Generally covariant quantization and the Dirac field

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Abstract

Canonical Hamiltonian field theory in curved spacetime is formulated in a manifestly covariant way. Second quantization is achieved invoking a correspondence principle between the Poisson bracket of classical fields and the commutator of the corresponding quantum operators. The Dirac theory is investigated and it is shown that, in contrast to the case of bosonic fields, in curved spacetime, the field momentum does not coincide with the generators of spacetime translations. The reason is traced back to the presence of second class constraints occurring in Dirac theory. Further, it is shown that the modification of the Dirac Lagrangian by a surface term leads to a momentum transfer between the Dirac field and the gravitational background field, resulting in a theory that is free of constraints, but not manifestly hermitian.

1 Introduction

Quantization in curved spacetime has a long history and there exists an equally long list of problems related to the subject. It is not our intention to give a review of those issues (see, e.g., [1] for a discussion of many of the related problems as well as for a list of relevant references). Here, instead, we wish, in a certain sense, to start from zero and investigate some of the consequences of a straightforward canonical quantization performed in curved spacetime that arise independently of eventual additional problems like those described in [1] and many other articles (e.g., the observer independent concept of a particle). We base our investigation on the principle of relativity, which states that, locally, we cannot distinguish between gravitational and inertial fields. Therefore, if we can put a given special relativistic theory into a manifestly generally covariant form and perform, always in a manifestly covariant way, the second quantization, then the incorporation of gravitational background fields will be trivial, since all our relations will remain identical in form. (This holds as long as we stick to the minimal coupling principle, avoiding thus explicit curvature couplings.) According to this procedure, the special relativistic theory dictates the form of the corresponding theory in gravitational background fields. If the
resulting theory is not free of problems, then there are essentially three possibilities: First, the problems might be solved in some way. This is the point of view we adopt here. In many cases, this involves difficulties concerning the interpretation of certain results, which might be straightforward in flat spacetime, but leads to ambiguities in curved spacetime. Second, the canonical quantization procedure might not be completely correct, and therefore leads to problems which, by chance, do not manifest themselves in the special relativistic limit. We do not investigate this possibility here. Finally, there is the possibility that it is simply not possible to perform second quantization in a manifestly covariant way, and thus, ultimately, in an observer independent way. Otherwise stated, in such a case, the quantization process does not commute with the change of the coordinate system, i.e., we will necessarily have to fix first the coordinate system, and then perform the quantization. Obviously, this is not a very attractive option, since it would essentially mean that at the quantum level, the principle of relativity ceases to be valid.

Assuming that the canonical quantization procedure is valid and can be performed in a generally covariant way, we proceed straightforwardly to the analysis of both bosonic and fermionic theories. We start by the formulation of classical field theory, where we use a direct generalization of the Poisson bracket used by Ozaki [2] in the framework of special relativistic theories, and perform the quantization of the theory by invoking the traditional correspondence principle between Poisson brackets and commutators (or anticommutators). The resulting quantum theory is essentially a curved spacetime generalization of Schwinger’s manifestly Lorentz covariant formulation of special relativistic field theory [3].

Of particular interest is the result that, in Dirac theory, the generators of spacetime translations are not given in terms of the field momentum operator. Thus, in particular, the time component of the field momentum, which is conventionally referred to as field Hamiltonian, does not generate the time evolution of the quantum fields, as is the case in flat spacetime. Instead, it turns out that the field momentum generates a kind of generalized translations which are directly related to the hermitian momentum operators $\tilde{p}_k$ derived in our previous article [4]. The reason is found in the occurrence of second class constraints in Dirac theory.

Further, an alternative way to quantize the Dirac field is presented, where the Lagrangian is modified by a surface term. The surface term is shown to lead to a change in the field momentum, that can be interpreted as a momentum transfer between the Dirac field and the gravitational background field. The resulting theory is free of constraints, and the generators of spacetime translations are now given directly in terms of the field momentum operator. However, the theory is not manifestly hermitian, and the symmetry between the field variables $\psi$ and $\bar{\psi}$ is broken, the latter playing merely the role of a Lagrange multiplier. Most interesting, in the special relativistic limit, both the hermitian and the non-hermitian formulations turn out to be equivalent, which is the reason why the issue of how we deal consistently with the constraints in Dirac theory is usually passed over in the related literature.

The article is organized as follows. In the next section, we explain our notations and give a few definitions that will be used throughout this article. In sections 3 and 4, classical Hamiltonian
field theory for bosonic fields is formulated in a manifestly covariant way, and in section 5 we proceed to second quantization. Finally, in section 6 we discuss the Dirac theory.

2 Preliminaries

Quite generally, our notations are identical to those used in [4]. In particular, we use latin letters from the middle of the alphabet $i, k, l, m \ldots$ to denote spacetime indices (e.g., the spacetime metric $g_{ik}$) and latin letters from the beginning of the alphabet $a, b, c \ldots$ to denote Lorentz vector indices (e.g., the flat tangent space metric $\eta_{ab}$, the Lorentz (or spin) connection $\Gamma_{ab}^i$, the tetrad field $e^i_a$, with $g_{ik} = e^i_a e^k_b \eta_{ab}$). Both spacetime and tangent space are four dimensional. At a later stage, we will also use spinor indices $L, M, N \ldots$ (running from 1 to 4) and write the Dirac spinor as $\psi^M$ (as well as $\bar{\psi}_M$ for the conjugate spinor, transforming with the inverse under a Lorentz gauge transformation).

Let $x \equiv x^i = (x^0, x^1, x^2, x^3)$ be spacetime coordinates such that a hypersurface element can be written as

$$d\sigma_i(x) = \begin{pmatrix} dx^1(\sigma) & dx^2(\sigma) & dx^3(\sigma) \\ dx^0(\sigma) & dx^2(\sigma) & dx^3(\sigma) \\ dx^0(\sigma) & dx^1(\sigma) & dx^3(\sigma) \\ dx^0(\sigma) & dx^1(\sigma) & dx^2(\sigma) \end{pmatrix},$$

where $dx^i(\sigma)$ means that $dx^i$ is restricted to some hypersurface $\sigma$ defined by $\Phi(x) = 0$. (E.g., for the hypersurface $x^0 = const$, we have $dx^0 = 0$ and $d\sigma_i(x) = \delta^0_i d^3x$.)

In the same coordinate system, we define

$$\delta^i(x - y) = \begin{pmatrix} \delta_{x^0 y^0} \delta(x^1 - y^1) & \delta(x^2 - y^2) & \delta(x^3 - y^3) \\ \delta_{x^1 y^1} \delta(x^0 - y^0) & \delta(x^2 - y^2) & \delta(x^3 - y^3) \\ \delta_{x^2 y^2} \delta(x^0 - y^0) & \delta(x^1 - y^1) & \delta(x^3 - y^3) \\ \delta_{x^3 y^3} \delta(x^0 - y^0) & \delta(x^1 - y^1) & \delta(x^2 - y^2) \end{pmatrix}. $$

The transformation behavior for $\delta^i(x - y)$ under a coordinate change is found from the known transformation behavior of $d\sigma_i(x)$ ($\sqrt{-g} d\sigma_i$ is a vector) by requiring $\delta^i(x - y) d\sigma_i$ to transform as scalar under general coordinate transformations. Thus, $\delta^i(x - y)$ transforms as vector density.

Next, consider a spacelike hypersurface $\sigma$ defined by $\Phi(x) = 0$, with the (timelike) normal vector $n_i = \Phi_\cdot$. For convenience, $\Phi(x)$ can be chosen such that $n^2 \equiv n_i n_k g^{ik} = 1$. Then, we have

$$\int_\sigma f(x) \delta^i(x - y) d\sigma_i(x) = f(y)$$

where the integration is carried out over the hypersurface $\sigma$ containing the point $y$. For the specific hypersurface $t = t_0 = const$, we find, e.g.,

$$\int_\sigma f(x) \delta^i(x - y) d\sigma_i(x) = \int_\sigma f(t, \vec{x}) \delta^{(3)}(\vec{x} - \vec{y}) \delta_{t t_0} d^3x$$
where \( t_x = t_0 \) is to be taken on the hypersurface in question. Thus, if \( t_y = t_0 \) (i.e., if \( y \) is on the hypersurface \( t = t_0 \)), the result is simply \( f(y) \), while else, we find zero. Thus, \( \delta^i(x - y) \) can be seen as covariant generalization of the three dimensional delta function.

Note that we have adopted the convention of [4] to use the index \( t \) for the time component of a spacetime vector and greek indices from the middle of the alphabet \( \mu, \nu \ldots \) for the spacelike components, i.e., e.g., \( A^m = (A^t, A^\mu) = (A^t, \vec{A}) \), \( \mu = 1, 2, 3 \). The same letter \( t \) is used for the time coordinate itself, \( x^m = (t, x^\mu) = (t, \vec{x}) \). If the time component of different events \( x^i, y^i \) is needed, we use the obvious notation \( t_x, t_y \) etc. It is important to have in mind that, whenever such a 3 + 1-split is used, it is understood that \( t \) is really a timelike coordinate, i.e., in particular \( g^{tt} > 0 \). This implies, of course, a certain restriction on the coordinate system. In a similar fashion, we use the index 0 for the time component of a Lorentz vector and greek indices from the beginning of the alphabet for the corresponding space components, e.g., \( A^a = (A^0, A^\alpha) \), \( \alpha = 1, 2, 3 \). (The only exception to those conventions was made in the expressions (1) and (2) where the spacetime index of \( x^i \) was given values from 0 to 3 for simplicity.)

Let us recall the following theorem for spacelike hypersurface integrals [3]

\[
0 = \int_\sigma f_i \, d\sigma_k - \int_\sigma f_k \, d\sigma_i. \tag{5}
\]

This results from the fact that the r.h.s. is independent of the choice of the hypersurface \( \sigma \) (see [3]), while for the specific choice \( t = t_0 \), it reduces to a two-dimensional surface integral which vanishes if an appropriate asymptotical behavior of \( f(x) \) is assumed (as will always be done throughout this article). From (5), we can farther deduce the following theorem

\[
\int_\sigma f(x) n_i \, d\sigma_k(x) = \int_\sigma f(x) n_k \, d\sigma_i(x). \tag{6}
\]

Note that in general, \( n_i = n_i(x) \), but we will omit the argument whenever there is no danger of confusion. The above results from \( f(x)n_i = f(x)\Phi_i(x) = (f(x)\Phi(x))_i - f_i\Phi(x) \), with \( \Phi(x) = 0 \) on the hypersurface we integrate over. Thus, we can apply the previous theorem.

In particular, we have

\[
\int_\sigma f(x) \delta^i(x - y) \, d\sigma_i = \int_\sigma f(x) \delta^i(x - y)n_k \, d\sigma_i = \int_\sigma f(x) \delta^i(x - y)n_i n_k \, d\sigma_k \tag{7}
\]

which is equal to \( f(y) \) if \( y \) lies on the hypersurface. (We take the convention that all quantities whose arguments are not written explicitely are to be taken at the point \( x \).) Let us introduce the following definitions

\[
d\sigma = n^i d\sigma_i \quad \tag{8}
\]

\[
\delta_\sigma(x - y) = \delta^i(x - y)n_i \quad \tag{9}
\]

\[
\delta_\sigma^m(x - y) = \delta_\sigma(x - y)n^i = \delta^m(x - y)n_m n^i. \quad \tag{10}
\]
We can thus write
\[
\int_{\sigma} f(x) \delta^i(x - y) d\sigma_i(x) = \int_{\sigma} f(x) \delta^i_\sigma(x - y) d\sigma_i(x)
\]
\[
= \int_{\sigma} f(x) \delta_\sigma(x - y) d\sigma = f(y)
\]
where for the last relation, it is assumed that $y$ lies on the hypersurface. Moreover, we have $\delta^i(x - y)n_i = \delta^i_\sigma(x - y)n_i$. Nevertheless, one should not confuse $\delta^i(x - y)$ which is given explicitly by (2), with $\delta^i_\sigma(x - y)$, which is defined with respect to a specific hypersurface. In particular, for $t = t_0$, we have $\delta^i_\sigma(x - y) = \delta t t_0 \delta^{(3)}(\vec{x} - \vec{y})g^{ij}/g^{tt}$. (The factor involving the metric components stems from the normalization $n^2 = 1$, which for $t = t_0$ (and thus $n_i = (n_t, 0, 0, 0)$ ) leads to $n_t = 1/\sqrt{g^{tt}}$ and $n^i = g^{ij}n_j$.) In particular, in flat spacetime, we see that $\delta^i_\sigma(x - y)$ has only one non-vanishing component, in contrast to (2).

For a functional
\[
F[\varphi, \sigma] = \int_{\sigma} f[\varphi(x)] d\sigma(x)
\]
we define the functional derivative by considering the variation of $F$ induced by a variation $\delta \varphi$, i.e., if
\[
\delta F = \int_{\sigma} \frac{\delta f}{\delta \varphi} \delta \varphi d\sigma,
\]
then we set by definition
\[
\frac{\delta F}{\delta \varphi(x)} \equiv \frac{\delta f}{\delta \varphi}(x).
\]

We are now ready to apply this formalism to classical field theory.

3 Covariant Poisson brackets and field momentum

Consider a Lagrangian density $L = L(\varphi, \varphi, m)$. The corresponding field equations read
\[
\pi^m_{,m} = \frac{\partial L}{\partial \dot{\varphi}},
\]
where
\[
\pi^m \equiv \frac{\partial L}{\partial \dot{\varphi}, m}
\]
will be called the generalized canonical momentum. Note that $\pi^m$ is actually a vector density (i.e., of the form $\sqrt{-g}$ times a vector). If there is more than one field, it is understood that there is an additional (suppressed) index labeling the different fields $\varphi^{(A)}$ and the corresponding momentum $\pi^{(A)m}$, and over which summation is to be carried out in all our expressions.
According to our definition of the functional derivative, we can write
\[
\frac{\delta \varphi(y)}{\delta \varphi(x)} = \delta'(x - y)n_i, \quad \frac{\delta \pi^i(y)}{\delta \pi^m(x)} = \delta^i_m \delta'(x - y)n_i. \tag{17}
\]
Next, we define the classical Poisson brackets by (see \cite{2})
\[
[A, B]_\sigma = \int_\sigma \left( \frac{\delta A}{\delta \varphi(z)} \frac{\delta B}{\delta \pi^m(z)} - \frac{\delta B}{\delta \varphi(z)} \frac{\delta A}{\delta \pi^m(z)} \right) d\sigma^m(z), \tag{18}
\]
where \(d\sigma^m = g^{mi}d\sigma_i\). A straightforward calculation using (17) as well as the theorem (6) leads to the following relations
\[
[\varphi(x), \varphi(y)]_\sigma = [\pi^m(x), \pi^i(y)]_\sigma = 0, \quad [\varphi(x), \pi^i(y)]_\sigma = \delta^i_\sigma(x - y) \tag{19}
\]
The field momentum is defined in the usual way as
\[
P^i_k = \int_\sigma \sqrt{-\gamma} t^i_k d\sigma_i, \tag{20}
\]
where \(t^i_k\) is the canonical stress-energy tensor. We find
\[
P^i_k = \int_\sigma (\pi^i_k \varphi - \delta^i_k \mathcal{L}) d\sigma_i. \tag{21}
\]
Note that \(P^i_k = P^i_k(\sigma)\), i.e., \(P^i_k\) depends in general on the choice of the hypersurface. In fact, it has been shown in \cite{3} that
\[
\frac{\delta}{\delta \sigma} \int_\sigma F^i d\sigma_i = F^i_i, \tag{22}
\]
meaning that the functional is independent of the hypersurface whenever the integrand is divergence free. (For the precise definition of \(\frac{\delta}{\delta \sigma}\), see \cite{3}.) In our case, this means that \(P^i_k\) is independent of \(\sigma\) if \((\sqrt{-g} t^i_k)_i = 0\), which is the case only in flat spacetime. If we restrict ourselves to the hypersurfaces \(t = t_0 = \text{const}\), parameterized by \(t_0\), this means that \(P^i_k\) is independent of \(t_0\) (i.e., it is conserved) whenever \((\sqrt{-g} t^i_k)_i = 0\), a well known result.

Next, we show that (for \(x \in \sigma\))
\[
\frac{\delta P^i_k}{\delta \pi^i(x)} = \varphi_{,k}(x)n_i \tag{22}
\]
and
\[
\frac{\delta P^i_k}{\delta \varphi_{,i}(x)} = -\pi^i_{,k}(x)n_i. \tag{23}
\]
We show this as follows. Starting from the explicit expression (20), we find
\[
\delta P^i_k = \int_\sigma \left[ \varphi_{,k} \delta \pi^i + \delta \varphi_{,k} \pi^i - \delta^i_k \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi - \delta^i_k \frac{\partial \mathcal{L}}{\partial \varphi_{,m}} \delta \varphi_{,m} \right] d\sigma_i
\]
\[
= \int_\sigma \varphi_{,k} \delta \pi^i d\sigma_i - \int_\sigma \pi^i_{,k} \delta \varphi d\sigma_i + \int_\sigma (\delta \varphi_{,i})_k d\sigma_i
\]
\[
- \int_\sigma \left[ \frac{\partial \mathcal{L}}{\partial \varphi} - \left( \frac{\partial \mathcal{L}}{\partial \varphi_{,m}} \right)_m \right] \delta \varphi d\sigma_k - \int_\sigma (\delta \varphi_{,m})_m d\sigma_k.
\]
The third integral cancels with the last one in view of the theorem (5), while the forth integral vanishes on-shell. For the first integral, we write
\[
\int_\sigma \varphi_{,k} \delta \pi^i n^m n_m d\sigma_i = \int_\sigma \varphi_{,k} \delta \pi^i n^m n_i d\sigma_m = \int_\sigma \varphi_{,k} \delta \pi^i n_i d\sigma,
\]
where theorem (6) has been used in the first step. This is now of the form (13) and therefore leads to (22). In the same way, (23) follows from the second integral.

Our expressions are identical in form to those derived by Schwinger [5], equation (2.91), for the quantized field in flat spacetime, where π was defined by π^i n_i (note that Schwinger assumed n_{i,k} = 0).

The Poisson bracket of \(\mathcal{P}_k\) with the canonical field variables can now be evaluated and the result is (assuming that \(x\) lies on the hypersurface \(\sigma\))
\[
\left[\mathcal{P}_k, \varphi(x)\right]_\sigma = -\varphi_{,k}(x), \quad \left[\mathcal{P}_k, \pi^i(x)\right]_\sigma = -\pi^{m}_{i,k}(x)n_m n^i
\]
which can be compared to equation (2.92) of [5].

Finally, we point out that an alternative approach to the definition of the Poisson bracket has been presented in [6].

4 Hamilton equations

It is very important to remark that the equations derived in the previous section should not be confused with the Hamilton equations of motion. The functional derivatives (22) and (23) are obtained from \(\mathcal{P}_k\) in the form (20) as it stands. It is in no way understood that \(\mathcal{P}_k\) should be expressed in terms of \(\pi^i\) and \(\varphi\) only. This cannot be done unambiguously anyway, because obviously, (20) is not a Legendre transformation performed to replace \(\varphi_{,i}\) by \(\pi^i\), or similar. If we write the first term in the form \(\int \pi^i \varphi_{,k} d\sigma_i = \int \pi^i n_i \varphi_{,k} d\sigma\), we see that \(\pi^i\) enters only as \(\pi^i n_i\).

Thus, at most one component of \(\varphi_{,k}\) could be Legendre transformed in that way.

The consistent way to set up a covariant Hamiltonian theory therefore consists in Legendre transforming the variable \(\varphi_{,i} n^i\) and replace it with \(\pi = \pi^i n_i\). Such a formalism has been worked out by Ozaki [2] for the flat space case, and we will adopt it here to curved spacetime. Since the formalism is manifestly covariant, there is not much to adopt. In fact, the relations of Ozaki should remain identical in form. The scope of our presentation is thus basically to check that this is indeed the case. Moreover, we will find a slight difference between our results and those of Ozaki, which is not directly related to curvature effects. Following [2], we define the Hamiltonian by
\[
h = \int_\sigma \sqrt{-g} t^i_k n^k d\sigma_i = \int_\sigma (\pi \varphi_{,k} n^k - \mathcal{L}) d\sigma,
\]
where \( \pi \) is defined by

\[
\pi = \frac{\partial L}{\partial \varphi_i} n_i = \pi^i n_i. \tag{26}
\]

Note that \( h \) should not be confused with the so-called De Donder-Weyl Hamiltonian, \( \pi^i \varphi,_{i} - L \), see, e.g., [7]. (See also the so-called polysymplectic Hamiltonian field theory [8].) The Hamiltonian \( h \) thus arises upon Legendre transforming the variable \( \varphi,_{i} n^i \) and replacing it with \( \pi \). The Hamiltonian \( h \) has first been introduced, in the context of special relativistic quantum theory, by Matthews [9]. For simplicity, we consider the case where (26) is solvable (in terms of \( \varphi,_{m}n^m \)). Using the formalism of the previous sections, the variation of \( h \) leads to the following equations

\[
\frac{\delta h}{\delta \pi} = \varphi,_{i} n^i \tag{27}
\]

\[
\frac{\delta h}{\delta \varphi} = -\left( \pi n^i,_{i} \right) + \pi n^i \left( n_{i,m} - n_{m,i} \right). \tag{28}
\]

To obtain those relations, theorems (5) and (6) have to be used several times, and in a final step the fact that \( n^i,_{k} n_i = -n^i n_{i,k} \) (from \( n^2 = 1 \)). Equations (27) and (28) are the Hamilton equations for classical field theory in a generally covariant form. They differ by the corresponding equations (11) and (12) of [2] by the second term in (28), as well as the part \( -\pi n^i,_{i} \) from the first term, which are absent in [2]. In flat spacetime, and choosing the hypersurface \( t = t_0 \), this term vanishes, but in curved spacetime, or even in flat spacetime, with a different hypersurface, this is not the case anymore. It is not completely clear to us where this difference comes from (see however the remarks on the conservation of \( h \) below).

To get an idea how this works in practice, consider the (real) scalar field Lagrangian

\[
L = \sqrt{-g} \left( \frac{1}{2} \varphi,_{m} \varphi^m - m^2 \varphi^2 \right). \tag{29}
\]

From (26), we find \( \pi = \sqrt{-g} \varphi,_{m} n^m \). In order to express (25) in terms of \( \pi \), we write

\[
\varphi,_{i} = \frac{\pi}{\sqrt{-g}} n_i \left[ \varphi,_{i} - \varphi,_{m} n^m n_i \right], \tag{30}
\]

which corresponds to a split in normal and tangential components. The Hamiltonian is then found in the form

\[
h = \int_{\sigma} \left[ \frac{1}{2} \frac{\pi^2}{\sqrt{-g}} - \frac{1}{2} \sqrt{-g} \varphi,_{m} \varphi^m + \frac{1}{2} \sqrt{-g} \left( \varphi,_{m} n^m \right)^2 + \frac{1}{2} m^2 \varphi^2 \right] d\sigma. \tag{31}
\]

Variation and comparison with (27) and (28) leads to the Hamilton equations

\[
\frac{\pi}{\sqrt{-g}} = \varphi,_{m} n^m \tag{32}
\]
\[
(\sqrt{-g} \, \varphi^m),_m + \sqrt{-g} \, m^2 \varphi - (\sqrt{-g} \, \varphi^i n_i n^m),_m \\
+ \sqrt{-g} \, \varphi^i n^m (n_{i,m} - n_{m,i}) \\
= -(\pi n^i),_i + \pi^i n^m (n_{i,m} - n_{m,i}).
\] (33)

As expected, the first equation leads back to the relation between \( \pi \) and \( \varphi, k n^k \), and inserting this into the second equation leads to the field equation for the scalar field in curved spacetime, 
\((\sqrt{-g} \, \varphi^m),_m + \sqrt{-g} \, m^2 \varphi = 0 \), or simply \( \Box g \varphi + m^2 \varphi = 0 \). Again, the derivation of the second equation involves several applications of the theorems \([5] \) and \([6] \), meaning that the formalism is not really comfortable, even for such simple applications. Greater ease could be provided by the definition of normal and tangential derivatives, as has been done in \([2] \), and observing that certain integrals over tangential divergences lead to two dimensional surface integrals that can be omitted. This has to be done with care, however, especially in curved spacetime. For instance, the following integral over the tangential divergence of a vector field \( A^i, \) \( \int_\sigma (A^i, m n^m n_i) d\sigma \) leads to \( \int_\sigma (A^i n^k, k - A^k n^i, k) d\sigma \), which is not always zero.

It is instructive to see the explicit expression of \( h \) for the hypersurface \( t = t_0 \), i.e., \( n_i = (1/\sqrt{-g} t, 0, 0, 0) \), and \( n^i = g^{ik} n_k = g^{it}/\sqrt{-g} t \), such that \( d\sigma = n^i d\sigma_i = \sqrt{-g} t^2 d^3x \). The scalar field Hamiltonian \((31)\) then takes the form (recall that spacetime indices are denoted \( m = (t, \mu) \), \( \mu = 1, 2, 3 \), where it is assumed that \( t \) is timelike, i.e., \( g^{tt} > 0 \))

\[
h = \int_\sigma \left[ \frac{1}{2} \frac{\sqrt{g}}{\sqrt{-g}} \pi^2 - \frac{1}{2} \sqrt{-g} g^{tt} \varphi, m \varphi^m \\
+ \frac{1}{2} \sqrt{-g} g^{tt} (\varphi, t \sqrt{g} t + \varphi, \mu g^{t \mu} \sqrt{g} t)^2 + \frac{1}{2} m^2 \varphi^2 \sqrt{-g} g^{tt} \right] d^3x.
\] (34)

The momentum in terms of \( \varphi, m \) is of the form

\[
\pi = \sqrt{-g} \varphi, mn^m = \sqrt{-g} \left( \varphi, t \sqrt{g} t + \varphi, \mu g^{t \mu} \sqrt{g} t \right).
\] (35)

Note that, since \( t \) is timelike, the three dimensional tensor \( g^{\mu \nu} \) is negative definite, and therefore \( h \) is positive. In flat spacetime, \( \pi \) reduces to \( \varphi, t \), and the Hamiltonian takes the conventional form \( h = \int_\sigma \frac{1}{2} (\pi^2 - \varphi, \mu \varphi^\mu + m^2 \varphi^2) d^3x \). It is hard to imagine how the Hamiltonian \((31)\) could have been guessed without having at our disposal a manifestly covariant formalism.

The above expressions take a simpler form if we use the Arnowitt-Deser-Misner (ADM) decomposition of the metric tensor, given by \( g_{tt} = N^2 - \tilde{g}_{\mu \nu} N^{\mu \nu} \), \( g_{t \mu} = -N_\mu \) and \( g_{\mu \nu} = -\tilde{g}_{\mu \nu} \). Then we have (for the same hypersurface as above) \( \pi = \sqrt{-g} (\varphi, t - \varphi, \mu N^\mu) \), and

\[
h = \int_\sigma \left[ \frac{1}{2} \frac{\pi^2}{\sqrt{-g}} + \frac{1}{2} \sqrt{-g} g^{\mu \nu} \varphi, \mu \varphi, \nu + \frac{1}{2} \sqrt{-g} m^2 \varphi^2 \right] d^3x,
\]

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where three dimensional indices are raised and lowered with \( \tilde{g}_{\mu\nu} \) and its inverse \( \tilde{g}^{\mu\nu} \).

Finally, let us derive the Poisson bracket of \( \hat{h} \) from (31) with the canonical variables \( \varphi \) and \( \pi \). For this, we need an expression for \( \delta \hat{h}/\delta \pi^i \). Consider some functional \( F = \int_\sigma f \delta \sigma \). According to (13), the variation with respect to \( \pi \) is given by

\[
\delta F = \int_\sigma \frac{\delta f}{\delta \pi} \delta \pi \delta \sigma = \int_\sigma \left( \frac{\delta f}{\delta \pi} \pi^i(n_i) \right) \delta \sigma = \int_\sigma \left( \frac{\delta f}{\delta \pi} \pi^i \right) \delta \sigma = \int_\sigma \frac{\delta f}{\delta \pi} \delta \pi^i \delta \sigma
\]

meaning that we have

\[
\frac{\delta F}{\delta \pi^i} = \frac{\delta F}{\delta \pi} \pi^i. \tag{36}
\]

We now easily find the following relations

\[
[h, \varphi(x)]_\sigma = -\varphi, (x)n^i, \quad [h, \pi(x)]_\sigma = -(\pi(x)n^i), i + \pi^i n^m (n_i,m - n_m,i).
\]

Again, the second relation differs by the terms \( \pi n^i \) and \( \pi^i n^m (n_i,m - n_m,i) \) from the result of [2]. The first relation suggests to interpret \( h \) as the generator of translations along \( n^i \), i.e., normal to the spacelike hypersurface. We will therefore occasionally refer to \( h \) as evolution operator. In particular, for \( t = t_0 \) and assuming a flat spacetime, \( h \) becomes equivalent to the time component of \( P_i \), and thus to the generator of time translations. It is important, however, that the direct interpretation of \( h \) as evolution operator is only valid for expressions that do only depend on the field \( \varphi \). For expressions involving \( \pi \), or \( n^i \) and \( g_{ik} \), the evolution is not directly given in terms of \( h \). One could eventually modify the Hamiltonian in order to get a relation of the form \([\hat{h}, \pi] = -\pi, m n^m \), but this would not really solve the problem for general expressions. Obviously, any Hamiltonian will satisfy \([\hat{h}, n_i] = [\hat{h}, g_{ik}] = 0 \) as long as the metric is treated as a background field, and therefore, it is not possible to construct an evolution operator satisfying \([\hat{h}, f] = -f, i n^i \) for a general expression \( f(\varphi, \pi, n^i, g_{ik}) \).

To illustrate what this means in practice, consider the case of the free electromagnetic field. We have \( \pi^{ik} = \frac{\partial L}{\partial A_{i,k}} \), which is antisymmetric, and thus, we have the primary constraints \( \Phi(x) = \pi^{ik} n_i n_k = 0 \). For consistency of the Hamiltonian theory, we have to require that \( \Phi(x) \) remains zero during the evolution of the system, i.e., \( \delta \Phi = \Phi(x^i + \varepsilon n^i) - \Phi(x^i) = \varepsilon \Phi, i n^i = 0 \) (where \( \varepsilon \) is an infinitesimal parameter), which leads to secondary constraints. Therefore, the secondary constraints are not simply obtained by requiring \([h, \Phi] = 0 \). Instead, in our case, we have \( \Phi(x) = \pi^{ik} n_i n_k = \pi, n^i \), and thus \( \delta \Phi = (\pi^{ik} n^k) n_i + \pi^i (n_i k n^k) \), where \( \pi^{ik} n^k \) is evaluated from equation (37) in the form \( \pi^{ik} n^k = -[h, \pi^i]_\sigma - \pi^i n^l + \pi^k n^m (n_k, m - n_m, k) \). We have checked that this leads to the secondary constraint \( \pi^{ik} n_i = (\sqrt{-g} F^{ik}) n_i = 0 \), which is the covariant expression for Gauss’ law. Any other expression, e.g., \( (\sqrt{-g} F^{ik}) n_i = 0 \) or similar would not be in accordance with the field equations. For the sake of completeness, we give the explicit form of the Maxwell Hamiltonian

\[
h = \int_\sigma \left[ -\frac{1}{2} \pi^i \pi_i + \pi^i A_{m,i} n^m + \frac{1}{4} \sqrt{-g} F^{lm} F_{lm} + \frac{1}{2} \sqrt{-g} F_{im} n^i F^{ml} n_m \right] \delta \sigma.
\]
with \( \pi^i = \pi^i_k n_k = \sqrt{-g} F^{ik} n_k \). The velocities have been eliminated using \( A_{i,k} = -\frac{\pi^i}{\sqrt{-g}} n_k + (A_{i,k} - F_{i|n} n^{|n}_k) \). The relations corresponding to (37) for \( A_i \) and \( \pi^i \) are easily established. They are identical to the scalar field case (just replace \( \phi \) with \( A_i \), \( \pi \) with \( \pi^i \) and \( \pi^k \) with \( \pi^{ik} \)). It is now an easy task to derive the secondary constraints.

Finally, we choose the hypersurface \( t = \text{const} \) and use the ADM parameterization. The Hamiltonian then simplifies to

\[
h = \int_\sigma \left[ \frac{1}{2} \frac{\pi^\mu \pi_\mu}{N^2 \sqrt{g}} + \pi^\mu A_{t,\mu} N^{-2} - \pi^\mu A_{\nu,\mu} N^\nu N^{-2} + \sqrt{g} \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] d^3x,
\]

where indices are raised and lowered with \( \tilde{g}_{\mu\nu} \). For the commutation relations, we remind that \( \delta_\sigma(x-y) = \delta^i(x-y) n_i = \delta^3(x-y) N \), and therefore we have \([A_i(x), \pi^k(y)]_\sigma = \delta^i_k N \delta^{(3)}(x-y) \) at equal times. The conventional choice for the momentum, in the framework of the ADM formalism, differs by a factor \( N \) from ours, namely \( \tilde{\pi}^i = \frac{\delta \hat{E}}{\delta A_{i,1}} \), instead of \( \pi^i = \frac{\delta \hat{E}}{\delta A_{i,k}} n_k \), resulting therefore in a simpler commutation relation, \([A_i(x), \pi^k(y)]_\sigma = \delta^k_i \delta^{(3)}(x-y) \). (In other words, we work with \( \pi^i = \pi^i_k n_k \), while the conventional choice is \( \pi^i = \pi^i_d \).) This is merely a matter of convenience. The conventional choice would appear unnatural when written in an explicitly covariant form.

It has also been claimed in [2] that (in flat spacetime), \( h \) is independent of the choice of the hypersurface \( \sigma \). We do not agree with that statement. According to Schwinger [3], an integral \( \int_\sigma f d\sigma_i \) is independent of \( \sigma \) whenever we have \( f_{;i} = 0 \). For \( h = \int_\sigma t^i_k n^i_k d\sigma_i \) this would be the case for \( (t^i_k n^i_k)_{;i} = 0 \), or, in view of the special relativistic conservation law for the canonical stress-energy tensor, for \( t^i_k n^i_k = 0 \). For a general \( t^i_k \), this can only be the case for \( n^i_k = 0 \), which holds only for specific hypersurfaces.

Similar, in curved spacetime, \( h \) is independent of \( \sigma \) for \( (\sqrt{-g} t^i_k n^i_k)_{;i} = 0 \). In the framework of general relativity, we have shown [10] that the canonical stress-energy tensor \( t^i_k \) is equivalent to the metric (Hilbert) tensor \( T^i_k \), i.e., they differ by a relocalization term and lead both to the same momentum vector \( P_k \). However, in general, \( t^i_k \) does not satisfy \( t^i_k = 0 \) and moreover, it is easy to show that \( t^i_k \) and \( T^i_k \) do not lead to the same \( h \). (Note that the canonical tensor is not a genuine tensor, and should be referred to as pseudo-tensor. However, in order to avoid confusion with the pseudo-stress-energy tensor of the gravitational field itself, we will stick to the expression canonical tensor, which is widely used in literature.) In the specific case of the scalar field, however, \( t^i_k \) incidentally coincides with the Hilbert tensor, and therefore satisfies the general relativity relation \( t^i_{k;\iota} = 0 \), or \((\sqrt{-g} t^i_k)_{;i} + \sqrt{-g} \hat{\Gamma}^k_{li} t^l_i = 0 \), where \( \hat{\Gamma}^k_{li} \) is the Christoffel connection. It is then straightforward to show that \((\sqrt{-g} t^i_k n^i_k)_{;i} = 0 \) (i.e., that \( h \) is independent of \( \sigma \)), whenever we have \( t^i_k (n^i_k + n^i_{k;i}) = 0 \), where we made use of the symmetry of \( t^i_k \). For this to hold in general, we must have \( n^i_k + n^i_{k;i} = 0 \). We thus must have a timelike Killing vector, orthogonal to a spacelike hypersurface, meaning that spacetime must be static. (In general, even if such a Killing vector \( \xi^i \) exists, it will not be normalized. In order to find a conserved quantity, one will have to replace \( h \) by \( \int_\sigma \sqrt{-g} t^i_k \xi^i d\sigma_i \).) Therefore, only for static spacetimes
can we choose $n_i$ (or rather $\xi_i$) in a way that $h$ is conserved. (Recall that this result is limited to the case where $t^{ik}_{i} = 0$.) In no way, however, is $h$ generally independent of the choice of $\sigma$.

In flat spacetime, for the hypersurface $t = t_0$, we have $h = P_t$, which justifies the conventional interpretation of $P_t$ as field Hamiltonian in special relativistic field theory. Nevertheless, they are fundamentally different quantities from a theoretical standpoint, and according to the above considerations, only $h$ should be referred to with the name Hamiltonian, since it is the quantity that, when expressed in terms of $\varphi$ and $\pi$, leads upon variation to the canonical Hamilton equations.

5 Second quantization

One of the major successes of the canonical Hamiltonian formulation of mechanics and field theory is its direct relation to quantum theory by means of a suitable correspondence principle. On the other hand, a special relativistic theory can unambiguously be written in a manifestly generally covariant form, if we exclude direct curvature couplings. In the presence of bosonic fields, this results in the replacement of the non-dynamical Minkowski metric by a spacetime dependent metric tensor $g_{ik}$. According to the principle of equivalence, this tensor is interpreted as dynamical gravitational field. In the presence of spinor fields, we need a tetrad field $e_i^a$ ($a, b, \ldots$ denote Lorentz vector indices) and the metric arises as derived quantity $g_{ik} = e_i^a e_k^b \eta_{ab}$. The theory is invariant under local Lorentz gauge transformations (in addition to general coordinate transformations) $e_i^a \rightarrow \Lambda^a_b(x) e_i^b$, and the special relativistic limit is given by $e_i^a = \delta_i^a$, i.e., when we have identification of the tangent Lorentz space with the spacetime manifold. Obviously, in that limit, the residual symmetry is the global Lorentz group, acting at the same time in tangent space and in spacetime, because in order to have $\delta_i^a = \tilde{e}_i^a(\tilde{x}) = \Lambda^a_b(x) e^b_k(\tilde{x}) \frac{\partial x^k}{\partial \tilde{x}^i}$, where $e_i^a = \delta_i^a$, $\Lambda^a_b$ must be the inverse of $\frac{\partial x^k}{\partial \tilde{x}^i}$, meaning in particular that the coordinate transformation $\frac{\partial x^k}{\partial \tilde{x}^i}$ has to be a Lorentz transformation. The important thing is that there is again only one covariant form to a given special relativistic theory. (Note, however, that this one to one correspondence does not hold for gravitational theories which involve additional variables, like the (independent) Lorentz connection in Poincaré gauge theory. In such a case additional tensors arise (e.g., torsion), that can be coupled at will to the matter fields, without destroying the general covariance. It turns out, however, that for the canonical quantization process, only the metric (or tetrad) structure is of importance.)

In summary, assuming the ideal case, to each classical special relativistic theory corresponds exactly one second quantized theory (correspondence principle), and to each special relativistic theory corresponds exactly one (eventually up to additional, non-metric gravitational tensor fields) generally covariant theory. It is therefore clear that, if there exists a manifestly covariant second quantized theory, then it can only be obtained by applying the correspondence principle to the special relativistic theory written in generally covariant form. If this does not lead to a consistent theory, this means that the quantization process cannot be done in a covariant, and
thus ultimately, in an observer independent way. One would then have to fix first the coordinate system, and perform the quantization afterwards. It is clear that this would ultimately mean that the principle of equivalence does not hold on a quantum level. Here, we assume that the second quantization can be performed in a manifestly covariant way. Taking into account gravitational effects (from the background curvature) is then, at least from a formal point of view, a trivial issue.

Our starting point is the Poisson bracket (18), and second quantization is achieved by replacing the field \( \varphi \) and the canonical momentum \( \pi \) by operators acting in a Hilbert space and satisfying the relations (19), where the Poisson bracket of the classical fields is replaced by the commutator of the corresponding operators,

\[
[A, B]_\sigma \rightarrow \frac{1}{i}[A, B] = \frac{1}{i}(AB - BA).
\]

Explicitly, the canonical commutation relations (CCR’s) read

\[
i[\varphi(x), \varphi(y)] = i[\pi^m(x), \pi^l(y)] = 0, \quad i[\pi^i(x), \varphi(y)] = \delta^i_\sigma(x - y),
\]

where \( x \) and \( y \) are on the hypersurface \( \sigma \). In particular, \( x - y \) is thus spacelike. (Recall that the commutation relations between fields at timelike separations can only be obtained with the help of the field equations for the specific theory under investigation (propagator), and explicit expressions are only available for free fields (interaction representation), see [3]. The direct application of Schwinger’s formalism to curved spacetime can only be achieved if we expand the metric around a flat background, \( g_{ik} = \eta_{ik} + h_{ik} \). Not only does this mean giving up general covariance, but moreover, in view of the inverse of \( g_{ik} \) occurring in standard matter Lagrangians, the explicit expressions in terms of \( h_{ik} \) contain infinite series already at this early stage of the theory.) The above CCR’s are identical to those used by Schwinger for the special relativistic theory [3] and are equivalent to those used in [12] on a curved background.

For the specific hypersurface \( t = t_0 \), we find from (2) and (10) that the last relation reduces to

\[
i[\pi^i(x), \varphi(y)] = \delta^{(3)}(\vec{x} - \vec{y})\delta_{t_xt_y},
\]

and \( i[\pi^m(x), \varphi(y)] = 0 \). Since both points are assumed to lie on \( \sigma \) anyway, we can omit the factor \( \delta_{t_xt_y} \), and we find the conventional equal time commutation relation at \( t_0 \) between the field and its canonical momentum. We see that, although (39) involves four components \( \pi^i \), there is only one component that plays the role of the canonically conjugate field, namely the component normal to the hypersurface, \( \pi = \pi^n n_i \). In terms of \( \pi \), we have the following CCR’s

\[
i[\varphi(x), \varphi(y)] = i[\pi(x), \pi(y)] = 0, \quad i[\pi(x), \varphi(y)] = \delta_{\sigma}(x - y),
\]

which is the form used in [12].
Let us apply this to the scalar field Lagrangian (29). The field momentum (20) is given by

\[ P_k = \int_\sigma (\pi^i \varphi_{,k} - \delta^i_k \mathcal{L}) d\sigma_i \]

\[ = \int_\sigma \pi^i \varphi_{,k} d\sigma_i - \frac{1}{2} \int_\sigma \sqrt{-g} \left( \varphi_{,m} \varphi^{,m} - m^2 \varphi^2 \right) d\sigma_k, \]  

(42)

where \( \pi^i = \sqrt{-g} \varphi^i \). It is important to recall that \( P_k \) has nothing to do with a Hamiltonian. Although \( P_t \) is conventionally referred to as field Hamiltonian, as we have seen in the previous section, \( P_k \) is not a Legendre transformation and is not a quantity which is used to determine equations of motion. In particular, it is not understood that \( \varphi_{,m} \) should be replaced by \( \pi^m \) in (12). From the form (12) and the CCR’s, we can deduce the commutation relations of \( P_k \) with \( \varphi \) and \( \pi^i \). Note that, for the evaluation of \( [P_k, \pi^i(x)] \), we have to write, e.g., for the first term from (12) \[ \int \pi^j(y) \varphi_{,k}(y), \pi^i(x) d\sigma_i(y) = i \int \pi^j(y) (\delta^i_k(y - x)) \, d\sigma_i(y) = -i \int \pi^j_k(y) \delta^i_k(y - x) d\sigma_i(y) + i \int (\pi^j(y) \delta^i_k(y - x)) \, d\sigma_i(y) \] (derivatives acting on \( y \)), where the second term cannot simply be omitted. (To our knowledge, there is no generalization of the rule \( \delta'(x-y) f(x) = -f'(x) \delta(x-y) \) for our \( \delta \) functions.) Instead, it cancels with an opposite term coming from the second term in (42), as can be shown using theorem (5) and the definition of \( \pi^m \). The final relations are found in the form

\[ i [P_k, \varphi(x)] = \varphi_{,k}(x), \quad i [P_k, \pi^i(x)] = \pi^m_k(x) n_m n^i, \]  

(43)

which are in direct correspondence to (24). It is understood that \( x \) lies on the same hypersurface \( \sigma \) (with normal vector \( n_i \)) that has been used in the definition of \( P_k \). Note that (43) are on-shell relations. In terms of \( \pi = \pi^i n_i \), the above relations read

\[ i [P_k, \varphi(x)] = \varphi_{,k}(x), \quad i [P_k, \pi(x)] = \pi^m_k(x) n_m, \]  

(44)

From its action on the field \( \varphi \), we can identify, as expected, \( P_k \) with the generator of spacetime translations. The same equations have been derived (assuming flat spacetime and \( n_{i,k} = 0 \)) by Schwinger [5], see equation (2.92).

On the other hand, for the evolution operator \( h (31) \), we find

\[ i [h, \varphi] = \frac{\pi}{\sqrt{-g}}, \]

\[ i [h, \pi] = - (\sqrt{-g} \varphi^m),_m - \sqrt{-g} m^2 \varphi \]

\[ + (\sqrt{-g} \varphi^i n^m),_m - \sqrt{-g} \varphi^i n^m (n_{i,m} - n_{m,i}). \]  

(45)

If we assume the correspondence principle and write the equations (37) in the form

\[ i [h, \varphi(x)] = \varphi_{,i}(x) n^i, \quad i [h, \pi(x)] = (\pi(x)n^i),_i - \pi^i n^m (n_{i,m} - n_{m,i}), \]  

(46)

then (45) leads consistently to the field equations. Those equations have been derived for the first time in [9] for flat spacetime (and \( n_{i,k} = 0 \)). We see that, thanks to the manifestly covariant formalism, the generalization to curved spacetime does not present any difficulties, at least on a formal level.
6 Dirac field

6.1 Classical theory

The Dirac equation in presence of gravitational fields (see e.g., [13]) has been studied in our previous article [4] with the focus on the relation between the Dirac Hamiltonian and the time component of the field momentum. Here, we will extend this analysis to the case of the quantized theory.

Recall the Dirac Lagrangian

$$L = e \left[ \frac{i}{2} (\bar{\psi} \gamma^i D_i \psi - D_i \bar{\psi} \gamma^i \psi) - m \bar{\psi} \psi \right], \quad (47)$$

with $e = \text{det} e^a_i = \sqrt{-g}$, $D_i \psi = \partial_i \psi - \frac{i}{4} \Gamma_{ab}^i \sigma_{ab} \psi$ and $\gamma^i = e^i_a \gamma^a$, where $\gamma^a$ are the constant Dirac matrices and $\sigma_{ab}$ the Lorentz generators. Recall that we use latin indices from the begin of the alphabet ($a, b, c \ldots$) for tangent space quantities and from the middle ($i, j, k \ldots$) for curved spacetime quantities. The Lorentz connection $\Gamma_{ab}^i$ ($\Gamma_{ab}^i = -\Gamma_{ba}^i$) and the tetrad $e^i_a$ transform under an (infinitesimal) local Lorentz transformation $\Lambda^a_b$ ($\Lambda^a_b = \delta^a_b + \varepsilon^a_b$) according to $\delta \Gamma_{ab}^i = -\partial_i \varepsilon_{ab} - \Gamma_{ci}^a e^c_b - \Gamma_{ci}^b e^c_a$, $\delta e^a_i = \varepsilon^a_b e^b_i$. The Lagrangian (47) is invariant under such transformations if $\psi$ undergoes the gauge transformation $\psi \rightarrow \exp \left[ (\frac{-i}{4}) \varepsilon_{ab} \sigma_{ab} \right] \psi$. The Dirac equation is derived in the form

$$i \gamma^i \nabla_i \psi = m \psi, \quad (48)$$

where $\nabla_i \psi = \partial_i \psi - \frac{i}{4} \tilde{\Gamma}_{ab}^i \sigma_{ab} \psi$. The connection $\tilde{\Gamma}_{ab}^i$ is the connection that effectively couples to the spinor field. If we work in the framework of general relativity, then $\Gamma_{ab}^i = \Gamma_{ab}^i(e)$, i.e., the connection is not an independent field and can be expressed in terms of the tetrad and its derivatives. In that case, $\tilde{\Gamma}_{ab}^i = \Gamma_{ab}^i$ (and $D_i = \nabla_i$). If the connection is considered to be an independent field (Poincaré gauge theory), then we have $\Gamma_{ab}^i = \Gamma_{ab}^i(e) + K^a_{ci}$, where $K^a_{ci}$ is the contortion tensor. However, only the totally antisymmetric part of $K^a_{ci}$ remains in the field equations. Therefore, the effective connection is given by $\tilde{\Gamma}_{ab}^i = \Gamma_{ab}^i + \tilde{K}^a_{ci}$, where $\tilde{K}^a_{ci} = e_{ci} K^{abc}$. Just as was the case in [4], those differences between general relativity and Poincaré gauge theory are not related to our specific discussion, and the use of the symbols $\nabla_i$ and $\tilde{\Gamma}_{ab}^i$ is a convenient way of treating both cases at the same time.

In Schroedinger form, the Dirac equation can be written as

$$H \psi = i \partial_t \psi \quad (49)$$

Using again greek indices $\alpha, \beta \ldots$ to denote the spatial part of $a, b \ldots$ (i.e., $a = (0, \alpha)$), and indices $\mu, \nu \ldots$ for the spatial part of the spacetime indices $i, j, k \ldots$ and $t$ for the time component, (e.g., $m = (t, \mu)$), we find

$$H = \frac{1}{g^{tt}} (\gamma^t m - i \gamma^t \gamma^\mu \nabla_\mu) - \frac{1}{4} \tilde{\Gamma}_{ab}^i \sigma_{ab}. \quad (50)$$
In the flat limit ($\bar{\Gamma}_{ab}^i = 0, \ e_a^i = \delta_a^i$), this reduces to the well known expression $H = \gamma^0 m - i\gamma^0 \gamma^\mu \partial_\mu = \beta m + \vec{\alpha} \cdot \vec{p}$.

Further, we define a manifestly covariant inner product in Dirac space 

$$
(\psi_1, \psi_2) = \int_\sigma \sqrt{-g} \bar{\psi}_1 \gamma^i \psi_2 d\sigma_i,
$$

(51)

where the integration is performed over a spacelike hypersurface. We have shown in [4] that $H$ is not in general hermitian with respect to this scalar product. Thus, by the (Dirac space) operator identity $H = i\partial_t$, the time evolution operator $i\partial_t$ is not hermitian. A similar situation holds for the translation operators $i\partial_\mu$, and altogether, it was shown that the following operator $(m = 0, 1, 2, 3)$

$$
\tilde{p}_m = i\partial_m + \frac{i}{2} \partial_m \ln \sqrt{-gg^{tt}},
$$

(52)

is hermitian. (Whenever expressions are used that are not manifestly covariant, as is the case with (52), it is understood that they refer to the hypersurface $t = t_0$, which has been used exclusively in [4]. The reason is that an explicit 3 + 1 split has already been performed by writing down equation (49). This can be avoided by writing $H_\sigma = im^\sigma \partial_m \psi$ instead, and then repeat the steps performed in [4]. The fact remains that the operator $i\partial_m$ is not hermitian. Moreover, in order to find the explicit form (52), it is assumed that $e_\alpha^i = 0$ for $\alpha = 1, 2, 3$, which can always be achieved by a suitable Lorentz rotation, since $e_0^i$ is a timelike Lorentz vector.) Moreover, it was shown that the expectation values of (52) are given by the field momentum, i.e.,

$$
P_k = (\psi, \tilde{p}_k \psi),
$$

(53)

where

$$
P_k = \int_\sigma \sqrt{-g} \bar{\psi}_k d\sigma_i = \int_\sigma \left[ \frac{\partial \mathcal{L}}{\partial (\partial_i \bar{\psi})} \partial_k \psi + \partial_k \bar{\psi} \frac{\partial \mathcal{L}}{\partial (\partial_i \psi)} - \delta_k^i \mathcal{L} \right] d\sigma_i.
$$

(54)

In particular, if $P_k$ is to play the role of the generator of spacetime translations in the quantum theory, then it seems strange that this operator does not coincide with the expectation value of the momentum $i\partial_k$ of the corresponding classical theory, as is the case in flat space (recall, e.g., the relation $P_t = \mathcal{H} = (\psi, H \psi) = \int \psi^\dagger H \psi d^3x$ of the special relativistic theory). Therefore, it is of interest to investigate the situation in the second quantized theory. Surprisingly, it will turn out that, in contrast to the case of the boson field (see equation (43)), the field momentum $P_k$ does not coincide with the generators of spacetime translations in curved spacetime.

6.2 Fermion Poisson Bracket and second quantization

The adaptation of the Poisson bracket (18) to the case of spinor fields is straightforward. Since we expect again a correspondence principle to lead to second quantization, and since fermions are quantized with anticommutators, the Poisson bracket for fermions, $\{A, B\}_\sigma$ should be symmetric
in A and B. Therefore, we simply replace the minus sign with a plus sign in (18) and, from a formal point of view, we are done. (The cases for fermion and boson brackets can also be formally treated at the same time, see, e.g., [2], where you can also find the generalized Jacobi identity for such brackets.) The larger problem concerns the correct choice of the canonical variables. Formally, we have in (17) two independent fields \( \psi \) and \( \bar{\psi} \), related by hermitian conjugation, giving us the two corresponding momentum variables \( \pi^i = \frac{i}{2} \sqrt{-g} \bar{\psi} \gamma^i \) and \( \bar{\pi}^i = -\frac{i}{2} \sqrt{-g} \gamma^i \psi \) (recall that only the component in \( n_i \) direction is the physical canonical momentum). Obviously, the variables \( (\psi, \bar{\psi}, \pi, \bar{\pi}) \) are not independent, and we have two constraints in the theory, which are easily shown to be second class. In order to get a consistent theory with such constraints, it is necessary to modify the classical Poisson bracket (in order to exclude non-physical degrees of freedom) before passing over to the second quantization (see Dirac [11] for details). Fortunately, since the quantization of the special relativistic Dirac theory is well known, we can directly inspire ourselves from the corresponding theory and put it into a generally covariant form. More details on the problems arising from the second class constraints will be given in section 6.4.

The consistent way is to use \( (\psi, \pi) \) as canonically conjugate variables, with \( \pi^i = \frac{i}{2} \sqrt{-g} \bar{\psi} \gamma^i \), and thus to define the following Poisson bracket

\[
\{A, B\}_\sigma = \int_\sigma \left( \frac{\delta A}{\delta \psi^M(z)} \frac{\delta B}{\delta \pi^M(z)} + \frac{\delta A}{\delta \pi^m(z)} \frac{\delta B}{\delta \psi^m(z)} \right) \, d\sigma^m(z),
\]

(55)

where we use capital letters \( K, L, M, N \ldots \) to denote spinor indices. More precisely, upper indices are used for spinors transforming like \( \psi \) under Lorentz gauge transformations, \( \psi \rightarrow \Lambda \psi \), while lower indices are used for quantities transforming with the inverse, like \( \bar{\psi} \rightarrow \bar{\psi} \Lambda^{-1} \). Thus, e.g., the Dirac matrices transform as \( \gamma^a \rightarrow \Lambda \gamma^b \Lambda^{-1} \Lambda^b \equiv \gamma^a \), where \( \Lambda^a_b \) is the corresponding Lorentz transformation in the vector representation. Thus, we write \( (\gamma^a)^M_N \). Further, if we denote with an upper dotted index a spinor transforming like \( \psi^\dagger \rightarrow \psi^\dagger \Lambda^\dagger \), and with a dotted lower index spinors transforming with the inverse \( (\Lambda^\dagger)^{-1} \), then the spinor metric \( \gamma^0 \) (used in \( \bar{\psi} \rightarrow \bar{\psi} \Lambda^{-1} \)) takes two lower indices, a dotted and an undotted, \( (\gamma^0)^M_N \) and is invariant under \( (\Lambda^\dagger)^{-1} \gamma^0 \Lambda^{-1} \). Note that \( (\gamma^0)^M_N \), although numerically identical with the zeroth component of \( \gamma^a \) is not a component of a Lorentz vector. In summary, the situation is identical to the special relativistic case, only that the Lorentz coordinate transformations are replaced by local Lorentz gauge transformations (not acting on the argument of \( \psi(x) \)).

The following Poisson brackets are easily derived (for \( x, y \) on \( \sigma \))

\[
\{\psi^N(x), \psi^M(y)\}_\sigma = 0, \quad \{\pi^i_N(x), \pi^i_M(y)\}_\sigma = 0, \\
\{\psi^N(x), \pi^i_M(y)\}_\sigma = \delta^N_M \delta^i_\sigma (x - y).
\]

(56)

Those relations are identical to those used (in the quantum theory) by Schwinger [2]. Note that the asymmetry between \( \psi \) and \( \bar{\psi} \) is only apparent. The corresponding relation for \( \bar{\psi} \) and
\[ \pi = \gamma^0 \pi^\dagger \] is easily found by hermitian conjugation of (56). In other words, we could equally well start with the canonical pair \((\tilde{\psi}, \pi)\) and construct the Poisson bracket accordingly.

Let us note that \(\pi^i = i\sqrt{-g} \tilde{\psi}\gamma^i\) can be inverted to \(\tilde{\psi} = -i(\sqrt{-g})^{-1}\pi^i \gamma^i \ni\), where the relation \(\gamma^i \ni \gamma^i \ni = 1\) has been used.

For the momentum vector \(\mathcal{P}_k = \int_\sigma \sqrt{-g} t^i_k d\sigma_i\), we find

\[
\mathcal{P}_k = \int_\sigma \frac{i}{2} \sqrt{-g} \tilde{\psi} \gamma^i \psi d\sigma_i - \int_\sigma \frac{i}{2} \sqrt{-g} \tilde{\psi}_k \gamma^i \psi d\sigma_i,
\]

where we have used the fact that \(\mathcal{L} = 0\) on shell. (Indeed, it is not hard to show that we can write \(\mathcal{L} = \frac{1}{2}(\tilde{\psi} \bar{\partial} \mathcal{L} + \bar{\partial} \mathcal{L} \psi)\), where the variations \(\delta \mathcal{L}/\delta \psi\) and \(\delta \mathcal{L}/\delta \tilde{\psi}\) vanish in virtue of the field equations. Thus, on the set of solutions of the field equations, \(\mathcal{L}\) is zero.) Next, we perform a partial integration of the second term, use the expression for \(\tilde{\psi}\) in terms of \(\pi^i\), as well as the fact that \(\int_\sigma (\sqrt{-g} \tilde{\psi} \gamma^i \psi) d\sigma_i = 0\), as is shown by using (5) and the on-shell relation \((\sqrt{-g} \tilde{\psi} \gamma^i \psi)_i = 0\) (see [13]). The result is

\[
\mathcal{P}_k = \int_\sigma \pi^i \psi_k d\sigma_i + \frac{1}{2} \int_\sigma \frac{1}{\sqrt{-g}} \pi^k \gamma^i \ni_i (\sqrt{-g} \gamma^i) \psi d\sigma_i.
\]

Performing another partial integration, we can also write

\[
\mathcal{P}_k = -\int_\sigma \pi^i \psi_k d\sigma_i + \frac{1}{2} \int_\sigma \frac{1}{\sqrt{-g}} \pi^k \gamma^i \ni_i (\sqrt{-g} \gamma^i) \psi d\sigma_i.
\]

Those expressions can be used to evaluate \(\delta \mathcal{P}_k/\delta \pi^i\) and \(\delta \mathcal{P}_k/\delta \psi\) respectively, which are needed to find the Poisson brackets between \(\mathcal{P}_k\) and the canonical field variables. The result is

\[
[p_k, \psi]_\sigma = -\psi_k - \frac{1}{2} \frac{1}{\sqrt{-g}} \gamma^m \ni_m (\sqrt{-g} \gamma^i) \psi
\]

\[
[p_k, \pi^i]_\sigma = -\pi^i_k \psi \psi^i + \frac{1}{2} \pi^k \ni_k (\gamma^m \psi \psi^m) (\sqrt{-g}) \psi \psi^i \psi^l.
\]

Note that at the left hand side, the Poisson bracket (18) has been used. This is because \(\mathcal{P}_k\) is bilinear in the spinor fields and as such, of bosonic nature. Comparing these results with (21), we see that they both differ in the appearance of an additional term. We will discuss this term shortly. For the moment, let us just remark that it vanishes for flat spacetime.

We now proceed to second quantization. We use again the correspondence principle

\[
\{A, B\}_\sigma \rightarrow \frac{1}{i} \{A, B\} = \frac{1}{i} (AB + BA),
\]

leading to the canonical anticommutation relations

\[
i\{\psi^N(x), \psi^M(y)\} = 0, \quad i\{\pi^i_N(x), \pi^k_M(y)\} = 0,
\]

\[
i\{\psi^N(x), \pi^i_N(y)\} = -\delta^N_M \delta^i_\sigma (x - y),
\]

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and from the expressions (58) and (59), we derive

\[ [P_k, \psi] = -i\psi_k - \frac{i}{2\sqrt{-g}} \gamma^m n_m (\sqrt{-g} \gamma^i)_{k n_i} \psi \]  
(64)

\[ [P_k, \pi^l] = -i\pi^l_{n i} n^l + \frac{i}{2} \pi^k n_k (\gamma^m n_m) (\sqrt{-g} \gamma^i)_{k n_i} \psi \]  
(65)

where commutators are used at the left hand side. Again, these relations differ from their bosonic counterparts (44) by the additional term at the right hand side.

Quite generally, for an operator \( \mathcal{O} \) that corresponds to the expectation value of a Dirac space operator \( \mathcal{O} = (\psi, O\psi) = -i \int_\sigma \pi O \psi d\sigma \) (see (51)), we obtain

\[ [\mathcal{O}, \psi] = -O\psi. \]  
(66)

In particular, for \( P_k \), we should thus have, according to (53)

\[ [P_k, \psi] = -\tilde{p}_k \psi, \]  
(67)

where \( \tilde{p}_k \) is the hermitian momentum operator. Indeed, for the hypersurface \( t = t_0 \) and assuming a gauge \( e^\alpha_\tau = 0, \alpha = 1, 2, 3 \), we find from (64)

\[ [P_k, \psi] = -i \left( \partial_k + \frac{1}{2} (\partial_k \ln \sqrt{-g} g^{tt}) \right) \psi, \]  
(68)

which is in perfect agreement with (52).

Similar relations are obtained for the evolution operator \( h = \int_\sigma \sqrt{-g} \pi^l n^l d\sigma \) defined in (25). The same additional term will appear in \([h, \psi]\) and \([h, \pi]\), both in the classical and in the quantum case. The analysis is straightforward and does not lead to new insight.

We conclude that \( P_k \) is not the generator of spacetime translations. (Neither is \( h \) the generator of translations along \( n_i \).) This is in direct correspondence with our result of [4], namely that \( P_k \) is not the expectation value of the operator \( p_k = i\partial_k \), but rather of the hermitian operator \( \tilde{p}_k \). Also, from (66) it is clear what operator corresponds to the generators of translations: It is the expectation value of \( p_k \). Thus, if we define \( \mathcal{P}_k^{(1)} = (\psi, p_k\psi) \), then we have \([\mathcal{P}_k^{(1)}, \psi] = -p_k \psi = -i\partial_k \psi \). There is only one problem: the operator \( p_k \) is not hermitian. Therefore, there is a second operator \( \mathcal{P}_k^{(2)} = (p_k \psi, \psi) \), which could equally well be used to define the expectation value of \( p_k \). How can we decide which of those operators, if any, corresponds to the physical field momentum? The reason for those ambiguities will become clear in the next section.

Finally, let us remark that the results are not an artifact of the quantization process. The same deviation from the bosonic case has been obtained in the classical case, namely equations (60) and (61). The reason is also obvious: It is the result of the fact that, due to the constraints in the theory, we were forced to deviate slightly from the canonical procedure. Thus, ultimately,
the problem originates from the fact that the Lagrangian theory contains two independent field variables $\psi$ and $\bar{\psi}$, but the corresponding Hamiltonian theory is constructed only from one canonically conjugate pair of variables, $\psi$ and $\pi^i$. This is not avoidable (anything else leads to inconsistencies), but it is also not unproblematic and leads to problems such as the above, concerning the interpretation of the field momentum. This will become very clear in the next section.

Let us also indicate that we can construct a Lorentz generator in the form

$$\Sigma_{ab} = -\frac{i}{4} \int \pi^i \sigma_{ab} \psi d\sigma_i,$$

with $\sigma_{ab} = \frac{i}{2} [\gamma_a, \gamma_b]$, which generates a Lorentz gauge transformation on the field $\psi$, i.e., we have

$$[\Sigma_{ab}, \psi]_{\sigma} = -\frac{i}{4} \sigma_{ab} \psi,$$

in consistence with the invariance of (47) under $\delta \psi = -\frac{i}{4} \varepsilon^{ab} \sigma_{ab} \psi$ with arbitrary (antisymmetric) gauge parameters $\varepsilon^{ab} = \varepsilon^{ab}(x)$.

Obviously, the same construction of $\Sigma_{ab}$ can be used in the quantum theory. Note that in contrast to the special relativistic theory, the above generator does not contain the orbital term $\sim \int \pi(x^k \partial_i - x^i \partial^k)\psi d^3x$. This is because in generally covariant theories, the spinor field $\psi$ transforms us a scalar under coordinate transformations, its spinor nature being exclusively related to the internal Lorentz transformations. In other words, the above operator generates gauge transformations, and not Lorentz coordinate transformations. In special relativity, both transformations are directly related, since the spinor transformation acts simultaneously on the argument of $\psi$, i.e., $\psi'(x') = S\psi(x)$, while in our case, we have $\psi'(x) = S(x)\psi(x)$, which, although a local transformation, acts at a given point $x$. (As is well known, the special relativistic spinor (Lorentz) transformations of $\psi$ cannot be generalized to the full diffeomorphism group, and therefore the only way is to treat $\psi$ as an invariant under those transformations, and to replace the spinor transformations by Lorentz gauge transformations, unrelated to coordinate transformations.) Note, however, that $\Sigma_{ab}$ is not a conserved quantity. A covariant conservation law for the angular momentum, based on the invariance of $\mathcal{L}$ under Lorentz gauge transformations, has been given in [10]. The orbital part is thereby expressed in terms of the Hilbert stress-energy tensor. This does still not lead, in the absence of gravitational fields, to the special relativistic expression. In fact, in order to obtain a generalization of the special relativistic conservation law, one will have to restrict the coordinate transformation group to the Lorentz group, such that, for the remaining class of coordinate systems, one can establish a one-to-one correspondence between internal Lorentz gauge transformations and the residual Lorentz coordinate transformations. Then, one can construct a complete set of 10 Poincaré generators, see, e.g., [14]. Such a construction, based on an explicit symmetry breaking, is not needed for our purpose which focuses on the momentum operator, and we will continue to treat the coordinate and gauge transformations separately.
6.3 Relocalization of field momentum

Consider the Dirac Lagrangian (47) in the following form

\[ L = \sqrt{-g} \left[ \frac{i}{2} (\bar{\psi} \gamma^i \partial_i \psi - \partial_i \bar{\psi} \gamma^i \psi) \right] + L_{\text{int}}, \tag{69} \]

where \( L_{\text{int}} \) contains the interaction terms between the spin connection and the spinor fields, as well as the mass term. This can equivalently be written in the form

\[ L = \sqrt{-g} \frac{i}{2} \bar{\psi} \gamma^i \partial_i \psi + i \frac{\bar{\psi}}{2} \left( \gamma^i \sqrt{-g} \right) \partial_i \psi - \left( \sqrt{-g} \bar{\psi} \gamma^i \psi \right) \partial_i + L_{\text{int}}, \tag{70} \]

Omitting the surface term, we find the following Lagrangian

\[ L^{(1)} = \sqrt{-g} \frac{i}{2} \bar{\psi} \gamma^i \partial_i \psi + i \frac{\bar{\psi}}{2} \left( \gamma^i \sqrt{-g} \right) \partial_i \psi + L_{\text{int}}, \tag{71} \]

which leads upon variation with respect to \( \psi \) and \( \bar{\psi} \) to the same equations as (69). Note that the role of the field variable \( \bar{\psi} \) has essentially been reduced to that of a Lagrange multiplier. Further, we recall that \( L \) is on-shell zero (see remark after (57)) and that \( (\sqrt{-g} \bar{\psi} \gamma^i \psi) \partial_i \) is also zero (charge conservation, see, e.g., [4]). As a result, \( L^{(1)} \) too is on-shell zero. The canonical stress-energy tensor

\[ \sqrt{-g} t^{(1)i}_{\ k} = \frac{\partial L^{(1)}}{\partial (\partial_i \psi)} \partial_k \psi + \frac{\partial L^{(1)}}{\partial (\partial_i \bar{\psi})} \partial_k \bar{\psi} - \delta^i_k L^{(1)} \tag{72} \]

therefore reduces to (note that \( L_{\text{int}} \) does not contain derivatives and therefore does not contribute in the first two terms of (72))

\[ \sqrt{-g} t^{(1)i}_{\ k} = \sqrt{-g} \bar{\psi} \gamma^i \psi_{,k}, \tag{73} \]

and the corresponding field momentum is of the form

\[ P_{k}^{(1)} = i \int_{\sigma} \sqrt{-g} \bar{\psi} \gamma^i \psi_{,k} d\sigma, \tag{74} \]

which differs from (58) by the second term in the latter. The fact that the omission of a surface term in \( L \) leads to a modification of the stress-energy tensor (a so-called relocalization) is well known. That it leads to a modification of the integrated momentum is a little bit more surprising. It is, however, quite natural. The reason can be traced back to the fact that a strict separation between the energy and momentum of the dynamical fields (in our case, the Dirac field) on one hand and the non-dynamical background fields (in our case, gravity) is devoid of physical sense. In other words, it is rather a matter of convention which amount of energy (momentum) is attributed to one or the other part, only the total energy (momentum) being of physical relevance.
We wish to point out that this is not a particularity of gravity. Instead of a general discussion, simply consider the following (albeit unrealistic) surface Lagrangian (in flat spacetime)

\[ \mathcal{L}_{\text{surf}} = B_{i,k} \ast F^{ik}, \]

where \( B_i \) is a dynamical vector field, and \( \ast F^{ik} \) is the dual of the Maxwell tensor \( F_{ik} = A_{k,i} - A_{i,k} \) for a given electrodynamic background field \( A_i \). Since \( \ast F^{ik} \) is identically zero, the above is indeed a total divergence. The stress-energy tensor for the dynamical field \( B_i \) takes an additional term \( \tilde{t}_{i}^{k} = \frac{\partial \mathcal{L}_{\text{surf}}}{\partial B_{l,i}} B_{l,k} - \delta_{i}^{k} \mathcal{L}_{\text{surf}} \) satisfying \( \tilde{t}_{i}^{k} = \ast F_{lm}^{k} B_{l,m} \), which is not identically zero, and thus the addition of \( \mathcal{L}_{\text{surf}} \) (to the Lagrangian for the dynamical field \( B_i \)) does not merely lead to a relocalization of \( t_{i}^{k} \), but rather to a modification of the integrated momentum. Exactly the opposite term in \( \mathcal{P}_{k} \) will be induced in the corresponding momentum vector of the field \( A_{m} \), as is easily shown, such that the surface term does indeed not contribute to the total momentum.

Thus, although usually, surface terms lead to a relocalization of the stress-energy, but leave the integrated momentum vector unchanged, in the presence of background fields, this is not true anymore. Apart from a relocalization, a momentum transfer between dynamical and background fields is induced by surface terms. Both relocalizations and momentum transfers, however, should not be physically relevant. It is quite a matter of convention whether we attribute, e.g., the potential energy of an electron in the Coulomb field of a proton either to the electron or to the electromagnetic field.

Having reduced \( \bar{\psi} \) to a Lagrange multiplier, we consider \( \psi \) as the only true field variable in \( \mathcal{L}^{(1)} \) and define

\[ \pi^{i} = \frac{\partial \mathcal{L}^{(1)}}{\partial \psi_{,i}} = i\sqrt{-g} \bar{\psi} \gamma^{i}, \]

and postulate again the anticommutation relations

\[ i\{\psi^{N}(x),\psi^{M}(y)\} = 0, \quad i\{\pi^{i}_{N}(x),\pi^{k}_{M}(y)\} = 0, \]
\[ i\{\psi^{N}(x),\pi^{i}_{M}(y)\} = -\delta^{N}_{M}\delta^{i}_{\sigma}(x - y). \]

According to (74), we have \( \mathcal{P}^{(1)} = \int_{\sigma} \pi^{i}\psi_{,i}d\sigma \), and the following commutators are straightforwardly evaluated

\[ [\mathcal{P}_{k}^{(1)},\psi] = -i\psi_{,k} = -p_{k}\psi \]
\[ [\mathcal{P}_{k}^{(1)},\pi^{i}] = -i\pi^{i}_{,k}n_{i}n_{l}, \]

which are now in complete correspondence to the bosonic case (43) and \( \mathcal{P}^{(1)}_{k} \) can be interpreted as generator of spacetime translations. Consistently, we also have that \( \mathcal{P}_{k}^{(1)} = (\psi, p_{k}\psi) \).

It is needless to say that instead of \( \bar{\psi} \), we can also eliminate \( \psi \) as dynamical variable and write

\[ \mathcal{L}^{(2)} = -\sqrt{-g} i\partial_{\gamma} \bar{\psi} \gamma^{i} \psi - \frac{i}{2} \bar{\psi} (\gamma^{i} \sqrt{-g})_{,i} \psi + \mathcal{L}_{\text{int}}, \]
where the same surface term with the opposite sign has been omitted this time. The canonical momentum is now given by  \( \pi^i = -i\sqrt{-g} \gamma^i \psi \), and assuming  \( i\{\bar{\psi}_N(x), \pi^i_M(y)\} = +\delta_M^N \delta_\sigma^i (x - y) \) (for consistency with (76), we have to use the opposite sign, such that one relation results from hermitian conjugation of the other) we find

\[
[P_{(2)}^k, \psi] = -i \bar{\psi}_{,k}, \quad [P_{(2)}^k, \pi^i] = -i \bar{\pi}^i_{,k} n^L,
\]

where we have  \( P_{(2)}^k = (p_k \psi, \psi) \).

### 6.4 The strictly canonical way

For completeness, we will briefly outline the problems that arise from the second class constraints if one tries to follow strictly the canonical procedure. The issue is not related to the covariant formalism, neither to the presence of the gravitational background fields and should be known to most readers from the corresponding special relativistic theory.

Formally, starting from the Dirac Lagrangian (47), one is let to define  \( \pi^i = \partial \mathcal{L}/\partial \dot{\psi}_{,i} = \sqrt{-g} (i/2) \bar{\psi} \gamma^i \) and  \( \bar{\pi}^i = \partial \mathcal{L}/\partial \dot{\bar{\psi}}_{,i} = -\sqrt{-g} (i/2) \gamma^i \psi \), and according to the general procedure, one assumes the following anticommutation relations

\[
i\{\psi^N(x), \psi^M(y)\} = 0, \quad i\{\bar{\psi}_N(x), \bar{\psi}_M(y)\} = 0, \\
i\{\psi^N(x), \bar{\pi}^i_M(y)\} = -\delta^N_M \delta_\sigma^i (x - y), \quad i\{\psi_N(x), \bar{\psi}_M(y)\} = 0, \\
i\{\bar{\pi}^{iN}(x), \bar{\pi}^{kM}(y)\} = 0, \quad i\{\bar{\psi}_N(x), \bar{\pi}^{iM}(y)\} = +\delta^N_M \delta_\sigma^i (x - y),
\]

where again, the sign difference is necessary in order for consistency, since hermitian conjugation of the relations in the first line leads to those in the second. There is also a classical realization for those relations, based on the Poisson bracket

\[
\{A, B\}_\sigma = \int_\sigma \left( \frac{\delta A}{\delta \psi^M(z)} \frac{\delta B}{\delta \bar{\psi}^M(z)} - \frac{\delta B}{\delta \bar{\psi}^M(z)} \frac{\delta A}{\delta \psi^M(z)} \right) \delta \sigma^M(z) d\sigma^m(z).
\]

Further, it is understood that  \( \{\bar{\psi}_N, \psi^M\} = \{\bar{\psi}^{iM}, \psi^{iN}\} = \{\bar{\psi}^{iM}, \psi^{iK}\} = \{\bar{\psi}^{iN}, \psi^M\} = 0 \), since they are anticommutators between different components of the canonical field variables  \( \Psi^A = (\psi, \bar{\psi}) \) and momentum variables  \( \Pi^a_A = (\pi^i, \bar{\pi}^i) \), where  \( A = 1, 2 \ldots 8 \). The momentum vector  \( P^{(can)}_k = \int_\sigma \sqrt{-g} t^i_k d\sigma_i \) takes the simple form (since  \( \mathcal{L} = 0 \))

\[
P^{(can)}_k = \int_\sigma (\pi^i \psi_{,k} + \bar{\psi}_{,k} \bar{\pi}^i) d\sigma_i
\]

which leads immediately to

\[
[P^{(can)}_k, \psi] = -i \psi_{,k}, \quad [P^{(can)}_k, \bar{\psi}] = -i \bar{\psi}_{,k}.
\]
as was to be expected, since we faithfully stucked to the canonical procedure.

Unfortunately, the above procedure is not correct. The reason, as we have pointed out, is found in the constraints of the theory. Indeed, from the expressions for \( \pi^i \) and \( \bar{\pi}^i \), we find two constraints

\[
\Phi_1^i = \pi^i - \sqrt{-g} \frac{i}{2} \bar{\psi} \gamma^i \equiv 0, \quad \Phi_2^i = \bar{\pi}^i + \sqrt{-g} \frac{i}{2} \gamma^i \psi \equiv 0.
\]

Note that we use here a manifestly covariant form of Dirac’s analysis [11] with a four component momentum \( \pi^i \). Actually, the physical constraints are found by considering the constraints along \( n_i \), i.e., \( \Phi_\alpha = \Phi_\alpha^i n_i = 0, \alpha = 1,2 \), involving only the true canonical momentum \( \pi^i n_i \). For our purposes, the component form can equally well be used, but one should have in mind that there are two (not eight) physical constraints (per point).

Using the anticommutation relations between the fields, we can evaluate

\[
\{ \Phi_1^i(x), \Phi_2^k(y) \} = -\frac{1}{2} \sqrt{-g} \left( \delta_\sigma^i (x-y) \gamma^k + \delta^k\sigma (x-y) \gamma^i \right),
\]

or, contracting with \( n_i n_k \), \( \{ \Phi_1^i(x), \Phi_2^k(y) \} = -\sqrt{-g} \delta_\sigma (x-y) \gamma^i n_i \neq 0 \). To be precise, the above relations, according to Dirac [11] have to be evaluated with the classical Poisson bracket, because we can proceed to the second quantization only after having dealt with the constraints. The result is of course the same. Such constraints, which do not possess a vanishing Poisson bracket, are referred to as second class constraints. In contrast to first class constraints (with vanishing Poisson brackets between each other), which are relatively easily dealt with (first class constraints are the kind that arises in gauge theories and are imposed on the physical states of the theory \( \Phi^\dagger \Psi >_{\text{phys}} = 0 \), recall, e.g., the Gupta-Bleuler quantization), second class constraints are more difficult to handle, and have, in some way, to be eliminated from the theory. According to Dirac, a necessary step, before the correspondence principle can be applied, is to modify the Poisson bracket. If this is not done, we have obviously an inconsistent theory (since \( \{ \Phi_1, \Phi_2 \} \neq 0 \) for \( \Phi_1 = \Phi_2 = 0 \)).

The inconsistency of the theory can also be seen directly, noting that, e.g., the anticommutation relation \( \{ \psi(x), \pi^i(y) \} = i \delta_\sigma^i (x-y) \) is not consistent with \( \{ \psi, \bar{\psi} \} = 0 \), as can be seen by inserting \( \pi^i = \sqrt{-g} (i/2) \bar{\psi} \gamma^i \) into the first relation. Also, for instance, the bracket \( \{ \mathcal{P}_k^{(\text{can})}, \psi \} \) gives a different result (namely zero) if we replace, e.g., \( \pi^i \) by \( \bar{\psi} \) in the expression for \( \mathcal{P}_k^{(\text{can})} \). Thus, quite obviously, we have too many canonical variables in our formalism. A general method of how one deals with second class constraints, removing the non-physical degrees of freedom from the theory, has been outlined in [11]. The crucial step is to replace the Poisson bracket \( \{ A, B \}_\sigma \) by a modified bracket \( \{ A, B \}_\sigma^* = \{ A, B \}_\sigma - \{ A, \Phi_{(m)} \}_\sigma c^{mn} \{ \Phi_{(n)}, B \}_\sigma \), where \( c^{mn} = \{ \Phi_{(m)}, \Phi_{(n)} \}_\sigma^{-1} \) is the inverse of the matrix formed from the second class constraints \( \Phi_{(m)} \). Second quantization is achieved upon application of the correspondence principle on this modified Poisson bracket. This method, when applied to the Dirac field, leads to the theories described in the previous sections, 6.2 and 6.3 depending on whether one starts from the manifestly hermitian Lagrangian, or from the surface term modified Lagrangian, respectively.
Here, instead, we were led to the same theories inspiring ourselves directly from the quantization of the special relativistic theory, since, we repeat, the problems described in this section are completely unrelated to the covariant formalism and to the eventual presence of gravitational fields.

6.5 Discussion

We have given two approaches to the canonical quantization for the Dirac field in curved spacetime. The first way, presented in section 6.2, is based on the manifestly hermitian Lagrangian and preserves the symmetry between the fields \( \psi \) and \( \bar{\psi} \). The field momentum \( P_k \) is equal to the expectation value of the hermitian momentum \( \tilde{p}_k \), but does not generate spacetime translations on the quantum fields. The second way, presented in section 6.3, starts from a non-hermitian Lagrangian, where one of the field variables \( \psi \) or \( \bar{\psi} \) has been eliminated by adding a suitable surface term to the Lagrangian. In this approach, the field momentum is equal to the expectation value of the non-hermitian momentum operator \( p_k = i\partial_k \) and corresponds to the generator of the spacetime translations.

On a classical level, we have traced back the differences between both approaches to a momentum transfer from the Dirac field to the gravitational background, induced by the surface term in the Lagrangian. Since the assignment of the potential (or interaction) energy to one or another of the interacting fields (Dirac and gravitational) is rather a matter of convention, both approaches should be physically equivalent. In order to establish this equivalence on the quantum level, further investigations are necessary. We suspect that both approaches can be related by a change of representation in operator space, involving a non-unitary transformation, very similar to the Dirac space transformation that has been presented in [4] to relate the hermitian momentum \( \tilde{p}_k \) to the non-hermitian translation operator \( p_k = i\partial_k \).

To illustrate this, we consider the manifestly hermitian theory of section 6.2, and choose again, for simplicity, the hypersurface \( t = t_0 \) and the gauge \( e^t_\alpha = 0 \), such that equation (68),

\[
[P_k, \psi] = -i \left( \partial_k + \frac{1}{2} (\partial_k \ln \sqrt{-g^{tt}}) \right) \psi,
\]

is valid. Next, we perform a transformation \( A \) in Hilbert space, acting on states \( |\Phi> \) and on operators \( O \)

\[
|\Phi> \rightarrow A|\Phi>, \quad O \rightarrow \tilde{O} = AOA^{-1},
\]

where

\[
A = (-g^{tt})^{-\frac{1}{2}}.
\]

This specific operator is diagonal in both Hilbert and Dirac space, and thus commutes, e.g., with \( \psi, \pi^i, \psi_k \) etc., as well as with \( P_k \) from (68). Therefore, we find \( \tilde{\psi} = A\psi A^{-1} = \psi \), and similarly, \( \tilde{\pi}^i = \pi^i \), \( \tilde{P}_k = P_k \), and so on. On the other hand, for \( p_k = i\partial_k \), we find

\[
\tilde{\partial}_k = A\partial_k A^{-1} = \partial_k + \frac{1}{2} (\partial_k \ln \sqrt{-g^{tt}}),
\]

25
leading to the hermitian operator $\tilde{p}_k$ from (52). Therefore, in the new representation, we can write

$$[\tilde{P}_k, \tilde{\psi}] = [P_k, \psi] = -i \left( \partial_k + \frac{1}{2} (\partial_k \ln \sqrt{-g} g^{tt}) \right) \psi = -i \tilde{\partial}_k \tilde{\psi},$$

showing that $P_k$ can indeed be interpreted as the generator of translations, but in a different representation. Those manipulations seem to indicate that there is indeed a direct connection between the hermitian formulation of the theory and the approach of section 6.3, where $P_k^{(1)}$ appears directly as generator of translations.

One can object that, if both approaches are related by a non-unitary transformation, they are still not physically equivalent, since such a transformation will not leave invariant, e.g., the expectation values of physical operators, and this will ultimately jeopardize the probability interpretation of the theory. This, however, is not necessarily the case, if the scalar product in Hilbert space is transformed simultaneously with the change of representation. In fact, on a quantum mechanical level, we have already shown in [4] how this works explicitly. Namely, we have to use the explicitly covariant scalar product $\langle \psi_1, \psi_2 \rangle = \int \sqrt{-g} \overline{\psi}_1 \gamma^i \psi_2 d\sigma_i$ in the hermitian case, while after the non-unitary transformation, we simply use the flat scalar product $<\psi_1, \psi_2> = \int \psi_1^\dagger \psi_2 d^3x$. In this way, we find the same expectation values before and after the non-unitary transformation that relates $p_i$ to $\tilde{p}_i$. In a sense, as we have outlined in [4], the non-unitarity of the transformation is absorbed by the change of the scalar product.

One can thus expect that a similar construction is possible on a field theoretical level, namely that both representations are equivalent if simultaneously with the non-unitary transformation, an appropriate change of the scalar product in Hilbert space is performed. (Technically, this means actually the introduction of a new Hilbert space.) Before one can conclude any further, one will first have to construct an explicit representation that satisfies the anticommutation relations $\{\pi, \psi\} = 0$. (This is not a trivial matter, since for anticommuting fields, we can not use the conventional representation of the bosonic case, where the field $\varphi$ is a multiplication operator and $\pi$ a functional derivative operator $-i \frac{\delta}{\delta \varphi}$. In the fermion case, one will have to use Grassman variables to construct a representation.)

Thus, both the so-called manifestly hermitian theory and the non-manifestly hermitian theory can turn out to be hermitian. The term hermitian is actually meaningless as long as we do not specify a Hilbert space (and in particular a scalar product) with respect to which the theory should be hermitian. (All we can say for now is that the manifestly hermitian theory is based on a (classically) real Lagrangian, but there is no direct correspondence between real and hermitian, except if restrict ourselves to specific scalar products. In this article, nevertheless, we have followed the usual convention to call real Lagrangians hermitian.)

Finally, it is interesting to remark that in flat spacetime, both approaches are equivalent, and the generator of translations in either approach is automatically given in terms of the field momentum. It is actually quite common, in standard textbooks, to use the non-hermitian version of the Dirac Lagrangian and to omit the discussion related to the second class constraints.
completely. Our analysis shows that the situation in curved spacetime needs to be treated with more care.

7 Conclusions

Canonical Hamiltonian field theory in curved spacetime has been formulated in a manifestly covariant way, and quantization has been achieved by a conventional correspondence principle. On a formal level, no problems related specifically to the presence of gravity arise. In the case of the bosonic theory (which was assumed to be free of constraints), we obtained the expected result that the field momentum operator generates spacetime translations on the field operators.

On the other hand, in Dirac theory, we have to deal with second class constraints and the situation is less straightforward. We first performed a manifestly hermitian quantization, where it turned out that the field momentum does not correspond to the generator of spacetime translations, but rather to a modified translational operator, which has been identified, in our previous work, as a generalized, hermitian momentum operator. An alternative, not manifestly hermitian quantization was achieved by modifying the Lagrangian by a surface term. In this approach, the field momentum corresponds directly to the generator of translations, which is, however, given in terms of a non-hermitian Dirac operator. The change of the field momentum induced by the surface term was interpreted on a classical level as momentum transfer between the Dirac field and the gravitational background field. Both approaches should be physically equivalent. In order to show this on the quantum level, further investigations are necessary. It is expected that the addition of a surface term to the Lagrangian results in a change of the representation, such that both approaches can be related by a non-unitary transformation in operator space. Equivalence between both representations is then achieved by a simultaneous change of the scalar product in Hilbert space.

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