Self-orthogonal codes from equitable partitions of association schemes

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Abstract
We give a method of constructing self-orthogonal codes from equitable partitions of association schemes. By applying this method, we construct self-orthogonal codes from some distance-regular graphs. Some of the obtained codes are optimal. Further, we introduce a notion of self-orthogonal subspace codes. We show that under some conditions equitable partitions of association schemes yield such self-orthogonal subspace codes and we give some examples from distance-regular graphs.

Keywords Association scheme · Equitable partition · Self-orthogonal code · Subspace code

Mathematics Subject Classification 05E30 · 05E18 · 94B05 · 94B60

1 Introduction
In this paper, we give a construction of self-orthogonal codes from equitable partitions of association schemes. The results presented in the paper can be seen as a generalization of results on construction of self-orthogonal codes from orbit matrices of 2-designs (see [5,10]) and strongly regular graphs (see [3]). Moreover, we study subspace codes obtained from equitable partitions of association schemes and introduce a notion of self-orthogonal subspace codes.

Self-orthogonal codes have wide applications in communications (see [16]), including, for example, in secret sharing. Further, some of the most interesting and best linear
codes known are self-orthogonal, e.g., the \([8, 4, 4]\) Hamming code, the \([24, 12, 8]\) extended binary Golay code and the \([12, 6, 6]\) ternary Golay code. These are the reasons why our interest is also in defining and constructing self-orthogonal subspace codes. Subspace codes are relatively new topic of interest (see [8]) giving a new approach to network coding.

Besides the theoretical results on a construction of self-orthogonal codes and self-orthogonal subspace codes from equitable partitions of association schemes, we also present examples obtained by applying the described methods on association schemes related to some distance-regular graphs. Some of the obtained linear codes are optimal.

The paper is outlined as follows. In the next section, we provide the relevant background information. In Sect. 3, we give a construction method. Following this, we describe our construction using equitable partitions of distance-regular graphs. In Sect. 5, we introduce a notion of self-orthogonal subspace codes, giving a method for obtaining such subspace codes. We conclude with our construction of self-orthogonal subspace codes via distance-regular graphs and give examples to demonstrate the construction.

In this work, we have used computer algebra systems GAP [18], Magma [1] and Sage [17], and Hanaki’s programs [9].

### 2 Preliminaries

We assume that the reader is familiar with the basic facts of theory of distance-regular graphs and association schemes. For background reading in theory of distance-regular graphs and association schemes we refer the reader to [2]. For further reading on the topic we refer the reader to [6] and [15]. We also assume a basic knowledge of coding theory (see [13]).

We will follow the definition of an association scheme given in [2], although some authors use a term a symmetric association scheme for such structure.

Let \(X\) be a finite set. An association scheme with \(d\) classes is a pair \((X, \mathcal{R})\) such that

1. \(\mathcal{R} = \{R_0, R_1, \ldots, R_d\}\) is a partition of \(X \times X\),
2. \(R_0 = \Delta = \{(x, x) | x \in X\}\),
3. \(R_i = R_i^T\) (i.e., \((x, y) \in R_i \Rightarrow (y, x) \in R_i\)) for all \(i \in \{0, 1, \ldots, d\}\),
4. there are numbers \(p_{ij}^k\) (the intersection numbers of the scheme) such that for any pair \((x, y) \in R_k\) the number of \(z \in X\) such that \((x, z) \in R_i\) and \((z, y) \in R_j\) equals \(p_{ij}^k\).

The relations \(R_i, i \in \{0, 1, \ldots, d\}\), of an association scheme can be described by the set of symmetric \((0, 1)\)-adjacency matrices \(\mathcal{A} = \{A_0, A_1, \ldots, A_d\}\), \([A_i]_{xy} = 1\) if \((x, y) \in R_i\), which generate \((d+1)\)-dimensional commutative and associative algebra over real or complex numbers called the Bose–Mesner algebra of the scheme. The matrices \(\{A_0, A_1, \ldots, A_d\}\) satisfy

\[
A_i A_j = \sum_{k=0}^d p_{ij}^k A_k = A_j A_i. \tag{1}
\]
Each of the matrices $A_i$, $i \in \{1, 2, ..., d\}$, represents a simple graph $\Gamma_i$ on the set of vertices $X$ (if $(x, y) \in R_i$ then vertices $x$ and $y$ are adjacent in $\Gamma_i$), and the graphs $\Gamma_i$ form an edge-coloring of the complete graph on $X$.

A $q$-ary linear code $C$ of dimension $k$ and length $n$, for a prime power $q$, is a $k$-dimensional subspace of a vector space $\mathbb{F}_q^n$. Elements of $C$ are called codewords. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be elements of $\mathbb{F}_q^n$. The Hamming distance between words $x$ and $y$ is the number $d(x, y) = |\{i : x_i \neq y_i\}|$. The minimum distance of the code $C$ is defined by $d = \min\{d(x, y) : x, y \in C, x \neq y\}$. The weight of a codeword $x$ is $w(x) = d(x, 0) = |\{i : x_i \neq 0\}|$. For a linear code $C$, $d = \min\{w(x) : x \in C, x \neq 0\}$.

The dual code $C^\perp$ of a code $C$ is the orthogonal complement of $C$ under the standard inner product $\langle \cdot, \cdot \rangle$, i.e., $C^\perp = \{v \in \mathbb{F}_q^n | \langle v, c \rangle = 0 \text{ for all } c \in C\}$. A code $C$ is self-orthogonal if $C \subseteq C^\perp$ and self-dual if $C = C^\perp$.

A linear code is called projective if no two columns of the generator matrix are linearly dependent. A code is projective if and only if the minimum distance of its dual code is at least three. A two-weight code is a code which has only two nonzero weights. Projective two-weight codes are related to strongly regular graphs. Let $w_1$ and $w_2$ (we suppose $w_1 < w_2$) be two weights of a projective two-weight code. The vertices of the strongly regular graph are identified by the codewords, and two vertices are adjacent if they are on distance $w_1$.

A $q$-ary linear code of length $n$, dimension $k$, and distance $d$ is called a $[n, k, d]_q$ code. An $[n, k]$ linear code $C$ is optimal if the minimum weight of $C$ achieves the Singleton bound on the minimum weight of $[n, k]$ linear codes, and near-optimal if its minimum weight is at most $1$ less than the largest possible value given by the Singleton bound. An $[n, k]$ linear code $C$ is said to be a best known linear $[n, k]$ code if $C$ has the highest minimum weight among all known $[n, k]$ linear codes. A catalogue of best known codes is maintained at [7], to which we compare the minimum weight of all codes constructed in this paper.

Linear codes over finite rings are defined similarly to linear codes over finite fields, where the codes are modules instead of vector spaces. $\mathbb{Z}_m$ denotes the ring of integers modulo $m$, where $m$ is a positive integer, $m \geq 2$. The most notable codes over rings are codes over $\mathbb{Z}_4$.

A subspace code $C_S$ is a nonempty set of subspaces of $\mathbb{F}_q^n$. R. Kötter and F. Kschischang proved (see [14]) that subspace codes are efficient for transmission in networks. For the parameters of a subspace code we will follow the notation from [11] and use a subspace distance given by

$$d_S(U, W) = \dim(U + W) - \dim(U \cap W),$$

where $U, W \in C_S$. The minimum distance of $C_S$ is given by

$$d = \min\{d_S(U, W) : U, W \in C_S, U \neq W\}.$$

A code $C_S$ is called an $(n, \#C_S, d; K)_q$ subspace code if the dimensions of the codewords of $C_S$ are contained in a set $K \subseteq \{0, 1, 2, ..., n\}$. In the case $K = \{k\}$, a subspace
code $C_S$ is called a \textit{constant dimension code} with the parameters $(n, \#C_S, d; k)_q$; otherwise, \textit{i.e.}, if all codewords do not have the same dimension, $C_S$ is called a \textit{mixed dimension code}. Such subspace code is denoted by $(n, \#C_S, d)_q$. For reading on recent results on subspace codes we refer the reader to \cite{8,12}.

In this paper, we give a construction of self-orthogonal codes using equitable partitions of association schemes. In the case of 2-class association schemes this method of construction coincides with the construction from strongly regular graphs given in \cite{3}. We apply this method to construct self-orthogonal codes from some distance-regular graphs of diameter $d$, $3 \leq d \leq 9$, \textit{i.e.}, Doro graph of diameter $d = 3$, Hadamard graph on 48 vertices of diameter $d = 4$, Doubled Gewirtz graph of diameter $d = 5$, Incidence graph of $GH(3, 3)$ of diameter $d = 6$, Doubled Odd graph $D(O_4)$ of diameter $d = 7$ and Foster graph of diameter $d = 8$. Some of the obtained self-orthogonal codes are optimal \textit{i.e.}, they reach the theoretical upper bound. Further, for obtaining self-orthogonal subspace codes, we apply the method on distance-regular graph called Doubled Higman–Sims graph of diameter $d = 5$.

### 3 Self-orthogonal linear codes from equitable partitions of association schemes

Suppose $A$ is a symmetric real matrix whose rows and columns are indexed by the elements of $X = \{1, \ldots, n\}$. Let $\{C_0, \ldots, C_{t-1}\}$ be a partition of $X$. The characteristic matrix $H$ is the $n \times t$ matrix whose $j$th column is the characteristic vector of $C_j$, where $j = 0, \ldots, t - 1$.

A partition $\Pi = \{C_0, C_1, \ldots, C_{t-1}\}$ of the $n$ vertices of a graph $G$ is \textit{equitable} (or regular) if for every pair of (not necessarily distinct) indices $i, j \in \{0, 1, \ldots, t - 1\}$ there is a nonnegative integer $b_{i,j}$ such that each vertex $v \in C_i$ has exactly $b_{i,j}$ neighbors in $C_j$, regardless of the choice of $v$. The $t \times t$ quotient matrix $B = (b_{i,j})$ is well-defined if and only if the partition $\Pi$ is equitable. An equitable (or regular) partition of an association scheme $(X, R)$ is a partition of $X$ which is equitable with respect to each of the graphs $\Gamma_i, i \in \{1, 2, \ldots, d\}$ corresponding to the association scheme $(X, R)$ with $d$ classes.

Let $\Pi$ be an equitable partition of $(X, R)$ with $t$ cells, and let $H$ be the characteristic matrix of the partition $\Pi$. Further, let $A_i$ be the adjacency matrix corresponding to a relation $R_i$. Then, the following holds:

$$A_i H = H M_i,$$  \hspace{1cm} (3)

where $M_i$ denotes the corresponding $t \times t$ quotient matrix of $A_i$ with respect to $\Pi$. The matrix $H^T H$ is diagonal and invertible and, therefore,

$$M_i = (H^T H)^{-1} H^T A_i H.$$  \hspace{1cm} (4)

The following theorem is implied in \cite[Section 3]{6}.
Theorem 1 Let $\Pi$ be an equitable partition of a $d$-class association scheme $(X, \mathcal{R})$ with $t$ cells, and let $M_i$, $i = 0, 1, \ldots, d$, denote the quotient matrix of the graph $\Gamma_i$ with respect to $\Pi$. Then

$$M_i M_j = \sum_{k=0}^{d} p_{i,j}^k M_k = M_j M_i,$$

where numbers $p_{i,j}^k$ are the intersection numbers of the scheme.

Proof Since $A_i A_j = A_j A_i = \sum_{k=0}^{d} p_{i,j}^k A_k$, it holds that

$$H M_i M_j = A_i H M_j = A_i A_j H = (\sum_{k=0}^{d} p_{i,j}^k A_k) H = H (\sum_{k=0}^{d} p_{i,j}^k M_k).$$

$\square$

Theorem 2 Let $\Pi$ be an equitable partition of a $d$-class association scheme $(X, \mathcal{R})$ with $n$ cells of the same length $|X|/n$ and let $p$ be a prime number. If there exists $i \in \{1, 2, \ldots, d\}$ such that for all $k \in \{0, 1, \ldots, d\}$ the prime $p$ divides $p_{i,i}^k$, then the rows of the matrix $M_i$ span a self-orthogonal code of length $n$ over the field $\mathbb{F}_q$, $q = p^m$, where $m$ is a positive integer.

Proof Since the partition $\Pi$ is equitable having all cells of the same size, and the matrices $A_i$, $i = 0, \ldots, d$, are symmetric, the matrices $M_i$, $i = 0, \ldots, d$, are symmetric too. Now, from (5) it follows that

$$M_i M_i^T = M_i M_i = \sum_{k=0}^{d} p_{i,i}^k M_k.$$

Since $p | p_{i,i}^k$ for all $k \in \{0, 1, \ldots, d\}$, $p_{i,i}^k = 0$ over $\mathbb{F}_q$, and the statement of the theorem holds. $\square$

A similar result can be proven for codes over rings $\mathbb{Z}_m$.

Theorem 3 Let $\Pi$ be a $d$-class equitable partition of an association scheme $(X, \mathcal{R})$ with $n$ cells of the same length $|X|/n$ and let $m$ be a positive integer. If there exists $i \in \{1, 2, \ldots, d\}$ such that for all $k \in \{0, 1, \ldots, d\}$ the integer $m$ divides $p_{i,i}^k$, then rows of the matrix $M_i$ span a self-orthogonal code of length $n$ over the ring $\mathbb{Z}_m$.

3.1 Codes from distance-regular graphs

Let $\Gamma$ be a graph with diameter $d$, and let $\delta(u, v)$ denote the distance between vertices $u$ and $v$ of $\Gamma$. The $i$-th-neighborhood of a vertex $v$ is the set $\Gamma_i(v) = \{w : \delta(v, w) = i\}$. Similarly, we define $\Gamma_i$ to be the $i$-th-distance graph of $\Gamma$, that is, the vertex set of $\Gamma_i$
is the same as for $\Gamma$, with adjacency in $\Gamma_i$ defined by the $i$th distance relation in $\Gamma$. The adjacency matrix of the graph $\Gamma_i$ is called the distance-$i$ matrix of $\Gamma$. We say that $\Gamma$ is distance-regular if the distance relations of $\Gamma$ give the relations of a $d$-class association scheme, that is, for every choice of $0 \leq i, j, k \leq d$, all vertices $v$ and $w$ with $\delta(v, w) = k$ satisfy $|\Gamma_i(v) \cap \Gamma_j(w)| = p_{ij}^k$ for some constant $p_{ij}^k$. In a distance-regular graph, we have that $p_{ij}^k = 0$ whenever $i + j < k$ or $k < |i - j|$. A distance-regular graph $\Gamma$ is necessarily regular with degree $p_{11}$; more generally, each distance graph $\Gamma_i$ is regular with degree $k_i = p_{0i}^0$. An equivalent definition of distance-regular graphs is the existence of the constants $b_i = p_{i+1,1}^i$ and $c_i = p_{1-i,1}^i$ for $0 \leq i \leq d$ (notice that $b_d = c_0 = 0$). The sequence \( \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d \} \), where $d$ is the diameter of $\Gamma$, is called the intersection array of $\Gamma$. Clearly, $b_0 = k$, $b_d = c_0 = 0$, and $c_1 = 0$.

Let $\Gamma$ be a distance-regular graph with diameter $d$ and the adjacency matrix $A$, and let $A_i$ denotes the distance-$i$ matrix of $\Gamma$, $i = 0, 1, \ldots, d$. Further, let $G$ be an automorphism group of $\Gamma$. In the sequel, the quotient matrix of $A_i$, $i = 1, 2, \ldots, d$, with respect to the orbit partition induced by $G$ will be denoted by $M_i$.

Since a distance-regular graph induces an association scheme, Theorem 2 implies the following corollary. This result is a generalization of the result on construction of self-orthogonal codes from orbit matrices of strongly regular graphs given in [3].

**Corollary 1** Let $\Gamma$ be a distance-regular graph with diameter $d$, and let $p$ be a prime number. Further, let an automorphism group $G$ acts on $\Gamma$ with $n$ orbits of the same length. If there exists $i \in \{1, 2, \ldots, d\}$ such that for all $k \in \{0, 1, \ldots, d\}$ the prime $p$ divides $p_{1,j}^i$, then the rows of the matrix $M_i$ span a self-orthogonal code of length $n$ over the field $\mathbb{F}_q$, $q = p^m$, where $m$ is a positive integer.

The following statement is a direct consequence of Theorem 3.

**Corollary 2** Let $\Gamma$ be a distance-regular graph with diameter $d$, and let $m$ be a positive integer. Further, let an automorphism group $G$ acts on $\Gamma$ with $n$ orbits of the same length. If there exists $i \in \{1, 2, \ldots, d\}$ such that for all $k \in \{0, 1, \ldots, d\}$ the integer $m$ divides $p_{1,j}^i$, then the rows of the matrix $M_i$ span a self-orthogonal code of length $n$ over the ring $\mathbb{Z}_m$.

**Remark 1** The trivial group acts on every distance-regular graph with all orbits of the length 1, and in that case the corresponding quotient matrix is actually the adjacency matrix of the graph.

## 4 Examples of self-orthogonal codes

In this section, we give examples of self-orthogonal codes obtained by applying the method described in Theorem 2 and Corollary 1. The examples of the construction from strongly regular graphs are given in [3]. Here, we apply the method by taking distance-regular graphs of diameter $d$, $3 \leq d \leq 8$, one example for each diameter. In each subsection, we give basic information on the distance-regular graphs, the coefficients $p_{1,i}^k$ of adjacency matrices and obtained self-orthogonal codes. More information about...
Table 1 The coefficients $p_{k_i}^i$ of adjacency matrices for the Doro graph $\Gamma_D$

|       | $A_0$ | $A_1$ | $A_2$ | $A_3$ |
|-------|-------|-------|-------|-------|
| $A_0 \cdot A_0$ | 1     | 0     | 0     | 0     |
| $A_1 \cdot A_1$ | 12    | 1     | 3     | 0     |
| $A_2 \cdot A_2$ | 40    | 20    | 24    | 24    |
| $A_3 \cdot A_3$ | 15    | 5     | 3     | 2     |

Table 2 Self-orthogonal codes obtained from the graph $\Gamma_D$

| $H \leq G_{\Gamma_D}$ | $i$ | The code |
|------------------------|-----|----------|
| $I$                    | 2   | $[68, 8, 32]_2^*$ |

the each graph can be seen in [2]. The obtained self-orthogonal codes marked with * are optimal. Some of the obtained codes are self-dual.

4.1 Doro graph, $d = 3$

The Doro graph, which we will denote by $\Gamma_D$, has 68 vertices and the intersection array {12, 10, 3; 1, 3, 8}. The distance-$i$ matrices $A_i$ of $\Gamma_D$ ($i = 0, 1, 2, 3$) determine an 3-class association scheme and form a basis of a Bose–Mesner algebra. The coefficients of $A_i \cdot A_i$, i.e., intersection numbers $p_{k_i}^i$, are given in Table 1.

The full automorphism group of the Doro graph, denoted by $G_{\Gamma_D}$, has order 16320. Table 1 shows that the only possibility to construct self-orthogonal codes from $\Gamma_D$ by applying Corollary 1 is a construction of codes over a field of characteristic 2 from the matrix $A_2$. The only subgroups of $G_{\Gamma_D}$ that act with all orbits of the same length are the trivial group and the groups of order 17 and 34, respectively. We constructed binary codes arising by applying Corollary 1, and in Table 2 we present the results obtained for the trivial group.

Remark 2 The code with parameters $[68, 8, 32]_2$ is a projective code with weights 32 and 40 which yields a strongly regular graph with parameters $(256, 187, 138, 132)$, the complement of a strongly regular graph with parameters $(256, 68, 12, 20)$.

4.2 Hadamard graph on 48 vertices, $d = 4$

The Hadamard graph on 48 vertices, which will be denoted by $\Gamma_H$, has the intersection array {12, 11, 6, 1; 1, 6, 11, 12}. The coefficients of $A_i \cdot A_i$ for the distance-$i$ matrices $A_i$ of $\Gamma_H$ ($i = 0, 1, \ldots, 4$) are given in Table 3.

The full automorphism group $G_{\Gamma_H}$ of $\Gamma_H$ has order 380160. Table 3 shows that by Corollary 1 one can construct binary codes from $A_1$, $A_2$ and $A_3$, and ternary codes form $A_1$ and $A_3$. For the subgroups of $G_{\Gamma_H}$ acting with all orbits of the same length we construct the quotient matrices $M_{i}^{\Gamma_H}$, $i = 1, 2$, which span self-orthogonal codes. Since the distance graphs $(\Gamma_H)_1$ and $(\Gamma_H)_3$ are isomorphic, we do not have to check quotient matrices of $(\Gamma_H)_3$. Table 4 presents the obtained results.
Table 3 The coefficients $p_{k,i}^{j}$ of adjacency matrices for the graph $\Gamma_{H}$

|      | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|------|-------|-------|-------|-------|-------|
| $A_0 \cdot A_0$ | 1     | 0     | 0     | 0     | 0     |
| $A_1 \cdot A_1$ | 12    | 0     | 6     | 0     | 0     |
| $A_2 \cdot A_2$ | 22    | 0     | 20    | 0     | 22    |
| $A_3 \cdot A_3$ | 12    | 0     | 6     | 0     | 0     |
| $A_4 \cdot A_4$ | 1     | 0     | 0     | 0     | 0     |

Table 4 Self-orthogonal codes obtained from the graph $\Gamma_{H}$

| $H \leq G_{\Gamma_{H}}$ | $i$ | The code | $H \leq G_{\Gamma_{H}}$ | $i$ | The code |
|-------------------------|-----|----------|-------------------------|-----|----------|
| $I$                     | 1   | [48, 24, 4]$_2$ | Z$_3$                   | 1   | [16, 8, 1]$_2$ |
| $I$                     | 2   | [48, 24, 2]$_2$ | Z$_3$                   | 1   | [16, 8, 4]$_2$ |
| $I$                     | 1   | [48, 34, 4]$_3$ | Z$_3$                   | 1   | [16, 4, 6]$_3$ |
| $Z_2$                   | 1   | [24, 2, 12]$_2$ | E$_4$                   | 2   | [12, 4, 2]$_2$ |
| $Z_2$                   | 2   | [24, 8, 2]$_2$  | E$_4$                   | 1   | [12, 5, 2]$_2$ |
| $Z_2$                   | 1   | [24, 10, 2]$_3$ | Z$_4$, E$_4$            | 1   | [12, 2, 6]$_3$ |
| $Z_2$                   | 2   | [24, 12, 2]$_2$ | E$_4$                   | 1   | [12, 3, 6]$_3$ |
| $Z_2$                   | 1   | [24, 12, 4]$_2$ | Z$_6$, S$_3$           | 1   | [8, 2, 4]$_2$ |
| $Z_2$                   | 1   | [24, 2, 12]$_3$ | Z$_6$, S$_3$           | 1   | [8, 4, 4]$_2$ * |
| $Z_2$                   | 1   | [24, 6, 6]$_3$  | Z$_6$, S$_3$           | 2   | [8, 4, 2]$_2$ |
| $Z_2$                   | 1   | [24, 7, 12]$_3$ | Z$_6$, S$_3$           | 1   | [8, 2, 6]$_3$ * |

**Remark 3** From the supports of all codewords of weight 4 of the code with parameters [24, 12, 4]$_2$, in a similar way as described in [4] we construct the triangular graph $T(12)$, *i.e.*, the strongly regular graph with parameters (66, 20, 10, 4), and from the codewords of weight 8 a strongly regular graph with parameters (495, 238, 109, 119). Further, we obtain a distance-regular graph of diameter 4, known as Johnson graph with 495 vertices having the intersection array [32, 21, 12, 5, 1, 4, 9, 16]. The code with parameters [8, 4, 4]$_2$ is the famous Hamming code.

**4.3 Doubled Gewirtz graph, $d = 5$**

The Doubled Gewirtz graph $\Gamma_{DG}$ has 112 vertices, the full automorphism group $G_{\Gamma_{DG}}$ of order 161280, and the intersection array [10, 9, 8, 2, 1; 1, 2, 8, 9, 10]. The coefficients of $A_i \cdot A_i$ for the distance-$i$ matrices $A_i$ of $\Gamma_{DG}$ ($i = 0, 1, \ldots, 5$) are given in Table 5. The group $G_{\Gamma_{DG}}$ contains subgroups acting on the graph $\Gamma_{DG}$ with all orbits of the same length. For the corresponding quotient matrices $M_i^H$ we obtain self-orthogonal codes presented in Table 6.

**Remark 4** From the supports of all codewords of weight 10 of the code with parameters [56, 20, 10]$_2$ we construct the Sims-Gewirtz graph, the unique strongly regular graph with parameters (56, 10, 0, 2), and from the codewords of weight 14, as explained in [4], we construct the unique strongly regular (120, 42, 8, 18) graph, known as the $L(3, 4)$ graph, defined on Baer subplanes.
Table 5: The coefficients $p_{k,i}^{(k)}$ of adjacency matrices for the graph $\Gamma_{DG}$

|       | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |
|-------|-------|-------|-------|-------|-------|-------|
| $A_0 \cdot A_0$ | 1     | 0     | 0     | 0     | 0     | 0     |
| $A_1 \cdot A_1$ | 10    | 0     | 2     | 0     | 0     | 0     |
| $A_2 \cdot A_2$ | 45    | 0     | 36    | 0     | 36    | 0     |
| $A_3 \cdot A_3$ | 45    | 0     | 36    | 0     | 36    | 0     |
| $A_4 \cdot A_4$ | 10    | 0     | 2     | 0     | 0     | 0     |
| $A_5 \cdot A_5$ | 1     | 0     | 0     | 0     | 0     | 0     |

Table 6: Self-orthogonal codes obtained from the graph $\Gamma_{DG}$

| $H \leq G_{\Gamma_{DG}}$ | $i$ | The code | $H \leq G_{\Gamma_{DG}}$ | $i$ | The code |
|--------------------------|-----|----------|--------------------------|-----|----------|
| $I$                      | 1,4 | [112, 40, 10] | $Z_2$                    | 2,3 | [56, 19, 18] |
| $I$                      | 2,3 | [112, 38, 18] | $E_4$                    | 1,4 | [28, 9, 8] |
| $Z_2$                    | 1,4 | [56, 18, 8] | $E_4$                    | 2,3 | [28, 9, 12] * |
| $Z_2$                    | 1,4 | [56, 20, 10] | $Z_7$                    | 1,4 | [16, 4, 2] |
| $Z_2$                    | 2,3 | [56, 18, 12] | $Z_7$                    | 2,3 | [16, 2, 6] |

Table 7: The coefficients $p_{k,i}^{(k)}$ of adjacency matrices for the graph $\Gamma_I$

|       | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |
|-------|-------|-------|-------|-------|-------|-------|
| $A_0 \cdot A_0$ | 1     | 0     | 0     | 0     | 0     | 0     |
| $A_1 \cdot A_1$ | 4     | 0     | 1     | 0     | 0     | 0     |
| $A_2 \cdot A_2$ | 12    | 0     | 2     | 0     | 1     | 0     |
| $A_3 \cdot A_3$ | 36    | 0     | 6     | 0     | 2     | 0     |
| $A_4 \cdot A_4$ | 108   | 0     | 18    | 0     | 33    | 0     |
| $A_5 \cdot A_5$ | 324   | 0     | 297   | 0     | 288   | 0     |
| $A_6 \cdot A_6$ | 243   | 0     | 162   | 0     | 162   | 0     |

4.4 Incidence graph of $GH(3, 3)$, $d = 6$

The incidence graph of $GH(3, 3)$, which we will denote by $\Gamma_I$, has 728 vertices, the intersection array $\{4, 3, 3, 3, 3, 3; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$, and the full automorphism group $G_{\Gamma_I}$ of order 8491392. The coefficients of $A_i \cdot A_i$ for the distance-$i$ matrices $A_i$ of $\Gamma_I$ ($i = 0, 1, \ldots, 6$) are given in Table 7. As in the previous examples, by applying Corollary 1 we obtain self-orthogonal codes which are presented in Table 8.

4.5 Doubled odd graph $D(O_4)$, $d = 7$

The Doubled Odd graph $D(O_4)$ has 70 vertices, the full automorphism group $G_{D(O_4)}$ of order 10080, and the intersection array $\{4, 3, 3, 2, 2, 1; 1; 1, 1, 2, 2, 3, 3, 4\}$. The coefficients of the distance-$i$ matrices $A_i$ of $D(O_4)$ ($i = 0, 1, \ldots, 7$) are given in
Table 8 Self-orthogonal codes obtained from the graph $\Gamma_I$

| $H \leq G_{\Gamma_I}$ | $i$ | The code | $H \leq G_{\Gamma_I}$ | $i$ | The code |
|------------------------|-----|----------|------------------------|-----|----------|
| $Z_7$                  | 3   | [104, 26, 12]_2 | $Z_13$ | 6 | [56, 6, 18]_3 |
| $Z_7$                  | 5   | [104, 26, 18]_3 | $D_{14}, Z_{14}$ | 3 | [52, 13, 12]_2 |
| $Z_7$                  | 6   | [104, 6, 36]_3 | $D_{14}, Z_{14}$ | 5 | [52, 13, 18]_3 |
| $Z_{13}$               | 3   | [56, 14, 8]_2 | $D_{14}, Z_{14}$ | 6 | [52, 3, 36]_3* |
| $Z_{13}$               | 5   | [56, 14, 9]_3 |

Table 9 The coefficients $p_{i,j}^k$ of adjacency matrices for the graph $D(O_4)$

|        | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| $A_0 \cdot A_0$ | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $A_1 \cdot A_1$ | 4     | 0     | 1     | 0     | 0     | 0     | 0     | 0     |
| $A_2 \cdot A_2$ | 12    | 0     | 5     | 0     | 4     | 0     | 0     | 0     |
| $A_3 \cdot A_3$ | 18    | 0     | 9     | 0     | 9     | 0     | 9     | 0     |
| $A_4 \cdot A_4$ | 18    | 0     | 9     | 0     | 9     | 0     | 9     | 0     |
| $A_5 \cdot A_5$ | 12    | 0     | 5     | 0     | 4     | 0     | 0     | 0     |
| $A_6 \cdot A_6$ | 4     | 0     | 1     | 0     | 0     | 0     | 0     | 0     |
| $A_7 \cdot A_7$ | 1     | 0     | 1     | 0     | 0     | 0     | 0     | 0     |

Table 10 Self-orthogonal codes obtained from the graph $D(O_4)$

| $H \leq G_{D(O_4)}$ | $i$ | The code |
|----------------------|-----|----------|
| $I$                  | 3,4 | [70, 26, 12]_3 |
| $Z_2$                | 3,4 | [35, 13, 12]_3 |
| $Z_5$                | 3,4 | [14, 2, 6]_3 |
| $Z_7$                | 3,4 | [10, 2, 3]_3 |

Table 9. By applying Corollary 1, in a similar way as in the previous examples, we obtain self-orthogonal codes and present the obtained results in Table 10.

4.6 Foster graph, $d = 8$

The Foster graph $\Gamma_F$ has 90 vertices, the full automorphism group $G_{\Gamma_F}$ of order 4320, and the intersection array $\{3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3\}$. The coefficients of the distance-$i$ matrices $A_i$ ($i = 0, 1, \ldots, 8$) are given in Table 11. In a similar way as in the previous examples, we obtain self-orthogonal codes by applying Corollary 1. The results are presented in Table 12.

Remark 5 The code with parameters $[45, 6, 20]_2$ is a projective code with weights 20 and 24 which yields a strongly regular graph with parameters $(64, 45, 32, 30)$. The code with parameters $[30, 8, 8]_2$ is a projective code with weights 8 and 16 which yields the unique strongly regular graph with parameters $(256, 30, 14, 2)$ and from the supports of codewords of weight 16 in a way explained in [4] we obtain the
Table 11 The coefficients $p_{i,j}^k$ of adjacency matrices for the Foster graph $\Gamma_F$

|       | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $A_0 \cdot A_0$ | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $A_1 \cdot A_1$ | 3     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     |
| $A_2 \cdot A_2$ | 6     | 0     | 1     | 0     | 1     | 0     | 0     | 0     | 0     |
| $A_3 \cdot A_3$ | 12    | 0     | 2     | 0     | 3     | 0     | 4     | 0     | 0     |
| $A_4 \cdot A_4$ | 24    | 0     | 12    | 0     | 12    | 0     | 12    | 0     | 24    |
| $A_5 \cdot A_5$ | 24    | 0     | 12    | 0     | 12    | 0     | 14    | 0     | 12    |
| $A_6 \cdot A_6$ | 12    | 0     | 2     | 0     | 4     | 0     | 1     | 0     | 6     |
| $A_7 \cdot A_7$ | 6     | 0     | 2     | 0     | 0     | 0     | 1     | 0     | 3     |
| $A_8 \cdot A_8$ | 2     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 1     |

Table 12 Self-orthogonal codes from the graph $\Gamma_F$

| $H \leq G_{\Gamma_F}$ | $i$ | The code | $H \leq G_{\Gamma_F}$ | $i$ | The code |
|------------------------|-----|----------|------------------------|-----|----------|
| $I$                    | 5   | $[90, 12, 20]_2$ | $Z_3$                  | 4   | $[30, 8, 8]_2$ |
| $I$                    | 4   | $[90, 8, 24]_2$  | $Z_5$                  | 5   | $[18, 4, 4]_2$ |
| $I$                    | 4   | $[90, 30, 3]_3$  | $Z_5$                  | 4   | $[18, 6, 3]_3$ |
| $Z_2$                  | 5   | $[45, 6, 20]_2$  | $S_3$                  | 4   | $[15, 4, 8]_2$ * |
| $Z_2$                  | 4   | $[45, 4, 24]_2$  | $D_{10}, Z_{10}$       | 5   | $[9, 2, 4]_2$ |
| $Z_2$                  | 4   | $[45, 15, 3]_3$  | $D_{10}, Z_{10}$       | 4   | $[9, 3, 3]_3$ |

unique strongly regular graph with parameters $(225, 28, 13, 2)$. From the supports of the codewords of weight 6 of the code $[18, 6, 3]_3$ we obtain the unique strongly regular graph with parameters $(15, 6, 1, 3)$, and from the supports of the codewords of weight 9 we obtain a distance-regular graph with diameter 3 having 20 vertices and intersection array $\{9, 4, 1; 1, 4, 9\}$. The code $[18, 4, 4]_2$ is not projective but it is two weight code with weights 4 and 8 which yields a strongly regular graph with parameters $(16, 6, 2, 2)$. From the supports of codewords of weight 8 of the same code we obtain the unique strongly regular graph with parameters $(9, 4, 1, 2)$.

5 Self-orthogonal subspace codes from equitable partitions of association schemes

As an analog of the definition of a self-orthogonal linear code we introduce the definition of a self-orthogonal subspace code as follows.

**Definition 1** Let $P_q(n)$ be the set of all subspaces of $\mathbb{F}_q^n$. The dual code of a subspace code $C_S \subseteq P_q(n)$ is the set $C_S^\perp$ of all vector spaces in $P_q(n)$ that are orthogonal to each vector space in $C_S$. If $C_S \subseteq C_S^\perp$, then $C_S$ is called a self-orthogonal subspace code. If $C_S = C_S^\perp$, then $C_S$ is called self-dual.

Let $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$, be the set of $n \times n$ adjacency matrices of the association scheme $(X, R)$, and let $q = p^m$ be a prime power. Let us consider the matrix algebra
\( \overline{\mathcal{A}} \) over \( \mathbb{F}_q \) generated by matrices \( A_{i_1}, A_{i_2}, \ldots, A_{i_t}, I = \{i_1, i_2, \ldots, i_t\} \subseteq \{0, 1, \ldots, d\}, t \geq 2. \) If \( p \mid p_{x,y}^k \) for all \( k \in \{0, 1, \ldots, d\} \) and all choices of \( x, y \in I \), then the row spaces of the elements of \( \overline{\mathcal{A}} \) are mutually orthogonal. Therefore, the set of the row spaces of elements of \( \overline{\mathcal{A}} \) forms a self-orthogonal subspace code \( C_S \subseteq \mathbb{F}_q^n \). Moreover, because of Theorem 1 the following theorem holds.

**Theorem 4** Let \( \Pi \) be an equitable partition of a \( d \)-class association scheme \( (X, \mathcal{R}) \) with \( n \) cells of the same length \( \frac{|X|}{n} \) and let \( p \) be a prime number. Further, let \( I = \{i_1, i_2, \ldots, i_t\} \subseteq \{0, 1, \ldots, d\} \) and \( p \mid p_{x,y}^k \), for all \( k \in \{0, 1, \ldots, d\} \) and all \( x, y \in I \). Then, the set of row spaces of elements of the matrix algebra generated by the matrices \( M_i, i \in I \), forms a self-orthogonal subspace code \( C_S \subseteq \mathbb{F}_q^n \), where \( q = p^m \) is a prime power.

### 5.1 Subspace codes from distance-regular graphs

Let \( \Gamma \) be a distance-regular graph with diameter \( d \) and adjacency matrix \( A \). Let \( A_i \) denotes the distance-\( i \) matrix of \( \Gamma \), for \( i = 0, 1, \ldots, d \). Let \( G \) be an automorphism group of \( \Gamma \). In the sequel, the quotient matrix of \( A_i \), where \( i = 1, 2, \ldots, d \), with respect to the orbit partition induced by \( G \) will be denoted by \( M_i \).

Since a distance-regular graph induces an association scheme, the next corollary follows from Theorem 4.

**Corollary 3** Let \( \Gamma \) be a distance-regular graph with diameter \( d \), and let an automorphism group \( G \) acts on \( \Gamma \) with \( n \) orbits of the same length. Further, let \( I = \{i_1, i_2, \ldots, i_t\} \subseteq \{0, 1, \ldots, d\} \) and \( p \) be a prime number such that \( p \mid p_{x,y}^k \), for all \( k \in \{0, 1, \ldots, d\} \) and all \( x, y \in I \). Then the set of row spaces of elements of the matrix algebra generated by the matrices \( M_i, i \in I \), forms a self-orthogonal subspace code \( C_S \subseteq \mathbb{F}_q^n \), where \( q = p^m \) is a prime power.

#### 5.1.1 Examples of self-orthogonal subspace codes

As an example, we apply Corollary 3 to a construction of self-orthogonal subspace codes from the Doubled Higman–Sims graph. The Doubled Higman–Sims graph, which we denote by with \( \tilde{\Gamma}_{dHS} \), is a distance-regular graph having 200 vertices, diameter \( d = 5 \), and intersection array \( \{22, 21, 16, 6, 1; 1, 6, 16, 21, 22\} \). See [2] for more information. The coefficients of \( A_1 \cdot A_i \) for the distance-\( i \) matrices \( A_i \) of \( \tilde{\Gamma}_{dHS} \) \( (i = 0, 1, \ldots, 5) \) are given in Table 13.

Table 13 shows that we have to check only the case \( I = \{1, 4\} \), i.e., the remaining product \( A_1 \cdot A_4 \). Since \( A_1 \cdot A_4 = 6A_3 + 22A_5 \), and \( p = 2 \) divides all coefficients \( p_{x,y}^k \) for \( k \in \{0, 1, \ldots, 5\} \) and \( x, y \in I \), the set of row spaces of the elements of the matrix algebra generated by matrices \( M_i, i \in I \), is a self-orthogonal subspace code.

The full automorphism group \( \Gamma_{dHS} \) of the Doubled Higman–Sims graph has order 177408000. For the subgroups of \( G_{dHS} \) acting on the graph \( \tilde{\Gamma}_{dHS} \) with all orbits of the same length we construct quotient matrices \( M_i, i \in I \). The self-orthogonal subspace codes obtained by applying Corollary 3 are presented in Table 14.
We believe that using the described methods one can obtain further interesting self-orthogonal subspace codes from the graph. We gave examples of self-orthogonal subspace codes obtained from distance-regular graphs. Some of the obtained codes are optimal. We also introduced self-orthogonal subspace codes and showed that under some conditions equitable partitions of association schemes yield such subspace codes. In this paper, we gave a method of constructing self-orthogonal codes from equitable partitions of association schemes and, as an illustration of the method, constructed self-orthogonal subspace codes from some distance-regular graphs. Some of the obtained codes are optimal. We also introduced self-orthogonal subspace codes and showed that under some conditions equitable partitions of association schemes yield such subspace codes. We gave examples of self-orthogonal subspace codes obtained from distance-regular graph. We believe that using the described methods one can obtain further interesting

| $A_0 \cdot A_0$ | $A_1 \cdot A_1$ | $A_2 \cdot A_2$ | $A_3 \cdot A_3$ | $A_4 \cdot A_4$ | $A_5 \cdot A_5$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1               | 22              | 77              | 77              | 22              | 1               |
| 0               | 0               | 0               | 0               | 0               | 0               |
| 0               | 6               | 60              | 60              | 6               | 0               |
| 0               | 0               | 56              | 56              | 0               | 0               |
| 0               | 0               | 0               | 0               | 0               | 0               |

Table 14: Self-orthogonal subspace codes from the graph $\Gamma_{dHS}$

| $H \leq G_{\Gamma_{dHS}}$ | The code |
|--------------------------|----------|
| $E_{25}$                 | (8, 8, 1, {0, 1, 2, 3, 4})$_2$ |
| $Z_5 \oplus Z_4$         | (10, 4, 1, {0, 1, 2})$_2$   |
| $Z_5 \oplus Z_4$         | (10, 5, 1, {0, 1, 2})$_2$   |
| $Z_{10}$                 | (20, 236, 1, {0, 1, 2, 3, 4, 5, 6})$_2$ |
| $Z_{10}$                 | (20, 107, 1, {0, 1, 2, 3, 4, 5, 6})$_2$ |
| $Z_{10}$                 | (20, 67, 1, {0, 1, 2, 3, 4})$_2$ |
| $Z_{10}$                 | (20, 3, 2, {0, 1, 2})$_2$   |
| $Z_4 \times Z_2$         | (25, 366, 1, {0, 1, 2, 3, 4, 5})$_2^*$ |
| $D_8$                    | (25, 83, 1, {0, 1, 2, 3, 4, 5})$_2$ |
| $E_8$                    | (25, 67, 1, {0, 1, 2, 3, 4})$_2$ |
| $E_8$                    | (25, 17, 1, {0, 1, 2, 3, 4})$_2$ |
| $D_8$                    | (25, 6, 1, {0, 1, 2, 3, 4})$_2$ |
| $Z_5$                    | (40, 27, 1, {0, 1, 2, 3, 4})$_2$ |
| $I$                      | (200, 3, 22, {0, 22, 44})$_2$ |

Remark 6: One of the codewords (i.e., subspaces) of the subspace code labeled with $*$ is an optimal linear code with parameters $[25, 4, 12]_2$, a two weight code with weights 12 and 16. It is not projective but, it yields the unique strongly regular graph with parameters $(16, 10, 6, 6)$, the complement of the strongly regular graph with parameters $(16, 5, 0, 2)$. From the supports of the codewords of weight 12 one obtains the Petersen graph, the unique strongly regular graph with parameters $(10, 3, 0, 1)$.

6 Conclusion

In this paper, we gave a method of constructing self-orthogonal codes from equitable partitions of association schemes and, as an illustration of the method, constructed self-orthogonal codes from some distance-regular graphs. Some of the obtained codes are optimal. We also introduced self-orthogonal subspace codes and showed that under some conditions equitable partitions of association schemes yield such subspace codes. We gave examples of self-orthogonal subspace codes obtained from distance-regular graph. We believe that using the described methods one can obtain further interesting
self-orthogonal codes and self-orthogonal subspace codes using association schemes, especially those arising from distance-regular graphs.

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