Nonfibered knots and representation shifts

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Abstract

A conjecture of [13] states that a knot is nonfibered if and only if its infinite cyclic cover has uncountably many finite covers. We prove the conjecture for a class of knots that includes all knots of genus 1, using techniques from symbolic dynamics.

Keywords: Knot, knot group, representation shift.

1 Introduction

Let $G$ be a finitely presented group with epimorphism $\chi : G \to \mathbb{Z}$. The kernel $K$ of $\chi$ need not be finitely generated. However, $K$ is finitely presented as a $\mathbb{Z}$-operator group [11]. In [12] [13] the authors exploited this structure to show that the representations of $K$ into a fixed finite group $\Sigma$ form a shift of finite type, a simple dynamical system described by a finite directed graph. We call this dynamical system the representation shift of $K$ in $\Sigma$. When $G$ is a knot or link group, representation shifts inform us about the algebraic topology of finite covering spaces from a purely dynamical perspective.

We review basic definitions of representation shifts and give a partial solution to Conjecture 4.4 of [13]. The complete solution would characterize nonfibered knots as knots with complicated representation shifts, where complexity is measured by topological entropy.

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2 Review of Representation Shifts

An augmented group system $\mathcal{G} = (G, \chi, x)$ is a triple consisting of a finitely presented group $G$, epimorphism $\chi : G \to \mathbb{Z}$ and distinguished element $x \in G$ such that $\chi(x) = 1$. Two such systems $\mathcal{G}_i = (G_i, \chi_i, x_i), \; i = 1, 2$ are equivalent (and regarded as the same) if there exists an isomorphism $f : G_1 \to G_2$ such that $f(x_1) = x_2$ and $\chi_1 = \chi_2 \circ f$.

Example 2.1. An augmented group system is associated to an oriented knot $k \subset S^3$ in a canonical manner. Let $G = \pi_1(S^3 \setminus k, p)$, where the base point $p$ is contained on the boundary $\partial N(k)$ of a tubular neighborhood $N(k) = S^1 \times D^2$ of $k$. Let $x$ be the homotopy class of a meridian $m \subset \partial N(k)$, with orientation acquired from $k$. Finally, let $\chi : G \to \mathbb{Z}$ be the abelianization homomorphism that sends $x$ to 1. It follows from the uniqueness of tubular neighborhoods that $\mathcal{G} = (G, \chi, x)$ is well defined.

We denote the kernel of $\chi$ by $K$. Given any finite group $\Sigma$, we consider the space $\text{Hom}(K, \Sigma)$ of representations $\rho : K \to \Sigma$. The basis for its topology is given by the sets

$$N_{a_1, \ldots, a_s}(\rho) = \{ \rho' \mid \rho'(a_i) = \rho(a_i), \; i = 1, \ldots, s \},$$

where $a_1, \ldots, a_s$ varies over all finite collections of elements of $K$. The topology is the compact-open topology where $K$ and $\Sigma$ are discrete spaces. Roughly speaking, representations are close in $\text{Hom}(K, \Sigma)$ if they agree on large finitely generated subgroups of $K$. The distinguished element $x$ induces a self-map $\sigma_x$ of $\text{Hom}(K, \Sigma)$ defined by

$$\sigma_x \rho(a) = \rho(x^{-1}ax) \; \forall a \in K.$$ 

It is easily seen that $\sigma_x$ is a homeomorphism.

The representation shift associated to $\mathcal{G} = (G, \chi, x)$ and $\Sigma$ is the pair $(\text{Hom}(K, \Sigma), \sigma_x)$. We denote it by $\Phi_{\Sigma}(\mathcal{G})$. It is a dynamical system well defined up to topological conjugacy. More precisely, if $\mathcal{G}_i, \; i = 1, 2$, are equivalent augmented group systems, then there exists a homeomorphism $F$ of the underlying spaces of $\Phi_{\Sigma}(\mathcal{G}_i)$ such that $F \circ \sigma_{x_1} = \sigma_{x_2} \circ F$.

The representation shift $\Phi_{\Sigma}(\mathcal{G})$ is an example of a shift of finite type, a special type of expansive 0-dimensional dynamical system, one that can be described by a finite directed graph. (See [4].) We use combinatorial group theory to construct such a graph for a representation shift.

Given an augmented group system $\mathcal{G} = (G, \chi, x)$, we can describe $G$ as an HNN extension $\langle x, B \mid x^{-1}ax = \phi(a), \; \forall a \in U \rangle$, where $B$ is a finitely
generated subgroup of $K$, and $U, V$ are isomorphic finitely generated subgroups of $B$ with isomorphism $\phi : U \to V$ (see [6]). The subgroup $B$ is an HNN base. One can choose $B$ so that it contains any prescribed finite subset of $K$ (see [10]).

**Example 2.2.** Let $G = (G, \chi, x)$ be an augmented group system associated to a knot, as in Example 2.1. An HNN decomposition for $G$ can be obtained in a natural way. Begin with a $\pi_1$-incompressible Seifert surface for $k$ meeting the exterior $E(k) = S^3 \setminus \text{int } N(k)$ in a connected surface $S$. Let $(W; S_0, S_1)$ be the resulting cobordism, with boundary comprising two copies $S_0, S_1$ of $S$ joined by an annulus $\partial S \times I$. Let $B = \pi_1(W, p)$, where the basepoint $p$ lies on the boundary of $S_0$. Let $U = \pi_1(S_0, p)$. The meridian $m$ appears as a path from $p \in S_0$ to a point $p_1 \in S_1$. Use the path to regard $\pi_1(S_1, p_1)$ as a subgroup $V$ of $B$. Clearly $G$ is described as $(B; U, V, \phi)$, where $\phi$ is induced by the gluing of $S_0$ to $S_1$ when recovering the exterior $E(k)$.

Conjugation by $x$ induces an automorphism of $K$. Let $B_j = x^{-j}Bx^j$, $U_j = x^{-j}Ux^j$ and $V_j = x^{-j}Vx^j$, for $j \in \mathbb{Z}$. Then $K$ is described as an infinite amalgamated free product

$$K = \langle B_j \mid V_j = U_{j+1}, \forall j \in \mathbb{Z} \rangle.$$

The vertex set of the graph $\Gamma$ consists of all representations $\rho_0 : U \to \Sigma$, a finite set since $U$ is finitely generated. If $\bar{\rho}_0$ is a representation from $B$ to $\Sigma$, then we draw a directed edge labeled $\bar{\rho}_0$ from the vertex $\rho_0 = \bar{\rho}_0|_U$ to the vertex $\rho_0 = \bar{\rho}_0|_V \circ \phi$. ($\Gamma$ may have parallel edges.) Consider a bi-infinite path in $\Gamma$ given by an edge sequence

$$\cdots \bar{\rho}_{-2} \bar{\rho}_{-1} \bar{\rho}_0 \bar{\rho}_1 \bar{\rho}_2 \cdots$$

The representations $B_j \to \Sigma$ given by $a \mapsto \bar{\rho}_j(\chi^j a \chi^{-j})$ have a unique common extension $\rho : K \to \Sigma$. Conversely, any representation $\rho : K \to \Sigma$ arises from such a path, and uniquely. Thus bi-infinite paths of the graph $\Gamma$ correspond bijectively to elements of $\text{Hom}(K, \Sigma)$. The map $\sigma_x$ acts as the left coordinate shift on the sequence of edges.

We may “prune” $\Gamma$ by removing any vertex or edge that is not contained in a bi-infinite path. The resulting graph has finitely many bi-infinite paths iff it consists of a collection of disjoint cycles. It contains uncountably many bi-infinite paths iff it contains two cycles with at least one common vertex.

A representation $\rho \in \Phi_\Sigma(G)$ has period $r$ if $\sigma_x^r(\rho) = \rho$. Such representations correspond to closed paths in $\Gamma$ with length dividing $r$. The set of
representations with period $r$ is denoted by $\text{Fix}(\sigma^r_{x})$. If $M_r$ is the $r$-fold cyclic cover of $\mathbb{S}^3$ branched over a knot $k$, then $\text{Fix}(\sigma^r_{x})$ is in natural bijective correspondence with $\text{Hom}(\pi_1 M_r, \Sigma)$ [13]. This correspondence connects dynamical properties of the representation shift with topological properties of $k$.

Topological entropy is one measure of complexity for a dynamical system. For a shift of finite type, it can be computed as the log of the spectral radius of the adjacency matrix $A$ of any directed graph that describes the shift. (Here $A_{i,j}$ is the number of edges from the $i$th vertex to the $j$th.) Consequently, the topological entropy of $\Phi_{\Sigma}(\mathcal{G})$, denoted by $h_{\Sigma}(\mathcal{G})$, is the exponential growth rate of $|\text{Hom}(\pi_1 M_r, \Sigma)|$ (see [13]). Notice that if $K$ is finitely generated, then $\Phi_{\Sigma}(\mathcal{G})$ is finite for all $\Sigma$, and so in this case $h_{\Sigma}(\mathcal{G})=0$.

Let $S_N$ denote the symmetric group on $\{1, \ldots, N\}$. It is well known that elements $\rho \in \text{Hom}(K, S_N)$ correspond in a finite-to-one manner with subgroups $H \leq K$ with index no greater than $N$. The correspondence is

$$\rho \mapsto \{g \in K \mid \rho(g)(1) = 1\}.$$ 

The preimage of a subgroup of index $N$ consists of $(N-1)!$ transitive representations. (A representation $\rho$ is transitive if $\rho(K)$ operates transitively on $\{1, \ldots, N\}$.) Note that if $\Phi_{S_N}(\mathcal{G})$ is uncountable, then $\Phi_{\Sigma}(\mathcal{G})$ is finite for all $\Sigma$, and so in this case $h_{\Sigma}(\mathcal{G})=0$.

We summarize the results of this section. Recall that any finite group embeds in a sufficiently large symmetric group.

**Proposition 2.3.** Let $k \subset \mathbb{S}^3$ be a knot with associated augmented group system $\mathcal{G}$. Then the following statements are equivalent.

1. The infinite cyclic cover of $k$ has uncountably many finite covers.
2. The representation shift $\Phi_{\Sigma}(\mathcal{G})$ is uncountable, for some finite group $\Sigma$.
3. The topological entropy $h_{\Sigma}(\mathcal{G})$ is positive, for some finite group $\Sigma$.
4. $\lim_{r \to \infty} \frac{1}{r} \log |\text{Hom}(\pi_1 M_r, \Sigma)|$ is positive, for some finite group $\Sigma$.

**3 Nonfibered knots**

We recall that a knot $k \subset \mathbb{S}^3$ is fibered if its exterior $E(k) = \mathbb{S}^3 \setminus \text{int} \ N(k)$ fibers over the circle. It is no loss of generality to assume that the fibration restricts to the standard projection $\partial N(k) \simeq k \times \mathbb{S}^1 \to \mathbb{S}^1$. Hence $E(k)$ is
seen to be homeomorphic to a mapping torus \( S \times I / F \), where \( F : S \to S \) is a homeomorphism of a minimal-genus Seifert surface \( S \) of \( k \).

If \( k \) is fibered, then the commutator subgroup \( G' \) of its group is finitely generated and free, isomorphic to \( \pi_1 S \). Conversely, a theorem of J. Stallings [15] implies that if \( k \) is a knot such that \( G' \) is finitely generated, then in fact \( G' \) is free and \( k \) is fibered.

If \( k \) is fibered and \( G \) is its associated augmented group system, then for any finite group \( \Sigma \), the representation shift \( \Phi_\Sigma(G) \) is finite. Its order is \( \left| \Sigma \right| 2^g \), where \( g \) is the genus of \( k \) (equal to the genus of its fiber). The trefoil and figure-eight knots are the only fibered knots of genus 1.

Conjecture 4.4 of [13] proposes a characterization of nonfibered knots. It states that \( k \) is nonfibered iff the entropy \( h_\Sigma(G) \) is positive for some finite group \( \Sigma \).

Remark 3.1. (1) In terms of the HNN base \( B \) described above, the condition that \( k \) is not fibered is equivalent to the condition that \( U \) is a proper subgroup of \( K \). Lemma 2.3 (Substitution Lemma) of [12] provides a strategy for showing that some \( \Phi_{S_n}(G) \) is uncountable: Find a periodic element of \( \Phi_{S_N}(G) \) such that some symbol, say \( N \), is fixed by every permutation in the image of \( U \) but moved by some element of \( \rho(K) \). Recall that periodic representations correspond to cycles in the graph \( \Gamma \). By introducing a new symbol (enlarging \( S_N \) to \( S_{N+1} \)), we can construct another periodic representation corresponding to a second cycle, one that branches from the first. Then \( \Phi_{S_{N+1}}(G) \) is uncountable.

(2) For our strategy, it suffices to find any representation \( \tilde{\rho} : G \to \Sigma \) such that \( \rho(U) \) is a proper subgroup of \( \rho(K) \). For given such a representation, and letting \( \rho : K \to \Sigma \) be the restriction, we enumerate the cosets of \( \rho(U) \) in \( \rho(K) \), say \( 1, \ldots, N \) \((N > 1)\). In a natural way, \( \rho \) determines an element of \( \Phi_{S_n}(G) \): \( a \in K \) is sent to the transitive permutation of cosets given by right multiplication by \( \rho(a) \). Note that if \( a \in U \), then such a permutation fixes the symbol corresponding to \( \rho(U) \). Finally, we note that if \( r \) is the order of \( \tilde{\rho}(x) \) in \( \Sigma \), then \( \sigma_r^a \rho = \rho \), since \( (\sigma_r^a \rho)(a) = \rho(x^{-r} ax^r) = \tilde{\rho}(x^{-r})^{\rho(a)} \tilde{\rho}(x)^r = \rho(a) \), for all \( a \in K \).

The representation \( \tilde{\rho} \) in the Remark 3.1 (2) “separates” the subgroup \( U \) from some element \( a \in K \).

In general, a subgroup \( U \) of a group \( G \) is separable if for any element \( a \in G \setminus U \), there exists a finite-index subgroup of \( G \) that contains \( U \) but not \( a \). Equivalently, there exists a finite representation \( \tilde{\rho} : G \to \Sigma \) such that \( \tilde{\rho}(a) \notin \tilde{\rho}(U) \). The strategy outlined in Remark 3.1 (2) requires only that \( U \) can be separated from some element of \( K \setminus U \).
Definition 3.2. An element $a \in G \setminus U$ is separable from $U$ if there exists a subgroup $H$ of finite index in $G$ containing $U$ but not $a$.

Question 15 of [16] asks if any finitely generated subgroup of a finitely-generated Kleinian group is separable. An affirmative answer would establish Theorem 3.4 for all hyperbolic knots. Although Thurston’s question remains open, a result of D. Long and G. Niblo [5] enables us to apply our strategy in the case of genus-1 knots (see also remarks that follow).

The theorem of Long and Niblo has been used by S. Friedl and S. Vidussi in [1] to show that twisted Alexander polynomials corresponding to finite representations decide if a genus-1 knot is fibered.

Theorem 3.3. (D. Long and G. Niblo [5]) Let $M$ be an orientable Haken 3-manifold. If $i : T \hookrightarrow M$ is an incompressibly embedded torus, then $i_* (\pi_1 T)$ is separable in $\pi_1 M$.

Theorem 3.4. Let $k$ be a knot of genus 1. Then $k$ is nonfibered iff the conclusions of Proposition 2.3 hold.

Proof. One implication of the theorem is clear: if the conclusion of Proposition 2.3 holds, then $k$ is nonfibered.

Assume that $k$ is nonfibered. Consider the 3-manifold $M$ obtained by 0-framed surgery on $k$; that is, by removing and replacing a tubular neighborhood $N(k) = k \times D^2$ in such a way that each disk $* \times D^2$ bounds a longitude of $k$. By results of [3], $M$ is irreducible. We denote the fundamental group of $M$ by $\hat{G}$.

The addition of a meridional disk converts a genus-1 Seifert surface $S$ for $k$ to a torus $\hat{S}$ in $M$. Since $\hat{S}$ is dual to a nontrivial cohomology class and $M$ is irreducible, we see that $\hat{S}$ is incompressible. Note in particular that $M$ is Haken.

Obtain an HNN decomposition $(\hat{B}; \hat{U}, \hat{V})$ for $\hat{G}$ much as we did for $G$, by splitting $M$ along $\hat{S}$. Here $\hat{U} = \pi_1 \hat{S}$. Since $k$ is not fibered, neither is $M$ [2]. Hence $\hat{U}$ must be a proper subgroup of $\hat{B}$. Select an element $\hat{a} \in B \setminus \hat{U}$. By Theorem 3.3 there exists a finite group $\Sigma$ and homomorphism $\hat{\rho} : \hat{G} \twoheadrightarrow \Sigma$ such that $\hat{\rho}(\hat{a}) \notin \hat{U}$.

The group $\hat{G}$ is a quotient of $G$. Let $p$ be the natural projection. Note that $p(\hat{U}) = \hat{U}$. Choose $a \in K$ such that $p(a) = \hat{a}$. Define $\rho = \hat{\rho} \circ p : G \twoheadrightarrow \Sigma$.

Remark 3.1(2) completes the proof.

Genus-1 knots are plentiful, the simplest examples being the twist knots (e.g. the knots $5_2, 6_1$) and doubled knots (obtained from a knot and any push-off by joining with a clasp). We extend the collection of nonfibered
knots with uncountable representation shifts by considering also any knot $k$ with group $G$ that maps homomorphically onto the group $\tilde{G}$ of a nonfibered genus-1 knot $\tilde{k}$. Examples of such knots $k$ include satellite knots with genus-1 pattern knot $[8]$.

**Corollary 3.5.** Let $k$ be a knot. Assume that the group of $k$ maps onto the group of a nonfibered knot $\tilde{k}$ of genus 1. Then $k$ is nonfibered and the conclusions of Proposition 2.3 hold.

**Proof.** Assume that $h : G \to \tilde{G}$ is an epimorphism, where $G$, $\tilde{G}$ are the groups of $k$, $\tilde{k}$, respectively. Let $K$, $\tilde{K}$ denote the respective commutator subgroups, and $x$, $\tilde{x}$ the meridianal generators of $k$, $\tilde{k}$.

Since $h(K) = \tilde{K}$ and $\tilde{K}$ is not finitely generated, we see at once that $K$ is not finitely generated. Hence $k$ is nonfibered.

If $h(x) = \tilde{x}$, then for any finite group $\Sigma$, the representation shift $\Phi_{\Sigma}(\tilde{G})$ corresponding to $\tilde{k}$ is a subshift of the representation shift $\Phi_{\Sigma}(G)$ corresponding to $k$; that is, $\text{Hom}(\tilde{K}, \Sigma)$ is a subspace of $\text{Hom}(K, \Sigma)$ with the shift map $\sigma_x$ restricting to $\sigma_{\tilde{x}}$. The epimorphism $h$ induces an embedding: $h^*\rho = \rho \circ h$. It follows that the topological entropy $h_{\Sigma}(\tilde{G})$ is no less than $h_{\Sigma}(G)$. By theorem 3.4, $h_{\Sigma}(\tilde{G}) > 0$ for some finite group $\Sigma$. Hence for such a group, $h_{\Sigma}(G)$ is also positive.

If $h(x) \neq \tilde{x}$, then there exists $a \in K$ such that $h(ax) = \tilde{x}^\epsilon$, where $\epsilon = \pm 1$. We may assume without loss of generality that $\epsilon = 1$. In this case, we replace $x$ by $ax$. Of course the augmented group system $\tilde{G}$ and associated representation shifts $\Phi_{\Sigma}(\tilde{G})$ change. However, by a result of [11], the topological entropy of the representation shift remains unchanged. As in the case in which $h(x) = \tilde{x}$, there exists a finite group $\Sigma$ such that $h_{\Sigma}(G) > 0$. \[\square\]

**References**

[1] S. Friedl and S. Vidussi, *Symplectic $S^1 \times N^3$ subgroup separability, and vanishing Thurston norm*, preprint, arXiv:math.GT/0701717

[2] D. Gabai, *Foliations and surgery on knots*, Bull. Amer. Math. Soc. 15 (1986), 83–87.

[3] D. Gabai, *Foliations and the topology of 3-manifolds. III*, J. Diff. Geom. 26 (1987), 479–536.

[4] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, Cambridge, 1995.
[5] D. Long and G.Niblo, *Subgroup separability and 3-manifold groups*, Math. Z. **207** (1991), 209–215.

[6] R.C. Lyndon and P.E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin, New York, 1977.

[7] D.S. Silver, *Augmented group systems and n-knots*, Math. Ann. **296** (1993), 585–593.

[8] W. Whitten and D.S. Silver, *Knot group epimorphisms*, Journal of Knot Theory and its Ramif. **15** (2006), 153–166.

[9] D.S. Silver and S.G. Williams, *Generalized n-colorings of links*, Proceedings of the 1995 Conference in Knot Theory at the Banach Center in Warsaw, 381–394.

[10] D. S. Silver, *HNN bases and high-dimensional knot groups*, Proc. Amer. Math. Soc. **124** (1996), 1247–1252.

[11] D.S. Silver and S.G. Williams, *Augmented group systems and shifts of finite type*, Israel J. Math. **95** (1996), 231–251.

[12] D.S. Silver and S.G. Williams, *On groups with uncountably many subgroups of finite index*, J. Pure and Appl Algebra **140** (1999), 75–86.

[13] D.S. Silver and S.G. Williams, *Knot invariants from symbolic dynamical systems*, Trans. Amer. Math. Soc. **351** (1999), 3243–3265.

[14] D.S. Silver and S.G. Williams, *Crowell’s derived group and twisted polynomials*, Journal of Knot Theory and its Ramif. **15** (2006), 1079–1094.

[15] J. Stallings, *On fibering certain 3-manifolds*, in Topology of 3-manifolds, Proc. 1961 Topology Institute Univ. Georgia (ed. M.K. Fort, Jr.), 95–100, Prentice-Hall Inc., Englewood Cliffs, NJ.

[16] W.P. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–381.

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