Schrödinger revisited: an algebraic approach

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Starting with the quantum Liouville equation, we write the density operator as the product of elements respectively in the left and right ideals of an operator algebra and find that the Schrödinger picture may be expressed through two representation independent algebraic forms in terms of the density and phase operators. These forms are respectively the continuity equation, which involves the commutator of the Hamiltonian with the density operator, and an equation for the time development of the phase operator that involves the anti-commutator of the Hamiltonian with this density operator. We show that this latter equation plays two important roles: (i) it expresses the conservation of energy in a system where energy is well defined and (ii) it provides a simple way to evaluate the gauge changes that occur in the Aharonov-Bohm, the Aharonov-Casher, and Berry phase effects. Both these operator (i.e. purely algebraic) equations also allow us to re-examine the Bohm interpretation, showing that it is in fact possible to construct Bohm interpretations in representations other than the $z$-representation. We discuss the meaning of the Bohm interpretation in the light of these new results in terms of non-commutative structures and this enables us to clarify its relation to standard quantum mechanics.

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I. INTRODUCTION

In a previous paper, Monk and Hiley [1] have suggested that instead of using the traditional Hilbert space description of quantum phenomena, one should give primary consideration to the algebraic structure, not only because it has a number of mathematical advantages that have already been pointed out by Dirac [2–4], but because it offers the possibility of a radically different interpretation of the quantum formalism. By algebraic structures, we mean exploiting the rich possibilities contained in, for example, C* and W* (von Neumann) algebras, which play an important role in field theory [5,6] as well as in equilibrium [7,8] and non-equilibrium statistical mechanics [9,10]. In spite of the potential richness of these methods, they have not been used in the general debate on the foundations of quantum theory mainly because of the abstract nature of the mathematics.

In their paper, Monk and Hiley [1] outlined how this mathematics can be simplified so that the approach becomes much more transparent. Furthermore, one can develop an interpretation for this formalism provided one is willing to give up the basic ideas of particles- and/or fields-in-interaction and instead, think in terms of process. Indeed, recent work [11–14] has shown how the algebraic approach does have a potential for taking the discussions of the meaning of the quantum formalisms into new domains. It is this background that provides the motivation for the present paper. However, we will not assume any detailed prior knowledge of this abstract algebraic structure. Our purpose here is to show how we can re-write the equations of elementary quantum mechanics in a purely algebraic way, which, as we show, yields some interesting new insights.

Traditionally, the algebraic approach has implied the use of the Heisenberg picture. Here the operators (or elements of the algebra, in our case) become time dependent and carry the dynamics of the quantum system. One clear advantage of this approach is that all the elements of the algebra are representation-independent and so we are not tied to any one particular representation. A further advantage is that the equations of motion have a close structural similarity to the classical equations of motion, viz, commutator brackets directly replace Poisson brackets [2,3].

In contrast to this, the Schrödinger picture has the time development entirely tied into the wave function, and as such is a representation dependent object which appears to exists only in Hilbert space and does not appear in the algebra. Thus, the Schrödinger picture does not seem to have the generality of the representation-independent Heisenberg picture.

In this paper, we will show that this representation dependence of the Schrödinger equation is only apparent and in section II we will show that it can be written in a representation independent form. This means writing the wave function as a “wave operator” (in the left ideal of the algebra) so that the Schrödinger equation becomes purely algebraic and independent of any representation in a Hilbert space.

The way to do this has been known for a long time [6,21]. Indeed, Monk and Hiley [1] have already shown in simple terms that the key step involves expressing the wave operator in the left ideal. This means that the wave function must be replaced by the density operator, $\rho$, even in the case of pure states. Here the density operator plays the role of an idempotent and it is this idempotency that is central in the whole approach.

Starting with the quantum Liouville equation and writing the density operator as the product of elements respectively in the left and right ideals in the algebra, we find that the Schrödinger picture may be expressed through two algebraic forms which are representation independent. These forms are the two equations (19)
and (20) for the density and phase operators. The first, expressed in terms of the commutator of the Hamiltonian and the density operator, is an alternative form of the quantum Liouville equation which, as is well known, describes the conservation of probability. The second describes the time dependence of the phase and is expressed in terms of the anti-commutator of the Hamiltonian and the density operator. This second equation, which does not appear in the literature as far as we are aware, becomes an equation for the conservation of energy in systems when energy is well defined.

In section III, the time dependent phase operator equation (20) is shown to be gauge invariant and reproduces some well known results of gauge theory in a very direct and simple way. For example, one can immediately derive the Aharonov-Bohm phase for a particle travelling in a vector potential while a trivial extension incorporates the Aharonov-Casher phase. This latter phase arises when a neutral particle with a magnetic moment passes a line-charge. In addition to these examples, the Berry phase and its associated energy follow almost trivially from the same equation.

In sections IV and V we use these equations to explore in more detail the Bohm interpretation [BI] [15,16]. Because the equations (19) and (20) are representation independent, we can construct a BI based on trajectories in any representation. To do this, we introduce a generalisation of the current density operator and demonstrate its use in both ordinary space and momentum space. In section V and through the examples in section VI we show in detail how one can construct a consistent BI in the $p$-representation. This is contrary to the assertion that this is not possible even in “the simplest case to construct an acceptable causal interpretation” [17] and we discuss the significance of this statement in the light of our examples.

Our examples not only remove one of serious criticisms of this interpretation, namely that it does not use the full symplectic symmetry of the quantum formalism, but also provide us with new insights into the meaning of BI and its relation to standard quantum mechanics. In all of this work no appeal is made to any classical formalism whatsoever, showing that the BI is quantum through-and-through.

Perhaps the most important conclusion of this work is to show the BI arises directly from the non-commutative structure of the quantum mechanical phase space. Non-commutative geometries are not built on any form of well defined continuous manifolds. We are forced to construct “shadow manifolds” [18,19]. As we show in section V, these shadow manifolds have the structure of a phase space. One is constructed using the $x$-representation and the other uses the $p$-representation. These spaces are different but converge to the same phase space in the classical limit.

In the final section VII we discuss the consequences of this construction for the BI. In fact we show that our approach more clearly illustrates the ideas that Bohm and Hiley presented in the final chapter of their book [16]. There it was argued that a new way of exploring the meaning of the quantum formalism required a new order, the implicate order, this order having its origins in the mathematics of non-commutative geometry [20]. The shadow phase spaces are examples of explicate orders. Finally, we briefly discuss how the BI fits into this general scheme.

II. THE ALGEBRAIC APPROACH TO THE SCHRÖDINGER EQUATION

We begin by writing the Schrödinger equation in a general representation

$$i\frac{\partial \psi(a_i, t)}{\partial t} = H\psi(a_i, t) \quad (\hbar = 1) \quad (1)$$

where the $a_i$ are the eigenvalues of $A$ an algebraic element or operator in the algebra$^1$. Equation (1) introduces the state vectors $\psi(a_i, t)$ that are not elements of the algebra; rather, they are elements of a separate vector (Hilbert) space and as such depend on a representation. In this form, the Schrödinger equation does not appear to be part of the algebra even though it uses elements of the operator algebra.

On the other hand, in the Heisenberg approach we write the Hamiltonian flow of the operator $A \in A$ as

$$A(0) \rightarrow A(t) = M(t)^{-1}A(0)M(t) \quad (2)$$

where $M(t) = \exp[-iHt]$, so giving rise to the Heisenberg equation of motion

$$\frac{dA}{dt} = \frac{1}{i}[A, H]_- \quad . \quad (3)$$

This means that the Heisenberg time evolution can be regarded as an inner automorphism in the algebra $M(t) : A \rightarrow A \forall A \in A$. The equation of motion can be generalised to include the explicit time dependence of $A$ giving

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{1}{i}[A, H]_- \quad . \quad (4)$$

Note that this equation is representation-free and the time evolution is discussed entirely within the algebra itself.

In the algebraic approach [21], the state function is introduced through a density operator, $\rho$. This operator is actually in the quantum algebra so that the Schrödinger time development must be implicit within the algebra itself. How, then, can the time evolution be algebraically

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$^1$Throughout this paper, we set $\hbar = 1$ and use the convention of representing operators by capitals and eigenvalues by lower case letters.
expressed within the Schrödinger picture, without reference to either Hilbert space or a particular representation in it?

In the usual approach to quantum mechanics, the density operator is, unfortunately, not introduced as a primitive notion in the theory. Rather, it is introduced almost as an after-thought when it is found necessary to deal with mixed states. But using the density operator as a starting point has the advantage of including both pure states and mixed states together and of satisfying the idempotent condition $\rho = \rho^2$. Moreover, if we adopt the further defining condition that the density operator must satisfy the Liouville theorem

$$\frac{d\rho}{dt} = 0,$$  \hspace{1cm} (5)

then, since the density operator is also an element of the algebra, equation (4) immediately leads to the quantum Liouville equation

$$\frac{\partial\rho}{\partial t} = \frac{1}{i}[H,\rho].$$  \hspace{1cm} (6)

If $\rho = \psi\langle\psi$, is an element of the algebra, then it must also be possible to identify the ket and the bra with particular elements of the algebra. A ket is an element of a vector space, which when multiplied from the left it must remain in that space. The algebraic equivalent of this vector space is a left ideal $I_L$. An element, $B\varepsilon$, of a left ideal, where $B$ is a wave operator and $\varepsilon$ a primitive idempotent, corresponds to a ket. Similarly an element, $\varepsilon C$, of a right ideal, $I_R$, corresponds to a bra. This suggests we write $\rho = B\varepsilon C = B\varepsilon$, which means that a pure state density operator corresponds to a two sided ideal, subject to the conditions $\rho = \rho^2$ and $\text{tr} \rho = 1$.

To see how to write the algebraic equivalent to the Schrödinger equation, let us substitute $\rho = B\varepsilon C$ into the equation of motion (6), so that

$$\frac{i\hbar}{\partial t} \varepsilon C + i\hbar B\varepsilon \frac{\partial C}{\partial t} = HB\varepsilon C - B\varepsilon CH. \hspace{1cm} (7)$$

Since $B$ and $C$ are operator elements outside the ideals of the algebra, it can be assumed that there exist $B^\dagger : B^\dagger B = 1$ and $C^\dagger : CC^\dagger = 1$. Multiplying the above equation from the left by $B^\dagger$ and from the right by $C^\dagger$, we find after re-arrangement, that

$$B^\dagger \left( i\hbar \frac{\partial B}{\partial t} - HB \right) \varepsilon = -\varepsilon \left( i\hbar \frac{\partial C}{\partial t} + CH \right) C^\dagger. \hspace{1cm} (8)$$

Since $B(C)$ is any non-null element of the algebra, we can write

$$i \left( \frac{\partial B}{\partial t} \right) \varepsilon = HB\varepsilon \hspace{1cm} (9)$$

and

$$-i\varepsilon \left( \frac{\partial C}{\partial t} \right) = \varepsilon CH. \hspace{1cm} (10)$$

We see immediately that equations (9) and (10), which are respectively in the left and right ideals of the algebra, have the same general form as the Schrödinger equation and its conjugate counterpart ($H$ is assumed to be Hermitian). We stress here again that $B$ and $C$ are elements of the algebra and not elements of a Hilbert space.

To see exactly how these two equations are related to the usual Hilbert space formalism, we specifically choose the wave operator $B$ to be a function of the position operator $X$ so that $B\varepsilon = B(X,t) \in I_L$ and then project $B(X,t)$ into a complex function belonging to $L^2(x,\mu)$ viz.$^2$

$$\eta : B(X,t) \rightarrow B(x,t) \hspace{1cm} (11)$$

so that

$$B(X,t)(x) = B(x,t). \hspace{1cm} (12)$$

This is the usual wave function$^3$, conventionally written as $\Psi(x,t)$. It is now straightforward to show that equation (9) becomes the Schrödinger equation

$$i\frac{\partial \Psi(x,t)}{\partial t} = H(x)\Psi(x,t). \hspace{1cm} (13)$$

The conjugate equation can be derived by first assuming the dual projection

$$\eta^* : C(X,t) \rightarrow C^*(x,t) \hspace{1cm} (14)$$

so that

$$C(X,t)(x) = C^*(x,t). \hspace{1cm} (15)$$

Again it is straightforward to show that equation (10) leads to the conjugate Schrödinger equation. Thus equations (9) and (10) are the algebraic, representation independent equivalents of the Schrödinger equation.

Now let us continue developing the general structure. We write the wave operators $B$ and $C$ in the mutually conjugate forms $B = \exp[iS_Q(t)]$ and $C = \exp[-iS_Q(t)]$, where $S_Q = S - i \ln R$. In this case equations (9) and (10) become the dual pair

$$-\frac{\partial S_Q}{\partial t} B\varepsilon = HB\varepsilon \hspace{1cm} (16)$$

and

$^2$L$^2(x,\mu)$ means square integrable complex functions with measure $\mu$.

$^3$Not all elements of a left ideal produce state functions that are physically meaningful. We will not discuss these restrictions here. (See Ballentine [22].)
\[-\varepsilon C \frac{\partial S_Q^I}{\partial t} = \varepsilon CH, \quad (17)\]

which are quantum algebraic equivalents of the Hamilton-Jacobi equation of classical mechanics

\[\frac{\partial S_{cl}}{\partial t} + H = 0, \quad (18)\]

where \(S_{cl}\) is the classical action. It is natural, therefore, to call \(S_Q\) the quantum action.

Equations (16) and (17) respectively evolve in the left and right ideals, which are mutually dual spaces, so reflecting the essential duality between the Schrödinger equation and its complex conjugate. This duality can be lifted out of the left and right ideals of the algebra and reflected in another pair of algebraic equations. Post- and pre-multiplying equations (16) and (17) by \(\varepsilon C\) and \(B\varepsilon\) respectively and then adding and subtracting the resulting equations, we find

\[\left( \frac{\partial S_Q}{\partial t} \rho + \rho \frac{\partial S_Q^I}{\partial t} \right) = [H, \rho]_-, \quad (19)\]

and

\[\left( \frac{\partial S_Q}{\partial t} \rho - \rho \frac{\partial S_Q^I}{\partial t} \right) = [H, \rho]_+, \quad (20)\]

Of note is the appearance of the commutator and the anti-commutator with the Hamiltonian on the RHS of these two equations. They are mathematically equivalent to the Schrödinger equation and its conjugate and are general in the sense that they are independent of a specific representation. Expressing equation (19) in Hermitian form, we immediately recover (6), the quantum Liouville equation. Equation (20) cannot in general be reduced to a simpler algebraic form but may be recognised as a symmetrised operator form of the Hamilton-Jacobi equation.

The meaning of equations (19) and (20) can be further clarified by looking at the diagonal elements in the \(a\)-representation. Thus,

\[\frac{\partial \rho_R(a)}{\partial t} + \frac{1}{i} [\rho, H]_- (a) = 0 \quad (21)\]

and

\[\rho_R(a) \frac{\partial S(a)}{\partial t} + \frac{1}{2} [\rho, H]_+ (a) = 0. \quad (22)\]

Here we have written \([\ldots, ](a) = \langle a | [\ldots] | a\rangle\) and \(\rho_R = R^2\). If we now choose \(A\) to be the position operator, then equation (21) takes the form

\[\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (23)\]

where \(\mathcal{P} = \mathcal{P}(x) = \rho_R(x) = \langle x | \rho | x \rangle\) and \(\mathbf{j}\) is a probability current. Thus, the Liouville equation (21) is identified with the conservation of probability as expected. Equation (22) expresses the time variation of the quantum phase and so we will call it (and the more general form (20)) the quantum phase equation. In a state in which the energy is well defined, this equation becomes

\[\frac{\partial S}{\partial t} = -E. \quad (24)\]

In this case, equation (22) expresses the conservation of energy in Hamilton-Jacobi form.

The Liouville equation (21) is well known and plays a prominent role in quantum statistical mechanics. The quantum phase equation (22) does not usually appear in the literature, although something similar has been used by George et al. [10] in their discussions of irreversible quantum processes. In their case, the anti-commutator is simply introduced by defining it to be the energy super-operator. What we show here is that this operator comes directly from the Schrödinger equation and although the extension to super operator status is possible, this generalisation is not necessary for the purposes of this paper.

In summary then, equations (19) and (20) are simply the algebraic equivalents of the Schrödinger equation when it is written in a way that does not depend on a specific representation. It is now easy to confirm that these two equations, when expressed in a particular representation, are simply the real and imaginary parts of the Schrödinger equation under a polar decomposition of the wave function written in that particular representation.

**III. THE QUANTUM PHASE EQUATION**

**A. Gauge invariance**

We will first examine equation (22) in some detail. Let us begin by looking at the form of this equation in the \(x\)-representation when we choose the Hamiltonian \(H = p^2/2m + V(x)\). Here equation (22) becomes

\[\frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V(x) - \frac{1}{2mR} \left( \frac{\partial^2 R}{\partial x^2} \right)^2 = 0 \quad (25)\]

This equation is the real part of the Schrödinger equation in the \(x\)-representation.

Before examining this equation in detail we must ensure that it is gauge invariant. To show that this is the case, let us first introduce the gauge transformation \(V'(x) = V(x) + V_0\). This must be accompanied by the phase transformation \(\psi'(x, t) = \phi(x) \exp[-i(E + V_0)t]\).

Since we are considering a well defined energy state, the transformed equation (20) will read

\[\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (26)\]

where \(\mathcal{P} = \mathcal{P}(x) = \rho_R(x) = \langle x | \rho | x \rangle\) and \(\mathbf{j}\) is a probability current. Thus, the Liouville equation (21) is identified with the conservation of probability as expected. Equation (22) expresses the time variation of the quantum phase and so we will call it (and the more general form (20)) the quantum phase equation. In a state in which the energy is well defined, this equation becomes

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so that
\[ ((-\frac{\partial S}{\partial t} + V_0)\rho - \rho(\frac{\partial S^\dag}{\partial t} + V_0)) = [H, \rho]_+ + [V_0, \rho]_+ \]

(26)

because \( \rho' = \rho, \, H' = H + V_0 \) and \( S_0' = S_Q + V_0 \).

Then, since \([V_0, \rho]_+ = 2V_0\rho\), we immediately recover equation (20), so establishing its gauge invariance and that of equation (22).

Gauge invariance in this case involves the phase change \( S' = S + S_0 \) since
\[ V'(x, t) = V(x) + V_0(t) \]
our equation gives
\[ \frac{\partial S_0}{\partial t} = V_0(t) \]
so that
\[ S_0 = \int_{t_0}^{t} V_0(t')dt' \]
It will immediately be recognised that this is the expression for the scalar part of the Aharonov-Bohm effect [23].

We can also obtain the magnetic phase shift from equation (22) by starting from the Hamiltonian
\[ H = \frac{1}{2m}(P - eA)^2 \quad (c = 1) \]
(30)
On expanding this Hamiltonian, we find
\[ H = \frac{1}{2m}P^2 - \frac{e}{2m}(P \cdot A + A \cdot P) + \frac{e^2}{2m}A^2 \]
(31)
\[ = H_{\text{free particle}} + H_{\text{int}} + H_{\text{free field}} \]
(32)
The corresponding phase will then consist of three terms
\[ S = S_{\text{free particle}} + S_{\text{int}} + S_{\text{free field}} \]
(33)
so that the diagonal form
\[ \rho_R \left( \frac{\partial S_{\text{int}}}{\partial t} \right) + \frac{1}{2} [H_{\text{int}}, \rho]_+ = 0 \]
(34)
gives
\[ \rho_R \frac{\partial S_{\text{int}}}{\partial t} = eA \cdot \jmath_x \]
(35)
in the \( x \)-representation. If we then write \( \jmath_x = \rho_R \nabla \), we find
\[ S_{\text{int}} = e \int_{t_0}^{t} A \cdot \nabla dt = e \int_{x_0}^{x} A \cdot dx \]
(36)
We immediately recognise this equation as the expression for the Aharonov-Bohm phase for the vector potential\(^5\).

\(^5\)This effect was derived in the \( x \)-representation from the ‘guidance’ condition by Philippidis et al [24]. A more recent discussion using this approach rather than the method we use can be found in Sjöqvist and Carlsen [25].

The phase for the Aharonov-Casher effect [26], which involves a neutral particle with a magnetic moment passing a line of electric charges, also follows trivially once the Hamiltonian
\[ H = \frac{1}{2m}(P - E \times \mu)^2 - \frac{\mu E^2}{m} \]
(37)
is assumed. The additional phase change also follows trivially from the same procedure used for the vector potential.

It should also be noted that the Berry phase [27,28] emerges directly from equation (22). In this case the behaviour of the quantum system depends on some additional cyclic parameter \( B(t) \). The phase now becomes a function of this parameter. Thus equation (22) becomes
\[ \rho_R \left( \frac{\partial S}{\partial t} + B \frac{\partial S}{\partial B} \right) + \frac{1}{2} [\rho, H]_+ = 0 \]
(38)
giving an extra phase factor \( B \frac{\partial S}{\partial B} \). Thus the contribution to the phase from this extra degree of freedom is
\[ S_{\text{Berry phase}} = \int_{t_0}^{t} B \frac{\partial S}{\partial B} dt \]
(39)
To evaluate this term, we need to consider specific problems which means going to a specific Hamiltonian in a specific representation. This representation is generally the \( x \)-representation. Berry [27] considered the case of the precession of nuclear spin in a magnetic field in his original paper, and showed that
\[ \frac{\partial S}{\partial B} = \Im \langle B(t), t | \nabla_B | B(t), t \rangle \]
(40)
so that
\[ S_{\text{Berry phase}} = \Im \int_{B_0}^{B} \langle B(t), t | \nabla_B | B(t), t \rangle dB \]
(41)
which is exactly the result obtained by Berry [27].

B. The \( x \)- and \( p \)-representations

Having seen how the additional phase changes arise for these simple gauge fields, we now return to examine the details of equation (25). To do this, let us consider the case of the harmonic oscillator with Hamiltonian \( H = \frac{p^2}{2m} + Kx^2/2 \). In this case equation (25) reads
\[ \frac{\partial S_x}{\partial t} + \frac{1}{2m} \left( \frac{\partial S_x}{\partial x} \right)^2 + \frac{Kx^2}{2} - \frac{1}{2mR_x} \left( \frac{\partial^2 R_x}{\partial x^2} \right) = 0 \]
(42)
where we have inserted the suffix \( x \) to emphasise that this is equation (22) expressed in the \( x \)-representation.
Now let us write down the corresponding equation in the \( p \)-representation. This takes the form

\[
\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + K \left( \frac{\partial S_p}{\partial p} \right)^2 - \frac{K}{2R_p} \left( \frac{\partial^2 R_p}{\partial p^2} \right) = 0 \quad (43)
\]

It should be noted that although the functional forms of these two equations are clearly different, they nevertheless have the same energy content. This can be very easily checked for the ground state of the harmonic oscillator. One can quickly show that both equations give the ground state energy to be \( \omega/2 \), the zero-point energy.

In spite of the differences in functional form, there are structural similarities between these two equations. These arise essentially because we have chosen a symmetric Hamiltonian. For example, instead of \( p \) that appears in what looks like a kinetic energy term in equation (43), we have \( (\partial S_x/\partial x) \) in equation (42), and instead of \( x \) in the potential energy term in equation (43), we have \( (\partial S_p/\partial p) \). The last term in each equation has the same general form except with the roles of \( x \) and \( p \) interchanged.

Since equation (42) is the real part of the Schrödinger equation, we can identify \(^6\)

\[
p_r = \Re \left[ \psi^*(x) \psi(x) \right] \frac{\partial S_x}{\partial x} \quad (44)
\]

Substituting this into equation (42) we find

\[
\frac{\partial S_x}{\partial t} + \frac{p^2}{2m} + K \left( \frac{\partial S_x}{\partial x} \right)^2 - \frac{1}{2mR_x} \left( \frac{\partial^2 R_x}{\partial x^2} \right) = 0 \quad (45)
\]

In the Bohm interpretation, \( p_r \) was identified with the “beable” momentum. With this identification equation (44) makes it now quite clear why the beable momentum is a function of \( x \), in contrast to the classical momentum which is always an independent variable.

If \( p_r \) is a momentum, then clearly equation (45) looks like an equation for the total energy of the quantum system. If we make the assumption that this is an expression for the conservation of energy then, in quantum theory, we must have an additional quality of energy represented by the last term on the RHS. This term is, of course, the quantum potential energy. As has been shown elsewhere this new quality of energy offers an explanation of quantum phenomena [16].

Notice that equation (44) allows the possibility of approaching the classical limit smoothly. In this limit \( S_x \rightarrow S_{cl} \), \( p_r \rightarrow p_{cl} = (\partial S_{cl}/\partial x) \) and the quantum potential energy becomes negligible so that equation (45) becomes the classical Hamilton-Jacobi equation.

Before continuing, we wish to stress a point that has not been often fully appreciated, namely, that equation (42) is a quantum equation and converting it to equation (45) requires no appeal whatsoever to classical physics. It is true that in the traditional approach to this equation, the BI has made use of the relation \( p = (\partial S_x/\partial x) \) by appealing to classical canonical theory. However, this is an unnecessary backward step.

It is because equations (22) and (42) are part of the quantum formalism that we were able to derive quantum effects such as the Aharonov-Bohm, Aharonov-Casher and Berry phases from equation (22). In passing it should also be noted that both the \( x \)- and \( p \)-representations of this equation (i.e., (42) and (43)) contain a term which we have called the quantum potential. This potential is modified by the presence of the gauge effects as was first shown by Philippidis, Bohm and Kaye [24]. The quantum potential is central to ensuring energy is conserved and, furthermore, it encapsulates quantum non-separability or quantum non-locality [30]. The quantum potential plays a key role in our approach and must be distinguished from Bohmian mechanics as advocated by Dürr et al. [31].

If we now turn to the \( p \)-representation, i.e., equation (43), we can write it in the form

\[
\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + K \left( \frac{\partial S_p}{\partial p} \right)^2 - \frac{K}{2R_p} \left( \frac{\partial^2 R_p}{\partial p^2} \right) = 0 \quad (46)
\]

by introducing

\[
x_r = \Re \left[ \psi^*(p) X \psi(p) \right] \frac{\partial S_p}{\partial p} \quad (47)
\]

Here \( x_r \) is the position “beable”, which now supplements the momentum \( p \). Again in the classical limit, we have \( S_p \rightarrow S_{cl} \), \( x_r \rightarrow x_{cl} = -(\partial S_{cl}/\partial p) \) and the last term on the RHS of (46) becomes negligible. It should be noted that in this limit equations (45) and (46) reduce to the same equation giving rise to a unique phase space, which is identical to the classical phase space.

All the above equations are part of standard quantum mechanics. Although we have drawn attention to the significance of equations (45) and (46) to the BI, we have yet to discuss the interpretation in any detail. To do this we first need to find a way to calculate “trajectories”. In the traditional approach to the BI this is done by regarding \( p = (\partial S_x/\partial x) \) as a “guidance” condition and then using \( \dot{x} = p/m \) from which one can calculate a set of trajectories. These trajectories are then integrals of the velocity associated with the probability current in the co-ordinate representation. However this is not the general way to do it as can be seen by considering the \( p \)-representation. The analogous expression in this representation is \( x = -(\partial S_p/\partial p) \) and this clearly cannot be regarded as a “guidance” condition. Something is not quite right here. In order to find out what is involved it is necessary to explore the Liouville equation (21) in more detail.

\(^6\)Holland [29] has called expressions of this type ‘local expectation values’.
IV. PROBABILITY CURRENTS

In this section we will focus our attention on the Liouville equation (21). In the $x$-representation, this equation gives rise to the conservation of probability equation (23) with the probability current defined by

$$\mathbf{J} = \frac{1}{2mi}[\psi^*(\nabla\psi) - (\nabla\psi^*)\psi].$$  \hspace{1cm} (48)

However, our aim is to find an expression for the current that is not representation specific. To do this we first consider the classical Liouville equation

$$\frac{\partial\rho}{\partial t} + \{\rho, H\} = 0$$  \hspace{1cm} (49)

where $\{\}$ is the Poisson bracket. It is easy to verify that this equation can be written in the form

$$\frac{\partial\rho}{\partial t} + \{\mathbf{J}_\rho, \mathbf{P}\} - \{\mathbf{J}_\mathbf{P}, \mathbf{X}\} = 0$$  \hspace{1cm} (50)

with

$$\mathbf{J}_\rho = \rho \nabla\mathbf{P}$$  \hspace{1cm} (51)

and

$$\mathbf{J}_\mathbf{P} = -\rho \nabla\mathbf{X}.$$  \hspace{1cm} (52)

(The Poisson bracket of two vector functions is defined here as $\{\mathbf{v}, \mathbf{w}\} = \sum_k \{v_k, w_k\}$.) If we now follow Dirac’s suggestion by respectively replacing classical variables and Poisson brackets with operators and commutators, we find

$$i \frac{\partial\rho}{\partial t} + \{\mathbf{J}_\rho, \mathbf{P}\} - \{\mathbf{J}_\mathbf{P}, \mathbf{X}\} = 0$$  \hspace{1cm} (53)

where

$$\mathbf{J}_\rho = \nabla\mathbf{P}(\rho\mathbf{H})$$  \hspace{1cm} (54)

and

$$\mathbf{J}_\mathbf{P} = -\nabla\mathbf{X}(\rho\mathbf{H}).$$  \hspace{1cm} (55)

In the simple case of a free particle of mass $m$ we have

$$\mathbf{J}_\rho = \frac{1}{2m}(\rho\mathbf{P} + \mathbf{P}\rho)$$  \hspace{1cm} (56)

and

$$\mathbf{J}_\mathbf{P} = 0.$$  \hspace{1cm} (57)

To see how this connects to the conventional results, let us evaluate equation (52) in the $x$-representation. Here we find

$$i \frac{\partial(x|\rho|x)}{\partial t} + (x|\mathbf{J}_\rho, \mathbf{P}|x) - (x|\mathbf{J}_\mathbf{P}, \mathbf{X}|x) = 0$$  \hspace{1cm} (58)

If $H = \frac{p^2}{2m} + V(\mathbf{X})$ then the first commutator gives

$$\langle x|\mathbf{J}_\rho, \mathbf{P}|x \rangle = \nabla_x \cdot \mathbf{j}_x$$

and the second commutator vanishes. Thus, equation (57) becomes

$$\frac{\partial\rho(x)}{\partial t} + \nabla_x \cdot \mathbf{j}_x = 0$$  \hspace{1cm} (59)

which is just equation (23) and

$$\mathbf{j}_x = \langle x|\nabla\rho(x)|x \rangle$$  \hspace{1cm} (60)

which gives an expression for the current that is identical to the usual expression given by equation (48). Furthermore, it is unique since it is independent of the form of the potential used in the Hamiltonian.

In the $p$-representation we find

$$\frac{\partial\rho(p)}{\partial t} + \nabla_p \cdot \mathbf{j}_p = 0$$  \hspace{1cm} (61)

Thus, we can now calculate probability currents in the $p$-representation. Unfortunately equation (61) does not give us a model independent expression for the probability current because the specific form of the current depends on the form of $V(x)$. On reflection this is not surprising because the rate of change of momentum must depend upon the externally applied potential. We will examine the consequences of these results for the Bohm interpretation in the next section.

V. RE-EXAMINATION OF THE BOHM APPROACH

Let us now re-appraise the Bohm approach in the light of the new results presented above.

It has been assumed that it is not possible to construct a BI using any representation other than the $x$-representation. This belief arises from an early correspondence between Epstein [33] and Bohm [17]. Epstein suggested that it should be possible to develop an alternative causal interpretation by starting in the momentum representation. Bohm replied agreeing that a new causal interpretation could possibly arise from such a procedure provided the canonical transformation on the particle variables were simultaneously accompanied by a corresponding linear transformation on the wave function. But, he concluded that this did not seem to lead, even in the simplest of cases, to an acceptable causal interpretation. He does not explain why he came to this conclusion but this position has remained the accepted wisdom.

The general results with the harmonic oscillator presented above show that, at least as far as the mathematics is concerned, it does seem possible to develop a causal
interpretation in the $p$-representation based upon equations (46), (47) and (61). We will illustrate how this can be done using specific examples in section VI, but here we will simply discuss the general principles involved.

We have already pointed out in section III that the so called “guidance” condition is assumed to play a pivotal role in what is known as “Bohmian mechanics” [31] does not generalise to the $p$-representation. However what does generalise is a method based on probability currents. Thus, in any $q$-representation we use

$$\frac{dq}{dt} = \frac{j_q}{P(q)}$$

(62)

which can be integrated immediately to find a set of trajectories in a general $(q,t)$ space.

In the $x$-representation we have

$$\frac{dx}{dt} = \frac{j_x}{P(x)}$$

(63)

which, when integrated, gives the particle trajectories of the BI. This approach is similar to that used in pragmatic quantum mechanics where the probability current is assumed to describe the flux of particles emerging from, say, a scattering process. Here the flux at a detector is interpreted as the rate of arrival of the scattered particles.

The additional assumption made in the BI is that particles exist with simultaneously well defined positions and momenta and each particle follows one of the one-parameter curves. Such an assumption is clearly excluded in standard quantum mechanics, but this leaves us with the difficulty of understanding how to incorporate the Born probability postulate and its role in the continuity equation (23) except in some abstract sense.

If we do follow the BI in the $x$-representation, then the position of the particle is clearly defined and the momentum, $p_r$, associated with the particle must be provided through the relation

$$p_r = \frac{\Re [\psi^*(x)P\psi(x)]}{|\psi(x)|^2} = \left( \frac{\partial S_x}{\partial x} \right) = m \frac{dx}{dt}$$

(64)

Here the “beable” momentum $p_r$ is wholly quantum in origin showing that the BI has its origins entirely within quantum mechanics.

Now in the $p$-representation we use

$$\frac{dp}{dt} = \frac{j_p}{P(p)}$$

(65)

to give a set of one-parameter curves in momentum space. In this approach the momentum of the particle has a clear meaning, while the position beable $x_r$ is given by equation (47), namely

$$x_r = \frac{\Re [\psi^*(p)X\psi(p)]}{|\psi(p)|^2} = -\left( \frac{\partial S_p}{\partial p} \right)$$

(66)

This means that the derivative in the current $j_r = -\left( \frac{\partial}{\partial x} \psi^*(x)\rho V(x)\psi(x) \right)$ given by equation (61) must be evaluated at $x = x_r$. Thus we again have a specification of the particle with a given momentum at a given “beable” position $x_r$.

Thus, the BI based on equations (45) and (46) leads to two distinct phase spaces, one constructed on each representation. Each phase space contains a set of trajectories, one derived from $j_x$ and the other from $j_p$. Although these phase spaces are actually different, they carry structures that are consistent with the content of the Schrödinger equation. This is in contrast to the classical limit where there is a unique phase space. However we have already noted that equations (45) and (46) reduce to a single equation in the classical limit so the existence of (at least) two phase spaces is a consequence of the quantum formalism.

Now the existence of at least two phase spaces may come as a surprise to those who see the BI as a return to classical or quasi-classical notions. What we have shown here is that the BI enables us to construct what we may call “shadow phase spaces”, a construct that is a direct consequence of the non-commutative nature of the quantum algebra. Giving ontological meaning to the non-commutative algebra implies a very radical departure from the way we think about quantum processes. This was the central theme of Bohm’s work on the implicate order [20]. The work presented in this paper fits directly into this conceptual structure, a point that will be discussed at length elsewhere.

Our present purpose is to clarify the structure of the mathematics lying behind the BI. To this end note that choosing a representation is equivalent to choosing an operator which is to be diagonal. Thus in the phase space described by equation (45) the position eigenvalues are used for the $x$ co-ordinates and we then construct the momentum co-ordinate through the condition $p_r = (\partial S/p/\partial x)$ to provide the “beable” momentum.

On the other hand, equation (46) describes a phase space constructed using the momentum eigenvalues together with the “beable” position $x_r$ defined by $x_r = -(\partial S_p/\partial p)$. In this way we see exactly how it is possible to construct two different phase spaces, one for each representation.

The fact that we can find a BI in the $p$-representation removes the criticism that the BI does not use the full symplectic symmetry of the quantum formalism. But removing this asymmetry might, at first sight, destroy the claim that the Bohm interpretation provides a unique ontological interpretation. This would only be true if we were insisting that the ontology demands a unique phase space. However, as we have already remarked

7Hereafter, the notation is restricted to one degree of freedom. Generalisation to many degrees of freedom is straightforward.
the quantum algebra is non-commutative and a unique phase space is not possible. It was for this reason that Bohm and one of us (BJH) began to explore the possibility of giving ontological significance to the algebra itself. This involves thinking in terms of process rather than particles- or fields-in-interaction and this leads, in turn, to introducing the implicate order mentioned above. This is a very different order from the one assumed by most physicists, which is essentially what we call the Cartesian order.

Once again we contrast our approach with Bohmian mechanics introduced by Dürr et al [31]. Their approach requires the \( x \)-representation to be taken as basic and the guidance relation to be taken as the defining equation of the approach. In view of the results presented here, we see we could have started from the \( p \)-representation. But here the relation \( p = (\partial S/\partial x) \) cannot play the role of a guidance condition. Hence making the guidance condition as the defining equation in the \( x \)-representation is arbitrary and contrary to what Bohm himself had in mind [34–36].

In regard to the lack of \( x-p \) symmetry in the traditional approach to the BI, Bohm and Hiley [16] found it necessary to discuss why \( x \) was the only intrinsic property of the particle, all others depended upon the context. This was certainly felt by one of us (BJH) to be a somewhat arbitrary imposition that did not seem to be a natural consequence of the symplectic invariance of the formalism itself. Had we started with the \( p \)-representation we would have found \( p \) to be the intrinsic property, while \( x \) depended upon some context. Thus the restoration of symmetry explains why particular variables become intrinsic and others not.

In the examples we give in this paper, we only consider the two operators \( X \) and \( P \). If we regard the change from the \( x \)-representation to the \( p \)-representation as a rotation of \( \pi/2 \) in phase space, we could think about exploring rotations through other angles. Such transformation exist and are known as fractional Fourier transformations which correspond to rotations through any angle \( \alpha \) in phase space [37,38]. These allow us to express equations (19) and (20) any arbitrary representation. This generalisation has been investigated and will be reported elsewhere [39].

All of this shows that the \( x \) variable is not special as far as the mathematics goes. The real question is why it is necessary to construct different phase spaces in the first place, but before we go into this question we want to present some examples where we can compare in more detail the results obtained from both \( x \)- and \( p \)-representations.

VI. SPECIFIC EXAMPLES: COMPARISONS OF \( X \)- AND \( P \)-REPRESENTATIONS

A. The free particle described by a Gaussian wave packet

We will start with the simplest case of a particle described by a Gaussian wave packet centred at position \( x = 0 \) with mean momentum zero.

The wave packet has the (normalised) Gaussian distribution

\[
\phi(p, t) = \left[ \frac{2(\Delta x)^2}{\pi} \right]^{\frac{1}{4}} \exp\left[ -p^2(\Delta x)^2 \right] \exp\left[ -\frac{ip^2}{2m}t \right] \tag{67}
\]

in the momentum representation. In this representation the current \( j_p = 0 = (dp/dt) \) so that the trajectories are of constant momentum. Equation (22) gives

\[
\frac{\partial S}{\partial t} + \frac{p^2}{2m} = 0 \tag{68}
\]

which shows that the quantum potential is zero, as is to be expected from the form of the wave function \( \phi(p, t) \).

In the \( x \)-representation, the wave packet spreads in the \( x \)-direction, having the wave function

\[
\psi(x, t) = \frac{1}{(2\pi(D(t)))^{\frac{1}{4}}} \exp \left[ -\frac{x^2}{4D(t)} + i \left( \frac{x^2t}{8m(\Delta x)^2D(t)} - \frac{1}{2} \arctan \left( \frac{t}{2m(\Delta x)^2} \right) \right) \right] \tag{69}
\]

where \( D(t) = (\Delta x)^2 + \left( \frac{t^2}{4m^2(\Delta x)^2} \right) \). The corresponding current is

\[
j_x = \frac{\mathcal{P}(x)}{m} \left( \frac{xt}{4m(\Delta x)^2D(t)} \right) \tag{70}
\]

and equation (22) yields

\[
\frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + \frac{1}{4mD(t)} - \frac{x^2}{8m[D(t)]^2} = 0, \tag{71}
\]

where the last two terms constitute the quantum potential.

This result can be easily understood since we are starting with the particle confined in a region \( \Delta x \) and, as time progresses, the wave packet spreads out as expected. The current \( j_x = \mathcal{P}(x)(dx/dt) \) and the trajectories calculated from this current fan out in a way that exactly reflects the spread of the wave packet. As the wave packet spreads, the quantum potential reduces eventually to zero. Thus, for a particular trajectory, the energy of the quantum potential is progressively converted to the kinetic energy of the particle, so accelerating it away from its initial position.
B. The quadratic potential

Here we will simply collect the results derived earlier in the paper for ease of comparison.

In the $x$-representation, where we write $\psi(x,t) = R_x \exp[iS_x]$, the energy equation becomes

$$\frac{\partial S_x}{\partial t} + \frac{1}{2m} \left( \frac{\partial S_x}{\partial x} \right)^2 + \frac{K}{2} x^2 - \frac{1}{2mR_x} \left( \frac{\partial^2 R_x}{\partial x^2} \right) = 0.$$  \hspace{1cm} (72)

While in the $p$-representation, where we now write $\psi(p,t) = R_p \exp[iS_p]$, the conservation of energy equation is

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + \frac{K}{2} \left( \frac{\partial S_p}{\partial p} \right)^2 - \frac{K}{2R_p} \left( \frac{\partial^2 R_p}{\partial p^2} \right) = 0.$$  \hspace{1cm} (73)

Now we turn to the probability currents and find

$$j_x = \frac{1}{2mi} \left[ \psi^*(x) \left( \frac{\partial \psi(x)}{\partial x} \right) - \left( \frac{\partial \psi^*(x)}{\partial x} \right) \psi(x) \right]$$  \hspace{1cm} (74)

$$j_p = \frac{K}{2i} \left[ \psi^*(p) \left( \frac{\partial \psi(p)}{\partial p} \right) - \left( \frac{\partial \psi^*(p)}{\partial p} \right) \psi(p) \right],$$  \hspace{1cm} (75)

in which the symmetry of the Hamiltonian is evident. That these currents are in fact different should not be too surprising as they arise in different spaces. Indeed, we can bring this out more clearly by using the respective polar forms of the $x$- and $p$-representation wave functions. In the $x$-representation

$$j_x = \frac{1}{m} R_x^2 \left( \frac{\partial S_x}{\partial x} \right)$$  \hspace{1cm} (76)

and so

$$\frac{dx}{dt} = \frac{1}{m} \left( \frac{\partial S_x}{\partial x} \right) = \frac{p_r}{m},$$  \hspace{1cm} (77)

whereas in the $p$-representation

$$j_p = KR_p^2 \left( \frac{\partial S_p}{\partial p} \right)$$  \hspace{1cm} (78)

so that

$$\frac{dp}{dt} = K \left( \frac{\partial S_p}{\partial p} \right) = - \left( \frac{\partial V}{\partial x} \right)_{x=x_r}.$$  \hspace{1cm} (79)

Thus, we see that the currents provide the mathematical means of constructing trajectories in the $x$-space and $p$-space respectively. It is a feature of both the linear potential and the quantum harmonic oscillator that $\frac{dp}{dt} = - \left( \frac{\partial V}{\partial x} \right)_{x=x_r}$, though this is not generally true.

C. The linear potential

Here the potential is $V(x) = ax$. In this case the current operators are

$$J_x = \frac{1}{2m} (pP + Pp)$$  \hspace{1cm} (80)

and

$$J_p = -ap.$$  \hspace{1cm} (81)

In the $p$-representation we find

$$j_p = (p|J_p|p) = -\mathcal{P}(p)a = \mathcal{P}(p)\frac{dp}{dt}.$$  \hspace{1cm} (82)

This result is identical to that obtained from classical mechanics through the equation

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = -a$$  \hspace{1cm} (83)

and suggests that the $p$-representation trajectories lie on the corresponding classical manifold. Indeed, equation (22) gives

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} - a\frac{\partial S_p}{\partial p} = 0.$$  \hspace{1cm} (84)

Now using $x_r = -\left( \frac{\partial S_p}{\partial p} \right)$, we find that the corresponding energy equation is

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + ax_r = 0,$$  \hspace{1cm} (85)

which has the same form as the classical Hamilton-Jacobi equation with $x_r = x$. This confirms that, for the $p$-representation of the linear potential, there is no quantum potential and that the trajectories are indeed classical.

We now compare these results with those for the $x$-representation. Here the corresponding Schrödinger equation is

$$\frac{d^2 \psi}{dx^2} - A^3 x \psi = 0$$  \hspace{1cm} (86)

where $A = (2ma)^{\frac{1}{2}}$. This equation has an Airy function

$$\psi(x) = C \text{Ai}(Ax)$$  \hspace{1cm} (87)

as a solution, which, being real, implies a zero probability density current

$$j_x = \langle x | J_x | x \rangle = \frac{\mathcal{P}(x)}{m} \frac{\partial S_x}{\partial x} = \mathcal{P}(x) \frac{dx}{dt} = 0.$$  \hspace{1cm} (88)

Using this result in equation (22) shows that in the $x$-representation the quantum potential is the negative of the classical potential and is not zero as in the $p$-representation. This example demonstrates that, while
they are consistent with the Schrödinger equation, Bohm trajectories may be representation dependent.

By way of explanation of the latter point, we observe that in the $p$-representation the wave function is complex, its incoming and outgoing components being separate on respectively the positive and the negative $p$-domains. On the other hand, in the $x$-representation, the incoming and outgoing waves combine to produce a real wave function. In particular, the $x$-representation solution may be split into incident and reflected components using the relation

$$
Ai(Ax) + \exp(-\frac{2}{3}i\pi)Ai(Ax\exp(-\frac{2}{3}i\pi))
+ \exp(\frac{2}{3}i\pi)Ai(Ax\exp(\frac{2}{3}i\pi)) = 0. \quad (89)
$$

Taking the incident and reflected wave function separately, one obtains non-zero probability density currents and a non-zero quantum potential. The resulting trajectories are classical at infinity but are non-classical near the origin, where reflection takes place with an instantaneous change of sign in velocity. This is in contrast to the classical trajectory which turns smoothly at the origin. It is important to note that the trajectories of the incident and reflected waves respectively do not embody the effects of interference. It is this interference, absent in the $p$-representation, that produces a stationary trajectory for the combined solution in the $x$-representation.

**D. The cubic potential**

The quantum phase equation (22) in the $x$-representation using $p_r = (\partial S_x/\partial x)$ gives

$$
\frac{\partial S_x}{\partial t} + \frac{p_r^2}{2m} + Ax^3 - \frac{1}{2mR_x} \frac{\partial^2 R_x}{\partial x^2} = 0, \quad (90)
$$

while in the $p$-representation

$$
\frac{\partial S_p}{\partial t} + \frac{p_r^2}{2m} + Ax_r^3 + 3A \frac{\partial^2 R_p}{\partial p^2} \left(\frac{\partial S_p}{\partial p}\right)
\frac{3A}{R_p} \left(\frac{\partial R_p}{\partial p}\right) \left(\frac{\partial^2 S_p}{\partial p^2}\right) + A \left(\frac{\partial^3 S_p}{\partial p^3}\right) = 0,
$$

where we have used $x_r = -(\partial S_p/\partial p)$. This clearly gives a far more complicated quantum potential. Nevertheless, the content is still consistent with Schrödinger’s equation. Both equations reduce to the same classical Hamilton-Jacobi equation when the quantum potential terms reduce to zero.

The respective currents are

$$
j_x = \frac{1}{2mi} \left[ \psi^*(x) \left(\frac{\partial \psi(x)}{\partial x}\right) - \left(\frac{\partial \psi^*(x)}{\partial x}\right) \psi(x) \right] \quad (91)
$$

and

$$
j_p = \frac{A}{2i} \left[ \psi(p) \frac{\partial^2 \psi^*(p)}{\partial p^2} + \psi^*(p) \frac{\partial^2 \psi(p)}{\partial p^2} - \frac{\partial \psi(p)}{\partial p} \frac{\partial \psi^*(p)}{\partial p} \right] \quad (92)
$$

which gives

$$
j_p = -R_p^2 \left(\frac{\partial V}{\partial x}\right)_{x=x_r}^2 + A \left[ 2R_p \left(\frac{\partial^2 R_p}{\partial p^2}\right) - \left(\frac{\partial R_p}{\partial p}\right)^2 \right] \quad (93)
$$

as opposed to the simple expression for $j_x$

$$
j_x = \frac{1}{m} R_x^2 \left(\frac{\partial S_x}{\partial x}\right). \quad (94)
$$

This clearly shows the limitation of using the condition $p_r = (\partial S_x/\partial x)$ as the guidance condition. It should also by now be quite clear that the Bohm trajectories in a particular representation are obtained from the probability current for that particular representation and not from any additional guidance condition.

**VII. CONCLUSIONS**

**A. Algebraic formulation of the Schrödinger picture**

In this paper we have shown how it is possible to write the content of the Schrödinger equation in algebraic form without reference to either Hilbert space or to any specific representation. The resulting two equations are respectively the Liouville equation, equation (19), and an equation that describes the time development of the phase, equation (20), which we have called the quantum phase equation. Furthermore, we have shown that this equation is gauge invariant and from it we calculated the Aharonov-Bohm, the Aharonov-Casher and the Berry phases in a simple and straightforward way.

We have also shown that it is possible to write the probability currents as algebraic operator forms. This allows us to define probability currents in any arbitrary representation. All of these results follow from the quantum formalism without the need to appeal to any classical formalism.

**B. The $x$ and $p$ representations: the quantum potential and the trajectories of probability current**

In sections III and IV, we expressed equations (21) and (22) in the $x$-representation (equations (58) and (45) respectively) and showed that they are identical to the two defining equations of the traditional Bohm interpretation [16]. The quantum potential emerges from
equation (42), which in turn comes directly from equation (22), showing that it cannot be “dismissed as artificial and obscuring the essential meaning of the Bohm approach” [40] without missing some of the essential novel features of quantum processes.

In particular, observed characteristic quantum phase or gauge effects come directly from equation (22). As Philippidis, Bohm and Kaye [24] have shown many years ago, the presence of the AB effect alters the quantum potential, which in turn accounts for the fringe shifts. Furthermore, it is the presence of the quantum potential that offers an explanation of Einstein-Podolsky-Rosen-type correlations [30], as well as quantum state teleportation [41].

In section V, we also showed that we can construct a BI in the p-representation. Comparing representations shows very clearly that the Bohm trajectories are simply the trajectories associated with the probability currents of the standard theory. The only assumption added to the standard quantum theory in the Bohm-Hiley [16] version of the BI is that particles have simultaneously well defined positions and momenta and actually follow these trajectories. Further, we claim that this position is implicit in pragmatic quantum mechanics in which the probability currents are assumed to be related to particle fluxes.

C. Shadow phase spaces

The central point that emerges from our approach is that we can construct two different phase spaces. In the example of the harmonic oscillator, the p-phase space is based on equations (44) and (45), while the p-phase space is built using equations (46) and (47). The explication of different phase spaces was further exemplified in section VI. The reason why we must resort to constructing different phase spaces is not too difficult to see once it is realised that we are dealing with a non-commutative structure.

For a commutative algebra, the Gel’fand construction allows us to start from the algebra and re-construct the underlying manifold [43]. Here the points, the topology and the metric structure of the manifold are all carried by the algebra. No such construction is possible for a non-commutative algebra. Thus, in our case there is no underlying phase space with points that can be specified by the pair of observables \((x, p)\). This is just what the uncertainty principle is telling us. This is the physicist’s way of explanation why there can be no single, unique, underlying continuous phase space.

Any attempt to produce a single phase space, such as is done in the Wigner-Moyal approach, must necessarily contain unacceptable features [44]. In this case, the probability distribution can be negative in certain situations. For these reasons, we must follow what is usually done in non-commutative geometry and construct shadow manifolds.

In this context, equation (42) provides an explanation as to why the energy can be conserved when we attribute to the particle at position \(x\), the beable momentum \(p_x = \partial S_x / \partial x\). Since it is a constructed momentum and not a measured momentum, the kinetic energy will not have the value necessary to conserve the total energy. Thus we need another term to “carry” this difference. Since equation (45) is the expression for the conservation of energy, the last term on the RHS of equation (45) is the place to “store” this energy difference. This shows that the quantum potential energy is an internal energy, and clearly does not have an external source.

D. Implications for the Bohm Interpretation of quantum mechanics

Finally, we will briefly comment on the implications of the above analysis on the BI. The traditional BI assumes the x-representation is special, but the reasons for this were never made clear. It was generally assumed that all physics must take place in an a priori given space-time arena, a point of view that we have called the Cartesian order. Hence the attempt to use the guidance condition as a defining equation for Bohmian mechanics [31]. However our mathematical analysis above shows that this condition is a contingent feature, which is dependent on the asymmetry of the Hamiltonian.

On the other hand, if we take the quantum formalism as primary then we must place our emphasis on the non-commutative structure of the algebra of formalism. If we do this then attempts to focus on a single phase space, which is equivalent to giving primary relevance to space-time, will fail. This in turn calls into question the way we think about quantum processes. Indeed Bohm has already argued that we must abandon the Cartesian order and replace it by a radically new approach to quantum phenomena which he called the implicate order [20]. Here the ontology is provided by the concept of process which is to be described by the non-commutative algebra. This is not a process in space-time, but a process from which space-time is to be abstracted. Abstraction here means to ‘make manifest’ and the order that is made manifest is called the explicate order.

The key point about this view is that there may be more than one explicate order and that these explicate orders cannot be made manifest together at the same time. This can be regarded as a direct consequence of the participatory nature of the quantum process. Thus the implicate order contains an ontological complementarity,
which is a necessary consequence of the non-commutative structure. In this picture the BI discussed above is said to contain two explicate orders, one depending on the \( x \)-representation and the other on the \( p \)-representation. These are the shadow phase spaces. Both are equally valid descriptions of the outward appearance of a quantum process within a given context.

Since our classical world is dominated by appearances in space-time, we would expect the most relevant explicate order to be that based on the \( x \)-representation, with the context being provided by the classical world. This is the world in which we place our apparatus and where our measurements take place. But clearly we need to explore these ideas further as a number of questions remain unanswered. We will leave this discussion for another paper.

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