Modular convergence in $H$-Orlicz spaces of Banach valued functions

Hemanta Kalita$^1$ · Bipan Hazarika$^2$

Received: 4 August 2022 / Accepted: 14 February 2023 / Published online: 13 March 2023 © The Author(s), under exclusive licence to Springer-Verlag Italia S.r.l., part of Springer Nature 2023

Abstract
In this article we develop the theory of $H$-Orlicz space generated by generalised Young function. Modular convergence of $H$-Orlicz space for the case of vector-valued functions and norm convergence in $H^\theta (X, \overline{\mu})$ where $X$ is any Banach space are discussed. Relationships of modular convergence and norm convergence of $H$-Orlicz spaces are discussed.

Keywords H-Orlicz space · Modular convergence · Norm convergence

Mathematics Subject Classification 26A39, 46B03, 46B20, 46B25.

1 Introduction

Z.W. Birnbaum and W. Orlicz to proposed a generalized space of $L^p$, later on it was known as Orlicz space. This space was later developed by Orlicz himself. The fundamental properties of Orlicz space with Lebesgue measure found in [11] also (see [12]). H. Nakano [14] introduced the concept of modular Orlicz space also (see [13]) for modular space. M.S. Skaff [17] developed generalized $N$-function. Gereralized $N$-function are generalization of the variable $N$-function (see [16]) and non decreasing $N$-function (see [21]). M.S. Skaff [18] discussed vector valued Orlicz spaces with generalised $N$-function. A. Kozek [10] studied Orlicz spaces of functions with values in Banach spaces. In his work, Kozek discussed Orlicz spaces from the modular spaces from point of view of generalised $N$-function. A. Kaminska and H. Hudzik [9] discussed the necessary condition of equality of the modular convergence and norm convergence in Orlicz spaces. In recent time, modular convergence theorem is a well known concept for many researcher in different areas of mathematics. Carlo Bardaro and Gianluca Vinti [5] studied modular convergence theorem for certain nonlinear integral

---

$^1$ Department of Mathematics, Assam Don Bosco University, Sonapur, Guwahati, Assam 782402, India

$^2$ Department of Mathematics, Gauhati University, Guwahati, Assam 781014, India
operators with homogeneous kernel, modular convergence in fractional Musielak-Orlicz Spaces (see [6]). Youssef Ahmida et al., discussed density of smooth functions on Musielak-Orlicz-Sobolev spaces (see [1]). Hazarika and Kalita [8] introduced $H$-Orlicz spaces with non absolute integrable functions in particular Henstock-Kurzweil integrable functions. The significance of $H$-Orlicz space is that the space $C^\infty_0$ is dense in $H$-Orlicz space which is not generally true in the case of Orlicz spaces. On the other side about Henstock-Kurzweil integral in 1957, Jaroslav Kurzweil discussed about a new integral in one of his publication, while unaware of the work of Kurzweil, Ralph Henstock published an article on integration theory in which he discussed the same integration as J. Kurzweil. This new integral can integrate a substantial type of functions compare to the Riemann or Lebesgue integral. In the theory in which he discussed the same integration as J. Kurzweil. This new integral can integrate a substantial type of functions compare to the Riemann or Lebesgue integral. In the honors of these mathematicians, nowadays this integral is called Henstock-Kurzweil integral in brief HK-integral. Measure theory is not essential in the definition of HK-integral. In this article in Sect. 2, we introduce a new norm which is equivalent to Alexiewicz norm. In Sect. 3, we extend the theory of H-Orlicz spaces with vector functions from the point of view of generalised Young-function. Finally in Sect. 3, we discuss various relationship of modular convergence as well as norm convergence of Orlicz spaces and H-Orlicz spaces.

Let $(J, \Omega, \overline{\mu})$ be a measure space, where $J$ consisting of a topological space, $\Omega$ is a $\sigma$-algebra of subsets of the set $J$, $\overline{\mu}$ is a $\sigma$-finite, positive, complete measure on $\Omega$. $(X, \tau)$ is a Banach space where $\tau$ is the topology. Consider $X^*$ is the continuous dual of $X$.

A Banach space $X$ is known to be weakly sequentially complete if every weak Cauchy sequence weakly converges in $X$ (see [2]). The weak topology $\tau_1$ on $X$ is defined to be the coarsest/ weakest topology under which each element of $X^*$ remains continuous on $X$. Let $M_X$ be a set of all $\overline{\mu}$-measurable functions defined on $J$ with values in $X$. The function $f \in M_X : f_i(t)$ is real valued functions where $t \in J$, $i = 1, 2, ..., n$. We will represent the function in $M_X$ by the vector notation $f : f(t) \ (t \in J)$ whenever it is convenient to do so.

For example: if $f, g$ are functions in $J$ and $a, b$ are real numbers, the symbol $af + bg$ denotes the function $af + bg : af(t) + bg(t) \ (t \in J)$.

Let us identify all functions $f$ in $M_X$ which are equal to zero for almost $t$ in $J$. Then we denote by the same symbol, $M_X$, the set of equivalence classes of functions defined by this identification.

**Definition 1.1** [9, Definition 2.1] A function $\theta : J \times X \to [0, \infty]$ is called a generalised Young function if it satisfies the following conditions: There exists a set $J_1 \in \Omega$, $\overline{\mu}(\Omega \setminus J_1) = 0$, such that

a) $\theta(\cdot, \cdot)$ is $B \times \Omega$ measurable, where $B$ denotes the $\sigma$-algebra of Borel subsets of $X$;
b) $\theta(t, f(t))$ is lower semi continuous on $X$ for every $t \in J_1$;
c) $\theta(t, f(t))$ is convex for every $t \in J_1$
d) $\theta(t, 0) = 0$ and $\theta(t, f(t)) = \theta(t, -f(t))$ for every $f \in M_X \ \ (t \in J_1)$.

A function $f : J \to X$ is $M$-measurable if there is a sequence of simple functions from $S_X(\Omega)$ converges to $f$ (M-a.e), where $S_X(\Omega)$ is a set of all $X$-valued simple functions.

$M$ denotes a $\sigma$-bounded family of positive measures defined on $\Omega$. This means for each $E \in \Omega$, there exists a pairwise disjoint collections $\{E_i\}_{i=1}^{\infty}$, $E_i \in \Omega$ such that $E = \bigcup_{i=1}^{\infty} E_i$ and $S_X(E)$ is a set of all $X$ valued simple functions.
Definition 1.2 [13] Given a linear space $X$, a functional $\rho(\cdot)$ defined on $X$ with values $-\infty < \rho(x) \leq \infty$ is called a modular if the following conditions hold:

(a) $\rho(x) = 0$ if and only if $x = 0$
(b) $\rho(-x) = \rho(x)$,
(c) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

Definition 1.3 [13] A sequence $(x_n) \subset X$ is said to be modular convergent to $x \in X$ if there exists a number $\alpha > 0$(depending on the sequence $(x_n)$) such that $\rho(\alpha (x_n - x)) \to 0$ as $n \to \infty$.

Definition 1.4 [18, Definition 2.2] Let $\theta(t, f(t))$ be a generalised Young function and let $L^\theta(X, \overline{\mu})$ be its associated Orlicz class. The closure $L^\theta(X, \overline{\mu})$ of $L^\theta(X, \overline{\mu})$ under positive scalar multiplication is a vector valued Orlicz space.

The space $L^\theta(X, \overline{\mu})$ is a Banach space with Luxemburg type norm

$$||f||_{L^\theta} = \inf \left\{ \alpha > 0 : (L) \int J \theta(t, f(t)) d\overline{\mu} \leq 1 \right\}.$$  

Recall that convergence of a sequence of functions $f_n$ converges to $f$ in the normed vector space $L^\theta(X, \overline{\mu})$ is defined in the following way:

$$f_n \to f \iff ||f_n - f||_{L^\theta} \to 0 \text{ as } n \to \infty.$$  

2 Henstock-Kurzweil integrable function spaces $H(X, \overline{\mu})$

In this section of the paper, we discuss about the Henstock-Kurzweil integrable function space. We have leave off the basic results of the space. We discuss only those results that are closely connected for next section.

A finite collection $D = \{D_i : i = 1, 2, \ldots, n\}$ of mutually disjoint elements of $\Omega$ is called sub-partition of $J$. A sub-partition $D = \{D_i : i = 1, 2, \ldots, n\}$ is called a partition of $J$ if $J = \bigcup_{i=1}^{n} D_i$. A sub-partition $D = \{D_i : i = 1, 2, \ldots, n\}$ of $J$ is said to be tagged if for each $D_i$, there is assigned a point $d_i \in D_i$ called a tag of $D_i$. Let $\widehat{D} = \{(D_i, d_i) : i = 1, 2, \ldots, n\}$ and $d_i \in D_i$. We assume the collection of all tagged sub-partitions of $J$ will be $\mathcal{D}$.

We can define the norm of the sub-partition $D = \{D_i\}_{i=1}^{n}$ as

$$||D||_{sp} = \sup\{\overline{\mu}(D_i) : i = 1, 2, \ldots, n\}.$$  

If $\widehat{D}_1, \widehat{D}_2 \in \mathcal{D}$, we say that $\widehat{D}_2$ is a refinement of $\widehat{D}_1$ and $\widehat{D}_2 \geq \widehat{D}_1$ if $||\widehat{D}_2|| \leq ||\widehat{D}_1||$ with $\bigcup \widehat{D}_1 \subset \bigcup \widehat{D}_2$. This relation does not depend on the tagging. It is easily verified that $\leq$ is reflexive and transitive on $\mathcal{D}$. If $\widehat{D}_1, \widehat{D}_2 \in \mathcal{D}$ then the sub partition

$$D_1 \cup D_2 = (D_1 \setminus \bigcup D_2) \cup \{A \cap B : A \in D_1, B \in D_2\}$$

is such that $\widehat{D}_1 \cup \widehat{D}_2 \in \mathcal{D}$ and $\widehat{D}_1 \cup \widehat{D}_2 \geq \widehat{D}_1, \widehat{D}_2$. Clearly ($\mathcal{D}$, $\geq$) is a directed set.

Let us consider a function $f : J \to X$. Assuming $\widehat{D} = \{(D_i, d_i) : i = 1, 2, \ldots, n\} \in \mathcal{D}$, then we construct the Riemann sum of $f$ corresponding to $\widehat{D}$ as

$$S(f, \widehat{D}) = \sum_{i=1}^{n} f(d_i)\overline{\mu}(D_i).$$
As $(\mathcal{Y}, \geq)$ is directed set, the function $\hat{D} = \{(D_i, d_i) : i = 1, 2, \ldots, n\} \rightarrow S(f, \hat{D})$ defines a net $S : \mathcal{Y} \rightarrow X$.

We are now ready to give our first definition of the Henstock-Kurzweil integral of an $X$-valued function as follows:

**Definition 2.1** A function $f : \mathcal{P} \subset \mathcal{Q} \rightarrow X$ is said to be Henstock-Kurzweil integrable on the set $\mathcal{P} \in \mathcal{Q}$ if there is an element $I_{\mathcal{P}} = \int_{\mathcal{P}} f d\mu$ such that for every $\epsilon > 0$, there exists $\hat{D}_0 \in \mathcal{Y}$ and $\hat{D} \geq \hat{D}_0$ in $\mathcal{Y}$, then

$$||S(f, \hat{D}) - \int_{\mathcal{Q}} f d\mu||_X \leq \epsilon.$$ 

We call net limits of the integral $f : \mathcal{Q} \rightarrow X$ as

$$\int_{\mathcal{Q}} f d\mu = \lim_{\hat{D} \in (\mathcal{Y}, \geq)} (S(f, \hat{D}))_{\hat{D} \in (\mathcal{Y}, \geq)}.$$ 

This integrals are unique, linear also for all $X$-valued Henstock-Kurzweil integrable. Let $HK(\mathcal{Q}, \mu, X)$ or in brief $HK(X, \mu)$ the set of all $X$-valued $\mu$-Henstock-Kurzweil integrable functions.

**Definition 2.2** For every function $f : J \rightarrow X$, $\mathcal{Y}$-variation of $f$ to be the number, finite or $+\infty$,

$$||f||_\mathcal{Y} = \sup\{|S(f, \hat{D})| : \hat{D} \in \mathcal{Y}\}$$

has bounded variation if $||f||_\mathcal{Y} < \infty$.

**Theorem 2.3** The expression (1) is a norm on $HK(X, \mu)$.

**Proof** Let $f \in HK(X, \mu)$ is any vector valued function.

(i) The non negativity of $f$, i.e., $||f||_\mathcal{Y} \geq 0$ are obvious.

(ii) Let $||f||_\mathcal{Y} = 0$. Then, $||f||_\mathcal{Y} = \sup\{|S(f, \hat{D})| : \hat{D} \in \mathcal{Y}\} = 0$. Now for a real value is equal to an element of $X$, we have

$$||f||_\mathcal{Y} = \sup\left\{|\sum_{i=1}^{n} f(d_i)\mu(D_i)| : \hat{D} = (D_i, d_i), i = 1, 2, \ldots\right\}$$

$$= \sum_{i=1}^{n} f(d_i)\mu(D_i)$$

$$= 0.$$ 

This implies, $f = 0$. Conversely, suppose $f = 0$, then $f(d_i)\mu(D_i) = 0$, where $\hat{D} = (D_i, d_i), i = 1, 2, \ldots$ Consequently, $\sum_{i=1}^{n} f(d_i)\mu(D_i) = 0$. Hence,

$$\sup\left\{|\sum_{i=1}^{n} f(d_i)\mu(D_i)| : \hat{D} = (D_i, d_i), i = 1, 2, \ldots\right\} = 0.$$ 

Therefore $||f||_\mathcal{Y} = 0$. 

\(\blacksquare\) Springer
(iii) For a scalar $\alpha$, we have
\[
\|\alpha f\|_\mathcal{Y} = \sup \left\{ \|S(\alpha f, \hat{D})\| : \hat{D} \in \mathcal{Y} \right\}
= \sup \left\{ \|\alpha S(f, \hat{D})\| : \hat{D} \in \mathcal{Y} \right\}
= |\alpha| \sup \left\{ \|S(f, \hat{D})\| : \hat{D} \in \mathcal{Y} \right\}
= |\alpha| \|f\|_\mathcal{Y}.
\]

(iv) Let $f_1, f_2 \in HK(X, \mu)$. Then,
\[
\|f_1 + f_2\|_\mathcal{Y} = \sup \left\{ \|S(f_1 + f_2, \hat{D})\| : \hat{D} \in \mathcal{Y} \right\}
= \sup \left\{ \|S(f_1, \hat{D}) + S(f_2, \hat{D})\| : \hat{D} \in \mathcal{Y} \right\}
\leq \sup \left\{ \|S(f_1, \hat{D})\| : \hat{D} \in \mathcal{Y} \right\} + \sup \left\{ \|S(f_2, \hat{D})\| : \hat{D} \in \mathcal{Y} \right\}
\leq \|f_1\|_\mathcal{Y} + \|f_2\|_\mathcal{Y}.
\]

Now we will going to prove that the norm (1) is equivalent to the Alexiecz norm for the Henstock integrable function spaces for finite dimensional case. Recalling that when $I = [a, b], -\infty < a < b < \infty$, $HK(I)$ is the space of Henstock-Kurzweil integrable functions defined on $I$. Alexiewicz norm is as follows:
\[
\|f\|_{HK} = \sup \left\{ \left| \int_a^t f \right| : a \leq t \leq b \right\}.
\]

The Henstock-Kurzweil integrable function spaces $\left( HK(I), \|\cdot\|_{HK} \right)$ is a linear space. If we identify functions which are a.e. in $I$, then $HK(I)$ is a normed space under Alexiewicz norm (2). For $f \in HK(\mathbb{R})$, Alexiewicz norm $\|f\|_{HK} = \sup_s \int_s^\infty |f(t)|d\mu$. If we replace $\mathbb{R}$ by $\mathbb{R}^n$, for $f \in HK(\mathbb{R}^n)$, we have Alexiewicz norm as follows:
\[
\|f\|_{HK} = \sup_{r > 0} \int_{B_r} f(x)d\mu < \infty,
\]
where $B_r$ is any closed cube of diagonal $r$ centered at the origin in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes.

**Remark 2.4** Every two norms on a finite dimensional (real or complex) vector space are equivalent, so if $X = \mathbb{R}^n$, then the norm (1) is equivalent to the Alexiecz norm (3) for the Henstock-Kurzweil integrable function spaces.

**Theorem 2.5** Let $f : \Omega \to X$. If $f = 0 \mu$-a.e. on $\Omega$, then $f \in HK(X, \mu)$ and $\int_{\Omega} f d\mu = 0$.

**Corollary 2.6** Let $f, g \in HK(X, \mu)$ be $\mu$-essentially equal, then $\int_{\Omega} f d\mu = \int_{\Omega} g d\mu$.

**Theorem 2.7** Let $(J, \Omega, \mu)$ be a finite measure, $X$ is a Banach space, then $(HK(X, \mu), \|\cdot\|_\mathcal{Y})$ is complete.
Proof Let us assume \((f_n)\) be a Cauchy sequence in \(HK(X, \overline{\mu})\), then for each \(\epsilon > 0\) we can find a natural number \(N\) such that \(m, n > N\) such that
\[
\sup_{\hat{D} \in \mathcal{D}} ||S(f_m - f_n, \hat{D})||_X \leq \epsilon.
\]
If \(\overline{w} \in \Omega\), under the assumption of sub-partition \(\hat{D} = \{(\Omega, \overline{w})\}\), we get
\[
||f_n(\overline{w}) - f_m(\overline{w})||_X \leq \overline{\mu}(\Omega)\epsilon.
\]
By hypothesis \(\epsilon > 0\), \(\overline{\mu}(\Omega) < \infty\), we find \((f_n(\overline{w}))\) is Cauchy sequence of \(X\). Again, since \(X\) is a Banach space, it is easy to define \(\overline{w} \rightarrow f(\overline{w}) = \lim_{n \rightarrow \infty} f_n(\overline{w}).\)

Using the concept of the Riemann sum over sub-partitions \(\hat{D}_1\) and \(\hat{D}_2\) in \(\mathcal{D}\) with \(\hat{D} = \hat{D}_1 \cup \hat{D}_2\) we get the following:
\[
\begin{align*}
||\int_{\Omega} f_n d\overline{\mu} - \int_{\Omega} f_m d\overline{\mu}||_X & \leq ||\int_{\Omega} f_n d\overline{\mu} - S(f_n, \hat{D})||_X + ||S(f_n, \hat{D}) - S(f_m, \hat{D})||_X \\
& + ||\int_{\Omega} f_m d\overline{\mu} - S(f_n, \hat{D})||_X \\
& < 3\epsilon.
\end{align*}
\]
This gives, for \(N \in \mathbb{N}\), \(\|\int_{\Omega} f_n d\overline{\mu} - \int_{\Omega} f_m d\overline{\mu}\|_X < \epsilon\) for \(m, n \geq N\). So, we can conclude \(\int_{\Omega} f_n d\overline{\mu}\) is Cauchy sequence in \(X\). Say \(\int_{\Omega} f_n d\overline{\mu}\) converges to \(x \in X\). Again,
\[
\begin{align*}
||S(f, \hat{D}) - x||_X & \leq ||S(f, \hat{D}) - S(f_n, \hat{D})||_X + ||S(f_n, \hat{D}) - S(f_m, \hat{D})||_X \\
& + ||S(f_m, \hat{D}) - \int_{\Omega} f_m d\overline{\mu}||_X + ||\int_{\Omega} f_m d\overline{\mu} - x||_X \\
& < 4\epsilon.
\end{align*}
\]
So, \(f \in HK(X, \overline{\mu})\) and \(x = \int_{\Omega} f d\overline{\mu}\). \(\square\)

Theorem 2.8 If \(X\) is weakly sequentially complete, then \(HK(X, \overline{\mu})\) is also weakly sequentially complete.

Proof Let \((f_n)\) be a weak Cauchy sequence in \(HK(X, \overline{\mu})\). Since \(f_n\) is essentially separably-valued in the Banach space. We may can assume \(X\) is separable. Since every Banach space are Fréchet space. Every Fréchet space satisfies the strict Mackey condition (see [15, 5.1.27–29]), so there exists a disc \(A \subset HK(X, \overline{\mu})\) containing the sequence \((f_n)\) and such that \(||.||_{A}\) induces on \((f_n)\) as same topology \(\tau_1\). Since every bounded subsets of \(HK(X, \overline{\mu})\) are part of \(HK(X, \overline{\mu})\), there exists a disc \(D \subset X\) such that the set defined by
\[
\mathcal{B} = \left\{ f \in HK(X, \overline{\mu}) : f(w) \in X_D \overline{\mu} - a.e., f \text{ is } ||.||_{D} - \text{ measurable and } \int f ||.||_X d\overline{\mu} \leq 1 \right\}
\]
contains \(A\). As \(A \subset \mathcal{B}\), we have \(||.||_{\mathcal{B}} \leq ||.||_{A}\) and this implies that \(||.||_{\mathcal{B}}\) also induces on \((f_n)\) with the same topology as \(\tau_1\). Using the [20, Proposition 1], \((f_n)\) is a weak Cauchy sequence also in \(HK(X, \overline{\mu})\). It is not hard to see that \(HK(X, \overline{\mu})_\mathcal{B}\) can be identified with \(HK(X_D, \overline{\mu})\). As \(X_D\) is a Banach space, so \((f_n)\) is weakly convergent.
Let \( f_n = g_n + h_n \), where the sequence \( (h_n) \) is weakly convergent to zero in \( HK(X_{\mathcal{D}}, \mu) \), hence also in \( HK(X, \mu) \). Again, each \((g_n)\) is in the convex hull \( \{f_n, f_{n+1}, \ldots\} \subset HK(X_{\mathcal{D}}, \mu) \) and the sequence \((g_n(w))\) is a weak Cauchy sequence in \( X_{\mathcal{D}} \), and in \( X \), for \( \mu \)-almost all \( w \in J \). Since \( X \) is weakly sequentially complete, we can define a \((\mu\text{-a.e.})\) function \( g \) by \( g(w) = \text{weak} - \lim g_n(w) \).

Again, for every \( x^* \in X^* \) the function \( \{g(\cdot), x^*\} \) is measurable as for \( \mu \)-almost all \( w \in J \), \( \{g(w), x^*\} = \lim\{g_n(w), x^*\} \). From our hypothesis \( X \) is separable, Pettis theorem gives \( g \) is \( \tau \)-measurable. Now let \( q \) be a \( \tau \)-continuous semi norm on \( X \) and \( r > 0 \) be such that \( q(x) \leq r, \forall x \in \mathbb{D} \). Using Fatou’s lemma \( \int_J q(g(w))d\mu(w) < \infty \). If we denote \( X_q \), the completion of the normed space \( \left( \frac{X}{\theta(q^{-1}(0))}, q \right) \) induced by the semi norm \( q \), the inequality above shows that the function \( g \) is in \( HK(X_q, \mu) \) and using [19, Lemma 8] we can have \((g_n)\) converges to \( g \) weakly in \( HK(X, \mu) \). Since \( g \in HK(X, \mu) \) and that \((g_n)\), hence also \((f_n)\), converges to \( g \) weakly in \( HK(X, \mu) \).

**Theorem 2.9** Let \( f : J \to X \) be any function of \( HK(X, \mu) \), then \( \theta(f) \) is in \( HK(X, \mu) \).

**Proof** Let \( f : I \subseteq J \to X \) be measurable (integrable) function of \( HK(X, \mu) \), then for \( \mathcal{P} \in \mathcal{Q} \), there is an element \( I_\mathcal{P} = \int_\mathcal{P} f d\mu \) such that

\[
\left| \left| S(f, \mathcal{D}) - \int_\mathcal{Q} f d\mu \right| \right|_X < \epsilon.
\]

As \( \theta \) is Young function. So, \( \theta(f(t)) \to \infty \) as \( t \to \infty \). We claim that \( \theta(f(t)) \) is in \( H(X, \mu) \). Since, Young function by definition, is an extended real Borel function. So, \( \theta(f(t)) \) is measurable. If \( D_1 \) and \( D_2 \) are both partitions of the same subset of \( I \), then for any subset \( I_0 \) of \( I \), we can write

\[
\mu(I) = \sum_{(I_0, I_0) \in D_2} \mu(I_0 \cap \bar{I}_0).
\]

So,

\[
\sum_{(I_0, I_0) \in \pi} \theta(t, f(t))(t)\mu(\bar{I}) = \sum_{(I_0, I_0) \in \pi} \sum_{(I_0, I_0) \in \pi} \theta(t, f(t))\mu(\bar{I}_0 \cap \bar{I}_0')
\]

i.e.,

\[
\left( \left| \left| \sum_{(I_0, I_0) \in \pi} \theta(t, f(t))\mu(\bar{I}) - \sum_{(I_0, I_0) \in \pi} \theta(t, f(t))\mu(\bar{I}) \right| \right|}_X < \epsilon.
\]

Thus, \( \theta(t, f(t)) \) is in \( HK(X, \mu) \). □

**Corollary 2.10** For all functions \( f \in M_X \) for which there exists a constant \( k > 0 \) such that

\[
\rho(kf) = \int_J \theta(t, kf(t))d\mu \in HK(X, \mu).
\]

### 3 H-Orlicz spaces \( \mathcal{H}^\theta(X, \mu) \)

In this section, we initiate to study of \( H \)-Orlicz spaces associate with modular of functions. Let \( M_X \) be the set of all real-valued (or complex-valued), \( \Omega \)-measurable and finite \( \mu \)-a.e.
functions on $J$, with equality $\mu$-a.e. Clearly $\theta(t, f(t))$ is $\Omega$—measurable function of $t \in J$ for every $f \in M_X$, we define

$$\rho(f) = (H) \int_J \theta(t, f(t)) d\mu.$$  \hspace{1cm} (4)

Clearly, $\rho(x)$ of the equation (4) is a modular in $M_X$.

We define the $H$-Orlicz class $H^\theta$ as follows:

$$H^\theta(X, \mu) = \{ f : J \to X \text{ measurable} : \int_J \theta(t, kf(t)) d\mu \in H(X, \mu), \text{ for some } k > 0 \}.$$  

It is very straightforward that:

$$H^\theta(X, \mu) = \{ f : J \to X \text{ measurable} : \int_J \theta(t, kf(t)) d\mu \in H(X, \mu) \} \to 0 \text{ as } k \to 0.$$  

**Theorem 3.1** The $H$-Orlicz class $H^\theta(X, \mu)$ is a convex set of functions. That is, for given $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, $\rho(\alpha f + \beta g) \leq \alpha \rho(f) + \beta \rho(g)$.

**Proof** The proof is similar as [8, Theorem 2.1]. \hfill \Box

**Theorem 3.2** The $H$-Orlicz class $H^\theta(X, \mu)$ is linear if and only if $H^\theta(X, \mu)$ is closed under positive scalar multiplication.

**Proof** Let $H^\theta(X, \mu)$ be a linear space then, clearly it is closed under positive scalar multiplication.

Conversely, assume if $H^\theta(X, \mu)$ is closed under positive scalar multiplication. We will prove $H^\theta(X, \mu)$ is linear space. From the definition of $\theta(t, f(t))$, we find $\theta(t, f(t)) = \theta(t, -f(t))$, this means $-f \in H^\theta(X, \mu)$. Assume $\alpha, \beta > 0$ are real numbers and $f, g \in H^\theta(X, \mu)$.

**Case I** Let $\alpha, \beta > 0$ then for each $t \in J$ and convexity of $\theta(t, f(t))$ we have

$$\theta(t, \frac{\alpha f(t) + \beta g(t)}{\alpha + \beta}) = \theta(t, \frac{|\alpha| f(t)}{|\alpha| + |\beta|}) + \theta(t, \frac{|\beta| g(t)}{|\alpha| + |\beta|}).$$  \hspace{1cm} (5)

By the assumption $\alpha + \beta > 0$. The right sides of the equation (5) is in $H(X, \mu)$, so we can conclude $\alpha f + \beta g \in H^\theta(X, \mu)$.

**Case II** If $\alpha \beta < 0$ with $\alpha < 0 < \beta$ then

$$\theta(t, \frac{|\alpha| (-f(t)) + \beta g(t)}{|\alpha| + \beta}) = \theta(t, \frac{|\alpha| |(-f(t))|}{|\alpha| + \beta}) + \theta(t, \frac{\beta g(t)}{|\alpha| + \beta}).$$  \hspace{1cm} (6)

By the assumption $|\alpha| + \beta > 0$. The right sides of the equation (6) is in $H(X, \mu)$ so we have $\alpha f + \beta g \in H^\theta(X, \mu)$. \hfill \Box

We define vector valued $H$-Orlicz space as below

**Definition 3.3** The closure of $H$-Orlicz class $H^\theta(X, \mu)$ under positive scalar multiplication of a vector valued generalised Young function $\theta(t, f(t))$ will be called vector valued $H$-Orlicz space, denoted as $\mathcal{H}^\theta(X, \mu)$, is the set of functions $f$ in $M_X$ for which there is some positive constant $k$ such that $kf$ is in $H^\theta(X, \mu)$. That is.

$$\mathcal{H}^\theta(X, \mu) = \{ f \in H^\theta(X, \mu) : kf \in H^\theta(X, \mu) \}.$$
We define the norm of $\mathcal{H}^\theta(X, \overline{\mu})$ as follows:

$$||f||_{(X, \overline{\mu})} = \inf\left\{ k > 0 : (H) \int_{J} \theta\left(t, \frac{|f(t)|}{k}\right) d\overline{\mu} \leq 1 \right\}. \tag{7}$$

It is very clear that $\left( \mathcal{H}^\theta(X, \overline{\mu}), ||.||_{(X, \overline{\mu})} \right)$ is a Banach spaces with the norm (7).

**Theorem 3.4** The classical Orlicz space $\mathcal{L}^\theta(X, \overline{\mu})$ is a dense subspace of $\mathcal{H}^\theta(X, \overline{\mu})$ as continuous dense embeddings. That is, $\mathcal{L}^\theta(X, \overline{\mu}) \hookrightarrow \mathcal{H}^\theta(X, \overline{\mu})$ is continuous dense embeddings.

**Proof** Let $h \in \mathcal{L}^\theta(X, \overline{\mu})$. Then $h \in L^1(X, \overline{\mu})$ with $||h||_{L^\theta} < \infty$. Then for some $k > 0$, we have

$$\inf \left\{ (H) \int_{J} \theta\left(t, \frac{|h(t)|}{k}\right) d\overline{\mu} \right\} \leq \inf \left\{ (L) \int_{J} \theta\left(t, \frac{|h(t)|}{k}\right) d\overline{\mu} \right\} \leq 1.$$ 

So, for some $k > 0$, $\inf \left\{ k > 0 : (L) \int_{J} \theta\left(t, \frac{|h(t)|}{k}\right) d\overline{\mu} \leq 1 \right\}$, we get the following

$$\inf \left\{ k > 0 : (H) \int_{J} \theta\left(t, \frac{|h(t)|}{k}\right) d\overline{\mu} \leq 1 \right\}.$$ 

Hence $h \in \mathcal{H}^\theta(X, \overline{\mu})$ with $||h||_{(X, \overline{\mu})} \leq ||h||_{\mathcal{L}}$. Hence the proof. \qed

**Theorem 3.5** Suppose $\overline{\mu}(X) < \infty$ and $\overline{\mu}$ is bounded, then $\mathcal{H}^\theta(X, \overline{\mu}) \hookrightarrow L^1(X, \overline{\mu})$ is continuous.

**Corollary 3.6** Suppose $\overline{\mu}(X) < \infty$ and $\overline{\mu}$ is bounded, then $\mathcal{H}^\theta(X, \overline{\mu}) \hookrightarrow HK(X, \overline{\mu})$ is continuous.

## 4 Modular and norm convergence of $\mathcal{H}^\theta(X, \overline{\mu})$

It is well known that norm convergence implies modular convergence in classical Orlicz spaces (see [12, page 9]). In this section, we discuss the relationship of modular and norm convergent of $\mathcal{H}^\theta(X, \overline{\mu})$ and $\mathcal{L}^\theta(X, \overline{\mu})$.

**Definition 4.1** We say that a sequence $(f_n) \in \mathcal{H}^\theta(X, \overline{\mu})$ is modular convergent to $f \in \mathcal{H}^\theta(X, \overline{\mu})$ if there exists a constant $k > 0$ such that $\rho(k(f_n - f)) \to 0$ as $n \to \infty$.

**Theorem 4.2** Modular convergent in $\mathcal{L}^\theta(X, \overline{\mu})$ implies modular convergent in $\mathcal{H}^\theta(X, \overline{\mu})$

**Proof** Let $f_n(\cdot) \in \mathcal{L}^\theta(X, \overline{\mu})$ be modular convergent to $f(\cdot) \in \mathcal{L}^\theta(X, \overline{\mu})$. Then there exists a constant $k > 0$ such that $\lim_{n \to \infty} \rho[k(f_n - f)] = 0$. Since $\mathcal{L}^\theta(X, \overline{\mu})$ is subset of $\mathcal{H}^\theta(X, \overline{\mu})$ as continuous dense embedding so, $f_n(\cdot) \in \mathcal{H}^\theta(X, \overline{\mu})$. Using the Definition 4.1, $f_n(\cdot)$ is modular convergent to $f(\cdot) \in \mathcal{H}^\theta(X, \overline{\mu})$. \qed

**Remark 4.3** The known fact the the norm convergence is modular convergence in $\mathcal{L}^\theta(X, \overline{\mu})$ along with the (Theorem 4.2), norm convergence in $\mathcal{L}^\theta(X, \overline{\mu})$ is modular convergence in $\mathcal{H}^\theta(X, \overline{\mu})$.  

\textcopyright Springer
Now we will check the relationship of norm convergence

**Theorem 4.4** Norm convergent in $\mathcal{L}^0(X, \mu)$ implies norm convergent in $\mathcal{H}^0(X, \mu)$.

**Proof** Let $f_n \in \mathcal{L}^0(X, \mu)$ such that $f_n \to f$ in $\mathcal{L}^0(X, \mu)$ in the way that $||f_n - f||_{\mathcal{L}^0} \to 0$ as $n \to \infty$. This means,

$$\inf \left\{ k > 0 : (L) \int_{\mathcal{J}} \theta(t, (f_n - f)(t)) d\mu \leq 1 \right\} \to 0$$

$$\implies (L) \int_{\mathcal{J}} \theta(t, (f_n - f)(t)) \to 0$$

i.e., $\theta(t, (f_n - f)(t)) = 0 \mu - a.e.$ as $n \to \infty$

$$\implies f_n - f = 0 \ a.e. \ as \ n \to \infty.$$

As, $f_n \in \mathcal{L}^0(X, \mu)$, this implies $f_n \in \mathcal{H}^0(X, \mu)$ and $f_n - f = 0$ as $\mu - a.e.$ Now using the Theorem 2.5,

$$(H) \int_{\mathcal{J}} (f_n - f) d\mu = 0.$$

Lastly, from Definition 1.1(d)

$$(H) \int_{\mathcal{J}} \theta(t, (f_n - f)(t)) = 0.$$

So,

$$\inf \left\{ k > 0 : (H) \int_{\mathcal{J}} \theta(t, (f_n - f)(t)) d\mu \leq 1 \right\} \to 0.$$

Hence, $||f_n - f||_{(X, \mu)} \to 0$ as $n \to \infty$. \qed

**Theorem 4.5** Modular convergence in $\mathcal{L}^0(X, \mu)$ is norm convergent in $\mathcal{H}^0(X, \mu)$.

**Proof** Let $f_n \in \mathcal{L}^0(X, \mu)$ be modular convergent to $f \in \mathcal{L}^0(X, \mu)$. Then for $k > 0$ (depending on the sequence of function $(f_n)$) such that $\rho(k(f_n - f)) \to 0$ as $n \to \infty$. That is,

$$(L) \int_{\mathcal{J}} \theta(t, k(f_n - f)) d\mu \to 0 \ as \ n \to \infty$$

i.e., $\lim_{n \to \infty} (L) \int_{\mathcal{J}} \theta(t, k(f_n - f)) d\mu = 0$

i.e., $\theta(t, k(f_n - f)) = 0 \mu \ a.e.$ when $n \to \infty$.

For $k > 0$,

$$\inf \left\{ \frac{1}{k} > 0 : (H) \int_{\mathcal{J}} \theta(t, \frac{|f_n - f|}{k}) d\mu \leq 1 \right\} \to 0 \mu - a.e. \ as \ n \to \infty$$

So, $||f_n - f||_{(X, \mu)} \to 0$ as $n \to \infty$. This completes the proof. \qed
Conclusion

In this article we have discussed Henstock-Kurzweil integrable function space with a new norm equivalent to Alexiecz norm. $HK(X, \mu)$ is Banach space with the new norm (1). $H$-Orlicz space has been discussed with Banach valued Henstock-Kurzweil integrable function. We have established the relationship of modular convergence and norm convergence of functions with values in Banach spaces in $H$-Orlicz space. We find modular convergence in $L^\theta(X, \mu)$ is norm convergence in $H^\theta(X, \mu)$. We conclude this article with an open problem as follows:

**Problem** Modular convergent in $H^\theta(X, \mu)$ does not implies modular convergent in $L^\theta(X, \mu)$.

Acknowledgements The authors would like to thank the reviewer for reading the manuscript carefully and making valuable suggestions that significantly improve the presentation of the paper.

Author Contributions All the authors have equal contribution for the preparation of the article.

Declarations

Conflict of interest The authors declare that there is no conflicts of interest. analysis.

References

1. Ahmida, Y., Chlebicka, I., Gwiazda, P., Youssfi, A.: Gossez’s approximation theorems in the Musielak-Orlicz-Sobolev spaces. J. Func. Anal. 275(9), 2538–2571 (2018)
2. Banach, S.: Théorie des Opérations Linéaires, Warsaw, Monografie Matematyczne, Vol. 1, (1932)
3. Boccuto, A., Minotti, A.M., Sambucini, A.R.: Set-valued Kurzweil-Henstock integral in Riesz spaces. Panameric Math. J. 23(1), 57–74 (2013)
4. Boccuto, A., Hazarika, B., Kalita, H.: Kuelbs-Steadman spaces for Banach space-valued measures. Mathematics 8(1005), 1–12 (2020)
5. Bardaro, C., Vinti, G.: A modular convergence theorem for certain nonlinear integral operators with homogeneous kernel. Collect. Math. 48(4–6), 393–407 (1997)
6. Bardaro, C., Vinti, G.: Modular convergence theorems in fractional Musielak-Orlicz spaces. Z. Anal. Anwend. 13(1), 155–170 (1994)
7. Candeloro, D., Di Piazza, L., Musial, K., Sambucini, A.R.: Some new results on integration for multifunctions. Ric. Math. 67, 361–372 (2018)
8. Hazarika, B., Kalita, H.: Henstock-Orlicz space and its dense space. Asian–Eur. J. Math. 14, 1–17 (2021)
9. Kaminska, A., Hudzik, H.: Some remarks on convergence in Orlicz space. Ann. Soc. Math. Pol., Ser. 1: Comment. Math., XX I, 81–88 (1979)
10. Kozek, A.: Orlicz spaces of functions with values in Banach spaces. Comm. Math. 19, 259–288 (1976)
11. Krasnosel’skii, M. A., Rutickii, Ja. B.: Convex functions and Orlicz spaces, Gosud. Izdat. Fiz.-Mat. Literat., Moskva (in Russian) (1958)
12. Maligranda, Lech: Orlicz spaces and Interpolation. Departamento de Matematica, Universidade Estadual de Campinas, Seminarios De Matematica (1989)
13. Musielak, J., Orlicz, W.: On modular spaces. Studia Math. 18, 49–65 (1959)
14. Nakano, H.: Generalized modular spaces. Studia Math. 31(5), 439–449 (1968)
15. Pérez Carreras, P., Bonet, J.: Barreled Locally Convex Spaces. In: Mathematics Studies 131. Amsterdam, New York (1987)
16. Portnov, V.R.: A contribution to the theory of Orlicz spaces generated by variable $N$-functions. Sov. Math. Dokl. 8, 857–860 (1967)
17. Skaff, M.S.: Vector valued Orlicz spaces generalized $N$-function. I. Pac. J. Mat. 28(1), 413–430 (1969)
18. Skaff, M.S.: Vector valued Orlicz spaces. II. Pac. J. Math. 28(2), 193–206 (1969)
19. Talagrand, M.: Weak Cauchy sequences in $L^1(X)$. Amer. J. Math. 106, 703–724 (1984)
20. Väth, M.: Fréchet spaces with no subspaces isomorphic to $l_1$. Math. Jpn. 38, 397–411 (1993)
21. Wang, S.W.: Convex functions of several variables and vectorvalued Orlicz spaces. Bull. Acad. Pol. Sci. Ser. Math. Astr. et Phys. 11, 279–284 (1963)
