The three point Pick problem on the Bidisk

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0 Introduction

The original Pick problem is to determine, given $N$ points $\lambda_1, \ldots, \lambda_N$ in the unit disk $\mathbb{D}$ and $N$ complex numbers $w_1, \ldots, w_N$, whether there exist a function $\phi$ in the closed unit ball of $H^\infty(\mathbb{D})$ (the space of bounded analytic functions on $\mathbb{D}$) that maps each point $\lambda_i$ to the corresponding value $w_i$. This problem was solved by G. Pick in 1916 [9], who showed that a necessary and sufficient condition is that the Pick matrix

$$\left( \frac{1 - \bar{w}_i w_j}{1 - \lambda_i \lambda_j} \right)_{i,j=1}^N$$

be positive semi-definite.

It is well-known that if the problem is extremal, i.e. the problem can be solved with a function of norm one but not with a function of any smaller norm, then the Pick matrix is singular, and the corresponding solution is a unique Blaschke product, whose degree equals the rank of the Pick matrix [6, 5].

In [1], the first author extended Pick’s theorem to the space $H^\infty(\mathbb{D}^2)$, the bounded analytic functions on the bidisk; see also [4, 3, 2]. It was shown in [2] that if the problem has a solution, then it has a solution that is a rational inner function; however the qualitative properties of general solutions are not fully understood. The example $\lambda_1 = (0,0)$, $\lambda_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$, $w_1 = 0$, $w_2 = \frac{1}{2}$ shows that even extremal problems do not always have unique solutions.

The two point Pick problem on the bidisk is easily analyzed. It can be solved if and only if the Kobayashi distance between $\lambda_1$ and $\lambda_2$ is greater than or equal to the hyperbolic distance between $w_1$ and $w_2$. On the bidisk, the Kobayashi distance is just the maximum

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of the hyperbolic distance between the first coordinates, and the hyperbolic distance between the second coordinates. A pair of points in $\mathbb{D}^2$ is called balanced if the hyperbolic distance between their first coordinates equals the hyperbolic distance between their second coordinates.

The two point Pick problem has a unique solution if and only if the Kobayashi distance between $\lambda_1$ and $\lambda_2$ exactly equals the hyperbolic distance between $w_1$ and $w_2$, and moreover $(\lambda_1, \lambda_2)$ is not balanced. In this case the solution is a M"{o}bius map in the coordinate function in which the Kobayashi distance is attained. If the distance between $\lambda_1$ and $\lambda_2$ equals the distance between $w_1$ and $w_2$, but the pair $(\lambda_1, \lambda_2)$ is balanced, then the function $\phi$ will be uniquely determined on the geodesic disk passing through $\lambda_1$ and $\lambda_2$, but will not be unique off this disk. (For the example $\lambda_1 = (0, 0)$, $\lambda_2 = (\frac{1}{2}, \frac{1}{2})$, $w_1 = 0$, $w_2 = \frac{1}{2}$, on the diagonal $\{(z, z)\}$ we must have $\phi(z, z) = z$; but off the diagonal any convex combination of the two coordinate functions $z^1$ and $z^2$ will work).

It is the purpose of this article to examine the three point Pick problem on the bidisk. Our main result is the following:

**Theorem 0.1** The solution to an extremal non-degenerate three point problem on the bidisk is unique. The solution is given by a rational inner function of degree 2. There is a formula for the solution in terms of two uniquely determined rank one matrices.

In the next section, we shall define precisely the terms “extremal” and “non-degenerate”, but roughly it means that the problem is genuinely two-dimensional, is really a 3 point problem not a 2 point problem, and the minimal norm of a solution is 1.

## 1 Notation and Preliminaries

We wish to consider the $N$ point Pick interpolation problem

$$\phi(\lambda_i) = w_i, \quad i = 1, \ldots, N$$

and

$$\|\phi\|_{H^\infty(\mathbb{D}^2)} \leq 1.$$  \hspace{1cm} (1.1)

We shall say that a solution $\phi$ to (1.1) is an extremal solution if $\|\phi\| = 1$, and no solution has a smaller norm.

For a point $\lambda$ in $\mathbb{D}^2$, we shall use superscripts to denote coordinates:

$$\lambda = (\lambda^1, \lambda^2).$$
Let $W$, $\Lambda^1$ and $\Lambda^2$ denote the $N$-by-$N$ matrices

$$W = (1 - \bar{w}_i w_j)_{i,j=1}^N,$$

$$\Lambda^1 = (1 - \bar{\lambda}^1_i \lambda^1_j)_{i,j=1}^N,$$

$$\Lambda^2 = (1 - \bar{\lambda}^2_i \lambda^2_j)_{i,j=1}^N.$$ 

A pair $\Gamma, \Delta$ of $N$-by-$N$ positive semi-definite matrices is called permissible if

$$W = \Lambda^1 \cdot \Gamma + \Lambda^2 \cdot \Delta.$$  \hspace{1cm} (1.2)

Here $\cdot$ denotes the Schur or entrywise product:

$$(A \cdot B)_{ij} := A_{ij} B_{ij}.$$ 

The main result of [1] is that the problem (1.1) has a solution if and only if there is a pair $\Gamma, \Delta$ of permissible matrices.

A kernel $K$ on $\{\lambda_1, \ldots, \lambda_N\} \times \{\lambda_1, \ldots, \lambda_N\}$ is a positive definite $N$-by-$N$ matrix

$$K_{ij} = K(\lambda_i, \lambda_j).$$ 

We shall call the kernel $K$ admissible if

$$\Lambda^1 \cdot K \geq 0$$

and

$$\Lambda^2 \cdot K \geq 0.$$ 

If the problem (1.1) has a solution and $K$ is an admissible kernel, then (1.2) implies that $K \cdot W \geq 0$. We shall call the kernel $K$ active if it is admissible and $K \cdot W$ has a non-trivial null-space. Notice that all extremal problems have an active kernel.

If one can find a pair of permissible matrices one of which is 0, then the Pick problem is really a one-dimensional problem because one can find a solution $\phi$ that depends only on one of the coordinate functions. If this occurs, we shall call the problem degenerate; otherwise we shall call it non-degenerate.

2 The three point problem

We wish to analyze extremal solutions to three point Pick problems. Fix three points $\lambda_1, \lambda_2, \lambda_3$ in $\mathbb{D}^2$, and three numbers $w_1, w_2, w_3$. Let notation be as in the previous section. We shall make the following assumptions throughout this section:
(a) The function $\phi$ is an extremal solution to the Pick problem of interpolating $\lambda_i$ to $w_i$, where $i$ ranges from 1 to 3.

(b) The function $\phi$ is not an extremal solution to any of the three two point Pick problems mapping two of the $\lambda_i$'s to the corresponding $w_i$'s.

(c) The three point problem is non-degenerate.

**Lemma 2.1** If $K$ is admissible, then $\text{rank}(K \cdot W) > 1$.

**Proof:** Suppose $(\Gamma, \Delta)$ is permissible. By (1.2), we have

$$K \cdot W = K \cdot \Lambda^1 \cdot \Gamma + K \cdot \Lambda^2 \cdot \Delta.$$ 

If $\text{rank}(K \cdot W) = 1$, then either $\Gamma = 0$ (which violates (c)), or there exists $t > 0$ such that

$$K \cdot \Lambda^1 \cdot \Gamma = tK \cdot W.$$

But then $(\frac{1}{t} \Gamma, 0)$ is permissible, violating assumption (c). \qed

**Lemma 2.2** If $K$ is an admissible kernel with a non-vanishing column, then $\text{rank}(K \cdot \Lambda^1) \geq 2$ and $\text{rank}(K \cdot \Lambda^2) \geq 2$.

**Proof:** Suppose that $\text{rank}(K \cdot \Lambda^1) = 1$. As no entry of $\Lambda^1$ can be 0, and some column of $K$ is non-vanishing, there is a column of $K \cdot \Lambda^1$ that is non-vanishing. As $K \cdot \Lambda^1$ is self-adjoint and rank one and has non-zero diagonal entries, the other two columns of $K \cdot \Lambda^1$ must be non-zero multiples of this non-vanishing column. So $Q := K \cdot \Lambda^1$ is a positive rank one matrix with no zero entries, and $K$ has no zero entries.

So

$$\Lambda^2 = \left( \frac{1}{K} \right) \cdot (K \cdot \Lambda^2)$$

$$= \left( \frac{1}{Q} \right) \cdot \Lambda^1 \cdot (K \cdot \Lambda^2)$$

$$= \left( \frac{1}{Q} \right) \cdot (K \cdot \Lambda^2) \cdot \Lambda^1,$$

where by $\frac{1}{K}$ and $\frac{1}{Q}$ is meant the entrywise reciprocal. Now $K \cdot \Lambda^2$ is positive by hypothesis, and $\frac{1}{Q}$ is positive because $Q$ is rank one and non-vanishing; moreover the Schur product of two positive matrices is positive [8, Thm 5.2.1]. Therefore $(\Gamma + \Delta \cdot \frac{1}{Q} \cdot K \cdot \Lambda^2, 0)$ is permissible, which violates assumption (c). \qed

**Lemma 2.3** If $K$ is an active kernel, it has a non-vanishing column.
Proof: By assumption (b), we cannot have both $K(\lambda_1, \lambda_2) = 0$ and $K(\lambda_1, \lambda_3) = 0$; for then $K$ restricted to $\{\lambda_2, \lambda_3\} \times \{\lambda_2, \lambda_3\}$ would be an active kernel for the two point problem on $\lambda_2, \lambda_3$, and so any solution to the two point problem would have norm at least one, so $\phi$ would be an extremal solution to the two point problem.

If neither of $K(\lambda_1, \lambda_2)$ or $K(\lambda_1, \lambda_3)$ are 0, we are done. So assume without loss of generality that the first is non-zero and the second equals zero. But then $K(\lambda_2, \lambda_3)$ cannot equal zero, for then $K$ restricted to $\{\lambda_1, \lambda_2\} \times \{\lambda_1, \lambda_2\}$ would be active, violating assumption (b). Thus we can conclude that the second column of $K$ is non-vanishing. \[\square\]

Lemma 2.4 If $(\Gamma , \Delta)$ is a permissible pair, then $\text{rank}(\Gamma) = 1 = \text{rank}(\Delta)$. 

Proof: Let $K$ be an active kernel. Then $K \cdot W$ is rank 2, and annihilates some vector $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$.

Moreover, by assumption (b), none of the entries of $\gamma$ are 0.

Suppose rank$(\Gamma) > 1$. We have 

$$K \cdot W = K \cdot \Lambda^1 \cdot \Gamma + K \cdot \Lambda^2 \cdot \Delta.$$ 

As $K \cdot \Lambda^1$ has non-zero diagonal terms, Oppenheim’s theorem [7, Thm 7.8.6] guarantees that rank$(K \cdot \Lambda^1 \cdot \Gamma) \geq \text{rank}(\Gamma)$. As $K \cdot W$ has rank 2, and $K \cdot \Lambda^2 \cdot \Delta \geq 0$, we must have rank$(\Gamma) = 2$. Write

$$\Gamma = \bar{u} \otimes \bar{u} + \bar{v} \otimes \bar{v},$$

where $\bar{u}$ and $\bar{v}$ are not collinear; if $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, then $\bar{u} \otimes \bar{u}$ denotes the matrix 

$$(\bar{u} \otimes \bar{u})_{ij} = u_i \bar{u}_j.$$ 

Let

$$K \cdot \Lambda^1 = \bar{w} \otimes \bar{w} + \bar{x} \otimes \bar{x}$$ 

if $K \cdot \Lambda^1$ is rank two, and

$$K \cdot \Lambda^1 = \bar{w} \otimes \bar{w} + \bar{x} \otimes \bar{x} + \bar{y} \otimes \bar{y}$$ 

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if it is rank three.

Notice that \((K \cdot \Lambda^1 \cdot \Gamma) \vec{\gamma} = 0\), because \(K \cdot \Lambda^1 \cdot \Gamma\) is positive and
\[\langle (K \cdot \Lambda^1 \cdot \Gamma) \vec{\gamma}, \vec{\gamma} \rangle = -\langle (K \cdot \Lambda^2 \cdot \Delta) \vec{\gamma}, \vec{\gamma} \rangle \leq 0.\]

Therefore all 4 of \((\vec{u} \otimes \vec{u}) \cdot (\vec{w} \otimes \vec{w}), (\vec{u} \otimes \vec{u}) \cdot (\vec{x} \otimes \vec{x}), (\vec{v} \otimes \vec{v}) \cdot (\vec{w} \otimes \vec{w}), (\vec{v} \otimes \vec{v}) \cdot (\vec{x} \otimes \vec{x})\) annihilate \(\vec{\gamma}\). Therefore
\[
\sum_{j=1}^{3} \vec{u}_j \vec{w}_j \gamma_j = 0
\]
\[
= \sum_{j=1}^{3} \vec{u}_j \vec{x}_j \gamma_j
\]
\[
= \sum_{j=1}^{3} \vec{v}_j \vec{w}_j \gamma_j
\]
\[
= \sum_{j=1}^{3} \vec{v}_j \vec{x}_j \gamma_j
\]

Therefore the vectors
\[
\begin{pmatrix}
\vec{w}_1 \gamma_1 \\
\vec{w}_2 \gamma_2 \\
\vec{w}_3 \gamma_3
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\vec{x}_1 \gamma_1 \\
\vec{x}_2 \gamma_2 \\
\vec{x}_3 \gamma_3
\end{pmatrix}
\]
are both orthogonal to both \(\vec{u}\) and \(\vec{v}\), and therefore are collinear (since \(\vec{u}\) and \(\vec{v}\) span a two-dimensional subspace of \(\mathbb{C}^3\)). As none of the entries of \(\vec{\gamma}\) are 0, it follows that \(\vec{w}\) and \(\vec{x}\) are collinear. Therefore \(\text{rank}(K \cdot \Lambda^1) = 1\), contradicting Lemmata (2.2) and (2.3).

**Lemma 2.5** The matrices \(\Gamma\) and \(\Delta\) are unique.

**Proof:** If both \((\Gamma_1, \Delta_1)\) and \((\Gamma_2, \Delta_2)\) were permissible, then \((\frac{1}{2}(\Gamma_1 + \Gamma_2), \frac{1}{2}(\Delta_1 + \Delta_2))\) would also be permissible. As all permissible matrices are rank one by Lemma 2.4, it follows that \(\Gamma_1\) and \(\Gamma_2\) are constant multiples of each other, and so are \(\Delta_1\) and \(\Delta_2\).

So suppose
\[
W = \Lambda^1 \cdot \Gamma + \Lambda^2 \cdot \Delta
\]
and
\[
W = \Lambda^1 \cdot t_1 \Gamma + \Lambda^2 \cdot t_2 \Delta,
\]
where both \( t_1, t_2 \) are positive, one is less than 1, and the other is bigger than 1. Then

\[
(1 - t_1) \Lambda_1 \cdot \Gamma + (1 - t_2) \Lambda_2 \cdot \Delta = 0.
\]

Assume without loss of generality that \( t_1 < 1 < t_2 \). Then \( \frac{t_2 - t_1}{1 - t_1} \Gamma, 0 \) is permissible, which contradicts Assumption (c). \( \square \)

**Theorem 2.6** The solution to an extremal non-degenerate three point problem satisfying Assumptions (a)-(c) is unique. It is given by a rational inner function of degree 2, and there is a formula in terms of \( \Gamma \) and \( \Delta \).

**Proof:** We have

\[
W = \Lambda_1 \cdot \Gamma + \Lambda_2 \cdot \Delta. \tag{2.7}
\]

Choose vectors \( \vec{a} \) and \( \vec{b} \) so that \( \Gamma = \vec{a} \otimes \vec{a} \) and \( \Delta = \vec{b} \otimes \vec{b} \).

Choose some point \( \lambda_4 \) in \( \mathbb{D}^2 \), distinct from the first three points. Let \( w_4 \) be the value attained at \( \lambda_4 \) by some solution \( \phi \) of the three point problem (1.1). Then the four point problem, interpolating \( \lambda_i \) to \( w_i \) for \( i = 1, \ldots, 4 \) has a solution, so we can find a pair of 4-by-4 permissible matrices \( \tilde{\Gamma} \) and \( \tilde{\Delta} \) satisfying (1.2). As the restriction of these matrices to the first three points satisfy (2.7), and \( \Gamma \) and \( \Delta \) are unique by Lemma 2.5, we get that \( \tilde{\Gamma} \) and \( \tilde{\Delta} \) are extensions of \( \Gamma \) and \( \Delta \). Therefore we have

\[
\begin{pmatrix}
W & 1 - \bar{w}_1 w_4 \\
* & 1 - \bar{w}_2 w_4 \\
* & 1 - w_4^2 \\
* & * & * & *
\end{pmatrix}
= \begin{pmatrix}
\Gamma \\
\bar{g}_1 \\
\bar{g}_2 \\
\bar{g}_3 \\
\bar{g}_4
\end{pmatrix} \cdot \begin{pmatrix}
\Lambda_1 \\
\lambda_1^1 \\
\lambda_1^2 \\
\lambda_1^3 \\
\lambda_1^4
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
\Delta \\
\bar{d}_1 \\
\bar{d}_2 \\
\bar{d}_3 \\
\bar{d}_4
\end{pmatrix} \cdot \begin{pmatrix}
\Lambda_2 \\
\lambda_2^1 \\
\lambda_2^2 \\
\lambda_2^3 \\
\lambda_2^4
\end{pmatrix}
\]

As \( \tilde{\Gamma} \) is positive, it must be that \( \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \) is in the range of \( \Gamma \), so

\[
\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = s \vec{a} = s \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}
\]

for some constant \( s \). Similarly,

\[
\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = t \vec{b} = t \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}
\]
for some \( t \).

Let

\[
\vec{v}_1 = \begin{pmatrix}
(1 - \bar{\lambda}_1^1 \lambda_1^1) a_1 \\
(1 - \bar{\lambda}_2^1 \lambda_1^1) a_2 \\
(1 - \bar{\lambda}_3^1 \lambda_1^1) a_3
\end{pmatrix}
\]

\[
\vec{v}_2 = \begin{pmatrix}
(1 - \bar{\lambda}_1^2 \lambda_2^2) b_1 \\
(1 - \bar{\lambda}_2^2 \lambda_2^2) b_2 \\
(1 - \bar{\lambda}_3^2 \lambda_2^2) b_3
\end{pmatrix}
\]

\[
\vec{v}_3 = \begin{pmatrix}
\bar{w}_1 \\
\bar{w}_2 \\
\bar{w}_3
\end{pmatrix}
\]

Looking at the first three entries of the last column of Equation (2.8), we get

\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} = s\vec{v}_1 + t\vec{v}_2 + w_4\vec{v}_3. \quad (2.9)
\]

Equation (2.9) has a unique solution for \( s, t \) and \( w_4 \) unless

\[
det \begin{pmatrix}
(1 - \bar{\lambda}_1^1 \lambda_1^1) a_1 & (1 - \bar{\lambda}_2^1 \lambda_1^2) b_1 & \bar{w}_1 \\
(1 - \bar{\lambda}_1^2 \lambda_1^1) a_2 & (1 - \bar{\lambda}_2^2 \lambda_2^1) b_2 & \bar{w}_2 \\
(1 - \bar{\lambda}_3^1 \lambda_1^1) a_3 & (1 - \bar{\lambda}_3^2 \lambda_2^1) b_3 & \bar{w}_3
\end{pmatrix} = 0. \quad (2.10)
\]

Notice that the determinant in (2.10) is analytic in \( \lambda_4 \). So if there is a single point \( \lambda_4 \) for which the determinant does not vanish, there is an open neighborhood of this point for which the determinant doesn’t vanish. Consequently \( w_4 \) (and hence \( \phi \)) would be determined uniquely on this open set, and hence on all of \( \mathbb{D}^2 \).

Suppose the determinant in (2.10) vanished identically. Then there is a set of uniqueness of \( \lambda_4 \)’s on which Equation (2.9) can be solved with either \( s \) or \( t \) equal to 0. (If both \( s \) and \( t \) were uniquely determined, then \( \vec{v}_3 \) would be 0, violating Assumption (a)). Without loss of generality, take \( t = 0 \). Moreover, we can also assume without loss of generality that \( w_1 \) and \( w_2 \) do not both vanish.

Then one can use the first component of Equation (2.9) to solve for \( s \), and the second one to get

\[
w_4 = \frac{(1 - \bar{\lambda}_1^1 \lambda_1^1) a_1 + (1 - \bar{\lambda}_2^1 \lambda_1^1) a_2}{(1 - \bar{\lambda}_1^1 \lambda_1^1) a_1 \bar{w}_2 + (1 - \bar{\lambda}_2^1 \lambda_1^1) a_2 \bar{w}_1}
\]

Then \( w_4 \) is given uniquely as a rational function of degree 1 of \( \lambda_4^1 \), violating both Assumptions (b) and (c).
Therefore we can assume that there is an open set on which Equation (2.9) has a unique solution, so by Cramer’s rule we get

\[
\phi(\lambda) = w_4 = \frac{\det\begin{pmatrix}
(1 - \bar{\lambda}_1 \lambda_1) a_1 & (1 - \bar{\lambda}_2 \lambda_3) b_1 & 1 \\
(1 - \bar{\lambda}_2 \lambda_1) a_2 & (1 - \bar{\lambda}_3 \lambda_4) b_2 & 1 \\
(1 - \bar{\lambda}_3 \lambda_1) a_3 & (1 - \bar{\lambda}_4 \lambda_2) b_3 & 1 
\end{pmatrix}
}{\det\begin{pmatrix}
(1 - \bar{\lambda}_1 \lambda_4) a_1 & (1 - \bar{\lambda}_2 \lambda_2) b_1 & w_1 \\
(1 - \bar{\lambda}_2 \lambda_4) a_2 & (1 - \bar{\lambda}_3 \lambda_1) b_2 & w_2 \\
(1 - \bar{\lambda}_3 \lambda_4) a_3 & (1 - \bar{\lambda}_4 \lambda_3) b_3 & w_3 
\end{pmatrix}}. \tag{2.11}
\]

Equation (2.11) gives a formula for \( \phi \) that shows that \( \phi \) is a rational function of degree at most 2, whose second order terms only involve the mixed product \( \lambda_1^1 \lambda_2^2 \).

To show that \( \phi \) is inner, we follow [2]. We can rewrite (1.2) as

\[
1 + \bar{\lambda}_i \lambda_j a_i \bar{a}_j + \bar{\lambda}_j \lambda_i b_i \bar{b}_j = \bar{w}_i w_j + a_i \bar{a}_j + b_i \bar{b}_j. \tag{2.12}
\]

Realizing both sides of (2.12) as Grammians, we get that there exists a 3-by-3 unitary \( U \) such that, for \( j = 1, 2, 3 \),

\[
U \begin{pmatrix}
\lambda_1^i a_i \\
\lambda_2^i b_i 
\end{pmatrix} = \begin{pmatrix}
w_j \\
\bar{a}_j \\
\bar{b}_j 
\end{pmatrix}. \tag{2.13}
\]

Writing

\[
U = \begin{pmatrix}
\mathbb{C} & \mathbb{C}^2 \\
\mathbb{C}^2 & \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\end{pmatrix},
\]

and letting

\[
E_\lambda = \begin{pmatrix}
\lambda_1^1 & 0 \\
0 & \lambda_2^2
\end{pmatrix},
\]

we can solve (2.13) to get

\[
w_j = A + B E_{\lambda_j} (1 - DE_{\lambda_j})^{-1} C.
\]

So the function

\[
\psi(\lambda) = A + B E_{\lambda} (1 - DE_{\lambda})^{-1} C
\]

interpolates the original data. Moreover \( \psi \) is inner, because a calculation shows that

\[
1 - \overline{\psi(\lambda) \psi(\lambda)} = ((1 - DE_{\lambda})^{-1} C)^*(1 - E_{\lambda}^* E_{\lambda})((1 - DE_{\lambda})^{-1} C),
\]

so \( |\psi| \) is less than 1 on \( \mathbb{D}^2 \) and equals 1 on the distinguished boundary. By uniqueness, we must have \( \psi = \phi \), and hence \( \phi \) is inner.
Finally, we must show that the degree of $\phi$ is exactly two. This is because an easy calculation shows that a rational function of degree one

$$\frac{c_1 + c_2 z^1 + c_3 z^2}{c_4 + c_5 z^1 + c_6 z^2}$$

is inner only if it is a function of either just $z^1$ or just $z^2$, i.e. either both $c_2$ and $c_5$ or both $c_3$ and $c_6$ can be chosen to be zero. This would violate Assumption (c).

\[\square\]

3 Finding $\Gamma$ and $\Delta$

Formula (2.11) works fine, provided one knows $\Gamma$ and $\Delta$ (or, equivalently, $a_1, a_2, a_3$ and $b_1, b_2, b_3$). Lemma 2.5 assures us that $\Gamma$ and $\Delta$ are unique; how does one find them?

First, let us make a simplifying normalization. One can pre-compose $\phi$ with an automorphism of $\mathbb{D}^2$, and post-compose it with an automorphism of $\mathbb{D}$; so one can assume that $\lambda_1 = (0, 0)$ and $w_1 = 0$. Write $\lambda_2 = (\alpha_2, \beta_2)$ and $\lambda_3 = (\alpha_3, \beta_3)$. Moreover, as $\Gamma_{ij} = a_i \bar{a}_j$ and $\Delta_{ij} = b_i \bar{b}_j$, we can choose $a_1 \geq 0$ and $b_1 \geq 0$; again without loss of generality we can assume that $b_1 > 0$. Thus we have

$$\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 - |w_2|^2 & 1 - \bar{w}_2 w_3 \\
1 & 1 - w_2 \bar{w}_3 & 1 - |w_3|^2
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & 1 \\
1 & |a_2|^2 & a_2 \bar{a}_3 \\
a_1 a_2 & \bar{a}_2 a_3 & |a_3|^2
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 - |\alpha_2|^2 & 1 - \bar{\alpha}_2 \alpha_3 \\
1 & 1 - \alpha_2 \bar{\alpha}_3 & 1 - |\alpha_3|^2
\end{pmatrix}

(3.1)

+ \begin{pmatrix}
b_1^2 & b_1 \bar{b}_2 & b_1 \bar{b}_3 \\
b_1 b_2 & |b_2|^2 & b_2 \bar{b}_3 \\
b_1 b_3 & \bar{b}_2 b_3 & |b_3|^2
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 - |\beta_2|^2 & 1 - \bar{\beta}_2 \beta_3 \\
1 & 1 - \beta_2 \bar{\beta}_3 & 1 - |\beta_3|^2
\end{pmatrix}

(3.2)

Looking at the first column of (3.1) we get

$$b_1 = \frac{1 - a_1^2}{\sqrt{1 - a_1^2}}$$
$$b_2 = \frac{1 - a_1 a_2}{\sqrt{1 - a_1^2}}$$
$$b_3 = \frac{1 - a_1 a_3}{\sqrt{1 - a_1^2}}$$

Thus we have three equations that uniquely determine $a_1, a_2, a_3$:

$$(1 - a_1^2)(1 - |w_2|^2) = (1 - a_1^2)|a_2|^2(1 - |\alpha_2|^2) + |1 - a_1 a_2|^2(1 - |\beta_2|^2)$$

(3.2)

$$(1 - a_1^2)(1 - \bar{w}_2 w_3) = (1 - a_1^2)a_2 \bar{a}_3(1 - \bar{\alpha}_2 \alpha_3) + (1 - a_1 a_2)(1 - \bar{a}_1 a_3)(1 - \bar{\beta}_2 \beta_3)$$

(3.3)

$$(1 - a_1^2)(1 - |w_3|^2) = (1 - a_1^2)|a_3|^2(1 - |\alpha_3|^2) + |1 - a_1 a_3|^2(1 - |\beta_3|^2)$$

(3.4)
Equation (3.3) can be used to solve for $a_3$ as a rational function of $a_1$ and $\bar{a}_2$; then one is left with two real algebraic equations in three real variables, $a_1, \Re(a_2)$ and $\Im(a_3)$. Provided the original data is really extremal, this system of two equations will have a unique solution with $a_1 \geq 0$. If the original data is not extremal, multiply $w_2$ and $w_3$ by a positive real number $t$, and choose the largest $t$ for which equations (3.2)–(3.4) can be solved. This will produce an inner function $\phi$ via (2.11); then the function $\frac{1}{t}\phi$ will be the unique function of minimal norm solving the original problem.

**Example 3.5** Let us consider a very symmetric special case. Let $\lambda_1 = (0,0)$, $\lambda_2 = (r,0)$, $\lambda_3 = (0,r)$, $w_1 = 0$, $w_2 = t$ and $w_3 = t$, where $t$ is to be chosen as large as possible and $r$ is a fixed *positive* number. Then by symmetry, we can assume that $a_1 = b_1 = \frac{1}{\sqrt{2}}$ and $a_2 = b_3 = \bar{a}_2$.

Equations (3.2)–(3.4) then reduce to:

\[
\begin{align*}
\frac{1}{2}(1 - t^2) &= \frac{1}{2}a_2^2(1 - r^2) + (1 - \frac{1}{\sqrt{2}}a_2)^2 \\
\frac{1}{2}(1 - t^2) &= a_2(\sqrt{2} - a_2)
\end{align*}
\]

Solving, one gets two solutions. One solution is

\[t = \frac{r}{2 - r}, \quad a_2 = \frac{\sqrt{2}}{2 - r};\]

the other is

\[t = \frac{r}{2 + r}, \quad a_2 = \frac{\sqrt{2}}{2 + r}.
\]

The first of these is clearly the extremal solution, and formula (2.11) then gives

\[\phi(z) = \frac{z^1 + z^2 - 2z^1z^2}{2 - z^1 - z^2}\]

as the extremal solution.

The second solution also corresponds to a pair of rank one matrices $\Gamma$ and $\Delta$ that satisfy (2.7), even though the problem is non-extremal. If one plugs in to (2.11) one gets the inner function

\[\phi_2(z) = \frac{z^1 + z^2 + 2z^1z^2}{2 + z^1 + z^2}.
\]

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