ON THE QUADRATIC VARIATION OF THE MODEL-FREE PRICE PATHS WITH JUMPS

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Abstract. We prove that the model-free typical (in the sense of Vovk) càdlàg price paths with mildly restricted downward jumps possess quadratic variation which does not depend on the specific sequence of partitions as long as these partitions are obtained from stopping times such that the oscillations of a path on the consecutive (half-open on the right) intervals of these partitions tend (in a specified sense) to 0. Finally, we also define quasi-explicit, partition independent quantities which tend to this quadratic variation.

1. Introduction

In this paper we deal with several properties of the quadratic variation of model-free, càdlàg price paths and integrals driven by such paths. In [11] Vovk proved the existence of quadratic variation of càdlàg price paths with mildly restricted jumps (along the sequence of so called Lebesgue partitions). In [6] this result was generalised and the existence of quadratic variation along the same sequence of Lebesgue partitions of càdlàg price paths with mildly restricted downward jumps was proven (this in particular proves the existence of quadratic variation of non-negative càdlàg price paths). Having these results in hand and a pathwise version of the BDG inequality proven in [1], it became possible to define a stochastic integral along càdlàg price paths with mildly restricted downward jumps for broad class of integrands. Moreover, there were also proven some continuity results for such integrals. Other approach for integrators with jumps whose absolute or relative size is bounded by some constant, was presented in [12]. In [12] (see also [13]) there was also introduced a very interesting notion of uniform on compacts quasi-always (ucqa) convergence, which roughly means that the trader becomes infinitely rich immediately after the convergence ceases to hold. Unfortunately we were able only to prove rather weak modes of convergence, for which continuity results (like Theorem 3 and Corollary 4) were available.

Since the existence of quadratic variation of price paths in model-free finance is of utmost importance (also in practical sense, as it corresponds to the well known notion of realized volatility), it is problematic that this fundamental object a priori depends on the choice of partitions. Fortunately, using the mentioned continuity results for model-free integrals, in this note we prove the independence of the quadratic variation of model-free price paths as long as the partitions are obtained from stopping times. This result is analogous to the results obtained for continuous (model-free) price paths by Vovk and Schafer in [13], for continuous semimartingales by Davis, Oblój and Siorka in [3] and for càdlàg semimartingales in Cont and Fournié [2]. Further, we prove another, partition-independent formula for the quadratic variation of model-free price paths with jumps in terms of the truncated variation. Finally, we deal with possibility of the extension of this formula for deterministic càdlàg functions possessing quadratic variation along some sequence of (nested) partitions in the sense
of Föllmer. We start with a technical result, which will be used in the sequel - integration by parts formula for model-free price paths perturbed by an adapted, finite variation path.

2. Notation, definitions and continuity of model-free, pathwise integrals

Let $d \in \{1, 2, \ldots \}$, $T \in (0, +\infty)$, $\mathbb{N} = \{1, 2, \ldots \}$, $\mathbb{R}_+ = [0, +\infty)$. For a topological space $\mathcal{E}$ by $D([0, T], \mathcal{E})$ we will denote the space of all càdlàg functions $\omega : [0, T] \to \mathcal{E}$. Now let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be some non-decreasing function. We will consider the underlying space $\Omega$ which is a subset of the set $\Omega_\psi \subset D \left([0, T], \mathbb{R}^d\right)$ of càdlàg functions with mildly restricted jumps directed downwards, that is $\omega = (\omega^1, \ldots, \omega^d) \in \Omega_\psi$ if it satisfies

$$\omega^i(t-) - \omega^i(t) \leq \psi \left( \sup_{s \in [0, t]} |\omega(s)| \right), \quad t \in (0, T),$$

where $\omega^i(t-) := \lim_{s \to t, s < t} \omega(s)$ for $i = 1, \ldots, d$. The following sample spaces are examples of $\Omega$:

1. $\Omega_\psi := C([0, T], \mathbb{R}^d)$, the space of all continuous functions $\omega : [0, T] \to \mathbb{R}^d$,
2. $\Omega_\psi := D([0, T], \mathbb{R}^d_+)$, the space of all non-negative càdlàg functions $\omega : [0, T] \to \mathbb{R}^d_+$ (here $\psi(x) = x$),
3. $\tilde{\Omega}_\psi$ which is defined as the subset of all càdlàg functions $\omega : [0, T] \to \mathbb{R}^d$ such that

$$|\omega(t) - \omega(t-)| \leq \psi \left( \sup_{s \in [0, t]} |\omega(s)| \right), \quad t \in (0, T),$$

and $\psi : \mathbb{R}_+ \to (0, \infty)$ is a fixed non-decreasing function.

A detailed financial interpretation of the last space can be found in [11] and a generalization of this space allowing for different bounds for jumps directed upwards resp. downwards was recently introduced in [12].

$\Omega$ will be our sample space and for each $t \in [0, T]$, $\mathcal{F}_t$ is defined to be the smallest $\sigma$-algebra on $\Omega$ that makes all functions $\omega \mapsto \omega(s), s \in [0, t]$, measurable and $\mathcal{F}_t$ is defined to be the universal completion of $\mathcal{F}_0$. An event is an element of the $\sigma$-algebra $\mathcal{F}_T$. Stopping times $\tau : \Omega \to [0, T] \cup \{\infty\}$ with respect to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ and the corresponding $\sigma$-algebras $\mathcal{F}_\tau$ are defined as usual. $S$ is the coordinate process, i.e. $S_t(\omega) := \omega(t)$.

Now a process $H : \Omega \times [0, T] \to \mathbb{R}^d$ is called a simple strategy if there exist stopping times $0 = \tau_0 \leq \tau_1 \leq \ldots$ and $\mathcal{F}_{\tau_k}$-measurable bounded functions $h_k : \Omega \to \mathbb{R}^d$, such that for every $\omega \in \Omega$, $\tau_k(\omega) = \tau_{k+1}(\omega) = \ldots \in [0, +\infty]$ from some $k \in \{1, 2, \ldots \}$ on, and

$$H_t(\omega) = h_k(\omega) \mathbf{1}_{[0, t)}(t) + \sum_{k=0}^{+\infty} h_k(\omega) \mathbf{1}_{(\tau_k(\omega), \tau_{k+1}(\omega])}(t).$$

For the simple strategy $H$ the corresponding integral process

$$(H \cdot S)_t(\omega) := \sum_{n=0}^{+\infty} h_n(\omega) \cdot (S_{\tau_{n+1} \wedge t}(\omega) - S_{\tau_n \wedge t}(\omega)) = \sum_{n=0}^{+\infty} h_n(\omega) S_{\tau_n \wedge t, \tau_{n+1} \wedge t}(\omega)$$

is well-defined for all $\omega \in \Omega$ and all $t \in [0, T]$; here we denote $S_{u,v} := S_v - S_u$ for $u, v \in [0, T]$.

The family of simple strategies will be denoted by $\mathcal{H}$. For $\lambda > 0$ a simple strategy $H$ will be called (strongly) $\lambda$-admissible if $(H \cdot S)_t(\omega) \geq -\lambda$ for all $\omega \in \Omega$ and all $t \in [0, T]$. The set of strongly $\lambda$-admissible simple strategies will be denoted by $\mathcal{H}_\lambda$. 

Definition 1. Vovk’s outer measure $\overline{\mathcal{P}}$ of a set $A \subseteq \Omega$ is defined as the minimal superhedging price for $1_A$, that is

$$\overline{\mathcal{P}}(A) := \inf \left\{ \lambda > 0 : \exists (H^n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ s.t. } \forall \omega \in \Omega : \liminf_{n \to \infty} (\lambda + (H^n \cdot S)_T(\omega)) \geq 1_A(\omega) \right\}.$$ 

A set $A \subseteq \Omega$ is called a null set if it has outer measure zero. A property $(P)$ holds for typical price paths if the set $A$ where $(P)$ is violated is a null set.

This definition differs slightly from original Vovk’s definition (see for example [10]). A detailed account on the difference of the original Vovk’s outer measure and the just defined measure is presented in [9, Subsection 2.3]. However, since we will need here some continuity results of the model-free integrals which were obtained for the just defined outer measure, we will use this measure instead of original Vovk’s outer measure.

By [6, Corollary 3.11] (and [11] for $\Omega = \tilde{\Omega}$) we know that there exist a sequence of partitions called the Lebesgue partitions, $\pi^n = \{\pi^n_k; k = 0, 1, \ldots\}, n \in \mathbb{N},$ (see [11, Sect. 5] or [6 Definition 3.1]) such that for typical price path $\omega \in \Omega$ the sequence of discrete quadratic (co)variations

$$Q^{i,j,n}_t(\omega) := \sum_{k=1}^\infty S^i_{\pi^n_{k-1} \wedge t} S^n_j(\omega)^2 S^j_{\pi^n_k \wedge t}(\omega), \quad t \in [0, T],$$

converges in uniform topology to some (càdlàg) function $[0, T] \ni t \mapsto [S^i, S^j]_t$.

We will use the following notation $||S||_T = \left( \sum_{i,j=1}^d [S^i, S^j]_T^2 \right)^{1/2}$.

Following [6] we will identify two processes $X, Y : \Omega \times [0, T] \to \mathbb{R}^d$ $(d = 1, 2, \ldots)$ if

$$\overline{\mathcal{P}}(\omega \in \Omega : ||X(\omega) - Y(\omega)||_\infty > 0) = 0,$$

where for $X, Y : \Omega \times [0, T] \to \mathbb{R}^d$ we define

$$||X(\omega) - Y(\omega)||_\infty := \sup_{0 \leq t \leq T} |X_t(\omega) - Y_t(\omega)|.$$ 

This defines an equivalence relation, and we will write $\mathcal{T}_0(\mathbb{R}^d)$ (or $\mathcal{T}_0$ in short) for the space of its equivalence classes. We equip the space $\mathcal{T}_0(\mathbb{R}^d)$ with the distance

$$d_\infty(X, Y) := E[||X - Y||_\infty \wedge 1],$$

where $E$ denotes an expectation operator defined for $Z : \Omega \to [0, \infty]$ by

$$E[Z] := \inf \left\{ \lambda > 0 : \exists (H^n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ s.t. } \forall \omega \in \Omega : \liminf_{n \to \infty} (\lambda + (H^n \cdot S)_T(\omega)) \geq Z(\omega) \right\}.$$ 

It can be shown that $(\mathcal{T}_0(\mathbb{R}^d), d_\infty)$ is a complete metric space and $(\overline{\mathcal{D}}(\mathbb{R}^d), d_\infty)$ is a closed subspace, where $\overline{\mathcal{D}}(\mathbb{R}^d)$ are those processes in $\mathcal{T}_0(\mathbb{R}^d)$ which have a càdlàg representative.

Definition 2. We will say that a sequence of $X_n \in \mathcal{T}_0$ converges in the outer measure $\overline{\mathcal{P}}$ on a set $A \subseteq \Omega$ to $X \in \mathcal{T}_0$ if for any $\varepsilon > 0$

$$\lim_{n \to +\infty} \overline{\mathcal{P}}(\omega \in A : ||X_n(\omega) - X(\omega)||_\infty > \varepsilon) = 0.$$ 

For $q, M > 0$ let us now define

$$\Omega_{q,M} := \{ \omega \in \Omega : [\omega]_T \leq q, ||\omega||_\infty \leq M \}.$$
It is easy to see that if for any \( q, M > 0 \), \( X_n \) converges in outer measure \( \overline{\mathbb{P}} \) on the set \( \Omega_{q,M}^{\infty} \) to \( X \in \overline{L}_0 \) then \( X \) is unique. For fixed \( q, M > 0 \) let us now introduce the (pseudo-)distance \( d_{\infty,b,M} \) on \( \overline{L}_0 \) which is given by

\[
d_{\infty,b,M}(X,Y) := \mathbb{E}[\|X - Y\|_\infty \wedge 1_{\Omega_{b,M}}].
\]

Now let us define for some fixed \( \epsilon \in (0,1) \) the following metric on \( \overline{L}_0 \)

\[
d^\epsilon_{\infty,\psi}(X,Y) := \sum_{n,m=1}^{\infty} 2^{-(n/2+m)(1+\epsilon)}(\psi/(2^m) \wedge 2^m)^{-1}d_{\infty,2^n,2^m}(X,Y)
\]

One of the main results of \([6]\) is the following (see \([6, \text{Theorem 4.2}]\)).

**Theorem 3.** There exists two metric spaces \((\overline{H}_1, d_{\overline{H}_1})\) and \((\overline{H}_2, d_{\overline{H}_2})\) such that the (equivalence classes of) step functions (simple strategies) are dense in \( \overline{H}_1, \overline{H}_2 \) embeds into \( \overline{\mathcal{D}}(\mathbb{R}^d) \) and the integral map \( I: H \rightarrow (H \cdot S) := \left( \int_{[0,t]} H_s dS_s \right)_{t \in [0,T]} \), defined for simple strategies in \( \mathcal{P}_C \), has a continuous extension that maps \((\overline{H}_1, d_{\overline{H}_1})\) to \((\overline{H}_2, d_{\overline{H}_2})\). Moreover, one has the following continuity estimate

\[
d_{\infty,\psi}((F \cdot S), (G \cdot S)) \lesssim d_{\infty}(F,G)^{1/3},
\]

and one can take \( d_{\overline{H}_1} = d_{\infty} \) and \( d_{\overline{H}_2} = d^\epsilon_{\infty,\psi} \) (defined in \((2)\)).

The metric space \((\overline{H}_1, d_{\overline{H}_1})\) can be chosen to contain the left-continuous versions of adapted càdlàg processes (see \([6, \text{Remark 4.3}]\)). If we replace the filtration \((\mathcal{F}_t)_{t \in [0,T]}\) by its right-continuous version, we can define \((\overline{H}_1, d_{\overline{H}_1})\) such that it contains at least the càglàd adapted processes and, furthermore, such that if \( (F_n) \subset \overline{H}_1 \) is a sequence with \( \sup_{\omega \in \Omega_{q,M}} |F_n(\omega) - F(\omega)|_\infty \rightarrow 0 \), then \( F \in \overline{H}_1 \) and there exists a subsequence \( (F_{n_k}) \) with

\[
\lim_{k \rightarrow \infty} \|(F_{n_k} \cdot S)(\omega) - (F \cdot S)(\omega)\|_\infty = 0
\]

for typical price paths \( \omega \in \Omega \). By \([3, \text{Corollary 4.9}]\) and \([3]\) we also get

**Corollary 4.** For \( a, q, m, M > 0 \) and any \( H \in \overline{H}_1 \) one has

\[
\overline{\mathbb{P}}\{\|(H \cdot S)\|_\infty \geq a\} \cap \{\|H\|_\infty \leq m\} \cap \{|[S]_T| \leq q\} \cap \{|[S]_\infty| \leq M\} \leq (1 + 3dM + 2d\psi(M))^{\frac{6\sqrt{q} + 2 + 2M}{a}}.
\]

### 3. Quadratic variation of model-free càdlàg price paths

#### 3.1. Integration by parts formula for model-free price paths.

The existence of quadratic variation for typical \( \omega \in \Omega \) in the sense outlined in the previous section is equivalent to the existence of quadratic variation in the sense of Norvaïša \([11, \text{Proposition 3}]\). This also allows (see \([11, \text{Sect. 7}]\)) to apply Föllmer’s pathwise Itô formula \([4]\) for typical price paths in \( \Omega \) and in particular to define the pathwise integral \( \int_{(0,T]} f'(\omega(s-))d\omega(s) \) (along the sequence of the Lebesgue partitions) for \( f \in C^2(\mathbb{R}^d, \mathbb{R}) \) when \( d = 1 \). To define the pathwise integral \( \int_{(0,T]} \nabla f(\omega(s-))d\omega(s) \) when \( d = 2, 3, \ldots \) and \( f \in C^2(\mathbb{R}^d, \mathbb{R}) \) or \( f \) is even more general, path-dependent functional, one may use results of Cont and Fournié \([2]\). We also have the following result which we will use to prove the next lemma.
Lemma 5. Let \( \omega, \tilde{\omega} : [0, T] \to \mathbb{R} \) be two càdlàg paths. Assume that \( \tilde{\omega} \) has finite total variation and let us consider two integrals

1. the integral \( \int_{[0,T]} \omega (t-) \, d\tilde{\omega} (t) \) understood as the Lebesgue-Stieltjes integral (with respect to the measure \( d\tilde{\omega} \) given by \( d\tilde{\omega} (a, b] := \tilde{\omega} (b) - \tilde{\omega} (a) \));
2. the integral \( (F) \int_{[0,T]} \omega (t-) \, d\tilde{\omega} (t) \) understood as Föllmer’s integral along the sequence of the Lebesgue partitions \( (\pi^n) \) for the two-dimensional path \((\omega, \tilde{\omega}), \) i.e.

\[
(F) \int_{[0,T]} \omega (t-) \, d\tilde{\omega} (t) := \lim_{n \to +\infty} \sum_{i=1}^{k_n} \omega \left( \tau^n_i \right) \left\{ \tilde{\omega} \left( \tau^n_i \right) - \tilde{\omega} \left( \tau^n_{i-1} \right) \right\},
\]

where \( \tau^n_i = \pi^n_i \land T, \) \( i = 0, 1, \ldots, k_n \) are points obtained from the \( n \)th Lebesgue partition \( \pi^n, \)

\[
\pi^n := \left\{ 0 = \pi^n_0 < \pi^n_1 < \ldots < \pi^n_{k_n-1} < T \leq \pi^n_{k_n} \ldots \right\}.
\]

These two integrals coincide, provided that the latter exists.

Proof. Step 1. First, let us notice that since \( \tilde{\omega} \) has finite total variation, for any \( \varepsilon > 0 \) we may uniformly approximate \( \omega \) by a step function

\[
\omega^\varepsilon (t) := \sum_{i=1}^{N} \omega \left( t_{i-1} \right) 1_{[t_{i-1}, t_i)} (t) + \omega \left( T \right) 1_{[T, T)} (t),
\]

where \( 0 = t_0 < t_1 < \ldots < t_N = T, \) such that

\[
\| \omega^\varepsilon - \omega \|_{\infty} \leq \varepsilon.
\]

To obtain such \( \omega^\varepsilon \) we may simply define \( t_0 := 0, \) \( t_i := \inf \{ t > t_{i-1} : |\omega (t) - \omega (t_{i-1})| > \varepsilon \} \) for \( i = 1, 2, \ldots \) such that \( t_{i-1} < +\infty \) (we apply the convention that \( \inf \emptyset = +\infty \)) and \( N := \max \{ i \in \{ 1, 2, \ldots \} : t_{i-1} < T \} \).

Step 2. We have the estimate

\[
\left| \int_{[0,T]} \omega (t-) \, d\tilde{\omega} (t) - \int_{[0,T]} \omega^\varepsilon (t-) \, d\tilde{\omega} (t) \right| \leq \int_{[0,T]} | \omega (t) - \omega^\varepsilon (t) | | d\tilde{\omega} (t) | \leq \varepsilon \int_{[0,T]} | d\tilde{\omega} (t) | = \varepsilon \text{TV}(\tilde{\omega}, [0, T]),
\]

where \( \text{TV}(\tilde{\omega}, [0, T]) \) denotes the total variation of \( \tilde{\omega} \). Moreover

\[
\int_{[0,T]} \omega^\varepsilon (t-) \, d\tilde{\omega} (t) = \sum_{i=1}^{N} \omega \left( t_{i-1} \right) (\tilde{\omega} (t_i) - \tilde{\omega} (t_{i-1})).
\]

We also have

\[
\left| \sum_{i=1}^{k_n} \omega \left( \tau^n_{i-1} \right) \left\{ \tilde{\omega} \left( \tau^n_{i} \right) - \tilde{\omega} \left( \tau^n_{i-1} \right) \right\} - \sum_{i=1}^{k_n} \omega^\varepsilon \left( \tau^n_{i-1} \right) \left\{ \tilde{\omega} \left( \tau^n_{i} \right) - \tilde{\omega} \left( \tau^n_{i-1} \right) \right\} \right| \\
\leq \sum_{i=1}^{k_n} \left| \omega \left( \tau^n_{i-1} \right) - \omega^\varepsilon \left( \tau^n_{i-1} \right) \right| \left| \tilde{\omega} \left( \tau^n_{i} \right) - \tilde{\omega} \left( \tau^n_{i-1} \right) \right| \\
\leq \varepsilon \text{TV}(\tilde{\omega}, [0, T]).
\]

(5)
Step 3. Now let us consider the difference
\[
\int_{(0,T]} \omega^\varepsilon (t-) \, d\tilde{\omega} (t) - \sum_{i=1}^{k_n} \omega^\varepsilon (\tau^n_{i-1}) \{ \tilde{\omega} (\tau^n_i) - \tilde{\omega} (\tau^n_{i-1}) \}
\]
\[
= \sum_{i=1}^{N} \omega^\varepsilon (t_{i-1}) (\tilde{\omega} (t_i) - \tilde{\omega} (t_{i-1})) - \sum_{i=1}^{k_n} \omega^\varepsilon (\tau^n_{i-1}) \{ \tilde{\omega} (\tau^n_i) - \tilde{\omega} (\tau^n_{i-1}) \}.
\]  
(6)

Let \( \pi^n (t) \) denotes the first point in the partition \( \pi^n \) such that \( \pi^n (t) \geq t \). From the definition of the \( n \)th Lebesgue partition it follows that for \( t < T \)
\[
\lim_{n \to +\infty} \pi^n (t) = \inf \{ u > t : \omega (u) \neq \omega (t-) \text{ or } \tilde{\omega} (u) \neq \tilde{\omega} (t-) \}.
\]
By the definition of times \( t_1, t_2, \ldots, t_{N-1} \) we have that for any \( t \in \{ t_1, t_2, \ldots, t_{N-1} \} \), \( \omega (t) \neq \omega (t-) \) or \( \omega (t) = \omega (t-) \) but \( \omega \) is not constant on any interval of the form \( [t, u] \), \( u \in (t, T] \) and \( \lim_{n \to +\infty} \pi^n (t) = t \). Thus, for sufficiently large \( n \)
\[
t_i \leq \pi^n (t_i) < t_{i+1}
\]
for \( i = 1, 2, \ldots, N - 1 \).

Now, since \( \omega^\varepsilon \) is constant on \( [t_{i-1}, t_i] \), \( i = 1, 2, \ldots, N \), denoting by \( k_n(t) \) such index that
\( \pi^n (t) = \tau^n_{k_n(t)} \) for sufficiently large \( n \) and \( i = 2, \ldots, N \) we have \( t_{i-2} \leq \tau^n_{k_n(t_{i-1})-1} < t_{i-1} \)
\[
\omega^\varepsilon (\tau^n_{k_n(t_{i-1})-1}) = \omega^\varepsilon (t_{i-2}) \]
and
\[
\sum_{i=1}^{N} \omega^\varepsilon (t_{i-1}) (\tilde{\omega} (t_i) - \tilde{\omega} (t_{i-1})) - \sum_{i=1}^{k_n} \omega^\varepsilon (\tau^n_{i-1}) \{ \tilde{\omega} (\tau^n_i) - \tilde{\omega} (\tau^n_{i-1}) \}
\]
\[
= \sum_{i=2}^{N} (\omega^\varepsilon (t_{i-1}) - \omega^\varepsilon (t_{i-2})) (\tilde{\omega} (\pi^n (t_{i-1})) - \tilde{\omega} (t_{i-1})).
\]
Since \( \lim_{n \to +\infty} \pi^n (t) = t \) for \( t \in \{ t_1, t_2, \ldots, t_{N-1} \} \) and \( \tilde{\omega} \) is càdlàg, we finally get that the difference \([\text{4}])\) tends to 0 as \( n \to +\infty \).

From this and \([\text{2}])\), \([\text{3}])\) (taking \( \varepsilon \) as close to 0 as we wish) we get the assertion.

Now, applying (multidimensional) Föllmer’s pathwise Itô formula to \( f (x_1, \ldots, x_d) = x_i x_j \)
\( i, j \in \{ 1, 2, \ldots, d \}, t \in [0, T] \) we obtain the integration by parts formula:
\[
\omega^i (t) \omega^j (t) - \omega^i (0) \omega^j (0) = (F) \int_{(0,t]} \omega^i (s-) \, d\omega^j (s) + (F) \int_{(0,t]} \omega^j (s-) \, d\omega^i (s)
\]
\[
+ [S^i (\omega), S^j (\omega)]_t,
\]
(7)
where \((F) \int_{(0,t]} \omega^i (s-) \, d\omega^j (s)\) denotes Föllmer’s pathwise integral (along the sequence of the Lebesgue partitions). Notice also that \((F) \int_{(0,t]} \omega^i (s-) \, d\omega^j (s)\) coincides with the model-free Itô integral \( \int_{(0,t]} S^i_{\omega^-} (\omega) \, dS^j_{\omega} (\omega) \) introduced in the previous section, since it is the limit of the sums of the form
\[
\sum_{k=1}^{\infty} S^i_{x_{k-1}^n}(\omega) S^j_{x_k^n \wedge t, x_{k-1}^n \wedge t}(\omega).
\]
and \( \sum_{k=1}^{\infty} S_{n_{k-1}^{P}}^{i}(\omega)1_{\{\tau_{n_{k-1}^{P},n_{k}^{P}}(t)\}}(t) \) converges uniformly to \( S_{n_{k-1}^{P}}^{i}(\omega) \) for \( t \in (0, T] \), thus, by Theorem 3 the distance \( d_{\infty,0} \) between Föllmer’s integral and the model-free Itô integral is 0, and this implies that for typical price paths they coincide. Thus (7) may be viewed as the integration by parts formula for the model-free Itô integral.

Next, we have the following result, which we will need in the sequel.

**Lemma 6.** Let \( \tilde{S} : \Omega \times [0, T] \to \mathbb{R} \) be such that the process \( \tilde{S} \) is adapted (to the filtration \((F_{t})_{t \in [0,T]}\)), and for any (one-dimensional) \( \omega \in \Omega, \tilde{\omega} := \tilde{S}(\omega) \) is finite variation, càdlàg path on \([0, T] \). Then for typical price paths \( \omega \in \Omega \) the following integration by parts formula holds

\[ \omega(T) \tilde{\omega}(T) - \omega(0) \tilde{\omega}(0) = \int_{(0,T]} S_{t-}^{\tilde{\omega}}(\omega) dS_{t}(\omega) + \int_{(0,T]} \omega(t-) d\tilde{\omega}(t) + \sum_{0 \leq t \leq T} \Delta \tilde{\omega}(t) \Delta \omega(t), \]

where \( \int_{(0,T]} S_{t-}^{\tilde{\omega}}(\omega) dS_{t}(\omega) \) denotes the model-free Itô integral and \( \int_{(0,T]} \omega(t-) d\tilde{\omega}(t) \) denotes the Lebesgue-Stieltjes integral which coincides with the Föllmer integral.

**Proof.** We will use Föllmer’s pathwise integration by parts formula (9). Since (as we have already noticed) the model-free Itô integral coincides for typical price paths with the Föllmer integral and since (by Lemma 5) Föllmer’s integral \( (F_{t})_{t \in [0,T]} \) \( \omega(t-) d\tilde{\omega}(t) \) along the sequence of Lebesgue partitions coincides with the classical Lebesgue-Stieltjes integral \( \int_{(0,T]} \omega(t-) d\tilde{\omega}(t) \), to obtain the thesis we need only to prove that \( \sum_{k=1}^{\infty} S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\tilde{\omega}) S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\omega) \) converges to \( \sum_{0 \leq t \leq T} \Delta \tilde{\omega}(t) \Delta \omega(t) \). The proof of this fact follows from the properties of Lebesgue partitions and the Schwartz inequality: let \( \varepsilon > 0 \) and let \( I^{\varepsilon,n}, n \in \mathbb{N} \) be the sequence of all indices \( k \in \mathbb{N} \) for which

\[ \left| \sum_{k=1}^{\infty} S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\tilde{\omega}) S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\omega) - \sum_{k \in I^{\varepsilon,n}} S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\tilde{\omega}) S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\omega) \right| \leq \sqrt{\varepsilon} \sum_{k=1}^{\infty} \left| S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\tilde{\omega}) \right| \left| S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\omega) \right| \]

Notice that \( \sum_{k=1}^{\infty} \left| S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\tilde{\omega}) \right| \) is bounded by the total variation of \( \tilde{\omega} \) while

\[ \sum_{k=1}^{\infty} \left| S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\omega) \right|^{2} \]

converges to the quadratic variation of \( \omega \) as \( n \to +\infty \). Finally notice that

\[ \lim_{n \to \infty} \sum_{k \in I^{\varepsilon,n}} S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\tilde{\omega}) S_{n_{k-1}^{P}}^{P} \wedge T, n_{k}^{P} \wedge T(\omega) = \sum_{0 \leq t \leq T, |\Delta \tilde{\omega}(t)| > \varepsilon} \Delta \tilde{\omega}(t) \Delta \omega(t) \]

and since \( \varepsilon > 0 \) may be as close to 0 as we wish, the result follows. \( \square \)
Moreover, let us define

\[ \text{We will say that the partition } \pi \text{ (with respect to a given filtration) if all } \tau_i, i = 0, 1, \ldots, \text{ is infinite, non-decreasing sequence of elements of } [0, T] \cup \{+\infty\} \text{ such that from some } i \in \{1, 2, \ldots\} \text{ on } \tau_i = \tau_{i+1} = \ldots = +\infty. \text{ To avoid redundancies, we may also assume (as in } [3], \text{ though this will not change the reasoning) that for all } i = 0, 1, \ldots, \tau_i < \tau_{i+1} \text{ whenever } \tau_i < +\infty. \text{ Similarly as Davis, Obloj and Siropaeas in } [3], \text{ for any } \omega \in \Omega \text{ we will denote}
\]

\[ O_T(\omega, \pi) := \sup \{|\omega(t) - \omega(s)| : s, t \in [\tau_{i-1}, \tau_i] \cap [0, T], i \in \{1, 2, \ldots\}\}. \]

Moreover, let us define

\[ [\omega]_T^\pi := \sum_{i=1}^{+\infty} (\omega(\tau_i \land t) - \omega(\tau_{i-1} \land t))^2 \text{ for } t \in [0, T]. \]

**Definition 8.** We will say that the partition \( \pi = \{0 = \tau_0 \leq \tau_1 \leq \ldots\} \) is an optional one (with respect to a given filtration) if all \( \tau_1 \leq \tau_2 \leq \ldots \) are stopping times (with respect to this filtration).

Let us recall also the definition of the convergence in outer measure on a given set (Definition 2). Now we are ready to state

**Theorem 9.** Let \( \pi^n, n = 1, 2, \ldots, \) be a sequence of optional partitions with respect to the filtration \( (\mathcal{F}_t)_{t \in [0, T]} \) of the interval \( [0, T] \) such that for all \( q, M > 0, O_T(\cdot, \pi^n) \) converges in outer measure to 0 as \( n \to 0 \) on \( \Omega_{q, M} \). Then for any \( q, M > 0 \) the sequence \( [\omega]_T^{\pi^n}, n = 1, 2, \ldots, \) converges in outer measure \( \mathbb{F} \) to \( [\omega] \) on the set \( \Omega_{q, M} \), where \( [\omega] \) is the quadratic variation obtained for the sequence of the Lebesgue partitions.

**Proof.** Let \( \pi^n = \{0 = \tau^n_0 \leq \tau^n_1 \leq \ldots\} \) and take

\[ F^n_1(\omega) := \sum_{i=1}^{+\infty} \omega(\tau^n_{i-1}) \cdot 1_{[\tau^n_{i-1}, \tau^n_i]}(t). \]

By the definition of \( O_T(\omega, \pi^n) \) we have

\[ \| F^n(\omega) - S(\omega) \|_{\infty} \leq O_T(\omega, \pi^n). \]

Now, using the integration by parts formula (7) for any \( t \in [0, T] \) we calculate

\[ [\omega]^{\pi^n}_t - [\omega]_t = 2 \int_{[0,t]} (S_{s-}(\omega) - F^n_{s-}(\omega)) \, dS_s(\omega), \tag{9} \]

where \( \int_{[0,t]} (S_{s-}(\omega) - F^n_{s-}(\omega)) \, dS_s(\omega) \) denotes the model-free Itô integral. (Note that this integral is well defined since the partition \( \pi^n \) is optional one.) Now, fixing \( \varepsilon, \delta > 0, \) and
applying Corollary 4 we get for some constant \( C_{q,M} \) depending on \( q \) and \( M \) (and \( \psi \)) only 
\[
\mathbb{P} \left( \omega \in \Omega_{q,M} : \left\| [\omega]_{t}^{n} - \left[ \omega \right] \right\| _{\infty} > \varepsilon \right) 
\]
\[
= \mathbb{P} \left( \omega \in \Omega_{q,M} : \left\| \left( \int (S_{s}^{n}(\omega) - F_{s}^{n}(\omega)) \, dS_{s}(\omega) \right) t \in [0,T] \right\| _{\infty} > \varepsilon/2 \right) 
\]
\[
\leq \mathbb{P} \left( \omega \in \Omega_{q,M} : O_{T}(\omega, \pi^{n}) > \delta \right) + \mathbb{P} \left( \omega \in \Omega_{q,M} : O_{T}(\omega, \pi^{n}) \leq \delta, \left\| \left( \int (S_{s}^{n}(\omega) - F_{s}^{n}(\omega)) \, dS_{s}(\omega) \right) t \in [0,T] \right\| _{\infty} > \varepsilon/2 \right) 
\]
\[
\leq \mathbb{P} \left( \omega \in \Omega_{q,M} : O_{T}(\omega, \pi^{n}) > \delta \right) + C_{q,M} \frac{\delta}{\varepsilon}.
\]
Since \( \mathbb{P} \left( \omega \in \Omega_{q,M} : O_{T}(\omega, \pi^{n}) > \delta \right) \to 0 \) as \( n \to +\infty \) and \( \delta/\varepsilon \) may be chosen arbitrary close to 0, we get the convergence result. \( \square \)

When we assume some stronger mode of convergence of \( O_{T}(\cdot, \pi^{n}) \) then, naturally, we may expect to obtain a stronger mode of convergence of \( [\omega]_{t}^{n} \). Let us recall the definition of distances \( d_{\infty} \) and \( d_{\infty,\psi} \) (defined in (2)). We have.

**Theorem 10.** Let \( \pi^{n}, n = 1, 2, \ldots, \) be a sequence of optional partitions of the interval \( [0,T] \) and let

\[
F_{t}^{n}(\omega) := \sum_{i=1}^{+\infty} \omega(\tau_{i-1}^{n}, \tau_{i}^{n}) (t).
\]

Assume that \( \lim_{n \to +\infty} d_{\infty}(F^{n}, S) = 0 \) then for any \( \epsilon \in (0, 1) \),

\[
d_{\infty,\psi}(\left[ \omega \right]_{T}^{n}, [\omega]) \to 0 \quad \text{as} \quad n \to 0.
\]

**Proof.** The assertion follows immediately from (9) and Theorem 3 applied to \( F_{t} = F_{t}^{n} \) and \( G_{t} = S_{t}^{n} \). \( \square \)

### 3.3. Quadratic variation expressed in terms of the truncated variation.

The theorem which we will prove in this subsection provides one more formula for the quadratic variation of model-free càdlàg price paths. This will be a model-free counterpart of the formula obtained for càdlàg semimartingales in 5. More precisely, we will obtain partition-independent formula for the continuous part of the quadratic variation, which we will denote by \( \langle \omega \rangle_{t} \). It is formally defined as

\[
\langle \omega \rangle_{t} = [\omega]_{t} - \sum_{0 < s \leq t} (\Delta \omega (t))^{2}
\]

where by \( [\omega]_{t} \) we mean the quadratic variation obtained along the sequence of the Lebesgue partitions (if it exists). To state our result we need to introduce the notion of the truncated variation (with the truncation parameter \( c \geq 0 \)) of a càdlàg path \( \omega : [0, T] \to \mathbb{R} \) which is defined as

\[
TV^{c}(\omega, [0, T]) := \sup_{n} \sup_{0 \leq t_{0} < t_{1} < \ldots < t_{n} \leq T} \sum_{i=1}^{n} \max \{ |\omega(t_{i}) - \omega(t_{i-1})| - c, 0 \}.
\]

Notice that \( TV^{c}(\omega, [0, T]) \) does not depend on any partition, since it is the supremum over all partitions of the interval \([0, T]\).
**Theorem 11.** Let us fix $q, M > 0$. For typical càdlàg price path in $\Omega$ the following convergence holds

$$c \cdot TV^c(\omega, [0, T]) \to F_{c \to 0^+} \langle \omega \rangle_T,$$

where $(\omega)_T$ denotes the continous part the quadratic variation defined along the sequence of Lebesgue partitions and $\to F_{c \to 0^+}$ denotes the convergence in the outer measure $F$ as $c \to 0^+$ on the set $\Omega_{q,M}$.

**Proof.** Using construction in [7 Sect. 2] we know that for any $c > 0$ there exists a càdlàg path $\omega^c : [0, T] \to \mathbb{R}$ such that

1. $\omega^c$ has finite total variation;
2. $\omega^c(0) = \omega(0)$;
3. for every $t \in [0, T]$, $|\omega(t) - \omega^c(t)| \leq c$;
4. for every $t \in [0, T]$, $|\Delta \omega^c(t)| := |\omega^c(t) - \omega^c(t-)| \leq |\Delta \omega(t)| := |\omega(t) - \omega(t-)|$;
5. the process $S^c(\omega) := \omega^c$ is adapted to the filtration $(\mathcal{F}_t)_{t \in [0,T]}$.

Moreover (see [7, Lemma 5.1]) we have

\begin{equation}
TV^{2c}(\omega, [0, T]) \leq TV(\omega^c, [0, T]) \leq TV^{2c}(\omega, [0, T]) + 2c
\end{equation}

and

\begin{equation}
c \cdot TV(\omega^c, [0, T]) = \int_{[0,T]} (\omega - \omega^c) \, d\omega^c,
\end{equation}

where $\int_{[0,T]} (\omega - \omega^c) \, d\omega^c$ denotes the standard Lebesgue-Stieltjes integral (recall that $\omega^c$ has finite total variation) and $TV(\omega^c, [0, T])$ denotes the total variation of $\omega^c$.

Similarly as in [E] proof of Theorem 1 we calculate

\begin{align*}
\int_{[0,T]} (\omega - \omega^c) \, d\omega^c &= \int_{[0,T]} (\omega(t) - \omega^c(t) + \Delta (\omega(t) - \omega^c(t))) \, d\omega^c(t) \\
&= \int_{[0,T]} \omega(t) \, d\omega^c(t) - \int_{[0,T]} \omega^c(t) \, d\omega^c(t) \\
&+ \sum_{0 < s \leq T} \Delta (\omega(t) - \omega^c(t)) \Delta \omega(t).
\end{align*}

By the integration by parts formula [E]

\begin{align*}
\int_{[0,T]} \omega(t) \, d\omega^c(t) &= \omega^c(T) \omega(T) - \omega^c(0) \omega(0) - \int_{[0,T]} S_{t-} (\omega^c) \, dS_t(\omega) \\
&- \sum_{0 < s \leq T} \Delta \omega^c(t) \Delta \omega(t).
\end{align*}

By $\to_{c \to 0^+}$ we will denote the convergence for typical $\omega \in \Omega$ as $c \to 0^+$. Using $|\Delta \omega^c(t)| \leq |\Delta \omega(t)|$ and $\sum_{0 < s \leq T} (\Delta \omega(t))^2 \leq [\omega]_T$, by the dominated convergence we get

$$\sum_{0 < s \leq T} \Delta \omega^c(t) \, \Delta \omega(t) \to_{c \to 0^+} \sum_{0 < s \leq T} (\Delta \omega(t))^2.$$

Now, by [E], the fact that $|\omega - \omega^c| \leq c$ and Corollary [E] we get the convergence

\begin{align*}
\int_{[0,T]} \omega(t) \, d\omega^c(t) &\to F_{c \to 0^+} (\omega(T))^2 - (\omega(0))^2 - \int_{[0,T]} S_{t-} (\omega) \, dS_t(\omega) - \sum_{0 < s \leq T} (\Delta \omega(t))^2.
\end{align*}
Recall also that \( \omega^c(t) \) has finite total variation, thus the Lebesgue-Stieltjes integral rules apply and we have
\[
\int_{[0,T]} \omega^c(t-) \, d\omega^c(t) = \frac{1}{2} \left( (\omega^c(T))^2 - (\omega^c(0))^2 \right) - \frac{1}{2} \sum_{0<s\leq T} (\Delta \omega^c(t))^2
\]
\[
\rightarrow_{c \to 0^+} \frac{1}{2} \left( (\omega(T))^2 - (\omega(0))^2 \right) - \frac{1}{2} \sum_{0<s\leq T} (\Delta \omega(t))^2.
\]
We also have
\[
\sum_{0<s\leq T} \Delta (\omega(t) - \omega^c(t)) \Delta \omega(t) \rightarrow_{c \to 0^+} 0.
\]
Thus, from [12] and the last three convergences we get
\[
\int_{(0,T]} (\omega - \omega^c) \, d\omega^c \rightarrow_{F_{c \to 0^+}} (\omega(T))^2 - (\omega(0))^2 - \int_{[0,T]} S_{t-} (\omega) \, dS_t (\omega) - \sum_{0<s\leq T} (\Delta \omega(t))^2
\]
\[
\frac{1}{2} \left( (\omega(T))^2 - (\omega(0))^2 \right) + \frac{1}{2} \sum_{0<s\leq T} (\Delta \omega(t))^2.
\]
By the Itô formula
\[
\int_{(0,T]} S_{t-} (\omega) \, dS_t (\omega) = \frac{1}{2} \left( (\omega(T))^2 - (\omega(0))^2 \right) - \frac{1}{2} [\omega]_T.
\]
Finally, from (14) and (15), recalling that \([\omega]_T = \langle \omega \rangle_T + \sum_{0<s\leq T} (\Delta \omega(t))^2\) we have
\[
\int_{0}^{T} (\omega - \omega^c) \, d\omega^c \rightarrow_{F_{c \to 0^+}} \frac{1}{2} [\omega]_T.
\]
and from (11) and (11) we get
\[
2c \cdot TV^{2c}(\omega, [0, T]) \rightarrow_{F_{c \to 0^+}} [\omega]_T.
\]
\[\square\]

**Remark 12.** From Theorem [17] an analogous result follows for any local martingale, since, for any probability measure \( \mathbb{P} \) such that the coordinate process \( S \) is a local martingale and any \( B \in \mathcal{F}_T \) we have \( \mathbb{P}(B) \leq \overline{\mathbb{P}}(B) \), see [6] Proposition 2.4. This, with the help of simple inequality
\[
c \cdot TV^c(S + A, [0, T]) = TV^c(S, [0, T]) \leq c \cdot TV(A, [0, T]),
\]
where \( A \) is a process with finite total variation (see [8] Fact 17 and ineq. (2.14)) proves similar result for any càdlàg semimartingale. However, for càdlàg semimartingales a stronger (almost sure) convergence may be obtained - see [3] Theorem 1.

### 3.4. Quadratic variation vs truncated variation of deterministic paths.

Let now \( \omega : [0, T] \to \mathbb{R} \) be a càdlàg deterministic path. Theorem [11] raises the question what is the relation between the existence of the limit \( c \cdot TV^c(\omega, [0, T]) \) as \( c \to 0^+ \) and the existence of the quadratic variation along some sequence of partitions of \( \omega \). It is well known that there exists continuous paths for which one may obtain arbitrary non-decreasing (and starting from 0) quadratic variations, as limits of discrete quadratic variations along appropriately chosen (refining) sequence of partitions (cf. [3] Theorem 7.1)].
In fact, trajectories of a standard (one-dimensional) Brownian motion $B$ provide (with probability 1) examples of such paths. Also, for any continuous function $\omega : [0, T] \to \mathbb{R}$, using the Darboux property it is easy to construct refining sequence of partitions such that the sequence of corresponding discrete quadratic variations tends to 0. These, together with the fact that for the standard Brownian motion $B$ one has almost surely
\[ c \cdot \text{TV}^c(B, [0, T]) \to_{c \to 0^+} T, \]
proves that the existence of the limit $c \cdot \text{TV}^c(\omega, [0, T])$ does not imply the existence of the same quadratic variation along a given sequence of partitions of $\omega$. And opposite, the existence of the quadratic variation along a given sequence of partitions of $\omega$ does not imply the existence of the finite limit $c \cdot \text{TV}^c(\omega, [0, T])$ as $c \to 0^+$ (this may for example tend to $+\infty$ for very irregular $\omega$, for example for $\omega$ which has $n^2$ oscillations of size $1/\sqrt{n}$ on the interval $[1/(n + 1), 1/n]$, $n \in \mathbb{N}$). However, it appears that it is possible to built up an Itô calculus based on the truncated variation (see [5, Sect. 4]).

**Remark 13.** An open question remains if the existence of the limit $c \cdot \text{TV}^c(\omega, [0, T])$ as $c \to 0^+$ implies the existence of the same quadratic variation along some sequence of partitions of $\omega$. A natural candidate for such a sequence would be the sequence of the Lebesgue partitions. For continuous $\omega$ this sequence is especially simple:
\[ \pi^k_0 = 0, \]
\[ \pi^k_n = \inf \{ t \in [0, T] : |\omega(t) - \omega(\pi^k_{n-1})| = 2^{-n} \}, \text{ for } k = 1, 2, \ldots. \]

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