Mean-field limits of particles in interaction with quantized radiation fields

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Dedicated to Herbert Spohn on the occasion of his 70th birthday.

Abstract

We report on a simple strategy to treat mean-field limits of quantum mechanical systems in which a large number of particles weakly couple to a second-quantized radiation field. Extending the method of counting, introduced in [21], with ideas inspired by [16] and [6] leads to a technique that can be seen as a combination of the method of counting and the coherent state approach. It is similar to the coherent state approach but might be slightly better suited to systems in which a fixed number of particles couple to radiation. The strategy is effective and provides explicit error bounds. As an instructional example we derive the Schrödinger-Klein-Gordon system of equations from the Nelson model with ultraviolet cutoff. Furthermore, we derive explicit bounds on the rate of convergence of the one-particle reduced density matrix of the non-relativistic particles in Sobolev norm. More complicated models like the Pauli-Fierz Hamiltonian can be treated in a similar manner [14].

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I Introduction

Quantum systems with many degrees of freedom are difficult to analyze. This is especially severe in the presence of quantized radiation fields which are described by Fock spaces with infinitely many degrees of freedom. The dynamics of such systems can, however, be studied in special situations by means of simpler effective evolution equations. These involve fewer degrees of freedom, are less exact but easier to investigate. Effective evolution equations for particles that interact with quantized radiation fields have rigorously been derived for example in [9, 6, 14, 27, 8, 11]. The general setting in these works is given by the tensor product of two Hilbert spaces

\[ \mathcal{H}^{(N)} = \mathcal{H}_p^{(N)} \otimes \mathcal{F}. \] (1)

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The space $\mathcal{H}_p^{(N)}$ describes $N$ non-relativistic particles and $\mathcal{F}$ (usually a bosonic Fock space) models the quantized radiation field in terms of gauge bosons. The dynamics of the system is governed by the Schrödinger equation with a Hamiltonian of the form

$$H^N := H_0^N + H_f + \sum_{j=1}^{N} H_{\text{int},j}. \quad (2)$$

Here, $H_0^N$ and $H_f$ (solely acting on $\mathcal{H}_p^{(N)}$ and $\mathcal{F}$) denote the free Hamiltonians of the particles and the radiation field. The term $H_{\text{int},j}$ establishes an interaction between the $j$-th particle and the radiation field. This couples the dynamics of the particles with the gauge bosons. A typical question of interest is, whether the quantized radiation field can be approximated by a classical field and the evolution of the whole system described by a system of simple effective equations. Usually one considers initial data $\Psi_{N,0} = \Phi_{N,0} \otimes W(\gamma^{1/2} \alpha_0)\Omega$ with no correlations between the particles and the gauge bosons, sometimes referred to as Pekar product state $\mathcal{F}$. The state $W(\gamma^{1/2} \alpha_0)\Omega \in \mathcal{F}$ denotes gauge bosons in the coherent state $\alpha_0$ with a mean particle number $\gamma ||\alpha_0||^2$, see (13). Hereby, $\gamma$ is a model dependent scaling parameter, for instance the number of particles $[9, 11, 13]$ or the strong coupling parameter in the Polaron model $[7, 8, 10]$. From physics literature it is commonly known that coherent states with a high occupation number of gauge bosons can approximately be described by a classical radiation field $[4, \text{Chapter III.C.4}]$. This allows us to describe the system in the limit $\gamma \to \infty$ (in a suitable sense, see Section III) effectively by the state of the particles $\Phi_{N,0}$ and a classical radiation field with mode function $\alpha_0$. The arising question is, if at later time $t$ one can still approximate the system by the pair $(\Phi_{N,t}, \alpha_t)$ which evolves according to a set of simple effective equations with initial datum $(\Phi_{N,0}, \alpha_0)$:

$$\Psi_{N,0} \xrightarrow{\gamma \to \infty} (\Phi_{N,0}, \alpha_0) \quad \text{Many-body dynamics} \quad \downarrow \quad \text{Effective dynamics} \quad \Psi_{N,t} \xrightarrow{\gamma \to \infty} (\Phi_{N,t}, \alpha_t). \quad (3)$$

This only holds, if the radiation sector of $\Psi_{N,t}$ is approximately given by a coherent state, i.e. if the gauge bosons, that are created during the time evolution, are either in a coherent state or subleading with respect to $\gamma$. The effect of the particles on the radiation field is typically negligible, if one considers a fixed number of particles, a coupling constant that tends to zero in a suitable sense and a coherent state, whose mean particle number scales with the parameter $\gamma [9]$. Otherwise, the state of the particles must have a special structure to ensure that the contributing gauge bosons are coherent $[1 \text{ Complement BIII}]$. This is expected, if one considers slow and heavy particles $[27]$ or a condensate of particles that weakly couple to the radiation field. In this work, we are interested in the latter situation. More explicitly, we study the dynamics of initial states $\Psi_{N,0} = \varphi_0^{\otimes N} \otimes W(N^{1/2} \alpha_0)\Omega$ with one particle wave function $\varphi_0$ in the limit $N = \gamma \to \infty$ where the fields in the interaction Hamiltonian $H_{\text{int},j}$ are multiplied by $N^{-1/2}$ (see Section III). We refer to this limit as mean-field limit, because it implies that the source term of the radiation field is replaced by its mean value in the effective description. So far, such kind of limits have been studied either by the coherent state approach $[3, 5, 6]$ or by means of Wigner measures $[1, 11]$. While the method of Wigner measures allows us to derive limiting equations for an extensive class of initial states it does in contrast to the coherent state approach not provide quantitative bounds on the rate of

\[ \text{These approaches usually embed the N particle states of } \mathcal{H}_p^{(N)} \text{ in a bosonic Fock space for the particles } \mathcal{F}_p \text{ and consider the Hilbert space } \mathcal{F}_p \otimes \mathcal{F}. \]
convergence. In the following, we present a strategy, similar to the coherent state approach, which is designed for systems with fixed particle number. Such systems usually arise in the non-relativistic limit when the creation and annihilation of the non-relativistic particles is suppressed.\(^2\) The method provides explicit bounds on the rate of convergence and can be seen as a combination of the method of counting and the coherent state approach. As an instructional example we derive the Schrödinger-Klein-Gordon system of equations from the Nelson model with ultraviolet cutoff. Our strategy seems general and we hope it will be useful in the treatment of more complicated models. It was already applied to derive the Maxwell-Schrödinger system of equations from the spinless Pauli-Fierz Hamiltonian [14].

\section{Setting of the problem}

We consider a system of \(N\) identical charged bosons interacting with a scalar field, described by a wave function \(\Psi_{N,t} \in \mathcal{H}^{(N)}\). The Hilbert space is given by

\[ \mathcal{H}^{(N)} := L^2(\mathbb{R}^3)^n \otimes \mathcal{F}, \]

where the scalar field is represented by elements of the Fock space \(\mathcal{F} := \bigoplus_{n \geq 0} L^2(\mathbb{R}^3)^{\otimes n}\). The subscript \(s\) indicates symmetry under interchange of variables. An element \(\Psi_N \in \mathcal{H}^{(N)}\) is a sequence \(\{\psi^{(n)}_N\}_{n \in \mathbb{N}_0}\) in \(L^2(\mathbb{R}^{3N+3n})\) with \(^4\)

\[ ||\Psi_N||^2 = \sum_{n=0}^{\infty} \int d^3x d^3k |\psi^{(n)}_N(x_1, \ldots, x_N, k_1, \ldots, k_n)|^2 < \infty. \]

On \(\mathcal{H}^{(N)}\), we define the (pointwise) annihilation and creation operators by \(^4\)

\[ a(k)\psi^{(n)}(X_N, k_1, \ldots, k_n) = (n+1)^{1/2} \psi^{(n+1)}_N(X_N, k, k_1, \ldots, k_n), \]
\[ a^*(k)\psi^{(n)}(X_N, k_1, \ldots, k_n) = n^{-1/2} \sum_{j=1}^n \delta(k - k_j)\psi^{(n)}_N(X_N, k_1, \ldots, \hat{k}_j, \ldots, k_n). \]

They are operator valued distributions and satisfy the commutation relations
\[ [a(k), a^*(l)] = \delta(k - l), \quad [a(k), a(l)] = [a^*(k), a^*(l)] = 0. \]

The time evolution of the wave function \(\Psi_{N,t}\) is governed by the Schrödinger equation
\[ i\partial_t \Psi_{N,t} = H_N \Psi_{N,t}. \]

Here,
\[ H_N = \sum_{j=1}^N \left( -\Delta_j + \frac{\Phi(x_j)}{\sqrt{N}} \right) + H_f \]

denotes the Nelson Hamiltonian and
\[ \hat{\Phi}_N(x) = \int d^3k \frac{\kappa(k)}{\sqrt{2\omega(k)}} \left( e^{ikx}a(k) + e^{-ikx}a^*(k) \right). \]

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\(^2\)For the sake of clarity, we want to stress that only the number of the non-relativistic particles is fixed while gauge bosons are created and destroyed during the time evolution.

\(^3\)Note that \(\psi^{(n)}_N\) is symmetric in the variables \(k_1, \ldots, k_n\). For notational convenience we will use the shorthand notation \(\psi^{(n)}_N(X_N, K_n) = \psi^{(n)}_N(x_1, \ldots, x_N, k_1, \ldots, k_n)\).

\(^4\)Here, \(\hat{k}_j\) means that \(k_j\) is left out in the argument of the function.
The scalar bosons evolve according to the dispersion relation $\omega(k) = (|k|^2 + m_b^2)^{1/2}$ with mass $m_b \geq 0$ and

$$\tilde{\kappa}(k) = (2\pi)^{-3/2} \mathbf{1}_{|k| \leq \Lambda}(k), \quad \text{with } \mathbf{1}_{|k| \leq \Lambda}(k) = \begin{cases} 1 & \text{if } |k| \leq \Lambda, \\ 0 & \text{otherwise}, \end{cases} \quad (11)$$

cuts off the high frequency modes of the radiation field. On the domain

$$\mathcal{D}(H_f) = \left\{ \Psi_N \in \mathcal{H}^{(N)} : \sum_{n=1}^{\infty} \int d^3x \, d^3k \sum_{j=1}^{n} w(k_j)|\Psi_N^{(n)}(X_N, K_n)|^2 < \infty \right\} \quad (12)$$

the free Hamiltonian of the scalar field is defined by

$$(H_f \Psi_N)^{(n)} = \sum_{j=1}^{n} w(k_j) \Psi_N^{(n)}. \quad (13)$$

By means of the creation and annihilation operators it can be written as

$$H_f = \int d^3k \, \omega(k) a^*(k) a(k). \quad (14)$$

The Nelson model was originally introduced to describe the interaction of non-relativistic nucleons with a meson field. By standard estimates of the field operator and Kato’s theorem it is easily shown that $H_N$ is a self-adjoint operator with $\mathcal{D}(H_N) = \mathcal{D}(\sum_{j=1}^{N} -\Delta_j + H_f)$ \[20\] \[24\]. The scaling in front of the interaction ensures that the kinetic and potential energy of $H_N$ are of the same order. For simplicity, we are first interested in the evolution of initial states of the product form

$$\varphi_0^\otimes N \otimes W(\sqrt{N} \alpha_0) \Omega. \quad (15)$$

Here, $\Omega$ denotes the vacuum in $\mathcal{F}$ and $W(f)$ is the Weyl operator

$$W(f) := \exp \left( \int d^3k \, f(k) a^*(k) - f^*(k) a(k) \right), \quad (16)$$

where $f \in L^2(\mathbb{R}^3)$. This choice of initial states corresponds to situations in which no correlations among the particles and the gauge bosons are present. Due to the interaction between the particles and the gauge bosons correlations take place and the time evolved state will no longer be of product form. However, for large $N$ and times of order one it can be approximated, in a sense more specified below, by a state of the form $\varphi_t^\otimes N \otimes W(\sqrt{N} \alpha_t) \Omega$, where $(\varphi_t, \alpha_t)$ solves the Schrödinger-Klein-Gordon equations\footnote{We use the shorthand notation $(\kappa * \Phi)(x, t) = \int d^3k \, e^{ikx} \tilde{\kappa}(k) \mathcal{F} \mathcal{T}\Phi(k, t)$, where $\mathcal{F} \mathcal{T}\Phi(k, t)$ denotes the Fourier transform of $\Phi(x, t)$.}

$$\begin{cases} i\partial_t \varphi_t(x) = H^{eff} \varphi_t(x) = [-\Delta + (\kappa * \Phi)(x, t)] \varphi_t(x), \\ i\partial_t \alpha_t(k) = \omega(k) \alpha_t(k) + (2\pi)^{3/2} \frac{\tilde{\kappa}(k)}{2\omega(k)} \mathcal{F} \mathcal{T} \left[ |\varphi_t|^2 \right](k), \\ \Phi(x, t) = \int d^3k \, (2\pi)^{-3/2} \frac{1}{\sqrt{2\omega(k)}} \left( e^{ikx} \alpha_t(k) + e^{-ikx} \alpha_t^*(k) \right), \end{cases} \quad (17)$$

with $(\varphi_0, \alpha_0) \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$. This system of equations determines the evolution of a single quantum particle in interaction with a classical scalar field. In the literature it is better known in its formally equivalent form

$$\begin{cases} i\partial_t \varphi_t(x) = [-\Delta + (\kappa * \Phi)(x, t)] \varphi_t(x), \\ \left[ \partial_t^2 - \Delta + m_b^2 \right] \Phi(x, t) = - (\kappa * |\varphi_t|^2)(x). \end{cases} \quad (18)$$
III Main result

The physical situation we are interested in is the dynamical description of a Bose-Einstein condensate of charges. We start initially with a product state \([15]\) and show that the condensate persists during the time evolution, i.e. correlations are small also at later times. Let

\[
N := \int d^3 k a^\dagger(k)a(k)
\]

be the number (of gauge bosons) operator with domain

\[
\mathcal{D}(N) = \left\{ \Psi_N \in \mathcal{H}^{(N)} : \sum_{n=1}^{\infty} n^2 \int d^3 x d^3 n |\Psi_N^{(n)}(X_N, K_n)|^2 < \infty \right\}
\]

and \(\Psi_{N,t} \in (L^2_\mathbb{R}^{3N}) \otimes \mathcal{F}) \cap \mathcal{H}^{(N)} \cap \mathcal{D}(N)\) such that \(\|\Psi_{N,t}\|_{\mathcal{H}^{(N)}} = 1\). On the Hilbert space \(L^2(\mathbb{R}^3)\) we define the "one-particle reduced density matrix of the charges" by

\[
\gamma_{N,t}^{(1,0)} := \text{Tr}_{2,\ldots,N} \otimes \text{Tr}_\mathcal{F}|\Psi_{N,t}\rangle\langle\Psi_{N,t}|,
\]

where \(\text{Tr}_{2,\ldots,N}\) denotes the partial trace over the coordinates \(x_2, \ldots, x_N\) and \(\text{Tr}_\mathcal{F}\) the trace over Fock space. Then, the charged particles of the many-body state \(\Psi_{N,t}\) are said to exhibit complete asymptotic Bose-Einstein condensation at time \(t\), if there exists \(\varphi_t \in L^2(\mathbb{R}^3)\) with \(\|\varphi_t\| = 1\), such that

\[
\text{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| \to 0,
\]

as \(N \to \infty\). Such \(\varphi_t\) is called the condensate wave function. For other indicators of condensation and their relation we refer to \([17]\). Moreover, we introduce the "one-particle reduced density matrix of the gauge bosons" with kernel

\[
\gamma_{N,t}^{(0,1)}(k, k') := N^{-1}\langle\Psi_{N,t}, a^\dagger(k')a(k)|\Psi_{N,t}\rangle_{\mathcal{H}^{(N)}}.
\]

\(\gamma_{N,t}^{(0,1)}\) is a positive trace class operator with \(\text{Tr}_{L^2(\mathbb{R}^3)}(\gamma_{N,t}^{(0,1)}) = N^{-1}\langle\Psi_{N,t}, N\Psi_{N,t}\rangle_{\mathcal{H}^{(N)}}\). It should be noted, that \([23]\) differs from the usual definition (e.g. \([25\ p.8]\)) by the weight factor \(\langle\Psi_N, N\Psi_N\rangle_{\mathcal{H}^{(N)}}/N\). Our choice ensures that we only measure deviations from the classical mode function that are at least of order \(N\). This is reasonable because Fock space vectors with a mean particle number smaller than of order \(N\) only have a subleading effect on the dynamics of the charged particles. We say the gauge bosons exhibit "asymptotic Bose-Einstein condensation", if there exists a state \(\alpha_t \in L^2(\mathbb{R}^3)\), such that

\[
\text{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|| \to 0,
\]

as \(N \to \infty\).

In order to derive our main result, the solutions of the Schrödinger-Klein-Gordon equations have to satisfy the following assumptions.

**Definition III.1.** Let \(m \in \mathbb{N}\), \(H^m(\mathbb{R}^3)\) denote the Sobolev space of order \(m\) and \(L^2_m(\mathbb{R}^3)\) a weighted \(L^2\)-space with norm \(\|\varphi\|_{L^2_m(\mathbb{R}^3)} = \|(1 + |\cdot|^2)^{m/2}\varphi\|_{L^2(\mathbb{R}^3)}\). We define two sets of solutions of the Schrödinger-Klein-Gordon equations:

\[
(\varphi_t, \alpha_t) \in \mathcal{G}_1 \iff (a) \quad (\varphi_t, \alpha_t) \text{ is a } L^2 \oplus L^2 \text{ solution of } (17) \text{ with } \|\varphi_t\|_{L^2(\mathbb{R}^3)} = 1
\]

\[
(b) \quad (\varphi_t, \alpha_t) \in H^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3).
\]

\[
(\varphi_t, \alpha_t) \in \mathcal{G}_2 \iff (a) \quad (\varphi_t, \alpha_t) \text{ is a } L^2 \oplus L^2 \text{ solution of } (17) \text{ with } \|\varphi_t\|_{L^2(\mathbb{R}^3)} = 1
\]

\[
(b) \quad (\varphi_t, \alpha_t) \in H^3(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3).
\]
These assumptions are expected to follow from appropriately chosen initial data.

**Conjecture III.2.** Let \((\varphi_0, \alpha_0) \in H^{2n}(\mathbb{R}^3) \oplus L^2_{\alpha}(\mathbb{R}^3)\) for \(1 \leq n \leq 2\). Then, there is a strongly differentiable \((H^{2n}(\mathbb{R}^3) \oplus L^2_{\alpha}(\mathbb{R}^3))\)-valued function \((\varphi(t), \alpha(t))\) on \([0, \infty)\) that satisfies \([17]\).

Our main theorem is the following.

**Theorem III.3.** Let \((\varphi_t, \alpha_t) \in \mathcal{G}_1\) and \(\Psi_{N,0} \in (L^2_{\alpha}(\mathbb{R}^{3N}) \otimes \mathcal{F}) \cap \mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N)\) with \(\|\Psi_{N,0}\| = 1\) such that

\[
\begin{align*}
a_N &= Tr_{L^2(\mathbb{R}^3)}[\gamma^{(1)}_{N,0}] - |\varphi_0\rangle \langle \varphi_0| \to 0 \quad \text{and} \\
N^{-1} |W^{-1}(\sqrt{N} \alpha_0)\Psi_{N,0}W^{-1}(\sqrt{N} \alpha_0)\Psi_{N,0}]_{\mathcal{H}(\mathcal{N})} &\to 0
\end{align*}
\]

as \(N \to \infty\). Let \(\Psi_{N,t}\) be the unique solution of \([8]\) with initial data \(\Psi_{N,0}\). Then, there exists a generic constant \(C\) independent of \(N\), \(\Lambda\) and \(t\) such that

\[
\begin{align*}
Tr_{L^2(\mathbb{R}^3)}[\gamma^{(1)}_{N,t}] - |\varphi_t\rangle \langle \varphi_t| &\leq a_N + b_N + N^{-1}e^{CA^2t}, \\
Tr_{L^2(\mathbb{R}^3)}[\gamma^{(0,1)}_{N,t}] - |\alpha_t\rangle \langle \alpha_t| &\leq a_N + b_N + N^{-1}e^{CA^2t}C(1 + ||\alpha_t||)
\end{align*}
\]

for any \(t \in \mathbb{R}_+\). In particular, for \(\Psi_{N,0} = \varphi_0^\otimes N \otimes W(\sqrt{N} \alpha_0)\Omega\) one obtains

\[
\begin{align*}
Tr_{L^2(\mathbb{R}^3)}[\gamma^{(1)}_{N,t}] - |\varphi_t\rangle \langle \varphi_t| &\leq N^{-1/2}e^{CA^2t}, \\
Tr_{L^2(\mathbb{R}^3)}[\gamma^{(0,1)}_{N,t}] - |\alpha_t\rangle \langle \alpha_t| &\leq N^{-1/2}e^{CA^2t}C(1 + ||\alpha_t||).
\end{align*}
\]

Moreover, let \((\varphi_t, \alpha_t) \in \mathcal{G}_2\) and \(\Psi_{N,0} \in (L^2_{\alpha}(\mathbb{R}^{3N}) \otimes \mathcal{F}) \cap \mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N) \cap \mathcal{D}(\mathcal{H}_N^2)\) such that

\[
\begin{align*}
c_N = \|\nabla_1 \left(1 - |\varphi_0\rangle \langle \varphi_0| \otimes 1_{L^2(\mathbb{R}^{3(N-1)})} \otimes 1_{\mathcal{F}}\right)\Psi_{N,0}\|_{\mathcal{H}(\mathcal{N})}^2 \to 0
\end{align*}
\]

as \(N \to \infty\). Then, there exists a positive monotone increasing function \(C(s)\) of the norms \(||\alpha_s||_{L^2(\mathbb{R}^3)}\) and \(||\varphi_s||_{H^1(\mathbb{R}^3)}\) such that

\[
Tr_{L^2(\mathbb{R}^3)}[\gamma^{(1)}_{N,t}] - |\varphi_t\rangle \langle \varphi_t| \sqrt{1 - \Delta} \leq a_N + b_N + c_N + N^{-1}C(t)e^{\Delta^2 J_0^t(s)ds}
\]

For \(\Psi_{N,0} = \varphi_0^\otimes N \otimes W(\sqrt{N} \alpha_0)\Omega\) one obtains

\[
Tr_{L^2(\mathbb{R}^3)}[\gamma^{(1)}_{N,t}] - |\varphi_t\rangle \langle \varphi_t| \sqrt{1 - \Delta} \leq N^{-1/2}C(t)e^{\Delta^2 J_0^t(s)ds}.
\]

**Remark III.4.** The convergence of the reduced density matrices in trace norm with rate \(N^{-1}\) was already shown in \([6]\) for special classes of initial states (coherent and product states).\(^\footnote{We suppose that Conjecture III.2 can be proven by a standard fixed-point argument. Especially due to the cutoff in the radiation field it seems possible to make use of Theorem X.74 in \([24]\).} \) \(^\footnote{Here, \(W^{-1}(\sqrt{N} \alpha_0) = W(-\sqrt{N} \alpha_0)\) is the inverse of the unitary Weyl operator \(W(\sqrt{N} \alpha_0)\), see Section I.1A.} \) \(^\footnote{To ease the presentation we have chosen for given \(t\) the scaling parameter \(N\) large enough such that \(0 \leq \beta(t) \leq 1\) and \(0 \leq \beta_2(t) \leq 1\) (see Subsections VIII.2 and VIII.3).} \) \(^\footnote{For a precise definition we refer to \([6]\) Theorem 3.} \)

\(^8\) \(^\footnote{For a precise definition we refer to \([6]\) Theorem 3.} \)

\(^9\) \(^\footnote{For a precise definition we refer to \([6]\) Theorem 3.} \)

\(^{10}\) \(^\footnote{For a precise definition we refer to \([6]\) Theorem 3.} \)
IV Comparison with the literature

In [9], Ginibre, Nironi and Velo derived the Schrödinger-Klein-Gordon system of equations from the Nelson model with cutoff. They considered a finite number of charged bosons, a coupling constant that tends to zero and a coherent state of gauge bosons whose particle number goes to infinity. The number of gauge bosons that are created during the time evolution is negligible in this case and it is possible to approximate the quantized scalar field by an external potential which evolves according to the Klein-Gordon equation without source term. Falconi [6] derived the Schrödinger-Klein-Gordon system of equations in the setting of the present paper by means of the coherent state approach. A comparison between his result and Theorem III.3 is given in Remark III.4. Making use of a Wigner measure approach Ammari and Falconi [1] were able to establish the classical limit (without quantitative bounds on the rate of convergence) of the renormalized Nelson model without cutoff. Teufel [27] considered the adiabatic limit of the Nelson model and showed that the interaction mediated by the quantized radiation field is well approximated by a direct Coulomb interaction. In [14] we used the strategy of the present paper to derive the Maxwell-Schrödinger equations from the spinless Pauli-Fierz Hamiltonian. Here, additional technical difficulties arise from the minimal coupling term in the Pauli-Fierz Hamiltonian.

V Notations

The Fourier transform of a function \( f \) is denoted by \( \hat{f} \) or \( FT[f] \). \( H^s(\mathbb{R}^3) \) stands for the Sobolev space with norm \( \| f \|_{H^s(\mathbb{R}^3)} = \left\| (1+|\cdot|^2)^{s/2} FT[f] \right\|_{L^2(\mathbb{R}^3)} \) and \( L^2_m(\mathbb{R}^3) \) is the weighted \( L^2 \) space with \( \| f \|_{L^2_m(\mathbb{R}^3)} = \left\| (1 + |\cdot|^2)^{m/2} f \right\|_{L^2(\mathbb{R}^3)} \). Moreover, we use \( \| A \|_{HS} = \sqrt{Tr A^* A} \) to denote the Hilbert-Schmidt norm. With a slight abuse of notation we write \( \Phi \) and \( F \) to indicate the scalar and auxiliary field but also their respective Fourier transforms. If we use \( \Phi(t) \) or \( F(t) \), we always refer to the coordinate representation of the fields. Furthermore, we apply the shorthand notation \( \Phi_\kappa(x,t) := (\kappa * \Phi)(x,t) \).

VI The strategy

We are interested in the evolution of product states of the form (15) under the dynamics (8). The scalar field in the Nelson Hamiltonian establishes an interaction between the charges and the field modes with wave vectors smaller than \( \Lambda \).10 This changes the state of the charges, leads to the creation and annihilation of gauge bosons and causes initially factorized states to build correlations between the charges, the gauge bosons as well as among charges and gauge bosons. To study the emergence of these correlations we combine the "method of counting", introduced in [21], with ideas from [16] and [6]. The result can be seen as a fusion of the "method of counting" and the coherent state approach, as used for instance in [6] [25]. The key idea is to prove condensation not in terms of reduced density matrices but to consider a different indicator of condensation. To study the correlations between the charges we introduce a functional \( \beta^a \), which counts the relative number of particles that are not in the state of the condensate wave function \( \varphi \).

Definition VI.1. For any \( N \in \mathbb{N} \), \( \varphi \in L^2(\mathbb{R}^3) \) with \( \| \varphi \| = 1 \) and \( 1 \leq j \leq N \) we define the

\[ ^{10} \text{One should note that the high frequency modes of the radiation field do not interact with the non-relativistic particles and evolve according to the free dynamics.} \]
time-dependent projectors \( p^{\varphi_i}_j : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N}) \) and \( q^{\varphi_i}_j : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N}) \) by

\[
p^{\varphi_i}_j f(x_1, \ldots, x_N) := \varphi_i(x_j) \int d^3x_j \varphi_i^*(x_j) f(x_1, \ldots, x_N) \quad \text{for all } f \in L^2(\mathbb{R}^{3N})
\]

and \( q^{\varphi_i}_j := 1 - p^{\varphi_i}_j \) \footnote{Occasionally, we use the bra-ket notation \( p^{\varphi_i}_j = |\varphi_i(x_j)\rangle\langle \varphi_i(x_j)| \).}. Let \( \Psi_{N,t} \in \mathcal{H}^{(N)} \). Then \( \beta^a : \mathcal{H}^{(N)} \times L^2(\mathbb{R}^3) \to \mathbb{R}^+_0 \) is given by

\[
\beta^a (\Psi_{N,t}, \varphi_t) := \langle \Psi_{N,t}, q^{\varphi_t}_1 \otimes 1_F \Psi_{N,t} \rangle.
\]

**Remark VI.2.** The functional \( \beta^a \) was already used in \cite{21,22,23,11,12,13,20} and others to derive the Hartree and Gross-Pitaevskii equation.

The situation is slightly different in the radiation sector because the number of gauge bosons is not preserved during the time evolution. Moreover, it is known from physics literature \cite{11} Chapter III.C.4] that the radiation field must be in a coherent state with a high occupation number of gauge bosons to behave classically. This is a state not only with little correlations but also a Poisson distributed number of gauge bosons. In order to investigate if the state of the radiation field is coherent we define a functional, referred to as \( \beta^b \), which measures the fluctuations of the field modes around the classical mode function \( \alpha_t \) for each time.

**Definition VI.3.** Let \( \alpha_t \in L^2(\mathbb{R}^3) \) and \( \Psi_{N,t} \in \mathcal{H}^{(N)} \cap D(\mathcal{N}) \). Then \( \beta^b : \mathcal{H}^{(N)} \cap D(\mathcal{N}) \times L^2(\mathbb{R}^3) \to \mathbb{R}^+_0 \) is given by

\[
\beta^b (\Psi_{N,t}, \alpha_t) := \int d^3k \langle \left( \frac{a(k)}{\sqrt{N}} - \alpha_t(k) \right) \Psi_{N,t}, \left( \frac{a(k)}{\sqrt{N}} - \alpha_t(k) \right) \Psi_{N,t} \rangle. \tag{38}
\]

**Remark VI.4.** Let \( \alpha_0 \in L^2(\mathbb{R}^3) \) and \( \Psi_{N,0} = W(\sqrt{N}\alpha_0)\Psi \) for some \( \Psi \in \mathcal{H}^{(N)} \cap D(\mathcal{N}) \). Then, the functional \( \beta^b \) can be written as

\[
\beta^b (\Psi_{N,t}, \alpha_t) = N^{-1} \langle \mathcal{U}_N(t; 0) \Psi, N \mathcal{U}_N(t; 0) \Psi \rangle, \tag{39}
\]

where \( \mathcal{U}_N(t; 0) = W^*(\sqrt{N}\alpha_t)e^{-ih_\mathcal{N}t}W(\sqrt{N}\alpha_0) \) denotes the fluctuation dynamics of the coherent state approach (as used for example in \cite{22} p.18)\footnote{This is a simple consequence of \( W(\sqrt{N}\alpha_t) \) being unitary and \( W^*(\sqrt{N}\alpha_t)a(k) = a(k)W^*(\sqrt{N}\alpha_t) + \sqrt{N}W^*(\sqrt{N}\alpha_t)\alpha_t(k) \), see \cite{24}.}. Thus, \( \beta^b \) measures the number of gauge boson fluctuations around the effective evolution.

**Remark VI.5.** It seems that \( \beta^a \) is the natural quantity to consider for condensates with fixed particle number. The functional \( \beta^b \), which usually arises in the coherent state approach as used in \cite{22,23,31} and others, is perfectly suited to keep track if the state of the radiation field remains coherent.

Finally, the counting functional is defined by

**Definition VI.6.** Let \( N \in \mathbb{N} \), \( \varphi_t \in L^2(\mathbb{R}^3) \) with \( ||\varphi_t|| = 1 \), \( \alpha_t \in L^2(\mathbb{R}^3) \) and \( \Psi_{N,t} \in \mathcal{H}^{(N)} \cap D(\mathcal{N}) \). Then \( \beta : \mathcal{H}^{(N)} \cap D(\mathcal{N}) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to \mathbb{R}^+_0 \) is defined by

\[
\beta (\Psi_{N,t}, \varphi_t, \alpha_t) := \beta^a (\Psi_{N,t}, \varphi_t) + \beta^b (\Psi_{N,t}, \alpha_t) . \tag{40}
\]

In summary, the functional has the following properties:

(i) \( \beta^a \) measures if the non-relativistic particles exhibit condensation.
(ii) \( \beta^b \) examines whether the radiation field is in a coherent state.

(iii) \( \beta (\Psi_{N,t}, \varphi_t, \alpha_t) \to 0 \) as \( N \to \infty \) implies condensation in terms of reduced density matrices (Lemma VII.1).

(iv) \( \beta (\Psi_{N,t}, \varphi_t, \alpha_t) = 0 \) if \( \Psi_{N,t} = \varphi_t^\otimes N \otimes W(\sqrt{N}\alpha_t)\Omega \) (see Lemma IX.2).

In order to show that the product structure (15) is preserved during the time evolution we apply the following strategy

1. We choose initial states \( \varphi_0, \alpha_0 \) and \( \Psi_{N,0} \) such that \( \beta (\Psi_{N,0}, \varphi_0, \alpha_0) \leq a_N + b_N \to 0 \) as \( N \to \infty \).

2. For each \( t \in \mathbb{R}_0^+ \) we estimate \( |d_t \beta (\Psi_{N,t}, \varphi_t, \alpha_t)| \leq C\Lambda^2 (\beta (\Psi_{N,t}, \varphi_t, \alpha_t) + N^{-1}) \) for some \( C \in \mathbb{R}_0^+ \). Then, Grönwall's Lemma establishes the bound \( \beta (\Psi_{N,t}, \varphi_t, \alpha_t) \leq e^{C\Lambda^2 t} (\beta (\Psi_{N,0}, \varphi_0, \alpha_0) + N^{-1}) \).

3. By means of property (iii) we conclude condensation in terms of reduced density matrices.

To show the convergence of \( \gamma^{(1,0)}_{N,t} \) to the projector onto the condensate wave function in Sobolev norm we include \( \beta^c (\Psi_{N,t}, \varphi_t) := ||\nabla_1 \Psi_{N,t}||^2 \) in the definition of the functional. This allows us to control the kinetic energy of the non-relativistic particles which are not in the condensate.

**Definition VI.7.** Let \( N \in \mathbb{N} \), \( \varphi_t \in H^2(\mathbb{R}^3) \) with \( ||\varphi_t|| = 1 \), \( \alpha_t \in L^2(\mathbb{R}^3) \) and \( \Psi_{N,t} \in \mathcal{D}(H_N) \cap \mathcal{D}(N) \). Then \( \beta_2 : \mathcal{D}(H_N) \cap \mathcal{D}(N) \times H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to \mathbb{R}_0^+ \) is defined by

\[
\beta_2 (\Psi_{N,t}, \varphi_t, \alpha_t) := \beta (\Psi_{N,t}, \varphi_t, \alpha_t) + \beta^c (\Psi_{N,t}, \varphi_t) = \beta (\Psi_{N,t}, \varphi_t, \alpha_t) + ||\nabla_1 \Psi_{N,t}||^2. \tag{41}
\]

We would like to remark, that the ultraviolet cutoff (11) is essential for the proof because:

1. The finiteness of \( ||\eta||_2 \) (see (57)) is needed to establish a connection between the difference of the radiation fields and the functional \( \beta^b \) by means of the auxiliary fields (64).

2. The cutoff \( \Lambda \) imposes regularity on the radiation fields which will be used to estimate the time derivative of \( ||\nabla_1 \Psi_{N,t}||^2 \). In spirit, this is opposite to the usual treatment of the polaron (15), where regularity of the electron state is used to obtain a sufficient decay in the field modes with large wave vectors.

**VII Relation to reduced density matrices**

In this section, we relate the functional \( \beta \) to the trace norm distance of the one-particle reduced density matrices.

**Lemma VII.1.** Let \( N \in \mathbb{N} \), \( \varphi_t \in L^2(\mathbb{R}^3) \) with \( ||\varphi_t|| = 1 \), \( \alpha_t \in L^2(\mathbb{R}^3) \) and \( \Psi_{N,t} \in \mathcal{H}(N) \cap \mathcal{D}(N) \). Then,

\[
\beta^a (\Psi_{N,t}, \varphi_t) \leq T_{L^2(\mathbb{R}^3)} |\gamma^{(1,0)}_{N,t}| - |\varphi_t\rangle\langle \varphi_t| \leq \sqrt{8\beta^a (\Psi_{N,t}, \varphi_t)}, \tag{42}
\]

\[\quad T_{L^2(\mathbb{R}^3)} |\gamma^{(0,0)}_{N,t}| - |\alpha_t\rangle\langle \alpha_t| \leq 3\beta^b (\Psi_{N,t}, \alpha_t) + 6 ||\alpha_t||_{L^2(\mathbb{R}^3)} \sqrt{\beta^b (\Psi_{N,t}, \alpha_t)}. \tag{43}\]
For $\varphi_t \in H^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$ and $\Psi_{N,t} \in H^{(N)} \cap \mathcal{D}(H_N)$, we have

$$Tr_{L^2(\mathbb{R}^3)}[\sqrt{1 - \Delta}(\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle \langle \varphi_t|) \sqrt{1 - \Delta}] \leq (1 + ||\varphi_t||^2_{H^1(\mathbb{R}^3)}) \times \left(\beta^a(\Psi_{N,t}, \varphi_t) + \beta^c(\Psi_{N,t}, \varphi_t)\right) + 2 ||\varphi_t||_{H^1(\mathbb{R}^3)} \sqrt{\beta^a(\Psi_{N,t}, \varphi_t) + \beta^c(\Psi_{N,t}, \varphi_t)}.$$ \hspace{1cm} (44)

**Proof.** The lower bound of (42) is proven by

$$\beta^a(t) = 1 - \langle \Psi_{N,t}, \gamma_{N,t}^{(1,0)} \rangle - \langle \varphi_t, \gamma_{N,t}^{(1,0)} \rangle = Tr_{L^2(\mathbb{R}^3)}(|\varphi_t\rangle \langle \varphi_t| - |\varphi_t\rangle \langle \varphi_t| \gamma_{N,t}^{(1,0)})$$

$$\leq ||p_1||_{op} Tr_{L^2(\mathbb{R}^3)}[\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle \langle \varphi_t|] = Tr_{L^2(\mathbb{R}^3)}[\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle \langle \varphi_t|].$$ \hspace{1cm} (45)

To obtain the upper bound we use that

$$Tr|\gamma - p| \leq 2 ||\gamma - p||_{HS} + Tr(\gamma - p)$$ \hspace{1cm} (46)

is valid for any one-dimensional projector $p$ and non-negative density matrix $\gamma$. The original argument of the proof was first observed by Robert Seiringer, see [25]. We present a version that is found in [2]: Let $(\lambda_n)_{n \in \mathbb{N}}$ be the sequence of eigenvalues of the trace class operator $A := \gamma - p$. Since $p$ is a rank one projection, $A$ has at most one negative eigenvalue. If there is no negative eigenvalue, $Tr[A] = Tr(A)$ and (46) holds. If there is one negative eigenvalue $\lambda_1$, we have $Tr[A] = |\lambda_1| + \sum_{n \geq 2} \lambda_n = 2|\lambda_1| + Tr(A)$. Inequality (46) then follows from $|\lambda_1| \leq ||A||_{op} \leq ||A||_{HS}$. This shows

$$Tr_{L^2(\mathbb{R}^3)}[\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle \langle \varphi_t|] \leq 2 ||\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle \langle \varphi_t||_{HS}$$ \hspace{1cm} (47)

because $Tr_{L^2(\mathbb{R}^3)}[\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle \langle \varphi_t|] = 0$. The upper bound of (42) is obtained by

$$Tr_{L^2(\mathbb{R}^3)}[\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle \langle \varphi_t|] \leq 2(1 - Tr_{L^2(\mathbb{R}^3)}(|\varphi_t\rangle \langle \varphi_t| \gamma_{N,t}^{(1,0)})) \leq 2(1 - Tr_{L^2(\mathbb{R}^3)}(|\varphi_t\rangle \langle \varphi_t| \gamma_{N,t}^{(1,0)})) = 2\beta^a(t).$$ \hspace{1cm} (48)

To prove (42) it is useful to write the kernel of $\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle \langle \alpha_t|$ as

$$\langle \gamma_{N,t}^{(0,1)} - |\alpha_t\rangle \langle \alpha_t|\rangle(k,l) = N^{-1}\langle \Psi_N, a^\dagger(l)a(k)\Psi_N - a^\dagger(l)a(k) \rangle$$

$$= \langle \left(N^{-1/2}a(l) - \alpha_t(l)\right)\Psi_N, \left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_N\rangle$$

$$= \alpha_t(k)\langle \left(N^{-1/2}a(l) - \alpha_t(l)\right)\Psi_N, \left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_N\rangle$$

$$= \alpha_t(l)\langle \left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_N, \left(N^{-1/2}a(l) - \alpha_t(l)\right)\Psi_N\rangle.$$ \hspace{1cm} (49)

By means of Schwarz’s inequality we have

$$||\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle \langle \alpha_t||^2 \leq \left|\left|\left(N^{-1/2}a(l) - \alpha_t(l)\right)\Psi_N\right|\right|^2 + \left|\left|\left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_N\right|\right|^2$$

$$+ |\alpha_t(l)|^2 \left|\left|\left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_N\right|\right|^2$$

$$+ |\alpha_t(k)|^2 \left|\left|\left(N^{-1/2}a(l) - \alpha_t(l)\right)\Psi_N\right|\right|^2.$$ \hspace{1cm} (50)
and
\[
\left| \gamma_{N,t}^{(0,1)} - |\alpha_t\rangle \langle \alpha_t| \right|^2_{HS} = \int d^3k \int d^3l \left| \langle \gamma_{N,t}^{(0,1)} - |\alpha_t\rangle \langle \alpha_t| \rangle(k, l) \right|^2 \\
\leq (\beta^b(t))^2 + 2 ||\alpha_t||^2_{L^2(\mathbb{R}^3)} \beta^b(t). \tag{51}
\]

Similarly, one obtains
\[
\text{Tr}_{L^2(\mathbb{R}^3)}(\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle \langle \alpha_t|) \leq \int d^3k \left| \langle \gamma_{N,t}^{(0,1)} - |\alpha_t\rangle \langle \alpha_t| \rangle(k) \right| \\
\leq \int d^3k \left| \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_N \right|^2_{\mathcal{H}(N)} \\
+ 2 \int d^3k ||\alpha_t(k)|| \left| \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_N \right|_{\mathcal{H}(N)}. \tag{52}
\]

Applying Schwarz’s inequality in the second line leads to
\[
\text{Tr}_{L^2(\mathbb{R}^3)}(\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle \langle \alpha_t|) \leq \beta^b(t) + 2 ||\alpha_t||_{L^2(\mathbb{R}^3)} \sqrt{\beta^b(t)}. \tag{53}
\]

Inequality \((53)\) follows from the monotonicity of the square root and \((10)\). The estimate \((11)\) originates from \((19)\). One starts with the relation
\[
\text{Tr}_{L^2(\mathbb{R}^3)}(\sqrt{1 - \Delta} |\phi_t\rangle \langle \phi_t|) \sqrt{1 - \Delta} = \sup_{||A_1|| \leq 1} \text{Tr}_{L^2(\mathbb{R}^3)}(A_1 \sqrt{1 - \Delta} |\phi_t\rangle \langle \phi_t|) \sqrt{1 - \Delta}), \tag{54}
\]

where the supremum is applied to all compact operators \(A_1\) on \(L^2(\mathbb{R}^3)\) with norm smaller or equal to one. Then, one continues with
\[
\text{Tr}_{L^2(\mathbb{R}^3)}\left( A_1 \sqrt{1 - \Delta} |\phi_t\rangle \langle \phi_t| \sqrt{1 - \Delta} \right) \\
= \langle \Psi_N, p_1^{\phi_t} \sqrt{1 - \Delta} A_1 \sqrt{1 - \Delta} p_1^{\phi_t} \Psi_N \rangle - \langle \phi_t, \sqrt{1 - \Delta} A_1 \sqrt{1 - \Delta} \phi_t \rangle \\
+ \langle \Psi_N, q_1^{\phi_t} \sqrt{1 - \Delta} A_1 \sqrt{1 - \Delta} q_1^{\phi_t} \Psi_N \rangle + \langle \phi_t, \sqrt{1 - \Delta} A_1 \sqrt{1 - \Delta} q_1^{\phi_t} \Psi_N \rangle. \tag{55}
\]

By means of
\[
\left| \sqrt{1 - \Delta} q_1^{\phi_t} \Psi_N \right|^2 = ||q_1^{\phi_t} \Psi_N||^2 + ||\nabla q_1^{\phi_t} \Psi_N||^2 \leq \beta^a(t) + \beta^c(t) \tag{59}
\]
and
\[
\left| \sqrt{1 - \Delta} p_1^{\phi_t} \right|^2_{op} \leq \langle \varphi_t, (1 - \Delta_1) \varphi_t \rangle = ||\varphi_t||^2_{H^1(\mathbb{R}^3)} \tag{60}
\]
we estimate
\[
|\phi_t, \sqrt{1 - \Delta} A_1 \sqrt{1 - \Delta} \phi_t \rangle \langle \Psi_N, p_1^{\phi_t} \Psi_N \rangle - 1 | \leq ||A_1||_{op} ||\varphi_t||^2_{H^1(\mathbb{R}^3)} \beta^a(t), \tag{56}
\]
\[
||A_1||_{op} ||\varphi_t||_{H^1(\mathbb{R}^3)} \sqrt{\beta^a(t) + \beta^c(t)}, \tag{57}
\]
\[
||A_1||_{op} \leq ||A_1||_{op} (\beta^a(t) + \beta^c(t)). \tag{61}
\]

This leads to
\[
\text{Tr}_{L^2(\mathbb{R}^3)}(\sqrt{1 - \Delta} \gamma_{N,t}^{(1,0)} - |\varphi_t\rangle \langle \varphi_t|) \sqrt{1 - \Delta} \leq \left( 1 + ||\varphi_t||^2_{H^1(\mathbb{R}^3)} \right) \left( \beta^a(t) + \beta^c(t) \right) \\
+ 2 ||\varphi_t||_{H^1(\mathbb{R}^3)} \sqrt{\beta^a(t) + \beta^c(t)}. \tag{62}
\]
VIII Estimates on the time derivative

VIII.1 Preliminary estimates

In the following, we control the change of \( \beta \) in time by separately estimating the time derivative of \( \beta^a \) and \( \beta^b \). On the one hand a change in \( \beta^a \) is caused by the fraction of particles which are not in the condensate state \( \varphi_t \). This behavior is analogous to the growth of diseases, where the infection rate of cells (or particles that will leave the condensate) at a given time is proportional to the number of already infected cells. On the other hand there will be a change due to the fact that the particles of the many-body system couple to the quantized radiation field, whereas the condensate wave function is in interaction with the classical field. To control the difference between the quantized and classical field by the functional \( \beta^b \) we will have to split the radiation fields in their positive and negative frequency parts.

\[
\hat{\Phi}_\kappa^+(x) := \int d^3 k \frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}} e^{ikx} a(k), \quad \hat{\Phi}_\kappa^-(x) := \int d^3 k \frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}} e^{-ikx} a^*(k),
\]

\[
\Phi_\kappa^+(x, t) := \int d^3 k \frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}} e^{ikx} \alpha_t(k), \quad \Phi_\kappa^-(x, t) := \int d^3 k \frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}} e^{-ikx} \alpha_t^*(k).
\] (63)

For technical reason it is then helpful to introduce the following (less singular) auxiliary fields

\[
\hat{F}_\kappa^+(x) := \int d^3 k \tilde{\kappa}(k) e^{ikx} a(k), \quad \hat{F}_\kappa^-(x) := \int d^3 k \tilde{\kappa}(k) e^{-ikx} a^*(k),
\]

\[
F_\kappa^+(x, t) := \int d^3 k \tilde{\kappa}(k) e^{ikx} \alpha_t(k), \quad F_\kappa^-(x, t) := \int d^3 k \tilde{\kappa}(k) e^{-ikx} \alpha_t^*(k).
\] (64)

By means of the cutoff function

\[
\hat{\eta}(k) := \frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}} = \frac{(2\pi)^{-3/2}}{\sqrt{2\omega(k)}} \mathbb{1}_{|k| \leq \Lambda}(k)
\] (65)

we are able to express the scalar fields in terms of the auxiliary fields.

**Lemma VIII.1.** Let \( \eta \) be the Fourier transform of \( \hat{\eta} \), then

\[
\hat{\Phi}_\kappa^+(x) = (\eta \ast \hat{F}_\kappa^+) (x), \quad \hat{\Phi}_\kappa^-(x) = (\eta \ast \hat{F}_\kappa^-) (x),
\]

\[
\Phi_\kappa^+(x, t) = (\eta \ast F_\kappa^+) (x, t), \quad \Phi_\kappa^-(x, t) = (\eta \ast F_\kappa^-) (x, t).
\] (66)

**Proof.** The proof is a simple application of convolutions theorem. \( \square \)

In the following, we will integrate the form-factor \( \eta \) of the radiation field and estimate the difference in the auxiliary fields. This requires that the \( L^2 \)-norms of the cutoff functions

\[
||\kappa||_2^2 = \Lambda^3/(6\pi^2) \quad \text{and} \quad ||\eta||_2^2 \leq \Lambda^2/(4\pi^2)
\] (67)

are finite. Subsequently, we use Plancherel’s theorem and estimate the difference in the positive frequency parts of the auxiliary fields by

\[
\int d^3 y \left| \langle N^{-1/2} \hat{F}_\kappa^+(y) - F_\kappa^+(y, t) \rangle \Psi_{N,t} \right|^2 = \int d^3 k \left| \langle N^{-1/2} \hat{F}_\kappa^+(k) - F_\kappa^+(k, t) \rangle \Psi_{N,t} \right|^2
\]

\[
= \int_{|k| \leq \Lambda} d^3 k \left( \langle N^{-1/2} a(k) - \alpha_t(k) \rangle \Psi_{N,t}, \langle N^{-1/2} a(k) - \alpha_t(k) \rangle \Psi_{N,t} \right| \leq \beta^b \langle \Psi_{N,t}, \alpha_t \rangle.
\] (68)

Pulling the pieces together we get
Lemma VIII.2. Let $\alpha_t \in L^2(\mathbb{R}^3)$ and $\Psi_{N,t} \in \mathcal{H}(N) \cap \mathcal{D}(N)$. Then, there exists a generic constant $C$ independent of $N$, $\Lambda$ and $t$ such that
\[
\left\| \left( N^{-1/2} \hat{\Phi}_\kappa(x) - \Phi_\kappa(x) \right) \Psi_{N,t} \right\|^2 \leq CA^2 \left( \beta^b(\Psi_{N,t}, \alpha_t) + N^{-1} \right),
\]
(69)
\[
\left\| \left( N^{-1/2} \hat{\Phi}_\kappa^-(x) - \Phi_\kappa^-(x) \right) \Psi_{N,t} \right\|^2 \leq CA^2 \left( \beta^b(\Psi_{N,t}, \alpha_t) + N^{-1} \right),
\]
(70)
\[
\left\| \left( N^{-1/2} \hat{\Phi}_\kappa^+(x) - \Phi_\kappa^+(x) \right) \Psi_{N,t} \right\|^2 \leq CA^2 \beta^b(\Psi_{N,t}, \alpha_t).
\]
(71)

Proof. From the canonical commutation relations (7), we obtain
\[
\left[ \left( N^{-1/2} \hat{\Phi}_\kappa^+(x) - \Phi_\kappa^+(x) \right), \left( N^{-1/2} \hat{\Phi}_\kappa^-(x) - \Phi_\kappa^-(x) \right) \right] = N^{-1} \left| \eta \right|_2^2
\]
(72)
and estimate
\[
\left\| \left( N^{-1/2} \hat{\Phi}_\kappa^+(x) - \Phi_\kappa^+(x) \right) \Psi_N \right\|^2 \leq 2 \left\| \left( N^{-1/2} \hat{\Phi}_\kappa^+(x) - \Phi_\kappa^+(x) \right) \Psi_N \right\|^2 + 2 \left\| \left( N^{-1/2} \hat{\Phi}_\kappa^+(x) - \Phi_\kappa^+(x) \right) \Psi_N \right\|^2 \leq 4 \left\| \left( N^{-1/2} \hat{\Phi}_\kappa^+(x) - \Phi_\kappa^+(x) \right) \Psi_N \right\|^2 + 2N^{-1} \left| \eta \right|_2^2.
\]
(73)

By means of Lemma VIII.3.1. we have
\[
\left\| \left( N^{-1/2} \hat{\Phi}_\kappa^+(x) - \Phi_\kappa^+(x) \right) \Psi_N \right\|^2 = \langle \int d^3y \eta(x_1-y) \left( N^{-1/2} \hat{F}_\kappa^+(y) - F_\kappa^+(y,t) \right) \Psi_N, \int d^3z \eta(x_1-z) \left( N^{-1/2} \hat{F}_\kappa^+(z) - F_\kappa^+(z,t) \right) \Psi_N \rangle \leq \int d^3y \int d^3z |\eta^*(x_1-z) \left( N^{-1/2} \hat{F}_\kappa^+(z) - F_\kappa^+(z,t) \right) \Psi_N, \eta^*(x_1-y) \left( N^{-1/2} \hat{F}_\kappa^+(z) - F_\kappa^+(z,t) \right) \Psi_N|.
\]
(74)

Cauchy-Schwarz inequality and the estimate $ab \leq 1/2 \left( a^2 + b^2 \right)$ give rise to
\[
\left\| \left( N^{-1/2} \hat{\Phi}_\kappa^+(x) - \Phi_\kappa^+(x) \right) \Psi_N \right\|^2 \leq \int d^3y \int d^3z \left| \eta^*(x_1-z) \left( N^{-1/2} \hat{F}_\kappa^+(y) - F_\kappa^+(y,t) \right) \Psi_N \right| \left| \eta^*(x_1-y) \left( N^{-1/2} \hat{F}_\kappa^+(z) - F_\kappa^+(z,t) \right) \Psi_N \right| \leq \int d^3y \int d^3z \left| \eta^*(x_1-z) \left( N^{-1/2} \hat{F}_\kappa^+(y) - F_\kappa^+(y,t) \right) \Psi_N \right|^2 = \int d^3y \right| \left( N^{-1/2} \hat{F}_\kappa^+(y) - F_\kappa^+(y,t) \right) \Psi_N, \int d^3z |\eta(x_1-z)|^2 \left( N^{-1/2} \hat{F}_\kappa^+(y) - F_\kappa^+(y,t) \right) \Psi_N \rangle = \left| \eta \right|_2^2 \int d^3y \left| \left( N^{-1/2} \hat{F}_\kappa^+(y) - F_\kappa^+(y,t) \right) \Psi_N \right|^2 \leq \left| \eta \right|_2^2 \beta^b(\Psi_{N,t}, \alpha_t).
\]
(75)

In total, we get
\[
\left\| \left( N^{-1/2} \hat{\Phi}_\kappa(x) - \Phi_\kappa(x) \right) \Psi_{N,t} \right\|^2 \leq \left| \eta \right|_2^2 \left( 4 \beta^b(\Psi_{N,t}, \alpha_t) + 2N^{-1} \right) \leq CA^2 \beta^b(\Psi_{N,t}, \alpha_t) + N^{-1}).
\]
(76)
The second and third inequality are shown analogously. Hereby, it is helpful to recall that $[p_1, \hat{F}_\kappa^+(y)] = [p_1, F_\kappa^+(y)] = 0$. □
VIII.2 Estimate on the time derivative of $\beta$

Subsequently, we control the change of $\beta(\Psi_{N,t}, \varphi_t, \alpha_t)$ in time.

**Lemma VIII.3.** Let $(\varphi_t, \alpha_t) \in G$ and $\Psi_{N,t}$ be the unique solution of (8) with initial data $\Psi_{N,0} \in (L^2(\mathbb{R}^{3N}) \otimes F) \cap D(N) \cap D(NH_N)$ such that $||\Psi_{N,0}|| = 1$. Then

$$
\begin{align*}
  d_t \beta^a(t) &= -2\text{Im}\langle \Psi_{N,t}, \left( N^{-1/2} \hat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1, t) \right) q_1^{\varphi_t} \Psi_{N,t} \rangle, \\
  d_t \beta^b(t) &= \text{Im}\langle \Psi_{N,t}, \left( \int d^3k \bar{\eta}(k) (2\pi)^{3/2} F^* |\varphi_t|^2(k) \left( N^{-1/2}a(k) - \alpha_t(k) \right) \right) \Psi_{N,t} \rangle \\
  &
\end{align*}
$$

where $q^{\varphi_t}$ is given by

$$
d_t q_1^{\varphi_t} = -i \left[ H^{eff}_1, q_1^{\varphi_t} \right],$$

where $H^{eff}_1 = -\Delta_1 + \Phi_\kappa(x_1, t)$ is the effective Hamiltonian acting on the first variable. This leads to

$$
\begin{align*}
  d_t \beta^a(t) &= d_t \langle \Psi_{N,t}, q_1^{\varphi_t} \Psi_{N,t} \rangle = i \langle \Psi_{N,t}, \left[ \left( H_N - H^{eff}_1 \right), q_1^{\varphi_t} \right] \Psi_{N,t} \rangle \\
  &= i \langle \Psi_{N,t}, \left[ N^{-1/2} \hat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1, t), q_1^{\varphi_t} \right] \Psi_{N,t} \rangle \\
  &= -2\text{Im}\langle \Psi_{N,t}, \left( N^{-1/2} \hat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1, t) \right) q_1^{\varphi_t} \Psi_{N,t} \rangle.
\end{align*}
$$

We calculate the commutator

$$
i \left[ H_N, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \right] = -i \omega(k)N^{-1/2}a(k) - iN^{-1} \sum_{j=1}^{N} \bar{\eta}(k)e^{-ikx_j}
$$

by means of the canonical commutation relations (7) and continue with

$$
\begin{align*}
  d_t \beta^b(t) &= \int d^3k d\lambda \left\langle N^{-1/2}a(k) - \alpha_t(k) \right| \Psi_{N,t}, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \rangle \\
  &= \int d^3k \left\langle \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t}, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \rangle \\
  &= \int d^3k \left\langle \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t}, i \left[ H_N, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \right] \Psi_{N,t} \rangle \\
  &= \int d^3k \left\langle \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t}, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \rangle \\
  &= 2 \int d^3k \text{Re}\left\langle \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t}, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \rangle \\
  &= 2 \int d^3k \text{Re}\left\langle \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t}, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \rangle
\end{align*}
$$

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\[ = 2 \int d^3 k \text{Re}\{i \omega(k) \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t}, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \}\]
\[+ 2 \int d^3 k \text{Re}\{i \sum_{j=1}^{N} \tilde{\eta}(k) e^{-ikx_j} \Psi_{N,t}, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \}\]
\[= 2 \int d^3 k \text{Re}\{i(2\pi)^{3/2} \tilde{\eta}(k) F T [\varphi_t^2](k) \Psi_{N,t}, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \}\}. \quad (81)\]

So if we use the symmetry of the wave function and \(\text{Re}\{iz\} = -\text{Im}\{z\}\), we get
\[d_t \beta^b(t) = - 2 \int d^3 k \text{Im}\{\omega(k) \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t}, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \}\]
\[+ 2 \int d^3 k \text{Im}\{i \tilde{\eta}(k) e^{-ikx_1} \Psi_{N,t}, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \}\]
\[+ 2 \int d^3 k \text{Im}\{(2\pi)^{3/2} \tilde{\eta}(k) F T [\varphi_t^2](k) \Psi_{N,t}, \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \}\]
\[= 2 \text{Im}\{\Psi_{N,t}, \left( \int d^3 k (2\pi)^{3/2} \tilde{\eta}(k) F T^*[\varphi_t^2](k) \left( N^{-1/2}a(k) - \alpha_t(k) \right) \Psi_{N,t} \}\}. \quad (82)\]

**Lemma VIII.4.** Let \((\varphi_t, \alpha_t) \in \mathcal{G}_t\) and \(\Psi_{N,t}\) be the unique solution of [8] with initial data \(\Psi_{N,0} \in (L_2(\mathbb{R}^3N) \otimes \mathcal{F}) \cap \mathcal{D}(N) \cap \mathcal{D}(NH_N)\) such that \(||\Psi_{N,0}|| = 1\). Then for any \(t \in \mathbb{R}_0^+\) there exists a generic constant \(C\) independent of \(N, \Lambda\) and \(t\) such that
\[|d_t \beta^a(t)| \leq C \Lambda^2 \left( \beta(\Psi_{N,t}, \varphi, \alpha_t) + N^{-1} \right), \quad (83)\]
\[\beta(\Psi_{N,t}, \varphi_t, \alpha_t) \leq C \Lambda^2 t \left( \beta(\Psi_{N,0}, \varphi_0, \alpha_0) + N^{-1} \right). \quad (84)\]

**Proof.** Schwarz’s inequality and \(ab \leq 1/2(a^2 + b^2)\) let us estimate the first line of Lemma VIII.3 by
\[|d_t \beta^a(t)| \leq 2 ||\Psi_{N,t}, \left( N^{-1/2} \tilde{\Phi}_N(x_1) - \Phi_N(x_1, t) \right) q_1^a || \Psi_{N,t}||\]
\[\leq \left| \left( N^{-1/2} \tilde{\Phi}_N(x_1) - \Phi_N(x_1, t) \right) \Psi_{N,t} \right|^2 + ||q_1^a \Psi_{N,t}||^2. \quad (85)\]

By Lemma VIII.2 we obtain
\[|d_t \beta^a(t)| \leq C \Lambda^2 \left( \beta(t) + N^{-1} \right). \quad (86)\]

In order to estimate \(d_t \beta^b(t)\) we notice that
\[\int d^3 k \tilde{\eta}(k) e^{ikx_1} \left( N^{-1/2}a(k) - \alpha(k, t) \right) = \int d^3 y \eta(x_1 - y) \left( N^{-1/2} \tilde{F}_N^+(y) - F_N^+(y, t) \right) \]
\[= \left( N^{-1/2} \tilde{\Phi}_N^+(x_1) - \Phi_N^+(x_1, t) \right) \quad (87)\]

and
\[\int d^3 k \tilde{\eta}(k)(2\pi)^{3/2} F T [\varphi_t^2](k) \left( N^{-1/2}a(k) - \alpha_t(k) \right) \]
\[= \int d^3 y \left( \eta * |\varphi_t|^2 \right)(y, t) \left( N^{-1/2} \tilde{F}_N^+(y) - F_N^+(y, t) \right). \quad (88)\]
follow from the convolution theorem. This gives
\[ d_t \beta^b(t) = -2 \text{Im} \int d^3 y \langle \Psi_{N,t}, \eta(x_1 - y) \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) \Psi_{N,t} \rangle + 2 \text{Im} \int d^3 y \langle \Psi_{N,t}, (\eta * |\varphi_t|^2)(y,t) \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) \Psi_{N,t} \rangle. \] 

We see that not only present gauge boson fluctuations around the coherent state lead to a growth in $\beta^b(t)$ but an additional change appears, because the second quantized radiation field couples to the mean particle density of the many-body system while the source of the classical field is given by the density of the condensate wave function. In order to estimate the difference between the densities by the functional $\beta^s(t)$ we insert the identity $1 = p_1^{\varphi_i} + q_1^{\varphi_i}$.

\[ d_t \beta^b(t) = -2 \text{Im} \int d^3 y \langle \Psi_{N,t}, p_1^{\varphi_i} \eta(x_1 - y)p_1^{\varphi_i} \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) \Psi_{N,t} \rangle + 2 \text{Im} \int d^3 y \langle \Psi_{N,t}, (\eta * |\varphi_t|^2)(y,t) \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) \Psi_{N,t} \rangle - 2 \text{Im} \int d^3 y \langle \Psi_{N,t}, q_1^{\varphi_i} \eta(x_1 - y)q_1^{\varphi_i} \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) \Psi_{N,t} \rangle - 2 \text{Im} \int d^3 y \langle \Psi_{N,t}, \eta(x_1 - y)q_1^{\varphi_i} \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) \Psi_{N,t} \rangle. \] 

The first two lines are the most important. They become small, because the mean particle density of the many-body system is approximately given by the density of the condensate wave function. From $\eta(-x) = \eta(x)$ we conclude

\[ p_1^{\varphi_i} \eta(x_1 - y)p_1^{\varphi_i} = p_1^{\varphi_i} \int d^3 z \eta(-z) |\varphi_t|^2(z,t) = p_1^{\varphi_i} (\eta * |\varphi_t|^2)(y,t) \] 

and continue with

\[ d_t \beta^b(t) = -2 \text{Im} \int d^3 y \langle \Psi_{N,t}, (p_1^{\varphi_i} - 1)(\eta * |\varphi_t|^2)(y,t) \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) \Psi_{N,t} \rangle - 2 \text{Im} \langle \Psi_{N,t}, q_1 \int d^3 y \eta(x_1 - y) \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) p_1^{\varphi_i} \Psi_{N,t} \rangle - 2 \text{Im} \langle \Psi_{N,t}, \int d^3 y \eta(x_1 - y) \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) q_1^{\varphi_i} \Psi_{N,t} \rangle = 2 \text{Im} \int d^3 y \langle \Psi_{N,t}, q_1^{\varphi_i} (\eta * |\varphi_t|^2)(y,t) \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) \Psi_{N,t} \rangle - 2 \text{Im} \langle \Psi_{N,t}, q_1^{\varphi_i} \left( N^{-1/2} \hat{F}^+_{\kappa}(x_1) - \Phi^+_{\kappa}(x_1,t) \right) p_1^{\varphi_i} \Psi_{N,t} \rangle - 2 \text{Im} \langle \Psi_{N,t}, \left( N^{-1/2} \hat{F}^+_{\kappa}(x_1) - \Phi^+_{\kappa}(x_1,t) \right) q_1^{\varphi_i} \Psi_{N,t} \rangle. \] 

In the following, we estimate each line separately.

\[ |(92)| \leq 2 \left| \int d^3 y \langle \eta * |\varphi_t|^2(y,t)q_1^{\varphi_i} \Psi_{N,t} \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) \Psi_{N,t} \rangle \right| \leq \int d^3 y \langle q_1^{\varphi_i} \Psi_{N,t} \left( \eta * |\varphi_t|^2(y,t) \right)^2 q_1^{\varphi_i} \Psi_N \rangle + \int d^3 y \left| \left( N^{-1/2} \hat{F}^+_{\kappa}(y) - F^+_{\kappa}(y,t) \right) \Psi_{N,t} \right|^2 \leq || \eta * |\varphi_t|^2 ||^2 \left\langle \Psi_{N,t}, q_1^{\varphi_i} \Psi_{N,t} \right\rangle + \beta^s(t) \leq CA^2 \beta(t). \]
Here we have used that
\[ \|\eta \|_2^2 \leq \|\eta \|_0^2 \|\varphi_t \|_2^2 \leq \|\eta \|_2^2 \|\varphi_t \|_2^2 = C \Lambda \] (96)
holds due to Young’s inequality and $\text{(67)}$. Lemma \textbf{VIII.2} leads to
\[ |\text{13}| \leq 2 |\left\langle q_1^2 \Psi_N, \left( N^{-1/2} \Phi_\kappa^+ (x_1) - \Phi_\kappa^+ (x_1, t) \right) p_1^2 \Psi_N \right\rangle| \]
\[ \leq \left\| \left( N^{-1/2} \Phi_\kappa^+ (x_1) - \Phi_\kappa^+ (x_1, t) \right) p_1^2 \Psi_N \right\|^2 + |q_1^2 \Psi_N|^2 \leq C \Lambda^2 \beta(t) \] (97)
and
\[ |\text{14}| \leq 2 |\left\langle \left( N^{-1/2} \Phi_\kappa^- (x_1) - \Phi_\kappa^- (x_1, t) \right) \Psi_N, q_1^2 \Psi_N \right\rangle| \]
\[ \leq \left\| \left( N^{-1/2} \Phi_\kappa^- (x_1) - \Phi_\kappa^- (x_1, t) \right) \Psi_N \right\|^2 + |q_1^2 \Psi_N|^2 \]
\[ \leq C \Lambda^2 \left( \beta(t) + N^{-1} \right). \] (98)
In total we have
\[ |d_t \beta^b (t)| \leq C \Lambda^2 \left( \beta(t) + N^{-1} \right). \] (99)
Now we can put the terms together to get
\[ d_t \beta (t) \leq |d_t \beta^a (t)| + |d_t \beta^b (t)| \leq C \Lambda^2 \left( \beta(t) + N^{-1} \right). \] (100)
Applying Gronwall’s lemma proves
\[ \beta(t) \leq \frac{e^{C \Lambda^2 t} \left( \beta(0) + N^{-1} \right)}{\Lambda^2}. \] (101)

\section{Control of the kinetic energy}

In order to prove the convergence of the one-particle reduced density matrix of the charges in Sobolev norm it is necessary to control the kinetic energy of the particles which are not in the condensate (see Section \textbf{VII}). To this end we include $\beta^c(\Psi_{N,t}, \varphi_t) := \| \nabla_1 q_1^2 \Psi_{N,t} \|^2$ in the definition of the functional and perform a Gronwall estimate for the redefined functional $\beta_2(\Psi_{N,t}, \varphi_t, \alpha_t)$.

\textbf{Lemma VIII.5.} Let $(\varphi_t, \alpha_t) \in G_2$ and $\Psi_{N,t}$ be the unique solution of (8) with initial data $\Psi_{N,0} \in (L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}) \cap D(\Lambda) \cap D(NH_N) \cap D(H_N^2)$ such that $\|\Psi_{N,0}\| = 1$. Then
\[ d_t \beta^c(\Psi_{N,t}, \varphi_t) = 2 \text{Im} \langle \left( N^{-1/2} \Phi_\kappa (x_1) - \Phi_\kappa (x_1, t) \right) \Psi_{N,t}, (-\Delta) q_1^2 \Psi_{N,t} \rangle \]
\[ - 2 \text{Im} \langle \left( N^{-1/2} \Phi_\kappa (x_1) - \Phi_\kappa (x_1, t) \right) p_1 \Psi_{N,t}, (-\Delta) q_1^2 \Psi_{N,t} \rangle \]
\[ - 2 \text{Im} \langle \left( N^{-1/2} \Phi_\kappa (x_1) q_1^2 \Psi_{N,t}, (-\Delta) q_1^2 \Psi_{N,t} \right) \rangle. \] (102)

\textbf{Proof.} We infer $\Psi_{N,t} \in (L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}) \cap D(\Lambda) \cap D(NH_N) \cap D(H_N^2)$ for all $t \in \mathbb{R}^+_0$ from $\Psi_{N,0} \in (L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}) \cap D(\Lambda) \cap D(NH_N) \cap D(H_N^2)$ by Stone’s Theorem and the invariance.
of $\mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N)$ during the time evolution (see [13] Appendix 2.11)). This ensures that the following expressions are well defined. The derivative of $\beta^c(t)$ is determined by

$$d_t \beta^c(t) = i\left\langle q^\xi_1 H_N \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle - i\left\langle q^\xi_1 \Psi_{N,t}, (-\Delta_1) q^\xi_1 H_N \Psi_{N,t} \right\rangle + i\left\langle H^{eff}_1, q^\xi_1 \right\rangle \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} - i\left\langle q^\xi_1 \Psi_{N,t}, (-\Delta_1) \left[ H^{eff}_1, q^\xi_1 \right] \Psi_{N,t} \right\rangle$$

$$= i\left\langle q^\xi_1 H_N \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle - i\left\langle (-\Delta_1) q^\xi_1 \Psi_{N,t}, q^\xi_1 H_N \Psi_{N,t} \right\rangle + i\left\langle H^{eff}_1, q^\xi_1 \right\rangle \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} - i\left\langle (-\Delta_1) q^\xi_1 \Psi_{N,t}, \left[ H^{eff}_1, q^\xi_1 \right] \Psi_{N,t} \right\rangle$$

$$= -2\text{Im}\left\langle q^\xi_1 H_N \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle - 2\text{Im}\left\langle H^{eff}_1, q^\xi_1 \right\rangle \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle.$$  \hspace{1cm} (103)

Since $\left\langle q^\xi_1 \left( -\Delta + N^{-1/2} \phi(x) \right) \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle$ and $\left\langle q^\xi_1 H_f \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle$ are real numbers for $i \in \{2, 3, \ldots, N\}$ this becomes

$$d_t \beta^c(t) = -2\text{Im}\left\langle q^\xi_1 \left( -\Delta + N^{-1/2} \phi(x) \right) \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle$$

$$+ 2\text{Im}\left\langle q^\xi_1 H^{eff}_1 \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle - 2\text{Im}\left\langle H^{eff}_1 q^\xi_1 \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle$$

$$= -2\text{Im}\left\langle q^\xi_1 \left( N^{-1/2} \phi(x) - \phi(x,t) \right) \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle - 2\text{Im}\Phi, N,t \left( q^\xi_1 \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right)$$

$$- 2\text{Im}\left\| (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\|^2$$

$$= -2\text{Im}\left\langle q^\xi_1 \left( N^{-1/2} \phi(x) - \phi(x,t) \right) \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle$$

$$- 2\text{Im}\left\langle \Phi, N,t \left( q^\xi_1 \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right) \right\rangle.$$ \hspace{1cm} (104)

The identity $q^\xi_1 \mathcal{O} = \mathcal{O} q^\xi_1 + \mathcal{O} q^\xi_1 - p^\xi_1 \mathcal{O}$ (for any operator $\mathcal{O}$) and

$$-\left\langle \Phi, N,t \left( q^\xi_1 \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right) \right\rangle = \left\langle \left( N^{-1/2} \phi(x) - \phi(x,t) \right) q^\xi_1 \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle$$

$$- \left\langle N^{-1/2} \phi(x) q^\xi_1 \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle$$ \hspace{1cm} (105)

lead to

$$d_t \beta^c(t) = 2\text{Im}\left\langle p^\xi_1 \left( N^{-1/2} \phi(x) - \phi(x,t) \right) \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle$$

$$- 2\text{Im}\left\langle \left( N^{-1/2} \phi(x) - \phi(x,t) \right) p^\xi_1 \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle$$

$$- 2\text{Im}\left\langle N^{-1/2} \phi(x) q^\xi_1 \Psi_{N,t}, (-\Delta_1) q^\xi_1 \Psi_{N,t} \right\rangle.$$ \hspace{1cm} (106)

**Lemma VIII.6.** Let $(\varphi_1, \alpha_2) \in G_2$ and $\Psi_{N,t}$ be the unique solution of \([5]\) with initial data $\Psi_{N,0} \in \left( L^2(\mathbb{R}^3)^N \right) \cap \mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N) \cap \mathcal{D}(H_N^2)$ such that $||\Psi_{N,0}|| = 1$. Then, there exists a positive monotone increasing function $C(s)$ of the norms $||\alpha_s||_{L^2(\mathbb{R}^3)}$ and $||\varphi_s||_{H^1(\mathbb{R}^3)}$ such that

$$|d_t \beta_2(\Psi_{N,t}, \varphi_1, \alpha_t)| \leq \Lambda^4 C(t) \left( \beta_2(\Psi_{N,t}, \varphi_1, \alpha_t) + N^{-1} \right),$$

$$\beta_2(\Psi_{N,t}, \varphi_1, \alpha_t) \leq e^{\Lambda^4 \int_0^t C(s) ds} \left( \beta_2(\Psi_{N,0}, \varphi_0, \alpha_0) + N^{-1} \right)$$ \hspace{1cm} (109)

hold for any $t \in \mathbb{R}^+_0$. 18
Proof. In order to estimate \(d_t \beta^c(t)\) by \(\beta\) and \(\|\nabla_1 q^{\pi^1}_N \|\) we will integrate by parts and apply Schwarz’s inequality. The gradient will hereby occasionally act on the radiation fields, which will give rise to the vector fields

\[
(\nabla \tilde{\Phi}_\kappa)(x) = \int d^3 k \tilde{\eta}(k) k i \left( e^{-ikx} a(k) - e^{-ikx} a^*(k) \right),
\]

\[
(\nabla \Phi_\kappa)(x, t) = \int d^3 k \tilde{\eta}(k) k i \left( e^{ikx} a_t(k) - e^{ikx} a^*_t(k) \right). \tag{110}
\]

We define the vector field \(\tilde{\Theta}(k) := \tilde{\eta}(k) k\) and its Fourier transform \(\Theta\) with \(\sum_{i=1}^3 \|\Theta^i\|_2^2 \leq \Lambda^4/(16\pi^2)\). This allows us to obtain the relation

\[
(\nabla \tilde{\Phi}_\kappa^+(x) = i \left( \Theta * \hat{F}_\kappa^+ \right)(x), \quad (\nabla \Phi_\kappa^+(x, t) = i \left( \Theta * F_\kappa^+ \right)(x) \tag{111}
\]

between the positive frequency part of the vector fields and the auxiliary fields \(\tilde{\Theta}\). In analogy to Lemma \(\text{VIII.2}\) one proves the estimates

\[
\left| \left( N^{-1/2} (\nabla \tilde{\Phi}_\kappa)(x_1) - (\nabla \Phi_\kappa)(x_1, t) \right) p_1 \Psi_N \right|^2 \leq \Lambda^4 \left( \beta^h(t) + N^{-1} \right),
\]

\[
\left| \left( N^{-1/2} (\nabla \tilde{\Phi}_\kappa)(x_1) - (\nabla \Phi_\kappa)(x_1, t) \right) q_1 \Psi_N \right|^2 \leq \Lambda^4 \left( \beta^h(t) + N^{-1} \right),
\]

\[
\left| \left( N^{-1/2} \Phi_\kappa(x_1) - \Phi_\kappa(x_1, t) \right) \nabla_1 p_1 \Psi_N \right|^2 \leq \Lambda^2 \|\nabla \varphi\|_2^2 \left( \beta^h(t) + N^{-1} \right). \tag{112}
\]

The first term of \(d_t \beta^c(t)\) is estimated by

\[
\left| \left( N^{-1/2} \Phi_\kappa(x_1) - \Phi_\kappa(x_1, t) \right) \nabla_1 p_1 \Psi_N \right|^2 \leq \Lambda^2 \|\nabla \varphi\|_2^2 \left( \beta^h(t) + N^{-1} \right) + \|\nabla_1 q_1 \Psi_N\|^2 \tag{113}
\]

Lemma \(\text{VIII.2}\) gives rise to

\[
\left| \left( N^{-1/2} \Phi_\kappa(x_1) - \Phi_\kappa(x_1, t) \right) \nabla_1 p_1 \Psi_N \right|^2 \leq \Lambda^2 C(\|\varphi\|_{H^1}) \left( \beta^2(t) + N^{-1} \right) \tag{114}
\]

Likewise, we estimate

\[
\left| \left( N^{-1/2} \Phi_\kappa(x_1) - \Phi_\kappa(x_1, t) \right) \nabla_1 p_1 \Psi_N \right|^2 \leq 2 \left| \left( N^{-1/2} \Phi_\kappa(x_1) - \Phi_\kappa(x_1, t) \right) \nabla_1 q_1 \Psi_N \right|^2 \tag{115}
\]

Due to triangular inequality, \((a + b)^2 \leq 2(a^2 + b^2)\) and \(\|\cdot\|\) this becomes

\[
\left| \left( N^{-1/2} \Phi_\kappa(x_1) - \Phi_\kappa(x_1, t) \right) \nabla_1 p_1 \Psi_N \right|^2 \leq \Lambda^4 C(\|\varphi\|_{H^1}) \left( \beta^2(t) + N^{-1} \right) \tag{116}
\]
Next, we consider line
\[
(108) = -2\text{Im}\langle \nabla_1 N^{-1/2} \hat{\Phi}_N(x_1) q_1^{\nu^1} \Psi_{N,t}, \nabla_1 q_1^{\nu^1} \Psi_{N,t} \rangle \\
= -2\text{Im}\langle N^{-1/2} (\nabla_1 \hat{\Phi}_N)(x_1) q_1^{\nu^1} \Psi_{N,t}, \nabla_1 q_1^{\nu^1} \Psi_{N,t} \rangle \\
- 2\text{Im}\langle N^{-1/2} \hat{\Phi}_N(x_1) \nabla_1 q_1^{\nu^1} \Psi_{N,t}, \nabla_1 q_1^{\nu^1} \Psi_{N,t} \rangle. 
\]

The scalar product in the last line is easily shown to be real. This yields
\[
(108) = -2\text{Im}\langle N^{-1/2} (\nabla_1 \hat{\Phi}_N)(x_1) - (\nabla_1 \Phi_N)(x_1, t) \rangle q_1^{\nu^1} \Psi_{N,t}, \nabla_1 q_1^{\nu^1} \Psi_{N,t} \rangle \\
- 2\text{Im}\langle (\nabla_1 \Phi_N)(x_1, t) q_1^{\nu^1} \Psi_{N,t}, \nabla_1 q_1^{\nu^1} \Psi_{N,t} \rangle. 
\]

and allows us to estimate
\[
|\langle 108 \rangle | \leq 2|\langle N^{-1/2} (\nabla_1 \hat{\Phi}_N)(x_1) - (\nabla_1 \Phi_N)(x_1, t) \rangle q_1^{\nu^1} \Psi_{N,t}, \nabla_1 q_1^{\nu^1} \Psi_{N,t} \rangle | \\
+ 2|\langle (\nabla_1 \Phi_N)(x_1, t) q_1^{\nu^1} \Psi_{N,t}, \nabla_1 q_1^{\nu^1} \Psi_{N,t} \rangle | \\
\leq \left|\left| N^{-1/2} (\nabla_1 \hat{\Phi}_N)(x_1) - (\nabla_1 \Phi_N)(x_1, t) \right| q_1^{\nu^1} \Psi_{N,t} \right| \left|\nabla_1 q_1^{\nu^1} \Psi_{N,t} \right| ^2 \\
+ 2 \left|\nabla_1 q_1^{\nu^1} \Psi_{N,t} \right| ^2 \leq C\Lambda^4 \left( \beta^b (t) + N^{-1} \right) + C\Lambda^4 \| \alpha_t \| ^2 \beta^a (t) + 2\beta^c (t) \\
\leq \Lambda^4 C(\| \alpha_t \| _2) \left( \beta_2 (t) + N^{-1} \right). 
\]

Here, we used (112) and the fact that
\[
\left|\langle (\nabla \Phi_N) (\cdot, t) \rangle \right| _\infty \leq C\Lambda^2 \| \alpha_t \| _2 
\]
holds because of Schwarz’s inequality. In total, we have
\[
|d_t \beta^c (t) | \leq \Lambda^4 C(\| \varphi_t \| _{H^1}, \| \alpha_t \|) \left( \beta_2 + N^{-1} \right). 
\]

With Lemma VIII.4 this implies
\[
|d_t \beta_2 [\Psi_{N,t}, \varphi_t, \alpha_t] | \leq \Lambda^4 C(\| \varphi_t \| _{H^1}, \| \alpha_t \|) \left( \beta_2 [\Psi_{N,t}, \varphi_t, \alpha_t] + N^{-1} \right) 
\]

Using the shorthand notation \( C(t) := C(\| \varphi_t \| _{H^1}, \| \alpha_t \|) \) we obtain
\[
\beta_2 [\Psi_{N,t}, \varphi_t, \alpha_t] \leq e^{\Lambda^4 \int_0^t C(s) da} \left( \beta_2 [\Psi_{N,0}, \varphi_0, \alpha_0] + N^{-1} \right) 
\]

by means of Gronwall’s lemma.

\[ \square \]

**IX Initial states**

Subsequently, we are concerned with the initial states of Theorem III.3.

**Lemma IX.1.** Let \( \Psi_{N,0} \in \left( L^2 (\mathbb{R}^3) \otimes \mathcal{F} \right) \cap \mathcal{D} (N) \) with \( \| \Psi_{N,0} \| = 1 \) and \((\varphi_0, \alpha_0) \in L^2 (\mathbb{R}^3) \oplus L^2 (\mathbb{R}^3) \) with \( \| \varphi_0 \| = 1 \). Then
\[
\beta^a (\Psi_{N,0}, \varphi_0) \leq Tr_{L^2 (\mathbb{R}^3)} \gamma_{N,0}^{(1,0)} - |\varphi_0\rangle \langle \varphi_0| = a_N, \\
\beta^b (\Psi_{N,0}, \alpha_0) = N^{-1} \langle \Psi_{N,0} | W^{-1} (\sqrt{N} \alpha_0) , N W^{-1} (\sqrt{N} \alpha_0) \Psi_{N,0} \rangle = b_N. 
\]
Proof. The first inequality is a consequence of Lemma VII.1. Before we prove the second relation we justify \[13\]. Therefore, is useful to note that the Weyl operator \((f \in L^2(\mathbb{R}^3))\)

\[
W(f) = \exp \left( \int d^3k \, f(k) a^*(k) - f^*(k) a(k) \right) \tag{125}
\]
is unitary

\[
W^{-1}(f) = W^*(f) = W(-f) \tag{126}
\]
and satisfies

\[
W^*(f) a(k) W(f) = a(k) + f(k), \quad W^*(f) a^*(k) W(f) = a^*(k) + f^*(k). \tag{127}
\]

This leads to

\[
\beta^b(\Psi_{N,t}, \alpha_t) = \int d^3k \left| \left( N^{-1/2} a(k) - \alpha_t(k) \right) \Psi_{N,t} \right|^2 \\
= \int d^3k \left| W^*(\sqrt{N} \alpha_t) \left( N^{-1/2} a(k) - \alpha_t(k) \right) W(\sqrt{N} \alpha_t) W^*(\sqrt{N} \alpha_t) \Psi_{N,t} \right|^2 \\
= \int d^3k \left| N^{-1/2} a(k) W^*(\sqrt{N} \alpha_t) \Psi_{N,t} \right|^2 \\
= N^{-1} \langle W^*(\sqrt{N} \alpha_t) e^{-iH_N t} \Psi_{N,0}, N W^*(\sqrt{N} \alpha_t) e^{-iH_N t} \Psi_{N,0} \rangle. \tag{128}
\]

Let

\[
\mathcal{U}_N(t; 0) := W^*(\sqrt{N} \alpha_t) e^{-iH_N t} W(\sqrt{N} \alpha_0) \tag{129}
\]
denote the fluctuation dynamics then

\[
\beta^b(\Psi_{N,t}, \alpha_t) = N^{-1} \langle \mathcal{U}_N(t; 0) W^{-1}(\sqrt{N} \alpha_0) \Psi_{N,0}, \mathcal{U}_N(t; 0) W^{-1}(\sqrt{N} \alpha_0) \Psi_{N,0} \rangle \tag{130}
\]
follows from the unitarity of the Weyl operator. In particular, we have

\[
\beta^b(\Psi_{N,0}, \alpha_0) = N^{-1} \langle W^{-1}(\sqrt{N} \alpha_0) \Psi_{N,0}, N W^{-1}(\sqrt{N} \alpha_0) \Psi_{N,0} \rangle = b_N. \tag{131}
\]

\[13\] More information is given for instance in \[25\], p.9.

Lemma IX.2. Let \((\varphi_0, \alpha_0) \in H^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \) with \(||\varphi_0|| = 1\) and \(\Psi_{N,0} = \varphi_0 \otimes W(\sqrt{N} \alpha_0) \Omega\). Then

\[
a_N = Tr_{L^2(\mathbb{R}^3)} \gamma^{(1,0)}_{N,0} \varphi_0 \varphi_0 = 0, \tag{132}
\]

\[
b_N = N^{-1} \langle W^{-1}(\sqrt{N} \alpha_0) \Psi_{N,0}, N W^{-1}(\sqrt{N} \alpha_0) \Psi_{N,0} \rangle = 0 \text{ and} \tag{133}
\]

\[
\Psi_{N,0} \in (L^2(\mathbb{R}^3)^{\otimes N}) \cap \mathcal{D}(N) \cap \mathcal{D}(NH_N). \tag{134}
\]

Let \((\varphi_0, \alpha_0) \in H^4(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \) with \(||\varphi_0|| = 1\) then

\[
e_N = ||\nabla_{1\mathbb{R}^3} \varphi_0 \Psi_{N,0}||^2 = 0, \tag{135}
\]

\[
\Psi_{N,0} \in (L^2(\mathbb{R}^3)^{\otimes N}) \cap \mathcal{D}(N) \cap \mathcal{D}(NH_N) \cap \mathcal{D}(H_N^2). \tag{136}
\]
Proof. From the definition of the one-particle reduced density matrix and (131) we directly obtain the relations (132) and (133). Equation (135) holds because \( \Psi_{N,0} \) is in the kernel of the projector \( q^0_1 \). In order to show (134) we point out that

\[
\Psi_{N,0}^{(n)}(X_N, K_n) = \prod_{i=1}^{N} \varphi_0(x_i) e^{-N||\alpha_0||^2/(2m)} - \frac{1}{2} \prod_{j=1}^{n} (N)^{1/2} \alpha_0(k_j)
\]  

(137)

follows from the definition of the the Weyl operators [25, p.8]. A direct calculation gives

\[
\sum_{n=1}^{\infty} n^2 \left| \Psi_{N,0}^{(n)} \right|^2 = N \|\alpha_0\|^2 + N^2 \|\alpha_0\|^4.
\]  

(138)

Hence, \( \Psi_{N,0}^{(n)} \in D(N) \) (see (20)). Moreover, we have \( \Psi_{N,0} \in D(\sum_{i=1}^{N} - \Delta_i) \) because \( \varphi_0 \in H^2(\mathbb{R}^3) \). A straightforward estimate leads to

\[
\sum_{n=1}^{\infty} \int d^3 x d^3 k \sum_{j=1}^{n} w(k_j)^2 \left| \Psi_{N,0}^{(n)}(X_N, K_n) \right|^2 \leq C(N, \|\alpha_0\|_{L^2(\mathbb{R}^3)}).
\]  

(139)

From (12) we then conclude \( \Psi_{N,0} \in D(H_f) \) and \( \Psi_{N,0}^{(n)} \in D(H_N) = D(\sum_{i=1}^{N} - \Delta_i) \cap D(H_f) \). Similarly, one derives

\[
\sum_{n=1}^{N} n^2 \left| (H_N \Psi_{N,0})^{(n)} \right|^2 \leq C \sum_{n=1}^{\infty} n^2 \left( \left| \sum_{j=1}^{N} \Delta_j \Psi_{N,0}^{(n)} \right|^2 + \left| \sum_{j=1}^{N} N^{-1/2} (\tilde{\Phi}_n(x_j) \Psi_{N,0})^{(n)} \right|^2 \right)
\]

\[
+ C \sum_{n=1}^{\infty} n^2 \left| (H_f \Psi_{N,0})^{(n)} \right|^2 \leq C(N, \Lambda, \|\varphi_0\|_{H^2(\mathbb{R}^3)}, \|\alpha_0\|_{L^2(\mathbb{R}^3)}).
\]  

(140)

and concludes \( \Psi_{N,0} \in D(NH_N) = \{ \Psi_N \in D(H_N) : H_N \Psi_N \in D(N) \} \). In order to show (133) we would like to note that \( (\varphi_0, \alpha_0) \in (H^4(\mathbb{R}^3), L^2_2(\mathbb{R}^3)), \|\cdot\|^2 \tilde{n} \in L^2(\mathbb{R}^3) \) and \( \tilde{n} \in L^2(\mathbb{R}^3) \) imply \( H_N \Psi_{N,0} \in D(\sum_{i=1}^{N} - \Delta_i) \). By means of the estimate

\[
\sum_{n=1}^{\infty} d^{3N} d^3 k \sum_{j=1}^{n} w(k_j)^2 \left| (H_N \Psi_{N,0})^{(n)}(X_N, K_n) \right|^2 \leq C(N, \Lambda, \|\varphi_0\|_{H^2(\mathbb{R}^3)}, \|\alpha_0\|_{L^2(\mathbb{R}^3)})
\]  

(141)

one obtains \( H_N \Psi_{N,0} \in D(H_f) \). In total, we have \( H_N \Psi_{N,0} \in D(H_N) \) and \( \Psi_{N,0} \in D(H^2_N) \). \( \square \)

X Proof of Theorem III.3

In order to finish the proof of Theorem III.3 we remark that Lemma X.1 leads to

\[
\beta(\Psi_{N,0}, \varphi_0, \alpha_0) \leq a_N + b_N,
\]

\[
\beta_2(\Psi_{N,0}, \varphi_0, \alpha_0) \leq a_N + b_N + c_N.
\]  

(142)

We then choose for a given time \( t \in \mathbb{R}_0^+ \) the number \( N \) of charged particles large enough such that the values of \( \beta(\Psi_{N,t}, \varphi_t, \alpha_t) \) in (84) and \( \beta_2(\Psi_{N,t}, \varphi_t, \alpha_t) \) in (109) are smaller than one and derive Theorem III.3 by means of Lemma VII.1.
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