Generalized scalar field cosmologies

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In this paper we review some well-known results and we prove some new theorems for general situations in the context of scalar field cosmologies with arbitrary potential (and with an arbitrary coupling to matter). We present some simple examples that violates one or more of the hypothesis of the Theorems proved, obtaining some counterexamples. In particular we incorporate cosine-like corrections with small phase, motivated by inflationary loop-quantum cosmology. We consider these corrections adapted to describe conventional inflationary scalar field cosmologies for FLRW metric and Bianchi I metrics. We use both local and global phase-space descriptions to find qualitative features and provide good approximations for solutions.

I. INTRODUCTION

Massless (i.e. long-ranged) and massive (i.e. short-ranged) scalar fields are present in most of the physical theories. Examples used in physics include the temperature distribution throughout space, the pressure distribution in a fluid, and spin-zero quantum fields, such as the Higgs field. These fields are the subject of scalar field theory. Mathematically, scalar fields is a real or complex-valued function or distribution on a region, which may be a set in some Euclidean space, Minkowski space, or more generally a subset of a manifold. The usual practice in mathematics is to impose further conditions on the field, such that it be continuous or often continuously differentiable to some order. In scalar theories of gravitation scalar fields are used to describe the gravitational field. Some scalar field theories with special interest are the cosmological models based on Scalar-tensor theories like Jordan theory \cite{[1]} as a generalization of the Kaluza-Klein theory; the Brans-Dicke theory \cite{[2]}; Horndeski theories \cite{[4]}; inflationary models \cite{[3]}; extended quintessence; modified gravity; Horava-Lifschitz and the Galileons, etc. \cite{[5]–[53]}. There are several studies in the literature that provides global an local dynamical systems analysis for scalar field cosmologies with arbitrary potentials and with arbitrary couplings. In the reference \cite{[54]} it was studied a very large, and natural class of scalar field models having an arbitrary non-negative potential function $V(\phi)$ with a flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric, yielding to a simple and regular past asymptotic structure which corresponds with the exactly integrable massless scalar field cosmologies; with the exception of a set that has measure zero. This model was generalized in \cite{[55]} for a flat and a negatively curved FLRW model and by adding a perfect fluid matter source. In particular, for potentials having a local zero minimum, flat and negatively curved FLRW models are ever expanding and the energy density asymptotically approaches zero, and the scalar field reaches asymptotically the minimum of the potential. Additionally, it was commented whether a closed FLRW model with ordinary matter can avoid re-collapse. The model by \cite{[55]} was extended in \cite{[56],[57]} for an scalar field non-minimally coupled to matter (this scenario incidentally contains as a particular realization the model of \cite{[58]}, that arises in the conformal frame of $f(R)$ theories nonminimally coupled to matter). By considering general potential $V(\phi)$ and a general coupling function $\chi(\phi)$, it was proved that under generic hypothesis the future attractor corresponds to the vacuum de Sitter solution. Using Hubble-normalized variables it was proved that the scalar field diverges into the past, extending the results by \cite{[54],[55]}. So, in order to study the dynamics close to the initial singularity it was studied the limit $\phi \rightarrow \infty$ by imposing some regularity conditions on the potential and on the coupling function. Interestingly, for a general class of models, which admits scaling solutions as in \cite{[54]}, the asymptotic structure of solutions towards the past is simple and regular, and it is independent of the features of the potential, the coupling function and the background matter. The dynamics of a nonminimally coupled scalar field model in case of a $F(\phi)R$–coupling with $F(\phi) = 1 - \xi \phi^2$ and the potentials $V(\phi) = V_0(1 + \phi^p)^2$ and $V(\phi) = V_0 e^{\lambda \phi^2}$ was studied in \cite{[59]}. Other nonminimally coupled scalar field models were studied, e.g., in \cite{[60],[70]}. In the reference \cite{[71]} is was studied the dynamics of homogeneous FLRW cosmological models with a self-interacting scalar field source, not only for the flat geometry but for negatively and positively curved models. The analysis incorporates a wide class of interaction potentials, requiring only the scalar field potential to be bounded from below and

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divergent when the field diverges. Thus, incorporating positive potentials which exhibit asymptotically polynomial or exponential behaviors. For potentials with negative inferior bound, was provided the analysis of the asymptotically anti de Sitter (AdS) states for such cosmologies.

In the reference [58], it was studied the evolution of a cosmological model corresponding to flat and negatively curved FLRW models with a perfect fluid matter source and an scalar field nonminimally coupled to matter with an exponential coupling $\chi$ (in the sense of [56, 57]). There were imposed flat and negatively curved FLRW models. It was proved in [59] that exists a very generic class of potentials, having an equilibrium corresponding to the non-negative local minimum for $V(\phi)$, which is asymptotically stable. The same happens for horizontal asymptotes approached from above by $V(\phi)$. Furthermore, in this reference were classified all flat models for which one of the matter constituents will eventually dominates. Particularly, if the barotropic matter index, $\gamma$, is larger than 1, generically there is an energy a transfer from the fluid to the scalar field, which eventually dominated.

In references [72–74] the original model by [54, 55] was extended to more complicated scenarios. In [74] it was characterized the asymptotic structure of the phase space of flat FLRW models in the conformal (Einstein) frame. In reference [81] it was studied the late time of a negatively curved FLRW model with a perfect fluid matter source and a scalar field nonminimally coupled to matter. As we have commented before, under mild assumptions on the potential, the equilibria corresponding to non-negative local minima for $V$ are asymptotically stable. For

1. Statement and proof of Proposition 1: For non-negative potentials with a local zero minimum at $\phi$; such that its derivative is bounded in some set where the potential is; and provided the derivative of the logarithm of the coupling function has an upper bound, then the energy densities of matter and of radiation, as well as the kinetic term, tend to zero when $t \to +\infty$. Thus, the Universe would expand forever in a de Sitter phase as $t \to +\infty$. This is an extension of the Proposition 2 of [55] to the non-minimal coupling context and an extension of Proposition 4 of [73] when the radiation is included in the cosmic budget.

2. Statement and proof of Proposition 2: Under the same hypotheses as in Proposition 1 and provided that $V(\phi)$ is strictly decreasing (increasing) for negative (positive) values of the scalar field, then the scalar field diverges into the future or it equals to zero (the last case holds only if the Hubble scalar vanish towards the future). This Proposition 2 is an extension of Proposition 3 of [55] and of Proposition 5 of [73] when the radiation is included in the background.

3. Statement and proof of Proposition 3: Assuming that the potential is non-negative (with not necessarily a local minimum at $0, 0)$, such that for $\phi \to -\infty, V(\phi)$ is unbounded. If its derivative is continuous and bounded on a set $A$ where the potential is bounded. Then the cosmological model evolves to a late-time de Sitter solution characterized by the divergence of the scalar field, provided that $V(\phi)$ is strictly decreasing. Additionally, if the potential vanish asymptotically, the Hubble scalar vanishes too. Proposition 3 is an extension of Proposition 6 discussed in [73] for $\rho_r > 0$.

4. Statement and proof of Proposition 4 (Proposition 3 of [73]) generalizing analogous result in [58]: If the potential $V(\phi)$ is such that the (possibly empty set) where it is negative is bounded and the (possibly empty) set of singular points of $V(\phi)$ is finite, then, the equilibrium point corresponding to a de Sitter solution, is an asymptotically stable singular point for the flow.

5. Implementation of a procedure for the analysis of Eqs. as $\phi \to \infty$ by using a suitable change of variables (based on [54]) to bring a neighborhood of $\phi \to \infty$ into a bounded set. The method have been exemplified for: (a) a double exponential potential $V(\phi) = V_1 e^{\alpha \phi} + V_2 e^{\beta \phi}, 0 < \alpha < \beta$ and a coupling function $\chi = \chi_0 e^{\frac{\lambda \phi}{\alpha}}$, where $\lambda$ is a constant, discussed in [75] and (b) a general class of potentials containing the cases investigated in [77, 78, 80], the so-called Albrecht-Skordis potential $V(\phi) = e^{-\mu \phi} (A - (\phi - B)^2)$, and a power-law coupling $\chi(\phi) = \left(\frac{\phi_0}{\lambda}\right)^{\frac{\alpha}{\beta}} \chi_0 (\phi - \phi_0)^{\frac{\alpha}{\beta}}$, $\alpha > 0$, constant, and $\phi_0 > 0$, which was originally investigated in [57] without the presence of radiation. As $\phi \to +\infty$ it can be found asymptotic solutions corresponding to: radiation-dominated cosmological solutions; power-law scalar-field dominated inflationary cosmological solutions; matter-kinetic-potential scaling solutions and radiation-kinetic-potential scaling solutions.

In [76] it was studied a flat FLRW model with a perfect fluid source and an scalar field with double exponential potential which is nonminimally coupled to matter. The coupling is derived from the formulation of the $f(R)$-gravity as an equivalent scalar-tensor theory. There were provided conditions for which $\rho_m \to 0, \phi \to 0$ and $\phi \to +\infty$ as $t \to \infty$ (see Proposition 1 of [76]), and conditions under which $H$ and $\phi$ blows-up in a finite time (see Proposition 2 of [76]).

In the reference [81] it was studied the late time of a negatively curved FLRW model with a perfect fluid matter source and a scalar field nonminimally coupled to matter. As we have commented before, under mild assumptions on the potential, the equilibria corresponding to non-negative local minima for $V$ are asymptotically stable. For
nondegenerate minima with zero critical value, it was proved that for \( \gamma > 2/3 \), there is a transfer of energy from the fluid and the scalar field to the energy density of the scalar curvature (changing the previous bound \( \gamma > 1 \) for flat FLRW model). Thus, the scalar curvature, if present, has a dominant effect on the late evolution of the universe and eventually dominates over both the perfect fluid and the scalar field. The analysis was complemented with the case where \( \lambda \) is exponential and therefore the scalar field diverges to infinity.

In \([82]\) was presented a generalized Brans-Dicke lagrangian including a non-minimally coupled Gauss-Bonnet term without imposing the vanishing torsion condition. The cosmological consequences of such model were studied. In \([43]\) it was studied the existence of exact solutions and integrable dynamical systems in multi-scalar field cosmology and more specifically in the so-called Chiral cosmology where nonlinear terms exists in the kinetic term of the scalar fields. Some exact analytic solutions for a system of N-scalar fields were presented. Some studies of cosmological effects of scalar fields and of their effects in multiple-field inflation are for example \([83–89]\).

In this paper we start by the simple scalar field model in vacuum, i.e., not additional matter is considered whatsoever, that is it is a particular case of the model studied in the references \([54–58, 71–75]\), when \( p_m = 0 \). Next we review some well-known results and prove some new theorems for more general situations than those studied in \([54–58, 71–75]\). We present some examples that violates one or more of the hypothesis of the Theorems proved, to see at which extent these conditions can be relaxed to obtain the same conclusions or to provide a counterexample. In particular we incorporate cosine-like corrections with small phase, but respecting the symmetries under the scalar field reversal \( \phi \to -\phi \) (if the original potential respect so). We have seen motivation for this kind of potential’s correction in the context of inflation in loop-quantum cosmology \([29]\). We choose the corrections best adapted to describe inflation in FLRW metric and in Bianchi I metrics. We use both local and global dynamical system variables, and smooth transformations of the scalar field which can be used in a combined way to provide qualitative features of the model at hand, and providing accurate schemes to find analytical approximations to the solutions following the Copeland, Liddle & Wands’s approach \([80]\), and the Alho & Uggla’s approach \([89]\).

The paper is organized as follows. In Section II it is studied a minimally coupled scalar field in vacuum. In Subsection II A are present some relevant results about the dynamics of the scalar field for very generic hypothesis over the free functions of the model. Furthermore, in Subsubsection II A 1 we follow the formulation by \([51]\) and we discuss the conditions under which the potential diverges to the past, which is a well-known generic fact as stated in Theorem II.4. Next in this Subsection, we prove Proposition II.5 in which we provide the center manifold calculation to analyze the stability of the de Sitter solution under mild hypothesis for the potential. The stability of the de Sitter solution is confirmed in Proposition II.6 for flat FLRW universe. Some discussions about this issue are presented in Subsection II B. In Subsection II C we discuss a method to generate exact solutions, and it is obtained some exact solutions for the scalar field model with exponential potential. In Subsection II D we proceed in more detail to examine qualitatively the scalar field model with potential \( V(\phi) = V_0 e^{-\lambda \phi} \), using the Copeland, Liddle & Wands’s approach \([80]\) in Subsubsection II C 1. The stability of the equilibrium points is discussed. This method is well-suited to investigate local stability features. However, it do not provide a global description of the phase space when generically \( \phi \) diverges, or when \( H \) tends to zero, in which case the method fails. For this reason in Subsubsection II C 2 we use the Alho & Uggla’s approach \([89]\) (first used by the authors for the monomial potential), which is well-suited for a global description of the phase space. We show the results for the vacuum case and for the exponential potential. The procedure will be helpful for more complicated situations. In Subsection II D we present a qualitative analysis for an scalar-field cosmology with generalized harmonic potential \( V_1(\phi) = \mu^3 \left[ \frac{\phi}{\mu} + b \cos \left( \delta + \frac{\phi}{\mu} \right) \right], \phi \neq 0 \); whereas in Subsection II E we present a qualitative analysis for an scalar-field cosmology with generalized harmonic potential \( V_2(\phi) = \mu^3 \left[ b f \left( \cos(\delta) - \cos \left( \delta + \frac{\phi}{\mu} \right) \right) + \phi^2 \right], \phi \neq 0 \). That is, we incorporate cosine-like corrections with small phase to the harmonic potential, motivated by inflationary loop-quantum cosmology. We use both local and global phase-space descriptions in Subsections II E 1 and II E 2 to find qualitative features and we present the asymptotic analysis as \( \phi \to \infty \) of each potential, respectively. In subsections II D 2 and II E 2 we investigate the oscillatory regime and we provide good approximations for solutions of the scalar field under the potentials \( V_1(\phi) \) and \( V_2(\phi) \), respectively. Substituting specific values of the free parameters of \( V_1(\phi) \) and \( V_2(\phi) \), we obtain some examples and counterexamples for Theorems II.1 and II.2 and II.3. Section III is devoted to the study of a minimally coupled scalar field in the presence of matter. In Subsection III A are annotated the main results: Propositions 2, 3, and 4 of \([55]\). These results were extended to Propositions 2.1, 2.2, 2.3 of \([56]\) for an scalar field nonminimally coupled to matter. Section IV is devoted to minimally coupled scalar field in FLRW and Bianchi I metrics. In Subsection IV A are presented the main results: Theorems IV.1, IV.2, IV.3. Section IV B is devoted to scalar field non-minimally coupled to matter in the FLRW and in the Bianchi I metrics. In subsections IV A are discussed the main results: Theorems V.2, V.3, V.4. In subsection V A it is studied the case positively curved FLRW model. In section VB it is summarized a global singularity theorem proved by one of the present authors. Section VII is devoted to conclusions.
II. MINIMALLY COUPLED SCALAR FIELD IN VACUUM.

In this section we study the scalar field cosmology with field equations

\[
\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0, \quad \dot{H} = -\frac{1}{2}\dot{\phi}^2, \quad 3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi).
\]

(1)

where the dot means derivative with respect to \( t \), \( H = \dot{a}/a \) denotes the Hubble expansion rate, \( \phi = \phi(t) \) is a scalar field and \( V(\phi) \), the scalar field self-interacting potential, is of class \( C^2 \).

A. Main results

We define the state variables \((H, y, \phi) \in \mathbb{R}^3\), where \( y = \dot{\phi} \). Therefore, we obtain the dynamical system

\[
\begin{align*}
\dot{H} &= -\frac{1}{2}y^2, \\
\dot{y} &= -3Hy - \frac{dV(\phi)}{d\phi}, \\
\dot{\phi} &= y.
\end{align*}
\]

(2a) (2b) (2c)

which defines a flow on the phase space

\[
\Omega = \{(H, y, \phi) \in \mathbb{R}^3 : 3H^2 = \frac{1}{2}y^2 + V(\phi)\}.
\]

(3)

First, we assume that the potential \( V(\phi) \) of class \( C^2 \) have the origin as a local minimum, with zero value, i.e., \( \phi = 0, \, V(0) = 0, \, V'(0) = 0, \, V''(0) < 0 \), say, for example, the monomial potential \( V(\phi) = m^2\phi^2 \). Therefore, \((0,0,0)\) is an equilibrium point of (2). Hence, if \( H(0) > 0 \) then \( H(t) > 0, \forall t > 0 \) because the set \( \{(H, y, \phi) \in \Omega : H = 0\} \), is invariant for the flow of (2), that is, an initially expanding universe will expand forever.

Theorem II.1 (Corollary of Proposition 2 of [55]). Let be \( V \in C^2(\mathbb{R}) \) such that

1. \( V(\phi) \geq 0 \), and \( V(\phi) = 0 \) if and only if \( \phi = 0 \).
2. \( V(\phi) \) is bounded on \( A \in \mathbb{R} \) if \( V(\phi) \) is bounded on \( A \).

Then, \( \lim_{t \to \infty} y = 0 \).

Proof. Let be \( O^+(x_0) \) the positive orbit passing by a regular point \( x_0 = (H, y, \phi) \in \Omega : H > 0 \) at the time \( t_0 \). Since \( H \) is positive and decreasing by Eq. (2a), we have \( \frac{1}{2}y(t)^2 + V(\phi(t)) = 3H(t)^2 \leq 3H(t_0)^2 \) for all \( t > t_0 \). From the non-negativity of each term in the left hand side of the above inequality we have that \( \frac{1}{2}y(t)^2 \) and \( V(\phi(t)) \) are bounded by \( 3H(t_0)^2 \) for all \( t > t_0 \).

Let be defined the set \( A = \{\phi \in \mathbb{R} : V(\phi) \leq 3H(t_0)^2\} \). Hence, the positive orbit \( O^+(x_0) \) is such that \( \phi \) belongs to the interior of the set \( A \).

Integrating Eq. (2a) it follows

\[
H(t_0) - H(t) = \int_{t_0}^{t_1} \frac{1}{2}y(s)^2 ds.
\]

(4)

Taking the limit \( t \to \infty \) in Eq. (4), we obtain

\[
\lim_{t \to \infty} H(t) = \lim_{t \to \infty} H(t_0) - \eta = \int_{t_0}^{\infty} \frac{1}{2}y(s)^2 ds,
\]

where we have used the fact that \( H(t) \to \eta \in \mathbb{R} \) as \( t \to \infty \), due to it is non-negative, decreasing, and bounded on the interval \( 0 \leq H(t) \leq H(t_0) \) for all \( t \geq t_0 \). Defining \( f(t) = \frac{1}{2}y(t)^2 \), and taking the derivative with respect to \( t \) we obtain

\[
\frac{d}{dt} f(t) = y\dot{y} = y \left( -3Hy - \frac{dV(\phi)}{d\phi} \right)
\]
Due to hypothesis 2., \( V(\phi) \) is bounded on \( A \), implies \( \frac{dV(\phi)}{d\phi} \) is bounded on \( A \), and let be \( K \) such that \( \left| \frac{dV(\phi)}{d\phi} \right| \leq K. \)

Hence, \( \left| \frac{df(t)}{dt} \right| \leq |3H\gamma^2| + \left| y \frac{dV(\phi)}{d\phi} \right| \leq 18H(t_0)^2 + 6KH(t_0). \) Therefore, \( \left| \frac{df(t)}{dt} \right| \) is bounded for all \( t > t_0 \) along \( O^+(x_0) \), additionally, from the convergence \( \int_{t_0}^{\infty} f(s)ds < \infty \), we have \( \lim_{t \to \infty} f(t) = 0 \). Then, \( \lim_{t \to \infty} y = 0. \)

This result is a particular case of the Proposition 2 of \([55]\) (annotated as Proposition III.1 in Sect. III) when \( \rho_m = 0. \)

**Theorem II.2** (Corollary of Proposition 3 of \([55]\)). Let be \( V \in C^2(\mathbb{R}) \) such that

1. \( V \geq 0, \) and \( V(\phi) = 0 \) if and only if \( \phi = 0. \)
2. \( V'(\phi) < 0 \) for \( \phi < 0 \) and \( V'(\phi) > 0 \) for \( \phi > 0. \)
3. \( V'(\phi) \) is bounded on \( A \subset \mathbb{R} \) if \( V(\phi) \) is bounded on \( A. \)

Then, \( \lim_{t \to \infty} \phi(t) \in \{ -\infty, 0, +\infty \}. \)

**Proof.** As before, let be taken the positive orbit \( O^+(x_0) \) passing by a regular point \( x_0 \in \{(H, y, \phi) \in \Omega : H > 0\} \) at the time \( t_0. \) From Theorem II.1, it follows that \( H(t) \) is positive, decreasing, and bounded along the orbit \( O^+(x_0), \) then, exists \( \lim_{t \to \infty} H(t) = \eta \in \mathbb{R}. \) Furthermore, from Theorem II.1 \( \lim_{t \to \infty} y(t) = 0. \) Now there are verified the following cases.

1. If \( \eta = 0, \) taking the limit \( t \to +\infty \) in \( 3H^2 = \frac{1}{2}y^2 + V(\phi), \) we obtain

\[
\lim_{t \to +\infty} V(\phi(t)) = 0.
\]

Additionally, \( V \) is continuous, and \( V(\phi) = 0 \) if and only if \( \phi = 0. \) Hence,

\[
\lim_{t \to +\infty} V(\phi(t)) = 0 \iff V\left( \lim_{t \to +\infty} \phi(t) \right) = 0 \iff \lim_{t \to +\infty} \phi(t) = 0.
\]

By continuity of \( V \) and hypothesis 1.

2. If \( \eta > 0, \) then \( \lim_{t \to +\infty} V(\phi(t)) = 3\eta^2 \) and exists \( t' \) such that \( V(\phi) > \frac{3}{2}\eta^2 \) for all \( t > t'. \) \( \phi \) cannot be identically zero for some \( t > t' \) because \( \phi = 0 \) if and only if \( V(\phi) = 0. \) Then, the sign of \( \phi \) is invariant for all \( t > t'. \) Let us assume that \( \phi(t) > 0 \) for all \( t > t'. \) Due to \( V \) is an increasing function for \( \phi \in (0, +\infty), \) we have

\[
\lim_{t \to +\infty} V(\phi(t)) = 3\eta^2 \leq \lim_{\phi \to +\infty} V(\phi).
\]

The equality is reached if and only if \( \lim_{t \to +\infty} \phi(t) = +\infty. \)

If

\[
\lim_{t \to +\infty} V(\phi(t)) = 3\eta^2 < \lim_{\phi \to +\infty} V(\phi),
\]

then, exists a unique \( \bar{\phi} > 0 \) such that

\[
\lim_{t \to +\infty} V(\phi(t)) = V(\bar{\phi}).
\]

Due to \( V \) is continuous, and strictly increasing,

\[
\lim_{t \to +\infty} \phi = \bar{\phi}.
\]

From Theorem II.1 it follows that \( \lim_{t \to +\infty} y = 0. \) Taking the limit \( t \to +\infty \) in Eq. (2b), we obtain

\[
\lim_{t \to +\infty} \dot{y} = -V'(\bar{\phi}) < 0
\]

Hence, exists \( t'' > t' \) such that \( \dot{y} < -\frac{V'(\bar{\phi})}{2} \) for all \( t \geq t''. \) This implies

\[
y(t) - y(t'') = \int_{t''}^{t} \dot{y}(s)ds < -\frac{V'(\bar{\phi})}{2}(t'' - t).
\]
This result is a particular case of the Proposition 3 of [55] (annotated as Proposition III.2 in Sect. III) when \( \rho > 0 \) for all \( t > t' \), we have \( \lim_{t \to +\infty} \phi(t) = +\infty \).

3. If \( \eta < 0 \), it can be proved in the same way that \( \phi < 0 \) for all \( t > t' \), and \( \lim_{t \to +\infty} \phi(t) = -\infty \).

This result is a particular case of the Proposition 4 by [55] (annotated as Proposition III.3 in Sect. III) when \( \rho_m = 0 \).

Furthermore, 

\[
\frac{3 \eta^2}{3 \eta^2} = \lim_{t \to +\infty} V(\phi(t)) = \lim_{\phi \to +\infty} V(\phi) > 3H(t_0)^2.
\]

This is impossible because \( H(t) \) is decreasing and \( H(t_0) \geq \eta \). In the same way, the assumption \( \lim_{t \to +\infty} \phi = -\infty \) leads to a contradiction. Then, \( \lim_{t \to +\infty} \phi = 0 \) and this implies \( \lim_{t \to +\infty} V(\phi(t)) = 0 \), and from [55] it follows \( \lim_{t \to +\infty} H(t) = 0 \).

Theorem II.3 (Corollary of Proposition 4 of [55]). Let be \( V \in C^2(\mathbb{R}) \) such that:

1. \( V \geq 0 \), and \( \lim_{\phi \to -\infty} V(\phi) = +\infty \).
2. \( V'(\phi) \) is continuous and \( V'(\phi) < 0 \) for all \( \phi \in \mathbb{R} \).
3. \( V'(\phi) \) is bounded on \( A \subset \mathbb{R} \) if \( V(\phi) \) is bounded on \( A \).

Then \( \lim_{t \to +\infty} y(t) = 0 \) and \( \lim_{t \to +\infty} \phi(t) = +\infty \).

Proof. As before, let be considered the positive orbit \( \mathcal{O}^+(x_0) \) passing at the time \( t_0 \) through a regular point \( x_0 \in \{(H, y, \phi) \in \Omega : H > 0\} \). As in the proof of Theorem II.1, we have \( H(t) \) is non-negative, along the orbit \( \mathcal{O}^+(x_0) \). By Eq. (1a), \( H(t) \) is decreasing and bounded along \( \mathcal{O}^+(x_0) \), then exists \( \lim_{t \to +\infty} H(t) = \eta \geq 0 \) and

\[
\frac{1}{2} \int_{t_0}^{+\infty} y(s)^2 ds = H(t_0) - \eta < +\infty.
\]

Furthermore, \( \frac{d}{dt} y(t)^2 \) is bounded, implying \( \lim_{t \to +\infty} y(t) = 0 \), and \( \lim_{t \to +\infty} V(\phi(t)) = 3 \eta^2 \). Due to \( V \) is strictly decreasing with respect to \( \phi \), it follows \( V(\phi) > \lim_{\phi \to +\infty} V(\phi) \) for any \( \phi \in \mathbb{R} \), then

\[
\lim_{t \to +\infty} V(\phi(t)) \geq \lim_{\phi \to +\infty} V(\phi).
\]

We have two cases:

1. If \( \lim_{t \to +\infty} V(\phi(t)) = \lim_{\phi \to +\infty} V(\phi) \), by continuity of \( V \) we have \( \lim_{t \to +\infty} \phi(t) = +\infty \).
2. If \( \lim_{t \to +\infty} V(\phi(t)) > \lim_{\phi \to +\infty} V(\phi) \), then exists a unique \( \bar{\phi} \) such that

\[
\lim_{t \to +\infty} V(\phi(t)) = V(\bar{\phi}).
\]

Due to \( V \) is continuous and strictly decreasing, then

\[
\lim_{t \to +\infty} \phi(t) = \bar{\phi}.
\]

Furthermore, from Eq. (25) it follows \( \lim_{t \to +\infty} \dot{y} = -V'(\bar{\phi}) > 0 \). Then, exists \( t' \) such that \( \dot{y} > -\frac{V'(\bar{\phi})}{2} \) for all \( t \geq t' \). Hence, we conclude that

\[
y(t) - y(t') > -\frac{V'(\bar{\phi})}{2} (t - t')
\]

a contradiction with \( \lim_{t \to +\infty} y(t) = 0 \).

This result is a particular case of the Proposition 4 by [55] (annotated as Proposition II.3 in Sect. III) when \( \rho_m = 0 \).
1. Alternative formulation.

In this section we follow the formulation by [54]. Let be defined the new variables

\[ w = \phi, \quad u = \frac{1}{H}, \quad v = \frac{\dot{\phi}}{\sqrt{6}H}, \quad (5) \]

and the new time derivative \( \frac{df}{dt} \equiv f' = H^{-1}f \). Then, we deduce the dynamical system

\[ \begin{align*}
  w' &= \sqrt{6}v, \\
  u' &= 3uv^2, \\
  v' &= 3v(v^2 - 1) - \frac{u^2V'(w)}{\sqrt{6}}, \quad (6a, 6b, 6c)
\end{align*} \]

defined on the phase space

\[ \Omega = \left\{ (w, u, v) \in \mathbb{R}^3 : u^2V(w) + 3v^2 = 3, V(w) \geq 0 \right\}. \quad (7) \]

**Theorem II.4** (Foster 1998, Theorem 1 in [54]). Let be \( V \in C^3 \). Consider the regular point \( x_0 \in \Omega \), with \( \Omega \) defined by (7), and let \( O^-(x_0) \) be the negative orbit of \( x_0 \) under the flow of (6). Then, \( w \) is almost always bounded along \( O^-(x_0) \).

**Proof** (adapted from [54]). Let be the regular point \( x_0 \in \Omega \), and let be \( O^-(x_0) \) the negative orbit of \( x_0 \) under Eqs. (6). Let be assumed that \( w \) is bounded along the negative orbit \( O^-(x_0) \). Then, \( O^-(x_0) \) it is contained on a compact subset of (the closure of) \( \Omega \), \( \Omega \setminus \Omega \), and there exists the \( \alpha \)-limit of \( x_0 \), denoted by \( \alpha(x_0) \), which is the union of orbits. Since \( u \) is monotonic increasing along \( O^-(x_0) \), then it must be constant on \( \alpha(x_0) \). Then, \( \alpha(x_0) \) is contained on the set \( v = 0 \). From Eqs. (6), it follows that the equilibrium point with \( u \neq 0 \), contained on the set \( v = 0 \), satisfies \( p = (w, u, v) = (w_0, u_0, 0) : V'(w_0) = 0 \), and \( u_0^2V(w_0) = 3 \).

Now we prove that no such equilibrium point can be the \( \alpha \)-limit \( \alpha(x_0) \).

The equilibrium points of Eqs. (6) corresponds to the equilibrium points of Eqs. (2), which is reduced to an unconstrained two dimensional dynamical system through the natural projection \((w, y) = (\phi, \dot{\phi})\), say

\[ \dot{w} = y, \quad \dot{y} = -\sqrt{\frac{3}{2}} y (y^2 + 2V(w))^\frac{3}{2} - V'(w). \quad (8) \]

Let be \( p_0 := (w_0, y_0) \) an equilibrium point of Eqs. (8) that satisfies \( y_0 = 0 \), \( V'(w_0) = 0 \), \( V(w_0) = V_0 > 0 \). The linearization have eigenvalues \( \{ \frac{1}{2} \left( -\sqrt{3}V_0 - 4V''(w_0) - \sqrt{3}V_0 \right), \frac{1}{2} \left( \sqrt{3}V_0 - 4V''(w_0) - \sqrt{3}V_0 \right) \} \). At least one of the above eigenvalues is always negative. Then, it follows from the invariant manifold theorem that there exists a local stable manifold of dimension one or two intersecting \( p_0 \), at which the solutions near to \( p_0 \) exponentially approaches \( p_0 \) as \( t \to +\infty \). The existence of an stable manifold of dimension \( n > 0 \) implies that all the solutions approaching \( p_0 \) as \( t \to -\infty \) must lie on an unstable or center manifold of dimension \( 2 - n < 2 \), therefore, contained on a set of Lebesgue measure zero.

The equilibrium points \( y = 0 \), \( V'(w) = 0 \), \( V(w) = V_0 > 0 \), corresponds to maximally symmetric solutions of the Einstein’s equations. When \( V_0 > 0 \), \( p_0 \) represents a de Sitter solution with \( \alpha(t) = a(t) = e^{\sqrt{\frac{2V_0}{3}}} \). The sign of the above eigenvalues depends on the sign of \( V''(w) \). If \( V''(w) > 0 \) (\( w_0 \) is a local minimum of \( V \)) or \( V''(w) < 0 \) (\( w_0 \) is a local maximum), then \( p_0 \) would be a sink or a saddle, respectively.

**Definition 1** (Degenerated minimum). The function \( V(\phi) \) have a local degenerated minimum of order \( n \) in \( \phi_* \) if \( V'(\phi_*) = V''(\phi_*) = \cdots = V^{(2n-1)}(\phi_*) = 0, V^{(2n)}(\phi_*) > 0 \), for some positive integer \( n \).

Going an step further of [54], we proceed to the calculation of the center manifold of the equilibrium point \( y = 0 \), \( V'(w) = 0 \), \( V(w) = V_0 > 0 \) when \( V''(w) = 0 \).

**Proposition II.5** (Leon & Franz-Silva, 2019). If \( V'(w_0) = 0 \), \( V(w_0) = V_0 > 0 \), \( V''(w_0) = 0 \), and \( V^{(3)}(w_0) \neq 0 \), i.e., the origin is an inflection point of \( V(\phi) \), then, the equilibrium point is unstable (saddle point). However, if \( w_0 \) is a degenerated
the results of the Proposition II.5. Equating to zero the coefficients of equal powers of $X$ by $h$ using Taylor series, we propose the expression $w = \frac{X^3}{3\sqrt{V_0}} + \frac{Y^2}{2\sqrt{3}\sqrt{V_0}} + \frac{X^2}{2\sqrt{3}\sqrt{V_0}} - \frac{X}{3\sqrt{V_0}} - \frac{Y_0}{2\sqrt{3}\sqrt{V_0}} + O(4)$, then, the center manifold is stable (respectively, unstable, saddle). Finally, we have the stability condition $V(w_0) = V_0 > 0, V'(w_0) = V(2)(w_0) = V(3)(w_0) = 0, V(4)(w_0) > 0,$ for a degenerated minimum of order two.

**Proof.** Due to $V''(w_0) = 0$, the linearization of $\rho_0$ have eigensystem \( \begin{pmatrix} 0 & -\sqrt{3}\sqrt{V_0} \\ 1,0 & -\frac{1}{3\sqrt{3}\sqrt{V_0}} \end{pmatrix} \). Therefore, the center manifold is tangent to the $w$-axis. Assuming that $V(w_0) = V_0 > 0, V'(w_0) = 0, V''(w_0)$, and $V(3)(w_0) \neq 0$ or $V(4)(w_0) \neq 0$, or both, and introducing the linear transformation $X = \frac{y}{\sqrt{3}\sqrt{V_0}} + w - w_0, Y = y,$ we obtain the Eqs.

\[ X' = \frac{-Y^3}{4V_0} + V(3)(w_0) \left( -\frac{Y^2}{6\sqrt{3}\sqrt{V_0}} - \frac{X^2}{2\sqrt{3}\sqrt{V_0}} + \frac{XY}{3V_0} \right) \\
+ V(4)(w_0) \left( -\frac{XY^2}{6\sqrt{3}\sqrt{V_0}} + \frac{Y^3}{54V_0^2} - \frac{X^3}{6\sqrt{3}\sqrt{V_0}} + \frac{X^2Y}{6V_0} + O(4), \right) \\
Y' = -\sqrt{3}\sqrt{V_0}Y - \frac{\sqrt{3}Y^3}{4V_0} + V(3)(w_0) \left( \frac{XY}{\sqrt{3}\sqrt{V_0}} - \frac{Y^2}{6V_0} - \frac{X^2}{2} \right) \\
V(4)(w_0) \left( \frac{Y^3}{18\sqrt{3}V_0^{3/2}} + \frac{X^2Y}{2\sqrt{3}\sqrt{V_0}} - \frac{XY^2}{6V_0} - \frac{X^3}{6} \right) + O(4), \]

where $O(4)$ denotes terms of fourth order in the vector norm, to be discarded. The behavior near $P_0 : (X, Y) = (0, 0)$ will be characterized by a local center manifold $(h(X))$ of the origin, tangent to the $X$-axis on the origin, where $h(X)$ is the solution of the differential equation

\[ \begin{align*}
-h'(X) & \left[ V(4)(w_0) \left( -\frac{Xh(X)^2}{6\sqrt{3}\sqrt{V_0}} + \frac{h(\xi)^3}{54V_0^2} - \frac{X^2h(X)}{6V_0} - \frac{X^3}{6\sqrt{3}\sqrt{V_0}} \right) \\
+ V(3)(w_0) & \left( -\frac{h(\xi)^2}{6\sqrt{3}\sqrt{V_0}} - \frac{Xh(\xi)}{3\sqrt{V_0}} - \frac{X^2}{2\sqrt{3}\sqrt{V_0}} - \frac{h(\xi)^3}{4V_0} \right) \\
+ V(4)(w_0) & \left( \frac{h(\xi)^3}{18\sqrt{3}\sqrt{V_0^{3/2}}} + \frac{X^2h(\xi)}{2\sqrt{3}\sqrt{V_0}} - \frac{X^3}{6V_0} - \frac{X^3}{6} \right) \\
+ V(3)(w_0) & \left( \frac{Xh(\xi)}{\sqrt{3}\sqrt{V_0}} - \frac{h(\xi)^2}{6V_0} - \frac{X^2}{2} \right) - \sqrt{3}h(X)^3 - \sqrt{3}\sqrt{V_0}h(X) = 0, \end{align*} \]

\[ h(0) = 0, h'(0) = 0. \]

Using Taylor series, we propose the expression $h(X) = aX^2 + bX^3 + O(X^4)$. The differential equation for $h(X)$ can be expressed as $X^2 \left( -\sqrt{3a}\sqrt{V_0} - \frac{1}{2}V(3)(w_0) \right) + X^3 \left( \frac{2aV(3)(w_0)}{\sqrt{3}\sqrt{V_0}} - \sqrt{3b}\sqrt{V_0} - \frac{1}{6}V(4)(w_0) \right) + O(X^4) = 0.$

Equating to zero the coefficients of equal powers of $X$, and solving for $a$ and $b$ we obtain $a = -\frac{V(3)(w_0)}{2\sqrt{3}\sqrt{V_0}}, \quad b = -\frac{V(4)(w_0) + 2V(3)(w_0)^2}{6\sqrt{3}\sqrt{V_0}}. \quad$ Hence, the local center manifold of the origin is given locally by the graph $Y = -\frac{X^2V(3)(w_0)}{2\sqrt{3}\sqrt{V_0}} - \frac{X^3V(4)(w_0) + 2V(3)(w_0)^2}{6\sqrt{3}\sqrt{V_0}} + O(X^4).$ Furthermore, the dynamics over the local center manifold of the origin is dictated by $X' = -\frac{X^2V(3)(w_0)}{2\sqrt{3}\sqrt{V_0}} - \frac{X^3V(4)(w_0) + 2V(3)(w_0)^2}{6\sqrt{3}\sqrt{V_0}} + O(X^4),$ from the analysis of the corresponding potential it follows the results of the Proposition II.5.
**Proposition II.6** (Collorary of Proposition 1 in [58]). Let be $V(\phi) \in C^2(\mathbb{R})$ such that [58]:

(i) The possibly empty set $\{\phi : V(\phi) < 0\}$ is bounded;

(ii) The possibly empty set of singular points of $V(\phi)$ is finite.

Let be $\phi_*$ a minimum strict of $V(\phi)$, possibly degenerated, with non-negative critical value. Then $p_* := (\phi, y, H) = (\phi_*, 0, \sqrt{\frac{V(\phi_*)}{3}})$ is an asymptotically stable equilibrium point of the flow of (2).

**Proof.** (We adapt the technique used in the proof of Proposition 1 in [58], that was first used in [71].) First, it is assumed that $V(\phi_*) > 0$. Let $\bar{V} > V(\phi_*)$ a regular value of $V$ such that the connected component of $V^{-1}(\bar{V})$ that contains $\phi_*$ as the only critical point of $V$, is a compact set in $\mathbb{R}$. Let be denoted this set by $A$ and define $\Psi = \{(\phi, y, H) \in \mathbb{R}^3 : \phi \in A, \frac{1}{2}y^2 + V(\phi) \leq \bar{V}\}$, where $\bar{V}$ is positive. It can be proved that $\Psi$ is a compact set as follows.

(i) $\Psi$ is closed in $\mathbb{R}^3$;

(ii) $V(\phi_*) \leq V(\phi) \leq \bar{V}$, $\forall \phi \in A$;

(iii) $\frac{1}{2}y^2 + V(\phi) \leq \frac{1}{2}y^2 + V(\phi) \leq \bar{V}$, then, $y$ is bounded;

(v) Finally, from (3), and the above facts, we have $\frac{V(\phi)}{H^2} \leq H^2 \leq \bar{V}$.

Let be $\Psi_+ \subseteq \Psi$ the connected component of $\Psi$ containing $p_*$. Following the same arguments as in [58, 71] it can be proved that $\Psi_+$ is positively invariant with respect to (1), i.e., all the solutions with initial state in $\Psi_+$ remains in $\Psi_+$ for all $t > 0$. Indeed, let be $x(t)$ such a solution and $\tilde{t} = \sup \{t > 0 : H(t) > 0\} \in \mathbb{R} \cup \{+\infty\}$. Defining

$$\epsilon = \frac{1}{2}y^2 + V(\phi), \quad \dot{\epsilon} = -3Hy^2. \quad (12)$$

When $\epsilon < \tilde{t}$, equation (12) implies that $\epsilon$ is monotonically decreasing. On the other hand, it can be proved by contradiction that $\phi(t) \in A$, $\forall t < \tilde{t}$. Otherwise, there would exists $t < \tilde{t}$ such that $V(\phi(t)) > \bar{V}$, but then

$$\dot{\bar{V}} < V(\phi(t)) \leq \frac{1}{2}y(t)^2 + V(\phi(t)) \leq \bar{V},$$

a contradiction. Due to $\frac{1}{2}y(t)^2 \geq 0$ along the flow, it follows

$$H(t)^2 \geq V(\phi(t)) \geq \frac{V(\phi_*)}{3} > 0.$$ 

We have proved that $H$ is strictly bounded away from zero; then, $\tilde{t} = +\infty$, and from this it is deduced that $x(t)$ remains on $\Psi_+$ for all $t > 0$. Therefore, $\Psi_+$ satisfies the hypothesis of LaSalle’s invariance Theorem (see Ref. [26]; and Theorem 8.3.1 of [71], pp. 111). Considering the monotonic function $\epsilon$ defined on $\Psi_+$ it follows that any solution with initial state on $\Psi_+$ must be such that $Hy^2 \to 0$ as $t \to +\infty$. Since $H$ is strictly bounded away from zero on $\Psi_+$ it follows that $y \to 0$ and $H^2 - \frac{V(\phi)}{3} \to 0$ as $t \to +\infty$. Due to $H$ is monotonic decreasing and it is bounded away zero, it must have a limit. This implies that $V(\phi)$ have also a limit. This limit must be $V(\phi_*)$; otherwise, $V'(\phi)$ would tend to a non zero value, and so would be the right hand side of (26), a contradiction. Therefore, the solution tends to $p_*$. If $V(\phi_*) = 0$, the set $\Psi$ is connected and we choose $\Psi_+$ as the subset of $\Psi$ with $H \geq 0$. The unique equilibrium point on $\Psi_+$ with $H = 0$ is the equilibrium point $p_*$, then, if $H(t) \to 0$ the solution is forced to tend to the equilibrium point due to $H$ is monotonic; on the contrary, if $H(t)$ would tend to a positive number, as before, we would have that $y \to 0$, $V(\phi) \to V(\phi_*) = 0$, hence, $H$ would necessarily tend to zero.

2. Results for $k = -1$ and for $k = 1$.

For scalar field in vacuum with non flat FLRW models, the Raychaudhuri equation, and the Friedmann equation becomes, respectively,

$$\dot{H} = -\frac{1}{2}y^2 + \frac{k}{a^2}, \quad (13)$$
and

\[ 3H^2 = \frac{1}{2} y^2 + V(\phi) - \frac{3k}{a^2}. \]  

(14)

Then, for negatively curved FLRW model, \( k = -1 \), the key features are that [71]:

1. From (13), \( H(t) \) is monotonic decreasing for \( t > t_0 \).

2. Let be defined \( W(\phi, y, H) = 3H^2 - \frac{1}{2} y^2 - V(\phi) \), such that \( \dot{W} = -2HW \). For \( k = -1, W \geq 0 \). Hence, if \( H > 0 \), \( W \) will be monotonic decreasing.

Let be given \( \phi_\ast \), a local minimum of \( V(\phi) \) with \( V(\phi_\ast) > 0 \). Let \( \dot{V} > V(\phi_\ast) \) be a regular value of \( V \) such that the connected component of \( V^{-1}\left((-\infty, \dot{V})\right) \) that contains \( \phi_\ast \) as the only critical point of \( V \), is a compact set in \( \mathbb{R} \). Let be considered a solution \( x(t) = (\phi(t), y(t), H(t)) \) such that \( \frac{1}{2} y(0)^2 + V(\phi(0)) \leq \dot{V} \), and let \( \dot{W} = W(0) = \frac{1}{a(0)} > 0 \). Define by \( \Psi_+ \) the connected component of the set

\[ \Psi := \left\{ (\phi, y, H) \in \mathbb{R}^3 : W(\phi, y, H) \in [0, \dot{W}], V(\phi_\ast) \leq \epsilon := \frac{1}{2} y^2 + V(\phi) \leq V \right\}, \]

containing the equilibrium point \( \bar{p}_\ast \). As before it can be proved that \( \Psi_+ \) is compact, positively invariant, and that \( H \) is strictly bounded away from zero on \( \Psi_+ \). Then, \( x(t) \) remains on \( \Psi_+ \) for all \( t > 0 \). Therefore, \( \Psi_+ \) satisfies the hypothesis of LaSalle’s invariance Theorem [96, 97]. Considering the monotonic function \( \epsilon \) defined on \( \Psi_+ \) it follows that any solution with initial state on \( \Psi_+ \) must be such that \( HY^2 \to 0 \) as \( t \to +\infty \). Considering the monotonic function \( W \) defined on \( \Psi_+ \) it follows that any solution with initial state on \( \Psi_+ \) must be such that \( HW^2 \to 0 \) as \( t \to +\infty \). Since \( H \) is strictly bounded away from zero on \( \Psi_+ \) it follows that \( y \to 0, W \to 0 \) and \( H^2 - \frac{V(\phi)}{3} \to 0 \) as \( t \to +\infty \). Due to \( H \) is monotonic decreasing and it is bounded away zero, it must have a limit. This implies that \( V(\phi) \) have also a limit. As before, this limit must be \( V(\phi_\ast) \). Therefore, the solution tends to \( \bar{p}_\ast \).

If \( V(\phi_\ast) = 0 \), the set \( \Psi \) is connected and we choose \( \Psi_+ \) as the subset of \( \Psi \) with \( H \geq 0 \). The unique equilibrium point on \( \Psi_+ \) with \( H = 0 \) is the equilibrium point \( \bar{p}_\ast \), then, if \( H(t) \to 0 \) the solution is forced to tend to the equilibrium point due to \( H \) is monotonic; on the contrary, if \( H(t) \) would tend to a positive number, as before, we would have that \( y \to 0, W \to 0, V(\phi) \to V(\phi_\ast) = 0 \), hence, \( H \) would necessarily tend to zero.

On the other hand, for \( W < 0 \), i.e., \( k = +1 \), it cannot be guaranteed the monotony of \( H \), so, we have to adapt the previous arguments in exactly the same ways in [71]. Let be given \( \phi_\ast \), a local minimum of \( V(\phi) \) with \( V(\phi_\ast) > 0 \). Let \( \dot{V} > V(\phi_\ast) \) be a regular value of \( V \) such that the connected component of \( V^{-1}\left((-\infty, \dot{V})\right) \) that contains \( \phi_\ast \) as the only critical point of \( V \), is a compact set in \( \mathbb{R} \). Let be considered a solution \( x(t) = (\phi(t), y(t), H(t)) \) such that \( \frac{1}{2} y(0)^2 + V(\phi(0)) \leq \dot{V} \), and let \( \dot{W} < 0 \) a value to determine, to act as a lower bound for \( W \). Taking the initial condition near the equilibrium point \( \bar{p}_\ast \), then \( H(0) > 0 \); since \( W(0) > \dot{W} \), from the equations \( \dot{W} = -2HW \) and \( \dot{V} = -3H^2 \), we have \( \dot{W} \geq \dot{W} \) (\( W \) now will be monotonic increasing and bounded by above by zero) and \( \epsilon \leq \dot{V} \). The last inequality implies that \( V(\phi(t)) \leq \dot{V} \). This implies that \( \phi(t) \) satisfies \( V(\phi_\ast) \leq V(\phi(t)) \leq \dot{V} \). Then,

\[ H^2 = \frac{\epsilon}{3} + \frac{W}{3} \geq \frac{V(\phi_\ast)}{3} + \frac{\dot{W}}{3} = H^2 + \frac{\dot{W}}{3} \implies H \geq \bar{H} := \left(\sqrt{1 + \frac{W}{3H^2}}\right) H_s, \]

where \( H_s = \sqrt{\frac{V(\phi_\ast)}{3}} \). Choosing \( \dot{W} \) small enough such that \( H_s^2 > -\frac{\dot{W}}{3} \), we have \( H(0) > 0 \implies H(t) \geq \bar{H} > 0 \). That is, \( H \) is bounded away zero which combined with the monotony of \( W, \epsilon \), and using the LaSalle’s invariance Theorem as in the case \( W \geq 0 \) leads to \( y \to 0, W \to 0 \) as \( t \to \infty \), and the equilibrium point \( \bar{p}_\ast \) is approached asymptotically.

If \( V(\phi_\ast) = 0 \), the equilibrium satisfy \( H_s = 0 \), and the nearby solutions may recollapse due the \( H \) change sign. Collapsing models were exhaustively studied, e.g., in [100, 109] for a wide class of self-interacting, self-gravitating homogeneous scalar field models.
B. Exact solutions

In this section we follow the formulation by [54]. Introducing the new time variable $dt = a^3 d\eta$, Eq. (2b) is rewritten as

$$\frac{d^2 \phi}{d\eta^2} + a^6 \frac{dV}{d\phi} = 0. \tag{15}$$

Now, defining $x = \phi$, $p_x = \frac{dx}{d\eta}$, $e^y = a^6$, $p_y = \frac{dy}{d\eta}$, we obtain the differential equations

$$\frac{dx}{d\eta} = p_x, \quad \frac{dy}{d\eta} = \frac{1}{6} \left[ \frac{1}{2} p_y^2 - 3 p_x^2 \right] \frac{dV(x)}{dx}, \quad \frac{dp_x}{d\eta} = \frac{1}{2} p_y^2 - 3 p_x^2. \tag{16}$$

For an exponential potential $V(x) = V_0 e^{-\lambda x}$, we have

$$\frac{dp_x}{d\eta} = \frac{\lambda}{6} \left[ \frac{1}{2} p_y^2 - 3 p_x^2 \right], \quad \frac{dp_y}{d\eta} = \frac{1}{2} p_y^2 - 3 p_x^2, \quad \frac{dx}{d\eta} = p_x, \quad \frac{dy}{d\eta} = p_y. \tag{17}$$

Therefore,

$$p_x = \frac{c_1 \lambda \left( \sqrt{6} \lambda \tanh (\Delta(\eta)) + 6 \right)}{6 \lambda^2 - 6}, \quad p_y = \frac{c_1 \lambda \left( \sqrt{6} \tanh (\Delta(\eta)) + \lambda \right)}{\lambda^2 - 6}, \tag{19a}$$

$$x = \frac{c_1 \lambda \left( \frac{12 \ln(\cosh(\Delta(\eta)))}{c_1} + 6 \eta \right)}{6 \lambda^2 - 6} + c_4, \quad y = \frac{c_1 \lambda \left( \frac{12 \ln(\cosh(\Delta(\eta)))}{c_1} + \lambda \eta \right)}{\lambda^2 - 6} + c_3, \tag{19b}$$

$$\Delta(\eta) = \frac{c_1 \lambda (\eta - 12 c_2 \lambda)}{2 \sqrt{6}}. \tag{19c}$$

We have to impose the compatibility condition:

$$12 \left( \lambda^2 - 6 \right) V_0 e^{c_3 - c_4 \lambda} - c_1 \lambda^2 = 0. \tag{20}$$

Finally, we obtain

$$a(\eta) = e^{\frac{r}{\sqrt{6}}} = e^{\frac{r}{\sqrt{6}} \left( \frac{c_1 \lambda^2}{\lambda^2 - 6} + c_3 \right) \cosh \frac{\eta}{\sqrt{6}} \left( \frac{c_1 \lambda (\eta - 12 c_2 \lambda)}{2 \sqrt{6}} \right)} \tag{21}$$

and the quadrature

$$t - t_0 = \int_1^{\eta} e^{\frac{r}{\sqrt{6}} \left( \frac{c_1 \lambda^2}{\lambda^2 - 6} + c_3 \right) \cosh \frac{\zeta}{\sqrt{6}} \left( \frac{c_1 \lambda (\zeta - 12 c_2 \lambda)}{2 \sqrt{6}} \right) } d\zeta. \tag{22}$$
| Label | $x$ | Existence | Stability |
|-------|-----|-----------|-----------|
| $P_0$ | 1   | $\forall \lambda$ | Non-hyperbolic for $\lambda = \frac{\sqrt{6}}{6}$, Sink for $\lambda > \sqrt{6}$, Source for $\lambda < \sqrt{6}$ |
| $P_1$ | $-1$ | $\forall \lambda$ | Non-hyperbolic for $\lambda = -\frac{\sqrt{6}}{6}$, Sink for $\lambda < -\sqrt{6}$, Source for $\lambda > -\sqrt{6}$ |
| $P_2$ | $\frac{\sqrt{6}}{6} \lambda$ | $-\sqrt{6} \leq \lambda \leq \sqrt{6}$ | Non-hyperbolic for $\lambda = \pm \sqrt{6}$, Sink for $-\sqrt{6} < \lambda < \sqrt{6}$ |

TABLE I: Existence conditions and stability conditions of the equilibrium points of Eqs. (26).

C. Qualitative analysis of the scalar field model with potential $V(\phi) = V_0 e^{-\lambda \phi}$.

1. Copeland, Liddle & Wands’s approach.

In this section we use the approach of [80]. Let be defined the new variables

$$x = \frac{\dot{\phi}}{\sqrt{6} H}, \quad y = \frac{\sqrt{V(\phi)}}{\sqrt{3} H}. \tag{23}$$

From the first integral $3H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi)$, we have for $H \neq 0$

$$\frac{1}{2} \dot{\phi}^2 + V(\phi) = 3H^2 \implies \frac{\dot{\phi}^2}{6H^2} + \frac{V(\phi)}{3H^2} = 1 \implies x^2 + y^2 = 1. \tag{24}$$

Let be defined the new time variable $\tau$, through $d\tau = H dt$. Then, for $H > 0$ we obtain the dynamical system:

$$\frac{dx}{d\tau} = 3x^3 - 3x + \sqrt{\frac{3}{2}} \lambda y^2, \quad \frac{dy}{d\tau} = 3xy \left( x - \frac{\sqrt{6}}{6} \lambda \right). \tag{25}$$

These Eqs. can be reduced in one dimension using the relation $x^2 + y^2 = 1$:

$$\frac{dx}{d\tau} = f(x) := 3 \left( x^2 - 1 \right) \left( x - \frac{\sqrt{6}}{6} \lambda \right). \tag{26}$$

Integrating the above equation we obtain

$$\tau := \ln a = c_1 + \ln \left[ (1 - x) \frac{\sqrt{6}}{6 \lambda} (x + 1) \frac{1}{\sqrt{x^2 + 6}} \left( 6x - \sqrt{6 \lambda} \right)^{\frac{x^2 - 6}{6}} \right]. \tag{27}$$

That is, the scale factor $a$ can be expressed as a function of $x$ by

$$a(x) = e^{c_1} (1 - x) \frac{\sqrt{6}}{6 \lambda} (x + 1) \frac{1}{\sqrt{x^2 + 6}} \left( 6x - \sqrt{6 \lambda} \right)^{\frac{x^2 - 6}{6}}, x \in [-1, 1]. \tag{28}$$

Furthermore,

$$\frac{d\phi}{dx} = \frac{dx}{d\tau} = -\frac{2\sqrt{6} x}{(1 - x^2) \left( 6x - \sqrt{6 \lambda} \right)}. \tag{29}$$
from which we have by integration
\[ \phi(x) = c_2 + \ln \left[ (1 - x)^{\frac{1}{\sqrt{6}} - \lambda} (x + 1)^{-\frac{1}{\sqrt{6}} + \lambda} \left( \sqrt{6} \lambda - 6x \right)^{\frac{2}{\sqrt{6}} - \frac{\lambda}{2}} \right]. \] (30)

On the other hand, from (23), with \( V(\phi) = V_0 e^{-\lambda \phi} \) we obtain
\[ H(x) = \sqrt{\frac{V_0}{3}} e^{-\frac{1}{2} \theta(x)} = \frac{V_0}{\sqrt{3}} e^{-\frac{\lambda}{2}} (1 - x)^{\frac{1}{\sqrt{6}} - \lambda} (x + 1)^{-\frac{1}{\sqrt{6}} + \lambda} \left( \sqrt{6} \lambda - 6x \right)^{\frac{2}{\sqrt{6}} - \frac{\lambda}{2}}. \] (31)

Finally, it is deduced that
\[ \frac{dt}{dx} = \frac{1}{H(x)} \frac{d^2}{dx^2} = \frac{2 \sqrt{3 - 3x^2} e^{-\frac{1}{2} \lambda}(1 - x)^{\frac{1}{\sqrt{6}} - \lambda}(x + 1)^{-\frac{1}{\sqrt{6}} + \lambda} \left( \sqrt{6} \lambda - 6x \right)^{-\frac{2}{\sqrt{6}} - \frac{\lambda}{2} - 1}}{V_0 (x^2 - 1)}, \] (32)
\[ t = \frac{2 e^{-\frac{1}{2} \lambda}}{\sqrt{3 - 3x^2}} \int \frac{\sqrt{3 - 3x^2} (1 - x)^{\frac{1}{\sqrt{6}} - \lambda} (x + 1)^{-\frac{1}{\sqrt{6}} + \lambda} \left( \sqrt{6} \lambda - 6x \right)^{-\frac{2}{\sqrt{6}} - \frac{\lambda}{2} - 1}}{x^2 - 1} \, dx. \] (33)

The equilibrium points of Eqs. (26) are \( P_0 : x = 1, P_1 : x = -1 \) and \( P_2 : x = \frac{2\sqrt{6}}{\lambda} \). \( P_2 \) exists \((1 \leq x \leq 1)\) for \( \lambda^2 \leq 6 \). Furthermore, \( \frac{df(x)}{dx} |_{P_0} = 6 - \sqrt{6} \lambda, \frac{df(x)}{dx} |_{P_1} = \sqrt{6} \lambda + 6, \frac{df(x)}{dx} |_{P_2} = \frac{1}{2} (\lambda^2 - 6) \), where \( f(x) \) denotes the right hand side of Eq. (26). According to the signs of \( df \) evaluated at the equilibrium points we obtain the following stability conditions. \( P_0 \) is Non-hyperbolic for \( \lambda = \frac{2\sqrt{6}}{\pi} \); it is a sink for \( \lambda > \sqrt{6} \); it is a source for \( \lambda < \sqrt{6} \). \( P_1 \) is Non-hyperbolic for \( \lambda = \frac{2\sqrt{6}}{\pi} \); it is a sink for \( \lambda > -\sqrt{6} \); it is a source \( \lambda < -\sqrt{6} \). \( P_2 \) is Non-hyperbolic for \( \lambda = \pm \sqrt{6} \); it is a sink for \( -\sqrt{6} < \lambda < \sqrt{6} \).

In table I are summarized the existence conditions and the stability conditions of \( P_0, P_1 \) and \( P_2 \).

2. Alho & Uggla’s approach.

In this section we use the approach by [90]. Let be defined the new variables
\[ T = \frac{m}{m + H}, \theta = \tan^{-1} \left( \frac{\dot{\phi}}{\sqrt{2} V(\phi)} \right), m > 0, \] (34)
and the new time derivative \( \dot{\tau} \) given by
\[ \frac{d\tau}{dt} = m + H, \] (35)
where we have assumed \( H > 0 \), to obtain the dynamical system
\[ \frac{dT}{d\tau} = 3(T - 1)^2 T \sin^2(\theta), \quad \frac{d\theta}{d\tau} = \frac{1}{2} (T - 1) \cos(\theta) \left( 6 \sin(\theta) - \sqrt{6} \lambda \right), \] (36)
defined in the finite cylinder \( S \) with boundaries \( T = 0 \) and \( T = 1 \). The variable \( T \) is suitable for global analysis [90], due to
\[ \frac{dT}{d\tau} \bigg|_{\sin \theta = 0} = 0, \quad \frac{d^2T}{d\tau^2} \bigg|_{\sin \theta = 0} = 0, \quad \frac{d^3T}{d\tau^3} \bigg|_{\sin \theta = 0} = 9 \lambda^2 (T - 1)^4 T. \] (37)

From the first Eq. of (36) and Eq. (37), \( T \) is a monotonically increasing function on \( S \). As a consequence, all orbits originate from the invariant subset \( T = 0 \) (which contains the \( \alpha \)-limit), which is classically related to the initial singularity with \( H \to \infty \), and ends on the invariant boundary subset \( T = 1 \), which corresponds asymptotically to \( H = 0 \).

In the Figs. 1 and 2 are presented some orbits of the flow of Eqs. (36) (left panel); and a projection over the cylinder
The equilibrium points of Eqs. (36) on \( \bar{S} \) (that is, with the boundary sets \( T = 0 \) and \( T = 1 \) attached) are the following.

1. The line \( P_0 = (1, \theta) \), with eigenvalues \( \{0, 0\} \). It is non-hyperbolic.

2. \( P_1 = (0, -\frac{\pi}{2} + 2n\pi), n \in \mathbb{N} \cup \{0\} \), with eigenvalues \( \{3, \frac{1}{2} (\sqrt{6} \lambda + 6)\} \). The stability conditions for \( P_1 \) are the following
   
   (a) Saddle for \( \lambda < -\sqrt{6} \).
   
   (b) Non-hyperbolic for \( \lambda = -\sqrt{6} \).
   
   (c) Source for \( \lambda > -\sqrt{6} \).

3. \( P_2 = (0, \frac{\pi}{2} + 2n\pi), n \in \mathbb{N} \cup \{0\} \), with eigenvalues \( \{3, \frac{1}{2} (6 - \sqrt{6} \lambda)\} \). The stability conditions of \( P_2 \) are the following

   (a) Saddle for \( \lambda > \sqrt{6} \).
   
   (b) Non-hyperbolic for \( \lambda = \sqrt{6} \).
   
   (c) Source for \( \lambda < \sqrt{6} \).
FIG. 2: Phase portrait of Eqs. (36) (left panel). Projection over the cylinder $S$ (right panel) for different values of $\lambda$.

4. $P_3 = (0, \arcsin \frac{\lambda}{\sqrt{6}})$, exists for $-\sqrt{6} \leq \lambda \leq \sqrt{6}$. The eigenvalues are $\left\{ \frac{\lambda^2}{2}, \frac{1}{2} (\lambda^2 - 6) \right\}$. The stability condition of $P_3$ are the following.
   (a) Non-hyperbolic for $\lambda = -\sqrt{6}$.
   (b) Saddle for $-\sqrt{6} < \lambda < 0$.
   (c) Non-hyperbolic for $\lambda = 0$.
   (d) Saddle for $0 < \lambda < \sqrt{6}$.
   (e) Non-hyperbolic for $\lambda = \sqrt{6}$.

The equilibrium points of Eqs. (36), as well as their existence conditions and stability conditions are summarized on table II.

Alternatively, we can use the new time variable $\tau = \ln a$, to obtain the unconstrained dynamical system:

$$
\frac{dT}{d\tau} = -3(T - 1)T \sin^2(\theta), \quad \frac{d\theta}{d\tau} = \frac{1}{2} \cos(\theta) \left( \sqrt{6} \lambda - 6 \sin(\theta) \right).
$$

The equilibrium points of Eqs. (38) are the following.
TABLE II: Existence conditions and stability conditions for the equilibrium points of Eqs. (36), \( n \in \mathbb{N} \cup \{0\} \).

| Label | \((T, \theta)\) | Existence | Stability |
|-------|----------------|------------|-----------|
| \(P_1\) | \((1, \theta)\) | \(\forall \lambda\) | Non-hyperbolic. |
| \(P_1\) | \((0, -\frac{\pi}{2} + 2n\pi)\) | \(\forall \lambda\) | Saddle for \(\lambda < -\sqrt{6}\). Non-hyperbolic for \(\lambda = -\sqrt{6}\). Source for \(\lambda > -\sqrt{6}\). |
| \(P_2\) | \((0, \frac{\pi}{2} + 2n\pi)\) | \(\forall \lambda\) | Saddle for \(\lambda > \sqrt{6}\). Non-hyperbolic for \(\lambda = \sqrt{6}\). Source for \(\lambda < \sqrt{6}\). |
| \(P_3\) | \((0, \arcsin \frac{\lambda}{\sqrt{6}})\) | \(-\sqrt{6} \leq \lambda \leq \sqrt{6}\) | Non-hyperbolic for \(\lambda \in \{-\sqrt{6}, 0, \sqrt{6}\}\). Saddle for \(-\sqrt{6} < \lambda < 0\) or \(0 < \lambda < \sqrt{6}\). |
| \(P_4\) | \((1, -\frac{\pi}{2} + 2n\pi)\) | \(\forall \lambda\) | Source for \(\lambda > \sqrt{6}\). |
| \(P_5\) | \((1, \frac{\pi}{2} + 2n\pi)\) | \(\forall \lambda\) | Saddle for \(\lambda > \sqrt{6}\). Non-hyperbolic for \(\lambda = \sqrt{6}\). Saddle for \(\lambda < \sqrt{6}\). |
| \(P_6\) | \((1, \arcsin \frac{\lambda}{\sqrt{6}})\) | \(-\sqrt{6} \leq \lambda \leq \sqrt{6}\) | Non-hyperbolic for \(\lambda \in \{-\sqrt{6}, 0, \sqrt{6}\}\). Sink for \(-\sqrt{6} < \lambda < 0\) or \(0 < \lambda < \sqrt{6}\). |

TABLE III: Existence conditions and stability conditions of the equilibrium points of Eqs. (38), \( n \in \mathbb{N} \cup \{0\} \).

1. \(P_1\) : \((T, \theta) = (0, -\frac{\pi}{2} + 2n\pi)\), \(n \in \mathbb{N} \cup \{0\}\), with eigenvalues \(\left\{3, \sqrt{\frac{3}{2}}\lambda + 3\right\}\). It is a saddle for \(\lambda < -\sqrt{6}\); Non-hyperbolic for \(\lambda = -\sqrt{6}\); source for \(\lambda > -\sqrt{6}\).

2. \(P_2\) : \((T, \theta) = (0, \frac{\pi}{2} + 2n\pi)\), \(n \in \mathbb{N} \cup \{0\}\), with eigenvalues \(\left\{3, 3 - \sqrt{\frac{3}{2}}\lambda\right\}\). It is a saddle for \(\lambda > \sqrt{6}\); Non-hyperbolic for \(\lambda = \sqrt{6}\); source for \(\lambda < \sqrt{6}\).

3. \(P_3\) : \((T, \theta) = \left(0, \arcsin \frac{\lambda}{\sqrt{6}}\right)\), with eigenvalues \(\left\{\frac{\lambda^2}{2}, \frac{1}{2} \left(\lambda^2 - 6\right)\right\}\). It is Non-hyperbolic for \(\lambda \in \{-\sqrt{6}, 0, \sqrt{6}\}\); saddle for \(-\sqrt{6} < \lambda < 0\) or \(0 < \lambda < \sqrt{6}\).

4. \(P_4\) : \((T, \theta) = (1, -\frac{\pi}{2} + 2n\pi)\), \(n \in \mathbb{N} \cup \{0\}\), with eigenvalues \(\left\{-3, \sqrt{\frac{3}{2}}\lambda + 3\right\}\). It is a sink for \(\lambda < -\sqrt{6}\); Non-hyperbolic for \(\lambda = -\sqrt{6}\); saddle for \(\lambda > -\sqrt{6}\).
5. \( P_5 : (T, \theta) = (1, \frac{\pi}{2} + 2n\pi), n \in \mathbb{N} \cup \{0\}, \) with eigenvalues \( \{ -3, 3 - \sqrt{3} \lambda \} \). It is a sink for \( \lambda > \sqrt{6} \); non-hyperbolic for \( \lambda = \sqrt{6} \); saddle for \( \lambda < \sqrt{6} \).

6. \( P_6 : (T, \theta) = (1, \arcsin \left( \frac{\lambda}{\sqrt{6}} \right)) \) with eigenvalues \( \{ -\frac{\lambda^2}{2}, \frac{1}{2} (\lambda^2 - 6) \} \). It is non-hyperbolic for \( \lambda \in \{ -\sqrt{6}, 0, \sqrt{6} \} \); sink for \(-\sqrt{6} < \lambda < 0 \) or \( 0 < \lambda < \sqrt{6} \).

In table III are summarized the existence conditions and stability conditions of the equilibrium points of Eqs. (38).

D. Qualitative analysis for an scalar-field cosmology with generalized harmonic potential \( V(\phi) = \mu^3 \left[ \frac{\phi^2}{\mu} + bf \cos \left( \delta + \frac{\phi}{\mu} \right) \right], \) \( b \neq 0. \)

In this section we proceed with the qualitative analysis of an scalar-field cosmology with generalized harmonic potential \( V(\phi) = \mu^3 \left[ \frac{\phi^2}{\mu} + bf \cos \left( \delta + \frac{\phi}{\mu} \right) \right], b \neq 0. \) In the Fig. 3 it is represented the potential \( V(\phi) \) and its derivative \( V'(\phi) \) for some values of the parameters \( (b, f, \delta, \mu) \). The condition for the existence of a local minimum at the origin is \( \delta = 0, \mu^3 \left( \frac{2}{\mu} - \frac{b}{f} \right) > 0; \) with \( V(0) = bf \mu^3 \). The condition for the existence of a local maximum at the origin is \( \delta = 0, \mu^3 \left( \frac{2}{\mu} - \frac{b}{f} \right) < 0; \) with \( V(0) = bf \mu^3 \). For \( \delta = 0, \mu = \frac{2f}{b}, \phi = 0, \) the origin is a degenerated local minimum of order two with \( V(0) = \frac{8f^4}{3b^2}. \)

Introducing the compact variables

\[
\begin{align*}
\tau &= \frac{2\pi}{k} t, \\
\frac{1}{\rho_c} &= \frac{3H^2}{t}, \\
\frac{u}{\sqrt{2\rho_c}} &= \frac{\phi}{f}, \\
v &= \frac{\phi}{f}.
\end{align*}
\]

we can rewrite the Friedmann equation as

\[
\frac{3H^2}{\rho_c} = \left[ u^2 + \frac{\mu^3 f}{\rho_c} \left( \frac{f v^2}{\mu} + b \cos (\delta + v) \right) \right].
\]

Introducing \( \tau = \frac{2\pi}{k} t \), we obtain the dynamical system

\[
\frac{du}{d\tau} = \frac{f \mu^3 (b \sin(\delta + v) - 2fv)}{2\rho_c} - \frac{3fuH}{\sqrt{2\rho_c}} \frac{dv}{d\tau} = u.
\]

Because we are interested in expanding universe we choose the branch \( H > 0; \)

\[
\frac{du}{d\tau} = \frac{f \mu^3 (b \sin(\delta + v) - 2fv)}{2\rho_c} - \frac{3fuH}{\sqrt{2\rho_c}} \frac{dv}{d\tau} = u,
\]

\[
\frac{dv}{d\tau} = u.
\]

redefining constants \( \rho_c = \frac{1}{2} bf \mu^3, \) \( k = \frac{2f}{b} \), we obtain the Eqs.

\[
\frac{du}{d\tau} = -\frac{1}{2} \sqrt{\frac{3}{2}} k \mu u \sqrt{k^2 v^2 + u^2 + 2 \cos(\delta + v) - k \mu v + \sin(\delta + v)}, \quad \frac{dv}{d\tau} = u.
\]

The origin \( (u, v) = (0, 0) \) is an equilibrium point if \( \delta = 0. \) Then, the eigenvalues of the linearization are \( \left\{ \frac{1}{4} \left( -\sqrt{k \mu (3b^2 k \mu - 16)} + 16 - \sqrt{3} bk \mu \right), \frac{1}{4} \left( \sqrt{k \mu (3b^2 k \mu - 16)} + 16 - \sqrt{3} bk \mu \right) \right\}. \)

The origin is a sink for

1. \( \mu < 0, k < \frac{1}{\mu}, b \geq \frac{4}{3} \sqrt{\frac{3b-3}{k^2 \mu^2}}, \) or

2. \( \mu > 0, k > \frac{1}{\mu}, b \geq \frac{4}{3} \sqrt{\frac{3b-3}{k^2 \mu^2}}. \)
FIG. 3: Generalized harmonic potential \( V(\phi) = \mu^3 \left[ \frac{\phi^2}{\mu} + bf \cos \left( \delta + \frac{\phi}{f} \right) \right] \) and its derivative.

FIG. 4: Phase portrait of Eqs. (43) for some choices of parameters \((b, f, \delta, \mu)\).

It is a source if

1. \( \mu < 0, k < \frac{1}{\mu} , b \leq - \frac{4}{3} \sqrt{\frac{3k\mu - 3}{k^2\mu^2}} \),

2. \( \mu > 0, k > \frac{1}{\mu} , b \leq - \frac{4}{3} \sqrt{\frac{3k\mu - 3}{k^2\mu^2}} \).

Finally, when \( \delta = 0 \) and \( k\mu > 1 \) the origin is a saddle.

Now, for \( k \neq 0 \) and \( |k\mu v_c| \leq 1 \), we have the equilibrium points \((u, v) = (0, v_c)\) such that \( \sin(\delta + v) - k\mu v = 0 \). To
obtain a real valued linearization matrix it is additionally required \(\frac{3\pi}{2k} \sin(\delta + \nu) + 3\cos(\delta + \nu) \geq 0\).

If \(\delta = 0\) and \(|\kappa \mu V| > 1\) there are not equilibrium points apart of the origin.

In general, for \(\delta \neq 0\) the system (43) admits no equilibrium points \((u, v) = (0, v_c)\), apart of the origin, for \(|\kappa \mu v_c| > 1\).

If \(|\kappa \mu v_c| \leq 1\), we have \(\delta = -v_c + \arcsin(\kappa \mu v_c)\), and we obtain the eigenvalues

\[
\left\{ -\frac{1}{2\sqrt{2}} \sqrt{k \mu \left( 3b^2 k \mu (2\sqrt{1-k^2 \mu^2 v_c^2 + k v_c^2}) - 32 \right) + 32 \sqrt{1-k^2 \mu^2 v_c^2 + b k \mu v_c}} \right\}.
\]

For the choice of parameters \((b, f, \delta, \mu) = (0.1, 0.33, 0, 0.9)\) we have \(\rho_c = \frac{24057}{20000000} \approx 0.0120285, k = \frac{22}{9} \approx 7.33333\).

The only equilibrium point is the origin with eigenvalues \(-0.285788 + 2.34911i, -0.285788 - 2.34911i\), which an stable spiral.

In Fig. 4(a) are presented some orbits of the flow of (43) for the choice of parameters \((b, f, \delta, \mu) = (0.1, 0.33, 0, 0.9)\). For this choice of parameters the hypothesis and the results of theorems II.1 and II.2 are verified.

For \(\delta \neq 0\) and \(|\kappa \mu v_c| \leq 1\), we have \(\delta = -v_c + \arcsin(\kappa \mu v_c)\), and we obtain the eigenvalues

\[
\left\{ -\frac{1}{2\sqrt{2}} \sqrt{k \mu \left( 3b^2 k \mu (2\sqrt{1-k^2 \mu^2 v_c^2 + k v_c^2}) - 32 \right) + 32 \sqrt{1-k^2 \mu^2 v_c^2 - b k \mu v_c}} \right\}.
\]

For the choice of parameters \((b, f, \delta, \mu) = (0.99, 0.09, 0, 0.9)\), we obtain \(\rho_c = \frac{649539}{2000000000} \approx 0.032477, k = \frac{20}{99} \approx 0.20202\).

The transcendental equation is \(\frac{2\pi}{11} - \sin(\nu) = 0\). Therefore, there exist three equilibrium points

1. \(A := (u, v) = (0, -2.64078)\). The linearization matrix is complex-valued with eigenvalues \(\{+0.997194\i, -1.06199\i\}\).
2. \(B := (u, v) = (0, 0)\), with eigenvalues \(\{-0.985828, 0.829944\}\). It is a saddle.
3. \(C := (u, v) = (0, 2.64078)\) The linearization matrix is complex-valued with eigenvalues \(\{0, +0.997194\i, 0. -1.06199\i\}\).

In this case the value of the potential at the stable equilibrium points has negative values. It is well- known that a constant negative potential generates an equilibrium state which is just the Anti - de Sitter (AdS) equilibrium solution.

In Fig. 4(b) are presented some orbits of the flow of (43) for \((b, f, \delta, \mu) = (0.99, 0.09, 0, 0.9)\). For these choices of parameters the hypothesis (i) of Theorem II.1 is violated, but the result \(\lim_{t \to +\infty} \phi = 0\) holds. The hypothesis (i) and (ii) of theorem II.2 are violated, and \(\lim_{t \to +\infty} \phi\) can be finite (rather than zero or infinity). Recall this theorem relies on the hypothesis that \(V'(\phi) < 0\) for \(\phi < 0\) and \(V'(\phi) > 0\) for \(\phi > 0\) (which is clearly violated in this example).

Finally, the hypothesis (i) and (ii) of II.3 are violated, and \(\lim_{t \to +\infty} \phi = 0, \lim_{t \to +\infty} \phi < \infty\).

1. **Asymptotic analysis as \(\phi \to \infty\).**

**Definition 2** (Class k WBI functions [54]).

1. Let \(V: \mathbb{R} \to \mathbb{R}\) a \(C^1\) function such that, there exist \(\phi_0 > 0\) with \(V(\phi) > 0\) for all \(\phi > \phi_0\). We say that \(V\) is WBI (well-behaved at infinity) of exponential order \(N\) if exists \(N < \infty\), such that

\[
W_V : \mathbb{R} \to \mathbb{R}, \quad \phi \mapsto \frac{V'(\phi)}{V(\phi)} - N
\]

is well defined, and satisfies

\[
\lim_{\phi \to \infty} W_V(\phi) = 0.
\]

2. A \(C^k\) function \(V(\phi)\) is of class \(k\)-WBI if it is WBI of exponential order \(N\), and if there exist \(\phi_0 > 0\) and a coordinate transformation \(\varphi = h(\phi)\) which maps the interval \([\phi_0, \infty)\) onto \((0, \epsilon]\), where \(\epsilon = h(\phi_0)\) and \(\lim_{h(\phi) \to +\infty} h(\phi) = 0\), with the following additional properties.

(a) \(h\) is \(C^{k+1}\) and strictly decreasing.

(b) The functions

\[
W_V = \begin{cases} 
\frac{V'(h^{-1}(\varphi))}{V(h^{-1}(\varphi))} - N, & \varphi > 0, \\
0, & \varphi = 0 
\end{cases}
\]

(46)
and
\[ h'(\phi) = \begin{cases} h'(h^{-1}(\phi)), & \phi > 0, \\ \lim_{\phi \to \infty} h'(\phi), & \phi = 0 \end{cases} \tag{47} \]
are C^k on the closed interval \([0, \epsilon]\); and
\[(c)\]
\[ \frac{dW_V}{d\phi}(0) = \frac{dh'}{d\phi}(0) = 0. \tag{48} \]

Given \( V(\phi) = \mu^3 \left[ \frac{\phi^2}{T} + bf \cos \left( \delta + \frac{\phi}{T} \right) \right] \), \( b \neq 0 \). Let be defined the transformation \( \phi = h(\phi) = \phi = \left( \delta + \frac{\phi}{T} \right)^{-\frac{1}{4}} \). \( \tag{49} \)

Observe that \( V(\phi) \) is \( C^2 \)- WBI with exponential order \( N = 0 \).
\[ \tilde{W}_V(\phi) = \begin{cases} -b\mu \phi^8 \sin \left( \frac{\phi}{4} \right) -2bf \phi^4, & \phi > 0, \\ 0, & \phi = 0 \end{cases}, \tag{50} \]

\[ h'(\phi) = \begin{cases} -\phi^5, & \phi > 0, \\ 0, & \phi = 0 \end{cases}, \tag{51} \]

that satisfy the conditions ii-(a), (b) and (c) of definition (2). Let be defined
\[ T = \frac{\mu}{\mu + H}, \theta = \tan^{-1} \left( \frac{\dot{\phi}}{\sqrt{2V(\phi)}} \right), \tag{52} \]

and the new time derivative \( \dot{} \) given by
\[ \frac{d\dot{\tau}}{dt} := \mu + H. \tag{53} \]

Hence, we have the dynamical system
\[ \frac{dT}{d\dot{\tau}} = 3(1 - T)^2 T \sin^2(\theta), \tag{54a} \]
\[ \frac{d\theta}{d\dot{\tau}} = -\frac{1}{2} (1 - T) \cos(\theta) \left( 6 \sin(\theta) + \sqrt{6W_V(\phi)} \right), \tag{54b} \]
\[ \frac{d\phi}{d\dot{\tau}} = \sqrt{6h'}(\phi)(1 - T) \sin(\theta), \tag{54c} \]

defined on a phase space which consist of the vector product \( S \times J \) of finite cylinder \( S \) with boundaries \( T = 0 \) and \( T = 1 \) with the interval \( J = \left[ 0, \left( \delta + \frac{\phi_0}{T} \right)^{-\frac{1}{4}} \right] \). The variable \( T \) is suitable for global analysis \[90\], due to
\[ \frac{dT}{d\dot{\tau}} \bigg|_{\sin \theta = 0} = 0, \quad \frac{d^2T}{d\dot{\tau}^2} \bigg|_{\sin \theta = 0} = 0, \]
\[ \frac{d^3T}{d\dot{\tau}^3} \bigg|_{\sin \theta = 0} = \frac{9(T - 1)^4 T v^8 \left( b \mu v^4 \sin \left( \frac{\delta}{v} \right) + 2 f \left( \delta v^4 - 1 \right) \right)^2}{f^2 \left( b \mu v^8 \cos \left( \frac{\delta}{v} \right) + f \left( \delta v^4 - 1 \right) \right)^2}. \tag{55} \]

From Eq. (54a) and Eq. (55), \( T \) is a monotonically increasing function on \( S \times J \). As a consequence, all orbits originate from the invariant subset \( T = 0 \) (which contains the \( \alpha \)-limit), which is classically related to the initial singularity
with $H \to \infty$, and ends on the invariant boundary subset $T = 1$, which corresponds to asymptotically $H = 0$. The (curves of) equilibrium points of (54) are the following:

1. $(T, \theta, \varphi) = (1, \theta, 0)$, with eigenvalues $0, 0, 0$. It is non-hyperbolic.

2. $(T, \theta, \varphi) = (T_c, 2n\pi, 0)$, with eigenvalues $\{3(T_c - 1), 0, 0\}$. It is non-hyperbolic.

3. $(T, \theta, \varphi) = (T_c, \pi + 2n\pi, 0)$, with eigenvalues $\{3(T_c - 1), 0, 0\}$. It is non-hyperbolic.

However, for local analysis, is more suitable the system

\[
\frac{dT}{d\tau} = 3(1 - T)T \sin^2(\theta),
\]

\[
\frac{d\theta}{d\tau} = -\frac{1}{2} \cos(\theta) \left(6\sin(\theta) + \sqrt{6}W_V(\varphi)\right),
\]

\[
\frac{d\varphi}{d\tau} = \sqrt{6}h'(\varphi) \sin(\theta),
\]

where we have used the new time variable $\tau = \ln a$.

The (curves of) equilibrium points of (56) are the following:

1. $(\bar{T}, \theta, \varphi) = (T_c, 2n\pi, 0)$, with eigenvalues $\{-3, 0, 0\}$. It is non-hyperbolic.

2. $(\bar{T}, \theta, \varphi) = (T_c, \pi + 2n\pi, 0)$, with eigenvalues $\{-3, 0, 0\}$. It is non-hyperbolic.

3. $(\bar{T}, \theta, \varphi) = (0, -\frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{3, 3, 0\}$. It is non-hyperbolic.

4. $(\bar{T}, \theta, \varphi) = (0, \frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{3, 3, 0\}$. It is non-hyperbolic.

5. $(\bar{T}, \theta, \varphi) = (1, -\frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{-3, 3, 0\}$. Behaves as saddle.

6. $(\bar{T}, \theta, \varphi) = (1, \frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{-3, 3, 0\}$. Behaves as saddle.

Alternatively, we can use the new time variable $\tau = \ln a$, the unbounded variable $\bar{T} = W = \frac{T}{1 - T}$, to obtain the unconstrained dynamical system:

\[
\frac{d\bar{T}}{d\tau} = 3\bar{T}\sin^2(\theta),
\]

\[
\frac{d\theta}{d\tau} = -\frac{1}{2} \cos(\theta) \left(6\sin(\theta) + \sqrt{6}W_V(\varphi)\right),
\]

\[
\frac{d\varphi}{d\tau} = \sqrt{6}h'(\varphi) \sin(\theta),
\]

defined on a phase space which consist of the vector product $S \times J$ of semi-infinite cylinder $S$ with one boundary $T = 0$, with the interval $J = \left[0, \left(\delta + \frac{\delta_0}{T}\right)^{-\frac{1}{2}}\right]$.

The (curves of) equilibrium points of (57) are the following

1. $(\bar{T}, \theta, \varphi) = (T_c, 2n\pi, 0)$, with eigenvalues $\{-3, 0, 0\}$. It is non-hyperbolic.

2. $(\bar{T}, \theta, \varphi) = (T_c, \pi + 2n\pi, 0)$, with eigenvalues $\{-3, 0, 0\}$. It is non-hyperbolic.

3. $(\bar{T}, \theta, \varphi) = (0, -\frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{3, 3, 0\}$. It is non-hyperbolic.

4. $(\bar{T}, \theta, \varphi) = (0, \frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{3, 3, 0\}$. It is non-hyperbolic.

2. **Oscillating regime.**

In the reference [99] it was studied oscillating scalar field models with potential $\frac{1}{2}\phi^2$ and potentials $\frac{1}{2}\phi^2 + W(\phi)$ with $W$ smooth and $W(\phi) = o(\phi^3)$. There were derived an improved asymptotic expansions for the solution in homogeneous and isotropic spaces. Various generalizations where obtained for non-linear massive scalar fields, $k$-
essence models and $f(R)$-gravity. In this section we investigate the potential $V(\phi) = \mu^3 \left[ \frac{\phi^2}{2} + bf \cos \left( \delta + \frac{\phi}{f} \right) \right]$, $b \neq 0$ looking for oscillatory behavior, as expected from the numerical investigations. We derive asymptotic expansions as well. Observe that the cosine corrections are $O(bf\mu^3)$, therefore, they do not fall in the potential class studied by [99].

The pair

$$
\begin{pmatrix}
\sqrt{2\mu} \phi \\
\sqrt{\dot{\phi}^2 + 2\mu \phi^2}
\end{pmatrix}
$$

(58)
defines a function of $t$ with values in the unit circle. Therefore, we define the angular function $\vartheta(t)$ that is unique under identification module $2\pi$, defined by

$$
\vartheta = \tan^{-1} \left( \frac{\dot{\phi}}{\sqrt{2\mu} \phi} \right),
$$

(59)

together with

$$
r = \sqrt{\dot{\phi}^2 + 2\mu \phi^2},
$$

(60)

with inverse

$$
\phi = \frac{r \cos(\vartheta)}{\sqrt{2\mu}}, \quad \dot{\phi} = r \sin(\vartheta).
$$

(61)

That satisfy

$$
-bf\mu^3 \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2f\mu}} \right) + 3H^2 - \frac{r^2}{2} = 0.
$$

(62)

For expanding universes ($H > 0$) we obtain the equations

\begin{align}
\dot{r} &= -\sqrt{\frac{3}{2}} r^2 \sin^2(\vartheta) \sqrt{2bf\mu^3 \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2f\mu}} \right) + r^2 + b\mu^3 \sin(\vartheta) \sin \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2f\mu}} \right) }, \\
\dot{\vartheta} &= -\sqrt{2\mu} - \sqrt{\frac{3}{2}} \sin(\vartheta) \cos(\vartheta) \sqrt{2bf\mu^3 \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2f\mu}} \right) + r^2 + b\mu^3 \sin(\vartheta) \sin \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2f\mu}} \right) }.
\end{align}

(63a, 63b)

Observe that for $b \to 0$, we obtain the equations

\begin{align}
\dot{r} &= -\sqrt{\frac{3}{2}} r^2 \sin^2(\vartheta), \\
\dot{\vartheta} &= -\sqrt{2\mu} - \sqrt{\frac{3}{2}} r \sin(\vartheta) \cos(\vartheta).
\end{align}

(64a, 64b)

The solutions of the limiting equation admits the asymptotic expansions [99]

$$
\vartheta(t) = -\sqrt{2\mu} t + O(\ln t), \quad r(t) = \frac{4}{\sqrt{6}t} + O(t^{-2} \ln t).
$$

(65)

Hence,

$$
\phi(t) = \frac{4 \cos t}{\sqrt{6}t} + O(t^{-2} \ln t), \quad \dot{\phi}(t) = \frac{4 \sin t}{\sqrt{6}t} + O(t^{-2} \ln t).
$$

(66)
These expansions can be improved up to the order $O(t^{-3} \ln t)$ as in [99]. Rather to obtain more accuracy in the asymptotic solution of the limiting problem as $b \to 0$, we use similar argument as in [99] to derive asymptotic expansions of the full problem ($b \neq 0$).

Note that

$$\dot{r} = b\mu^3 \sin(\delta) \sin(\theta) + \left( \frac{b\mu^2 \cos(\delta) \cos(\theta) \sin(\theta)}{\sqrt{2f}} - \sqrt{3}b f \mu^3 \cos(\delta) \sin^2(\theta) \right) r + O(r^2), \quad (67a)$$

$$\dot{\varphi} = \sqrt{2} \mu = \frac{b\mu^3 \sin(\delta) \cos(\varphi)}{r} + \left( \frac{b\mu^2 \cos(\delta) \cos^2(\varphi)}{\sqrt{2f}} - \sqrt{3} \sin(\theta) \cos(\varphi) \sqrt{b f \mu^3 \cos(\delta)} \right) r + O(r^2). \quad (67b)$$

To obtain an approximated solution near the oscillatory regime we can take the average with respect to $\varphi$ over any orbit of period $2\pi$, given by

$$\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) d\varphi, \quad \cos(c) \geq 0, \quad (68)$$

FIG. 5: Phase portrait of Eqs. (67) (left panel). Projection over the cylinder $S$ (right panel) for $(b, f, \delta) = (0.1, 0.33, 0)$ and different values of $\mu$. 

(a)

(b)

(c)

(d)
to $f = (\dot{r}, \dot{\theta})$, leading to

$$
\begin{align*}
\dot{r} &= -\frac{1}{2} \sqrt{3} r \sqrt{b f \mu^3 \cos(\delta)}, \\
\dot{\theta} &= \frac{b \mu^2 \cos(\delta)}{2 \sqrt{2} f} - \sqrt{2} \mu.
\end{align*}
$$

(69)

The averaged equations have solution

$$
\begin{align*}
r(t) &= r_0 e^{-\frac{1}{2} \sqrt{3} \sqrt{b f \mu^3 \cos(\delta)}}, \\
\theta(t) &= \left( \frac{b \mu^2 \cos(\delta)}{2 \sqrt{2} f} - \sqrt{2} \mu \right) t + \theta_0.
\end{align*}
$$

(70)

Introducing along with $\theta$, and $r$, the new variable

$$
\varepsilon = \frac{H}{\mu + H},
$$

(71)

with inverse

$$
H = \frac{\mu \varepsilon}{1 - \varepsilon},
$$

(72)
satisfying
\[ -bf\mu^3(\varepsilon - 1)^2 \cos \left( \delta + \frac{r \cos(\delta)}{\sqrt{2}f\mu} \right) - \frac{1}{2}r^2(\varepsilon - 1)^2 + 3\mu^2\varepsilon^2 = 0, \] (73)
along with the time derivative \( \dot{\tau} \) given by
\[ \frac{d\dot{\tau}}{dt} := \mu + H, \] (74)
we obtain the Eqs.
\[ \varepsilon' = -\frac{r^2(1 - \varepsilon)^3 \sin^2(\vartheta)}{2\mu^2}, \] (75a)
\[ r' = -3r\varepsilon \sin^2(\vartheta) + b\mu^2(1 - \varepsilon) \sin(\vartheta) \sin \left( \delta + \frac{r \cos(\delta)}{\sqrt{2}f\mu} \right), \] (75b)
\[ \vartheta' = -\sqrt{2}(1 - \varepsilon) - 3\varepsilon \sin(\vartheta) \cos(\vartheta) + \frac{b\mu^2(1 - \varepsilon) \cos(\vartheta) \sin \left( \delta + \frac{r \cos(\delta)}{\sqrt{2}f\mu} \right)}{r}. \] (75c)

To obtain an approximated solution near the oscillatory regime we take the average with respect to \( \vartheta \) over any orbit of period \( 2\pi \), \( \langle (\varepsilon', r', \vartheta') \rangle \), leading to
\[ \varepsilon' = -\frac{r^2(1 - \varepsilon)^3}{4\mu^2}, \] (76a)
\[ r' = -\frac{3r\varepsilon}{2}, \] (76b)
\[ \vartheta' = \sqrt{2}(1 - \varepsilon) - \left\langle \frac{b\mu^2(1 - \varepsilon) \cos(\vartheta) \sin \left( \delta + \frac{r \cos(\delta)}{\sqrt{2}f\mu} \right)}{r} \right\rangle. \] (76c)

But, as \( r \to 0 \),
\[ \frac{1}{2\pi} \int_0^{2\pi} b\mu^2(1 - \varepsilon) \cos(\vartheta) \sin \left( \delta + \frac{r \cos(\delta)}{\sqrt{2}f\mu} \right) \frac{d\vartheta}{r} \sim \frac{b\mu(1 - \varepsilon) \cos(\delta)}{2\sqrt{2}f} - \frac{r^2(b(1 - \varepsilon) \cos(\delta))}{32(\sqrt{2}f^3\mu)} + O(r^3). \] (77)

Finally, we have
\[ \varepsilon' = -\frac{r^2(1 - \varepsilon)^3}{4\mu^2}, \] (78a)
\[ r' = -\frac{3r\varepsilon}{2}, \] (78b)
\[ \vartheta' = \left[ -\sqrt{2} + \frac{b\mu \cos(\delta)}{2\sqrt{2}f} - \frac{r^2(b \cos(\delta))}{32(\sqrt{2}f^3\mu)} \right] (1 - \varepsilon), \] (78c)
with the averaged constraint
\[ \mu^2 \left( 3\varepsilon^2 - bf\mu(1 - \varepsilon)^2 \cos(\delta) \right) + \frac{r^2(1 - \varepsilon)^2(b\mu \cos(\delta) - 4f)}{8f} = 0, \] (79)
as \( r \to 0 \).
The above system is integrable yielding

\[ r(\varepsilon) = \frac{\sqrt{2} \sqrt{c_1 (\varepsilon - 1)^2 + \mu^2 (6\varepsilon - 3)}}{1 - \varepsilon}, \tag{80a} \]

\[ \vartheta(\varepsilon) = \mu \tanh^{-1} \left( \frac{c_1 (\varepsilon - 1) + \mu^2}{\mu \sqrt{9\mu^2 - 3c_1}} \right) \left( b\mu \cos(\delta) - 4f \right) \frac{-b\mu \cos(\delta)}{8\sqrt{2}\mu^3 (1 - \varepsilon)} + c_2, \tag{80b} \]

and

\[ 3(t - t_0) = \ln \left( \frac{(1 - \varepsilon)^2}{c_1 (\varepsilon - 1)^2 + \mu^2 (6\varepsilon - 3)} \right) - 6\mu^2 \varepsilon + O(\varepsilon^2), \tag{80c} \]

as \( \varepsilon \to 0. \)

In the figure it is presented the phase portrait of Eqs. (67) (left panel) and the projection over the cylinder S (right panel) for \((b, f, \delta) = (0.1, 0.33, 0)\) and different values of \(\mu.\) Figure it is presented the phase portrait of Eqs. (67) (left panel) and the projection over the cylinder S (right panel) for \((b, f, \delta) = (0.99, 0.09, 0)\) and different values of \(\mu.\) The plots show the oscillatory behavior of the solutions.

E. Qualitative analysis for a scalar-field cosmology with generalized harmonic potential

\[ V(\phi) = \mu^3 \left( bf \left( \cos(\delta) - \cos \left( \frac{\delta}{f} \right) \right) + \frac{\phi^2}{\rho_c} \right), b \neq 0. \]

In this section we proceed with the qualitative analysis study of an scalar-field cosmology with generalized harmonic potential \(V(\phi) = \mu^3 \left( bf \left( \cos(\delta) - \cos \left( \frac{\delta}{f} \right) \right) + \frac{\phi^2}{\rho_c} \right), b \neq 0.\) In the Fig. it is represented the generalized harmonic potential \(V(\phi)\) and its derivative \(V'(\phi)\) for some values of the parameters \((b, f, \delta, \mu).\)

Introducing the compact variables

\[ u = \frac{\dot{\phi}}{\sqrt{2}\rho_c}, v = \frac{\dot{\phi}}{f}, \tag{81} \]

and the new time variable \(\tau = \frac{\sqrt{2\rho_c}}{f} t,\) we obtain the dynamical system

\[ \frac{du}{d\tau} = -f \mu^2 \left( b\mu \sin(\delta + v) + 2fv \right) \frac{\sqrt{2}\rho_c}{\sqrt{2}\rho_c}, \quad \frac{dv}{d\tau} = u. \tag{82} \]

where

\[ 3H^2 = \mu^3 \left( \frac{f^2 v^2}{\mu} - b f \cos(\delta + v) \right) + b f \mu^3 \cos(\delta) + \rho_c u^2. \tag{83} \]

Because we are interested in expanding universe, \(H > 0,\) we choose the branch

\[ \frac{du}{d\tau} = -\frac{\sqrt{2} fu \sqrt{b f \mu^3 (\cos(\delta) - \cos(\delta + v)) + f^2 \mu^2 v^2 + \rho_c u^2}}{\sqrt{\rho_c}} - \frac{bf \mu^3 \sin(\delta + v)}{2\rho_c} - \frac{f^2 \mu^2 v}{\rho_c}, \quad \frac{dv}{d\tau} = u. \tag{84a} \]
redefining constants $\rho_c = \frac{1}{2} bf \mu^3$, $k = \frac{2f}{\rho_c}$, we obtain the Eqs.

$$\frac{du}{d\tau} = -\frac{1}{2} \sqrt{\frac{3}{2}} bk \mu u \sqrt{2 \cos(\delta) + kv^2 + u^2 - 2 \cos(\delta + v)} - kv - \sin(\delta + v),$$  \hspace{1cm} (85a)\\
$$\frac{dv}{d\tau} = u.$$  \hspace{1cm} (85b)\\

The origin $(u, v) = (0, 0)$ is an equilibrium point if $\delta = 0$. Then, the eigenvalues of the linearization are $\{-\sqrt{-k - 1}, \sqrt{-k - 1}\}$. The origin is a saddle for $k < -1$ and a center for $k > -1$.

Now, for $k \neq 0$ and $|kv_c| \leq 1$, we have the equilibrium points $(u, v) = (0, v_c)$ such that $-kv - \sin(\delta + v) = 0$. To obtain
a real valued linearization matrix it is additionally required $3 \cos(\delta) - \frac{3}{2} v \sin(\delta + v) - 3 \cos(\delta + v) \geq 0$. 
If $\delta = 0$ and $|kv| > 1$ there are not equilibrium points apart of the origin.
In general, for $\delta \neq 0$ the system admits no equilibrium points $(u, v) = (0, v_c)$ apart of the origin, for $|kv_c| > 1$.
If $|kv_c| \leq 1$, we have the equilibrium points $(u, v) = (0, v_c)$ where $v_c$ are the roots of the transcendental equation $-kv - \sin(\delta + v) = 0$.
For $\delta \neq 0$ and $|kv_c| \leq 1$, we have $\delta = -\sin^{-1}(kv) - v$, and we obtain the eigenvalues
\[
\left\{ -\frac{1}{k} \sqrt{6b^2k^2\mu^2 - 2k^2{v}^2 + k^2v^2 + 2\cos(\sin^{-1}(kv) + v)} - 64 \left(1 - k^2v^2 + \frac{3k^2v^2}{2} + 3\cos(\sin^{-1}(kv) + v)\right), \\
\frac{1}{k} \sqrt{6b^2k^2\mu^2 - 2k^2{v}^2 + k^2v^2 + 2\cos(\sin^{-1}(kv) + v)} - 64 \left(1 - k^2v^2 + \frac{3k^2v^2}{2} + 3\cos(\sin^{-1}(kv) + v)\right), \\
-\frac{1}{k} \sqrt{6b^2k^2\mu^2 - 2k^2{v}^2 + k^2v^2 + 2\cos(\sin^{-1}(kv) + v)} - 64 \left(1 - k^2v^2 + \frac{3k^2v^2}{2} + 3\cos(\sin^{-1}(kv) + v)\right) \right\}.
\]
For the choice of parameters $(b, f, \delta, \mu) = (0.1, 0.33, 0, 0.9)$ we have $\rho_c = \frac{24057}{2000000000} \approx 0.0120285, k = \frac{22}{3} \approx 7.33333$. The only equilibrium point is the origin with eigenvalues \[\left\{ \frac{5}{\sqrt{3}}, -\frac{5i}{\sqrt{3}} \right\}. \]
Is is a nonlinear center. In Fig. 8(a) are presented some orbits of the flow of (84) for the choice of parameters $(b, f, \delta, \mu) = (0.1, 0.33, 0, 0.9)$. For this choice of parameters are verified the hypothesis and the results of theorems \[\text{II.1 and II.2} \text{ (limit}_{t \to \infty} \phi = 0, \text{ and } \text{lim}_{t \to \infty} \phi = 0). \]
Substituting the values $(b, f, \delta, \mu) = (0.99, 0.09, 0, 0.9)$, we obtain $\rho_c = \frac{649539}{2000000000} \approx 0.032477, k = \frac{20}{99} \approx 0.20202$. The transcendental equation is $-\frac{20}{99} - \sin(v) = 0$. The equilibrium points are:
1. $A : (u, v) = (0, -4.88035)$, eigenvalues $-0.140267 - 0.591197i, -0.140267 + 0.591197i$, stable spiral.
2. $B : (u, v) = (0, -4.12769)$, eigenvalues $-0.749132, 0.467117$, saddle.
3. $C : (u, v) = (0, 0)$, eigenvalues $-1.09637i, 0. + 1.09637i$, center.
4. $D : (u, v) = (0, 4.12769)$, eigenvalues $-0.749132, 0.467117$, saddle.
5. $E : (u, v) = (0, 4.88035)$, eigenvalues $-0.140267 - 0.591197i, -0.140267 + 0.591197i$, stable spiral.
In Fig. 8(b) are presented some orbits of the flow of (84) for $(b, f, \delta, \mu) = (0.99, 0.09, 0, 0.9)$. For this choice both hypothesis of theorem \[\text{II.1} \text{ hold}, \text{ and the result } \text{lim}_{t \to +\infty} \phi = 0 \text{ is attained. The hypothesis } (ii) \text{ of theorem } \text{II.2} \text{ is violated and } \text{lim}_{t \to +\infty} \phi \text{ can be zero, or finite. Recall this theorem relies on the monotonicity of } V(\phi). \text{ The hypothesis } (i) \text{ of theorem } \text{II.3} \text{ is violated. The hypothesis } (i) \text{ and } (iii) \text{ are satisfied, and } \text{lim}_{t \to +\infty} \phi = 0, \text{lim}_{t \to +\infty} \phi < \infty. \text{ This theorem relies on that } V(\phi) \text{ is monotonic decreasing.}\]

1. **Asymptotic analysis as } \phi \to \infty.\]

Given $V(\phi) = \mu^3 \left[ b f \left( \cos(\delta) - \cos \left( \delta + \phi \right) \right) + \frac{2}{b^2} \right], b \neq 0$. Let be defined the transformation

$$
\varphi = h(\phi) = \varphi = \left( \delta + \frac{\phi}{f} \right)^{-\frac{1}{2}}. \hspace{0.5cm} \text{(86)}
$$

Observe that $V(\phi)$ is $C^2$- WBI with exponential order $N = 0$.

$$
\tilde{W}_V(\varphi) = \left\{ \begin{array}{ll}
\frac{b h_{\phi} \varphi^8 \sin \left( \frac{\phi}{f^2} \right) - 2 \delta \varphi^3 f^2 \varphi^4}{f \left( b h_{\phi} \varphi \left( \cos(\delta) - \cos \left( \frac{\phi}{f^2} \right) \right) + f(\delta \varphi^3 - 1)^2 \right)}, & \varphi > 0, \\
0, & \varphi = 0
\end{array} \right., \hspace{0.5cm} \text{(87)}
$$

Let be defined

$$
\tilde{H}(\varphi) = \left\{ \begin{array}{ll}
-\frac{\varphi^5}{f}, & \varphi > 0, \\
0, & \varphi = 0
\end{array} \right.. \hspace{0.5cm} \text{(88)}
$$

$$
T = \frac{\mu}{\mu + H}, \theta = \tan^{-1} \left( \frac{\phi}{\sqrt{2V(\phi)}} \right), \hspace{0.5cm} \text{(89)}
$$
and the new time derivative $\dot{\tau}$ given by

$$\frac{d\tau}{dt} := \mu + H. \quad (90)$$

Hence, we have the dynamical system

$$\frac{dT}{d\tau} = 3(1 - T)T \sin^2(\theta), \quad (91a)$$
$$\frac{d\theta}{d\tau} = -\frac{1}{2}(1 - T) \cos(\theta) \left( 6 \sin(\theta) + \sqrt{6} \dot{W}_V(\varphi) \right), \quad (91b)$$
$$\frac{d\varphi}{d\tau} = \sqrt{6} h'(\varphi)(1 - T) \sin(\theta), \quad (91c)$$

defined on a phase space which consist of the vector product $S \times J$ of finite cylinder $S$ with boundaries $T = 0$ and $T = 1$ with the interval $J = \left[0, \left(\delta + \frac{2\alpha}{T}\right)^{-1}\right]$. The variable $T$ is suitable for global analysis [90], due to

$$\frac{dT}{d\tau}\bigg|_{\sin \theta = 0} = 0, \quad \frac{d^2T}{d\tau^2}\bigg|_{\sin \theta = 0} = 0.$$ $\frac{d^3T}{d\tau^3}\bigg|_{\sin \theta = 0} = \frac{9(T - 1)^4T^8 \left( h\mu^4 \sin \left( \frac{1}{a} \right) + 2f(\delta v^4 - 1) \right)^2}{f^2 \left( h\mu^8 \cos \left( \frac{1}{a} \right) + f(\delta v^4 - 1) \right)^2}. \quad (92)$$

From the first of Eq. (91) and Eq. (92), $T$ is a monotonically increasing function on $S \times J$. As a consequence, all orbits originate from the invariant subset $T = 0$ (which contains the $\alpha$-limit), which is classically related to the initial singularity with $H \to \infty$, and ends on the invariant boundary subset $T = 1$, which corresponds to asymptotically $H = 0$. The (curves of) equilibrium points of (91) are the following:

1. $(T, \theta, \varphi) = (1, \theta_c, 0)$, with eigenvalues $0, 0, 0$. It is non-hyperbolic.
2. $(T, \theta, \varphi) = (T_c, 2n\pi, 0)$, with eigenvalues $3(T_c - 1, 0, 0)$. It is non-hyperbolic.
3. $(T, \theta, \varphi) = (T_c, \pi + 2n\pi, 0)$, with eigenvalues $3(T_c - 1, 0, 0)$. It is non-hyperbolic.

However, for local analysis, is more suitable the system

$$\frac{dT}{d\tau} = 3(1 - T)T \sin^2(\theta), \quad (93a)$$
$$\frac{d\theta}{d\tau} = -\frac{1}{2} \cos(\theta) \left( 6 \sin(\theta) + \sqrt{6} \dot{W}_V(\varphi) \right), \quad (93b)$$
$$\frac{d\varphi}{d\tau} = \sqrt{6} h'(\varphi) \sin(\theta), \quad (93c)$$

where we have used the new time variable $\tau = \ln \alpha$.

The (curves of) equilibrium points of (93) are the following

1. $(T, \theta, \varphi) = (T_c, 2n\pi, 0)$, with eigenvalues $\{-3, 0, 0\}$. It is non-hyperbolic.
2. $(T, \theta, \varphi) = (T_c, \pi + 2n\pi, 0)$, with eigenvalues $\{-3, 0, 0\}$. It is non-hyperbolic.
3. $(T, \theta, \varphi) = (0, -\frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{3, 3, 0\}$. It is non-hyperbolic.
4. $(T, \theta, \varphi) = (0, \frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{3, 3, 0\}$. It is non-hyperbolic.
5. $(T, \theta, \varphi) = (1, -\frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{-3, 0, 0\}$. Behaves as saddle.
6. $(T, \theta, \varphi) = (1, \frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{-3, 0, 0\}$. Behaves as saddle.
Alternatively, we can use the new time variable $\tau = \ln a$, the unbounded variable $\bar{T} = \frac{m}{T} = \frac{T}{1-T}$, to obtain the unconstrained dynamical system:

\begin{align}
\frac{d\bar{T}}{d\bar{\tau}} &= 3\bar{T}\sin^2(\theta), \\
\frac{d\theta}{d\bar{\tau}} &= -\frac{1}{2} \cos(\theta) \left( 6\sin(\theta) + \sqrt{6}W(\varphi) \right), \\
\frac{d\varphi}{d\bar{\tau}} &= \sqrt{6}\dot{\rho}(\varphi)\sin(\theta),
\end{align}

defined on a phase space which consist of the vector product $S \times J$ of semi-infinite cylinder $S$ with one boundary $T = 0$, with the interval $J = \left[ 0, \left( \delta + \frac{\phi}{T} \right)^{-\frac{1}{2}} \right]$.

The (curves of) equilibrium points of (94) are the following

1. $(\bar{T}, \theta, \varphi) = (T_c, 2n\pi, 0)$, with eigenvalues $\{-3, 0, 0\}$. It is non-hyperbolic.
2. $(\bar{T}, \theta, \varphi) = (T_c, \pi + 2n\pi, 0)$, with eigenvalues $\{-3, 0, 0\}$. It is non-hyperbolic.
3. $(\bar{T}, \theta, \varphi) = (0, -\frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{3, 3, 0\}$. It is non-hyperbolic.
4. $(\bar{T}, \theta, \varphi) = (0, \frac{\pi}{2} + 2n\pi, 0)$, with eigenvalues $\{3, 3, 0\}$. It is non-hyperbolic.

2. Oscillating regime.

In this section we investigate the potential $V(\phi) = \mu^3 \left[ \frac{bf}{\mu} \left( \cos(\delta) - \cos \left( \delta + \frac{2\phi}{\mu} \right) \right) + \frac{\phi^2}{\mu} \right], b \neq 0$ looking for oscillatory behavior, as expected from the numerical investigations. We derive asymptotic expansions as well. Observe that the cosine corrections are $O(bf\mu^3)$, therefore, they do not fall in the potential class studied by \[99\]. As before, we define the pair

\begin{equation}
\left( \frac{\sqrt{2\mu} \phi}{\sqrt{\dot{\phi}^2 + 2\mu\phi^2}}, \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2 + 2\mu\phi^2}} \right)
\end{equation}

defines a function of $t$ with values in the unit circle. Therefore, we define the angular function $\vartheta(t)$ that is unique under identification module $2\pi$, defined by

\begin{equation}
\vartheta = \tan^{-1} \left( \frac{\phi}{\sqrt{2\mu} \phi} \right),
\end{equation}

together with

\begin{equation}
r = \sqrt{\dot{\phi}^2 + 2\mu\phi^2},
\end{equation}

with inverse

\begin{equation}
\phi = \frac{r \cos(\vartheta)}{\sqrt{2\mu}}, \quad \dot{\phi} = r \sin(\vartheta).
\end{equation}

That satisfy

\begin{equation}
bf\mu^3 \left( \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2f \mu}} \right) - \cos(\delta) \right) + 3H^2 - \frac{r^2}{2} = 0.
\end{equation}
For expanding universes \((H > 0)\) we obtain the equations

\[
\begin{align*}
\dot{r} &= -\sqrt{\frac{3}{2}} r \sin^2(\vartheta) \left( \sqrt{2} b f \mu^3 \left( \cos(\delta) - \cos \left( \frac{r \cos(\vartheta)}{\sqrt{2} f \mu} \right) \right) \right) + r^2 \\
- b \mu^3 \sin(\vartheta) \sin \left( \frac{r \cos(\vartheta)}{\sqrt{2} f \mu} \right), \\
\dot{\vartheta} &= -\sqrt{2} \mu - \sqrt{\frac{3}{2}} \sin(\vartheta) \cos(\vartheta) \left( \sqrt{2} b f \mu^3 \left( \cos(\delta) - \cos \left( \frac{r \cos(\vartheta)}{\sqrt{2} f \mu} \right) \right) \right) + r^2 \\
- b \mu^3 \cos(\vartheta) \sin \left( \frac{r \cos(\vartheta)}{\sqrt{2} f \mu} \right) \\
\end{align*}
\] (100a)

Observe that for \(b \to 0\), the solutions of the limiting equation admits the asymptotic expansions [99]

\[
\begin{align*}
\vartheta(t) &= -\sqrt{2} \mu t + O(t) \\
r(t) &= \frac{4}{\sqrt{6} t} + O(t^{-2} \ln t).
\end{align*}
\] (101)

Hence, when \(b = 0\),

\[
\begin{align*}
\phi(t) &= \frac{4 \cos t}{\sqrt{6} t} + O(t^{-2} \ln t), \\
\phi(t) &= \frac{4 \sin t}{\sqrt{6} t} + O(t^{-2} \ln t).
\end{align*}
\] (102)

Now we derive asymptotic expansions of the full problem \((b \neq 0)\).

Note that

\[
\begin{align*}
\dot{r} &= -b \mu^3 \sin(\vartheta) \sin(\vartheta) - b \mu^2 \cos(\vartheta) \cos(\vartheta) \sin(\vartheta) \sin(\vartheta) \sin(\vartheta) r \\
- \sqrt{2} \mu \sqrt{b \mu^2 \cos(\vartheta) \sin(\vartheta) \sin(\vartheta) \sin(\vartheta) r^3/2} + b \mu^2 \sin(\vartheta) \sin(\vartheta) \sin(\vartheta) \sin(\vartheta) r^2 \\
+ O \left( r^{5/2} \right),
\end{align*}
\] (103a)

\[
\begin{align*}
\dot{\vartheta} &= -\sqrt{3} \cos(\vartheta) \sqrt{b \mu^2 \cos(\vartheta) \sin(\vartheta) \sin(\vartheta) \sin(\vartheta) r^3/2} + b \mu^3 \cos(\vartheta) \sin(\vartheta) r \\
- \sqrt{3} \cos(\vartheta) \left( \frac{b \mu \cos(\vartheta) \cos(\vartheta) \sin(\vartheta) + 2 \sin(\vartheta)) \sin(\vartheta)) \right) r^{3/2} + b \cos(\vartheta) \cos(\vartheta) \sin(\vartheta) r^2 \\
+ O \left( r^{5/2} \right).
\end{align*}
\] (103b)

To obtain an approximated solution near the oscillatory regime we can take the average with respect to \(\vartheta\) over any orbit of period \(2\pi\), \(\langle \dot{r}, \dot{\vartheta} \rangle\), leading to

\[
\begin{align*}
\dot{r} &= k r^{3/2}, \\
k &= \frac{2^{3/4} \sqrt{3} \sqrt{b} \mu \left( E \left( \frac{\pi}{2} \right) - E \left( \frac{\pi}{2} + \pi \right) \right) \sqrt{\sin(\delta)}}{5\pi},
\end{align*}
\] (104)

\[
\dot{\vartheta} + \sqrt{2} \mu = \frac{b \cos(\vartheta) (r - 4f \mu) (r + 4f \mu)}{32 \sqrt{2} f^3}. 
\] (105)

where \(E(\phi|m)\) gives the elliptic integral of the second kind

\[
E(\phi|m) = \int_0^\phi \left( 1 - m \sin^2(\theta) \right)^{1/2} d\theta, \quad \frac{\pi}{2} < \phi < \frac{\pi}{2}.
\] (106)

The averaged equations have solution

\[
\begin{align*}
r(t) &= \frac{4}{(c_1 + kt)^2}, \\
\vartheta(t) &= -\frac{b \cos(\delta) \left( 3 f^2 \mu^2 (c_1 + kt) + \frac{1}{(c_1 + kt)} \pi \right) + 12 f^3 \mu (c_1 + kt)}{6 \sqrt{2} f^3 k} + \vartheta_0.
\end{align*}
\] (107)
Introducing along with $\vartheta$, and $r$, the new variable

$$
\varepsilon = \frac{H}{\mu + H},
$$

(108)

with inverse

$$
H = \frac{\mu \varepsilon}{1 - \varepsilon},
$$

(109)

satisfying

$$
b f \mu^3 (\varepsilon - 1)^2 \left( \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2 f} \mu} \right) - \cos(\delta) \right) - \frac{1}{2} r^2 (\varepsilon - 1)^2 + 3 \mu^2 \varepsilon^2 = 0,
$$

(110)

along with the time derivative $\dot{r}$ given by

$$
\frac{d\dot{r}}{dt} := \mu + H,
$$

(111)
we obtain the Eqs.

\begin{align}
\varepsilon' &= \frac{r^2(\varepsilon - 1)^3 \sin^2(\vartheta)}{2\mu^2}, \\
r' &= -3\varepsilon r \sin^2(\vartheta) + b\mu^2(\varepsilon - 1) \sin(\vartheta) \sin \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2f\mu}} \right), \\
\vartheta' &= -\sqrt{2}(1 - \varepsilon) - 3\varepsilon \sin(\vartheta) \cos(\vartheta) + \frac{b\mu^2(\varepsilon - 1) \cos(\vartheta) \sin \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2f\mu}} \right)}{r}.
\end{align}

To obtain an approximated solution near the oscillatory regime we take the average with respect to \(\vartheta\) over any orbit of period \(2\pi\), \(\langle (\varepsilon', r', \vartheta') \rangle\), leading to

\begin{align}
\varepsilon' &= \frac{(\varepsilon - 1)^3 r^2}{4\mu^2}, \\
r' &= -\frac{3\varepsilon r}{2}, \\
\vartheta' &= \frac{(\varepsilon - 1)(4f + b\mu \cos(\delta))}{2\sqrt{2f}} - \frac{(b(\varepsilon - 1) \cos(\delta))r^2}{32 \left(\sqrt{2f^3}\mu\right)}.
\end{align}
with the averaged constraint
\[ 3\mu^2 \epsilon^2 - \frac{r^2 ((\epsilon - 1)^2(b\mu \cos(\delta) + 4f))}{8f} = 0, \]  
(114)
as \( r \to 0 \).
The above system is integrable yielding
\[ r(\epsilon) = \frac{\sqrt{2}\sqrt{c_1(\epsilon - 1)^2 + \mu^2(6\epsilon - 3)}}{1 - \epsilon}, \]  
(115a)
\[ \vartheta(\epsilon) = \frac{\mu (b\cos(\delta) - \frac{8f^2 \tanh^{-1} \left( \frac{c_1(\epsilon - 1) + 3\mu^2}{\mu \sqrt{9\mu^2 - 3c_1}} \right)}{\sqrt{9\mu^2 - 3c_1}}) (b\mu \cos(\delta) + 4f)}{8\sqrt{2}f^3} + c_2, \]  
(115b)
and
\[ 3(t - t_0) = \ln \left( \frac{(1 - \epsilon)^2 \left( \frac{3\mu \sqrt{3\mu^2 - c_1 + \sqrt{3}\epsilon}}{3\mu \sqrt{3\mu^2 - c_1 - \sqrt{3}\epsilon}} \right)}{c_1(\epsilon - 1)^2 + \mu^2(6\epsilon - 3)} \right) \]
\[ \sim \ln \left( \frac{2\mu (\sqrt{9\mu^2 - 3c_1 + 3\mu^2}) - c_1}{c_1 - 3\mu^2} \right) \frac{\mu}{\sqrt{\mu^2 - 4}} \right) - \frac{6\mu^2 \epsilon}{c_1 - 3\mu^2} + O(\epsilon^2), \]  
(115c)
as \( \epsilon \to 0 \).

Figure 9 shows the phase portrait of Eqs. (100) (left panel) and the projection over the cylinder \( S \) (right panel) for \((b, f, \delta) = (0.1, 0.33, 0)\) and different values of \( \mu \). In Figure 10 it is presented the phase portrait of Eqs. (100) (left panel) and the projection over the cylinder \( S \) (right panel) for \((b, f, \delta) = (0.99, 0.09, 0)\) and different values of \( \mu \). The plots show the periodic nature of the solutions.

### III. MINIMALLY COUPLED SCALAR FIELD IN THE PRESENCE OF MATTER

In this section we consider an scalar field model in the presence of matter with field equations [80]:
\[ \ddot{\phi} + 3H \dot{\phi} + \frac{dV(\phi)}{d\phi} = 0, \quad \dot{\rho}_m + 3\gamma H \rho_m = 0, \quad 3H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi) + \rho_m. \]  
(116)
where the dot means derivative with respect to \( t \), \( H = \dot{a}/a \) denotes the Hubble expansion rate, \( \phi = \phi(t) \) is a scalar field and \( V(\phi) \), the scalar field self-interacting potential, is of class \( C^2 \). \( \rho_m \) corresponds to the energy density of matter, with equation of state (EoS) \( w_m = \frac{p_m}{\rho_m} = -1 \), where \( \gamma \in [0, 2] \) denotes a constant barotropic index.

#### A. Main results

In this section we investigate the space of solutions of Eqs. (116) using qualitative tools of the dynamical systems theory. We assume that the universe currently undergoes an accelerated expansion phase, i.e., \( H(0) > 0 \). We define
the phase space variables \((H, \rho_m, y, \phi) \in \mathbb{R}^4\), where \(y = \dot{\phi}\), to obtain the dynamical system

\[
\begin{align*}
\dot{H} &= -\frac{1}{2} \left( \gamma \rho_m + y^2 \right), \\
\dot{\rho}_m &= -3\gamma H \rho_m, \\
\dot{y} &= -3Hy - \frac{dV(\phi)}{d\phi}, \\
\dot{\phi} &= y
\end{align*}
\]  

(117a) to (117d)

defined on the phase space

\[
\Omega = \left\{ (H, \rho_m, y, \phi) \in \mathbb{R}^5 : 3H^2 = \rho_m + \frac{1}{2} y^2 + V(\phi) \right\}.
\]  

(118)

First, we assume that the potential \(V(\phi)\) have a local minimum at the origin, with zero value, i.e., \(\phi = 0, V(0) = 0, V''(0) < 0\). Therefore, \((0, 0, 0, 0)\) is an equilibrium point of (116) due to the invariance of the sign of \(H(t)\), as discussed in Sect. II A.

**Proposition III.1** (Miritzis 2003. Proposition 2 of [55]). Let be \(V \in C^2(\mathbb{R})\) such that

1. \(V(\phi) \geq 0, \text{ and } V(\phi) = 0 \text{ if and only if } \phi = 0.\)

2. \(V'(\phi)\) is bounded on \(A \subset \mathbb{R}\) if \(V(\phi)\) is bounded on \(A.\)

Then, \(\lim_{t \to \infty} (\rho_m, y) = (0, 0).\)

**Proof.** See Ref. [55].

**Proposition III.2** (Miritzis 2003. Proposition 3 of [55]). Let \(V \in C^2(\mathbb{R})\) such that

1. \(V \geq 0, \text{ and } V(\phi) = 0 \text{ if and only if } \phi = 0.\)

2. \(V'(\phi) < 0 \text{ for } \phi < 0 \text{ and } V'(\phi) > 0 \text{ for } \phi > 0.\)

3. \(V'(\phi)\) is bounded on \(A \subset \mathbb{R}\) if \(V(\phi)\) is bounded on \(A.\)

Then, \(\lim_{t \to \infty} \phi(t) \in \{-\infty, 0, +\infty\}.\)

**Proof.** See Ref. [55].

**Proposition III.3** (Miritzis 2003. Proposition 4 of [55]). Let be \(V \in C^2(\mathbb{R})\) such that:

1. \(V \geq 0, \text{ and } \lim_{\phi \to -\infty} V(\phi) = +\infty.\)

2. \(V'(\phi)\) is continuous and \(V'(\phi) < 0\) for all \(\phi \in \mathbb{R}.\)

3. \(V'(\phi)\) is bounded on \(A \subset \mathbb{R}\) if \(V(\phi)\) is bounded on \(A.\)

Then, \(\lim_{t \to \infty} y(t) = 0 \text{ and } \lim_{t \to +\infty} \phi(t) = +\infty.\)

**Proof.** See Ref. [55].

### IV. MINIMALLY COUPLED SCALAR FIELD: FLRW AND BIANCHI I METRICS

We investigate the scalar field cosmology with field equations [91, 98]:

\[
\begin{align*}
\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} &= 0, \quad (119a) \\
\dot{\rho}_m + 3\gamma H \rho_m &= 0, \quad (119b) \\
\dot{a} &= aH, \quad (119c) \\
\dot{H} &= -\frac{1}{2} \left( \gamma \rho_m + \dot{\phi}^2 \right) + \frac{1}{6} aG'_0(a), \quad (119d) \\
3H^2 &= \frac{1}{2} \dot{\phi}^2 + V(\phi) + \rho_m + \Lambda + G_0(a), \quad (119e)
\end{align*}
\]
where the dot means derivative with respect to $t$, $H = \dot{a}/a$ denotes the Hubble expansion rate, $\phi = \phi(t)$ is a scalar field and $V(\phi)$, the scalar field self-interacting potential, is of class $C^2$. $\rho_m$ corresponds to the energy density of matter, with equation of state (EoS) $w_m = \frac{p_m}{\rho_m} := \gamma - 1$, where $\gamma \in [0, 2]$ denotes a constant barotropic index. To include FLRW and Bianchi I metrics we have used the auxiliary function:

$$G_0(a) = \begin{cases} -3 \frac{k}{a^2}, k = 0, \pm 1, & \text{FLRW}, \\ \frac{a^2}{a^2}, & \text{Bianchi I} \end{cases}$$

(120)

Integrating equation (119b) we obtain $\rho_m = \frac{a_0^2}{a^2}$.

### A. Main results

In this section we study the solution space of Eqs. (119) by means of dynamical system s tools. We assume that initially $H(0) > 0$. Let us assume that $\Lambda \geq 0$. Defining the state variables $(H, \rho_m, a, y, \phi) \in \mathbb{R}^5$, with $y = \dot{\phi}$, we obtain the dynamical system

$$
\begin{align*}
\dot{H} &= -\frac{1}{2} (\gamma \rho_m + y^2) + \frac{1}{6} a G_0'(a), \\
\dot{\rho}_m &= -3 \gamma H \rho_m, \\
\dot{a} &= a H, \\
\dot{y} &= -3 H y - \frac{dV(\phi)}{d\phi}, \\
\dot{\phi} &= y
\end{align*}
$$

(121)

defined on the phase space

$$\left\{(H, \rho_m, a, y, \phi) \in \mathbb{R}^5 : 3H^2 = \rho_m + \frac{1}{2} y^2 + V(\phi) + \Lambda + G_0(a) \right\}.$$  

(122)

First we study the cases $G_0(a) = -3 \frac{k}{a^2}, k = 0, -1$, and $G_0(a) = \frac{a^2}{a^2}$. In the Sect. IV A 1 we study the case $G_0(a) = \frac{3}{a^2}$. The first two cases belongs to the more general situation $G_0(a) = \frac{K^2}{a^p} \geq 0$, with $K = 0$ for the flat FRW metric $K^2 = 1, p = 2$ for the negatively curved FLRW metric, and $K^2 = a_0^2, p = 6$ for the Bianchi I metric. Let be defined

$$\Omega = \left\{(H, \rho_m, a, y, \phi) \in \mathbb{R}^5 : 3H^2 = \rho_m + \frac{1}{2} y^2 + V(\phi) + \Lambda + \frac{K^2}{a^p} \right\}.$$  

(123)

and following the same line of reasoning of [74], we consider non-negative potential $V(\phi)$ of class $C^2$. In the first place, we assume that the potential have a local minimum at $\phi = 0$, $V(0) = 0$. Hence, $(0, 0, a_s, 0, 0)$, $a_s \in \mathbb{R} \cup \{+\infty\}$, is an equilibrium point of (121). This implies that $H(0) > 0$ implies $H(t) > 0, \forall t > 0$; that is, the set $\{(H, \rho_m, a, y, \phi) \in \Omega : H = 0\}$, is invariant for the flow of (121). On the contrary, if $H$ changes sign, this would imply that the origin would be crossed by such a solution, violating the existence and uniqueness theorem for the solutions of a $C^1$ differential equation.

**Theorem IV.1** (Leon & Franz-Silva, 2019). Let be $V \in C^2$, such that

1. $V \geq 0$, and $V(\phi) = 0$ if and only if $\phi = 0$.
2. $V'(\phi)$ is bounded on $A \subset \mathbb{R}$ if $V(\phi)$ is bounded on $A$.

Let be $\Lambda \geq 0$, and suppose that $G_0(a) \geq 0$ have the negative powerlaw form $G_0(a) = \frac{K^2}{a^p}, p > 0$. Then, $\lim_{t \to \infty} \left(\rho_m, y, \frac{K^2}{a^p}\right) = (0, 0, 0)$ and $a_s = +\infty$.

**Proof.** Let be $O^+(x_0)$ the positive orbit passing at $t_0$ through a regular point $x_0 \in \{(H, \rho_m, a, y, \phi) \in \Omega : H > 0\}$. Due to $H$ is positive, decreasing, and bounded along the orbit $O^+(x_0)$, there exists the limit $\lim_{t \to \infty} H(t)$ and it is a non-negative number $\eta$. Additionally, $H(t) \leq H(t_0)$ for all $t > t_0$. Then, $\rho_m(t) + \frac{1}{2} y(t)^2 + V(\phi(t)) + \Lambda + \frac{K^2}{a(t)^p} = 3H^2 \leq $
Suppose that \( O^+(x_0) \) is such that the variable \( \phi \) remains at the interior of \( A \). Given \( G_0(a) = \frac{K^2}{\alpha}, p > 0 \), equation (121a) can be written as

\[
\dot{H} = -\frac{1}{2}(\gamma \rho_m + y^2) - \frac{K^2 p}{6a e},
\]

such that

\[
H(t_0) - H(t) = \int_{t_0}^{t} \left( \frac{1}{6}K^2 \rho_m(s)^{-p} + \frac{1}{2} \gamma \rho(s) + \frac{1}{2} y(s)^2 \right) ds.
\]

Taking the limit as \( t \to +\infty \) in the above expression we obtain

\[
H(t_0) - \eta = H(t_0) - \lim_{t \to \infty} H(t) = \int_{t_0}^{\infty} \left( \frac{1}{6}K^2 \rho_m(s)^{-p} + \frac{1}{2} \gamma \rho(s) + \frac{1}{2} y(s)^2 \right) ds.
\]

This implies that

\[
\int_{t_0}^{\infty} \left( \frac{1}{6}K^2 \rho_m(s)^{-p} + \frac{1}{2} \gamma \rho(s) + \frac{1}{2} y(s)^2 \right) ds < \infty.
\]

On the other hand, defining \( f(t) = \left( \frac{1}{6}K^2 \rho_m(t)^{-p} + \frac{1}{2} \gamma \rho(t) + \frac{1}{2} y(t)^2 \right) \), we have

\[
\frac{d}{dt}f(t) = -yV'(\phi) + H (-\frac{1}{6}K^2 \rho_m(t)^{-p} - \frac{1}{2} \gamma \rho_m(t) + 2y^2) \] 

\[
= -yV'(\phi) + H \left( -\frac{1}{6}K^2 \rho_m(t)^{-p} \frac{1}{2} \gamma \rho_m(t) + 2y^2 \right).\]

Hence, \( \frac{d}{dt}f(t) \leq y|V'(\phi)| + \frac{1}{6}p^2H \left[ K^2 \rho_m(t)^{-p} + \frac{1}{2} \gamma \rho_m(t) + 2y^2 \right] \leq \sqrt{6} \eta H(t_0) |V'(\phi(t))| + \frac{1}{2} H(t_0)^3(9\gamma^2 + p^2 + 36) \), for all \( t > t_0 \), due to \( \rho_m, \frac{1}{2} y(t)^2 \), and \( \frac{K^2}{\alpha(t)} \) are bounded by \( 3H(t_0)^2 \) for all \( t > t_0 \). Finally, given \( V(\phi) \) is bounded on \( A \), \( V'(\phi) \) is also bounded on \( A \), from which we have \( \frac{d}{dt}f(t) \leq \infty \) along the positive orbit \( O^+(x_0) \). Summarizing, \( f(t) \) is non-negative, with bounded derivative along \( O^+(x_0) \), and \( \int_{t_0}^{\infty} f(s) ds \) is convergent, implies \( \lim_{t \to \infty} f(t) = 0 \), from which we deduce the limits \( \lim_{t \to \infty} \left( \rho_m, y, \frac{K^2}{\alpha} \right) = (0, 0, 0) \).

Following the same strategy as in the proof of Proposition 2 of [74], we obtain an analogous result stated as

**Theorem IV.2** (Leon & Franz-Silva, 2019). Let \( V \in C^2 \), such that

1. \( V \geq 0 \), and \( V(\phi) = 0 \) if and only if \( \phi = 0 \).
2. \( V'(\phi) < 0 \) for \( \phi < 0 \) and \( V'(\phi) > 0 \) for \( \phi > 0 \).
3. \( V'(\phi) \) is bounded on \( A \subset \mathbb{R} \) if \( V(\phi) \) is bounded on \( A \).

Let be \( \Lambda \geq 0 \) and assume that \( G_0(a) = \frac{K^2}{\alpha}, p > 0 \). Then, \( \lim_{t \to \infty} \phi \in \{-\infty, 0, +\infty\} \).

**Proof.** As before let be considered the positive orbit \( O^+(x_0) \) passing at the time \( t_0 \) through the regular point \( x_0 \in \{(H, \rho_m, a, y, \phi) \in \Omega : H > 0\} \). Using the same argument as in the proof of Theorem IV.1, we have \( \lim_{t \to \infty} H(t) = \eta \) along the orbit \( O^+(x_0) \). If \( 3\eta^2 > \Lambda \), then by the restriction (123) we have \( \lim_{t \to \infty} V(\phi(t)) = 0 \). Because \( V \) is continuous \( V(\phi) = 0 \) this implies that \( \lim_{t \to \infty} \phi(t) = 0 \).

Suppose that \( 3\eta^2 > \Lambda \). From Eq. (123) we obtain \( \lim_{t \to \infty} V(\phi(t)) = 3\eta^2 - \Lambda > 0 \), because by theorem IV.1, \( \lim_{t \to \infty} \left( \rho_m, y, \frac{K^2}{\alpha} \right) = (0, 0, 0) \). Hence, there exists \( t' \) such that \( V(\phi) > (3\eta^2 - \Lambda)/2 \) for all \( t > t' \). From this fact it follows that \( \phi \) cannot be zero for \( t > t' \) due to \( \phi = 0 \Leftrightarrow V(\phi) = 0 \). Then, the sign of \( \phi \) is invariant for all \( t > t' \).

Suppose that \( \phi \) is positive for all \( t > t' \). Due to \( V \) is an increasing function of \( \phi \) in \((0, +\infty)\), we have \( \lim_{t \to \infty} V(\phi(t)) = (3\eta^2 - \Lambda) \leq \lim_{t \to \infty} V(\phi) \). By continuity and monotonicity of \( V \) it follows that the equality holds if and only if \( \lim_{t \to \infty} \phi(t) = +\infty \).

If \( \lim_{t \to \infty} V(\phi(t)) < \lim_{t \to \infty} V(\phi) \), then exists \( \tilde{\phi} > 0 \) such that

\[
\lim_{t \to \infty} V(\phi(t)) = V(\tilde{\phi}).
\]

Due to \( V \) is continuous and strictly increasing we have

\[
\lim_{t \to \infty} \phi = \tilde{\phi}.
\]
By theorem IV.1, \( \lim_{t \to \infty} \left( \rho_m, y, \frac{K^2}{a^p} \right) = (0, 0, 0) \). Taking the limit \( t \to \infty \) on (121d) we have

\[
\lim_{t \to \infty} \frac{d}{dt} y = -V'(\tilde{\phi}) < 0.
\]

Hence, exists \( t'' > t' \) such that \( \frac{d}{dt} y < -\frac{V'(\tilde{\phi})}{2} \) for all \( t \geq t'' \). This implies

\[
y(t) - y(t'') = \int_{t''}^{t} \left( \frac{d}{dt} y \right) dt < -\frac{1}{2} V'(\tilde{\phi}) (t - t''),
\]

that is, \( y(t) \) takes negative values large enough as \( t \to \infty \), which is impossible because \( \lim_{t \to \infty} y(t) = 0 \). Henceforth, if \( \phi > 0 \) for all \( t > t' \), we have \( \lim_{t \to \infty} \phi = + \infty \). In the same way, for \( \phi < 0 \) for all \( t > t' \), we have \( \lim_{t \to \infty} \phi = - \infty \).

If initially \( 3H(t_0)^2 < \min \{ \lim_{\phi \to \infty} V(\phi), \lim_{\phi \to -\infty} V(\phi) \} \), then, \( \lim_{t \to \infty} H(t) = \frac{\Lambda}{3} \). Indeed, from the above theorem \( \lim_{t \to \infty} \phi = \pm \infty \), from the restriction (123), it follows

\[
3\eta^2 - \Lambda = \lim_{t \to \infty} V(\phi(t)) = \lim_{\phi \to \infty} V(\phi) > 3H(t_0)^2.
\]

This is impossible because \( H(t) \) is decreasing and \( H(t_0) \geq \eta, \Lambda \geq 0 \). In the same way, the assumption \( \lim_{t \to \infty} \phi = - \infty \) leads to a contradiction. Then, \( \lim_{t \to \infty} \phi = 0 \) and this implies \( \lim_{t \to \infty} V(\phi(t)) = 0 \), and from (123) it follows \( \lim_{t \to \infty} H(t) = \frac{\Lambda}{3} \).

**Theorem IV.3** (Leon & Franz-Silva, 2019). Let be \( V \in C^2 \) such that

1. \( V \geq 0 \), and \( \lim_{\phi \to -\infty} V(\phi) = + \infty \).
2. \( V'(\phi) \) is continuous and \( V'(\phi) < 0 \).
3. \( V'(\phi) \) is bounded in \( A \subset \mathbb{R} \) if \( V(\phi) \) is bounded on \( A \).

Let \( \Lambda \geq 0 \). Assume that \( G_0(a) = \frac{K^2}{a^p}, p > 0 \). Then, \( \lim_{t \to \infty} \left( \rho_m, y, \frac{K^2}{a^p} \right) = (0, 0, 0) \), and \( \lim_{t \to \infty} (a, \phi) = (+ \infty, + \infty) \).

**Proof.** As before, let be \( O^+(x_0) \) the positive orbit passing at \( t_0 \) through a regular point \( x_0 \in \{ (H, \rho_m, a, y, \phi) \in \Omega : H > 0 \} \). From Eq. (121b), it follows that the set \( \rho > 0 \) is invariant for the flow of (121) with restriction (123) along the orbit \( O^+(x_0) \); besides \( \rho \) is different from zero if initially \( \rho(t_0) \) it is so. This implies that \( H \) is never zero, because by (123), \( 3H(t)^2 \geq \rho(t) > 0 \) for all \( t > t_0 \), then, \( H \) is always non-negative if initially it is non-negative. Furthermore, from equation (IV.4), it follows that \( H \) is decreasing, then \( \exists \lim_{t \to \infty} H(t) = \eta \geq 0 \) and

\[
\int_{t_0}^{\infty} \left( \frac{1}{6} K^2 \rho^2(a)^{-p} + \frac{1}{2} \gamma \rho + \frac{1}{2} y(s)^2 \right) ds = H(t_0) - \eta < + \infty.
\]

As in Theorem IV.1, the total derivative of \( f(t) = \left( \frac{1}{6} K^2 \rho^2(a)^{-p} + \frac{1}{2} \gamma \rho + \frac{1}{2} y(t)^2 \right) \) is bounded, the improper integral \( \int_{t_0}^{\infty} f(t) dt \) is convergent. Then, \( \lim_{t \to + \infty} f(t) = 0 \), and this implies

\[
\lim_{t \to \infty} \left( \rho_m, y, \frac{K^2}{a^p} \right) = (0, 0, 0).
\]

It can be proved that \( \lim_{t \to \infty} \phi = + \infty \) in the same way as it was proved for Theorem IV.2.

From Eq. (123) we have \( \lim_{t \to \infty} V(\phi) = 3\eta^2 - \Lambda \). Because \( V \) is strictly decreasing with respect to \( \phi \), then \( V(\phi) > \lim_{\phi \to \infty} V(\phi) \) for all \( \phi \). Hence, \( \lim_{t \to \infty} V(\phi(t)) \geq \lim_{\phi \to \infty} V(\phi) \). Let be considered two cases:

1. If \( \lim_{t \to \infty} V(\phi(t)) = \lim_{\phi \to \infty} V(\phi) \), by continuity of \( V \) it follows \( \lim_{t \to \infty} \phi = + \infty \).

2. If \( \lim_{t \to \infty} V(\phi(t)) > \lim_{\phi \to \infty} V(\phi) \), then, exists a unique \( \phi \) such that

\[
\lim_{t \to \infty} V(\phi(t)) = V(\phi).
\]
Because $V$ is continuous and strictly decreasing it follows

$$\lim_{t \to \infty} \phi = \bar{\phi}.$$  

From Eq. (121d) we have

$$\lim_{t \to \infty} \frac{d}{dt} y = -V'(\bar{\phi}) > 0.$$  

Hence, exists $t'$ such that $\frac{d}{dt} y > -V'(\bar{\phi})/2$ for all $t \geq t'$. Therefore,

$$y(t) - y(t') > -\frac{V'(\bar{\phi})}{2} (t - t'),$$

which is impossible because $\lim_{t \to \infty} y(t) = 0$. Finally, $\lim_{t \to \infty} \phi = +\infty$.

If additionally, $\lim_{\phi \to \infty} V(\phi) = 0$, then $H \to \sqrt{\frac{2}{a}}$ as $t \to \infty$.

**Proposition IV.4** (Giambo & J. Miritzis 2010. Proposition 1 of [58]). Let be $V(\phi) \in C^2(\mathbb{R})$ such that [58]:

(i) The possibly empty set $\{\phi : V(\phi) < 0\}$ is bounded;

(ii) The possibly empty set of singular points of $V(\phi)$ is finite.

Set $\Lambda = 0$. Let be $\phi_*$ a minimum strict of $V(\phi)$, possibly degenerated, with non-negative critical value. Then $p_* := (\phi_*, y, \rho_m, H) = \left(\phi_*, 0, 0, \sqrt{\frac{V(\phi_*)}{a}}\right)$ is an asymptotically stable equilibrium point of the flow of (2) for $k = 0$ (flat models), and for $k = -1$ (negatively curved models).

**Proof.** See Ref. [58].

1. The case $G_0(a) = \frac{3}{a^2}$.

In this case the Raychaudhuri equation, and the Friedmann equation becomes, respectively,

$$\dot{H} = -\frac{1}{2}(\gamma \rho_m + y^2) + \frac{1}{a^2},$$

and

$$3H^2 = \frac{1}{2} y^2 + V(\phi) + \Lambda + \rho_m - \frac{3}{a^2}.$$  

We assume $\Lambda \geq 0$. Defining

$$W = 3H^2 - \frac{1}{2} y^2 - V(\phi) - \Lambda - \rho_m, \quad \dot{W} = -2HW.$$  

and

$$\epsilon = \frac{1}{2} y^2 + V(\phi) + \rho_m, \quad \dot{\epsilon} = -3H \left(\gamma \rho_m + y^2\right).$$  

For $k = +1$ we have $W < 0$, i.e., it cannot be guaranteed the monotony of $H$ from (124), so, we have to adapt the previous arguments in exactly the same way as in [21]. Let be given $\phi_*$, a local minimum of $V(\phi)$ with $V(\phi_*) > 0$. Let $\bar{V} = V(\phi_*)$ be a regular value of $V$ such that the connected component of $V^{-1}(\{-\infty, \bar{V}\})$ that contains $\phi_*$ as the only critical point of $V$, is a compact set in $\mathbb{R}$. Let be considered a solution $x(t) = (\phi(t), y(t), \rho_m(t), H(t))$ such that $\frac{1}{2} y(0)^2 + V(\phi(0)) + \rho_m(0) \leq \bar{V}$, and let $\bar{W} < 0$ a value to determine, to act as a lower bound for $W$.

Taking the initial condition near the equilibrium point $p_* := (\phi, y, \rho_m, H) = (\phi_*, 0, 0, H_*)$, where $H_* = \sqrt{\frac{V(\phi_*)}{a}}$, then
\( H(0) > 0 \); since \( W(0) > \bar{W}, \) from the equations (126) and (127), we have \( W \geq \bar{W} \) (\( W \) will be monotonic increasing and bounded by above by zero) and \( \epsilon \leq \bar{V}. \) The last inequality implies that \( V(\phi(t)) \leq \bar{V}. \) This implies that \( \phi(t) \) satisfies \( V(\phi) \leq V(\phi(t)) \leq \bar{V}. \) Then,

\[
H^2 = \frac{\epsilon}{3} + \frac{W}{3} \geq \frac{V(\phi)}{3} + \frac{\bar{W}}{3} = H_*^2 + \frac{\bar{W}}{3} \implies H \geq \bar{H} := \left( 1 + \frac{\bar{W}}{3 H_*^2} \right) H_.*
\]

Choosing \( \bar{W} \) small enough such that \( H_*^2 > -\frac{\bar{W}}{3}, \) we have \( H(0) > 0 \implies H(t) \geq \bar{H} > 0. \) That is, \( H \) is bounded away zero which combined with the monotony of \( W, \epsilon, \) and using the LaSalle’s invariance Theorem as in the case \( W \geq 0 \) leads to \( y \to 0, \rho_m \to 0, W \to 0 \) as \( t \to \infty, \) and the equilibrium point \( p_* \) is approached asymptotically.

If \( V(\phi) = 0, \) the equilibrium satisfy \( H_* = 0, \) and the nearby solution may recollapse due the \( H \) change sign.

V. SCALAR FIELD NON-MINIMALLY COUPLED TO MATTER: FLRW AND BIANCHI I METRICS

In this section we consider a scalar field cosmology, with scalar field coupled to matter given by the field equations (74, 95):

\[
\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = \frac{1}{2}(4 - 3\gamma)\rho_m H a \frac{d\ln \chi}{da},
\]

(128a)

\[
\dot{\rho}_m + 3\gamma H \rho_m = -\frac{1}{2}(4 - 3\gamma)\rho_m H a \frac{d\ln \chi}{da},
\]

(128b)

\[
\dot{a} = aH,
\]

(128c)

\[
\dot{H} = -\frac{1}{2} \left( \gamma \rho_m + \dot{\phi}^2 \right) + \frac{1}{6} \dot{a} G_0(a),
\]

(128d)

\[
3H^2 = \rho_m + \frac{1}{2} \dot{\phi}^2 + V(\phi) + \Lambda + G_0(a),
\]

(128e)

where we assume \( \Lambda \geq 0 \) and

\[
G_0(a) = \begin{cases} 
-\frac{3k}{\pi^2}, & k = 0, \pm 1, \text{ FLRW}, \\
\frac{a^2}{a^2}, & \text{ Bianchi I}
\end{cases}
\]

(129)

Integrating (128b) we obtain

\[
\rho_m = \rho_{m,0} \frac{a^{3\gamma}}{\chi(a)^{2 + \frac{3\gamma}{2}}}.
\]

(130)

A. Main results

In this section we study qualitatively the solution space of Eqs. (128). We assume that the universe currently undergoing an accelerated expansion phase, i.e., we assume \( H(0) > 0. \) We define the state variables \((H, \rho_m, a, y, \phi) \in \mathbb{R}^5, \) with \( y = \dot{\phi}, \) whose evolution is given by

\[
\dot{H} = -\frac{1}{2} \left( \gamma \rho_m + y^2 \right) + \frac{1}{6} \dot{a} G'_0(a),
\]

(131a)

\[
\dot{\rho}_m = -3\gamma H \rho_m - \frac{1}{2}(4 - 3\gamma)\rho_m H a y \frac{d\ln \chi}{da},
\]

(131b)

\[
\dot{a} = aH,
\]

(131c)

\[
\dot{y} = -3Hy - \frac{dV(\phi)}{d\phi} + \frac{1}{2}(4 - 3\gamma)\rho_m \dot{\phi} H a \frac{d\ln \chi}{da},
\]

(131d)

\[
\dot{\phi} = y
\]

(131e)
defined on the phase space

\[
\left\{(H, \rho_m, a, y, \phi) \in \mathbb{R}^5 : 3H^2 = \rho_m + \frac{1}{2}y^2 + V(\phi) + \Lambda + G_0(a)\right\}.
\]  

(132)

We first study the cases \(G_0(a) = -3\frac{\kappa_0^2}{a^2}, k = 0, -1, \) and \(G_0(a) = \frac{\sigma_0^2}{a^2}\). In the Sect. [74], we study the case \(G_0(a) = \frac{\kappa_0^2}{a^2} \geq 0\), with \(K = 0\) for the flat FRW metric, \(K^2 = 1, p = 2\) for the negatively curved FLRW metric, or \(K^2 = \sigma_0^2, p = 6\) for the Bianchi I metric. Define,

\[
\Omega = \left\{(H, \rho_m, a, y, \phi) \in \mathbb{R}^5 : 3H^2 = \rho_m + \frac{1}{2}y^2 + V(\phi) + \Lambda + K^2 \right\}.
\]

(133)

Following the same line of reasoning as in [74], we consider non-negative scalar field potential \(V(\phi)\) of class \(C^2\). First we consider potentials with a local minimum at \(\phi = 0\), \(V(0) = 0\). In this case \((0, 0, a_s, 0, 0), a_s \in \mathbb{R} \cup \{+\infty\},\) is an equilibrium point of is an equilibrium point of the flow of [131]. This implies that if initially \(H(0) > 0\) then \(H(t) > 0, \forall t > 0\). This is due to the set

\[
\{(H, \rho_m, a, y, \phi) \in \Omega : H = 0\},
\]

(134)

is invariant for the flow of [131]. That is, \(H\) do not changes sign; on the contrary, if there exists an orbit with \(H(0) > 0\) and \(H(t_1) < 0\) for some \(t_1 > 0\), this solution would passes through the origin, violating the existence and uniqueness of the solutions of a \(C^1\) flow.

**Theorem V.1** (Leon & Franz-Silva, 2019). Let be \(V \in C^2\) such that

1. \(V \geq 0\), and \(V(\phi) = 0\) if and only if \(\phi = 0\).
2. \(V'(\phi)\) is bounded on \(A \subset \mathbb{R}\) if \(V(\phi)\) is bounded on \(A\).

Let \(\chi \in C^2\) such that for all \(A \subset \mathbb{R}\) there exists \(K_1\), possibly depending on \(A\), such that \(\left|\frac{\chi'(\phi)}{\chi(\phi)}\right| < K_1\) for all \(\phi \in A\). Let \(\Lambda \geq 0\), and assume that \(G_0(a) \geq 0\) have a negative powerlaw functional form \(G_0(a) = \frac{K^2}{a^p}, p > 0\). Then, \(\lim_{t \to \infty} \left(\rho_m, y, \frac{K^2}{a^p}\right) = (0, 0, 0)\).

**Proof.** Let be the positive orbit \(O_+(x_0)\), passing at the time \(t_0\) through the regular point \(x_0 \in \{(H, \rho_m, a, y, \phi) \in \Omega : H > 0\}\). Since \(H\) is positive and decreasing along \(O_+(x_0)\), there exists \(\lim_{t \to \infty} H(t)\) and it is a non-negative number \(\eta\). Furthermore, \(H(t) \leq H(t_0)\) for all \(t > t_0\). Then, \(\rho_m(t) + \frac{1}{2}y(t)^2 + V(\phi(t)) + \Lambda + \frac{K^2}{a^p} = 3H^2 \leq 3H(t_0)^2\), for all \(t > t_0\), and from the non-negative of all the above terms it follows that \(\rho_m, \frac{1}{2}y(t)^2, \Lambda, \frac{K^2}{a^p}\) are bounded by \(3H(t_0)^2\) for all \(t > t_0\). Let be defined the set \(A = \{\phi \in \mathbb{R} : V(\phi) \leq 3H(t_0)^2\}\). Then, the orbit \(O_+(x_0)\) is such that \(\phi\) remains at the interior of \(A\). Given \(G_0(a) = \frac{K^2}{a^p}, p > 0\), the equation [128d] can be written as

\[
\dot{H} = -\frac{1}{2} \left(\gamma \rho_m + y^2\right) - \frac{K^2 p}{6a^p},
\]

(135)

such that

\[
H(t_0) - H(t) = \int_{t_0}^{t} \left(\frac{1}{6}K^2 pa(s)^{-p} + \frac{1}{2}\gamma \rho(s) + \frac{1}{2}y(s)^2\right) ds.
\]

Taking limit as \(t \to +\infty\) we obtain

\[
H(t_0) - \eta = H(t_0) - \lim_{t \to \infty} H(t) = \int_{t_0}^{\infty} \left(\frac{1}{6}K^2 pa(s)^{-p} + \frac{1}{2}\gamma \rho(s) + \frac{1}{2}y(s)^2\right) ds.
\]

From which we find the convergent improper integral

\[
\int_{t_0}^{\infty} \left(\frac{1}{6}K^2 pa(s)^{-p} + \frac{1}{2}\gamma \rho(s) + \frac{1}{2}y(s)^2\right) ds < \infty.
\]
Define \( f(t) = \left( \frac{1}{2} K \rho a(t)^{-p} + \frac{1}{2} p(t) + \frac{1}{2} y(t)^2 \right) \).

\[
\frac{d}{dt} f(t) = -y V'(\phi) + H \left( -\frac{1}{2} K \rho a(t)^{-p} - \frac{3}{2} (\gamma^2 \rho + 2y^2) \right) + \frac{(\gamma - 2)(3\gamma - 4) \rho y}{4x(a)} = -y V'(\phi) + H \left( -\frac{1}{2} K \rho a(t)^{-p} - \frac{3}{2} (\gamma^2 \rho + 2y^2) \right) + \frac{(\gamma - 2)(3\gamma - 4) \rho y}{4x(a)}.
\]

Hence, \( |\frac{d}{dt} f(t)| \leq y |V'(\phi)| + \frac{1}{p^2} H |K \rho a(t)^{-p}| + \frac{3}{2} \gamma^2 \rho y + 3y^2 H + \frac{(\gamma - 2)(3\gamma - 4)}{4} \rho y^2 \leq \sqrt{6} H(t_0)^{2/3} |V'(\phi(t))| + \frac{1}{2} H(t_0)^{2}(9\gamma^2 + p^2 + 36) + \frac{3}{2} H(t_0)^{4} |(\gamma - 2)(3\gamma - 4)| \), for all \( t > t_0 \) along the positive orbit \( O^+(x_0) \). For deducing the above we have used the results that \( \rho_m, \frac{1}{2} y(t)^2 \), and \( \frac{K^2}{a(t)^p} \) are bounded by \( 3H(t_0)^2 \) for all \( t > t_0 \), and the hypothesis for \( \chi \). Finally, due to \( V(\phi) \) is bounded on \( A \), \( V'(\phi) \) will be bounded on \( A \) too, results that \( |\frac{d}{dt} f(t)| < \infty \) along the positive orbit \( O^+(x_0) \). Given that \( f(t) \) is non-negative, it has bounded derivative along the orbit \( O^+(x_0) \), and \( \int_{t_0}^{\infty} f(s) \, ds \) is convergent. We have \( \lim_{t \to \infty} \left( \rho_m, y, \frac{K^2}{a(t)^p} \right) = (0, 0, 0) \).

Following the same procedures as in the proof of Proposition 2 of [74], and theorem IV.2, we obtain the following result:

**Theorem V.2** (Leon & Franz-Silva, 2019). Let be \( V \in C^2 \), such that

1. \( V \geq 0 \), and \( V(\phi) = 0 \) if and only if \( \phi = 0 \).
2. \( V'(\phi) < 0 \) for \( \phi < 0 \) and \( V'(\phi) > 0 \) for \( \phi > 0 \).
3. \( V'(\phi) \) is bounded on \( A \subset \mathbb{R} \) if \( V(\phi) \) is bounded on \( A \).

Let be \( \chi \in C^2 \) such that for any set \( A \) there exists a constant \( K_1 \), possibly depending on \( A \), such that \( |\chi'(\phi)| < K_1 \) for all \( \phi \in A \). Let be \( \Lambda \geq 0 \) and assume that \( G_0(a) = \frac{K^2}{a^p}, p > 0 \). Then, \( \lim_{t \to \infty} \phi \in \{-\infty, 0, +\infty\} \).

**Proof.** Follows the same stages of the proof of theorem IV.2.

**Theorem V.3** (Leon & Franz-Silva, 2019). Let be \( V \in C^2 \) such that

1. \( V \geq 0 \), and \( \lim_{\phi \to -\infty} V(\phi) = +\infty \).
2. \( V'(\phi) \) is continuous and \( V''(\phi) < 0 \).
3. \( V'(\phi) \) is bounded on \( A \subset \mathbb{R} \) if \( V(\phi) \) is bounded on \( A \).

Let be \( \chi \in C^2 \) such that for all \( A \subset \mathbb{R} \) there exists a constant \( K_1 \), possibly depending on \( A \), such that \( |\chi'(\phi)| < K_1 \) for all \( \phi \in A \). Let be \( \Lambda \geq 0 \) and assume \( G_0(a) = \frac{K^2}{a^p}, p > 0 \). Then, \( \lim_{t \to \infty} \left( \rho_m, y, \frac{K^2}{a^p} \right) = (0, 0, 0) \), and \( \lim_{t \to \infty} \phi = +\infty \).

**Proof.** Follows the same stages of the proof of theorem IV.8.

**Theorem V.4** (Leon & Franz-Silva, 2019). Let be \( V(\phi) \in C^2(\mathbb{R}) \) such that:

(i) The possibly empty set \( \{ \phi : V(\phi) < 0 \} \) is bounded;

(ii) The possibly empty set of singular points of \( V(\phi) \) is finite.

Let \( \Lambda = 0 \), and assume that \( G_0(a) \geq 0 \) such that \( \frac{aG_0(a)}{G_0(a)} \leq -p < 0 \), for all \( a > 0 \).

Let be \( \phi_* \) a minimum strict of \( V(\phi) \), possibly degenerated, with non-negative critical value. Then \( p_* := (\phi, y, \rho_m, H) = \left( \phi_*, 0, 0, \sqrt{\frac{V(\phi_*)}{3}} \right) \) is an asymptotically stable equilibrium point of the flow of (128).

**Proof.** Let be defined

\[
W(\phi, y, \rho_m, H) = H^2 - \frac{1}{3} \left( \frac{1}{2} y^2 + V(\phi) + \rho_m \right) := \frac{1}{3} G_0(a),
\]

that satisfies

\[
\dot{W} = HW \frac{aG_0(a)}{G_0(a)} < -pHW.
\]

\[1 \text{ Observe that in [88], was studied the case } G_0(a) = -\frac{3k}{2\gamma}, \text{ leading to } W = -2HW. \]
That is, \( W \) is non-negative and decreasing with respect to \( t \).

Let be defined
\[
\epsilon = \frac{1}{2}y^2 + V(\phi) + \rho_m, \quad \dot{\epsilon} = -3H \left( \gamma \rho_m + y^2 \right).
\]

This implies that \( \epsilon \) is decreasing too.

First, it is assumed that \( V(\phi_\ast) > 0 \). Let \( \tilde{V} > V(\phi_\ast) \) a regular value of \( V \) such that the connected component of \( V^{-1} \left( (-\infty, \tilde{V}) \right) \) that contains \( \phi_\ast \) is a compact set in \( \mathbb{R} \). Let be denoted this set by \( A \) and define \( \Psi \) as
\[
\Psi = \left\{ (\phi, y, \rho_m, H) \in \mathbb{R}^4 : \phi \in A, \epsilon \leq \tilde{V}, \rho_m \geq 0, W(\phi, y, \rho_m, H) \in [0, \tilde{W}] \right\},
\]

where \( \tilde{W} \) is positive. It can be proved that \( \Psi \) is a compact set as follows.

1. \( \Psi \) is a closed set in \( \mathbb{R}^4 \).
2. \( V(\phi_\ast) \leq V(\phi) \leq \tilde{V} \), for all \( \phi \in A \).
3. \( \frac{1}{2}y^2 + V(\phi_\ast) \leq \frac{1}{2}y^2 + V(\phi) + \rho_m = \epsilon \leq \tilde{V} \), therefore, \( y \) is bounded.
4. \( \rho_m \leq \tilde{V} - \frac{1}{2}y^2 - V(\phi) \leq \tilde{V} - V(\phi_\ast) \), therefore, \( \rho_m \) is bounded.
5. From (136), and the above facts, it follows that
\[
\frac{V(\phi_\ast)}{3} \leq \frac{V(\phi)}{3} \leq H^2 = W + \frac{1}{3} \left( \frac{1}{2}y^2 + V(\phi) + \rho_m \right) \leq \tilde{W} + \frac{\tilde{V}}{3}.
\]

That is, \( H \) is also bounded.

Let be \( \Psi_+ \subseteq \Psi \) the connected component of \( \Psi \) containing \( p_\ast \). Following the same arguments as in [58, 71], and in the proof of Theorem II.6 we prove that \( \Psi_+ \) is positively invariant with respect to (128). Let \( x(t) \) any solution starting at \( \Psi_+ \), and define \( \bar{t} = \sup \left\{ t > 0 : H(t) > 0 \right\} \subseteq \mathbb{R} \cup \{ +\infty \} \). When \( t < \bar{t} \), (137) and (138) implies that both \( W \) and \( \epsilon \) decreases. Moreover, let us assume that exists \( t < \bar{t} \) such that \( \phi(t) \not\in A \). Hence, \( V(\phi(t)) > \tilde{V} \), but then
\[
\dot{V} < V(\phi(t)) \leq \frac{1}{2}y(t)^2 + V(\phi(t)) + \rho_m(t) = \epsilon(t) \leq \tilde{V},
\]
a contradiction. Therefore, \( \phi(t) \in A, \forall t < \bar{t} \). But, \( W \geq 0 \) along the flow under (128), due to \( \Lambda \geq 0 \), and \( G_0(a) \geq 0 \), by hypothesis. Hence, it follows
\[
H(t)^2 \geq \frac{1}{3} \left( \frac{1}{2}y(t)^2 + V(\phi(t)) + \rho_m(t) \right) \geq \frac{V(\phi(t))}{3} \geq \frac{V(\phi_\ast)}{3}.
\]

That is, as long as \( H \) remains positive, it is strictly bounded away from zero; thus, \( \bar{t} = +\infty \). Therefore, \( \Psi_+ \) satisfies the hypothesis of LaSalle’s invariance Theorem [96, 97]. Considering the monotonic functions \( \epsilon \) and \( W \) defined on \( \Psi_+ \), it follows that any solution with initial state on \( \Psi_+ \) must be such that \( H y^2 \to 0, H \gamma \rho_m \to 0 \) as \( t \to +\infty \). Since \( H \) is strictly bounded away from zero on \( \Psi_+ \) it follows that \( y \to 0, \rho_m \to 0 \) (recall \( 1 \leq \gamma \leq 2 \)) and \( H^2 - \frac{V(\phi)}{3} \to 0 \) as \( t \to +\infty \). But from the hypotheses \( G_0(a) \geq 0 \) and \( \frac{aG_0'(a)}{G_0(a)} \leq -p < 0 \), for all \( a > 0 \), we have
\[
\dot{H} = -\frac{1}{2} \left( \gamma \rho_m + y^2 \right) + \frac{1}{6} aG_0'(a) \leq -\frac{1}{2} \left( \gamma \rho_m + y^2 \right) - \frac{p}{6} G_0(a) \leq 0.
\]

Due to \( H \) is monotonic decreasing and it is bounded away zero, it must have a limit. This implies that \( V(\phi) \) have also a limit. This limit must be \( V(\phi_\ast) \); otherwise, \( V'(\phi) \) would tend a non zero value, and so would be the right hand side of (131d), a contradiction. Therefore, the solution tends to \( p_\ast \).

If \( V(\phi_\ast) = 0 \), the set \( \Psi \) is connected and we choose \( \Psi_+ \) as the subset of \( \Psi \) with \( H \geq 0 \). The unique equilibrium point on \( \Psi_+ \) with \( H = 0 \) is the equilibrium point \( p_\ast \), then, if \( H(t) \to 0 \) the solution is forced to tend to the equilibrium point due to \( H \) is monotonic; on the contrary, if \( H(t) \) would tend to a positive number, as before, we would have that \( y \to 0, \rho_m \to 0, W \to 0, V(\phi) \to V(\phi_\ast) = 0 \), hence, \( H \) would necessarily tend to zero.
1. The case \( G_\alpha(a) = \frac{1}{a^7} \).

The equations of motion are the same as in Sect. \[V.A\], with the exception of the Raychaudhuri equation and Friedmann equation that are \((124)\) and \((125)\). Using the same arguments as in Sect. \[IV.A\], we have that if \( \phi_* \) is a local minimum of \( V \) with \( V(\phi_*) > 0 \), then, \( y \to 0, \rho_m \to 0, W \to 0 \) as \( t \to \infty \), and the equilibrium point \( p_* \) is approached asymptotically.

B. A global singularity theorem

Finally, for flat models we have a global singularity theorem presented in \[57\], which is an extension of Theorem 6 in \[54\] (page 3501).

Let be denoted by \( \mathcal{E}^k \) the set of class \( C^k \) functions well behaved in both \(+\infty\) and \(-\infty\). Latin uppercase letters with subscripts \(+\infty\) and \(-\infty\), respectively indicate the exponential order of \( \mathcal{E}^k \) functions in \(+\infty\) and in \(-\infty\). Then we have the following theorem.

**Theorem V.5.** Let be \( V \in \mathcal{E}^2 \) such that \( N^2_{\pm\infty} < 6 \) and \( \chi \in \mathcal{E}^2 \) such that

i) \( 0 < \gamma < \frac{4}{3} \) and \( M_{\pm\infty} > \frac{-\sqrt{6(\gamma-2)}}{3\gamma-4} \) or

ii) \( \frac{4}{3} < \gamma < 2 \) and \( M_{\pm\infty} < \frac{-\sqrt{6(\gamma-2)}}{3\gamma-4} \)

Then, it is verified asymptotically that:

\[
H = \frac{1}{3t} + O\left(\epsilon^\pm V(t)\right),
\]

\[
\phi = \pm \frac{2\sqrt{2}}{3} \ln \frac{t}{\bar{c}} + O\left(t\epsilon^\pm V(t)\right),
\]

\[
\dot{\phi} = \pm \frac{2\sqrt{2}}{3} t^{\frac{1}{2}} + O\left(\epsilon^\pm V(t)\right),
\]

\[
\rho_m = \chi_0^2 t^{-\gamma} \chi \left( \pm \frac{2\sqrt{2}}{3} \ln \frac{t}{\bar{c}} \right)^{\frac{3\gamma-2}{2}} \left(1 + O\left(t\epsilon^\pm V(t)\right)\right),
\]

where \( \epsilon^\pm V(t) = tV \left( \pm \sqrt{\frac{2}{3}} \ln \frac{t}{\bar{c}} \right) \), and \( \bar{c} \) is an integration constant.

VI. CONCLUSIONS

In this paper we have reviewed some well-known results and we have proved new results like Proposition \[II.5\] and Theorems \[IV.1, IV.2, IV.3, V.2, V.3, and V.4\] valid for general situations in the context of scalar field cosmologies with arbitrary potential and/or with arbitrary couplings to matter. We presented some examples that violates one or more of the hypothesis of the Theorems proved, to see at which extent these conditions can be relaxed to obtain the same conclusions or to provide a counterexample. In particular we incorporated cosine-like corrections with small phase, but respecting the symmetries under the scalar field reversal \( \phi \to -\phi \) (if the original potential respect so). We have seen motivation for this kind of potential’s correction in the context of inflation in loop-quantum cosmology \[79\]. We chosen the corrections best adapted to describe inflation in FLRW metric and in Bianchi I metrics. We use both local and global dynamical system variables, and smooth transformations of the scalar field which can be used in a combined way to provide qualitative features of the model at hand, and providing accurate schemes to find analytical approximations to the solutions following the Copeland, Liddle & Wands’s approach \[80\], and the Alho & Uggla’s approach \[90\]. We have provided good approximations for solutions.

In particular, we have started with the simple scalar field model in vacuum, which is a particular case of the model studied in the references \[54–58, 71–75\], when \( \rho_m = 0 \). We have adapted the arguments developed in the proofs of Propositions 2, 3, and 4 of \[55\]. It is well noticing that the Propositions 2, 3, and 4 of \[55\] were extended to Propositions 2.2.1, 2.2.2, 2.2.3 of \[56\] for an scalar field nonminimally coupled to matter. The results of \[56, 57\] as well were extended by \[72, 75\] when additional matter sources, like radiation, are taken into account.
In Section I, we have studied a minimally coupled scalar field in vacuum. In Subsection II A, we have presented some relevant results about the dynamics of the scalar field. Basically, we have discussed the conditions of an scalar field potential \( V \in C^2(\mathbb{R}) \) under which \( \lim_{t \to \infty} \phi = 0 \). These conditions are very general: non-negativity of the potential which zero only on the origin and boundedness of both \( V'(\phi) \) and \( V(\phi) \) (Theorem II.1). Additionally, we have presented some extra conditions for having \( \lim_{t \to \infty} \phi(t) \in \{ -\infty, 0, +\infty \} \). They are the previous conditions, with the addition of \( V'(\phi) > 0 \) for \( \phi > 0 \) and \( V'(\phi) < 0 \) for \( \phi < 0 \) (Theorem II.2). We have considered mild conditions under the potential (satisfied by the exponential potential with negative slope) for having \( \lim_{t \to +\infty} \phi = 0 \) and \( \lim_{t \to +\infty} \phi(t) = +\infty \) (Theorem II.3). Furthermore, in Subsubsection II A 1, we have followed the formulation by [54], and we discussed the conditions under which the potential diverges to the past, which are a well-known generic fact as stated in Theorem II.4. Next in this Subsection, we have proved Proposition II.5 in which we provide the center manifold calculation for de Sitter solution, which results stable under mild hypothesis for the potential. The stability of the de Sitter solution is confirmed in Proposition II.6 for flat FLRW universe. Some discussions about this issue was presented in Subsection II A 2. In Subsection II B, we have discussed a method to generate exact solutions, and we have applied it to obtain some exact solutions for the scalar field model with exponential potential \( V(\phi) = V_0 e^{-\lambda \phi} \).

In Section III, we have studied a minimally coupled scalar field in the presence of matter. In Subsection III A, we have presented Proposition 2, 3, and 4 of [55]. These results were extended as well by [72–75] when additional matter sources, like radiation, are taken into account.

Section IV was devoted to minimally coupled scalar field in FLRW and Bianchi I metrics. In Subsection IV A, we have discussed the main results: Theorems IV.1, IV.2, IV.3. Section V was devoted to investigate an scalar field non-minimally coupled to matter in the FLRW and in the Bianchi I metrics. In subsection V A, we have discussed the main results: Theorems V.2, V.3, V.4. In Subsection V A 1, we have studied the case positively curved FLRW model. In section V B, we have annotated a global singularity theorem proved by one of the present authors in [57].

Finally, the aim of this paper was to review some well-known results and to prove new results, valid for general situations in the context of scalar field cosmologies with arbitrary potential and/or with arbitrary couplings to matter, and presenting some new examples and counterexamples.
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