\
(\lambda_1 g - V)v^{p-1} & \text{in } (\Omega \setminus \overline{K_1}) \cup (U_1 \cap K_1), \\
-|F'(v)|^{p-2}[(p-1)F''(v)|\nabla v|^p_A + F'(v)\Delta_{p,A}v] & \text{otherwise.}
\end{cases}
$$

Define the potential $\mathcal{V}$ by setting

$$
\mathcal{V} = \frac{|Q'_{p,A,V - \lambda_1 |g|}[\bar{v}]|}{\bar{v}^{p-1}}.
$$
Clearly, \( V \in L^1_+(\Omega) \) which for \( p > N \) means that \( V \in \mathcal{M}_c^p(p; \Omega) \). Moreover, for \( p \leq N \), Hypothesis (H0) implies that \( V \in \mathcal{M}_c^p(p; \Omega) \). Therefore, \( \bar{v} \) is a weak positive supersolution of \( Q'_{p,A,V-\lambda_1|g|+V}[u] = 0 \) in \( \Omega \). Hence, \( Q_{p,A,V-\lambda_1|g|+V} \geq 0 \) in \( \Omega \), and the Claim is proved.

We set \( \lambda_t = t\lambda_1 + (1-t)\lambda_0 \). By using [28], it follows that the set

\[
\{ (t,s) \in [0,1] \times \mathbb{R} \mid Q_{p,A,V-\lambda_t|g|+sV} \geq 0 \text{ in } \Omega \}
\]

is a convex set. Hence, the function \( \nu : [0,1] \rightarrow \mathbb{R} \) defined by

\[
\nu(t) := \min\{ s \in \mathbb{R} \mid Q_{p,A,V-\lambda_t|g|+sV} \geq 0 \text{ in } \Omega \}
\]

is convex. Since \( \mathcal{V} \) has compact support, [28, Proposition 4.19] implies that \( Q_{p,A,-\lambda_t|g|+\nu(t)V} \) is critical for all \( t \in [0,1] \). We note that by definition \( \nu(t) > 0 \) for all \( t \in (0,1) \), while \( \nu(0) \leq 0 \). Since \( \nu \) is convex, we have \( \nu(0) = 0 \), and hence, \( Q_{p,A,V-S_g(\Omega)|g|} \) is critical in \( \Omega \). □

**Definition 5.3 (Null sequence and ground state).** A nonnegative sequence \((\phi_n) \in W^{1,p}(\Omega) \cap C_c(\Omega)\) is called a null-sequence with respect to the nonnegative functional \( Q_{p,A,V} \) if

- there exists a subdomain \( O \subseteq \Omega \) such that \( \|\phi_n\|_{L^p(\Omega)} \approx 1 \) for all \( n \in \mathbb{N} \),
- \( \lim_{n \to \infty} Q_{p,A,V}(\phi_n) = 0 \).

We call a positive function \( \Phi \in W^{1,p}(\Omega) \cap C(\Omega) \) a ground state of \( Q_{p,A,V} \) if \( \phi \) is an \( L^p_{loc}(\Omega) \) limit of a null-sequence.

**Remark 5.4.** A nonnegative functional \( Q_{p,A,V} \) is critical in \( \Omega \) if and only if \( Q_{p,A,V} \) admits a null-sequence in \( \Omega \). Moreover, any null-sequence converges weakly in \( L^p_{loc}(\Omega) \) to the unique (up to a multiplicative constant) positive (super)solution of the equation \( Q'_{p,A,V}[u] = 0 \) in \( \Omega \). Furthermore, there exists a null-sequence which converges locally uniformly in \( \Omega \) to the ground state, and the ground state is a positive solution of the equation \( Q'_{p,A,V}[u] = 0 \) in \( \Omega \) which has minimal growth in a neighborhood of infinity in \( \Omega \) [15, 28].

**Lemma 5.5.** Let \((\phi_n) \in W^{1,p}(\Omega) \cap C_c(\Omega)\) be a null-sequence with respect to the nonnegative functional \( Q_{p,A,V} \), and let \( \Phi \in W^{1,p}(\Omega) \cap C(\Omega) \) be the corresponding ground state. For each \( n \in \mathbb{N} \), let \( \hat{\phi}_n = \min\{\phi_n, \Phi\} \). Then \((\hat{\phi}_n)\) is a null-sequence.

**Proof.** Clearly, \( \hat{\phi}_n \in W^{1,p}(\Omega) \cap C_c(\Omega) \) and \( Q_{p,A,V}[\hat{\phi}_n] \geq 0 \). We claim \( Q_{p,A,V}(\hat{\phi}_n) \leq Q_{p,A,V}(\phi_n) \). Consider the open set \( O_n := \{ x \in \Omega \mid \phi_n(x) > \Phi(x) \} \). Since \( Q_{p,A,V} \) is nonnegative and \( \phi \) is a ground state, it follows that \( Q'_{p,A,V}[\Phi] = 0 \) in \( \Omega \). By testing this equation with the nonnegative function \((\phi_n^p - \Phi^p)/\Phi^{p-1}\), we obtain

\[
0 \leq \int_{O_n} |\nabla \phi_n|^p_A \nabla \Phi \cdot \nabla \left( \frac{\phi_n^p}{\Phi^{p-1}} \right) \, dx + \int_{O_n} V \phi_n^p \, dx - Q_{p,A,V}(\Phi|_{O_n})
\]

\[
= p \int_{O_n} \left( \frac{\phi_n}{\Phi} \right)^{p-1} |\nabla \phi_n|^p_A \nabla \Phi \cdot \nabla \phi_n \, dx - (p-1) \int_{O_n} \left( \frac{\phi_n}{\Phi} \right)^p |\nabla \Phi|^p_A \, dx + \int_{O_n} V \phi_n^p \, dx - Q_{p,A,V}(\Phi|_{O_n})
\]

\[
(5.2)
\]

Now, by the Picone-type identity [15, Lemma 4.9] for \( Q_{p,A,V} \), we have

\[
L(\phi_n, \Phi) := |\nabla \phi_n|^p_A + (p-1) \frac{\phi_n^p}{\Phi^p} |\nabla \Phi|^p_A - p \left( \frac{\phi_n}{\Phi} \right)^{p-1} |\nabla \Phi|^p_A \nabla \Phi \cdot \nabla \phi_n \geq 0.
\]

\[
(5.3)
\]
Using the above Picone identity in (5.2) to get
\[
0 \leq \int_{O_n} |\nabla \phi_n|^p dx - \int_{O_n} L(\phi_n, \Phi)dx + \int_{O_n} V\phi_n^p dx - Q_{p,A,V}(\Phi|_{O_n})
\]
\[
\leq \int_{O_n} |\nabla \phi_n|^p dx + \int_{O_n} V\phi_n^p dx - Q_{p,A,V}(\Phi|_{O_n}) = Q_{p,A,V}(\phi_n|_{O_n}) - Q_{p,A,V}(\Phi|_{O_n}).
\]
Thus, \(Q_{p,A,V}(\hat{\phi}_n|_{O_n}) = Q_{p,A,V}(\Phi|_{O_n}) \leq Q_{p,A,V}(\phi_n|_{O_n})\). On the other hand, since \(\hat{\phi}_n = \phi_n\) on \(\Omega \setminus O_n\), our claim follows.

Now, since \(\lim_{n \to \infty} Q_{p,A,V}(\phi_n) = 0\), it follows that \(\lim_{n \to \infty} Q_{p,A,V}(\hat{\phi}_n) = 0\). By homogeneity of \(Q_{p,A,V}\), we may assume that \(\|\phi_n\|_{L^p(O)} = 1\) where \(O \Subset \Omega\) is the given open set. The dominated convergence theorem implies that \(\lim_{n \to \infty} \|\phi_n\|_{L^p(O)} = 1\), and \(\hat{\phi}_n \to \Phi\) in \(L^p_{\text{loc}}(\Omega)\). Thus, \((\hat{\phi}_n)\) is a null-sequence.

**Proof of Theorem 5.1.** Lemma 5.2 implies that \(Q_{p,A,V-S_g(\Omega)|g|}\) is critical in \(\Omega\), let \(\Phi\) be its ground state satisfying \(\Phi(x_0) = 1\), where \(x_0 \in \Omega\). By Lemma 5.5, the functional \(Q_{p,A,V-S_g(\Omega)|g|}\) admits a null-sequence \((\hat{\phi}_n)\) such that \(\hat{\phi}_n \leq \Phi\) for all \(n \geq 1\), and \(\hat{\phi}_n \to \Phi\) in \(L^p_{\text{loc}}(\Omega)\). We have
\[
\int_{\Omega} |\nabla \hat{\phi}_n|^p_A dx + \int_{\Omega} V_+|\hat{\phi}_n|^p_A dx = Q_{p,A,V-S_g(\Omega)|g|}(\hat{\phi}_n) + \int_{\Omega} V|\hat{\phi}_n|^p dx + S_g(\Omega) \int_\Omega |g|\hat{\phi}_n^p dx. \tag{5.4}
\]
Recall that \(\lim_{n \to \infty} Q_{p,A,V-S_g(\Omega)|g|}(\hat{\phi}_n) = 0\). Moreover, for \(S_g(\Omega) < s < S^\infty_g\) there exists \(K \Subset \Omega\) such that \(Q_{p,A,V-s|g|} \geq 0\) in \(\Omega \setminus K\). Therefore, by the Kovařík-Pinchover necessary condition (see Proposition 3.4), we obtain that \(\int_\Omega |g|\Phi^p dx < \infty\). Our assumption on \(V^-\) implies that \(\int_\Omega V^-\Phi^p dx < \infty\). Consequently, (5.4) implies that the sequence
\[
\left(\int_{\Omega} \left[|\nabla \hat{\phi}_n|^p_A + V_+|\hat{\phi}_n|^p_A\right] dx\right)
\]
is a bounded sequence, i.e., \(\hat{\phi}_n\) is bounded in \(D^{1,p}_{A,V+}(\Omega)\). Hence, \(\hat{\phi}_n \rightharpoonup \phi\) in \(D^{1,p}_{A,V+}(\Omega)\) for some \(\phi \in D^{1,p}_{A,V+}(\Omega)\). Further, since \(\hat{\phi}_n \to \Phi\) in \(L^p_{\text{loc}}(\Omega)\), it follows that \(\Phi = \phi \in D^{1,p}_{A,V+}(\Omega)\). Now, by letting \(n \to \infty\) in (5.4) and recalling (4.3), we get
\[
S_g(\Omega) \int_\Omega |g|\Phi^p dx \leq \int_\Omega \left[|\nabla \Phi|^p_A + V|\Phi|^p\right] dx \leq S_g(\Omega) \int_\Omega |g|\Phi^p dx.
\]
Consequently, the best constant \(S_g(\Omega)\) is attained at \(\Phi \in D^{1,p}_{A,V+}(\Omega)\). \(\square\)

**Lemma 5.6.** Let \(g \in H_p(\Omega, V) \cap M^q_{\text{loc}}(\Omega), g \neq 0\), satisfy \(S_g(\Omega) = S^\infty_g(\Omega)\). Then \(g \not\in H_{p,0}(\Omega, V)\). Moreover, if \(Q_{p,A,V}\) is subcritical in \(\Omega\), and \(p = 2\) or \(V = 0\), then a Hardy-weight \(g\) satisfying \(S_g(\Omega) = S^\infty_g(\Omega)\) exists and \(H_p(\Omega, V) \neq H_{p,0}(\Omega, V)\).

**Proof.** Without loss of generality, we may assume that \(g \geq 0\) and \(S_g(\Omega) = 1\). Further, by Theorem 1.2, we may take \(B_g(\Omega, V)\) as the norm on \(H_p(\Omega, V)\). Suppose that there is a nonnegative sequence \(H_p(\Omega, V) \cap L^\infty_c(\Omega)\) such that
\[
\lim_{n \to \infty} \|g_n - g\|_{H_p(\Omega, V)} = 0.
\]
Consider an exhaustion \((\Omega_n)\) of \(\Omega\) satisfying \(\text{supp} g_n \subseteq \Omega_n\). Recall that \(B_g(\Omega_1, V) \leq B_g(\Omega_2, V)\) if \(\Omega_1 \subset \Omega_2\). Therefore, in \(\Omega^*_n := \Omega \setminus \Omega_n\), we have
\[
\|g_n - g\|_{H_p(\Omega, V)} \geq \|g_n - g\|_{H_p(\Omega^*_n, V)} = \|g\|_{H_p(\Omega^*_n, V)} = B_g(\Omega^*_n, V),
\]
which contradicts the assumption on \(g_n\).

Now, if \(p = 2\) or \(V = 0\), then the optimal Hardy-weights \(W\) constructed in [10, 11, 39] satisfy the assumption \(S_W(\Omega) = S^*_W(\Omega)\).

For \(x \in \overline{\Omega}\) and \(g \in H_p(\Omega, V)\), define
\[
S_g(x, \Omega) := \liminf_{r \to 0} \left\{ Q_{p,A,V}(\phi) \mid \phi \in D_{A,V}^1(\Omega \cap B_r(x)), \int_{\Omega \cap B_r(x)} |g| |\phi|^p dx = 1 \right\},
\]
and let \(\Sigma_g := \{x \in \overline{\Omega} \mid S_g(x, \Omega) < \infty\}\), and \(S_g^*(\Omega) := \inf_{x \in \overline{\Omega}} S_g(x, \Omega)\). The following lemma claims that if \(g \in \mathcal{M}^q_{\text{loc}}(p, \Omega)\), then \(\Sigma_g \cap \Omega = \phi\).

**Lemma 5.7.** Let \(g \in \mathcal{M}^q_{\text{loc}}(p; \Omega)\), then for any \(x \in \Omega\) we have \(S_g(x, \Omega) = \infty\). In particular, \(\Sigma_g \cap \Omega = \phi\).

**Proof.** It is well known [18, Theorem 13.19] that
\[
C_0 r^{-p} = \inf \left\{ \int_{B_r(0)} |\nabla \phi|^p dx \mid \phi \in W^{1,p}(B_r(0)) \cap C^c(B_r(0)) \setminus \{0\} \right\},
\]
for some \(C_0 > 0\). Let \(g, V \in \mathcal{M}^q_{\text{loc}}(p; \Omega)\) and \(x \in \Omega\). Choose \(r > 0\) such that \(B_r := B_r(x) \subseteq \Omega\). Then, by Adams-Morrey inequality (Proposition 2.4), for any \(\varepsilon, \delta > 0\) there exists \(C(N, p, q), > 0\) such that for any \(\phi \in W^{1,p}(B_r(0)) \cap C^c(B_r(0))\)
\[
\int_{B_r} |V| |\phi|^p dx \leq \delta \int_{B_r} |\nabla \phi|^p dx + \frac{C(N, p, q)}{\varepsilon} \|V\|_{L^q_B(B_r(0))} \int_{B_r} |\phi|^p dx,
\]
and
\[
\int_{B_r} |g| |\phi|^p dx \leq \varepsilon \int_{B_r} |\nabla \phi|^p dx + \frac{C(N, p, q)}{\delta} \|g\|_{L^q_B(B_r)} \int_{B_r} |\phi|^p dx.
\]
Now, for any \(M > 0\), choose \(\varepsilon = \frac{\theta_{B_1}}{2M}\), where \(\theta_{B_1} > 0\) is the local uniform ellipticity constant of \(A\) on \(B_1\). Then, by (5.7), we have
\[
\int_{B_r} |g| |\phi|^p dx \leq \frac{\theta_{B_1}}{2M} \int_{B_r} |\nabla \phi|^p dx + \frac{C(N, p, q)}{\frac{\theta_{B_1}}{2M}} \|g\|_{L^q_B(B_r)} \int_{B_r} |\phi|^p dx.
\]
Use the local uniform ellipticity of \(A\), (5.6), and (5.8) to obtain
\[
\int_{B_r} |\nabla \phi|^p dx + \int_{B_r} V |\phi|^p dx - M \int_{B_r} |g| |\phi|^p dx \geq \left[\frac{\theta_{B_1}}{2} - \delta\right] \int_{B_r} |\nabla \phi|^p dx
\]
\[
- \frac{C(N, p, q)}{\delta} \|V\|_{L^q_B(B_r)} \int_{B_r} |\phi|^p dx - \frac{MC(N, p, q)}{\frac{\theta_{B_1}}{2M}} \|g\|_{L^q_B(B_r)} \int_{B_r} |\phi|^p dx
\]
\[
\geq \left(C_0 r^{-p} \left[\frac{\theta_{B_1}}{2} - \delta\right] - \frac{C(N, p, q)}{\delta} \|V\|_{L^q_B(B_r)} - \frac{MC(N, p, q)}{\frac{\theta_{B_1}}{2M}} \|g\|_{L^q_B(B_r)}\right) |\phi|^p_{L^p(B_r)},
\]
where the latter inequality follows from (5.5) and Remark 2.2-(ii). Choosing $0 < \delta < \frac{\theta B_1}{2}$, it follows that for any $M > 0$ there exists $C_1 > 0$ such that the right hand side of the above inequality is greater than $C_1 r^{-p} \| \phi \|_{L^p(B_r)}^p$ for sufficiently small $r > 0$. Hence, $S_g(x, \Omega) = \infty$. □

6. Attainment of the best constant II

In this section, we use concentration compactness arguments to provide another sufficient condition for the best constant in (1.2) to be attained in $D_{A,V}^{1,p}(\Omega)$.

Throughout this section we assume the following condition on $V$:

\((H1)\) : $V \in \mathcal{M}_q^{\text{loc}}(p, \Omega)$ is such that $V^- \in \mathcal{H}_{p,0}(\Omega, V^+)$.

By Theorem 4.3, if (H1) holds, then $T_{V^-}$ is compact in $D_{A,V}^{1,p}(\Omega)$. In addition, by (4.3), we have

$$
\frac{1}{B_g(\Omega)} = S_g(\Omega) = \inf \{ Q_{p,A,V}(\phi) \mid \phi \in D_{A,V}^{1,p}(\Omega), \int_\Omega |g| |\phi|^p \, dx = 1 \}.
$$

Let $\bigcup_n \Omega_n = \Omega$ be a compact smooth exhaustion of $\Omega$. Recall the definitions of $S_g(x, \Omega)$, $S_g^*(\Omega)$, and $\Sigma_g$. Motivated by [37, 38], we define

$$
S_g^\infty(\Omega) := \lim_{R \to \infty, B_R^c} \inf \left\{ Q_{p,A,V}(\phi) \mid \phi \in D_{A,V}^{1,p}(\Omega \cap B_R^c), \int_\Omega |g| |\phi|^p \, dx = 1 \right\},
$$

Clearly, $S_g(\Omega) \leq \min\{S_g^*(\Omega), S_g^\infty(\Omega)\}$. We make the following assumption on $g$:

\((H2)\) : $g \in \mathcal{H}_p(\Omega, V)$ with $|\Sigma_g| = 0$.

Now we state our result.

**Theorem 6.1.** Let $g, V$ satisfy (H1), (H2) with $S_g(\Omega) < \min\{S_g^*(\Omega), S_g^\infty(\Omega)\}$. Then $B_g(\Omega)$ is attained in $D_{A,V}^{1,p}(\Omega)$.

In order to prove Theorem 6.1, we need a $(g,V)$-depended concentration compactness lemma. For $V = 0$ and $A = I_{N \times N}$, analogous results are obtained in [5, 37, 38].

### 6.1. A variant of concentration compactness lemma.

Let $\mathcal{M}(\mathbb{R}^N)$ be the space of all Radon measures (i.e., regular, finite, Borel signed-measures). Recall that $\mathcal{M}(\mathbb{R}^N)$ is a Banach space with respect to the norm $||\mu|| = |\mu|(\mathbb{R}^N)$ (total variation of the measure $\mu$). By the Riesz representation theorem [2, Theorem 14.14], $\mathcal{M}(\mathbb{R}^N)$ is the dual of $C_0(\mathbb{R}^N) := \mathcal{C}_c(\mathbb{R}^N)$ in $L^\infty(\mathbb{R}^N)$. A sequence $(\mu_n)$ is said to be weak* convergent to $\mu$ in $\mathcal{M}(\mathbb{R}^N)$ if

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} \phi \, d\mu_n = \int_{\mathbb{R}^N} \phi \, d\mu \quad \forall \phi \in C_0(\mathbb{R}^N),
$$

and we denote $\mu_n \overset{*}{\rightharpoonup} \mu$. It follows from the Banach-Alaoglu theorem that if $(\mu_n)$ is a bounded sequence in $\mathcal{M}(\mathbb{R}^N)$, then (up to a subsequence) there exists $\mu \in \mathcal{M}(\mathbb{R}^N)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$. 

A function in $\mathcal{D}^{1,p}_A(\Omega)$ can be considered as a function in $\mathcal{D}^{1,p}_{A,V^+}(\mathbb{R}^N)$ by its zero extension. Following this convention, for $u_n, u \in \mathcal{D}^{1,p}_A(\Omega)$ and a Borel set $E$ in $\mathbb{R}^N$, we define the following sequences of measures:

$$\nu_n(E) := \int_E \|g\| u_n - u|^p dx, \quad \Gamma_n(E) := \int_E (|\nabla (u_n - u)|^p_A + V|u_n - u|^p) dx,$$

$$\tilde{\Gamma}_n(E) := \int_E (|\nabla u_n|^p_A + V|u_n|^p) dx,$$

where $g \in \mathcal{H}_p(\Omega, V)$. If $u_n \rightharpoonup u$ in $\mathcal{D}^{1,p}_A(\Omega)$, then by assumptions (H1) and (H2), the sequences $\nu_n, \Gamma_n, \text{ and } \tilde{\Gamma}_n$ have weak* convergent subsequences. Let

$$\nu_n \rightharpoonup^* \nu; \quad \Gamma_n \rightharpoonup^* \Gamma; \quad \tilde{\Gamma}_n \rightharpoonup^* \tilde{\Gamma} \quad \text{in } \mathcal{M}(\mathbb{R}^N).$$

**Proposition 6.2.** Then, for $p \in (1, \infty)$, there exists $C(\varepsilon, p) > 0$ such that

$$\|a + b|^p_A - |a|^p_A \leq \varepsilon |a|^p_A + C(\varepsilon, p)|b|^p \quad \forall a, b \in \mathbb{R}^N.$$  

**Proof.** Let $\varepsilon > 0$ and $p \in (1, \infty)$. Then there exists $C(\varepsilon, p) > 0$ such that

$$\|s + t|^p - |s|^p \leq \varepsilon |s|^p + C(\varepsilon, p)|t|^p \quad \forall s, t \in \mathbb{R}$$

(see [19], page 22). Now, for any $\theta \in [-1, 1]$, using (6.1) we obtain

$$\|s^2 + 2\theta s + 1|^\frac{p}{2} - s^p \leq \varepsilon |s|^p + C(\varepsilon, p) \quad \forall s \in \mathbb{R}$$

for some $C(\varepsilon, p) > 0$. By taking $s = \frac{|a|^p_A}{|b|^p_A}$ and $\theta = \frac{<A, b, b >}{|a|^p_A |b|^p}$ we obtain our claim.

**Lemma 6.3.** Let $\Phi \in C^1_b(\Omega)$ be such that $\nabla \Phi$ has compact support and $(u_n) \in W^{1,p}(\Omega) \cap C_c(\Omega)$ be such that $u_n \rightharpoonup u$ in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$. Then

$$\lim_{n \to \infty} \int_{\Omega} |\nabla ((u_n - u)\Phi)|^p_A dx = \lim_{n \to \infty} \int_{\Omega} |\nabla (u_n - u)|^p_A |\Phi|^p dx.$$

**Proof.** Let $\varepsilon > 0$ be given. Using Proposition 6.2,

$$\left| \int_{\Omega} |\nabla ((u_n - u)\Phi)|^p_A dx - \int_{\Omega} |\nabla (u_n - u)|^p_A |\Phi|^p dx \right| \leq \varepsilon \int_{\Omega} |\nabla (u_n - u)|^p_A |\Phi|^p dx + C(\varepsilon, p) \int_{\Omega} |u_n - u|^p |\nabla \Phi|^p_A dx.$$

Since $\nabla \Phi$ is compactly supported, it follows from a similar arguments as in Remark 4.1-(iii) that $\int_{\Omega} |u_n - u|^p |\nabla \Phi|^p_A dx \to 0$ as $n \to \infty$. Further, as $(u_n)$ is bounded in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$ and $\varepsilon > 0$ is arbitrary, we obtain the desired result.

Next, we prove the absolute continuity of the measure $\nu$ with respect to $\Gamma$.

**Lemma 6.4.** Let $V$ satisfy (H1), $g \in \mathcal{H}_p(\Omega, V)$, and $u_n \in W^{1,p}(\Omega) \cap C_c(\Omega)$ be such that $u_n \rightharpoonup u$ in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$. Then, for any Borel set $E$ in $\mathbb{R}^N$,

$$\mathbb{S}^*_g \nu(E) \leq \Gamma(E), \quad \text{where } \mathbb{S}^*_g \nu = \inf_{x \in \Omega} \mathbb{S}^*_g(x).$$

In particular, $\nu$ is supported on $\mathbb{S}^*_g$. 


Proof. For \( \Phi \in C_c^\infty(\mathbb{R}^N) \), \((u_n - u)\Phi \in \mathcal{D}^{1,p}_{A,V,+}(\Omega)\). Therefore, since \( g \in \mathcal{H}_p(\Omega,V) \), using Remark 4.2, we obtain

\[
\int_{\mathbb{R}^N} |\Phi|^p d\nu_n = \int_{\Omega} |g||(u_n - u)\Phi|^p dx \leq \mathcal{B}_g(\Omega) \int_{\Omega} [||\nabla((u_n - u)\Phi)||_A^p + V|(u_n - u)\Phi|^p] dx \\
= \mathcal{B}_g(\Omega) \int_{\mathbb{R}^N} [||\nabla((u_n - u)\Phi)||_A^p + V|(u_n - u)\Phi|^p] dx. \tag{6.2}
\]

By Proposition 6.2

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla((u_n - u)\Phi)|_A^p dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla(u_n - u)|_A^p dx.
\]

Thus, by taking \( n \to \infty \) in (6.2), we obtain

\[
\int_{\mathbb{R}^N} |\Phi|^p d\nu \leq \mathcal{B}_g(\Omega) \int_{\mathbb{R}^N} |\Phi|^p d\Gamma.
\]

Thus,

\[
\nu(E) \leq \mathcal{B}_g(\Omega) \Gamma(E) \quad \forall E \text{ Borel set in } \mathbb{R}^N.
\]

In particular, \( \nu \ll \Gamma \) and hence by Radon-Nikodym theorem,

\[
\nu(E) = \int_E \frac{d\nu}{d\Gamma} d\Gamma \quad \forall E \text{ Borel in } \mathbb{R}^N. \tag{6.3}
\]

Further, by Lebesgue differentiation theorem [12], we obtain

\[
\frac{d\nu}{d\Gamma}(x) = \lim_{r \to 0} \frac{\nu(B_r(x))}{\Gamma(B_r(x))}. \tag{6.4}
\]

Now replacing \( g \) by \( g\chi_{B_r(x)} \) and proceeding as before,

\[
\nu(B_r(x)) \leq \mathcal{B}_g(\Omega \cap B_r(x)) \Gamma(B_r(x)).
\]

Thus from (6.4) we get

\[
\frac{d\nu}{d\Gamma}(x) \leq \frac{1}{\mathcal{S}_g^*(x)} \leq \frac{1}{\mathcal{S}_g^*}. \tag{6.5}
\]

Hence, \( \|\frac{d\nu}{d\Gamma}\|_\infty \leq \frac{1}{\mathcal{S}_g^*} \), and using (6.3), \( \mathcal{S}_g^* \nu(E) \leq \Gamma(E) \) for all Borel subsets \( E \) of \( \mathbb{R}^N \). \( \square \)

The next lemma gives a lower estimate for the measure \( \tilde{\Gamma} \). For \( V = 0 \) and \( A = I_{N\times N} \), similar estimate is obtained in [37, Lemma 2.1].

**Lemma 6.5.** Let \( V, g \) satisfy (H1), (H2). If \( u_n \in W^{1,p}(\Omega) \cap C(\Omega) \) is such that \( u_n \rightharpoonup u \) in \( \mathcal{D}^{1,p}_{A,V,+}(\Omega) \), then \( \tilde{\Gamma} \geq |\nabla u|_A^p + V|u|^p + \mathcal{S}_g^* \nu \).

**Proof.** Our proof splits into three steps.

**Step 1:** \( \tilde{\Gamma} \geq |\nabla u|_A^p + V|u|^p \). Indeed, let \( \phi \in C_c^\infty(\mathbb{R}^N) \) with \( 0 \leq \phi \leq 1 \), we need to show that \( \int_{\mathbb{R}^N} \phi d\tilde{\Gamma} \geq \int_{\mathbb{R}^N} \phi [||\nabla u||_A^p + V|u|^p] dx \). Notice that,

\[
\int_{\mathbb{R}^N} \phi d\tilde{\Gamma} = \lim_{n \to \infty} \int_{\mathbb{R}^N} \phi d\tilde{\Gamma}_n = \lim_{n \to \infty} \int_{\Omega} \phi [||\nabla u_n||_A^p + V|u_n|^p] dx \\
= \lim_{n \to \infty} \int_{\Omega} [F(x,u_n(x),\nabla u_n(x)) - V^-|u_n|^p] dx,
\]
where \( F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is defined as \( F(x, r, z) = \phi(x)[|z|_{A(x)}^p + V^+(x)r^p] \). Clearly, \( F \) is a Carathéodory function and \( F(x, r, \cdot) \) is convex for almost every \((x, r) \in \Omega \times \mathbb{R}\). Hence, by [34, Theorem 2.11], we have
\[
\int_{\mathbb{R}^N} \phi \left[ |\nabla u|_A^p + V^+|u|^p \right] dx = \int_{\Omega} \phi \left[ |\nabla u|_A^p + V^+|u|^p \right] dx \\
\leq \liminf_{n \to \infty} \int_{\Omega} \phi \left[ |\nabla u_n|_A^p + V^+|u_n|^p \right] dx.
\]

On the other hand, \( V^- \in \mathcal{H}_{p,0}(\Omega, V^+) \) implies that \( V^- \phi \in \mathcal{H}_{p,0}(\Omega, V^+) \). Hence, by Theorem 4.3 we have \( \lim_{n \to \infty} \int_{\Omega} V^- \phi |u_n|^p dx = \int_{\Omega} V^- \phi |u|^p dx \). Combining these two facts we obtain Claim 1.

**Step 2:** \( \tilde{\Gamma} = \Gamma \) on \( \Sigma_g \). Indeed, let \( E \subset \Sigma_g \) be a Borel set. Thus, for each \( m \in \mathbb{N} \), there exists an open subset \( \Omega_m \) containing \( E \) such that \( |\Omega_m| = |\Omega_m \setminus E| < \frac{1}{m} \). Let \( \varepsilon > 0 \) be given. Using Lemma 6.3 it follows that for any \( \phi \in C^\infty_c(\Omega_m) \) with \( 0 \leq \phi \leq 1 \), we have
\[
\left| \int_{\Omega} \phi \, d\Gamma - \int_{\Omega} \phi \, d\tilde{\Gamma} \right| = \left| \int_{\Omega} \phi \left[ |\nabla (u_n - u)|_A^p + V|u_n - u|^p \right] dx - \int_{\Omega} \phi \left| |\nabla u_n|_A^p + V|u_n|^p \right| dx \right| \\
\leq \int_{\Omega} \phi \left[ |\nabla (u_n - u)|^p_A - |\nabla u_n|^p_A \right] dx + \int_{\Omega} \phi |V^+| |u_n - u|^p dx + \int_{\Omega} \phi |V^-| |u_n - u|^p dx - \int_{\Omega} \phi |u_n|^p dx \\
\leq \left[ \varepsilon \int_{\Omega} \phi |\nabla u_n|^p dx + C_1(\varepsilon, p) \int_{\Omega} \phi |\nabla u|^p dx \right] + \left[ \varepsilon \int_{\Omega} \phi |V^+| |u_n|^p dx + C_2(\varepsilon, p) \int_{\Omega} \phi |V^+| |u|^p dx \right] \\
+ \left[ \varepsilon \int_{\Omega} \phi |V^-| |u|^p dx + C_3(\varepsilon, p) \int_{\Omega} \phi |V^-| |u|^p dx \right] \leq \varepsilon C_4 L + C(\varepsilon, p) \int_{\Omega_m} [|\nabla u|^p_A + V^+|u|^p] dx,
\]
where \( L = \sup_{x \in \Omega} \{|\nabla u|^p_A + V^+|u|^p| dx\} \), and in the latter inequality we used the fact that \( V^- \in \mathcal{H}_{p,0}(\Omega, V^+) \). Now letting \( n \to \infty \), we obtain
\[
\left| \int_{\Omega} \phi \, d\Gamma - \int_{\Omega} \phi \, d\tilde{\Gamma} \right| \leq \varepsilon C_4 L + C(\varepsilon, p) \int_{\Omega_m} [|\nabla u|^p_A + V^+|u|^p] dx.
\]
Therefore,
\[
\left| \Gamma(\Omega_m) - \tilde{\Gamma}(\Omega_m) \right| = \sup \left\{ \left| \int_{\Omega} \phi \, d\Gamma - \int_{\Omega} \phi \, d\tilde{\Gamma} \right| \mid \phi \in C^\infty_c(\Omega_m), 0 \leq \phi \leq 1 \right\} \\
\leq \varepsilon C_4 L + C(\varepsilon, p) \int_{\Omega_m} [|\nabla u|^p_A + V^+|u|^p] dx,
\]
Now as \( m \to \infty \), \( |\Omega_m| \to 0 \) (since \( |\Sigma_g| = 0 \)), and hence, \( |\Gamma(E) - \tilde{\Gamma}(E)| \leq \varepsilon C_4 L \). Since \( \varepsilon > 0 \) is arbitrary, we conclude \( \Gamma(E) = \tilde{\Gamma}(E) \).

**Step 3:** \( \tilde{\Gamma} \geq |\nabla u|^p_A + V|u|^p + S_g^* \nu \). From Lemma 6.4 we have \( \Gamma \geq S_g^* \nu \). Furthermore, by Lemma 6.4, \( \nu \) is supported on \( \Sigma_g \). Hence Step 1 and Step 2 yield the following:
\[
\tilde{\Gamma} \geq \left\{ \begin{array}{ccc}
\left| \nabla u|_A^p + V|u|^p, \\
S_g^* \nu.
\end{array} \right\}
\] (6.6)
Since $|\Sigma_\nu| = 0$, the measure $|\nabla u|^p_A + V|u|^p$ is supported in $\Sigma_\nu^c$, and hence from (6.6) we easily obtain $\tilde{\Gamma} \geq |\nabla u|^p_A + V|u|^p + S_\nu^g \nu$.

**Lemma 6.6.** Suppose that $V$ satisfies (H1), $g \in H_p(\Omega, V)$ is nonnegative. Let $u_n \in W^{1,p}(\Omega) \cap C_c(\Omega)$ be such that $u_n \rightharpoonup u$ in $D_{A,V}^{1,p}(\Omega)$, and $\Phi_R \in C_b^{\infty}(\mathbb{R}^N)$ with $0 \leq \Phi_R \leq 1$, $\Phi_R = 0$ on $B_R$ and $\Phi_R = 1$ on $B_R^c$. Then,

(A) $\lim_{R \to \infty} \lim_{n \to \infty} \int_{\Omega \cap B_R^c} g|u_n|^p \, dx = \lim_{R \to \infty} \lim_{n \to \infty} \int_{\Omega} \nabla (\Phi_R \nu_n) \, \nu_n \, dx = \lim_{R \to \infty} \lim_{n \to \infty} \int_{\Omega} \Phi_R \, d\nu_n$,

(B) $\lim_{R \to \infty} \lim_{n \to \infty} \int_{\Omega \cap B_R^c} \left[ |\nabla u|^p + V|u|^p \right] \, dx = \lim_{R \to \infty} \lim_{n \to \infty} \int_{\Omega} \nabla (\Phi_R \Gamma_n) \, \nu_n \, dx = \lim_{R \to \infty} \lim_{n \to \infty} \int_{\Omega} \Phi_R \, d\Gamma_n$.

**Proof.** By Brezis-Lieb lemma [19, Theorem 1.9],

$$\left| \lim_{n \to \infty} \nu_n (\Omega \cap B_R^c) - \lim_{n \to \infty} \int_{\Omega \cap B_R^c} g|u_n|^p \, dx \right| \leq \lim_{n \to \infty} \left| \nu_n (\Omega \cap B_R^c) - \int_{\Omega \cap B_R^c} g|u_n|^p \, dx \right| \leq \lim_{n \to \infty} \int_{\Omega \cap B_R^c} |\nabla u|^p \, dx - \int_{\Omega \cap B_R^c} g|u_n|^p \, dx = \int_{\Omega \cap B_R^c} g|u|^p \, dx.$$  

By (4.3), $g|u|^p \in L^1(\Omega)$, therefore, the right-hand side integral goes to 0 as $R \to \infty$. Thus, we get the first equality in (A). For the second equality, it is enough to observe that

$$\int_{\Omega \cap B_R^c} g|u_n - u|^p \, dx \leq \int_{\Omega} g|u_n - u|^p \Phi_R \, dx \leq \int_{\Omega \cap B_R^c} g|u_n - u|^p \, dx.$$  

Now by taking $n \to \infty$, and then $R \to \infty$, we get the required equality.

Now we proceed to prove (B). For $\varepsilon > 0$, using Lemma 6.3 we estimate

$$\lim_{n \to \infty} \left| \Gamma_n (\Omega \cap B_R^c) - \int_{\Omega \cap B_R^c} \left[ |\nabla u|^p_A + V|u|^p \right] \, dx \right|$$  

$$= \lim_{n \to \infty} \left| \int_{\Omega \cap B_R^c} \left[ |\nabla (u_n - u)|^p_A + V|u_n - u|^p \right] \, dx - \int_{\Omega \cap B_R^c} \left[ |\nabla u|^p_A + V|u|^p \right] \, dx \right|$$  

$$\leq \lim_{n \to \infty} \left( \int_{\Omega \cap B_R^c} \left[ |\nabla (u_n - u)|^p_A - |\nabla u|^p_A \right] \, dx + \int_{\Omega \cap B_R^c} \left[ |\nabla u|^p - V\Phi_R |u|^p \right] \, dx \right)$$  

$$= \left( \varepsilon \lim_{n \to \infty} \int_{\Omega \cap B_R^c} |\nabla u|^p_A \, dx + C_1(\varepsilon, p) \int_{\Omega \cap B_R^c} |\nabla u|^p_A \, dx \right)$$  

$$+ \left( \varepsilon \lim_{n \to \infty} \int_{\Omega \cap B_R^c} (V^+ + V^-)|u|^p \, dx + C_2(\varepsilon, p) \int_{\Omega \cap B_R^c} (V^+ + V^-)|u|^p \, dx \right)$$  

$$\leq \varepsilon C_3 L + C_4(\varepsilon, p) \int_{\Omega \cap B_R^c} \left[ |\nabla u|^p_A + V^+ |u|^p \right] \, dx,$$

where $L = \sup_n \left\{ \int_{\Omega} \left[ |\nabla u|^p_A + V^+ |u|^p \right] \, dx \right\}$. The latter inequality uses once again the fact that $V^+ \in H_p(\Omega, V^+)$. Thus, due to the subadditivity of the limsup, and by taking $R \to \infty$ and then $\varepsilon \to 0$, we obtain the first equality of (B). The second equality of part (B) follows from the same argument as the corresponding part of (A). \qed
Lemma 6.7. Let $V$ satisfy $(H1)$, $g \in \mathcal{H}_p(\Omega, V)$ be nonnegative, and $u_n \in W^{1,p}(\Omega) \cap C_c(\Omega)$ be such that $u_n \to u$ in $\mathcal{D}^{1,p}_{A,V+}(\Omega)$. Set

$$\nu_\infty = \lim_{R \to \infty} \lim_{n \to \infty} \nu_n(\Omega \cap B_R^c) \quad \text{and} \quad \Gamma_\infty = \lim_{R \to \infty} \lim_{n \to \infty} \Gamma_n(\Omega \cap B_R^c).$$

Then

(i) $\mathcal{S}_g^\infty \nu_\infty \leq \Gamma_\infty$,

(ii) $\lim_{n \to \infty} \int_{\Omega} g|u_n|^p dx = \int_{\Omega} g|u|^p dx + \|\nu\|_\infty + \nu_\infty$.

(iii) Further, if $|\Sigma_g| = 0$, then we have

$$\lim_{n \to \infty} \int_{\Omega} [\nabla u_n]^p_A + V|u_n|^p dx \geq \int_{\Omega} [\nabla u]^p_A + V|u|^p dx + \mathcal{S}_g^\infty \|\nu\|_\infty + \Gamma_\infty.$$

Proof. (i): For $R > 0$, choose $\Phi_R \in C^1_b(\mathbb{R}^N)$ satisfying $0 \leq \Phi_R \leq 1$, $\Phi_R = 0$ on $\overline{B_{R+1}}$, and $\Phi_R = 1$ on $B_{R+1}^c$. Clearly, $(u_n - u)\Phi_R \in \mathcal{D}^{1,p}_{A,V+}(\Omega \cap \overline{B_R})$, and since $g \in \mathcal{H}_p(\Omega, V)$, we have

$$\int_{\Omega \cap \overline{B_R}} g((u_n - u)\Phi_R)^p dx \leq \mathcal{B}_g(\Omega \cap \overline{B_R}) \left[ \int_{\Omega \cap \overline{B_R}} \nabla ((u_n - u)\Phi_R)^p dx + \int_{\Omega \cap \overline{B_R}} V|u_n - u|^p dx \right].$$

By Lemma 6.3

$$\lim_{n \to \infty} \int_{\Omega \cap \overline{B_R}} |\nabla ((u_n - u)\Phi_R)|_A^p dx = \lim_{n \to \infty} \int_{\Omega \cap \overline{B_R}} |\nabla (u_n - u)|_A^p \Phi_R^p dx.$$

Therefore, letting $n \to \infty$, $R \to \infty$, and using Lemma 6.6 successively in the above inequality, we obtain $\mathcal{S}_g^\infty \nu_\infty \leq \Gamma_\infty$.

(ii): Choosing $\Phi_R$ as in Lemma 6.6 and using Brezis-Lieb lemma [19, Theorem 1.9], we have

$$\lim_{n \to \infty} \int_{\Omega} g|u_n|^p dx = \lim_{n \to \infty} \int_{\Omega} g|u|^p (1 - \Phi_R) dx + \int_{\Omega} g|u_n|^p \Phi_R dx$$

$$= \lim_{n \to \infty} \left[ \int_{\Omega} g|u|^p (1 - \Phi_R) dx + \int_{\Omega} g|u_n - u|^p (1 - \Phi_R) dx + \int_{\Omega} g|u_n|^p \Phi_R dx \right]$$

$$= \int_{\Omega} g|u|^p (1 - \Phi_R) dx + \int_{\Omega} (1 - \Phi_R) d\nu + \lim_{n \to \infty} \int_{\Omega} g|u_n|^p \Phi_R dx \quad (6.7)$$

The last equality uses the facts that $g|u_n - u|^p \xrightarrow{a} \nu$, and $\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$ for any real sequences $(a_n), (b_n)$ with $(a_n)$ being convergent. Now, by taking $R \to \infty$ in (6.7) and using Lemma 6.6-(A), we obtain

$$\lim_{n \to \infty} \int_{\Omega} g|u_n|^p dx = \int_{\Omega} g|u|^p dx + \|\nu\|_\infty + \nu_\infty.$$

(iii): Notice that

$$\lim_{n \to \infty} \int_{\Omega} [\nabla u_n]^p_A + V|u_n|^p dx$$

$$= \lim_{n \to \infty} \left[ \int_{\Omega} [\nabla u_n]^p_A + V|u_n|^p (1 - \Phi_R) dx + \int_{\Omega} [\nabla u_n]^p_A + V|u_n|^p \Phi_R dx \right]$$

$$= \int_{\Omega} (1 - \Phi_R) \Gamma + \lim_{n \to \infty} \int_{\Omega} [\nabla u_n]^p_A + V|u_n|^p \Phi_R dx.$$
The last equality uses the facts that \(|\nabla u_n|^p + V|u_n|^p \leq \tilde{\Gamma}\), and \(\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n\) for any real sequences \((a_n), (b_n)\) with \((a_n)\) being convergent. By taking \(R \to \infty\) and using part (B) Lemma 6.6 we get
\[
\lim_{n \to \infty} \int_{\Omega} \|\nabla u_n\|_A^p + V|u_n|^p \, dx = \|\tilde{\Gamma}\| + \Gamma_\infty.
\]
Now, using Lemma 6.5, we obtain
\[
\lim_{n \to \infty} \int_{\Omega} \|\nabla u_n\|_A^p + V|u_n|^p \, dx \geq \int_{\Omega} \|\nabla u\|_A^p + V|u|^p \, dx + \mathbb{S}_g^* \|\nu\| + \Gamma_\infty. \tag{6.8}
\]

**Proof of Theorem 6.1.** Recall that, for \(g \in H_p(\Omega, V)\), the best constant \(B_g(\Omega)\) in (1.2) is given by the Rayleigh-Ritz variational principle
\[
\frac{1}{B_g(\Omega)} = \inf \left\{ \mathcal{R}_V(\phi) := \frac{\int_{\Omega} \|\nabla \phi\|_A^p + V|\phi|^p \, dx}{\int_{\Omega} g||\phi||^p \, dx} \mid \phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \setminus \{0\} \right\}
\]
\[
= \inf \left\{ \mathcal{R}_V(\phi) \mid \phi \in W^{1,p}(\Omega) \cap C_c(\Omega), \int_{\Omega} \|\nabla \phi\|_A^p + V|\phi|^p \, dx = 1 \right\},
\]
where in the latter equality we used the homogeneity of the Rayleigh quotient. Let \((u_n)\) be a sequence that minimizes \(\mathcal{R}_V(\phi)\) over \(\{\phi \mid \phi \in W^{1,p}(\Omega) \cap C_c(\Omega), \int_{\Omega} \|\nabla \phi\|_A^p + V|\phi|^p = 1\}\).

Then, one can see that
\[
\limsup_{n \to \infty} \int_{\Omega} g||u_n||^p \, dx \geq B_g(\Omega) \lim_{n \to \infty} \int_{\Omega} \|\nabla u_n\|_A^p + V|u_n|^p \, dx. \tag{6.8}
\]
Since \(\int_{\Omega} \|\nabla u_n\|_A^p + V|u_n|^p \, dx = 1\) for all \(n \in \mathbb{N}\), i.e., \((u_n)\) is bounded in \(D_{A,V}^{1,p}(\Omega)\), it follows that \(u_n \rightharpoonup u\) in \(D_{A,V}^{1,p}(\Omega)\) (up to a subsequence). \(V^- \in H_{p,0}(\Omega, V^+)\) implies that
\[
\lim_{n \to \infty} \int_{\Omega} V^-|u_n|^p \, dx = \int_{\Omega} V^-|u|^p \, dx \quad \text{(by Remark 4.4-(i))}. \]
Thus, we obtain
\[
\lim_{n \to \infty} \int_{\Omega} \|\nabla u_n\|_A^p + V|u_n|^p \, dx = \lim_{n \to \infty} \int_{\Omega} \|\nabla u_n\|_A^p + V|u_n|^p \, dx - \lim_{n \to \infty} \int_{\Omega} V^-|u_n|^p \, dx. \tag{6.9}
\]
Recall our assumptions that \(u_n \rightharpoonup u\) in \(D_{A,V}^{1,p}(\Omega)\) and that up to a subsequence,
\[
\|\nabla u_n - \nabla u\| + V|u_n - u|^p \rightharpoonup \tilde{\Gamma}, \quad \|\nabla u_n|^p + V|u_n|^p \rightharpoonup \tilde{\Gamma}, \quad |g||u_n - u|^p \rightharpoonup \nu \text{ in } M(\mathbb{R}^N).
\]
Using Lemma 6.7 we get
\[
\limsup_{n \to \infty} \int_{\Omega} g||u_n||^p \, dx = \int_{\Omega} g||u||^p \, dx + \|\nu\| + \nu_\infty. \tag{6.10}
\]
Suppose that \(\|\nu\| + \nu_\infty \neq 0\). Recalling that the Hardy-type inequality holds for \(u \in D_{A,V}^{1,p}(\Omega)\) (by Remark 4.2-(i)), and using (6.9), Lemma 6.7 (iii), the spectral gap assumption,
and finally (6.10), we obtain
\[
\lim_{n \to \infty} \int_{\Omega} |g||u_n|^p \, dx \geq B_g(\Omega) \lim_{n \to \infty} \int_{\Omega} ||\nabla u_n|^p + V|u_n|^p| \, dx
\]
\[
\geq B_g(\Omega) \lim_{n \to \infty} \int_{\Omega} ||\nabla u_n|^p + V|u_n|^p| \, dx \geq B_g(\Omega) \left( \int_{\Omega} ||\nabla u|^p + V|u|^p| \, dx + S^*_g ||\nu|| + \Gamma_\infty \right)
\]
\[
\geq B_g(\Omega) \left( \frac{1}{B_g(\Omega)} \int_{\Omega} |g||u|^p \, dx + ||\nu|| + \nu_\infty \right) = \lim_{n \to \infty} \int_{\Omega} |g||u_n|^p \, dx,
\]
which is a contradiction. Thus \( ||\nu|| = \nu_\infty = 0 \). Therefore, \( \lim_{n \to \infty} \int_{\Omega} |g||u_n|^p \, dx = \int_{\Omega} |g||u|^p \, dx \).

Further, since \( u_n \to u \) in \( D^{1,p}_A(\Omega) \), we have
\[
\int_{\Omega} ||\nabla u|^p + V^+|u|^p| \, dx \leq \lim_{n \to \infty} \int_{\Omega} ||\nabla u_n|^p + V^+|u_n|^p| \, dx.
\]
Also, the assumption \( V^- \in \mathcal{H}_{p,0}(\Omega, V^+) \) implies that \( \lim_{n \to \infty} \int_{\Omega} V^-|u_n|^p \, dx = \int_{\Omega} V^-|u|^p \, dx \) (by Remark 4.4-(iii)). Consequently, \( \mathcal{R}(u) \), the Rayleigh-Ritz quotient of \( u \), satisfies
\[
\frac{1}{B_g(\Omega)} \leq \mathcal{R}(u) \leq \frac{\lim_{n \to \infty} \int_{\Omega} ||\nabla u_n|^p + V|u_n|^p| \, dx}{\lim_{n \to \infty} \int_{\Omega} |g||u_n|^p \, dx} \leq \lim_{n \to \infty} \mathcal{R}(u) = \frac{1}{B_g(\Omega)}.
\]
Hence, \( B_g(\Omega) \) is attained at \( u \).

**Remark 6.8.** Theorems 5.1 and 6.1 provide different sufficient conditions for the attainment of the best constant of Hardy-type equalities for \( Q_{p,A,V} \). Let us compare these conditions. In Theorem 6.1, we assume that \( V \in \mathcal{M}_{\text{loc}}^0(p; \Omega) \) with \( V^- \in \mathcal{H}_{p,0}(\Omega, V^+) \). Consider for example the operator \( -\Delta_p(u) - \lambda|x|^{-p} \) in \( \Omega = \mathbb{R}^N \setminus \{0\} \) with \( 0 < \lambda < ((p-1)/p)^p \), and take \( 0 \leq g \in C_c^\infty(\Omega) \). Then \( \mathcal{R}(u) = \frac{\lim_{n \to \infty} \int_{\Omega} ||\nabla u_n|^p + V|u_n|^p| \, dx}{\lim_{n \to \infty} \int_{\Omega} |g||u_n|^p \, dx} \leq \lim_{n \to \infty} \mathcal{R}(u) \). On the other hand, in view of Lemma 5.7, if \( g \in \mathcal{M}_{\text{loc}}^0(p; \Omega) \), then \( \Sigma^*_g \cap \Omega = \emptyset \). Thus, Theorem 6.1 allows stronger local singularities of \( g \) than considered on Theorem 5.1.

Next, we characterize the Hardy-weights \( g \) such that \( T_g \) is compact on \( D^{1,p}_{A,V}(\Omega) \) in the case \( V = V^+ \geq 0 \).

**Theorem 6.9.** Let \( V \) be nonnegative, and \( g \in \mathcal{H}_p(\Omega, V) \), \( g \neq 0 \). Then \( T_g \) is compact on \( D^{1,p}_{A,V} \) if and only if \( \Sigma^*_g = \Sigma^\infty_g = \infty \).

**Proof.** Assume that \( T_g \) is compact on \( D^{1,p}_{A,V}(\Omega) \), and on the contrary there exists \( x \in \overline{\Omega} \) such that \( \Sigma_g(x) < \infty \). We may assume that \( |g| > 0 \) in a neighborhood of \( \Omega \cap B_1/n_0(x) \). By the definition of \( \Sigma_g(x) \), for each \( n \in \mathbb{N} \) large enough there exists \( \phi_n \in D^{1,p}_{A,V}(\Omega \cap B_1/n(x)) \) such that
\[
Q_{p,A,V}(\phi_n) < \Sigma_g(x) + \frac{1}{n}, \quad \text{and} \quad \int_{\Omega \cap B_1/n(x)} |g||\phi_n|^p \, dx = 1.
\]
This implies that \((\phi_n)\) is a bounded sequence in \(D_{A,V}^{1,p}(\Omega)\), and hence, \(\phi_n \rightharpoonup \phi\) in \(D_{A,V}^{1,p}(\Omega)\) (up to a subsequence). Clearly, \(\phi = 0\) as the supports of \(\phi_n\) are shrinking to a singleton set \(\{x\}\). Thus, by the compactness of \(T_g\), it follows that \(\lim_{n \to \infty} \int_\Omega |g||\phi_n|^pdx = 0\), which is a contradiction. Therefore, \(S_g(x) = \infty\). Since \(x \in \overline{\Omega}\) is arbitrary, we infer that \(S_g = \infty\).

Following a similar arguments, one shows that \(S_\infty = \infty\).

Conversely, assume that \(g \in \mathcal{H}_p(\Omega, V)\) satisfies \(S_\infty = S_g = \infty\). Let \((\phi_n)\) be a sequence in \(D_{A,V}^{1,p}(\Omega)\) such that \(\phi_n \rightharpoonup \phi\) in \(D_{A,V}^{1,p}(\Omega)\). Then, using Lemma 6.4, we conclude that \(\|\nu\| = 0\) and using Lemma 6.7-(i), we have \(\nu_\infty = 0\). Thus, by Lemma 6.7-(ii) and Fatou’s lemma, we obtain

\[
\lim_{n \to \infty} \int_\Omega |g||\phi_n|^pdx = \int_\Omega |g||\phi|^pdx \leq \lim_{n \to \infty} \int_\Omega |g||\phi_n|^pdx.
\]

Hence, \(\lim_{n \to \infty} \int_\Omega |g||\phi_n|^pdx = \int_\Omega |g||\phi|^pdx\). This proves that \(T_g\) is compact on \(D_{A,V}^{1,p}(\Omega)\).

Finally, we provide a way of producing Hardy-weights for which the best constant in (1.2) is attained.

**Proposition 6.10.** Let \(V\) satisfy (H1), and let \(g_0 \in \mathcal{H}_p(\Omega, V)\) be a nonnegative and nonzero function satisfying \(S_{g_0}^* = S_{g_0}^\infty = \infty\). Then, for any nonnegative and nonzero \(g \in \mathcal{H}_p(\Omega, V)\), and \(\varepsilon > 0\) satisfying \(\varepsilon > S_{g_0}/S_g\), we have \(S_{g + \varepsilon g_0} < \min\{S_{g + \varepsilon g_0}^*, S_{g + \varepsilon g_0}^\infty\}\).

Moreover, if for such an \(\varepsilon\), the potential \(g + \varepsilon g_0\) satisfies (H2), then \(S_{g + \varepsilon g_0}\) is achieved at some \(\psi \in D_{A,V}^{1,p}(\Omega)\).

**Remark 6.11.** If \(g_0\) and \(g\) are nonzero, nonnegative functions in \(\mathcal{M}_{loc}^q(p;\Omega) \cap \mathcal{H}_p(\Omega, V)\), and \(g_0\) has a compact support in \(\Omega\), then the first part of the theorem holds for \(\varepsilon > S_{g_0}/S_g\).

**Proof of Proposition 6.10.** First, we claim that \(S_g^* = S_{g + g_0}^*\). For any \(x \in \overline{\Omega}\), we have

\[
\int_\Omega (g + g_0)|\phi|^pdx \leq \frac{1}{S_{g_0}(\Omega \cap B_r(x))} + \frac{1}{S_{g_0}(\Omega \cap B_r(x))} Q_{p,A,V}(\phi) \quad \forall \phi \in D_{A,V}^{1,p}(\Omega \cap B_r(x)).
\]

Thus,

\[
\frac{1}{S_{g+g_0}(\Omega \cap B_r(x))} \leq \frac{1}{S_{g_0}(\Omega \cap B_r(x))} + \frac{1}{S_{g_0}(\Omega \cap B_r(x))}.
\]

By taking \(r \to 0\), we get \(S_g(x, \Omega) \leq S_{g+g_0}(x, \Omega)\) (as \(S_{g_0}(x, \Omega) = \infty\) for all \(x\)). Obviously, \(S_{g+g_0}(x, \Omega) \leq S_g(x, \Omega)\) (as \(g, g_0\) are nonnegative). Hence, \(S_g(x, \Omega) = S_{g+g_0}(x, \Omega)\) for all \(x \in \mathbb{R}^N\). Consequently, \(S_g^* = S_{g+g_0}^*\). Repeating similar arguments, it follows that \(S_g^\infty = S_{g+g_0}^\infty\).

Now, we are ready to prove the proposition. By Theorem 6.1, \(S_{g_0}\) is achieved at some \(\varphi \in D_{A,V}^{1,p}(\Omega)\). For \(\varepsilon > S_{g_0}/S_g\), we have

\[
S_{g+\varepsilon g_0} \leq \frac{Q_{p,A,V}(\varphi)}{\varepsilon \int_\Omega (g + \varepsilon g_0)|\varphi|^pdx} \leq \frac{Q_{p,A,V}(\varphi)}{\varepsilon \int_\Omega g_0|\varphi|^pdx} = \frac{S_{g_0}}{\varepsilon} < S_g.
\]

Therefore,

\[
S_{g+\varepsilon g_0} < S_g \leq \min\{S_g^*, S_g^\infty\} = \min\{S_{g+\varepsilon g_0}^*, S_{g+\varepsilon g_0}^\infty\}.
\]

Moreover, if \(g + \varepsilon g_0\) satisfies (H2), then the second assertion of the proposition follows since \(g + \varepsilon g_0\) satisfies the assumptions of Theorem 6.1. □
Appendix A. Auxiliary Lemmas

The following lemma is an extension of [11, Lemma 2.10] to the \((p, A)\)-Laplacian case.

**Lemma A.1.** Let \(0 < u \in W^{1,p}_\text{loc} \cap C(\Omega)\). Let \(F \in C^2(\mathbb{R}_+)\) satisfy \(F' \geq 0\), and for \(p < 2\) assume further that \(F'(s)^{p-2}F''(s) \to 0\) as \(s \to s_0\) where \(s_0\) is any critical point of \(F\). Then the following formula holds in the weak sense:

\[
- \Delta_{p,A}(F(u)) = -|F'(u)|^{p-2} [(p-1)F''(u)|\nabla u|^p_A + F'(u)\Delta_{p,A}(u)].
\]  

(A.1)

Moreover, if \(\Delta_{p,A}(u) \in L^1_{\text{loc}}(\Omega)\), then \(\Delta_{p,A}(F(u)) \in L^1_{\text{loc}}(\Omega)\).

**Proof.** Denote \(g := -\Delta_{p,A}(u)\), and let \(\varphi \in C_0^\infty(\Omega)\). By the product and chain rules, we have

\[
\int_\Omega |\nabla F(u)|^{p-2}_A \nabla F(u) \cdot \nabla \varphi \, d\nu = \int_\Omega |\nabla u|^{p-2}_A \nabla u \cdot \nabla \left(|F'(u)|^{p-2}F'(u)\right) \varphi \, d\nu + \int_\Omega |\nabla u|^{p-2}_A \nabla u \cdot \nabla \left(|F'(u)|^{p-2}F'(u)\varphi\right) \, d\nu
\]

Note that for \(p \geq 2\), the function \(\psi(s) := |s|^{p-2}s\) is continuously differentiable, and \(\psi'(s) := (p-1)|s|^{p-2}\). Moreover, by our assumption on \(F\) it follows that \(|F'(u)|^{p-2}F'(u)\varphi \in W^{1,p}_c(\Omega)\). Consequently, the second term of the right hand side in the above equality equals

\[
\int_\Omega |\nabla u|^{p-2}_A \nabla u \cdot \nabla \left(|F'(u)|^{p-2}F'(u)\varphi\right) \, d\nu = \int_\Omega g|F'(u)|^{p-2}F'(u)\varphi \, d\nu.
\]

Therefore,

\[
\int_\Omega |\nabla F(u)|^{p-2}_A \nabla F(u) \cdot \nabla \varphi \, d\nu = \int_\Omega |\nabla u|^{p-2}_A \nabla u \cdot \nabla \left(|F'(u)|^{p-2}F'(u)\right) \varphi \, d\nu + \int_\Omega g|F'(u)|^{p-2}F'(u)\varphi \, d\nu.
\]

Consequently, in the weak sense we have

\[
-\Delta_{p,A}(F(u)) = -|\nabla u|^{p-2}_A \nabla u \cdot \nabla \left(|F'(u)|^{p-2}F'(u)\right) - \Delta_{p,A}(u)|F'(u)|^{p-2}F'(u).
\]

Since \(\psi'(s) := (p-1)|s|^{p-2}\) for \(s \neq 0\), and \(\psi'\) is integrable at 0, we have that in the weak sense

\[
|\nabla u|^{p-2}_A \nabla u \cdot \nabla \left(|F'(u)|^{p-2}F'(u)\right) = (p-1)|F'(u)|^{p-2}F''(u)|\nabla u|^p_A.
\]

This implies (A.1), and hence clearly completes the proof of Lemma A.1. \(\square\)

**Acknowledgments**

The authors thank F. Gesztesy and B. Simon for helpful comments on the literature. The authors acknowledge the support of the Israel Science Foundation (grant 637/19) founded by the Israel Academy of Sciences and Humanities.
References

[1] Adimurthi, N. Chaudhuri, and M. Ramaswamy. An improved Hardy-Sobolev inequality and its application. *Proc. Amer. Math. Soc.*, 130(2):489–505 (electronic), 2002.

[2] C. D. Aliprantis and K. C. Border. *Infinite dimensional analysis. A hitchhiker’s guide.* Springer, Berlin, third edition, 2006.

[3] W. Allegretto. Principal eigenvalues for indefinite-weight elliptic problems in $\mathbb{R}^n$. *Proc. Amer. Math. Soc.*, 116(3):701–706, 1992.

[4] T. Anoop. On weighted eigenvalue problems and applications. *Ph.D Thesis*, 2011.

[5] W. Allegretto. Principal eigenvalues for indefinite-weight elliptic problems in $\mathbb{R}^n$. *Proc. Amer. Math. Soc.*, 116(3):701–706, 1992.

[6] T. Anoop and U. Das. The compactness and the concentration compactness via $p$-capacity. *Annali di Matematica Pura ed Applicata (1923 -)*, 200(6):2715–2740, 2021.

[7] V. Benci and D. Fortunato. Discreteness conditions of the spectrum of Schrödinger operators. *J. Math. Anal. Appl.*, 64(3):695–700, 1978.

[8] B. Bianchini, L. Mari, and M. Rigoli. Yamabe type equations with a sign-changing nonlinearity, and the prescribed curvature problem. *J. Differential Equations*, 260(10):7416–7497, 2016.

[9] H. Brezis and J. L. Vázquez. Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Complut. Madrid*, 10(2):443–469, 1997.

[10] B. Devyver, M. Fraas, and Y. Pinchover. Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon. *J. Funct. Anal.*, 266(7):4422–4489, 2014.

[11] B. Devyver and Y. Pinchover. Optimal $L^p$ Hardy-type inequalities. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(1):93–118, 2016.

[12] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[13] S. Filippas and A. Tertikas. Optimizing improved Hardy inequalities. *J. Funct. Anal.*, 192(1):186–233, 2002.

[14] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original.

[15] Y. Hou, Y. Pinchover, and A. Rasila. Positive solutions of the $A$-Laplace equation with a potential, 2021. arXiv: 2112.01755.

[16] H. Kovářík and Y. Pinchover. On minimal decay at infinity of Hardy-weights. *Commun. Contemp. Math.*, 22(5):1950046, 18, 2020.

[17] P. D. Lamberti and Y. Pinchover. $L^p$ Hardy inequality on $C^{1,\gamma}$ domains. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 19(3):1135–1159, 2019.

[18] G. Leoni. *A first course in Sobolev spaces*, volume 181 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2017.

[19] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[20] G. M. Lieberman. Sharp forms of estimates for subsolutions and supersolutions of quasilinear elliptic equations involving measures. *Comm. Partial Differential Equations*, 18(7-8):1191–1219, 1993.

[21] P. L. Lions. The concentration-compactness principle in the calculus of variations: The locally compact cases I & II. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 223–283, 1984.

[22] P. L. Lions. The concentration-compactness principle in the calculus of variations: The limit cases I & II. *Rev. Mat. Iberoamericana*, 1(1):45–121, 145–201, 1985.

[23] J. Malý and W. P. Ziemer. *Fine regularity of solutions of elliptic partial differential equations*, volume 51 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[24] A. Manes and A. M. Micheletti. Un’estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine. *Boll. Un. Mat. Ital. (4)*, 7:285–301, 1973.
[25] V. G. Maz’ya. *Sobolev spaces*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.

[26] V. G. Maz’ya. Lectures on isoperimetric and isocapacitary inequalities in the theory of sobolev spaces. *Contemp. Math.*, 338:307–340, 01 2003.

[27] Y. Pinchover. Large scale properties of multiparameter oscillation problems. *Comm. Partial Differential Equations*, 15(5):647–673, 1990.

[28] Y. Pinchover and G. Psaradakis. On positive solutions of the \((p, A)\)-Laplacian with potential in Morrey space. *Analysis & PDE*, 9(6):1317–1358, 2016.

[29] Y. Pinchover and N. Regev. Criticality theory of half-linear equations with the \((p,a)\)-Laplacian. *Nonlinear Analysis: Theory, Methods & Applications*, 119:295–314, 2015.

[30] Y. Pinchover, A. Tertikas, and K. Tintarev. A Liouville-type theorem for the \(p\)-Laplacian with potential term. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(2):357–368, 2008.

[31] Y. Pinchover and K. Tintarev. Ground state alternative for \(p\)-Laplacian with potential term. *Calc. Var. Partial Differential Equations*, 28(2):179–201, 2007.

[32] Y. Pinchover and K. Tintarev. On positive solutions of minimal growth for singular \(p\)-Laplacian with potential term. *Adv. Nonlinear Stud.*, 8(2):213–234, 2008.

[33] P. Pucci and J. Serrin. *The maximum principle*, volume 73 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 2007.

[34] F. Rindler. *Calculus of variations*. Universitext. Springer, Cham, 2018.

[35] H. L. Royden. *Real analysis*. The Macmillan Company, New York; Collier-Macmillan Ltd., London, Second Edition.

[36] E. Sawyer and R. L. Wheeden. Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Amer. J. Math.*, 114(4):813–874, 1992.

[37] D. Smets. A concentration-compactness lemma with applications to singular eigenvalue problems. *J. Funct. Anal.*, 167(2):463–480, 1999.

[38] A. Tertikas. Critical phenomena in linear elliptic problems. *J. Funct. Anal.*, 154(1):42–66, 1998.

[39] I. Versano. Optimal Hardy-weights for the \((p, A)\)-Laplacian with a potential term, to appear in *Proc. Roy. Soc. Edinburgh Sect. A*. arXiv: 2112.04449.

[40] N. Visciglia. A note about the generalized Hardy-Sobolev inequality with potential in \(L^{p,d}(\mathbb{R}^n)\). *Calc. Var. Partial Differential Equations*, 24(2):167–184, 2005.

[41] A. C. Zaanen. *An introduction to the theory of integration*. North-Holland Publishing Company, Amsterdam, 1958.