On the Tolman and Møller mass-energy formulae in general relativity

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Abstract. Two of the most satisfactory descriptions of mass–energy and its localization in general relativity are given in terms of the Tolman and Møller formulae. The form of these two equations, as well as their derivation, are so utterly different that for thirty–five years, there was never even the slightest suspicion that there was any relationship between them. Yet, as it will be shown in this talk, for time–independent fields the two formulae are in fact identical.

1. Introduction
The problem that has bedeviled the general theory of relativity most, from the early days of the theory to the present, is the problem of mass-energy and of its localization. Despite the immense effort of many relativists there is still no satisfactory, universally accepted formula for the total (that is, matter plus field) mass-energy of either the whole or part of a physical system.

As is well known there is an infinite number of widely different formulae purporting to give the total mass-energy of a physical system. The one thing that all these formulae have in common is that, for an isolated physical system which is time–independent and which, asymptotically, is strictly the Minkowski space-time expressed in Cartesian coordinates, they all give the correct value for the total mass-energy of the whole of the system. Unless these conditions are strictly satisfied almost all the formulae give values for the mass-energy of the whole system which are, more often than not, totally absurd. The most striking absurdity is that for the Minkowski space–time, when expressed in spherical Polar coordinates, the Einstein formula (given by equations (4–6) below), gives an infinite amount of energy for the whole of the 3–space \( t = \text{constant} \). Exactly the same absurd result is obtained for the Schwarzschild (exterior) space–time when it is expressed in the usual curvature coordinates. The next, and related problem, concerns the localization of mass-energy. That is the problem of finding the (total) mass-energy within a given region of some 3-space \( x^0 \equiv t = \text{constant} \). Again, almost all of the mass-energy formulae are unable to deal with this problem.

To the best of my knowledge the two mass-energy formulae which, for time independent fields at any rate, can deal satisfactorily with both the total mass-energy of the whole system and the localization of mass-energy are the Tolman and Møller formulae. These formulae are discussed briefly in section two where the main steps of the derivation of the Møller formula is also given. Despite the totally different derivations of the two formulae and their totally different form, it is shown in section three that, for time independent fields, the two
formulae are, in fact, completely equivalent. This equivalence is discussed briefly in section four. The equivalence of the two formulae was first derived by the author in references [1, 2]. The present talk was prompted by Dr E. C. Vagenas whose work [3], and references therein, showed that the Møller formula is still a topic of active research.

2. Tolman and Møller formulae

We shall denote by $m_T$ the Tolman mass-energy of a physical system. It is given by the Tolman formula [4, 5]

$$m_T = \int_V \left( T_1^1 + T_2^2 + T_3^3 - T_0^0 \right) \sqrt{-g} dV \equiv \int_V \left( T_\alpha^\alpha - T_0^0 \right) \sqrt{-g} dV.$$  \hspace{1cm} (1)

Here, $T_j^i$ is the energy-momentum-stress tensor of the system, $g$ the determinant of the space-time metric $g_{ij}$, $V$ the region of the 3-space $x^\alpha = \text{constant}$ occupied by matter or electromagnetic fields, and $dV$ is the coordinates 3-volume element. {Throughout this paper Latin indices take the values, 0,1,2,3 and Greek indices the values 1,2,3 with Einstein’s summation and range conventions applied in each case. Relativistic units are used so that $c = G = 1$, and the signature of the matrix is +2. Comma (,) denotes partial differentiation $(f, i = \frac{\partial f}{\partial x^i}).$} This formula has, as Tolman remarked, “the great advantage that it can be evaluated by integrating over the region occupied by matter or electromagnetic energy”.

Since Tolman’s formula is widely known it should suffice to say that Tolman’s derivation of formula (1) is based on Einstein’s energy pseudotensor, and so too is the much simpler derivation due to Papapetrou [6]. By far the simplest derivation, and one which circumvents the use of pseudotensors, was given by Landau and Lifshitz in their book on Classical Fields [7]. Another derivation, for static fields, was given by Whittaker [8].

We come now to the Møller formula. Since this is not as well-known as Tolman’s formula it will be helpful to outline the main steps by which Møller arrived at his equation. Møller starts with the Einstein energy complex $\Theta^a_b$ which is defined by

$$\Theta^a_b = \sqrt{-g} (T^a_b + t^a_b),$$  \hspace{1cm} (2)

$t^a_b$ being Einstein’s energy pseudotensor. This complex satisfies the divergence equation

$$\Theta^\alpha_{b,\alpha} = 0,$$  \hspace{1cm} (3)

which leads, in the usual way, to the definition of the energy-momentum (four) vector of the system

$$(-E, P_\alpha) = \int_V \Theta^\alpha_0 dV.$$  \hspace{1cm} (4)

A simpler and more useful form of the Einstein complex is in terms of the so-called Freud [9] superpotentials $h^{ac}_b$ defined by

$$h^{ac}_{b} = -h^{ca}_{b} = \frac{g_{bi}}{2\kappa \sqrt{-g}} \left[ (-g)(g^{ai} g^{cj} - g^{ci} g^{aj}) \right]_{ij}, (\kappa = 8\pi),$$  \hspace{1cm} (5)

that is,

$$\Theta^a_b = h^{ac}_{b,c}.$$  \hspace{1cm} (6)
We note that because of the antisymmetry of $h_{bc}^a$ in $a$ and $c$, equation (3) is satisfied automatically.

It must be emphasised again that Einstein’s equation (4) is meaningful only when $V$ is the whole 3-space $x^\alpha=$constant of a physical system which is time–independent and, asymptotically, its field is strictly the Minkowski space-time expressed in Cartesian coordinates. It was precisely the attempt to remove this last restriction, the adherence to Cartesian coordinates, that lead Møller to his mass-energy formula.

Møller argued [10] that equation (3) determines $\Theta_{bc}^a$ only to within a quantity $S_{bc}^a$ with identically vanishing divergence, i.e. $S_{bc,a}^a = 0$. He uses this freedom to define a new energy complex $J_{bc}^a$ by

$$J_{bc}^a = \Theta_{bc}^a + S_{bc}^a,$$

where $S_{bc}^a$ behaves like a tensor density of weight $-1$ under all linear transformations, and is chosen so that

(i) $S_{bc,a}^a = 0$

(ii) $\int_V S_{bc}^a dx^1 dx^2 dx^3 = 0$ when the coordinates $x^1, x^2, x^3$ are quasi-Galilean and $V$ is the whole of the 3-space $x^0=$constant,

(iii) $J_{bc}^a$ behaves like a 4-vector density under all spatial coordinate transformations

$$x^\alpha \rightarrow \vec{x}^\alpha = \vec{x}^\alpha(x^1, x^2, x^3), \quad x^\alpha \rightarrow \vec{x}^\alpha = x^\alpha, \quad (\alpha = 1, 2, 3),$$

These three requirements lead Møller to the expression

$$S_{bc}^a = h_{bc}^{ac} - \delta_{bc}^a h_{ij}^{ij} + \delta_{bc}^{ja} h_{ji}^{ja},$$

where $h_{bc}^{ac}$ are defined by equation (5). Furthermore, Møller showed that $J_{bc}^a$ can be written in the form

$$J_{bc}^a = \chi_{bc}^{ac},$$

where the (Møller) superpotentials $\chi_{bc}^{ac}$ are given by

$$\chi_{bc}^{ac} = \frac{\sqrt{-g}}{\kappa} g^{ai} g^{c3}(g_{bj,i} - g_{bi,j}).$$

Because of the antisymmetry of $\chi_{bc}^{ac}$ in $a$ and $c$ we get

$$J_{bc,a}^a = 0,$$

from which follows the usual interpretation that the energy-momentum is given by

$$(-m_M, P_\alpha) = \int_V J_{bc}^a dV,$$

having written $m_M$ for the Møller mass-energy of the system under consideration.

This is, in essence, Møller’s theory. The advantages of Møller’s formula (12) over Einstein’s formula (4) are obvious. Because of condition (ii) above, Møller’s formula agrees with Einstein’s whenever the latter is applicable. In contradistinction to Einstein’s formula,
Møller’s formula can, in view of condition (iii), be used in any spatial coordinate system. Furthermore Møller’s formula can be used to find the total energy and momentum within any region of a 3-space $x^α=$-constant. Thus, unlike Einstein’s theory, it is meaningful to talk about the localization of energy in Møller’s theory.

Over the years the author applied Møller’s formula on many occasions to many different physical systems. On all occasions the results were meaningful and interesting. It was, for example, on the basis of Møller’s formula that the author discussed, for the first time, quantitatively the contribution of the electromagnetic energy of a charged sphere to the total gravitational mass of the sphere. For further details and applications of Møller’s formula see [11, 12].

Let us concentrate on the zeroth component of (12), namely the Møller mass-energy $m_M$. Remembering that $χ^α_β = -χ^β_α$ equations (10) and (12) give

$$m_M = -\int_V J^o_α dV = -\int_V χ^{αα}_o dV \quad (α = 1, 2, 3),$$

(13)

where

$$χ^{αα}_o = \frac{\sqrt{-g}}{κ} g^{αo} g^{αj} (g_{o,j} - g_{o,j}).$$

(14)

Recall, also, that the Tolman mass-energy $m_T$ is given by equation (1), i.e.

$$m_T = \int_V (T^o_α - T^o_o)\sqrt{-g}dV.$$

(15)

The expressions for $m_T$ and $m_M$, as well as their derivation, are so utterly different that, for thirty-five years, there was never even the slightest suspicion that there was any relationship between them. Yet, whenever the two formulae were applied by the author to time-independent spherically symmetric fields, the results were always identical. In view of these results the author ventured to ask the question: “Is it possible that these totally different looking formulae are in fact identical for general time-independent fields?”.

Once the question was asked it was easy to give the answer. Surprisingly, almost in disbelief, it is proved in the following section that the two formulae are indeed identical.

### 3. The equivalence of the Tolman and Møller formulae

The proof of the equivalence of the two formulae (13) and (15) is based on the following simple formula due to Landau and Lifshitz [7]:

$$R^o_α = -\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{αo}Γ^o_{αi})_α,$$

(16)

$R^o_α$ being the (0,0) component of the Ricci tensor $R^i_j$. The formula holds for arbitrary time independent fields. Now, from Einstein’s field equations, written in the form $R^i_j = -κ(T^i_j - \frac{1}{2}δ^i_j T^o_o), \quad κ = 8\pi$, we get

$$R^o_α = \frac{κ}{2}(T^o_α - T^o_o).$$

(17)

Using this in (16) we get

$$(T^o_α - T^o_o)\sqrt{-g} = -\frac{2}{κ}(\sqrt{-g}g^αoΓ^o_{αi})_α.$$
Now, from the definition of the Christoffel symbols, we get
\[ g^{i\alpha} \Gamma_{\alpha}^{i} = g^{i\alpha} g^{\alpha j} (g_{oj,i} + g_{ij,o} - g_{oi,j}), \]
or, since by assumption \( g_{ij} \) is time-independent,
\[ g^{i\alpha} \Gamma_{\alpha}^{i} = \frac{1}{2} g^{i\alpha} g^{\alpha j} (g_{oj,i} - g_{oi,j}). \] (19)
Substituting this in equation (18) we get
\[ (T_{\alpha}^{\alpha} - T_{\alpha}^{\alpha}) \sqrt{-g} = -\frac{1}{\kappa} \sqrt{-g} g^{i\alpha} g^{\alpha j} (g_{oj,i} - g_{oi,j}), \]
or, by equation (14),
\[ (T_{\alpha}^{\alpha} - T_{\alpha}^{\alpha}) \sqrt{-g} = -\chi_{\alpha\alpha}^{\alpha\alpha}, \] (20)
\( \chi_{\alpha\beta}^{\alpha\beta} \) being the Møller superpotential. We have thus arrived, quite unexpectedly, at the result that the integrands of the Tolman and Møller mass-energy formulae are identical. The two formulae are therefore completely equivalent. However different the dresses of the two formulae, they cover one and the same body.

4. Discussion
It is, of course, satisfying to see the number of mass-energy formulae reduced even by one; even if the original number is infinite! The equivalence of the Tolman \( (m_T) \) and Møller \( (m_M) \) formulae, however, is also important in at least two further aspects: (i) Because the integrand of \( m_M \) is in the form of plain divergence it is often simpler to calculate the mass-energy of a system using the formula \( m_M \) (and Gauss’s theorem) rather than \( m_T \). (ii) The mass-energy of a system should appear as the zeroth component of an energy-momentum 4-vector. Up until now the Tolman mass-energy \( m_T \) stood entirely on its own. From the equivalence \( m_T = m_M \) established in the paper, \( m_T \) (just as \( m_M \)) is now the zeroth component of the Møller 4-vector \( (-m_M, P_{\alpha}) \) defined by equation (12); strictly speaking \( (-m_M, P_{\alpha}) \) is a 4-vector under the transformation (8).

For a critical re-appraisal of Tolman’s (and, therefore, Møller’s) formula the reader is referred to papers [13]-[15].

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