ON SUBGROUPS OF AN AUTOMORPHISM GROUP OF AN IRREDUCIBLE SYMPLECTIC MANIFOLD

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Abstract. Let $X$ be an irreducible symplectic manifold and $L$ a nef line bundle on $X$ which is isotropic with respect to the Beauville-Bogomolov quadratic form. It is known that a subgroup $\text{Aut}(X, L)$ of an automorphism group of $X$ which fix $L$ is almost abelian. We give a formula of the rank of $\text{Aut}(X, L)$ in terms of MBM divisors. We also prove that the nef cone of $X$ cut out MBM classes, which is a generalization of Kovac’s structure theorem of nef cones of $K3$ surfaces.

1. Introduction

We start with recalling definitions of a Neron-Severi group, an ample cone and a nef cone.

Definition 1.1. Let $X$ be a compact Kähler manifold. A Neron-Severi group $\text{NS}(X)$ is a subgroup of $H^2(X, \mathbb{Z})$ defined by

$$\text{NS}(X) := H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z}).$$

We denote by $\text{NS}_\mathbb{R}(X)$ the $\mathbb{R}$-vector space generated by $\text{NS}(X)$. An ample cone $\text{Amp}(X)$ of $X$ is the cone in $\text{NS}_\mathbb{R}(X)$ defined by

$$\text{Amp}(X) := \text{NS}_\mathbb{R}(X) \cap \mathcal{K}(X),$$

where $\mathcal{K}(X)$ is the Kähler cone of $X$. The closure of $\text{Amp}(X)$ in $\text{NS}_\mathbb{R}(X)$ is said to be nef cone and denoted by $\text{Nef}(X)$.

We also recall the definition of irreducible symplectic manifolds.

Definition 1.2. A compact Kähler manifold $X$ is said to be irreducible symplectic if $X$ has the following three properties:

1. $X$ is simply connected;
2. $X$ carries a holomorphic symplectic form and;
3. $\dim H^0(X, \Omega^2_X) = 1$.

A $K3$ surface has above three properties and gives a plain example of irreducible symplectic manifolds. It is expected that $K3$ surfaces and irreducible symplectic manifolds share many geometric properties. One of the biggest geometric features of $K3$ surfaces is Global Torelli theorem, which was obtained in [8], [12] and [17].

Theorem 1.1 (Global Torelli Theorem for projective $K3$ surfaces). Let $X$ and $X'$ be projective $K3$ surfaces. Assume that there exists an isometry $\phi : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ with respect to the cup products. If $\phi$ respects the Hodge structure and $\phi(\text{Amp}(X')) \cap \text{Amp}(X) \neq \emptyset$, there exists an automorphism $\Phi$ such that the induced morphism $\Phi^*$ on cohomologies coincides with $\phi$.

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A higher dimensional analogue of Global Torelli Theorem was obtained by Verbitsky in [21, Theorem 1.18]. After introducing the definition of monodromy groups, we state it in a form suitable for use in this paper according to [13, Theorem 1.3 (2)].

**Definition 1.3.** Let $X$ be an irreducible symplectic manifold. We denote by $q_X$ the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{Z})$. Let $O(H^2(X, \mathbb{Z}), q_X)$ be an isometry group with respect to $q_X$. Let us consider smooth morphisms $\mathcal{X} \to (B, o)$ such that $(B, o)$ is an analytic space with the reference point $o$ and the fibre at $o$ is isomorphic to $X$. We note that $B$ may have any kind of singularities. For such a smooth morphism, we have a natural representation $\pi_1(B, o) \to O(H^2(X, \mathbb{Z}))$. A subgroup $\text{Mon}(X)$ of $O(H^2(X, \mathbb{Z}), q_X)$ is the subgroup generated by the images of all such representations.

**Theorem 1.2** (Global Torelli Theorem for projective irreducible symplectic manifolds). Let $X$ be a projective irreducible symplectic manifold. Assume that there exists an element $\phi$ of $\text{Mon}(X)$ which respects the Hodge structures and $\phi(\text{Amp}(X)) \cap \text{Amp}(X) \neq \emptyset$. Then there exists an automorphism $\Phi$ of $X$ such that the induced automorphism $\Phi^*$ on $H^2(X, \mathbb{Z})$ coincides with $\phi$.

By the above theorem, we will have automorphisms of projective irreducible symplectic manifolds if we construct elements of $\text{Mon}(X)$ which satisfies the assumptions of Theorem 1.2. In this note, we will construct such elements of $\text{Mon}(X)$ and give three applications. The first application concerns with the structure of nef cones of projective irreducible symplectic manifolds. By [10] and [7], we have the following structure theorem of a Kähler cone of an irreducible symplectic manifold.

**Theorem 1.3.** Let $X$ be an irreducible symplectic manifold and $\mathcal{N}(X)$ the set of rational curves on $X$. We define the positive cone $\mathcal{C}(X)$ in $H^{1,1}(X, \mathbb{R})$ by

$$\mathcal{C}(X) := \{ x \in H^{1,1}(X, \mathbb{R}) | q_X(x) > 0, q_X(x, \kappa) > 0 \}$$

where $q_X$ is the Beauville-Bogomolov form and $\kappa$ is a Kähler class. Then

$$\mathcal{N}(X) = \{ x \in \mathcal{C}(X) | \forall e \in \mathcal{N}(X), x.e > 0 \}$$

We denote by $\mathcal{C}_{\text{NS}}(X)$ the intersection of $\text{NS}_\mathbb{R}(X)$ and $\mathcal{C}(X)$. A nef cone $\text{Nef}(X)$ of an irreducible symplectic manifold $X$ can be described as follows:

$$\text{Nef}(X) = \{ x \in \mathcal{C}_{\text{NS}}(X) | \forall e \in \mathcal{N}(X), x.e \geq 0 \}$$

where $-$ stands for the closure in $\text{NS}_\mathbb{R}(X)$. On the other hand, Kovacs gave a description of an effective cone of a K3 surface in [11], whose dual cone with respect to the cup product is a nef cone.

**Theorem 1.4** ([11, Corollary 1]). Let $X$ be a projective K3 surface whose Picard number is greater than two. We denote by $\mathcal{N}(X)$ the set of $(-2)$-curves on $X$. If $\mathcal{N}(X) = \emptyset$, then an effective cone $\text{Eff}(X)$ of $X$ coincides with $\mathcal{C}_{\text{NS}}(X)$. If $\mathcal{N}(X) \neq \emptyset$, then

$$\text{Eff}(X) = \sum_{e \in \mathcal{N}(X)} \mathbb{R}_+ e.$$

If we consider the dual statement of the above theorem, we find that $\text{Nef}(X)$ coincides with $\mathcal{C}_{\text{NS}}(X)$ if $\mathcal{N}(X) = \emptyset$. If $\mathcal{N}(X) \neq \emptyset$,

$$\text{Nef}(X) = \{ x \in \text{NS}_\mathbb{R}(X) | \forall e \in \mathcal{N}(X), x.e \geq 0 \}$$
It is a natural question whether a nef cone of an irreducible symplectic manifold has a similar structure. We give a positive answer of this question. To state our result, we recall monodromy birationally minimal classes, which is introduced in [1 Definition 1.13].

**Definition 1.4** (Monodromy Birationally Minimal Class). Let $X$ be an irreducible symplectic manifold. A cohomology class $e$ of $H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Q})$ is said to be Monodromy birationally minimal if there exists an element $\gamma$ of $\text{Mon}(X)$ such that $\gamma(e) \perp \varpi(X)$ is an open set of $\gamma(e)^\perp$. We denote by $\text{MBM}(X)$ the set of Monodromy birationally minimal classes of $X$.

**Remark 1.1.** If $X$ is a K3 surface, then

$$\text{MBM}(X) = \{ e \in H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z}) | \langle e, e \rangle = -2 \}.$$ 

By the above remark, we can restate the structure of the nef cone of a K3 surface in the following form. If $\text{MBM}(X) = \emptyset$, then $\text{Nef}(X)$ coincides with $\overline{\mathcal{N}}_{\text{NS}}(X)$. If $\text{MBM}(X) \neq \emptyset$, there exists a subset $\mathcal{N}(X)$ of $\text{MBM}(X)$ such that

$$\text{Nef}(X) = \{ x \in \text{NS}_\mathbb{R}(X) | \forall e \in \mathcal{N}(X), x.e \geq 0 \}$$

Now we state the first application.

**Theorem 1.5.** Let $X$ be a projective irreducible symplectic manifold whose Picard number is greater than two. If $\text{MBM}(X) = \emptyset$, then $\text{Nef}(X)$ coincides with $\overline{\mathcal{N}}_{\text{NS}}(X)$. If $\text{MBM}(X) \neq \emptyset$, there exists a subset $\mathcal{N}(X)$ of $\text{MBM}(X)$ such that

$$\text{Nef}(X) = \{ x \in \text{NS}_\mathbb{R}(X) | \forall e \in \mathcal{N}(X), q_X(x, e) \geq 0 \}$$

where $q_X$ is the Beauville-Bogomolov quadratic form.

**Remark 1.2.** If $X$ is a K3 surface, $\mathcal{N}(X)$ coincides with the set of smooth rational curves. In the above theorem, $\mathcal{N}(X)$ seems to coincide with the set of rational cohomology class corresponding to smooth rational curves.

The second application concerns with the rank of a subgroup of an automorphism group of an irreducible symplectic manifold which fixes a line bundle. We recall the definition of an almost abelian group according to [14].

**Definition 1.5.** A group $G$ is said to be almost abelian of rank $r$ if $G$ has a normal subgroup $G^{(0)}$ such that $|G : G^{(0)}| < \infty$ and $G^{(0)}$ sits in the following exact sequence.

$$1 \to K \to G^{(0)} \to \mathbb{Z}^r \to 0$$

where $K$ is a finite group.

**Theorem 1.6.** Let $X$ be an irreducible symplectic manifold and $L$ an isotropic nef line bundle with respect to Beauville-Bogomolov quadratic form. We define the subset $\text{MBM}(X)^\circ$ of $\text{MBM}(X)$ by

$$\text{MBM}(X)^\circ := \{ e \in \text{MBM}(X) | e^{\perp} \cap \text{Nef}(X) \text{ is an open set of } e^{\perp} \}.$$ 

Let $W_\mathbb{R}$ be a sub linear space in generated by $c_1(L)$ and $c_1(L)^\perp \cap \text{MBM}(X)^\circ$. We denote by $\text{Aut}(X, L)$ the subgroup of $\text{Aut}(X)$ defined by

$$\text{Aut}(X, L) := \{ g \in \text{Aut}(X) | g^*L \cong L \}$$

Then $\text{Aut}(X, L)$ is almost abelian of rank $\dim \text{NS}_\mathbb{R}(X) - \dim W_\mathbb{R} - 1$. 

Remark 1.3. Let $X$ be a K3 surface. Assume that $X$ admits an elliptic fibration $\pi : X \to \mathbb{P}^1$. We denote by $L$ the pull back of the tautological bundle of $\mathbb{P}^1$. For a point $t$ of $\mathbb{P}^1$, we let $n_t$ be the number of irreducible components of the fibre at $t$. In this case, $c_1(L^\perp) \cap \text{MBM}(X)^{\circ}$ consists of irreducible components of reducible singular fibres of $\pi$. Since cohomology classes of irreducible components of a reducible singular fibre has only one relation in $H^2(X, \mathbb{Z})$, $\dim W_{\mathbb{R}} = 1 + \sum_{t \in \mathbb{P}^1}(n_t - 1)$ and Shioda-Tate formula in [18], [20] and [19] asserts that the rank of Mordell-Weil group of $\pi$ coincides with $\dim \text{NS}_{\mathbb{R}}(X) - \dim W_{\mathbb{R}} - 1$. Since Mordell-Weil group of $\pi$ can be considered as a subgroup of $\text{Aut}(X, L)$, Theorem 1.6 can be considered as a generalization of Shioda-Tate formula.

By Theorem 1.6 the rank of $\text{Aut}(X, L)$ is less than or equal $\dim H^2(X, \mathbb{R}) - 2$. It is a natural question whether this bound is sharp. The third application is that the bound is attained after deforming the pair $(X, L)$.

Definition 1.6. Let $X$ and $X'$ be compact Kähler manifolds. We also let $L$ and $L'$ be line bundles on $X$ and $X'$, respectively. Two pairs $(X, L)$ and $(X', L')$ are deformation equivalent if there exists a smooth morphism $\pi : \mathcal{X} \to B$ over an analytic space $B$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$ which has the following two properties:

1. There exist two points $p$ and $p'$ of $B$ such that $\iota : \pi^{-1}(p) \cong X$ and $\iota' : \pi^{-1}(p') \cong X'$, respectively.
2. The restriction of $\mathcal{L}$ to $\pi^{-1}(p)$ is isomorphic to $L$ via $\iota$ and the restriction of $\mathcal{L}$ to $\pi^{-1}(p')$ is isomorphic to $L'$ via $\iota'$.

Theorem 1.7. Let $X$ be an irreducible symplectic manifold whose second Betti number is greater than five and $L$ an isotropic line bundle with respect to Beauville-Bogomolov quadratic form. Then there exists an irreducible symplectic manifold $X'$ and a line bundle $L'$ on $X'$ such that $(X, L)$ is deformation equivalent to $(X', L)$ in the sense of Definition 1.6 and the rank of $\text{Aut}(X', L')$ is equal to $\dim H^2(X', \mathbb{R}) - 2$.

This note is organized as follows. In section 1, We will construct special elements of $O(H^2(X, \mathbb{Z}))$, which is a key of the proof of Theorems 1.5, 1.6 and 1.7. In sections 2, 3, and 4, we give a proof of Theorem 1.5, 1.6 and 1.7 respectively.

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2. Construction of elements of the monodromy group

We recall a standard properties of isometry group of a lattice due to the step 2 of the proof of [3, Proposition 3.2].

Lemma 2.1. Let $\Lambda$ be a lattice and $\Lambda'$ a sublattice of $\Lambda$ such that $|\Lambda : \Lambda'| < \infty$. We also let $O(\Lambda)$ and $O(\Lambda')$ be isometry groups of $\Lambda$ and $\Lambda'$, respectively. The groups $O(\Lambda)$ and $O(\Lambda')$ can be considered as subgroups of $O(\Lambda \otimes \mathbb{Z} \mathbb{Q})$. Moreover

$$|O(\Lambda') : O(\Lambda) \cap O(\Lambda')|$$

is finite.
Proof. Since \(|\Lambda : \Lambda'| < \infty\), there exists a positive integer \(N\) such that \(\Lambda \subset \frac{1}{N}\Lambda'\). Since \(O(\Lambda')\) preserves \(\frac{1}{N}\Lambda'\), \(O(\Lambda')\) acts on \(\frac{1}{N}\Lambda'/\Lambda'\). Then \(O(\Lambda) \cap O(\Lambda')\) is the stabilizer group of \(\Lambda/\Lambda'\). Since \(\Lambda/\Lambda'\) has only finitely many element, we are done. \(\square\)

Proposition 2.1. Let \(\Lambda\) be a lattice of rank \(n\) whose index is \((1, n - 1)\). Assume that \(\Lambda\) contains an isotropic element \(\ell\). Let \(W\) be a negative definite sub lattice contained in \(\ell\). Assume that \(n - \text{rank}(W) > 2\).

1. We define the subgroup \(\tilde{\Gamma}\) of the isometry group \(O(\Lambda)\) of \(\Lambda\) by
   \[ \tilde{\Gamma} := \{ g \in O(\Lambda) | g(\ell) = \ell, \forall w \in W, g(w) = w \} \]
   Then \(\tilde{\Gamma}\) contains a subgroup \(\tilde{\Gamma}_0\) which is isomorphic to \(\mathbb{Z}^{n-\text{rank}(W)-2}\) and \(|\tilde{\Gamma} : \tilde{\Gamma}_0| < \infty\).

2. We denote by \(\Lambda_{\mathbb{R}}\) the linear space \(\Lambda \otimes_{\mathbb{Z}} \mathbb{R}\) and define a positive cone \(\mathcal{C}(\Lambda_{\mathbb{R}})\)
   \[ \mathcal{C}(\Lambda_{\mathbb{R}}) := \{ x \in \Lambda_{\mathbb{R}} | \langle x, x \rangle > 0 \} \]
   For every element \(g\) of \(\Gamma_0\) and every element \(x\) of \(\mathcal{C}(\Lambda_{\mathbb{R}})\),
   \[ \lim_{m \to \infty} g^m x = \ell \quad \text{in} \quad \mathbb{P}(\Lambda_{\mathbb{R}}). \]

Proof. (1) Let \(W^\perp\) be the orthogonal lattice of \(W\). The restriction
   \[ \tilde{\Gamma} \ni g \mapsto g|_{W^\perp} \in O(W^\perp) \]
   is injective, because \(g\) acts on \(W\) trivially. We identify \(\tilde{\Gamma}\) and its image. Since the index of \(W^\perp\) is \((1, n - \text{rank}(W) - 1)\), by [14, Proposition 2.9], \(\tilde{\Gamma}\) contains a subgroup \(\tilde{\Gamma}_0\) which is isomorphic to \(\mathbb{Z}^m\), \((0 \leq m \leq n - \text{rank}(W) - 2)\). Moreover \(|\tilde{\Gamma} : \tilde{\Gamma}_0| < \infty\). Let us assume that we have a subgroup \(\tilde{\Gamma}_0\) of \(\tilde{\Gamma}\) which is isomorphic to \(\mathbb{Z}^{n-\text{rank}(W)-2}\). Since \(|\tilde{\Gamma}_0 : \tilde{\Gamma}_0 \cap \tilde{\Gamma}_0'| \leq |\tilde{\Gamma} : \tilde{\Gamma}_0| < \infty\), \(m \geq n - \text{rank}(W) - 2\) and we are done. Hence we will construct a subgroup \(\tilde{\Gamma}_0\) which is isomorphic to \(\mathbb{Z}^{n-\text{rank}(W)-2}\). Let us consider the projection \(r : \ell^\perp \cap \Lambda \to \ell^\perp \cap \Lambda / \mathbb{Z}\ell\). Replacing \(W\) by its saturation in \(\Lambda\), we may assume that \(W\) is primitive. Since \(W\) is negative definite, \(W \cong r(W)\). Moreover, \(r(W)\) is primitive. We choose \(\text{rank}(W)\) elements \(\{u_1, \ldots, u_{\text{rank}(W)}\}\) of \(W\) such that the set of the residue classes \(\{\bar{u}_1, \ldots, \bar{u}_{\text{rank}(W)}\}\) forms a generator of \(r(W)\). Since \(r(W)\) is primitive, we have \(n - \text{rank}(W) - 2\) elements \(\{u_{\text{rank}(W)+1}, \ldots, u_{n-2}\}\) such that the residue classes \(\{\bar{u}_1, \ldots, \bar{u}_{n-2}\}\) forms a generator of \(\ell^\perp / \mathbb{Z}\ell\). Then \(\{\ell, u_1, \ldots, u_{n-2}\}\) forms a generator of \(\ell^\perp\). Since \(\ell^\perp\) is a primitive sub lattice of \(\Lambda\), we have an element \(\ell'\) of \(\Lambda\) such that \(\{\ell, u_1, \ldots, u_{n-2}, \ell'\}\) forms a generator of \(\Lambda\). The gram matrix \(G_{\Lambda}\) of the bilinear form of \(\Lambda\) with respect to the basis \(\{\ell, u_1, \ldots, u_{n-2}, \ell'\}\) can be described as follows;

\[ G_{\Lambda} = \begin{pmatrix} 0 & 0 & a \\ 0 & A & b \\ a & b & c \end{pmatrix} \]

where \(A\) is a negative definite symmetric matrix and \(a\) is a nonzero integer. We put \(d = \det A\). For an integer \(i\) with \(\text{rank}(W) + 1 \leq i \leq n - 2\), we define a \(n - 2\) row vector \(\gamma_i\) by

\[ (\text{The } j\text{-th column of } \gamma_i) = \begin{cases} d & j = i \\ 0 & j \neq i \end{cases} \]
Let $\gamma$ be a linear combination of $\gamma_i$, $(\text{rank}(W) + 1 \leq i \leq n - 2)$. We define the matrix $T(\gamma)$ by

$$g(\gamma) = \begin{pmatrix} 1 & -2\gamma & -2a\gamma(A^{-1})^t\gamma & -2\gamma b \\ 0 & E & 2a(A^{-1})^t\gamma & 1 \end{pmatrix},$$

where $E$ is the $(n-2) \times (n-2)$ identity matrix. Since $^tT(\gamma)G_\Lambda T(\gamma) = G_\Lambda$, there exists an element $g(\gamma)$ of $O(\Lambda)$ whose matrix of representation with respect to the basis $\{\ell, u_1, \ldots, u_{n-2}, \ell'\}$ coincides with $T(\gamma)$. By definition

1. $g(\gamma)(\ell) = \ell$
2. $g(\gamma)(u_i) = u_i - 2a_i\ell$ $(1 \leq i \leq n - 2)$
3. $g(\gamma)(\ell') = \ell' + 2a\sum_{i=1}^{n-2} b_i u_i - (2a(\gamma(A^{-1})^t\gamma) + 2\gamma b)\ell$

where $a_i$ is the $i$-th column of $\gamma$ and $b_i$ is the $i$-th row of $(A^{-1})^t\gamma$. By definition, $a_i = 0$, $(1 \leq i \leq \text{rank}(W))$. Hence $g(\gamma)$ is an element of $\Gamma$. Moreover

$$g(\gamma + \gamma') = g(\gamma)g(\gamma') = g(\gamma')g(\gamma)$$

for all linear combinations $\gamma$ and $\gamma'$ of $\gamma_i$, $(\text{rank}(W) + 1 \leq i \leq n - 2)$. By definition, $g(\gamma) = E$ if and only if $\gamma = 0$. We define the subgroup $\bar{\Gamma}_0$ of $\bar{\Gamma}$ generated by $g(\gamma_i)$, $(\text{rank}(W) + 1 \leq i \leq n - 2)$. By construction, $\bar{\Gamma}_0$ is isomorphic to $\mathbb{Z}^{n - \text{rank}(W) - 2}$ and we are done.

(2) We will use the same notation as in the proof of part (1). For an element $x$ of $\mathcal{O}(\Lambda_R)$, we have the following expression.

$$x = \alpha_0\ell + \sum_{i=1}^{n-2} \alpha_i u_i + \beta\ell'.$$

By the equations 1, 2 and 3, we have

$$g(m\gamma)(x) = \left(\alpha_0 - \sum_{i=1}^{n-2} 2m\alpha_i a_i - \beta(2am^2(\gamma(A^{-1})^t\gamma) + 2m\gamma b)\right)\ell$$

$$+ \sum_{i=1}^{n-2} (\alpha_i + 2\beta am b_i)u_i$$

$$+ \beta\ell'.$$

Since $\langle x, x \rangle > 0$ and the index of the induced bilinear form on $\ell^\perp$ is $(0, 0, n - 2)$, $\beta \neq 0$. Hence the order of growth of the coefficient of $\ell$ is $m^2$, while the order of growth of other coefficients are at most $m$. This implies that

$$\lim_{m \to \infty} g(m\gamma)(x) = \ell \text{ in } \mathbb{P}(\Lambda_R)$$

and we are done.

\[ \square \]

**Corollary 2.1.** Let $X$ be a projective symplectic manifold. Assume that there exists an element $\ell$ of $\text{NS}(X)$ which is isotropic with respect to Beauville-Bogomolov quadratic form. We denote by $\text{Mon}(X)$ the monodromy group of $X$ and by $n$ the Picard number of $X$. Let $W$ be a negative definite sublattice of $\text{NS}(X)$ which is contained in $\ell^\perp$. Assume that $n - \text{rank}(W) > 2$. Then $\text{Mon}(X)$ contains a subgroup $\Gamma$ which has the following three properties:
(1) $\Gamma$ is isomorphic to $\mathbb{Z}^{n-\text{rank}W-2}$;

(2) The action of $\Gamma$ respects the Hodge structure of $H^2(X, \mathbb{Z})$ and $\Gamma$ acts on the transcendental lattice of $H^2(X, \mathbb{Z})$ trivially;

(3) For every element $g$ of $\Gamma$, $g(\ell) = \ell$ and $g(w) = w$ for all elements of $W$ and;

(4) For every element $g$ of $\Gamma$ and every element $x$ of $\mathcal{C}_{\text{NS}}(X)$,

$$\lim_{m \to \infty} g^m x = \ell \text{ in } \mathbb{P}(\text{NS}_R(X)).$$

Proof. By Proposition 2.1, we have a subgroup $\bar{\Gamma}$ of $O(\text{NS}(X))$ which has the following three properties:

(1) $\bar{\Gamma}$ is isomorphic to $\mathbb{Z}^{n-\text{rank}W-2}$;

(2) For every element $g$ of $\bar{\Gamma}$, $g(\ell) = \ell$ and $g(w) = w$ for all elements of $W$ and;

(3) For every element $g$ of $\bar{\Gamma}$ and every element $x$ of $\mathcal{C}_{\text{NS}}(X)$,

$$\lim_{m \to \infty} g^m x = \ell \text{ in } \mathbb{P}(\text{NS}_R(X)).$$

Let $\text{NS}(X)^\perp$ be the orthogonal lattice of $\text{NS}(X)$ in $H^2(X, \mathbb{Z})$ with respect to Beauville-Bogomolov quadratic form. We recall $\text{NS}(X)^\perp$ is nothing but the transcendental lattice. We define a subgroup $\Gamma'$ of $O(\text{NS}(X) \oplus \text{NS}(X)^\perp)$ by

$$\Gamma' := \{g + \text{id}_{\text{NS}(X)}: g \in \bar{\Gamma}\}.$$

Since $|H^2(X, \mathbb{Z}) : \text{NS}(X) \oplus \text{NS}(X)^\perp| < \infty$,

$$|O(\text{NS}(X) \oplus \text{NS}(X)^\perp) \cap O(H^2(X, \mathbb{Z}))| < \infty$$

by Lemma 2.1. By the definition, the action of $\Gamma' \cap O(H^2(X, \mathbb{Z}))$ respects the Hodge structure of $H^2(X, \mathbb{Z})$ and $\Gamma' \cap O(H^2(X, \mathbb{Z}))$ acts on the transcendental lattice of $H^2(X, \mathbb{Z})$ trivially. By [21] Theorem 7.2 and [3] Theorem 2.6,

$$|O(H^2(X, \mathbb{Z})) : \text{Mon}(X)| < \infty.$$

Hence $|\Gamma' : \Gamma' \cap \text{Mon}(X)| < \infty$. If we define $\Gamma$ by $\Gamma' \cap \text{Mon}(X)$, we are done. □

3. Proof of Theorem 1.5

Before starting to prove Theorem 1.5, we prepare Lemma 3.1.

Lemma 3.1. Let $X$ be an irreducible symplectic manifold. Assume that the nef cone $\text{Nef}(X)$ contains an open set $U$ of $\partial\mathcal{C}_{\text{NS}}(X)$ and $\text{MBM}(X) \neq \emptyset$. Then $\partial\mathcal{C}_{\text{NS}}(X) \cap \text{NS}(X) \neq \emptyset$.

Proof. Let $e$ be an element of $\text{MBM}(X)$ such that $e^\perp \cap \text{Nef}(X)$ is an open set of $e^\perp$. We choose a 2-plane $H$ in $\text{NS}_R(X)$ as $H$ contains $e$, $H \cap U \neq \emptyset$ and $H$ is defined over $\text{NS}_R(X)$. The restriction $\text{Nef}(X) \cap H$ is generated by two rays $\ell_1$ and $\ell_2$. Since $H \cap U \neq \emptyset$, we may assume that $q_X(\ell_1) = 0$, where $q_X$ is the Beauville-Bogomolov quadratic form of $X$. Let $\pi: \mathfrak{X} \to \text{Def}(X)$ be a Kuranishi family of $X$. We define the subset $\Omega$ in $\mathbb{P}(H^2(X, \mathbb{C}))$ by

$$\Omega := \{x \in \mathbb{P}(H^2(X, \mathbb{C})) | q_X(x) = 0, q_X(x + \tilde{x}) > 0\}.$$

By [5] Théorème 5], we have a morphism $p: \text{Def}(X) \to \Omega$, which is locally isomorphic. We choose a point $t$ of $\text{Def}(X)$ such that $(p(t)^\perp \cap H^2(X, \mathbb{Z})) \otimes_\mathbb{Z} \mathbb{R} = H$. Let $\mathfrak{X}_t$ be the fibre of $\pi$ at $t$. Then $\text{NS}(\mathfrak{X}_t) = H$. We have an induced diffeomorphism $\iota: \mathfrak{X}_t \cong X$. By [11] Corollary 5.13, $\text{MBM}(\mathfrak{X}_t) = \iota^*(\text{MBM}(X) \cap H)$. Hence $\text{Nef}(\mathfrak{X}_t) \supset \iota^*(\text{Nef}(X) \cap H)$ and $\iota^*(\ell_1)$ is a ray of $\text{Nef}(\mathfrak{X}_t)$. Since $e \in H$,
MBM(\mathcal{H}) \neq \emptyset. By [1] Theorem 1.19, Nef(\mathcal{H}) \neq \overline{\text{NS}}(\mathcal{H}). By [10] Theorem 1.3 (1), two rays of Nef(\mathcal{H}) are rational, especially \nu^*(\ell_1) is rational. Since \nu^* preserves rationalities, we are done.

**Proof of Theorem 1.5.** If MBM(X) = \emptyset, Nef(X) = \overline{\text{NS}}(X) by [1] Theorem 1.19, and we are done. Assume that MBM(X) \neq \emptyset. We choose a Kähler class \kappa of H^2(X, \mathbb{R}) We define the subset \mathcal{N}(X) of MBM(X) by

\[ \mathcal{N}(X) := \{ e \in \text{MBM}(X) | e^\perp \cap \text{Nef}(X) \text{ is an open set of } e^\perp, q_X(e, \kappa) > 0 \}. \]

We also define the cone D in NS_R(X) by

\[ D := \{ x \in \text{NS}_R(X) | \forall e \in \mathcal{N}(X), q_X(e, x) \geq 0 \}. \]

If Nef(X) = D, we are done. We derive a contradiction assuming Nef(X) \neq D. By [1] Theorem 1.19, D \cap \overline{\text{NS}}(X) = \text{Nef}(X). Hence D contains an element e of NS_R(X) such that q_X(x) < 0. This implies that Nef(X) \cap \partial \overline{\text{NS}}(X) contains an open set of \partial \overline{\text{NS}}(X). Since MBM(X) \neq \emptyset, \partial \overline{\text{NS}}(X) \cap \text{NS}(X) \neq \emptyset by Proposition 3.1. The boundary \partial \overline{\text{NS}}(X) is defined by a rational quadratic form. Hence \partial \overline{\text{NS}}(X) \cap \text{NS}(X) forms a dense subset of \partial \overline{\text{NS}}(X). The intersection \partial \overline{\text{NS}}(X) \cap Nef(X) contains an open set of \partial \overline{\text{NS}}(X) and we have a nonzero element \ell of \partial \overline{\text{NS}}(X) \cap \text{Nef}(X) \cap \text{NS}(X) such that \[ e^\perp \cap \text{MBM}(X) = \emptyset. \]

Let \Gamma be a subgroup of Mon(X) obtained by Corollary 2.1. For an element g of \Gamma, by [1] Theorem 1.19 and [13] Lemma 5.7, g(\text{Amp}(X)) is an connected component of \overline{\text{NS}}(X) \setminus \bigcup_{e \in \text{MBM}(X)} e^\perp. Hence if g(\text{Amp}(X)) \neq \text{Amp}(X), then there exists an element e of MBM(X) such that the hyperplane e^\perp separates Amp(X) and g(\text{Amp}(X)), that is,

\[ \text{Amp}(X) \subset e^{>0}, g(\text{Amp}(X)) \subset e^{<0}, \]

where e^{>0} := \{ x \in \text{NS}_R(X) | q_X(x, e) > 0 \}. Since g(\ell) = \ell, e should be an element of \ell^\perp \cap \text{MBM}(X). By the choice of \ell, such a class does not exist and g(\text{Amp}(X)) = \text{Amp}(X). By Theorem 1.2, there exists an automorphism \Phi of X such that \Phi^* = g. By Proposition 2.1 \lim_{m \to \infty} g^m(x) = \ell \in \mathbb{P}(\text{NS}_R(X)) for all \ell \in \mathbb{P}(\text{NS}_R(X)). Hence, for every element x of \overline{\text{NS}}(X), there exists a positive integer N such that (\Phi^N)^* x \in \text{Amp}(X). This implies that Nef(X) = \overline{\text{NS}}(X). By [1] Theorem 1.19, MBM(X) = \emptyset. This contradicts the first assumption that MBM(X) \neq \emptyset. \qed

**4. Proof of Theorem 1.6**

**Proof.** First we will prove that Aut(X, L) is an almost abelian group and its rank is at most \text{dim NS}_R(X) - \text{dim } W_R - 1. Let \Gamma be the image of the natural representation \rho : Aut(X, L) \to O(\text{NS}(X)). By [15] Corollary 2.7, the kernel of \rho is finite. Hence it is enough to prove that \Gamma is an almost abelian group of rank at most \text{dim NS}_R(X) - \text{dim } W_R - 1 by [15] Proposition 9.3 (2)]. Let us consider the natural projection

\[ r : c_1(L)^\perp \to c_1(L)^\perp / \mathbb{Z}c_1(L). \]

We define the lattice \hat{W} by

\[ \hat{W} := r(\text{W}_R) \cap (c_1(L)^\perp \cap \text{NS}(X) / \mathbb{Z}c_1(L)). \]
We choose elements \( \{e_1, \ldots, e_k\} \) of \( c_1(L)^+ \cap \text{MBM}(X)^0 \) as their residue classes give a generator of \( W \). We note that \( k = \text{rank}(W) = \dim W_{\mathbb{R}} - 1 \). Let \( W \) be the sub lattice of \( \text{NS}(X) \) generated by \( \{e_1, \ldots, e_k\} \). Then there exists a natural isomorphism \( W \cong \bar{W} \). Since the induced bilinear form on \( c_1(L)^+ / \mathbb{R}c_1(L) \) is negative definite, \( \bar{W} \) is negative definite and hence \( W \) also is. Let \( W^\perp \) be the orthogonal lattice of \( W \) with respect to the Beauville-Bogomolov quadratic form. Since \( \Gamma \) preserves \( c_1(L) \) and \( c_1(L)^+ \cap \text{MBM}(X)^0 \), \( \Gamma \) preserves \( W \) and \( W^\perp \). We consider the following homomorphism

\[
\mu_1 : \Gamma \ni g \to g|_W \oplus g|_{W^\perp} \in O(W) \oplus O(W^\perp),
\]

and the projection \( \mu_2 : O(W) \oplus O(W^\perp) \to O(W^\perp) \). Since \( [\text{NS}(X) : W \oplus W^\perp < \infty] \), \( \mu_1 \) is injective. Since \( W \) is negative definite, \( O(W) \) is finite and the kernel of \( \mu_2 \) is finite. Hence the kernel of \( \Gamma \ni \mu_2 \circ \mu_1(\Gamma) \) is finite. Therefore it is enough to prove that \( \mu_2 \circ \mu_1(\Gamma) \) is almost abelian of rank at most

\[
\dim \text{NS}_{\mathbb{R}}(X) - \dim W_{\mathbb{R}} - 1 = \dim \text{NS}_{\mathbb{R}}(X) - \text{rank}(W) - 2
\]

by \([15] \) Proposition 9.3 (2)]. Since \( W^\perp \) is a lattice whose index is \( (1, n - \text{rank}(W) - 1) \) and \( \Gamma \) preserves \( c_1(L) \), \( \mu_2 \circ \mu_1(\Gamma) \) is an almost abelian group whose rank is at most \( n - \text{rank}(W) - 2 \) by \([14] \) Proposition 2.9 and we are done.

Next we will prove that

\[
\text{rank}(\text{Aut}(X, L)) = \dim \text{NS}_{\mathbb{R}}(X) - \text{rank}(W) - 2.
\]

Since \( W \) is negative definite and contained in \( c_1(L)^+ \), we have a subgroup \( \Gamma_0 \) of \( \text{Mon}(X) \) by Corollary 2.1. We note that \( \Gamma_0 \) is isomorphic to \( \mathbb{Z}^{\dim \text{NS}_{\mathbb{R}}(X) - \text{rank}(W) - 2} \). Let \( g \) be an element of \( \Gamma_0 \). By \([1] \) Theorem 1.19 and \([13] \) Lemma 5.17, \( g(\text{Amp}(X)) \) coincides with a connected component of \( \mathbb{C}^{\text{NS}(X)} \setminus \bigcup_{e \in \text{MBM}(X)^0} e^\perp \) whose closure contains \( c_1(L) \). Hence, if \( g(\text{Amp}(X)) \neq \text{Amp}(X) \), there exists an element \( e \) of \( c_1(L)^+ \cap \text{MBM}(X)^0 \) such that \( g(\text{Amp}(X)) \subset e^{>0} \) and \( \text{Amp}(X) \subset e^{<0} \). By the definition of \( W \) and Corollary 2.1, \( g \) fixes \( c_1(L) \) and all elements of \( c_1(L)^+ \cap \text{MBM}(X)^0 \). Hence there are no such elements in \( c_1(L)^+ \cap \text{MBM}(X)^0 \). Therefore \( g(\text{Amp}(X)) = \text{Amp}(X) \) and there exists an element \( \Phi \) of \( \text{Aut}(X, L) \) such that \( \Phi^* = g \) by Theorem 1.2. This implies that \( \Gamma_0 \) is a subgroup of \( \text{Aut}(X, L) \) and the rank of \( \text{Aut}(X, L) \) coincides with \( \dim \text{NS}_{\mathbb{R}}(X) - \text{rank}(W) - 2 \). \( \square \)

5. Proof of Theorem 1.7

Lemma 5.1. Let \( \overline{\Lambda} \) be a lattice whose index is \( (2, \text{rank}(\overline{\Lambda}) - 2) \). We fix a positive integer \( N \) and define

\[
\overline{\Lambda}_N := \{x \in \overline{\Lambda} - N < \langle x, x \rangle < 0\}
\]

We denote by \( \text{Gr}^+(2, \overline{\Lambda}_{\mathbb{R}}) \) the open set of Grassmanian \( \text{Gr}(2, \overline{\Lambda}_{\mathbb{R}}) \) which consists of positive 2-planes in \( \overline{\Lambda}_{\mathbb{R}} \). Let \( \text{Gr}^+(2, \overline{\Lambda}_{\mathbb{R}})^0 \) be a subset of \( \text{Gr}^+(2, \overline{\Lambda}_{\mathbb{R}}) \) defined by

\[
\text{Gr}^+(2, \overline{\Lambda}_{\mathbb{R}})^0 := \{\sigma \in \text{Gr}^+(2, \overline{\Lambda}_{\mathbb{R}}) \mid \forall x \in \overline{\Lambda}_N, \sigma^ans not contained in \( \overline{\Lambda}_{\mathbb{R}} \} \}
\]

Then \( \text{Gr}^+(2, \overline{\Lambda}_{\mathbb{R}})^0 \) is open in \( \text{Gr}^+(2, \overline{\Lambda}_{\mathbb{R}}) \).

Proof. Let \( w \) be an element of \( \overline{\Lambda}_{\mathbb{R}} \) such that \( \langle w, w \rangle > 0 \). It is enough to prove that the subset \( V \) of \( w^\perp \) defined by

\[
V := \{v \in w^\perp : \langle v, v \rangle > 0, \forall x \in \overline{\Lambda}_N \cap w^\perp, \langle v, x \rangle \neq 0\}
\]
is open in \( w^\perp \). It is obvious that \( V \neq \emptyset \). We choose an element \( v \) of \( V \). For a positive number \( \epsilon \), we consider the subset of \( w^\perp \) defined by

\[
\{ x \in w^\perp | \langle v, x \rangle \leq \epsilon, -N \leq \langle x, x \rangle \leq 0 \}
\]

Since the signature of the bilinear form on \( w^\perp \) is \( (1, \dim w^\perp - 1) \), the above set is compact. Hence the set

\[
\{ x \in \Lambda_N \cap v^\perp | \langle v, x \rangle \leq \epsilon \}
\]

is finite. Therefore

\[
\min_{x \in \Lambda_N \cap w^\perp} |\langle v, x \rangle|
\]

is positive and we are done. \( \square \)

**Corollary 5.1.** Let \( \Lambda \) be a lattice whose index is \( (3, \text{rank}(\Lambda) - 3) \). We fix a positive integer \( N \). Assume that \( \Lambda \) has an isotropic element \( \ell \). Let \( \Lambda_N \) be the subset of \( \Lambda \) defined by

\[
\Lambda_N := \{ x \in \Lambda | -N < \langle x, x \rangle < 0 \}.
\]

We denote by \( \text{Gr}_{++}(2, \ell^\perp) \) the open set of Grassmanian \( \text{Gr}(2, \ell^\perp) \) which consists of positive 2-planes in \( \ell^\perp \). Then the subset \( \text{Gr}_{++}(2, \ell^\perp)^\circ \) defined by

\[
\text{Gr}_{++}(2, \ell^\perp)^\circ := \{ x \in \Lambda_N \cap \ell^\perp, \sigma \nsubseteq \sigma^\perp \}
\]

is open.

**Proof.** We denote by \( \overline{\Lambda} \) the quotient lattice \( \Lambda \cap \ell^\perp / \mathbb{Z} \). The symbols \( \overline{\Lambda}_R, \overline{\Lambda}_N, \text{Gr}_{++}(2, \overline{\Lambda}_R) \) and \( \text{Gr}_{++}(2, \overline{\Lambda}_N)^\circ \) represent the same objects in Lemma 5.1. Let us consider the projection \( \pi : \ell^\perp \rightarrow \overline{\Lambda}_R = \ell^\perp / \mathbb{R} \). Since \( \pi \) respect the bilinear forms on \( \ell^\perp \) and \( \overline{\Lambda}_R \), we have the induced morphism \( \pi' : \text{Gr}_{++}(2, \ell^\perp) \rightarrow \text{Gr}_{++}(2, \overline{\Lambda}_R) \). By definition, \( (\pi')^{-1}(\text{Gr}_{++}(2, \overline{\Lambda}_N)^\circ) = \text{Gr}_{++}(2, \ell^\perp)^\circ \). By Lemma 5.1 \( \text{Gr}_{++}(2, \overline{\Lambda}_N)^\circ \) is open and we are done. \( \square \)

We recall the definition of marked irreducible symplectic manifolds, their moduli and the global period map.

**Definition 5.1.** Let \( \Lambda \) be a lattice whose index is \( (3, \text{rank}(\Lambda) - 3) \). A marked irreducible symplectic manifold \((X, \varphi)\) is a pair of an irreducible symplectic manifold \( X \) and an isometry \( \varphi : H^2(X, \mathbb{Z}) \rightarrow \Lambda \). Two marked irreducible symplectic manifold \((X, \varphi)\) and \((X', \varphi')\) are isomorphic if there exists an isomorphism \( \Phi : X \cong X' \) such that \( \varphi' = \varphi \circ \Phi^* \), where \( \Phi^* \) is the induced isometry \( H^2(X', \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z}) \). A moduli space of marked irreducible symplectic manifold \( \mathfrak{M}_\Lambda \) is the set of isomorphic classes of marked irreducible symplectic manifolds. We define the global period map \( \mathcal{P} : \mathfrak{M}_\Lambda \rightarrow \mathbb{P}(\Lambda_C) \) by

\[
\mathcal{P} : \mathfrak{M}_\Lambda \ni (X, \varphi) \rightarrow \varphi(H^{2,0}(X)) \in \mathbb{P}(\Lambda_C),
\]

where \( \Lambda_C = \Lambda \otimes \mathbb{C} \).

The following lemma is well-known for specialist and we add it for readers convenience.

**Lemma 5.2 ([11 (1.18)])**. The symbols \((X, \varphi), \Lambda, \mathfrak{M}_\Lambda \) and \( \mathcal{P} \) represent the same objects in Definition 5.1. We define the subset of \( \mathbb{P}(\Lambda) \) by

\[
\Omega_\Lambda := \{ x \in \mathbb{P}(\Lambda_C) | \langle x, x \rangle = 0, \langle x, \bar{x} \rangle > 0 \}
\]

Then \( \mathfrak{M}_\Lambda \) is a complex manifold and \( \mathcal{P} \) is a holomorphic morphism. The image of \( \mathcal{P} \) is contained in an open set of \( \Omega_\Lambda \).
Proof. Let \( \mathcal{X} \to \text{Def}(X) \) be the Kuranishi family of \( X \). By [1 Théorème 5], we have a holomorphic morphism \( p_X : \text{Def}(X) \to \Omega_A \). If \( (X', \varphi') \) is another marked irreducible symplectic manifold which is isomorphic to \( (X, \varphi) \) in the sense of Definition [5,2] \( p_X : \text{Def}(X') \to \Omega_A \) can be patched \( p_X \) by universality of the Kuranishi space. Hence \( \mathfrak{M}_A \) carries a structure of a complex manifold and \( \mathcal{P} \) is holomorphic. Since \( p_X \) is locally isomorphic, the image of \( \mathcal{P} \) is an open set of \( \Omega_A \). \( \square \)

We prove a property of fibres of the global period map.

**Lemma 5.3.** The symbols \((X, \varphi), \mathfrak{M}_A, \mathcal{P} \) and \( \Lambda \) represent the same objects in Definition [5.7]. Let \( \mathfrak{M}_A^\Lambda \) be a connected component of \( \mathfrak{M}_A \) which contains \((X, \varphi)\). We denote by \( t \) the point \( \mathcal{P}(X, \varphi) \). Assume that \( \Lambda \cap t^\perp \) has an isotropic element \( \ell \). Then there exists a marked irreducible symplectic manifold \((X', \varphi')\) such that \((X', \varphi') \in \mathfrak{M}_A^\Lambda\), \( X' \) carries a nef line bundle \( L' \) with \( \varphi'(c_1(L')) = \ell \) and \( \mathcal{P}(X', \varphi') = t \).

**Proof.** Let \( \mathcal{C}(X) \) be the positive cone of \( H^{1,1}(X, \mathbb{R}) \) and \( \text{MBM}(X) \) the set of the Monodromy birationally minimal classes. We choose a connected component \( C \) of \( \mathcal{C}(X) \setminus \bigcup_{e \in \text{MBM}(X)} e^\perp \) such that \( \varphi^{-1}(\ell) \) is contained in \( C \), where \( \overline{C} \) is the closure of \( C \). We define the subgroup \( \text{Mon}^{\text{Hdg}}(X) \) by

\[
\text{Mon}^{\text{Hdg}}(X) := \{ \gamma \in \text{Mon}(X) \mid \gamma(H^{2,0}(X)) = H^{2,0}(X) \}
\]

By [1 Definition 6.1 and Theorem 6.2], there exists an irreducible symplectic manifold \( X' \), a bimeromorphic map \( f : X \to X' \) and an element \( \gamma \) of \( \text{Mon}^{\text{Hdg}}(X) \) such that \( \gamma \circ f^*(\mathcal{K}(X')) = C \), where \( f^* \) is the induced morphism \( H^2(X', \mathbb{R}) \to H^2(X, \mathbb{R}) \) and \( \mathcal{X}(X') \) is the Kähler cone of \( X' \). By definition of \( \text{Mon}(X) \), \((X, \varphi)\) and \((X, \varphi \circ \gamma)\) belong to a same connected component of \( \mathfrak{M}_A \). By [10 Theorem 2.5], \((X', \varphi \circ \gamma)\) and \((X', \varphi \circ \gamma \circ f^*)\) belong to a same connected component of \( \mathfrak{M}_A \). Hence if we define \( \varphi' = \varphi \circ \gamma \circ f^* \), \((X', \varphi')\) belongs to \( \mathfrak{M}_A^{\Lambda} \). Since \( \gamma \circ f^*(H^{2,0}(X')) = H^{2,0}(X) \), \( \mathcal{P}(X', \varphi') = t \).

We finish the proof of Lemma if we have proved that \( X' \) carries a line bundle \( L' \) such that \( L' \) is nef and \( c_1(L') = (\varphi')^{-1}(\ell) \). Since \( (\varphi')^{-1}(\ell) \in H^{1,1}(X', \mathbb{R}) \cap H^2(X', \mathbb{Z}) \), there exists a line bundle \( L' \) on \( X' \) such that \( c_1(L') = (\varphi')^{-1}(\ell) \). By definition, \( \varphi^{-1}(\ell) \in \overline{C} \). Thus \( (\varphi')^{-1}(\ell) \) is contained in the closure of \( \mathcal{K}(X') \). Hence \( L' \) is nef and we are done. \( \square \)

The following Proposition is the punch line of the proof of Theorem [1.7a]

**Proposition 5.1.** Let \( X \) be an irreducible symplectic manifold whose Betti number is greater than five and \( L \) a line bundle on \( X \) with \( q_X(c_1(L)) = 0 \), where \( q_X \) is the Beauville-Bogomolov quadratic form. Then there exists an irreducible symplectic manifold \( X' \) and a line bundle \( L' \) which has the following properties:

1. The pairs \((X, L)\) and \((X', L')\) are deformation equivalent in the sense of Definition [1.6];
2. The line bundle \( L' \) is nef;
3. The Picard number of \( X' \) is equal to \( \dim H^2(X', \mathbb{R}) - 2 \); and
4. The intersection \( c_1(L')^\perp \cap \text{MBM}(X') \) is empty.

**Proof.** Let \( \Lambda \) be a lattice isomorphic to \((H^2(X, \mathbb{Z}), q_X)\), where \( q_X \) is the Beauville-Bogomolov quadratic form. The symbols \( \mathfrak{M}_A \) and \( \mathcal{P} \) represent the same objects in Definition [5.7]. We put \( \ell = \varphi(c_1(L)) \). Let \( \Omega_{A, \ell} \) be a subset of \( \Omega_A \) defined by \( \Omega_{A, \ell} := \{ x \in \Omega_A \mid \langle x, \ell \rangle = 0 \} \).
We will define two subsets of $\Omega_{\Lambda,\ell^+}$. The first one is defined by
\[ \Omega_{\Lambda,\ell^+}^{\text{max}} := \{ x \in \Omega_{\Lambda,\ell^+} | \text{rank}(x^\perp \cap \Lambda) = \text{rank}(\Lambda) - 2 \} \]

\textbf{Claim 5.1.} The subset $\Omega_{\Lambda,\ell^+}^{\text{max}}$ is dense.

\textbf{Proof.} We choose an element $t$ of $\Omega_{\Lambda,\ell^+}$. There exist sequences $a_m$ and $b_m$ in $\Lambda_Q$ such that
\[ \lim_{m \to \infty} a_m = \text{Re}(t), \quad \lim_{m \to \infty} b_m = \text{Im}(t) \]
Since $t \in \Omega_{\Lambda,\ell^+}$, $\langle \text{Re}(t), \text{Im}(t) \rangle > 0$, $\langle \text{Im}(t), \text{Im}(t) \rangle > 0$ and $\langle \text{Re}(t), \text{Im}(t) \rangle = 0$. Hence, we may assume that $\langle a_m, a_m \rangle > 0$ and $\langle b_m, b_m \rangle > 0$ for all $m$. Moreover, we may assume that $\lim_{m \to \infty} \langle a_m, b_m \rangle = 0$. We define other sequences $c_m$ and $d_m$ in $\Lambda_Q$ by
\[ c_m = b_m - \frac{\langle a_m, b_m \rangle}{\langle a_m, a_m \rangle} a_m \]
\[ d_m = \sqrt{\frac{\langle a_m, a_m \rangle}{\langle c_m, c_m \rangle}} c_m \]
Then $a_m + \sqrt{-1}d_m \in \Omega_{\Lambda,\ell^+}^{\text{max}}$. By definition, $\lim_{m \to \infty} a_m + \sqrt{-1}d_m = t$ and we are done. \qed

Let $N$ be a positive integer and $\Lambda_N$ represents the same object in Corollary 5.1. We define the second subset of $\Omega_{\Lambda,\ell^+}$ by
\[ \Omega_{\Lambda,\ell^+}^\circ := \{ x \in \Omega_{\Lambda,\ell^+} | x^\perp \cap \Lambda_N = \emptyset \} \]

\textbf{Claim 5.2.} The subset $\Omega_{\Lambda,\ell^+}^\circ$ is open and dense.

\textbf{Proof.} For a very general point $x$ of $\Omega_{\Lambda,\ell^+}$, $x^\perp \cap \Lambda = \mathbb{Z}\ell$. Hence $x \in \Omega_{\Lambda,\ell^+}^\circ$ and $\Omega_{\Lambda,\ell^+}^\circ$ is dense. We have a natural identification
\[ \Omega_{\Lambda,\ell^+} \cong \text{Gr}_{++}(2, \ell^+) \]
where $\text{Gr}_{++}(2, \ell^+)$ is the set of positive 2-planes in $\ell^+$. The correspondence is given by
\[ \Omega_{\Lambda,\ell^+} \ni t \mapsto (\text{Re}(t), \text{Im}(t)) \in \text{Gr}_{++}(2, \ell^+) \]
where $(\text{Re}(t), \text{Im}(t))$ is the 2-plane spanned by $\text{Re}(t)$ and $\text{Im}(t)$. Under this identification, $\Omega_{\Lambda,\ell^+}^\circ$ corresponds to
\[ \text{Gr}_{++}(2, \ell^+) \cap \{ x \in \Lambda_N \cap \ell^+, \sigma \not\subset x^\perp \} \]
By Corollary 5.1, the above set is open in $\text{Gr}_{++}(2, \ell^+)$ and we are done. \qed

Let $\pi : \mathcal{K} \to \text{Def}(X)$ be a Kuranishi family of $X$. For a point $t$ of $\text{Def}(X)$, $\pi$ gives a natural marking $\varphi_t : H^2(\mathcal{K}_t, \mathbb{Z}) \to \Lambda$, where $\mathcal{K}_t$ is the fibre at $t$. We consider the subset of $\text{Def}(X)$ defined by
\[ \text{Def}(X, L) := \{ t \in \text{Def}(X) | \mathcal{P}(\mathcal{K}_t, \varphi_t) \in \Omega_{\Lambda,\ell^+} \} \]
and the restriction family $\mathcal{K}_L \to \text{Def}(X, L)$. By [6 Corollaire 1], $\mathcal{K}_L$ carries a line bundle $\mathcal{L}$ such that the restriction of $\mathcal{L}$ to $X$ is isomorphic to $L$. Since the Betti number is greater than five, by [2 Corollary 1.4], there exists a positive integer $N$ such that $\varphi_t(\text{MBM}(X_t)) \subset \Lambda_N$ for all $t \in \text{Def}(X)$. By Claim 5.1 and Claim 5.2, there exists a point $t_0$ of $\text{Def}(X)$ such that $\mathcal{P}(\mathcal{K}_{t_0}, \varphi_{t_0}) \in \Omega_{\Lambda,\ell^+}^{\text{max}} \cap \Omega_{\Lambda,\ell^+}^\circ$. By
Lemma 5.3 we have a marked irreducible symplectic manifold \((X', \varphi')\) such that \(X'\) carries a nef line bundle \(L'\) with \(\varphi'(c_1(L')) = \ell\) and \(\mathcal{P}(X', \varphi') = t_0\). Since \(\text{rank}(\mathcal{H}^1_\varphi) = \text{rank}(\mathcal{A}) - 2\), the Picard number of \(X'\) is equal to \(\dim H^2(X', \mathbb{R}) - 2\). Since \(X_0^+ \cap t_0^+ \cap \Lambda_0 = 0\), \(c_1(L')^{+} \cap \text{MBM}(X') = 0\). Thus we are done if we prove that \((X, L)\) and \((X', L')\) are deformation equivalent in the sense of Definition 1.6. Let \(\pi': \mathcal{P}' \to \text{Def}(X')\) be a Kuranishi family of \(X'\). We consider the restriction family \(\mathcal{P}'_t \to \text{Def}(X', L')\) which is obtained by the same manner of \(\mathcal{P}'_t \to \text{Def}(X, L)\). Since \(t_0 \in \mathcal{P}(\text{Def}((X, L))) \cap \mathcal{P}(\text{Def}(X', L'))\), \(\mathcal{P}(\text{Def}((X, L))) \cap \mathcal{P}(\text{Def}(X', L'))\) is a non empty open subset of \(\Omega_{X, L}^{\pm}\). Hence there exists a point \(t_1\) of \(\mathcal{P}(\text{Def}(X, L)) \cap \mathcal{P}(\text{Def}(X', L'))\) such that \(t_1^+ \cap \Lambda = \mathbb{Z}\ell\). Let \(\mathcal{P}'_{t_1}\) be the fibre of \(\mathcal{P}' \to \text{Def}(X, L)\) at \(t_1\) and \(\mathcal{P}'_{t_1}^+\) the fibre of \(\mathcal{P}'_{t_1} \to \text{Def}(X', L')\) at \(t_1\). We denote by \(\varphi'_{t_1}\) the induced marking on \(H^2(\mathcal{P}'_{t_1}, \mathbb{Z})\) and by \(\varphi'_{t_1}\) the induced marking on \(H^2(\mathcal{P}'_{t_1}, \mathbb{Z})\). Then \((\mathcal{P}'_{t_1}, \varphi'_{t_1})\) and \((\mathcal{P}'_{t_1}, \varphi'_{t_1})\) are isomorphic by [13, Theorem 2.2 (5)]). We denote by \(\Phi_t\) an isomorphism between \((\mathcal{P}'_{t_1}, \varphi'_{t_1})\) and \((\mathcal{P}'_{t_1}, \varphi'_{t_1})\). Since \(\varphi'_{t_1}(\ell) = c_1(L'_{t_1})\) and \((\varphi'_{t_1})^{-1}(\ell) = c_1(L'_{t_1})\), \(\Phi^\ast_{t_1} L'_{t_1} \cong L'_{t_1}\). Hence \((X, L)\) and \((X', L')\) are deformation equivalent in the sense of Definition 1.6. \(\square\)

**Proof of Theorem 1.7.** By Proposition 5.1, we have a pair \((X', L')\) with deformation equivalent to \((X, L)\) such that \(L'\) is nef and \(c_1(L')^{+} \cap \text{MBM}(X') = 0\). By Theorem 1.6, \(\text{Aut}(X', L')\) is almost abelian whose rank is equal to \(\dim H^2(X', \mathbb{R}) - 2\). \(\square\)

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