Einstein - Bianchi system with sources

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Abstract

We write a first order symmetric hyperbolic system coupling the Riemann tensor with the derivative of an electromagnetic 2-form and the dynamical acceleration of a perfect relativistic fluid. We determine the associated, coupled, Bel-Robinson type energy, and the integral equality that it satisfies.

Dedicated to Guy Boillat with deep esteem and affection.

1 Introduction.

The interest of formulating fundamental physical laws as first order symmetric hyperbolic systems has been stressed, after the foundational work of K. O. Friedrichs, by various authors. Particularly important developments in the case of systems of conservation laws have been obtained by Lax, Boillat, Ruggeri and Strumia, Anile, Dafermos. They have used the existence, in many physical cases, of an additional convex conserved density, associated to an entropy functional. They have obtained symmetric first order hyperbolic systems in many cases of physical interest, by introducing auxiliary variables similar to Lagrange multipliers.

In the case of Einstein’s gravitation theory, the problem seems more delicate. There is no local density of gravitational energy, nor entropy. The effective strength of the gravitational field lies in the Riemann tensor of the spacetime metric. Its evolution is governed by the so-called higher order equations (Bel 1958, Lichnerowicz 1964), deduced from the Bianchi identities. The system satisfied by the trace free part of the Riemann tensor, the
Weyl tensor, was some time ago recognized as a linear, first order symmetrizable hyperbolic system (FOSH), with constraints, homogeneous in vacuum. See H. Friedrich (1996) and references therein. The evolution equations for the Riemann tensor itself, now renamed Bianchi equations, have also been written (CB-Yo 1997, 2001) as a FOSH system, made explicit in terms of four two-tensors, introduced by Bel (1958), the electric and magnetic gravitational fields and corresponding inductions relative to a Cauchy adapted frame. A FOSH system has also been written for the Bianchi equations (no longer homogeneous) and the dynamical acceleration of a perfect fluid (CB-Yo2002). In this article, dedicated to one of the best specialist of these FOSH systems, we couple the Bianchi equations with the equations satisfied by the dynamical acceleration of a charged fluid and the derivatives of the associated Maxwell field.

2 Einstein equations with sources.

2.1 Definitions.

The Einstein equations, on a 4-dimensional manifold $V$, link the Ricci tensor of a pseudo Riemannian metric $g$ of Lorentzian signature with the stress energy tensor $T$ of sources. These equations read

\[ \text{Ricci}(g) = \rho = T - \frac{1}{2} g(tr g T), \quad (2.1) \]

that is, in local coordinates,

\[ R_{\alpha\beta} = \rho_{\alpha\beta}, \quad \text{with} \quad \rho_{\alpha\beta} \equiv T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^{\lambda}. \]

The stress energy tensor of an electrically charged, relativistic, perfect fluid is the sum of the fluid stress energy tensor:

\[ t_{\alpha\beta} \equiv (\mu + p) u_{\alpha} u_{\beta} + pg_{\alpha\beta} \]

and the Maxwell tensor:

\[ \tau_{\alpha\beta} \equiv F^{\alpha}_{\lambda} F^{\beta\lambda} - \frac{1}{4} g_{\alpha\beta} F^{\lambda\mu} F_{\lambda\mu}. \quad (2.2) \]
In these formulas $u^\alpha$ is the unit kinematical velocity satisfying $u^\alpha u_\alpha = -1$; $\mu, p, S$ are the specific energy, pressure and entropy. These thermodynamic quantities are assumed to be positive and linked by an equation of state:

$$\mu = \mu(p, S).$$

(2.3)

The entropy $S$ satisfies a conservation equation\(^1\) along the flow lines:

$$u^\alpha \partial_\alpha S = 0.$$  \hspace{1cm} (2.4)

The tensor $F$ is the antisymmetric electromagnetic closed 2-form. It satisfies the Maxwell equations:

$$dF = 0, \quad \text{and} \quad \delta F = J,$$

(2.5)

where $d$ and $\delta$ are respectively the differential and codifferential operator (in the metric $g$), while $J$ is the electric current. We suppose $J$ to be a convection current, that is we have:

$$J = qu, \quad \text{i.e.} \quad J^\alpha = qu^\alpha,$$

(2.6)

with $q$ the electric charge density.

2.2 Fluid equations.

The conservation laws, consequence of the Einstein equations, read:

$$\nabla_\alpha T^{\alpha\beta} \equiv \nabla_\alpha(t^{\alpha\beta} + \tau^{\alpha\beta}) = 0.$$  \hspace{1cm} (2.7)

The Maxwell equations imply that, with $J^\lambda F^\beta_\lambda$ called the Lorentz force,

$$\nabla_\alpha \tau^{\alpha\beta} = J^\lambda F^\beta_\lambda.$$  \hspace{1cm} (2.8)

The index $f$ of the fluid, is given by the identity

$$f(p, S) := \exp \int_{p_0}^{p} \frac{dp}{\mu(p, S) + p}.$$  \hspace{1cm} (2.9)

\(^1\)Such an equation can be deduced as usual from a particle number density conservation and the Gibbs relation.
We introduce the dynamical velocity, tangent vector to the flow lines, defined by
\[ C^\alpha := f u^\alpha, \text{ hence } C^\alpha C_\alpha = -f^2. \] (2.10)
The dynamical velocity incorporates information on the kinematic velocity \( u^\alpha \) and the thermodynamic quantities. The definitions of \( C \) and \( f \) imply that:
\[ -C^\beta \nabla_\alpha C_\beta \equiv f \frac{\partial f}{\partial x^\alpha} = f^2 \frac{\partial p}{\mu + p} + f \frac{\partial f}{\partial S} \partial_\alpha S. \] (2.11)
The entropy equation 2.4 and the tensor \( t_{\alpha\beta} \) become:
\[ C^\alpha \partial_\alpha S = 0, \] (2.12)
\[ t_{\alpha\beta} \equiv (\mu + p)f^{-2}C_\alpha C_\beta + g_{\alpha\beta}p. \] (2.13)
Therefore the equations 2.7 can be written:
\[ C^\beta \nabla_\alpha [(\mu + p)f^{-2}C^\alpha] + (\mu + p)f^{-2}C^\alpha \nabla_\alpha C^\beta + \partial^\beta p = -J^\lambda F^\beta_\lambda. \] (2.14)
We contract with \( C_\beta \) these equations and we use the equation 2.11 and the entropy conservation 2.12 to obtain the following continuity equation (under our hypothesis on \( J \), the Lorentz force \( J^\lambda F^\beta_\lambda \) is orthogonal to \( C \))
\[ \nabla_\alpha [(\mu + p)f^{-2}C^\alpha] = 0, \] (2.15)
which can be written:
\[ \nabla_\alpha C^\alpha + (\mu'_p - 1) \frac{C^\alpha C^\beta}{C^\lambda C_\lambda} \nabla_\alpha C_\beta = 0, \] (2.16)
where \( \mu'_p := \frac{\partial \mu}{\partial p} \) is a given function of \( p \) and \( S \).
One deduces from 2.14 and 2.15 the following equations of motion
\[ C^\alpha \nabla_\alpha C^\beta + f^2 \frac{\partial^\beta p}{\mu + p} = -f^2 \frac{J^\lambda F^\beta_\lambda}{\mu + p}. \] (2.17)
Using 2.11 we see that the equations of motion may be written:
\[ C^\alpha \{ \nabla_\alpha C_\beta - \nabla_\beta C_\alpha + (\mu + p)^{-1}(-C^\lambda C_\lambda)^\frac{1}{2} q F_\beta_\alpha \} + \frac{1}{2} \frac{\partial f^2}{\partial S} \partial_\beta S = 0. \] (2.18)
Remark 2.1 These equations are not independent, their left hand side is orthogonal to $C$.

In these equations the fluid unknowns are the four components of the vector $C^\alpha$, and the scalar $S$. The specific pressure $p$ is a known function of $f$ (i.e. of $C^\alpha$, by 2.10) and of $S$

$$p \equiv p(C^\alpha C_\alpha, S),$$

(2.19)
determined by inverting the relation 2.3.

We suppose that $S$ is initially constant, it remains then constant during the evolution and the term $\partial_\alpha S$ disappears from the equations. It can be proved, by adapting the Rendall symmetrization$^2$ of the perfect uncharged fluid, that the equations 2.16 and 2.17 can be written as a first order symmetric hyperbolic system for the dynamical velocity $C$. The electromagnetic field $F$ appears as underivated in these equations.

3 Bianchi equations.

In contradistinction with the fluid and electromagnetic sources, the writing of the Einstein equations 2.1 as a first order system for the derivatives of the metric requires a choice of gauge. It was done straightforwardly by Fisher and Marsden using harmonic (now called wave) coordinates. In these coordinates the Ricci tensor takes the form of a quasidiagonal, quasilinear system of wave equations for the metric $g$. Though the non linear stability of Minkowski spacetime has now been proved (Lindblad and Rodnianski 2005) using these coordinates, the fundamental geometrical object signaling the existence of a gravitational field remains the Riemann tensor.

The Bianchi equations satisfied by the Riemann tensor of a metric $g$, solution of the Einstein equations 2.1, are:

$$\nabla_0 R_{hi,\lambda\mu} + \nabla_i R_{0h,\lambda\mu} - \nabla_h R_{0i,\lambda\mu} = 0$$

(3.1)

and

$$-\nabla_0 R_{i,\lambda\mu}^h - \nabla_h R_{::i,\lambda\mu}^h = -J_{\lambda,0\mu} \equiv -\nabla_0 \rho_{\mu\lambda} + \nabla_\mu \rho_{0\lambda}$$

(3.2)

$^2$General techniques, using an additional convex entropy function, of Lax 1957, Boillat 1994, Ruggeri and Strumia 1981 (reported in the book by Anile 1982), could also be used.
We recall that a sliced Lorentzian manifold \((V, g)\) is a product \(M \times R\) endowed with a Lorentzian metric \(g\) which induces on each \(M_t := M \times \{t\}\) a Riemannian metric \(\bar{g}_t\). The metric \(g\) takes in a Cauchy adapted moving frame, i.e. a frame with its timelike axis orthogonal to the space slices, the usual 3+1 form
\[
ds^2 = -N^2 dt^2 + g_{ij}(\bar{\theta}^i + \beta^i dt)(\bar{\theta}^j + \beta^j dt). \tag{3.3}
\]
The \(g_{ij}\) are the components of \(\bar{g}_t\) in a coframe \(\bar{\theta}^i\) on \(M\). Particular cases are the natural coframe \(\bar{\theta}^i = dx^i\), and an orthonormal coframe \(\bar{\theta}^i = a^i_j dx^j\) where \(g_{ij} = \delta_{ij}\). We denote by \(g^{hk}\) the contravariant components of \(\bar{g}_t\), they are, in our spacetime frames, also the \(hk\) contravariant components of \(g\). The derivatives \(\partial_\alpha\) are the Pfaff derivatives in the coframe \(\theta^0 = dt, \theta^i = \bar{\theta}^i + \beta^i dt\), that is
\[
\partial_0 = \frac{\partial}{\partial t} - \beta^i \partial_i, \quad \partial_i = \bar{\partial}_i = A^j_i \frac{\partial}{\partial x^j}, \tag{3.4}
\]
with \(A\) the inverse matrix of \(a\) whose elements are \(a^i_j\).

**Theorem 3.1** The equations 3.1 and 3.2. are a first order symmetric hyperbolic system for the components of the Riemann tensor in a Cauchy adapted frame, with \(\bar{\theta}^i\) an orthonormal frame.

**Proof.** The principal operator of these equations is diagonal by blocks, a block corresponding to the derivatives of the components \(R_{hi,\lambda\mu}\) and \(R_{0h,\lambda\mu}\) for a given pair \(\lambda, \mu, \lambda < \mu\). Such a block is the following symmetric matrix, if \(\bar{\theta}^i\) is an orthonormal coframe:
\[
\begin{pmatrix}
N^{-2} \partial_0 & 0 & 0 & -\partial_1 & 0 \\
0 & N^{-2} \partial_0 & 0 & 0 & -\partial_2 \\
0 & 0 & N^{-2} \partial_0 & -\partial_3 & 0 \\
\partial_2 & 0 & -\partial_3 & \partial_0 & 0 \\
-\partial_1 & \partial_3 & 0 & 0 & \partial_0 \\
0 & -\partial_2 & \partial_1 & 0 & 0
\end{pmatrix}.
\]

Each block is symmetric, hence the full principal operator is also symmetric.

The matrix \(M^t\) of the coefficients of derivatives \(\frac{\partial}{\partial t}\) is identical to the matrix \(M^0\) of the coefficients of the derivatives \(\partial_0\). It is diagonal, with coefficients either 1 or \(N^{-2}\), it is therefore positive definite.

The energy on a \(t = \text{constant} \) submanifold \(M_t\), constant \(t\), associated to the Bianchi equations is the so-called Bel Robinson energy, integral on \(M_t\).
of the positive definite quadratic form $M^t$ in the Riemann tensor. If one introduces the electric and magnetic fields and inductions space 2- tensors associated with the Riemann tensor given by

$$E_{ij} \equiv R^0_{i,0j}, \quad D_{ij} \equiv \frac{1}{4}\eta_{ihk}\eta_{jlm}R^{hk,lm},$$

$$H_{ij} \equiv \frac{1}{2}N^{-1}\eta_{hkk}R^{hk}_{,0j}, \quad B_{ji} \equiv \frac{1}{2}N^{-1}\eta_{hhk}R_{0j,}^{hk},$$

the Bel - Robinson energy density on $M_t$ is given by:

$$\mathcal{E}_g \equiv \frac{1}{2}(|E|^2 + |H|^2 + |D|^2 + |B|^2).$$

In the case that we are considering of equations with sources, the Bianchi equations contain, in addition to the Riemann tensor and its gradient, the dynamical acceleration $\nabla C$ and the gradient $\nabla F$ of $F$, but no derivative of these quantities.

4 Equations for $\nabla F$.

The Maxwell equations imply

$$\delta dF + d\delta F = dJ$$

that is, using the Ricci identity and some manipulation of indices, the following semilinear wave equation for $F$:

$$\nabla^\alpha \nabla_\alpha F_\beta\gamma = f_\beta\gamma,$$

with

$$f_\beta\gamma := R_\beta^\lambda F_\gamma\lambda - R_\gamma^\lambda F_\lambda\beta - 2R_\beta^\lambda \gamma^\alpha F_\alpha\lambda + \nabla_\beta J_\gamma - \nabla_\gamma J_\beta.$$  (4.3)

It is easy to deduce a symmetric hyperbolic first order system for $\nabla F$ from the wave equation satisfied by $F$. We set

$$F_\gamma,\alpha\beta := \nabla_\gamma F_{\alpha\beta},$$

In a Cauchy adapted frame with orthonormal $\bar{\partial}^i$, the equations to satisfy read:

$$N^{-2}\nabla_0 F_{0,\alpha\beta} - \delta^{ij}\nabla_i F_{j,\alpha\beta} = f_{\alpha\beta}$$

(4.5)
and
\[ \nabla_0 F_{i,\alpha \beta} - \nabla_i F_{0,\alpha \beta} = R_{0i,\alpha} \lambda F_{\lambda \beta} + R_{0i,\beta} \lambda F_{\alpha \lambda}. \] (4.6)

The left hand sides of these equations are a symmetric hyperbolic first order operator for the four unknowns \((F_{0,\alpha \beta}, F_{i,\alpha \beta})\) for each pair \((\alpha, \beta), \alpha < \beta\). The right hand sides contain \(\nabla C\) and \(\text{Riemann}(g)\), but no derivative of these quantities.

The energy density associated to the above system is a positive quadratic form, \(\mathcal{E}_{\nabla F}\) in \(\nabla F\). It is sometimes called a "superenergy".

5 Equations for \(\nabla C\).

The dynamical acceleration \(\nabla C\) satisfies the following equations obtained by covariant differentiation of 2.16. and 2.17, and use of the Ricci identities:

\[ M_{\gamma \beta} \equiv C^\alpha (\nabla_\alpha C_{\gamma \beta} - \nabla_\beta C_{\gamma \alpha}) + a_{\gamma \beta} = 0 \] (5.1)

and

\[ g^{\alpha \beta} \nabla_\alpha C_{\gamma \beta} + (\mu'_p - 1) \frac{C^\alpha C^\beta}{C^\lambda C^\gamma} \nabla_\alpha C_{\gamma \beta} + b_\gamma = 0 \] (5.2)

where we have set
\[ C_{\gamma \beta} \equiv \nabla_\gamma C_{\beta}, \] (5.3)

\[ a_{\gamma \beta} \equiv C_{\gamma}^\alpha (C_{\alpha \beta} - C_{\beta \alpha}) + C^\alpha C^\lambda R_{\gamma \alpha \beta} + \nabla_\gamma \{(\mu + p)^{-1} (-C^\lambda C^\gamma)^{\frac{1}{2}} q C^\alpha F_{\beta \alpha}\}, \] (5.4)

\[ b_\gamma \equiv -R_{\gamma \lambda} C^\lambda + \nabla_\gamma \{(\mu'_p - 1) \frac{C^\alpha C^\beta}{C^\lambda C^\gamma} \} C_{\alpha \beta} \] (5.5)

The last term in \(b_\gamma\) is a quadratic form in \(C_{\alpha \beta}\) whose coefficients are functions of the \(C^\alpha\) and \(S\). These functions can be computed by using the identity (we use \(S = \text{constant}\))
\[ \nabla_\gamma \mu'_p \equiv \mu''_{p'} \partial_\gamma p. \]

By the definition of \(f\) and the identity 2.4 it holds that
\[ \partial_\gamma p = (\mu + p)^{-1} \partial_\gamma f = -(\mu + p)(C^\gamma C^\gamma)^{-1} C^\alpha C_{\gamma \alpha}. \]
The equations 5.1. are not independent, because they satisfy the identities

\[ C^\beta M_{\gamma \beta} \equiv 0, \quad (5.6) \]

the equations 5.1 and 5.2. are not a well posed system. Instead of the 4 × 4 equations 4.1 we consider the 4 × 3 ones:

\[ \tilde{M}_{\gamma i} \equiv M_{\gamma i} - \frac{C_i}{C_0} M_{\gamma 0} = 0 \quad (5.7) \]

The terms in derivatives of \( C_{\gamma \lambda} \) in these equations can be written in the following form:

\[ C^\alpha \partial_\alpha (C_{\gamma i} - \frac{C_i}{C_0} C_{\gamma 0}) - (\partial_i - \frac{C_i}{C_0} \partial_0)(C^\alpha C_{\gamma \alpha}) \quad (5.8) \]

**Lemma 5.1** The system 5.2,5.7. is equivalent to a FOS (First Order Symmetric) system for \( C_{\gamma \alpha} \) with right hand side function of the Riemann tensor and \( \nabla F \), not of their derivatives.

**Proof.** The system is quasi diagonal by blocks, each block corresponding to a given value of the index \( \gamma \). We will write the principal operator of a block by omitting this index. We set

\[ U_i \equiv C_{\gamma i} - \frac{C_i}{C_0} C_{\gamma 0}, \quad U_0 \equiv C^\alpha C_{\gamma \alpha} \quad (5.9) \]

and we define the differential operators \( \tilde{\partial}_\alpha \) as follows:

\[ \tilde{\partial}_0 \equiv C^\alpha \partial_\alpha, \quad \tilde{\partial}_i = \partial_i - \frac{C_i}{C_0} \partial_0 \quad (5.10) \]

The principal terms (derivatives of \( C_{\gamma \alpha} \)) in the equations 5.7 with index \( \gamma \) are

\[ \tilde{\partial}_0 U_i - \tilde{\partial}_i U_0. \quad (5.11) \]

We have by inverting 5.9:

\[ C_{\gamma 0} \equiv \frac{C_0(U_0 - C^i U_i)}{C^\lambda C_\lambda} \]

\(^4\)An analogous procedure is used for the symmetrization of the Euler equations in K.O. Friedrichs 1969 and in Rendall 1992.
\[ C_{\gamma i} \equiv U_i + \frac{C_i (U_0 - C^j U_j)}{C^\lambda C_\lambda} \]

The principal terms of 5.2. read, using the above formulae

\[ \frac{\mu'_p C^\alpha \partial_{\alpha} U_0}{C^\lambda C_\lambda} + (g^{ij} - \frac{C^i C^j}{C^\lambda C_\lambda}) \partial_i U_j - \frac{C^0 C^i}{C^\lambda C_\lambda} \partial_0 U_i \quad (5.12) \]

We introduce the positive definite (if \( C \) is timelike) quadratic form

\[ \tilde{g}^{ij} \equiv g^{ij} - \frac{C^i C^j}{C^\lambda C_\lambda} \quad (5.13) \]

Then we find that

\[ \frac{\tilde{g}^{ij} C_j}{C_0} \equiv \frac{C^0 C^i}{C^\lambda C_\lambda} \]

The principal terms 5.12 are therefore

\[ \frac{\mu'_p \tilde{\partial}_0 U_0}{C^\lambda C_\lambda} + \tilde{g}^{ij} \tilde{\partial}_i U_j \quad (5.14) \]

The matrix of the coefficients of the derivatives \( \tilde{\partial}_\alpha \) in the equations deduced from the system 5.2, 5.7 is

\[
\begin{pmatrix}
-\frac{\mu'_p}{C^\lambda C_\lambda} \tilde{\partial}_0 & -\tilde{\partial}^1 & -\tilde{\partial}^2 & -\tilde{\partial}^3 \\
-\tilde{\partial}_1 & \tilde{\partial}_0 & 0 & 0 \\
-\tilde{\partial}_2 & 0 & \tilde{\partial}_0 & 0 \\
-\tilde{\partial}_3 & 0 & 0 & \tilde{\partial}_0
\end{pmatrix}
\]

This matrix is symmetrized by taking the product with the \( 4 \times 4 \) matrix

\[
\begin{pmatrix}
1 & 0 \\
0 & g_{ij}
\end{pmatrix}
\]

**Hyperbolicity.**

In the case we are considering the matrix \( \tilde{M}^0 \) is diagonal, with positive elements if \( \mu'_p > 0 \) and \( C^\alpha \) is timelike. The principal matrix written with natural coordinates is also symmetrizable. The corresponding matrix \( \tilde{M}^t \) is not diagonal, and it is not obvious that it is positive definite. In fact it will be so only if \( \mu'_p \geq 1 \). We will prove the following lemma.

**Lemma 5.2** The system 5.2, 5.7 is FOSH if \( \mu'_p \geq 1 \) and \( C \) is timelike.
Proof. It is simpler to compute directly the energy density for the considered system: its positivity is equivalent to the positivity of the matrix $M_t$. Multiplying 5.2. and 5.7 respectively by $U_0$ and $\tilde{g}^{ij}U_j$ gives equations of the form

$$\frac{1}{2} \frac{\mu'_p \tilde{\partial}_0 (U_0)^2}{f^2} - \tilde{g}^{ij}U_0 \tilde{\partial}_i U_j = U_0 \Phi_0 \quad (5.15)$$

$$\tilde{g}^{ij}U_j \tilde{\partial}_0 U_i - \tilde{g}^{ij}U_i \tilde{\partial}_j U_0 = \tilde{g}^{ij}U_j \Phi_i \quad (5.16)$$

where the $\Phi_\alpha$ contain only non differentiated terms in $U$, Riemann and $\nabla F$. We add these two equations, replace the operators $\tilde{\partial}$ by the operators $\partial$ and carry out some manipulations using the expression for $\tilde{g}^{ij}$ and the Leibniz rule. We obtain that

$$\partial_0 \mathcal{E}_m + \nabla_i \mathcal{H}_m = Q_m \quad (5.17)$$

The function $Q_m$ is a quadratic form in $C_{\gamma\alpha}$, $F_{\alpha,\beta\gamma}$ and the Riemann tensor, while $\mathcal{E}_m$ is the energy density on $M_t$ of the dynamical acceleration $\nabla C$. It is the quadratic form given by:

$$\mathcal{E}_m \equiv f^{-2}[(\mu'_p - 1)U_0^2 + (U_0 - C^i U_i)^2] + g^{ij}U_i U_j \quad (5.18)$$

It is positive definite if $\mu'_p \geq 1$ and $C^\alpha$ is timelike. \[\Box\]

Remark 5.3 The system is hyperbolic in the sense of Leray if $\mu'_p > 0$, but the submanifolds $x^0 =$constant are 'spacelike' with respect to the fluid wave cone only if the fluid sound speed is less than the speed of light, i.e., $\mu'_p \geq 1$.

The quantity $\mathcal{E}_m$ is called the fluid "acceleration energy" density.

6 Coupled system.

The previous results give, for $\text{Riem}(g)$, $\nabla F$ and $\nabla C$, when $g$, a lorentzian metric, $F$, a 2 form and $C$, a timelike vector, are known, a first order symmetric system, quasi diagonal by blocks, hyperbolic if $\mu'_p \geq 1$.

By choosing a densitized lapse (time wave gauge), as in [CB-Yo97] and [CB-Yo01] one can obtain a full symmetric first order system containing, in addition, $g$ and its connection, $F$ and $C$. 

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The superenergy density $E$ for the full system is the sum of the Bel-Robinson energy density of the gravitational field, the energy density of $\nabla F$, supernergy of $F$, and the energy density of the dynamical acceleration:

$$E \equiv E_g + E_F + E_m.$$  

Using the expression of $\partial_0$ and the mean extrinsic curvature $\tau \equiv g^{ij}K_{ij}$ of the space slices $S_t$, whose volume element we denote by $\mu_g$, we obtain by integration an integral equality whose right hand side couples all these superenergies:

$$\int_{S_t} E \mu_g = \int_{S_{t_0}} E \mu_g + \int_{t_0}^t \int_{S_0} \{-N\tau E + Q\} \mu_g dt$$

where $Q$ is a quadratic form in $\text{Riemann}(g)$, $\nabla F$ and $\nabla C$, which could be estimated in terms of $E$ if the other unknowns, $C$, $g$ and its connection, were estimated.

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