NUMERICAL METHODS FOR PDE MODELS RELATED TO PRICING AND EXPECTED LIFETIME OF AN EXTRACTION PROJECT UNDER UNCERTAINTY

María Suárez-Taboada
Department of Mathematics, University of A Coruña and CITIC, Campus Elviña s/n
15071 – A Coruña, Spain

Carlos Vázquez*
Department of Mathematics, University of A Coruña CITIC and ITMATI, Campus Elviña s/n
15071 – A Coruña, Spain

(Communicated by Tomas Caraballo)

Abstract. Numerical techniques for solving some mathematical models related to a mining extraction project under uncertainty are proposed. The mine valuation is formulated as a complementarity problem associated to a degenerate second order partial differential equation (PDE), which incorporates the option to abandon the project. The probability of completion and the expected lifetime of the project are the respective solutions of problems governed by similar degenerated PDE operators. In all models, the underlying stochastic factors are the commodity price and the remaining resource. After justifying the required boundary conditions on the computational bounded domain, the proposed numerical techniques mainly consist of a Crank-Nicolson characteristics method for the time discretization to cope with the convection dominating setting and Lagrange finite elements for the discretization in the commodity and resource variables, with the additional use of an augmented Lagrangian active set method for the complementarity problem. Some numerical examples are discussed to illustrate the performance of the methods and models.

1. Introduction. At some point in their activities, mining companies need to take the decision to initiate or not an extraction project from a mineral reserve, thus facing several uncertainties upfront. The sources of these uncertainties can be related to political or legal aspects, as well as to labor market or supply and demand changes, that may occur during the future development of the project. Usually, it is assumed that all these uncertainties can be captured by the fluctuations of the commodity price and/or the remaining mineral in the reserve from which the commodity is obtained. Also the uncertain variations of the ore grade can be considered.

2010 Mathematics Subject Classification. Primary: 91G80, 65M25; Secondary: 91G60, 65M22.
Key words and phrases. Investment under uncertainty, PDEs, complementarity problems, characteristics methods, Crank-Nicolson, ALAS algorithm.

This article has been funded by Spanish MINECO (Projects MTM2013-47800-C2-1-P and MTM2016-76497-R) and Xunta de Galicia (Grant GRC2014/044), including FEDER funds.

* Corresponding author: Carlos Vázquez.

3503
Moreover, in the present approach we assume that the company has always the option to close down the mine when the value or cost-benefit of the extraction project is below certain threshold, which is usually related to low prices in the commodity, thus making the extraction not profitable anymore. This option to abandon also motivates the interest in computing the probability of project completion and the expected lifetime of the project.

The option to abandon an investment project under uncertainty can be found in the classical book of Dixit and Pindyck [10], among other works in the literature. In the mining valuation setting, it was first treated in [5], where a contingent claim valuation approach is used. However, the determination of the probability of completion and the expected lifetime of the project has been first addressed in [13]. Note that the probability of completion can be understood as a measure of the risk involved. The consideration of probability of completion requires the use of an additional stochastic factor, that could be the remaining reserve size or the ore grade (or both), for example. In this setting of several underlying stochastic factors, the contingent claims approximation leads to PDE models that require the use of appropriate numerical methods, as the analytical solution is not available. For example, the work by Evatt et al. [12] considers the ore grade and the commodity price as stochastic factors, where the results illustrate the minor effect of ore grade uncertainty. Later in [13] the authors just considered the commodity price and the resource size as underlying uncertain factors. In this alternative model, the authors pose PDE formulations of the models satisfied by the extraction project value, the probability of completion and the expected lifetime. For the numerical solution, they propose the use of a first order semi-Lagrangian scheme for time discretization, combined with an implicit finite differences scheme in the spatial-like variables (i.e., the commodity price and the remaining resource size). For the inequality constraint (obstacle problem) associated to the PDE problem of the mine value, the authors propose a classical projected successive over relaxation (PSOR) scheme.

In this paper we consider the PDE formulations from [13] and mainly propose several improvements in the numerical solution of the problems governing the mine valuation, the probability of the project completion and the expected lifetime, as well as some insights and improvements in the boundary conditions. Thus, after the usual localization procedure to truncate the unbounded domain to a large enough bounded one, we analyze the required boundary conditions for the different problems. In this respect, we first identify the boundaries requiring these conditions by following the methodology in [19], and then use some ideas first introduced in [9] to establish a Dirichlet boundary condition on some new boundaries of the truncated domain. These ideas have been later used in [7], for example. Moreover, we propose a second order semi-Lagrangian-Crank Nicolson time discretization scheme which was introduced and numerically analyzed in [2]. This scheme has been applied in finance for pricing Asian options [4] and stock loans [21], for example. This time discretization scheme is combined with piecewise quadratic Lagrange finite elements for spatial-like variables, so that we are in the so called Lagrange-Galerkin methods setting. Here, these methods are used to discretize in time and space the PDEs associated to the mine valuation, probability of completion and expected lifetime of the project. Concerning the specific solution of the mine valuation problem, in order to deal with the non linearity associated to this obstacle problem, the augmented Lagrangian active set (ALAS) method proposed in [17] is used, which was first applied in finance to price Asian options with early exercise opportunity (Amerasian
options) in [4]. The ALAS method provides the approximation of the mine value at any time before the horizon of the investment, as well as the multiplier associated to the unilateral constraint and the optimal abandonment boundary (free boundary) which separates the region where the company should abandon from the region where it results better to continue with the extraction project. In the proposed global algorithm, we first compute the value of the mine and the quoted regions and optimal abandonment boundary by solving the obstacle problem. Once these values are obtained, the probability of completion and the expected lifetime of the project are obtained by solving the corresponding initial-boundary value problems associated to the PDEs. Thus, while the mine value problem exhibits analogies with pricing of American options or early exercise derivatives, the probability of completion and expected lifetime problems are close to the pricing of European style options, when these latter problems are posed on the non abandonment region.

The article is organized as follows. In Section 2, the mathematical models for the mine project value, the probability of completion and the expected lifetime are stated. Section 3 is devoted to the localization in a bounded domain and boundary conditions. In Section 4, the numerical methods for solving the models are presented. In Section 5, some numerical results for an academic example and for a real mine problem are shown to illustrate the performance of the proposed methods and models. Finally, some conclusions and ideas of future research are discussed.

2. Mathematical models. The statement of mine valuation models under uncertainty has been addressed in the literature by means of two main approaches: contingent claims and probabilistic based methods. The contingent claims approach is based on a dynamic hedging methodology to build up riskless portfolios following the seminal ideas in [6]. These ideas have been applied in the mine valuation under uncertainty setting in [5] and [12]. The second approach uses hedging arguments to specify the stochastic dynamics of the risk-adjusted commodity price process and applies a modified Feynman-Kac equation. The second approach seems more suitable to obtain the models for the probability of completion and the expected lifetime. For this main reason, this is the approach in [13] and the one we follow here.

For this purpose, we first consider a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})\), as the framework where we define the dynamics of the stochastic processes involved in these mathematical models. The underlying stochastic factors to be considered are the commodity price and the remaining resource of the ore to be extracted from the mine, the values of which at time \(t\) are represented by \(S_t\) and \(Q_t\), respectively. We assume that the mine value \(V_t = V(t, S_t, Q_t)\), the probability of completion \(P_t = P(t, S_t, Q_t)\) and the expected lifetime \(D_t = D(t, S_t, Q_t)\) processes depend on \((t, S_t, Q_t)\) through the corresponding functions \(V\), \(Q\) and \(D\), respectively. Therefore, we can denote by \(u_t = U(t, S_t, Q_t)\) any process that depends on \((t, S_t, Q_t)\) through a generic function \(U\), as the mine value, probability of completion and expected lifetime are in this situation. Note that the ore grade \(G\) is not considered here as a relevant underlying factor as it has been proved in [12] that its effects are minor.

Before applying the general Feynman-Kac theorem that relates the expectation in the appropriate probability measure to the solution of a PDE, we will introduce the dynamics of the underlying stochastic factors. More precisely, we assume that the risk neutral price of the commodity evolves according to a geometric Brownian motion dynamics, thus satisfying the stochastic differential equation (SDE)

\[
    dS_t = (r - \delta)S_t \, dt + \sigma S_t \, dW_t, \tag{1}
\]
where $W$ represents a standard real Brownian motion on the previously introduced filtered probability space, while the constant parameters $r$, $\delta$ and $\sigma$ are the risk free interest rate, the convenience yield of the commodity and the volatility of the commodity price, respectively. For simplicity, hereafter we consider the notation $\mu = r - \delta$.

Note that other Itô diffusion processes (to account with mean reversion properties) or even jump-diffusion processes (to incorporate the possible presence of jumps) could be considered for the dynamics of the commodity price.

We assume that the dynamics of the ore remaining resource $Q_t$ satisfies the SDE

$$dQ_t = -q dt,$$  \hspace{1cm} (2)

where $q$ represents the rate of extraction of ore-bearing material from the mine, being greater than zero and bounded above by $q_m$ due to physical operation constraints. Although $q$ could be a function of $(t, S_t, Q_t)$, in the present work it will be considered as constant. Note that the mining extraction will finish either at expiry date or when the remaining reserve is exhausted (i.e., when $Q_t = 0$).

Now, using modified versions of the Feynman-Kac theorem (see [18] or [20], for example) for a generic process $u_t = U(t, S_t, Q_t)$ which depends on $(t, S_t, Q_t)$, such that $u_T = f(T, S_T, Q_T)$ is given, we obtain that the function $U : \Omega_T \to \mathbb{R}$ satisfies the following PDE in the unbounded domain $\Omega_T = (0, T) \times \mathbb{R}_+^2$:

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} - q \frac{\partial U}{\partial Q} + \mu S \frac{\partial U}{\partial S} - cu + g = 0,$$  \hspace{1cm} (3)

jointly with the final condition

$$U(T, S, Q) = f(T, S, Q),$$  \hspace{1cm} (4)

where the parameter $c$ represents the constant discount rate, $g$ corresponds to a nonhomogeneous source term and $f$ is the final condition.

The characterization of $U$ in terms of the expectations is given by

$$U(t, S, Q) = \mathbb{E}_P^{t, S, Q_T} \left[ \exp(-c(T-t))f(T, S_T, Q_T) + \int_t^T \exp(-c(T-s))g(s, S_s, Q_s) ds \right],$$  \hspace{1cm} (5)

where $\mathbb{E}_P$ denotes the expectation under the probability measure $P$.

Note that both formulations, the one in terms of an expectation (5) and the one in terms of a PDE (3), correspond to any derivative of European style depending on $(t, S_t, Q_t)$ with a final payoff given by (4). In the following subsections, the specific mathematical models associated to the mine valuation, the probability of completion and the expected lifetime are presented.

2.1. Mine valuation model. In the valuation of the extraction project, we have to incorporate the possibility of abandonment of the project. The decision of the company to abandon or to continue with the mine project will depend on the mine value. Although we will assume the abandonment when the mine value is equal to zero and abandonment costs are not taken into account, in the initial notation we consider abandonment when the value reaches a general abandonment value $\Lambda$, so that $\Lambda(t, S, Q)$ is the abandonment value at time $t$, with asset price $S$ and remaining ore reserve $Q$. The presence of this opportunity to stop the extraction project by the company implies that we are in an analogous setting as American options or early exercise derivatives pricing problems.
Therefore, the formulation in terms of expectations leads to an optimal stopping time problem. More precisely, if we denote by $T(t, T)$ the set of stopping times with values in $[t, T]$, then the mine value with option to abandon is given by

$$V(t, S, Q) = \sup_{\tau \in T(t, T)} \mathbb{E}_{(t, S, Q)}^p \left[ \exp(-c(\tau - t)) f(\tau, S_\tau, Q_\tau) + \int_t^\tau \exp(-c(\tau - s)) g(S_s, Q_s) \, ds \right].$$

(6)

In expression (6) we must define the parameter $c$ and the function $g$. The discount parameter $c$ is taken equal to the interest rate $r$. Moreover, for the instantaneous cashflow term $g$ generated during the extraction, we consider the expression

$$g(S, Q) = qGS - (\epsilon_M(Q) + \epsilon_P(Q)),$$

(7)

where $\epsilon_M = \epsilon_M(Q)$ and $\epsilon_P = \epsilon_P(Q)$ represent the extraction costs and the processing costs for refining the ore-bearing material, respectively. Moreover, $qGS$ represents the amount of saleable ore with $G$ the amount of ore obtained per unit of extracted material (ore grade). Note that in [12] the effect of ore-grade uncertainty has been studied. In fact, they conclude that the effect provided by the stock price is greater than the one added by the ore-grade and consequently, the dependence on $G$ is not considered in this work.

Next, by taking into account the equivalence between the optimal stopping time formulation and the complementarity problem associated to the appropriate PDE, the equivalent formulation to (6) consists of finding $V : \Omega_T \to \mathbb{R}$, such that:

$$\begin{cases}
\max \{ \mathcal{L}[V] - f, \Lambda - V \} = 0 & \text{in } \Omega_T, \\
V(T, S, Q) = \Lambda(T, S, Q) & (S, Q) \in \mathbb{R}_+^2,
\end{cases}$$

(8)

where $f(S, Q) = \epsilon_P + \epsilon_M - qGS$, $\Lambda$ is the obstacle function and the operator $\mathcal{L}$ is

$$\mathcal{L}[V] = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - q \frac{\partial V}{\partial Q} + \mu S \frac{\partial V}{\partial S} - rV.$$ 

(9)

Note that the mine will be worthless at abandonment and at the expiry date of the mine’s lease, denoted by $T$, which is translated into

$$\Lambda(t, S, Q) = 0, \quad \text{in } \Omega_T.$$ 

(10)

Boundary conditions are analyzed in next section, once the localized bounded domain is defined.

The first equation in formulation (8) can be equivalently written in the form:

$$\mathcal{L}[V] \geq f, \quad V \geq \Lambda, \quad (\mathcal{L}[V] - f) \cdot (V - \Lambda) = 0, \quad \text{in } \Omega_T.$$ 

(11)

Thus, for each time $t \in [0, T]$ we distinguish two regions in the $(S, Q)$-plane: one corresponding to $V > \Lambda$ (non abandonment region) and denoted by $\Omega_t^+$, where it is optimal to continue the extraction project, and another one with $V = \Lambda$ (abandonment region) denoted by $\Omega_t^0$, where the project is cancelled. The boundary separating both regions in the $(S, Q)$-plane is the optimal abandonment boundary (and represents the classical free boundary in obstacle problems). As a consequence of (11), assuming enough regularity in the solution and in the free boundary, at the abandonment boundary the following so called smooth pasting conditions hold:

$$V = \Lambda \quad \text{and} \quad \nabla V \cdot \vec{n} = \nabla \Lambda \cdot \vec{n}, \quad \text{on} \quad \partial \Omega_t^+ \cap \partial \Omega_t^0,$$
where \( \mathbf{n} \) denotes the unitary normal vector to the free boundary in the \((S,Q)\)-plane pointing towards \( \Omega^+_t \).

Note that the optimal abandonment boundary is an additional unknown of the mine value problem, which has a relevant financial interest.

### 2.2. Probability of project completion model

In order to state the model governing the probability of the project completion, we first consider that the probability depends on the same stochastic factors as the mine value, so that the general PDE (3) is the departure point and we need to precise the suitable choices of \( c \) and \( g \).

As argued in [13], in this case we are not dealing with a monetary quantity but with a probability, so that no discount is applied and \( c = 0 \). Moreover, the source term \( g \) vanishes as there are no costs or benefits involved in the probability. Moreover, we note that the probability of completion in the abandonment region is equal to zero. Therefore, we can pose a PDE problem in the non-abandonment region once this has been obtained as part of the solution of the mine valuation problem. More precisely, if we define the set

\[
\Omega^+_{[0,T]} = \bigcup_{t \in [0,T]} (\{t\} \times \Omega^+_t),
\]

then the function \( P : \Omega^+_{[0,T]} \to \mathbb{R} \) satisfies:

\[
\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - q \frac{\partial P}{\partial Q} + \mu S \frac{\partial P}{\partial S} = 0, \quad \text{in } \Omega^+_{[0,T]},
\]

jointly with the obvious final condition

\[
P(T,S,Q) = 1, \quad (S,Q) \in \Omega^+_T.
\]

Boundary conditions are analyzed in next section, once an appropriate bounded domain is considered. In practice and in the numerical solution, we will extend the formulation to the whole domain \( \Omega_T \) by considering at each time \( t \) that \( P = 0 \) in the abandonment region \( \Omega^0_T \).

### 2.3. Expected lifetime model

As the expected lifetime of the project \( D \) depends on time, commodity price and ore reserve, then the general equation (3) applies. Moreover, as we are dealing with lifetime, the source term and discount parameter vanish (i.e., \( c = g = 0 \)). Thus, the function \( D : \Omega^+_{[0,T]} \to \mathbb{R} \) satisfies the PDE

\[
\frac{\partial D}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 D}{\partial S^2} - q \frac{\partial D}{\partial Q} + \mu S \frac{\partial D}{\partial S} = 0, \quad \text{in } \Omega^+_{[0,T]},
\]

jointly with the obvious final condition

\[
D(T,S,Q) = T, \quad (S,Q) \in \Omega^+_T.
\]

As in the previous models, boundary conditions are analyzed in next section. Also in this model we will extend the formulation to the whole domain \( \Omega_T \) by considering at each time \( t \) that \( D = t \) in the abandonment region \( \Omega^0_T \).

### 3. Divergence form and localization in a bounded domain

All the previous models are initially posed in an unbounded domain, so that for their numerical solutions a localization procedure will be applied to define an appropriate bounded computational domain.
3.1. Mine valuation. As we will apply a finite elements method for the spatial discretization and this method is based on the variational formulation, we first rewrite the operator (9) in divergence form. In order to handle more initial value problems we introduce the time variable \( \tau = T - t \) and pose the equivalent problem:

\[
\mathcal{L}[V] \geq f, \quad V \geq \Lambda, \quad \mathcal{L}[V] - f \cdot (V - \Lambda) = 0, \quad \text{in } \Omega_T, \tag{16}
\]

where the operator in divergence form is given by

\[
\mathcal{L}[V] = \frac{\partial}{\partial \tau} V + \vec{v} \cdot \nabla V - \text{div}(A \nabla V) + rV, \tag{17}
\]

with the involved matrix and vector being

\[
A(S,Q) = \begin{pmatrix} \frac{1}{2} \sigma^2 S^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{v}(S,Q) = \begin{pmatrix} (\sigma^2 - \mu)S \\ q \end{pmatrix}. \tag{18}
\]

The complementarity problem (16) is completed with the generic initial condition

\[
V(0,S,Q) = \Lambda(0,S,Q) \tag{19}
\]

Note that from (10) the function \( \Lambda \) is equal to zero and therefore it does not depend on time. In certain abuse of notation we maintain the notation \( V \) for the unknown in spite of the change in the time variable.

We will proceed in the same way for the problems related to the probability of completion and expected lifetime.

As in most problems in finance, the numerical solution with finite elements requires the approximation of the original problem in an unbounded domain by another one in a bounded domain. This localization procedure has to be performed so that the truncation by the bounded domain and the associated boundary conditions do not affect the solution in the region of financial interest. For the problem of European vanilla options and Dirichlet boundary conditions, a rigorous analysis appears in [16]. Usually, the boundary conditions at the new boundaries of the bounded domain are obtained from financial and/or mathematical arguments.

For the localization procedure, let us consider \( S^\infty \) and \( Q^\infty \) large enough real numbers suitably chosen and let the bounded domain in variables \( S \) and \( Q \) be denoted by \( \Omega = (0,S^\infty) \times (0,Q^\infty) \), with Lipschitz boundary \( \Gamma = \partial \Omega \), such that it can be decomposed as \( \Gamma = \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_1^- \cup \Gamma_2^- \), where

\[
\Gamma_1^+ = \Gamma \cap \{ S = S^\infty \}, \quad \Gamma_1^- = \Gamma \cap \{ S = 0 \}, \quad \Gamma_2^+ = \Gamma \cap \{ Q = Q^\infty \}, \quad \Gamma_2^- = \Gamma \cap \{ Q = 0 \}. \tag{20}
\]

Next, by applying the theory of second order partial PDEs that can be found in [19] based on the theory of Fichera [15], and taking into account the expression of the matrix \( A \) and the vector \( \vec{v} \), only boundary conditions at \( \Gamma_1^+ \) and \( \Gamma_2^- \) are required. Thus, following the ideas in [19], for simplicity let us introduce the notation

\[
x_1 = S, \quad x_2 = Q. \tag{21}
\]

Then, the operator associated to the Cauchy problem can be written in the form:

\[
\mathcal{L}^* = \sum_{i,j=1}^{2} a_{ij}^* \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{2} b_j^* \frac{\partial}{\partial x_j} + l^* + \frac{\partial}{\partial t}, \tag{22}
\]

where the involved matrix and vector are defined as follows

\[
A^*(x_1, x_2) = (a_{ij}^*) = \begin{pmatrix} \frac{1}{2} \sigma^2 x_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{v}^*(x_1, x_2) = (b_j^*) = \begin{pmatrix} \mu x_1 \\ -q \end{pmatrix}. \tag{23}
\]
and \( l^*(x_1, x_2) = -r \). Thus, in terms of the inwards normal vector to the boundary of \( \Omega \), \( \vec{m} = (m_1, m_2) \), we introduce the following subsets of \( \Gamma \):

\[
\Sigma_1 = \{(x_1, x_2) \in \Gamma , \sum_{i,j=1}^{2} a_{ij}^* m_i m_j > 0 \},
\]

\[
\Sigma_2 = \left\{ (x_1, x_2) \in \Gamma - \Sigma_1 , \sum_{i=1}^{2} \left( b_i^* - \sum_{j=1}^{2} \frac{\partial a_{ij}^*}{\partial x_j} \right) m_i < 0 \right\}.
\]

As indicated in [19], for the initial boundary value problem associated to (22), the boundary conditions at \( \Sigma_1 \cup \Sigma_2 \) are required. So, by considering each part of \( \Gamma \) defined in (20), we get:

- On boundary \( \Gamma_1^+ \): \( x_1 = x_1^\infty, \ 0 \leq x_2 \leq x_2^\infty, \ \vec{m} = (-1, 0) \),

\[
\sum_{i,j=1}^{2} a_{ij}^* m_i m_j = a_{11}^* m_1^2 = \frac{1}{2} (q x_1)^2 > 0.
\]

- On boundary \( \Gamma_2^- \): \( 0 \leq x_1 \leq x_1^\infty, \ x_2 = x_2^\infty, \ \vec{m} = (0, -1) \),

\[
\sum_{i,j=1}^{2} a_{ij}^* m_i m_j = a_{11}^* m_1^2 = 0, \ \sum_{i=1}^{2} \left( b_i^* - \sum_{j=1}^{2} \frac{\partial a_{ij}^*}{\partial x_j} \right) m_i = q > 0 \quad (q \in (0, q_m)).
\]

- On boundary \( \Gamma_1^- \): \( x_1 = 0, \ 0 \leq x_2 \leq x_2^\infty, \ \vec{m} = (1, 0) \),

\[
\sum_{i,j=1}^{2} a_{ij}^* m_i m_j = a_{11}^* m_1^2 = 0, \ \sum_{i=1}^{2} \left( b_i^* - \sum_{j=1}^{2} \frac{\partial a_{ij}^*}{\partial x_j} \right) m_i = (\mu - \sigma^2) x_1 = 0.
\]

- On boundary \( \Gamma_2^- \): \( 0 \leq x_1 \leq x_1^\infty, \ x_2 = 0, \ \vec{m} = (0, 1) \),

\[
\sum_{i,j=1}^{2} a_{ij}^* m_i m_j = a_{11}^* m_1^2 = 0, \ \sum_{i=1}^{2} \left( b_i^* - \sum_{j=1}^{2} \frac{\partial a_{ij}^*}{\partial x_j} \right) m_i = -q < 0.
\]

Therefore, we obtain that \( \Sigma_1 = \Gamma_1^+ \) and \( \Sigma_2 = \Gamma_2^- \), so that we have to impose boundary conditions on \( \Sigma_1 \cup \Sigma_2 = \Gamma_1^+ \cup \Gamma_2^- \).

Boundary \( \Gamma_2^- \) corresponds to \( Q = 0 \), thus meaning that the ore reserve is exhausted. As in [13], we will assume that the value of the mine is equal to zero when this happens, i.e, we consider the boundary condition:

\[
V(\tau, S, 0) = 0. \quad (26)
\]

Although in [13] the authors pose that if the stock price tends to infinity then the mine value is taken equal to the stock price and this condition could be used on \( \Gamma_1^+ \), we prefer to follow the arguments first used in [9] and also followed in [7] to deduce Dirichlet boundary conditions for analogous problems.

If we divide by \( S^2 \) the PDE associated to (16) in the region \( V > \Lambda \), we obtain

\[
\frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} - \frac{1}{S^2} \frac{\partial V}{\partial \tau} - \frac{q}{S^2} \frac{\partial V}{\partial Q} + \frac{\mu}{S} \frac{\partial V}{\partial S} - \frac{r}{S^2} V = \frac{qG}{S} - \frac{\epsilon_f + \epsilon_M}{S} \quad (27)
\]

Then, when \( S \) tends to infinity, we get

\[
\lim_{S \to \infty} \frac{\partial^2 V}{\partial S^2} = 0. \quad (28)
\]
Therefore, using (28) to neglect the second order derivative term in (27) when $S$ is large, we consider that
\[- \frac{\partial V}{\partial \tau} - q \frac{\partial V}{\partial Q} + \mu S \frac{\partial V}{\partial S} - rV = qGS - (\epsilon_P + \epsilon_M), \quad S \to \infty \tag{29}\]
Therefore, assuming that for large enough values of $S$ both (28) and (29) hold, we try a solution to (29) in the form
\[V(\tau, S, Q) = H_1(\tau)S + H_2(\tau)Q + H_3(\tau),\]
thus obtaining the following system of ordinary differential equations:
\[
\begin{align*}
H_1'(\tau) &= (\mu - r)H_1(\tau) + qG \\
H_2'(\tau) &= -rH_2(\tau) \\
H_3'(\tau) &= -qH_2(\tau) - rH_3(\tau) - (\epsilon_M + \epsilon_P) 
\end{align*}
\tag{30}\]
By solving the previous system and using $V(0, S, Q) = 0$ to determine the involved constants, the following boundary condition is obtained:
\[V(\tau, S^\infty, Q) = V_S^\infty(\tau, Q) = qGS^\infty \left(e^{(\mu-r)\tau} - 1\right) + \frac{\epsilon_M + \epsilon_P}{r} \left(e^{-r\tau} - 1\right). \tag{31}\]
We also propose a mixed formulation for the complementarity problem by introducing the multiplier $M : [0, T] \times \Omega \to \mathbb{R}$, so that we can replace (16) by
\[
\frac{\partial V}{\partial \tau} - \text{div}(A\nabla V) + \bar{v} \cdot \nabla V + rV + M = f \quad \text{in } (0, T) \times \Omega, \tag{32}\]
jointly with the complementarity conditions
\[V \geq \Lambda, \quad M \leq 0, \quad (V - \Lambda) \cdot M = 0 \quad \text{in } (0, T) \times \Omega. \tag{33}\]
This kind of mixed formulations has been previously used in the pricing of early exercise Asian options [4] or stock loans [21], for example. As in these previous works, we will apply the mixed formulation (32)-(33) to the fully discretized problem.

### 3.2. Probability of completion of the project.
As in the mine valuation problem, we localize the initial unbounded domain in the same bounded domain $\Omega$, change the time variable and write the PDE (12) in divergence form by using the operator
\[\mathcal{L}[P] = \frac{\partial P}{\partial \tau} + \bar{v} \cdot \nabla P - \text{div}(A\nabla P) + rP, \tag{34}\]
with matrix $A$ and vector $\bar{v}$ given by (18), while we take $r = 0$.

Therefore, PDE (12) is replaced by the following one:
\[\mathcal{L}[P] = 0 \quad \text{in } \Omega^+_0[0, T]. \tag{35}\]
Also $P$ satisfies the initial condition
\[P(0, S, Q) = 1, \quad \text{in } \Omega^+_0. \tag{36}\]
Concerning the boundary conditions, as the differential operator has the same first and second order derivatives terms as in the mine valuation problem, we need to impose boundary conditions at the same parts of the boundary, i.e. on $\Gamma_1^+ \cup \Gamma_2^+$. On the boundary $\Gamma_2^+$ the ore reserve is exhausted and therefore the probability of completion is one, so that we impose the same condition as in [13]:
\[P(\tau, S, 0) = 1. \tag{37}\]
On the boundary $\Gamma^+_1$, we follow the same methodology as in the mine value problem. First, we divide the PDE by $S^2$ and taking the limit we get
\[
\lim_{S \to \infty} \frac{\partial^2 P}{\partial S^2} = 0.
\]
Next, for a large value of $S$ we try the solution:
\[
P(\tau, S, Q) = H_1(\tau) S + H_2(\tau) Q + H_3(\tau)
\]
to the corresponding PDE without the second order derivative term. Then, solving the resulting ODE system and using (36), the following boundary condition on $\Gamma^+_1$ is posed:
\[
P(\tau, S^\infty, Q) = 1. \tag{38}
\]
Note that this is the same boundary condition proposed in [13].

**Remark 1.** Although the PDE problem associated to the probability of completion is initially posed in the non abandonment region, as we indicate in the corresponding forthcoming section about the spatial discretization, we will solve this PDE problem in the fixed spatial domain $\Omega$ by extending this probability to $P = 0$ in the abandonment region. Analogous extensions will be carried out for the expected lifetime problem. This is the reason why we just analyze the boundary conditions related to the required boundaries of $\Omega$.

### 3.3. Expected lifetime of the project.

As the differential operator is the same as in the probability of completion problem, we follow the same steps and consider
\[
\overline{L}[D] = \frac{\partial D}{\partial \tau} + \bar{v} \cdot \nabla D - \text{div}(A \nabla D) + rD, \tag{39}
\]
with $A$ and $\bar{v}$ given by (18), and we take $r = 0$. Next, PDE (14) is replaced by
\[
\overline{L}[D] = 0 \quad \text{in} \quad \Omega^+_0. \tag{40}
\]
Moreover, $D$ satisfies the initial condition
\[
D(0, S, Q) = T, \quad \text{in} \quad \Omega^+_0. \tag{41}
\]

Concerning the boundary conditions, as the differential operator is the same as in the probability of completion problem, we need to impose boundary conditions at the same parts of the boundary, i.e. on $\Gamma^+_1 \cup \Gamma^-_2$. On the boundary $\Gamma^-_2$ the ore reserve is exhausted at time $t$, so that we impose the same condition as in [13]:
\[
D(\tau, S, 0) = T - \tau. \tag{42}
\]

At the boundary $\Gamma^+_1$, we initially follow the same methodology as in the previous problems. Thus, we divide the PDE by $S^2$ and take the limit when $S \to \infty$ to get
\[
\lim_{S \to \infty} \frac{\partial^2 D}{\partial S^2} = 0.
\]
Additionally, we impose continuity at the intersection $\Gamma^+_1 \cap \Gamma^-_2$, so that condition $D(S^\infty, 0) = 0$ must be satisfied. Thus, if we impose
\[
D(\tau, S^\infty, Q) = (T - \tau) + \frac{Q}{q}, \tag{43}
\]
both conditions are satisfied. Furthermore, the PDE without the second order derivative term is also satisfied. However, we note than the duration cannot exceed
the project horizon \( T \), which clearly is not guaranteed for all \( \tau \) with expression (43). Therefore, we propose the boundary condition at \( \Gamma_{-2} \):

\[
D(\tau, S^\infty, Q) = \min \left( (T - \tau) + \frac{Q}{q}, T \right).
\] (44)

This condition on \( \Gamma_{+1} \) differs from the condition \( D(\tau, S^\infty, Q) = T \), which is proposed in [13]. Note that condition (44) seems more realistic and avoids a discontinuity at the point \((S^\infty, 0)\). Actually, according to (44), for large values of \( S \) the project is always completed: at time \( T - \tau \) when the mine is exhausted \((Q = 0)\); at time \( T \) when the remaining ore reserve is large enough \((Q \geq q\tau)\) so that at the rate of extraction \( q \) it is not exhausted before \( T \); or at time \( T - \tau + Q/q \) when the remaining ore reserve is \( Q < q\tau \). In this latter case, the mine will be exhausted when reaching this time and the probability of completion is equal to one, as assumed on \( \Gamma_{+1} \).

4. Numerical methods. In this section we describe the numerical methods for solving the stated PDE problems for the mine value, the probability of completion and the expected lifetime of the project. Note that the three problems involve the same differential operator defined in (9), in which the terms corresponding to some second order derivatives are missing and therefore it can be classified as a degenerate second order operator. Thus, it can also be understood as an extreme case of convection dominated operator, where the terms containing the first order derivatives dominate the ones containing the second order ones in certain parts of the domain. In this respect, the use of classical discretization schemes can give rise to spurious oscillations in the numerical solution, which are not present in the exact solution. Among the possible alternatives to cope with convection dominated problems, the methods of characteristics (also known as semi-Lagrangian schemes) for the time discretization have been successfully used in the literature. These methods have been introduced in [1] and [11], for example. Also, a first order version of these methods has been proposed in [13] for the here treated problems related to mine extraction projects, where it has been combined with an implicit finite differences scheme for the spatial discretization.

In the present article, we propose a Crank-Nicolson characteristics time discretization scheme combined with Lagrange finite element methods introduced in [2], where the analysis to prove the convergence properties is addressed, as well as the use of appropriate numerical quadrature formulas. This combination has been applied in different financial problems in [4, 21], among others. For the inequality constraints associated to the early opportunity of abandonment in the mine valuation problem, we propose a mixed formulation and the adaption of an augmented Lagrangian active set technique introduced in [17] for double obstacle problems.

4.1. Discretization in time. As previously indicated, as the differential operator governing the PDE of the three problems is degenerated, we consider it as a extreme case of convection dominated differential operator. Therefore, we propose the Crank-Nicolson Lagrange-Galerkin method introduced in [2] for the time discretization. For this purpose, we introduce the material derivative

\[
\frac{D}{D\tau} = \frac{\partial}{\partial\tau} + \vec{v} \cdot \nabla,
\] (45)

which contains the time derivative plus the term of the first order spatial derivatives. Next, associated to the vector field \( \vec{v} \), we define the characteristics curve through
We apply this scheme to the respective problems for \( u \) around the point \( \tau \), which solves the final value problem:

\[
\partial_\tau X_e((S, Q), \tilde{\tau}; \tau) = \tilde{v}(X_e((S, Q), \tilde{\tau}; \tau)), \quad X_e((S, Q), \tilde{\tau}; \tau) = (S, Q) .
\]  

(46)

Note that due to the expression of \( \tilde{v} \), the final value problem (46) can be analytically solved, so that the two components of the characteristics curve are given by:

\[
X^1_e((S, Q), \tilde{\tau}; \tau) = Se^{-(\sigma^2-\mu)(\tilde{\tau}-\tau)}, \quad X^2_e((S, Q), \tilde{\tau}; \tau) = Q - q(\tilde{\tau} - \tau).
\]  

(47)

In case that the exact expressions were not available, we need to use appropriate numerical methods to solve (46) and obtain their approximations.

The idea of the characteristics scheme for time discretization is to approximate the material derivative (45) of the unknown of the problem by an upwinded approximation following the characteristics curves.

Once the characteristics curves have been obtained or approximated, let us consider the time step \( \Delta \tau = \frac{T}{N} \) for a given \( N > 0 \), and the time meshpoints \( \tau^n = n\Delta \tau \), \( n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N \). Then, at time \( \tau^{n+\frac{1}{2}} \), the material derivative approximation with the characteristics method is given by:

\[
\frac{Du}{D\tau} \approx \frac{u^{n+1} - u^n \circ X^n_e}{\Delta \tau} ,
\]

where \( X^n_e(S, Q) := X_e((S, Q), \tau^{n+\frac{1}{2}}; \tau) \), the components of which are given by

\[
X^{n,1}_e(S, Q) = Se^{-(\sigma^2-\mu)\Delta \tau}, \quad X^{n,2}_e(S, Q) = Q - q\Delta \tau.
\]

We apply this scheme to the respective problems for \( u = V, P \) and \( D \).

4.1.1. Time discretization of the mine valuation problem. In the Crank-Nicolson characteristics time discretization method, we consider a Crank-Nicolson scheme around the point \( (X_e((S, Q), \tau^{n+\frac{1}{2}}; \tau)) \) with \( \tau = \tau^{n+\frac{1}{2}} \) for \( n = 0, \ldots, N - 1 \), then the time discretized PDE operator (17) to obtain the approximation at time \( \tau^{n+1} \) can be written as follows:

\[
\mathcal{Z}[V] \left( X_e((S, Q), \tau^{n+1}; \tau^{n+\frac{1}{2}}), \tau^{n+\frac{1}{2}} \right) \approx \frac{V^{n+1}(S, Q) - V^n(X^n_e(S, Q))}{\Delta \tau}
\]

\[
-\frac{1}{2} \text{div}(AVV^{n+1})(S, Q) - \frac{1}{2} \text{div}(AVV^n)(X^n_e(S, Q)) + \frac{1}{2} (rV^{n+1}(S, Q))
\]

\[
+ \frac{1}{2} (f^{n+1}(S, Q) + f^n(X^n_e(S, Q))).
\]  

(48)

Note that the terms evaluated at the previous time step \( n \) are applied to the point moved backwards through the characteristics \( (X^n_e(S, Q)) \). For simplicity, for a function \( u \) let us introduce the notation \( \mathcal{Z}[u]^{n+\frac{1}{2}}(S, Q) = \mathcal{Z}[u] \left( X_e((S, Q), \tau^{n+1}; \tau^{n+\frac{1}{2}}), \tau^{n+\frac{1}{2}} \right) \).

4.1.2. Time discretization for the probability of completion and expected lifetime problems. As in the mine valuation problem, we consider a Crank-Nicolson scheme around the particular point \( (X_e((S, Q), \tau^{n+\frac{1}{2}}; \tau)) \) with \( \tau = \tau^{n+\frac{1}{2}} \) for \( n = 0, \ldots, N - 1 \), then the time discretized PDE operators (34) and (39) can be written as follows:

\[
(\mathcal{Z}[u]^{n+\frac{1}{2}}(S, Q) \approx \frac{u^{n+1}(S, Q) - u^n(X^n_e(S, Q))}{\Delta \tau}
\]

\[
-\frac{1}{2} \text{div}(AVu^{n+1})(S, Q) - \frac{1}{2} \text{div}(AVu^n)(X^n_e(S, Q)).
\]  

(50)

with \( u = P \) or \( D \) for the respective problems.
4.2. Spatial discretization.

4.2.1. Mine valuation problem. In order to pose the spatial discretization with finite elements, we first state the weak formulation for the time discretized problem. For this, we consider the functional spaces:

\[ H^1_{V,Γ}(Ω) = \{ ψ ∈ H^1(Ω)/ψ|_{Γ^+} = V_{S∞}, \; ψ|_{Γ^-} = 0 \}, \]

\[ H^1_{0,Γ}(Ω) = \{ ψ ∈ H^1(Ω)/ψ|_{Γ^+} + ψ|_{Γ^-} = 0 \}. \]

First, by considering that \( V^{n+1} ∈ H^1_{V,Γ}(Ω) \), multiplying the terms in (48) by \( ψ ∈ H^1_{0,Γ}(Ω) \) and integrating in \( Ω \), we have:

\[
\left( [L[V]]^{n+\frac{1}{2}}, ψ \right) ≈ \int_Ω V^{n+1} - V^n ∙ X^n_ψ dSdQ - \frac{1}{2} \int_Ω \text{div}(A∇V^{n+1})ψ dSdQ
- \frac{1}{2} \int_Ω (\text{div}(A∇V^n)) ∙ X^n_ψ dSdQ + \frac{1}{2} \int_Ω rV^{n+1}ψ dSdQ
+ \frac{1}{2} \int_Ω (rV^n) ∙ X^n_ψ dSdQ - \frac{1}{2} \int_Ω (f^{n+1} + f^n ∙ X^n_ψ) dSdQ. \tag{51}
\]

Next, by applying Lemma 3.1. in [3] to the third term on the right hand side in (51) and by considering the usual Green’s formula to the second term, we get:

\[
\left( [L[V]]^{n+\frac{1}{2}}, ψ \right) ≈ \int_Ω V^{n+1} - V^n ∙ X^n_ψ dSdQ + \frac{1}{2} \int_Ω A∇V^{n+1}∇ψ dSdQ
+ \frac{1}{2} \int_Ω (F^n_e)^{-1}(A∇V^n) ∙ X^n_ψ dSdQ + \frac{1}{2} \int_Ω (\text{div}(F^n_e)^{-t}(A∇V^n)) ∙ X^n_ψ dSdQ
- \frac{1}{2} \int_Γ \vec{n} ∙ A∇V^{n+1}ψ dA - \frac{1}{2} \int_Γ ((F^n_e)^{-t}\vec{n} ∙ (A∇V^n)) ∙ X^n_ψ dA \tag{52}
+ \frac{1}{2} \int_Ω rV^{n+1}ψ dSdQ + \frac{1}{2} \int_Ω (rV^n) ∙ X^n_ψ dSdQ - \frac{1}{2} \int_Ω (f^{n+1} + f^n ∙ X^n_ψ) dSdQ,
\]

where notation \( dA \) is used for the integration measure on the boundary \( Γ \). The tensor \((F^n_e)^{-t}(S, Q) = (∇X^n_e((S, Q), τ_{n+1}; τ_n))^{-t}\) takes the form

\[
(F^n_e)^{-t} = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix},
\]

with \( b = \exp((σ^2 - μ)Δτ) \), so that \((F^n_e)^{-t}\) is independent of \( S \) and \( Q \).

Next, let us compute the boundary integrals appearing in (53). Note that we have \( \vec{n} ∙ A∇V^{n+1} = 0 \) on \( Γ^-_1 ∪ Γ^+_2 \) and that \( ψ = 0 \) on \( Γ^+_1 \). Therefore, we obtain

\[
\int_Γ \vec{n} ∙ A∇V^{n+1}ψ dA = \int_{Γ^+_1} \vec{n} ∙ A∇V^{n+1}ψ dA = 0.
\]

Moreover, for the second boundary integral, we have

\[
\int_Γ ((F^n_e)^{-t}\vec{n} ∙ (A∇V^n)) ∙ X^n_ψ dA = \int_{Γ^+_1} ((F^n_e)^{-t}\vec{n} ∙ (A∇V^n)) ∙ X^n_ψ dA = 0. \tag{53}
\]
Therefore, expression (53) becomes
\[
\left( (\mathcal{L}[V])^{n+\frac{1}{2}}, \psi \right) \approx \int_{\Omega} V_h^{n+1} - V^n \circ X^n_{\psi} dSdQ + \frac{1}{2} \int_{\Omega} A\nabla V_h^{n+1} \psi dSdQ + \frac{1}{2} \int_{\Omega} (F^n_{-\psi})^{-1} (A \nabla V^n) \circ X^n_{\psi} dSdQ + \frac{1}{2} \int_{\Omega} r^{n+1} \psi dSdQ + \frac{1}{2} \int_{\Omega} (r^{n+1}_\psi) \circ X^n_{\psi} dSdQ - \frac{1}{2} \int_{\Omega} (f^{n+1} + f^n \circ X^n_{\psi}) \psi dSdQ,
\]
(54)

In order to obtain the fully discretized problem, we combine the previously described time discretization with a finite elements spatial discretization. Thus, we consider a family of quadrangular meshes \( \{ \tau_h \} \) of the domain \( \Omega \). Associated to each mesh \( \tau_h \), let \( (T_Q, \Sigma_T) \) be a family of piecewise quadratic Lagrangian finite elements, where \( Q_2 \) denotes the space of polynomials defined in \( T \in \tau_h \) with degree less or equal than two in each spatial variable and \( \Sigma_T \) the subset of nodes of the element \( T \). More precisely, let us define the finite elements space \( V_h \):
\[
V_h = \{ \varphi_h \in C^0(\Omega) : \varphi_{hT} \in Q_2, \forall T \in \tau_h \},
\]
(55)
where \( C^0(\Omega) \) denotes the space of continuous functions on \( \Omega \). Moreover, we consider
\[
\begin{align*}
\mathcal{V}_{h,0} &= \{ \varphi_h \in V_h : \varphi_h = 0 \text{ on } \Gamma^+_1 \cup \Gamma^-_2 \}, \\
\mathcal{V}_{h,V} &= \{ \varphi_h \in V_h : \varphi_h = V_{S^\infty} \text{ on } \Gamma^+_1 \text{ and } \varphi_h = 0 \text{ on } \Gamma^-_2 \},
\end{align*}
\]
(56) (57)

Therefore, if \( V^n_h \) denotes the finite element approximation of \( V \in H^1(\Omega) \), then the spatial discretization of (54) can be written in the form
\[
\left( (\mathcal{L}[V_h])^{n+\frac{1}{2}}, \psi_h \right) \approx \int_{\Omega} V_h^{n+1} - V^n \circ X^n_{\psi_h} dSdQ + \frac{1}{2} \int_{\Omega} A\nabla V_h^{n+1} \psi_h dSdQ + \frac{1}{2} \int_{\Omega} (F^n_{h,\psi})^{-1} (A \nabla V^n_h) \circ X^n_{\psi_h} dSdQ + \frac{1}{2} \int_{\Omega} r^{n+1}_h \psi_h dSdQ + \frac{1}{2} \int_{\Omega} (r^{n+1}_{h,\psi}) \circ X^n_{\psi_h} dSdQ - \frac{1}{2} \int_{\Omega} (f^{n+1} + f^n \circ X^n_{\psi_h}) \psi_h dSdQ, \forall \psi_h \in \mathcal{V}_{h,0}.
\]
(58)

Once the discretizations in time and in space have been applied, we are led to the following fully discretized complementarity problem at each time step \( n \):
\[
A_h V^n_h \geq b^n_h - 1, \quad V^n_h \geq \Lambda_h, \quad (A_h V^n_h - b^n_h - 1) \cdot (V^n_h - \Lambda_h) = 0,
\]
(59)
where \( A_h \) denotes the matrix that is obtained after the discretization with the finite element space \( V_h \), \( V^n_h \) denotes de vector containing the values of the solution at the nodes of the finite element mesh and \( \Lambda_h \) is the vector of the node values of the function \( \Lambda \). Thus, the corresponding mixed formulation of the complementarity problem (59) can be written in the form
\[
A_h V^n_h + M^n_h = b^n_{h-1},
\]
(60)
jointly with the complementarity conditions
\[
V^n_h \geq \Lambda_h, \quad M^n_h \leq 0, \quad M^n_h \cdot (V^n_h - \Lambda_h) = 0,
\]
(61)
where \( M^n_h \) denotes the vector containing the nodal values of the multiplier.

The augmented Lagrangian active set algorithm has been introduced in [17] and mainly builds sequences aiming to converge to the unknowns \( V^n_h \) and \( M^n_h \), i.e. to the mine values and the multiplier; as well as sequences of subsets of \( \Omega \) aimed to converge to the regions where it is optimal to maintain (inactive region) and to abandon the project (active region). Thus, each iteration mainly contains two steps. In the
first one, the domain is decomposed into active and inactive regions (depending on whether the constraints are active or not). In the second step, a reduced linear system associated to the inactive part is solved. Although the algorithm can be used under bilateral constraints (in case of upper and lower constraints), we use the algorithm under a unilateral constraint. Among other applications in finance, the method has been already successfully used when pricing early exercise Asian options \[4\] and stock loans \[21\]. Besides to the seminal work \[17\], we address the reader to these articles for further details on the algorithm.

4.2.2. Probability of completion and expected lifetime of the project. In both problems the equations at each time step are posed on the non abandonment domain, which has been previously obtained from the solution of the mine value problem. Therefore, in order to state the weak formulation for the semidiscretized problem associated to the probability of completion, we introduce the functional spaces:

\[
H^1_{P,0}\Omega^+_{n+1} = \{ \psi \in H^1(\Omega^+_{n+1})/\psi|_{\Gamma^+_n} = 1, \psi|_{\Gamma^-_n} = 0 \}
\]

\[
H^1_{0,0}\Omega^+_{n+1} = \{ \psi \in H^1(\Omega^+_{n+1})/\psi|_{\Gamma^+_n} = \psi|_{\Gamma^-_n} = 0 \}
\]

First, by considering that \( P^{n+1} \in H^1_{P,\partial\Omega^+_{n+1}} (\Omega^+_{n+1}) \), multiplying the terms in (48) by \( \psi \in H^1_{0,\partial\Omega^+_{n+1}} (\Omega^+_{n+1}) \) and integrating in \( \Omega^+_{n+1} \), we have:

\[
\int_{\Omega^+_{n+1}} P^{n+1} - P^n \circ X^n \psi dSdQ - \frac{1}{2} \int_{\Omega^+_{n+1}} \text{div}(A\nabla(P^{n+1})) \psi dSdQ
\]

\[
- \frac{1}{2} \int_{\Omega^+_{n+1}} \text{div}(A\nabla(P^n) \circ X^n) \psi dSdQ = 0 .
\] (62)

By applying in (62) Lemma 3.4 from \[3\] and the usual Green’s formula, we get

\[
\int_{\Omega^+_{n+1}} \frac{P^{n+1} - P^n \circ X^n}{\Delta \tau} \psi dSdQ + \frac{1}{2} \int_{\Omega^+_{n+1}} A\nabla P^{n+1} \psi dSdQ
\]

\[
+ \frac{1}{2} \int_{\Omega^+_{n+1}} (F^n)^{-1}A\nabla(P^n) \circ X^n \nabla \psi dSdQ
\]

\[
+ \frac{1}{2} \int_{\Omega^+_{n+1}} \text{div}((F^n)^{-1}A\nabla(P^n) \circ X^n) \psi dSdQ = 0
\]

\[
= \frac{1}{2} \int_{\partial\Omega^+_{n+1}} \vec{n} \cdot A\nabla P^{n+1} \psi \, dA + \frac{1}{2} \int_{\partial\Omega^+_{n+1}} (F^n)^{-1} \vec{n} \cdot A\nabla(P^n) \circ X^n \psi \, dA .
\] (63)

Next, concerning the boundary integrals appearing in the right hand side of formulation (63), first note that \( \vec{n} \cdot A\nabla u^{n+1} = 0 \) on \( \Gamma^-_1 \cup \Gamma^+_1 \), and that \( \psi = 0 \) on \( (\Gamma^+_1 \cup \Gamma^-_2 \cup \partial\Omega^0_{n+1}) \cap \partial\Omega^+_{n+1} \), so that the first boundary integral vanishes. Analogously, the second boundary integral also vanishes. Furthermore, we also have

\[
\int_{\Omega^+_{n+1}} \text{div}(F^n)^{-1}(A\nabla P^n) \circ X^n \psi dSdQ = 0 .
\]

Therefore, equation (63) becomes

\[
\int_{\Omega^+_{n+1}} \frac{P^{n+1} - P^n \circ X^n}{\Delta \tau} \psi dSdQ + \frac{1}{2} \int_{\Omega^+_{n+1}} A\nabla P^{n+1} \psi dSdQ
\]

\[
+ \frac{1}{2} \int_{\Omega^+_{n+1}} (F^n)^{-1} \vec{n} \cdot A\nabla(P^n) \circ X^n \psi dSdQ = 0 .
\] (64)
However, if we apply a finite elements spatial discretization to the previous formulation we need to compute at each time step a different spatial domain $\Omega_{n+1}$, with several involved drawbacks: we need different meshes for different times, thus requiring additional interpolation, among others. In order to overcome these difficulties, we propose to extend the definition of the probability of completion to the abandonment region at each time by $P = 0$, which holds in this region. In this way, we can deal with a fixed domain $\Omega$, which is the same one as in the mine value problem and therefore we can consider the same finite elements mesh in this domain. In order to impose the condition $P = 0$ on the abandonment region in the finite elements discretization, we just force the nodes of this region to take the value $P = 0$ in the same way we deal with Dirichlet boundary conditions to prescribe the given values of the unknown at the nodes of the boundaries with these conditions.

Thus, in terms of the finite elements space defined in (55), we introduce:

$$V_{h,P}^{n+1} = \{ \varphi_h \in V_h : \varphi_h = 1 \text{ on } \Gamma_1^+ \cup \Gamma_2^- \text{ and } \varphi_h = 0 \text{ on } \Omega_0^n \}, \quad (65)$$

so that the fully discretized problem consists of finding $P^{n+1} \in V_{h,P}^{n+1}$, such that

$$\int_\Omega \frac{P_{n+1}^h - P_n^h \circ X_n^\tau}{\Delta \tau} \psi_h \, ds \, dq + \frac{1}{2} \int_\Omega A \nabla P_{n+1}^h \nabla \psi_h \, ds \, dq + \frac{1}{2} \int_\Omega (F_{n+1}^\tau)^{-1} (A \nabla P_n^h) \circ X_n^\tau \nabla \psi_h \, ds \, dq = 0, \quad \forall \psi_h \in V_{h,0}. \quad (66)$$

Analogously, in the expected lifetime problem we introduce

$$V_{h,D}^{n+1} = \{ \varphi_h \in V_h : \varphi_h = T - \tau \text{ on } \Gamma_1^+ \cup \Gamma_2^- \text{ and } \varphi_h = T - \tau \text{ on } \Omega_0^n \}, \quad (67)$$

so that the fully discretized problem consists of finding $D^{n+1} \in V_{h,D}^{n+1}$, such that

$$\int_\Omega \frac{D_{n+1}^h - D_n^h \circ X_n^\tau}{\Delta \tau} \psi_h \, ds \, dq + \frac{1}{2} \int_\Omega A \nabla D_{n+1}^h \nabla \psi_h \, ds \, dq + \frac{1}{2} \int_\Omega (F_{n+1}^\tau)^{-1} (A \nabla D_n^h) \circ X_n^\tau \nabla \psi_h \, ds \, dq = 0, \quad \forall \psi_h \in V_{h,0}. \quad (68)$$

We use a product three point Gauss-Legendre quadrature formula to approximate all the integral terms appearing in the fully discretized problems corresponding to mine value, probability of completion and expected lifetime.

5. Numerical results. In this section we show some numerical results in order to illustrate the performance of the proposed numerical methods and the PDE models. For this purpose, we first build up an academic example with analytical solution in order to illustrate the order of convergence of the numerical method. Next, we apply the proposed numerical methods to the case of a real mine.

For the finite elements space, in all examples the quadrangular meshes are structured and uniform with the edges of the elements being parallel to the boundaries of the rectangular domain. In Table 1, main data of the different meshes are presented.

|          | Mesh 8 | Mesh 16 | Mesh 32 | Mesh 64 |
|----------|--------|---------|---------|---------|
| Number of nodes | 289    | 1089    | 4225    | 16641   |
| Number of elements | 64     | 256     | 1024    | 4096    |

Table 1. Data of the quadrangular finite element meshes.
5.1. **Academic test with analytical solution.** We first consider an example
with analytical solution to illustrate the order of convergence of the method. Thus,
we consider the following PDE in the spatial domain \( \Omega = [0, 2] \times [0, 1] \):

\[
\frac{\partial U}{\partial \tau} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + q \frac{\partial U}{\partial Q} - \mu S \frac{\partial U}{\partial S} + cU = f, 
\]

where the function \( f \) is chosen so that \( U(\tau, S, Q) = \exp(\tau(S + Q)) \) satisfies (69).
Moreover, the initial condition and the boundary conditions on \( \Gamma^+_1 \) and \( \Gamma^+_2 \) are
chosen accordingly. The parameters are in the corresponding column of Table 2.

| Parameter                          | Academic test | Real mine   |
|-----------------------------------|---------------|-------------|
| Extraction costs ($\epsilon_M$)   | 1             | 1 $\text{tonne}^{-1}$ |
| Processing costs ($\epsilon_P$)   | 4             | 4 $\text{tonne}^{-1}$ |
| Interest rate ($r$)               | 0.1           | 10 $\text{yr}^{-1}$ |
| Dividend yield ($\delta$)         | 0.1           | 10 $\text{yr}^{-1}$ |
| Volatility ($\sigma$)             | 0.3           | 30 $\text{yr}^{-1}$ |
| Maximum duration extraction ($T$)  | 1             | 14 yr       |
| $q$                               | 1             | 1           |
| $G$                               | 9.74          | 9.74 $\text{g tonne}^{-1}$ |

**Table 2.** Parameter values for the academic test with analytical
solution and the real mine

Errors between the exact and numerical solutions in \( l^\infty((0, T); l^2(\Omega)) \) discrete
norm are presented in Table 3. Clearly, these errors illustrate the second order
convergence with respect to the time step \( \Delta \tau \) and to the spatial step \( h \). Note that
in the quadrangular meshes we use, the parameter \( h \) is associated to the length of the
edges of each quadrangular element of the mesh. This parameter \( h \) is successively
divided by two with each mesh refinement in meshes shown in Table 1.

| Mesh  | $\Delta \tau = 10^{-1}$ | $\Delta \tau = 10^{-2}$ | $\Delta \tau = 10^{-3}$ | $\Delta \tau = 10^{-4}$ |
|-------|-------------------------|-------------------------|-------------------------|-------------------------|
| 8     | 4.3913 $\times 10^{-3}$ | 5.4307 $\times 10^{-3}$ | 2.8440 $\times 10^{-6}$ | 2.8942 $\times 10^{-6}$ |
| 16    | 5.4574 $\times 10^{-3}$ | 5.4133 $\times 10^{-3}$ | 4.9435 $\times 10^{-6}$ | 4.4366 $\times 10^{-6}$ |
| 32    | 7.8917 $\times 10^{-3}$ | 7.9003 $\times 10^{-3}$ | 2.5526 $\times 10^{-6}$ | 3.0282 $\times 10^{-6}$ |
| 64    | 8.9779 $\times 10^{-3}$ | 9.0633 $\times 10^{-5}$ | 6.5549 $\times 10^{-7}$ | 1.0258 $\times 10^{-6}$ |

**Table 3.** Relative errors in \( l^\infty((0, T); l^2(\Omega)) \) discrete norm
between the exact and numerical solutions for the academic test

The numerical results are in agreement with the theoretical analysis developed
in [2], where the order of convergence of the here proposed method is analyzed for a
more general PDE problem and a general family of Lagrange finite elements spaces.
In particular, for piecewise quadratic elements the second order of convergence with
respect to the time step and the spatial mesh step is theoretically proved.

On the other hand, we note that the second order convergence exhibited for
the proposed method represents an advantage with respect to the semi-Lagrangian
method proposed in [12, 13], which has first order convergence in time. Thus, the
method we propose reduces the time error for the same time step and allows the
use of less number of time steps for a given accuracy.
5.2. Real mine problem. In this section we apply the model and numerical methods to a real mine case. Main data come from [13], where in turn they are provided from Gemcom Software International (a large mine solutions provider). Thus, the data for this case are indicated in the corresponding column of Table 2.

Concerning the numerical methods, we consider a computational spatial domain \([0, 20]\times[0, 20]\) while the time interval is \([0, 14]\). The numerical results we present in this section are obtained with 10000 time steps and for the Mesh 64 in Table 1.

The computed mine value price for time \(t = 0\) is shown in Figure 1. Besides obtaining the mine value, the proposed numerical methods allow to compute the abandonment and non abandonment regions for time \(t = 0\), which are represented in black and white, respectively, in Figure 2. These regions are identified through the ALAS algorithm. A zoom near the optimal abandonment boundary is shown in Figure 2 (right). In Figure 2 we show that the optimal abandonment boundary at time \(t = 0\), which can be understood as a curve \(S^* = S^*(Q,t)\) at time \(t\), is almost a vertical line in the \((S,Q)\)-plane, except for small values of the ore reserve. The nonconstant behaviour for small values of the ore reserve can be distinguished with an appropriate zoom. Also in relation with the abandonment boundary, we note that in [13] a constant approximation of the value \(S^*\) is obtained. Actually, this approximation is obtained as the commodity price for which the mine value is equal to zero when the option to abandon is not considered. More precisely, in [13] an average argument is considered and the obtained constant approximation, \(\overline{S}^*\), is

\[
\overline{S}^* = \frac{\delta (\epsilon_P + \epsilon_M)}{rqG}
\]  

and estimates the value of \(S^*\). This approximation is valid for \(r \approx \delta\), as it is the case with the data in Table 2. Moreover, when the data in Table 2 are considered then the value \(\overline{S}^* = 0.51\) is obtained, which is close to and a bit larger than the observed optimal abandonment prices in Figure 2 for most of the values of \(Q\).

In Figure 3 the corresponding computed values of the probability of completion (left) and the expected lifetime (right) of the project at time \(t = 0\) are shown. Moreover, in Figure 4 we represent the evolution of probability of completion with respect to time to expiry when \(Q = 0.5\) (left) and \(Q = 10\) (right). This is shown in three curves associated to different values of \(S\), all of them starting with \(P = 1\) at expiry date and decreasing with time to expiry. For the higher prices the curves
Concerning the computing time, it takes around 12 minutes in a Intel(R)Core (TM) i5-3337U CPU @1.8 GHz 64 bits, 4 Gb de RAM.

6. Conclusions. In this article, we consider the initial model proposed in [13] for the mine value, probability of completion and expected life time in an extraction project under uncertainty. The mathematical analysis is addressed to clarify and propose adequate boundary conditions to the associated PDE problems. In the three models the boundary conditions at the upper boundary in the direction of the commodity price, $S$, are changed or made more precise. Concerning the numerical methods, we propose a second order semi-Lagrangian Crank-Nicolson scheme that improves the first order semi-Lagrangian method considered in [13]. Moreover, a piecewise quadratic finite elements method is proposed for the spatial discretization instead of an implicit finite differences technique. In order to cope with the complementarity problem associated to the mine valuation, the ALAS method from [17] is proposed, which results more efficient than a projected SSOR method. The proposed method is based on a mixed formulation by introducing a Lagrange multiplier associated to the inequality constraints. Once the mine valuation problem has been...
solved and the non abandonment region has been identified, the probability of completion and the expected lifetime problems are initially posed on the time dependent non abandonment region, which is extended to the same fixed computational domain where the mine valuation problem is posed. In this way, in both problems the numerical solution is carried out in the same fixed domain as for the mine valuation problem, thus taking advantage of using the same spatial mesh instead of a mesh that should be adapted to the non abandonment domain at each time step. The extension of the probability of completion and expected lifetime values to the fixed domain is made at the level of the fully discretized problems in the involved linear systems. In order to validate the numerical methods, an academic example with known analytical solution is solved to illustrate the second order convergence in time and space. Finally, an extraction problem with real data is considered and the numerical results are in agreement with those ones presented in [13], with certain variations mainly due to the changes in the more realistic boundary conditions here considered.

In a future work, the authors are considering the case in which the extraction rate is proportional to the commodity price, which makes sense, instead of being constant. Another possible extension is in line with [14], where the extraction project owner can handle the extraction rate as a control variable inside the operating system that can be used to maximize the expected future discounted cash flows. In this setting, an appropriate Hamilton-Jacobi-Bellman (HJB) equation can be posed and the here proposed numerical techniques can be used inside the optimal control problem. In this case, the determination of the optimal extraction rate $q^*$ in the interval of feasible extraction rates and the corresponding mine value $V^*$ are coupled each other, in a nonlinear programming setting. Analogous numerical difficulties have been addressed in [8] in the context of optimal gas storage problems.

REFERENCES

[1] M. Bercovier, O. Pironneau and V. Sastry, Finite elements and characteristics for some parabolic-hyperbolic problems, Applied Mathematical Modelling, 7 (1983), 89–96.

[2] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Numerical analysis of convection-diffusion-reaction problems with higher order characteristics finite elements. Part II: Fully discretized scheme and quadrature formulas, SIAM Journal Numerical Analysis, 44 (2006), 1854–1876.

[3] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Numerical solution of variational inequalities for pricing Asian options by higher order Lagrange-Galerkin methods, Applied Numerical Mathematics, 56 (2006), 1256–1270.
[4] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Comparison of two algorithms to solve a fixed-strike Amerasian options pricing problem, in Free Boundary Problems, International Series in Numerical Mathematics, 154 (eds. I.N. Figueiredo, J.F. Rodrigues and L. Santos), Birkhäuser, (2007), 95–106.

[5] M. J. Brennan and E. S. Schwartz, Evaluating natural resources investments, Journal of Business, 58 (1985), 135–157.

[6] F. Black and M. Scholes, The pricing of option and corporate liabilities, Journal Political Economy, 81 (1973), 637–654.

[7] D. Castillo, A. M. Ferreiro, J. A. García-Rodríguez and C. Vázquez, Numerical methods to solve PDE models for pricing business companies in different regimes and implementation in GPUs, Applied Mathematics and Computation, 219 (2013), 11233–11257.

[8] Z. Cheng and P. A. Forsyth, A semi-Lagrangian approach for natural gas storage, SIAM Journal on Scientific Computing, 30 (2007), 339–368.

[9] Y. D’Halluin, P. A. Forsyth and G. Labahn, A semi-Lagrangian approach for American Asian options under jump diffusion, SIAM Journal on Scientific Computing, 27 (2005), 315–345.

[10] A. K. Dixit and R. S. Pindyck, Investment Under Uncertainty, Princeton University Press, Princeton, NJ, 1994.

[11] J. Douglas, Jr. and T. F. Russell, Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures, SIAM Journal on Numerical Analysis, 19 (1982), 871–885.

[12] G. W. Evatt, P. V. Johnson, P. W. Duck and S. D. Howell, Mine valuations in the presence of a Stochastic ore-grade, Int. Assoc. Eng., 3 (2010), 1811–1816.

[13] G. W. Evatt, P. V. Johnson, P. W. Duck, S. D. Howell and J. Moriarty, The expected lifetime of an extraction project, Proceedings of the Royal Society, 467 (2011), 244–263.

[14] G. W. Evatt, P. V. Johnson, P. W. Duck and S. D. Howell, Optimal costless extraction rate changes from a non-renewable resource, European Journal of Applied Mathematics, 25 (2014), 681–705.

[15] G. Fichera, On a Unified theory of boundary value problems for elliptic-parabolic equations of second order in boundary value problems, University of Wisconsin Press, 1960.

[16] R. Kangro and R. Nicolaides, Far field boundary conditions for Black-Scholes equations, SIAM Journal Numerical Analysis, 38 (2000), 1357–1368.

[17] T. Kärkkäinen, K. Kunisch and P. Tarvainen, Augmented Lagrangian active set methods for obstacle problems, Journal Optimization Theory Applications, 119 (2003), 499–533.

[18] B. Øksendal, Stochastic Differential Equations, 5th edition, Springer, Berlin, 2003.

[19] O. A. Oleinik and E. V. Radkevic, Second Order Equations with Nonnegative Characteristic Form, A.M.S. and Plenum Press, Providence, 1973.

[20] A. Pascucci, PDE and Martingale Methods in Option Pricing, Bocconi & Springer Series, Springer-Verlag, New York, 2011.

[21] A. Pascucci, M. Suárez-Taboada and C. Vázquez, Mathematical analysis and numerical methods for a PDE model of a stock loan pricing problem, Journal of Mathematical Analysis and Applications, 403 (2013), 38–53.

Received March 2018; revised April 2018.

E-mail address: mariasuarez@udc.es
E-mail address: carlosv@udc.es