Modular curves, invariant theory and $E_8$

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Abstract

The $E_8$ root lattice can be constructed from the modular curve $X(13)$ by the invariant theory for the simple group PSL(2, 13). This gives a different construction of the $E_8$ root lattice. It also gives an explicit construction of the modular curve $X(13)$.

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1. Introduction

The $E_8$ root lattice (see [14]) occurs in: the equation of the $E_8$-singularity (theory of singularities), the Barlow surface, which is homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 8 \mathbb{CP}^2$ (differential topology of 4-manifolds), and the automorphism group of the configuration of 120 tritangent planes of Bring’s curve (representation theory and classical algebraic geometry) (see section two for more details).

In the present paper, we will give a different relation which connects $E_8$ with the modular curve $X(13)$. We will show that the $E_8$ root lattice can be constructed from the modular curve $X(13)$ by the invariant theory for the simple group PSL(2, 13).

Let us begin with the invariant theory for PSL(2, 13). Recall that the six-dimensional representation (the Weil representation) of the finite simple
group $\text{PSL}(2, 13)$ of order 1092, which acts on the five-dimensional projective space $\mathbb{P}^5 = \{(z_1, z_2, z_3, z_4, z_5, z_6) : z_i \in \mathbb{C} \ (i = 1, 2, 3, 4, 5, 6)\}$. This representation is defined over the cyclotomic field $\mathbb{Q}(e^{2\pi i/13})$. Put

$$S = -\frac{1}{\sqrt{13}} \begin{pmatrix}
\zeta^{12} - \zeta & \zeta^{10} - \zeta^3 & \zeta^4 - \zeta^9 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 \\
\zeta^{10} - \zeta^3 & \zeta^4 - \zeta^9 & \zeta^{12} - \zeta & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 \\
\zeta^4 - \zeta^9 & \zeta^{12} - \zeta & \zeta^{10} - \zeta^3 & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} \\
\zeta^5 - \zeta^8 & \zeta^{12} - \zeta & \zeta^4 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 \\
\zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 \\
\zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11}
\end{pmatrix}$$

and

$$T = \text{diag}(\zeta^7, \zeta^{11}, \zeta^8, \zeta^6, \zeta^2, \zeta^5)$$

where $\zeta = \exp(2\pi i/13)$. We have

$$S^2 = T^{13} = (ST)^3 = 1. \quad (1.1)$$

Let $G = \langle S, T \rangle$, then $G \cong \text{PSL}(2, 13)$. We construct some $G$-invariant polynomials in six variables $z_1, \ldots, z_6$. Let

$$w_\infty = 13A_0^2, \quad w_\nu = (A_0 + \zeta^\nu A_1 + \zeta^{4\nu} A_2 + \zeta^{9\nu} A_3 + \zeta^{3\nu} A_4 + \zeta^{12\nu} A_5 + \zeta^{10\nu} A_6)^2$$

(1.2)

for $\nu = 0, 1, \ldots, 12$, where the senary quadratic forms (quadratic forms in six variables) $A_j$ $(j = 0, 1, \ldots, 6)$ are given by

$$A_0 = z_1z_4 + z_2z_5 + z_3z_6, \quad A_1 = z_1^2 - 2z_3z_4, \quad A_2 = -z_5^2 - 2z_3z_4, \quad A_3 = z_2^2 - 2z_1z_5, \quad A_4 = z_3^2 - 2z_2z_6, \quad A_5 = -z_4^2 - 2z_1z_6, \quad A_6 = -z_6^2 - 2z_3z_5. \quad (1.3)$$

Then $w_\infty, w_\nu$ for $\nu = 0, \ldots, 12$ are the roots of a polynomial of degree fourteen. The corresponding equation is just the Jacobian equation of degree fourteen (see [23], pp.161-162). On the other hand, set

$$\delta_\infty = 13^2G_0, \quad \delta_\nu = -13G_0 + \zeta^\nu G_1 + \zeta^{2\nu} G_2 + \cdots + \zeta^{12\nu} G_{12} \quad (1.4)$$
for $\nu = 0, 1, \ldots, 12$, where the senary sextic forms (i.e., sextic forms in six variables) $G_j \ (j = 0, 1, \ldots, 12)$ are given by

\[
\begin{align*}
G_0 &= D_0^2 + D_\infty^2, \\
G_1 &= -D_7^2 + 2D_0D_1 + 10D_\infty D_1 + 2D_2D_{12} + \\
& \quad -2D_3D_{11} + 4D_4D_{10} - 2D_5D_5, \\
G_2 &= -2D_1^2 - 4D_0D_2 + 6D_\infty D_2 - 2D_4D_{11} + \\
& \quad + 2D_5D_{10} - 2D_6D_9 - 2D_7D_8, \\
G_3 &= -D_8^2 + 2D_0D_3 + 10D_\infty D_3 + 2D_6D_{10} + \\
& \quad -2D_7D_7 - 4D_{12}D_4 - 2D_1D_2, \\
G_4 &= -D_2^2 + 10D_0D_4 - 2D_\infty D_4 + 2D_3D_{12} + \\
& \quad -2D_5D_8 - 4D_1D_3 - 2D_{10}D_7, \\
G_5 &= -2D_9^2 - 4D_0D_5 + 6D_\infty D_5 - 2D_{10}D_8 + \\
& \quad + 2D_6D_{12} - 2D_7D_3 - 2D_{11}D_7, \\
G_6 &= -2D_3^2 - 4D_0D_6 + 6D_\infty D_6 - 2D_{12}D_7 + \\
& \quad + 2D_3D_4 - 2D_5D_1 - 2D_8D_{11}, \\
G_7 &= -2D_{10}^2 + 6D_0D_7 + 4D_\infty D_7 - 2D_1D_6 + \\
& \quad -2D_2D_5 - 2D_8D_{12} - 2D_9D_{11}, \\
G_8 &= -2D_4^2 + 6D_0D_8 + 4D_\infty D_8 - 2D_3D_5 + \\
& \quad -2D_6D_2 - 2D_{11}D_{10} - 2D_1D_7, \\
G_9 &= -D_{11}^2 + 2D_0D_9 + 10D_\infty D_9 + 2D_5D_4 + \\
& \quad -2D_1D_8 - 4D_{10}D_{12} - 2D_3D_6, \\
G_{10} &= -D_5^2 + 10D_0D_{10} - 2D_\infty D_{10} + 2D_6D_4 + \\
& \quad -2D_3D_7 - 4D_9D_1 - 2D_{12}D_{11}, \\
G_{11} &= -2D_{12}^2 + 6D_0D_{11} + 4D_\infty D_{11} - 2D_9D_2 + \\
& \quad -2D_5D_6 - 2D_7D_4 - 2D_3D_8, \\
G_{12} &= -D_6^2 + 10D_0D_{12} - 2D_\infty D_{12} + 2D_2D_{10} + \\
& \quad -2D_1D_{11} - 4D_5D_9 - 2D_4D_8.
\end{align*}
\]

Here, the senary cubic forms (cubic forms in six variables) $D_j \ (j = 0, 1, \ldots, 12)$
are given as follows:

\[
\begin{align*}
D_0 &= z_1 z_2 z_3, \\
D_1 &= 2z_2 z_3^2 + z_2^2 z_6 - z_4^2 z_5 + z_1 z_5 z_6, \\
D_2 &= -z_6^3 + z_2^2 z_4 - 2z_2 z_5^2 + z_1 z_4 z_5 + 3z_3 z_5 z_6, \\
D_3 &= 2z_1 z_2^2 + z_1^2 z_5 - z_4 z_6^2 + z_3 z_4 z_5, \\
D_4 &= -z_2^2 z_3 + z_1 z_6^2 - 2z_1^2 z_6 - z_1 z_3 z_5, \\
D_5 &= -z_3^2 + z_3^2 z_5 - 2z_3 z_6 + 2z_1 z_5 z_6, \\
D_6 &= -z_5^3 + z_5^2 z_6 - 2z_5 z_4^2 + z_3 z_4 z_5 + 3z_2 z_4 z_5, \\
D_7 &= -z_3^2 + z_2 z_4 z_6 - z_1 z_3^2 z_6 - 3z_1 z_2 z_5 + 2z_1^2 z_4, \\
D_8 &= -z_1 z_2^2 + z_2 z_3 z_5 - 3z_1 z_3 z_4 + 2z_3^2 z_6, \\
D_9 &= 2z_1^2 z_3 + z_3^2 z_4 - z_2^2 z_6 + z_2 z_4 z_6, \\
D_{10} &= -z_1 z_3^2 + z_2 z_4^2 - 2z_2 z_5^2 - z_1 z_2 z_6, \\
D_{11} &= -z_3^2 + z_1 z_2^2 - z_1 z_2 z_4 - 3z_2 z_3 z_6 + 2z_2 z_5 z_6, \\
D_{12} &= -z_1^2 z_2 + z_3 z_5^2 - 2z_5 z_6^2 - z_2 z_3 z_4, \\
D_{18} &= z_4 z_5 z_6.
\end{align*}
\]

Then \(\delta_\infty, \delta_\nu\) for \(\nu = 0, \ldots, 12\) are the roots of a polynomial of degree fourteen. The corresponding equation is not the Jacobian equation. Now, a family of invariants for \(G\) is given as follows: put

\[
\Phi_4 = \sum_{\nu=0}^{12} w_\nu + w_\infty, \quad \Phi_8 = \sum_{\nu=0}^{12} w_\nu^2 + w_\infty^2, \quad (1.6)
\]

\[
\Phi_{12} = -\frac{1}{13 \cdot 52} \left( \sum_{\nu=0}^{12} \delta_\nu^2 + \delta_\infty^2 \right), \quad \Phi'_{12} = -\frac{1}{13 \cdot 30} \left( \sum_{\nu=0}^{12} w_\nu^3 + w_\infty^3 \right), \quad (1.7)
\]

\[
\Phi_{16} = \sum_{\nu=0}^{12} w_\nu^4 + w_\infty^4, \quad \Phi_{18} = \frac{1}{13 \cdot 6} \left( \sum_{\nu=0}^{12} \delta_\nu^3 + \delta_\infty^3 \right), \quad (1.8)
\]

\[
\Phi_{20} = \frac{1}{13 \cdot 25} \left( \sum_{\nu=0}^{12} w_\nu^5 + w_\infty^5 \right), \quad \Phi_{30} = -\frac{1}{13 \cdot 1315} \left( \sum_{\nu=0}^{12} \delta_\nu^5 + \delta_\infty^5 \right), \quad (1.9)
\]
and \( x_i(z) = \eta(z)a_i(z) \) (1 \( \leq i \leq 6 \)), where

\[
\begin{align*}
&\begin{cases}
  a_1(z) := e^{-\frac{11\pi i}{6}} \theta \left[ \begin{array}{c}
    13 \\
    1
  \end{array} \right] (0, 13z), \\
  a_2(z) := e^{-\frac{7\pi i}{6}} \theta \left[ \begin{array}{c}
    13 \\
    1
  \end{array} \right] (0, 13z), \\
  a_3(z) := e^{-\frac{5\pi i}{6}} \theta \left[ \begin{array}{c}
    13 \\
    1
  \end{array} \right] (0, 13z), \\
  a_4(z) := -e^{-\frac{3\pi i}{6}} \theta \left[ \begin{array}{c}
    13 \\
    1
  \end{array} \right] (0, 13z), \\
  a_5(z) := e^{-\frac{\pi i}{3}} \theta \left[ \begin{array}{c}
    9 \\
    1
  \end{array} \right] (0, 13z), \\
  a_6(z) := e^{-\frac{\pi i}{3}} \theta \left[ \begin{array}{c}
    1 \\
    1
  \end{array} \right] (0, 13z)
\end{cases},
\end{align*}
\tag{1.11}
\]

are theta constants of order 13 and \( \eta(z) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^n) \) with \( q = e^{2\pi i z} \) is the Dedekind eta function which are all defined in the upper-half plane \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \). Our main theorem is the following:

**Theorem 1.1.** The \( G \)-invariant polynomials \( \Phi_4, \ldots, \Phi_{30} \) in \( x_1(z), \ldots, x_6(z) \) can be identified with modular forms as follows:

\[
\begin{align*}
&\begin{cases}
  \Phi_4(x_1(z), \ldots, x_6(z)) = 0, \\
  \Phi_8(x_1(z), \ldots, x_6(z)) = 0, \\
  \Phi_{12}(x_1(z), \ldots, x_6(z)) = \Delta(z), \\
  \Phi'_8(x_1(z), \ldots, x_6(z)) = \Delta(z), \\
  \Phi_{16}(x_1(z), \ldots, x_6(z)) = 0, \\
  \Phi_{18}(x_1(z), \ldots, x_6(z)) = \Delta(z)E_6(z), \\
  \Phi_{20}(x_1(z), \ldots, x_6(z)) = \eta(z)^8 \Delta(z)E_4(z), \\
  \Phi_{30}(x_1(z), \ldots, x_6(z)) = \Delta(z)^2 E_6(z).
\end{cases}
\end{align*}
\tag{1.12}
\]

Theorem 1.1 has many consequences. The first one comes from the theory of singularities: there exists at least two kinds of constructions of the equation of the \( E_8 \)-singularity: one is given by the icosahedral group in the celebrated book of Klein [23], i.e., the icosahedral singularity (see [5], p. 107), the other is given by the group \( \text{PSL}(2, 13) \).
**Theorem 1.2** (A different construction of the $E_8$-singularity: from $X(13)$ to $E_8$). The equation of the $E_8$-singularity can be constructed from the modular curve $X(13)$ as follows:

$$\Phi_{30}^3 - \Phi_{20}^2 = 1728\Phi_{12}^5,$$  \hspace{1cm} (1.13)

where $\Phi_j = \Phi_j(x_1(z), \ldots, x_6(z))$ for $j = 12, 20$ and $30$. As polynomials in six variables $z_1, \ldots, z_6$, $\Phi_{12}$, $\Phi_{20}$ and $\Phi_{30}$ are $G$-invariant polynomials.

In fact, in his talk at ICM 1970 [4], Brieskorn showed how to construct the singularity of type $ADE$ directly from the simple complex Lie group of the same type. At the end of that paper [4] Brieskorn says: “Thus we see that there is a relation between exotic spheres, the icosahedron and $E_8$. But I still do not understand why the regular polyhedra come in.” (see also [16], [17] and [4]). As a consequence, Theorem 1.2 shows that the $E_8$ root lattice is not necessarily constructed from the icosahedron. That is, the icosahedron does not necessarily appear in the triple (exotic spheres, icosahedron, $E_8$) of Brieskorn [4]. The group $\text{PSL}(2,13)$ can take its place and there is the other triple (exotic spheres, $\text{PSL}(2,13)$, $E_8$). The higher dimensional liftings of these two distinct groups and modular interpretations on the equation of the $E_8$-singularity give the same Milnor’s standard generator of $\Theta_7$.

The second consequence of Theorem 1.1 comes from differential topology of 4-manifolds. The manifold $\mathbb{CP}^2 \# 8 \mathbb{CP}^2$ has two distinct differentiable structures, both of which come from algebraic surfaces: one is the eight-fold blow-up of the projective plane carrying the standard smooth structure, the other is the Barlow surface [2], which is a simply connected minimal surface of general type with $q = p_g = 0$ and $K^2 = 1$. In fact, up to now, only two kinds of such surfaces are known: the first example is the Barlow surface [2], the second examples are given by Lee and Park in the appendix of [29], both of them are simply connected, minimal, complex surfaces of general type with $p_g = 0$ and $K^2 = 1$. The Barlow surface comes from a certain Hilbert modular surface associated to the icosahedral group (see [15]). On the other hand, the Lee-Park surfaces are constructed by a rational blow-down surgery and a $\mathbb{Q}$-Gorenstein smoothing theory (see [29]). The Barlow surface is homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 8 \mathbb{CP}^2$ (see [28]). We prove the following:

**Theorem 1.3** (A different construction of the Barlow surface: from $X(13)$ to $E_8$). The Barlow surface can be constructed from the modular curve $X(13)$.
The third consequence of Theorem 1.1 comes from representation theory
and classical algebraic geometry: the automorphism group of the configu-
ration of 120 tritangent planes of Bring’s curve is the quotient of the Weyl
group \( W(E_8) \) of the root system of type \( E_8 \) by the normal subgroup \( \{ \pm 1 \} \)
with order \( 2^{13} \cdot 3^5 \cdot 5^2 \cdot 7 \), which is just the group \( G_{8,2} \) studied by Coble in p.
356 of his paper [7]. Note that Bring’s curve has an analogue: Fricke’s octavic
curve, both of them arise from the resolution of the equation of the fifth
degree (see [23]). Hence, they are intimately connected with the icosahedron.

As a consequence of Theorem 1.1, we prove the following:

**Theorem 1.4** (A different construction of Bring’s curve and Fricke’s octavic curve: from \( X(13) \) to \( E_8 \)). Both Bring’s curve and Fricke’s octavic curve can be constructed from the modular curve \( X(13) \).

Theorem 1.2, Theorem 1.3 and Theorem 1.4 show that there exist two
distinct constructions of the \( E_8 \) root lattice from its three different appear-
ances. Hence, the fact that the \( E_8 \) root lattice is not necessarily constructed
from the icosahedron, can be realized not only from the theory of singular-
ities and exotic spheres, but also from differential topology of 4-manifolds,
representation theory and classical algebraic geometry.

The fourth consequence of Theorem 1.1 comes from the explicit con struc-
tion of modular curves, which is a classical problem studied by Klein (see
[24], [25], [26] and [27]).

**Problem 1.5.** Let \( p \geq 7 \) be a prime number. Give an explicit con-
struction of the modular curve \( X(p) \) of level \( p \) from the invariant theory for
\( \text{PSL}(2, p) \) using projective algebraic geometry.

For \( p = 7 \), the modular curve \( X(7) \) is given by the celebrated Klein quartic
curve (see [24])

\[
x^3 y + y^3 z + z^3 x = 0.
\]

For \( p = 11 \), the modular curve \( X(11) \) leads to the study of the Klein cubic
threefold (see [25])

\[
v^2 w + w^2 x + x^2 y + y^2 z + z^2 v = 0.
\]

Following Klein’s method for the cubic threefold, Adler and Ramanan (see
[1]) studied Problem 1.5 when \( p \) is a prime congruent to 3 modulo 8 by some
cubic hypersurface invariant under \( \text{PSL}(2, p) \). However, their method can
not be valid for \( p = 13 \). As a consequence of Theorem 1.1, we find an explicit
construction of the modular curve \( X(13) \). Let

\[
\phi_{12}(z_1, \ldots, z_6) = \Phi_{12}(z_1, \ldots, z_6) - \Phi'_{12}(z_1, \ldots, z_6).
\]
Theorem 1.6 (An explicit construction of the modular curve $X(13)$).

There is a morphism

$$\Phi : X(13) \to C \subset \mathbb{CP}^5$$

with $\Phi(z) = (x_1(z), \ldots, x_6(z))$, where $C$ is an algebraic curve given by a family of $G$-invariant equations

$$\begin{align*}
\Phi_4(z_1, \ldots, z_6) &= 0, \\
\Phi_8(z_1, \ldots, z_6) &= 0, \\
\phi_{12}(z_1, \ldots, z_6) &= 0, \\
\Phi_{16}(z_1, \ldots, z_6) &= 0,
\end{align*}$$

(1.15)

where $\phi_{12}$ is given as in (1.14).

This paper consists of five sections. In section two, we revisit the standard construction of the $E_8$ root lattice by means of the icosahedron. This includes the $E_8$-singularity, the Barlow surface, Bring’s curve and Fricke’s octavic curve. In section three, we explain the invariant theory for $\text{PSL}(2, 13)$. In particular, we construct the senary quadratic forms $A_j$ ($0 \leq j \leq 6$), the senary cubic forms $D_j$ ($j = 0, 1, \ldots, 12, \infty$) and the senary sextic forms $G_j$ ($0 \leq j \leq 12$). From $A_j$ we construct the Jacobian equation of degree fourteen. From $D_j$ and $G_j$ we construct another equation of degree fourteen. Combining Jacobian equation with that equation, we obtain a family of polynomials which are invariant under the action of $\text{PSL}(2, 13)$. Together with theta constants of order thirteen, this gives the modular parametrization of these invariant polynomials. Therefore, we obtain Theorem 1.1. In section four, we give three constructions of the $E_8$ root lattice, and prove Theorem 1.2, Theorem 1.3 and Theorem 1.4. In section five, we give an explicit construction of the modular curve $X(13)$.

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2. Standard construction: from the icosahedron to $E_8$

2.1. $E_8$-singularity: from the icosahedron to $E_8$

Let us recall some classical result on the relation between the icosahedron and the $E_8$-singularity (see [31]). Starting with the polynomial invariants of the finite subgroup of $\text{SL}(2, \mathbb{C})$, a surface is defined from the single syzygy
which relates the three polynomials in two variables. This surface has a
singularity at the origin; the singularity can be resolved by constructing a
smooth surface which is isomorphic to the original one except for a set of
component curves which form the pre-image of the origin. The components
form a Dynkin curve and the matrix of their intersections is the negative of
the Cartan matrix for the appropriate Lie algebra. The Dynkin curve is the
dual of the Dynkin graph. For example, if $\Gamma$ is the binary icosahedral group,
the corresponding Dynkin curve is that of $E_8$, and $C^2/\Gamma \subset C^3$ is the set of
zeros of the equation
\[ x^2 + y^3 + z^5 = 0. \] (2.1)
The link of this $E_8$-singularity, the Poincaré homology 3-sphere (see [22]),
has a higher dimensional lifting:
\[ z_1^5 + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0, \quad \sum_{i=1}^5 z_i\bar{z}_i = 1, \quad z_i \in \mathbb{C} \quad (1 \leq i \leq 5), \] (2.2)
which is the Brieskorn description of one of Milnor’s exotic 7-dimensional
spheres. In fact, it is an exotic 7-sphere representing Milnor’s standard gen-
erator of $\Theta_7$ (see [3], [4] and [20]).
In his celebrated book [23], Klein gave a parametric solution of the above
singularity (2.1) by homogeneous polynomials $T$, $H$, $f$ in two variables of
degrees 30, 20, 12 with integral coefficients, where
\[ f = z_1 z_2 (z_1^{10} + 11 z_1^5 z_2^5 - z_2^{10}), \]
\[ H = \frac{1}{121} \left| \begin{array}{cc} \frac{\partial^2 f}{\partial z_1^2} & \frac{\partial^2 f}{\partial z_1 \partial z_2} \\ \frac{\partial^2 f}{\partial z_2 \partial z_1} & \frac{\partial^2 f}{\partial z_2^2} \end{array} \right| = -(z_1^{20} + z_2^{20}) + 228(z_1^{15} z_2^5 - z_1^5 z_2^{15}) - 494 z_1^{10} z_2^{10}, \]
\[ T = -\frac{1}{20} \left| \begin{array}{cc} \frac{\partial f}{\partial z_1} \frac{\partial f}{\partial H} \\ \frac{\partial f}{\partial z_2} \frac{\partial f}{\partial H} \end{array} \right| = (z_1^{30} + z_2^{30}) + 522(z_1^{25} z_2^5 - z_1^5 z_2^{25}) - 10005(z_1^{20} z_2^{10} + z_1^{10} z_2^{20}). \]
They satisfy the famous (binary) icosahedral equation
\[ T^2 + H^3 = 1728 f^5. \] (2.3)
In fact, $f$, $H$ and $T$ are invariant polynomials under the action of the binary
icosahedral group. The above equation (2.3) is closely related to Hermite’s
celebrated work (see [19], pp.5-12) on the resolution of the quintic equations.
Essentially the same relation had been found a few years earlier by Schwarz
(see [33]), who considered three polynomials \( \varphi_{12}, \varphi_{20} \) and \( \varphi_{30} \) whose roots correspond to the vertices, the midpoints of the faces and the midpoints of the edges of an icosahedron inscribed in the Riemann sphere. He obtained the identity \( \varphi_{30}^3 - 1728 \varphi_{12}^5 = \varphi_{30}^2 \). Thus we see that from the very beginning there was a close relation between the \( E_8 \)-singularity and the icosahedron. Moreover, the icosahedral equation (2.3) can be interpreted in terms of modular forms which was also known by Klein (see [26], p. 631). Let \( x_1(z) = \eta(z) a(z) \) and \( x_2(z) = \eta(z) b(z) \), where

\[
a(z) = e^{- \frac{4 \pi i}{w} } \theta \left[ \frac{3}{1} \right] (0, 5z), \quad b(z) = e^{- \frac{4 \pi i}{w} } \theta \left[ \frac{1}{1} \right] (0, 5z)
\]

are theta constants of order five and \( \eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \) with \( q = e^{2 \pi i z} \) is the Dedekind eta function which are all defined in the upper-half plane \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \). Then

\[
\begin{cases}
f(x_1(z), x_2(z)) = - \Delta(z), \\
H(x_1(z), x_2(z)) = - \eta(z)^8 \Delta(z) E_4(z), \\
T(x_1(z), x_2(z)) = \Delta(z)^2 E_6(z),
\end{cases}
\]

where

\[
E_4(z) := \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (m,n)=1} \frac{1}{(mz+n)^4}, \quad E_6(z) := \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (m,n)=1} \frac{1}{(mz+n)^6}
\]

are Eisenstein series of weight 4 and 6, and \( \Delta(z) = \eta(z)^{24} \) is the discriminant. The relations

\[
j(z) := \frac{E_4(z)^3}{\Delta(z)} = \frac{H(x_1(z), x_2(z))^3}{f(x_1(z), x_2(z))^5},
\]

\[
j(z) - 1728 = \frac{E_6(z)^2}{\Delta(z)} = - \frac{T(x_1(z), x_2(z))^2}{f(x_1(z), x_2(z))^5}
\]

give the icosahedral equation (2.3) in terms of theta constants of order five.

2.2. The Barlow surface: from the icosahedron to \( E_8 \)

In [28], Kotschick showed that the manifold \( \mathbb{CP}^2 \# 8 \mathbb{CP}^2 \) has two distinct differentiable structures, both of which come from algebraic surfaces. The surfaces are the eight-fold blow-up of the projective plane carrying the standard smooth structure and the Barlow surface (see [2]), which is a simply-connected minimal surface of general type with \( q = p_g = 0 \) and \( K^2 = 1 \).
Kotschick proved that the Barlow surface $B$ is homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$.

The Barlow surface $B$ (see [2]) is obtained as the minimal desingularization of $Y/D_{10}$, where $Y$ is a certain Hilbert modular surface (see [15]) and $D_{10}$ acts with finite fixed locus. Let $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ be the upper half-plane. Denote by $K$ the real quadratic field $\mathbb{Q}(\sqrt{21})$ and by $\mathcal{O}_K$ its ring of integers. The Hilbert modular group $\text{SL}(2, \mathcal{O}_K)$ acts on $\mathbb{H} \times \mathbb{H}$ by

$$
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \circ (z_1, z_2) = \begin{pmatrix}
\alpha z_1 + \beta' z_2 + \beta' \\
\gamma z_1 + \delta'
\end{pmatrix},
$$

where $x \mapsto x'$ denotes conjugation over $\mathbb{Q}$ in $K$. Consider the 2-congruence subgroup $\Gamma \subset \text{SL}(2, \mathcal{O}_K)$, where

$$
\Gamma = \left\{ \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \in \text{SL}(2, \mathcal{O}_K) : \alpha \equiv \delta \equiv 1, \beta \equiv \gamma \equiv 0 (\text{mod } 2) \right\}
$$

is the principal congruence subgroup of $\text{SL}(2, \mathcal{O}_K)$ for the prime ideal generated by 2. The surface $(\mathbb{H} \times \mathbb{H})/\text{SL}(2, \mathcal{O}_K)$ is the quotient of $(\mathbb{H} \times \mathbb{H})/\Gamma$ by the group

$$
\text{SL}(2, \mathcal{O}_K)/\Gamma \cong \text{SL}(2, \mathcal{O}_K/2\mathcal{O}_K) \cong \text{SL}(2, \mathbb{F}_4) \cong A_5.
$$

The surface $Y$ is the minimal desingularisation of the resolution of the compactification of $(\mathbb{H} \times \mathbb{H})/\Gamma$. It is a simply-connected surface of general type with $p_g = 4$ and $K^2 = 10$. It can be proved (see [15]) that the canonical map of $Y$ is 2 : 1 onto the 20-nodal quintic $Q \subset \mathbb{CP}^4$ given by

$$
\sum_{i=0}^{4} z_i = 0, \quad \sum_{i=0}^{4} z_i^5 - \frac{5}{4} \sum_{i=0}^{4} z_i^2 \cdot \sum_{i=0}^{4} z_i^3 = 0.
$$

(2.5)

The icosahedral group $A_5$ acts on $\mathbb{CP}^4$ by the standard action on the coordinates. The quintic $Q$ is $A_5$-invariant and its 20 nodes are the $A_5$-orbit of the point $(2, 2, 2, -3 - \sqrt{-7}, -3 + \sqrt{-7})$. The $A_5$-action on $Q$ is covered by an action on $Y$, so that we have an action of $A_5 \times \mathbb{Z}/2\mathbb{Z} = A_5 \cup A_5\sigma$ on $Y$, where the generator $\sigma \in \mathbb{Z}/2\mathbb{Z}$ is the covering involution. Elements of $A_5 \times \mathbb{Z}/2\mathbb{Z}$ acting on $Y$ are denoted like the corresponding elements acting on $Q$. Let $\Phi : Y \to Q$ be the quotient map. The main result from [2] can be summarized as follows:
Proposition 2.1 (see [2] and [28]). Let $\alpha = (02)(34)\sigma$, $\beta = (01234)$. Then $\beta$ acts freely on $Y$ and $\alpha$ has 4 fixed points. The resolution of the nodes of $Y/D_{10}$, where $D_{10} = \langle \alpha, \beta \rangle$, gives a minimal surface $B$ of general type with $\pi_1 = 0$, $q = p_g = 0$ and $K^2 = 1$.

2.3. Bring’s curve and Fricke’s octavic curve: from the icosahedron to $E_8$

Both Bring’s curve and Fricke’s octavic curve arise from the resolution of the equation of the fifth degree (see [23]). Hence, they are intimately connected with the icosahedron.

Let us recall some basic facts from classical enumerative geometry (see [18]). We will use the notation that $[n|a_1, a_2, \ldots, a_k]$ means the (not necessarily complete) intersection of $k$ polynomials of degrees $a_1, \ldots, a_k$ respectively in $\mathbb{P}^n$. In 1863, Clebsch found that the canonical sextic curve of genus four has exactly 120 tritangent planes (i.e., planes which are tangent to the curve at precisely three points). This curve can be realized as $[4|1, 2, 3]$, i.e., the intersection of a hyperplane, a quadric and a cubic in Fermat form in the five homogeneous coordinates of $\mathbb{P}^4$, giving us the so-called Bring’s curve (see [9] and [10]).

Specially, Bring’s curve can be realized as the Fermat cubic, sliced by the Fermat quadric, and then the line, in the homogeneous coordinates of $\mathbb{P}^4$:

$$B = \left\{ \sum_{i=0}^{4} x_i^3 = \sum_{i=0}^{4} x_i^2 = \sum_{i=0}^{4} x_i = 0 \right\} \subset \mathbb{P}^4. \quad (2.6)$$

This classic result is well-understood in terms of del Pezzo surfaces of degree one (see [20]). The canonical model of a del Pezzo surface of degree one is the double cover of a quadratic cone, branched over a canonical space curve of genus 4 and degree 6 given by the complete intersection of the cone with a unique cubic surface. The 240 lines on the del Pezzo surface arise in pairs from the 120 tritangent planes to the canonical curve, which can be identified with its odd theta characteristics.

The automorphism group of the 240 lines is the Weyl group $W(E_8)$ of the root system of type $E_8$. Its order is $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$. The automorphism group of the 120 tritangent planes is the quotient by the normal subgroup $\{\pm 1\}$ with order $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$, which is just the group $G_{8,2}$ studied by Coble in p. 356 of his paper [7]. In fact, Coble proved that $G_{8,2}$ is isomorphic with the group of the tritangent planes of a space sextic of genus 4 on a quadric cone (see [7], p. 359).
Note that Bring’s curve has a natural modular interpretation (see [8], p. 500). Namely, it is isomorphic to the modular curve $\mathbb{H}/\Gamma$, where $\Gamma = \Gamma_0(2) \cap \Gamma(5)$. It is also realized as the curve of fixed points of the Bertini involution on the del Pezzo surface of degree one obtained from the elliptic modular surface $S(5)$ of level 5 by blowing down the zero section.

Moreover, there is even a correspondence between the above classical enumerative geometrical problem and the Monster simple group $\mathbb{M}$ due to [18], observation 1: for the Monster $\mathbb{M}$, we have the following sums for the cusp numbers $C_g$ over the 172 rational conjugacy classes:

$$\sum_g C_g = 360 = 3 \cdot 120, \quad \sum_g C_g^2 = 1024 = 2^{10}.$$ 

The 360 is thrice 120, which is the number of tritangent planes to Bring’s curve. Furthermore, in analogy to Bring’s sextic curve, there is the octavic of Fricke of genus nine (see [11] and [13]), the Fermat $[4|1, 2, 4]$ defined as

$$F = \left\{ \sum_{i=0}^4 x_i^4 = \sum_{i=0}^4 x_i^2 = \sum_{i=0}^4 x_i = 0 \right\} \subset \mathbb{P}^4. \quad (2.7)$$

The number of tritangent planes on $F$ is precisely $2048 = 2 \cdot 1024$, twice the sum of square of the cusps.

3. Modular curve $X(13)$ and invariant theory for $\text{PSL}(2, 13)$

At first, we will study the six-dimensional representation of the finite simple group $\text{PSL}(2, 13)$ of order 1092, which acts on the five-dimensional projective space $\mathbb{P}^5 = \{(z_1, z_2, z_3, z_4, z_5, z_6) : z_i \in \mathbb{C} \ (i = 1, 2, 3, 4, 5, 6)\}$. This representation is defined over the cyclotomic field $\mathbb{Q}(e^{2\pi i/13})$. Put

$$S = -\frac{1}{\sqrt{13}} \begin{pmatrix} \zeta^{12} - \zeta & \zeta^{10} - \zeta^3 & \zeta^4 - \zeta^9 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 \\ \zeta^{10} - \zeta^3 & \zeta^4 - \zeta^9 & \zeta^{12} - \zeta & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 \\ \zeta^4 - \zeta^9 & \zeta^{12} - \zeta & \zeta^{10} - \zeta^3 & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} \\ \zeta^5 - \zeta^8 & \zeta^{2} - \zeta^{11} & \zeta^{6} - \zeta^7 & \zeta^{12} - \zeta & \zeta^{10} & \zeta^9 - \zeta^4 \\ \zeta^2 - \zeta^{11} & \zeta^{6} - \zeta^7 & \zeta^{5} - \zeta^8 & \zeta^3 - \zeta^{10} & \zeta^9 - \zeta^4 & \zeta - \zeta^{12} \\ \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^9 - \zeta^4 & \zeta - \zeta^{12} & \zeta^{3} - \zeta^{10} \end{pmatrix} \quad (3.1)$$
and
\[ T = \text{diag}(\zeta^7, \zeta^{11}, \zeta^8, \zeta^6, \zeta^2, \zeta^5), \tag{3.2} \]
where \( \zeta = \exp(2\pi i/13) \). We have
\[ S^2 = T^{13} = (ST)^3 = 1. \tag{3.3} \]

Let \( G = \langle S, T \rangle \), then \( G \cong \text{PSL}(2, 13) \) (see [34], Theorem 3.1).

Put \( \theta_1 = \zeta + \zeta^3 + \zeta^9, \theta_2 = \zeta^2 + \zeta^6 + \zeta^5, \theta_3 = \zeta^4 + \zeta^{12} + \zeta^{10}, \) and \( \theta_4 = \zeta^8 + \zeta^{11} + \zeta^7 \). We find that
\[
\begin{align*}
\theta_1 + \theta_2 + \theta_3 + \theta_4 &= -1, \\
\theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4 &= 2, \\
\theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4 &= 4, \\
\theta_1\theta_2\theta_3\theta_4 &= 3.
\end{align*}
\]

Hence, \( \theta_1, \theta_2, \theta_3 \) and \( \theta_4 \) satisfy the quartic equation \( z^4 + z^3 + 2z^2 - 4z + 3 = 0 \), which can be decomposed as two quadratic equations
\[
\left( z^2 + \frac{1 + \sqrt{13}}{2} z + \frac{5 + \sqrt{13}}{2} \right) \left( z^2 + \frac{1 - \sqrt{13}}{2} z + \frac{5 - \sqrt{13}}{2} \right) = 0
\]
over the real quadratic field \( \mathbb{Q}(\sqrt{13}) \). Therefore, the four roots are given as follows:
\[
\begin{align*}
\theta_1 &= \frac{1}{4} \left( -1 + \sqrt{13} + \sqrt{-26 + 6\sqrt{13}} \right), \\
\theta_2 &= \frac{1}{4} \left( -1 - \sqrt{13} + \sqrt{-26 - 6\sqrt{13}} \right), \\
\theta_3 &= \frac{1}{4} \left( -1 + \sqrt{13} - \sqrt{-26 + 6\sqrt{13}} \right), \\
\theta_4 &= \frac{1}{4} \left( -1 - \sqrt{13} - \sqrt{-26 - 6\sqrt{13}} \right).
\end{align*}
\]

Moreover, we find that
\[
\begin{align*}
\theta_1 + \theta_3 + \theta_2 + \theta_4 &= -1, \\
\theta_1 + \theta_3 - \theta_2 - \theta_4 &= \sqrt{13}, \\
\theta_1 - \theta_3 - \theta_2 + \theta_4 &= -\sqrt{13} + 2\sqrt{13}, \\
\theta_1 - \theta_3 + \theta_2 - \theta_4 &= \sqrt{13} - 2\sqrt{13}.
\end{align*}
\]
Let us study the action of $ST^\nu$ on $\mathbb{P}^5$, where $\nu = 0, 1, \ldots, 12$. Put
\[
\alpha = \zeta + \zeta^{12} - \zeta^5 - \zeta^8, \quad \beta = \zeta^3 + \zeta^{10} - \zeta^2 - \zeta^{11}, \quad \gamma = \zeta^9 + \zeta^4 - \zeta^6 - \zeta^7.
\]
We find that
\[
13ST^\nu(z_1) \cdot ST^\nu(z_4) = \beta z_1 z_4 + \gamma z_2 z_5 + \alpha z_3 z_6 + \gamma \zeta^2 + \alpha \zeta^{9\nu} z_2^2 + \beta \zeta^{3\nu} z_3^2 - \gamma \zeta^{12\nu} z_4^2 - \alpha \zeta^{4\nu} z_5^2 - \beta \zeta^{10\nu} z_6^2 + (\alpha - \beta) \zeta^{5\nu} z_1 z_2 + (\beta - \gamma) \zeta^{6\nu} z_2 z_3 + (\gamma - \alpha) \zeta^{2\nu} z_1 z_3 + (\beta - \alpha) \zeta^{8\nu} z_4 z_5 + (\gamma - \beta) \zeta^{7\nu} z_5 z_6 + (\alpha - \gamma) \zeta^{11\nu} z_4 z_6 - (\alpha + \beta) \zeta^{12\nu} z_1 z_6 - (\beta + \gamma) \zeta^{9\nu} z_1 z_5 - (\gamma + \alpha) \zeta^{3\nu} z_2 z_6 + (\alpha + \beta) \zeta^{12\nu} z_1 z_6 - (\beta + \gamma) \zeta^{4\nu} z_2 z_4 - (\gamma + \alpha) \zeta^{10\nu} z_3 z_5.
\]
\[
13ST^\nu(z_2) \cdot ST^\nu(z_5) = \gamma z_1 z_4 + \alpha z_2 z_5 + \beta z_3 z_6 + \alpha \zeta^2 + \alpha \zeta^{9\nu} z_2^2 + \alpha \zeta^{3\nu} z_3^2 - \alpha \zeta^{12\nu} z_4^2 - \beta \zeta^{4\nu} z_5^2 - \gamma \zeta^{10\nu} z_6^2 + (\beta - \gamma) \zeta^{5\nu} z_1 z_2 + (\alpha - \beta) \zeta^{6\nu} z_2 z_3 + (\gamma - \alpha) \zeta^{2\nu} z_1 z_3 + (\gamma - \beta) \zeta^{7\nu} z_4 z_5 + (\gamma - \alpha) \zeta^{8\nu} z_5 z_6 + (\alpha - \gamma) \zeta^{11\nu} z_4 z_6 + (\beta + \gamma) \zeta^{9\nu} z_1 z_5 - (\alpha + \beta) \zeta^{3\nu} z_2 z_6 + (\beta + \gamma) \zeta^{12\nu} z_1 z_6 - (\gamma + \alpha) \zeta^{4\nu} z_2 z_4 - (\alpha + \beta) \zeta^{10\nu} z_3 z_5.
\]
\[
13ST^\nu(z_3) \cdot ST^\nu(z_6) = \alpha z_1 z_4 + \beta z_2 z_5 + \gamma z_3 z_6 + \beta \zeta^2 + \gamma \zeta^{9\nu} z_2^2 + \gamma \zeta^{3\nu} z_3^2 - \gamma \zeta^{12\nu} z_4^2 - \beta \zeta^{4\nu} z_5^2 - \alpha \zeta^{10\nu} z_6^2 + (\gamma - \alpha) \zeta^{5\nu} z_1 z_2 + (\beta - \gamma) \zeta^{6\nu} z_2 z_3 + (\alpha - \beta) \zeta^{2\nu} z_1 z_3 + (\gamma - \beta) \zeta^{7\nu} z_4 z_5 + (\beta - \gamma) \zeta^{8\nu} z_5 z_6 + (\alpha - \beta) \zeta^{11\nu} z_4 z_6 + (\gamma + \alpha) \zeta^{9\nu} z_1 z_5 - (\beta + \gamma) \zeta^{3\nu} z_2 z_6 + (\gamma + \alpha) \zeta^{12\nu} z_1 z_6 - (\alpha + \beta) \zeta^{4\nu} z_2 z_4 - (\beta + \gamma) \zeta^{10\nu} z_3 z_5.
\]

Note that $\alpha + \beta + \gamma = \sqrt{13}$, we find that
\[
\sqrt{13} \left[ ST^\nu(z_1) \cdot ST^\nu(z_4) + ST^\nu(z_2) \cdot ST^\nu(z_5) + ST^\nu(z_3) \cdot ST^\nu(z_6) \right] = (z_1 z_4 + z_2 z_5 + z_3 z_6) + (\zeta^2 + \zeta^{9\nu} z_2^2 + \zeta^{3\nu} z_3^2 - (\zeta^{12\nu} z_4^2 + \zeta^{4\nu} z_5^2 + \zeta^{10\nu} z_6^2) + - 2(\zeta^{9\nu} z_1 z_5 + \zeta^{3\nu} z_2 z_6) - 2(\zeta^{12\nu} z_1 z_6 + \zeta^{4\nu} z_2 z_4 + \zeta^{10\nu} z_3 z_5). \]

15
Let 
\[ \varphi_\infty(z_1, z_2, z_3, z_4, z_5, z_6) = \sqrt{13}(z_1z_4 + z_2z_5 + z_3z_6) \] 
(3.4)
and 
\[ \varphi_\nu(z_1, z_2, z_3, z_4, z_5, z_6) = \varphi_\infty(ST^\nu(z_1, z_2, z_3, z_4, z_5, z_6)) \] 
(3.5)
for \( \nu = 0, 1, \ldots, 12 \). Then 
\[ \varphi_\nu = (z_1z_4 + z_2z_5 + z_3z_6) + \zeta^\nu(z_1^2 - 2z_3z_4) + \zeta^{4\nu}(-z_5^2 - 2z_2z_4) + \zeta^{9\nu}(z_2^2 - 2z_1z_5) + \zeta^{3\nu}(z_3^2 - 2z_2z_6) + \zeta^{12\nu}(-z_4^2 - 2z_1z_6) + \zeta^{10\nu}(-z_6^2 - 2z_3z_5). \] 
(3.6)
This leads us to define the following senary quadratic forms (quadratic forms in six variables):
\[
\begin{align*}
A_0 &= z_1z_4 + z_2z_5 + z_3z_6, \\
A_1 &= z_1^2 - 2z_3z_4, \\
A_2 &= -z_5^2 - 2z_2z_4, \\
A_3 &= z_2^2 - 2z_1z_5, \\
A_4 &= z_3^2 - 2z_2z_6, \\
A_5 &= -z_4^2 - 2z_1z_6, \\
A_6 &= -z_6^2 - 2z_3z_5.
\end{align*}
\] 
(3.7)
Hence,
\[ \sqrt{13}ST^\nu(A_0) = A_0 + \zeta^\nu A_1 + \zeta^{4\nu} A_2 + \zeta^{9\nu} A_3 + \zeta^{3\nu} A_4 + \zeta^{12\nu} A_5 + \zeta^{10\nu} A_6. \] 
(3.8)
Let \( H := Q^5P^2 \cdot P^2Q^6P^8 \cdot Q^5P^2 \cdot P^3Q \) where \( P = ST^{-1}S \) and \( Q = ST^3 \).
Then (see [35], p.27)
\[ H = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \] 
(3.9)
Note that \( H^6 = 1 \) and \( H^{-1}TH = -T^4 \). Thus, \( \langle H, T \rangle \cong \mathbb{Z}_{13} \times \mathbb{Z}_6 \). Hence, it is a maximal subgroup of order 78 of \( G \) with index 14. We find that \( \varphi_\infty^2 \) is invariant under the action of the maximal subgroup \( \langle H, T \rangle \). Note that 
\[ \varphi_\infty = \sqrt{13}A_0, \quad \varphi_\nu = A_0 + \zeta^\nu A_1 + \zeta^{4\nu} A_2 + \zeta^{9\nu} A_3 + \zeta^{3\nu} A_4 + \zeta^{12\nu} A_5 + \zeta^{10\nu} A_6 \]
for \( \nu = 0, 1, \ldots, 12 \). Let \( w = \varphi^2 \), \( w_\infty = \varphi_\infty^2 \) and \( w_\nu = \varphi_\nu^2 \). Then \( w_\infty, w_\nu \) for \( \nu = 0, \ldots, 12 \) form an algebraic equation of degree fourteen, which is just the Jacobian equation of degree fourteen (see [23], pp.161-162), whose roots are these \( w_\nu \) and \( w_\infty \):

\[
 w^{14} + a_1 w^{13} + \cdots + a_{13} w + a_{14} = 0.
\]

On the other hand, we have

\[
 -13 \sqrt{13ST'(z_1) \cdot ST'(z_2) \cdot ST'(z_3)} = -r_4(\z_1^{8}z_1^3 + \z_1^{7}z_2^3 + \z_1^{11}z_3^3) - r_2(\z_4^5z_4^3 + \z_5^6z_5^3 + \z_6^2z_6^3) - r_3(\z_1^{12}z_1^3z_2^3 + \z_2^4z_2^3z_3^3 + \z_3^{10}z_3^2z_1^3) - r_1(\z_4^{13}z_4^2z_5^2 + \z_5^{10}z_5^2z_6^2 + \z_6^{13}z_6^2z_4^2) \\
 + 2r_1(\z_4^{13}z_4^2z_5^2 + \z_5^{10}z_5^2z_6^2 + \z_6^{13}z_6^2z_4^2) - 2r_2(\z_6^{10}z_6^2z_5^2 + \z_5^{13}z_5^2z_6^2) \\
 + 2r_1(\z_4^{13}z_4^2z_5^2 + \z_5^{10}z_5^2z_6^2 + \z_6^{13}z_6^2z_4^2) + \\
 + r_0(\z_1^{11}z_1^{12}z_2^2z_3^2 + \z_2^{10}z_2^{11}z_3^2z_4^2 + \z_3^{13}z_3^{11}z_4^2z_5^2) + \\
 + r_0(\z_1^{11}z_1^{12}z_2^2z_3^2 + \z_2^{10}z_2^{11}z_3^2z_4^2 + \z_3^{13}z_3^{11}z_4^2z_5^2) +
\]

where

\[
 r_0 = 2(\theta_1 - \theta_3) - 3(\theta_2 - \theta_4), \quad r_\infty = 2(\theta_4 - \theta_2) - 3(\theta_1 - \theta_3),
\]

\[
 r_1 = \sqrt{-13 - 2\sqrt{13}}, \quad r_2 = \sqrt{-13 + 3\sqrt{13}/2}, \quad r_3 = \sqrt{-13 + 2\sqrt{13}}, \quad r_4 = \sqrt{-13 - 3\sqrt{13}/2}.
\]

This leads us to define the following senary cubic forms (cubic forms in six
variables):

\[
\begin{align*}
\nu & = 1, \quad \delta
\end{align*}
\]

\[
\begin{align*}
D_0 &= z_1 z_2 z_3, \\
D_1 &= 2 z_2 z_3^2 + z_2^2 z_6 - z_1^2 z_5 + z_1 z_5 z_6, \\
D_2 &= -z_6^2 + z_2 z_4 - 2 z_2 z_5 + z_1 z_4 z_5 + 3 z_3 z_5 z_6, \\
D_3 &= 2 z_1 z_2^2 + z_1 z_5 - z_4 z_6^2 + z_3 z_4 z_5, \\
D_4 &= -z_2^2 z_3 + z_1 z_6^2 - 2 z_1 z_6 - z_1 z_3 z_5, \\
D_5 &= -z_4^2 + z_3^2 z_5 - 2 z_3 z_6^2 + z_2 z_5 z_6 + 3 z_1 z_4 z_5, \\
D_6 &= -z_3^2 + z_1^2 z_6 - 2 z_1 z_4^2 + z_3 z_4 z_6 + 3 z_2 z_4 z_5, \\
D_7 &= -z_2^2 + z_3 z_4^2 - z_1 z_3 z_6 - 3 z_1 z_2 z_5 + 2 z_1^2 z_4, \\
D_8 &= -z_1^2 + z_2 z_6^2 - z_2 z_3 z_5 - 3 z_1 z_3 z_4 + 2 z_3 z_6, \\
D_9 &= 2 z_1 z_3^2 + z_3 z_4 - z_5 z_6 + z_2 z_4 z_6, \\
D_{10} &= -z_1 z_3^2 + z_2 z_4^2 - 2 z_4 z_6^2 - z_1 z_2 z_6, \\
D_{11} &= -z_3^2 + z_1 z_6^2 - z_1 z_2 z_4 - 3 z_2 z_3 z_6 + 2 z_2 z_5, \\
D_{12} &= -z_2 z_3^2 + z_3 z_5^2 - 2 z_5 z_6^2 - z_2 z_3 z_4, \\
D_\infty &= z_4 z_5 z_6. \\
\end{align*}
\]

Then

\[
-13 \sqrt{13} ST' \nu (D_0) = r_0 D_0 + r_1 \zeta' D_1 + r_2 \zeta' D_2 + r_1 \zeta' D_3 + r_3 \zeta' D_4 + \\
+ r_2 \zeta' D_5 + r_2 \zeta' D_6 + r_4 \zeta' D_7 + r_4 \zeta' D_8 + \\
+ r_1 \zeta' D_9 + r_3 \zeta' D_{10} + r_4 \zeta' D_{11} + r_3 \zeta' D_{12} + r_\infty D_\infty.
\]

\[
-13 \sqrt{13} ST' \nu (D_\infty) = r_0 D_0 - r_3 \zeta' D_1 - r_4 \zeta' D_2 - r_3 \zeta' D_3 + r_1 \zeta' D_4 + \\
- r_4 \zeta' D_5 - r_4 \zeta' D_6 + r_4 \zeta' D_7 + r_4 \zeta' D_8 + \\
- r_3 \zeta' D_9 + r_1 \zeta' D_{10} + r_2 \zeta' D_{11} + r_1 \zeta' D_{12} - r_0 D_\infty.
\]

Let

\[
\delta_\infty (z_1, z_2, z_3, z_4, z_5, z_6) = 13^2 (z_1^2 z_2^2 z_3^2 + z_4^2 z_5^2 z_6^2)
\]

(3.11)

and

\[
\delta_\nu (z_1, z_2, z_3, z_4, z_5, z_6) = \delta_\infty (ST' \nu (z_1, z_2, z_3, z_4, z_5, z_6))
\]

(3.12)

for \( \nu = 0, 1, \ldots, 12 \). Then

\[
\delta_\nu = 13^2 ST' \nu (G_0) = -13 G_0 + \zeta' G_1 + \zeta' G_2 + \cdots + \zeta' G_{12},
\]

(3.13)
where the senary sextic forms (i.e., sextic forms in six variables) are given as follows:

\[
\begin{align*}
G_0 &= D_0^2 + D_\infty^2, \\
G_1 &= -D_7^2 + 2D_0D_1 + 10D_\infty D_1 + 2D_2D_{12} + 2D_3D_{11} - 4D_4D_{10} - 2D_5D_9 - 2D_7D_8, \\
G_2 &= -2D_1^2 - 4D_0D_2 + 6D_\infty D_2 - 2D_4D_{11} + 2D_6D_{10} - 2D_6D_9 - 2D_7D_8, \\
G_3 &= -D_8^2 + 2D_0D_3 + 10D_\infty D_3 + 2D_6D_{10} + 2D_9D_7 - 4D_{12}D_4 - 2D_1D_2, \\
G_4 &= -D_2^2 + 10D_0D_4 - 2D_\infty D_4 + 2D_5D_{12} + 2D_9D_8 - 4D_1D_3 - 2D_{10}D_7, \\
G_5 &= -2D_9^2 - 4D_0D_5 + 6D_\infty D_5 - 2D_{10}D_8 + 2D_9D_{12} - 2D_2D_3 - 2D_{11}D_7, \\
G_6 &= -2D_3^2 - 4D_0D_6 + 6D_\infty D_6 - 2D_{12}D_7 + 2D_3D_4 - 2D_5D_1 - 2D_8D_{11}, \\
G_7 &= -2D_{10}^2 + 6D_0D_7 + 4D_\infty D_7 - 2D_1D_6 + 2D_2D_5 - 2D_8D_{12} - 2D_9D_{11}, \\
G_8 &= -2D_4^2 + 6D_0D_8 + 4D_\infty D_8 - 2D_3D_5 + 2D_6D_2 - 2D_{11}D_{10} - 2D_1D_7, \\
G_9 &= -D_{11}^2 + 2D_0D_9 + 10D_\infty D_9 + 2D_3D_4 + 2D_1D_8 - 4D_{10}D_{12} - 2D_3D_6, \\
G_{10} &= -D_5^2 + 10D_0D_{10} - 2D_\infty D_{10} + 2D_6D_4 + 2D_3D_7 - 4D_9D_1 - 2D_{12}D_{11}, \\
G_{11} &= -2D_{12}^2 + 6D_0D_{11} + 4D_\infty D_{11} - 2D_9D_2 + 2D_3D_6 - 2D_7D_4 - 2D_3D_8, \\
G_{12} &= -D_6^2 + 10D_0D_{12} - 2D_\infty D_{12} + 2D_2D_{10} + 2D_1D_{11} - 4D_3D_9 - 2D_4D_8.
\end{align*}
\] (3.14)

We have that $G_0$ is invariant under the action of $\langle H, T \rangle$, a maximal subgroup of order 78 of $G$ with index 14. Note that $\delta_\infty$, $\delta_\nu$ for $\nu = 0, \ldots, 12$ form an algebraic equation of degree fourteen. However, we have $\delta_\infty + \sum_{\nu=0}^{12} \delta_\nu = 0$. Hence, it is not the Jacobian equation of degree fourteen.

Recall that the theta functions with characteristic $\begin{bmatrix} \epsilon & \epsilon' \end{bmatrix} \in \mathbb{R}^2$ is defined
by the following series which converges uniformly and absolutely on compact subsets of \( \mathbb{C} \times \mathbb{H} \) (see [12], p. 73):

\[
\theta \left[ \frac{\epsilon}{e} \right] (z, \tau) = \sum_{n \in \mathbb{Z}} \exp \left\{ 2\pi i \left[ \frac{1}{2} \left( n + \frac{\epsilon}{2} \right)^2 \tau + \left( n + \frac{\epsilon}{2} \right) \left( z + \frac{\epsilon'}{2} \right) \right] \right\}.
\]

The modified theta constants (see [12], p. 215) \( \varphi_l(\tau) := \theta[\chi_l](0, k\tau) \), where the characteristic \( \chi_l = \left[ \frac{2l+1}{k} \right], \ l = 0, \ldots, \frac{k-3}{2}, \) for odd \( k \) and \( \chi_l = \left[ \frac{2l}{k} \right], \ l = 0, \ldots, \frac{k}{2}, \) for even \( k \). We have the following:

**Proposition 3.1.** (see [12], p. 236). For each odd integer \( k \geq 5 \), the map \( \Phi : \tau \mapsto (\varphi_0(\tau), \varphi_1(\tau), \ldots, \varphi_{k-5}(\tau), \varphi_{k-3}(\tau)) \) from \( \mathbb{H} \cup \mathbb{Q} \cup \{ \infty \} \) to \( \mathbb{C}^{\frac{k-1}{2}} \), defines a holomorphic mapping from \( \mathbb{H}/\Gamma(k) \) into \( \mathbb{C}P^{\frac{k-1}{2}} \).

In our case, the map \( \Phi : \tau \mapsto (\varphi_0(\tau), \varphi_1(\tau), \varphi_2(\tau), \varphi_3(\tau), \varphi_4(\tau), \varphi_5(\tau)) \) gives a holomorphic mapping from the modular curve \( X(13) = \mathbb{H}/\Gamma(13) \) into \( \mathbb{C}P^5 \), which corresponds to our six-dimensional representation, i.e., up to the constants, \( z_1, \ldots, z_6 \) are just modular forms \( \varphi_0(\tau), \ldots, \varphi_5(\tau) \). Let

\[
\begin{align*}
\left\{ 
\begin{array}{l}
    a_1(z) := e^{-\frac{11\pi i}{36}} \theta \left[ \frac{11}{13} \right] (0, 13z) = q^{\frac{11}{13}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2+11n)}, \\
    a_2(z) := e^{-\frac{7\pi i}{36}} \theta \left[ \frac{7}{15} \right] (0, 13z) = q^{\frac{7}{15}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2+7n)}, \\
    a_3(z) := e^{-\frac{5\pi i}{36}} \theta \left[ \frac{5}{13} \right] (0, 13z) = q^{\frac{5}{13}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2+5n)}, \\
    a_4(z) := -e^{-\frac{3\pi i}{36}} \theta \left[ \frac{3}{13} \right] (0, 13z) = -q^{\frac{3}{13}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2+3n)}, \\
    a_5(z) := -e^{-\frac{9\pi i}{36}} \theta \left[ \frac{9}{13} \right] (0, 13z) = -q^{\frac{9}{13}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2+9n)}, \\
    a_6(z) := e^{-\frac{13\pi i}{36}} \theta \left[ \frac{13}{13} \right] (0, 13z) = q^{\frac{13}{13}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2+n)}.
\end{array}
\right.
\end{align*}
\]

be the theta constants of order 13 and

\[
A(z) := (a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))^T.
\]

The significance of our six dimensional representation of PSL(2,13) comes from the following:
Proposition 3.2 (see [35], Proposition 2.5). If \( z \in \mathbb{H} \), then the following relations hold:

\[
A(z + 1) = e^{-\frac{2\pi i}{3}} T A(z), \quad A \left( -\frac{1}{z} \right) = e^{\frac{2\pi i}{3}} \sqrt{z} S A(z),
\]

where \( S \) and \( T \) are given in (3.1) and (3.2), and \( 0 < \arg \sqrt{z} \leq \pi / 2 \).

Recall that the principal congruence subgroup of level 13 is the normal subgroup \( \Gamma(13) \) of \( \Gamma = \text{PSL}(2, \mathbb{Z}) \) defined by the exact sequence \( 1 \rightarrow \Gamma(13) \rightarrow \Gamma(1) \xrightarrow{f} G \rightarrow 1 \) where \( f(\gamma) \equiv \gamma \pmod{13} \) for \( \gamma \in \Gamma = \Gamma(1) \). There is a representation \( \rho : \Gamma \rightarrow \text{PGL}(6, \mathbb{C}) \) with kernel \( \Gamma(13) \) defined as follows: \( \Gamma(1) \) and \( \Gamma(13) \) are given in (3.1) and (3.2), and \( \rho \) should be obvious.

Put \( x_i(z) = \eta(z) a_i(z) \) and \( y_i(z) = \eta^3(z) a_i(z) \) (\( 1 \leq i \leq 6 \)). Let

\[
X(z) = (x_1(z), \ldots, x_6(z))^T \quad \text{and} \quad Y(z) = (y_1(z), \ldots, y_6(z))^T.
\]

Then \( X(z) = \eta(z) A(z) \) and \( Y(z) = \eta^3(z) A(z) \). Recall that \( \eta(z) \) satisfies the following transformation formulas \( \eta(z + 1) = e^{\frac{2\pi i}{3}} \eta(z) \) and \( \eta \left( -\frac{1}{z} \right) = e^{-\frac{2\pi i}{3}} \sqrt{z} \eta(z) \). By Proposition 3.2, we have

\[
X(z + 1) = e^{\frac{2\pi i}{3}} \rho(t) X(z), \quad X \left( -\frac{1}{z} \right) = z \rho(s) X(z),
\]

\[
Y(z + 1) = e^{\frac{2\pi i}{3}} \rho(t) Y(z), \quad Y \left( -\frac{1}{z} \right) = e^{-\frac{2\pi i}{3}} z^2 \rho(s) Y(z).
\]

Define \( j(\gamma, z) := cz + d \) if \( z \in \mathbb{H} \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \). Hence, \( X(\gamma(z)) = u(\gamma) j(\gamma, z) \rho(\gamma) X(z) \) and \( Y(\gamma(z)) = v(\gamma) j(\gamma, z)^2 \rho(\gamma) Y(z) \) for \( \gamma \in \Gamma(1) \), where
\( u(\gamma) = 1, \omega \text{ or } \omega^2 \) with \( \omega = e^{2\pi i} \) and \( v(\gamma) = \pm 1 \) or \( \pm i \). Since \( \Gamma(13) = \ker \rho \), we have \( X(\gamma(z)) = u(\gamma)j(\gamma, z)X(z) \) and \( Y(\gamma(z)) = v(\gamma)j(\gamma, z)^2Y(z) \) for \( \gamma \in \Gamma(13) \). This means that the functions \( x_1(z), \ldots, x_6(z) \) are modular forms of weight one for \( \Gamma(13) \) with the same multiplier \( u(\gamma) = 1, \omega \text{ or } \omega^2 \) and \( y_1(z), \ldots, y_6(z) \) are modular forms of weight two for \( \Gamma(13) \) with the same multiplier \( v(\gamma) = \pm 1 \) or \( \pm i \).

From now on, we will use the following abbreviation

\[
A_j = A_j(a_1(z), \ldots, a_6(z)) \quad (0 \leq j \leq 6),
\]

\[
D_j = D_j(a_1(z), \ldots, a_6(z)) \quad (j = 0, 1, \ldots, 12, \infty)
\]

and

\[
G_j = G_j(a_1(z), \ldots, a_6(z)) \quad (0 \leq j \leq 12).
\]

We have

\[
\begin{align*}
A_0 &= q^{11}(1 + O(q)), \\
A_1 &= q^{37}(2 + O(q)), \\
A_2 &= q^{27}(2 + O(q)), \\
A_3 &= q^{47}(1 + O(q)), \\
A_4 &= q^{35}(-1 + O(q)), \\
A_5 &= q^{87}(-1 + O(q)), \\
A_6 &= q^{17}(-1 + O(q)),
\end{align*}
\]

and

\[
\begin{align*}
D_0 &= q^{15}(1 + O(q)), \\
D_\infty &= q^2(-1 + O(q)), \\
D_1 &= q^{39}(2 + O(q)), \\
D_2 &= q^{35}(-1 + O(q)), \\
D_3 &= q^{11}(1 + O(q)), \\
D_4 &= q^{19}(-2 + O(q)), \\
D_5 &= q^{47}(-1 + O(q)),
\end{align*}
\]

\[
\begin{align*}
D_6 &= q^{35}(-1 + O(q)), \\
D_7 &= q^{37}(1 + O(q)), \\
D_8 &= q^{51}(3 + O(q)), \\
D_9 &= q^{57}(-2 + O(q)), \\
D_{10} &= q^{71}(1 + O(q)), \\
D_{11} &= q^{77}(-4 + O(q)), \\
D_{12} &= q^{87}(-1 + O(q)).
\end{align*}
\]
Hence,

\[
\begin{align*}
G_0 &= q_7^2 (1 + O(q)), \\
G_1 &= q_{32}^4 (13 + O(q)), \\
G_2 &= q_{32}^4 (-22 + O(q)), \\
G_3 &= q_{32}^4 (21 + O(q)), \\
G_4 &= q_{32}^4 (-1 + O(q)), \\
G_5 &= q_{32}^4 (2 + O(q)), \\
G_6 &= q_{32}^4 (2 + O(q)), \\
G_7 &= q_{32}^4 (-2 + O(q)), \\
G_8 &= q_{32}^4 (-8 + O(q)), \\
G_9 &= q_{32}^4 (6 + O(q)), \\
G_{10} &= q_{32}^4 (1 + O(q)), \\
G_{11} &= q_{32}^4 (-8 + O(q)), \\
G_{12} &= q_{32}^4 (17 + O(q)).
\end{align*}
\]

Note that

\[
\begin{align*}
w_\nu &= (A_0 + \zeta^{4\nu} A_2 + \zeta^{9\nu} A_3 + \zeta^{3\nu} A_4 + \zeta^{12\nu} A_5 + \zeta^{10\nu} A_6)^2 \\
&= A_0^2 + 2 (A_1 A_5 + A_2 A_3 + A_4 A_6) + \\
&+ 2 \zeta^{4\nu} (A_0 A_1 + A_2 A_6) + 2 \zeta^{3\nu} (A_0 A_4 + A_2 A_5) + \\
&+ 2 \zeta^{9\nu} (A_0 A_3 + A_5 A_6) + 2 \zeta^{12\nu} (A_0 A_5 + A_3 A_4) + \\
&+ 2 \zeta^{10\nu} (A_0 A_6 + A_1 A_3) + 2 \zeta^{4\nu} (A_0 A_2 + A_1 A_4) + \\
&+ \zeta^{2\nu} (A_1^2 + 2 A_4 A_5) + \zeta^{5\nu} (A_3^2 + 2 A_1 A_2) + \\
&+ \zeta^{6\nu} (A_4^2 + 2 A_3 A_6) + \zeta^{11\nu} (A_5^2 + 2 A_1 A_6) + \\
&+ \zeta^{8\nu} (A_2^2 + 2 A_3 A_5) + \zeta^{7\nu} (A_6^2 + 2 A_4 A_2),
\end{align*}
\]

where

\[
\begin{align*}
A_0^2 + 2 (A_1 A_5 + A_2 A_3 + A_4 A_6) &= q_7^2 (-1 + O(q)), \\
A_0 A_1 + A_2 A_6 &= q_{32}^4 (-3 + O(q)), \\
A_0 A_4 + A_2 A_5 &= q_{32}^4 (-3 + O(q)), \\
A_0 A_3 + A_5 A_6 &= q_{32}^4 (1 + O(q)), \\
A_0 A_5 + A_3 A_4 &= q_{32}^4 (-1 + O(q)), \\
A_0 A_6 + A_1 A_3 &= q_{32}^4 (-1 + O(q)), \\
A_0 A_2 + A_1 A_4 &= q_{32}^4 (-1 + O(q)),
\end{align*}
\]

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and

\[
\begin{align*}
A_1^2 + 2A_4 A_5 &= q^{\frac{17}{26}} (6 + O(q)), \\
A_3^2 + 2A_1 A_2 &= q^{\frac{21}{26}} (8 + O(q)), \\
A_4^2 + 2A_3 A_6 &= q^{\frac{23}{26}} (-1 + O(q)), \\
A_5^2 + 2A_1 A_6 &= q^{\frac{5}{26}} (-3 + O(q)), \\
A_2^2 + 2A_3 A_5 &= q^{\frac{29}{26}} (2 + O(q)), \\
A_6^2 + 2A_4 A_2 &= q^{\frac{1}{26}} (1 + O(q)).
\end{align*}
\]

**Proof of Theorem 1.1.** We divide the proof into three parts (see also [36]). The first part is the calculation of $\Phi_{20}$ and $\Phi'_{12}$. Let

$$
\Phi_{20} = w_0^5 + w_1^5 + \cdots + w_{12}^5 + w_\infty^5.
$$

As a polynomial in six variables, $\Phi_{20}(z_1, z_2, z_3, z_4, z_5, z_6)$ is a $G$-invariant polynomial. Moreover, for $\gamma \in \Gamma(1)$,

$$
\Phi_{20}(Y(\gamma(z))^T) = \Phi_{20}(v(\gamma)j(\gamma, z)^2(\rho(\gamma)Y(z))^T) \\
= v(\gamma)^{20} j(\gamma, z)^{40} \Phi_{20}((\rho(\gamma)Y(z))^T) = j(\gamma, z)^{40} \Phi_{20}((\rho(\gamma)Y(z))^T).
$$

Note that $\rho(\gamma) \in \langle (\rho(s), \rho(t)) = G$ and $\Phi_{20}$ is a $G$-invariant polynomial, we have

$$
\Phi_{20}(Y(\gamma(z))^T) = j(\gamma, z)^{40} \Phi_{20}(Y(z))^T, \quad \text{for } \gamma \in \Gamma(1).
$$

This implies that $\Phi_{20}(y_1(z), \ldots, y_6(z))$ is a modular form of weight 40 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. In fact,

$$
\Phi_{20}(a_1(z), \ldots, a_6(z)) = 13^5q^{\frac{2}{5}} (1 + O(q))^5 + \\
+ \sum_{\nu=0}^{12} [q^{\frac{1}{2}} (1 + O(q)) + \\
+ 2\zeta^\nu q^{\frac{21}{26}} (-3 + O(q)) + 2\zeta^{3\nu} q^{\frac{33}{26}} (-3 + O(q)) + 2\zeta^{9\nu} q^{\frac{35}{26}} (1 + O(q)) + \\
+ 2\zeta^{12\nu} q^{\frac{29}{26}} (-1 + O(q)) + 2\zeta^{10\nu} q^{\frac{1}{26}} (-1 + O(q)) + 2\zeta^{4\nu} q^{\frac{17}{26}} (-1 + O(q)) + \\
+ \zeta^{2\nu} q^{\frac{19}{26}} (6 + O(q)) + \zeta^{4\nu} q^{\frac{23}{26}} (8 + O(q)) + \zeta^{6\nu} q^{\frac{25}{26}} (-1 + O(q)) + \\
+ \zeta^{11\nu} q^{\frac{27}{26}} (-3 + O(q)) + \zeta^{8\nu} q^{\frac{29}{26}} (2 + O(q)) + \zeta^{7\nu} q^{\frac{1}{26}} (1 + O(q))]^5.
$$

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We will calculate the \( q^{1/2} \)-term which is the lowest degree. For the partition \( 13 = 4 \cdot 1 + 9 \), the corresponding term is

\[
\left( \frac{5}{4, 1} \right) (\zeta^7 q^{1/6})^4 \cdot (-3) \zeta^{11} q^{2/3} = -15 q^{1/2}.
\]

For the partition \( 13 = 3 \cdot 1 + 2 \cdot 5 \), the corresponding term is

\[
\left( \frac{5}{3, 2} \right) (\zeta^7 q^{1/6})^3 \cdot (2 \zeta^{9} q^{2/3})^2 = 40 q^{1/2}.
\]

Hence, for \( \Phi_{20}(y_1(z), \ldots, y_6(z)) \) which is a modular form for \( \Gamma(1) \) with weight 40, the lowest degree term is given by

\[
(-15 + 40)q^{1/2} \cdot q^{2/3} = 25q^{3/2}.
\]

Thus,

\[
\Phi_{20}(y_1(z), \ldots, y_6(z)) = q^{3}(13 \cdot 25 + O(q)).
\]

The leading term of \( \Phi_{20}(y_1(z), \ldots, y_6(z)) \) together with its weight 40 suffice to identify this modular form with \( \Phi_{20}(y_1(z), \ldots, y_6(z)) = 13 \cdot 25\Delta(z)^3 E_4(z) \).

Consequently,

\[
\Phi_{20}(x_1(z), \ldots, x_6(z)) = 13 \cdot 25\Delta(z)^3 E_4(z) / \eta(z)^{40} = 13 \cdot 25\eta(z)^8 \Delta(z) E_4(z).
\]

Let

\[
\Phi_{12}' = w_0^3 + w_1^3 + \cdots + w_{12}^3 + w_{\infty}^3.
\]

The calculation of \( \Phi_{12}' \) is similar as that of \( \Phi_{20} \). We find that

\[
\Phi_{12}'(x_1(z), \ldots, x_6(z)) = -13 \cdot 30\Delta(z).
\]

The second part is the calculation of \( \Phi_4, \Phi_8 \) and \( \Phi_{16} \). The calculation of \( \Phi_4 \) has been done in [35], Theorem 3.1. We will give the calculation of \( \Phi_{16} \).

Let

\[
\Phi_{16} = w_0^4 + w_1^4 + \cdots + w_{12}^4 + w_{\infty}^4.
\]

Similar as the above calculation for \( \Phi_{20} \), we find that \( \Phi_{16}(y_1(z), \ldots, y_6(z)) \) is a modular form of weight 32 for the full modular group \( \Gamma(1) \). Moreover, we
will show that it is a cusp form. In fact, 
\[ \Phi_{16}(a_1(z), \ldots, a_6(z)) = 13^4 q^2 (1 + O(q))^4 + \]
\[ + \sum_{j=0}^{12} q^j (-1 + O(q)) + \]
\[ + 2\zeta^7 q^{216} (-3 + O(q)) + 2\zeta^9 q^{216} (-3 + O(q)) + 2\zeta^{10q} q^{216} (1 + O(q)) + \]
\[ + 2\zeta^{12q} q^{216} (-1 + O(q)) + 2\zeta^{14q} q^{216} (-1 + O(q)) + 2\zeta^{15q} q^{216} (-1 + O(q)) + \]
\[ + \zeta^{16q} q^{216} (-3 + O(q)) + \zeta^{17q} q^{216} (8 + O(q)) + \zeta^{18q} q^{216} (-1 + O(q)) + \]
\[ + \zeta^{19q} q^{216} (2 + O(q)) + \zeta^{20q} q^{216} (1 + O(q)))^4. \]

We will calculate the \( q \)-term which is the lowest degree. For example, consider the partition \( 26 = 3 \cdot 1 + 23 \), the corresponding term is

\[ \left( \frac{4}{3, 1} \right) (\zeta^{7q} q^{216})^3 \cdot 8\zeta^5 q^{216} = 32q. \]

For the other partitions, the calculation is similar. In conclusion, we find that the coefficients of the \( q \)-term is an integer. Hence, for \( \Phi_{16}(y_1(z), \ldots, y_6(z)) \) which is a modular form for \( \Gamma(1) \) with weight 32, the lowest degree term is given by

\[ \text{some integer} \cdot q \cdot q^{216-16} = \text{some integer} \cdot q^3. \]

This implies that \( \Phi_{16}(y_1(z), \ldots, y_6(z)) \) has a factor of \( \Delta(z)^3 \), which is a cusp form of weight 36. Therefore, \( \Phi_{16}(y_1(z), \ldots, y_6(z)) = 0 \). The calculation of \( \Phi_8 \) is similar as that of \( \Phi_{16} \).

The third part is the calculation of \( \Phi_{12}, \Phi_{18} \) and \( \Phi_{30} \). Let

\[ \Phi_{12} = \delta_0^2 + \delta_1^2 + \cdots + \delta_{12}^2 + \delta_{\infty}^2. \]

As a polynomial in six variables, \( \Phi_{12}(z_1, z_2, z_3, z_4, z_5, z_6) \) is a \( G \)-invariant polynomial. Moreover, for \( \gamma \in \Gamma(1) \),

\[ \Phi_{12}(X(\gamma(z))^T) = \Phi_{12}(u(\gamma)j(\gamma, z)(\rho(\gamma)X(z))^T) \]
\[ = u(\gamma)^{12} j(\gamma, z)^{12} \Phi_{12}(\rho(\gamma)X(z))^T = j(\gamma, z)^{12} \Phi_{12}(\rho(\gamma)X(z))^T. \]

Note that \( \rho(\gamma) \in \langle \rho(s), \rho(t) \rangle = G \) and \( \Phi_{12} \) is a \( G \)-invariant polynomial, we have

\[ \Phi_{12}(X(\gamma(z))^T) = j(\gamma, z)^{12} \Phi_{12}(X(z))^T, \quad \text{for } \gamma \in \Gamma(1). \]
This implies that $\Phi_{12}(x_1(z), \ldots, x_6(z))$ is a modular form of weight 12 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. In fact,

$$\Phi_{12}(a_1(z), \ldots, a_6(z)) = 13^4 q^{\frac{7}{2}} (1 + O(q))^2 +$$

$$+ \sum_{\nu=0}^{12} [-13q^{\frac{7}{2}} (1 + O(q)) +$$

$$+ \zeta\nu q^{\frac{1}{2}} (13 + O(q)) + \zeta^{2\nu} q^{\frac{7}{2}} (-22 + O(q)) + \zeta^{3\nu} q^{\frac{11}{2}} (-21 + O(q)) +$$

$$+ \zeta^{4\nu} q^{\frac{3}{2}} (-1 + O(q)) + \zeta^{5\nu} q^{\frac{7}{2}} (2 + O(q)) + \zeta^{6\nu} q^{\frac{11}{2}} (2 + O(q)) +$$

$$+ \zeta^{7\nu} q^{\frac{15}{2}} (-2 + O(q)) + \zeta^{8\nu} q^{\frac{19}{2}} (-8 + O(q)) + \zeta^{9\nu} q^{\frac{23}{2}} (6 + O(q)) +$$

$$+ \zeta^{10\nu} q^{\frac{27}{2}} (1 + O(q)) + \zeta^{11\nu} q^{\frac{31}{2}} (-8 + O(q)) + \zeta^{12\nu} q^{\frac{35}{2}} (17 + O(q))^2].$$

We will calculate the $q^{\frac{1}{2}}$-term which is the lowest degree. For the partition $26 = 3 + 23$, the corresponding term is

$$\left( \begin{array}{c} 2 \\ 1,1 \end{array} \right) \zeta^{4\nu} q^{\frac{3}{2}} \cdot (-1) \cdot \zeta^{9\nu} q^{\frac{23}{2}} \cdot 6 = -12q^{\frac{1}{2}}.$$

For the partition $26 = 7 + 19$, the corresponding term is

$$\left( \begin{array}{c} 2 \\ 1,1 \end{array} \right) \zeta^{5\nu} q^{\frac{7}{2}} \cdot 2 \cdot \zeta^{8\nu} q^{\frac{19}{2}} \cdot (-8) = -32q^{\frac{1}{2}}.$$

For the partition $26 = 11 + 15$, the corresponding term is

$$\left( \begin{array}{c} 2 \\ 1,1 \end{array} \right) \zeta^{6\nu} q^{\frac{11}{2}} \cdot 2 \cdot \zeta^{7\nu} q^{\frac{15}{2}} \cdot (-2) = -8q^{\frac{1}{2}}.$$

Hence, for $\Phi_{12}(x_1(z), \ldots, x_6(z))$ which is a modular form for $\Gamma(1)$ with weight 12, the lowest degree term is given by $(-12 - 32 - 8)q^{\frac{1}{2}} \cdot q^{\frac{23}{2}} = -52q$. Thus,

$$\Phi_{12}(x_1(z), \ldots, x_6(z)) = q(-13 \cdot 52 + O(q)).$$

The leading term of $\Phi_{12}(x_1(z), \ldots, x_6(z))$ together with its weight 12 suffice to identify this modular form with

$$\Phi_{12}(x_1(z), \ldots, x_6(z)) = -13 \cdot 52 \Delta(z).$$
Let
\[ \Phi_{18} = \delta_0^3 + \delta_1^3 + \cdots + \delta_{12}^3 + \delta_{\infty}^3. \]
The calculation of \( \Phi_{18} \) is similar as that of \( \Phi_{12} \). We find that
\[ \Phi_{18}(x_1(z), \ldots, x_6(z)) = 13 \cdot 6\Delta(z)E_6(z). \]

Let
\[ \Phi_{30} = \delta_0^5 + \delta_1^5 + \cdots + \delta_{12}^5 + \delta_{\infty}^5. \]
As a polynomial in six variables, \( \Phi_{30}(z_1, z_2, z_3, z_4, z_5, z_6) \) is a \( G \)-invariant polynomial. Similarly as above, we can show that \( \Phi_{30}(x_1(z), \ldots, x_6(z)) \) is a modular form of weight 30 for the full modular group \( \Gamma(1) \). Moreover, we will show that it is a cusp form. In fact,
\[ \Phi_{30}(a_1(z), \ldots, a_6(z)) = 13^{10} q^{\frac{35}{4}} (1 + O(q))^5 + \]
\[ + \sum_{\nu=0}^{12}[-13 q^{\frac{\nu}{4}} (1 + O(q)) + \]
\[ + \zeta^\nu q^{\frac{43}{4}} (13 + O(q)) + \zeta^{2\nu} q^{\frac{47}{4}} (-22 + O(q)) + \zeta^{3\nu} q^{\frac{51}{4}} (-21 + O(q)) + \]
\[ + \zeta^{4\nu} q^{\frac{55}{4}} (-1 + O(q)) + \zeta^{5\nu} q^{\frac{59}{4}} (2 + O(q)) + \zeta^{6\nu} q^{\frac{63}{4}} (2 + O(q)) + \]
\[ + \zeta^{7\nu} q^{\frac{67}{4}} (-2 + O(q)) + \zeta^{8\nu} q^{\frac{71}{4}} (-8 + O(q)) + \zeta^{9\nu} q^{\frac{75}{4}} (6 + O(q)) + \]
\[ + \zeta^{10\nu} q^{\frac{79}{4}} (1 + O(q)) + \zeta^{11\nu} q^{\frac{83}{4}} (-8 + O(q)) + \zeta^{12\nu} q^{\frac{87}{4}} (17 + O(q))^5. \]

We will calculate the \( q^{\frac{3}{4}} \)-term which is the lowest degree. (1) For the partition 39 = 4 \cdot 3 + 27, the corresponding term is
\[ \left( \begin{array}{c} 5 \\ 4, 1 \end{array} \right) (\zeta^{4\nu} q^{\frac{27}{4}} \cdot (-1))^4 \cdot \zeta^{10\nu} q^{\frac{35}{4}} = 5q^{\frac{3}{4}}. \]

(2) For the partition 39 = 3 \cdot 3 + 7 + 23, the corresponding term is
\[ \left( \begin{array}{c} 5 \\ 3, 1, 1 \end{array} \right) (\zeta^{4\nu} q^{\frac{3}{4}} \cdot (-1))^3 \cdot \zeta^{5\nu} q^{\frac{7}{4}} \cdot 2 \cdot \zeta^{9\nu} q^{\frac{23}{4}} \cdot 6 = -240q^{\frac{3}{4}}. \]

(3) For the partition 39 = 3 \cdot 3 + 11 + 19, the corresponding term is
\[ \left( \begin{array}{c} 5 \\ 3, 1, 1 \end{array} \right) (\zeta^{4\nu} q^{\frac{3}{4}} \cdot (-1))^3 \cdot \zeta^{6\nu} q^{\frac{13}{4}} \cdot 2 \cdot \zeta^{8\nu} q^{\frac{19}{4}} \cdot (-8) = 320q^{\frac{3}{4}}. \]
(4) For the partition $39 = 3 \cdot 3 + 2 \cdot 15$, the corresponding term is
\[
\binom{5}{3,2} (\zeta^{4\nu} q^{\frac{3}{12}} \cdot (-1))^3 \cdot (\zeta^{7\nu} q^{\frac{15}{12}} \cdot (-2))^2 = -40q^{\frac{3}{2}}.
\]

(5) For the partition $39 = 2 \cdot 3 + 3 \cdot 11$, the corresponding term is
\[
\binom{5}{2,3} (\zeta^{4\nu} q^{\frac{3}{12}} \cdot (-1))^2 \cdot (\zeta^{6\nu} q^{\frac{11}{12}} \cdot 2)^3 = 80q^{\frac{3}{2}}.
\]

(6) For the partition $39 = 2 \cdot 3 + 2 \cdot 7 + 19$, the corresponding term is
\[
\binom{5}{2,2,1} (\zeta^{4\nu} q^{\frac{3}{12}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{12}} \cdot 2)^2 \cdot \zeta^{7\nu} q^{\frac{19}{12}} \cdot (-2) = -960q^{\frac{3}{2}}.
\]

(7) For the partition $39 = 2 \cdot 3 + 7 + 11 + 15$, the corresponding term is
\[
\binom{5}{2,1,1,1} (\zeta^{4\nu} q^{\frac{3}{12}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{12}} \cdot 2)^2 \cdot (\zeta^{6\nu} q^{\frac{11}{12}} \cdot 2) \cdot (\zeta^{7\nu} q^{\frac{15}{12}} \cdot (-2) = -480q^{\frac{3}{2}}.
\]

(8) For the partition $39 = 1 \cdot 3 + 3 \cdot 7 + 15$, the corresponding term is
\[
\binom{5}{1,3,1} (\zeta^{4\nu} q^{\frac{3}{12}} \cdot (-1) \cdot (\zeta^{5\nu} q^{\frac{7}{12}} \cdot 2)^3 \cdot (\zeta^{7\nu} q^{\frac{15}{12}} \cdot (-2) = 320q^{\frac{3}{2}}.
\]

(9) For the partition $39 = 1 \cdot 3 + 2 \cdot 7 + 2 \cdot 11$, the corresponding term is
\[
\binom{5}{1,2,2} (\zeta^{4\nu} q^{\frac{3}{12}} \cdot (-1) \cdot (\zeta^{5\nu} q^{\frac{7}{12}} \cdot 2)^2 \cdot (\zeta^{6\nu} q^{\frac{11}{12}} \cdot 2)^2 = -480q^{\frac{3}{2}}.
\]

(10) For the partition $39 = 4 \cdot 7 + 11$, the corresponding term is
\[
\binom{5}{4,1} (\zeta^{5\nu} q^{\frac{7}{12}} \cdot 2)^4 \cdot (\zeta^{6\nu} q^{\frac{11}{12}} \cdot 2 = 160q^{\frac{3}{2}}.
\]

Hence, for $\Phi_{30}(x_1(z), \ldots, x_6(z))$ which is a modular form for $\Gamma(1)$ with weight 30, the lowest degree term is given by
\[
(5 - 240 + 320 - 40 + 80 - 960 - 480 + 320 - 480 + 160)q^{\frac{3}{2}} \cdot q^{\frac{30}{24}} = -1315q^2.
\]

Thus,
\[
\Phi_{30}(x_1(z), \ldots, x_6(z)) = q^2(-13 \cdot 1315 + O(q)).
\]
The leading term of $\Phi_{30}(x_1(z), \ldots, x_6(z))$ together with its weight 30 suffice to identify this modular form with

$$\Phi_{30}(x_1(z), \ldots, x_6(z)) = -13 \cdot 1315 \Delta(z)^2 E_6(z).$$

Up to a constant, we revise the definition of $\Phi_{12}$, $\Phi'_{12}$, $\Phi_{18}$, $\Phi_{20}$ and $\Phi_{30}$ as given by (1.8), (1.9) and (1.10). Consequently,

$$
\begin{align*}
\Phi_{12}(x_1(z), \ldots, x_6(z)) &= \Delta(z), \\
\Phi'_{12}(x_1(z), \ldots, x_6(z)) &= \Delta(z), \\
\Phi_{18}(x_1(z), \ldots, x_6(z)) &= \Delta(z)E_6(z), \\
\Phi_{20}(x_1(z), \ldots, x_6(z)) &= \eta(z)^8 \Delta(z)E_4(z), \\
\Phi_{30}(x_1(z), \ldots, x_6(z)) &= \Delta(z)^2E_6(z).
\end{align*}
$$

(3.17)

This completes the proof of Theorem 1.1.

\[ \square \]

4. A different construction: from the modular curve $X(13)$ to $E_8$

4.1. $E_8$-singularity: from $X(13)$ to $E_8$

In this section, we will give a different construction of the equation of the $E_8$-singularity: the symmetry group is the simple group $\text{PSL}(2,13)$ and the equation has a modular interpretation in terms of theta constants of order thirteen.

Put

$$\Phi_j = \Phi_j(x_1(z), \ldots, x_6(z)) \quad \text{for } j = 12, 18, 20, 30.$$ 

By Theorem 1.1, the relations

$$
\begin{align*}
j(z) := \frac{E_4(z)^3}{\Delta(z)} &= \frac{\Phi_{20}^3}{\Phi_{12}^2}, \quad j(z) - 1728 = \frac{E_6(z)^2}{\Delta(z)} = \frac{\Phi_{30}^2}{\Phi_{12}^2}
\end{align*}
$$

(4.1)

give the equation

$$\Phi_{20}^3 - \Phi_{30}^2 = 1728\Phi_{12}^5. \quad (4.2)$$

Hence, we have the following:

**Theorem 4.1** (A different construction of the $E_8$-singularity: from $X(13)$ to $E_8$). The equation of the $E_8$-singularity can be constructed from the modular curve $X(13)$ as follows:

$$\Phi_{20}^3 - \Phi_{30}^2 = 1728\Phi_{12}^5.$$
where $\Phi_{12}$, $\Phi_{20}$ and $\Phi_{30}$ are $G$-invariant polynomials.

Let us recall some facts about exotic spheres (see [20]). A $k$-dimensional compact oriented differentiable manifold is called a $k$-sphere if it is homeomorphic to the $k$-dimensional standard sphere. A $k$-sphere not diffeomorphic to the standard $k$-sphere is said to be exotic. The first exotic sphere was discovered by Milnor in 1956 (see [32]). Two $k$-spheres are called equivalent if there exists an orientation preserving diffeomorphism between them. The equivalence classes of $k$-spheres constitute for $k \geq 5$ a finite abelian group $\Theta_k$ under the connected sum operation. $\Theta_k$ contains the subgroup $bP_{k+1}$ of those $k$-spheres which bound a parallelizable manifold. $bP_{4m}$ ($m \geq 2$) is cyclic of order $2^{2m-2}(2^{2m-1} - 1)$ numerator $(4B_m/m)$, where $B_m$ is the $m$-th Bernoulli number. Let $g_m$ be the Milnor generator of $bP_{4m}$. If a $(4m - 1)$-sphere $\Sigma$ bounds a parallelizable manifold $B$ of dimension $4m$, then the signature $\tau(B)$ of the intersection form of $B$ is divisible by 8 and $\Sigma = \frac{\tau(B)}{8}g_m$. For $m = 2$ we have $bP_8 = \Theta_7 = \mathbb{Z}/28\mathbb{Z}$. All these results are due to Milnor-Kervaire (see [21]). In particular,

$$
\sum_{i=0}^{2m} z_i \overline{z_i} = 1, \quad z_0^3 + z_1^{6k-1} + z_2^2 + \cdots + z_{2m}^2 = 0
$$

is a $(4m-1)$-sphere embedded in $S^{4m+1} \subset \mathbb{C}^{2n+1}$ which represents the element $(-1)^m k \cdot g_m \in bP_{4m}$. For $m = 2$ and $k = 1, 2, \cdots, 28$ we get the 28 classes of 7-spheres. Theorem 4.1 shows that the higher dimensional liftings of two distinct symmetry groups and modular interpretations on the equation of the $E_8$-singularity give the same Milnor’s standard generator of $\Theta_7$.

### 4.2. The Barlow surface: from $X(13)$ to $E_8$

In this section, we will construct the Barlow surface from the modular curve $X(13)$ by the method of transversal linear sections.

By Theorem 1.1, we have

$$
\Phi_{12}(x_1(z), \ldots, x_6(z)) \cdot \Phi_{18}(x_1(z), \ldots, x_6(z)) - \Phi_{30}(x_1(z), \ldots, x_6(z)) = 0.
$$

(4.3)

This gives a morphism

$$
\Phi : X(13) \to X \subset \mathbb{CP}^5
$$

(4.4)

with $\Phi(z) = (x_1(z), \ldots, x_6(z))$, where the variety $X$ is given by an equation in six variables $z_1, \ldots, z_6$ of degree 30:

$$
\Phi_{12}(z_1, \ldots, z_6) \cdot \Phi_{18}(z_1, \ldots, z_6) - \Phi_{30}(z_1, \ldots, z_6) = 0.
$$

(4.5)
Moreover, we have the following morphism

$$\varphi : X \to Z \subset \mathbb{CP}^{13},$$

$$(z_1, \ldots, z_6) \mapsto (\delta_0, \delta_1, \ldots, \delta_{12}, \delta_\infty),$$

where $\delta_0, \delta_1, \ldots, \delta_{12}, \delta_\infty$ are given by (1.4). The variety $Z$ is given by

$$\{ \delta_0 + \cdots + \delta_{12} + \delta_\infty = 0, \quad (\delta_0^2 + \cdots + \delta_\infty^2)(\delta_0^3 + \cdots + \delta_\infty^3) - \frac{4056}{1315}(\delta_0^5 + \cdots + \delta_\infty^5) = 0 \},$$

which is a Fano 11-fold. Let

$$W = Z \cap H_1 \cap H_2 \cap \cdots \cap H_9$$

be a transversal linear section, where the hyperplane $H_i := \{ \delta_{4+i} = 0 \}$ ($1 \leq i \leq 8$) and $H_9 := \{ \delta_\infty = 0 \}$. Now, let us recall the following:

**Proposition 4.2** (see [15]). In the family of quintic surfaces $S_{(\lambda, \mu)}$ in $\mathbb{P}^4$ given by

$$S_{(\lambda, \mu)} = \{(x_0 : \ldots : x_4) \in \mathbb{P}^4 : \sigma_1 = 0, \lambda \sigma_2 \sigma_3 + \mu \sigma_5 = 0 \},$$

(where $\sigma_k$ denotes the $k$-th elementary symmetric polynomial in $x_0, \ldots, x_4$ and $\lambda, \mu \in \mathbb{C}$), all but the following six are non-singular:

(i) $\sigma_5 = 0$ (reducible, consisting of 5 planes, meeting along 10 lines which in turn meet 3 at a time in 10 points),

(ii) $\sigma_2 \sigma_3 = 0$ (reducible, consisting of a quadric and a cubic surface meeting along a non-singular sextic curve),

(iii) $2 \sigma_5 + \sigma_2 \sigma_3 = 0$ (20 singularities, namely the $S_5$-orbit of $(-2 : -2 : -2 : 3 + \sqrt{-7} : 3 - \sqrt{-7}))$,

(iv) $25 \sigma_5 - 12 \sigma_2 \sigma_3 = 0$ (10 singularities, namely the $S_5$-orbit of $(-2 : -2 : -2 : 3 : 3))$,

(v) $50 \sigma_5 + \sigma_2 \sigma_3 = 0$ (5 singularities, namely the $S_5$-orbit of $1 : 1 : 1 : 1 : 1$),

(vi) $2 \sigma_5 - \sigma_2 \sigma_3 = 0$ (15 singularities, namely the $S_5$-orbit of $0 : 1 : -1 : 1 : -1)$.

Using the notation in Theorem 4.2, we find that $W$ is equivalent to the quintic surface $S_{(1 : -\frac{476}{417})}$, which can be deformed in the family $S_{(\lambda, \mu)}$ to the surface (iii) in Theorem 4.2. Note that the surface (iii) in Theorem 4.2 is
equivalent to the quintic surface $Q$ in (2.5). Let $\Phi : Y \to Q$ be a covering map, where $Y$ is 2 : 1 onto the 20-nodal quintic $Q \subset \mathbb{CP}^4$. The icosahedral group $A_5$ acts on $\mathbb{CP}^4$ by the standard action on the coordinates. The quintic $Q$ is $A_5$-invariant and its 20 nodes are the $A_5$-orbit of the point $(2, 2, 2, -3 - \sqrt{-7}, -3 + \sqrt{-7})$. The $A_5$-action on $Q$ is covered by an action on $Y$, so that we have an action of $A_5 \times \mathbb{Z}/2\mathbb{Z} = A_5 \cup A_5 \sigma$ on $Y$, where the generator $\sigma \in \mathbb{Z}/2\mathbb{Z}$ is the covering involution. Elements of $A_5 \times \mathbb{Z}/2\mathbb{Z}$ acting on $Y$ are denoted like the corresponding elements acting on $Q$. Now, we can apply the main result from [2]:

**Proposition 4.3** (see [2] and [28]). Let $\alpha = (02)(34)\sigma$, $\beta = (01234)$. Then $\beta$ acts freely on $Y$ and $\alpha$ has 4 fixed points. The resolution of the nodes of $Y/D_{10}$, where $D_{10} = \langle \alpha, \beta \rangle$, gives a minimal surface $B$ of general type with $\pi_1 = 0$, $q = p_g = 0$ and $K^2 = 1$.

By [28], $B$ is homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$. Therefore, we have proved the following:

**Theorem 4.4.** The Barlow surface can be constructed from the modular curve $X(13)$.

### 4.3. Bring’s curve and Fricke’s octavic curve: from $X(13)$ to $E_8$

In this section, we will construct both Bring’s curve and Fricke’s octavic curve from the modular curve $X(13)$ by the method of transversal linear sections.

By Theorem 1.1, we have

\[
\begin{align*}
\Phi_4(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_8(x_1(z), \ldots, x_6(z)) &= 0, \\
\phi_{12}(x_1(z), \ldots, x_6(z)) &= 0,
\end{align*}
\]

which gives a morphism from $X(13)$ to an algebraic surface $S_1 \subset \mathbb{CP}^5$ given by the following $G$-invariant equations

\[
\begin{align*}
\Phi_4(z_1, \ldots, z_6) &= 0, \\
\Phi_8(z_1, \ldots, z_6) &= 0, \\
\phi_{12}(z_1, \ldots, z_6) &= 0.
\end{align*}
\]

Moreover, there is a morphism

\[
\varphi_1 : S_1 \to Y_1 \subset \mathbb{CP}^{27}, \quad (z_1, \ldots, z_6) \mapsto (w_0, w_1, \ldots, w_{12}, w_\infty, \delta_0, \delta_1, \ldots, \delta_{12}, \delta_\infty),
\]
where \( w_0, w_1, \ldots, w_{12}, w_\infty \) are given by (1.2) and \( \delta_0, \delta_1, \ldots, \delta_{12}, \delta_\infty \) are given by (1.4), and the variety \( Y_1 \) is given by

\[
\begin{cases}
w_0 + w_1 + \cdots + w_{12} + w_\infty = 0, \\
w_0^2 + w_1^2 + \cdots + w_{12}^2 + w_\infty^2 = 0, \\
w_0^3 + w_1^3 + \cdots + w_{12}^3 + w_\infty^3 = \frac{15}{26}(\delta_0^2 + \delta_1^2 + \cdots + \delta_{12}^2 + \delta_\infty^2),
\end{cases}
\]

which is a 24-dimensional Fano variety. Let

\[ C_1 = Y_1 \cap H_1 \cap H_2 \cap \cdots \cap H_{23} \]

be a transversal linear section, where the hyperplane \( H_i := \{ w_{4+i} = 0 \} \) (1 \( \leq \) \( i \) \( \leq \) 8), \( H_9 := \{ w_\infty = 0 \} \), \( H_j := \{ \delta_{j-10} = 0 \} \) (10 \( \leq \) \( j \) \( \leq \) 22) and \( H_{23} := \{ \delta_\infty = 0 \} \). Then \( C_1 \) is equivalent to Bring’s curve.

Similarly, by Theorem 1.1, we have

\[
\begin{cases}
\Phi_4(x_1(z), \ldots, x_6(z)) = 0, \\
\Phi_8(x_1(z), \ldots, x_6(z)) = 0, \\
\Phi_{16}(x_1(z), \ldots, x_6(z)) = 0,
\end{cases}
\]

which gives a morphism from \( X(13) \) to an algebraic surface \( S_2 \subset \mathbb{CP}^5 \) given by the following \( G \)-invariant equations

\[
\begin{cases}
\Phi_4(z_1, \ldots, z_6) = 0, \\
\Phi_8(z_1, \ldots, z_6) = 0, \\
\Phi_{16}(z_1, \ldots, z_6) = 0.
\end{cases}
\]

Moreover, there is a morphism

\[ \varphi_2 : S_2 \rightarrow Y_2 \subset \mathbb{CP}^{13}, \]

\[ (z_1, \ldots, z_6) \mapsto (w_0, w_1, \ldots, w_{12}, w_\infty), \]

where \( w_0, w_1, \ldots, w_{12}, w_\infty \) are given by (1.2) and the variety \( Y_2 \) is given by

\[
\begin{cases}
w_0 + w_1 + \cdots + w_{12} + w_\infty = 0, \\
w_0^2 + w_1^2 + \cdots + w_{12}^2 + w_\infty^2 = 0, \\
w_0^4 + w_1^4 + \cdots + w_{12}^4 + w_\infty^4 = 0,
\end{cases}
\]

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which is a Fano 10-fold. Let
\[ C_2 = Y_2 \cap H_1 \cap H_2 \cap \cdots \cap H_9 \]  
be a transversal linear section, where the hyperplane \( H_i := \{ w_{4+i} = 0 \} \) (1 \( \leq i \leq 8 \)) and \( H_9 := \{ w_{\infty} = 0 \} \). Then \( C_2 \) is equivalent to Fricke’s octavic curve. Therefore, we have proved the following:

**Theorem 4.5.** Both Bring’s curve and Fricke’s octavic curve can be constructed from the modular curve \( X(13) \).

5. An explicit construction of the modular curve \( X(13) \)

In this section, we will study the following classical problem in the theory of modular curves, which goes back to F. Klein (see [24] and [25]).

**Problem 5.1.** Let \( p \geq 7 \) be a prime number. Give an explicit construction of the modular curve \( X(p) \) of level \( p \) from the invariant theory for \( PSL(2, p) \) using projective algebraic geometry.

**Example 5.2.** The equation of the modular curve \( X(7) \) of level 7 is given by the Klein quartic curve (see [24])
\[ x^3 y + y^3 z + z^3 x = 0 \] 
in \( \mathbb{P}^2 \). The left hand side of (5.1) is the unique quartic invariant for \( PSL(2, 7) \) in this representation.

**Example 5.3.** The matrix
\[
\begin{pmatrix}
w & v & 0 & 0 & z \\
v & x & w & 0 & 0 \\
0 & w & y & x & 0 \\
0 & 0 & x & z & y \\
z & 0 & 0 & y & v
\end{pmatrix}
\]
is (up to a factor) the Hessian matrix of a cubic invariant for \( PSL(2, 11) \), namely the Klein cubic threefold (see [25])
\[ v^2 w + w^2 x + x^2 y + y^2 z + z^2 v = 0. \]  
(5.2)

The modular curve \( X(11) \) of level 11 is the singular locus of the Hessian of this cubic threefold. In fact, there are only 10 distinct quartics defining the locus, namely
\[ v^2 w z - v w y^2 - x^2 y z = 0, \quad v y^3 + w^3 z + w x^3 = 0 \]
and their images under successive applications of the cyclic permutation $(vwxyz)$.

In general, the locus of the modular curve $X(p)$ of level $p$ can be defined by $\binom{(p-1)/2}{3}$ quartics (see [1], p. 59). Following Klein’s method for the cubic threefold (5.2), Adler and Ramanan (see [1]) studied Problem 5.1 when $p$ is a prime congruent to 3 modulo 8 by some cubic hypersurface invariant under $\text{PSL}(2, p)$. However, their method can not be valid for $p = 13$. In this case, the number of the locus is 20. As Ramanan has remarked, one can actually reduce the number further, namely to $(p - 1)/2$, but this does not lead to explicit equations (see [1], p. 59).

As a consequence of Theorem 1.1, we find an explicit construction of the modular curve $X(13)$.

**Theorem 5.4.** There is a morphism

$$\Phi : X(13) \to C \subset \mathbb{CP}^5$$

with $\Phi(z) = (x_1(z), \ldots, x_6(z))$, where $C$ is an algebraic curve given by a family of $G$-invariant equations

$$\begin{align*}
\Phi_4(z_1, \ldots, z_6) &= 0, \\
\Phi_8(z_1, \ldots, z_6) &= 0, \\
\phi_{12}(z_1, \ldots, z_6) &= 0, \\
\Phi_{16}(z_1, \ldots, z_6) &= 0.
\end{align*}$$

**Proof.** Theorem 1.1 implies that

$$\begin{align*}
\Phi_4(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_8(x_1(z), \ldots, x_6(z)) &= 0, \\
\phi_{12}(x_1(z), \ldots, x_6(z)) &= 0, \\
\Phi_{16}(x_1(z), \ldots, x_6(z)) &= 0.
\end{align*}$$

\[ \square \]
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