Komar energy and Smarr formula for noncommutative Schwarzschild black hole

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We calculate the Komar energy $E$ for a noncommutative Schwarzschild black hole. A deformation from the conventional identity $E = 2ST_H$ is found in the next to leading order computation in the noncommutative parameter $\theta$ (i.e. $O(\sqrt{\theta}e^{-M^2/\theta})$) which is also consistent with the fact that the area law now breaks down. This deformation yields a nonvanishing Komar energy at the extremal point $T_H = 0$. We then work out the Smarr formula, clearly elaborating the differences from the standard result $M = 2ST_H$, where the mass ($M$) of the black hole is identified with the asymptotic limit of the Komar energy. Similar conclusions are also shown to hold for a deSitter–Schwarzschild geometry.

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The deep connection between gravity and thermodynamics has been known for a long time. For instance, in a thermodynamical system the entropy $S$, temperature $T$ and energy $E$ are related by the first law of thermodynamics

$$dE = TdS$$

(1)

where, for simplicity, we ignore work terms.

A similar relation exists for black holes where $E$ is identified with the Komar energy and $T$ is the Hawking temperature. Indeed there is an integral version of the above equation

$$E = 2TS$$

(2)

Specifically, for a Schwarzschild black hole, the Komar energy equals the mass ($M$) of the black hole, so that

$$M = 2TS$$

(3)

This identity is easily verified by putting the well known expressions for the entropy and the Hawking temperature

$$T = \frac{1}{8\pi M}, \quad S = \frac{A}{4} = 4\pi M^2$$

(4)

The relations (2) and (3) are quite fundamental in black hole thermodynamics. The first one (2) is very general and holds for any black hole. General discussions and different derivations of eq. (2) have been provided earlier in the literature. The effective Komar energy appearing in the left hand side of eq. (2) is basically identified with the conserved charge corresponding to the Killing vector defined at the event horizon. The relation (3), on the contrary, is valid only for a Schwarzschild black hole. It is known as the Smarr formula that connects the macroscopic parameters of a black hole with $2TS$ (the right hand side of eq. (3)). The connection between eq. (2) and the generalised Smarr formula valid for any black hole has been recently elaborated in a series of papers.

In this paper we first present a short derivation of eq. (2) for the specific case of a spherically symmetric metric, where the role of the area law ($S = A/4$) is emphasised. Then we consider the status of eqs. (2, 3) in the context of noncommutative Schwarzschild black hole. It is well known that the noncommutativity may be introduced in different ways in the context of black holes. Here we invoke noncommutativity through a coherent state formalism, which implies the replacement of the ordinary point wise multiplication by a Voros multiplication. The analysis is performed by carrying out an expansion in the noncommutative parameter $\theta$. We show that both eqs. (2, 3) are deformed. This deformation starts from the next to leading order in the expansion process (i.e. from $O(\sqrt{\theta}e^{-M^2/\theta})$) in case of relation (2). Indeed it is precisely at this order that the area law also breaks down. Consequently the implication of the area law in the proof of eq. (2), as shown here, gets highlighted. Deformations in eq. (3), on the contrary, are found from the leading order correction itself. We find that the deviations from eqs. (2, 3) begin to play an important role at the extremal point ($T_H = 0$) of these black holes. We observe that as we go nearer to the extremal point (which corresponds to a minimum value of the horizon radius $r_h$), the correction term involving the noncommutative parameter $\theta$ leads to a nonvanishing Komar energy.

The final part of this paper is dedicated to studying similar issues in the context of deSitter–Schwarzschild geometry. The reason is the similarity between this
geometry with the noncommutative Schwarzschild geometry. Both avoid the singularity problems, though in different ways, present in the usual Schwarzschild black hole. Also, as was shown in [25], the noncommutative Schwarzschild geometry in the limit \( r \ll \sqrt{\theta} \) passes over to the deSitter metric. Our calculations show that for deSitter–Schwarzschild black holes [30], deviations from the expected relations [24,25] occur, quite analogous to the noncommutative case. This is reassuring in the sense that it gives us confidence in the construction of noncommutative black holes.

In [25], it has been shown that the expression for the Komar energy \( E \) of a spherically symmetric stationary black hole metric

\[
ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\]

is given by

\[
E = r_h^2 \kappa = 2\pi r_h^2 T_H
\]

where

\[
\kappa = \frac{1}{2} \left[ \partial_r f(r) \right]_{r=r_h}
\]

is the surface gravity of the black hole and

\[
T_H = \kappa / (2\pi)
\]

is the Hawking temperature for a general static and spherically symmetric spacetime. Now assuming that the area law \( S = A / 4 = \pi r_h^2 \) holds, it is easy to see that the above relation for the Komar energy can be written as

\[
E = 2ST_H
\]

which is the sought relation.

We now proceed to investigate the status of the above relation in the case of noncommutative black holes. We start with the noncommutative Schwarzschild metric [12,28]

\[
f(r) = f_0(r) = 1 - \frac{2M(\theta)}{r}
\]

\[
M(\theta) = \frac{2M}{\sqrt{\theta}} \gamma \left( \frac{3}{2}, \frac{r^2}{4\theta} \right)
\]

where

\[
\gamma(3/2, r^2/4\theta) = \int_0^{r^2/(4\theta)} dt t^{1/2} e^{-t}
\]

is the lower incomplete gamma function. The Komar energy for this spherically symmetric spacetime can be computed from eq. (10) and yields

\[
E = M(\theta) - \frac{Mr^3}{2\theta \sqrt{\pi} \theta} e^{-r^2/(4\theta)}
\]

in the limit \( r \to \infty \) gives \( E = M \). Also, expectedly, for \( \theta \to 0 \), we reproduce \( E = M \). Now since the Komar energy in the asymptotic limit \( (r \to \infty) \) gives the mass of the black hole, therefore we identify \( M \) (and not \( M(\theta) \)) as the mass of the noncommutative Schwarzschild black hole [11]. This identification will play an important role in the subsequent discussion.

The event horizon of the black hole can be found by setting \( f_0(r_h) = 0 \) in eq. (11), which yields

\[
r_h = \frac{4M}{\sqrt{\pi}} \gamma \left( \frac{3}{2}, \frac{r_h^2}{4\theta} \right).
\]

Since this equation cannot be solved in a closed form, we take the large radius regime \( (r^2/4\theta >> 1) \) where we can expand the incomplete gamma function to solve \( r_h \) by iteration. We keep terms up to the leading \( (1/\sqrt{\theta} e^{-M^2/\theta}) \) and next to leading \( (\sqrt{\theta} e^{-M^2/\theta}) \) orders, to obtain

\[
r_h \approx 2M \left[ 1 - \frac{2M}{\sqrt{\pi} \theta} \left( 1 + \theta / 2M^2 \right) e^{-M^2/\theta} \right].
\]

Using eqs. (8), (9) and (11), the Hawking temperature for the noncommutative Schwarzschild black hole is found to be

\[
T_H = \frac{1}{4\pi} \left[ \frac{1}{r_h} - \frac{r_h^2}{4\theta^{3/2}} e^{-r_h^2/(4\theta)} \right].
\]

To write the Hawking temperature in the regime \( r^2/4\theta >> 1 \) as a function of \( M \) we use (15). Keeping, as before, terms up to the order \( \sqrt{\theta} e^{-M^2/\theta} \), we get

\[
T_H = \frac{1}{8\pi M} \left[ 1 - \frac{4M^3}{\theta \sqrt{\pi} \theta} \left( 1 - \frac{\theta / 2M^2 - \theta^2 / 4M^4} {e^{-M^2/\theta}} \right) \right].
\]

We shall now use the first law of black hole thermodynamics to calculate the Bekenstein-Hawking entropy. This law is given by

\[
dS = \frac{dM}{T_H}.
\]

Hence the Bekenstein-Hawking entropy up to the order \( \sqrt{\theta} e^{-M^2/\theta} \) is found to be

\[
S = \int \frac{dM}{T_H} = 4\pi M^2 - 16 \sqrt{\frac{\pi}{\theta}} M^3 \left( 1 + \frac{\theta}{2M^2} \right) e^{-M^2/\theta}.
\]

In order to express the entropy in terms of the noncommutative horizon area \( (A_\theta) \), first we use eq. (13) to obtain

\[
A_\theta = 4\pi r_h^2 = 16\pi M^2 - 64 \sqrt{\frac{\pi}{\theta}} M^3 \left( 1 + \frac{\theta}{2M^2} \right) e^{-M^2/\theta} + O(\theta^3 / e^{-M^2/\theta}).
\]
Comparing eqs. (19) and (23), we find that at the leading order in $\theta$ (i.e. $O(\frac{1}{\theta} e^{-M^2/\theta})$), the noncommutative black hole entropy satisfies the area law

$$S = \frac{A_0}{4}.$$  \hspace{1cm} (21)

However, it is easy to note that there is a deviation from the area law in the next to leading order in $\theta$, i.e. at $O(\sqrt{\theta} e^{-M^2/\theta})$.

Now we compute the Komar energy at the horizon using eqs. (14, 15, 17) up to the the order $\sqrt{\theta} e^{-M^2/\theta}$, to get

$$E = M \left[ 1 - \frac{2M}{\sqrt{\pi\theta}} \left( \frac{2M^2}{\theta} + 1 \right) e^{-M^2/\theta} - \frac{1}{M} \sqrt{\frac{\theta}{\pi}} e^{-M^2/\theta} \right].$$  \hspace{1cm} (22)

Finally, using eqs. (17), (19) and (22), we obtain

$$E = 2ST_H + 2\sqrt{\frac{\theta}{\pi}} e^{-M^2/\theta} + O(\theta^{3/2} e^{-M^2/\theta})$$

$$= 2ST_H + 2\sqrt{\frac{\theta}{\pi}} e^{-S/(4\pi\theta)} + O(\theta^{3/2} e^{-S/(4\pi\theta)})$$  \hspace{1cm} (23)

where in the second line we have used eq. (19) to replace $M^2$ by $S/(4\pi)$ in the exponent.

The above relation can also be written with $M$ being expressed in terms of the black hole parameters $S$ and $T_H$ using eq. (22)

$$M = 2ST_H + \frac{1}{2\pi\sqrt{\pi\theta}} \left( S + \frac{S^2}{2\pi\theta} + 6\pi\theta \right) e^{-S/(4\pi\theta)} + O(\theta^{3/2} e^{-S/(4\pi\theta)}).$$  \hspace{1cm} (24)

Since $M$ has been identified earlier to be the mass of the black hole, we name eq. (24) as the Smarr formula for noncommutative Schwarzschild black hole. The above relations (23, 24) are the main results of this paper.

We now make some observations on eq. (23). Interestingly, we have once again been able to write the deformed relation (involving the noncommutative parameter $\theta$) in terms of the Komar energy $E$, entropy $S$ and the Hawking temperature $T_H$. The above result (23) shows that the relation (10) is satisfied up to the leading order in $\theta$ (i.e. $O(\frac{1}{\theta} e^{-M^2/\theta})$). This is indeed consistent with the fact that the area law (21) also holds at this order in $\theta$. However, the validity of the relation (10) breaks down in the next to leading order in $\theta$ (i.e. $O(\sqrt{\theta} e^{-M^2/\theta})$) in accordance with the fact that there is a deviation from the area law at this order in $\theta$ as seen above. The noncommutative Smarr formula (24), on the contrary, deviates from the usual one (3) right from the leading correction itself.

Further, we note that for a black hole with an entropy $S >> 4\pi\theta$, the second term on the right hand side of eq. (23) is small compared to the first term and hence the relation (10) is once again satisfied. This is in conformity with an observation made in (25) where it was shown that the area law (21) is satisfied for $r_h \geq 4.8\sqrt{\theta}$ implying (from our present analysis) that the relation (10) holds.

To see this, we rewrite eq. (23) by replacing $S$ by $A_0/4$ in the exponent, to get

$$E = 2ST_H + 2\sqrt{\frac{\theta}{\pi}} e^{-A_0/(16\pi\theta)} + O(\theta^{3/2} e^{-A_0/(16\pi\theta)}).$$  \hspace{1cm} (25)

Therefore, when $A_0 >> 16\pi\theta$, i.e. $r_h >> 2\sqrt{\theta}$, the second term on the right hand side of eq. (25) is small compared to the first term and hence the relation (10) gets satisfied which implies that the area law (21) holds in this regime.

Finally, note that there is a lower bound to the horizon area $A_0$ ($A_0 = k\theta$, $k \geq 1$) from purely physical grounds. This is also compatible with the fact that there is a lower bound to the horizon radius $r_h$ ($r_h \geq 3\sqrt{\theta}$) since the Hawking temperature becomes negative for $r_h < 3\sqrt{\theta}$. This yields $k \geq 36\pi$. Further, for $r_h = 3\sqrt{\theta}$, the contribution to the Komar energy comes from the second term in eq. (25) only since $T_H = 0$. Putting the minimum value of $k$ (corresponding to which $T_H = 0$) in eq. (25), we obtain

$$E = 2\sqrt{\frac{\theta}{\pi}} e^{-9/4}. $$  \hspace{1cm} (26)

Interestingly, the above expression gives a nonvanishing Komar energy for noncommutative Schwarzschild black holes at their extremal point ($T_H = 0$) which vanishes in the limit $\theta \rightarrow 0$. The nonvanishing Komar energy thus owes its origin to the noncommutative parameter $\theta$ appearing in the metric of the noncommutative Schwarzschild black hole (11).

To get further insights in the results obtained so far, we now consider the metric of deSitter-Schwarzschild geometry (given by Dymnikova(22) having similar features as the noncommutative Schwarzschild metric (11) at $r \rightarrow \infty$ and $r \rightarrow 0$ limits. The metric coefficient of this spherically symmetric black hole reads

$$g(r) = 1 - \frac{r_g}{r}(1 - e^{-r^2/(r_g^2)})$$

$$r_g = 2M, r_g^2 = 3/\Lambda.$$  \hspace{1cm} (27)

The above metric has the property that at $r \rightarrow \infty$, it behaves like Schwarzschild geometry and at $r \rightarrow 0$, it behaves like deSitter geometry and hence there is no singularity at $r = 0$. These properties are similar to the metric (11) of the noncommutative Schwarzschild black hole.

Once again the Komar energy for this spherically symmetric spacetime (27) can be computed from eq. (13) and yields

$$E = \frac{r_g}{2}(1 - e^{-r^2/(r_g^2)}) - \frac{3r_g^3}{2}r_e e^{-r^2/(r_g^2)}$$  \hspace{1cm} (28)
which in the limit $r \to \infty$ gives $E = M$. Therefore we identify $M$ as the mass of this black hole.

The event horizon of the black hole can be found by setting $g(r_h^{(D)}) = 0$, which leads to

$$r_h^{(D)} = r_g(1 - e^{-\frac{r_g^3}{r_0^3}}).$$  \hspace{1cm} (29)

This equation once again cannot be solved for $r_h^{(D)}$ in a closed form. However, in the large radius regime $(r_h^{(D)} / (r_0^3 r_g) \gg 1)$, we can solve (29) by iteration. Keeping terms up to the order $e^{-r_g^3 / r_0^3} = e^{-4M^2 / r_0^2}$, we find

$$r_h^{(D)} = r_g(1 - e^{-r_g^3 / r_0^3}).$$  \hspace{1cm} (30)

Using eqs. (8), (27), (29), the Hawking temperature for the deSitter-Schwarzschild black hole is found to be

$$T_H^{(D)} = \frac{1}{4 \pi} \left[ \frac{r_g}{r_h^{(D)2}} (1 - e^{-\frac{r_g^3}{r_0^3}}) + \frac{3r_h^{(D)}}{r_0^3} e^{-\frac{r_g^3}{r_0^3}} \right].$$  \hspace{1cm} (31)

Keeping, as before, terms up to the order $e^{-r_g^3 / r_0^3}$, we get

$$T_H^{(D)} = \frac{1}{4 \pi r_g} \left[ 1 + e^{-r_g^3 / r_0^3} - \frac{3r_h^{(D)}}{r_0^3} e^{-r_g^3 / r_0^3} \right].$$  \hspace{1cm} (32)

The first law of black hole thermodynamics [13] is now used to compute the entropy which up to the order $e^{-4M^2 / r_0^2}$

$$S^{(D)} = 4\pi M^2 - 2\pi r_g^2 e^{-4M^2 / r_0^2} - 12\pi M^2 e^{-4M^2 / r_0^2}.$$  \hspace{1cm} (33)

In order to express the entropy in terms of the horizon area $(A^{(D)})$, we use [40] to obtain

$$A^{(D)} = 16\pi M^2 (1 - 2r_g^2 e^{-4M^2 / r_0^2} - 2\pi M^2 e^{-4M^2 / r_0^2}).$$  \hspace{1cm} (34)

Comparing equations (33) and (34), we find that the area law is not satisfied at the order $e^{-4M^2 / r_0^2}$. As in the case of the noncommutative Schwarzschild black hole, there is a deviation in this case from the area law at the order $e^{-4M^2 / r_0^2}$.

Now to obtain the relation between the Komar energy $E$, entropy $S$ and the Hawking temperature $T_H$, we compute the Komar energy at the horizon using eqs. [41], [41] up to the order $e^{-4M^2 / r_0^2}$, to get

$$E^{(D)} = M \left( 1 - e^{-4M^2 / r_0^2} - \frac{12M^2}{r_0^3} e^{-4M^2 / r_0^2} \right).$$  \hspace{1cm} (35)

Finally, using eqs. (31), (33), (35) and replacing $M^2$ by $S/(4\pi)$, we obtain

$$E^{(D)} = 2S^{(D)}T_H^{(D)} + \sqrt{\frac{S^{(D)}}{4\pi}} \left[ 1 + \frac{2\pi r_g^2}{S^{(D)}} \right] e^{-S^{(D)}/(\pi r_0^3)} + O(e^{-2S^{(D)}/(\pi r_0^3)}).$$  \hspace{1cm} (36)

The above relation can also be written with $M$ being expressed in terms of the black hole parameters $S^{(D)}$ and $T_H^{(D)}$ using eqs. (35), (33)

$$M = 2ST_H + \sqrt{\frac{S^{(D)}}{4\pi}} \left[ 1 + 2\pi r_g^2 \right] e^{-S^{(D)}/(\pi r_0^3)} + O(e^{-2S^{(D)}/(\pi r_0^3)}).$$  \hspace{1cm} (37)

The above relations (36), (37) are the analogues of eqs. (24), (21). Note that (as in the case of the noncommutative Schwarzschild black hole (23)) once again we obtain a deformed relation involving the Komar energy $E$, entropy $S$ and the Hawking temperature $T_H$ at the order $e^{-4M^2 / r_0^2}$. This is consistent with the fact that the area law in this case is not satisfied at this order.

Now from eq. (31), we find that the Hawking temperature $T_H^{(D)}$ vanishes when $r_h^{(D)} = 0$, $\infty$ for any $r_g$. Considering the physically interesting solution $r_h^{(D)} = 0$ (for which $T_H^{(D)} = 0$), which also satisfies (29) for any $r_g$, we find that the relation (36) up to order $O(e^{-S^{(D)}/(\pi r_0^3)})$ reduces to

$$E^{(D)} = \sqrt{\frac{S^{(D)}}{4\pi}} \left( 1 + \frac{2\pi r_g^2}{S^{(D)}} \right) e^{-S^{(D)}/(\pi r_0^3)}.$$  \hspace{1cm} (38)

Hence we find that there exists a nonvanishing Komar energy in case of deSitter-Schwarzschild black holes when the Hawking temperature $T_H^{(D)} = 0$. This feature is exactly similar with the results obtained in case of noncommutative Schwarzschild black holes. However, in the latter case, there is a nonvanishing lower bound of the horizon radius at which the Hawking temperature vanishes whereas the horizon radius goes to zero when the Hawking temperature vanishes in the former case.

To summarise, we have computed the Komar energy for a noncommutative Schwarzschild black hole governed by the metric (11). This is used to obtain (23) where a deformation from the standard result (2) is found in the next to leading order in the expansion involving the noncommutative parameter $\theta$. Next, the mass of the black hole is identified by taking the asymptotic infinity limit of the Komar energy. This is found to be identical with the usual Schwarzschild mass $M$. The noncommutative version of the Smarr formula (22) is obtained. Here the deformation from the usual formula (3) begins at the leading noncommutative correction. Similar results are derived for a deSitter-Schwarzschild geometry governed by the metric (24). Since the noncommutative Schwarzschild metric, in an appropriate limit, passes over to this metric, such a similarity bolsters our confidence in carrying out computations in a noncommutative formulation. Finally, we conclude by noting that the presence of a nonvanishing Komar energy when the Hawking temperature vanishes in case of noncommutative Schwarzschild black holes can also be found in the deSitter-Schwarzschild geometry, thereby concretizing our results. Further, in the former case, the result is due
to a combined effect of the noncommutative parameter $\theta$ and the entropy of the black hole whereas the result in case of deSitter-Schwarzschild metric is due to the cosmological constant and the entropy of the black hole.