The Conformal Group $SO(4, 2)$ and Robertson-Walker spacetimes

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Abstract

The Robertson-Walker spacetimes are conformally flat and so are conformally invariant under the action of the Lie group $SO(4, 2)$, the conformal group of Minkowski spacetime. We find a local coordinate transformation allowing the Robertson-Walker metric to be written in a manifestly conformally flat form for all values of the curvature parameter $k$ continuously and use this to obtain the conformal Killing vectors of the Robertson-Walker spacetimes directly from those of the Minkowski spacetime. The map between the Minkowski and Robertson-Walker spacetimes preserves the structure of the Lie algebra $so(4, 2)$. Thus the conformal Killing vector basis obtained does not depend upon $k$, but has the disadvantage that it does not contain explicitly a basis for the Killing vector subalgebra. We present an alternative set of bases that depend (continuously) on $k$ and contain the Killing vector basis as a sub-basis (these are compared with a previously published basis). In particular, bases are presented which include the Killing vectors for all Robertson-Walker spacetimes with additional symmetry, including the Einstein static spacetimes and the de Sitter family of spacetimes, where the basis depends on the Ricci scalar $R$.

Keywords: Robertson-Walker spacetimes; conformal group; symmetry

1 Introduction

We are interested in the geometrical symmetry properties of the Robertson-Walker spacetimes. These models are a special case of the general conformally flat spacetimes, or to be precise, conformally Minkowski spacetimes. The conformal group of Minkowski spacetime is the Lie group $SO(4, 2)$. Thus any spacetime conformally related to Minkowski spacetime will be conformally invariant under the action of $SO(4, 2)$. As a result, every conformally flat spacetime will admit a Lie algebra of conformal Killing vectors, that Lie algebra being $so(4, 2)$. The Robertson-Walker spacetimes are spatially homogeneous and isotropic and so admit, at least, a 6-dimensional Lie
algebra of Killing vector fields. In some cases there is additional symmetry i.e. the Einstein static spacetimes and the de Sitter family of spacetimes.

Levine [1] discusses isometries and conformal motions in conformally flat spaces and determines the conditions which such a space must satisfy in order that it may admit an isometry group. Levine [2] classifies conformally flat spaces according to these conditions. Maartens and Maharaj [3] have obtained a basis for the conformal Killing vector algebra of Robertson-Walker space-times for \( k \pm 1 \) by generalising a Minkowski conformal Killing vector and then commuting this with the Killing vectors to generate a complete set. However, some of the commutation relations presented in the paper are incorrect for the cases \( k = \pm 1 \) and we have rectified this.

In this paper we derive the conformal Killing vectors of the Robertson-Walker spacetimes: we find a coordinate system in which the general Robertson-Walker metric takes on a manifestly conformally flat form in which case the conformal Killing vectors of the Robertson-Walker spacetimes are simply the conformal Killing vectors of the Minkowski spacetime. The transformation we present is valid for all values of curvature \( k \) continuously. We discuss the properties of the conformal Killing vectors and the structure of the Lie algebra \( so(4,2) \) and present alternative bases according to the value of \( k \). We show why the basis presented in Maartens and Maharaj [3] in the case \( k = \pm 1 \) is only valid for these cases. We then present a basis which contains a basis for the Killing vector subalgebra for general Robertson-Walker models and which is legitimate for all values of curvature \( k \) continuously. The de Sitter family of spacetimes, which are completely homogeneous and isotropic in four dimensions, are a special case of the general Robertson-Walker spacetimes. These can similarly be written in a conformally flat form characterised by the (constant) Ricci scalar \( R \). Thus we can make similar statements regarding the conformal structure of these spacetimes.

There are a number of reasons why it is advantageous to have a basis for the algebra that explicitly contains a sub-basis for the Killing vectors. Firstly, it enables the dimension of the isometry group to be gleaned immediately in any case by examining the conformal factors (Tables I and II). Perhaps more important, though, is the fact that in any application where symmetry information is useful, such as the study of particle motion (as in Maharaj and Maartens [4] and Maartens and Maharaj [5]) or of perturbations of Robertson-Walker spacetimes, the ability to easily identify the Killing vectors (which give rise to conserved quantities) can greatly simplify the problem.

In the following greek indices take on the values 0, 1, 2, 3 and latin indices 1, 2, 3, unless otherwise indicated.

## 2 Conformal Killing vectors

Let \( M \) be a spacetime manifold with metric tensor \( g \) of Lorentz signature. Any vector field \( X \) which satisfies

\[
\mathcal{L}_X g = 2 \phi(x^\alpha) g
\]  \hspace{1cm} (1)
is said to be a \textit{conformal Killing vector} (CKV) of $g$. If $\phi$ is not constant on $M$ then $X$ is called a \textit{proper conformal Killing vector}, if $\phi$ is constant on $M$ then $X$ is called a \textit{homothetic Killing vector} (HKV) and if $\phi$ is constant and $\phi \neq 0$ on $M$ then $X$ is called \textit{proper homothetic}. If $\phi ; \alpha \beta = 0$ then $X$ is called a \textit{special conformal Killing vector} (SCKV) and if $\phi = 0$ then $X$ is said to be a \textit{Killing} vector (KV).

The maximum dimension of the algebra of CKV on $M$ is 15 and this is achieved if $M$ is conformally flat. The maximum dimension of the KV algebra is 10 and this occurs if $M$ is of constant curvature. If $M$ is not of constant curvature then this algebra has dimension at most 7. The algebra of HKV has dimension equal to or at most one greater than that of the KV algebra. If this algebra has its maximum dimension of 11 then $M$ is flat. Any CKV field in a flat spacetime is a SCKV and so the maximum dimension of the SCKV algebra is 15. If this occurs $M$ is flat whilst if $M$ is non-flat its maximum dimension is 8. For details and proofs see Hall [6] and references therein. It will be assumed throughout this paper that the spacetimes considered admit no local (nonglobalizable) conformal Killing vector fields.

Now, if we have a conformally related metric tensor $f^2(x^\alpha)g$ then we can investigate the Lie derivative of this metric tensor with respect to the CKV of $g$, defined in (1). We have that
$$
\mathcal{L}_X(f^2g) = \mathcal{L}_X(f^2)g + f^2\mathcal{L}_X(g) = X(f^2)g + 2f^2\phi g = 2[X(f^2)/2f^2 + \phi](f^2g) = 2\phi'(x^\alpha)(f^2g) .
$$

It is now obvious that a CKV $X$ of $g$ is necessarily a CKV of $f^2g$ with conformal factor $\phi' = [X(f^2)/2f^2 + \phi]$.

Let us investigate the variation of the quantities $X \cdot p$ along a geodesic with tangent vector $p$. It is straightforward to show that
$$
\nabla_p(X \cdot p) = \phi \ g(p, p) .
$$

Thus, if $X$ is a KV then $X \cdot p$ is a conserved quantity along geodesics. If the CKV $X$ is not a KV, then $X \cdot p$ is conserved along null geodesics only.

3 Robertson-Walker Spacetimes

The metric of a 3-dimensional space of constant curvature is
$$
ds^2 = \frac{dr^2}{(1 - kr^2/R^2)} + r^2 d\Omega^2 \quad \text{where} \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 .
$$

where $K = +1, 0, -1$ indicating positive, zero or negative curvature respectively and $R$ is the radius of curvature. We define $k = K/R^2$ to be the curvature, that is, the parameter $k$ takes on \textbf{all} values
including zero in the metric

\[ ds^2 = \frac{dr^2}{(1 - kr^2)} + r^2 \Omega^2 . \]  

(5)

Consider the line element (5) and note that when \( k = -1, 0, +1 \) we can write

\[ ds^2 = d\chi^2 + r^2(\chi) \Omega^2 \]  

(6)

where \( d\chi = \frac{dr}{(1 - kr^2)^{1/2}} \). The function \( r(\chi) \) is \( \sin \chi \) for spherical spatial geometry, \( \chi \) for flat spatial geometry and \( \sinh \chi \) for hyperbolic spatial geometry. The line element for the general Robertson-Walker spacetime can be written

\[ ds^2_{\text{RW}} = -dt^2 + S^2(t) ds^2 . \]  

(7)

The function \( S(t) \) is the cosmological scale factor.

3.1 Conformally Flat Spacetime

3.1.1 Transformation to conformal time \( \tau \)

First we write the general Robertson-Walker metric as

\[ ds^2_{\text{RW}} = S^2(t) \left( -\frac{dt^2}{S^2(t)} + ds^2 \right) \]  

(8)

\[ = S^2(t) (-d\tau^2 + ds^2) \text{ where } \tau = \int \frac{dt}{S(t)} . \]  

(9)

The line element \( ds^2_E = -d\tau^2 + ds^2 \) in the case \( k > 0 \) is normally referred to as the Einstein static spacetime and in the case \( k < 0 \), the anti-Einstein static spacetime. However, from now on we shall refer to the three cases \( k > 0, k = 0 \) and \( k < 0 \) collectively as the Einstein static spacetimes.

3.1.2 Einstein static spacetimes in conformally flat form

We now present a coordinate transformation of the form

\[ ds^2_E = C^2(\tau', r', k) ds^2_M(\tau', r', \theta', \phi') \]  

(10)

where \( C(\tau', r', k) \) is some function (to be determined) depending on the value of \( k \) and \( ds^2_M \) is the line element of Minkowski spacetime

\[ ds^2_M = -d\tau'^2 + dr'^2 + r'^2(\theta'^2 + \sin^2 \theta' d\phi'^2) . \]  

(11)

We first demonstrate that the Einstein static spacetime with \( k = +1 \) can be written in a conformally flat form (we do this by considering null coordinates in the Minkowski spacetime as described in appendix A). Then we generalise this to all Einstein static spacetimes of arbitrary curvature \( k \).

For the case \( k = +1 \) the coordinate transformation is as follows

\[ \tau' = \frac{2 \sin \tau}{(\cos \tau + \cos \chi)} ; \]

\[ r' = \frac{2 \sin \chi}{(\cos \tau + \cos \chi)} . \]  

(12)
The line element (11) becomes

$$ds^2_M = \frac{4}{(\cos \tau + \cos \chi)^2} (-d\tau^2 + d\chi^2 + \sin^2 \chi d\Omega^2)$$

(13)

which is conformal to the Einstein static spacetime in the case where \(k = +1\). We use the transformation (12) and line element (13) as our specimen. We generalise the transformation (12) to accommodate spacetimes with arbitrary values of curvature \(k\).

Now, we can define a new time coordinate \(\epsilon\) via

$$d\tau = d\epsilon / (1 - k \epsilon^2)^{1/2}$$

and radial coordinate \(r\) via

$$d\chi = dr / (1 - kr^2)^{1/2}$$

for \(-\infty < k < \infty\). Note that when \(k = +1, 0, -1\) we have

$$\epsilon = \begin{cases} 
\sin \tau & k = +1 \\
\tau & k = 0 \\
\sinh \tau & k = -1 
\end{cases}$$

$$r = \begin{cases} 
\sin \chi & k = +1 \\
\chi & k = 0 \\
\sinh \chi & k = -1 
\end{cases}$$

(14)

Thus we have

$$ds^2_M = \frac{4}{F^2} \left( -\frac{d\epsilon^2}{(1 - k\epsilon^2)} + \frac{dr^2}{(1 - kr^2)} + r^2 d\Omega^2 \right)$$

(15)

where

$$F(\tau, r) = (1 - k\tau^2)^{1/2} + (1 - kr^2)^{1/2}.$$ 

(16)

The coordinate transformation is

$$\tau' = \frac{2 \epsilon}{F(\tau, r)}; \quad r' = \frac{2 r}{F(\tau, r)}.$$ 

(17)

We note that

$$F(\tau', r') = 4 \left( [(1 + k\tau'^2)/2] + (1 + kr'^2/2)]^2 - k^2\tau'^2r'^2 \right)^{-1/2}.$$ 

(18)

The inverse transformations to (17) are

$$\epsilon = F(\tau', r') \tau'/2; \quad r = F(\tau', r') r'/2.$$ 

(19)

We note that, with the exception of the case \(k = 0\), our coordinate transformations are only valid locally. There does not exist a global coordinate transformation mapping all of Robertson-Walker spacetime conformally into Minkowski spacetime. However, the geometrical objects exist independently of any particular coordinate system and so we shall not mention these issues any further.

4 The CKV of Minkowski spacetime.

The line element for Minkowski spacetime (11) can be written in cartesian coordinates

$$x' = r' \sin \theta' \cos \phi', \quad y' = r' \sin \theta' \sin \phi', \quad z' = r' \cos \theta',$$

(20)

as follows

$$ds^2_M = -d\tau^2 + dx'^2 + dy'^2 + dz'^2.$$ 

(21)
The isometry group of this spacetime is the 10-parameter Poincaré group (inhomogeneous Lorentz group) \( ISO(3,1) \), the generators being the 6 generators of the homogeneous Lorentz group \( SO(3,1) \) and the 4 spacetime translations, giving 10 independent Killing vectors.

The conformal isometry group of Minkowski spacetime is the 15-parameter conformal group \( SO(4,2) \) and includes the isometry group \( ISO(3,1) \). The 15 CKV are

4 translations: \( T_\alpha = \frac{\partial}{\partial x^\alpha} \)

6 rotations: \( M_{\alpha\beta} = x'_{\alpha} \frac{\partial}{\partial x'^\beta} - x'_{\beta} \frac{\partial}{\partial x'^\alpha} \)

1 dilation: \( D = x'^\alpha \frac{\partial}{\partial x'^\alpha} \)

4 inversions: \( K_\alpha = 2x'^\alpha x'^\beta \frac{\partial}{\partial x'^\beta} - (x'^\beta x'^\beta) \frac{\partial}{\partial x'^\alpha} \) \( (22) \)

where \( x'^\alpha = \eta_{\alpha\beta} x'^\beta \) and we shall refer to this as the Minkowski basis. We can see from Table I that with this choice of basis for the conformal algebra all the CKV are SCKV and there are only 4 proper CKV. The remaining 11 are HKV, and of course, only one of these is not a KV. The 15 CKV form a basis for the Lie algebra \( so(4,2) \). The commutation relations for these vector fields are as follows

\[
\begin{align*}
[ M_{\alpha\beta}, M_{\gamma\delta} ] &= \eta_{\alpha\delta} M_{\beta\gamma} + \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\gamma\delta} M_{\beta\alpha} + \eta_{\alpha\beta} M_{\gamma\delta}; \\
[ T_\alpha, T_\beta ] &= 0; \quad [ T_\alpha, M_{\beta\gamma} ] = \eta_{\beta\alpha} T_\gamma - \eta_{\gamma\alpha} T_\beta; \\
[ K_\alpha, K_\beta ] &= 0; \quad [ K_\alpha, M_{\beta\gamma} ] = \eta_{\beta\alpha} K_\gamma - \eta_{\gamma\alpha} K_\beta; \\
[ D, K_\alpha ] &= K_\alpha; \quad [ D, M_{\alpha\beta} ] = 0; \quad [ T_\alpha, D ] = T_\alpha; \\
[ T_\alpha, K_\beta ] &= 2(\eta_{\alpha\beta} D - M_{\alpha\beta}); \quad (23)
\end{align*}
\]

which is isomorphic to the Lie algebra \( so(4,2) \) presented in appendix C. The subalgebra consisting of rotations \( M_{ij} \) and boosts \( M_{0k} \) is isomorphic to the Lie algebra \( so(3,1) \). This subalgebra, together with the translations \( T_\alpha \) form the Lie algebra \( iso(3,1) \). It is obvious that the subalgebra of \( so(4,2) \) consisting of rotations \( M_{\alpha\beta} \) and the SCKV \( K_\alpha \) is also isomorphic to the Lie algebra \( iso(3,1) \). The group of translations is an invariant subgroup of \( ISO(3,1) \) and so the Lie group \( ISO(3,1) \) is nonsemi-simple. The de Sitter and anti-de Sitter groups \( SO(4,1) \) and \( SO(3,2) \) are also subgroups of the conformal group \( SO(4,2) \), however, they are not isometry groups of the Minkowski spacetime. Thus, the algebra \( so(4,2) \) contains the subalgebras \( so(4,1) \) and \( so(3,2) \) which is shown explicitly in appendix C and section 6.

In order to implement the transformation (17) to obtain the CKV for Robertson-Walker spacetimes we re-write the Minkowski CKV (22) in terms of spherical polar coordinates as follows

\[
\begin{align*}
T_0 &= \frac{\partial}{\partial r'} \\
T_1 &= \sin \theta \cos \phi \frac{\partial}{\partial r'} + \frac{1}{r'} \left( \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \frac{\partial}{\partial \phi} \right) \\
T_2 &= \sin \theta \sin \phi \frac{\partial}{\partial r'} + \frac{1}{r'} \left( \cos \theta \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \frac{\partial}{\partial \phi} \right)
\end{align*}
\]
\[ T_3 = \cos \theta \frac{\partial}{\partial \nu'} - \frac{\sin \theta}{r'} \frac{\partial}{\partial \theta} \]

\[ M_{01} = \sin \theta \cos \phi \left( -r' \frac{\partial}{\partial \nu'} - r' \frac{\partial}{\partial \tau} \right) - \frac{\tau'}{r'} \left( \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \frac{\sin \phi}{\sin \theta} \frac{\partial}{\partial \phi} \right) \]

\[ M_{02} = \sin \theta \sin \phi \left( -r' \frac{\partial}{\partial \nu'} - r' \frac{\partial}{\partial \tau} \right) - \frac{\tau'}{r'} \left( \cos \theta \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \frac{\partial}{\partial \phi} \right) \]

\[ M_{03} = \cos \theta \left( -r' \frac{\partial}{\partial \nu'} - r' \frac{\partial}{\partial \tau} \right) + \frac{\tau'}{r'} \sin \theta \frac{\partial}{\partial \theta} \]

\[ M_{12} = \frac{\partial}{\partial \phi} \]

\[ M_{13} = -\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \]

\[ M_{23} = -\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \]

\[ D = \frac{\tau'}{r'} + r' \frac{\partial}{\partial r} \]

\[ K_0 = -2r' D - (-\tau^2 + r^2) T_0 \]

\[ K_1 = 2r' \sin \theta \cos \phi D - (-\tau^2 + r^2) T_1 \]

\[ K_2 = 2r' \sin \theta \sin \phi D - (-\tau^2 + r^2) T_2 \]

\[ K_3 = 2r' \cos \theta \cos \phi D - (-\tau^2 + r^2) T_3 \]

\[ (24) \]

5 The CKV of Robertson-Walker spacetimes

We now apply the transformation \( (17) \) to the Minkowski CKV \((24)\). In doing this we will have simply re-written the Minkowski CKV in a different coordinate system so that the algebra remains unchanged. However we will also obtain a convenient basis for the Robertson-Walker CKV. From \( (15) \) we find that

\[ \frac{\partial}{\partial \tau'} = \frac{1}{2} \left( A \frac{\partial}{\partial \tau} - k r \left( 1 - k r^2 \right)^{\frac{1}{2}} \frac{\partial}{\partial r} \right) \]

\[ \frac{\partial}{\partial \nu'} = \frac{1}{2} \left( A \frac{\partial}{\partial \tau} + k r \left( 1 - k r^2 \right)^{\frac{1}{2}} \frac{\partial}{\partial r} \right) \]

\[ (25) \]

where \( A = 1 + (1 - k r^2)^{\frac{1}{2}} \). The relations \( (25) \) are essentially all we need to construct the CKV for the RW spacetimes. They are as follows

\[ T_0 = \frac{1}{2} \left( A \frac{\partial}{\partial \tau} - k r \left( 1 - k r^2 \right)^{\frac{1}{2}} \frac{\partial}{\partial r} \right) \]

\[ T_1 = \frac{1}{2} \sin \theta \cos \phi \left( A \left( 1 - k r^2 \right)^{\frac{1}{2}} \frac{\partial}{\partial r} - k r \frac{\partial}{\partial \tau} \right) + \frac{F}{2r} \left( \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \frac{\partial}{\partial \phi} \right) \]

\[ T_2 = \frac{1}{2} \sin \theta \sin \phi \left( A \left( 1 - k r^2 \right)^{\frac{1}{2}} \frac{\partial}{\partial r} - k r \frac{\partial}{\partial \tau} \right) + \frac{F}{2r} \left( \cos \theta \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \frac{\partial}{\partial \phi} \right) \]

\[ T_3 = \frac{1}{2} \cos \theta \left( A \left( 1 - k r^2 \right)^{\frac{1}{2}} \frac{\partial}{\partial r} - k r \frac{\partial}{\partial \tau} \right) - \frac{F}{2r} \sin \theta \frac{\partial}{\partial \theta} \]

\[ M_{01} = -\sin \theta \cos \phi \left( \left( 1 - k r^2 \right)^{\frac{1}{2}} \frac{\partial}{\partial \tau} + \left( 1 - k r^2 \right) r \frac{\partial}{\partial r} \right) - \frac{\epsilon}{r} \left( \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \frac{\partial}{\partial \phi} \right) \]

\[ M_{02} = -\sin \theta \sin \phi \left( \left( 1 - k r^2 \right)^{\frac{1}{2}} r \frac{\partial}{\partial \tau} + \left( 1 - k r^2 \right) r \frac{\partial}{\partial r} \right) - \frac{\epsilon}{r} \left( \cos \theta \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \frac{\partial}{\partial \phi} \right) \]
\[ M_{03} = - \cos \theta \left( (1 - k\epsilon^2) \frac{1}{r} \frac{\partial}{\partial \tau} + (1 - k\epsilon^2) \epsilon \frac{\partial}{\partial \tau} \right) + \epsilon \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \]

\[ M_{12} = \frac{\partial}{\partial \phi} \]

\[ M_{13} = - \cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \]

\[ M_{23} = - \sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \]

\[ D = \left( 1 - kr^2 \right)^{\frac{1}{2}} \left( \epsilon \frac{\partial}{\partial \tau} + (1 - k\epsilon^2) \frac{1}{r} \frac{\partial}{\partial r} \right) \]

\[ K_0 = - \frac{4\epsilon}{F} D - \frac{4}{F^2} (-\epsilon^2 + r^2) T_0 \]

\[ K_1 = \frac{4r}{F} \sin \theta \cos \phi D - \frac{4}{F^2} (-\epsilon^2 + r^2) T_1 \]

\[ K_2 = \frac{4r}{F} \sin \theta \sin \phi D - \frac{4}{F^2} (-\epsilon^2 + r^2) T_2 \]

\[ K_3 = \frac{4r}{F} \cos \theta D - \frac{4}{F^2} (-\epsilon^2 + r^2) T_3 \]

The rotations \( M_{ij} \) are obviously invariant under the map. Of course, the commutation relations are still as in the Lie algebra \((23)\). These CKV can be expressed in terms of the pseudo-isotropic coordinates (appendix \[3\]) and can be written as

\[ T_0 = \frac{1}{2} (P_0 + H); \]

\[ T_i = \frac{1}{2} (P_i + \tilde{Q}_i) \]

where

\[ P_0 = \frac{\partial}{\partial \tau} \]

\[ P_i = K_+ \frac{\partial}{\partial x^i} + \frac{k}{2} \bar{x}^i \frac{\partial}{\partial \bar{x}^j} \left( \bar{x}^j \frac{\partial}{\partial \bar{x}^j} \right) \]

\[ H = K_+^{-1} (1 - k\epsilon^2) \frac{\partial}{\partial \tau} - k\epsilon \left( \bar{x}^j \frac{\partial}{\partial \bar{x}^j} \right) \]

\[ \tilde{Q}_i = -k\epsilon K_+^{-1} \bar{x}^i \frac{\partial}{\partial \tau} + (1 - k\epsilon^2) \frac{1}{2} \left( -P_i + 2 \frac{\partial}{\partial \bar{x}^i} \right) \]

and

\[ M_{0i} = - (1 - k\epsilon^2) \frac{1}{2} K_+^{-1} \bar{x}^i \frac{\partial}{\partial \tau} - \epsilon \left( -P_i + 2 \frac{\partial}{\partial \bar{x}^i} \right) \]

\[ M_{ij} = \bar{x}^i \frac{\partial}{\partial \bar{x}^j} - \bar{x}^j \frac{\partial}{\partial \bar{x}^i} \]

\[ D = \epsilon K_+^{-1} \frac{\partial}{\partial \tau} + (1 - k\epsilon^2) \frac{1}{2} \left( \bar{x}^j \frac{\partial}{\partial \bar{x}^j} \right) \]

\[ K_0 = - \frac{4\epsilon}{((1 - k\epsilon^2) - K_+^{-1})} D - \frac{4}{((1 - k\epsilon^2) + K_+^{-1})^2} (-\epsilon^2 + r^2 k_+^{-2}) T_0 \]

\[ K_i = \frac{4K_+^{-1} \bar{x}^i}{((1 - k\epsilon^2) + K_+^{-1})} D - \frac{4}{((1 - k\epsilon^2) + K_+^{-1})^2} (-\epsilon^2 + r^2 k_+^{-2}) T_i \]

The conformal factors for the CKV \((28)\) are presented in Table \[1\]. Thus we can see that, with the exception of the rotations \( M_{ij} \), the SCKV of Minkowski spacetime are mapped into proper CKV.
in the Robertson-Walker spacetimes. Only in special cases will some of these proper CKV reduce
to SCKV, see sections 5.2 and 3. Of course, the structure consta nts in the Lie algebra (23) do not
involve the curvature parameter $k$. Thus if we wish to recover, say, the 6-dimensional isometry
subalgebras (so(4), iso(3), or so(3,1) for positive, zero and negative values of $k$) whose structure
constants depend upon the value of $k$, it will be necessary to make a change of basis. The form
of the CKV (27) is very suggestive as it is well known that the vector fields $P_i$ are Killing vector
fields. In the following sections we will construct such bases.

5.1 The basis of Maartens and Maharaj

Now, it can be shown that

$$kK_0 = 2(-P_0 + H)$$
$$kK_i = 2(P_i - \bar{Q}_i^*).$$

(30)

Taking into account the relations (27) and (30) we can write

$$P_0 = T_0 - \frac{k}{4}K_0$$
$$H = T_0 + \frac{k}{4}K_0$$
$$P_i = T_i + \frac{k}{4}K_i$$
$$\bar{Q}_i^* = T_i - \frac{k}{4}K_i$$

(31)

The conformal factors for these CKV are shown in Table III. Let us consider the basis

$$P_0; P_i; H; \bar{Q}_i^*; M_{0i}; M_{ij}; D;$$

(32)

as an alternative basis for the Lie algebra so(4,2). Now, it can be easily seen that when $k = 0$ we
have $P_0 = H$ and $P_i = \bar{Q}_i^*$. Thus when $k = 0$ the vector fields (32) will not span the 15-dimensional
vector space. Thus the basis (32) is only legitimate in the case where $k \neq 0$. However, we shall
consider this basis further. Table IV shows the structure of the the Lie algebra in terms of this
basis. The subalgebra formed by the elements $\{H; P_i; M_{0i}; M_{ij}\}$ and the subalgebra formed by
the elements $\{P_0; \bar{Q}_i^*; M_{0i}; M_{ij}\}$ are complimentary in the sense that when we scale out $k$, these
have the structure so(4,1) and so(3,2) respectively for the case $k > 0$ and so(3,2) and so(4,1)
respectively when $k < 0$. The subalgebra $\{M_{0i}; M_{ij}\}$ has the structure so(3,1). The subalgebra
formed by $\{P_i, M_{jk}\}$ has the structure so(4) or so(3,1) according to whether $k > 0$ or $k < 0$ and
the subalgebra formed by $\{\bar{Q}_i^*, M_{jk}\}$ has the structure so(3,1) or so(4) according to whether $k > 0$
or $k < 0$. Of course, the former is the KV algebra of the spacetime. The subalgebra $\{P_0, D, H\}$ has
the structure constants of the Lie algebra so(2,1). The subalgebra structure is shown schematically
in Figure 1. Let us consider the CKV in the case where $k = \pm 1$. We find that the CKV presented
above are related to those in Maartens and Maharaj [3] in this case by the following

$$\bar{Q}_i^* = -kQ_i^*$$
\[ M_{0i} = k Q_i \]
\[ D = -k H^* \]  \hspace{1cm} (33)

where they have \( h(\tau) = (1 - k\tau^2)^{\frac{1}{2}} \). Thus the Lie algebra for the cases \( k = \pm 1 \) can be written as follows

\[
\begin{align*}
[P_0, P_i] &= 0; & [P_0, M_{ij}] &= 0; & [P_i, P_j] &= -k M_{ij}; \\
[M_{ij}, M_{mn}] &= \eta_{jm} M_{in} + \eta_{jn} M_{mi} + \eta_{im} M_{nj} + \eta_{in} M_{jm}; \\
[P_i, M_{jk}] &= \eta_{ij} P_k - \eta_{ik} P_j; & [P_i, H] &= Q_i; & [P_i, H^*] &= Q_i^*; \\
[P_0, H] &= H^*; & [P_0, H^*] &= -k H; & [P_0, Q_i] &= Q_i^*; & [P_0, Q_i^*] &= -k Q_i; \\
[P_1, Q_3] &= -k \eta_{ij} H; & [P_i, Q_i^*] &= -k \eta_{ij} H^*; & [M_{ij}, H] &= 0; & [M_{ij}, H^*] &= 0; \\
[Q_1, M_{jk}] &= \eta_{ij} Q_k - \eta_{ik} Q_j; & [Q_i^*, M_{jk}] &= \eta_{ij} Q_k - \eta_{ik} Q_j; \\
[H, H^*] &= -k P_0; & [Q_i^*, H] &= 0; & [Q_i, H^*] &= 0; & [Q_i, H] &= k P_i; \\
[Q_i^*, H^*] &= P_i; & [Q_i, Q_j] &= M_{ij}; & [Q_i^*, Q_j^*] &= k M_{ij}; & [Q_i, Q_j^*] &= -\eta_{ij} P_0. \hspace{1cm} (34)
\end{align*}
\]

It should be noted that some of the commutation relations in the Lie algebra (3.4) presented in Maartens and Maharaj [3] are incorrect - the commutation relations \([M_{ij}, M_{mn}], [H, H^*], [Q_i, H] \) and \([Q_i^*, Q_j^*] \) are incorrect and the relation \([Q_i, Q_j^*] \) is missing. We have rectified these in the Lie algebra above.

### 5.2 The “continuous” basis

Alternatively, we can choose

\[
P_0; P_i; M_{0i}; M_{ij}; D; K_0; K_i;
\]

as a basis for the algebra \( so(4, 2) \). Again the full KV algebra is a subalgebra of the complete Lie algebra. However, in this case the choice of basis is valid for all values of curvature parameter \( k \) including the case \( k = 0 \). The Lie algebra is shown in Table 6. The subalgebra formed by the elements \( \{P_i, M_{jk}\} \) corresponds to the KV subalgebras \( so(4), iso(3) \) or \( so(3, 1) \) for \( k > 0 \), \( k = 0 \) and \( k < 0 \) respectively. The subalgebra formed by the elements \( \{M_{0i}, M_{jk}\} \) corresponds to the Lie algebra \( so(3, 1) \). The subalgebra formed by \( \{K_i, M_{jk}\} \) corresponds to the algebra \( iso(3) \). However, \( M_{0i} \) is not a KV in general and the \( K_i \) are never KV. The subalgebra structure is shown schematically in Figure 4. The \( P_0 \) becomes a HKV when \( S(t) = C t \) where \( C = constant \) and becomes a KV when \( S(t) = C \) and in this case we recover the Einstein static spacetimes. However, in the case when \( k = 0 \) and \( S(t) = C \) the \( M_{0i} \) are necessarily KV also and so there are an extra 4 KV. See Maartens and Maharaj [3] for further discussion.
6 De Sitter and anti-de Sitter spacetimes

Let us now consider spacetimes with constant curvature. These are a special case of the Robertson-Walker spacetimes. In this case the Riemann curvature tensor is determined by the Ricci scalar alone. These spacetimes are Einstein spaces i.e.

\[ R_{\alpha\beta} = \frac{1}{4} R g_{\alpha\beta} . \]  

(36)

The spacetime of constant curvature with \( R = 0 \) is Minkowski spacetime, \( R > 0 \) is de Sitter spacetime and in the case \( R < 0 \), anti-de Sitter spacetime. Of course, the spacetimes of constant curvature are maximally symmetric i.e. admit 10 independent Killing vector fields. Thus they are homogeneous and isotropic spacetimes. The generic de Sitter spacetime can be written (see Hawking and Ellis [7])

\[ ds^2 = -dt^2 + (1 + R\sigma^2) \left( \frac{dr^2}{1 - Rr^2} + r^2 d\Omega^2 \right) \]  

(37)

for arbitrary values of curvature \( R \), where \( dt/d\sigma = 1/(1 + R\sigma^2)^{1/2} \). It is obvious that this metric takes on the same form as that of a Robertson-Walker spacetime with \( S(t) = (1 + R\sigma^2)^{1/2} \). However, these coordinates are distinct from those used in section 3 to describe the Robertson-Walker spacetimes. We note that the constants \( k \) and \( R \) are related by the following

\[ R = 6S^{-2}(S \partial^2 S/\partial t^2 + (\partial S/\partial t)^2 + k) . \]  

(38)

The correspondence noted above allows us to read off the CKV for the de Sitter family of spacetimes. Of course the 6 KV \( P_i; M_{jk} \) will be inherited and in addition the 4 CKV \( H \) and \( M_{0i} \) are also KV for the de Sitter spacetimes. The quantity

\[ \Phi = S^{-1}\partial S/\partial \tau(1 - R\epsilon)^{1/2} - R\epsilon \]  

(39)

is zero for all \( R \) and as a result the conformal factors for \( H \) and \( M_{0i} \) become zero, see Tables I and II. It is easily verified that \( \Phi = 0 \) for \( R = 0 \). For \( R > 0 \) we can scale out \(|R|\) and write \( S = \cosh t \) and from (\[H\]), \( \tan \tau = \sinh t \). We can also write \((1 - R\epsilon^2)^{1/2} = \cos \tau \) and \(-R\epsilon = -\sin \tau \) and it follows that \( \Phi = 0 \). Similarly, for \( R < 0 \) we can scale out \(|R|\) and write \( S = \cos t \) and from (\[H\]), \( \cosh \tau = \sec t \). We can also write \((1 - R\epsilon^2)^{1/2} = \cosh \tau \) and \(-R\epsilon = \sinh \tau \) and it follows that \( \Phi = 0 \).

Thus we now choose

\[ H; P_i; M_{0i}; M_{ij}; D; K_0; K_i; \]  

(40)

as a basis for the algebra \( so(4,2) \) and we shall refer to this as the “de Sitter” basis. The full KV algebra is a subalgebra of the complete algebra for all values of curvature parameter \( R \) including the case \( R = 0 \). The Lie algebra is shown in Table VI and schematically in Figure 3.

The KV subalgebra \{\(H; P_i; M_{0i}; M_{ij}\)\} is the general de Sitter algebra i.e. \(so(4,1), iso(3,1)\) or \(so(3,2)\) according to whether \(R > 0\), \(R = 0\) or \(R < 0\). The subalgebra of the de Sitter algebra with
elements \{M_{0i}; M_{ij}\} form the Lie algebra $so(3, 1)$ thus de Sitter spacetime will always possess a Lorentz (isometry) subgroup. The subalgebra of $so(4, 2)$ with elements \{K_i; M_{ij}\} forms the Lie algebra $iso(3)$.

Of course, the de Sitter family of spacetimes can also be written in terms of the curvature parameter $k$. It can easily be seen from equation (38) that the sign of $k$ is constrained by the sign of $R$. In the case $R > 0$ there are three distinct forms corresponding to $k > 0$, $k = 0$ and $k < 0$, and for $R = 0$ there are two forms given by $k = 0$ and $k < 0$. For the case $R < 0$ there is only one possibility, $k < 0$, see Torrence and Couch [8]. Maartens and Maharaj [3] list all possible forms for the scale factors $S(t)$ for values of curvature $k$ in the Robertson-Walker models in order that the spacetime be of constant curvature. All of these cases are encapsulated in the formalism presented above.

7 Conclusions

The mapping between Minkowski spacetime and the Robertson-Walker spacetimes provides a convenient way of analysing their conformal structure. In particular the structure of the Lie algebra of conformal Killing vector fields is preserved by the mapping. This has allowed us to make informed decisions regarding the choice of basis in particular situations of interest. For both the Robertson-Walker and de Sitter models we have presented bases which always contain the Killing vector basis and have shown that the Lie algebra structure can be continuously parameterised by the curvature $k$ (or the Ricci scalar $R$).

Of course, this type of analysis is valid for any spacetime which can be written in a manifestly conformally flat form although the bases presented here will not necessarily contain the Killing vector basis (if any).

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A Einstein static spacetimes in conformally flat form

We can write the Einstein static spacetime in a conformally flat form by considering null coordinates in the Minkowski spacetime as described in Hawking and Ellis [7].

We define the advanced and retarded null coordinates $v$ and $w$ as follows

$$v = \frac{1}{4}(\tau' + \tau'); \quad w = \frac{1}{4}(\tau' - \tau').$$

(41)
Note that Hawking and Ellis \[7\] omit the factor 1/4 in their definition. The Minkowski line element (11) becomes

$$ds^2_M = 16 (-dvdw + \frac{1}{4}(v-w)^2d\Omega^2)$$ \hspace{1cm} (42)$$

where \(-\infty < v < \infty, -\infty < w < \infty\) and \(v \geq w\). We can then define new null coordinates in which \(v\) and \(w\) are transformed to finite values i.e. \(\tan p = v, \tan q = w\). Thus \(-\pi/2 < p < \pi/2, -\pi/2 < q < \pi/2\) and \(p \geq q\). The line element (42) becomes

$$ds^2_M = 16 \sec^2 p \sec^2 q (-dpdq + \frac{1}{4} \sin^2(p-q) d\Omega^2).$$ \hspace{1cm} (43)$$

Define \(\tau = p + q, \chi = p - q\) where \(-\pi < \tau + \chi < \pi, -\pi < \tau - \chi < \pi, \chi \geq 0\). Finally, (43) becomes

$$ds^2_M = 4 \sec^2 \left(\frac{\tau + \chi}{2}\right) \sec^2 \left(\frac{\tau - \chi}{2}\right) (-d\tau^2 + d\chi^2 + \sin^2 \chi d\Omega^2)$$ \hspace{1cm} (44)$$

and the coordinates are related by

$$\tau' = 2 \tan \left(\frac{\tau + \chi}{2}\right) + 2 \tan \left(\frac{\tau - \chi}{2}\right); \hspace{0.5cm} r' = 2 \tan \left(\frac{\tau + \chi}{2}\right) - 2 \tan \left(\frac{\tau - \chi}{2}\right).$$ \hspace{1cm} (45)$$

Note that the coordinate transformation (45) can be re-expressed as

$$\tau' = 2 \sin \tau \sec \left(\frac{\tau + \chi}{2}\right) \sec \left(\frac{\tau - \chi}{2}\right) = \frac{2 \sin \tau}{(\cos \tau + \cos \chi)},$$
$$r' = 2 \sin \chi \sec \left(\frac{\tau + \chi}{2}\right) \sec \left(\frac{\tau - \chi}{2}\right) = \frac{2 \sin \chi}{(\cos \tau + \cos \chi)}. \hspace{1cm} (46)$$

This is the transformation presented in section 25.4 of Stephani [9]. The line element (44) becomes

$$ds^2_M = \frac{4}{(\cos \tau + \cos \chi)^2} (-d\tau^2 + d\chi^2 + \sin^2 \chi d\Omega^2)$$ \hspace{1cm} (47)$$

and we recognise this as being conformal to the Einstein static spacetime in the case where \(k = +1\).

**B Conformally flat spatial sections**

We can use the transformation

$$\bar{r} = \frac{2r}{1 + (1 - kr^2)^{\frac{1}{2}}} \hspace{1cm} \text{with inverse} \hspace{0.5cm} r = \bar{r} K^{-1}_+$$ \hspace{1cm} (48)$$

where \(K_+ = 1 \pm kr^2/4\). This transformation is valid for all values of \(k\). The spatial sections of the Robertson-Walker spacetimes (3) then take on the conformally flat form

$$ds^2 = K_+^{-2}(d\bar{r}^2 + \bar{r}^2d\Omega^2).$$ \hspace{1cm} (49)$$

Further, defining pseudo-cartesian coordinates \(\{\vec{x}^i\}\) such that \(\vec{x} = \bar{r} \sin \theta \cos \phi, \vec{y} = \bar{r} \sin \theta \sin \phi, \vec{z} = \bar{r} \cos \theta\), we have

$$ds^2 = K_+^{-2}(d\vec{x}^2 + d\vec{y}^2 + d\vec{z}^2)$$ \hspace{1cm} (50)$$

where \(\bar{r} = (\vec{x}^2 + \vec{y}^2 + \vec{z}^2)^{\frac{1}{2}}\).
C Conformal group $SO(4, 2)$

The orthogonal group $O(p, q)$ leaves invariant the metric $\eta$ on $\mathbb{R}^n$ with signature

$$\begin{pmatrix} + + \cdots & - - \cdots \\ p & q \end{pmatrix}$$  (51)

where $p + q = n$. The special orthogonal group $SO(p, q)$ is a Lie group which has group elements $g$ satisfying $\det(g) = 1$. The associated Lie algebra, which we shall denote $so(p, q)$, has Killing vector generators

$$L_{AB} = x_A \frac{\partial}{\partial x^B} - x_B \frac{\partial}{\partial x^A},$$  (52)

where $A, B = 1, 2, \ldots n$ and $x_A = \eta_{AB} x^B$. The Lie algebra has the form

$$[L_{AB}, L_{CD}] = \eta_{AD} L_{BC} + \eta_{BC} L_{AD} + \eta_{AC} L_{DB} + \eta_{BD} L_{CA}.$$  (53)

In particular the Lie group $SO(4, 2)$ is the isometry group of the flat space with metric signature $(+ + + - - -)$ and leaves Minkowski spacetime, with metric $(- + + +)$, conformally invariant.

This Lie group is called the conformal group. We can choose the following basis

$$M_{\alpha\beta} = L_{\alpha\beta};$$
$$T_\alpha = L_{\alpha 5} + L_{\alpha 6};$$
$$K_\alpha = L_{\alpha 5} - L_{\alpha 6};$$
$$D = L_{56}$$  (54)

where $\alpha = 0, 1, 2, 3$ and $\alpha = 0 \equiv A = 4$. We can write the Lie algebra in terms of the basis (54) as

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \eta_{\alpha\delta} M_{\beta\gamma} + \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\alpha\gamma} M_{\beta\delta} + \eta_{\beta\delta} M_{\alpha\gamma};$$
$$[T_\alpha, T_\beta] = 0; \quad [T_\alpha, M_{\beta\gamma}] = \eta_{\beta\alpha} T_\gamma - \eta_{\gamma\alpha} T_\beta;$$
$$[K_\alpha, K_\beta] = 0; \quad [K_\alpha, M_{\beta\gamma}] = \eta_{\beta\alpha} K_\gamma - \eta_{\gamma\alpha} K_\beta;$$
$$[D, K_\alpha] = K_\alpha; \quad [D, M_{\alpha\beta}] = 0; \quad [T_\alpha, D] = T_\alpha;$$
$$[T_\alpha, K_\beta] = 2(\eta_{\alpha\beta} D - M_{\alpha\beta}).$$  (55)

The Poincare group $ISO(3, 1)$ is a subgroup of the conformal group $SO(4, 2)$. In the basis (54) it is clear that the subalgebra consisting of elements $T_\alpha$ and $M_{\beta\gamma}$ is isomorphic to $iso(3, 1)$. In addition, the subalgebra of $so(4, 2)$ consisting of $K_\alpha$ and $M_{\beta\gamma}$ is also isomorphic to the algebra $iso(3, 1)$. The group of translations is an invariant subgroup of $ISO(3, 1)$ and so the Lie group $ISO(3, 1)$ is nonsemi-simple.

The de Sitter and anti-de Sitter groups $SO(4, 1)$ and $SO(3, 2)$ are also subgroups of the conformal group $SO(4, 2)$. These are simply the subgroups preserving the flat metrics with signatures $(+ + + + -)$ and $(+ + + - -)$ respectively. Thus, the algebra $so(4, 2)$ contains the subalgebras $so(4, 1)$ and $so(3, 2)$, see appendix D.
Thus we have, for example, the following embedded subalgebras

\[
\begin{align*}
so(4,2) & \supset iso(3,1) \supset iso(3) \supset so(3) \\
& \supset so(3,1) \supset so(3) \\
so(4,2) & \supset so(4,1) \supset so(4) \supset so(3) \\
& \supset iso(3) \supset so(3) \\
& \supset so(3,1) \supset so(3) \\
so(4,2) & \supset so(3,2) \supset so(3,1) \supset so(3)
\end{align*}
\]

(56)

For more details of the subgroups of the conformal group \(SO(4,2)\), see Beckers et al [10] and references therein.

D  The de Sitter and anti-de Sitter Lie algebras

Let us consider the Lie algebras of the isometry groups of the de Sitter and anti-de Sitter spacetimes.

For the Lie algebra \(so(4,1)\) we can choose the following basis

\[
\begin{align*}
M_{ij} &= L_{ij}; \\
T_i &= L_{i4} + L_{i5}; \\
K_i &= L_{i4} - L_{i5}; \\
D &= L_{45}
\end{align*}
\]

(57)

where \(i = 1, 2, 3\). We can write the Lie algebra in terms of the basis \([57]\) as

\[
\begin{align*}
[M_{ij}, M_{kl}] &= \eta_{il}M_{jk} + \eta_{jk}M_{il} + \eta_{ik}M_{jl} + \eta_{jl}M_{ik}; \\
[T_i, T_j] &= 0; \quad [T_i, M_{jk}] = \eta_{ij}T_k - \eta_{ik}T_j; \\
[K_i, K_j] &= 0; \quad [K_i, M_{jk}] = \eta_{ij}K_k - \eta_{ik}K_j; \\
[D, K_i] &= K_i; \quad [D, M_{ij}] = 0; \quad [T_i, D] = T_i; \\
[T_i, K_j] &= 2(\eta_{ij}D - M_{ij}).
\end{align*}
\]

(58)

In this basis it is clear that the elements \(\{T_i, M_{jk}\}\) form the Lie algebra \(iso(3)\). We can also define

\[
\begin{align*}
T_i &= L_{i0} \\
K_i &= L_{i5} \\
D &= L_{05}
\end{align*}
\]

(59)

where \(i = 1, 2, 3\) and \(\alpha = 0 \equiv A = 4\). We can then write the de Sitter and anti-de Sitter Lie algebras as

\[
\begin{align*}
[T_i, L_{jk}] &= \eta_{ij}T_k - \eta_{ik}T_j; \quad [T_i, T_j] = \mp L_{ij};
\end{align*}
\]
\[
[L_{ij}, L_{kl}] = \eta_{il} L_{jk} + \eta_{jk} L_{il} + \eta_{ik} L_{lj} + \eta_{jl} L_{ki};
\]
\[
[K_i, L_{jk}] = \eta_{ij} K_k - \eta_{ik} K_j; \quad [K_i, K_j] = L_{ij};
\]
\[
[D, L_{ij}] = 0; \quad [T_i, D] = \pm K_i; \quad [K_i, D] = T_i;
\]
\[
[T_i, K_j] = -\eta_{ij} D;
\]

where we take the upper signs for \(so(4,1)\) and the lower signs for \(so(3,2)\). Thus we can see that the elements \(T_i\) and \(L_{jk}\) form the subalgebra \(so(4)\) or \(so(3,1)\) in the cases \(so(4,1)\) and \(so(3,2)\) respectively. In both cases the elements \(K_i\) and \(L_{jk}\) form the subalgebra \(so(3,1)\).

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| CKV | conformal factor | type |
|-----|-----------------|------|
| T_α | \( \phi = 0 \)    | KV   |
| M_{αβ} | \( \phi = 0 \)  | KV   |
| D   | \( \phi = 1 \)  | HKV  |
| K_α | \( \phi = 2\kappa_α \) | SCKV |

Table I: The conformal factors for the CKV of Minkowski spacetime.

| CKV | conformal factor | type |
|-----|-----------------|------|
| T_0 | \( \phi = S^{-1/2} \partial S/\partial \tau - k/2 \epsilon K^{-1}_- K^{-1}_+ \) | CKV |
| T_i | \( \phi = -kK^{-1}_- \bar{x}^i (\epsilon S^{-1} \partial S/\partial \tau + (1 - k\epsilon^2)^{1/2}) \) | CKV |
| M_{0i} | \( \phi = K^{-1}_- \bar{x}^i (-1 - k\epsilon^2)^{1/2} S^{-1} \partial S/\partial \tau + k\epsilon \) | CKV |
| M_{ij} | \( \phi = 0 \) | KV |
| D   | \( \phi = K^{-1}_- (\epsilon S^{-1} \partial S/\partial \tau + (1 - k\epsilon^2)^{1/2}) \) | CKV |
| K_0 | \( \phi = -2S^{-1} \partial S/\partial \tau F^{-2} (2\epsilon^2 K K^{-1}_- + (\epsilon^2 + r^2)A) - 2\epsilon K K^{-1}_- \) | CKV |
| K_i | \( \phi = 2S^{-1} \partial S/\partial \tau F^{-2} K^{-1}_+ \epsilon \bar{x}^i (2FK K^{-1}_- + k(-\epsilon^2 + r^2)) + 2\bar{x}^i K^{-1}_- \) | CKV |

Table II: The conformal factors for the CKV of Robertson-Walker spacetimes.

| CKV | conformal factor | type |
|-----|-----------------|------|
| P_0 | \( \phi = S^{-1} \partial S/\partial \tau \) | CKV |
| H   | \( \phi = K^{-1}_- (S^{-1} \partial S/\partial \tau (1 - k\epsilon^2)^{1/2} - k\epsilon) \) | CKV |
| P_i | \( \phi = 0 \) | KV |
| Q_i^* | \( \phi = -kK^{-1}_- \bar{x}^i (\epsilon S^{-1} \partial S/\partial \tau + (1 - k\epsilon^2)^{1/2}) \) | CKV |

Table III: The conformal factors for certain linear combinations of Robertson-Walker CKV.
|     | \(H\) | \(P_k\) | \(M_{0k}\) | \(M_{kl}\) | \(\bar{Q}_k^*\) | \(P_0\) | \(D\) |
|-----|-------|--------|-----------|-----------|----------------|-------|------|
| \(H\) | 0     | \(-kM_{0k}\) | \(-P_k\) | 0         | 0              | \(kD\) | \(P_0\) |
| \(P_i\) | \(kM_{0i}\) | \(-kM_{ik}\) | \(-\eta_{ik}H\) | \(\eta_{ik}P_l - \eta_{il}P_k\) | \(-k\eta_{ik}D\) | 0 | \(\bar{Q}_i^*\) |
| \(M_{0i}\) | \(P_i\) | \(\eta_{ik}H\) | \(M_{ik}\) | \(\eta_{ik}M_{0l} - \eta_{il}M_{0k}\) | \(\eta_{ik}P_0\) | \(\bar{Q}_i^*\) | 0 |
| \(M_{ij}\) | 0 | \(\eta_{jk}P_j - \eta_{ik}P_j\) | \(\eta_{jk}M_{0j} - \eta_{ik}M_{0j}\) | \(so(3)\) | \(\eta_{jk}\bar{Q}_i^* - \eta_{ik}\bar{Q}_j^*\) | 0 | 0 |
| \(\bar{Q}_i^*\) | 0 | \(k\eta_{ik}D\) | \(-\eta_{ik}P_0\) | \(\eta_{ik}\bar{Q}_i^* - \eta_{il}\bar{Q}_k^*\) | \(kM_{ik}\) | \(-kM_{0i}\) | \(P_i\) |
| \(P_0\) | \(-kD\) | 0 | \(-\bar{Q}_k^*\) | 0 | \(kM_{0k}\) | 0 | \(H\) |
| \(D\) | \(-P_0\) | \(-\bar{Q}_k^*\) | 0 | 0 | \(-P_k\) | \(-H\) | 0 |

Table IV: The Lie algebra in terms of the basis of Maartens and Maharaj. \(so(3)\) is shorthand for the commutation relations 

\[ [M_{ij}, M_{mn}] = \eta_{jm}M_{in} + \eta_{jn}M_{mi} + \eta_{im}M_{nj} + \eta_{in}M_{jm}. \]
Table V: The Lie algebra in terms of the continuous basis.

|     | D             | P₀          | K₀          | K₁          | M₀          | M₁          | M₂          |
|-----|---------------|-------------|-------------|-------------|-------------|-------------|-------------|
| D   | (P₀ - \(\frac{1}{2}K₀\)) | 0           | 0           | -2M₀k       | η₀k(P₀ + \(\frac{1}{2}K₀\)) | η₀kP₀ - \(\frac{1}{2}k\)P₀ | η₀kM₀ - \(\frac{1}{2}k\)M₀ |
| K₀  | 0             | -2M₀k       | η₀kK₀       | -2M₀K₀      | 0           | 0           | 0           |
| K₁  | -2M₀k         | η₀kK₀       | 2(M₀K₀ - \(\frac{1}{2}k\)D) | η₀kK₀ - \(\frac{1}{2}k\)K₀ | M₀ - \(\frac{1}{2}k\)M₀ | η₀kM₀ - \(\frac{1}{2}k\)M₀ | η₀kK₀ - \(\frac{1}{2}k\)K₀ |
| K₂  | -2D           | 0           | 0           | 0           | K₀          | 0           | 0           |
|     | (P₀ + \(\frac{1}{2}K₀\)) | 0           | -K₀         | η₀kP₀ - \(\frac{1}{2}k\)P₀ | η₀kM₀ - \(\frac{1}{2}k\)M₀ | 0           | 0           |
|     | 0             | -K₀         | -K₀         | 0           | 0           | 0           | \(\frac{1}{2}K₀\) | D             |
Table VI: The Lie algebra in terms of the de Sitter basis.

|    | H         | P_k       | M_{0k}   | M_{kl}   | K_k       | K_0      | D                  |
|----|-----------|-----------|----------|----------|-----------|----------|--------------------|
| H  | 0         | −RM_{0k}  | −P_k     | 0        | −2M_{0k}  | −2D      | H − \frac{R}{2}K_0|
| P_i| RM_{0i}   | −RM_{ik}  | −\eta_{ik}(P_0 + \frac{R}{2}K_0) | \eta_{ik}P_l − \eta_{il}P_k | 2(\eta_{ik}D − M_{ik}) | 2M_{0i} | P_i − \frac{R}{2}K_i |
| M_{0i}| P_i | \eta_{ik}(P_0 + \frac{R}{2}K_0) | M_{ik}   | \eta_{ik}M_{0l} − \eta_{il}M_{0k} | \eta_{ik}K_0 | K_i     | 0                  |
| M_{ij}| 0        | \eta_{ik}P_i − \eta_{ik}P_j | \eta_{ik}M_{0i} − \eta_{ik}M_{0j} | so(3) | \eta_{ik}K_i − \eta_{ik}K_j | 0       | 0                  |
| K_i| 2M_{0i}  | 2(M_{ki} − \eta_{ik}D) | −\eta_{ik}K_0 | \eta_{ik}K_l − \eta_{il}K_k | 0       | 0       | −K_i               |
| K_0| 2D       | −2M_{0k}  | −K_k     | 0        | 0         | 0       | −K_0               |
| D | −(H − \frac{R}{2}K_0) | −(P − \frac{R}{2}K_k) | 0        | 0         | K_k      | K_0     | 0                  |
Figure 1: The subalgebra structure is shown schematically for the basis of Maartens and Maharaj [32]. The algebras appearing in square brackets apply in the case $k < 0$. 
Figure 2: The subalgebra structure is shown schematically for the continuous basis (35). The algebras appearing in square brackets apply in the cases $k = 0$ and $k < 0$ respectively.
Figure 3: The subalgebra structure is shown schematically for the de Sitter basis (40). The 10 vector fields $\mathbf{H}; \mathbf{P}_i; \mathbf{M}_{0i}; \mathbf{M}_{ij}$ form the KV subalgebra of the de Sitter basis. The algebras appearing in square brackets apply in the cases $R = 0$ and $R < 0$ respectively.