Adaptivity vs Postselection

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Abstract

We study the following problem: with the power of postselection (classically or quantumly), what is your ability to answer adaptive queries to certain languages? More specifically, for what kind of computational classes $C$, we have $P^C$ belongs to $\text{PostBPP}$ and $\text{PostBQP}$? While a complete answer to the above question seems impossible given the development of present computational complexity theory. We study the analogous question in query complexity, which sheds light on the limitation of relativized methods (the relativization barrier) to the above question.

Informally, we show that, for a partial function $f$, if there is no efficient small bounded-error algorithm for $f$ classically or quantumly, then there is no efficient postselection bounded-error algorithm to answer adaptive queries to $f$ classically or quantumly. Our results imply a new proof for the classical oracle separation $P \subsetneq \text{NP} \subsetneq \text{PP}$. They also lead to a new oracle separation $P^{\text{SZK}} \subsetneq \text{PP}$, which is close to an oracle separation between $\text{SZK}$ and $\text{PP}$—an open problem in the field of oracle separations.

1 Introduction

1.1 Background

The idea of postselection has been surprisingly fruitful in theoretical computer science and quantum computing [AA11, DdW09, LMGP+11, BJS11]. Philosophically, it addresses the following question: if you believe in the Many-worlds interpretation\(^2\) and can condition on a rare event (implemented by killing yourself when seeing the undesired outcomes), then what would you be able to compute in a reasonable amount of time? The complexity classes $\text{PostBPP}$ [HHT97] and $\text{PostBQP}$ [Aar05] are defined to represent the computational problems you can solve with the ability of postselection in a classical world or a quantum world.

However, even with that seemingly omnipotent power of postselection, your computational power is still bounded. It is known that $\text{PostBPP} \subseteq \text{PH}$ [HHT97], and (surprisingly) $\text{PostBQP} = \text{PP}$ [Aar05]. Hence, it seems quite plausible that even with the postselection power, you are still not able to solve a PSPACE-complete problem, as it is widely believed that $\text{PH}$ and $\text{PP}$ are strictly contained in PSPACE.

Another more non-trivial (and perhaps unexpected) weakness of those postselection computation classes, is their inability to simulate adaptive queries to certain language. For example, it is known that $P^{\text{NP}[O(\log n)]}$ is contained in $\text{PostBPP}$, and this result relativizes. But there is an oracle separation between $P^{\text{NP}[\omega(\log n)]}$ and $\text{PostBQP}$ [Bei94]. In other words, there is no relativized $\text{PostBQP}$ algorithm can simulate $\omega(\log n)$ adaptive queries to a certain language in $\text{NP}$. In contrast, we know that $P^{[\text{NP}]} \subseteq \text{PostBPP} \subseteq \text{PP}$, hence they are capable of simulating non-adaptive queries to $\text{NP}$.

Then a natural question follows:

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\(^*\)This work was done when the author was visiting MIT.

\(^1\)In the world of query complexity, being efficient means using $O(\text{polylog}(n))$ time.

\(^2\)https://en.wikipedia.org/wiki/Many-worlds_interpretation

\(^3\)O(\log n) stands for the P algorithm can only make $O(\log n)$ queries to the oracle.
**Question 1.1.** What is the limit of the abilities of these postselection classes on simulating adaptive queries to certain languages? More specifically, is there any characterization of the complexity class $C$ such that $P^C$ is contained in $PostBPP$ or $PostBQP$?

Arguably, a complete answer to this problem seems not possible at the present time: even determining whether $P^{NP} \subseteq PP$ is already extremely hard, as showing $P^{NP} \subseteq PP$ probably requires some new non-relativized techniques, and proving $P^{NP} \not\subseteq PP$ implies $PH \not\subseteq PP$, which is a long-standing open problem.

**1.2 Relativization and the analogous question in query complexity**

So in this paper, inspired by the oracle separation in [Bei94], we study this problem from a relativization point of view. **Relativization**, or **oracle separations** are ultimately about the query complexity. Given a complexity class $C$, there is a canonical way to define its analogue in query complexity: partial functions which are computable by a non-uniform $C$ machine with polylog($n$) queries to the input. For convenience, we will use $C_{dt}$ to denote the query complexity version of $C$. We adopt the convention that $C_{dt}$ denotes the query analogue of $C$, while $C^{dt}(f)$ denotes the $C_{dt}$ complexity of the partial function $f$.

For a partial function $f$, we use $\text{len}(f)$ to denote its input length. We say a family of partial function $f \in C^{dt}$, if $C^{dt}(f) = O(\text{polylog}(\text{len}(f)))$ for all $f \in f$.

In order to study this question in the query complexity setting, given a partial function $f$, we need to define its adaptive version.

**Definition 1.2** (Adaptive Construction). Given a function $f : D \rightarrow \{0, 1\}$ with $D \subseteq \{0, 1\}^M$ and an integer $d$, we define $\text{Ada}_{f,d}$, its depth $d$ adaptive version, as follows:

$$\text{Ada}_{f,0} := f,$$
and

$$\text{Ada}_{f,d} : D \times D_{d-1} \times D_{d-1} \rightarrow \{0, 1\}$$

$$\text{Ada}_{f,d}(w, x, y) := \begin{cases} 
\text{Ada}_{f,d-1}(x) & f(w) = 0 \\
\text{Ada}_{f,d-1}(y) & f(w) = 1 
\end{cases}$$

where $D_{d-1}$ denotes the domain of $\text{Ada}_{f,d-1}$.

The input to $\text{Ada}_{f,d}$ can be encoded as a string of length $(2^{d+1} \cdot 1 - 1) \cdot M$. Thus, $\text{Ada}_{f,d}$ is a partial function from $D^{(2^{d+1} - 1)} \rightarrow \{0, 1\}$.

Then, given a family of partial function $f$, we define $\text{Ada}_f := \{\text{Ada}_{f,d} \mid f \in f, d \in \mathbb{N}\}$.

Notice that when you have the ability to adaptively solve $d + 1$ queries to $f$ (or with high probability), then it is easy to solve $\text{Ada}_{f,d}$. Conversely, in order to solve $\text{Ada}_{f,d}$, you need to be able to adaptively answer $d + 1$ questions to $f$, as even knowing what is the right $i$th question to answer requires you to correctly answer all the previous $i - 1$ questions.

Now, everything is ready for us to state the analogous question in query complexity.

**Question 1.3.** What is the characterization of the partial functions family $f$ such that $\text{Ada}_f \in PostBPP^{dt}$ ($PostBQP^{dt}$)?

There are at least two reasons to study Question 1.3. First, it is an interesting question itself in query complexity. Second, an answer to Question 1.3 also completely characterizes the limitation on the relativized techniques for answering Question 1.1 i.e., the limitation of relativized methods for simulating adaptive queries to certain complexity classes with the power of postselection.

This paper provides some interesting results toward resolving Question 1.3.
Our results

Despite that we are not able to give a complete answer to Question 1.3. We provide some interesting lower bounds showing that certain functions’ adaptive versions are hard for these postselection classes.

Formally, we prove the following theorems.

**Theorem 1.4 (Classical Case).** For a family of partial function \( f \), \( \text{Ada}_f \notin \text{PostBPP}^{dt} \) if \( f \notin \text{SBP}^{dt} \cap \text{coSBP}^{dt} \).

**Theorem 1.5 (Quantum Case).** For a family of partial function \( f \), \( \text{Ada}_f \notin \text{PostBQP}^{dt}(\text{PP}^{dt}) \) if \( f \notin \text{SBQP}^{dt} \cap \text{coSBQP}^{dt} \).

Roughly speaking, \( \text{SBP} \) is a relaxation of \( \text{BPP} \), it is the set of languages \( L \) such that there exists a \( \text{BPP} \) machine \( M \), which accepts \( x \) with probability \( \geq 2\alpha \) if \( x \in L \); and with probability \( \leq \alpha \) if \( x \notin L \) for a positive real number \( \alpha \). And \( \text{SBQP} \) is the quantum analogue of \( \text{SBP} \), where you are allowed to use a polynomial time quantum algorithm instead.

Our theorems show that, for a partial function \( f \), if there is no efficient classical (quantum) algorithm which accepts all the 1-inputs with a slightly better chance than all the 0-inputs, then there is no efficient PostBPP (PostBQP) algorithm that can answer adaptive queries to \( f \).

Applications in oracle separations

Our results have several applications in oracle separations.

- A new proof for \( \text{P}^{\text{NP}^O} \not\subset \text{P}^{\text{P}^O} \):

  We prove that \( \text{SBQP}^{dt}(f) \) is indeed equivalent to one-sided low-weight approximate degree, denoted by \( \deg_+(f) \), which is defined in Definition 2.19 and lower bounded by one-sided approximate degree \( \deg_+(f) \).

  Using the fact that \( \deg_+(\text{OR}_n) \geq \Omega(\sqrt{n}) \), Theorem 1.5 implies that \( \text{AdaOR} \notin \text{P}^{\text{dt}} \). Which yields an alternative proof for the classical oracle separation between \( \text{P}^{\text{NP}} \) and \( \text{PP} \) in [Bez94].

  Our proof is arguably more simple and elegant. Also, unlike the seemingly artificial problem ODD-MAX-BIT in [Bez94], \( \text{AdaOR} \) seems a more natural hard problem in \( \text{P}^{\text{NP}} \).

- The new oracle separation \( \text{P}^{\text{SZK}^O} \not\subset \text{P}^{\text{P}^O} \):

Since the Permutation Testing Problem, denoted by \( \text{PTP}_n \) (see Problem 2.23 for a formal definition), satisfies \( \deg_+(\text{PTP}_n) \geq \Omega(n^{1/3}) \) and has a \( \log(n) \)-time SZK protocol. Theorem 1.5 implies that \( \text{Ada}_{\text{PTP}} \not\subset \text{P}^{\text{dt}} \), which in turn shows an oracle separation between \( \text{P}^{\text{SZK}} \) and \( \text{PP} \).

It has been an open problem [Ani12] that whether there exists an oracle separation between \( \text{SZK} \) and \( \text{PP} \), our result is pretty close to an affirmative answer to that.

A toy example

We use \( \text{AdaOR}_n := \text{AdaOR}_{n,\log n} \) as a toy example to illustrate the techniques we used to prove lower bounds against \( \text{PP}^{dt} \). The approach for \( \text{PostBPP}^{dt} \) is similar. In the following we sketch a proof for \( \text{AdaOR} \not\subset \text{P}^{\text{dt}} \).

We say a polynomial \( p \) computes a partial function \( f \), if \( p(x) \geq 1 \) when \( f(x) = 1 \), and \( p(x) \leq 1 \) when \( f(x) = 0 \). We can show that if \( \text{AdaOR} \in \text{P}^{\text{dt}} \), then there is a \( \text{o}(\sqrt{n}) \)-degree polynomial \( p \) computing \( \text{AdaOR}_n \) with \( ||p||_{\infty} = \max_{x} |p(x)| \leq 2^{\text{polylog}(n)} \) (Lemma 1.5). We are going to show this is impossible.

By the fact that \( \deg_+(\text{OR}_n) \geq \Omega(\sqrt{n}) \) [NS94] and a clever use of minimax theorem (Lemma 1.4), there exist two distributions \( D_0 \) and \( D_1 \) supported on \( \text{OR}_n^{-1}(0) \) and \( \text{OR}_n^{-1}(1) \), such that for any \( \text{o}(\sqrt{n}) \)-degree polynomial \( p \) computing \( \text{OR}_n \), we have

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4 For the formal definitions of \( \text{SBP} \), \( \text{PostBPP} \), \( \text{PostBQP} \), \( \text{SBQP} \) and their equivalents in query complexity, see the preliminaries.

5 We slightly abuse notation by using \( \text{AdaOR} \) to denote \( \{\text{AdaOR}_n\}_{n=1}^\infty \).
−p(D_0) > 2 \cdot p(D_1).

Now, let f_d := \text{AdaOR}_{w,d}, we define the input distributions D^d_0 and D^d_1 to f_d recursively as follows:

\[
D^d_0 := \begin{cases} D_0 & d = 0 \\
D_0 \times D^{d-1}_0 \times D^{d-1}_0 & d \geq 1 \end{cases}
\quad D^d_1 := \begin{cases} D_1 & d = 0 \\
D_1 \times D^{d-1}_1 \times D^{d-1}_1 & d \geq 1 \end{cases}
\]

Observe that D^d_1 is supported on i-inputs to f_d for i \in \{0, 1\}.

Then we show that for any o(\sqrt{n})-degree polynomial p computing f_d, it must satisfy \(-p(D^d_0) > 2^{2^d} \cdot p(D^d_1)\).

Hence, for d = \log n, we have \(-p(D^d_0) > 2^n\), which means \(\|p\|_\infty > 2^n\), contradiction.

Now it remains to prove the above claim. The proof is a surprisingly elegant induction. The base case \(d = 0\) already follows from the definition. When \(d \geq 1\), let the input to f_d be a triple \((w, x, y)\) as in the definition.

Let \(p(w, x, y)\) be a polynomial computing f_d. We define \(p(D_w, D_x, D_y) := E_{w \sim D_w, x \sim D_x, y \sim D_y}[p(w, x, y)]\) for simplicity. For any fixed \(W \in \text{support}(D_0)\) and \(Y \in \text{support}(D^{d-1}_0)\), the polynomial in x defined by \(p_L(x) := p(W, x, Y)\) computes \(f_{d-1}\), hence \(-p_L(D^{d-1}_0) > 2^{2^{d-1}} \cdot p_L(D^{d-1}_1)\). By linearity, we have \(-p(D_0, D^{d-1}_0, D^{d-1}_0) > 2^{2^d-1} \cdot p(D_0, D^{d-1}_1, D^{d-1}_0)\).

Afterwards, for any fixed X \in \text{support}(D^{d-1}_1) and Y \in \text{support}(D^{d-1}_0), the polynomial in w defined by \(p_M(w) := -p(w, X, Y)\) computes \(f\), hence \(-p_M(D_0) > 2 \cdot p_M(D_1)\). Again by linearity, we have \(p(D_0, D^{d-1}_1, D^{d-1}_1) > -2 \cdot p(D_1, D^{d-1}_1, D^{d-1}_1)\).

Finally, for any fixed W \in \text{support}(D_1) and X \in \text{support}(D^{d-1}_1), the polynomial in y defined by \(p_R(y) := p(W, X, y)\) computes \(f_{d-1}\), hence \(-p_R(D^{d-1}_0) > 2^{2^{d-1}} \cdot p_R(D^{d-1}_1)\). Still by linearity, we have \(-p(D_1, D^{d-1}_1, D^{d-1}_0) > 2^{2^d-1} \cdot p(D_1, D^{d-1}_1, D^{d-1}_1)\).

Putting everything together, we have

\[-p(D^d_0) = -p(D_0, D^{d-1}_0, D^{d-1}_0) > 2^{2^d} \cdot p(D_1, D^{d-1}_1, D^{d-1}_1) = 2^{2^d} \cdot p(D^d_1).\]

This completes the proof.

We remark that it actually gives a lower bound on the threshold weight of AdaOR.

1.6 Paper organization

In Section 2 we introduce some preliminaries and the formal definitions of those partial function classes in query complexity. We prove Theorem 1.3 in Section 3 and Theorem 1.5 in Section 4. In fact, we provide two tighter lower bounds Theorem 3.5 and Theorem 4.3 for that purpose. Then in Section 5 we provide formal proofs for the two oracle separation results we mentioned in the introduction.

2 Preliminaries

2.1 Decision trees and quantum query algorithms

A decision tree is the analogue of a deterministic algorithm in the query complexity world, and a quantum query algorithm is the analogue of a quantum algorithm. See [BDW02] for a nice survey on query complexity.

Let T be a randomized decision tree, we use C(T) to denote the maximum number of queries incurred by T in the worst case (i.e. the maximum height of a decision tree in the support of T).

Let Q be a quantum query algorithm, we use C(Q) to denote the number of queries taken by Q.

We assume a randomized decision tree T (or a quantum query algorithm Q) outputs a result in \{0, 1\}, and we use T(x) (Q(x)) to denote the (random) output of T (Q) given an input x.
2.2 Conical juntas

We introduce the definition for conical juntas [GLM+15], which will be used frequently throughout this paper.

Let \( x = x_1 \ldots x_M \in \{0, 1\}^M \) be a string. Then a literal is a term of the form \( x_i \) or \( 1 - x_i \), and a \( k \)-term is a product of \( k \) literals (each involving a different \( x_i \)), which is 1 if the literals all take on prescribed values and 0 otherwise.

**Definition 2.1.** A \( T \)-conical junta \( h \) is a non-negative linear combination of \( T \)-terms, i.e.,

\[
    h(x) := \sum_{i} \alpha_i \cdot C_i(x),
\]

where for each \( i \) we have \( \alpha_i \geq 0 \) and \( C_i \) is a \( T \)-term. We also define \( \text{weight}(h) := \sum_{i} \alpha_i \).

The following lemma shows that conical juntas is more powerful than randomized decision tree.

**Lemma 2.2** (Essentially Theorem 15 in [BDW02]). The acceptance probability of a \( T \)-query randomized decision tree \( T \) can be represented by a \( T \)-conical junta \( h \) with \( \text{weight}(h) \leq 2^T \).

2.3 Complexity classes and their query complexity analogues

We assume familiarity with some standard complexity classes like \( \text{PP} \). For completeness, we introduce those less well known complexity classes: \( \text{SBP} \), \( \text{SBQP} \), \( \text{A0PP} \), \( \text{PostBPP} \) (\( \text{BPP} \) path) and \( \text{PostBQP} \), and define their analogues in query complexity along the way.

Recall that \( C_{dt} \) is the set of the partial function family \( f \) such that \( C_{dt}(f) = O(\text{polylog}(\text{len}(f))) \) for all \( f \in f \), hence we only need to define \( C_{dt}(f) \) for a partial function \( f \).

2.3.1 \( \text{PP}_{dt} \)

We first define \( \text{PP}_{dt}(f) \).

**Definition 2.3.** Let \( f : D \rightarrow \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function. Let \( T \) be a randomized decision tree which computes \( f \) with a probability better than \( \frac{1}{2} \). Let \( \alpha \) be the maximum real number such that

\[
    \Pr[T(x) = f(x)] \geq \frac{1}{2} + \alpha
\]

for all \( x \in D \).

Then we define \( \text{PP}_{dt}(T; f) := C(T) + \log_2(1/\alpha) \), and \( \text{PP}_{dt}(f) \) as the minimum of \( \text{PP}_{dt}(T; f) \) over all \( T \) computing \( f \) with a probability better than \( \frac{1}{2} \).

2.3.2 \( \text{SBP} \) and \( \text{SBP}_{dt} \)

Now we recall the definition of \( \text{SBP} \), there are several equivalent definitions for \( \text{SBP} \) in [BGM06] (see Proposition 21), we use the most convenient one here.

**Definition 2.4.** \( \text{SBP} \) (defined by Böhler, Glaßer and Meister [BGM06]) is the class of languages \( L \subseteq \{0, 1\}^* \) for which there exists a \( \text{BPP} \) machine \( M \) and a polynomial \( p \), such that for all inputs \( x \):

(i) \( x \in L \Rightarrow \Pr[M(x) \text{ accepts}] \geq 2^{-p(|x|)} \).

(ii) \( x \notin L \Rightarrow \Pr[M(x) \text{ accepts}] \leq 2^{-p(|x|)-1} \).

Then we define the query complexity analogue of \( \text{SBP} \) in the standard way.
Definition 2.5. Let \( f : D \to \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function. We say a randomized decision tree \( T \) \text{SBP}-computes \( f \) if

\[
\Pr[T(x) = 1] \geq 2 \cdot \Pr[T(y) = 1] \quad \text{and} \quad \Pr[T(x) = 1] > 0
\]

for all \( x \in f^{-1}(1) \) and \( y \in f^{-1}(0) \).

We define \( \text{SBP}^d(f) \) as the minimum of \( C(T) \) over all \( T \) \text{SBP}-computing \( f \).

And we simply let \( \text{coSBP}^d(f) := \text{SBP}^d(\neg f) \).

It may seem strange at first that there is no log\(_2\)(1/\(\alpha\)) term in our definition of \( \text{SBP}^d(f) \). Actually, one can show that having the log\(_2\)(1/\(\alpha\)) term or not would not change the partial function class \( \text{SBP}^d \): the following lemma shows that whenever we have a randomized decision tree \( T \) \text{SBP}-computing a function \( f \), \( T \) can be made to \text{SBP}-compute \( f \) with a reasonable probability gap.

Lemma 2.6 (Proposition 33 in \[GLM^*15\]). Let \( f : D \to \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function. Suppose \( d = \text{SBP}^d(f) \). Then there is a randomized decision tree \( T \) \text{SBP}-computing \( f \) and a real number \( \alpha \), such that

\[
\Pr[T(x) = 1] \geq 2\alpha \quad \text{and} \quad \Pr[T(y) = 1] \leq \alpha \quad \text{and} \quad \alpha \geq 2^{-(d+1)} \left( \frac{n}{d} \right)^{-1}
\]

for all \( x \in f^{-1}(1) \), \( y \in f^{-1}(0) \).

### 2.3.3 PostBPP and PostBPP\text{dt}

In this subsection we review the definition of \text{PostBPP}, and define its analogue in query complexity.

Roughly speaking, \text{PostBPP} consists of the computational problems can be solved in probabilistically polynomial time, given the ability to \text{postselect} on an event (which may happen with a very small probability). Formally:

**Definition 2.7.** \text{PostBPP} (defined by Han, Hemaspaandra, and Thierauf) is the class of languages \( L \subseteq \{0, 1\}^* \) for which there exists a BPP machine \( M \), which can either “succeed” or “fail” and conditioned on succeeding either “accept” or “reject,” such that for all inputs \( x \):

(i) \( \Pr[M(x) \text{ succeeds}] > 0 \).

(ii) \( x \in L \implies \Pr[M(x) \text{ accepts } | M(x) \text{ succeeds}] \geq \frac{2}{3} \).

(iii) \( x \notin L \implies \Pr[M(x) \text{ accepts } | M(x) \text{ succeeds}] \leq \frac{1}{3} \).

\text{PostBPP}^d(f) \text{ can be defined similarly.}

**Definition 2.8.** Now we allow a randomized decision tree to output a failure mark \(*\) besides 0 and 1.

Let \( f : D \to \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function. We say a randomized decision tree \( T \) \text{PostBPP}-computes \( f \) if

\[
\Pr[T(x) = 1] \geq 2 \cdot \Pr[T(x) = 0] \quad \text{and} \quad \Pr[T(y) = 0] \geq 2 \cdot \Pr[T(y) = 1]
\]

for all \( x \in f^{-1}(1) \) and \( y \in f^{-1}(0) \).

Fix a \( T \) \text{PostBPP}-computing \( f \), let \( \alpha \) be the maximum real number such that

\[
\Pr[T(x) \neq *] \geq \alpha
\]

for all \( x \in D \).

Then we define \( \text{PostBPP}^d(T; f) = C(T) + \log_2(1/\alpha) \) for \( T \) \text{PostBPP}-computing \( f \), and \( \text{PostBPP}^d(f) \) as the minimum of \( \text{PostBPP}^d(T; f) \) over all \( T \) \text{PostBPP}-computing \( f \).

\(^6\)In the original paper it is called \text{BPP}_\text{path}. 

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2.3.4 SBQP and SBQP$^{dt}$

In this subsection we review the definition of SBQP, and define its analogue in query complexity. Roughly speaking, SBQP is just the quantum analogue of BQP.

**Definition 2.9.** SBQP (defined by Kuperberg [Kup09]) is the class of languages $L \subseteq \{0, 1\}^*$ for which there exists a polynomial-time quantum algorithm $M$ and a polynomial $p$, such that for all inputs $x$:

(i) $x \in L \implies \Pr[M(x) \text{ accepts}] \geq 2^{-p(|x|)}$.

(ii) $x \notin L \implies \Pr[M(x) \text{ accepts}] \leq 2^{-p(|x|)-1}$.

Then we define its query complexity analogue.

**Definition 2.10.** Let $f : D \to \{0, 1\}$ with $D \subseteq \{0, 1\}^M$ be a partial function. We say a quantum query algorithm $Q$ SBQP-computes $f$ if

$$\Pr[Q(x) = 1] \geq 2 \cdot \Pr[Q(y) = 1] \text{ and } \Pr[Q(x) = 1] > 0$$

for all $x \in f^{-1}(1)$ and $y \in f^{-1}(0)$.

Fix a $Q$ SBQP-computing $f$, let $\alpha$ be the maximum real number such that

$$\Pr[Q(x) = 1] \geq 2\alpha \text{ and } \Pr[Q(y) = 1] \leq \alpha$$

for all $x \in f^{-1}(1)$ and $y \in f^{-1}(0)$.

Then we define $SBQP^{dt}(Q; f) = C(Q) + \log_2(1/\alpha)$ for $Q$ SBQP-computing $f$ and $SBQP^{dt}(f)$ as the minimum of $SBQP^{dt}(Q; f)$ over all $Q$ SBQP-computing $f$.

And we simply let $coSBQP^{dt}(f) := SBQP^{dt}(-f)$.

2.3.5 A0PP and A0PP$^{dt}$

In this subsection we review the definition of A0PP, and define its analogue in query complexity.

There are several equivalent definitions for A0PP, we choose the most convenient one here.

**Definition 2.11.** A0PP (defined by Vyalyi [Vya03]) is the class of languages $L \subseteq \{0, 1\}^*$ for which there exists a BPP machine $M$ and a polynomial $p$, such that for all inputs $x$:

(i) $x \in L \implies \Pr[M(x) \text{ accepts}] \geq \frac{1}{2} + 2^{-p(|x|)}$.

(ii) $x \notin L \implies \Pr[M(x) \text{ accepts}] \in \left[\frac{1}{2}, \frac{1}{2} + 2^{-p(|x|)-1}\right]$.

**Definition 2.12.** Let $f : D \to \{0, 1\}$ with $D \subseteq \{0, 1\}^M$ be a partial function. We say a randomized decision tree $T$ A0PP-computes $f$ if

- $\Pr[T(x) = 1] \geq 1/2$ for all $x \in D$.
- $\Pr[T(x) = 1] - 1/2 \geq 2 \cdot (\Pr[T(y) = 1] - 1/2)$ for all $x \in f^{-1}(1)$ and $y \in f^{-1}(0)$.

Fix a $T$ A0PP-computing $f$, let $\alpha$ be the maximum real number such that $\Pr[T(x) = 1] - 1/2 \geq 2\alpha$ and $\Pr[T(y) = 1] - 1/2 \leq \alpha$ for all $x \in f^{-1}(1)$ and $y \in f^{-1}(0)$.

Then we define $A0PP^{dt}(T; f) = C(T) + \log_2(1/\alpha)$ for $T$ A0PP-computing $f$ and $A0PP^{dt}(f)$ as the minimum of $A0PP^{dt}(T; f)$ over all $T$ A0PP$^{dt}$-computing $f$.

And we simply let $coA0PP^{dt}(f) := A0PP^{dt}(-f)$.

In [Kup09], Kuperberg showed that SBQP is in fact equal to A0PP.

**Theorem 2.13 (Kup09).** SBQP = A0PP.

It is easy to see its proof relativizes, hence we have the following corollary in query complexity.

**Corollary 2.14.** $SBQP^{dt} = A0PP^{dt}$.
2.4 PostBQP and PostBQP\textsuperscript{dt}

PostBQP is defined similarly as PostBPP, just replaced the BPP machine by a polynomial time quantum algorithm. And PostBQP\textsuperscript{dt}(f) is defined in the same way as PostBPP(f) except for changing the randomized decision tree \(T\) to a quantum query algorithm \(Q\).

In [Aar05], Aaronson showed that PostBQP is indeed PP in disguise.

**Theorem 2.15** ([Aar05]). PostBQP = PP.

This result relativizes, therefore we have the following corollary.

**Corollary 2.16.** PostBQP\textsuperscript{dt} = PP\textsuperscript{dt}.

2.5 Approximation degrees

In this subsection we introduce a new notion of one-sided approximate degree, which is closely connected to A0\textsuperscript{PP\textsuperscript{dt}}(f).

**Definition 2.17.** We say a polynomial \(p\) one-sided approximates a partial function \(f : D \rightarrow \{0, 1\}\) with approximation constant \(0 \leq \epsilon < 1\), if \(p(x) \in [0, \epsilon]\) when \(f(x) = 0\), and \(p(x) \geq 1\) when \(f(x) = 1\). Write \(p(x) := \sum_{i=1}^{m} a_i \cdot M_i(x)\) as a sum of monomials, we define weight\((p) := \sum_{i=1}^{m} |a_i|\).

**Remark 2.18.** Our definition of one-sided approximation is slightly different from the standard one [She14, BT13, She15], but it greatly simplifies several discussions in our paper.

**Definition 2.19.** The one-sided approximate degree of a partial function \(f\) with approximation constant \(0 \leq \epsilon < 1\), denoted by \(\hat{\deg}_+(f)\), is the minimum degree of a polynomial one-sided approximating \(f\).

The one-sided low-weight approximate degree of a partial function \(f\) with approximation constant \(0 \leq \epsilon < 1\), denoted by \(\hat{\deg}_+^\epsilon(f)\), is defined by

\[
\hat{\deg}_+^\epsilon(f) := \min_p \max \{\deg(p), \log_2(\text{weight}(p))\},
\]

where \(p\) goes over all polynomials which one-sided approximates \(f\).

We simply let \(\hat{\deg}_-(f) := \hat{\deg}_+(-f)\) and \(\hat{\deg}_-^\epsilon(f) := \hat{\deg}_+^\epsilon(-f)\).

We also define \(\hat{\deg}_+^\epsilon(f)\) and \(\hat{\deg}_-^\epsilon(f)\) as \(\hat{\deg}_+^{1/2}(f)\) and \(\hat{\deg}_-^{1/2}(f)\). \(\hat{\deg}_-\) and \(\hat{\deg}_-\) are defined similarly.

Clearly \(\hat{\deg}_+^\epsilon(f) \geq \hat{\deg}_+^\epsilon(f)\). And the choice of constant 1/2 is arbitrary, as we can reduce the approximation error by the following lemma.

**Lemma 2.20.** For any \(0 < \epsilon_1 < \epsilon_2 < 1\), \(\hat{\deg}_+^\epsilon_1(f) \leq \left(\frac{\ln \epsilon_1}{\ln \epsilon_2}\right)^{\log_2} \cdot \hat{\deg}_+^\epsilon_2(f)\).

**Proof.** We can just take the \(\left[\frac{\ln \epsilon_1}{\ln \epsilon_2}\right]^\log_2\) power of the polynomial corresponding to \(\hat{\deg}_+^\epsilon_2(f)\).

We show that \(\hat{\deg}_+^\epsilon(f)\) is in fact equivalent to \(A0\text{PP}^{\text{dt}}(f)\) up to a constant factor.

**Theorem 2.21.** Let \(f : D \rightarrow \{0, 1\}\) with \(D \subseteq \{0, 1\}^M\) be a partial function, then

\[
\hat{\deg}_+(f) \leq 2 \cdot A0\text{PP}^{\text{dt}}(f),
\]

and

\[
A0\text{PP}^{\text{dt}}(f) \leq 2 \cdot \hat{\deg}_+(f) + 2.
\]
Proof. For the first claim, suppose $\text{A0PP}^{dt}(f) = d$, then there exists a $T$-query randomized decision tree $T$ and a constant $\alpha > 0$, such that

- $\Pr[T(x) = 1] \geq 1/2$ for all $x \in D$.
- $\Pr[T(x) = 1] - 1/2 \geq 2\alpha$ and $\Pr[T(y) = 1] - 1/2 \leq \alpha$ for all $x \in f^{-1}(1)$ and $y \in f^{-1}(0)$.
- $T + \log_2(1/\alpha) = d$.

Let $h$ be the conical junta representing the acceptance probability of $T$. We have weight($h$) $\leq 2^T$.

By expanding every $T$-term into $2^T$ monomials, we can further represent $h$ by a polynomial $p_h$ with weight($p_h$) $\leq 2^{2T}$.

Now, we define the polynomial $p(x) := \frac{1}{2\alpha} \cdot (p_h(x) - 1/2)$. We claim that $p$ one-sided approximates $f$. Indeed, when $f(x) = 0$, we have $p_h(x) \in [1/2, 1/2 + \alpha]$, hence $p(x) \in [0, 1/2]$; and when $f(x) = 1$, we have $p_h(x) \geq 1/2 + 2\alpha$, hence $p(x) \geq 1$.

Moreover, weight($p$) $\leq (\text{weight}(p_h) + 1/2) \cdot \frac{1}{2\alpha} \leq 2^{2T}/\alpha$, as $\alpha < 1/4$.

Hence $\hat{\deg_+}(f) \leq \max\{\deg(p), \log_2(\text{weight}(p))\} \leq \max\{T, 2T + \log_2(1/\alpha)\} \leq 2d = 2 \cdot \text{A0PP}^{dt}(f)$.

For the second claim, suppose $\hat{\deg_+}(f) = d$, then there exists a $T$-degree polynomial $p$ one-sided approximating $f$, where $T \leq d$ and $\log_2(\text{weight}(p)) \leq d$.

Let $p(x) = \sum_i a_i \cdot M_i(x)$ and $S = \text{weight}(p) = \sum_i |a_i|$, where $a_i \in \mathbb{R}$ and $M_i$ is a unit monomial.

Consider the following algorithm:

- Pick a random unit monomial $M$ by selecting $M_i$ with probability $|a_i|/S$.
- If $M$ evaluates to 1 on the given input, accept if $a_i > 0$ and reject otherwise.
- If $M$ evaluates to 0 on the given input, accept with probability $1/2$.

Clearly, as $p$ is of degree $T$, the above algorithm can be implemented by a $T$-query randomized decision tree $T$. Now we analyze the acceptance probability of $T$ on an input $x$. We can see $\Pr[T(x) = 1] = \sum_i \frac{1 + M_i(x) \cdot a_i/|a_i|}{2} \cdot \frac{|a_i|}{S} = \frac{1}{2} + p(x) \cdot \frac{1}{2S}$.

Which means, when $f(x) = 0$, we have $p(x) \in [0, 1/2]$, hence $\frac{1}{2} \leq \Pr[T(x) = 1] \leq \frac{1}{2} + \frac{1}{4S}$; and when $f(x) = 1$, we have $p(x) \geq 1$, therefore $\frac{1}{2} \leq \Pr[T(x) = 1] \leq \frac{1}{2} + \frac{1}{2S}$. So we can take $\alpha = \frac{1}{4S}$.

So $\text{A0PP}^{dt}(f) \leq \text{A0PP}^{dt}(f; T) \leq T + \log_2(4S) = T + 2 + \log_2(\text{weight}(p)) \leq 2 \cdot \hat{\deg_+}(f) + 2$.

And the following corollary follows from the definitions.

**Corollary 2.22.** Let $f : D \to \{0, 1\}$ with $D \subseteq \{0, 1\}^M$ be a partial function, then

\[
\hat{\deg_-}(f) \leq 2 \cdot \co\text{A0PP}^{dt}(f),
\]

and

\[
\co\text{A0PP}^{dt}(f) \leq 2 \cdot \hat{\deg_-}(f) + 2.
\]
2.6 Problem

Problem 2.23 (Permutation Testing Problem or PTP). Given black-box access to a function \( f : [n] \to [n] \), and promised that either

(i) \( f \) is a permutation (i.e., is one-to-one), or

(ii) \( f \) differs from every permutation on at least \( n/8 \) coordinates.

The problem is to accept if (i) holds and reject if (ii) holds.

Assume \( n \) is a power of 2, we use \( \text{PTP}_n \) to denote the Permutation Testing Problem on functions from \([n] \to [n] \). \( \text{PTP}_n \) can be viewed as a partial function \( D \to \{0,1\} \) with \( D \subseteq \{0,1\}^{n \log_2 n} \).

We will make use of the following results in [Aar12].

Theorem 2.24 (Essentially Theorem 8 in [Aar12]). \( \deg^+ (\text{PTP}_n) \geq \Omega(n^{1/3}) \).

Proposition 2.25 (Proposition 2 in [Aar12]). \( \text{PTP}_n \) has an \( O(\log n) \) time \( \text{SZK} \) protocol.

3 Classical case

In this section we prove Theorem 1.4.

3.1 \( \text{SBP}^\text{dt} \) by conical juntas

We first show when considering the \( \text{SBP}^\text{dt} \), we can work with a conical junta instead of a randomized decision tree.

Proposition 3.1. The definition of \( \text{SBP}^\text{dt}(f) \) is unchanged if we replace the \( T \)-query randomized decision tree by a \( T \)-conical junta.

Proof. We are going to show the existence of a \( T \)-query randomized decision tree \( T \) \( \text{SBP} \)-computing \( f \) is equivalent to the existence of a \( T \)-conical junta \( h \) \( \text{SBP} \)-computing \( f \).

Suppose there exists a \( T \)-query randomized decision tree \( T \) \( \text{SBP} \)-computing \( f \), then the acceptance probability of \( T \) can be presented as a \( T \)-conical junta by Lemma 2.2.

For the other direction, suppose there exists a \( T \)-conical junta \( h \) \( \text{SBP} \)-computing \( f \), let \( h(x) := \sum_i \alpha_i \cdot C_i(x) \). Consider the following algorithm: let \( P = \sum_i \alpha_i \), then we pick a random \( T \)-term by selecting \( C_i \) with probability \( \alpha_i/P \) and accept if \( C_i \) evaluates to 1 on the given input. It is not hard to see the above algorithm can be represented by a \( T \)-query randomized decision tree, and it \( \text{SBP} \)-computes \( f \). \( \square \)

3.2 A dual characterization for \( \text{SBP}^\text{dt} \)

We first establish an equivalent dual condition of a function having large \( \text{SBP}^\text{dt} \) complexity.

Lemma 3.2. Let \( f : D \to \{0,1\} \) with \( D \subseteq \{0,1\}^M \) be a partial function, \( T \) be a positive integer, \( \text{SBP}^\text{dt}(f) > T \) if and only if there exist two distributions \( D_0 \) and \( D_1 \) supported on \( f^{-1}(0) \) and \( f^{-1}(1) \) respectively, such that

\[
C(D_0) > \frac{1}{2} \cdot C(D_1)
\]

for any \( T \)-term \( C \),

where \( C(D_i) \) is defined as \( \mathbb{E}_{x \sim D_i}[C(x)] \) for \( i \in \{0,1\} \).
**Proof.** Let $H_T$ be the set of all $T$-conical juntas on $\{0, 1\}^M$, and $f^i := f^{-1}(i)$ for $i \in \{0, 1\}$, by Proposition 3.1, $\text{SBP}^d(f) > T$ is equivalent to 

$$
\min_{h \in H_T} \max_{(x, y) \in f^0 \times f^1} \left( h(x) - \frac{1}{2} \cdot h(y) \right) > 0.
$$

Then by the minimax theorem, the above is again equivalent to 

$$
\max_{D_{xy}} \min_{h \in H_T} \mathbb{E}_{(x, y) \sim D_{xy}} \left( h(x) - \frac{1}{2} \cdot h(y) \right) > 0.
$$

where $D_{xy}$ is a distribution on $f^0 \times f^1$.

Observe that we can further take $D_{xy}$ to be a product distribution and we can assume $h$ is just a $T$-term. Putting everything together, $\text{SBP}^d(f) > T$ is equivalent to 

$$
\max_{D_0 \text{ on } f^0} \max_{D_1 \text{ on } f^1} \min_{C \text{ is a } T\text{-term}} \mathbb{E}_{x \sim D_0, y \sim D_1} \left( C(x) - \frac{1}{2} \cdot C(y) \right) > 0,
$$

where $D_i$ is a distribution on $f^i$ for $i \in \{0, 1\}$. This completes the proof. 

\[\square\]

**Remark 3.3.** Another way to prove the above lemma is to use strong duality in linear programming directly. We feel our proof by minimax theorem is conceptually cleaner.

The following corollary follows from the definition.

**Corollary 3.4.** Let $f : D \to \{0, 1\}$ with $D \subseteq \{0, 1\}^M$ be a partial function, $T$ be a positive integer, $\text{coSBP}^d(f) > T$ if and only if there exist two distributions $D_0$ and $D_1$ supported on $f^{-1}(0)$ and $f^{-1}(1)$ respectively, such that 

$$
C(D_1) > \frac{1}{2} \cdot C(D_0) \text{ for any } T\text{-term } C.
$$

### 3.3 Proof for Theorem 1.4

We will prove the following tighter lower bound for $\text{PostBPP}^d(\text{Ada}_{f, d})$, from which Theorem 1.4 follows easily.

**Theorem 3.5.** Let $f : D \to \{0, 1\}$ with $D \subseteq \{0, 1\}^M$ be a partial function and $d$ be a non-negative integer. Suppose $\text{SBP}^d(f) > T$ or $\text{coSBP}^d(f) > T$, then we have 

$$
\text{PostBPP}^d(\text{Ada}_{f, d}) > \min(T/5, (2^d - 1)/5).
$$

We first show Theorem 3.5 implies Theorem 1.4

**Proof of Theorem 1.4.** Suppose $f \notin \text{SBP}^d$, the case that $f \notin \text{coSBP}^d$ is similar.

Then there exists a sequence of function $\{f_i\}_{i=1}^\infty \subseteq f$ such that $\text{SBP}^d(f_i) > \log(\text{len}(f_i))^4$. Then we consider the partial function sequence $\{\text{Ada}_{f_i, \lceil \log(\text{len}(f_i)) \rceil}\}_{i=1}^\infty \subseteq \text{Ada}_f$.

By Theorem 3.5, we have 

$$
\text{PostBPP}^d(\text{Ada}_{f_i, \lceil \log(\text{len}(f_i)) \rceil}) > \min(\log(\text{len}(f_i))^4/5, (\text{len}(f_i) - 1)/5).
$$

Note that $\text{len}(\text{Ada}_{f_i, \lceil \log(\text{len}(f_i)) \rceil}) \leq 2 \cdot \text{len}(f_i)^2$, we can see $\text{Ada}_f \notin \text{PostBPP}^d$ due to the above sequence. 

Now we are going to prove Theorem 3.5. We say a pair of conical juntas $a(x)$ and $r(x)$ computes a function $f$ if it satisfies the following two conditions:
• When \( f(x) = 1 \), \( a(x) \geq 5 \cdot r(x) \) and \( a(x) \geq 1 \).
• When \( f(x) = 0 \), \( r(x) \geq 5 \cdot a(x) \) and \( r(x) \geq 1 \).

In order to lower bound the \( \text{PostBPP}^{dt} \) complexity of some functions, we introduce some consequences of a function having low \( \text{PostBPP}^{dt} \) complexity.

**Lemma 3.6.** Let \( f : D \to \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function, \( T \) be a positive integer. Suppose \( \text{PostBPP}^{dt}(f) \leq T \), then there exist two \( 5T \)-conical juntas \( a(x) \) and \( r(x) \) such that

- The pair of \( a(x) \) and \( r(x) \) computes \( f \).
- \( \max_{x \in \{0, 1\}^M} a(x) \leq 2^{5T+1} \) and \( \max_{x \in \{0, 1\}^M} r(x) \leq 2^{5T+1} \).

**Proof.** Amplifying the probability gap by taking the majority of 5 independent runs, we get a randomized decision tree \( T \) such that

- \( \text{PostBPP}^{dt}(T; f) \leq 5T \).
- \( \Pr[T(x) = 1] \geq 5 \cdot \Pr[T(x) = 0] \) and \( \Pr[T(y) = 0] \geq 5 \cdot \Pr[T(y) = 1] \) for all \( x \in f^{-1}(1) \) and \( y \in f^{-1}(0) \).

Then we simply define \( a(x) \) (\( r(x) \)) as \( 2^{5T+1} \) multiplies the acceptance (reject) probability of \( T \). Clearly \( a(x) \) and \( r(x) \) can be represented by \( 5T \)-conical juntas, as \( T \) makes at most \( 5T \) queries.

Now we show \( a(x) \) and \( r(x) \) satisfy our two conditions. The second condition follows directly from their definitions. And for the first condition, when \( f(x) = 1 \), we have \( a(x) \geq 5 \cdot r(x) \) by their definitions, and since \( \Pr[T(x) \in \{0, 1\}] = 2^{-5T} \) for all \( x \in D \), \( a(x) \geq 2^{5T+1} \cdot \frac{5}{6} \cdot 2^{-5T} \geq 1 \). The case when \( f(x) = 0 \) can be verified in the same way. This completes the proof. \( \square \)

Our proof relies on the following two key lemmas.

**Lemma 3.7.** Let \( f : D \to \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function with \( \text{SBP}^{dt}(f) > T \). Then for each integer \( d \), there exist two distributions \( D_1^d \) and \( D_0^d \) supported on \( \text{Ada}_{f, d}^{-1}(1) \) and \( \text{Ada}_{f, d}^{-1}(0) \) respectively, such that \( r(D_0^d) > 2^d \cdot a(D_1^d) \) for any \( T \)-conical juntas \( a(x) \) and \( r(x) \) computing \( \text{Ada}_{f, d} \).

**Lemma 3.8.** Let \( f : D \to \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function with \( \text{coSBP}^{dt}(f) > T \). Then for each integer \( d \), there exist two distributions \( D_1^d \) and \( D_0^d \) supported on \( \text{Ada}_{f, d}^{-1}(1) \) and \( \text{Ada}_{f, d}^{-1}(0) \) respectively, such that \( a(D_1^d) > 2^d \cdot a(D_1^d) \) for any \( T \)-conical juntas \( a(x) \) and \( r(x) \) computing \( \text{Ada}_{f, d} \).

Before proving Lemma 3.7 and Lemma 3.8, we show they imply Theorem 3.5.

**Proof of Theorem 3.5.** We first prove the case when \( \text{SBP}^{dt}(f) > T \). Suppose that \( \text{PostBPP}^{dt}(\text{Ada}_{f, d}) \leq \min(T/5, (2^d - 1)/5) \), by Lemma 3.6, there is a pair of \( T \)-conical juntas \( a(x) \) and \( r(x) \) computing \( \text{Ada}_{f, d} \) such that \( \max_{x} r(x) \leq 2^d \).

By Lemma 3.7 we have \( r(D_0^d) > 2^d \cdot a(D_1^d) \geq 2^d \). Hence there must exist an \( x \) such that \( r(x) > 2^d \), contradiction.

Then case for \( \text{coSBP}^{dt}(f) > T \) follows from exactly the same argument and Lemma 3.8. \( \square \)

### 3.4 Proof for Lemma 3.7

Finally we prove Lemma 3.7, the proof for Lemma 3.8 is completely symmetric.
Proof of Lemma 3.7: We are going to construct those distributions by an induction on \(d\). Let \(f_d := A_d a_{f,d}\).

As \(\text{SBP}^d(f) > T\), by Lemma 3.2, there exist two distributions \(D_0\) and \(D_1\) supported on \(f^{-1}(0)\) and \(f^{-1}(1)\) respectively, such that

\[
h(D_0) > \frac{1}{2} \cdot h(D_1)\]

for any \(T\)-conical junta \(h\).

Let \(a(x)\) and \(r(x)\) be a pair of \(T\)-conical juntas computing \(f_d\).

The base case \(d = 0\) is very simple. \(f_0\) is just the \(f\) itself. We let \(D_0^0 = D_1\) and \(D_0^0 = D_0\). Then we have \(a(D_0) > \frac{1}{2} \cdot a(D_1)\). Also, \(r(D_0) > 5 \cdot a(D_0)\) as \(D_0\) is supported on \(f^{-1}(0)\). Putting these facts together, we have \(r(D_0) > 2 \cdot a(D_1)\), which means \(r(D_0^0)/a(D_0^0) > 2 = 2^0\) and completes the case for \(d = 0\).

For \(d > 0\), suppose that we have already constructed distributions \(D_0^{d-1}\) and \(D_1^{d-1}\) on inputs of \(f_{d-1}\), we are going to construct \(D_0^d\) and \(D_1^d\) based on them.

We first decompose the input to \(f_d\) as a triple \((w, x, y) \in D \times D_{d-1} \times D_{d-1}\) as in its definition, in which \(D\) denotes the domain of \(f\), and \(D_{d-1}\) denotes the domain of \(f_{d-1}\).

For a pair of \(T\)-conical juntas \(a(w, x, y)\) and \(r(w, x, y)\) computing \(f_d\), consider the following two \(T\)-conical juntas on \(x\):

\[
a_L(x) := a(D_0, x, D_0^{d-1}) = \mathbb{E}_{w \sim D_0, y \sim D_0^{d-1}}[a(w, x, y)],
\]

and

\[
r_L(x) := r(D_0, x, D_0^{d-1}) = \mathbb{E}_{w \sim D_0, y \sim D_0^{d-1}}[r(w, x, y)].
\]

Note that for any fixed \(W \in \text{support}(D_0)\) and any \(Y \in \text{support}(D_0^{d-1})\), by the definition of \(f_d\), the \(T\)-conical junta pair \(a(W, x, Y)\) and \(r(W, x, Y)\) must compute \(f_{d-1}\). It is not hard to verify by linearity, that their expectations \(a_L(x)\) and \(r_L(x)\) also compute \(f_{d-1}\).

Therefore, plugging in \(D_0^{d-1}\) and \(D_1^{d-1}\), we have \(r_L(D_0^{d-1}) > 2^{d-1} \cdot a_L(D_0^{d-1})\), which means

\[
r(D_0, D_0^{d-1}, D_0^{d-1}) > 2^{d-1} \cdot a(D_0, D_1^{d-1}, D_0^{d-1}).
\]

Then, for each fixed \(X, Y\), the polynomial \(a_M(w) := a(w, X, Y)\) is a \(T\)-conical junta, so we have \(a_M(D_0) > \frac{1}{2} \cdot a_M(D_1)\). Hence by linearity,

\[
a(D_0, D_1^{d-1}, D_0^{d-1}) > \frac{1}{2} \cdot a(D_1, D_1^{d-1}, D_0^{d-1}).
\]

Now, notice that \(D_1\) is supported on \(f^{-1}(1)\), and \(D_0^{d-1}\) is supported on \(f_{d-1}^0\), so \((D_1, D_1^{d-1}, D_0^{d-1})\) is supported on \(f_{d-1}^1\), therefore

\[
a(D_1, D_1^{d-1}, D_0^{d-1}) \geq 5 \cdot r(D_1, D_1^{d-1}, D_0^{d-1}).
\]

Finally, consider the polynomials on \(y\) defined by

\[
a_R(y) := a(D_1, D_1^{d-1}, y) \text{ and } r_R(y) := r(D_1, D_1^{d-1}, y).
\]

By the same augment as above, they are also a pair of \(T\)-conical juntas computing \(f_{d-1}\), so plugging in \(D_0^{d-1}\) and \(D_1^{d-1}\) again, we have \(r_R(D_0^{d-1}) > 2^{d-1} \cdot a_R(D_1^{d-1})\), which means

\[
r(D_1, D_1^{d-1}, D_0^{d-1}) > 2^{d-1} \cdot a(D_1, D_1^{d-1}, D_1^{d-1}).
\]

Putting everything together, we have

\[
r(D_0, D_0^{d-1}, D_0^{d-1}) > (2^{d-1} \cdot \frac{1}{2} \cdot 5 \cdot 2^{d-1}) \cdot a(D_1, D_1^{d-1}, D_1^{d-1}) > 2^{2d} \cdot a(D_1, D_1^{d-1}, D_1^{d-1}).
\]

So we can just take \(D_0^d = (D_1, D_1^{d-1}, D_1^{d-1})\) and \(D_0^0 = (D_0, D_0^{d-1}, D_0^{d-1})\). It is not hard to see that these distributions are supported on \(f_{d-1}^1(1)\) and \(f_{d-1}^1(0)\) respectively. This completes the proof. \(\square\)
4 Quantum case

In this section we prove Theorem 4.1.6

Let \( f : D \to \{0,1\} \) with \( D \subseteq \{0,1\}^M \) be a partial function, we say a polynomial \( p \) on \( M \) variables computes \( f \), if \( p(x) \geq 1 \) whenever \( f(x) = 1 \), and \( p(x) \leq 1 \) whenever \( f(x) = 0 \).

4.1 Existence of the hard distributions

In this section we show that if \( \deg_+(f) \) is large, there must exist some input distributions with nice properties.

**Lemma 4.1.** Let \( f : D \to \{0,1\} \) with \( D \subseteq \{0,1\}^M \) be a partial function and \( T \) be a non-negative integer, if \( \deg_+(f) > T \), then there exist two distributions \( D_0 \) and \( D_1 \) supported on \( f^{-1}(0) \) and \( f^{-1}(1) \) respectively, such that

\[
-p(D_0) > 2 \cdot p(D_1)
\]

for all degree-\( T \) polynomial \( p \) computing \( f \) and satisfying \( \text{weight}(p) \leq 2^T \).

In order to prove the above lemma, we need the following simple lemma.

**Lemma 4.2.** For any degree-\( T \) polynomial \( p \) computing \( f \) and satisfying \( \text{weight}(p) \leq 2^T \), there must exist \( x \in f^{-1}(0) \) and \( y \in f^{-1}(1) \) such that \( -p(x) > 2 \cdot p(y) \).

**Proof.** Suppose not, let \( p \) be a degree-\( T \) polynomial computing \( f \), and satisfies \( \text{weight}(p) \leq 2^T \) and \( \max_{x \in f^{-1}(0)} -p(x) \leq 2 \cdot \min_{y \in f^{-1}(1)} p(y) \).

Let \( C = \max_{x \in f^{-1}(0)} -p(x) \), consider the following polynomial \( q(x) := \frac{2}{3} \cdot \frac{p(x)}{C} + 1 \). We can see that when \( f(x) = 0 \), we have \( p(x) \in [-C, -1] \), hence \( q(x) \in [0, \frac{2}{3}] \), and when \( f(x) = 1 \), we have \( p(x) \geq \frac{1}{2} \cdot C \), therefore \( q(x) \geq 1 \). Which means \( q \) one-sided approximates \( f \) with error constant \( 2/3 \).

Also, we have \( \text{weight}(q) \leq \text{weight}(p) \cdot \frac{2}{3C} + \frac{2}{3} \leq \text{weight}(p) \) as \( C \geq 1 \). So \( \max \{\deg(q), \log_2(\text{weight}(q))\} \leq T \), contradiction to the fact that \( \deg_+(f) > T \).

Then we prove Lemma 4.1.

**Proof of Lemma 4.1.** By Lemma 4.2, we have

\[
\min_p \max_{(x,y) \in f^0 \times f^1} -p(x) - 2 \cdot p(y) > 0,
\]

where \( p \) is a degree-\( T \) polynomial \( p \) computing \( f \) and satisfying \( \text{weight}(p) \leq 2^T \), \( f^0 := f^{-1}(0) \) and \( f^1 := f^{-1}(1) \). By the minimax theorem, and noting that all the valid polynomials form a convex set, there exists a distribution \( D_{xy} \) on \( f^0 \times f^1 \) such that for any degree-\( T \) polynomial \( p \) computing \( f \) and satisfying \( \text{weight}(p) \leq 2^T \), we have

\[
\mathbb{E}_{(x,y) \sim D_{xy}} [-p(x) - 2 \cdot p(y)] > 0.
\]

Then we simply let \( D_0 \) (\( D_1 \)) be the marginal distribution of \( D_{xy} \) on \( f^0 \) (\( f^1 \)), which completes the proof.

And the following corollary follows by the definition of \( \deg_+ \).
Corollary 4.3. Let \( f : D \to \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function and \( T \) be a non-negative integer, if \( \deg_-(f) > T \), then there exist two distributions \( D_0 \) and \( D_1 \) supported on \( f^{-1}(0) \) and \( f^{-1}(1) \) respectively, such that

\[
p(D_1) > -2 \cdot p(D_0)
\]

for all degree-\( T \) polynomial \( p \) computing \( f \) and satisfying \( \text{weight}(p) \leq 2^T \).

4.2 Proof for Theorem 1.5

In fact we provide the following tighter lower bound for \( \text{PP}^d_t(\text{Ada}_{f,d}) \), from which Theorem 1.5 follows trivially.

Theorem 4.4. Let \( f : D \to \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function and \( T \) be a non-negative integer. Suppose \( \deg_+(f) > T \) or \( \deg_-(f) > T \), then we have

\[
\text{PP}^d_t(\text{Ada}_{f,d}) > \min(T/4, 2^{d-1}).
\]

We first show Theorem 3.5 implies Theorem 1.4.

Proof of Theorem 1.5. Suppose \( f \notin \text{SBQP}^d_t \), the case that \( f \notin \text{coSBQP}^d_t \) is similar.

By Corollary 2.4 and Theorem 2.21, there exists a sequence of function \( \{f_i\}_{i=1}^\infty \subseteq f \) such that \( \deg_+(f_i) > \log(\text{len}(f_i))^i \). Then we consider the partial function sequence \( \{\text{Ada}_{f_i,\lceil \log(\text{len}(f_i))\rceil}\}_{i=1}^\infty \subseteq \text{Ada}_f \).

By Theorem 4.3, we have

\[
\text{PP}^d_t(\text{Ada}_{f_i,\lceil \log(\text{len}(f_i))\rceil}) > \min(\log(\text{len}(f_i))^i/4, \text{len}(f_i)/2).
\]

Note that \( \text{len}(\text{Ada}_{f_i,\lceil \log(\text{len}(f_i))\rceil}) \leq \text{len}(f_i)^2 \), we can see \( \text{Ada}_f \notin \text{PP}^d_t \) due to the above sequence. \(\square\)

Now, we are going to prove Theorem 4.4. We begin by introducing some consequences of a function having low \( \text{PP}^d_t \) complexity.

Lemma 4.5. Let \( f : D \to \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function, \( T \) be a positive integer. Suppose \( \text{PP}^d_t(f) \leq T \), then there exists a degree \( T \)-polynomial \( p \) such that

- \( p \) computes \( f \).
- \( \text{weight}(p) \leq 2^{2T} \).

Proof. By our assumption, there exists a \( t \)-query randomized decision tree \( T \) and a real number \( \alpha > 0 \) such that

- when \( f(x) = 1 \), \( \Pr[T(x) = 1] \geq \frac{1}{2} + \alpha \).
- when \( f(x) = 0 \), \( \Pr[T(x) = 1] \leq \frac{1}{2} - \alpha \).
- \( t + \log_2(1/\alpha) \leq T \).

Let \( h \) be the conical junta representing the accepting probability of \( T \). We have \( \text{weight}(h) \leq 2^t \).

By expanding every \( t \)-term into \( 2^t \) monomials, we can further represent \( h \) by a polynomial \( p_h \) with \( \text{weight}(p_h) \leq 2^t \). Now we define \( p(x) := (p_h(x) - \frac{1}{2})/\alpha \). Clearly \( p \) computes \( f \).

Moreover, \( \text{weight}(p) \leq (2^{2t} + \frac{1}{2}) \cdot (1/\alpha) \leq 2^{2T} \), which completes the proof. \(\square\)
Our proof proceeds by a similar fashion as in the previous section, it again relies on the following two key lemmas.

**Lemma 4.6.** Let \( f : D \to \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function with \( \deg_+(f) > T \). Then for each integer \( d \), there exist two distributions \( D_0^d \) and \( D_1^d \) supported on \( Ada_{f,d}(1) \) and \( Ada_{f,d}(0) \) respectively, such that \( -p(D_0) > 2^d \cdot p(D_1) \) for any degree-\( T \) polynomial \( p \) computing \( f \) and satisfying \( \text{weight}(p) \leq 2^T \).

**Lemma 4.7.** Let \( f : D \to \{0, 1\} \) with \( D \subseteq \{0, 1\}^M \) be a partial function with \( \deg_+(f) > T \). Then for each integer \( d \), there exist two distributions \( D_0^d \) and \( D_1^d \) supported on \( Ada_{f,d}(1) \) and \( Ada_{f,d}(0) \) respectively, such that \( p(D_1) > 2^d \cdot p(D_0) \) for any degree-\( T \) polynomial \( p \) computing \( f \) and satisfying \( \text{weight}(p) \leq 2^T \).

We first show these two lemmas imply Theorem 4.4 in a straightforward way.

**Proof of Theorem 4.4.** We only prove for the case when \( \deg_+(f) > T \), the case for \( \deg_-(f) > T \) is symmetric.

Otherwise, suppose \( \text{PP}^d(\text{Ada}_{f,d}) \leq \min(T/4, 2^{d-1}) \). By Lemma 2.20, we have a degree-\( T/4 \) polynomial \( p \) computing \( \text{Ada}_{f,d} \) with \( \text{weight}(p) \leq \min(2^{T/2}, 2^{2d}) \). From Lemma 2.20, \( \deg_+(f) = \deg_+(f) \leq 2 \cdot \deg_+(f) \), hence \( \deg_+(f) > T/2 \). Then by Lemma 4.6, there exist two distributions \( D_0^d \) and \( D_1^d \) supported on \( Ada_{f,d}(1) \) and \( Ada_{f,d}(0) \) respectively, such that \( -p(D_0) > 2^d \cdot p(D_1) \) as \( p \) is of degree at most \( T/4 \) and satisfies \( \text{weight}(p) \leq 2^{T/2} \).

But this means that \( -p(D_0) > 2^d \), which implies there exists an \( x \) such that \( p(x) < -2^d \), therefore \( \text{weight}(p) > 2^{2d} \), contradiction. \( \square \)

### 4.3 Proof for Lemma 4.6

Finally we prove Lemma 4.6, the proof for Lemma 4.7 is completely symmetric.

**Proof of Lemma 4.6.** We say a polynomial \( p \) is valid, if it is of degree at most \( T \), and satisfies \( \text{weight}(p) \leq 2^T \). Let \( f_d := Ada_{f,d} \). We are going to construct those distributions by an induction.

By Lemma 4.1 there exist two distributions \( D_0 \) and \( D_1 \) supported on \( f^{-1}(0) \) and \( f^{-1}(1) \) respectively, such that

\[
-p(D_0) > 2 \cdot p(D_1)
\]

for all valid polynomial \( p \) computing \( f \).

For the base case \( d = 0 \), since \( f_0 \) is just \( f \), we simply let \( D_0^0 = D_0 \) and \( D_1^0 = D_1 \). Then for all valid polynomial \( p \) computing \( f_0 \), we have \( -p(D_0^0) > 2 \cdot p(D_1^0) = 2^0 \cdot p(D_1^0) \).

When \( d > 0 \), suppose that we have already constructed the required distributions \( D_0^{d-1} \) and \( D_1^{d-1} \) for \( f_{d-1} \). Decompose the input to \( f_d \) as \((w, x, y) \in D \times D_{d-1} \times D_{d-1} \) as in the definition, we claim that \( D_0^d = (D_0^0, D_0^{d-1}, D_0^{d-1}) \) and \( D_1^d = (D_1^0, D_1^{d-1}, D_1^{d-1}) \) satisfy our conditions.

Let \( p(w, x, y) \) be a valid polynomial computing \( f_d \). We define \( p(D_w, D_x, D_y) := \mathbb{E}_{w \sim D_w, x \sim D_x, y \sim D_y} [p(w, x, y)] \) for simplicity. By the definition, we can see that for any fixed \( W \in \text{support}(D_0) \) and \( Y \in \text{support}(D_0^{d-1}) \), the polynomial in \( x \) defined by \( p_L(x) := p(W, x, Y) \) is a valid polynomial computing \( f_{d-1} \), hence \( -p_L(D_0^{d-1}) > 2^{d-1} \cdot p_L(D_1^{d-1}) \). By linearity, we have

\[
-p(D_0^d, D_0^{d-1}, D_0^{d-1}) > 2^{d-1} \cdot p(D_0, D_1^{d-1}, D_1^{d-1}).
\]

Now, for any fixed \( X \in \text{support}(D_1^{d-1}) \) and \( Y \in \text{support}(D_0^{d-1}) \), by the definition, we can see that the polynomial in \( w \) defined by \( p_{M}(w) := p(w, X, Y) \) is a valid polynomial computing \( f \), hence \( -p_M(D_0) > 2 \cdot p_M(D_1) \). Again by linearity, we have

\[
p(D_0^d, D_1^{d-1}, D_0^{d-1}) > -2 \cdot p(D_1, D_1^{d-1}, D_0^{d-1}) > -p(D_1, D_1^{d-1}, D_0^{d-1}).
\]
Finally, for any fixed $W \in \text{support}(D_1)$ and $X \in \text{support}(D_{d-1}^d)$, the polynomial in $y$ defined by $p_R(y) := p(W, X, y)$ is a polynomial computing $f_{d-1}$, hence $-p_R(D_{0}^{d-1}) > 2^{2^{d-1}} \cdot p_R(D_{1}^{d-1})$ from the hypothesis. By linearity, we have

$$-p(D_1, D_{1}^{d-1}, D_{0}^{d-1}) > 2^{2^{d-1}} \cdot p(D_1, D_{1}^{d-1}, D_{1}^{d-1}).$$

Putting everything together, we have

$$-p(D_0^d) = -p(D_0, D_{0}^{d-1}, D_{0}^{d-1}) > 2^{2^{d}} \cdot p(D_1, D_{1}^{d-1}, D_{1}^{d-1}) = 2^{2^{d}} \cdot p(D_{1}^{d}).$$

This completes the proof.

5 Formal proofs for the oracle separations

We begin with a famous lower bound on $\deg^+(\text{OR}_n)$ by Nisan and Szegedy.

**Theorem 5.1** ([NS94]). $\deg^+(\text{OR}_n) \geq \Omega(\sqrt{n})$.

Then we consider the problem $\text{AdaOR}_n := \text{Ada}_{\text{OR}_n, \log_2 n}$.

By Theorem [1.3], we have

$$\text{PP}^{dt}(\text{AdaOR}_n) \geq \Omega(\sqrt{n}).$$

On the other hand, there is a simple polylog($n$)-time $\text{P}^{\text{NP}}$ algorithm for $\text{AdaOR}$. By a standard diagonalization argument, we have the following corollary.

**Corollary 5.2.** There exists an oracle $O$ such that $\text{P}^{\text{NP}}^O \nsubseteq \text{PP}^O$.

Similarly, for the problem $\text{AdaPTP}_n := \text{Ada}_{\text{OR}_n, \log_2 n}$.

by Theorem [2.24] and Theorem [1.5], we have

$$\text{PP}^{dt}(\text{AdaOR}_n) \geq \Omega(n^{1/3}).$$

By Proposition [2.26], we can see $\text{AdaPTP}$ admits a polylog($n$)-time $\text{P}^{\text{SZK}}$ algorithm, hence again by a standard diagonalization argument, we have the following corollary.

**Corollary 5.3.** There exists an oracle $O$ such that $\text{P}^{\text{SZK}}^O \nsubseteq \text{PP}^O$.

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