Rare K decays in the Standard Model

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The very clean theoretical predictions for the rare decays \( K \rightarrow \pi \nu \bar{\nu} \) and \( K_L \rightarrow \pi^0 \ell^+ \ell^- \) are reviewed, and their various theoretical inputs summarized. The less favorable situation for \( K_L \rightarrow \mu^+ \mu^- \) is also commented.

I. INTRODUCTION

The rare decays \( K \rightarrow \pi \nu \bar{\nu} \) and \( K_L \rightarrow \pi^0 \ell^+ \ell^- \), driven by semi-leptonic flavor-changing neutral currents (FCNC), are exceptionally clean probes of the flavor structure of the Standard Model, or of the still elusive New Physics. Concerted theoretical efforts have brought the SM predictions to an impressive level of accuracy (for the current experimental situation, see [1]). In this section, the main theoretical ingredients are briefly reviewed, while the situation for each mode is summarized in the following sections.

A. FCNC electroweak structure

FCNC arise at one loop in the electroweak theory. The processes driving the rare semi-leptonic K decays are the W box, Z and \( \gamma \) penguins\(^2\), see Fig.1, and lead to the amplitudes

\[
A(K_L \rightarrow \pi^0 X) = \sum_{q=u,c,t} (\text{Im} \lambda_q + \varepsilon \text{Re} \lambda_q) y_q^X (m_q),
\]

\[
A(K^+ \rightarrow \pi^+ X) = \sum_{q=u,c,t} (\text{Re} \lambda_q + i \text{Im} \lambda_q) y_q^X (m_q),
\]

with \( X = \nu \bar{\nu}, \ell^+ \ell^- \) and \( \lambda_q = V_{qe} V_{qd}^* \). In standard terminology, the \( \varepsilon \) part is the indirect CP-violating piece (ICPV), while the \( \text{Im} \lambda_q \) part is called direct CP-violating (DCPV). The amplitude for \( K_S \) is obtained from the \( K_L \) one by interchanging \( \text{Im} \lambda_q \leftrightarrow \text{Re} \lambda_q \).

Without the dependence of the loop functions \( y_q^X \) on the quark masses, CKM unitarity would imply vanishing \( K_S \) amplitudes. For \( X = \nu \bar{\nu} \), only the Z penguin and W box enter, \( y_{u/e}^{\nu \bar{\nu}} \sim m_e^2 \), and light-quark contributions are suppressed. Since, in addition, \( \varepsilon \sim 10^{-3} \) and \( \text{Re} \lambda_t \sim \text{Im} \lambda_t \), ICPV is very small. For \( K^+ \), the c-quark contribution is suppressed from the loop, but enhanced by \( \text{Re} \lambda_c \gg \lambda_t \), and ends up being comparable to the t-quark contribution.

For \( X = \ell^+ \ell^- \), the photon penguin also enters with its scaling \( y_q^{\ell^+ \ell^-} \sim \log(m_q) \) for \( m_q \rightarrow 0 \). In the standard CKM phase-convention, DCPV is still short-distance dominated thanks to \( \text{Im} \lambda_u = 0 \), but not ICPV, completely dominated by the long-distance u-quark photon penguin, \( K_1 \rightarrow \pi^0 \gamma^* \rightarrow \pi^0 \ell^+ \ell^- \). The same holds for \( K^+ \rightarrow \pi^+ \ell^+ \ell^- \), completely dominated by long-distance and therefore not very interesting for New Physics search.

For \( K_L \rightarrow \ell^+ \ell^- \), there is no photon penguin, and the electroweak structure is similar to \( K \rightarrow \pi \nu \bar{\nu} \), up to the change \( \text{Im} \lambda_q \leftrightarrow \text{Re} \lambda_q \).

Along with these contributions, there can be two-loop, third order electroweak contributions, if the extra suppression is compensated by non-perturbative long-distance enhancement. This occurs for modes with charged leptons, where the double-photon penguin gives a CP-conserving contribution \( \sim \text{Re} \lambda_q \) to \( K_L \rightarrow \pi^0 \ell^+ \ell^- \) and \( K_L \rightarrow \ell^+ \ell^- \), and is completely dominated by long-distance \( (u-) \) quark), Fig.2c.

B. QCD corrections

Having identified the relevant electroweak structures, QCD effects have now to be included. This is done in three main steps:

Step 1: Integration of heavy degrees of freedom (top, \( W, Z \), including perturbative QCD effects above \( M_W \)). This generates local FCNC operators (Fig.1 with \( t \)-quark), and Fermi-type four-fermion local operators.

Step 2: Resummation of QCD corrections (running down). At the \( \epsilon \) threshold (similar for \( b, \tau \)), four-fermion operators are combined to form closed c-loops, which are then replaced by a tower of effective interactions in increasing powers of \( (\text{external momentum})/(\text{charm mass}) \), Fig.2a. The lowest order consists again of the dimension-six FCNC operators, while dimension-eight operators are corrections scaling naively like \( m_W^2/m_c^2 \sim 15\% \).

These first two steps (the OPE) can, in principle, be achieved to any desired level of precision within perturbative QCD, though the computation of the required multiloop diagrams represents a formidable task at higher orders. Still, this is unavoidable in order to reduce theoretical errors, in particular scale dependencies. At this stage, one has obtained the complete Hamiltonian, i.e. all the effective operators, with the short-distance physics encoded in their Wilson coefficients.

Step 3: To get the amplitudes, the matrix elements of these operators between meson states remain to be

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estimated. To this end, one makes use of the symmetries of QCD, as embodied in Chiral Perturbation Theory (ChPT), to relate the desired matrix elements to experimentally known quantities.

For the most interesting dimension-six semi-leptonic operators, the matrix elements can be related to those of $K_{22}, K_{33}$ decays (taking into account isospin-breaking corrections). Contributions from four light-quark operators ($Q_1, ..., Q_6$) are represented directly in terms of meson fields in ChPT, such that non-local $u,c$-quark loops are represented as meson loops (Fig.2b,c). The price to pay is the introduction of some unknown low-energy constants ($G_{8,27,...}$), to be extracted from experiment. In particular, $G_8$ is fixed from $B(K\to\pi\pi)$, accounting for the large non-perturbative $\Delta I = 1/2$ effects. For dimension-eight operators, an approximate matching is done with the ChPT representation of the $u,c$-quark contributions.

**II. THE $K^+\to\pi^+\nu\bar{\nu}$ AND $K_L\to\pi^0\nu\bar{\nu}$ DECAYS**

Thanks to the suppression of light-quark effects, these modes are the cleanest and their rates are precisely predicted within the SM. Their branching ratios read:

$$B_{+}^{\nu\bar{\nu}} = \kappa_+ \left( \frac{\text{Im} \lambda_1}{\lambda^5} X_t \right)^2 + \left| \frac{\text{Re} \lambda_1}{\lambda^5} X_t + \frac{\text{Re} \lambda_2}{\lambda} P_{u,c} \right|^2,$$

$$B_L^{\nu\bar{\nu}} = \kappa_L \left| \frac{\text{Im} \lambda_1}{\lambda^5} X_t \right|^2,$$

with $P_{u,c} = P_c + \delta P_{u,c}$. The Wilson coefficient of the dimension-six FCNC operator $Q^\nu = (sd)_{V-A} (\bar{\nu}\nu)_{V-A}$ arising from the top-quark loop is known at NLO, $X_t = 1.646 \pm 0.041 \pm 0.03 \cdot 10^{-10}$ and $\kappa_+ = 5.26 \pm 0.06 \cdot 10^{-11}$ for $\lambda = 0.225$. Finally, for $K_L \to \pi^0\nu\bar{\nu}$, ICPV is of about 1% while the CP-conserving contribution arising from box diagrams is less than 0.01%.

The SM predictions are then

$$B(K_L \to \pi^0\nu\bar{\nu}) = (2.7 \pm 0.4) \cdot 10^{-11},$$

$$B(K^+ \to \pi^+\nu\bar{\nu}) = (8.4 \pm 1.0) \cdot 10^{-11}.$$  

The error on $K_L \to \pi^0\nu\bar{\nu}$ is dominated by $\text{Im} \lambda_t$, while for $K^+ \to \pi^+\nu\bar{\nu}$, it breaks down to scales (13%), $m_c(22\%)$, CKM, $\alpha_s$, $m_t(37\%)$ and matrix-elements from $K_{33}$ and light-quark contributions (28%). Further improvements are thus possible through a better knowledge of $m_c$, of the isospin-breaking in the $K \to \pi$ form-factors, or by a lattice study of higher-dimensional operators. 

As the determination of $\lambda_t$ from general UT fits to B physics data is already very precise, and expected to be further improved in the near future, the main interest of the $K \to \pi\nu\bar{\nu}$ decays is to test the CKM paradigm for CP-violation in the SM. Indeed, these modes do offer a
produced at LO through the finite two-loop process and identifies its precise nature \[11\]. In ChPT, loops are small and one just needs to fix a counterterm, (Fig.2b). In ChPT, loops are small and one just needs to fix a counterterm. To estimate them, though unknown counterterms. To estimate them, though, one can use the experimental information on \(K_L \to \gamma \gamma\) together with the perturbative behavior of the \(s d\to uu\) \(\gamma\gamma\) loop \[20\]. Finally, SD and LD produce the same 0^- state and thus interfere. This

| \(\ell\) | \(C_{\text{dir}}\) | \(C_{\text{int}}\) | \(C_{\text{mix}}\) | \(C_{\gamma\gamma}\) |
|-------|----------------|-------------|--------------|----------------|
| e     | \((4.62 \pm 0.24) \,(y_{7V}^2 + y_{A}^2)\) | \((11.3 \pm 0.3) \,y_{7V}\) | \(14.5 \pm 0.5\) | \(\approx 0\) |
| \(\mu\) | \((1.09 \pm 0.05) \,(y_{7V}^2 + 2.32y_{A}^2)\) | \((2.63 \pm 0.06) \,y_{7V}\) | \(3.36 \pm 0.20\) | \(5.2 \pm 1.6\) |

III. THE \(K_L \to \pi^0 \ell^+ \ell^-\) DECAYS

Here the situation is more involved. The \(t\) and \(c\)-quark contributions generate both the dimension-six vector \(Q_{7V} = (s d)_V (\ell\ell)_V\) and axial-vector \(Q_{7A} = (s d)_V (\ell\ell)_A\) operators, whose Wilson coefficients \(y_{7V}, y_{7A}\) are known to NLO\[2\]. The former produces the \(\ell^+ \ell^-\) pair in a 1^- state, the latter in a 1^+ state and, in addition, for \(\ell = \mu\), in a helicity-suppressed 0^- state. Indirect CP-violation is related to \(K_S \to \pi^0 \ell^+ \ell^-\), for which the long-distance photon penguin dominates (Fig.2b). In ChPT, loops are small and one just needs to fix a counterterm, \(a_S\) \[13\]. This can be done up to a sign from NA48 measurements as \(a_S = 1.2 \pm 0.2\) \[13\]. Producing \(\ell^+ \ell^-\) in a 1^- state, it interferes with the contribution from \(Q_{7V}\), arguably constructively \[14\].

The CP-conserving (CPC) contribution from \(Q_{1,\ldots,6}\) proceeds through two-photons, i.e. produces the lepton pair in either a helicity-suppressed 0^+ or phase-space suppressed 2^+ state. Only the 0^+ state is produced at LO through the finite two-loop process \(K_L \to \pi^0 \ell^+ P^- \to \pi^0 \gamma \gamma \to \pi^0 \ell^+ \ell^-\), \(P = \pi, K\) (Fig.2c). Higher order corrections are estimated using \(K_L \to \pi^0 \gamma \gamma\) experimental data for both the 0^+ and 2^+ contributions \[14\].

Altogether, the branching ratios are

\[\mathcal{B}^{\ell^+ \ell^-} = (C_{\text{dir}}^\ell \pm C_{\text{int}}^\ell |a_S| + C_{\text{mix}}^\ell |a_S|^2 + C_{\gamma\gamma}^\ell) \times 10^{-12},\]

together with the coefficients given in Table I. Interestingly, these coefficients obey \(C_i^\ell / C_{\gamma\gamma}^\ell \approx 0.23\) due to the phase-space suppression, but for the helicity-suppressed contributions arising from the \(Q_{7A}\) operator (DCPV) and from \(\gamma\gamma\) (CPC). This maintains the sensitivity of \(\mathcal{B}^{\ell^+ \mu^-}\) on the interesting short-distance physics at the same level as \(\mathcal{B}^{e^+ e^-}\). Further, it allows in principle to disentangle the \(Q_{7V}\) and \(Q_{7A}\) contributions from the measurements of both modes. This is illustrated in Fig.3a, where the hyperbola corresponds to a common rescaling of both \(y_{7V}\) and \(y_{7A}\) \[16\]. As discussed in \[16, 17\], this plane is particularly interesting to look for signals of New Physics, and identify its precise nature \[11\].

In the SM, \(y_{7A} (M_W) = -0.68 \pm 0.03\) and \(y_{7V} (\mu \approx 1\text{ GeV}) = 0.73 \pm 0.04\) \[2\], and the predicted rates are \[14, 16, 17\]:
interference, which depends on the sign of \( A(K_L \to \gamma\gamma) \), is presumably constructive \([\ref{21}]\). Better measurements of \( K_S \to \pi^0\gamma\gamma \) or \( K^+ \to \pi^+\gamma\gamma \) could settle this sign.

\( K_L \to \mu^+\mu^- \) is thus obviously not as clean as \( K \to \pi\nu\bar{\nu} \) or \( K_L \to \pi^0\ell^+\ell^- \). Nevertheless, being measured precisely, it can still lead to interesting constraints in some specific scenarios like SUSY at large \( \tan\beta \)[\ref{11}].

V. CONCLUSION

Thanks to the numerous theoretical efforts, the four rare decays, \( K \to \pi\nu\bar{\nu} \) and \( K_L \to \pi^0\ell^+\ell^- \), now provide for one of the cleanest and most sensitive tests of the Standard Model. These modes are promising not only to get clear signals of New Physics – or to severely constrain it –, but also to uncover the nature of the possible New Physics at play through the specific pattern of deviations they would exhibit with respect to the SM predictions.

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