Bubbling nodal solutions for a large perturbation of the Moser-Trudinger equation on planar domains

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Abstract
In this work we study the existence of nodal solutions for the problem
\[- \Delta u = \lambda u e^{u^2 + |u|^p} \]
in \( \Omega \), \( u = 0 \) on \( \partial \Omega \),
where \( \Omega \subseteq \mathbb{R}^2 \) is a bounded smooth domain and \( p \to 1^+ \).

If \( \Omega \) is ball, it is known that the case \( p = 1 \) defines a critical threshold between
the existence and the non-existence of radially symmetric sign-changing solutions.
In this work we construct a blowing-up family of nodal solutions to such problem
as \( p \to 1^+ \), when \( \Omega \) is an arbitrary domain and \( \lambda \) is small enough. As far as we
know, this is the first construction of sign-changing solutions for a Moser-Trudinger
critical equation on a non-symmetric domain.

1 Introduction
Let us consider the equation
\[ \Delta u + \lambda u e^{u^2 + a|u|^p} = 0 \in \Omega, \ u = 0 \text{ on } \partial \Omega, \]
where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^2 \), \( \lambda \) is a positive parameter and the nonlinear
term \( h(u) := u e^{a|u|^p} \), with \( a \in \mathbb{R} \) and \( p \in [0, 2) \), is a lower-order perturbation of \( e^{u^2} \)
according to the definition given by Adimurthi in [2].

The nonlinearity \( f(u) = h(u) e^{u^2} \) is critical from the view point of the Trudinger
imbedding. Indeed, in view of the Moser-Trudinger inequality (see [20, 29, 24])
\[
\sup \left\{ \int_{\Omega} e^{u^2} \, dx : u \in H_0^1(\Omega), \|u\|_{H_0^1(\Omega)}^2 \leq 4\pi \right\} < +\infty, \tag{2}
\]
the functional
\[ J_\lambda(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} F(u) dx, \quad u \in H^1_0(\Omega), \] (3)

where \( F(t) = \int_0^t f(s) ds \), is well defined and its critical points are solutions to problem (1). Adimurthi in [2] proved that \( J_\lambda \) satisfies the Palais-Smale condition in the infinite energy range \((-\infty, 2\pi)\) but, as observed by Adimurthi and Prashant in [5], the critical nature of \( f(u) \) reflects in the failure of the Palais-Smale condition at the sequence of energy levels \( 2\pi k \) with \( k \in \mathbb{N} \) (see also [7]).

In [2] Adimurthi proved the existence of a critical point of \( J_\lambda \) if the perturbation \( h \) is large, i.e. \( a \geq 0 \), and if \( 0 < \lambda < \lambda_1(\Omega) \), where \( \lambda_1(\Omega) \) is the first eigenvalue of \(-\Delta\) with Dirichlet boundary condition ((see also [4])). Such a critical point is a positive solution to problem (1). Successively, Adimurthi and Prashant in [6] showed that the condition \( a \geq 0 \) is necessary to get a positive solution to (1). Indeed, they proved that if the perturbation \( h \) is small, i.e. \( a < 0 \), then there are no positive solutions to problem (1) when the domain \( \Omega \) is a ball provided \( \lambda \) is small. The case \( a = 0 \) in a general domain \( \Omega \) has been studied by Del Pino, Musso and Ruf [14] using a perturbative approach. Indeed they find multiplicity of positive solutions which blow-up in one or more points of \( \Omega \) (depending on the geometry) as \( \lambda \to 0 \). We point out that a general qualitative analysis of blowing-up families of positive solutions to problem (1) has been obtained by Druet in [13] (see also [8, 17, 16]).

As far as it concerns the existence of sign-changing solutions, Adimurthi and Yadava in [8] proved that problem (1) has a nodal solution when \( \lambda \) is small if there is the further restriction \( p > 1 \) on the growth of the large perturbation \( h \) (i.e. \( a > 0 \)). Actually, this condition turns out to be optimal for the existence of nodal radial solutions in a ball. Indeed Adimurthi and Yadava in [9] proved that if \( a > 0 \) and \( \Omega \) is a ball, problem (1) does not have any radial sign-changing solution when \( \lambda \) is small and \( p \in [0,1] \). If one drops the radial requirement, Adimurthi and Yadava in [8] proved the existence of infinitely many sign-changing solutions in a ball whatever \( \lambda > 0 \) is. We point out that, in the case \( a = 0 \), the approach of Del Pino, Musso and Ruf [14] allows to find sign-changing solutions which blow-up positively and negatively at least at two different points in any domain \( \Omega \) as \( \lambda \to 0 \) (even if this is not explicitly said in their work).

According to the previous discussion, it turns out that when \( a > 0 \) the case \( p = 1 \) defines a critical threshold for the existence of radial sign-changing solutions in the ball. Indeed, when \( \Omega = B(0,1) \), (1) has radially symmetric sign-changing solutions which blow-up as \( p \to 1^+ \). The precise behavior of such solutions was studied by Grossi and Naimen in [19]. Therefore, when \( a > 0 \), it is natural to ask whether it is possible to find sign-changing solutions to problem (1) on an arbitrary planar domain \( \Omega \) which blow-up at one point in \( \Omega \) as \( p \to 1^+ \).

In this paper we give a positive answer. More precisely, let us consider the problem
\[
\begin{align*}
-\Delta u &= \lambda u e^{u^2+|u|^{4-\varepsilon}} \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\] (4)
where $\varepsilon$ is a positive small parameter. Set
\[ f_\varepsilon(t) = t e^{t^2 + |t|^{1+\varepsilon}}. \] (5)

For a given $0 < \lambda < \lambda_1(\Omega)$, let $u_0$ be a positive solution of the problem
\[
\begin{aligned}
-\Delta u_0 &= \lambda f_0(u_0) \quad \text{in } \Omega, \\
u_0 &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\] (6)
whose existence has been established by Adimurthi in [2]. We make the following assumptions:

(A1) $u_0$ is non-degenerate, i.e. there is no non-trivial solution $\varphi \in H_0^1(\Omega)$ of the equation
\[-\Delta \varphi = \lambda f'_0(u_0) \varphi \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on } \partial \Omega.
\] (7)

(A2) $u_0$ has a $C^1$-stable critical point $\xi_0 \in \Omega$ such that $u_0(\xi_0) > \frac{1}{2}$.

Then, we will show that [6] admits a family of sign-changing solutions which blow-up at $\xi_0$ with residual mass $-u_0$ as $\varepsilon \to 0$, namely:

**Theorem 1.1** For $0 < \lambda < \lambda_1(\Omega)$, let $u_0$ be a solution of (6) such that [A1] and [A2] are satisfied. Let also $\xi_0$ be as in [A2]. Then there exist $\varepsilon_0 > 0$ and a family $(u_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ of sign-changing solutions to (4) such that:

- $\max_{B(\xi_0, r)} u_\varepsilon \to +\infty$ as $\varepsilon \to 0$, for any $0 < r < d(\xi_0, \partial \Omega)$.
- $u_\varepsilon \rightharpoonup -u_0$ weakly in $H_0^1(\Omega)$ and in $C^1(\overline{\Omega} \setminus \{\xi_0\})$.

Let us make some comments about assumptions [A1] and [A2].

**Remark 1.2**

- The solution $u_0$ to problem (6) turns out to be non-degenerate when $\Omega$ is the ball as proved by Adimurthi, Karthik and Giacomoni in [4]. In a work in progress, Grossi and Naimen are going to prove that the solution is also non-degenerate when $\Omega$ is convex and symmetric (see [20]). Actually, we believe that the non-degeneracy condition holds true for most domains $\Omega$ and positive parameters $\lambda$. Indeed, one could use similar arguments to those used by Micheletti and Pistoia in [23] for a class of singularly perturbed equations.

- We remind that $\xi_0$ is a $C^1$-stable critical point of $u_0$ if the Brouwer degree $\deg(\nabla u_0, B(\xi_0, r), 0) \neq 0$. In particular, any strict local maximum point of $u_0$ is $C^1$-stable. We point out that by Adimurthi and Druet [3] we can deduce that assumption [A2] holds true when the parameter $\lambda$ is small enough.
We strongly believe that the condition \( u_0(\xi_0) > \frac{1}{2} \) is not purely technical, but it is necessary to build a solution which blows-up at \( \xi_0 \). Indeed, we conjecture that, if \( u_0(\xi_0) \leq \frac{1}{2} \), there does not exist any sign-changing solution which blows-up at \( \xi_0 \) with non-trivial residual mass \( u_0 \) as \( \varepsilon \to 0 \). We point out that, in a different setting, a similar condition was proved by Mancini and Thizy [22] for problem \( \text{[1]} \) on a ball with \( p = 1 \) and \( a < 0 \): in fact, they show that the value at the origin of the residual mass of any non-compact sequence of radially symmetric positive solutions must be equal to \(-\frac{a}{2}\) (and we get \( \frac{1}{2} \), when \( a = -1 \)).

Actually, we can give a more precise description of the asymptotic behavior of the solution \( u_\varepsilon \) as \( \varepsilon \to 0 \), since it is build via a Lyapunov-Schmidt procedure. For \( \delta, \mu > 0 \), and \( \xi \in \mathbb{R}^n \), let us consider the functions

\[
U_{\delta, \mu, \xi}(x) = \log \left( \frac{8\mu^2\delta^2}{(\mu^2\delta^2 + |x - \xi|^2)^2} \right),
\]

which describe the set of all the solutions to the Liouville equation

\[
-\Delta U = e^U \quad \text{in } \mathbb{R}^2,
\]

under the condition \( e^U \in L^1(\mathbb{R}^2) \) (see [21], [12]). We further consider the projection \( PU_{\delta, \mu, \xi} := (-\Delta)^{-1} e^{U_{\delta, \mu, \xi}} \), where \( (-\Delta)^{-1} : L^2(\Omega) \to H^1_0(\Omega) \) is the inverse of \( -\Delta \). Namely, \( PU_{\delta, \mu, \xi} \) is defined as the unique solution to

\[
\begin{align*}
-\Delta PU_{\delta, \mu, \xi} &= -\Delta U_{\delta, \mu, \xi} = e^{U_{\delta, \mu, \xi}} \quad \text{in } \Omega, \\
PU_{\delta, \mu, \xi} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Intuitively, we want to look for solutions of \( \text{[4]} \) that look like \( \alpha PU_{\delta, \mu, \xi} - u_0 \) for suitable choices of the parameters \( \alpha, \delta, \mu, \xi \). Unfortunately, in order to successfully perform Lyapunov-Schmidt reduction, a more precise ansatz is necessary and we are forced to replace \( u_0 \) with a better approximation of the solutions. First, the non-degeneracy assumption \( \text{[A1]} \) allows to find a positive solution \( v_\varepsilon \in C^1(\Omega) \) of \( \text{[4]} \) such that

\[
v_\varepsilon \to u_0 \quad \text{in } C^1(\Omega),
\]
as \( \varepsilon \to 0 \). Then, we consider the function

\[
V_{\varepsilon, \alpha, \xi} := v_\varepsilon + \alpha w_{\varepsilon, \xi} + \alpha^2 z_{\varepsilon, \xi},
\]

where \( \alpha \in (0, 1) \) is a small positive parameter depending on \( \varepsilon, \mu, \xi \) such that \( \alpha \to 0 \) as \( \varepsilon \to 0 \), and \( w_{\varepsilon, \xi} \) and \( z_{\varepsilon, \xi} \) are defined as the unique solutions to the couple of linear problems

\[
\begin{align*}
\Delta w_{\varepsilon, \xi} + \lambda f'_\varepsilon(v_\varepsilon)w_{\varepsilon, \xi} &= 8\pi \lambda G_\xi f'_\varepsilon(v_\varepsilon) \quad \text{in } \Omega, \\
w_{\varepsilon, \xi} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

and

\[
\begin{align*}
\Delta z_{\varepsilon, \xi} + \lambda f'_\varepsilon(v_\varepsilon)z_{\varepsilon, \xi} &= \frac{\lambda}{2} f''_\varepsilon(-v_\varepsilon)(8\pi G_\xi - w_\varepsilon)^2 \quad \text{in } \Omega, \\
z_{\varepsilon} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
with $G_\xi$ denoting the Green function of $\Omega$ with singularity at $\xi$, namely the distributional solution to
\[
\begin{cases}
-\Delta G_\xi = \delta_\xi & \text{in } \Omega, \\
G_\xi = 0 & \text{on } \partial\Omega.
\end{cases}
\] (14)

Problems (12) and (13) are nothing but the linearization of problem (4) around the solution $v_\varepsilon$ and the R.H.S.’s are the terms of the second order Taylor’s expansion with respect to $\alpha$ of $f_\varepsilon(\alpha PU_{\delta,\mu,\xi} - V_\varepsilon,\alpha,\xi)$ far away from the concentration point $\xi$ (indeed $PU_{\delta,\mu,\xi} \sim 8\pi G_\xi$ because of (23)).

Theorem 1.1 follows at once by the following result:

**Theorem 1.3** Let $\lambda$, $u_0$, $\xi_0$ be as in Theorem 1.1. There exists $\varepsilon_0 > 0$ and functions $\alpha, \delta, \mu : (0, \varepsilon_0) \to (0, +\infty)$, $\xi : (0, \varepsilon_0) \to \Omega$ and $\varphi : (0, \varepsilon_0) \to H^1_0(\Omega)$ such that:

- $u_\varepsilon := \alpha(\varepsilon) PU_{\delta(\varepsilon),\mu(\varepsilon),\xi(\varepsilon)} - V_\varepsilon,\alpha(\varepsilon),\xi(\varepsilon) + \varphi(\varepsilon)$ is a solution (4).
- $\alpha(\varepsilon) \to 0$, $\delta(\varepsilon) \to 0$, $\mu(\varepsilon) \to \sqrt{8\pi e^{-1}}$, $\xi(\varepsilon) \to \xi_0$, and $u_\varepsilon(\xi(\varepsilon)) \to +\infty$ as $\varepsilon \to 0$.
- $\|\varphi(\varepsilon)\|_{H^1_0(\Omega)} + \|\varphi(\varepsilon)\|_{L^\infty(\Omega)} = O(e^{-\log(2u_0(\xi_0))\varepsilon})$.

Let us briefly sketch the main steps of the proof of Theorem 1.3. First, in Section 2, we choose $\alpha = \alpha(\varepsilon, \mu, \xi)$ and $\delta = \delta(\varepsilon, \mu, \xi)$ such that the function
\[
\omega_{\varepsilon,\mu,\xi} := \alpha PU_{\delta(\varepsilon),\mu(\varepsilon),\xi(\varepsilon)} - V_\varepsilon,\alpha(\varepsilon),\xi(\varepsilon) + \varphi(\varepsilon)
\] (15) is an approximate solution of (4). Then, we look for solutions of (4) of the form $\omega_{\varepsilon,\mu,\xi} + \varphi$ with $\varphi \in H^1_0(\Omega)$. Clearly, (4) can be written in terms of $\varphi$ as
\[
-\Delta \varphi - \lambda f'(\omega_{\varepsilon,\mu,\xi}) \varphi = R + N(\varphi), \tag{16}
\]
where the error term $R$ is defined by
\[
R = R_{\varepsilon,\mu,\xi} := \Delta \omega_{\varepsilon,\mu,\xi} + \lambda f'(\omega_{\varepsilon,\mu,\xi}), \tag{17}
\]
and the higher order term $N$ by
\[
N(\varphi) = N_{\varepsilon,\mu,\xi}(\varphi) := \lambda \left(f_\varepsilon(\omega_{\varepsilon,\mu,\xi} + \varphi) - f_\varepsilon(\omega_{\varepsilon,\mu,\xi}) - f'_\varepsilon(\omega_{\varepsilon,\mu,\xi}) \varphi\right). \tag{18}
\]
Equivalently, introducing the linear operator
\[
L \varphi = L_{\varepsilon,\mu,\xi} \varphi := \varphi - (-\Delta)^{-1}(\lambda f'(\omega_{\varepsilon,\mu,\xi}) \varphi), \tag{19}
\]
we need to solve
\[
L \varphi = (-\Delta)^{-1}(R + N(\varphi)). \tag{20}
\]
A careful and delicate estimate of the error $R$ will be given in Section 3. The behaviour of the operator $L$ will be studied in Section 4. On the one hand, for functions supported away from a suitable shrinking neighborhood of $\xi$, we will show that $L$ is close to the
operator $L_1 \varphi := \varphi - (-\Delta)^{-1}(\lambda f_0')(u_0)\varphi$, which is invertible on $H^1_0(\Omega)$ because of the non-degeneracy assumption \( [A1] \). On the other hand, near the point $\xi$, $L$ is close to the operator $L_0 \varphi := \varphi - (-\Delta)^{-1}(e^{i\beta_{\mu,\xi}})\varphi$. This operator appears in the analysis of several critical problems in dimension 2 (see for example \([10\, 13\, 18]\)) and its behavior is well known: although $L_0$ is not invertible, it is possible to find an approximate threedimensional kernel $K_{\delta,\mu,\xi}$ for $L_0$ by projecting on $H^1_0(\Omega)$ the three functions

$$
Z_{0,\delta,\mu,\xi}(x) = \frac{\delta^2\mu^2 - |x - \xi|^2}{|x - \xi|^2 + \delta^2\mu^2}, \\
Z_{i,\delta,\mu,\xi}(x) = \frac{-2\delta\mu(x_i - \xi_i)}{|x - \xi|^2 + \delta^2\mu^2}, \quad i = 1, 2.
$$

Such properties transfer to the operator $L$, which turns out to be invertible on the subspace $K_{\delta,\mu,\xi}$ orthogonal to $K_{\delta,\mu,\xi}$ in $H^1_0(\Omega)$. More precisely, denoting by $\pi$ and $\pi^\perp$ the projections of $H^1_0(\Omega)$ respectively on $K_{\delta,\mu,\xi}$ and $K_{\delta,\mu,\xi}^\perp$, we will show that $\pi^\perp L$ is invertible on $K_{\delta,\mu,\xi}^\perp$. Then, it is natural to split equation (20) as

$$
\begin{align*}
\varphi &= (\pi^\perp L)^{-1}\pi^\perp (-\Delta)^{-1}(R + N(\varphi)), \\
\pi L\varphi &= \pi (-\Delta)^{-1}(R + N(\varphi)).
\end{align*}
$$

The first equation of (21) will be solved in Section 5 where for any $\mu > 0$, $\xi$ close to $\xi_0$ and any small $\varepsilon > 0$, we will find a solution $\varphi_{\varepsilon,\mu,\xi}$ via a contraction mapping argument on a sufficiently small ball in $K_{\delta,\mu,\xi}^\perp \cap L^\infty(\Omega)$. Then, recalling that $\dim K_{\delta,\mu,\xi} = 3$ and using assumption \( [A2] \), we will show in Section 5 that it is possible to choose the three parameters $\mu = \mu(\varepsilon)$ and $\xi = \xi(\varepsilon) = (\xi_1(\varepsilon), \xi_2(\varepsilon))$ so that the second equation in (21) is also fulfilled. Clearly, for such choice of $\mu$ and $\xi$, the function $\varphi_{\varepsilon,\mu(\varepsilon),\xi(\varepsilon)}$ solves both the equations in (21) (or, equivalently \([10]\) and (20)), and $u_\varepsilon := \omega_{\varepsilon,\mu(\varepsilon),\xi(\varepsilon)} + \varphi_{\varepsilon,\mu(\varepsilon),\xi(\varepsilon)}$ is a solution of \([11]\).

It is important to point out that choice of the concentration point $\xi(\varepsilon)$ is extremely delicate since the scaling parameter $\delta$ turns out to be much smaller than the parameter $\alpha$, whose powers control all the error terms. To overcome this difficulty, we introduce a new argument based on a precise Pohozaev-type identity. This allows us to bypass global a priori gradient estimates on the solution $\varphi_{\varepsilon,\mu,\xi}$, which are hard to obtain for Moser-Trudinger critical problems. Our argument requires a very precise ansatz of the approximate solution $\omega_{\varepsilon,\mu,\xi}$. In particular, the presence of the correction terms $\omega_{\varepsilon,\mu,\xi}^\perp$ and $z_{\varepsilon,\xi}$ in the expression of $V_{\varepsilon,\alpha,\xi}$ is not merely technical, but plays a crucial role both in the estimates of the error term $R$ and in the choice of $\xi(\varepsilon)$.

### 2 Construction of the approximate solution

In this section we give the detailed construction of the approximate solution $\omega_{\varepsilon,\mu,\xi}$. Here and in the rest of the paper, we will assume that $(\mu, \xi) \in \mathcal{U} \times B(\xi_0, \sigma)$, where $\mathcal{U} \subseteq \mathbb{R}^+$ is an open interval containing $\mu_0 := \sqrt{5}d^{-1}$, $\xi_0$ is as in the assumption \( [A2] \), and $0 < \sigma < \frac{1}{2}d(\xi_0, \partial \Omega)$. By \( [A2] \) we can also assume

$$
\inf_{B(\xi_0, \sigma)} u_0(\xi) > \frac{1}{2}.
$$

(22)
2.1 The main terms of the ansatz

Let us introduce the main property of the projection of the bubble $PU_{\delta,\mu,\xi}$ defined in (10), which gives the main term of the approximate solution close to the blow-up point $\xi$. Let $G(\cdot) = G(\cdot, \xi)$ be the Green’s function of $-\Delta$ with Dirichlet boundary conditions introduced in (14) and let $H(\cdot, \xi)$ be its regular part, i.e.

$$H(x, \xi) := G_{\xi}(x) - \frac{1}{2\pi} \log \frac{1}{|x - \xi|}.$$ 

**Lemma 2.1** We have

$$PU_{\delta,\mu,\xi}(x) = U_{\delta,\mu,\xi}(x) - \log(8\mu^2\delta^2) + 8\pi H(x, \xi) + \psi_{\delta,\mu,\xi}(x),$$

where

$$||\psi_{\delta,\mu,\xi}||_{C^1(\Omega)} = O(\delta^2),$$

uniformly with respect to $\mu \in U$, $\xi \in B(\xi_0, \sigma)$. In particular,

$$PU_{\delta,\mu,\xi} \to 8\pi G_{\xi} \text{ in } C^1_{\text{loc}}(\Omega \setminus \{\xi\}).$$  \hspace{1cm} (23)

**Proof.** See for example [11, Proposition 5.1]. \hfill \square

Next, let us define the main term of the approximate solution in the whole domain as $\alpha PU_{\delta,\mu,\xi} - \nu_\epsilon$ where $\alpha$ is a positive parameter approaching zero as $\epsilon \to 0$ and $\nu_\epsilon$ is a non-degenerate solution to (4), whose existence is proved in the following lemma.

**Lemma 2.2** Let $\lambda$ and $u_0$ be as in Theorems 1.1 and 1.3. There exists $\epsilon_0 > 0$, and a family of functions $(\nu_\epsilon)_{0<\epsilon<\epsilon_0} \subseteq C^1(\Omega)$ such that:

i. $\nu_\epsilon$ is a non-degenerate weak solution of (1) for any $\epsilon \in (0, \epsilon_0)$.

ii. $\nu_\epsilon \to u_0$ in $C^1(\Omega)$ as $\epsilon \to 0$.

iii. There exists $c > 0$ such that $\nu_\epsilon(x) \geq cd(x, \partial \Omega)$ for any $x \in \Omega$, $\epsilon \in (0, \epsilon_0)$.

**Proof.** Let $F : (-1, 1) \times H^1_0(\Omega) \to H^1_0(\Omega)$ be defined by

$$F(\epsilon, u) = F_\epsilon(u) := u - (-\Delta)^{-1}(\lambda f_\epsilon(u)), \hspace{1cm} (24)$$

where $f_\epsilon$ is defined as in (5). $F$ is well defined because the Moser-Trudinger inequality implies that $f_\epsilon(u) \in L^p(\Omega)$ for any $1 \leq p < +\infty$ and $u \in H^1_0(\Omega)$. Moreover, it is a $C^1$-map and its partial derivative $DF_\epsilon(u) : H^1_0(\Omega) \to H^1_0(\Omega)$ defined by

$$DF_\epsilon(u)[\varphi] = \varphi - (-\Delta)^{-1}(\lambda f'_\epsilon(u)\varphi)$$

is a Fredholm operator of index 0 (since the embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ is compact).

Now, let $u_0$ be a non-degenerate weak solution of (10) such that (A1) holds true. In particular, $F_0(u_0) = 0$ and $DF_0(u_0)$ is invertible. Therefore, by the implicit function theorem, we can construct a $C^1$ curve $\epsilon \mapsto \nu_\epsilon \in H^1_0(\Omega)$, defined for $|\epsilon| < \epsilon_0$ such that $\nu_0 = u_0$, $F_\epsilon(\nu_\epsilon) = 0$, and $DF_\epsilon(\nu_\epsilon)$ is invertible for $|\epsilon| < \epsilon_0$. Then (4) holds.
Applying the Moser-Trudinger inequality (2) and standard elliptic estimates, we obtain ii.

Hopf’s lemma and the compactness of \( \partial \Omega \) give \( \partial u_0 \leq -2c \) on \( \partial \Omega \) for some \( c > 0 \). Then, for \( \varepsilon \) sufficiently small, we have \( \partial u_\varepsilon \leq -c \) on \( \partial \Omega \), which in turn gives \( v_\varepsilon \geq c_0(x, \partial \Omega) \) for \( x \) in a neighborhood of \( \partial \Omega \). Finally, since \( v_\varepsilon \to u_0 \) uniformly in \( \Omega \), and \( u_0 > 0 \) in \( \Omega \), we get iii.

\[ \square \]

2.2 The correction of the ansatz

We need to correct the ansatz in the whole domain by solving the following two linear problems (12) and (13):

\[
\begin{align*}
\Delta w_\varepsilon + \lambda f'_\varepsilon(v_\varepsilon)w_\varepsilon &= 8\pi \lambda G_\xi f'_\varepsilon(v_\varepsilon) \quad \text{in } \Omega, \\
w_\varepsilon &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

and

\[
\begin{align*}
\Delta z_\varepsilon + \lambda f'_\varepsilon(v_\varepsilon)z_\varepsilon &= 8\pi \lambda G_\xi f''(-v_\varepsilon)(8\pi G_\xi - w_\varepsilon)^2 \quad \text{in } \Omega, \\
z_\varepsilon &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Lemma 2.3 For any \( 0 < \varepsilon < \varepsilon_0 \) and any \( \xi \in \Omega \), there exist \( w_\varepsilon, z_\varepsilon \) such that (12) and (13) hold. Moreover, there exists \( C > 0 \) such that

\[ \|w_\varepsilon\|_{C^1(\Omega)} + \|z_\varepsilon\|_{C^1(\Omega)} \leq C \] (25)

for \( \varepsilon \in (0, \varepsilon_0), \xi \in \Omega \).

Proof. The existence of the solutions immediately follows from the non-degeneracy of the function \( v_\varepsilon \) proved in Lemma 2.2. Moreover, since for any \( p \in [1, +\infty) \) one has

\[ \sup_{\xi \in \Omega} \|G_\xi\|_{L^p(\Omega)} < +\infty \quad \text{and} \quad \sup_{0 < \varepsilon < \varepsilon_0} \|v_\varepsilon\|_{C^1(\Omega)} < +\infty, \]

(24) follows by standard elliptic estimates. \( \square \)

Finally, we introduce the corrected ansatz as

\[ \omega_\varepsilon := \alpha PU_{\delta, \mu, \xi} - V_{\varepsilon, \alpha, \xi} \] (26)

with

\[ V_{\varepsilon, \alpha, \xi} := v_\varepsilon + \alpha w_\varepsilon + \alpha^2 z_\varepsilon, \] (27)

where \( v_\varepsilon \) is defined in Lemma 2.2 and \( w_\varepsilon, z_\varepsilon \) as in Lemma 2.3

2.3 The choice of parameters

It will be necessary to choose the parameters \( \alpha = \alpha(\varepsilon, \mu, \xi) \) and \( \delta = \delta(\varepsilon, \mu, \xi) \) such that \( \lambda f_\varepsilon(\omega_\varepsilon) \sim \alpha e^{U_{\delta, \mu, \xi}} \) when \( |x - \xi| \sim \delta \). We point out that one of the main difficulties in this problem is that this estimates holds true only at a very small scale.

Let us fix the values of \( \alpha \) and \( \delta \) according to the next lemma. The proof is based on the contraction mapping theorem and is postponed to the appendix.
Lemma 2.4 There exist $\varepsilon_0 > 0$ and functions $\alpha = \alpha(\varepsilon, \mu, \xi)$, $\beta = \beta(\varepsilon, \mu, \xi)$ and $\delta = \delta(\varepsilon, \mu, \xi)$, defined in $(0, \varepsilon_0) \times \mathcal{U} \times B(\xi_0, \sigma)$ and continuous with respect to $\mu$ and $\xi$, such that

\[
\begin{cases}
\lambda \beta e^{\beta + 1 + \varepsilon} = \frac{\alpha}{\delta}, \\
2\alpha \beta + \alpha \beta^2 + \varepsilon \alpha \beta^2 = 1, \\
\beta = 4\alpha \log \frac{1}{\delta} - V_{\varepsilon, \alpha, \xi} + \alpha c_{\mu, \xi},
\end{cases}
\]  

where $c_{\mu, \xi} \equiv -\log(8\mu^2) + 8\pi H(\xi, \xi)$ and $V_{\varepsilon, \alpha, \xi}$ is defined in (11).

Moreover, as $\varepsilon \to 0$, we have that

\[
\alpha(\varepsilon, \mu, \xi) = \frac{1}{2} e^{-\log(2u_0(\xi)) + \alpha(1)},
\]

\[
\beta(\varepsilon, \mu, \xi) = \frac{1}{2\alpha} - u_0(\xi) + o(1),
\]

\[
\log \delta(\varepsilon, \mu, \xi) = 1 + o(1),
\]

where $o(1) \to 0$ as $\varepsilon \to 0$, uniformly for $\mu \in \mathcal{U}$ and $\xi \in B(\xi_0, \sigma)$.

Remark 2.5 Note that (29-31) give $\alpha(\varepsilon, \mu, \xi), \delta(\varepsilon, \mu, \xi) \to 0$ and $\beta(\varepsilon, \mu, \xi) \to +\infty$ as $\varepsilon \to 0$, uniformly for $\mu \in \mathcal{U}$ and $\xi \in B(\xi_0, \sigma)$.

From now on we let $\alpha = \alpha(\varepsilon, \mu, \xi)$, $\beta = \beta(\varepsilon, \mu, \xi)$ and $\delta = \delta(\varepsilon, \mu, \xi)$ be as in Lemma 2.4.

It will be convenient to work on the scaled domain $\Omega_{\xi, \delta} := \left\{ \frac{x - \xi}{\delta}, \ x \in \Omega \right\}$. Note that we have the scaling relation

\[
U_{\delta, \mu, \xi}(x) = \bar{U}_{\mu} \left( \frac{x - \xi}{\delta} \right) - 2 \log \delta,
\]

where

\[
\bar{U}_{\mu}(y) = U_{1, \mu, 0}(y) = \log \left( \frac{8\mu^2}{(\mu^2 + |y|^2)^2} \right).
\]

Lemma 2.6 As $\varepsilon \to 0$, we have

\[
\omega_{\varepsilon, \mu, \xi}(\xi + \delta y) = \beta + \alpha \bar{U}_{\mu}(y) + O(\delta |y|) + O(\delta^2),
\]

uniformly for $y \in B(0, \frac{\sigma}{2})$, $\mu \in \mathcal{U}$ and $\xi \in B(\xi_0, \sigma)$. Moreover, for any $R > 0$ it holds also true that

\[
\lambda f_{\varepsilon}(\omega_{\varepsilon, \mu, \xi})(\xi + \delta y) = \alpha e^{U_{\delta, \mu, \xi}(\xi + \delta y)} (1 + O(\alpha^2)),
\]

as $\varepsilon \to 0$ uniformly for $y \in B(0, R)$, $\mu \in \mathcal{U}$ and $\xi \in B(\xi_0, \sigma)$. 

9
Then estimate (34) is proved.

By Lemmas 2.2 and 2.3, we know that

\[ V_{\varepsilon, \alpha, \mu}(\xi + \delta y) = V_{\varepsilon, \alpha, \mu}(\xi) + O(\delta) \]

which proves (35).

Similarly, since

\[ H(\xi + \delta y) = H(\xi) + O(\delta) \]

Then estimate (34) is proved.

Now, let us prove (35). Note that (22)–(31) yield \( \beta = O \left( \frac{1}{\alpha} \right) \), \( \delta = O(e^{-\frac{1+o(1)}{\alpha^2}}) \), and \( \beta^\varepsilon = 2u_0(\xi) + o(1) = O(1) \). For \( |y| \leq R \), (34) implies

\[ \omega_{\varepsilon, \mu, \xi}(\xi + \delta y) = \beta + \alpha \tilde{U}_\mu(y) + O(\delta) \]

In particular

\[ \omega_{\varepsilon, \mu, \xi}(\xi + \delta y)^2 = \beta^2 + 2\alpha \beta \tilde{U}_\mu(y) + O(\beta \delta) \]

and

\[ \omega_{\varepsilon, \mu, \xi}(\xi + \delta y)^{1+\varepsilon} = (\beta + \alpha \tilde{U}_\mu(y) + O(\delta))(\beta + \alpha \tilde{U}_\mu(y) + O(\delta))^\varepsilon \]

\[ = (\beta + \alpha \tilde{U}_\mu(y) + O(\delta)) \beta^\varepsilon \left(1 + \frac{\alpha}{\beta} \tilde{U}_\mu(y) + O(\alpha \delta)\right)^\varepsilon \]

\[ = \beta^{1+\varepsilon} + \alpha \beta^\varepsilon \tilde{U}_\mu(y) + \varepsilon \alpha \beta^2 \tilde{U}_\mu(y) + O(\varepsilon \alpha^3) \]

Then, using (28) we get

\[ \lambda f_\varepsilon(\omega_{\varepsilon, \mu, \xi})(\xi + \delta y) = \lambda \omega_{\varepsilon, \mu, \xi}(\xi + \delta y)e^{\omega_{\varepsilon, \mu, \xi}(\xi + \delta y)} \]

\[ = \lambda \beta(1 + O(\alpha^2))e^{\beta^2 + \beta^{1+\varepsilon} + (2\alpha \beta + \alpha \beta^\varepsilon + \alpha \varepsilon \beta^3) \tilde{U}_\mu(y) + O(\alpha^2)} \]

\[ = \lambda \beta e^{\beta^2 + \beta^{1+\varepsilon}} \frac{1}{\beta} \left(1 + O(\alpha^2)\right)e^{O(\alpha^2)} \]

which proves (35).

It is also useful to point out the following result which will be used in the next sections.
Remark 2.7 Lemma 2.1 and Lemma 2.4 give

\[ 0 \leq \alpha PU_{\delta, \mu, \xi} \leq \beta + u_0(\xi) + o(1), \]

and

\[ -V_{\alpha, \varepsilon, \xi} \leq \omega_{\varepsilon, \mu, \xi} \leq \beta + o(1), \]

uniformly for \( x \in \Omega, \varepsilon \in (0, \varepsilon_0), \mu \in U, \xi \in B(\xi_0, \sigma) \).

Notation: In order to simplify the notation, we will write \( U_\varepsilon, \tilde{U}, V_\varepsilon, \omega_\varepsilon, \varpi_\varepsilon \) and \( z_\varepsilon \) instead of \( U_{\delta, \mu, \xi}, \tilde{U}_\mu, V_{\varepsilon, \alpha, \xi}, \omega_{\varepsilon, \mu, \xi}, \varpi_{\varepsilon, \xi} \) and \( z_{\varepsilon, \xi} \), without specifying explicitly the dependence on the parameters. It is important to point out that all the estimates of the next sections will be uniform with respect to \( \mu \in U \) and \( \xi \in B(\xi_0, \sigma) \). This will allow us to choose freely the values of \( \mu \) and \( \xi \) in Section 6. Consistently, the notation \( O(f(x, \varepsilon, \alpha, \beta, \delta)) \) and \( o(f(x, \varepsilon, \alpha, \beta, \delta)) \) will be used for quantities depending on \( \varepsilon, \mu \) (and the parameters \( \alpha, \beta, \delta \) of Lemma 2.4) and satisfying respectively

\[ |O(f(x, \varepsilon, \mu, \xi, \alpha, \beta, \delta))| \leq C_f(x, \varepsilon, \mu, \xi, \alpha, \beta, \delta) \quad \text{and} \quad \frac{o(f(x, \varepsilon, \mu, \xi, \alpha, \beta, \delta))}{f(x, \varepsilon, \mu, \xi, \alpha, \beta, \delta)} \to 0, \]
as \( \varepsilon \to 0 \), uniformly for \( \mu \in U \) and \( \xi \in B(\xi_0, \sigma) \).

3 The estimate of the error term

In this section we give estimates for the error term \( R \) defined in (17)

\[ R = R_{\varepsilon, \mu, \xi} := \Delta \omega_{\varepsilon, \mu, \xi} + \lambda f_\varepsilon(\omega_{\varepsilon, \mu, \xi}). \]

It will be convenient to split \( \Omega \) into four different regions:

\[ \Omega = B(\xi, \rho_0) \cup \left( B(\xi, \rho_1) \setminus B(\xi, \rho_0) \right) \cup \left( B(\xi, \rho_2) \setminus B(\xi, \rho_1) \right) \cup \left( \Omega \setminus B(\xi, \rho_2) \right), \]

(38)

where \( \rho_0 = \rho_0(\varepsilon, \mu, \xi), \rho_1 = \rho_1(\varepsilon, \mu, \xi), \rho_2 = \rho_2(\varepsilon, \mu, \xi) \), are defined by

\[ \rho_0 = \delta e^{\frac{\xi}{\nu}}, \quad \rho_1 = e^{-\frac{u_0(\xi)}{2\nu}} \quad \text{and} \quad \rho_2 = e^{-\frac{\xi}{\nu}}. \]

(39)

Note that

\[ \delta \ll \rho_0 \ll \rho_1 \ll \rho_2 \ll 1, \quad \text{as} \quad \varepsilon \to 0, \]

by (29) and (31). Roughly speaking, we have to split the error into four parts: in \( B(\xi, \rho_0) \) we have \( \lambda f_\varepsilon(\omega_\varepsilon) = \alpha e^{\nu_\varepsilon}(1 + o(1)) \) (see (29)) and we can use a blow-up argument to get a uniform weighted estimate on \( R \). This estimate does not hold anymore in the set \( \Omega \setminus B(\xi, \rho_0) \), which we further split into three parts: the region \( \Omega \setminus B(\xi, \rho_2) \), where \( \alpha G_\xi = O(\varepsilon) \) and a uniform estimate on \( R \) can be obtained via a Taylor expansion of \( f_\varepsilon(\omega_\varepsilon) \) (using that \( \omega_\varepsilon = -V_\varepsilon + 8\pi \alpha G_\xi + o(\varepsilon^2) \)), and the two annuli \( B(\xi, \rho_1) \setminus B(\xi, \rho_0) \) and \( B(\xi, \rho_2) \setminus B(\xi, \rho_1) \), where we give quite delicate integral estimates. The last two regions are treated separately since \( \omega_\varepsilon \geq c_0 > 0 \) in \( B(\xi, \rho_1) \setminus B(\xi, \rho_0) \), while \( \omega_\varepsilon \) changes sign in \( B(\xi, \rho_2) \setminus B(\xi, \rho_1) \) (cfr. Lemma 3.2 and Lemma 3.11).
3.1 A uniform expansion in \( B(\xi, \rho_1) \)

In this section we give a more precise version of the expansions in (36)-(37).

**Lemma 3.1** For any \( \varepsilon \in (0, 1) \) and \( x \geq -1 \), we have

\[
|(1 + x)^{1+\varepsilon} - 1 - (1 + \varepsilon)x| \leq \varepsilon x^2.
\]

**Proof.** According to Bernoulli’s inequality we have

\[
(1 + x)\varepsilon \leq 1 + \varepsilon x \quad (40)
\]

and

\[
(1 + x)^{1+\varepsilon} \geq 1 + (1 + \varepsilon)x. \quad (41)
\]

Since \( x \geq -1 \), thanks to (40) we have that

\[
(1 + x)^{1+\varepsilon} \leq (1 + x)(1 + \varepsilon x) = 1 + (1 + \varepsilon)x + \varepsilon x^2. \quad (42)
\]

Then, the conclusion follows from (41) and (42). \qed

**Lemma 3.2** Set \( c_0 := \frac{1}{2} \inf_{\xi \in B(\xi_0, \sigma)} u_0(\xi) \). For \( x \in B(\xi, \rho_1) \), we have that

\[
\beta + \alpha \bar{U}(x - \xi \delta) \geq c_0, \quad (43)
\]

for sufficiently small \( \varepsilon \). In particular, we have

\[
c_0 \leq \omega_\varepsilon \leq \beta(1 + o(1)). \quad (44)
\]

**Proof.** The definitons of \( \bar{U} \) and \( \rho_1 \) (see (33) and (39)), and (30)-(31) give

\[
\beta + \alpha \bar{U}\left(\frac{x - \xi}{\delta}\right) \geq \beta + \alpha \bar{U}\left(\frac{\rho_1}{\delta}\right) = \beta - 4\alpha \log \frac{\rho_1}{\delta} + o(1) = u_0(\xi) + o(1),
\]

which implies (43) for sufficiently small \( \varepsilon \). To get (44), it is sufficient to apply Lemma 2.6 and Remark 2.7. \qed

**Lemma 3.3** For \( x \in B(\xi, \rho_1) \), we have

\[
\omega^2_\varepsilon(x) + \omega^{1+\varepsilon}_\varepsilon(x) = \beta^2 + \beta^{1+\varepsilon} + \bar{U}\left(\frac{x - \xi}{\delta}\right) + \alpha^2 \bar{U}^2\left(\frac{x - \xi}{\delta}\right) + O\left(\varepsilon \alpha^3 \left(1 + \bar{U}^2\left(\frac{x - \xi}{\delta}\right)\right)\right).
\]
Proof. Set \( y = \frac{x - \xi}{\delta} \in B(0, \frac{\alpha}{\beta}) \). Noting that \( \overline{U}(y) = O(\alpha^{-2}) \) and using Lemma 2.6, we get

\[
\omega_2^2(x) = \omega_2^2(\xi + \delta y) = (\beta + \alpha \overline{U}(y) + O(\rho_1))^2
= \beta^2 + 2\alpha\beta \overline{U}(y) + \alpha^2 \overline{U}(y)^2 + O(\beta \rho_1).
\]

Similarly, since Lemma 3.2 gives \( \frac{du}{d\gamma}(y) \geq -1 + \frac{\alpha}{\beta} \geq -1 \), by Lemma 3.1 we infer

\[
|\omega_\varepsilon|^1(\varepsilon)(x) = \beta^{1+\varepsilon} \left( 1 + \frac{\alpha}{\beta} \overline{U}(y) + O(\alpha \rho_1) \right)^{1+\varepsilon}
= \beta^{1+\varepsilon} \left( 1 + (1 + \varepsilon) \left( \frac{\alpha}{\beta} \overline{U}(y) + O(\alpha \rho_1) \right) + O \left( \varepsilon \left( \frac{\alpha}{\beta} \overline{U}(y) + O(\alpha \rho_1) \right)^2 \right) \right)
= \beta^{1+\varepsilon} + (1 + \varepsilon) \alpha \beta \overline{U}(y) + O(\varepsilon^3(1 + \overline{U}^2(y))).
\]

Then the conclusion follows from the second equation in (28). \( \Box \)

3.2 Expansions in \( B(\xi, \rho_0) \)

Let us now restrict our attention to the smaller ball \( B(\xi, \rho_0) \). This allows to control the term \( \alpha^2 \overline{U}^2 \) appearing in the expansion of Lemma 3.3. Indeed, since \( |\overline{U}(y)| = -4 \log |y| + O(1) \) as \( |y| \to +\infty \), we have that

\[
\overline{U} \left( \frac{x - \xi}{\delta} \right) = O \left( \frac{\varepsilon}{\alpha} \right) \quad \text{and} \quad \alpha^2 \overline{U}^2 \left( \frac{x - \xi}{\delta} \right) = O(\varepsilon^2) \quad \text{for} \quad x \in B(\xi, \rho_0).
\] (45)

Lemma 3.4 For \( x \in B(\xi, \rho_0) \), we have

\[
R(x) = \alpha^3 e^{U_\varepsilon(x)} \left( 2 \overline{U} \left( \frac{x - \xi}{\delta} \right) + \overline{U}^2 \left( \frac{x - \xi}{\delta} \right) \right) + \alpha^4 e^{U_\varepsilon(x)} O \left( 1 + \overline{U}^4 \left( \frac{x - \xi}{\delta} \right) \right).
\]

Proof. Set \( y = \frac{x - \xi}{\delta} \). First by Lemma 2.6, Lemma 3.3 and (28)-(32), we get that

\[
\lambda f_\varepsilon(\omega_\varepsilon(x)) = \lambda \beta \left( 1 + \frac{\alpha}{\beta} \overline{U}(y) + O(\rho_1) \right) e^{\omega_\varepsilon^2(x) + \omega_1^{1+\varepsilon}(x)}
= \frac{\alpha}{\delta^2} \left( 1 + 2\alpha^2 \overline{U}(y) + O(\alpha^3(1 + |\overline{U}(y)|))) \right) e^{\overline{U}(y) + \alpha^2 \overline{U}^2(y) + O(\varepsilon^3(1 + \overline{U}^2(y)))}
= \alpha e^{U_\varepsilon(x)} \left( 1 + 2\alpha^2 \overline{U}(y) + O(\alpha^3(1 + |\overline{U}(y)|))) \right) e^{\alpha^2 \overline{U}^2(y) + O(\varepsilon^3(1 + \overline{U}^2(y)))}.
\]

Now, by (16), we can expand the last exponential term, and find

\[
e^{\alpha^2 \overline{U}^2(y) + O(\varepsilon^3(1 + \overline{U}^2(y)))} = 1 + \alpha^2 \overline{U}^2(y) + O(\varepsilon^3(1 + \overline{U}^2(y))) + O(\alpha^4(1 + \overline{U}^4(y)))
= 1 + \alpha^2 \overline{U}^2(y) + O(\varepsilon^3(1 + \overline{U}^2(y))).
\]

We can so conclude that

\[
\lambda f_\varepsilon(\omega_\varepsilon(x)) = \alpha e^{U_\varepsilon(x)} + \alpha^3 e^{U_\varepsilon(x)} (2 \overline{U}(y) + \overline{U}(y)^2) + \alpha^4 e^{U_\varepsilon(x)} O(1 + \overline{U}^4(y)).
\] (46)
Moreover, by (10)-(13), and Lemmas 2.2-2.3 we have
\[ \Delta \omega_\varepsilon = -\alpha e^{U_\varepsilon} + O(1) = -\alpha e^{U_\varepsilon} (1 + O(\alpha) e^{-U_\varepsilon}) = -\alpha e^{U_\varepsilon} (1 + o(\alpha^3)), \] (47)
where in the last equality we used that
\[ e^{-U_\varepsilon(x)} = \frac{(\delta^2 \mu^2 + |x-\xi|^2)^2}{8\delta^2 \mu^2} = O(\delta^2 e^{\frac{\mu}{\varepsilon}}) = o(\alpha^3), \]
for \( x \in B(\xi, \rho_0) \). Thanks to (46) and (47), we conclude that
\[ R(x) = \alpha^3 e^{U_\varepsilon(x)} (2 \bar{U}(y) + \bar{U}^2(y)) + \alpha^4 e^{U_\varepsilon(x)} O(1 + \bar{U}^4(y)). \]

As an immediate consequence of the previous lemma we obtain the estimate:

**Corollary 3.5** We have that
\[ R = O\left( \alpha^3 e^{U_\varepsilon} \left( 1 + \bar{U}^4 \left( \frac{\cdot - \xi}{\delta} \right) \right) \right) \]
in \( B(\xi, \rho_0) \).

### 3.3 Estimates on \( B(\xi, \rho_1) \setminus B(\xi, \rho_0) \)

In this region, it is difficult to provide pointwise estimates of \( R \) because the term \( \alpha^2 \bar{U}^2 \) appearing in the expansion of Lemma 3.3 becomes very large. Then, we will look for integral estimates. Specifically we will show that \( R \) is (very) small in \( L^p(B(\xi, \rho_1) \setminus B(\xi, \rho_0)) \), for a suitable choice of \( p = p(\alpha) \) such that \( p \to 1 \) as \( \varepsilon \to 0 \), uniformly with respect to \( \xi \in B(\xi_0, \sigma), \mu \in \mathcal{U} \).

**Lemma 3.6** There exists \( c_1 > 0 \) such that
\[ 0 \leq \lambda f_\varepsilon(\omega_\varepsilon) \leq \alpha e^{U_\varepsilon} + \alpha^2 (1 + c_1 \varepsilon \alpha) \bar{U}^2 \left( \frac{\xi}{\mu} \right) , \]
in \( B(\xi, \rho_1) \setminus B(\xi, \rho_0) \).

**Proof.** Since \( 0 \leq \omega_\varepsilon \leq \beta \) in \( B(\xi, \rho_1) \setminus B(\xi, \rho_0) \), from Lemma 3.3 and (28) we get
\[ \lambda f_\varepsilon(\omega_\varepsilon) \leq \lambda \beta e^{\beta^2 + \beta^{1+e} + \bar{U}(\frac{\xi}{\mu}) + \alpha^2 \bar{U}^2(\frac{\xi}{\mu})(1+O(\varepsilon \alpha))} \]
\[ = \frac{\alpha}{\delta^2} e^{\bar{U}(\frac{\xi}{\mu}) + \alpha^2 \bar{U}^2(\frac{\xi}{\mu})(1+O(\varepsilon \alpha))} \]
\[ = \alpha e^{U_\varepsilon} + \alpha^2 \bar{U}^2(\frac{\xi}{\mu})(1+O(\varepsilon \alpha)). \]

For \( c_1 \) as in Lemma 3.6 let us consider the function
\[ \Gamma_\varepsilon(x) := e^{U_\varepsilon(x) + \alpha^2 \bar{U}^2(\frac{\xi}{\mu})(1+\varepsilon \alpha)}. \]
Lemma 3.7 Set $p := 1 + \alpha^2$. There exists $c_2 > 0$ such that

$$\|\Gamma_\varepsilon\|_{L^p(B(\xi, \rho_1) \setminus B(\xi, \rho_0))} = O \left( \alpha^{-1} e^{-\frac{4c_0}{\varepsilon^2}} \right).$$

Proof. First of all, we observe that for $q \in \left(\frac{1}{2}, +\infty\right)$, $R > 0$, one has

$$\int_{\mathbb{R}^2 \setminus B(0, R)} e^{\rho \varepsilon} dy \leq \int_{\mathbb{R}^2 \setminus B(0, R)} \frac{(8\mu^2)^q}{|y|^{4q}} dy = \frac{\pi(8\mu^2)^q}{(2q - 1)R^{4q - 2}}. \tag{49}$$

For $x \in B(\xi, \rho_1) \setminus B(\xi, \rho_0)$, set $y = \frac{\xi - x}{\rho_0 \delta} \in B(0, \frac{\rho_0}{\delta}) \setminus B(0, \frac{\rho_1}{\delta})$. Clearly we have

$$\|\Gamma_\varepsilon\|_{L^p(B(\xi, \rho_1) \setminus B(\xi, \rho_0))} = \delta^{\frac{2-2q}{p}} \left( \int_{B(0, \frac{\rho_0}{\delta}) \setminus B(0, \frac{\rho_1}{\delta})} e^{pU(1 + \alpha^2U(1 + c_1 \varepsilon))} dy \right)^{\frac{1}{p}}. \tag{50}$$

Set $\tilde{\rho} = \delta e^{\frac{1}{\varepsilon^2}}$, so that $\rho_0 \ll \tilde{\rho} \ll \rho_1$. For $\frac{\rho_1}{\delta} \leq |y| \leq \frac{\rho_0}{\delta}$, we have

$$p(1 + \alpha^2U(1 + \varepsilon c_1 \alpha)) = 1 + O(\sqrt{\alpha}) \geq \frac{2}{3}.$$

Then, for $\varepsilon$ small enough, (49) yields

$$\int_{B(0, \frac{\rho_0}{\delta}) \setminus B(0, \frac{\rho_1}{\delta})} e^{pU(1 + \alpha^2U(1 + c_1 \varepsilon \alpha))} dy \leq \int_{\mathbb{R}^2 \setminus B(0, \frac{\rho_1}{\delta})} e^{\frac{2}{3}U} dy = O \left( \left( \frac{\rho_1}{\delta} \right)^{-\frac{2}{3}} \right) = O(e^{-\frac{2c_0}{\varepsilon^2}}). \tag{51}$$

For $\frac{\rho_1}{\delta} \leq |y| \leq \frac{\rho_0}{\delta}$, by (30) and Lemma 3.2, we have

$$1 + \alpha^2U(1 + c_1 \varepsilon \alpha) = 1 + \alpha(\beta + \alpha U(y))(1 + c_1 \varepsilon \alpha) - \alpha \beta (1 + c_1 \varepsilon \alpha)$$

$$\geq \frac{1}{2} + (c_0 + u_0(\xi))\alpha + o(\alpha)$$

$$\geq \frac{1}{2} + c_0 \alpha.$$

Hence, we get

$$\int_{B(0, \frac{\rho_0}{\delta}) \setminus B(0, e^{\frac{1}{\varepsilon^2}})} e^{pU(1 + \alpha^2U(1 + c_1 \varepsilon \alpha))} dy \leq \int_{\mathbb{R}^2 \setminus B(0, e^{\alpha - \frac{3}{2}})} e^{(\frac{1}{2} + c_0 \alpha)U(y)} dy = O \left( \alpha^{-\frac{1}{p}} e^{-\frac{4c_0}{\varepsilon^2}} \right). \tag{52}$$

Thus, by (30), (51), (52), we obtain

$$\|\Gamma_\varepsilon\|_{L^p(B(\xi, \rho_1) \setminus B(\xi, \rho_0))} = O \left( \delta^{\frac{2-2q}{p}} \alpha^{-\frac{1}{p}} e^{-\frac{4c_0}{\varepsilon^2}} \right).$$

Since (29) - (31) give

$$\delta^{\frac{2-2q}{p}} = \delta^{-\frac{2a_2}{1 + \alpha^2}} = O(1), \quad \alpha^{\frac{1}{p}} = \alpha^{\frac{1}{1-p}} = \alpha(1 + o(1)), \quad e^{-\frac{4c_0}{\varepsilon^2}} = O(e^{-\frac{4c_0}{\varepsilon^2}}),$$

we get the conclusion. \(\square\)
Lemma 3.8 Let $p$ and $c_2$ be as in Lemma 3.7, then

$$
\|R\|_{L^p(B(\xi, \rho_1) \setminus B(\xi, \rho_0))} = O(e^{-\frac{c_2}{\sqrt{\alpha}}}).
$$

Proof. By Lemma 3.6 and Lemma 3.7 we get that

$$
\|\lambda f_\epsilon(\omega_\epsilon)\|_{L^p(B(\xi, \rho_1) \setminus B(\xi, \rho_0))} = O(e^{-\frac{c_2}{\sqrt{\alpha}}}).
$$

On the other hand, we have

$$
\Delta \omega_\epsilon(x) = -\alpha e^{U_\epsilon(y)} + O(1),
$$

so that

$$
\|\Delta \omega_\epsilon\|_{L^p(B(\xi, \rho_1) \setminus B(\xi, \rho_0))} \leq \alpha \|e^{U_\epsilon}\|_{L^p(B(\xi, \rho_1) \setminus B(\xi, \rho_0))} + O(\rho_1^{\frac{2}{p}})
\leq \alpha \delta^{-2p} \|e^{U}\|_{L^p(\mathbb{R}^2 \setminus (0, \rho_0^2))} + O(\rho_1^{\frac{2}{p}})
= O\left(\frac{\alpha \delta^2}{\rho_0^2}\right) + O(\rho_1^{\frac{2}{p}})
= o(e^{-\frac{c_2}{\sqrt{\alpha}}}).
$$

\square

3.4 Estimates in $B(\xi, \rho_2) \setminus B(\xi, \rho_1)$

In $B(\xi, \rho_2) \setminus B(\xi, \rho_1)$ we can only say that $\omega_\epsilon$ and $R$ are uniformly bounded. Since $\rho_2$ is very small, we still get integral bounds for $R$.

Lemma 3.9 We have $\omega_\epsilon = O(1)$ and $R = O(1)$ in $\Omega \setminus B(\xi, \rho_1)$. In particular,

$$
\|R\|_{L^2(B(\xi, \rho_2) \setminus B(\xi, \rho_1))} = O(\rho_2) = O(e^{-\frac{\alpha}{\rho_0^2}}).
$$

Proof. Let us recall that $\omega_\epsilon = \alpha PU_\epsilon - V_\epsilon$ with $V_\epsilon = V_{\epsilon, \alpha, \xi}$ defined as in (11). According to Lemma 2.2 and Lemma 2.3 we have $V_\epsilon = O(1)$ in $\Omega$. Besides Lemma 2.4 gives

$$
\alpha PU_\epsilon = \alpha \log \left(\frac{1}{(\mu^2 \delta^2 + |x - \xi|^2)^2}\right) + O(\alpha) = O(1)
= O(\alpha \log \frac{1}{\rho_1}) + O(\alpha) = O(1),
$$

for $x \in \Omega \setminus B(\xi, \rho_1)$. Then, $\omega_\epsilon = O(1)$ and $f_\epsilon(\omega_\epsilon) = O(1)$ in $\Omega \setminus B(\xi, \rho_1)$. Similarly

$$
\Delta \omega_\epsilon = -\alpha e^{U_\epsilon} + O(1) = \frac{\alpha \delta^2 \mu^2}{(\delta^2 \mu^2 + |x - \xi|^2)^2} + O(1)
= O(\delta^2 \rho_1^{-4}) + O(1) = O(1).
$$

Therefore $R = O(1)$. \square
3.5 Estimates in $\Omega \setminus B(\xi, \rho_2)$

In $\Omega \setminus B(\xi, \rho_2)$ we will use that $\omega_\varepsilon \sim 8\pi \alpha G_\xi - V_\varepsilon$. Our choice of $V_\varepsilon$ will make $R$ uniformly small, namely of order $\alpha^3$. Note further that the choice of $\rho_2$ gives $\alpha G_\xi = O(\varepsilon)$ on $\Omega \setminus B(\xi, \rho_2)$.

**Lemma 3.10** As $\varepsilon \to 0$ we have

$$\|P_{\varepsilon} - 8\pi G_\xi\|_{C^1(\bar{\Omega} \setminus B(\xi, \rho_2))} = O(\delta^2 \rho_2^{-3}).$$

**Proof.** By Lemma 2.1 we have

$$P_{\varepsilon}(x) = \log \frac{1}{(\delta^2 \mu^2 + |x - \xi|^2)^2} + 8\pi H(x, \xi) + \psi_{\delta, \mu, \xi}$$

$$= -4 \log |x - \xi| + 8\pi H(x, \xi) - 2 \log \left(1 + \frac{\delta^2 \mu^2}{|x - \xi|^2}\right) + \psi_{\delta, \mu, \xi}$$

$$= 8\pi G_\xi(x) - 2 \log \left(1 + \frac{\delta^2 \mu^2}{|x - \xi|^2}\right) + \psi_{\delta, \mu, \xi}$$

Since $\|\psi_{\delta, \mu, \xi}\|_{C^1(\bar{\Omega})} = O(\delta^2)$ as $\varepsilon \to 0$, it is sufficient to observe that

$$\|\log \left(1 + \frac{\delta^2 \mu^2}{|x - \xi|^2}\right)\|_{C^1(\bar{\Omega} \setminus B(\xi, \rho_2))} = O(\delta^2 \rho_2^{-3}).$$

□

**Lemma 3.11** There exists a constant $c > 0$ such such that

$$\omega_\varepsilon(x) \leq -c d(x, \partial \Omega) < 0,$$

for any $x \in \Omega \setminus B(\xi, \rho_2)$, provided $\varepsilon$ is sufficiently small.

**Proof.** By Lemma 2.2, Lemma 2.3 and (11) we have

$$V_\varepsilon(x) \geq c(1 + O(\alpha))d(x, \partial \Omega) \quad \forall x \in \Omega,$$

for some $c > 0$. Then, Lemma 3.10 implies that

$$\omega_\varepsilon(x) \leq -c(1 + O(\alpha))d(x, \partial \Omega)$$

in a neighborhood of $\partial \Omega$. By definiton of $\rho_2$, we have that $P_{\varepsilon} = G_\xi + o(1) = O(\frac{\varepsilon}{\alpha})$ in $\Omega \setminus B(\xi, \rho_2)$. Then, using again Lemma 2.2 and Lemma 2.3, we get $\omega_\varepsilon = -u_0 + o(1)$ uniformly in $\Omega \setminus B(\xi, \rho_2)$. Since $u_0 > 0$ in $\Omega$, this toghether with (53) yields the conclusion. □

**Lemma 3.12** In $\Omega \setminus B(\xi, \rho_2)$, we have $R = O(\alpha^3(1 + G^3_\xi))$. In particular,

$$\|R\|_{L^2(\Omega \setminus B(\xi, \rho_2))} = O(\alpha^3).$$
Since $v_\epsilon > 0$ in $\Omega$, $\omega_\epsilon < 0$ in $\Omega \setminus B(\xi, \rho_2)$, and $f_\epsilon \in C^3((-\infty, 0))$, for any $x \in \Omega \setminus B(\xi, \rho_2)$ we can find $\theta(x) \in [0, 1]$ such that

$$f_\epsilon(\omega_\epsilon) = f_\epsilon(-v_\epsilon + \alpha PU_\epsilon - \alpha w_\epsilon - \alpha^2 z_\epsilon)$$

$$= f_\epsilon(-v_\epsilon) + f'_\epsilon(-v_\epsilon)(\alpha PU_\epsilon - \alpha w_\epsilon - \alpha^2 z_\epsilon) + \frac{1}{2} f''_\epsilon(-v_\epsilon)(\alpha PU_\epsilon - \alpha w_\epsilon - \alpha^2 z_\epsilon)^2$$

$$+ \frac{1}{6} f'''_\epsilon(-v_\epsilon + \theta(\alpha PU_\epsilon - \alpha w_\epsilon - \alpha^2 z_\epsilon))(\alpha PU_\epsilon - \alpha w_\epsilon - \alpha^2 z_\epsilon)^3$$

According to Lemma 2.3 and Lemma 3.10, we have

$$|z_\epsilon| + |w_\epsilon| = O(G_\xi) \quad \text{and} \quad \alpha PU_\epsilon = 8\pi \alpha G_\xi(1 + o(\alpha^3)).$$

Thus we get

$$f_\epsilon(\omega_\epsilon) = -f_\epsilon(v_\epsilon) + \alpha f'_\epsilon(v_\epsilon)(8\pi G_\xi - w_\epsilon) + \alpha^2 \left( \frac{1}{2} f''_\epsilon(-v_\epsilon)(8\pi G_\xi - w_\epsilon)^2 - f'_\epsilon(v_\epsilon)z_\epsilon \right)$$

$$+ O(\alpha^3(1 + G_\xi^3)) + O(\alpha^3|f'''_\epsilon(-v_\epsilon + \theta(\alpha PU_\delta - \alpha w_\delta - \alpha^2 z_\delta))(G_\xi^3)|).$$

A direct computation shows the existence of a constant $C > 0$ such that

$$|f'''_\epsilon(t)| \leq C(|t|^{\epsilon-1} + t^4)e^{t^2+|t|^{1+\epsilon}} \quad \forall t \neq 0.$$ 

Since $-v_\epsilon + \theta(\alpha PU_\epsilon - \alpha w_\epsilon - \alpha^2 z_\epsilon) = O(1)$ uniformly in $\Omega \setminus B(\xi, \rho_2)$, and since Lemma 3.10 implies $-v_\epsilon + \theta(\alpha PU_\epsilon + \alpha w_\epsilon + \alpha^2 z_\epsilon) \leq -c d(\cdot, \partial \Omega)$ in a neighborhood of $\partial \Omega$, we get

$$|f'''_\epsilon(-v_\epsilon + \theta(\alpha PU_\delta - \alpha w_\delta - \alpha^2 z_\delta))| = O(1 + d(\cdot, \partial \Omega)^{\epsilon-1}).$$

Since $G_\xi = O(d(\cdot, \partial \Omega))$ near $\partial \Omega$, we deduce that

$$f_\epsilon(\omega_\epsilon) = -f_\epsilon(v_\epsilon) + \alpha f'_\epsilon(v_\epsilon)(8\pi G_\xi - w_\epsilon) + \alpha^2 \left( \frac{1}{2} f''_\epsilon(-v_\epsilon)(8\pi G_\xi - w_\epsilon)^2 - f'_\epsilon(v_\epsilon)z_\epsilon \right)$$

$$+ O(\alpha^3(1 + G_\xi^3)).$$

Since by construction we have $\Delta \omega_\epsilon = -\alpha e^{U_\epsilon} - \Delta v_\epsilon - \alpha \Delta w_\epsilon - \alpha^2 \Delta z_\epsilon$, with $v_\epsilon$, $w_\epsilon$, $z_\epsilon$ solving (4) and (12)-(13), we conclude that

$$R = -\alpha e^{U_\epsilon} + O(\alpha^3(1 + G_\xi^3))$$

$$= O(\delta^2 \rho_2^{-4}) + O(\alpha^3(1 + G_\xi^3))$$

$$= O(\alpha^3(1 + G_\xi^3)).$$
3.6 The final estimate of the error in a mixed norm

We can summarize the estimates of the previous sections as follows:

In \( B(\xi, \rho_0) \), Corollary 3.5 gives

\[
|R| \leq \alpha^3 j_\varepsilon(x)
\]

where

\[
j_\varepsilon(x) := e^{\bar{U}(x)} \left( 1 + |\bar{U}(x - \xi)|^4 \right).
\]

In \( B(\xi, \rho_1) \setminus B(\xi, \rho_0) \), Lemma 3.8 shows that the norm of \( R \) in \( L^{1 + \alpha^2} \) is exponentially small in \( \alpha \).

Finally, in \( \Omega \setminus B(\xi, \rho_1) \), Lemma 3.9 and Lemma 3.12 give \( L^2 \) estimates on \( R \). This suggests to introduce the norm

\[
\|f\|_\varepsilon := \|j_\varepsilon^{-1} f\|_{L^\infty(B(\xi, \rho_0))} + \frac{1}{\alpha^2} \|f\|_{L^{1 + \alpha^2}(B(\xi, \rho_1) \setminus B(\xi, \rho_0))} + \|f\|_{L^2(\Omega \setminus B(\xi, \rho_1))}.
\]

The coefficient \( \frac{1}{\alpha^2} \) is chosen in order to match the norm of \( (-\Delta)^{-1} \) as a linear operator from \( L^{1 + \alpha^2}(B(\xi, \rho_1) \setminus B(\xi, \rho_0)) \) into \( L^{\infty}(B(\xi, \rho_1) \setminus B(\xi, \rho_0)) \) (see Corollary B.4).

According to the estimates above we have:

**Proposition 3.13** There exists \( D_1 > 0, \varepsilon_0 > 0 \) such that

\[
\|R\|_\varepsilon \leq D_1 \alpha^3,
\]

for any \( \varepsilon \in (0, \varepsilon_0) \), \( \mu \in U \), \( \xi \in B(\xi_0, \sigma) \).

We conclude this section by stating some simple properties of the norm \( \| \cdot \|_\varepsilon \) and the weight \( j_\varepsilon \).

**Lemma 3.14** There exists a constant \( C > 0 \) such that

\[
\| \cdot \|_{L^1(\Omega)} \leq C \| \cdot \|_\varepsilon
\]

for any \( \varepsilon > 0 \), \( \mu \in U \), \( \xi \in B(\xi_0, \sigma) \).

**Proof.** Let \( f : \Omega \rightarrow \mathbb{R} \) be a Lebesgue measurable function. Then

\[
\|f\|_{L^1(B(\xi, \rho_0))} \leq \|f\|_\varepsilon \int_{B(\xi, \rho_0)} j_\varepsilon dx = \|f\|_\varepsilon \int_{B(0, \rho_0 / 2)} e^\bar{U}(1 + \bar{U}^4) dy \leq C \|f\|_\varepsilon.
\]

By Hölder’s inequality

\[
\|f\|_{L^1(B(\xi, \rho_1) \setminus B(\xi, \rho_0))} \leq \|f\|_{L^{1 + \alpha^2}(B(\xi, \rho_1) \setminus B(\xi, \rho_0))} \rho_1^{\frac{2\alpha^2}{1 + \alpha^2}} \leq C \|f\|_\varepsilon,
\]

and

\[
\|f\|_{L^1(\Omega \setminus B(\xi, \rho_1))} \leq \|f\|_{L^2(\Omega \setminus B(\xi, \rho_1))} \Omega \setminus B(\xi, \rho_1)^{\frac{1}{2}} \leq C \|f\|_\varepsilon.
\]

Hence, the conclusion follows.\( \square \)
Lemma 3.15 For any $\varepsilon > 0$ let $\rho_\varepsilon, \sigma_\varepsilon$ be such that $\rho_2 \leq \sigma_\varepsilon \leq \sigma$ and $\delta \ll \rho_\varepsilon \leq \rho_0$ as $\varepsilon \to 0$. Let $\varphi_\varepsilon$ be the solution to

\[
\begin{align*}
-\Delta \varphi_\varepsilon &= j_\varepsilon & \text{in } B(\xi, \sigma_\varepsilon) \setminus B(\xi, \rho_\varepsilon), \\
\varphi_\varepsilon &= 0 & \text{on } \partial B(\xi, \sigma_\varepsilon) \setminus B(\xi, \rho_\varepsilon).
\end{align*}
\]

As $\varepsilon \to 0$, we have

$$
\|\varphi_\varepsilon\|_{\mathcal{L}^\infty(B(\xi, \sigma_\varepsilon) \setminus B(\xi, \rho_\varepsilon))} = o(1).
$$

Proof. Let us first note that there exists a constant $c > 0$, such that

\[
\delta^2 j_\varepsilon(\xi + \delta \cdot) = e^C (1 + U^4) = \frac{8\mu^2}{(\mu^2 + |\cdot|^2)^2} \left(1 + \log 4 + \frac{8\mu^2}{(\mu^2 + |\cdot|^2)^2}\right) \leq c \frac{\mu}{(\mu^2 + |\cdot|^2)^2}
\]

in $\mathbb{R}^2$. Then, by the maximum principle, we have

$$
|\varphi_\varepsilon| \leq c\psi \left(\frac{\cdot - \xi}{\delta}\right) \quad \text{in } B(\xi, \sigma_\varepsilon) \setminus B(\xi, \rho_\varepsilon),
$$

(56)

where $\psi$ satisfies

\[
\begin{align*}
-\Delta \psi &= \frac{\mu}{(\mu^2 + |\cdot|^2)^2} & \text{in } A_\varepsilon := B(0, \frac{\rho_\varepsilon}{\delta}) \setminus B(0, \frac{\rho_0}{\delta}), \\
\psi &= 0 & \text{on } \partial A_\varepsilon.
\end{align*}
\]

Since the function $W := -\log(\mu + \sqrt{\cdot^2 + \mu^2})$ satisfies $-\Delta W = \frac{\mu}{(\mu^2 + |\cdot|^2)^2}$, we have

$$
\psi = a + b \log |\cdot| + W,
$$

for suitable constants $a, b \in \mathbb{R}$. Denoting $R_1 = \frac{\rho_\varepsilon}{\delta}$ and $R_2 = \frac{\rho_0}{\delta}$, one can verify that

$$
a = \frac{W(R_2) \log R_1 - W(R_1) \log R_2}{\log R_2 - \log R_1} \quad \text{and} \quad b = \frac{W(R_1) - W(R_2)}{\log R_2 - \log R_1}.
$$

Since

$$
|W + \log |\cdot|| \leq \frac{C\mu}{\delta} = O \left(\frac{1}{R_1}\right),
$$

uniformly in $A_\varepsilon$, one has $a = O \left(\frac{\log R_2}{R_1(\log R_2 - \log R_1)}\right)$ and $b = 1 + O \left(\frac{1}{R_1(\log R_2 - \log R_1)}\right)$. Then

\[
\begin{align*}
\psi &= a + (b - 1) \log |\cdot| + O \left(\frac{1}{R_1}\right) \\
&= O \left(\frac{1}{R_1} \frac{\log R_2}{\log R_2 - \log R_1}\right) + O \left(\frac{1}{R_1}\right) \\
&= O \left(\frac{1}{R_1} \frac{1}{1 - \frac{\log R_1}{\log R_2}}\right) + O \left(\frac{1}{R_1}\right).
\end{align*}
\]

Since

$$
\frac{\log R_1}{\log R_2} = \frac{\log \rho_0}{\log \sigma_\varepsilon - \log \delta} \leq \frac{\log \rho_0}{\log \rho_2 - \log \delta} = O(\alpha),
$$

we conclude that $\psi_\varepsilon = O(\frac{1}{R_1}) = o(1)$, uniformly in $A_\varepsilon$. Then, the conclusion follows by (56).
4 The Linear Theory

Let us consider the linear operator

\[ L \varphi = \varphi - (-\Delta)^{-1}(\lambda f'(\omega_\varepsilon)\varphi) \]

introduced in \([16]\). In this section we give a priori estimates for the operator \(L\) and we prove its invertibility on a suitable subspace of \(H^1_0(\Omega)\).

**Lemma 4.1** The following expansions hold:

1. \( \lambda f'(\omega_\varepsilon) = e^{U_\varepsilon}(1 + O(\varepsilon^2)) \) in \( B(\xi, \rho_0) \).
2. \( \lambda f'(\omega_\varepsilon) = O(\Gamma_\varepsilon) \) in \( B(\xi, \rho_1) \), with \( \Gamma_\varepsilon \) as in \([18]\).
3. \( \lambda f'(\omega_\varepsilon) = O(1) \) in \( \Omega \setminus B(\xi, \rho_1) \).
4. \( \|\lambda f'(\omega_\varepsilon)\chi_{B(\xi,\rho_1)} - e^{U_\varepsilon}\|_\varepsilon = o(1) \) as \( \varepsilon \to 0 \).

**Proof.** For \( x \in B(\xi, \rho_0) \), using \([25-32]\), Lemma \([3.3]\), \([3.9]\), and \([4.5]\), we have that

\[ \lambda f'(\omega_\varepsilon) = \lambda (1 + 2\varepsilon^2 + (1 + \varepsilon)\omega_\varepsilon^{1+\varepsilon})e^{2+\omega_\varepsilon^{1+\varepsilon}} = \lambda \beta^2 (2 + O(\alpha))e^{\beta^2 + \beta^{1+\varepsilon} + U(\varepsilon)} + O(\varepsilon^2) = e^{U_\varepsilon} (1 + O(\varepsilon^2)). \]

For \( x \in B(\xi, \rho_1) \), using Remark \([2.7]\) Lemma \([3.3]\) we have

\[ \lambda f'(\omega_\varepsilon) = \lambda (1 + 2\varepsilon^2 + (1 + \varepsilon)\omega_\varepsilon^{1+\varepsilon})e^{2+\omega_\varepsilon^{1+\varepsilon}} = \lambda \beta^2 (2 + O(\alpha))e^{\beta^2 + \beta^{1+\varepsilon} + U(\varepsilon)} + U(\varepsilon) + O(\varepsilon^2) = O(\Gamma_\varepsilon). \]

Claim 3 follows directly from Lemma \([3.9]\). Finally, claim 4 follows by claims 1 and 2, using also Lemma \([4.7]\) and the estimates

\[ \|e^{U_\varepsilon}\|_{L^{1+\alpha}(B(\xi,\rho_1)\setminus B(\xi,\rho_0))} = o(1), \quad \|e^{U_\varepsilon}\|_{L^{2}(\Omega\setminus B(\xi,\rho_1))} = o(1). \]

According to Lemma \([4.1]\) for \( |x - \xi| \leq \rho_0 \), \( L \) approaches the operator \( L_0 \phi := \varphi - (-\Delta)^{-1}(e^{U_\varepsilon}\phi) \). Note that

\[ L_0 \varphi = 0 \quad \text{in} \; \Omega \quad \Leftrightarrow \quad -\Delta \phi = e^{U_\varepsilon} \phi \quad \text{in} \; \Omega \]

\[ \Leftrightarrow \quad -\Delta \Phi = e^{U_\varepsilon} \Phi \quad \text{in} \; \frac{\Omega - \xi}{\delta}, \text{ where } \Phi = \varphi(\xi + \delta \cdot). \]

Let us recall the following known fact about \( L_0 \) (see for example \([10]\)).
Proposition 4.2 All bounded weak solutions of the problem
\[ -\Delta \Phi = e^{\bar{U}} \Phi \quad \text{in } \mathbb{R}^2 \] (57)
have the form
\[ \Phi = c_0 Z_0 + c_1 Z_1 + c_2 Z_2, \]
where \( c_0, c_1, c_2 \in \mathbb{R} \) and
\[
Z_0(y) := \frac{\mu^2 - |y|^2}{\mu^2 + |y|^2}, \quad Z_1(y) := \frac{2\mu y_1}{\mu^2 + |y|^2}, \quad Z_2(y) := \frac{2\mu y_2}{\mu^2 + |y|^2}.
\]

Remark 4.3 The functions \( Z_0, Z_1, Z_2 \) are orthogonal in \( D^{1,2}(\mathbb{R}^2) \), that is
\[
\int_{\mathbb{R}^2} \nabla Z_i \cdot \nabla Z_j dy = \int_{\mathbb{R}^2} e^{\bar{U}} Z_i Z_j dy = \frac{8}{3} \pi \delta_{i,j}. \quad (58)
\]

In the following we denote
\[ Z_{i,\varepsilon}(x) := Z_i \left( \frac{x - \xi}{\delta} \right) \quad \text{and} \quad PZ_{i,\varepsilon} = (-\Delta)^{-1} Z_{i,\varepsilon}, \quad i = 0, 1, 2. \]

Lemma 4.4 It holds true that
\[ PZ_{0,\varepsilon} = Z_{0,\varepsilon} + 1 + O(\delta^2) \] and \( PZ_{i,\varepsilon} = Z_{i,\varepsilon} + O(\delta), \quad i = 1, 2, \]
uniformly with respect to \( \mu \in U, \xi \in B(\xi_0, \sigma) \).

Proof. See for example Appendix A in [18]. \( \square \)

Lemma 4.4 shows the smallness of \( PZ_{i,\varepsilon} - Z_{i,\varepsilon} \) for \( i = 1, 2 \), but not for \( i = 0 \). For this reason, in many cases it is convenient to replace \( PZ_{0,\varepsilon} \) with the function
\[
\tilde{Z}_\varepsilon := \begin{cases} Z_{0,\varepsilon} & \text{if } |x - \xi| \leq \rho_0, \\ Z_{0,\varepsilon}(\rho_0)(\log \frac{\rho_1 - \log |x - \xi|}{\log \rho_1 - \log \rho_0}) & \text{if } \rho_0 \leq |x - \xi| \leq \rho_1, \\ 0 & \text{if } |x - \xi| \geq \rho_1. \end{cases} \quad (59)
\]

Lemma 4.5 The function \( \tilde{Z}_\varepsilon \) satisfies the following properties:
\begin{itemize}
  \item \( \tilde{Z}_\varepsilon \in H^1_0(\Omega) \) and \( |\tilde{Z}_\varepsilon| \leq 1 \) in \( \Omega \).
  \item \( \|\nabla (\tilde{Z}_\varepsilon - Z_{0,\varepsilon})\|_{L^2(\Omega)} \to 0 \), uniformly for \( \mu \in U \) and \( \xi \in B(\xi_0, \sigma) \).
\end{itemize}

Proof. The first property follows trivially from the definition. Moreover we have
\[
\|\nabla (\tilde{Z}_\varepsilon - Z_{0,\varepsilon})\|_{L^2(\Omega)}^2 \leq \frac{Z_{0,\varepsilon}(\rho_0)^2}{(\log \rho_1 - \log \rho_0)^2} \int_{B(\xi_0) \setminus B(\xi,\rho_0)} \frac{1}{|x - \xi|^2} dx + \|\nabla Z_{0,\varepsilon}\|_{L^2(\Omega \setminus B(\xi,\rho_0))}^2 \\
\leq \frac{2\pi Z_{0,\varepsilon}(\rho_0)^2}{\log \rho_1 - \log \rho_0} + \|\nabla Z_0\|_{L^2(\mathbb{R}^2 \setminus B(0, \rho_0))}^2 \\
= O(\alpha^2) + O(e^{-\alpha}) \to 0,
\]

22
Proof. We assume by contradiction that there exists $h_\varepsilon$ for any $\varepsilon > 0$. Proposition 4.6 There exist $\varepsilon > 0$ and a constant $D_0 > 0$ such that
\[
\|\varphi\|_{H^1_0(\Omega)} + \|\varphi\|_{L^\infty(\Omega)} \leq D_0 \|h_\varepsilon\|,
\]
for any $\varepsilon \in (0, \varepsilon_0)$, $\mu \in U$, $\xi \in B(\xi_0, \sigma)$, $h \in Y_\varepsilon$ and $\varphi \in K^\perp_\varepsilon$ satisfying
\[
\pi^\perp \{ L\varphi - (-\Delta)^{-1} h \} = 0.
\]
Proof. We assume by contradiction that there exists $\varepsilon_n \to 0$, $\mu_n \in U$, $\xi_n \in B(\xi_0, \sigma)$, $h_n \in Y_\varepsilon$ and a solution $\varphi_n \in K^\perp_{\varepsilon_n}$ of (61) such that
\[
\frac{\|\varphi_n\|_{H^1_0(\Omega)} + \|\varphi_n\|_{L^\infty(\Omega)}}{\|h_n\|_{\varepsilon_n}} \to +\infty.
\]
Let $\delta_n, \alpha_n, \beta_n$ be the parameters in Lemma 4.5 corresponding to $\varepsilon_n, \mu_n$ and $\xi_n$. Let also $\rho_{0,n}, \rho_{1,n}, \rho_{2,n}$ be defined as in (39). We denote $\omega_n := \omega_{\varepsilon_n}, U_n := U_{\varepsilon_n}, Z_{i,n} := Z_{i,\varepsilon_n}$ and $f_n := f_{\varepsilon_n}$. W.l.o.g we can assume that $\|\varphi_n\|_{H^1_0(\Omega)} + \|\varphi_n\|_{L^\infty(\Omega)} = 1$ and $\|h_n\|_{\varepsilon_n} \to 0$. Since $\varphi_n$ satisfies (61), there exist $c_{i,n} \in \mathbb{R}$, $i = 0, 1, 2$, such that
\[
-\Delta \varphi_n - \lambda f'_n(\omega_n) \varphi_n = h_n + \sum_{i=0}^{2} c_{i,n} e^{U_n} Z_{i,n}.
\]
Step 1 We have $c_{i,n} \to 0$ as $n \to +\infty$, $i = 0, 1, 2$.

Let $\tilde{Z}_n := \tilde{Z}_{\varepsilon_n}$ be the function defined in (59). Testing equation (62) against $\tilde{Z}_n$, we get
\[
\sum_{j=0}^{2} c_{j,n} \int_{\Omega} e^{U_n} Z_{j,n} \tilde{Z}_n dx = \int_{\Omega} \nabla \tilde{Z}_n \cdot \nabla \varphi_n dx - \int_{\Omega} \lambda f'_n(\omega_n) \varphi_n \tilde{Z}_n dx - \int_{\Omega} h_n \tilde{Z}_n dx.
\]
Since $\|\varphi_n\|_{H^1_0(\Omega)} \leq 1$ and $\varphi_n \in K^\perp_{\varepsilon_n}$, using Lemma 2.2 we get
\[
\int_{\Omega} \nabla \tilde{Z}_n \cdot \nabla \varphi_n dx = \int_{\Omega} \nabla Z_{0,n} \cdot \nabla \varphi_n dx + o(1) = \int_{\Omega} e^{U_n} Z_{0,n} \varphi_n dx + o(1) + o(1) = o(1),
\]
23
as \( n \to +\infty \). By Lemma 4.1 and Lemma 3.7, we find
\[
\int_{\Omega} \lambda f_n'(\omega_n) \varphi_n \tilde{Z}_n dx = \int_{B(\xi_n, \rho_0, n)} e^{U_n} \varphi_n Z_{0,n} dx + O(\varepsilon_n^2) + O \left( \| \Gamma \varphi \|_{L^1(B(\xi_n, \rho_0, n) \setminus B(\xi_n, \rho_0, n))} \right)
\]
\[
= \int_{\Omega} e^{U_n} \varphi_n Z_{0,n} dx + o(1) = o(1).
\]

Finally, Lemma 4.5 and Lemma 3.14 give
\[
| \int_{\Omega} h_n \tilde{Z}_n dx | \leq \| h_n \|_{L^1(\Omega)} \leq C \| h_n \|_{\varepsilon_n} = o(1).
\]

Then (63) rewrites as
\[
\sum_{j=0}^{2} c_{j,n} \int_{\Omega} e^{U_n} Z_{j,n} \tilde{Z}_n dx = o(1).
\] (64)

With similar arguments, testing equation (62) against \( PZ_{i,n} \) for \( i = 1, 2 \), we get that
\[
\sum_{j=0}^{2} c_{j,n} \int_{\Omega} e^{U_n} Z_{j,n} PZ_{i,n} dx = - \int_{\Omega} \lambda f_n'(\omega_n) \varphi_n PZ_{i,n} dx - \int_{\Omega} h_n PZ_{i,n} dx
\]
\[
= \int_{\Omega} e^{U_n} \varphi_n Z_{i,n} dx + o(1) = o(1).
\] (65)

Note that, as in (58), we have
\[
\int_{\Omega} e^{U_n} Z_{j,n} \tilde{Z}_n dx = \int_{B(\xi_n, \rho_0, n)} e^{U_n} Z_{j,n} Z_{0,n} dx + O \left( \int_{\mathbb{R}^2 \setminus B(\xi_n, \rho_0, n)} e^{U_n} \right)
\]
\[
= \int_{B(0, \frac{\rho_0}{n})} e^{U} Z_j Z_0 dy + o(1)
\]
\[
= \frac{8}{3} \pi \delta_{ij} + o(1),
\]
for \( j = 0, 1, 2 \). Similarly
\[
\int_{\Omega} e^{U_n} Z_{j,n} PZ_{i,n} dx = \int_{\Omega} e^{U_n} Z_{j,n} Z_{i,n} dx + o(1)
\]
\[
= \frac{8}{3} \pi \delta_{ij} + o(1),
\]
for \( i = 1, 2, j = 0, 1, 2 \). Then, (63) and (64) rewrite as
\[
\sum_{j=0}^{2} c_{j,n} (\delta_{ij} + o(1)) = o(1),
\]
which implies the conclusion.
Step 2 If \( \bar{h}_n := h_n + (\lambda f_n'(\omega_n)\chi_{B(\xi_n,\rho_{1,n})} - e^{U_n}) \varphi_n + \sum_{j=0}^2 c_{j,n} e^{U_n} Z_{j,n} \), then

\[-\Delta \varphi_n = e^{U_n} \varphi_n + \lambda f_n'(\omega_n)\chi_{\Omega \setminus B(\xi_n,\rho_{1,n})} \varphi_n + \bar{h}_n \text{ in } \Omega, \quad \text{and } \|\bar{h}_n\|_{\infty} \to 0. \quad (66)\]

Since \( \|h_n\|_{\infty} \to 0, |Z_{i,n}| \leq 1, \) and \( \|\lambda f_n'(\omega_n)\chi_{B(\xi_n,\rho_{1,n})} - e^{U_n}\|_{\infty} \to 0 \) by Lemma 4.1, it is sufficient to observe that \( \|e^{U_n}\|_{\infty} = O(1) \) and apply Step 1.

Step 3 There exists \( \delta_n \ll \rho_n \leq \rho_{0,n} \) such that, up to a subsequence, \( \|\varphi_n\|_{L^\infty(B(\xi_n,\rho_{0,n}))} \to 0 \) as \( n \to +\infty. \)

Let us consider the sequence \( \Phi_n(y) := \varphi_n(\xi_n + \delta_n y), y \in \Omega - \xi_n. \) By (66) \( \Phi_n \) satisfies

\[-\Delta \Phi_n = e^{\bar{U}} \Phi_n + \delta_n^2 \bar{h}_n(\xi + \delta_n \cdot) \text{ in } B(0, \frac{\rho_{1,n}}{\delta_n}). \]

We know that

\[|e^{\bar{U}}(\Phi_n(y))| \leq e^{\bar{U}}(y) \leq \frac{8}{\mu^2},\]

and, for \( y \in B(0, \frac{\rho_{0,n}}{\delta_n}) \), that

\[\delta_n^2 |\bar{h}_n(\xi + \delta_n y)| \leq \delta_n^2 j_{\xi_n}(\xi + \delta_n y)\|\bar{h}_n\|_{\infty} = e^{\bar{U}}(y)(1 + |\bar{U}(y)|^4)\|\bar{h}_n\|_{\infty} \leq C\|\bar{h}_n\|_{\infty} \to 0.\]

In particular \( \Phi_n \) and \( \Delta \Phi_n \) are uniformly bounded in \( B(0, \frac{\rho_{0,n}}{\delta_n}) \). By standard elliptic estimates, we can find \( \Phi_0 \in C(\mathbb{R}^2) \cap H^1_{\text{loc}}(\mathbb{R}^2) \) and a sequence \( R_n \to +\infty, R_n \leq \frac{\rho_{0,n}}{\delta_n} \), such that, up to a subsequence, \( \|\Phi_n - \Phi_0\|_{L^\infty(B(0,R_n))} \to 0. \) Moreover, \( |\Phi_0| \leq 1 \) and \( \Phi_0 \) is a weak solution to

\[-\Delta \Phi_0 = e^{\bar{U}} \Phi_0 \text{ in } \mathbb{R}^2.\]

According to Proposition 4.2, we must have \( \Phi_0 = \kappa_0 Z_0 + \kappa_1 Z_1 + \kappa_2 Z_2 \), for some \( \kappa_i \in \mathbb{R}, \) \( i = 0,1,2. \) Keeping in mind (58) and using that \( e^{\bar{U}} \in L^1(\mathbb{R}^2) \), we obtain

\[0 = \int_{\Omega} e^{U_n} Z_{i,n} \varphi_n \, dx = \int_{\Omega - \xi_n} e^{\bar{U}} Z_{i,n} \Phi_n \, dy = \int_{B(0,R_n)} e^{\bar{U}} Z_{i,n} \Phi_n \, dy + O\left(\int_{\mathbb{R}^2 \setminus B(0,R_n)} e^{\bar{U}} \, dy\right) \to \frac{8}{3} \pi \kappa_i, \]

for \( i = 0,1,2. \) This implies \( \kappa_i = 0, i = 0,1,2. \) Then \( \Phi_0 \equiv 0 \) and we get the conclusion with \( \rho_n = \delta_n R_n. \)

Step 4 Up to a subsequence, \( \xi_n \to \bar{\xi} \in \Omega \) and \( \varphi_n \to 0 \) in \( L^\infty_{\text{loc}}(\Omega \setminus \{\bar{\xi}\}), \) as \( n \to \infty. \)

We know that \( \varphi_n \) satisfies (66) in \( \Omega. \) Since \( |\varphi_n| \leq 1, \) \( \|e^{U_n}\|_{L^\infty(\Omega \setminus B(\xi_n,\rho_{1,n}))} \to 0, \) \( \|h_n\|_{L^2(\Omega \setminus B(\xi_n,\rho_{1,n}))} \to 0, \) and \( \|f_n'(\omega_n)\|_{L^\infty(\Omega \setminus B(\xi_n,\rho_{1,n}))} = O(1), \) by elliptic estimates we find that \( \varphi_n \) is bounded in \( C^0_{\text{loc}}(\Omega \setminus \{\bar{\xi}\}), \) for some \( \gamma \in (0,1). \) Therefore, there exists \( \varphi_0 \in C(\bar{\Omega}) \cap H^1_0(\Omega), \) such that \( \varphi_n \to \varphi_0 \) locally uniformly on \( \bar{\Omega} \setminus \{\bar{\xi}\} \) and weakly in
\( H^1_0(\Omega) \). Noting that \( \omega_n \to -u_0 \) locally uniformly in \( \Omega \setminus \{\xi\} \) and that \( f'_n \) is even, we see that \( \varphi_0 \) satisfies \( \Delta \varphi_0 + f'_0(u_0)\varphi_0 \) in \( \Omega \setminus \{\xi\} \). Actually, since \( \varphi_0, \Delta \varphi_0 \in L^\infty(\Omega) \), \( \varphi_0 \) is a weak solution of \( \Delta \varphi_0 + f'_0(u_0)\varphi_0 = 0 \) in \( \Omega \). Then, the non-degeneracy of \( u_0 \) implies \( \varphi_0 \equiv 0 \).

**Step 5** \( \|\varphi_n\|_{L^\infty(\Omega)} \to 0 \).

By Step 4, we can find a sequence \( \sigma_n \geq \rho_{2,n} \) such that \( \|\varphi_n\|_{L^\infty(\Omega\setminus B(\xi,\sigma_n))} \to 0 \) as \( n \to +\infty \), up to a subsequence. Then, it is sufficient to show that \( \|\varphi_n\|_{L^\infty(A_n)} \to 0 \), where \( A_n := B(\xi_n, \sigma_n) \setminus B(\xi_n, \rho_n) \) and \( \rho_n \) is as in Step 3. We can split \( \varphi_n = \varphi_n^{(0)} + \varphi_n^{(1)} + \varphi_n^{(2)} + \varphi_n^{(3)} \), where

\[
\begin{cases}
\Delta \varphi_n^{(0)} = 0 & \text{in } A_n, \\
\varphi_n^{(0)} = \varphi_n & \text{on } \partial A_n,
\end{cases}
\quad
\begin{cases}
-\Delta \varphi_n^{(i)} = f_i,n & \text{in } A_n, \\
\varphi_n^{(i)} = 0 & \text{on } \partial A_n, \quad \text{for } i = 1, 2, 3,
\end{cases}
\]

with

\[
\begin{align*}
f_{1,n} &:= e^{U_n} \varphi_n + \tilde{h}_n \chi B(\xi_n, \rho_{0,n}), \\
f_{2,n} &:= \tilde{h}_n \chi B(\xi_n, \rho_{1,n}) \setminus B(\xi_n, \rho_n), \\
f_{3,n} &:= \tilde{h}_n \chi B(\xi_n, \xi_n, \rho_{1,n}) \setminus B(\xi_n, \rho_{1,n}) \varphi_n.
\end{align*}
\]

By the maximum principle

\[
\|\varphi_n^{(0)}\|_{L^\infty(A_n)} \leq \|\varphi_n\|_{L^\infty(\partial A_n)} \to 0.
\]

Since

\[
|f_{1,n}| \leq e^{U_n} + \|\tilde{h}_n\|_{\infty} \leq j_{\varepsilon_n}(1 + o(1)) \leq 2j_{\varepsilon_n},
\]

we get that \( |\varphi_n^{(1)}| \leq 2\psi_n \), where \( \psi_n \) satisfies

\[
\begin{cases}
-\Delta \psi_n = j_{\varepsilon_n} & \text{in } A_n, \\
\psi_n = 0 & \text{on } \partial A_n.
\end{cases}
\]

Lemma \ref{lemma:maximum_principle} implies \( \|\psi_n\|_{L^\infty(A_n)} \to 0 \), hence \( \|\varphi_n^{(1)}\|_{L^\infty(A_n)} \to 0 \). Finally, since \( |A_n| \) is uniformly bounded, elliptic estimates (see Corollaries \ref{corollary:elliptic_estimates_1} and \ref{corollary:elliptic_estimates_2}) give

\[
\|\varphi_n^{(2)}\|_{L^\infty(A_n)} \leq C \|f_{2,n}\|_{L^{1+\alpha/2}(A_n)} = \frac{C}{\alpha^2} \|\tilde{h}_n\|_{L^{1+\alpha/2}(B(\xi_n, \rho_{1,n}) \setminus B(\xi_n, \rho_n))} \leq \|\tilde{h}_n\|_{\infty} \to 0,
\]

and

\[
\|\varphi_n^{(3)}\|_{L^\infty(A_n)} \leq C \|f_{3,n}\|_{L^2(A_n)} = O(\|\tilde{h}_n\|_{\infty}) + O(\sqrt{\sigma_n}) \to 0.
\]

**Step 6** *Conclusion of the proof.*

By Step 5, we have that \( \|\varphi_n\|_{H^1_0(\Omega)} = 1 - \|\varphi_n\|_{L^\infty(\Omega)} \to 1 \). But \eqref{eq:elliptic_estimate} gives

\[
\|\varphi_n\|_{H^1_0(\Omega)}^2 = \int e^{U_n} \varphi_n^2 \, dx + \int_{\Omega \setminus B(\xi_n, \rho_{1,n})} \lambda f'_n(\omega_n) \varphi_n^2 \, dx + \int \tilde{h}_n \varphi_n \, dx
\]

\[
= O(\|\varphi_n\|_{L^\infty(\Omega)}^2) + o(\|\varphi_n\|_{L^2(\Omega)}) \to 0.
\]

Then, we get a contradiction. \( \square \)

As a consequence we have that \( \pi^\perp L \) is invertible on \( K_{\varepsilon}^\perp \).
Corollary 4.7 \( \pi^\perp L : K^\perp_{\varepsilon} \rightarrow K^\perp_{\varepsilon} \) is invertible.

Proof. This follows by standard Fredholm theory. Indeed, for any \( \varepsilon > 0 \) the map \( F(\varphi) := \pi^\perp(-\Delta)^{-1}(f'(\omega_\varepsilon)\varphi) \) defines a compact operator on \( K^\perp_{\varepsilon} \) (in fact on \( H^1_0(\Omega) \)). Then \( \pi^\perp L = \text{Id}_{K^\perp_{\varepsilon}} - F \) is a Fredholm operator of index 0. Proposition 4.6 implies that \( \pi^\perp L \) is injective, hence it is invertible on \( K^\perp_{\varepsilon} \).

\[ \square \]

5 The reduction to a finite dimensional problem

This section is devoted to reduce the problem to a finite dimensional one. More precisely, we prove:

Proposition 5.1 There exist \( \varepsilon_0 > 0 \) and a map \( (\varepsilon, \mu, \xi) \rightarrow \varphi_{\varepsilon, \mu, \xi} \in K^\perp_{\varepsilon} \cap L^\infty(\Omega) \) defined in \( (0, \varepsilon_0) \times U \times B(\xi_0, \sigma) \) and continuous with respect to \( \mu \) and \( \xi \), such that for some \( D > 0 \)

\[
\| \varphi_{\varepsilon, \mu, \xi} \|_{H^1_0} + \| \varphi_{\varepsilon, \mu, \xi} \|_{L^\infty} \leq D\alpha^3,
\]

and

\[
\pi^\perp \left\{ L\varphi_{\varepsilon, \mu, \xi} - (-\Delta)^{-1}(R + N(\varphi_{\varepsilon, \mu, \xi})) \right\} = 0,
\]

where the linear operator \( L \) is defined in (19), the error term \( R \) is defined in (17) and the quadratic term \( N \) is defined in (18).

5.1 Estimates on \( N(\varphi) \)

For a function \( \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega) \), let \( N(\varphi) \) be defined as in (18), i.e.

\[
N(\varphi) = N_{\varepsilon, \mu, \xi}(\varphi) := \lambda \left( f_\varepsilon(\omega_\varepsilon, \mu, \xi) + \varphi - f_\varepsilon(\omega_\varepsilon, \mu, \xi) - f'_\varepsilon(\omega_\varepsilon, \mu, \xi) \varphi \right).
\]

Let us estimate \( \| N(\varphi) \|_\varepsilon \), where \( \| \cdot \|_\varepsilon \) is defined as in (55). Let us define

\[
B_\alpha := \{ \varphi \in L^\infty(\Omega) : \| \varphi \|_{L^\infty(\Omega)} \leq \alpha \}.
\]

Lemma 5.2 There exists \( D_2 > 0 \) such that

\[
\| N(\varphi_1) - N(\varphi_2) \|_\varepsilon \leq D_2\alpha^{-1} \left( \| \varphi_1 \|_{L^\infty(\Omega)} + \| \varphi_2 \|_{L^\infty(\Omega)} \right) \| \varphi_1 - \varphi_2 \|_{L^\infty(\Omega)},
\]

for any \( \varphi_1, \varphi_2 \in B_\alpha \).

Proof. First, for any \( x \in \Omega \) we can find \( \theta_1 = \theta_1(x) \in [0, 1] \) such that

\[
N(\varphi) - N(\varphi_1) = \lambda \left( f_\varepsilon(\omega_\varepsilon + \varphi_2 - \varphi_1) - f_\varepsilon(\omega_\varepsilon + \varphi_1) \right)
\]

\[
= \lambda \left( f'_\varepsilon(\omega_\varepsilon + \theta_1 \varphi_2 + (1 - \theta_1) \varphi_1) \varphi_2 - f'_\varepsilon(\omega_\varepsilon + \varphi_2) \varphi_2 - f'_\varepsilon(\omega_\varepsilon + \varphi_1) \varphi_2 - \varphi_1 \right)
\]

\[
= \lambda \left( f'_\varepsilon(\omega_\varepsilon + \varphi_3) - f'_\varepsilon(\omega_\varepsilon) \right) \varphi_2 - \varphi_1,
\]

where \( \varphi_3 := \theta_1 \varphi_2 + (1 - \theta_1) \varphi_1. \) Furthermore, there exists \( \theta_2 = \theta_2(x) \) such that

\[
f'_\varepsilon(\omega_\varepsilon + \varphi_3) = f'_\varepsilon(\omega_\varepsilon) + f''_\varepsilon(\omega_\varepsilon + \theta_2 \varphi_3) \varphi_3.
\]
Thus, we obtain
\[
|N(\varphi_1) - N(\varphi_2)| = \lambda |f''_\epsilon(\omega_\varepsilon + \theta_2 \varphi_3)||\varphi_3||\varphi_1 - \varphi_2|
\leq \lambda |f''_\epsilon(\omega_\varepsilon + \theta_2 \varphi_3)|(\|\varphi_1\|_{L^\infty(\Omega)} + \|\varphi_2\|_{L^\infty(\Omega)}) \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)}.
\] (70)

Then, in order to conclude the proof, we shall bound \(|f''_\epsilon(\omega_\varepsilon + \theta_2 \varphi_3)||_\epsilon\). Note that, there exists a universal constant \(C_0 > 0\) such that
\[
|f''_\epsilon(t)| \leq C_0 (1 + |t|^3)e^{t^2 + |t|^{1+\epsilon}}, \quad \forall t \in \mathbb{R}.
\]

By Remark 2.7 we have \(\omega_\varepsilon = O(\beta) = O(\alpha^{-1})\). Since \(|\varphi_3| \leq |\varphi_1| + |\varphi_2| \leq 2\alpha\), we get
\[
(\omega_\varepsilon + \theta_2 \varphi_3)^3 \leq \omega_\varepsilon^2 + 2|\omega_\varepsilon||\varphi_3| + \varphi_3^2 = \omega_\varepsilon^2 + O(1).
\] (71)

By convexity, we also have
\[
|\omega_\varepsilon + \theta_2 \varphi_3|^3 \leq (|\omega_\varepsilon| + |\varphi_3|)^3 \leq 4(|\omega_\varepsilon|^3 + |\varphi_3|^3) \leq 4(|\omega_\varepsilon|^3 + \alpha^3).
\] (72)

In \(B(\xi, \rho_1)\) we have \(\omega_\varepsilon \geq c_0\) by Lemma 3.2 so that
\[
(\omega_\varepsilon + \theta_2 \varphi_3)^{1+\epsilon} \leq \omega_\varepsilon^{1+\epsilon} + O(1).
\] (73)

Clearly (71)- (73) yield the existence of a constant \(C_1 > 0\) such that
\[
|f''_\epsilon(\omega_\varepsilon + \theta_2 \varphi_3)| \leq C_1 \alpha^{-2} \omega_\varepsilon e^{\omega_\varepsilon^2 + |\omega_\varepsilon|^{1+\epsilon}} = C_1 \alpha^{-2} f_\varepsilon(\omega_\varepsilon),
\]
in \(B(\xi, \rho_1)\). Arguing as in Lemma 3.4 (see (46)) we get
\[
\lambda |f''_\epsilon(\omega_\varepsilon + \theta_2 \varphi)| \leq C_0 \alpha^{-1} f_\varepsilon \quad \text{in} \ B(\xi, \rho_0).
\] (74)

Lemma 3.6 and Lemma 3.7 yield
\[
\lambda \|f''_\epsilon(\omega_\varepsilon + \theta_2 \varphi)\|_{L^{1+\alpha^2}(B(\xi, \rho_0) \setminus B(\xi, \rho_0))} = O(\alpha^{-2} e^{-\frac{c_2}{\alpha}}).
\] (75)

Finally, thanks to Lemma 3.5 we know that
\[
\lambda f''_\epsilon(\omega_\varepsilon + \theta_2 \varphi_3) = O(1) \quad \text{in} \ \Omega \setminus B(\xi, \rho_1).
\] (76)

Thanks to (74)- (76) we infer
\[
\lambda \|f''_\epsilon(\omega_\varepsilon + \theta_2 \varphi_3)\|_\epsilon = O(\alpha^{-1}),
\]
and the conclusion follows from (10).
\[\square\]

**Remark 5.3** Applying Lemma 5.2 with \(\varphi_2 = 0\), we obtain that
\[
\|N(\varphi)\|_\epsilon \leq D_2 \alpha^{-1} \|\varphi\|_{L^\infty(\Omega)},
\]
for any \(\varphi \in B_\alpha\).

**Remark 5.4** The proof of Proposition 5.2 and Lemma 3.9 also shows that
\[
\|N(\varphi)\|_{L^\infty(\Omega \setminus B(\xi, \rho_1))} \leq D_3 \|\varphi\|_{L^\infty(\Omega)},
\]
for any \(\varphi \in B_\alpha\).
5.2 Proof of Proposition 5.1: a fixed point argument

Let us consider the operator

$$
T = T_{\varepsilon, \mu, \xi} := \left( \pi_L^{-1} \pi L^{-1} \left[ (-\Delta)^{-1} \left( N(\phi) + R \right) \right] \right)
$$

(77)
on the space $$X := K_\varepsilon^\perp \cap L^\infty(\Omega)$$, which is a Banach space with respect to the norm

$$
\| \cdot \|_X = \| \cdot \|_{H^1_0(\Omega)} + \| \cdot \|_{L^\infty(\Omega)}.
$$

Let $$D_1$$ and $$D_0$$ be the constants defined in Proposition 3.13 and Proposition 4.6. Let us set

$$
E_\varepsilon := \{ \phi \in X : \| \phi \|_X \leq D_0 (D_1 + 1) \alpha^3 \}.
$$

Proposition 5.1 is an immediate consequence of the following result.

**Proposition 5.5** There exists $$\varepsilon_0 > 0$$ such that, for any $$\varepsilon \in (0, \varepsilon_0)$$, $$\mu \in U$$, $$\xi \in B(\xi_0, \sigma)$$, $$T$$ has a fixed point $$\phi_{\varepsilon, \mu, \xi} \in E_\varepsilon$$, which depends continuously on $$\mu$$ and $$\xi$$.

**Proof.** Since $$E_\varepsilon$$ is a closed subspace of $$X$$ and $$T$$ depends continuously on $$\mu$$ and $$\xi$$, it is sufficient to verify that

1. $$T$$ maps $$E_\varepsilon$$ into itself.

2. $$T$$ is a contraction, i.e. $$\| T(\phi_1) - T(\phi_2) \|_{H^1_0(\Omega)} \leq \theta \| \phi_1 - \phi_2 \|_{H^1_0(\Omega)}$$ for some positive constant $$\theta < 1$$ and for any $$\phi_1, \phi_2 \in E_\varepsilon$$.

Then the conclusion follows by the contraction mapping theorem.

**Step 1** $$T$$ maps $$E_\varepsilon$$ into itself.

Let us denote $$C_0 := D_0 (D_1 + 1)$$. Take $$\phi \in E_\varepsilon$$ and set

$$
h(\phi) := R + N(\phi).
$$

If $$\varepsilon$$ is small enough, we have that $$\alpha^2 C_0 \leq 1$$, so that $$E_\varepsilon \subseteq B_\alpha$$ (see (69)). By Proposition 3.13 and Remark 5.3 we get

$$
\| h(\phi) \|_\varepsilon \leq \| R \|_\varepsilon + \| N(\phi) \|_\varepsilon \\
\leq D_1 \alpha^3 + D_2 \alpha^{-1} \| \phi \|_{L^\infty(\Omega)}^2 \\
\leq D_1 \alpha^3 + C_0^2 D_2 \alpha^5,
$$

for any $$\phi \in E_\varepsilon$$. Then, if we take $$\varepsilon$$ small enough so that $$C_0^2 D_2 \alpha^2 \leq 1$$, we get that

$$
\| h(\phi) \|_\varepsilon \leq (D_1 + 1) \alpha^3.
$$

Since by definition

$$
\pi L(T(\phi)) = \pi (-\Delta)^{-1} h(\phi),
$$

we have by Proposition 4.6 that

$$
\| T(\phi) \|_X \leq D_0 \| h(\phi) \|_\varepsilon \leq D_0 (D_1 + 1) \alpha^3,
$$

that is $$T(\phi) \in E_\varepsilon$$. 

29
Step 2  \( T \) is a contraction mapping in \( E_\varepsilon \).

Let us take \( \varepsilon \) small enough so that \( D_0 D_2 C_0 \alpha^2 \leq \frac{1}{4} \) and \( E_\varepsilon \subseteq B_\alpha \). By Propositions 4.6 and 5.2 we have
\[
\|T(\varphi_1) - T(\varphi_2)\|_X \leq D_0 \|h(\varphi_1) - h(\varphi_2)\|_\varepsilon \\
= D_0 \|N(\varphi_1) - N(\varphi_2)\|_\varepsilon \\
\leq D_0 D_2 \alpha^{-1} (\|\varphi_1\|_{L^\infty(\Omega)} + \|\varphi_2\|_{L^\infty(\Omega)}) \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)} \\
\leq 2 C_0 D_0 D_2 \alpha^2 \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)},
\]
for any \( \varphi_1, \varphi_2 \in E_\varepsilon \). Then, \( T \) is a contraction mapping on \( E_\varepsilon \). \( \square \)

6  The reduced problem: proof of Theorem 1.3 completed

Let \( \varphi_\varepsilon := \varphi_{\varepsilon, \mu, \xi} \) be as in Proposition 5.1. By (68), we can find \( \kappa_{\varepsilon,i} = \kappa_{\varepsilon,i}(\mu, \xi) \), \( i = 0, 1, 2 \) (which depend continuously on \( \mu \) and \( \xi \)), such that
\[
- \Delta \varphi_\varepsilon = \lambda f'_\varepsilon(\varphi_\varepsilon) \varphi_\varepsilon + R + N(\varphi_\varepsilon) + \sum_{j=0}^{2} \kappa_{\varepsilon,j} e^{U_\varepsilon} Z_{\varepsilon,j} . \quad (78)
\]
Equivalently, setting \( u_\varepsilon := \omega_\varepsilon + \varphi_\varepsilon \),
\[
- \Delta u_\varepsilon = \lambda f'_\varepsilon(u_\varepsilon) + \sum_{j=0}^{2} \kappa_{\varepsilon,j} e^{U_\varepsilon} Z_{\varepsilon,j} . \quad (79)
\]
Our aim is to find the parameter \( \mu = \mu(\varepsilon) \) and the point \( \xi = \xi(\varepsilon) \) so that the \( \kappa_{\varepsilon,i} \)'s are zero provided \( \varepsilon \) is small enough.

Proposition 6.1  It holds true that
\[
\kappa_{0,\varepsilon} = 6 \pi \alpha^3 \left( 2 - \log \left( \frac{8}{\mu^2} \right) + o(1) \right) , \quad (80)
\]
and
\[
\kappa_{i,\varepsilon} = -\kappa_{0,\varepsilon} a_{i,\varepsilon} + \frac{3 \mu}{2} \frac{\partial v_\varepsilon}{\partial x_i}(\xi) + O(\alpha \delta) , \quad i = 1, 2 \quad (81)
\]
as \( \varepsilon \to 0 \) uniformly with respect to \( \mu \in \mathcal{U} \) and \( \xi \in B(\xi_0, \sigma) \). Here, the \( a_{i,\varepsilon} \)'s are continuous functions of \( \mu \) and \( \xi \) and \( a_{i,\varepsilon} = O(\alpha^2) \) uniformly for \( (\mu, \xi) \in \mathcal{U} \times B(\xi_0, \sigma) \).

Proof.
Step 1 Let us prove that
\[ \kappa_{i,\epsilon} = O(\alpha^3) \quad \text{for} \quad i = 0, 1, 2 \quad (82) \]
and
\[ \|\varphi_\epsilon\|_{C^1(\Omega \setminus B(\xi, 2\sigma))} = O(\alpha^3). \quad (83) \]

First, since (67) gives \( \|\varphi\|_{L^\infty(\Omega)} = O(\alpha^3) \), Proposition 3.13, Lemma 3.14, Remark 5.3 and Lemma 4.1 yield
\[ \|R\|_{L^1(\Omega)} = O(\alpha^3), \quad \|N(\varphi_\epsilon)\|_{L^1(\Omega)} = O(\alpha^5), \quad \|\lambda f'_\epsilon(\omega_\epsilon)\varphi_\epsilon\|_{L^1(\Omega)} = O(\alpha^3). \]

Recalling that
\[ \int_{\Omega} e^{U_\epsilon} Z_{j,n} PZ_{i,n} dx = \frac{8}{3} \pi \delta_{ij} + O(\delta), \quad \text{for} \quad i, j = 0, 1, 2, \]
by Lemma 4.4 and (58), we get (82) by testing equation (78) with \( PZ_{i,n} \), \( i = 0, 1, 2 \).

By Lemma 3.12, Remark 5.4, and Lemma 4.1, one has
\[ \lambda f'_\epsilon(\omega_\epsilon) = O(1), \quad R = O(\alpha^3), \quad N(\varphi_\epsilon) = O(\alpha^6), \]
uniformly in \( \Omega \setminus B(\xi, \frac{\sigma}{2}) \). Then
\[ \|\Delta \varphi_\epsilon\|_{L^\infty(\Omega \setminus B(\xi, \frac{\sigma}{2}))} + \|\varphi_\epsilon\|_{L^\infty(\Omega)} = O(\alpha^3), \]
and we get (83) by standard elliptic estimates.

Step 2 Proof of (80).

Let \( \tilde{Z}_\epsilon \) be the function defined in (59). We shall test equation (78) against \( \tilde{Z}_\epsilon \). With the same arguments of the proof of Proposition 4.6 (Step 1), we obtain
\[ \int_{\Omega} \nabla \varphi_\epsilon \cdot \nabla \tilde{Z}_\epsilon dx = \int_{\Omega} \nabla \varphi_\epsilon \cdot \nabla Z_{0,\epsilon} dx + o(\|\varphi_\epsilon\|_{H^1_0(\Omega)}) = o(\alpha^3). \]

Moreover
\[ \int_{\Omega} \lambda f'_\epsilon(\omega_\epsilon)\varphi_\epsilon \tilde{Z}_\epsilon dx = \int_{B(\xi, \rho_0)} e^{U_\epsilon} Z_{0,\epsilon} \varphi_\epsilon dx + O(\epsilon^2 \alpha^3) + O(\alpha^3\|\Gamma_\epsilon\|_{L^1(B(\xi, \rho_0) \setminus B(\xi, \rho_0))}) = o(\alpha^3), \]
and
\[ \int_{\Omega} e^{U_\epsilon} Z_{j,\epsilon} \tilde{Z}_\epsilon dx = \int_{\mathbb{R}^2} e^0 Z_{j,0} dy + O\left( \int_{\mathbb{R}^2 \setminus B(0, \frac{\rho_0}{2})} e^0 dx \right) = \frac{8}{3} \pi \delta_{ij} + O(\delta^2 \rho_0^{-2}). \]
By Lemma 3.4 and Lemma 3.8, we get
\[
\int_{\Omega} R\tilde{Z}_n dx = \int_{B(\xi, \rho_0)} RZ_{0,n} dx + O\left(\|R\|_{L^1(B(\xi, \rho_1) \setminus B(\xi, \rho_0))}\right)
\]
\[
= \alpha^3 \int_{B(0, \rho_0)} e^{\tilde{U}} (2\tilde{U} + \tilde{U}^2) Z_0 dy + O\left(\alpha^4 \int_{\mathbb{R}^2} e^{\tilde{U}} (1 + \tilde{U}^4) dy\right) + o(\alpha^4)
\]
\[
= 16\pi\alpha^3 \left(\log \left(\frac{8}{\mu^2}\right) - 2\right) + O(\alpha^4).
\]
Finally, we have that
\[
\int_{\Omega} N(\varphi)\tilde{Z}_\varepsilon dx = O\left(\|N(\varphi)\|_\varepsilon\right) = O(\alpha^5).
\]
Then, testing (78) against \(\tilde{Z}_\varepsilon\) and using (82), one gets
\[
0 = 16\pi\alpha^3 \left(\log \left(\frac{8}{\mu^2}\right) - 2\right) + \frac{8}{3}\pi k_{0, \varepsilon} + o(\alpha^3),
\]
from which we get (80).

**Step 3** Let us prove
\[
\sum_{j=0}^{2} \kappa_{j, \varepsilon} \int_{\Omega} e^{U_{\varepsilon}} Z_{j, \varepsilon} \frac{\partial u_{\varepsilon}}{\partial x_i} dx = -8\pi\alpha \frac{\partial v_{\varepsilon}}{\partial \nu}(\xi) + O(\alpha^2), \quad i = 1, 2, \quad (84)
\]
We multiply (79) and \(\frac{\partial u_{\varepsilon}}{\partial x_i}\). Applying the Pohozaev identity (see e.g. [27, Proposition 2, Proof of Step 1]), we obtain
\[
-\frac{1}{2} \int_{\partial \Omega} \frac{\partial u_{\varepsilon}}{\partial \nu} \frac{\partial u_{\varepsilon}}{\partial x_i} d\sigma = \lambda \int_{\Omega} f_{\varepsilon}(u_{\varepsilon}) \frac{\partial u_{\varepsilon}}{\partial x_i} dx + \sum_{j=0}^{2} \kappa_{j, \varepsilon} \int_{\Omega} e^{U_{\varepsilon}} Z_{j, \varepsilon} \frac{\partial u_{\varepsilon}}{\partial x_i} dx, \quad (85)
\]
Since \(u_{\varepsilon} = 0\) on \(\partial \Omega\), the divergence theorem yields
\[
\int_{\Omega} f_{\varepsilon}(u_{\varepsilon}) \frac{\partial u_{\varepsilon}}{\partial x_i} dx = \int_{\Omega} \frac{d}{dx_i} \left(\int_{0}^{u_{\varepsilon}(x)} f_{\varepsilon}(t) dt\right) dx
\]
\[
= \int_{\partial \Omega} \nu_i \left(\int_{0}^{u_{\varepsilon}(x)} f_{\varepsilon}(t) dt\right) d\sigma = 0. \quad (86)
\]
By (83), the definition of \(u_{\varepsilon}\) and \(\omega_{\varepsilon}\), Lemma 2.8, Lemma 3.10, we have
\[
\frac{\partial u_{\varepsilon}}{\partial \nu} = -\frac{\partial v_{\varepsilon}}{\partial \nu} + \alpha \frac{\partial}{\partial \nu}(8\pi G_{\xi} - w_{\varepsilon}) + O(\alpha^2)
\]
on \(\partial \Omega\). Thus, keeping in mind that \(|\nabla v_{\varepsilon}|, |\nabla w_{\varepsilon}|\) and \(|\nabla G_{\xi}|\) are uniformly bounded on \(\partial \Omega\) (see Lemma (2.2) and (2.3)) and that \(\frac{\partial u_{\varepsilon}}{\partial x_i} = \frac{\partial}{\partial \nu} v_i\), we obtain
\[
\int_{\partial \Omega} \frac{\partial u_{\varepsilon}}{\partial x_i} \frac{\partial u_{\varepsilon}}{\partial \nu} d\sigma = \int_{\partial \Omega} \frac{\partial v_{\varepsilon}}{\partial x_i} \frac{\partial v_{\varepsilon}}{\partial \nu} d\sigma + 2\alpha \int_{\partial \Omega} \frac{\partial v_{\varepsilon}}{\partial x_i} \frac{\partial}{\partial \nu}(w_{\varepsilon} - 8\pi G_{\xi}) d\sigma + O(\alpha^2). \quad (87)
\]
Applying the Pohozaev identity to \( v_\varepsilon \) and arguing as in (86), we get that
\[
\int_{\partial \Omega} \frac{\partial v_\varepsilon}{\partial x_i} \frac{\partial v_\varepsilon}{\partial \nu} \, d\sigma = -2\lambda \int_{\Omega} f_\varepsilon(v_\varepsilon) \frac{\partial v_\varepsilon}{\partial x_i} \, dx = 0. \tag{88}
\]
Integrating by parts and noting that \(-\Delta \frac{\partial v_\varepsilon}{\partial x_i} = \lambda f_\varepsilon'(v_\varepsilon) \frac{\partial v_\varepsilon}{\partial x_i}\) in \( \Omega \), we get
\[
\int_{\partial \Omega} \frac{\partial v_\varepsilon}{\partial x_i} \frac{\partial}{\partial \nu} (w_\varepsilon - 8\pi G_\xi) \, d\sigma = \int_{\Omega} \left( \frac{\partial v_\varepsilon}{\partial x_i} \Delta w_\varepsilon - w_\varepsilon \frac{\partial v_\varepsilon}{\partial x_i} \right) \, dx + 8\pi \frac{\partial v_\varepsilon}{\partial x_i}(\xi)
\]
\[= 0 \text{ by (12)} \]
This together with (87)-(88) gives
\[
\frac{1}{2} \int_{\partial \Omega} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial \nu} \, d\sigma = 8\pi \alpha \frac{\partial v_\varepsilon}{\partial x_i}(\xi) + O(\alpha^2). \tag{89}
\]
Finally, (84) follows by (85)-(86) and (89).

**Step 4** For \( i = 1, 2, j = 0, 1, 2 \), we have
\[
\int_{\Omega} e^{U_\varepsilon} Z_{j,\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} \, dx = -\frac{\alpha}{\delta} \left( \frac{16}{\delta \mu} \pi \delta_{ij} + O(\alpha^2) \right). \tag{90}
\]
For \( i = 1, 2 \) and \( j = 0, 1, 2 \). Note that we have the identity
\[
\frac{\partial}{\partial x_i} e^{U_\varepsilon} Z_{j,\varepsilon} = \frac{e^{U_\varepsilon}}{\delta \mu} \left( \delta_{ij}(Z_{0,\varepsilon} + 1) - \delta_{j0}Z_{i,\varepsilon} - 3Z_{i,\varepsilon}Z_{j,\varepsilon} \right).
\]
Setting \( \Psi_{ij} := \delta_{ij}(Z_0 + 1) - \delta_{j0}Z_i - 3Z_iZ_j \) and applying the divergence theorem, we find
\[
\int_{\Omega} e^{U_\varepsilon} Z_{j,\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} \, dx = -\int_{\Omega} \frac{d}{dx_i} \left( e^{U_\varepsilon} Z_{j,\varepsilon} \right) \, dx
\]
\[= -\frac{1}{\delta \mu} \int_{\Omega} e^{U_\varepsilon} \left( \delta_{ij}(Z_{0,\varepsilon} + 1) - \delta_{j0}Z_{i,\varepsilon} - 3Z_{i,\varepsilon}Z_{j,\varepsilon} \right) \, dx
\]
\[= -\frac{1}{\delta \mu} \int_{\Omega} e^{U_\varepsilon}(\xi + \delta y)e^\delta \Psi_{ij} \, dy
\]
\[= -\frac{1}{\delta \mu} \int_{B(0,\delta)} e^{U_\varepsilon}(\xi + \delta y)e^\delta \Psi_{ij} \, dy + O(\beta \delta^2),
\]
where in the last equality we used that
\[
u_\varepsilon = O(\beta) \quad \text{and} \quad e^\delta \Psi_{ij} = O(|\nu_\varepsilon|^{-5}), \tag{91}
\]
for \( |\nu| \geq \frac{\delta}{\beta} \). By Lemma 2.6 we have
\[
u_\varepsilon(\xi + \delta y) = \beta + \alpha \tilde{U}(y) + O(\alpha^3) + O(\delta |\nu|), \tag{92}
\]
for \( |\nu| \geq \frac{\delta}{\beta} \).
for \( y \in B(0, \frac{\alpha}{2}) \). Using again (91), we get that
\[
\int_{B(0, \frac{\alpha}{2})} e^U \Psi_{ij} dy = \int_{\mathbb{R}^2} e^U \Psi_{ij} dy + O(\delta^3).
\]
Similarly, we have
\[
\int_{B(0, \frac{\alpha}{2})} e^U \Psi_{ij} dy = \int_{\mathbb{R}^2} e^U \Psi_{ij} dy + O(\beta^2 \delta^3)
\]
\[
= \frac{16}{3} \pi \delta_{ij} + O(\beta^2 \delta^3),
\]
and (90) is proved.

**Step 5 Proof of (81).**

Let us set
\[
a_{ij, \varepsilon} = a_{ij, \varepsilon}(\xi, \mu) := -\frac{3\mu}{16\pi} \int_{\Omega} e^{U_{\varepsilon}^j Z_j_{\varepsilon}} \frac{\partial u_{\varepsilon}}{\partial x_i} d\sigma.
\]
According to Step 4, we have \( a_{0, \varepsilon} = O(\alpha^2) \) if \( i = 1, 2 \). Moreover, the matrix \( A = (a_{ij, \varepsilon})_{i,j \in \{1,2\}} \) is invertible and its inverse \( A^{-1} = (a_{ij}^\varepsilon)_{i,j \in \{1,2\}} \) satisfies
\[
a_{ij}^\varepsilon = \delta_{ij} + O(\alpha^2), \quad i, j = 1, 2.
\]
Then (81) follows by (84), just setting
\[
a_{i, \varepsilon} := \sum_{j=1}^{2} a_{ij}^\varepsilon a_{0j, \varepsilon}.
\]

\[\square\]

It is important to point out that (81) cannot be considered a precise uniform expansion of \( \kappa_{i, \varepsilon} \). Indeed, (80) and the rough (but difficult to improve) estimate \( a_{i, \varepsilon} = O(\alpha^2) \) yield only \( \kappa_{0, \varepsilon} a_{i, \varepsilon} = O(\alpha^5) \). Since \( \delta \ll \alpha^5 \) it is not possible to identify the leading term in the RHS of (81). However, it is clear that the term involving \( \frac{\partial u_{\varepsilon}}{\partial x_i} \) becomes dominant when \( \kappa_{0, \varepsilon} \) vanishes. This is enough for our argument.

**Proof of Theorem 1.3 completed**

**Proof.** Let us consider the vector field
\[
B_{\varepsilon}(\mu, \xi) = \left( \frac{1}{6\pi \alpha^2} \kappa_{0, \varepsilon}, \frac{2}{3 \delta \mu} (\kappa_{1, \varepsilon} + \kappa_{0, \varepsilon} a_{1, \varepsilon}), \frac{2}{3 \delta \mu} (\kappa_{2, \varepsilon} + \kappa_{0, \varepsilon} a_{2, \varepsilon}) \right).
\]
By construction, for any \( \varepsilon > 0 \), \( B_{\varepsilon} \) depends continuously on \( \mu \) and \( \xi \). Moreover, thanks to (80), (81) and Lemma 2.2 we have
\[
B_{\varepsilon} \to \bar{B}(\mu, \xi) := \left( 2 - \log \left( \frac{8}{\mu^2} \right), \nabla u_0(\xi) \right)
\]
34
as \( \varepsilon \to 0 \), uniformly for \( \mu \in \mathcal{U} \) and \( \xi \in B(\xi_0, \sigma) \). By assumption [A2], \( \bar{B} \) has a \( C^0 \)-stable zero at the point \((\mu_0, \xi_0)\), with \( \mu_0 = \sqrt{8}e^{-1} \). Then, for \( \varepsilon \) small enough, there exist \( \xi = \xi(\varepsilon) \to \xi_0, \mu = \mu(\varepsilon) \to \mu_0 \) as \( \varepsilon \to 0 \) such that \( B_\varepsilon(\mu(\varepsilon), \xi(\varepsilon)) = 0 \). Clearly, this is equivalent to \( \kappa_{i, \varepsilon, \mu(\varepsilon), \xi(\varepsilon)} = 0, \ i = 0, 1, 2 \). That concludes the proof.

\[\square\]

**Appendix A. The proof of Lemma 2.4**

Proof. The third equation in (28) allows to write \( \delta \) as a function of \( \alpha, \beta, \varepsilon, \mu, \xi \):

\[
\log \frac{1}{\delta^2} = \frac{\beta}{2\alpha} + \frac{V_{\varepsilon, \alpha, \xi}(\xi)}{2\alpha} - \frac{c_{\mu, \xi}}{2},
\]

and the second equation in (28) gives \( \alpha \) as a function of \( \beta, \varepsilon, \mu, \xi \):

\[
\alpha = \frac{2}{\beta + \beta^\varepsilon + \beta^\varepsilon}.
\]

Then, (after a simple computation) it is sufficient to prove that there exists \( \beta = \beta(\varepsilon, \mu, \xi) \) such that

\[
\frac{1}{\beta} \left( \log \lambda + \frac{c_{\mu, \xi}}{2} \right) + 2 \frac{\log \beta}{\beta} + \left( \frac{1}{2} \beta^\varepsilon - u_0(\xi) \right) - (V_{\varepsilon, \alpha, \xi}(\xi) - u_0(\xi)) \tag{93}
\]

\[
+ \frac{\log \left( 2 + \beta^\varepsilon - \varepsilon \beta^\varepsilon \right)}{\beta} - \frac{1}{2} \beta^\varepsilon - \frac{1}{2} V_{\varepsilon, \alpha, \xi}(\xi) \left( \beta^\varepsilon - \varepsilon \beta^\varepsilon \right) = 0.
\]

Now, we choose \( \beta^\varepsilon := 2u_0(\xi) + \theta_\varepsilon(\xi, \mu) \) with \( \|\theta_\varepsilon\|_{C^0(B(\xi_0, \sigma) \times \mathcal{U})} \) so small that

\[
2u_0(\xi) + \theta_\varepsilon(\xi, \mu) \geq \eta > 1 \quad \text{in} \quad B(\xi_0, \sigma) \times \mathcal{U}.
\]

This is possible because of (22). With this choice we have \( \frac{1}{\beta} = O \left( \eta^{-\frac{1}{2}} \right) \). It is easy to show that (93) has a solution \( \theta_\varepsilon \) because of a simple fixed point argument. Indeed (93) rewrites as \( \theta_\varepsilon = T(\theta_\varepsilon) \) where \( T \) is a contraction mapping on the ball

\[
\left\{ \theta_\varepsilon \in C^0(B(\xi_0, \sigma) \times \mathcal{U}) : \|\theta_\varepsilon\|_{C^0(B(\xi_0, \sigma) \times \mathcal{U})} \leq \rho_\varepsilon \right\},
\]

where \( \rho_\varepsilon := \rho \min \left\{ \frac{1}{2} \eta^{-\frac{1}{2}}, \|v_0 - u_0\|_{C^0(\Omega)} \right\} \) for some \( \rho > 0 \) and \( \rho_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). Here we use the expression of \( V_{\varepsilon, \alpha, \xi}(\xi) \) in (11) and (ii) of Lemma 2.2.

\[\square\]

**Appendix B. A Stampacchia type estimate**

In this section we prove domain-independent estimates for solutions of the Poisson equation \( -\Delta u = f \), under Dirichlet boundary conditions, with \( f \in L^p(\Omega) \) and \( p \) approaching 1. Our strategy is based on the Stampacchia method.
Lemma B.1 ([28], Lemma 4.1) Let \( \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be a nonincreasing function. Assume that there exist \( M > 0, \gamma > 0, \delta > 1 \) such that

\[
\psi(h) \leq \frac{M \psi(k)^\delta}{(h-k)^\gamma} \quad \forall \ h > k > 0.
\]

Then \( \psi(d) = 0 \), where \( d = M \frac{\frac{1}{\gamma} \psi(0)^\frac{1}{\gamma}}{\frac{\delta-1}{\delta}} \).

Let \( \Omega \subseteq \mathbb{R}^2 \) be a bounded smooth domain. For any \( q > 1 \), let \( S_q(\Omega) \) be the Sobolev’s constant for the embedding of \( H^1_0(\Omega) \) in \( L^q(\Omega) \), namely

\[
S_q(\Omega) = \inf_{u \in H^1_0(\Omega)} \frac{\|u\|_{H^1_0(\Omega)}}{\|u\|_{L^q(\Omega)}}.
\]

It is known that \( 0 < S_q(\Omega) < +\infty \) and that (see [26] Lemma 2.2)

\[
\lim_{q \to +\infty} \sqrt{q} S_q(\Omega) = \sqrt{8\pi e}.
\]

Theorem B.2 Let \( \Omega \) be a bounded smooth domain. For \( p > 1 \), \( f \in L^p(\Omega) \), the unique solution \( u \in H^1_0(\Omega) \) of the equation \(-\Delta u = f\) satisfies

\[
\|u\|_{L^\infty(\Omega)} \leq 4S_{\frac{3p+1}{p-1}}(\Omega)^{-2}\|f\|_{L^p(\Omega)}\frac{2}{3p+1}.
\]

Proof. We want to apply the previous lemma to the function \( \psi(k) := |A_k|, \ A_k := \{x \in \Omega : |u(x)| > k\} \).

For any \( k > 0 \), let us consider the function

\[
v_k(x) := \begin{cases} 0 & |u(x)| \leq k, \\ u(x) - k & u(x) > k, \\ -u(x) - k & u(x) < -k. \end{cases}
\]

Note that \( v_k \in H^1_0(\Omega) \) and \( |\nabla v_k| = |\nabla u| \chi_{A_k} \). If we test the equation against \( v_k \) we get

\[
\int_{\Omega} \nabla u \cdot \nabla v_k \, dx = \int_{\Omega} f v_k \, dx. \tag{94}
\]

For any \( q \in (1, p) \) H"older’s inequality gives

\[
\int_{\Omega} f v_k \, dx = \int_{A_k} f v_k \, dx \leq \|f\|_{L^q(A_k)} \|v_k\|_{L^{\frac{q}{q-1}}(A_k)} \leq \|f\|_{L^p|\Omega|} \frac{p-q}{p} \|v_k\|_{L^{\frac{q}{q-1}}(A_k)}. \tag{95}
\]

By Sobolev’s inequality, we have that

\[
\int_{\Omega} \nabla u \cdot \nabla v_k \, dx = \int_{A_k} |\nabla v_k|^2 \, dx \geq S_{\frac{q}{q-1}}(\Omega)^2 \|v_k\|_{L^{\frac{q}{q-1}}}^2. \tag{96}
\]
By (94)- (96), we have
\[ \|v_k\|_{L^q(\Omega)} \leq S_{\frac{q}{q-1}}(\Omega)^{-2}\|f\|_{L^p}|A_k|^{\frac{p-q}{pq}}. \]

Now, for any \( h > k \), we have that \( A_h \subseteq A_k \) and \( v_k \geq (h - k) \) in \( A_h \), hence
\[ \int_{\Omega} |v_k|^\frac{q}{q-1} dx = \int_{A_k} v_k^\frac{q}{q-1} dx \geq \int_{A_h} v_h^\frac{q}{q-1} dx \geq (h - k)^\frac{q}{q-1}|A_h|. \]

In conclusion, we find
\[ (h - k)|A_h|^\frac{q-1}{q} \leq S_{\frac{q}{q-1}}(\Omega)^{-2}\|f\|_{L^p}|A_k|^{\frac{p-q}{pq}}, \]
or, equivalently,
\[ \psi(h) \leq \frac{S_{\frac{q}{q-1}}(\Omega)^{-2}\|f\|_{L^p}}{(h - k)^{\frac{q-1}{q}}} \psi(k)^{\frac{p-q}{pq}}. \]

Then, we are in position to apply Lemma 1.1 to \( \psi \) with \( M = S_{\frac{q}{q-1}}(\Omega)^{-2}\|f\|_{L^p} \), \( \gamma = \frac{q}{q-1} \), and \( \delta = \frac{p-q}{pq} \). For this, we need to impose that \( \delta = \frac{p-q}{p(q-1)} \), that is \( q < \frac{2p}{p+1} \).

Note that \( 1 < \frac{2p}{p+1} < p \). According to Stampacchia’s Lemma, we have
\[ \psi(d) = 0 \quad \text{where} \quad d = M\psi(0) = S_{\frac{q}{q-1}}^2\|f\|_{L^p}|\Omega|^{-\frac{2p-q(p+1)}{pq}} 2^{\frac{p-q}{pq}}. \]

This implies that
\[ \|u\|_{L^\infty(\Omega)} \leq S_{\frac{q}{q-1}}(\Omega)^{-2}\|f\|_{L^p}|\Omega|^{-\frac{2p-q(p+1)}{pq}} 2^{\frac{p-q}{pq}}. \]

This is true for any choice of \( q \in (1, \frac{2p}{p+1}) \). If we take for example \( p \) the midpoint of \( (1, \frac{2p}{p+1}) \), that is \( q = \frac{1}{2} + \frac{p}{p+1} = \frac{3p+1}{2p+1} \), then we find that
\[ \frac{q}{q-1} = \frac{3p+1}{p-1}, \quad \frac{2p-q(p+1)}{pq} = \frac{3p-1}{3p^2+p}, \quad \frac{p-q}{2p-q(p+1)} = \frac{2p+1}{p+1} \leq 2, \]
and we get the conclusion.

**Corollary B.3** Given \( K > 0 \) and \( p > 1 \), there exists a constant \( C = C(K, p) \) such that, for any domain \( \Omega \subseteq \mathbb{R}^2 \) with \( |\Omega| \leq K \) and any \( f \in L^p(\Omega) \) the unique solution \( u \in H^1_0(\Omega) \) of \(-\Delta u = f\) satisfies
\[ \|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \]

**Corollary B.4** Given \( K > 0 \), there exist \( p_0 = p_0(K) \) and \( C = C(K) \) such that, for any \( 1 < p < p_0 \), any domain \( \Omega \subseteq \mathbb{R}^2 \) with \( |\Omega| \leq K \), and any \( f \in L^p(\Omega) \), the unique solution \( u \in H^1_0(\Omega) \) of \(-\Delta u = f\) satisfies
\[ \|u\|_{L^\infty(\Omega)} \leq \frac{C}{p-1}\|f\|_{L^p(\Omega)}. \]
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