Transformation Semigroups with the Deformed Multiplication

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Pairwise non-isomorphic semigroups obtained from the finite inverse symmetric semigroup $\mathcal{IS}_n$, finite symmetric semigroup $\mathcal{T}_n$ and bicyclic semigroup by the deformed multiplication proposed by Ljapin are classified.

Key Words: symmetric semigroup, inverse symmetric semigroup, bicyclic semigroup, deformed multiplication.

1 Introduction

In a famous Ljapin’s monograph [1, 393] there is the following problem. Let $\Omega_1, \Omega_2$ be arbitrary nonempty sets, $S$ be a set of maps from $\Omega_1$ to $\Omega_2$. Fix a map $\alpha : \Omega_2 \to \Omega_1$ and define the new multiplication of maps from $S$ via: $\varphi \circ \psi = \varphi \cdot \alpha \cdot \psi$, where the symbol of $\cdot$ denotes a usual composition of the maps (we perform multiplication from left to right). It is easy to verify that the defined operation is associative. Ljapin proposed to investigate the properties of this semigroup with respect to the restrictions applied to $S$ and $\alpha$. In particular, there appears an interesting case if $\Omega_1 = \Omega_2 = \Omega$, $S$ is some transformation semigroup on $\Omega$, and $\alpha \in S$. Further this case might be generalized to an arbitrary semigroup $S$: for a fixed $a$ from $S$ define the operation $*_a$ on $S$ via: $x*_a y = xay$. The set $S$ with this operation is, obviously, a semigroup which we denote $(S, *_a)$. The operation $*_a$ is called the multiplication deformed by $a$ (or just the deformed multiplication).

In the present paper we give the full classification of pairwise non-isomorphic semigroups obtained from the finite inverse symmetric semigroup $\mathcal{IS}_n$ of all partial injections of an $n$–element set, finite symmetric semigroup $\mathcal{T}_n$ of all transformations of an $n$–element set, and bicyclic semigroup by the deformed multiplication. We follow terminology and notation from [2].
2 $\mathcal{IS}_n$ with the deformed multiplication

For a partial transformation $\alpha \in \mathcal{IS}_n$ denote by $\text{dom}(\alpha)$ the domain of $\alpha$, and denote by $\text{ran}(\alpha)$ its image. The value $|\text{ran}(\alpha)|$ is called the rank of $\alpha$ and is denoted by $\text{rank}(\alpha)$.

**Lemma 1.** For an arbitrary $\alpha \in \mathcal{IS}_n$ the number of idempotents in $(\mathcal{IS}_n, *_{\alpha})$ is equal to $2^{\text{rank}(\alpha)}$.

**Proof.** Let $\text{rank}(\alpha) = k$. The equality $\varepsilon = \varepsilon *_{\alpha} \varepsilon = \varepsilon \alpha \varepsilon$ implies that $\text{dom}(\varepsilon) \subseteq \text{ran}(\alpha)$ and $\text{ran}(\varepsilon) \subseteq \text{dom}(\alpha)$, moreover, for any $x \in \text{dom}(\varepsilon)$ we have $\varepsilon(x) = \alpha^{-1}(x)$. Therefore an idempotent $\varepsilon$ is completely defined by its domain $\text{dom}(\varepsilon)$. On the other hand for any subset $A \subseteq \text{ran}(\alpha)$ the element, $\varepsilon_A$, such that $\text{dom}(\varepsilon_A) = A$ and $\varepsilon_A(x) = \alpha^{-1}(x)$ for all $x \in A$, satisfies the equality $\varepsilon_A \alpha \varepsilon_A = \varepsilon_A$. In other words, $\varepsilon_A$ is an idempotent in $(\mathcal{IS}_n, *_{\alpha})$. Hence there is a one-to-one correspondence between idempotents in $(\mathcal{IS}_n, *_{\alpha})$ and the subsets of $\text{ran}(\alpha)$ which completes the proof.

**Theorem 1.** Semigroups $(\mathcal{IS}_n, *_{\alpha})$ and $(\mathcal{IS}_n, *_{\beta})$ are isomorphic if and only if $\text{rank}(\alpha) = \text{rank}(\beta)$.

**Proof.** Lemma 1 provides the necessity of the condition. Conversely, let $\text{rank}(\alpha) = \text{rank}(\beta) = k$. Then there exist permutations $\tau$ and $\pi$ in $S_n$ such that $\beta = \tau \alpha \pi$. Define the map $f : (\mathcal{IS}_n, *_{\alpha}) \to (\mathcal{IS}_n, *_{\beta})$ by $f(\xi) = \pi^{-1} \xi \tau^{-1}$. Obviously, $f$ is bijective, moreover, for arbitrary $\xi, \eta \in \mathcal{IS}_n$ we have:

$$f(\xi *_{\alpha} \eta) = \pi^{-1} \xi *_{\alpha} \eta \tau^{-1} = \pi^{-1} \xi \tau^{-1} \alpha \pi \pi^{-1} \eta \tau^{-1} =
= \pi^{-1} \xi \tau^{-1} \beta \pi^{-1} \eta \tau^{-1} = f(\xi) *_{\beta} f(\eta).$$

**Corollary 1.** There are $(n + 1)$ pairwise non-isomorphic semigroups obtained from $\mathcal{IS}_n$ by the deformed multiplication.

**Proof.** Follows from theorem 1 and the fact, that the rank of the element in $\mathcal{IS}_n$ can be equal to any integer from the interval $[0, n]$.

3 $\mathcal{T}_n$ with the deformed multiplication

Let $\mathcal{T}_n$ be the symmetric semigroup of all transformations of the set $N = \{1, 2, \ldots, n\}$. We call the type of transformation $a \in \mathcal{T}_n$ a set $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_k$ is the number of those elements $y \in N$, whose full inverse image $a^{-1}(y)$ contains exactly $k$ elements. Obviously, $1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \cdots + n \cdot \alpha_n = n$, and the sum $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ is equal to the cardinality of the image of $a$. 

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Theorem 2. Semigroups \((T_n,*_a)\) and \((T_n,*_b)\) are isomorphic if and only if transformations \(a\) and \(b\) have the same type.

Proof. Necessity. Let \((T_n,*_a)\) and \((T_n,*_b)\) be isomorphic semigroups. On semigroup \((T_n,*_a)\) define an equivalence relation \(\sim_a\) in the following way: \(x \sim_a y\) if and only if \(x *_a u = y *_a u\) for all \(u \in T_n\). In the same vein define an equivalence relation \(\sim_b\) on \((T_n,*_b)\). First we prove that arbitrary isomorphism \(\varphi : (T_n,*_a) \rightarrow (T_n,*_b)\) is in accordance with these equivalence relations, that is, \(\varphi(x) \sim_b \varphi(y)\) if and only if \(x \sim_a y\).

Indeed, let \(\varphi : (T_n,*_a) \rightarrow (T_n,*_b)\) be an isomorphism and let \(x \sim_a y\). Then for all \(u \in T_n\) we have \(x *_a u = y *_a u\), and further \(\varphi(x) *_b \varphi(u) = \varphi(y) *_b \varphi(u)\). However \(\varphi(u)\) runs over whole set \(T_n\), so \(\varphi(x) \sim_b \varphi(y)\). The inverse map \(\varphi^{-1} : (T_n,*_b) \rightarrow (T_n,*_a)\) is also isomorphic, therefore \(\varphi(x) \sim_b \varphi(y)\) implies \(x \sim_a y\). Consequently, \(x \sim_a y\) if and only if \(\varphi(x) \sim_b \varphi(y)\).

Therefore any isomorphism between \((T_n,*_a)\) and \((T_n,*_b)\) maps equivalence classes of the relation \(\sim_a\) into corresponding equivalence classes of \(\sim_b\). Hence for the relations \(\sim_a\) and \(\sim_b\) the cardinalities and the numbers of the equivalence classes must be equal. We show that by the cardinalities of equivalence classes of the relation \(\sim_a\) we can find the type \((\alpha_1,\alpha_2,\ldots,\alpha_n)\) of \(a\) uniquely.

Lemma 2. \(x \sim_a y\) if and only if \(xa = ya\).

Proof. Obviously, the equality \(xa = ya\) implies \(x \sim_a y\). Now, let \(xa \neq ya\). Then there exists \(k\) in \(N\) such that \((xa)(k) \neq (ya)(k)\). Chose an element \(u\) from \(T_n\), which has different meanings in points \((xa)(k)\) and \((ya)(k)\). Then \(x *_a u = xau\) and \(y *_a u = yau\) have different meanings in \(k\). Hence, \(x *_a u \neq y *_a u\) and \(x \sim_a y\). \(\Box\)

Denote by \(\rho_a\) the partition of the set \(\{1,2,\ldots,n\}\) induced by \(a\) (that is, \(x\) and \(y\) belong to the same block of the partition \(\rho_a\) if and only if \((a(x) = a(y))\). Count the cardinality of the equivalence class \(\overline{x_0} = \{x \mid xa = x_0a\}\) of the relation \(\sim_a\) for a fixed element \(x_0\). First consider the element \(y := x_0a = \begin{pmatrix} 1 & 2 & \ldots & n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix}\). Obviously, every \(y_i\) belongs to the image of \(a\), \(i = 1,\ldots,n\). Denote by \(N_a(a_i)\) the block of the partition \(\rho_a\), defined by the element \(a_i\) from the image of \(a\). By \(n_a(a_i)\) we denote the cardinality of this block. The equality \(xa = y\) is satisfied if and only if \((xa)(i) = y_i\) for every \(i\) or, what is the same, \(x(i) \in N_a(y_i)\). So \(x(i)\) can be chosen in \(n_a(y_i)\) ways. The images of \(x\) in different points are chosen independently, therefore transformation \(x\) can be chosen in

\[
\prod_{i=1}^{n} n_a(y_i) \tag{1}
\]
ways and we have the cardinality of the class \( \overline{x}_0 \).

Denote by \( m \) the least cardinality of the blocks of the partition \( \rho_a \). The cardinality of the equivalence class of the relation \( \sim_a \) is the least if all multipliers in (1) are equal to \( m \), and this cardinality is \( m^n \). Now we find the number of equivalence classes \( \overline{x}_0 \) of the relation \( \sim_a \) which have cardinality \( m^n \). To make all multipliers in (1) equal to \( m \), there should be \( n_a(y_i) = m \) for all \( i \). However \( \{ t \mid n_a(t) = m \} = \alpha_m \). So every \( y_i \) can be chosen in \( \alpha_m \) ways. Since the meanings of \( y_i \) for different \( i \) are chosen independently, there are \( \alpha_m^n \) different \( y = x_0 \alpha \) such that the corresponding class \( \overline{x}_0 \) contains \( m^n \) elements. Therefore by the relation \( \sim_a \) we may find the index \( m \) and the value \( \alpha_m \) of the first non zero component of the type \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) of \( a \).

Now the components \( \alpha_l \) for \( l > m \) can be found recursively. Assume that the components \( \alpha_1, \alpha_2, \ldots, \alpha_{l-1} \) are already known. For the relation \( \sim_a \) denote by \( C \) the number of equivalence classes of the cardinality \( l \cdot m^{n-1} \). Then \( C \) is equal to the number of sets \((i_1, i_2, \ldots, i_n)\) where some of \( i_1, i_2, \ldots, i_n \) may coincide in general, such that:

\[
n_a(y_{i_1})n_a(y_{i_2}) \ldots n_a(y_{i_n}) = l \cdot m^{n-1}
\]

(2)

Since by assumption \( \alpha_1, \alpha_2, \ldots, \alpha_{l-1} \) are known, we may find the number \( A \) of those sets \((i_1, i_2, \ldots, i_n)\) for which all multipliers in the left hand side of (2) are less than \( l \). This number equals \( \prod_{k=1}^{n} \alpha_{m_k} \), where \( m \leq m_k < l \), and \( m_1 \cdot m_2 \cdots m_n = l \cdot m^{n-1} \). The number \( B \) of all sets \((i_1, i_2, \ldots, i_n)\) such that one of the multipliers in the left hand side of (2) is equal to \( l \) and other \((n-1)\) multipliers equal \( m \), is \( n \cdot \alpha_l \cdot \alpha_{m-1}^n \). Therefore \( \alpha_l \) can be found from the equality \( A + B = C \).

Applying the same reasoning to semigroup \((T_n, *_b)\) we can find the type \((\beta_1, \beta_2, \ldots, \beta_n)\) of \( b \) via the cardinalities of the equivalence classes of the relation \( \sim_b \). For isomorphic semigroups \((T_n, *_a)\) and \((T_n, *_b)\) the numbers of the equivalence classes of the same cardinality coincide, and the values \( \alpha_k \), \( \beta_k \), \( k = 1, \ldots, n \) are defined by the numbers of equivalence classes. So for all \( k \) we have \( \alpha_k = \beta_k \), that is, elements \( a \) and \( b \) have the same type.

**Sufficiency.** Let elements \( a \) and \( b \) have the type \((\alpha_1, \alpha_2, \ldots, \alpha_n)\). Then there exist permutations \( \pi \) and \( \tau \) in \( S_n \), such that \( b = \tau a \pi \). The map \( f : (T_n, *_a) \to (T_n, *_b) \) such that \( f(x) = \pi^{-1} x \tau^{-1} \) defines the isomorphism between \((T_n, *_a)\) and \((T_n, *_b)\). Indeed, \( f \) is bijective and

\[
f(x *_a y) = \pi^{-1} x *_a y \tau^{-1} = \pi^{-1} x \tau^{-1} \tau a \pi \pi^{-1} y \tau^{-1} = \pi^{-1} x \tau^{-1} b \pi^{-1} y \tau^{-1} = f(x) *_b f(y).
\]

\(\square\)
Corollary 2. Let \( p(n) \) denote the number of ways in which we can split positive integer \( n \) into non ordered sum of the natural integers. Then there are \( p(n) \) pairwise non-isomorphic semigroups obtained from \( T_n \) by the deformed multiplication.

**Proposition 1.** In \( T_n \) there are

\[
n! \left( \binom{n}{\alpha_1} \right) \left( \binom{n-\alpha_1}{\alpha_2} \right) \cdots \left( \binom{n-\sum_{i=1}^{n-1} \alpha_i}{\alpha_n} \right) \prod_{i=1}^{n} (i!)^{\alpha_i}
\]

transformations of the type \((\alpha_1, \alpha_2, \ldots, \alpha_n)\).

**Proof.** To define transformation \( a \in T_n \) of the type \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) first we chose in \( \alpha_1 \) elements in \( \mathbb{N} \) which have 1-element inverse images. This can be done in \( \binom{n}{\alpha_1} \) ways. Further from elements which left we chose \( \alpha_2 \) elements which have 2-element inverse images, and so on. So the image of \( a \) can be defined in \( \binom{n}{\alpha_1} \binom{n-\alpha_1}{\alpha_2} \cdots \binom{n-\sum_{i=1}^{n-1} \alpha_i}{\alpha_n} \) ways. Now we write these elements in some order and we write elements which were chosen at \( k \) step exactly \( k \) times. Then every of \( n! \) permutations \( i_1, \ldots, i_n \) of numbers 1, 2, \ldots, \( n \) defines transformation \( a = \left( \begin{array}{ccc} i_1 & i_2 & \cdots & i_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{array} \right) \) of the type \((\alpha_1, \alpha_2, \ldots, \alpha_n)\). However in this way every transformation \( a \) is counted for several times, since permutations shifting elements with the same images define the equal transformation. Hence to find the number of transformations of type \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) we need to divide the value \( n! \binom{n}{\alpha_1} \binom{n-\alpha_1}{\alpha_2} \cdots \binom{n-\sum_{i=1}^{n-1} \alpha_i}{\alpha_n} \) by the repetition factor \( \prod_{i=1}^{n} (i!)^{\alpha_i} \) with which every transformation is received. \( \square \)

### 4 Bicyclic semigroup with the deformed multiplication

Bicyclic semigroup is a semigroup \( B = \langle a, b | ab = 1 \rangle \). It is known [3] that \( B \) is the inverse semigroup and each element in \( B \) can be uniquely written in the canonical form \( b^m a^k \), \( m, k \geq 0 \). Moreover, \( (b^m a^k)^{-1} = b^k a^m \).

**Proposition 2.** For every \( \alpha \in B \), \( \alpha = b^m a^k \), \( \{b^{k+i} a^{k+i}, i \geq 0\} \) is the set of idempotents in the deformed semigroup \((B, \ast_\alpha)\). Moreover, idempotents form the infinite decreasing chain with respect to a natural partial order on the set of idempotents.
Proof. Element $\varepsilon_i = b^{k+i}a^{m+i}$ is an idempotent in semigroup $(\mathcal{B}, \ast_\alpha)$. Really, $\varepsilon_i \ast_\alpha \varepsilon_i = b^{k+i}a^{m+i}b^ka^{k+i}a^{m+i} = b^{k+i}a^{k+i}b^{k+i}a^{m+i} = b^{k+i}a^{m+i} = \varepsilon_i$.

Now let $\varepsilon = b^ia^s$ be an idempotent of $(\mathcal{B}, \ast_\alpha)$. Assume, that $s < m$. Then $b^ia^s = \varepsilon = \varepsilon \ast_\alpha \varepsilon = b^ia^s b^ka^sb^i = b^{l+m-s}a^kb^ia^s$. To make the powers of $a$ in the canonical form of the left and right hand sides of this equality equal, we need $k \leq t$. Then $b^{l+m-s-t-k}a^s = b^ia^s$, and $2t + m - s - k = t$, $m - s = k - t$. However under assumption, $m - s > 0$ and $k - t \leq 0$, so the latter equality is impossible. Hence, $s \geq m$. Then $\varepsilon \ast_\alpha \varepsilon = b^ia^{s-m+k}b^ia^s$, and $s - m + k = t$, $s - m = t - k$. Denote $i = s - m$. Then $\varepsilon = b^{k+i}a^{m+i}$.

Let $\varepsilon_i = b^{k+i}a^{m+i}$ and $\varepsilon_j = b^{k+j}a^{m+j}$ be two idempotents and without loss of generality let $i \geq j$. Then $\varepsilon_i \ast_\alpha \varepsilon_j = b^{k+i}a^{m+i}b^ka^{k+j}a^{m+j} = b^{k+i}a^{m+j} = b^{k+i}a^{m+i} = \varepsilon_i$. Analogously $\varepsilon_j \ast_\alpha \varepsilon_i = \varepsilon_i$, therefore $\varepsilon_i \leq \varepsilon_j$ if and only if $i \geq j$. So the set of idempotents is linearly ordered, and $\varepsilon_0 = b^{k+0}a^{m+0} = b^ka^m$ is maximal idempotent in $(\mathcal{B}, \ast_\alpha)$.

Theorem 3. For different $\alpha$ and $\beta$ semigroups $(\mathcal{B}, \ast_\alpha)$ and $(\mathcal{B}, \ast_\beta)$ are not isomorphic.

Proof. Let $\alpha = b^n a^k$, $\beta = b^u a^v$. Take idempotent $\varepsilon_i = b^{k+i}a^{m+i}$, $i \geq 0$ in semigroup $(\mathcal{B}, \ast_\alpha)$ and consider the sets:

$$P_i^\alpha = \{ \xi \in \mathcal{B} \mid \varepsilon_i \ast_\alpha \xi \neq \xi \} \quad \text{and} \quad Q_i^\alpha = \{ \xi \in \mathcal{B} \mid \xi \ast_\alpha \varepsilon_i \neq \xi \}.$$ 

If element $\xi = b^ia^s$ does not belong to $P_i^\alpha$ then $\xi = \varepsilon_i \ast_\alpha \xi$, that is, $b^{k+i}a^{m+i}b^ka^sb^ia^s = b^ia^s$ and $b^{k+i}a^{k+i}b^ia^s = b^ia^s$.

If $k + i > t$ then $b^{k+i}a^{k+i}b^ia^s = b^{k+i}a^{s+k+i-t} \neq b^ia^s$. Therefore $k + i \leq t$.

On the other hand, if $k + i \leq t$ then $b^{k+i}a^{k+i}b^ia^s = b^{k+i}a^{l-(k+i)}a^s = b^ia^s$.

Hence, element $\xi = b^ia^s \notin P_i^\alpha$ if and only if $k + i \leq t$.

In the same way we can shown that $\xi = b^ia^s \notin Q_i^\alpha$ if and only if $m + i \leq s$.

Thus $\xi = b^ia^s \in P_i^\alpha \cap Q_i^\alpha$ if and only if $t < k + i$ and $s < m + i$. Then the cardinality of an intersection $P_i^\alpha \cap Q_i^\alpha$ equals $|P_i^\alpha \cap Q_i^\alpha| = (k + i)(m + i)$.

It is relatively easy to show that by these cardinalities the powers of the element $\alpha = b^n a^k$ can be found. Indeed, $|P_1^\alpha \cap Q_1^\alpha| - |P_1^\alpha \cap Q_0^\alpha| = (k + 1)(m + 1) - (k + 1)m = km + k + m + 1 - km = k + 1$, and $|P_0^\alpha \cap Q_1^\alpha| - |P_0^\alpha \cap Q_0^\alpha| = (k + 1)(m + 1) - k(m + 1) = m + 1$.

Applying the same reasoning to $(\mathcal{B}, \ast_\beta)$, $\beta = b^u a^v$ we can show that $|P_1^\beta \cap Q_1^\beta| - |P_1^\beta \cap Q_0^\beta| = v + 1$ and $|P_0^\beta \cap Q_1^\beta| - |P_0^\beta \cap Q_0^\beta| = u + 1$.

Hence if semigroups $(\mathcal{B}, \ast_\alpha)$ and $(\mathcal{B}, \ast_\beta)$ are isomorphic then the cardinalities of corresponding sets must be equal, that is, $u = m$ and $v = k$, and $\alpha = \beta$.

Corollary 3. With respect to isomorphism by the deformed multiplication we get infinitely many different semigroups from the bicycle semigroup.
Theorem 4. Semigroups \((\mathcal{B}, \ast_\alpha)\) and \((\mathcal{B}, \ast_\beta)\) are anti-isomorphic if and only if \(\alpha\) and \(\beta\) are inverse.

Proof. Let semigroups \((\mathcal{B}, \ast_\alpha)\) and \((\mathcal{B}, \ast_\beta)\) be such that elements \(\alpha\) and \(\beta\) are inverse in \(\mathcal{B}\). It is known that the bicycle semigroup is inverse, and for an element \(\alpha = b^m a^k\) there is a unique inverse element written in a canonical form as \(\beta = b^k a^m\).

Consider the map \(\varphi : (\mathcal{B}, \ast_\alpha) \to (\mathcal{B}, \ast_\beta)\), \(\varphi(b^x a^y) = b^y a^x\). Then \(\varphi\) is a bijection and \(\varphi(\alpha) = \varphi(b^m a^k) = b^k a^m = \beta\). Let \(b^x a^y, b^a\) be arbitrary elements in \(\mathcal{B}\). If \(y \geq t\) then \(\varphi(b^x a^y b^a) = \varphi(b^x a^{y-t+s}) = b^y a^{y-s} a^x = b^s a^t b^y a^x = \varphi(b^a) \varphi(b^y a^y)\). The analogous equality is received if \(y < t\). First we prove that \(\varphi(b^{x_1} a^{y_1} \cdots b^{x_n} a^{y_n}) = b^{y_n} a^{x_n} \cdots b^{y_1} a^{x_1}\). In fact,

\[
\varphi(b^{x_1} a^{y_1} b^{x_2} a^{y_2} \cdots b^{x_n} a^{y_n}) = \varphi(b^{x_2} a^{y_2} \cdots b^{x_n} a^{y_n}) \varphi(b^{x_1} a^{y_1}) = \varphi(b^{x_2} a^{y_2} \cdots b^{x_n} a^{y_n}) b^{y_1} a^{x_1} = \cdots = b^{y_n} a^{x_n} \cdots b^{y_1} a^{x_1}.
\]

Now for arbitrary \(\xi = b^x a^y, \eta = b^a a^v\) in \(\mathcal{B}\) we have

\[
\varphi(\xi \ast_\alpha \eta) = \varphi(b^x a^y b^m a^k b^u a^v) = b^v a^u b^k a^m b^y a^x = \varphi(\eta) \ast_\beta \varphi(\xi).
\]

Hence, \(\varphi\) is anti-isomorphism between \((\mathcal{B}, \ast_\alpha)\) and \((\mathcal{B}, \ast_\beta)\). However the semigroup anti-isomorphic to the given one is unique with respect to isomorphism, so if semigroups \((\mathcal{B}, \ast_\alpha)\) and \((\mathcal{B}, \ast_\beta)\) are anti-isomorphic then elements \(\alpha\) and \(\beta\) are inverse. □

References

[1] Ljapin Y.S., Semigroups, Moscow, Fizmatgiz, 1960 (Russian).

[2] Artamonov V.A, Salij V.N., Skornyakov L.A. and others, General Algebra, Moscow, Nauka, 1991, vol. 1 (Russian).

[3] Clifford A.H., Preston G.B., The Algebraic Theory of Semigroups, Moscow, Mir, 1972 (Russian translation)

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