Zero-One Law for random uniform hypergraphs

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1 Introduction

In this work limit probabilities of first-order properties of the random \( s \)-uniform hypergraph in the binomial model \( G^s(n, p) \) are studied. We give a complete discription of all positive \( \alpha \) such that \( G^s(n, n^{-\alpha}) \) obeys Zero-One Law. Moreover, for any rational \( \rho \geq 1/(s-1) \) we prove the existence of a strictly balanced \( s \)-uniform hypergraph with the density \( \rho \).

An \( s \)-uniform hypergraph, or \( s \)-hypergraph, \( G \) is a pair \((V(G), E(G))\) consisting of two sets, the vertices \( V(G) \) and edges \( E(G) \) of \( G \), where each edge \( e \in E(G) \) is a set of \( s \) elements of \( V(G) \). In particular, 2-hypergraph is an ordinary graph.

We consider the binomial model \( G^s(n, p) \) (see, e.g., [1]-[4]) of random \( s \)-hypergraph on \( n \) vertices with the probability of appearing of an edge \( p = p(n) \in [0, 1] \). We say that an \( s \)-hypergraph property \( A \) holds almost surely if

\[
\lim_{n \to \infty} \Pr[G^s(n, p(n)) \models A] = 1
\]

and almost never if

\[
\lim_{n \to \infty} \Pr[G^s(n, p(n)) \models A] = 0.
\]

To the best of our knowledge, one of the most studied class of properties in the sense of almost sure theory is the class of first-order properties. It involves all \( s \)-hypergraph properties, that could be expressed by first-order formulae. We restrict ourselves by considering only first-order properties in this work.

The random \( s \)-hypergraph \( G^s(n, p(n)) \) obeys Zero-One Law (see [5]-[9]) if every first-order property \( A \) holds almost surely or almost never.

In 1988 J. Spencer and S. Shelah gave the complete description of all \( \alpha > 0 \), such that the random graph \( G(n, n^{-\alpha}) \) obeys Zero-One Law.

**Theorem 1** (J. Spencer, S. Shelah, [5]). Let \( p(n) = n^{-\alpha} \), \( \alpha > 0 \).

1. If \( \alpha \in \mathbb{Q} \cap (0, 1] \) or \( \alpha = \frac{k+1}{k}, \ k \in \mathbb{N} \), then \( G(n, p(n)) \) does not obey Zero-One Law.
2. If \( \alpha \in (\mathbb{R} \setminus \mathbb{Q}) \cap (0, 1] \), then \( G(n, p(n)) \) obeys Zero-One Law.
3. If \( \alpha > 1 \), \( \alpha \neq \frac{k+1}{k} \), then \( G(n, p(n)) \) obeys Zero-One Law.
In this paper, we generalize (1) and (2) of Theorem 1 to the case of $s$-hypergraphs. In the proof of the generalization of (1) we exploit the following theorem (that is also proved in this paper): densities of strictly balanced $s$-hypergraphs occupy full spectrum $[1/(s-1), \infty) \cap \mathbb{Q}$. In the proof of the generalization of (2) we use Duplicator’s look-ahead strategy, that was proposed by Spencer (see [7]-[9]).

Clause (3) has already been generalized by N.C. Sandalha and M. Telles. Their result is the following.

**Theorem 2** (N.C. Saldanha, M. Telles, [4]). Let $p = n^{-\alpha}$, $\alpha > s - 1$, $\alpha \notin \{s - 1 + \frac{1}{k}, k \in \mathbb{N}\}$. Then $G^s(n, p)$ obeys Zero-One Law.

In Section 2 we review the results on distribution of small subgraphs in the random hypergraph. In Section 3 we state new results. Their proofs are given in Section 4 and Section 6. The last proof exploits the look-ahead strategy which is based on constructions described in Section 5.

## 2 Small sub-hypergraphs

Let us introduce some notations. For an arbitrary uniform hypergraph $G = (V, E)$ denote $V(G) = V$, $E(G) = E$, $v(G) = |V|$, $e(G) = |E|$. Denote the density $e(G)/v(G)$ of $G$ by $\rho(G)$.

We say that $G$ is strictly balanced if its density is greater than a density of any its proper sub-hypergraph. Obviously, if $G$ is strictly balanced, then it is connected. Moreover, $G$ is strictly balanced if and only if for any its proper connected sub-hypergraph $H$ the inequality $\rho(G) > \rho(H)$ holds.

**Theorem 3** (A. Ruciński, A. Vince, [10]). Let $\rho \in \mathbb{Q}$, $\rho \geq 1$. Then there exists a strictly balanced graph with the density $\rho$.

Obviously, if $\rho < 1$, then such a strictly balanced graph exists if and only if $\rho = k/(k+1)$ for some $k \in \mathbb{N}$. Indeed, any strictly balanced graph with a density less than 1 is a tree, and all trees are strictly balanced.

Let $G$ be a strictly balanced $s$-hypergraph with $v$ vertices, $e$ edges and $a$ automorphisms. Denote by $N_G$ the number of copies of $G$ in $G^s(n, p)$. The following theorem is an obvious generalization of a classical result of Bollobás. We believe, this result is well-known. However, we do not know if it has been published. Therefore, we adapted the proof of Bollobás’s theorem from [11] to the case of uniform hypergraphs.

**Theorem 4.** If $p = n^{-v/e}$ then

$$N_G \xrightarrow{d} \text{Pois} \left( \frac{1}{a} \right).$$
Proof. Consider the factorial moments of $N_G$, defined as

\[ E(N_G)_k = E[N_G(N_G - 1) \ldots (N_G - k + 1)]. \]

We have for $k = 1, 2, \ldots$

\[ E(N_G)_k = \sum_{G_1, \ldots, G_k} \Pr(I_{G_1} \cdots I_{G_k} = 1) = E'_k + E''_k, \]

where the summation is over all ordered $k$-tuples of distinct copies of $G$ in $K_n$, and $E'_k$ is a partial sum where the copies in a $k$-tuple are mutually vertex disjoint. It is easy to see that

\[ E'_k \sim (EN_G)^k \sim (1/a)^k. \]

It remains to prove that $E''_k = o(1)$. This fact is a consequence of the following claim.

Claim. Let $e_t$ be the minimum number of edges in a $t$-vertex union of $k$ not mutually vertex disjoint copies of $G$. Then for every $k \geq 2$ and $v \leq t \leq kv - 1$ we have $e_t > t \rho(G)$.

The proof of Claim in the case $s = 2$ can be found in [11], for arbitrary $s$ the proof is the same.

Denote by $L_G$ the property of containing a copy of $s$-hypergraph $G$. The following theorem is a generalization of a classical result of Erdős, Rényi [12] and Bollobás [13], stated and proved by A.G. Vantsyan.

Theorem 5 ([14]). Fix a finite $s$-hypergraph $G$. If $p(n) \ll n^{-1/\rho_{\max}(G)}$ then

\[ \lim_{n \to \infty} \Pr(G^s(n, p(n)) \models L_G) = 0. \]

If $p(n) \gg n^{-1/\rho_{\max}(G)}$ then

\[ \lim_{n \to \infty} \Pr(G^s(n, p(n)) \models L_G) = 1. \]

3 New results

Our objective is to prove a generalization of (1) and (2) of Theorem 1 for random uniform hypergraphs.
3.1 Generalization of (1)

In order to deal with positive rational $\alpha \leq s - 1$ and $\alpha \in \{ s - 1 + \frac{1}{k}, k \in \mathbb{N} \}$ we extended Theorem 3 to the case of uniform hypergraphs.

**Theorem 6.** Let $s \in \mathbb{N}$, $s \geq 2$, $\rho \in \mathbb{Q}$. Then there exists a strictly balanced $s$-hypergraph with a density $\rho$ if and only if $\rho \geq \frac{1}{s-1}$ or $\rho = \frac{k}{1+k(s-1)}$ for some $k \in \mathbb{N}$.

As the property of containing a fixed sub-hypergraph $H$ could be expressed by a first order formula, the following theorem is a corollary of Theorems 4 and 6.

**Theorem 7.** If $\alpha \in (0, s - 1] \cap \mathbb{Q}$ or $\alpha = s - 1 + \frac{1}{k}$ for some $k \in \mathbb{N}$ then the random $s$-hypergraph $G^s(n, n^{-\alpha})$ does not obey Zero-One Law.

3.2 Generalization of (2)

**Theorem 8.** If $\alpha$ is a positive irrational number then the random $s$-hypergraph $G^s(n, n^{-\alpha})$ obeys Zero-One Law.

Thus, according to Theorems 4, 7 and 8 the random $s$-hypergraph $G^s(n, n^{-\alpha})$ obeys Zero-One Law if and only if $\alpha \in (0, s - 1] \setminus \mathbb{Q}$ or $\alpha \in (s - 1, \infty) \setminus \{ s - 1 + 1/k, k \in \mathbb{N} \}$.

The scheme of the proofs is the following. In Section 4 we prove Theorem 6. In Section 6 we prove Theorem 8. Main tools for this proof are given in Section 5.

4 Proof of Theorem 6

We divide the proof into two parts. In the first part we will prove that if $\rho \geq \frac{1}{s-1}$ then there exists a strictly balanced $s$-hypergraph with the density $\rho$. In the second part the case $\rho < \frac{1}{s-1}$ is considered. We will prove that in this case there exists a strictly balanced $s$-hypergraph with the density $\rho$ if and only if $\rho = \frac{k}{1+k(s-1)}$ for some $k \in \mathbb{N}$.

4.1 Case $\rho \geq \frac{1}{s-1}$

**Lemma 1.** Fix rational $\rho > 0$. If there exists a strictly balanced $s$-hypergraph $(s \geq 2)$ with the density $\rho$, then there exists a strictly balanced $(s+1)$-hypergraph with the density $\rho$.

**Proof of lemma** Let $G$ be a strictly balanced $s$-hypergraph with the density $\rho$. Denote by $\tilde{G}$ the union of two disjoint copies $G_1, G_2$ of the hypergraph $G$. Choose one vertex in each copy: $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. Consider the $(s+1)$-hypergraph $H$ defined as follows.

$$V(H) = V(\tilde{G}), \quad E(H) = \{ e \cup \{ v_2 \}, e \in E(G_1) \} \cup \{ e \cup \{ v_1 \}, e \in E(G_2) \}.$$
Let us prove that \( H \) is strictly balanced. Consider an arbitrary proper sub-hypergraph \( K \subset H \). Denote \( W := G|_{V(K)} \) — induced sub-hypergraph of \( G \). As \( G \) is strictly balanced we have \( \rho(G) \geq \rho(W) \), and the equality holds only if \( W = G_1 \) or \( W = G_2 \). Under the definition of \( H \) we have \( e(K) \leq e(W) \). Moreover, if \( W = G_1 \) or \( W = G_2 \) then \( e(K) = 0 \), and the inequality is strict. Therefore, \( \rho(K) < \rho(G) = \rho(H) \). Hence \( H \) is strictly balanced.

\[\square\]

Now, by Theorem 3 and Lemma 11 it is sufficient to prove Theorem 6 only for the case \( s \geq 3 \) and \( \frac{1}{s-3} > \rho \geq \frac{1}{s-1} \).

Let \( s \geq 3, \rho = \frac{m}{n}, n = (s-1)m - r \), \( r \in \{0, 1, \ldots, m-1\} \). We assume that \( m \geq 3 \). In the cases \( m = 1 \) and \( m = 2 \) consider the fraction \( \frac{3m}{n} \) instead (in that way all fractions would be considered). Consider the set \( I = \{[k \cdot m/r], k = 1, 2, \ldots, r\} \) (hereinafter, \([\cdot]\) denotes the floor function). Consider the \( s \)-hypergraph \( G \) with \( V(G) = \{1, 2, \ldots, n\} \) and \( E(G) \) defined as follows. There would be \( m \) edges in \( E(G) \). The first edge is \( \{1, 2, \ldots, s\} \). Further, if the \( k \)-th edge is \( \{x, x+1, \ldots, x+s-1\} \), then the \((k+1)\)-st edge is either \( \{x+s-2, x+s-1, \ldots, x+2s-3\} \) if \((k+1) \in I\), or \( \{x+s-1, x+s, \ldots, x+2s-2\} \) if \((k+1) \notin I\). While building the edges we identify vertices 1 and \( n \), and \( 2 \) and \( n+2 \), etc. (i.e. we suppose that vertices are located on a circle). From the equality \( n = (s-1)m - r \) it follows that if \( 1 \notin I \) then the \( m \)-th edge intersects with the 1-st edge by one vertex, and if \( 1 \in I \) then they intersect by two vertices. Thus every edge has one or two vertices in the intersection with the previous one, and there are exactly \( r \) intersections consisting of two vertices.

Let us prove that the defined above \( s \)-hypergraph \( G \) is strictly balanced. Consider an arbitrary proper sub-hypergraph \( H \subset G \). We will prove that \( \rho(H) < \rho(G) \). We may assume without loss of generality that \( H \) is connected, because its density is not greater than the maximum of densities of its connected components. In this case, \( V(H) \) is a set of neighbouring (going in a row) vertices of the hypergraph \( G \). We may assume that \( H = G|_{V(H)} \), because adding some edges to \( H \) increases its density. Let the edges of \( H \) have numbers \( i, i+1, \ldots, j \), where \( 1 \leq i \leq j \leq 2m \). We may assume that if the intersection of the first or the last edge (edge with the number \( i \) or \( j \) respectively) and the next one consists of two vertices. Indeed, if that intersection consists of one vertex and \( \rho(H) \geq \rho(G) \), then we delete this first or last edge and \((s-1)\) its vertices, that are not in the intersection and get a hypergraph with a density not less than \( \rho(G) \). Thus we may assume that both the first and the last intersections contain two vertices. Let \( i = [km/r] - 1 \), \( j = [lm/r] \) \((1 \leq k \leq l \leq 2r)\). Then \( e(H) = [lm/r] - [km/r] + 2 \), \( v(H) = e(H)s - (e(H) - 1) - (l - k + 1) \). Let us prove that \( mv(H) - ne(H) > 0 \), and so \( m/n > \rho(H) \).

\[
mv(H) - ne(H) = m(e(H)s - (e(H) - 1) - (l - k + 1)) - (m(s-1) - r)e(H) = \\
= m - m(l - k + 1) + re(H) = -m(l - k) + r([lm/r] - [km/r] + 2) > \\
> -m(l - k) + r(lm/r - km/r + 1) = r \geq 0.
\]

The inequality is proved. Thus \( G \) is strictly balanced with \( \rho(G) = m/n \).
4.2 Case $\rho < \frac{1}{s-1}$

Let $G$ be an arbitrary $s$–hypergraph. Let $v_1, \ldots, v_m \in V(G)$, $e_1, \ldots, e_m \in E(G)$. We say that the sequence $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1} = v_1)$ is a cycle if for any $i = 1, \ldots, m$

- $v_i \neq v_{i+1}$
- $e_i \neq e_{i+1}$, where $e_{m+1} = e_1$
- $\{v_i, v_{i+1}\} \in e_i$.

We say that $s$–hypergraph is a tree if it is connected and has no cycles. It is easy to see that a tree with $k$ edges has exactly $1 + k(s - 1)$ vertices, and so its density is $\frac{k}{1 + k(s - 1)}$. Let $G$ be an $s$–hypergraph with a density less than $\frac{1}{s-1}$. Our goal is to prove that $G$ is a strictly balanced if and only if $G$ is a tree.

Note that every connected nonempty sub-hypergraph of a tree is a tree itself. Hence if $H$ is a connected proper sub-hypergraph of $G$ then

$$\rho(H) = \frac{e(H)}{1 + e(H)(s - 1)} < \frac{e(G)}{1 + e(G)(s - 1)} = \rho(G).$$

Thus $G$ is strictly balanced.

Now, let $G$ be a strictly balanced $s$–hypergraph with a density $\rho < \frac{1}{s-1}$. Obviously, $G$ is connected. Note that if $G$ has a cycle $(v_1, e_1, v_2, e_2, \ldots, v_n, e_m, v_1)$ then the induced $s$–hypergraph $G|_{e_1 \cup \ldots \cup e_m}$ has at least $m$ edges and at most $m(s - 1)$ vertices, so its density is not less than $\frac{1}{s-1}$. So $G$ is acyclic and connected, hence $G$ is a tree.

5 Main tools

In order to prove Theorem 8 we use the methods introduced by J.H. Spencer (see [8], [9]). In this section we generalize some definitions and theorems from [8], [16]. The results obtained in this section will be used in the proof of Theorem 8 in Section 6.

5.1 Extensions

In this subsection we generalize some well known definitions and review some theorems from [8], [16], that are concerned with rooted graphs and their extensions.

Let $s \geq 2$ and irrational $\alpha > 0$ be fixed throughout this section. A rooted $s$-hypergraph is a pair $(R, H)$ where $H$ is an $s$-hypergraph on vertex set, say, $V(H) = \{a_1, \ldots, a_r, b_1, \ldots, b_v\}$ and $R = \{a_1, \ldots, a_r\} \subseteq V(H)$ is a specified subset of $V(H)$. Note that $R$ could be empty. The vertices of $R$ are called roots. Let $v = v(R, H)$ denote the number of vertices that are not roots and let $e = e(R, H)$ denote the number of edges of $H$ excluding those edges with
all $s$ vertices being roots. We call $(v, e)$ the type of $(R, H)$. We say that $(R, H)$ is dense if $v - e\alpha < 0$ and sparse if $v - e\alpha > 0$. As $\alpha$ is irrational, then every $(R, H)$ is either dense or sparse. If $R \subseteq S \subset V(H)$, then we call $(R, H|_S)$ a subextension of $(R, H)$. If $R \subset S \subseteq V(H)$ we call $(S, H)$ a nailextension of $(R, H)$. We call $(R, H)$ rigid if all of its nailextensions are dense. We call $(R, H)$ safe if all of its subextensions are sparse. We call $(R, H)$ minimally safe if it is safe and has no safe nailextensions. We sometimes write $(R, S)$ for $(R, H|_S)$ when the hypergraph $H$ is understood.

Review some properties of rooted hypergraphs (their proofs in the case $s = 2$ can be found in [8], for arbitrary $s$ the proofs are the same).

Claim 1. Let $(R, H)$ be not safe. Then it has a rigid subextension.

Claim 2. Let $(R, H)$ be not rigid. Then it has a safe nailextension.

Claim 3. If $(R, H)$ is minimally safe and $R \subset S \subset V(H)$ then $(S, H)$ is rigid.

Claim 4. Let $(R, H)$ be minimally safe of type $(v, e)$ with $v > 1$. Then $v - e\alpha < 1$.

Let $G$ be an arbitrary $s$-hypergraph. Let $x = (x_1, \ldots, x_r)$ be an $r$-tuple of distinct vertices of $G$ and let $y = (y_1, \ldots, y_v)$ be a $v$-tuple of distinct vertices of $G$ such that $x \cap y = \emptyset$. We say that $y$ is an $(R, H)$-extension of $x$ if $\{x_{i_1}, \ldots, x_{i_k}, y_{i_{k+1}}, \ldots, y_s\} \in E(G)$ whenever $\{a_{i_1}, \ldots, a_{i_k}, b_{i_{k+1}}, \ldots, b_s\} \in E(H)$ where $0 \leq k < s$. Note that $G|_{x \cup y}$ should have all edges of $H$ which contain at least one nonroot, but there are no restrictions for containing some additional edges. If there are no additional edges, then we say that $y$ is an exact $(R, H)$-extension of $x$. For any $x = (x_1, \ldots, x_r)$ of distinct vertices in $G^s(n,p)$ let $N_x$ denote the number of $(R, H)$-extensions $y$ of $x$.

Review the results on the distribution of the number of extensions in $G^s(n,p)$ (their proofs in the case $s = 2$ can also be found in [8], for arbitrary $s$ the proofs are the same; the proofs are based on the properties of rooted hypergraphs and Janson’s Inequality (see [15])).

Claim 5. Let $(R, H)$ be minimally safe and have type $(v, e)$ with $r$ roots. Let $x$ be a fixed $r$-tuple from $G^s(n,p)$ with $p = n^{-\alpha}$. Set $\mu = E N_x$. Then $\Pr[N_x = 0] = e^{-\mu(1+o(1))}$.

The proof of the following theorem is based on Claim 5.

Theorem 9 (Counting Extensions Theorem). Let $(R, H)$ be safe and have type $(v, e)$ with $r$ roots. Then almost surely

$$N_x \sim \mu$$

for all $x = (x_1, \ldots, x_r)$ of distinct vertices in $G^s(n, n^{-\alpha})$. 
5.2 The t-closure

Let $G$ be an arbitrary $s$-hypergraph and let irrational $\alpha > 0$ be fixed. A rigid $t$-chain in $G$ is a sequence $x = x_0 \subset x_1 \subset \ldots \subset x_k$ of subsets of $V(G)$ with all $(x_{i-1}, G|_{x_i})$ rigid (with respect to $\alpha$) and all $|x_i \setminus x_{i-1}| \leq t$. The $t$-closure of $x$, denoted by $\text{cl}_t(x)$, is the maximal $y$ for which there exists a rigid $t$-chain (of arbitrary length) $x = x_0 \subset x_1 \subset \ldots \subset x_k = y$. When there are no such rigid $t$-chains we define $\text{cl}_t(x) = x$. To see that $t$-closure is well defined we note that if $x = x_0 \subset x_1 \subset \ldots \subset x_k = z$ and $y = y_0 \subset y_1 \subset \ldots \subset y_l = y$ are rigid $t$-chains then so is $x = x_0 \subset x_1 \subset \ldots \subset x_k \cup y_1 \subset \ldots \subset x_k \cup y_l = z \cup y$. Alternatively, the $t$-closure $\text{cl}_t(x)$ is the minimal set containing $x$ which has no rigid extensions of $\leq t$ vertices.

The following result is a corollary of Theorem 5 (its proof in the case $s = 2$ can be found in [8], [9], for arbitrary $s$ the proof is the same).

Theorem 10 (Finite Closure Theorem). Fix positive integers $t$ and $r$. Set $\varepsilon$ equal to the minimal value of $(e\alpha - v)/v$ over all integers $v$, $e$ with $1 \leq v \leq t$ and $e\alpha - v > 0$. Let $K$ be such that $r - K\varepsilon < 0$. Then in $G^s(n, n^{-\alpha})$ almost surely

$$|\text{cl}_t(X)| \leq K + r$$

for all $X \subset V(G)$ with $|X| = r$.

5.3 Generic extension

Let $\alpha > 0$ be fixed, let $(R, H)$ be a rooted graph, and let $t$ be a positive integer. We say that $y = (y_1, \ldots, y_v)$ is a $t$-generic $(R, H)$-extension of $x$ if the following two properties hold.

- $y$ is an exact $(R, H)$-extension of $x$.
- If any $z = (z_1, \ldots, z_s)$ with $s \leq t$ forms a rigid extension over $x \cup y$ then there are no edges in $G|_{x \cup y \cup z}$ containing at least one vertex from $y$ and at least one vertex from $z$.

The next theorem (its proof for graphs can be found in [8]) which is a corollary of Theorems 9, 10 and properties of rooted hypergraphs (see Section 5.1) states the existence of a generic extension.

Theorem 11 (Generic Extension Theorem). If $(R, H)$ is safe, relative to $\alpha$, then in $G \sim G^s(n, p)$ with $p = n^{-\alpha}$ almost surely every $x$ has a $t$-generic extension $y$. 
5.4 Ehrenfeucht game

Let us define the *Ehrenfeucht Game* $\text{EHR}(G_1, G_2; k)$ with two disjoint $s$-hypergraphs $G_1$ and $G_2$ ($V(G_1) \cap V(G_2) = \emptyset$), two players (Spoiler and Duplicator) and a fixed number of rounds $k$. Each round has two parts, Spoiler’s move followed by Duplicator’s move. On the $i$-th round Spoiler selects either a vertex $x_i \in V(G_1)$ or a vertex $y_i \in V(G_2)$. Then Duplicator must select a vertex from the other graph. If Spoiler chooses previously chosen vertex, say, $x_i = x_j \in V(G_1)$ ($j < i$), then Duplicator must choose the corresponding vertex $y_j \in V(G_2)$. If Spoiler chooses a vertex $x_i \notin \{x_1, \ldots, x_{i-1}\}$, then Duplicator must choose a vertex $y_i \notin \{y_1, \ldots, y_{i-1}\}$. If Duplicator can not do it then Spoiler wins the game. At the end of the game the vertices $x_1, x_2, \ldots, x_k \in V(G_1)$ and $y_1, y_2, \ldots, y_k \in V(G_2)$ are chosen. For Duplicator to win she must assure that, for all $1 \leq i \leq j \leq k$, $x_i$, $x_j$ are adjacent if and only if $y_i$, $y_j$ are adjacent. If Duplicator does not win then Spoiler wins.

**Theorem 12.** The random $s$-hypergraph $G^s(n, p(n))$ obeys Zero-One Law if and only if for every positive integer $k$

$$\lim_{n,m \to \infty} \Pr\{\text{Duplicator wins EHR}(G_1, G_2; k)\} = 1,$$

where $G_1 \sim G^s(n, p(n))$, $G_2 \sim G^s(m, p(m))$ are independently chosen and have disjoint vertex sets.

6 Proof of Theorem

Let us prove Theorem. We fix an irrational $\alpha > 0$. All probabilities are with respect to the random $s$-hypergraph $G^s(n, p)$ with $p = n^{-\alpha}$.

Our approach is through the Ehrenfeucht game. We fix the number $k$ of moves. We will give a strategy for Duplicator so that, as $n, m \to \infty$, she almost surely wins EHR($G_1, G_2, k$) where $G_1 \sim G^s(n, n^{-\alpha})$ and $G_2 \sim G^s(m, m^{-\alpha})$ are independently chosen.

Let $a_1, a_2, \ldots, a_k = 0$ be nonnegative integers. Duplicator uses $(a_1, \ldots, a_k)$-look-ahead strategy which is defined in the following way. Duplicator makes any moves in response to Spoiler so that at the end of $r$-th move the $a_r$-types of the vertices chosen are the same in both hypergraphs. That is, if $x_1, \ldots, x_r \in G_1$, $y_1, \ldots, y_r \in G_2$ have been chosen then there is a hypergraph isomorphism from $cl_{a_r}(x_1, \ldots, x_r)$ to $cl_{a_r}(y_1, \ldots, y_r)$ sending each $x_j$ to its corresponding $y_j$. Note that if Duplicator is able to keep to this strategy then at the end of the game the 0-closures are the same and she has won. Following, we define define $a_1, \ldots, a_k$ by reverse induction so that Duplicator will almost surely be able to keep to $(a_1, \ldots, a_k)$-look-ahead strategy. Suppose, inductively, that $a_{r+1}$ has been defined. We define $a_r$ to be any integer satisfying

1. $a_r \geq a_{r+1}$.
2. almost surely $|c_{a_{r+1}}(W)| - r \leq a_r$ for all sets $W$ of size $r + 1$.

The existence of such $a$ is a consequence of Theorem 10. In [8] it is proved that almost surely this strategy works. In the original proof, only graphs are considered. However, this proof is based on Theorem 11 only, which is also true for hypergraphs. So, by Theorem 12 $G^*(n, n^{-\alpha})$ obeys Zero-One Law when $\alpha$ is a positive irrational number.

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