Non-perturbative results for the spectrum
of surface-disordered waveguides

N. M. Makarov and A. V. Moroz

Institute for Radiophysics and Electronics, 12 Acad. Proskura St., Kharkov, 310085, Ukraine

Abstract

We calculated the spectrum of normal scalar waves in a planar waveguide with absolutely soft randomly rough boundaries beyond the perturbation theories in the roughness heights and slopes, basing on the exact boundary scattering potential. The spectrum is proved to be a nearly real non-analytic function of the dispersion $\zeta^2$ of the roughness heights (with square-root singularity) as $\zeta^2 \to 0$. The opposite case of large boundary defects is summarized.

In finite systems a long-distance (waveguide) signal propagation is caused by multiple reflections of the signal from opposite lateral boundaries. If the boundaries are irregular, each of the reflections is accompanied by a non-coherent scattering of the travelling wave. The multiple successive scattering events lead to substantial dephasing and attenuation of the primary signal.

As far as we know, this effect was first consistently treated in works. That simple and physically clear approach was based on the perturbation theory in the squared r.m.s. height $\zeta$ of the boundary roughness and therefore required small enough $\zeta$, viz. $(k_z\zeta)^2 \ll 1$, $(\zeta/R_c)^2 \ll 1$, $k\zeta^2/R_c \ll 1$. Here $k_z$ is the transverse (normal to the waveguiding direction) component of the wavevector $\vec{k}$ and $R_c$ is the mean length of the boundary defects. In more recent papers, the theory of wave scattering from statistically rough surfaces was extended to arbitrary values of the Rayleigh parameter $(k_z\zeta)^2$. However, the other two inequalities of the above set were still necessary, which made it impossible to deal with, e.g., steep
roughness slopes ($\left(\zeta/R_c\right)^2 \gtrsim 1$) and/or the shadowing effect ($k\zeta^2/R_c \gtrsim 1$).

In this Letter we put forward the approach which is non-perturbative in the roughness heights and slopes. It is based instead on the exploitation of the exact boundary scattering operator. Due to this fact, we managed to extend the waveguide theory up to the quite general conditions of weak scattering [8], which are much less restrictive than the above approximations. The most impressive advantage of our method is that it leads to new physical results even for the region of small heights $\zeta$ ($\left(k\zeta\right)^2 \ll 1$), where the waveguide propagation is believed to be well studied. The most surprising result is a non-analytic (square-root) dependence of the waveguide spectrum on the dispersion $\zeta^2$ (i.e. on the coefficient of correlation) of the roughness heights. This non-analyticity can not be in principle derived perturbatively. It means that the exact randomly rough boundary can not be reduced to the smooth random-impedance one even for the arbitrarily small irregularities.

We consider a 2D (planar) strip confined to a region of the $xz$-plane defined by $\xi(x) \leq z \leq d$, where $\xi(x)$ is a Gaussian distributed random function of the longitudinal coordinate $x$ characterized by the properties

$$\langle \xi(x) \rangle = 0, \quad \langle \xi(x)\xi(x') \rangle = \zeta^2 W(|x - x'|).$$  \hfill (1)

The angular brackets denote an average over the ensemble of realizations of the profile function $\xi(x)$. The binary coefficient of correlation $W(|x|)$ has the unit amplitude ($W(0) = 1$) and the typical width of $R_c$. The spatial distribution of a scalar wave field inside the strip is governed by the Helmholtz equation and the temporal dependence by the factor $\exp(-i\omega t)$, $k = \omega/c$. Both lateral boundaries $z = \xi(x)$ and $z = d$ are supposed to be absolutely soft, i.e. the field vanishes upon them. We seek the averaged Green’s function $\langle G(x, x'; z, z') \rangle$ to this Dirichlet boundary value problem.

The exact integral equation for $G(x, x'; z, z')$ can be obtained through the use of Green’s theorem:

$$G(x, x'; z, z') = G_0(|x - x'|; z, z') + \int_{-\infty}^{\infty} dx_s dz_s G_0(|x - x_s|; z, z_s) \hat{\Sigma}(x_s, z_s) G(x_s, x'; z_s, z'),$$  \hfill (2)
where \( \hat{\Xi}(x_s, z_s) \) is the effective scattering operator,

\[
\hat{\Xi}(x_s, z_s) = \delta [z_s - \xi(x_s)] \left[ \frac{\partial}{\partial z_s} - \frac{d\xi(x_s)}{dx_s} \frac{\partial}{\partial x_s} \right],
\]

and \( G_0(|x - x'|; z, z') \) is the Green’s function for the ideal (smooth) strip with \( \xi(x) \equiv 0 \).

To perform averaging of Eq. (2) we apply the elegant technique derived in works \(^8\), \(^9\). Although the averaging procedure is rather straightforward, it has few fine points which will be discussed elsewhere \(^10\). As the result, we find the averaged Green’s function:

\[
\langle G(x, x'; z, z') \rangle \equiv G(x, x'; z, z') = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \exp \left[ ik_x(x - x') \right] G(k_x; z, z'),
\]

where longitudinal Fourier transform \( G(k_x; z, z') \) is given by

\[
G(k_x; z, z') \sim \frac{G_0(k_x; z, z')}{1 - k_z \cot(k_z d) M(k_x)}.
\]

Here \( k_z = k_z(k_x) = \sqrt{k^2 - k_x^2} \) and \( G_0(k_x; z, z') \) is the Fourier transform (similar to Eq. (3)) of \( G_0(|x - x'|; z, z') \). The self-energy \( M(k_x) \) is defined via the binary correlator of the scattering operator (3):

\[
M(k_x) = \int_{-\infty}^{\infty} dz_s dz_s' dx_s \frac{\sin(k_z z_s)}{k_z} \exp(-ik_x x_s) \times \langle \hat{\Xi}(x_s, z_s) G_0(|x_s - x_s'|; z_s, z_s') \hat{\Xi}(x_s', z_s') \rangle \exp(ik_x x_s') \frac{\sin(k_z z_s')}{k_z}.
\]

By equating the denominator of Eq. (5) to zero, we obtain the dispersion equation for the rough-bounded strip. The solution to this equation in the lowest (linear) order in \( M(k_x) \) is \( k_x = k_n + \delta k_n \). Here \( k_n = \sqrt{k^2 - (\pi n/d)^2} \) is the unperturbed longitudinal wavenumber of an \( n \)-th propagating normal waveguide mode and \( \delta k_n \) being the complex modification to \( k_n \) caused by the wave scattering from the lower irregular boundary,

\[
\delta k_n = \gamma_n + i(2L_n)^{-1} = -M(k_n)(\pi n/d)^2/k_n d.
\]

The real part \( \gamma_n \) of \( \delta k_n \) is responsible for variation of the phase velocity (for dephasing), while \( L_n \) has the meaning of the attenuation length for the \( n \)-th mode.
In fact, a form of the solution (5), (7) is common and well-known. Our improvement lies in the self-energy $M(k_x)$ (6). This expression is nothing else but a first non-vanishing (quadratic) term in an expansion of $M(k_x)$ in powers of the exact scattering operator (3). We stress that such an approximation is essentially different from and is much more general than an extensively exploited first term in an expansion of $M(k_x)$ in powers of the dispersion $\zeta^2$.

To find the domain of validity for Eqs. (6), (7) one can use ideas proposed in the book. We have proved that this domain coincides with the two natural requirements of weak wave scattering from a rough boundary, which are equations themselves with respect to the external dimensionless parameters $(k\zeta)^2$, $kR_c$, $kd/\pi$, and $n$,

$$\left|\delta k_n\right|\Lambda_n \ll 1, \quad \left|\delta k_n\right|R_c \ll 1.$$ (8)

The first of these implies smallness of the complex phase shift over the distance $\Lambda_n = 2k_n d/(\pi n/d)$ passed by an $n$-th mode between two successive reflections from the rough boundary. Under this condition the mode experiences a large number $L_n/\Lambda_n \gg 1$ of the reflections before its amplitude will be substantially attenuated. Also this ensures smallness of $\delta k_n$ in comparison with the unperturbed wavenumber $k_n$, because the inequality $k_n\Lambda_n \gtrsim 1$ always holds.

The second of Eqs. (8) indicates that the phase shift must remain small over the typical variation scale $R_c$ (effective correlation radius) of the boundary scattering potential. Obviously, this is the necessary and sufficient condition for correctness of statistical averaging over $\xi(x)$. The quantity $R_c$ is defined as the typical width of the correlator $\langle \hat{\xi}(x_s, z_s)\hat{\xi}(x'_s, z'_s) \rangle$ in Eq. (8) as a function of $x_s - x'_s$. It does not generally coincide with the mean length $R_c$ of boundary defects.

We next calculate the correlator and the integrals over $z_s$ and $z'_s$ in Eq. (8). Then we substitute the result into Eq. (7) and extract real and imaginary parts of $\delta k_n$. Finally, we get explicit formulae for $\gamma_n$ and $L_n$: 

4
\[
\gamma_n = \frac{\zeta^2 (\pi n/d)^2}{2 k_n d} \sum_{n'=1}^{n_d} \frac{(\pi n'/d)^2}{k_{n'} d} \left[ \tilde{W}_S(k_n, k_{n'}) - \tilde{W}_S(k_n, -k_{n'}) \right] - \frac{(\pi n/d)^2}{k_n d} M_2(k_n)
\]
\[
+ 2\zeta^2 \frac{(\pi n/d)^2}{k_n d} \sum_{n'=n_d+1}^{\infty} \frac{(\pi n'/d)^2}{|k_{n'} d|} \int_0^\infty dx \exp(-|k_{n'}| x) \text{Re} \left[ \exp(-ik_n x) \tilde{W}(k_n, i |k_{n'}|; x) \right],
\tag{9}
\]
\[
L_n^{-1} = \zeta^2 \frac{(\pi n/d)^2}{k_n d} \sum_{n'=1}^{n_d} \frac{(\pi n'/d)^2}{k_{n'} d} \left[ \tilde{W}_C(k_n, k_{n'}) + \tilde{W}_C(k_n, -k_{n'}) \right].
\tag{10}
\]

Here integer \( n_d = [kd/\pi] \) is the number of the propagating normal modes in the smooth strip. The function \( \tilde{W}(k_x, q_x; x) \) is the generalized coefficient of correlation \((\tilde{W}(k_x, q_x; x) \simeq \mathcal{W}(|x|)) \) as \( \zeta^2 \to 0 \),

\[
\tilde{W}(k_x, q_x; x) = \left(4k_x q_x \zeta^2\right)^{-1} \times \left\{ \left[ (k_x + q_x)^2 + (k_x + q_x)(k_x - q_x) \right] \left( \frac{k_x}{k_x} - \frac{q_x}{q_x} \right) - (k_x - q_x)^2 \frac{k_x q_x}{k_x q_x} \right\} S(k_x + q_x, k_x + q_x; x)
\]
\[
\left[ (k_x - q_x)^2 + (k_x - q_x)(k_x - q_x) \right] \left( \frac{k_x}{k_x} + \frac{q_x}{q_x} \right) + (k_x - q_x)^2 \frac{k_x q_x}{k_x q_x} \right\} S(k_x - q_x, k_x - q_x; x)
\]
\[
+ 2(k_x - q_x) \left[ q_x \frac{k_x}{q_x} - k_x \frac{q_x}{k_x} + (k_x - q_x) \frac{k_x q_x}{k_x q_x} \right] S(k_x + q_x, k_x - q_x; x) \right\};
\tag{11}
\]
\[
S(t_1, t_2; x) = (t_1 t_2)^{-1} \sin \left[ t_1 t_2 \zeta^2 \mathcal{W}(|x|) \right] \exp \left[ -(t_1^2 + t_2^2) \zeta^2/2 \right],
\tag{12}
\]

where \( q_x = k_x(q_x) \) and the functions \( \tilde{W}_{S(C)}(k_x, q_x) \) stand for sine and cosine Fourier transforms of \( \tilde{W}(k_x, q_x; x) \) respectively. The component \( M_2(k_x) \) of the self-energy is given by

\[
M_2(k_x) = k_x^2 \frac{2k_x d}{k_x^2} \sum_{n'=1}^{\infty} \left\{ 2S(k_x + \pi n'/d, k_x - \pi n'/d; 0)
\right.
\left[ S(k_x + \pi n'/d, k_x + \pi n'/d; 0) - S(k_x - \pi n'/d, k_x - \pi n'/d; 0)
\right.
\left[ \frac{k_x}{k_x} \frac{\pi n'}{d} \right] \left[ S(k_x + \pi n'/d, k_x + \pi n'/d; 0) - S(k_x - \pi n'/d, k_x - \pi n'/d; 0) \right].
\tag{13}
\]

An essential distinction between \( \gamma_n \) (9) and \( L_n^{-1} \) (10) is that the latter is formed by scattering of a given \( n \)-th propagating mode into propagating waveguide modes with \( n' \leq n_d \) only, while the former has much more complicated structure due to contributions of both propagating and evanescent \((n' > n_d)\) modes. This feature is a basis for surprising properties of \( \gamma_n \).
Brief analysis and discussions: We start with a relatively simple and widely used limiting case of small boundary perturbations, when

\[(k\zeta)^2 \ll 1. \tag{14}\]

Here Eq. (10) for \(L_n\) is simply reduced to the standard result from the earlier works by replacing \(\tilde{W}(k_x, q_x; x)\) with \(W(|x|)\). On the contrary, \(\gamma_n\) shows an unconventional type of behavior. The reason is that the last two terms of Eq. (9) are mainly formed by those evanescent modes whose normal wavelengths \((\pi n'/d)^{-1}\) are of the order of the roughness height \(\zeta\). Each of such ‘resonant’ modes gives a contribution to \(\gamma_n\) proportional to \(\zeta^2\), while the number \(n'\) of those modes is \(\sim d/\zeta \gg n_d\), i.e. it is inversely proportional to \(\zeta\). All this gives a linear dependence of \(\gamma_n\) on the roughness height \(\zeta\),

\[\gamma_n \sim \zeta (\pi n/d)^2/k_n d. \tag{15}\]

This formula is the main result of the Letter. It leads to the following significant conclusions:

1. Since \(L_n^{-1} \propto \zeta^2\) as \(\zeta^2 \to 0\), then \(\gamma_n \gg L_n^{-1}\) and, hence, the entire spectrum shift \(\delta k_n\) turns out to be nearly real, i.e. \(\delta k_n \simeq \gamma_n \propto \zeta\). This means that a signal propagating through a nearly smooth waveguide is dephased (chaotized) much earlier (over much shorter distances) than its initial amplitude is damped.

2. From Eq. (15) and item 1 it follows that \(\delta k_n\) is a non-analytic (square-root) function of the dispersion \(\zeta^2\), or of the binary correlator \(\langle \rangle\): \(\delta k_n \propto (\zeta^2)^{1/2}\).

3. Usually it is believed that the condition (14) is sufficient to infer that any long-wave normal mode with \((k_2\zeta)^2 \ll 1\) (i.e. with \((\pi n/d)^{-1} \gg \zeta\)) is mainly scattered into the long-wave modes as well, \((\pi n'/d)^{-1} \gg \zeta\). This assumption allows immediate replacement of the exact Dirichlet boundary condition, formulated on a randomly rough boundary, by an approximate impedance one laid down on the averaged (deterministic) boundary \(z = 0\) with the random impedance \(\xi(x)\). We have proved that such a reduction is groundless, because the ‘resonant’ evanescent modes with \((\pi n'/d)^{-1} \sim \zeta\) dominate. Thus, the problem of wave propagation through a waveguide with an absolutely soft random boundary can not
be reduced to that with the smooth random-impedance boundary even for the arbitrarily weak perturbations.

An important step in analyzing the case (14) is to obtain the explicit weak-scattering condition. To this end, we substitute Eq. (15) into Eqs. (8) and apply the asymptotic \( \tilde{R}_c \simeq R_c \), which is valid for \( (k\zeta)^2 \ll 1 \). As a result, we find that Eqs. (8) can be rewritten as

\[
(k_z\zeta)^2 = (\pi n/d)^2\zeta^2 \ll \min \left\{ 1, (\Lambda_n/R_c)^2 \right\}.
\]

(16)

This inequality is automatically satisfied within Eq. (14) if successive reflections of an \( n \)-th mode from the rough boundary are not correlated \( R_c \ll \Lambda_n \). However, if the correlations are strong \( \Lambda_n \ll R_c \), then Eq. (16) supplements Eq. (14) and may even become more restrictive than Eq. (14). Note that the roughness slope \( \zeta/R_c \) may far exceed unity within the limit (14), (14).

The situation with large boundary defects, when

\[
(k\zeta)^2 \gg 1,
\]

(17)

is much more diverse and complicated than the case (14). For the most part it is analyzable only numerically. Therefore we postpone the details until the longer paper and list here only few of intriguing results:

1. In contrast to the case (14), the imaginary part of \( \delta k_n \) (7) may well compete with its real part. Moreover, the situation with \( L_n^{-1} \gtrsim |\gamma_n| \) is rather typical.

2. The real spectrum shift \( \gamma_n \) (4) reverses the sign as \( k\zeta \) reaches some threshold value \( k\zeta \approx 1.5 \div 2.5 \), which is slightly dependent on the other parameters. So, the large boundary defects (17) may not only decrease, but also increase the phase velocity of a propagating wave.

3. As \( k\zeta \lesssim 2 \), \( \delta k_n \) is almost insensitive to the slope \( \zeta/R_c \) and to the presence of the shadowing effect, which is controlled by the Fresnel parameter \( k\zeta^2/R_c \). However, as \( k\zeta \gtrsim 2 \), the weak-scattering conditions (8) are fulfilled for not too steep slopes only \( (\zeta/R_c \lesssim 2 \div 3) \), while the strong shadowing effect is still allowed \( (k\zeta^2/R_c \lesssim 8 \div 10) \).
We mention that the approach presented in this Letter can be extended to vector wave fields.

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