INJECTIVE ENVELOPES OF SEPARABLE C*-ALGEBRAS

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Abstract. Characterisations of those separable C*-algebras that have type I injective envelopes or W*-algebra injective envelopes are presented.

An operator system $I$ is injective if for every inclusion $E \subseteq F$ of operator systems each completely positive linear map $\omega : E \rightarrow I$ has a completely positive extension to $F$. An injective envelope of an operator system $E$ is an injective operator system $I$ such that $E \subseteq I$ and $I$ is minimal among all injective operator systems that contain $E$. That is, if $E \subseteq I_0 \subseteq I$, with $I_0$ injective, then $I_0 = I$. Hamana [11] proved that every operator system $E$ has an injective envelope and that all injective envelopes of $E$ are completely isometric. Because every injective operator system is completely order isomorphic to an injective C*-algebra [5], and because two C*-algebras are *-isomorphic if and only if they are completely order isomorphic [4], one can unambiguously refer to “the” injective envelope of $E$, which is an injective C*-algebra $I(E)$ that contains $E$ as an operator system.

If $E$ is a C*-algebra $A$, then $A$ is contained in $I(A)$ as a C*-subalgebra. The purpose of the present paper is to study how properties of a C*-algebra $A$ determine properties of its injective envelope, especially in the case of separable C*-algebras $A$.

The injective envelope $I(A)$ of any C*-algebra $A$ is a monotone complete C*-algebra. Thus, $I(A)$ is a direct sum of AW*-algebras of types I, II, and III. Herein we show that if $A$ is separable, then $I(A)$ has no direct summand that is finite and of type II. Further, we show that a separable C*-algebra $A$ has a type I injective envelope if and only if $A$ has a liminal essential ideal. We also characterise those separable C*-algebras $A$ for which $I(A)$ is a W*-algebra.

There are a number of other useful enveloping structures that contain a given C*-algebra $A$ as a C*-subalgebra. Of these, the local multiplier algebra $M_{\text{loc}}(A)$ [1, 10, 21] and the regular monotone completion $\overline{A}$ [23, 17, 12] have important roles in arriving at our results. These structures, together with the injective envelope, are discussed in the following preliminary section.

This research is supported in part by the Natural Sciences and Engineering Research Council of Canada.

2000 Mathematics Subject Classification: Primary 46L05; Secondary 46L07.

Keywords and Phrases: injective envelope, local multiplier algebra, regular monotone completion, C*-algebra, AW*-algebra.

June 15, 2005.
1. Preliminary Results

1.1. Terminology and notation. As usual, we will denote by $B(H)$ and $K(H)$ the set of bounded and compact operators on a Hilbert space $H$. Because the algebras under study are not represented in any particular way as acting on a Hilbert space, we shall employ the following terminology. A $C^*$-algebra $A$ is said to be a $W^*$-algebra if, as a Banach space, $B$ is the dual space $X^*$ of some (in fact, unique) Banach space $X$. It is a classical fact [23, Theorem III.3.5] that a $C^*$-algebra $A$ is a $W^*$-algebra if and only if $A$ has a representation as a von Neumann algebra of operators acting on some Hilbert space. A $C^*$-algebra $A$ is called an AW*-algebra if the left annihilator of each right ideal in $A$ is of the form $Ap$ for some projection $p \in A$. Although every $W^*$-algebra is an AW*-algebra, the converse is not true: there exist AW*-algebras that fail that have any faithful representation as a von Neumann algebra.

If $B$ is an AW*-algebra, then $p \sim q$ denotes the Murray–von Neumann equivalence of projections $p$ and $q$ in $B$. Thus, a projection $p \in B$ is finite if $q \sim p$ and $q \leq p$ only if $q = p$; otherwise $p$ is an infinite projection. If the identity $1 \in B$ is a finite projection, then $B$ is said to be finite algebra. A projection $p \in B$ is abelian if the AW*-algebra $pBp$ is commutative.

An AW*-algebra $B$ is said to be: of type I if every direct summand of $B$ has an abelian projection; of type II if $B$ has no abelian projections but every direct summand has a finite projection; and of type III if all projections in $B$ are infinite. If the centre $Z(B)$ of an AW*-algebra is $C$, then $B$ is a factor. Type I AW*-algebras are of considerable interest herein. In particular, type I AW*-algebras are injective $C^*$-algebras [12] and type I AW*-factors are of the form $B(H)$ [13].

AW*-algebras differ from $W^*$-algebras in that the former can fail to have any normal states. An AW*-algebra $B$ is wild [26] if the only normal positive linear functional $\varphi$ on $B$ is $\varphi = 0$. Every AW*-factor is either a $W^*$-algebra or a wild AW*-algebra [26].

A $C^*$-algebra $A$ is said to be postliminal (or type I, or GCR) if every representation of $A$ generates a type I von Neumann algebra, and $A$ is liminal (or CCR) if every irreducible representation $\pi : A \to B(H)$ satisfies $\pi(A) = K(H)$. An elementary $C^*$-algebra is one that $\ast$-isomorphic to $K(H)$ for some Hilbert space $H$.

We shall employ the following notation from [11]. If $\{E_\alpha\}_{\alpha \in \Lambda}$ is a family of operator systems, then

$$\prod_{\alpha \in \Lambda} E_\alpha = \{(e_\alpha)_{\alpha} : e_\alpha \in E_\alpha \text{ and } \sup_{\alpha} \|e_\alpha\| < \infty\};$$

$$\bigoplus_{\alpha \in \Lambda} E_\alpha = \{(e_\alpha)_{\alpha} : e_\alpha \in E_\alpha \text{ and } \forall \varepsilon > 0 \text{ only finitely many } e_\alpha \text{ satisfy } \|e_\alpha\| > \varepsilon\}.$$
1.2. Injective envelopes. An operator system \( I \) is injective if for every inclusion \( E \subset F \) of operator systems each completely positive linear map \( \omega : E \to I \) has a completely positive extension to \( F \). Arveson’s extension theorem \([2]\) for completely positive linear maps with values in \( B(H) \) demonstrates that \( B(H) \) is injective. This fact can be used to show that if an operator system \( I \) is represented as a unital, \(*\)-closed subspace of \( B(H) \), then \( I \) is injective if and only if \( I \) is the range of some completely positive linear map \( \phi : B(H) \to B(H) \) for which \( \phi^2 = \phi \). Such maps \( \phi \) are commonly referred to as projections, or conditional expectations. A theorem of Choi and Effros \([3]\) demonstrates that if \( I \) is an injective operator system given by the range of a projection \( \phi \) on \( B(H) \), then \( I \) is completely order isomorphic to a \( C^*\)-algebra, obtained by changing the product of \( I \) to \( x \circ y = \phi(xy) \).

An injective envelope of an operator system \( E \) is an injective operator system \( I \) and a complete isometry \( \kappa : E \to I \) such that, if \( I_0 \) is an injective operator system with \( \kappa(E) \subseteq I_0 \subseteq I \), then \( I_0 = I \). The existence and uniqueness (up to complete isometry) of the injective envelope was established by Hamana \([11]\); thus, it is a common practice to drop reference to \( \kappa \) and assume that \( E \) is already realised as an operator system in \( I \). The following proposition of Hamana is a useful criterion for determining when an injective operator system \( I \) containing \( E \) is an injective envelope.

** Proposition 1.1. **(\([11]\) Lemma 3.7) Consider an inclusion \( E \subseteq I \) of operator systems, where \( I \) is injective. The following statements are equivalent.

1. \( I \) is an injective envelope of \( E \).
2. The only completely positive linear map \( \omega : I \to I \) for which \( \omega|_E = \text{id}_E \) is the identity map \( \omega = \text{id}_I \).

We note below a property that we shall make frequent use of.

** Lemma 1.2. **If \( \{E_\alpha\}_{\alpha \in \Lambda} \) is a family of operator systems, then \( \prod_\alpha E_\alpha \) is injective if and only if \( E_\alpha \) is injective for every \( \alpha \in \Lambda \).

** Proof. **Fix an inclusion \( E \subset F \) of operator systems.

Assume that \( \prod_\alpha E_\alpha \) is injective. If \( \phi : E \to E_\beta \) is completely positive, define \( \tilde{\phi} : E \to \prod_\alpha E_\alpha \) by \( (\tilde{\phi}(x))_\beta = \phi(x) \) and \( (\tilde{\phi}(x))_\alpha = 0 \) if \( \alpha \neq \beta \). Then there exists \( \psi : F \to \prod_\alpha E_\alpha \) extending \( \tilde{\phi} \). So \( \pi_\beta \circ \psi \) is a completely positive extension of \( \phi \).

Conversely, if \( E_\alpha \) is injective for every \( \alpha \), and \( \phi : E \to \prod_\alpha E_\alpha \) is completely positive, then for each \( \alpha \) the map \( \pi_\alpha \circ \phi : E \to E_\alpha \) is completely positive, and so there exists \( \psi_\alpha : F \to E_\alpha \) completely positive extension. Thus the map \( \prod_\alpha \psi_\alpha : F \to \prod_\alpha E_\alpha \) is a completely positive extension of \( \phi \). \( \square \)

1.3. Regular monotone completions. A \( C^*\)-algebra \( B \) is monotone complete if every bounded increasing net \( \{h_n\}_n \) in \( B_{sa} \) has a least upper bound in \( B_{sa} \), where \( B_{sa} \) denotes the real vector space of hermitian elements of \( B \). The least upper bound of a bounded increasing net \( \{h_n\}_n \) in \( B_{sa} \) is denoted by \( \text{sup}_n h_n \). A \( C^*\)-algebra \( B \) is monotone \( \sigma\)-complete if every bounded increasing sequence \( \{h_n\}_{n \in \mathbb{N}} \) in \( B_{sa} \) has a least upper bound in \( B_{sa} \). (The terminology “monotone complete” is called “monotone closed” in some of the standard texts, such as \([19]\) and \([23]\). We follow
Hamana [12] by using the term “monotone closed” in a sense different from [19] and [23]; this is explained below.)

Monotone complete C*-algebras are unital [23] and if B is monotone σ-complete and satisfies the countable chain condition (namely, for each for each S ⊂ Bsa that is bounded above in Bsa there is a countable subset S₀ ⊆ S such that any upper bound for S₀ is also an upper bound for S), then B is monotone complete [25]. Every W*-algebra is monotone complete and a C*-algebra B is an AW*-algebra if and only if each maximal abelian C*-subalgebra D ⊆ B is monotone complete. However, it is not known whether every AW*-algebra is monotone complete. A well-known theorem of Tomiyama [24] for conditional expectations between C*-algebras, which is proved below for operator systems, implies in particular that the injective envelope of an operator system is monotone closed.

**Proposition 1.3.** Let E ⊆ M be operator systems, with M monotone complete. If there exists a positive linear map φ : M → E such that φ_E = id_E, then E is monotone complete.

**Proof.** Let \{h_α\}_α be a bounded increasing net in E. It is in particular an increasing bounded net in M, so there exists \( \hat{h} \in M \), \( \hat{h} = \sup_α h_α \). Let \( h = \phi(\hat{h}) \). Then \( h - h_α = \phi(\hat{h} - h_α) \geq 0 \), for every α, so that h is an upper bound for \{h_α\}_α. If \( k \in E \) and \( h_α \leq k \) for every α, then because \( k \in M \) we have that \( \hat{h} \leq k \). Thus, \( k - h = \phi(k - \hat{h}) \geq 0 \), which implies that h is the supremum of \{h_α\}_α in E. \qed

**Corollary 1.4.** The injective envelope I(A) of any C*-algebra A is monotone complete. In particular, I(A) is an AW*-algebra.

If B is a monotone complete C*-algebra, then a subset \( S \subseteq B_{sa} \) is monotone closed in B if, for every bounded increasing net \{s_α\}_α in S, \( \sup_α s_α \) (which exists in B) is contained in S. In particular, if A is a C*-subalgebra of B and if m-cl_B A_{sa} denotes the smallest subset of \( B_{sa} \) that contains A_{sa} and is monotone closed in B, then the monotone closure of A in B is defined to be the set

\[ m-cl_B A = m-cl_B A_{sa} + i m-cl_B A_{sa}. \]

It so happens that m-cl_B A is a monotone complete C*-subalgebra of B [12, Lemma 1.4].

A C*-subalgebra C of B is called a monotone closed C*-subalgebra of B if m-cl_B C = C. Because the property of C being monotone closed in B involves both C and B, it is possible for a C*-subalgebra C of B to be monotone complete yet fail to be monotone closed in B. In fact, it is frequently the case that a von Neumann algebra \( M \subset B(H) \) is not monotone closed in \( B(H) \).

A C*-subalgebra A of a C*-algebra B is said to be order dense in B if

\[ h = \sup\{k \in A^+ : k \leq h\}, \quad \forall h \in B^+. \]

For example, \( K(H) \) is order dense in \( B(H) \).

A regular monotone completion of a C*-algebra A is a C*-algebra B such that

1. A is a C*-subalgebra of B,
2. B is monotone complete,
Injective Envelopes of Separable C*-algebras

In [12], Hamana proved that a regular monotone completion exists for every C*-algebra $A$ and any two regular monotone completions of $A$ are $*$-isomorphic. Thus, $\overline{A}$ is used to denote “the” regular monotone completion of $A$. Hamana’s construction of $\overline{A}$ is via the injective envelope of $A$. Namely, $\overline{A}$ is the monotone closure of $A$ in $I(A)$.

The regular monotone $\sigma$-completion $\overline{A}\sigma$ of a C*-algebra $A$ was introduced by Wright [25]. Hamana recovers $\overline{A}\sigma$ via the injective envelope by considering monotone $\sigma$-closure of $A$ in $I(A)$ (the definitions are analogous to earlier ones, but with sequences in place of nets).

For each C*-algebra $A$ there is a representation in which
\[ A \subseteq \overline{A}\sigma \subseteq \overline{A} \subseteq I(A), \]
where each containment is as a C*-subalgebra. We shall assume this representation in our work herein. An important feature of this sequence of containments is:

$\overline{A}$ is monotone closed in $I(A)$.

**Theorem 1.5.** Assume that $A$ is a separable C*-algebra.

1. (Wright) $\overline{A}\sigma = \overline{A}$.
2. (Ozawa–Saitô) [17] The AW*-algebra $\overline{A}$ has no type II direct summand.
3. (Hamana) [12] If $A$ is postliminal, then $\overline{A}$ is of type I.
4. (Saitô) [20] If $K \subseteq A$ is an essential ideal of $A$, then $K = \overline{A}$.
5. If $K \subseteq A$ is an essential ideal of $A$, then then $I(K) = I(A)$.

**Proof.** Only the proof of (5) need be given, as it is not explicitly stated in the literature. By [1], $\overline{K} = \overline{A}$ if $K \subseteq A$ is an essential ideal of $A$. Furthermore, $I(\overline{A}) = I(A)$, by [12, Lemma 3.7]. Hence, $I(K) = I(\overline{K}) = I(\overline{A}) = I(A)$.

1.4. **Local multiplier algebras.** The multiplier algebra of a C*-algebra $A$ is the C*-subalgebra $M(A)$ of the enveloping von Neumann algebra $A^{**}$ that consists of all $x \in A^{**}$ for which $xa \in A$ and $ax \in A$, for all $a \in A$. If $J \subseteq A$ is an ideal, then $J^{**}$ is identified with the closure of $J$ in $A^{**}$ with respect to the strong operator topology. Thus, if $J$ and $K$ are ideals of $A$, and if $J \subseteq K$, then $M(J) \supseteq M(K) \supseteq M(A)$.

An ideal $K$ of $A$ is said to be essential if $K \cap J \neq \{0\}$ for every nonzero ideal $J \subseteq A$. Any essential ideal is necessarily nonzero. Consider the multiplier algebra $M(J)$ of any essential ideal $J$ of $A$. If $\mathcal{E}(A)$ is the set of essential ideals of $A$, partially ordered by reverse inclusion, then the set $\mathcal{E}(A)$ of multiplier algebras $M(K)$ of $K \in \mathcal{E}(A)$ is a directed system of C*-algebras. $M_{\text{loc}}(A)$ is then defined to be the C* direct limit of the directed system $K \in \mathcal{E}(A)$. In [11], Ara and Mathieu give a systematic account of the theory of local multiplier algebras of C*-algebras. Their book is our basic reference on the topic.

There are various ways to realise $M_{\text{loc}}(A)$ “concretely” as a C*-subalgebra of some other C*-algebra:

1. as a C*-subalgebra of a quotient of $A^{**}$ [11];
(ii) as a $C^*$-subalgebra of a quotient of $A^{**}$, where the quotient is monotone
$\sigma$-complete [21];
(iii) as a $C^*$-subalgebra of $I(A)$ [10].

In this final case, $M_{\text{loc}}(A)$ is realised by idealisers in $I(A)$ of essential ideals of $A$.
Specifically, by [10, Corollary 4.3],

$$M_{\text{loc}}(A) = \left( \bigcup_{K \in \mathcal{E}(A)} \{ x \in I(A) : xK + Kx \subseteq K \} \right)^{\sim},$$

where the closure is with respect to the norm topology of $I(A)$. Thus,

$$A \subseteq M_{\text{loc}}(A) \subseteq I(A)$$

is an inclusion of $C^*$-subalgebras. In [9], Frank showed an additional sequence of inclusions as $C^*$-subalgebras:

$$A \subseteq M_{\text{loc}}(A) \subseteq M_{\text{loc}}(M_{\text{loc}}(A)) \subseteq \overline{A} \subseteq I(A).$$

1.5. Injective envelopes of separable and prime $C^*$-algebras.

**Proposition 1.6.** If $A$ is a separable $C^*$-algebra, then $I(A)$ does not have a finite type II direct summand.

*Proof.* It is enough to show that if $I(A)$ has a finite direct summand, then this summand is of type I. Because $I(A)e = I(A)e$ for any central projection $e \in I(A)$ [12, Lemma 6.2], and since the $C^*$-algebra $Ae$ is separable, we may assume without loss of generality that $I(A)$ itself is a finite algebra. Thus, the identity $1 \in I(A)$ is a finite projection, and so $1$ is a finite projection in $\overline{A}$ as well. Therefore, $\overline{A}$ is of type I [17, Theorem 2]. But type I algebras are injective; hence $\overline{A} = I(A)$. \hfill $\Box$

The next proposition, which builds on work of Hamana, determines which $C^*$-algebras lead to factors.

**Proposition 1.7.** The following statements are equivalent for any $C^*$-algebra $A$.

1. $\overline{A}$ is a factor.
2. $I(A)$ is a factor.
3. $A$ is prime.

*Proof.* The equivalence of [11] and [3] was established by Hamana [12] Theorem 7.1. To prove that [11] and [2] are equivalent, note that $Z(\overline{A}) = Z(I(\overline{A}))$, because $\overline{A}$ is monotone complete [12, Theorem 6.3]. Further, because $I(\overline{A}) = I(A)$ [12, Lemma 3.7], we conclude that $Z(\overline{A}) = Z(I(A))$. Thus, $\overline{A}$ is a factor if and only if $I(A)$ is a factor. \hfill $\Box$
2. W*-algebra Injective Envelopes

The injective envelope $I(A)$ of any C*-algebra $A$ is an AW*-algebra. However, in rare instances $I(A)$ is known in fact to be a W*-algebra. This is so if $A$ can be represented as acting on a Hilbert space in such a way as to contain every compact operator \[3, 11\]. In this section we characterise those separable C*-algebras $A$ for which $I(A)$ is a W*-algebra.

**Lemma 2.1.** If $A$ is a C*-algebra for which $I(A)$ is a W*-algebra, then $\overline{A}$ is a W*-algebra.

**Proof.** Without loss of generality we may assume that $I(A)$ is represented as a von Neumann algebra acting on a Hilbert space. Let \( \{h_\alpha\}_\alpha \) be any bounded increasing net in $\overline{A}_{sa}$. Because $I(A)$ is a von Neumann algebra, \( \{h_\alpha\}_\alpha \) has a least upper bound $h$ such that $h = \lim_\alpha h_\alpha$ in the strong-operator-topology. Note that the supremum of \( \{h_\alpha\}_\alpha \) in $I(A)$ necessarily coincides with $h$ and, because $A$ is monotone closed in $I(A)$, $h \in A$. Thus, $\overline{A}$ is a C*-algebra of operators for which the strong-operator limit of every bounded increasing net of hermitian elements of $A$ belongs to $A$. By \[14\, \text{Lemma 1}\], this implies that $\overline{A}$ is a von Neumann algebra. \[\Box\]

**Lemma 2.2.** The following statements are equivalent for a von Neumann algebra $M$.

1. $M$ is a direct product of type I factors.
2. $M$ is generated by its minimal projections.

The lemma above is well known. However, as it is important for our work, the ideas that underlie the proof are worth noting here briefly. First of all, if $M$ is a direct product of type I factors, then $M$ is generated by the family of all the minimal projections of all the factors. Conversely, if $M$ is generated by minimal projections, then it cannot have a type II nor type III direct summand. Indeed, if $Me$ is type II or type III, with $e$ a central projection in $M$, consider $q \in M$ a minimal projection such that $qe \neq 0$. Such a projection exists because otherwise $e = 0$. Since $q$ is minimal in $M$, $q = qeq = q$ and so $q \in Me$. But then $Me$ admits a minimal projection, which is a contradiction. Thus $M$ is type I, and it can be expressed as a direct integral over a type I factor-valued measure. The diffuse part of this measure has to be zero, because any projection in the diffuse part will not be minimal, and we can reason as before. Therefore, the measure is atomic and $M$ is a direct product of type I factors.

**Corollary 2.3.** If $M$ is a von Neumann algebra generated by minimal projections, then $M$ is injective.

**Proof.** Type I factors are injective, by Arveson’s theorem \[2\] on the injectivity of $B(H)$. Lemma \[22\] asserts that $M$ is a direct product of type I factors; by Lemma \[12\] every direct product of injective C*-algebras is injective. Hence, $M$ is injective. \[\Box\]
Lemma 2.4. Suppose that $A$ is a $C^*$-subalgebra of a von Neumann algebra $M$ and that $M = A''$.

(1) If $M$ is generated by its minimal projections, each of which is contained in $A$, then $A$ is order dense in $M$.

(2) If $A$ is separable and if $A$ is order dense in $M$, then $M$ is generated by its minimal projections, each of which is contained in $A$.

Proof. For the proof of (1), choose a nonzero $h \in M^+$ and consider the set

$$\mathcal{F} = \{ (k_i) \subset A^+ : \sum_{\text{finite}} k_i \leq h \}.$$ 

There is a strictly positive $\lambda$ in the spectrum $\sigma(h)$ of $h$. Let $e \in M$ be the spectral projection $e = e^h ([\lambda, \infty))$, where $e^h$ denotes the spectral resolution of $h$. Thus, $0 \neq \lambda e \leq he$. Moreover, $e$ majorises a minimal projection $p$ of $M$; by hypothesis, $p \in A$. Thus, $0 \neq \lambda p = e(\lambda p)e \leq e(\lambda)e = \lambda e \leq he \leq h$, and so $\lambda p \in \mathcal{F}$, which proves that $\mathcal{F} \neq \emptyset$. It is clear that $\mathcal{F}$ is inductive under inclusions of those families and so, by Zorn’s Lemma, $\mathcal{F}$ has a maximal family $W$. Since every finite sum of this family is less than $h$,

$$y = \sup \left\{ \sum_{k \in K} k : K \text{ is finite and } K \subset W \right\} \leq h.$$ 

If $y \neq h$, then $h - y \in M^+$, and so by the first paragraph there exists $k \in A^+$ such that $k \leq h - y$. If it were true that $k \in W$, then for each net $(h_i)$ of those finite sums of elements in $W$ such that $h_i \nearrow y$, the net $(h_i + k) \nearrow y + k$, which contradicts the fact that $y$ is the supremum. Hence, $k \not\in W$. But if $k \not\in W$, then the family $W$ is not maximal, which is again a contradiction. Therefore, it must be that $y = h$, which proves that $A$ is order dense in $M$.

For the proof of (2), note that because $A$ is separable and $A'' = M$, to prove that $M$ is generated by its minimal projections, each of which is contained in $A$, it is enough, by [23, p. 139], to prove that any normal state $\omega \in M_*$ is faithful precisely when its restriction $\omega|_A$ to $A$ is faithful. Thus, let $\omega$ be a normal state on $M$ that is faithful on $A$. Assume that $\omega(h) = 0$, where $h \in M^+$. Because $h = \sup\{k \in A^+ : k \leq h\}$, we have that $0 \leq \omega(k) \leq \omega(h) = 0$ for each $k \in A^+$ with $k \leq h$. Thus, $\omega(k) = 0$, which implies that $k = 0$ because $\omega$ is faithful on $A$. Hence, $h = 0$ and so $\omega$ is faithful on $M$. \[
\]

The following theorem is the main result of the present section.

Theorem 2.5. The following statements are equivalent for a separable $C^*$-algebra $A$.

(1) $I(A)$ is a $W^*$-algebra.

(2) $I(A)$ is a discrete type I $W^*$-algebra.

(3) There exists a faithful representation $\pi : A \to B(H)$ such that the von Neumann algebra $\pi(A)''$ is generated by its minimal projections, each of which is contained in $\pi(A)$.

(4) There exists an ideal $K$ of $A$ such that
(a) $K$ is a minimal essential ideal and
(b) $K \cong_\pi \bigoplus_n K(H_n)$, for some sequence of Hilbert spaces $H_n$.

Proof. Assume that $I(A)$ is a W*-algebra. Then there is a faithful representation $\tilde{\pi} : I(A) \to B(H)$ such that $\tilde{\pi}(I(A))$ is a von Neumann algebra and $\pi(A)$ is a C*-subalgebra of $\tilde{\pi}(I(A))$, where $\pi = \tilde{\pi}|_A$. Without loss of generality, we assume that $I(A)$ is a von Neumann algebra acting on a Hilbert space. Consider the regular monotone completion $\overline{A}$ of $A$, which can be realised as the monotone closure of $A$ in $I(A)$ by Hamana’s theorem [12, Theorem 3.1]. Furthermore, because $I(A)$ is a von Neumann algebra, $\overline{A}$ is a von Neumann algebra, by Lemma 2.1. Thus, $A'' \subseteq \overline{A''} = \overline{A}$. As $A$ is separable and order dense in $A''$, the von Neumann algebra $A''$ is generated by its minimal projections, each of which is contained in $A$ (Lemma 2.4). Furthermore, by Lemma 2.2 $A''$ is a direct product of type I factors, which implies that $A''$ is injective by Corollary 2.3. Because $A \subseteq A'' \subseteq I(A)$, we conclude that $A'' = \overline{A} = I(A)$, by minimality of the injective envelope. This proves that (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

We next show that (3) $\Rightarrow$ (4). Assume there exists a faithful representation $\pi : A \to B(H)$ such that the von Neumann algebra $\pi(A)' = \overline{\pi(A)}$ is generated by its minimal projections, each of which is contained in $\pi(A)$. Without loss of generality, assume that $A$ is already represented as a subalgebra of $B(H)$ and that $M = A''$ is generated by its minimal projections, each of which lie in $A$.

Let $K \subseteq A$ be the ideal of $A$ generated by the minimal projections of $M$. We first show that $K$ is an essential ideal, minimal among all essential ideals of $A$. Suppose that $J \subseteq A$ is a nonzero ideal. Choose any nonzero $h \in J^+$. As shown in the proof of (1) of Lemma 2.4 there is a $\lambda > 0$ and a spectral projection $e \in M$ of $h$ such that $\lambda e \leq he$, and there is a minimal projection $p$ of $M$ such that $ep = pe = p$ and $0 \neq \lambda p \leq php \in J \cap K$. That is, $J \cap K \neq \{0\}$, which proves that $K$ is an essential ideal of $A$.

Because $M = A''$ is generated by its minimal projections, $M$ is a discrete type I von Neumann algebra, by Lemma 2.2. Hence, there is a faithful normal $*$-representation $\varrho$ of $M$ on a Hilbert space $H$ of the form $H = \bigoplus_n H_n$ such that $\varrho(K) \subseteq \varrho(A) \subseteq \varrho(M) = \prod_n B(H_n)$. Obviously, the minimal projections of any $B(H_n)$ are minimal projections of $\varrho(M)$ and are, hence, elements of $\varrho(K)$. On the other hand, if $e$ is a minimal projection of $\prod_n B(H_n)$, then $e \in B(H_n)$ for some $n \in \mathbb{N}$ (for otherwise $e$ is cut by some minimal central projection). Therefore, $\bigoplus_n K(H_n) \subseteq \varrho(K)$. However, $\varrho(K)$ is the smallest C*-algebra that contains the minimal projections of $\varrho(M)$; hence $\varrho(K) = \bigoplus_n K(H_n)$. Since $K \cong_\pi \bigoplus_n K(H_n)$, it is a minimal essential ideal.

We now prove that (4) $\Rightarrow$ (1). Suppose that $A$ has a minimal essential ideal $K$ such that $K \cong_\pi \bigoplus_n K(H_n)$. Therefore, by [1, Lemma 1.2.1],

$$M(K) = M\left(\bigoplus_n K(H_n)\right) = \prod_n M(K(H_n)) = \prod_n B(H_n),$$

which shows that $M(K)$ is a (type I) W*-algebra. Furthermore, because $K$ is a minimal essential ideal of $A$, $M(K) = M_{loc}(A)$ by [1, Remark 2.3.7]. Hence, $M_{loc}(A)$
is an injective $W^*$-algebra. However, $A \subseteq M_{\text{loc}}(A) \subseteq I(A)$ as $C^*$-subalgebras, and so by definition of the injective envelope, it must be that $M_{\text{loc}}(A) = I(A)$, which proves that $I(A)$ is a $W^*$-algebra.

3. Type I Injective Envelopes

One extension of Arveson’s fundamental theorem [2] on the injectivity of $B(H)$ is a result of Hamana [12, Proposition 5.2] that states that every type I $AW^*$-algebra is injective. The following theorem describes those separable $C^*$-algebras that have type I injective envelopes.

Theorem 3.1. If $A$ is a separable $C^*$-algebra $A$, then $I(A)$ is a type I $AW^*$-algebra if and only if $A$ has a liminal essential ideal. If this is the case, then $\overline{A} = I(A)$.

Proof. Assume that $A$ is separable and has a liminal essential ideal $K$. Because $\overline{A}$ and $\overline{K}$ are isomorphic [20, Corollary 2.1] and because $K$ is liminal, $\overline{A}$ is a type I $AW^*$-algebra [12, Theorem 6.6]. Hence, $\overline{A} = I(A)$ and $I(A)$ is a type I $AW^*$-algebra.

Conversely, assume that $I(A)$ is a type I $AW^*$-algebra. Because $\overline{A} \subseteq I(A)$ and because $\overline{A}$ and $I(A)$ have the same type I direct summands [12, Corollary 6.5], we conclude that $\overline{A} = I(A)$. Thus, $A$ is order dense in $I(A)$.

Because $I(A)$ is of type I, the $C^*$-subalgebra $J \subset I(A)$ generated by the abelian projections of $I(A)$ is a liminal ideal of $I(A)$ [13, Theorem 2]. We aim to prove that $K = A \cap J$ is a liminal essential ideal of $A$.

Suppose that $\alpha_0$ is an irreducible representation of $J$ on a Hilbert space $H_{\alpha_0}$. As $J$ is an ideal of $I(A)$, $\alpha_0$ extends uniquely to an irreducible representation $\alpha$ of $I(A)$ on the same Hilbert space $H_{\alpha_0}$. Thus, $\alpha(I(A)) \supseteq \alpha(J) = \alpha_0(J) = K(H_{\alpha_0})$.

If $\hat{J}$ denotes the spectrum of $J$ (unitary equivalence classes of irreducible representations of $J$) and if, for each $\alpha_0 \in \hat{J}$, $\alpha$ denotes the unique extension of $\alpha_0$ to an irreducible representation of $I(A)$, we consider the representation $\rho$ of $I(A)$ defined by

$$\rho = \bigoplus_{\alpha_0 \in \hat{J}} \alpha.$$

By construction, $\rho|_J$ is a faithful representation of $J$. We next show that $\rho|_A$ is a faithful representation of $A$. Suppose that $a \in A^+$ satisfies $\rho(a) = 0$. If $e \in I(A)$ is any abelian projection, then $eae \in J$ and $\rho(eae) = \rho(e)\rho(a)\rho(e) = 0$. Because $\rho|_J$ is a faithful representation of $J$, $eae = 0$; so, $a^{1/2}e = 0$. Thus, $a^{1/2}e = 0$ for all abelian projections of $I(A)$. Because $I(A)$ is a type I $AW^*$-algebra,

$$1 = \sup \{e : e \in I(A) \text{ is an abelian projection}\}.$$

Therefore, by [12, Lemma 1.9],

$$a = a^{1/2}1a^{1/2} = \sup \{a^{1/2}e a^{1/2} : e \in I(A) \text{ is an abelian projection}\} = 0,$$

which proves that $\rho|_A$ is a faithful representation of $A$.

(Indeed $\rho$ is a faithful representation of $I(A)$ as well. To prove this, suppose that $h \in I(A)^+ = \overline{A}^+$ satisfies $\rho(h) = 0$. Thus, $\rho(a) = 0$ for all $a \in A^+$ for which $a \leq h$. Since $\rho|_A$ is a faithful representation of $A$, $\rho(a) = 0$ only if $a = 0$. Because
Let $s \in J^+$ be nonzero and choose any $a_0 \in \mathcal{J}$. Then $\alpha(a)$ is compact for every $a \in A^+$ such that $a \leq s$. To verify this, fix $a \in A^+$ for which $a \leq s$; thus, $\alpha(a) \leq \alpha(s) = \alpha_0(s)$. Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a sequence in the unit sphere of the Hilbert space $H_{\alpha_0}$. By the compactness of $\alpha(s)^{1/2}$, there is a subsequence $\{\xi_{n_k}\}_{k \in \mathbb{N}}$ such that $\{\alpha(s)^{1/2}\xi_{n_k}\}_{k \in \mathbb{N}}$ is convergent. This implies that the sequence $\{\alpha(a)^{1/2}\xi_{n_k}\}_{k \in \mathbb{N}}$ is a Cauchy sequence, for

$$\|\alpha(a)^{1/2}\xi_{n_j} - \alpha(a)^{1/2}\xi_{n_m}\|^2 = \langle \alpha(a) (\xi_{n_j} - \xi_{n_m}), (\xi_{n_j} - \xi_{n_m}) \rangle \leq \langle \alpha(s) (\xi_{n_j} - \xi_{n_m}), (\xi_{n_j} - \xi_{n_m}) \rangle = \|\alpha(s)^{1/2}\xi_{n_j} - \alpha(a)^{1/2}\xi_{n_m}\|^2.$$

Hence, $\{\alpha(a)^{1/2}\xi_{n_k}\}_{k \in \mathbb{N}}$ is convergent, which yields $\alpha(a)$ compact. Since the choice of $\alpha_0 \in \mathcal{J}$ is arbitrary, this shows that $\rho(a) \in \rho(J)$ if $a \in A^+$ satisfies $a \leq s$. Because $\rho$ is faithful, this is to say that $a \in J$ if $a \in A^+$ satisfies $a \leq s$. Furthermore, since $s$ is nonzero and $A$ is order dense in $I(A)$, there is a nonzero $a \in A^+$ such that $a \leq s$. In particular, this nonzero $a$ belongs to $J$, thereby proving that $K = A \cap J \neq \{0\}$.

The previous paragraph establishes the following identity:

$$s = \sup \{a \in K^+ : a \leq s\}, \quad \forall s \in J^+.$$

This fact will now be used to prove that $K$ is an essential ideal of $A$. To this end, let $L$ be any ideal of $A$ for which $L \cap K = \{0\}$. Thus if $b \in L^+$, then $bab = 0$ for every $a \in K^+$. Now, if $e \in I(A)$ is any abelian projection, then $e \in J^+$ and

$$e = \sup \{a \in K^+ : a \leq e\}.$$

Therefore, again by [12] Lemma 1.9,

$$beb = \sup \{bab \in K^+ : a \leq e\} = 0.$$

Thus, $eb = be = 0$ for every abelian projection $e \in I(A)$, which implies that $b = 0$ (as demonstrated earlier in this proof). Hence, $L \cap K = \{0\}$ only if $L = \{0\}$ and so $K$ is an essential ideal of $A$.

The final point to verify is that $K$ is liminal. But this follows from the fact that every $C^*$-subalgebra of a liminal $C^*$-algebra is liminal [6] Proposition 4.2.4], and by noting that $K$ is a $C^*$-subalgebra of the liminal ideal $J$ of $I(A)$.

4. Applications

**Theorem 4.1.** The following statements hold for every separable $C^*$-algebra $A$.

1. $A$ has a liminal essential ideal if and only if $A$ has postliminal essential ideal.
2. If $A$ is abelian, then $I(A)$ is a $W^*$-algebra if and only if there exists a finite or countably infinite set $\Gamma$ such that $I(A) = l^\infty(\Gamma)$ and $c_0(\Gamma) \subseteq A \subseteq l^\infty(\Gamma)$.
3. If $A$ is simple and $I(A)$ is a $W^*$-algebra, then $A = K(H)$ for some Hilbert space $H$.
4. $I(A)$ admits a faithful state.
(5) If $A$ is prime, then exactly one of the following two statements holds:
   (a) $I(A) \cong B(H)$, for some separable Hilbert space $H$;
   (b) $I(A)$ is a wild type III $AW^*$-factor.

In particular, if $A$ has no postliminal essential ideal, then $I(A)$ is a wild type III $AW^*$-factor.

Proof. For the proof of (1), every liminal ideal is postliminal, by definition. Thus, assume that $A$ has a postliminal essential ideal, say $K$. As $A$ and $K$ are separable and $K$ is an essential ideal, $\overline{K} = \overline{A}$ (Theorem 1.5). Because $K$ is liminal, $\overline{K}$ is type I, and so $\overline{A} = I(A)$ is of type I. By Theorem 3.1 $A$ has a minimal essential ideal, which proves (1).

To prove (2), suppose now that $A$ is abelian and $I(A)$ is a $W^*$-algebra. By Theorem 2.5 $A$ has a minimal essential ideal $K$ for which $K \cong \bigoplus_{n \in \Gamma} K(H_n)$, for some finite or countable infinite set $\Gamma$; however, as $K$ is abelian, $K(H_n) = \mathbb{C}$ for every $n$, whence $K = c_0(\Gamma)$. As $K$ contains all minimal projections in $A$, we deduce that $A \subseteq l^\infty(\Gamma)$. Finally, $I(c_0(\Gamma)) = l^\infty(\Gamma)$ (because $c_0(\Gamma)$ is order dense in $l^\infty(\Gamma)$), so that $I(A) = l^\infty(\Gamma)$. The converse is a direct application of Theorem 2.5 where the minimal essential ideal of $A$ is $c_0(\Gamma)$.

To prove (3), assume that $A$ is simple and that $I(A)$ is a $W^*$-algebra. By Theorem 2.5 $A$ has a minimal essential ideal of the form $K = \bigoplus_n K(H_n)$. Being simple, $A = K$; and for $K$ to be simple, there can be only one summand. Thus, $A = K(H_1)$.

For the proof of (4), note that because $A$ is separable, $A$ has a faithful representation as a $C^*$-subalgebra of $B(H)$, where $H$ is a separable Hilbert space. Thus, by Hamana’s construction of the injective envelope, there is a projection $\phi : B(H) \to B(H)$ such that $\phi(B(H)) = I(A)$. The separability of $H$ implies that $B(H)$ has a faithful state $\omega$. This state is also faithful on the $C^*$-algebra representation of $I(A)$. To prove this, recall that the product $\circ$ on $I(A)$ is given by $x \circ y = \phi(xy)$, for all $x, y \in I(A)$. Suppose $x \in I(A)$ is such that $\omega(x^* \circ x) = 0$. Then $\omega(\phi(x^*x)) = 0$ and so $\phi(x^*x) = 0$, as $\omega$ is a faithful state on $B(H)$. Therefore, by the Schwarz inequality for completely positive maps, $0 \leq \phi(x^*x) \leq \phi(x^*x) = 0$. This implies that $\phi(x) = 0$. However, on $I(A)$ the map $\phi$ acts as the identity. Thus, $x = \phi(x) = 0$, which proves that $\omega$ is a faithful state on the $C^*$-algebra representation of $I(A)$.

To prove (5), assume now that $A$ is prime. By Proposition 1.7, $I(A)$ is a factor. But this factor cannot be of type II for the following reasons. Proposition 1.6 already excludes the case of finite type II $AW^*$-factors. By [7], every type II$_\infty$ $AW^*$-factor that admits a faithful state is a $W^*$-factor. Since $I(A)$ admits a faithful state and since $I(A)$ is a $W^*$-algebra only in the case where $I(A)$ is of type I (Theorem 2.5), it is impossible for $I(A)$ to be a type II$_\infty$ $AW^*$-factor. Hence, $I(A)$ is a factor of either type I or type III.

In the case where $I(A)$ is of type I we have $I(A) \cong B(H)$ for some Hilbert space $H$, because all type I $AW^*$-factors have this form [15, Theorem 2]. Indeed, in this case, $\overline{A}^\sigma = I(A) \cong B(H)$; since $\overline{A}^\sigma$ is countably decomposable, $H$ can be chosen to be separable.
If $I(A)$ is not of type I, then the type III AW*-factor $I(A)$ is cannot be a W*-algebra, by Theorem 2.5. Every AW*-factor that is not W*-algebra is wild [26]; hence, $I(A)$ is wild. □

We wish to remark that statement (4) of Theorem 4.1 above was previously noted (without proof) and employed in [12, Corollary 3.8].

Turning now to the local multiplier algebra, in most cases the precise determination of $M_{\text{loc}}(A)$ is difficult, and so one is interested to know what properties $M_{\text{loc}}(A)$ might exhibit. In particular, the following questions have been raised in the literature.

(Q1) For which C*-algebras $A$ is $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$? (11 21)
(Q2) For which C*-algebras $A$ is $M_{\text{loc}}(A)$ injective? (9 10)

Partial answers to these questions are listed in the theorem below.

**Theorem 4.2.** Assume that $A$ is a separable C*-algebra.

1. If $A$ has a liminal essential ideal, then $M_{\text{loc}}(M_{\text{loc}}(A))$ is an injective C*-algebra of type I and

   $$M_{\text{loc}}(M_{\text{loc}}(A)) = \overline{A} = I(A).$$

2. If $A$ has a minimal essential ideal that is *-isomorphic to a C*-algebraic direct sum of elementary C*-algebras, then $M_{\text{loc}}(A)$ is an injective W*-algebra of type I and

   $$M_{\text{loc}}(A) = M_{\text{loc}}(M_{\text{loc}}(A)) = \overline{A} = I(A);$$

   Proof. To prove (1), let $K$ be a liminal essential ideal of $A$. As $A$ and $K$ are separable and $K$ is an essential ideal, $\overline{K} = \overline{A}$. Because $K$ is liminal, $\overline{K}$ is type I, and so $\overline{A} = I(A)$ is of type I. Again using that $A$ and $K$ are separable and that $K$ is an essential ideal, conclude that from [21] Theorem 2.8 that $M_{\text{loc}}(M_{\text{loc}}(A)) = \overline{A}$. Hence, $M_{\text{loc}}(M_{\text{loc}}(A))$ is an injective C*-algebra of type I.

   For the proof of (2), note that Theorem 2.5 and its proof imply there is a minimal essential ideal $K$ of $A$ such that $K \cong \bigoplus K(H_n)$ and $M(K) = M_{\text{loc}}(A) = \overline{A} = I(A)$. Every AW*-algebra is its own local multiplier algebra [11 Theorem 2.3.8], and so

   $$M_{\text{loc}}(A) = M_{\text{loc}}(M_{\text{loc}}(A)) = I(A).$$

This completes the proof of (2). □

There is an unresolved issue: is $M_{\text{loc}}(A)$ injective if $A$ is separable and has a liminal essential ideal? Recall that if $K$ is an essential ideal of $A$, then $\overline{K} = \overline{A}$. Thus, it is sufficient to ask: is $M_{\text{loc}}(A)$ injective if $A$ is separable and liminal? This question is at present open.

**5. Nonseparable C*-algebras**

The focus of this paper has been on separable C*-algebras. For example, Proposition 1.6 and Theorem 2.5 do not hold for nonseparable C*-algebras. More specifically, if $R$ denotes the hyperfinite II$_1$ factor $R$, then $R$ is injective and, thus, $R = I(R)$ is a W*-factor of type II. However, this leads to another question of
interest: if $M$ is a nonhyperfinite $\text{II}_1$ factor, then what is the injective envelope of $M$? Because $M$ is simple, $I(M)$ is an AW*-algebra factor; is $I(M)$ a finite AW*-factor? More generally, does the passage from $M$ to $I(M)$ preserve type if $M$ is a von Neumann algebra?

Although Proposition 1.6 and Theorem 2.5 do not hold for nonseparable $C^*$-algebras, the necessity part of Theorem 3.1 was established without recourse to separability. Thus, the following theorem holds.

**Theorem 5.1.** If the injective envelope of a $C^*$-algebra $A$ is of type I, then $A$ has a liminal essential ideal.

The original motivation for the concept of injectivity is Arveson’s Hahn–Banach Extension Theorem [2] for completely positive linear maps, and the idea of an injective envelope stems from Arveson’s theory of boundary representations [3]. In the work on boundary representations, the algebras under consideration need not have been separable, but frequently the algebras were assumed to have nontrivial intersection with the compact operators. In this spirit we have the following result, which generalises one form the “boundary theorem” from $B(H)$ to discrete type I von Neumann algebras and which shows that statement (3) of Theorem 2.5 holds for nonseparable $C^*$-algebras as well.

**Theorem 5.2.** If $\pi : A \to B(H)$ is a faithful representation of a $C^*$-algebra $A$ on a Hilbert space $H$ such that $\pi(A)^\prime\prime$ is generated by its minimal projections, each of which is contained in $\pi(A)$, then $\pi(A)^\prime\prime = I(A)$.

**Proof.** Without loss of generality, we may assume that $A$ is already faithfully represented as a $C^*$-subalgebra of $B(H)$ such that $M = A^\prime\prime$ is generated by its minimal projections, each of which is contained in $A$. Because $M$ is generated by minimal projections, $M$ is an injective von Neumann algebra, by Corollary 28. To show that $M$ is the injective envelope of $A$, it is sufficient, by Proposition 1.1, to show that any completely positive linear map $\varphi : M \to M$ that fixes $A$ must be the identity map on $M$. If this is indeed so, then $M$ is an injective envelope for $A$ and, by the uniqueness of the injective envelope, we deduce that $M = I(A)$. If $\varphi : M \to M$ is a completely positive map such that $\varphi|_A = \text{id}_A$, then we will show that $\varphi = \text{id}_M$.

To this end, observe that because $\varphi : M \to M$ is a unital completely positive map that preserves $A$, $\varphi$ has the following property:

$$\varphi(xk) = \varphi(x)k, \text{ for every } k \in A.$$  

This fact follows from the Cauchy-Schwarz inequality and from the fact that $A$ is in the multiplicative domain of $\varphi$ (see [22, 9.2] or [16, Corollary 2.6]). Using this fact we shall deduce below that

$$x \geq 0 \text{ if and only if } \varphi(x) \geq 0. \tag{5.1}$$

Indeed, one implication is obvious from the positivity of $\varphi$. To prove the other implication, assume that $\varphi(x) \geq 0$. Thus, $\varphi(\text{Im}(x)) = \text{Im}(\varphi(x)) = 0$. Let $z = \text{Im}(x)$ and write $z = z^+ - z^-$, where $z^+, z^- \in M^+$ are such that $z^+z^- = z^-z^+ = 0$. 

Our first goal is to prove that \( z^+ = 0 \). Suppose, on the contrary, that \( z^+ \neq 0 \). Thus, there is a strictly positive \( \lambda \) in the spectrum of \( z^+ \); hence, there is a spectral projection \( p \in M \) such that \( 0 \neq \lambda p \leq p z^+ = z^+ p \). Note that \( z^- p = 0 \), as the projection \( p \) is in the von Neumann algebra generated by \( z^+ \) and \( z^- z^- = z^- z^+ = 0 \). Let \( q \in A \) be an arbitrary minimal projection of \( M \) and consider the projection \( p \wedge q \in M \). Because \( p \wedge q \leq q \) and \( q \) is minimal, either \( p \wedge q = 0 \) or \( p \wedge q = q \). We will show that the latter case cannot occur (under the conventional assumption that minimal projections are defined to be nonzero). Assume that it is true that \( p \wedge q = q \). Then \( 0 \neq q = p \wedge q \leq p \). Pre- and post-multiply the inequality \( \lambda q \leq \lambda p \leq z^+ p = zp \) by \( q \) to obtain \( \lambda q \leq q(zp)q \leq qzq \). Note that \( \varphi(zq) = \varphi(z)q \) (because \( A \) is in the multiplicative domain of \( \varphi \)) and that \( \varphi(z) = 0 \) (by hypothesis). Likewise, for any hermitian \( y \in M \), \( \varphi(qy) = \varphi(yq)^* = q \varphi(y) \). Thus, \( \varphi(qzq) = q \varphi(z)q = 0 \) and \( 0 \leq \lambda q = \varphi(\lambda q) \leq q \varphi(z)q = 0 \). This implies that \( q = 0 \), which contradicts the fact that \( q \) is minimal and, thus, nonzero. Therefore, it must be that \( p \wedge q = 0 \), for every minimal projection \( q \) of \( M \). Because every nonzero projection in \( M \) majorises a minimal projection, we conclude that \( p = 0 \), in contradiction to the fact that \( p \) is a nonzero spectral projection of \( z^+ \). Hence, it must be that \( z^+ = 0 \).

A similar argument shows that \( z^- = 0 \). We can find a nonzero \( \lambda \in \mathbb{R}^+ \) and a minimal projection \( q \in A \) such that \( qzq \leq -\lambda q \); thus \( -\lambda q = \varphi(-\lambda q) \geq \varphi(qzq) = q \varphi(z)q = 0 \), and again \( q = 0 \).

We conclude that \( z = 0 \), which implies that \( x \) is selfadjoint. It remains to show that \( x \) is positive. Assume that \( x \) is not positive. Thus, there exists a nonzero spectral projection in the negative part of \( \sigma(x) \); by taking once again a suitable minimal subprojection \( q \), we can find \( \lambda > 0 \) such that \( qxq \leq -\lambda q \). But then \( \varphi(qxq) \leq -\lambda q \); and on the other hand, \( \varphi(qxq) = q \varphi(x)q \geq 0 \). The contradiction implies that no such \( q \) can exist, and so \( x \geq 0 \).

From \( (5.1) \) and the fact the \( \varphi \) preserves \( A \), we have that \( k \in A \), \( k \leq x \) if and only if \( k \leq \varphi(x) \). Statement \( (1) \) of Lemma \( (2.4) \) asserts that \( A \) is order dense in \( M \). Hence, \( \varphi(x) = x \) for every \( x \in M^+ \), which implies that \( \varphi \) is the identity map on \( M \).

6. Open Questions

Although this paper is mainly concerned with type I injective envelopes of separable \( C^* \)-algebras, there are a number of unresolved questions that underscore the limits of our current state of knowledge concerning injective envelopes in general. A few such questions are listed here.

(1) Suppose that \( A \) is a separable \( C^* \)-algebra.
   (a) Is \( M_{\text{loc}}(A) \) an AW*-algebra?
   (b) Is \( \overline{A}_I = I(A) \) if \( I(A) \) is of type III?

(2) Suppose that \( \bigotimes_{1}^{\infty} M_2 \) and \( \bigotimes_{1}^{\infty} M_3 \) denote the UHF \( C^* \)-algebras obtained through the tensor products of the matrix algebras \( M_2 \) and \( M_3 \) respectively.
The injective envelope of each of these C\(^*\)-algebras is a wild type III AW\(^*\)-factor. Is it true that
\[
I \left( \bigotimes^\infty_1 M_2 \right) = I \left( \bigotimes^\infty_1 M_3 \right) ?
\]

(3) Suppose that \( M \) is a von Neumann algebra.
(a) What is \( I(M) \) if \( M \) is not injective?
(b) If \( M \) is a non-injective type \( \Pi_1 \) factor, then is the AW\(^*\)-factor \( I(M) \) also of type \( \Pi \)?

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