A new explicit expansion approach to Mersenne primes

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Abstract

This paper first proves what the author called the Eight Levels Theorem and then highlights a new explicit expansion approach to Lucas-Lehmer primality test for Mersenne primes and gives a new criterion for Mersenne compositeness. Also, we prove four new combinatorial identities.

1 Notations

For a natural number $n$, we define $\delta(n) = n \pmod 2$. For an arbitrary real number $x$, $\lfloor \frac{x}{2} \rfloor$ is the highest integer less than or equal to $\frac{x}{2}$. We also need the following notation

$$
\prod_{\lambda=0}^{-1} n^2 - (4\lambda)^2 := 1
$$

2 Introduction

Primes of special form have been of perennial interest, [5]. Among these, the primes of the form

$$
2^p - 1
$$

which are called Mersenne prime. It is outstanding in their simplicity.

- Clearly, if $2^p - 1$ is prime then $p$ is prime. The number $2^p - 1$ is Mersenne composite if $p$ is prime but $2^p - 1$ is not prime.

- Mathematics is kept alive by the appearance of challenging unsolved problems. The current paper gives new expansions to the following two major open questions in number theory (see [1], [3], [4], [5], [7]): Are there infinitely many Mersenne primes? Are there infinitely many Mersenne composite?

- Mersenne primes have a close connection to perfect numbers, which are numbers that are equal to the sum of their proper divisors. It is known that Euclid and Euler proved that a number $N$ is even perfect number if and only if $N = 2^{p-1}(2^p - 1)$ for some prime $p$, and $2^p - 1$ is prime. Euclid proved only that this statement was sufficient. Euler,
2000 years later, proved that all even perfect numbers are of the form \(2^{p-1}(2^p - 1)\) where \(2^p - 1\) is a Mersenne prime \(M_p\) (see [4, 10]). Thus the theorems of Euclid and Euler characterize all even perfect numbers, reducing their existence to that of Mersenne primes.

- The odd perfect numbers are quite a different story (see [10, 8, 2]). It is unknown whether there is any odd perfect number. Recently, [8] showed that odd perfect numbers, if exist, are greater than \(10^{1500}\).

- It is well-known the following two theorems (see [4], [1], [7]):

**Theorem 1. (Lucas-Lehmer)**

\(2^p - 1\) is Mersenne prime **if and only if**

\[
2n - 1 \mid (1 + \sqrt{3})^n + (1 - \sqrt{3})^n
\]

where \(n := 2^{p-1}\).

**Theorem 2. (Euclid-Euler-Lucas-Lehmer)**

A number \(N\) is even perfect number **if and only if** \(N = 2^{p-1}(2^p - 1)\) for some prime \(p\), and

\[
2n - 1 \mid (1 + \sqrt{3})^n + (1 - \sqrt{3})^n
\]

where \(n := 2^{p-1}\).

### 2.1 The purpose of the paper

The aim of this paper is to study some arithmetical properties of the coefficients of the expansion

\[
x^n + y^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n)(xy)^{\lfloor \frac{n}{2} \rfloor - k}(x^2 + y^2)^k
\]

for \(n \equiv 0, 2, 4, 6 \pmod{8}\). Then we study the expansion

\[
\frac{x^n + y^n}{x + y} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n)(xy)^{\lfloor \frac{n}{2} \rfloor - k}(x^2 + y^2)^k
\]

for \(n \equiv 1, 3, 5, 7 \pmod{8}\). Then we show that the numbers of the form

\[
\prod_{\lambda} n^2 - (4\lambda)^2
\]

arise up naturally in the coefficients of the expansions (3), (4), and enjoy some arithmetical properties. We then be able to find three new expansions (three different versions) for Theorem (1), and a new expansion for Theorem (2), which should give a better theoretical understanding for two major questions about integers; whether there are infinitely many Mersenne primes or Mersenne composites. We also prove four new combinatorial identities about the nature of integers.
3 Summary for the main results of the paper

This paper proves all of the following new theorems for the primality of Mersenne numbers, even perfect numbers, proves a criterion for compositeness of Mersenne numbers, and gives four combinatorial identities related to the nature of numbers:

**Theorem 3. (Lucas-Lehmer-Moustafa)(Version 1)**

Given prime \( p \geq 5 \). \( 2^p - 1 \) is prime if and only if

\[
2n - 1 \mid \sum_{\substack{k=0, \\ k \text{ even}}}^{|n/2|} \phi_k(n) \tag{6}
\]

where \( n := 2^p - 1 \), \( \phi_k(n) \) are defined by the double index recurrence relation

\( \phi_k(m) = 4 \phi_{k-1}(m - 2) - \phi_k(m - 4) \)

and the initial boundary values satisfy

\[
\phi_0(m) = \begin{cases} 
+2 & m \equiv 0 \pmod{8} \\
0 & m \equiv \pm 2 \pmod{8} \\
-2 & m \equiv 4 \pmod{8}
\end{cases}, \quad \phi_1(m) = \begin{cases} 
+2 & m \equiv 2 \pmod{8} \\
0 & m \equiv 0, 4 \pmod{8} \\
-2 & m \equiv 6 \pmod{8}
\end{cases}. \tag{7}
\]

**Theorem 4. (Lucas-Lehmer-Moustafa)(Version 2)**

Given prime \( p \geq 5 \), \( n := 2^p - 1 \). The number \( 2^p - 1 \) is prime if and only if

\[
2n - 1 \mid \sum_{\substack{k=0, \\ k \text{ even}}}^{|n/2|} (-1)^{[n/2]} \prod_{\lambda=0}^{[n/2]-1} \frac{n^2 - (4\lambda)^2}{k!}. \tag{8}
\]

**Theorem 5. (Lucas-Lehmer-Moustafa)(Version 3)**

Given prime \( p \geq 5 \). The number \( 2^p - 1 \) is prime if and only if

\[
2n - 1 \mid \sum_{\substack{k=0, \\ k \text{ even}}}^{|n/2|} \phi_k(n) \tag{9}
\]

where \( n := 2^p - 1 \), \( \phi_k(n) \) are generated by the double index recurrence relation

\[
\frac{\phi_k(n)}{\phi_{k-2}(n)} = - \frac{n^2 - (2k - 4)^2}{k (k - 1)},
\]

and we can choose either of the following initial values to generate \( \phi_k(n) \) from the starting term \( \phi_0(n) \) or the last term \( \phi_{[n/2]}(n) \) :

\[
\phi_0(n) = +2 \quad \text{and} \quad \phi_{[n/2]}(n) = 4^{[n/2]} \tag{10}
\]
Theorem 6. (Euclid-Euler-Lucas-Lehmer-Moustafa)

A number \(N\) is even perfect number if and only if \(N = 2^{p-1}(2^p - 1)\) for some prime \(p\), and

\[
2n - 1 \mid \sum_{k=0, \text{k even}}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{k}{2} \rfloor} \prod_{\lambda=0}^{\lfloor \frac{n}{2} \rfloor} n^2 - (4\lambda)^2 \tag{11}
\]

where \(n := 2^{p-1}\).

Theorem 7. (Criteria for compositeness of Mersenne numbers)

Given prime \(p \geq 5\). The number \(2n - 1 = 2^p - 1\) is Mersenne composite number if

\[
2n - 1 \nmid \sum_{k=0, \text{k even}}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{k}{2} \rfloor} \prod_{\lambda=0}^{\lfloor \frac{n}{2} \rfloor} [(4\lambda)^2 - 4^{-1}] \tag{12}
\]

Theorem 8. (Four new combinatorial identities)

For any natural number \(n\), the following combinatorial identities are correct

\[
\begin{align*}
&n \equiv 0 \pmod{4} \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = 2 \prod_{\lambda=0}^{\lfloor \frac{n}{4} \rfloor} n^2 - (4\lambda)^2 \\
&n \equiv 1 \pmod{4} \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = \prod_{\lambda=1}^{\lfloor \frac{n-1}{2} \rfloor} (n+1)^2 - (4\lambda - 2)^2 \\
&n \equiv 2 \pmod{4} \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = 2n \prod_{\lambda=1}^{\lfloor \frac{n-2}{2} \rfloor} n^2 - (4\lambda - 2)^2 \\
&n \equiv 3 \pmod{4} \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = (n+1) \prod_{\lambda=1}^{\lfloor \frac{n-3}{2} \rfloor} (n+1)^2 - (4\lambda)^2
\end{align*}
\]

\[4\text{ The eight levels theorem}\]

Now we prove what we call the Eight Levels Theorem for the expansion of the polynomial

\[
\frac{x^n + y^n}{(x + y)^{3(n)}}
\]

in terms of the symmetric polynomials \(xy\) and \(x^2 + y^2\). Then we investigate some properties for the coefficients of that expansions, and give some applications and prove my results in the summary.
4.1 The statement of the eight levels theorem

**Theorem 9. (The Eight Levels Theorem)**

For any complex numbers \( x, y \), any non-negative integers \( n, k \), the coefficients \( \Psi_k(n) \) of the expansion

\[
\frac{x^n + y^n}{(x + y)^{\delta(n)}} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n)(xy)^{\lfloor \frac{n}{2} \rfloor - k}(x^2 + y^2)^k
\]

are integers and

\[
\Psi_0(n) = \begin{cases} 
+2 & n \equiv 0 \pmod{8} \\
+1 & n \equiv \pm 1 \pmod{8} \\
0 & n \equiv \pm 2 \pmod{8} \\
-1 & n \equiv \pm 3 \pmod{8} \\
-2 & n \equiv \pm 4 \pmod{8}
\end{cases}
\]

and, for each \( 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \), the coefficients satisfy the following statements

- For \( n \equiv 0, 2, 4, 6 \) (mod 8):

  \[n \equiv 0 \pmod{8}\]

  \[
  \Psi_k(n) = \begin{cases} 
0 & \text{for } k \text{ odd} \\
2 (-1)^{\lfloor \frac{k}{2} \rfloor} \frac{\prod_{\lambda=0}^{\lfloor \frac{k}{2} \rfloor - 1} n^2 - (4\lambda)^2}{4^k k!} & \text{for } k \text{ even}
\end{cases}
\]

  \[n \equiv 2 \pmod{8}\]

  \[
  \Psi_k(n) = \begin{cases} 
0 & \text{for } k \text{ even} \\
2 (-1)^{\lfloor \frac{k}{2} \rfloor} n \frac{\prod_{\lambda=1}^{\lfloor \frac{k}{2} \rfloor} n^2 - (4\lambda - 2)^2}{4^k k!} & \text{for } k \text{ odd}
\end{cases}
\]

  \[n \equiv 4 \pmod{8}\]

  \[
  \Psi_k(n) = \begin{cases} 
0 & \text{for } k \text{ odd} \\
2 (-1)^{\lfloor \frac{k}{2} \rfloor + 1} \frac{\prod_{\lambda=0}^{\lfloor \frac{k}{2} \rfloor} n^2 - (4\lambda)^2}{4^k k!} & \text{for } k \text{ even}
\end{cases}
\]

  \[n \equiv 6 \pmod{8}\]

  \[
  \Psi_k(n) = \begin{cases} 
0 & \text{for } k \text{ even} \\
2 (-1)^{\lfloor \frac{k}{2} \rfloor + 1} n \frac{\prod_{\lambda=1}^{\lfloor \frac{k}{2} \rfloor} n^2 - (4\lambda - 2)^2}{4^k k!} & \text{for } k \text{ odd}
\end{cases}
\]
• For \( n \equiv 1, 3, 5, 7 \pmod{8} \):

\[
\begin{align*}
\Psi_k(n) & = (-1)^\left\lfloor \frac{n}{2} \right\rfloor (n+1)^2 - (4\lambda - 2)^2 \\
\prod_{\lambda=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (n+1)^2 & - (4\lambda - 2)^2 \\
\frac{4^k k!}{4^k k!} & \\
\end{align*}
\]

\[
\begin{align*}
\Psi_k(n) & = (-1)^\left\lfloor \frac{n}{2} \right\rfloor (n+1)(n+1-2k)^{\delta(k)} \\
\prod_{\lambda=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor} (n+1)^2 & - (4\lambda - 2)^2 \\
\frac{4^k k!}{4^k k!} & \\
\end{align*}
\]

\[
\begin{align*}
\Psi_k(n) & = (-1)^\left\lfloor \frac{n}{2} \right\rfloor + 1 (n+1)(n+1-2k)^{\delta(k)} \\
\prod_{\lambda=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor} (n+1)^2 & - (4\lambda - 2)^2 \\
\frac{4^k k!}{4^k k!} & \\
\end{align*}
\]

\[
\begin{align*}
\Psi_k(n) & = (-1)^\left\lfloor \frac{n}{2} \right\rfloor + \delta(k) (n+1)(n+1-2k)^{\delta(k-1)} \\
\prod_{\lambda=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor} (n+1)^2 & - (4\lambda - 2)^2 \\
\frac{4^k k!}{4^k k!} & \\
\end{align*}
\]

**Proof.** Put \((x, y) = (1, \sqrt{-1})\) in (13), we immediately get (14). To prove the statements of Theorem (9), we need to prove the following lemmas.

**Lemma 10.** For each natural number \( n, 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \), the coefficients \( \Psi_k(n) \) of the expansion of (13) are integers, unique and satisfy

\[
\Psi_k(n) = \Psi_{k-1}(n-2) - \Psi_k(n-4).
\]

**Proof.** From the Fundamental Theorem on Symmetric Polynomials, [9, 6], we have a sequence of integers \( \Psi_k(n) \) satisfy the expansion of (13). From the algebraic independence of \( xy, x^2 + y^2 \), the coefficients \( \Psi_k(n) \) of the expansion of (13) are unique. Now, multiply (13) by \( xy(x^2 + y^2) \), and noting \((x + y)^{\delta(n+2)} = (x + y)^{\delta(n)} = (x + y)^{\delta(n-2)} \) and \( \left\lfloor \frac{n+2}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and \( \left\lfloor \frac{n-2}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor - 1 \) we get

\[
\begin{align*}
\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor + 1} \Psi_{k-1}(n)(xy)\left(\frac{n}{2}\right)^{2-k}(x^2 + y^2)^k &= \\
\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor + 1} \Psi_k(n+2)(xy)\left(\frac{n}{2}\right)^{2-k}(x^2 + y^2)^k + \sum_{k=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor - 1} \Psi_k(n-2)(xy)\left(\frac{n}{2}\right)^{2-k}(x^2 + y^2)^k
\end{align*}
\]

Again from the algebraic independence of \( xy, x^2 + y^2 \), and from (16), we get the following identity for any natural number \( n \)
\(\Psi_k(n + 2) = \Psi_{k-1}(n) - \Psi_k(n - 2).\) \hspace{1cm} (17)

Replace \(n\) by \(n - 2\) in (17), we get (15).

**Lemma 11.** For every even natural number \(n\), \(0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor\), the following statements are true for each case:

\[
\begin{align*}
\text{If } n &\equiv 0 \text{ or } 4 \pmod{8} \quad \text{then } \Psi_k(n) = 0 \quad \text{for } k \text{ odd}, \\
\text{if } n &\equiv 2 \text{ or } 6 \pmod{8} \quad \text{then } \Psi_k(n) = 0 \quad \text{for } k \text{ even}. 
\end{align*}
\] \hspace{1cm} (18)

**Proof.** Consider \(n\) even natural number, and replace \(\delta(n) = 0\) in (13) to get

\[
x^n + y^n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi_k(n)(xy)^{\left\lfloor \frac{n}{2} \right\rfloor - k}(x^2 + y^2)^k
\] \hspace{1cm} (19)

Then replace \(x\) by \(-x\) in (19), we get

\[
x^n + y^n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \Psi_k(n)(-xy)^{\left\lfloor \frac{n}{2} \right\rfloor - k}(x^2 + y^2)^k
\] \hspace{1cm} (20)

From the algebraic independence of \(xy, x^2 + y^2\), and from (19), (20) we get the proof. \(\square\)

From (13), we get the following initial information about \(\Psi_k(n)\).

**Lemma 12.**

- \(\Psi_0(0) = +2, \quad \Psi_1(0) = 0, \quad \Psi_2(0) = 0, \quad \Psi_3(0) = 0,\)
- \(\Psi_0(2) = 0, \quad \Psi_1(2) = +1, \quad \Psi_2(2) = 0, \quad \Psi_3(2) = 0,\)
- \(\Psi_0(4) = -2, \quad \Psi_1(4) = 0, \quad \Psi_2(4) = +1, \quad \Psi_3(4) = 0,\)
- \(\Psi_0(6) = 0, \quad \Psi_1(6) = -3, \quad \Psi_2(6) = 0, \quad \Psi_3(6) = +1.\)

**Lemma 13.** For any odd natural number \(n\), the coefficients \(\Psi_k(n)\) of the expansion of (13) satisfy the following property

\[
\Psi_k(n) = \frac{2(k + 1)}{(n + 1)} \Psi_{k+1}(n + 1) + \frac{\left\lfloor \frac{n+1}{2} \right\rfloor - k}{(n + 1)} \Psi_k(n + 1)
\] \hspace{1cm} (21)

**Proof.** Consider \(n\) odd, then \(n + 1\) even. Then from the expansion of (13) we get the following

\[
x^{n+1} + y^{n+1} = \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \Psi_k(n + 1)(xy)^{\left\lfloor \frac{n+1}{2} \right\rfloor - k}(x^2 + y^2)^k
\] \hspace{1cm} (22)
Now acting the differential operator \( \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \) on (22) and noting that

\[
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) (x^{n+1} + y^{n+1}) = (n + 1)(x^n + y^n),
\]

\[
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) xy = x + y,
\]

\[
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) (x^2 + y^2) = 2(x + y),
\]

and equating the coefficients, we get the proof.

4.2 The proof of Theorem (9) for \( n \equiv 0, 2, 4, 6 \pmod{8} \)

Now in this section we prove Theorem (9) for \( n \) even. The values of \( \Psi_k(n) \) that comes from the formulas of Theorem (9) for \( n = 0, 2, 4, 8 \) are identical with the correct values that come from Lemma (12). So, Theorem (9) is correct for the initial values \( n = 0, 2, 4, 8 \). Now we assume the validity of Theorem (9) for each \( 0 \leq m < n \), with \( m \equiv 0, 2, 4, 6 \pmod{8} \) and need to prove that Theorem (9) for \( n \). Lemma (11) proves the validity of Theorem (9) for \( n \equiv 0 \) or \( 4 \pmod{8} \) if \( k \) odd, and for \( n \equiv 2 \) or \( 6 \pmod{8} \) if \( k \) even. Therefore it remains to prove the validity of the following cases:

- \( n \equiv 0 \pmod{8}, k \text{ even} \)
- \( n \equiv 2 \pmod{8}, k \text{ odd} \)
- \( n \equiv 4 \pmod{8}, k \text{ even} \)
- \( n \equiv 6 \pmod{8}, k \text{ odd} \)

We prove these cases one by one as following.

Consider \( n \equiv 0 \pmod{8} \) and \( k \text{ even} \)

\text{Proof.} \ In \ this \ case, \( n - 2 \equiv 6 \pmod{8} \) and \( n - 4 \equiv 4 \pmod{8} \) and from Lemma (10), we
get
\[
\Psi_k(n) = \Psi_{k-1}(n-2) - \Psi_k(n-4)
\]
\[= 2 \left(-1\right)^{\frac{k+1}{2}+1} (n-2) \frac{\prod_{\lambda=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor} (n-2)^{2-(4\lambda-2)^2}}{4^{k-1} (k-1)!} - 2 \left(-1\right)^{\frac{k+1}{2}+1} \frac{\prod_{\lambda=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} (n-4)^{2-(4\lambda)^2}}{4^k k!}
\]
\[= 2 \left(-1\right)^{\frac{k+1}{2}} (n-2) \frac{\prod_{\lambda=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} (n-4\lambda+4)}{4^{k-1} (k-1)!} + 2 \left(-1\right)^{\frac{k+1}{2}} \frac{\prod_{\lambda=0}^{\left\lfloor \frac{k+1}{2} \right\rfloor} (n+4\lambda-4)}{4^k k!}
\]
\[= 2 \left(-1\right)^{\frac{k+1}{2}} \frac{\prod_{\lambda=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} (n-4\lambda) \prod_{\lambda=0}^{\left\lfloor \frac{k+1}{2} \right\rfloor} (n+4\lambda) \left[4k(n-2) + (n-4\left\lfloor \frac{k}{2} \right\rfloor)(n-4)\right]}{4^k k!}
\]
\[= 2 \left(-1\right)^{\frac{k+1}{2}} \frac{\prod_{\lambda=0}^{\left\lfloor \frac{k+1}{2} \right\rfloor} (n+4\lambda) \left(n(n+4) + 4\left\lfloor \frac{k}{2} \right\rfloor - 1\right)}{4^k k!}
\]
\[= 2 \left(-1\right)^{\frac{k+1}{2}} \frac{\prod_{\lambda=0}^{\left\lfloor \frac{k+1}{2} \right\rfloor} n^2 - (4\lambda)^2}{4^k k!}
\]

Consider \(n \equiv 2 \pmod{8}\) and \(k\) odd

Proof. In this case, \(n-2 \equiv 0 \pmod{8}\) and \(n-4 \equiv 6 \pmod{8}\) and from Lemma (10), we get
\[
\Psi_k(n) = \Psi_{k-1}(n-2) - \Psi_k(n-4)
\]
\[= 2 \left(-1\right)^{\frac{k+1}{2}} \frac{\prod_{\lambda=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor} (n-2)^{2-(4\lambda-2)^2}}{4^{k-1} (k-1)!} - 2 \left(-1\right)^{\frac{k+1}{2}} \frac{\prod_{\lambda=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} (n-4)^{2-(4\lambda-2)^2}}{4^k k!}
\]
\[= 2 \left(-1\right)^{\frac{k+1}{2}} \frac{\prod_{\lambda=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} (n-4\lambda-2) \prod_{\lambda=0}^{\left\lfloor \frac{k+1}{2} \right\rfloor} (n+4\lambda-2) \left[4k(n-2) + (n-4)(n-4\left\lfloor \frac{k}{2} \right\rfloor - 2)\right]}{4^k k!}
\]
As
\[
4k(n-2) + (n-4)(n-4\left\lfloor \frac{k}{2} \right\rfloor - 2) = n(n+2k-4),
\]
we get the following
\[
\Psi_k(n) = 2 \left(-1\right)^{\frac{k}{2}} \frac{\prod_{\lambda=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (n-4\lambda+2) \prod_{\lambda=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (n+4\lambda-2)}{4^k k!}
\]
\[= 2 \left(-1\right)^{\frac{k}{2}} \frac{\prod_{\lambda=1}^{\left\lfloor \frac{k}{2} \right\rfloor} n^2 - (4\lambda-2)^2}{4^k k!}
\]

\[\square\]
Consider \( n \equiv 4 \pmod{8} \) and \( k \) even

**Proof.** In this case, \( n - 2 \equiv 2 \pmod{8} \) and \( n - 4 \equiv 0 \pmod{8} \) and from Lemma (10), we get

\[
\Psi_k(n) = \Psi_{k-1}(n-2) - \Psi_k(n-4)
\]

\[
= 2 (-1)^{\lceil \frac{k}{2} \rceil - 1} (n-2) \frac{\prod_{\lambda=0}^{k-2} (n-2)^2 - (4\lambda - 2)^2}{4^{k-1} (k-1)!} - 2 (-1)^{\lceil \frac{k}{2} \rceil} \frac{\prod_{\lambda=0}^{(n-4)^2 - (4\lambda)^2}}{4^k k!}
\]

\[
= 2 (-1)^{\lceil \frac{k}{2} \rceil + 1} (n-2) \frac{\prod_{\lambda=0}^{(n-4)^2 - (4\lambda-4)^2}}{4^{k-1} (k-1)!} + 2 (-1)^{\lceil \frac{k}{2} \rceil + 1} \frac{\prod_{\lambda=0}^{(n-4\lambda-4)^2} \prod_{\lambda=0}^{(n+4\lambda-4)}}{4^k k!}
\]

Hence

\[
\Psi_k(n) = 2 \frac{(-1)^{\lceil \frac{k}{2} \rceil + 1}}{4^k k!} \prod_{\lambda=0}^{\lceil \frac{k}{2} \rceil - 1} (n-4\lambda) \prod_{\lambda=0}^{\lceil \frac{k}{2} \rceil - 2} (n+4\lambda) n (n+ 4(\lceil \frac{k}{2} \rceil - 1))
\]

\[
= 2 \frac{(-1)^{\lceil \frac{k}{2} \rceil + 1}}{4^k k!} \prod_{\lambda=0}^{\lceil \frac{k}{2} \rceil - 1} (n-4\lambda) \prod_{\lambda=0}^{\lceil \frac{k}{2} \rceil - 1} (n+4\lambda) = 2 (-1)^{\lceil \frac{k}{2} \rceil + 1} \frac{\prod_{\lambda=0}^{\lceil \frac{k}{2} \rceil - 1} n^2 - (4\lambda)^2}{4^k k!}
\]

Consider \( n \equiv 6 \pmod{8} \) and \( k \) odd

**Proof.** In this case, \( n - 2 \equiv 4 \pmod{8} \) and \( n - 4 \equiv 2 \pmod{8} \) and from Lemma (10), we get

\[
\Psi_k(n) = \Psi_{k-1}(n-2) - \Psi_k(n-4)
\]

\[
= 2 (-1)^{\lceil \frac{k}{2} \rceil + 1} \frac{\prod_{\lambda=0}^{\lceil \frac{k}{2} \rceil - 1} (n-2)^2 - (4\lambda - 2)^2}{4^{k-1} (k-1)!} - 2 (-1)^{\lceil \frac{k}{2} \rceil} \frac{\prod_{\lambda=0}^{(n-4)^2 - (4\lambda-2)^2}}{4^k k!}
\]

\[
= 2 (-1)^{\lceil \frac{k}{2} \rceil + 1} \frac{\prod_{\lambda=0}^{(n-4\lambda-2)} \prod_{\lambda=0}^{(n+4\lambda-2)}}{4^{k-1} (k-1)!} + 2 (-1)^{\lceil \frac{k}{2} \rceil + 1} \frac{\prod_{\lambda=0}^{(n-4\lambda-2)^2} \prod_{\lambda=0}^{(n+4\lambda-6)}}{4^k k!}
\]

Hence

\[
\Psi_k(n) = 2 \frac{(-1)^{\lceil \frac{k}{2} \rceil + 1}}{4^k k!} n \prod_{\lambda=1}^{\lceil \frac{k}{2} \rceil} (n - 4\lambda + 2) \prod_{\lambda=1}^{\lceil \frac{k}{2} \rceil} (n + 4\lambda - 2)
\]

\[
= 2 n (-1)^{\lceil \frac{k}{2} \rceil + 1} \frac{\prod_{\lambda=1}^{\lceil \frac{k}{2} \rceil} n^2 - (4\lambda - 2)^2}{4^k k!}
\]
4.3 The proof of Theorem (9) for \( n \equiv 1, 3, 5, 7 \pmod{8} \)

With the help of Lemma (13), we rely on Theorem (9) for the even case that we already proved, together with Lemma (11), to prove Theorem (9) for the odd cases \( n \equiv 1, 3, 5, 7 \pmod{8} \), one by one for each parity for \( k \).

Consider \( n \equiv 1 \pmod{8} \) and \( k \) odd

Proof. In this case, \( n + 1 \equiv 2 \pmod{8} \) and from Lemma (11), we get \( \Psi_{k+1}(n + 1) = 0 \). Hence, from Lemma (13), and from Theorem (9), we get the following relation

\[
\Psi_k(n) = \frac{\lfloor \frac{n+1}{2} \rfloor - k}{(n + 1)} \Psi_k(n + 1) = (-1)^{\lfloor \frac{n}{2} \rfloor} (n + 1 - 2k) \frac{\prod_{\lambda=1}^{\lfloor \frac{n}{2} \rfloor} (n + 1)^2 - (4\lambda - 2)^2}{4^k k!}
\]

\( \square \)

Consider \( n \equiv 1 \pmod{8} \) and \( k \) even

Proof. In this case, \( n + 1 \equiv 2 \pmod{8} \) and from Lemma (11), we get \( \Psi_k(n + 1) = 0 \). From Lemma (13), and from Theorem (9), we get the following relation

\[
\Psi_k(n) = 2 \frac{k + 1}{(n + 1)} \Psi_{k+1}(n + 1) = (-1)^{\lfloor \frac{n}{2} \rfloor} \frac{\prod_{\lambda=1}^{\lfloor \frac{n}{2} \rfloor} (n + 1)^2 - (4\lambda - 2)^2}{4^k k!}
\]

\( \square \)

Consider \( n \equiv 3 \pmod{8} \) and \( k \) odd

Proof. In this case, from Lemma (11), we get \( \Psi_k(n + 1) = 0 \). Hence, from Lemma (13), and from Theorem (9), we get the following relation

\[
\Psi_k(n) = 2 \frac{k + 1}{(n + 1)} \Psi_{k+1}(n + 1) = (-1)^{\lfloor \frac{n}{2} \rfloor} (n + 1) \frac{\prod_{\lambda=1}^{\lfloor \frac{n}{2} \rfloor} (n + 1)^2 - (4\lambda)^2}{4^k k!}
\]

\( \square \)

Consider \( n \equiv 3 \pmod{8} \) and \( k \) even

Proof. In this case, \( \Psi_{k+1}(n + 1) = 0 \). From Lemma (13), and from Theorem (9), we get the following relation

\[
\Psi_k(n) = \frac{\lfloor \frac{n+1}{2} \rfloor - k}{(n + 1)} \Psi_k(n + 1) = (-1)^{\lfloor \frac{n}{2} \rfloor + 1} (n + 1)(n + 1 - 2k) \frac{\prod_{\lambda=1}^{\lfloor \frac{n}{2} \rfloor - 1} (n + 1)^2 - (4\lambda)^2}{4^k k!}
\]

\( \square \)
Consider \( n \equiv 5 \pmod{8} \) and \( k \) odd

Proof. In this case, from Lemma (11), we get \( \Psi_{k+1}(n+1) = 0 \). Hence, from Lemma (13), and from Theorem (9), we get the following relation

\[
\Psi_k(n) = \frac{\left(\begin{array}{c} n+1 \\ 2 \end{array}\right) - k}{n+1} \Psi_k(n+1) = (-1)^{\frac{k}{2}+1} (n+1 - 2k) \frac{\prod_{\lambda=1}^{\frac{k}{2}} (n+1)^2 - (4\lambda - 2)^2}{4^k k!}
\]

Consider \( n \equiv 5 \pmod{8} \) and \( k \) even

Proof. In this case, \( \Psi_k(n+1) = 0 \). From Lemma (13), and from Theorem (9), we get the following relation

\[
\Psi_k(n) = 2 \left(\begin{array}{c} k+1 \\ 2 \end{array}\right) \Psi_{k+1}(n+1) = (-1)^{\frac{k}{2}+1} \frac{\prod_{\lambda=1}^{\frac{k}{2}} (n+1)^2 - (4\lambda - 2)^2}{4^k k!}
\]

Consider \( n \equiv 7 \pmod{8} \) and \( k \) odd

Proof. In this case, from Lemma (11), we get \( \Psi_k(n+1) = 0 \). Hence, from Lemma (13), and from Theorem (9), we get the following relation

\[
\Psi_k(n) = 2 \left(\begin{array}{c} k+1 \\ 2 \end{array}\right) \Psi_{k+1}(n+1) = (-1)^{\frac{k}{2}+1} (n+1 - 2k) \frac{\prod_{\lambda=1}^{\frac{k}{2}-1} (n+1)^2 - (4\lambda - 2)^2}{4^k k!}
\]

Consider \( n \equiv 7 \pmod{8} \) and \( k \) even

Proof. In this case, \( \Psi_{k+1}(n+1) = 0 \). From Lemma (13), and from Theorem (9), we get the following relation

\[
\Psi_k(n) = \frac{\left(\begin{array}{c} n+1 \\ 2 \end{array}\right) - k}{n+1} \Psi_k(n+1) = (-1)^{\frac{k}{2}+1} (n+1)(n+1 - 2k) \frac{\prod_{\lambda=1}^{\frac{k}{2}-1} (n+1)^2 - (4\lambda - 2)^2}{4^k k!}
\]

This completes the proof of Theorem (9).
5 General characteristics for $\Psi$–sequence

5.1 Examples for $\Psi$–Sequence.

From Theorem (9), we list some examples that show the splendor of the natural factorization of $\Psi_k(n)$ for $k = 0, 1, 2, 3, 4, 5, 6, 7$:

$$\Psi_0(n) = \begin{cases} +2 & n \equiv 0 \pmod{8} \\ +1 & n \equiv 1 \pmod{8} \\ 0 & n \equiv 2 \pmod{8} \\ -1 & n \equiv 3 \pmod{8} \\ -2 & n \equiv 4 \pmod{8} \\ -1 & n \equiv 5 \pmod{8} \\ 0 & n \equiv 6 \pmod{8} \\ +1 & n \equiv 7 \pmod{8} \end{cases}$$

$$\Psi_1(n) = \begin{cases} 0 & n \equiv 0 \pmod{8} \\ +\frac{(n-1)}{4} & n \equiv 1 \pmod{8} \\ +\frac{n}{4} & n \equiv 2 \pmod{8} \\ +\frac{(n+1)}{4} & n \equiv 3 \pmod{8} \\ 0 & n \equiv 4 \pmod{8} \\ -\frac{(n-1)}{4} & n \equiv 5 \pmod{8} \\ -\frac{n}{4} & n \equiv 6 \pmod{8} \\ -\frac{(n+1)}{4} & n \equiv 7 \pmod{8} \end{cases}$$

$$\Psi_2(n) = \begin{cases} -2 \frac{n^2}{2^2} & n \equiv 0 \pmod{8} \\ -\frac{(n-1)(n+3)}{4^2 2^2} & n \equiv 1 \pmod{8} \\ 0 & n \equiv 2 \pmod{8} \\ +\frac{(n-3)(n+1)}{4^2 2^2} & n \equiv 3 \pmod{8} \\ +\frac{n^2}{2^2} & n \equiv 4 \pmod{8} \\ +\frac{(n-1)(n+3)}{4^2 2^2} & n \equiv 5 \pmod{8} \\ 0 & n \equiv 6 \pmod{8} \\ -\frac{(n-3)(n+1)}{4^2 2^2} & n \equiv 7 \pmod{8} \end{cases}$$

$$\Psi_3(n) = \begin{cases} 0 & n \equiv 0 \pmod{8} \\ -\frac{(n-5)(n-1)(n+3)}{4^3 3!} & n \equiv 1 \pmod{8} \\ -2 \frac{(n-2) n (n+2)}{4^3 3!} & n \equiv 2 \pmod{8} \\ -\frac{(n-3)(n+1)(n+5)}{4^3 3!} & n \equiv 3 \pmod{8} \\ 0 & n \equiv 4 \pmod{8} \\ +\frac{(n-5)(n-1)(n+3)}{4^3 3!} & n \equiv 5 \pmod{8} \\ +2 \frac{(n-2) n (n+2)}{4^3 3!} & n \equiv 6 \pmod{8} \\ +\frac{(n-3)(n+1)(n+5)}{4^3 3!} & n \equiv 7 \pmod{8} \end{cases}$$
\[ \Psi_4(n) = \begin{cases} 
+2 \frac{(n-4) n^2 (n+4)}{4^4 4!} & n \equiv 0 \pmod{8} \\
+ (n-5)(n-1)(n+3)(n+7) & n \equiv 1 \pmod{8} \\
0 & n \equiv 2 \pmod{8} \\
- \frac{(n-7)(n-3)(n+1)(n+5)}{4^4 4!} & n \equiv 3 \pmod{8} \\
-2 \frac{(n-4) n^2 (n+4)}{4^4 4!} & n \equiv 4 \pmod{8} \\
- \frac{(n-5)(n-1)(n+3)(n+7)}{4^4 4!} & n \equiv 5 \pmod{8} \\
0 & n \equiv 6 \pmod{8} \\
+ \frac{(n-7)(n-3)(n+1)(n+5)}{4^4 4!} & n \equiv 7 \pmod{8} 
\end{cases} \tag{25} \]

and

\[ \Psi_5(n) = \begin{cases} 
0 & n \equiv 0 \pmod{8} \\
+ \frac{(n-9)(n-5)(n-1)(n+3)(n+7)}{4^5 5!} & n \equiv 1 \pmod{8} \\
+2 \frac{(n-6)(n-2) n (n+2)(n+6)}{4^5 5!} & n \equiv 2 \pmod{8} \\
+ \frac{(n-7)(n-3)(n+1)(n+5)(n+9)}{4^5 5!} & n \equiv 3 \pmod{8} \\
0 & n \equiv 4 \pmod{8} \\
- \frac{(n-9)(n-5)(n-1)(n+3)(n+7)}{4^5 5!} & n \equiv 5 \pmod{8} \\
-2 \frac{(n-6)(n-2) n (n+2)(n+6)}{4^5 5!} & n \equiv 6 \pmod{8} \\
- \frac{(n-7)(n-3)(n+1)(n+5)(n+9)}{4^5 5!} & n \equiv 7 \pmod{8} 
\end{cases} \tag{26} \]
For \( n \) part first then go around the center factor to fill the left part and so on as explained in the.

\( 5.2 \) Right tendency for \( \Psi_k(n) \)

For \( n \equiv \pm 1 \pmod{8} \), and from the data above, and from the formulas of \( \Psi_k(n) \), we can observe that the factors of the numerators for \( k = 1, 2, 3, 4, 5, 6, \ldots \) always fill the right part first then go around the center factor to fill the left part and so on as explained in the.
5.3 Left tendency for $\Psi_k(n)$

However, for $n \equiv \pm 3 \pmod{8}$, and from the data above, and the formulas of $\Psi_k(n)$, we can also observe that the factors of the numerators for $k = 1, 2, 3, 4, 5, 6, \cdots$ always fill the left part first then go around the center factor to fill the right part and so on as explained in the following pattern and table

| $k$  | The Center                |
|------|---------------------------|
| 1    | $(n - 1)$                 |
| 2    | $(n - 1) \quad (n + 3)$  |

Right Tendency For $\Psi_k(n)$

| $k$  | $n$  | $n$  | $n$  |
|------|------|------|------|
| 3    | $(n - 5)$ | $(n - 1)$ | $(n + 3)$ |
| 4    | $(n - 5)$ | $(n - 1)$ | $(n + 3)$ | $(n + 7)$ |
| 5    | $(n - 9)$ | $(n - 5)$ | $(n - 1)$ | $(n + 3)$ | $(n + 7)$ |
| 6    | $(n - 9)$ | $(n - 5)$ | $(n - 1)$ | $(n + 3)$ | $(n + 7)$ | $(n + 11)$ |

(29)

5.4 The signs phenomena of $\Psi-$integers

For $k \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{8}$, we get the following data

| $k \pmod{8}$ | $-1 | \left[ \frac{k}{2} \right] + 1$ | $-1 | \left[ \frac{k}{2} \right] + \delta(k)$ | $-1 | \left[ \frac{k}{2} \right] + \delta(k - 1)$ |
|--------------|-----------------|-----------------|-----------------|
| 0            | +1              | +1              | +1              |
| 1            | +1              | -1              | +1              |
| 2            | -1              | +1              | -1              |
| 3            | -1              | +1              | +1              |
| 4            | +1              | -1              | -1              |
| 5            | +1              | -1              | +1              |
| 6            | -1              | +1              | +1              |
| 7            | -1              | +1              | -1              |

(30)

Therefore, the signs of $\Psi-$sequence get periodically every 8 steps. Table (32) shows that three plus are followed by zero then followed by three minus then followed by zero and so
on. For $n \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{8}$ and for $k \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{8}$ the following data is useful

| $n \pmod{8}$ | $\Psi_0(n)$ | $\Psi_1(n)$ | $\Psi_2(n)$ | $\Psi_3(n)$ | $\Psi_4(n)$ | $\Psi_5(n)$ | $\Psi_6(n)$ | $\Psi_7(n)$ |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0           | +           | 0           | -           | 0           | +           | 0           | -           | 0           |
| 1           | +           | +           | -           | -           | +           | +           | -           | -           |
| 2           | 0           | +           | 0           | -           | 0           | +           | 0           | -           |
| 3           | -1          | +           | +           | -           | -           | +           | +           | -           |
| 4           | -10         | 0           | +           | 0           | -           | 0           | +           | 0           |
| 5           | -10         | -           | +           | +           | -           | -           | +           | +           |
| 6           | 0           | 0           | -           | 0           | +           | -           | 0           | +           |
| 7           | +           | -           | -           | -           | +           | +           | -           | -           |

5.5 General property for the $\Psi$–integers

Whether $n \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{8}$, it is rather surprising that the quantity

$$\Psi_k(n) \div \Psi_{k-2}(n)$$

always gives

$$-\frac{n^2 - (2k - 4)^2}{16k(k-1)} \quad \text{or} \quad -\frac{(n \pm 1)^2 - (2k - 2)^2}{16k(k-1)} \quad \text{or} \quad 0 \div 0.$$ 

Therefore, computing these ratios, at the eight levels $n \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{8}$, immediately proves the following desirable theorem that gives a fraction with a difference of two squares divided by a multiplication of the factors 16, $k$, and $k - 1$.

**Theorem 14. (Generating The $\Psi$–integers From The Previous Term)**

If $\Psi_k(n)$ not identically zero, then

$$\frac{\Psi_k(n)}{\Psi_{k-2}(n)} = \frac{1}{16k(k-1)} \left[ n + (-1)^{\frac{n}{8}}k\delta(n) \right]^2 - \frac{1}{16k(k-1)} \left[ 2k - 2 - 2\delta(n-1) \right]^2,$$

and we can choose either of the following initial values to generate $\Psi_k(n)$ from the starting term $\Psi_0(n)$ or the last term $\Psi_{\frac{n}{8}}(n)$:

$$\Psi_0(n) = \begin{cases} 
+2 & n \equiv 0 \pmod{8} \\
+1 & n \equiv \pm 1 \pmod{8} \\
0 & n \equiv \pm 2 \pmod{8} \\
-1 & n \equiv \pm 3 \pmod{8} \\
-2 & n \equiv \pm 4 \pmod{8} 
\end{cases}, \quad \Psi_{\frac{n}{8}}(n) = 1.$$
6 The emergence of $\phi$–sequence

Another natural sequence that emerges naturally from $\Psi_k(n)$ is the integer sequence $\phi_k(n)$ which is defined as follows

Definition 15.  
$$\phi_k(n) := 4^k \Psi_k(n)$$

6.1 Recurrence relation of order 4 to generate $\phi$–sequence

From (10), we get the following recurrence relation

Lemma 16. For each natural number $n$, $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, the integers $\phi_k(n)$ satisfy the following property

$$\phi_k(n) = 4 \phi_{k-1}(n-2) - \phi_k(n-4)$$  \hspace{1cm} (33)

From (23), we easily get the following initial values

Lemma 17. For each natural number $n$

$$\phi_0(n) = \begin{cases} +2 & n \equiv 0 \pmod{8} \\ +1 & n \equiv \pm 1 \pmod{8} \\ 0 & n \equiv \pm 2 \pmod{8} \\ -1 & n \equiv \pm 3 \pmod{8} \\ -2 & n \equiv \pm 4 \pmod{8} \end{cases}$$

$$\phi_1(n) = \begin{cases} 0 & n \equiv 0 \pmod{8} \\ +(n-1) & n \equiv 1 \pmod{8} \\ +2n & n \equiv 2 \pmod{8} \\ +(n+1) & n \equiv 3 \pmod{8} \\ 0 & n \equiv 4 \pmod{8} \\ -(n-1) & n \equiv 5 \pmod{8} \\ -2n & n \equiv 6 \pmod{8} \\ -(n+1) & n \equiv 7 \pmod{8} \end{cases}$$  \hspace{1cm} (34)

6.2 Explicit formulas For $\phi$–sequence

Now, from Theorem (9), we get the following explicit formulas for the integer sequence $\phi_k(n)$.

Lemma 18. For any non negative integers $n, k$, the sequences $\phi_k(n)$ satisfy the following statements

$$n \equiv 0 \pmod{8}$$

$$\phi_k(n) = \begin{cases} 0 & \text{for } k \text{ odd} \\ 2(-1)^{\left\lfloor \frac{k}{2} \right\rfloor} \prod_{\lambda=0}^{\left\lfloor \frac{k}{2} \right\rfloor-1} \frac{n^2-(4\lambda)^2}{k!} & \text{for } k \text{ even} \end{cases}$$
\[ n \equiv 1 \pmod{8} \]
\[
\phi_k(n) = (-1)^{\left\lfloor \frac{k}{2} \right\rfloor} (n+1)^2 (n+1-2k)^{\delta(k)} \prod_{\lambda=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (n+1)^2 - (4\lambda-2)^2 \]

\[ n \equiv 2 \pmod{8} \]
\[
\phi_k(n) = \begin{cases} 
0 & \text{for } k \text{ even} \\
2 (-1)^{\left\lfloor \frac{k}{2} \right\rfloor} n \prod_{\lambda=1}^{\left\lfloor \frac{k}{2} \right\rfloor} n^2 - (4\lambda-2)^2 & \text{for } k \text{ odd}
\end{cases}
\]

\[ n \equiv 3 \pmod{8} \]
\[
\phi_k(n) = (-1)^{\left\lfloor \frac{k}{2} \right\rfloor + \delta(k-1)} (n+1)(n+1-2k)^{\delta(k-1)} \prod_{\lambda=1}^{\left\lfloor \frac{k}{2} \right\rfloor-\delta(k-1)} (n+1)^2 - (4\lambda)^2 \]

\[ n \equiv 4 \pmod{8} \]
\[
\phi_k(n) = \begin{cases} 
0 & \text{for } k \text{ odd} \\
2 (-1)^{\left\lfloor \frac{k}{2} \right\rfloor+1} \prod_{\lambda=0}^{\left\lfloor \frac{k}{2} \right\rfloor-1} n^2 - (4\lambda)^2 & \text{for } k \text{ even}
\end{cases}
\]

\[ n \equiv 5 \pmod{8} \]
\[
\phi_k(n) = (-1)^{\left\lfloor \frac{k}{2} \right\rfloor+1} (n+1-2k)^{\delta(k)} \prod_{\lambda=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (n+1)^2 - (4\lambda-2)^2 \]

\[ n \equiv 6 \pmod{8} \]
\[
\phi_k(n) = \begin{cases} 
0 & \text{for } k \text{ even} \\
2 (-1)^{\left\lfloor \frac{k}{2} \right\rfloor+1} n \prod_{\lambda=1}^{\left\lfloor \frac{k}{2} \right\rfloor} n^2 - (4\lambda-2)^2 & \text{for } k \text{ odd}
\end{cases}
\]

\[ n \equiv 7 \pmod{8} \]
\[
\phi_k(n) = (-1)^{\left\lfloor \frac{k}{2} \right\rfloor+\delta(k)} (n+1)(n+1-2k)^{\delta(k-1)} \prod_{\lambda=1}^{\left\lfloor \frac{k}{2} \right\rfloor-\delta(k-1)} (n+1)^2 - (4\lambda)^2 \]

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6.3 Nonlinear recurrence relation to generate $\phi$–sequence

To study the arithmetic properties of $\phi$–Sequence, we need to generate $\phi_k(n)$ from the previous one, $\phi_{k-2}(n)$, or generate $\phi_k(n)$ from the next one, $\phi_{k+2}(n)$. As

$$\frac{\Psi_k(n)}{\Psi_{k-2}(n)} = \frac{\phi_k(n)}{4^2 \phi_{k-2}(n)},$$

from Theorem (14), we get the following desirable theorem.

**Theorem 19. (Generating The $\phi$–integers From The Previous Term)**

*If $\phi_k(n)$ not identically zero, $n$ even, then*

$$\frac{\phi_k(n)}{\phi_{k-2}(n)} = -\frac{n^2 - (2k - 4)^2}{k(k - 1)},$$

and depending on the parity of $k$ we can choose either of the following initial values to generate $\phi_k(n)$ from the starting terms $\phi_0(n)$ or $\phi_1(n)$:

$$\phi_0(n) = \begin{cases} +2 & n \equiv 0 \pmod{8} \\ -2 & n \equiv 4 \pmod{8} \end{cases}, \quad \phi_1(n) = \begin{cases} +2 & n \equiv 2 \pmod{8} \\ -2 & n \equiv 6 \pmod{8} \end{cases}, \quad (35)$$

7 An overview of applications towards Mersenne primes

From Lemmas (16), (17),(18), we ready to prove the following theorem.

7.1 Primality tests for Mersenne numbers

**Theorem 20. (Lucas-Lehmer-Moustafa)(Version 1)**

*Given prime $p \geq 5$, $2^p - 1$ is prime if and only if*

$$2^p - 1 \mid \sum_{\substack{k=0, \\ k \text{ even}}}^{\lfloor \frac{n}{4} \rfloor} \phi_k(n) \quad \text{(36)}$$

*where $n := 2^p - 1$, $\phi_k(n)$ are defined by the double index recurrence relation*

$$\phi_k(m) = 4 \phi_{k-1}(m - 2) - \phi_k(m - 4)$$

*and the initial boundary values satisfy*

$$\phi_0(m) = \begin{cases} +2 & m \equiv 0 \pmod{8} \\ 0 & m \equiv \pm 2 \pmod{8} \\ -2 & m \equiv 4 \pmod{8} \end{cases}, \quad \phi_1(m) = \begin{cases} +2 & m \equiv 2 \pmod{8} \\ 0 & m \equiv 0, 4 \pmod{8} \\ -2 & m \equiv 6 \pmod{8} \end{cases}, \quad (37)$$
Proof. Given prime $p \geq 5$, let $n := 2^{p-1}$. From Lucas-Lehmer-Test, see [4] and [1], we have

$$2^p - 1 \text{ is prime } \iff 2^p - 1 \mid (1 + \sqrt{3})^n + (1 - \sqrt{3})^n.$$ 

Hence, as $n \equiv 0 \pmod{8}$, replace $x = 1 + \sqrt{3}$, $y = 1 - \sqrt{3}$ in Theorem (9), we get the following equivalent statements:

$$2^p - 1 \text{ is prime } \iff 2^p - 1 \mid \sum_{k=0, \ k \text{ even}}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n) (-2)^{\lfloor \frac{n}{2} \rfloor - k} (8)^k$$

$$\iff 2^p - 1 \mid 2^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0, \ k \text{ even}}^{\lfloor \frac{n}{2} \rfloor} \Psi_k(n) 4^k$$

$$\iff 2^p - 1 \mid \sum_{k=0, \ k \text{ even}}^{\lfloor \frac{n}{2} \rfloor} \phi_k(n).$$

Lemmas (16), (17) already proved the rest of Theorem (20). □

From Lemmas (16), (17), we should observe that the recurrence relation of $\phi_k(n)$ is always even integer for $n \equiv 0 \pmod{8}$. Hence from Lemma (18), we get the following theorem.

**Theorem 21. (Lucas-Lehmer-Moustafa)(Version 2)**

*Given prime $p \geq 5$, $n := 2^{p-1}$. The number $2^p - 1$ is prime if and only if*

$$2 n - 1 \mid \sum_{k=0, \ k \text{ even}}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor} \prod_{\lambda=0}^{\lfloor \frac{n}{2} \rfloor - 1} n^2 - (4 \lambda)^2 \frac{k!}{k!}.$$  \hfill (38)

### 7.2 Generating $\phi$–integers

When we generate the $\phi$–integers needed for Mersenne numbers, we should notice that for $n := 2^p, p \geq 5$, we have

$$n \equiv 0 \pmod{8}.$$ 

Hence we get the following theorem.

**Theorem 22. (Generating $\phi$–integers without changing $n$)**

*For $n := 2^{p-1}, p \geq 5$, then*

$$\frac{\phi_k(n)}{\phi_{k-2}(n)} = - \frac{n^2 - (2k - 4)^2}{k (k - 1)},$$  \hfill (39)
and we can choose either of the following initial values to generate $\phi_k(n)$ from the starting term $\phi_0(n)$ or the last term $\phi_{\lfloor \frac{n}{2} \rfloor}(n)$:

$$\phi_0(n) = +2, \quad \phi_{\lfloor \frac{n}{2} \rfloor}(n) = 4^{\lfloor \frac{n}{2} \rfloor}.$$ 

**Theorem 23. (Lucas-Lehmer-Moustafa)(Version 3)**

Given prime $p \geq 5$, $n := 2^{p-1}$. The number $2^p - 1$ is prime if and only if

$$2n - 1 \mid \sum_{k=0, \ k \text{ even}}^{\lfloor \frac{n}{2} \rfloor} \phi_k(n)$$

where $\phi_k(n)$ are defined and generated by the double index recurrence relation

$$\frac{\phi_k(n)}{\phi_{k-2}(n)} = -\frac{n^2 - (2k - 4)^2}{k(k-1)},$$

and we can choose either of the following initial values to generate $\phi_k(n)$ from the starting term $\phi_0(n)$ or the last term $\phi_{\lfloor \frac{n}{2} \rfloor}(n)$:

$$\phi_0(n) = +2, \quad \phi_{\lfloor \frac{n}{2} \rfloor}(n) = 4^{\lfloor \frac{n}{2} \rfloor}.$$ 

### 7.3 Criteria for compositeness of Mersenne numbers

The following theorem is an immediate consequence of Theorem (21)

**Theorem 24. (Criteria for compositeness of Mersenne numbers)**

Given prime $p \geq 5$. The number $2n - 1 = 2^p - 1$ is Mersenne composite number if

$$2n - 1 \mid \sum_{k=0, \ k \text{ even}}^{\lfloor \frac{n}{2} \rfloor} \phi_k(n)$$

$$\prod_{\lambda=0}^{\lfloor \frac{n}{2} \rfloor - 1} [(4\lambda)^2 - 4^{-1}] \quad \frac{k!}{k!}.$$ 

### 8 Some formulas and possible scenario

Now, consider $p$ prime greater than 3, and $n := 2^{p-1}$. The previous sections give various explicit formulas and techniques to compute and generate all of the terms $\phi_k(n)$ needed for checking the primality of the Mersenne number $2^p - 1$. 

22
8.1 Formulas

Now we compute the summation
\[ \sum_{k=0, k \text{ even}}^{\left\lfloor \frac{n}{2} \right\rfloor} \phi_k(n) \]  
for the first few terms from the starting and from the end. From Theorem (22), we get the following explicit terms
\[ \phi_0(n) = +2, \]
Now, compute \( \phi_2(n) \) as following
\[ \phi_2(n) = -\frac{n^2 - (4 - 4)^2}{2(2 - 1)} \phi_0(n) = -n^2. \]
Proceeding this way, we get
\[ \phi_4(n) = -\frac{n^2 - (8 - 4)^2}{4(4 - 1)} \phi_2(n) = +\frac{n^2(n^2 - 4^2)}{12}. \]
Similarly
\[ \phi_6(n) = -\frac{n^2(n^2 - 4^2)(n^2 - 8^2)}{360}, \]
\[ \phi_8(n) = +\frac{n^2(n^2 - 4^2)(n^2 - 8^2)(n^2 - 12^2)}{20160}, etc \]
Now, compute \( \phi_k(n) \) from the end, and from Theorem (22), we get
\[ \phi_{k-2}(n) = -\frac{k(k - 1)}{n^2 - (2k - 4)^2} \phi_k(n). \]
Initially
\[ \phi_{\left\lfloor \frac{n}{2} \right\rfloor}(n) = 2^n, \]
Hence
\[ \phi_{\left\lfloor \frac{n}{2} \right\rfloor - 2}(n) = -n \ 2^{n-5}, \]
and
\[ \phi_{\left\lfloor \frac{n}{2} \right\rfloor - 4}(n) = +n \ (n - 6) \ 2^{n-11}. \]
Similarly
\[ \phi_{\left\lfloor \frac{n}{2} \right\rfloor - 6}(n) = -\frac{n(n - 8)(n - 10)}{3} \ 2^{n-16}, \]
\[ \phi_{\left\lfloor \frac{n}{2} \right\rfloor - 8}(n) = +\frac{n(n - 10)(n - 12)(n - 14)}{3} \ 2^{n-23}, etc. \]
The author feels that we need a clever way to evaluate the sum (42). We may like to add the terms in a way reflects some elegant arithmetic. Remember that we do not need to compute the sum (42) exactly; but we just need to find the sum modulo $2n - 1$. According to the following theorem, and working modulo $2n - 1$, the last term always gives the value of the first term.

**Theorem 25.** For $p \geq 5$ prime, and $n := 2^{p-1}$,

$$\phi(k|n)(n) \equiv 2 \pmod{2n - 1}. \quad (43)$$

**Proof.** We should observe that if $p \geq 5$ prime, then $n = 2^{p-1} = 1 + \zeta p$, for some positive integer $\zeta$. Then

$$\phi(k|n)(n) = 2^n = 2^{1+\zeta p} = 2^1 (2^p)^\zeta \equiv 2 \pmod{2n - 1}. \qed$$

Hence this encourages one to compute the partial sums of

$$\sum_{k=0, k \text{ even}}^{\frac{n}{2}} \phi_k(n) \pmod{2n - 1}, \quad (44)$$

in the following order.

### 8.2 The 5 scenario

For example, take $p = 5$, then $n = 2^4$. Hence

$$\phi_0(31) \equiv +2 \pmod{31}, \quad \phi_2(31) \equiv -2^3 \pmod{31},$$

$$\phi_4(31) \equiv +2^2 + 1 \pmod{31}, \quad \phi_6(31) \equiv -1 \pmod{31}, \quad (45)$$

and $\phi_8(31) \equiv +2 \pmod{31}$. Hence we get the partial sums

$$\phi_8(31) \equiv +2,$$

$$\phi_8(31) + \phi_0(31) \equiv +2^2,$$

$$\phi_8(31) + \phi_0(31) + \phi_2(31) \equiv -2^2,$$

$$\phi_8(31) + \phi_0(31) + \phi_2(31) + \phi_4(31) \equiv +1,$$

$$\phi_8(31) + \phi_0(31) + \phi_2(31) + \phi_4(31) + \phi_6(31) \equiv 0. \quad (46)$$

As we ended up with zero, this shows that $2^5 - 1 = 31$ is Mersenne prime. This particular example should motivate us for more theoretical investigations for other similar scenarios for this particular pattern that may occur in general for other cases for the partial sums

$$\phi(k|n)(n),$$

$$\phi(k|n)(n) + \phi_0(n),$$

$$\phi(k|n)(n) + \phi_0(n) + \phi_2(n),$$

$$\phi(k|n)(n) + \phi_0(n) + \phi_2(n) + \phi_4(n), \quad (47)$$

$$\cdots$$
9 Factoring factorial in terms of difference of squares

Choosing $k = \lfloor \frac{n}{2} \rfloor$ in the Eight Levels Theorem (9), and noting $\Psi(\lfloor \frac{n}{2} \rfloor)(n) = 1$, we surprisingly get the following eight combinatorial identities which reflect some unexpected facts about the nature of numbers.

\[
\begin{align*}
 n \equiv 0 \pmod{8} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = 2 \prod_{\lambda=0}^{\lfloor \frac{n}{4} \rfloor - 1} n^2 - (4\lambda)^2 \\
 n \equiv 1 \pmod{8} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = \prod_{\lambda=1}^{\lfloor \frac{n+1}{2} \rfloor} (n+1)^2 - (4\lambda-2)^2 \\
 n \equiv 2 \pmod{8} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = 2 n \prod_{\lambda=1}^{\lfloor \frac{n-2}{4} \rfloor} n^2 - (4\lambda-2)^2 \\
 n \equiv 3 \pmod{8} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = (n+1) \prod_{\lambda=1}^{\lfloor \frac{n-3}{4} \rfloor} (n+1)^2 - (4\lambda)^2 \\
 n \equiv 4 \pmod{8} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = 2 \prod_{\lambda=0}^{\lfloor \frac{n-1}{4} \rfloor} n^2 - (4\lambda)^2 \\
 n \equiv 5 \pmod{8} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = \prod_{\lambda=1}^{\lfloor \frac{n-1}{2} \rfloor} (n+1)^2 - (4\lambda-2)^2 \\
 n \equiv 6 \pmod{8} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = 2 n \prod_{\lambda=1}^{\lfloor \frac{n-2}{4} \rfloor} n^2 - (4\lambda-2)^2 \\
 n \equiv 7 \pmod{8} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = (n+1) \prod_{\lambda=1}^{\lfloor \frac{n+1}{4} \rfloor} (n+1)^2 - (4\lambda)^2
\end{align*}
\]

Writing only the different identities, which are 4, we get the following theorem which gives formulas and links to factorize any factorial in terms of a product of difference of squares.

**Theorem 26.** (New combinatorial identities) For any natural number $n$, the following combinatorial identities are correct

\[
\begin{align*}
 n \equiv 0 \pmod{4} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = 2 \prod_{\lambda=0}^{\lfloor \frac{n+1}{4} \rfloor - 1} n^2 - (4\lambda)^2 \\
 n \equiv 1 \pmod{4} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = \prod_{\lambda=1}^{\lfloor \frac{n+1}{2} \rfloor} (n+1)^2 - (4\lambda-2)^2 \\
 n \equiv 2 \pmod{4} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = 2 n \prod_{\lambda=1}^{\lfloor \frac{n-2}{4} \rfloor} n^2 - (4\lambda-2)^2 \\
 n \equiv 3 \pmod{4} & \quad 4^{\lfloor \frac{n}{2} \rfloor} (\lfloor \frac{n}{2} \rfloor)! = (n+1) \prod_{\lambda=1}^{\lfloor \frac{n+3}{4} \rfloor} (n+1)^2 - (4\lambda)^2
\end{align*}
\]

**Supplementary information**

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.
Conflict of interest

The author declares that he has no conflict of interest.

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