New Families of Single–Qubit Control Fields: An Algorithm

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Abstract. The dynamics of a two-level system interacting with a new class of analytically solvable driving fields is analyzed. Such fields are obtained using an inverse-engineering approach, which allows to exactly factorize the correspondent time-evolution operator via the Wei-Norman theorem. This technique is presented as an algorithm and using an example we show that the free parameters involved can be used to tune some aspects of the dynamics on demand such as the atomic population inversion.

1. Introduction

Precise quantum control operations are required in several applications of quantum mechanics such as quantum computing and quantum information. Dealing with two-level atoms (or qubits) usually time-dependent driving fields are demanded to achieve such control operations. However the number of exactly solvable models is very limited. Recently a novel procedure to generate exactly solvable families of driving fields for one qubit was proposed in Ref. [1]. The method lies on the Wei-Norman context and produces fields for which the correspondent time-evolution operator factorizes exactly as the product of three exponentials whose argument involves a single element of the $su(2)$ algebra. Within this framework it is shown that the dynamics is closely related to the solutions of a parametric oscillator-like equation. Departing from such an equation some particular cases have been studied in the literature previously. For instance, starting with a harmonic oscillator equation a generalization of the circularly polarized field is obtained. In the same context, a decaying field is acquired if the correspondent frequency is a purely imaginary number [1]. Besides, we have shown that the dynamics is sensitive to the nature of the Mathieu functions when one departs from the Mathieu equation [2].

In this work we first revisit the procedure proposed in [1] and then a new class of exactly solvable driving fields is obtained. We also show that the correspondent dynamics is susceptible to the nature of Airy functions. These functions were first studied by George B. Airy in 1838 when calculating the light intensity in the neighborhood of a caustic [3], and are relevant in many branches of Physics. For instance, they have been used in fluid mechanics, elasticity, and in non-linear solitons through the non-linear Korteweg-de Vries equation, to mention a few [3, 4, 5]. However, since the seminal paper of M. V. Berry and N. L. Balazs published in 1979 [6], in which it is discussed the existence of nonspreading wave packets as solutions...
to the Schrödinger equation, the application of Airy functions have been exploited in Optics, from the introduction of accelerating optical beams that do not propagate in a straight line but in parabolic trajectories [7, 8, 9], to the study of light bullets [10] and the Goos-Hänchen and Imbert-Fedorov shifts of light reflected by a dielectric surface [11], among others.

This contribution is organized as follows. In Sec. 2 the general procedure discussed in Ref. [1] is revisited and presented as an algorithm. A new class of exactly solvable driving fields is obtained in Sec. 3 and the correspondent dynamics discussed. Finally, we provide some conclusions and perspectives of our work in Sec. 4.

2. Analytically solvable driving fields

We are interested in solving the time-evolution problem

\[ i \frac{dU(t)}{dt} = H_2(t) \cdot U(t), \quad U(0) = \mathbb{I}, \]  

where the single two-level system Hamiltonian reads

\[ H_2(t) = \Delta \sigma_0 + V(t)\sigma_+ + \overline{V}(t)\sigma_- , \]  

here \( \Delta \in \mathbb{R} \) and \{\( \sigma_0, \sigma_\pm \)\} are the three generators of the \( su(2) \) algebra defined in terms of the three conventional Pauli matrices as follows: \( \sigma_0 = \frac{1}{2}\sigma_z, \sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y) \). Besides, the complex-valued function \( V \) describes the interaction of the qubit with a semiclassical field and will be referred as interaction term. A convenient way to tackle the problem is using the Wei-Norman theorem, which allows to write the time-evolution operator \( U \) as the following product of exponentials

\[ U(t) = e^{\alpha(t)\sigma_+}e^{\Delta f(t)\sigma_0}e^{\beta(t)\sigma_-} , \]

the complex-valued functions \( \alpha, f, \) and \( \beta \) are called factorization functions and a direct substitution of (2) into equation (1) turns the evolution problem into a non-linear system of coupled equations

\[
\begin{align*}
\alpha' - \alpha \Delta f' - \beta' e^{-\Delta f} &= -iV, \\
\Delta f + 2\alpha \beta' e^{-\Delta f} &= -i\Delta, \\
\beta' e^{\Delta f} &= -i\overline{V},
\end{align*}
\]

with the initial conditions \( \alpha(0) = f(0) = \beta(0) = 0 \). Solutions to the previous system are found in a very limited number of cases [12]. Thus one has to seek for alternative methods to find exactly solvable solutions. In Ref. [1] it has been shown that the disentangling problem has analytical solutions for certain classes of interaction terms \( V \) related to an Ermakov equation. In this work we present the method discussed there as an algorithm. The meaning of the functions and parameters involved will be clarified in the following. The factorization functions \( \alpha, f \) and \( \beta \) as well as the interaction term \( V \) are determined by two functions \( \varphi \) and \( \mu \) obtained as follows:

(i) We must start choosing the function \( \Omega \) that can be real-valued or purely-imaginary-valued. Besides, the solutions of the parametric oscillator-like equation

\[ \varphi''(t) + \Omega^2(t)\varphi(t) = 0, \]  

are known with the initial conditions

\[ \lim_{t \to 0} \frac{1}{R(t)} \left[ \frac{\varphi(t)}{\varphi(t)} + \frac{1}{2} \frac{R'(t)}{R(t)} \right] = 0, \quad \varphi(0) = 1, \]

where \( R_0 := R(0) \) and \( \frac{R'(0)}{R(0)} = (\ln R_0)' \) are provided as free parameters.
The driving field

(ii) Using the Pinney-Ermakov method we should find a particular solution to the nonlinear equation

\[ \mu''(t) + \Omega^2(t)\mu(t) = \frac{\Omega_0^2}{\mu^3(t)}, \quad \Omega_0 = \left[ |R_0|^2 + \frac{\lambda^2}{4} \right]^{1/2}, \]  

with \( \mu \) a real-valued function. Besides the correspondent initial conditions read

\[ \mu(0) := \mu_0 = 1, \quad (\ln R_0)' = i\lambda - 2\mu'_0. \]  

Note that if \((\ln R_0)' = \delta_1 + i\delta_2\) with \(\delta_1, \delta_2 \in \mathbb{R}\) it is found that \(\lambda = \delta_2\) and \(\mu'_0 = -\delta_1/2\). A convenient solution is given in Ref. [13]. Providing two linear independent functions \(z\) and \(v\), a solution of (7) fulfills

\[ \mu^2(t) = av^2(t) + bv(t)z(t) + cz^2(t), \]  

the constants \(a, b\) and \(c\) read

\[ a = \frac{c_1}{w_0^2}, \quad c = \frac{\Omega_0^2 + c_0^2}{c_1}, \quad b^2 - 4ac = -4\frac{\Omega_0^2}{w_0^2}, \]  

besides \(c_0\) and \(c_1\) are integration constants and \(W(v, z) = w_0\) is the constant Wronskian of the functions \(v\) and \(z\).

(iii) The driving field \(V\) is given by the following expression

\[ V(t) = e^{-i\Delta t R(t)}, \quad \text{where} \quad R(t) = \frac{R_0}{\mu^2(t)} \exp \left[ i\lambda \int_0^t ds \frac{ds}{\mu^2(s)} \right]. \]  

The function \(R\) is also referred as the interaction term. Besides, the factorization functions read

\[ \alpha(t) = i R_0 \mu^2(t) \exp \left[ -i\Delta t - i\lambda_1(t) \right] \frac{\varphi'(t)}{\varphi(t)} - \mu'(t) \mu(t) + \frac{i\lambda}{2\mu^2(t)}, \]  

\[ \beta(t) = -i R_0 \int_0^t ds \frac{\varphi^2(s)}{\varphi'(s)}, \]  

\[ \Delta f(t) = \ln \left[ \frac{\mu^2(t)}{\varphi^2(t)} \right] - i\lambda_1(t) - i\Delta t, \]  

where

\[ \mu_1(t) = \int_0^t ds \frac{\varphi^2(s)}{\varphi'(s)}. \]  

Given the initial state as the upper energy level, \(|\psi(0)\rangle = |0\rangle\), the former expressions allow to compute its time-evolution as

\[ |\psi(t)\rangle = e^{-\Delta f(t)/2}[e^{\Delta f(t)} + \alpha(t)\beta(t)|0\rangle + e^{-\Delta f(t)/2}\beta(t)|1\rangle]. \]  

Hence, the atomic population inversion can be calculated

\[ P(t) = \mu^2(t)|\varphi(t)\beta(t)|^2 \left\{ \frac{1}{\beta(t)\varphi^2(t)} + \frac{\alpha(t)}{\mu^2(t)} e^{i[\Delta t + \lambda_1(t)]} - \frac{1}{\mu^2(t)} \right\}. \]  

We stress the fact that the evolution problem (1) is analytically solvable with the driving field (11). On the other hand, the initial conditions \(R_0 = ig\) and \((\ln R_0)' = i\delta\) with \(g \in \mathbb{C}\) and \(\delta \in \mathbb{R}\) retrieve the usual initial conditions of the circularly polarized field case [12]. Indeed, the constant \(\Omega_0 = \sqrt{|g|^2 + \delta^2/4}\) matches with the Rabi frequency. Analytically solvable driving fields have been found taking \(\Omega^2(t) = \Omega_1^2\) with \(\Omega_1 \in \mathbb{R}\) [1] and \(\Omega^2(t) = \omega_0 - \omega_1 \cos^2(t)\) with \(\omega_0, \omega_1 \in \mathbb{R}\) [2].
3. Oscillating amplitude non-periodic driving fields

In this Section the formalism described previously is presented taking the case $\Omega^2(t) = -\chi^3 t$ with $\chi \in \mathbb{R}$ to construct single-qubit radiation fields. We follow the steps 1-3 considering the free parameters $R_0$ and $(\ln R_0)'$ given by $R_0 = -ig$ and $(\ln R_0)' = i\delta$ with $g \in \mathbb{C}$ and $\delta \in \mathbb{R}$.

(i) In this case the parametric oscillator-like equation reduces to a conventional Airy equation

$$\varphi''(t) - \chi^3 t \varphi(t) = 0,$$

whose solution can be written in terms of the functions $Ai$ and $Bi$ [3] as

$$\varphi(t) = a_\chi Ai(-\chi t) + b_\chi Bi(-\chi t).$$

The values $a_\chi$ and $b_\chi$ are determined by the initial condition (6):

$$b_\chi = -\frac{a_\chi}{\sqrt{3}} = \frac{i\delta \Gamma(\frac{1}{3})}{4 \cdot 3^{1/6} \chi},$$

where $\Gamma$ stands for the Gamma function.

(ii) On the other hand, the solution to the corresponding Ermakov equation with the circularly polarized field initial conditions reads

$$\mu^2(t) = x_\chi Ai(-\chi t) + y_\chi Ai(-\chi t)Bi(-\chi t) + z_\chi Bi^2(-\chi t),$$

where the constants

$$x_\chi = \frac{9\chi^2 \Gamma(\frac{2}{3})}{4 \cdot 3^{2/3} \chi^2 \Gamma(\frac{2}{3})}, \quad y_\chi = \frac{9\chi^2 \Gamma(\frac{2}{3})}{6 \cdot 3^{1/6} \chi^2 \Gamma(\frac{2}{3})}, \quad z_\chi = \frac{9\chi^2 \Gamma(\frac{2}{3})}{12 \cdot 3^{2/3} \chi^2 \Gamma(\frac{2}{3})},$$

were determined using the proper initial conditions and $W(Ai, Bi) = -\chi/\pi$ has been used [3].

(iii) The family of driving fields (11) is thus determined by the set of parameters $\chi$, $g$ and $\delta$. Besides the nature of such control fields is sensitive to the nature of the Airy functions. For instance, the fact that these are non-periodic is reflected in Fig. 1 (a), (b) and (c) as the correspondent trajectories do not close. The three parameters $g$, $\delta$ and $\chi$ ($\Delta$ is a global phase factor) can be used to tune the dynamics on demand. The number $g$ is related to the amplitude of the driving field, $\delta$ multiplies the phase of the driving field, whereas $\chi$ is related to the frequency of the oscillations of the amplitude of the driving field, since it multiplies the argument of the Airy functions present in $\mu$. In Figs. 1a, b, c, we depict the driving field $R$ for several values of the parameters $g$, $\delta$ and $\chi$. On the other hand, the time-evolution of the atomic population inversion is shown in Figs. 1d, e, f. A color scale from blue to red shows the time flow correspondence between the driving field and population inversion plots of Fig. 1 for equal time intervals. Fig. 1 shows that the amplitude of the oscillations of the driving field increases as time passes in any case. This can be understood in terms of the nature of the Airy functions; $Ai(-\chi t)$ and $Bi(-\chi t)$ are oscillating decaying functions of $t$ for $\chi > 0$, hence $\mu$. Therefore, the amplitude of the oscillations driving field increases as time flows since it is inversely proportional to $1/\mu$. 
Figure 1. The driving field $R$ in the complex plane for $\Delta = 1$, (a) $g = 1, \delta = 0.5, \chi = 0.5$, (b) $g = 2, \delta = 2, \chi = 5$, (c) $g = \sqrt{5}, \delta = 4, \chi = 3$. The corresponding population inversion is shown in (d-f). Time flows from blue color to red in the same scale between the corresponding driving field and population inversion plots.
Fig. 2 shows the behavior of the population inversion by varying independently the control parameters $g$, $\delta$ and $\chi$. As can be observed in Fig. 2a, an increase in $g$ increases the amplitude of the population inversion until it reaches the value $P = -1$. It can also be observed that, as $g$ increases, the time interval in which $P$ is close to -1 increases without changing the position of the maxima of $P$. On the other hand, an increase in $\delta$ reduces the amplitude of the oscillations of $P$, causing the minima of the latter to move away from -1 (see 2b). Interestingly, Fig. 2c shows that the increase of the frequency related parameter $\chi$, increases the frequency of the oscillations of $P$ without changing its amplitude.

4. Conclusions

We have presented the study of the behavior of a two-level atom in presence of driving fields generated by applying the inverse approach shown in Ref. [1, 2], through the analysis of the population inversion. We have proposed a linear parametric oscillator-like equation that leads to solutions to the associated Ermakov equation, driving fields and population inversion that can be written in terms of the Airy functions. We have shown that there are three relevant free parameters characterizing the system and how to control the system properties, namely, the amplitude, frequency and width of the oscillations of the driving fields as well as of the population inversion. We have also shown that the oscillation amplitude of this kind of driving fields increases with time in general. We hope that the features of the system discussed in this work contribute to the knowledge of interesting systems in quantum control.

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