On finite complete rewriting systems, finite derivation type, and automaticity for homogeneous monoids

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ABSTRACT

The class of finitely presented monoids defined by homogeneous (length-preserving) relations is considered. The properties of admitting a finite complete rewriting system, having finite derivation type, being automatic, and being bi-automatic, are investigated for monoids in this class. The first main result shows that for any possible combination of these properties and their negations there is a homogeneous monoid with exactly this combination of properties. We then extend this result to show that the same statement holds even if one restricts attention to the class of \( n \)-ary multihomogeneous monoids (meaning every side of every relation has fixed length \( n \), and all relations are also content preserving).

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1 INTRODUCTION

There has been a lot of interest recently in finitely presented algebras $A$ over a field $K$ defined by homogeneous semigroup relations, that is, semigroup algebras $K[S]$ where $S$ is a semigroup given by a finite presentation where all relations are length preserving. Numerous interesting semigroup algebras arise in this way, including: algebras yielding set-theoretic solutions to the Yang–Baxter equation and quadratic algebras of skew type (see for example [ESS99, JO07, CJO10c]), algebras related to Young diagrams, representation theory and algebraic combinatorics such as the Plactic and Chinese algebras (see [Lot02, Ch. 5], [CO04, LS81] and [CEK+01, JO11, CO13]), and algebras defined by permutation relations (see [CJO10b, CJO10a, CK12]). In these examples, there are strong connections between the structure of the algebra $K[S]$ and that of the underlying semigroup $S$. Further motivation for studying this class comes from other important semigroups in the literature that admit homogeneous presentations, such as the hypoplactic monoid [Nov00], shifted Plactic monoid [Ser10], monoids with the same multihomogeneous growth as the Plactic monoid [DK94], trace monoids [DM97], and the positive braid monoid [BKL98, DP99].

One of the fundamental first steps that is necessary when investigating a semigroup $S$ defined by homogeneous relations, and its associated semigroup algebra $K[S]$, is to find a good set of normal forms (canonical representatives over the generating set) for the elements of the monoid, and thus for elements of the algebra. (See the list of open problems in [CJO10b, Section 3] for more on the importance of this problem in the context of semigroups defined by permutation relations.) Specifically we would like a set of normal forms that is a regular language, and we want to be able to compute effectively with these normal forms. Two situations where such a good set of normal forms does exist are for monoids that admit presentations by finite complete rewriting systems (see [BO93]), and for monoids and semigroups that are automatic (see [ECH+92, CRRT01]). Each of these properties also has implications for properties of the corresponding semigroup algebra. Indeed, if the semigroup admits a finite complete rewriting system, then the semigroup algebra admits a finite Gröbner–Shirshov basis (see [Hey00] for an explanation of the connection between Gröbner–Shirshov bases and complete rewriting systems), while the automaticity of the semigroup implies that the algebra is an automaton algebra in the sense of Ufarnovskij; see [Ufn95] and [CO12, Section 1].

Many of the examples of homogeneous semigroups mentioned above have been shown to admit presentations by finite complete rewriting systems, and have been shown to be biautomatic; see for example [CGM, CGMar, CO12, GK10, KO10, CQ08]. It is natural to ask to what extent these results generalise to arbitrary homogeneous semigroups. One can ask: Does every homogeneous semigroup admit a presentation by a finite complete rewriting system? Is every such semigroup biautomatic? Within the class of homogeneous semigroups, what is the relationship between admitting a finite complete rewriting system and being biautomatic? (For general semigroups, these properties are independent; see [OSKM98]). The aim of this paper is to make a comprehensive investigation of these questions. In fact, we shall consider two different strengths of automaticity, called automaticity and biautomaticity, and we shall also investigate the homotopical finiteness property of finite derivation type (FDT) in the sense of Squier [SOK94b], which is a key finiteness property that is satisfied by monoids that admit presentations by finite complete rewriting
systems (full definitions of all of these concepts will be given in Section 2).

There are various degrees of homogeneity that one can impose on a semigroup presentation. We shall consider finite presentations \(\langle A \mid R \rangle\) which are:

- homogeneous: relations are length preserving;
- multihomogeneous: for each letter \(a\) in the alphabet \(A\), and for every relation \(u = v\) from \(R\), the number of occurrences of the letter \(a\) in \(u\) equals the number of occurrences of the letter \(a\) in \(v\);
- \(n\)-ary homogeneous: there is a fixed global constant \(n\) such that for every relation \(u = v\) in \(R\) we have \(|u| = |v| = n\);
- \(n\)-ary multihomogeneous: simultaneously \(n\)-ary homogeneous and multihomogeneous.

Of course, the most restricted class listed here is the class of \(n\)-ary multihomogeneous presentations.

For brevity, we introduce the following terminology for the four properties we are interested in: a monoid is

- \textsf{fcrs} if it admits a presentation via a finite complete rewriting system (with respect to some generating set);
- \textsf{fdt} if it has finite derivation type;
- \textsf{biauto} if it is biautomatic;
- \textsf{auto} if it is automatic.

We will also use the natural negated terms: \textsf{non-fcrs}, \textsf{non-fdt}, \textsf{non-biauto}, and \textsf{non-auto}.

We are interested in which combinations of these properties a homogeneous monoid can have. Since in general \textsf{fcrs} implies \textsf{fdt}, and \textsf{biauto} implies \textsf{auto}, not all combinations will be possible. But, subject to these restrictions, our main results show that any possible combination is possible within the class of homogeneous monoids. We then extend this result showing that for every possible combination of these properties, one can construct an \(n\)-ary multihomogeneous monoid with the given combination of properties (see Theorem 5.1). These results are summarised in Table 1 and the relationship between the various examples is illustrated in Figure 1.

The paper is structured as follows: in Section 2 we give the basic definitions and results needed in the paper. In Section 3 we present the fundamental

| \(\textsf{FCRS} \implies \textsf{FDT}\) | \(\textsf{BIAUTO} \implies \textsf{AUTO}\) | Example |
| --- | --- | --- |
| Y | Y | Y | Y | Plastic monoid |
| Y | Y | N | Y | Example 3.1 |
| Y | Y | N | N | Example 3.5 |
| N | Y | Y | Y | Example 3.6 |
| N | Y | N | Y | Example 4.5 |
| N | Y | N | N | Example 4.6 |
| N | N | Y | Y | Example 3.8 |
| N | N | N | Y | Example 4.7 |
| N | N | N | N | Example 4.8 |

Table 1. A catalogue of \(n\)-ary multihomogeneous monoids exhibiting all possible combinations of allowable properties.
examples from which all of our other examples will be built, namely the Plactic monoid, and Examples 3.1, 3.5, 3.6 and 3.8. Section 4 contains general results about the behaviour of the properties under free products of monoids, which we then use to construct further examples (as illustrated in Figure 1). In Section 5 we prove some general results which are then applied to extend our result for homogeneous monoids to the more restricted class of \( n \)-ary multihomogeneous monoids.

2 PRELIMINARIES

The subsection below on derivation graphs, homotopy bases and finite derivation type is self-contained, but it can be complemented with [KO97, KO01]. For further information on automatic semigroups, see [CRRT01]. We assume familiarity with basic notions of automata and regular languages (see, for example, [HU79]) and transducers and rational relations (see, for example, [Ber79]). For background on string rewriting systems we refer the reader to [BN98, BO93].

2.1 Words, rewriting systems, and presentations

We denote the empty word (over any alphabet) by \( \varepsilon \). For an alphabet \( A \), we denote by \( A^* \) the set of all words over \( A \). When \( A \) is a generating set for a monoid \( M \), every element of \( A^* \) can be interpreted either as a word or as an element of \( M \). For words \( u, v \in A^* \), we write \( u = v \) to indicate that \( u \) and \( v \) are equal as words and \( u =_M v \) to denote that \( u \) and \( v \) represent the same element of the monoid \( M \). The length of \( u \in A^* \) is denoted \( |u| \), and, for any \( a \in A \), the number of symbols \( a \) in \( u \) is denoted \( |u|_a \). We denote by \( u^{\text{rev}} \) the reversal of a word \( u \); that is, if \( u = a_1 \cdots a_{n-1} a_n \) then \( u^{\text{rev}} = a_n a_{n-1} \cdots a_1 \), with \( a_i \in A \). If \( \mathcal{R} \) is a relation on \( A^* \), then \( \mathcal{R}^\# \) denotes the congruence generated by \( \mathcal{R} \).

We use standard terminology and notation from the theory of string rewrit-
ing systems; see \[BO93\] or \[BNo8\] for background reading.

A presentation is a pair \(\langle A \mid R \rangle\) that defines [any monoid isomorphic to] \(A^*/R\). The presentation \(\langle A \mid R \rangle\) is homogeneous (respectively, multihomogeneous) if for every \((u, v) \in R\), also known as defining relations, and \(a \in A\), we have \(|u| = |v|\) (respectively, \(|u|_a = |v|_a\)). That is, in a homogeneous presentation, defining relations preserve length; in a multihomogeneous presentation, defining relations preserve the numbers of each symbol. A monoid is homogeneous (respectively, multihomogeneous) if it admits a homogeneous (respectively, multihomogeneous) presentation.

A string rewriting system, or simply a rewriting system, is a pair \(\langle A, R \rangle\), where \(A\) is a finite alphabet and \(R\) is a set of pairs \((\ell, r)\), usually written \(\ell \rightarrow r\), known as rewriting rules or simply rules, drawn from \(A^* \times A^*\). The single reduction relation \(\rightarrow_R\) is defined as follows: \(u \rightarrow_R v\) (where \(u, v \in A^*\)) if there exists a rewriting rule \((\ell, r) \in R\) and words \(x, y \in A^*\) such that \(u = x\ell y\) and \(v = xry\). That is, \(u \rightarrow_R v\) if one can obtain \(v\) from \(u\) by substituting the word \(r\) for a subword \(\ell\) of \(u\), where \(\ell \rightarrow r\) is a rewriting rule. The reduction relation \(\rightarrow^*_R\) is the reflexive and transitive closure of \(\rightarrow_R\). The subscript \(R\) is omitted when it is clear from context. The process of replacing a subword \(\ell\) by a word \(r\), where \(\ell \rightarrow r\) is a rule, is called reduction by application of the rule \(\ell \rightarrow r\); the iteration of this process is also called reduction. A word \(w \in A^*\) is reducible if it contains a subword \(\ell\) that forms the left-hand side of a rewriting rule in \(R\); it is otherwise called irreducible.

The rewriting system \(\langle A, R \rangle\) is finite if both \(A\) and \(R\) are finite. The rewriting system \(\langle A, R \rangle\) is noetherian if there is no infinite sequence \(u_1, u_2, \ldots\) of words from \(A^*\) such that \(u_i \rightarrow u_{i+1}\) for all \(i \in \mathbb{N}\). That is, \(\langle A, R \rangle\) is noetherian if any process of reduction must eventually terminate with an irreducible word. The rewriting system \(\langle A, R \rangle\) is confluent if, for any words \(u, u', u'' \in A^*\) with \(u \rightarrow^* u'\) and \(u \rightarrow^* u''\), there exists a word \(v \in A^*\) such that \(u' \rightarrow^* v \) and \(u'' \rightarrow^* v\). It is well known that a noetherian system is confluent if and only if all critical pairs resolve, where critical pairs are obtained by considering overlaps of left hand sides of the rewrite rules \(R\); see \[BO93\] for more details. A rewriting system that is both confluent and noetherian is complete. If a monoid admits a presentation with respect to some generating set that forms a finite complete rewriting system, the monoid is fcrs.

The Thue congruence \(\leftrightarrow_R\) is the equivalence relation generated by \(\rightarrow_R\). The elements of the monoid presented by \(\langle A \mid R \rangle\) are the \(\leftrightarrow_R\)-equivalence classes. The relations \(\leftrightarrow_R\) and \(R\) coincide.

Let \(M\) be a homogeneous monoid. Let \(\langle A \mid R \rangle\) be a homogeneous presentation for \(M\). Since none of the generators represented by \(A\) can be non-trivially decomposed, the alphabet \(A\) represents a unique minimal generating set for \(M\), and any generating set must contain this minimal generating set. Any two words over \(A\) representing the same element of \(M\) must be of the same length. So there is a well-defined function \(\lambda : M \rightarrow \mathbb{N}^0\) where \(x\lambda\) is defined to be the length of any word over \(A\) representing \(x\). It is easy to see that \(\lambda\) is a homomorphism.

2.2 Derivation graphs, homotopy bases, and finite derivation type

Given a monoid presentation \(\langle A \mid R \rangle\) one builds a (combinatorial) 2-complex \(D\), called the Squier complex, whose 1-skeleton has vertex set \(A^*\) and edges corresponding to applications of relations from \(R\), and that has 2-cells
adjoined for each instance of “non-overlapping” applications of relations from $R$ (see below for a formal definition of non-overlapping relations). There is a natural action of the free monoid $\Lambda^*$ on the Squier complex $\mathcal{D}$. A collection of closed paths in $\mathcal{D}$ is called a homotopy base if the complex obtained by adjoining cells for each of these paths, and those that they generate under the action of the free monoid on the Squier complex, has trivial fundamental groups.

A monoid defined by a presentation is said to have finite derivation type (or FDT for short) if the corresponding Squier complex admits a finite homotopy base. Squier [SOK94] proved that the property FDT is independent of the choice of finite presentation, so we may speak of FDT monoids. The original motivation for studying this notion is Squier’s result [SOK94] which says that if a monoid admits a presentation by a finite complete rewriting system then the monoid must have finite derivation type. Further motivation for the study of these concepts comes from the fact that the fundamental groups of connected components of Squier complexes, which are called diagram groups, have turned out to be a very interesting class of groups; see [GS97].

In more detail, with any monoid presentation $\mathcal{P} = (\Lambda \mid R)$ we associate a graph (in the sense of Serre [Sero03]) as follows. The derivation graph of $\mathcal{P}$ is an infinite graph $\Gamma = \Gamma(\mathcal{P}) = (V, E, \iota, \tau, -1)$ with vertex set $V = \Lambda^*$, and edge set $E$ consisting of the collection of 4-tuples

$$\{(w_1, r, e, w_2) : w_1, w_2 \in \Lambda^*, r \in \mathbb{R}, e \in \{+1, -1\}\}.$$ 

The functions $\iota, \tau : E \rightarrow V$ associate with each edge $E = (w_1, r, e, w_2)$ (with $r = (r_1, r_0, r_{-1}) \in \mathbb{R}$) its initial and terminal vertices $\iota E = w_1 r e w_2$ and $\tau E = w_1 r_{-e} w_2$, respectively. The mapping $-1 : E \rightarrow E$ associates with each edge $E = (w_1, r, e, w_2)$ an inverse edge $E^{-1} = (w_1, r, -e, w_2)$.

A path is a sequence of edges $\mathcal{P} = E_1 \circ E_2 \circ \ldots \circ E_n$ where $\tau E_i = \iota E_{i+1}$ for $i = 1, \ldots, n-1$. Here $\mathcal{P}$ is a path from $\iota E_1$ to $\tau E_n$ and we extend the mappings $\iota$ and $\tau$ to paths by defining $\iota \mathcal{P} = \iota E_1$ and $\tau \mathcal{P} = \tau E_n$. The inverse of a path $\mathcal{P} = E_1 \circ E_2 \circ \ldots \circ E_n$ is the path $\mathcal{P}^{-1} = E_n^{-1} \circ E_{n-1}^{-1} \circ \ldots \circ E_1^{-1}$, which is a path from $\tau \mathcal{P}$ to $\iota \mathcal{P}$. A closed path is a path $\mathcal{P}$ satisfying $\iota \mathcal{P} = \tau \mathcal{P}$. For two paths $\mathcal{P}$ and $\mathcal{Q}$ with $\tau \mathcal{P} = \iota \mathcal{Q}$ the composition $\mathcal{P} \circ \mathcal{Q}$ is defined.

We denote the set of paths in $\Gamma$ by $P(\Gamma)$, where for each vertex $w \in V$ we include a path $1_w$ with no edges, called the empty path at $w$. The free monoid $\Lambda^*$ acts on both sides of the set of edges $E$ of $\Gamma$ by

$$x \cdot E \cdot y = (xw_1, r, e, w_2y)$$

where $E = (w_1, r, e, w_2)$ and $x, y \in \Lambda^*$. This extends naturally to a two-sided action of $\Lambda^*$ on $P(\Gamma)$ where for a path $\mathcal{P} = E_1 \circ E_2 \circ \ldots \circ E_n$ we define

$$x \cdot \mathcal{P} \cdot y = (x \cdot E_1 \cdot y) \circ (x \cdot E_2 \cdot y) \circ \ldots \circ (x \cdot E_n \cdot y).$$

If $\mathcal{P}$ and $\mathcal{Q}$ are paths such that $\iota \mathcal{P} = \iota \mathcal{Q}$ and $\tau \mathcal{P} = \tau \mathcal{Q}$ then we say that $\mathcal{P}$ and $\mathcal{Q}$ are parallel, and write $\mathcal{P} \parallel \mathcal{Q}$. We use $\parallel \subseteq P(\Gamma) \times P(\Gamma)$ to denote the set of all pairs of parallel paths.

An equivalence relation $\sim$ on $P(\Gamma)$ is called a homotopy relation if it is contained in $\parallel$ and satisfies the following four conditions.

1. If $E_1$ and $E_2$ are edges of $\Gamma$, then

$$(E_1 \cdot \iota E_2) \circ (\tau E_1 \cdot E_2) \sim (\iota E_1 \cdot E_2) \circ (E_1 \cdot \tau E_2).$$
2. For any $P, Q \in P(\Gamma)$ and $x, y \in A^*$

$$P \sim Q \text{ implies } x \cdot P \sim x \cdot Q \cdot y.$$  

3. For any $P, Q, R, S \in P(\Gamma)$ with $\tau R = \tau P = \tau Q$ and $\iota S = \tau P = \tau Q$

$$P \sim Q \text{ implies } R \circ P \circ S \sim R \circ Q \circ S.$$  

4. If $P \in P(\Gamma)$ then $P^{-1} \sim 1_P$, where $1_P$ denotes the empty path at the vertex $1_P$.

The idea behind condition 1 is the following. Suppose that a word $w$ has two disjoint occurrences of rewriting rules in the sense that $w = \alpha_\epsilon \beta \alpha'_r \epsilon' \beta'$ where $\alpha, \beta, \alpha', \beta' \in A^*$, $r, r' \in R$ and $\epsilon, \epsilon' \in \{-1, +1\}$. Let $E_1 = (\alpha, r, \epsilon, \beta)$ and $E_2 = (\alpha', r', \epsilon', \beta')$. Then the paths

$$P = (E_1 \cdot tE_2) \circ (\tau E_1 \cdot E_2), \quad P' = (tE_1 \cdot E_2) \circ (E_1 \cdot \tau E_2)$$

give two different ways of rewriting the word $w = \alpha_\epsilon \beta \alpha'_r \epsilon' \beta'$ to the word $w = \alpha_{-\epsilon} \beta \alpha'_{-r'} \epsilon' \beta'$, where in $P$ we first apply the left-hand relation and then the right-hand, while in $P'$ the relations are applied in the opposite order; see Figure 2. We want to regard these two paths as being essentially the same, and this is achieved by condition 1. This is usually called pull-down and push-up.

For a subset $C$ of $\parallel$, the homotopy relation $\sim_C \text{ generated by } C$ is the smallest (with respect to inclusion) homotopy relation containing $C$. The relation $\sim_0 = \sim_\emptyset \text{ generated by the empty set } \emptyset$ is the smallest homotopy relation. If $\sim_C$ coincides with $\parallel$, then $C$ is called a homotopy base for $\Gamma$. The presentation $\langle A \mid R \rangle$ is said to have finite derivation type if the derivation graph $\Gamma$ of $\langle A \mid R \rangle$ admits a finite homotopy base. A finitely presented monoid $S$ is said to have finite derivation type, or to be fdt, if some (and hence any by [SOK94a, Theorem 4.3]) finite presentation for $S$ has finite derivation type.

It is not difficult to see that a subset $C$ of $\parallel$ is a homotopy base of $\Gamma$ if and only if the set

$$\{(P \circ Q^{-1}, 1_P) : (P, Q) \in C\}$$

is a homotopy base for $\Gamma$. Thus we say that a set $D$ of circuits is a homotopy base if the corresponding set $\{(P, 1_P) : P \in D\}$ is a homotopy base.
2.3 Automaticity and biautomaticity

**Definition 2.1.** Let $A$ be an alphabet and let $\$\$ be a new symbol not in $A$. Define the mapping $\delta_R : A^* × A^* \to ((A \cup \{\$\}) × (A \cup \{\$\}))^*$ by

$$(u_1 \cdots u_m, v_1 \cdots v_n) \mapsto\begin{cases} (u_1, v_1) \cdots (u_m, v_n) & \text{if } m = n, \\ (u_1, v_1) \cdots (u_m, v_n)(u_{n+1}, \$) \cdots (u_m, \$) & \text{if } m > n, \\ (u_1, v_1) \cdots (u_m, v_m)(\$, v_{m+1}) \cdots (\$, v_n) & \text{if } m < n, \end{cases}$$

and the mapping $\delta_L : A^* × A^* \to ((A \cup \{\$\}) × (A \cup \{\$\}))^*$ by

$$(u_1 \cdots u_m, v_1 \cdots v_n) \mapsto\begin{cases} (u_1, v_1) \cdots (u_m, v_n) & \text{if } m = n, \\ (u_1, \$) \cdots (u_{m-n}, \$)(u_{m-n+1}, v_1) \cdots (u_m, v_n) & \text{if } m > n, \\ (\$, v_1) \cdots (\$, v_{n-m})(u_1, v_{n-m+1}) \cdots (u_m, v_n) & \text{if } m < n, \end{cases}$$

where $u_i, v_i ∈ A$.

**Definition 2.2.** Let $M$ be a monoid. Let $A$ be a finite alphabet representing a set of generators for $M$ and let $L ⊆ A^*$ be a regular language such that every element of $M$ has at least one representative in $L$. For each $a ∈ A \cup \{\varepsilon\}$, define the relations

$$L_a = \{(u, v) : u, v ∈ L, u a =_M v\}$$
$$a L = \{(u, v) : u, v ∈ L, a u =_M v\}.$$  

The pair $(A, L)$ is an automatic structure for $M$ if $L_a \delta_R$ is a regular language over $(A \cup \{\$\}) \times (A \cup \{\$\})$ for all $a ∈ A \cup \{\varepsilon\}$. A monoid $M$ is automatic, or auto, if it admits an automatic structure with respect to some generating set.

The pair $(A, L)$ is a biautomatic structure for $M$ if $L_a \delta_R$, $a L \delta_R$, $L_a \delta_L$, and $a L \delta_L$ are regular languages over $(A \cup \{\$\}) \times (A \cup \{\$\})$ for all $a ∈ A \cup \{\varepsilon\}$. A monoid $M$ is biautomatic, or biauto, if it admits a biautomatic structure with respect to some generating set. [Note that biauto implies auto.]

Hoffmann & Thomas have made a careful study of biautomaticity for semigroups [HT05]. They distinguish four notions of biautomaticity for semigroups that require at least one of $L_a \delta_R$ and $L_a \delta_L$ and at least one of $a L \delta_R$ and $a L \delta_L$ to be regular. These notions are all equivalent for groups and more generally for cancellative semigroups [HT05, Theorem 1] but distinct for semigroups [HT05, Remark 1 & § 4]. biauto clearly implies all four Hoffmann–Thomas notions of biautomaticity. However, we shall shortly prove that, within the class of homogeneous monoids, any of the Hoffmann–Thomas notions of biautomaticity implies biauto (see **Proposition 2.5**).

In proving that $R \delta_R$ or $R \delta_L$ is regular, where $R$ is a relation on $A^*$, a useful strategy is to prove that $R$ is a rational relation (that is, a relation recognized by a finite transducer [Ber79, Theorem 6.1]) and then apply the following result, which is a combination of [FS93, Corollary 2.5] and [HT05, Proposition 4]:

**Proposition 2.3.** If $R \subseteq A^* × A^*$ is rational relation and there is a constant $k$ such that $|u| − |v| ≤ k$ for all $(u, v) ∈ R$, then $R \delta_R$ and $R \delta_L$ are regular.
Now we shall prove some results on automaticity and biautomaticity for the class of homogeneous monoids.

Unlike the situation for groups, both automaticity and biautomaticity for monoids and semigroups are dependent on the choice of generating set [CRRT01, Example 4.5]. However, for monoids, biautomaticity and automaticity are independent of the choice of semigroup generating sets [DRR99, Theorem 1.1]. However, in the case of homogeneous monoids, we do have independence of the choice of generating set:

**Proposition 2.4.** Let \( M \) be a homogeneous monoid that is auto (respectively, bi-auto). Then for any finite alphabet \( C \) representing a generating set for \( M \) there is a language \( K \) over \( C \) such that \((C, K)\) is an automatic (respectively, biautomatic) structure for \( M \).

**Proof of 2.4.** We first consider the case for auto. Suppose \((B, L)\) is an automatic structure for \( M \). Assume without loss that \((B, L)\) is an automatic structure with uniqueness for \( M \).

Notice that both the alphabet \( B \) and the alphabet \( C \) must contain a subalphabet representing the unique minimal generating set of \( M \). Without loss of generality, assume that they both contain the alphabet \( A \) representing this minimal generating set. For each \( b \in B \), let \( w_b \in A^\ast \) be such that \( w_b =_M b \).

Let \( \Omega \subseteq B^\ast \times A^\ast \) be the relation

\[
\{(b_1b_2\cdots b_k, w_{b_1}w_{b_2}\cdots w_{b_k}) : k \in \mathbb{N} \cup \{0\}, b_i \in B\}.
\]

It is easy to see that \( \Omega \) is a rational relation. Let

\[
K = L \circ \Omega = \{v \in A^\ast : (\exists u \in L)((u,v) \in \Omega)\}.
\]

Let \( a \in A \cup \{\varepsilon\} \). Then

\[
(u, v) \in K_a \iff u \in K \land v \in K \land u =_M v
\]

\[
\iff (\exists u', v' \in L)((u, u') \in \Omega \land (v, v') \in \Omega \land u' =_M v')
\]

\[
\iff (\exists u', v' \in L)((u, u') \in \Omega \land (v, v') \in \Omega \land (u', v') \in L_a)
\]

\[
\iff (u, v) \in \Omega^{-1} \circ L_a \circ \Omega.
\]

Hence \( K_a = \Omega^{-1} \circ L_a \circ \Omega \) is a rational relation. Furthermore

\[
(u, v) \in K_a \implies u =_M v
\]

\[
\implies (ua)\lambda = v\lambda
\]

\[
\implies u\lambda + a\lambda = v\lambda
\]

\[
\implies |u| + 1 = |v|;
\]

thus \( K_a\delta_R \) is a regular language by Proposition 2.3.

For \( c \in C - A \), let \( u = u_1 \cdots u_m \in A^\ast \) be such that \( u = c \); then \( K_c \delta_R = (K_{u_1} \circ K_{u_2} \circ \cdots \circ K_{u_m})\delta_R \) is regular by [CRRT01, Proposition 2.3] and similarly \( K_c\delta_L \) is regular. Hence \((C, K)\) is an automatic structure for \( M \).

For biauto, assume \((B, L)\) is a biautomatic structure for \( M \) and follow the above reasoning to show that each of the languages \( K_a\delta_R, aK\delta_R, K_a\delta_L, \) and \( aK\delta_L \) are regular.

**Proposition 2.5.** Let \( M \) be a homogeneous monoid, let \( B \) be a finite alphabet representing a generating set for \( M \), and let \( L \) be a regular language over \( B \) such that for each \( b \in B \cup \{\varepsilon\} \), at least one of \( L_b\delta_R \) and \( L_b\delta_L \) and at least one of \( bL\delta_R \) and \( bL\delta_L \) is regular. Then \( M \) is biauto.
Proof of 2.5. Suppose \( L_b \delta_R \) and \( bL \delta_R \) are regular; the other cases are similar. Then \( M \) admits a right-biautomatic structure \((B, L)\). Thus \( L_b \delta_R \) and \( bL \delta_R \) are regular languages for all \( b \in B \cup \{\varepsilon\} \).

As in the proof of Proposition 2.4, the alphabet \( B \) must contain a subalphabet representing the unique minimal generating set of \( M \). Without loss of generality, assume that \( B \) contains the alphabet \( A \) representing this minimal generating set.

Construct the relation \( \Omega \subseteq B^* \times A^* \) as in the proof of Proposition 2.4. Let \( K = L \circ \Omega \).

Let \( a \in A \subseteq B \). Then at least one of \( L_a \delta_R \) and \( L_a \delta_L \) and at least one of \( aL \delta_R \) and \( aL \delta_L \) is regular. In particular, \( L_a \) and \( aL \) are rational relations. So \( K_a = \Omega^{-1} \circ L_a \circ \Omega \) and \( aK = \Omega^{-1} \circ aL \circ \Omega \) are rational relations. If \( \{u, v\} \) is in \( K_a \) or \( aK \), then \(|u| + 1 = |v|\). Hence \( K_a \delta_R, K_a \delta_L, aK \delta_R, \) and \( aK \delta_L \) are all regular by Proposition 2.3. Since \( a \in A \cup \{\varepsilon\} \) was arbitrary, this proves that \((A, K)\) is a biautomatic structure for \( M \).

Despite the positive results obtained so far, note that \textsc{auto} does not imply \textsc{biauto} in the class of homogeneous monoids, as we shall see below in Example 3.1.

3 FUNDAMENTAL EXAMPLES

3.1 An fcrs, fdt, non-biauto, auto homogeneous monoid

In this subsection we present a homogeneous monoid that is \textsc{fcrs} and thus \textsc{fdt}, is \textsc{auto}, but is not \textsc{biauto}. By considering the reversal semigroup of this example we will get a homogeneous monoid that admits a finite complete rewriting system but is not automatic.

Example 3.1. Let \( M \) be the monoid defined by the presentation \( \langle A \mid \mathcal{R} \rangle \), where \( A = \{a, b, c\} \) and \( \mathcal{R} \) consists of the rewriting rules
\[
\begin{align*}
\text{cbab} &\rightarrow \text{cbcb}, & \text{cbbb} &\rightarrow \text{cbcb}, & \text{cbca} &\rightarrow \text{cacb}, \\
\text{cbaa} &\rightarrow \text{cbca}, & \text{cbba} &\rightarrow \text{cbca}, \\
\text{caab} &\rightarrow \text{cacb}, & \text{cabb} &\rightarrow \text{cacb}, \\
\text{caaa} &\rightarrow \text{caca}, & \text{caba} &\rightarrow \text{caca},
\end{align*}
\]

Proposition 3.2. The monoid \( M \) of Example 3.1 is \textsc{fcrs}.

Proof of 3.2. The rewriting system \((A, \mathcal{R})\) is noetherian because every rewriting rule either decreases the number of non-c symbols or decreases the number of symbols b to the left of symbols a. To see that it is confluent, notice that the only overlaps are those between the left-hand side of \text{cbca} \rightarrow \text{cacb} and the left-hand side of a rule of the form \text{caxy} \rightarrow \text{cacy}, where \( x, y \in \{a, b\} \).

However, we have
\[
\begin{align*}
\text{cbca}x &\rightarrow \begin{cases} 
\text{cacb}x &\rightarrow \text{cabcxa} \rightarrow \text{cabcxa} \rightarrow \text{cacacb} \\
\text{cbcaca} &\rightarrow \text{cacbca} \rightarrow \text{cacacb}
\end{cases} \\
\text{cbca}x &\rightarrow \begin{cases} 
\text{cabcxb} &\rightarrow \text{cacbcb} \\
\text{bcacac} &\rightarrow \text{cabcacb}.
\end{cases}
\end{align*}
\]

Therefore \((A, \mathcal{R})\) is confluent.
**Proposition 3.3.** The monoid $M$ of Example 3.1 is auto, but non-biauto.

**Proof of 3.3.** Let $L$ be the language of normal forms of $(A, R)$. Since $(A, R)$ is finite, $L$ is regular. Let $u \in L$. Consider the following cases separately:

1. $uc$ must also be in normal form, since no left-hand side of a rewriting rule ends in $c$. Hence
   \[ L_c = \{ (u, uc) : u \in L \}. \]

2. If $ub$ is not in normal form, then $ub$ must end with the left-hand side of a rewriting rule. Hence $u = u'cxy$ for some $x, y \in \{ a, b \}$, then $ub = u'cxyb \rightarrow u'cxb$. This word $u'cxb$ is in normal form since $u'c$ (which is a prefix of $u$) is in normal form and no rewriting rule has left-hand side $cxb$ for any $x \in \{ a, b \}$. Thus
   \[ L_b = \{ (u, ub) : ub \in L \} \cup \{ (u'cxy, u'cxb) : u'cxy \in L, x, y \in \{ a, b \} \}. \]

3. If $ua$ is not in normal form, then $ua$ must end with the left-hand side of a rewriting rule and so either $u = u'cxc$ or $u = u'cxy$ for some $x, y \in \{ a, b \}$.
   (a) If $u = u''(cb)\alpha c$ where $\alpha \geq 1$ is maximal. Then $ua = u''(cb)\alpha ca \rightarrow u''ca(cb)^\alpha$, which is in normal form since $u''$ and $ca(ab)^\alpha$ are in normal form and the only left-hand side of a rewriting rule of that ends in $ca$ is $cbca$, and $\alpha$ is maximal.
   (b) If $u = u''cay$ then $ua = u''caya \rightarrow u''caaca$ and this word is in normal form since $u''ca$ is in normal form.
   (c) If $u = u''(cb)\alpha y$ where $\alpha \geq 1$ is maximal then $ua = u''(cb)\alpha y \rightarrow u''(cb)\alpha ca \rightarrow u''ca(cb)^\alpha$ and this word is in normal form since $u''ca$ is in normal form and $\alpha$ is maximal.

Therefore
\[ L_a = \{ (u, ua) : ua \in L \} \cup \{ (u''(cb)\alpha c, u''ca(cb)^\alpha) : \alpha \in \mathbb{N}, u''(cb)\alpha c \in L, u'' \notin A^*cb \} \cup \{ (u''cay, u''caaca) : y \in \{ a, b \}, u''cay \in L \} \cup \{ (u''(cb)\alpha y, u''ca(cb)^\alpha) : y \in \{ a, b \}, \alpha \in \mathbb{N}, u''(cb)\alpha y \in L, u'' \notin A^*cb \}. \]

Note also that $L_x = \{ (u, u) : u \in L \}$. It is easy to see that $L_x$ is a rational relation for any $x \in A \cup \{ \varepsilon \}$. If $(u, v)$ lies in one of these relations, then $|u| - |v| \leq 1$ and so $L_x \delta_R$ is regular for all $x \in A \cup \{ \varepsilon \}$ by Proposition 2.3. Hence $M$ is auto.

Suppose, with the aim of obtaining a contradiction, that $M$ is biauto. Then by Proposition 2.4 it admits a biautomatic structure $(A, L)$. In particular, $(cL \circ cL^{-1})\delta_R$ is regular. Let $n$ be an even number exceeding the number of states in an automaton recognizing $(cL \circ cL^{-1})\delta_R$. Observe that
\[ cL \circ cL^{-1} = \{ (u, v) \in L : cu =_M cv \}. \]

Notice that $a^n b^{n+1}$ is not represented by any other word over $A$ and similarly $b^n a^{n} b$ is not represented by any other word over $A$. So $a^n b^{n+1}, b^n a^n b \in L$. Furthermore,
\[ ca^n b^{n+1} \rightarrow^* (ca)^{n/2} (cb)^{(n/2)+1} \]
and
\[cb^n a^n b \rightarrow^* (cb)^{n/2}(ca)^{n/2}cb\]
\[\rightarrow^* (ca)^{n/2}(cb)^{(n/2)+1}\]

and so \((a^n b^{n+1}, b^n a^n b) \in c \Lambda c \Lambda^{-1}\). Since \(n\) exceeds the number of states in an automaton recognizing \((c \Lambda c \Lambda^{-1})\delta_R\), we can pump within the first \(n\) letters of \((a^n b^{n+1}, b^n a^n b)\delta_R\) to see that \((a^{n+ik} b^{n+1}, b^{n+ik} a^n b) \in c \Lambda c \Lambda^{-1}\) for some \(k \geq 1\) and for all \(i \in \mathbb{N}\). But
\[ca^{n+2k} b^{n+1} \rightarrow^* (ca)^{n/2+k}(cb)^{(n/2)+1}\]

and
\[cb^{n+2k} a^n b \rightarrow^* (cb)^{n/2+k}(ca)^{n/2}cb\]
\[\rightarrow^* (ca)^{n/2}(cb)^{(n/2)+k+1}\]

which is a contradiction. So \(M\) is not \(biauto\).

3.2 An fcrs, fdt, non-biauto, non-auto homogeneous monoid

Definition 3.4. Let \(S\) be a semigroup defined by a presentation \(\langle A | R \rangle\). Denote by \(S^{rev}\) the semigroup defined by the presentation \(\langle A | R^{rev} \rangle\), where \(R^{rev} = \{(1^{rev}, r^{rev}) : (1, r) \in R\}\), that is called the reversal semigroup of \(S\).

Example 3.5. Consider the monoid \(M^{rev}\) defined by the presentation \(\langle A | R^{rev} \rangle\), where \(\langle A | R \rangle\) is the presentation defining Example 3.1.

It is clear that \(M^{rev}\) is also fcrs and thus fdt. Now, by [HT03, Lemma 3.4] we conclude that \(M^{rev}\) is non-auto and thus non-biauto.

3.3 A non-fcrs, fdt, biauto, auto homogeneous monoid

The following homogeneous monoid was introduced by Katsura and Kobayashi [KK97, Example 3], who showed that it is non-fcrs, but is fdt. We shall prove that it is biauto and thus auto.

Example 3.6. Let \(A = \{a, b_i, c_1, d_1 : i = 1, 2, 3\}\) and let \(R\) consist of the rewriting rules
\[b_i a \rightarrow a b_i\]
\[i \in \{1, 2, 3\}\]
(3.1)
\[c_j b_j \rightarrow c_1 b_1\]
\[j \in \{2, 3\}\]
(3.2)
\[b_j d_j \rightarrow b_1 d_1\]
\[j \in \{2, 3\}\]
(3.3)

Let \(M = \langle A | R \rangle\). Then \(M\) is fdt [KK97, § 4] but is non-fcrs [KK97, Proposition 3].

Proposition 3.7. The monoid \(M\) of Example 3.6 is biauto and thus auto.

Proof of 3.7. Let \(S\) consist of the following rewriting rules:
\[b_i a \rightarrow a b_i\]
\[i \in \{1, 2, 3\}\]
(3.4)
\[c_j a^k b_j \rightarrow c_1 a^k b_1\]
\[j \in \{2, 3\}, k \in \mathbb{N} \cup \{0\}\]
(3.5)
\[c_1 a^k b_1 d_1 \rightarrow c_1 a^{k+c_1} b_1 d_1\]
\[j \in \{2, 3\}, k, c_1 \in \mathbb{N} \cup \{0\}\]
(3.6)
\[b_j d_j \rightarrow b_1 d_1\]
\[j \in \{2, 3\}\]
(3.7)
It is easy to see that every rule in $S$ is a consequence of those in $R$. In particular, using rules in $R$, we have

$$c_ja^k b_j \leftrightarrow c_ja^{k-1} b_j a \leftrightarrow \ldots \leftrightarrow c_jb_j a^k \leftrightarrow c_1b_1a^k \leftrightarrow c_1ab_1a^{k-1} \leftrightarrow \ldots \leftrightarrow c_1a^kb_1;$$

and

$$c_1a^kb_1a^\ell d_j \leftrightarrow \ldots \leftrightarrow c_1b_1a^{k+\ell} d_j \leftrightarrow \ldots \leftrightarrow c_jb_j a^{k+\ell} d_j \leftrightarrow c_1a^{k+\ell} b_1 d_1.$$ 

The rewriting system $(A, S)$ is noetherian since rules (3.5) decrease the number of symbols $b_j$ with $j \in \{2, 3\}$; rules (3.6) decrease the number of symbols $d_j$ with $j \in \{2, 3\}$ and do not increase the number of symbols $b_j$ with $j \in \{2, 3\}$; rules (3.7) decrease the numbers of symbols $b_j$ with $j \in \{2, 3\}$; and rules (3.4) decrease the number of symbols $a$ to the right of symbols $b_1$. Hence any rewriting using a rule in $S$ decreases a word with respect to the right-to-left length-plus-lexicographic order induced by any ordering of $A$ satisfying $b_1 < b_j < a < d_1 < d_j$ (for $j \in \{2, 3\}$).

To see that the $(A, S)$ is confluent, notice that there are three possible overlaps of left-hand sides of rewriting rules: an overlap of (3.4) and (3.5), an overlap of (3.7) and (3.5), and an overlap of (3.4) and (3.6). However, critical pairs resolve, since

$$c_ja^kb_j a \rightarrow \begin{cases} 
c_j a^{k+1} b_j \rightarrow c_1 a^{k+1} b_1 \\
c_1 a^kb_1 a \rightarrow c_1 a^{k+1} b_1 
\end{cases}$$

and

$$c_ja^kb_j d_j \rightarrow \begin{cases} 
c_j a^kb_1 d_1 \\
c_1a^kb_1 d_j \rightarrow c_ja^kb_1 d_1 
\end{cases}$$

and finally

$$c_1 a^kb_1a^\ell d_j \rightarrow \begin{cases} 
c_1a^{k+\ell} b_1 d_1 \\
c_1 a^{k+1} b_1 a^{\ell-1} d_j \rightarrow c_1 a^{k+\ell} b_1 d_1. 
\end{cases}$$

So $(A, S)$ is confluent.

Let $L$ be the language of $S$-irreducible words. That is,

$$L = A^* - A^* \left( \left\{ b_1 a, b_2 a, b_3 a \right\} \cup \left\{ c_2, c_3 \right\} a^* \left\{ b_2, b_3 \right\} \cup c_1 a^* b_1 a^* \left\{ d_2, d_3 \right\} \cup \left\{ b_2d_2, b_3d_3 \right\} \right) A^*;$$

thus $L$ is regular. If $w \in L$ and $x \in A$, then either $wx$ is irreducible, or $wx$ can be rewritten to an irreducible word by a single application of a rule from $S$. Since a single rewriting step can be carried out by a synchronous regular automaton, the languages $L_x \delta_R$ are regular. Similarly the languages $L \delta_R$ are regular and so $(A, L)$ is a biautomatic structure for $M$. 

\[ \]
In this section we will give an example of a homogeneous monoid that is non-fdt and thus non-fcrs, but which is biauto and thus auto.

**Example 3.8.** Let \( A = \{a, b\} \) and let \( \mathcal{R} \) be the rewriting system on \( A \cup \{c\} \) consisting of the three rules:

\[
\begin{align*}
K_a : \quad ac & \rightarrow ca \\
K_b : \quad bc & \rightarrow cb \\
C : \quad cab & \rightarrow cbb.
\end{align*}
\]

Let \( \mathcal{P} \) be the presentation \( \langle A \cup \{c\} | \mathcal{R} \rangle \).

**Theorem 3.9.** Let \( M \) be the monoid defined by the presentation \( \mathcal{P} \) in Example 3.8. The monoid \( M \):

1. has a set of normal forms \( A^* \cup c^+ b^* a^* \);
2. is biauto and thus auto;
3. is non-fdt and thus non-fcrs.

Part 1 of Theorem 3.9 will follow from Lemma 3.11 below, and part 2 is proved in Lemma 3.12. Then, the rest of the subsection will be devoted to proving that \( M \) is non-fdt, thus establishing part 3.

**Remark 3.10.** The methods we use here to prove that Example 3.8 is not-fdt are similar to those used in the proof of [GMP11, Theorem 1]. In particular we will use the notion of critical peaks, and resolution of critical peaks, in our proof. We refer the reader to [GMP11, Section 2] for the definitions of these concepts, and their connection with complete rewriting systems and fdt.

Let us begin by fixing some of the notation. We start by adding to \( \mathcal{P} \) infinitely many rules of the form

\[
\mathcal{C}_u : \quad cuab \rightarrow cubb \quad (u \in A^*)
\]

and denote by \( \mathcal{R}' \) the set of all these rules. Notice first that \( \mathcal{C}_c \) is precisely the rule \( C \) defined above and that, for any word \( u \in A^* \) the words \( cuab \) and \( cubb \) represent the same element of the monoid \( M \), since in the word \( cuab \) we can use relations of the form \( K_x \) to pass the letter \( c \) through the word \( u \) from left to right, then replace \( cab \) by \( cbb \) using the relation \( C \), and finally move the \( c \) back through \( u \) again from right to left using the relations \( K_x \). It follows that the presentations \( \mathcal{P} = \langle A \cup \{c\} | \mathcal{R} \rangle \) and \( \overline{\mathcal{P}} = \langle A \cup \{c\} | \mathcal{R} \cup \mathcal{R}' \rangle \) are equivalent presentations, in the sense that two words \( u, v \in (A \cup \{c\})^* \) are equivalent modulo the relations \( \mathcal{R} \) if and only if they are equivalent modulo the relations \( \mathcal{R} \cup \mathcal{R}' \). In particular, the monoid \( M \) is also defined by the infinite presentation

\[
\overline{\mathcal{P}} = \langle A \cup \{c\} | \mathcal{R} \cup \mathcal{R}' \rangle.
\]

**Lemma 3.11.** The infinite presentation \( \overline{\mathcal{P}} \) is a complete presentation of \( M \). The set of irreducible words with respect to this complete rewriting system is \( A^* \cup c^+ b^* a^* \).

**Proof of 3.11.** The fact that \( \overline{\mathcal{P}} \) is a presentation for \( M \) follows from the comments made before the statement of the lemma. By considering the (left-to-right) length-plus-lexicographic ordering on \( \{a, b, c\}^* \) induced by \( a > b > c \)
one sees that the rewriting system $\mathcal{F}$ is noetherian. Also, the set of irreducible words under this rewriting system is easily seen to be equal to the set $A^* \cup c^{-1}b^*a^*$. Finally, to prove that $\mathcal{F}$ is confluent it suffices to consider all possible overlaps between left-hand sides of the rewriting rules $K_\times (x \in A)$ and $C_u (u \in A^*)$, showing that all critical peaks arising from these overlaps resolve (see [GMP11, Section 2]). There are three different ways in which these rewrite rules can overlap, giving rise to three types of critical peaks, all of which can be resolved; see Figure 3. This proves that $\mathcal{F}$ is confluent and thus completes the proof of the lemma.

LEMMA 3.12. The monoid $M$ is biauto.

Proof of 3.12. Let $L = A^* \cup c^{-1}b^*a^*$. We will prove that $(A \cup \{c\}, L)$ is a biautomatic structure for $M$. By the previous lemma, $L$ is a language of unique representatives for $M$, so $L_\epsilon$ is regular. Next, notice that

$$L_u = \{(u, ua) : u \in L\}$$
$$L_b = \{(u, ub) : u \in A^* \cup \{c^i b^j a^k, c^i b^{j+k+1} : i \geq 1, j, k \geq 0\}\}$$
$$L_c = \{(a^k, ca^k) : k \geq 0\} \cup \{(uba^k, cb^{i+1}a^k) : u \in A^*, k \geq 0\}$$
$$\cup \{(c^i b^j a^k, c^{i+1} b^j a^k) : i \geq 1, j, k \geq 0\}$$
$$aL = \{(u, au) : u \in A^* \cup \{c^i a^k, c^i a^{k+1} : i \geq 1, k \geq 0\}$$
$$\cup \{(c^i b^j a^k, c^{i+1} b^j a^k) : i \geq 1, j, k \geq 0\}\}$$
$$bL = \{(u, bu) : u \in A^* \cup \{c^i b^j a^k, c^i b^{j+1} a^k : i \geq 1, j, k \geq 0\}\}$$
$$cL = \{(a^k, ca^k) : k \geq 0\} \cup \{(uba^k, cb^{i+1}a^k) : u \in A^*, k \geq 0\}$$
$$\cup \{(c^i b^j a^k, c^{i+1} b^j a^k) : i \geq 1, j, k \geq 0\}$$.

All of these relations are rational, and so, since $M$ is homogeneous, their images under $\delta_R$ and $\delta_L$ are regular by Proposition 2.3. So $(A, L)$ is a biautomatic structure for $M$.

Let $\Gamma$ denote the derivation graph of $\mathcal{P}$, and $\overline{\Gamma}$ the derivation graph of $\mathcal{F}$. Let $\Gamma_Z$ denote the connected components of $\Gamma$ with vertex set the set of all words in $A \cup \{c\}$ with at least two occurrences of the letter $c$. Likewise let $\overline{\Gamma}_Z$ be the connected component of $\overline{\Gamma}$ with the same vertex set as $\Gamma_Z$.

Let us denote by $\mathcal{C}$ the set of critical circuits of the form $(\mathcal{CT})$ and $\mathcal{Z}$ the critical circuits of the form $(\mathcal{CT}2)$. Observe that the critical circuits in $\mathcal{Z}$ only appear in $\overline{\Gamma}_Z$ since the words labelling vertices in $(\mathcal{CT}2)$ all contain two occurrences of the letter $c$. The set of critical circuits $\mathcal{C} \cup \mathcal{Z}$ forms an infinite homotopy base for $\overline{\Gamma}$ (see [GMP11, Lemma 3]).

We now want to use the infinite homotopy base $\mathcal{C} \cup \mathcal{Z}$ for $\mathcal{F}$ to obtain an infinite homotopy base for $\Gamma$. In order to do this we need to take the critical circuits $(\mathcal{CT})$–$(\mathcal{CT}3)$ and transform them into circuits in the derivation graph $\Gamma$ by replacing each occurrence of an edge $\mathcal{C}_u$ by a corresponding path in $\Gamma$.

As mentioned above when proving that $\mathcal{P}$ and $\mathcal{F}$ are equivalent presentations, the edges $\mathcal{C}_u$ can be realized in $\Gamma$ by paths $\mathcal{C}_u$ which are defined inductively as follows: we first set $C_c$ to be the rule $C$, and then for $u = xu'$, with $x \in A$ and $u' \in A^*$ we set $\mathcal{C}_u$ to be the path

$$\mathcal{C}_u : \quad cxu'ab \xrightarrow{K_{x \cdot u'} ab} xcu'ab \xrightarrow{K_{x \cdot u'} b b} xcu'bb \xrightarrow{K_{x \cdot u'} bb} cxu'bb. \quad (3.8)$$
So, $C_u$ is the path in $\Gamma$ from $cuab$ to $cubb$ given by commuting $c$ through $u$ using the relations $K_x$, applying the relation $C$ to transform $ucab$ into $ucbb$, and then commuting $c$ back through $u$ again using the relations $K_x$, ending at the vertex $cubb$.

Now let us define a mapping $\varphi$ from the set of paths $P(\Gamma)$ in $\Gamma$ to the set of paths $P(\Gamma)$ in $\Gamma$. Let $\varphi : P(\Gamma) \to P(\Gamma)$ be the map given by $(\alpha \cdot C_u \cdot \beta) \varphi = \alpha \cdot C_u \cdot \beta$, for all $\alpha, \beta \in (A \cup \{c\})^*$ and $u \in A^*$, and defined to be the identity on every other edge of $\Gamma$. Let $\mathcal{E} = (\mathcal{E})\varphi$ and $\mathcal{Z} = (\mathcal{Z})\varphi$. Since $\mathcal{E} \cup \mathcal{Z}$ forms a homotopy base for $\Gamma$ it follows that $\mathcal{E} \cup \mathcal{Z}$ is an infinite homotopy base for $\Gamma$ (see [GMP11, Lemma 4]).

Observe that the infinite homotopy base $\mathcal{E} \cup \mathcal{Z}$ for $\Gamma$ is nothing more than the set of circuits in $\Gamma$ obtained by taking the set of circuits $(CT1) - (CT3)$ and replacing each occurrence of the edge $C_u$ by the path $C_u$ defined in (3.8). Let us denote this corresponding set of circuits $\mathcal{E} \cup \mathcal{Z}$ in $\Gamma$ by $(\mathcal{E})\varphi$.

The monoid $M$ is presented by the finite presentation $\mathcal{P}$ and associated to $\Gamma$, the derivation graph of $\mathcal{P}$, we have an infinite homotopy base $\mathcal{E} \cup \mathcal{Z}$. Notice that if $M$ was FDT then we would have a finite subset $\mathcal{E}_0 \cup \mathcal{Z}_0$ of $\mathcal{E} \cup \mathcal{Z}$ which would be a finite homotopy base for $\Gamma$. Our aim now is to show that this leads to a contradiction, and thus conclude that $M$ is not FDT. In order to do this we shall now define a mapping from the set of paths $P(\Gamma)$ in $\Gamma$ into the integral monoid ring $\mathbb{Z}M$. Define $\Phi : P(\Gamma) \to \mathbb{Z}M$ to be the unique map which extends the mapping:

- $(\alpha \cdot K_a \cdot \beta)\Phi = \overline{\alpha}$,
- $(\alpha \cdot K_b \cdot \beta)\Phi = \overline{\alpha}$, and
- $(\alpha \cdot C \cdot \beta)\Phi = \overline{\alpha}$,

where $\alpha, \beta \in (A \cup \{c\})^*$ and $\overline{\alpha} \in M$ denotes the element of $M$ represented by the word $\alpha$, to paths in such a way that

$$(P \circ Q)\Phi = (P)\Phi + (Q)\Phi \quad \text{and} \quad (P^{-1})\Phi = -(P)\Phi.$$
The following basic properties of $\Phi$ are then easily verified for all paths $P, Q \in P(\Gamma)$ and words $\alpha, \beta \in (A \cup \{c\})^*$:

1. $(\alpha \cdot P \cdot \beta)\Phi = \tau \cdot (P)\Phi$
2. $(P \circ (P^{-1})\Phi = 0$
3. $([P, Q])\Phi = 0$ where
   \[ [P, Q] = (P \cdot tQ) \circ (\tau P \cdot Q) \circ (P^{-1} \cdot \tau Q) \circ (tP \cdot Q^{-1}). \]
4. If $P \sim_0 Q$ then $(P)\Phi = (Q)\Phi$.

Here, property 4 follows from properties 1, 2 and 3. Note that property 4 implies that $\Phi$ induces a well-defined map on the homotopy classes of paths of $\Gamma$.

In what follows we shall often omit bars from the top of words in the images under $\Phi$ and simply write words from $(A \cup \{c\})^*$ with the obvious intended meaning. Recall that $\mathcal{C}$ is the set of critical circuits of the form $(\mathcal{CT}T)$ and $(\mathcal{CT}3)$, and that $\mathcal{C} = (\mathcal{C})\phi$ is the corresponding set of circuits in $\Gamma$. Let $F = \{a, b\}^*$ denote the free monoid on the alphabet $\{a, b\}$.

**Lemma 3.13.** If $M$ is fdt then the submodule $\langle (\mathcal{C})\Phi \rangle_{ZF}$, of the left ZF-module $ZF$, generated by $(\mathcal{C})\Phi$ is a finitely generated left ZF-module.

**Proof of 3.13.** Assume that $M$ is fdt and therefore $P$ is fdt. Since $\mathcal{C} \subseteq Z$ is a homotopy base for its derivation graph $\Gamma$, it follows that there are finite subsets $\mathcal{C}_0 \subseteq \mathcal{C}$ and $Z_0 \subseteq Z$ such that $\mathcal{C}_0 \cup Z_0$ is a finite homotopy base for $\Gamma$. Let $\mathcal{C} \in \mathcal{C}$ be arbitrary. We claim that $(\mathcal{C})\Phi \in \langle (\mathcal{C}_0)\Phi \rangle_{ZF}$. Once established, this will prove the lemma, since $(\mathcal{C}_0)\Phi$ is a finite subset of $\langle (\mathcal{C})\Phi \rangle_{ZF}$.

By [GMP11, Lemma 2], since $\mathcal{C}$ is a closed path in $\Gamma$ and $\mathcal{C}_0 \cup Z_0$ is a homotopy base for $\Gamma$, we can write

\[ \mathcal{C} \sim_0 P_1^{-1} \circ (\alpha_1 \cdot Q_1 \cdot \beta_1) \circ P_1 \circ \cdots \circ P_n^{-1} \circ (\alpha_n \cdot Q_n \cdot \beta_n) \circ P_n, \quad (3.9) \]

where each $P_i \in P(\Gamma)$, $\alpha_i, \beta_i \in (A \cup \{c\})^*$ and $Q_i \in (\mathcal{C}_0 \cup Z_0)^{+1}$.

Since the vertices of $\mathcal{C}$ have exactly one $c$, and all the relations in the presentation $P$ involve the letter $c$, it follows that $\alpha_i, \beta_i \in A^*$ and $Q_i \in \mathcal{C}_0$, for all $i \in \{1, \ldots, n\}$. Applying $\Phi$ gives

\[ (\mathcal{C})\Phi = \alpha_1 (Q_1)\Phi + \cdots + \alpha_n (Q_n)\Phi \in \langle (\mathcal{C}_0)\Phi \rangle_{ZF} \]

as claimed.\[ \square \]

To complete our proof, it remains to compute the subset $(\mathcal{C})\Phi$ of $ZF$ and then prove that the submodule $\langle (\mathcal{C})\Phi \rangle_{ZF}$ of $ZF$ is not finitely generated as a left ZF-module, where $F = \{a, b\}^*$. Recall that $\mathcal{C}$ is the set of critical circuits of the form $(\mathcal{CT}T)$ and $(\mathcal{CT}3)$, and that $\mathcal{C} = (\mathcal{C})\phi$. So $\mathcal{C}$ is the set of closed paths $(CT1)$ and $(CT3)$ obtained by applying the mapping $\phi$ to the closed paths $(\mathcal{CT}T)$ and $(\mathcal{CT}3)$, that is, obtained by taking each occurrence of $\mathcal{C}_u$ and replacing it by the path $\mathcal{C}_u$. One can easily deduce from equation (3.8) that for any word $u \in A^*$ we have $C_u \Phi = u$. Using this fact, the result of computing $(\mathcal{C})\Phi$ is given in Table 2.

**Lemma 3.14.** The submodule $\langle (\mathcal{C})\Phi \rangle_{ZF}$ of $ZF$, where

\[ (\mathcal{C})\Phi = \{u(ab - bb)v \mid u, v \in A^*\} \cup \{0_{XM}\}, \]

is not finitely generated as a left ZF-module, and therefore $M$ is not fdt.
Proposition 4.1. Let $M_1$ and $M_2$ be homogeneous monoids. Then $M_1 \ast M_2$ is auto (respectively, biauto) if and only if $M_1$ and $M_2$ are auto (respectively, biauto).

Proof of 4.1. Let $\langle A_1 \mid \mathcal{R}_1 \rangle$ and $\langle A_2 \mid \mathcal{R}_2 \rangle$ be homogeneous presentations for $M_1$ and $M_2$, respectively.

Suppose $M_1 \ast M_2$ is auto; the proof for biauto is similar. By Proposition 2.4, $M_1 \ast M_2$ admits an automatic structure $\langle A_1 \cup A_2, L \rangle$. Since $M_1$ is homogeneous, there is no non-empty word over $A_1$ representing the identity of $M_1$; hence every word in $(A_1 \cup A_2)^*$ representing an element of $M_2$ must lie in $A_2^*$. Thus $\langle A_2, L \cap A_2^* \rangle$ is an automatic structure for $M_2$; similarly, $\langle A_1, L \cap A_1^* \rangle$ is an automatic structure for $M_1$.

On the other hand, suppose $M_1$ and $M_2$ are auto; again, the proof for biauto is similar. Then there are automatic structures $\langle A_1, L_1 \rangle$ and $\langle A_2, L_2 \rangle$ for $M_1$ and $M_2$, respectively. Assume furthermore that $\langle A_1, L_1 \rangle$ and $\langle A_2, L_2 \rangle$
are automatic structures with uniqueness for $M_1$ and $M_2$. Let
\[ L = L_1((L_2 - \varepsilon)(L_1 - \varepsilon))^* L_2. \]

Following the reasoning in [CRRT01, Proof of Theorem 6.2], we see that $(A_1 \cup A_2, L)$ is an automatic structure for $M_1 \ast M_2$.

Now we consider the interaction of free product with fcrs within the class of homogeneous monoids.

**Theorem 4.2 ([PWoo, Theorem D]).** Let $M_1$ and $M_2$ be monoids. Suppose that $M_2$ has no non-trivial left- or right-invertible elements. Then if $M_1 \ast M_2$ is fcrs, then $M_1$ is fcrs.

**Corollary 4.3.** Let $M_1$ and $M_2$ be homogeneous monoids. Then $M_1 \ast M_2$ is fcrs if and only if $M_1$ and $M_2$ are fcrs.

**Proof of 4.3.** Suppose $M_1 \ast M_2$ is fcrs. Since $M_1$ and $M_2$ are homogeneous, neither contains any non-trivial left- or right-invertible elements. So $M_1$ and $M_2$ are both fcrs by Theorem 4.2.

Now suppose that $M_1$ and $M_2$ are fcrs; then they are presented by finite complete rewriting systems $(A_1, R_1)$ and $(A_2, R_2)$, respectively. Assume without loss of generality that $A_1$ and $A_2$ are disjoint. Then $M_1 \ast M_2$ admits a presentation via the rewriting system $(A_1 \cup A_2, R_1 \cup R_2)$, which is easily seen to be complete.

Finally, we recall the following result on the interaction of the free product and fdt:

**Theorem 4.4 ([Ott99a, Ott99b]).** Let $M_1$ and $M_2$ be monoids. Then $M_1 \ast M_2$ is fdt if and only if $M_1$ and $M_2$ is fdt.

Now it is straightforward to use free products to construct the remaining examples. In each example, Corollary 4.3, Theorem 4.4, and Proposition 4.1 together show that the free product has the desired properties.

**Example 4.5.** Let $M_1$ be the fcrs, fdt, non-biauto, auto homogeneous monoid from Example 3.1, and let $M_2$ be the non-fcrs, fdt, biauto, auto homogeneous monoid from Example 3.6. Then the homogeneous monoid $M_1 \ast M_2$ is non-fcrs, fdt, non-biauto, and auto.

**Example 4.6.** Let $M_1$ be the fcrs, fdt, non-biauto, non-auto homogeneous monoid from Example 3.5, and let $M_2$ be the non-fcrs, fdt, biauto, auto homogeneous monoid from Example 3.6. Then the homogeneous monoid $M_1 \ast M_2$ is non-fcrs, fdt, non-biauto, and non-auto.

**Example 4.7.** Let $M_1$ be the fcrs, fdt, non-biauto, auto homogeneous monoid from Example 3.1, and let $M_2$ be the non-fcrs, non-fdt, biauto, auto homogeneous monoid from Example 3.8. Then the homogeneous monoid $M_1 \ast M_2$ is non-fcrs, non-fdt, non-biauto, and auto.

**Example 4.8.** Let $M_1$ be the fcrs, fdt, non-biauto, non-auto homogeneous monoid from Example 3.5, and let $M_2$ be the non-fcrs, non-fdt, biauto, auto homogeneous monoid from Example 3.8. Then the homogeneous monoid $M_1 \ast M_2$ is non-fcrs, non-fdt, non-biauto, and non-auto.
Figure 1 shows the relationship between the various examples. Corollary 4.3, Theorem 4.4, and Proposition 4.1 together show that by taking a free product, one obtains a new monoid whose properties are given by taking the logical conjunction (that is, the ‘and’ operation) of corresponding properties. Thus forming a free product corresponds to the meet operation in the illustrated semilattice. Since the fundamental examples given in Section 3 correspond to the elements of this semilattice that have no non-trivial decomposition, one sees that these examples are the smallest necessary set required to obtain all examples of every possible combination of properties using the free product.

5 FROM HOMOGENEOUS TO n-ARY MULTIHOMOGENEOUS MONOIDS

Thus far we have proved that for every possible combination of the properties fcrs, fdt, biauto, auto, and their negations, there exists a homogeneous monoid with exactly those properties. In this section, we exhibit two constructions that allow us to turn a homogeneous monoid with a particular combination of these properties into an n-ary multihomogeneous with the same combination of properties. The upshot of this is that we have the following result, which follows by combining Theorem 5.13 and Corollary 5.6.

**Theorem 5.1.** For each possible combination of the properties fcrs, fdt, biauto, auto, and their negations, there exists an n-ary multihomogeneous monoid with exactly that combination of properties.

5.1 Rees-ideal-commensurable homogeneous monoids

We begin by introducing a new definition which is inspired by the notions of abstractly commensurable groups [dlHoo, §§ iv.27ff] and Rees index for semigroups. A subsemigroup $T$ of a given semigroup $S$ has finite Rees index if $S \setminus T$ is finite. In that case the semigroup $S$ is said to be a small extension of $T$, and $T$ a large subsemigroup of $S$. The main interest is that such semigroups share many important properties (see [CM] for a survey).

**Definition 5.2.** Two semigroups $S_1$ and $S_2$ are said to be Rees-commensurable if there are finite Rees index subsemigroups $T_i \subseteq S_i$ (for $i = 1, 2$) that are isomorphic.

Is is easy to verify that Rees-commensurability is an equivalence relation on semigroups.

This notion can be naturally extended to ideals.

**Definition 5.3.** Two semigroups $S_1$ and $S_2$ are said to be Rees-ideal-commensurable if there are finite Rees index ideals $U_i \subseteq S_i$ (for $i = 1, 2$) that are isomorphic.

The idea behind these notions is that abstract Rees-ideal-commensurable semigroups share many important properties, such as fcrs, fdt, biauto, and auto.

**Proposition 5.4.** fcrs, fdt, biauto, and auto are preserved under abstract Rees-ideal-commensurability:
Proof of 5.4. 1. It is known that fcrs is inherited by small extensions [Wang98] and by large subsemigroups [KWPW11]. Although the result for small extensions was stated in the context of monoids, it can be naturally extended to semigroups. These two results imply that fcrs is preserved under abstract Rees-(ideal)-commensurability.

2. It is known that fdt is inherited by small extensions [Wang98] of monoids, and by large semigroup ideals [Mal09]. We recall that the notion of finite derivation type was first introduced for monoids, but it was naturally extended to the semigroup case [Mal06].

The result on small extensions can be easily adapted for the semigroup case. Indeed, let T be a semigroup and consider the monoid $T^1$ obtained from T by adding an identity. If T is fdt the derivation graph of $T^1$ can be obtained from the derivation graph of T by adding an extra connected component with a single vertex corresponding to the empty word. Thus $T^1$ is fdt. Now, if S is a small extension of the semigroup T, it turns out that the monoid $S^1$ is a small extension of the monoid $T^1$. Therefore, by [Wang98, Theorem 2] the monoid $S^1$ is fdt. But S is a large ideal of $S^1$, and by [Mal09, Theorem 1] we conclude that S is fdt.

The two results on small extensions and large ideals of semigroups show that fdt is preserved under abstract Rees-ideal-commensurability.

3. By the analogue for biauto of [HTR02, Theorem 1.1], biauto is inherited by small extensions and by large subsemigroups, and therefore biauto is preserved under abstract Rees(-ideal)-commensurability.

4. By [HTR02, Theorem 1.1], auto is inherited by small extensions and by large subsemigroups, and therefore auto is preserved under abstract Rees(-ideal)-commensurability.

[fcrs, biauto, and auto are actually preserved under abstract Rees-commensurability, but for this paper, only the result for abstract Rees-ideal-commensurability is needed.]

The preceding result is important because, in the case of (multi)homogeneous and n-ary (multi)homogeneous monoids, the following result holds:

**Proposition 5.5.** Every finitely presented (multi)homogeneous monoid is Rees-ideal-commensurable to an n-ary (multi)homogeneous monoid, where n can be chosen arbitrarily as long as it is greater than or equal to the length of the longest relation in $\mathcal{R}$.

**Proof of 5.5.** Let $S = \langle A \mid \mathcal{R} \rangle$ be a finite homogeneous presentation of a monoid M. Let n be greater than or equal to the maximum length of a relation in $\mathcal{R}$. Let $I = \{[u] \in M : |u| \geq n\}$. Note that I is an ideal of finite Rees index in M.

Let $\mathcal{R}' = \{(u\ell v, urv) : (\ell, r) \in \mathcal{R}, u, v \in A^*, |u\ell v| = n\}$. Then $S' = \langle A \mid \mathcal{R}' \rangle$ is an n-ary (multi)homogeneous presentation.

Let $I' = \{[u] \in S' : |u| \geq n\}$. The set $I'$ is a finite Rees index ideal of $S'$.

Since $\mathcal{R}'$ is contained in the Thue congruence generated by $\mathcal{R}$, we can define a map $\varphi : I' \to I$, with $[u]_{\mathcal{R}'} \varphi = [u]_{\mathcal{R}}$. This mapping is clearly surjective, but also injective since, for any $[u]_{\mathcal{R}'}, [v]_{\mathcal{R}'} \in I'$ such that $([u]_{\mathcal{R}'}) \varphi = ([v]_{\mathcal{R}'}) \varphi$, we get $[u]_{\mathcal{R}} = [v]_{\mathcal{R}}$, with $|u| = |v| \geq n$, and therefore $[u]_{\mathcal{R}'} = [v]_{\mathcal{R}'}$. It is routine to check that $\varphi$ is a homomorphism. Thus I and $I'$ are isomorphic finite Rees index ideals of M and $M'$. So M and $M'$ are Rees-ideal-commensurable.
Corollary 5.6. Let \( P \) be a list of properties preserved under abstract Rees-ideal-commensurability. Then there exists a (multi)homogeneous monoid satisfying every property in \( P \) if and only if there exists an \( n \)-ary (multi)homogeneous monoid satisfying every property in \( P \).

Note that the list of properties \( P \) can contain ‘negative’ properties like ‘not finitely generated’. As an immediate consequence, we obtain the result we require:

Theorem 5.7. For each possible combination of the properties FCRS, FDT, BIAUTO, and their negations, there exists an \( n \)-ary homogeneous monoid with exactly that combination of properties.

5.2 Embedding into a multihomogeneous monoid

Let \( \langle A | R \rangle \) be a presentation, where \( A = \{a_1, \ldots, a_n\} \). Define a homomorphism

\[
\phi : A^* \to (x, y)^*, \quad a_i \mapsto x^2y^i xy^{n+1-i}.
\]

Proposition 5.8. If \( \langle A | R \rangle \) is homogeneous, then \( \langle x, y | R \phi \rangle \) is multihomogeneous.

Proof of 5.8. Suppose \( \langle A | R \rangle \) is homogeneous. Let \( (u, v) \in R \). Then \( |u| = |v| \). Since \( a_i \phi \) contains 3 symbols \( x \) and \( n+1 \) symbols \( y \) for all \( i \in \{1, \ldots, n\} \), it follows that \( |u\phi|_x = 3|u| = 3|v| = |v\phi|_x \) and \( |u\phi|_y = (n+1)|u| = (n+1)|v| = |v\phi|_y \). Hence \( \langle x, y | R \phi \rangle \) is multihomogeneous.

The set \( A\phi \) is a code in the alphabet \( \{x, y\}^* \) in the sense that \( A\phi \) is a set of free generators for \( (A\phi)^* \). Furthermore, a word \( u \in \{x, y\}^* \) has a unique decomposition of the form \( z_0u_1z_1 \cdots z_{n-1}u_nz_n \), where \( u_i \in (A\phi)^* \) and any other factor of \( u \) belonging to \( (A\phi)^* \) is a factor of some \( u_i \). Notice that \( z_0 \) and \( z_n \) may be empty, and if \( u_1, \ldots, u_n \) are non empty then also \( z_1, \ldots, z_{n-1} \) are non empty.

Proposition 5.9. The monoid \( \langle A | R \rangle \) embeds into the monoid \( \langle x, y | R \phi \rangle \) via the map \( u \mapsto u\phi \), and the words over \( \{x, y\} \) representing elements of \( \langle A | R \phi \rangle \) are precisely the words in \( A^\phi \).

Proof of 5.9. Note first that \( \phi \) is a well-defined homomorphism from \( \langle A | R \rangle \) to \( \langle x, y | R \phi \rangle \) since words over \( A \) related by the congruence \( R^\# \) generated by \( R \) are obviously mapped to words related by \( (R\phi)^\# \).

Now let us prove that the map is injective. Let \( w, w' \in A^* \) and suppose that \( (w\phi, w'\phi) \in (R\phi)^\# \). Then there is a sequence of elementary transitions

\[
w\phi = t_0 \leftrightarrow t_1 \leftrightarrow \ldots \leftrightarrow t_k = w'\phi,
\]

where \( t_i \in \{x, y\}^* \) and \( t_i \) is obtained from \( t_{i-1} \) by applying a relation in \( R\phi \) for all \( i \). The aim is to prove by induction on \( i \) that \( t_i = s_i\phi \) for some \( s_i \in A^* \) and \( s_i \) is obtained from \( s_{i-1} \) by applying a relation in \( R \) for all \( i \).

This is trivially true for \( i = 0 \). So suppose it is true for \( i - 1 \); we will prove it for \( i \). By the induction assumption, \( t_{i-1} = s_{i-1}\phi \) for some \( s_{i-1} \in A^* \). Now, either

- \( t_{i-1} = p(u\phi)q \) and \( t_i = p(v\phi)q \) for some \( (u, v) \in R \) and \( p, q \in \{x, y\}^* \), or
- \( t_{i-1} = p(v\phi)q \) and \( t_i = p(u\phi)q \) for some \( (u, v) \in R \) and \( p, q \in \{x, y\}^* \).
Finally, note that $p$ completes the induction step. For a set $\{u\}_{R\phi}$ of closed paths in $\Gamma(A | R\phi)$, let $C\phi$ denote the set $\{P\phi : P \in C\}$. It is clear that, given a closed path $P$ in $\Gamma(A | R\phi)$, we have $P \sim_C 1_P$ if, and only if, $P\phi \sim_{C\phi} 1_{i(P\phi)}$.

Let $P$ be a non-empty closed path in $\Gamma(A | R\phi)$. By (5.2) there are closed paths $Q_i$ in $\Gamma(A | R\phi)$ such that $P \sim_C 1_P$ if, and only if, $P\phi \sim_{C\phi} 1_{i(P\phi)}$.

So if $C$ is a finite homotopy base for $\Gamma(A | R\phi)$ then $P_i \sim_{C\phi} 1_{i(P\phi)}$, which in turn implies that $P \sim_{C\phi} 1_{i(P\phi)}$. Consequently, $C\phi$ is a finite homotopy base for $\Gamma(A | R\phi)$.  

The two cases are exactly parallel, so assume the first case holds. Now, $t_{i-1} = s_{i-1} \phi$, where $s_{i-1} \in A^*$. So $t_{i-1}$ decomposes as a concatenation of words in $A\phi$; that is, words of the form $x^2y^{|x|^2+j}$ for various $j \in \{1, \ldots, n\}$. Since subwords $x^2$ only occur at the start of such words in $A\phi$, the $x^2$ at the start of $u\phi$ lies at the start of a word from $A\phi$ in the decomposition of $t_{i-1}$. Since all words in $A\phi$ have the same length, $u\phi$ must also finish at the end of some word from $A\phi$ in the decomposition of $t_{i-1}$. Hence $p$ and $q$ are (possibly empty) concatenations of words in $A\phi$, and so $p = p' \phi$ and $q = q' \phi$ for some $p', q' \in A^*$. Thus $t_i = p(\phi)q = (p'\phi q')\phi = s_i \phi$, where $s_i = p'q' \in A^*$. Finally, note that $s_i$ is obtained from $s_{i-1}$ by applying the relation $(u, v)$. This completes the induction step.

Finally, notice that this reasoning also shows that any word related to $w\phi$ by $(R\phi)^n$ must lie in $A^* \phi$. This proves the last part of the proposition.  

The mapping $\phi$ can be naturally extended to the corresponding presentations, and from those to the corresponding associated Squier complexes. Also, a word $t$ can be obtained from a word $a$ by a single application of a defining relation in $R\phi$ if, and only if,

\[
s = z_0u_1z_1 \cdots z_{i-1}(p\phi)z_i \cdots z_{n-1}u_n z_n,
\]

\[
t = z_0u_1z_1 \cdots z_{i-1}(q\phi)z_i \cdots z_{n-1}u_n z_n,
\]

where $q$ can be obtained from $p$ by a single application of a defining relation in $R\phi$ if, and only if,

\[
\#u_1 \# \cdots \#(E\phi)_1 \# \cdots \#u_n \# = 1
\]

(5.1)

the edge corresponding to the application of this defining relation from $R\phi$ to obtain $t$ from $s$, where $E$ is the edge corresponding to the application of the corresponding defining relation in $R$ to obtain $q$ from $p$.

Therefore, if $P$ is a nonempty path in the Squier complex $\Gamma(A | R\phi)$, by using pull-down and push-up, we have

\[
P \sim_C 1_P \circ P_1 \circ P_2 \circ \cdots \circ P_n,
\]

(5.2)

where each $P_i$ is a path of the form (5.1).

Proposition 5.10. The monoid $\langle A | R \rangle$ is FDT if and only if the monoid $\langle x, y | R\phi \rangle$ is FDT.

Proof of 5.10. For a set $C$ of closed paths in $\Gamma(A | R)$, let $C\phi$ denote the set $\{P\phi : P \in C\}$. It is clear that, given a closed path $P$ in $\Gamma(A | R)$, we have $P \sim_C 1_P$ if, and only if, $P\phi \sim_{C\phi} 1_{i(P\phi)}$.

Let $P$ be a non-empty closed path in $\Gamma(A | R\phi)$. By (5.2) there are closed paths $Q_i$ in $\Gamma(A | R\phi)$ such that

\[
P \sim_C 1_P \circ P_1 \circ P_2 \circ \cdots \circ P_n,
\]

and each $P_i$ has the form $\#u_1 \# \cdots \#(Q_i\phi)_i \# \cdots \#u_n \#$.

So if $C$ is a finite homotopy base for $\Gamma(A | R)$ then $P_i \sim_{C\phi} 1_{i(P\phi)}$, which in turn implies that $P \sim_{C\phi} 1_{i(P\phi)}$. Consequently, $C\phi$ is a finite homotopy base for $\Gamma(A | R\phi)$.
Conversely, let $D$ be a finite homotopy base (of closed paths) for $\Gamma(\langle x, y \mid R \phi \rangle)$. For each $Q$ in $D$, arguing as before, there are closed paths $Q_1, \ldots, Q_n$ in $\Gamma(\langle A \mid R \rangle)$, such that

$$Q \sim D \circ P_1 \circ P_2 \circ \cdots \circ P_n,$$

and $P_i = \#u_1 \# \cdots \#(Q_i) \# \cdots \#u_n \#$. Denote by $C$ the finite set of all $Q_i$’s, for all $Q \in D$. From [GMP11, Lemma 2] we conclude that $\sim_D = \sim_C \phi$.

Finally, from the first observation, we know that any closed path $P$ in $\Gamma(\langle A \mid R \rangle)$ satisfies $P \sim C \cup \mathcal{L}_\text{intr}$, since $C \phi$ is a homotopy base of $\Gamma(\langle x, y \mid R \phi \rangle)$.

**Proposition 5.11.** The monoid $\langle A \mid R \rangle$ is auto (respectively, biauto) if and only if the monoid $\langle x, y \mid R \phi \rangle$ is auto (respectively, biauto).

**Proof of 5.11.** For convenience, let $M$ be the monoid $\langle A \mid R \rangle$ and $N$ be the monoid $\langle x, y \mid R \phi \rangle$. We will prove the result for biauto; the result for auto follows by considering multiplication only on one side.

Suppose that $N$ is biauto. By Proposition 2.4, there is a biautomatic structure $(\langle x, y \rangle, K)$ for $N$. By Proposition 5.9, words over $\{x, y\}$ representing elements of the image under $\phi$ of $M$ are precisely those in $(A \phi)^*$. So $K \cap (A \phi)^*$ must map onto the image of $M$.

Let $L = K \phi^{-1}$. The map $\phi$ and its converse $\phi^{-1}$ are rational relations, so $L \subseteq A^*$ is regular. Since $K \cap (A \phi)^*$ must map onto the image of $M$, the language $L$ maps onto $M$. For any $a \in A \cup \{\varepsilon\}$,

$$[u, v] \in L_a \iff u \in L \land v \in L \land u a =_M v$$

$$\iff (\exists u', v' \in K)((u, u') \in \phi \land (v, v') \in \phi \land u'(a \phi) =_N v')$$

$$\iff (\exists u', v' \in K)((u, u') \in \phi \land (v, v') \in \phi \land (u', v') \in K_a \phi)$$

$$\iff (u, v) \in \phi \circ K_{a \phi} \circ \phi^{-1}.$$  

Since $K_{a \phi}$ is rational relation (since $(\langle x, y \rangle, K)$ is an automatic structure for $N$), and $\phi$ is a rational relation, if follows that $L_a$ is a rational relation. Since $M$ is homogeneous, $(u, v) \in L_a \implies |u| - |v| \leq 1$, and so $L_\varepsilon \delta_R$ and $L_\varepsilon \delta_L$ are regular by Proposition 2.3. Symmetrical reasoning shows that $a L_\varepsilon \delta_R$ and $a L_\varepsilon \delta_L$ are regular. Hence $(A, L)$ is a biautomatic structure for $M$.

Now suppose that $M$ is biauto. By Proposition 2.4, there is a biautomatic structure $(A, L)$ for $M$. Let $J = \{x, y\}^* - L \phi$; so $J$ consists of all words over $\{x, y\}^*$ that represent elements of $N - M \phi$. Notice in particular that $\varepsilon \notin J$. Let

$$K = K_1 \cup K_2 \cup K_3 \cup K_4,$$

where

$$K_1 = (J(L \phi - \{\varepsilon\}))^*$$

$$K_2 = (J(L \phi - \{\varepsilon\}))^* J$$

$$K_3 = (J(L \phi - \{\varepsilon\}))(J(L \phi - \{\varepsilon\}))^*$$

$$K_4 = ((L \phi - \{\varepsilon\}))^*.$$

So $K$ consists of alternating products of elements of $J$ and elements of $L \phi - \{\varepsilon\}$. Notice further that $K$ is regular. The aim is to prove that $(\langle x, y \rangle, K)$ is a biautomatic structure for $N$.

First, let $w \in \{x, y\}^*$ be a representative of some element of $N$. Then $w$ is either the empty word, in which case it lies in $K_1$ and so in $L$, or $w$ uniquely factors in one of the four ways

$$z_0 u_1 z_1 \cdots z_{k-1} u_k z_k,$$

$$z_0 u_1 z_1 \cdots z_{k-1} u_k z_k,$$

$$u_1 z_1 \cdots z_{k-1} u_k z_k,$$

$$u_1 z_1 \cdots z_{k-1} u_k z_k.$$
where $z_i \in J$ and $u_i \in (A\Phi)^* - \{\epsilon\}$. For each $i$, there is a word $u_i' \in A^*$ such that $u_i' \Phi = u_i$. Since $(A, L)$ is a bi-automatic structure for $M$, there is a word $u_i'' \in L$ such that $u_i'' \equiv_{M} u_i'$. Let $v_i = u_i'' \Phi \in L\Phi$ for each $i$. Then $v_i = \equiv_{N} u_i$ and so $w$ is equal to one of the four words

$$z_0 v_1 z_1 \cdots z_{k-1} v_{k}, \quad z_0 v_1 z_1 \cdots z_{k-1} v_k z_k,$$

$$v_1 z_1 \cdots z_{k-1} v_k, \quad v_1 z_1 \cdots z_{k-1} v_k z_k,$$

which lie, respectively, in $K_1$, $K_2$, $K_3$, and $K_4$. So every element of $N$ has a representative in $K$.

The next step is to prove that $K_t \delta_R, K_t \delta_L, tK \delta_R, tK \delta_L$ are regular for all $t \in \{x, y, \epsilon\}$. We will show that $K_t \delta_R$ and $K_t \delta_L$ are regular; the other cases are proved by symmetrical reasoning.

So let $w \in K$ and consider right-multiplying $w$ by $t \in \{x, y, \epsilon\}$. The word $w$ is either empty or it uniquely factorizes in one of the forms

$$z_0 u_1 z_1 \cdots z_{k-1} u_k, \quad z_0 u_1 z_1 \cdots z_{k-1} u_k z_k,$$

$$u_1 z_1 \cdots z_{k-1} u_k z_k,$$

where $z_i \in J$ and $u_i \in L\Phi - \{\epsilon\}$. Consider the corresponding factorizations of $w t$:

$$z_0 u_1 z_1 \cdots z_{k-1} u_k t = z_0 u_1 z_1 \cdots z_{k-1} u_k z_k',$$

where $z_k' = t \in J$;

$$z_0 u_1 z_1 \cdots z_{k-1} u_k z_k t = \begin{cases} 
    z_0 u_1 z_1 \cdots z_{k-1} u_k z_k', & \text{where } z_k' = z_k t, \text{ if } z_k t \in J; \\
    z_0 u_1 z_1 \cdots z_{k-1} u_k', & \text{where } (u_k, u_k') \in L(z_k t)\Phi^{-1} \Phi, \text{ if } z_k t \in A\Phi \\
    & \text{where } (\epsilon, u_k') \in L_{vt}\Phi^{-1} \Phi, \text{ if } z_k = z_k' v, \\
    & \text{where } z_k' \in J \text{ and } vt \in A\Phi \\
\end{cases}$$

$$u_1 z_1 \cdots z_{k-1} u_k z_k t = \begin{cases} 
    u_1 z_1 \cdots z_{k-1} u_k z_k', & \text{where } z_k' = z_k t, \text{ if } z_k t \in J; \\
    u_1 z_1 \cdots z_{k-1} u_k', & \text{where } (u_k, u_k') \in L(z_k t)\Phi^{-1} \Phi, \text{ if } z_k t \in A\Phi \\
    & \text{where } (\epsilon, u_k') \in L_{vt}\Phi^{-1} \Phi, \text{ if } z_k = z_k' v, \\
    & \text{where } z_k' \in J \text{ and } vt \in A\Phi \\
\end{cases}$$

Hence

$$K_t = \left( J_\epsilon(L\Phi - \{\epsilon\})_\epsilon \right)^* (\epsilon, t)$$

$$\cup \left( J_\epsilon(L\Phi - \{\epsilon\})_\epsilon \right)^* J_t$$

$$\cup \bigcup_{z \in \{x, y\}} (J_\epsilon(L\Phi - \{\epsilon\})_\epsilon)^* j_\epsilon(L\Phi - \{\epsilon\})_{zt} \Phi^{-1} (z, \epsilon)$$

$$\cup \bigcup_{v \in \{x, y\}} (J_\epsilon(L\Phi - \{\epsilon\})_\epsilon)^* j_v^{-1} (L_{vt}\Phi^{-1} \Phi \cap \{\epsilon\} \times \{x, y\})^*$$
\[ \{ (L\Phi - \{ \varepsilon \})_x (J_\phi (L\Phi - \{ \varepsilon \})_x)^*(\varepsilon, t) \] 
\[ \cup \{ (L\Phi - \{ \varepsilon \})_y (L\Phi - \{ \varepsilon \})_y \} J_t \] 
\[ \cup \bigcup_{z \in \{ x, y \}^*} (\{ (L\Phi - \{ \varepsilon \})_x \}^* (L\Phi - \{ \varepsilon \})_x J_z (L\Phi - \{ \varepsilon \})_{zt} \phi^{-1}(z, \varepsilon) \] 
\[ \cup \bigcup_{v \in \{ x, y \}^*} (\{ (L\Phi - \{ \varepsilon \})_x \}^* (L\Phi - \{ \varepsilon \})_x J_v^{-1}(L\phi\phi^{-1} \phi \cap \{ \varepsilon \} \times \{ x, y \}^* ) \].

Noting that the various unions are all finite, this proves that \( K_t \) is a rational relation. Since \( N \) is homogeneous, \( (u, v) \in K_t \implies \|u\| - \|v\| \leq 1 \), and so \( K_t \delta_R \) and \( K_t \delta_L \) are regular by Proposition 2.3.

Hence \( \langle \{ x, y \}, K \rangle \) is a bi-automaton structure for \( N \).

Recall that if a monoid is defined by a homogeneous presentation \( \langle A \mid R \rangle \), then the alphabet \( A \) represents a unique minimal generating set, and hence any generating set must contain it.

**Proposition 5.12.** If \( \langle B \mid Q \rangle \) is another finite presentation for \( \langle A \mid R \rangle \), with \( A \subseteq B \), then

\[ \mathcal{P} = \langle x, y, B \mid Q, (a\phi, a) \mid (\forall a \in A) \rangle \]

is a finite presentation defining \( \langle x, y \mid R\phi \rangle \). Moreover, the presentation \( \langle B \mid Q \rangle \) is complete if and only if the presentation \( \mathcal{P} \) is complete. Hence \( \langle A \mid R \rangle \) is fcrs if and only if \( \langle x, y \mid R\phi \rangle \) is fcrs.

**Proof of 5.12.** For convenience, let \( M \) be the monoid \( \langle A \mid R \rangle \) and \( N \) be the monoid \( \langle x, y \mid R\phi \rangle \).

Using Tietze transformations we obtain a new presentation for \( N \). Indeed, for each \( a \in A \) we insert in the presentation \( \langle x, y \mid R\phi \rangle \) a generator \( a \) and a relation \( (a\phi, a) \), thus obtaining a Tietze equivalent presentation

\[ \langle x, y, A \mid R\phi, (a\phi, a) \mid (\forall a \in A) \rangle . \]

Since \( \phi \) is an homomorphism and again by Tietze transformations we get another presentation defining the same monoid:

\[ \langle x, y, A \mid R, (a\phi, a) \mid (\forall a \in A) \rangle . \]

As it is possible to obtain from the presentation \( \langle A \mid R \rangle \) the presentation \( \langle B \mid Q \rangle \) using finitely many Tietze transformations, we can obtain from the presentation \( \langle x, y, A \mid R, (a\phi, a) \mid (\forall a \in A) \rangle \) the presentation \( \mathcal{P} \) by using the same Tietze transformations.

Suppose that \( \langle B \mid Q \rangle \) is also complete. Observe that \( \Omega \) relates words from the alphabet \( B \), and that a relation from the set \( E = \{ (a\phi, a) : a \in A \} \) has left hand side in \( \{ x, y \}^* \) and right hand side in \( A \). Thus, if \( w \rightarrow_{\Omega} w' \rightarrow_{\varepsilon} w'' \), we can find \( \overline{w} \) such that \( w \rightarrow_{\varepsilon} \overline{w} \rightarrow_{\Omega} w'' \). Hence, \( \rightarrow_{\varepsilon} \) quasi-commutes over \( \rightarrow_{\Omega} \), that is, \( \rightarrow_{\Omega} \circ \rightarrow_{\varepsilon} \subseteq \rightarrow_{\varepsilon} \circ \rightarrow_{\Omega} \). By [BD86, Theorem 1], the rewriting system \( \langle \{ x, y \} \cup A, \Omega \cup \varepsilon \rangle \) is terminating if, and only if, both \( \Omega \) and \( \varepsilon \) are terminating. By assumption \( \Omega \) is terminating and it is easy to verify that \( \varepsilon \) is also, by length-reduction.

Attending to the previous observation on the relations from \( \Omega \) and \( \varepsilon \), we can also deduce that whenever \( w' \leftarrow_{\Omega} w \rightarrow_{\varepsilon} w'' \), there exists \( \overline{w} \) such that \( w' \rightarrow_{\varepsilon} \overline{w} \leftarrow_{\Omega} w'' \). Therefore, the relations \( \rightarrow_{\varepsilon} \) and \( \leftarrow_{\Omega} \) commute, that is,
by \[\text{BN98}, \text{Lemma 2.7.10}\], if \(\mathcal{E}\) and \(\mathcal{Q}\) are confluent and \(\rightarrow^*_\mathcal{E}\) and \(\rightarrow^*_\mathcal{Q}\) commute, then \(\mathcal{Q} \cup \mathcal{E}\) is also confluent. Since by assumption \(\mathcal{Q}\) is confluent it remains to show that \(\mathcal{E}\) is confluent.

Observe that, given left hand side of rules in \(\mathcal{E}\), \(a_i\phi, a_j\phi\), they only overlap trivially, that is, when \(a_i\phi = a_j\phi\). Since \(\phi\) is injective we get \(a_i = a_j\). Therefore, \(\mathcal{E}\) has no critical pairs and thus is also confluent.

Conversely, suppose that \(\mathcal{P}\) is a complete presentation. It is then clear that \(\langle B \mid \mathcal{Q}\rangle\) is terminating. Confluence also holds from the fact that in \(\mathcal{P}\) all critical pairs are resolved, in particular, those arising from relations in \(\mathcal{Q}\). Hence, each resolution associated to relations from \(\mathcal{Q}\), can only involve relations from \(\mathcal{Q}\), since left hand sides of rules in \(\mathcal{E}\) belong to \(\{x, y\}^*\). Therefore, \(\langle B \mid \mathcal{Q}\rangle\) is complete.

Combining the preceding three propositions gives the desired result:

**Theorem 5.13.** For each possible combination of the properties \(\text{ECKRS, FDT, BIAuto, AUTO, and their negations}\), there exists a multihomogeneous monoid with exactly that combination of properties.

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