POSITIVE AND SIGN-CHANGING CLUSTERS AROUND SADDLE POINTS OF THE POTENTIAL FOR NONLINEAR ELLIPTIC PROBLEMS

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Abstract. We study the existence and asymptotic behavior of positive and sign-changing multipeak solutions for the equation

$$-\varepsilon^2 \Delta v + V(x)v = f(v) \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon$ is a small positive parameter, $f$ is a superlinear, subcritical and odd nonlinearity, $V$ is a uniformly positive potential. No symmetry on $V$ is assumed. It is known ([19]) that this equation has positive multipeak solutions with all peaks approaching a local maximum of $V$. It is also proved that solutions alternating positive and negative spikes exist in the case of a minimum (see [9]). The aim of this paper is to show the existence of both positive and sign-changing multipeak solutions around a nondegenerate saddle point of $V$.

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1. Introduction

This paper deals with the following nonlinear perturbed elliptic equation

$$-\varepsilon^2 \Delta v + V(x)v = |v|^{p-2}v \quad \text{in } \mathbb{R}^N$$

where $N \geq 2$, $\varepsilon$ is a small parameter, the potential $V \in C^1(\mathbb{R}^N, \mathbb{R})$ is bounded from below away from zero, the exponent $p$ satisfies $2 < p < \frac{2N}{N-2}$ if $N \geq 3$ and $p > 2$ if $N = 2$. This equation arises when one looks for standing waves of the nonlinear Schrödinger equation

$$i\varepsilon \frac{\partial \psi}{\partial t} = -\varepsilon^2 \Delta \psi + V(x)\psi - |\psi|^{p-2}\psi,$$

which appears in different problems in nonlinear optics, in plasma physics, etc.

Equation (1.1) has attracted much attention: a large number of works are concerned with the question of semiclassical limit, that is, the behaviour of solutions when $\varepsilon$ tends to zero. This has an important physical interest since letting $\varepsilon$ go to zero formally describes the transition from Quantum Mechanics to Classical Mechanics. It has been shown that if $P_0$ is a nondegenerate or, more generally, a topologically nontrivial critical point of $V$, there exists a family of solutions $v_\varepsilon$ which develops a single spike near $P_0$ as $\varepsilon \to 0$ ([2], [1], [10], [12], [17], [20], [23], see [3] for further references). Also, when $V$ has several critical points, multi-peaks have been constructed with each peak concentrating at a separate critical point (see [11], [14], [18], [22] and references therein).

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In this paper we are interested in a special kind of solution for equation (1.1), the so called \textit{cluster}, i.e. a combination of several interacting peaks concentrating at the same point as \( \varepsilon \to 0^+ \). In [19] Kang and Wei construct this kind of solution: more precisely, given \( \ell \geq 1 \) and \( P_0 \) a strict local maximum of \( V \), there exists a positive \textit{cluster} with \( \ell \) peaks concentrating at \( P_0 \). They also prove that such solutions do not exist around nondegenerate minimum points of \( V \). After that, several papers have addressed the question of existence of multibump solutions concentrating around a minimum of \( V \). This result has become known first for exactly one positive and one negative peak ([1], [5]), and later under polygonal symmetries of \( V \) ([8]), or in the one-dimensional case ([13]). In a recent paper \textit{clusters} with at most 6 mixed positive and negative peaks have been found, see [9].

All previous results are concerned with the existence of a \textit{clustered} solution localized around a minimum or a maximum point of \( V \). So the question of whether other critical points of \( V \) may generate a \textit{cluster} or not arises naturally. The aim of this paper is to construct both positive and sign-changing \textit{clusters} around a nondegenerate saddle point of \( V \).

In order to provide the exact formulation of our results let us fix some notation. We point out that the class of nonlinearities \( f \) satisfying (f1)-(f3) includes, and it is not restricted to, the model \( f(v) = |v|^{p-2}v \) with \( p > 1 + \sqrt{2} \) if \( N = 1, 2 \) and \( p \in \left(1 + \sqrt{2}, \frac{2N}{N-2}\right) \) if \( N \in [3, 11] \) (if \( N \geq 12 \) the interval is empty). Other nonlinearities can be found in [6].
Let us now state the hypotheses on the potential $V$ that will be used.

V1) $V \in C^2(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V > 0$.

V2) $V$ has a nondegenerate saddle point at $P_0$ and, without loss of generality, we may assume $V(P_0) = 1$. We define $r \in \{1, \ldots, N-1\}$ as the number of positive eigenvalues of $D^2V(P_0)$, counted with their multiplicity.

As already mentioned, in this paper we give two results. First, for any fixed positive integer $\ell$ there exists a $\ell$-peak positive clustered solution concentrating at $P_0$. Furthermore each peak has a profile similar to $w$ suitably rescaled. More precisely we will prove the following theorem.

**Theorem 1.1.** Assume that hypotheses (f1)–(f3) and (V1)–(V2) hold and let $\varepsilon > 0$ sufficiently small, the equation (1.2) has a positive solution $v_\varepsilon \in H^1(\mathbb{R}^N)$.

Furthermore there exist $P_1^\varepsilon, \ldots, P_\ell^\varepsilon \in \mathbb{R}^N$ such that, as $\varepsilon \to 0^+$,

(i) $v_\varepsilon(x) = \sum_{i=1}^{\ell} w\left(\frac{x-P_i^\varepsilon}{\varepsilon}\right) + o(\varepsilon)$ uniformly for $x \in \mathbb{R}^N$;

(ii) $|P_i^\varepsilon - P_j^\varepsilon| \geq 2\beta \varepsilon \log \frac{1}{\varepsilon}$ (i $\neq$ j) and $|P_i^\varepsilon - P_0| \leq \varepsilon^\beta$ for any fixed $\beta \in (0,1)$.

Secondly, we prove that the equation (1.2) possesses a cluster with $h$ positive peaks and $k$ negative peaks approaching $P_0$, where $h$ and $k$ are integers under some restrictions. The exact formulation of the result is the following.

**Theorem 1.2.** Assume that $N \geq 2$ and hypotheses (f1)–(f3) and (V1)–(V2) hold. Let $h, k$ satisfying

\[ h, k \geq 1, \quad \ell := h + k \leq 6. \]

(i) If $r \geq 2$, then, for $\varepsilon > 0$ sufficiently small, the equation (1.2) has a solution $v_\varepsilon \in H^1(\mathbb{R}^N)$.

Furthermore there exist $P_1^\varepsilon, \ldots, P_\ell^\varepsilon \in \mathbb{R}^N$ such that, as $\varepsilon \to 0^+$,

- $\bullet$ $v_\varepsilon(x) = \sum_{i=1}^{h} w\left(\frac{x-P_i^\varepsilon}{\varepsilon}\right) - \sum_{i=h+1}^{\ell} w\left(\frac{x-P_i^\varepsilon}{\varepsilon}\right) + o(\varepsilon)$ uniformly for $x \in \mathbb{R}^N$;

- $\bullet$ $|P_i^\varepsilon - P_j^\varepsilon| \geq 2\beta \varepsilon \log \frac{1}{\varepsilon}$ (i $\neq$ j) and $|P_i^\varepsilon - P_0| \leq \varepsilon^\beta$ for any fixed $\beta \in (0,1)$.

(ii) If $r = 1$, the same result as (i) holds with the additional assumption $k \in \{h-1, h, h+1\}$.

We point out that positive clustered solutions have also been found for the following equation

\[-\varepsilon^2 \Delta v + v = Q(x)|v|^{p-2}v \quad \text{in} \quad \mathbb{R}^N\]

around critical points of $Q$ ([7], [21]). In particular, as far as we know, the only work regarding clusters concentrating near a saddle point is [4]. However we are unaware of cluster phenomena with mixed positive and negative peaks near a saddle point. Theorem 1.2 seems to be the first result in this line.

The proofs of Theorem 1.1 and 1.2 rely on perturbation arguments, which combine the variational approach with a Lyapunov-Schmidt type procedure. A sketch of this procedure is given in Section 2. Throughout the paper we will need some asymptotic estimates, made in detail in Appendix A. With this estimates in hand and thanks to the non-degeneracy condition (1.4), we can use the contraction mapping principle to solve the auxiliary equation. Since the computations are quite technical, they have been postponed to Appendix B.
In Section 3 we are concerned with the finite dimensional bifurcation equation. Alternatively, we look for critical points of an associated reduced functional. This is the main difficulty of our problem; here the reduced functional has a quite involved behaviour due to the different interactions of the potential and the bumps.

It seems not easy to find the exact position of the bumps in a direct way, if no symmetry assumptions are made. In this paper we use a max-min technique applied to the reduced functional in the spirit of [9]. This max-min argument is far from obvious, specially in the case of sign-changing solutions. It takes into account the interaction among the bumps (which depends on their respective sign) and the effect of $V$ on each bump (which depends on its spatial displacement).

**NOTATION:** Throughout the paper we will often use the notation $C$ to denote generic positive constants. The value of $C$ is allowed to vary from place to place.

2. The reduction process: sketch of the proof

In this section we outline the main steps of the so called finite dimensional reduction, which reduces the problem to finding a critical point for a functional on a finite dimensional space. We postpone the proofs and details to Appendix A and Appendix B.

Associated to (1.2) is the following energy functional:

$$I_{\varepsilon} : H^1_V(\mathbb{R}^N) \to \mathbb{R}, \quad I_{\varepsilon}[v] := \frac{1}{2} \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla v|^2 + V(x)|v|^2) \, dx - \int_{\mathbb{R}^N} F(v) \, dx.$$  

where $F(t) = \int_0^t f(s) \, ds$ and

$$H^1_V(\mathbb{R}^N) = \{ v \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)|v|^2 \, dx < \infty \}.$$  

Let us equip $H^1_V(\mathbb{R}^N)$ with the following scalar product:

$$(u, v)_\varepsilon = \int_{\mathbb{R}^N} (\varepsilon^2 \nabla u \nabla v + V(x)uv) \, dx.$$  

It is well known that $I_{\varepsilon} \in C^2(H^1_V(\mathbb{R}^N), \mathbb{R})$ and the critical points of $I_{\varepsilon}$ are the finite-energy solutions of (1.2).

Without loss of generality we assume throughout the paper that $P_0 = 0$. Moreover, after suitably rotating the coordinate system, we may assume that in a small neighborhood of 0 the following expansion holds:

$$V(x) = 1 + \frac{1}{2} \sum_{n=1}^N \lambda_n x_n^2 + o(|x|^2) \text{ as } x \to 0,$$

where $\lambda_n > 0$ for $n = 1, \ldots, r$, $\lambda_n < 0$ for $n = r + 1, \ldots, N$.

Consider $M^+, \ M^- \in \mathbb{R}^{N^2}$ the following diagonal matrices

$$M^+ = \text{diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0) \quad M^- = \text{diag}(0, \ldots, 0, |\lambda_{r+1}|, \ldots, |\lambda_N|),$$

and set

$$M = M^+ - M^- = D^2V(0) = \text{diag}(\lambda_1, \ldots, \lambda_N), \quad \overline{M} = M^+ + M^- = \text{diag}(|\lambda_1|, \ldots, |\lambda_N|).$$
Next for $\ell \geq 2$ define the configuration space:

$$\Gamma_\varepsilon = \left\{ \mathbf{P} = (P_1, \ldots, P_\ell) \in \mathbb{R}^{N\ell} \mid \sum_{i} (\varepsilon^2 \lambda_i)^{1/2} \phi_i < \varepsilon \beta, \quad \varepsilon \left( \frac{P_i - P_j}{\varepsilon} \right) < \varepsilon^{2\beta} \quad \text{for } i \neq j \right\},$$

where $\beta \in (0, 1)$ is a number sufficiently close to 1.\footnote{Observe that $\Gamma_\varepsilon$ is nonempty, since for $\varepsilon$ sufficiently small $\{\mathbf{P} \mid |P_i| \leq \varepsilon \log^2 \frac{1}{\varepsilon}, |P_i - P_j| \geq 2\beta \varepsilon \log \frac{1}{\varepsilon} \quad \text{for } i \neq j \} \subset \Gamma_\varepsilon$ thanks to assumption (V2) and (1.3).}

Observe that, according to (1.3),

$$\Gamma_\varepsilon \subset \left\{ \mathbf{P} = (P_1, \ldots, P_\ell) \in \mathbb{R}^{N\ell} \mid |P_i| \leq \left( \min_i |\lambda_i| \right)^{-1/2} \varepsilon^\beta \quad \text{for } i \neq j \right\}.$$

For $\mathbf{P} = (P_1, \ldots, P_\ell) \in \Gamma_\varepsilon$ set

$$w_{\mathbf{P}}(x) = w \left( \frac{x - P_i}{\varepsilon} \right), \quad w_{\mathbf{P}} = \sum_{i=1}^{\ell} \tau_i w_{P_i}, \quad \tau_i \in \{-1, +1\}.$$

Let $\chi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function such that $\chi(x) = 1$ if $|x| < 1$, so that one has $\chi w_{\mathbf{P}} \in H^1_V(\mathbb{R}^N)$.

We look for a solution to (1.2) in a small neighbourhood of the first approximation $\chi w_{\mathbf{P}}$, i.e. a solution of the form as $v := \chi w_{\mathbf{P}} + \phi$, where the rest term $\phi$ is small. To this aim we introduce the following functions:

$$Z_{P_i, n} = (V(x) - \varepsilon^2 \Delta) \frac{\partial (\chi w_{P_i})}{\partial x_n}, \quad i \in \{1, \ldots, \ell\}, \quad n \in \{1, \ldots, N\}.$$

The object is to solve the following nonlinear problem: given $\mathbf{P} = (P_1, \ldots, P_\ell) \in \Gamma_\varepsilon$, find $(\phi, \alpha_{in})$ such that

$$\begin{cases} S_\varepsilon[\chi w_{\mathbf{P}} + \phi] = \sum_{i,n} \alpha_{in} Z_{P_i, n}, \\ \phi \in H^2(\mathbb{R}^N) \cap H^1_V(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \phi Z_{P_i, n} \, dx = 0, \quad i = 1, \ldots, \ell, \quad n = 1, \ldots, N, \end{cases}$$

where

$$S_\varepsilon[v] = \varepsilon^2 \Delta v - V(x)v + f(v).$$

**Lemma 2.1.** Set $\eta = \beta^2(1 + \sigma)$. Provided that $\varepsilon > 0$ is sufficiently small, for every $\mathbf{P} \in \Gamma_\varepsilon$ there is a pair $(\phi_{\mathbf{P}}, \alpha_{in}(\mathbf{P})) \in \left( H^2(\mathbb{R}^N) \cap H^1_V(\mathbb{R}^N) \right) \times \mathbb{R}^{N\ell}$ satisfying (2.3) and

$$\|\phi_{\mathbf{P}}\|_{H^1} \leq \varepsilon \eta, \quad (\phi_{\mathbf{P}}, \phi_{\mathbf{P}})_\varepsilon \leq \varepsilon^{N + 2\eta}, \quad |\alpha_{in}(\mathbf{P})| \leq \varepsilon^{1+\eta}.$$

Moreover the map $\mathbf{P} \in \Gamma_\varepsilon \mapsto \phi_{\mathbf{P}} \in H^1_V(\mathbb{R}^N)$ is $C^1$.

We refer to Appendix B for the proof.

For $\varepsilon > 0$ sufficiently small consider the reduced functional

$$J_\varepsilon : \Gamma_\varepsilon \to \mathbb{R}, \quad J_\varepsilon[\mathbf{P}] := \varepsilon^{-N} I_\varepsilon[\chi w_{\mathbf{P}} + \phi_{\mathbf{P}}] - c_1,$$

where $\phi_{\mathbf{P}}$ has been constructed in Lemma 2.1 and $c_1 = \frac{\ell}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) \, dx - \ell \int_{\mathbb{R}^N} F(w) \, dx$. The next proposition contains the key expansions of $J_\varepsilon$ and $\nabla J_\varepsilon$ (see Appendix B for the proof.).
Proposition 2.2. The following expansions hold:

\begin{equation}
J_\varepsilon[P] = \frac{c_2}{2} \sum_{i=1}^{\ell} M[P_i]^2 - \frac{1}{2} \varepsilon^{-N} \sum_{i \neq j} \tau_i \tau_j \int_{\mathbb{R}^N} f(w_{P_i}) w_{P_j} \, dx + o(\varepsilon^{2\beta}),
\end{equation}

\begin{equation}
\frac{\partial J_\varepsilon}{\partial P_i}[P] = c_2 M[P_i]^2 - \varepsilon^{-N} \sum_{j \neq i} \tau_i \tau_j \frac{\partial}{\partial P_i} \left[ \int_{\mathbb{R}^N} f(w_{P_i}) w_{P_j} \, dx \right] + o(\varepsilon^3), \quad i = 1, \ldots, \ell
\end{equation}

uniformly for $P \in \Gamma_\varepsilon$, where $c_2 = \frac{1}{2} \int_{\mathbb{R}^N} w^2 \, dx$.

By Lemma A.2 (see Appendix A), we have also the following expansion:

\begin{equation}
J_\varepsilon[P] = \frac{c_2}{2} \sum_{i=1}^{\ell} M[P_i]^2 - \frac{c_3}{2} \sum_{i \neq j} \tau_i \tau_j w \left( \frac{P_i - P_j}{\varepsilon} \right) + o(\varepsilon^{2\beta}),
\end{equation}

uniformly for $P \in \Gamma_\varepsilon$, where $c_3 = \int_{\mathbb{R}^N} f(w) e^{x_1} \, dx$.

Finally the next lemma concerns the relation between the critical points of $J_\varepsilon$ and those of $I_\varepsilon$. It is quite standard in singular perturbation theory; its proof can be found in [3], for instance.

Lemma 2.3. Let $P_\varepsilon \in \Gamma_\varepsilon$ be a critical point of $J_\varepsilon$. Then, provided that $\varepsilon > 0$ is sufficiently small, the corresponding function $v_\varepsilon = \chi_{P_\varepsilon} + \phi_{P_\varepsilon}$ is a solution of (1.2).

So, we conclude the proof by showing the existence of a critical point of $J_\varepsilon$. This will be accomplished in next section.

3. A max-min argument: proof of Theorem 1.1 and Theorem 1.2

In this section we apply a max-min argument to characterize a topologically nontrivial critical value of $J_\varepsilon$. More precisely we will construct sets $D_\varepsilon$, $K$, $K_0 \subset \mathbb{R}^{N \ell}$ satisfying the following properties:

(P1) $D_\varepsilon$ is an open set, $K_0$ and $K$ are compact sets, $K$ is connected and $K_0 \subset K \subset D_\varepsilon \subset \overline{D_\varepsilon} \subset \Gamma_\varepsilon$;

(P2) if we define the complete metric space $\mathcal{F}$ by

$$\mathcal{F} = \{ \eta : K \rightarrow D_\varepsilon \mid \eta \text{ continuous}, \eta(P) = P \forall P \in K_0 \},$$

then

\begin{equation}
J^*_\varepsilon := \sup_{\eta \in \mathcal{F}} \min_{P \in K_0} J_\varepsilon[\eta(P)] < \min_{P \in K_0} J_\varepsilon[P].
\end{equation}

(P3) For every $P \in \partial D_\varepsilon$ such that $J_\varepsilon[P] = J^*_\varepsilon$, we have that $\partial D_\varepsilon$ is smooth at $P$ and there exists a vector $\tau_P$ tangent to $\partial D_\varepsilon$ at $P$ so that $J'_\varepsilon[P](\tau_P) \neq 0$.

Under these assumptions a critical point $P_\varepsilon \in D_\varepsilon$ of $J_\varepsilon$ with $J_\varepsilon[P_\varepsilon] = J^*_\varepsilon$ exists, as a standard deformation argument involving the gradient flow of $J_\varepsilon$ shows.

We define

$$D_\varepsilon = \left\{ P \in \mathbb{R}^{N \ell} \mid \left| \sum_{i=1}^{\ell} M[P_i]^2 + \sum_{i \neq j} \tau_i \tau_j \frac{P_i - P_j}{\varepsilon} \right| < c_4 \varepsilon^{2\beta} \right\}.$$
where \( c_4 = \min\{c_2, c_3\} \). We immediately get \( \Gamma \subset \Gamma_\varepsilon \). In the following we will denote by \( A \) and \( B \) the subspaces associated to the positive and negative eigenvalues of \( M \) respectively, whose direct sum is \( \mathbb{R}^N \), i.e.

\[
A = \text{span}\{e_1, \ldots, e_r\}, \quad B = \text{span}\{e_{r+1}, \ldots, e_N\},
\]

where \( e_1, \ldots, e_N \) is the standard basis in \( \mathbb{R}^N \).

### 3.1. Definition of \( K, K_0 \), and proof of (P1)-(P2)

In this subsection we define the sets \( K, K_0 \) for which properties (P1)-(P2) hold. In addition, we will prove that

\[
J_\varepsilon = o(\varepsilon^{2\beta}).
\]

For the sake of clarity we distinguish the case of positive peaks from that of mixed positive and negative peaks.

#### 3.1.1. I case: \( k = 0, \ell = h \geq 1 \)

We have \( \tau_i = 1 \) for all \( i = 1, \ldots, \ell \). Let us fix \( b_1, \ldots, b_\ell \in B \) such that

\[
|b_i| \leq 2\ell \varepsilon \log \frac{1}{\varepsilon}, \quad \forall i,
\]

and define the following convex open set \( U \) of \( A^\ell \):

\[
U = \left\{ (a_1, \ldots, a_\ell) \in A^\ell \left| c_2 \sum_{i=1}^{\ell} M^+[a_i]^2 < \frac{c_4}{2} \varepsilon^{2\beta} \right. \right\},
\]

and

\[
K = \left\{ P = (a_1 + b_1, \ldots, a_\ell + b_\ell) \in \mathbb{R}^{N\ell} \left| (a_1, \ldots, a_\ell) \in U \right. \right\},
\]

\[
K_0 := \left\{ P = (a_1 + b_1, \ldots, a_\ell + b_\ell) \in \mathbb{R}^{N\ell} \left| (a_1, \ldots, a_\ell) \in \partial U \right. \right\}.
\]

\( K \) is clearly isomorphic to \( U \) by the immediate isomorphism

\[
(a_1 + b_1, \ldots, a_\ell + b_\ell) \in K \longmapsto (a_1, \ldots, a_\ell) \in U
\]

and \( K_0 \approx \partial U \). \( K_0 \) and \( K \) are compact sets, \( K \) is connected (since \( U \) is convex) and \( K_0 \subset K \). Furthermore \( M[a_i + b_i]^2 = M^+[a_i]^2 - M^-[b_i]^2 = M^+[a_i]^2 + O(\varepsilon^2 \log^2 \frac{1}{\varepsilon}) \), analogously \( M[a_i + b_i]^2 = M^+[a_i]^2 + M^-[b_i]^2 = M^+[a_i]^2 + O(\varepsilon^2 \log^2 \frac{1}{\varepsilon}) \) and, since \( w \) is decreasing in \( |x| \),

\[
w(a_i + b_i - a_j - b_j) \leq w \left( \frac{b_i - b_j}{\varepsilon} \right) = o(\varepsilon^2)
\]

for \( i \neq j \).

Then we deduce \( K \subset \mathcal{D}_\varepsilon \) and, by Proposition 2.2

\[
J_\varepsilon(P) = \frac{c_2}{2} \sum_{i=1}^{\ell} M^+[a_i]^2 + o(\varepsilon^{2\beta}) \text{ uniformly on } K,
\]

by which, since \( c_2 \sum_{i=1}^{\ell} M^+[a_i]^2 = \frac{c_4}{4} \varepsilon^{2\beta} \) if \( (a_1, \ldots, a_\ell) \in \partial U \),

\[
J_\varepsilon(P) = \frac{c_4}{4} (1 + o(1)) \varepsilon^{2\beta} \text{ uniformly on } K_0.
\]

Let \( \eta \in \mathcal{F} \), namely \( \eta : K \rightarrow \mathcal{D}_\varepsilon \) is a continuous function such that \( \eta(P) = P \) for any \( P \in K_0 \). Then we can compose the following maps

\[
A^\ell \supset U \longmapsto K \xrightarrow{\eta} \eta(K) \subset \mathcal{D}_\varepsilon \xrightarrow{(\pi_A)^\ell} A^\ell,
\]
denoting by \( \pi_A \) the orthogonal projection of \( \mathbb{R}^N \) onto \( A \), and we call \( T : U \to A^\ell \) the resulting composition. \( T \) is a continuous map. We claim that \( T = id \) on \( \partial U \). Indeed, if \((a_1, \ldots, a_\ell) \in \partial U \), then \((a_1 + b_1, \ldots, a_\ell + b_\ell) \in K_0\), consequently \( \eta(a_1 + b_1, \ldots, a_\ell + b_\ell) = a_1 + b_1, \ldots, a_\ell + b_\ell \), by which
\[
T(a_1, \ldots, a_\ell) = (\pi_A)^*(a_1 + b_1, \ldots, a_\ell + b_\ell) = (a_1, \ldots, a_\ell).
\]
Since \( 0 = (0, \ldots, 0) \in U \), hence the theory of the topological degree ensures that \( \text{deg}(T, U, 0) = \text{deg}(id, U, 0) = 1 \). Then there exists \((\bar{a}_1, \ldots, \bar{a}_\ell) \in U \) such that \( T(\bar{a}_1, \ldots, \bar{a}_\ell) = 0 \), i.e. \( P^\eta := \eta(\bar{a}_1 + b_1, \ldots, \bar{a}_\ell + b_\ell) \in B^\ell \). Using Proposition 2.2 we get
\[
\min_{P \in K} J_\varepsilon[\eta(P)] \leq J_\varepsilon[P^\eta] = \frac{C_2}{2} \sum_{i=1}^\ell M^- |P_i^\eta|^2 - \frac{C_3}{2} \sum_{i \neq j} \| (P_i^\eta - P_j^\eta) \| + o(\varepsilon^{2\beta}) \leq o(\varepsilon^{2\beta}).
\]
Hence
\[
J_\varepsilon^* = \sup_{\eta \in F} \min_{P \in K} J_\varepsilon[\eta(P)] \leq o(\varepsilon^{2\beta}).
\]
On the other hand, by taking \( \eta = id \) and using \( 3.16 \),
\[
J_\varepsilon^* \geq \min_{P \in K} J_\varepsilon[P] \geq o(\varepsilon^{2\beta}).
\]
Combining last two estimates we get \( 3.15 \). Finally comparing \( 3.15 \) with \( 3.17 \), the max-min inequality \( 3.14 \) follows.

3.1.2. II case: \( h, k \geq 1, \ell : h + k \leq 6 \). For the sake of simplicity assume \( h \geq k \). In order to define \( K \) and \( K_0 \), we need to consider special type of configurations \( P(a, r) \in \mathbb{R}^{N\ell} \). To this aim it is convenient to distinguish three cases.

(i) \( k = h \) or \( k = h - 1 \).

We set \( \tau_i = (-1)^i + 1 \). Let us fix \( v \in A \) such that \( |v| = 1 \). Then we consider the configurations \( P \) which lie in \( A \) and are aligned in the direction \( v \) with alternating sign, i.e. configurations of the form
\[
P = P(a, r) = \left( \begin{array}{c} a \\ a + r_2v \\ a + (r_2 + r_3)v \\ \vdots \\ a + (r_2 + \ldots + r_\ell)v \end{array} \right) \in \mathbb{R}^{N\ell},
\]
where \( a \in A \) and \( r = (r_2, \ldots, r_\ell) \in (0, +\infty)^{\ell-1} \). Observe that by construction we have \( |P_i - P_j| = r_{j+1} + r_{j+2} + \ldots + r_i \) if \( i > j \), therefore
\[
r_i = |P_i - P_{i-1}| = \min_{j \leq i} |P_i - P_j| = \min_{j \leq i, \tau_j = -\tau_i} |P_i - P_j| \quad \forall i = 2, \ldots, \ell.
\]
Moreover, if \( i > j \) and \( \tau_i \tau_j = 1 \), then \( i \geq j + 2 \), and consequently \( |P_i - P_j| \geq r_i + r_{i-1} \), by which
\[
|P_i - P_j| \geq 2 \min_{2 \leq s \leq \ell} r_s \quad \text{if } \tau_i \tau_j = 1, i \neq j.
\]

(ii) \( k = 1, 2 \leq h \leq 5, r \geq 2 \)

We set \( \tau_1 = -1, \tau_i = 1 \) for \( i = 2, \ldots, \ell \). Let us fix \( v_2, \ldots, v_\ell \in A \) such that \( |v_i| = 1 \) and each \( v_i \) points at the vertex of a regular \( h \) polygon. Then we consider the configurations \( P \) which lie
in $A$ and such that $P_i$ for $i \geq 2$ is located on the half-line starting from $P_1$ in the $v_i$ direction; more precisely

$$(3.21) \quad P = P(a, r) = \left( \begin{array}{c} a \\ a + r_2 v_2 \\ \vdots \\ a + r_\ell v_\ell \end{array} \right) \in \mathbb{R}^\ell,$$

where $a \in A$ and $r = (r_2, \ldots, r_\ell) \in (0, +\infty)^{\ell-1}$. We point out that

$$(3.22) \quad r_i = |P_i - P| = \min_{j < i, \tau_j = -\tau_i} |P_i - P_j| \quad \forall i = 2, \ldots, \ell. \quad \text{Moreover, if } \tau_i \tau_j = 1, \text{ then } i, j \geq 2 \text{ and we have } |P_i - P_j|^2 = r_i^2 + r_j^2 - 2r_ir_j \cos \frac{2\pi(i-j)}{h}, \text{ which implies}$$

$$(3.23) \quad |P_i - P_j| \geq \sqrt{2 - 2 \cos \frac{2\pi}{h} \min_{2 \leq s \leq \ell} r_s} \quad \text{if } \tau_i \tau_j = 1, \ i \neq j. \quad \text{Taking into account that } \sqrt{2 - 2 \cos \frac{2\pi}{h}} > 1 \text{ if } 2 \leq h \leq 5, \text{ by (3.22) and (3.23) we immediately get}$$

$$(3.24) \quad |P_i - P_j| \geq \min_{2 \leq s \leq \ell} r_s \quad \text{if } i \neq j.$$

(iii) $h = 4, \ k = 2, \ r \geq 2$

We set $\tau_1 = \tau_3 = -1, \ \tau_2 = \tau_4 = \tau_5 = \tau_6 = 1$. Let us fix two orthogonal vectors $v, w \in A$ such that $|v| = |w| = 1$. Then consider the configurations of the type

$$(3.25) \quad P(a, r) = \left( \begin{array}{c} a \\ a + r_2 v \\ a + (r_2 + r_3)v \\ a + (r_2 + r_3 + r_4)v \\ a + r_5 w \\ a - r_6 w \end{array} \right) \in \mathbb{R}^\ell,$$

where $a \in A$ and $r = (r_2, \ldots, r_\ell) \in (0, +\infty)^{\ell-1}$. It is immediate to check that

$$(3.26) \quad r_i = \min_{j < i} |P_i - P_j| = \min_{j < i, \tau_j = -\tau_i} |P_i - P_j| \quad \text{and}$$

$$(3.27) \quad |P_i - P_j| \geq \sqrt{2} \min_{2 \leq s \leq \ell} r_s \quad \text{if } \tau_i \tau_j = 1, \ i \neq j.$$

Observe that (i)-(ii)-(iii) cover all cases $(h, k)$ with the assumptions of Theorem 1.2.

We now define:

$$S = \left\{ P \in \mathbb{R}^\ell \left| c_2 \sum_{i=1}^\ell M_i |P_i|^2 + c_3 \sum_{i \neq j} w \left( \frac{P_i - P_j}{\varepsilon} \right) < \frac{c_4}{2} \varepsilon^{2\beta} \right\}$$

and

$$\bar{U} = \{(a, r) \in A \times (0, +\infty)^{\ell-1} : P(a, r) \in S\}. $$
\( \tilde{U} \) is an open set. In principle, we do not know whether \( \tilde{U} \) is connected or not, so we will define \( U \) as a conveniently chosen connected component. We claim that \( (0, r_\varepsilon) \in \tilde{U} \), where \( r_\varepsilon \) is defined as:

\[
r_\varepsilon = (2\varepsilon \log \frac{2}{\varepsilon}, \ldots, 2\varepsilon \log \frac{1}{\varepsilon}) \in \mathbb{R}^\ell - 1.
\]

Indeed, setting \( \mathbf{P}(0, r_\varepsilon) = (P_0^0, \ldots, P_0^\ell) \), according to (3.19), (3.21) and (3.24) we have \( |P_i^0 - P_j^0| \geq 2\varepsilon \log \frac{1}{\varepsilon} \) for \( i \neq j \) and, by (1.5), we immediately check \( w(\frac{P_i^0 - P_j^0}{\varepsilon}) = o(\varepsilon^2) \) for \( i \neq j \). Furthermore, according to (3.18)-(3.21) one has \( |P_i^0| \leq 2\varepsilon \log \frac{1}{\varepsilon} \), then we infer \( M^+[P_i^0]^2 = O(\varepsilon^2 \log^2 \frac{1}{\varepsilon}) \).

Now we are in conditions of defining \( U, K \) and \( K_0 \):

\[
U = \text{the connected component of } \tilde{U} \text{ containing } (0, r_\varepsilon),
\]

\[
K = \{ \mathbf{P}(a, r) \in \mathbb{R}^{N\ell} : (a, r) \in \tilde{U} \},
\]

\[
K_0 = \{ \mathbf{P}(a, r) \in \mathbb{R}^{N\ell} : (a, r) \in \partial U \}.
\]

\( K \) is clearly isomorphic to \( \tilde{U} \) by the obvious isomorphism, and \( K_0 \approx \partial U \). In particular \( K \) and \( K_0 \) are compact sets and \( K \) is connected. Moreover we have \( K_0 \subset K \subset \mathcal{D}_\varepsilon \).

If \( \mathbf{P} = \mathbf{P}(a, r) \in K \), by (2.7) and (3.19), (3.22), (3.24) we get \( r_i \geq 2\beta^2 \varepsilon \log \frac{1}{\varepsilon} \) for all \( i = 2, \ldots, \ell \); then (3.20), (3.23), (3.27) imply \( |P_i - P_j| \geq 2\varepsilon \log \frac{1}{\varepsilon} \) if \( \tau_i = \tau_j = 1 \) and \( i \neq j \) for \( \beta \) sufficiently close to 1 (observe that \( \sqrt{2 - 2 \cos \frac{2\pi}{h}} > 1 \) if \( 2 \leq h \leq 5 \)). Then by (1.5) it follows that

\[
w\left( \frac{P_i - P_j}{\varepsilon} \right) = o(\varepsilon^2) \text{ if } \tau_i \tau_j = 1, \ i \neq j, \ \text{uniformly on } K.
\]

Roughly speaking, the configurations in \( K \) have the crucial property that the mutual distance between the points \( P_i, P_j \) with \( \tau_i = \tau_j, \ i \neq j \), is sufficiently large so that their interaction term \( w\left( \frac{P_i - P_j}{\varepsilon} \right) \) becomes negligible; moreover, since \( K \subset \mathcal{A}_\ell \), then \( M|P_i|^2 = M^+[P_i|^2 \), and consequently the main terms which appear in \( J_\varepsilon \) are positive. Indeed by Proposition 2.2 we deduce

\[
J_\varepsilon[\mathbf{P}] = \frac{c_2}{2} \sum_{i=1}^\ell M^+[P_i]^2 + \frac{c_3}{2} \sum_{i \neq j} w\left( \frac{P_i - P_j}{\varepsilon} \right) + o(\varepsilon^{2\beta}) \text{ uniformly on } K,
\]

by which, since \( c_2 \sum_{i=1}^\ell M^+[P_i]^2 + c_3 \sum_{i \neq j} w\left( \frac{P_i - P_j}{\varepsilon} \right) = \frac{c_4}{4} \varepsilon^{2\beta} \) if \( \mathbf{P} \in K_0 \),

\[
J_\varepsilon[\mathbf{P}] = \frac{c_4}{4} (1 + o(1)) \varepsilon^{2\beta} \text{ uniformly on } K_0.
\]

Let \( \eta \in \mathcal{F} \), namely \( \eta : K \rightarrow \mathcal{D}_\varepsilon \) is a continuous function such that \( \eta(\mathbf{P}) = \mathbf{P} \) for any \( \mathbf{P} \in K_0 \). Then we can compose the following maps

\[
A \times (0, +\infty)^{\ell - 1} \supset \bar{U} \longleftarrow K \xrightarrow{\eta} \eta(K) \subset \mathcal{D}_\varepsilon \xrightarrow{\mathcal{H}} A \times (0, +\infty)^{\ell - 1}
\]

where \( \mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_\ell) : \mathbb{R}^{N\ell} \rightarrow A \times (0, +\infty)^{\ell - 1} \) is defined by

\[
\mathcal{H}_1(P_1, \ldots, P_\ell) = \pi_A(P_1), \ \mathcal{H}_i(P_1, \ldots, P_\ell) = \min_{j < i, \tau_j = \tau_i} |P_i - P_j| \text{ for } i \geq 2,
\]

denoting by \( \pi_A \) the orthogonal projection of \( \mathbb{R}^N \) onto \( A \). We set

\[
T : \bar{U} \rightarrow A \times (0, +\infty)^{\ell - 1}
\]
the resulting composition. Clearly \( T \) is a continuous map. We claim that \( T = id \) on \( \partial U \). Indeed, if \((a,r) \in \partial U\), then by construction \( \mathbf{P}(a,r) \in K_0\); consequently \( \eta(\mathbf{P}(a,r)) = \mathbf{P}(a,r) \), by which, using the definitions (3.18)-(3.21)-(3.25),
\[
\mathcal{H}_1(\mathbf{P}(a,r)) = \pi_A(a) = a
\]
while, using (3.19)-(3.22)-(3.26),
\[
\mathcal{H}_i(\mathbf{P}(a,r)) = r_i \quad \text{for} \quad i \geq 2.
\]
This proves that \( T = id \) on \( \partial U \).

The theory of the topological degree assures that \( \deg(T,U,(0,r_2)) = \deg(id,U,(0,r_2)) = 1 \); then there exists \((a_\eta,r_\eta) \in U\) such that \( T(a_\eta,r_\eta) = (0,r_2) \), i.e., setting \( \mathbf{P}_\eta := \eta(\mathbf{P}(a_\eta,r_\eta)) \in \eta(K) \),
\[
\pi_A(P^\eta_1) = 0, \quad \min_{j,i,\tau_j = -\tau_i} |P^\eta_j - P^\eta_i| = 2\varepsilon \log \frac{1}{\varepsilon} \quad \text{for} \quad i \geq 2.
\]
In particular this implies
\[
P^\eta_1 \in B, \quad |P^\eta_j - P^\eta_i| \geq 2\varepsilon \log \frac{1}{\varepsilon} \quad \text{if} \quad \tau_i = -\tau_j,
\]
which gives
\[
w\left(\frac{P^\eta_i - P^\eta_j}{\varepsilon}\right) = o(\varepsilon^2) \quad \text{if} \quad \tau_i = -\tau_j.
\]
Moreover, by the second of (3.30), recalling that \( \tau_1 = -\tau_2 \), it is not difficult to check that \( |P^\eta_i - P^\eta_1| \leq 2\varepsilon \log \frac{1}{\varepsilon} \) for all \( i \); then \( |\pi_A(P^\eta_1)| = |\pi_A(P^\eta_i - P^\eta_1)| \leq 2\varepsilon \log \frac{1}{\varepsilon} \) and consequently
\[
M[\pi^\eta] = -M[\pi^\eta]^2 + O\left(\varepsilon^2 \log^2 \frac{1}{\varepsilon}\right).
\]
By Proposition 2.2 we infer
\[
\min_{\mathbf{P} \in K} J_\varepsilon[\eta(\mathbf{P})] \leq J_\varepsilon[\eta] = -\frac{c_2}{2} \sum_{i=1}^\ell M[\pi^\eta_i]^2 - \frac{c_3}{2} \sum_{i \neq j, \tau_i = \tau_j} w\left(\frac{P^\eta_i - P^\eta_j}{\varepsilon}\right) + o(\varepsilon^{2\beta}) \leq o(\varepsilon^{2\beta}).
\]
By taking the supremum for all the maps \( \eta \in \mathcal{F} \) we obtain
\[
J^* = \sup_{\eta \in \mathcal{F}} \min_{\mathbf{P} \in K} J_\varepsilon[\eta(\mathbf{P})] \leq o(\varepsilon^{2\beta}).
\]
On the other hand, by taking \( \eta = id \) and using (3.23),
\[
J^* \geq \min_{\mathbf{P} \in K} J_\varepsilon[\mathbf{P}] \geq o(\varepsilon^{2\beta}).
\]
Last two estimates yield (3.15). Finally, comparing (3.15) with (3.29), the max-min inequality (3.14) follows.

3.2. Proof of (P3). Let us define
\[
\Phi_\varepsilon : \Gamma_\varepsilon \to \mathbb{R}, \quad \Phi_\varepsilon(\mathbf{P}) = \frac{c_2}{2} \sum_{i=1}^\ell \mathcal{M}[P_i]^2 + \frac{c_3}{2} \sum_{i \neq j} w\left(\frac{P_i - P_j}{\varepsilon}\right).
\]
We shall prove (P3) by contradiction: assume that there exist \( \varepsilon_n \to 0 \), \( \mathbf{P}_\varepsilon = (P_1^\varepsilon, \ldots, P_\ell^\varepsilon) \in \partial \mathcal{D}_\varepsilon \), and a vector \((\mu_{\varepsilon,1}, \mu_{\varepsilon,2})\) in the unit circle, i.e., \( \mu_{\varepsilon,1}^2 + \mu_{\varepsilon,2}^2 = 1 \), such that:
\[
\Phi_\varepsilon(\mathbf{P}_\varepsilon) = \frac{c_1}{2} \varepsilon_n^{2\beta},
\]
\[
J_\varepsilon[\mathbf{P}_\varepsilon] = J^*_\varepsilon.
\]
\[
\mu_{\varepsilon,n} J'_{\varepsilon} [P_{\varepsilon}] + \mu_{\varepsilon,n} \Phi'_{\varepsilon} [P_{\varepsilon}] = 0.
\]

Last expression can be read as \( J'_{\varepsilon} [P_{\varepsilon}] \) and \( \Phi'_{\varepsilon} [P_{\varepsilon}] \) are linearly dependent. Observe that this contradicts either the smoothness of \( \partial D_{\varepsilon,n} \) or the nondegeneracy of \( J'_{\varepsilon} [P_{\varepsilon}] \) on the tangent space.

For the sake of clarity, in what follows we will drop the subscript \( n \). Moreover, at many steps of the arguments we will pass to a subsequence, without further notice. By using Proposition 2.2 and (3.15), we have:

\[
\frac{c_2}{2} \sum_{i=1}^{\ell} M_i [P_i^\varepsilon]^2 + \frac{c_3}{2} \sum_{i \neq j} w \left( \frac{P_i^\varepsilon - P_j^\varepsilon}{\varepsilon} \right) = \frac{c_4}{2} \varepsilon^{2\beta},
\]

(3.31)

\[
\frac{c_2}{2} \sum_{i=1}^{\ell} M_i [P_i^\varepsilon]^2 - \frac{c_3}{2} \sum_{i \neq j} \tau_i \tau_j w \left( \frac{P_i^\varepsilon - P_j^\varepsilon}{\varepsilon} \right) = o(\varepsilon^{2\beta})
\]

(3.32)

\[
\frac{c_2}{2} \sum_{i=1}^{\ell} M_i [P_i^\varepsilon]^2 + c_3 \sum_{\tau_i = -\tau_j} w \left( \frac{P_i^\varepsilon - P_j^\varepsilon}{\varepsilon} \right) = \frac{c_4}{2} \varepsilon^{2\beta} + o(\varepsilon^{2\beta}),
\]

(3.34)

\[
\frac{c_2}{2} \sum_{i=1}^{\ell} M_i [P_i^\varepsilon]^2 - c_3 \sum_{\tau_i = -\tau_j} w \left( \frac{P_i^\varepsilon - P_j^\varepsilon}{\varepsilon} \right) = \frac{c_4}{2} \varepsilon^{2\beta} + o(\varepsilon^{2\beta}).
\]

(3.35)

Combining (3.31) and (3.32) we obtain:

(3.33)

Motivated by (3.33), we distinguish two cases:

**Case 1**: There exists \( C > 0 \) independent of \( \varepsilon \) such that \( \sum_{i=1}^{\ell} |\mu_{\varepsilon,2} + \mu_{\varepsilon,1}| M_i [P_i^\varepsilon]^2 + |\mu_{\varepsilon,2} - \mu_{\varepsilon,1}| M_i [P_i^\varepsilon]^2 \geq C\varepsilon^{2\beta} \).

For instance, we can assume that \( \sum_{i=1}^{\ell} |\mu_{\varepsilon,2} + \mu_{\varepsilon,1}| M_i [P_i^\varepsilon]^2 \geq C\varepsilon^{2\beta} \). In particular, recalling that \( M_i [P_i^\varepsilon]^2 < \varepsilon^{2\beta} \), this implies that \( |\mu_{\varepsilon,2} + \mu_{\varepsilon,1}| \to 0 \) and there exists \( i_0 \in \{1, \ldots, \ell\} \) such that \( M_i [P_i^\varepsilon]^2 \geq C\varepsilon^{2\beta} \).

The idea is the following: we make the derivative \( \mu_{\varepsilon,1} J'_{\varepsilon} [P_{\varepsilon}] + \mu_{\varepsilon,2} \Phi'_{\varepsilon} [P_{\varepsilon}] \) along the same direction for all points “close” to \( P_{i_0}^\varepsilon \). Since the direction is the same, the derivative of the interaction among those points should be zero. And this direction will be chosen conveniently to get a contradiction.

Take \( \beta' \in (\beta, 1) \) fixed; let us define

\[
I = \{ i = 1, \ldots, \ell : |P_i^\varepsilon - P_{i_0}^\varepsilon| = o(\varepsilon^{\beta'}) \}.
\]

We take \( P_{i_0}^\varepsilon = (P_{i_0,1}^\varepsilon, \ldots, P_{i_0,r}^\varepsilon, 0, \ldots, 0) \) the projection of \( P_{i_0}^\varepsilon \) onto \( A \) (here \( P_{i_0,n}^\varepsilon \) denotes the \( n \)-th component of \( P_{i_0}^\varepsilon \)). Recall that \( |P_i^\varepsilon| = O(\varepsilon^\beta) \). By multiplying (3.33) by \( P_{i_0}^\varepsilon \) and adding in \( i \in I \), we...
have:

\[
\sum_{i \in I} c_2(\mu_{\varepsilon,2} + \mu_{\varepsilon,1})M^+[P_i^r, P_{i_0}^r]
\]

(3.36)

\[
+ \sum_{i \in I, j \neq i} \left[ \mu_{\varepsilon,2} \frac{c_3}{\varepsilon} w'\left(\frac{|P_i^r - P_j^r|}{\varepsilon}\right) \frac{P_i^r - P_j^r}{|P_i^r - P_j^r|} - \mu_{\varepsilon,1} \frac{\tau_i \tau_j}{\varepsilon^N} \partial_{P_i} \int_{\mathbb{R}^N} f(w_{P_i^r}) w_{P_j^r} \, dx \right] \cdot P_{i_0}^r = o(\varepsilon^{2\beta}).
\]

We now estimate each of the above terms in order to get a contradiction. First, observe that

\[
M^+[P_i^r, P_{i_0}^r] = M^+[P_{i_0}^r, P_i^r] + M^+[P_i^r - P_{i_0}^r, P_{i_0}^r] \geq C\varepsilon^{2\beta} + o(\varepsilon^{3\beta}).
\]

Recall also that \(|\mu_{\varepsilon,2} + \mu_{\varepsilon,1}| \sim 0\). So, it suffices to show that the rest of the terms in (3.36) are negligible to obtain a contradiction.

We split the second sum in two terms; those with \(j \in I\) and those with \(j \notin I\). Let us start with the latter; by using Lemma A.2, we have:

(3.37)

\[
\sum_{i \in I, j \notin I} \left[ \mu_{\varepsilon,2} \frac{c_3}{\varepsilon} w'\left(\frac{|P_i^r - P_j^r|}{\varepsilon}\right) \frac{P_i^r - P_j^r}{|P_i^r - P_j^r|} - \mu_{\varepsilon,1} \frac{c_3 + o(1)}{\varepsilon} \frac{\tau_i \tau_j}{\varepsilon^N} \partial_{P_i} \int_{\mathbb{R}^N} f(w_{P_i^r}) w_{P_j^r} \, dx \right] \cdot P_{i_0}^r.
\]

Observe that, by definition of \(I\), \(|P_j^r - P_i^r| \geq C\varepsilon^{\beta'}\) for \(i \in I, j \notin I\). This implies that \(w'\left(\frac{|P_i^r - P_j^r|}{\varepsilon}\right) = o(e^{-C\varepsilon^{-\beta'}})\), and then (3.37) is negligible.

We now consider the following sum in \(j \in I\):

\[
\sum_{i \in I} \sum_{j \in I, j \neq i} \left[ \mu_{\varepsilon,2} c_3 \partial_{P_i} w\left(\frac{|P_i - P_j|}{\varepsilon}\right) - \mu_{\varepsilon,1} \frac{\tau_i \tau_j}{\varepsilon^N} \partial_{P_i} \int_{\mathbb{R}^N} f(w_{P_i^r}) w_{P_j^r} \, dx \right].
\]

By a change of variables we deduce that \(\int_{\mathbb{R}^N} f(w_{P_i^r}) w_{P_j^r} \, dx\) is a function of \(|P_i - P_j|\). And it is easy to conclude that for any \(\xi \in C^1(\mathbb{R})\),

\[
\sum_{i \in I} \sum_{j \in I, j \neq i} \partial_{P_i} \xi(|P_i - P_j|) = \sum_{i \in I} \sum_{j \in I, j \neq i} \xi'(|P_i - P_j|) \frac{P_i - P_j}{|P_i - P_j|} = 0.
\]

So we get a contradiction in Case 1.

**Case 2:** \(\sum_{i=1}^r |\mu_{\varepsilon,2} + \mu_{\varepsilon,1}|M^+[P_i^r]^2 + |\mu_{\varepsilon,2} - \mu_{\varepsilon,1}|M^{-}[P_i^r]^2 = o(\varepsilon^{2\beta})\).

In a sense, here the effect of \(M[P_i^r]^2\) is negligible and the interaction among the bumps is important. But in \(\exists\) it was proved that the bumps cannot reach an equilibrium by themselves (see Lemma 3.1), and this gives us the desired contradiction.

Since \(\mu_{\varepsilon,1}^2 + \mu_{\varepsilon,2}^2 = 1\), then at least one between \(\mu_{\varepsilon,1} + \mu_{\varepsilon,2}\) and \(\mu_{\varepsilon,1} - \mu_{\varepsilon,2}\) does not go to 0. If \(\mu_{\varepsilon,2} + \mu_{\varepsilon,1} \not\to 0 \Rightarrow \sum_i M^+[P_i^r]^2 = o(\varepsilon^{2\beta})\), and by (3.36), \(\sum_{\tau_i = -\tau_j} w(\frac{P_i^r - P_j^r}{\varepsilon}) \geq C\varepsilon^{2\beta}\). Analogously, if \(\mu_{\varepsilon,2} - \mu_{\varepsilon,1} \not\to 0\) we can use (3.36) to conclude \(\sum_{\tau_i = -\tau_j} w(\frac{P_i^r - P_j^r}{\varepsilon}) \geq C\varepsilon^{2\beta}\).

In any case, we have:

\[
\sum_{i \neq j} |\mu_{\varepsilon,2} - \mu_{\varepsilon,1} \tau_i \tau_j| w\left(\frac{P_i^r - P_j^r}{\varepsilon}\right) \geq C\varepsilon^{2\beta}.
\]

So, there exist \(i_0 \neq j_0\) so that

\[
|\mu_{\varepsilon,2} - \mu_{\varepsilon,1} \tau_{i_0} \tau_{j_0}| w\left(\frac{P_{i_0}^r - P_{j_0}^r}{\varepsilon}\right) \geq C\varepsilon^{2\beta}.
\]
Define:

\[ I = \left\{ i = 1, \ldots, \ell : \frac{|P_i^\varepsilon - P_j^\varepsilon|}{\varepsilon \log(1/\varepsilon)} \text{ is bounded} \right\}. \]

Observe that, at least, \( i_0, j_0 \in I \). For any \( i \in I \), we can pass to the limit on the following expressions:

\[ \frac{P_i^\varepsilon - P_j^\varepsilon}{2\varepsilon \log(1/\varepsilon)} \rightarrow Q_i \in \mathbb{R}^N, \]

and

\[ \varepsilon^{-2\beta}(\mu_{\varepsilon,2} - \mu_{\varepsilon,1}\tau_i \tau_j) w \left( \frac{P_i^\varepsilon - P_j^\varepsilon}{\varepsilon} \right) \rightarrow a_{i,j} \in \mathbb{R}, \; j = 1, \ldots, \ell. \]

We point out that \( P_{i_0} = 0 \) and \( a_{i_0,j_0} \neq 0 \). We recall that \( w \left( \frac{P_i^\varepsilon - P_j^\varepsilon}{\varepsilon} \right) = O(\varepsilon^{2\beta}) \); hence, from (3.39), we obtain:

\[ Q_i - Q_j = \lim_{\varepsilon \rightarrow 0} \frac{P_i^\varepsilon - P_j^\varepsilon}{2\varepsilon \log(1/\varepsilon)} \Longrightarrow |Q_i - Q_j| \geq 1, \; i, j \in I. \]

Before going on, we are interested in extracting consequences from \( a_{i,j} \neq 0 \), with \( i \in I \). In such case there exist \( c, c' \) positive constants such that \( c \varepsilon^{2\beta} \geq w \left( \frac{P_i^\varepsilon - P_j^\varepsilon}{\varepsilon} \right) \geq c' \varepsilon^{2\beta} \). Then,

\[ \frac{|P_i^\varepsilon - P_j^\varepsilon|}{2\varepsilon \log(1/\varepsilon)} \rightarrow 1. \]

In particular, \( j \in I \). Moreover, similarly as in (3.38), we obtain that \( |Q_i - Q_j| = 1 \).

By using (3.38) together with Lemma A.2, we get:

\[ c_2 \left( (\mu_{\varepsilon,2} + \mu_{\varepsilon,1})M^+[P_i^\varepsilon] + (\mu_{\varepsilon,2} - \mu_{\varepsilon,1})M^-[P_i^\varepsilon] \right) + \]

\[ \frac{c_3}{\varepsilon} \sum_{j, j \neq i} (\mu_{\varepsilon,2} - \mu_{\varepsilon,1}(1 + o(1)) \tau_i \tau_j) w' \left( \frac{P_i^\varepsilon - P_j^\varepsilon}{\varepsilon} \right) \frac{P_i^\varepsilon - P_j^\varepsilon}{|P_i^\varepsilon - P_j^\varepsilon|} = o(\varepsilon^\beta), \; i \in I. \]

We multiply by \( \varepsilon^{1-2\beta} \) and use (3.5) to obtain

\[ c_3 \sum_{j, j \neq i} \varepsilon^{-2\beta}(\mu_{\varepsilon,2} - \mu_{\varepsilon,1} \tau_i \tau_j) w \left( \frac{P_i^\varepsilon - P_j^\varepsilon}{\varepsilon} \right) \frac{P_i^\varepsilon - P_j^\varepsilon}{|P_i^\varepsilon - P_j^\varepsilon|} = o(1), \; i \in I. \]

Recall that \( a_{i,j} = 0 \) for any \( j \notin I \). Passing to the limit:

\[ \sum_{j \in I, j \neq i} a_{i,j} \frac{Q_i - Q_j}{|Q_i - Q_j|} = 0, \; i \in I. \]

In other words, the points \( Q_i \in \mathbb{R}^N, \; i \in I \), satisfy that \( Q_{i_0} = 0, \; |Q_i - Q_j| \geq 1 \) and \((Q_i)_{i \in I}\) is a critical point of the function:

\[ (Z_i)_{i \in I} \rightarrow \sum_{i,j \in I, i \neq j} a_{i,j} |Z_i - Z_j|, \; Z_i \in \mathbb{R}^N. \]

where \( a_{i,j} = a_{j,i}, \; a_{i,j} = 0 \) for points \( Q_i, Q_j \) such that \( |Q_i - Q_j| > 1 \), and \( a_{i_0,j_0} \neq 0 \).

We finish the proof by showing that this is impossible. For that we need to distinguish between the case of positive peaks and the case of mixed positive and negative peaks.

In the first case \( \tau_i = 1 \) for all \( i = 1, \ldots, \ell \). By the definition of \( a_{i,j} \) and the fact \( a_{i_0,j_0} \neq 0 \), we conclude that \( \mu_{\varepsilon,2} - \mu_{\varepsilon,1} \rightarrow 0 \). Moreover, \( a_{i,j} \) have all the same sign as \( \mu_{\varepsilon,2} - \mu_{\varepsilon,1} \). Assume, for instance, that
\(a_{i,j} \geq 0\). But in such case \((Q_i)_{i \in I}\) cannot be a critical point of the map given by \((3.39)\), as can be seen using dilatations. More specifically, if we multiply \((3.39)\) by \(Q_i\) and make the addition, we get:

\[
\sum_{i \in I} \sum_{j \in I, j \neq i} a_{i,j} \frac{Q_i - Q_j}{|Q_i - Q_j|} \cdot Q_i = \sum_{i,j} a_{i,j}|Q_i - Q_j| \geq a_{i_0,j_0} > 0.
\]

The case in which there are peaks of different sign is excluded thanks to the next lemma, proved in [9]. We point out that the restriction \(\ell \leq 6\) is needed only at this point.

**Lemma 3.1.** Let \(\ell \geq 2\) and consider the function:

\[
\Phi : (Z_1, \ldots, Z_\ell) \in \mathbb{R}^{N\ell} \to \sum_{i \neq j} a_{ij}|Z_i - Z_j|,
\]

where \(a_{ij} = a_{ji}\). Suppose that \(\Phi\) is not identically zero and that there exists a critical point \((Q_1, \ldots, Q_\ell)\) of \(\Phi\) satisfying:

\[
|Q_i - Q_j| \geq 1 \text{ for } i \neq j \quad \text{and} \quad |Q_i - Q_j| = 1 \text{ if } a_{i,j} \neq 0.
\]

Then \(\ell \geq 7\).

**Proof of Theorems 1.1 and 1.2 completed.** According to Lemma 2.3 for \(\varepsilon > 0\) sufficiently small \(\chi w_{P_\varepsilon} + \phi_{P_\varepsilon}\) solves the equation \((1.2)\), where \(P_\varepsilon = (P_1^\varepsilon, \ldots, P_\ell^\varepsilon) \in \Gamma_\varepsilon\) is the critical point of \(J_\varepsilon\) with critical value \(J_\varepsilon^*\). The construction of the family \(P_\varepsilon\) depends on the particular \(\beta \in (0, 1)\) chosen at the beginning of Section 2. To emphasize this fact we denote this family as \(P_{\varepsilon, \beta}\). Let \(\beta_k \subset (0, 1)\) be any sequence such that \(\beta_k \to 1\). Then there is a decreasing sequence of positive numbers \(\varepsilon_k\) such that for all \(0 < \varepsilon < \varepsilon_k\) one has:

1. \(\chi w_{P_{\varepsilon, \beta_k}} + \phi_{P_{\varepsilon, \beta_k}}\) solves \((1.2)\),
2. by \((2.7)\), \(|P_{i, \beta_k}^\varepsilon| \leq (\min_i |\lambda_i|)^{-1/2} \varepsilon^{\beta_i}, |P_{i, \beta_k}^\varepsilon - P_{j, \beta_k}^\varepsilon| \geq 2\beta \varepsilon^2 \log \frac{1}{\varepsilon} \) for \(i \neq j\), and
3. \(|\phi_{P_{\varepsilon, \beta_k}}| \leq \varepsilon^{\beta_k(1+\sigma)}\).

We define \(P_\varepsilon = P_{\varepsilon, \beta_k}\) and \(w_\varepsilon = \chi w_{P_{\varepsilon, \beta_k}} + \phi_{P_{\varepsilon, \beta_k}}\) if \(\varepsilon_{k+1} < \varepsilon < \varepsilon_k\) and we clearly have that the theses of Theorem 1.1 and Theorem 1.2 hold.

**Appendix A. Key energy estimate**

Consider the configuration set \(\Gamma_\varepsilon\) and the approximate solutions \(\chi w_P\) defined in Section 2. In this Appendix we will derive some crucial estimates. We note that by assumption (V2) and \((2.7)\) we have \(|\nabla V(P_i)| \leq C\varepsilon^\beta\) for \(P \in \Gamma_\varepsilon\); then by \((1.3)\) we deduce

\[
|V(x)\chi w_P - V(P_i)w_P_i| \leq |\nabla V(P_i)||x - P_i|w_P_i + C|x - P_i|^2w_P_i \leq C\varepsilon^{1+\beta}w_P_i^{2/3},
\]

by which

\[
(A.41) \quad V(x)\chi w_P - V(P_i)w_P_i = O(\varepsilon^{1+\beta})w_P_i^{2/3}, \quad V(x)\chi w_P - w_P = O(\varepsilon^{2\beta})w_P_i^{2/3}
\]

uniformly for \(P \in \Gamma_\varepsilon\).

**Remark A.1.** Observe that by \((1.3)\) it follows that

\[
\frac{w(z + \xi)}{w(\xi)} \leq C \left(\frac{|\xi|}{1 + |z + \xi|}\right)^{N-1} e^{e^{\xi|z|}} \quad \forall z, \xi \in \mathbb{R}^N.
\]
Lemma A.2. For \( i \neq j \) the following expansions hold uniformly for \( \mathbf{P} \in \Gamma_{\varepsilon} \):

\[
\int_{\mathbb{R}^N} f(w_{P_i})w_{P_j} \, dx = c_3 \varepsilon^N (1 + o(1)) w\left(\frac{P_i - P_j}{\varepsilon}\right),
\]

\[
\frac{\partial}{\partial P_i} \left[ \int_{\mathbb{R}^N} f(w_{P_i})w_{P_j} \, dx \right] = c_3 \varepsilon^{N-1} (1 + o(1)) w'\left(\frac{|P_i - P_j|}{\varepsilon}\right) \frac{P_i - P_j}{|P_i - P_j|}.
\]

where \( c_3 = \int_{\mathbb{R}^N} f(w)e^{x_1} \, dx \).

Proof. First consider the function

\[
\xi(\rho) = \int_{\mathbb{R}^N} f(w)w(x + \rho e_1) \, dx, \quad \rho > 0,
\]

where \( e_1 \) is the first vector of the standard basis of \( \mathbb{R}^N \), i.e. \( e_1 = (1,0,\ldots,0) \). According to (1.5) for every \( x \in \mathbb{R}^N \) we have

\[
\lim_{\rho \to \infty} \frac{w(x + \rho e_1)}{w(\rho)} = \lim_{\rho \to \infty} e^{-|x + \rho e_1| + \rho} = e^{-x_1}.
\]

Thanks to (A.42) the Dominated Convergence Theorem applies and gives \( \frac{\xi(\rho)}{w(\rho)} \to \int_{\mathbb{R}^N} f(w)e^{-x_1} \, dx \). Next compute

\[
\xi'(\rho) = \int_{\mathbb{R}^N} f(w)w'(x + \rho e_1) \frac{x_1 + \rho}{|x + \rho e_1|} \, dx.
\]

Using (1.5) and proceeding as above we get

\[
\frac{\xi'(\rho)}{w'(\rho)} \to \int_{\mathbb{R}^N} f(w)e^{-x_1} \, dx.
\]

Since

\[
\int_{\mathbb{R}^N} f(w_{P_i})w_{P_j} \, dx = \varepsilon^N \int_{\mathbb{R}^N} f(w)w\left(\frac{P_i - P_j}{\varepsilon}\right) \, dx = \varepsilon^N \xi\left(\frac{|P_i - P_j|}{\varepsilon}\right),
\]

and

\[
\frac{\partial}{\partial P_i} \left[ \int_{\mathbb{R}^N} f(w_{P_i})w_{P_j} \, dx \right] = \varepsilon^{N-1} \xi'(\frac{|P_i - P_j|}{\varepsilon}) \frac{P_i - P_j}{|P_i - P_j|},
\]

then the thesis follows. \( \square \)

Lemma A.3. For every \( i = 1, \ldots, \ell \) the following asymptotic expansion holds uniformly for \( \mathbf{P} \in \Gamma_{\varepsilon} \):

\[
\int_{\mathbb{R}^N} V(x)\chi^2 w_{P_i} \nabla w_{P_i} \, dx = -\frac{\varepsilon^N}{2} M[P_i] \int_{\mathbb{R}^N} w^2 \, dx + o(\varepsilon^{N+\beta}).
\]
Proof. Observe that
\[|\nabla(\varepsilon^2 V(x)) - \nabla (P_1) - D^2 V(P_1)(x - P_1)|w_{P_1} \leq C|x - P_1|^2 w_{P_1} \leq C\varepsilon^2 w_{P_1}^{1/2}\]
by which, using that \(\int_R D^2 V(P_1)(x - P_1)w_{P_1}^2 dx = \varepsilon^N \int_R D^2 V(P_1)y w^2(y)dy = 0\),
\[
2 \int_R V(x)\varepsilon^2 w_{P_1} \nabla w_{P_1} dx = - \int_R \nabla (\varepsilon^2 V(x)) w_{P_1}^2 dx = - \int_R \nabla V(P_1)w_{P_1}^2 dx + O(\varepsilon^{N+2})
\]
\[= -\varepsilon^N \nabla V(P_1) \int_R w^2 dx + O(\varepsilon^{N+2})\]
uniformly for \(P \in \Gamma_\varepsilon\), and, since \(\nabla V(P) = M|P| + o(|P|)\) as \(P \to 0\), we obtain the thesis. \(\blacksquare\)

The next proposition provides an estimate of the error up to which the functions \(\chi w_P\) satisfy (1.2).

**Lemma A.4.** There exists a constant \(C > 0\) such that for every \(\varepsilon > 0\) and \(P \in \Gamma_\varepsilon\):
\[|S_\varepsilon[\chi w_P]| \leq C\varepsilon^{\beta(\beta + \sigma)} \sum_{i=1}^\ell w_{P_i}^{1-\beta}\]
where \(S_\varepsilon\) is the operator defined in (2.9).

**Proof.** By (A.41) we deduce
\[
\varepsilon^2 \Delta (\chi w_P) - V(x)\chi w_P + f(\chi w_P) = \varepsilon^2 \Delta w_P - w_P + f(w_P) + O(\varepsilon^{2\beta}) \sum_{i=1}^\ell w_{P_i}^{2/3}
\]
\[= f(w_P) - \sum_{i=1}^\ell \tau_i f(w_{P_i}) + O(\varepsilon^{2\beta}) \sum_{i=1}^\ell w_{P_i}^{2/3}\]
uniformly for \(P \in \Gamma_\varepsilon\). Given \(P \in \Gamma_\varepsilon\), in the following we will make use of the following sets \(A_{\varepsilon,i}\)
\[A_{\varepsilon,i} = \{x \in \mathbb{R}^N \mid w_{P_i} > a\varepsilon^{\beta}\}\]
where \(a > 0\) is chosen such that, according to (A.43),
\[A_{\varepsilon,i} \cap A_{\varepsilon,j} = \emptyset \ \forall i \neq j.\]
Observe that
(A.45) \[w_{P_i} \leq a\varepsilon^{\beta} \leq w_{P_j} \text{ on } A_{\varepsilon,i} \text{ for } j \neq i.\]
Then, by using assumption (f1), we get
\[|f(w_P) - \tau_i f(w_{P_i})| \leq C w_{P_i}^{\sigma} \sum_{j \neq i} w_{P_j} \text{ on } A_{\varepsilon,i},\]
by which
\[|f(w_P) - \tau_i f(w_{P_i})| \leq C\varepsilon^{\beta(\beta - \sigma)} \sum_{j \neq i} w_{P_i} w_{P_j}^{\sigma} w_{P_j}^{1-\beta} \leq C\varepsilon^{\beta(\beta + \sigma)} \sum_{j \neq i} w_{P_j}^{1-\beta} \text{ on } A_{\varepsilon,i}.\]
On the other hand
\[|f(w_{P_j})| \leq C w_{P_j}^{1+\sigma} \leq C\varepsilon^{\beta(\beta + \sigma)} w_{P_j}^{1-\beta} \text{ on } \mathbb{R}^N \setminus A_{\varepsilon,i},\]
\[|f(w_P)| \leq C \sum_{j=1}^\ell w_{P_j}^{1+\sigma} \leq C\varepsilon^{\beta(\beta + \sigma)} \sum_{j=1}^\ell w_{P_j}^{1-\beta} \text{ on } \mathbb{R}^N \setminus \cup_{j=1}^\ell A_{\varepsilon,j}.\]
Since $\beta(\beta + \sigma) < 2\beta$ we obtain the thesis.

With the help of Lemma A.2 and Lemma A.3 we derive the following key energy estimate.

**Proposition A.5.** The following asymptotic expansions hold uniformly for $\mathbf{P} = (P_1, \ldots, P_\ell) \in \Gamma_\varepsilon$:

\begin{equation}
I_\varepsilon[\chi w \mathbf{P}] = c_1 \varepsilon^N + \frac{c_2}{2} \varepsilon^N \sum_{i=1}^\ell M[P_i]^2 - \frac{1}{2} \sum_{i \neq j} \tau_i \tau_j \int_{\mathbb{R}^N} f(w_{P_i})w_{P_j} \, dx + o(\varepsilon^{N+2\beta}),
\end{equation}

\begin{equation}
\frac{\partial}{\partial P_i} \left( I_\varepsilon[\chi w \mathbf{P}] \right) = c_2 \varepsilon^N M[P_i] - \sum_{j, j \neq i} \tau_i \tau_j \frac{\partial}{\partial P_i} \left( \int_{\mathbb{R}^N} f(w_{P_j})w_{P_j} \, dx \right) + o(\varepsilon^{N+\beta}), \quad i = 1, \ldots, \ell,
\end{equation}

where the constants $c_1, c_2$ are given by

\[ c_1 = \frac{\ell}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) \, dx - \ell \int_{\mathbb{R}^N} F(w) \, dx, \quad c_2 = \frac{1}{2} \int_{\mathbb{R}^N} w^2 \, dx. \]

**Proof.** We begin by estimating the potential term: by (A.41) we derive

\begin{align*}
\int_{\mathbb{R}^N} V(x)|\chi w \mathbf{P}|^2 \, dx &= \sum_{i=1}^\ell \int_{\mathbb{R}^N} V(x)|\chi w_{P_i}|^2 \, dx + \sum_{i \neq j} \tau_i \tau_j \int_{\mathbb{R}^N} V(x)\chi w_{P_i}\chi w_{P_j} \, dx \\
&= \sum_{i=1}^\ell V(P_i) \varepsilon^N \int_{\mathbb{R}^N} w^2 \, dx + \sum_{i \neq j} \tau_i \tau_j V(P_i) \int_{\mathbb{R}^N} w_{P_i}w_{P_j} \, dx + o(\varepsilon^{N+2\beta}) \\
&= \sum_{i=1}^\ell \left( 1 + \frac{1}{2} M[P_i]^2 \right) \varepsilon^N \int_{\mathbb{R}^N} w^2 \, dx + \sum_{i \neq j} \tau_i \tau_j \int_{\mathbb{R}^N} w_{P_i}w_{P_j} \, dx + o(\varepsilon^{N+2\beta})
\end{align*}

uniformly for $\mathbf{P} \in \Gamma_\varepsilon$, where the last equality follows by assumption (V2) and (A.43).

Next we compute

\begin{align*}
\frac{\varepsilon^2}{2} \int_{\mathbb{R}^N} |\nabla(\chi w \mathbf{P})|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} F(\chi w \mathbf{P}) \, dx &= \frac{\varepsilon^2}{2} \int_{\mathbb{R}^N} |\nabla w \mathbf{P}|^2 \, dx - \int_{\mathbb{R}^N} F(w) \, dx \\
&= \frac{\ell \varepsilon^N}{2} \int_{\mathbb{R}^N} |\nabla w|^2 \, dx - \varepsilon^N \int_{\mathbb{R}^N} F(w) \, dx \\
&+ \frac{\varepsilon^2}{2} \sum_{i \neq j} \tau_i \tau_j \int_{\mathbb{R}^N} \nabla w_{P_i} \nabla w_{P_j} \, dx - \int_{\mathbb{R}^N} \left( F(w) - \sum_{j=1}^\ell F(w_{P_j}) \right) \, dx + o(\varepsilon^{N+2})
\end{align*}

uniformly for $\mathbf{P} \in \Gamma_\varepsilon$.

Combining (A.47) with (A.48), and using equation (1.3), we get

\begin{align*}
I_\varepsilon[\chi w \mathbf{P}] &= c_1 \varepsilon^N + \frac{c_2}{2} \varepsilon^N \sum_{i=1}^\ell M[P_i]^2 + \frac{1}{2} \sum_{i \neq j} \tau_i \tau_j \int_{\mathbb{R}^N} f(w_{P_i})w_{P_j} \, dx \\
&- \int_{\mathbb{R}^N} \left( F(w) - \sum_{i=1}^\ell F(w_{P_i}) \right) \, dx + o(\varepsilon^{N+2\beta}) \\
&= c_1 \varepsilon^N + \frac{c_2}{2} \varepsilon^N \sum_{i=1}^\ell M[P_i]^2 - \frac{1}{2} \sum_{i \neq j} \tau_i \tau_j \int_{\mathbb{R}^N} f(w_{P_i})w_{P_j} \, dx - H(\mathbf{P}) + o(\varepsilon^{N+2\beta}),
\end{align*}
uniformly for $P \in \Gamma_{\varepsilon}$, where we have set

$$H(P) = \int_{\mathbb{R}^N} F(w_P) dx - \sum_{i=1}^{\ell} \int_{\mathbb{R}^N} F(w_{P_i}) dx - \sum_{i \neq j} \tau_i \tau_j \int_{\mathbb{R}^N} f(w_{P_i}) w_{P_j} dx, \quad P \in \Gamma_{\varepsilon}.$$ 

Consider the sets $A_{\varepsilon,i}$ defined in Lemma A.4 by assumption (f1) we have

$$|H(P)| \leq \sum_{i=1}^{\ell} \int_{A_{\varepsilon,i}} \left| F(w_P) - F(w_{P_i}) - f(w_{P_i}) \sum_{j \neq i} \tau_i \tau_j w_{P_j} \right| dx + C \sum_{i=1}^{\ell} \int_{\mathbb{R}^N \setminus A_{\varepsilon,i}} \left( w_{P_i}^2 + w_{P_j}^{1+\sigma} \sum_{j \neq i} w_{P_j} \right) dx.$$

By (A.45) we get

$$\left| F(w_P) - F(w_{P_i}) - f(w_{P_i}) \sum_{j \neq i} \tau_i \tau_j w_{P_j} \right| \leq C w_{P_i}^\sigma \sum_{j \neq i} w_{P_j}^2 = C \sum_{j \neq i} (w_{P_i} w_{P_j})^\sigma w_{P_j}^{2-\sigma} \text{ on } A_{\varepsilon,i}.$$

Taking into account of (A.43) and (A.45), the above inequalities imply $H(P) = o(\varepsilon^{N+2})$ uniformly for $P \in \Gamma_{\varepsilon}$. Then by (A.49) we obtain (A.46).

We now estimate the error term $o(\varepsilon^{N+2})$ in (A.46) in the $C^1$ sense. To this aim, fix $i \in \{1, \ldots, \ell\}$; by definition we have $\frac{\partial w_P}{\partial P_i} = -\tau_i \nabla w_{P_i}$. Then, by using Lemma A.3 we can compute

$$\tau_i \frac{\partial L[\chi w_P]}{\partial P_i} = -\langle T_\varepsilon'[\chi w_P], \chi \nabla w_{P_i} \rangle = \int_{\mathbb{R}^N} \left( \varepsilon^2 \Delta (\chi w_{P}) - V(x) \chi w_{P} + f(\chi w_{P}) \right) \chi \nabla w_{P_i} dx$$

$$= \varepsilon^2 \int_{\mathbb{R}^N} \sum_{j=1}^{\ell} \Delta w_{P_j} - \sum_{j \neq i} \tau_j \chi V(x) w_{P_j} + f(w_{P_j}) \right) \nabla w_{P_i} dx + o(\varepsilon^{N+\beta})$$

uniformly for $P \in \Gamma_{\varepsilon}$. Since $|\nabla w_{P_i}| \leq C \varepsilon^{-1} w_{P_i}$, using (A.41) and (A.43), for $i \neq j$ we have

$$\int_{\mathbb{R}^N} \chi V(x) w_{P_j} \nabla w_{P_i} dx = \int_{\mathbb{R}^N} w_{P_j} \nabla w_{P_i} + o(\varepsilon^{N+\beta}),$$

while $\int_{\mathbb{R}^N} w_{P_i} \nabla w_{P_i} dx = \frac{1}{2} \int_{\mathbb{R}^N} \nabla w_{P_i}^2 dx = 0$. Then, using (1.3) we arrive to

$$\tau_i \frac{\partial L[\chi w_P]}{\partial P_i} = \varepsilon^2 \int_{\mathbb{R}^N} \sum_{j=1}^{\ell} \tau_j \Delta w_{P_j} - \sum_{j \neq i} \tau_j \chi V(x) w_{P_j} + f(w_{P_j}) \right) \nabla w_{P_i} dx + o(\varepsilon^{N+\beta})$$

$$= \varepsilon^2 \int_{\mathbb{R}^N} \left( f(w_{P_i}) - \sum_{j \neq i} \tau_j f(w_{P_j}) \right) \nabla w_{P_i} dx + o(\varepsilon^{N+\beta})$$

$$= c_2 \varepsilon^2 \int_{\mathbb{R}^N} \left( \frac{f(w_{P_i})}{w_{P_i}} - \sum_{j \neq i} \tau_j f(w_{P_j}) \right) \nabla w_{P_i} dx + K(P) + o(\varepsilon^{N+\beta})$$

uniformly for $P \in \Gamma_{\varepsilon}$, where we have set

$$K(P) = \int_{\mathbb{R}^N} \left( f(w_{P_i}) - \sum_{j \neq i} \tau_j f(w_{P_j}) - f'(w_{P_j}) \sum_{j \neq i} \tau_j w_{P_j} \right) \nabla w_{P_i} dx.$$
By assumption (fl) it follows
\[
|K(P)| \leq \int_{A_{\varepsilon,i}} \left| f(w_P) - \tau_i f(w_{P_i}) - f'(w_{P_i}) \sum_{j \neq i} \tau_j w_{P_j} \right| \nabla w_{P_i} \, dx \\
+ \sum_{j \neq i} \int_{A_{\varepsilon,j}} |f(w_P) - \tau_j f(w_{P_j})| \nabla w_{P_i} \, dx \\
+ C \sum_{j=1}^\ell \int_{\mathbb{R}^N \setminus A_{\varepsilon,j}} w_{P_j}^{1 + \sigma} |\nabla w_{P_i}| + C \sum_{j \neq i} \int_{\mathbb{R}^N \setminus A_{\varepsilon,i}} \nabla w_{P_i} \nabla w_{P_j} \, dx.
\]
(A.50)

Observe that
\[
|f(w_P) - \tau_i f(w_{P_i}) - f'(w_{P_i}) \sum_{j \neq i} \tau_j w_{P_j}| \nabla w_{P_i} | \leq C \varepsilon^{-1} \sum_{j \neq i} w_{P_j}^{1 + \sigma} w_{P_i} = C \varepsilon^{-1} \sum_{j \neq i} (w_{P_i} w_{P_j}) w_{P_j}.
\]

Next fix \( j \neq i \): by (A.45) we have
\[
|f(w_P) - \tau_j f(w_{P_j})| \nabla w_{P_i} | \leq C \varepsilon^{-1} w_{P_j}^{1 + \sigma} w_{P_i} \sum_{k \neq j} w_{P_k} = C \varepsilon^{-1} (w_{P_i} w_{P_j}) w_{P_j}^{1 + \sigma} \sum_{k \neq j} w_{P_k} \text{ on } A_{\varepsilon,j}.
\]

Inserting (A.51)-(A.52) into (A.50), and using (A.43) and (A.45), we deduce \( K(P) = o(\varepsilon^{N+1}) \) uniformly for \( P \in \Gamma_\varepsilon \). Thus we have obtained
\[
\frac{\partial I_f[\chi w_P]}{\partial P_i} = c_2 \varepsilon^N M[P] + \sum_{j \neq i} \tau_i \tau_j \int_{\mathbb{R}^N} f'(w_{P_i}) w_{P_j} \nabla w_{P_i} \, dx + o(\varepsilon^{N+\beta}) \\
= c_2 \varepsilon^N M[P] - \sum_{j \neq i} \tau_i \tau_j \frac{\partial}{\partial P_i} \int_{\mathbb{R}^N} f(w_{P_i}) w_{P_j} \, dx + o(\varepsilon^{N+\beta})
\]
uniformly for \( P \in \Gamma_\varepsilon \), and the second part of the thesis follows. \( \square \)

**APPENDIX B. LYAPUNOV-SCHMIDT REDUCTION**

In this appendix we carry out the reduction procedure sketched in Section 3. In particular we will prove Lemma 2.1 and Proposition 2.2. A large part of the proofs follows in a standard way but we include some details here for completeness.

**B.1. The linearized equation.** Consider the functions \( Z_{P_i,n} \) defined in Section 2. Observe that by proceeding as in the proof of (A.44) we deduce
\[
Z_{P_i,n} = (1 - \varepsilon^2 \Delta) \frac{\partial w_{P_i}}{\partial x_n} + O(\varepsilon^{2\beta - 1}) w_{P_i}^{2/3} = f'(w_{P_i}) \frac{\partial w_{P_i}}{\partial x_n} + O(\varepsilon^{2\beta - 1}) w_{P_i}^{2/3}
\]
uniformly for \( P \in \Gamma_\varepsilon \). After integration by parts it is immediate to prove that
\[
\left( \phi, \frac{\partial (\chi w_{P_i})}{\partial x_n} \right)_\varepsilon = \int_{\mathbb{R}^N} \phi Z_{P_i,n} \, dx \quad \forall \phi \in H^1(\mathbb{R}^N),
\]
then orthogonality to the functions \( \frac{\partial (\chi w_{P_i})}{\partial x_n} \) in \( H^1(\mathbb{R}^N) \) with respect to the scalar product \( \langle \cdot, \cdot \rangle_\varepsilon \) is equivalent to orthogonality to \( Z_{P_i,n} \) in \( L^2(\mathbb{R}^N) \). Hence we easily get
\[
\int_{\mathbb{R}^N} Z_{P_i,n} \frac{\partial (\chi w_{P_i})}{\partial x_m} \, dx = \delta_{ij} \delta_{mn} \varepsilon^{N-2} \left\| \frac{\partial w_{P_i}}{\partial x_n} \right\|_{H^1(\mathbb{R}^N)}^2 + o(\varepsilon^{N-2}),
\]
uniformly for \( P \in \Gamma_\varepsilon \) (\( \delta_{ij} \) and \( \delta_{nm} \) denoting the Kronecker’s symbols), where \( \|v\|_{H^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (|\nabla v|^2 + |v|^2) \, dx \).
Let $\mu \in (0, \sigma)$ be a sufficiently small number and introduce the following weighted norm:

$$\|\phi\|_{*, P} := \sup_{x \in \mathbb{R}^N} \left( \sum_{i=1}^\ell w_{P_i}(x) \right)^{-\mu} |\phi(x)|.$$  

We first consider a linear problem: given $P \in \Gamma_\varepsilon$ and $\theta \in L^2(\mathbb{R}^N)$, find a function $\phi$ and constants $\alpha_{in}$ satisfying

$$\left\{ \begin{array}{l} \mathcal{L}_P[\phi] = \theta + \sum_{i,n} \alpha_{in} Z_{P_{i,n}}, \\ \phi \in H^2(\mathbb{R}^N) \cap H^1_V(\mathbb{R}^N), \int_{\mathbb{R}^N} \phi Z_{P_{i,n}} \, dx = 0 \text{ for } i = 1, \ldots, \ell, \ n = 1, \ldots, N, \end{array} \right.$$  

where

$$\mathcal{L}_P[\phi] := \varepsilon^2 \Delta \phi - V(x) \phi + f'(\chi w_P) \phi.$$  

**Lemma B.1.** There exists a constant $C > 0$ such that, provided that $\varepsilon$ is sufficiently small, if $P \in \Gamma_\varepsilon$ and $(\phi, \theta, \alpha_{in})$ satisfies (B.57), then

$$|\alpha_{in}| \leq C(\varepsilon^{1+\sigma} \|\phi\|_{*, P} + \varepsilon \|\theta\|_{*, P}).$$

**Proof.** By multiplying the equation in (B.57) by $\frac{\partial(\chi w_{P_i})}{\partial x_m}$ and integrating over $\mathbb{R}^N$, we get

$$\sum_{i,n} \alpha_{in} \int_{\mathbb{R}^N} Z_{P_{i,n}} \frac{\partial(\chi w_{P_i})}{\partial x_m} \, dx = - \int_{\mathbb{R}^N} \theta \frac{\partial(\chi w_{P_i})}{\partial x_m} \, dx + \int_{\mathbb{R}^N} \mathcal{L}_P[\phi] \frac{\partial(\chi w_{P_i})}{\partial x_m} \, dx.$$  

First examine the left hand side of (B.58). By using (B.55)

$$\left| \sum_{i,n} \alpha_{in} \int_{\mathbb{R}^N} Z_{P_{i,n}} \frac{\partial(\chi w_{P_i})}{\partial x_m} \, dx \right| \geq C \varepsilon^{N-2} |\alpha_{jm}| + o(\varepsilon^{N-2}) \sum_{i,n} |\alpha_{in}|.$$  

The first term on the right hand side of (B.58) can be estimated as

$$\int_{\mathbb{R}^N} |\theta \frac{\partial(\chi w_{P_i})}{\partial x_m}| \, dx \leq C \|\theta\|_{*, P} \int_{\mathbb{R}^N} |\nabla w_{P_i}| \, dx \leq C \varepsilon^{N-1} \|\theta\|_{*, P}.$$  

Finally, by using (B.58),

$$\left| \int_{\mathbb{R}^N} \mathcal{L}_P[\phi] \frac{\partial(\chi w_{P_i})}{\partial x_m} \, dx \right| = \left| \int_{\mathbb{R}^N} \phi \left[ - Z_{P_{i,m}} + f'(\chi w_P) \frac{\partial(\chi w_{P_i})}{\partial x_m} \right] \, dx \right| \leq C \|\phi\|_{*, P} \int_{\mathbb{R}^N} \left| (f'(w_P) - f'(w_{P_j})) \frac{\partial w_{P_i}}{\partial x_m} \right| \, dx + C \varepsilon^{N+2\beta-2} \|\phi\|_{*, P}$$

$$\leq C \varepsilon^{-1} \|\phi\|_{*, P} \sum_{i \neq j} \int_{\mathbb{R}^N} w_{P_i}^2 w_{P_j} \, dx + C \varepsilon^{N+2\beta-2} \|\phi\|_{*, P} \leq C \|\phi\|_{*, P} (\varepsilon^{N+2\beta-1} + \varepsilon^{N+2\beta-2}).$$

where last inequality follows from (A.38). Combining this with (B.58), (B.59) and (B.60), we achieve the thesis. 

Now we prove the following a priori estimate for (B.57).
Lemma B.2. There exists a constant $C > 0$ such that, provided that $\varepsilon$ is sufficiently small, if $P \in \Gamma_\varepsilon$ and $(\phi, \theta, \alpha_{in})$ satisfies (B.57), the following holds:

$$\|\phi\|_{*,P} \leq C\|\theta\|_{*,P}.$$  

Proof. We argue by contradiction. Assume the existence of a sequence $\varepsilon_k \to 0^+$, $P_k \in \Gamma_{\varepsilon_k}$ and $(\phi_k, \theta_k, \alpha_{in}^k)$ satisfying (B.57) such that

$$\|\phi_k\|_{*,P_k} = 1, \quad \|\theta_k\|_{*,P_k} = o(1).$$

By Lemma [3.1] we deduce $\alpha_{in}^k = o(\varepsilon)$ for every $(i, n)$, by which $\|\theta_k + \sum_{i,n} \alpha_{in}^k \Delta P_{i,n}\|_{*,P_k} = o(1)$ and, consequently,

(B.61)  

$$\|\varepsilon_k^2 \Delta \phi_k - V(x)\phi_k + f'(\chi \omega_{\varepsilon_k})\phi_k\|_{*,P_k} = o(1).$$

We claim that

(B.62)  

$$\|\phi_k\|_{L^\infty(B_{R_k}(P_k))} = o(1) \quad \forall R > 0.$$  

Otherwise, we may assume that $\|\phi_k\|_{L^\infty(B_{R_k}(P_k))} \geq c > 0$ for some $R > 0$. By multiplying the equation in (B.62) by $\phi_k$ and integrating by parts we immediately get that the sequence $\phi_k(\varepsilon_k x + P_k^k)$ is bounded in $H^1(\mathbb{R}^N)$. Therefore, possibly passing to a subsequence, $\phi_k(\varepsilon_k x + P_k^k) \rightharpoonup \phi_0$ weakly in $H^1(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$, and $\phi_0$ satisfies

$$\Delta \phi_0 - \phi_0 + f'(\omega)\phi_0 = 0, \quad |\phi_0(x)| \leq \omega^\mu(x).$$

According to elliptic regularity theory we may assume $\phi_k(\varepsilon_k x + P_k^k) \to \phi_0$ uniformly on compact sets, then $\|\phi_0\|_{\infty} \geq c$. By assumption (3) $\phi_0 = \sum_{n=1}^N a_n \frac{\partial w}{\partial x_n}$. On the other hand for $m = 1, \ldots, N$, using (B.53),

$$0 = \int_{\mathbb{R}^N} \phi_k(\varepsilon_k x + P_k^k)Z_{m,n}(\varepsilon_k x + P_k^k) \to \sum_{n=1}^N a_n \int_{\mathbb{R}^N} \frac{\partial w}{\partial x_n} = a_m \|\frac{\partial w}{\partial x_m}\|_{H^1(\mathbb{R}^N)},$$

which implies $a_m = 0$, that is $\phi_0 = 0$. The contradiction follows.

Hence we have proved (B.62), by which we immediately obtain

$$\|f'(\chi \omega_{\varepsilon_k})\phi_k\|_{*,P_k} = o(1)$$

and, by (B.61),

$$\|\varepsilon_k^2 \Delta \phi_k - V(x)\phi_k\|_{*,P_k} = o(1).$$

Observe that by (1.19), if we set $\Phi_k(x) = \frac{1}{2} (\sum_{i=1}^\ell w_{P_k})^\mu$, it follows that, provided that $\mu$ is chosen sufficiently small, for every $k$:

$$\varepsilon_k^2 \Delta \Phi_k - V(x)\Phi_k \leq \frac{\inf_{\mathbb{R}^N} V}{2} \Phi_k \text{ in } \mathbb{R}^N.$$  

Then one has

$$\varepsilon_k^2 \Delta (\Phi_k \pm \phi_k) - V(x)(\Phi_k \pm \phi_k) \leq 0 \text{ in } \mathbb{R}^N.$$  

By the comparison principle it follows that $\Phi_k \pm \phi_k \geq 0$. Then we have $|\phi_k| \leq \frac{1}{2}(\sum_{i=1}^\ell w_{P_k})^\mu$, by which $\|\phi_k\|_{*,P_k} \leq \frac{1}{2}$, in contradiction with $\|\phi_k\|_{*,P_k} = 1$.  

Now we are in position to provide the existence of a solution for the system (B.57).
Lemma B.3. For $\varepsilon > 0$ sufficiently small, for every $P \in \Gamma_\varepsilon$ and $\theta \in L^2(\mathbb{R}^N)$, there exists a unique pair $(\phi, \alpha_{in})$ solving (B.57). Furthermore
\[
\|\phi\|_{*, P} \leq C\|\theta\|_{*, P}, \quad |\alpha_{in}| \leq C(\varepsilon^{1+\sigma}\|\theta\|_{*, P} + \varepsilon\|\theta\|_{*, P}).
\]

Proof. The existence follows from Fredholm’s alternative. For every $P \in \Gamma_\varepsilon$ let us consider $\mathcal{H}_P$ the closed subset of $H^1_\varepsilon(\mathbb{R}^N)$ defined by
\[
\mathcal{H}_P = \left\{ \phi \in H^1_\varepsilon(\mathbb{R}^N) \mid \left( \phi, \frac{\partial(\chi w_P)}{\partial x_i} \right)_\varepsilon = 0 \quad \forall i = 1, \ldots, \ell, \forall n = 1, \ldots N \right\}.
\]
Notice that, by (B.54), $\phi \in \mathcal{H}_P$ solves the equation $\mathcal{L}_P[\phi] = \theta + \sum_{i,n} \alpha_{in} Z_{P,n}$ if and only if
\[
(B.63) \quad (\phi, \psi)_\varepsilon - \int_{\mathbb{R}^N} f'(\chi w_P)\phi \psi dx = - \int_{\mathbb{R}^N} \theta \psi dx \quad \forall \psi \in \mathcal{H}_P.
\]
Indeed, once we know $\phi$, we can determine the unique $\alpha_{in}$ from the linear system of equations
\[
\int_{\mathbb{R}^N} f'(\chi w_P)\phi \frac{\partial(\chi w_P)}{\partial x_m} dx = \int_{\mathbb{R}^N} \theta \frac{\partial(\chi w_P)}{\partial x_m} dx + \sum_{i,n} \alpha_{in} \int_{\mathbb{R}^N} Z_{P,n} \frac{\partial(\chi w_P)}{\partial x_m} dx,
\]
for $j = 1, \ldots, \ell, m = 1, \ldots, N$, which is uniquely solvable according to (B.55). By standard elliptic regularity, $\phi \in H^2(\mathbb{R}^N)$.

Thus it remains to solve (B.63). According to Riesz’s representation theorem, take $K_P(\phi), \bar{\theta} \in \mathcal{H}_P$ such that
\[
(K_P(\phi), \psi)_\varepsilon = - \int_{\mathbb{R}^N} f'(\chi w_P)\phi \psi dx \quad (\bar{\theta}, \psi)_\varepsilon = - \int_{\mathbb{R}^N} \theta \psi dx \quad \forall \psi \in \mathcal{H}_P.
\]
Then problem (B.63) consists in finding $\phi \in \mathcal{H}_P$ such that
\[
(B.64) \quad \phi + K_P(\phi) = \bar{\theta}.
\]
It is easy to prove that $K_P$ is a linear compact operator from $\mathcal{H}_P$ to $\mathcal{H}_P$. Using Fredholm’s alternatives, (B.64) has a unique solution for each $\bar{\theta}$, if and only if (B.64) has a unique solution for $\bar{\theta} = 0$. Let $\phi \in \mathcal{H}_P$ be a solution of $\phi + K_P(\phi) = 0$; then $\phi$ solves the system (B.57) with $\theta = 0$ for some $\alpha_{in} \in \mathbb{R}$. Lemma B.2 implies $\phi \equiv 0$. The remaining part of the Lemma follow by Lemma B.1 and Lemma B.2. \qed

B.2. Lyapunov-Schmidt Reduction. To complete the Lyapunov-Schmidt Reduction, it remains to prove Lemma 2.1 and Proposition 2.2.

Proof of Lemma 2.1. We write the equation in (2.8) in the following form:
\[
(B.65) \quad \mathcal{L}_P[\phi] = -S_\varepsilon[\chi w_P] - N_P[\phi] + \sum_{i,n} \alpha_{in} Z_{P,n}.
\]
and use contraction mapping theorem. Here
\[
N_P[\phi] = f(\chi w_P + \phi) - f(\chi w_P) - f'(\chi w_P)\phi.
\]
Consider the metric space $B_P = \{ \phi \in L^2(\mathbb{R}^N) \mid \|\phi\|_{*, P} \leq \varepsilon^\eta \}$ endowed with the norm $\|\cdot\|_{*, P}$. Given $\phi_1, \phi_2 \in B_P$, by assumption (f1) we have
\[
(B.66) \quad \|N_P[\phi_1] - N_P[\phi_2]\|_{*, P} \leq C\varepsilon^\eta \|\phi_1 - \phi_2\|_{*, P}.
\]
For every $\phi \in B_P$ we define $A_P[\phi] \in H^2(\mathbb{R}^N) \cap H_1^1(\mathbb{R}^N)$ to be the unique solution to the system (B.57) given by Lemma [3.3] with $\theta = \theta_P[\phi] := -S_{\epsilon}[\chi w_P] - N_P[\phi]$. By (B.66), Lemma [4.4] Lemma [3.3]

$$
\|A_P[\phi]\|_{L^p} \leq C \|\theta_P[\phi]\|_{L^p} \leq C(\epsilon^{(\beta+\sigma)} + \epsilon^{(1+\eta)}) < \epsilon^\eta
$$

at least for small $\epsilon$, and hence $A_P[\phi] \in B_P$. Moreover, since $A_P[\phi_1] - A_P[\phi_2]$ solves the system (B.57) with $\theta = -N_P[\phi_1] + N_P[\phi_2]$, by (B.66) and Lemma [3.3] we also have that

$$
\|A_P[\phi_1] - A_P[\phi_2]\|_{L^p} \leq C\|N_P[\phi_1] - N_P[\phi_2]\|_{L^p} \leq \|\phi_1 - \phi_2\|_{L^p} \quad \forall \phi_1, \phi_2 \in B_P, \quad \forall \epsilon \in \Gamma,
$$

i.e. the map $A_P$ is a contraction map from $B_P$ to $B_P$. By the contraction mapping theorem, (2.8) has a unique solution $(\phi_P, \alpha_{in}(P)) \in B_P \times \mathbb{R}^{N^l}$.

Finally, by multiplying the equation in (B.65) by $\phi_P$ and integrating over $\mathbb{R}^N$ we immediately obtain $(\phi_P, \alpha_{in}) \leq C\epsilon^{N+2\eta}$. By Lemma [3.1] we get

$$
|\alpha_{in}(P)| \leq C(\epsilon^{1+\eta} \|\phi_P\|_{L^p} + \epsilon \|\theta_P[\phi_P]\|_{L^p}) \leq C\epsilon^{1+\eta}.
$$

The fact that the map $P \in \Gamma \rightarrow \phi_P \in H^1_\epsilon(\mathbb{R}^N)$ is $C^1$ follows from the Implicit Function Theorem. See [3], for instance.

**Proof of Proposition 2.2** We compute

$$
I_\epsilon[\chi w_P + \phi_P] = \frac{1}{2} \int_{\mathbb{R}^N} (\epsilon^2|\nabla(\chi w_P + \phi_P)|^2 + V(x)(\chi w_P + \phi_P)^2)dx - \int_{\mathbb{R}^N} F(\chi w_P + \phi_P)dx
$$

$$
= I_\epsilon[\chi w_P] - \int_{\mathbb{R}^N} S_{\epsilon}[\chi w_P] \phi_P dx + \frac{1}{2}(\phi_P, \phi_P)_\epsilon
$$

$$
- \int_{\mathbb{R}^N} (F(\chi w_P + \phi_P) - F(\chi w_P) - f(\chi w_P)\phi_P)dx.
$$

By Lemma [4.4] we have $|S_{\epsilon}[\chi w_P]| \leq \epsilon^\eta \sum_{i=1}^{\ell} w_i^{-\beta}$ for small $\epsilon$, while $|F(\chi w_P + \phi_P) - F(\chi w_P) - f(\chi w_P)\phi_P| \leq C|\phi_P|^2$; hence, by using (2.11) we get

$$
I_\epsilon[\chi w_P + \phi_P] = I_\epsilon[\chi w_P] + O(\epsilon^{N+2})
$$

uniformly for $P \in \Gamma$. (2.11) follows from Proposition [3.4] Next, denoting by $P_{i,n}$ the $n$-th component of $P_i$, since $\frac{\partial w_P}{\partial x_n} = -\tau_i \frac{\partial w_P}{\partial x_n}$, we compute

$$
\frac{\partial}{\partial P_{i,n}} I_\epsilon[\chi w_P + \phi_P] = -\int_{\mathbb{R}^N} S_{\epsilon}[\chi w_P + \phi_P] \frac{\partial(\chi w_P + \phi_P)}{\partial x_n} dx
$$

$$
= \frac{\partial}{\partial P_{i,n}} I_\epsilon[\chi w_P] - \tau_i(\phi_P, \phi_P) \frac{\partial(\chi w_P)}{\partial x_n} - \int_{\mathbb{R}^N} S_{\epsilon}[\chi w_P + \phi_P] \frac{\partial \phi_P}{\partial P_{i,n}} - \int_{\mathbb{R}^N} (f(\chi w_P + \phi_P) - f(\chi w_P)) \frac{\partial(\chi w_P)}{\partial P_{i,n}}
$$

$$
= \frac{\partial}{\partial P_{i,n}} I_\epsilon[\chi w_P] - \sum_{j,m} \alpha_{jm}(P) \int_{\mathbb{R}^N} Z_{P_{j,m}} \frac{\partial \phi_P}{\partial P_{i,n}} + \tau_i \int_{\mathbb{R}^N} (f(\chi w_P + \phi_P) - f(\chi w_P)) \frac{\partial w_P}{\partial x_n}.
$$

Since $\int_{\mathbb{R}^N} Z_{P_{j,m}} \frac{\partial \phi_P}{\partial P_{i,n}} dx = 0$, by differentiation we get

$$
(B.67) \quad \int_{\mathbb{R}^N} Z_{P_{j,m}} \frac{\partial \phi_P}{\partial P_{i,n}} dx = -\int_{\mathbb{R}^N} \frac{\partial Z_{P_{j,m}}}{\partial P_{i,n}} \phi_P = O(\epsilon^{N+\eta-2}),
$$

\footnote{Observe that $|\frac{\partial Z_{P_{j,m}}}{\partial P_{i,n}}| = \delta_{ij} |(V(x) - \epsilon^2\Delta)(\frac{\partial w_P}{\partial x_m})| \leq C\epsilon^{-2} w_P$ by (18).}
by which, using Lemma 2.1,

$$\sum_{j,m} \alpha_{jm}(P) \int_{\mathbb{R}^N} Z_{P_j,m} \frac{\partial \varphi_P}{\partial x_n} \, dx = O(\varepsilon^{N+2\eta-1}).$$  \hspace{1cm} (B.68)

By assumption (f1) we have $|f(\chi w_P + \phi_P) - f(\chi w_P) - f'(\chi w_P)\phi_P| \leq C|\phi_P|^{1+\sigma}$; consequently

$$\int_{\mathbb{R}^N} (f(\chi w_P + \phi_P) - f(\chi w_P) - f'(\chi w_P)\phi_P) \chi \frac{\partial w_P}{\partial x_n} = O(\varepsilon^{N+(1+\sigma)-1}).$$  \hspace{1cm} (B.69)

Finally, by (A.43) and (B.53),

$$\left| \int_{\mathbb{R}^N} f'(\chi w_P)\phi_P \chi \frac{\partial w_P}{\partial x_n} \right| = \left| \int_{\mathbb{R}^N} f'(\chi w_P) \left( \chi \frac{\partial w_P}{\partial x_n} - Z_{P_i,n} \right) \phi_P \, dx \right| \leq C\varepsilon \int_{\mathbb{R}^N} \left| f'(w_P) - f'(w_{P_i}) \right| \left| \frac{\partial w_P}{\partial x_n} \right| \, dx + C\varepsilon^{N+2\beta+\eta-1}$$

$$\leq C\varepsilon \sum_{j \neq i} \int_{\mathbb{R}^N} w_{P_i}^\sigma \left| \frac{\partial w_P}{\partial x_n} \right| \, dx + C\varepsilon^{N+2\beta+\eta-1} \leq C\varepsilon^{N+2\beta+\eta-1}$$  \hspace{1cm} (B.70)

where in the last inequality we have used (A.43). Combining (B.68), (B.69), and (B.70), we deduce

$$\frac{\partial}{\partial P_i,n} I_\varepsilon[\chi w_P + \phi_P] = \frac{\partial}{\partial P_i,n} I_\varepsilon[\chi w_P] + O(\varepsilon^{N+(1+\sigma)^2-1})$$

uniformly for $P \in \Gamma_r$. By applying Proposition A.3, we obtain (2.12), using that $\beta(1+\sigma)^2 - 1 > \beta$ thanks to assumption (f1) if $\beta$ is close to 1. \hfill \Box

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