1. Introduction

In this paper we will be concerned with Einstein manifolds with controlled geometry at infinity. In Riemannian signature, the prime example is the hyperbolic space, namely, the ball $\mathbb{B}^{n+1}$ endowed with its Riemannian metric $g_{\mathbb{H}^{n+1}}$ of negative constant sectional curvature. The point here is that, if one rescales the hyperbolic metric as $\tilde{g}_{\mathbb{H}^{n+1}} := x^2 g_{\mathbb{H}^{n+1}}$, this metric can be continuously extended to the closed
ball $\mathbb{B}^{n+1}$ and its pullback to $\partial\mathbb{B}^{n+1}$ is conformal to the canonical metric $g_{\mathbb{R}^n}$. One then says that the conformal class $[g_{\mathbb{R}^n}]$ is the conformal infinity of $g_{\mathbb{R}^{n+1}}$. Here $x$ a boundary-defining function of the ball, that is, a positive function in $\mathbb{B}^{n+1}$ that is smooth up to the boundary and vanishes to first order on $\partial\mathbb{B}^{n+1}$.

An important theorem of Graham and Lee [19] asserts that we can “tickle” the hyperbolic space at infinity and still obtain a Riemannian Einstein metric on the ball. More precisely, their result says that, if $\tilde{g}$ is a metric on the sphere that is close enough to the canonical one in $C^{2,\alpha}(S^n)$, then there is a Riemannian metric $g$ in $\mathbb{B}^{n+1}$ that satisfies the Einstein equation

$$\text{Ric}(g) = -ng$$

and whose conformal infinity is $[\tilde{g}]$ (notice that the normalization of the constant in the right-hand side of the equation is inessential). To put it differently, the rescaled metric $\bar{g} := x^2 g$ extends continuously to the closure $\overline{\mathbb{B}}^{n+1}$ and satisfies the boundary condition

$$(j_{\partial\mathbb{B}^n})_* \bar{g} = \tilde{g},$$

possibly up to multiplication by a positive scalar function. Here $j_{\partial\mathbb{B}^n}$ is the inclusion map $\partial\mathbb{B}^n \to \mathbb{B}^n$, so $(j_{\partial\mathbb{B}^n})_* g$ is the pullback of $\tilde{g}$ to the boundary. Moreover, $\tilde{g}$ and $g_{\mathbb{B}^{n+1}}$ are close in $C^{2,\alpha}(\mathbb{B}^n)$. Significant refinements of this result can be found in [7, 25, 3] and references therein.

Our objective in this paper is to prove a Lorentzian analog of the Graham–Lee theorem, thereby showing the existence of Einstein metrics of Lorentzian signature with prescribed conformal infinity. In this context it is clear that the role of the hyperbolic space must be played by its Lorentzian counterpart, namely, the anti-de Sitter space $\text{AdS}_{n+1}$. This is simply the solid cylinder $\mathbb{R} \times \mathbb{B}^n$ endowed with the Lorentzian metric of negative constant sectional curvature. Its conformal infinity is given by $[g_{\mathbb{R} \times S^{n-1}}]$, with

$$g_{\mathbb{R} \times S^{n-1}} := -dt^2 + g_{\mathbb{R}^{n-1}}$$

being the canonical Lorentzian metric of the cylinder. This means that, if $x$ is a boundary defining function, the rescaled metric $\tilde{g}_{\text{AdS}} := x^2 g_{\text{AdS}}$ can be continuously extended to the closure $\overline{\mathbb{R} \times \mathbb{B}^n}$ and its pullback to the boundary is given by $g_{\mathbb{R} \times S^{n-1}}$ up to multiplication by a positive function.

Interestingly, the motivation to consider the problem of constructing Lorentzian Einstein manifolds with prescribed conformal infinity is not only academic. In fact, as is discussed in [2, 11], the problem appears in the context of the AdS/CFT correspondence in string theory [26]. This is a conjectural relation which posits that a gravitational field on a Lorentzian $(n+1)$-manifold endowed with an asymptotically anti-de Sitter metric can be recovered from a conformal gauge field defined on the conformal boundary of the manifold. The gravitational field is typically modeled as a Lorentzian metric satisfying the Einstein equation and the conformal gauge field corresponds to the conformal infinity $[\tilde{g}]$ of the metric. In this setting, the holographic principle asserts that the boundary data (which in the context of the Einstein equation would be the boundary metric $\tilde{g}$), defined on defined on the $n$-dimensional boundary, propagates through a suitable $(n+1)$-manifold (which here is the solid cylinder $\mathbb{R} \times \mathbb{B}^n$ and is referred to as the bulk in the physics literature) to determine the field (here the metric $g$) via a locally well-posed problem.
The main theorem of this paper asserts that, analogously to what happened in the case of the hyperbolic space, we can obtain a Lorentzian Einstein metric $g$ whose conformal infinity corresponds to any small enough perturbation of that of the anti-de Sitter space AdS$_{n+1}$. Before we can state the theorem, let us formulate the problem carefully. We aim to find a solution of the Einstein equation (1.1) on $(-T, T) \times \mathbb{B}^n$ whose pullback to the boundary is

$$
(j_{(-T, T) \times \partial \mathbb{B}^n})^* \bar{g} = \hat{g},
$$

where $\hat{g}$ is a Lorentzian metric, $j_{(-T, T) \times \partial \mathbb{B}^n}$ is the inclusion map of the boundary and we have normalized the conformal factor to 1 without loss of generality. Throughout the paper we will use the notation $\bar{g} := x^2 g$ for the rescaled metric.

Furthermore, since the Einstein equation in Lorentzian signature is basically a nonlinear wave equation, it is clear that one must also take initial conditions for the metric. To avoid necessary complications at this stage, we will simply write the initial conditions as

$$
g|_{t=0} = g_0, \quad \partial_t g|_{t=0} = g_1,
$$

although the admissible choices of $g_0$ and $g_1$ are severely constrained. In fact, it is well-known that choosing initial conditions to the Einstein equation is essentially equivalent to prescribing a Riemannian metric and a second fundamental form (which must satisfy the constraint equations) for the spacelike hypersurface $\{0\} \times \mathbb{B}^n$. Additionally, the initial and boundary conditions $(g_0, g_1, \hat{g})$ must satisfy certain nontrivial compatibility conditions. We discuss these conditions in Appendix A but, for the purposes of this Introduction, it would be enough to consider the simplest case: $g_0 := g_{\text{AdS}}, g_1 := 0$ and $\hat{g}$ any Lorentzian metric that is identically equal with $g_{\mathbb{R} \times S^{n-1}}$ for $t \leq 0$. This corresponds to the physically relevant situation of how the spacetime departs from the AdS geometry as nontrivial boundary data are switched on at the conformal infinity.

The main theorem of this paper can then be stated as follows. For simplicity, we will not make precise the assumptions on the initial conditions here; full details (including the definition and meaning of the quantities $C_{s,r}$) will be given later on. The result will be proved in any dimension $n \geq 3$.

**Theorem 1.1.** Let us take a Lorentzian metric $\hat{g}$ that is close to $g_{\mathbb{R} \times S^{n-1}}$ in the sense that

$$
\|\hat{g} - g_{\mathbb{R} \times S^{n-1}}\|_{C^p(I \times S^{n-1})} < \delta,
$$

where $I := (-T_0, T_0)$ is some fixed time interval and $p$ is large enough (i.e., larger than some dimensional constant $p_0(n)$). We also take compatible initial conditions $(g_0, g_1)$, which are assumed to be close to those of the anti-de Sitter space in the sense that certain quantities $C_{s,r}$ are smaller than $\delta$ for some large enough numbers $s$ and $r$.

If $\delta$ is small enough, then there is some $T \in (0, T_0)$, depending only on $n$ and $\delta$, and a Lorentzian metric $g$ on $(-T, T) \times \mathbb{B}^n$ which satisfies the Einstein equation (1.1) and the boundary and initial conditions (1.2)-(1.3). Furthermore, the difference between $g$ and the anti-de Sitter metric can be estimated as

$$
\|\hat{g} - g_{\text{AdS}}\|_{C^{n-2}((-T, T) \times \mathbb{B}^n)} < C\delta.
$$

Some comments are now in order. Firstly, an important observation is that the metric $g$ is only guaranteed to exist only locally in time (that is, for $|t| < T$).
Naively, one would expect this to be optimal, as this is heuristically related to the fact that the anti-de Sitter space is not expected to enjoy the good stability properties of the Minkowski \cite{11} or de Sitter spacetimes \cite{17}, which makes it natural to speculate that singularities should appear in finite time even for “small” initial and boundary data. Unfortunately there are no rigorous results in this direction. Secondly, the $C^{n-2}$ bound is known to be optimal thanks to a power-series analysis of the solutions in a neighborhood of the boundary which goes back to Fefferman and Graham (see e.g. \cite{16}). Thirdly, it is worth emphasizing that, for the simple choice of initial and boundary conditions discussed right before the statement of the theorem, only the condition $\|\hat{g} - g_{\mathbb{R} \times S^{n-1}}\|_{C^p(I \times S^{n-1})} < \delta$ needs to be taken into account because all the quantities $C_{s,r}$ are identically zero.

Let us now discuss how the main theorem is related to the existing results in the literature. As we have already mentioned, the Riemannian case is reasonably well understood thanks to the work of Graham–Lee \cite{19}, Anderson \cite{1,3}, Biquard \cite{7} and others. Lacking results on the Lorentzian case, the Riemannian problem has therefore been used as the basic model to understand the holographic prescription in the context of the AdS/CFT correspondence since Witten’s influential paper \cite{30}.

The situation for the holographic prescription problem in Lorentzian signature is much less clear-cut, since both the available analytical techniques and the expected results are necessarily different. The study on wave equations on asymptotically anti-de Sitter spaces has attracted much attention in the last few years, though. To the best of our knowledge, the wave equation on AdS$_4$ was first considered in \cite{8} using the strong symmetry of the problem to separate variables. Again for AdS$_4$, Choquet-Bruhat \cite{9,10} proved global existence for the Yang–Mills equation under a radiation condition, and Ishibashi and Wald \cite{24} gave a proof of the well-posedness of the Cauchy problem for the Klein–Gordon equation in AdS$_{n+1}$ using spectral theory. More refined results for the Klein–Gordon equation in an AdS space were developed by Bachelot \cite{4,5,6}, who used energy methods and dispersive estimates to study the decay of the solutions and prove Strichartz estimates and some results on the propagation of singularities.

In \cite{28}, Vasy established fine results on the propagation of singularities are proved for the Klein–Gordon equation on asymptotically AdS spaces using microlocal analysis. Holzegel and Warnick, both independently and in joint work \cite{21,23,22}, used energy methods to prove the well-posedness of the Cauchy problem for this equation in asymptotically AdS$_4$ space-times and discussed the boundedness of solutions to the Klein–Gordon equation in stationary AdS black hole geometries. The local well-posedness for semilinear Klein–Gordon equations in asymptotically anti-de Sitter spaces with nontrivial boundary conditions at infinity was established in \cite{15}. Spherically symmetric Einstein–Klein–Gordon systems have been considered in \cite{23}.

2. Strategy of the proof

In this section we will present the overall strategy of the proof of Theorem \ref{thm:main}. We will also point out where the main points of the argument can be found in the article, so this section also serves as a guide to the paper.

In a way, Theorem \ref{thm:main} is a local well-posedness result for the Einstein equation with “small” initial and boundary data, where “small” means close to those of the anti-de Sitter space. This is analogous to the classical result of Graham and
Lee [19], which is also a small data theorem for the Riemannian Einstein equation that one applies to consider perturbations of the conformal geometry at infinity of the hyperbolic space.

In the Riemannian case, first of all one replaces the Einstein equation by an equivalent quasilinear elliptic system using DeTurck’s trick [12, 13]. The resulting elliptic equation has singularities at the boundary \((x = 0)\), but the usual elliptic estimates provide suitable weighted estimates for the equation. Armed with these estimates, the theorem is proved using the inverse function theorem and Fredholm operators. (As an aside, it should be noticed that many algebraic computations that will be needed in this paper are just as in Graham–Lee [19], even though the signature of the metric is different, so we will rely on them in many places.)

As is well-known, in the Lorentzian case the equation presents a completely different behavior, since the modified Einstein equation is now a quasilinear hyperbolic system. However, the key difficulties that arise in the proof of a local well-posedness result for these equations are well understood under the assumption of global hyperbolicity since the foundational result of Choquet-Bruhat [27].

However, Theorem 1.1 presents three additional difficulties that make its proof rather involved, both technically and conceptually. The first difficulty is that asymptotically anti-de Sitter metrics are not globally hyperbolic, so the classical result of Choquet-Bruhat does not apply. This fact, which is due to the existence of timelike geodesics that escape to infinity in finite time, is closely related to the way that the data propagates in the equation. Furthermore, just as in the Riemannian case, there are terms in the equation that are strongly singular at the boundary. Contrary to what happens there, however, the usual hyperbolic estimates are not enough to control the behavior of the equation in an asymptotically anti-de Sitter space. As we will see, it forces us to introduce a functional framework adapted to the geometry of these spaces.

The second difficulty is that, under the global hyperbolicity assumption, the modified Einstein equation is a quasi-diagonal system, meaning that the leading part of the hyperbolic system (that is, the terms involving second-order derivatives) is given by a scalar differential operator (essentially the wave operator \(g^{\mu\nu}\partial_\mu\partial_\nu\)). Here this is not exactly so. This is because, for an asymptotically anti-de Sitter metric, the leading terms of the equation (that is, the ones that cannot be absorbed into constants in the estimates) are not only the second-order derivatives, which are indeed given by the wave operator, but also the terms that are most singular at \(x = 0\). When these terms are taken into account, the equation is no longer quasi-diagonal, so one must approximately diagonalize the operators and take into account that the estimates that we obtain in different “eigenspaces” are not equivalent. It is remarkable, though, that the various powers of \(x\) that appear in scattered through the equations work together to allow us to prove Theorem 1.1.

The third difficulty is that, in general, it is notoriously hard to impose boundary conditions in the Einstein equations (see e.g. [18] and references therein). The way that we circumvent this problem is by constructing the solution metric \(g\) as a sum of two terms, one that is “large” at infinity and which we construct using essentially algebraic methods and one whose existence must be proved using analytic techniques and which is “small” at infinity, so that for all practical purposes one does not need to consider the boundary conditions here.
Hence we are led to considering the following strategy to tackle the problem:

**Step 1: The modified Einstein equation.** In Section 3 we discuss how one can replace the Einstein equation (1.1) by a quasilinear hyperbolic system $Q(g) = 0$ using DeTurck’s trick. It should be noticed, however, that the lack of global hyperbolicity makes nontrivial some arguments in the proof that both equations are equivalent. This is established in Section 10 using ideas developed in the paper (Theorem 10.1).

**Step 2: Peeling off the metric.** In Section 4 we construct asymptotically anti-de Sitter metrics $\gamma_l$ that are “approximate solutions” to the modified Einstein equation $Q(g) = 0$ and satisfy the desired boundary conditions (Theorem 4.5). These metrics have the property that $Q(\gamma_l)$ is suitably small and are obtained from the boundary datum $\hat{g}$ in an essentially algebraic way that can be understood as peeling off the leading “layers” of the solution at $x = 0$, step by step. The parameter $l$ corresponds to the number of steps that one considers and is related to the norms in which $\gamma_l$ is an approximate solution of the modified Einstein equation.

**Step 3: Setting an iteration within a suitable functional framework.** To construct the metric $g$ that solves the modified Einstein equation, we write it as

$$g = \gamma + x^n u,$$

where we have set $\gamma = \gamma_l$ for a large enough $l$. There $\gamma$ is going to be the “large” part at $x = 0$ and the other terms is going to be “small”.

To try to construct $u$, we set up an iteration in Section 5. The convergence of this iteration will not be proved until Section 9 however. Before that, we need to defined suitable Sobolev spaces adapted to the geometry of the anti-de Sitter space in which we can derive suitable estimates for $u$. In Sections 6 and 7 we consider two related scales of Sobolev spaces, $H^{m,r}_\alpha$ and $H^{m,r}$, and derive several key estimates for them. It should be noticed that not only the proofs of these estimates are different from those of the usual Sobolev spaces $H^k(\mathbb{R}^n)$, but in fact so is the range of parameters for which e.g. we have pointwise estimates (Corollary 6.3) or can obtain estimates for the product of two functions (Theorem 7.1).

**Step 4: Linear estimates and convergence of the iteration.** Using the above adapted Sobolev spaces, in Section 8 we obtain estimates for the linear operators that appear in the iteration under certain assumptions about the structure of the metric. Here the way that the various powers of $x$ appear is crucial to derive estimates that are analogous (although the spaces and range of parameters are different) to the usual ones obtained for globally hyperbolic quasilinear wave equations. It should be emphasized, though, that the fact that the equation is effectively not quasi-diagonal and the fall-off of the nonlinearities at the boundary make the analysis of the linear equations and the treatment of the functional spaces much subtler than in our previous paper [15].

With these estimates in hand and equipped with the results about the adapted Sobolev spaces established in the previous step, the proof of the convergence of the iteration goes along the lines of the classical result for globally hyperbolic spaces. The details are presented in Section 9 although, as we have already mentioned, to show that these metrics are in fact Einstein one has to wait until Theorem 10.1.
The paper concludes with two appendices. In Appendix A we discuss the constraint and compatibility conditions that must be imposed on the initial and boundary data. In Appendix B we record some results about the integral operators $A_\alpha$ and $A^*_\alpha$, defined in (6.7), that we established in [13]. These operators play an important role in Sections 6 and 7. For the benefit of the reader, we also include a sketch of the proof.

3. The modified Einstein equation

When dealing with the Einstein equation, a first difficulty, well understood by now, is that the gauge invariance of the Einstein equation under the diffeomorphism group makes it a very degenerate system. A standard way of solving this difficulty is using DeTurck’s trick [12, 13]. To explain this method, we will make use of some reference metric $\gamma_0$ that we will define shortly. To avoid unnecessary repetitions, let us introduce the following

**Definition 3.1.** A metric $g$ on $I \times \mathbb{B}^n$ is called *weakly asymptotically AdS* if the following conditions hold:

(i) The rescaled reference metric $\bar{g} := x^2 g$ is of class $C^2$ up to the boundary.

(ii) If one takes coordinates $(t, x, \theta)$ in a neighborhood of a point at the boundary, with $\theta \equiv (\theta^1, \ldots, \theta^{n-1})$ being local coordinates in $\mathbb{S}^{n-1}$, then on the boundary the metric coefficients read as

$$\bar{g}_{xx} = 0, \quad \bar{g}_{xt} = \bar{g}_{x\theta^i} = 0.$$ 

In particular, $|dx|^2_{\bar{g}} = 1$ on $I \times \partial \mathbb{B}^n$.

This definition is motivated by the formal calculations of Graham and Lee in [19], many of which carry over verbatim to the case of Lorentzian signature. The definition should be compared with that of an asymptotically AdS metric, cf. [20].

We will choose the reference metric $\gamma_0$ to be an weakly asymptotically AdS metric whose pullback to $I \times \partial \mathbb{B}^n$ is $\hat{g}$. A way of doing this is to take the AdS metric $g_{\text{AdS}}$ and extend the tensor $\hat{g}_\mu^\nu$ on $T_p(I \times \partial \mathbb{B}^n)$ to a tensor $G^\mu_\nu$ on $T_p(I \times \mathbb{B}^n)$ for each point $p \in I \times \partial \mathbb{B}^n$ by setting

$$G_{ij} := \hat{g}_{ij}, \quad G_{\mu\nu}(\bar{g}_{\text{AdS}})^{\nu\lambda}x_\lambda = x_\mu.$$

In this equation the latin (resp. Greek) indices correspond to the $n$ coordinates in $I \times \partial \mathbb{B}^n$ (resp. over the $n+1$ coordinates in $I \times \mathbb{B}^n$), $(\bar{g}_{\text{AdS}})^{\mu\nu}$ are the components of the inverse of $\bar{g}_{\text{AdS}}$ and we are using the notation

$$x_\mu := \partial_\mu x.$$ 

We record here that, with the particular choice of $x := 1 - r$, where $r$ is the Euclidean distance to the center of the ball, the AdS$_{n+1}$ metric can be written in a neighborhood of the boundary as $g_{\text{AdS}} = x^{-2} \bar{g}_{\text{AdS}}$ with

$$\bar{g}_{\text{AdS}} = -(1 + x^2)^2 dt^2 + dx^2 + (1 - x^2)^2 g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the standard metric on the sphere. Therefore on the conformal boundary we have

$$\bar{g}_{\text{AdS}}|_{\mathbb{R} \times \partial \mathbb{B}^n} = dx^2 + g_{\mathbb{R} \times \mathbb{S}^{n-1}}.$$
and, of course, its pullback to the boundary is

\[ (j_{(-T,T) \times \partial B^n})^* \bar{g}_{\text{AdS}} = g_{\mathbb{R} \times S^{n-1}}. \]

We can now extend \( G \) to a tensor field \( E(G) \) defined on a small neighborhood of \( I \times \partial B^n \), for instance by parallel transport along radial geodesics with respect to the metric \( g_{\text{AdS}} \). The reference metric can then be taken as

\[ \gamma_0 := \chi g_{\text{AdS}} + (1 - \chi) E(G), \]

with \( \chi \) a suitable cutoff function that is equal to 1 in a neighborhood of the boundary. Notice that the reference metric depends on the boundary datum \( \hat{g} \), and that it is a (non-degenerate) Lorentzian metric because \( \hat{g} \) is close to \( g_{\mathbb{R} \times S^{n-1}} \). As usual, we will use the notation \( \bar{\gamma}_0 := x^2 \gamma_0 \).

Let us denote by \( \Gamma_{\lambda \rho}^\nu \) and \( \tilde{\Gamma}_{\lambda \rho}^\nu \) the Christoffel symbols of the metrics \( g \) and \( \gamma_0 \), respectively. DeTurck’s trick consists in looking for solutions to the modified Einstein equation

\[ Q(g) = 0, \]

where the components of the tensor \( Q(g) \) are given in terms of those of the Ricci tensor, \( R_{\mu \nu} \), by

\[ Q_{\mu \nu} := R_{\mu \nu} + ng_{\mu \nu} + \frac{1}{2} (\nabla_\mu W_\nu + \nabla_\nu W_\mu). \]

Here the covariant derivatives and the Ricci tensor are those of the metric \( g \) and the 1-form \( W \) is

\[ W_\mu := g_{\mu \nu} g^\lambda \nu (\Gamma_\nu^\lambda - \tilde{\Gamma}_\nu^\lambda). \]

We will discuss the relationship between the solutions of the Einstein equation (1.1) and those of the modified equation (3.2) in Section 10 as the lack of global hyperbolicity introduces some peculiarities. Notice that \( Q(g) \) also depends on the initial and boundary conditions through the reference metric \( \gamma_0 \).

It is well-known that the advantage of Eq. (3.2) over the Einstein equation is that the nondegeneracy has been taken care of; indeed, (3.2) is a quasilinear wave equation because

\[ Q_{\mu \nu} = -\frac{1}{2} g^{\lambda \nu} \partial_\lambda \partial_\mu g_{\mu \nu} + B_{\mu \nu}(g, \partial g), \]

with the second term quadratic in \( \partial g \). Our goal now is to solve the modified Einstein equation (3.2) together with the initial and boundary conditions

\[ g|_{t=0} = g_0, \quad \partial_t g|_{t=0} = g_1, \quad (j_{(-T,T) \times \partial B^n})^* \bar{g} = \hat{g}. \]

For the class of metrics that we are considering, the coefficients are strongly singular at \( x = 0 \). Indeed, it essentially follows from a computation by Graham and Lee [19, Eq. (2.19)] that for a weakly asymptotically AdS metric \( g \) one can express (3.5) in terms of \( \bar{g} \) as

\[ Q_{\mu \nu} = \frac{1}{x^2} \left( n(1 - \bar{g}^{\lambda \nu} x_\lambda x_\nu) \bar{g}_{\mu \nu} - \frac{1}{2} (B_\mu x_\nu + B_\nu x_\mu) \right) + \frac{1}{x} \mathcal{P}^1(\bar{g}) + \mathcal{P}^2(\bar{g}), \]

where

\[ B_\mu := \bar{g}^{\lambda \nu}(\bar{\gamma}_0)_{\lambda \rho} \bar{g}_{\mu \nu}(\bar{\gamma}_0)_{\rho \lambda} x_\lambda - (n + 1) x_\mu. \]
and $\mathcal{P}^1(\bar{g})$ (resp. $\mathcal{P}^2(\bar{g})$) stands for terms that depend smoothly on $x$, $\bar{g}$, $\bar{\gamma}_0$ and $\partial \bar{\gamma}_0$ and are linear in $\partial \bar{g}$ (resp. linear in $\partial^2 \bar{g}$ and quadratic in $\partial \bar{g}$). Here all the indices are raised and lowered using the metric $\bar{g}_{\mu \nu}$ but $(\bar{\gamma}_0)^{\mu \nu}$, which is the inverse of $\bar{\gamma}_0$.

In view of Eq. (3.6), we can immediately make the following important observation:

**Proposition 3.2.** Suppose that $g$ is weakly asymptotically AdS. Then $Q(g) = O(x^{-1})$ if and only if the following relations hold true on $(-T,T) \times \partial B^n$:

\[ \bar{g}^{\mu \nu} (\bar{\gamma}_0)^{\mu \nu} = n + 1 \quad \text{and} \quad \bar{g}^{\mu \nu} x_\nu = (\bar{\gamma}_0)^{\mu \nu} x_\nu. \]

### 4. Peeling off the metric

Throughout we will use the notation $B^n_R$ for the $n$-dimensional ball centered at $0$ of radius $R$, dropping the subscript in the case of the unit ball. The defining function $x$ will be a $C^\infty$ positive function on the $B^n$ that vanishes to first order at the boundary. For $R$ close to $1$, this ensures that one can take $x$ as a coordinate in the annulus

\[ A := B^n \setminus B^n_R. \]

To parametrize $A$ we will always take coordinates $(x, \theta)$, where $\theta = (\theta^1, \ldots, \theta^{n-1})$ are local coordinates in the sphere $S^{n-1}$. Since a weakly asymptotically AdS metric is only problematic in a neighborhood of the boundary, these will be the most convenient coordinates to carry out the key estimates that will be needed in this paper.

Let us denote by $S^2$ the space of symmetric covariant 2-tensors in $I \times \overline{B^n}$, with $I := (-T_0, T_0)$ an interval. In the following proposition we provide a convenient decomposition of this space at any point close to, or lying on, the conformal boundary $I \times \partial B^n$. **Throughout the section, we will assume that $g$ is a weakly asymptotically AdS metric.**

**Proposition 4.1.** In $I \times \overline{A}$, the space of symmetric tensors can be decomposed as

\[ S^2 = \mathcal{V}^g_0 \oplus \mathcal{V}^g_1 \oplus \mathcal{V}^g_2 \oplus \mathcal{V}^g_3, \]

where

\[ \mathcal{V}^g_0 := \{ H \in S^2 : H_{\mu \nu} = \varphi \bar{g}_{\mu \nu} \text{ with } \varphi \text{ scalar} \}, \]

\[ \mathcal{V}^g_1 := \{ H \in S^2 : H_{\mu \nu} \bar{g}^{\nu \lambda} x_\lambda = 0 \text{ and } H_{\mu \nu} \bar{g}^{\mu \nu} = 0 \}, \]

\[ \mathcal{V}^g_2 := \{ H \in S^2 : H_{\mu \nu} = \varphi [(n+1)x_\mu x_\nu - \bar{g}_{\mu \nu}] \text{ with } \varphi \text{ scalar} \}, \]

\[ \mathcal{V}^g_3 := \{ H \in S^2 : H_{\mu \nu} = a_\mu x_\nu + a_\nu x_\mu \text{ with } \bar{g}^{\lambda \rho} a_\lambda x_\rho = 0 \}. \]

**Proof.** Since the 1-form $dx$ does not vanish in $I \times \overline{A}$, it is easy to check that $\mathcal{V}^g_i \cap \mathcal{V}^g_j = \{0\}$ if $i \neq j$ and that the dimensions of the spaces $\mathcal{V}^g_i$ at each point of $I \times \overline{A}$ are

\[ 1, \quad \frac{n(n+1)}{2} - 1, \quad 1 \quad \text{and} \quad n, \]

respectively. The sum of these numbers gives

\[ \frac{(n+1)(n+2)}{2}, \]

that is, the dimension of $S^2$ at any point. The proposition the follows. \qed
In what follows we will need more information about the structure of the modified Einstein operator $Q(g)$ in a neighborhood of the boundary. To analyze $Q(g)$, we will restrict our attention to the set $A$ and use coordinates $(t, x, \theta)$, where $\theta$ are local spherical coordinates on $\mathbb{S}^{n-1}$. It was computed by Graham and Lee [19, Proposition 2.10] that the action of the differential of the map (4.3) on a symmetric tensor $h = h_0 + h'$, with $h_0 \in \mathcal{V}_0^g$ and $h' \in \mathcal{V}_1^g \oplus \mathcal{V}_2^g \oplus \mathcal{V}_3^g$, is of the form

\[
(DQ)_g(h) = -\frac{1}{2}((\Box_g - 2n)h_0 + (\Box_g + 2)h') + xL^1 h,
\]

where $\Box_g h_{\mu\nu} := g^{\lambda\rho}\nabla_\lambda \nabla_\rho h_{\mu\nu}$ is the Laplacian on tensor fields and we henceforth use the notation $L^m$ for a matrix $m$th order linear differential operator in the conormal derivatives $(x\partial_x, \partial_\theta, \partial_t)$ whose coefficients are smooth functions of $(x, \bar{g}, \partial \bar{g})$ up to $x = 0$.

In particular, the part with second-order derivatives of the linearized operator $(DQ)_g$ is the same as that of the wave operator $-\frac{1}{2}\Box_g$. Regarding the terms that are most singular at $x = 0$, it was shown in [19, Proposition 2.7] that, in terms of the coordinates $(t, x, \theta)$, the Laplacian on a symmetric tensor $h$ can be expanded in $x$ as

\[
\Box_g h_{\mu\nu} = (x^2 \partial_x^2 + (1-n)x\partial_x)h_{\mu\nu} + 2h_{\mu\rho}g^{\lambda\lambda}g^{\rho\rho}x_\lambda x_\rho g_{\mu\nu} \\
- (n+1)(h_{\mu\lambda}g^{\rho\rho}x_\rho x_\nu + h_{\nu\lambda}g^{\rho\rho}x_\rho x_\mu) + 2g^{\rho\rho}h_{\lambda\rho}x_\mu x_\nu \\
+ xL^1(h)_{\mu\nu} + x^2L^2(h)_{\mu\nu}.
\]

To further simplify this expression, let us define the quadratic polynomials

\[
p_j(s) := -\frac{1}{2}\left(s - \frac{n}{2} + \alpha_j\right)\left(s - \frac{n}{2} - \alpha_j\right),
\]

where $0 \leq j \leq 3$ and $\alpha_j$ are the constants

\[
\alpha_0 := \frac{\sqrt{n(n+8)}}{2}, \quad \alpha_1 := \frac{n}{2}, \quad \alpha_2 := \alpha_0, \quad \alpha_3 := \frac{n}{2} + 1.
\]

In the following lemma, which we borrow from [19, Lemma 2.9] with a minor change the notation, we use the subspaces $\mathcal{V}_j^g$ to effectively diagonalize $(DQ)_g$ up to terms that are smaller at $x = 0$. Here $p_j(x\partial_x)$ has the obvious meaning.

**Lemma 4.2 ([19]).** If $h \in \mathcal{V}_j^g$, we have that

\[
(DQ)_g(h)_{\mu\nu} = p_j(x\partial_x)h_{\mu\nu} + x(L^1 h)_{\mu\nu} + x^2(L^2 h)_{\mu\nu}.
\]

We will also need some information on the second derivative $(D^2Q)_g$, understood as a quadratic form. For our purposes, it will be enough to have the following symbolic description of $(D^2Q)_g(h)$, where we are not displaying indices for the ease of notation:

**Lemma 4.3.** The second derivative of $Q$ is of the form

\[
(D^2Q)_g(h) = O(1)\tilde{h} \partial^2 \tilde{h} + O(1) \partial \tilde{h} \partial \tilde{h} + O(x^{-1}) \tilde{h} \partial \tilde{h} + O(x^{-2}) \tilde{h} \tilde{h}.
\]

Here $O(1)$ denotes a smooth function of $(x, \bar{g}, \partial \bar{g})$, $O(x^s) := x^s O(1)$ and we are using the notation $\tilde{h} := x^2h$. 
Proof. Ignoring the indices, we can use Eqs. (3.5) and (3.6) to symbolically write the structure of $Q(g)$ as

$$Q(g) = g^{-1} \partial^2 g + a_1(\bar{g}) \partial \bar{g} \bar{g} + \frac{a_2(\bar{g})}{x} \partial \bar{g} + \frac{a_3(\bar{g})}{x^2},$$

where $a_j(\bar{g})$ stands for a smooth function of $x$ and $\bar{g}$. Since

$$Q(g + \epsilon h) = Q(g) + \epsilon (DQ)_g(h) + \frac{1}{2} \epsilon^2 (D^2 Q)_g(h) + \mathcal{O}(\epsilon^3),$$

an elementary computation using that

$$(g + \epsilon h)^{-1} = g^{-1} - \epsilon g^{-1} \bar{g}^{-1} + \epsilon^2 g^{-1} \bar{g}^{-1} \bar{g}^{-1} - \mathcal{O}(\epsilon^3)$$

readily yields the desired expression for $(D^2 Q)_g$. □

We will also need the following elementary fact:

**Lemma 4.4.** For any integers $\sigma \geq 0$ and $s$ there is a polynomial $f$ of degree $\sigma$ or $\sigma + 1$ such that

$$p_j(x \partial_x)(x^s f(x)) = x^s (\log x)^\sigma f.$$  

Furthermore, $f$ has degree $\sigma + 1$ if and only if $p_j(s) = 0$.

**Proof.** Since $p_j(0) \neq 0$, it is clear that

$$p_j(x \partial_x) \left(\frac{1}{p_j(0)}\right) = 1.$$  

We now proceed by induction on $s$ and $\sigma$. Indeed, assume that the statement holds true for all $s \leq s_0$ and $\sigma \leq \sigma_0$. The key observation is that

$$(4.4) \quad p_j(x \partial_x)(x^s (\log x)^\sigma) = p_j(s)x^s (\log x)^\sigma - \frac{\sigma(2s + 4 - n)}{2} x^2 (\log x)^\sigma - \frac{\sigma(\sigma - 1)}{2} x^2 (\log x)^{\sigma - 2}.$$  

If $p_j(s_0 + 1) \neq 0$, by the induction hypothesis there is a polynomial $F$ of degree at most $\sigma_0$ such that

$$p_j(x \partial_x) \varphi = x^{s_0 + 1}(\log x)^{\sigma_0}$$

with

$$\varphi := x^{s_0 + 1} \left(\frac{(\log x)^{\sigma_0}}{p_j(s_0 + 1)} + F(\log x)\right).$$

On the other hand, if $p_j(s_0 + 1) = 0$ we have that $s_0$ is $\frac{n}{2} \pm \alpha_j$, and in this case $2s_0 + 2 - n$ is always nonzero. Hence the induction hypothesis and the identity (4.4) ensure that we can then take a polynomial $F$ of degree at most $\sigma_0$ such that

$$p_j(x \partial_x) \varphi = x^{s_0 + 1}(\log x)^{\sigma_0}$$

with

$$\varphi := x^{s_0 + 1} \left(\frac{2(\log x)^{\sigma_0 + 1}}{(\sigma_0 + 1)(2s_0 + 2 - n)} + F(\log x)\right).$$

The same argument yields analogous functions $\psi$ with

$$p_j(x \partial_x) \psi = x^{s_0}(\log x)^{\sigma_0 + 1}$$

and deals with the case of negative $s$, thereby completing the induction argument. □
To state the following theorem, we will introduce the space $C^r_m(I \times \mathbb{B}^n)$ of functions with $m + r$ continuous derivatives, with the peculiarity that the last $r$ derivatives with respect to $x$ are regularized by multiplying by $x$. This way, for instance, for all $k, l, m \geq 1$ we have that

$$x^m \log x \frac{d^l}{dx^l}$$

is in $C^m_k$ but not in $C^{m+k-1}$. To define it, we will also use a smooth nonnegative function $\chi_k$ of $x$ that vanishes in $I \times \mathbb{B}^n_R$ and is equal to 1 for $I \times (\mathbb{B}^n \setminus \mathbb{B}_R/2)$. With these objects, we can now define $C^m_r(I \times \mathbb{B}^n)$ as the space of functions $\varphi$ such that

$$\|\varphi\|_{C^m(I \times \mathbb{B}^n)^k} := \|(1 - \chi_k)\varphi\|_{C^{m+r}(I \times \mathbb{B}^n)} + \sum_{|\beta|+j+k \leq r} \|(x\partial_x)^j \partial_x^k \partial_{\beta} (\chi_k \varphi)\|_{C^m(I \times \mathbb{B}^n)}$$

is finite. The space $C^r_p(\mathbb{B}^n)$ is defined analogously. (Of course, the notation $\partial_{\beta}$ is somewhat heuristic as $\mathbb{S}^{n-1}$ is not covered by a global chart. To define it rigorously, it is standard that one can resort to either covering the sphere with a fixed finite collection of charts or to taking vector fields $X_1, \ldots, X_M$ on $\mathbb{S}^{n-1}$ that span the whole tangent space $T_p \mathbb{S}^{n-1}$ at each point $p \in \mathbb{S}^{n-1}$ and replacing $\partial_{\beta} \varphi$ by

$$X_1^{\beta_1} \cdots X_M^{\beta_M} \varphi,$$

with $|\beta| = \beta_1 + \cdots + \beta_M$. For notational simplicity, we will stick to the heuristic notation $\partial_{\beta} \varphi$, which must be interpreted in the aforementioned sense.)

We shall next present the main result of this section, which is a procedure to obtain asymptotically anti-de Sitter metrics $\gamma$ that satisfy the boundary condition $(I \times \partial \mathbb{B}^n)_* \gamma = \hat{g}$ and for which $Q(\gamma)$ is suitably small.

To state the theorem, we need to introduce some notation. Given nonnegative integers $s$ and $\sigma$, we will say that a symmetric tensor field $q$ is in $O_j(x^s \log^{\leq \sigma} x)$ if it can be written in $\mathbb{A}$ as

$$q = x^s \sum_{\sigma' = 0}^\sigma (\log x)^{\sigma'} B^{\sigma'},$$

where $B^{\sigma'}$ is a smooth symmetric tensor field in $I \times \overline{\mathbb{B}^n}$ satisfying the bounds

$$\|B^{\sigma'}\|_{C^k(I \times \mathbb{B}^n)} \leq F_k(\|\hat{g}\|_{C^{k+j}(I \times \mathbb{S}^{n-1})})$$

for each $k$ and $r$, provided that the difference

$$\|\hat{g} - g_{\mathbb{R} \times \mathbb{S}^{n-1}}\|_{C^0(I \times \mathbb{S}^{n-1})}$$

is small enough. Here $F_k$ is a polynomial with $F_k(0) = 0$. Although we will not say it explicitly hereafter, it is important that in all the terms of the form $O_j(x^s \log^{\leq \sigma} x)$ that will appear in this section, the coefficients of the corresponding polynomials $F_k$ will be bounded independently of $\hat{g}$.

**Theorem 4.5.** Let us take a nonnegative integer $l$ and suppose that

$$\|\hat{g} - g_{\mathbb{R} \times \mathbb{S}^{n-1}}\|_{C^p(I \times \mathbb{S}^{n-1})} < \delta$$

for some small enough $\delta > 0$ and $p \geq l$. Then there is a weakly asymptotically AdS metric $\gamma_l$ on $I \times \mathbb{B}^n$ of the form

$$\gamma_l = \sum_{k=0}^l O_j(x^{k-2} \log^{\leq \sigma} x)$$
whose pullback to the boundary of $\tilde{\gamma}_l := x^2\gamma_l$ is

$$(j_{I\times\partial B^n})^*\tilde{\gamma}_l = \hat{g}$$

and such that

$$(4.7) \quad Q(\gamma_l) = O(\log x) + O(\log^2 x),$$

where $\sigma_k$ is a nonnegative integer that is equal to zero for all $k \leq n-2$. Furthermore, the metric $\gamma_l$ is close to $g_{AdS}$ in the sense that

$$(4.8) \quad \|\tilde{\gamma}_l - \hat{g}_{AdS}\|_{C^r(I\times\partial B^n)} < C_\delta$$

with $p' := \min\{p-l,n-2\}$ and $r := p - l - p'$, while $Q(\gamma_l)$ is bounded by

$$(4.9) \quad \|x^{p'-l}Q(\gamma_l)\|_{C^0_{p'-l-2}(I\times\partial B^n)} < C_\delta.$$

Proof. It stems from Proposition 3.2 that choosing as $\gamma_0$ the reference metric (3.1) proves the result for $l = 0$. To see how things work for $l = 1$, let us write the $O(1)$ terms that appears in $Q(\gamma_0) = O(1) + O(2)$ as

$$O(1) = \frac{H_1}{x} + O(1),$$

where the tensor field $H_1$ is defined in terms of this quantity as

$$(4.10) \quad H_1 := E(x\gamma_0(x^{-1}))|_{x=0}$$

and is $O(1)$. Here $E$ denotes the extension operator that we introduced in Eq. (3.1).

Let us now use the direct sum decomposition of $S^2$ proved in Proposition 4.1 to write in a unique way

$$H_1 = \sum_{j=0}^3 H_{1j},$$

with $H_{1j} \in \mathcal{V}_j^{\gamma_0}$. We will take now

$$\gamma_1 := \gamma_0 - \sum_{j=0}^3 f_{1j}(x)H_{1j}$$

with suitably chosen functions $f_{1j}(x)$. By Lemma 4.2 and Taylor’s formula,

$$Q(\gamma_1) = Q(\gamma_0) + (DQ)_\gamma(\gamma_1 - \gamma_0) + I_1$$

$$= \sum_{j=0}^3 \left(\frac{1}{x} - p_j(x\partial_x)f_{1j}\right)H_{1j} + O_2(1) + (x\mathcal{L}^1 + x^2\mathcal{L}^2)(\gamma_1 - \gamma_0) + I_1,$$

where the error term is

$$I_1 := \int_0^1 (D^2Q)_{(1-s)\gamma_0 + s(\gamma_1 - \gamma_0)} ds.$$

Since $p_j(-1) \neq 0$, Lemma 4.4 ensures that we can take functions $f_{1j} = O(x^{-1})$ (indeed, $f_{1j}(x) = x^{-1}/p_j(-1)$) such that

$$p_j(x\partial_x)f_{1j} = \frac{1}{x}.$$
Since $H_1$, in principle, is only defined in a neighborhood of the boundary, we should include in $f_{ij}$ a suitable cut-off function, which we henceforth omit for the simplicity of notation. In any case, with this choice of $f_{ij}$ and Lemma 4.3 we obtain that the error term is controlled by

$$I_1 = O_2(1) + O_3(x),$$

which immediately implies that

$$Q(\gamma_1) = O_2(1) + O_3(x).$$

The general case follows by an induction argument that also relies on Taylor’s formula and Lemmas 4.1, 4.2, 4.3. To sketch the proof, let us assume that the claim holds for all integers up to $l - 1$. To prove it for $l$, we argue as above to write

$$Q(\gamma_{l-1}) = x^{l-2} \sum_{j=0}^{3} \sum_{k=0}^{\sigma_{l-1}} (\log x)^{k} H_{l k j} + O_{l+1}(x^{l-1} \log^{\leq \sigma_{l-1}} x),$$

with $H_{l k j} = O_l(1)$ a tensor field in $V^\wedge_{l-1}$. Indeed, $H_{l k j}$ can be assumed to be related to the extension via the operator $E$ of a suitable tensor field defined on the boundary, in an analogous fashion to (4.10). Lemma 4.4 allows us to take polynomials $f_{l k j}$, of degree $k$ if $p_j(l - 2) \neq 0$ and $k + 1$ otherwise, so that

$$p_j(x \partial_x)(x^{l-2} f_{l k j}(\log x)) = x^{l-2}(\log x)^k.$$

If we now set

$$\tilde{\gamma}_l := \gamma_{l-1} - x^l \sum_{j=0}^{3} \sum_{k=0}^{\sigma_{l-1}} f_{l k j}(\log x) H_{l k j},$$

a computation analogous to the one for $\gamma_1$ then shows that

$$Q(\gamma_l) = O_{l+1}(x^{l-1} \log^{\leq \sigma_l} x) + O_{l+2}(x^l \log^{\leq \sigma_l} x)$$

for some integer $\sigma_l$. This integer remains zero as long as $\sigma_{l-1} = 0$ and $p_j(l - 2) \neq 0$, that is, until $l = n - 1$ (cf. Eq. (4.3)).

It is apparent from the construction that the tensor fields $H_{l k j}$ that appear at the $l^{th}$ step of the induction that the coefficients depend continuously on $\tilde{\gamma}$ and its $l^{th}$ order derivatives, so that in particular their difference from from those $H_{l k j}$ corresponding to a different metric $\tilde{\gamma}'$ on $\mathbb{R} \times S^{n-1}$ is bounded by

$$\|H_{l k j} - H'_{l k j}\|_{C^k(\mathbb{R} \times S^n)} \leq C\|\tilde{\gamma} - \tilde{\gamma}'\|_{C^{k+1}(\mathbb{R} \times S^{n-1})}.$$

For the canonical metric $g_{\mathbb{R} \times S^{n-1}}$ we have $\gamma_l = g_{\text{AdS}}$ for all $l$, so this immediately yields the estimate (4.3). The reason for which this estimate is valid in $C^m$ only for $m \leq n - 2$ if $l \geq n - 3$ is that in the expression for $\tilde{\gamma}_l$ we then get logarithmic terms starting with $x^{n-3} \log x$. This is not a problem for the $C^m$ norm of the statement, for the reasons discussed in Eq. (4.5). The fact that $\tilde{\gamma}$ is close enough to $g_{\mathbb{R} \times S^{n-1}}$ in $C^l$ ensures that the symmetric tensor $\gamma_l$ remains a Lorentzian metric, which is weakly asymptotically AdS because $\gamma_0$ is and $\tilde{\gamma}_l$ differs from $\gamma_0$ by terms that vanish at $x = 0$.

In view of the structure of the metric $\gamma_l$, the bound (4.9) is immediate upon realizing that $Q(g_{\text{AdS}}) = 0$ and that, by (4.7), $x^{2-l} Q(\gamma_l)$ is of the form

$$x^{2-l} Q(\gamma_l) = O_{l+2}(x \log^{\leq \sigma} x).$$
Indeed, a straightforward calculation using this formula then shows that
\[
\|x^{2-\ell}Q(\gamma_l)\|_{C^{p-1}_0} = \|x^{2-\ell}Q(\gamma_l) - Q(g_{\text{AdS}})\|_{C^{p-1}_0} \leq C\|\hat{g} - g_{\mathbb{R} \times S^{n-1}}\|_{C^p} < C\delta,
\]
thereby completing the proof of the theorem. \(\square\)

5. Setting the iteration

Our goal in this section is to set up an iterative procedure that will eventually lead to a solution of the equation \(Q(g) = 0\) with the desired initial and boundary conditions. To this end, let us write the solution as
\[
g = \gamma + h,\]
where
\[
\gamma = \gamma_l
\]
is the metric constructed in Theorem 4.5 with some large enough value of the parameter \(l\) that we will specify later. Intuitively, the weakly asymptotically AdS metric \(\gamma\) is the part of the metric that is “large” at the boundary and \(h\) is “smaller”.

Let us recall from Eq. (3.5) that one can write \(Q(g)\) in local coordinates as
\[
Q(g) = \tilde{P}_g g + B(g),
\]
where we define the \(g\)-dependent linear differential operator \(\tilde{P}_g\) as
\[
(\tilde{P}_g g')_{\mu\nu} := -\frac{1}{2}g^{\lambda\rho}\partial_\lambda\partial_\rho g'_{\mu\nu}
\]
and \(B(g)\) depends on \(g\) and quadratically on \(\partial g\). Taylor’s formula ensures that
\[
B(g) = B(\gamma) + (DB)\gamma h - \tilde{E}(h),
\]
where the error term is
\[
\tilde{E}(h) := -\int_0^1 (D^2 B)_{\gamma + sh}(h) \, ds
\]
and the second order differential of \(B\) is understood as a quadratic form. The equation \(Q(g) = 0\) can then be written as
\[
\tilde{P}_g h + (DB)g h + (\tilde{P}_g \gamma - \tilde{P}_g \gamma) + Q(\gamma) - \tilde{E}(h) = 0.
\]

Let us now define a linear operator, depending on \(g\), as
\[
T_g h := -3h(\nabla(\gamma) x, \nabla(\gamma) x) \tilde{g},
\]
where \(\nabla(\gamma)\) stands for the connection associated with the metric \(\gamma\). As easy computation shows that \(T_g\) is the differential of the function \(g \mapsto \tilde{P}_g \gamma - \tilde{P}_g \gamma\) at \(g = \gamma\). Hence we will set
\[
\tilde{F}(h) := T_g h + \tilde{P}_g \gamma - \tilde{P}_g \gamma,
\]
which, in view of (3.3), allows us to write the equation \(Q(g) = 0\) as
\[
\tilde{P}_g h + (DB)g h + T_g h = -Q(\gamma) + \tilde{F}(h) + \tilde{E}(h).
\]
Let us now define another \(g\)-dependent linear differential operator \(P_g\) by setting
\[
\tilde{P}_g h + (DB)g h + T_g h =: x^{\frac{2}{2}} P_g u,
\]
where we have introduced the new unknown $u$ as

$$h =: x^{\frac{2}{n}} u.$$ 

Full details about the structure of the differential operator will be given in Section 8.

In terms of $u$, the equation $Q(g) = 0$ can be finally written as

$$P_g u = \mathcal{F}_0 + \mathcal{G}(u),$$

where

$$\mathcal{G}(u) := \mathcal{F}(u) + \mathcal{E}(u)$$

and

$$\mathcal{F}_0 := -x^{-\frac{2}{n}} Q(\gamma), \quad \mathcal{F}(u) := x^{-\frac{2}{n}} 2 \bar{\mathcal{F}}(h), \quad \mathcal{E}(u) := x^{-\frac{2}{n}} 2 \bar{\mathcal{E}}(h).$$

As we will see in forthcoming sections, our objective will be to solve this equation using an iterative procedure that will produce $u$ as the limit of a sequence $u^n$, with $u^1 := 0$ and

$$P_g u^{n+1} = \mathcal{F}_0 + \mathcal{G}(u^n).$$

Of course, here $g^n := \gamma + x^{\frac{2}{n}} u^n$ and the initial conditions that we need to impose are

$$u^{n+1}|_{t=0} = u_0, \quad \partial_t u^{n+1}|_{t=0} = u_1,$$

where we have set

$$u_j := x^{-\frac{2}{n}} (g_j - \partial_t \gamma|_{t=0}) \quad \text{for} \ j = 0, 1.$$

6. ADAPTED SObOLEV SPACES

In this section we will introduce some twisted Sobolev spaces that are adapted to the AdS geometry near the conformal boundary. They will be key in our derivation of the estimates that will allow us to prove the convergence of the iteration presented in the previous section. Specifically, we will consider two kinds of adapted Sobolev spaces, $\mathcal{H}_\alpha^n$ and $\mathcal{H}^m$, as well as certain modifications of them, $\mathcal{H}_\alpha^{m,r}$ and $\mathcal{H}^{m,r}$, that play a role somewhat similar to that of the spaces $C_{\alpha}^m$ introduced in (4.6). The first kind of adapted spaces depends on a parameter $\alpha$ that in our applications will ultimately be one of the quantities $\alpha_j$ defined in (4.3), so we will assume throughout that $\alpha > 1$ without further mention. The properties of these spaces for $\alpha < 1$ are quite different, as discussed in [15].

To define the spaces $\mathcal{H}_\alpha^n$, let us begin by introducing the twisted derivative with parameter $\alpha$ as

$$D_{x, \alpha} \varphi := \partial_x \varphi + \frac{\alpha}{x} \varphi.$$

Its formal adjoint in the Hilbert space

$$L^2_x := L^2((0, \infty), x \, dx)$$

is

$$D_{x, \alpha}^* \varphi := -\partial_x \varphi + \frac{\alpha - 1}{x} \varphi,$$

and we will set

$$D_{x, \alpha}^{(k)} \varphi := \begin{cases} (D_{x, \alpha}^* D_{x, \alpha})^\frac{k}{2} \varphi & \text{if } k \text{ is even}, \\ D_{x, \alpha} (D_{x, \alpha}^* D_{x, \alpha})^\frac{k-1}{2} \varphi & \text{if } k \text{ is odd}, \end{cases}$$

with the proviso that $D_{x, \alpha}^{(0)} \varphi := \varphi$. 
The twisted Sobolev space $H^m_\alpha \equiv H^m_\alpha (\mathbb{B}^n)$ is defined as follows. Let us suppose that the function $u$ is supported in a small neighborhood of the boundary $\partial \mathbb{B}^n$, which we can assume to be the annulus $A$ defined in (4.1). We can then define its $H^m_\alpha$ norm as

$$\|u\|_{H^m_\alpha (A)}^2 := \sum_{j+|\beta| \leq m} \int_{S^{n-1}} \int_0^1 |D_x^{(j)} \partial_\theta^\beta u|^2 x \, dx \, d\theta$$

where $d\theta$ is the canonical measure on the sphere and the twisted derivative acts on $u$ in the obvious way. Using a suitable cutoff function that is equal to 1 in a neighborhood of the boundary and vanishes outside $A$, for a function $u$ defined on the ball we can then set

$$\|u\|_{H^m_\alpha} := \|\chi u\|_{H^m_\alpha (A)} + \| (1 - \chi) u\|_{H^m(\mathbb{B}^n)} ,$$

where $H^m$ is the usual Sobolev space. The space $H^m_\alpha$ can then be defined as the closure in this norm of the space of smooth functions compactly supported in the ball, the definition being also applicable to tensor-valued functions using standard arguments. For $m = 0$ the norm, which does not depend on $\alpha$, will be simply denoted by $\|u\|_{L^2}$ or occasionally $\|u\|$.

For real $s > 0$, we can use interpolation to define the space $H^s_\alpha \equiv H^s_\alpha (\mathbb{B}^n)$. Equivalently, since $D_{x,\alpha} D_x^\alpha$ is an essentially self-adjoint operator in $L^2(\mathbb{R}^+, x \, dx)$ with the domain $C^\infty_0(\mathbb{R}^+)$, we can write

$$\|u\|_{H^s_\alpha} := \|\Lambda_\alpha^s (\chi u)\|_{L^2} + \| (1 - \chi) u\|_{H^s(\mathbb{B}^n)} ,$$

where

$$\Lambda_\alpha^s := (1 - \Delta_{S^{n-1}} + D_{x,\alpha} D_x^\alpha)^{s/2}$$

is defined using the spectral theorem. As we did in (4.6), we can also consider the space with $m$ derivatives as above and $r$ “regularized” derivatives. For this we use the norm that is defined as

$$\|u\|_{H^m_\alpha} := \sum_{j+|\beta| \leq r} \| (x \partial_x)^j \partial_\theta^\beta (\chi u)\|_{H^m_\alpha} + \| (1 - \chi) u\|_{H^m+r(\mathbb{B}^n)} .$$

Closely related scales of Sobolev spaces are $\mathcal{H}^m \equiv \mathcal{H}^m(\mathbb{B}^n)$ and $\mathcal{H}^{m,r} \equiv \mathcal{H}^{m,r}(\mathbb{B}^n)$, which do not depend on any parameters and are respectively defined as the closure of $C^\infty(\mathbb{B}^n)$ in the norm

$$\|u\|_{\mathcal{H}^m} := \sum_{j+|\beta| \leq m} \|x^{j-m} \partial_x^\beta \partial_\theta^\beta (\chi u)\|_{L^2} + \| (1 - \chi) u\|_{H^{m+r}(\mathbb{B}^n)} ,$$

$$\|u\|_{\mathcal{H}^{m,r}} := \sum_{j=0}^r \|x^{j} u\|_{\mathcal{H}^{m+r}} ,$$

in each case. Notice that these norm are constructed by including in each derivative a singular weight that depends on the number of $x$-derivatives that one is taking. These spaces can also be defined for non-integer values using interpolation or, denoting by $\partial_x^* := -\partial_x - 1/x$ the formal adjoint of $\partial_x$ with respect to the $L^2$ product, directly through the formula

$$\|u\|_{\mathcal{H}^s} := \left\| \left( \frac{1 - \Delta_{S^{n-1}}}{x^2} + \partial_x^* \partial_x \right)^{s/2} (\chi u) \right\|_{L^2} + \| (1 - \chi) u\|_{H^s(\mathbb{B}^n)} .$$
In particular, this ensures that the usual interpolation formulas are valid for these scales of Sobolev spaces.

We shall need estimates relating the various adapted Sobolev spaces that we have introduced. A simple observation is the following, which shows how multiplication by powers of \(x\) can help us redistribute the “standard” and “regularized” derivatives in the spaces \(H^{m,r}_{\alpha}\) and \(H^{m,r}\):

**Proposition 6.1.** Given nonnegative integers \(m, r\) and an integer \(l \in [-m, r]\), we have the inequality

\[
\|x^l u\|_{H^{m,r}_{\alpha}} \leq C \|u\|_{H^{m+l,r-l}_{\alpha}}.
\]

**Proof.** It is enough to expand the various terms appearing in the definitions of the norm and use some elementary algebra. \(\square\)

To explore the properties of these spaces we will make use of the integral operators

\[
A_{\alpha} \varphi(x) := x^{-\alpha} \int_0^x y^\alpha \varphi(y) \, dy,
\]

\[
A_{\alpha}^* \varphi(x) := x^{\alpha-1} \int_x^1 y^{1-\alpha} \varphi(y) \, dy,
\]

which act on functions of one variable and will play an essential role in the rest of this section. Notice that these operators are right inverses of \(D_{x,\alpha}\) and \(D_{x,\alpha}^*\) in the sense that

\[
D_{x,\alpha}(A_{\alpha} \varphi) = D_{x,\alpha}^*(A_{\alpha}^* \varphi) = \varphi;
\]

in particular, \(A_{\alpha}^*\) is the adjoint of \(A_{\alpha}\) in \(L^2_x\). Obviously \(A_{\alpha}, A_{\alpha}^*\) also act on functions defined on \(A\). In Appendix B we record some important properties of these operators, extracted from [15].

A simple but important estimate is the following, which gives an \(L^\infty\) bound for functions belonging to an adapted Sobolev space. Notice that, contrary to what happens in the usual Sobolev embedding theorem, we are not asking for the square-integrability of \(\frac{n}{2} + \epsilon\) derivatives but actually of \(\frac{n}{2} + 1 + \epsilon\):

**Theorem 6.2.** Let \(u \in H^{1,r}_{\alpha}\) with \(r > \frac{n-1}{2}\). Then we have the pointwise estimate in the ball

\[
\|u\|_{L^\infty} \leq C \|u\|_{H^{1,r}_{\alpha}}.
\]

**Proof.** By the definition of the norm and the Sobolev embedding, it is obviously enough to prove the result for \(u\) supported in \(A\). But for a.e. \((x, \theta)\) in \(A\) we then have

\[
|u(x, \theta)| = |A_{\alpha}^*(D_{x,\alpha} u)(x, \theta)|
\leq C \|D_{x,\alpha} u(\cdot, \theta)\|_{L^2_x}
\leq C \|D_{x,\alpha} u\|_{L^2_x H^{r}_{\theta}}
\leq C \|u\|_{H^{1,r}_{\alpha}},
\]

where \(H^{r}_{\theta} = H^r(S^{n-1})\) is the Sobolev space in the sphere of order \(r\) and to pass to the first, second and third lines we have respectively used the properties (i) and (iii) in Theorem B.1 and the Sobolev embedding. The theorem then follows. \(\square\)
Corollary 6.3. For any \( \rho > \frac{n-1}{2} \), \( \|u\|_{\mathcal{H}^m(B^n)} \leq C\|u\|_{\mathcal{H}^{m+1,r + \rho}} \). Furthermore, in \( \mathbb{A} \) we have the bound
\[
\|x^{-m+i}(x \partial_x)^j \partial_\rho^\beta u\|_{L^\infty} \leq C\|u\|_{\mathcal{H}^{m+1,r + \rho}}
\]
for all indices with \( i \leq m \) and \( i + j + |\beta| \leq m + r \).

Proof. It stems from Theorem 6.2 and the fact that there is a converse to this inequality, so that the norms \( \mathcal{H}^m \) follow from an elementary computation. That for some range of parameters we will do here, because we have that \( u \in \mathcal{H}^{m+1,r + \rho} \).

\[ \square \]

The connection between the spaces \( \mathcal{H}^{m,r}_\alpha \) and \( \mathcal{H}^{m,r} \) is subtler. Of course, the estimate
\[
(6.8) \quad \|u\|_{\mathcal{H}^{m,r}_\alpha} \leq C\|u\|_{\mathcal{H}^{m,r}}
\]
follows from an elementary computation. That for some range of the parameters there is a converse to this inequality, so that the norms \( \mathcal{H}^{m,r}_\alpha \) and \( \mathcal{H}^{m,r} \) are equivalent, is more sophisticated. The following theorem is the partial converse to the inequality (6.8) that we need:

Theorem 6.4. For any \( k \leq m \), if \( \alpha > k - 1 \),
\[
\|u\|_{\mathcal{H}^{k.r + m - k}} \leq C\|u\|_{\mathcal{H}^{m,r}_\alpha}.
\]

In particular, both norms are equivalent.

Proof. Since \( u \in \mathcal{H}^{m,r}_\alpha \) if and only if \( (x \partial_x)^j \partial_\rho^\beta u \in \mathcal{H}^m_\alpha \) for all \( j + |\beta| \leq r \), it is clearly enough to prove that
\[
\|u\|_{\mathcal{H}^{k,m-1}} \leq C\|u\|_{\mathcal{H}^m_\alpha}
\]
whenever \( \alpha > k - 1 \). There is no loss of generality in proving the result for functions supported in \( \mathbb{A} \), since away from the boundary both norms are equivalent.

With \( m = 1 \), it suffices to see that one can write
\[
u = A_\alpha(D_{x,\alpha}u)
\]
as a consequence of Theorem 6.1 and that, due to this theorem,
\[
\left\| \frac{u}{x} \right\|_{L^2} \leq C\|D_{x,\alpha}u\|_{L^2} \leq C\|u\|_{\mathcal{H}^1_\alpha}.
\]

Hence
\[
\left\| \partial_x u \right\|_{L^2} = \left\| D_{x,\alpha}u - \frac{\alpha u}{x} \right\|_{L^2} \leq \left\| D_{x,\alpha}u \right\|_{L^2} + \alpha \left\| \frac{u}{x} \right\|_{L^2} \leq C\|u\|_{\mathcal{H}^1_\alpha},
\]
as we wanted to prove.

Let us now consider the case \( m = 2 \). A moment’s thought reveals that it is enough to keep track of derivatives with respect to \( x \) in the argument, which is what we will do here, because we have that \( \partial_\rho^\beta u \in \mathcal{H}^{m-|\beta|}_\alpha \). Hence let us start by using Theorem 6.1 to write
\[
D_{x,\alpha}u = A_\alpha^*(D_{x,\alpha}^{(2)}u) + x^{\alpha-1}f_1(\theta),
\]
where \( f_1(\theta) \) is a function on the sphere satisfying \( \|f_1\|_{L^2} \leq C\|u\|_{\mathcal{H}^2_\alpha} \). Here we are using the notation \( L_\theta^2 \equiv L^2(S^{n-1}) \). Again by Theorem 6.1 this implies
\[
u = A_\alpha^{(2)}(D_{x,\alpha}^{(2)}u) + cx^\alpha f_1(\theta),
\]
where $c$ is a constant and we are using the notation

$$A^{(l)}_\alpha \varphi := \begin{cases} (A^*_\alpha A_\alpha)^{\frac{l}{2}} \varphi & \text{if } l \text{ is even,} \\ A_\alpha (A^*_\alpha A_\alpha)^{\frac{l-1}{2}} \varphi & \text{if } l \text{ is odd.} \end{cases}$$

The desired estimates follow from this formula and the properties of the operators $A_\alpha$ and $A^*_\alpha$ listed in Theorem B.1. In order to see this, we start by noticing that

$$\left\| \frac{A^{(2)}_\alpha \varphi}{x^2} \right\|_{L^2} = \left\| \frac{1}{x^{\alpha+2}} \int_0^x y^\alpha A^*_\alpha \varphi(y) \, dy \right\|_{L^2}$$

$$\leq C \left\| \frac{A^*_\alpha \varphi}{x} \right\|_{L^2} \leq C \| \varphi \|_{L^2},$$

(6.9)

which readily yields

$$\left\| \frac{u}{x^2} \right\|_{L^2} \leq \left\| \frac{A^{(2)}_\alpha (D^{(2)}_{x,\alpha} u)}{x^2} \right\|_{L^2} + |c| \| x^{\alpha-2} f_1(\theta) \|_{L^2}$$

$$\leq C \| D^{(2)}_{x,\alpha} u \|_{L^2} + |c| \| x^{\alpha-2} \|_{L^2}^2 \| f_1 \|_{L^2}^2 \leq C \| u \|_{H^2},$$

provided $\alpha > 1$, which is the condition for $x^{\alpha-2}$ to be in $L^2$. If $\alpha \in (0, 1]$, one can easily fix the argument by multiplying by a factor of $x$, which yields the estimate

$$\| u \|_{H^{1,1}} \leq C \| u \|_{H^2},$$

for $\alpha$ in this range. A similar argument shows that

$$\left\| \frac{\partial_x u}{x} \right\|_{L^2} \leq \left\| \frac{A^*_\alpha (D^{(2)}_{x,\alpha} u)}{x} \right\|_{L^2} + C \left\| \frac{u}{x^2} \right\|_{L^2} \leq C \| u \|_{H^2},$$

$$\left\| \frac{\partial_x^2 u}{x^2} \right\|_{L^2} \leq \| D^{(2)}_{x,\alpha} u \| + C \left\| \frac{\partial_x u}{x} \right\|_{L^2} + C \left\| \frac{u}{x^2} \right\|_{L^2} \leq C \| u \|_{H^2},$$

provided $\alpha > 1$. This proves the claim for $m = 2$.

The general case follows by induction using the same argument using that, if $u \in H^m_\alpha$, one can write it as

$$u = A^{(m)}_\alpha (D^{(m)}_{x,\alpha} u) + \sum_{0 \leq j \leq m/2} x^{\alpha+2(j-1)} f_j(\theta),$$

with $\| f_j \|_{L^2} \leq C \| u \|_{H^m_\alpha}$. As before, the constraint on $\alpha$ appears from the fact that, for $u$ to be in $H^{k,j}$, $x^{\alpha-k}$ must be in $L^2$, which forces $\alpha > k - 1$. The only aspect that is slightly different than above is that the way in which the powers of $x$ must the distributed when we have an expression of the form $x^{-l} A^{(l)}_\alpha$ is by recursively using the formulas

$$\| x^{-l} A^*_\alpha \varphi \|_{L^2} \leq C \| x^{-l-1} \varphi \|_{L^2}, \quad x^{-l} A_\alpha \varphi = \frac{1}{x} A_{\alpha+l-1} (x^{-l} \varphi).$$

Combining Theorem 6.3 with Proposition 6.1, we arrive at the following useful
Corollary 6.5. If $\alpha > m - l - 1$,
$$
\|u\|_{H^{m,r}} \leq C\|x^l u\|_{H^m_r}.
$$

Proof. It is enough to consider $l \leq m$. We then have
$$
\|x^l u\|_{H^{m,r}} \leq C\|u\|_{H^{m-l,r+i}} \leq C\|u\|_{H^m_{r+i}},
$$
where we have used Theorem 6.4 to pass to the second inequality.

7. Nonlinear estimates for adapted Sobolev spaces

We shall next provide estimates that help us deal with nonlinear functions of elements of an adapted Sobolev space. To obtain estimates for products of functions in adapted Sobolev spaces, a basic result will be the following. To state it, we will use the notation

$$
D_{k,\beta} := (x\partial_x)^k \partial_{\theta}^\beta.
$$

Theorem 7.1. Given $r > \frac{n-1}{2}$, consider functions $w_1, \ldots, w_{m-1} \in H^{1,r}$ and $u \in H^{0,r}$, which we can assume to be supported in $A$. Then, given multiindices with
$$
\sum_{i=1}^m (k_i + |\beta_i|) \leq r,
$$
we have that
$$
\|(D_{k_1,\beta_1} w_1) \cdots (D_{k_{m-1},\beta_{m-1}} w_{m-1}) (D_{k_m,\beta_m} u)\|_{L^2} \leq C \|u\|_{H^{0,r}} \prod_{i=1}^{m-1} \|w_i\|_{H^{1,r}}.
$$

Proof. Notice that for any $\alpha > 1$ we have
$$
\left\| \left( \prod_{j=1}^{m-1} D_{k_j,\beta_j} w_j \right) D_{k_m,\beta_m} u \right\|_{L^2}^2 = \int \left( \prod_{j=1}^{m-1} (D_{k_j,\beta_j} w_j)^2 \right) (D_{k_m,\beta_m} u)^2 x \, dx \, d\theta
\leq \int \left( \prod_{j=1}^{m-1} \sup_{x'} |D_{k_j,\beta_j} w_j(x', \theta)| \right)^2 (D_{k_m,\beta_m} u)^2 x \, dx \, d\theta
\leq \int \left( \prod_{j=1}^{m-1} \|D_{x,\alpha} D_{k_j,\beta_j} w_j(\cdot, \theta)\|_{L^2_x} \right)^2 (D_{k_m,\beta_m} u)^2 x \, dx \, d\theta
\leq \int_{S^{n-1}} \prod_{j=1}^m V_j^2 \, d\theta,
$$
where we have defined
$$
V_m := \|D_{k_m,\beta_m} u\|_{L^2_x} \quad \text{and} \quad V_j := \|D_{x,\alpha} D_{k_j,\beta_j} w_j(\cdot, \theta)\|_{L^2_x}
$$
for $1 \leq j \leq m-1$ and in order to pass to the third line we have used that, by Theorem [B.1], for any one-variable function $\varphi(x) \in H^1_{\alpha}$ with $\alpha > 1$ we have the inequality:
$$
\|\varphi\|_{L^2_x} = \|A_\alpha(D_{x,\alpha} \varphi)\|_{L^2_x} \leq C \|D_{x,\alpha} \varphi\|_{L^2_x}.
$$
By definition and the Sobolev embedding, when \( r - k_j - |\beta_j| < \frac{n-1}{2} \) we have

\[
V_j \in H^{-k_j - |\beta_j|}_g \subset L^{p_j}_g, \quad p_j := \frac{2n - 2}{n - 1 - 2r - 2k_j - 2|\beta_j|},
\]

while for \( r - k_j - |\beta_j| > \frac{n-1}{2} \) the function \( V_j \) is in \( L^\infty_g \). For convenience, we will also relabel the functions \( V_j \) so that \( r - k_j - |\beta_j| > \frac{n-1}{2} \) if and only if \( j > m' \), so that \( V_j \in L^{p_j}_g \) with \( p_j = \infty \) for \( j > m' \). We will also relabel the functions so that \( r - k_j - \beta_j = \frac{n-1}{2} \) exactly for \( m'' < j \leq m' \), and for this range of \( j \)'s we will take \( p_j \) to be any finite but very large number. Of course, these last two sets can obviously be empty. Since \( S^{n-1} \) is compact, the generalized Schwartz inequality ensures that the integral (7.2) can be estimated as

\[
\int_{S^{n-1}} \prod_{j=1}^m V_j^2 \, d\theta \leq \prod_{j=1}^m \|V_j\|_{L^2_{p_j}}^2 \leq C \prod_{j=1}^m \|V_j\|^2_{H^{-k_j - |\beta_j|}_g} \leq C ||u||^2_{H^{m''}} \prod_{j=1}^{m-1} \|v_j\|_{H^{m''}}^2,
\]

provided

\[
\sum_{j=1}^m \frac{2}{p_j} \leq 1. \tag{7.3}
\]

Let us show that the condition (7.3) holds, which completes the proof of the theorem. For this, let us write

\[
r = (1 + \rho)\frac{n - 1}{2},
\]

where \( \rho > 0 \) by hypothesis. Since \( p_j = \infty \) for \( m > m' \) and \( p_j \) is arbitrarily large for \( m'' < j \leq m' \), we can then take an arbitrarily small constant \( \delta \) such that

\[
\sum_{j=1}^m \frac{2}{p_j} \leq \sum_{j=1}^{m'} \frac{2}{p_j} + \delta = \frac{1}{n - 1} \sum_{j=1}^{m'} (n - 1 - 2r + 2k_j + 2|\beta_j|) + \delta
\]

\[
= \frac{1}{n - 1} \left( m''(n - 1 - 2r) + 2 \sum_{j=1}^{m'} (k_j + |\beta_j|) \right) + \delta
\]

\[
\leq m'' - \frac{2r(m'' - 1)}{n - 1} + \delta
\]

\[
= 1 - (m'' - 1)\rho + \delta. \tag{7.4}
\]

Therefore, the claim follows for \( m'' \geq 2 \) by taking \( \delta \) smaller than \( (m'' - 1)\rho \). To conclude the proof, let us discuss the remaining cases. When \( m'' = 0 \), the claim is immediate. For \( m'' = 1 \) one can go over the proof of (7.3) and observe that the only problematic case is when \( k_1 + |\beta_1| = r \). But in this case \( k_j + |\beta_j| = 0 \) for all
j > 1, which implies that there are not any \( j \)'s for which \( r - k_j - |\beta_j| = \frac{n-1}{2} \) and thus one can take \( \delta = 0 \). The theorem then follows.

Theorem 7.1 will be key in the rest of the paper. It should be noticed that this theorem provides a wide range of estimates for nonlinear functions of elements of an adapted Sobolev space. In particular, we have the following result, where, although we do not emphasize it notationally, here the function \( F(w_1, \ldots, w_N) \) can also depend on the space variables:

**Corollary 7.2.** Let \( u \in \mathcal{H}^{0,r} \) and \( w_1, \ldots, w_m \in \mathcal{H}^{1,r} \) with \( r > \frac{n-1}{2} \). Then, if \( F \) is a \( C^r \) function of \( w_j \) and a \( C^0 \) function of the space variables (whose dependance will not be made explicit), we have

\[
\| F(w_1, \ldots, w_m) u \|_{\mathcal{H}^{0,r}} \leq C \| u \|_{\mathcal{H}^{0,r}},
\]

where \( C \) depends on \( \| w_1 \|_{\mathcal{H}^{1,r}} + \cdots + \| w_m \|_{\mathcal{H}^{1,r}} \).

**Proof.** The result follows by applying Theorem 7.1 to the various terms that appear after using the Leibnitz rule on \( D_{k,\beta} [F(w_1, \ldots, w_m) u] \) with \( k + |\beta| \leq r \). \( \square \)

8. Estimates for the linearized equation

For future convenience, we will assume that the metric \( g \) possesses the following properties, which will be needed in the following section to prove the convergence of the iteration set in Section 5. While some parameters could have been chosen in a different range for the purposes of this section, this way the application of these results in the following section will be transparent.

**Assumptions on the metric.** Throughout this section we will assume that the metric \( g \) satisfies the following hypotheses:

1. The metric \( g \) is weakly asymptotically hyperbolic and can be written as
   \[
   \bar{g} = \bar{\gamma} + x w,
   \]
   with \( \gamma \equiv \gamma_l \) is the metric constructed in Theorem 4.5 with \( l \geq \frac{n}{2} + s + 2 \), for some integer \( s \) satisfying
   \[
   2 \leq s < \frac{n}{2} + 2.
   \]

2. The tensor field \( w \) is bounded as
   \[
   \sum_{k=0}^{s-1} \| \partial_k^r w \|_{L^\infty_t \mathcal{H}^{2,r+s+k-2}} + \| \partial_t^s w \|_{L^\infty_t \mathcal{H}^{1,r-1}} < \Lambda,
   \]
   for some integer \( r > \frac{n-1}{2} \) and some small constant \( \Lambda \).

3. The boundary datum \( \bar{g}_0 \), which appears in the problem through \( \bar{\gamma} \), satisfies
   \[
   \| \bar{g} - g_{\mathbb{R} \times S^{n-1}} \|_{C^p(I \times S^{n-1})} < \delta
   \]
   for some \( p \geq l + r + s + 1 \) and some small enough \( \delta \).
Using the formula \[12\], which ensures that the principal part of \( P_0 \) is \( \tilde{g}^{\mu \nu} \partial_{\mu} \partial_{\nu} \), together with the small-\( x \) behavior described in Lemma 4.2 and the fact that \( \tilde{g} \) is weakly asymptotically AdS, it is easy to derive a manageable expression for \( P_0 u \). Specifically, if we take \( u \in \mathcal{V}^0 \), a direct calculation shows that \( P_0 u \) can be written in \( A \) using local coordinates as

\[
(8.2) \quad (P_0 u)_{\mu \nu} = \frac{1}{2} \tilde{g}^{\mu \nu} \left( \partial_i^2 + \partial_{\theta^i} \tilde{G}^{ij} \partial_{\theta^j} + \tilde{D}_{x,\alpha_i} b^1 \tilde{D}_{x,\alpha_j} + x \partial_{\theta^i} (b^4)^j \partial_{\theta^j} \right) u_{\mu \nu} + \left( \tilde{b}^5 \partial_x u + \tilde{b}^5 \partial_t u + \tilde{b}^7 \partial_\theta u + \frac{\tilde{b}^8}{x} u \right)_{\mu \nu},
\]

where as usual the local coordinates \( \theta = (\theta^1, \ldots, \theta^{n-1}) \) parametrize the sphere \( S^{n-1} \), the star denotes the formal adjoint of a differential operator computed with respect to the scalar product of \( L^2 \), and the quantities \( b^i \) are scalar functions or tensor fields that depend smoothly on \( \tilde{g} \) and \( \partial \tilde{g} \). Observe that the principal part of \( P_0 \) is scalar. Although we do not make explicit the tensorial structure of the tensor fields \( b^i \) appearing in the non-principal part of the operator, their action must be understood in the obvious fashion, e.g.,

\[
(b^6 \partial_x u)_{\mu \nu} \equiv (b^6)^{\lambda \rho}_{\mu \nu} \partial_\lambda u_{\rho}.
\]

Notice that, in particular,

\[
(8.3) \quad b^1 = \frac{\tilde{g}^{xx}}{\tilde{g}^{tt}}, \quad G^{ij} = \frac{\tilde{g}^{i \theta^j}}{\tilde{g}^{tt}}, \quad x (b^2)^i = \frac{2 \tilde{g}^{x \theta^i}}{\tilde{g}^{tt}}, \quad x b^3 = \frac{2 \tilde{g}^{xt}}{\tilde{g}^{tt}}, \quad (b^4)^i = \frac{2 \tilde{g}^{\theta^i i}}{\tilde{g}^{tt}}.
\]

Since the metric is weakly asymptotically AdS, all the quantities \( b^i \) are of order \( \mathcal{O}(1) \), and in fact we can write

\[
(8.4) \quad b^1 = 1 + \tilde{x} b^1,
\]

with \( \tilde{b}^1 \) a differentiable function of \( w \) and the spacetime variables. In particular, when \( \tilde{g} \) is the anti-de Sitter metric \( g_{AdS} \), the only terms that appear in the principal part of the operator and are nonzero are

\[
\tilde{g}^{00}_{AdS} = -\frac{1}{(1 + x^2)^2}, \quad b^1_{AdS} = (1 + x^2)^2, \quad G^{ij}_{AdS} = \left( \frac{1 + x^2}{1 - x^2} \right)^2 (g_{S^{n-1}})^{ij}.
\]

We shall next derive estimates for a function satisfying the scalar equation

\[
(8.5) \quad L_{g,\alpha} u = F, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1,
\]

where

\[
L_{g,\alpha} u := \left( \partial_t^2 + \partial_{\theta^i} G^{ij} \partial_{\theta^j} + D_{x,\alpha} b^1 D_{x,\alpha} + x \partial_{\theta^i} (b^4)^j \partial_{\theta^j} + x \partial_x b^3 \partial_t + \partial_{\theta^i} (b^4)^j \partial_t \right) u.
\]

Taking \( \alpha = \alpha_j \), \( L_{g,\alpha} \) would be the part of \( P_0 u \) containing both the highest order derivatives and the more singular terms at \( x = 0 \), which is a scalar differential operator for \( u \in \mathcal{V}^1 \). The metric \( g \) is assumed to satisfy the above hypotheses, and we will also assume that \( \alpha \geq n/2 \). The reason for which we introduce this auxiliary equation is to postpone the treatment of the tensorial nature of the equation until the end of this section, but we have chosen to keep the notation \( u \) for the unknown as we will eventually replace \( u \) by a tensor field satisfying \( P_0 u = F \).
In the following theorem we provide a priori estimates for the problem \((8.5)\). To state the theorem in an economic way, let us denote by 

\[
(8.6) \quad u_k := \partial_t^k u|_{t=0}, \quad 2 \leq k \leq s,
\]

the value of the \(k\)th time derivative of \(u\) at \(t = 0\). Notice that, isolating the term with the highest number of time derivatives in \((8.5)\) and differentiating \(k - 2\) times with respect to \(t\), one can write \(u_k\) in terms of derivatives of the initial data and source term \((u_0, u_1, F)\). The functions \(u_k\) will often appear in arguments via the quantity

\[
(8.7) \quad C_{s,r} := \sum_{k=0}^{s-1} \| u_k \|_{L^\infty_t \mathcal{H}^{1,r+s-k-1}} + \| u_s \|_{\mathcal{H}^{0,r}}.
\]

For the tensor-valued equation \(P_g u = F\), this quantity will correspond to the quantity that appears in the statement of Theorem \(1.1\).

To state the results, we will make use of the following norms (here the prime does not refer to any sort of duality):

\[
(8.8a) \quad \| u \|_{s,r} := \sum_{k=0}^{s-1} \| \partial_t^k u \|_{L^\infty_t \mathcal{H}^{1,r+s-k-1}} + \| \partial_t^s u \|_{L^\infty_t \mathcal{H}^{0,r}},
\]

\[
(8.8b) \quad \| F \|_{s,r} := \sum_{k=0}^{s-1} \| \partial_t^k F \|_{L^\infty_t \mathcal{H}^{0,r+s-k-1}}.
\]

Throughout, we will use the notation \(C_0\) for constants depending only on \(\delta\) and \(\Lambda\).

**Theorem 8.1.** For any \(F \in L^\infty_t L^2\) there is a unique solution \(u \in L^\infty_t \mathcal{H}^1 \cap W^{1,\infty}_t L^2\) to the Cauchy problem \((8.5)\), which satisfies the following estimate in \((-T,T) \times \mathbb{R}^n\):

\[
\| u \|_{s,r} \leq C_0 e^{C_0 T} (C_{s,r} + T \| F \|_{s,r}).
\]

**Proof.** There is no loss of generality in assuming that \(u\) is supported in \(\Lambda\), since the estimate is known to hold for \(u\) supported away from the boundary. Let us then define the energy functional

\[
(8.9) \quad E_1[v] := \frac{1}{2} \int_{\mathbb{R}^n} \left( (\partial_t v)^2 + G^{ij} \partial_i v \partial_j v + b^1 (D_{x,\alpha} v)^2 + x(b^2)^i \partial_x v \partial_i v \right) dx \, d\theta,
\]

where in the rest of this section we will write \(\partial_i \equiv \partial_{x^i}\). Since \(b^2\) is small and \(G^{ij}\) is close to the canonical metric on the sphere by the assumptions (i)–(iii), it is apparent that at any time \(E_1[v]^{\frac{1}{2}}\) is equivalent to the norm \(\| v \|_{\mathcal{H}^1} + \| \partial_t v \|_{L^2}\) (which is in turn equivalent to \(\| v \|_{\mathcal{H}^1} + \| \partial_t v \|_{L^2}\) by Theorem \(6.4)\) in the sense that

\[
(8.10) \quad \frac{1}{C} E_1[v]^{\frac{1}{2}} \leq \| v \|_{\mathcal{H}^1} + \| \partial_t v \|_{L^2} \leq C E_1[v]^{\frac{1}{2}}
\]

where the constant \(C\) only depends on

\[
\| \bar{\theta} g \|_{C^1} + \| \partial_t \bar{\theta} g \|_{C^1} + \| \partial^2_t \bar{\theta} \|_{C^0}.
\]

In particular, by Corollary \(6.3\) \(C \equiv C_0\) only depends on \(\Lambda\) via \(w\) and on \(\delta\) via \(\bar{\gamma}\).

Now let us use the energy functional \((8.9)\) to define

\[
E_{1,r} \equiv [\mathcal{D}_{k,\beta} v] := \sum_{k+|\beta| \leq r'} E_1[\mathcal{D}_{k,\beta} v],
\]
where again we are using the shorthand notation \( D_{k,\beta} := (x\partial_x)^k \partial^\beta_x \). In view of the norm equivalence \( \text{(8.10)} \), it is clear that \( E_{1,\rho}[v] \) is equivalent to the norm

\[
\|v\|_{H^{1,\rho}} + \|\partial_t v\|_{H^{0,\rho}}
\]

with a constant that only depends on \( \Lambda \) and \( \delta \). We can now define a higher analog of the energy \( E_1 \) by setting

\[
(8.11) \quad E_{s,\rho}[v] := \sum_{k=0}^{s-1} E_{1,s+k-1}[\partial_t^k v].
\]

In view of the norm equivalence \( \text{(8.10)} \), it is clear that \( E_{s,\rho}[v]^{1/2} \) is equivalent to the norm

\[
(8.12) \quad \sum_{k=0}^{s-1} \|\partial_t^k v\|_{H^{1,s+k-1}} + \|\partial_t^s v\|_{H^{0,\rho}}
\]

in the same sense as above, which implies that

\[
\sup_{|t| \leq T} E_{s,\rho}[v]^{1/2}
\]

is equivalent to \( \|v\|_{s,\rho} \).

Our goal now is to show that, if \( u \) is a solution of \( \text{(8.5)} \), the energy \( E_{s,\rho}[u] \) satisfies the differential inequality

\[
(8.13) \quad \partial_t E_{s,\rho}[u] \leq C_0 E_{s,\rho}[u] + C_0 E_{s,\rho}[u]^{1/2} \sum_{k=0}^{s-1} \|\partial_t^k F\|_{H^{0,\rho,s+k-1}}.
\]

Indeed, it is standard that this implies

\[
E_{s,\rho}[u](t)^{1/2} \leq C_0 e^{C_0 t} \left( E_{s,\rho}[u](0)^{1/2} + \sum_{k=0}^{s-1} \int_{-|t|}^{|t|} \|\partial_t^k F\|_{H^{0,\rho,s+k-1}} \right).
\]

Since clearly

\[
E_{s,\rho}[u](0)^{1/2} \leq C_0 C_{s,\rho},
\]

the theorem then follows from the above inequality.

Armed with Theorems \( \text{6.2} \) and \( \text{7.1} \), the proof of \( \text{(8.13)} \) is now standard. Let us begin by computing the evolution of \( E_{1,\rho+s-1}[u] \). One readily finds that it is given by

\[
(8.14) \quad \partial_t E_{1,\rho+s-1}[u] = \sum_{k+|\beta| \leq \rho+s-1} \left[ \int \partial_t (D_{k,\beta}u) L_{g,\alpha}(D_{k,\beta}u) + \int \partial_t D_{k,\beta} u \partial_x \left( b^3 \partial_t D_{k,\beta} u \right) - \int \partial_t D_{k,\beta} u \partial_t \left( (b^4)^i \partial_i D_{k,\beta} u \right) + \int \mathcal{O}(1) \partial_t D_{k,\beta} u \partial \partial D_{k,\beta} u + \int \frac{\mathcal{O}(1)}{x} D_{k,\beta} u \partial \partial D_{k,\beta} u + \int \frac{\mathcal{O}(1)}{x} (D_{k,\beta} u)^2 + \int \mathcal{O}(1) \left( \partial D_{k,\beta} u \right)^2 \right],
\]
where all the integrals hereafter correspond to integration over the ball with respect to the natural measure \( \pi \, dx \, d\theta \) and we are denoting by \( O(1) \) well-behaved functions of \( \bar{\gamma}, w \) and \( \bar{\partial}w \). We claim that this can be estimated as

\[
\partial_t E_{1, r+s-1}[u] \leq C_0 E_{1, r+s-1}[u] + C_0 E_{1, r+s-1}[u] \sum_{k+|\beta| \leq r+s-1} \| L_{g, \alpha}(D_{k, \beta} u) \|
\]

where \( \| \cdot \| \) stands for the \( L^2 \) norm. Indeed, for \( k + |\beta| \leq r + s - 1 \) the first term in \((8.14)\) is bounded as

\[
\int \partial_t D_{k, \beta} u L_{g, \alpha} D_{k, \beta} u \leq C_0 E_{1, r+s-1}[u] \| L_{g, \alpha} D_{k, \beta} u \|
\]

and the last for summands can be easily upper bounded by

\[
C_0 E_{1, r+s-1}[u]
\]

using Theorems 6.2 and 7.1. Let us now consider the first of the two remaining terms. We have that

\[
\left| \int x \partial_t D_{k, \beta} u \partial_x (b^\beta \partial_t D_{k, \beta} u) \right| = \left| \int (\partial_t D_{k, \beta} u)^2 x \partial_x b^\beta + \frac{1}{2} \int b^\beta x \partial_x [(\partial_t D_{k, \beta} u)^2] \right|
\]

\[
\leq \left| \int \frac{1}{2} x \partial_x b^\beta - b^\beta (\partial_t D_{k, \beta} u)^2 \right|
\]

\[
\leq C_0 E_{1, r+s-1}[u]
\]

and an analogous argument shows that

\[
\left| \int \partial_t D_{k, \beta} u \partial_x ((b^\beta)^i \partial_t D_{k, \beta} u) \right| \leq C_0 E_{1, r+s-1}[u].
\]

Putting everything together, this yields \((8.15)\). To conclude, we can now estimate the commutator using Theorems 6.2 and 7.1 to infer that

\[
\| L_{g, \alpha}(D_{k, \beta} u) \| \leq \| D_{k, \beta}(L_{g, \alpha} u) \| + \| [L_{g, \alpha}, D_{k, \beta} u] \|
\]

\[
\leq \| D_{k, \beta} F \| + \| [L_{g, \alpha}, D_{k, \beta} u] \|
\]

\[
\leq \| F \|_H^{r_s,r+s-1} + C_0 E_{r,s}[u] \frac{1}{2},
\]

which shows that

\[
\partial_t E_{1, r+s-1}[u] \leq C_0 E_{r,s}[u] + C_0 E_{s,r}[u] \frac{1}{2} \| F \|^r_s.
\]

The computation of the time evolution of the other quantities \( E_{1, r+s-k-1} [\partial_t^k u] \) appearing in the definition of \( E_{s,r}[u] \) (cf. Eq. \((8.11)\)) is similar, the only difference being that one needs to control the commutator

\[
\| L_{g, \alpha}(D_{j, \beta} \partial_t^k u) \| \leq \| D_{j, \beta} \partial_t^k F \| + \| [L_{g, \alpha}, D_{j, \beta} \partial_t^k] u \|
\]

\[
\leq \| \partial_t^k F \|_H^{r_s,r+s-k-1} + C_0 E_{r,s}[u] \frac{1}{2}.
\]

Summing over \( k \), this readily yields the differential inequality \((8.13)\).

\[ \square \]

\textbf{Remark 8.2.} Notice that we are not imposing that \( u(t) \in H^2_\alpha \) for a.e. \( t \), so Eq. \((8.5)\) is obviously understood using the energy formulation, as it is customary.
Promoting the estimates proved in Theorem 8.1 to estimates for the tensor-valued equation

\begin{equation}
Pg u = F, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1,
\end{equation}

is now immediate as the norms (8.8) can be trivially extended to tensor-valued functions. As before, we will state the theorem in terms of the quantity $C_{s,r}$, which we can still define in terms of the initial data and source term at $t = 0$ as in Eq. (8.7).

**Theorem 8.3.** For all times $T < T_0$, if $u$ solves the problem (8.16) one has the estimates

\[
\|u\|_{s,r} \leq C_0(T\|F\|'_{s,r} + e^{C_0 T}\|F\|', C_{s,r}),
\]

where the constants $C_0$ depends only on $\Lambda$ and $\delta$.

A final simple result that will come in handy in the following section is the following, which controls the difference between the solution to two Cauchy problems of the form (8.16) with different metrics and source terms. For concreteness we will control the difference in the $\| \cdot \|_{1,0}$ norm and assume that we have the same initial conditions $(u_0, u_1)$, but we could have used any norm $\| \cdot \|'_{s,r}$ with $s' \leq s - 1$ and allowed for distinct initial conditions. It is worth emphasizing that estimating the difference is not completely trivial a priori because the leading part of the equation, as represented by the operator $Pg$, is not scalar: we have seen that the parameter $\alpha = \alpha_j$ takes a different value depending on the subspace $V^g_j$ that $u$ is assumed to belong to. However, the structure of the metrics under consideration allows to prove the result quite easily.

**Proposition 8.4.** Let

\[
\bar{g} := \bar{\gamma} + xw \quad \text{and} \quad \bar{g}' := \bar{\gamma} + xw'
\]

be metrics satisfying the assumptions (i)–(iii) above. Suppose that $u, u' \in L^\infty H^1 \cap H^1 L^2$ satisfy the equations

\[
Pg u = F \quad \text{and} \quad Pg' u' = F'
\]

with the same initial conditions $(u_0, u_1)$. Then the difference is bounded by

\[
\|u - u'\|_{1,0} \leq Ce^{CT}(\|F - F'\|'_{1,0} + \|w - w'\|_{1,0}),
\]

where the constant $C$ only depends on $\Lambda, \delta, \|F\|', C_{s,r}$.

**Proof.** A short computation using the expression for $Pg$ shows that the differential operator $Pg$, whose leading part at $x = 0$ is not scalar, can be symbolically written in a neighborhood of $x = 0$ as

\begin{equation}
Pg u = A_2(\bar{g}) \partial^2 u + \left( \frac{A_1(\bar{g})}{x} + A'_1(\bar{g}, \partial \bar{g}) \right) \partial u + \left( \frac{A_0(\bar{g})}{x^2} + \frac{A'_0(\bar{g}, \partial \bar{g})}{x} \right) u,
\end{equation}

where $A_j, A'_j$ are tensor-valued functions. Furthermore, we know that the term with second-order derivatives is scalar, and given by (8.2) (cf. also (8.4))
With \( \bar{g} = \bar{\gamma} + xw \), it then follows that \( P_{g} \) agrees with \( P_{\gamma} \) modulo terms that are subdominant at \( x = 0 \). More precisely, Theorem 7.1 yields
\[
\| (P_{g} - P_{g'})u \| \leq \sum_{k=0}^{2} \left( \left\| \frac{A_{k}(\bar{g}) - A_{k}(\bar{g}')}{x^{2-k}} \right\| + \frac{1}{x} \left\| \frac{A_{k}^{'}(\bar{g}, \bar{\theta} \bar{g}) - A_{k}^{'}(\bar{g}', \bar{\theta} \bar{g}')}{} \right\| \right) \leq C\|w - w'\|_{1,0},
\]
with \( \| \cdot \| \) denoting the \( L^{2} \) norm and the constant \( C \) depending only on the quantities discussed at the statement as a consequence of the estimates for \( u \) proved in Theorem 8.3.

To see why this is true, let us consider a term that does not depend on \( \partial \bar{g} \), such as \( A_{2}(\bar{g}) \partial^{2}u \). Observe that, as the \( L^{\infty} \) norm of \( w \) and \( \partial w \) is bounded by a constant that depends on \( \Lambda \) by Theorem 6.2, it is standard that we have
\[
|A(\bar{g}, \partial \bar{g}) - A(\bar{g}', \partial \bar{g}')| \leq C_{0}(\|w - w'| + |x\partial w - \partial w'|) .
\]

Therefore,
\[
\| (A_{2}(\bar{g}) - A_{2}(\bar{g}')) \partial^{2}u \| \leq \| (xw - xw')H(xw, xw') \partial^{2}u \|
\leq C\|\partial^{2}u\|
\leq C\|w - w'\|_{L^{\infty}}\|w\|_{L^{1, r'}}
\leq C\|w - w'\|_{1, 0} .
\]

Here \( H \) is a smooth tensor-valued function, \( r' \) is any number larger in \( \frac{n-1}{2} \) and the constant \( C \) is as above. When derivatives of \( \bar{g} \) are involved, the argument is similar. For instance,
\[
\| (A_{1}^{'}(\bar{g}, \partial \bar{g}) - A_{1}^{'}(\bar{g}', \partial \bar{g}')) \partial u \| \leq \| (w - w')H_{1} \partial u \| + \| x(\partial w - \partial w')H_{2} \partial u \|
\leq C\|\partial u\| + C\|x\partial w - \partial w'\|\|x\partial u\|_{L^{\infty}}
\leq C\|w - w'\|_{H^{1, r'}}
\leq C\|w - w'\|_{1, 0} .
\]

To conclude the proof of the proposition, let us notice that
\[
P_{g'}(u - u') = F' - F + (P_{g} - P_{g'})u .
\]
Since
\[
\| (P_{g} - P_{g'})u \|_{1,0} = \| (P_{g} - P_{g'})u \|_{L^{1,0}} \leq C\|w - w'\|_{1,0}
\]
by (8.18), Theorem 8.3 then provides the desired control for the difference \( u - u' \).}

9. Convergence of the iteration

We are now ready to prove the existence of solutions to the equation \( Q(g) = 0 \) with the desired initial and boundary conditions. With the technical tools that we have already developed, the argument is now standard.

To present the result, let us introduce a new norm that is stronger than \( \| u \|_{s, r} \) is the sense that it also includes additional (adapted) derivatives with respect to the
variable $x$. To define it, we can assume that the tensor field $u$ is supported in $\mathcal{A}$ and consider its decomposition

$$u = u^0 + u^1 + u^2 + u^3,$$

where $u^i \in V^i$. The norm is then defined using the metric $\gamma$ as

$$\|u\|_{s,r} := \|u\|_{s,r} + \sum_{j=0}^{3} \sum_{i+k+m \leq s-2} \|D^{(2+i)}x,\alpha_j u^j\|_{H^{0,r+k}}.$$

For $s = 1$ we simply take $\|u\|_{1,r} := \|u\|_{1,r}$. By Theorem 6.4 and the fact that $\alpha_j \geq n^2$, for $s < \frac{n}{2} + 1$ this is equivalent to

$$\|u\|_{s,r} := \|u\|_{s,r} + \sum_{i+k+m \leq s-2} \|\partial^k u\|_{H^{2+i,r+m}},$$

so in particular it does not depend on $\gamma$. Likewise, for $s \in \left[\frac{n}{2} + 1, \frac{n}{2} + 2\right)$ one can write

$$\|u\|_{s,r} := \|u\|_{s,r} + \sum_{j=0}^{3} \sum_{i+k+m \leq s-2} \|D^{(s)}x,\alpha_j u^j\|_{H^{0,r}} + \sum_{i+k+m \leq s-2} \|\partial^k u\|_{H^{2+i,r+m}},$$

Of course, when $u$ is not supported in $\mathcal{A}$ one defines its triple norm using a compactly supported function $\chi$ e.g. as in Eq. (6.3). It should be noticed that we will not only estimate $u$, but also $x^\rho u$, as in the bound (9.2) below. The reason for this is that this not only amounts to redistributing standard and regularized derivatives as in Proposition 6.1, but in fact allows us to control $\rho$ additional time derivatives of $u$. This will be useful to prove Theorem 1.1.

**Theorem 9.1.** Let us choose numbers $r$, $l$ and $p$ and take $\gamma \equiv \gamma_l$ as in the assumptions (i)–(iii) of Section 8. Suppose that the initial and boundary data are such that

$$\|\hat{g} - g_{2\times S^{n-1}}\|_{C^p(I \times S^{n-1})} + C_{s,r} < \delta,$$

where $C_{s,r}$ is defined by (8.7) and $\delta$ is sufficiently small. Then there is some time $T > 0$ and a function $u$ such that

$$g := \gamma + x^{\frac{n}{2}}u$$

solves the modified Einstein equation $Q(g) = 0$ in $(-T,T) \times \mathbb{B}^n$ with the specified initial and boundary conditions and is bounded as

$$\|u\|_{s,r} < C\delta.$$  \hspace{1cm} (9.1)

Furthermore, if $r > \frac{n-1}{2} + \rho$ with $\rho$ a positive integer,

$$\|x^\rho u\|_{s+\rho,r-\rho} < C\delta.$$  \hspace{1cm} (9.2)

**Proof.** For simplicity we will divide the proof in four steps. As usual, it is enough to prove the estimates in a small neighborhood $\mathcal{A}$ of the boundary. As before, we will write the metric as $\tilde{g} = \tilde{\gamma} + x^{\frac{n}{2}+2}u$ and write the equation $Q(g) = 0$ in the convenient form (5.5).

**Estimates for the source terms.** Let us begin by deriving some estimates for the functions $F(u)$ and $E(u)$ under the assumption that

$$\|u\|_{s,r} \leq \Lambda,$$  \hspace{1cm} (9.3)
where $\Lambda$ is some small constant. Notice that this ensures that we can write the metric as in Section 8 that is, as $\bar{g} = \bar{\gamma} + xw$ with $w := x^{\frac{n}{2}+1}u$ bounded in the norm $\mathcal{S}^{1}$. Throughout, we will denote by $C_0$ a constant that only depends on $\Lambda$ and $\delta$ and we will use without further mention the properties of the adapted Sobolev spaces that we established in Sections 6 and 7.

A close look at Eq. (5.4) reveals that the function $F(u)$ can be written as

$$F(u) = \frac{F(\bar{g})}{x},$$

where $F(\bar{g})$ is a smooth function of $\bar{g} := \bar{\gamma} + x^{\frac{n}{2}+2}u$ (in particular, $F(u)$ does not involve any derivatives of $u$). Hence at any fixed time we have

$$\|F(u) - F(u')\| \leq C_0 \left\| \frac{u - u'}{x} \right\| \leq C_0 \|u - u'\|_{\mathcal{H}^1},$$

where $\| \cdot \|$ again stands for the $L^2$ norm, which implies

$$\|F(u) - F(u')\|_{1,0} \leq C_0 \|u - u'\|_{1,0}.$$

Furthermore, by the elementary inequality $\|v/x\|_{\mathcal{H}^k} \leq C \|v\|_{\mathcal{H}^{k+1}}$,

$$\|F(u)\|_{s,r} = \sup_{|t| < T} \sum_{k=0}^{s-1} \left\| \partial^k_x F(u) \right\|_{\mathcal{H}^{k+1+s-r-1}} \leq C_0 \sup_{|t| < T} \sum_{k=1}^{s-1} \left\| \partial^k_t u \right\|_{\mathcal{H}^{1+s-r-k-1}} \leq C_0 \|u\|_{s,r}.$$

Using the formula for $E(u)$ given in Eq. (5.2) and computing the second derivative of $B$ as in Lemma 4.3, we infer that $E(u)$ can be symbolically written as

$$E(u) = \int_0^1 x^{\frac{n}{2}} B(u, x \partial u) \, d\sigma,$$

where $B$ is a quadratic form whose coefficients are smooth functions of $\bar{\gamma} + \sigma x^{\frac{n}{2}+2}u$ and the integral is with respect to the parameter $\sigma$. Using this formula and arguing essentially as in the case of $F(u)$ one can prove the analogous estimates

$$\|E(u) - E(u')\|_{1,0} \leq C_0 \|u - u'\|_{1,0},$$

$$\|E(u)\|_{s,r} \leq C_0 \|u\|_{s,r}.$$

Hence it stems that the function $G(u) := F(u) + E(u)$ that appears in Eq. (5.5) satisfies the same bounds, that is,

$$\|G(u) - G(u')\|_{1,0} \leq C_0 \|u - u'\|_{1,0},$$

$$\|G(u)\|_{s,r} \leq C_0 \|u\|_{s,r}.$$  \hfill (9.4)

$$\|G(u) - G(u')\|_{1,0} \leq C_0 \|u - u'\|_{1,0},$$

$$\|G(u)\|_{s,r} \leq C_0 \|u\|_{s,r}.$$  \hfill (9.5)

**Convergence in the low norm.** Our objective will be to solve the equation using the iteration

$$P_g u^{m+1} = F_0 + G(u^m),$$

where $g^m := \gamma + x^{\frac{n}{2}}u^m$ and the initial conditions that we impose are

$$u^{m+1}|_{t=0} = u_0, \quad \partial_t u^{m+1}|_{t=0} = u_1.$$  \hfill (9.6a)

$$u^{m+1}|_{t=0} = u_0, \quad \partial_t u^{m+1}|_{t=0} = u_1,$$  \hfill (9.6b)
where of course \( u_j := x^{-\frac{n}{2}}(g_j - \partial^i_j \gamma|_{t=0}) \). We can start the iteration with \( u^1 := 0 \) and the desired solution to the equation \( Q(u) = 0 \) will arise as the limit of \( u^m \) as \( m \to \infty \). Notice that we are using superscripts both for the sequence of iterates and for the components of \( u \) in the space \( \mathcal{V}_\gamma \), but this should not cause any confusion because only the former will appear in the study of the convergence of the sequence.

Let us assume that the condition (9.3) is satisfied. To prove the convergence of the sequence in the norm \( \| \cdot \|_{1,0} \), then we can use Proposition 8.4 and the estimate (9.4) to write, for \( T < T_0 \),

\[
\| u^{m+1} - u^m \|_{1,0} \leq C_0 T \| \mathcal{G}(u^m) - \mathcal{G}(u^{m-1}) \|_1 \leq C_0 T \| u^m - u^{m-1} \|_{1,0}.
\]

(9.7)

It then follows that the sequence \( (u^m)_{m=1}^{\infty} \) converges in the norm \( \| \cdot \|_{1,0} \) to some \( u \in L_\infty t \mathcal{H}_1 \cap W_1^1 t L_2 \), provided that \( T \) is smaller than some constant depending only on \( \Lambda \) and \( \delta \).

**Boundedness in the high norm.** Let us assume that the bound (9.3) is satisfied. Applying Theorem 8.3 to Eq. (9.6) immediately yields, for \( T < T_0 \),

\[
\| u^{m+1} \|_{s,r} \leq C_0 (C_{s,r} + T \| \mathcal{F}_0 \|_{s,r} + T \| \mathcal{G}(u^m)' \|_{s,r}).
\]

(9.8)

Plugging the choice of the parameters \( s, r \) and \( l \) into Theorem 4.5 one obtains that

\[
\| \mathcal{F}_0 \|_{s,r} \leq C \| \mathcal{F}_0 \|_{C^{l+r-1}(I \times \mathbb{B}^n)} \leq C \| \hat{g} - g_\mathcal{R} \|_{C^p(I \times \mathbb{B}^n-1)} < C \delta,
\]

where we have used that \( l \geq s + \frac{n}{2} + 2 \) and \( p \geq l + s + r + 1 \). If we substitute this in the inequality (9.8) and use the estimate (9.5), we arrive at

\[
\| u^{m+1} \|_{s,r} \leq C_0 (\delta + T \| u^m \|_{s,r}) \leq \Lambda/2
\]

(9.9)

(9.10)

provided that \( \delta \) and \( T \) are chosen small enough (specifically, \( T < \Lambda/(4C_0) \) and \( T < 1/(4C_0) \) will do).

Since the sequence \( (u^m) \) is bounded in \( \| \cdot \|_{s,r} \) by (9.10) and converges to \( u \) in \( \| \cdot \|_{1,0} \) by (9.7), together with the fact that these spaces possess good interpolation properties (essentially as a consequence of the formula (6.6)), we immediately obtain that \( u^m \to u \) in \( \| \cdot \|_{s',r} \) for any real \( s' < s \) and that \( u \) also satisfies the bound \( \| u \|_{s,r} \leq \Lambda/2 \). The usual argument then shows (cf. e.g. [27, Chapter 9]) that \( u \) is indeed a solution of the equation \( Q(g) = 0 \) in \( (-T, T) \times \mathbb{B}^n \), with \( T \) small enough, and that \( u \) is bounded by

\[
\| u \|_{s,r} < C \delta
\]

(9.11)

as a consequence of (9.9).

**Higher spatial regularity.** Our goal now is to show that, if \( u \) satisfies the equation \( Q(g) = 0 \), one can readily show that up to \( s \) adapted derivatives of \( u \) can then be
controlled in terms of the energy $E_{s,r}[u]$. More precisely, need to prove that

$$\sum_{j=0}^{3} \|D_{x,a_j}^{(2+j)} u^j\|_{L^\infty H^{0,r+m}} \leq C_0 \|u\|_{s,r} + C_0 \sum_{i+k+m \leq s-2} \|\partial_i^k f_0\|_{H^{i,r+m}}.$$  

(9.12)

Since $l \geq s + \frac{n}{2} + 2$ and $p \geq l + s + r + 1$, Theorem 4.5 then asserts that

$$\sum_{i+k+m \leq s-2} \|\partial_i^k f_0\|_{H^{i,r+m}} \leq C \|x^{2-s} f_0\|_{C_0^{p-1}(I \times \mathbb{R}^n)}.$$  

$$\leq C \|\tilde{g} - g_{\mathbb{R} \times \mathbb{R}^n}\|_{C_0^{p}(I \times \mathbb{R}^n)} < C \delta.$$  

Hence the desired bound (9.1) follows from the inequality (9.12) and the estimate (9.11).

The estimates (9.12) are proved by isolating the term $D_{x,a}^{(2)} u$ in the equation $Q(g) = 0$, which we write as

$$P_g u = f_0 + g(u)$$  

with $g = \gamma + x^2 u$. Once the term $D_{x,a}^{(2)} u$ has been isolated, we can take the necessary number of $x$-derivatives for which we need a priori estimates. For concreteness, let us spell out the details for the first quantity, namely the norm $\|D_{x,a}^{(2)} u\|_{L^\infty H^{0,r+s-2}}$.

From Eq. (8.5) we can write

$$D_{x,a}^{(2)} u = \frac{1}{b} \left( f_0 + \gamma + D_{x,a}^{(2)} u + (\partial_i b^1) D_{x,a}^{(2)} u - \partial_i^* (G^{ik} \partial_k u) - \frac{1}{2} \partial_i [(b^2)^i \partial_k u] \right.$$  

$$\left. - x \partial_i (b^3 \partial_k u) - \partial_i [(b^4)^i \partial_k u] + l.o.t. \right),$$  

where the superscript $j$ indicates the component in $V_j^\gamma$ and we have employed the identity (8.14) to write

$$P_g u = P_{\gamma} u + l.o.t.$$  

using the same ideas as in the proof of Proposition 8.18. Besides, we have used that, as

$$\|\tilde{g} - \tilde{\gamma}\|_{C^0} \leq \|x^2 u\|_{L^\infty} \leq C \|u\|_{s,r} < C_0 \delta,$$

Eq. (8.3) guarantees that we can indeed divide by $b^1$ to solve the equation for $D_{x,a}^{(2)} u$. To compute the norm $\|D_{x,a}^{(2)} u\|_{H^{0,r+s-2}}$ we must now consider the action of the differential operator $D_{k,\beta}$ on this equation, with $l + |\beta| \leq r + s - 2$ and $D_{k,\beta}$ defined as in (7.1). Given the dependence on $u$ of the various terms that appear in the equation, a straightforward computation shows that in fact shows that the terms that appear cannot indeed be controlled using the norm $\|u\|_{s,r}$. The norms $\|D_{x,a}^{(2)} u\|_{H^{0,r+s-2}}$ are of the symbolic form

$$\|F(u)\|_{H^{0,r+s-2}} + \|F(u) x \partial_i u\|_{H^{0,r+s-2}} + \|F(u) x \partial_i \partial_j u\|_{0,r+s-2},$$  

and these are clearly controlled by $\|u\|_{s,r}$.
Now that we have estimated \( \| \mathbf{D}_{x,a}^{(2)} u^j \|_{H^0, r + s - 2} \), which gives control over \( \| u \|_{L^\infty H^2, r + s - 2} \), we can easily obtain bounds for \( \| \mathbf{D}_{x,a}^{(2)} \partial_t^k u \|_{L^\infty H^0, r + s - 1} \) by taking time derivatives in Eq. (9.13) and repeating the argument. Estimates for the other terms \( \| \mathbf{D}_{x,a}^{(2)} \partial_t^k u \|_{L^\infty H^0, r + s} \) are then obtained by successively acting with \( \mathbf{D}_{x,a}^{(i)} \) on Eq. (9.13), with \( i = 1, 2, \ldots, s - 2 \). The only difference being that one has to use that, by the choice of the range of parameters made in the assumptions (i)-(iii), the norms \( \| \cdot \|_{H^{k,r}} \) and \( \| \cdot \|_{H^{k,r'}} \) are equivalent by Theorem 6.4 for all \( s' < \frac{n}{2} + 1 \).

**Additional time derivatives.** The proof of the a priori estimate (9.2) is, in a way, analogous to that of (9.12). If we now isolate \( \partial_t^i u \) in Eq. (9.13), we find that the component \( u^j \in V_j^i \) satisfies the equation

\[
(9.15) \quad \partial_t^2 u^j = \mathcal{G}(u^j) - \mathbf{D}_{x,a}^{(2)} u^j - (\partial_x b^1) \mathbf{D}_{x,a} u^j - \partial_t^* (G^i k \partial^i u^j) - x \partial_t^* ([b^1]^i) \partial_x u^j - \partial_t^* ([b^1]^i) \partial_t^j u^j + l.o.t.
\]

Multiplying by \( x^p \), taking \( s - 1 \) derivatives with respect to \( t \) and using the bound \( \| u \|_{s,r} < C \delta \), we immediately find that \( x^p \partial_t^{s+1} u \) satisfies

\[
\| x^p \partial_t^{s+1} u \|_{L^\infty H^0, r - 1} < C_0 \delta.
\]

Likewise, by successively taking \( 1, 2, \ldots, s - 2 + i \) time derivatives in (9.15) and repeating the argument, we readily obtain the bound

\[
\| x^p \partial_t^{s+i} u \|_{L^\infty H^0, r - i} < C_0 \delta
\]

for \( 2 \leq i \leq \rho \), which completes the proof of the theorem.

In the following corollary we show how to derive the bound in \( C^{n-2} \) that appears in the statement of Theorem 1.1.

**Corollary 9.2.** Let us denote by \( \left[ \frac{n}{2} \right] \) the largest integer smaller than \( \frac{n}{2} \). Let us take

\[
s := \left\lfloor \frac{n}{2} \right\rfloor + 2, \quad r := n + 3, \quad l := n + 4, \quad p := 2n + \left\lceil \frac{n}{2} \right\rceil + 10.
\]

Then, under the same hypotheses as Theorem 9.1, the metric \( g \) satisfies the bound

\[
\| \tilde{g} - \tilde{g}_{\text{AdS}} \|_{C^{n-2}((-T,T) \times \mathbb{R}^n)} < C \delta.
\]

**Proof.** By Theorem 4.5 we have

\[
\| \tilde{g} - \tilde{g}_{\text{AdS}} \|_{C^{n-2}} \leq \| \tilde{\gamma} - \tilde{g}_{\text{AdS}} \|_{C^{n-2}} + \| x^{\frac{n}{2} + 2} u \|_{C^{n-2}} < C \delta + \| x^{\frac{n}{2} + 2} u \|_{C^{n-2}},
\]

so it only remains to estimate \( \| x^{\frac{n}{2} + 2} u \|_{C^{n-2}} \). This can be done as follows:

\[
\| x^{\frac{n}{2} + 2} u \|_{C^{n-2}} \leq C \| x^{\frac{n}{2} + 2} u \|_{C^{n-2}} \leq C \| [x^{\frac{n}{2} + 2} u]_{H^{n-1, \frac{2n}{2} + 1}} \leq C \| [x^{\frac{n}{2} + 2} u]_{s + \left\lfloor \frac{n}{2} \right\rfloor + 2, r - \left\lfloor \frac{n}{2} \right\rfloor - 2} \|.
\]

Here we have used that \( \left\lfloor \frac{n}{2} \right\rfloor + 1 > \frac{2n}{3} \) and Corollary 6.3 to pass to the second line and we have exploited the choice of the parameters to pass to the third one.
10. DeTurck’s trick revisited

Corollary 9.2 provides a weakly asymptotically AdS metric $g$ that solves the equation $Q(g) = 0$ in $(-T,T) \times \mathbb{B}^n$, satisfies the desired initial and boundary conditions and is bounded as

$$
\|g - \bar{g}_{\text{AdS}}\|_{C^{n-2}((-T,T) \times \mathbb{B}^n)} < C\delta.
$$

Our objective in this section is to show that $g$ is also a solution of the Einstein equation $\text{Ric}(g) = -ng$, which completes the proof of Theorem 1.1. The standard way of proving this is via DeTurck’s trick. A textbook presentation of this method can be found in [27, Chapter 14], so we will only sketch the main ideas and refer to this book for further details. It should be noticed, however, that the lack of global hyperbolicity and the fact that the equations that appear are singular at the conformal boundary ensure that an additional effort is necessary to show that DeTurck’s method actually works in the situation that we are considering. Fortunately, the estimates that we have derived in the previous sections of this paper are well suited for this task.

The key idea in DeTurck’s method is that, if $Q(g) = 0$, the 1-form $W$ introduced in (3.4) to break the gauge invariance of the Einstein equation must satisfy the linear hyperbolic equation

$$
\Delta g W_\mu + R^\nu_\mu W_\nu = 0,
$$

where $R^\nu_\mu := g^{\nu\lambda}R_{\mu\lambda}$ is the tensor obtained by raising an index of the Ricci tensor of the metric $g$. When the metric $g$ is globally hyperbolic, it is immediate that if $W_\mu = 0$ and $\partial_\mu W_\mu = 0$ at $t = 0$, then $W \equiv 0$ for all time, which readily implies that the metric satisfies the Einstein equation $\text{Ric}(g) = -ng$ because of the structure of the operator $Q$.

The difficulty here is that Eq. (10.1) is not globally hyperbolic. In fact, since $g$ is weakly asymptotically AdS (which ensures that $g = x^{-2}\bar{g}$ for some $\bar{g}$ smooth enough up to the boundary and such that $|dx|_{\bar{g}} = 1$ on $(-T,T) \times \mathbb{B}^n$), a tedious computation shows that in $A$ Eq. (10.1) read as

$$
\left(3 - n\right)\frac{\partial_x W_\mu}{x} - nW_\mu + (n - 1)\bar{g}^{\nu\lambda}x_\lambda W_\nu x_\mu = \text{l.o.t.},
$$

where l.o.t. stand for terms with at most on derivative of $W$ that are smaller at $x = 0$ (i.e., they are of the form $O(1) \partial W + O(x^{-1})W$).

Let us now write $W := W^0 + W^3$, with

$$
(W^0)_\mu := \frac{\bar{g}^{\lambda\nu}x_\lambda W_\nu}{|dx|_{\bar{g}}}x_\mu.
$$

This decomposition diagonalizes (10.2) in the sense that the leading terms of the equation (both in terms of derivatives and singular behavior at the boundary) are now controlled by a scalar operator:

$$
\mathcal{L}_0 W_0 := \left(\bar{g}^{\lambda\nu}\partial_\lambda \partial_\nu + \frac{3 - n}{x} \partial_x - \frac{3n - 1}{x^2}\right)W_0 = \text{l.o.t.},
$$

$$
\mathcal{L}_3 W_3 := \left(\bar{g}^{\lambda\nu}\partial_\lambda \partial_\nu + \frac{3 - n}{x} \partial_x - \frac{2n}{x^2}\right)W_3 = \text{l.o.t.}.
$$
where again l.o.t. stands for lower-order terms that are smaller at \( x = 0 \). Setting \( W_j =: x^2 V_j \) for \( j = 0, 3 \), we can now write

\[
\mathcal{L}_j W_j =: x^2 P_j V_j ,
\]

where in \( \mathcal{A} \) the linear operator \( P_j \) reads as

\[
P_j V_j = \tilde{g}^{00} \left( \partial_t^2 + \partial_\theta^k G^{ik} \partial_\theta + x \partial_t \tilde{b}^2 \partial_x 
+ x \partial_\theta \tilde{b}^3 \partial_\theta + x \partial_x \tilde{b}^4 \partial_\theta + x \partial_t \tilde{b}^5 \partial_\theta \right) V_j 
+ \left( \tilde{b}^6 \partial_x V_j + x \tilde{b}^7 \partial_t V_j + x \tilde{b}^8 \partial_\theta V_j + \tilde{b}^9 V_j \right)
\]

with \( \alpha_0 \) and \( \alpha_3 \) defined in Eq. (4.3)

Since this has the same structure as the operator \( P_g \) considered in (8.2), a minor variation of Theorem 8.3 proves, in particular, that any solution \( V := V_0 + V_3 \) must vanish identically in \((-T, T) \times B^n\) if it has zero boundary and initial conditions. The compatibility conditions for the initial and boundary conditions guarantee that this is indeed the case (cf. Appendix [A]), so we have proved the following

**Theorem 10.1.** The metric \( g \) constructed in Theorem [9.7] (or Corollary [9.2]) solves the Einstein equation \( \text{Ric}(g) = -ng \) in \((-T, T) \times B^n\).

The main result of the paper (Theorem [1.1]) then follows.

**Appendix A. Constraint equations and compatibility conditions**

In this appendix we will explicitly write down the constraints that must be satisfied by the initial and boundary data \((g_0, g_1, \tilde{g})\) and sketch their proof.

Let us begin with the initial conditions. As is well known (see e.g. [27]), specifying the initial data \((g_0, g_1)\) is equivalent to fixing a Riemannian metric \( \tilde{g} \) on \( B^n \) and a two-tensor \( K \) which plays the role of a second fundamental form. In our case, \( \tilde{g} \) and \( K \) must be such that \( x^2 \tilde{g} \) and \( x^2 K \) can be extended continuously to the boundary.

Additionally, the following constraint equations are satisfied:

\[
\begin{align}
\tilde{R}_{\tilde{g}} - |K|^2_{\tilde{g}} + (\text{tr}_{\tilde{g}} K)^2 &= -n(n - 1), \\
\tilde{g}^{ij} \tilde{\nabla}_i K_{ji} - \tilde{\nabla}_i (\text{tr}_{\tilde{g}} K) &= 0.
\end{align}
\]

Here the quantities with tildes are computed using the Riemannian metric \( \tilde{g} \), \( \tilde{R}_{\tilde{g}} \) stands for the scalar curvature of \( \tilde{g} \) and we use latin indices to label the spatial coordinates. The proof goes exactly as in [27].

The spacetime metric at \( t = 0 \) is then given by

\[
g_0 = -\frac{dt^2}{x^2} + \tilde{g}.
\]

The condition that \( K \) be the second fundamental form of the spatial hypersurface \( \{t = 0\} \) translate into

\[
\partial_t g_{ij} \big|_{t=0} = 2K_{ij}
\]
while the time derivatives of the coefficients $g_{t\mu}$ at 0 are chosen so as to ensure that the 1-form $W$ (cf. Eq. (3.4)) vanishes at $t = 0$:

$$\partial_t g_{tt}|_{t=0} = -2\tilde{\Gamma}_0|_{t=0} - 2 \text{tr}_{\tilde{g}} K,$$

$$\partial_t g_{ti}|_{t=0} = -\tilde{\Gamma}_i|_{t=0} + \frac{1}{2} \tilde{g}^{jk}(2\partial_j \tilde{g}_{ik} - \partial_i \tilde{g}_{jk}).$$

Here $\tilde{\Gamma}_\mu := g_{\mu\nu} g^{\lambda\rho} \tilde{\Gamma}_{\lambda\rho}^{\nu}$, so its value at 0 is determined by (A.2). It is known that the constraint equations together with the above choice of initial conditions imply that $\nabla_\mu W_\nu|_{t=0} = 0$. The above equations provide the structure of $g_1$ in terms of $\tilde{g}$ and $K$.

Additionally, there are compatibility conditions between the initial conditions $(g_0, g_1)$ and the boundary datum $\tilde{g}$. As is well-known, solving the Einstein equation in a bounded domain with nontrivial boundary conditions on the boundary is usually problematic (see e.g. [18] and references therein). Fortunately, in this paper we can exploit the fact that the metric we want to construct is asymptotically anti-de Sitter to obtain a manageable, sufficient set of compatibility conditions: we only need to impose that the functions $u_k$, defined in (5.6) and (8.6), belong to $H^1_{\infty}$ for $0 \leq k \leq s - 1$ and to $L^2$ for $k = s$. Indeed, this integrability condition at infinity is enough to ensure that the arguments in the paper make sense, essentially because we can integrate by parts in the proof of Theorem 8.3.

Of course, there is some extra room to refine this condition because we have used the metric $g_{\text{AdS}}$ to define the tensor field $\gamma$: natural though it is, we could have taken a very small perturbation of $g_{\text{AdS}}$ instead. Still, these sufficient compatibility conditions are enough to show that there are many admissible initial and boundary conditions $(\tilde{g}, g_0, g_1)$. The most interesting case is probably the one mentioned in the Introduction: $g_0 = g_{\text{AdS}}$, $g_1 = 0$ and $\tilde{g}$ identical with $g_{\mathbb{R} \times S^{n-1}}$ for $t \leq 0$.

**Appendix B. Some estimates for the operators $A_\alpha$ and $A_\alpha^*$**

The integral operators $A_\alpha$ and $A_\alpha^*$, defined in (6.7), play a key role in some arguments presented in Sections 6 and 7. Therefore we will record here some estimates the we proved in [15, Theorem 3.1 and Proposition 3.3], where as usual we assume that $\alpha > 1$. For the benefit of the reader, we also include a sketch of the proof.

**Theorem B.1** ([15]). The following statements hold:

(i) Acting on one-variable functions, the operators $A_\alpha$ and $A_\alpha^*$ define continuous maps

$$L^2_x \to L^\infty_x.$$  

(ii) The operators $\frac{1}{x} A_\alpha$ and $\frac{1}{x} A_\alpha^*$ are continuous maps

$$L^2_x \to L^2_x$$ and $$L^2 \to L^2.$$  

(iii) If $u$ is a function in $L^2(\mathbb{A})$ with $D_{x,\alpha} u$ in $L^2(\mathbb{A})$, then

$$u(x, \theta) = (A_\alpha D_{x,\alpha} u)(x, \theta).$$  

(iv) If $u$ is a function in $L^2(\mathbb{A})$ with $D_{x,\alpha} u$ in $L^2(\mathbb{A})$, then

$$u(x, \theta) = (A_\alpha^* D_{x,\alpha}^* u)(x, \theta) + f(\theta) x^{\alpha-1}.$$
the function \( f(\theta) \) being bounded in \( L^2_\theta \equiv L^2(S^{n-1}) \) by
\[
\|f\|_{L^2_\theta} \leq C(\|u\|_{L^2} + \|D_{x,\alpha}u\|_{L^2})
\]

**Proof.** We can assume that \( u \) is smooth and supported in the region \( 0 < x < 1 \).

Let us begin analyzing the mapping properties of \( A^*_\alpha \).

In view of the expression for \( A^*_\alpha \), we will use the Hardy inequality
\[
(B.1) \quad \int_0^1 x^{2\alpha-2r-1} \left( \int_x^1 y^{1-\alpha} \varphi(y) \, dy \right)^2 \, dx \leq C \int_0^1 x^{3-2r} \varphi(x)^2 \, dx,
\]
where \( r = 0,1 \). To prove this, let us set
\[
\psi(x) := \int_x^1 y^{1-\alpha} \varphi(y) \, dy.
\]

Then integrating by parts and using the Cauchy–Schwarz inequality we find
\[
\int_0^1 x^{2\alpha-2r-1} \psi^2 \, dx = \frac{1}{\alpha-1} \int_0^1 \varphi \psi x^{\alpha-2r-1} \, dx
\]
\[
= \frac{1}{\alpha-r} \int_0^1 (x^{\alpha-r-2} \psi) (x^{\frac{3}{2}-r} \varphi) \, dx
\]
\[
\leq \frac{1}{\alpha-r} \left( \int_0^1 x^{2\alpha-2r-1} \psi^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^1 x^{3-2r} \varphi^2 \, dx \right)^{\frac{1}{2}}.
\]

This proves \( (B.1) \). This implies that, with \( r = 0,1 \), \( \frac{1}{\pi} A^*_\alpha \) is a bounded map
\[
L^2((0,1),x^{3-2r} \, dx) \rightarrow L^2((0,1),x^{1-2r} \, dx),
\]
and with \( r = 1 \) this implies that \( \frac{1}{\pi} A^*_\alpha : L^2_x \rightarrow L^2_x \). Since the star denotes the adjoint with respect to the \( L^2_x \) product, a standard duality argument then ensures that \( A^*_\alpha \) is a bounded map
\[
L^2((0,1),x^{1+2r} \, dx) \rightarrow L^2((0,1),x^{2r-1} \, dx),
\]
which with \( r = 0 \) implies that \( \frac{1}{\pi} A^*_\alpha : L^2_x \rightarrow L^2_x \). The fact that this also corresponds to \( L^2 \rightarrow L^2 \) bounds is immediate.

Let us now pass to the pointwise bounds. To prove (i) for \( A^*_\alpha \), we utilize the Cauchy-Schwarz inequality to write
\[
\left| A^*_\alpha \varphi(x) \right| = x^{\alpha-1} \left| \int_x^1 y^{1-\alpha} \varphi(y) \, dy \right|
\]
\[
\leq x^{\alpha-1} \left( \int_x^1 y^{1-2\alpha} \, dy \right)^{\frac{1}{2}} \left( \int_x^1 y \varphi(y)^2 \, dy \right)^{\frac{1}{2}}
\]
\[
\leq \|\varphi\|_{L^2_x} \left( \frac{1-x^{\alpha-1}}{2-2\alpha} \right)^{1/2}
\]
\[
\leq (2-2\alpha)^{-\frac{1}{2}} \|\varphi\|_{L^2_x}.
\]

The \( L^\infty_x \) estimate for \( A^*_\alpha \) is similar.

To prove (iv), notice that if \( u_1 := D^*_{x,\alpha} u \in L^2 \), we can solve the ODE
\[
D^*_{x,\alpha} u = u_1
\]
to write

\[ u = A^*_\alpha(u_1) + f(\theta) x^{\alpha-1} \]

for some function \( f(\theta) \). Moreover,

\[ \|f\|_{L^2} \leq C \|f(\theta) x^{\alpha-1}\|_{L^2} \leq C(\|u\|_{L^2} + \|A^*_\alpha(u_1)\|_{L^2}) \leq C(\|u\|_{L^2} + \|u_1\|_{L^2}) , \]

where we have used that \( A^*_\alpha : L^2 \to L^2 \) by (ii). To prove (iii), the reasoning is analogous: again we can solve the ODE

\[ D_{x,\alpha}u = u_2 \]

to write

\[ u = A_\alpha(u_2) + f_2(\theta) x^{-\alpha} , \]

but we infer that \( f_2 \) must be 0 because \( x^{-\alpha} \) is not in \( L^2_x \). The theorem then follows. \( \square \)

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