Coupled Möbius Maps as a Tool to Model Kuramoto Phase Synchronization

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We propose Möbius maps as a tool to model synchronization phenomena in coupled phase oscillators. Not only does the map provide fast computation of phase synchronization, it also reflects the underlying group structure of the sinusoidally coupled continuous phase dynamics. We study map versions of various known continuous-time collective dynamics, such as the synchronization transition in the Kuramoto-Sakaguchi model of non-identical oscillators, chimeras in two coupled populations of identical phase oscillators, and Kuramoto-Battogtokh chimeras on a ring, and demonstrate similarities and differences between the iterated map models and their known continuous-time counterparts.

I. INTRODUCTION

Ensembles of sinusoidally coupled phase oscillators are widely adopted as canonical models for synchronization in various scientific and engineering inquiries. For instance, models of coupled phase oscillators have been successfully applied to functional connectivity of the human brain, neuronal oscillatory behaviors, and neural encoding. Increasingly, they also serve as a computational tool in machine learning and artificial intelligence based on oscillatory neural networks, which opens up a new perspective for hardware implementations.

The most popular models in the field, the Kuramoto-Sakaguchi model and the Winfree model are formulated as systems of ordinary differential equations for coupled phase oscillators. The goal of this paper is to propose a discrete-time analogue of the Kuramoto-Sakaguchi model, a system of globally coupled maps that has similar dynamical properties, and which also provides fast and exact computation of the dynamics in discrete-time steps. Globally coupled maps have been intensively studied in the literature, often with emphasis on the collective dynamics of intrinsically chaotic units.

Among the existing coupled maps, globally coupled circle maps are ideal for studying synchronization phenomenon due to their periodic domains. The simplest and most widely used circle map is the sine circle map \( \theta \rightarrow \theta + \Omega + \varepsilon \sin \theta \), which has been explored in the context of global coupling as well as in non-trivial coupling networks such as computational neural networks. However, the coupled sine circle maps have several properties different from that of the Kuramoto-Sakaguchi model. For example, a known property of the continuous-time Kuramoto-Sakaguchi model is that clustering, i.e., the formation of several distinct synchronized groups, cannot occur. However, for the sine circle maps, when the map parameters are tuned to regions which do not produce chaos in a single map (i.e., the mapping remains one-to-one), the iterated map dynamics for identical units governed by the same mean-field nevertheless produces various complex cluster states.

As shown in previous literature, the propagator of continuous-time sinusoidally forced phase oscillators has the form of a Möbius map (MM). The Möbius transform lies at the heart of the low dimensional dynamical theory for globally forced populations of continuous-time phase oscillators formulated by Watanabe and Strogatz (WS). There, the Möbius transform is used to convert the original phase variables to new conserved quantities, such that the time-varying transformation parameters obey a simple low-dimensional system of ordinary differential equations.

In this paper, we implement a Möbius map, inspired from the aforementioned Möbius transform, as the basic circle map. The main arguments for studying synchronization using Möbius map are threefold. First, one distinct feature of a Möbius map over other circle maps is that a single map does not possess higher-order phase-locked states (higher-order Arnold tongues), and instead exhibits only the simplest first-order Arnold tongue. Therefore by using a Möbius map, complex clustering can be avoided by default. Second, coupled Möbius maps under positive coupling qualitatively agrees with Kuramoto-type coupled dynamics in continuous time, and can be used for fast computation of desired Kuramoto dynamics by making larger computational steps. Third, Möbius maps under negative coupling produce new and interesting dynamics, different from the continuous-time counterparts.

By using an MM model, the fundamental properties of the continuous-time models are preserved and many synchronization phenomena can be reproduced. One such
property is the WS integrability mentioned above. Another property is the applicability of the Ott-Antonsen (OA) ansatz [32], under which the existence of a low-dimensional manifold can be established for an infinite population of globally coupled continuous-time phase oscillators. Similarly for maps, we can establish a low dimensional dynamics in an ensemble of phases oscillators evolved under identical Möbius mapping.

The plan of the paper is as follows. In Sec. II we first review the general form of the complex Möbius map and discuss its group properties. We discuss its single-map dynamics under function iteration and fixed parameters. Then we present a real form of the MM to be used in numerical calculations. Next, by allowing the parameters of the map to vary in time and applying the group properties, we study the low dimensional dynamics in globally coupled identical maps, and make a connection to the WS and OA theory. We also briefly discuss the connection of Möbius maps to homographic maps. In Sec. III we find the Möbius map which solves the Adler equation with piece-wise constant parameters, and suggest a model of globally coupled, non-identical Möbius maps analogous to the Kuramoto-Sakaguchi model. Lastly, in Sec. IV we compute synchronizing dynamics using the map analogues of several systems which have been studied extensively before in the context of the Kuramoto-Sakaguchi dynamics. We discuss globally coupled MMs with frequency heterogeneity, chimera states in two populations of identical phase oscillators with different intra- and inter-population coupling [33], and chimeras on a periodic lattice of identical oscillators with non-local coupling [33] [36]. In all examples, known dynamical behaviour of the smooth dynamics can be qualitatively reproduced under positive coupling, and an interesting new dynamical behaviour can be found for negative coupling, under which the familiar continuous-time dynamics would simply be incoherent or asynchronous.

II. MÖBIUS MAP AND ITS PROPERTIES

A. Basic properties

In order to model the phase dynamics in discrete-time steps we are looking for a map from the unit circle $S^1$ to itself. Examples of global coupling of such one dimensional maps on a periodic domain include the sine circle map [18] [23] [25] [37], more general circle maps [19], as well as chaotic maps such as the Bernoulli map [22]. The Möbius map we propose is a one-to-one mapping on the open unit disc $\mathbb{D}$ in the complex plane and its boundary $S^1$. The set of these maps is a Lie group, the Möbius group, with parameters $(q, \psi) \in \mathbb{D} \times S^1$, i.e., $|q| < 1$ and $\exp(i\psi) \in S^1$. The map for any complex number $z$ in the closed unit disk, $z \in \{ \mathbb{D} \cup S^1 \}$, can be written as a composition of a rotation by an angle $\psi$, and a directional contraction

$$ C_q : z \rightarrow \frac{q + z}{1 + q^*z} \quad (2) $$

in the direction of complex parameter $q$, and with the degree of contraction controlled by $|q|$. We adopt the standard form of the MM $\mathbb{M}$ as

$$ \mathcal{M}_{q,\psi} (z) = C_q \circ \mathcal{R}_\psi (z) = \frac{q + e^{i\psi}z}{1 + q^*e^{i\psi}z} \quad (3) $$

with $q^*$ denoting the complex conjugate of $q$. The phase $\varphi$ on the unit circle, $\exp(i\varphi) = z$, $z \in S^1$, is therefore transformed by

$$ \mathcal{M}_{q,\psi} (e^{i\varphi}) = \frac{q + e^{i(\varphi + \psi)}}{1 + q^*e^{i(\varphi + \psi)}} \quad (4) $$

The rotational actions commute: $\mathcal{R}_{\psi_1} \circ \mathcal{R}_{\psi_2} = \mathcal{R}_{\psi_2} \circ \mathcal{R}_{\psi_1} = \mathcal{R}_{\psi_1 + \psi_2}$, with the inverse of the rotation $\mathcal{R}_\psi^{-1} = \mathcal{R}_{-\psi}$. The inverse of the contraction is $C_q^{-1} = C_{-q}$ such that

$$ \mathcal{M}_{q,\psi}^{-1} = \mathcal{R}_{-\psi} \circ C_{-q} \quad (5) $$

Rotational symmetry is expressed as

$$ C_q = \mathcal{R}_{-\psi} \circ C_{qe^{i\psi}} \circ \mathcal{R}_\psi \quad (6) $$

The identity map is $\mathcal{M}_{0,0}$.

Under a functional composition of two MMs, a very useful property is that the new map parameter of the composite map is itself expressed by a MM using the parameters of the component maps:

$$ \mathcal{M}_{q_2,\psi_2} (z) = \mathcal{M}_{q_1,\psi_1} \circ \mathcal{M}_{q_0,\psi_0} (z) = \frac{\mathcal{M}_{q_1,\psi_1} (q_0) + C_{q_1 q_0} e^{i\psi_1} e^{i\psi_0} z}{1 + \mathcal{M}_{q_1,\psi_1}^* (q_0) C_{q_1 q_0} e^{i\psi_1} e^{i\psi_0}} \quad (7) $$

or equivalently,

$$ q_2 = \mathcal{M}_{q_1,\psi_1} (q_0) \quad e^{i\psi_2} = C_{q_1 q_0} e^{i\psi_1} \cdot e^{i\psi_0} \quad (8) $$

Hence MM forms a group under functional composition.

B. Dynamics of the iterated Möbius map

Here we shortly discuss the iterated map dynamics of a single MM with constant parameters $q$ and $\psi$:

$$ z^{(n+1)} = \mathcal{M}_{q,\psi} (z^{(n)}) \quad (9) $$

where $n = 0, 1, \ldots$ is a discrete time index.

To find the fixed points of the discrete dynamics under Eq. (9), we solve the quadratic equation

$$ z^2 - \frac{e^{i\psi} - 1}{q^*e^{i\psi}} z - \frac{q}{q^*e^{i\psi}} = 0 \quad (10) $$




Eq. (10) has two solutions \( z_1 \) and \( z_2 \) with the properties
\[
z_1 z_2 = -\frac{q}{q^*} e^{-i\psi}, \quad z_1 + z_2 = \frac{e^{i\psi} - 1}{q^* e^{i\psi}}. \quad (11)
\]
From the first property it follows that \(|z_1||z_2| = 1\), which means that either the two fixed points are on the unit circle, or one fixed point is inside and the other outside the unit circle. We make the general ansatz
\[
z_1 = \kappa e^{i(\xi + \Delta)}, \quad z_2 = \frac{1}{\kappa} e^{i(\xi - \Delta)}. \quad (12)
\]
Denoting \( q = \rho \cdot \exp(i\varphi) \) with \( 0 \leq \rho < 1 \), we obtain from (11) the following two relations:
\[
\xi = \varphi - \frac{\psi - \pi}{2}, \quad (13)
\sin \frac{\psi}{2} = \frac{\rho}{2} \left[ \left( \kappa + \frac{1}{\kappa} \right) \cos \Delta + i \left( \kappa - \frac{1}{\kappa} \right) \sin \Delta \right]. \quad (14)
\]
The two fixed points do not uniquely determine the Möbius group parameters \( q \) and \( \psi \). In the first regime, the two fixed points are on the unit circle, which means \( \kappa = 1 \). As a result, the second relation (14) is simplified to
\[
\rho \cos \Delta = \sin \frac{\psi}{2}. \quad (15)
\]
The condition for fixed points on the unit circle is therefore
\[
\rho > \left| \sin \frac{\psi}{2} \right|. \quad (16)
\]
One of the fixed points is stable and the other unstable, so the dynamics of the single Möbius map is trivial, and the rotation number is 0. When equality holds in Eq. (16), it corresponds to the tangent bifurcation point, where the two fixed points merge into one.

In the second regime, \( \kappa < 1 \), i.e., \( z_1 \) is inside the unit circle, then Eq. (14) yields two results
\[
\Delta = 0 \quad (17)
\kappa = \rho^{-1} \left( \sin \frac{\psi}{2} \pm \sqrt{\sin^2 \frac{\psi}{2} - \rho^2} \right). \quad (18)
\]
For \( \kappa \) to be a real number, \( \rho \leq |\sin(\psi/2)| \) must be satisfied, which is the exact opposite condition from Eq. (16).

Under this set of map parameters, i.e., \( \rho \leq |\sin(\psi/2)| \), map (9) shows rotational dynamics, which can be reduced to a pure rotation by virtue of a transformation which is also an MM
\[
y^{(n)} = C_{-z_1} \left( z^{(n)} \right) \quad (19)
\]
The resulting pure rotational dynamics is
\[
y^{(n+1)} = C_{-z_1} \circ \mathcal{M}_{\eta, \psi} \circ C_{z_1} \left( y^{(n)} \right) = \mathcal{R}_\eta \left( y^{(n)} \right), \quad (20)
\]
with the fixed point \( z_1 = \kappa \cdot \exp(i\xi) \) as the group parameter, and the rotation number is
\[
\eta = 2 \arctan \left( \frac{\tan \frac{\psi}{2} \sqrt{1 - \rho^2 \sin^2 \frac{\psi}{2} \frac{1}{2}}}{\frac{\rho}{2} \sin^2 \frac{\psi}{2}} \right). \quad (21)
\]
Eq. (21) shows that in this second regime, the rotation number \( \eta \) is a smooth function of the map parameters \( \psi \) and \( \rho \). Altogether, the above analysis shows that the iterated single (uncoupled) MM dynamics with fixed parameters has only one Arnold tongue given by (16), a region of synchronous phase locking, with rotation number zero (and when rotational map parameter \( \psi \) takes its values on a real line, it has an Arnold tongue with an integer rotation number). For all other non-zero rational values of \( \eta \) there are no extended regions of phase locking as found typically in other types of circle maps.

Explicitly, this implies that clustering dynamics cannot occur under an iterated map dynamics under MM with constant parameters.

Incidentally, like all invertible circle maps (those mapping the unit circle to itself in a one-to-one way), chaotic phase dynamics cannot occur under the iterated map dynamics of MM, regardless of whether the map parameters are constant or time-varying, even including chaotic sequences of map parameters.

C. Real form of Möbius map

Compared to the complex form of the Möbius map (3), the following real form of MM is more suitable for numerical implementation in programming languages that do not natively support a data type for complex numbers. Denoting \( z^{(n)} = \exp(i \varphi^{(n)}) \), \( q = \rho \cdot \exp(i \varphi) \), and using the identity \( \exp(is) = (1 + i \tan \frac{s}{2})(1 - i \tan \frac{s}{2})^{-1} \), we obtain from (3)
\[
\tan \frac{\varphi^{(n+1)} - \varphi}{2} = \frac{1 - \rho \tan \frac{\varphi^{(n)} + \psi - \varphi}{2}}{1 + \rho \tan \frac{\varphi^{(n)} + \psi - \varphi}{2}}. \quad (22)
\]
A further transformation yields still another form of the Möbius map
\[
\sin(\varphi^{(n+1)} - \varphi) = \frac{(1 - \rho^2) \sin(\varphi^{(n)} + \psi - \varphi)}{1 + \rho^2 + 2 \rho \cos(\varphi^{(n)} + \psi - \varphi)},
\cos(\varphi^{(n+1)} - \varphi) = \frac{(1 + \rho^2) \cos(\varphi^{(n)} + \psi - \varphi) + 2 \rho}{1 + \rho^2 + 2 \rho \cos(\varphi^{(n)} + \psi - \varphi)}. \quad (23)
\]
This allows for a straightforward numerical implementation of the Möbius map via the ATAN2 function:
\[
\varphi^{(n+1)} = \varphi + \text{ATAN2} \left[ (1 - \rho^2) \sin(\varphi^{(n)} + \psi - \varphi), (1 + \rho^2) \cos(\varphi^{(n)} + \psi - \varphi) + 2 \rho \right]. \quad (24)
\]
D. Low dimensional evolution of oscillator ensembles under MM

The group property of the MM, as shown by Eqs. (7) and (8), means that the evolution under MM dynamics from any set of initial states is reducible to a three-dimensional evolution of the parameters of the map $q$ and $\psi$. In particular, consider single-map dynamics with discrete sequence of parameters $q^{(n)}, \psi^{(n)}$ that vary in time

$$e^{i\varphi^{(n)}} = M_{q^{(n)}, \psi^{(n)}} \left( e^{i\varphi^{(n-1)}} \right).$$

(25)

The evolution over any time interval from the initial state $\exp(i\varphi^{(0)})$ to the final state $\exp(i\varphi^{(n)})$ can be expressed as an MM

$$e^{i\varphi^{(n)}} = M_{Q^{(n)}, \Psi^{(n)}} \left( e^{i\varphi^{(0)}} \right),$$

(26)

with the composite group parameters governed by the following MM

$$Q^{(n)} = M_{q^{(n)}, \psi^{(n)}} \left( Q^{(n-1)} \right),$$

$$e^{i\Psi^{(n)}} = C_{q^{(n)}Q^{(n-1)}} \left( e^{i\psi^{(n)}} \right) e^{i\psi^{(n-1)}},$$

(27)

due to the group property (7). We note that the MM governing $Q$ is the same as the original MM that governs the phase (25). The evolution of any ensemble of oscillators governed by an identical sequence of M"{o}bius map is always restricted to a three-dimensional manifold described by (27) parametrized by $(Q^{(n)}, \Psi^{(n)})$. Hence the discrete-time dynamics (27) for an ensemble of oscillators identically forced (i.e., under common forcing) is fully analogous to the Watanabe-Strogatz quasi-mean-field equations in the continuous-time case (also see Sec. III A below).

In fact, the transformation of any measure on the unit circle via the Ruelle-Frobenius-Perron operator corresponding to the mapping of the phases is restricted to such a three-dimensional manifold. In the special case of a continuous, uniform phase density $\mu_0(\varphi) = 1/(2\pi)$, its invariant manifold (MM transformed density) is the family of wrapped Cauchy distributions, a.k.a. a univariate Poisson kernel, as shown by Ref. 27. The invariant manifold exactly corresponds to the Ott-Antonsen invariant manifold in the continuous-time case, so we shall call this family the OA manifold. Consider $\mu_Q = M_{Q, \Psi} [\mu_0] = C_Q [\mu_0]$, where the bracket denotes the functional transformation of the density of phases, via Ruelle-Perron-Frobenius operator of the maps $M_{Q, \Psi}$ and $C_Q$, both transforming the phases on the unit circle. Because $\mu_0$ is uniform, and is invariant under rotation, i.e., $\mathcal{R}_\Psi [\mu_0] = \mu_0$, the density $\mu_Q$ is independent of $\Psi$. Then the following holds for the characteristic function or Fourier transform of the phase density

$$\langle e^{ik\varphi} \rangle_{\mu_Q} = \int_0^{2\pi} e^{ik\varphi} \mu_Q(\varphi) d\varphi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left( M_{Q, \Psi} (e^{i\varphi}) \right)^k d\varphi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{Q + e^{i(\Psi + \varphi)}}{1 + Q^* e^{i(\Psi + \varphi)}} \right)^k d\varphi$$

$$= \frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{z} \left( \frac{Q + z}{1 - Q^* z} \right)^k dz = Q^k$$

(28)

where the last integral is a complex contour integral with a simple pole $z = 0$ inside and a $k$th-order pole $z = (Q^*)^{-1}$ outside the unit circle. In the derivation above we have also used the fact that the integral over the unit circle with respect to the transformed density $\mu_Q$ is equal to the integral of the transformed circle $M_{Q, \Psi} (S^1)$ with respect to the uniform density $\mu_0$.

Eq. (28) shows that the first circular moment of the phase distribution is $\langle e^{ik\varphi} \rangle_{\mu_Q} = Q$, and all higher moments are integer powers of $Q$. Therefore, on the OA manifold, the MM parameter $Q$ that transforms a uniform density to the wrapped Cauchy density is the first circular moment of the phase distribution, i.e., the usual Kuramoto order parameter

$$Q = Z = \langle e^{i\varphi} \rangle_{\mu_Q}.$$

(29)

Incidentally, the explicit form of the wrapped Cauchy probability density, corresponding to the set of the moments (28), reads

$$\mu_Q(\varphi) = \frac{1}{2\pi} \left( 1 - |Q|^2 \right) e^{-|Q|e^{i\varphi}}.$$

(30)

From result (29) and the group property (27), the evolution of the mean-field $Z$ on the unit disc $\mathbb{D} \cup S^1$ can then be expressed by the same MM that transforms the phases on the unit circle $S^1$

$$Z^{(n)} = \frac{q^{(n)} + e^{i\psi^{(n)}} Z^{(n-1)}}{1 + q^{*^{(n)}} e^{i\psi^{(n)}} Z^{(n-1)}}.$$

(31)

Eq. (31) is in fact a discrete analogue of the Ott-Antonsen equation in continuous-time dynamics. It is interesting to note that, map (31) has the exact same form as the the map describing the dynamics of one MM (25). This is similar to the fact, that for the continuous-time dynamics, the OA equation for the mean field has the same form as the equation for the individual oscillator, written in terms of $e^{i\varphi}$.

As a side note, both time-varying parameters $q$ and $\psi$ in Eq. (25) can contain noisy components. In this way, more complicated noisy dynamics can also be studied with the discrete map model proposed here for which all the results above still hold.
E. Relation to homographic maps

Griniasty and Hakim [38] studied a family of homographic maps, defined for real $x$
\[ x^{(n+1)} = a - \frac{b}{x^{(n)}}. \]  
(32)

This map leaves a Cauchy distribution density invariant, in the same sense as MM leaves a wrapped Cauchy distribution invariant. A homographic map (32) can be shown to be equivalent to MM, as the latter (Eq. 22) can be rewritten as a fractional linear transformation of the variable $x^{(n)} = \tan (\varphi^{(n)}/2)$ as follows
\[ x^{(n+1)} = \frac{(1 + \rho)A + (1 - \rho)B + x^{(n)}[(1 - \rho) - (1 + \rho)AB]}{1 - \rho - (1 - \rho)AB - x^{(n)}[(1 + \rho)B + (1 - \rho)A]}, \]
\[ A = \tan \frac{\vartheta}{2}, \quad B = \tan \frac{\psi - \vartheta}{2}. \]  
(33)

The Möbius map (33) can be considered as a shifted homographic map (32).

III. RELATION TO ADLER EQUATION AND CONSTRUCTION OF GLOBALLY COUPLED MÖBIUS MAPS

A. Möbius map as a solution to the Adler equation

The Adler equation [39] with constant parameters has the form
\[ \dot{\varphi} = A [\lambda - \sin (\varphi - \beta)], \]  
(34)

where the real-valued parameters consist of the amplitude $A$, the ratio $\lambda$ between the constant bias term and the amplitude of the sinusoidal forcing and the phase shift $\beta$. It is known for $|\lambda| \leq 1$ that the Adler equation has a steady state solution, and for $|\lambda| > 1$, it yields phase rotations.

The solution of the Adler equation over a time interval $\tau$ can be shown to be an MM, where the fixed parameters consist of $A, \lambda, \beta$ and $\tau$, but in the end only the product $A \cdot \tau$ enters the solution. Denoting
\[ \sigma = \sqrt{1 - \lambda^2}, \quad \Gamma = \tanh \left( \frac{A \tau}{2} \sigma \right), \]  
(35)

and using the conventions $\sqrt{-1} = i$ and $\tanh(ix) = \tan(x)$, we can show the solution of Eq. (34) over $\tau$ is an MM
\[ e^{i\varphi(\tau)} = \frac{(\sigma + i\lambda \Gamma)e^{i\varphi(0)} + e^{i\beta \Gamma}}{(\sigma - i\lambda \Gamma) + e^{i\varphi(0)}e^{-i\beta \Gamma}} = M_{q,\varphi} \left( e^{i\varphi(0)} \right) \]  
(36)

with the group parameters
\[ q = \frac{e^{i\beta \Gamma}}{\sigma - i\lambda \Gamma}, \quad e^{i\psi} = \frac{\sigma + i\lambda \Gamma}{\sigma - i\lambda \Gamma}. \]  
(37)

If the solution to the Adler equation after $\tau$ is an MM, then the evolution under iterated MMs is an MM again, as shown by the group property in Sec. II A. Therefore the solution to the problem (34) is an MM after any time interval for fixed parameters $A, \lambda$ and $\beta$. Additionally, for time-dependent parameters $A(t), \lambda(t)$ and $\beta(t)$, which are arbitrary functions of time, the solution to the problem (34) is still a MM.

Consequently, all basic properties of the Adler equation are inherited by the MM. In particular, it is known that for a periodic solution of the Adler equation there is only one Arnold tongue, corresponding to an integer rotation number [40,41]. This matches exactly the property of MM as discussed in Section II B i.e., MM has at most one stable fixed point in the synchronized state.

It is important to note, that Eq. (36) can be viewed as an ideal numerical scheme to simulate the continuous-time Adler equation with small time step $\tau$. In fact, while a standard Euler scheme, which to the linear order in $\tau$ coincides with the MM, breaks the Watanabe-Strogatz partial integrability of the Adler equation [29], the Möbius map (36) preserves this partial integrability, similar to the symplectic schemes for Hamiltonian equations.

In the special case where the amplitude $A = A(t)$ has explicit time dependence, the solution is still an MM. Under this condition, the Adler equation (34) has the form of a linear phase response to a time-dependent forcing, $\dot{\varphi} = H(\varphi)A(t)$. Separation of variables shows that in this case the product $\sigma \tau$ from before is to be replaced by $P$, which is the integral of $A(t)$ over the time interval $\tau$
\[ P = \int_0^\tau A(t)dt. \]  
(38)

The solution to Adler equation with time varying parameter $A(t)$ is hence the “kick map”
\[ K_{\lambda,P,\beta}(z) = \frac{(\sigma + i\lambda \Gamma)z + e^{i\beta \Gamma}}{(\sigma - i\lambda \Gamma) + ze^{-i\beta \Gamma}}, \]  
(39)

with $\Gamma = \tanh(\sigma P/2)$, mapping a phase under a “kick” $P$. This way, $A(t)$ can be any generic function, regardless if it a delta pulse or a constant force.

B. Globally coupled Möbius maps

To formulate a model of globally coupled MMs, we can use the result from Sec. II A where we show a MM as the solution to an Adler equation. We start with the Kuramoto-Sakaguchi model in continuous time
\[ \dot{\varphi}_j = \omega_j + \varepsilon R \sin (\Theta - \varphi_j - \alpha), \quad j = 1, \ldots, N. \]  
(40)

Here $\{\varphi_j\}$ is the set of phases of oscillators in a population of size $N$, indexed by oscillator index $j$. Their natural frequencies $\omega_j$ are generally different. The Kuramoto mean-field of the population is defined according
to

\[ Z = R e^{i\Theta} = \frac{1}{N} \sum_{j=1}^{N} e^{i\varphi_j}. \]  

(41)

One can see that for a constant mean-field parameter $R$ and $\Theta$, the dynamics of each oscillator is governed by the Adler equation [41], where they are effectively decoupled from each other and are only coupled to the constant mean-field. However, because we do not assume a constant mean-field, we must look for ways to implement the changing mean-field via time-dependent parameters of the map. We notice that the parameters $\lambda$ and $A$ enter the resulting MM (36) in a rather complex manner. Therefore, to get a simple discrete-time model that nevertheless carries essential properties of the Kuramoto-Sakaguchi model, we split the evolution in two stages. In stage one, the phases are transformed according to the kick map $K_{0,\varepsilon R(n),\alpha}(\varphi(n), \Theta(n))$, i.e., with a constant kick $P(n) = \varepsilon R(n)$, setting the kick duration $\tau = 1$ without loss of generality. In the second stage phases undergo free rotation over a time interval $T$ according to their natural frequencies $\omega_j$. Combining stage one and two, the resulting model of globally coupled Möbius maps reads

\[ e^{i\varphi_j(n+1)} = e^{i\omega_j T} e^{i\varphi_j(n)} + e^{i(\Theta(n)-\alpha)} \tanh \left( \frac{\varepsilon R(n)}{2} \right), \]

(42)

\[ R(n) e^{i\Theta(n)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\varphi_j(n)}. \]

This discrete-time system has, similar to the Kuramoto-Sakaguchi model [40], the natural frequencies $\omega_j$, the coupling strength $\varepsilon$ and the coupling phase shift $\alpha$ as parameters, with an additional parameter time interval $T$.

IV. MEAN-FIELD DYNAMICS FOR PHASES EVOLVED UNDER COUPLED MÖBIUS MAPS

A. Globally coupled population of Kuramoto-Sakaguchi type

Here we consider the simplest case of one population of globally coupled MMs (42) in the thermodynamic limit. On the Ott-Antonsen manifold, the phase density $\mu_Q(\omega)$ for each value of $\omega$ is a wrapped Cauchy distribution with mean-field $Q(\omega) = \langle e^{i\varphi} \rangle_{\mu_Q(\omega)}$, as derived before from Eq. (42). $Q$ then evolves according to the same map for the individual phase with frequency $\omega$ [42], i.e.,

\[ Q(n+1)(\omega) = e^{i\omega T} \frac{Q(n)(\omega) + e^{i(\Theta(n)-\alpha)} \tanh \left( \frac{\varepsilon R(n)}{2} \right)}{1 + Q(n)(\omega) e^{-i(\Theta(n)-\alpha)} \tanh \left( \frac{\varepsilon R(n)}{2} \right)}. \]

(43)

as derived in Sec. [11]. Integrating over the frequency distribution, we obtain the global mean-field

\[ Z(n) = R(n) e^{i\Theta(n)} = \int_{-\infty}^{\infty} Q(n)(\omega) g(\omega) d\omega. \]

(44)

Similar to the approach of Ott and Antonsen [32], we can assume that $Q(\omega)$ is analytic in the upper half-plane, which allows us to calculate the integral via the residue theorem. For a Lorentzian frequency distribution of mean $\omega_0$ and scale parameter $\gamma$

\[ g(\omega) = \frac{1}{\pi \gamma (\omega - \omega_0)^2 + \gamma^2} \]

we have $Z = Q(\omega_0 + i\gamma)$. Accordingly, the global mean-field evolves as

\[ Z(n+1) = e^{i(\omega_0 - \gamma) T} \left[ \frac{Z(n) + e^{i(\Theta(n) - \alpha)} \tanh \left( \frac{\varepsilon R(n)}{2} \right)}{1 + Z(n) e^{-i(\Theta(n) - \alpha)} \tanh \left( \frac{\varepsilon R(n)}{2} \right)} \right]. \]

(46)

The free rotation $R_{\omega T}(z) = \exp(i\omega T) z$ for identical oscillators is replaced by a rotation with the mean frequency $\omega_0$ of the ensemble and a decay of the mean-field due to population heterogeneity $\gamma$, which denotes the width of the natural frequency distribution.

For this model we can calculate the steady state order parameter $\mathcal{R} = R \exp(\gamma T)$ after each kick implicitly. Because we can always go into the co-rotating frame with the mean frequency $\omega_0$, we can set it to 0 without loss of generality. Setting the order parameter $\mathcal{R}$ equal on both sides of (46)

\[ \mathcal{R}^2 = \left[ \frac{\Gamma e^{-i\alpha} + \mathcal{R} e^{-\gamma T}}{1 + \Gamma e^{i\alpha} \mathcal{R} e^{-\gamma T}} \right]^2 = \]

\[ = \frac{\Gamma^2 + \mathcal{R}^2 e^{-2\gamma T} + 2\mathcal{R} e^{-\gamma T} \cos \alpha}{1 + \Gamma^2 \mathcal{R}^2 e^{-2\gamma T} + 2\Gamma \mathcal{R} e^{-\gamma T} \cos \alpha} \]

(47)

where $\Gamma = \tanh \left( \varepsilon e^{-\gamma T} \mathcal{R}/2 \right)$, we solve a quadratic equation for $\Gamma$, and obtain

\[ \Gamma = \frac{\mathcal{R}}{1 - \mathcal{R}^4 e^{-2\gamma T}} \left[ -(1 - \mathcal{R}^2) e^{-\gamma T} \cos \alpha \right. \]

\[ \pm \sqrt{(1 - \mathcal{R}^2)^2 e^{-2\gamma T} \cos^2 \alpha + (1 - e^{-2\gamma T}) \left( 1 - \mathcal{R}^4 e^{-2\gamma T} \right)} \right]. \]

(48)

Inverting the expression for $\varepsilon$ we obtain

\[ \varepsilon = \frac{2}{\mathcal{R} e^{-\gamma T}} \text{arctanh} (\Gamma). \]

(49)

Eq. (48) and Eq. (49) together allow us to express coupling strength $\varepsilon$ explicitly as a function of the steady state synchronization order parameter $\mathcal{R}$ and to plot them in a bifurcation diagram, as shown in Fig. [1].
The first notable limit of the expression of the bifurcation curve is the existence of two critical coupling strengths for $\tilde{R} \to 0$

$$\varepsilon_{cr} = 2 \left( -\cos \alpha \pm \sqrt{\cos^2 \alpha + e^{2\gamma T}} - 1 \right).$$

(50)

This implies that there is always a positive and a negative critical coupling strength for the incoherent state in globally coupled MMs. The second limit is the limit of identical oscillators $\gamma \to 0$. Then $\tilde{R} = R$ and

$$\Gamma = \tanh \left( \frac{\varepsilon R}{2} \right) = \frac{-\cos \alpha \pm \frac{\cos \alpha}{1 + \varepsilon R^2}}{1 + \frac{\cos \alpha}{1 + \varepsilon R^2}}.$$

(51)

This indicates two lines of fixed points connecting incoherence at $R = 0$ and complete synchronization $R = 1$.

Under negative coupling and identical frequency, there are several regimes for a transition to synchrony. At $\varepsilon_0 = 0$, the stability of complete synchronization and incoherence is exchanged instantly. At $\varepsilon_1 = -4\cos \alpha$, incoherence $R = 0$ becomes unstable, and at $\varepsilon_2 = \log [(1 - \cos \alpha)/(1 + \cos \alpha)]$, complete synchronization $R = 1$ becomes unstable.

The existence of a synchronization transition for strongly repulsively coupled oscillators under discrete time stands in stark contrast to the continuous-time Kuramoto-Sakaguchi model (10). In the continuous case, the order parameter $R$ decreases to zero continuously under negative coupling, whereas in the coupled-maps system a negative forcing strong enough can invert the orientation of the mean-field during one step, and even increases its amplitude.

**B. Two-population chimera**

Here we consider a setup similar to the one studied in Ref. [22] where two populations of identical continuous-time oscillators interact, with each population more strongly coupled to itself than to the other population. To formulate the corresponding MM model, we denote coupled phases in the two populations by their complex exponentials as before, $z_{1,j} = \exp(i\varphi_{1,j})$ and $z_{2,j} = \exp(i\varphi_{2,j})$, and the corresponding mean-field of each population as

$$Z_1 = \frac{1}{N_1} \sum_j z_{1,j}, \quad Z_2 = \frac{1}{N_2} \sum_j z_{2,j}.$$

The forces acting on the populations are linear combinations of these mean-fields

$$Y_1 e^{i\Psi_1} = pZ_1 + (1 - p)Z_2,$$

$$Y_2 e^{i\Psi_2} = pZ_2 + (1 - p)Z_1,$$

(52)

where parameter $p$ defines relative strengths of intra- and inter-population couplings. The resulting MMs for the phase variables are

$$z_{1,j}^{(n+1)} = \frac{z_{1,j}^{(n)} + e^{i(\Psi_{1,j}^{(n)} - \alpha)} \tanh(\frac{\gamma_1}{2})}{1 + z_{1,j}^{(n)} e^{-i(\Psi_{1,j}^{(n)} - \alpha)} \tanh(\frac{\gamma_1}{2})},$$

$$z_{2,j}^{(n+1)} = \frac{z_{2,j}^{(n)} + e^{i(\Psi_{2,j}^{(n)} - \alpha)} \tanh(\frac{\gamma_2}{2})}{1 + z_{2,j}^{(n)} e^{-i(\Psi_{2,j}^{(n)} - \alpha)} \tanh(\frac{\gamma_2}{2})},$$

(53)

where $\alpha$ is the common phase shift and $\varepsilon$ is the common coupling strength. Here we set the identical frequency to zero by going into a co-rotating frame with the common natural frequency.

In the thermodynamical limit, i.e., $N_1, N_2 \to \infty$, assuming that both systems are on the OA manifold, we can write the dynamics of the coupled system as two coupled maps of the order parameters $Z_{1,2}$ (according to

![FIG. 1: Steady state order parameter $\tilde{R}$ as a function of coupling strength (i.e., the bifurcation curve) in the attractively (a)-(b) ($\varepsilon > 0$) or repulsively coupled (c)-(d) ($\varepsilon < 0$) Möbius map model. Without loss of generality we assume the time interval between discrete kicks to be $T = 1.0$. Linearly unstable and stable partially synchronized states are marked by dotted and solid lines, respectively. In (a) and (c), we keep $\alpha = \pi/4$ constant and vary the natural frequency heterogeneity parameter $\gamma$ from zero to 0.5 (from top to bottom). In (b) and (d) we set $\gamma$ to a constant value, $\gamma = 0.05$ in (b) and $\gamma = 0$ in (d), and vary the parameter $\alpha$. In (a)-(b) we see the typical second-order synchronization transition as in the classical Kuramoto-Sakaguchi model with frequency heterogeneity. For negative coupling strengths as in (c)-(d) there can be several transitions, both continuous and discontinuous, even for identical oscillators in (d) with $\gamma = 0$.](image)
FIG. 2: Bifurcation diagram illustrating the stability of the chimera states of the two coupled maps of the mean-fields \( \text{(54)} \). Scatter plots depict the stable solutions (after transient) obtained from the direct simulation of the coupled maps \( \text{(54)} \) showing \( |Z_1| \) (orange) and \( |Z_2| \) (green). The dashed line is the fixed point of the coupled map dynamics found via numerical solver while assuming one of the mean-field is 1 (at full synchrony).

\[
Z_1^{(n+1)} = \frac{Z_1^{(n)} + e^{i(\Psi_1 - \alpha)} \Gamma_1}{1 + Z_1^{(n)} e^{-i(\Psi_1 - \alpha)} \Gamma_1},
\]

\[
Z_2^{(n+1)} = \frac{Z_2^{(n)} + e^{i(\Psi_2 - \alpha)} \Gamma_2}{1 + Z_2^{(n)} e^{-i(\Psi_2 - \alpha)} \Gamma_2},
\]

where \( \Gamma_1 = \tanh(\varepsilon Y_1/2) \), \( \Gamma_2 = \tanh(\varepsilon Y_2/2) \), \( Y_{1,2} \) and \( \Psi_{1,2} \) expressed by Eq. \( \text{(52)} \).

As in Ref. \( \text{[42]} \) we choose \( \alpha = 0.5\pi - 0.025 \), in-group coupling ratio \( p = 0.6 \), and start iterations at initial order parameters \( Z_{1,2}^{(0)} \) with small initial amplitudes, either close to in-phase or to anti-phase. We first evolve the coupled maps \( \text{(54)} \) according to various positive coupling strength \( \varepsilon \). We found it to be qualitatively similar to the continuous limit solutions. Namely, at low coupling strength, chimera states, where one of the population in full synchrony and the other in partial synchrony are obtained. At intermediate coupling strength, depending on the initial conditions \( Z_{1,2}(0) \), either chimera or full synchrony of the two populations is a possible steady state solution, where it is appropriate to speak of a basin of attraction, similar to the continuous case \( \text{(42)} \). At high coupling strength, both populations are in full synchrony at steady state.

For negative \( \varepsilon \), we see four regimes. At \( \varepsilon > \varepsilon_{\text{cr}} \), corresponding to the negative critical coupling strength (derived in Sec. \( \text{[15A]} \)), we observe only the complete asynchronous case with vanishing order parameter. As we decrease \( \varepsilon \), we see first a regime where only chimeras are obtained, then a regime where either chimera or full synchrony is possible depending on the initial conditions, and finally the regime where only full synchrony of both populations is obtained.

Therefore it is possible for negative coupling strength under discrete dynamics to create steady states that are either partially (chimera state) or completely synchronized. This is in contrast with continuous-time dynamics, where under negative coupling both order parameters can only decrease to zero at steady state.

To probe the stability of the chimera state of the coupled maps, we conduct a bifurcation analysis across both positive and negative \( \varepsilon \) regime (Fig. \( \text{[2]} \)). The stable chimera state, as found via integrating the coupled maps, numerically agrees with the fixed points found by numerical solver for stable chimera states. At \( \varepsilon \approx -0.07 \) a period-doubling bifurcation of the chimera amplitude occurs corresponding to a periodic or quasi-periodic mean-field. As \( \varepsilon \) continues to increase to about \(-0.06\) the quasi-periodic orbit collides with full synchronization and both disappear. The asynchronous state becomes stable. In the limit \( \varepsilon \to 0^+ \) the map dynamics corresponds to the continuous-time dynamics discussed in \( \text{[42]} \). The loss of stability of the chimera for a larger positive coupling strength at \( \varepsilon \approx 0.16 \) is again an effect of the discrete map dynamics.

C. Chimera on a ring

The first example of a chimera state for continuous-time oscillators was on a one-dimensional ring with non-local coupling \( \text{[34 35]} \), first explored by Kuramoto and Battogtokh. The oscillators are coupled via a kernel function, which determines the spatial extend of the interactions with their neighbors. We can find similar chimeras with the coupled map model.

The oscillators on the ring have positions \( x_j = 2\pi j/N \), where \( N \) is the total number. Following Ref. \( \text{[54]} \), we have chosen the kernel as \( g_{jm} = 1 + B \cos(x_j - x_m) \), so that the field acting on oscillator \( j \) is calculated as

\[
U_j = R_j e^{i\Theta_j} = \sum_{m=1}^{N} g_{jm} \sin(\varphi_j - \varphi_m - \alpha).
\]

The phases are driven by these local fields according to the Möbius map

\[
z_j^{(n+1)} = \frac{z_j^{(n)} + e^{i(\Theta_j^{(n)} - \alpha)} \tanh(\varepsilon R_j^{(n)})}{1 + z_j^{(n)} e^{-(i(\Theta_j^{(n)} - \alpha))} \tanh(\varepsilon R_j^{(n)})},
\]

where, as before, \( z_j = \exp(i\varphi_j) \).

Similar to the continuous dynamics, we can obtain a stable chimera pattern for a range of positive values of \( \varepsilon \) (e.g. \( \varepsilon = 0.025 \)) (not shown). Same as in the two-population case before, under discrete map dynamics, there exists a regime under large negative coupling
V. CONCLUSION

In this paper we propose a method of modelling synchronizing phase dynamics using a Möbius map. This map precisely reproduces the dynamics of phase oscillators in discrete time. It can be an ideal choice for fast simulation of phase synchronization, since it inherits all the properties of continuous-time phase dynamics. In particular, neither clustering nor chaos under the iteration of a sequence of Möbius maps can occur. All continuous-time models based on the Adler phase equation (i.e., with a pure sinusoidal coupling) can be equivalently studied via Möbius maps. We mention here, that also phase coupling models with pure higher-harmonics couplings can be modelled with correspondingly modified Möbius maps.

With the proposed Möbius map, we have studied map analogues of known continuous-time models for oscillator ensembles with various connection topologies: the globally coupled Kuramoto-Sakaguchi model, two coupled populations of identical oscillators, and identical oscillators on a ring with non-local coupling via cosine or square distance kernel. For small coupling strengths and small free rotation time step, the coupled maps reproduce the dynamics of their continuous-time dynamical counterparts. Especially, we have reproduced known chimera states with the coupled maps under non-local couplings. For large coupling strength, and in particular for large repulsive coupling, the discrete time dynamics can lead to new synchronization phenomena with continuous and discontinuous bifurcations to synchrony. This phenomenon is not observed in the equivalent continuous-time models.

Beyond deterministic models, we can also use the discrete Möbius map approach to simulate stochastic models by simply adding noisy components to the time-varying parameters of the maps.

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