State generaings for Jones and Kauffman-Jones polynomials

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Abstract. A state generating is introduced to determine the Jones polynomial of a link. Formulae for two infinite families of knots are shown by applying this method, the second family of which are proved to be non-alternating. Moreover, the method is generalized to compute the Jones-Kauffman polynomial of a virtual link. As examples, formulae for one infinite family of virtual knots are given.

Keywords: Link; virtual link; state generating; embedding presentation

1. Introduction

Given a diagram \( L \) of a link in \( \mathbb{R}^3 \) (or \( S^3 \)), denote a crossing by a letter, regard \( e = (u^r, v^s) \) as an edge if no any other crossings along the line between \( u^r \) and \( v^s \), then an embedding presentation \( L = (V, E) \) with a rotation \( \mathcal{P} = \sum_{u \in V} \sigma_u \) is obtained [17]. Here, \( V \) is the set of all crossings and \( E \) is the set of edges. \( \sigma_u \) is an anticlockwise rotation of edges incident with \( u \). If \( e \) is an overcrossing at \( u \), then \( r = + \) (omitted for brevity), otherwise \( r = - \). Throughout this paper a link \( L \) (or a virtual link) is always a corresponding embedding presentation, also a marked diagram (or a marked virtual diagram) unless otherwise specified. The link equivalent class \([L]\) is the corresponding link in \( \mathbb{R}^3 \) (or \( S^3 \)) and the virtual link equivalent class \([L]\) is the corresponding virtual link in \( S \times I \).

The Jones polynomial is an invariant of \([L]\) which brought on major advances in knot theory [9]. The Kauffman bracket polynomial of a link was introduced, which is a simple definition to calculate the corresponding Jones polynomial [10]. Based on the Kauffman bracket polynomial, several methods were proposed to compute Jones polynomials of links via Tutte polynomials [16] [11] and Bollobás-Riordan polynomials [1] [2] for some graphs. A spanning tree expansion of Jones polynomial was first introduced by constructing a signed graph in [16]. This method was extended in [11]. The Jones polynomial of any link equivalent class can also be calculated from the Bollobás-Riordan polynomial of the ribbon graph via a certain oriented ribbon graph [7]. In addition, a matrix for calculating the Jones polynomial of a knot equivalent class was given [18]. However, since determining the Tutte polynomial of a graph is \#P-hard, it is still tough to calculate the Jones polynomial of a link equivalent class \([L]\) [8], especially a non-alternating link \( L \) with a large crossings.

A virtual link in \( S \times I \) and its Kauffman-Jones polynomial were introduced in [12], which are the generalizations of a link in \( \mathbb{R}^3 \) (or \( S^3 \)) and its Jones polynomial. Similarly, given a virtual diagram \( L \) of a virtual link in \( S \times I \), denote a crossing by a letter, regard \( e = (u^r, v^s) \) as an edge if no any other crossings along the line between \( u^r \) and \( v^s \), then an embedding presentation \( L = (V, E) \) with a rotation

\[ \mathcal{P} = \sum_{u \in V} \sigma_u \]
Thus, the Jones polynomial of \([L]\) is given below as a polynomial of a virtual link.

Correspondingly, approaches for the Jones polynomial of a link equivalent class \([L]\) were extended to compute the Kauffman-Jones polynomial of a virtual link equivalent class. Firstly, the Kauffman-Jones polynomial of a checkerboard colorable virtual link \(L\) can be computed from the signed ribbon graph polynomial of its Seifert ribbon graph \([5]\). In fact, the Jones polynomial of the Tutte polynomial can be used to compute the Kauffman-Jones polynomials of some virtual links \([4]\). Secondly, a relative variant of the other generalization of links based on their bracket polynomials and generalizes this approach to calculate the Kauffman-Jones class can be computed from the signed ribbon graph polynomial of any of their signed ribbon graphs \([3]\).

This paper introduces a new method called a state generating to calculate Jones polynomials of links based on their bracket polynomials and generalizes this approach to calculate the Kauffman-Jones polynomial of a virtual link.

\[
\mathcal{P} = \sum_{u \in \mathcal{V}} \sigma_u \text{ is obtained} \quad [17]. \quad \text{Here,} \ \mathcal{V} \text{ is the set of all crossings and} \ E \text{ is the set of edges.} \ \sigma_u \text{ is an anticyclewise rotation of edges incident with} \ u. \ \text{If} \ u \ \text{is a classical crossing, then} \ u^+ \text{ (omitted for brevity) and} \ u^- \text{ represent an overcrossing and an undercrossing at} \ u \ \text{respectively, otherwise} \ u^+ \text{ and} \ u^- \text{ represent two occurrences of} \ u. \ \text{Throughout this paper a virtual link} \ L \ \text{is always a corresponding embedding presentation, also a marked virtual diagram unless otherwise specified. The virtual link equivalent class} \ [L] \ \text{is the corresponding virtual link in} \ S \times I. \]

Correspondingly, approaches for the Jones polynomial of a link equivalent class \([L]\) were extended to compute the Kauffman-Jones polynomial of a virtual link equivalent class. Firstly, the Kauffman-Jones polynomial of a checkerboard colorable virtual link \(L\) can be calculated via the Bollobás-Riordan polynomial of the corresponding ribbon graph \([4]\). Secondly, a relative variant of the other generalization of the Tutte polynomial can be used to compute the Kauffman-Jones polynomials of some virtual links equivalent classes \([6]\). Thirdly, the Kauffman-Jones polynomial of a virtual link equivalent class was computed via the the signed ribbon graph polynomial of its Seifert ribbon graph \([5]\). In fact, the Jones polynomial of a link equivalent class and the Kauffman-Jones polynomial of a virtual link equivalent class can be computed from the signed ribbon graph polynomial of any of their signed ribbon graphs \([3]\).

This paper introduces a new method called a state generating to calculate Jones polynomials of links based on their bracket polynomials and generalizes this approach to calculate the Kauffman-Jones polynomial of a virtual link.

![Fig.0: Splitting](image)

Given a link \(L\) with \(n\) crossings for \(n \geq 2\), let \(\sigma_u = (e_1, e_2, e_3, e_4)\) be the rotation at \(u \in V(L)\) where \(e_1 = (u, x_1^1), e_2 = (u^-, x_2^2), e_3 = (u, x_3^3), e_4 = (u^-, x_4^4)\), \(r_i \in \{+, -\}\) for \(1 \leq i \leq 4\). If one replaces passes \(x_1^1 u x_3^3, x_2^2 u x_4^4\) with passes \(x_1^1 u x_4^4, x_2^2 u x_3^3\) respectively, then gets a state \(A\) at \(u\) denoted by \(s_u = A\). Otherwise, if one replaces \(x_1^1 u x_3^3, x_2^2 u x_4^4\) with \(x_1^1 u^- x_2^2, x_3^3 u^- x_4^4\) respectively, then gets a state \(A^{-1}\) at \(u\) denoted by \(s_u = A^{-1}\) (See Fig.0). By assigning one and only one state of \(A\) and \(A^{-1}\) to each \(u \in V(L)\), one obtains a state \(s\) of \(L\) and a corresponding graph called the state graph \(G(s)\) of \(s\) which consists of loops. Two states \(s\) and \(s'\) of \(L\) are distinct if and only if there exists a crossing \(u\) such that \(s_u \neq s'_u\). Set \(S_L\) to be the set of all states of \(L\). It is obvious that \(S_L\) contains \(2^n\) elements. Let \(c(s), b(s)\) and \(l(s)\) denote the number of crossings, \(A^{-1}\) and loops in a state \(s\) of \(L\) respectively. Then \(a(s) = c(s) - b(s)\) is the number of state \(A\) in \(s\). Let \(p_i(L) = \sum_{s \in S_L, l(s) = i} A^{a(s) - b(s)}\). Then the Kauffman bracket polynomial is given below

\[
< L := \sum_{i \geq 1} p_i(L)(-A^2 - A^{-2})^{i-1}. \]

Thus, the Jones polynomial of \([L]\) is deduced as follow

\[
V_L(t) = (-A)^{-\Delta(L)} < L >
\]
where $\omega(L)$ is the writhe of $L$ and $t = A^{-4}$. Let $\rho_i(V_L(t))$ and $\rho_i(V_L(t))$ denote the highest and lowest powers of $t$ occurring in $V_L(t)$ respectively. Then the value $br(V_L(t)) = \rho_i(V_L(t)) - \rho_i(V_L(t))$ is called the \textit{breath of $V_L(t)$}. Obviously, it is enough to calculate $p_i(L)$ and $\omega(L)$ in order to obtain $V_L(t)$.

Similarly, set $L$ to be a virtual link and set $\Gamma L$ to be a set of its classical crossings with $|\Gamma L| = n$ for $n \geq 1$. Let $\sigma_u = (e_1,e_2,e_3,e_4)$ be the rotation at $u \in V(\Gamma L)$ where $e_1 = (u,x_1^r), e_2 = (u^-,x_2^r), e_3 = (u,x_3^r), e_4 = (u^-,x_4^r)$, $r_i \in \{+, -\}$ for $1 \leq i \leq 4$. If one replaces passes $x_1^rux_4^r, x_2^rux_3^r$ respectively, then gets a state $A$ at $u$ denoted by $s_u = A$. Otherwise, if one replaces $x_1^ru^rux_3^r, x_2^ru^-x_4^r$ with passes $x_1^rux_3^r, x_2^ru^-x_4^r$ respectively, then gets a state $A^{-1}$ at $u$ denoted by $s_u = A^{-1}$ (See Fig.0). By assigning one and only one state of states $A$ and $A^{-1}$ to each $u \in V(\Gamma L)$, one obtains a state $s$ of $L$ and then gets the corresponding state graph which consists of components. Two states $s$ and $s'$ of $L$ are distinct if and only if there exists a crossing $u \in V(\Gamma L)$ such that $s_u \neq s_u'$. Set $S_L$ to be the set of all states of $L$. It is obvious that $S_L$ contains $2^n$ elements. Let $c(s), b(s)$ and $l(s)$ denote the number of classical crossings, $A^{-1}$ and connected components in a state $s$ of $L$ respectively. Then $a(s) = c(s) - b(s)$ is the number of state $A$ in a state $s$. Let $p_i(L) = \sum_{s \in S_L, l(s)=i} A^{a(s)-b(s)}$. So the Kauffman-Jones polynomial $f_L(A)$ for a virtual link is given below

$$f_L(A) = (-A)^{-3\omega(L)} \sum_{i \geq 1} p_i(L)(-A^2 - A^{-2})^{i-1}$$

where $\omega(L)$ is the writhe of $L$.

![Fig.1: Jones polynomial of the right handed trefoil $RT_0$](image)

Now we introduce a state generating to calculate the Jones polynomial of a link and the Kauffman-Jones polynomial for a virtual link. In order to calculate the Jones polynomial of a link $L$ (or a Kauffman-Jones polynomial of a virtual link $L$), choose a link $L_1$ (or a virtual link $L_1$) with $|V(L_1)| < |V(L)|$ (or $|V(\Gamma L_1)| < |V(\Gamma L)|$) such that each state of $L$ is generated by some state of $L_1$. This method is called a state generating. If a state $s_1$ of $L_1$ generates a state of $s$ of $L$, then $s_1$ is called the parent of $s$ denoted by $par(s)$.

For example, in order to calculate the Jones polynomial of the right handed trefoil $RT_0$, we choose the unknot $O$ shown in Fig.1. $O$ contains two distinct states $s_j$ whose state graphs are $(x_1)(x_1)$ and
Theorem 1 obtained for shown in Fig. 2. The Jones polynomials of $RT_\omega$ sequence for $1 \leq j \leq 4$. The state $s_1$ generates four distinct states $s_j(0)$ of $RT_0$ for $1 \leq j \leq 4$. The state $s_2$ generates four distinct states $s_j(0)$ of $RT_0$ for $5 \leq j \leq 8$ (See Fig. 1). We show their loops of state graphs of $s_j(0)$ with loop number in brackets in sequences below for $1 \leq j \leq 8$

\[
(x_1 x_3 x_2)(x_1 x_2 x_3)\{2\} \quad (x_1 x_2 x_1 x_3 x_2 x_3)\{1\} \quad (x_1 x_2 x_3 x_2 x_1 x_3)\{1\} \\
(x_1 x_2 x_1 x_3)(x_2 x_3)\{2\} \quad (x_1 x_3 x_2 x_1 x_2)\{1\} \quad (x_1 x_3 x_2 x_3)(x_1 x_2)\{2\} \\
(x_1 x_3 x_2 x_3 x_2)\{2\} \quad (x_1 x_3 x_2)(x_1 x_2 x_3)\{3\} 
\]

Obviously,

\[
\begin{align*}
p_1(RT_0) &= A^2 p_1(O) + 2 p_2(O) = A + 2A = 3A, \\
p_2(RT_0) &= 2 p_1(O) + (A^2 + A^{-2}) p_2(O) = 3A^{-1} + A^3, \\
p_3(RT_0) &= A^{-2} p_1(O) = A^{-3}.
\end{align*}
\]

Then

\[
< RT_0 > = 3A + (3A^{-1} + A^3)(-A^2 - A^{-2}) + A^{-3}(-A^2 - A^{-2})^2 = A^{-7} - A^{-3} - A^5.
\]

Since $\omega(RT_0) = 3$,

\[
V_{RT_0}(t) = (-A)^{-9}(A^{-7} - A^{-3} - A^5) = A^{-4} + A^{-12} - A^{-16} = t + t^3 - t^4.
\]

Consider $RT_0$. Add $2n$ crossings $y_i$ on $(x_1, x_2)$ in sequence and add $2n$ crossings $z_i$ on $(x_1, x_3)$ in sequence for $1 \leq i \leq 2n$, delete edges $(x_1, x_2), (x_1, x_2), (x_1, x_3)$, and then add edges $(x_2, y_{2n}), (x_2, z_{2n}), (x_3, z_{2n}), (y_{2k}, y_{2k-1}), (y_{2k}, y_{2k-1}), (z_{2k}, z_{2k-1}), (z_{2k}, z_{2k-1}), (y_{2k}, z_{2k+1}), (z_{2k}, z_{2k+1})$ and $(y_{2k}, y_{2k+1})$ for $1 \leq k \leq n$ where $y_{2n+1} = x_2, y_{2n+1} = x_2$ and $z_{2n+1} = x_3$. A type of knots $RT_n$ are obtained for $n \geq 1$, which belong to the first type of knots called $2$-string alternating knots. $RT_3$ is shown in Fig. 2. The Jones polynomials of $RT_n$ are obtained for $n \geq 1$.

Theorem 1.1 For $n \geq 1$

\[
V_{RT_n}(t) = \frac{t^{3n}}{\alpha - \tilde{\alpha}}(t + t^3 - t^4)(\alpha^{n+1} - \tilde{\alpha}^{n+1}) - (1 + t - t^2)(\alpha^n - \tilde{\alpha}^n)
\]

where

\[
\begin{align*}
\alpha + \tilde{\alpha} &= t^{-2} - t^{-1} + 2 - t^2; \\
\alpha \cdot \tilde{\alpha} &= 1.
\end{align*}
\]

Given a knot $KV_0$ in Fig. 3, delete edges $(x_1, x_2)$, add $2n$ crossings $y_i$ on $(x_2, x_3)$ in sequence, $2n$ crossings $z_i$ on $(x_6, x_4)$ in sequence for $1 \leq i \leq 2n$, add edges $(x_2, y_1), (x_6, y_1), (x_6, z_1), (y_{2k-1}, y_{2k}), (y_{2k-1}, y_{2k+1}), (y_{2k}, z_{2k-1}), (z_{2k-1}, z_{2k}), (z_{2k-1}, z_{2k}), (z_{2k}, z_{2k+1})$ for $1 \leq k \leq n$ where $y_{2n+1} = x_3, y_{2n+1} = x_3$ and $z_{2n+1} = x_4$. Then the second type of knots $KV_n$ are given for $n \geq 1$. $KV_1$ is the knot $10_{152}$ [13]. Each $KV_n$ is non-alternating and its Jones polynomial is shown for $n \geq 1$.

Theorem 1.2 $KV_n$ are non-alternating knots for $n \geq 1$. 

Theorem 1.3 For \( n \geq 1 \),

\[ V_{KV_n}(A) = A^{(12n+18)} \sum_{i=1}^{3} g_i(n) \]

where

\[ g_1(n) = (A^4 + 1 + A^{-4}) A^{-4n-6} + \sum_{i=0}^{n-1} A^{-4i} ((\alpha_n^{-i} - \bar{\alpha}_1^{-n-i}) - (2A^4 - A^{-4})(\alpha_1^{-n-1-i} - \bar{\alpha}_1^{-n-1-i})) \]

\[ + \ (A^{-2} - 2A^{-6} + A^{-10}) \sum_{i=0}^{n-1} A^{-4i} (\alpha_1^{-n-1-i} - \bar{\alpha}_1^{-n-1-i}) \]

\[ + \ (A^{-6} - A^{-10}) \sum_{j=0}^{n-1} A^{-4j} (1 + A^{8n-8j-4}) + \frac{A^2 - 2A^{-2} + A^{-6}}{1 - A^8} \sum_{j=0}^{n} A^{-4j} (1 - A^{8n-8j}), \]

\[ g_2(n) = \frac{A^2 - A^{-2} + A^{-6}}{A^4 + 1} (1 - A^{8n}) + \frac{A^6 + A^{-6}}{A^4 + 1} (A^{8n+4} - 1), \]

\[ g_3(n) = \frac{(A^2 + A^{-2})(A^4 - 1 + A^{-4})}{\alpha_2 - \bar{\alpha}_2} ((1 - A^{12} + A^6 - A^2)(\alpha_2^{n+1} - \bar{\alpha}_2^{n+1}) \]

\[ + (A^{12} - A^8 + A^4 - A^2)(\alpha_2^n - \bar{\alpha}_2^n)), \]

\[ \left\{ \begin{array}{l}
\alpha + \bar{\alpha} = A^8 + 2A^4 + 1 - 2A^{-4}; \\
\alpha \cdot \bar{\alpha} = A^{12} + 2A^{8} - 2 - A^{-4} + A^{-8},
\end{array} \right. \]

\[ \left\{ \begin{array}{l}
\alpha + \bar{\alpha} = A^8 + 4 - 1 - A^{-4}; \\
\alpha \cdot \bar{\alpha} = A^{12} - 2A^4 - 2A^{-4} - 2A^{-8}.
\end{array} \right. \]

Let \( x_1 \) be a virtual crossing in \( RT_n \) for \( n \geq 0 \). Then a type of virtual knots \( RT'_n \) are obtained. Their Kauffman-Jones polynomials are as follows for \( n \geq 1 \).

Theorem 1.4 For \( n \geq 1 \)

\[ f_{RT'_n}(A) = \frac{A^{-12n}}{\alpha - \bar{\alpha}} ((2A^{-4} - A^{-10})(\alpha^{n+1} - \bar{\alpha}^{n+1}) - (1 - A^{-2} + A^{-6} + A^{-8} - A^{-10})(\alpha^n - \bar{\alpha}^n)) \]

where

\[ \left\{ \begin{array}{l}
\alpha + \bar{\alpha} = A^8 - 4 + 2 - A^{-4} + A^{-8}; \\
\alpha \cdot \bar{\alpha} = 1.
\end{array} \right. \]

This paper is organized as follows. In Section 2, we use the state generating method introduced in Section 1 to study the properties of \( RT_n \), and then prove Theorem 1.1 for \( n \geq 1 \). In Section 3, we prove Theorems 1.2 and 1.3. In Section 4 we prove Theorem 1.4 by applying the state generating method for an infinite family of virtual links \( RT'_n \) for \( n \geq 1 \). Finally some open problems are given in Section 5.

2. Jones polynomials of \( RT_n \)

In this section, we divide the set \( S(RT_n) \) of all of states for \( RT_n \) into four set \( S_j(n) \) for \( j \in \{I, II, III, IV\} \) for \( n \geq 1 \). We study the recursive formulae for \( S_j(n) \) and then prove Theorem 1.1.
RT₀ has eight distinct states $s_j(0)$ shown in Fig.1 for $1 \leq j \leq 8$. Each state $s(0)$ of RT₀ generates sixteen distinct states of RT₁ according to distinct states of $y_i$ and $z_i$ for $1 \leq i \leq 2$. Generally, each state $s(n-1)$ of RTₙ₋₁ generates sixteen distinct states of RTₙ for $n \geq 1$ according to distinct states of $y_i$ and $z_i$ for $2n-1 \leq i \leq 2n$. Let $S(\text{RT}_n)$ denote the set of all of distinct states of RTₙ for $n \geq 0$ and set

$$S_I(n) = \{ s \in S(\text{RT}_n), s_{x_2} = s_{x_3} = A \mid \exists 1 \leq k \leq n \text{ such that } s_{y_2k-1}, s_{y_2k} \neq A^2, s_{y_i} = A, s_{z_{2k-1}} = s_{z_k} = A^- \text{ for } 2k + 1 \leq i \leq 2n \text{ or } s_{z_1} = s_{z_i} = A^-, s_{y_i} = A \text{ for } 1 \leq i \leq 2n \}$$

$$S_{II}(n) = \{ s \in S(\text{RT}_n), s_{x_2} = s_{x_3} = A \mid \exists 1 \leq k \leq n \text{ such that } s_{z_{2k-1}}, s_{z_{2k}} \neq A^2, s_{y_i} = A, s_{z_i} = A^- \text{ for } 2k + 1 \leq i \leq 2n \text{ or } s_{z_1} = s_{z_i} = A^-, s_{y_i} = A \text{ for } 1 \leq i \leq 2n \}$$

$$S_{III}(n) = \{ s \in S(\text{RT}_n), s_{z_{2k-1}} \cdot s_{z_{2k}} \neq A^2 \mid \exists 1 \leq k \leq n \text{ such that } s_{y_2k-1}, s_{y_2k} \neq A^2, s_{y_i} = A, s_{z_{2k-1}} = s_{z_k} = A^- \text{ for } 2k + 1 \leq i \leq 2n \text{ or } s_{z_1} = s_{z_i} = A^-, s_{y_i} = A \text{ for } 1 \leq i \leq 2n \}$$

$$S_{IV}(n) = \{ s \in S(\text{RT}_n), s_{x_2} \cdot s_{x_3} \neq A^2 \mid \exists 1 \leq k \leq n \text{ such that } s_{z_{2k-1}}, s_{z_{2k}} \neq A^2, s_{y_i} = A, s_{z_i} = A^- \text{ for } 2k + 1 \leq i \leq 2n \text{ or } s_{z_1} = s_{z_i} = A^-, s_{y_i} = A \text{ for } 1 \leq i \leq 2n \}$$

Obviously, there exists one and only one $j \in \{I, II, III, IV\}$ such that $s \in S_J(n)$ for each $s \in S(\text{RT}_n)$. Given $s(n-1) \in S(\text{RT}_{n-1})$, it generates sixteen distinct states $s_j(n)$ of $S(\text{RT}_n)$ as follows for $1 \leq j \leq 16$:

- $s_1(n)$ with $s_{y_{2n-1}} = s_{y_{2n}} = s_{z_{2n-1}} = s_{z_{2n}} = A$
- $s_2(n)$ with $s_{y_{2n-1}} = A^- \text{ and } s_{y_{2n}} = s_{z_{2n-1}} = s_{z_{2n}} = A$
- $s_3(n)$ with $s_{y_{2n}} = A^- \text{ and } s_{y_{2n-1}} = s_{z_{2n-1}} = s_{z_{2n}} = A$
- $s_4(n)$ with $s_{z_{2n-1}} = A^- \text{ and } s_{y_{2n-1}} = s_{y_{2n}} = s_{z_{2n}} = A$
- $s_5(n)$ with $s_{z_{2n}} = A^- \text{ and } s_{y_{2n-1}} = s_{y_{2n}} = s_{z_{2n-1}} = A$
- $s_6(n)$ with $s_{y_{2n-1}} = s_{y_{2n}} = A^- \text{ and } s_{z_{2n-1}} = s_{z_{2n}} = A$
- $s_7(n)$ with $s_{z_{2n-1}} = s_{z_{2n}} = A^- \text{ and } s_{y_{2n}} = s_{z_{2n}} = A$
- $s_8(n)$ with $s_{y_{2n-1}} = s_{z_{2n-1}} = A^- \text{ and } s_{y_{2n}} = s_{z_{2n}} = A$
- $s_9(n)$ with $s_{y_{2n}} = s_{z_{2n-1}} = A^- \text{ and } s_{y_{2n}} = s_{z_{2n}} = A$
- $s_{10}(n)$ with $s_{y_{2n}} = s_{z_{2n-1}} = A^- \text{ and } s_{y_{2n}} = s_{z_{2n}} = A$
- $s_{11}(n)$ with $s_{z_{2n-1}} = s_{z_{2n}} = A^- \text{ and } s_{y_{2n}} = s_{z_{2n}} = A$
- $s_{12}(n)$ with $s_{y_{2n-1}} = s_{y_{2n}} = s_{z_{2n-1}} = A^- \text{ and } s_{z_{2n}} = A$
Moreover, the state graph of \( s \leq 14 \) and \( s \leq 13 \) are states of \( j \) and \( y \) with loop number in brackets in sequence for \( 1 \leq j \leq 16 \) and \( 1 \leq i \leq 2n \) in \( s_{11}(n) \).

Because \( s_{22n-1} = s_{22n} = A^- \) and \( s_{y2n-1} \neq A^2 \) at \( s_j(n) \) for \( 14 \leq j \leq 16 \),

\[ s_j(n) \in S_{11} \]

Moreover, the state graph of \( s_j(n) \) has loops with loop number in brackets in sequence for \( 1 \leq j \leq 14 \) and \( 1 \leq i \leq 2n \):

\[
\begin{align*}
(x_1^2 x_3 x_2 y_2 y_{2n-1} y_{2n-2} A z_{2n-2} y_{2n-1} z_{2n-1} z_{2n-2} z_{2n-2} B)C\{i\} \\
(y_{2n-1} y_{2n-2} A z_{2n-2} y_{2n-1} z_{2n-1} z_{2n-2} B)C\{i+1\} \\
(y_{2n-1} z_{2n-2} A z_{2n-2} y_{2n-1} z_{2n-1} z_{2n-2} z_{2n-2} B)C\{i+2\}
\end{align*}
\]

Thus the result is clear.

(2) Because \( s_{22n-1} \cdot s_{22n} \neq A^2 \) for \( 1 \leq j \leq 10 \) and \( 1 \leq i \leq 13 \),

\[ s_j(n) \in S_{11} \]

Moreover, the state graph of \( s_j(n) \) has loops with loop number in brackets in sequence for \( 1 \leq j \leq 12 \) and \( 1 \leq i \leq 13 \):

\[
\begin{align*}
(z_{2n-1} z_{2n} y_{2n} x_{2n} x_{2n-1} z_{2n-2} y_{2n-1} z_{2n-1} z_{2n-2} z_{2n-2} B)C\{i+2\} \\
(z_{2n-1} z_{2n-1} y_{2n-1} y_{2n-2} A z_{2n-2} y_{2n-1} z_{2n-1} z_{2n-2} B)C\{i+3\} \\
(z_{2n-1} z_{2n-2} z_{2n-2} y_{2n-2} A z_{2n-2} y_{2n-1} z_{2n-1} z_{2n-2} z_{2n-2} B)C\{i+1\} \\
(z_{2n-1} z_{2n-2} z_{2n-2} y_{2n-2} A z_{2n-2} y_{2n-1} z_{2n-1} z_{2n-2} z_{2n-2} B)C\{i+1\}
\end{align*}
\]
Moreover, the state graph of each \( s_j \) have i - 1 loops for 14 ≤ j ≤ 15 and the state graph of \( s_{16} \) has i loops.

(2) Otherwise \( s_j \) ∈ \( S_{II} \). Moreover, the state graph of each \( s_j \) has i loops for 7 ≤ j ≤ 11, the state graph of each \( s_j \) has i + 1 loops for each 2 ≤ j ≤ 5 and 12 ≤ j ≤ 13 and the state graph of \( s_j \) has i + 2 loops for j = 1, 6.

Proof. Set \( s(n - 1) \) ∈ \( S_{II}(n - 1) \). Without loss of generality, suppose that there exists some 1 ≤ k ≤ n - 1 such that \( s_{2k} = s_{yi} = A \), \( s_{2i} = A^{-} \) for 2k + 1 ≤ l ≤ 2n - 2. Other cases are left to readers to verify.

Assume that the state graph of \( s(n - 1) \) has the loops \((x_{1}x_{2}x_{y_{2n-2}}A)\{(z_{2n-2}x_{y_{2n-2}})B\}C(1)\) where \( A \) and \( B \) are linear sequences, \( C \) is the product of \( i - 2 \) loops and \( \epsilon \) ∈ \{ +, -, \}. (1) Because there exists a \( n \) such that \( s_{2n-1} = A^{-}, s_{2n-1} \neq A^{2} \) for 14 ≤ j ≤ 16, \( s_j(n) \) ∈ \( S_{I}(n) \).

Moreover, the state graph of \( s_j(n) \) has loops with loop number in brackets in sequence below for 14 ≤ j ≤ 16:

\[
\begin{align*}
&\{(z_{2n-2}x_{y_{2n-2}})B\}C(i - 1) \\
&\{(z_{2n-2}x_{y_{2n-2}})B\}C(i - 1) \\
&\{y_{2n-1}x_{y_{2n-2}}B\}C(i - 1) \\
&\{y_{2n-1}x_{y_{2n-2}}B\}C(i - 1)
\end{align*}
\]

(2) Because \( s_{2n-1} \neq A^{-} \) for 1 ≤ j ≤ 13 and \( j \neq 11\), \( s_j(n) \) ∈ \( S_{II}(n) \).

Because there exists a \( k \) such that \( s_{2k} = s_{yi} = A \) and that \( s_{2i} = A^{-} \) for 2k + 1 ≤ l ≤ 2n in \( s_{11}(n) \),

\( s_{11}(n) \) ∈ \( S_{II}(n) \).

Moreover, \( s_j(n) \) has loops with loop number in brackets in sequence below for 1 ≤ j ≤ 13:

\[
\begin{align*}
&(z_{2n-2}x_{y_{2n-2}})B\}C(i + 1) \\
&(z_{2n-2}x_{y_{2n-2}})B\}C(i + 1) \\
&(z_{2n-2}x_{y_{2n-2}})B\}C(i + 1)
\end{align*}
\]

Lemma 2.2 Let \( s(n - 1) \) ∈ \( S_{II}(n - 1) \) and its state graph with loops i for \( n \) ≥ 1 and \( i \) ≥ 2. Suppose that \( s_j(n) \) are states of \( RT_n \) above and that \( |s_j(n)| = s(n - 1) \) for 1 ≤ j ≤ 16.

(1) If 14 ≤ j ≤ 16, then \( s_j(n) \) ∈ \( S_{I}(n) \). Moreover, the state graph of \( s_{11}(n) \) has i loops, the state graph of each \( s_j \) have i - 1 loops for 14 ≤ j ≤ 15 and the state graph of \( s_{16}(n) \) has i loops.

(2) Otherwise \( s_j \) ∈ \( S_{II} \). Moreover, the state graph of each \( s_j \) has i loops for 7 ≤ j ≤ 11, the state graph of each \( s_j \) has i + 1 loops for each 2 ≤ j ≤ 5 and 12 ≤ j ≤ 13 and the state graph of \( s_j \) has i + 2 loops for j = 1, 6.
Lemma 2.3 Let $s(n - 1) \in S_{III}(n - 1)$ and let its state graph $i$ loops for $n \geq 1$ and $i \geq 2$. Suppose that $s_j(n)$ are states of $RT_n$ above and that $par(s_j(n)) = s(n - 1)$ for $1 \leq j \leq 16$.

1. If $j = 11$ and $14 \leq j \leq 16$, then $s_j \in S_{III}(n)$. Moreover, the state graph of $s_{11}(n)$ has $i$ loops, the state graph of each $s_j(n)$ have $i + 1$ loops for $14 \leq j \leq 15$ and the state graph of $s_{16}(n)$ has $i + 2$ loops.

2. Otherwise $s_j(n) \in S_{IV}(n)$. Moreover, the state graph of each $s_j(n)$ has $i - 1$ loops for $4 \leq j \leq 5$, the state graph of each $s_j(n)$ has $i$ loops for $j = 1$ and $7 \leq j \leq 10$, the state graph of each $s_j(n)$ has $i + 1$ loops for $2 \leq j \leq 3$ and $12 \leq j \leq 13$ and the state graph of $s_6(n)$ has $i + 2$ loops.

Lemma 2.4 Let $s(n - 1) \in S_{IV}(n - 1)$ with loops $i$ for $n, i \geq 1$. Suppose that $s_j(n)$ are states of $RT_n$ above and that $par(s_j(n)) = s(n - 1)$ for $1 \leq j \leq 16$.

1. If $14 \leq j \leq 16$, then $s_j(n) \in S_{III}(n)$. Moreover, the state graph of each $s_j(n)$ has $i + 1$ loops for $14 \leq j \leq 15$ and the state graph of $s_{16}(n)$ has $i + 2$ loops.

2. Otherwise $s_j(n) \in S_{IV}(n)$. Moreover, the state graph of each $s_j(n)$ has $i$ loops for $7 \leq j \leq 11$, the state graph of each $s_j(n)$ has $i + 1$ loops for $2 \leq j \leq 5$ and $12 \leq j \leq 13$ and the state graph of $s_j(n)$ has $i + 2$ loops for $j = 1, 6$.

Recursive relations are given below.

Lemma 2.5 Let $p_{1, I}(RT_0) = A$, $p_{2, II}(RT_0) = A^3$, $p_{2, III}(RT_0) = 2A^2$, $p_{1, III}(RT_0) = A^{-3}$, $p_{1, IV}(RT_0) = 2A$ and $p_{2, IV}(RT_0) = A^{-1}$. Set $p_{1, I}(RT_n) = \sum_{s \in S_I(RT_n), l(s) = i} A^{u(s) - b(s)}$, $p_{1, II}(RT_n) = \sum_{s \in S_{II}(RT_n), l(s) = i} A^{u(s) - b(s)}$. 

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\[ p_{i,III}(RT_n) = \sum_{s \in S_{III}(RT_n), l(s) = i} A^{a(s)-b(s)}, \quad p_{i,IV}(RT_n) = \sum_{s \in S_{IV}(RT_n), l(s) = i} A^{a(s)-b(s)}. \] Then for \( n \geq 1 \)

\[
\begin{align*}
 p_{i,1}(RT_n) &= p_{i,1}(RT_{n-1}) + 2A^{-2}p_{i-1,1}(RT_{n-1}) + A^{-4}p_{i-2,1}(RT_{n-1}) \\
 &\quad + 2A^{-2}p_{i+1,1}(RT_{n-1}) + A^{-4}p_{i,II}(RT_{n-1}); \\
 p_{i,II}(RT_n) &= 2A^2p_{i-1,1}(RT_{n-1}) + (A^4 + 4)p_{i,III}(RT_{n-1}) \\
 &\quad + 2(A^2 + 2A^{-2})p_{i-3,1}(RT_{n-1}) + p_{i-4,1}(RT_{n-1}) + 5p_{i,II}(RT_{n-1}) \\
 &\quad + (4A^2 + 2A^{-2})p_{i-1,II}(RT_{n-1}) + (A^4 + 1)p_{i-2,II}(RT_{n-1}); \\
 p_{i,III}(RT_n) &= p_{i,III}(RT_{n-1}) + 2A^{-2}p_{i-1,III}(RT_{n-1}) + A^{-4}p_{i-2,III}(RT_{n-1}) \\
 &\quad + 2A^{-2}p_{i-1,IV}(RT_{n-1}) + A^{-4}p_{i-2,IV}(RT_{n-1}); \\
 p_{i,IV}(RT_n) &= 2A^2p_{i+1,III}(RT_{n-1}) + (A^4 + 4)p_{i,IV}(RT_{n-1}) \\
 &\quad + (2A^2 + 2A^{-2})p_{i-1,III}(RT_{n-1}) + p_{i-2,III}(RT_{n-1}) + 5p_{i,IV}(RT_{n-1}) \\
 &\quad + (4A^2 + 2A^{-2})p_{i-1,IV}(RT_{n-1}) + (A^4 + 1)p_{i-2,IV}(RT_{n-1}).
\end{align*}
\]

Proof. Based on Lemmas 2.1 and 2.2, for \( n \geq 1 \),

Similarly, the following result is clear from Lemmas 2.3 and 2.4.

\[
\begin{align*}
 p_{i,III}(RT_n) &= p_{i,III}(RT_{n-1}) + 2A^{-2}p_{i-1,III}(RT_{n-1}) + A^{-4}p_{i-2,III}(RT_{n-1}) \\
 &\quad + 2A^{-2}p_{i-1,IV}(RT_{n-1}) + A^{-4}p_{i-2,IV}(RT_{n-1}); \\
 p_{i,IV}(RT_n) &= 2A^2p_{i+1,III}(RT_{n-1}) + (A^4 + 4)p_{i,IV}(RT_{n-1}) + 2(A^2 + 2A^{-2})p_{i-1,III}(RT_{n-1}) \\
 &\quad + p_{i-2,III}(RT_{n-1}) + 5p_{i,IV}(RT_{n-1}) + (4A^2 + 2A^{-2})p_{i-1,IV}(RT_{n-1}) \\
 &\quad + (A^4 + 1)p_{i-2,IV}(RT_{n-1}).
\end{align*}
\]

Proof of Theorem 1.1. Set

\[
F_1(x, y) = \sum_{i \geq 1, n \geq 0} p_{i,1}(RT_n)x^{i-1}y^n, \quad F_2(x, y) = \sum_{i \geq 1, n \geq 0} p_{i,II}(RT_n)x^{i-1}y^n, \quad F_3(x, y) = \sum_{i \geq 1, n \geq 0} p_{i,III}(RT_n)x^{i-1}y^n, \quad F_4(x, y) = \sum_{i \geq 1, n \geq 0} p_{i,IV}(RT_n)x^{i-1}y^n.
\]

Let \( f_1(x) = \sum_{i \geq 1} p_{i,1}(RT_n)x^{i-1} \), let \( f_2(x) = \sum_{i \geq 1} p_{i,II}(RT_n)x^{i-1} \), let \( f_3(x) = \sum_{i \geq 1} p_{i,III}(RT_n)x^{i-1} \) and let \( f_4(x) = \sum_{i \geq 1} p_{i,IV}(RT_n)x^{i-1} \). It fol-
lows from equations (1-2) that

\[
\begin{aligned}
\sum_{i \geq 1, n \geq 1} p_{1, i}(R T_n) x^i y^n &= \sum_{i \geq 1, n \geq 1} p_{1, i}(R T_{n - 1}) x^i y^n + 2 A^{-2} \sum_{i \geq 1, n \geq 1} p_{i - 1, i}(R T_{n - 1}) x^i y^n \\
&+ A^{-4} \sum_{i \geq 1, n \geq 1} p_{i - 1, i}(R T_{n - 1}) x^i y^n + 2 A^{-2} \sum_{i \geq 1, n \geq 1} p_{i + 1, i}(R T_{n - 1}) x^i y^n \\
&+ A^{-4} \sum_{i \geq 1, n \geq 1} p_{1, i}(R T_{n - 1}) x^i y^n;
\end{aligned}
\]

(5)

Since \( p_{1, 1}(R T_0) = A \) and \( p_{2, i1}(R T_0) = A^3 \), the set (5) of equations is reduced to the following set of equations

\[
\begin{aligned}
(x y + 2 A^{-2} x^2 y + A^{-4} x^3 y - x) F_1(x, y) + (2 A^{-2} y + A^{-4} x y) F_2(x, y) &= -A x; \\
(2 A^2 x y + (A^4 + 4) x^2 y + (2 A^2 + 2 A^{-2}) x^3 y + x^4 y) F_1(x, y) + ((5 y + 4 A^2 + 2 A^{-2}) x y + (A^4 + 1) x^2 y - 1) F_2(x, y) &= -A^3 x.
\end{aligned}
\]

Let

\[
D = \begin{vmatrix}
xy + 2 A^{-2} x^2 y + A^{-4} x^3 y - x & 2 A^{-2} y + A^{-4} x y \\
2 A^2 x y + (A^4 + 4) x^2 y + (2 A^2 + 2 A^{-2}) x^3 y + x^4 y & (5 y + 4 A^2 + 2 A^{-2}) x y + (A^4 + 1) x^2 y - 1
\end{vmatrix}
= -x + (6 x - 5 x^3 + x^5) y - x y^2.
\]

Then,

\[
F_1(x, y) = \frac{1}{D} \begin{vmatrix}
-A x & 2 A^{-2} y + A^{-4} x y \\
-A^3 x & (5 y + 4 A^2 + 2 A^{-2}) x y + (A^4 + 1) x^2 y - 1
\end{vmatrix} = \frac{A(1 - 3 y - A^{-2} x y - 4 A^2 x y - (A^4 + 1) x^2 y)}{1 - (6 - 5 x^2 + x^4) y + y^2}
\]

and

\[
F_2(x, y) = \frac{1}{D} \begin{vmatrix}
xy + 2 A^{-2} x^2 y + A^{-4} x^3 y - x & -A x \\
2 A^2 x y + (A^4 + 4) x^2 y + (2 A^2 + 2 A^{-2}) x^3 y + x^4 y & -A^3 x
\end{vmatrix} = \frac{A^2 x + A^2 x y + (A^4 + 2) x^2 y + (2 A^2 + A^{-2} x^3) y + x^4 y}{1 - (6 - 5 x^2 + x^4) y + y^2}
\]

Suppose that \( 1 + (6 - 5 x^2 + x^4) y + y^2 = (1 - \alpha y)(1 - \alpha y) \) where

\[
\begin{aligned}
\alpha + \bar{\alpha} &= 6 - 5 x^2 + x^4; \\
\alpha \cdot \bar{\alpha} &= 1.
\end{aligned}
\]

The following equalities can be obtained

\[
F_1(x, y) = \frac{A(1 + (-3 - A^{-2} x - 4 A^2 x - (A^4 + 1) x^2) y)}{(1 - \alpha y)(1 - \alpha y)}
\]

\section*{11}
Thus, for

\[ n \geq 0 \]

where

\[ n \]

and combining with the equalities (6-9), we conclude the following results for

\[ f \]

Since

\[ RT \]

contains 4 crossings and

\[ \omega(v) = 1 \]

for each

\[ v \in V(RT) \]

by setting

\[ x = -A^2 - A^{-2} \]

and combining with the equalities (6-9), we conclude the following results for

\[ n \geq 1 \]

Thus, for

\[ n \geq 1 \]

\[ f_1(x) = \frac{A}{\alpha - \bar{\alpha}}((\alpha^{n+1} - \bar{\alpha}^{n+1}) + (3 + A^{-2} - A^{-2}x + (A^4 + 1)x^2)(\alpha^n - \bar{\alpha}^n)) \] (6)

and

\[ f_2(x) = \frac{A^3x}{\alpha - \bar{\alpha}}((\alpha^{n+1} - \bar{\alpha}^{n+1}) + (1 + (A^2 + 2A^{-2})x + (2 + A^{-4})x^2 + A^{-2}x^3)(\alpha^n - \bar{\alpha}^n)) \] (7)

By a similar way, the following equalities can be concluded for

\[ n \geq 1 \]

\[ f_3(x) = \frac{2A^{-3}x + A^{-3}x^2}{\alpha - \bar{\alpha}}((\alpha^{n+1} - \bar{\alpha}^{n+1}) - (3 + (4A^2 + A^{-2})x + (A^4 + 1)x^2)(\alpha^n - \bar{\alpha}^n)) \] (8)

and

\[ f_4(x) = \frac{2A + A^{-3}x}{\alpha - \bar{\alpha}}((\alpha^{n+1} - \bar{\alpha}^{n+1}) + (1 + (A^2 + 2A^{-2})x + (2 + A^{-4})x^2 + A^{-2}x^3)(\alpha^n - \bar{\alpha}^n)) \] (9)

Since

\[ RT \]

contains 4n + 3 crossings and

\[ \omega(v) = 1 \]

for each

\[ v \in V(RT) \]

by setting

\[ x = -A^2 - A^{-2} \]

and combining with the equalities (6-9), we conclude the following results for

\[ n \geq 1 \]

\[ V_{RT_n}(t) = (-A)^{-12n+9} \sum_{j=1}^{4} f_j(x) \]

\[ = \frac{A^{-12n}}{\alpha - \bar{\alpha}}((A^{-4} + A^{-12} - A^{-16})(\alpha^{n+1} - \bar{\alpha}^{n+1}) - (1 + A^{-4} - A^{-8})(\alpha^n - \bar{\alpha}^n)) \]

\[ = \frac{t^{12n}}{\alpha - \bar{\alpha}}((t + t^3 - t^4)(\alpha^{n+1} - \bar{\alpha}^{n+1}) - (1 + t - t^2)(\alpha^n - \bar{\alpha}^n)) \]

where

\[ \begin{aligned}
\alpha + \bar{\alpha} &= t^{-2} - t^{-1} + 2 - t + t^2; \\
\alpha \cdot \bar{\alpha} &= 1.
\end{aligned} \]

\[ \square \]

3. Jones polynomials of \( KV_n \)
In this section, for each $KV_n$ with $n \geq 1$, we divide the set $S(KV_n)$ of all of its states into $S^{(i)}(n)$ for $1 \leq j \leq 3$ and obtain some recursive relations. Based on these relations, $KV_n$ is proved to be non-alternating and Theorem 1.2 is concluded.

Let $S(KV_n)$ be the set of all of states of $KV_n$. Denote three sets below

$$S^{(1)}(n) = \{ s \in S(KV_n) | s_{x_i} = A^- \text{ for } 1 \leq i \leq 6 \}$$

$$S^{(2)}(n) = \{ s \in S(KV_n) | s_{x_i} = A^- \text{ for } 1 \leq i \leq 3, \text{ and } \prod_{i=4}^{6} s_{x_i} = A^3, \prod_{i=4}^{6} s_{x_i} = A \text{ or } \prod_{i=4}^{6} s_{x_i} = A^- \}$$

$$S^{(3)}(n) = S(KV_n) \setminus \bigcup_{i=1}^{2} S^{(i)}(n).$$

Set $p_i^{(j)}(n) = \sum_{s \in S^{(i)}(n), l(s) = i} A^{a(s) - b(s)}$ for $1 \leq j \leq 3$ and $i \geq 1$. Obviously,

$$p_i(KV_n) = \sum_{i=1}^{3} p_i^{(j)}(n).$$

**Case 1.** $s \in S^{(1)}(n)$.

Set

$$S_{II}^{(1)}(n) = \{ s | s_{y_l} = s_{z_i} = A^- \text{ for } 1 \leq i \leq 2n \},$$

$$S_{III}^{(1)}(n) = \{ s | \exists l \leq k_0 \leq k_1 \leq n \text{ such that } s_{z_{2k_1-1}} \cdot s_{z_{2k_0}} \neq A^{-2}, s_{z_{2k_1-1}} \cdot s_{z_{2k_1}} \neq A^{-2},$$

$$s_{y_{2k_0-1}} = s_{y_{2k_0}} = s_{y_l} = s_{z_l} = A^{-}, \text{ for } 1 \leq l \leq 2k_0 - 2 \text{ and } 2k_1 + 1 \leq l \leq 2n \},$$

$$S_{IV}^{(1)}(n) = \{ s | \exists l \leq k_0 \leq k_1 - 1 \leq n \text{ such that } s_{z_{2k_1-1}} \cdot s_{z_{2k_0}} \neq A^{-2}, s_{y_{2k_1-1}} \cdot s_{y_{2k_1}} \neq A^{-2},$$

$$s_{z_{2k_1-1}} = s_{z_{2k_1}} = s_{y_{2k_0-1}} = s_{y_{2k_0}} = s_{y_l} = s_{z_l} = A^{-}, \text{ for } 1 \leq l \leq 2k_0 - 2 \text{ and } 2k_1 + 1 \leq l \leq 2n \},$$

$$S_{V}^{(1)}(n) = \{ s | \exists l \leq k_0 \leq k_1 \leq n \text{ such that } s_{y_{2k_0-1}} \cdot s_{y_{2k_0}} \neq A^{-2}, s_{z_{2k_1-1}} \cdot s_{z_{2k_1}} \neq A^{-2},$$

$$s_{y_l} = s_{z_l} = A^{-}, \text{ for } 1 \leq l \leq 2k_0 - 2 \text{ and } 2k_1 + 1 \leq l \leq 2n \}. $$

By a similar way in the argument of the proof in Lemma 2.5, the following recursive relations are shown.
Lemma 3.1 Let \( p_{3,1}^{(1)}(0) = A^{-6} \). Set \( p_{I,1}^{(1)}(n) = \sum_{s \in S_1^{(1)}, l(s) = i} A^{a(s) - b(s)}, \)
\( p_{I,II}^{(1)}(n) = \sum_{s \in S_{II}^{(1)}, l(s) = i} A^{a(s) - b(s)}, \)
\( p_{I,III}^{(1)}(n) = \sum_{s \in S_{III}^{(1)}, l(s) = i} A^{a(s) - b(s)}, \)
\( p_{I,IV}^{(1)}(n) = \sum_{s \in S_{IV}^{(1)}, l(s) = i} A^{a(s) - b(s)}, \) and \( p_{I,V}^{(1)}(n) = \sum_{s \in S_{V}^{(1)}, l(s) = i} A^{a(s) - b(s)}. \)
Then
\[
\begin{align*}
p_{I}^{(1)}(n) &= p_{I,1}^{(1)}(n) + p_{I,II}^{(1)}(n) + p_{I,III}^{(1)}(n) + p_{I,IV}^{(1)}(n) + p_{I,V}^{(1)}(n)
\end{align*}
\]
where
\[
\begin{align*}
p_{I,1}^{(1)}(n) &= A^{-4} p_{I,1}^{(1)}(n - 1); \\
p_{I,II}^{(1)}(n) &= 2A^{-2} p_{I+1,II}^{(1)}(n - 1) + p_{I,II}^{(1)}(n - 1) \\
&\quad + (A^4 + 4)p_{I+1,II}^{(1)}(n - 1) + (4A^2 + 2A^{-2})p_{I-1,II}^{(1)}(n - 1) + (A^4 + 1)p_{I-2,II}^{(1)}(n - 1) \\
&\quad + 2A^{-2} p_{I-1,IV}^{(1)}(n - 1) + 5p_{I-2,IV}^{(1)}(n - 1) \\
&\quad + 4A^2 p_{I-3,IV}^{(1)}(n - 1) + A^4 p_{I-4,IV}^{(1)}(n - 1); \\
p_{I,III}^{(1)}(n) &= 2A^{-2} p_{I+1,III}^{(1)}(n - 1) + p_{I,III}^{(1)}(n - 1) + A^{-4} p_{I+1,III}^{(1)}(n - 1) \\
&\quad + 2A^{-2} p_{I-1,III}^{(1)}(n - 1) + p_{I-2,III}^{(1)}(n - 1) \\
&\quad + 2A^{-2} p_{I-1,IV}^{(1)}(n - 1) + p_{I-2,IV}^{(1)}(n - 1); \\
p_{I,IV}^{(1)}(n) &= 2A^{-2} p_{I+1,IV}^{(1)}(n - 1) + p_{I,IV}^{(1)}(n - 1) \\
&\quad + A^4 p_{I-1,IV}^{(1)}(n - 1) + 2A^2 p_{I-1,IV}^{(1)}(n - 1) + p_{I-2,IV}^{(1)}(n - 1); \\
p_{I,V}^{(1)}(n) &= 4p_{I+2,IV}^{(1)}(n - 1) + 4A^2 p_{I+1,IV}^{(1)}(n - 1) + A^4 p_{I-1,IV}^{(1)}(n - 1) + 2A^{-2} p_{I+1,III}^{(1)}(n - 1) \\
&\quad + 5p_{I-1,IV}^{(1)}(n - 1) + 4A^2 p_{I-1,III}^{(1)}(n - 1) + A^4 p_{I-2,III}^{(1)}(n - 1) + (4 + A^{-2}) p_{I-2,IV}^{(1)}(n - 1) \\
&\quad + (4A^2 + 2A^{-2}) p_{I-1,IV}^{(1)}(n - 1) + (A^4 + 1) p_{I-2,IV}^{(1)}(n - 1).
\end{align*}
\]

Case 2. \( s \in S^{(2)}(n) \).

Let
\[
S_1^{(2)}(n) = \{ s | \exists 1 \leq k \leq n \text{ such that } s_{2k+1} \cdot s_{2k} \neq A^{-2}, s_{2k-1} = s_{2k} = s_{2i} = s_{2i-1} = A^-, \text{ for } 2k + 1 \leq l \leq 2n \},
\]
\[
S_{II}^{(2)}(n) = \{ s | \text{ either } s_{yi} = s_{zi} = A^- \text{ or } \exists 1 \leq k \leq n \text{ such that } s_{2k-1} \cdot s_{2k} \neq A^{-2}, \text{ for } 1 \leq i \leq 2n, 2k + 1 \leq l \leq 2n \}.
\]

Lemma 3.2 Set \( p_{2,II}^{(2)}(0) = 3A^{-4}, p_{3,III}^{(2)}(0) = 3A^{-2}, p_{4,II}^{(2)}(0) = 1 \). Set \( p_{I,1}^{(2)}(n) = \sum_{s \in S_{I,1}^{(2)}, l(s) = i} A^{a(s) - b(s)}, \)
\( p_{I,II}^{(2)}(n) = \sum_{s \in S_{II,1}^{(2)}, l(s) = i} A^{a(s) - b(s)}, \) and \( p_{I,IV}^{(2)}(n) = \sum_{s \in S_{IV,1}^{(2)}, l(s) = i} A^{a(s) - b(s)}. \) Then
\[
\begin{align*}
p_{I}^{(2)}(n) &= p_{I,1}^{(2)}(n) + p_{I,II}^{(2)}(n)
\end{align*}
\]
where
\[
\begin{align*}
p_{I,1}^{(2)}(n) &= A^{-4} p_{I,1}^{(2)}(n - 1) + 2A^{-2} p_{I-1,1}^{(2)}(n - 1) + p_{I-2,1}^{(2)}(n - 1) \\
&\quad + 2A^{-2} p_{I+1,II}^{(2)}(n - 1) + p_{I,II}^{(2)}(n - 1); \\
p_{I,II}^{(2)}(n) &= 2A^{-2} p_{I+1,II}^{(2)}(n - 1) + 5p_{I-2,II}^{(2)}(n - 1) + 4A^2 p_{I-3,II}^{(2)}(n - 1) \\
&\quad + A^4 p_{I-4,II}^{(2)}(n - 1) + (4 + A^{-4}) p_{I-2,II}^{(2)}(n - 1) \\
&\quad + (4A^2 + 2A^{-2}) p_{I-1,II}^{(2)}(n - 1) + (A^4 + 1) p_{I-2,II}^{(2)}(n - 1).
\end{align*}
\]
Case 3. $s \in S(3)(n)$.

Let

$$S(3)_I(n) = \{s| \text{ either } s_{x_j} = A^{-3} \prod_{i=1}^{3} s_{x_i} \neq A^{-3} \text{ for } 4 \leq j \leq 6 \text{ or } \exists l \leq k \leq n \text{ such that}$$

$$s_{y_{2k-1}} \cdot s_{y_{2k}} \neq A^{-2}, s_{22k-1} = s_{22k} = s_{y_l} = s_{z_l} = A^{-1}, \text{ for } 2k + 1 \leq l \leq 2n\}$$

$$S(3)_II(n) = \{s| \text{ either } s_{y_i} = s_{z_i} = A^{-2} \prod_{i=1}^{3} s_{x_i} \neq A^{-3} \text{ or } \exists l \leq k \leq n \text{ such that}$$

$$s_{22k-1} \cdot s_{22k} \neq A^{-2}, s_{y_l} = s_{z_l} = A^{-1}, \text{ for } 1 \leq i \leq 2n, 2k + 1 \leq l \leq 2n\}$$

**Lemma 3.3** Set $p_{2,1}^{(3)}(0) = 3A^{-4}$, $p_{3,1}^{(3)}(0) = 3A^{-2}$, $p_{4,1}^{(3)}(0) = 1$, $p_{1,1,II}^{(3)}(0) = 9A^{-2}$, $p_{2,II}^{(3)}(0) = 18$, $p_{3,II}^{(3)}(0) = 15A^{2}$, $p_{4,II}^{(3)}(0) = 6A^{4}$, $p_{5,II}^{(3)}(0) = A^{6}$. Set $p_{i,II}^{(3)}(n) = \sum_{s \in S(3)_II(n), l(s) = i} A^{a(s) - b(s)}$, $p_{i,II}^{(3)}(n) = \sum_{s \in S(3)_II(n), l(s) = i} A^{a(s) - b(s)}$. Then

$$p_{i,II}^{(3)}(n) = p_{i,1}^{(3)}(n) + p_{i,II}^{(3)}(n)$$

where

$$p_{1,II}^{(3)}(n) = A^{-4}p_{1,1}^{(3)}(n-1) + 2A^{-2}p_{1,1,II}^{(3)}(n-1) + p_{1,2,1}^{(3)}(n-1) + 2A^{-2}p_{1,II}^{(3)}(n-1) + p_{II}^{(3)}(n-1)$$

$$p_{i,II}^{(3)}(n) = 2A^{-2}p_{i,1,II}^{(3)}(n-1) + 5p_{i,1,II}^{(3)}(n-1) + 4A^{2}p_{i,1,II}^{(3)}(n-1)$$

As an example, we calculate the Jones polynomial of $KV_1$ (also $10_{152}$). By Lemma 3.1, the following results are obtained

$$\begin{align*}
p_{3,1}^{(1)}(1) &= A^{-10}, p_{2,1}^{(1)}(1) = p_{1,1}^{(1)}(1) = 2A^{-8}, \\
p_{3,1}^{(1)}(1) &= p_{1,1}^{(1)}(1) = A^{-6}, p_{1,1,II}^{(1)}(1) = 4A^{-6}, \\
p_{2,II}^{(1)}(1) &= 4A^{-4}, p_{3,II}^{(1)}(1) = A^{-2}.
\end{align*}$$

(10)

By Lemma 3.2, the following conclusions are given

$$\begin{align*}
p_{1,1}^{(2)}(1) &= 6A^{-6}, p_{2,1}^{(2)}(1) = 9A^{-4}, p_{3,1}^{(2)}(1) = 5A^{-2}, p_{4,1}^{(2)}(1) = 1, \\
p_{2,II}^{(2)}(1) &= 12A^{-4} + 3A^{-8}, p_{3,II}^{(2)}(1) = 24A^{-2} + 9A^{-6}, p_{4,II}^{(2)}(1) = 19 + 10A^{-4}, \\
p_{2,II}^{(2)}(1) &= 7A^{2} + 5A^{-2}, p_{6,II}^{(2)}(1) = A^{4} + 1.
\end{align*}$$

(11)

By Lemma 3.3, we get

$$\begin{align*}
p_{2,II}^{(3)}(1) &= 18A^{-4} + 3A^{-8}, p_{3,1}^{(3)}(1) = 9A^{-6} + 45A^{-2}, p_{4,1}^{(3)}(1) = 48 + 10A^{-4}, \\
p_{5,II}^{(3)}(1) &= 27A^{2} + 5A^{-2}, p_{6,II}^{(3)}(1) = 8A^{4} + 1, p_{1,II}^{(3)}(1) = A^{6}, p_{4,II}^{(3)}(1) = 36A^{-2} + 15A^{-6}, \\
p_{2,II}^{(3)}(1) &= 108 + 57A^{-4}, p_{3,II}^{(3)}(1) = 141A^{2} + 89A^{-2}, p_{4,II}^{(3)}(1) = 74 + 102A^{4}, \\
p_{5,II}^{(3)}(1) &= 43A^{6} + 35A^{2}, p_{6,II}^{(3)}(1) = 10A^{8} + 9A^{4}, p_{7,II}^{(3)}(1) = A^{10} + A^{6}.
\end{align*}$$

(12)
By combining with the equalities (10 – 12), we have

\[
\begin{align*}
p_1(KV_1) &= 36A^{-2} + 25A^{-6}, \\
p_2(KV_1)(A^2 - A^{-2}) &= -108A^2 - 208A^{-2} - 10A^{-6} - 10A^{-10}, \\
p_3(KV_1)(A^2 - A^{-2})^2 &= 141A^6 + 446A^2 + 499A^{-2} + 205A^{-6} + 22A^{-10} + A^{-14}, \\
p_4(KV_1)(A^2 - A^{-2})^3 &= -102A^{10} - 448A^6 - 752A^2 - 588A^{-2} - 202A^{-6} - 20A^{-10}, \\
p_5(KV_1)(A^2 - A^{-2})^4 &= 43A^{14} + 241A^{10} + 337A^6 + 626A^2 + 379A^{-2} + 109A^{-6} + 10A^{-10}, \\
p_6(KV_1)(A^2 - A^{-2})^5 &= -10A^{18} - 68A^{14} - 192A^{10} - 290A^6 - 250A^2 \\
&- 120A^{-2} - 28A^{-6} - 2A^{-10}, \\
p_7(KV_1)(A^2 - A^{-2})^6 &= A^{22} + 8A^{18} + 27A^{14} + 50A^{10} + 55A^6 + 36A^2 + 14A^{-2} + 2A^{-6}
\end{align*}
\]

By applying the equalities (1) and (13), the Kauffman bracket polynomial of \(KV_1\) is

\[
<KV_1> = \sum_{i=1}^{p} p_i(KV_1)(A^2 - A^{-2})^{i-1} = A^{22} - 2A^{18} + 2A^{14} - 3A^{10} + 2A^6 - 2A^{-2} + A^{-6} + A^{-14}.
\]

Since \(\omega(KV_1) = -10\), the Jones polynomial of \(RV_1\) is deduced

\[
\begin{align*}
V_{KV_1}(t) &= A^{10}(A^{22} - 2A^{18} + 2A^{14} - 3A^{10} + 2A^6 - 2A^{-2} + A^{-6} + A^{-14}) \\
&= A^{22} - 2A^{18} + 2A^{14} - 3A^{10} + 2A^6 - 2A^{32} + A^{28} + A^{24} + A^{16} \\
&= t^{-13} - 2t^{-12} + 2t^{-11} - 3t^{-10} + 2t^{-9} - 2t^{-8} + t^{-7} + t^{-6} + t^{-4}.
\end{align*}
\]

It is obvious that \(RT_1\) is non-alternating. In order to prove that \(RT_n\) are non-alternating for \(n \geq 2\).

We consider the highest power and the lowest power of \(A\) in \(f_i(n) = p_i(KV_n)(A^2 - A^{-2})^{i-1}\) for \(i \geq 1\).

**Lemma 3.4** Set \(f_i(n) = p_i(KV_n)(A^2 - A^{-2})^{i-1}\) for \(n, i \geq 1\). Let \(\rho_h(f_i(n))\) and \(\rho_l(f_i(n))\) denote the highest power and the lowest power of \(A\) in \(f_i(n)\) respectively. Then for \(n, i \geq 1\)

\[
\rho_h(f_i(n)) \leq 8k + 14.
\]

and

\[
\rho_l(f_i(n)) \geq -4k - 10.
\]

**Proof.** This conclusion will be verified by induction on \(n\). By equalities of (13), the result is obvious for \(n = 1\).

Assume that the result holds for the integer \(k(k \geq 2)\). That is for \(i \geq 1\)

\[
\rho_h(f_i(k)) \leq 8n + 14 \text{ and } \rho_l(f_i(k)) \geq -4n - 10.
\]

This implies for \(i \geq 1\)

\[
\rho_h(f_i^{(v)}(k)) \leq 8n + 14 \text{ and } \rho_l(f_i^{(v)}(k)) \geq -4n - 10 \quad (14)
\]
where \( f_{ij}^{(r)}(k) = p_i^{(r)}(KV_k)(-A^2 - A^{-2})^{i+1} \) for \( j \in \{I, II, III, IV\} \) and \( 1 \leq r \leq 3 \). By Lemma 3.6,

\[
\begin{align*}
\left\{
\begin{array}{ll}
p_{i, I}^{(1)}(k + 1)(-A^2 - A^{-2})^{i-1} = A^{-4}p_{i, I}^{(1)}(k)(-A^2 - A^{-2})^{i-1}; \\
p_{i, II}^{(1)}(k + 1)(-A^2 - A^{-2})^{i-1} = (2A^2 - p_{i+1, I}^{(1)}(k) + p_{i, I}^{(1)}(k) + (4A^2 + 2A^{-2})p_{i-1, II}^{(1)}(k) + (A^4 + 1)p_{i-2, II}^{(1)}(k) + 2A^{-2}p_{i-1, I}^{(1)}(k) + 5p_{i-2, IV}^{(1)}(k) + 4A^2p_{i-3, IV}^{(1)}(k) + (A^2 - A^{-2})^{i-1}; \\
p_{i, III}^{(1)}(k + 1)(-A^2 - A^{-2})^{i-1} = (2A^2p_{i+1, III}^{(1)}(k) + p_{i, III}^{(1)}(k) + A^{-4}p_{i-1, III}^{(1)}(k) + 2A^2p_{i-1, III}^{(1)}(k) + p_{i-2, III}^{(1)}(k) + p_{i-2, IV}^{(1)}(k) + 4A^2p_{i-3, IV}^{(1)}(k) + (A^4 + 1)p_{i-2, V}^{(1)}(k)(-A^2 - A^{-2})^{i-1}; \\
p_{i, IV}^{(1)}(k + 1)(-A^2 - A^{-2})^{i-1} = (2A^2p_{i+1, IV}^{(1)}(k) + p_{i, IV}^{(1)}(k) + 4A^2p_{i+1, IV}^{(1)}(k) + (A^4 + 1)p_{i-2, V}^{(1)}(k)(-A^2 - A^{-2})^{i-1}; \\
p_{i, V}^{(1)}(k + 1)(-A^2 - A^{-2})^{i-1} = (4p_{i+1, V}^{(1)}(k) + 4A^2p_{i+1, V}^{(1)}(k) + A^4p_{i, V}^{(1)}(k) + 2A^2p_{i+1, III}^{(1)}(k) + 4A^2p_{i-1, V}^{(1)}(k) + A^4p_{i-2, III}^{(1)}(k) + 2A^2p_{i-1, IV}^{(1)}(k) + (A^4 + 1)p_{i-2, V}^{(1)}(k)(-A^2 - A^{-2})^{i-1}. \\
\end{array}
\right.
\end{align*}
\]

This implies the following equalities

\[
\begin{align*}
\left\{
\begin{array}{ll}
f_{i, I}^{(1)}(k + 1) = A^{-4}f_{i, I}^{(1)}(k); \\
f_{i, II}^{(1)}(k + 1) = (2A^2f_{i+1, II}^{(1)}(k)(-A^2 - A^{-2})^{i-1} + f_{i, II}^{(1)}(k) + (A^4 + 4)f_{i+1, II}^{(1)}(k) + 4A^2f_{i-1, II}^{(1)}(k)(-A^2 - A^{-2}) + 4A^2f_{i-2, II}^{(1)}(k)(-A^2 - A^{-2})^2 + 2A^2f_{i+1, II}^{(1)}(k)(-A^2 - A^{-2}) + 5f_{i-2, IV}^{(1)}(k)(-A^2 - A^{-2})^2 + 4A^2f_{i-3, IV}^{(1)}(k)(-A^2 - A^{-2})^3 + A^4f_{i-4, IV}^{(1)}(k)(-A^2 - A^{-2})^4; \\
f_{i, III}^{(1)}(k + 1) = 2A^{-2}f_{i+1, III}^{(1)}(k)(-A^2 - A^{-2})^{i-1} + f_{i, III}^{(1)}(k) + A^{-4}f_{i+1, III}^{(1)}(k) + 2A^{-2}f_{i+1, III}^{(1)}(k)(-A^2 - A^{-2}) + f_{i-2, IV}^{(1)}(k)(-A^2 - A^{-2})^2 + 2A^{-2}f_{i+1, IV}^{(1)}(k)(-A^2 - A^{-2}) + f_{i-2, V}^{(1)}(k)(-A^2 - A^{-2})^2; \\
f_{i, IV}^{(1)}(k + 1) = 2A^{-2}f_{i+1, IV}^{(1)}(k)(-A^2 - A^{-2})^{i-1} + f_{i, IV}^{(1)}(k) + A^4f_{i-1, IV}^{(1)}(k)(-A^2 - A^{-2}) + 2A^{-2}f_{i+1, IV}^{(1)}(k)(-A^2 - A^{-2})^2 + 4A^2f_{i-1, IV}^{(1)}(k)(-A^2 - A^{-2})^2; \\
f_{i, V}^{(1)}(k + 1) = 4f_{i+1, V}^{(1)}(k)(-A^2 - A^{-2})^{i-2} + 4A^2f_{i+1, V}^{(1)}(k)(-A^2 - A^{-2})^{i-1} + A^4f_{i, V}^{(1)}(k) + 2A^2f_{i+1, III}^{(1)}(k)(-A^2 - A^{-2}) + 5f_{i-2, IV}^{(1)}(k)(-A^2 - A^{-2})^2 + 4A^2f_{i-3, IV}^{(1)}(k)(-A^2 - A^{-2}) + A^4f_{i-2, IV}^{(1)}(k)(-A^2 - A^{-2})^2 + 4A^2f_{i-2, V}^{(1)}(k)(-A^2 - A^{-2}) + (A^4 + 1)f_{i-2, V}^{(1)}(k)(-A^2 - A^{-2})^2.
\end{array}
\right.
\end{align*}
\]

By combining with (14-15), we get for \( i \geq 1 \)

\[
\rho_i(f_{ij}^{(1)}(k + 1)) \leq 8n + 14 \quad \text{and} \quad \rho_i(f_{ij}^{(1)}(k + 1)) \geq -4n - 10 \tag{16}
\]

where \( j \in \{I, II, III, IV\} \). By applying Lemma 3.2 – 3 with a similar way in the argument of the proof of (16), we obtain for \( i \geq 1 \)

\[
\rho_i(f_{ij}^{(r)}(k + 1)) \leq 8n + 14 \quad \text{and} \quad \rho_i(f_{ij}^{(r)}(k + 1)) \geq -4n - 10 \tag{17}
\]
where \( j \in \{I, II, III, IV\} \) and \( 2 \leq r \leq 3 \).

Thus, it is obvious by combining (16-17) that for \( i \geq 1 \)
\[
\rho_h(f_i(k+1)) \leq 8(k+1) + 14 \text{ and } \rho_l(f_i(k+1)) \geq -4(k+1) - 10.
\]

Hence, the conclusion is implied. \( \square \)

In 1987, Kauffman, Thistlethwaite and Murasugi independently proved the following result.

**Lemma 3.5** ([15][10][13]) If \( L \) is connected, irreducible, alternating link, then the breadth of \( V_L(t) \) is precisely \( m \).

**Proof of Theorem 1.2.** Since
\[
< KV_n > = \sum_{i \geq 1} f_i(n),
\]
it is obvious by Lemma 3.4 that for \( i \geq 1 \)
\[
\rho_h(< KV_n >) \leq 8n + 14 \text{ and } \rho_l(< KV_n >) \geq -4n - 10.
\]

Thus,
\[
\rho_h(V_{KV_n}(t)) \leq \frac{-3\omega(KV_n) + 8n + 14}{4} \text{ and } \rho_l(V_{KV_n}(t)) \geq \frac{-3\omega(KV_n) - 4n - 10}{4}.
\]

Then
\[
br(V_{KV_n}(t)) = \rho_h(V_{KV_n}(t)) - \rho_l(V_{KV_n}(t)) \leq 3n + 6. \quad (18)
\]

Because \( KV_n \) is connected and irreducible with \( 4n + 6 \) crossings, the result is implied by Lemma 3.5 and the inequality (18). \( \square \)

**Proof Theorem 1.3.** Since \( \omega(KV_n) = -4n - 6 \) and \( p_i(n) = \sum_{k=1}^{3} p_i^k(n) \), the result is implied by applying a similar way in the argument of the proof of Theorem 1.1. \( \square \)

4. Kauffman-Jones polynomials for a type of virtual links

Fig 4: A type of virtual knots \( RT_n' \)

![Fig 4: A type of virtual knots RTₙ'](image)
In this section, we calculate Kauffman-Jones polynomials of the infinite family of virtual knots $RT'_n$ for $n \geq 1$. Set $S(RT'_n)$ to be the set of all states of $RT'_n$ for $n \geq 0$. Denote the following set.

\[ S_1(RT'_n) = \{ s \in S(RT'_n), s_{2\mathbf{k}} = A \} \text{ either } \exists 1 \leq k < n \text{ such that } s_{2k-1} \cdot s_{2k} \neq A^2, s_{y_i} = A, \]

\[ s_{2k-1} = s_{2k} = s_{z_i} = A^* \text{ for } 2k + 1 \leq i \leq 2n \text{ or } s_{z_i} = s_{z_i} = A^*, s_{y_i} = A \text{ for } 1 \leq i \leq 2n \}, \]

\[ S_II(RT'_n) = \{ s \in S(RT'_n), s_{2\mathbf{k}} = A \} \text{ either } \exists 1 \leq k < n \text{ such that } s_{2k-1} \cdot s_{2k} \neq A^2, s_{y_i} = A, \]

\[ s_{2k-1} = s_{2k} = s_{z_i} = A^* \text{ for } 2k + 1 \leq i \leq 2n \text{ or } s_{z_i} = s_{y_i} = A \text{ for } 1 \leq i \leq 2n \}, \]

\[ S_III(RT'_n) = \{ s \in S(RT'_n), s_{2\mathbf{k}} = A^* \} \text{ either } \exists 1 \leq k < n \text{ such that } s_{2k-1} \cdot s_{2k} \neq A^2, s_{y_i} = A, \]

\[ s_{2k-1} = s_{2k} = s_{z_i} = A^* \text{ for } 2k + 1 \leq i \leq 2n \text{ or } s_{z_i} = s_{z_i} = A^*, s_{y_i} = A \text{ for } 1 \leq i \leq 2n \}. \]

\[ S_IV(RT'_n) = \{ s \in S(RT'_n), s_{2\mathbf{k}} = A^* \} \text{ either } \exists 1 \leq k < n \text{ such that } s_{2k-1} \cdot s_{2k} \neq A^2, s_{y_i} = A, \]

\[ s_{2k-1} = A^* \text{ for } 2k + 1 \leq i \leq 2n \text{ or } s_{z_i} = s_{y_i} = A \text{ for } 1 \leq i \leq 2n \} \].

**Lemma 4.1** Let $p_{1, II}(RT'_0) = 1$, $p_{1, III}(RT'_0) = A^2$, $p_{1, IV}(RT'_0) = A^{-2}$, $p_{1, IV}(RT'_0) = 1$. Set $p_{1, I}(RT'_n) = \sum_{s \in S_1(RT'_n), (l, s) = i} A^{a(s) - b(s)}$, $p_{1, II}(RT'_n) = \sum_{s \in S_{II}(RT'_n), (l, s) = i} A^{a(s) - b(s)}$, $p_{1, III}(RT'_n) = \sum_{s \in S_{III}(RT'_n), (l, s) = i} A^{a(s) - b(s)}$, $p_{1, IV}(RT'_n) = \sum_{s \in S_{IV}(RT'_n), (l, s) = i} A^{a(s) - b(s)}$. Then for $n \geq 1$

\[
\begin{align*}
    p_{1, I}(RT'_n) &= p_{1, I}(RT'_{n-1}) + 2A^{-2}p_{1-1, I}(RT'_{n-1}) + A^{-4}p_{-2, 1}(RT'_{n-1}) \\
    &\quad + 2A^{-2}p_{2, 1}(RT'_{n-1}) + A^{-4}p_{-1, II}(RT'_{n-1}); \\
    p_{1, II}(RT'_n) &= 2A^2p_{1, I}(RT'_{n-1}) + (A^4 + 4)p_{-1, I}(RT'_{n-1}) \\
    &\quad + (2A^2 + 2A^{-2})p_{-2-1, II}(RT'_{n-1}) + p_{-3, 1}(RT'_{n-1}) + 5p_{1, II}(RT'_{n-1}) \\
    &\quad + (4A^2 + 2A^{-2})p_{-2-1, II}(RT'_{n-1}) + (A^4 + 1)p_{-2, 1}(RT'_{n-1}); \\
    p_{1, III}(RT'_n) &= 2A^2p_{1, I}(RT'_{n-1}) + (A^4 + 4)p_{-1, III}(RT'_{n-1}) \\
    &\quad + (2A^2 + 2A^{-2})p_{-2-1, III}(RT'_{n-1}) + A^{-4}p_{-1-2, II}(RT'_{n-1}); \\
    p_{1, IV}(RT'_n) &= 2A^2p_{1, I}(RT'_{n-1}) + (A^4 + 4)p_{-1, IV}(RT'_{n-1}) \\
    &\quad + (2A^2 + 2A^{-2})p_{-2-1, IV}(RT'_{n-1}) + p_{-2, 1}(RT'_{n-1}) + 5p_{1, IV}(RT'_{n-1}) \\
    &\quad + (4A^2 + 2A^{-2})p_{-2-1, IV}(RT'_{n-1}) + (A^4 + 1)p_{-2, 1}(RT'_{n-1}).
\end{align*}
\]

**Proof of Theorem 1.4** By a similar way in the argument of the proof of Theorem 1.1, the result holds. \(\square\)

### 5. Further study

In this section, we introduce general m-string alternating links (or virtual links) and m-string tangle links (or virtual links) for $m \geq 2$. Several problems are proposed.

Set $n$ to be a positive integer in this section. Generally, given a link (or virtual link) $L_0$ with $m + 1$ parallel edges, denote one of them by $e_0$ and denote others by $e_i = (u^{r_i}_i, v^{r_i}_i)$ in sequence for $1 \leq i \leq m$. Here $r_i \in \{+, -\}$, $r_0 = r$. Assume that $e_0$ is the leftmost edge and leave other readers to get in a similar way. Add $2n$ crossings $x_{i,j}$ on $e_i$ in sequence for $1 \leq i \leq m$ and $1 \leq j \leq 2n$ respectively. Let $(u^{r_1}_{x_{1,1}}, x^{r_1}_{x_{1,1}}, x^{r_1}_{x_{1,2}})$, $(x^{r_1}_{x_{1,2}}, x^{r_1}_{x_{1,3}})$, $\cdots$, $(x^{r_1}_{x_{2n-1}}, x^{r_1}_{x_{2n-2}})$, $(x^{r_1}_{x_{2n-2}}, v^{r_1}_1)$ be a subdivision of $e_1$ for odd $1 \leq i \leq m$ and let $(u^{r_1}_{x_{1,1}}, x^{r_1}_{x_{1,1}}, x^{r_1}_{x_{1,2}})$, $(x^{r_1}_{x_{1,2}}, x^{r_1}_{x_{1,3}})$, $\cdots$, $(x^{r_1}_{x_{2n-1}}, x^{r_1}_{x_{2n-2}})$, $(x^{r_1}_{x_{2n-2}}, v^{r_1}_1)$ be a subdivision of $e_i$ for even $1 \leq i \leq m$.
Add edges \((u_i^r, x_{i,1}^{-r})\) and \((v_i^r, x_{i,2n}^r)\) for odd \(1 \leq i \leq m\), add edges \((u_i^r, x_{i,1}^r)\) and \((v_i^r, x_{i,2n}^{-r})\) for even \(1 \leq i \leq m\), add edges \((x_{i,l}^{-r}, x_{i+1,l}^r), (x_{i,l}^r, x_{i+1,l}^{-r})\) for odd \(1 \leq i \leq m\) and odd \(1 \leq l \leq 2n\), \((x_{i,l}^{-r}, x_{i+1,l}^{-r})\) for odd \(1 \leq i \leq m\) and even \(1 \leq l \leq 2n\), and then add edges \((x_{m,l}^r, x_{n,l+1}^{-r})\) for even \(m\) and odd \(1 \leq l \leq 2n\). A link (or a virtual link) \(AL_n\) is constructed which is called an \(m\)-string alternating link (or virtual link). Here, \(x_{m+1,l}^r = x_{m,l+1}^{-r}\) and \(x_{0,l}^r = x_{1,l+1}^{-r}\) for odd \(1 \leq l \leq 2n\), \(x_{0,1}^r = u_0^r, x_{0,2n+1}^{-r} = v_0^{-r}\). An example is shown in Fig.5 (b) for \(m = 3\) and \(n = 2\).

\[
\begin{array}{cccc}
 u_0^r & u_1^r & u_2^r & u_3^r \\
v_0^r & v_1^r & v_2^r & v_3^r \\
\end{array}
\]

(a) \(L_0\)  
(b) \(AL_4\)  
(c) \(TL_4\)  

Fig.5: \(L_0\), \(AL_4\) and \(TL_4\)

Similarly, let \(L_0\) be a link (or virtual link) above. Add \(2n\) crossings \(x_{i,l}\) on \(e_i\) in sequence for \(1 \leq i \leq m\) and \(1 \leq l \leq 2n\) respectively. Let \((u_i^r, x_{i,1}^r), (x_{i,1}^r, x_{i,2}^{-r}), \ldots, (x_{i,2n-1}^r, x_{i,2n}^{-r}), (x_{i,2n}^r, v_i^r)\) be a subdivision of \(e_i\). Add edges \((u_i^r, x_{i,1}^r)\) and \((v_i^r, x_{i,2n}^{-r})\) for \(1 \leq i \leq l\), add edges \((x_{i,l}^r, x_{i+1,l}^{-r}), (x_{i,l}^r, x_{i+1,l}^r), (x_{i,l}^{-r}, x_{i+1,l}^{-r})\) for odd \(1 \leq i \leq m\) and odd \(1 \leq l \leq 2n\), \((x_{i,l}^{-r}, x_{i+1,l}^r)\) for odd \(1 \leq i \leq m\) and even \(1 \leq l \leq 2n\), and then add edges \((x_{m,l}^r, x_{n,l+1}^{-r})\) for even \(m\) and odd \(1 \leq l \leq 2n\). A link (or virtual link) \(TL_n\) is constructed which is called an \(m\)-string tangle link (or virtual link). Here, \(x_{m+1,l}^r = x_{m,l+1}^{-r}\) and \(x_{0,l}^r = x_{1,l-1}^{-r}\) for odd \(1 \leq l \leq 2n\), \(x_{m+1,l}^r = x_{m+1,l-1}^{-r}\) and \(x_{0,l}^r = x_{1,l+1}^{-r}\) for even \(1 \leq l \leq 2n\), \(x_{0,1}^r = u_0^r, x_{0,2n+1}^{-r} = v_0^{-r}\). An example is also shown in Fig.5 (c) for \(m = 3\) and \(n = 2\).

**Problem 5.1.** Given a link \(L_0\), let \(AL_n\) is an \(m\)-string alternating link constructed from \(L_0\) for \(m \geq 3\). Determine \(V_{AL_n}(t)\).

**Problem 5.2.** Given a link \(L_0\), let \(TL_n\) is an \(m\)-string tangle link constructed from \(L_0\) for \(m \geq 3\). Determine \(V_{TL_n}(t)\).

**Conjecture 5.3.** Suppose that \(L_0\) is connected and irreducible and that \(TL_n\) is an \(m\)-string tangle link constructed from \(L_0\) for \(m \geq 2\). If \(L_0\) is non-alternating, then \(TL_n\) is also non-alternating.

**Conjecture 5.4.** Suppose that a link \(L_0\) is prime and that \(L_n\) is an \(m\)-string alternating (or tangle) link constructed from \(L_0\) for \(m \geq 2\). Then \(L_n\) is prime.

**References**

[1] B. Bollobás, O. Riordan, A polynomial of graphs on orientable surfaces, *Proc. London Math. Soc.*, 83 (2001), 513-531.
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