A transformed rational function method and exact solutions to the $3 + 1$ dimensional Jimbo-Miwa equation

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Abstract

A direct approach to exact solutions of nonlinear partial differential equations is proposed, by using rational function transformations. The new method provides a more systematic and convenient handling of the solution process of nonlinear equations, unifying the tanh-function type methods, the homogeneous balance method, the exp-function method, the mapping method, and the $F$-expansion type methods. Its key point is to search for rational solutions to variable-coefficient ordinary differential equations transformed from given partial differential equations. As an application, the construction problem of exact solutions to the $3 + 1$ dimensional Jimbo-Miwa equation is treated, together with a Bäcklund transformation.

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Key words. The tanh-function type method, The exp-function method, The $F$-expansion type method, The $3 + 1$ dimensional Jimbo-Miwa equation

1 Introduction

Partial differential equations (PDEs) describe various nonlinear phenomena in natural and applied sciences such as fluid dynamics, plasma physics, solid state physics, optical fibers,

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acoustics, mechanics, biology and mathematical finance. Their solution spaces are infinite-dimensional and contain diverse solution structures. It is of significant importance to solve nonlinear PDEs from both theoretical and practical points of view. Due to the nonlinearity of differential equations and the high dimension of space variables, it is a difficult job for us to determine whatever exact solutions to nonlinear PDEs.

It has been a successful idea to generate exact solutions of nonlinear wave equations by reducing PDEs into ordinary differential equations (ODEs). Many approaches to exact solutions in the literature follow such an idea, which contain the tanh-function method \[1\]-\[5\], the sech-function method \[6\]-\[9\], the homogeneous balance method \[10, 11\] the extended tanh-function method \[12\]-\[14\], the sine-cosine method \[15\], the tanh-coth method \[16\], the Jacobi elliptic function method \[17\], the exp-function method \[18\], the F-expansion method \[19\], the mapping method \[20\], and the extended F-expansion method \[21, 22\]. Given an ODE of differential polynomial type, either constant-coefficient or variable-coefficient, one can always adopt computer algebra systems to search for rational solutions pretty systematically. This is one of the main reasons why those reduction methods work well.

Based on this observation, we would, in this paper, like to propose a direct and systematical approach to exact solutions of nonlinear equations by using rational function transformations. The method is very suitable for an easier and more effective handling of the solution process of nonlinear equations, unifying the existing solution methods mentioned above. Its key point is to find rational solutions to variable-coefficient ODEs transformed from given nonlinear PDEs. Together with an auto-Bäcklund transformation, an application of our approach will generate new exact solutions to the 3 + 1 dimensional Jimbo-Miwa equation \[23\] (called the Jimbo-Miwa equation in \[24\]):

\[
P_{JM}(u) := u_{xxxx} + 3u_yu_{xx} + 3u_xu_{xy} + 2u_{yt} - 3u_{xx} = 0
\]  \(1.1\)

where \(u = u(x, y, z, t)\), \(u_x = \frac{\partial u}{\partial x}\), etc.

The 3 + 1 dimensional Jimbo-Miwa equation \(1.1\) is the second member in the entire Kadomtsev-Petviashvili hierarchy \[23\], originally defined by a Hirota bilinear equation

\[
[(D_x^3 + 2D_t)D_y - 3D_xD_z)]\tau \bullet \tau = 0, \quad 1.2
\]

with the link being \(u = 2(\ln \tau)_x\). The Jimbo-Miwa equation \(1.1\) passes the Painlevé test only for a subclass of solutions \[24\] and its symmetry algebra does not have a Kac-Moody-Virasoro structure \[25\]. Nevertheless, different type solutions to the Jimbo-Miwa equation \(1.1\) are found (see, say, \[26\]-\[30\]). The Hirota perturbation technique yields one- and two-soliton solutions \[24\] and dromion type solutions \[28\], and the truncated Painlevé series leads to a special Bäcklund transformation \[31\]. Obviously, the Jimbo-Miwa equation \(1.1\) has the following \(x\)- or \(y\)-independent solutions:

\[
u = f(x, t) + h(z, t), \quad u = g(y, z) + h(z, t), \quad 1.3
\]

where \(f\), \(g\) and \(h\) can be arbitrary functions in the indicated variables. These solutions contain more special solutions: \(u = f(x, t), \quad u = g(y, z)\) and \(u = h(z, t)\), which are independent of
two variables of $x, y, z, t$. Starting with such solutions, various variable separated solutions have been presented by using the truncated Painlevé series [31] [32] and abundant nonlinear coherent structures have been exhibited [32]. We will concentrate primarily on constructing travelling wave solutions to the Jimbo-Miwa equation (1.1) by formulating a problem of finding rational solutions to a transformed Jimbo-Miwa equation.

The paper is organized as follows. In Section 2 a unified formulation for getting exact solutions to nonlinear equations is proposed, by using rational function transformations. In Section 3 an application and an auto-Bäcklund transformation are made to solve the $3 + 1$ dimensional Jimbo-Miwa equation (1.1). A few of concluding remarks are given in the final section, along with some polynomial solutions.

## 2 A transformed rational function method

To describe our solution process, let us focus on a scalar $3 + 1$ dimensional partial differential equation

$$P(x, y, z, t, u_x, u_y, u_z, u_t, \cdots) = 0,$$

(2.1)

though the solution process also works for systems of nonlinear equations. We assume that there are exact solutions to the differential equation (2.1):

$$u(x, y, z, t) = u(\xi), \quad \xi = \xi(x, y, z, t).$$

(2.2)

Usually, we can have

$$\xi(x, y, z, t) = ax + by + cz - \omega t,$$

(2.3)

where $a, b, c$ and $\omega$ are constants, in the constant-coefficient case, and

$$\xi(x, y, z, t) = a(t)x + b(t)y + c(t)z - \omega(t),$$

(2.4)

where $a(t), b(t), c(t)$ and $\omega(t)$ are functions of $t$, in the $t$-dependent-coefficient case. Under the transformation (2.2), the partial differential equation (2.1) is put into an ordinary differential equation:

$$Q(x, y, z, t, u^{(r)}, u^{(r+1)}, \cdots) = 0,$$

(2.5)

where $u^{(i)} = \frac{du}{d\xi}, \ i \geq 1$, and $r$ is the least order of derivatives in the equation. To keep the solution process as simple as possible, the function $Q$ should not be a total $\xi$-derivative of another function. Otherwise, taking integration with respect to $\xi$ further reduces the transformed equation.

An important step in the solution process is to introduce a new variable $\eta = \eta(\xi)$ by a solvable ordinary differential equation, for example, a first-order differential equation:

$$\eta' = T = T(\xi, \eta),$$

(2.6)

where $T$ is a function of $\xi$ and $\eta$, and the prime denotes the derivative with respect to $\xi$. In case that we have a general second-order differential equation to begin with, we should first
obtain its first integrals \[33\] and then use the method of planar dynamical systems to solve \[34\]. Two simple solvable cases of the above function \(T\) are as follows:

\[
T = T(\eta) = \eta, \quad T = T(\eta) = \alpha + \eta^2, \quad \alpha = \text{const.}
\]

(2.7)

The corresponding first-order equations have the particular solutions \(\eta = e^\xi\) and

\[
\eta = \begin{cases} 
-\frac{1}{\xi}, & \text{when } \alpha = 0, \\
-\sqrt{-\alpha} \tanh \sqrt{-\alpha} \xi \text{ or } -\sqrt{-\alpha} \coth \sqrt{-\alpha} \xi, & \text{when } \alpha < 0, \\
\sqrt{\alpha} \tan \sqrt{\alpha} \xi \text{ or } -\sqrt{\alpha} \cot \sqrt{\alpha} \xi, & \text{when } \alpha > 0,
\end{cases}
\]

(2.8)

respectively \[12\]. Those two cases correspond to the exp-function method and the extended tanh-function method, respectively.

More general assumptions than (2.7) can engender special function solutions to nonlinear wave equations. For instance, taking \((\eta')^2 = S(\eta)\) with some fourth-order polynomials \(S(\eta)\) in \(\eta\) (or equivalently, \(\eta'' = R(\eta)\) with some third-order polynomials \(R(\eta)\) in \(\eta\)) can yield Jacobi elliptic function solutions; and such assumptions are the bases for the extended tanh-function method, the \(F\)-expansion method and the extended \(F\)-expansion method, and work for many particular nonlinear wave equations.

The basic idea of using solvable ordinary differential equations was successfully used to solve the 2+1 dimensional Korteweg-de Vries-Burgers equation in \[35\], based on \(a\eta'' + b\eta' + c\eta^2 + d\eta = 0\) (\(a, b, c, d = \text{const.}\)), and the Kolmogorov-Petrovskii-Piskunov equation in \[12\], based on \(\eta' = 1 \pm \eta^2\). Later it was broadly adopted in the extended tanh-function method \[13, 14\], the tanh-coth method \[16\], the \(F\)-expansion method \[19\], the mapping method \[20, 36\], and the extended \(F\)-expansion method \[21, 22\].

Let us proceed to consider rational functions

\[
v(\eta) = \frac{p(\eta)}{q(\eta)} = \frac{p_m \eta^m + p_{m-1} \eta^{m-1} + \cdots + p_0}{q_n \eta^n + q_{n-1} \eta^{n-1} + \cdots + q_0},
\]

(2.9)

where \(m\) and \(n\) are two natural numbers, and \(p_i, \ 0 \leq i \leq m\) and \(q_i, \ 0 \leq i \leq n\) are normally constants but could be functions of the independent variables as in the \(F\)-expansion type method. All Laurent polynomial and polynomial functions are only special examples of rational functions. We search for travelling wave solutions determined by

\[
u^{(r)}(\xi) = v(\eta) = \frac{p(\eta)}{q(\eta)},
\]

(2.10)

where \(p(\eta)\) and \(q(\eta)\) are polynomials as indicated above. It is direct to compute that

\[
u^{(r+1)} = Tv', \quad \nu^{(r+2)} = T\partial_\eta \nu^{(r)} = T\nu'' + T\nu', \quad \cdots,
\]

(2.11)

which is based on \(\partial_\xi = T\partial_\eta\). Note that by the prime, we denote the derivatives with respect to the involved variable, for instance, \(u' = \frac{du}{d\xi}, \ u' = \frac{du}{d\eta}, \) and \(u'' = \frac{d^2u}{d\eta^2}\).
Now we assume that the transformed equation (2.5) is a rational function equation of \( \eta \) with a given pair of \( m \) and \( n \). This can be achieved for all nonlinear equations of differential polynomial type, when \( T \) is, for example, a rational function in \( \eta \). Then we just need to force the numerator of the resulting rational function in the transformed equation to be zero. This yields a system of algebraic equations on all variables \( a, b, c, \omega, p_i, 0 \leq i \leq m \) and \( q_i, 0 \leq i \leq n \), and solve this system (which could be a differential system as in the \( F \)-expansion type method) to determine \( p(\eta), q(\eta) \) and \( \xi \). Finally, integrating \( v(\eta) \) with respect to \( \xi \), \( r \) times, we obtain a class of travelling wave solutions

\[
\begin{align*}
    u(x, y, z, t) &= u(\xi) = \int \cdots \int_{r} \frac{p(\eta(\xi))}{q(\eta(\xi))} d\xi \cdots d\xi \\
    &= \int_{0}^{\xi} \int_{0}^{\xi_r} \cdots \int_{0}^{\xi_2} \frac{p(\eta(\xi_1))}{q(\eta(\xi_1))} d\xi_1 \cdots d\xi_{r-1} d\xi_r + \sum_{i=1}^{r} d_i \xi^{r-i},
\end{align*}
\]

where \( d_i, 1 \leq i \leq r, \) are arbitrary constants. If \( r = 1 \), there is only the last definite integral over \([0, \xi]\) in (2.12). The resulting solutions will definitely contain a polynomial part in \( \xi \), when \( r > 1 \).

**The case of the exp-function method:**

If we take \( \eta = e^{\xi} \), then the solution function is determined by

\[
    u(\xi) = \int \cdots \int_{r} \frac{p_m e^{m\xi} + p_{m-1} e^{(m-1)\xi} + \cdots + p_0}{q_n e^{n\xi} + q_{n-1} e^{(n-1)\xi} + \cdots + q_0} d\xi \cdots d\xi.
\]

This can yield solutions generated by the exp-function method [18].

**The case of the extended tanh-function method:**

If we take \( \eta = -\tanh(\xi) \), then the solution function is determined by

\[
    u(\xi) = \int \cdots \int_{r} \frac{(-1)^m p_m \tanh^m(\xi) + (-1)^{m-1} p_{m-1} \tanh^{m-1}(\xi) + \cdots + p_0}{(-1)^n q_n \tanh^n(\xi) + (-1)^{n-1} q_{n-1} \tanh^{n-1}(\xi) + \cdots + q_0} d\xi \cdots d\xi.
\]

This can yield solitary wave solutions. If we take \( \eta = \tan(\xi) \), then the solution function is determined by

\[
    u(\xi) = \int \cdots \int_{r} \frac{p_m \tan^m(\xi) + p_{m-1} \tan^{m-1}(\xi) + \cdots + p_0}{q_n \tan^n(\xi) + q_{n-1} \tan^{n-1}(\xi) + \cdots + q_0} d\xi \cdots d\xi.
\]

This can yield periodic wave solutions. The other choices of \( \eta = -\coth(\xi) \) and \( \eta = -\cot(\xi) \) generate similar type exact solutions. Note that we only use the selection of \( \alpha = 1 \) and a general nonzero value of \( \alpha \) may lead to more general solutions. The selection of \( \alpha = 0 \) presents rational solutions. If \( v \) is a polynomial and \( r = 0 \), then \( u \) gives an exact solution in...
the form of a finite series in \( \tanh \xi \) or \( \tan \xi \), which is obtainable by the \( \tanh \)-function type method \[9, 12, 13\].

The solution process described above unifies the existing methods using \( \tanh \)-function type functions, \( \tan \)-function type functions, the exponential functions and the Jacobi elliptic functions, and allows us to carry out the involved computation more systematically and conveniently by powerful computer algebra systems such as Maple, Mathematica, MuPAD and Matlab. While applying to construction of special function solutions to nonlinear equations, we have to pay special attention to the particular forms of given nonlinear equations to get workable transformed equations.

3 Solving the 3 + 1 dimensional Jimbo-Miwa equation

To generate travelling wave solutions

\[
\begin{align*}
  u(x, y, z, t) &= u(\xi), \quad \xi = ax + by + cz - \omega t, \\
  \text{where } a, b, c, \text{ and } \omega &\text{ are the angular wave numbers and wave frequency, we only need to solve the reduced 3 + 1 dimensional Jimbo-Miwa equation \((1.1)\):}
\end{align*}
\]

\[
\begin{align*}
  a^3bu'''' + 6a^2bu''u'' - (2b\omega + 3ac)u'' = 0,
\end{align*}
\]

where the prime denotes the derivatives with respect to \( \xi \). Integrate it once with respect to \( \xi \) to obtain

\[
\begin{align*}
  a^3bu'''' + 3a^2b(u')^2 - (2b\omega + 3ac)u' = 0.
\end{align*}
\]

We set \( r = 1 \) and \( u' = v \), and then, we have the transformed Jimbo-Miwa equation

\[
\begin{align*}
  a^3bT^2v'' + a^3bTT'v' + 3a^2bv^2 - (2b\omega + 3ac)v = 0,
\end{align*}
\]

where the prime denotes the derivatives with respect to \( \eta \).

3.1 The case of \( \eta' = \eta \)

In this case, the transformed Jimbo-Miwa equation \((3.3)\) becomes

\[
\begin{align*}
  a^3b\eta^2v'' + a^3b\eta v' + 3a^2bv^2 - (2b\omega + 3ac)v = 0.
\end{align*}
\]

We try to search for a rational solution \( v \) with \( m = n = 3 \). A direct computation with Maple tells that there are only two choices with a non-constant \( v \) and \( abc \neq 0 \):

\[
\begin{align*}
  v(\eta) &= \frac{4aq_1q_2\eta}{4q_1^2\eta^2 + 4q_1q_2\eta + q_1^2}, \quad \omega = \frac{a(a^2b - 3c)}{2b},
\end{align*}
\]
\[
v(\eta) = \frac{p_0(p_1^2\eta^2 + 16p_0p_1\eta + 16p_0^2)}{q_0(p_1^2\eta^2 - 8p_0p_1\eta + 16p_0^2)}, \quad a = \frac{3p_0}{q_0}, \quad \omega = \frac{9p_0(3bp_0^2 + cq_0^2)}{2bq_0^3}.
\]

Accordingly, we have the following travelling wave solutions to the 3 + 1 dimensional Jimbo-Miwa equation (3.3):

\[
u(x, y, z, t) = -\frac{2aq_1}{q_1 + 2q_2e^\xi} + d, \quad \xi = ax + by + cz - \frac{a(a^2b - 3c)}{2b}t,
\]

and

\[
u(x, y, z, t) = \frac{24p_0^2}{q_0(4p_0 - p_1e^\xi)} + \frac{p_0}{q_0}\xi + d, \quad \xi = -\frac{3p_0}{q_0}x + by + cz - \frac{9p_0(3bp_0^2 + cq_0^2)}{2bq_0^3}t,
\]

where the involved constants are all arbitrary. Note that the second class of travelling wave solutions, defined by (3.8), contain a linear function in \(\xi\).

We point out that all exact solutions generated in [29] belong to the above first class of travelling wave solutions, defined by (3.7), which can be observed by multiplying common factors, re-scaling the involved constants and checking the rational form of the \(v\)-function. Only for the solution in the formula (30) of [29], one first needs to cancel one common factor \(e^\xi + b_1\).

### 3.2 The case of \(\eta' = \alpha + \eta^2\)

In this case, the transformed Jimbo-Miwa equation (3.3) becomes

\[
a^3b\eta^4v'' + 2a^3b\eta^3v' + 2a^3b\alpha\eta^2v'' + a^3\alpha^2\eta^2v'' + 2a^3b\co\eta' + 3a^2b^2 - (2b\omega + 3ac)v = 0.
\]

We try to search for a rational solution \(v\) with \(m = 3\) and \(n = 1\). Similarly, a direct computation with Maple tells that there are only two choices with a non-constant \(v\) and \(abc \neq 0\):

\[
v(\eta) = -2a\alpha - 2a\eta^2, \quad \omega = -\frac{a(4a^2\alpha + 3c)}{2b},
\]

and

\[
v(\eta) = -\frac{2}{3}a\alpha - 2a\eta^2, \quad \omega = a(4a^2b\alpha - 3c).
\]

Through re-scaling the involved constants, we can know that the general nonzero value of \(\alpha\) does not engender more general solutions in the above two cases. Therefore, we select \(\alpha = 1\) and take \(\eta = \tan \xi\) and \(\eta = -\cot \xi\), we obtain the corresponding travelling wave solutions to the 3 + 1 dimensional Jimbo-Miwa equation (3.3):

\[
\begin{align*}
u(x, y, z, t) &= -2a \tan \xi + d, \quad \xi = ax + by + cz + \frac{a(4a^2 + 3c)}{2b}t, \\
u(x, y, z, t) &= -2a \tan \xi + \frac{4a\xi}{3} + d, \quad \xi = ax + by + cz - \frac{a(4a^2b - 3c)}{2b}t,
\end{align*}
\]

\[\text{(3.12)}\]
and
\[
\begin{cases}
 u(x, y, z, t) = 2a \cot \xi + d, \quad \xi = ax + by + cz + \frac{a(4b^2 + 3c)}{2b} t, \\
 u(x, y, z, t) = 2a \cot \xi + \frac{4a\xi}{3} + d, \quad \xi = ax + by + cz - \frac{a(4a^2b - 3c)}{2b} t,
\end{cases}
\]
(3.13)
where the involved constants are all arbitrary. Note that each second class of travelling wave solutions contain a linear function in $\xi$. The selection of $\alpha = 0$ leads to a class of rational solutions
\[
u(x, y, z, t) = 2a \frac{\xi}{\xi} + d = 4ab^2 + 3bcz + a(4a^2b - 3c) + d,
\]
(3.14)
where the involved constants are all arbitrary.

It is also direct to see that if we take $\eta = -\tanh \xi$ and $\eta = -\coth \xi$, we will obtain the same travelling wave solutions as those presented in the sub-section 3.1. Our solutions generated from (3.5) and (3.10) contain the solutions given in [27], and reduce to the solutions in the 2 + 1 dimensional case of $u = u(x, y, t)$ in [37].

### 3.3 Bäcklund transformation

Let $u = u(x, y, z, t)$ be a solution to the 3 + 1 dimensional Jimbo-Miwa equation (1.1). Evidently, if a function $v = v(x, y, z, t)$ satisfies
\[
3u_yv_{xx} + 3u_{xx}v_y + 3u_{xy}v_x + 3u_xv_{xy} + P_{JM}(v) = 0,
\]
(3.15)
where $P_{JM}$ is the Jimbo-Miwa operator defined in (1.1), then the sum of the two functions, $w = u + v$, gives another solution to the Jimbo-Miwa equation (1.1). Therefore, once we find a function $v$ satisfying (3.15), we get a new solution $w = u + v$ from a known one $u$. This forms a general auto-Bäcklund transformation for the Jimbo-Miwa equation (1.1).

**Sub-BT1:** First, it follows directly from the above Bäcklund transformation that if two solutions $u$ and $v$ of the Jimbo-Miwa equation (1.1) satisfy
\[
u_xv_{xx} + u_{xx}v_y + u_{xy}v_x + u_xv_{xy} = 0,
\]
(3.16)
then $w = u + v$ is a third solution to the Jimbo-Miwa equation (1.1). Further, we easily see that any solution $u = u(x, y, z, t)$ plus an arbitrary function $h(z, t)$ gives a new solution for the Jimbo-Miwa equation (1.1).

**Sub-BT2:** Second, if we take
\[
v = 2(\ln \phi)_x = 2\frac{\phi_x}{\phi},
\]
(3.17)
then $w = u + 2(\ln \phi)_x$ solves the Jimbo-Miwa equation (1.1), when $\phi = \phi(x, y, z, t)$ satisfies
\[
\begin{cases}
 \phi_{xxx} + 3u_x\phi_{xy} + 3u_y\phi_{xx} + 2\phi_{yt} - 3\phi_{xz} = 0, \\
 \phi_{xxx}\phi_y + 3\phi_x\phi_{xy} - 3\phi_{xx}\phi_{xy} + 3u_x\phi_x\phi_y + 3u_y\phi_x^2 + 2\phi_y\phi_t - 3\phi_x\phi_z = 0,
\end{cases}
\]
(3.18)
where \( u \) is a solution to the Jimbo-Miwa equation (1.1). This special Bäcklund transformation was introduced through the truncated Painlevé series for the Jimbo-Miwa equation (1.1) in [31] and for the derivative Jimbo-Miwa equation \((P_{JM}(u))_x = 0\) in [38]. It was successfully used to present diverse variable separated solutions involving arbitrary functions of two variables for the Jimbo-Miwa equation in [31] [32].

One one hand, if we take a solution
\[
\phi = a_0 + a_1 f(x) + a_2 q(x, t) + a_3 p(y, z),
\]
where \(a_i, 0 \leq i \leq 3\), and \(c_0\) are constants, then the two conditions in (3.18) will be satisfied when
\[
2q_t + q_{xxx} - q_x(c_0 - 3f_x) = 0,
\]
which determines the function \(f\). All the involved constants and the functions \(h, p, q\) are arbitrary. This generates the class of exact solutions to the Jimbo-Miwa equation (1.1), presented in [32]. More specially, when \(a_1 = a_2 = 0\), we obtain a sub-class of solutions given in [31].

On the other hand, if we take a solution
\[
\phi = a_1 + a_2 p(y, z)q(x, t),
\]
where \(a_1\) and \(a_2\) are constants, then the two conditions in (3.18) will be satisfied when
\[
(q_x)^2 - q_{xx}q_x = 0, \quad p_y(q_{xxx} + 2q_t) + 3g_y p_{xx} - 3p_z q_x = 0.
\]
A particular case is given by
\[
q(x, t) = e^{c_1x + c_2t}, \quad g(y, z) = -\frac{1}{3c_1^2} \int \frac{(c_1^2 + 2c_2)p_y - 3c_1 p_z}{p} dy,
\]
where \(c_1\) and \(c_2\) are constants, and thus the resulting new solution reads
\[
w = \frac{2a_2 c_1 p(y, z)e^{c_1x + c_2t}}{a_1 + a_2 p(y, z)e^{c_1x + c_2t}} - \frac{1}{3c_1^2} \int \frac{(c_1^2 + 2c_2)p_y - 3c_1 p_z}{p} dy + h(z, t),
\]
where all the involved constants and functions are all arbitrary.

**Sub-BT3:** Third, if we take a travelling wave solution \(u = u(x, y, z, t) = u(ax + by + cz - \omega t)\) to the Jimbo-Miwa equation (1.1), then the function
\[
w(x, y, z, t) = u(ax + by + cz - \omega t) + a'dx + b'y + c'z - \omega't + d',
\]
where \(a', b', c', d', \omega'\) are all constants satisfying
\[
a'b' + a'b = 0,
\]
presents a new solution to the Jimbo-Miwa equation (1.1).

Now taking advantage of this feature in (3.25), it is direct to check by multiplying common factors and re-scaling the involved constants, particularly the angular wave numbers and the wave frequency, that all exact solutions presented in [30] are special examples of combination solutions of the travelling wave solutions, defined in (3.7), (3.8), (3.12) and (3.13), and some linear function solutions.
A new systematical solution procedure to constructing exact solutions to nonlinear partial differential equations, both constant-coefficient and variable-coefficient, is proposed, based upon rational function transformations. Its key point is to search for rational solutions to variable-coefficient ordinary differential equations transformed from given partial differential equations. Together with an auto-Bäcklund transformation, an application of our method leads to various new travelling wave solutions to the 3 + 1 dimensional Jimbo-Miwa equation (1.1).

Our method allows us carry out the solution process of nonlinear wave equations more systematically and conveniently by computer algebra systems such as Maple, Mathematica, MuPAD and Matlab, unifying the tanh-function method, the sech-function method, the homogeneous balance method, the extended tanh-function method, the sine-cosine method, the F-expansion method, the mapping method and the extended F-expansion method. The presented method also works for systems of nonlinear wave equations, and it can be applied to construction of special function solutions, like the extended F-expansion method [21] [22], the extended mapping method [39] and the improved extended F-expansion method [40] [41] [42] [43].

Besides travelling wave solutions and variable separated solutions, the 3 + 1 dimensional Jimbo-Miwa equation (1.1) has the following class of polynomial solutions:

\[ u_1 = a_0 + a_1 x + a_2 y + a_3 z + a_4 t + a_5 x z + a_6 x t + a_7 y z + \frac{3}{2} a_5 y t + a_8 z t, \]  
(4.1)

where \( a_i, \quad 0 \leq i \leq 8 \), are arbitrary constants. These are all polynomial solutions found with Maple among the class of polynomial functions with individual degrees of the independent variables less than two. Letting \( u = u_1 \), the condition (3.16) becomes

\[ a_2 v_{xx} + a_1 v_{xy} = 0, \]  
(4.2)

and thus, one can easily generate other solutions, for example,

\[ u_2 = u_1|_{a_2=0} + f(x, t) + h(z, t), \quad u_3 = u_1 + g(y, z) + h(z, t), \]  
(4.3)

where \( u_1 \) is defined by (4.1) and \( f, g, h \) can be arbitrary functions. Taking \( f, g, h \) as polynomials in the indicated variables engender two other classes of polynomial solutions.

All the presented solutions, including travelling wave solutions, variable separated solutions and polynomial solutions, show the remarkable richness of the solution space of the 3 + 1 dimensional Jimbo-Miwa equation (1.1), though the equation itself is expected to be non-integrable since it has solutions which are not single-valued functions in the neighborhood of their singularity surfaces [24]. A similar diversity situation of exact solutions can be found in solution spaces of typical nonlinear wave equations, for instance, the 1 + 1 dimensional
equations: the Korteweg-de Vries equation [44, 45], the Boussinesq equation [46, 47], the nonlinear Schrödinger type equation [48] and the Hirota-Satsuma coupled KdV equation [49], and the 2 + 1 dimensional equations: the Kadomtsev-Petviashvili equation [50], the Davey-Stewartson equation [51, 52] and the Boiti-Leon-Pempinelli dispersive long-wave system [53].

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References

[1] Lan HB and Wang KL. Exact solutions for two nonlinear equations: I. J Phys A: Math Gen 1990;23:3923–8.
[2] Malfliet W. Solitary wave solutions of nonlinear wave equations. Amer J Phys 1992;60:650–4.
[3] Malfliet W and Hereman W. The tanh method I: Exact solutions of nonlinear evolution and wave equations. Phys Scripta 1996;54:563–8.
[4] Parkes EJ and Duffy BR. An automated tanh-function method for finding solitary wave solutions to non-linear evolution equations. Comput Phys Commun 1996;98:288-300.
[5] Li ZB and Liu YP. RATH: A Maple package for finding travelling solitary wave solutions to nonlinear evolution equations. Comput Phys Commun 2002;148:256–66.
[6] Ma WX. Travelling wave solutions to a seventh order generalized KdV equation. Phys Lett A 1993;180:221–4.
[7] Ma WX and Zhou DT. Solitary wave solutions to a generalized KdV equation. Acta Phys Sinica 1993;42:1731–4.
[8] Duffy BR and Parkes EJ. Travelling solitary wave solutions to a seventh-order generalized KdV equation. Phys Lett A 1996;214:271–2.
[9] Parkes EJ, Zhu Z, Duffy BR and Huang HC. Sech-polynomial travelling solitary-wave solutions of odd-order generalized KdV equations. Phys Lett A 1998;248:219–24.
[10] Wang ML. Solitary wave solutions for variant Boussinesq equations. Phys Lett A 1995;199:169–72.
[11] Wang ML. Exact solutions for a compound KdV-Burgers equation. Phys Lett A 1996;213:279–87.
[12] Ma WX and Fuchssteiner B. Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation. Internat J Non-Linear Mech 1996;31:329–38.
[13] Fan EG. Extended tanh-function method and its applications to nonlinear equations. Phys Lett A 2000;277:212–8.
[14] Han TW and Zhuo XL. Rational form solitary wave solutions for some types of high order nonlinear evolution equations. Ann Differential Equations 2000;16:315–9.
[15] Yan CT. A simple transformation for nonlinear waves. Phys Lett A 1996;224:77–84.
[16] Wazwaz AM. The tanh-coth method for solitons and kink solutions for nonlinear parabolic equations. Appl Math Comput 2007;188:1467–75.
[17] Liu SK, Fu ZT, Liu SD and Zhao Q. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys Lett A 2001;289:69–74.
[18] He JH and Wu XH. Exp-function method for nonlinear wave equations. Chaos Solitons Fractals 2006;30:700–8.
[19] Zhou Y, Wang ML and Wang YM. Periodic wave solutions to a coupled KdV equations with variable coefficients. Phys Lett A 2003;308:31–6.
[20] Peng YZ. A Mapping Method for obtaining exact travelling wave solutions to nonlinear evolution equations. Chinese J Phys 2003;41:103–10.
[21] Liu JB and Yang KQ. The extended F-expansion method and exact solutions of nonlinear PDEs. Chaos Solitons Fractals 2004;22:111–21.
[22] Chen Y and Yan ZY. New exact solutions of (2+1)-dimensional Gardner equation via the new sine-Gordon equation expansion method. Chaos Solitons Fractals 2005;26:399–406.
[23] Jimbo M and Miwa T. Solitons and infinite-dimensional Lie algebras. Publ Res Inst Math Sci 1983;19:943–1001.
[24] Dorizzi B, Grammaticos B, Ramani A and Winternitz P. Are all the equations of the Kadomtsev-Petviashvili hierarchy integrable? J Math Phys 1986;27:2848–52.
[25] Rubin J and Winternitz P. Point symmetries of conditionally integrable nonlinear evolution equations. J Math Phys 1990;31:2085–90.
[26] Tian B and Gao YT. Beyond travelling waves: a new algorithm for solving nonlinear evolution equations. Comput Phys Commun 1996;95:139–42.
[27] Senthilvelan M. On the extended applications of homogeneous balance method. Appl Math Comput 2001;123:381–8.
[28] Wazwaz AM. Multiple-soliton solutions for the Calogero-Bogoyavlenskii-Schiff, Jimbo-Miwa and YTSF equations. Appl Math Comput 2008;203:592–7.
[29] Öziş T and Aslan I. Exact and explicit solutions to the (3 + 1)-diemnsional Jimbo-Miwa equation via the Exp-function method. Phys Lett A 2008;372:7011–5.
[30] Wazwaz AM. New solutions of distinct physical strcutures to high-dimensional nonlinear evolution equations. Appl Math Comput 2008;196:363–70.
[31] Liu XQ and Jiang S. New solutions of the 3 + 1 dimnesional Jimbo-Miwa equation. Appl Math Comput 2004;158:177–84.
[32] Tang XY and Liang ZF. Variable separated solutions for the (3 + 1)-dimensional Jimbo-Miwa equation. Phys Lett A 2006;351:398–402.
[33] Feng ZS. On explicit exact solutions to the compound Burgers-KdV equation. Phys Lett A 2002;293:57–66.
[34] Li JB, Wu JH and Zhu HP. Traveling waves for an integrable higher order KdV type wave equations. Internat J Bifur Chaos Appl Sci Engrg 2006;16:2235–60.
[35] Ma WX. An exact solution to two-dimensional Korteweg-de Vries-Burgers equation. J Phys A: Math Gen 1993;26:L17–20.
[36] Yomba E. On exact solutions of the coupled Klein-Gordon-Schrödinger and the complex coupled KdV equations using mapping method. Chaos Solitons Fractals 2004;21:209–29.

[37] Chen YZ and Ding XW. Exact travelling wave solutions of nonlinear evolution equations in (1 + 1) and (2 + 1) dimensions. Nonlinear Anal 2005;61:1005–13.

[38] Tian B, Gao YT and Hong W. The solitonic features of a nonintegrable (3 + 1)-dimensional Jimbo-Miwa equation. Comput Math Appl 2002;44:525–8.

[39] El-Wakil SA and Abdou MA. The extended mapping method and its applications for nonlinear evolution equations. Phys Lett A 2006;358:275–82.

[40] Wang D and Zhang HQ. Further improved $F$-expansion method and new exact solutions of Konopelchenko-Dubovsky equation. Chaos Solitons Fractals 2005;25:601–10.

[41] Zhang S and Xia TC. A generalized $F$-expansion method and new exact solutions of Konopelchenko-Dubovsky equations. Appl Math Comput 2006;183:1190–200.

[42] Abdou MA. The extended $F$-expansion method and its application for a class of nonlinear evolution equations. Chaos Solitons Fractals 2007;31:95–104.

[43] Gao L, Ma WX and Xu W. Travelling wave solutions to Zufiria’s higher-order Boussinesq type equations. Nonlinear Anal (2008) to appear.

[44] Ma WX and You Y. Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions. Trans Amer Math Soc 2005;357:1753–78.

[45] Aktosun T and van der Mee C. Explicit solutions to the Korteweg-de Vries equation on the half line. Inverse Problems 2006;22:2165–74.

[46] Li CX, Ma WX, Liu XJ and Zeng YB. Wronskian solutions of the Boussinesq equation—solitons, negatons, positons and complexitons. Inverse Problems 2007;23:279–96.

[47] Ma WX, Li CX and He JS. A Second Wronskian Formulation of the Boussinesq Equation. Nonlinear Anal (2008) to appear.

[48] Lee JH, Lin CK and Pashaev OK. Shock waves, chiral solitons and semiclassical limit of one-dimensional anyons. Chaos Solitons Fractals 2004;19:109–28.

[49] Tam HW, Ma WX, Hu XB and Wang DL. The Hirota-Satsuma coupled KdV equation and a coupled Ito system revisited. J Phys Soc Jpn 2000;69:45–52.

[50] Biondini G and Kodama Y. On a family of solutions of he Kadomtsev-Petviashvili equation which also satisfy the Toda lattice hierarchy. J Phys A: Math Gen 2003;36:10519–36.

[51] Hietarinta J and Hirota R. Multidromion solutions to the Davey-Stewartson equation. Phys Lett A 1990;145:237–44.

[52] Lou SY and Lu JZ. Special solutions from the variable separated approach: the Davey-Stewartson equation. J Phys A: Math Gen 1996;29:4209–15.

[53] Ma WX. Diversity of exact solutions to a restricted Boiti-Leon-Pempinelli dispersive long-wave system. Phys Lett A 2003;319:325–33.