THE STATEMENT OF MOCHIZUKI’S COROLLARY 3.12, INITIAL THETA DATA, AND THE FIRST TWO INDETERMINACIES

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Abstract. This paper does not give a proof of Mochizuki’s Corollary 3.12. It is the first in a series of three papers concerning Mochizuki’s Inequalities. The present paper concerns the setup of Corollary 3.12 and the first two indeterminacies, the second [DH20a] concerns log-Kummer correspondences and ind3, and the third [DH20b] concerns applications to Diophantine inequalities (in the style of IUT4). These manuscripts are designed to provide enough definitions and background to give readers the ability to apply Mochizuki’s statements in their own investigations. Along the way, we have faithfully simplified a number of definitions, given new auxiliary definitions, and phrased the material in a way to maximize the differences between Theorem 1.10 of IUT4 and Corollary 3.12 of IUT3. It is our hope that doing so will enable creative readers to derive interesting and perhaps unforeseen consequences of Mochizuki’s inequality.

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1. Introduction

It has been almost seven years since Mochizuki first released his manuscripts online and the content of his inequality remains poorly understood today. In fact, at the time of the Oxford workshop in December 2015, things were so opaque that Brian Conrad famously asked during one of the sessions whether Mochizuki’s inequality even represented an inequality of two real numbers. We have come a long way since then. (Let us begin by stating unambiguously that Mochizuki’s inequality is an inequality of real numbers.)

The purpose of this paper and its sequels [DH20b] and [DH20a] is to put Mochizuki’s inequality in a user friendly context for working mathematicians. While these manuscripts do indicate in some places how certain parts of Mochizuki’s constructions work, they do not attempt to give a proof of [Moc15c, Corollary 3.12]. Moreover we black-box and suppress

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the anabelian geometry as much as possible (at some junctures this is simply not possible)\footnote{Unlike many papers, here, the black-boxing is a feature and not a bug!}. By the end of [DH20b] we will rigorously derive a variant of Theorem 1.10 of [Moc15d], (an effective version of Szpiro’s inequality for elliptic curves in “initial theta data”). In this, all of the assumptions will be made transparent — including the statement of Corollary 3.12 and how to apply it.

Now about the black-boxes. Traditionally, Mochizuki’s work has been very difficult to discuss because of our inability to black-box his constructions. We have found a way to do this using model theory. In this manuscript (and its sequels) we work under the hypothesis that all of Mochizuki’s “functorial algorithms” can be expressed in terms of interpretations in the sense of model theory. We refer the reader to [Hod97, §4.3] for the basics of interpretations (the more topos-minded readers might be inclined to read [Car18, Definition 6.12] which provides a more categorical framework). We just mention in passing that for this to work we need to abandon classical finitary logic and allow for countable conjunctions of formulas and countably many sorts (this is by default done in the topos theory literature but is atypical of classical first order model theory literature). A development of “hard” model theory in this setting can be found, for example, in [Mar14]. For readers wanting a glimpse of how seriously Mochizuki’s theory is interconnected with model theory we point them to the large number of interpretation tables in the appendix of [DH20a] which document his handiwork. These tables (of \(\approx 100\) interpretations) account for about half of the interpretations occurring in IUT\footnote{This appendix has been extracted and put on the first author’s webpage in the “Notes” section. This footnote will be removed once [DH20a] is put on the arxiv. Also, for the experts, we remark that we consider it an open problem to determine the minimal signatures for these interpretations.}.

In order to begin the black-boxing process, one needs to develop the structures (in the sense of model theory) that are then to be interpreted in developments towards [Moc15c, Corollary 3.12]. This manuscript does just that. We have broken down many of the definitions in IUT3 which provide a sort of skeleton that require anabelian fuel to run. While unfortunately we do not give complete accounts of these interpretations (and how they interact), we give many remarks often labeled “connections to IUT papers” indicating where our definitions are coming from and where the reader can read more. For readers seeking a more qualitative understanding of how outer automorphisms of absolute galois groups of \(p\)-adic fields can affect invariants of elliptic curves related to Szpiro’s conjecture we refer them to [Jos20].

We begin by stating Mochizuki’s inequality unambiguously. Mochizuki’s Inequality ([Moc15c, Corollary 3.12]) states that for an elliptic curve \(E\) over a number field \(F\) sitting in initial theta data

\[
(F/F, E_F, l, M, V, V_{mod}, \varepsilon)
\]

the following inequality holds

\[
\tilde{\deg}(P_q) \leq \tilde{\deg}_{g_{hp}}(P_{\text{null}}(U_\Theta)).
\] (1.1)
In [3] we set up definitions and notation necessary to make (1.1) precise. In particular we set up an “adelic” abelian group $\mathbb{L}$ which is abstractly isomorphic to a space that appears in [Moc15c, Proposition 3.9]. In Mochizuki’s notation $\mathbb{L} \cong \mathbb{Z}^\mathbb{Q}((S^+ F^{(\mathcal{O} \circ D^-)}))_{\psi_0}$. The group $\mathbb{L}$ contains two very important regions $U_\Theta$ and hull($U_\Theta$) involved in (1.1). The region $U_\Theta$ is explicitly given by

$$U_\Theta = \text{Ind}2 \cdot (\text{Ind}1 \cdot (\mathcal{O}_L(-P_\Theta))^{\text{Ind}3}) \subset \mathbb{L}$$

The symbols Ind1, Ind2, and Ind3 are Mochizuki’s infamous indeterminacies. Here Ind1, Ind2 are (factor-wise) $\mathbb{Q}_p$-vector space automorphisms (possibly $p = \infty$) and Ind3, which is the most complicated of the indeterminacies, is the subject of our second paper, and is a map on power sets. The order of operations in (1.2) is intentional and correct. We note that Mochizuki prefers to consider Ind1 and Ind2 as operating “at the same time” omitting this composition. We also note that Mochizuki’s definition of Ind3, is in terms of a containment of a particular region in a “log-shell” and omits any derivation of the kind we provide in [DH20a]. As stated before, the meaning of the symbol $(\mathcal{O}_L(-P_\Theta))^{\text{Ind}3}$ is a bit nuanced and is not just the application of the function Ind3 on a set $\mathcal{O}_L(-P_\Theta)$. The region $\mathcal{O}_L(-P_\Theta)$ is built from (“Gaussian”) monoid actions and the operation Ind3 also uses these module structure in its definition in addition to a process Mochizuki calls log-linking.

To make the relation between the degrees appearing in (1.1) and “log-volumes” on $\mathbb{L}$ precise we state that $\mathbb{L}$ is equipped with a collection of subsets $\mathcal{N}(\mathbb{L})$ and a map $\overline{\nu} : \mathcal{N}(\mathbb{L}) \to \mathbb{R}$ such that

$$\overline{\ln} \nu_L(\text{hull}(U_\Theta)) = -\overline{\deg}_{lgp} (P_{\text{hull}(U_\Theta)})$$

As there has been some confusion regarding this matter in the past, we would like to state unequivocally that the “log-volumes” in [Moc15c] and [Moc15d] are not actually logarithms of volumes but rather averages of weighted sums of logs of normalized $p$-adic and archimedean Haar measures. To streamline the presentation we have found it convenient to think probabilistically in terms of random measure spaces and random measurable sets. This is new. In our formulation $\mathbb{L} = \prod L_p$ (as abelian groups) and $\overline{\ln} \nu_L(\text{hull}(U_\Theta)) := \sum_p \overline{\nu}_{L_p}(\text{hull}(U_\Theta)_p)$. For $p \neq \infty$, a random measurable set $A_p \in \mathcal{N}(\mathbb{L}_p)$ can be viewed as $A_p = \prod_{\vec{v}} A_{\vec{v}} \subset \prod_{\vec{v}} L_{\vec{v}}$ where each $L_{\vec{v}}$ is a finite dimensional $p$-adic vector space with a (normalized) Haar measure $\mu_{\vec{v}}$ and $\vec{v}$ runs over sets of tuples of places of $\mathbb{Q}(j_E)$. Once the set indices $\vec{v}$ is given the

3There are actually 4 interpretations of this structure in [Moc15c] and various comparisons of these interpretations (and their expansions) are central to Mochizuki’s claims. There is

- the “holomorphic/frobenius” (hol/fr) interpretation [Moc15c] c.f. Theorem 3.11.ii.a (left terms)],
- the “holomorphic/etale” (hol/et) interpretation [Moc15c] c.f. Theorem 3.10.ii (forth display)],
- the “mono-analytic/frobenius” (ma/fr) interpretation [Moc15c] c.f. Thm 3.11.ii.a, (middle terms]),
- the “mono-analytic/etale” (ma/et) interpretation [Moc15c] Theorem 3.11.i.a].

These interpretations have been listed in decreasing strength. We say that a structure $S$ is stronger than another structure $S'$ if and only if $S$ interprets $S'$. We say that an interpretation $I : [T_1] \to [S]$ is stronger than an interpretation $I' : [T_2] \to [S]$ if $T_1$ interprets $T_2$ as structures. Here we, for a structure $S$ we are letting $[S]$ denote a small connected groupoid of structures isomorphic to $S$. 3
structure of a probability space we define
\[ \ln \nu_{L_p}(A_p) := \mathbb{E}(\ln \mu_{\vec{v}}(A_{\vec{v}})/\dim(L_{\vec{v}})) = \sum_{\vec{v}} (\ln \mu_{\vec{v}}(A_{\vec{v}})/\dim(L_{\vec{v}})) \Pr(\vec{v}). \]

In other words \( \ln \nu_{L_p}(A_p) \) the expectation of the random variable
\[ \vec{v} \mapsto \ln \mu_{\vec{v}}(A_{\vec{v}})/\dim(L_{\vec{v}}), \]
as \( \vec{v} \) varies over our sample space. These spaces, the conversion between degrees of divisors, and “measures” are exposited in detail in the sections surrounding §3.6. See Theorem [3.10.1] for a proof of how these divisor/region conversions work. In the sequel this formalism allows us to state a probabilistic version of Szpiro’s inequality [DH20b, Theorem 1.0.3]. This inequality provides a nice milestone between [Moc15c, Theorem 1.10] (Mochizuki’s version of Szpiro’s conjecture under the initial theta data hypotheses) and [Moc15c, Corollary 3.12].

In [Moc15d, Definition 3.1, pg 61—63] Mochizuki introduces a notion of initial theta data (using several pages):
\[ (\mathcal{F}/F, E_F, l, C_K, V_{V^\text{had}}(K), \mathfrak{e}). \]

In §5 we demystify this definition breaking it into digestable pieces. In this section we relate the definition of initial theta data to the theory of the classical modular curve \( X_0(l) \) and its Fricke involution. After giving the definitions we prove an existence theorem which uses the classical theory of transvections. Finally, once general existence is established, we construct an explicit tuple of initial theta data for the elliptic curve \( E/\mathbb{Q} \) with Cremona label E11a1. We note that the methods here could, in principle, be used to generate databases of initial theta data, although it is unclear at the time of writing this if division field computations make it prohibitively expensive to run experiments.

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2. Background and Notation

2.1. General Notation.

2.1.1. $p$ will always denote a prime.

2.1.2. Sets denotes the category of sets, Ab denotes the category of abelian groups, Grp denotes the category of groups, Mon denotes the category of monoids, and Vect$_F$ the category of $F$-vector spaces for some field $F$. Given a category $C$ and objects $a, b$ we let $C(a, b)$ denote the hom set of those two objects in that category.

2.1.3. If $C$ is a category and $C \in C$ we will let $[C]$ denote the connected groupoid consisting of the subcategory of $C$ of all objects isomorphic to $C$ where the morphisms are isomorphisms.

2.1.4. If $A$ is an abelian group we will let $A_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} A$. 
2.1.5. We now define $M^{pf}$ for a commutative monoid $M$ with unit. Let $J$ is the divisibility category whose object are natural numbers and whose morphisms keep track of divisibility: $n \rightarrow m$ if $m = dn$; call this morphism $d$. We let $M^{pf} := \text{colim} D$ where $D : J \rightarrow \text{Mon}$ where $D(n) = M$ for every $n$ and the image of each morphism $d$ is $m \mapsto m^d$.

In the case that $M$ is an abelian group $M^{pf} \cong M \otimes \mathbb{Z} \mathbb{Q}$.

2.2. **Finite Probability Spaces.** By a finite probability space we mean a pair $(S, \Pr_S)$ where $S$ is a finite set and $\Pr_S : S \rightarrow [0, 1]$ is a function satisfying $\sum_{s \in S} \Pr_S(s) = 1$. When the context is clear we will use the notation $S = (S, \Pr_S)$ and just refer to all probability functions as $\Pr$ rather than $\Pr_S$. We may also use the notation $(S, \{w_s\}_{s \in S})$ for $(S, \Pr)$ where we have let $w_s = \Pr(s)$.

If $f(x)$ is a vector valued function on $S$ we define the expectation to be 

$$E(f(x) : x \in S) := \sum_{s \in S} f(s) \Pr(s).$$

When the space over which the expectation being taken is clear we may also just write $E(f(x))$.

Given two such space $(S_1, \Pr_1)$ and $(S_2, \Pr_2)$ we form the independent product $(S_1 \times S_2, \Pr_1 \times \Pr_2)$ by declaring $\Pr(s_1, s_2) = \Pr(s_1) \Pr(s_2)$.

2.3. **Varieties and Galois Theory.**

2.3.1. If $L$ is a field then $\overline{L} \supset L$ will denote an algebraic closure. Also $G_L = G(\overline{L}/L)$ will denote its absolute Galois group.

2.3.2. For an abelian variety $A$ and an integer $m$ we let $A[m]$ denote its group-scheme of $m$-torsion points. Suppose now that $A$ is defined over a field $L$. We may abusively write $A[m]$ for the $G_L$-module $A[m](\overline{L})$. We will let $\rho_m : G_L \rightarrow \text{Aut}(A[m])$ denote the natural $G_L$-representation. For $m$ coprime to the characteristic of $L$ we refer to $\rho_m$ as the mod $m$ Galois representation and we have $A[m] \cong (\mathbb{Z}/m)^{2g}$ as abelian groups.

2.3.3. If $Z$ is a hyperbolic curve with compactification $\overline{Z}$ we will let $Z^c = Z \setminus \overline{Z}$ denote the collection of cusps. We will also let $\pi_1^{\text{alg}}(Z, \overline{z})$ denote the étale fundamental group of $Z$. When confusion doesn’t arise we may just write $\pi_1$ instead of $\pi_1^{\text{alg}}$.

2.4. **Local Fields.**

2.4.1. We will let $\mathbb{C}_p$ denote the $p$-adic completion of $\overline{\mathbb{Q}}_p$.

2.4.2. Let $K/\mathbb{Q}_p$ be a finite extension. Let $\pi$ be a uniformizer of $K$. By the normalized valuation on $K$ we mean the valuation $\text{ord}_K$ satisfying $\text{ord}_K(\pi) = 1$. If $L/K$ is a finite extension the we will let $e(L/K)$ denote the ramification degree of the extension. Note that normalized valuations satisfy $\text{ord}_L = e(L/K) \text{ord}_K$. We will let $|x|_K = p^{-\text{ord}_K(x)}$.

2.4.3. If $L/\mathbb{Q}_p$ is a finite extension then $I_{L/L}$ will denote the inertia group of $L$. 
2.4.4. Also we let \( r_p = p^{-1/(p-1)} \in \mathbb{R} \). This is the radius at which the \( p \)-adic logarithm and \( p \)-adic exponential become mutually inverse to each other.

2.4.5. Let \( K_1, \ldots, K_m \) be finite extensions of another field \( K_0 \) all of which are discretely valued. Let \( L = K_1 \otimes_{K_0} \cdots \otimes_{K_0} K_m \). Let \( T = \mathcal{O}_{K_1} \otimes \mathcal{O}_{K_0} \cdots \otimes \mathcal{O}_{K_0} \mathcal{O}_{K_m} \). We will let \( \mathcal{O}_L \) denote the integral closure/normalization of \( T \) in the total ring of fractions (which turns out to be \( L \)). If we decompose \( L \) using the Chinese Remainder Theorem then \( L = \bigoplus_{j=1}^r L_j \) and after a bit of work, one can check that \( \mathcal{O}_L = \bigoplus \mathcal{O}_{L_j} \) (see [Eis13, pg60; Exercises 3.14, 3.15; pg 251]).

Haar measures on tensor products are normalized so that \( \mu_L(\mathcal{O}_L) = 1 \).4

2.4.6. Let \( L \) be a finite extension of \( \mathbb{Q}_p \). We give \( L \) two norms. The first norm is \( |x|_p := p^{-\operatorname{ord}_p(x)} \) where \( \operatorname{ord}_p \) is the valuation normalized so that \( \operatorname{ord}_p(p) = 1 \). The second norm is \( \|x\|_L = |N_{L/\mathbb{Q}_p}(x)|_p = |x|_{K/\mathbb{Q}_p}^{|K:\mathbb{Q}_p|} \).

The field \( L \) has a Haar measure \( \mu_L \) which we normalize to give \( \mathcal{O}_L \) measure 1. Since \( a \in L \) acts \( \mathbb{Q}_p \)-linearly by scalar multiplication we have \( \mu_L(aU) = |\det(a)|_p \mu_L(U) = \|a\|_L \mu_L(U) \) for every measurable set \( U \).

2.4.7. If \( L \) is a complete archimedean field (\( \cong \mathbb{R} \) or \( \mathbb{C} \)). We give \( L \) two norms (the second of which is actually a semi-norm in the complex case). We first let \( |x|_\infty \) denote its usual absolute value and we let \( \|x\|_L = |x|_\mathbb{R}^{[L:\mathbb{R}]} \).

We again note that \( L \) admits a Haar measure \( \mu_L \) and for every \( a \in L^* \) and every \( U \subset L \) measurable we have \( \mu_L(aU) = \|a\|_L \mu_L(U) \). This follows from the change of variables formula for integration, for example.

2.5. **Global Fields.** In this section \( L \) will be a number field. We will let \( V(L) \) denote the collection of places of \( L \). We let \( V(L)_0 \) denote the finite places an \( V(L)_\infty \) denote the infinite places. If \( f^* : L_0 \to L \) is an inclusion of fields there is a natural map \( f : V(L) \to V(L_0) \). If \( v_0 \in V(L_0) \) and \( f^* \) as above we will use the notation \( V(L)_{v_0} = \{ v \in V(L) : f(v) = v_0 \} \).

2.5.1. For \( w \in V(L) \) we let \( L_w \) denote the completion of \( L \) with respect to this place. We will let \( G_w = G_{L_w} \) and often make a choices of embeddings \( \overline{L} \subset \overline{L}_w \) so that \( G_w \subset G_L \).

2.5.2. If \( Z \) is a scheme over a number field \( L \) and \( w \in V(L) \) we will use the notation \( Z_w = Z \times_{\operatorname{Spec} L} \operatorname{Spec} L_w = Z \otimes_L L_w = Z_{L_w} \).

2.5.3. If \( v \in V(L)_0 \) we let \( \|x\|_v = \|x\|_{L_v} \) for \( x \in L \) or \( L_v \). We will also let \( \mathcal{O}_v = \mathcal{O}_{L_v} \) denote the ring of integers and \( \kappa(v) \) denote the residue field.

4Mochizuki makes different choices for this normalization in IUT3 and IUT4. We will fix this convention once and for all.
For an Arakelov divisor \( D = \sum_{v \in V(L)} a_v[v] \in \hat{\text{Div}}(L) \) then the Arakelov degree is given by
\[
\hat{\deg}_L(D) = \sum_{v \mid \infty} n_v \ln |\kappa(v)| + \sum_{v \not\mid \infty} n_v,
\]
and the normalized Arakelov degree is \( \hat{\deg}_L(D) = \hat{\deg}_L(D)/[F : \mathbb{Q}] \). If \( L \subset L' \) is a field extension and \( \phi : V(L') \to V(L) \) is the induced map then \( \hat{\deg}_{L'}(\phi^*D) = \hat{\deg}_L(D) \). We will let \( \hat{\text{Div}}(L)_0 \) denote the divisors supported on \( V(L)_0 \).

If \( f \in L^\times \) then
\[
\text{div}(f) = \sum_{v \in V(L)_0} \text{ord}_v(f) - \sum_{v \in V(L)_\infty} \ln \|f\|_v[v].
\]

If \( a = (a_v)_{v \in V(L)} \in \mathbb{A}_L \), the ring of adeles, then
\[
\text{div}((a_v)) = \sum_{v \mid \infty} \text{ord}_v(a_v)[v] - \sum_{v \not\mid \infty} \ln \|a_v\|_v[v].
\]

For all \( D \in \hat{\text{Div}}(L)_0 \) there always exists some \( t = (t_v) \in \mathbb{A}_L \) such that \( D = \text{div}(t) \). Such a \( t \) is unique up to \( \mathcal{O}_{\mathbb{A}_L}^\times \), and locally, we have the following conversion between Arakelov degrees and log volumes:
\[
\ln \mu_L(t_v\mathcal{O}_v) = \ln \|t_v\|_v = -\hat{\deg}((\text{ord}_v(t_v)[v]).
\]

3. Fake Adeles, Random Measurable Sets, and Pilot Objects

Mochizuki’s inequality is stated in terms of averages of degrees of tuples of Arakelov divisors (or equivalently averages of log volumes of tuples of adelic regions). In what follows we define the “measure spaces” \( \mathbb{L} \) isomorphic to \( \mathcal{I}(\mathbb{Q}(\mathbb{S}^2 \mathcal{F}(V_{\infty} D_{\infty})))_{V_0} \) from \cite{Moc15c} Prop 3.9. These spaces and their “measures” are necessary for defining the right hand side of (1.1). As an abstract structure, the space \( \mathbb{L} \) depends only on an extension of number fields \( F_0 \subset K \) and a section \( V \subset V(K) \) of the natural map \( V(K) \to V(F_0) \). In what follows for \( v \in V(F_0) \) we will use the notation \( v \) for its lift to \( V \). The data \((F_0, V)\) is a fragment of “initial theta data”, a definition which will be given in full in \[5\]

### 3.1. lgp Divisors
Two relevant objects for us will be the \( q \) and Theta pilot divisors associated to an elliptic curve. The \( q \)-pilot divisor is an Arakelov divisor and the theta pilot divisor is an “lgp-divisor”, something we now define.

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5 The reason for this is an artifact of the anabelian interpretations of 1) the Kummer class of (a root of) the Jacobi Theta function and 2) a collection of “evaluation points”. We suppress these details and refer the reader to the notes \cite{Mor17} and the slides \cite{Hos15} for an introduction to these ideas.

6 Again there are actually 4 interpretations of these spaces in his theory which we suppress. A large part of the anabelian theory is about passing between these interpretations with various module structures.
Definition 3.1.1. Let $r$ be a natural number. An lgp-divisor for a number field $L$ is a tuple $(P_j)_{j=1}^r$ where $P_j \in \text{Div}(L)_\mathbb{Q}$. The degree of an lgp-divisor is simply the uniform average of the degrees of its components:

$$\hat{\deg}_{\text{lgp}}(P) = \frac{1}{r} \sum_{j=1}^r \hat{\deg}_L(P_j) = \mathbb{E}(\hat{\deg}_L(P_j) : 1 \leq j \leq r).$$

To define the theta and $q$-pilot divisors we need to recall some basic facts about Tate uniformization.

3.2. Tate Uniformizations. Recall that for all elliptic curves $A$ over a field $L$, a finite extension of $\mathbb{Q}_p$, with $|j_A|_L > 1$ we have an isomorphism $A(L) \cong E_q(L)$ where $E_q$ is the Tate curve for some unique $q \in \mathcal{L}$ called the Tate parameter (a geometric invariant which can be written as a formal power series in $1/j_E$ with integer coefficients). These parameters are important to us because of their relation to the minimal discriminant: $\text{ord}_L(q) = \text{ord}_L(\Delta_{E_q})$ and each $E_q$ is a minimal Weierstrass model. All of this is in [Sil13, Ch V], and some proofs are given in [DH20b, §3.5]. Later in §5 will we make use of the rigid analytic uniformizations $\phi : \mathcal{L}^\times / q^\mathbb{Z} \to E_q(L)$ to study torsion points.

3.3. $q$-Pilot and Theta-Pilot. Let $E$ be an elliptic curve over a number field $F$ and let $S \subset V(\mathbb{Q}(j_E))$ be a non-empty collection of places of bad multiplicative reduction (for any model of $E$ over the field of moduli). Since the places are multiplicative, $E$ has a Tate uniformization with Tate parameter $q_v$ at all places $v \in S$. After fixing $l$ a prime number we let $q_v^{1/2l}$ be a choice of $2l$th root. The $q$-pilot divisor associated to $(E, S, l)$ is then

$$P_q := \sum_{v \in S} \text{ord}_v(q_v) [v] \in \hat{\text{Div}}(\mathbb{Q}(j_E))_\mathbb{Q}.$$  

The degree of this divisor is related to the minimal discriminant (see [DH20b, §3.8]) which plays a roll in the Diophantine applications.

The theta pilot divisor is the lgp-divisor $P_\Theta = (P_{\Theta,j})_{j=1}^{(l-1)/2} \in \text{Div}_{\text{lgp}}(\mathbb{Q}(j_E))_\mathbb{Q}$, where

$$P_{\Theta,j} = \sum_{v \in S} \text{ord}_v(q_v^{j^2})[v].$$

Remark 3.3.1. Using Theorem 3.10.1 below we define the theta pilot region to be the region associated to this lgp-divisor $\mathcal{O}_L(-P_\Theta) \subset \mathbb{L}$; a “messed-up” version of this region play a role in the right hand side of (1.1).

The discussion of the space $\mathbb{L}$ begins with a discussion of log-volumes.

---

7lgp stands for Log Gaussian Procession
3.4. Normalizations of Log Measures. Let $W$ be a finite dimensional $\mathbb{Q}_p$-vector space. Let $\mu_W$ denote a Haar measure and $\mathcal{M}(W)$ its measurable sets. We define the normalized log-measure $\ln \mu_W : \mathcal{M}(W) \to \mathbb{R}$ by

$$\ln \mu_W(\Omega) := \log \mu_W(\Omega) / \dim_{\mathbb{Q}_p}(W). \quad (3.1)$$

As we will make use of conversions between regions and divisors it is convenient to restate (2.1) in normalized log-volume notation: Let $F$ be a number field and $\nu$ a non-archimedean place of $F$. Let $a_v \in F_{0,v}$. We have

$$\ln \mu_{F_{0,v}}(a_v \mathcal{O}_v) = \ln |a_v|_p = \frac{-\deg(\text{ord}_v(a_v)[v])}{[F_{0,v} : \mathbb{Q}_p]} \quad (3.2)$$

This is will be used often in our conversions, and we include it because the normalizations are easy to mess up.

3.5. Random Measurable Sets. The right hand side of (1.1) is the expectation of a random measurable set. We give the necessary definitions to make sense of this presently.

**Definition 3.5.1.** Let $S$ be a finite probability space. A random measure space over $S$ is a tuple $X = (X_s)_{s \in S}$ where for each $s \in S$, $X_s = (X_s, \mu_{X_s})$ is a measure space. A random measurable set is a set $U = (U_s)_{s \in S} \in \prod_{s \in S} \mathcal{M}(X_s)$. We will let $\mathcal{N}(X) = \prod_{s \in S} \mathcal{M}(X_s)$ denote the collection of random measurable sets. The expected measure of a random measurable set is

$$\mathbb{E}(U) = \mathbb{E}(\mu_{X_s}(U_s) : s \in S).$$

The expected normalized log measure of a random measurable set is

$$\ln \nu(U) = \mathbb{E}(\ln \mu(U_s) : s \in S).$$

**Remark 3.5.2.** In our applications, the components of our random measure spaces will be vector spaces or modules and it will be convenient to abusively use a symbol $X$ for both the tuple $X = (X_s)_{s \in S}$ and the product $X = \prod_{s \in S} X_s$. Similarly since there is a bijection between random measurable sets and subsets of the product $X$ which are products of measurable sets. In notation:

$$\mathcal{N}(X) \leftrightarrow \left\{ \prod_{s \in S} U_s \subset \prod_{s \in S} X_s : U_s \in \mathcal{M}(X_s) \right\};$$

$$(U_s)_{s \in S} \leftrightarrow \prod_{s \in S} U_s.$$  

Because of this bijection we will conflate $(U_s)_{s \in S} \in \mathcal{N}(X)$ with $\prod_{s \in S} U_s \subset X$ when convenient. In this same vein if $(Y_s)_{s \in S}$ a random measure space then for $X = \prod_{s \in S} Y_s$ we write $\mathcal{N}(X) = \prod_{s \in S} \mathcal{N}(Y_s)$ and $\ln \nu(U) = \mathbb{E}(\ln \nu(U_s) : s \in S)$ for $U \in \mathcal{N}(X)$.

**Example 3.5.3.** Suppose $W$ is finite dimensional vector space over a complete local field $L$ which admits a direct sum decomposition $W = \bigoplus_{i=1}^m W_i$. Define a probability measure $\Pr : \{1, \ldots, m\} \to \mathbb{R}$ by $\Pr(i) = w_i := \dim(W_i) / \dim(W)$ so that $(\{1, \ldots, m\}, \{w_i\}_{i=1}^m)$ becomes a discrete probability space. One can define a collection of random measurable sets
\( \mathcal{N}(W) \) on \( W \) of the form \( U = \bigoplus_i U_i \subset W \) by asking \( U_i \in \mathcal{M}(W_i) \). That is the collection of random measurable sets on \( W \) is \( \mathcal{N}(W) = \bigoplus_{i=1}^m \mathcal{M}(W_i) \).

A \( p \)-adic Mochizuki log-measure on \( W = \bigoplus_{i=1}^m W_i \) is a function \( \ln \nu : \mathcal{N}(W) \to \mathbb{R} \) defined on \( U = \bigoplus_{i=1}^m U_i \) by
\[
\ln \nu_W(U) = \mathbb{E}(\ln \mu_{W_i}(U_i) : i \in \{1, \ldots, m\}) := \sum_{i=1}^m w_i \ln \mu_{W_i}(U_i). \tag{3.3}
\]

In effect this is just expected measure of a random measurable set. We will call the tuple \( W = (W = \bigoplus_{i=1}^m W_i, \Pr : \{1, \ldots, m\} \to \mathbb{R}, \ln \nu_W : \mathcal{M}(W) \to \mathbb{R}) \) a \( p \)-adic Mochizuki measure space.

**Remark 3.5.4.** Let \( L \) over \( \mathbb{Q}_p \) be a finite extension. If \( W = \bigoplus W_i \) is a \( p \)-adic Mochizuki measure space and each \( W_i \) is a \( L \)-vector space then we say \( \Pr : \{1, \ldots, m\} \to \mathbb{R} \) is Mochizuki normalized if for every \( x \in L \) and every \( U = \bigoplus_{i=1}^m U_i \in \mathcal{M}(W) \) we have
\[
\ln \nu(xU) = \log |x|_p + \ln \nu(U). \tag{3.4}
\]

All of our expected log measures of \( p \)-adic vector space will be Mochizuki normalized. The property of being Mochizuki normalized does not uniquely specify the weights of our expectations. This is a common misconception.

It turns out that for every direct sum decomposition of a \( p \)-adic vector space the normalized log-measure has an interpretation as a probability measure after choosing the appropriate weights. This will play a roll in our construction of “fake adeles”.

**Lemma 3.5.5.** If \( W \) is a \( \mathbb{Q}_p \)-vector space which admits a decomposition \( W = \bigoplus_{i=1}^m W_i \) then
\[
\ln \mu_W(U) = \ln \nu_W(U), \tag{3.5}
\]
where we define \( \Pr : \{1, \ldots, m\} \to \mathbb{R} \) by \( \Pr(i) = \dim(W_i)/\dim(W) \).

In what follows, the “fake adeles” will be a construction where we replace the normalized log-measure in the expectations \( \mathbb{E}(\ln \mu_{W_i}(U_i)) \) with normalized log-measures of larger vector spaces \( \tilde{W}_i \supset W_i \) and then consider expectations of the form \( \mathbb{E}(\ln \mu_{\tilde{W}_i}(\tilde{U}_i)) \) where \( \Pr : \{1, \ldots, m\} \to \mathbb{R} \) is still defined as \( \Pr(i) = \dim(W_i)/\dim(W) \).

3.6. **Fake Adeles.** Mochizuki’s inequality is stated in terms of “log measures” on tensor powers of rings of “fake adeles”. We define these presently.

**Definition 3.6.1.** Fix an extension of number fields \( F_0 \subset K \) and a section \( V \subset V(K) \) of the map \( V(K) \to V(F_0) \). The ring of fake adeles is the additive topological group \( \hat{A}_V = \prod_{v \in V(F_0)} K_v \) with the restricted product topology. At times we view this simply as a topological abelian group (forgetting the ring structure)\(^8\)

\(^8\)The only purpose of the topology is to induce topologies on the factors. We make no use of local compactness and only consider “measures” of products of open sets.
We will make use of two modifications of the fake adele construction. The first variant is the tensor power construction where for a non-negative integer \( r \) we define
\[
\mathcal{A}_{\mathcal{V}}^{\otimes r+1} := \prod_p \mathcal{A}_{\mathcal{V},p}^{\otimes r+1}
\]
where \( \mathcal{A}_{\mathcal{V},p}^{\otimes r+1} = \bigoplus_{(v_0,\ldots,v_r) \in V(F_0)^{r+1}} K_{v_0}^{\otimes \cdots \otimes K_{v_r}} \) and the product over \( p \) is given the restricted product topology. Here the tensor products are taken over \( \mathbb{Q}_p \) (observe that this definition works even for \( p = \infty \)). The second variant uses algebraic closures. We define the ind topological ring \( \overline{\mathcal{A}}_{\mathcal{V}}^{\otimes r+1} \) containing \( \mathcal{A}_{\mathcal{V}}^{\otimes r+1} \) by the formula
\[
\overline{\mathcal{A}}_{\mathcal{V}}^{\otimes r+1} = \prod_p \prod_{v \in V(F_0)^{r+1}} \mathcal{K}_{v_0}^{\otimes \cdots \otimes \mathcal{K}_{v_r}}
\]
where again for the product over \( p \) we use the restricted product topology.

In each of these constructions we have a ring of integers. The ring of fake adeles is defined to be the ring
\[
\mathcal{O}_{\mathcal{V}} = \mathcal{O}_{\mathcal{A}_{\mathcal{V}}} := \prod_{v \in V(F_0)} \mathcal{O}_v \subset \mathcal{A}_{\mathcal{V}},
\]
and for tensor powers the ring of integers is
\[
\mathcal{O}_{\mathcal{A}_{\mathcal{V}}^{\otimes r+1}} := \prod_{p \in V(\mathbb{Q})} \mathcal{O}_{\mathcal{A}_{\mathcal{V},p}^{\otimes r+1}}.
\]
To expand \( \mathcal{O}_{\mathcal{A}_{\mathcal{V},p}^{\otimes r+1}} \) as a direct sum of domains we apply the discussion from §2.4.5 (which we pursue in more detail in [DH20b, §3]). We will make use of the notation
\[
\mathcal{O}_{\mathcal{w}} = \mathcal{O}(\mathcal{w}_0,\ldots,\mathcal{w}_r)
\]
to denote the ring of integers in \( K_{\mathcal{w}} := K_{\mathcal{w}_0}^{\otimes \cdots \otimes K_{\mathcal{w}_r}} \) for \( \mathcal{w} \in \mathcal{V}_{r+1}^p \). We will also let \( \ln \mu_{(\mathcal{w}_0,\ldots,\mathcal{w}_r)} = \ln \mu_{K_{\mathcal{w}_0}^{\otimes \cdots \otimes K_{\mathcal{w}_r}}} \) be the unique normalized log Haar measure on the tensor product of fields \( K_{\mathcal{w}_0}^{\otimes \cdots \otimes K_{\mathcal{w}_r}} \) satisfying
\[
\ln \mu_{\mathcal{w}}(\mathcal{O}_{\mathcal{w}}) = 0.
\]

Remark 3.6.2 (Connections to IUT papers). In [Moc15c, Proposition 3.9.i] normalizations of measures in tensor products rely on definability of \( \mathcal{O}_{\mathcal{w}} \) which requires knowledge of \( K_{\mathcal{w}} \) as a ring and not merely as an additive topological group. In the case of a fixed non-archimedean prime \( p \) the main difference between interpretations of \( \mathcal{A}_{\mathcal{V},p}^{\otimes r+1} \) using fundamental groups ([DH20a, Corollary 1.10 Appendix]) and interpretations using absolute Galois groups of \( p \)-adic fields ([DH20a, Interpretations in G Appendix]) is that the fundamental group interpretations give the ring structure while the interpretations in absolute Galois groups only give topological groups. This makes \( \mathcal{O}_{\mathcal{A}_{\mathcal{V},p}^{\otimes r+1}} \) not definable in interpretations in absolute Galois groups of \( p \)-adic fields. Remarkably, although one can only interpret the multiplicative monoid of a \( p \)-adic field using class field theory\footnote{Jarden-Ritter gives examples of non-isomorphic \( p \)-adic fields whose absolute Galois groups are the isomorphic.} the additive Haar measures are still
interpretable ([DH20a, Interpretations in G Appendix]). Also [Moc15c, Proposition 3.9] says that we can still enforce condition (3.6) on our Haar measure in the situation of [DH20a, Interpretations in G Appendix] since in the tensor product of $p$-adic fields $K_1 \otimes \cdots \otimes K_m$ (viewed as a just a finite dimensional $p$-adic vector space) by imposing a volume of any of the sets $\mathcal{O}_{K_1 \otimes \cdots \otimes K_m}$, $\mathcal{O}_{K_1} \otimes \cdots \otimes \mathcal{O}_{K_m}$, or $\log(\mathcal{O}_{K_1}^r) \otimes \cdots \otimes \log(\mathcal{O}_{K_m}^r)$ consistent with $\mu_{K_1 \otimes \cdots \otimes K_m}(\mathcal{O}_{K_1 \otimes \cdots \otimes K_m}) = 1$.

Finally, in order to define random measurable sets, we need to give the index sets $V(F_0)_p^{r+1}$ the structure of a probability space. First, the set $(V(F_0)_p, \Pr : V(F_0)_p \to \mathbb{R})$ defines a finite probability space with

$$\Pr(v) = [F_{0,v} : \mathbb{Q}_p]/[F_0 : \mathbb{Q}].$$

Since products of probability spaces are probability spaces the sets $V(F_0)_p^{r+1}$ also have the natural structure of a probability space. This gives $A_{\mathbb{L}_p}^{\otimes r+1}$ the structure of a random measure space with $\mathcal{N}(A_{\mathbb{L}_p}^{\otimes r+1}) = \prod_{\bar{v} \in V(F_0)_p^{r+1}} \mathcal{M}(K_{\bar{v}})$. Explicitly if $U = \bigoplus_{\bar{v} \in V(F_0)_p^{r+1}} U_{\bar{v}}$ we have

$$\ln \nu_{A_{\mathbb{L}_p}^{\otimes r+1}}(U) = \mathbb{E}(\ln \nu_{\mathbb{L}_p}(U_{\bar{v}}) : \bar{v} \in V(F_0)_p^{r+1}).$$

We can now define $\mathbb{L}$, which was one of the aims of this section.

**Definition 3.6.3.** The space $\mathbb{L}$ and its Mochizuki measure $\ln \nu_{\mathbb{L}} : \mathcal{N}(\mathbb{L}) \to \mathbb{R}$ are characterized by the following formulas:

$$\mathbb{L} = \prod_p \mathbb{L}_p,$$

$$\ln \nu_{\mathbb{L}}(B) = \sum_p \ln \nu_{\mathbb{L}_p}(B_p),$$

$$\mathbb{L}_p = \bigoplus_{j=1}^{(l-1)/2} A_{\mathbb{L}_p}^{\otimes j+1},$$

$$\ln \nu_{\mathbb{L}_p}(B) = \mathbb{E} \left( \ln \nu_{A_{\mathbb{L}_p}^{\otimes j+1}}(B_p^{(j+1)})) \right),$$

$$A_{\mathbb{L}_p}^{\otimes j+1} = (\bigoplus_{\mathbb{F}_p} A_{\mathbb{L}_p})^{\otimes j+1},$$

$$\log \nu_{A_{\mathbb{L}_p}^{\otimes j+1}}(B) = \mathbb{E}(\log \nu_{\mathbb{L}_p}(B_{\bar{v}}) : \bar{v} \in V(F_0)_p^{j+1}).$$

Here $B = \prod_p B_p \subset \mathbb{L}$ where $B_p = \prod B_p^{(j+1)}$ where $B_p^{(j+1)} = \prod B_{\bar{v}}$ and in terms of random measurable sets we have $\mathcal{N}(\mathbb{L}_p) = \prod_j \mathcal{N}(A_{\mathbb{L}_p}^{\otimes j+1})$ and $\mathcal{N}(A_{\mathbb{L}_p}^{\otimes j+1}) = \prod_{\bar{v} \in V(F_0)_p^{j+1}} \mathcal{M}(K_{\bar{v}})$. We will abusively write $\mathcal{N}(\mathbb{L})$ for the products of sets from $\mathcal{N}(\mathbb{L}_p)$ and refer to $\ln \nu_{\mathbb{L}}$ as the expectation of a random measurable set.

It remains to define interesting regions in $\mathbb{L}$. These will be defined via an action of $\mathbb{A}_{\mathbb{L}}$ given in the next sections.

**3.7. Action through a Tensor Factor.** Let $K_1, \ldots, K_r$ be finite extensions of $\mathbb{Q}_p$ contained in a common algebraic closure. Let $K_0$ denote their common intersection. The action of $a \in K_{r-1}$ on $K_1 \otimes \cdots \otimes K_{r-1} = \prod_{j=1}^r L_j$ through the $r$th tensor factor is a semi-linear diagonal action in the $L_j$ coordinates (see [DH20b, §3.3]). This means that $K_{r-1} \hookrightarrow L_j$ for each $j$ and that $a$ acts via scaling by $\sigma_j(a)$ for some automorphism $\sigma_j$ of $L_j$ on the $j$th factor.
In our application we will let \( K_i = K_{\underline{\nu}} \) and take \( a_{\underline{\nu} - 1} \in K_{\underline{\nu} - 1} \) and \( U \in \mathcal{M}(K_{\underline{\nu}} \otimes \cdots \otimes K_{\underline{\nu} - 1}) \) and get
\[
\ln \mu(\underline{\nu}, \ldots, \underline{\nu} - 1)(a_{\underline{\nu} - 1} \cdot U) = \ln |a_{\underline{\nu} - 1}|_p + \ln \mu(U), \tag{3.7}
\]
where the action of \( a_{\underline{\nu} - 1} \) on \( K_{\underline{\nu}} \otimes \cdots \otimes K_{\underline{\nu} - 1} \) is through the \( r \)th tensor factor.

3.8. Peel Decomposition. For each \( i \in \{0, \ldots, r\} \) we can decompose \( A_{\underline{\nu},p}^{\otimes r + 1} \) in the following way:
\[
A_{\underline{\nu},p}^{\otimes r + 1} = \bigoplus_{(v_0, \ldots, v_r) \in V(F_0)^{r+1}} K^{(v_0, \ldots, v_r)}
= \bigoplus_{v \in V(F_0)_p} \left( \bigoplus_{((v_0, \ldots, v_{r-1}), (v_{r+1}, \ldots, v_{r})) \in V(F_0)^r_p} K^{(v_0, \ldots, v_{r-1})} \otimes K^{(v_{r+1}, \ldots, v_{r})} \right).
\]
Here we have “peeled off” all the summands of \( A_{\underline{\nu},p}^{\otimes r + 1} \) where the \( i \)th tensor factor is equal to \( v \in V_p \). We call these summands \( \text{Peel}_i A_{\underline{\nu},p}^{\otimes r + 1} \). When convenient we will just write \( \text{Peel}_i \) for \( \text{Peel}_i A_{\underline{\nu},p}^{\otimes r + 1} \). We also will dispose of decompositions by the same name on \( A_{\underline{\nu}}^{\otimes r + 1} \). Also observe that for each \( 0 \leq i \leq r \) we have
\[
A_{\underline{\nu},p}^{\otimes r + 1} = \bigoplus_{v \in V(F_0)_p} \text{Peel}_i v.
\tag{3.8}
\]
We call the decomposition \((3.8)\) the peel decomposition.

We remark that the peel decomposition allows us to define a \( A_{\underline{\nu}} \)-algebra structure on \( \text{peel}_i \colon A_{\underline{\nu}} \times A_{\underline{\nu}}^{\otimes r + 1} \rightarrow A_{\underline{\nu}}^{\otimes r + 1} \) for each \( r \in \mathbb{Z}_{\geq 0} \) and each \( i \) with \( 0 \leq i \leq r \). The map \( \text{peel}_i \colon K_{\underline{\nu}} \rightarrow \text{Peel}_i \) is defined by (extending linearly)
\[
\text{peel}_i(a_{\underline{\nu}}) \cdot (b_{\underline{\nu}_0} \otimes \cdots \otimes b_{\underline{\nu}_i} \otimes b_{\underline{\nu}_{i+1}} \otimes b_{\underline{\nu}}) = (b_{\underline{\nu}_0} \otimes \cdots \otimes b_{\underline{\nu}_i} \otimes a_{\underline{\nu}} b_{\underline{\nu}_i} \otimes b_{\underline{\nu}_{i+1}} \otimes b_{\underline{\nu}}). \tag{3.9}
\]
Here \( a_{\underline{\nu}} \) just acts on each summand of \( \text{Peel}_i \) through the same tensor index.

Remark 3.8.1 (Connections to IUT papers). (1) The “peel map” and “peel decomposition” describe the situation of \([\text{Moc15c}, \text{Proposition 3.1}]\). In terms of Mochizuki’s notation when \( \alpha = i \) and \( A = S_{j+1}^\pm \) then we have a commutative diagram between his notation and ours

\[
\begin{array}{ccc}
\log^{(\alpha F_{\underline{\nu}})} & \longrightarrow & \log^{(A,\alpha F_{\underline{\nu}})} \\
\downarrow & & \downarrow \sim \\
K_{\underline{\nu}} & \xrightarrow{\text{peel}_i} & \text{Peel}_i A_{\underline{\nu}}^{\otimes j+1}
\end{array}
\]

(2) Mochizuki has good reason to work with finite sets \( A \) and elements \( \alpha \in A \) rather than standard integers. In IUT3 \( A \) is a set of conjugacy classes of decomposition groups and \( \alpha \in A \) is a particular class. From the theory of Kummer classes, these classes index “evaluation points”. When restricting our Kummer class of (a root of) the Jacobi theta function to these group we are anabelianly evaluating the theta
function at certain special points which return powers of the Tate parameter. See \[\text{Mor17}, \text{St11}2\].

(3) The difference between the 4th display of [Moc15c, Proposition 3.2] and the morphism $\log(a)F_L \to \log(A,a)F_L$ above (as in [Moc15c, Proposition 3.1]) is again the interpreting structure. The structures $\log(A,a)F_L \subset \log(A)F_L$, for example, have a similar commutative diagram relating to Peel$^i_L \subset A_{\log}^{r+1}$ but morphisms in that diagram are only morphisms of ind-topological abelian groups with a random measure structure and not ind-topological rings. This echoes Remark 3.6.2.

3.9. **Divisor/Modules Conversion.** Let $D \in \hat{\text{Div}}(F_0)_{0,Q} = (\text{Div}(F_0)_{0,Q}$ and let $a = (a_v)_{v \in V} \in A_V$ be such that $D = \text{div}(a)$. Fix some index $i$ where $0 \leq i \leq r$ and define

$$O_{L^{(r)}}(-D) = \bigoplus_p O_{L^{(r)}}(-D)$$

where

$$O_{L^{(r)}}(-D) = \bigoplus \text{peel}^i_j((a_v)_{v \in V(F_0)}) \cdot O_{\text{peel}^i_j L^r}.$$  

We make the following variant for lgp divisors: If $D = (D_j) \in \text{Div}(F_0)_{0,\text{lgp}}$ then

$$O_{L}(-D) := \bigoplus_{j=1}^r O_{L^{(j)}}(-D_j).$$

Finally, for the next section, in the special case that $F_0 = K$, we use the notation $O_{A_{\hat{F}_0}^{r+1}}(-D) := O_{L^{(r)}}(-D)$. In the very special case $r = 0$ we will just write $O_{A_{\hat{F}_0}}(-D)$.

**Remark 3.9.1 (Connections to IUT papers).** The definition of the regions $O_L(-D)$ differ slightly from Mochizuki’s treatment as he uses certain monoid actions on the ind topological groups $\overline{A_{\log}^{r+1}}$ rather than an action of $A_V$. In [Moc15c, Proposition 3.4] he describes an action

$$\Psi_{\text{F}_{\log}^{(0,0)}(H)} \times \prod_{j \in S^{s+1}_{j+1} \setminus \{0\}} T_{j}^{0} (S_{j+1}^{s+1,j};(0,1)F_L) \to \prod_{j \in S^{s+1}_{j+1} \setminus \{0\}} T_{j}^{0} (S_{j+1}^{s+1,j};(0,1)F_L).$$

where in our terminology

$$T_{j}^{0} (S_{j+1}^{s+1,j};(0,1)F_L) \cong \text{Peel}^j_L A_{\log}^{s+1},$$

$$\Psi_{\text{F}_{\log}^{(0,0)}(H)} \cong \bigotimes_{j=1}^{(l-1)/2} (\text{Peel}^j_L A_{\log}^{s+1}).$$

Where $O_{K_{\text{log},\Delta}} \cong \{(a, a, \ldots, a) \in K_{\text{log}}^{(l-1)/2} : a \in O_{K_{\text{log}}}\}$. We observe that while the interpretation in [Moc15c, Proposition 3.4] has a ring structure (given by the construction in our appendix of the sequel [DH20a, Corollary 1.10 Appendix]) later, he passes on to reducts which are not rings (these use the interpretations here: [DH20a, Interpretations in G Appendix]).

In Moc15c Proposition 3.5.i, 3.5.ii] Mochizuki furnishes interpretations $(k_{\text{mod}})_a \subset (k_{\text{mod}})_A \subset \log(A,F_V)$ here $(k_{\text{mod}}) = \overline{A_{\log}} \cong F_0 = Q(j_E)$. The interpretation of these structures pass
through the theory of so-called “kappa coric functions” which are the global analog of the
more familiar etale theta functions; these interpretations are developed in [Moc15a, §5].

3.10. Degree/Volume Conversion. The following Theorem records conversions between
divisors and regions. The normalizations are tedious so we wanted to record them here.

Theorem 3.10.1. Let $F_0 \subset K$ be an extension of number fields. Let $V \subset V(K)$ be a section
of the natural map $V(K) \to V(F_0)$. Let $d = [K : F_0]$. In what follows all divisors will be
supported only at finite places.

(1) (Conversion for Adeles) If $D \in \hat{\text{Div}}(F_0)$ then for every $r \geq 1$ we have
\[ -\hat{\deg}_{F_0}(D) = \ln \mu_{\hat{A}_{F_0}}(\mathcal{O}_{\hat{A}_{F_0}}(-D)) = \ln \mu_{\hat{A}_{F_0}}(\mathcal{O}_{\hat{A}_{F_0}}^\otimes(-D)). \]

(2) (Conversion for Fake Adeles) If $D \in \hat{\text{Div}}(F_0)_{Z[1/d]}$ then for every $r \geq 1$ we have
\[ -\hat{\deg}_{F_0}(D) = \ln \nu_{\hat{A}_{K}}(\mathcal{O}_{\hat{A}_{K}}(-D)) = \ln \nu_{\hat{A}_{K}}(\mathcal{O}_{\hat{A}_{K}}^\otimes(-D)). \]

(3) (Conversion for lgp Divisors) If $D = (D_j) \in \hat{\text{Div}}(F_0)_{\text{lgp},Z[1/d]}$ then
\[ -\hat{\deg}_{\text{lgp},F_0}(D) = \ln \nu_{\hat{A}_L}(\mathcal{O}_{\hat{A}_L}(-D)). \]

Proof. (1) We give a proof of the first equality in (1). Let $t = (t_v)_{v \in V(F_0)} \in \hat{A}_{F_0}$ be such
that $\text{ord}_v(t_v) = \text{ord}_v(D)$. It suffices to prove this equality for divisors supported over
a finite place $p$ and hence consider the analogous statements for $\hat{A}_{F_0,p}$. We have
\[ \ln \mu_{\hat{A}_{F_0,p}}(\mathcal{O}_{\hat{A}_{F_0,p}}(-D)) = \mathbb{E}(\ln \mu_{\hat{A}_{F_0,p}}(t_v \mathcal{O}_v) : v \in V(F_0)_p) \]
\[ = \mathbb{E}(\hat{\deg}(\text{ord}_v(t_v)[v]) / [F_0,v : Q_p]) \]
\[ = \frac{-1}{[F_0,v]} \sum_{v \mid p} \hat{\deg}(\text{ord}_v(D)[v]) = -\hat{\deg}(D). \]

The second equality follows from (2.1).

(2) Let $t \in \hat{A}_{F_0}$ be such that $\text{div}(t) = D$. Then without loss of generality we can suppose
that $\mathcal{O}_{\hat{A}_{K}}^\otimes(-D)$ is defined via the $r$th peel map:
\[ \mathcal{O}_{\hat{A}_{K}}^\otimes(r-1)(-D) = \bigoplus_{(v_0,\ldots,v_{r-1})} \text{peel}^{r-1}_{v_{r-1}}(t_{v_{r-1}}) \cdot \mathcal{O}_{(v_0,\ldots,v_{r-1})}. \]

This means
\[ \ln \mu_{\hat{A}_{F_0,p}}^\otimes(r-1)(\mathcal{O}_{\hat{A}_{K}}^\otimes(r-1)(-D)) = \mathbb{E}(\ln \mu_{\hat{A}_{F_0,p}}^\otimes(r-1)(\text{peel}^{r-1}_{v_{r-1}}(t_{v_{r-1}}) \cdot \mathcal{O}_{(v_0,\ldots,v_{r-1})})) \]
\[ = \mathbb{E}(\ln |t_{v_{r-1}}|_p : (v_0,\ldots,v_{r-1}) \in V(F_0)_p^r) \]
\[ = \mathbb{E}(\ln |t_v|_p : v \in V(F_0)_p) \]
\[ = \mathbb{E}(\frac{-\hat{\deg}(\text{ord}_v(t_v)[v])}{[F_0,v : Q_p]}) = -\hat{\deg}(D). \]

The second line follows from (3.7), the last line follows from (2.1), and the last
equality follows exactly the same reasoning as the proof in item (1) for $r = 1$. 

(3) We will now prove the first equality of the second assertion. Fix \( t = (t_{\nu}) \in \mathbb{A}_V \) such that \( \text{ord}_v(t_{\nu}) = \text{ord}_v(D) \). We have the following equalities

\[
\ln \nu_{\mathbb{A}_V,p}(\mathcal{O}_{L,p}(-D)) = \ln \nu(\bigoplus_{\mu|p} t_{\mu} \mathcal{O}_{\nu}) \\
= \mathcal{E}(\ln \mu_{\nu}(t_{\mu} \mathcal{O}_{\nu}) : v \in V(F_0)_p) = \mathcal{E}(\ln |t_{\nu}|_p : v \in V(F_0)_p) \\
= \mathcal{E}(\deg(\text{ord}_v(t_{\nu})[v])/[F_{0,v} : \mathbb{Q}_p] : v \in V(F_0)_p) = -\hat{\deg}(D).
\]

Note that the second equality is exactly the reason we defined \( \ln \nu \) in this “fake adelic” way.

(4) Again, fix \( t = (t_{\nu}) \in \mathbb{A}_V \) so that \( \text{ord}_v(t_{\nu}) = \text{ord}_v(D) \). The proof here follows exactly as in \( \text{(2)} \) only starting with \( \ln \nu^{(r)}_{\mathbb{A}_V,p}(\mathcal{O}_{L^{(r-1)},p}(-D)) \) and replacing all the \( \nu \)'s in the argument of the expectations with \( \nu^{(r)} \).

(5) We now prove the assertion in \( \text{(3)} \). We have \( \mathcal{O}_L(-D) = \bigoplus_{j=1}^{(l-1)/2} \mathcal{O}_{L^{(j)},p}(-D_j) \) and

\[
\ln \nu_L(\mathcal{O}_L(-D)) = \mathcal{E}(\ln \nu_{L^{(j)}_{\nu}}(\mathcal{O}_{L^{(j)},p}(-D_j)) : 1 \leq j \leq (l - 1)/2) = -\hat{\deg}(D)
\]

where the second equality follows from assertion \( \text{(2)} \).

\[
\square
\]

4. Indeterminacies and \( U_{\Theta} \)

In this section we discuss the indeterminacies Ind1 and Ind2 appearing in \( (1.1) \). Readers who just want a formulas are referred to \( \text{§4.6} \) and \( \text{§4.8} \). Unpacking the definition of Ind1 from [Moc15a, Proposition 6.9] for the purpose of explicit computations is laborious. At the level of \( \mathbb{I}_{\nu,j}^{(j)} := \mathbb{A}_{V,p}^{\otimes j+1} \) elements of Ind1 are just automorphisms of the finite dimensional \( \mathbb{Q}_p \)-vector spaces \( \mathbb{I}_{\nu,j}^{(j)} \) induced by automorphisms of the \( \mathbb{Z}_p \)-lattice \( \mathcal{I}_{V,p}^{\otimes j+1} \subset \mathbb{I}_{\nu,j}^{(j)} \) defined by

\[
\mathcal{I}_{V,p}^{\otimes j+1} := \bigoplus_{\nu \in V(F_0)^{j+1}} \mathcal{I}_{\nu}^{j+1},
\]

where if \( \nu = (\nu_0, \ldots, \nu_j) \), then \( \mathcal{I}_{\nu} = \mathcal{I}_{\nu_0} \otimes \cdots \otimes \mathcal{I}_{\nu_j} \) and for \( \nu \in V \) we define \( \mathcal{I}_{\nu} = \frac{1}{z_{\nu}} \log(\mathcal{O}_L^\times) \). The \( \mathcal{I}_{\nu} \) are Mochizuki’s so-called log-shells. We will also see that the Ind2 indeterminacies also preserve this lattice. Finally, we give a formula for an upper bound on the Ind3 indeterminacies punting to \( \text{[DF20a]} \).

4.1. Isomorphisms vs Automorphisms; trivializations. For a connected groupoid \( C \) it will be convenient to select a particular object \( C \in C \) and an isomorphism \( \varphi_X : X \to C \) for each \( X \in C \). Doing this will allow us to switch between the viewpoint of considering objects up to isomorphism and objects up to automorphism: if \( \psi : X_1 \to X_2 \) then \( \gamma \in \text{Aut}(C) \) is given by \( \gamma := \varphi_{X_2}^{-1} \psi \varphi_{X_1} \). Conversely, given \( \gamma \in \text{Aut}(C) \) we isomorphism between \( X_1 \) and \( X_2 \) is \( \psi := \varphi_{X_2}^{-1} \gamma \varphi_{X_1} \). We will often call a fixed isomorphism like this a trivialization.
4.2. **Polymorphisms.** If \( \mathcal{C} \) is a category we let \( \mathcal{C}^{\text{poly}} \) denote the *poly-category* where objects are the same as \( \mathcal{C} \) but morphisms are now subsets of morphisms \( \mathcal{C}^{\text{poly}}(X, Y) = 2^{\mathcal{C}(X,Y)} \). Composition of morphisms are done element-wise. Mochizuki uses these constructions to identify objects. In our construction we will identify objects.

4.3. **Capsules.** The following notions are introduced in [Moc15a, §0, pg33].

4.3.1. Let \( \mathcal{C} \) be a category (usually a connected groupoid). A *capsule* is a collection of objects \( \{A_i\}_{i \in I} \) indexed by a finite set \( I \). We will sometimes just write \( A = \{A_i\}_{i \in I} \) and view \( A : I \to \mathcal{C} \) as a diagram. A morphism of capsules

\[
A = \{A_i\}_{i \in I} \to B = \{B_j\}_{j \in J}
\]

is a pair \((f, (F_i)_{i \in I})\) consisting of an injective map of finite sets \( f : I \to J \) and a collection of morphisms \( F_i : A_i \to B_{f(i)} \). We will sometimes just denote these by \((f, F)\). The category of capsules will be denoted by \( \text{Cap}(\mathcal{C}) \). An *n-capsule* is a capsule with index set of cardinality \( n + 1 \).

**Remark 4.3.1.** David Roberts (the Australian one) has pointed out to us that capsules are a special case of what SGA4 calls a semirepresentable objects. See [Aut19, Tag0DBB] for a modern treatment.

4.3.2. A morphism \( A \to B \) is a *capsule-full poly-morphism* if for all \( i \in I \) the map \( F_i : A_i \to B_{f(i)} \) is a full polymorphism.\(^{10}\)

4.3.3. A morphism \((f, F) : A \to B\) is a *capsule-full poly-isomorphism* if is capsule full polymorphisms and \( f \) is a bijection.

4.4. **Processions.** This definition comes from [Moc15a, Definition 4.10]. Let \( \mathcal{C} \) connected groupoid. Let \( J \) be a finite index set. Given a capsule \( X_J \) we will define a procession \( \text{Prc}(X_J) \) for an object \( X \).

4.4.1. A *procession* is a diagram \( D : (1 \to 2 \to \cdots \to n) \to \text{Cap}(\mathcal{C}) \) where \( D^{(j)} \) is an \((j + 1)\)-capsule and \( D^{(j)} \to D^{(j+1)} \) is a capsule-full polymorphism for \( 1 \leq j < n \).

4.4.2. A morphism of processions \((D^{(1)} \to D^{(2)} \to \cdots \to D^{(n)}) \to (E^{(1)} \to E^{(2)} \to \cdots \to E^{(m)})\) is a pair \((g, G)\) consisting of an order preserving map

\[
g : \{1, \ldots, n\} \to \{1, \ldots, m\}
\]

with a a collection \( G^{(j)} : D^{(j)} \to E^{(g(j))} \) of capsule-full poly-morphisms.

\(^{10}\)Equivalently this is a polymorphisms in the category of capsule morphisms which is full over the map of index sets \( f : I \to J \). This is how Mochizuki does it.
4.5. Local Frobenioids = Representations of Topological Groups On Monoids.

This section is used to replace Mochizuki’s Frobenioids by simpler objects. See [Mor17, Obus’s section] for a brief introduction to Frobenioids. In the non-archimedean case the Frobenioids $\mathcal{F}_+^* \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times 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where \(1 \leq j \leq n\) and \(0 \leq i \leq j\).

In what follows we will abusively conflate the automorphism \(\gamma\) with the isomorphism \(G\) using \(\S 4.1\). At the level of the single tensor factor \(K_{v_0} \otimes K_{v_1} \cdots \otimes K_{v_j}\) of \(L_p^{(j)}\) the capsule automorphism \(\gamma^{(j)} = (\gamma^{(j)}_{i,v})\), which we abusively write as \(\gamma = (\gamma_{i,v})\) can be described as (extending linearly) a permutation of tensor factors followed by a tensor product of automorphisms

\[
x_{0,v_0} \otimes x_{1,v_1} \otimes \cdots \otimes x_{j,v_j} \\
\mapsto \gamma_{g^{-1}(0),v_{g^{-1}(0)}} \otimes \gamma_{g^{-1}(1),v_{g^{-1}(1)}} \otimes \cdots \otimes \gamma_{g^{-1}(j),v_{g^{-1}(j)}} \otimes x_{g^{-1}(j+1),v_{g^{-1}(j+1)}}.
\]

We observe that the codomain of this map is \(K_{v_0} \otimes \cdots \otimes K_{v_j}\) to \(K_{v_{g^{-1}(0)}} \otimes \cdots \otimes K_{v_{g^{-1}(j)}}\), and unfortunately, the multiple subscripts are necessary. If confusion arises the material around \(\S 4.2\) should help.

4.7. **Procession Automorphism for \(n = 3\)**. We will form an automorphism of \(\text{Prc}(X_{0,1,2,3})\) for some object \(X\) in an unspecified category under the convention that whenever a permutation choice needs to be made, we select the permutation to be the standard cyclic permutation which increases indices by one. This is pictured in Figure 4.7.

Under this convention: a 3-procession consists of \(2 + 3 + 4\) object isomorphic to \(X\),

\[
X_{1,0}, X_{1,1} \\
X_{2,0}, X_{2,1}, X_{2,2} \\
X_{3,0}, X_{3,1}, X_{3,2}, X_{3,3}
\]

together with maps \(2 + 3\) maps between objects (again consult Figure 4.7)

\[
F_0^{(1)} : X_{1,0} \to X_{2,1}, \\
F_1^{(1)} : X_{1,1} \to X_{2,2}, \\
F_0^{(2)} : X_{2,0} \to X_{3,1}, \\
F_1^{(2)} : X_{2,1} \to X_{3,2}, \\
F_2^{(2)} : X_{2,2} \to X_{3,3}.
\]

The 3-procession \(\text{Prc}(X_{0,1,2,3})\) actually consists three capsules indexed by \(j = 1, 2, 3\) where the \(j\)th capsule is the collection \((X_{j,i})\) where \(0 \leq i \leq j\). For each \(j\) the collection \(F_0^{(j)}, F_1^{(j)}, \ldots, F_{j-1}^{(j)}\) are the capsule morphisms constituting the procession (again we chose a fixed permutation for exposition).

We now consider the automorphism \(G\). The simplest morphism of processions is pictured in Figure 4.7 and it has \(2+3+4\) maps:

\[
G_0^{(1)} : X_{1,0} \to X_{1,1}, \\
G_1^{(1)} : X_{1,1} \to X_{1,0}, \\
G_0^{(2)} : X_{2,0} \to X_{2,1}, \\
G_1^{(2)} : X_{2,1} \to X_{2,2}, \\
G_2^{(2)} : X_{2,2} \to X_{2,3}.
\]
These morphisms are clumped into three morphisms of capsules. The morphisms $G^{(1)}_i$ are morphisms from the first capsule, the morphisms $G^{(2)}_i$ make up a morphism from the second capsule and $G^{(3)}_i$ make up a morphism from the third capsule.

We now look at the special case when $X$ is a split prime strip $\mathcal{F}^+ = (\mathcal{F}_u^+)$. Since the Frobenioids $\mathcal{F}_u^+$ are equivalent to objects $(G_u, \overline{M}_u) \in (G_u, O^*_{\mathbb{K}_u})$ we will view these mono-analytic prime strips as an infinite collection of monoids $\overline{M}_u$ with groups and group actions understood. Any morphism of strips $(\overline{M}_u) \to (\overline{M}_u)$ is just a collection of morphisms of monoids (with Galois actions) for each $\nu$. In what follows the roman $M$'s without the bars will denote Galois invariants; $M_{\nu} := M_{\nu}/(\text{torsion}) (\cong \mathbb{K}_u$ as an additive topological group) we introduce

$$K_{j,i,u} := \text{klog}_+ (M_{\nu}),$$

which we are only viewing as additive topological groups.

If we suppose (for simplicity of exposition) that $V_p = \{u, v, w, x\}$, so that there are only four places above $p$ and will consider $j = 2$. The fragment

$$L_p^{(2)} = \bigoplus_{\langle \alpha, \beta, \gamma \rangle} K_{2,0,\alpha} \otimes K_{2,1,\beta} \otimes K_{2,2,\gamma}$$

then has $64 = 4 \cdot 4 \cdot 4$ direct summands, and on each factor we will have induced maps

$$G^{(2)}_{0,\alpha} : K_{2,0,\alpha} \to K_{2,1,\beta},$$

$$G^{(2)}_{1,\beta} : K_{2,1,\beta} \to K_{2,2,\gamma},$$

$$G^{(2)}_{2,\gamma} : K_{2,2,\gamma} \to K_{2,0,\alpha},$$

which further induce maps

$$G^{(2)}_{0,\alpha} \otimes G^{(2)}_{1,\beta} \otimes G^{(2)}_{2,\gamma} : K_{2,0,\alpha} \otimes K_{2,1,\beta} \otimes K_{2,2,\gamma} \to K_{2,0,\alpha} \otimes K_{2,0,\alpha} \otimes K_{2,2,\beta}$$

given by (extending linearly) the map

$$\alpha \otimes \beta \otimes \gamma \mapsto G^{(2)}_{2,\gamma}(\gamma) \otimes G^{(2)}_{0,\alpha}(\alpha) \otimes G^{(2)}_{1,\beta}(\beta).$$

(4.2)

To get the full maps $G^{(2)} : \mathbb{L}_p^{(2)} \to \mathbb{L}_p^{(2)}$, we just take the direct sum of the $64 = 4 \cdot 4 \cdot 4$ component morphisms described above. That is the simplest procession we can write down.
Figure 1. A diagram containing the morphisms involved in a general automorphism of \( \text{Prc}(X_{\{0,1,2,3\}}) \) for some object \( X \) of a category. Each bullet represents an object isomorphic to \( X \) (an object of some category). Each arrow represents an isomorphism of object. The inner square boxes represent a capsule. The outer square boxes represent the collection of capsules constituting a procession. Arrows coming out of a capsule constitute a capsule morphisms. Note that there are two types in this definition: ones that are inter-procession and ones that are intra-procession. The inner-procession morphisms are part of the definition of a procession and are necessarily order preserving. The intra-processions morphisms are the collection of capsule morphisms that constitute a morphism of processions. Note that for the domain procession we have omitted the inter-processions morphisms involved in the definition of the procession; they are the same as the ones on the right if the map is an automorphism and they can be another collection of order preserving capsule morphisms if we want our procession morphism to be between distinct processions (really, all inter-precession morphisms are the same up to a permutation of the index set so it doesn’t matter so much).
4.8. \(p\)-adic Ind2. The theory of the second indeterminacy is well-documented in [Hos18, Theorem 7.6].

These indeterminacies are referred to as “isometries” in [Moc15b]. They act through the group

\[
\text{Aut}_{\mathbb{Q}_p}(K_\mathfrak{a}^e \cap I_\mathfrak{a}) = \{ \varphi \in \text{Vect}_{\mathbb{Q}_p}(K_\mathfrak{a}^e, K_\mathfrak{a}^e) : \varphi(I_\mathfrak{a}) \subset I_\mathfrak{a} \}.
\]

Another way to see these are as \(\mathbb{Q}_p\)-vector space automorphisms which arise as \(\mathbb{Z}_p\)-lattice isomorphisms of \(I_\mathfrak{a}\). Abusively, we will let \(\text{Ind}2\) denote the collection of automorphisms induced by these automorphisms on \(L_p(j)\), \(L_p\), and \(L\) (which are given as products of the automorphisms on each of the summands).

This means that there exists automorphisms of \((G_\mathfrak{a}, \mathcal{M}^{\mathfrak{a}}_\mathfrak{a}) \in [G_{K_\mathfrak{a}}, (\mathcal{K}_\mathfrak{a}^e, +)]\) which do not come from automorphisms of \(G_\mathfrak{a}\). It turns out [Moc15b].

**Remark 4.8.1.** The second indeterminacy is local and arises (at non-archimedean primes) from the fact that the functor

\[
[G_{K_\mathfrak{a}}, \mathcal{O}_{\mathcal{K}_\mathfrak{a}}^{x, \mu}] \rightarrow [G_{K_\mathfrak{a}}]
\]

is not faithful. This is relevant in the passage from ma/fr interpretations to ma/et interpretations.

4.9. \(p\)-adic Ind3. This is covered in detail [DH20a]. This is a map \(\text{Ind}3 : 2^L \rightarrow 2^L\) where \(2^L\) denotes the power set. Implicit in this indeterminacy are certain monoid actions on (factors of) \(L\). Also, we assume a bound on \(\mathcal{O}_L(-P_\Theta)^{\text{Ind}3}\) which is given locally on the \(p\)-part in \(\text{Peel}_{\mathfrak{a}}\) component by

\[
(q^{j^2/2i})^N \cdot \text{Peel}_{\mathfrak{a}}(I_\mathfrak{a}^{\otimes j+1})
\]

(4.3)

4.10. The (Coarse) Multiradial Representation of the Theta Pilot Region. The **multiradial representation of the theta pilot region** is the region

\[
U_\Theta := \text{Ind}2(\text{Ind}1((\mathcal{O}_L(-P_\Theta))^{\text{Ind}3})) \subset \mathbb{L}.
\]

(4.4)

**Remark 4.10.1.** The notation \(U_\Theta\) follows [Tan18] who was the first to give this very important object notation. We remark that the order of operations and formulas for the indeterminacies do not appear in [Tan18] or [Moc15c, Moc15d].

4.11. **Hulls.** In this section we define \(\text{hull}(\Omega)\) for \(\Omega \subset \mathbb{L}\). As \(\mathbb{L} = \prod_p \mathbb{L}_p\) as abelian groups and each \(\mathbb{L}_p\) is the direct sum of finitely many local fields it suffices to define hulls locally—that is if \(\Omega = \prod_p \Omega_p\) then

\[
\text{hull}(\Omega) := \prod_p \text{hull}(\Omega_p).
\]

Now fix \(p \in V(\mathbb{Q})\). Let \(L = L_1 \oplus \cdots \oplus L_s\) be a product of local fields. The **hull** of \(\Omega \subset L\) is then defined to be the smallest polydisc containing \(\Omega\). That is,

\[
\text{hull}(\Omega) := \bigcap\{D_{L_1}(r_1) \cap \cdots \cap D_{L_s}(r_s) : D_{L_1}(r_1) \oplus \cdots \oplus D_{L_s}(r_s) \supset \Omega\}.
\]

\(^{13}\)Invertible linear maps on finite dimensional \(\mathbb{Q}_p\)-vector spaces have determinant one. This means the distortion factor (the determinant) is 1 and the measure of sets are preserved under these maps.
Remark 4.11.1. The idea here is that regions like the local factors of \( U_\Theta \) do not have module structures. The hull construction repairs this by taking the smallest possible module containing this regions. This construction is allegedly needed in order to relate the multiradial representation to a gluing of (interpreted) realified \( q \) and theta pilots encountered in the so-called Theta link\(^{14}\). The purpose of doing this is to encode our region further into another interpretation of Arakelov divisors suitable for comparison to another construction.

5. Global Multiplicative Subspaces, Initial Theta Data, and \( E_{11a1} \)

In this section we explain the meaning of the “Initial Theta Data” parameters

\[(F,l,E,S,V,M,\epsilon),\]

appearing in Mochizuki’s theory. At a 0th order approximation, these are just constants satisfying enough hypotheses so that a zoo of anabelian interpretations apply simultaneously. Our exposition breaks the tuple \((F,l,E,S,V,M,\epsilon)\) into two parts: \((F,l,E,S)\) and \((V,M,\epsilon)\).

The first chunk, \((F,l,E,S)\), which we call “pre theta data” consists of an elliptic curve \( E \) over a field \( F \) together with a rational prime \( l \) and a collection of places of bad multiplicative reduction \( S \). These parameters are then asked to satisfy some conditions related to the \( l \)-torsion that allow Mochizuki’s anabelian machine to run. These should be thought of “non-isotriviality conditions over the field with one element”.

The second part of the tuple, \((V,M,\epsilon)\), serves as a simulation for a “global multiplicative subspace” and “global canonical generator” (the precise definitions are given in §5.4). For Mochizuki, these serve as a fix to a failure of his approach to the ABC conjecture in his Hodge-Arakelov paper \([Moc99]\). There, his inequalities suffered from so-called “Gaussian poles” rendering them useless.

Finally, after the definition of initial theta data is given, we construct an explicit tuple \((F,l,E,S,V,M,\epsilon)\) where \( E \) is a base change of the elliptic curve over \( \mathbb{Q} \) with Cremona label E11a1.

Remark 5.0.1. We would like to acknowledge the notes \([Mok15]\), which were an initial attempt to simplify Mochizuki’s three page definition of initial theta data in \([Moc15a]\). The authors found these notes useful when first learning the theory. Also, we would like to state that much of our understanding of the second part of the tuple, \((V,M,\epsilon)\), benefitted greatly from conversations with Mochizuki.

5.1. Fricke Involutions. In the process of demystifying Mochizuki’s “initial theta data” we make use of the classical theory of the Fricke involution, which we now recall. Let \( Y_0(l) \) be the moduli stack\(^{13}\) of elliptic curves with \( l \)-torsion subgroups isomorphic to the constant

---

\(^{14}\)There are three versions of the theta links appearing in IUT: one in \([Moc15a]\), one in \([Moc15b]\), and one in \([Moc15c]\). The one that is relevant in the LGP×\( \mu \) theta link of \([Moc15c]\). The construction in \([Moc15c]\) depends critically on the auxiliary constructions surrounding \([Moc15b]\) so much of material can’t be skipped.

\(^{15}\) We note that pairs \((E,M)\) consisting of an elliptic curve together with an \( l \)-torsion subgroup have an involution which implies the moduli problem is not fine. One can define a course space as a quotient of \( Y_1(N) \) which is fine. See for example \([Par03]\ §7.1] \).
group scheme $\mathbb{Z}/l\mathbb{Z}$. For each field $L$, the $L$-rational points have the form $[(E, C)] \in Y_0(l)(L)$ where $[(E, C)]$ is an isomorphism classes of an elliptic curve $E$ over $L$ together with an order $l$ cyclic subgroup $C(L) \subset E[l](L)$ stable under $G(L/L)$.

**Definition 5.1.1.** The Frickc involution is the endomorphism $Y_0(l) \to Y_0(l)$ is given by

$$(E, M) \mapsto (\overline{E}, \overline{M}) := (E/M, E[l]/M).$$

We note that $E$ and $\overline{E}$ are connected by an isogeny $f : E \to \overline{E} := E/M$ given by modding out by the cyclic subgroup $M \subset E$. We will call this isogeny the Frickc isogeny or Frickc cover. Note that $E/M \cong M$ under these identifications.

This shows that the Frickc involution is indeed an involution.

Finally, we remark that in the stable compactification $X_0(l)$ of $Y_0(l)$ there are exactly two cusps. One which is a nodal cubic and the other which is a chain of $l$ copies of $\mathbb{P}^1$'s in a loop. It in the extension of the Frickc involution to the compactification, these two curves in involution.

5.2. Shimura Curves. For two subgroups $\Gamma_1$ and $\Gamma_2$ of $\text{PSL}_2(\mathbb{R})$ we say $\Gamma_1 \sim \Gamma_2$ if and only if $[\Gamma_1 : \Gamma_1 \cap \Gamma_2] < \infty$ and $[\Gamma_2 : \Gamma_1 \cap \Gamma_2] < \infty$. Also for a subgroup $\Gamma_0$ we write $\text{Comm}(\Gamma_0) = \{ \gamma \in \text{PSL}_2(\mathbb{R}) : \gamma \Gamma_0 \gamma^{-1} \sim \Gamma_0 \}$.

Let $Y$ be a pointed hyperbolic curve over $C$. Let $\tilde{Y} = \mathcal{H}$ be its universal cover and let $\Gamma_Y$ be the image of $\pi_1(Y) \to \text{Aut}(\mathcal{H}) = \text{PSL}_2(\mathbb{R})$.

**Definition 5.2.1** ([Moc98, Definition 2.3]). We say that $Y$ is a Shimura Curve if and only if there exists some $\mathcal{O} \subset Q$ an order in a quaternion algebra over a totally real field (which we regard as embedded in $M_2(\mathbb{R})$) such that

$$\overline{\mathcal{O}} \cap \text{SL}_2(\mathbb{R}) \sim \Gamma_Y.$$ 

Here the overline denotes the image of this group in $\text{PSL}_2(\mathbb{R})$.

Shimura curves are rare. A result of Take cited at the end of §2 of [Moc98] states that for each $(g, r)$ there are only finitely many hyperbolic Shimura curves of type $(g, r)$.

**Remark 5.2.2.** The avoidance of Shimura curves is important for anabelian reasons. Mochizuki needs algebraic curves $Z$ over a number fields $K$ which admits a so-called $K$-cores (terminal objects in a category $\text{Loc}_K(Z)$). This is used (say) in the reconstructions at the beginning of [Moc15b] and later applied in Mochizuki’s theta evaluation procedure. The existence of a $K$-core for $Z_K$ is implied by $Z_C$ being non-Shimura.

For an excellent exposition of the category $\text{Loc}_K(Z)$ we refer the reader to [Sza15]. See also [Moc02, Remark 2.1.2].
5.3. Initial Theta Data (Long Form Part 1). Initial Theta Data will consist of a large tuple

\[(F, l, E, S, V, M, \epsilon)\].

We will break this tuple into two parts. First, fix the following.

- Let \(F\) be a number field with algebraic closure \(\overline{F}\), and \(l\) a rational prime.
- Let \(E\) be an elliptic curve over \(F\)\(^16\).
- Let \(K = F(E[l])\)\(^17\).
- Fix \((E, M) \in Y_0(l)(F)\) and \(\{\epsilon, -\epsilon\} \in E[l](K) / \pm 1\).

We now give conditions of the first part of the theta data.

**Definition 5.3.1.** We say that \((F, l, E, S)\) is pre theta data if and only if

1. (Non-Shimura) \(E \mathcal{C} \setminus o\) is not a Shimura curve.
2. (Non-isotrivial) \(F/\mathbb{Q}(j_E)\) is Galois and \(\rho_l(G_F) \subset \text{Aut}(E[l](\overline{F}))\) contains a subgroup isomorphic to \(\text{SL}_2(F_l)\).
3. (Torsion Conditions) \(\sqrt{-1} \in F\) and \(E(F) \supset E(\overline{F})[30]\)
4. (Places of Multiplicative Reduction) \(S \subset V(\mathbb{Q}(j_E))_0\) is a non-empty set of places such that every \(v \in S\) has odd residue characteristic and every model \(E_{\mathbb{Q}(j_E)}\) of \(E\) over \(\mathbb{Q}(j_E)\) has multiplicative reduction at \(v\).
5. (Congruence Conditions) Let \(E_{\mathbb{Q}(j_E)}\) be a model over the field of moduli. Let \(S \subset V(\mathbb{Q}(j_E))\) be an admissible collection of bad places as above. For \(l\) the prime above, and all \(v \in S\) we require
   - (a) \(l \nmid \text{char}(v)\) for all \(v \in S\),
   - (b) \(l \nmid \text{ord}_v(q_v)\),
   - (c) \(l \geq 5\),
   - (d) \(l \nmid [F : \mathbb{Q}(j_E)]\).

We give a couple remarks to aid the reader.

**Remark 5.3.2.**
1. If we follow [Moc15a, Definition 3.1] directly, condition 3 should use 6-torsion rather than 30-torsion. Since [Moc15d] later imposes a condition on 30-torsion, we have decided to impose this condition presently.
2. Condition 2 precludes elliptic curves having CM geometrically (as Galois representations of such curves have abelian image).
3. This data is free of conditions on \(V\). This means at this point we are free to choose \(V\) to be whatever.

In order to formulate the conditions on the remaining part of the tuple \((V, M, \epsilon)\) we need to recall some fact about torsion in the Tate uniformization.

\(^{16}\)Mochizuki uses \(X_F = X = E \setminus o\) the punctured elliptic curve. Since elliptic curves have identities this data is implicit in \(E\).

\(^{17}\)Which is also \(\cong \overline{F}^{\text{ker}(\rho_l)}\).
5.4. Multiplicative Subspaces and Canonical Generators. The following “canonical splitting” of the torsion representation for Tate curves is needed both to make definitions and for explicit computations of initial theta data in \[5.7\].

\textbf{Lemma 5.4.1 (Sil13 Chapter V)\textbf{].} Let \(L/Q_p\) be a \(p\)-adic field. Let \(\text{ord}_L\) be its normalized valuation. Let \(E/L\) be an elliptic curve with \(\text{ord}_L(j_E) < 0\) (so that it does not have potentially good reduction). Let \(l \geq 3\) be a prime not dividing \(\text{ord}_L(j_E)\). There exists some \(\sigma \in I_{\mathfrak{T}/L}\) and generators \(P_1, P_2 \in E[l](\overline{L})\) such that

\[
\sigma(P_1) = P_1,
\]

\[
\sigma(P_2) = P_1 + P_2.
\]

\textit{Proof.} The basic strategy is to 1) show we can reduce to the case where we have split multiplicative reduction and \(\zeta \in L\), and 2) prove the split case. The proof of the split case goes as follows: Let \(q\) be the Tate parameter of \(E/L\) and let \(Q = q^{1/l} \in \overline{L}\) be an \(l\)th root. Then \(L(Q, \zeta)/L(\zeta)\) is a Kummer extension and \(G(L(Q, \zeta)/L(\zeta)) = \langle \sigma \rangle\) with \(\sigma(Q) = \zeta Q\).

We will make use of the fact that the Tate uniformization \(\phi: \mathfrak{T}^\times /q^Z \to E(\overline{L})\) is an isomorphism of \(G_L\)-modules (using this isomorphism we have explicit description of torsion via 
\[E(\overline{L})[l] \xleftarrow{\phi} (\zeta^2 Q^2)/q^Z\]. We set

\[
P_1 = \phi(\zeta), P_2 = \phi(Q),
\]

and compute:

\[
\sigma(P_1) = \sigma(\phi(\zeta)) = \phi(\sigma(\zeta)) = \sigma(\zeta) = P_1,
\]

\[
\sigma(P_2) = \sigma(\phi(Q)) = \phi(\sigma(Q)) = \phi(\zeta Q) = \phi(\zeta) + \phi(Q) = P_1 + P_2.
\]

This completes the split multiplicative reduction case.

The reduction step says the following: If \(L'/L\) is a finite extension with \(l \nmid [L' : L]\) and there exists some \(\sigma \in I_{\mathfrak{T}/L'}\) and \(P_1, P_2 \in E[l](\overline{L})\) such that \(\sigma(P_1) = P_1\) and \(\sigma(P_2) = P_1 + P_2\) whenever \(l \nmid \text{ord}_{L'}(j_{E'})\) then the same holds for \(L\).

We give a proof of the reduction step. If \(l \nmid \text{ord}_{L'}(q)\) then there exists some \(\sigma \in I_{\mathfrak{T}/L'}\) with \(P_1\) and \(P_2\) giving a transvection. Suppose now \(l \nmid \text{ord}_{L}(q)\). Since \(\text{ord}_{L'}(q) = e(L'/L) \text{ord}_L(q)\) we get that \(l \nmid \text{ord}_{L'}(q)\) unless \(l|e(L'/L)\). By hypothesis, there exists some \(\sigma \in I_{\mathfrak{T}/L'}\) and \(P_1, P_2 \in E(\overline{L})\) giving a transvection. Since \(I_{\mathfrak{T}/L'} \subset I_{\mathfrak{T}/L}\) this gives the result.

General case: suppose \(l \nmid \text{ord}_{L}(q)\). Let \(L' = L(\sqrt{\gamma}, \zeta)\) so that \(E_{L'}\) has split multiplicative reduction and a root of unity (as in case (1)). We have that \([L' : L]|2(l - 1)\) and hence \(l \nmid [L' : L]\). The general result now follows from the reduction step. \(\square\)

\(18\gamma = -e_4/e_6\)
The point here is we can find a unipotent matrix in the representation. That is there exists some $P_1, P_2$ and $\sigma$ such that

$$\rho_l(\sigma) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

with respect to the basis

$$P_1 \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad P_2 \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}. $$

**Definition 5.4.2.** An element $\sigma$ together with a choice of basis $P_1$ and $P_2$ is called a transvection.

This now brings us to the main point of this subsection, the “local canonical splitting” (which Mochizuki attempts to globalize): If $L/\mathbb{Q}_p$ is a finite extension, and $E/L$ is an elliptic curve with multiplicative reduction then there exists a canonical inclusion $\mu_l(L) \xrightarrow{\text{can}} E(L)[l]$ of $G_L$-modules given by the following diagram.

$$\begin{array}{cccccccccc}
1 & \longrightarrow & \mu_l(L) & \longrightarrow & E(L)[l] & \longrightarrow & \mathbb{Z}/l & \longrightarrow & 1 \\
\sim & \quad & \sim & \quad & \sim & \quad & \sim & \quad & \\
1 & \xrightarrow{\phi} & \zeta_l^2 & \longrightarrow & \zeta_l^2Q^2/q^2 & \longrightarrow & \zeta_l^2Q^2/\zeta_l^2q^2 & \longrightarrow & 1
\end{array} \quad (5.1)
$$

We make two important remarks. First, the sequence (5.1) does not depend on the choices of $Q$ and $\zeta_l$. Second, the generator of the quotient is

$$Q \equiv q^{1/l} \mod \langle \zeta_l, q \rangle \quad (5.2)$$

and this class is uniquely determined. We record this situation using the following definition.

**Definition 5.4.3.** Let $L$ be a finite extension of $\mathbb{Q}_p$ and $E$ an elliptic curve over $L$ with multiplicative reduction.

1. We call the $G_L$-submodule isomorphic to $\mathbb{F}_l(1) := \mu_l(L)$ in (5.1) the (local) multiplicative subspace.

2. We call the generator of the quotient of the $G_L$-module $E[l]$ isomorphic to $\mathbb{Z}/l$ (as a $G_L$-module) given in (5.2) the (local) canonical generator. (In terms of the Tate uniformization, this is just the image of an $l$th root of the Tate parameter.)

Mochizuki wishes to globalize this situation. Consider the situation now an elliptic curve $E$ over a number field $F$ and let $K = F(E[l])$ be the $l$-division field of $F$ associated to $E$. Fix $M \subset E[l](K)$ isomorphic to $\mathbb{Z}/l$ as a subgroup, which we may or may not view it as a $G(K/F)$-module (most of the time it will be a subgroup which is not preserved by $G(K/F)$).

Let $w \in V(K)_0$ and let $v$ denote the place in $V(F)$ below $w$. Completing $K$ at the place $w$ gives the field $K_w$ determines which naturally contains the completion of $F$ at $v$; this gives
a diagram of inclusions

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(2) Let $\epsilon \in M$. We say that $\epsilon$ is a \textit{(spin) global canonical generator} for $w$ if the image of $\epsilon_w$ in the $M_w$ ($\cong \mathbb{Z}/l$ by the first part) is $\pm$ the local canonical canonical generator at $w$.

If $S \subset V(K)$ we will say that $M$ is a \textit{global multiplicative subspace for $S$} if it is for each $w \in S$. Similarly, we will say $\epsilon$ is a \textit{global canonical generator for $S$} if it is for each $w \in S$.

5.5. \textbf{Initial Theta Data: “Fake” Global Multiplicative Subspaces and Canonical Generators.} Let $E$ be an elliptic curve over a number field $F$. Fix $l$ a prime. Let $K = F(E[l]) = \overline{F}_{\ker(\rho_l)}$ be the field obtained by adjoining the $l$-torsion. Consider now the tuple $(V, M, \epsilon)$ where

- $V \subset V(K)$ is a lift of $V(\mathbb{Q}(j_E))$.
- $M \subset E[l](K)$ is a subgroup. ($\leftrightarrow M \subset E[l](K)$ by Fricke)
- $\epsilon \in M$ (which we only care about up to $\pm 1$)

We can now state the definition of initial theta data.

\textbf{Definition 5.5.1.} Fix $(F, l, E, S)$ is a tuple of pre theta data. Let $K = F(E[l])$. \textit{Initial theta data} is a tuple

$$(V, M, \epsilon)$$

where

- $V \subset V(K)$ is a lift of $V(\mathbb{Q}(j_E))$.
- $M \subset E[l](K)$ is a subgroup. ($\leftrightarrow M \subset E[l](K)$ by Fricke)
- $\epsilon \in M$ (which we only care about up to $\pm 1$)

We can now state the definition of initial theta data.

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\textbf{Definition 5.5.1.} Fix $(F, l, E, S)$ is a tuple of pre theta data. Let $K = F(E[l])$. \textit{Initial theta data} is a tuple

$$(V, M, \epsilon)$$

where $M$ is the Fricke involute of $M$ a global multiplicative subspace (with respect to $V \cap V(K)_S$) and $\epsilon \in M(F)$ is a global canonical generator (with respect to $V \cap V(K)_S$).

If $E$ is a member of a tuple of initial theta data we say $E$ \textit{sits in initial theta data}.

5.6. \textbf{Existence of Initial Theta Data.} We now prove existence of initial theta data. We begin by giving initial theta data over a single place of bad reduction.

\textbf{Lemma 5.6.1 (Simple Initial Theta Data).} Let $F$ be a number field. Let $E/F$ be an elliptic curve. Suppose that $j_E \notin \mathcal{O}_F$. Let $K = F(E[l])$ for some rational prime $l$. For all but finitely many choices of $l$, there exists some $P_1, P_2 \in E[l](K)$ and some $\sigma \in G_F$ with such that

$$\rho_l(\sigma) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

with respect to this basis.

\textbf{Proof.} Compare what follows to [Sil13, V.6.2]. Let $w \in V(F)_0$ with $|j_E|_w > 1$. Fix an inclusions $F \subset F_w$ so that we have $G_w \subset G_F$. By the existence of local transvections there exists some $\sigma \in G_w$ and $P_{w,1}, P_{w,2} \in E[l](K_w)$ so that $\rho_l(\sigma)$ is unipotent. We need now some $M \subset E[l](K)$ and some $\epsilon \in M(K)$ (up to sign) such that after base change they identify with the $q^{1/l}$. The identifications of $G_w \subset G_K$ and $E[l](F) \subset E(F_w)$ give the result letting $S = \{w\}$.

$\square$
One claim of \cite{Moc15d} Proposition 2.2 is that there exists a global multiplicative subspace \( M \subset E[l] \) relative to some \( V \subset V(K) \) a section of \( V(F_0) \) (in the case when we have more than one bad prime). The lemma below verifies this assertion.

**Lemma 5.6.2** (Existence of Global Multiplicative Subspaces). For all \( M \subset E(\mathcal{F})l \) and all \( v \in V(F_0) \) there exists some \( \underline{v} \in V(K)_v \) such that \( M_{\underline{v}} \cong \mu_\ell(\underline{v}) \). Performing this operations allows us to construct a collection \( \underline{V} \) for a given subspace \( M \) so that \( M \) is a global multiplicative subspace with respect to \( \underline{V} \).

**Proof.** Call a pair \( (w, M) \) consisting of a place \( w \in V(K)_v \) and a subspace \( M \subset E[l] \) good if \( M = \mu(w) \). Note that if a pair \( (w, M) \) is good and \( \sigma \in G_F \) then \( (\sigma(w), \sigma(M)) \) is good. This is because if \( \sigma(w) = w' \) then \( \sigma G(w/v) \sigma^{-1} = G(w'/v) \) and we can write \( \tau' = \sigma \tau \sigma^{-1} \) for each \( \tau' \in G(w'/v) \). This then shows that the action of \( \tau' \) on \( \sigma(M) \) performs as advertized since for each \( m \in M \)

\[
\tau' \cdot \sigma(m) = \sigma \tau \sigma^{-1} m = \sigma \tau(m) = \sigma(\chi_l(\tau)m) = \sigma(\chi_l(m)) \sigma(m).
\]

This proves \( \sigma(\mu(w)) \cong \mu(\sigma(w)) \).\(^{19}\)

We will now apply the above to show that a collection of \( \underline{V} \) exists. First let \( v \mapsto \tilde{v} \) be any set theoretic section \( V(F) \to V(K) \) of the natural projection \( V(K) \to V(F) \). Consider good pairs \( (\tilde{v}, \mu(\tilde{v})) \) where \( \mu(\tilde{v}) \subset E[l](K) \) is the unique subspace which is cyclotomic for \( G(\tilde{v}/v) \subset G(K/F) \). Note that \( G_F \) acts transitively on both \( V(K)_v \) and the subspaces of \( E[l](K) \cong \mathbb{P}_F^2 \) by the hypothesis that \( \rho_l(G_F) \subset \text{SL}_2(\mathbb{F}_l) \). For each \( \tilde{v} \) select some \( \sigma_v \in G(K/F) \) such that \( \sigma_v(\mu(v)) = M \). Defining a section \( \underline{V} \subset V(K) \) by

\[
\underline{v} := \sigma_v(\tilde{v}),
\]

completes the proof. To define a section \( V(F_0) \to V(K) \) take any section \( V(F_0) \to V(F) \) and compose it with a section of the type described above. \( \Box \)

**Remark 5.6.3.** In \cite{Moc15a} Definition 3.1 Mochizuki uses \( \underline{C}_K \) instead of \( M \) and defines \( \underline{\xi} \) as a cusp of \( \underline{C}_K \) rather that as an element of \( M(K) \). Here \( \underline{C}_K \) is a hyperbolic orbicurve defined over \( K \) (a stack), defined to be the quotient of \( \underline{X}_K \) by the elliptic involution. We claim that these are equivalent.

First suppose we are given the data of \( \underline{C}_K \) together with a cusp \( \underline{\xi} \in \underline{C}_K \). Since we have the maps \( \underline{C}_K \to C_K \) and \( X_K \to C_K \) we can form the fiber product \( \underline{X}_K \coloneqq \underline{C}_K \times_{C_K} X_K \) which has the isogeny \( \underline{X}_K \to X_K \). The kernel of the compactification of this map is \( M \).

Conversely, suppose that we have \( M \) as (a quotient of \( E[l] \)) and \( \underline{\xi} \). This determines the cover \( \underline{E} \) as in the second part of \( \S 5.1 \) via the Galois correspondence.

**Remark 5.6.4.** We parrot \cite{Moc15a} Remark 3.1.3.

1. \( (F, E, l, S, V, \underline{M}, \underline{\xi}) \) is completely determined by \( (F, E, S, V, M) \).
2. \( C_K \) is determined up to \( K \)-isomorphism by \( (F, E, l, V) \).
3. Given choices of \( (E, l, S) \) the choices of \( V \) don’t affect the theory.

\(^{19}\)Although the cyclotomic character might be modified slightly. The reader may wish to consult Chapter 2, Lemma 1.3 of Mochizuki’s Hodge Arakelov Theory paper for a criteria for determining when a subspace is multiplicative.
5.7. Example: Initial Theta Data for E11a1. We now produce initial theta data for the elliptic curve \( E/\mathbb{Q} \) with Cremona label \( \text{E11a1} \) (see [http://www.lmfdb.org/EllipticCurve/Q/11/a/2](http://www.lmfdb.org/EllipticCurve/Q/11/a/2)). This elliptic curve is given by

\[
E : y^2 + y = x^3 - x^2 - 7820x - 263580,
\]

and its basic invariants are

\[
c_4(E) = 496 = 2^4 \cdot 31,
\]
\[
c_6(E) = 20008 = 2^3 \cdot 41 \cdot 61,
\]
\[
j_E = -1 \cdot 2^{12} \cdot 11^{-5} \cdot 31^3.
\]

We now go through the “Simple Initial Theta Data” computation (Lemma 5.6.1) for this curve explicitly.

5.7.1. Non-empty Collection of Places of Bad Reduction \( S = V_{\text{bad}}^{\text{mod}} \). There is only one place of bad reduction. That is at \( p = 11 \) and \( E \) has split multiplicative reduction at this place. This means \( S = \{11\} \subset V(\mathbb{Q}) \).

5.7.2. Minimal Weierstrass Model at \( p = 11 \). The local Weierstrass minimal model at \( p = 11 \) of \( E \) is

\[
E_{11}^{\min} : y^2 + y = x^3 - x^2 - 10x - 20.
\]

As \( E \) is defined over \( \mathbb{Q} \) which has class number 1 we have that this is a global minimal model. We record its invariants:

- Minimal discriminant valuation : 5
- Conductor exponent : 1
- Kodaira Symbol: I5
- Tamagawa Number: \( |E/\mathbb{E}^0| = |\mathbb{Z}/q| = \text{ord}(q) = 5 \)

5.7.3. The Tate Uniformization at \( p = 11 \). The Tate model of this curve is

\[
E_{q_{11}} : y^2 + xy = x^3 + s_4(q_{11})x + s_6(q_{11}),
\]

where \( q_{11} \) is the Tate parameter. We have computed these out to \( O(11^{25}) \) using Sage\(^{20}\):

\[
q = q_{11} = 0.0, 0, 0, 0, 10, 2, 6, 6, 5, 4, 4, 1, 4, 1, 0, 5, 9, 9, 3, 3, 1, 3, 4, \ldots
\]
\[
s_4 = 0.0, 0, 0, 0, 5, 7, 1, 0, 5, 9, 1, 9, 2, 10, 2, 0, 1, 6, 2, 6, 4, 10, 10, \ldots
\]
\[
s_6 = 0.0, 0, 0, 0, 1, 8, 4, 4, 5, 5, 10, 2, 1, 8, 2, 7, 10, 9, 6, 3, 3, 8, 5, \ldots
\]

Here the decimal is the beginning of the integral digits and commas separate 11-adic digits. We will make use of the the uniformization map

\[
\mathbb{Q}_{11}^\times / q_{11}^\mathbb{Z} \xrightarrow{\phi} E(\mathbb{Q}_{11}).
\]

\(^{20}\)See [http://doc.sagemath.org/html/en/reference/padics/sage/rings/padics/padic_extension_generic.html](http://doc.sagemath.org/html/en/reference/padics/sage/rings/padics/padic_extension_generic.html) and [http://sporadic.stanford.edu/reference/curves/sage/schemes/elliptic_curves/ell_tate_curve.html](http://sporadic.stanford.edu/reference/curves/sage/schemes/elliptic_curves/ell_tate_curve.html).
In particular for each \( n \) if we let \( \zeta_n \) be a primitive \( n \)th root of unity and \( Q_n = q^{1/n} \) be a choice of \( n \)th root we have

\[
P_{1,n} = \varphi(\zeta_n), \quad P_{2,n} = \varphi(Q_n) \in E[n](\overline{\mathbb{Q}}).
\]

5.7.4. Field \( F \). We will let \( F = \mathbb{Q}(\sqrt{-1}, E(\overline{\mathbb{Q}})[30]) \). To obtain \( E(\overline{\mathbb{Q}})[30] \) “numerically” (the method we describe seems not practically computable) in \( \overline{\mathbb{Q}} \) we fix an embedding \( \mathbb{Q} \subset \mathbb{Q}_{11} \). We than take the coordinates of the Tate uniformization and map them to our model over \( \mathbb{Q} \). Observe that \( \varphi = f \circ \phi \) where \( f : E_q \to E \) is the morphism defined over \( \mathbb{Q}_{11} \) (since \( E \) has split multiplicative reduction over \( \mathbb{Q}_{11} \) — with a global minimal Weierstrass model defined over \( \mathbb{Q} \) and \( \phi \) the Tate uniformization of \( E_q \)). This gives \( E_q(\mathbb{Q}_{11})[30] \cong \zeta^2 Q^2/q_{11}^2 \) where \( Q = q^{1/30} \) is some choice of 30th root of \( q \) and \( \zeta \) is a primitive \( l \)th root of unity and taking the image of these points under \( f \) gives \( \overline{\mathbb{Q}} \)-points.

5.7.5. Condition on \( l \). The prime \( l = 13 \) satisfies the divisibility hypotheses of initial theta data. We have seen that \( \text{ord}(q) = 5 \), the only bad place is \( p = 11 \), and that the order of \([ F : \mathbb{Q}(j_E) ] \) (here \( \mathbb{Q}(j_E) = \mathbb{Q} \) can be computed by viewing \( F \) as a succession of prime torsion extensions. The order of the \( r \)-torsion extension divides \( |GL_2(F_r)| \) for a given prime \( r \). The relevant orders for 30-torsion are given below:

| \( r \) | \( |GL_2(F_r)| = (r^2 - 1)(r^2 - 1 - (r - 1)) \) |
|-----|------------------------------------------------------------------|
| 2   | 6 = 2 \cdot 3                                                    |
| 3   | 48 = 2^4 \cdot 3                                                 |
| 5   | 480 = 2^5 \cdot 3 \cdot 5                                        |

5.7.6. Surjectivity of \( \rho_l \). We require that \( \rho_l : G_F \to \text{Aut}(E[l]) \) contains a copy of \( \text{SL}_2(F_l) \) relative to some basis. It can be checked, for example, using Sage that the mod \( l \) Galois representation for \( l > 5 \) is surjective. We did this for \( l = 13 \).\(^{21}\)

5.7.7. Computation of the Field \( K \). We have \( K = F(E_K[l](\overline{F})) \) for \( l = 13 \); There are two approaches to explicit computations. One may use the Weierstrass uniformization (at archimedean places) and the Tate uniformization (at both archimedean and non-archimedean places).

5.7.8. The generator \( \zeta \). We use (the image of) \( q^{1/l} \) where \( q \) is the Tate parameter at \( p = 11 \) and \( l = 13 \). These values can be gleaned from the Tate uniformization: \( \zeta = (q_{11}/q^2)^{1/13} \). So the torsion field at a local place has the form \( K_v = \mathbb{Q}_{11}(\zeta_6, \zeta_{13}, q_{11}^{1/13}, q_{11}^{1/6}) \).

\(^{21}\)We note that \( l = 7 \) also works.
Remark 5.7.1. One could hope to generate simple example of initial theta data where $K$ is small by looking for some $E$ over $\mathbb{Q}$ with many $\mathbb{Q}$-rational torsion points. This approach is hopeless. Recall that Mazur’s Theorem says that if $E$ is an elliptic curve defined over $\mathbb{Q}$ then the torsion then the possible torsion subgroups of $E(\mathbb{Q})$ are $C_1, C_2, \ldots , C_{10}, C_2 \oplus C_4, C_2 \oplus C_6, C_2 \oplus C_8$. Here $C_n$ denotes the cyclic group of order $n$.

We would also like to point out that Szpiro’s inequality implies a uniform computable bound on the number of torsion points in a field $F$. [Sh13, Exercise 5.16].

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