A BOURGAIN-PISIER CONSTRUCTION FOR GENERAL BANACH SPACES

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Abstract. We prove that every Banach space, not necessarily separable, can be isometrically embedded into a $L_\infty$-space in a way that the corresponding quotient has the Radon-Nikodym and the Schur properties. As a consequence, we obtain $L_\infty$ spaces of arbitrary large densities with the Schur and the Radon-Nikodym properties. This extends the result by J. Bourgain and G. Pisier in [Bo-Pi] for separable spaces.

1. Introduction

The main question considered in this paper is the largeness of the class of $L_\infty$ spaces in terms of embeddability. Recall that a Banach space $X$ is called $L_{\infty,\lambda}$ when for every finite dimensional subspace $F$ of $X$ there is a subspace $G$ of $X$ $\lambda$-isomorphic to $\ell_{\dim G}^\infty$ containing $F$. $L_\infty$ just means $L_{\infty,\lambda}$ for some $\lambda$. There are two remarkable results for the class of separable $L_\infty$ spaces. The first by J. Bourgain and G. Pisier in [Bo-Pi] states that every separable Banach space $X$ can be isometrically embedded into a $L_\infty$-space $Y_X$ in such a way that the corresponding quotient space $Y_X/X$ has the Radon-Nikodym property (RNP) and the Schur property. The second, more recent one, by D. Freeman, E. Odell and Th. Schlumprecht [Fr-Od-Schl] tells that every space with separable dual can be isomorphically embedded into a $L_\infty$-space with separable dual (therefore an $\ell_1$-predual). Both constructions are the natural extensions of the work of J. Bourgain and F. Delbaen [Bo-De] and Bourgain [Bo]. There are several other recent examples. Perhaps the most impressive one is the $L_\infty$-space by S. A. Argyros and R. G. Haydon [Ar-Ha] where every operator is the sum of a multiple of the identity and a compact one.

In the non-separable context much less is known. Spaces of functions on a non-metrizable compactum, or non-separable Gurarij spaces are non-separable $L_\infty$-spaces. There is a wide variety of structures in the non-separable level for spaces in these two classes. Based on combinatorial axioms outside ZFC, there are non-separable spaces in these two classes without uncountable biorthogonal systems or where every operator is the sum of a multiple of the identity and an operator with separable range (see [Lo-To] for more information). On the other hand, the separable structure of the known examples is too simple: either they are $c_0$-saturated, that is, every infinite dimensional subspace of it contains an isomorphic copy of $c_0$, or universal for the separable spaces. So it is natural to ask if there are examples of non-separable $L_\infty$-space without isomorphic copies of $c_0$, with the (RNP) or with the Schur property. Our main result in Theorem 3.1 is that the embedding Theorem by Bourgain and Pisier reminds valid for any density. In

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particular, by embedding $\ell_1(\kappa)$ for an infinite cardinal number $\kappa$, we obtain examples of $L_\infty$ spaces of arbitrary density with the Radon-Nikodym and the Schur properties.

For a given separable space $X$, the corresponding Bourgain-Pisier superspace $Y_X$ of it is built in such a way that $Y_X$ and the quotient $Y_X/X$ are both the inductive limit of linear systems $(Z_n, j_n)_{n \in \mathbb{N}}$ of a special type of isometrical embedding $j_n : Z_n \to Z_{n+1}$ (η-admissible embeddings, see Definition 2.2), and such that, in addition, the corresponding $Z_n$’s are finite dimensional for the quotient space $Y_X/X$. The key fact to get the Radon-Nikodym and the Schur properties of the quotient space $Y_X/X$ is the metric property of η-admissible embeddings exposed here in Lemma 1 and its consequence to inductive limits as above for finite dimensional spaces (see Bo-Pi, Theorem 1.6).

In contrast to the separable case, the main difficulty in the non-separable case is the construction the appropriate inductive limit. Indeed, if $X$ is non-separable, then it is unlikely to find a nice linear system having the space $Y_X$ as the corresponding limit. In general, every Banach space $X$ is naturally represented as the inductive limit of its finite dimensional subspaces together with the corresponding inclusions between them. Our inductive system $((E_s)_{s \in I}, (j_{t,s})_{t \subseteq s})$ to represent $Y_X$ is also based on the inclusion relation over the index set $I$ consisting on all finite subsets of the density of the space $X$. This provides a natural way to isometrically embed $X$ into $Y_X$. In addition, our inductive system is constructed in a way that its linear subsystems $((E_{s,n})_{n \in \mathbb{N}}, (j_{s,n,s_{n+1}})_{n \in \mathbb{N}})$ are Bourgain-Pisier linear systems as above. In other words, every separable subspace $Z$ of $Y_X$ can be isometrically embedded into the separable Bourgain-Pisier extension $Y_Z$. So, having into account that the Radon-Nikodym and Schur properties are separably determined, we readily have that the quotient $Y_X/X$ has the desired properties.

To construct the spaces $E_s$ and the corresponding embeddings $j_{s,t} : E_s \to E_t$ we define first finite linear systems $((E_1^{(s)})_{t \subseteq s}, (j_{u,t})_{u \prec t})$ of η-admissible embeddings $j_{u,t}^{(s)} : E_{u}^{(s)} \to E_{t}^{(s)}$, where $\prec$ is a natural well ordering extending the inclusion relation. Obviously, this raises a problem of coherence, since given $s \subseteq p \subseteq q$ we will have defined two “s-extensions” $E_{s}^{(p)}$ and $E_{s}^{(q)}$ of $E_{s} = X$ and therefore two isometric embeddings $X \to E_{s}^{(p)}$ and $X \to E_{s}^{(q)}$. This is corrected by defining simultaneously an infinite directed system $((E_{s}^{(p)})_{s \subseteq p}, (k_{s}^{(p,q)})_{s \subseteq p \subseteq q})$ of η-admissible embeddings making the appropriate diagrams commutative.

Finally, let us point out that nothing is known in how to skip the separability assumption in the Freeman-Odell-Schlumprecht embedding Theorem, or, even more basic, if a non-separable Bourgain-Delbaen exists, i.e. a non-separable $L_\infty$-space not containing isomorphic copies of $c_0$ or $\ell_1$.

The paper is organized as follows. The Section 2 is a survey of basic facts concerning the Kisliakov’s extension method and η-admissible, in particular we present a new extension fact concerning these embeddings in Lemma 2.5. The last section is devoted to the proof of the Theorem 3.1.

2. Background and basic facts

We use standard terminology in Banach space theory from the monographs [Al-Ka] and [Li-Tz]. The goal of this section is to present the basic notions of η-admissible diagrams and η-admissible embeddings introduced by Bourgain and Pisier. To complete the information we
give here, specially for some proofs, we refer the reader to the original paper \[Bo-Pi\] or to the recent book by P. Dodos \[Do\].

Recall the Kisliakov’s extension method \[Ki\]: Given Banach spaces \(S \subseteq B\) and \(E\) and an operator \(u : S \to E\) such that \(\|u\| \leq \eta \leq 1\), let

\[
N_u := \{(s, -u(s)) \in B \times E : s \in S\},
\]

\[
i_B : B \to (B \oplus_1 E)/N_u
\]

\[
b \mapsto i_B(b) = (b, 0) + N_u
\]

\[
i_E : E \to (B \oplus_1 E)/N_u
\]

\[
e \mapsto i_E(e) = (0, e) + N_u
\]

Then the diagram \((K)\)

\[
\begin{array}{c}
B \xrightarrow{i_B} (B \oplus_1 E)/N_u \\
\downarrow (K) \\
S \xrightarrow{u} E \\
\end{array}
\]

is commutative, \(i_E\) is an isometrical embedding, and \(\|i_B\| \leq 1\). This diagram has several categorical properties such as minimality and uniqueness.

**Definition 2.1.** We say that a diagram

\[
\begin{array}{c}
B \xrightarrow{\bar{u}} E_1 \\
\downarrow j \\
S \xrightarrow{u} E \\
\end{array}
\]

is a \(\eta\)-admissible diagram when there is an isometry \(T : (B \oplus_1 E)/N_u \to E_1\) such that \(j = T \circ i_E\) and \(\bar{u} = T \circ i_B\). The canonical \(\eta\)-admissible diagram associated to the triple \((S, B, u)\) is the Kisliakov’s diagram \((K)\) above.

An isometrical embedding \(j : E \to E_1\) is called \(\eta\)-admissible embedding when there are \(S \subseteq B\), \(E_1\), \(u : S \to E\), \(\bar{u} : B \to E_1\) forming together with \(j : E \to E_1\) an \(\eta\)-admissible diagram.

Observe that \(\eta\)-admissible diagrams are always commutative.

**Definition 2.2.** \[Bo-Pi\] A surjective operator \(\pi : E \to F\) is called a metric surjection when the associated isomorphism \(\bar{\pi} : E/\text{Ker}(\pi) \to F\) is an isometry.

The following are useful known characterizations, not difficult to prove.
Proposition 2.3. (a) Let $S \subseteq B$, $E$ and $E_1$ be normed spaces, and $\eta \leq 1$. A diagram $\Delta$ 

\[
\begin{array}{c}
B \xrightarrow{\bar{u}} E_1 \\
\downarrow \Delta \\
S \xrightarrow{u} E
\end{array}
\]

is an $\eta$-admissible diagram if and only if

(a.1) $j$ is an isometry and $\|u\| \leq \eta$,

(a.2) $\pi : B \oplus_1 E \to E_1$ defined for $(b, e) \in B \times E$ by $\pi(b, e) := \bar{u}(b) + j(e)$ is a metric surjection.

(a.3) $\text{Ker}(\pi) = N_u = \{(s, -u(s)) : s \in S\}$.

(b) An isometrical embedding $j : E \to E_1$ is $\eta$-admissible iff there is some Banach space $B$ and a metric surjection $\pi : B \oplus_1 E \to E_1$ such that $\pi(0, e) = j(e)$ for every $e \in E$ and $\|\pi(b, e)\| \geq \|e\| - \eta \|b\|$ for every $(e, b) \in E \times B$. □

It follows from (b) above that the composition of two $\eta$-admissible embeddings is also $\eta$-admissible. Although we are not going to used them directly, two metric properties of $\eta$-admissible embeddings crucial for the Radon-Nikodym and Schur properties of the Bourgain-Pisier quotient $Y_X/X$.

Lemma 1. Suppose that the diagram $\Delta$ above is $\eta$-admissible. Then,

(a) $\|\pi(b)\| = \|(b, 0) + \text{Ker}(\pi)\| = \inf_{s \in S} \|b + s\| + \|u(s)\|$ for every $b \in B$. Consequently, $\|\pi\| \leq 1$, and if there is $\delta \leq 1$ such that $\|u(s)\| \geq \delta \|s\|$ for every $s \in S$, then $\|\pi(b)\| \geq \delta \|b\|$ for every $b \in B$. In other words, if $u$ is an isomorphic embedding then so is $\bar{u}$ with better isomorphic constant.

(b) Let $q : E_1 \to E_1/j(E)$ be the natural quotient map. Suppose that $x_0, \ldots, x_n \in E_1$ are such that $x_0 + \cdots + x_n \in j(E)$. Then

$$\sum_{i=0}^{n} \|x_i\| \geq \| \sum_{i=0}^{n} x_i \| + (1-\eta) \sum_{i=0}^{n} \|q(x_i)\|.$$ 

The fact in (b) is taken from [Do] and it has an equivalent probabilistic reformulation in [Bo-Pi]. It is the key to prove the following.

Theorem 2.4. [Bo-Pi, Theorem 1.6.] Suppose that $(E_n)_n$ is a sequence of finite dimensional spaces, and suppose that $j_n : E_n \to E_{n+1}$ is an $\eta$-admissible embedding for each $n$. Then the inductive limit of $(E_n, j_n)_n$ has the Schur and the Radon-Nikodym properties.

2.1. One step extension. We finish this section with the following result, somehow stating that an appropriate composition of $\eta$-admissible diagrams is again $\eta$-admissible.
Lemma 2.5. Suppose that

\[ B_0 \xrightarrow{\bar{u}_0} X_0 \xrightarrow{j_2} X_2 \]

\[ S_0 \xrightarrow{u_0} E \xrightarrow{j_1} X_1 \]

is a commutative diagram such that:

1. \((\Delta.0), (\Delta.1)\) and \((\Delta.2)\) are \(\eta\)-admissible diagrams.
2. \(j : X_1 \to X_2\) is an isometry.

Then the diagram

\[ B_0 \xrightarrow{j_2 \circ \bar{u}_0} X_2 \]

\[ S_0 \xrightarrow{j_1 \circ u_0} X_1 \]

is \(\eta\)-admissible.

Proof. Let \(\pi : B_0 \oplus_1 X_1 \to X_2, \pi(b_0, x) := j_2(\bar{u}_0(b_0)) + j(x)\), and for \(i = 0, 1, 2\), let \(\pi_i : B_i \otimes E_i \to X_i\) be defined by \(\pi_i(b, e) := \bar{u}_i(b) + j_i(e)\), where \(E_0 = E_1 = E, E_2 = X_0\) and \(B_2 = B_1\). We have to check that \((\alpha.1), (\alpha.2)\) and \((\alpha.3)\) in Proposition 2.3 (a) hold. By hypothesis \(j\) is isometry and clearly \(\|j_1 \circ u_0\| = \|u_0\| \leq \eta\), so we get \((\alpha.1)\).

Claim 1. \(\pi(b_0, \pi_1(b_1, e)) = \pi_2(b_1, \pi_0(b_0, e))\) for every \(b_0 \in B_0, b_1 \in B_1\) and \(e \in E\).

Proof of Claim:

\[ \pi(b_0, \pi_1(b_1, e)) = \pi_2(b_1, \pi_0(b_0, e)) = j_2(\bar{u}_0(b_0)) + j(\pi(b_1, e)) = j_2(\bar{u}_0(b_0)) + j(\bar{u}_1(b_1) + j_1(e)) = j_2(\bar{u}_0(b_0) + j_0(e)) + j(\bar{u}_1(b_1)) = j_2(\bar{u}_0(b_0) + j_0(e)) + j_1(b_1) = \pi_2(b_1, \pi_0(b_0, e)). \]

It follows from this that \(\pi\) is onto.

Claim 2. \(\text{Ker}(\pi) = \{(s_0, -j_1(u_0(s_0))) : s_0 \in S_0\} = \{(b_0, \pi_1(0, -u_0(b_0))) : b_0 \in S_0\}\).

Proof of Claim: The last equality follows from the fact that by definition, \(\pi_1(0, -u_0(b_0)) = -j_1(u_0(b_0))\). We prove now the first equality. Fix \(s_0 \in S_0\), and we work to prove that
\(\pi(s_0, -j_1(u_0(s_0))) = 0\). Using the commutativity of the diagram we obtain
\[
\pi(s_0, -j_1(u_0(s_0))) = j_2(\pi_0(s_0)) - j_1(u_0(s_0)) = j_2(\pi_0(s_0)) - j_2(j_0(u_0(s_0))) = j_2(\pi_0(s_0)) - j_0(u_0(s_0))) = j_2(0) = 0.
\]
Now suppose that \(\pi(b_0, g) = 0\). Let \((b_1, e) \in B_1 \times E\) be such that \(\pi_1(b_1, e) = g\). Then, by Claim 1 it follows that
\[
(b_1, \pi_0(b_0, e)) \in \text{Ker}(\pi_2).
\]
And hence, \(b_1 \in S_1\) and \(\pi_0(b_0, e) = -u_2(b_1)\). It follows that
\[
0 = \bar{u}_0(b_0) + j_0(e) + u_2(b_1) = \bar{u}_0(b_0) + j_0(e) + j_0(u_1(b_1)) = \pi_0(b_0, e + u_1(b_1))
\]
So, \(b_0 \in S_0\) and \(e + u_1(b_1) = -u_0(b_0)\). By applying \(j_1\) to the last equality, we obtain that
\[
g := j_1(e) + \bar{u}_1(b_1) = -j_1(u_0(b_0)),
\]
as desired. \(\square\)

It follows readily that (\(\alpha.3\)) holds. It rests to prove the property (\(\alpha.2\)).

**Claim 3.** \(\|\pi(b_0, g)\| = \inf_{s_0 \in S_0} \|b_0 + s_0\| + \|g - j_1(u_0(s_0))\|\).

**Proof of Claim:** Fix \((b_1, e) \in B_1 \times E\) such that \(\pi_1(b_1, e) = g\). Then, by Claim 1 it follows that \(\pi(b_0, g) = \pi_2(b_1, \pi_0(b_0, e))\). Hence,
\[
\|\pi(b_0, g)\| = \|\pi_2(b_1, \pi_0(b_0, e))\| = \|(b_1, \pi_0(b_0, e)) + \text{Ker}(\pi_2)\| = \inf_{s_1 \in S_1} \left(\|b_1 + s_1\| + \|\pi_0(b_0, e) - u_2(s_1)\|\right) = \inf_{s_1 \in S_1} \left(\|b_1 + s_1\| + \|\pi_0(b_0, e) - j_0(u_1(s_1))\|\right) = \inf_{s_1 \in S_1} \left(\|b_1 + s_1\| + \|\pi_0(b_0, e - u_1(s_1))\|\right) = \inf_{s_1 \in S_1} \left(\|b_1 + s_1\| + \inf_{s_0 \in S_0} \left(\|b_0 + s_0\| + \|e - u_1(s_1) - u_0(s_0)\|\right)\right) = \inf_{s_0 \in S_0} \left(\|b_0 + s_0\| + \inf_{s_1 \in S_1} \left(\|s_1 + b_1\| + \|e - u_1(s_1) - u_0(s_0)\|\right)\right) = \inf_{s_0 \in S_0} \left(\|b_0 + s_0\| + \|b_1, e - u_0(s_0)\| + \text{Ker}(\pi_1)\|\right) = \inf_{s_0 \in S_0} \left(\|b_0 + s_0\| + \pi_1(b_1, e) + \pi_1(0, -u_0(s_0))\|\right) = \inf_{s_0 \in S_0} \left(\|b_0 + s_0\| + \|g - j_1(u_0(s_0))\|\right).
\]
\(\square\)
From this we prove that $\pi$ is a metric surjection: Fix $(b_0, g) \in B_0 \times G$, and let $(b_1, e) \in B_1 \times E$ be such that $g = \pi_1(b_1, e)$. Then by the Claim 3 it follows that
\[
\| (b_0, g) - \text{Ker} (\pi) \| = \| (b_0, \pi_1(b_1, e)) + \text{Ker} (\pi) \| = \\
= \inf_{s_0 \in S_0} \left( \| b_0 + s_0 \| + \| \pi_1(b_1, e) + \pi_1(0, -u_0(s_0)) \| \right) = \\
= \inf_{s_0 \in S_0} \left( \| b_0 + s_0 \| + \| \pi_1(b_1, e - u_0(s_0)) \| \right) = \\
= \inf_{s_0 \in S_0} \left( \| b_0 + s_0 \| + \inf_{s_1 \in S_1} (\| b_1 + s_1 \| + \| e - u_0(s_0) - u_1(s_1) \|) \right) = \\
= \inf_{s_0 \in S_0} \left( \| b_0 + s_0 \| + \| g - j_1(u_0(s_0)) \| \right) = \| \pi(b_0, g) \|,
\]
the last equality by Claim 3.

3. The main result

Our goal is to isometrically embed a given Banach space, not necessarily separable, into a $\mathcal{L}_\infty$-space in such a way that the corresponding quotient has the Schur and the Radon-Nikodym properties. Extending the approach of Bourgain and Pisier, we will find the $\mathcal{L}_\infty$-space as a direct, not necessarily linear, limit of $\eta$-admissible embeddings. The following is our main result.

**Theorem 3.1.** Every infinite dimensional Banach space $X$ can be isometrically embedded into a $\mathcal{L}_\infty$-space $Y$ of the same density that $X$ such that the quotient $Y/X$ has the Radon-Nikodym and the Schur properties.

**Corollary 3.2.** For every infinite cardinal number $\kappa$ there is a $\mathcal{L}_\infty$-space of density $\kappa$ with the Radon-Nikodym and the Schur properties.

**Proof.** For a fixed infinite cardinal number $\kappa$, apply the Theorem 3.1 to $X = \ell_1(\kappa)$. Then the corresponding superspace $Y$ is the desired space, since the required properties are three space properties.

For the proof of Theorem 3.1 we need the following two concepts.

**Definition 3.3.** Recall that the **anti-lexicographical** ordering $\prec$ on the family $[\kappa]^{<\omega}$ of finite subsets of $\kappa$ is defined recursively as follows: $\emptyset \prec s$ for every non empty $s$, and
\[
t \prec s \text{ if and only if } \begin{cases} \max t < \max s \text{ or } \\
\max t = \max s \text{ and } t \setminus \{ \max t \} \prec s \setminus \{ \max s \} \end{cases}
\]

This is a well-ordering on $[\kappa]^{<\omega}$ that extends the inclusion relation $\subseteq$. We introduce some notation: For each $\emptyset \subsetneq t \subset s$, we denote by $\bar{t}(s)$ the immediate $\prec$-predecessor of $t$ in the family $\mathcal{P}(s)$ of subsets of $s$, i.e.
\[
\bar{t}(s) := \max \{ u \subset s : u \prec t \}.
\]
Obviously this is well defined since $\mathcal{P}(s)$ is finite. We write $\bar{t}$ to denote $\bar{t}(t)$.

**Definition 3.4.** Recall that a **directed system** is $((X_i)_{i \in I}, (j_{i_0,i_1})_{i_0 \leq i_1})$, where $X_i$ are Banach spaces, $<_I$ is a directed partial ordering, $j_{i_0,i_1} : X_{i_0} \to X_{i_1}$ are isometrical embeddings, such that if $i_0 \leq I i_1 \leq I i_2$, then $j_{i_0,i_2} = j_{i_1,i_2} \circ j_{i_0,i_1}$, and such that $j_{i,i} = \text{Id } X_i$. 


From now on we fix an infinite dimensional Banach space $X$ of density $\kappa$, and a dense subset $D = \{d_\alpha : \alpha < \kappa\}$ of it. For each $s \in [\kappa]^{<\omega}$, let $X_s$ be the linear span of $\{d_\alpha\}_{\alpha \in s}$. Fix also $\lambda > 1$ and $\eta < 1$ such that $\lambda \cdot \eta < 1$.

**Lemma 3.5.** There is a direct system $((E_s)_{s \in [\kappa]^{<\omega}}, (j_{s,t})_{s \subseteq t, s,t \in [\kappa]^{<\omega}})$ and $(G_s)_{s \in [\kappa]^{<\omega}}$ such that:

1. $G_s \subseteq E_s$ are Banach spaces, $E_\emptyset = X$.
2. Each $j_{s,t} : E_s \to E_t$ is an $\eta$-admissible isometrical embedding such that $j_{s,t} E_s$ has finite codimension in $E_t$.
3. $G_s$ is $\lambda$-isomorphic to $\ell^{\dim G_s}_\infty$.
4. $\bigcup_{t \subseteq s} j_{t,s}(G_t) \cup j_{\emptyset,s}(X_s) \subseteq G_s$.

We are ready now to give a proof of Theorem 3.1 from this lemma.

**Proof of Theorem 3.1.** Fix $((E_s)_{s \in [\kappa]^{<\omega}}, (j_{s,t})_{s \subseteq t, s,t \in [\kappa]^{<\omega}})$ and $(G_s)_{s \in [\kappa]^{<\omega}}$ as in Lemma 3.5. Let $E$ be the completion of the inductive limit of $((E_s)_{s \in [\kappa]^{<\omega}}, (j_{s,t})_{s \subseteq t, s,t \in [\kappa]^{<\omega}})$. Because of property (4) in Lemma 3.5, it follows that $((G_s)_{s \in \mathcal{F}}, (j_{t,s} | G_t)_{t \subseteq s, s,t \in [\kappa]^{<\omega}})$ is also a directed system of finite dimensional normed spaces $G_s$ which are $\lambda$-isomorphic to $\ell^{\dim G_s}_\infty$. Let $Y$ be the completion of the corresponding direct limit $\lim_{s \in [\kappa]^{<\omega}} G_s$. It is clear that $Y$ can be isometrically imbedded into $E$, while there is a natural isometric embedding of $X$ into $Y$: $X$ is the completion of the direct limit $((X_s)_{s \in [\kappa]^{<\omega}}, (i_{t,s})_{t \subseteq s, s \in [\kappa]^{<\omega}})$, where $i_{t,s} : X_t \to X_s$ is the inclusion map. For each finite subset subset $s$ of $\kappa$, let $g_s : X_s \to G_s$ be $g_s := j_{\emptyset,s} | X_s$, which is well defined by (4). This is obviously an isometric embedding such that $i_{t,s} \circ g_s = g_s \circ i_{t,s}$ for every $t \subseteq s$, and hence $X$ isometrically embeds into $Y$.

If we denote by $j_{s,\infty} : G_s \to Y$ the corresponding limit of $(j_{s,t})_{s \subseteq t}$, then $\bigcup_{s \in [\kappa]^{<\omega}} j_{s,\infty}(G_s)$ is dense in $Y$. It follows that $Y$ is a $\mathcal{L}_{\infty,\lambda}$-space. Since each $G_s$ is finite dimensional, it follows that $Y$ has density at most $|[\kappa]^{<\omega}| = \kappa$. Since $X$ isometrically embeds into $Y$, the density of $Y$ has to be $\kappa$. Let us see that $Y/X$ has the Radon-Nikodym and the Schur properties: We use that $Y/X$ is naturally isometrically embedded into $E/X$, and we prove that $E/X$ has these two properties. Observe that these two properties are properties of separable subspaces of $E/X$. So let $Z \subseteq E/X$ be a separable subspace of $E/X$. By construction, we can find a sequence $(s_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{F}$ such that $s_n \subseteq s_{n+1}$, and such that $Z$ is a subspace of the closure of the quotient

$$
\left(\lim_{n \to \infty} (E_{s_n})_{n \in \mathbb{N}}, (j_{s_n,s_{n+1}})_{n \in \mathbb{N}})\right)/X.
$$

This quotient can be naturally isometrically identified with the inductive limit of finite dimensional spaces $((E_{s_n}/j_{\emptyset,s_n} X)_{n \in \mathbb{N}}, (j_{s_n,s_{n+1}})_{n \in \mathbb{N}})$, which, by Theorem 2.4, has the two required properties. □

The existence of the direct system in Lemma 3.5 is based on the following local construction.

**Lemma 3.6.** For every finite subset $s$ of $\kappa$ there are $E_t(s), G_t(s), j_{u,t}(s), k_u(s)$ such that

$$
(E_t(s))_{t \subseteq s}, (G_t(s))_{t \subseteq s}, j_{u,t}(s)_{u \subseteq t, u,t \subseteq s} \text{ and } (k_u(s))_{u \subseteq s}
$$

such that
For each finite subset \( s \) of \( \kappa \) one has that
\[
((E_t(s))_{t \leq s}, (j_{t,u})_{u \prec t, u \leq s})
\]
is a (finite) system of \( \eta \)-admissible isometrical embeddings such that \( j_{u,t}^{(s)} E_t^{(s)} \) has finite codimension in \( E_t^{(s)} \).

(B) (Transition directed system) For every finite subset \( u \) of \( \kappa \) one has that
\[
((E_u^{(s)})_{s \leq \kappa, \kappa < \omega}, (k_u^{(t,s)})_{u \leq t \leq s})
\]
is a system of \( \eta \)-admissible isometrical embeddings such that \( k_u^{(t,s)} E_t^{(t)} \) has finite codimension in \( E_u^{(s)} \).

(C) (Coherence property) For every \( v \subseteq u \subseteq t \subseteq s \) one has that
\[
k_u^{(t,s)} \circ j_{v,u}^{(t)} = k_v^{(t,s)} \circ j_{v,u}^{(v)}.
\]

(D) For every \( t \subseteq s \in [\kappa]^{< \omega} \) one has that \( G_t^{(s)} \subseteq E_t^{(s)} \) is finite dimensional and
\[
d(G_t^{(s)}, \ell_\infty^{\dim G_t^{(s)}}) \leq \lambda.
\]

(E) For every \( u \subseteq t \subseteq s \) one has that
\[
k_u^{(t,s)} (G_u^{(t)}) = G_u^{(s)}.
\]

(F) For every \( \emptyset \not\subseteq t \subseteq s \) one has that
\[
j_{t,u}^{(s)} (G_t^{(s)}) \cup j_{u,t}^{(s)} (X_t) \subseteq G_t^{(s)}.
\]

We postpone its proof, and we pass to prove Lemma 3.5.

**Proof of Lemma 3.5.** Fix \( (E_t^{(s)})_{t \leq s}, (G_t^{(s)})_{t \leq s}, (j_{u,t})_{u \prec t, u \leq s} \) and \( (k_u^{(t,s)})_{u \leq t \leq s} \) as in Lemma 3.6. For each finite subset \( s \) of \( \kappa \) set \( E_s := E_s^{(s)} \) and \( G_s := G_s^{(s)} \). Given \( t \subseteq s \) let \( j_{t,s} : E_t \to E_s \) be
\[
j_{t,s} := j_{t,s}^{(s)} \circ k_t^{(s,t)}.
\]

It follows from properties (A) and (B) in Lemma 3.6 that \( j_{t,s} \) is an \( \eta \)-admissible embedding such that \( j_{t,s} E_t \) has finite codimension in \( E_s \).

**Claim 4.** \( ((E_s)_{s \in [\kappa]^{< \omega}}, (j_{t,s})_{t \leq s \in [\kappa]^{< \omega}}) \) is a directed system of \( \eta \)-admissible isometrical embeddings.

**Proof of Claim:** Suppose that \( u \subseteq t \subseteq s \). Then
\[
j_{t,u} \circ j_{t,u} = j_{t,s}^{(s)} \circ (k_t^{(t,s)} \circ j_{u,t}^{(t)}) \circ k_t^{(u,t)} = j_{t,s}^{(s)} \circ (j_{u,t}^{(s)} \circ k_t^{(t,s)}) \circ k_t^{(u,t)} = = (j_{t,s}^{(s)} \circ j_{t,u}^{(s)}) \circ (k_t^{(u,t)} \circ k_t^{(t,s)}) = j_{u,t}^{(s)} \circ k_u^{(u,t)} = j_{u,t} = j_{u,s}.
\]

For each \( s \in [\kappa]^{< \omega} \), let \( G_s = G_s^{(s)} \).

**Claim 5.**
(a) \( d(G_s, \ell_\infty^{\dim G_s}) \leq \lambda \), i.e. \( G_s \) is \( \lambda \)-isomorphic to \( \ell_\infty^{\dim G_s} \).
(b) For every \( t \subseteq s \) one has that \( j_{t,s} E_t \subseteq G_s \).
Proof of Claim: (a) follows from (7) in (D). (b): By (8) and (9) one has that

\[ j_{t,s}(G_t) = j^{(s)}_{t,s} \circ k^{(t,s)}_t(G_t) = j^{(s)}_{t,s}G_t \subseteq G^{(s)}_s = G_s. \]

It only rests to give a proof of Lemma 3.6.

Proof of Lemma 3.6 Fix \( s \in [k]^{<\omega} \). We define \( \vartriangleleft \)-recursively on \( s \) all the objects in (5) together with an integer \( n_t \in \mathbb{N} \), \( S_t \subseteq \ell_{\infty}^n \) and \( \nu_t : S_t \to E^{(t)}_t \) for each \( \emptyset \subsetneq t \subseteq s \) such that

(a) \( E^{(t)}_0 = X \), \( k^{(t,s)}_t = \text{Id}_X \).

(b) \( \nu_t : S_t \to E^{(t)}_t \) is an isomorphism with \( \|\nu_t\| \leq \eta_t \), \( \|\nu_t^{-1}\| \leq \lambda_t \), and

\[ \nu_t(S_t) = \langle j^{(t)}_{u,t}(G^{(t)}_u) \cup j^{(t)}_{u,t}(X_t) \rangle \subseteq E^{(t)}_t, \]  

where \( u = \overline{t}^{(t)} \) is the \( \vartriangleleft \)-penultimate element in \( P(t) \), if \( |t| > 1 \) and \( u = \emptyset \), if \( |t| = 1 \) (and hence \( \overline{t} = \emptyset \)).

(c) \( E^{(s)}_t = (\ell^m_t \oplus_1 E^{(s)}_t)/N^{(s)}_{v^{(s)}_t} \); the diagram

is commutative, (\( \Delta \)) is a canonical \( \eta \)-admissible diagram, and

\[ G^{(s)}_t = v^{(s)}_t(\ell^m_t) = \{ (x, 0) + N^{(s)}_t : x \in \ell^m_t \}. \]  

(d) For every \( u \prec t \) subset of \( s \), we have that

\[ j^{(s)}_{u,t} = j^{(s)}_{u,t} \circ j^{(t,s)}_{u,t}, \]  

(c) For every \( u \subseteq t \subseteq s \), \( k^{(t,s)}_u : E^{(t)}_u \to E^{(s)}_u \) satisfies that for every \( (x, y) \in \ell_{\infty}^m \times E^{(t)}_{\overline{u}(t)} \) by

\[ k^{(t,s)}_u((x, y) + N^{(t)}_u) = (x, j^{(s)}_{u,\overline{u}(t)} \circ k^{(t,s)}_{\overline{u}(t)}(y)) + N^{(s)}_u. \]
The requirement in (e) can be fulfilled because the commutativity of the following diagram:

$$\begin{array}{c}
v_u(t) \quad E^{(u)}_v \quad j^{(t)}_{u,v} \quad E^{(s)}_u \\
E^{(u)}_v \quad j^{(t)}_{u,v} \quad E^{(s)}_u \quad j^{(s)}_{u,v} \quad E^{(u)}_v \\
E^{(u)}_v \quad j^{(t)}_{u,v} \quad E^{(s)}_u \quad j^{(s)}_{u,v} \quad E^{(u)}_v \\
\end{array}$$

It rests to check that the conditions (A)–(F) hold:
(A): It is clear from the definition of $j^{(t)}_{u,v}$ is an $\eta$-admissible isometrical embedding, and it follows from the equality in [13] that $j^{(t)}_{u,v} = j^{(t)}_{v,u} \circ j^{(t)}_{v,s}$ for every $u, v \subseteq s$ with $u \preceq v \preceq t$.
(C): Let $v \subseteq u \subseteq t \subseteq s$. We want to prove that $k^{(t,s)}_{v} \circ j^{(t)}_{v,u} = j^{(s)}_{u,v} \circ k^{(t,s)}_{u}$. Using that, by inductive hypothesis, the left side of the following diagram is commutative,

$$\begin{array}{c}
E^{(s)}_v \quad j^{(s)}_{v,w} \quad E^{(s)}_u \\
E^{(s)}_v \quad j^{(s)}_{v,w} \quad E^{(s)}_u \\
\end{array}$$

it suffices to prove that $k^{(t,s)}_{v} \circ j^{(t)}_{u,v} = j^{(s)}_{u,v} \circ k^{(t,s)}_{u}$. Let $x \in E^{(t)}_{u,v}$ Then by (c),

$$k^{(t,s)}_{v} \circ j^{(t)}_{u,v}(x) = k^{(t,s)}_{v}(0, x + N^{(t)}_{u}) = (0, j^{(s)}_{u,v}(0, x + N^{(t)}_{u}) + N^{(s)}_{u})$$

(B): Suppose that $v \subseteq u \subseteq t \subseteq s$. We have to see that $k^{(u,s)}_{v} = k^{(t,s)}_{v} \circ k^{(u,t)}_{v}$. Recall that from (e) it follows that for every $(x, y) \in \ell^{n^u}_\infty \times E^{(u)}_{v}$ and for every $(x, z) \in \ell^{n^u}_\infty \times E^{(t)}_{v}$ one has that

$$k^{(u,s)}_{v}((x, y) + N^{(u)}_{v}) = (x, j^{(t)}_{v,w}(y) + N^{(t)}_{v})$$

$$k^{(u,s)}_{v}((x, y) + N^{(u)}_{v}) = (x, j^{(s)}_{v,w}(y) + N^{(s)}_{v})$$

$$k^{(t,s)}_{v}((x, z) + N^{(t)}_{v}) = (x, j^{(s)}_{v,w}(z) + N^{(t)}_{v})$$

(15)    (16)    (17)
Hence, using inductively (C),

\[ k_{\nu}^{(t,s)} \circ k_{\nu}^{(u,t)}((x, y) + N_v^{(u)}) = (x, j^{(s)}_{\nu t(s), \nu t(s)} \circ j^{(t)}_{\nu t(s), \nu t(s)} \circ k_{\nu t(s)}^{(u,t)}(y)) + N_v^{(t)} = \\
= (x, j^{(s)}_{\nu t(s), \nu t(s)} \circ j^{(s)}_{\nu t(s), \nu t(s)} \circ k_{\nu t(s)}^{(u,t)}(y)) + N_v^{(t)} \\
= (x, j^{(s)}_{\nu t(s), \nu t(s)} \circ k_{\nu t(s)}^{(u,s)}(y)) + N_v^{(t)} \\
= k_{\nu}^{(u,s)}((x, y) + N_v^{(u)}). \]

We now prove that \( k_{u}^{(t,s)} \) is an \( \eta \)-admissible isometrical embedding: By inductive hypothesis the composition \( j : E^{(t)}_{\tilde{u}(t)} \to E^{(s)}_{\tilde{u}(s)}, j := j^{(s)}_{\tilde{u}(t), \tilde{u}(s)} \circ k_{\tilde{u}(t)}^{(t,s)}, \) is an \( \eta \)-admissible isometrical embedding.

We then fix \( S \subseteq B, \nu : S \to E^{(t)}_{\tilde{u}(t)} \) and \( \bar{\nu} : B \to E^{(s)}_{\tilde{u}(s)} \) such that

\[
\begin{array}{c}
B \\
\downarrow \bar{\nu}
\end{array}
\begin{array}{c}
E^{(s)}_{\tilde{u}(s)} \\
\downarrow (\Delta_0)
\end{array}
\begin{array}{c}
E^{(t)}_{\tilde{u}(t)} \\
\downarrow \nu
\end{array}
\begin{array}{c}
S \\
\downarrow j
\end{array}

\text{is an} \ \eta \text{-admissible diagram. It follows by (C), that the following diagram is commutative:}

\[
\begin{array}{ccc}
B & \xrightarrow{\tilde{u}_0} & E^{(s)}_{\tilde{u}(s)} \\
\downarrow (\Delta_0) & & \downarrow j^{(s)}_{\tilde{u}(s), \tilde{u}(s)} \\
S & \xrightarrow{u_0} & E^{(t)}_{\tilde{u}(t)} \\
\downarrow & & \downarrow j^{(t)}_{\tilde{u}(t), \tilde{u}(t)} \\
\ell_{\infty}^{(u_0)}/N_{\tilde{u}(t)}^{(u_0)} & \xrightarrow{\nu_0(s)} & \ell_{\infty}^{(u_0)} \\
\downarrow & & \downarrow \nu_0(t) \\
\ell_{\infty}^{(u_0)} & \xleftarrow{k_{\tilde{u}(t)}^{(t,s)}} & E^{(t)}_{\tilde{u}(t)} \\
\end{array}
\]

Since (\( \Delta_0 \)), (\( \Delta_1 \)) and (\( \Delta_2 \)) are \( \eta \)-admissible diagram, we conclude from Lemma 2.3 that \( k_{u}^{(t,s)} \) is an \( \eta \)-admissible embedding.

(D) is clear by definition of \( G_t^{(s)} \).

(E): Fix \( u \subseteq t \subseteq s \). Then

\[ k_{u}^{(t,s)}(G_t^{(s)}) = k_{u}^{(t,s)}((x) + N_t^{(t)}) : x \in \ell_{\infty}^{(u)} = \{ (x) + N_t^{(s)} : x \in \ell_{\infty}^{(s)} \} = G_t^{(s)}. \]
(F): Let $t \subseteq s$. We have to prove the inclusion in (9). Notice that the diagram

$$
\begin{array}{ccc}
\ell_\infty^t & \xrightarrow{\ell \ell(s)} & E_t^{(s)} = (\ell_\infty^t \oplus_1 E_{\pi(s)}^t)/N_t^{(s)} \\
\downarrow & & \downarrow \\
S_t & \xrightarrow{\ell \ell(s)} & E_{\pi(s)}^t \\
\end{array}
$$

is commutative. Let $u \subseteq t$ be the immediate $\prec$-predecessor of $\pi$ in $t$, if $|t| > 1$, and let $u = \emptyset$ otherwise. Then by (b),

$$
G_t^{(s)} = \{(x,0) + N_t^{(s)} : s \in S_t\} = \ell \ell^{-1}(E_t^{(s)}) \supseteq \\
\supseteq \ell \ell^{-1}(S_t) = \ell \ell^{-1}(S_t) = \\
= j_{t,s}^{(s)} \circ j_{t,t}^{(s)} \circ k_{t,s}^{(s)} \circ \ell \ell(s)(S_t) = j_{t,s}^{(s)} \circ k_{t,s}^{(s)} \circ \ell \ell(s)(S_t) = k_{t,s}^{(t,s)} \circ \ell \ell(s)(S_t) = \\
= k_{t,s}^{(t,s)} \circ \ell \ell(s)(G_u^{(t)}) \cup j_{t,s}^{(t)}(X_t) = \\
= \left\langle k_{t,s}^{(t,s)} \circ \ell \ell(s)(G_u^{(t)}) \cup j_{t,s}^{(t)}(X_t) \right\rangle = \\
= \left\langle j_{u,t}^{(s)} \circ k_{t,s}^{(s)}(X_t) \right\rangle = \\
= \left\langle j_{u,t}^{(s)}(G_u^{(t)}) \cup j_{t,s}^{(t)}(X_t) \right\rangle.
$$

\[\square\]

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