A generalized pricing and hedging framework for multi-currency fixed income desks

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Abstract

It is well known that traded foreign exchange forwards and cross currency swaps (CCS) cannot be priced applying cash and carry arguments. This paper proposes a generalized multi-currency pricing and hedging framework that allows the flexibility of choosing the perspective from which funding is managed for each currency. When cross currency basis spreads collapse to zero, this method converges to the well established single currency setting in which each leg is funded in its own currency. A worked example tests the quality of the method.

1 Introduction

Before the financial crisis started in July 2007 with Bear Stearns default, interest rate desks would essentially use a unique interest rate curve for each currency to price and hedge all derivative products. The crisis showed that big investment banks could default and therefore lending and borrowing activities became severely restricted, no longer allowing for certain hedging strategies. Ever since, basis spreads of tenor swaps where no longer negligible.

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After some debate, today there exists a broad consensus about how this issue must be handled for a single currency setting. Regarding different tenors, i.e. 3 and 6 month FRA (Forward Rate Agreement), as independent entities, leaves enough room for new variables to fit the arbitrage free equations that traded products must satisfy. In the single currency setting (see [1]), the discount curve for each currency is build out of overnight indexed swaps (OIS) through a bootstrapping algorithm. Forward Libor estimation curves for each different tenor frequency are thereafter calculated out of par and tenor swap quotes by another bootstrapping algorithm given the previously calculated discount curve. Some theoretical models have been proposed for the dynamics of these curves (e.g. see [2], [3] and [4]).

In the cross currency swap market a similar situation exists. The price of traded foreign exchange (FX) forwards and cross currency swaps (CCS) can not be exactly derived from arbitrage free models calibrated exclusively to the Swap Markets on each currency and the FX spot price, (see [5] for a heuristic analytic formula to predict basis tenor and cross currency swaps). This is so, because FX forward and CCS prices are, among other things, driven by trading flows and as long as the differences between the “Theoretical Price” and the market price remain low, the expected return will not be enough to compensate the bank capital expenses. Unfortunately, for the multi-currency setting, it is not easy to introduce new variables in a similar way as it was done for the single currency setting (see [13]).

The paper proposes a generalized framework to price and hedge cross currency swaps. This framework makes it possible to map the entity funding structure while still matching liquidly traded products i.e. it allows centralizing funding or simply fund each leg in its own currency. Starting from the single currency setting in each currency, it is based on the decomposition of customized cross currency swaps as a combination of market quoted currency swaps plus a minor amount of additional cash flows which embed the pricing and hedging inaccuracies which are model-dependent. These minor additional cash-flows can then be OIS-discounted (or funded) in their own currency or converted to another currency through foreign exchange forwards and thereafter OIS-discounted or funded in that currency. This naturally allows choosing the currency perspective from which funding is managed. When cross currency basis spreads tend to zero, this multi-currency pricing converges to the valuation of cash flows in each currency according to the single currency setting. Valuation of complex multi-currency exotic products such as callable swaps are beyond the scope of this framework.
The paper assumes full collateralization of trades (see [1]) and a funding structure between Front Office desks and the balance sheet of the bank or ALCO (Asset and Liability Committee) at the overnight index-rate at least on one currency, provided that desks do not have a consistent liquidity imbalance between borrowing and lending (see [6], [7] and [8] for more information about discounting and collateralized derivative pricing). When dealing with uncollateralized pricing and even if these hypotheses are not fulfilled, the collateralized framework will always provide a reference “risk-free” pricing on top of which funding and credit value adjustments can be applied (see [9], [10], [11], [12], [13] for an overview of these adjustments).

The paper starts with section 2 which defines the products involved in the discussion. Section 3 presents a study of foreign exchange and cross currency markets to illustrate their driving forces and motivate the proposed methodology. Section 4 shows how the funding currency is chosen for each currency. The proposed decomposition method to price cross currency swaps and foreign exchange forwards is presented in sections 5 to 7. Section 8 compares the proposed method with a benchmark using a worked example. Finally, section 9 concludes.

2 Product definition

This section describes and defines the products and the notation which will be used across the paper. Figure 1 presents the structure of payments of a floating rate note. The issuer of the note receives the notional (represented by 1) on the start date and pays floating interest, \( L_i = L(t_i, t_{i+1}) \) (e.g. Libor or Euribor index), for each period from \( t_i \) to \( t_{i+1} \), according to the frequency of the note. The fixing of the Libor-index on \( t_i \), \( L_i \), is multiplied by the year fraction, \( \tau_i \), of the period from \( t_i \) to \( t_{i+1} \). On expiry date, \( t_N \), the holder of the note pays the floating interest of the last period and returns the notional. The floating indices fix their value usually two business days before the start of each period and are paid two business days after the end of the period depending on the conventions. The notation \( FRN_{t_0, t_N}^C \) represents the remaining cashflows of a FRN that occur beyond \( t_0 \) and ending on \( t_N \), \( s^C \) is a fixed spread added to the floating leg and \( C \) is the currency in which fixed and floating payments are denominated (\( C \) corresponds to USD in figure 1).

Figure 2 presents the structure of a non-resettable cross currency swap (NCS). The main purpose of this structure is to transfer funding from one
Figure 1: Cash flow structure of a floating rate note (FRN).

Figure 2: Cash flow structure of a non-resettable currency swap (NCS).

\[
FRN_{t_0, t_N}^{s_0, s_N}
\]
currency to another (e.g. giving a loan in domestic currency and financing it by borrowing money in a foreign currency plus entering into a cross currency swap). This structure will be denoted $NCS_{t_0 \sim t_N}^{s_{Cd}, s_{Cf}}$, where $C_d$ is the domestic currency (EUR in figure 2), $C_f$ is the foreign currency (USD in figure 2), $X_i$ is the exchange rate fixing on date $t_i$ in units of foreign currency per unit of domestic currency, $s_{Cf}$ is the fixed spread added to the foreign floating leg and $s_{Cd}$ is the spread added to the domestic floating leg.

![Diagram](image.png)

**Figure 3: Cash flow structure of a resettable or market cross currency swap (CCS).**

To reduce counterparty exposure due to foreign exchange risk, banks usually trade resettable currency swaps (CCS) of figure 3 instead of the non-resettable structure of figure 2. The foreign notional is reset at the end of each period according to the foreign exchange rate at that moment. The upper dotted arrows of 1 EUR correspond to the notional returned at the end of each period and the 1 EUR lower dotted arrows correspond to the notional received at the beginning of each period starting on that date. Both lines...
are dotted because they cancel each other. This structure will be denoted by \( CCS_{t_0, t_N}^{s_d, s_f} \).

\[
NCS_{t_0, t_N}^{s_d, s_f} = FRN_{t_0, t_N}^{s_d} - X_{t_0} \cdot FRN_{t_0, t_N}^{s_f}
\]

(1)

\[
NCS_{t_0, t_N}^{s_d, s_f} = CCS_{t_0, t_N}^{s_d, s_f} + \sum_{i=1}^{N-1} \left( X_{t_i} - X_{t_{i-1}} \right) FRN_{t_i, t_N}^{s_f}
\]

(2)

Equation (1) presents the decomposition of a non-resettable NCS into a sum of two floating rate notes and equation (2) shows how a NCS can be decomposed into a resettable CCS plus a sum of floating rate notes, where \( X_{t_i} \) denotes the foreign exchange fixing on \( t_i \).

3 Study of FX forward and CCS markets

Under the assumption of no arbitrage, a foreign exchange forward should be priced using a simple cash and carry reasoning according to equation (3), where \( X_{t,T} \) denotes the forward foreign exchange rate from present time, \( t \), to expiry, \( T \).

\[
X_{t,T} = X_0 \cdot \frac{DF_{d}^{t_T}}{DF_{f}^{t_T}}
\]

(3)

Considering that fixed income desks fund collateralized derivatives at the overnight-index rate, the discount factors of equation (3) should be apparently calculated according to OIS curves. However, foreign exchange markets are also shared by other participants such as money market desks, which will apply the same cash and carry reasoning but with a completely different set of market instruments such as deposit rates. In addition, resettable currency swap (CCS) structures of different maturities may also imply foreign exchange forwards. Before the crisis, the difference among these three methodologies was negligible. However, at this moment it is not. This means that a careful analysis is needed to know where real prices come from and why.

Figure 4 compares the historical difference in basis points of the foreign exchange forward quoted in the market and those implied from deposit rates, OIS curves and CCS for maturities of 3, 6 and 9 months and 1 year. The first two methods (deposit and OIS) use equation (3) with different discount
Figure 4: Evolution of difference in basis points between market and calculated foreign exchange forwards varying method (“depo”, “OIS” and “CCS”) and maturity (3m, 6m, 9m and 1y).
factors coming from either deposit rates \((DF_{t,T} = \frac{1}{1+r_T}, \text{ where } r_T \text{ is the deposit rate with maturity } T)\) and OIS curves bootstrapped from OIS swaps. The third method (CCS) estimates forward foreign exchange rates out of a bootstrapping method which calculates foreign exchange forwards implied from forcing zero valuation of market currency swaps.

It is clear from figure 4 that for the four maturities, foreign exchange forwards estimated from OIS curves show the highest difference with respect to actual traded forwards, followed by those estimated by deposit rates. Foreign exchange forwards estimated from CCS are very aligned with market. These plots suggest that the foreign exchange market is shared by participants financed with deposit (money market) and overnight rates (derivatives), having those financed with deposits a bigger weight. The reason why CCS and foreign exchange forwards are aligned is because both are zero cost contracts which are collateralized (this does not happen with deposits).

The misalignment of the foreign exchange forward rates does not come out of wrong discounting. Theoretically, the foreign exchange forward could be arbitraged with the cash and hold argument (foreign exchange forwards are normally collateralized). However, this would imply altering the bank cash balance forcing borrowing huge amounts of one currency and lending them into another. This situation is not sustainable as the ALCO would not provide funding at the overnight index rate for big amounts of cash consistently in the same direction. Therefore, the challenge is to build pricing and hedging models in this situation of misaligned markets.

4 Choosing valuation perspective

Most of the foreign exchange risk from multi-currency fixed income desks comes out of cross currency swaps. Sections 5 and 6 show how customized cross currency swaps can be decomposed in terms of market quoted currency swaps, whose value will be zero, plus a marginal set of additional cash flows which have to be properly valued too.

Consider the evaluation operator, \(V_t^{C_1} [1_{\{t=T\}}^{C_2}]\), where the indicator function, \(1_{\{t=T\}}^{C_2}\), represents a cash flow payment in currency \(C_2\) at time \(T\). This operator assigns to this cash flow, a value at time \(t\) in currency \(C_1\) without saying anything about how the cash flow is funded. For that purpose, the expectation operator \(E_t^{C_1} [1_{\{t=T\}}^{C_2}]\) is introduced. It represents the value in currency \(C_1\) at time \(t\) of the cash flow, funded in currency \(C_1\). Equation 4
shows how this expectation is calculated depending on the funding currency, where $DF_{t,T}^C$ is the OIS discount factor between $t$ and $T$ of currency $C$ and $X_{t,T}$ is the forward exchange rate at time $t$ to change one unit of currency $C_1$ to $C_2$ at time $T$ (these forward exchange rates will be calculated according to section 7).

$$E_t^C[1_{(t=T)}^C] = DF_{t,T}^C$$

$$E_t^{C_1}[1_{(t=T)}^{C_2}] = X_{t,T}^{-1}DF_{t,T}^{C_1}$$

(4)

Since current regulation [16] enforces inter-bank trades to be liquidated through Clearing Counter Parties (CCP) e.g. London Clearing House [17] and collateral interest payments are indexed to overnight rates i.e Eonia, Fed Funds, Sonia ... the authors suggest valuing these remaining cashflows applying the corresponding overnight discount curve on each currency according to equation (5), where $X_t$ is the spot exchange rate to change one unit of $C_1$ into $C_2$.

$$V_t^{C_1}[1_{(t=T)}^{C_2}] = X_t^{-1}E_t^{C_2}[1_{(t=T)}^{C_2}] = X_t^{-1}DF_{t,T}^{C_2}$$

(5)

However, some institutions might decide to fund them in another currency. In this situation, cash flows can be exchanged to that currency with foreign exchange forwards and discounted with the OIS curve of that currency according to equation (6). This way, the cross currency basis risk is well taken into account for pricing and hedging and the funding currency can be easily chosen.

$$V_t^{C_1}[1_{(t=T)}^{C_2}] = E_t^{C_1}[1_{(t=T)}^{C_2}] = X_t^{-1}DF_{t,T}^{C_1}$$

(6)

$$E_t^{C_d}[FRN_{t_0,t_N}^{C_f}] \approx \frac{DF_{t,t_0}^{C_d}}{X_{t,t_0}} - \sum_{i=0}^{N-1} \left( L_{t,i}^{C_f} + s_{C_f} \right) \tau_{t,i}^{C_f} \frac{DF_{t,t_{i+1}}^{C_d}}{X_{t,t_{i+1}}} - \frac{DF_{t,t_N}^{C_d}}{X_{t,t_N}}$$

(7)

Equation (7) shows the valuation of an FRN denominated in foreign currency, $C_f$, from the domestic perspective (valued and funded in domestic currency, $C_d$), where $L_{t,i}^{C_f}$ is the forward Libor index rate of currency $C_f$ at time $t$ of the period from $t_i$ to $t_{i+1}$, $X$ will represent from now on the exchange rate from domestic to foreign currencies and $\tau_{t,i}^{C_f}$ is the year fraction from $t_i$ to $t_{i+1}$ according to the conventions of currency $C_f$. See that the result of equation (7) will not be equal to $E_t^{C_f}[FRN_{t_0,t_N}^{C_f}] X_t^{-1}$, unless the cross currency basis spread is equal to zero.
The heuristic equation (7) is indeed an approximation as it is assuming that Libor and foreign exchange rates are independent. However, this fact might not have much impact, because the floating cash flows which do not net out in the derivative portfolio will be hedged and therefore swapped into fixed cash flows which will be properly priced according to equation (4).

This second approach makes the proposed framework equivalent to a popular class of heuristic models described on [15] and known as “four curve model”. Basically, once domestic and foreign estimation and discount curves have been calibrated to the overnight and reference swaps on each currency, the foreign discount curve will be “recalibrated” to fit cross currency quotes. The main drawback of this procedure is that foreign swaps are no longer correctly priced.

5 Pricing fwd start market currency swaps

This section presents how to calculate the market spread of a forward starting resettable currency swap which makes its present value equal to zero. The method is based on pricing each leg according to the single currency setting. As this valuation is only correct when the cross currency basis spread is zero, it is corrected with market quoted currency swaps as similar as possible to the one considered.

\[ p(T) = 1 + \min(k) \text{ such that } T - t \leq \sum_{j=0}^{k} \tau_{mkt}^j \]  
\[ q(T) = 1 + \min(k) \text{ such that } T - t_U \leq \sum_{j=0}^{k} \tau_{prd}^j \]  
\[ t_{mkt}^i = t + \sum_{j=0}^{i-1} \tau_{mkt}^j \quad t_{prd}^i = t_U + \sum_{j=0}^{i-1} \tau_{prd}^j \]

Consider \( t_{mkt}^i \) the market schedule of payments considered at present time, \( t \), for a given tenor frequency (e.g. 3 months, 6 months, etc) with year fractions \( \tau_{mkt}^i \) corresponding to periods from \( t_{mkt}^i \) to \( t_{mkt}^{i+1} \). Equation (8) returns for a given time \( T \), the period number ending at or beyond \( T \) (see that \( t_{mkt}^{mkt} = t_{mkt}^0 \)). The sequence of market periods start at present time \( t \). The first period goes from \( t_{mkt}^{mkt} = t \) to \( t_{mkt}^1 \) and has a year fraction of \( \tau_{mkt}^0 \). Similarly, \( t_{prd}^i \), is the schedule of payments of a customized product starting at
and ending at \( t_V \). Equation (9) returns for time \( T \), the period number of the product schedule ending at or beyond \( T \).

\[
t_{\text{mkt}}^m = t + \sum_{j=0}^{p(t_H)-1} \tau_{\text{mkt}}^m \\
t_{\text{pr}}^m = t + \sum_{j=0}^{p(t_H)-2} \tau_{\text{pr}}^m
\] (11)

Equation (11) shows the nearest times of the market schedule before \( (t_{\text{mkt}}^m) \) and after \( (t_{\text{mkt}}^m) \) a given time, \( t_H \). Equation (12) shows the same thing for the schedule of the customized product considered for valuation. Resettable currency swaps quoted in the market start at present time, \( t \), in terms of the moving payment date schedule defined by the conventions of the currency considered.

\[
V^C_{t}[CCS_{t,t_V}^{C_{d}}] = V^C_{t}[CCS_{t,t_U}^{C_{d}}] + V^C_{t}[CCS_{t,U,V}^{C_{d}}] \] (13)

\[
p(t_V) \sum_{i=1}^{s_{V}} s_{V}^{mkt} \tau_{i-1}^{mkt} DF_{t,t_i}^{C_d} \]
\[
p(t_U) \sum_{i=1}^{s_{U}} s_{U}^{mkt} \tau_{i-1}^{mkt} DF_{t,t_i}^{C_d} + \sum_{i=p(t_U)+1}^{t_{pr}+p(t_V)-2} s_{U,V}^{mkt} \tau_{i-1}^{mkt} DF_{t,t_i}^{C_d}\] (14)

\[
E^C_{t} \left[ FRN_{t_{U},t_{V}}^{C_{d},s_{U}^{SCS}} \right] - X_{t}^{-1} \sum_{j=1}^{q(t_V)} X_{t,pr,j}^{SCS} E^C_{t} \left[ FRN_{t_{U},t_{V}}^{C_{d},s_{U}^{SCS},C_{f}=0} \right] = 0 \] (15)

Equation (13) shows the relation between spot and forward starting CCS. A quoted spot starting CCS expiring at \( t_V \) can be expressed as the composition of a spot starting CCS expiring at \( t_U \) and a forward starting CCS starting at \( t_U \) and expiring at \( t_V \). See that the fixed spreads are set on the domestic currency leg (this case would correspond to a domestic currency different from USD). The notional and floating payment structure is completely equivalent from both sides of equation (13). Only the fixed spreads added to the floating legs differ between both sides. Notice that \( s_{V}^{mkt} \), \( s_{U}^{mkt} \)
are quoted in the market and $s_{UV}^{mkt}$ is not but can be derived from them through equation (14).

Equation (15) shows the pricing equation of the forward starting CCS alone, starting at $t_U$ and expiring at $t_V$ in which each leg is priced according to the single currency setting (each leg is funded in its own currency). The spread, $s_{UV}^{SCS}$, is calculated such that equation (15) is satisfied. The first expectation is the valuation of an FRN with the payments of the domestic leg of the CCS funded in the domestic currency. The second expectation is the valuation of the resettable foreign leg from the foreign perspective. After the valuation under the single currency setting, it is converted to domestic currency through the spot exchange rate. The resettable structure (second expectation) is expressed as a sum of forward starting single period floating rate notes whose notional is dependent on the forward exchange rate $X_{t,U,t_V}^{SCS}$ calculated according to section 7. Equation (15) only prices the forward starting currency swap correctly when the cross currency basis spread is zero.

$$e_{UV} = s_{UV}^{mkt} - s_{UV}^{SCS}$$

Equation (16) shows the spread difference between the market and single currency setting (SCS). This is the error made by the SCS model. However, this difference is not calculated for the actual forward starting CCS from $t_U$ to $t_V$, but for two similar forward CCS with the same maturity but starting on $t_U$, the previous date to $t_U$ out of the market schedule $\{t_i^{mkt}\}$, and $t_U$, the following date to $t_U$ from this schedule. It is clear that the estimation of the error $e_{UV}$ would be very accurate as $s_{UV}^{mkt}$ obtained from (14) exactly corresponds to a forward CCS derived from direct market quoted swaps. The same happens for $e_{UV}$.

$$e_{UV} = \frac{t_U}{t_{mkt}^{U}} - \frac{t_{mkt}^{U}}{t_U} e_{UV} + \frac{t_U - t_{mkt}^{U}}{t_{mkt}^{U}} e_{UV}$$

1It is assumed that the frequency of the floating legs is 3 months. If the floating leg frequency is different from 3 months in either leg, a synthetic market quote for the spot CCS should be obtained introducing the quotes of tenor swaps.

2See that independence of foreign exchange and Libor rates has been assumed: $E[X_{t_U} FRN] = X_{t,t_U} E[FRN]$. This approximation is not a major problem since $s_{UV}^{SCS}$ is only calculated for interpolation purposes.
Equation (17) obtains the pricing error of the SCS model for the customized forward CCS as a linear interpolation between \( e_{UV} \) and \( e_{UV} \). This interpolation has good accuracy as it involves only the pricing error of the SCS model in between two well-calculated market points. On the other hand, the SCS model calculates the correct spread when the cross currency basis is zero. Therefore, the accuracy is not significantly lost by the error interpolation because it is indeed small.

\[
{s_{UV}^{mkt} = s_{UV}^{SCS} + e_{UV}}
\]  

Equation (18) finally obtains the forward CCS spread from the single currency setting out of equation (15), corrected by an estimation of the error, \( e_{UV} \), which is obtained through linear interpolation between two accurately-estimated errors. This spread makes the net present value of the structure equal to zero.

This way of interpolating allows sticking the valuation as much as possible to market. Approximations are only carried out on a very small contribution given by the error of the single currency setting. This error would disappear as the cross currency basis spread approaches to zero. Therefore, this pricing method naturally reduces to the single currency setting when the cross currency basis spread disappears.

6 Pricing customized currency swaps

This second step prices a customized resettable currency swap through a decomposition procedure in which the currency swap is expressed as the sum of a forward starting market currency swap (priced according to section 5) plus some additional small contributions priced according to section 4 which allow flexibility to choose the funding and hedging perspective. Pricing inaccuracies in these small contributions do not usually have a significant impact in pricing.

Consider the customized currency swap of figure 3 where the domestic and foreign currencies are EUR and USD and the start and end dates are \( t_U = t_0 \) and \( t_V = t_N \). The swap is valued in between two fixing dates at time \( t \), where \( t_{L}^{prd} \) is the date of the last floating rate fixing, \( t_{L}^{prd} \) is the following date in the product schedule \( \{t_{i}^{prd}\} \) and \( t_{L}^{prd} < t < t_{L}^{prd} \). According to equation (1), \( t_q(t_L) = t_L \) and \( t_q(t_L) = t_L \).
Equation (19) shows the currency swap decomposition where the left hand side is the customized swap with fixed spreads in both legs, $s_{Cd}^{UV}$ and $s_{Cf}^{UV}$. The first term of the right hand side is a forward starting resettable currency swap whose spread, $s_{mkt}^{LV}$, is obtained according to equation (18). The second and third terms represent the swap structure of the already started hub period (exchange of notionals and floating payments at the end of the period) and the sum adds fixed payments to the domestic leg, equal to the difference between the customized and market spreads, and the fixed spread, $s_{Cf}^{UV}$, of the foreign curve.

No valuation has been performed yet, only a payoff decomposition. At this point, the only multi-currency piece of the decomposition is the forward starting CCS whose valuation is zero ($s_{mkt}^{LV}$ is calculated to satisfy this condition) and does not need any funding. The rest of the pieces involve single currency fixed cash flows. The first two, given by equation (20), must be priced assuming a common funding currency (either domestic or foreign) taking expectations, either $E_{t}^{Cd} [\cdot]$ or $E_{t}^{Cf} [\cdot]$, of the indicator functions according to equation (5). This common funding avoids the inconsistency between the market forward foreign exchange rate, $X_{t_t}^{t_{prd}}$, and the rate given by equation (23) of OIS discount factors. Equation (21) shows the joint valuation of these two cash flows expressed in units of domestic currency. Foreign exchange forwards are interpolated according to section 5.

3The FRN cash flows in equation (20) change signs with respect to equation (19) because the notation for a long position of FRN was defined returning the notional and paying the floating rate at expiry.

4This inconsistency could allow for an arbitrage, as these cash flows are big enough (they involve a Libor rate instead of a spread) and the term between their fixing and payment is very liquid (just 3 months).
\[
CF^{FRN} = CF^{C_f} \{t = t_{prd}^f\} - CF^{C_d} \{t = t_{prd}^d\}
\]
\[
CF^{C_d} = \left[1 + \left(L^{C_d}_{t_{prd}^d} + s^{C_d}_{UV}\tau_{q(t_{prd}^d)}\right)\right]
\]
\[
CF^{C_f} = X\{t_{prd}^f\} \left[1 + \left(L^{C_f}_{t_{prd}^d} + s^{C_f}_{UV}\tau_{q(t_{prd}^d)}\right)\right]
\]  

(20)

\[
V^C_d[CF^{FRN}] = E^C_d[CF^{FRN}] = (CF^{C_f}X_{t_{t_{prd}^d}}^{-1} - CF^{C_d})DF^{C_d}_{t,t_{t_{prd}^d}}
\]

\[
V^C_f[CF^{FRN}] = \frac{E^C_f[CF^{FRN}]}{X_t} = X_t^{-1}(CF^{C_f} - CF^{C_d}X_{t_{t_{prd}^d}})DF^{C_f}_{t,t_{t_{prd}^d}}
\]  

(21)

Similarly, the rest of cash flows of equation (19) should also be jointly funded in a chosen centralized currency. However, as these payments are very small (just some basis points) and are spread out along many maturities, they can be priced assuming funding in the currency in which they are denominated (instead of a centralized currency) or any other currency, calculating the expectation of the indicator functions using equation (4).

\[
E^C_d[X_{t_{j}}FRN^{C_f}_{t_{t_{j}}t_{N}}] \approx X_{t_{t_{j}}}E^C_d[FRN^{C_f}_{t_{t_{j}}t_{N}}]
\]  

(22)

Non-resettable currency swaps can be priced according to the decomposition of equation (22). The first term of the right hand side is a customized resettable cross currency swap which can already be priced. The second term is a sum of expectations of floating rate notes. If these terms are chosen to be funded in a different currency from which they are denominated (e.g. the domestic currency, \(C_d\), or the collateral currency), each of these expectations is approximated according to equation (22), assuming that the evolution of foreign exchange and swap rates are independent of each other.

7 Pricing foreign exchange forwards

In order to interpolate forward foreign exchange rates on a particular date \(t_{t_{j}}^{prd} \in t_{t_{j}}^{prd}\) as needed for equations (4) to (7) and equation (21), a similar method is used as section 5. Consider that \(X^{mkt\_j}_{t_{t_{j}}t_{j}^{mkt}}\) are the market quoted foreign exchange rates from the swap point quotes.

\[
X^{SCS}_{t,T} = X_t \frac{DF^{C_d}_{t,T}}{DF^{C_f}_{t,T}}
\]  

(23)
Equation (24) shows the errors of the forward foreign exchange rates on dates \( t_{mkt} \) and \( t_{mkt} \) when the single currency setting (SCS) is used. The forward foreign exchange rates, \( X_{SCS}^{t,T} \), are calculated according to equation (23), where the discount factors are taken from OIS curves. The dates from the market schedule, \( t_{mkt} \) and \( t_{mkt} \), are in between date \( t_{prd} \) of the product schedule. The exchange rate error is well known on dates \( t_{mkt} \) and \( t_{mkt} \) as they belong to the market schedule. To estimate the error on \( t_{prd} \), a linear interpolation is carried out as shown by equation (25).

\[
e_{t_{mkt}} = X_{t_{mkt}}^{mkt} - X_{t_{mkt}}^{SCS} \quad e_{t_{mkt}} = X_{t_{mkt}}^{mkt} - X_{t_{mkt}}^{SCS}
\]

\[
e_{t_{prd}} = \frac{t_{mkt} - t_{prd}}{t_{mkt} - t_{mkt}} e_{t_{mkt}} + \frac{t_{mkt} - t_{pdata}}{t_{mkt} - t_{mkt}} e_{t_{mkt}}
\]

Equation (26) shows how the exchange rate on \( t_{tdata} \) is finally priced with the single currency setting and corrected with the interpolated error \( e_{t_{data}} \). See that when the cross currency basis spread disappears, the error is equal to zero and this pricing methodology smoothly converges to the single currency setting.

8 Worked example

This section presents a worked example comparing two methods with different financing schemes: all remaining foreign cash flows are funded in domestic currency\(^5\) (four-curve method), or each remaining cash flow is discounted in its own currency (Multi-Funding using proposed method). This will be done for the resettable (CCS) and non-resettable (NCS) currency swaps of figures 2 and 3 with legs denominated in USD and EUR and notional amount of 100 million EUR. Market data has been taken on January 29th 2014, with spot and forward foreign exchange rates in USD per EUR of {Spot: 1.3533, 1y: 1.3543, 2y: 1.3610, 3y: 1.3741, 4y: 1.3928, 5y: 1.4143, 7y: 1.4589, 10y:

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\(^5\)For the proposed method, this is equivalent to convert cash flows to domestic currency through foreign exchange forwards and discount with domestic OIS curve.
1.5145}, and a cross currency basis spread curve in basis points added to the EUR floating leg of \{1y: -4, 2y: -5.5, 3y: -6.25, 4y: -7, 5y: -7, 7y: -6.75, 10y: -5.75, 15y: -4.75, 20y: -4.5\} with zero spread on the USD floating leg.

The four-curve method assumes centralized funding in EUR and the proposed method, instead of considering centralized funding in either EUR or USD, it will consider that each leg is funded in its own currency. Each case considers five curves: Eonia ("EO"), 3 month Euribor ("E3M"), Federal Funds ("FF"), 3 month US Libor ("U3M") and cross currency basis spread ("CCB").

| Mat | CCS | Four-curve method | Multi-Funding |
|-----|-----|-------------------|---------------|
|     | EO  | E3M   | FF  | U3M | CCB | EO  | E3M | FF  | U3M | CCB |
| 1y  | 0   | 0     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 5y  | 0   | 0     | 0   | 0   | -1  | 0   | 0   | 0   | 0   | 0   |
| 9y  | 0   | 0     | 0   | 0   | -2  | 0   | 0   | 0   | 0   | 0   |
| 10y | 0   | 0     | 0   | 0   | 108 | 0   | 0   | 0   | 0   | 96  |

Table 1: 10 year CCS deltas (thousand EUR) for four-curve and proposed (Multi-Funding) methods moving interest rates by one basis point for each curve.

| Mat | NCS | Four-curve method | Multi-Funding |
|-----|-----|-------------------|---------------|
|     | EO  | E3M   | FF  | U3M | CCB | EO  | E3M | FF  | U3M | CCB |
| 1y  | 0   | 0     | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 5y  | -1  | 1     | 0   | 0   | 0   | 0   | 0   | -1  | 1   | 0   |
| 9y  | -2  | 2     | 0   | 0   | 0   | 0   | 0   | -2  | 2   | 0   |
| 10y | 15  | -15   | 0   | 0   | 97  | 0   | 0   | 14  | -14 | 96  |

Table 2: 10 year NCS deltas (thousand EUR) for four-curve and proposed (Multi-funding) methods moving interest rates by one basis point for each curve.

Tables 1 and 2 show delta sensitivities in thousand EUR of each curve for 10 year CCS and NCS under rate movements of one basis point of the four-curve and proposed methods. The spread added to the EUR floating leg is -5.75 bp. The four-curve approach calculates sensitivities by finite differences using a propagation algorithm which before pricing again the perturbed scenario, re-calibrates the basis USD discount curve after moving either interest or foreign exchange rates.
For the CCS of table 1, both methods provide zero sensitivities to every curve except for the cross currency curve, “CCB”. This is as expected because the proposed algorithm decomposes the CCS in terms of direct quotes of CCB curve and the four-curve uses the propagation method to calculate sensitivities. Some minor sensitivities appear in CCB curve for the four-curve method for 5 and 9 years. There is a slight mismatch for the four-curve method at 10 year maturity (108 versus 96) which does not happen for the NCS of table 2. This mismatch is much lower for a 5 year CCS (52 versus 50.1) and it might be explained by slight numerical problems of the propagation sensitivity algorithm (this does not happen with the proposed method).

According to equation (2), the NCS is decomposed in a CCS plus a series of USD cash flows. As the forward foreign exchange curve is increasing, these cash flows are always paid. Table 2 shows the sensitivities of the NCS. The proposed method yields sensitivities to these USD cash flows for “FF” and “U3M” curves as they are funded and discounted with “FF” curve. See that sensitivities to “EO” and “E3M” curves are zero as every EUR cash flow is incorporated into the CCS (the sensitivity appears in the CCB curve). The four-curve method only provides sensitivities to EUR curves of the same sign as the corresponding USD sensitivities of the proposed method. The sensitivities are of the same order for both “EO” and “FF” curves as they have similar levels.

\[
\Delta^s = \frac{\partial(V^s X_t^{-1})}{\partial(X_t^{-1})} = -\Delta^e X_t + V^s \\
\Delta^e = \frac{\partial V^s}{\partial X_t} = -\Delta^s X_t^{-1} + V^s X_t^{-1}
\]

Equation (27) shows foreign exchange deltas as seen from Europe (\(\Delta^s\) USD) and the United States (\(\Delta^e\) EUR), where \(V^s\) is the price in USD and \(X_t\) is the spot foreign exchange rate. Pricing systems based in Europe usually report \(\Delta^e - V^s X_t^{-1}\) as FX delta, because the deal premium is already in EUR and this amount has to be subtracted from the total amount, \(\Delta^e\), of

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6Rising “U3M” curve increases USD paid cash flows and so a negative sensitivity is reported. Rising “FF” curve decreases USD payments providing a positive sensitivity.

7Rising “EO” curve reduces value of EUR leg of the calibrated CCS, which has to be compensated by an increase of USD basis discount curve after propagation (market CCS re-calibration) providing a positive sensitivity. Rising “E3M” curve lowers USD basis discount curve yielding a negative sensitivity.
EUR to cancel, in order to be delta hedged. See that this cancelling EUR transaction, \( \Delta^s X_t^{-1} \), is carried out against a quantity of USD, \(-\Delta^s\), which cancels the open US position, \( \Delta^s \). For a US investor, the delta reported by the system would be \( \Delta^s - \Delta^s \).

\[
\begin{array}{c|cc|cc}
\text{FX Delta} & \text{Four-curve} & \text{Multi-Funding} \\
\hline
\Delta^€ \text{ EUR} & 39 & -1,380 & -37 & -2,623 \\
\Delta^$ USD & 26 & 1,884 & 52 & 3,417 \\
\end{array}
\]

Table 3: 10 year foreign exchange deltas (thousands) of CCS and NCS for four-curve and proposed methods.

Table 3 shows the foreign exchange deltas of CCS and NCS for four-curve and proposed methods according to equation (27). Values corresponding to \( \Delta^€ \) are in thousand EUR and to \( \Delta^s \) in thousand USD. To understand why they are different, see that for the NCS, \( \Delta^s = E_t^s \left[ -X_{t_0} F R N_{t_0, 10y}^{s=0} \right] \), according to equation (1) and for the CCS, \( \Delta^s = E_t^s \left[ -X_{t_0} F R N_{t_0, 3m}^{s=0} \right] \), as only the first 3 month period contributes. The difference between both methods arises because the proposed method discounts less with the “FF” curve (USD funding) and the four-curve approach more with the cross currency basis discount curve (EUR funding). If cross currency basis spreads were zero, both foreign exchange deltas would be equal. CCS EUR deltas (\( \Delta^€ \)) change signs between four-curve and proposed methods (39 versus -37), because both CCS premium and delta are small and comparable (see equation (27) to relate EUR and USD deltas).

\[
\begin{align*}
\Delta^s_{\text{CCS}} &= X_t E_t^€ \left[ -X_{t_0} F R N_{t_0, 3m}^{s=0} \right] = 26 \\
\Delta^s_{\text{NCS}} &= X_t E_t^€ \left[ -X_{t_0} F R N_{t_0, 10y}^{s=0} \right] = 1,906
\end{align*}
\]  

(28)

If the proposed method would choose the EUR (instead of USD) as funding currency for the USD leg, the valuation of the USD leg would be according to equation (7) and the foreign exchange deltas would be given (in thousand USD) by equation (28). See that they are very close to those provided by the four-curve method (see second row, left side of table 3).

8This is because cash flows do not depend on \( X_t \) and differentiating the USD leg multiplied by \( X_t^{-1} \) (to convert to EUR) with respect to \( X_t^{-1} \) yields the present value of the USD leg.

9Each cash flow would be divided by the forward exchange rate at each maturity to
9 Conclusions

A generalized model-independent pricing and hedging framework for multi-currency fixed-income desks has been proposed. This methodology allows choosing the funding currency of each leg, irrespective of the currency in which it is denominated. This way, each leg can be funded in the currency of denomination, the collateral currency or allowing switching to a common currency for consolidation purposes if needed.

This methodology naturally converges to the valuation of each leg according to the single currency setting when cross currency basis spreads tend to zero. A worked example compares this new approach with a benchmark.

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convert to EUR. When differentiating with respect to $X_t^{-1}$ to get delta, the $X_t$ factor in the denominator of each cash flow disappears. This is equivalent to multiplying the EUR valuation by $X_t$, to cancel the $X_t$ factor dividing in each cash flow.
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