ON A PROOF OF A CONJECTURE OF MARINO-VAFA ON
HODGE INTEGRALS

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ABSTRACT. We outline a proof of a remarkable formula for Hodge integrals
conjectured by Mariño and Vafa [23] based on large N duality.

1. Introduction

Let \( \overline{\mathcal{M}}_{g,n} \) denote the Deligne-Mumford moduli stack of stable curves of genus \( g \)
with \( n \) marked points. Let \( \pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n} \) be the universal curve, and let \( \omega_\pi \)
be the relative dualizing sheaf. The Hodge bundle
\[
E = \pi_* \omega_\pi
\]
is a rank \( g \) vector bundle over \( \overline{\mathcal{M}}_{g,n} \) whose fiber of over \( [C, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n} \) is
\( H^0(C, \omega_C) \). Let \( s_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1} \) denote the section of \( \pi \) which corresponds to
the \( i \)-th marked point, and let
\[
L_i = s_i^* \omega_\pi
\]
be the line bundle over \( \overline{\mathcal{M}}_{g,n} \) whose fiber over \( [C, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n} \) is the cotan-
gent line \( T_{x_i}^* C \) at the \( i \)-th marked point \( x_i \). A Hodge integral is an integral of the form
\[
\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g}
\]
where \( \psi_i = c_1(L_i) \) is the first Chern class of \( L_i \), and \( \lambda_j = c_j(E) \) is the \( j \)-th Chern
class of the Hodge bundle.

Hodge integrals arise in the calculations of Gromov-Witten invariants by local-
ization techniques [14, 7]. The explicit evaluation of Hodge integrals is a difficult
problem. The Hodge integrals involving only \( \psi \) classes can be computed recursively
by Witten’s conjecture [26] proven by Kontsevich [13]. Algorithms of computing
Hodge integrals are described in [2].

In [23], M. Mariño and C. Vafa obtained a closed formula for a generating func-
tion of certain open Gromov-Witten invariants, some of which has been reduced to
Hodge integrals by localization techniques which are not fully clarified mathemati-
cally. This leads to a conjectural formula of Hodge integrals. To state this formula,
we introduce some notation, following [28]. Let
\[
\Lambda^g_j(u) = u^g - \lambda_1 u + \cdots + (-1)^g \lambda_g
\]
be the Chern polynomial of \( \mathbb{E}^g \), the dual of the Hodge bundle. For a partition \( \mu \)
given by
\[
\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l(\mu) > 0,
\]
let $|\mu| = \sum_{i=1}^{l(\mu)} \mu_{i}$, and define

$$C_{g,\mu}(\lambda; \tau) = -\sqrt{-1}^{l(\mu)+l(\mu)}\left[\lambda(\tau+1)\right]^{l(\mu)-1} \left[\prod_{i=1}^{l(\mu)} (1-\mu_{i}\tau+a) \prod_{i=1}^{l(\mu)} \psi_{\lambda_{i}}(\tau+1)\right]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \psi_{\lambda_{i}}(\tau+1) \prod_{i=1}^{l(\mu)} \psi_{\lambda_{i}}(\tau+1).$$

$$C_{\mu}(\lambda; \tau) = \sum_{g \geq 0} \lambda^{2g-2+l(\mu)} C_{g,\mu}(\lambda; \tau)$$

Note that

$$\int_{\mathcal{M}_{0,l(\mu)}} \frac{\Delta_{\mu}^{2}(1)\Delta_{\nu}^{2}(-\tau-1)\Delta_{\nu}^{2}(\tau)}{\prod_{i=1}^{l(\mu)} (1-\mu_{i}\psi_{\lambda_{i}})} = \frac{1}{\prod_{i=1}^{l(\mu)} (1-\mu_{i}\psi_{\lambda_{i}})} = |\mu|^{l(\mu)-3}$$

for $l(\mu) \geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_{1}, p_{2}, \ldots, p_{n}, \ldots)$, and define

$$p_{\mu} = p_{\mu_{1}} \cdots p_{\mu_{l(\mu)}}$$

for a partition $\mu = (\mu_{1} \geq \cdots \geq \mu_{l(\mu)} > 0)$. Define generating functions

$$C(\lambda; \tau; p) = \sum_{|\mu| \geq 1} C_{\mu}(\lambda; \tau)p_{\mu},$$

$$C(\lambda; \tau; p)^{*} = e^{C(\lambda; \tau; p)}.$$

As pointed out in [23], by comparing computations in [23] with computations in [12], one obtains a conjectural formula for $C_{\mu}(\tau)$. This formula is explicitly written down in [28].

(1)

$$C(\lambda; \tau; p) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \sum_{\mu} \left( \sum_{|\mu'| = |\mu|} \prod_{i=1}^{n} \chi_{\lambda_{i}}(C(\mu_{i})) \frac{\chi_{\lambda_{i}}(C(\mu_{i}))}{z_{\mu_{i}}} e^{-n(\tau+\frac{1}{2})\kappa_{\nu_{i}}\lambda/2} V_{\nu_{i}}(\lambda) \right) p_{\mu},$$

(2)

$$C(\lambda; \tau; p)^{*} = \sum_{|\mu| \geq 0} \left( \sum_{|\nu| = |\mu|} \chi_{\lambda}(C(\mu)) \frac{\chi_{\lambda}(C(\mu))}{z_{\mu}} e^{-n(\tau+\frac{1}{2})\kappa_{\nu}\lambda/2} V_{\nu}(\lambda) \right) p_{\mu},$$

where

$$V_{\nu}(\lambda) = \prod_{1 \leq a < b \leq l(\nu)} \frac{\sin[(\nu_{a} - \nu_{b} + b - a)\lambda/2]}{\sin[(b-a)\lambda/2]} \prod_{i=1}^{l(\nu)} \frac{1}{\prod_{1 \leq a < b \leq l(\nu)} \sin[(v - i + l(\nu))\lambda/2]}.$$

We now explain the notation on the right-hand sides of (1) and (2). For a partition $\mu$ given by

$$\mu_{1} \geq \mu_{2} \geq \cdots \geq \mu_{l(\mu)} > 0,$$

$\chi_{\mu}$ denotes the character of the irreducible representation of $S_{d}$ indexed by $\mu$, where $d = |\mu| = \sum_{i=1}^{l(\mu)} \mu_{i}$. The number $\kappa_{\mu}$ is defined by

$$\kappa_{\mu} = |\mu| + \sum_{i} (\mu_{i}^{2} - 2l_{\mu_{i}}).$$
For each positive integer $i$,
\[ m_i(\mu) = |\{ j : \mu_j = i \}|. \]
Denote by $C(\nu)$ the conjugacy class of $S_d$ corresponding to the partition $\nu$, and by $\chi_\mu(C(\nu))$ the value of the character $\chi_\mu$ on the conjugacy class $C(\nu)$. Finally,
\[ z_\mu = \prod_j m_j(\mu)!^{m_j(\mu)}. \]

In this paper, we will call (1) the Mariño-Vafa formula.

The third author proved in [27] some special cases of the Mariño-Vafa formula and found several interesting applications. He showed in [29] that calculation of BPS numbers in the local $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ geometries can be reduced to the Mariño-Vafa formula, and proved in [30] a special case of a conjecture by A. Iqbal [11] assuming the Mariño-Vafa formula.

We now describe our approach to the Mariño-Vafa formula (1). Denote the right-hand sides of (1) and (2) by $R(\lambda; \tau; p)$ and $R^\cdot(\lambda; \tau; p)$ respectively. In [28], the third author proved the following two equivalent cut-and-join equations similar to the one satisfied by Hurwitz numbers [6], [20], [10, Section 15.2].

**Theorem 1.**
\[
\partial R \over \partial \tau = \frac{\sqrt{-1} \lambda}{2} \sum_{i,j \geq 1} \left( ij p_{i+j} \frac{\partial^2 R}{\partial p_i \partial p_j} + ij p_{i+j} \frac{\partial R}{\partial p_i} \frac{\partial R}{\partial p_j} + (i + j)p_ip_j \frac{\partial R}{\partial p_{i+j}} \right),
\]
\[
\partial R^\cdot \over \partial \tau = \frac{\sqrt{-1} \lambda}{2} \sum_{i,j \geq 1} \left( ij p_{i+j} \frac{\partial^2 R^\cdot}{\partial p_i \partial p_j} + (i + j)p_ip_j \frac{\partial R^\cdot}{\partial p_{i+j}} \right).
\]

Here is a crucial observation: One can rewrite (6) as a sequence of systems of ordinary equations, one for each positive integer $d$, hence if $C(\lambda; \tau; p)^\cdot$ satisfies (6), then it is determined by the initial value $C(\lambda; 0; p)^\cdot$. To prove (1) or (2), it suffices to prove the following two statements:

(a) Equation (4) is satisfied by $C(\lambda; \tau; p)$.
(b) $C(\lambda; 0; p) = R(\lambda; 0; p)$.

Or equivalently,

(a)' Equation (5) is satisfied by $C(\lambda; \tau; p)^\cdot$.
(b)' $C(\lambda; 0; p)^\cdot = R(\lambda; 0; p)^\cdot$.

It is shown in [28] that (b) holds. Therefore, the Mariño-Vafa formula (1) follows from the following theorem.

**Theorem 2.**
\[
\frac{\partial C}{\partial \tau} = \frac{\sqrt{-1} \lambda}{2} \sum_{i,j \geq 1} \left( ij p_{i+j} \frac{\partial^2 C}{\partial p_i \partial p_j} + ij p_{i+j} \frac{\partial C}{\partial p_i} \frac{\partial C}{\partial p_j} + (i + j)p_ip_j \frac{\partial C}{\partial p_{i+j}} \right)
\]

The rest of the paper is organized as follows. In Section 2, we give a proof of the initial condition (b). In Section 3, we give the proof of Theorem 1 in [28]. In Section 4, we outline the proof of Theorem 2 in [22]. The details we omit here are straightforward calculations which will be given in [22]. Complete lists of relevant references will be given in [28, 22].
2. Initial Condition

The proof of the initial condition (b) needs the following two theorems.

**Theorem 2.1.** We have

\[
C(\lambda; 0; p) = - \sum_{n>0} \frac{\sqrt{-1}^{n+1} p_n}{2n \sin(n\lambda/2)}.
\]

**Proof.** When \( l(\mu) > 1 \), we clearly have

\[
C_{\mu}(\lambda; 0) = 0.
\]

When \( \mu = (n) \) we have

\[
C_{(n)}(\lambda; 0) = - \sum_{g \geq 0} \lambda^{2g-1} \sqrt{-1}^{n+1} \prod_{a=1}^{n-1} (n \cdot 0 + a) \int_{X_{g,1}} \frac{\Lambda^*_{g}(1) \Lambda^*_{g}(0) \Lambda^*_{g}(-1)}{1 - n\psi_1}.
\]

In the second equality we have used the Mumford's relations \[24, 5.4\]:

\[
\Lambda^*_{g}(1) \Lambda^*_{g}(0) \Lambda^*_{g}(-1) = (-1)^g.
\]

In the third equality we have used \[3, Theorem 2\]. This proves (7). \( \square \)

**Theorem 2.2.** We have the following identity:

\[
\log \left( \sum_{n \geq 0} \sum_{|\rho|=n} \prod_{e \in \rho} \frac{e^{\frac{i}{2} e h(e) \lambda}}{2 \sin(h(e) \lambda/2)} \chi_{\rho}(\eta) \frac{1}{z_{\eta}} p_{\eta} \right) = - \sum_{d \geq 1} \frac{\sqrt{-1}^{d+1} p_d}{2d \sin(d\lambda/2)}.
\]

For a partition \( \eta \),

\[
n(\eta) = \sum_i (i - 1) \eta_i = \sum_i \left(\eta_i^2 \right) / 2.
\]

For any box \( e \in \eta \), denote by \( h(e) \) its hook length. Then

\[
\sum_{x \in \eta} h(x) = n(\eta) + n(\eta') + |\eta|.
\]

**Lemma 2.1.** Introducing formal variables \( x_1, \ldots, x_n, \ldots \) such that

\[
p_i(x_1, \ldots, x_n, \ldots) = x_1^i + \cdots + x_n^i + \cdots.
\]

Then for any positive integer \( n \), we have

\[
\sum_{n \geq 0} \sum_{|\rho|=n} \frac{q^{n(\rho)} \chi_{\rho}(\eta)}{z_{\eta}} p_{\eta} = \frac{1}{\prod_{i,j} (1 - tx_i q_j^{-1})}.
\]
Proof. Recall the following facts about Schur polynomials:

\[ s_\rho(x) = \sum_\eta \frac{\chi_\rho(\eta)}{z_\eta} p_\eta(x), \]
\[ s_\rho(1, q, q^2, \ldots) = \frac{q^{q(\rho)}}{\prod_{e \in \rho} (1 - q^{h(e)})}, \]
\[ \sum_{n \geq 0} t^n \sum_{|\rho|=n} s_\rho(x) s_\rho(y) = \frac{1}{\prod_{i,j} (1 - tx_i y_j)}. \]

Combining these two identities, one gets:

\[ \sum_{n \geq 0} t^n \sum_{|\rho|=n} \frac{q^{q(\rho)}}{\prod_{e \in \rho} (1 - q^{h(e)})} s_\rho(x) = \frac{1}{\prod_{i,j} (1 - tx_i q^{j-1})}. \]

The proof is completed by (10).

As a corollary we now prove (8). First we need the following:

**Lemma 2.2.** For any partition \( \rho \) we have

\[ \frac{1}{2} \sum_{e \in \rho} h(e) - n(\rho) = \frac{1}{4} \kappa_\rho + \frac{1}{2} |\rho|. \]

**Proof.**

\[
\begin{align*}
\frac{1}{2} \sum_{e \in \rho} h(e) - n(\rho) &= \frac{1}{2} (n(\rho') - n(\rho) + |\rho|) \\
&= \frac{1}{2} \left( \sum_i \left( \frac{p_i}{2} \right) - \sum_i (i-1) p_i + |\rho| \right) \\
&= \frac{1}{4} \left( \sum_i p_i (p_i - 1) - 2 \sum_i i p_i + 4 |\rho| \right) \\
&= \frac{1}{4} \kappa_\rho + \frac{1}{2} |\rho|.
\end{align*}
\]

\[ \square \]

Let \( q = e^{-\sqrt{-1} \lambda} \) and \( t = \sqrt{-1} q^{1/2} \), then we have

\[
\begin{align*}
\sum_{n \geq 0} t^n \sum_{|\rho|=n} \frac{q^{q(\rho)}}{\prod_{e \in \rho} (1 - q^{h(e)})} \frac{\chi_\rho(\eta)}{z_\rho} p_\eta \\
&= \sum_{n \geq 0} \sqrt{-1} q^{n/2} \sum_{|\rho|=n} \frac{q^{q(\rho)} e^{-\frac{1}{2} \sum_{e \in \rho} h(e)}}{\prod_{e \in \rho} (q^{h(e)}/2 - q^{h(e)/2})} \frac{\chi_\rho(\eta)}{z_\rho} p_\eta \\
&= \sum_{n \geq 0} \sqrt{-1} q^{n/2} \sum_{|\rho|=n} e^{\frac{1}{2} f(\rho) \sqrt{-1} \lambda} \frac{q^{q(\rho)} e^{-\frac{1}{2} \sum_{e \in \rho} h(e)}}{\prod_{e \in \rho} (q^{h(e)}/2 - q^{h(e)/2})} \frac{\chi_\rho(\eta)}{z_\rho} p_\eta \\
&= \sum_{n \geq 0} \sum_{|\rho|=n} \frac{e^{\frac{1}{2} f(\rho) \sqrt{-1} \lambda}}{\prod_{e \in \rho} 2 \sin(h(e) \lambda/2)} \frac{\chi_\rho(\eta)}{z_\rho} p_\eta.
\end{align*}
\]
Hence by (9),

\[
\begin{align*}
\log \left( \sum_{n \geq 0} \sum_{\rho | \rho = n} \prod_{e \in \rho} e^{\pm \sqrt{\lambda} \rho / \sqrt{\lambda}} \chi_{\rho}(\eta) \zeta_{\rho} p_{\eta} \right) &= \log \left( \sum_{n \geq 0} \sum_{\rho | \rho = n} \prod_{e \in \rho} \frac{q^{n(\rho)}}{1 - q^{h(e)}} \chi_{\rho}(\eta) \zeta_{\rho} p_{\eta} \right) \\
&= \log \left( \prod_{i,j} (1 - tx_{i}q^{j-1}) \right) = \sum_{i,j \geq 1} \frac{1}{d} q^{d(j-1)} x_{i}^{d} \\
&= \sum_{j \geq 1} \sum_{d \geq 1} \frac{1}{d} q^{d(j-1)} p_{d} = \sum_{d \geq 1} \frac{p_{d}}{d} \frac{t^{d}}{1 - q^{d}} \\
&= - \sum_{d \geq 0} \frac{\sqrt{1-t}^{d+1} p_{d}}{2d \sin(d\lambda/2)}.
\end{align*}
\]

By (5) we have

\[
\begin{align*}
R(\lambda; 0; p) &= \log \left( \sum_{n \geq 0} \sum_{\rho | \rho = n} \chi_{\rho}(\eta) e^{\pm \sqrt{\lambda} \rho} \lambda V_{\rho} p_{\eta} \right) = - \sum_{d \geq 1} \frac{\sqrt{1-t}^{d+1} p_{d}}{2d \sin(d\lambda/2)},
\end{align*}
\]

where we have used the following identity proved in [28]:

\[
V_{\mu} = \frac{1}{2^l \prod x_{\in \mu} \sin[h(x)\lambda/2]}.
\]

Hence (b) is proved.

3. Proof of Theorem 1

Recall

\[
e_{\mu} = \sum_{g \in C_{\mu}} g
\]

lies in the center of the group algebra \( \mathbb{C}S_{d} \), hence it acts as a scalar \( f_{\nu}(\mu) \) on any irreducible representation \( R_{\nu} \). In other words, let \( \rho : S_{d} \rightarrow \text{End} R_{\nu} \) be the representation indexed by \( \nu \), then

\[
\sum_{g \in C(\mu)} \rho_{\nu}(g) = f_{\nu}(\mu) \text{id}.
\]

We need the following interpretation of \( \kappa_{\nu} \) in terms of character:

\[
\kappa_{\nu} = 2 f_{\nu}(C(2)).
\]

See e.g. [28 (5)]. Here we use \( C(2) \) to denote the class of transpositions. We need the following result:

**Lemma 3.1.** Suppose \( h \in S_{d} \) has cycle type \( \mu \). The product \( C(2) \cdot h \) is a sum of elements of \( S_{d} \) whose type is either a cut or a join of \( \mu \). More precisely, there are \( ijn_{i}(\mu)m_{j}(\mu) \) elements obtained from \( h \) by joining an \( i \)-cycle in \( h \) to a \( j \)-cycle in \( h \), and there are \( (i+j)m_{i+j}(\mu) \) elements obtained from \( h \) by cutting an \( (i+j) \)-cycle into an \( i \)-cycle and a \( j \)-cycle.
Proof. Denote by \([s_1, \ldots, s_k]\) a \(k\)-cycle. Then
\[
[s, t] \cdot [s, s_2, \ldots, s_i, t, t_2, \ldots, t_j] = [s, s_2, \ldots, s_i][t, t_2, \ldots, t_j],
\]
i.e., an \((i + j)\)-cycle is cut into an \(i\)-cycle and a \(j\)-cycle. Conversely,
\[
[s, t] \cdot [s, s_2, \ldots, s_i][t, t_2, \ldots, t_j] = [s, s_2, \ldots, s_i, t, t_2, \ldots, t_j],
\]
i.e., an \(i\)-cycle and a \(j\)-cycle is joined to an \((i + j)\)-cycle. Hence for a permutation \(h\) of type \(\mu, \eta\) is a sum of all elements obtained from \(h\) by either a cut or a join. Fix a pair of \(i\)-cycle and \(j\)-cycle of \(h\), there are \(i \cdot j\) different ways to join them to an \((i + j)\)-cycle. Taking into the account of \(m_i(\mu)\) choices of \(i\)-cycles, and \(m_j(\mu)\) choices of \(j\)-cycles, we get
\[
ijm_i(\mu)m_j(\mu)
\]
different ways to obtain an element from \(h\) by joining an \(i\)-cycle in \(h\) to a \(j\)-cycle in \(h\). Similarly, fix an \((i + j)\)-cycle of \(h\), there are \(i + j\) different ways to cut it into an \(i\)-cycle and a disjoint \(j\)-cycle in \(h\). And taking into account the number of \((i + j)\)-cycles in \(h\), we get
\[
(i + j)m_{i+j}(\mu)
\]
different ways to obtain an element from \(h\) by cutting an \((i + j)\)-cycle into an \(i\)-cycle and a \(j\)-cycle. \(\square\)

Now we have for any \(h \in S_d\) of cycle type \(\mu\)
\[
\sum_{\mu} f_{\nu}(2) \chi_{\nu}(\mu) p_{\mu} = \sum_{\mu} \text{tr}[f_{\nu}(2) \cdot \rho_{\nu}(h)] \cdot \prod_i p_{i}^{m_i(\mu)}
\]
\[
= \sum_{\mu} \text{tr}[\sum_{g \in C(2)} \rho_{\nu}(g) \cdot \rho_{\nu}(h)] \cdot \prod_i p_{i}^{m_i(\mu)}
\]
\[
= \sum_{\mu} \text{tr} \rho_{\nu}(\sum_{g \in C(2)} g \cdot h) \cdot \prod_i p_{i}^{m_i(\mu)}
\]
\[
= \sum_{\mu} \left( \sum_{\eta \in J_{i,j}(\mu)} ijm_i(\mu)m_j(\mu)\chi_{\nu}(\eta) + \sum_{\eta \in C_{i,j}(\mu)} (i + j)m_{i+j}(\mu)\chi_{\nu}(\eta) \right) \cdot \prod_i p_{i}^{m_i(\mu)}
\]
\[
= \left( ij p_{i+j} + (i + j) p_i p_j \right) \sum_{\eta} \chi_{\nu}(\eta) p_{\eta}.\]

Here we have the following notations. Let \(\mu, \eta\) be two partitions, both represented by Young diagrams. We write \(\eta \in J_{i,j}(\mu)\) and \(\mu \in C_{i,j}(\eta)\) if \(\eta\) is obtained from \(\mu\) by remove a row of length \(i\) and a row of length \(j\), then adding a row of length...
Two such morphisms are isomorphic if they differ by an isomorphism of the domain and an automorphism of the bubble components of the automorphism group in the stability condition (4) above. This induces an action \( P \) on and the trivial actions on the bubble components. This in turn induces an action \( P \) on the moduli space of relative morphisms.  

4. Proof of Theorem 2

4.1. Moduli space of relative morphisms. We first describe the moduli space of stable relative morphisms to \( \mathbb{P}^1 \) used in [19]. The moduli spaces of stable relative morphisms are constructed by J. Li [15]. The construction in symplectic geometry was carried out independently by Li-Ruan [18] and Ionel-Parker [9, 10].

Let \( \mathbb{P}^1|m| \) be a chain of \( m+1 \) copies \( \mathbb{P}^1 \), such that the \( i \)-th copy is glued to the \((i+1)\)-th copy at the point \( p_1^{(i)} \) for \( i \leq m \). The first copy will be referred to as the root component, and the other components will be called the bubble components. A point \( p_1^{(m)} \) is fixed on the \((m+1)\)-th component. Denote by \( \pi|m| : \mathbb{P}^1|m| \to \mathbb{P}^1 \) the map which is identity on the root component and contracts all the bubble components to \( p_1^{(0)} \).

Let \( \mu \) be a partition of \( d > 0 \). Let \( \overline{M}_{g,0}(\mathbb{P}^1, \mu) \) be the moduli space of morphisms

\[
 f : (C, x_1, \ldots, x_{l(\mu)}) \to \mathbb{P}^1|m|,
\]

such that

1. \((C, x_1, \ldots, x_{l(\mu)})\) is a prestable curve of genus \( g \) with \( l(\mu) \) marked points.
2. \( f^{-1}(p_1^{(m)}) = \sum_{i=1}^{l(\mu)} \) as Cartier divisors, and \( \deg(\pi|m| \circ f) = d \).
3. The preimage of each node in \( \mathbb{P}^1|m| \) consists of nodes of \( C \). If \( f(y) = p_1^i \) and \( C_1 \) and \( C_2 \) are two irreducible components of \( C \) which intersects at \( y \), then \( f|_{C_1} \) and \( f|_{C_2} \) has the same contact order to \( p_1^i \) at \( y \).
4. The automorphism group of \( f \) is finite.

Two such morphisms are isomorphic if they differ by an isomorphism of the domain and an automorphism of the bubble components of \( \mathbb{P}^1|m| \). In particular, this defines the automorphism group in the stability condition (4) above.

In [15][18], J. Li showed that \( \overline{M}_{g,0}(\mathbb{P}^1, \mu) \) is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

\[
 r = 2g - 2 + |\mu| + l(\mu),
\]

so it has a virtual fundamental class of degree \( r \).

4.2. Torus action. Consider the \( \mathbb{C}^* \)-action

\[
 t \cdot [z^0 : z^1] = [tz^0 : z^1]
\]
on \( \mathbb{P}^1 \). It has two fixed points \( p_0 = [0 : 1] \) and \( p_1 = [1 : 0] \). This induces an action on \( \mathbb{P}^1|m| \) by the action on the root component induced by the isomorphism to \( \mathbb{P}^1 \), and the trivial actions on the bubble components. This in turn induces an action on \( \overline{M}_{g,0}(\mathbb{P}^1, \mu) \).
4.3. The branch morphism. There is a branch morphism

\[ \text{Br} : \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu) \to \text{Sym}^r \mathbb{P}^1 \cong \mathbb{P}^r. \]

Note that \( \mathbb{P}^r \) can be identified with \( \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(r))) \), and the isomorphism

\[ \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(r))) \cong \text{Sym}^r \mathbb{P}^1 \]

is given by \( [s] \mapsto \text{div}(s) \). The \( \mathbb{C}^* \) action on \( \mathbb{P}^1 \) induces a \( \mathbb{C}^* \) action on \( H^0(\mathbb{P}^1, \mathcal{O}(r)) \) by

\[ t \cdot (z^0)^k(z^1)^{r-k} = t^{-k}(z^0)^k(z^1)^{r-k}. \]

So \( \mathbb{C}^* \) acts on \( \mathbb{P}^r \) by

\[ t \cdot [a_0, a_1, \ldots, a_r] = [a_0, t^{-1}a_1, \ldots, t^{-r}a_r], \]

where \( (a_0, a_1, \ldots, a_r) \) corresponds to \( \sum_{k=0}^{r} a_k(z_0)^k(z_1)^{r-k} \in H^0(\mathbb{P}^1, \mathcal{O}(r)) \). With this action, the branch morphism is \( \mathbb{C}^* \)-equivariant.

4.4. The Obstruction Bundle. In [19], J. Li and Y. Song constructed an obstruction bundle over the stratum where the target is \( \mathbb{P}^1[0] = \mathbb{P}^1 \), and proposed an extension over the entire \( \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu) \). Here we use a different extension which is equivalent to the one used in [19].

Let \( \pi[m] : \mathbb{P}^1[m] \to \mathbb{P}^1 \) be the contraction to the root component, and denote \( \tilde{f} = \pi[m] \circ f \). Dual to the obstruction space at a map \( f : (C, x_1, \ldots, x_{l(\mu)}) \to \mathbb{P}^1[m] \), consider the vector bundle \( V \) with fiber at \( f \) given by

\[ H^1(C, \mathcal{O}_C(-D)) \oplus H^1(C, \tilde{f}^* \mathcal{O}_{\mathbb{P}^1}(-1)), \]

where \( D = x_1 + \ldots + x_{l(\mu)} \). It is a direct sum of two vector bundles \( V_D \) and \( V_{D_d} \).

Note that

\[ H^0(C, \mathcal{O}_C(-D)) = H^0(C, \tilde{f}^* \mathcal{O}_{\mathbb{P}^1}(-1)) = 0, \]

so the ranks of \( V_D \) and \( V_{D_d} \) are, by Riemann-Roch, \( l(\mu) + g - 1 \) and \( d + g - 1 \), respectively.

We lift \( \mathbb{C}^* \) action on \( \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu) \) to \( V_D \) and \( V_{D_d} \) as follows. The action on \( V_{D_d} \) comes from an action on \( \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1 \) with weights \( p \) and \( p+1 \) at the two fixed points \( p_0 \) and \( p_1 \), respectively, where \( p \in \mathbb{Z} \). The fiber of \( V_D \) does not depend on the map \( f \), so the fibers over two points in the same orbit of the \( \mathbb{C}^* \) action can be canonically identified. The action of \( \lambda \in \mathbb{C}^* \) on \( V_D \) is multiplication by \( \lambda^{-p-1} \).

4.5. Functorial localization. Let \( T = \mathbb{C}^* \). We will compute

\[ \text{Br}_* e_T(V) = \sum_{l=0}^{r} a_l(p) H^l_{T} u^{r-l}. \]

by virtual functorial localization [21].

Let \( F(p, x) = \sum_{l=0} a_l(p) x^l \). We have

\[ \frac{\tilde{f}_* \text{Br}_* e_T(V)}{e_T(T_{p, \mathbb{P}^r})} = \frac{F(p, k)}{(-1)^{r-k} k!(r-k)!}. \]

By functorial localization, we have

\[ \int_{p_{r-k}} \frac{f_{r-k}^* \text{Br}_* e_T(V)}{e_T(T_{p_{r-k}, \mathbb{P}^r})} = \sum_{F \in \text{Br}^{-1}(p_{r-k})} \int_{[F]_{virt}} \frac{e_T(V)}{e_T(N_F^{virt})}. \]
for $k = 0, \ldots, r$, where $N_F$ is the virtual normal bundle of the fixed loci $F$ in $\mathcal{M}_{g,0}(\mathbb{P}^1, \mu)$. It is computed in [22] that

$$\sum_{F \subset \text{Br}^{-1}(p-r-k)} \int_{[F]_{vir}} \frac{e_T(V)}{e_T(N_F^{vir})} = (p + 1)^k J_{g,\mu}^k(p),$$

where $J_{g,\mu}^k(p)$ is a degree $r - k$ polynomial in $p$, and

$$J_{g,\mu}^k(-p - 1) = (-1)^{d-l(\mu)} + k J_{g,\mu}^k(p).$$

Moreover, we have

$$J_{g,\mu}^0(p) = -1^{(\mu)-|\mu|} C_{g,\mu}(p),$$

$$J_{g,\mu}^1(p) = -1^{(\mu)-|\mu|-1} \left( \sum_{\nu \in J(\mu)} I_1(\nu) C_{g,\nu}(p) + \sum_{\nu \in C(\mu)} I_2(\nu) C_{g,\nu}(p) + \sum_{g_1+g_2=|\nu|, \nu \in C(\mu)} I_3(\nu^1, \nu^2) C_{g_1,\nu^1}(p) C_{g_2,\nu^2}(p) \right).$$

Here we use the notation in [20]. The set $J(\mu)$ (join) consists of partitions of $d$ of the form

$$\nu = (\mu_1, \ldots, \hat{\mu}_i, \ldots, \hat{\mu}_j, \ldots, \mu_{l(\mu)}; \mu_i + \mu_j)$$

and the set $C(\mu)$ (cut) consists of partitions of $d$ of the form

$$\nu = (\mu_1, \ldots, \hat{\mu}_i, \ldots, \mu_{l(\mu)}; j, k)$$

where $j + k = \mu_i$. The precise definitions of $I_1$, $I_2$, and $I_3$ can be found in [20]. It follows from the definition that (6) in Theorem 2 is equivalent to

$$\frac{d}{d\tau} J_{g,\mu}^0(\tau) = -J_{g,\mu}^1(\tau).$$

Since

$$F(p, x) = \sum_{k=0}^{r} (-1)^{r-k}k!(r-k)! x(x-1) \cdots (x-k+1)(x-k-1) \cdots (x-r)$$

$$= \sum_{k=0}^{r} (p+1)^{r-k} J_{g,\mu}^{r-k}(p) x(x-1) \cdots (x-k+1)(x-k-1) \cdots (x-r)$$

$$= \sum_{k=0}^{r} (p+1)^{r-k} J_{g,\mu}^{k}(p) x(x-1) \cdots (x-(r-k+1))(x-(r-k+1)) \cdots (x-r),$$

therefore,

$$\text{Br}_*: e_T(V) = \sum_{k=0}^{r} (p+1)^{r-k} J_{g,\mu}^{k}(p) H(H-u) \cdots (H-(r-k+1)u)(H-(r-k+1)u) \cdots (H-ru).$$
4.6. Final Calculations. Let $\tau = -p - 1$, then

$$
\text{Br}_*e_T(V) = \sum_{k=0}^{r} (-\tau)^k J_{g,\mu}^k(-\tau - 1)H(H - u) \cdots (H - (r - k - 1)u) \\
\cdot (H - (r - k + 1)u) \cdots (H - ru)
$$

Therefore,

$$
\text{Br}_*e_T(V) = (-1)^{d-l(\mu)} \sum_{k=0}^{r} \tau^k J_{g,\mu}^k(\tau)H(H - u) \cdots (H - (r - k - 1)u)(H - (r - k + 1)u) \cdots (H - ru).
$$

For $i = 0, \ldots, r - 1$, we have

$$
H^i H(H - u) \cdots (H - (r - k - 1)u)(H - (r - k + 1)u) \cdots (H - ru)
$$

which is of degree no more than $i$. Therefore,

$$
\int_{\mathbb{P}^r} \text{Br}_*e_T(V)H^i = (-1)^{d-l(\mu)} u^i \sum_{k=0}^{r} (r - k)^i \tau^k J_{g,\mu}^k(\tau).
$$

Let $J_{g,\mu}^k(\tau) = \sum_{j=0}^{r-k} a_j^k \tau^j$. We have

$$
u^{-i} \int_{\mathbb{P}^r} \text{Br}_*e_T(V)H^i = (-1)^{d-l(\mu)} \sum_{l=0}^{r} \left( \sum_{j+k=l} (r - k)^j a_j^k \right) \tau^l.$$

Here is a crucial observation: as a polynomial in $\tau$, $u^{-i} \int_{\mathbb{P}^r} \text{Br}_*e_T(V)H^i$ is of degree no more than $i$. Therefore,

$$
\sum_{j+k=l} (r - k)^j a_j^k = 0
$$

for $0 \leq i < l \leq r$. Now fix $l$ such that $1 \leq l \leq r$. We have

$$(14) \quad \sum_{k=0}^{l} (r - k)^i a_{i-k}^k = 0, \quad 0 \leq i < l,$$

which is a system of $l$ linear equations of the $l + 1$ variables $\{a_k^k : k = 0, \ldots, l\}$.

Both

$$(r - t)^i : i = 0, \ldots, l - 1$$

and

$$\{1, t, t(t - 1), \ldots, t(t - 1) \cdots (t - l + 2)\}$$

are bases of the vector space

$$\{ f(t) \in \mathbb{Q}[t] : \deg(f) \leq l - 1 \}.$$
so there exists an invertible \( l \times l \) matrix \( (A_{ij})_{0 \leq i,j \leq l-1} \) such that

\[
t(t-1) \cdots (t-i+1) = \sum_{j=0}^{l-1} A_{ij} (r-t)^j.
\]

In particular,

\[
k(k-1) \cdots (k-i+1) = \sum_{j=0}^{l} A_{ij} (r-k)^j.
\]

for \( k = 0, 1, \ldots, l \), so (13) is equivalent to

\[
\sum_{k=0}^{l} k(k-1) \cdots (k-i+1) a_{l-k}^k = 0, \quad 0 \leq i < l,
\]

i.e.,

\[
\sum_{k=i}^{l} \frac{k!}{(k-i)!} a_{l-k}^k = 0, \quad 0 \leq i < l.
\]

The above equations can be rewritten as

\[
\begin{pmatrix}
1 & 1 & \cdots & \cdots & \cdots & 1 \\
0 & 1! & 2 & \cdots & \cdots & l \\
0 & 0 & 2! & 3 & \cdots & l(l-1) \\
0 & 0 & 0 & 3! & \cdots & l(l-1)(l-2) \\
0 & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & (l-1)! & l(l-1)\cdots 2 \\
\end{pmatrix}
\begin{pmatrix}
a_0^0 \\
a_1^1 \\
a_2^2 \\
\vdots \\
\vdots \\
a_{l-1}^{l-1} \\
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
\vdots \\
0 \\
\end{pmatrix}.
\]

The kernel is clearly one dimensional. One can check that the kernel is given by

\[(15) \quad a_{l-k}^k = (-1)^k \frac{l!}{k!(l-k)!} a_0^0.\]

Note that (15) for \( l = 1, \ldots, r \) is equivalent to

\[
J^k_{g,\mu}(\tau) = \frac{(-1)^k}{k!} \frac{d^k}{d\tau^k} J^0_{g,\mu}(\tau)
\]

for \( k = 0, \ldots, r \). In particular,

\[
J^1_{g,\mu}(\tau) = -\frac{d}{d\tau} J^0_{g,\mu}(\tau)
\]

which is equivalent to the cut-and-join equation (3) in Theorem 2.

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