A numerical retro-action model relates rocky coast erosion to percolation theory

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Abstract

We discuss various situations where the formation of rocky coast morphology can be attributed to the retro-action of the coast morphology itself on the erosive power of the sea. Destroying the weaker elements of the coast, erosion can create irregular seashores. In turn, the geometrical irregularity participates in the damping of sea-waves, decreasing their erosive power. There may then exist a mutual self-stabilization of the wave amplitude together with the irregular morphology of the coast. A simple model of this type of stabilization is discussed. The resulting coastline morphologies are diverse, depending mainly on the morphology/damping coupling. In the limit case of weak coupling, the process spontaneously builds fractal morphologies with a dimension close to $4/3$. This provides a direct connection between the coastal erosion problem and the theory of percolation. For strong coupling, rugged but non-fractal coasts may emerge during the erosion process, and we investigate a geometrical characterization in these cases. The model is minimal, but can be extended to take into account heterogeneity in the rock lithology and various initial conditions. This allows to mimic coastline complexity, well beyond simple fractality. Our results suggest that the irregular morphology of coastlines as well as the stochastic nature of erosion are deeply connected with the critical aspects of percolation phenomena.

Key words:
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1 Introduction

Coastline morphology is of current interest in geophysical research and coastline erosion may have important economic consequences [1,2]. Even more, the concern about global warming has increased the demand for a better understanding of coastal evolution. This paper deals specifically with the erosion of rocky coasts.

Rocky coasts have been estimated to represent 75% of the world’s shorelines [3]. They are found in different contexts, and there exists a rich bibliography on the subject, see for example [4] and the references therein, as well as [5] for an update bibliography. However, this estimation strongly depends on the very definition of what constitutes a rocky coast. Many cliffed coasts are fronted by beaches, with many different morphologies and several different dynamical processes in action. The morphology of these sea-shores may result from several different processes; tectonicity and various erosion mechanisms (sea, rivers, wind) acting on different soils as well as the possible role of sediment transport. Sea erosion can be imperceptibly slow, but nevertheless shapes coastal morphology. It can also be observed over human timescale, being of concern to planners. For instance, a study of cliff coasts in New Zealand reports sea erosion rates with peaks as high as 9 meters in one year [6,7,8] (with typical rates ranging from 0.02 to 0.5 m/yr). Eroding cliffed shorelines account for 20% of the entire New Zealand coast [8].

Accordingly to the definition of rocky coasts given in [5], here we address coasts dynamics in the limiting case where the role of sediment transport is considered to be negligible. For instance, tectonically active coasts often display rocky coasts with very limited sediment deposited by rivers (as in Peru and Chile or along the North America cordillera). The rugged appearance of these coasts is usually considered as an extension of the rugged mountains characterizing the nearby landscape, however it is difficult to exclude that sea wave erosion does not play a role in their morphology. Collision coasts also tend to be rocky, containing few depositional features. Because of their relative youth, neo-trailing edge coasts, such as the Arabian coast along the Red Sea, are also rugged and mostly rocky. Furthermore, there are many sites throughout the world where rocky and rugged coasts are found in tectonically passive margins, such as South Africa, parts of Argentina and Brazil, eastern Canada, southern Australia and a section of north-west Europe.

One might think that wave erosion can play a role in relatively low rocky coasts, but the height of the cliff is not a general contraindication for erosive sea dynamics. Very often these coasts exhibit some kind of irregular morphology. In the last decades, attempts to describe global geometry of sea-coasts have been made using the tools of fractal geometry to the point that the coast
of Britain has been taken as an introductory archetype of self-similarity in nature [9,10,11]. Since then, many tentative applications of fractal concepts to geomorphology [12,13] have been published but at the same time there has been some debate about the fractality of coastlines [14,15,16].

In other words, coasts and rocky coasts may be fractal or not, depending possibly on the scale on which the coast is observed. Often but not always, different scales exhibit different shapes. This corresponds to the variety of possible contributions to coast morphology mentioned above. Nevertheless, the observation of geometrical similarities and the presence of “some sort” of scale invariance in the morphology of rocky coasts, may suggest the existence of a common mechanism which, when in action, shapes this type of coastline.

A qualitative model for the appearance of fractal sea-coasts had been suggested in [17]. The idea is that irregular coasts contribute to the damping of sea waves with the consequence that the resulting erosion is weaker. More recently, a numerical model of such coastal dynamics was developed [18] and studied. It was found that a mechanism based on the retro-action of coastal shape on wave damping, leads necessarily, for the specific case worked in this paper, to the self-stabilization of a fractal coastline. Interestingly, it was found that the self-stabilized geometry belongs to a well defined universality class, precisely characterized in the mathematical theory of percolation [19]. In that sense, the notion of fractal geometry for rocky sea-coasts should no more appear as a curiosity, but as a necessary consequence of percolation theory.

In this paper we advocate the model, discussing the statistical analysis of earth geomorphological data, the possible role of sediments in erosion, the role of large scale heterogeneity in the lithology of the eroded coasts, and we address the problem of statistical characterization of different non fractal morphologies predicted by the model. The structure of the paper is the following.

In Section 2 we present some statistical field geomorphological data that suggest that coastlines geometry differs from the general earth geomorphology.

Our specific erosion model, based on the aforementioned retro-action mechanism, is presented in Section 3. There we discuss the fundamental connection between our model and the general theory of percolation.

The erosion dynamics, short and long term evolutions, are discussed in Section 4. We also discuss how the critical nature of percolation, results in a very irregular, or episodic, erosion process.

The question of fractal versus non-fractal sea-coast is discussed in Section 5. More precisely we give statistical arguments that could help to distinguish transitory from final morphology.
In Section 6.2 we address the fundamental question of the role of geology in the frame of our retro-action model. The qualitative results is that, very often, coastal morphology should be the results of both retro-action and geological constraints.

In Section 7 we discuss the possible role of sediments and rubbles in the erosion process. We show that in various cases the same scaling and morphologies should be obtained.

In Section 8 we present some detailed data analysis of some rocky coasts, unfolding the complexity of the real morphology of coastlines as well as of the inland coastal regions.

In Section 9 we give the summary and conclusion of the paper.

2 Is coastline morphology ”distinct” in global geomorphology?

A question that can be raised, observing rocky coasts, is whether their geometry is simply inherited by the morphology of the inland [11], or whether the interaction with the sea changes and shapes the coast in a distinct way. In order to disentangle and recognize specific features of coastlines, with respect to higher earth surface isolines, it is interesting to use tools which may help to reveal general universal features of their geometry.

2.1 Coastline morphology in global geomorphology

The following analysis of the world coastlines is part of a more general and detailed analysis, published elsewhere [20] (more details in the Appendix A). Topographic data for Earth have been obtained from the SRTM30-plus set [21]. The data consists in the earth surface elevation over a grid of points. From the data we computed earth isolines, the coastline belonging to the 0 elevation isoline. Next, the whole Earth surface is divided in squares of 4 degrees latitude x 4 degrees longitude and the fractal dimension is computed for each isoline portion in each square. Fig. 1 shows the result of the coastline fractal analysis. The Bottom left panel of Fig. 1 represents the distribution of the measured fractal dimensions for the zero elevation isoline. The world coastline average dimension is found to be slightly above 1.2, but rocky coasts have often higher dimensions.

Moreover, one can consider the behavior of the global isolines as a function of elevation near the sea level and compare their corresponding dimension. In the bottom right panel of Fig. 1 we show the average fractal dimension
measured between $-700$ and $700$ meters. Interestingly, exactly at $0$ elevation a rapid change in the measured average fractal dimension is observed. This gives an indication that the interaction between sea and land is responsible for the complex geometry of coastlines. In this sense, a model for the geometry of rocky coastlines should explicitly take in account the main physical processes

Fig. 1. Statistical analysis of coastlines fractal dimension. Top: color map of earth coastline, the color shows the local measured fractal dimension. Bottom left: Distribution of measured fractal dimension on the Earth coastlines. Bottom right: average fractal dimension for Earth isolines as a function of elevation (in meters) between $-700m$ and $700m$; the world average fractal dimension for coastlines (elevation $0$) is slightly above $1.2$. The horizontal dash-dotted line corresponds to $4/3$. 

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taking places at the interface between sea and land, that is the dynamics of coastal erosion.

2.2 The specific case of plateau coasts

The global analysis presented so far, suggests that the coastal morphology is not the sole reflect of the inland morphology. Large scale geometry of coastline may be the result of many different characteristic phenomena. Sand deposits usually smooth the irregularity of rocky coasts, filling bays, or may display specific patterns [22]. On the other hand the rough geometry of glacial valleys gives the very convoluted coastline typical of fjords at large absolute latitudes.

However, in many cases, rocky coasts present a very steep slope (or terrain gradient) with respect to the inland profile. This is the case of cliffs, or what we call "plateau" coasts. In our mind, a plateau coast is characterized by an extreme situation where a flat landscape becomes steep at the coast. A photographic example is shown in Fig. 2. In this case one expects: first, that sea erosion is the most relevant shaping mechanism, and second, that a 2D model (as the one presented in the next section) could be sufficient to approach such morphology evolution.

Of course, the case of plateau coasts is a limiting case. Discussion of several other examples of coastline complexity will be given in Section 8 below. Through these more detailed examples, we wish to put forward the idea that those shores with high local slopes (or terrain gradients) at the shore may be reasonably attributed to marine erosion, acting in a given geological context.

3 The 2D model of rocky coasts erosion

Rocky coasts erosion is the product of marine and atmospheric causes [23]. There exist many different erosion processes: wave quarrying, abrasion, wetting and drying, frost shattering, thermal expansion, salt water corrosion, carbonation, hydrolysis. In the same time, the mechano-chemical properties of the rocks constituting the coast, which are linked to structure, composition and aging defining their “lithology”, exhibit an unknown dispersion.

On the other hand, erosion is the consequence of the existence of an “erosion power”. It is a selective mechanism, which progressively eliminates the weaker parts of the surface. The remaining shore is then hardened as compare to the initial shore. But the erosion power is not constant and may change during erosion. In particular, damping mechanisms caused by the erosion itself could
arise, establishing a self-stabilizing mechanism. In other words, erosion is the product of three ingredients: erosion power, rocks lithology, and a damping depending on the shore morphology. The interplay between these three factors is discussed here in various conditions, through the numerical implementation of a simple model [18].

The specific damping mechanism considered here, relies on the studies of irregular or fractal acoustic cavities. They show that viscous damping is increased on a longer, irregular surface [24,25,26,27]. These considerations have been applied practically in the conception of efficient acoustic road absorber now installed along several roads in France [28].

An other example of the proposed selection mechanism, is the case of the dynamics of pit corrosion of thin aluminum films [29]. There, fractal geometries spontaneously appears at the interface of the corroded solid. The phenomenon can be understood by means of a minimal model [30], which disregards many atomic details of the corrosion process. The analogy between erosion and corrosion is less artificial than one could think a priori (see discussion in Section 7).
In the following we make an arbitrary distinction between “rapid” mechanical erosion (namely wave quarrying) and “slow” weakening of the rocks due to the action of the elements (weathering processes). These “slow” weakening events trigger, from time to time, new “rapid” erosion sequences. The justification is that mechanical erosion generally occurs rapidly, mainly during storms, after rocks has been slowly altered and weakened. We first study this supposedly rapid erosion mechanisms. Then we show that the full complex dynamics, involving fast and slow processes, changes the shape of the coast on a longer time scale keeping its gross geometrical characteristics. This dynamics is reminiscent of the quasi-equilibrium evoked by Trenhaile [31] (see below).

Our model schematize the sea, the land, and their interaction in the following way.

3.1 The sea-coast system as a damped resonator

In analogy with the acoustic oscillations in a cavity, the sea, together with the coast, is considered to constitute a resonator [24,25]. It is assumed that there exists an average excitation power of the waves $P_0$. The “force” acting on the unitary length of the coast is measured by the square of the wave amplitude $\Psi^2$. This wave amplitude is related to $P_0$ by a relation of the type $\Psi^2 \sim P_0 Q$ where $Q$ is the morphology dependent quality factor of the system: the smaller the quality factor, the stronger the damping of the sea-waves. There are several different causes for damping. Since the different loss mechanisms occur independently, the quality factor satisfies a relation of the type

$$\frac{1}{Q} = \frac{1}{Q_{\text{coast}}} + \frac{1}{Q_{\text{other}}},$$

(1)

where $Q_{\text{coast}}$ is the quality factor due to the “viscous” dissipation of the fluid moving along the coast and the nearby islands and $Q_{\text{other}}$ is related to other damping mechanisms (e.g. bulk viscous damping). Studies of fractal or irregular acoustic cavities [24,25,26,27] have shown that the viscous damping increases roughly proportionally to the cavity perimeter.

This model uses, as a working hypothesis, the idea that sea-waves are more damped along an irregular coast much in the same way as acoustic modes in irregular cavities. This appears to be an empirically known effect used to build efficient break-waters that are based on hierarchical accumulations of tetrapods piled over layers of smaller and smaller rocks, in close analogy with fractal geometry (see Fig. 3, left, and the many descriptions of breakwaters in Ref. [32]).
Fig. 3. Artificial and natural break-waters. Left: Break-water made of concrete tetrapods on layers of rocks (picture from [http://www.sys.com.my](http://www.sys.com.my)). Right: Picture taken at Pacific Grove, in the Monterey peninsula, California. Our measure of the fractal dimension of such coast is close to $4/3$.

To check the relation between irregularity and damping for the particular shapes of a sea-coast during erosion, we consider the properties of four 2D resonators, with the upper side corresponding to four successive coast shapes. For each time and morphology we solve numerically the Helmoltz wave equation with Neumann boundary conditions. We compute the eigenmodes in a given frequency range, assuming weak losses on the eroding profile. For each eigenmode we compute the energy dissipation which is supposed to take place on the coast. In other words we study the wave dissipation due to viscous forces acting on the boundary of irregular swimming pools. Results are given in Fig. 4, showing that the average losses of the computed eigenmodes in the four resonators increase roughly proportionally to the coast perimeter. The fact that the velocity of sea waves depends on the sea-floor depths would not modify the general link between perimeter and damping.

Therefore, one can, in first approximation, assume that $Q_{\text{coast}}$ is inversely proportional to the coast perimeter $L_p(t)$ whereas $Q_{\text{other}}$ is independent of the coast morphology. In other words, the sea exerts a homogeneous erosion power $f(t)$ on each coast element proportional to $\Psi^2(t)$:

$$f(t) = \frac{f_0}{1 + \frac{g}{L_0} L_p(t)},$$

(2)

where $L_p(t)$ is the total length of the coast at time $t$ (then $L_p(t = 0) = L_0$). The factor $g$ measures the relative contribution to damping of a flat shore as compared with the total damping. The quantity $f_0$ is the renormalized value of $P_0$ such that $f(t) < 1$ at all $t$. Small $g$ factor correspond to weak coupling between the erosion power and the coast length and large $g$ correspond to strong coupling.

Note that for the higher frequency waves with short wavelengths that contains a large erosion power, it is clear that their damping is proportional to the
3.2 The land as a disordered solid

The “resisting” random earth is modeled by a square lattice of random units of global width $L_0$. Each site represents a small portion of the earth, named a rock here. The sea acts on a shoreline constituted of these rocks, each one characterized by a random number $l_i$, between 0 and 1, representing its lithology. The erosion model should also take into account that a site surrounded by the sea is weaker than a site surrounded by earth sites. Hence, the resistance
to erosion \( r_i \) of a site depends on both its lithology and the number of sides exposed to the action of the sea. This is implemented here through the following weakening rule: sites surrounded by three earth sites have a resistance \( r_i = l_i \). If in contact with 2 sea sites the resistance is assumed to be equal to \( r_i = l_i^2 \). And, if site \( i \) is attacked by 3 or 4 sides, it has zero resistance. The iterative evolution rule is simple: at computer time step \( t \), all coast sites with \( r_i < f(t) \) are eroded (sometimes exposing new sites to erosion), and then \( L_p(t) \) and \( f(t) \) are updated together with the resistances of the earth sites in contact with the sea. Then, from one step to the next, some sites are eroded because they present a “weak lithology” while some strong sites are eroded due to their weaker stability due to sea neighboring. An example of local evolution is shown in Fig. 5. Note that our variable \( t \) simply denotes a number of computer steps and therefore is not a real time.

Fig. 5. Illustration of the erosion process. The thick number at the square center represent the lithology \( \{l_i\} \). The numbers in the corners are the corresponding resistances \( r_i \) which depend on the local environment as explained in the text. The sites marked with 1 are earth sites with no contact with the sea. Left and right: situations before and after an erosion step with \( f(t) = 0.5 \). After this step resistances are updated due to the new sea environment.

3.3 Results

To exemplify the intrinsic properties of the model we consider an artificial situation where erosion would start on a flat sea-shore. The computer implementation of the above dynamic model leads to a spontaneous evolution of the smooth seashore towards geometrical irregularity as shown in Fig. 6. The figure exhibits the time evolution of an initially flat coastline towards geometrical irregularity. The left column describes the case of weak coupling, the right column the case of strong coupling.

In the case of "weak coupling" the terminal morphology is highly irregular and it looks much like some of the irregular morphologies observed on the field. Consider for instance, the north eastern coast of Sardinia: the fractal dimension of this coast found to be very close to \( 4/3 \), as shown in Fig. 7. There, we compare the coastline fractal dimension (0m isoline) and the isolines
Fig. 6. Time evolution of the coastline morphology starting with a flat sea-shore. Left and right columns respectively weak and strong coupling. Top to bottom: successive morphologies with the final morphologies at the bottom. Note that case (b), transitory shape with weak coupling and case (f), final shape with strong damping appear to be similar but it is shown below that there exist statistical means to distinguish one from the other.

closest to the coast, which is quite steep. On the opposite, the inland does not present very high reliefs. The picture at the bottom-left panel shows the coast near Palau.

Indeed, in the weak coupling case, our model produces a fractal terminal morphology with a dimension very close to $4/3$ (see Fig. 8, left). Note, however, that the fractal morphology extends up to a maximum scale, of the order of the transverse width of the artificial coastline $\sigma$ (depicted in the last snapshots of Fig. 6). The precise mathematical definition of this statistical width $\sigma$ is given below (see Eq. 4).

Here we want to stress that the average width is the distance below which the coast geometry is fractal. In other words we expect to have a geometrical scale invariance up to a distance $\sigma$. At larger scale the seashore can be considered as the reunion of independent (uncorrelated) fractals of size $\sigma$.

It turns out that $\sigma$ depends directly through a power law of the coupling parameter $g$. This is shown also in the right panel of Fig. 8. One observes a power law with exponent close to $−4/7$. As we will discuss below, the fractal dimension $D_f = 4/3$ and the $\sigma$ scaling exponent $−4/7$ are the signature of the deep connection of our model with percolation theory. In the language of
Fig. 7. Northern coast of Sardinia (near Palau). Top right: Box-counting measure of coastal isolines, compared with the ideal box-counting for a fractal with 4/3 dimension. Top right inset: Coastal isolines, several elevations. Bottom left inset: A picture of the coast.

statistical physics, the model belongs to the universality class of percolation, as will be discussed in the next section.

For “strong coupling”, the erosion ends on rugged morphologies, as shown in Fig. 6(f). This can be understood as the limit of small $\sigma$, i.e. where the geometrical correlation of the coast is short ranged. Note that such rugged morphology resembles to transitory morphologies observed with weak coupling as that of Fig. 6(b). We will see below in Section 5 that there exists statistical methods to distinguish transitory (young coasts) from final morphologies (old coasts).

In summary, we discuss a model which although based on very few ingredients, gives rise to a variety of coastline morphologies, fractal or simply rugged, transitory or final.

3.4 Relation between coastal morphology and percolation theory

Our study indicates the existence of a connection between coastal erosion of rocky coasts and percolation theory. Percolation is a cornerstone of the theory of disordered systems, which has brought new understanding and techniques to a broad range of topics in physics, materials science, complex networks,
Fig. 8. Top left: Box-counting determination of the model coast fractal dimension. The straight line is a power law with a slope -4/3. The best fit gives: $D_f = 1.332(3)$. The data refer to a large system with $L_0 = 10^4$, and small damping $g = 10^{-4}$. Top right: Scaling behaviour of the coast width $\sigma$. The straight-line is a power law with the GP exponent -4/7 (each point is an average over 400 samples with $L_0 = 5000$). Bottom: Dependence of the erosion force $f(t)$ as a function of the number of computer time steps $t$. The left figure shows the evolution of the sea erosion force acting on the coastline during a “rapid” sea-erosion process (different values of the scale gradients $g$ and of $f(0)$). This dynamics spontaneously stops at a value weakly dependent on $g$ (systems with $L_0 = 1000$, averaged over ten different realisations). The right plot is the erosion force $f(t)$ during the complete dynamics (“slow” weathering process triggering “rapid” erosion) illustrated in Fig. 9.

epidemiology, etc. [19]

Percolation theory deals with the following statistical problem. Consider a lattice, like the square lattice for instance, and independently assign to each
site a random number obtained from a uniform distribution between 0 and 1. Now select the set of sites which happen to have a number smaller than some fixed arbitrary value \( p \) and occupy them. If these selected sites are first nearest neighbors, they define so-called clusters. Of course if \( p \) is small, the clusters themselves will be of small size. However, strictly above a critical value \( p_c \), there exists an infinite cluster that crosses the lattice from left to right and from top to bottom. The most important characteristics of percolation phenomena is that near criticality, that is when \( p \) is close to \( p_c \), the scaling properties of the system are independent of the lattice geometry. Although the value of the percolation threshold \( p_c \) does depend on the lattice under consideration the exponents of the so-called scaling laws that describe the properties of clusters and the geometry of the infinite cluster at percolation are independent of the lattice. In particular the external frontier of the percolation cluster is fractal with a fractal dimension exactly equal to \( 7/4 \) and the so-called accessible perimeter has a fractal dimension equal to \( 4/3 \) [33,34,35,36].

As the reader can suspect, there is a similarity between the percolation cluster and the coastline produced by our model. In fact, at the end of the erosion process, all earth sites at the interface with the sea have a resistance larger than the wave erosive power. So, in some sense, our erosion model spontaneously identifies, and stops at, a percolating interface constituted of “strong” sites.

Indeed, there exist a direct relation between our model and the theory of percolation. In particular, besides the fractal dimension of the coast, the behavior of the width \( \sigma \) as a function of the coupling factor \( g \) exhibit a power law dependence, as shown in Fig. 8. This power law is characteristic of a specific variant of percolation, called gradient percolation [37,38,39]. In the Appendix B we explain gradient percolation and how it can be related to our erosion model.

At this stage it is important to recall briefly the concept of universality in statistical physics. Universality means that several very different phenomena can exhibit analogous macroscopic properties described by the same power law exponents. The phenomena which exhibits the same exponents belong to a unique universality classes. The models described in this paper, percolation, gradient percolation, our model of coastal erosion, all belongs to the percolation universality class, irrespectively of many details. For example, if we change the lattice geometry, or the distribution of the rocks lithology, the model still belongs to the percolation universality class. Here it means that shorelines made of rocks of different nature and sizes, subjected to different external climate, can exhibit the same large scale geometrical properties.

Recent studies inspired by our model, corroborates the deep connection between coastal morphology and percolation. A remarkable property of such real coasts has been observed: they have been found to be conformally invariant [40]. This geometrical property stands for itself but there exists a
mathematical demonstration that it exists for the so-called accessible perimeter of the percolation cluster. (It should be stressed that not every fractal with dimension $4/3$ is conformally invariant. [33,34,35]).

4 Dynamical evolution of the coast morphology

Let now describe in more detail the erosion dynamics resulting from the minimal rules defined by our model. For the sake of simplicity, we consider first the case of a flat (smooth) initial coastline submitted to the erosion action of the sea. As discussed later (Section 6.1), the dynamics with different initial morphologies can be understood from these results.

4.1 Erosion dynamics

In the first steps of the dynamics the erosion front keeps quite smooth and it roughens progressively as shown in Fig. 6. During the process, finite clusters are detached from the infinite earth, creating islands. At any time, both the islands and the coastline perimeters contribute to the damping. As the total coastline length $L_p(t)$ increases, the sea force becomes weaker. At a certain time step $t_f$, the weakest point of the coast is stronger than $f(t_f)$ and the “rapid” dynamics stops. This indicates that erosion reinforces the coast by preferential elimination of its weakest elements until the coast is strong enough to resist further erosion. Whatever the dynamics, at the stopping time $t_f$ the coastline is irregular (see Fig.6) up to a characteristic width $\sigma$. This width $\sigma$ is defined as the standard deviation of the final coastline depth. More precisely, defined $n_f$ as the mean number of points of the front lying on the line $x$, and $x_f$ as the average position of the front, that is

$$x_f = \frac{\sum_{x=0}^{L_g} xn_f(x)}{\sum_{x=0}^{L_g} n_f(x)},$$

then $\sigma$ is

$$\sigma^2 = \frac{\sum_{x=0}^{L_g} (x - x_f)^2 n_f(x)}{\sum_{x=0}^{L_g} n_f(x)}.$$  

The (averaged) time evolution of $f(t)$ is shown in Fig. 8 (left). The dynamics depends strongly on the value of $g$. If $g$ is large enough the dynamics is rapid and the erosion stops on an irregular but non-fractal sea-shore (see the strong coupling case $g = 0.1$ in Fig. 8). On the opposite, if $g$ is small enough, the
dynamics last much longer and it finally stops on a fractal sea-shore (\textit{weak coupling} case). Note that the final values of the sea-power are different from the classical percolation threshold. This is linked to the weakening rule implemented in the model.

It is however important to stress that the time here is a computer step, not directly comparable with physical time. A unit of computer step corresponds to the duration between erosion events. Such a duration is \textit{directly related to the strength or fragility of the coast itself}.

### 4.2 Long term dynamics

Of course, the real dynamics of the coasts are more complex than the “rapid” processes considered above. They result from the interplay with the slow weathering processes, generally attributed to carbonation or hydrolysis \cite{23}. These processes act on longer, geological, time scales.

In order to mimic this long term evolution after the ending of ”rapid” erosion, the lithology parameter $l_i$ of all the coast sites is decreased by a small fraction $\epsilon$, \textit{i.e.} $l_i' = (1 - \epsilon) l_i$ with $\epsilon \ll 1$ after the erosion has stopped at $t_f$. One or a few coast sites then become weaker than $f(t_f)$ and they are eroded. This exposes new sites, previously protected, to erosion, triggering possibly a new start of the rapid erosion dynamics up to a next arrest. This process can then be iterated. Snapshots of the coastline at successive arrest times are shown in Fig. 9 together with their measured fractal dimension. Note that the measured fractal dimension fluctuates around $4/3$, which is the expected fractal dimension for a very large coast with a vanishing or very small coupling $g$ (as in Fig. 9).

Moreover, at each restart of erosion, a finite and strongly fluctuating portion of the earth is eroded. The “slow” weathering mechanisms induces also small fluctuations of $f(t)$ (see Fig. 8 right). In the language of coastal studies \cite{41}, the system state evolves through a dynamical equilibrium where small perturbations may stimulate large fluctuations and avalanche dynamics. The coastal engineering community has recently suggested the use of a stochastic description of the dynamics of rocky coast erosion, characterized by episodic, discontinuous events, more than simple constant erosion rate processes \cite{42}. Field inspections \cite{43} confirmed that the local variability in rock resistances, or in general ”geological contingency influences the nature and scale of erosion processes and thresholds” \cite{44}.

In our model such fluctuating dynamics is due to the underlying criticality of percolation systems. A detailed statistical analysis of the episodic erosion events is out of the scope of the present paper. Nevertheless, we wish to point
out that the statistics of such events should follow power laws (similarly to what has been computed for the etching of disordered solids at low temperature [45]). Interestingly, power laws have been observed in the statistics of field coastal soft-cliff erosion [46]. In [47], using a high temporal resolution rockfall statistics, the episodic character of the dynamics is debated, in opposition to a continuum activity. Within our approach, a power law statistics is expected in the framework of percolation theory.

Note that the fluctuating dynamics of $f(t)$ in our model, are not due to fluctuations of the sea incoming power $f_0$. These could also be included, in order to mimic storms, for instance. Such fluctuations could give raise to reactivations of fast eroding events, as for the weathering mechanism. In general, we don’t expect a change in the overall set of morphologies generated by the model.

5 Fractal versus non-fractal seacoasts

As explained above, our model of coastal erosion shows how the feedback between the lithology heterogeneity of shores and the damping effect of geometric irregularity may lead, in the weak coupling case, the coast towards a fractal geometric shape with a dimension characteristic of percolation. This case can be qualified as “canonic” since it corresponds to well established percolation theory results.
However, under different conditions, the model may generate coastlines with complex irregular shapes, not necessarily fractal. This happens in the case of strong coupling morphology of “old coasts”, or in the case of transient morphologies of the weak coupling ”young coasts”. A measure that can result useful to characterize the morphology in this case is the following.

First, one has to recall that the exponent which plays a role in the geometry is that of the accessible perimeter, namely $4/3$. So an empirical way to determine if an observed coast may result of such erosion is to measure the length of the coast contained in a square box of side $l$. This is the classical mass method in fractal studies. In our case, this length should be proportional to $l$ to the power $4/3$ up to a box size of order $\sigma$. For larger boxes, the mass should be linear as a function of $l$. This is shown in Fig. 10 for various computed final or “old” coasts obtained for different, but quite large, values of $g$. In this case, as can be seen in Fig. 8 left, $\sigma$ is much smaller then 100, and the coasts appear simply rough, rather than fractal (see Fig. 6(f)). Nevertheless, a clear flattening of the curve, for $l \approx \sigma$ is visible. (The fact that the signature of the fractal exponent can be observed in a non fractal front has been also investigated mathematically for gradient percolation in [48,49]).

Interestingly, for young transient coasts as that of Fig. 6(b), which also appears rough, non fractal as in Fig 6(f), the flattening is much less evident. In Fig. 10 we show the measure of $m(l)$ during the fast erosion process, which eventually leads to red curve in Fig. 10. We note that the flattening arise quite late in the erosion process. This suggest that with respect to this measure, transient structures behave differently than final structures.
6 Application to more general geological situations

Our model, as discussed until now, may appear too simplistic. Here we discuss how our results applies more generally.

6.1 Starting from an irregular coastline

In the above results, the general trend is to reach a rugged morphology starting from a flat one, an obviously artificial situation. Even without including specific mechanisms, mimicking a differential erosion due to convergence of sea waves (due to local topography or bathymetry, as well as specific wind directions), we can wonder what would happen in our model if one would start from an initial coast with a salient geometry.

Let call $\sigma_f$ the typical scale length of the irregularity produced by the dynamics starting with an initial flat coast (for instance measured as the average of $\sigma$, defined in Eq. 4, on several dynamics at large times). This quantity depends on $g$ (it is proportional to $g^{-4/7}$). Suppose now the case of an initial non flat coast, that is a coast with an initial characteristic irregularity scale $d$. Then, if $d > \sigma_f$, the erosion dynamics will initially change (decorate) the coast on the smallest scale $\sigma_f$, keeping the larger irregularity $d$. Then, the slow erosion dynamics will eventually lose memory of the initial geometry, leading to a shoreline irregular up to time $\sigma_f$. This case is shown in Fig. 11, where a “toothed” coastline is submitted to erosion according to our model in the case of strong coupling (small $\sigma_f$). Otherwise, if $d < \sigma_f$, the erosion will increase the irregularity up to length $\sigma_f$, which becomes the dominant irregularity scale. In Fig. 12 we show the results of the equivalent dynamics (same $g$ and $f(0)$) of two initially different irregular coasts, respectively with $d < \sigma_f$ and $d > \sigma_f$. After some time, which depends on the dynamics parameters, the depth of the coast shows a fluctuating dynamics around $\sigma_f$, irrespectively from the initial value of $d$. This, we think, resembles what Trenhaile had in mind drawing a figure in his paper [31] (reproduced here as an inset in Fig. 12). Several observation are however in order. Here $d_{eq}$ corresponds to $\sigma_{eq}$, which is related to the correlation length of the underlying percolation process. It depends on $g$, the coupling between (geometrical) damping and erosion. There’s no need to invoke an a priori differential erosion rate between headlands and bays. Finally, the quasi-equilibrium regime evoked by Trenhaile, corresponds here to the stationary, critical dynamics, where avalanches are triggered by slow erosion, local, events.

We think that this kind of dynamics blends two apparently contrasting ideas: by one side the quasi-equilibrium dynamics predicted by Trenhaile, on the
Fig. 11. Long term erosion, strong coupling starting from an irregular geometry (shore flattening): a) initial configuration; b) after 50 erosion cycles; c) after 100 erosion cycles; d) after 150 erosion cycles.

Fig. 12. Depth of the coast, measured as the length scale \( \sigma \) of its irregularity, defined in Eq. 4, as a function of computer time during the slow erosion process. Two different initial conditions are considered: large \( d \equiv \sigma(0) \) (red curve) and small \( d \) (black curve), compared with the average depth obtained starting from a initially flat shore \( \sigma_f \) (dashed line). Both simulations are performed with \( g = 0.1 \), \( L = 1000 \) and \( f(0) = 0.6 \). In the inset, reproduction of Fig. 5 from [31].

Other the image of an episodic dynamics [50,42]. At the same time it supports the continuum activity scenario recently proposed in [47], where the magnitude-frequency distribution corresponds to the avalanche statistics of our critical dynamics, expected in our model since the connection with percolation and observed in similar cases [45].
6.2 Geological heterogeneity

It is of general consent that marine erosion processes acts on an earth that possess its own geological identity and this obviously influence the observed morphologies. In the above calculations and discussion, the lithology distribution has been considered totally random, without any spatial correlations. By this, we mean that the lithology of neighboring “rocks” are independent random numbers. There is no correlation in the disorder.

This is a limit case. A more realistic description should include the concept of “geological heterogeneity”. In this case the lithology exhibits a dispersion around a local average value, which changes only on large distances. The scale of variation of this local average is called the correlation distance of the random lithology.

In Fig. 13, the earth is constituted at the beginning of a collection of different patches, some of which contains long distance correlations. It then presents some regions of weak lithology and some regions of strong lithology while other regions are uncorrelated. The final morphology retains this heterogeneity, with part of the shore very irregular while other regions are smoother (similar phenomena could be invoked in order to interpret several detailed studies of the observed self-similarity properties of seacoasts [14,15,16]).

Such irregular coastlines, which retain some strong resemblance with real coasts, do not enter a simple fractal or scaling category. The important fact here is that, whatever the lithological conditions, there exists a spontaneous evolution towards irregularity that stops spontaneously. This is again a consequence of the concept of percolation but of course of a percolation problem asked on a spatially correlated randomness. Moreover, even in this case, the initial flat coast is an idealization, and some memory of ancient shapes could also influence the coastline morphology, as shown in Fig. 13.

7 The role of sediments

Up to now, the results described here where obtained under the assumption that sediments play no role, either because they do not modify damping on the surface, either because they disappear by a rapid transport effect. There exists, indeed, situations in which sediment transport can be neglected. See, for instance, the discussion in Chapt. 18 in [3].

However, when sediments stay on site, they can produce two different effects in terms of rock erosion (the role of sediments produced by erosion as determinant
for shaping sediment rich shorelines, which has witnessed recent modeling efforts [22,51], is not discussed here). First if they are small enough they can play the role of little hammers accelerated by the waves. On the opposite they can fall on the sea floor and contribute to the damping of waves. To simplify, the small pieces increase erosion while the large heavy pieces increase damping. The erosion increase has been neglected here because its role should be only transitory.

Now, suppose, in an extreme scenario, that the damping is not dominated, as was assumed above, by the interaction with the coast perimeter but by the damping effect of the sediments. In fact, it is known, and however poorly understood, that the shore waves are partially damped by their interaction with the sea floor. In first approximation, the more sediments the more damping. This means that the equivalent quality factor would be inversely proportional to the total amount of eroded material at time $t$ that we call $M(t)$, rather than
to the coast perimeter length. In that case the erosion power would evolve as

\[ f(t) = \frac{f_0}{1 + g' \frac{M(t)}{M_0}}, \quad (5) \]

where the factor \( g' \) measures the relative contribution to damping of accumulated sediments shore as compared with the total damping. It might also take care of the fraction of the sediments transported away from the coast. In particular erosion of very high cliffs, producing large amounts of heavy rubble, would consequently induce a strong coupling effect, leading to rugged but not fractal sea shores.

Note that this damping mechanism, in which the eroding force decreases with the amount of eroded mass, leads to a one to one correspondence with the aluminum corrosion experiments and models mentioned above [30] (where the corrosive power decreases with the amount of corroded material). Then, the theory developed there, which again makes connection to gradient percolation, applies also in our case. This means that erosion will lead also to fractal or scaling interfaces with the same scaling geometries.

The two mechanisms may occur simultaneously, giving \( f_0 / f(t) = 1 + gL_p(t)/L_0 + g'M(t)/M_0 \), without affecting the result: more generally we expect that any model that express some kind of link between random erosion and increased damping would lead to the same type of self-organized fractality or scaling.

### 8 Coastline complexity

The analysis presented in Section 2, suggests that the coastal morphology is not the sole reflect of the inland morphology. Before discussing specific examples in detail one should recall that other phenomenon are known to play a role. For example sand deposits usually smooth the irregularity of rocky coasts, filling bays, or may display specific patterns [22]. Also the rough geometry of glacial valleys gives the very convoluted coastline typical of fjords at large absolute latitudes. Note also that our model has used implicitly the fact that the power giving rise to wave excitation was uniform. Of course this may be not true on a too large scale (think for instance to different wind or current conditions).

In this section we consider several specific locations, and we analyze the coastline fractal morphology, in comparison with the fractal geometry of the closest inland isolines. This is made possible by high resolution SRTM3 data-set, which provide altimetric data of the earth surface in a grid of 3 arc-seconds (that is a resolution of about 90m) and with a vertical error smaller than 9
meters [52]. Thanks to this new tools, it is possible to unfold the complexity of real coastline geometry, which is the result of the interplay of several physical phenomena, and, consequently might go beyond the simple model proposed here.

In the following field analysis, we restrict to coasts where the terrain gradient at the coast is much larger than the average gradient in the inland, that is what we named plateau coasts, in Section 2.2.

An example of such coasts is the coast of Brittany, as can be seen in Fig. 14: the inland is very flat (most of the coastal land is lower than 100m), and the shores are usually very steep. The coast is subject to severe storms, which can have impressive effects on rocky cliffs [53]. However, the overall measured fractal dimension of the coastline is smaller than $4/3$. This could be due to the presence of beaches or in general sand deposits. If this would be the case, isolines slightly higher should not be affected by this and they should better show the effect of pure sea erosion. An increase of fractal dimension is in fact observed for isolines slightly higher then sea level. However a very interesting phenomena occurs, which can be clearly observed from the box counting curve of the 100m isoline: there, two appreciably different slopes appear, one at short scale and another, steeper, for longer range. This observation, which can not be explained by our simple model, could be the result of different geophysical processes, acting at different length scale and/or on different time scales. In the lower inbox of the figure we show the fractal dimension measured restricting the range of the fit respectively at short and long range. This effect seems maximum at about 100m of elevation. Interestingly, the short range fit is very close to $4/3$ for isolines around 70m (again note that above 150m the range for box-counting is too small to be conclusive).

Such a large scale bending of isoline box-counting curves seems common in other coastal regions, for instance in the northern coast of California, as shown in Fig. 15. (Previous measures of the coastline fractal dimension of American coasts are reviewed for instance here [54]).

In the inset of the left panel, the long range and small range slopes are compared as a function of isoline elevation. While the short range slope is quite constant around $4/3$ (which is quite interesting, view the uplifting nature of this coast), at long range a wide variation is observed.

In order to test the generality of this observation, we perform the analysis on the whole American western coast, from $29^\circ N$ to $40^\circ N$ of latitude, parceled in $1^\circ \times 1^\circ$ coastal cells. For each cell, we computed the short and long range slopes as a function of isoline elevation. In the right panel of Fig. 15 we plot the average of the measured slopes. One can see that, even if weaker, the effect is still visible. At the moment, we don’t have any explanation for this phenomena,
Fig. 14. Analysis of Brittany coast. Scaling of box counting shows complex features, not a simple power law. This is clearly visible for the $h = 100$ m isoline. Upper inset: map of the region analysed. Lower inset: slopes of box counting curves at short and long range, as a function of elevation.

Fig. 15. Left: Analysis of northern Californian coast. Scaling of box counting shows complex features, not a simple power law. This is clearly visible for the $h = 400$ m isoline. Upper inset: map of the region analysed. Lower inset: slopes of box counting curves at short and long range, as a function of elevation. Right: The average fractal dimension of West coast of north america as a function of elevation, computed at short and large range.

even if the multifractal nature of the terrain could possibly be invoked [56]. A more detailed investigation, including the eastern north American coast, which behaves differently, will be presented in a future publication [57].
9 Summary and Conclusion

In this work we have discussed a model for the formation of irregular rocky coastal morphology. This model links the reciprocal evolution of the erosion power with the topography of the coast submitted to that erosion. The model reproduces at least qualitatively some of the features of real coasts using only simple ingredients: the randomness of the lithology and the decrease of the erosion power of the sea.

Despite the simplicity of the model, a complex phenomenology emerges. This is not surprising in complex systems research. As stated by Murray et al.: "The analytical lens of emergent phenomena highlights the idea that studying the building blocks of a system the small-scale processes within a landscape, for example may not be sufficient to understand the way the system works on much larger scales. The collective behaviors of small-scale components synthesize into effectively new interactions that produce large-scale structures and behaviors". [58]

In our model, depending on time, on the damping strength and on possible correlations between the lithological properties, different coastline characteristics emerge. In the simplest case, "weak coupling", the retro-action leads to the spontaneous formation of a fractal seacoast with a fractal dimension $D_f = 4/3$.

The appearance of this specific fractal dimension uncovers the close, deep connection with percolation theory and the "universality class" of percolation in the physics of critical systems.

The dynamics in the weak-coupling case leads to a self-organized fractality, since the fractal geometry plays the role of a morphological attractor: whatever its initial shape, a rocky shore will end up fractal when submitted to this type of erosion. For this reason this case can be qualified as "canonic".

For larger coupling, and/or depending on the erosion stage, various irregular coastlines emerge from the same model. Between these irregular but non-fractal morphologies we have been able to distinguish "young" or transient from "old" or final coastlines. Future work should then include field observation, in the hope to find such morphologies and to confront these ideas with geologic knowledge. Obviously, one of the first goals of field studies will be to try to read world coastline morphology in terms of gradient percolation in the limit of strong gradients. For these "old" coasts, one should observe power laws with percolation exponents although the coasts may not be fractal [48].

At this point, one should stress that, between the enormous geometrical variety and complexity of sea-coasts, fractal sea-coasts and specially those with
dimension close to 4/3 should no longer be considered anymore as "complex". On the contrary, they may in fact be the "most simple" consequences of uncorrelated randomness and retro-action whatever the details of retro-action. In such a frame, they are necessary consequences of percolation phenomena. Since percolation possess the universality properties of phase transitions [19], the scaling properties of these coasts should not depend on more specific processes. By this, we do not mean that the specificity of the process no longer plays a role, for instance on the real time scale of erosion, but that they do not determine the global large scale geometry and scaling behaviors. More precisely:

- Any retro-action damping acting on any distribution of lithology will lead to the spontaneous self-stabilization of an irregular coast. In particular, nonlinear damping effects, possibly including the role of turbulence, would modify the time history of erosion but not the scaling properties of the coast geometry.
- Once the coast is stabilized the long term evolution will be stochastic. It will be triggered even by small events and will produce some kind of avalanche statistics. This is related to the fact that a stabilized sea-shore may hide "weak" lithology regions or more fragile patches.
- In the field, the islands which have resisted erosion under a power larger than the final power \( f(t_f) \), should be stronger that the coast itself. This could be verified on the historical data of known seacoasts and the evolution of neighboring islands.

In summary, the present work provides a rationale that connects damping, as illustrated in Fig. 3 left, with rocky coast morphology, as illustrated in Fig. 3 right. A simple feedback mechanism that relates the large scale morphology with the sea wave erosion power (usually noted with \( F_W \) in the coastal literature [59]), together with a local variability of rock resistance (\( F_R \)), naturally lead to the formulation of our minimal model, which points out how both the irregular morphology of coastlines as well as the episodic and stochastic erosion dynamics may both be the effect of an underlying critical (percolation) point. Of course, such a feedback process, does not exclude the existence of other processes acting at more local, or meso-scales [60].

Nevertheless, our framework confirms the idea that the final coast emerges from a natural selection process, which eliminates the weaker part of the coast. The resulting shoreline constitutes a strong, but possibly fragile, barrier to sea erosion. To the extent that this idea applies, natural coasts should be "preserved" and managed with care.
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Appendices

A Isoline analysis

Topographic data for Earth have been obtained from the SRTM30-plus set [21]. The data consists in the earth surface elevation over a grid of points. The resolution of the grid is 2 minutes of degree for latitude and longitude, which corresponds, in the region of interest, to about four kilometers. We have extracted the coastline as the set of points at zero elevation. In fact, we use a generalized method to extract “isolines” at arbitrary elevation: to draw the isoline of level $h$, we identify on the topographic grid all the nearest neighbor sites whose elevations, $h_1$ and $h_2$, satisfying $h_1 \geq h \geq h_2$. Using coordinates and elevations of such points, the coordinates of the isoline point are computed via a simple linear interpolation.

Once the isolines are found, the whole Earth surface is divided in squares of 4 degrees latitude x 4 degrees longitude. In fact the square regions are separated by only two degrees as to have an overlapping covering of the total surface. Then we proceed in computing the fractal dimension in each square, via the classical box counting procedure [61]: the fractal dimension $D_f$ has been measured through a least squares fit of the exponent of the box counting plot in the range of $(0.06 : 0.6)$ degrees, which roughly correspond to a range from few kilometers to several tenths of kilometers. We disregard fractal dimensions computed on isoline sets with less than $N_m = 500$ points. The regression error in the fractal dimension $D_f$ so estimated never exceeded 4%. The whole numerical analysis has been repeated in the following cases: (i) full resolution for continental land (30 seconds instead of 2 minutes); (ii) different interpolation schemes for the definitions of isolines; (iii) different minimum number of points ($N_m = 100,1000,2000$). The values of the corresponding measured fractal dimensions are only slightly affected, and the main results do not change.
**B Gradient percolation**

In order to relate the erosion model to the theory of percolation, we need to introduce the Gradient Percolation model (GP) [37]. In this model, each site \((x,y)\) of the lattice is occupied with a probability which changes linearly from 1 to 0 in a given direction \(p(x) = 1 - x/L_g\) \((L_g\) is the size of the lattice in the \(x\) direction). There is then a *gradient* in the occupation probability (not to be confused with the terrain gradient in geomorphology).

In GP there is always an infinite cluster of occupied sites as there is a region where \(p\) is larger than the standard percolation (SP) threshold \(p_c\). There is also an infinite cluster of empty sites as there is a region where \(p\) is smaller than \(p_c\). The object of interest is the GP front, i.e. the external limit (or frontier) of the infinite occupied cluster.

This front is a random fractal object with \(D_f = 7/4\) but the accessible part of it is a random fractal with \(D_f = 4/3\) [33,34,35,36]. It has an average position \(x_f\) and a statistical width \(\sigma\) defined as follows. For \(0 \leq x \leq L_g\), \(n_f(x)\) is the mean number, per unit horizontal length, of points of the front lying on the line \(x\). It measures the front density at distance \(x\). The position \(x_f\) and the width \(\sigma\) of the front are then defined in terms of the \(n_f(x)\) by Eq.3 and 4.

It was found in [37] that the mean front is located at a distance where the density of occupation is very close to \(p_c\) or \(p(x_f) \approx p_c\) [38,62,63]. It was also found that the width \(\sigma\) depends on \(L_g\) through a power law \(\sigma \propto (L_g)^{\nu/(1+\nu)}\) where \(\nu = 4/3\) is the correlation length exponent [19] so that \(\sigma \propto (L_g)^{4/7}\). The width \(\sigma\) was also shown to be a percolation correlation length [64].

As we can see, the fractal dimension of the accessible GP front coincide with the value measured in the erosion model. Moreover, the width of the front scales with respect to the gradient exactly as the width of our coastlines do with respect to the parameter \(g\), i.e. through an exponent equal to 4/7. See Fig. 10 right)

The reason for this is that \(g\) is proportional to a gradient of occupation probability by the sea from the following argument. At time \(t\), the erosion power is \(f(t)\) while the sea has eroded the earth up to an average depth \(X(t)\), an increasing function of \(t\). Inverting this function, \(f\) can be written as \(f(t(X))\).

There exists then a spatial gradient of the occupation probability by the sea. For small enough \(g\) one can write \(|df/dx| = |(g/L_0)(dL_p(X)/dX)|\). The quantity \(dL_p(X)/dX\) is a function of \(g\) but to the lowest order it is a constant independent of \(g\) since even with \(g = 0\), there will be an erosion due to randomness and a consequent perimeter evolution \(L_p(t)\). Then to lowest order, the real gradient \(df/dX\) is linear in \(g\), the coupling factor in the erosion model.
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