Radiation from an electron bunch flying over a surface wave

A. A. Saharian\textsuperscript{1}, A. R. Mkrtchyan\textsuperscript{2}, L. A. Gevorgian, L. Sh. Grigorian\textsuperscript{3}, B. V. Khachatryan,

Institute of Applied Problems in Physics
25 Nersessian Str., 375014 Yerevan, Armenia

Abstract. Radiation generated by the passage of a monoenergetic electron bunch above the surface wave excited in plane interface between homogeneous media with different dielectric constants is investigated. For the surface wave of general profile the radiation intensity is expressed via the radiated power from a single charge and bunch form factor. Various types of transverse and longitudinal distributions of electrons in the bunch have been considered including Gaussian, asymmetrical Gaussian, two Gaussian and rectangular distribution with asymmetrical exponential tails. Conditions are specified under which the coherent radiation essentially exceeds the incoherent part.

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1 Introduction

Recently a great deal of attention has been devoted to the investigations of the radiation from charged particles in periodic structures. Such radiation has a number of remarkable properties and is widely used in various regions of science and technology (see, for example, \cite{1} - \cite{4}, and references therein). The possible physical applications include the generation of the electromagnetic radiation in various wavelength ranges by beams of charged particles and the determination of the characteristics of emitting particles by using the properties of the radiation field. As an example one can mention here the transition radiation from a charge traversing a stack of plates or moving in a medium with periodically varying dielectric permittivity \cite{3}, \cite{5}, \cite{6}. Another important example is the Smith-Purcell radiation, which arises when charged particles are in flight near a diffraction grating (see e.g.\cite{1}, \cite{2}, \cite{7}, \cite{8}). This radiation is one of the main mechanisms for the generation of electromagnetic waves in the millimeter and submillimeter wavelength range (see \cite{7}).

Recent theoretical and experimental developments \cite{13}, \cite{14} have shown that the use of external (ultrasonic, temperature gradient) fields have a potential to control various parameters of emitted radiation. The modification of the crystal lattice by external fields can result in radiation intensity enhancement. In papers \cite{13}, \cite{15} we have considered the radiation from a charge flying over a surface acoustic wave excited on the plane interface between homogeneous media with different dielectric constants. It is shown that the use

\textsuperscript{1}E-mail: saharyan@www.physdep.r.am
\textsuperscript{2}E-mail: malpic@iappp.sci.am
\textsuperscript{3}E-mail: levonshg@iappp.sci.am
of surface waves essentially simplifies the control of angular-frequency characteristics of the emitted radiation.

The present paper deals with the consideration of the similar issue for relativistic electron bunches. It also generalizes similar results in [18] for coherent diffraction radiation from an electron bunch. In Section 2 we derive a formula for the radiation intensity of an electron bunch flying over the surface wave of an arbitrary profile. In Section 3 we consider various types of transverse and longitudinal distribution of electrons in the bunch and specify conditions under which the coherent part of the radiation essentially exceeds its incoherent part.

2 Radiation intensity

Assume that a monoenergetic bunch of \( N \) particles moves at constant velocity \( v \) along the \( z \) axis, parallel to the propagation direction of a surface wave excited on plane boundary between homogeneous media with permittivities \( \varepsilon_1 \) and \( \varepsilon_2 \). If \( x \) and \( y \) axes of the Cartesian system of coordinates are directed perpendicular and parallel to the interface respectively, then the relation between coordinates \( x, y, z \) of the interface may be written as

\[
x = -d + f(\xi), \quad \xi \equiv k_0 z \mp \omega_0 t
\]

where \( k_0 \) and \( \omega_0 \) are the wave number and frequency of the surface wave, \( d \) is the distance of the non-excited interface from origin of coordinate system and \( f \) is a function describing the surface wave profile. The minus/plus signs correspond to the propagation of wave in the positive/negative direction of \( z \) axis.

The radiation spectrum does not depend on surface wave profile and can be obtained by using the following arguments. In the bunch rest frame the radiation frequency \( \nu' \) is multiple to the frequency of passing one wavelength \( \lambda'_0 \) of surface wave: \( \nu' = mv'_0/\lambda'_0 \), \( m \) is an integer, \( v'_0 = (v_0 \mp v)/(1 - v_0v/c^2) \) is the velocity of surface wave in the bunch rest frame, and \( v_0 = \omega_0/k_0 \). If \( \lambda_0 \) is the corresponding wavelength in the laboratory system then \( \lambda'_0 = \lambda_0 \sqrt{1 - v^2/c^2}/(1 - v_0v/c^2) \). By substituting these relations into the standard transformation formula for the frequency, one obtains the relation between the radiation frequency \( \omega \) and emission angle in the laboratory system. For the medium with permittivity \( \varepsilon_\alpha \) this relation has the form (see, for example, [14])

\[
\omega = \frac{m(k_0v \mp \omega_0)}{1 - \beta \sqrt{\varepsilon_\alpha \cos \theta}}, \quad \beta = v/c
\]

where \( \theta \) is the angle between the bunch velocity and the radiation direction. When \( \omega_0 = 0 \) we obtain the standard relation for the static case [1]. Note that in the case \( v \gg v_0 \) one has \( \omega \gg m\omega_0 \), whereas \( \omega \sim m\omega_0 \) for \( v \sim v_0 \). The relation (2) can be derived also from the conditions of system periodicity under the transformations

\[
t \rightarrow t + \frac{\lambda_0}{v \mp v_0}, \quad z \rightarrow z + \frac{\lambda_0v}{v \mp v_0}
\]

(3)

In particular, it follows from here that the projection of the wave vector on the axis \( z \), \( k_z = \omega \sqrt{\varepsilon_\alpha \cos \theta}/c \), is equal to

\[
k_{zm} = \frac{\omega}{v} - m \left( \frac{k_0 \mp \omega_0}{v} \right)
\]

(4)
The relation (2) is a direct consequence of this formula. It can be easily seen that Eq. (4) corresponds to that the waves emitted from the two neighboring humps of the surface wave are in phase.

As seen from (2) in the case of infinite media the photons radiated in fixed direction $\theta$ have frequencies multiple to the main harmonic with $m = 1$. If the media have a finite size, $2z_p$, in direction of bunch moving ($z$ axis) then in the formula for the radiation intensity instead of $\delta$-function one has $\sin [z_p(k_z - k_{zm})] / \pi (k_z - k_{zm})$ with $k_{zm}$ from Eq.(4). Now one has a sharp peak at frequency given by (2) with width at half maximum equal to

$$\Delta \omega = \frac{2\nu}{z_p (1 - \beta \sqrt{\varepsilon_\alpha \cos \theta})}$$

(5)

For $z_p \gg \lambda_0$ the distance between the neighboring harmonics is much larger than the width of the spectral line.

The electric and magnetic fields $\vec{E}_\alpha$ and $\vec{H}_\alpha$ in the regions with permittivity $\varepsilon_\alpha$ ($\alpha = 1, 2$) are determined by the Maxwell equations with corresponding boundary conditions on the interface (3). We can write the fields in the first medium as

$$\vec{E}_1 = \vec{E}_0 + \vec{E}_1', \quad \vec{H}_1 = \vec{H}_0 + \vec{H}_1',$$

(6)

where $\vec{E}_0, \vec{H}_0$ are the fields of bunch at its motion in the homogeneous medium with permittivity $\varepsilon_1$. It is convenient to present the expressions for these fields in the form

$$\vec{F}_0 = \sum_{j=1}^{N} \int d\omega dk_y F_0(\omega, k_y) e^{-i\vec{k}_j \cdot \vec{R}_j} e^{i(k_{zm} - \omega t)}, \quad F = E, H$$

(7)

$$\vec{E}_0(\omega, k_y) = \frac{q}{2\pi v g_0} \left( \frac{k_y}{\varepsilon_1} - \frac{\omega}{c^2 v^2} \right), \quad \vec{H}_0(\omega, k_y) = \frac{q}{2\pi v g_0} \left[ \frac{\vec{v}}{c k_{zm}} \right]$$

where the vector $\vec{R}_j = (X_j, Y_j, Z_j)$ gives the position of the $j$-th particle in the bunch at the initial moment $t = 0$. In Eq. (7) the factor depending on the particle number is separated and the following notations are introduced:

$$g_0 = \left( \frac{\omega^2}{c^2} \varepsilon_1 - k_y^2 - \frac{\omega^2}{v^2} \right)^{1/2}, \quad \vec{k} = (g_0 \text{sgn} (x - X_j), k_y, \omega/v)$$

(8)

The fields $\vec{E}_1', \vec{H}_1', \vec{E}_2, \vec{H}_2$ are solutions of homogeneous Maxwell equations. With due regard for the system periodicity these fields can be written in the form of Fourier expansions

$$\vec{F}_\alpha = \sum_{j=1}^{N} \sum_{m=-\infty}^{+\infty} \int d\omega dk_y \bar{F}_{\alpha m}(\omega, k_y) e^{i(\vec{k}_{\alpha} \cdot \vec{r} - \omega t)}, \quad \alpha = 1, 2$$

(9)

where $k_z$ is determined by the relation (4),

$$\vec{k}_{\alpha} = (k_{xz}(\alpha), k_y, k_{z\alpha}(\alpha)), \quad k_{xz}(\alpha) = \left( \frac{\omega^2}{c^2} \varepsilon_\alpha - k_y^2 - k_{zm}(\alpha)^2 \right)^{1/2}$$

(10)
\[ F_{\alpha m}^{(j)}(\omega, k_y) = \frac{v + v_0}{(2\pi)^2 \lambda_0 v} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\xi \int_0^{+\infty} d\tau \hat{F}_{\alpha m}^{(j)}(t, \tau) e^{-i(\bar{\kappa}^{(\alpha)} \bar{\tau} - \omega t)} \]  

and \( \hat{F}_{\alpha m}^{(j)}(t, \tau) \) is the corresponding field for the \( j \)-th particle.

In the expansions (11) the Fourier components are determined by corresponding boundary conditions for the fields on the interface (2). To reveal the structure of corresponding set of boundary conditions does not depend on the particle number \( j \). By using Eq. (2) the formula for the fields spectral components for the reason that the normal \( \vec{n} \) is time dependent. By substituting the corresponding expressions (5) and (4) for the fields into (12), multiplying by \( \exp[-ik_y y + i\omega t(t \mp v_0/v)] \) and integrating over \( y \) and \( t \) for fixed \( z \mp v_0 t \) one obtains

\[ \vec{n} \cdot \sum_{m=-\infty}^{+\infty} \left\{ \varepsilon_\alpha \hat{E}_{\alpha m}^{(j)}(\omega, k_y) \exp \left[ ik_x^{(\alpha)} (-d + f(\xi)) \right] \right\}_{\alpha=1}^{\alpha=2} e^{-2\pi imz/\lambda_0} |_{\omega=\omega'\mp m\omega_0} = \]

\[ \vec{n} \cdot \varepsilon_1 \hat{E}_0(\omega, k_y) \exp \left[ -ig_0 (\mp 1/v) \right] \exp \left[ i\omega_1 X_j - i k_y Y_j - \frac{i\omega}{v} Z_j \right] |_{\omega=\omega'} \]  

It follows from here that for the new functions

\[ \hat{E}_{\alpha m}^{(1)}(\omega, k_y) = \hat{E}_{\alpha m}^{(j)}(\omega, k_y) \exp \left( -ig_0(\omega_1)X_j + ik_y Y_j + \frac{i\omega_1}{v} Z_j \right) , \quad \omega_1 = \omega \pm m\omega_0 \]

the corresponding set of boundary conditions does not depend on the particle number \( j \) :

\[ \vec{n} \cdot \sum_{m=-\infty}^{+\infty} \varepsilon_\alpha \hat{E}_{\alpha m}^{(1)}(\omega, k_y) \exp \left[ ik_x^{(\alpha)} (-d + f(\xi)) \right]_{\alpha=1}^{\alpha=2} e^{-2\pi imz/d} |_{\omega=\omega'\mp m\omega_0} = \]

\[ \varepsilon_1 \vec{n} \cdot \hat{E}_0(\omega, k_y) \exp \left[ ig_0 (d - f(\xi)) \right] |_{\omega=\omega'} \]  

The values \( \hat{E}_{\alpha m}^{(1)} \) determine the field of a single charge with coordinates \( X_j = Y_j = Z_j = 0 \) at the initial moment \( t = 0 \). The other boundary conditions can be considered analogously. By using Eq. (2) the formula for \( \omega_1 \) can be written as

\[ \omega_1 = m k_0 v \frac{1 \mp (v_0/c)\sqrt{\varepsilon_\alpha} \cos \theta}{1 - \beta \sqrt{\varepsilon_\alpha} \cos \theta} \approx \frac{m k_0 v}{1 - \beta \sqrt{\varepsilon_\alpha} \cos \theta} \]  

where we have used the fact that surface wave velocity is much less than \( c \). For example, in case of surface wave excited in \( SiO_2 \) one has \( v_0 \approx 3 \cdot 10^5 cm/sec \) and \( v_0/c \approx 10^{-5} \).

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As it follows from the above analysis the Fourier components of the bunch field can be presented in the form

\[ F_{\alpha m} = N \sum_{j=1}^{N} F_{\alpha m}^{(j)} = N \sum_{j=1}^{N} \exp \left( i g_0 (\omega_1) X_j - i k_y Y_j - i \frac{\omega_1}{v} Z_j \right) \] (18)

At large distances \( \rho = \sqrt{x^2 + y^2} \) from the bunch trajectory we find by means of the stationary phase method

\[ F_{\alpha m} (\omega, \vec{r}) = \sqrt{\frac{2\pi |\omega|}{ic\rho}} \sqrt{\varepsilon_\alpha} \sin \theta \cos \varphi \vec{F}_{\alpha m} (\omega, k_y) e^{i \vec{k}(\alpha) \vec{r}} \] (19)

The density of radiation energy flux during the whole time of emission is determined as

\[ \frac{c}{4\pi} \int_{-\infty}^{+\infty} dt \left[ \vec{E} \vec{H} \right] = c \int_{0}^{\infty} d\omega \sum_{m} \text{Re} \left[ \vec{E}_m (\omega, \vec{r}) \vec{H}_m (\omega, \vec{r}) \right] = \int_{0}^{\infty} d\omega \sum_{m} \vec{P}_m (N) (\omega) \] (20)

Using the expressions (19) for the fields we find the spectral density of the radiation energy flux in the medium \( \alpha \) for given \( m \) as

\[ \vec{P}_m (N) (\omega) = 2\pi \varepsilon_\alpha \sin \theta \cos^2 \varphi \left| \vec{E}_{\alpha m} (\omega, k_y) \right|^2 \vec{F}(\alpha) / k(\alpha) \] (21)

It follows from (18) that for an arbitrary periodic function \( f \) in (1) this expression can be written in the form

\[ \vec{P}_m (N) (\omega) = \vec{P}_m (1) (\omega) S_N, \quad S_N = \left| N \sum_{j=1}^{N} \exp \left( i g_0 (\omega_1) X_j - i k_y Y_j - i \frac{\omega_1}{v} Z_j \right) \right|^2 \] (22)

where \( \vec{P}_m (1) (\omega) \) is the corresponding function for the radiation of a single charge with coordinates \( x = y = z = 0 \) at the initial moment \( t = 0 \). The formula for \( \vec{P}_m (1) (\omega) \) is derived in [13]-[15] and the dependence of the corresponding radiation intensity on various parameters is investigated. For this reason here we will mostly concerned with bunch form factor.

As it directly follows from Eq. (16) the dependence of the radiation intensity in the medium \( \alpha \) on the distance of the electron trajectory from the non-excited interface, \( d \), is determined by the factor

\[ \exp \left( - \frac{2\omega}{v} d \text{Re} \sigma \right), \quad \sigma = \left( 1 - \beta^2 \varepsilon_1 \right) \frac{\omega^2}{\omega_1^2} + \beta^2 \varepsilon_\alpha \sin^2 \theta \sin^2 \varphi \right]^{1/2}, \quad g_0 = i \frac{\omega}{v} \sigma \] (23)

This factor does not depend on the specific form of the wave profile in (1) and is determined by the dependence of the field spectral components for uniformly moving particle on distance \( d \). As mentioned above for relativistic particles with \( \beta^2 \varepsilon_1 \gtrsim 1 \) one has \( \omega_1 \approx \omega \) and the radiation at large azimuthal angles is exponentially suppressed and the radiation
distribution is strongly anisotropic. For the radiation in the vacuum at \( \varphi \approx 0 \) the factor is equal to \( \exp(-2\omega_{1}d/v\gamma) \), with \( \gamma = 1/\sqrt{1-\beta^{2}} \) been the Lorentz factor.

In the case of \( v\sqrt{\varepsilon_{1}} < c \), the quantity \( \sigma \) is real and the intensity exponentially decreases with increasing distance. The same is the case for the directions of radiations satisfying the condition

\[
\sin^{2} \theta \sin^{2} \varphi > 1 - c^{2}/v^{2}\varepsilon_{1}
\]

when \( v\sqrt{\varepsilon_{1}} > c \). For the last case and directions off the region (24), the radiation intensity does not depend on particle distance from a surface of the periodic structure in the absence of absorption. This corresponds to the reflection of Cherenkov radiation emitted in the first medium.

3 Bunch form factor and coherent effects

We shall assume that the coordinates of the \( j \)-th particle are independent random variables and the location probability of the \( j \)-th particle at a given point is independent of the particle number. By averaging the quantity (22) over the positions of a particle in the bunch and making a summation similar to [16], we obtain

\[
\langle \vec{P}_{m}^{(N)} \rangle = \langle S_{N} \rangle \vec{P}_{m}^{(1)}
\]

\[
\langle S_{N} \rangle = Nh + N(N - 1) |h_{x}h_{y}h_{z}|^{2}
\]

Here we have introduced the notations

\[
h = \langle \exp\left(-\frac{2\omega}{v}XRe\sigma\right) \rangle, \quad h_{l} = \langle \exp(iK_{l}l) \rangle, \quad l = x, y, z,
\]

\[
K_{x} = -i\frac{\omega}{v}\sigma, \quad K_{y} = k_{y} = \frac{\omega}{c}\sqrt{\varepsilon_{0}}\sin \theta \sin \varphi, \quad K_{z} = \frac{\omega_{1}}{v}
\]

where \( \sigma \) and \( \omega_{1} \) are given by relations (23) and (15). In particular, when \( \omega_{0} = 0 \) we obtain the case of Smith-Purcell radiation, generated at the passage of bunched electrons above the surface of a diffraction grating. The functions \( |h_{l}|^{2} \) determine bunch form factors in corresponding directions. In the right hand side of the expression (25) the summand proportional to \( N^{2} \) determines the contribution of coherent effects. The advantage of coherent radiation is that we can generate intense radiation by a beam of a low average current. The intensity of this radiation is determined by a bunch shape. Conventionally it is assumed that the coherent radiation is produced at wavelengths equal and longer than the electron bunch length. However as we shall see below this conclusion depends on the distribution of electrons in the bunch and can not be valid for a number of realistic distributions.

As it follows from (26) the form factors in \( y \) and \( z \) directions are determined by the Fourier transforms of the corresponding bunch distributions. First let us consider a Gaussian distribution. For this distribution of electrons,

\[
f_{l} = \frac{1}{\sqrt{2\pi b_{l}^{2}}} \exp\left(-\frac{l^{2}}{2b_{l}^{2}}\right), \quad l = x, y, z
\]

with \( b_{x}, b_{y} \) and \( b_{z} \) being corresponding characteristic sizes of the bunch, we have after averaging (assuming that all particles of the bunch are in a medium with permittivity \( \varepsilon_{1} \)
and therefore $b_x < d - a$, with $a$ being the surface wave amplitude)

$$h = \exp \left( \frac{2\omega^2}{v^2} (\Re \sigma)^2 b_x^2 \right), \quad \vert h_x \vert^2 = \exp \left( \frac{\omega^2}{v^2} \sigma^2 b_x^2 \right), \quad h_l = \exp \left( -K^2 b_l^2/2 \right), l = y, z \quad (29)$$

By substituting these relations into (25) one obtains the following expression of the form factor

$$\langle S_N \rangle = N \exp \left( \frac{2\omega^2}{v^2} (\Re \sigma)^2 b_x^2 \right) \left[ 1 + (N - 1) \exp \left( -\frac{\omega^2}{v^2} |\sigma|^2 b_x^2 - k_y b_y^2 \right) \right] \quad (30)$$

where the second summand in square brackets determines the relative contribution of coherent effects into the radiation intensity. For non-relativistic electrons when $b_y \gtrsim b_{x,z}$ the corresponding exponent is equal to

$$\exp[-(2\pi m/\lambda_0)^2 (b_x^2 + b_z^2) - (2\pi/\lambda)^2 b_y^2 \sin^2 \theta \sin^2 \varphi]$$

with $\lambda_0$ being surface wave wavelength (here we consider the case $\varepsilon_1 = 1$). Note that in this case for the wavelength of radiation one has $\lambda \sim \lambda_0/\beta m$ and the coherent effects are exponentially suppressed when $\lambda \gtrsim b_l/\beta$. For a relativistic bunch the relative contribution of coherent effects is equal to

$$N \exp \left\{ -(2\pi/\lambda)^2 \left[ (b_x^2 + b_y^2) \sin^2 \theta \sin^2 \varphi + b_z^2 \right] \right\} \quad (31)$$

for $\sin \theta \sin \varphi > \gamma^{-1}$, and

$$N \exp \left\{ -(2\pi b_x/\lambda \gamma)^2 \left[ b_x^2 + b_y^2 \sin^2 \theta \sin^2 \varphi \right] \right\} \quad (32)$$

for the radiation with $\sin \theta \sin \varphi \lesssim \gamma^{-1}$. As we see in this case the transverse form factor is strongly anisotropic. It follows from (31) that for a real $\sigma$, fixed electron number and fixed distance of the bunch’s center from surface wave the radiation intensity exponentially increases with increasing $b_x$. This is because the number of electrons passing close to the surface wave increases. The number of electrons with long distances will also increase. But the contribution of close electrons surpasses the decrease of the intensity due to distant ones.

As we see for a Gaussian distribution the relative contribution of coherent effects is exponentially suppressed in the case $b_i > \lambda$, with $\lambda$ being the radiation wavelength. This result is a consequence of the mathematical fact that in the case of a function $f(x) \in C^\infty (R)$ one has the following estimate for the integral (see, for example, [14])

$$F(u) \equiv \int_{-\infty}^{+\infty} f(l) e^{iu l} dl = O(u^{-\infty}), \quad u \rightarrow +\infty \quad (33)$$

where $u = 2\pi b_x/\lambda$ in the case of longitudinal form factor of the relativistic bunch.

Up to now we have considered the case of Gaussian distribution in the bunch. However it should be noted that due to various beam manipulations the bunch shape can be highly non-Gaussian (see, for instance, [20]). In (33) the continuity condition for the function $f(l)$ and infinite number of its derivatives is essential. It can be easily seen that when
\( f(l) \in C^{n-1}(R) \), and the derivative \( f^{(n)}(l) \) is discontinuous at point \( l_1 \), then the asymptotic estimate

\[
F(u) = (-iu)^{-n-1} \left[ f^{(n)}(l_1+) - f^{(n)}(l_1-) \right], \quad u \to +\infty
\]  

(34)
takes place. Unlike the case of (33) now the form factor for the short wavelengths decreases more slowly, as power-law, \( (2\pi b_l/\lambda)^{-n-1} \). In the coherent part of the radiation this form factor is multiplied by a large number, \( N \), of particles per bunch and coherent effects can dominate for

\[
2\pi b_l/\lambda < N^{1/(n+1)}
\]  

(35)
and the radiation intensity is enhanced by the factor \( N(\lambda/2\pi b_l)^{2(n+1)} \). Since conventionally there are \( 10^8 - 10^{10} \) electrons per bunch (see e.g. [20]) the condition (35) can be easily meet even in the case \( 2\pi b_l/\lambda > 1 \). For instance, in the case of \( n = 1, N \sim 10^{10} \) coherent radiation is dominant for \( 100\lambda > b_l \).

As an example we shall consider the electron distribution function having the asymmetric Gaussian form (about the asymmetric distribution of electrons in a bunch see, for example, [21])

\[
f(z) = \frac{2}{\sqrt{2\pi(1+p)b_l}} \left[ \exp\left(-\frac{l^2}{2p^2b_l^2}\right) \theta(-l) + \exp\left(-\frac{l^2}{2b_l^2}\right) \theta(l) \right],
\]  

(36)
where \( \theta(l) \) is the unit step function, \( l_0 = (1+p)b_l \) is the characteristic bunch length, parameter \( p \) determines the degree of bunch asymmetry. The corresponding function \( F(u) \) can be easily found (see also [16], [17])

\[
F(u) = \frac{1}{p+1} \left\{ e^{-i\omega^2} + pe^{-\omega^2t^2} - \frac{2i}{\sqrt{\pi}} W(t) - pW(\omega t) \right\},
\]  

(37)
where

\[
W(t) = \int_0^t \exp(l^2 - t^2) dl, \quad t = \frac{ub_l}{\sqrt{2}}
\]  

(38)
Note that the expression \( |F(u)|^2 \) is invariant with respect to the replacement \( p \to 1/p, b_l \to b_l/p \) that corresponds to the mirror reversal of the bunch. When the electron distribution is symmetric \( (p = 1) \), the second summand in square brackets of (37) is equal to zero and, as was mentioned earlier, the form factor exponentially decreases for the short wavelengths \( \lambda < 2\pi b_l/\beta \). For \( pt \gg 1 \) the asymptotic behaviour of the function \( F(u) \) can be easily found from (37) and has the form

\[
F(u) \sim i\sqrt{\frac{2}{\pi}} \frac{1-p}{u^3b_l^3p^2}
\]  

(39)
Note that this formula can be obtained also directly from (34) by taking into account that in the case of (33) one has \( f(l) \in C^1(R) \) and the second derivative is discontinuous \( (n = 2 \) in (34)).

Let us discuss in more detail the case when the electrons are normally distributed in the \( x \) and \( y \) directions and the distribution function in the \( z \) direction has an asymmetric Gaussian form (34). Now the expression (24) for \( \langle S_N \rangle \) can be written as

\[
\langle S_N \rangle = N \exp\left(\frac{2\omega^2}{v^2}b_x^2\right) \left[ 1 + (N-1) \exp\left(-\frac{\omega^2}{v^2}b_x^2 - k_y^2b_y^2\right) |F(\omega_1/v)|^2 \right]
\]  

(40)
where the second summand in the square brackets determines the contribution of coherent effects into the radiation intensity. In case of relativistic bunch and \( \varphi > \gamma^{-1} \), the exponent in this summand is of an order of \((2\pi b_i/\lambda)^2, i = x, y\) and when the transverse size of the bunch is shorter than the radiation wavelength, then the relative contribution of coherent effects is \( \sim N |F(\omega_1/v)|^2 \). For an asymmetrical bunch this contribution can be dominant even in case when the bunch length is greater than the wavelength. Indeed, according to (39) even for weakly asymmetrical bunch we have \( N |F(\omega_1/v)|^2 \sim N u / \omega_1 b )^6 \) and the radiation is coherent if \( b_z \lesssim \lambda N^{1/6} / 2\pi \) where we have taken into account that for an relativistic bunch \( \omega \gg m \omega_0 \) and therefore \( \omega_1 \approx \omega \), as was mentioned above. In the case of bunch with \( N \sim 10^{10} \) it follows hence that for \( 2\pi b_z / \lambda \lesssim 10 \) the radiation is coherent. For the radiation \( \varphi \lessgtr \gamma^{-1} \), the exponent of the second summand in square brackets of Eq. (40) at \( \varepsilon_1 = 1 \) is of an order of \((2\pi b_x / \lambda \gamma)^2 \). It follows hence that for a relativistic bunch the radiation in directions \( \varphi \lessgtr \gamma^{-1} \) can be coherent even in the case when the transverse size of the bunch is greater than the wavelength. For this it is sufficient that the following conditions

\[
b_x \lesssim \frac{\gamma \lambda}{2\pi}, \quad b_z \lesssim \frac{\lambda N^{1/6}}{2\pi}
\]

were met. The second of these conditions is written for weakly asymmetrical bunches. In the case of strongly asymmetrical bunch the corresponding conditions are much less restrictive: \( b_z \lesssim \lambda N^{1/2}/2\pi \) for \( pb_z \ll \lambda \).

Let \( f(l, a) \) be a continuous distribution function depending on parameter \( a \), and \( \lim_{a \to 0} f(l, a) = f(l) \). The integral \( F(u, a) \) for \( f(l, a) \) uniformly converges and hence \( \lim_{a \to 0} F(u, a) = F(u) \).

It follows from here that the estimate presented above is valid for continuous functions as well if they are sufficiently close to the corresponding discontinuous function (the corresponding derivative is sufficiently large, see below). Aiming to illustrate this we shall consider the asymmetric distribution function

\[
f(z, a_l, b_l, l_0) = \frac{1}{4 l_0} \left[ \text{th} \left( \frac{l + l_0}{a_l} \right) - \text{th} \left( \frac{l - l_0}{b_l} \right) \right]
\]

For \( l_0 > a_l, b_l \) this function describes a rectangular bunch having exponentially decreasing asymmetric tails with characteristic sizes \( a_l \) and \( b_l \). In the limit \( a_l, b_l \to 0 \) one obtains the rectangular distribution with the bunch length \( 2l_0 \). The explicit evaluation of the expression (33) with the function (42) leads to

\[
F(u, a_l, b_l, l_0) = \frac{i}{2 u l_0} \left( \frac{a_l e^{-i u l_0}}{\sin \varpi_l} - \frac{b_l e^{i u l_0}}{\sinh \varpi_l} \right), \quad \varpi_l \equiv \pi u c_l / 2
\]

It follows from here that if \( \varpi_l \sim 1 \) then \( F \sim (u l_0)^{-1} \) for \( u l_0 \gg 1 \). In the limit \( a_l, b_l \to 0 \) from (43) one obtains the well known form factor for the rectangular distribution:

\[
F_{\text{rect}}(u, l_0) = \frac{\sin u l_0}{u l_0}
\]

As we see the rectangular distribution is a good approximation for (42) if \( a_l, b_l \ll \lambda \). The main contribution to (43) comes from the bunch tails, i.e. from the parts of bunch with large
derivatives of the distribution function. This is the case for the general case of distribution function as well: if \( u_0 \gg 1 \) the main contribution comes from the parts of the bunch where \( df/d(l/l_0) \sim u \) and in this case \( F(u) \sim 1/u, u \to \infty \). This can be generalized for higher derivatives as well: if \( d^i f/d(l/l_0) \ll u, i = 1, \ldots, n-1 \), and \( d^n f/d(l/l_0)^n \sim u \) then \( F(u) \sim (u_0)^{-n} \).

For a relativistic bunch one has \( u \sim 2\pi/\lambda \) for the form factors in \( y \) and \( z \) directions and the conclusion can be formulated as follows. If for the distribution function one has

\[
\lambda \frac{d^i f}{d(l/l_0)^i} < 2\pi, \quad i = 1, \ldots, n-1; \quad \lambda \frac{d^n f}{d(l/l_0)^n} \sim 2\pi
\]  

then the relative contribution of coherent effects into the radiation intensity is proportional to \( N(\lambda/2\pi l_0)^{2n} \) and the radiation can be partially coherent in the case \( \lambda < l_0 \) but \( \lambda > 2\pi l_0 N^{-1/2n} \), with \( l_0 \) being the characteristic bunch size in corresponding direction. In this case the main contribution into the radiation intensity comes from the parts of the bunch with large derivatives of the distribution function in the sense of second condition in (45).

For example in the case of asymmetric distribution (36) when \( ub \) with large derivatives of the distribution function in the sense of second condition in (45). Now the relative contribution of coherent effects is proportional to

\[
\frac{\lambda}{l_0} \sim 1 \quad \text{and} \quad \frac{\lambda}{l_0} < 1 \quad \text{the main contribution comes from the left Gaussian tail with} \quad l < 0.
\]

For this tail \( df/d(l/b_l) \sim u/(pb_l) \sim u, \quad l \sim pb \) and therefore \( F(u) \sim 1/(ul) \). This can be seen directly from the exact relation (37) as well by using the asymptotic formula for the function \( W(t) \).

As an another example of the bunch distribution let us consider the superposition of two Gaussian functions

\[
f(l) = \exp(-l^2/2a_l^2) + \exp(\alpha - (l - l_1)^2/2b_l^2) \sqrt{2\pi / (a_l + e^\alpha b_l)}
\]  

where \( l_1 \) is the displacement of the centers, and \( e^\alpha \) characterizes the relative heights. (As it have been noted in (22) the measured spectrum of transition radiation generated by femtosecond electron bunches agrees much better with a distribution consisting of a short Gaussian core and a Gaussian tail than a Gaussian or rectangular ones (see also (20))). The corresponding form factor have the form

\[
F(u) = \frac{a_l \exp(-u^2 a_l^2/2) + b_l \exp(\alpha + iul_1 - u^2 b_l^2/2)}{a_l + e^\alpha b_l}
\]  

For \( \alpha < 0 \) and \( a_l > b_l \) bunch length and the main contribution into the number of particles are determined by the first summand in (46). However the form factor is determined by the contribution of the second one

\[
F(u) \sim \frac{b_l e^\alpha}{a_l} \exp(iul_1 - u^2 b_l^2/2)
\]  

when \( ub_l \sim 1 \) and \( ua_l \gg 1 \). As in the previous case the main contribution into the bunch form factor is due to the parts where the derivative of the distribution function is large (\( n = 1 \) in (45)). Now the relative contribution of coherent effects is proportional to \( N e^{2\alpha}(b_l/a_l)^2 \) assuming that \( a_l \gg \lambda, b_l \sim \lambda \).

By using the formulas presented above one can obtains the transversal and longitudinal form factors (26) for various combinations of the considered distributions of electrons. We
have to choose \( u = \omega_1/v \) for the longitudinal form factor, \( u = k_y \) and \( u = i\omega \sigma/v \) for the factors corresponding to the \( y \) and \( x \) directions respectively. Note that to obtain the formulas (25), (26) we have assumed that bunch moves in the media with \( \varepsilon = \varepsilon_1 \) and therefore the distribution function in \( x \) direction have to be zero or tend zero sufficiently fast for \( x < -d + a \) (with \( a \) being the surface wave amplitude), when the contribution of bunch tail with \( x < -d + a \) is negligible, \( f(x) \exp(-\omega \sigma x/v) \to 0, x \to -\infty \). This is the case for a Gaussian bunch, and is not the case for (42).

As an example of combination of various distributions let us consider the case when the transverse distribution in the bunch have rectangular form with characteristic sizes \( x_0 \) and \( y_0 \) in the \( x \) and \( y \) directions correspondingly. By using the formulas presented above the bunch form factor can be written in the form

\[
\langle S_N \rangle = N \frac{\sinh(2\omega \sigma x_0/v)}{2\omega \sigma x_0/v} \left\{ 1 + (N - 1) \frac{\tanh(\omega \sigma x_0/v)}{\omega \sigma x_0/v} \frac{\sin^2 k_y y_0}{k_y^2 y_0^2} |h_z|^2 \right\}
\]

where \( k_y = (\omega \sqrt{\varepsilon_1}/c) \sin \theta \sin \varphi \). The longitudinal form factor \( |h_z|^2 = |F(\omega_1/v)|^2 \) and is determined by relations (29), (37), (13), (14), (17) for various distributions. For example in the case of symmetric distribution function \( a_z = b_z \)

\[
|h_z|^2 = \left( \frac{\pi a_z}{2 \varepsilon_0} \right)^2 \left[ \frac{\sin(\omega_1 z_0/v)}{\sinh(\pi \omega_1 a_z/2v)} \right]^2
\]

Taking into account that the surface wave velocity is much less than the light velocity, we arrive at the expression

\[
\frac{\omega_1 a_z}{v} = \frac{2\pi m a_z}{\lambda_0 (1 - \beta \sqrt{\varepsilon_1} \cos \theta)}
\]

with \( \lambda_0 \) being the wavelength of surface wave. In the case of relativistic electrons this expression is equal to \( 2\pi a_z / \lambda \), where \( \lambda \) is the wavelength of the radiation.

4 Conclusion

The present paper is devoted to the radiation from an electron bunch of arbitrary structure flying over the surface wave excited in plane interface between media with different dielectric constants. The radiation intensity at given direction and harmonic is presented in the form of product of the corresponding quantity for a single electron and bunch form factor. In a general case of a monoenergetic bunch this form factor is averaged by the statistical distribution function of electrons.

The various examples of transverse and longitudinal distribution functions are considered, including Gaussian, asymmetric Gaussian, two Gaussian, rectangular distribution with exponential tails and their combinations. For fixed number of particles in the bunch and fixed distance of the bunch’s center from surface wave the radiation intensity exponentially increases with increasing \( b_x \), bunch size in direction perpendicular to the interface. This is because the number of electrons passing close to the surface wave increases. The number of electrons with long distances increases as well. But the contribution of close electrons surpasses the decrease of the intensity due to distant ones.
It is shown that the radiation from a bunch can be partially coherent in the range of wavelengths much shorter than the characteristic longitudinal size of the bunch and the main contribution into the radiation intensity comes from the parts of the bunch with large derivatives of the distribution function in the sense of (45). In this case for short wavelengths the relative contribution of coherent effects decreases as power-law instead of exponentially decreasing. The corresponding conditions for the distribution function are specified (similar conditions for the specific case of diffraction radiation and asymmetric Gaussian distribution were derived in [18]). The coherent effects lead to an essential increase in the intensity of emitted radiation.

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