GOOD GEODESICS SATISFYING THE TIMELIKE CURVATURE-DIMENSION CONDITION

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Abstract. Let \((M, d, m, \ll, \preceq, \tau)\) be a causally closed, \(K\)-globally hyperbolic, regular measured Lorentzian geodesic space satisfying the weak timelike curvature-dimension condition \(wTCD_p(K, N)\) in the sense of Cavalletti and Mondino. We prove the existence of geodesics of probability measures on \(M\) which satisfy the entropic semiconvexity inequality defining \(wTCD_p(K, N)\) and whose densities with respect to \(m\) are additionally uniformly \(L^\infty\) in time. This holds apart from any nonbranching assumption. We also discuss similar results under the timelike measure-contraction property.

1. Introduction

Lott–Sturm–Villani theory. Almost two decades ago, descriptions of synthetic Ricci curvature bounds by \(K \in \mathbb{R}\) for metric measure spaces \((M, d, m)\) via optimal transport were set up by Sturm [39, 40], and independently Lott and Villani [27].

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This leads to the so-called CD(K, ∞) spaces. Combined with the works of Ambrosio, Gigli, and Savaré [3, 4, 5], these have become a research area which is highly active today. The strongest results in this framework have been obtained for CD(K, N) spaces, i.e., CD(K, ∞) spaces admitting a synthetic upper bound N ∈ [1, ∞) on their “dimension” [6, 7, 13, 16, 20, 27, 40]. The literature is too large to be cited exhaustively; let us only mention the splitting theorem [19] for RCD(0, N) spaces, i.e., infinitesimally Hilbertian CD(0, N) spaces, which in turn has lead to a good structure theory for RCD(K, N) spaces [9, 21, 31].

In [35, 36] Rajala has proven the existence of W_{2}-geodesics of measures whose densities are uniformly $L^\infty$ in time under CD(K, ∞) and CD(K, N), respectively, once their endpoints have bounded support and $L^\infty$-densities with respect to m. In addition, by [2, 36] such a geodesic can be constructed to satisfy the defining entropic semiconvexity inequality. This generalizes results from the smooth case [14, 28, 37], and remarkably does not rely on any nonbranching assumption.

Rajala’s work has been an important ingredient to establish cornerstone results for the theory of CD spaces. We only quote a few: a metric Brenier theorem [3], existence of “many” test plans which in turn are used to axiomatize Sobolev calculus [3, 20], the Sobolev-to-Lipschitz property (without using heat flow [4]) which links this Eulerian calculus to the Lagrangian side [19], the proof of constancy of the dimension for CD(K, N) spaces [9], the weak Poincaré inequality [36], stability of super-Ricci flows [41], etc.

**Nonsmooth Lorentzian geometry.** Recently, ideas inspired in particular by [16] have lead to synthetic *timelike* lower Ricci bounds in nonsmooth general relativity by Cavalletti and Mondino [12] in terms of the (weak) *timelike curvature-dimension conditions* TCD$_p^\ell(K, N)$ and wTCD$_p^\ell(K, N)$, $p \in (0, 1)$, in the “entropic” sense.

Here, the relevant structures are *Lorentzian pre-length spaces* $(M, d, \ll, \leq, \tau)$ [26], see also [38] for a related approach. In the same way metric measure spaces generalize smooth Riemannian manifolds, these provide singular counterparts to smooth Lorentzian spacetimes. They come with a chronological relation $\ll$ and a notion of causality $\leq$ between points in $M$. The *time separation function* $\tau$ takes over the role of a metric, in the sense that parallel to the smooth case $[24, 29, 33]$ it allows for notions of length, geodesics, strong causality, etc.

Developing such a singular theory within general relativity aims to include spacetimes with low regularity metrics. In turn, this would allow one to address the Cauchy initial value problem for the Einstein equation, the cosmic censorship conjectures, and physically relevant models in wider generality than currently possible. See e.g. the introductions of [12, 26] for related literature.

The TCD and wTCD conditions are defined by asking for the convexity, see Definition 2.13 and Definition 2.16, of the exponentiated relative entropy

\[
U_N(\mu) := e^{-\text{Ent}_m(\mu)/N},
\]

on the space of probability measures on $M$, along some chronological optimal mass transport from past to future located distributions. Here, optimality — and the notion of *geodesics* with respect to which convexity of $U_N$ is formulated — are quantified by the $p$-*Lorentz–Wasserstein distance*

\[
\ell_p(\mu, \nu) := \sup \|\tau\|_{L^p(M^2, \pi)},
\]

$p \in (0, 1]$, first introduced in [15] and further studied in [12, 25, 29, 32, 42]. The supremum is taken over all causal couplings $\pi$ of $\mu$ and $\nu$; see Section 2.2 for details.

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1 Though $\tau(x, y)$ should not be interpreted as a distance, cf. Remark 2.2, but rather as the maximal proper time a spacetime point $x \in M$ needs to travel to $y \in M$. 

Notably, TCD\( \rho \)\( (0, N) \) and wTCD\( \rho \)\( (0, N) \) are nonsmooth analogues of the strong energy condition of Hawking and Penrose [22, 23, 34]. The latter is a nonnegative definiteness condition on the stress-energy tensor. For a vanishing cosmological constant, this boils down to nonnegativity of the Ricci tensor in every timelike direction by the Einstein equation. In turn, the latter property can be characterized by convexity of \( U_N \) along suitable \( \ell_\rho \)-geodesics. This was discovered in [29, 32] and in fact motivated the authors of [12] to introduce the TCD and wTCD conditions for singular measured Lorentzian pre-length spaces \( (M, d, m, \ll, \leq, \tau) \), i.e. Lorentzian pre-length spaces endowed with a reference measure \( m \).

**Objective.** In this article, we provide a nonsmooth Lorentzian version of Rajala’s results [35, 36], namely the rich existence of good \( \ell_\rho \)-geodesics. We hope our results to be an equally useful contribution to the young theory of TCD and wTCD spaces as [35, 36] was for CD spaces; possible applications of Theorem 1.2 and related future work we attack soon are described below.

Before stating our major Theorem 1.2, we specify what we mean by “good”. A detailed account of our notation is postponed to Chapter 2.

All over this paper, given \( \pi \in \mathcal{P}(M^2) \) we use the abbreviation

\[
T_\pi := \|\pi\|_{L^2(M, \pi)}.
\]

**Definition 1.1.** Let \( (M, d, m, \ll, \leq, \tau) \) be a measured Lorentzian pre-length space, and let \( p \in (0, 1) \), \( K \in \mathbb{R} \), and \( N \in (0, \infty) \). A timelike proper-time parametrized \( \ell_\rho \)-geodesic \( (\mu_t)_{t \in [0, 1]} \), in a sense made precise in Subsection 2.2.4, is called good if the following conditions hold.

a. There exists some \( \ell_\rho \)-optimal coupling \( \pi \in \Pi_{\ll}(\mu_0, \mu_1) \) with \( \tau \in L^2(M^2, \pi) \), and for every \( t \in [0, 1] \), \( \mu_t = \rho_t m \in \mathcal{D}(\text{Ent}_m) \) and

\[
U_N(\mu_t) \geq \sigma_{K,N}^{(1-t)}(T_\pi) U_N(\mu_0) + \sigma_{K,N}^{(t)}(T_\pi) U_N(\mu_0).
\]

b. We have the uniform \( L^\infty \)-bound

\[
\sup\{\|\mu_t\|_{L^\infty(M, m)} : t \in [0, 1]\} < \infty.
\]

**Theorem 1.2.** Assume that \( (M, d, m, \ll, \leq, \tau) \) is a causally closed, \( \mathcal{K} \)-globally hyperbolic, regular Lorentzian geodesic space satisfying wTCD\( \rho \)\( (K, N) \) for \( p \in (0, 1) \), \( K \in \mathbb{R} \), and \( N \in (0, \infty) \). Let \( (\mu_0, \mu_1) = (\rho_0 m, \rho_1 m) \in \mathcal{P}_w^\infty(M, m)^2 \) be strongly timelike \( p \)-dualizable, and suppose that the density \( \rho_0, \rho_1 \in L^\infty(M, m) \). Then there exists a good timelike proper-time parametrized \( \ell_\rho \)-geodesic from \( \mu_0 \) to \( \mu_1 \).

More precisely, there exists a timelike proper-time parametrized \( \ell_\rho \)-geodesic \( (\mu_t)_{t \in [0, 1]} \) from \( \mu_0 \) to \( \mu_1 \) satisfies, for every \( t \in [0, 1] \), \( \mu_t = \rho_t m \in \mathcal{D}(\text{Ent}_m) \) and

\[
\|\rho_t\|_{L^\infty(M, m)} \leq e^{D\sqrt{K-N}/2} \max\{\|\rho_0\|_{L^\infty(M, m)}, \|\rho_1\|_{L^\infty(M, m)}\},
\]

where \( D := \sup \tau(\text{spt} \mu_0 \times \text{spt} \mu_1) \).

We also establish the subsequent variations of Theorem 1.2.

- Assuming the stronger TCD\( \rho \)\( (K, N) \) condition in place of wTCD\( \rho \)\( (K, N) \), one may relax the hypotheses on \( \mu_0 \) and \( \mu_1 \), cf. Remark 3.15.
- A version of it holds for a dimension-independent wTCD\( \rho \)\( (K, \infty) \) condition, newly introduced in Definition 4.1 below following [39], cf. Theorem 4.5 and Remark 4.6. A key message here is that the upper dimension bound, apart from which timelike Ricci bounds have not been studied thus far, is not strictly required, but of course gives quantitatively better results.
- Our proof can be adapted to the study of more general timelike convex functionals on \( \mathcal{P}(M) \), cf. Theorem 4.7.
Lastly, we present a version for the timelike measure-contraction property TMCP_{\ell}^{p}(K, N) from [12], following the work [11] for metric measure spaces, cf. Theorem 4.11. Here, by nature of the TMCP condition (the terminal measure is a Dirac measure) the r.h.s. of the obtained inequality

$$\|\rho_{t}\|_{L^{\infty}(M, m)} \leq \frac{1}{(1 - p)^{N}} e^{D_{t}K - N} \|\rho_{0}\|_{L^{\infty}(M, m)}$$

is not bounded in \(t \in [0, 1]\), but blows up as \(t\) approaches 1.

In the wTCD context, Theorem 1.2 seems optimal with respect to which endpoints \(\mu_{0}\) and \(\mu_{1}\) are allowed, i.e. strongly timelike \(p\)-dualizable ones according to Definition 2.7. Having covered this framework, Theorem 1.2 is expected to be relevant for stability questions, since the limit of a sequence of TCD spaces is only known to be wTCD in general [12, Thm. 3.14]. We point out that local causal closedness and \(K\)-global hyperbolicity, two main assumptions of Theorem 1.2, are precisely the regularity conditions used to set up the corresponding notion of weak convergence of measured Lorentzian geodesic spaces in [12]. Moreover, the author is indebted to Robert McCann for pointing out to add the assumption of regularity, cf. Subsection 2.1.6, to Theorem 4.5 and to Corollary 1.3 below.

As in [35, 36] we avoid any assumption on (timelike) nonbranching [12, Def. 1.10], which makes Theorem 1.2 convenient for spacetimes with low regularity. Indeed, while spacetimes with \(C^{1,1}\)-metrics are timelike nonbranching [12], this property is expected to fail e.g. for lower regularity of the metric and for closed cone structures [30]. Nevertheless, already in the timelike nonbranching case, an interesting byproduct of Theorem 1.2 and the uniqueness of chronological \(\ell_{p}\)-optimal couplings and \(\ell_{p}\)-geodesics [12, Thm. 3.19, Thm. 3.20] is the following.

**Corollary 1.3.** Assume that \((M, d, m, \ll, \leq, \tau)\) is a timelike nonbranching, causally closed, \(K\)-globally hyperbolic, regular Lorentzian geodesic space obeying wTCD_{\ell}^{p}(K, N) for \(p \in (0, 1)\), \(K \in \mathbb{R}\), and \(N \in (0, \infty)\). Let the pair \((\mu_{0}, \mu_{1}) = (\rho_{0} m, \rho_{1} m) \in \mathcal{P}_{c}^{\infty}(M, m)^{2}\) be strongly timelike \(p\)-dualizable, and suppose that \(\rho_{0}, \rho_{1} \in L^{\infty}(M, m)\). Then the unique timelike proper-time parametrized \(\ell_{p}\)-geodesic \((\mu_{t})_{t \in [0, 1]}\) from \(\mu_{0}\) to \(\mu_{1}\) is good.

**Remark 1.4.** One can replace timelike nonbranching by the weaker condition of timelike \(p\)-essential nonbranching [8, Def. 2.21, Rem. 2.22] in Corollary 1.3. In fact, in this case, independently of the results in this paper we have recently shown the equivalence of the conditions TCD_{\ell}^{p}(K, N) and wTCD_{\ell}^{p}(K, N) [8, Thm. 3.35] (and to the reduced TCD conditions in terms of Rényi’s entropy from [8, Def. 3.2]). The above mentioned uniqueness results still hold [8, Thm. 4.15, Thm. 4.16, Rem. 4.17].

**Proof strategy for Theorem 1.2.** The proof of Theorem 1.2 is fully carried out in Chapter 3. We were mostly inspired by the clever strategy [36] whose arguments we adapt to the Lorentzian setting at various instances.

Roughly speaking, the proof is based on a bisection argument. Given \(\mu_{0}\) and \(\mu_{1}\) as in the assumptions, we choose an \(\ell_{p}\)-midpoint \(\mu_{1/2}\) of them at which \(U_{N}\) is maximal, see Lemma 3.4. By wTCD_{\ell}^{p}(K, N), it obeys (1.2) for time 1/2. Then we choose \(\ell_{p}\)-midpoints \(\mu_{1/4}\) of \(\mu_{0}\) and \(\mu_{1/2}\), and \(\mu_{3/4}\) of \(\mu_{1/2}\) and \(\mu_{1}\), in the same manner. Iteratively, we thus construct a collection of measures \(\mu_{t} \in D(\text{Ent}_{m})\) for every dyadic \(t \in [0, 1]\). Based on a crucial property of the distortion coefficients \(\sigma_{K,N}^{(r)}\), stated in Lemma 3.5, the inequality (1.2) — which is a priori only true for \(\mu_{t}\) at time 1/2 with \(\mu_{0}\) and \(\mu_{1}\) replaced by its ancestors — extends to every dyadic \(t \in [0, 1]\) with \(\mu_{0}\) and \(\mu_{1}\) on the r.h.s. This inequality is stable under weak completion to a timelike proper-time parametrized \(\ell_{p}\)-geodesic \((\mu_{t})_{t \in [0, 1]}\) defined on all of \([0, 1]\), obtained by intermediate gluing of timelike \(\ell_{p}\)-optimal geodesic plans, and passage to the limit.
GOOD GEODESICS SATISFYING THE TCD CONDITION

As afterwards discussed in Section 3.4, maximality of $U_N$ already suffices to guarantee the desired density bounds for $(\mu_t)_{t \in [0,1]}$. The reason is that by the curvature-dimension condition, for appropriate endpoints mass has to be spread along some timelike proper-time parametrized $\ell_p$-geodesic in a certain way, see Lemma 3.8. If the density of some midpoint in our construction was too large, we could use the latter geodesic to shuffle mass around and to build a midpoint with strictly smaller entropy, contradicting the maximality of $U_N$ at our chosen midpoint. The proofs of the key Proposition 3.11 and Proposition 3.13 are based on this principle.

Contrary to [35, 36], however, we do not really work with bare $\ell_p$-intermediate points, but with intermediate slices of timelike $\ell_p$-optimal geodesic plans, cf. Subsection 2.2.4. An $\ell_p$-intermediate point does not need to admit a chronological coupling to any of its endpoints, which is however required to invoke the wTCD condition. On the other hand, this is clear if the chosen intermediate point already lies on a timelike $\ell_p$-optimal geodesic plan. In fact, the notion of strong timelike $p$-dualizability is preserved along such plans, see Lemma 3.1. (For timelike $p$-dualizability arising in the definition of $\text{TCD}_p^e(K, N)$, however, this is unclear, cf. Remark 3.15.)

**Organization.** In Chapter 2, we shortly review the theories of Lorentzian pre-length spaces, Lorentzian optimal transport, and timelike curvature-dimension conditions. Chapter 3 contains the proof of Theorem 1.2, while Chapter 4 outlines, and partially proves, the above mentioned extensions of Theorem 1.2.

## 2. Optimal transport on Lorentzian spaces

This chapter recalls recent progress in nonsmooth Lorentzian geometry and optimal transport theory on such spaces. The reader is invited to consult the main references [12, 26] for more details, proofs, and especially examples.

### 2.1. Lorentzian geodesic spaces.

#### 2.1.1. Basic assumptions and notation.

Everywhere in this paper, let $(M, d)$ be a proper — hence complete and separable — metric space. All topological and measure-theoretic properties are understood with respect to the topology induced by $d$ and its induced Borel $\sigma$-algebra, respectively.

Moreover, we always fix a nontrivial Radon measure $m$ on $M$. For simplicity, we assume that $m$ is fully supported, in symbols $\text{spt } m = M$.

Let $\mathcal{P}(M)$ be the set of all probability measures on $M$. Let $\mathcal{P}_c (M)$ and $\mathcal{P}^{\text{ac}}(M, m)$ be its subsets of all elements with compact support and $m$-absolutely continuous measures, respectively, and set $\mathcal{P}^{\text{ac}}_c(M, m) := \mathcal{P}_c (M) \cap \mathcal{P}^{\text{ac}}(M, m)$. Whenever we say that a specified property is satisfied “subject to the decomposition $\mu = \rho \, m + \mu_\perp$”, we mean that $\mu_\perp$ is the $m$-singular part in the Lebesgue decomposition of $\mu \in \mathcal{P}(M)$ with respect to $m$, and that $\mu = \mu_\perp = \rho \, m \in \mathcal{P}^{\text{ac}}(M, m)$.

For a Borel map $F : M \to M'$ into a metric space $(M', d')$ as well as $\mu \in \mathcal{P}(M)$, $F_\sharp \mu \in \mathcal{P}(M')$ designates the usual push-forward of $\mu$ under $F$ given by $F_\sharp \mu(B) := \mu[F^{-1}(B)]$ for every Borel set $B \subseteq M'$.

Let $\mathcal{P}(M)$, let $\Pi(\mu, \nu)$ be the set of all couplings of $\mu$ and $\nu$, i.e. all $\pi \in \mathcal{P}(M^2)$ with $\pi[\cdot \times M] = \mu$ and $\pi[M \times \cdot] = \nu$.

Let $C([0, 1]; M)$ denote the set of all continuous curves $\gamma : [0, 1] \to M$, endowed with the uniform topology. For $t \in [0, 1]$, the evaluation map $e_t : C([0, 1]; M) \to M$ is defined by $e_t(\gamma) := \gamma_t$. 
2.1.2. Chronology and causality. Throughout, we fix a preorder ≤ and a transitive relation ≪, contained in ≤, on \( M \). We say that \( x, y \in M \) are in timelike or causal relation if \( x ≪ y \) or \( x ≤ y \), respectively. The triple \( (M, ≪, ≤) \) is called causal space [26, Def. 2.1]. We write \( x < y \) provided \( x ≤ y \) yet \( x ≠ y \). Let us set
\[
M^2_≤ := \{(x, y) ∈ M^2 : x ≤ y\},
M^2_≪ := \{(x, y) ∈ M^2 : x ≪ y\}.
\]

**Definition 2.1.** We term a causal space \( (M, ≪, ≤) \) causally closed if \( ≤ \) is closed, i.e. \( M^2_≤ \) is closed in \( M^2 \).

Given a (not necessarily Borel) set \( A ⊂ M \), we define [26, Def. 2.3] the chronological future \( I^+(A) \subset M \) and the causal future \( J^+(A) \subset M \) of \( A \) by
\[
I^+(A) := \{y ∈ M : x ≪ y \text{ for some } x ∈ A\},
J^+(A) := \{y ∈ M : x ≤ y \text{ for some } x ∈ A\}.
\]

Analogously, the chronological past \( I^-(A) \) and the causal past \( J^-(A) \) of \( A \) are defined. By a slight abuse of notation, given \( x ∈ M \) we write \( I^x := I^+(\{x\}) \) and \( J^x := J^-(\{x\}) \). For \( µ ∈ P(M) \), we write \( I^x(µ) := I^+(spt µ) \) and \( J^x(µ) := J^-(spt µ) \). For all these objects, we set \( I(A, B) := I^+(A) ∩ I^-(B) \), and we define \( I(x, y), J(µ, v), J(A, B), J(x, y), J(µ, µ) \) analogously.

2.1.3. Lorentzian pre-length spaces. A function \( τ : M^2 → [0, ∞] \) is a time separation function [26, Def. 2.8] if it is lower semicontinuous, and for every \( x, y, z ∈ M \),

a. \( τ(x, y) = 0 \) if \( x ≲ y \),
b. \( τ(x, y) > 0 \) if and only if \( x ≲ y \) — in other words, \( M^2_≤ = \{τ = 0\} \) — and
c. if \( x ≲ y ≲ z \) we have the reverse triangle inequality
\[
(τ(x, z) ≥ τ(x, y) + τ(y, z)). \tag{2.1}
\]

The existence of such a \( τ \) implies that ≪ is an open relation [26, Prop. 2.13], whence \( I^+(A) \) is open for every \( A ⊂ M \) [26, Lem. 2.12].

**Remark 2.2.** Besides (2.1), unlike the metric in metric geometry \( τ \) is only symmetric in pathological cases: for every \( x ∈ M \), either \( τ(x, x) = 0 \) or \( τ(x, x) = ∞ \), and if \( τ(x, y) ∈ (0, ∞) \), then \( τ(y, x) = 0 \) [26, Prop. 2.14].

**Definition 2.3.** A Lorentzian pre-length space is a quintuple \( (M, d, ≪, ≤, τ) \) consisting of a causal space \( (M, ≪, ≤) \) equipped with a proper metric \( d \) and a time separation function \( τ \) as above.

2.1.4. Length of curves. Let \( (M, d, ≪, ≤, τ) \) be a Lorentzian pre-length space. A curve is a continuous map \( γ : [a, b] → M, a, b ∈ R \) with \( a < b \). Such a curve \( γ \) is called (future-directed) timelike or (future-directed) causal if it is Lipschitz continuous with respect to \( d \), and \( γ_s ≪ γ_t \) or \( γ_s ≤ γ_t \) for every \( s, t ∈ [a, b] \) with \( s < t \), respectively. It is null if it is causal and \( γ_s ≺ γ_b \). Analogous notions make sense for past-directed curves and their causal character (i.e. their property of being chronological, causal, or null). Unless stated otherwise, every curve of a specified causal character is assumed future-directed.

The length of a curve \( γ : [a, b] → M \) is defined through
\[
Len_γ := \inf\{τ(γ_{t_0}, γ_{t_1}) + … + τ(γ_{t_{n-1}}, γ_{t_n})\},
\]
where the infimum is taken over all \( n ∈ N \) and all \( t_0, …, t_n ∈ [a, b] \) with \( t_0 = 0, t_n = 1, \) and \( t_i < t_{i+1} \) for every \( i ∈ \{0, …, n - 1\} \) [26, Def. 2.24]. It is additive with respect to restriction to disjoint partitions [26, Lem. 2.25]. Reparametrizations do neither change causal characters [26, Lem 2.27] nor the τ-length [26, Lem. 2.28].
2.1.5. Geodesics. A future-directed causal curve \( \gamma : [a, b] \to M \) is a geodesic (or maximal) provided \( \text{Len}_\tau(\gamma) = \tau(\gamma_a, \gamma_b) \) [26, Def. 2.33]. The spaces of all such curves is denoted by \( \text{Geo}(M) \subset \text{Lip}([0, 1]; M) \), and its subset of timelike curves is called \( \text{TGeo}(M) \).

Recall that every element of \( \text{TGeo}(M) \) has a weak parametrization [26, Def. 3.31] by \( \tau \)-arclength if \( \tau \) is continuous and \( \tau(x, x) = 0 \) for every \( x \in M \) [26, Cor. 3.35]. This induces a natural reparametrization map \( r : \text{TGeo}(M) \to C([0, 1]; M) \) which is continuous [8, Lem. B.6]. In particular, all elements \( \eta \) of

\[
\text{TGeo}^\tau(M) := r(\text{TGeo}(M))
\]

are timelike and proper-time parametrized, i.e. for every \( s, t \in [0, 1] \) with \( s < t \),

\[
\tau(\eta_s, \eta_t) = (t - s) \tau(\eta_0, \eta_1) > 0.
\]

In general, elements of \( \text{TGeo}^\tau(M) \) are not Lipschitz continuous [26, p. 424]. Yet, as \( \text{TGeo}^\tau(M) \) is the continuous image of \( \text{TGeo}(M) \) which, as subset of \( \text{Lip}([0, 1]; M) \), will have better compactness properties under Assumption 3.2, cf. Subsection 2.1.7, the latter transfer to \( \text{TGeo}^\tau(M) \); compare with [8, Sec. B.3] and Lemma 2.11.

We call \( (M, d, \ll, \leq, \tau) \) geodesic if for every \( x, y \in M \) with \( x < y \) there exists a future-directed causal geodesic \( \gamma \in \text{Geo}(M) \) with initial point \( x \) and final point \( y \).

2.1.6. Regularity. We will later assume \( (M, d, \ll, \leq, \tau) \) to be regular (ly localizable) according to [26, Def. 3.16]. Instead of giving this rather technical definition, let us list its most important consequences which will be relevant for our purposes. Under regularity, geodesy, and global hyperbolicity, cf. Subsection 2.1.7 below, the \( \tau \)-length \( \text{Len}_\tau \) is upper semicontinuous in the following sense [26, Prop. 3.17, Thm. 3.26]: if \( (\gamma_n)_{n \in \mathbb{N}} \) is a sequence of causal curves \( \gamma_n : [0, 1] \to M \) converging uniformly to a causal curve \( \gamma : [0, 1] \to M \), then \( \text{Len}_\tau(\gamma) \geq \limsup_{n \to \infty} \text{Len}_\tau(\gamma_n) \).

Moreover, in regular Lorentzian pre-length spaces, causal geodesics have a causal character, i.e. they are either timelike or null (hence do not switch from timelike to causal or vice versa) [26, Thm. 3.18]. In this case, a causal geodesic \( \gamma : [0, 1] \to M \) is timelike if and only if \( \tau(\gamma_0, \gamma_1) > 0 \). In particular, on regular Lorentzian geodesic spaces, every \( x, y \in M \) with \( x \ll y \) can be connected by a timelike geodesic — which does not follow from geodesy alone — and the set of all such geodesics is closed for fixed \( x \) and \( y \) [8, Lem. B.1].

2.1.7. Global hyperbolicity. Next, we introduce a useful condition which ensures both that the time separation function \( \tau \) behaves nicely, and that geodesics exist.

Following [12, Sec. 1.1], we call \( (M, d, \ll, \leq, \tau) \) non-totally imprisoning if for every compact \( C \subset M \) there exists a constant \( c > 0 \) such that the arclength of every causal curve in \( C \) with respect to \( d \) is bounded from above by \( c \).

**Definition 2.4.** A Lorentzian pre-length space \( (M, d, \ll, \leq, \tau) \) is globally hyperbolic if it is non-totally imprisoning and \( J(x, y) \) is compact for every \( x, y \in M \). It is \( \mathcal{K} \)-globally hyperbolic if \( J(K_0, K_1) \) is compact for all compact \( K_0, K_1 \subset M \).

**Remark 2.5.** \( \mathcal{K} \)-global hyperbolicity is not much more restrictive than global hyperbolicity: if in addition to the latter, \( (M, d, \ll, \leq, \tau) \) is locally causally closed [26, Def. 3.4] and \( J^\pm(x) \neq \emptyset \) for every \( x \in M \), \( \mathcal{K} \)-global hyperbolicity holds true [12, Lem. 1.5].

On the other hand, every locally causally closed, \( \mathcal{K} \)-globally hyperbolic Lorentzian geodesic space is in fact causally closed [12, Lem. 1.6].

Thanks to [26, Def. 3.25, Thm. 3.26], global hyperbolicity implies the nonsmooth analogue of the well-known strong causality condition for smooth Lorentzian spacetimes [33, Def. 14.11]. On every globally hyperbolic Lorentzian length space (see
### Lorentz–Wasserstein distance

Smooth Lorentzian theories of optimal transport have been studied in [15, 25, 29, 32, 42]. We review the cornerstones of the accompanying nonsmooth theory recently developed in [12] now.

#### Chronological and causal couplings

Let \((M, d, \ll, \leq, \tau)\) be a Lorentzian prelength space, and \(\mu, \nu \in \mathcal{P}(M)\). We define the set \(\Pi_{\leq}(\mu, \nu)\) of all chronological couplings of \(\mu\) and \(\nu\) to consist of all \(\pi \in \Pi(\mu, \nu)\) with \(\pi[M^2_{\leq}] = 1\). Similarly, the set \(\Pi_{\leq}(\mu, \nu)\) of all causal couplings of \(\mu\) and \(\nu\) is defined. If \((M, d, \ll, \leq, \tau)\) is causally closed, clearly \(\pi \in \Pi_{\leq}(\mu, \nu)\) if and only if \(\pi \in \Pi(\mu, \nu)\) and \(\text{spt} \pi \subset M^2_{\leq}\).

Intuitively, a chronological or causal coupling of \(\mu\) and \(\nu\) describes a law for transporting an infinitesimal mass portion \(d\mu(x)\) to an infinitesimal mass portion \(d\nu(y)\) in such a way that \(x \ll y\) or \(x \leq y\), respectively.

#### The \(\ell_p\)-optimal transport problem

In the following, given any \(p \in (0, 1)\) and following [12, 29] we adopt the conventions

\[
\sup \emptyset := -\infty, \\
(-\infty)^p := (\infty)^{1/p} := -\infty, \\
\infty - \infty := -\infty.
\]

The total transport cost function \(\ell_p: \mathcal{P}(M)^2 \to [0, \infty] \cup \{-\infty\}\) is given by

\[
\ell_p(\mu, \nu) := \sup \left\{ \|\pi\|_{L^p(M^2, \pi)} : \pi \in \Pi_{\leq}(\mu, \nu) \right\} \\
= \sup \left\{ \|l\|_{L^p(M^2, \pi)} : \pi \in \Pi(\mu, \nu) \right\}.
\]

Here, the function \(l: M^2 \to [0, \infty] \cup \{-\infty\}\) is defined by

\[
l^p(x, y) := \begin{cases} 
\tau^p(x, y) & \text{if } x \leq y, \\
-\infty & \text{otherwise}.
\end{cases}
\]

**Remark 2.6.** The sets of maximizers for both suprema defining \(\ell_p(\mu, \nu)\) coincide. One advantage of the second formulation is that under (local) causal closedness and global hyperbolicity assumptions, \(l^p\) is upper semicontinuous on \(M^2\). In this case, standard optimal transport techniques [1, 43] can be applied to study the second problem, which in turn yields results for the first [12, Rem. 2.2]. Moreover, the preimages \(l^{-1}([0, \infty))\) and \(l^{-1}((0, \infty))\) conveniently encode causality and chronology of points in \(M^2\), respectively.

A coupling \(\pi \in \Pi(\mu, \nu)\) of \(\mu, \nu \in \mathcal{P}(M)\) is \(\ell_p\)-optimal if \(\pi \in \Pi_{\leq}(\mu, \nu)\) and

\[
\ell_p(\mu, \nu) = \|\pi\|_{L^p(M^2, \pi)} = \|l\|_{L^p(M^2, \pi)}.
\]

If \((M, d, \ll, \leq, \tau)\) is locally causally closed and globally hyperbolic, and if \(\mu, \nu \in \mathcal{P}_c(M)\) with \(\Pi_{\leq}(\mu, \nu) \neq \emptyset\), then there exists an \(\ell_p\)-optimal coupling \(\pi\) of \(\mu\) and \(\nu\), and its total transport cost \(\|\pi\|_{L^p(M^2, \pi)}\) is finite [12, Prop. 2.3].

Lastly, an important property of \(\ell_p\) is the reverse triangle inequality [12, Prop. 2.5] strongly reminiscent of (2.1): for every \(\mu, \nu, \sigma \in \mathcal{P}(M)\),

\[
\ell_p(\mu, \sigma) \geq \ell_p(\mu, \nu) + \ell_p(\nu, \sigma).
\]

\[\text{This holds in more generality, but the named case will be the only relevant in our work.}\]
2.2.3. Timelike $p$-dualizability. Next, we review the concept of (strong) timelike $p$-dualizability, $p \in (0, 1]$, of pairs $(\mu, \nu) \in \mathcal{P}(M)$. It originates in [12] and generalizes the notion of $p$-separation from [29, Def. 4.1]. Pairs satisfying this condition allow for a good duality theory [12, Prop. 2.19, Prop. 2.21, Thm. 2.26], which has been used to characterize $\ell_p$-geodesics in the smooth case [29, Thm. 4.3, Thm. 5.8]. In our case, it is needed to set up the timelike curvature-dimension condition from Definition 2.16 below.

In view of the subsequent Definition 2.7 taken from [12, Def. 2.18, Def. 2.27], we refer to [12, Def. 2.6] for the inherent definition of cyclical monotonicity of a subset of $M^2_\infty$ with respect to $L^p$, which generalizes the standard concept of cyclical monotonicity with respect to any given cost function [43, Def. 5.1]. It will only be relevant in Lemma 3.1 below.

As usual, given any $a, b : M \to \mathbb{R}$ we define the function $a \oplus b : M^2 \to \mathbb{R}$ by $(a \oplus b)(x, y) := a(x) + b(y)$.

**Definition 2.7.** Given $p \in (0, 1]$, a pair $(\mu, \nu) \in \mathcal{P}(M)$ is termed

- timelike $p$-dualizable by $\pi \in \Pi_<(\mu, \nu)$ if $\pi$ is an $\ell_p$-optimal coupling, and there exist Borel functions $a, b : M \to \mathbb{R}$ with $a \oplus b \in L^1(M^2, \mu \otimes \nu)$ and $L^p \subseteq a \oplus b$ on $\text{spt} \mu \times \text{spt} \nu$,
- strongly timelike $p$-dualizable by $\pi \in \Pi_<(\mu, \nu)$ provided $(\mu, \nu)$ is timelike $p$-dualizable by $\pi$, and there exists some $L^p$-cyclically monotone Borel set $\Gamma \subset M^2_\infty \cap (\text{spt} \mu_0 \times \text{spt} \mu_1)$ such that any given coupling $\sigma \in \Pi_\leq(\mu_0, \mu_1)$ is $\ell_p$-optimal if and only if $\sigma(\Gamma) = 1$, and
- timelike $p$-dualizable if $(\mu, \nu)$ is timelike $p$-dualizable by some $\pi \in \Pi_<(\mu, \nu)$; analogously for strong timelike $p$-dualizability.

Moreover, any $\pi$ as above is called timelike $p$-dualizing.

In this framework, we define

$$
\text{TD}_p(M) := \{ (\mu, \nu) \in \mathcal{P}(M)^2 : (\mu, \nu) \text{ timelike } p\text{-dualizable} \},$
$$
$$
\text{STD}_p(M) := \{ (\mu, \nu) \in \mathcal{P}(M)^2 : (\mu, \nu) \text{ strongly timelike } p\text{-dualizable} \}.
$$

**Remark 2.8.** It will be useful to keep in mind that by definition, every $\ell_p$-optimal coupling of a strongly timelike $p$-dualizable pair is concentrated on $M^2_\infty$.

**Example 2.9.** Evidently, if $(\mu, \nu) \in \mathcal{P}_c(M)$, then $(\mu, \nu)$ is timelike $p$-dualizable if and only if there exists an $\ell_p$-optimal coupling $\pi \in \Pi_<(\mu, \nu)$ concentrated on $M^2_\infty$.

A relevant example (already in the smooth case, cf. [29, Lem. 4.4, Thm. 7.1]) for the strong version is the following. If $(\mu, \nu) \in \mathcal{P}_c(M)$ on a locally causally closed, globally hyperbolic Lorentzian geodesic space $(M, d, \ll, \leq, \tau)$ with $\text{spt} \mu \times \text{spt} \nu \subset M^2_\infty$, then the pair $(\mu, \nu)$ is strongly timelike $p$-dualizable, $p \in (0, 1]$ [12, Cor. 2.29].

2.2.4. Geodesics revisited. Given $p \in (0, 1]$, following [8, Subsec. 2.3.6, App. B], see also [12, Def. 2.31] and [29, Def. 1.1], we now recall the nonsmooth notion of timelike proper-time parametrized $\ell_p$-geodesics.

Recall the continuous reparametrization map $r$ for elements of $\text{TGeo}(M)$ introduced in Subsection 2.1.5. For $\mu_0, \mu_1 \in \mathcal{P}(M)$, we define

$$
\text{OptTGeo}_{\ell_p}(\mu_0, \mu_1) := \{ \pi \in \mathcal{P}(\text{Geo}(M)) : (e_0, e_1)_{\ell_p} \pi \in \Pi_<(\mu_0, \mu_1) \text{ is } \ell_p\text{-optimal} \},$
$$
$$
\text{OptTGeo}_{\ell_p}^+(\mu_0, \mu_1) := r \text{OptTGeo}_{\ell_p}(\mu_0, \mu_1).
$$

All elements of the latter class are concentrated on the set $\text{TGeo}^+(M)$ from (2.2).

In the following definition, we say that $\pi \in \mathcal{P}(C([0, 1]; M))$ represents a curve $(\mu_t)_{t \in [0, 1]}$ in $\mathcal{P}(M)$ if $\mu_t = (e_t)_{\ell_p} \pi$ holds for every $t \in [0, 1]$. 
Definition 2.10. A collection \((\mu_t)_{t\in[0,1]}\) of elements of \(\mathcal{P}(M)\) is termed timelike proper-time parametrized \(\ell_p\)-geodesic if it is represented by some element \(\pi\) belonging to \(\text{OptTGeo}^p_\ell(\mu_0, \mu_1)\). Any such \(\pi\) will be called timelike \(\ell_p\)-optimal geodesic plan.

Note that every curve \((\mu_t)_{t\in[0,1]}\) as in Definition 2.10 obeys
\[
\ell_p(\mu_s, \mu_t) = (t-s) \ell_p(\mu_0, \mu_1) > 0.
\]
Thus, \((\mu_t)_{t\in[0,1]}\) is an \(\ell_p\)-geodesic in the sense of [12, Def. 2.13] and [29, Def. 1.1] provided \(\ell_p(\mu_0, \mu_1) < \infty\). By regularity and geodesy of \((M, \mathcal{d}, \ll, \leq, \tau)\) and a standard measurable selection argument, cf. Lemma 2.11 and Assumption 3.2 below, timelike proper-time parametrized \(\ell_p\)-geodesics exist in great generality.

We then have the following result from [8, Prop. B.11] used at many occasions below. The compactness of \(M\) assumed therein is only made for notational simplicity and is not restrictive, as Lemma 2.11 will always be applied within causal diamonds which are compact by assumption. Note that the chronology assumption on the limit marginals in the last clause therein is essential. Indeed, unlike \(\ell_p\)-optimality [12, Sec. 2.3], chronology is in general not stable under weak limits (in contrast to causality, which will be a closed condition by assumption).

To formulate the lemma, given \(s, t \in [0,1]\) with \(s < t\), let \(\text{restr}^t_s: C([0,1]; M) \to C([0,1]; M)\) be the restriction map defined by
\[
\text{restr}^t_s(\gamma)_r := \gamma((1-r)s + rt).
\]

Lemma 2.11. Let \(p \in (0,1]\) and suppose that \((M, \mathcal{d}, \ll, \leq, \tau)\) is a compact, causally closed, \(\mathcal{K}\)-globally hyperbolic, regular Lorentzian geodesic space. Suppose \((\mu_0, \mu_1) \in \text{TD}_p(M)\). Then the following properties hold.

(i) For every \(\ell_p\)-optimal \(\pi \in \Pi_{\ll}(\mu_0, \mu_1)\), there is \(\pi \in \text{OptTGeo}^p_\ell(\mu_0, \mu_1)\) such that \(\pi = (e_0, e_1)\# \pi\).

(ii) There is at least one proper-time parametrized \(\ell_p\)-geodesic from \(\mu_0\) to \(\mu_1\).

(iii) For every \(\pi \in \text{OptTGeo}^p_\ell(\mu_0, \mu_1)\) and every \(s, t \in [0,1]\) with \(s < t\),
\[
(\text{restr}^t_s)\# \pi \in \text{OptTGeo}^p_\ell((e_s, e_t), (e_t, e_1)\# \pi).
\]

(iv) If \(\pi \in \text{OptTGeo}^p_\ell(\mu_0, \mu_1)\) and if \(\sigma\) is any nontrivial measure on \(C([0,1]; M)\) with \(\sigma \leq \pi\), then \(\sigma[C([0,1]; M)]^{-1}\sigma\) is an element of \(\text{OptTGeo}^p_\ell(\sigma_0, \sigma_1)\), where \(\sigma_i := \sigma[C([0,1]; M)]^{-1}(e_i, e_1)\# \sigma \in \mathcal{P}(M), i \in \{0,1\}\).

(v) If \((\mu_0, \mu_1) \in \text{STD}_p(M)\) is the weak limit of a given sequence \((\mu^n_0, \mu^n_1)_{n \in \mathbb{N}}\) in \(\mathcal{P}(M)^2\), then every sequence \((\pi^n)_{n \in \mathbb{N}}\) satisfying \(\pi^n \in \text{OptTGeo}^p_\ell(\mu^n_0, \mu^n_1)\) for every \(n \in \mathbb{N}\) has an accumulation point, and any such point belongs to \(\text{OptTGeo}^p_\ell(\mu_0, \mu_1)\).

2.3. Entropic timelike curvature-dimension condition.

Definition 2.12. A sextuple \((M, \mathcal{d}, m, \ll, \leq, \tau)\) consisting of a Lorentzian pre-length space \((M, \mathcal{d}, \ll, \leq, \tau)\) endowed with a Radon measure \(m\) as hypothesized in Subsection 2.1.1 will be called measured Lorentzian pre-length space.

For measured Lorentzian pre-length spaces, all notions from Section 2.1 are understood with respect to the inherent Lorentzian pre-length structure.

2.3.1. Timelike \((K, N, p)\)-convexity. For later convenience, we introduce the following Definition 2.13 leaned on [29, Def. 6.5]. With a slight abuse of notation compared to (1.1), given a functional \(E: \mathcal{P}(M) \to [-\infty, \infty]\) with sufficiently large finiteness domain \(\mathcal{D}(E) \subset \mathcal{P}(M)\) and \(N \in (0, \infty)\), define \(U_N: \mathcal{P}(M) \to [0, \infty]\) by \(U_N(\mu) := e^{-E(\mu)/N}\). In our work, the most relevant functional \(E\) is the relative entropy \(\text{Ent}_m\) introduced in Subsection 2.3.2 below, but see also Section 4.2.
For $K \in \mathbb{R}$, $r \in [0, 1]$, and $\vartheta \in [0, \infty]$, we consider the distortion coefficients

$$
\sigma_{K,N}(\vartheta) := \begin{cases} 
\frac{\sin(r\vartheta \sqrt{K/N})}{\sin(\vartheta \sqrt{K/N})} & \text{if } 0 < K \vartheta^2 < N\pi^2, \\
\frac{\sinh(r\vartheta \sqrt{-K/N})}{\sinh(\vartheta \sqrt{-K/N})} & \text{if } K \vartheta^2 = 0, \\
\infty & \text{if } K \vartheta^2 > \pi^2.
\end{cases}
$$

Here we employ the convention $0 \cdot \infty := 0$.

**Definition 2.13.** Let $p \in (0, 1)$, $K \in \mathbb{R}$, and $N \in (0, \infty)$. We say that a functional $E : \mathcal{P}(M) \to [-\infty, \infty]$ on a Lorentzian pre-length space $(M, d, \ll, \leq, \tau)$ is $(K, N, p)$-convex relative to $\mathcal{Q} \subset \mathcal{P}(M)^2$ if the following holds. For every $\mu_0, \mu_1 \in \mathcal{Q} \cap \mathcal{D}(E)^2$ with $\ell_p(\mu_0, \mu_1) < \infty$ there exist an $\ell_p$-optimal coupling $\pi \in \Pi_\infty(\mu_0, \mu_1)$ and a timelike proper-time parametrized $\ell_p$-geodesic $(\mu_t)_{t \in [0,1]}$ from $\mu_0$ to $\mu_1$ such that for every $t \in [0,1]$, we have

$$
\mathcal{U}_N(\mu_t) \geq \sigma_{K,N}(1-t) \mathcal{U}_N(\mu_0) + \sigma_{K,N}(t) \mathcal{U}_N(\mu_1).
$$

**Remark 2.14.** Unlike the metric definition of $(K, N)$-convex functions [16, Def. 2.7], in Definition 2.13 the pathological situation $T_\tau = \infty$ might occur. This either reduces to a trivial condition $(K < 0)$, does not involve any $L^2$-norm of $\tau$ at all $(K = 0)$ or — for $K > 0$ and in the relevant case $E = \text{Ent}_\mu$ — cannot hold by the timelike Bonnet–Myers theorem [12, Prop. 3.6].

**Remark 2.15.** In the framework of Definition 2.13, define $e : [0, 1] \to [-\infty, \infty]$ by $e(t) := E(\mu_t)$. Then $(K, N, p)$-convexity of $E$ is equivalent to the following. For every $\mu_0, \mu_1 \in \mathcal{Q} \cap \mathcal{D}(E)^2$ with $\ell_p(\mu_0, \mu_1) < \infty$ there exist an $\ell_p$-optimal coupling $\pi \in \Pi_\infty(\mu_0, \mu_1)$ and a timelike proper-time parametrized $\ell_p$-geodesic $(\mu_t)_{t \in [0,1]}$ such that if $e^{-1}((-\infty))$ is empty and $\|\tau\|_{L^2(M^2, \pi)} < \infty$, then $e$ is semiconvex on $(0, 1)$ and satisfies

$$
\tilde{e} - \frac{1}{\tilde{N}} \tilde{e}^2 \geq K \|\tau\|_{L^2(M^2, \pi)}^2
$$

in the distributional sense on $(0, 1)$.

2.3.2. **Relative entropy.** We define $\text{Ent}_\mu : \mathcal{P}(M) \to [-\infty, \infty]$ by

$$
\text{Ent}_\mu(\mu) = \begin{cases} 
\int_M \rho \log \rho \, dm & \text{if } \mu = \rho m \ll m, \ (\rho \log \rho)^+ \in L^1(M, m), \\
\infty & \text{otherwise}.
\end{cases}
$$

This functional possesses the following properties, details of which can be found in [12, 29, 39]. By Jensen’s inequality, $\text{Ent}_\mu(\mu) \geq - \log m[\text{spt} \mu] > -\infty$ for every $\mu \in \mathcal{P}(M)$. Moreover, $\text{Ent}_\mu$ is weakly lower semicontinuous in the following form: if a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{P}(M)$ converges weakly to $\mu \in \mathcal{P}(M)$ and there is a Borel set $C \subset M$ with $m[C] < \infty$ and $\text{spt} \mu_n \subset C$ for every $n \in \mathbb{N}$, then $\text{Ent}_\mu(\mu) \leq \liminf_{n \to \infty} \text{Ent}_\mu(\mu_n)$.

2.3.3. **The curvature-dimension condition.** Now we finally come to the main definition from [12], namely [12, Def. 3.2], based on the groundbreaking results [29, Cor. 6.6, Cor. 7.5] and [32, Cor. 4.4].

Recall the definition (1.1) of the exponentiated relative entropy $\mathcal{U}_N$.

**Definition 2.16.** Let $(M, d, m, \ll, \leq, \tau)$ be a measured Lorentzian pre-length space, and let $p \in (0, 1)$, $K \in \mathbb{R}$, and $N \in (0, \infty)$. We say that the former satisfies
a. the (entropic) timelike curvature-dimension condition $\text{TCD}^p_p(K, N)$ if $\mathcal{U}_N$ is $(K, N, p)$-convex relative to $\mathcal{T}D_p(M)$, and

b. the weak (entropic) timelike curvature-dimension condition $\text{wTCD}^p_p(K, N)$ if $\mathcal{U}_N$ is $(K, N, p)$-convex relative to $\text{STD}_p(M) \cap \mathcal{P}_c(M)^2$.

Remark 2.17. If the space $(M, d, m, \ll, \leq, \tau)$ is $\mathcal{K}$-globally hyperbolic and satisfies the $\text{wTCD}^p_p(K, N)$ condition (in fact, $\text{TMCP}^p_p(K, N)$ according to Definition 4.9 suffices), then it is timelike geodesic. If in addition, it is causally path connected [26, Def. 3.4] — in particular, if $(M, d, m, \ll, \leq, \tau)$ is a Lorentzian length space — then it is geodesic [12, Rem. 3.9]. Hence, we may and will always assume the geodesic property with no restriction.

In Section 4.1, we introduce an “infinite-dimensional” analogue of the $\text{wTCD}$ condition in the spirit of [39].

Among the many properties of these TCD and $\text{wTCD}$ conditions proven in [12], let us quote: the timelike Brunn–Minkowski inequality [12, Prop. 3.4], the timelike Bishop-Gromov inequality [12, Prop. 3.5], the timelike Bonnet–Myers inequality [12, Prop. 3.6], consistency and scaling properties [12, Lem. 3.10], or nonsmooth Hawking–Penrose singularity theorems [12, Thm. 5.6, Cor. 5.13]. The limit of a sequence of measured Lorentzian geodesic $\text{TCD}^p_p(K, N)$ spaces converging weakly, in a certain sense, is (only) $\text{wTCD}^p_p(K, N)$ [12, Thm. 3.12]. Lastly, under the additional assumption of timelike nonbranching, the following hold. Given $\mu_0 \in \mathcal{D}(\text{Ent}_m)$ and $\mu_1 \in \mathcal{P}(M)$ admitting an $\ell_p$-optimal coupling in $\Pi_\leq(\mu_0, \mu_1)$, we have uniqueness of $\ell_p$-optimal couplings of $\mu_0$ to $\mu_1$ [12, Thm. 3.19]; similarly, they are connected by a unique timelike proper-time parametrized $\ell_p$-geodesic [12, Thm. 3.20].

Remark 2.18. Except for the Bonnet–Myers inequality, all preceding results are in fact valid under the weaker [12, Prop. 3.11] timelike measure contraction property from Definition 4.9 below.

3. Existence of good geodesics

In this chapter, we prove Theorem 1.2. We show every intermediate result under the most general assumptions, possibly beyond those of Theorem 1.2. Together, however, these reduce precisely to the hypotheses of our main result.

3.1. Strong timelike $p$-dualizability along $\ell_p$-geodesics. The main argument for the construction of the timelike proper-time parametrized $\ell_p$-geodesic for Theorem 1.2 is based on bisection by iteratively selecting appropriate midpoints of timelike proper-time parametrized $\ell_p$-geodesics. To this aim, we have to ensure that strong timelike $p$-dualizability behaves well along these.

The proof of the corresponding nontrivial Lemma 3.1 is grounded on a private communication of the author with Fabio Cavalletti and Andrea Mondino.

Lemma 3.1. Let $(M, d, \ll, \leq, \tau)$ be a globally hyperbolic, regular Lorentzian geodesic space, $p \in (0, 1]$, and $(\mu_0, \mu_1) \in \mathcal{P}(M)^2$. Moreover, let $\pi \in \text{OptTGeo}^p_p(\mu_0, \mu_1)$ and define $\mu_t := (e_t)^* \pi \in \mathcal{P}(M)$, $t \in [0, 1]$. If the pair $(\mu_0, \mu_1)$ is (strongly) timelike $p$-dualizable, so is $(\mu_s, \mu_t)$ for every $s, t \in [0, 1]$ with $s < t$.

Proof. We assume that $s, t \in (0, 1)$, the case $\{s, t\} \cap \{0, 1\} \neq \emptyset$ is analogous. Note that $(e_0, e_t)^* \pi$ is concentrated on $M^2_\pi$, and so is $(e_s, e_t)^* \pi \in \Pi(\mu_s, \mu_t)$. Since the latter is $\ell_p$-optimal and the total cost

$$\ell_p(\mu_s, \mu_t) = (t - s) \ell_p(\mu_0, \mu_1)$$

is positive and finite, the pair $(\mu_s, \mu_t)$ is timelike $p$-dualizable.
GOOD GEODESICS SATISFYING THE TCD CONDITION

To show strong timelike $p$-dualizability of $(\mu_1, \mu_2)$ if $(\mu_0, \mu_1)$ has this property, we have to construct an $L^p$-cyclically monotone Borel set $\Gamma_{s,t} \subset M_\leq^2 \cap (\text{spt } \mu_s \times \text{spt } \mu_t)$ such that $\pi[\Gamma_{s,t}] = 1$ for every $\ell^p$-optimal coupling $\pi \in \Pi_\leq(\mu_s, \mu_t)$. To this aim, let $\Gamma \subset M_\leq^2 \cap (\text{spt } \mu_0 \times \text{spt } \mu_1)$ be an $L^p$-cyclically monotone Borel set on which every $\ell^p$-optimal coupling belonging to $\Pi_\leq(\mu_0, \mu_1)$ is concentrated, and define

$$\Gamma_{s,t} := (e_s, e_t)[(e_0, e_1)^{-1}(\Gamma)].$$

To show that $\Gamma_{s,t}$ is $L^p$-cyclically monotone, we follow the proof of [17, Lem. 4.4]. Let $n \in \mathbb{N}$ and $(x^1, y^1), \ldots, (x^n, y^n) \in \Gamma_{s,t}$, and select $\gamma^1, \ldots, \gamma^n \in \text{TGeo}^e(M)$ with $(x^i, y^i) = (\gamma^1, \gamma^i)$ for every $i \in \{1, \ldots, n\}$. Since $\Gamma$ is $L^p$-cyclically monotone and since $\gamma^i \in \text{TGeo}^e(M)$ for every $i \in \{1, \ldots, n\}$, the empirical measure $\sigma$ of $\gamma^1, \ldots, \gamma^n$ is a timelike $\ell^p$-optimal geodesic plan interpolating its endpoints [12, Prop. 2.8]. Therefore, $(e_s, e_t)_*\sigma$ is an $\ell^p$-optimal coupling of its marginals, and applying [12, Prop. 2.8] again yields the $L^p$-cyclical monotonicity of

$$\text{spt}(e_s, e_t)_*\sigma = \bigcup_{i=1}^n \{ (\gamma^i, \gamma^i) \} = \bigcup_{i=1}^n \{ (x^i, y^i) \}.$$

Given any $\ell^p$-optimal coupling $\pi \in \Pi_\leq(\mu_s, \mu_t)$, by gluing and a measurable selection argument as in the proof of [1, Thm. 2.10], using that $\mu_s$ and $\mu_t$ lie on a timelike proper-time parametrized $\ell^p$-geodesic, we find $\alpha \in \text{OptTGeo}^e_\ell(\mu_0, \mu_1)$ with $(e_s, e_t)_*\alpha = \pi$. Noting that

$$\pi[\Gamma_{s,t}] = (e_s, e_t)_*[\Gamma_{s,t}] = (e_0, e_1)_*[\Gamma_{s,t}] = 1$$

then terminates the proof.

3.2. Construction of a candidate. In this section, we construct an appropriate timelike proper-time parametrized $\ell^p$-geodesic $(\mu_t)_{t \in [0,1]}$ for which we verify in Section 3.3 and Section 3.4 that it satisfies the goodness properties from Definition 1.1.

Assumption 3.2. From now on, until the end of this article, and unless explicitly stated otherwise we assume $(M, d, m, \ll , \leq, \tau)$ to be a causally closed, $\mathcal{X}$-globally hyperbolic, regular Lorentzian geodesic space.

Given $N \in (0, \infty)$, let $\mathbb{U}_N$ be as in (1.1), and for $t \in (0, 1)$ define the functional

$$\mathcal{V}_N : \mathcal{P}(C([0, 1]; M)) \to [0, \infty]$$

through

$$\mathcal{V}_N(\tau) := \mathbb{U}_N((e_t)_*\tau).$$

Except for Section 4.3, we mostly work with the functional

$$\mathcal{V}_N := \mathcal{V}_N^{1/2}. \quad (3.1)$$

Remark 3.3. $\mathcal{V}_N$ only depends on a single slice of its argument, and one is tempted to follow the CD-treatise [35] more closely instead and consider the functional $\mathbb{U}_N$ on the set of $\ell^p$-midpoints of $\mu_0$ and $\mu_1$ similar to [35, Ch. 3] or [36, Sec. 3.2]. However, in our case it is more convenient to work with timelike $\ell^p$-geodesic plans. For instance, for $\pi \in \text{OptTGeo}^e_\ell(\mu_0, \mu_1), \mu_0, \mu_1 \in \mathcal{P}(M)$, the pairs $(\mu_0, (e_{1/2})_*\pi)$ and $((e_{1/2})_*\pi, \mu_1)$ inherit the dualizability and chronology properties of $(\mu_0, \mu_1)$ — needed e.g. for Proposition 3.11 below (and recall Lemma 3.1) — while this seems unclear for general $\ell^p$-midpoints.

Lemma 3.4. Let $p, t \in (0, 1), N \in (0, \infty)$, as well as $\mu_0, \mu_1 \in \text{STD}_p(M) \cap \mathcal{P}_c(M)^2$. Then $\mathcal{V}_N$ has a maximizer in $\text{OptTGeo}^e_\ell(\mu_0, \mu_1)$ with finite value. Moreover, if $\text{wTCD}^e(K, N)$ holds for some $K \in \mathbb{R}$ and $N \in (0, \infty)$, and if the pair $(\mu_0, \mu_1) \in (\mathcal{P}_c(M) \cap \mathcal{D}(\text{Ent}_{\text{Ah}}))^2$ is strongly timelike $p$-dualizable, for every maximizer $\pi \in \text{OptTGeo}^e_\ell(\mu_0, \mu_1)$ of $\mathcal{V}_N$ the measure $(e_t)_*\pi \in \mathcal{P}_c(M)$ has finite entropy; in particular $(e_t)_*\pi \ll m$. 

Proof. First, recall from Section 2.2 that \( \text{OptTGeo}_{p}^{\ell_{p}}(\mu_{0}, \mu_{1}) \neq \emptyset \). As \( \text{spt}(e_{t})_{\pi} = \{ \gamma_{t} : \gamma \in \text{spt } \pi \} \subset J(\mu_{0}, \mu_{1}) \) for every \( \pi \in \text{OptTGeo}_{p}^{\ell_{p}}(\mu_{0}, \mu_{1}) \), we have
\[
\mathcal{V}_{N}^{\ell_{p}}(\pi) \leq \mu \left[ J(\mu_{0}, \mu_{1}) \right]^{1/N}
\]
by Jensen’s inequality. Thus, \( \mathcal{V}_{N}^{\ell_{p}} \) is bounded on \( \text{OptTGeo}_{p}^{\ell_{p}}(\mu_{0}, \mu_{1}) \).

Moreover, \( \mathcal{V}_{N}^{\ell_{p}} \) is weakly upper semicontinuous on \( \text{OptTGeo}_{p}^{\ell_{p}}(\mu_{0}, \mu_{1}) \). Since the latter is weakly compact by Lemma 2.11, the existence of a maximizer for \( \mathcal{V}_{N}^{\ell_{p}} \) follows from the direct method.

The last claim follows by taking the \( t \)-slice of a timelike \( \ell_{p} \)-optimal geodesic plan representing a timelike proper-time parametrized \( \ell_{p} \)-geodesic from \( \mu_{0} \) to \( \mu_{1} \) witnessing the \((K, N, p)\)-convexity inequality of \( \mathcal{U}_{N} \) as a competitor. Hence, the maximum of \( \mathcal{V}_{N}^{\ell_{p}} \) is strictly positive. \( \square \)

We construct \( (\mu_{t})_{t \in [0, 1]} \) as follows. Let the pair \( \mu_{0}, \mu_{1} \in \mathcal{P}_{c}(M) \cap \mathcal{D}(\text{Ent}_{m}) \) be strongly timelike \( p \)-dualizable. Initially, set \( \mu_{1/2} := (e_{1/2})_{\pi_{1}} \in \mathcal{P}_{c}(M) \cap \mathcal{D}(\text{Ent}_{m}) \), where \( \pi_{1} \in \text{OptTGeo}_{p}^{\ell_{p}}(\mu_{0}, \mu_{1}) \) is a maximizer of \( \mathcal{V}_{N}^{\ell_{p}} \) according to Lemma 3.4.

By induction, suppose that for a given \( n \in \mathbb{N} \) we have defined \( \mu_{k-2n} \in \mathcal{P}_{c}(M) \in \mathcal{D}(\text{Ent}_{m}) \) for every \( k \in \{1, \ldots, 2^{n+1} - 1\} \). For every odd \( k \in \{1, \ldots, 2^{n+1} - 1\} \), by construction the pair \( (\mu_{k-12^{-n-1}}), (\mu_{(k+1)2^{-n-1}}) \in (\mathcal{P}_{c}(M) \cap \mathcal{D}(\text{Ent}_{m}))^{2} \) is strongly timelike \( p \)-dualizable thanks to Lemma 3.1. Let
\[
\pi_{n+1}^{k} \in \text{OptTGeo}_{p}^{\ell_{p}}(\mu_{(k-1)2^{-n-1}}, (k+1)2^{-n-1})
\]
maximize \( \mathcal{V}_{N}^{\ell_{p}} \) on the latter set, cf. Lemma 3.4. We glue together these timelike \( \ell_{p} \)-optimal geodesic plans \( \pi_{n+1}^{0}, \ldots, \pi_{n+1}^{2^{n+1}} \) and obtain \( \pi_{n+1} \in \text{OptTGeo}_{p}^{\ell_{p}}(\mu_{0}, \mu_{1}) \).

Inductively, we thus get a sequence \( (\pi_{n})_{n \in \mathbb{N}} \) in \( \text{OptTGeo}_{p}^{\ell_{p}}(\mu_{0}, \mu_{1}) \) which, by Lemma 2.11, has a weak limit \( \pi \in \text{OptTGeo}_{p}^{\ell_{p}}(\mu_{0}, \mu_{1}) \) along a nonrelabeled subsequence. In turn, the plan \( \pi \) induces a timelike proper-time parametrized \( \ell_{p} \)-geodesic \( (\mu_{t})_{t \in [0, 1]} \) defined by
\[
\mu_{t} = (e_{t})_{\pi}.
\]
In the rest of this chapter, we refer to \( (\mu_{t})_{t \in [0, 1]} \) as the candidate (but we may use the notation \( (\mu_{t})_{t \in [0, 1]} \) at other occasions as well, whenever convenient).

Let \( \mathcal{D} \subset \mathcal{Q} \) henceforth denote the set of dyadic numbers.

### 3.5. \((K, N, p)\)-convexity inequality

Now we start proving that the candidate \( (\mu_{t})_{t \in [0, 1]} \) is good according to Definition 1.1: it has to obey the \((K, N, p)\)-convexity inequality for \( \mathcal{U}_{N} \) defining \( \text{wTCD}_{p}(K, N) \) for \( p \in (0, 1), K \in \mathbb{R} \), and \( N \in (0, \infty) \), and \( \| \mu_{t} \|_{L_{\infty}(\gamma, m)} \) has to be uniformly bounded in \( t \in [0, 1] \) subject to the decomposition \( \mu_{t} = \rho_{t} m \). (Recall that \( \mu_{t} \in \mathcal{D}(\text{Ent}_{m}) \) for every \( t \in [0, 1] \) by Lemma 3.4, weak lower semicontinuity of \( \text{Ent}_{m} \), and Jensen’s inequality.) We start with the former.

The subsequent identities taken from [35, Lem. 3.2] are crucial in the proof of the main Proposition 3.7.

**Lemma 3.5.** Let \( K \in \mathbb{R} \) and \( N \in (0, \infty) \), and let \( t_{1}, t_{2}, t_{3} \in [0, 1] \) with \( t_{1} < t_{2} \) as well as \( \theta \geq 0 \). Then
\[
\sigma_{K, N}^{(1-t_{3})^{t_{1}}+t_{3}t_{2}}(\theta) = \sigma_{K, N}^{(1-t_{3})^{t_{1}}}(t_{2} - t_{1}) \theta + \sigma_{K, N}^{(1-t_{3})^{t_{1}}}(t_{2} - t_{1}) \theta \sigma_{K, N}^{(1-t_{3})^{t_{2}}}(\theta),
\]
\[
\sigma_{K, N}^{(1-t_{3})^{t_{1}}+t_{3}t_{2}}(\theta) = \sigma_{K, N}^{(1-t_{3})^{t_{1}}}(t_{2} - t_{1}) \theta + \sigma_{K, N}^{(1-t_{3})^{t_{1}}}(t_{2} - t_{1}) \theta \sigma_{K, N}^{(1-t_{3})^{t_{2}}}(\theta).
\]

**Remark 3.6.** Recall that analogous formulas are not valid for the distortion coefficients used to set up the finite-dimensional CD condition for metric measure spaces.
3.5 is the main reason for the local-to-global property of its reduced counterpart [7, Thm. 5.1], see also [8, Thm. 3.45].

Proposition 3.7. Assume \( w\text{TCD}_p^ε(K, N) \) for some \( p \in (0, 1) \), \( K \in \mathbb{R} \) and \( N \in (0, \infty) \). Let \((\mu_0, \mu_1) \in \mathcal{P}_c(M) \cap \mathcal{D}(\text{Ent}_m)\) be strongly timelike \( p \)-dualizable. Then there exists some timelike \( p \)-dualizing coupling \( \pi \in \Pi_{\leq}(\mu_0, \mu_1) \) such that the candidate \((\mu_t)_{t \in [0, 1]} \) associated to \( \mu_0 \) and \( \mu_1 \) from Section 3.2 obeys, for every \( t \in [0, 1] \),

\[
\mathcal{U}_N(\mu_t) \geq \sigma_{K,N}^{(1-\epsilon)}(T_{\tau}) \mathcal{U}_N(\mu_0) + \sigma_{K,N}^{(1)}(T_{\tau}) \mathcal{U}_N(\mu_1).
\]

Proof. Given the above candidate, we have to construct \( \pi \). To this aim, we loosely follow [35, Sec. 3.1], but have to perform a subtle modification. The curvature-dimension condition in [35, Def. 1.1] might be different from its entropic counterpart for metric measure spaces which are not essentially nonbranching [16, Def. 3.1, Thm. 3.12], and the \( \text{TCD}_p^ε(K, N) \) condition from Definition 2.16 is formulated in the spirit of [16]. In particular, the timelike proper-time parametrized \( \ell_\pi \)-geodesic and the \( \ell_\pi \)-optimal coupling of \( \mu_0 \) and \( \mu_1 \) therein have a priori nothing to do with each other, unlike [35, Def. 1.1]. We thus have to keep track of all couplings appearing in the TCD condition at every step of the dyadic construction from Section 3.2.

This is done by a monotonicity argument by gradually selecting the plan with respect to which the \( L^2 \)-norm of \( \pi \) is maximal if \( K < 0 \) or minimal if \( K \geq 0 \), and a tightness argument justifying the final passage to the limit. For simplicity, let us assume that \( K < 0 \), the other case is treated analogously.

**Step 1.** Approximate \((K, N, p)\)-convexity inequality for dyadic times. By maximality of \( \mathcal{V}_{\pi} \), the \( w\text{TCD}_p^ε(K, N) \) condition, and Lemma 2.11 there is an \( \ell_\pi \)-optimal coupling \( \pi_1 \in \Pi_{\leq}(\mu_0, \mu_1) \) such that

\[
\mathcal{U}_N(\mu_{1/2}) \geq \sigma_{K,N}^{(1/2)}(T_{\pi_1}) \mathcal{U}_N(\mu_0) + \sigma_{K,N}^{(1/2)}(T_{\pi_1}) \mathcal{U}_N(\mu_1).
\]

Now suppose that for every \( n \in \mathbb{N} \) there exists \( \pi_n \in \Pi_{\leq}(\mu_0, \mu_1) \) such that for every \( k \in \{1, \ldots, 2^n - 1\} \),

\[
\mathcal{U}_N(\mu_{k2^{-n}}) \geq \sigma_{K,N}^{(1-k2^{-n})}(T_{\pi_n}) \mathcal{U}_N(\mu_0) + \sigma_{K,N}^{(k2^{-n})}(T_{\pi_n}) \mathcal{U}_N(\mu_1). \tag{3.2}
\]

Let \( k \in \{1, \ldots, 2^{n+1} - 1\} \) be an odd number. Arguing as above and noting that the ancestors \( \mu_{(k-1)2^{-n-1}} \) and \( \mu_{(k+1)2^{-n-1}} \) of \( \mu_{k2^{-n-1}} \) are strongly timelike \( p \)-dualizable, there is an \( \ell_\pi \)-optimal coupling \( \omega_{\pi_n} \in \Pi_{\leq}(\mu_{(k-1)2^{-n-1}}, \mu_{(k+1)2^{-n-1}}) \) such that

\[
\mathcal{U}_N(\mu_{k2^{-n-1}}) \geq \sigma_{K,N}^{(1/2)}(T_{\omega_{\pi_n}^{k+1}}) \mathcal{U}_N(\mu_{(k-1)2^{-n-1}}) + \sigma_{K,N}^{(1/2)}(T_{\omega_{\pi_n}^{k+1}}) \mathcal{U}_N(\mu_{(k+1)2^{-n-1}}) \]

\[
\geq \sigma_{K,N}^{(1/2)}(T_{\omega_{\pi_n}^{k+1}}) \sigma_{K,N}^{(1-(k-1)2^{-n-1})}(T_{\pi_n}) \mathcal{U}_N(\mu_0) + \sigma_{K,N}^{(1/2)}(T_{\omega_{\pi_n}^{k+1}}) \sigma_{K,N}^{(1-(k-1)2^{-n-1})}(T_{\pi_n}) \mathcal{U}_N(\mu_1) \]

\[
+ \sigma_{K,N}^{(1/2)}(T_{\omega_{\pi_n}^{k+1}}) \sigma_{K,N}^{(1-(k+1)2^{-n-1})}(T_{\pi_n}) \mathcal{U}_N(\mu_0) + \sigma_{K,N}^{(1/2)}(T_{\omega_{\pi_n}^{k+1}}) \sigma_{K,N}^{(1-(k+1)2^{-n-1})}(T_{\pi_n}) \mathcal{U}_N(\mu_1).
\]

In the second inequality, we have used our induction hypothesis. By Lemma 2.11 and arguing as for [1, Thm. 2.11] we now construct a plan \( \alpha_{n+1}^k \in \text{Opt} \text{Geo}_p^\pi(\mu_0, \mu_1) \), which is henceforth fixed, with the property

\[
(\epsilon_{(k-1)2^{-n-1}}, \epsilon_{(k+1)2^{-n-1}}) \tau \alpha_{n+1}^k = \omega_{n+1}^k.
\]
Having at our disposal these timelike \( \ell_p \)-optimal geodesic plans \( \alpha^k_{n+1} \) for every odd index \( k \in \{1, \ldots, 2^{n+1} - 1\} \), employing that
\[
T_{n+1} = 2^{-n} T_{n+1}^k
\]
for \( \pi^k_{n+1} := (e_0, e_1) \alpha^k_{n+1} \), considering the \( \ell_p \)-optimal coupling
\[
\pi^k_{n+1} := \text{argmax} \{ \mathcal{T}_\pi : \pi \in \{ \pi_n, \pi^k_{n+1}, \ldots, \pi^k_{2^{n+1}-1} \} \}
\]
of \( \mu_0 \) and \( \mu_1 \), and that the function \( \sigma^{(r)}_{K,N}(\theta) \) is increasing in \( \theta \geq 0 \) for every \( r \in [0,1] \), from the above inequalities we obtain
\[
\mathcal{U}_N(\mu_{k2^{-n-1}}) \geq \sigma^{(1/2)}_{K',N}(2^{-n} T_{\pi_{n+1}}) \sigma^{(1-(k-1)/2^{n-1})}_{K,N}(T_{\pi_{n+1}}) \mathcal{U}_N(\mu_0) + \sigma^{(1/2)}_{K',N}(2^{-n} T_{\pi_{n+1}}) \sigma^{(1-(k+1)/2^{n-1})}_{K,N}(T_{\pi_{n+1}}) \mathcal{U}_N(\mu_1) + \sigma^{(1/2)}_{K',N}(2^{-n} T_{\pi_{n+1}}) \sigma^{(1-(k+1)/2^{n-1})}_{K,N}(T_{\pi_{n+1}}) \mathcal{U}_N(\mu_0) + \sigma^{(1/2)}_{K',N}(2^{-n} T_{\pi_{n+1}}) \sigma^{(1-(k+1)/2^{n-1})}_{K,N}(T_{\pi_{n+1}}) \mathcal{U}_N(\mu_1).
\]

In the last step, we have used Lemma \ref{lemma:uniform_density_bounds}.

**Step 2. Construction of \( \pi \) and conclusion.** By induction, we have thus obtained a sequence \( (\pi_n)_{n \in \mathbb{N}} \) of \( \ell_p \)-optimal couplings of \( \mu_0 \) and \( \mu_1 \) such that \( \pi_n \) satisfies (3.2) for every \( n \in \{1, \ldots, 2^n - 1\} \), \( n \in \mathbb{N} \). Since \( \pi_n \subset \text{spt} \mu_0 \times \text{spt} \mu_1 \) is compact for every \( n \in \mathbb{N} \), Prokhorov’s theorem, stability \([12, \text{Thm. 2.14}]\) and strong timelike \( p \)-dualizability of \( \mu_0 \) and \( \mu_1 \) imply weak convergence of a nonrelabeled subsequence of \( (\pi_n)_{n \in \mathbb{N}} \) to an \( \ell_p \)-optimal coupling \( \pi \in \Pi_{\mathcal{M}}(\mu_0, \mu_1) \). Since \( \tau \) is continuous and bounded on \( \text{spt} \mu_0 \times \text{spt} \mu_1 \), we have \( T_{\pi_n} \to T_{\pi} \) as \( n \to \infty \). Sending \( n \to \infty \) in the inequality for \( \mathcal{U}_N \) from Step 1 and employing weak upper semicontinuity of \( \mathcal{U}_N \) in the case \( t \in \{0,1\} \setminus \mathcal{D} \) thus gives the desired inequality.

**Step 3. Properties of \( \pi \).** By \([12, \text{Thm. 2.14}]\), \( \pi \) constitutes in fact an \( \ell_p \)-optimal coupling of \( \mu_0 \) and \( \mu_1 \). As such, it is concentrated on \( M'_{\infty} \) thanks to Remark \ref{remark:concentration_of_measure}, whence it is timelike \( p \)-dualizing. \( \square \)

### 3.4. Uniform density bounds

Now we show that the candidate \( (\mu_t)_{t \in [0,1]} \) from Section 3.2 satisfies the desired \( L^\infty \)-bounds for its densities with respect to \( m \). This requires some preliminary work culminating in Proposition 3.13 below, where it turns out that maximizers of \( \nu_N \) directly admit the correct density bounds.

#### 3.4.1. Spread of mass

First, we examine how the wTCD condition spreads mass along appropriate timelike proper-time parametrized \( \ell_p \)-geodesics. In view of Proposition 3.11, Corollary 3.12, and Proposition 3.13 this is the key result which provides us with the critical threshold for the \( L^\infty \)-norm of \( \rho_t \) subject to the decomposition \( \mu_t = \rho_t m \), \( t \in [0,1] \).

**Lemma 3.8.** Let \( (M, d, m, \ll, \leq, \tau) \) satisfy wTCD\( p(K, N) \) for some \( p \in (0,1) \), \( K \in \mathbb{R} \) and \( N \in (0, \infty) \). Suppose \( (\mu_0, \mu_1) = (\rho_0 m, \rho_1 m) \in D_{\mathbb{C}}(M, m)^2 \) is strongly timelike \( p \)-dualizable, and that \( \rho_0, \rho_1 \in L^\infty(M, m) \), \( i \in \{0,1\} \). Then there exists a timelike proper-time parametrized \( \ell_p \)-geodesic \( (\mu_t)_{t \in [0,1]} \) connecting \( \mu_0 \) and \( \mu_1 \) such that \( \mu_t = \rho_t m \in D(\text{Ent}_m) \) for every \( t \in (0,1) \), and
\[
m(\{\rho_{1/2} > 0\}) \geq e^{-D\sqrt{\tau} / 2} \max\{\|\rho_0\|_{L^\infty(M, m)}, \|\rho_1\|_{L^\infty(M, m)}\}^{-1},
\]
where \( D := \sup \tau(\text{spt} \mu_0 \times \text{spt} \mu_1) \).

**Proof.** First, note that \( \mu_0, \mu_1 \in D(\text{Ent}_m) \). Moreover, \( \sup \tau(\text{spt} \mu_0 \times \text{spt} \mu_1) < \infty \) by \( K \)-global hyperbolicity. Lastly, as wTCD\( p(K, \infty) \) implies wTCD\( p(-K, \infty) \), we may and will assume without restriction that \( K \leq 0 \).
Let $\pi \in \Pi_{c<}(\mu_0, \mu_1)$ be a timelike $p$-dualizing coupling for $(\mu_0, \mu_1)$ and $(\mu_1)_{t \in [0,1]}$ be a timelike proper-time parametrized $\ell_p$-geodesic from $\mu_0$ to $\mu_1$ along which $U(t)$ obeys the $(K, N, p)$-convexity property from Definition 2.16. By Lemma 2.11, $(\mu_1)_{t \in [0,1]}$ is represented by some plan $\pi \in \text{OptTGeo}^p_\mathbb{R}(\mu_0, \mu_1)$. Since $\mu_0 = \{t_1 : \gamma \in \text{spt} \pi \in \mathbb{J}(\mu_0, \mu_1)\}$ is compact, the TCD-property implies that $\mu_1 \in \mathcal{D}(\text{Ent}_m)$ for every given $t \in (0,1)$. Moreover,

$$E := \{\rho_{1/2} > 0\}$$

is contained in a compact set, whence $m[E] \in (0, \infty)$. Set

$$R := \max\{\|\rho_0\|_{L^\infty(M,\mu_0)}, \|\rho_1\|_{L^\infty(M,\mu_1)}\}$$

and note that for every $\theta \in [0, \infty)$ and every given $t \in (0,1)$, we have

$$\sigma_{K,N}^{(t)}(\theta) \geq t e^{-(1-t)\theta \sqrt{-K/N}},$$

see e.g. [11, Rem. 2.3]. Therefore

$$U_N(\mu_{1/2}) \geq \sigma_{K,N}^{(1/2)}(T_N) U_N(\mu_0) + \sigma_{K,N}^{(1/2)}(T_N) U_N(\mu_1) \geq \frac{1}{2} e^{-D\sqrt{-K/N}} U_N(\mu_0) + \frac{1}{2} e^{-D\sqrt{-K/N}} U_N(\mu_1) \geq e^{-D\sqrt{-K/N}/2 R^{-1/N}}.$$ 

Here we used that $\pi$-esssup $\tau(M^2) \leq D$. On the other hand, $U_N(\mu_{1/2}) \leq m[E]^{1/N}$ as in the proof of Lemma 3.4. The claim follows.

Remark 3.9. Of course, the same reasoning yields

$$m[\{\rho_t > 0\}] \geq e^{-D\sqrt{-K/N}} \max\{\|\rho_0\|_{L^\infty(M,\mu_0)}, \|\rho_1\|_{L^\infty(M,\mu_1)}\}^{-1}$$

for every $t \in (0,1)$ in the situation of Lemma 3.8. Note that for $t = 1/2$, which is the relevant case in the sequel, Lemma 3.8 provides a better constant, though.

3.4.2. Mass excess functional. Now we study the mass excess functional we deal with later especially in Proposition 3.11, Corollary 3.12, and Proposition 3.13. It has already been considered in [35, 36] in the context of metric measure spaces. It measures how much its input deviates from satisfying our density requirements for a good geodesic. Given $c \geq 0$ as well as $t \in (0,1)$, define $\mathcal{F}_c : \mathcal{P}(M) \to [0,1]$ by

$$\mathcal{F}_c(\mu) := \|(\rho - c)^+\|_{L^1(M,\mu)} + \mu_{\perp}[M] \quad (3.3)$$

subject to the decomposition $\mu = \rho \mathfrak{m} + \mu_{\perp}$, and $\mathcal{E}_c^t : \mathcal{P}(C([0,1]; M)) \to [0,1]$ by

$$\mathcal{E}_c^t(\pi) := \mathcal{F}_c((e_t^c)\pi).$$

Except for Section 4.3 below, we mostly work with the functional

$$\mathcal{E}_c := \mathcal{E}_c^{1/2}. \quad (3.4)$$

Lemma 3.10. Let $p \in [0,1]$, $t \in (0,1)$, as well as $c \geq 0$. Suppose that $(\mu_0, \mu_1) \in \text{STD}_p(M) \cap \mathcal{P}_c(M)^2$. Then $\mathcal{E}_c^t$ has a minimizer in $\text{OptTGeo}^p_\mathbb{R}(\mu_0, \mu_1)$.

Proof. Since $J(\mu_0, \mu_1)$ is compact, the functional $\mathcal{F}_c$ is weakly lower semicontinuous on $\mathcal{P}(J(\mu_0, \mu_1))$, cf. e.g. [43, Thm. 30.6] or [36, Lem. 3.6]. Hence, $\mathcal{E}_c^t$ is weakly lower semicontinuous on $\text{OptTGeo}^p_\mathbb{R}(\mu_0, \mu_1)$. The claim follows as for Lemma 3.4. □
3.4.3. $\mathcal{L}^\infty$-bounds for minimizers of $\mathcal{E}_c$. In this section, we study the minimal values of $\mathcal{E}_c$ for all real $c$ no smaller than the critical threshold

$$\text{thr} := e^{D_N/2} \max \left\{ \| \rho_0 \|_{L^\infty(M,m)}, \| \rho_1 \|_{L^\infty(M,m)} \right\}, \quad (3.5)$$

where $D := \sup \tau(\text{spt } \mu_0 \times \text{spt } \mu_1)$ for every $\mu_0, \mu_1 \in \mathcal{P}_c(M)$ as hypothesized in Theorem 1.2 (recall Lemma 3.8). In fact, in this case it turns out that the minimal value of $\mathcal{E}_c$ is always 0. To prove this, we first go strictly above the threshold in Proposition 3.11, which is where most of the work has to be done. Corollary 3.12 establishes the analogous result for the precise threshold $\text{thr}$.

**Proposition 3.11.** Suppose $\text{wTCD}_p^\infty(K,N)$ for some $p \in (0,1)$, $K \in \mathbb{R}$ and $N \in (0,\infty)$. Let $(\mu_0, \mu_1) = (\rho_0 m, \rho_1 m) \in \mathcal{P}^\infty_c(M,m)^2$ be strongly timelike p-dualizable, and assume that $\rho_0, \rho_1 \in L^\infty(M,m)$. Finally, let $c > \text{thr}$. Then

$$\min \mathcal{E}_c(\text{OptTGeo}_p^\infty(\mu_0, \mu_1)) = 0.$$ 

**Proof.** We roughly follow the strategy of the proof of [36, Prop. 3.11], up to several modifications required since we work entirely with timelike $\ell_p$-optimal geodesic plans and not with $\ell_p$-intermediate points. We argue by contradiction. Suppose that

$$\min \mathcal{E}_c(\text{OptTGeo}_p^\infty(\mu_0, \mu_1)) > 0.$$ 

Let $\text{Min}_c \subset \text{OptTGeo}_p^\infty(\mu_0, \mu_1)$ be the set of minimizers of $\mathcal{E}_c$ on $\text{OptTGeo}_p^\infty(\mu_0, \mu_1)$, which is nonempty by Lemma 3.10. Since all midpoints of elements of $\text{Min}_c$ have support in the compact set $J(\mu_0, \mu_1)$, there exists $\pi \in \text{Min}_c$ such that

$$m[\{ \rho_\nu > c \}] \geq \frac{\text{thr}^{1/4}}{e^{1/4}} \sup \{ m[\{ \rho_\omega > c \}] : \pi \in \text{Min}_c \} \quad (3.6)$$

subject to the decompositions $\nu = \rho_\nu m + \nu_\perp$ and $\omega = \rho_\omega m + \omega_\perp$, employing the abbreviations $\nu := (e_{1/2})_2^\pi$ and $\omega := (e_{1/2})_2^{\pi \perp}$. In the sequel, our strategy is to shuffle around mass from $\pi$ which contributes towards the positivity of $\mathcal{E}_c(\pi)$ to build a timelike $\ell_p$-optimal geodesic plan from $\mu_0$ to $\mu_1$ with less energy. That way, we will arrive to a contradiction.

**Step 1. Detection of the set of midpoints with large density.** We will first assume that $m[\{ \rho_\nu > c \}] > 0$, in which case the supremum on the r.h.s. of (3.6) is strictly positive. Fix $\delta > 0$ such that

$$m[\{ \rho_\nu > c + \delta \}] \geq \frac{\text{thr}^{1/2}}{e^{1/2} c^{1/2}} m[\{ \rho_\nu > c \}] \quad (3.7)$$

Henceforth using the abbreviations

$$A := \{ \rho_\nu > c \},$$
$$A_\delta := \{ \rho_\nu > c + \delta \},$$
$$G_\delta := (e_{1/2})^{-1}(A_\delta),$$

we define $\kappa_0, \kappa_1 \in \mathcal{P}_c(M)$ by

$$\kappa_0 := \nu[A_\delta]^{-1}(e_0)_i^\pi [\pi L G_\delta],$$
$$\kappa_1 := \nu[A_\delta]^{-1}(e_1)_i^\pi [\pi L G_\delta].$$

In other words, we take the portion of curves in $\text{spt } \pi$ which hits $A_\delta$ at time 1/2 and both trace it back to $\text{spt } \mu_0$ and follow it forward to $\text{spt } \mu_1$, up to normalization. It is straightforward to verify that $\kappa_1 \ll m$, and that the density of $\kappa_i$ with respect to $m$ is $m$-essentially bounded, $i \in \{0,1\}$. Lastly, $(\kappa_0, \kappa_1)$ is (strongly, by restriction) timelike $p$-dualizable by $\pi := \nu[A_\delta]^{-1}(e_0, e_1)_i^\pi [\pi L G_\delta] \in \mathcal{P}(M^2)$. Indeed, $\pi$ constitutes a coupling of $\kappa_0$ and $\kappa_1$ which is supported on $M^2_\infty$ by Remark 2.8. As $\kappa_0$ and $\kappa_1$ are compactly supported, the claim thus follows from [12, Rem. 2.20].
Step 2. Construction of a new geodesic. By Step 1, Lemma 3.8 and Lemma 2.11, and as \( \sup \tau(\text{spt } \kappa_0 \times \text{spt } \kappa_1) \leq D \), there exists \( \beta \in \text{OptTGeo}^x_\rho(\kappa_0, \kappa_1) \) representing a timelike proper-time parametrized \( \ell_\rho \)-geodesic from \( \kappa_0 \) to \( \kappa_1 \) such that

\[
\mathfrak{m}\left[\{\rho > 0\}\right] \geq \frac{1}{\text{thr}}
\]

subject to the decomposition \((e_{1/2})_\beta = \rho \mathfrak{m}\). Set

\[
\alpha := \pi_L(\text{TGeo}^x(M) \setminus G_\delta) + \frac{c}{c + \delta} \pi_L G_\delta + \frac{\delta}{c + \delta} \nu[A_\delta] \beta.
\]

We verify that \( \alpha \in \text{OptTGeo}^x_\rho(\mu_0, \mu_1) \). Clearly, \( \alpha \) is supported on \( \text{TGeo}^x(M) \), and \((e_0, e_1)_\alpha \) is a chronological coupling of \( \mu_0 \) and \( \mu_1 \). We claim that the latter is in fact \( \ell_\rho \)-optimal. To demonstrate this, we first note that

\[
\int_{M^2} \tau^P d(e_0, e_1)_\beta \alpha = \int_{\text{TGeo}^x(M)} \tau^P \circ (e_0, e_1) d\alpha
\]

\[
= \int_{\text{TGeo}^x(M) \setminus G_\delta} \tau^P \circ (e_0, e_1) d\pi + \frac{c}{c + \delta} \int_{\text{TGeo}^x(M) \cap G_\delta} \tau^P \circ (e_0, e_1) d\pi
\]

\[
+ \frac{\delta}{c + \delta} \nu[A_\delta] \int_{\text{TGeo}^x(M)} \tau^P \circ (e_0, e_1) d\beta
\]

\[
= \int_{\text{TGeo}^x(M) \setminus G_\delta} \tau^P \circ (e_0, e_1) d\pi - \frac{\delta}{c + \delta} \int_{\text{TGeo}^x(M) \cap G_\delta} \tau^P \circ (e_0, e_1) d\pi
\]

\[
+ \frac{\delta}{c + \delta} \nu[A_\delta] \int_{\text{TGeo}^x(M)} \tau^P \circ (e_0, e_1) d\beta
\]

\[
= \ell^P_\rho(\mu_0, \mu_1) - \frac{\delta}{c + \delta} \int_{M^2} \tau^P d(e_0, e_1)_\beta [\pi_L G_\delta]
\]

\[
+ \frac{\delta}{c + \delta} \nu[A_\delta] \ell^P_\rho(\kappa_0, \kappa_1).
\]

In the last step, we used that \( \pi \in \text{OptTGeo}^x_\rho(\mu_0, \mu_1) \) and \( \beta \in \text{OptTGeo}^x_\rho(\kappa_0, \kappa_1) \).

Now note that \( \pi \mathfrak{L} G_\delta \leq \pi \), and given that \( \pi \in \text{OptTGeo}^x_\rho(\mu_0, \mu_1) \), by Lemma 2.11 \( \nu[A_\delta]^{-1} \pi \mathfrak{L} G_\delta \) constitutes a timelike \( \ell_\rho \)-optimal geodesic plan interpolating its marginals. The latter are precisely \( \kappa_0 \) and \( \kappa_1 \), whence

\[
\int_{M^2} \tau^P d(e_0, e_1)_\beta [\pi \mathfrak{L} G_\delta] = \nu[A_\delta] \ell^P_\rho(\kappa_0, \kappa_1),
\]

which proves the \( \ell_\rho \)-optimality of \((e_0, e_1)_\beta \alpha \).

Step 3. Energy excess of \( \alpha \). We decompose \( \theta = \rho \mathfrak{m} + \theta_\perp \), where \( \theta := (e_{1/2})_\beta \alpha \), and compute

\[
\mathcal{E}_\nu(\pi) - \mathcal{E}_\nu(\alpha) = \int_M (\rho_\nu - c)^+ dm + \nu_\perp[M] - \int_M (\rho_\nu - c)^+ dm - \theta_\perp[M]
\]

\[
= \int_{M \setminus A_\delta} \left[ (\rho_\nu - c)^+ - \left[ \rho_\nu + \frac{\delta}{c + \delta} \nu[A_\delta] \rho - c \right]^+ \right] dm
\]

\[
+ \int_{A_\delta} \left[ (\rho_\nu - c)^+ - \left[ \rho_\nu + \frac{\delta}{c + \delta} \nu[A_\delta] \rho - c \right]^+ \right] dm
\]

\[
= \int_{M \setminus A_\delta} \left[ (\rho_\nu - c)^+ - \left[ \rho_\nu + \frac{\delta}{c + \delta} \nu[A_\delta] \rho - c \right]^+ \right] dm
\]

\[
+ \frac{\delta}{c + \delta} \int_{A_\delta} [\rho_\nu - \nu[A_\delta] \rho] dm
\]

\[
= \int_{M \setminus A_\delta} \left[ (\rho_\nu - c)^+ - \left[ \rho_\nu + \frac{\delta}{c + \delta} \nu[A_\delta] \rho - c \right]^+ \right] dm
\]
The r.h.s. converges to zero as \( n \) and assume that \( \rho \), which yields the desired contradiction.

Analogously to above, we can shuffle this singular portion to the section, recall the definitions (3.7) and (3.8) and giving \( \nu \) has a nontrivial singular part with respect to \( m \) since \( \mathcal{E}_c(\pi) > 0 \). Analogously to above, we can shuffle this singular portion to the \( \beta \)-part of the timelike proper-time parametrized \( \ell_p \)-geodesic \( \alpha \) constructed in Step 2 using (3.8) and giving \( \mathcal{E}_c(\alpha) < \mathcal{E}_c(\pi) \), which leads to a contradiction.

**Step 4. Treatise of the singular part.** In the remaining case \( m(\{\rho_\nu > c\}) = 0 \), the measure \( \nu \) has a nontrivial singular part with respect to \( m \) since \( \mathcal{E}_c(\pi) > 0 \). Analogously to above, we can shuffle this singular portion to the \( \beta \)-part of the timelike proper-time parametrized \( \ell_p \)-geodesic \( \alpha \) constructed in Step 2 using (3.8) and giving \( \mathcal{E}_c(\alpha) < \mathcal{E}_c(\pi) \), which leads to a contradiction.

**Corollary 3.12.** Under the same assumptions as in Proposition 3.11,

\[
\min \mathcal{E}_{\text{thr}}(\text{OptTGeo}^*_p(\mu_0, \mu_1)) = 0.
\]

**Proof.** By Proposition 3.11, we get that that for every \( n \in \mathbb{N} \),

\[
\min \mathcal{E}_{\text{thr}}(\text{OptTGeo}^*_p(\mu_0, \mu_1)) \\
\leq \min \mathcal{E}_{\text{thr} + 2^{-n}}(\text{OptTGeo}^*_p(\mu_0, \mu_1)) + 2^{-n} m[J(\mu_0, \mu_1)] \\
= 2^{-n} m[J(\mu_0, \mu_1)].
\]

The r.h.s. converges to zero as \( n \to \infty \) by compactness of \( J(\mu_0, \mu_1) \). □

3.4.4. **Maximizers of \( \mathcal{V}_N \) have zero excess.** For the subsequent main result of this section, recall the definitions (3.1) of \( \mathcal{V}_N \) and (3.5) of thr, respectively.

**Proposition 3.13.** Suppose \( \text{wTCD}_p^p(K, N) \) for some \( p \in (0, 1) \), \( K \in \mathbb{R} \), and \( N \in (0, \infty) \). Let \( (\rho_0, \rho_1, m) \in \mathcal{T}^{\text{loc}}_p(M, m)^2 \) be strongly timelike \( p \)-dualizable, and assume that \( \rho_0, \rho_1 \in L^\infty(M, m) \). Then

\[
\mathcal{E}_{\text{thr}}(\pi) = 0
\]

for every maximizer \( \pi \in \text{OptTGeo}^*_p(\mu_0, \mu_1) \) of \( \mathcal{V}_N \).
Proof. As for Proposition 3.11, our ansatz is a contradiction argument, i.e. we assume the existence of a maximizer $\pi \in \text{OptTGeo}^\gamma_{\ell_p}(\mu_0, \mu_1)$ with
$$\mathcal{E}_{\text{thr}}(\pi) > 0.$$ Our proof follows the one of [35, Prop. 3.5] and for Proposition 3.11.

**Step 1.** Detection of the set of midpoints with large density. We recall from Lemma 3.4 that by the hypothesized wTCD condition and since all timelike proper-null geodesics are parametrized by the hypothesized wTCD condition and since all timelike proper-null geodesics from $\mu_0$ to $\mu_1$ have support in the compact set $J(\mu_0, \mu_1)$, we must have $\mathbb{V}_N(\pi) > 0$. In particular, $\nu := (e_{1/2})_2 \pi \ll \mathcal{m}$. Now let $\eta > 0$ such that
$$\mathcal{m}[\{\rho_\nu > \text{thr} + \eta\}] \geq \mathcal{m}[\{\rho_\nu > \text{thr} + 2\eta\}] > 0$$ subject to the decomposition $\nu = \rho_\nu \mathcal{m}$, and define
$$c_1 := \frac{4}{\eta} \mathcal{m}[\{\rho_\nu > \text{thr} + \eta\}] - \frac{4}{\eta} \mathcal{m}[\{\rho_\nu > \text{thr} + 2\eta\}].$$ Given any $\phi \in (0, \eta/3)$ there exists $\delta \in (\eta, 2\eta)$ such that
$$\mathcal{m}[A_\delta] < \mathcal{m}[A'_\delta] + c_1 \phi,$$ where we have defined
$$A_\delta := \{\rho_\nu > \text{thr} + \delta\},$$ $$A'_\delta := \{\rho_\nu > \text{thr} + \delta - 3\phi\}.$$

**Step 2.** Construction of a new geodesic. Let $\kappa_0, \kappa_1 \in \mathcal{P}_\mathcal{c}(M)$ be defined as in Step 1 of the proof of Proposition 3.11. Using Corollary 3.12, there exists $\beta \in \text{OptTGeo}^\gamma_{\ell_p}(\mu_0, \mu_1)$ such that
$$\|\rho\|_{L^\infty(M, \mathcal{m})} \leq \frac{\text{thr}}{\nu[A_\delta]}$$ subject to the decomposition $(e_{1/2})_2 \beta = \rho_\mathcal{m}$. Setting $G_\delta := (e_{1/2})^{-1}(A_\delta)$, define $\alpha \in \text{OptTGeo}^\gamma_{\ell_p}(\rho_0, \rho_1)$ through
$$\alpha := \pi \mathcal{L}(\text{TGeo}^\gamma(M) \setminus G_\delta) + \frac{\text{thr} + \delta - \phi}{\text{thr} + \delta} \pi \mathcal{L} G_\delta + \frac{\phi}{\text{thr} + \delta} \nu[A_\delta]\beta.$$ **Step 3.** Energy excess of $\alpha$. We decompose $\theta = \rho_\mathcal{m} \mathcal{m} + \theta_\perp$, where $\theta := (e_{1/2})_2 \alpha$, and let $\rho$ denote the density of $(e_{1/2})_2 \beta$ with respect to $\mathcal{m}$. Then the subsequent estimates are readily verified.

- On $A_\delta$, we have
  $$\rho_\theta\leq \frac{\text{thr} + \delta - \phi}{\text{thr} + \delta} \rho_\nu + \frac{\phi}{\text{thr} + \delta}\nu[A_\delta]\rho_\theta \leq \frac{(\text{thr} + \delta - \phi)\rho_\nu + \text{thr} + \phi}{\text{thr} + \delta},$$
  $$\rho_\theta\leq \frac{(\text{thr} - \rho_\nu)\phi}{\text{thr} + \delta} < \rho - \frac{\delta \phi}{\text{thr} + \delta},$$
  $$\rho_\theta\geq \frac{\text{thr} + \delta - \phi}{\text{thr} + \delta} \rho_\nu > \text{thr} + \delta - \phi.$$ 

- On $A'_\delta \setminus A_\delta$, we have
  $$\rho_\theta\leq \rho_\nu + \frac{\phi}{\text{thr} + \delta}\nu[A_\delta]\rho \leq \rho_\nu + \frac{\text{thr} \phi}{\text{thr} + \delta} < \text{thr} + \delta + \phi.$$ 

- On $M \setminus A'_\delta$, we have
  $$\rho_\theta \leq \rho_\nu + \frac{\phi}{\text{thr} + \delta}\nu[A_\delta]\rho \leq \text{thr} + \delta - 3\phi + \frac{\text{thr} \phi}{\text{thr} + \delta} \leq \text{thr} + \delta - 2\phi.$$
Moreover, we define and estimate the mass differences
\[
\kappa_{A_{\delta}} := \int_{A_{\delta}} (\rho_\nu - \rho_\mu) \, dm \geq c_2 \phi,
\]
\[
\kappa_{A'_{\delta} \setminus A_{\delta}} := \int_{A'_{\delta} \setminus A_{\delta}} (\rho_\theta - \rho_\mu) \, dm < c_1 \phi^2,
\]
\[
\kappa_{M \setminus A'_{\delta}} := \int_{M \setminus A'_{\delta}} (\rho_\theta - \rho_\mu) \, dm,
\]
where \( c_2 := \delta m[A_{\delta}]/(\text{thr} + \delta) \). Approximating \( \text{Ent}_m \) by a Rényi-type entropy [39, Lem. 4.1] and with analogous computations as for [35, Prop. 3.5], for every \( \varepsilon > 0 \) there exists \( N_\varepsilon \geq N \) such that:
\[
\text{Ent}_m(\theta) - \text{Ent}_m(\nu) \leq \varepsilon - N_\varepsilon \int_M \rho_\theta^{-1/N_\varepsilon} \, dm + N_\varepsilon \int_M \rho_\nu^{-1/N_\varepsilon} \, dm
\]
\[
\leq \varepsilon - N_\varepsilon \int_{A_{\delta}} \rho_\theta^{-1/N_\varepsilon} (\rho_\nu - \rho_\theta) \, dm + N_\varepsilon \int_{M \setminus A_{\delta}} \rho_\nu^{-1/N_\varepsilon} (\rho_\nu - \rho_\theta) \, dm
\]
\[
\leq \varepsilon - N_\varepsilon \kappa_{A_{\delta}} (\text{thr} + \delta - \phi)^{-1/N_\varepsilon} - N_\varepsilon \kappa_{M \setminus A'_{\delta}} (\text{thr} + \delta - 2\phi)^{-1/N_\varepsilon}
\]
\[
- N_\varepsilon \kappa_{A'_{\delta} \setminus A_{\delta}} (\text{thr} + \delta + \phi)^{-1/N_\varepsilon}
\]
\[
= N_\varepsilon \kappa_{A_{\delta}} [(\text{thr} + \delta - \phi)^{-1/N_\varepsilon} - (\text{thr} + \delta - 2\phi)^{-1/N_\varepsilon}]
\]
\[
+ N_\varepsilon \kappa_{A'_{\delta} \setminus A_{\delta}} [(\text{thr} + \delta - 2\phi)^{-1/N_\varepsilon} - (\text{thr} + \delta + \phi)^{-1/N_\varepsilon}]
\]
\[
\leq \varepsilon - \frac{c_2 - c_1 \phi}{(\text{thr} + \delta - 2\phi)} \phi^2 + c_2 c_3 \phi^3.
\]
Here \( c_3 > 0 \) is some constant independent of \( \varepsilon \), and the last inequality is computed as in the proof of [35, Prop. 3.5]. Choosing \( \varepsilon \) and \( \phi \) small enough, the r.h.s. becomes strictly negative. Therefore \( \mathcal{V}_N(\alpha) > \mathcal{V}_N(\pi) \), which is the desired contradiction. \( \square \)

The following consequence thus terminates the proof of Theorem 1.2.

**Corollary 3.14.** Retain the assumptions and the notation from Proposition 3.13. Then the candidate \((\mu_t)_{t \in [0,1]}\) constructed in Section 3.2 satisfies, for every \( t \in [0,1] \), \( \mu_t = \rho_t \, m \in \mathcal{D}(\text{Ent}_m) \) as well as
\[
\|\rho_t\|_{L^\infty(M,m)} \leq e^{D \sqrt{\kappa_{\text{max}}}} \max \{ \|\rho_0\|_{L^\infty(M,m)}, \|\rho_1\|_{L^\infty(M,m)} \}.
\]

**Remark 3.15.** Minor modifications of the above arguments give a TCD version of Theorem 1.2, namely assuming \( \text{TCD}_p^\varepsilon(K,N) \) instead of \( \text{wTCD}_p^\varepsilon(K,N) \) therein, and that — instead of being strongly timelike \( p \)-dualizable — every \( \ell_p \)-optimal coupling of \( \mu_0 \) and \( \mu_1 \) is chronological.

To see this, first note that item (v) of Lemma 2.11 merely needs all \( \ell_p \)-optimal couplings of \( \mu_0 \) and \( \mu_1 \) to be chronological [8, Prop. B.11]. Second, similarly as in Lemma 3.1, the property of pairs admitting only chronological \( \ell_p \)-optimal couplings propagates through proper-time parametrized \( \ell_p \)-geodesics \((\mu_t)_{t \in [0,1]}\). Indeed, if one \( \ell_p \)-optimal coupling of \((\mu_s, \mu_t)\) is not chronological for some \( s, t \in [0,1] \) with \( s < t \), restricting it to null related point pairs and using a gluing procedure we could produce a measure \( \pi \) on \( \mathcal{P}(\Omega([0,1]; M)) \) concentrated on maximizing causal curves such that \((e_0, e_1) \sharp \pi \) is \( \ell_p \)-optimal. But the latter must be chronological by
assumption, and since $\pi$-a.e. curve changes its character from timelike to null and back to timelike, this contradicts regularity of $(M, d, \ll, \leq, \tau)$.

Remark 3.16. Similar arguments as above give the existence of good geodesics for arbitrary metric measure spaces obeying $\text{CD}^p(K, N)$. While in the essentially nonbranching case, this partly follows from [36, Thm. 1.2] by [16, Thm. 3.12], we are not aware of such a general result for the *entropic* CD condition.

4. Variations of the main result

Finally, we discuss various extensions of Theorem 1.2. In all cases, the proof of Theorem 1.2 can then mostly be adapted to the respective situation up to some details which we highlight below.

4.1. The infinite-dimensional case. The following is a Lorentzian analogue of Sturm’s $\text{CD}(K, \infty)$ condition for metric measure spaces [39, Def. 4.5] (see also [27]); the counterpart of Theorem 4.5 for metric measure spaces is due to [35, Thm. 1.3] whose strategy we loosely follow.

**Definition 4.1.** A measured Lorentzian pre-length space $(M, d, m, \ll, \leq, \tau)$ is termed to obey the weak timelike curvature-dimension condition $\text{wTCD}^p(K, \infty)$ for $p \in (0, 1)$ and $K \in \mathbb{R}$ if for every strongly timelike $p$-dualizable pair $(\mu_0, \mu_1) \in (\mathcal{P}_c(M) \cap \mathcal{D}(\text{Ent}_m))^2$, there exists a timelike $p$-dualizing coupling $\pi \in \Pi_{\ll}(\mu_0, \mu_1)$ and a timelike proper-time parametrized $\ell_p$-geodesic $(\mu_t)_{t \in [0, 1]}$ such that, for every $t \in [0, 1],$

$$\text{Ent}_m(\mu_t) \leq (1 - t) \text{Ent}_m(\mu_0) + t \text{Ent}_m(\mu_1) - \frac{K}{2} t(1 - t) T^2.$$  

Remark 4.2. In an evident way, one may define the $\text{CD}^p(K, \infty)$ condition as an infinite-dimensional analogue of $\text{CD}^p(K, N)$. Taking Remark 3.15 into account, similar results as those presented below for $\text{wTCD}^p(K, \infty)$ hold for this curvature-dimension condition as well, up to minor modifications.

Before turning to the main Theorem 4.5 of this section, independently of it we examine elementary properties of the $\text{wTCD}^p(K, \infty)$ condition just introduced. The reader may directly go over to Subsection 4.1.2 at first reading.

4.1.1. From finite to infinite dimension. It is clear that $\text{wTCD}^p(K, \infty)$ has analogous consistency and scaling properties as its finite-dimensional Lorentzian counterpart [12, Lem. 3.10]. Moreover, as already indicated in Remark 4.2 it can be regarded as an “infinite-dimensional” analogue of the $\text{wTCD}^p(K, \infty)$ condition from Definition 2.16 by the following result.

**Proposition 4.3.** The condition $\text{wTCD}^p_{\infty}(K, N)$ implies $\text{wTCD}^p(K, \infty)$ for every $p \in (0, 1)$ and every $K \in \mathbb{R}$.

**Proof.** We follow the argument for [16, Lem. 2.12]. By nestedness of the weak timelike curvature-dimension condition [12, Lem. 3.10], given any strongly timelike $p$-dualizing pair $(\mu_0, \mu_1) \in (\mathcal{P}_c(M) \cap \mathcal{D}(\text{Ent}_m))^2$, there exists a timelike $p$-dualizing $\pi \in \Pi_{\ll}(\mu_0, \mu_1)$ and a $\mathcal{D}(\text{Ent}_m)$-valued timelike proper-time parametrized $\ell_p$-geodesic $(\mu_t)_{t \in [0, 1]}$ from $\mu_0$ to $\mu_1$ such that, for every $N' \geq N$, we have

$$\mathcal{U}_{N'}(\mu_1) \geq \sigma_{K,N'}^{1-t}(T_\pi) \mathcal{U}_{N'}(\mu_0) + \sigma_{K,N'}^{t}(T_\pi) \mathcal{U}_{N'}(\mu_1).$$  

(4.1)

Note that $\pi$ and $(\mu_t)_{t \in [0, 1]}$ can be chosen independently of $N'$. Using that

$$\sigma_{K,N'}^{t}(\vartheta) = t - \frac{K}{6N'}(t^3 - t) \vartheta^2 + o((N')^{-1}),$$

$$\mathcal{U}_{N'}(\mu) = 1 - \frac{1}{N'} \text{Ent}_m(\mu) + o((N')^{-1})$$
for every $\mu \in \mathcal{D}(\text{Ent}_m)$ as $N' \to \infty$, the claim follows by subtracting 1 at both sides of (4.1), multiplying the resulting inequality by $N'$, and letting $N' \to \infty$. \hfill \square

4.1.2. Geodesics with uniformly bounded densities. Now we turn to our actual goal, namely Theorem 4.5. Its statement holds in a stronger form, cf. Remark 4.6, but we prefer to present a slightly different proof for the sole existence of timelike proper-time parametrized $\ell_p$-geodesics with bounded densities. This underlines the role that $\ell_p$-geodesics play also in the $w\text{TCD}_p(K, \infty)$ case and is exemplary for a similar result in the next Section 4.2 where, however, the relevant exponential term does not appear.

Reproducing the proof of Lemma 3.8 gives the following.

**Lemma 4.4.** Let $(M, d, m, \ll, \leq, \tau)$ obey $w\text{TCD}_p(K, \infty)$ for some $p \in (0, 1)$ and $K \in \mathbf{R}$. Assume that $(\mu_0, \mu_1) = (\rho_0 m, \rho_1 m) \in \mathcal{P}_c(M, m)^2$ is strongly timelike $p$-dualizable with $\rho_0, \rho_1 \in L^\infty(M, m)$. Then there is a timelike proper-time parametrized $\ell_p$-geodesic $(\mu_t)_{t \in [0, 1]}$ from $\mu_0$ to $\mu_1$ such that $\mu_t = \rho_t m \in \mathcal{D}(\text{Ent}_m)$ for every $t \in (0, 1)$, and

$$m\left[\left\{\rho_{t/2} > 0\right\}\right] \geq e^{-K \cdot D^2/8} \max\left\{\|\rho_0\|_{L^\infty(M, m)}, \|\rho_1\|_{L^\infty(M, m)}\right\}^{-1},$$

where $D \geq \sup\tau(spt \mu_0 \times spt \mu_1).

**Theorem 4.5.** Suppose $w\text{TCD}_p(K, \infty)$ for $p \in (0, 1)$ and $K \in \mathbf{R}$. Let $(\mu_0, \mu_1) = (\rho_0 m, \rho_1 m) \in \mathcal{P}_c(M, m)^2$ be strongly timelike $p$-dualizable, and assume that $\rho_0, \rho_1 \in L^\infty(M, m)$. Then there is a timelike proper-time parametrized $\ell_p$-geodesic $(\mu_t)_{t \in [0, 1]}$ connecting $\mu_0$ and $\mu_1$ such that for every $t \in [0, 1], \mu_t \in \mathcal{D}(\text{Ent}_m)$ and

$$\|\rho_t\|_{L^\infty(M, m)} \leq e^{-K \cdot D^2/12} \max\left\{\|\rho_0\|_{L^\infty(M, m)}, \|\rho_1\|_{L^\infty(M, m)}\right\},$$

where $D := \sup\tau(spt \mu_0 \times spt \mu_1).

**Proof.** We only outline the proof and highlight the necessary changes compared to our arguments in Chapter 3. Let us redefine

$$\text{thr} := e^{-K \cdot D^2/8} \max\left\{\|\rho_0\|_{L^\infty(M, m)}, \|\rho_1\|_{L^\infty(M, m)}\right\}.$$

Unlike Section 3.2, here we directly construct a candidate by selecting our midpoints as minimizers of the functional $E_{\text{thr}}$ from (3.4). Let $\pi \in \text{OptTGeo}_{\ell_p}(\mu_0, \mu_1)$ be a minimizer of $E_{\text{thr}}$ according to Lemma 3.10. Observe that Corollary 3.12 still holds in this case, where the modified threshold $\text{thr}$ comes from Lemma 4.4. Define $\mu_{1/2} := (e_{1/2}) \pi \in \mathcal{P}_c(M)$. By Corollary 3.12, we have $\mu_{1/2} \in \mathcal{D}(\text{Ent}_m)$ and

$$\|\rho_{1/2}\|_{L^\infty(M, m)} \leq e^{-K \cdot D^2/8} \max\left\{\|\rho_0\|_{L^\infty(M, m)}, \|\rho_1\|_{L^\infty(M, m)}\right\}$$

subject to the decomposition $\mu_{1/2} = \rho_{1/2} m$. By Lemma 3.1, the pairs $(\mu_0, \mu_{1/2})$ and $(\mu_{1/2}, \mu_1)$ are strongly timelike $p$-dualizable. The construction of $\mu_{1/2}$ yields that $\sup\tau(spt \mu_0 \times spt \mu_{1/2})$ and $\sup\tau(spt \mu_{1/2} \times spt \mu_1)$ are no larger than $D/2$. Moreover, $\mu_{1/2}$ is an $1/2$-midpoint with respect to $\ell_p$. Next, we construct $\mu_{1/4} \in \mathcal{P}_c(M)$ and $\mu_{3/4} \in \mathcal{P}_c(M)$ as above as midpoints of some element of $\text{OptTGeo}_{\ell_p}(\mu_0, \mu_{1/2})$ and $\text{OptTGeo}_{\ell_p}(\mu_{1/2}, \mu_1)$ according to Lemma 3.10, Corollary 3.12, and Lemma 3.1, respectively. Proceeding iteratively in this way after gluing, as in Section 3.2, we get a timelike proper-time parametrized $\ell_p$-geodesic $(\mu_t)_{t \in [0, 1]}$ with the following properties. For every $t \in [0, 1] \cap \mathbf{N}$ written as $t = k 2^{-n}, n \in \mathbf{N}$ and odd $k \in \{1, \ldots, 2^n - 1\}$, we have $\mu_t \in \mathcal{D}(\text{Ent}_m)$ with

$$\|\rho_t\|_{L^\infty(M, m)} \leq e^{-K \cdot D^2/8} \max\left\{\|\rho_{(k-1)2^{-n}}\|_{L^\infty(M, m)}, \|\rho_{(k+1)2^{-n}}\|_{L^\infty(M, m)}\right\},$$

subject to the decomposition $\mu_s = \rho_s m$ for all $s \in [0, 1]$ under consideration. Here we have used that by our midpoint construction along timelike $\ell_p$-optimal geodesic
plans, for every $n \in \mathbb{N}$ and every odd $k \in \{0, \ldots, 2^n\}$ the function $\tau$ is no larger than $2^{-n+1} D$ on $\text{spt}\mu_{(k-1)2^{-n}} \times \text{spt}\mu_{(k+1)2^{-n}}$. Inductively, (4.2) holds for every $t \in [0, 1] \cap D$.

By weak lower semicontinuity of the functional $\mathcal{F}_{\text{thr}}$ from (3.3) on $\mathcal{P}(J(\mu_0, \mu_1))$, see the proof of Lemma 3.10, and (4.3) we get $\mathcal{F}_{\text{thr}}(\mu_t) = 0$ for every $t \in [0, 1]$. This implies $\mu_t \in \mathcal{D}(\text{Ent}_m)$ and (4.3) for its density with respect to $m$.

Remark 4.6. Combining the arguments of [2, Ch. 4] with our strategy in Chapter 3, one can construct timelike proper-time parametrized $\ell_r$-geodesics satisfying the conclusion of Theorem 4.5 along which, in addition, the semiconvexity inequality for $\text{Ent}_m$ defining $\text{wTCD}_p(K, \infty)$ holds.

4.2. General timelike convex functionals. Similar conclusions as in Theorem 4.5 can also be drawn for any kind of functional obeying a certain timelike convexity property as follows. Let $f : [0, \infty) \to \mathbb{R}$ be convex with $f(0) = 0$, and define the functional $E : \mathcal{P}(M) \to [-\infty, \infty]$ by

$$E(\mu) := \begin{cases} \int_M f(\rho) \, dm & \text{if } \mu = \rho \, m \ll m, \ f^+(\rho) \in L^1(M, m), \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 4.7. Let $E$ be weakly $(0, N, p)$-convex relative to $\mathcal{P}_c(M)^2 \cap \text{STD}_p(M)$ according to Definition 2.13, $p \in (0, 1)$ and $N \in (0, \infty]$, and assume that $r \mapsto f(r)/r$ is strictly increasing on $(0, \infty)$. Then for every strongly timelike $p$-dualizable pair $(\mu_0, \mu_1) = (\rho_0 \, m, \rho_1 \, m) \in \mathcal{P}_c(M, m)$ with $\rho_0, \rho_1 \in L^\infty(M, m)$, there exists a timelike proper-time parametrized $\ell_r$-geodesic $(\mu_t)_{t \in [0, 1]}$ such that for every $t \in (0, 1)$, we have $\mu_t = \rho_t \, m \in \mathcal{D}(E)$ and

$$\|\rho_t\|_{L^\infty(M, m)} \leq \max\{\|\rho_0\|_{L^\infty(M, m)}, \|\rho_1\|_{L^\infty(M, m)}\}.$$

Again, the proof of Theorem 4.7 is mainly based on the following result. The rest is treated analogously to Chapter 3 and Section 4.1.

Lemma 4.8. Let $E$ be as hypothesized in Theorem 4.7. Let $(\mu_0, \mu_1) = (\rho_0 \, m, \rho_1 \, m) \in \mathcal{P}_c(M, m)^2$ be strongly timelike $p$-dualizable with $\rho_0, \rho_1 \in L^\infty(M, m)$. Then there is a timelike proper-time parametrized $\ell_r$-geodesic $(\mu_t)_{t \in [0, 1]}$ connecting $\mu_0$ and $\mu_1$ such that for every $t \in (0, 1)$, $\mu_t = \rho_t \, m \in \mathcal{D}(E)$ and

$$m\{\{\mu_t \gg 0\}\} \geq \max\{\|\rho_0\|_{L^\infty(M, m)}, \|\rho_1\|_{L^\infty(M, m)}\}^{-1}.$$

Proof. By our assumptions on $\rho_0, \rho_1$, and $f$ it is clear that $\mu_0, \mu_1 \in \mathcal{D}(E)$.

Let $p \in \Pi_{\ll}(\mu_0, \mu_1)$ be a timelike $p$-dualizing coupling and $(\mu_t)_{t \in [0, 1]}$ be a timelike proper-time parametrized $\ell_r$-geodesic with respect to which $E$ satisfies its defining convexity inequality. As in the proof of Lemma 3.8, we infer that $\mu_t \in \mathcal{D}(E)$ for every $t \in (0, 1)$, and that

$$E := \{\rho_t \gg 0\},$$

subject to the decomposition $\mu_t = \rho_t \, m$, obeys $m[E] \in (0, \infty)$. Setting

$$R := \max\{\|\rho_0\|_{L^\infty(M, m)}, \|\rho_1\|_{L^\infty(M, m)}\}$$

then yields, on the one hand,

$$E(\mu_t) \leq (1 - t) \int_{\{\rho_0 \gg 0\}} f(\rho_0) \rho_0 \, dm + t \int_{\{\rho_1 \gg 0\}} f(\rho_1) \rho_1 \, dm \leq \frac{f(R)}{R}.$$

On the other hand, $E(\mu_t) \geq m[E] f(m[E])^{-1}$ by Jensen’s inequality. Employing the strict monotonicity of $r \mapsto f(r)/r$ on $(0, \infty)$ yields the claim. □
4.3. Timelike measure-contraction property. Lastly, following [11] for metric measure spaces, we establish a version of Theorem 1.2 for the subsequent timelike measure contraction property TMCP\(p\)(\(K, \infty\)) \cite[Def. 3.7]{12} as follows.

**Definition 4.9.** A measured Lorentzian pre-length space \((M, d, m, \ll, \leq, \tau)\) satisfies TMCP\(p\)(\(K, N\)) for \(p \in (0, 1), K \in \mathbb{R}\) and \(N \in (0, \infty)\) if for every \(\mu_0 \in \mathcal{P}_c(M) \cap \mathcal{D}(\text{Ent}_m)\) and every \(x_1 \in I^+(\mu_0)\) there exists a timelike proper-time parametrized \(\ell_p\)-geodesic \((\mu_t)_{t \in [0,1]}\) from \(\mu_0\) to \(\mu_1 := \delta_{x_1}\) such that for every \(t \in (0,1)\),

\[
\mathcal{M}\left\{\{\rho_t > 0\}\right\} \geq (1 - t)N e^{-tD\sqrt{K - N}} \|\rho_0\|_{L^\infty(M,m)}^{-1},
\]

where \(D \geq \sup \tau(\text{spt}\mu_0 \times \{x_1\})\).

**Lemma 4.10.** Let \((M, d, m, \ll, \leq, \tau)\) satisfy TMCP\(p\)(\(K, N\)) for \(p \in (0, 1), K \in \mathbb{R}\), and \(N \in (0, \infty)\). Suppose that \(\mu_0 = \rho_0 m \in \mathcal{P}_c^c(M, m)\) with \(\rho_0 \in L^\infty(M, m)\). Lastly, let \(x_1 \in I^+(\mu_0)\). Then there exists some timelike proper-time parametrized \(\ell_p\)-geodesic \((\mu_t)_{t \in [0,1]}\) from \(\mu_0\) to \(\mu_1 := \delta_{x_1}\) such that for every \(t \in (0,1)\), we have \(\mu_t = \rho_t \in \mathcal{D}(\text{Ent}_m)\) and

\[
\mathcal{M}\left\{\{\rho_t > 0\}\right\} \geq (1 - t)N e^{-tD\sqrt{K - N}} \|\rho_0\|_{L^\infty(M,m)}^{-1},
\]

where \(D \geq \sup \tau(\text{spt}\mu_0 \times \{x_1\})\).

This terminates the proof.

**Theorem 4.11.** Assume TMCP\(p\)(\(K, N\)) for some \(p \in (0, 1), K \in \mathbb{R}\), and \(N \in (0, \infty)\). Let \(\mu_0 = \rho_0 m \in \mathcal{P}_c(M)\) with \(\rho_0 \in L^\infty(M, m)\). Lastly, let \(x_1 \in I^+(\mu_0)\). Then there exists a timelike proper-time parametrized \(\ell_p\)-geodesic \((\mu_t)_{t \in [0,1]}\) from \(\mu_0\) to \(\mu_1 := \delta_{x_1}\) satisfying the following two properties for every \(t \in [0,1]\).

(i) We have \(\mu_t = \rho_t m \in \mathcal{D}(\text{Ent}_m)\) with

\[
\mathcal{U}_N(\mu_t) \geq \sigma_{K,N}(\mu_t) \mathcal{U}_N(\mu_0).
\]

(ii) Setting \(D := \sup \tau(\text{spt}\mu_0 \times \{x_1\})\), we have

\[
\|\rho_t\|_{L^\infty(M,m)} \leq \left(1 - t\right)^N e^{D\sqrt{K - N}} \|\rho_0\|_{L^\infty(M,m)}.
\]

**Proof.** The bisection argument from Chapter 3 does not work under TMCP\(p\)(\(K, N\)) since \(\mu_1 \not\ll m\), while every intermediate measure should be absolutely continuous with respect to \(m\). We rather follow the proof of \cite[Thm. 3.1]{11}.

Let \((M, d, \ll^+, \leq, \tau^+)\) denote the causally reversed structure of \((M, d, \ll, \leq, \tau)\) \cite[Def. 1.2]{12}, i.e. \(x \ll^+ y\) if and only if \(y \ll x, x \leq^+ y\) if and only if \(y \leq x,\) and \(\tau^+(x, y) := \tau(y, x)\), \(x, y \in M\). Let \(\ell^+_p\) be the cost function associated to \(\tau^+\).

**Step 1.** Construction of a “backward” geodesic. Given any \(n, k \in \mathbb{N}\), we set 

\[
s^+_n := (1 - 2^{-n})^k.\text{ Fix } n \in \mathbb{N}, \text{ and assume that } \beta^+_k \in \text{OptGeo}^{\tau^+}_p(\mu_1, \mu_0) \text{ has been defined such that for every } i \in \{1, \ldots, k\}, (e_{s^+_n})_i \beta^+_k \in \mathcal{P}_c^c(M, m) \text{ and }
\]

\[
\sup \tau^+(\{x_1\} \times \text{spt}(e_{s^+_n})_i \beta^+_k) \leq 2^{-n} s^+_n D.
\]

Let the functional \(V^2_{n, \tau^+}\) be defined as in \((3.4)\). By Lemma 3.4, the latter admits a maximizer \(\pi^{k+1}_n \in \text{OptGeo}^{\tau^+}_p((e_{s^+_n})_i \beta^+_k, \mu_1)\). Let \(\sigma^{k+1}_n \in \text{OptGeo}^{\tau^+}_p(\mu_1, (e_{s^+_n})_i \beta^+_k)\)
be the timelike $\ell_\rho$-optimal geodesic plan obtained by “time-reversal” of $\pi_{n+1}^{k+1}$. By a gluing argument, we construct a measure $\beta_{n+1}^{k+1} \in \text{OptTGeo}_{T^{-}}(\mu_1, \mu_0)$ with

\[(\text{restr}_{0}^{s_{n}})_{0} \beta_{n+1}^{k+1} = \pi_{n+1}^{k+1},\]
\[(\text{restr}_{1}^{s_{n}})_{0} \beta_{n}^{k} = (\text{restr}_{n}^{1})_{0} \beta_{n}.\]

Using the induction hypothesis, the bound

$$\sup \tau^{-}(\{x\} \times \text{spt}(e_{n+1}^{k+1}) \beta_{n+1}^{k+1}) \leq 2^{-n} s_{n}^{k} D$$

obtained by construction, using Lemma 4.10 and following the lines of Proposition 3.11, Corollary 3.12 and Proposition 3.13 for the threshold

$$\ell_{n+1}^{k+1} := \frac{1}{(1 - 2^{-n})^{N}} e^{2^{-n} s_{n}^{k} D \sqrt{N - N}} \|\rho_{0}\|_{L^{\infty}(M, M)}$$

for every $n, k \in \mathbb{N}$ we obtain $(e_{n+1}^{k+1})_{0} \beta_{n+1}^{k+1} = \rho_{n+1}^{k+1} m \in \mathcal{D}(\text{Ent}_{M}) \cap \mathcal{P}_{c}(M)$ with

$$\|\rho_{n+1}^{k+1}\|_{L^{\infty}(M, M)} \leq \frac{1}{(1 - 2^{-n})^{N}} e^{2^{-n} s_{n}^{k+1} D \sqrt{N - N}} \|\rho_{n+1}^{k}\|_{L^{\infty}(M, M)}.$$

Inductively, by geometric summation this yields

$$\|\rho_{n+1}^{k}\|_{L^{\infty}(M, M)} \leq \frac{1}{(s_{n})^{N}} e^{(1-s_{n}) D \sqrt{N - N}} \|\rho_{0}\|_{L^{\infty}(M, M)}.$$  \hspace{1cm} (4.5)

**Step 2.** Construction of the geodesic and verification of its properties. We iteratively construct a family

$$\{\beta_{n}^{k} : n \in \mathbb{N}, k \in \mathbb{N}_{0}\} \subset \text{OptTGeo}_{T^{-}}(\mu_{1}, \mu_{0})$$

according to Step 1. Let $(\beta_{n}^{k})_{n \in \mathbb{N}}$ be an enumeration of the elements of this class. By Lemma 2.11, the latter admits a weak limit $\beta \in \text{OptTGeo}_{T^{-}}(\mu_{1}, \mu_{0})$ along a nonrelabeled subsequence. Let $\alpha \in \text{OptTGeo}_{T^{-}}(\mu_{0}, \mu_{1})$ be the “time-reversal” of $\beta$, which induces a timelike proper-time parametrized $\ell_\rho$-geodesic $(\mu_{1}, t_{0})$ from $\mu_{0}$ to $\mu_{1}$.

By weak lower semicontinuity of $\mathcal{F}_{c}$ in $\mathcal{P}(J(\mu_{0}, \mu_{1}))$ for appropriate values $c > 0$, we get $\mu_{1} = \rho_{1} m \in \mathcal{D}(\text{Ent}_{M}) \cap \mathcal{P}_{c}(M)$, and as in the last step of the proof of Theorem 4.5 we obtain the weak stability of (4.5), whence $\|\rho_{k}\|_{L^{\infty}(M, M)}$ obeys the desired estimate for every $t \in [0, 1]$. The proof is terminated.

**Remark 4.12.** If in addition $(M, d, m, c, \leq, \tau)$ is timelike nonbranching in the above Theorem 4.11, as for Corollary 1.3 $\mu_{0}$ and $\mu_{1}$ are connected by a unique timelike proper-time parametrized $\ell_\rho$-geodesic, which thus automatically satisfies the conclusions of Theorem 4.11.

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