PERIOD AND INDEX IN THE BRAUER GROUP OF AN ARITHMETIC SURFACE

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ABSTRACT. In this paper we introduce two new ways to split ramification of Brauer classes on surfaces using stacks. Each splitting method gives rise to a new moduli space of twisted stacky vector bundles. By studying the structure of these spaces we prove new results on the standard period-index conjecture. The first yields new bounds on the period-index relation for classes on curves over higher local fields, while the second can be used to relate the Hasse principle for forms of moduli spaces of stable vector bundles on pointed curves over global fields to the period-index problem for Brauer groups of arithmetic surfaces. We include an appendix by Daniel Krashen showing that the local period-index bounds are sharp.

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1. INTRODUCTION

Let $K$ be a field and $D$ a central division algebra over $K$. There are two basic numerical invariants one can attach to $D$. First, it is a standard fact that the $K$-dimension of $D$ is a square, say $\dim_K D = n^2$, from which one can extract $n$, the index of $D$. Second, $D$ gives rise to an element of the Brauer group of $K$, which is a torsion group. We can thus read off the order of $D$ in $\text{Br}(K)$, known as the period of $D$. Unless noted otherwise, throughout this paper, we will only work with Brauer classes $\alpha \in \text{Br}(K)$ of period prime to the characteristic exponent of $K$. 
The principal results of this paper are the following. Given a positive integer $M$, let $M'$ denote the submonoid of $N$ consisting of natural numbers relatively prime to $M$.

**Definition 1.1.** Let $M$ and $a$ be positive integers. A field $k$ has $M'$-Brauer dimension at most $a$ if for every finitely generated field extension $L/k$ of transcendence degree $t$ at most 1 and every Brauer class $\alpha \in \text{Br}(L)$ with period contained in $M'$ one has $\text{ind}(\alpha)|\text{per}(\alpha)^{a-1+t}$.

The case of particular interest is when $M = p$ for some prime $p$ (usually the characteristic exponent of $K$ or of a residue field of $K$ under some valuation).

**Notation 1.2.** Throughout this paper, when we use the phrase “Brauer dimension at most $a$” we will always tacitly assume that $a$ is a positive integer.

**Theorem (Theorem 6.3).** Suppose $K$ is the fraction field of an excellent Henselian discrete valuation ring with residue field $k$ of characteristic exponent $p$. If $k$ has $M'$-Brauer dimension at most $a$ then $K$ has $(Mp)^{\ell}$-Brauer dimension at most $a + 1$.

We can get an inductive corollary as follows. By a $d$-local field with residue field $k$ we will mean a sequence of fields $k_d, \ldots, k_0 = k$ such that for $i = 1, \ldots, d$, $k_i$ is the fraction field of an excellent Henselian discrete valuation ring with residue field $k_{i-1}$. Natural examples are given by iterated Laurent series fields $k((x_1)) \cdots ((x_d))$.

**Corollary 1.3.** If $K$ is a $d$-local field with residue field $k$ of $M'$-Brauer dimension at most $a$ and characteristic exponent $p$, then $K$ has $(Mp)^{\ell}$-Brauer dimension at most $a + d$.

A concrete application of this result yields various instances of the standard conjecture (recalled below).

**Corollary 1.4.** Let $L/K$ be a field extension of transcendence degree 1 and $\alpha \in \text{Br}(L)$ a Brauer class.

1. If $K = k((x_1)) \cdots ((x_d))$ is the field of iterated Laurent series over an algebraically closed field and $\text{per}(\alpha)$ is relatively prime to the characteristic exponent of $k$ then $\text{ind}(\alpha)|\text{per}(\alpha)^{d}$.

2. If $K$ is $d$-local with finite residue field of characteristic $p$ and $\text{per}(\alpha)$ is relatively prime to $p$ then $\text{ind}(\alpha)|\text{per}(\alpha)^{d+1}$.

3. If $K$ is an algebraic extension of the maximal unramified extension of a local field of residue characteristic $p$ and $\text{per}(\alpha)$ is relatively prime to $p$, then $\text{ind}(\alpha) = \text{per}(\alpha)$.

Linking local global, we also prove the following. Let $K$ be a global field and $C/K$ a proper, smooth, geometrically connected curve with a rational point $P$. For the purposes of this paper, we will refer to the following statement as the Colliot-Thélène conjecture. (It is a special case of a more general conjecture which may be found in [5].)

**Conjecture (Colliot-Thélène).** If $X$ is a smooth projective geometrically rational variety over a global field $K$ with geometric Picard group $\mathbb{Z}$, then $X$ has a 0-cycle of degree 1 if and only $X \otimes_K K_{\nu}$ has a 0-cycle of degree 1 for every place $\nu$ of $K$.

**Theorem (Theorem 7.2).** Assuming the Colliot-Thélène conjecture, any $\alpha \in \text{Br}(K(C))$ of odd period such that

1. $\alpha$ is unramified over primes dividing $\text{per}(\alpha)$
2. $\alpha|_{K} \in \text{Br}(K)$ is trivial

satisfies $\text{ind}(\alpha)|\text{per}(\alpha)^2$.

When these surfaces are equicharacteristic, i.e., they are surfaces over finite fields, then this is part of the standard conjecture for $C_3$-fields. Moreover, as shown in Corollary 7.4 below, one can deduce the standard conjecture for classes of odd period over fields of transcendence degree 2 over finite fields from the above theorem.

The key idea underlying these results is the observation that one can split ramification of Brauer classes using stacks and produce moduli problems – twisted sheaves over various kinds of stacky curves – whose solution gives bounds on the period-index relation.
After some stack-theoretic preliminaries in Section 2, we describe two methods for splitting ramification by stacks in Section 3. We study the Brauer groups of certain finite classifying stacks in Section 4 in preparation for a discussion of stacky moduli problems in Section 5. Finally, in Sections 6 and 7, we use the two different stacky splitting methods to study the local and global period-index results described above. At the end is an appendix by Daniel Krashen showing that our local results are sharp.

To put this work in context, let us make a few historical remarks and remind the reader of standard conjectures on the relation between period and index. Basic Galois cohomology shows that \( \text{per}(D) | \text{ind}(D) \) and that both have the same prime factors, so that \( \text{ind}(D) | \text{per}(D)^\ell \) for some power \( \ell \). The exponent \( \ell \) is still poorly understood, but the following “conjecture” has gradually evolved from its study. (The reader is referred to page 12 of [6] for what appears to be the first recorded statement of the conjecture, in the case of function fields of complex algebraic varieties.)

**Standard Conjecture.** If \( K \) is a “\( d \)-dimensional” field and \( \alpha \in \text{Br}(K) \), then \( \text{ind}(\alpha) | \text{per}(\alpha)^{d-1} \).

The phrase “\( d \)-dimensional” is left purposely vague. It is perhaps best to illustrate what it could mean by summarizing which cases of the standard conjecture have been proven. In [35], Tsen proved that the Brauer group of the function field of a curve over an algebraically closed field is trivial. (More generally, his proof shows that any \( C_1 \)-field has trivial Brauer group.) The next progress in this direction was not made until de Jong proved in [10] that the period and index are equal for Brauer classes over the function field of a surface over an algebraically closed field. (By contrast, it is still unknown whether this equality holds for any curve over a \( C_1 \)-field, or for any \( C_2 \)-field.)

Threefolds have so far been quite difficult to handle, and in fact no one has even proven that there is a bound on the period-index relation for classes in \( \text{Br}(C(x, y, z)) \) (or the function field of any other particular threefold). The methods used to prove these theorems have no inductive structure as one increases the dimension of the ambient variety; this is in part responsible for the difficulties faced in making progress along these lines. For example, de Jong’s proof makes essential use of the deformation theory of Azumaya algebras on surfaces, something which one can prove becomes far more complicated on an ambient threefold.

In a slightly different direction (and between Tsen and de Jong in history), Saltman proved in [34] and [35] that \( \text{ind} | \text{per}^2 \) for curves over \( p \)-adic fields, and he provided a method for showing that if \( K \) is the fraction field of an excellent Henselian discrete valuation ring with residue field \( k \), and if \( \text{ind} | \text{per}^a \) for every Brauer class on every curve over \( k \), then one has that \( \text{ind} | \text{per}^{a+2} \) for every Brauer class on every curve over \( K \). His techniques provide for a kind of induction as one proceeds up a ladder of Henselian fields, with the exponent in the relation increasing by 2 each time.

The key to Saltman’s method is splitting the ramification of Brauer classes by making branched covers, which allows one to reduce questions on curves over the fraction field to curves over the residue field. Because of the singularities which appear on branched covers, in order to completely split the ramification of a Brauer class Saltman is forced to make two covers, which is responsible for the increase of the exponent by 2.

The methods used in this paper are in some sense a hybrid of the ideas of Saltman and de Jong: we make some branching (either with schemes or stacks) and then use moduli theory to reduce a period-index relation to a rational point problem.

Connections between the Hasse principle and the period-index problem are also discussed in [26], where this connection is used to *disprove* the Hasse principle (and the Colliot-Thélène conjecture) for smooth projective geometrically rational varieties over \( C(x, y) \) with geometric Picard group \( \mathbb{Z} \). It should be interesting to investigate this connection in more detail over global fields. Some preliminary results in this direction are described in [23].

**Notation**

We adopt the following notations:

1. Given a local ring \( A \) with residue field \( \kappa \) and a separable closure \( \kappa \subset \kappa^{\text{sep}} \), we let \( A^{h,a} \) denote the strict Henselization of \( A \) with respect to \( \kappa^{\text{sep}} \). We may occasionally abuse the notation and leave the choice of \( \pi \) ambiguous.
(2) Given an abelian group \( G \), we let \( G' \) denote the subgroup consisting of elements of order prime to \( p \), where \( p \) is the characteristic exponent of the base field.

(3) Given a (possibly generic) point \( p \) of a scheme \( X \), we will write \( \kappa(p) \) for the residue field of \( p \).

If \( X \) is integral, we will also write \( \kappa(X) \) for the function field (the residue field at the generic point).

Some additional stack-specific notation will be introduced in Section \[2.1\]

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2. Stack-theoretic preliminaries

2.1. Notation and basic results. The following notations will be essential.

(1) Given an algebraic space \( X \), an invertible sheaf \( \mathcal{L} \), and an integer \( n \), the \( \mu_n \)-gerbe of \( n \)th roots of \( \mathcal{L} \) will be denoted \([\mathcal{L}]^{1/n}\). This is a stack on \( X \) which solves the moduli problem given by pairs \((\mathcal{M}, \varphi)\) with \( \mathcal{M} \) an invertible sheaf and \( \varphi: \mathcal{M} \otimes n \to \mathcal{L} \) is an isomorphism. It is easy to check that the class of this gerbe in \( \text{H}^2(X_{\text{fppf}}, \mu_n) \) is the image of the isomorphism class of \( \mathcal{L} \) under the map \( \text{Pic}(X) \to \text{H}^2(X_{\text{fppf}}, \mu_n) \) induced by the Kummer sequence (in the fppf topology). In particular, there is a \([\mathcal{L}]^{1/n}\)-twisted invertible sheaf (see Section \[2.2\] below). \( \mathcal{M} \) with \( \mathcal{M} \otimes n \cong \mathcal{L} \).

(2) A surface is a quasi-compact excellent algebraic space of equidimension 2.

(3) An orbisurface is a quasi-compact excellent Deligne-Mumford stack of equidimension 2 with trivial generic stabilizer.

(4) Given an integer \( n \), an \( A_{n-1} \)-orbisurface (resp. Zariski \( A_{n-1} \)-orbisurface) is a separated regular tame orbisurface \( \mathcal{X} \) with a coarse moduli space \( \pi: \mathcal{X} \to X \) such that for each point \( x \in X \), the localization \( \mathcal{X} \times_X \text{Spec} \mathcal{O}_{x,X} \) (resp. \( \mathcal{X} \times_X \text{Spec} \mathcal{O}_{x,X} \)) has the form \([R/\mu_n]\), where \( R \) is a regular local ring of dimension 2 and \( \mu_n \) acts on \( m_R/m_R^2 \) by \( \sigma \otimes \sigma^v \), where \( \sigma \) is the standard character of \( \mu_n \).

(5) Given a stack \( \mathcal{X} \), the inertia stack is the relative group-stack \( \mathcal{I}(\mathcal{X}) \to \mathcal{X} \) representing the functor which to an object \( \varphi: X \to \mathcal{X} \) assigns the group \( \text{Aut}(\varphi) \) of automorphisms of \( \varphi \) in the fiber category of \( \mathcal{X} \) over \( X \). If \( \mathcal{X} \) is an Artin stack locally of finite presentation over an algebraic space \( S \), then it is a standard fact that \( \mathcal{I}(\mathcal{X}) \to \mathcal{X} \) is of finite presentation.

(6) Given a stack \( \mathcal{X} \), the stacky locus is the stack-theoretic support of the inertia stack.

(7) Given a Deligne-Mumford stack \( \mathcal{X} \) with coarse moduli space \( \pi: \mathcal{X} \to X \) and a point \( p \in X \), the residual gerbe at \( p \) is the reduced structure on the preimage of \( p \) in \( \mathcal{X} \). (This is a special case of a more general notion described in section 11 of \[22\].) If \( \mathcal{X} \) is an \( A_{n-1} \)-orbisurface, then the residual gerbe at any point \( p \) is either isomorphic to \( \text{Spec} \kappa(p) \) (which we will call trivial) or is a \( \mu_n \)-gerbe over \( \kappa(p) \); for a Zariski \( A_{n-1} \)-orbisurface, the non-trivial residual gerbes have trivial cohomology class in \( \text{H}^2(\kappa(p), \mu_n) \).

(8) Given a residual gerbe \( \xi \) lying over \( p \) with residue field \( \kappa \), we will write \( \xi \) for the base change of \( \xi \) to an algebraic closure \( \overline{\kappa} \). (As \( \overline{\kappa} \) is often the case, we may abuse notation – as we have already – and leave the choice of embedding \( \kappa \to \overline{\kappa} \) out of the notation.)

(9) Given a Deligne-Mumford stack \( \mathcal{X} \) with coarse moduli space \( \pi: \mathcal{X} \to X \) and a residual gerbe \( \xi \in \mathcal{X} \) over \( p \in X \), we will call the base change \( \mathcal{X} \times_X \text{Spec} \mathcal{O}_{p,X}^{h} \) (resp. \( \mathcal{X} \times_X \text{Spec} \mathcal{O}_{p,X}^{hs} \)) the Henselization (resp. strict Henselization) of \( \mathcal{X} \) at \( \xi \).

For basic results on gerbes, the reader is referred to \[11\] or, for a condensed discussion, \[26\] or \[24\]. When blowing up \( A_{n-1} \)-orbisurfaces, we will need the following generalization of their local structure:

Definition 2.1.1. Given a regular local ring \( A \) of dimension 2, an action of \( \mu_n \) on \( A \) has type \((a,b)\) if \( \mathcal{L} / \mathcal{L} \otimes Z/nZ^{\otimes 2} \) if the induced representation of \( \mu_n \) on \( m_A/m_A^2 \) splits as \( \sigma^a \otimes \sigma^b \). We will also say that the quotient stack \([\text{Spec} A/\mu_n]\) has type \((a,b)\).
Given any tame regular Deligne-Mumford surface, a residual gerbe of the form $B\mu_n$ has a corresponding type: it is the type of the Henselization of the stack at that gerbe.

2.2. Twisted sheaves on a topos and deformation theory. Let $(X, \mathcal{O})$ be a ringed topos. For the sake of convenience, we will assume that there is a covering $U$ of the final object such that $(U, \mathcal{O}_U)$ and $(U \times U, \mathcal{O}_{U \times U})$ are isomorphic to big fpqc topos of schemes. (Among other things, this hypothesis ensures that quasi-coherent sheaves are an abelian category, and it is satisfied by all algebraic stacks.)

The references for the results in this section are [26] and [24] and the references therein.

The period-index problem is the following, which may be found in Proposition 3.1.2.1 of [26].

**Definition 2.2.1.** An $\mathcal{A}$-twisted sheaf is an $\mathcal{O}_\mathcal{A}$-module $\mathcal{F}$ such that the natural action $\mathcal{F} \times A \to \mathcal{F}$ agrees with the right $A$-action induced by the (left) module structure on $\mathcal{F}$.

As usual, one can define coherent and quasi-coherent twisted sheaves, and one can speak of locally free (quasi-coherent) twisted sheaves. The basic fact underlying the usefulness of twisted sheaves for the period-index problem is the following, which may be found in Proposition 3.1.2.1 of [26].

**Lemma 2.2.2.** If $\mathcal{X} \to X$ is a $G_m$-gerbe over a regular algebraic space of dimension at most 2 then there is a locally free $\mathcal{X}$-twisted sheaf of rank $n$ if and only if the index of the class $[\mathcal{X}] \in H^2(X, G_m)$ divides $n$.

More concretely, Azumaya algebras on $X$ with Brauer class $[\mathcal{X}]$ are precisely those algebras which may be written as the direct images of trivial algebras $O_{\mathcal{V}}(\mathcal{V})$ on $\mathcal{X}$ with $\mathcal{V}$ a locally free $\mathcal{X}$-twisted sheaf.

Thus, if we wish to show that the period equals the index, we need only look for a locally free twisted sheaf whose rank equals the period; this places a geometric overlay on the seemingly algebraic period-index problem. This apparently trivial fact has many useful consequences, some of which we will see below (and various others of which are described in [26]).

Let $0 \to I \to \mathcal{O} \to \mathcal{O}_0 \to 0$ be a small extension of rings in $X$. Given an $A$-gerbe $\mathcal{X} \to (X, \mathcal{O})$, define the reduction of $\mathcal{X}$ to be the $A(\mathcal{O}_0)$-gerbe $\mathcal{X}_0 \to X_0 := (X, \mathcal{O}_0)$ given by applying Proposition 3.1.5 of [11] to the map $A(\mathcal{O}) \to A(\mathcal{O}_0)$ of sheaves of abelian groups. (This is just a stacky rigidification of the pullback map on cohomology $H^2(X, A(\mathcal{O})) \to H^2(X, A(\mathcal{O}_0))$ induced by the map of ringed topoi $(X, \mathcal{O}_0) \to (X, \mathcal{O})$.)

**Proposition 2.2.3.** With the above notation, let $F_0$ be an $\mathcal{X}_0$-twisted sheaf.

1. There is an obstruction in $\text{Ext}^2_\mathcal{X}(F_0, I \otimes F_0)$ to deforming $F_0$ to an $I$-flat $X$-twisted sheaf.
2. If $X$ is a scheme of dimension 1 and $F_0$ is locally free, then for any $I$ we have $\text{Ext}^2(F_0, I \otimes F_0) = H^2(X, \mathcal{X}\text{om}(F_0, I \otimes F_0)) = 0$.
3. If $X$ is an algebraic stack, the Grothendieck existence theorem holds for formal deformations of coherent twisted sheaves.

For further details and proofs, we refer the reader to [24] and [26]. The last statement is not found any of the references; it follows from the Grothendieck existence theorem for stacks [31] applied to the results of Section 2.3 below, which shows that gerbes on big sites of algebraic stacks are representable by algebraic stacks.

2.3. Moduli spaces of twisted sheaves. We briefly summarize a few facts about the moduli space of twisted sheaves on a curve. The purpose of this section is mostly to fix notation. We refer the reader to [24] for further details and proofs.

Let $X/K$ be a smooth proper curve over a field and $s \to X$ a $\mu_n$-gerbe. Given $L \in \text{Pic}(X)$, there is a moduli stack of stable twisted sheaves of rank $n$ and determinant $L$, written $\text{Tw}_{s/K}(n, L)$. It is a $\mu_n$-gerbe over its coarse moduli space $\text{Tw}_{s/K}(n, L)$. Moreover, $\text{Tw}_{s/K}(n, L)$ is a form of the space of stable sheaves $\text{St}_X(n, L')$ for an appropriate $L' \in \text{Pic}(X)$. (In particular, it arises as a twist by automorphisms in $\text{Pic}(X)$.) Thus, $\text{Tw}_{s/K}(n, L)$ is geometrically (separably) unirational; if $\deg L$ is
appropriately chosen, then it is in fact proper and geometrically rational, with (geometric) Picard group $Z$.

2.4. **A stack on a stack is a stack.** We assume for the sake of simplicity that all categories in this section admit arbitrary (non-empty) coproducts and all finite fibered products (whose indexing sets are non-empty), but not necessarily that they have final objects. Given a category $S$, we will interchangeably use the terminology “$S$-groupoid” and “category fibered in groupoids over $S$.”

2.4.1. Let $S$ be a category and $\pi_X : X \to S$ and $\pi_Y : Y \to S$ two categories fibered in groupoids.

**Definition 2.4.2.** Given a functor $F : Y \to X$ of fibered categories, the fibered replacement of $F$ is the fibered category $(Y, F) \to X$ consisting of triples $(y, x, \varphi)$ with $y \in Y$, $x \in X$, and $\varphi : F(y) \to x$ an isomorphism. The arrows $(y, x, \varphi) \to (y', x', \varphi')$ are arrows $y \to y'$ and $x \to x'$ such that the diagram

\[
\begin{array}{ccc}
F(y) & \longrightarrow & x \\
\downarrow & & \downarrow \\
F(y') & \longrightarrow & x'
\end{array}
\]

commutes.

There is a natural functor $Y \to (Y, F)$ given by sending $y$ to $(y, F(y), \text{id})$ and a natural functor $(Y, F) \to Y$ sending $(y, x, \varphi)$ to $y$.

**Lemma 2.4.3.** Given $F : Y \to X$ as above,

1. the natural functor $(Y, F) \to X$ is a category fibered in groupoids;
2. the natural functor $(Y, F) \to S$ is a category fibered in groupoids;
3. the natural functors $Y \to (Y, F)$ and $(Y, F) \to Y$ define an equivalence of fibered categories over $S$.

**Proof:** To check that $(Y, F) \to X$ is a category fibered in groupoids, suppose $(y, x, \varphi)$ is an object of $(Y, F)$ and $\alpha : x' \to x$ is an arrow in $X$. Since $Y$ is a fibered category, there is an arrow $\beta : y' \to y$ such that $\pi_Y(\beta) = \pi_X(\alpha)$. Since every arrow in a category fibered in groupoids is Cartesian, there is an isomorphism $\varphi' : F(y') \to x'$ such that the diagram

\[
\begin{array}{ccc}
F(y') & \longrightarrow & x' \\
\downarrow & & \downarrow \\
F(y) & \longrightarrow & x
\end{array}
\]

commutes. But this says precisely that $(y', x', \varphi') \to (y, x, \varphi)$ lies over $\alpha$. We will be finished once we check that every arrow $(y', x', \varphi') \to (y, x, \varphi)$ is Cartesian. Thus, suppose $(y'', x', \varphi'') \to (y, x, \varphi)$ is another arrow such that the underlying arrow $x' \to x$ is the same. In this case, we know that $y'' \to y$ and $y' \to y$ have the same image in $S$, whence there is a unique arrow $y'' \to y'$ over $y$. This results in a diagram

\[
\begin{array}{ccc}
F(y') & \longrightarrow & F(y) \\
\downarrow & & \downarrow \\
x' & \rightarrow & x
\end{array}
\]

Since the arrow $F(y') \to x'$ is uniquely determined by the arrows $F(y'') \to F(y)$ and the arrow $x' \to x$, we see that there results a map $(y'', x', \varphi'') \to (y', x', \varphi')$, as desired.

That $(Y, F) \to S$ is a category fibered in groupoids is a formal consequence of the first part and the fact that $X \to S$ is a category fibered in groupoids.
It is immediate that the composition $Y \to (Y, F) \to Y$ is equal (!) to the identity. In the other direction, the composition $(Y, F) \to Y \to (Y, F)$ sends $(y, x, \varphi)$ to $(y, F(y), \text{id})$. The diagram

\[
\begin{array}{ccc}
F(y) & \xrightarrow{\varphi} & x \\
\downarrow\text{id} & & \downarrow\varphi^{-1} \\
F(y) & \xrightarrow{\text{id}} & F(y)
\end{array}
\]

shows that the map $(\text{id}_y, \varphi^{-1})$ is an isomorphism $(y, x, \varphi) \sim (y, F(y), \text{id})$, which shows that the composition of the two functors is naturally isomorphic to $\text{id}_{(Y, F)}$. This gives the last statement. \hfill \square

**Proposition 2.4.4.** Given a category fibered in groupoids $X \to S$, the forgetful functor and the functor $\nu : (F : Y \to X) \mapsto (Y, F)$ give an equivalence between the slice 2-category of $S$-groupoids over $X$ and the 2-category of $X$-groupoids.

**Proof.** It is clear by construction that the (strong) equivalence $Y \cong (Y, F)$ of $S$-groupoids proven in Lemma 2.4.3 is functorial in $Y$ and $F$. In particular, it follows that given two functors $F : Y \to X$ and $F' : Y' \to X$, the functor $\text{Hom}_X(Y, Y') \to \text{Hom}_X((Y, F), (Y', F'))$ is an equivalence of groupoids, so that $\nu$ is fully faithful (in the strong sense). On the other hand, any $X$-groupoid $H : Z \to X$ is naturally an $S$-groupoid (as $X$ is an $S$-groupoid), and we see from Lemma 2.4.3(iii) that the natural functor $Z \to (Z, H)$ is an equivalence of categories over $X$. It follows that $\nu$ is essentially surjective, and is therefore an equivalence of 2-categories. \hfill \square

2.4.5. Now suppose that $S$ is a site and $X \to S$ and $Y \to S$ are stacks in groupoids. The fibered category underlying $X$ has a natural site structure as follows: a morphism $y \to x$ is a covering if and only if its image in $S$ is a covering. Equivalently, a sieve over $x$ is a covering sieve if and only if its image in $S$ is a covering sieve. (To align this with the way one usually thinks of sites in a geometric context, one could equally well let the objects of the site in question consist of morphisms $\varphi : V \to X$, with $V$ an object of $S$ thought of as a fibered category over $S$. It is easy to see that this yields a naturally equivalent site.) Write $X^s$ for the resulting site.

**Lemma 2.4.6.** Let $F : Y \to X$ and $G : X \to S$ be two categories fibered in groupoids. Suppose $S$ is a site and $X$ is given the induced site structure. If $F$ and $G$ are stacks, then $GF$ is a stack.

**Proof.** Upon replacing $X$ and $Y$ by equivalent fibered categories if necessary, we may assume that $F$ and $G$ have compatible cleavages over $S$, so that the concept of “pullback” is defined and functorial with respect to $F$.

We first claim that $Y$ is a prestack over $S$, i.e., that the Hom presheaves are sheaves. Let $z' \to z$ be a covering in $S$ and $y_1, y_2$ objects of $Y_z$ with images $x_1$ and $x_2$ in $X_z$. Consider the map of presheaves on $S_{/z}$

\[\iota : \text{Hom}_z(y_1, y_2) \to \text{Hom}_z(x_1, x_2).\]

Since $X$ is a stack over $S$, the target is a sheaf. We claim that the pullback of $\iota$ via any map $\varphi : t \to \text{Hom}_z(x_1, x_2)$ over a map $t \to z \in S$ is a sheaf on $S_{/t}$. Let $\tilde{\varphi} : x_{1|t} \to x_{2|t}$ be the map corresponding to $\varphi$. Since $Y$ is a category fibered in groupoids over $X$, the pullback is a torsor under $\text{Hom}_{x_{1|t}}((y_1|t), \tilde{\varphi}^* y_2|t)$, which is a sheaf since $Y$ is a (pre)stack over $X$. Thus, $\iota$ is a map of presheaves with the codomain a sheaf and which is relatively representable by sheaves; it follows that the domain is a sheaf, which shows that $Y$ is a prestack over $S$.

Now let $\overline{Y}$ be the stackification of $Y$ as a fibered $S$-category, constructed in §II.2 of [11] or in Lemme 3.2 of [22]. Since $Y$ is a prestack over $S$, the map of fibered categories $Y \to \overline{Y}$ is fully faithful on fiber categories (a monomorphism of fibered categories). The universal property of stackification yields a map of $S$-stacks $Q : \overline{Y} \to X$; letting $\overline{Y} = (\overline{Y}, Q)$ (which is still an $S$-stack, as it is equivalent to $\overline{Y}$), there
results a diagram

```
  Y
 /\
/  \\  
 X
  \\
S
```

of fibered categories over $S$. Let $z \in S$, $x \in X_z$, and $t \in \tilde{Y}_x$. By the construction of $\tilde{Y}$ (and the equivalence $\tilde{Y} \cong \tilde{Y}$), there is a covering $z' \rightarrow z$, an object $y' \in Y_{z'}$, and an arrow $\sigma : y' \rightarrow t|_{z'}$ over the identity of $z'$. Let $\rho : x'' \rightarrow x|_{z'}$ be the image of $\sigma$ in $X$, so that $\rho$ is an arrow over the identity of $z'$, hence has an inverse. There results an isomorphism $(\rho^{-1})^*y' \rightarrow t|_{z'}$ over $x|_{z'}$. Thus, $Y \rightarrow \tilde{Y}$ is a morphism of $X$-stacks which is fully faithful and such that every object of $\tilde{Y}$ is locally in the image of $Y$. It follows that it is a 1-isomorphism of stacks, and thus that $Y$ is 1-isomorphic to a stack on $X$ which is a stack over $S$. The result follows.

Lemma 2.4.7. Suppose $X$ and $Y$ are $S$-stacks and $F : Y \rightarrow X$ is a functor over $S$. The fibered replacement $(Y,F)$ is a stack on $X$ and on $S$.

Proof: We can describe $(Y,F)$ as the graph of $F$ in the following way: the second projection and the functor $F$ yield two maps $pr_2 : Y \times Y \rightarrow X$. Given an object $s \in S$ and a map $f : s \rightarrow Y \times X$, we thus get a sheaf $\text{Isom}_s(pr_2 \circ f, F \circ pr_1 \circ f)$ on $s$. It is easy to see that this is a “construction locale” in the sense of §14 of [22]; in fact, this defines a sheaf on $(Y \times X)^s$. It is well-known that sheaves $\mathcal{Z}$ on $(Y \times X)^s$ are in bijection with representable morphisms of $S$-stacks $Z \rightarrow Y \times X$. The $S$-stack in question here is easily seen to be $(Y,F)$. To see that $(Y,F)$ is an $X$-stack, we note that we have a diagram $(Y,F) \rightarrow Y \times X \rightarrow X$ where each arrow is a stack (the first is even a sheaf). Thus, the result follows from Lemma 2.4.6.

Proposition 2.4.8. The equivalence of Proposition 2.4.7 restricts to an equivalence of $S$-stacks over $X$ and $X^s$-stacks.

Proof: This follows immediately from Lemma 2.4.7.

Corollary 2.4.9. Suppose $\mathcal{X}$ is an algebraic stack. Given $\alpha \in H^2(\mathcal{X}_{fppf}, \mu_n)$, there is a morphism of algebraic stacks $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ and a map $\mu_n \rightarrow \mathcal{Y}(\mathcal{Y})$ such that

1. for any algebraic space $\varphi : V \rightarrow \mathcal{X}$, $\mathcal{Y} \times \mathcal{X} V \rightarrow V$ is a $\mu_n$-gerbe representing $\alpha|_V$; 
2. the natural morphism of sites associated to $\pi$ is a $\mu_n$-gerbe representing $\alpha$.

Proof: This follows immediately from Proposition 2.4.8 once we note that a relative $\mu_n$-gerbe over an algebraic stack is itself algebraic (as algebraicity is local in the smooth topology).

This allows us to carry over the formalism of twisted sheaves (including the Grothendieck existence theorem) to algebraic stacks without any significant changes.

Remark 2.4.10. The reader uncomfortable with the techniques sketched above can translate the results of this paper into the language of Azumaya algebras and their deformations in the étale topos of a Deligne-Mumford stack, at the expense of knowing a bit more about the étale cohomology of such stacks. We will not pursue this point.

2.5. Vector bundles on a stack and descent of formal extensions. Let $\mathcal{X}$ be a tame Deligne-Mumford stack separated and of finite type over a locally Noetherian algebraic coarse moduli space $\pi : \mathcal{X} \rightarrow X$. Define a stack $\mathcal{Y}_{\mathcal{X}/X}$ (or simply $\mathcal{Y}$ when $\mathcal{X}$ is understood) on the big fppf site of $X$ whose fiber category $\mathcal{Y}_T$ over $T = \text{Spec} A \rightarrow X$ is the groupoid of locally free sheaves on $\mathcal{X} \times_X T$.

Proposition 2.5.1. With the above notation, the stack $\mathcal{Y}_{\mathcal{X}/X}$ is an Artin stack smooth over $X$. 

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Proof. In the general case, it is easiest to proceed by Artin’s representability theorem [3]. (This requires descending $\mathcal{X}$ over an excellent base, which is easily done by the standard techniques in §8 of [14].) This makes significantly easier by the fact that the natural deformation and obstruction modules for any object $V \in \mathcal{V}_T$ (over affine $T = \text{Spec} A$) with respect to the trivial infinitesimal extension $A + M$ (in the notation of [3]) vanish. Indeed, these are described by the higher cohomologies of $\mathcal{V}_T(V) \otimes M$, which vanishes because the course moduli space $T$ of $\mathcal{X}_T$ is affine and $\mathcal{X}_T$ is tame. The Schlessinger conditions (condition S1 in [3]) then follow trivially, as the set of deformations of an object $a \in \mathcal{V}(A)$ over any infinitesimal extension $A^\prime \to A$ is a singleton (!). The infinitesimal automorphisms of $V \in \mathcal{V}(A)$ with respect to the extension $A + M$ are given by $\text{End}(V) \otimes_A M$, which is a finite $A$-module compatible with arbitrary base change. Finally, the Grothendieck existence theorem for $\mathcal{V}$ follows immediately from Proposition 2.1 of [31]. Putting this all together easily yields the result; the reader who is interested in this level of generality is encouraged to work out the rest of the details.

In this paper we will only actually need this result when $\pi$ is a flat morphism, and in this case it is easy to give a direct proof: it suffices to work étale locally on $X$, so we may assume that $X = \text{Spec} B$ and $\mathcal{X} = [\text{Spec} C/G]$, with $C$ a finite locally free $B$-algebra and $G$ a finite group. In this case, the stack $\mathcal{V}$ is identified with the substack of $G$-equivariant locally free $B$-modules which admit an equivariant $C$-action. It is trivial that locally free $G$-equivariant $B$-modules form an algebraic stack locally of finite presentation over $B$, as adding a $G$-structure amounts to choosing sections of the automorphism sheaf satisfying algebraic conditions coming from a presentation of $G$. Since everything is finite and flat, it is also easy to see that the addition of a $C$-structure is an algebraic condition (described in terms of homomorphisms between tensor products). The result follows.

Corollary 2.5.2. Let $\mathcal{X}$ be a tame separated Deligne-Mumford stack of finite type over a locally Noetherian algebraic space with coarse moduli space $\pi : \mathcal{X} \to X$. Suppose $X = \text{Spec} A$ is affine and $I \subset A$ is an ideal. Let $X' = \text{Spec} A/I$, $U = X \setminus Z(I)$, and $U' = U \times_X X'$. Given a locally free sheaf $V$ on the open substack $\mathcal{X} \times_X U$ and an extension $V'$ of $V|_U$ to $\mathcal{X} \times_X X'$, there is a locally free sheaf $W$ on $\mathcal{X}$ extending $V$ and such that $W|_{X'} \cong V'$.

Proof. This follows from the previous proposition combined with Theorem 6.5 of [30]. The history surrounding this type of descent problem and various situations in which it holds are also described in [ibid.].

3. Splitting ramification

3.1. Splitting by cyclic orbifold covers. We prove a result in the spirit of Saltman [34, 35] on moving the ramification in a cyclic cover. Let $X \to T$ be a regular surface flat over a local Dedekind scheme with infinite perfect residue field. Denote the reduced special fiber by $X_0$. Let $\alpha \in \text{Br}(\kappa(X))'$. By standard methods [27], we may blow up $X$ and assume that $\text{Ram}(\alpha) + X_0$ is a strict normal crossings divisor, say $\text{Ram}(\alpha) = D = D_1 \cup \cdots \cup D_r$.

Proposition 3.1.1. Given $n > 0$, there is a $T$-very ample normal crossings divisor $C$ such that

1. $C + D$ has normal crossings;
2. there is an ample divisor $H$ with $C + D \sim nH$.

Proof. First, note that $X$ is projective over $T$ (as any multisection which intersects every component of the closed fiber is $T$-ample). It is easy to see that if $H$ is “sufficiently very ample” then $C = nH - D$ is very ample, thus proving second statement.

The crossings of $D$ occur only in the closed fiber. Since $k$ is perfect and infinite, it follows from Bertini’s theorem that there is a section of $nH - D$ over the reduced special fiber which contains only smooth points of $D_0$. This hyperplane lifts over all of $T$, giving rise to $C$. Since the intersection of $C$ with $D$ is reduced, it follows that $C$ has regular components. Thus, $C + D$ has normal crossings.

Let $X' \to X$ denote the cyclic cover resulting from extracting an $n$th root of the divisor $C + D$, as in section 2.4 of [17]. By local calculations, one can see that $X'$ is normal with finite quotient singularities, so that the local rings at singular points look like $R[z]/(z^n - xy)$, where $R$ is a local ring of $X$ and $x$ and $y$ are a system of parameters.
Lemma 3.1.2. There is a Zariski $A_{n-1}$-orbisurface $\mathcal{X}$ with coarse moduli space $\pi : \mathcal{X} \to X'$ such that $\pi$ is an isomorphism over the regular locus of $X'$.

Proof. This is made quite easy by the fact that the singularities of $X'$ are quotient singularities in the Zariski topology. Let $x_1, \ldots, x_m \in X'$ be the (closed) singular points and let $V = X' \setminus \{x_1, \ldots, x_m\}$ be the regular locus. For each $x_i$, there is a regular $T$-scheme $U_i$ with a $\mu_n$ action and a $\mu_n$-invariant morphism $U_i \to X'$ such that $\mu_n$ acts freely in codimension 1 and fixes a single closed point. It follows that the induced map from the (regular) quotient stack $Y_i := [U_i/\mu_n] \to X'$ is an isomorphism over $V \times_X [U_i/\mu_n]$. Gluing the $Y_i$ and $V$ together yields $\mathcal{X}$. □

The following result is the key to the rest of this paper. By using an orbifold resolution of the quotient singularities of $X'$ instead of the standard minimal resolution in the sense of Lipman [27] (as Saltman used), we avoid having to think about additional ramification divisors. This results in an optimally efficient way of splitting the ramification of $\alpha$, in the sense that the function field extension has minimal degree.

Proposition 3.2.1. Suppose $\alpha$ Uniqueness follows from the fact that the map $n$ has period $\mathcal{A}$ since the degeneracy locus of the trace is of codimension $\mathcal{A}$ be reflexive, hence locally free (as $X$ contains a primitive $\mathcal{D}$ which is an isomorphism over $\mathcal{X}$ excellen scheme of dimension $\mathcal{A}$). Given an Azumaya algebra $\mathcal{A}$ over $\mathcal{X}$ there is a Zariski singularities of $\mathcal{X}$, again, we know that the restriction map $\mathcal{A} |_{\mathcal{X}} \to \mathcal{X}$ is an Azumaya algebra.

Lemma 3.1.2. There is a Zariski $\mathcal{X}$-orbisurface $\mathcal{X}$ with coarse moduli space $\pi : \mathcal{X} \to X'$ such that $\pi$ is an isomorphism over the regular locus of $X'$.

Proof. This is made quite easy by the fact that the singularities of $X'$ are quotient singularities in the Zariski topology. Let $x_1, \ldots, x_m \in X'$ be the (closed) singular points and let $V = X' \setminus \{x_1, \ldots, x_m\}$ be the regular locus. For each $x_i$, there is a regular $T$-scheme $U_i$ with a $\mu_n$ action and a $\mu_n$-invariant morphism $U_i \to X'$ such that $\mu_n$ acts freely in codimension 1 and fixes a single closed point. It follows that the induced map from the (regular) quotient stack $Y_i := [U_i/\mu_n] \to X'$ is an isomorphism over $V \times_X [U_i/\mu_n]$. Gluing the $Y_i$ and $V$ together yields $\mathcal{X}$. □

The following result is the key to the rest of this paper. By using an orbifold resolution of the quotient singularities of $X'$ instead of the standard minimal resolution in the sense of Lipman [27] (as Saltman used), we avoid having to think about additional ramification divisors. This results in an optimally efficient way of splitting the ramification of $\alpha$, in the sense that the function field extension has minimal degree.

Proposition 3.2.1. Let $\mathcal{X}$ be the stack constructed in Lemma 3.1.2. There is an Azumaya algebra $\mathcal{A}$ on $\mathcal{X}$ with Brauer class $\alpha$ $\mathcal{X}$ in the sense of Lipman [27] (as Saltman used), we avoid having to think about additional ramification divisors. This results in an optimally efficient way of splitting the ramification of $\alpha$, in the sense that the function field extension has minimal degree.
So we may suppose \( R \) is a dvr with uniformizer \( t \) and residue field \( \kappa \). The ramification of \( \alpha \) is given by some cyclic extension \( \kappa(\mathfrak{t}^{1/n})_\kappa \), where \( a \in R^\times \) is a unit. Letting \( (a, t)_n \) denote the cyclic algebra of degree \( n \), we see that the class \( \alpha - (a, t)_n \) is unramified. Thus, to prove the result, it suffices to prove it for the algebra \( (a, t)_n \). Note that a local model for \( X\{D^{1/n}\} \) is given by the stack quotient \([\text{Spec } R[t^{1/n}] / \mu_n]\). To show that \( (a, t)_n \) extends to an unramified class over \( X\{D^{1/n}\} \), it thus suffices to find a \( \mu_n \)-equivariant Azumaya order in \( (a, t)_n \otimes K(R)(t^{1/n}) \).

Recall that \( (a, t)_n \) is generated by \( x \) and \( y \) such that \( x^n = a \), \( y^n = t \), and \( xy = \zeta yx \). We can choose new generators for \( (a, t)_n \otimes K(R)(t^{1/n}) \) by letting \( \bar{y} = y/y^{1/n} \). The \( R[t^{1/n}] \)-order generated by \( x \) and \( \bar{y} \) is then Azumaya (as its reduction modulo \( t^{1/n} \) is the split central simple algebra \((\mathfrak{t}, 1)_n\)), and has a \( \mu_n \)-action arising from the action on \( t^{1/n} \).

The reduced preimage of \( D \) in \( X\{D^{1/n}\} \) is a residual curve of the type studied in Section 5 above. Thus, \( \alpha \) gives rise to a Brauer class of period \( n \) on this residual curve. The moduli of uniform twisted sheaves on such a residual curve developed in Section 5.1 will give us information that can be deformed off of the ramification locus and back to the function field of \( X \) (in favorable circumstances).

4. SOME COHOMOLOGY

In this section, we gather together the cohomological computations which we will need throughout the rest of the paper. This includes a study of the Brauer group and second cohomology of \( \mu_n \), as well as a discussion of the second cohomology of certain stacky surfaces which will play a fundamental role starting in Section 3.

4.1. Computation of \( \text{Br}(\mathbb{B}_{\mu_n}^{\text{r}}) \).

Lemma 4.1.1. Let \( Y \) be a scheme such that \( \text{Br}(Y) = H^2(Y, G_m)_{\text{tors}} \). The natural map
\[
\text{Br}(\mathbb{B}_{\mu_n,Y}) \to H^2(\mathbb{B}_{\mu_n,Y}, G_m)_{\text{tors}}
\]
is an isomorphism.

Proof. The natural map \( Y \to \mathbb{B}_{\mu_n,Y} \) is finite locally free of degree \( n \). Thus, an element \( \alpha \in H^2(\mathbb{B}_{\mu_n,Y}, G_m) \) lies in \( \text{Br}(\mathbb{B}_{\mu_n,Y}) \) if and only if \( \alpha|_Y \) lies in \( \text{Br}(Y) \) (see e.g. 3.1.4.5 of [26]). But this is ensured by the hypothesis.

Remark 4.1.2. The lemma notably holds when \( Y \) is the spectrum of a field. By Gabber’s theorem [9], we thus see that the Brauer group and cohomological Brauer group of \( \mathbb{B}_{\mu_n,Y} \) coincide whenever \( Y \) is a quasi-compact separated scheme admitting an ample invertible sheaf.

Lemma 4.1.3. If \( \mathcal{T} \) is separably closed and \( n \) is invertible in \( \mathcal{T} \) then \( \text{Br}(\mathbb{B}_{\mu_n}^{\text{r}}) = 0 \).

Proof. In this case, \( \mathcal{T} \) contains a primitive \( n \)th root of unity, so that it is enough to show \( \text{Br}((\mathbb{Z}/n\mathbb{Z})) = 0 \). Thus, it suffices to show that any \( \mathbb{Z}/n\mathbb{Z} \)-equivariant central simple \( \mathcal{T} \)-algebra \( A \) of degree invertible in \( \mathcal{T} \) is isomorphic to the endomorphism algebra of a \( \mathbb{Z}/n\mathbb{Z} \)-equivariant vector space \( V \). Since \( \mathcal{T} \) is separably closed, the algebra underlying \( A \) is simply \( M_\nu(\mathcal{T}) \) for some integer \( \nu \) invertible in \( L \). By the Skolem-Noether theorem, the generator of \( \mathbb{Z}/n\mathbb{Z} \) acts on \( M_\nu(\mathcal{T}) \) via conjugation by some element \( \varphi \in \text{GL}_\nu(\mathcal{T}) \) such that \( \varphi^n \) is a scalar \( \gamma \). Since \( n \in L^\times \), there is some \( \varepsilon \) such that \( \varepsilon^n = \gamma^{-1} \). Setting \( \varphi' = \varepsilon \varphi \) yields an element of order \( n \) of \( \text{GL}_\nu(\mathcal{T}) \), i.e., a \( \mathbb{Z}/n\mathbb{Z} \)-equivariant structure on \( \mathcal{T} \), giving rise to the same equivariant structure on \( M_\nu(\mathcal{T}) \). The result follows.

Proposition 4.1.4. Let \( L \) be a field. There is a split exact sequence
\[
0 \to \text{Br}(L)^r \to \text{Br}(\mathbb{B}_{\mu_n})^r \to H^1(\text{Spec } L, \mathbb{Z}/n\mathbb{Z}) \to 0
\]
compatible with base extension.

Proof. This comes from the Leray spectral sequence for the projection \( \pi : \mathbb{B}_{\mu_n} \to L \), using (1) \( R^1\pi_* G_m = \mathbb{Z}/n\mathbb{Z} \) and (2) \( R^2\pi_* G_m = 0 \). The first fact is simply the computation of the character group of \( \mu_n \), while the second follows from Lemma 4.1.3 and Lemma 4.1.1. The splitting comes from the natural map \( \text{Spec } L \to \mathbb{B}_{\mu_n} \).
Corollary 4.1.5. Let \( L \) be a field and suppose \( n \in \mathbb{L}^\times \). Suppose that there is a positive integer \( d \) such that every \( \alpha \in \text{Br}(L)[n] \) satisfies \( \text{ind}(\alpha) = \text{per}(\alpha)^d \). Then every \( \beta \in \text{Br}(\mathbb{B}\mu_n)[n] \) satisfies \( \text{ind}(\beta) = \text{per}(\beta)^{d+1} \).

**Proof.** Suppose \( \beta \in \text{Br}(\mathbb{B}\mu_n) \) has order \( m \), where \( m \) is a positive divisor of \( n \). Using, we can write \( \beta = \beta' + \beta'' \) with \( \beta' \in \text{Br}(L)[m] \) and \( \beta'' \in \mathbb{H}^1(\text{Spec} \ L/\mathbb{Z}/m\mathbb{Z}) \subset \mathbb{H}^1(\text{Spec} \ L, \mathbb{Z}/m\mathbb{Z}) \). By assumption, there is a field extension \( L'/L \) of degree \( m'^d \), annihilating \( \beta' \). As there is clearly a finite étale extension \( L''/L \) of degree \( m \) annihilating \( \beta'' \), we conclude from the compatibility of Proposition 4.1 with field extension there is a finite étale morphism \( e : \text{Spec} \ L'' \to \text{Spec} \ L \) of degree \( m^{d+1} \) such that \( \beta |_{\mathbb{B}\mu_n \otimes_{\mathbb{L}} L''} = 0 \). \( \square \)

4.2. Computation of \( \mathbb{H}^2(\mathbb{B}\mu_n, \mu_n) \).

**Proposition 4.2.1.** Let \( L \) be a field. The Kummer sequence and the natural covering \( \text{Spec} \ L \to \mathbb{B}\mu_n \) give rise to a canonical isomorphism
\[
\mathbb{H}^2(\mathbb{B}\mu_n, \mu_n) \cong \mathbb{Z}/n\mathbb{Z} \oplus \text{Br}(L)[n] \oplus H^1(\text{Spec} \ L, \mu_n) = \text{Pic}(\mathbb{B}\mu_n) \oplus \text{Br}(\mathbb{B}\mu_n)
\]
in which projection to the first factor corresponds to base change to \( \overline{L} \) and inclusion of the first factor corresponds to the first Chern class map \( \text{Pic}(\mathbb{B}\mu_n) \to \mathbb{H}^2(\mathbb{B}\mu_n, \mu_n) \). This isomorphism is compatible with base extension.

**Proof.** This follows readily from Proposition 4.1. \( \square \)

**Remark 4.2.2.** Let us make explicit the boundary map \( \delta : \mathbb{H}^1(\mathbb{B}\mu_n, \text{PGL}_n) \to \mathbb{H}^2(\mathbb{B}\mu_n, \mu_n) \) when the base field is separably closed (and \( n \) is invertible). Fix an isomorphism \( \mathbb{Z}/n\mathbb{Z} \cong \mu_n \). The cohomology of \( \mathbb{B}\mu_n \) is then simply the group cohomology of \( \mathbb{Z}/n\mathbb{Z} \). Given a homomorphism \( \mathbb{Z}/n\mathbb{Z} \to \text{PGL}_n \) (corresponding to a class \( \varphi' \in \mathbb{H}^1(\mathbb{B}\mu_n, \text{PGL}_n) \)), choose a lift \( \alpha \) of \([1]\) to \( \text{SL}_n \) (which is possible as \( L \) is separably closed). Concretely, given a \( \mathbb{Z}/n\mathbb{Z} \)-equivariant structure on \( M_n(L) \), choose a matrix \( A \) with \( A^n = 1 \) such that conjugation by \( A \) defines the action of \([1]\) on \( M_n(L) \). Now, \( \det A \) is an \( n \)-th root of 1, say \( \zeta \). Setting \( \alpha \) equal to \( \zeta^{-1/n} A \) yields a lift of \([1]\) to \( \text{SL}_n \). We see that \( \alpha^n \) equals scalar multiplication by \( \zeta \), which is identified with an element of \( \mathbb{Z}/n\mathbb{Z} \) via the chosen isomorphism. This element is precisely \( \delta([\varphi']) \).

4.2.3. When \( n \) is prime and the base field contains a primitive \( n \)-th root of unity, then we can explicitly describe the image of the map
\[
\mathbb{H}^1(\mathbb{B}\mu_n, \text{PGL}_n)_{\text{triv}} \to \mathbb{H}^2(\mathbb{B}\mu_n, \mu_n)
\]
where \( \mathbb{H}^1(\mathbb{B}\mu_n, \text{PGL}_n)_{\text{triv}} \) consists of elements whose associated Brauer class is trivial when pulled back along \( \text{Spec} \ L \to \mathbb{B}\mu_n \). (More generally, such a description is available for \( \mathbb{BZ}/n\mathbb{Z} \) rather than \( \mathbb{B}\mu_n \).) In order to avoid confusion, we fix a prime \( \ell \), invertible in \( L \), and assume that \( L \) contains a primitive \( \ell \)-th root of unity. Via a choice of primitive root, we get an isomorphism \( \mathbb{Z}/\ell\mathbb{Z} \cong \mu_\ell \), which we implicitly use in what follows.

**Definition 4.2.4.** Let \( L \) be a field. Given a \( G_m \)-gerbe \( \pi : \mathcal{G} \to \mathbb{B}\mu_n \), a locally free \( \mathcal{G} \)-twisted sheaf \( \mathcal{W} \) is regular if for some (and hence any) section \( \tau : \mathbb{B}\mu_n \otimes L \to \mathcal{G} \otimes L \) of \( \pi \otimes L \), the pullback \( \tau^* \mathcal{W} \) is the sheaf associated to the regular representation of \( \mu_n \). A locally free \( \mathcal{G} \)-twisted sheaf \( \mathcal{W} \) is totally regular if it has the form \( \mathcal{W} \cong \mathbb{B} \) for some regular locally free \( \mathcal{G} \)-twisted sheaf \( \mathcal{W} \).

A twisted sheaf on a \( \mu_n \)-gerbe over \( \mathbb{B}\mu_n \) will be called regular if the pushforward to the associated \( G_m \)-gerbe is regular.

**Lemma 4.2.5.** Suppose \( \ell \) is prime and invertible in \( L \) and \( \pi : \mathcal{G} \to \mathbb{B}\mu_\ell \) is a non-trivial \( G_m \)-gerbe whose pullback to \( \text{Spec} \ L \) by the natural map \( \text{Spec} \ L \to \mathbb{B}\mu_\ell \) is trivial. If \( \mathcal{V} \) is a \( \mathcal{G} \)-twisted sheaf of rank \( \ell \), then \( \mathcal{V} \) is regular.

**Proof.** Fix an isomorphism \( \mathbb{Z}/\ell\mathbb{Z} \cong \mu_\ell \). Using this identification and the triviality of \( \mathcal{G} \mid \ell \), if we given \( \mathcal{V} \), the algebra \( \pi_* : \delta \mathcal{V} \mid \ell \) is identified with a \( \mathbb{Z}/\ell\mathbb{Z} \)-equivariant structure on \( M_\ell(L) \), which is in turn identified with an element of order \( \ell \) in \( \text{PGL}_\ell \). This can be lifted to a matrix \( A \in \text{GL}_\ell(L) \) such that \( A^\ell = b \) for some scalar \( b \in \mathbb{L}^\times \) with non-zero image in \( L^\times /\left(L^\times \right)^\ell \). Extending scalars to \( L' := L(b^{1/\ell}) \), we see that we can detect regularity by looking at the eigenspace decomposition of \( A' := b^{-1/\ell} A \): we have that \( (A')^\ell = 1 \), so that \( (L')^\ell \) corresponds to a representation of \( \mu_\ell \). This is simply the pullback of \( \mathcal{V} \).
along a section of $\mathcal{G} \to B\mu_ℓ$; the different choices of the $ℓ$th root of $b$ correspond to the different choices of sections of the gerbe and have no effect on regularity, as they only permute the eigenspaces.

By elementary Galois theory, we know that the action of $A$ on the vector space $V = L^ℓ$ has the property that $V \otimes L^{sep}$ breaks up as a direct sum of distinct one-dimensional eigenspaces for the action of $A \otimes L^{sep}$. But these are the same as the eigenspaces for $A' \otimes L, L^{sep}$. This implies that $(L')^ℓ$ is the regular representation of $μ_ℓ$, as desired.

\[ \square \]

**Corollary 4.2.6.** Suppose $L$ is finite. Let $\mathcal{G} \to B\mu_ℓ$ be a $μ_ℓ$-gerbe with cohomology class $(a, 0, β ≠ 0)$. Any two $\mathcal{G}$-twisted sheaves of rank $ℓ$ are isomorphic.

**Proof.** Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two such sheaves. By Lemma 4.2.5 both $\mathcal{G}_1$ and $\mathcal{G}_2$ are regular, so that the scheme $\text{Isom}(\mathcal{G}_1, \mathcal{G}_2)$ is a torsor under a torus. By Lang’s theorem, it has a rational point.

\[ \square \]

**Proposition 4.2.7.** Let $\mathcal{A}$ be an Azumaya algebra of degree $ℓ$ over $B\mu_ℓ$. Suppose that the Brauer class of $\mathcal{A}$ is non-trivial but that $\mathcal{A}|_{\text{Spec} L} = 0$. Via the isomorphism of Proposition 4.2.1, the class of $\mathcal{A}$ in $H^2(B\mu_ℓ, μ_ℓ)$ has the form

1. $(1, 0, α)$ for some $α ≠ 0$ if $ℓ = 2$;
2. $(0, 0, α)$ for some $α ≠ 0$ if $ℓ$ is odd.

Thus, given a $μ_ℓ$-gerbe $\mathcal{G} \to B\mu_ℓ$ with cohomology class $(a, 0, α ≠ 0)$, every locally free $\mathcal{G}$-twisted sheaf of rank $ℓ$ has determinant $a - δℓ ∈ Z/ℓZ$, where $δ$ is the Kronecker delta function.

**Proof.** Let $\mathcal{G} \to B\mu_ℓ$ be the $G_m$-gerbe of trivializations of $\mathcal{A}$. We have that $\mathcal{A}|_{\mathcal{G}} = \mathcal{G}(\mathcal{G})$ for some locally free $\mathcal{G}$-twisted sheaf of rank $ℓ$. By Lemma 4.2.5 we know that $\mathcal{G}$ is regular. If we represent $\mathcal{G}$ by an element $\overline{\mathcal{G}}$ of order $ℓ$ in $\text{PGL}_ℓ$, then the regularity implies that any lift $A'$ of $\overline{\mathcal{G}}$ in $\text{GL}_ℓ \otimes L^{sep}$ such that $A' = 1$ gives the regular representation of $Z/ℓZ$ on $(L^{sep})^ℓ$, and thus has trivial determinant if $ℓ$ is odd, or determinant $[1]$ if $ℓ = 2$.

\[ \square \]

### 4.3. The Brauer group of $B(μ_ℓ × μ_ℓ)$ and Saltman’s meteorology.

In this section, we compute the Brauer group of the classifying stack $B(μ_ℓ × μ_ℓ)$ and compare the results to Saltman’s recent “meteorology” of crossings point in the ramification divisor of Brauer classes on surfaces $[33]$.

Let $L$ be a field containing a chosen primitive $ℓ$th root of unity; we will use this to identify $μ_ℓ$ with $Z/ℓZ$ in what follows (although we will keep the notation canonical). Write $\text{Br}_0(B(μ_ℓ × μ_ℓ))$ for the kernel of the natural map $\text{Br}(B(μ_ℓ × μ_ℓ)) → \text{Br}(L)$ induced by pullback to the point $u : \text{Spec} L → B(μ_ℓ × μ_ℓ)$. The structural morphism of $B(μ_ℓ × μ_ℓ)$ and the point $u$ induce a natural splitting $\text{Br}(B(μ_ℓ × μ_ℓ)) ≃ \text{Br}(L) ⊕ \text{Br}_0(B(μ_ℓ × μ_ℓ))$.

**Lemma 4.3.1.** The group $\text{Br}_0(B(μ_ℓ × μ_ℓ))$ classifies group scheme extensions

\[ 1 → G_m → G → μ_ℓ × μ_ℓ → 1. \]

In particular, the functor sending the listed extension to the 1-morphism $BG → B(μ_ℓ × μ_ℓ)$ together with the inertial trivialization $G_m → \mathcal{J}(BG/\mathcal{G}(μ_ℓ × μ_ℓ))$ defines a bijection between the set of isomorphism classes of extensions and the set of isomorphism classes of $G_m$-gerbes with Brauer classes in $\text{Br}_0(B(μ_ℓ × μ_ℓ))$.

**Proof.** It is standard that the morphism $BG → B(μ_ℓ × μ_ℓ)$ is a $G_m$-gerbe. Pulling back the associated Brauer class to $L$ is equivalent to pulling back the group extension along the map $B1 → B(μ_ℓ × μ_ℓ)$, where $1$ is the singleton group scheme. But this pullback is just $G_m$, resulting in the trivial $G_m$-gerbe $BG_m$ over $L$. Thus, the gerbe $BG → B(μ_ℓ × μ_ℓ)$ has Brauer class in $\text{Br}_0(B(μ_ℓ × μ_ℓ))$, as claimed.

On the other hand, if $\mathcal{G} → B(μ_ℓ × μ_ℓ)$ is a $G_m$-gerbe representing a class in $\text{Br}_0(B(μ_ℓ × μ_ℓ))$, then we know that (1) $\mathcal{G} → \text{Spec} L$ is a gerbe (as it is a composition of gerbes), and (2) there is a map $\text{Spec} L → \mathcal{G}$ over $L$. Thus, $\mathcal{G} ≃ BG$ for some group scheme $G$ of finite type over $L$. Moreover, since $\mathcal{G}$ is a $G_m$-gerbe over $B(μ_ℓ × μ_ℓ)$, we get a natural expression of $G$ as an extension

\[ 1 → G_m → G → μ_ℓ × μ_ℓ → 1 \]

(using the inertial trivialization $G_m ≃ \mathcal{J}(\mathcal{G})$).
To see that the extensions in question are central, note that we can identify sheaves on \( \mathcal{G} \) with representations of \( G \). In particular, a locally free \( \mathcal{G} \)-twisted sheaf of (positive) rank \( N \) yields a morphism of sequences

\[
\begin{array}{cccc}
1 & \rightarrow & G_m & \rightarrow \mu_\ell \times \mu_\ell & \rightarrow 1 \\
\downarrow & & \downarrow \mathrm{id} & & \\
1 & \rightarrow & G_m & \rightarrow \GL_N & \rightarrow \PGL_N & \rightarrow 1.
\end{array}
\]

Tensoring any locally free \( \mathcal{G} \)-twisted sheaf of positive rank with the pullback of the sheaf on \( \mathcal{B}(\mu_\ell \times \mu_\ell) \) corresponding to the regular representation of \( \mu_\ell \times \mu_\ell \), we can find such a diagram in which the right vertical arrow is injective. (This corresponds to tensoring a \( \mu_\ell \times \mu_\ell \)-equivariant Azumaya algebra with the sheaf of endomorphisms of the regular representation, which is itself a power of the regular representation and thus faithful.) It follows that the middle vertical arrow is injective and then that the extension is central.

Using the isomorphism \( \mu_\ell \times \mu_\ell \cong \mathbb{Z}/\ell \times \mathbb{Z}/\ell \mathbb{Z} \) determined by the chosen primitive \( \ell \)th root of unity, a projective representation \( \mu_\ell \times \mu_\ell \rightarrow \PGL_N \) is determined by two elements \( X \) and \( Y \) in \( \GL_N(L) \) such that \( X^\ell \in L^\times, Y^\ell \in L^\times \) and \([X,Y] = \gamma \in \mu_\ell(L)\). Since \( X \) and \( Y \) are determined up to elements of \( G_m(L) \), the triple \((X^\ell, Y^\ell, \gamma)\) yields an element of \( L^\times/\{\ell\}^\times \times L^\times/\{\ell\}^\times \times \mu_\ell \), and it is clear that this is an invariant of the isomorphism class of the underlying extension \( 1 \rightarrow G_m \rightarrow G \rightarrow \mu_\ell \times \mu_\ell \rightarrow 1 \), yielding a set-theoretic morphism

\[i : \Br_0(\mathcal{B}(\mu_\ell \times \mu_\ell)) \rightarrow (L^\times/\{\ell\}^\times)^2 \times \mu_\ell(L).\]

**Proposition 4.3.2.** The map \( i \) is an isomorphism of abelian groups.

**Proof:** We first show that \( i \) is a map of groups. Recall that the group law on \( \Br_0 \) is induced by tensor product of \( \mu_\ell \times \mu_\ell \)-equivariant matrix algebras. Given two actions \( \varphi_1, \varphi_2 \) of \( \mu_\ell \times \mu_\ell \) on \( M_N(L) \) and \( M_N(L) \), we compute the sum in \( \Br_0 \), by taking the tensor product representation \( M_N(L) \otimes M_N(L) \). The elements \( X \) and \( Y \) then act diagonally, and we easily see that \( i(\varphi_1 \otimes \varphi_2) = i(\varphi_1)i(\varphi_2) \), as desired.

We now claim that \( i \) is surjective. Let \((A,B,\gamma)\) be a triple. Consider the (possibly commutative!) \( L \)-algebra \((A,B,\gamma)\) with generators \( x \) and \( y \) subject to the relations \( x^\ell = A, y^\ell = B, \) and \( xy = \gamma yx \). The elements \( x \) and \( y \) are units, and their action on \((A,B,\gamma)\) by conjugation gives a projective representation \( \varphi \) of \( \mu_\ell \times \mu_\ell \) (by looking at the induced action on the endomorphism algebra of the vector space underlying \((A,B,\gamma)\)) such that \( i(\varphi) = (A,B,\gamma) \).

Finally, we claim that \( i \) is injective. Suppose given an action of \( \mu_\ell \times \mu_\ell \) on \( M_N(L) \) whose invariants are \((1,1,1)\), we get commuting elements \( X,Y \in \GL_N(L) \) such that \( X^\ell = \text{id} \) and \( Y^\ell = \text{id} \). This gives a representation \( \mu_\ell \times \mu_\ell \rightarrow \GL_N \) whose image in \( \PGL_N \) is the given action, which shows that the equivariant matrix algebra is the endomorphisms of an equivariant vector space, thus trivializing the Brauer class.

**Corollary 4.3.3.** Every class in \( \Br_0(\mathcal{B}(\mu_\ell \times \mu_\ell)) \) has period \( \ell \).

**4.3.4.** Of utmost importance will be various indices of the classes in \( \Br_0 \). The natural point \( \Spec L \rightarrow \mathcal{B} \mu_\ell \) yields two maps \( r_i : \mathcal{B} \mu_\ell \rightarrow \mathcal{B}(\mu_\ell \times \mu_\ell), i = 1, 2 \), which we will call the restriction maps. Throughout this section, we fix a \( G_m \)-gerbe \( \mathcal{G} \rightarrow \mathcal{B}(\mu_\ell \times \mu_\ell) \) in \( \Br_0(\mathcal{B}(\mu_\ell \times \mu_\ell)) \) with invariant \( i(\mathcal{G}) = (A,B,\gamma) \). We will abusively write \( r_i : \mathcal{G} \rightarrow \mathcal{G}_{G\mu_\ell} \) for the map induced by \( r_i \); these maps pull back twisted sheaves to twisted sheaves.

**Definition 4.3.5.** A \( \mathcal{G} \)-twisted sheaf \( V \) is **totally biregular** if \( r_i^* V \) is totally regular for \( i = 1, 2 \).

**Definition 4.3.6.** The minimal rank of a non-zero totally biregular \( \mathcal{G} \)-twisted sheaf will be called the **global index**.

**Proposition 4.3.7.** Suppose the field \( L \) is finite.

1. If \( \gamma \neq 1 \) then both the index and global index of \( \mathcal{G} \) are \( \ell \).
2. If \( \gamma = 1 \) but \( A \neq 1 \) and \( B \neq 1 \) then both the index and global index of \( \mathcal{G} \) are \( \ell \).
3. If \( \gamma = 1 \) and \( A = 1 \) but \( B \neq 1 \), then the index of \( \mathcal{G} \) is \( \ell \), while the global index of \( \mathcal{G} \) is \( \ell^2 \).
Proof. For the first statement, note that since $\gamma \neq 0$, the cyclic algebra $(A, B)_\gamma$ is a central simple algebra of index $\ell$. Since $L$ is finite, there is an isomorphism $(A, B)_\gamma \cong M_\ell(L)$, thus yielding a projective representation $\varphi : \mu_\ell \times \mu_\ell \to \operatorname{PGL}_\ell$ with $i(\varphi) = (A, B, \gamma)$. This shows that the index is $\ell$; in fact, it realizes the given $\mu_\ell \times \mu_\ell$-action on $M_\ell(L)$ as the endomorphisms of some $\mathcal{G}$-twisted sheaf $V$ of rank $\ell$. Moreover, the restriction of the Brauer class via the $r_i$ is given by restricting $\varphi$ to each factor $\mu_\ell$. To see that the sheaf $V$ is totally biregular, we can use the explicit description of $(A, B, \gamma)$, as follows. Extending scalars to $\mathbb{L}$ and replacing $x$ by $x/A^{1/\ell}$ and $y$ by $y/B^{1/\ell}$ yields an isomorphism $(A, B)_\gamma \otimes \mathbb{L} \cong (1, 1)_\gamma \otimes \mathbb{L}$ of $\mu_\ell \times \mu_\ell$-equivariant $\mathbb{L}$-algebras which have the same projective representation. On the other hand, we can realize $(1, 1)_\gamma$ explicitly using the elements

$$X = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \gamma & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma^{\ell-1} \end{pmatrix}$$

and

$$Y = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

from which it is obvious that each of $X$ and $Y$ acts on $L^\ell$ as the regular representation of $\mu_\ell$. This shows that the twisted sheaf $V$ is totally biregular, as desired.

To prove the second statement, note that since both $A$ and $B$ are non-zero, they generate $L^\times/(L^\times)^{\ell} \cong \mathbb{Z}/\ell\mathbb{Z}$, so that there is some $n$ relatively prime to $\ell$ with $A^n = B \in L^\times/(L^\times)^{\ell}$. Consider the field extension $L' := L[x]/(x^\ell - A)$. Scalar multiplication by $x$ and by $x^n$ yield an $L$-linear action on $L'$ corresponding to a map $G \to \operatorname{GL}_L$; the image in $\operatorname{PGL}_L$ will be a projective representation with invariant $(A, B, 1)$. To check biregularity we can again work over $\mathbb{L}$ and replace $x$ by $x/A^{1/\ell}$, whence we see that the resulting map $\mu_\ell \to \operatorname{GL}_L$ is now the regular representation. Thus, the global index is $\ell$, and this implies that the index is $\ell$.

To see the last statement, note that the algebra $L[x, y]/(x^\ell - 1, y^\ell - B)$ admits an action as in the previous paragraph, yielding a $\mathcal{G}$-twisted sheaf of rank $\ell^2$ that is totally biregular. On the other hand, since the action of the first factor lifts to an element of $\operatorname{GL}_{L^\ell}$, the restriction of $\mathcal{G}$ via $r_1$ is trivial; since $\mathcal{B} \mu_\ell \to \mathcal{B} \mu_\ell \times \mathcal{B} \mu_\ell$ has degree $\ell$, this shows that $\mathcal{G}$ has index $\ell$. If the global index were $\ell$, there would be a pair of commuting elements $X$ and $Y$ in $\operatorname{GL}_L(L)$ such that $X^\ell = 1$ and $Y^\ell = B$. By the double centralizer theorem (applied to the subalgebra generated by $Y), X$ would have to lie in the subalgebra of $\operatorname{M}_L(L)$ generated by $Y$ (i.e., this algebra is its own centralizer). But the only $\ell$th roots of unity in this subalgebra are those in $L$, which means that the action of $X$ is not (geometrically) conjugate to the regular representation. \hfill \Box

Corollary 4.3.8. Suppose the triple of $\mathcal{G}$ is $(A, B, 1)$. Then there is a locally free $\mathcal{G}$-twisted sheaf $V$ of rank $\ell^2$ such that for any invertible $\mathcal{G} \otimes \mathcal{L}$-twisted sheaf $\Lambda$, the sheaf $V_\mathcal{L} \otimes \Lambda^\gamma$ is the sheaf on $\mathcal{B}(\mu_\ell \times \mu_\ell) \subset \mathcal{G}$ associated to the regular representation of $\mu_\ell \times \mu_\ell$.

Proof. Since $\gamma = 1$, we see that the cohomology class of $\mathcal{G}$ has the form $\text{pr}^\gamma_1 \alpha + \text{pr}^\gamma_2 \beta$, where $\alpha$ is the class in $\operatorname{Br}(\mathcal{B} \mu_\ell)$ associated to $A$ and $\beta$ is the class associated to $B$. Letting $W_1$ be a regular $\alpha$-twisted sheaf and $W_2$ a regular $\beta$-twisted sheaf, we see that $\text{pr}^\gamma_1 W_1 \otimes \text{pr}^\gamma_2 W_2$ is a $\mathcal{G}$-twisted sheaf with the requisite properties. \hfill \Box

4.3.9. The reader familiar with [33] might notice a resemblance between Proposition 4.3.7 and Saltman’s “meteorology” of points on ramification divisors for Brauer classes on surfaces. In this paragraph we will explain the connection (at least in the special case in which we have proved it; it is easy to extend the method above to the general case and achieve total parity with Saltman’s setup). We will be content to simply sketch the correspondence, as it is a digression from our primary purpose here.
Let us briefly remind the reader of Saltman’s terminology. Suppose $R$ is a regular local ring of dimension 2 with residue field $\kappa$, and suppose $s$ and $t$ are generators of the maximal ideal of $R$. For any domain $B$ arising in the following, we will write $K(B)$ for its fraction field.

Given a Brauer class $\alpha$ over $K$ of period $\ell$ invertible in $R$ which is ramified only over the (transversely intersecting) divisors cut out by $s$ and $t$, the ramification theory of Brauer classes (reviewed in Section 3 below) produces two ramification extensions $L_1/K(R/sR)$ and $L_2/K(R/tR)$ of degree $\ell$ which have the same order of ramification over the closed point $\Spec \kappa$ lying in both $\Spec R/sR$ and $\Spec R/tR$.

There are three possibilities for the configuration of $L_1$ and $L_2$ (up to the symmetry of $s$ and $t$ in the notation):

1. both $L_1$ and $L_2$ ramify over $\Spec \kappa$;
2. both $L_1$ and $L_2$ are unramified over $\Spec \kappa$ and are non-trivial;
3. $L_1$ is trivial and $L_2$ is non-trivial and unramified over $\Spec \kappa$.

In Saltman’s terminology, the first case is called “cold,” the second case is called “chilly”, and the third is called “hot”.

To connect this with the Brauer group of $\mathbb{B}(\mu_\ell \times \mu_\ell)$, we give a preview of results which will be fully explained in Section 3 (The reader may find it easier to read this after reading that section.) Let $\mathcal{X} \to \Spec R$ be the stack arising by taking the $\ell$th roots of the divisors $\Spec R/sR$ and $\Spec R/tR$ in $\Spec R$ (see Section 3.2 for definitions). By 3.2.1, the class $\alpha$ extends to a class in $\Br(\mathcal{X})$. In fact, we have an isomorphism between $\Br(\Spec R \setminus (\Spec R/sR \cup \Spec R/tR))[\ell]$ and $\Br(\mathcal{X})[\ell]$. Now $\mathcal{X}$ contains $\mathbb{B}(\mu_\ell \times \mu_\ell)$ as a closed substack (the reduced stack structure on the preimage of $\Spec \kappa$ under the natural map $\mathcal{X} \to \Spec R$). Reduction defines a morphism $\Br(\mathcal{X})[\ell] \simeq \Br(\mathbb{B}(\mu_\ell \times \mu_\ell))[\ell]$; when the residue field $\kappa$ is finite, this finally results in a morphism

$$\Br(\Spec R \setminus (\Spec R/sR \cup \Spec R/tR))[\ell] \simeq \Br(\mathcal{X})[\ell] \to \Br(\mathbb{B}(\mu_\ell \times \mu_\ell)) = Br_0(\mathbb{B}(\mu_\ell \times \mu_\ell)).$$

If $R$ is complete, the composition is in fact an isomorphism and $\Br(\mathcal{X})$ is itself $\ell$-torsion (so the left side becomes $\Br(\mathcal{X})$).

Via this map, the three ramification situations described above (cold, chilly, and hot) correspond precisely to the three types of Brauer classes described in Proposition 4.3.7. It is interesting to note that ramification information for classes over the complicated field $K$ gets translated into information about Brauer classes over the classifying stack $\mathbb{B}(\mu_\ell \times \mu_\ell)$ that only sees the finite residue field. The techniques explained here are a demonstration that the geometry of stacks makes this connection less silly than it might at first seem, as one can import flexible global geometric methods into ramified situations.

5. Residual curves and moduli of uniform twisted sheaves

Let $k$ be a field. Fix an odd prime $\ell$.

**Definition 5.1.** A stack $\mathcal{C} \to k$ is a residual curve if

1. $\mathcal{C}$ is a connected tame Deligne-Mumford stack of dimension 1 whose coarse moduli space $C$ is a simple normal crossing;
2. if $C_1, \ldots, C_r$ are the (smooth) irreducible components of $C$, then there are Zariski-locally trivial $\mu_\ell$-gerbes $\mathcal{C}_i' \to C_i$ such that $\mathcal{C} = \mathcal{C}_1' \times_C \mathcal{C}_2' \times_C \cdots \times_C \mathcal{C}_r'$.

The curves $\mathcal{C}_i := \mathcal{C} \times_C C_i$ are then smooth Deligne-Mumford stacks of dimension 1 with generic stabilizer $\mu_\ell$, and a finite closed substack over which the stabilizer is $\mu_\ell \times \mu_\ell$. We will write $\mathcal{C}^o$ for $\mathcal{C} \setminus \cup_{i \neq j} \mathcal{C}_j$; it is the intersection of the smooth locus of $\mathcal{C}$ with $\mathcal{C}$, and is a $\mu_\ell$-gerbe over a the dense open subscheme $C_i \cap C^o \subset C_i$.

The way such curves usually arise is as the stacky loci of the root construction [4] applied to a simple normal crossing divisor in a smooth surface, or to a simple normal crossing divisor in a fiber of an arithmetic surface. That is certainly how they will arise for us. In this section we fix a residual curve $\mathcal{C}$, with irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_r$. Write $\kappa_i$ for $\Gamma(\mathcal{O}_{\mathcal{C}_i}, \mathcal{O}_{\mathcal{C}_i})$; we thus have that $\mathcal{C}_i$ is a geometrically integral smooth curve over $\kappa_i$.

Fix a $\mu_\ell$-gerbe $\mathcal{G} \to \mathcal{C}$. Write $\mathcal{G}_i := \mathcal{G} \times_{\mathcal{C}} \mathcal{C}_i$. We will write $\pi$ for the natural map $\mathcal{G} \to \mathcal{C}$ and, by abuse of notation, for the natural map $\mathcal{G}_i \to \mathcal{C}_i$. Finally, we will write $\mathcal{G}_i^o$ for $\mathcal{G}_i \times_{\mathcal{C}_i} \mathcal{C}_i^o$. 


We assume that for each geometric point \( c \to C^{sm} \), the restriction \( \mathcal{G} \otimes \kappa(c) \) is trivial in \( H^2(\mathcal{G} \otimes \kappa(c), \mu_\ell) \). The following lemma shows that this is a harmless assumption if one is concerned only with the Brauer class of \( [\mathcal{G}] \).

**Lemma 5.2.** Given a class \( \alpha \in Br(\mathcal{C})[\ell] \), there is a \( \mu_\ell \)-gerbe \( \mathcal{G} \to \mathcal{C} \) such that for every geometric point \( c \to C \) the restriction \( \mathcal{G}|_{\mathcal{C} \otimes \kappa(c)} \) has trivial class in \( H^2(\mathcal{C} \otimes \kappa(c), \mu_\ell) \).

**Proof.** Given any collection of characters \( \chi^{i_1}_i \) of the generic geometric stabilizers \( B_{\mu_\ell} \) on \( \mathcal{C}^\circ \), there is an invertible sheaf \( L \) on \( \mathcal{C} \) such that for any geometric residual gerbe \( x : B_{\mu_\ell} \to \mathcal{C}^\circ \) the character associated to \( x^*L \) is \( \chi^{i_1}_i \). Indeed, the existence of an invertible sheaf \( L_i \) on \( \mathcal{C}_i \) with character \( \chi^{i_1}_i \) follows from the assumption that \( \mathcal{C}^\circ \to \mathcal{C}^\circ \) is a Zariski-locally trivial \( \mu_\ell \)-gerbe. A local calculation then shows that the invertible sheaf \( L_i := L_i(-\sum_{j \neq i} t_j \mathcal{C}_j \cap \mathcal{C}_i) \) has the property that at a residual gerbe \( \xi = B_{\mu_\ell} \times B_{\mu_\ell} \hookrightarrow \mathcal{C}_i \times \mathcal{C}_j \), we have \( L_i|_{\xi} \) as character \( \chi^{i_1}_i \otimes \chi^{i_2}_j \). Gluing the \( L_i \) together yields \( L \), as desired.

Now let \( c_i \to \mathcal{C}^\circ \) be a set of geometric points, and let the image of \( \mathcal{G} \otimes \kappa(c_i) \) be \( n_i \in H^2(\mathcal{C}_i \otimes \kappa(c), \mu_\ell) \). Let \( L \) be the invertible sheaf as in the previous paragraph with characters \( \chi^{n_1}_i \). Replacing \( \mathcal{G} \) by a \( \mu_\ell \)-gerbe representing the class \( [\mathcal{G}] - [L]^i/\ell \) (where \( [L]^i/\ell \) denotes the image of \( L \) via the Kummer map \( H^1(\mathcal{C}_i, G_m) \to H^2(\mathcal{C}_i, \mu_\ell) \)) yields a gerbe \( \mathcal{G}' \) with the same Brauer class and whose cohomology class vanishes at all geometric points in \( C^{sm} \), as desired.

In this section, we will describe certain stacks of \( \mathcal{G} \)-twisted sheaves.

**Definition 5.3.** Given an element \( a \in \mathbb{Z}/\ell \mathbb{Z} \), a sheaf \( \mathcal{F} \) on \( \mathcal{C}^\circ \) will be called isotypic of type \( a \) if the geometric generic fiber \( \mathcal{F} \otimes \kappa(C_t) \) has the form \( (\chi^a)^{\otimes N} \) for some \( N \) as sheaves on \( B_{\mu_\ell} \otimes \kappa(C_t) \), where \( \chi : \mu_\ell \to G_m \) is the natural inclusion character.

**Definition 5.4.** Given a locally free \( \mathcal{G}_i \)-twisted sheaf \( V_i \), an eigendecomposition of \( V \) is a direct decomposition \( V = V_0 \oplus V_1 \oplus \cdots \oplus V_{s-1} \) such that for all \( s \) and \( i \), the sheaf \( \pi_* \mathcal{H}om(V_s, V_i) \mathcal{C}^\circ \) is isotypic of type \( [\ell - s] \).

**Lemma 5.5.** Let \( V \) be a locally free \( \mathcal{G} \)-twisted sheaf. Choose an algebraic closure \( \overline{\mathbb{K}} \) of \( k \).

1. There is at most one eigendecomposition of \( V \otimes \overline{\mathbb{K}} \), up to isomorphism.
2. If \( V \otimes \overline{\mathbb{K}} \) admits an eigendecomposition, then the set of numbers \( \{\deg \mathcal{H}om(V_s, V_i)\} \) is independent of the choice of \( \overline{\mathbb{K}} \) and of eigendecomposition over \( \overline{\mathbb{K}} \).

**Proof.** We may assume \( k = \overline{\mathbb{K}} \). Suppose \( V = V_0 \oplus \cdots \oplus V_{s-1} = W_0 \oplus \cdots \oplus W_{s-1} \) are two eigendecompositions of \( V \). Write \( \overline{\mathbb{K}} \) for an algebraic closure of the function field of \( C_t \). Let \( M \) be an invertible \( \mathcal{G} \otimes \mathcal{K} \)-twisted sheaf. Restricting the two decompositions to the geometric generic point yields two eigendecompositions \( \oplus (V_i \otimes M') = \oplus (W_i \otimes M') \) of the \( \overline{\mathbb{K}} \)-linear \( \mu_\ell \)-representation \( V \otimes \overline{\mathbb{K}} \). It follows from the standard representation theory of \( \mu_\ell \) that there is a (cyclic) permutation \( i \to j(i) \) such that \( V_i \otimes \overline{\mathbb{K}} = W_{j(i)} \otimes \overline{\mathbb{K}} \). Re-indexing, we may assume that \( j(i) = i \). We conclude by faithful flatness that \( V_i \otimes K = W_i \otimes K \) as subsheaves of \( V \otimes K \).

Now consider the quotient \( V \to V_i \). For \( j \neq i \), the fact that \( V_j \otimes K = W_j \otimes K \) implies that the natural surjection \( \oplus W_j \to V_i \) factors through a surjection \( W_i \to V_i \). Since \( W_i \) and \( V_i \) are of finite rank, it follows that this surjection is an isomorphism.

The set of degrees \( \{\deg \mathcal{H}om(V_s, V_i)\} \) will be denoted \( d(V) \). The set \( d(V) \) always contains 0 and is symmetric with respect to negation. Similarly, the set \( r(V) := \{\text{rk} \mathcal{H}om(V_s, V_i)\} \) is independent of the geometric eigendecomposition.

**Definition 5.6.** A locally free \( \mathcal{G} \)-twisted sheaf \( V \) is **uniform of determinant \( \mathcal{O} \)** if

1. \( \det V \cong \mathcal{O} \);
2. for all \( i \), if \( V|_{\mathcal{G}_i} \) admits a geometric eigendecomposition then \( d(V) = \{0\} \) and \( r(V) \) is a singleton.

In particular, we have the following lemma.

**Lemma 5.7.** If \( V \) is a uniform \( \mathcal{G} \)-twisted sheaf of rank \( \ell^2 \) then
(1) for every residual gerbe $\xi : \mathcal{B} \mu_{\ell} \to \mathcal{G}$ in the smooth locus of $\mathcal{C}$, the restriction $V|\xi$ is totally regular;
(2) for each residual gerbe $\xi'$ of $\mathcal{C}$ of the form $\mathcal{B} \mu_{\ell} \times \mathcal{B} \mu_{\ell}$, the restriction $V|\xi'$ is totally biregular.

Proof. It suffices to prove this after passing to an algebraically closed base field, so we may assume that $\kappa$ is algebraically closed and therefore that $V|\xi$ has an eigendecomposition $V|\xi = V_0 + \cdots + V_{r-1}$.

Choosing an invertible $\mathcal{G}|\xi$-twisted sheaf $L$, the sheaf $V|\xi \otimes L^\vee$ is thus identified with a representation $W$ of $\mu_{\ell}$, admitting a direct sum decomposition $W = W_0 + \cdots + W_{r-1}$ such that $\mathcal{Hom}(W_s, W_t)$ is of constant rank and is isotypic of weight $t - s$ for all $s$ and $t$.

It follows that each $W_s$ is isotypic of a fixed rank $r$, from which it follows that $V|\xi \cong \rho^{\otimes r}$, giving the first statement. The second statement follows from the first: the two factors of the inertia stack $\mu_{\ell} \times \mu_{\ell}$ of $\xi'$ are limits of the inertia stacks $\mu_{\ell}$ on the two branches of $\mathcal{C}$ meeting at $\xi'$. Since restriction to a factor corresponds to deformation into the smooth locus of the corresponding branch, we see that total biregularity implies total regularity on the branches.

\section{5.1. Moduli of uniform twisted sheaves.} Write $\xi_1, \ldots, \xi_m$ for the set of singular residual gerbes of $\mathcal{C}$; thus, each $\xi_i$ is isomorphic to $\mathcal{B} \mu_{\ell} \times \mathcal{B} \mu_{\ell}$. Write $\mathcal{G}(i)$ for the restriction of $\mathcal{G}$ to $\xi_i$. By Corollary 4.3.8 for each $i$ there is a totally biregular $\mathcal{G}(i)$-twisted sheaf $V(i)$ which is geometrically isomorphic to a twist of the regular representation of $\mu_{\ell} \times \mu_{\ell}$. We fix such a choice for each $i = 1, \ldots, m$ in this section.

Definition 5.1.1. Let $\mathcal{U}$ denote the $k$-stack whose objects over $T \to \text{Spec} \ k$ are locally free $\mathcal{G}_T$-twisted sheaves $\mathcal{V}$ of rank $\ell^2$ together with an isomorphism $\det \mathcal{V} \cong \mathcal{O}$ such that for each geometric point $t \to T$,

1. the fiber $\mathcal{V}_t$ is uniform as a $\mathcal{G}_t$-twisted sheaf;
2. for each $i$, the restriction of $\mathcal{V}$ to $(\xi_i)_t$ is isomorphic to $V(i)_t$.

The primary goal of this section is to prove the following theorem.

Theorem 5.1.2. The stack $\mathcal{U}$ is a smooth Artin stack over $k$ with quasi-affine stabilizers. If $\ell$ is odd, $\mathcal{U}$ is geometrically connected.

The primary use of the theorem in this paper will be the following corollary.

Corollary 5.1.3. Suppose $\ell$ is odd. If $\mathcal{C}$ is a residual curve over a finite field $L$ and $\mathcal{G} \to \mathcal{C}$ is a $\mu_{\ell}$-gerbe, then for some $N > 1$ relatively prime to $\ell$, there is a locally free $\mathcal{G}_L$-twisted sheaf of rank $N\ell^2$ and trivial determinant.

Proof. Since $\mathcal{U}$ is integral with quasi-affine stabilizers, there is a dense open substack $\mathcal{V} \subset \mathcal{U}$ of the form $[P/\text{GL}_n]_\mathcal{G}$ with $P$ an algebraic space (Proposition 3.5.9 of [20]). Since $\mathcal{U}$ is geometrically integral, so is $\mathcal{V}$ and thus $P$. By the Lang-Weil estimates, $P$ has a point over an extension $L/k$ of degree prime to $\ell$. Taking the image of this point in $\mathcal{U}$ yields a locally free $\mathcal{G}_L \otimes_k L$-twisted sheaf of rank $\ell^2$ and trivial determinant. The proof follows from the following Lemma.

Lemma 5.1.4. Let $X$ be a proper stack over a field $k$ and $L/k$ a finite étale algebra of degree $d$. Write $\pi : X \otimes L \to X$ for the natural projection. If $V$ is a locally free sheaf of rank $r$ on $X \otimes L$ of determinant $\mathcal{O}$ then $\pi_*V$ is locally free on $X$ of rank $r$ and determinant $\mathcal{O}$.

Proof. To check that the determinant is trivial, it suffices to work over $k$ (as $X$ is proper), and thus we may assume that $\text{Spec} \ L = \bigcup_{i=1}^N \text{Spec} \ k$ is a finite product of copies of $k$. Now $\pi_*V = \otimes V|_{p_i}$, so that $\det \pi_*V = \otimes \det V|_{p_i}$. If $\det V \cong \mathcal{O}$, then for each $i$ we have $\det V|_{p_i} \cong \mathcal{O}$, giving the result.

5.1.5. Before proving Theorem 5.1.2 we first describe a geometric avatar of the ramification extensions. Consider the stack $\mathcal{P}_i := \mathcal{P}_{\mathcal{C}_{\ell}}^{(1)} \mathcal{G}_i$ parametrizing $C_{\ell}$-flat families of $\mathcal{G}_i$-twisted sheaves on $\mathcal{C}_{\ell}$.

Standard methods show that $\mathcal{P}_i \to C_{\ell}$ is a $G_m$-gerbe over a $R_i \to C_{\ell}$.

Proposition 5.1.6. With the preceding notation, there is a cyclic étale covering $R_i \to C_{\ell}$ such that $\mathcal{P}_i$ is a $G_m$-gerbe over $R_i$.

Intrinsically, $R_i$ is the rigidification of $\mathcal{P}_i$ with respect to $G_m$, in the notation of [1].
Proof. It is standard that \( \mathcal{P}_i \) is a \( \mathbb{G}_m \)-gerbe over an algebraic space \( R_i \). Moreover, elementary deformation theory shows that the relative Picard stack of \( C_i^\circ \) is smooth, so that \( R_i \to C_i^\circ \) is also smooth. On the other hand, the relative Picard stack of \( C_i^\circ \) over \( C_i^\circ \) is a \( \mathbb{G}_m \)-gerbe over the constant group scheme \( \mathbb{Z}/\ell \mathbb{Z} \). Tensoring defines an action \( \mathbb{Z}/\ell \mathbb{Z} \times R_i \to R_i \) over \( C_i^\circ \) which is simply transitive on geometric fibers. We conclude that \( R_i \to C_i^\circ \) is naturally a \( \mathbb{Z}/\ell \mathbb{Z} \)-torsor, as desired. \( \Box \)

Remark 5.1.7. It is not too difficult to see that \( R_i \) is precisely the ramification extension of \( \alpha \) at \( C_i \). The easiest proof goes by noting that they split one another.

There is an elementary criterion for the existence of an eigendecomposition.

**Proposition 5.1.8.** Given a field extension \( L/\kappa_i \), a non-zero locally free \( \mathcal{G}_i \)-twisted sheaf \( V \) has an eigendecomposition if and only if \( R_i \otimes_{\kappa_i} L \) is disconnected.

**Proof.** It suffices to assume \( \kappa_i = L \) and to work at the generic point \( \text{Spec} \, \kappa \) of \( C_i \). We first claim that there exists a \( V \) with an eigendecomposition if and only if the Brauer class of \( \mathcal{G} \) is pulled back from \( \text{Spec} \, \kappa \).

To see this, suppose \( V = V_0 \oplus \cdots \oplus V_{t-1} \) is an eigendecomposition. Then \( \mathcal{H}om(V_0, V_0) \) is an Azumaya algebra on \( 
\mathcal{B}\mu_{\ell,K} \) with the same Brauer class as \( \mathcal{G} \) and on which \( \mu_\ell \) acts trivially. It follows that the Brauer class of \( \mathcal{G} \) is pulled back from \( \text{Spec} \, \kappa \). On the other hand, if the class is a pullback, then let \( \lambda \) be an Azumaya algebra over \( \text{Spec} \, K \) representing \( [\mathcal{G}] \). The category of \( \mathcal{G} \)-twisted sheaves is naturally equivalent to the category of \( \mu_\ell \)-equivariant right \( \lambda \)-modules. But the representation theory of \( \mu_\ell \) on vector spaces over a division ring is the same as that of a field, so we see that any module admits an eigendecomposition.

Thus, to prove the Proposition it suffices to prove that \( R_i \) is split if and only if \( [\mathcal{G}] \) is pulled back from \( \text{Spec} \, \kappa \). Suppose \( R_i \) is split, and let \( s : \text{Spec} \, \kappa \to R_i \) be any section. By definition, the obstruction to lifting \( s \) to an invertible \( \mathcal{G} \)-twisted sheaf vanishes if and only if \( [\mathcal{G}] \) vanishes. This obstruction is thus (at the very least) an element of \( \text{Br}(K) \) whose pullback under the injective map \( \text{Br}(K) \hookrightarrow \text{Br}(\mathcal{B}\mu_{\ell,K}) \) generates the same cyclic subgroup as \( [\mathcal{G}] \). It follows that \( \mathcal{G} \) is a pullback. Conversely, assume that \( \mathcal{G} \) is a pullback from \( K \), say \( \mathcal{G} = \overline{\mathcal{G}} \times_{\text{Spec} \, K} \mathcal{B}\mu_{\ell,K} \). The \( \mathbb{G}_m \)-gerbe associated to \( \overline{\mathcal{G}} \) over \( \text{Spec} \, K \) is naturally isomorphic to the relative twisted Picard stack of \( \mathcal{G} \) over \( K \), so that \( \text{Pic}_{\overline{\mathcal{G}}/\text{Spec} \, K} = \text{Spec} \, K \). Given a \( K \)-scheme \( T \), it is easy to see any \( \mathcal{G}_T \)-twisted sheaf \( L \) has the form \( L \otimes \Lambda \) with \( \Lambda \) an invertible \( \mathcal{G}_T \)-twisted sheaf and \( \Lambda \) an invertible sheaf on \( \mathcal{B}\mu_{\ell,T} \). This shows that the natural map \( \text{Pic}_{\mathcal{B}\mu_{\ell}/\text{Spec} \, K} \times \text{Pic}_{\overline{\mathcal{G}}/\text{Spec} \, K} \to \text{Pic}_{\mathcal{G}/\text{Spec} \, K} \) is an isomorphism, which shows that \( R_i \) is split, as desired. \( \Box \)

### 5.1.9

We now prove Theorem 5.1.2 Since the statement is geometric, it suffices to work over the algebraic closure of the base field. Thus, **we will assume that the base field \( k \) is algebraically closed for the rest of this section.** Without loss of generality, we may assume that \( i = 1, \ldots, s \) are the indices such that the coverings \( R_i \to C_i^\circ \) are split. For each \( i = 1, \ldots, s \), we fix an invertible \( \mathcal{G}_i \)-twisted sheaf \( \Lambda_i \).

The idea of the proof is the following. Basic deformation theory shows that \( \mathcal{W} \) is smooth, so it suffices to show that it is connected. Let \( V \) and \( W \) be two objects of \( \mathcal{W} \) over \( k \). A general map \( V \to W(k) \) has as cokernel \( Q \) an invertible twisted sheaf supported on a finite étale subscheme of \( \mathcal{F}^\circ \). The uniformity conditions ensure that \( Q \) moves in an irreducible family \( I \). The vector bundle over \( I \) parametrizing extensions of \( Q \) by \( V \) then surjects onto \( \mathcal{W} \), showing that it is connected. Of course, this outline is hopelessly naïve, but there is a version of it that works, which we now describe.

**Definition 5.1.10.** Let \( \mathcal{C} \to \mathcal{G} \) be a finite flat covering by a scheme. The **\( \mathcal{C} \)-regularity of a \( \mathcal{G} \)-twisted sheaf \( \mathcal{F} \)** is the Castelnuovo-Mumford regularity of \( \mathcal{F}|_{\mathcal{C}} \) with respect to the polarization pulled back from \( \mathcal{C} \).

**Lemma 5.1.11.** Given a positive integer \( N \), there is a quasi-compact open substack \( \mathcal{W}_N \subset \mathcal{W} \) parametrizing uniform locally free \( \mathcal{G} \)-twisted sheaves of \( \mathcal{C} \)-regularity at most \( N \). Moreover, the natural inclusion \( \cup_N \mathcal{W}_N \hookrightarrow \mathcal{W} \) is an isomorphism.

**Proof.** By Proposition 3.2.2 of [25], the pullback morphism \( \mathcal{W} \to \text{Sh}(\mathcal{C}) \) is of finite presentation, where \( \text{Sh}(\mathcal{C}) \) denotes the stack of flat families of locally free sheaves on \( \mathcal{C} \). On the other hand, we know
that the open substack of $\mathcal{S}(\mathbb{C})$ parametrizing sheaves of regularity at most $N$ is quasi-compact. We conclude that $\mathcal{V}_N$ is quasi-compact, as desired.

Since $\mathcal{V}$ is the union of the quasi-compact open substacks $\mathcal{V}_N$, it suffices to prove that each $\mathcal{V}_N$ is connected. Thus, in what follows we will fix $N$ and write $\mathcal{V}$ in place of $\mathcal{V}_N$.

**Proposition 5.1.12.** There exists a positive integer $n$ such that for any two objects $V$ and $W$ in $\mathcal{V}_{\text{spec}k}$, a general map $V \to W(n)$ has cokernel $Q$ satisfying the following conditions.

1. The support $S := \text{Supp} Q$ is a finite reduced substack of $\mathcal{V}^0$ and $Q$ is identified with a $\mathcal{G}_S$-twisted invertible sheaf.
2. For any $i$ we have $|S \cap C_i| = \ell^n H \cdot C_i$.
3. For $i = 1, \ldots, s$, there is a partition $S \cap C_i = S_0 \bigcup \cdots \bigcup S_{s-1}$ such that for each $j$, $|S_j| = \ell^n H \cdot C_i$, and $Q|_{S_j} \cong \Lambda_{S_j} \otimes \chi^j$.

In other words, the cokernel of a general map of uniform sheaves is itself “uniformly distributed” with respect to the characters of $\mu_i$.

**Proof:** Using methods similar to those of Lemma 3.1.4.8ff of [24], for any $n$ we have an exact sequence of sets

$$(1) \quad \text{Hom}(V, W(n)) \longrightarrow \prod_i \text{Hom}(V|_{C_i}, W(n)|_{C_i}) \longrightarrow \prod_j \text{Hom}(V|_{\xi_j}, W(n)|_{\xi_j}).$$

For $n$ sufficiently large, we have that the composition $\text{Hom}(V, W(n)) \to \prod \text{Hom}(V|_{\xi_j}, W(n)|_{\xi_j})$ and the projections $\text{Hom}(V, W(n)) \to \text{Hom}(V|_{C_i}, W(n)|_{C_i})$ are surjective. Fixing identifications $V|_{\xi_j} \cong V(j)$ and $W|_{\xi_j} \cong W(j)$, we consider only those maps $V \to W(n)$ and $V|_{C_i} \to W(n)|_{C_i}$ which induce the identity on $V(j)$ for each $j$; these form linear subspaces which we will denote $H$ and $H_i$, respectively. For sufficiently large $n$, the natural maps $H \to H_i$ are surjective.

We first claim that it suffices to prove the statement for the restrictions $V_i$ and $W(n)_i$ to $C_i$. If this statement is proven for each $i$ then we can use Diagram (1) to glue such maps together. Since the conditions are clearly open, this will show that a general map $V \to W(n)$ has this property. Thus, we will restrict our attention to $C_i$ in what follows.

If $i = 1, \ldots, s$, then $V$ and $W$ have eigendecompositions, and maps between $V$ and $W(n)$ must preserve the eigensheaves, each of which has rank $\ell$. Thus, to show the result we may assume that for $i = 1, \ldots, s$ the sheaves $V$ and $W$ are generically isomorphic eigensheaves of degree $0$ and rank $\ell$. Suppose that $\Lambda_{\xi} \otimes W$ is isotypic of type $\chi^j$. Then any simple quotient of $W(n)$ supported on a reduced residual gerbe $x$ in $\mathcal{C}_i^\ell$ will be of the form $(\Lambda_{\xi})_x \otimes \chi^j$, and we see that to prove the result it suffices to prove that a general map between $V$ and $W(n)$ which induces an isomorphism at each $\xi_j$ has cokernel which is an invertible sheaf on a finite reduced substack of $C_i^\ell$.

Twisting down by $\Lambda_{\xi} \otimes \chi^j$, we reduce to proving the following statement. Let $C$ be a smooth projective curve over an algebraically closed field and let $D \subset C$ be a reduced effective Cartier divisor. Given two locally free sheaves $V$ and $W$ of rank $\ell$ on the root stack $C(D^{1/\ell})$ such that $\det V \cong \det W \cong \mathcal{O}_C$ and $V_D \cong W_D$, a general map $V \to W(n)$ with prescribed value over $D$ has cokernel an invertible sheaf supported on a finite reduced subscheme of $C \setminus D$.

Let $\Phi : V|_H \to W(n)|_H$ be the universal map on $C \times H$. We are interested in the locus over $H$ over which $\text{coker} \Phi$ is smooth over $C$ with fibers of rank at most $1$. This is clearly an open condition on $H$, and the complement of this locus forms a cone $B$. We will investigate the fiber dimension of this cone over $C$.

Given a point $c \in C$, we have that the restriction map

$$H \to \text{Hom}(V_{\text{Spec}(\mathcal{O}_{C,c}/m_c^2)}, W(n)_{\text{Spec}(\mathcal{O}_{C,c}/m_c^2)}) = M_{\ell}(\mathcal{O}_{C,c}/m_c^2)$$

is surjective. Writing a matrix as $A + \varepsilon B$, we have by the Jacobi formula that

$$\det(A + \varepsilon B) = \det A + \varepsilon \text{Tr}(\text{adj}(A)B),$$

where $\text{adj}$ denotes the classical adjoint (arising from cofactor matrices). The vanishing of this expression is clearly of codimension at least 3, so the fiber of $B$ over $c$ is of codimension at least 3 in $H$. This shows that $B$ has codimension at least 2 in $H$, whence $H$ contains a point $h$ such that $\text{coker} \Phi_h$ is smooth. To show that a general such point has fiber rank 1, it suffices by a similar argument to show
that the locus of element in \(M_t(k)\) in which the rank drops by at least 2 has codimension at least 2. This can be settled by a trick: a perturbation argument shows that the locus in question is the boundary of the locus of non-injective maps, which is itself of codimension 1. \(\square\)

For each \(i\), define a \(k\)-stack \(\mathcal{D}_i\) as follows. The objects of \(\mathcal{D}_i\) over \(T\) are pairs \((E, L)\) with \(E \subset C_i^2 \times T\) a closed subscheme which is finite étale over \(T\) of degree \(n^2 H \cdot C_i\) and \(L\) an invertible \(\mathcal{G}_E\)-twisted sheaf such that if \(i = 1, \ldots, s\), then there is a partition \(E = E_0 \sqcup \cdots \sqcup E_{t-1}\) with \(L|_{E_j} \otimes \Lambda^j_i\) isotypic of type \(\chi^j\). (If \(i > s\) there is no condition, and \(\mathcal{D}_i\) parametrizes pairs \((E, L)\) with \(L\) any invertible \(\mathcal{G}_E\)-twisted sheaf.)

**Proposition 5.1.13.** The stack \(\mathcal{D}_i\) is irreducible.

**Proof.** First suppose \(i = 1, \ldots, s\), so that \(R_i \to C_i^2\) is split. We claim that \(\mathcal{D}_i\) is a torus-gerbe over the dense open subscheme of \(\prod_{j=0}^{t-1} \left( \text{Sym}^{n^2 H \cdot C_i}, C_i^2 \right)\) parametrizing (unordered) tuples of pairwise distinct points. Indeed, given a pair \((E, L)\) with partition \(E = E_0 \sqcup \cdots \sqcup E_{t-1}\), the partition defines a point of the product of symmetric powers which lies in the locus parametrizing pairwise distinct points. Moreover, this point completely characterizes the partition of \(E\) (including the weight assigned to the restriction of \(E\) to each element of the partition). Thus, the fibers of this map are precisely given by stacks of line bundles on each \(E_i\). These are gerbes banded by tori of rank \(n^2 H \cdot C_i\), since each \(E_i\) consists of \(\ell\) points. Since the symmetric powers are irreducible, it follows that \(\mathcal{D}_i\) is irreducible.

Now suppose that \(i > s\). We claim that \(\mathcal{D}_i\) is a gerbe over a dense open subscheme of \(\text{Sym}^{n^2 H \cdot C_i}; R_i\) banded by a torus of rank \(n^2 H \cdot C_i\). To see that there is a map \(\mathcal{D}_i \to \text{Sym}^{n^2 H \cdot C_i}; R_i\), note first that if \(E = T \sqcup \cdots \sqcup T \hookrightarrow (C_i^2)_{T}\) is the trivial finite étale cover then the invertible sheaf \(L\) yields \(n^2 H \cdot C_i\) maps from \(T\) to \(R_i\), as \(R_i\) is the relative twisted Picard space. By composition, this gives a point of \(\text{Sym}^{n^2 H \cdot C_i}; R_i\). Moreover, acting by an automorphism of the pair \((T \sqcup \cdots \sqcup T \hookrightarrow (C_i^2)_{T}, L)\) (i.e., an automorphism of \(L\)) does not change the image point in \(\text{Sym}^{n^2 H \cdot C_i}; R_i\). By descent theory, any pair \((E, L)\) gives rise to a point of \(\text{Sym}^{n^2 H \cdot C_i}; R_i\), as desired. Since the (geometric) points of \(E\) are pairwise distinct in \(C_i^2\), the image of \((E, L)\) lies in the open subscheme \(O\) of \(\text{Sym}^{n^2 H \cdot C_i}; R_i\) parametrizing points with pairwise distinct images in \(C_i^2\).

The statement that \(\mathcal{D}_i\) is a torus gerbe over \(O\) can be checked étale-locally on \(T\), so we may assume that \(E = T \sqcup \cdots \sqcup T\). Suppose two pairs \((T \sqcup \cdots \sqcup T \hookrightarrow (C_i^2)_{T}, L_1)\) and \((T \sqcup \cdots \sqcup T \hookrightarrow (C_i^2)_{T}, L_2)\) give rise to the same point of \(\text{Sym}^{n^2 H \cdot C_i}; R_i\). Since each summand \(T\) maps to a distinct section of \((C_i^2)_{T}\), it follows that upon possibly reordering the summands, for each summand \(T\) the two restrictions \((L_1)_{T}\) and \((L_2)_{T}\) must map to the same section of \(R_i\) and thus differ by an invertible sheaf on \(T\). Shrinking \(T\) if necessary, we see that any two pairs with the same image in \(\text{Sym}^{n^2 H \cdot C_i}; R_i\) are locally isomorphic. In addition, since any point of \(\text{Sym}^{n^2 H \cdot C_i}; R_i\) parametrizing distinct points comes étale locally from a finite collection of disjoint sections of \(R_i\) and \(R_i\) is the relative Picard space, we see that any section of \(O\) locally gives a point of \(\mathcal{D}_i\). Thus, \(\mathcal{D}_i \to O\) is a gerbe. Finally, the band of \(\mathcal{D}_i\) is a torus because automorphisms of a pair \((E, L)\) are given by automorphisms of \(L\), which is an invertible sheaf over a finite étale \(T\)-scheme of rank \(n^2 H \cdot C_i\).

Since \(R_i\) is connected and smooth, it is irreducible, whence its symmetric powers are irreducible. But then, since a torus \(T\) is irreducible, we see that any \(T\)-gerbe over \(\text{Sym}^{n^2 H \cdot C_i}; R_i\) is also irreducible, as desired. \(\square\)

Let \(\mathcal{D} = \prod_i \mathcal{D}_i\). Define a geometric vector bundle over \(\mathcal{D}\) as follows. Let \(\Omega\) be the universal object on \(\mathcal{G} \times \mathcal{D}\).

**Lemma 5.1.14.** The complex \(\mathbf{R} pr_{pr_2}\), \(\mathbf{R} \mathcal{H}om(\Omega, pr_1^*V)[1]\) is quasi-isomorphic to a locally free sheaf \(\mathcal{F}\) on \(\mathcal{D}\). Moreover, its sheaf has the property that for any affine scheme \(T\) and any morphism \(\psi: T \to \mathcal{D}\), the set \(\mathcal{F}_T(T)\) parametrizes extensions \(0 \to V \to W \to Q \to 0\) with \(Q\) the object of \(\mathcal{D}\) corresponding to \(\psi\).

**Proof.** By standard nonsense, the complex \(\mathbf{R} pr_{pr_2}\), \(\mathbf{R} \mathcal{H}om(\Omega, pr_1^*V)\) is perfect and commutes with arbitrary base change on \(\mathcal{D}\). Moreover, since \(\Omega\) has torsion fibers (over \(pr_2\)) and \(V\) is locally free, we have that for each geometric point \(q: \text{Spec} \mathfrak{R} \to \mathcal{D}\) the sheaf \(\mathbf{H}^0(L_q^*\mathbf{R} pr_{pr_2}, \mathbf{R} \mathcal{H}om(\Omega, pr_1^*V))\) is zero, and
Indeed, we can glue the sheaves together at the nodes of \( \mathcal{C} \) to get a uniform sheaf on the entire residual curve. (This is trivial because \( k \) is now assumed to be algebraically closed.)

**Proposition 5.1.17.** The stack \( \mathcal{U} \) is nonempty.

**Proof:** Using Lemma [5.1.16] we may assume \( C = C_i \) for a fixed \( i \). If the ramification cover \( R_i \to C_i \) is connected then by definition we just need a locally free \( \mathcal{G}_i \)-twisted sheaf with the given local structures \( V(j) \) and trivial determinant. Choosing any locally free \( \mathcal{G}_i \)-twisted sheaf \( W \) of rank \( \ell^2 \) with totally regular fiber, we can glue on the local structures \( V(j) \) as follows. Write \( x_j \in C_i \) for the image of \( \xi_j \). Since it is reductive, infinitesimal deformation of \( V(j) \) over \( \text{Spec}(\mathcal{O}_{C_i, x_j}) \) is unobstructed. Since \( \mathcal{G}_i \to C_i \) is proper, we can apply the Grothendieck existence theorem to any such infinitesimal obstruction to yield a locally free \( \mathcal{G}_i \otimes \text{Spec}(\mathcal{O}_{C_i, x_j}) \)-sheaf \( \tilde{V} \) whose restriction to \( \xi_j \) is isomorphic to \( V(j) \). On the other hand, writing \( K \) for the fraction field of \( \mathcal{O}_{C_i, x_j} \), we have that \( W_K \) and \( \tilde{V}_K \) are isomorphic (as there is only one totally regular structure of a given rank over a field). Choosing a generic isomorphism and applying Corollary [2.5.2] yields a locally free \( \mathcal{G}_i \)-twisted sheaf \( V \) which is totally biregular and isomorphic to \( V(j) \) over \( \xi_j \) for each \( j \).

It remains to handle the case in which \( R_i \to C_i \) is split. In this case, since \( \mathcal{U} \) is geometrically trivial at residual gerbes (by assumption), there is an invertible \( \mathcal{G}_i \)-twisted sheaf \( \Lambda \) such that \( \Lambda \otimes \ell \) is the pullback of an invertible sheaf \( \lambda \) on \( C_i \). Let \( V_0, V_1, \ldots, V_{\ell-1} \) be locally free sheaves on \( \mathcal{C}_i \) such that

1. for each \( s = 0, \ldots, \ell - 1 \) the sheaf \( V_s \) is generically isotypic on the \( j \)th power of the generic stabilizer \( \mu_\ell \) of \( \mathcal{C}_i \),
2. for each \( j \) the restriction \( (V_s)(j) \) is isomorphic to the representation of \( \mu_\ell \times \mu_\ell \) which is the regular representation of the second factor tensored with the \( s \)-power character of the first factor, and
3. the determinant of \( \Lambda \otimes V_s \) is trivial for each \( s \).

This is possible because \( \ell \) is odd, so the regular representation has trivial determinant. The sheaf \( \Lambda \otimes (V_0 \oplus \cdots \oplus V_{\ell-1}) \) is then the desired uniform \( \mathcal{G}_i \)-twisted sheaf of rank \( \ell^2 \).

**6. Curves over Higher Local Fields**

In this section we fix a positive integer \( M \).

Let \( R \) be an excellent Henselian discrete valuation ring with fraction field \( K \), uniformizer \( t \), and residue field \( k \) of characteristic exponent \( p \). Suppose \( n \) is invertible in \( k \) and that \( R \) contains a primitive \( n \)th root of unity.

**Lemma 6.1.** There is an inclusion \( R \subset R' \) such that

1. \( R' \) is an excellent Henselian discrete valuation ring with uniformizer \( t \);
(2) \(R'\) is a colimit of finite free extensions of \(R\) of \(p\)-power degree;
(3) the morphism of residue fields \(k \hookrightarrow k'\) identifies \(k'\) with a perfect closure of \(k\).

Proof. If \(p = 1\), we \(R' = R\); for the rest of the proof, we assume that \(p > 1\). Given an element \(x \in k'\),
we can choose a lift \(x \in R\) and consider the extension \(S = R[y]/(y^p - x)\). It is easy to see that \(S\) is finite
and free over \(R\) of degree \(p\) and that \(\bar{t}\) generates a maximal ideal, so that \(S\) is a dvr with uniformizer \(t\).
The result follows by transfinite induction. \(\square\)

Lemma 6.2. If \(k\) has \(M'\)-Brauer dimension at most \(a\) and \(C\) is a nodal (possibly non-proper) curve over
\(k\), then for any \(\mu_n\)-gerbe \(\mathcal{G} \to C\) with \(n \in M'\), there is a locally free \(\mathcal{G}\)-twisted sheaf of rank \(n^a\).

Proof. The conclusion holds when \(C\) is smooth by the definition of the Brauer dimension and the fact
that a torsion-free coherent twisted sheaf is locally free (so that any coherent torsion-free extension of
a generic twisted sheaf will satisfy the conclusion of the lemma). Thus, the conclusion holds for
the normalization of \(C\). Moreover, at a node of \(C\), the restrictions of the locally free \(\mathcal{G}\)-twisted sheaves on
the two branches are isomorphic. It follows by a basic descent argument that the locally free sheaves
on the components of \(C\) may be glued at the nodes to achieve the desired result. \(\square\)

Theorem 6.3. If \(k\) has \(M'\)-Brauer dimension at most \(a\) then \(K\) has \((Mp)'\)-Brauer dimension at most \(a + 1\).

Proof. We have to check two things: (1) for any finite extension \(J\) of \(K\) and any \(\alpha \in \text{Br}'(J)\) we have
that \(\text{ind}(\alpha)|\per(\alpha)^a\), and (2) for any finitely generated extension \(J(D)\) of transcendence degree 1 and
any \(\alpha \in \text{Br}'(J(D))\) we have \(\text{ind}(\alpha)|\per(\alpha)^{a+1}\).

First, let \(J\) be a finite extension of \(K\) with ring of integers \(S\) and uniformizer \(u\). Given a Brauer class
\(\alpha \in \text{Br}(J)\) of period \(n\) prime to \(p\), it is clear that \(\alpha\) will be unramified over \(S[u^{1/n}]\), which is finite free of
degree \(n\) over \(S\). Let \(\mathcal{G} \to \text{Spec} S[u^{1/n}]\) be a \(\mu_n\)-gerbe representing \(\alpha\). The residue field \(L\) has Brauer
dimension at most \(a\) by hypothesis, so that there is a locally free \(\mathcal{G} \otimes L\)-twisted sheaf of rank \(n^a-1\).
Deformation theory lifts this to a locally free \(\mathcal{G}\)-twisted sheaf of rank \(n^a-1\), which pushes forward at
the generic point to yield a locally free twisted sheaf of rank \(n^a\) on a gerbe over \(\text{Spec} J\) representing \(\alpha\).
This completes the proof of (1).

It remains to verify (2). Any finitely generated extension of transcendence degree 1 has the form
\(J(D)\) with \(J\) finite over \(K\) and \(D\) a geometrically integral curve over \(J\). Given \(\alpha \in \text{Br}'(J(D))\), we may
normalize \(R\) in \(J\) and thus assume that \(J = K\). We may also assume that \(\text{per}(\alpha) = n\) is prime (and
relatively prime to \(p\) by assumption). By the standard finite-presentation tricks, we see that we can
replace \(R\) with the \(R'\) of Lemma 6.1 and thus assume that \(k\) is perfect.

Let \(X \to \text{Spec} R\) be a regular model of \(D\) over \(R\), which exists by [27]. After blowing up \(X\) if necessary,
we may assume that the union of the ramification divisor of \(\alpha\) and the (reduced) special fiber of \(X/R\) is a strict normal crossings divisor. Applying Corollary 3.1.4, we see that there is an orbisurface \(\mathcal{X}/T\)
with a generically finite map \(\mathcal{X} \to X\) of degree \(n\) over which \(\alpha\) is unramified. It thus suffices to show
that the index of \(\alpha\) restricted to \(\mathcal{X}/T\) divides \(n^a\). Let \(\mathcal{G} \to \mathcal{X}\) be a \(\mu_n\)-gerbe representing \(\alpha\) and write \(\mathcal{X}_0\)
for the reduced structure on the special fiber of \(\mathcal{X}\) over \(T\).

In order to prove (2), it suffices to find a locally free \(\mathcal{G}\)-twisted sheaf of rank \(n^a\). Since \(\mathcal{X}_0\) is 1-
dimensional and tame, we see from Proposition 2.2.3 that any locally free \(\mathcal{G}|_{\mathcal{X}_0}\)-twisted sheaf of rank \(n^a\)
vanishes obstruction space and thus deforms to a locally free formal \(\mathcal{G}\)-twisted sheaf. Applying
the Grothendieck existence theorem (Proposition 2.2.3(3)), we can effectivize the deformation to yield
a locally free \(\mathcal{G}\)-twisted sheaf of rank \(n^a\). Thus, it suffices to show that there is a locally free \(\mathcal{G}|_{\mathcal{X}_0}\)-twisted sheaf of rank \(n^a\).

Write \(\pi : \mathcal{X}_0 \to Y\) for the coarse moduli space of the stack \(\mathcal{X}_0\). There are finitely many closed points
\(y_1, \ldots, y_r \in Y\) such that (1) \(\pi\) is an isomorphism over \(Y \setminus \{y_1, \ldots, y_r\}\) and (2) the reduced structure
\(\xi_i := (\mathcal{X}_0 \times_Y y_i)_{\text{red}}\) is isomorphic to \(E_{\mu_n,\kappa(y_i)}\) for each \(i\). Applying Corollary 4.1.5 and the hypothesis
that \(k\) has Brauer dimension at most \(a\), we see that for each \(i = 1, \ldots, r\), there is a locally free \(\mathcal{G}_{\xi_i}\)-
twisted sheaf \(\mathcal{F}_i\) of rank \(n^a\). Moreover, since \(\mu_n\) is reductive, it follows from Proposition 2.2.3 that
infinitesimal deformations of \(\mathcal{F}_i\) are universally unobstructed (the obstruction space vanishes). Thus,
writing $\mathcal{X}_0^{(i)} = \mathcal{X}_0 \times_Y \text{Spec} \mathcal{O}_{Y,y_i}$, it follows from Proposition 2.2.3 that for each $i = 1, \ldots, r$, there is a locally free $\mathcal{G}|_{\mathcal{X}_0^{(i)}}$-twisted sheaf of rank $n^a$.

On the other hand, $C := \mathcal{X}_0 \setminus \{\xi_1, \ldots, \xi_s\} \cong Y \setminus \{y_1, \ldots, y_r\}$ is a nodal curve over $k$, so by Lemma 6.2 there is a locally free $\mathcal{G}_C$-twisted sheaf of rank $n^a$. Applying Corollary 2.5.2 we see that we can glue the twisted sheaf over $C$ to the twisted sheaves on each $\mathcal{X}_0^{(i)}$ to find a locally free $\mathcal{G}|_{\mathcal{X}_0}$-twisted sheaf of rank $n^a$, as desired. $\square$

**Corollary 6.4.** Let $C$ be a regular curve over a field $K$ and $\alpha \in \text{Br}(K(C))$.

1. If $K$ is a $d$-local field with finite residue field of characteristic $p$ and $\text{per}(\alpha) \in p'$ then $\text{ind}(\alpha)|\text{per}(\alpha)^{d+1}$.
2. If $K$ is a $d-1$-local with residue field of transcendence degree 1 over an algebraically closed field (e.g., $K = k(x_1)\cdot \cdots \cdot (x_d)$) of characteristic exponent $p$ and $\text{per}(\alpha) \in p'$, then $\text{ind}(\alpha)|\text{per}(\alpha)^d$.
3. If $K$ is an algebraic extension of a local field of residual characteristic exponent $p$ which contains the maximal unramified extension and $\text{per}(\alpha) \in p'$, then $\text{ind}(\alpha) = \text{per}(\alpha)$.

**Proof:** The first two statements follow by induction, starting with class field theory or de Jong’s theorem [10], while the third is a direct application of the theorem (after reducing to the case where $K$ is finite over the maximal unramified extension). $\square$

7. CURVES OVER GLOBAL FIELDS

Let $C/K$ be a proper smooth curve over a global field with a point $P \in C(K)$. In this section we link the period-index problem for certain classes in $\text{Br}(C)$ to the Hasse principle for geometrically rational varieties over $K$.

We first recall a well-known conjecture of Colliot-Thélène. What is written here is a special case. For the general case (which just says that the Brauer-Manin obstruction is the only one for 0-cycles of degree 1 on smooth proper varieties) the reader is referred to Conjecture 2.5 of [5].

**Conjecture 7.1** (Colliot-Thélène). Suppose $V/K$ is a smooth projective variety such that $V \otimes \overline{K}$ is rational and $\text{Pic}(V \otimes \overline{K}) = \mathbb{Z}$. If there is a 0-cycle of degree 1 on $V \otimes K$, for all places $v$ of $K$ then there is a 0-cycle of degree 1 on $V$.

We will link this to the period-index problem as follows. Write $\text{Br}_p(C)$ for the kernel of the restriction map $P^* : \text{Br}(C) \rightarrow \text{Br}(K)$. Given an integer $n$, say that a place of $K(C)$ divides $n$ if the restriction of the valuation on $K(C)$ to the prime field has residue field of characteristic dividing $n$. (In geometric language, the center of the valuation on a proper model of $C$ over a scheme of integers of $K$ has image contained in the closed subscheme cut out by the function $n \cdot 1$.)

**Theorem 7.2.** If Conjecture 7.1 holds, then any $\alpha \in \text{Br}_p(C)$ of odd period which is unramified at all places dividing $\text{per}(\alpha)$ satisfies $\text{ind}(\alpha)|\text{per}(\alpha)^2$.

**Proof of Theorem 7.2 when $\alpha$ has prime period $\ell$.** Let $\text{per}(\alpha) = \ell$. If $K$ has characteristic 0, let $T$ be the scheme of integers in $K$; if $K$ has positive characteristic, choose a transcendence basis for $K$ and let $T$ be the normalization of $\mathbb{P}^1$ in the resulting function field extension. We will call $T$ “the” scheme of integers of $K$. Note that when $K$ has infinite places $\infty$, the assumption that $\text{per}(\alpha)$ is odd implies that $\alpha|_{\mathcal{O}_{\infty}} = 0$.

Taking a branched cover if necessary, we may assume that $C$ has genus at least 2. Choose a regular proper model $\mathcal{C} \rightarrow T$ and let $D \subset \mathcal{C}$ be the ramification divisor of $\alpha$. Since $\alpha \in \text{Br}(C)$, we see that $D$ is supported in finitely many fibers of $\mathcal{C} \rightarrow T$, and the assumption on the ramification not dividing the period of $\alpha$ shows that the residue fields of the components of $D$ have characteristics prime to $\ell$. Using Lipman resolution [27], we may assume that the fibers containing the ramification divisor of $\alpha$ form an snc divisor which we will abusively write as $D \subset \mathcal{C}$. Applying Proposition 3.2.1 we see that $\alpha$ extends to a class in $\text{Br}(C(D^{1/\ell}))$. Moreover, the scheme $D$ is a union of finitely many residual curves $D_1, \ldots, D_b$. Choosing a $\mu_{N^2}$-gerbe $\mathcal{G} \rightarrow \mathcal{C}(D^{1/\ell})$ such that $\mathcal{G}|_{\mathcal{C}_0} = 0$, we see that for each $i = 1, \ldots, b$, there is a locally free $\mathcal{G}_{|D_i}$-twisted sheaf of trivial determinant and rank $N^2\ell^2$ with $\ell \mid N$. Taking direct sums if necessary, we may assume that $N$ is the same for all $i$.

Let $\mathcal{V} \rightarrow \text{Spec} K$ be the stack of stable locally free $\mathcal{G}_K$-twisted sheaves of rank $N^2\ell^2$ with determinant $\mathcal{O}(P)$. It follows from Proposition 3.1.1.6 of [24] and [16] that $\mathcal{V}$ is a $\mu_{N^2\ell^2}$-gerbe over a smooth projective
geometrically rational $K$-variety $V$ with $\text{Pic}(V \otimes \overline{K}) = \mathbb{Z}$. We claim that there exists an $N$ relatively prime to $\ell$ such that for all places $\nu$ of $K$, we have $\mathcal{V}(K_\nu) \neq \emptyset$. Assuming we have done this, we can then apply Conjecture 7.1 to conclude that $V$ has a 0-cycle of degree 1. Moreover, since we will prove that the stack (and not merely the coarse space) has local points, it follows from a simple argument with the Leray spectral sequence that $\text{Br}(K) \to \text{Br}(V)$ is surjective and then that the Brauer class of $\mathcal{V} \to V$ is trivial. Thus, the 0-cycle of degree 1 on $V$ lifts to $\mathcal{V}$. But then there is a locally free $\mathcal{G}_\nu$-twisted sheaf of rank $M\ell^2$ for some $M$ relatively prime to $\ell$, so that $\text{ind}(\alpha) = M\ell^2$. Since $\text{per}(\alpha) = \ell$, we conclude by standard methods that $\text{ind}(\alpha)|\ell^2$, as desired.

Thus, it remains to find local objects in $\mathcal{V}(K_\nu)$. Since adjoining a primitive $\ell$th root of unity is a prime-to-$\ell$ extension, we may assume by Lemma 5.1.4 that $K$ contains a primitive $\ell$th root of unity. If $\nu$ is not in the image of the ramification locus (including infinite $\nu$), we claim that $\alpha|_{K_\nu} = 0$. If $\nu$ is infinite, this is an elementary parity argument. If $\nu$ is finite, then this follows from the fact that the Brauer group of a proper curve over a complete dvr with finite residue field is finite. In these cases, the existence of an object of $\mathcal{V}(K_\nu)$ follows immediately from the fact that stable vector bundles of arbitrary invariants exist on curves of genus at least 2 over any infinite field.

Now suppose $\nu$ has center under one of the residual curves $D_i$. By Corollary 5.1.3 there is some $N$ relatively prime to $\ell$ and a locally free $\mathcal{G}_{|D_i}$-twisted sheaf $V_i$ of rank $\ell^2$ and trivial determinant. Taking suitable direct sums, we may assume that $N$ is independent of $i$. The deformation theory of twisted sheaves in Proposition 2.2.3 shows that $V_i$ deforms to yield a locally free $\mathcal{G} \times K_\nu$-twisted sheaf $V_i$ of rank $N\ell^2$ with trivial determinant. Since $\alpha|_P = 0$, we can make an elementary transform to yield a locally free $\mathcal{G} \times K_\nu$-twisted sheaf $W_i$ of rank $N\ell^2$ and determinant $\mathcal{O}(P)$. Let $\mathcal{F}$ denote the stack of all locally free $\mathcal{G} \times K_\nu$-twisted sheaves of rank $N\ell^2$ and determinant $\mathcal{O}(P)$. We know that $\mathcal{V} \subset \mathcal{F}$ is a dense open immersion of irreducible smooth stacks. Thus, any algebraic deformation of $W_i$ over a smooth base which is versal at $[W_i]$ will contain a dense open parametrizing stable sheaves. But any smooth connected $K_\nu$-variety with a rational variety has a dense set of rational points, so we conclude that there is a rational point parametrizing a stable twisted sheaf, yielding an object of $\mathcal{V}(K_\nu)$, as desired.

**Proof of Theorem 7.2 in the general case.** The Schur decomposition of $\alpha$ immediately reduces to the situation where $\text{per}(\alpha)$ is a prime power, say $\text{per}(\alpha) = \ell^n$. The class $\alpha' := \ell^{n-1}\alpha$ has period $\ell$, so assuming that we have treated the case of prime period (done below), we can find a covering $X' \to X$ of degree $\ell^2$ splitting $\alpha'$. Since the pullback of $\alpha$ to $X'$ has period $\ell^{n-1}$, we are done by induction, given the following (crucial) lemma.

**Lemma 7.3.** If $\beta \in \text{Br}(\mathcal{O}_{X,P})$ is a class whose restriction to $P$ is trivial, then there is a splitting field $K'/K(X)$ such that the normalization of $X$ in $K'$ has a $K$-rational point lying over $P$.

**Proof.** Let $P \to \text{Spec} \mathcal{O}_{X,P}$ be a Brauer-Severi scheme representing $\beta$. Since $P \otimes \kappa(P)$ is trivial, there is a $\kappa(P)$-rational point $Q \in P_P$. Let $U$ be an affine neighborhood of $Q$ in $P$. Choose a regular sequence $\overline{f}_1, \ldots, \overline{f}_d$ cutting out $Q$ in the fiber $P_P$ and choose arbitrary lifts $f_1, \ldots, f_d \in \mathcal{O}_P(U)$. Let $Z$ be the subscheme of $U$ determined by $f_1, \ldots, f_d$. By construction, $Z$ is flat over $\mathcal{O}_{X,P}$ and normal at $Q$ with residue field $K = \kappa(P)$. It follows that the generic fiber of $Z$ gives rise to the desired splitting field.

Applying the lemma in our situation, we see that there is some $X' \to X$ splitting $\alpha'$, possibly of large degree, such that there is a rational point $Q \in X'(K)$ over $P$. This property clearly holds for any intermediate curve $X' \to X'' \to X$. Since $\alpha'$ has index dividing $\ell^2$, there will be such an intermediate curve $X''$ of degree $\ell^2$ over $X$, and we know that there is a rational point $Q \in X''(K)$ such that $\alpha|_Q = 0$. Thus, we can proceed by induction.

We can also get the first “proof” of the standard conjecture for a class of $C_3$-fields.

**Corollary 7.4.** Suppose $Y$ is a proper smooth surface over a finite field $\mathbb{F}_q$. Given $\alpha \in \text{Br}(\mathbb{F}_q(Y))'$, if Conjecture 7.1 holds then $\text{ind}(\alpha)|\text{per}(\alpha)^2$.

**Sketch of proof.** We first record an argument due to de Jong for fibering a birational model $Y$ so that the ramification of $\alpha$ ends up in a fiber. We can replace $\mathbb{F}_q$ with its maximal prime-to-$q$ extension and thus assume that the base field is an infinite algebraic extension of its finite prime field.
We may assume that the ramification divisor of $\alpha$ is a strict normal crossings (snc) divisor $D = D_1 + \cdots + D_m \subset Y$. For each component $D_i$, there is a finite ramification extension $D'_i \to D_i$. By the Chebotarev density theory, there are closed points $d_i \in D_i$ of arbitrarily high degree such that $D'_i$ is totally split over $d_i$. It is an exercise to check (by examining the Henselian local structure of $\alpha$ near $d_i$ as in e.g. Proposition 1.2 of [34]) that the ramification divisor of $\alpha$ in the blow up $Y$ at $d_i$ is the strict transform of $D$.

Blowing up enough such $d_i$, we may assume that $D$ is contained in a snc divisor $E$ whose components have a negative definite intersection matrix. By Theorem 2.9(B) of [2], there is a morphism to a normal projective surface $Y \to \bar{Y}$ which is an isomorphism outside of $E$. In particular, taking a pencil of very ample divisors on $\bar{Y}$ which meet transversely in the smooth locus, we see that a blow-up of $\bar{Y}$ in finitely many smooth points fibers over $\mathbb{P}^1$ with a section passing entirely through the smooth locus. Blowing up the corresponding locus in $Y$ yields a blow up $\bar{Y} \to Y$ and a proper flat morphism $\pi : \bar{Y} \to \mathbb{P}^1$ with a section $P : \mathbb{P}^1 \to \bar{Y}$ such that

1. $\pi$ has smooth geometrically connected generic fiber;
2. the ramification of $\alpha$ is entirely contained in a fiber of $\pi$;
3. $\alpha|_P = 0$.

The last statement follows from the fact that any base point of the pencil gives rise to a section, and that $\mathrm{Br}(\mathbb{P}^1) = 0$.

Writing $C$ for the generic fiber of $\pi$, we have that $\alpha \in \mathrm{Br}_P(C)$. If $q$ is odd or $q$ is even and the period of $\alpha$ is odd then the period-index relation now follows immediately from Theorem 7.2 If $q$ is even and $\text{per}(\alpha)$ is 2 then the the result follows from the fact that the absolute Frobenius $F : Y \to Y$ is finite locally free of degree 4 and acts as multiplication by 2 on the Brauer group. The general case follows from the Schur decomposition and induction on the power of 2 dividing the period. \hfill $\square$

APPENDIX A. PERIOD-INDEX EXAMPLES

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In this appendix to the paper of Max Lieblich, we will construct examples of division algebras over certain fields in order to demonstrate that the bounds given in corollary [14] are sharp. The main tool in doing this is the use of valuation theory for division algebras as in [39]. I would like to thank A. Wadsworth for pointing out some results in the literature which helped to shorten the exposition of these examples.

A.1. CURVES OVER ITERATED LAURENT SERIES FIELDS. Let $k$ be an algebraically closed field, $\text{char}(k) \neq p$. Let $K = k((x_1)) \cdot \cdots \cdot (x_r)(t)$. In this section, we will construct a division algebra over $K$ with period $p$ and index $p^r$.

Let $K' = k(t)((x_1)) \cdot \cdots \cdot (x_r))$. Note that $K'$ is a Henselian valued field with values in $\mathbb{Z}^r$ and with residue field $k(t)$. Also note that we have inclusions $k(t) \subset K \subset K'$. Let $\ell /k(t)$ be abelian Galois with group $(\mathbb{Z}/p)^r$. By Kummer theory, we may write $\ell = \ell_1 \otimes_{k(t)} \cdots \otimes_{k(t)} \ell_r$, with $\ell_1 = k(t)(\alpha_1)$, $\alpha^p_1 = \alpha_1 \in k(t)$. Using [39], example 3.6, we note that the inertial lifts $L'_i = K'/(\alpha_i)$ of the $\ell_i$ are cyclic Galois, say with generators $\sigma_i$ of the Galois group, and that the algebra

$$A' = (L'_1/K', \sigma_1, x_1) \otimes_K \cdots \otimes_K (L'_r/K', \sigma_r, x_r)$$

is a (nicely semi-ramified) division algebra. Therefore, if we set $L_i = K(\alpha_i)$ and

$$A = (L_1/K, \sigma_1, x_1) \otimes_K \cdots \otimes_K (L_r/K, \sigma_r, x_r),$$

we have $A \subset A \otimes_K K' = A'$. Since $A'$ is division, $A$ contains no zerodivisors and therefore must also be division. Therefore $A$ is a division algebra of index $p^r$ and period $p$ as desired.

A.2. CURVES OVER HIGHER LOCAL FIELDS. In this section we will construct division algebras of period $p$ and index $p^{r+1}$ over fields of the form $F(t)$, where $F$ is a $r$-local field in the following two cases:

$F = F_q((x_1)) \cdot \cdots \cdot (x_r))$ with $q \neq p$ or $F = \mathbb{Q}_q((x_1)) \cdot \cdots \cdot (x_{r-1}))$ with $p|(q - 1)$. 

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**Lemma A.2.1.** Let $k$ be a global field, and choose $\ell/k$ Galois with group $(\mathbb{Z}/p)^r = \langle \sigma_1, \ldots, \sigma_r \rangle$. Then we may choose a cyclic division algebra $D/k$ of index $p$ such that $D \otimes \ell$ is still division.

**Proof:** To do this we simply note that by Chebotarev density (see [36], or [15]), there are infinitely many primes such that our field extension $\ell$ is completely split. Let $\mathfrak{p}$ be such a prime in $k$. By Cassels-Fröhlich page 187 corollary 9.8, a central simple algebra with a nontrivial Hasse invariant at $\mathfrak{p}$ must remain nontrivial upon extending scalars to $\ell$. Therefore, we may simply choose an arbitrary division algebra with a Hasse invariant of order $p$ at $\mathfrak{p}$. Since the period is equal to the index over a global field, this algebra also has index $p$. □

Recall that we have either:

1. $K = \mathbb{F}_q((x_1)) \cdots ((x_r))(t)$ with $q \neq p$ or
2. $K = \mathbb{Q}_q((x_1)) \cdots ((x_{r-1}))(t)$ with $p | (q - 1)$.

In the second case, we let $x_r = q$, and in each case $K$ posesses a valuation into the (lexicographically) ordered abelian group $\mathbb{Z}^r$ with uniformizers $x_1, \ldots, x_r$. Let $\overline{K} \cong \mathbb{F}_q(t)$ be the residue field.

Since $\overline{K}$ is global, we may choose a extension $\ell/\overline{K}$ and a symbol algebra $D$ as in lemma A.2.1.

**Lemma A.2.2.** We may find an unramified abelian extension $L/K$ of dimension $p^r$ and an inertial symbol algebra $D/K$ of degree $p$ such that $\mathcal{L} = \ell$ and $\mathcal{D} = D$.

Using this lemma, it follows that if we write $L$ as the compositum of the cyclic algebras $L_i$ with cyclic Galois groups generated by $\sigma_i$, then the algebra

$$A = D \otimes_K (L_1/K, \sigma_1, x_1) \otimes_K \cdots \otimes_K (L_r/K, \sigma_1, x_r)$$

is a (valued) division algebra by [29], theorem 1. Therefore we will have the desired algebra of index $p^r+1$ and period $p$.

**Proof:**

Case (1). $K = \mathbb{F}_q((x_1)) \cdots ((x_r))(t)$.

We have natural inclusions $\overline{K} \subset K \subset K((x_1)) \cdots ((x_r))$. It is well known that $\overline{L} = \ell \otimes_{\overline{K}} K((x_1)) \cdots ((x_r))$ and $\overline{D} = D \otimes_{\overline{K}} K((x_1)) \cdots ((x_r))$ are both division (for example, this follows from lemma A.2.3 below). Since $K((x_1)) \cdots ((x_r))$ is Henselian valued, it follows that the valuation extends to $\overline{D}$ and $\overline{L}$ and one may check that these are unramified. Therefore, if we define $L = \ell \otimes K$ and $D = \mathcal{D} \otimes K$, we have $L \subset \overline{L}$ and $D \subset \overline{D}$, and in particular both are domains and hence division. These clearly have the desired properties.

Case (2). $K = \mathbb{Q}_q((x_1)) \cdots ((x_{r-1}))(t)$ with $p | (q - 1)$.

By the assumption on $p$ and $q$, we know that by Hensel's lemma, $\overline{K}$ and $\mathbb{Q}_q(t)$ (and hence $K$) contain a primitive $p^r$'th root of unity which we will always denote by $\omega$. Therefore Kummer theory applies and we may write $\ell = \ell_1 \otimes_{\overline{K}} \cdots \otimes_{\overline{K}} \ell_r$, with each $\ell_i = K(\alpha_i)$, $\alpha_i^p = \alpha_i \in \overline{K}$. Also, we may write $D$ as a symbol algebra $\mathcal{D} = (a, b)_{\overline{K}, \omega}$.

Choose lifts $\bar{a}, \bar{b} \in \mathbb{Z}_q(t)^* \subset K$ of $a, b$ respectively. Set $L_i = K(\bar{a}_i)$, $\bar{a}_i^p = \bar{a}_i$, and let $\bar{D} = (\bar{a}, \bar{b})_{\overline{K}, \omega}$. We will now demonstrate that $L = L_1 \otimes_K \cdots \otimes_K L_r$, $D$, and $L \otimes \overline{K}$ are division algebras. To check this it suffices to show that these $K$ algebras are division algebras after being tensored over $K$ with $\mathbb{Q}_p(t)((x_1)) \cdots ((x_{r-1}))$. In other words, if we let $F = \mathbb{Q}_q(t)((x_1)) \cdots ((x_{r-1}))$, we may replace $L_i$ with $L'_i = F(\bar{a}_i)$ and $D$ with $D' = (\bar{a}, \bar{b})_{F, \omega}$. Let $\mathcal{O}_F = \mathbb{Z}_q(t)[[x_1, \ldots, x_{r-1}]]$. Note that $\mathcal{O}_F$ is a valuation ring with quotient field $F$ and residue field $\overline{F} = \mathbb{F}_p(t) = \overline{K}$. If we let $\mathcal{O}_{D'}$ be the algebra $(\bar{a}, \bar{b})_{\mathcal{O}_F, \omega}$, and $\mathcal{O}_{L'_i} = \mathcal{O}_F(\bar{a}_i)$, then we have

$$\mathcal{O}_{D'} \otimes_{\mathcal{O}_F} F = D, \quad \mathcal{O}_{D'} \otimes_{\mathcal{O}_F} \overline{F} = \overline{D}$$

$$\mathcal{O}_{L'_i} \otimes_{\mathcal{O}_F} F = L'_i, \quad \mathcal{O}_{L'_i} \otimes_{\mathcal{O}_F} \overline{F} = \ell_i$$

$$\mathcal{O}_{L'_i} \otimes_{\mathcal{O}_F} \cdots \otimes_{\mathcal{O}_F} \mathcal{O}_{L'_i} \otimes_{\mathcal{O}_F} F = L'_i \otimes_F \cdots \otimes_F L'_r,$$

$$\mathcal{O}_{L'_i} \otimes_{\mathcal{O}_F} \cdots \otimes_{\mathcal{O}_F} \mathcal{O}_{L'_i} \otimes_{\mathcal{O}_F} \overline{F} = \ell_1 \otimes_{\overline{K}} \cdots \otimes_{\overline{K}} \ell_r.$$
The fact that the desired algebras are division now follows from lemma A.2.3.

It remains to show that the division algebra $D = (a, b)$ and the field $L = K(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r)$ possess unramified valuations. For each of the given algebras, this can be checked directly. Let $\mathfrak{B}$ be the standard basis of one of these - in the first case consisting of expressions of the form $\alpha_i^j \beta^l$, $0 \leq i, j \leq p - 1$ where $\alpha$ and $\beta$ are the standard generators of the symbol algebra, or in the second case consisting of expressions of the form $\alpha_1^{n_1} \cdots \alpha_r^{n_r}$, $0 \leq n_i \leq p - 1$. We may then define our valuation via

$$v(\sum_{b \in \mathfrak{B}} \lambda_b b) = \min_{\lambda \in \mathfrak{B}} \{v(\lambda)\}.$$  

One may now observe that this is an unramified valuation.

\[\square\]

**Lemma A.2.3.** Let $R$ be a valuation ring with residue field $k$ and field of fractions $F$. Suppose that $A$ is an $R$-algebra which is free of finite rank as an $R$-module, and that $A \otimes_R k$ is a division algebra. Then $A \otimes_R F$ is also a division algebra.

**Proof.** The norm polynomial on $A$ defines a projective and hence proper scheme $X$ over $\operatorname{Spec}(R)$. It is easy to see that $A \otimes_R L$ is a domain if and only if $X(L) = \emptyset$. In particular, if $A \otimes_R F$ was not a domain, we would have a point in $X(F)$, and by the valuative criterion for properness, we would thereby obtain a point in $X(k)$, contradicting the fact that $A \otimes_R k$ is a domain. The conclusion now follows from the observation that a finite dimensional domain is a division algebra. \[\square\]

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