ON FANO INDICES OF $\mathbb{Q}$-FANO 3-FOLDS

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Abstract. We shall give the best possible upper bound of the Fano indices together with a characterization of those $\mathbb{Q}$-Fano 3-folds which attain the maximum in terms of graded rings.

0. Introduction

$\mathbb{Q}$-Fano 3-folds play important roles in birational algebraic geometry. They have been studied by several authors since G. Fano. In this paper, we study $\mathbb{Q}$-Fano 3-folds from the view of their Fano indices (See definition 0.2 below) and graded rings. More concretely, we give an optimal upper bound for the Fano indices and also characterize those $\mathbb{Q}$-Fano 3-folds which attain the maximum in terms of graded rings (Theorem 0.3). Throughout this paper, we work over the complex number field $\mathbb{C}$.

Definition 0.1. Let $X$ be a normal projective 3-fold. We call $X$ a $\mathbb{Q}$-Fano 3-fold if:

1. $X$ has only $\mathbb{Q}$-factorial terminal singularities;
2. the anti-canonical (Weil) divisor $-K_X$ is ample; and
3. $\rho(X) = 1$, $\rho(X)$ is the Picard number of $X$.

Let $X$ be a $\mathbb{Q}$-Fano 3-fold. There are two important indices of $X$:

Definition 0.2. We define the Gorenstein index $r = r(X)$ and the Fano index $f = f(X)$ by

$$r(X) := \min \{ n \in \mathbb{Z}_{>0} | nK_X \text{ is Cartier} \};$$

$$f(X) := \max \{ m \in \mathbb{Z}_{>0} | K_X = mA \text{ for some integral Weil divisor } A \}.$$ 

Here the equality $K_X = mA$ means that $K_X - mA$ is linear equivalent to 0. If $-K_X = f(X)A$, we call $A = A_X$ a primitive Weil divisor.

In earlier works of Shokurov, Alexeev, Iskovskikh, Prokhorov, Sano, Mella and others the Fano index was defined in a different way, as the maximal rational such that $-K_X \equiv rH$ for some ample Cartier divisor $H$. Note that our definition is different from the one used by previous authors.

Although the Gorenstein indices do not appear in the statement of main results, they play crucial roles in the proof (See section 2).

Our main result is as follows:

Theorem 0.3. Set

$$\mathcal{F} := \{ n \in \mathbb{Z}_{>0} | 1 \leq n \leq 11, \text{ or } 13, 17, 19 \} = \{1, 2, \cdots, 9, 10, 11, 13, 17, 19\}.$$
Let $X$ be a $\mathbb{Q}$-Fano 3-fold of Fano index $f(X)$ and $A = A_X$ a primitive Weil divisor. Then:

1. $f(X) \in \mathcal{F}$. In particular, $f(X) \leq 19$.
2. If $X$ is the $\mathbb{P}(3,4,5,7)$, $f$ attains the maximum 19.

   In addition, for any $X$ with $f(X) = 19$, the Hilbert series of $(X, A_X)$ coincides with that of $(\mathbb{P}(3,4,5,7), \mathcal{O}(1))$, i.e.
   \[ \sum_{n \geq 0} h^0(X, \mathcal{O}_X(nA_X)) t^n = \frac{1}{(1-t^3)(1-t^4)(1-t^5)(1-t^7)}. \]

3. Each element of $\mathcal{F}$ except possibly 10 is realized as a Fano index.

We expect the uniqueness of $X$ with $f(X) = 19$, to which the second statement of (2) provides a supporting evidence. For the statement (3), we shall construct desired examples as hypersurfaces in suitable weighted projective spaces (Section 2). As a by-product, it turns out that for each $f \in \mathcal{F} - \{10\}$, there is a $\mathbb{Q}$-Fano 3-fold $X$ of Fano index $f$ with only cyclic quotient terminal singularities. We expect that there is no Fano 3-fold of Fano index 10. We hope to return back this problem in future.

Our proof is based on an (effective version of) boundedness theorem of $\mathbb{Q}$-Fano 3-folds due to Kawamata [Ka3] (Theorem 1.7), the singular Riemann-Roch formula by Reid [Re] (Theorem 1.4). In order to make our estimate optimal, we also use computer programs called Magma [Ma] at the final stage. We emphasize that our use of computer programs involves nothing more than addition, subtraction, multiplication and division of reasonable amount of positive integers, which, in principle, can be done also by hand. We collect the necessary programs in the appendix for interesting readers.

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1. Preliminaries

In this section, we recall the notion called the basket of singularities after Mori [Mo] and Reid [Re], and two fundamental theorems, namely, the singular Riemann-Roch theorem due to Reid [Re] and the boundedness theorem of $\mathbb{Q}$-Fano 3-folds due to Kawamata [Ka3]. These two theorems will be essential for our study.
Let \((U, P)\) be a germ of a 3-dimensional terminal singularity of index \(r = r_P > 1\). It is known by \([Mo]\) that if \((U, P)\) is not a quotient singular point, \((U, P)\) can be deformed to a (unique) collection of a finite number of terminal quotient singularities, say \(\{P_k\}_{k=1}^{m}\). Write a type of singularity of \(P_k\) as

\[
\frac{1}{r_{P,k}}(1, a_{P,k}, -a_{P,k}) \text{ or } [r_{P,k}, a_{P,k}].
\]

Then \((r_{P,k}, a_{P,k}) = 1\) and \(r_k \geq 2\), \(n_P \geq 2\) and \(r_P = \text{lcm}(r_{P,k})\) hold\(^1\). We call the set \(B(U, P) := \{P_k\}_{k=1}^{m}\) the basket of singularities of \((U, P)\). When \((U, P)\) is already a quotient singular point, we regard \(\{(U, P)\}\) itself as the basket of singularities of \((U, P)\). Since the 3-dimensional terminal singularities are isolated singularities, we can speak of the basket of singularities in the global case:

**Definition 1.1.** Let \(X\) be a terminal 3-fold. Let \(\{P_i\}_{i=1}^m\) be the set of singular points of \(X\) with \(r(P_i) \geq 2\) and \(P_i \in U_i(\subset X)\) be a small analytic neighborhood of \(P_i\). Then, we call the disjoint union \(\bigcup_{i=1}^m B(U_i, P_i)\) the basket of singularities of \(X\).

We often describe the basket of singularities of \(X\) by listing up the type of each point in the basket, like

\[
B(X) = \{[2, 1], [2, 1], [2, 1], [5, 2], [7, 2]\}.
\]

As it will be reviewed below, several important invariants of \(\mathbb{Q}\)-Fano 3-folds depend only on the basket of singularities (but not on the actual set of singularities of \(X\)). However, we should also notice that the basket of singularities encodes no information about Gorenstein singular points. For instance, if

\[
B(X) = \{[3, 1], [4, 1], [5, 2], [7, 2]\},
\]

then one can deduce that \(X\) has four quotient singular points of indicate types by using the facts explained above and in the footnote. However one can not say anything about Gorenstein singular points of \(X\).

Let \(X\) be a \(\mathbb{Q}\)-factorial terminal 3-fold and \(P \in X\) be a singular point of local Gorenstein index \(r_P > 1\). Here the local Gorenstein index \(r_P\) is defined to be the smallest positive integer \(r_P\) such that \(r_PK_X\) is Cartier at \(P\). In particular, by \([Ka2, \text{Corollary 5.2}]\), the local class group at \(P\) is isomorphic to the cyclic group \(C_{r_P} := \mathbb{Z}/r_P\) and \(K_X\) is a generator of the local class group. Then, for each Weil divisor \(D\) on \(X\), there is a unique integer \(i := i(P, D) \in [0, r_P - 1]\) such that \(D = iK_X\). We call the integer \(i = i(P, D)\) the local index of \(D\) at \(P\). When \(D\) can be deformed under the general deformation \(\mathcal{U} \to \Delta\) of the germ \(P \in \mathcal{U}\), we can write a similar equation in terms of the basket of singularities \(\{P_k = [r_k, a_k]\}\) of \((U, P)\), i.e. there is a unique integer \(i := i(P_k, D_k) \in [0, r_k - 1]\) such that \(D_k = iK_{\mathcal{U}_k}\) around \(P_k \in \mathcal{U}_k\). By abuse of notation, we also call the integer \(i := i(P_k, D_k)\) the local index of \(D\) at \((\text{the basket point})\) \(P_k\).

\(^1\)By \([Mo]\), it is known that except one exceptional case where \(r_P = r_{P,1} = 4\) and \(r_{P,i} = 2\ (i \geq 2)\), one has also \(r_{P,k} = r_P\) for all \(k\).
Lemma 1.2. Let $X$ be a $\mathbb{Q}$-Fano 3-fold. Let $\mathcal{B}(X) = \{[r_k,a_k]\}_{k=1}^m$ be the basket of singularities of $X$, $r := r(X)$ be the Gorenstein index of $X$, $f = f(X)$ be the Fano index and $A$ be a primitive Weil divisor. Then

1. $r$ is the least common multiple of $\{r_k\}_{k=1}^m$, i.e. $r = \text{lcm}(r_k)_{k=1}^m$.
2. $A^3 = n/r$ for some positive integer $n$.
3. $r_k$ and $f$ are coprime, i.e. $(r_k,f) = 1$. In particular, the local index $i_{k,1}$ of $A$ is the integer which is uniquely determined by $i_{k,1} \in [0,r_k - 1]$ and $-i_{k,1}f \equiv 1 \mod r_k$.

Proof of (1). By definition and by the fact that $K_X$ is a generator of the local class group at each singular point of $X$ (cited above), the Gorenstein index $r$ is the least common multiple of the orders of the local class groups. Now the assertion (1) follows from the local version cited above.

Proof of (2). Let $N^r$ be any large multiple of $r$. Since $X$ has only isolated singularities and $N^r K_X$ is very ample, by using Bertini’s theorem we can find a smooth element $S$ in $|N^r A|$. Since $A$ is $\mathbb{Q}$-Cartier and integral, $A|S$ is an well-defined integral Cartier divisor on a smooth $S$ and $N^r A^3 = (A.A.S) = (A|S)^2 \in \mathbb{Z}$.

Replacing $N$ by $N + 1$, we also have that $(N + 1)r A^3 \in \mathbb{Z}$. Hence $r A^3 \in \mathbb{Z}$. The positivity of $n$ follows from the ampleness of $A$.

Proof of (3). Let $P$ be a singular point of $X$ with local Gorenstein index $r_P$. For the same reason as before, we can write $A = i(P,A)K_X$ in the local class group $G_P$ at $P$. Since $-K_X = f A$, we then have $K_X = -i(P,A)f K_X$ in $G_P$. Since $G_P = (K_X) \simeq \mathbb{Z}/r_P$, we have $i(P,A)f \equiv 1 \mod r_P$. Thus, $(f,r_P) = 1$ at each $P$. Therefore $(f,r) = 1$ and $(f, r_k) = 1$ by $r = \text{lcm}(r_P)_{P \in \text{Sing } X} = \text{lcm}(r_k)_{k=1}^m$. Since $-K_X = f A$ at $P_k$, we then have that $-K_X = f i_{k,1} K_X$, i.e. $f i_{k,1} \equiv -1 \mod r_k$.

This implies the last assertion.

Remark 1.3. For each $n$, the local indices $i_{k,n}$ of $n A$ satisfy two obvious relations

$$i_{k,n} \equiv n i_{k,1} \mod r_k \text{ and } 0 \leq i_{k,n} \leq r_k - 1.$$ 

These two relations (together with necessary division and subtraction) determine the value $i_{k,n}$ from $i_{k,1}$.

Now we can write down the singular Riemann-Roch formula for $\mathbb{Q}$-Fano 3-folds:

Theorem 1.4 (Kawamata, KMM, Reider). Let $X$ be a $\mathbb{Q}$-Fano 3-fold of Fano index $f = f(X)$ and $A$ be a primitive Weil divisor (so that $-K_X = f A$). Let $\mathcal{B}(X) = \{[r_k,a_k]\}_{k=1}^m$ be the basket of singularities of $X$. Put $P_n(X) := \chi(n A)$. We define the Hilbert series of $X$ as the formal power series $P(X,t) = \sum_{n=0}^{\infty} P_n(X)t^n$. Then

1. \[ \chi(O_X) = 1 - \frac{K_X.c_2(X)}{24} + \sum_{k=1}^{m} \frac{r_k^2 - 1}{24r_k}. \]
(2) \( P_n(X) = h^0(nA) \) for all \( n > -f \) and \( P_n(X) = 0 \) for all \(-f < n < 0\).

(3) 

\[
\chi(\mathcal{O}_X(nA)) = \chi(\mathcal{O}_X) + \frac{n(n+f)(2n+f)}{12}A^3 \\
+ \frac{nA.c_2(X)}{12} + \sum_{k=1}^{m} \left( -i_{k,n} \frac{r_k^2 - 1}{12r_k} + \sum_{j=1}^{i_{k,n}-1} \frac{b_{k,j}(r_k - b_{k,j})}{2r_k} \right),
\]

where \( i_{k,n} \in [0, r_k - 1] \) is the local index of \( nA \) at \( P_k \), \( b_k \in [0, r_k - 1] \) is the integer such that \( a_{k,b} \equiv 1 \mod r_k \) and \( b_{k,j} \in [0, r_k - 1] \) is the integer such that \( b_{k,j} \equiv b_{k,j} \mod r_k \). In particular,

\[
P(X, t) = \frac{1}{1-t} + \frac{(f^2 + 3f + 2)t + (-2f^2 + 8)t^2 + (f^2 - 3f + 2)t^3}{12(1-t)^4}A^3 \\
+ \frac{t}{(1-t)^2} A.c_2(X) \\
+ \sum_{k=1}^{m} \frac{1}{1-t^{r_k}} \left( \sum_{l=1}^{r_k-1} \left( -i_{k,l} \frac{r_k^2 - 1}{12r_k} + \sum_{j=1}^{i_{k,l}-1} \frac{b_{k,j}(r_k - b_{k,j})}{2r_k} \right) t^l \right).
\]

Proof. The statement (1) is proved in [Ka1, Section 2] (See also [Re, Corollary 10.3]). Since \( A \) is ample and \( nA = K_X + (n+f)A \), by using Kawamata-Viehweg vanishing theorem (see e.g. [KMM, Theorem 1.2.5]), we have \( h^i(\mathcal{O}_X(nA)) = 0 \) for \( n > -f \) and for \( i > 0 \). This implies the first equality of (2). For the second equality, we may now note that \( h^0(nA) = 0 \) for \( n < 0 \). The first equality of (3) is the so-called singular Riemann-Roch formula. This is shown by [Re, Theorem 10.2] for an arbitrary projective terminal 3-fold. Observe that

\[
\frac{t^N}{(1-t)^{N+1}} = (-1)^N \sum_{L \geq N} \frac{L!}{N!(L-N)!} t^L
\]

and that \( i_{k,n} = i_{k,n+r_k} \). Here the last equality is a direct consequence of the definition of the index. Now the second equality in (3) follows from the first equalities.

\[\square\]

Corollary 1.5. Under the same notation as in Theorem 1.4, if \( f = f(X) \geq 3 \), then

\[
A^3 = \frac{12}{(f-1)(f-2)} \\
\times \left( 1 - \frac{A.c_2(X)}{12} + \sum_{k=1}^{m} \left( -i_{k,-1} \frac{r_k^2 - 1}{12r_k} + \sum_{j=1}^{i_{k,-1}-1} \frac{b_{k,j}(r_k - b_{k,j})}{2r_k} \right) \right).
\]

Proof. Since \( f \geq 3 \), we have \( \chi(\mathcal{O}_X(-A)) = 0 \) by the theorem 1.4 (2). Now, substituting \( n = -1 \) into the first equality of the theorem 1.4 (3), we get the result.

\[\square\]

Next, we recall the boundedness theorem due to Kawamata. In his paper [Ka3], he shows the following:
Theorem 1.6 ([Ka3, Proposition 1 (see also Theorem 2)]). There is a universal constant \( b > 0 \) such that
\[ (-K_X)^3 \leq b(-K_X.c_2(X)) \]
holds for all \( \mathbb{Q} \)-Fano 3-folds \( X \). In particular, \( 0 < (-K_X.c_2(X)) \).

However, he shows more in the course of proof, as we shall now explain. Let \( X \) be a \( \mathbb{Q} \)-Fano 3-fold and \( E := (\Omega^1_X)^{**} \) be the double dual of the sheaf of Kähler differentials of \( X \). If \( E \) is not \( \mu \)-semistable (with respect to \(-K_X\)), then one can take the so-called maximal destabilizing sheaf \( F \) of \( E \), i.e. a (unique) \( \mu \)-semistable subsheaf \( F \subset E \), which is necessarily reflexive and of rank \( s = 1 \) or 2, such that
\[ (c_1(F).(-K_X)^2)_s > (K_X.(-K_X)^2)_3. \]

Set \( c_1(F) = tK_X \). For this expression, we used the fact that \( \rho(X) = 1 \) and \( X \) is \( \mathbb{Q} \)-factorial. It is shown that \( 0 < t < s/3 \) [Ka3, Pages 442-443]. Under these notations, one can say what he showed as in the following more effective form:

Theorem 1.7 ([Ka3, Proposition 1 (see also the proof there)]). Under the above setting, one has:

1. If \( E \) is \( \mu \)-semistable, then
\[ (-K_X)^3 \leq 3(-K_X.c_2(X)). \]

2. If \( E \) is not \( \mu \)-semistable and \( s = 1 \), then one of the following holds:
\[ (1-t)(1+3t)(-K_X)^3 \leq 4(-K_X.c_2(X)), \]

or
\[ (tu+(t+u)(1-t-u))(-K_X)^3 \leq (-K_X.c_2(X)) \]
for some rational number \( u \) such that \( t < u < 1-t-u \).

3. If \( E \) is not \( \mu \)-semistable and \( s = 2 \), then
\[ t(4-3t)(-K_X)^3 \leq 4(-K_X.c_2(X)). \]

2. Fano indices of \( \mathbb{Q} \)-Fano 3-folds

In this section, we shall show Theorem 0.3. Throughout this section, we assume that \( X \) is a \( \mathbb{Q} \)-Fano 3-fold of Fano index \( f = f(X) \geq 3 \), \( B(X) = \{ [r_k, a_k] \}_{k=1}^m \) is the basket of singularities of \( X \), \( r := r(X) := \text{lcm}(r_k)_{k=1}^m \) is the Gorenstein index and \( A \) be a primitive Weil divisor.

The following quantity is important in the sequel:

Definition 2.1.
\[ B(X) := r \left( 24 - \sum_{k=1}^m \left( r_k - \frac{1}{r_k} \right) \right). \]

Using Theorem 1.7, we shall first deduce the following inequality:

Proposition 2.2.
\[ (4f^2 - 3f)A^3 \leq 4(-K_X.c_2(X)). \]
Proof. We show the inequality by dividing into the four cases in Theorem 1.7.

1. the case where \((-K_X)^3 \leq 3(-K_X.c_2(X))\).
   In this case we have
   \[
   \frac{4f^3}{3} A^3 \leq 4(-K_X.c_2(X))
   \]
   by \(-K_X = fA\). Since \(f \geq 3\), one has also
   \[
   4f^2 - 3f \leq 4f^2 \leq \frac{4f^3}{3}.
   \]
   Combining these two inequalities, we get the desired inequality.

2. the case where \((1-t)(1+3t)(-K_X)^3 \leq 4(-K_X.c_2(X))\).
   In this case we have
   \[
   (1-t)(1+3t)f^3 A^3 \leq 4(-K_X.c_2(X)).
   \]
   by \(-K_X = fA\). Since the function \((1-x)(1+3x)\) is increasing in the range \(0 \leq x \leq 1/3\) and since \(0 < t < 1/3\), we have
   \[
   f^3 A^3 \leq 4(-K_X.c_2(X)).
   \]

   One has also
   \[
   4f^2 - 3f \leq 4f^2 \leq f^3 (f \geq 4) \text{ and } 4f^2 - 3f = f^3 (f = 3).
   \]
   Thus, we get the desired inequality in this case, too.

3. the case where \((tu + (t+u)(1-t-u))(-K_X)^3 \leq (-K_X.c_2(X))\) for some rational number \(u\) such that \(t < u < 1-t-u\).
   By \(0 < t < u < 1-t-u\), we have also \(0 < t < 1-2t\), i.e. \(0 < t < 1/3\) and \(t < u < (1-t)/2\). Since
   \[
   tu + (t+u)(1-t-u) = -\left(u - \frac{1-t}{2}\right)^2 + \frac{3t^2 + 2t + 1}{4},
   \]
   \(tu + (t+u)(1-t-u)\) is increasing with respect to \(u\) in the range \((t, (1-t)/2)\).
   Thus
   \[
   -3t^2 + 2t = t^2 + 2t(1-2t) \leq tu + (t+u)(1-t-u),
   \]
   and therefore
   \[
   (-12t^2 + 8t)f^3 A^3 \leq 4(-K_X.c_2(X)).
   \]

   Since
   \[
   t \in \left\{\frac{1}{f}, \frac{2}{f}, \ldots, \frac{n}{f} \right\} \cap \left(0, \frac{1}{3}\right),
   \]
   we have
   \[
   -12t^2 + 8t = -12 \left(t - \frac{1}{3}\right)^2 + \frac{4}{3} \geq \frac{12}{f^2} + \frac{8}{f}.
   \]
   Therefore
   \[
   (-12t^2 + 8t)f^3 \geq 8f^2 - 12f \geq 4f^2 - 3f
   \]
   for \(f \geq 3\). From this inequality, we obtain
   \[
   (4f^2 - 3f)A^3 \leq 4(-K_X.c_2(X)).
   \]

4. the case where \(t(4 - 3t)(-K_X)^3 \leq 4(-K_X.c_2(X))\).
   By \(-K_X = fA\), we have
   \[
   t(4 - 3t)f^3 A^3 \leq 4(-K_X.c_2(X)).
   \]
Again by $-K_X = fA$ and by $c_1(\mathcal{F}) = tK_X$ (by the definition of $t$), one has $c_1(\mathcal{F}) = -ftA$ in the Weil divisor class group (in the numerical sense). Since $A$ is a generator of this group (by the $\mathbb{Q}$-factoriality of $X$ and $\rho(X) = 1$), we have $ft \in \mathbb{Z}$. Therefore

$$t \in \left\{ \frac{1}{f}, \frac{2}{f}, \cdots, \frac{n}{f}, \cdots \right\} \cap \left( 0, \frac{2}{3} \right).$$

Using this, we obtain

$$t(4 - 3t) = -3 \left( t - \frac{1}{3} \right)^2 + \frac{1}{3} \geq \frac{1}{f} \left( 4 - \frac{3}{f} \right).$$

Therefore

$$(4f^2 - 3f)A^3 = \frac{1}{f} \left( 4 - \frac{3}{f} \right)f^3A^3 \leq t(4 - 3t)f^3A^3 \leq 4(-K_X.c_2(X)).$$

Now we are done. \hfill \Box

The next inequality is crucial for us.

**Corollary 2.3.**

$$4f^2 - 3f \leq 4B(X).$$

**Proof.** We have $A^3 \geq 1/r$ by Lemma 1.2 (2). Substituting this inequality and the equality in Theorem 1.4 (1) into the inequality of Proposition 2.2, we obtain the result. \hfill \Box

First we shall bound $B(X)$ from the above, then one can also estimate $f$.

**Proposition 2.4.**

1. ([Ka3, Proof of Theorem 2], [KMMT, Proof of 1.2 (1)])

$$\sum_{k=1}^{m} \left( r_k - \frac{1}{r_k} \right) < 24.$$  

2. 

$$0 < B(X) \leq 2489.$$ 

Moreover, the right equality in (2) holds if and only if $(m = 4$ and $)$

$$\{r_1, r_2, r_3, r_4\} = \{3, 4, 5, 7\}$$

**Remark 2.5.** For a $\mathbb{Q}$-Fano 3-fold $\mathbb{P}(3, 4, 5, 7)$, we have

$$\{r_k\}_{k=1}^{m} = \{3, 4, 5, 7\} \text{ and } B(\mathbb{P}(3, 4, 5, 7)) = 2489.$$ 

Note also that $f(\mathbb{P}(3, 4, 5, 7)) = 19$. (c.f. Proposition 2.13.) This already indicates that the value $f = 19$ is something special.

**Proof.** Our argument here is suggested by T. Katsura.

Since $(-K_X.c_2(X)) > 0$ by Theorem 1.6 and since $\chi(\mathcal{O}_X) = 1$, we have the first inequality

$$\sum_{k=1}^{m} \left( r_k - \frac{1}{r_k} \right) < 24$$

by Theorem 1.4(1). This is also equivalent to $0 < B(X)$. In what follows, we seek the maximum value of $B(X)$ together with the basket which attains the maximum. For this purpose, it is more convenient to observe
first the following purely arithmetical claim (apart from \(\mathbb{Q}\)-Fano 3-folds for a moment):

**Lemma 2.6.** Let \(\{r_k\}_{k=1}^m\) be a finite sequence of integers such that \(r_k \geq 2\) for all \(k\) and such that

\[
\sum_{k=1}^{m} \left( r_k - \frac{1}{r_k} \right) < 24.
\]

Set \(r = \text{lcm}(r_k)_{k=1}^m\). Then

\[
B(\{r_k\}_{k=1}^m) := r \left( 24 - \sum_{k=1}^{m} \left( r_k - \frac{1}{r_k} \right) \right) \leq 2489
\]

and the equality holds if and only if (\(m = 4\) and ) \(\{r_1, r_2, r_3, r_4\} = \{3, 4, 5, 7\}\).

**Proof of Lemma.** By the second condition of \(\{f_k\}_{r=1}^m\), we have

\[
r_k \leq 24 \text{ for all } k \text{ and } m \cdot \frac{3}{2} \leq \sum_{k=1}^{m} \left( r_k - \frac{1}{r_k} \right) < 24.
\]

Thus \(m \leq 15\). In particular, there are only finitely many sequences \(\{r_k\}_{k=1}^m\) which satisfy the initial two conditions. So, there is certainly the maximum of \(B(\{r_k\}_{k=1}^m)\), say \(M\), when \(\{r_k\}_{k=1}^m\) varies. In what follows, we seek the value \(M\) as well as the sequences which attain the maximum.

**Claim 2.7.** The sequence \(\{3, 4, 5, 7\}\) satisfies the initial conditions and \(B(\{3, 4, 5, 7\}) = 2489\). In particular, \(2489 \leq M\).

**Proof.** This follows from a direct calculation. \(\square\)

**Claim 2.8.** If \(r_j = p^a q^b\) (\(p\) and \(q\) are different prime numbers, \(a \geq 1\) and \(b \geq 1\) for some \(j\), then \(B(\{r_k\}_{k=1}^m) < M\).

**Proof.** We may assume that \(j = m\), i.e. \(r_m = p^a q^b\). Consider a new sequence \(\{s_k\}_{k=1}^{m+1}\) defined by

\[
s_k = r_k \text{ for } k \leq m - 1 \text{ and } s_m = p^a, s_{m+1} = q^b.
\]

Then \(\text{lcm}(s_k)_{k=1}^{m+1} = \text{lcm}(r_k)_{k=1}^m\) and

\[
\left( p^a q^b - \frac{1}{p^a q^b} \right) - \left( \left( p^a - \frac{1}{p^a} \right) + \left( q^b - \frac{1}{q^b} \right) \right)
\]

\[
= (p^a - 1)(q^b - 1) \left( 1 - \frac{1}{p^a q^b} \right) > 0.
\]

Thus, the sequence \(\{s_k\}_{k=1}^{m+1}\) satisfies the initial conditions and

\[
B(\{r_k\}_{k=1}^m) < B(\{s_k\}_{k=1}^{m+1})\).
\]

\(\square\)

**Claim 2.9.** If there are two numbers \(i \neq j\) such that \(r_i = p^a\) and \(r_j = p^b\) (\(p\) is a prime number and \(a \geq b\) are positive integers ), then \(B(\{r_k\}_{k=1}^m) < M\).
Proof. As before, we may assume that \( r_{m-1} = p^a \) and \( r_m = p^b \). Consider a new sequence \( \{s_k\}_{k=1}^{m-1} \) defined by

\[ s_k = r_k. \]

Then \( \text{lcm} (s_k)_{k=1}^{m-1} = \text{lcm} (r_k)_{k=1}^{m} \) and

\[ \sum_{k=1}^{m-1} \left( s_k - \frac{1}{s_j} \right) < \sum_{k=1}^{m} \left( r_k - \frac{1}{r_k} \right). \]

Thus, the sequence \( \{s_k\}_{k=1}^{m-1} \) satisfies the condition and \( B(\{r_k\}_{k=1}^{m}) < B(\{s_k\}_{k=1}^{m-1}) \).

By Claims 2.8 and 2.9, we may now assume that all \( r_i \) are primary, i.e. \( r_i = p_i^a \) where \( p_i \) is a prime number, and \( r_i \) are coprime to one another, i.e. \( p_i \neq p_j \) if \( i \neq j \). In particular, \( \text{lcm} (r_k)_{k=1}^{m} = \prod_{k=1}^{m} r_k \).

Claim 2.10. If \( m \neq 4 \), then \( B(\{r_k\}_{k=1}^{m}) < M \).

Proof. If \( m \geq 5 \), then by \( r_k \leq 24 \) and by the coprime conditions above (now assumed), we have

\[ \sum_{k=1}^{m} \left( r_k - \frac{1}{r_k} \right) \geq \left( 2 - \frac{1}{2} \right) + \left( 3 - \frac{1}{3} \right) + \left( 5 - \frac{1}{5} \right) + \left( 7 - \frac{1}{7} \right) + \left( 11 - \frac{1}{11} \right) \geq 2 + 3 + 5 + 7 + 11 - \frac{1}{2} \cdot 5 > 24, \]

a contradiction to the initial conditions. Therefore \( m \leq 4 \).

Next assume that \( m \leq 3 \). Since the value \( B(\{r_k\}_{k=1}^{m}) \) is invariant even if we formally add terms 1 into the sequence, we may assume that the sequence is of the form

\[ \{r_k\}_{k=1}^{m} = \{a, b, c\} \]

in which some of \( a, b, c \) are allowed to be 1. By the coprime condition, we have \( \text{lcm}(r_k)_{k=1}^{m} = abc \) and

\[ B(\{r_k\}_{k=1}^{m}) = abc \left( 24 - a - b - c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \leq abc(27 - a - b - c) \leq abc(28 - a - b - c) \leq \left( \frac{28}{4} \right)^4 = 2401 < 2489 \leq M. \]

Hence the claim follows.

Now we may furthermore assume that \( m = 4 \) and (without loss of generality) that

\[ r_1 > r_2 > r_3 > r_4. \]

Claim 2.11. If \( r_1 \geq 11 \), then \( B(\{r_k\}_{k=1}^{4}) < M \).
Proof. By coprime condition and $r_1 \geq 11$, we have
\[
B(\{r_k\}_{k=1}^4) = r_1 r_2 r_3 r_4 \left( 24 - r_1 - r_2 - r_3 - r_4 + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right) \\
\leq r_1 r_2 r_3 r_4 (28 - r_1 - r_2 - r_3 - r_4) \\
\leq r_1 \left( \frac{28 - r_1}{3} \right)^3 \leq 11 \cdot 6^3 = 2376 < 2489 \leq M.
\]
This implies the claim. □

Now we can complete the proof of Lemma 2.6. By Claim 2.11 and coprime conditions, we have $r_1 \leq 9$ for the maximum $B$. Now there are exactly 6 sequences $r_1 > r_2 > r_3 > r_4$ which satisfy the coprime conditions and $r_1 \leq 9$. They are:

\[(9, 8, 7, 5), (9, 7, 5, 4), (9, 7, 5, 2)\]

\[(8, 7, 5, 3), (7, 5, 4, 3), (7, 5, 3, 2).\]

Among these six candidates, the first two sequences do not satisfy the initial condition $\sum_{k=1}^{m}(r_k - \frac{1}{r_k}) < 24$. Now, by calculating $B(\{r_k\}_{k=1}^4)$ for the other four sequences, we obtain the desired result as in Lemma 2.6. □

Now Proposition 2.4 follows from Lemma 2.6. □

By combining Corollary 2.3 and Proposition 2.4 (2), we obtain a rough estimate of $f$:

**Corollary 2.12.** $f \leq 50$.

*Proof.*** By Corollary 2.3 and Proposition 2.4 (2), we have
\[
4f^2 - 3f \leq 4 \cdot 2489 = 9956.
\]
This implies $f \leq 50$. □

In order to obtain an optimal estimate $f \leq 19$, we need one more work. What we will do from now is to seek integral solutions of a system of equalities and inequalities which the baskets $B(X) = \{[r_k, a_k]\}_{k=1}^m$ of $\mathbb{Q}$-Fano 3-folds $X$ must satisfy.

If there is a $\mathbb{Q}$-Fano 3-fold $X$ of Fano index $f (\geq 3)$, for which we now know $f \leq 50$, there must be integer solutions $m$, $r_k$, $a_k$ (or equivalently $b_k$), $i_{k,n}$ ($1 \leq k \leq m$) of the following equations and inequalities:

1. By Proposition 2.4 (2)
\[
\sum_{k=1}^{m} \left( r_k - \frac{1}{r_k} \right) < 24 \text{ and } (r_k, a_k) = 1.
\]

2. By Lemma 1.2 (3), for all $k$,
\[
(f, r_k) = 1.
\]
(3) By Corollary 1.5,
\[ A^3 = \frac{12}{(f-1)(f-2)} \]
\[ \times \left( 1 - \frac{A.c_2(X)}{12} + \sum_{k=1}^{m} \left( -i_{k,1} \frac{r_k^2 - 1}{12r_k} + \sum_{j=1}^{i_{k,1}-1} \frac{b_k j (r_k - b_k j)}{2r_k} \right) \right) \]
> 0.

(4) By Theorem 1.4 (2), for all \( n = -1, -2, \cdots, -(f-1) \)
\[ \chi(O_X(nA)) = 1 + \frac{n(n+f)(2n+f)}{12} A^3 + \frac{nA.c_2(X)}{12} \]
\[ + \sum_{k=1}^{m} \left( -i_{k,n} \frac{r_k^2 - 1}{12r_k} + \sum_{j=1}^{i_{k,n}-1} \frac{b_k j (r_k - b_k j)}{2r_k} \right) = 0. \]

(5) By Proposition 2.2
\[ (4f^2 - 3f) A^3 \leq 4(-K_X.c_2(X)). \]

Here by 1.4(1), we have
\[ -K_X.c_2(X) = 24 - \sum_{k=1}^{m} \left( r_k - \frac{1}{r_k} \right). \]

As we remarked before, there are only finitely many integers \( \{r_k\}_{k=1}^{m} \) which satisfy the inequality (1). For each \( f \) and \( \{r_k\}_{k=1}^{m} \), there is a unique integer of \( i_{k,1} \) (whence \( i_{k,n} \)) by Lemma 1.2 (3) and Theorem 1.4, and finitely many integers \( b_k \) (or equivalently \( a_k \)) by \( b_k \in [0, r_k - 1] \). For each such possibility in the range \( 9 \leq f \leq 50 \), we check if it satisfies (2)–(5), by additions and multiplications. In principle, we can do this by hand. However, it is a little messy to do so and we use a computer program Magma. Among 5 conditions, the condition (4) seems fairly strong. As a result, we actually find that there are no integer solutions when \( f \geq 20 \). Thus we have \( f \leq 19 \).

For instance, our programs give the following list (c.f. Table 1):

In order to make this process clear, we gave programs we used in the appendix.

The next proposition shows the optimality of the estimate \( f \leq 19 \):

**Proposition 2.13.** The weighted projective space \( \mathbb{P}(3, 4, 5, 7) \) is a \( \mathbb{Q} \)-Fano 3-fold of Fano index 19.

**Proof.** Recall that \( X := \mathbb{P}(3, 4, 5, 7) \) is an abelian quotient of \( \mathbb{P}^4 \) by an obvious action by the abelian group \( C_3 \times C_4 \times C_5 \times C_7 \). Therefore, \( X \) is \( \mathbb{Q} \)-factorial, the Weil divisor class group is generated by the Serre’s twisting sheaf \( O_X(1) \), and that \( \text{Sing}(X) = \{[3, 1], [4, 1], [5, 2], [7, 2]\} \), which are terminal. In addition, by the canonical bundle formula (See [Do]), we have
\[ O_X(K_X) = O_X(-19). \]
Thus the Fano index of \( X \) is 19. \( \square \)
Table 1. The number of decreasing

| f  | ∆ (1) | ∆ (2) | ∆ (3) | ∆ (4) | ∆ (5) |
|----|-------|-------|-------|-------|-------|
| 13 | 25161 | 23187 | 6622  | 6     | 2     |
| 19 | 25161 | 24972 | 7173  | 1     | 1     |
| 20 | 25161 | 714   | 417   | 0     | 0     |
| 23 | 25161 | 25139 | 9261  | 0     | 0     |
| 24 | 25161 | 478   | 329   | 0     | 0     |
| 50 | 25161 | 714   | 167   | 0     | 0     |

However, since we obtain all the solutions of (1)–(5) for $3 \leq f \leq 50$, we can say more about $X$ for each possible $3 \leq f \leq 19$. For instance, We also find that if $f = 19$, then $X$ necessarily satisfies

$$A^3 = \frac{1}{3 \cdot 4 \cdot 5 \cdot 7},$$

$$B(X) = \{[3, 1], [4, 1], [5, 2], [7, 2]\},$$

and

$$P(X, t) = \frac{1}{(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^7)}.$$

In this way, we can obtain the assertion (1) and (2) in the Theorem 0.3. The assertion (3) is now easily proved. Let $f$ be an integer in the assertion (3). Then one can actually construct explicit examples of $\mathbb{Q}$-Fano 3-folds of index $f$ as general hypersurfaces in weighted projective spaces. For instance, we have the following simple examples (in which the equations are chosen to be general) with indicated singular points:

Now we are done. Q.E.D. for the Theorem 0.3.

Remark 2.14. Similarly, using Magma program, (but use $\chi(A) \geq 0$ instead of $A^3 > 0$ when $f = 1, 2$), we obtain

$$-K_X^3 \leq \frac{2 \cdot 5^3}{3},$$

for a $\mathbb{Q}$-Fano 3-folds $X$. Unfortunately, we do not know whether this estimate is optimal or not.

Finally, we pose three interesting unsettled problems which are closely related to our theorem:

Question 1. Is there a more intrinsic reason why $f(X) \leq 19$ should hold?

Question 2. Classify all $X$ with $f(X) = 19$ up to isomorphism. $X \simeq \mathbb{P}(3, 4, 5, 7)$ if $f(X) = 19$?

Question 3. Is there a $\mathbb{Q}$-Fano 3-fold of Fano index 10?
### 3. Appendix: Magma Program

This is a program which we used at the final step of the proof for the calculation of \(1/12A_c^2(X), A^3\) and \(P_n(X)\) and the Hilbert series \(P(X,t)\) from Fano index, local indices, and the baskets of singularities, i.e. from the values \(f, i_{n,k}, [r_k, a_k]\).

Here, BB is a list of Baskets generated automatically by computer under the condition of \(\sum_{k=1}^m (r_k - 1/r_k) < 24\).

```magma
forward Ac2over12_is, contribution;
intrinsic FanoHilbertSeries(f::RngIntElt,B::SeqEnum) -> RngElt
{The Hilbert series of a Fano 3-fold of Fano index f and basket B}

K := RationalFunctionField(Rationals());
t := K.1;
I := 1/(1-t);
II := 1/12*A3_is(f,B)*
```
\[(f^2+3*f+2)t + (-2*f^2+8)*t^2 + (f^2-3*f+2)*t^3) / (1-t)^4;\]

\[\text{III} := \text{Ac2over12}\_is(f,B)\star t / (1-t)^2;\]

\[\text{IV} := \& + \left[ \text{Parent}(t) \mid \& + \left[ \text{Parent}(t) \mid \text{contribution}(f,r,a,n)\star t^n : n \in [1..r-1] \right] / (1-t^r) \right.\]

where \(r\) is \(p[1]\)
where \(a\) is \(p[2] : p \in B \];

return I + II + III + IV;
end intrinsic;

///////////////////////////////////////////////////////
// Auxiliary functions
///////////////////////////////////////////////////////

function i\_is(f,r,n)
    h,u,v := XGCD(f,r);
    return \((-n*u) \mod r\);
end function;

bar := func< m,r | m \mod r >;

inv := func< a,r | i\_is(a,r,1) >;

function contribution(f,r,a,n)
    i := i\_is(f,r,n);
    b := inv(a,r);
    first := \(-i*(r^2-1)/(12*r)\);
    if i in \{0,1\} then
        extra := 0;
    else
        extra := \& + \left[ \text{bar}(b\star j,r)(r-\text{bar}(b\star j,r))/(2*r) : j \in [0..i-1] \right];
    end if;
    return first + extra;
end function;

function Ac2over12\_is(f,B)
    sumpart := \& + \left[ \text{Rationals()} \mid (r^2-1)/(12*r) \right. where r is \(p[1] : p \in B \];
    return \((2-\text{sumpart})/f\);
end function;

// require \(f \geq 3: \ldots\)
function A3\_is(f,B)
    factor := \(12/((f-1)*(f-2))\);
    c2\_part := \text{Ac2over12}\_is(f,B);
periodic := &+[ Rationals() | contribution(f,r,a,-1)
  where a is p[2]
  where r is p[1] : p in B ];
return factor * (1 - c2_part + periodic);
end function;

intrinsic FanoCoefficient(f::RngIntElt,B::SeqEnum,n::RngIntElt)
  -> RngElt
{The n-th coefficient of the Hilbert series of Fano
  with Fano index f and basket B}
V := 1+1/12*A3_is(f,B)*n*(n+f)*(2n+f)+n*Ac2over12_is(f,B)+
  &+[Rationals()| contribution(f,r,a,n)
  where a is p[2]
  where r is p[1] : p in B];
vprintf User1: "tP_(%o) = %o\n",n,V;
return V;
end intrinsic;

BB := Baskets(24);
B1 := [ B : B in BB | &and[ GCD(p[1],f) eq 1 : p in B ] ];
B2 := [ B : B in B1 | A3_is(f,B) gt 0 ];
B3 := [ B : B in B2 | A3_is(f,B)*(4*f^2-3)
  le 48*f*Ac2over12_is(f,B) ];
Bfinal := [ B : B in B3 | &and[ coeff(f,B,n) eq 0 :
  n in [-(f-1)..-1]] ];

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