The Singular Angle of Nonlinear Systems

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Abstract—In this paper, we introduce an angle notion, called the singular angle, for stable nonlinear systems from an input-output perspective. The proposed system singular angle, based on the angle between $L_2$-signals, describes an upper bound for the “rotating effect” from the system input to output signals. It is, thus, different from the recently appeared nonlinear system phase which adopts the complexification of real-valued signals using the Hilbert transform. It can quantify the passivity and serve as an angular counterpart to the system $L_2$-gain. It also provides an alternative to the nonlinear system phase. A nonlinear small angle theorem, which involves a comparison of the loop system angle with $\pi$, is established for feedback stability analysis. When dealing with multi-input multi-output linear time-invariant (LTI) systems, we further come up with the frequency-wise and $H_\infty$ singular angle notions based on the matrix singular angle, and develop corresponding LTI small angle theorems.

Index Terms—Small angle theorem, singular angle, nonlinear systems, passivity, input-output system analysis.

I. INTRODUCTION

In classical control theory, the gain (or magnitude) and phase (or angle) are two fundamental and parallel concepts for single-input single-output (SISO) linear time-invariant (LTI) systems. Over the past half century, various efforts have been invested into generalizing these gain and phase notions for more general systems. It is well accepted that the $H_\infty$-norm [1] of multi-input multi-output (MIMO) LTI systems and the $L_2$-gain [2], [3] of nonlinear systems serve as the roles of the gain. However, consensus about the phase (or angle) counterpart is lacking among researchers, even for MIMO LTI systems. Notable attempts at exploring such a counterpart for MIMO LTI systems include the principal phase [4], phase uncertainties [5], [6] and phase margin [7], [8]. In addition, the authors of [9] and [10] recently proposed a suitable MIMO LTI system phase definition based on the matrix numerical range.

When systems become more complex, e.g., nonlinear systems, the notion of phase (or angle) is not well understood. The well-known passivity [2], [3] has been considered as a description of phasic flavor in nonlinear systems by some researchers [11] for a long time. Nevertheless, we think that the passivity is only qualitatively phase-related [12]. Recently, we developed a phase definition [12], [13] for a class of stable nonlinear systems from an input-output perspective. The key idea behind this definition is to complexify real-valued signals by using the analytic signal and Hilbert transform [14], since the notion of phase (or angle) arises naturally in a complex domain.

This paper aims to explore a new angle notion, called the singular angle, for stable nonlinear systems from an input-output perspective. The idea of the new notion comes from the angle between nonzero vectors in the Euclidean space, namely, $\theta(x, y) := \arccos \left( \frac{\langle x, y \rangle}{|x||y|} \right)$ for $0 \neq x, y \in \mathbb{R}^n$. The phrase “singular angle”, coined by H. Wielandt in his lecture notes [15, Section 23] in 1967, was originally defined for complex matrices using the angle between nonzero complex vectors. Concretely, the singular angle $\theta(A)$ of a nonzero matrix $A \in \mathbb{C}^{n \times n}$ is defined by the formula [15, Section 23]

$$\cos \theta(A) := \inf_{x \neq 0, x \in \mathbb{C}^n} \frac{\Re \{x^*Ax\}}{|x||Ax|}.$$  

The matrix singular angle also has some alternative names given by other mathematicians, such as the operator angle [16, Chapter 3], antieigenvalue [17], operator deviation [18] and real-part angle [19]. It is worth noting that, three of these ideas, i.e., the singular angle, operator angle and operator deviation, were all conceived independently in different contexts in the late 1960s. We adopt the appellation “singular angle” since, to the best of our knowledge, [15, Section 23] is the earliest work involving the matrix singular angle.

In this paper, we first adopt and study the angle between nonzero $L_2$-signals, which is a Hilbert space angle [16, Chapter 3]. Subsequently, inspired by the matrix singular angle, we define the singular angle of a nonlinear system using the angles between the input and output signals. The system singular angle can quantify the passivity, namely, the singular angle of a passive system is no greater than $\pi/2$. Meanwhile, it is related to, but distinct from, the existing input-output passivity indices [20], [21] which also quantify the passivity. In contrast to these indices, the system singular angle offers an alternative quantity from a purely angular viewpoint.

The system singular angle serves as an alternative counterpart to the system $L_2$-gain on account of the following similarities:

1) The $L_2$-gain is an operator norm induced by the $L_2$-signal norm, and the singular angle is also an “induced” notion rooted in signals;
2) the $L_2$-gain describes an upper bound for the “stretching effect” from the system input to output signals, while the
singular angle correspondingly provides an upper bound for the “rotating effect”.

A nonlinear small angle theorem is next developed for feedback stability analysis in terms of the loop system angle being less than π. This theorem complements the celebrated nonlinear small gain theorem [22] which involves the loop system $L_2$-gain being less than 1. It also generalizes a version of the passivity theorem [2], [3]. This theorem is further extended by using multipliers. Moreover, two variations of this theorem, an $L_2$-version and an incremental version, are proposed for the purpose of enriching the system singular angle theory.

When specializing to MIMO LTI systems, to reduce conservatism, we further propose the frequency-wise and $H_{\infty}$ singular angles on the basis of the matrix singular angle. We also discuss these angles defined in the frequency domain with the aforesaid singular angle defined in the time domain. An LTI frequency-wise small angle theorem is finally established for feedback stability.

The proposed system singular angle and the recent system phase [10], [12] are generally not the same, and they, having different strengths, are both worthy of investigation and development. The former stems from the Euclidean space angle, while the latter generalizes the phase of a complex number; The former has an advantage in studying cascaded interconnections, while the latter in investigating parallel interconnections. In short, this paper provides an alternative perspective of exploring the notion of phase (or angle) in nonlinear systems.

The angle between signals in the $L_2$ space has been used in the field of control by the leading works [23] and [24], in which it is utilized to define the secant gain for output strictly passive systems. Very recently, the incremental angle between signals in the $L_2$ space was presented as a part of the scaled relative graph of nonlinear operators [25], nonlinear systems [26], [27] and linear operators [28].

The remainder of the paper is structured as follows. In Section II, the preliminaries on signals and systems are included. In Section III, the angle between signals is introduced, and the singular angle of a nonlinear system is defined with several useful properties. Section IV is dedicated to a nonlinear small angle theorem for feedback stability analysis. Then, this theorem is applied to cyclic systems. Section V provides a link between the singular angle and passivity, and an application to Lur’e sections. Section VI obtains the singular angle of closed-loop systems from that of open-loop ones. In Section VII, we further propose the frequency-wise singular angle theory for MIMO LTI systems for the sake of reducing conservatism. Section VIII presents three variations of the singular angle, and compares the singular angle with the recent nonlinear system phase. Section IX concludes this paper.

II. Notation and Preliminaries

Let $F = \mathbb{R}$ or $\mathbb{C}$ be the field of real or complex numbers, and $F^n$ be the linear space of $n$-dimensional vectors over $F$. Denote $\mathbb{C}_+$ as the closed complex right half-plane. For $x, y \in F^n$, denote $\langle x, y \rangle$ and $|x| = \sqrt{\langle x, x \rangle}$ as the Euclidean inner product and norm, respectively. The conjugate, transpose and conjugate transpose of matrices are denoted by $(\cdot)^*$, $(\cdot)^T$ and $(\cdot)^{T*}$, respectively. The real and imaginary parts of a complex number $z \in \mathbb{C}$ are denoted by $\text{Re}(z)$ and $\text{Im}(z)$, respectively. The angle of a nonzero $z \in \mathbb{C}$ in the polar form $|z| e^{j \Omega}$ is denoted by $\angle z$. If $z = 0$, then $\angle z$ is undefined. Denote $\mathcal{RH}_{n \times n}$ as the space consisting of $n \times n$ real rational proper matrix-valued functions with no poles in $\mathbb{C}_+$ and $\mathcal{RH}_{1 \times 1} = \mathbb{R}$.

Let the $L_2$ space be the set of all energy-bounded $\mathbb{R}^n$-valued signals:

$$L_2^2 := \left\{ u : \mathbb{R} \to \mathbb{R}^n \mid \|u\|^2_2 := \int_{-\infty}^{\infty} |u(t)|^2 dt < \infty \right\}.$$  

The superscript $n$ is often dropped when the dimension is clear from the context. The causal subspace of $L_2$ is denoted by

$$L_2[0, \infty) := \left\{ u \in L_2 \mid u(t) = 0 \text{ for } t < 0 \right\}.$$  

For $T \geq 0$, define the truncation $\Gamma_T$ on all $u : \mathbb{R} \to \mathbb{R}^n$ by

$$\left( \Gamma_T u \right)(t) := \begin{cases} u(t), & t \leq T, \\ 0, & t > T. \end{cases}$$  

For simplicity, we denote $u_T := \Gamma_T u$ for any $T \geq 0$. The extended space of $L_2[0, \infty)$ is then denoted by

$$L_{2e}[0, \infty) := \left\{ u : \mathbb{R} \to \mathbb{R}^n \mid u_T \in L_2[0, \infty), \forall T \geq 0 \right\}.$$  

Let $\hat{u}$ denote the Fourier transform of a signal $u \in L_2$. By the well-known Plancherel’s theorem, for all $u, v \in L_2$, we have

$$\langle u, v \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(j\omega) \hat{v}(j\omega) d\omega.$$  

An operator $P : L_{2e}[0, \infty) \to L_{2e}[0, \infty)$ is said to be causal if $\Gamma_T P = \Gamma_T P \Gamma_T$ for all $T \geq 0$, and is said to be noncausal if it is not causal. We always assume that an operator $P$ maps the zero signal to the zero signal, i.e., $P0 = 0$. We view a system as an operator from real-valued input signals to real-valued output signals. Additionally, we consider only “square” systems with the same number of inputs and outputs, and assume that these systems are nonzero, i.e., $P \neq 0$.

A nonlinear system is represented by a causal operator $P : L_{2e}[0, \infty) \to L_{2e}[0, \infty)$. The $L_2$ domain of $P$, namely, the set of all its input signals in $L_2[0, \infty)$ such that the output signals are in $L_2[0, \infty)$, is denoted by $\text{dom}(P) := \{ u \in L_2[0, \infty) \mid Pu \in L_2[0, \infty) \}$. Such a causal system $P$ (operator, resp.) is said to be stable (bounded, resp.) if $\text{dom}(P) = L_2[0, \infty)$ and

$$\|P\| := \sup_{0 \neq u \in L_2[0, \infty)} \frac{\|Pu\|_2}{\|u\|_2} < \infty. \quad (1)$$  

Here, $\|P\|$ is called the $L_2$-gain of $P$ and it is the key quantity used in the gain-based input-output nonlinear system control theory. In addition, it holds that [3, Proposition 1.2.3]

$$\|P\| = \sup_{u \in L_2[0, \infty), T > 0} \frac{\|Pu\|_2}{\|u\|_2}.$$  

A causal stable system $P$ is called passive [3] if

$$\langle u_T, (Pu)_T \rangle \geq 0, \quad \forall u \in L_{2e}[0, \infty) \text{ and } \forall T > 0,$$  

and...
which is equivalent to [3, Proposition 2.2.5]
\[ \langle u, Pu \rangle \geq 0, \quad \forall u \in L_2[0, \infty). \]

III. THE ANGLE BETWEEN SIGNALS AND THE SINGULAR ANGLE OF A NONLINEAR SYSTEM

This section is devoted to establishing an angle notion, called the singular angle, for nonlinear systems. It manifests that the angle between $L_2$-signals is a pseudometric function, endowed with a triangle inequality. In addition, the system singular angle captures an upper bound of the “rotating effect” from the system input to output signals.

A. The Angle Between Signals

For any $u, v \in L_2[0, \infty)$, we define the angle $\theta(u, v) \in [0, \pi]$ between $u$ and $v$ by the following formula:
\[ \cos \theta(u, v) := \frac{\langle u, v \rangle}{\|u\|_2 \|v\|_2}, \quad \text{if } u, v \in L_2[0, \infty) \setminus \{0\}, \quad (2) \]
and $\theta(u, v) = 0$, otherwise. In light of the Cauchy-Schwarz inequalities, i.e., $|\langle u, v \rangle| \leq \|u\|_2 \|v\|_2$ for all $u, v \in L_2[0, \infty)$, the ratio in (2) takes values in $[-1, 1]$, and thus $\theta(u, v)$ is always well defined. The angle between signals is a natural extension of the Euclidean space angle between vectors.

The following lemma introduces a useful triangle inequality of the angles between signals, which plays a significant role in the feedback stability analysis in Section IV. This lemma is a modification of [16, Lemma 3.3-1] and the result in [18]. To make the content self-contained, we include the proof in Appendix A.

Lemma 1: For all $u, w \in L_2[0, \infty)$ and $v \in L_2[0, \infty) \setminus \{0\}$, it holds that
\[ \theta(u, w) \leq \theta(u, v) + \theta(v, w). \]

Formula (2) and Lemma 1 indicate that the angle between signals $\theta: (L_2[0, \infty) \setminus \{0\}) \times (L_2[0, \infty) \setminus \{0\}) \rightarrow [0, \pi]$ is a pseudometric on account of the following three properties:

1) Pseudo identity of indiscernibles: $\theta(u, v) = 0$ if and only if $u = kv$ for some scalars $k > 0$;
2) symmetry: $\theta(u, v) = \theta(v, u)$;
3) triangle inequality: $\theta(u, w) \leq \theta(u, v) + \theta(v, w)$
for all $u, v, w \in L_2[0, \infty) \setminus \{0\}$. Additionally, the following property is as expected:
\[ \theta(ku, v) = \begin{cases} \theta(u, v), & k > 0, \\ \pi - \theta(u, v), & k < 0. \end{cases} \]

B. The Singular Angle of a Nonlinear System

Having introduced the angle between signals, we proceed to define the singular angle of a nonlinear system associated with the input and output signal pairs. Consider a causal stable system $P: L_2[0, \infty) \rightarrow L_2[0, \infty)$. The singular angle of $P$, denoted by $\theta(P) \in [0, \pi]$, is defined via the formula
\[ \cos \theta(P) := \inf_{0 \neq u \in L_2[0, \infty), \|u\|_2 \neq 0} \frac{\langle u, Pu \rangle}{\|u\|_2 \|Pu\|_2}. \]

It is well known that the $L_2$-gain defined in (1) describes an upper bound for the “stretching effect” of a system from the input to output signals. Analogously, the singular angle proposed in (3) can be interpreted as an upper bound for the “rotating effect” of a system from the input to output signals. The imagination of this “rotating effect” is naturally borrowed from the Euclidean space angle.

Based on (3), an equivalent characterization for a stable passive system $P$ is $\theta(P) \in [0, \pi/2]$. Apparently, the system singular angle can quantify the passivity from a new angular perspective. We will elaborate on connections of the singular angle and passivity in Section V.

The system singular angle has its advantages in studying cascaded interconnected systems. For two given systems $P_1$ and $P_2$, a cascaded interconnected system $P$ is defined to be $P = P_2P_1$. The following proposition presents how $\theta(P)$ is related to $\theta(P_1)$ and $\theta(P_2)$, respectively.

Proposition 1: For causal stable systems $P_1: L_2[0, \infty) \rightarrow L_2[0, \infty)$ and $P_2: L_2[0, \infty) \rightarrow L_2[0, \infty)$, the cascaded interconnected system $P = P_2P_1$ satisfies
\[ \theta(P) \leq \theta(P_1) + \theta(P_2). \]

Proof: Let $y_1 = P_1u_1$, $y_2 = P_2u_2$. Since $y_1 = u_2$, for all $u_1, P_1u_1, P_2P_1u_1 \in L_2[0, \infty) \setminus \{0\}$, it holds that
\[ \theta(u_1, y_2) = \theta(u_1, P_2P_1u_1) \leq \theta(u_1, P_1u_1) + \theta(P_1, P_2P_1u_1) \]
according to Lemma 1. Thus, we have
\[ \theta(P_2P_1) \leq \sup_{u_1, P_1u_1 \neq 0} \sup_{0 \neq u_2 \in L_2[0, \infty), P_2u_2 \neq 0} \theta(u_1, P_1u_1) + \theta(u_2, P_2u_2) \]

which completes the proof.

IV. A NONLINEAR SMALL ANGLE THEOREM

This section presents the main result of this paper, a nonlinear small angle theorem. The theorem states a new feedback stability condition in terms of the “loop system singular angle” being less than $\pi$. It can be regarded as an alternative angular complement to the famous nonlinear small gain theorem.

Consider the feedback system shown in Fig. 1, where $P: L_2[0, \infty) \rightarrow L_2[0, \infty)$ and $C: L_2[0, \infty) \rightarrow L_2[0, \infty)$
are two causal stable systems, $e_1$ and $e_2$ are external signals, and $u_1, u_2, y_1$ and $y_2$ are internal signals. Let $P \# C$ denote this feedback system. Algebraically, we have the following equations:

$$u = e - \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} y \quad \text{and} \quad y = \begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix} u,$$

where $u = [u_1, u_2]^T$, $e = [e_1, e_2]^T$ and $y = [y_1, y_2]^T$. We assume that all the feedback systems in this paper are well-posed in the following sense.

**Definition 1:** The feedback system $P \# C$ is said to be well-posed if

$$F_{P,C} := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} : L_2(0, \infty) \to L_2(0, \infty) = \begin{bmatrix} I & C \\ -P & I \end{bmatrix}$$

has a causal inverse on $L_2(0, \infty)$.

The input-output feedback stability is defined as follows.

**Definition 2:** A well-posed feedback system $P \# C$ is said to be stable if there exists a constant $c > 0$ such that

$$\|\Gamma_T u\|_2 \leq c \|\Gamma_T e\|_2$$

for all $T \geq 0$ and for all $e \in L_2(0, \infty)$. We, in the following, consider a special structure of $P \# C$, namely, a single-loop feedback system obtained by setting the external signal $e_2 = 0$ in Fig. 1. Let $P \# C|_{e_2=0}$ denote this single-loop feedback system. The stability of $P \# C|_{e_2=0}$ can be defined in a similar fashion to Definition 2. The motivation is that very often it is sufficient to study $P \# C|_{e_2=0}$ when investigating nonlinear feedback systems [29]. In particular, if $C$ is linear, then $e_2$ in $P \# C$ can be included in $e_1$ [30, Section 8]. In such a case, $P \# C$ is equivalent to $P \# C|_{e_2=0}$.

We revisit a basic version of the classical nonlinear small gain theorem [22], [3, Section 2.1]: The well-posed feedback system $P \# C|_{e_2=0}$ is stable if

$$\|P\| \|C\| < 1. \quad (4)$$

In what follows, we in parallel establish the so-called nonlinear small angle theorem which ensures feedback stability using the singular angles.

**Theorem 1 (Nonlinear small angle theorem):** The well-posed feedback system $P \# C|_{e_2=0}$ is stable if

$$\theta(P) + \theta(C) < \pi. \quad (5)$$

**Proof:** When special zero signals are involved in the feedback loop, namely, $u_1 = 0$, $u_2 = 0$ or $y_2 = 0$, the stability of $P \# C|_{e_2=0}$ can be shown separately. When $u_1 = 0$, it holds that $y_1 = u_2 = y_2 = 0$; When $u_2 = 0$, the stability of $P \# C|_{e_2=0}$ is deduced from the open-loop stability of $P$; When $y_2 = 0$, $P \# C|_{e_2=0}$ becomes a cascaded open-loop system $CP$. Since $C$ and $P$ are stable, then $CP$ is stable. Therefore, it suffices to show the proof for the case $u_1, u_2, y_2 \in L_2(0, \infty) \setminus \{0\}$. The singular angle as a system property is defined on $L_2(0, \infty)$, while the well-posedness and closed-loop stability are defined on $L_2(0, \infty)$. To deal with this gap, we use a homotopy method $\tau \in [0, 1]$ and several steps to prove the result, which are inspired by [29].

**Step 1:** For all $u \in L_2[0, \infty) \setminus \{0\}$ and $\tau \in [0, 1]$, show that there exists $c_0 > 0$, independent of $\tau$, such that

$$\|u\|_2 \leq c_0 \|F_{P,\tau-u}\|_2.$$

Let $y_1 = Pu_1$ and $y_2 = \tau Cy_2$. When $\tau = 0$, $P \# (\tau C)|_{e_2=0}$ is stable by the open-loop stability. We only need to consider the case $\tau \in (0, 1]$ and note that $\theta(C) = \theta(\tau C)$ for all $\tau \in (0, 1]$. Since $\cos(\cdot)$ is a decreasing function on $[0, \pi]$, by hypothesis (5), we have

$$\cos(\theta(P) + \theta(\tau C)) > -1.$$

The above inequality implies

$$\cos(\theta(u_1, y_1) + \theta(u_2, y_2)) > -1$$

for all $u_1, u_2 \in L_2(0, \infty) \setminus \{0\}$ and $y_1, y_2 \neq 0$. By Lemma 1 and $y_1 = u_2$, we have

$$-1 < \cos(\theta(u_1, y_1) + \theta(u_2, y_2)) \leq \cos \theta(u_1, y_1) \leq 1$$

for all $u_1, u_2 \in L_2(0, \infty) \setminus \{0\}$ and $y_1, y_2 \neq 0$. Note that

$$\|F_{P,\tau-u}\|_2^2 = \left\|\begin{bmatrix} u_1 + y_2 \\ 0 \end{bmatrix}\right\|_2^2 = \|u_1\|^2_2 + \|y_2\|^2_2 + 2 \|u_1\|_2 \|y_2\|_2 \cos \theta(u_1, y_2)$$

for all $u_1, u_2 \in L_2(0, \infty) \setminus \{0\}$ and $y_1, y_2 \neq 0$. Noting [29] Firstly, we assume that $\theta(u_1, y_1) \in [0, \pi/2]$, and thus $\cos \theta(u_1, y_2) \in [0, 1]$. In this case, discarding the nonnegative terms $\|y_1\|_2$ and $2 \|u_1\|_2 \|y_2\|_2 \cos \theta(u_1, y_2)$ in (6) yields

$$\|u_1\|_2 \leq \|F_{P,\tau-u}\|_2.$$

Since $P$ is stable and $u_2 = y_1$, we have

$$\|u_2\|_2 = \|Pu_1\|_2 \leq \|P\| \|u_1\|_2 \leq \|P\| \|F_{P,\tau-u}\|_2.$$
where \( \pi/2 < \theta(u_1, y_2) \leq \theta(P) + \theta(\tau C) < \pi \) and \( \theta(\tau C) = \theta(C) \) when \( \tau \in (0, 1] \). Moreover, since \( P \) is stable, there exists a constant \( c_2 := c_1 \| P \| > 0 \) such that

\[
\| u_2 \|_2 = \| P u_1 \|_2 \leq \| P \| \| u_1 \|_2 \leq c_2 \| F_{P, \tau C} u_2 \|_2.
\]

Note that \( c_1 > 1 \). Thus, we can unify the two cases \( \theta(u_1, y_2) \in [0, \pi/2] \) and \( \theta(u_1, y_2) \in (\pi/2, \pi) \) in the following inequality:

\[
\| u \|_2 \leq \| u_1 \|_2 + \| u_2 \|_2 \leq (c_1 + c_2) \| F_{P, \tau C} u \|_2.
\]

Then, there exists a constant \( c_0 := c_1 + c_2 > 0 \), independent of \( \tau \), such that for all \( u \in \mathcal{L}_2[0, \infty) \setminus \{0\} \) and for all \( \tau \in [0, 1] \),

\[
\| u \|_2 \leq c_0 \| F_{P, \tau C} u \|_2.
\]

**Step 2:** Show that the stability of \( P \# (\tau C) \big|_{\tau=0} \) implies the stability of \( P \# (\tau + \nu C) \big|_{\tau=0} \) for all \( |\nu| < \mu = 1/(c_0 \| C \|) \), where \( \mu \) is independent of \( \tau \).

By the well-posedness assumption, the inverse \( (F_{P, \tau C})^{-1} \) is well-defined on \( \mathcal{L}_{2\epsilon}([0, \infty)) \). By hypothesis, \( (F_{P, \tau C})^{-1} \) is bounded on \( \mathcal{L}_2([0, \infty)) \). Given \( u \in \mathcal{L}_{2\epsilon}([0, \infty)) \), we define

\[
u_T := (F_{P, \tau C})^{-1} \Gamma_T (F_{P, \tau C} u) \in \mathcal{L}_2([0, \infty)).
\]

Then, we have

\[
\| \Gamma_T u \|_2 = \| \Gamma_T \nu_T \|_2 \leq \| \nu_T \|_2 \leq c_0 \| (F_{P, \tau C} u) \|_2
\]

Note that \( c_0 > 1 \). Thus, we can unify the two cases \( \theta(u_1, y_2) \in [0, \pi/2] \) and \( \theta(u_1, y_2) \in (\pi/2, \pi) \) in the following inequality:

\[
\| u \|_2 \leq \| u_1 \|_2 + \| u_2 \|_2 \leq (c_0 + c_2) \| (F_{P, \tau C} u) \|_2.
\]

Then, there exists a constant \( c_0 := c_1 + c_2 > 0 \), independent of \( \tau \), such that for all \( u \in \mathcal{L}_2[0, \infty) \setminus \{0\} \) and for all \( \tau \in [0, 1] \),

\[
\| u \|_2 \leq c_0 \| (F_{P, \tau C} u) \|_2.
\]

**A. An Application to the Cyclic Systems**

Consider the feedback system with a special structure shown in Fig. 2, which is called a cyclic system, where \( P_1, P_2, \ldots, P_N \) are \( N \) causal stable subsystems defined on \( \mathcal{L}_{2\epsilon}([0, \infty)) \). The cyclic system and its stability condition have been studied in the biochemistry [31], [32], biology and control [23], [24] disciplines. A notable stability condition called the secant condition is given in [23], [24]. Concretely, for an output strictly passive system \( P : \mathcal{L}_{2\epsilon}([0, \infty)) \rightarrow \mathcal{L}_{2\epsilon}([0, \infty)) \), there exists \( \gamma > 0 \) such that

\[
\gamma < \mu < 1/(c_0 \| C \|) := \mu.
\]

**Step 3:** Show that \( P \# (\tau C) \big|_{\tau=0} \) is stable when \( \tau = 1 \).

When \( \tau = 0 \), \( (F_{P,\tau C})^{-1} \) is bounded since \( P \) is open-loop stable. It has been shown in Step 2 that \( (F_{P,\tau C})^{-1} \) is bounded for \( \tau < \mu \), and then it is bounded for \( \tau < 2\mu \) using the iterative process, etc. By induction, \( (F_{P,\tau C})^{-1} \) is bounded for all \( \tau \in [0, 1] \). We conclude that \( P \# C \big|_{\tau=0} \) is stable by setting \( \tau = 1 \).

We call (5) the small angle condition, which serves as an alternative counterpart to the elegant small gain condition (4). Specifically, the former involves a comparison of the loop singular angle \( \theta(P) + \theta(\tau C) \) with \( \pi \), while the latter a comparison of the loop \( \mathcal{L}_{2\epsilon} \)-gain \( \| P \| \| C \| \) with 1.

Theorem 1 also holds for the other single-loop feedback system \( P \# C \big|_{\tau=0} \) defined in the sense of setting the other external signal \( e_1 = 0 \) in Fig. 1. Observe that the stronger feedback stability with an “infinite gain margin” is guaranteed in Theorem 1. Concretely, if condition (5) holds, then the well-posed \((\tau P) \# C \) is also stable for all \( \tau > 0 \). A one-line proof follows from \( \theta(\tau P) = \theta(P) \) for all \( \tau > 0 \). This coincides with the infinite gain margin concept in classical control theory.

When \( P \) is stable passive and \( C \) is very strictly passive, Theorem 1 reduces to a version of the passivity theorem by noting \( \theta(P) \leq \pi/2 \) and \( \theta(C) < \pi/2 \). To make this clear, we build a link between the system singular angle and very strict passivity, as detailed in Section V.

In Theorem 1, only two systems \( P \) and \( C \) are involved in the standard feedback loop. This does not fully reveal the advantage of the system singular angle in terms of the cascaded property presented in Proposition 1. To this end, in the next subsection, we apply Theorem 1 to a commonly-seen cyclic feedback system.

**Fig. 2. A cyclic feedback system.**

\[
\gamma(u, P u) \geq \| P u \|_2^2, \quad \forall u \in \mathcal{L}_2[0, \infty).
\]
Proof: According to the triangle inequality in Lemma 1, we have $\theta(u_1, y_N) > -\pi$. This corollary then can be proved using the same arguments as in the proof of Theorem 1.

V. RELATION TO THE PASSIVITY

In this section, we establish connections between the system singular angle and the well-known notion of passivity in nonlinear systems. As a prologue, we first estimate the singular angle of very strictly passive systems. Second, we demonstrate that a system with a singular angle greater than $\pi/2$ can be equivalently represented by an input-feedforward-output-feedback passive system with a group of constraints. Finally, we apply the nonlinear small angle theorem to a Lur'e system by estimating the singular angle of a sector bounded static nonlinearity.

A common practice for quantifying passive systems is to introduce the input-output passivity indices [20], [21]. Specifically, a causal stable system $P: L_2[0, \infty) \to L_2[0, \infty)$ is called very strictly passive if there exist $\nu, \rho > 0$ such that

$$\langle u, Pu \rangle \geq \nu \|u\|_2^2 + \rho \|Pu\|_2^2, \quad \forall u \in L_2[0, \infty),$$

where $\nu$ and $\rho$ are called the input passivity index and output passivity index, respectively. In addition, $P$ is called input-feedforward-output-feedback passive if (9) holds for certain $\nu, \rho < 0$ for all $u \in L_2[0, \infty)$.

The following proposition shows that the singular angle of a very strictly passive system can be well estimated.

**Proposition 2:** For a very strictly passive system $P$ with input and output indices $\nu, \rho > 0$, it holds that

$$\theta(P) \leq \arccos 2\sqrt{\nu \rho} < \pi/2.\quad (10)$$

Proof: For all $u \in L_2[0, \infty) \setminus \{0\}$ with $Pu \neq 0$, rearranging inequality (9) yields

$$\frac{\langle u, Pu \rangle}{\|u\|_2^2} \geq \nu \|Pu\|_2^2 + \rho \|Pu\|_2^2.$$

The right-hand side of the above inequality is bounded from below by a constant according to the geometric and arithmetic means inequality, namely,

$$\nu \|Pu\|_2^2 + \rho \|Pu\|_2^2 \geq 2\sqrt{\nu \rho \|Pu\|_2^2} \geq 2\sqrt{\nu \rho}.$$ 

This gives

$$\cos \theta(P) = \inf_{0 \neq u \in L_2[0, \infty), \|u\|_2^2 \neq 0} \frac{\langle u, Pu \rangle}{\|u\|_2^2 \|Pu\|_2^2} \geq 2\sqrt{\nu \rho}$$

and $\theta(P) \leq \arccos 2\sqrt{\nu \rho} < \pi/2$.

Note that the domain of $\arccos(\cdot)$ is $[-1, 1]$, which implies that, in (10), $\nu$ and $\rho$ should first satisfy $\nu \rho < 1/4$. In fact, this is an implicit constraint for very strictly passive systems. See [33, Lemma 2.6].

The next proposition reveals that a system whose singular angle is greater than $\pi/2$ is an input-feedforward-output-feedback passive system satisfying a group of constraints.

**Proposition 3:** Let $P$ be a causal stable system and $\alpha \in (\pi/2, \pi]$. The following two statements are equivalent:

1) $\theta(P) \in (\pi/2, \alpha]$;
2) For all $\nu, \rho > 0$ satisfying $\nu \rho = (\cos \alpha)^2/4$, it holds that

$$\langle u, Pu \rangle \geq \nu \|u\|_2^2 + \rho \|Pu\|_2^2, \quad \forall u \in L_2[0, \infty).$$

Proof: 1) $\rightarrow$ 2): By singular angle definition (3), for all $u \in L_2[0, \infty)$, we have

$$\langle u, Pu \rangle \geq \cos \theta(P) \|u\|_2^2 \|Pu\|_2 \geq \cos \alpha \|u\|_2 \|Pu\|_2 \geq \cos \alpha \frac{c \|u\|_2^2 + \rho \|Pu\|_2^2}{2}, \quad \forall c > 0,$$

where the second and last inequalities use the assumption $0 > \cos \theta(P) \geq \cos \alpha$ and the geometric and quadratic means (GM-QM) inequality, respectively. Therefore, the indices $\nu$ and $\rho$ are parameterized by $c$ in the following way:

$$\nu = \frac{c \cos \alpha}{2} < 0 \quad \text{and} \quad \rho = \frac{\cos \alpha}{2c} < 0, \quad \forall c > 0.$$

2) $\rightarrow$ 1): We need to show is that, for all $u \in L_2(0, \infty) \setminus \{0\}$ with $Pu \neq 0$, we have

$$\frac{\langle u, Pu \rangle}{\|u\|_2 \|Pu\|_2} \geq \cos \alpha.$$

To this end, let $u$ be arbitrary in $L_2(0, \infty) \setminus \{0\}$ with $Pu \neq 0$. Then, we choose that

$$\nu = \frac{\cos \alpha \|Pu\|_2}{2 \|u\|_2^2} < 0 \quad \text{and} \quad \rho = \frac{\cos \alpha \|u\|_2^2}{2 \|Pu\|_2^2} < 0.$$

Clearly, $\nu \rho = (\cos \alpha)^2/4$. Then, assertion 2) tells that

$$\langle u, Pu \rangle \geq \frac{\cos \alpha \|Pu\|_2^2}{2 \|u\|_2^2} + \frac{\cos \alpha \|Pu\|_2^2 \|Pu\|_2^2}{2} \|u\|_2 = \cos \alpha \|Pu\|_2 ^2$$

This completes the proof.

Note that the indices $\nu$ and $\rho$ in Proposition 3 should meet $\nu \rho = (\cos \alpha)^2/4$; i.e., their product is a constant. This gives an input-feedforward-output-feedback passive system satisfying a set of constraints parameterized by $\alpha$. Roughly speaking, a system singular angle greater than $\pi/2$ can be understood via merging a group of the two constrained indices together. But can a system singular angle smaller than $\pi/2$ be interpreted in a similar way? The current answer is half-right: A very strictly passive system satisfying a convex combination condition of the indices implies a singular angle smaller than $\pi/2$. This is elaborated in the following proposition as an extension of Proposition 2.

**Proposition 4:** Let $P$ be a causal stable system and a group of numbers $\nu_k, \rho_k > 0$ with $\nu_k \rho_k \leq 1/4$, where $k = 1, 2, \ldots, \infty$. Suppose that, for all $\tau_k \geq 0$ with $\sum_{k=1}^{\infty} \tau_k = 1$, we have

$$\langle u, Pu \rangle \geq \sum_{k=1}^{\infty} \tau_k \nu_k \|u\|_2^2 \|Pu\|_2^2, \quad \forall u \in L_2[0, \infty).$$

Then, $\theta(P) \leq \arccos 2\sqrt{\max_k \nu_k \rho_k}$.
Proof: For all \( u \in \mathcal{L}_2[0, \infty) \setminus \{0\} \), we have
\[
\langle u, Pu \rangle \geq \sum_{k=1}^{\infty} \tau_k \nu_k \| u \|_2^2 + \sum_{k=1}^{\infty} \tau_k \rho_k \| Pu \|_2^2
= \sum_{k=1}^{\infty} \tau_k \left( \nu_k \| u \|_2^2 + \rho_k \| Pu \|_2^2 \right)
\geq \sum_{k=1}^{\infty} \tau_k \left( 2\sqrt{\nu_k \rho_k} \| u \|_2 \| Pu \|_2 \right),
\]
where the last inequality uses the GM-QM inequality. Since \( \tau_k \geq 0 \) and \( \sum_{k=1}^{\infty} \tau_k = 1 \), the maximum of the right-hand side of (11) is given by \( 2\sqrt{\max_k \nu_k \rho_k} \| u \|_2 \| Pu \|_2 \). This implies
\[
\frac{\langle u, Pu \rangle}{\| u \|_2 \| Pu \|_2} \geq 2\sqrt{\max_k \nu_k \rho_k}
\]
for all \( u \in \mathcal{L}_2[0, \infty) \setminus \{0\} \) with \( Pu \neq 0 \). It follows from definition (3) that \( \theta(P) \leq \arccos 2\sqrt{\max_k \nu_k \rho_k} \).

A. An Application to the Lur’e Systems

Consider a scalar static nonlinear system \( N : \mathcal{L}_2[0, \infty) \to \mathcal{L}_2[0, \infty) \) defined by
\[
(Nu)(t) = h(u(t)),
\]
where \( h : \mathbb{R} \to \mathbb{R} \) satisfies
\[
(h(x) - ax)(h(x) - bx) \leq 0, \quad \forall x \in \mathbb{R}
\]
with \( b > a > 0 \). The graphical representation of \( h \), illustrated by Fig. 3, belongs to a sector whose boundaries are the two lines \( y = ax \) and \( y = bx \). Consequently, we say \( N \) is a sector bounded static nonlinearity, and it belongs to the nonlinearity sector from \( a \) to \( b \). It is known that such a nonlinearity is a special very strictly passive system. To observe this, according to (12), for all \( u \in \mathcal{L}_2[0, \infty) \) \( \setminus \{0\} \) and for all \( t \geq 0 \), we have
\[
(a + b)u(t)(Nu)(t) \geq ab |u(t)|^2 + |(Nu)(t)|^2.
\]
Rearranging and integrating both sides of the above inequality yield
\[
\langle u, Nu \rangle \geq \frac{ab}{a + b} \| u \|_2^2 + \frac{1}{a + b} \| Nu \|_2^2
\]
since \( b > a > 0 \). Note that, by (9), \( N \) is a very strictly passive system by specifying \( \nu = ab/(a + b) \) and \( \rho = 1/(a + b) \). Therefore, we can estimate \( \theta(N) \) from the nonlinearity sector parameters \( a \) and \( b \), as detailed in the following proposition.

The proof follows directly from invoking Proposition 2 and is thus omitted.

Proposition 5: For a static nonlinear system \( N \) satisfying (12), we have
\[
\theta(N) \leq \arccos \frac{2\sqrt{ab}}{a + b}.
\]

It is noteworthy that, the estimation of the system singular angle \( \theta(N) \) in Proposition 5 is equal to that of the nonlinear system phase \( \theta(N) \) proposed in our recent paper [12, Proposition 1], namely,
\[
\arccos \frac{2\sqrt{ab}}{a + b} = \arcsin \frac{b - a}{a + b}.
\]

This equality shows one connection between the system singular angle and the system phase [12], although these two notions are generally different. A more detailed comparison of these two notions is provided in Section VIII-D.

Fig. 4. A SISO Lur’e system.

Fig. 5. The interpretations of the circle criterion with the disk \( D(a, b) \) (blue) and the nonlinear small angle theorem with two rays (dashed) for a SISO Lur’e system when \( b > a > 0 \).

The sector bounded static nonlinearity is widely adopted in the modeling of nonlinear feedback systems. For example, consider the simple Lur’e system \( P \# N \) [2, Section 5.6] illustrated by Fig. 4. It consists of a SISO LTI system \( P \) with \( P(s) \in \mathcal{RH}_\infty \) and a scalar static system \( N \) satisfying nonlinearity sector condition (12). The aim is to derive a stability condition on \( P(s) \) against all static nonlinearities contained in a sector from \( a \) to \( b \). For such a \( P \# N \), the celebrated circle criterion [34], [2, Section 6.6.1] has stood out as being endowed with a nice geometric interpretation. Specifically, \( P(s) \) is required to meet
\[
\inf_{\omega \in [-\infty, \infty), z \in D(a, b)} |P(j\omega) - z| > 0,
\]
where \( D(a, b) \) denotes the disk defined by
\[
D(a, b) := \left\{ z \in \mathbb{C} \mid z + \frac{a + b}{2ab} \leq \frac{b - a}{2ab} \right\}.
\]
In other words, the Nyquist plot of \( P(s) \) is bounded away from the disk \( D(a,b) \). See Fig. 5 for an illustration.

It is known that both magnitude and angle information of \( P(s) \) is utilized in the circle criterion. However, only angle information is considered in Theorem 1. For a fair comparison of the circle criterion with Theorem 1, we rule out the magnitude part by imposing a stronger stability requirement on \( P \neq N \); namely, \((\tau P) \neq N \) should be stable for all \( \tau > 0 \).
Roughly speaking, an “infinite gain margin” is required. In such a case, the disk \( D(a,b) \) will be enlarged and become a cone formed by infinitely many disks. This cone, shown in Fig. 5, is equivalently characterized by the angular condition in the following corollary, which is rooted in Theorem 1. The proof is provided in Appendix A.

\textit{Corollary 2:} The well-posed Lur’e system \((\tau P) \neq N \) is stable for all \( \tau > 0 \) if
\[
\angle P(j\omega) \in \left( \arccos \frac{2\sqrt{ab}}{a+b} - \pi, \pi - \arccos \frac{2\sqrt{ab}}{a+b} \right)
\]
for all \( \omega \in [-\infty, \infty] \) and \( P(j\omega) \neq 0 \).

\section{VI. THE SINGULAR ANGLES OF FEEDBACK INTERCONNECTED SYSTEMS}

A complex nonlinear network is often composed of a large number of subsystems. For such a network, it is desirable to have scalable analysis; i.e., the property of the network can be deduced from the properties of the subsystems. We have shown in Proposition 1 the advantage of the singular angle in studying cascaded interconnected systems. In what follows, we further study the singular angle of feedback interconnected systems by connecting the singular angles of open-loop and closed-loop systems, respectively.

For the well-posed feedback system \( P \neq C|_{e_2=0} \) in Fig. 1, denote the closed-loop map from \( e_1 \) to \( y_1 \) as
\[
G := e_1 \rightarrow y_1 : L_{2e}[0, \infty) \rightarrow L_{2e}[0, \infty).
\]
Given \( \alpha \in [0, \pi/2) \), let \( \mathcal{P}_\alpha \) be the set of angle-bounded systems defined by
\[
\mathcal{P}_\alpha := \{ P : L_{2e}[0, \infty) \rightarrow L_{2e}[0, \infty] \mid \theta(P) \leq \alpha \}.
\]
Apparently, the set \( \mathcal{P}_\alpha \) is a cone, namely, \( kP \in \mathcal{P}_\alpha \) for \( P \in \mathcal{P}_\alpha \) and \( k > 0 \). In addition, the set \( \mathcal{P}_\alpha \) is closed under the feedback map \( G \), as detailed in the next proposition.

\textit{Proposition 6:} For \( P, C \in \mathcal{P}_\alpha \), it holds that \( G \in \mathcal{P}_\alpha \).

\textit{Proof:} By hypothesis, small angle condition (5) is satisfied. According to Theorem 1, \( G \) is stable. The hypothesis also gives
\[
\langle u_1, y_1 \rangle \geq \cos \alpha \| u_1 \|_2 \| y_1 \|_2,
\]
\[
\langle u_2, y_2 \rangle \geq \cos \alpha \| u_2 \|_2 \| y_2 \|_2
\]
for all \( u_1, u_2 \in L_2[0, \infty) \setminus \{0\} \) and \( y_1, y_2 \neq 0 \). Adding the above two inequalities together and using \( u_1 = e_1 - y_2, u_2 = y_1 \) yield
\[
\langle e_1, y_1 \rangle \geq \cos \alpha (\| e_1 - y_2 \|_2 + \| y_1 \|_2 \| y_2 \|_2).
\]
Note that \( \cos \alpha > 0 \), and we have
\[
\frac{\langle e_1, y_1 \rangle}{\| y_1 \|_2} \geq \cos \alpha (\| e_1 - y_2 \|_2 + \| y_2 \|_2) \geq \cos \alpha \| e_1 \|_2,
\]
where the last inequality uses the triangle inequality. Then,
\[
\cos \theta(G) = \inf_{0 \neq e_1 \in L_2[0, \infty), \| e_1 \|_2 \| G e_1 \|_2} \geq \cos \alpha
\]
and \( \theta(G) \leq \alpha \). We thus conclude that \( G \in \mathcal{P}_\alpha \).

When the open-loop systems \( P \) and \( C \) have distinct singular angles, the following proposition indicates that \( \theta(G) \) can be well estimated from \( \theta(P) \) and \( \theta(C) \).

\textit{Proposition 7:} For a stable feedback system \( P \neq C|_{e_2=0} \), we assume \( \theta(P) + \theta(C) \leq \pi \). It holds that
\[
\theta(G) \leq \max \{ \theta(P), \theta(C) \}.
\]

\textit{Proof:} By hypothesis, we know that \( G \) is stable and the following inequalities
\[
\langle u_1, y_1 \rangle \geq \cos \theta(P) \| u_1 \|_2 \| y_1 \|_2,
\]
\[
\langle u_2, y_2 \rangle \geq \cos \theta(C) \| u_2 \|_2 \| y_2 \|_2
\]
hold for all \( u_1, u_2 \in L_2[0, \infty) \setminus \{0\} \) and \( y_1, y_2 \neq 0 \). Adding the two inequalities together and using \( u_1 = e_1 - y_2, u_2 = y_1 \) yield
\[
\frac{\langle e_1, y_1 \rangle}{\| y_1 \|_2} \geq \cos \theta(P) \| e_1 - y_2 \|_2 + \cos \theta(C) \| y_2 \|_2.
\]
The condition \( \theta(P) + \theta(C) \leq \pi \) implies that \( \cos \theta(P) + \cos \theta(C) \leq 0 \). We next prove three possible cases separately.

Case 1: \( \theta(P) > \pi/2 \). This gives \( \cos \theta(P) < 0 \) and \( \cos \theta(C) > 0 \). Therefore, we obtain
\[
\cos \theta(P) \| e_1 - y_2 \|_2 + \cos \theta(C) \| y_2 \|_2
\]
\[
= \cos \theta(P) (\| e_1 - y_2 \|_2 - \| y_2 \|_2)
\]
\[
+ [\cos \theta(C) + \cos \theta(P)] \| y_2 \|_2
\]
\[
\geq \cos \theta(P) (\| e_1 - y_2 \|_2 - \| y_2 \|_2) \geq \cos \theta(P) \| e_1 \|_2,
\]
where the first inequality is from discarding the positive term and the last inequality uses the triangle inequality \( \| e_1 - y_2 \|_2 \leq \| e_1 \|_2 + \| y_2 \|_2 \). Combining (13) and (14) gives
\[
\cos \theta(G) = \inf_{0 \neq e_1 \in L_2[0, \infty), \| e_1 \|_2 \| e_1 \|_2} \geq \cos \theta(P).
\]
It follows that \( \theta(G) \leq \theta(P) \).

Case 2: \( \theta(C) > \pi/2 \). By the same reasoning, we obtain
\[
\cos \theta(P) \| e_1 - y_2 \|_2 + \cos \theta(C) \| y_2 \|_2
\]
\[
= [\cos \theta(P) + \cos \theta(C)] \| e_1 - y_2 \|_2
\]
\[
+ \cos \theta(C) (\| y_2 \|_2 - \| e_1 - y_2 \|_2)
\]
\[
\geq \cos \theta(C) (\| y_2 \|_2 - \| e_1 - y_2 \|_2) \geq \cos \theta(C) \| e_1 \|_2,
\]
where the last inequality uses \( -\| e_1 - y_2 \|_2 \leq \| e_1 \|_2 - \| y_2 \|_2 \). This gives \( \theta(G) \leq \theta(C) \).

Case 3: \( \theta(C), \theta(P) \leq \pi/2 \). Without loss of generality, let \( \theta(P) \geq \theta(C) \). This gives \( 0 \leq \cos \theta(P) \leq \cos \theta(C) \). Then,
\[
\cos \theta(P) \| e_1 - y_2 \|_2 + \cos \theta(C) \| y_2 \|_2 \geq \cos \theta(P) \| e_1 - y_2 \|_2 \| y_2 \|_2.
\]
By the same reasoning as in the proof of Proposition 6, we have \( \theta(G) \leq \theta(P) \). Therefore, for all three cases, we conclude that \( \theta(G) \leq \max \{ \theta(P), \theta(C) \} \).
Propositions 6 and 7 also hold for the other well-posed feedback system $P \neq C_{|e_1=0}$. In light of Propositions 6 and 7, the singular angle of a nonlinear network, which consists of subsystems through suitable cascaded and feedback interconnections, can be well estimated by the singular angles of the subsystems.

VII. THE SINGULAR ANGLE OF LTI SYSTEMS

An LTI system can be viewed as a convolution operator in the time domain or as a transfer function matrix in the frequency domain. In general, there are two routes to defining the singular angle of LTI systems. One is to directly inherit the time-domain singular angle definition in Section III, with the system being LTI. The other is to define the frequency-domain singular angle of transfer function matrices by means of the matrix singular angle. But are these two approaches equivalent? The answer is generally negative, even in the simplest case of SISO LTI systems.

Given a stable SISO LTI system $P$ with transfer function $P(s) \in \mathcal{RH}_\infty$, we define the $\mathcal{H}_\infty$ singular angle of $P(s)$ via the following form:

$$\cos \theta_\infty(P) := \inf_{P(j\omega) \neq 0} \frac{\operatorname{Re}(P(j\omega))}{|P(j\omega)|}.$$  

To avoid any ambiguity, we use the subscript $\infty$ to indicate the $\mathcal{H}_\infty$ singular angle $\theta_\infty(P)$ in the frequency domain, which distinguishes it from the singular angle $\theta(P)$ defined by (3) in the time domain. For a fixed $\omega_0$, $P(j\omega_0)$ is a complex number and $\angle P(j\omega)$ gives the classical phase response of $P(s)$. The ratio $\operatorname{Re}(P(j\omega))/|P(j\omega)|$ equals $\cos \angle P(j\omega)$ when $P(j\omega) \neq 0$. Roughly speaking, the $\mathcal{H}_\infty$ singular angle of $P(s)$ returns the “largest phase” of $P(s)$. The appellation “$\mathcal{H}_\infty$ singular angle” is motivated by the well-known inequalities and the fact that $\cos \theta_P(P(s)) < 0$. Rearranging (16) and using the Plancherel’s theorem yield

$$\frac{\langle u, P\hat{u} \rangle}{\|u\|_2 \|P\hat{u}\|_2} \geq \cos \theta_P(P),$$

which implies

$$\cos \theta(P) = \inf_{P \neq 0} \frac{\langle u, P\hat{u} \rangle}{\|u\|_2 \|P\hat{u}\|_2} \geq \cos \theta_\infty(P)$$

and $\theta(P) \leq \theta_\infty(P)$.

Proposition 8 indicates that, for SISO LTI systems, when the $\mathcal{H}_\infty$ singular angle is smaller than $\pi/2$, it is actually less than the singular angle, i.e., $\theta_\infty(P) < \theta(P)$. We provide a concrete example in Appendix B to support this claim.

From the above discussion on SISO LTI systems, one can see that it is meaningful to develop a pure frequency-domain singular angle theory for LTI systems. In the remainder of this section, we first lay the mathematical foundations by investigating the matrix singular angle defined in [15, Section 23]. We then address MIMO LTI systems on the shoulders of the matrix foundation.

A. The Singular Angle of Complex Matrices

The angle definition between elements in $L_2[0, \infty)$ can be naturally extended to an arbitrary complex or real Hilbert space $\mathcal{H}$. Specifying $\mathcal{H} = \mathbb{C}^n$ endowed with a real inner product will bring the following angle between complex vectors.

For any vectors $x, y \in \mathbb{C}^n$, the angle $\theta(x, y) \in [0, \pi]$ between $x$ and $y$ is defined via

$$\cos \theta(x, y) := \frac{\operatorname{Re}(x^*y)}{|x||y|}, \quad \text{if } x, y \neq 0,$$

and $\theta(x, y) := 0$, otherwise. Recall that, for a nonzero matrix $A \in \mathbb{C}^{n \times n}$, the singular angle $\theta(A) \in [0, \pi]$, which is
introduced in [15, Section 23] and [16, Chapter 3], is defined by the formula

\[
\cos \theta(A) := \inf_{0 \neq x \in \mathbb{C}^n, Ax \neq 0} \frac{\text{Re}(x^*Ax)}{|x| |Ax|}.
\]  

(17)

The above matrix singular angle definition has a graphical interpretation in a complex plane. It is closely related to the matrix normalized numerical range \([35, 36]\) defined as

\[
W_N(A) := \left\{ \frac{x^*Ax}{|x||Ax|} \in \mathbb{C} \mid 0 \neq x \in \mathbb{C}^n, Ax \neq 0 \right\}.
\]

By the Cauchy-Schwarz inequalities, for all \(z \in W_N(A)\), we have \(|z| \leq 1\). Thus, \(W_N(A)\) is contained in the unit disk in \(\mathbb{C}\). Additionally, if \(z \in W_N(A)\), then \(|z| = 1\) if and only if \(z = \lambda(A)/|\lambda(A)|\) for some eigenvalues \(\lambda(A)\) of \(A\). By introducing \(W_N(A)\), we can reformulate definition (17) as follows:

\[
\cos \theta(A) = \inf_{z \in W_N(A)} \text{Re}(z).
\]

See a graphical illustration of \(W_N(A)\) and \(\theta(A)\) in Fig. 6.

**Proof:** By hypothesis, for all \(x \in \mathbb{C}^n \setminus \{0\}\), we have

\[
0.5 (x^*Ax + x^*A^*x) > -|x||Ax|.
\]

Let \(x_i \in \mathbb{C}^n\) be chosen such that \(x_i\) is the \(i\)-th eigenvector corresponding to \(\lambda_i(A)\). It follows that, for all \(x_i\), we have

\[
0.5 (x_i^*\lambda_i(A)x_i + (\lambda_i(A)x_i)^*x_i) > -|x_i||\lambda_ix_i|.
\]

This gives \(\text{Re}(\lambda_i(A)) + |\lambda_i(A)| > 0\). When \(\lambda_i(A) = 0\), the angle is undefined. It follows that \(\cos \angle \lambda_i(A) > -1\) and \(-\pi < \angle \lambda_i(A) < \pi\) hold for all \(\lambda_i(A) \neq 0\).

The proof of Lemma 3 and definition (17) imply that the matrix singular angle provides a certain upper bound for the angles of the eigenvalues, i.e, \(|\angle \lambda_i(A)| \leq \theta(A)\).

For matrices \(A, B \in \mathbb{C}^{n \times n}\), the singularity of \(I + BA\) is an essential issue when we study MIMO LTI feedback systems. Combining Lemmas 2 and 3 yields an angular condition for the purpose of determining the singularity of \(I + BA\), as detailed in the following theorem.

**Theorem 2 (Matrix small angle theorem):** For nonzero \(A, B \in \mathbb{C}^{n \times n}\), it holds that \(\det(I + BA) \neq 0\) if

\[
\theta(A) + \theta(B) < \pi.
\]

**Proof:** Using Lemma 2, we have \(\theta(BA) \leq \theta(A) + \theta(B) < \pi\). According to Lemma 3, we obtain \(\angle \lambda(BA) \in (-\pi, \pi)\), which gives \(\det(I + BA) \neq 0\).

We next briefly discuss the computational issue of the matrix singular angles from two perspectives. First, the author of [35] proposes a systematic approach to generating the normalized numerical range analogously to the standard numerical range. Thus, the matrix singular angle can be readily calculated from the normalized numerical range as we adopt in Fig. 6. Second, formula (17) leads to a nonconvex optimization problem, and we leave this for future research.

Equipped with the core matrix singular angle theory, we are ready to cope with MIMO LTI systems.

**B. The Singular Angles of MIMO LTI Systems**

Consider a causal stable MIMO LTI system \(P\) with transfer function matrix \(P(s) \in \mathcal{RH}_{\infty}^{n \times n}\). Based on the matrix singular angle, we define the frequency-wise singular angle \(\theta(P(j\omega)) \in [0, \pi]\) for each \(\omega \in [-\infty, \infty]\) via the formula

\[
\cos \theta(P(j\omega)) := \inf_{0 \neq x \in \mathbb{C}^n, P(j\omega)x \neq 0} \frac{\text{Re}(x^*P(j\omega)x)}{|x||P(j\omega)x|}, \quad \text{if } P(j\omega) \neq 0,
\]

and \(\theta(P(j\omega)) = 0\), otherwise. Additionally, the \(\mathcal{H}_{\infty}\) singular angle \(\theta_{\mathcal{H}_{\infty}}(P) \in [0, \pi]\) is then defined by

\[
\theta_{\mathcal{H}_{\infty}}(P) := \sup_{\omega \in [-\infty, \infty]} \theta(P(j\omega)).
\]

(18)

Clearly, (18) generalizes the SISO case presented in (15).

In what follows, we investigate the feedback stability of MIMO LTI systems. For LTI systems \(P\) and \(C\) with \(P(s) \in \mathcal{RH}_{\infty}^{n \times n}\) and \(C(s) \in \mathcal{RH}_{\infty}^{n \times n}\), the well-posedness and feedback stability definitions in Section IV need further clarification and simplification. Specifically, the well-posedness of LTI feedback system \(P \# C\) is equivalent to \((I + C(s)P(s))^{-1}\) existing and being proper [1, Lemma 5.1].
Since \( P(s), C(s) \in \mathcal{RH}_{0}^{\infty} \), then \( P \# C \) is stable if and only if \((I + C(s)P(s))^{-1} \in \mathcal{RH}_{0}^{\infty} \) [1, Corollary 5.6]. The following frequency-wise stability feedback stability result stays the foundation of the LTI system singular angle theory.

**Theorem 3 (Frequency-wise LTI small angle theorem):** Let \( P \) and \( C \) be stable LTI systems with \( P(s) \in \mathcal{RH}_{0}^{\infty} \) and \( C(s) \in \mathcal{RH}_{0}^{\infty} \). Then, the well-posed \( P \# C \) is stable if
\[
\theta(P(j\omega)) + \theta(C(j\omega)) < \pi
\]
for all \( \omega \in [-\infty, \infty] \).

*Proof:* It suffices to show that \( \det(I + C(s)P(s)) \neq 0 \) for all \( s \in \mathcal{C} \cup \left\{ \infty \right\} \). We adopt a homotopy method by letting \( t \) be an arbitrary number in [0,1]. Note that \( \theta(C(j\omega)) = \theta(\tau C(j\omega)) \) when \( \tau \in (0,1) \) and \( \theta(C(j\omega)) \geq \theta(\tau C(j\omega)) = 0 \) when \( \tau = 0 \). By Lemma 2 and hypothesis, for all \( \omega \in [-\infty, \infty] \) and all \( \tau \in [0,1] \), we have
\[
\theta(\tau C(j\omega)P(j\omega)) \leq \theta(P(j\omega)) + \theta(C(j\omega)) \leq \theta(P(j\omega)) + \theta(C(j\omega)) < \pi.
\]
It follows from Lemma 3 that, for all \( \tau \in [0,1] \), we have
\[
-\pi < \angle(\tau C(j\omega)P(j\omega)) < \pi, \quad \forall \omega \in [-\infty, \infty]\,
\]
for \( i = 1, 2, \cdots, n \). This gives that, for all \( \tau \in [0,1] \), we have
\[
det(I + \tau C(s)P(s)) \neq 0, \quad \forall s \in \mathcal{C} \cup \left\{ \infty \right\}.
\]
We then extend the result on the imaginary axis to the closed right half-plane. When \( \tau = 0 \), it holds that
\[
det(I + C(s)P(s)) \neq 0, \quad \forall s \in \mathcal{C} \cup \left\{ \infty \right\}
\]
According to the continuity of the closed-loop system poles and by (19), we obtain
\[
det(I + C(s)P(s)) \neq 0, \quad \forall s \in \mathcal{C} \cup \left\{ \infty \right\}
\]
for all \( \tau \in [0,1] \). The proof is completed by setting \( \tau = 1 \).

For MIMO LTI systems, it follows from Proposition 8 and (18) that the frequency-wise singular angle \( \theta(P(j\omega)) \) is less than the others, \( \theta_{\infty}(P) \) and \( \theta(P) \), i.e.,
\[
\theta(P(j\omega)) \leq \theta_{\infty}(P) \leq \theta(P), \quad \forall \omega \in [-\infty, \infty].
\]
Consequently, the use of \( \theta(P(j\omega)) \), in contrast to that of \( \theta(P) \), in formulating the LTI small angle theorem reduces conservatism. The following corollary involves the \( \mathcal{H}_{\infty} \)-singular angle, which follows directly from Theorem 3 by noting (20).

**Corollary 3:** Let \( P \) and \( C \) be stable LTI systems with \( P(s) \in \mathcal{RH}_{0}^{\infty} \) and \( C(s) \in \mathcal{RH}_{0}^{\infty} \). Then, the well-posed \( P \# C \) is stable if
\[
\theta_{\infty}(P) + \theta_{\infty}(C) < \pi.
\]

**VIII. EXTENSIONS AND FURTHER DISCUSSIONS**

The purpose of this section is twofold. The first is dedicated to several notable extensions and variations of the system singular angle. We come up with the generalized singular angle via the use of multipliers, which can reduce the conservatism of the small angle theorem. An \( \mathcal{L}_{2,e} \)-version and an incremental version are also proposed. The second is to draw a comparison between the system singular angle and the recently appeared nonlinear system phase [12], since both of the papers aim at analyzing nonlinear systems from an angular or phasic viewpoint.

### A. The Generalized Singular Angle

Over the past half century, the multiplier approach has been widely adopted in feedback system analysis. The key idea behind the method is to insert suitable multipliers into a feedback system so that the new feedback system can satisfy some stability conditions. This certainly reduces the conservatism of feedback system analysis. Representative works including the multiplier theorem [37], QSR-dissipativity theory [38] and integral quadratic constraint (IQC) theory [29]. These works inspire us to incorporate the use of multipliers into the system singular angle theory.

Let multipliers \( M_{1} \) and \( M_{2} \) : \( \mathcal{L}_{2} \rightarrow \mathcal{L}_{2} \) be linear bounded invertible operators. For all \( u, v \in \mathcal{L}_{2}[0, \infty) \) and \( 0 \neq v \in \mathcal{L}_{2}[0, \infty) \), it holds that
\[
\theta_{M_{1}, M_{2}}(u, w) \leq \theta_{M_{1}, M_{2}}(u, v) + \theta_{M_{2}, M_{3}}(v, w).
\]

*Proof:* For all \( u, w \in \mathcal{L}_{2}[0, \infty) \) and \( v \in \mathcal{L}_{2}[0, \infty) \), we denote \( \bar{u} := M_{1}u, \bar{w} := M_{2}w \) and \( \bar{v} := M_{3}v \). Since \( M_{1}, M_{2} \) and \( M_{3} \) are bounded, it holds that \( \bar{u}, \bar{w} \in \mathcal{L}_{2} \) and \( \bar{v} \in \mathcal{L}_{2} \). Following the same reasoning as in the proof of Lemma 1, we then obtain \( \theta(\bar{u}, \bar{w}) \leq \theta(\bar{u}, \bar{v}) + \theta(\bar{v}, \bar{w}) \), where the definition of the angle is slightly modified to fit \( \mathcal{L}_{2} \). This gives \( \theta_{M_{1}, M_{2}}(u, w) \leq \theta_{M_{1}, M_{2}}(u, v) + \theta_{M_{2}, M_{3}}(v, w) \).

Given a nonlinear system \( P : \mathcal{L}_{2}[0, \infty) \rightarrow \mathcal{L}_{2}[0, \infty) \), using the multipliers, we define the generalized singular angle \( \theta_{M_{1}, M_{2}}(P) \in [0, \pi) \) with respect to \( M_{1} \) and \( M_{2} \) via
\[
\cos \theta_{M_{1}, M_{2}}(P) := \inf_{\theta \neq u \in \mathcal{L}_{2}[0, \infty)} \frac{\langle M_{1}u, M_{2}P(u) \rangle}{\|M_{1}u\|_{2} \|M_{2}P(u)\|_{2}}.
\]

The conservatism of Theorem 1 can be reduced to a large extent by the use of appropriate multipliers. We state the following generalized version of Theorem 1.

**Theorem 4 (Generalized nonlinear small angle theorem):**

The well-posed feedback system \( P \# C_{\epsilon_{e}=0} \) is stable if there exist a unitary multiplier \( M_{1} \) and a multiplier \( M_{2} \) such that
\[
\theta_{M_{1}, M_{2}}(P) + \theta_{M_{1}, M_{2}}(C) < \pi.
\]

*Proof:* The proof can be shown by following an analogous procedure to the proof of Theorem 1, except for the following differences. By hypothesis, we have
\[
\cos \theta_{M_{1}, M_{2}}(P) + \theta_{M_{1}, M_{2}}(C) > -1,
\]
which gives
\[
\cos \theta_{M_{1}, M_{2}}(u_{1}, y_{1}) + \theta_{M_{2}, M_{3}}(u_{2}, y_{2}) > -1
\]
for all \( u_{1}, u_{2} \in \mathcal{L}_{2}[0, \infty) \). Applying Lemma 4 and noting \( y_{1} = u_{2} \) yield
\[
-1 < \cos \theta_{M_{1}, M_{2}}(u_{1}, y_{1}) + \theta_{M_{2}, M_{3}}(u_{2}, y_{2}) \leq \cos \theta_{M_{1}, M_{2}}(u_{1}, y_{2}) \leq 1
\]
for all \( u_1, u_2 \in L_2[0, \infty) \setminus \{0\} \). Since \( M_1 \) is unitary, i.e., \( M_1^* = M_1^{-1} \), then \( \cos \theta_{M_1, M_1}(u_1, y_2) = \cos \theta(u_1, y_2) \). The proof is then completed by the same reasoning as in the proof of Theorem 1.

The following useful corollary is directly obtained by specifying \( M_1 = I \) in Theorem 4.

**Corollary 4:** The well-posed feedback system \( P \# C\vert_{\epsilon_2 = 0} \) is stable if there exists a multiplier \( M \) such that
\[
\theta_{I, M}(P) + \theta_{M, I}(C) < \pi.
\]

**B. The \( L_2e \), Singular Angle**

Nonlinear system properties can be defined not only on \( L_2[0, \infty) \) but also on \( L_2e[0, \infty) \). In Section II, it is shown that, for causal stable nonlinear systems, the definition of \( L_2e \)-gain is equivalent to that of \( L_2e \)-gain, and an analogous equivalence applies to the definition of passivity. This motivates us to define the system singular angle on \( L_2e[0, \infty) \) and to explore the existence of a similar equivalence.

For a system \( P : L_2e[0, \infty) \to L_2e[0, \infty) \), the \( L_2e \), singular angle \( \theta_e(P) \in [0, \pi] \) is defined by the formula
\[
\cos \theta_e(P) := \inf_{u \in L_2e[0, \infty), \ T > 0, \ u_T \not\equiv 0} \frac{\langle u_T, (Pu)_T \rangle}{\|u_T\|_2 \|P(u)_T\|_2}. \tag{21}
\]

The following proposition indicates that \( \theta(P) \) and \( \theta_e(P) \) are generally unequal.

**Proposition 9:** For a causal stable system \( P \), we have
\[
\theta(P) \leq \theta_e(P).
\]

In addition, if \( \theta(P) \in [0, \pi/2] \), then \( \theta(P) = \theta_e(P) \).

**Proof:** The case \( \theta(P) \leq \theta_e(P) \) is proved by taking \( T \to \infty \) in (21). We next show \( \theta_e(P) \leq \theta(P) \) when \( \theta(P) \in [0, \pi/2] \). Since \( P \) is causal, then \( (Pu)_T = (Pu_T)_T \) holds. By the \( L_2e \), singular angle definition, we have
\[
\cos \theta_e(P) = \inf_{u \in L_2e[0, \infty), \ T > 0, \ u_T \not\equiv 0} \frac{\langle u_T, (Pu)_T \rangle}{\|u_T\|_2 \|P(u)_T\|_2} = \inf_{u \neq 0 \in L_2e[0, \infty), \ P(u) \neq 0} \frac{\langle u_T, P(u)_T \rangle}{\|u_T\|_2 \|P(u)_T\|_2}
\]
which gives \( \theta(P) \leq \theta_e(P) \).

The following corollary follows directly from Theorem 1.

**Corollary 6:** The well-posed feedback system \( P \# C\vert_{\epsilon_2 = 0} \) is stable if
\[
\theta_1(P) + \theta_1(C) < \pi.
\]

**C. The Incremental Singular Angle**

In large-scaled nonlinear network analysis, the concept of incremental properties plays a crucial role for consensus and synchronization issues, e.g., incremental \( L_2e \)-gain \([3], [39]\), incremental passivity \([39]–[41]\) and incremental IQC \([42], [43]\). To this end, we develop an analogous version for the system singular angle.

The **incremental singular angle** \( \theta_1(P) \) for a causal stable system \( P : L_2e[0, \infty) \to L_2e[0, \infty) \) is defined by
\[
\cos \theta_1(P) := \inf_{u, v \in L_2e[0, \infty), \ u \not\equiv 0, \ P(u) \not\equiv 0} \frac{\langle u - v, P(u) - P(v) \rangle}{\|u - v\|_2 \|P(u) - P(v)\|_2}. \tag{22}
\]

In Corollary 6, in fact, one can further conclude a stronger result in terms of the incremental feedback stability \([3], [39]\) by using the incremental singular angles.

**D. A Comparison with the Recent Nonlinear System Phase**

A practical nonlinear system can only accept real-valued signals. To define a phase or angle notion in nonlinear systems, we have two feasible paths. The first is what we have done in this paper, inspired by the Euclidean space angle between vectors. The second is to introduce complex elements to nonlinear systems on the grounds that phases are naturally defined for complex numbers. To this end, in our recent works \([12]\) and \([13]\), we have proposed the notion of the nonlinear system phase through complexifying real-valued signals using the Hilbert transform. Concretely, for a causal stable system \( P \), we first define the **angular numerical range** of \( P \) to be
\[
W'(P) := \left\{ \frac{1}{2} \left( u + jH(u), P(u) \right) \mid u \in L_2[0, \infty) \right\},
\]
where \( H \) denotes the Hilbert transform \([14]\) frequently used in signal processing. Such a system \( P \) is said to be **semi-sectorial**
if $W'(P)$ is contained in a closed complex half-plane. Then, for semi-sectorial systems, the phase of $P$, denoted by $\Phi(P)$, is defined to be the phase sector

$$\Phi(P) := \left[\phi(P), \widetilde{\phi}(P)\right],$$

where $\phi(P)$ and $\widetilde{\phi}(P)$ are called the phase infimum and phase supremum of $P$, respectively, and are defined by

$$\phi(P) := \inf_{0 \neq z \in W'(P)} \angle z$$

and

$$\widetilde{\phi}(P) := \sup_{0 \neq z \in W'(P)} \angle z.$$

The system singular angle and the system phase are, in general, different. On the one hand, consider a positive definite matrix $A = \text{diag}\{1, 2\}$ which is semi-sectorial. Its phase $\Phi(A) = [0, 0]$ is zero [10], while its singular angle $\Theta(A) \approx 19.4712\pi/180$ is nonzero and conservative. On the other hand, for those matrices which are not semi-sectorial, their phases are undefined. Their singular angles, however, can still be computed. Consequently, we believe that both the singular angle and the phase are worthy of investigation and development. We refer the reader to [12] for more details of the nonlinear system phase. We end this section by briefly comparing the system singular angle with the system phase [10], [12] with the summary in Table I.

### IX. CONCLUSION

In this paper, we propose an angle notion, the singular angle, for stable nonlinear systems from an input-output perspective. The system singular angle plays the role of a new angular counterpart to the system $L_2$-gain, and serves as an alternative to the recent nonlinear system phase. Subsequently, we establish the main result, a nonlinear small angle theorem, for feedback stability analysis. We then further develop several notable extensions and variations of the system singular angle and small angle theorem. It is hoped that this paper provides an alternative starting point for bringing the recent renaissance of the classical phase (or angle) notion in MIMO LTI systems [10] into nonlinear systems [12].

### APPENDIX A

#### THE PROOFS OF LEMMA 1 AND COROLLARY 2

**Proof of Lemma 1:** Suppose that $u, v, w \in L_2(0, \infty) \setminus \{0\}$. Construct the following Gramian matrix:

$$G = \begin{bmatrix}
\langle u, u \rangle & \langle u, v \rangle & \langle u, w \rangle \\
\langle v, u \rangle & \langle v, v \rangle & \langle v, w \rangle \\
\langle w, u \rangle & \langle w, v \rangle & \langle w, w \rangle
\end{bmatrix}.$$  

By the facts that $G$ is positive semi-definite and $\det G \geq 0$, we have

$$\det G = \langle u, u \rangle \left[\langle v, v \rangle \langle w, w \rangle - \|v, w\|^2 \right]$$

$$+ \langle u, v \rangle \left[\langle w, u \rangle \langle v, w \rangle - \langle u, w \rangle \langle v, v \rangle \right]$$

$$+ \langle u, w \rangle \left[\langle v, u \rangle \langle w, v \rangle - \langle u, v \rangle \langle w, w \rangle \right] \geq 0,$$

which is equivalent to

$$\|u\|_2^2 \|v\|_2^2 \|w\|_2^2 \left[1 - \cos \theta(u, v) \cos \theta(v, w) \right]$$

$$- \cos \theta(u, v) \cos \theta(v, w) \geq 0.$$  

The above inequality implies

$$1 - \cos \theta(u, v) \cos \theta(v, w) \geq 0.$$  

Rewrite the above inequality as

$$\left\{1 - \cos \theta(u, v) \right\} \left\{1 - \cos \theta(v, w) \right\} \geq \cos \theta(u, v) \cos \theta(v, w) - \cos \theta(u, w).$$

Note that $\theta(u, v), \theta(v, w) \in [0, \pi]$. Taking the square roots of both sides of the inequality yields

$$\sin \theta(u, v) \sin \theta(v, w) \geq |\cos \theta(u, v) \cos \theta(v, w) - \cos \theta(u, w)|$$

$$\geq \cos \theta(u, v) \cos \theta(v, w) - \cos \theta(u, w).$$

This gives $\cos \theta(u, v) + \theta(v, w) \leq \cos \theta(u, w)$. Since $\theta(u, v) + \theta(v, w) \in [0, 2\pi]$ and $\theta(u, w) \in [0, \pi]$, by the property of the cosine function, we conclude $\theta(u, w) \leq \theta(u, v) + \theta(v, w)$. Suppose that one of $u$ and $w$ is zero. Let $u$ be so. Then $\theta(u, w) = \theta(u, v) = 0$ by (2). The triangle inequality still holds since $\theta(v, w) \geq 0$.  

#### TABLE I

| Property                      | Singular angle $\theta(P)$ | Phase $\Phi(P) = [\phi(P), \widetilde{\phi}(P)]$ |
|-------------------------------|----------------------------|--------------------------------------------------|
| Value                         | A quantity $\theta(P) \in [0, \pi]$ | A $\pi$-length interval $\Phi(P) \subset [-3\pi/2, 3\pi/2]$ |
| Applicable scope              | All systems                | Semi-sectorial systems |
| Feedback stability condition  | $\theta(P) + \theta(C) < \pi$ | $\Phi(P) + \widetilde{\phi}(C) < \pi$, $\phi(P) + \phi(C) > -\pi$ |
| Cascaded interconnection      | $\theta(P_2P_1) \leq \theta(P_1) + \theta(P_2)$ | $\Phi(P_1), \Phi(P_2) \subseteq [\alpha, \beta] \Rightarrow \Phi(P_1 + P_2) \subseteq [\alpha, \beta]$ |
| Parallel interconnection      | $\theta(G) \leq \max(\theta(P), \theta(C))$ | $\Phi(G) \subseteq \left\{\left[\phi(P), -\widetilde{\phi}(C)\right], \max\left\{\Phi(P), -\phi(C)\right\}\right\}$ |
| Feedback interconnection      | Nonequivalent              | Equivalent |
| Time domain vs frequency domain in LTI systems | Yes                         | Yes |
| Interpretation                | The rotating effect from system input to output signals | The tradeoff between the real energy and reactive energy |

A COMPARISON BETWEEN THE SYSTEM SINGULAR ANGLE AND SYSTEM PHASE
Proof of Corollary 2: By Proposition 5, we have \( \theta(N) \leq \arccos \frac{2\sqrt{ab}}{a+b} < \pi/2 \). By hypothesis, when \( \angle P(j\omega) > \pi/2 \) or \( \angle P(j\omega) < -\pi/2 \) holds for some \( \omega \in (\infty, \infty) \), it holds that \( \theta_\infty(P) \in (\pi/2, \pi - \arccos \frac{2\sqrt{ab}}{a+b}) \). Then, for all \( \tau > 0 \), we have
\[
\theta(\tau P) + \theta(N) = \theta(P) + \theta(N) = \theta_\infty(P) + \theta(N) < \pi,
\]
where the second equality uses \( \theta_\infty(P) = \theta(P) \) by invoking Proposition 8. In this case, by Theorem 1, \( (\tau P) \# N \) is stable for all \( \tau > 0 \). When \( |\angle P(j\omega)| \leq \pi/2 \) holds for all \( \omega \in [-\infty, \infty] \), \( \tau P \) is passive, and \( N \) is very strictly passive, as stated before. In this case, by the passivity theorem [2, Section 6.6], \( (\tau P) \# N \) is stable for all \( \tau > 0 \).

APPENDIX B
AN EXAMPLE FOR SISO LTI SYSTEMS

This appendix complements Proposition 8. When \( \theta_\infty(P) \in [0, \pi/2) \), we show in the following that \( \theta_\infty(P) < \theta(P) \) via constructing a concrete example. Construct a signal \( f \in L^2(0, \infty) \) with its \( \hat{f} \) chosen such that
\[
|\hat{f}(j\omega)| = \begin{cases} 
  c_0 & \text{if } |\omega + \omega_0| < \epsilon \text{ or } |\omega - \omega_0| < \epsilon, \\
  c_1 & \text{if } |\omega + \omega_1| < \epsilon \text{ or } |\omega - \omega_1| < \epsilon, \\
  0 & \text{otherwise},
\end{cases}
\]
where \( \epsilon > 0 \) is a small positive number, \( \omega_0, \omega_1 \geq 0 \), and \( c_0, c_1 > 0 \) are chosen such that \( \hat{f} \) has unit 2-norm, i.e.,
\[
\frac{c_0^2 + c_1^2}{2\epsilon} = \frac{\pi}{2\epsilon}.
\]
(23)

In light of definition (3), using this particular \( f \) yields
\[
\cos \theta(P) \leq \frac{\langle f, Pf \rangle}{\|f\|_2 \|Pf\|_2} = \frac{\|\hat{f} \|_2 \|\hat{P}\|_2}{\|\hat{f}\|_2 \|\hat{P}\|_2} = \frac{2\epsilon (ReP(j\omega_0)c_0^2 + ReP(j\omega_1)c_1^2)}{\sqrt{2\epsilon \pi} \left( |P(j\omega_0)|^2 c_0^2 + |P(j\omega_1)|^2 c_1^2 \right)}
\]
\[
= \frac{2\epsilon \pi \sqrt{c_0^2 ReP(j\omega_0) + (1 - 2\epsilon c_0^2) ReP(j\omega_1)}}{\sqrt{2\epsilon \pi} \left( |P(j\omega_0)|^2 + (1 - 2\epsilon c_0^2) |P(j\omega_1)|^2 \right)},
\]
(24)
where the second-last equality follows from (23). Assume that \( \theta_\infty(P) \) is attained at \( \omega_0 \), namely,
\[
\cos \theta_\infty(P) = \frac{Re(P(j\omega_0))}{|P(j\omega_0)|}.
\]
Denote \( P(j\omega_0) = z_0 := a_0 + jb_0 \in \mathbb{C} \) and \( P(j\omega_1) = z_1 := a_1 + jb_1 \in \mathbb{C} \) with \( a_0, a_1 > 0 \). Since \( \theta_\infty(P) \) is attained at \( \omega_0 \), then \( z_1, z_2 \) should further satisfy \( |\angle z_0| \geq |\angle z_1| \). Denote \( \tau := 2\epsilon c_0^2/\pi \). It follows from (23) that \( c_0^2 < \pi/(2\epsilon) \) and \( 0 < 2\epsilon c_0^2/\pi < \pi \). Thus, \( \tau \in (0, 1) \). We rewrite (24) as
\[
\cos \theta(P) \leq \frac{\tau a_0 + (1 - \tau) a_1}{\sqrt{\tau(a_0^2 + b_0^2) + (1 - \tau)(a_1^2 + b_1^2)}}
\]
and
\[
\cos \theta_\infty(P) = \frac{a_0}{\sqrt{a_0^2 + b_0^2}}.
\]
We aim to find feasible \( \tau, z_0 \) and \( z_1 \) for some systems \( P \) to show that \( \theta_\infty(P) < \theta(P) \). The following is an example.

Example 1: Let \( \tau = 0.4 \) and
\[
P(s) = \frac{(s + 5)(s^2 + 3s + 102.3)}{(s + 1)(s^2 + 6s + 109)}.
\]
One can verify that \( \theta_\infty(P) \) is attained at \( \omega_0 = 2.613 \) and thus \( a_0 = 1.3092, b_0 = -1.332 \). Let \( \omega_1 = 10 \) so that \( a_1 = 0.5286, b_1 = -0.1577 \). Clearly, \( |\angle z_0| \geq |\angle z_1| \). Then, we have
\[
\cos \theta(P) \leq \frac{0.4 \times 1.3092 + 0.6 \times 0.5286}{\sqrt{0.4 \times 3.4882 \times 0.6 \times 0.3043}} = 0.6694,
\]
\[
\cos \theta_\infty(P) = \frac{1.3092}{\sqrt{3.4882}} = 0.701.
\]
This implies that \( \cos \theta(P) < \cos \theta_\infty(P) \) and \( \theta_\infty(P) < \theta(P) \).

ACKNOWLEDGMENT

The authors would like to thank Axel Ringh of University of Gothenburg and Chalmers University of Technology, and Dan Wang of The Hong Kong University of Science and Technology for useful discussions.

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