No greedy bases for matrix spaces with mixed $\ell_p$ and $\ell_q$ norms *

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Abstract

We show that none of the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_\ell_q$, $1 \leq p \neq q < \infty$ have a greedy basis. This solves a problem raised by Dilworth, Freeman, Odell and Schlumprecht. Similarly, the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{c_0}$, $1 \leq p < \infty$, and $(\bigoplus_{n=1}^{\infty} c_0)_{\ell_q}$, $1 \leq q < \infty$, do not have greedy bases. It follows from that and known results that a class of Besov spaces on $\mathbb{R}^n$ lack greedy bases as well.

1 Introduction

Given a (say, real) Banach space $X$ with a Schauder basis $\{x_i\}$, an $x \in X$ and an $n \in \mathbb{N}$ it is useful to determine the best $n$-term approximation to $x$ with respect to the given basis. I.e., to find a set $A \subset \mathbb{N}$ with $n$ elements and coefficients $\{a_i\}_{i \in A}$ such that

$$\|x - \sum_{i \in A} a_i x_i\| = \inf \{\|x - \sum_{i \in B} b_i x_i\| ; |B| = n, b_i \in \mathbb{R}\}$$

or, given a $C < \infty$, at least to find such an $A \subset \mathbb{N}$ and coefficients $\{a_i\}_{i \in A}$ with

$$\|x - \sum_{i \in A} a_i x_i\| \leq C \inf \{\|x - \sum_{i \in B} b_i x_i\| ; |B| = n, b_i \in \mathbb{R}\}.$$
This problem attracted quite an attention in modern Approximation Theory. Of course one would also like to have a simple algorithm to find such a set $\{a_i\}_{i \in A}$. It would be nice if we could take $\{a_i\}_{i \in A}$ to be just the set of the $n$ largest, in absolute value, coefficients in the expansion of $x$ with respect to the basis $\{x_i\}$. Or, if this set is not unique, any such set. The basis $\{x_i\}$ is called Greedy if for some $C$ this procedure works; i.e., for all $x = \sum_{i=1}^{\infty} a_i x_i$, all $n \in \mathbb{N}$ and all $A \subset \mathbb{N}$, $|A| = n$, satisfying $\min\{|a_i|; i \in A\} \geq \max\{|a_i|; i \notin A\}$,

$$
\|x - \sum_{i \in A} a_i x_i\| \leq C \inf\{\|x - \sum_{i \in B} b_i x_i\| ; |B| = n, b_i \in \mathbb{R}\}.
$$

Konyagin and Temlyanov [KT] provided a simple criterion to determine whether a basis is greedy: $\{x_i\}$ is greedy if and only if it is unconditional and democratic.

Recall that $\{x_i\}$ is said to be unconditional provided, for some $C < \infty$, all eventually zero coefficients $\{a_i\}$ and all sequences of signs $\{\varepsilon_i\}$,

$$
\| \sum_{i \in A} \varepsilon_i a_i x_i \| \leq C \| \sum_{i \in A} a_i x_i \|.
$$

$\{x_i\}$ is said to be democratic provided for some $C < \infty$ and all finite $A, B \subset \mathbb{N}$ with $|A| = |B|$,

$$
\| \sum_{i \in A} x_i \| \leq C \| \sum_{i \in B} x_i \|.
$$

We refer to [DFOS] for a survey of what is known about space that have or do not have greedy bases. In [DFOS] Dilworth, Freeman, Odell and Schlumprecht determined which of the spaces $X = (\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}$, $1 \leq p \neq q \leq \infty$ (with $c_0$ replacing $\ell_\infty$ in case $q = \infty$) have a greedy basis. It turns out that this happens exactly when $X$ is reflexive. They also raise the question of whether $(\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}$, $1 < p \neq q < \infty$ have greedy bases. Here we show that these spaces (as well as their non-reflexive counterparts) do not have greedy bases. By the Konyagin-Temlyanov characterization it is enough to prove that each unconditional basis of $(\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}$, $1 \leq p \neq q \leq \infty$ (with $c_0$ replacing $\ell_\infty$ in case $p$ or $q$ are $\infty$) has two subsequences, one equivalent to the unit vector basis of $\ell_p$ ($c_0$ if $p = \infty$) and one to the unit vector basis of $\ell_q$ ($c_0$ if $q = \infty$).

**Theorem 1** Each normalized unconditional basis of the spaces $(\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}$, $1 \leq p \neq q < \infty$ has a subsequence equivalent to the unit vector basis of $\ell_p$. 
and another one equivalent to the unit vector basis of $\ell_q$. Similarly, each unconditional basis of the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{c_0}$, $1 \leq p < \infty$ (resp. $(\bigoplus_{n=1}^{\infty} c_0)_{\ell_q}$, $1 \leq q < \infty$) has a subsequence equivalent to the unit vector basis of $\ell_p$ (resp. $c_0$) and another one equivalent to the unit vector basis of $c_0$ (resp. $\ell_q$). Consequently, none of these spaces have a greedy basis.

For $1 \leq p, q < \infty$ the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$ are isomorphic to certain Besov spaces on $\mathbb{R}^n$. We refer to [Me] for the definition of the Besov spaces $B^{s,q}_p$ and for the fact that they are isomorphic to $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$. See in particular [Me, Section 6.10, Proposition 7] (and also [Me, Section 2.9, Proposition 4]). We thank P. Wojtaszczyk for this reference.

**Corollary 1** Let $1 \leq p \neq q < \infty$ and $s$ any real number then the space $B^{s,q}_p$ does not have a greedy basis.

Recall that this stand in contrast with the main result in [DFOS] which states that, in the reflexive cases, the corresponding Besov spaces on $[0,1]$ do have greedy bases.

In the special case of $1 < q < \infty$ and $p = 2$ the theorem above was actually proved in [Sc]. There the isomorphic classification of the span of unconditional basic sequences in $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$, $1 < q < \infty$, which span complemented subspaces were characterize. Although it is not stated there, the proof actually established the theorem above in these special cases. Shortly after [Sc] appeared Odell [Sc] strengthened the result and classified all the complemented subspace of $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$ (thus there is no wonder that [Sc] was forgotten...). We remark in passing that this special case of $p = 2$ was of particular interest since $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$ is isomorphic to a complemented subspace of $L_q[0,1]$.

The first step in the proof in [Sc] is to reduce the case of a general unconditional basic sequence in $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$ whose span is complemented to one which is also a block basis of the natural basis of $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$. This reduction no longer hold for $p \neq 2$. The complications in the present note stems from this fact. The way we overcome it is by transferring the problem to a larger space (of arrays $\{a_{i,j,k}\}$) of mixed $q,p$ and 2 norms. Unfortunately, this makes the notations quite cumbersome.
2 Preliminaries

$Z_{q,p}, 1 \leq p, q < \infty$ will denote here the space of all matrices $a = \{a(i, j)\}_{i,j=1}^{\infty}$ with norm

$$\|a\| = \|a\|_{q,p} = \left(\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a(i, j)|^p\right)^{q/p}\right)^{1/q}.$$ 

If $p$ or $q$ are $\infty$ we replace the corresponding $\ell_p$ or $\ell_q$ norm by the $\ell_\infty$ norm and continue to denote by $Z_{q,p}$ the completion of the space of finitely supported matrices under this norm. (Thus, $c_0$ replacing $\infty$ would be a more precise notation in this case but, since it would complicated our statements, we prefer the above notation.) The spaces $Z_{q,p}$ are the subject of investigation of this paper. They are more commonly denoted by $\ell_q(\ell_p)$ or $\bigoplus_{n=1}^{\infty} \ell_p\ell_q$ (as we have done in the introduction) but since we shall be forced to also consider more complicated spaces with mixed norms we prefer the notation above.

If $\{k_n\}_{n=1}^{\infty}$ is any sequence of positive integers, we shall denote by $Z_{q,p};\{k_n\}$, the subspace of $Z_{q,p}$ consisting of matrices $a$ satisfying $a(i, j) = 0$ for all $i > k_j$.

We also denote by $Z_{q,p,r}$ (we’ll use this only for $r = 2$) the spaces of arrays $a = \{a(u, i, j)\}_{u,i,j=1}^{\infty}$ with norm

$$\|a\| = \|a\|_{q,p,r} = \left(\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \left(\sum_{u=1}^{\infty} |a(u, i, j)|^r\right)^{p/r}\right)^{q/p}\right)^{1/q},$$

with the same convention as above when one of $p, q$ (or $r$) is $\infty$. Similarly, $Z_{q,p};\{k_n\},r$ denotes the subspace of $Z_{q,p,r}$ consisting of arrays $a$ satisfying $a(u, i, j) = 0$ for all $i > k_j$.

By $P_n$ we denote the natural projection onto the $n$-th column in $Z_{q,p}$; i.e, $P_n(\{a(i, j)\}) = \{\bar{a}(i, j)\}$, where $\bar{a}(i, j) = a(i, j)$ if $j = n$ and $\bar{a}(i, j) = 0$ otherwise. Similarly, $P_k^n$ denotes the natural projection onto the first $k$ elements in the $n$-th column. $Q_N$ denotes $\sum_{n=1}^{N} P_n$.

Given a Banach lattice $X$, an $1 < r < \infty$ and $x_1, x_2, \cdots \in X$ one can define the operation $(\sum |x_n|^r)^{1/r}$ in a manner consistent with what we usually mean by such an operation (when $X$ is a lattice of functions or sequences, for example). See e.g. [LT2 Section 1.d] for this and what follows.

In particular if $X$ has a 1-unconditional basis $\{e_i\}$ (which is the only kind of lattices we’ll consider here) then for $x_n = \sum_{i=1}^{N} a_i^n e_i$, $n = 1, 2, \ldots, N$, $(\sum |x_n|^r)^{1/r} = \sum_{i=1}^{N} (\sum_{n=1}^{N} |a_i^n|^r)^{1/r} e_i$. 

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Recall that $X$ is said to be $r$-convex (resp. $r$-concave) with constant $K$ if for all $n$ and all $x_1, x_2, \ldots, x_n \in X$

$$\|\left(\sum_{i=1}^{n} |x_i|^r\right)^{1/r}\| \leq K\left(\sum_{i=1}^{n} \|x_i\|^r\right)^{1/r} \quad \text{(resp.} \quad \left(\sum_{i=1}^{n} \|x_i\|^r\right)^{1/r} \leq K\left(\sum_{i=1}^{n} |x_i|^r\right)^{1/r}\text{)}).$$

$X$ is said to be $r$-convex (resp. $r$-concave) if it is $r$-convex (resp. $r$-concave) with some constant $K < \infty$. $Z_{q,p}$ is easily seen to be $\min\{p, q\}$-convex with constant 1 and $\max\{p, q\}$-concave with constant 1.

It is also known that $X$ is $r$-convex (resp. $r$-concave) if and only if its dual $X^*$ is $r'$-concave (resp. $r'$-convex) where $r' = r/(r - 1)$.

Given a Banach lattice $X$ we denote by $X(\ell_2)$ the (completion of the) space of (finite) sequences $x = (x_1, x_2, \ldots)$ of elements of $X$ for which the norm

$$\|x\| = \|(\sum_j |x_j|^2)^{1/2}\|$$

is finite. If $X$ has a 1-unconditional basis $\{e_j\}$ then this is just the (completion of the) space of matrices $a = \{a(i, j)\}$ (with only finitely many non-zero entries) with norm

$$\|a\| = \|\sum_j (\sum_i |a(i, j)|^2)^{1/2} e_i\|.$$

The following two lemmas are well known but maybe hard to find so we reproduce their proofs.

**Lemma 1** Let $\{x_i\}_{i=1}^\infty$ be a normalized unconditional basic sequence in $Z_{q,p}$, $1 \leq p < q \leq \infty$. If for some $\varepsilon > 0$ and $N \in \mathbb{N}$ $\|Q_N x_i\| > \varepsilon$ for all $i$ then $\{x_i\}_{i=1}^\infty$ has a subsequence equivalent to the unit vector basis of $\ell_p$.

**Proof:** Assume first $p > 1$. Given a sequence of positive $\varepsilon_i$-s and passing to a subsequence (which without loss of generality we assume is the all sequence) we can assume that there is a sequence of $\{y_i\}$ of vectors disjointly supported with respect to the natural basis of $Z_{q,p}$ such that $\|x_i - y_i\| < \varepsilon_i$ for all $i$. (Use the fact that $\{x_i\}$ doesn’t have a subsequence equivalent to the unit vector basis of $\ell_1$ and the argument for Proposition 1.a.12 in [LT1], for example). $\{y_i\}$ is 1-dominated by the unit vector basis of $\ell_p$ and dominates $\{Q_N y_i\}$
which in turn $C$-dominates the unit vector basis of $\ell_p$ for $C = 1/(\varepsilon - \sup \varepsilon_i)$; i.e.,

$$\left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p} \geq \left\| \sum_{i=1}^{\infty} a_i y_i \right\| \geq (\varepsilon - \sup \varepsilon_i)\left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p}$$

for all scalars $\{a_i\}$. If the $\varepsilon_i$'s are small enough a similar inequality holds for the (sub)sequence $\{x_i\}$.

If $p = 1$ then given a sequence of positive $\varepsilon_i$'s and passing to a subsequence (which without loss of generality we assume is the all sequence) we can assume that there is a vector $y$ and sequence of $\{y_i\}$ of vectors all disjointly supported with respect to the natural basis of $Z_{q,p}$ such that $\|x_i - y - y_i\| < \varepsilon_i$ for all $i$. If $y \neq 0$ and the $\{\varepsilon_i\}$ are small enough then, using the unconditionality $\{x_i\}$ is clearly equivalent to the unit vector basis of $\ell_1$. If $y = 0$ the same argument as for $p > 1$ works here too.

**Lemma 2** Let $\{x_i\}$ be a $K$-unconditional basic sequence in a Banach lattice which is $r$-concave for some $r < \infty$ Let $\bar{x}_i \in X(\ell_2)$ be defined by $(0, \ldots, 0, x_i, 0, \ldots)$, $x_i$ in the $i$-th place. Then the sequences $\{x_i\}$ in $X$ and $\{\bar{x}_i\} in X(\ell_2)$ are equivalent.

If in addition $X$ is also $s$-convex for some $s > 1$ and $[x_i]$, the closed linear span of $\{x_i\}$, is complemented in $X$ then $[\bar{x}_i]$ is complemented in $X(\ell_2)$.

**Proof:** The first assertion, due to Maurey, can be found in [Ma] or [LT2, Theorem 1.d.6(i)]. The second is probably harder to find so we reproduce it. Let $P = \sum_{i=1}^{\infty} x_i^* \otimes x_i$, with $x_i^* \in X^*$, be the projection onto $[x_i]$; i.e.,

$$P(x) = \sum_{i=1}^{\infty} x_i^*(x)x_i \quad x \in X.$$ 

Define $\bar{P} = \sum_{i=1}^{\infty} \bar{x}_i^* \otimes \bar{x}_i \ (\bar{x}_i^* \in X^*(\ell_2) = X(\ell_2)^*)$; i.e.,

$$\bar{P}(x) = \sum_{i=1}^{\infty} \bar{x}_i^*(x)\bar{x}_i \quad x \in X(\ell_2).$$

Using the facts that $\{\bar{x}_i\}$ is equivalent to $\{x_i\}$, $\{\bar{x}_i^*\}$ is equivalent to $\{x_i^*\}$, and $\{\bar{x}_i^*, \bar{x}_i\}$ is a biorthogonal sequence, it is easy to see that $\bar{P}$ is a bounded projection on $X(\ell_2)$ with range $[\bar{x}_i]$. 

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3 Proof of the main result, the reflexive case

Since the non-reflexive cases (i.e., when \( p \) or \( q \) are 1 or \( \infty \)) of Theorem [1] require a bit different treatment and since the problem raised in [DFOS] was restricted to the reflexive cases only, we prefer to delay the proof of the non-reflexive cases to the next section.

**Proposition 1** Let \( \{x_i\}_{i=1}^{\infty} \) be a normalized unconditional basic sequence in \( Z_{q,p} \), \( 1 < p, q < \infty \) such that \( \{x_i\}_{i=1}^{\infty} \) is complemented in \( Z_{q,p} \). If no sub-sequence of \( \{x_i\}_{i=1}^{\infty} \) is equivalent to the unit vector basis of \( \ell_p \) then \( \{x_i\}_{i=1}^{\infty} \) isomorphically embeds in \( Z_{q,p}, \{n\}, 2 \) as a complemented subspace.

**Proof:** We may clearly assume \( p \neq q \) and by duality that \( q > p \). Let \( \{\varepsilon_n\}_{n=1}^{\infty} \) be a sequence of positive numbers. By Lemma [1] for all \( n \) only finitely many of the \( x_i \)-s satisfy \( \|P_n x_i\| \geq \varepsilon_n \). Consequently, for each \( n \in \mathbb{N} \) there is a \( k_n \in \mathbb{N} \) such that \( \|(P_n - P_{k_n}^n)x_i\| < \varepsilon_n \) for all \( i \). We denote \( Q = \sum_{n=1}^{\infty} P_n^{k_n} \).

In the case \( p = 2 \) we showed in [Sc] that without loosing generality we can assume that \( \{x_i\} \) is a block basis of the natural basis of \( Z_{q,p} \) and then \( \{Qx_i\} \) and \( \{(I - Q)x_i\} \) are also unconditional basic sequences. This is no longer true when \( p \neq 2 \). We overcome this difficulty by switching to the larger space \( Z_{q,p}, 2 \). Define for each \( i \) \( \bar{x}_i \in Z_{q,p}, 2 \) by

\[
\bar{x}_i(w, u, v) = \begin{cases} x_i(u, v), & \text{if } w = i; \\ 0, & \text{if } w \neq i. \end{cases}
\]

Let the projection \( P \) from \( Z_{q,p} \) onto \( [x_i] \) be given by

\[
P x = \sum_{i=1}^{\infty} x_i^*(x)x_i
\]

where \( \{x_i^*\} \in Z_{q', p'} \) (\( p' = p/(p-1), q' = q/(q-1) \)) satisfy \( x_i^*(x_j) = \delta_{i,j} \), \( i, j = 1, 2, ... \). Then by Lemma [2]

\[
\bar{P} x = \sum_{i=1}^{\infty} \bar{x}_i^*(x)\bar{x}_i
\]

is a bounded projection from \( Z_{q,p}, 2 \) onto \( [\bar{x}_i] \) and \( \{x_i\}_{i=1}^{\infty} \) is equivalent to \( \{\bar{x}_i\}_{i=1}^{\infty} \).

We denote by \( \bar{P}_n = P_n \otimes I_{\ell_2} \) on \( Z_{q,p}, 2 \); i.e., \( \bar{P}_n(x)(w, u, v) = P_n(x(w, \cdot, \cdot))(u, v) \).

We also similarly denote \( \bar{P}_n^k = P_n^k \otimes I_{\ell_2}, \bar{Q}_N = Q_N \otimes I_{\ell_2}, \) and \( \bar{Q} = Q \otimes I_{\ell_2} \).
Note that now \( \{ \hat{Q} \hat{x}_i \} \) and \( \{ (I - \hat{Q}) \hat{x}_i \} \) are also unconditional basic sequences. We would like to show that if \( \varepsilon_n \to 0 \) fast enough, then \( \{ \hat{Q} \hat{x}_i \} \) is equivalent to \( \{ \hat{x}_i \} \) and thus to \( \{ x_i \} \) and that \( \{ \hat{Q} \hat{x}_i \} \) is complemented.

Now,

\[
(I - \hat{Q}) \hat{P} \hat{Q} \hat{x}_n = \sum_{i=1}^{\infty} \hat{x}_i^*(\hat{Q} \hat{x}_n)(I - \hat{Q}) \hat{x}_i, \hspace{1em} n = 1, 2, \ldots.
\]

The operator \( (I - \hat{Q}) \hat{P} \) sends the span of the unconditional basic sequence \( \{ \hat{Q} \hat{x}_n \} \) to the span of the unconditional basic sequence \( \{ (I - \hat{Q}) \hat{x}_n \} \) thus the diagonal operator \( D \) defined by

\[
D \hat{Q} \hat{x}_n = \hat{x}_n^*(\hat{Q} \hat{x}_n)(I - \hat{Q}) \hat{x}_n, \hspace{1em} n = 1, 2, \ldots.
\]

is bounded (see e.g. [10] or [11, Proposition 1.c.8]). If we show that \( \hat{x}_n^*(\hat{Q} \hat{x}_n) \) are uniformly bounded away from zero this will show that \( \{ \hat{Q} \hat{x}_n \} \) dominates \( \{ (I - \hat{Q}) \hat{x}_n \} \) and thus also \( \{ \hat{x}_n \} = \{ (I - \hat{Q}) \hat{x}_n + \hat{Q} \hat{x}_n \} \). That \( \{ \hat{Q} \hat{x}_n \} \) is dominated by \( \{ \hat{x}_n \} \) is clear from the boundedness of \( \hat{Q} \). This will show that \( \{ \hat{Q} \hat{x}_n \} \) is equivalent to \( \{ x_n \} \). To show that \( \hat{x}_n^*(\hat{Q} \hat{x}_n) \) are uniformly bounded away from zero note that

\[
\hat{x}_n^*(\hat{Q} \hat{x}_n) = 1 - \hat{x}_n^*((I - \hat{Q}) \hat{x}_n)
\]

and that

\[
|\hat{x}_n^*((I - \hat{Q}) \hat{x}_n)| \leq \| \hat{P}((I - \hat{Q}) \hat{x}_n) \| \leq \| \hat{P} \| \sum_{i=1}^{\infty} \varepsilon_i.
\]

So, if \( \| \hat{P} \| \sum_{i=1}^{\infty} \varepsilon_i < 1/2 \), then \( \hat{x}_n^*(\hat{Q} \hat{x}_n) \geq 1/2 \) for all \( n \).

We still need to show that \( \{ \hat{Q} \hat{x}_n \} \) is complemented. Note that \( \{ \hat{x}_n^*(\hat{Q} \hat{x}_n), \hat{Q} \hat{x}_n \} \) is a biorthogonal sequence such that \( \{ \hat{Q} \hat{x}_n \} \) is equivalent to \( \{ x_n \} \) and \( \{ \hat{x}_n^*(\hat{Q} \hat{x}_n) \} \) is dominated by \( \{ x_n^* \} \). It follows that

\[
x \to \sum_{n=1}^{\infty} \frac{\hat{x}_n^*(x)}{\hat{x}_n^*(\hat{Q} \hat{x}_n)} \hat{Q} \hat{x}_n
\]

defines a bounded projection with range \( \{ \hat{Q} \hat{x}_n \} \).

We have shown that \( \{ x_i \} \) embeds complementably into \( Z_{q,p;\{k_n\},2} \) for some sequence of positive integers \( \{ k_n \} \). This last space is clearly isometric to a norm one complemented subspace of \( Z_{q,p;\{n\},2} \).

\[ \square \]
In the case $p = 2$ the argument above simplifies and actually shows that under the assumptions of Proposition 1 we can strengthen the conclusion to:

$[x_i]$ embeds complementably in $Z_{q,2;\{n\}}$ (which is isomorphic to $\ell_q$). We will not dwell on it farther as this is contained in [Sc]. The next proposition combined with the previous one will show in particular that any unconditional basis of $Z_{q,p}$ contains a subsequence equivalent to the unit vector basis of $\ell_p$. We’ll need to use this also in the next section so we include the non-reflexive cases here as well.

**Proposition 2** Let $1 \leq p, q \leq \infty$. If $p \neq 2, q$, then $\ell_p$ (or $c_0$ in case $p = \infty$) does not embed into $Z_{q,p;\{n\},2}$.

**Proof:** Assume $\ell_p$ or $c_0$ embeds into $Z_{q,p;\{n\},2}$. Passing to a subsequence of the image of the unit vector basis of $\ell_p$ or $c_0$, taking successive differences (this is needed only in the case $p = 1$) and using a simple perturbation argument, we may assume that some normalized block basis $\{x_i\}$ of the natural basis of $Z_{q,p;\{n\},2}$ is equivalent to the unit vector basis of $\ell_p$ ($c_0$ if $p = \infty$). Let $P_{n,m}$, $m = 1, 2, \ldots, 1 \leq n \leq m$, denote the canonical projection onto the $n,m$ copy of $\ell_2$ in $Z_{q,p;\{n\},2}$:

$$P_{n,m}(\{a(w, u, v)\}) = \{\bar{a}(w, u, v)\}$$

where

$$\bar{a}(w, u, v) = \begin{cases} a(w, u, v), & \text{if } u = n, \ v = m; \\ 0, & \text{otherwise.} \end{cases}$$

Assume first $p > 2$. For each $n, m$ $P_{n,m}$ acts as a compact operator from $[x_i]$ to $\ell_2$ as every bounded operator from $\ell_p$, $p > 2$ or $c_0$ to $\ell_2$ do. Consequently, given a sequence of positive numbers $\{\varepsilon_{n,m}\}$, we can find $k_{n,m} \in \mathbb{N}$ such that $\|(P_{n,m} - P_{n,m}^{k_{n,m}})[x_i]\| < \varepsilon_{n,m}$ for all $n, m$. Then, if $\sum_{n,m} \varepsilon_{n,m}$ is small enough

$$(\sum_{n,m} P_{n,m}^{k_{n,m}})[x_i]$$

is an isomorphism and we get that $[x_i]$ embeds into $Z_{q,p;\{n\},2;\{k_{n,m}\}}$. Now for each finite $m$ and $k$ the $\ell_p^m$ sum of $\ell_2^k$-s-embeds into $\ell_p^N$ for some $N$ depending only on $p, m$ and $k$. It thus follows that $[x_i]$ embeds into $Z_{q,p;\{k_n\}}$ for some sequence of positive integers $\{k_n\}$. Passing to a farther subsequence of $\{x_i\}$, we get that the unit vector basis of $\ell_p$ (or $c_0$ in the case $p = \infty$) is equivalent to that of $\ell_q$ which is a contradiction.
The case $1 \leq p < 2$ is just a bit more complicated. Here $P_{n,m}$ doesn’t act as a compact operator from $[x_i]$ to $\ell_2$ but it is still strictly singular. Consequently, for each $n, m$ and $l$ we can find a normalised block basis of $\{x_i\}_{i=l}^\infty$ such that $\| (P_{n,m})_{[x_i]} \| < \varepsilon_{n,m}$ and consequently there is a block basis of $\{x_i\}$ whose first $l - 1$ terms are just $x_1, \ldots, x_{l-1}$, and $k_{n,m,l}$ such that

$$\| (P_{n,m} - P_{k_{n,m,l}}^{k_{n,m,l}})_{[x_i]} \| < \varepsilon_{n,m}.$$ 

A simple diagonal argument will now produce a normalised block basis $\{z_i\}$ of $\{x_i\}$ and natural numbers $k_{n,m,l}$s such that

$$\sum_{n,m} P_{n,m}^{k_{n,m,l}}_{[z_i]}$$

an isomorphism. Since $\{z_i\}$ is equivalent to the unit vector basis of $\ell_p$ we get that $\ell_p$ embeds into $Z_{q,p} \{n\}, 2 \{k_{n,m}\}$. The rest of the proof in this case is the same as in the case $p > 2$.

We are now aiming at proving that every normalized unconditional basis of $Z_{q,p}$ contains a subsequence equivalent to the unit vector basis of $\ell_q$.

**Proposition 3** Let $\{x_i\}_{i=1}^\infty$ be a normalized unconditional basic sequence in $Z_{q,p}$, $1 < p, q < \infty$ such that $\{x_i\}_{i=1}^\infty$ is complemented in $Z_{q,p}$. If no subsequence of $\{x_i\}_{i=1}^\infty$ is equivalent to the unit vector basis of $\ell_q$ then $\{x_i\}_{i=1}^\infty$ isomorphically embeds in $Z_{p,2}$ as a complemented subspace.

**Proof:** We may assume $q < p$. We first claim that for each $\varepsilon > 0$ there is an $N$ such that $\| (I - Q_N) x_i \| < \varepsilon$ for each $i = 1, 2, \ldots$ Indeed if this is not the case then there is an $\varepsilon > 0$, a sequence $0 = N_1 < N_2 < \cdots$ in $\mathbb{N}$ and a subsequence $\{y_i\}$ of $\{x_i\}$ such that $\| (Q_{i+1} - Q_i) y_i \| \geq \varepsilon$ for all $i$. Passing to a further subsequence and a small perturbation we may assume that $\{y_i\}$ is a block basis of the natural basis of $Z_{q,p}$. Then, since $q < p$, for all scalars $\{a_i\}$,

$$\sum_{i=1}^\infty |a_i|^q \geq \left\| \sum_{i=1}^\infty a_i y_i \right\| \geq \left\| \sum_{i=1}^\infty a_i (Q_{i+1} - Q_i) y_i \right\| \geq \varepsilon \left( \sum_{i=1}^\infty |a_i|^q \right)^{1/q}$$

in contradiction to the fact that no subsequence of the $\{x_i\}$ is equivalent to the unit vector basis of $\ell_q$. 

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The rest of the proof is now similar to that of Proposition \( \boxed{} \), only a bit simpler. Fix an \( \varepsilon > 0 \) and let \( N \) be as in the beginning of this proof. Let \( P = \sum_{i=1}^{\infty} x_i^* \otimes x_i \) be the projection onto \([x_i]\) and let \( \bar{x}_i \) (in \( Z_{q,p,2} \)), \( \bar{P} \) and \( \bar{Q}_N \) be as in the proof of Proposition \( \boxed{} \). Consider the operator \((I - \bar{Q}_N)\bar{P}\) as acting from the span of the unconditional basic sequence \( \{\bar{Q}_N\bar{x}_i\} \) to the span of the unconditional sequence \( \{(I - \bar{Q}_N)\bar{x}_i\} \):

\[
(I - \bar{Q}_N)\bar{P}\bar{Q}_N\bar{x}_n = \sum_{i=1}^{\infty} \bar{x}_i^*(\bar{Q}_N\bar{x}_n)(I - \bar{Q}_N)\bar{x}_i, \ n = 1, 2, \ldots .
\]

Its diagonal defined by

\[
D\bar{Q}_N\bar{x}_n = \bar{x}_n^*(\bar{Q}_N\bar{x}_n)(I - \bar{Q}_N)\bar{x}_n, \ n = 1, 2, \ldots
\]

is bounded (\( \boxed{} \) or \( \boxed{} \)). So if we show that \( \bar{x}_n^*(\bar{Q}_N\bar{x}_n) \) are bounded away from zero then the sequence \( \{\bar{Q}_N\bar{x}_i\} \) will dominate the sequence \( \{(I - \bar{Q}_N)\bar{x}_i\} \) and thus also \( \{\bar{x}_i\} \) and \( \{x_i\} \). This will also show that

\[
x \to \sum_{n=1}^{\infty} \frac{\bar{x}_n^*(x)}{\bar{x}_n^*(\bar{Q}_N\bar{x}_n)} \bar{Q}_N\bar{x}_n
\]

defines a bounded projection from \( \bar{Q}_N Z_{q,p,2} \) (which is isomorphic to \( Z_{p,2} \)) onto \([\bar{Q}_N \bar{x}_i]\) (which is isomorphic to \([x_i]\)).

To show that \( \bar{x}_n^*(\bar{Q}_N\bar{x}_n) \) are bounded away from zero note that

\[
\bar{x}_n^*(\bar{Q}_N\bar{x}_n) = 1 - \bar{x}_n^*((I - \bar{Q}_N)\bar{x}_n)
\]

and that

\[
|\bar{x}_n^*((I - \bar{Q}_N)\bar{x}_n)| \leq \|\bar{P}(I - \bar{Q}_N)\bar{x}_n\| \leq \|\bar{P}\|\varepsilon.
\]

So, if \( \|\bar{P}\|\varepsilon < 1/2 \), then \( \bar{x}_n^*(\bar{Q}_N\bar{x}_n) \geq 1/2 \) for all \( n \).

**Remark 1** With a bit more effort one can strengthen the conclusion of Proposition \( \boxed{} \) to: \([x_i]_{i=1}^{\infty} \) is isomorphic to \( \ell_p \). This is done by first showing that one can embed \([x_i]_{i=1}^{\infty} \) as a complemented subspace in \( Z_{p,2;\{n\}} \) which is isomorphic to \( \ell_p \) and using the fact that any infinite dimensional complemented subspace of \( \ell_p \) is isomorphic to \( \ell_p \).
Proof of Theorem I in the reflexive case: Propositions 1 and 2 show that any normalized unconditional basis of $Z_{q,p}$, $1 < p, q < \infty$, has a subsequence equivalent to the unit vector basis of $\ell_p$. To show that any such basis also has a subsequence equivalent to the unit vector basis of $\ell_q$, we need, in view of Proposition 3, only prove that $Z_{q,p}$ doesn’t embed complementably into $Z_{p,2}$ for $1 < q \neq p < \infty$. This can probably be done directly (especially in the case $q \neq 2$ in which case it is also true that $\ell_q$ does not embed into $Z_{p,2}$) but it also follows from the main theorems of [Sc] and [Od] in which the complemented subspaces of $Z_{p,2}$ (in [Sc] only those with unconditional basis) where characterized.

4 Proof of the main result, the non-reflexive case

Recall that the subscript $\infty$ in $Z_{\infty,p}$ refers, by our convention, to the $c_0$ (rather than $\ell_\infty$) sum. Similarly, the subscript $\infty$ in $Z_{q,\infty}$ refers to the $q$ sum of $c_0$. We are going to show that any unconditional basis of each of the spaces $Z_{q,p}$, $p \neq q$, when at least one of $p$ or $q$ is 1 or $\infty$ contains a subsequence equivalent to the unit vector basis of $\ell_p$ ($c_0$ if $p = \infty$) and another subsequence equivalent to the unit vector basis of $\ell_q$ ($c_0$ if $q = \infty$).

The spaces $Z_{1,\infty}$ and $Z_{\infty,1}$ (as well as $Z_{1,2}$ and $Z_{\infty,2}$) have unique, up to permutation, unconditional bases [BCLT]. These bases clearly contain a subsequence equivalent to the unit vector basis of $c_0$ and another equivalent to the unit vector basis of $\ell_1$, so we only need to deal with the spaces $Z_{\infty,p}$, $1 < p < \infty$, and their duals $Z_{1,p'}$ and with $Z_{q,\infty}$, $1 < q < \infty$, and their duals $Z_{q',1}$.

We shall need some classical results concerning unconditional bases and duality. These can be found conveniently in sections 1.b. and 1.c. of [LT1]. $\ell_1$ does not isomorphically embed into $Z_{\infty,p}$, $1 < p < \infty$, (resp. into $Z_{q,\infty}$, $1 < q < \infty$) (this follows for example from the fact that these spaces are $p$ (resp. $q$) convex). It thus follows from a theorem of James [Ja] or see [LT1, Theorem 1.c.9] that any unconditional basis of these spaces is shrinking. See [LT1, Proposition 1.b.1] for the definition of a shrinking basis as well as for the fact that then the biorthogonal basis is an unconditional basis of the dual space $Z_{1,p'}$, $1 < p < \infty$, (resp. $Z_{q',1}$, $1 < q < \infty$). Thus, in order to prove Theorem I in the non-reflexive cases, if would be enough to show
that any normalized unconditional basis of $Z_{1,p}$, $1 < p < \infty$, (resp. $Z_{q,1}$, $1 < q < \infty$) contains a subsequence equivalent to the unit vector basis of $\ell_1$ and another subsequence equivalent to the unit vector basis of $\ell_p$ (resp. $\ell_q$).

Let $\{x_n\}$ be a normalized unconditional basis of $X^* = Z_{1,p}$, $1 < p < \infty$, (resp. $X^* = Z_{q,1}$, $1 < q < \infty$) such a basis is boundedly complete and its biorthogonal basis spans a space isomorphic to $X = Z_{\infty,p'}$ (resp. $X = Z_{q',\infty}$).

We begin with a proposition which replaces Propositions 1 and 2 for the current cases.

Proposition 4 Let $\{x_n\}$ be a normalized unconditional basis of $Z_{1,p}$, $1 < p < \infty$, (resp. $Z_{q,1}$, $1 < q < \infty$). Then $\{x_n\}$ contains a subsequence equivalent to the unit vector basis of $\ell_1$ (resp. $\ell_p$).

Proof: By proposition 2 $\ell_p$ does not embed into $Z_{1,p}\{n\},2$ for $1 < p < \infty$ and $\ell_1$ does not embed into $Z_{q,1}\{n\},2$ for $1 < q < \infty$. It is thus enough to show that if $\{x_n\}$ contains no subsequence equivalent to the unit vector basis of $\ell_p$ (resp. $\ell_1$) then $[x_n]$ embeds in $Z_{1,p}\{n\},2$ (resp. $Z_{q,1}\{n\},2$).

The case of $Z_{q,1}$, $1 < q < \infty$: We proceed as in the proof of Proposition 1. Since $q > 1$ the beginning of the proof works for $p = 1$ as well. The problem arise when we need to show that $\bar{P}$ is bounded as this no longer follow from Lemma 2. But here we can use instead [LT2, Theorem 1.d.6(ii)] to prove that $\bar{P}$ is bounded in a very similar way to the proof of Lemma 2. The rest of the proof of Proposition 1 carries over.

The case of $Z_{1,p}$, $1 < p < \infty$: Assume $\{x_n\}$ be a basis of $Z_{1,p}$, $1 < p < \infty$. Let $\{x_n^*\}$ be the biorthogonal basis (of $Z_{\infty,p'}$). By the assumption that $\{x_n\}$ doesn’t contain a subsequence equivalent to the unit vector basis of $\ell_p$, $[x_n^*]$ doesn’t contain a subsequence equivalent to the unit vector basis of $\ell_{p'}$. The proof of Proposition 1 works for $Z_{\infty,p'}$, $1 < p' < \infty$, as well, with the same modification for the proof that $\bar{P}$ is bounded as in the previous paragraph, to show that in this case $[x_n^*]$ embeds (even complementably) into $Z_{\infty,p'\{n\}},2$.

The next proposition replaces Proposition 3 in the non-reflexive case.

Proposition 5 (i) Let $\{x_n\}$ be a normalized unconditional basis of $Z_{1,p}$, $1 < p < \infty$. Then the unit vector basis of $\ell_1$ is equivalent to a subsequence of $\{x_n\}$.

(ii) Let $\{x_n\}$ be a normalized unconditional basis of $Z_{q,1}$, $1 < q < \infty$. Then the unit vector basis of $\ell_q$ is equivalent to a subsequence of $\{x_n\}$.  

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Proof: The proof of Proposition 3 works for $Z_{q,p}$ also in the case $q = 1 < p < \infty$ and we get that under the assumption of $(i)$, if no subsequence of $\{x_n\}$ is equivalent to the unit vector basis of $\ell_1$ then $[x_n]$ embeds into $Z_{p,2}$ but this space has type $\min\{p,2\}$ so $\ell_1$ and thus also $Z_{1,p}$, $1 < p < \infty$, do not embed into it. This proves $(i)$.

$(ii)$ It is enough to show that the unit vector basis of $\ell_{q'}$ is equivalent to a subsequence of $\{x^*_n\}$ (the biorthogonal basis to $\{x_n\}$) which is an unconditional basis of $Z_{q',\infty}$. The proof of Proposition 3 gives that if this is not the case then $Z_{q',\infty}$ isomorphically embeds as a complemented subspace in $Z_{\infty,2}$. Now if $Z_{q',\infty}$ isomorphically embeds as a complemented subspace in $Z_{\infty,2}$ then an easy application of Pelczynski’s decomposition method gives that $Z_{q',\infty} \oplus Z_{\infty,2}$ is isomorphic to $Z_{\infty,2}$ but this immediately presents an unconditional basis for $Z_{\infty,2}$ which is not equivalent to a permutation of the canonical basis of $Z_{\infty,2}$. This stands in contradiction to a result from [BCLT] and thus proves $(ii)$.

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