On badly approximable vectors

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Motivated by a wonderful paper [7] where a powerful method was introduced, we prove a criterion for a vector \( \alpha \in \mathbb{R}^d \) to be a badly approximable vector. Moreover we construct certain examples which show that a more general version of our criterion is not valid.

1. Badly approximable real numbers and continued fractions.

Let \( ||x|| = \min_{a \in \mathbb{Z}} |x - a| \) denote the distance from a real \( x \) to the nearest integer. A real irrational number \( \alpha \) is called badly approximable if

\[
\inf_{q \in \mathbb{Z}^+} q ||q\alpha|| > 0.
\]

It is a well known fact that \( \alpha \) is a badly approximable number if and only if the partial quotients in continued fraction expansion

\[
[a_0; a_1, a_2, ..., a_\nu, ...] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_\nu + \cdots}}},
\]

are bounded, that is

\[
\sup_{\nu \geq 1} a_\nu < \infty
\]

(see, for example Theorem 5F from Chapter I from [10]). Let us consider the sequence of the best approximations to \( \alpha \), that is the sequence of integers

\[
q_1 < q_2 < ... < q_\nu < q_{\nu+1} < ...
\]

such that

\[
||q_\nu \alpha|| = |q_\nu \alpha - p_\nu| < ||q \alpha||,
\]

for all positive integers \( q < q_\nu \).

By Lagrange’s theorem all the best approximations \( (q_\nu, p_\nu) \) with \( q_\nu > 1 \) are just the convergents

\[
\frac{p_\nu}{q_\nu} = [a_0; a_1, ..., a_\nu]
\]

for the continued fraction expansion (1). For the convergents’ denominators and for the remainders \( \xi_\nu = ||q_\nu \alpha|| \) we have recurrent formulas

\[
q_{\nu+1} = a_{\nu+1} q_\nu + q_{\nu-1}, \quad \xi_{\nu+1} = \xi_{\nu-1} - a_{\nu+1} \xi_\nu.
\]

So by taking integer parts we have

\[
a_{\nu+1} = \left[ \frac{q_{\nu+1}}{q_\nu} \right] = \left[ \frac{\xi_{\nu-1}}{\xi_\nu} \right],
\]

and the following obvious statement is valid.

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Research is supported by the Russian Science Foundation under grant 19-11-00001.
Proposition 1. An irrational number $\alpha$ is badly approximable if and only if
\[ \sup_{\nu \geq 1} \frac{q_{\nu+1}}{q_{\nu}} < \infty \]
and if and only if
\[ \inf_{\nu \geq 1} \frac{\xi_{\nu+1}}{\xi_{\nu}} > 0. \]

In the present paper we deal with a generalization of Proposition 1 to simultaneous Diophantine approximation for several real numbers and to Diophantine approximation for one linear form. In the next section we recall all the necessary definitions and in Section 3 we formulate our main results.

2. Simultaneous approximation to $d$ numbers and linear forms.

We consider a real vector $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{R}^d$ such that $1, \alpha_1, ..., \alpha_d$ are linearly independent over $\mathbb{Z}$. Vector $\alpha$ is called badly approximable if
\[ \inf_{q \in \mathbb{Z}_+} q^{1/d} \max_{1 \leq j \leq d} ||q\alpha_j|| > 0 \] (2)
By the famous Perron-Khintchine’s transference theorem (see Theorem 5B from Chapter IV from [10]) condition (2) is equivalent to
\[ \inf_{m=(m_1, ..., m_d) \in \mathbb{Z}^d \setminus \{0\}} \left( \max_{1 \leq j \leq d} |m_j| \right)^d \max_{1 \leq j \leq d} ||m_1 \alpha_1 + ... + m_d \alpha_d|| > 0. \] (3)
We consider the best approximation vectors for simultaneous approximation $z_\nu = (q_\nu, a_{1,\nu}, ..., a_{d,\nu}), \ \nu = 1, 2, 3, ...,$ (4)
satisfying
\[ q_1 < q_2 < ... < q_\nu < q_{\nu+1} < ... , \]
\[ \xi_\nu = \max_{1 \leq j \leq d} ||q_\nu \alpha_j|| = \max_{1 \leq j \leq d} |q_\nu \alpha_j - a_{j,\nu}| < \max_{1 \leq j \leq d} ||q\alpha_j||, \ \forall q < q_\nu. \]
\[ \xi_1 > \xi_2 > ... > \xi_\nu > \xi_{\nu+1} > ... , \] (5)
as well as the best approximation vectors in the sense of the linear form $m_\nu = (m_{0,\nu}, m_{1,\nu}, ..., m_{d,\nu}), \ \nu = 1, 2, 3, ...$ (6)
Namely, if we define $M_\nu = \max_{1 \leq j \leq d} |m_{j,\nu}|$, we have
\[ M_1 < M_2 < ... < M_\nu < M_{\nu+1} < ... . \] (7)
At the same time for the values of linear form
\[ L_\nu = ||m_{1,\nu} \alpha_1 + ... + m_{d,\nu} \alpha_d|| = |m_{0,\nu} + m_{1,\nu} \alpha_1 + ... + m_{d,\nu} \alpha_d| \]
the inequalities
\[ L_\nu < ||m_1 \alpha_1 + ... + m_d \alpha_d||, \ \forall (m_1, ..., m_d) \in \mathbb{Z}^d \setminus \{0\} \text{ with } \max_{1 \leq j \leq d} |m_j| < M_\nu, \]
and

\[ L_1 > L_2 > \ldots > L_\nu > L_{\nu+1} > \ldots \]

are valid. Basic facts about best approximation vectors can be found for example in [1] and [6]. In particular, from the Minkowski convex body theorem it follows that

\[ \xi_\nu \leq \frac{1}{q_{\nu+1}^{1/d}} \] (8)

and

\[ L_\nu \leq \frac{1}{M_{\nu+1}^d} \] (9)

3. Main results.

Our first result is the following criterium of badly approximability.

**Theorem 1.** Suppose that \( \alpha_1, \ldots, \alpha_d, 1 \) are linearly independent over \( \mathbb{Q} \). Then the following three statements are equivalent:

(i) \( \alpha \) is badly approximable;

(ii) \( \sup_j \frac{q_{j+1}}{q_j} < \infty \);

(iii) \( \inf_j \frac{L_{j+1}}{L_j} > 0 \).

We prove the implication (ii) \( \implies \) (i) in Sections 6, 7. A proof of the implication (iii) \( \implies \) (i) will be given in Section 8. Here we should note that the implications (i) \( \implies \) (ii) and (i) \( \implies \) (iii) are obvious.

Indeed from the definition (2) and inequality (8) we immediately get

\[ \frac{\gamma}{q_{\nu}^{1/d}} \leq \xi_\nu \leq \frac{1}{q_{\nu+1}^{1/d}} \quad \forall \nu \]

for some positive \( \gamma \) and so \( \frac{q_{\nu+1}}{q_\nu} \leq \gamma^{-d} \), that is (ii). Similarly from (3) we get

\[ L_{\nu+1} \geq \frac{\gamma}{M_{\nu+1}^d} \quad \forall \nu \]

with some positive \( \gamma \) and together with (9) this gives

\[ \frac{L_{\nu+1}}{L_\nu} \geq \gamma, \]

and this is (iii).

In fact for badly approximable \( \alpha \) we can say something more, by the same argument.

**Remark 1.** If \( \alpha \in \mathbb{R}^d \) is badly approximable then besides the inequalities (ii) and (iii) the inequalities

\[ \inf_j \frac{\xi_{j+1}}{\xi_j} > 0, \quad \text{and} \quad \sup_j \frac{M_{j+1}}{M_j} < \infty \] (10)

are also valid.

Indeed, we can easily get the first inequality from (10) by combining inequality \( \xi_{\nu+1} > \gamma/q_{\nu+1}^{1/d} \) and (8); the second inequality from (10) can be obtained by combining \( L_\nu > \gamma M_\nu^{-d} \) and (9). However the converse statements are not true. Our second result is given by the following statement. For
the simplicity reason we formulate and prove this result for two-dimensional case only. However the 
construction may be easily generalized to the case of simultaneous approximation to $d$ numbers.

**Theorem 2.** There exists uncountably many $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that 
- $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Z}$;
- $\inf \xi_{\nu+1} \xi_{\nu} > 0$;
- $\alpha$ is not badly approximable.

The idea of the construction from the proof of Theorem 2 is quite simple. It is related to a 
construction from our earlier paper [5]. One should construct a vector $\alpha \in \mathbb{R}^2$ such that the best 
approximation vectors to it for long times lie in two-dimensional subspaces. Moreover, for the 
integer approximations from these two-dimensional subspaces we should ensure some kind of "one-
dimensional badly approximability". However a complete proof for Theorem 2 is rather cumbersome.
We give our proof of Theorem 2 in Sections 9, 10 and 11.

We would like to note that very recently during the refereeing process of this paper an alternative 
construction to prove Theorem 2 by means of Parametric Geometry of Numbers based on a deep 
theorem due to D. Roy [8] was obtained by W.M. Schmidt [12].

In the present paper we would like to announce a theorem dual to Theorem 2 which deals with 
the best approximations in the sense of a linear form. The formulation of this result is below.

**Theorem 3.** There exist uncountably many $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that 
- $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Z}$;
- $\sup M_{\nu+1} M_{\nu} < \infty$;
- $\alpha$ is not badly approximable.

In this paper we do not give a proof of Theorem 3 but just announce it. The proof we have is 
based on the same idea as the proof of Theorem 2 but it is even more technical and cumbersome. 
Moreover, it is related to some general phenomenon, and we suppose to consider it in a separate 
paper which now is in preparation.

### 4. On Diophantine exponents.

For a real $\alpha \in \mathbb{R}^d$ we recall the definitions of the *ordinary* Diophantine exponent $\omega(\alpha)$ and the *uniform* Diophantine exponent $\hat{\omega}(\alpha)$ in the sense of simultaneous Diophantine approximation. The 
ordinary Diophantine exponent $\omega(\alpha)$ is defined as the supremum of those $\gamma \in \mathbb{R}$ for which there 
exists an unbounded sequence of values of $T \in \mathbb{R}_+$ such that the system 

$$
\begin{align*}
&\max_{1 \leq j \leq d} ||q\alpha_j|| \leq T^{-\gamma}, \\
&1 \leq q \leq T
\end{align*}
$$

has an integer solution $q \in \mathbb{Z}$. The uniform Diophantine exponent $\hat{\omega}(\alpha)$ is defined as the supremum 
of those $\gamma \in \mathbb{R}$ for which there exists $T_0$ such that for every $T \geq T_0$ the system \((\mathbf{11})\) has an integer 
solution $q \in \mathbb{Z}$. Equivalently in terms of the best approximation vectors, $\hat{\omega}(\alpha)$ can be defined as the 
supremum of those $\gamma \in \mathbb{R}$ for which the inequality 

$$
\xi_\nu \leq q_{\nu+1}^{-\gamma}
$$

is valid for all $\nu$ large enough.

It is well known that 

$$
\frac{1}{d} \leq \hat{\omega}(\alpha) \leq 1
$$
for every $\alpha \in \mathbb{R}^d \setminus \mathbb{Q}^d$ and obviously

$$\hat{\omega}(\alpha) \leq \omega(\alpha) \leq +\infty.$$  

As it was discovered by V. Jarník [13], the first trivial inequality here can be improved. The optimal lowed bound for $\omega(\alpha)$ in terms of $\hat{\omega}(\alpha)$ was obtained in [3] where the authors solve a problem by W.M. Schmidt and L. Summerer [11]. In the case when the numbers $1, \alpha_1, ..., \alpha_d$ are linearly independent over $\mathbb{Q}$ in the paper [3] the authors establish the inequality

$$\frac{\omega(\alpha)}{\hat{\omega}(\alpha)} \geq G_d(\hat{\omega}(\alpha)), \quad (13)$$

where $G_d(\hat{\omega}(\alpha)) \geq 1$ is the positive root of the equation

$$t^{d-1} = \frac{\hat{\omega}(\alpha)}{1 - \hat{\omega}(\alpha)}(1 + t + ... + t^{d-2}). \quad (14)$$

The main argument of the proof from [3] is that there exist infinitely many $\nu$ with

$$q_{\nu+1} \geq q^{G_d(\hat{\omega}(\alpha))}_\nu. \quad (15)$$

Here we should note that the wonderful paper [7] deals with a simple and elegant proof of this result as well as with some other related problems.

If $\alpha \in \mathbb{R}^d$ is a badly approximable vector we have $\omega(\alpha) = \hat{\omega}(\alpha) = \frac{1}{d}$.

However, Theorem 2 shows that for $d \geq 2$ the condition

$$\inf_\nu \frac{\xi_{\nu+1}}{\xi_\nu} > 0 \quad (16)$$

may be satisfied for $\alpha$ which is not badly approximable. Moreover the construction from the proof of Theorem 2 gives $\alpha$ with $\hat{\omega}(\alpha) = \frac{1}{2}$ and $\omega(\alpha) = 1$. We would like to give a comment on this, and formulate the following statement.

**Proposition 2.** Suppose that among the numbers $\alpha_1, ..., \alpha_d$ there exist at least two numbers linearly independent together with 1 over $\mathbb{Q}$, and suppose that $\alpha$ satisfies condition (16). Then

$$\hat{\omega}(\alpha) \leq \frac{1}{2}. \quad (17)$$

**Proof.** Jarník [13] proved that under the conditions of Proposition 1 there exist infinitely many linearly independent triples $z_{\nu-1}, z_\nu, z_{\nu+1}$ of consecutive best approximation vectors. Moreover for such a triple there exist indices $j_1, j_2$ such that

$$D = \begin{vmatrix} q_{\nu-1} & a_{j_1, \nu-1} & a_{j_2, \nu-1} \\ q_\nu & a_{j_1, \nu} & a_{j_2, \nu} \\ q_{\nu+1} & a_{j_1, \nu+1} & a_{j_2, \nu+1} \end{vmatrix} = \begin{vmatrix} q_{\nu-1} & a_{j_1, \nu-1} - q_{\nu-1} \alpha_{j_1} & a_{j_2, \nu-1} - q_{\nu-1} \alpha_{j_2} \\ q_\nu & a_{j_1, \nu} - q_\nu \alpha_{j_1} & a_{j_2, \nu} - q_\nu \alpha_{j_2} \\ q_{\nu+1} & a_{j_1, \nu+1} - q_{\nu+1} \alpha_{j_1} & a_{j_2, \nu+1} - q_{\nu+1} \alpha_{j_2} \end{vmatrix} \neq 0.$$  

But from the definition of values $\xi_\nu$ and (16) we see that

$$1 \leq |D| \leq 6\xi_\nu q_{\nu+1} \leq \xi^2_\nu q_{\nu+1}.$$  

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(of course here the constant in the sign \(\ll\) may depend on \(\alpha\)). The last inequality together with the definition of \(\hat{\omega}(\alpha)\) in terms of the inequality (12) gives (17). □

It is clear that the bound (17) is optimal for \(d = 2\). However what are admissible values of \(\hat{\omega}(\alpha)\) and \(\omega(\alpha)\) for general \(d\) under the condition (16) for the numbers \(1, \alpha_1, ..., \alpha_d\) which are linearly independent over \(\mathbb{Q}\) seems to be an open question.

In addition, here we would like to give the following remark. We should note that if

\[ bq_\nu^{-\beta} < \xi_\nu < aq_\nu^{-\alpha}, \]

with some positive \(a, b\) and \(\alpha, \beta\) satisfying \(\beta \geq \alpha \geq 1/d\), then

\[ q_{\nu+1} < C q_\nu^{\beta/\alpha} \quad \text{with} \quad C = \left(\frac{a}{b}\right)^{1/\alpha}, \]

in particular

\[ q_{\nu+1} < C' q_\nu^{d\beta} \quad \text{with} \quad C' = \frac{1}{b^d}. \]

Consider the exponent

\[ \tau(\alpha) = \limsup_{\nu \to \infty} \frac{\log q_{\nu+1}}{\log q_\nu} \]

which contain information about the growth of the best approximation vectors to \(\alpha\). Then the observation mentioned above can be summarized as

**Proposition 3.** Suppose that the numbers \(1, \alpha_1, ..., \alpha_d\) are linearly independent over \(\mathbb{Q}\). Then

\[ G_d(\hat{\omega}(\alpha)) \leq \tau(\alpha) \leq \frac{\omega(\alpha)}{\hat{\omega}(\alpha)} \leq d\omega(\alpha). \]

Moreover

\[ \hat{\omega}(\alpha) \leq \frac{1}{\sum_{j=0}^{d-1} \tau(\alpha)^{-j}}. \] (21)

Proof. Lower bound for \(\tau(\alpha)\) in (20) immediately follows from (15). Upper bound comes from (19) under the condition (18). Inequality (21) follows from (15) and (14). □

5. Some notation.

We use the following notation. Together with the best approximation vectors (4) which we have denoted by \(z_\nu\) we consider the points

\[ Z_\nu = (q_\nu, q_\nu \alpha_1, ..., q_\nu \alpha_d). \] (22)

By \(|\xi|\) we denote the Euclidean norm of the vector \(\xi \in \mathbb{R}^k\) in any dimension \(k\). By

\[ |\eta|_\infty = \max_{1 \leq j \leq d} |\eta_j| \]

we denote the sup-norm of the vector \(\eta \in \mathbb{R}^d\). In the case \(x = (x_0, x_1, ..., x_d) \in \mathbb{R}^{d+1}\) we will use the notation

\[ |x|_\infty = \max_{1 \leq j \leq d} |x_j| \]
to deal with the sup-norm of the shortened vector \( \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \). So for \( \xi_\nu = \mathbf{Z}_\nu - \mathbf{z}_\nu \) we have \( \xi_\nu = |\xi_\nu|_\infty \).

It is clear that

\[
\xi_\nu = |\mathbf{Z}_\nu - \mathbf{z}_\nu| \leq \sqrt{d} \xi_\nu
\]  

(23)

Let

\[
\rho(\mathcal{A}, \mathcal{B}) = \inf_{a \in \mathcal{A}, b \in \mathcal{B}} |a - b|
\]

be the Euclidean distance between sets \( \mathcal{A} \) and \( \mathcal{B} \).

6. Main geometric lemma.

We define inductively a special collection of \( d + 1 \) linearly independent best approximation vectors. Let \( \nu_1 = \nu, \nu_2 = \nu + 1 \). Then, if \( \mathbf{z}_{\nu_1}, \mathbf{z}_{\nu_2}, \ldots, \mathbf{z}_{\nu_j+1} \) are defined we find the smallest \( \mu \geq \nu_j+1 \) such that the vectors \( \mathbf{z}_{\nu_1}, \mathbf{z}_{\nu_2}, \ldots, \mathbf{z}_{\nu_j+1}, \mathbf{z}_\mu \) are independent and put \( \mathbf{z}_{\nu_j} = \mathbf{z}_\mu \). At the end of the procedure we have \( d + 1 \) independent vectors \( \mathbf{z}_{\nu_1}, \mathbf{z}_{\nu_2}, \ldots, \mathbf{z}_{\nu_{d+1}} \).

(24)

We define linear subspaces

\[
\pi_j = \langle \mathbf{z}_{\nu_1}, \mathbf{z}_{\nu_2}, \ldots, \mathbf{z}_{\nu_j} \rangle_{\mathbb{R}}, \quad j = 1, \ldots, d + 1
\]

(25)

and lattices

\[
\Gamma_j = \pi_j \cap \mathbb{Z}^{d+1}.
\]

(26)

In particular \( \Gamma_1 = \langle \mathbf{z}_{\nu} \rangle_{\mathbb{Z}} \) and \( \Gamma_{d+1} = \mathbb{Z}^{d+1} \). By \( \Delta_j \) we denote the \( j \)-dimensional fundamental volume of lattice \( \Gamma_j \). In particular \( \Delta_1 = |\mathbf{z}_{\nu}| \) and \( \Delta_{d+1} = 1 \).

Here we should note that by Minkowski Convex Body Theorem applied for the two-dimensional lattice \( \Gamma_2 \) we have

\[
\xi_\nu \nu_{\nu+1} \leq \Delta_2,
\]

(27)

and also

\[
\xi_\nu \nu_{\nu+1} \geq K \Delta_2, \quad \text{where} \quad K = \frac{1}{2\sqrt{d(1 + \alpha_1^2 + \ldots + \alpha_d^2)}}
\]

(28)

(for the details see for example [6] or Theorem 1.5 from [2]). Moreover, (28) together with (8) for every best approximation \( \nu \geq 1 \) gives

\[
K \Delta_2 \leq \nu_{\nu+1}^{d-1}
\]

or

\[
(K \Delta_2)^{\frac{d-1}{d}} \leq \nu_{\nu+1}.
\]

(29)

Lemma 1. For every \( j \) one has

\[
\frac{\Delta_{j+1}}{\Delta_j} \leq 2\sqrt{d} \frac{\nu_{\nu+1}}{\nu_{\nu+1}^{d-1}} \xi_{\nu+1}.
\]

(23)

Proof. Let \( \mathbf{w} \in \Gamma_{j+1} \setminus \Gamma_j \) be a primitive vector such that

\[
\Gamma_{j+1} = \langle \Gamma_j, \mathbf{w} \rangle_{\mathbb{Z}}
\]
It is clear that the lattice \( \Gamma_{j+1} \) splits into a union of affine sublattices with respect to \( \Gamma_j \):

\[
\Gamma_{j+1} = \bigcup_{k \in \mathbb{Z}} (\Gamma_j + k\mathbf{w}).
\]

We consider affine \( j \)-dimensional subspaces

\[
\pi_{j,k} = \pi_j + k\mathbf{w} \supset \Gamma_j + k\mathbf{w}.
\]

It is clear that the Euclidean distance between each two neighboring subspaces \( \pi_{j,k} \) and \( \pi_{j,k+1} \) is equal to \( \frac{\Delta_{j+1}}{\Delta_j} \). So in the case \( k \neq 0 \) we have

\[
\rho(\pi_{j,k}, \pi_{j,k+1}) = |k| \cdot \frac{\Delta_{j+1}}{\Delta_j} \geq \frac{\Delta_{j+1}}{\Delta_j}. \quad (30)
\]

Define \( k_* \) from the condition

\[
z_{\nu_{j+1}} \in \pi_{j,k_*}.
\]

As \( z_{\nu_{j+1}} \notin \pi_j \) we have \( k_* \neq 0 \). As \( z_{\nu_{j+1}} \in \pi_j \) from (28) we get

\[
\rho(z_{\nu_{j+1}}-1, \pi_{j}) \leq \sqrt{d} \xi_{\nu_j-1}.
\]

As

\[
|z_{\nu_{j+1}}| = \frac{q_{\nu_{j+1}}}{q_{\nu_{j+1}-1}}
\]

we deduce

\[
\rho(z_{\nu_{j+1}}-1, \pi_{j}) = \frac{q_{\nu_{j+1}}}{q_{\nu_{j+1}-1}} \cdot \rho(z_{\nu_{j+1}}, \pi_{j}) \leq \frac{q_{\nu_{j+1}}}{q_{\nu_{j+1}-1}} \cdot \sqrt{d} \xi_{\nu_{j+1}}. \quad (31)
\]

As \( z_{\nu_{j+1}} \in \pi_{j,k_*} \) we see that

\[
\rho(z_{\nu_{j+1}}, \pi_{j,k_*}) \leq \sqrt{d} \xi_{\nu_{j+1}}. \quad (32)
\]

From (30), triangle inequality, formulas (31,32) and the inequalities \( \xi_{\nu_{j+1}} < \xi_{\nu_{j+1}-1} \) and \( q_{\nu_{j+1}} > q_{\nu_{j+1}-1} \) we get

\[
\frac{\Delta_{j+1}}{\Delta_j} \leq \rho(z_{\nu_{j+1}}, \pi_{j,k_*}) \leq \rho(z_{\nu_{j+1}}, \pi_{j}) + \rho(z_{\nu_{j+1}}, \pi_{j,k_*}) \leq \sqrt{d} \frac{q_{\nu_{j+1}}}{q_{\nu_{j+1}-1}} \xi_{\nu_{j+1}-1} + \sqrt{d} \xi_{\nu_{j+1}} \leq 2 \sqrt{d} \frac{q_{\nu_{j+1}}}{q_{\nu_{j+1}-1}} \xi_{\nu_{j+1}-1}.
\]

Everything is proved.\( \square \)

7. Proof of Theorem 1: simultaneous approximation.

Let \( \alpha_1, \ldots, \alpha_d \) be given. We suppose that (ii) is valid and deduce (i). For a given \( \nu \) from (28) and \( \Delta_{d+1} = 1 \) we get the inequality

\[
\xi_{\nu} q_{\nu+1} \geq K \Delta_2 = K \cdot \frac{\Delta_2}{\Delta_3} \cdot \frac{\Delta_3}{\Delta_4} \cdots \frac{\Delta_d}{\Delta_{d+1}}. \quad (33)
\]

Now we deduce from (ii) the condition (i). Lemma 1 gives

\[
\xi_{\nu} q_{\nu+1} \geq \frac{K}{(2\sqrt{d})^{d-1}} \prod_{j=3}^{d+1} \frac{q_{\nu_j-1}}{q_{\nu_j}} \cdot \frac{1}{\prod_{j=3}^{d+1} \xi_{\nu_j-1}}. \quad (34)
\]

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As we supposed that (ii) is valid, there exists $M$ such that

$$\frac{q_{\nu+1}}{q_{\nu}} \leq M \quad \forall \nu.$$  

Moreover from (8) we have $\xi_{\nu j - 1} \leq \xi_{\nu} \forall j = 3, ..., d + 1$. Now we continue with (34) and get

$$\xi_{\nu} q_{\nu+1} \geq \frac{K}{(2\sqrt{dM\xi_{\nu}})^{d-1}}.$$  

As $q_{\nu+1} \leq M q_{\nu}$ we get

$$q_{\nu}^{1/d} \xi_{\nu} \geq \frac{K^{1/d}}{(2\sqrt{d})^{(d-1)/d} M} \quad \forall \nu$$

and (i) is proved.

8. Proof of Theorem 1: linear form.

We suppose that (iii) is valid and deduce (i). We follow the same argument as in Sections 5, 6, but we need to make some changes. We use a standard trick which reduces the problem for linear forms to the problem for simultaneous approximation. This trick was used in [3], Section 5.2.

The proof is quite similar so we will give just a sketch of a proof. First of all we need a generalization of Lemma 1. Suppose that $\Lambda$ be a full-dimensional lattice in $\mathbb{R}^{d+1}$ with coordinates $(x_0, x_1, ..., x_d)$.

We consider the best simultaneous approximations of the line

$$\ell = \{x \in \mathbb{R}^{d+1} : x_1 = x_2 = ... = x_d = 0\}$$

by the points of the lattice $\Lambda$. Here by the best approximation point we mean a point $z = (z_0, z_1, ..., z_d) \in \Lambda$ such that in the parallelepiped

$$\Pi_z = \{z' = (z_0', z_1', ..., z_d') \in \mathbb{R}^{d+1} : |z_0'| \leq |z_0|, |z'|_{\infty} \leq |z|_{\infty}\}$$

there is no lattice points different from the points $0, \pm z$, that is

$$\Pi_z \cap \Lambda = \{0, z, -z\}.$$
besides the points 0, ±z_ν, ±z_{ν+1}:

$$\Pi_ν \cap \Lambda = \{0, z_ν, -z_ν, z_{ν+1}, -z_{ν+1}\}.$$ 

The sequence of the best approximation vectors z_ν is infinite if there is no non-zero lattice points on the axis ℓ. If there is a non-zero point z ∈ Λ ∩ ℓ then the sequence of the best approximation vectors is finite. In our proof we need to consider the case when this sequence is finite. We suppose that our lattice Λ and the best approximation vector z_1 satisfy one more condition (b) the sequence of the best approximation vectors z_ν, ν ≥ 1 does not lie in a proper linear subspace of \(\mathbb{R}^{d+1}\).

Now for the lattice Λ satisfying conditions (a) and (b) we are able to define points z_ν_j, ν_1 = 1 < ν_2 < ... < ν_{d+1} from (24) and subspaces π_j from (25). In the definition of lattice Γ_j there will be a slight difference. Instead of (26) we put

$$Γ_j = π_j \cap Λ.$$ 

Again by Δ_j we define the fundamental volumes of j-dimensional lattices Γ_j. In particular

$$Δ_{d+1} = \det Λ.$$ 

(35)

The inequality (28) transforms now into the following statement.

**Lemma 2.** Suppose that for a certain ν we have

$$|z_ν|_∞ \cdot z_{0,ν+1} \geq 1$$

(36)

and

$$|z_ν|_∞ \leq 1.$$ 

(37)

Then

$$|z_ν|_∞ \cdot z_{0,ν+1} \geq \frac{Δ_2}{2√2d}.$$ 

(38)

Proof. In fact, this lemma follows from inequality (59) of Lemma 10 from [3]. For the sake of completeness we give here a proof. Consider the 2 × (d + 1) matrix

$$M = \begin{pmatrix}
  z_{0,ν} & z_{1,ν} & z_{2,ν} & \ldots & z_{d,ν} \\
  z_{0,ν+1} & z_{1,ν+1} & z_{2,ν+1} & \ldots & z_{d,ν+1}
\end{pmatrix}.$$ 

Then Δ_2^2 is just the sum of squares of all 2 × 2 minors

$$M_{i,j} = \left| \begin{array}{cc}
  z_{i,ν} & z_{j,ν} \\
  z_{i,ν+1} & z_{j,ν+1}
\end{array} \right|$$

of matrix M, that is

$$Δ_2^2 = \sum_{0 \leq i < j \leq d} M_{i,j}^2.$$ 

As z_{0,ν} < z_{0,ν+1} and |z_ν|_∞ > |z_{ν+1}|_∞ we have

$$|M_{0,j}| \leq 2|z_ν|_∞ \cdot z_{0,ν+1}, \quad ∀ j = 1, 2, ..., d.$$
From (37) we see that 
\[ |M_{i,j}| \leq 2, \quad \forall i, j = 1, 2, ..., d. \]
So by (36) we get 
\[ \Delta_2^2 \leq 4d(z_{0,\nu+1})^2 + 4d^2(z_{0,\nu+1})^2, \]
and Lemma 2 follows. □

Instead of Lemma 1 now we have the following statement.

**Lemma 1’**. Suppose that the lattice \( \Lambda \) and the best approximation vector \( \mathbf{z}_1 \) satisfy properties (a) and (b) and consider the best approximation vectors (24). Then for every \( j \) one has
\[
\frac{\Delta_{j+1}}{\Delta_j} \leq 2\sqrt{d} \frac{z_{0,\nu+1}}{z_{0,\nu+1}} |\mathbf{z}_{\nu+1} - 1|_{\infty}.
\]

The proof of Lemma 1’ just follows the steps of the proof of Lemma 1. The only difference is that instead of the points \( \mathbf{Z}_\nu \) defined in (22) which lie on the line \( \langle (1, \alpha_1, ..., \alpha_d) \rangle_\mathbb{R} \) one should consider the points
\[ \mathbf{Z}_\nu = (z_{0,\nu}, 0, ..., 0) \in \ell. \]
We left the proof to the reader. □

Now we are ready to deduce badly approximability of \( \mathbf{a} \) from the condition (iii). Let us consider best approximation vectors (6). It may happen that there exists \( \nu_0 \) and a proper linear subspace \( \mathcal{L} \subset \mathbb{R}^{d+1} \) of dimension \( 3 \leq l = \dim \mathcal{L} < d + 1 \) such that \( \mathbf{m}_\nu \in \mathcal{L} \) for all \( \nu \geq \nu_0 \) (see [4] for the first result in this direction and [6] and the literature therein for a survey and related results). But we will show later that under condition (iii) this is not possible.

So first of all we consider the case when for any \( \nu_0 \) the best approximation vectors \( \mathbf{m}_\nu, \nu \geq \nu_0 \) do not lay in a proper linear subspace of \( \mathbb{R}^{d+1} \). Suppose that vectors
\[ \mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_\mu \]
do not lay in a proper linear subspace of \( \mathbb{R}^{d+1} \).

We consider the lattice
\[
\Lambda_\mathbf{a} = \begin{pmatrix}
1 & \alpha_1 & \alpha_2 & ... & \alpha_d \\
0 & 1 & ... & 0 & 0 \\
0 & 0 & ... & 0 & 0 \\
0 & 0 & ... & 1 & 0 \\
0 & 0 & ... & 0 & 1
\end{pmatrix} \mathbb{Z}^{d+1},
\]
a parameter \( T > 0 \) and the lattice
\[
\Lambda_\mathbf{a}^{[\mu]} = G \Lambda_\mathbf{a}, \quad G = \begin{pmatrix}
T^{-d} & 0 & ... & 0 & 0 \\
0 & T & ... & 0 & 0 \\
0 & 0 & ... & T & 0 \\
0 & 0 & ... & 0 & T
\end{pmatrix}, \quad \det \Lambda_\mathbf{a}^{[\mu]} = 1.
\]

As \( \alpha_1, ..., \alpha_d \) are linearly independent over \( \mathbb{Z} \), the lattice \( \Lambda_\mathbf{a}^{[\mu]} \) satisfies condition (a). For the lattice \( \Lambda_\mathbf{a}^{[\mu]} \) the points
\[ \mathbf{z}_\nu = \pm G \mathbf{m}_{\mu-\nu+1}, \quad \nu = 1, ..., \mu \quad (39) \]
are the best approximation points in the sense of this section, and the condition (b) is satisfied. We choose the signs \( \pm \) in (39) to have \( z_{0, \nu} = +1 \), and \( 0 < z_{0, 1} < z_{0, 2} < \ldots < z_{0, \mu} \). We see that
\[
  z_{0, \nu} = T^{-\nu} L_{\nu-\nu+1}, \quad |z_{\nu}|_\infty = M_{\nu-\nu+1} T.
\] (40)

If we take \( T \leq M_\nu^{-1} \) we see that (37) is satisfied for all \( \nu = 1, \ldots, \mu \). We can take \( T \) small enough to get
\[
  |z_{\nu}|_\infty z_{0, \nu+1} > |z_{\nu}|_\infty z_{0, \nu} = T^{1-\nu} L_{\nu-\nu+1} M_{\nu-\nu+1} > 1 \quad \forall \nu = 1, \ldots, \mu.
\]

So the conditions of Lemma 2 are satisfied. Now we apply Lemma 2 and Lemma 1’ to show that
\[
  |z_1|_\infty z_{0, 2} > \frac{\Delta_2}{2\sqrt{2}d} = \frac{1}{2\sqrt{2}d} \prod_{j=2}^{d} \frac{\Delta_j}{\Delta_{j+1}} \det \Lambda_{\nu}\| = \frac{1}{2\sqrt{2}d} \prod_{j=2}^{d} \frac{\Delta_j}{\Delta_{j+1}} \prod_{j=2}^{d} \frac{1}{\prod_{j=2}^{d} |z_{\nu, j+1-1}|_\infty}.
\]

We have assumed (iii), so
\[
  \frac{L_{j+1}}{L_j} > \gamma > 0, \quad \forall j
\]
and by the first formula from (40) we get
\[
  \frac{z_{0, \nu+1}}{z_{0, \nu}} = \frac{L_{\nu-\nu+1+2}}{L_{\nu-\nu+1}} > \gamma.
\]

From the other hand, for \( j \geq 2 \) we have \( |z_{\nu, j+1-1}|_\infty < |z_1|_\infty \), because of \( \nu_j+1-1 \geq \nu_{j-1} \geq \nu_2 > \nu_1 = 1 \) and
\[
  |z_{\nu, j+1-1}|_\infty = M_{\nu-\nu_j+1+2} T, \quad |z_1|_\infty = M_\mu T
\]
(see (40)) and \( M_{\mu+1} < M_{\mu+\nu_j+1} \) (see (37)). We conclude with
\[
  |z_1|_\infty z_{0, 2} \gg \frac{1}{|z_1|_\infty},
\]
or
\[
  |z_1|_\infty z_{0, 2} > \gamma |z_1|_\infty z_{0, 2} \gg \frac{1}{|z_1|_\infty},
\]
as \( z_{0, 2} > \gamma \). We apply (40) again to see that \( L_\mu M_\mu \gg \frac{1}{\gamma} \). The last inequality holds for all \( \mu \) large enough and this means that \( \alpha \) is badly approximable.

Now we suppose that there exists \( \nu_0 \) and a proper linear subspace \( \mathcal{L} \subset \mathbb{R}^{d+1} \) of dimension \( 3 \leq l = \dim \mathcal{L} < d + 1 \) such that \( m_\nu \in \mathcal{L} \) for all \( \nu \geq \nu_0 \). We may suppose that \( \mathcal{L} \) has the minimal dimension among all such subspaces. Then \( \mathcal{L} \) is a rational subspace and inside \( \mathcal{L} \) we have an irrational subspace
\[
  \mathcal{L}_1 = \{ x = (x_0, x_1, \ldots, x_d) \in \mathcal{L} : x_0 + x_1 \alpha_1 + \ldots + x_d \alpha_d = 0 \} \subset \mathcal{L}.
\]
But then all the best approximations vectors \( m_\nu \) will be all the best approximation vectors of the lattice \( \mathcal{L} \cap \mathbb{Z}^{d+1} \) to \( \mathcal{L}_1 \) in the induced norm, and this means that the values \( L_\nu \) are proportional to the values \( \rho(m_\nu, \mathcal{L}_1) \). From the other hand the argument behind shows that the \((l-1)\)-dimensional subspace \( \mathcal{L}_1 \) is badly approximable in \( \mathcal{L} \), that is
\[
  \inf_{m \in \mathcal{L} \setminus \mathbb{Z}^{d+1}} \rho(m, \mathcal{L}_1) |m|^{l-1} > 0.
\]

But then all the best approximations vectors \( m_\nu \) will be all the best approximation vectors of the lattice \( \mathcal{L} \cap \mathbb{Z}^{d+1} \) to \( \mathcal{L}_1 \) in the induced norm, and this means that the values \( L_\nu \) are proportional to the values \( \rho(m_\nu, \mathcal{L}_1) \) and hence
\[
  \inf_{\nu} L_\nu |m_\nu|^{l-1} > 0.
\]
This is not possible, because for $l \leq d$ this contradicts (9).

So the proof is completed. □

**Remark 3.** In the last part of the proof we deal with the situation when the subspace of best approximations for a linear form has dimension smaller than $d + 1$. In particular we proved that this is not possible for badly approximable $\alpha$. Such type of problems were discussed in a recent paper [9].

9. Construction of approximations in two-dimensional subspace.

The following obvious lemma will be very useful.

**Lemma 3.** Let $v = (p, b_1, b_2) \in \mathbb{Z}^3$, $p \geq 1$ be a primitive integer vector and $V = \left(\frac{b_1}{p}, \frac{b_2}{p}\right)$ be the corresponding rational vector. Suppose that $\delta = \delta(v) = \frac{1}{2p^2}$. Then for all $x$ under the condition

$$|x - V|_\infty < \delta$$

the vector $v$ is a best approximation vector for $x$.

**Proof.** Let us assume for two independent vectors $v = (p, b_1, b_2)$ and $v' = (p', b_1', b_2') \in \mathbb{Z}^3$ with $0 < p' \leq p$ the induced vectors $V = \left(\frac{b_1}{p}, \frac{b_2}{p}\right)$ and $V' = \left(\frac{b_1'}{p'}, \frac{b_2'}{p'}\right)$ both have distance smaller that $\delta$ from $x$. Then

$$|V - V'|_\infty \leq |V - x|_\infty + |V' - x|_\infty < \frac{1}{p^2}.$$  

On the other hand, since by linear independence $V \neq V'$ and both coordinates in the difference $V - V'$ have common denominator $pp' \leq p^2$, we have the reverse bound $|V - V'|_\infty \geq p^{-2}$, and this is a contradiction. Hence $v'$ is linearly dependent to $v$. Finally since $v$ is primitive, there is no such integer vector $v' \neq v$ with $p' \leq p$. □

**Lemma 4.** Suppose that two independent integer points

$$v_0 = (p_0, b_{1,0}, b_{2,0}), \quad v_1 = (p_1, b_{1,1}, b_{2,1}) \in \mathbb{Z}^3$$

with

$$p_1 > p_0 \geq 1 \quad (41)$$

and the corresponding rational points

$$V_0 = \left(\frac{b_{1,0}}{p_0}, \frac{b_{2,0}}{p_0}\right), \quad V_1 = \left(\frac{b_{1,1}}{p_1}, \frac{b_{2,1}}{p_1}\right) \in \mathbb{Q}^2 \cap [0, 1]^2$$

satisfy the following conditions.

(i) the lattice $\Lambda = \langle v_0, v_1 \rangle_\mathbb{Z}$ is complete, that is

$$\langle v_0, v_1 \rangle_\mathbb{Z} = \pi \cap \mathbb{Z}^3$$

where

$$\pi = \langle v_0, v_1 \rangle_\mathbb{R}$$

is a two-dimensional plane spanned by $v_0$ and $v_1$; by $\Delta$ we denote the fundamental volume of two-dimensional lattice $\Lambda = \langle v_0, v_1 \rangle_\mathbb{Z}$.
(ii) points \( \mathbf{V}_0 \) and \( \mathbf{V}_1 \) satisfy
\[
|\mathbf{V}_0 - \mathbf{V}_1|_\infty \leq \frac{1}{2} \min \left( \frac{1}{p_0 \Delta}, \delta(\mathbf{v}_0) \right),
\] (42)
where \( \delta(\mathbf{v}_0) \) is defined in Lemma 3.

Consider the vectors \( \mathbf{v}_i = (p_i, b_{1,i}, b_{2,i}), 2 \leq i \leq k \) defined recursively by
\[
\mathbf{v}_i = \mathbf{v}_{i-1} + \mathbf{v}_{i-2},
\] (43)
and the corresponding rational points
\[
\mathbf{V}_i = \left( \frac{b_{1,i}}{p_i}, \frac{b_{2,i}}{p_i} \right) \in \mathbb{Q}^2
\]
such that
\[
p_k \geq \kappa = \kappa(\mathbf{v}_0, \mathbf{v}_1) = \max \left( \Delta^2, \sqrt{\frac{\Delta}{\delta(\mathbf{v}_0)}}, \sqrt{\frac{p_1 \Delta}{p_0 |\mathbf{V}_2 - \mathbf{V}_1|_\infty}}, \left(1 + \frac{p_1}{p_0} \right) \right).
\] (44)
Then for any \( \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \) satisfying
\[
|x - \mathbf{V}_k|_\infty \leq \frac{\Delta}{100p_k^2}
\] (45)
either
\[
\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{k-2}, \mathbf{v}_{k-1}, \mathbf{v}_k,
\] (46)
or
\[
\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{k-2}, \mathbf{v}_k
\] (47)
is the sequence of all consecutive best approximation vectors from \( \mathbf{v}_0 \) to \( \mathbf{v}_k \), that is all the best approximation vectors \( \mathbf{z} = (q, a_1, a_2) \) to \( \mathbf{x} \) with \( p_0 \leq q \leq p_k \).

Moreover for every \( \mathbf{x} \) under the consideration we have
\[
\frac{|p_i \mathbf{x} - \mathbf{y}_i|_\infty}{|p_{i-1} \mathbf{x} - \mathbf{y}_{i-1}|_\infty} \geq \frac{1}{4}, \quad i = 1, 2, \ldots, k - 1.
\] (48)

Proof. Let us start with any \( \mathbf{x} \) satisfying (45). For \( 0 \leq i \leq k \) consider points
\[
\mathbf{Z}_i = (p_i, p_i x_1, p_i x_2) \quad \text{and} \quad \mathbf{z}_i = \left( \frac{b_{1,k}}{p_k}, \frac{b_{2,k}}{p_k} \right)
\]
and the remainder vectors
\[
\mathbf{\eta}_i = \mathbf{Z}_i - \mathbf{v}_i \quad \text{and} \quad \mathbf{\eta}_i = \mathbf{z}_i - \mathbf{v}_i.
\]
More generally, for a vector \( \mathbf{v} = (p, b_1, b_2) \in \pi \) we write
\[
\mathbf{\eta}(\mathbf{v}) = \left( 0, p \frac{b_{1,k}}{p_k} - b_1, p \frac{b_{2,k}}{p_k} - b_2 \right).
\]
We should note here that as all the vectors \( \mathbf{\eta}(\mathbf{v}) \) are parallel, their sup-norms \( |\mathbf{\eta}(\mathbf{v})|_\infty \) are proportional to Euclidean norms \( |\mathbf{\eta}(\mathbf{v})| \), that is for vectors \( \mathbf{v} = (p, b_1, b_2), \mathbf{v}' = (p', b'_1, b'_2) \in \pi \) we have
\[
\frac{|\mathbf{\eta}(\mathbf{v})|_\infty}{|\mathbf{\eta}(\mathbf{v}')|_\infty} = \frac{|\mathbf{\eta}(\mathbf{v})|}{|\mathbf{\eta}(\mathbf{v}')|}.
\] (49)
It is clear that
\[ |\eta(v)|_\infty \geq |\eta(v)|/\sqrt{2}. \]

From (43) it follows that
\[ \eta_{i+1} = \eta_i + \eta_{i-1} \quad \text{and} \quad \eta_{i+1} = \eta_i + \eta_{i-1}. \]

In addition we may note that vectors \( \eta_i \) are parallel and
\[ \eta_i = -\frac{|\eta_i|}{|\eta_i - 1|} \cdot \eta_{i-1}. \]

So
\[ |\eta_k| = 0, \quad |\eta_{k-1}| = |\eta_{k-2}|, \quad |\eta_{i-1}| = |\eta_i| + |\eta_{i+1}|, \]

and we can write the ratio \( \frac{|\eta_{i-1}|}{|\eta_i|} \) as the continued fraction and get the estimates
\[ \frac{|\eta_{i-1}|}{|\eta_i|} = \left[ 1; 1, \ldots, 1 \right]_{k-i} \leq 2, \quad 1 \leq i \leq k-1 \]
and
\[ \frac{|\eta_{i-1}|}{|\eta_i|} \geq \frac{3}{2}, \quad 1 \leq i \leq k-2 \]

So by (49),
\[ \frac{|\eta_i|_\infty}{|\eta_{i-1}|_\infty} = \frac{|\eta_i|}{|\eta_{i-1}|} \geq \frac{1}{2}, \quad 1 \leq i \leq k-1, \]
and
\[ \frac{|\eta_i|_\infty}{|\eta_{i-1}|_\infty} \leq \frac{2}{3}, \quad 1 \leq i \leq k-2. \]

We should note that the point \( V_k \) belongs to the segment with endpoints \( V_0, V_1 \) which belong to the plane \( \pi \). From (12) we see that \( |V_k - V_0| < \frac{\delta(v_0)}{2} \). So by Lemma 3, \( v_0 \) is a best approximation vector to \( V_k \). Moreover, the integer lattice \( \mathbb{Z}^3 \) splits into two-dimensional sublattices parallel to \( \pi \). The Euclidean distances between the corresponding neighboring two-dimensional planes is equal to \( \Delta^{-1} \). So from (12) we see that for any integer point \( (p'', b''_1, b''_2) \in \mathbb{Z}^3 \setminus \pi \) one has \( |p''V_k - b''|_\infty > |p_0V_k - b|_\infty \). So we deduce that all the best approximations to \( V_k \) with denominator greater than \( p_0 \) lie in the plane \( \pi \).

Now we consider an approximation to \( V_k \) from subspace \( \pi \). For any \( i = 1, \ldots, k \) the points \( v_{i-1}, v_i \in \pi \) form a basis of \( \Lambda \). Moreover the points \( v_{i-1}, v_i \) lie on the opposite sides from the line \( \langle v_k \rangle_\mathbb{R} \). (Here we should note that for the case \( i = k \) the point \( v_i \) lies just on the line \( \langle v_k \rangle_\mathbb{R} \), however our argument remains valid.) So there is no vectors \( v = (p, b_1, b_2) \in \pi \) satisfying
\[ p_{i-1} < p < p_i, \quad \text{and} \quad v = \lambda v_{i-1} + \mu v_i, \quad \lambda \in \{0, 1\}, \mu \in \mathbb{Z}. \]

We see that for any vector \( v = (p, b_1, b_2) \in \pi \) with \( p_{i-1} < p < p_i \) we have
\[ v = \lambda v_{i-1} + \mu v_i, \quad \lambda \neq 0, 1, \mu \in \mathbb{Z}. \]

Consider the lines
\[ \ell = \ell(\lambda) = \{ x = \lambda v_{i-1} + \mu v_i, \mu \in \mathbb{R} \} \quad \lambda \in \mathbb{Z}. \]
We should note that if points $\mathbf{v} = (p, b_1, b_2) \in \ell(\lambda)$ and $\mathbf{v}' = (p, b'_1, b'_2) \in \ell(\lambda')$ with the same first coordinate $p \in (p_{i-1}, p_i)$ belong to two parallel lines $\ell(\lambda)$ and $\ell(\lambda')$ with integers $\lambda \neq \lambda'$ then

$$|\eta(\mathbf{v} - \mathbf{v}')| = |\mathbf{v} - \mathbf{v}'| \geq \min_{\mathbf{v} \in \ell(0), \mathbf{v}' \in \ell(1)} |\mathbf{v} - \mathbf{v}'| = \sigma_i |\eta_{i-1}|,$$

where

$$\sigma_i = \left(1 + \frac{|\eta_i|}{|\eta_{i-1}|}, \frac{p_{i-1}}{p_i}\right).$$

(52)

We would like to give a comment on the last equality in (52). To obtain this inequality one should note that

$$\min_{\mathbf{v} \in \ell(0), \mathbf{v}' \in \ell(1)} |\mathbf{v} - \mathbf{v}'| = |\mathbf{v}_{i-1} - \mathbf{z}'_{i-1}| = |\mathbf{v}_{i-1} - \mathbf{z}_{i-1}| + |\mathbf{z}_{i-1} - \mathbf{z}'_{i-1}|,$$

where

$$\mathbf{z}'_{i-1} = \left(p_{i-1} - \frac{b_{1,i}}{p_i}, p_{i-1} - \frac{b_{2,i}}{p_i}\right).$$

But $|\mathbf{v}_{i-1} - \mathbf{z}_{i-1}| = |\eta_{i-1}|$ and $|\mathbf{z}_{i-1} - \mathbf{z}'_{i-1}| = \frac{p_{i-1}}{p_i}|\eta_i|$, and (52) follows.

So (52) shows that for all $i = 1, \ldots, k$ and for all $\mathbf{v} = (p, b_1, b_2) \in \pi$ with $p_{i-1} < p < p_i$ we have

$$|\eta(\mathbf{v})| \geq \sigma_i |\eta_{i-1}|.$$  

(53)

Now from (50) and the inequality $\frac{p_{i-1}}{p_i} \geq \frac{1}{2}$, $2 \leq i \leq k - 1$ we see that

$$\sigma_1 \geq 1 + \frac{p_0}{2p_1} > 1; \quad \sigma_i \geq \frac{5}{4} \quad \text{for} \quad 2 \leq i \leq k - 1$$

and (53) transforms into

$$|\eta(\mathbf{v})|_\infty \geq \left(1 + \frac{p_0}{2p_1}\right) |\eta_0|_\infty, \quad \text{for all} \quad \mathbf{v} = (p, b_1, b_2) \in \pi \quad \text{with} \quad p_0 < p < p_1;$$

(54)

$$|\eta(\mathbf{v})|_\infty \geq \frac{5}{4} |\eta_{i-1}|_\infty \quad \text{for all} \quad \mathbf{v} = (p, b_1, b_2) \in \pi \quad \text{with} \quad p_{i-1} < p < p_i; \quad 2 \leq i \leq k - 1.$$ 

(55)

By the same argument

$$|\eta(\mathbf{v})|_\infty \geq |\eta_{k-2}|_\infty + |\eta_{k-1}|_\infty \frac{p_{k-2}}{p_{k-1}} \geq \frac{3}{2} |\eta_{k-2}|_\infty \quad \text{for all} \quad \mathbf{v} = (p, b_1, b_2) \in \pi \quad \text{with} \quad p_{k-1} < p < p_k.$$ 

(56)

Now we see that (47) is the sequence of all best approximation vectors to $\mathbf{V}_k$ with denominators between $p_0$ and $p_k$. As for the point $\mathbf{v}_{k-1}$, it is not a best approximation vector because $|\eta_{k-1}| = |\eta_{k-2}|$ and so $|\eta_{k-1}|_\infty = |\eta_{k-2}|_\infty$ and $p_{k-1} > p_{k-2}$.

Now we need to estimate $|\eta_i|_\infty, i = 0, \ldots, k - 1$ from below. We consider the lattice $\Lambda$ and the parallelogram

$$\Pi = \left\{(x, y_1, y_2) \in \mathbb{R}^3 : |x| \leq p_{i+1}, \max_{j=1,2} \left|x \frac{b_{j,k}}{p_k} - y_j\right| \leq |\eta_i|_\infty \right\} \cap \pi.$$

For its area we have

$$\Delta \leq \text{area}(\Pi) \leq 2\sqrt{2}|\eta_i|_\infty \times \sqrt{1 + \left(\frac{b_{1,k}}{p_k}\right)^2 + \left(\frac{b_{2,k}}{p_k}\right)^2} p_{i+1} \leq 2\sqrt{6}|\eta_i|_\infty p_{i+1}.$$ 

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we see that

\[ |\eta_i|_\infty \geq \frac{\Delta}{2\sqrt{6}p_{i+1}}, \quad i = 0, \ldots, k - 1. \tag{57} \]

Now we prove the statement of the lemma about points \( \mathbf{x} \) under the condition \( (45) \). From \( (45) \) we see that

\[ |p\mathbf{x} - p\mathbf{V}_k|_\infty = \max_{j=1,2} |px_j - \frac{b_{j,k}}{p_k}| \leq \frac{\Delta}{100p_k} \cdot \frac{p}{p_k}, \tag{58} \]

and in particular for \( p \leq p_k \) one has

\[ |p\mathbf{x} - p\mathbf{V}_k|_\infty \leq \frac{\Delta}{100p_k}. \tag{59} \]

First of all we show that the vectors \( \mathbf{v} = (p, b_1, b_2) \in \mathbb{Z}^3 \setminus \Lambda \) with \( p > p_0 \) cannot be best approximation vectors for \( \mathbf{x} \). Indeed, \( \mathbf{V}_k \) belongs to the segment with endpoints \( \mathbf{V}_0, \mathbf{V}_1 \) and inequality \( (42) \) show that

\[ |\mathbf{V}_k - \mathbf{V}_0| < \frac{1}{2p_0\Delta}. \]

Now \( (59) \) together with the inequality \( (44) \) written as \( p_k \geq \Delta^2 \) give the bound

\[ |p_0\mathbf{x} - \mathbf{v}_0|_\infty \leq |p_0\mathbf{V}_k - \mathbf{v}_0|_\infty + |p_0\mathbf{x} - p_0\mathbf{V}_k|_\infty \leq \frac{1}{2\Delta} + \frac{\Delta}{100p_k} \leq \frac{1}{\Delta} \leq |p\mathbf{x} - \mathbf{v}|_\infty. \]

Then we show that \( \mathbf{v}_0 \) is a best approximation for \( \mathbf{x} \). Indeed, from \( (12, 45) \) and \( (44) \) in the form

\[ p_k \geq \sqrt{\frac{\Delta}{\delta(\mathbf{v}_0)}} \]

we have

\[ |\mathbf{V}_0 - \mathbf{x}| \leq |\mathbf{V}_k - \mathbf{V}_0| + |\mathbf{V}_k - \mathbf{x}| \leq \delta(\mathbf{v}_0). \]

So \( \mathbf{v}_0 \) is the best approximation vector for \( \mathbf{x} \).

Now we study approximation to \( \mathbf{x} \) by vectors \( \mathbf{v}_i, i = 0, 1, \ldots, k \).

From the triangle inequality and \( (59) \) for vectors \( p_i\mathbf{x} \) and \( \mathbf{v}_i = (b_{1,i}, b_{2,i}) \) we deduce for \( |\eta_i|_\infty = |p_i\mathbf{V}_k - \mathbf{v}_i|_\infty \) the inequalities

\[ |\eta_i|_\infty - \frac{\Delta}{100p_k} \leq |p_i\mathbf{x} - \mathbf{v}_i|_\infty \leq |\eta_i|_\infty + \frac{\Delta}{100p_k}, \quad i = 0, \ldots, k. \tag{60} \]

We should note that from \( (57) \) we have

\[ \frac{\Delta}{100p_k|\eta_i|_\infty} \leq \frac{\sqrt{6}}{50}, \quad 0 \leq i \leq k - 1. \tag{61} \]

So from the last inequality and \( (51) \) for \( i = 1, \ldots, k - 2 \) we get

\[ \frac{|p_i\mathbf{x} - \mathbf{v}_i|_\infty}{|p_{i-1}\mathbf{x} - \mathbf{v}_{i-1}|_\infty} \leq \frac{|\eta_i|_\infty}{|\eta_{i-1}|_\infty} \cdot \frac{1 + \frac{\Delta}{100p_k|\eta_i|_\infty}}{1 - \frac{\Delta}{100p_k|\eta_{i-1}|_\infty}} \leq \frac{3}{4}, \quad 1 \leq i \leq k - 2. \tag{62} \]

In addition from \( (45) \), \( (60) \) and \( (61) \) we deduce

\[ \frac{|p_k\mathbf{x} - \mathbf{v}_k|_\infty}{|p_\nu\mathbf{x} - \mathbf{v}_\nu|_\infty} \leq \frac{\Delta}{100p_k|\eta_k|_\infty} \cdot \frac{1 + \frac{\Delta}{100p_k|\eta_k|_\infty}}{1 - \frac{\Delta}{100p_k|\eta_\nu|_\infty}} \leq \frac{1}{2}, \quad \nu = k - 2, k - 1. \tag{63} \]
Let us show that there is no best approximations \( \mathbf{v} = (p, b_1, b_2) \) with \( p_{i-1} < p < p_i \) for all \( i = 1, 2, \ldots, k \).

First of all we consider the case \( i = 1 \) that is \( p_0 < p < p_1 \). In this case we will take into account the inequality
\[
\| \mathbf{v} \|_\infty \geq |p_0(\mathbf{V}_2 - \mathbf{V}_0)|_\infty,
\]
as well as the inequalities
\[
\left| \| p_0 \mathbf{x} - \mathbf{v}_0 \|_\infty - \| \mathbf{v}_0 \|_\infty \right| \leq \frac{p_0 \Delta}{100p_k^2}, \quad \left| \| p \mathbf{x} - \mathbf{v} \|_\infty - \| \mathbf{v}(\mathbf{x}) \|_\infty \right| \leq \frac{p_1 \Delta}{100p_k^2},
\]
which follow from (58). Three last inequalities together with (44) in the form
\[
p_k \geq k \geq \sqrt{\frac{p_1 \Delta}{100|p_0(\mathbf{V}_2 - \mathbf{V}_1)|_\infty (1 + \frac{p_1}{p_0})}},
\]
and (54) lead to
\[
\frac{\| p \mathbf{x} - \mathbf{v} \|_\infty}{\| p_0 \mathbf{x} - \mathbf{v}_0 \|_\infty} \geq \frac{\| \mathbf{v}(\mathbf{x}) \|_\infty - \frac{p_1 \Delta}{100p_k^2}}{\| \mathbf{v}_0 \|_\infty + \frac{p_1 \Delta}{100p_k^2}} \geq \frac{\| \mathbf{v}(\mathbf{x}) \|_\infty}{\| \mathbf{v}_0 \|_\infty} \cdot \frac{1 - \frac{p_1 \Delta}{100p_k^2 \| \mathbf{v}_0 \|_\infty}}{1 + \frac{p_1 \Delta}{100p_k^2 \| \mathbf{v}_0 \|_\infty}} > 1,
\]
and we proved everything what we need in the case \( p_0 < p < p_1 \).

Next, suppose that \( p_{i-1} < p < p_i \) and \( 2 \leq i \leq k - 1 \). Then (55) and (61) give
\[
\frac{\| p \mathbf{x} - \mathbf{v} \|_\infty}{\| p_{i-1} \mathbf{x} - \mathbf{v}_{i-1} \|_\infty} \geq \frac{\| \mathbf{v}(\mathbf{x}) \|_\infty - \frac{p_1 \Delta}{100p_k^2}}{\| \mathbf{v}_{i-1} \|_\infty} \cdot \frac{1 - \frac{p_1 \Delta}{100p_k^2 \| \mathbf{v}_{i-1} \|_\infty}}{1 + \frac{p_1 \Delta}{100p_k^2 \| \mathbf{v}_{i-1} \|_\infty}} > 1,
\]
and everything is done in the case \( p_1 < p < p_{k-1}, p \neq p_i \) also.

By similar argument using (56) and (61) for \( p_{k-1} < p < p_k \) we see that
\[
\frac{\| p \mathbf{x} - \mathbf{v} \|_\infty}{\| p_{i-2} \mathbf{x} - \mathbf{v}_{i-1} \|_\infty} > 1.
\]

We see from (62), (63) and the lower bounds for \( \| p \mathbf{x} - \mathbf{v} \|_\infty \) that \( \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{k-2} \) and \( \mathbf{v}_k \) are the best approximation vectors for \( \mathbf{x} \), and \( \mathbf{v}_{k-1} \) may be a best approximation vector or may be not. So all the best approximations for \( \mathbf{x} \) form either the sequence (46) or the sequence (47).

To finish the proof of Lemma 4 we need to show (48). But this can be done analogously to (62), as from (50) and (61) we see that
\[
\frac{\| p_i \mathbf{x} - \mathbf{v} \|_\infty}{\| p_{i-1} \mathbf{x} - \mathbf{v}_{i-1} \|_\infty} \geq \frac{\| \mathbf{v} \|_\infty}{\| \mathbf{v}_{i-1} \|_\infty} \cdot \frac{1 - \frac{\Delta}{100p_k^2 \| \mathbf{v}_{i-1} \|_\infty}}{1 + \frac{\Delta}{100p_k^2 \| \mathbf{v}_{i-1} \|_\infty}} \geq \frac{1}{4}, \quad 1 \leq i \leq k - 1.
\]

\[\Box\]

10. Three-dimensional subspaces.

Lemma 5. Consider two independent integer points
\[
\mathbf{w}^0 = (p_0', b_1', b_2'), \quad \mathbf{w}_0'' = (p_0'', b_1'', b_2'')
\]

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and the two-dimensional subspace \( \pi = \langle \mathbf{w}'_0, \mathbf{w}''_0 \rangle_{\mathbb{R}} \). Suppose that for the corresponding rational points we have
\[
\mathbf{W}'_0 = \left( \frac{b'_1,0}{p'_0}, \frac{b'_1,0}{p'_0} \right), \quad \mathbf{W}''_0 = \left( \frac{b''_1,0}{p''_0}, \frac{b''_1,0}{p''_0} \right) \in [0,1]^2.
\]
Suppose that \( \mathbf{w}'_0 \) and \( \mathbf{w}''_0 \) form a basis of the lattice \( \Lambda = \pi \cap \mathbb{Z}^3 \), that is
\[
\Lambda = \langle \mathbf{w}'_0, \mathbf{w}''_0 \rangle_{\mathbb{Z}},
\]
and \( \Delta \) is the two-dimensional fundamental volume of \( \Lambda \). Suppose that parameters \( \gamma_1 \) and \( \gamma_2 \) satisfy the inequalities
\[
\gamma_2 \geq \gamma_1^2, \quad \gamma_1 \geq 50.
\] (64)
Consider the point
\[
\mathbf{w}_0 = \mathbf{w}'_0 + \mathbf{w}''_0 = (p_0, b_{1,0}, b_{2,0})
\]
and the corresponding rational point and \( \mathbf{W}_0 = \left( \frac{b_{1,0}}{p_0}, \frac{b_{1,0}}{p_0} \right) \in [0,1]^2 \). Suppose that
\[
p_0 \geq \gamma_1 \Delta^2.
\] (65)
Let \( \mathbf{n} \) be an orthogonal vector to \( \pi \) and \( |\mathbf{n}| = 1 \). Consider the point
\[
\mathbf{r}_0 = (x_0, y_{1,0}, y_{2,0}) = \mathbf{w}_0 + \mathbf{n} \cdot \frac{\Delta}{\gamma_1 p_0} \in \mathbb{R}^3
\] (66)
and the corresponding two-dimensional point
\[
\mathbf{x}_0 = (x_{1,0}, x_{2,0}) = \left( \frac{y_{1,0}}{x_0}, \frac{y_{2,0}}{x_0} \right) \in \mathbb{R}^2.
\] (67)
Suppose that for all \( \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \) satisfying
\[
|\mathbf{x} - \mathbf{x}_0|_\infty = \max_{j=1,2} |x_j - x_{j,0}| \leq \frac{\Delta}{\gamma_1 p_0}
\] (68)
the vector \( \mathbf{w}_0 \) is a best approximation vector to \( \mathbf{x} \).

Then there exists an integer point \( \mathbf{w}_1 = (p_1, b_{1,1}, b_{2,1}) \) with the following properties:

(i) \( \mathbf{w}_1 \) belongs to the affine subspace \( \pi_1 = \pi + \frac{\Delta}{\gamma_1} \cdot \mathbf{n} \);

(ii) both triples
\[
\mathbf{w}'_0, \mathbf{w}_0, \mathbf{w}_1
\]
and
\[
\mathbf{w}''_0, \mathbf{w}_0, \mathbf{w}_1
\]
form bases in \( \mathbb{Z}^3 \);

(iii) vectors \( \mathbf{w}_0, \mathbf{w}_1 \) form a basis of the two-dimensional lattice
\[
\Lambda_1 = \langle \mathbf{w}_0, \mathbf{w}_1 \rangle_{\mathbb{R}} \cap \mathbb{Z}^3
\]
with two-dimensional fundamental volume \( \Delta_1 \);
(iv) the inequalities
\[
\left( \gamma_1 - \frac{2}{\gamma_1} \right) \cdot \left( \frac{p_0}{\Delta} \right)^2 \leq p_1 \leq \left( \gamma_1 + \frac{2}{\gamma_1} \right) \cdot \left( \frac{p_0}{\Delta} \right)^2
\]

and
\[
\frac{1}{4} \cdot \frac{p_0}{\Delta} \leq \Delta_1 \leq 12 \cdot \frac{p_0}{\Delta}
\]
are valid;

(v) define \( W_1 = (b_{11} p_1, b_{21} p_1) \), then for any \( x = (x_1, x_2) \in \mathbb{R}^2 \) satisfying
\[
|x - W_1|_{\infty} = \max_{j=1,2} \left| x_j - \frac{b_{j,1}}{p_1} \right| \leq \frac{\Delta}{\gamma_2 p_0 p_1}
\]
either the vectors
\[
w_0, \ w_1
\]
are two consecutive best approximation vectors to \( x \) or the vectors
\[
w_0, \ w_1 - w_0, \ w_1
\]
are three consecutive best approximation vectors to \( x \)

Remark 4. For the point \( x_0 \) one has
\[
|x_0 - W_0| \leq \frac{\Delta}{\gamma_1 p_0} + (k - 1) \sqrt{1 + \left( \frac{b_{1,0}}{p_0} \right)^2 + \left( \frac{b_{2,0}}{p_0} \right)^2 + \left( \frac{\Delta}{\gamma_1 p_0^2} \right)^2} \leq \frac{2\Delta}{\gamma_1 p_0^2},
\]
where \( 1 \leq k = \frac{p_0}{\gamma_1 p_0^2 \sin \psi} \) and \( \psi \) is the angle between \( n \) and \( e = (1, 0, 0) \). (We take into account that \( \frac{b_{1,0}}{p_0} \in [0, 1] \).)

Remark 5. From inequalities (64, 69, 70) it follows that
\[
\frac{1}{8} \cdot \frac{\Delta}{p_0} \leq \frac{\Delta_1}{p_1} \leq 24 \cdot \frac{\Delta}{p_0}.
\]

Remark 6. From inequalities (69) and (29) it follows that
\[
p_1 \geq \frac{\gamma_1}{2} p_0 \frac{p_0}{\Delta^2} \geq \frac{\gamma_1}{2} p_0.
\]

Proof of Lemma 5. We should note that the parallelogram
\[
\Pi = \{ z \in \mathbb{R}^3 : \ z = \lambda w'_0 + \mu w''_0, \ 0 \leq \lambda, \mu \leq 1 \}
\]
is a fundamental domain with respect to \( \Lambda \) and the two-dimensional affine subspace \( \pi_1 \) contains a lattice \( \Lambda_1 \subset \mathbb{Z}^2 \) congruent to \( \Lambda \). Then any shift of parallelogram \( \Pi \) which belongs to \( \pi_1 \) contains an integer point. Consider the point
\[
X_0 = (X_0, Y_{1,0}, Y_{2,0}) = (x_0)_R \cap \pi_1
\]

\[2\]It is important that the constants in (70) do not depend on \( \gamma_1 \).
and the parallelogram $\Pi + X_0$. By the discussion above it contains an integer point. We denote this point by $w_1 = (p_1, b_{1,1}, b_{2,1})$. This is just the integer point what we need. Indeed, properties (i) and (ii) are clearly satisfied. As vector $w_0$ is primitive and there is no integer points between subspaces $\pi$ and $\pi_1$, property (iii) is satisfied also. From the construction we see that

$$X_0 = x_0 \cdot \frac{\gamma_1 p_0}{\Delta^2}, \quad |p_0 - x_0| \leq \frac{\Delta}{\gamma_1 p_0}, \quad |p_1 - X_0| \leq p_0. \quad (72)$$

So

$$|p_1 - \frac{\gamma_1 p_0^2}{\Delta^2}| \leq p_0 + \frac{1}{\Delta} < 2p_0$$

and we get (69) by taking into account (65). To get (70) we will estimate the area $\Delta_1$ of parallelogram $P = \{ z = aX_0 + bw_1, \ a, b \in [0, 1) \}$.

Observe that

$$\Delta_1 = \text{area } P = \text{area } P_0 + \lambda_* \text{area } P' + \mu_* \text{area } P'',$$

with some $\lambda_*, \mu_* \in (-1, 1)$, where

$$P_0 = \{ z = aw_0 + bw_1, \ a, b \in [0, 1) \}$$

and

$$P' = \{ z = aw'_0 + bw_1, \ a, b \in [0, 1) \}, \ P'' = \{ z = aw''_0 + bw_1, \ a, b \in [0, 1) \}.$$ 

It is clear that

$$\text{area } P', \text{area } P'' \leq \Delta,$$

and

$$\text{area } P_0 = |X_0| \rho(\mathbf{x}_0, \langle w_0 \rangle_{\mathbb{R}}) = |X_0| \cdot \frac{\Delta}{\gamma_1 p_0},$$

where

$$\left( p_0 - \frac{\Delta}{\gamma p_0} \right) \frac{\gamma p_0}{\Delta^2} \leq |X_0| = |x_0| \cdot \frac{\gamma p_0}{\Delta^2} \leq \left( \sqrt{3} p_0 + \frac{\Delta}{\gamma p_0} \right) \frac{\gamma p_0}{\Delta^2}.$$

So

$$|X_0| \cdot \frac{\Delta}{\gamma_1 p_0} - 2\Delta \leq \Delta_1 \leq |X_0| \cdot \frac{\Delta}{\gamma_1 p_0} + 2\Delta,$$

and together with (69) the last two formulas give (70).

To finish the proof it remains to explain (v).

If $x$ satisfies (71) then it satisfies (68). Indeed, as $w_1 \in \Pi + X_0$, the point $W_1 = \left( \frac{b_{1,1}}{p_1}, \frac{b_{2,1}}{p_1} \right)$ belongs to a convex polygon with vertices

$$x_0 = (x_{1,0}, x_{2,0}), \quad x_{0,0} = \left( \frac{Y_{1,0} + b_{1,0}}{X_0 + p_0}, \frac{Y_{2,0} + b_{2,0}}{X_0 + p_0} \right), \quad x_{0,1} = \left( \frac{Y_{1,0} + b'_{1,0}}{X_0 + p'_0}, \frac{Y_{2,0} + b'_{1,0}}{X_0 + p'_0} \right), \quad x_{0,2} = \left( \frac{Y_{1,0} + b''_{1,0}}{X_0 + p''_0}, \frac{Y_{2,0} + b''_{1,0}}{X_0 + p''_0} \right)$$

and sup-norm diameter

$$2 \max_{i=0,1,2} |x_0 - x_{0,j}| \leq 2 \max_{j=1,2} \left( \left| \frac{Y_{j,0}}{X_0} - \frac{Y_{j,0} + b'_{j,0}}{X_0 + p'_0} \right|, \left| \frac{Y_{j,0}}{X_0} - \frac{Y_{j,0} + b''_{j,0}}{X_0 + p''_0} \right|, \left| \frac{Y_{j,0}}{X_0} - \frac{Y_{j,0} + b_{j,0}}{X_0 + p_0} \right| \right) \leq$$
\[
\leq \frac{8\Delta}{X_0 p_0} = \frac{8\Delta^3}{\gamma_1 x_0 p_0^2} \leq \frac{16\Delta}{\gamma_1 p_0^2},
\] (73)

The last inequalities in (73) should be explained. Indeed,

\[
|Y_{j,0} - b_{j,0}| = \frac{|Y_{j,0}(X_0 + p_0) - X_0(Y_{j,0} + b_{j,0})|}{X_0(X_0 + p_0)} = \frac{|Y_{j,0}p_0 - X_0b_{j,0}|}{X_0(X_0 + p_0)} < \frac{|Y_{j,0}p_0 - X_0b_{j,0}|}{X_0^2} = \frac{1}{X_0} \left| \frac{p_0' Y_{j,0}}{X_0} - b_{j,0}' \right|, \quad j = 1, 2.
\]

But

\[
\left| \frac{p_0' Y_{j,0}}{X_0} - b_{j,0}' \right| \leq \rho(p_0' W_0', p_0' x_0) \leq \rho(p_0' W_0', p_0' W_0) + p_0' \cdot \rho(W_0, x_0), \quad j = 1, 2.
\] (74)

For the two summands in the right hand side here we have the bound

\[
\rho(p_0' W_0', p_0' W_0) = \rho(w_0', \langle w_0 \rangle_\mathbb{R} \cap \{x_0 = p_0'\}) \leq 2\rho(w_0', \langle w_0 \rangle_\mathbb{R}) \leq \frac{2\Delta}{|w_0|}
\]

and the bound of Remark 4, respectively. So we continue (74) with

\[
\left| \frac{p_0' Y_{j,0}}{X_0} - b_{j,0}' \right| \leq \frac{2\Delta}{|w_0|} + p_0' \cdot \frac{2\Delta}{\gamma_1 p_0^2} \leq \frac{4\Delta}{p_0}.
\]

Quite similar bounds are valid for \( \left| \frac{p_0 Y_{j,0}}{X_0} - b_{j,0}' \right| \) and \( \left| \frac{p_0 Y_{j,0}}{X_0} - b_{j,0}' \right|, \quad j = 1, 2 \). This gives the first inequality in (73). To get the last inequality in (73) we use (72) and (64). So we explained how to prove (73).

So as \( W_1 \in \text{conv}(x_0, x_{0,0}, x_{0,1}, x_{0,2}) \) from (73) we deduce the inequality

\[
|W_1 - x_0| \leq \frac{16\Delta}{\gamma_1 p_0^2}. \tag{75}
\]

This gives

\[
|x - x_0| \leq |W_1 - x_0| + |W_1 - x| \leq \frac{16\Delta}{\gamma_1 p_0^2} + \frac{\sqrt{2}\Delta}{\gamma_2 p_0^2} \leq \frac{\Delta}{2\gamma_1 p_0^2}, \tag{76}
\]

(we used the triangle inequality, conditions (71) with bound \( p_1 \geq p_0 \) and (64)) and we have (64).

So \( w_0 \) is a best approximation vector for \( x \). In (70) we have an upper bound for \( \Delta_1 \) which does not depend on \( \gamma \). This means that for any \( w \in \mathbb{Z}^3 \setminus \langle w_0, w_1 \rangle_\mathbb{R} \) we have

\[
\rho(w, \langle w_0, w_1 \rangle_\mathbb{R}) \geq \frac{1}{\Delta_1} \geq \frac{\Delta}{12p_0}.
\]

For large \( \gamma_1 \) the point \( w_0 \) is essentially closer to the line \( \langle x \rangle_\mathbb{R} \) than the points \( w = (p, b_1, b_2) \in \mathbb{Z}^3 \setminus \langle w_0, w_1 \rangle_\mathbb{R} \) with \( p \leq p_1 \). Indeed, put \( W = \left( \frac{w}{p}, \frac{b}{p} \right) \), then by the previous inequality and (71) we see that

\[
\sqrt{2}|px - pW| \geq |px - pW| \geq \frac{1}{\Delta_1} - p|x - W_1| \geq \frac{\Delta}{12p_0} - \frac{\Delta}{\gamma_2 p_0} \geq \frac{\Delta}{13p_0}.
\]

At the same time

\[
|p_0 x - p_0 W_0| \leq \frac{2\Delta}{\gamma_1 p_0}.
\]

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by (76) and Remark 4. As \( \gamma_1 \geq 50 \) we see that there \( x \) has no best approximations \( w = z^3 \setminus \langle w_0, w_1 \rangle_R \) with \( p_0 \leq p \leq p_1 \) So for all \( x \) satisfying (71) all the best approximations between \( p_0 \) and \( p_1 \) lie in the two-dimensional subspace \( \langle w_0, w_1 \rangle_R \). We see from (71) that

\[
|p_1x - p_1W_1|_\infty \leq \frac{\Delta}{\gamma_2p_0}.
\]

But from the construction (66) and (76) we have

\[
\sqrt{2}|p_0x - p_0W_0|_\infty \geq |p_0x - p_0W_0| \geq \frac{\Delta}{\gamma_1p_0} - \frac{\Delta}{2\gamma_1p_0} = \frac{\Delta}{2\gamma_1p_0}.
\]

So

\[
|p_0x - p_0W_0|_\infty > |p_1x - p_1W_1|_\infty.
\]

We have the following situation. For any \( x \) satisfying (71) vectors \( w_0, w_1 \) are best approximation vectors, and we do not have best approximation vectors \( w = (p, b_1, b_2) \in Z^3 \setminus \langle w_0, w_1 \rangle_R \) with \( p_0 \leq p \leq p_1 \). The parallelogram with vertices \( 0, w_0, w_1 - w_0, w_1 \) is a fundamental parallelogram for the lattice \( \langle w_0, w_1 \rangle_Z \). So the distances from its vertices \( w_0, \) and \( w_1 - w_0 \) to the diagonal \( \langle w_1 \rangle_R \) are equal. This means that for a point \( x \) which is close to \( W_1 \), the only one possible opportunity for a vector \( w = (p, b_1, b_2) \in Z^3 \) with \( p_0 < p < p_1 \) to be a best approximation to \( x \) is \( w = w_1 - w_0 \). Of course we cannot say that the vector \( w_1 - w_0 \) is a best approximation for sure. It depends on which of the vectors \( w_0 \) and \( w_1 - w_0 \) is closer to the line spanned by the point \( (1, x_1, x_2) \).

We see that for all \( x \) satisfying (71) all the best approximations with denominators between \( p_0 \) and \( p_1 \) should be among the vectors \( w_0, w_1 - w_0, w_1, \) and everything is proved. \( \square \)

Here we should note that from (75) and Remark 4 by the triangle inequality immediately follows

**Remark 7.** For the rational points \( W_0 \) and \( W_1 \) from Lemma 5 one has

\[
|W_0 - W_1| \leq \frac{3\Delta}{\gamma_1p_0^2}.
\]

11. Proof of Theorem 2.

We construct a sequence of integer vectors

\[
z_\nu = (q_\nu, a_{1,\nu}, a_{2,\nu}) \in Z, \quad \nu \in Z_+ \tag{77}
\]

which will be "almost" best approximation vectors to the limit point

\[
a = \lim_{\nu \to \infty} A_\nu \tag{78}
\]

where

\[
A_\nu = \begin{pmatrix} a_{1,\nu} & a_{2,\nu} \\ q_\nu & q_\nu \end{pmatrix}
\]

are the corresponding rational points. For these vectors and \( x = (x_1, x_2) \) we consider the values

\[
\xi_\nu = \max_{j=1,2} |q_\nu x_j - a_{j,\nu}|,
\]

which of course depend on \( x \).
First of all we consider the lattice
\[ \Lambda_1 = \langle e_1, e_2 \rangle \mathbb{Z}, \quad e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0). \]

We put \( i_1 = 1 \) and take
\[ z_{i_1} = z_1 = (q_1, a_{1,1}, a_{2,1}), \quad z_{i_1+1} = z_2 = (q_2, a_{1,2}, a_{2,2}) \]
to be a basis of \( \Lambda_1 \) in such a way that \( q_2 > q_1 \) and all the conditions \((41, 42)\) of Lemma 4 are satisfied for \( v_0 = z_1, v_1 = z_2 \). (In particular, the condition \((42)\) is satisfied if the angle between the basis vectors \( z_1, z_2 \) is small.) We take
\[ \gamma = \max(400, q_2/q_1), \]
so \( q_2 \leq \gamma q_1 \). Now we define vectors \((77)\) by inductive procedure. Let vectors \((77)\) be defined up to \( z_{i_t+1} \) and the following conditions are valid
\[ (A) \text{ two last vectors } v_0 = z_{i_t}, v_1 = z_{i_t+1} \text{ satisfy all the conditions } (41, 42) \text{ of Lemma 4 where } p_0 = q_{i_t}, p_1 = q_{i_t+1} \text{ and } \Delta = \Delta_t \text{ is the fundamental volume of two-dimensional lattice } \Lambda_t = \langle z_{i_t}, z_{i_t+1} \rangle \mathbb{Z}, \]
moreover
\[ |V_0 - V_1|_{\infty} = |A_{i_t} - A_{i_{t+1}}|_{\infty} \leq \frac{\Delta_t}{30\gamma^2 q_{i_t}^2}; \tag{79} \]
\[ (B) \text{ for all } x = (x_1, x_2) \text{ satisfying } \]
\[ |x - A_{i_t}|_{\infty} \leq \frac{\Delta_t}{24\gamma^2 q_{i_t}^2} \tag{80} \]
all the best approximation vectors \( z = (q, a_1, a_2) \) with \( q_1 \leq q \leq q_{i_t} \) are among the vectors from the sequence
\[ z_1, z_2, \ldots, z_{i_t}; \tag{81} \]
\[ (C) \text{ among every two consecutive vectors } z_{\nu}, z_{\nu+1} \text{ from } (81) \text{ at least one vector is a best approximation vector for every } x \text{ satisfying } (80); \]
\[ (D) \text{ for all } x \text{ satisfying } (80) \text{ and for every } \nu \leq i_t - 1 \text{ one has } \frac{\xi_{\nu}}{\xi_{\nu+1}} \geq \frac{1}{16\sqrt{6}(50\gamma^2 + 2)}. \]

When vectors \((77)\) will be defined, the limit point \((78)\) will satisfy
\[ |\alpha - A_{i_t}|_{\infty} \leq \frac{\Delta_t}{24\gamma^2 q_{i_t}^2} \quad \forall t \in \mathbb{Z}_+, \tag{82} \]
as the inequality \((80)\) for \( t + 1 \) leads to the inequality \((80)\) for \( t \). This limit vector \( \alpha \) will be just the vector we need for Theorem 2.

Here we should note that for \( t = 1 \) the conditions \((B)\) is satisfied automatically as \( z_1 \) is a best approximation vector for all \( x \) satisfying \((80)\). At the same time for \( t = 1 \) conditions \((C)\) and \((D)\) are empty, because we have only one vector \( z_1 \).

Now we explain how to construct next vectors
\[ z_{\nu}, \quad i_t + 2 \leq \nu \leq i_{t+1} + 1. \tag{83} \]
satisfying conditions \((A), (B), (C), (D)\) of the next step. We start with the explanation of the construction and then we will verify the conditions \((A), (B), (C), (D)\).
First of all we apply Lemma 4 with
\[ v_0 = z_{i_t}, \quad v_1 = z_{i_{t+1}} \]
and take vectors
\[ z_{i_t+\nu} = v_\nu, \quad 2 \leq \nu \leq k_t \]
where \( v_\nu \) are defined in (13). We take \( k = k_t \) large enough to satisfy (14) as well as the inequalities
\[ q_{i_t+k_t} \geq \gamma \Delta_t^2 \quad (84) \]
and
\[ q_{i_t+k_t} \geq \gamma q_{i_t}, \quad (85) \]
We define
\[ i_{t+1} = i_t + k_t + 2, \]
so
\[ i_{t+1} - 2 = i_t + k_t. \]
Then for
\[ \gamma_1 = \gamma, \quad \gamma_2 = \gamma^2 \]
and vectors
\[ w_0' = z_{i_{t+1}-4}, \quad w_0'' = z_{i_{t+1}-3} \]
we apply Lemma 5. Of course we have
\[ w_0 = w_0' + w_0'' = z_{i_{t+1}-2}. \]
We need to check the condition (65) and the condition on \( x \) satisfying (68). But (65) follows from (84). As for the condition on \( x \) we will check it right now. In our situation \( p_0 = q_{i_{t+1}-2} \) and (68) means that
\[ |x - x_0|_\infty \leq \frac{\Delta_t}{\gamma q_{i_{t+1}-2}^2}. \]
Remark 4 with \( W_0 = A_{i_{t+1}-2} \) gives
\[ |x_0 - W_0|_\infty = |x_0 - A_{i_{t+1}-2}|_\infty \leq \frac{2\Delta_t}{\gamma q_{i_{t+1}-2}^2}. \]
So by the triangle inequality
\[ |x - A_{i_{t+1}-2}|_\infty \leq \frac{3\Delta_t}{\gamma q_{i_{t+1}-2}^2} \leq \frac{\Delta_t}{100 q_{i_{t+1}-2}^2}. \]
So \( x \) satisfies (45) and the condition on \( x \) follows from the conclusion of Lemma 4, as in both sequences (46) and (47) the last vector is \( v_k = w_0 = z_{i_{k+1}-2} \). We verified the possibility of application of Lemma 5. Lemma 5 gives us the vector
\[ z_{i_{t+1}} = w_1. \]
Then we define
\[ z_{i_{t+1}-1} = w_1 - w_0. \]
Now we should define $z_{i_{t+1}+1}$. First of all we define the next two-dimensional lattice $\Lambda_{t+1} = \langle z_{i_{t+1}} - z_{i_{t+1}+1} \rangle z$ with fundamental volume $\Delta_{t+1}$. Then we define

$$z_{i_{t+1}+1} = z_{i_{t+1}} - az_{i_{t+1}}, \quad \text{where} \quad a = [50 \gamma^2] + 1. \quad (86)$$

It is clear that $\Lambda_{t+1} = \langle z_{i_{t+1}}, z_{i_{t+1}+1} \rangle z$.

So all the vectors (83) are defined and we must check the conditions (A), (B), (C), (D) of the new inductive step.

Condition (A) is satisfied because of

$$|a_{j,i_{t+1}}q_{i_{t+1}} - a_{j,i_{t+1}}q_{i_{t+1}}| \leq \Delta_{t+1},$$

and for (79) with $t$ replaced by $t + 1$ we have

$$|A_{i_{t+1}} - A_{i_{t+1}+1}|_\infty = \max_{j=1,2} \left| \frac{a_{j,i_{t+1}}}{q_{i_{t+1}} - q_{i_{t+1}+1}} \right| \leq \frac{\Delta_{t+1}}{q_{i_{t+1}} q_{i_{t+1}+1}} \leq \frac{\Delta_{t+1}}{A q_{i_{t+1}}^2} \leq \frac{\Delta_{t+1}}{50 \gamma^2 q_{i_{t+1}}^2}. \quad (87)$$

Let us check the conditions of Lemma 4. Inequality (41) is clear. As for (42), we should show that

$$|A_{i_{t+1}} - A_{i_{t+1}+1}|_\infty \leq \frac{1}{2q_{i_{t+1}} \Delta_{t+1}} \quad (88)$$

and

$$|A_{i_{t+1}} - A_{i_{t+1}+1}|_\infty \leq \frac{\delta(z_{i_{t+1}})}{2}. \quad (89)$$

To get (88) we use (29) for the best approximation vector $z_{i_{t+1}}$ with $\Delta_{t+1}$ instead of $\Delta$. Then

$$q_{i_{t+1}} \geq (K \Delta_{t+1})^2 \geq \frac{\Delta_{t+1}^2}{\gamma^2},$$

and this deduces (88) from (87).

From condition (v) of Lemma 5 we see that $w_1 = z_{i_{t+1}}$ will be a best approximation vector for all $x$ satisfying the condition (71). So we have

$$\frac{\Delta_t}{\gamma^2 q_{i_{t+1}-2} q_{i_{t+1}}} \leq \delta(z_{i_{t+1}}),$$

as $w_0 = z_{i_{t+1}}$ is always a best approximation vector under the assumption (71). The last inequality together with (87) and Remark 5 (where $\Delta = \Delta_t, \Delta_1 = \Delta_{t+1}, p_0 = q_{i_{t+1}-2}, p_1 = q_{i_{t+1}}$) gives

$$|A_{i_{t+1}} - A_{i_{t+1}+1}|_\infty \leq \frac{\Delta_{t+1}}{50 \gamma^2 q_{i_{t+1}}^2} \leq \frac{\Delta_t}{2 \gamma^2 q_{i_{t+1}-2} q_{i_{t+1}}} \leq \frac{\delta(z_{i_{t+1}})}{2},$$

and this is just (89).

So condition (A) is satisfied.

Now we verify conditions (B) and (C). Suppose that $x$ satisfies (80) for the next step, that is

$$|x - A_{i_{t+1}}|_\infty \leq \frac{\Delta_{t+1}}{24 \gamma^2 q_{i_{t+1}}^2} \quad (90)$$
From (90) and Remark 5 we see that
\[ |x - A_{i+1}|_\infty \leq \frac{\Delta_i}{\gamma^2 d_{i+1}^{-1} q_{i+1}}. \]
So by Lemma 5 either
\[ z_{i+1-2}, z_{i+1-1}, z_{i+1} \]
or
\[ z_{i+1-2}, z_{i+1} \]
are successive best approximations to \( x \).
Then from Remark 7 (with \( W_0 = A_{i+1-2}, W_1 = A_{i+1}, \Delta = \Delta_i, \gamma_1 = \gamma, p_0 = q_{i+1-2}^2 \)) we have
\[ |A_{i+1-2} - A_{i+1}|_\infty \leq \frac{3\Delta_i}{400 q_{i+1-2}^2}. \]
This inequality together with (90) leads to
\[ |x - A_{i+1-2}|_\infty \leq |x - A_{i+1}|_\infty + |A_{i+1-2} - A_{i+1}|_\infty \leq \frac{\Delta_i}{100 q_{i+1-2}^2}. \]
So by Lemma 4 we see that either
\[ z_{i+1}, z_{i+1-1}, z_{i+1-4}, z_{i+1-3}, z_{i+1-2} \]
or
\[ z_{i+1}, z_{i+1-1}, z_{i+1-4}, z_{i+1-2} \]
is the sequence of successive best approximations to \( x \).
Again from (90) and Remark 6 (\( q_{i+1} = p_0 = q_{i+1-2}, p_1 = q_{i+1} \)) which now states that \( q_{i+1} \geq \frac{\gamma}{2} q_{i+1-2} \) we deduce
\[ |x - A_{i+1}|_\infty \leq \frac{\Delta_i}{24 \gamma^2 q_{i+1}^2} \leq \frac{\Delta_i}{96 \gamma^2 q_{i+1}^2}. \]
Then, by Remark 7 (\( W_0 = A_{i+1-2}, W_1 = A_{i+1}, p_0 = q_{i+1-2} = q_{i+1+k} \)) and (85) we see that
\[ |A_{i+1} - A_{i+1-2}|_\infty \leq \frac{3\Delta_i}{\gamma q_{i+1}^2} \leq 3\Delta_i \leq \frac{3\Delta_i}{100 \gamma^2 q_{i+1}^2}. \]
In the notation of Lemma 4 we have \( V_0 = A_{i+1}, V_1 = A_{i+1}, V_k = A_{i+1-2} \). So
\[ |A_{i+1-2} - A_{i+1}|_\infty = |V_k - V_0|_\infty \leq |V_1 - V_0|_\infty = |A_{i+1} - A_{i+1}|_\infty \leq \frac{\Delta_i}{30 \gamma^2 q_{i+1}^2}, \]
by (79) from condition (A). So last three inequalities lead to
\[ |x - A_{i+1}|_\infty \leq |x - A_{i+1}|_\infty + |A_{i+1} - A_{i+1-2}|_\infty + |A_{i+1-2} - A_{i+1}|_\infty \leq \frac{\Delta_i}{24 \gamma^2 q_{i+1}^2}, \]
and by inductive assumption we have the required properties for all the best approximations \( z = (q, a_1, a_2) \) with \( q_i \leq q \leq q_i \).
By the way, we see that condition (80) for \( t + 1 \)-th step ensures condition (80) for \( t \)-th step, and we proved the inequality (82).
We see that we have established conditions (B) and (C) for all the best approximations $z = (q, a_1, a_2)$ in the range $q_1 \leq q \leq q_{i+1}$.

Let us verify condition (D) for $i_t \leq \nu \leq i_{t+1} - 1$. We consider the cases

1) $\nu = i_t$,
2) $i_t < \nu \leq i_{t+1} - 3$,
3) $\nu = i_{t+1} - 2$,
4) $\nu = i_{t+1} - 1$

separately.

1) First of all we need lower bound for the approximation $\xi_{i_t} = q_{i_t} |\alpha - A_{i_t}|_\infty$. We use the notation of Lemma 4 with

$$y_0 = q_{i_t} \mathbf{x} - z_{i_t}, \quad \xi_{i_t} = |y_0|_\infty.$$

By (86) of the previous inductive step we have

$$q_{i_t+1} \leq (50\gamma^2 + 2)q_{i_t}.$$

Remark 5 with $p_0 = q_{i_t}, p_1 = q_{i_t+1}, \Delta = \Delta_{t-1}, \Delta_1 = \Delta_t$ for Lemma 5 applied on the previous inductive step gives

$$\Delta_t \geq \frac{q_{i_t}}{8q_{i_t-2}} \Delta_{t-1}$$

Now from (57) with $i = 0, \Delta = \Delta_t$, we get

$$\xi_{i_t} \geq \frac{\Delta_t}{2\sqrt{6q_{i_t+1}}} \geq \frac{\Delta_{t-1}}{16\sqrt{6}(50\gamma^2 + 2)q_{i_t-2}}. \quad (91)$$

From (27) with $\nu = i_t - 2, \Delta_2 = \Delta_{t-1}$ we see that

$$\xi_{i_t-2} \leq \frac{\Delta_{t-1}}{q_{i_t-1}} \leq \frac{\Delta_{t-1}}{q_{i_t-2}}. \quad (92)$$

Points $0, z_{i_t-2}, z_{i_t-1}, z_{i_t}$ form a parallelogram and so

$$\xi_{i_t-1} = \xi_{i_t-2} - \xi_{i_t} < \xi_{i_t-2}.$$

Now (91,92) give us

$$\frac{\xi_{i_t}}{\xi_{i_t-1}} \geq \frac{\xi_{i_t}}{\xi_{i_t-2}} \geq \frac{1}{16\sqrt{6}(50\gamma^2 + 2)},$$

and this is what we need.

2) For $\nu$ from the interval $i_t < \nu \leq i_{t+1} - 3$ from (48) of Lemma 4 follows

$$\frac{\xi_{\nu}}{\xi_{\nu-1}} \geq \frac{1}{4}.$$

3) Let $x_0$ be the point form Lemma 5 applied on $(t + 1)$-th step. In the notation of Lemma 5 we have $W_0 = A_{i_{t+1}-2}, W_1 = A_{i_{t+1}}, p_0 = q_{i_{t+1}-2}, p_1 = q_{i_{t+1}}, \Delta = \Delta_t, \Delta_1 = \Delta_{t+1}$. Then

$$\xi_{i_{t+1}-2} \geq q_{i_{t+1}-2} |A_{i_{t+1}-2} - x_0|_\infty - q_{i_{t+1}-2} |x - x_0|_\infty. \quad (93)$$

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But from the construction (66) we have
\[ q_{i_{t+1}-2} |A_{i_{t+1} - x_0}|_\infty = p_0 |W_0 - x_0|_\infty \geq \frac{\Delta_t}{2 \gamma q_{i_{t+1} - 2}}. \quad (94) \]
Then,
\[ q_{i_{t+1}-2} |x - x_0|_\infty \leq q_{i_{t+1}-2} |A_{i_{t+1}} - x_0|_\infty + p_0 |x - A_{i_{t+1}}|_\infty \leq \frac{17 \Delta_t}{\gamma^2 q_{i_{t+1} - 2}} \quad (95) \]
(we use inequalities (75) for the first summand and \((t + 1)\)-th step of (80), Remark 5 for the second summand). Now (93,94,95) gives
\[ \xi_{i_{t+1} - 2} \geq \frac{\Delta_t}{4 \gamma q_{i_{t+1} - 2}}. \]
Together with (27) for \(\nu = i_{t+1} - 3\) this gives
\[ \frac{\xi_{i_{t+1} - 2}}{\xi_{i_{t+1} - 3}} \geq \frac{1}{4\gamma}. \]

4) As in the case 1) the points \(0, z_{i_{t+1} - 2}, z_{i_{t+1} - 1}, z_{i_{t+1}}\) form a parallelogram and so
\[ \xi_{i_{t+1} - 1} = \xi_{i_{t+1} - 2} - \xi_{i_{t+1}}. \]
As \(\xi_{i_{t+1}}\) is much smaller than \(\xi_{i_{t+1} - 2}\) we immediately have
\[ \frac{\xi_{i_{t+1} - 1}}{\xi_{i_{t+1} - 2}} = 1 - \frac{\xi_{i_{t+1}}}{\xi_{i_{t+1} - 2}} \geq \frac{1}{2}. \]

We see that condition (D) is valid in the range \(i_t < \nu \leq i_{t+1}\).

Now we have constructed the vectors (77) satisfying the conditions (A), (B), (C), (D) for every \(t\) and Theorem 2 follows. \(\square\).

**Acknowledgements.**

The authors thank the anonymous referee for careful reading of the manuscript and for important suggestions.

The second named is a winner of the “Leader” contest conducted by Theoretical Physics and Mathematics Advancement Foundation “BASIS” and would like to thank the foundation and jury.

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