Resonance states $0^+$ of the Boron isotope $^8$B from the Jost-matrix analysis of experimental data

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Abstract

The available $R$-matrix parametrization of experimental data on the excitation functions for the elastic and inelastic $p^7$Be scattering at the collision energies up to 3.4 MeV is used to generate the corresponding partial-wave cross sections in the states with $J^\pi = 0^+$. Thus obtained data are considered as experimental partial cross sections and are fitted using the semi-analytic two-channel Jost matrix with proper analytic structure and some adjustable parameters. Then the spectral points are sought as zeros of the Jost matrix determinant (which correspond to the $S$-matrix poles) at complex energies. The correct analytic structure makes it possible to calculate the fitted Jost matrix on any sheet of the Riemann surface whose topology involves not only the square-root but also the logarithmic branching caused by the Coulomb interaction. In this way, two overlapping $0^+$ resonances at the excitation energies $\sim 1.79$ MeV and $\sim 1.96$ MeV have been found.

1 Introduction

The spectrum of the eight-nucleon system $^8$B was established as a result of many experimental and theoretical studies (an extensive list of publications related to $^8$B, can be found in Ref. [1]). There are excited states that appear in all of them more or less at the same energies. These well established levels (taken from Ref. [1]) are schematically depicted in Fig. 1. However,
Figure 1: Low-lying excited states of the nuclear system $^8$B and the two-body thresholds for the decays $^8$B $\to$ $^7$Be$(\frac{3}{2}^-)$ + $p$ and $^8$B $\to$ $^7$Be$^*(\frac{1}{2}^-)$ + $p$. The data are taken from Refs. [1,2].
there are also few unconfirmed levels that follow from some of the studies and not from the others. In the present work, we do an attempt to clarify the existence and parameters of the resonance level with the quantum numbers $J^\pi = 0^+$. Using available experimental data, we construct the two-channel Jost matrices and then analytically continue them onto the Riemann surface of the energy. The spectral points are sought as the zeros of the Jost-matrix determinant, which correspond to the poles of the $S$-matrix.

As the experimental data, we use the partial cross sections with $J^\pi = 0^+$ for all four possible transitions between the states $p^7\text{Be}(\frac{3}{2}^-)$ and $p^7\text{Be}^*(\frac{1}{2}^-)$, where the second channel involves the first excited state of the Beryllium isotope. In order to obtain these cross sections, we use the $R$-matrix given in Ref. [3]. That $R$-matrix was constructed by parametrizing the measured excitation functions for the elastic and inelastic $p^7\text{Be}$ scattering.

We fit these data using the two-channel Jost matrix. Since the cross sections are extracted from the available $R$-matrix parametrization of a large collection of experimental data, we indirectly fit the original data. After the fitting at real energies, the Jost matrix is considered at complex $E$, where the zeros of its determinant correspond to the poles of the $S$-matrix.

The multi-channel Jost matrix is taken in a special representation suggested in Refs. [5,6], where it is given as a sum of two terms. Each of these terms is factorized in a product of two matrices, one of which is an unknown analytic single-valued function of the energy and the other is given explicitly as a function of the channel momenta. The explicitly given factors are responsible for the branching of the Riemann surface. The unknown single-valued matrices are parametrized and the parameters are found via fitting the available experimental data.

With the semi-analytic representation of the Jost matrix, where the factors responsible for the topology of the Riemann surface are given explicitly, it is easy to explore the behaviour of the Jost matrix on all the sheets of the Riemann surface. In this way we are able to accurately locate the resonance poles and to examine the possibility that the so called “shadow” poles exist on the other sheets of the surface.

2 Jost matrices

The Jost matrices are only defined for the binary reactions, where the colliding particles may either change their internal states ($a + b \rightarrow a^* + b^*$) or transit to another pair of particles ($a + b \rightarrow c + d$). In general, the masses of the particles may change. This means that the channels $ab$, $a^*b^*$, $cd$, etc. have different thresholds.

For a given two-body system of particles, $a$ and $b$, there are infinite number of possible combinations of their orbital angular momentum $\ell$ and the two-body spin $s = s_a + s_b$. However, not all combinations of $\ell$ and $s$ are coupled to each other. For the conserving total angular momentum $J$ and the parity $\pi$, only few transitions of the type $(\ell, s) \leftrightarrow (\ell', s')$ are possible.
In the present paper, we consider the low-energy \((E < 3.4 \text{ MeV})\) collision of proton with the nucleus \(^7\text{Be}\). In its ground state, this nucleus has \(J^\pi = 3/2^-\). As a result of such a collision, the target nucleus may be excited to the state with \(J^\pi = 1/2^-\) at the energy \(0.4291 \text{ MeV}\) (see Ref. [2]). The other excited states of \(^7\text{Be}\) are too high as compared to the maximal collision energy and thus can be safely ignored. Therefore we deal with the following (elastic and inelastic) coupled processes:

\[
\begin{align*}
\text{channel 1:} & \quad p + ^7\text{Be}(\frac{3}{2}^-) \quad \rightarrow \quad p + ^7\text{Be}(\frac{3}{2}^-), \\
\text{channel 2:} & \quad p + ^7\text{Be}^*(\frac{1}{2}^-) \quad \rightarrow \quad p + ^7\text{Be}^*(\frac{1}{2}^-)
\end{align*}
\]

where the circle in the middle is either the intermediate scattering state of the direct reaction or the compound resonance state of \(^8\text{B}\). It is easy to see that the state \(0^+\) of our eight-body system can only be formed if \(\ell = 1\) and \(s = 1\) in both channels. This means that we deal with a two-channel problem.

The \(N\)-channel Jost matrices \(f^{(\text{in})}\) and \(f^{(\text{out})}\) are defined as the energy-dependent \((N \times N)\)-“amplitudes” of the incoming and outgoing multi-channel (diagonal-matrix) spherical waves, \(H^{(-)}\) and \(H^{(+)}\), in the asymptotic behaviour of the regular solution, \(\phi(E,r)\), of the radial Schrödinger equation,

\[
\phi(E,r) \rightarrow r \rightarrow \infty \quad H^{(-)}(E,r) f^{(\text{in})}(E) + H^{(+)}(E,r) f^{(\text{out})}(E).
\]

A more detailed description of their meaning and properties can be found in Refs. [4–7]. It is worthwhile to write Eq. (2) in the explicit form for the case of two coupled channels \((N=2)\):

\[
\begin{align*}
\phi(E,r) \rightarrow r \rightarrow \infty & \quad \begin{bmatrix}
H^{(-)}_1(\eta_1,k_1r)e^{i\sigma_1} & 0 \\
0 & H^{(-)}_2(\eta_2,k_2r)e^{i\sigma_2}
\end{bmatrix}
\begin{bmatrix}
f^{(\text{in})}_{11}(E) & f^{(\text{in})}_{12}(E) \\
f^{(\text{in})}_{21}(E) & f^{(\text{in})}_{22}(E)
\end{bmatrix}
+ \\
& \quad \begin{bmatrix}
H^{(+)}_1(\eta_1,k_1r)e^{-i\sigma_1} & 0 \\
0 & H^{(+)}_2(\eta_2,k_2r)e^{-i\sigma_2}
\end{bmatrix}
\begin{bmatrix}
f^{(\text{out})}_{11}(E) & f^{(\text{out})}_{12}(E) \\
f^{(\text{out})}_{21}(E) & f^{(\text{out})}_{22}(E)
\end{bmatrix}
\end{align*}
\]

where

\[
H^{(\pm)}_\ell(\eta,kr) = F_\ell(\eta,kr) \mp iG_\ell(\eta,kr) \rightarrow r \rightarrow \infty \quad \mp i \exp \left\{ \pm i \left[ kr - \eta \ln(2kr) - \frac{\ell \pi}{2} + \sigma_\ell \right] \right\}.
\]
In these equations, \( k_n, \ell_n, \eta_n, \) and \( \sigma_{\ell_n} \) are the momentum, angular momentum, Sommerfeld parameter, and the pure Coulomb phase-shift in the channel \( n \); the functions \( F_\ell \) and \( G_\ell \) are the standard regular and irregular Coulomb solutions of the Schrödinger equation (see, for example, Ref. [8]).

### 2.1 Observables

The \( N \) columns of the matrix \( \phi(E,r) \) constitute a regular basis. Therefore a physical wave function, i.e. a column \( u(E,r) \), is their linear combination:

\[
u(E,r) = \phi(E,r)c,
\]

where \( c \) is a column matrix of the combination coefficients. These coefficients are to be chosen to satisfy certain physical boundary conditions at infinity. For a spectral point (either bound or a resonant state) the physical wave function should only have the outgoing waves in its asymptotic behaviour,

\[
u(E,r) \xrightarrow{r \to \infty} H^{(-)}(E,r)f^{(\text{in})}(E)c + H^{(+)}(E,r)f^{(\text{out})}(E)c.
\]

This can only be achieved if the first term in this equation is zero, i.e. if the unknown combination coefficients \( c_n \) obey the homogeneous system of linear equations,

\[
f^{(\text{in})}(E)c = \begin{bmatrix} f_{11}^{(\text{in})}(E) & f_{12}^{(\text{in})}(E) \\ f_{21}^{(\text{in})}(E) & f_{22}^{(\text{in})}(E) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0,
\]

which has a non-zero solution if and only if

\[
\det \begin{bmatrix} f_{11}^{(\text{in})}(E) & f_{12}^{(\text{in})}(E) \\ f_{21}^{(\text{in})}(E) & f_{22}^{(\text{in})}(E) \end{bmatrix} = 0.
\]

The roots \( E = \mathcal{E}_n \) of this equation are the spectral points. At real negative energies (\( \mathcal{E}_n < 0 \)) they correspond to the bound states, and at the complex energies (\( \mathcal{E}_n = E_r - i\Gamma/2 \)) they give us the resonances.

It is not difficult to shown (see, for example, Refs. [4,5]) that the scattering is determined by the “ratio” of the amplitudes of the out-going and in-coming waves, i.e. by the \( S \)-matrix,

\[
S(E) = f^{(\text{out})}(E) \left[ f^{(\text{in})}(E) \right]^{-1},
\]
whose poles correspond to the roots of eq. (8). The partial cross section that describes the transition between any two particular channels, can be obtained via the corresponding elements of the $S$-matrix (see, for example, Ref. [9]),

$$\sigma^J(n' \leftarrow n) = \pi \mu_n \mu_{n'} \left| \frac{S^J_{n'n} - \delta_{n'n}}{k_n} \right|^2,$$

(10)

where $\mu_n$ is the reduced mass in the channel $n$.

The partial widths of a resonance can be found using the method developed in Ref. [10]:

$$\Gamma_n = \frac{\text{Re}(k_n)|A_n|^2 \Gamma}{\sum_{n'=1}^{N} \mu_{n'} \text{Re}(k_{n'})|A_{n'}|^2},$$

(11)

where $A_1$ and $A_2$ are the asymptotic amplitudes (see Ref. [10]) of the channels, given by

$$A_1 = f^{(out)}_{11} - \frac{f^{(in)}_{11}}{f^{(in)}_{12}} f^{(out)}_{12}, \quad A_2 = f^{(out)}_{21} - \frac{f^{(in)}_{21}}{f^{(in)}_{22}} f^{(out)}_{22}.$$

(12)

In these equations the Jost matrices are taken at the complex resonant energy.

## 2.2 Analytic properties

The Jost matrices (and thus the $S$-matrix) are multi-valued complex functions of the energy-variable $E$. They can be treated as single-valued, if considered on a multi-layered Riemann surface. Each threshold is a branch point of such a surface. The multi-valuedness and thus the branching stem from the fact that the Jost matrices depend on the energy via the channel momenta,

$$k_n = \pm \sqrt{\frac{2\mu_n}{\hbar^2} (E - E_n)}, \quad n = 1, 2, \ldots, N,$$

(13)

where $E_n$ are the threshold energies. There are $2^N$ possible combinations of the signs in front of the $N$ square roots (13), and thus for a single value of $E$ there are $2^N$ different values of the Jost matrices. If the interacting particles are charged, there is an additional uncertainty in calculating the Jost matrices for a given $E$. This is because the Coulomb spherical waves (4) and thus their amplitudes, $f^{(in/out)}(E)$, in the asymptotic behaviour (2) depend on the logarithms, $\ln k_n$, of the channel momenta. The complex function $\ln k_n$ has infinitely many different values,

$$\ln k_n = \ln \{|k_n| e^{i \arg(k_n) + 2\pi m_n}\} = \ln |k_n| + i \arg(k_n) + i 2\pi m_n,$$

(14)

$$m_n = 0, \pm 1, \pm 2, \ldots,$$
corresponding to different choices of $m_n$. This implies that the Jost matrices are defined on a “spiral” Riemann surface with infinitely many layers (for more details see Ref. [7]). At each threshold, this surface is branching due to both the square-root and the logarithm multi-valuedness. The layers are identified by the signs of $\text{Im} k_n$ and the logarithmic indices $m_n$. For the two-channel problem, the layers can be denoted by the symbols of the type $(\pm \pm)^{m_1m_2}$. The layers with $m_n \neq 0$ are far away from the real axis, where the physical scattering energies belong to. This means that such layers may be safely ignored, and we should only consider the “principal” layers corresponding to $m_1 = m_2 = 0$.

For our two-body problem, the Riemann surface is schematically depicted in Fig. 2. Each sheet of this surface is cut along its own real axis and the interconnections among the cuts are done in such a way that one full circle around the threshold $E_n$ changes the sign of $\text{Im} k_n$, two full circles around $E_n$ change the logarithmic index $m_n$. If we go around both thresholds, then both momenta and both logarithmic indices do change. This is illustrated in Fig. 3.

2.3 Analytic structure

The multi-channel Riemann surface has a very complicated topology. The intricate interconnections of its layers should be kept in mind not only in pure theoretical considerations, but also in the analysis of experimental data when one tries to extract the information on the resonances. The reason is that the resonance poles of the $S$-matrix lie on one of the Riemann sheets. In order to reach them, one has to do the analytic continuation of the $S$-matrix, starting from the real axis. In doing this, one should be careful, and especially when such a continuation is done near a branch point.

All the complications caused by the branching of the Riemann surface, can be circumvented by using the semi-analytic representations of the Jost matrices suggested in Refs. [5, 6]. In these representations, the factors responsible for the branching of the Riemann surface are given explicitly. For the charged-particle case, it was shown [6] that the Jost matrices have the following structure

$$f^{(\text{in/out})} = Q^{(\pm)} [D^{-1} AD - (M \pm i) K^{-1} DBD] ,$$

where the unknown matrices $A(E)$ and $B(E)$ are single-valued functions of the energy and are defined on the simple energy-plane without any branch points. All the troubles with the branching stem from the explicitly given factors (diagonal matrices):

$$Q^{(\pm)} = \text{diag} \left\{ \frac{e^{\pi \eta_1/2 \ell_1!}}{\Gamma(\ell_1 + 1 \pm i \eta_1)}, \frac{e^{\pi \eta_2/2 \ell_2!}}{\Gamma(\ell_2 + 1 \pm i \eta_2)}, \ldots, \frac{e^{\pi \eta_N/2 \ell_N!}}{\Gamma(\ell_N + 1 \pm i \eta_N)} \right\} ,$$

$$D = \text{diag} \left\{ C_{\ell_1}(\eta_1) k_{\ell_1+1}^{\ell_1+1}, C_{\ell_2}(\eta_2) k_{\ell_2+1}^{\ell_2+1}, \ldots, C_{\ell_N}(\eta_N) k_{\ell_N+1}^{\ell_N+1} \right\} ,$$

7
Figure 2: Interconnections of the Riemann sheets for the two-channel problem, where the Coulomb potential is present in both channels: (a) the interconnections between the thresholds $E_1$ and $E_2$; (b) the interconnections above the highest threshold. The symbols $(\pm\pm)_{m_1m_2}$ label the sheets where $\text{Im}(k_1)$ and $\text{Im}(k_2)$ are either positive or negative, and the subscripts $m_1m_2$ are the numbers of $i2\pi$ in Eq. (14) for the channels.
Figure 3: Two-circle path around both thresholds ($E_1$ and $E_2$) on the Riemann surface of a two-channel problem with the Coulomb forces in both channels. If the starting point is on the sheet $(-+)_00$, then the final point is on the sheet $(+-)_{11}$.

$$M = \text{diag} \left\{ \frac{2\eta_1 h(\eta_1)}{C^2_0(\eta_1)}, \frac{2\eta_2 h(\eta_2)}{C^2_0(\eta_2)}, \ldots, \frac{2\eta_N h(\eta_N)}{C^2_0(\eta_N)} \right\},$$

$$K = \text{diag} \{ k_1, k_2, \ldots, k_N \}. \tag{19}$$

They involve the Coulomb barrier factor $C_\ell$ and the function $h(\eta)$ that is responsible for the logarithmic branching:

$$C_\ell(\eta) = \frac{2^\ell e^{-\pi \eta/2}}{(2\ell)!!} \exp \left\{ \frac{1}{2} \left[ \ln \Gamma(\ell + 1 + i\eta) + \ln \Gamma(\ell + 1 - i\eta) \right] \right\} \xrightarrow{\eta \to 0} 1, \tag{20}$$

$$h(\eta) = \frac{1}{2} [\psi(i\eta) + \psi(-i\eta)] - \ln \eta, \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \eta = \frac{e^2 Z_1 Z_2 \mu}{\hbar^2 k}. \tag{21}$$

In the explicit form for the matrix elements, Eq. (15) can be written as

$$f_{mn}^{(\text{in}/\text{out})}(E) = \frac{e^{\pi \eta_m/2} \ell_m!}{\Gamma(\ell_m + 1 \pm i\eta_m)} \left\{ \frac{C_{\ell_m}(\eta_m) k_{\ell_m+1}^{\ell_m+1}}{C_{\ell_m}(\eta_m) k_{\ell_m+1}^{\ell_m+1}} A_{mn}(E) - \left[ \frac{2\eta_m h(\eta_m)}{C^2_0(\eta_m)} \pm i \right] C_{\ell_m}(\eta_m) C_{\ell_m}(\eta_n) k_{\ell_m}^{\ell_m} k_{\ell_n}^{\ell_n+1} B_{mn}(E) \right\}. \tag{22}$$

The matrices $A(E)$ and $B(E)$ are the same for both $f^{(\text{in})}$ and $f^{(\text{out})}$, and they are real for real energies.

Apparently, the analytic structure of the $S$-matrix (2) is even more complicated than that of the Jost matrices. This means that none of the simplified phenomenological formulæ for
the multi-channel $S$-matrix (that very often are used to fit experimental data) can guarantee the correct topology of the Riemann surface. The consequences of such a simplification for the analytic continuation of the $S$-matrix are unclear and unpredictable.

2.4 Approximation and analytic continuation

In the exact expressions (22), the only unknowns are the matrices $A(E)$ and $B(E)$, which are single-valued and analytic. They can be expanded in Taylor series around an arbitrary complex energy $E_0$,

$$A(E) = a^{(0)} + a^{(1)}(E - E_0) + a^{(2)}(E - E_0)^2 + \cdots,$$

$$B(E) = b^{(0)} + b^{(1)}(E - E_0) + b^{(2)}(E - E_0)^2 + \cdots. \tag{23}$$

Here $a^{(m)}(E_0)$ and $b^{(m)}(E_0)$ are the $(N \times N)$-matrices (for a two-channel case, $N = 2$) depending on the choice of the center $E_0$ of the expansion. The matrix elements of $a^{(m)}(E_0)$ and $b^{(m)}(E_0)$ are the unknown parameters. We can take the first several terms of these expansions and find the unknown parameters by fitting some available experimental data. As a result, we obtain approximate analytic expressions (22) for the Jost matrices.

It is convenient to choose the central point $E_0$ on the real axis. Such a choice makes the matrices $a^{(m)}(E_0)$ and $b^{(m)}(E_0)$ real. After adjusting the parameters (via fitting the data) we can consider the same Jost matrices (22) at complex energies and thus can locate the resonances, as is schematically illustrated in Fig. 4.

When looking at complex $E$, we can choose the appropriate sheet of the Riemann surface. The single-valued functions $A(E)$ and $B(E)$ are the same on all the sheets. The differences
only stem from the explicit factors depending on $k_n$ and $\ln k_n$ in Eq. (22). For a given energy $E$, we calculate the square roots (13) and $\ln k_n$ for all the channel momenta. Choosing appropriate signs in front of the square roots and adding appropriate number of $2\pi$ in Eq. (14), we can place the point on any Riemann sheet that we need. In other words, the analytic continuation of the Jost matrices from the real axis (where the fitting is done) to a chosen Riemann sheet is always done correctly despite the approximations (23,24).

### 3 The data and fitting procedure

The Jost matrices describe the two-body states with definite quantum numbers, namely, $(J^\pi, \ell, s)$. If we were trying to fit “raw” experimental data, we would need to sum up several states with different $J$ and many partial waves. This would result in too many free parameters and the task would become unmanageable. To avoid such a difficulty, we consider partial cross sections (for a given $J^\pi$) separately.

In the present work, we deal with the state $0^+$ of the system $p^7\text{Be}$. This state involves only one partial wave, namely, $(\ell, s) = (1, 1)$ in both channels. In order to obtain the partial cross sections, one has to do the partial-wave analysis of the “raw” data. This is a very complicated task by itself. We therefore rely on the existing $R$-matrix analysis of the system $p^7\text{Be}$, published in Ref. [3], where the experimental data on the excitation functions for the elastic and inelastic $p^7\text{Be}$ scattering were parametrized. As a result of this analysis, the authors of Ref. [3] reported three new low-energy resonances with the quantum numbers $0^+, 1^+$ and $2^+$.

Using the parameters given in Ref. [3], we construct the $R$-matrix and then the corresponding $S$-matrix, from which any partial cross section can be calculated. Since the $R$-matrix of Ref. [3] was obtained by fitting the “raw” data, the partial cross sections we obtain from this $R$-matrix, can be considered as experimental. In a sense, such an approach is similar to treating the scattering phase-shifts as experimental data despite the fact that nobody measures them directly and they are obtained from a complicated partial-wave analysis of the “raw” data.

Thus obtained partial cross sections for the four processes (11) are given in Figs. 5-8 where they are shown by the dots. We consider these dots as the (indirectly obtained) experimental points, which we fit by varying the Jost matrices.

As the basis for parametrizing the Jost matrices, we use the semi-analytic expressions (22), where the unknown matrices $A(E)$ and $B(E)$ are analytic and single-valued functions of the
$\sigma^J (\text{mb})$

$^7\text{Be} \left[ \frac{3}{2}^- \right] (p, p) ^7\text{Be} \left[ \frac{3}{2}^- \right]$  

$J^\pi = 0^+$  

$(\ell, s)_{\text{in}} = (1, 1)$  

$(\ell, s)_{\text{out}} = (1, 1)$

dots: $R$-matrix  

curve: Jost-matrix fit  

$E = E_{\text{cm}} - E_{p^7\text{Li}}^{\text{th}}$

Figure 5: Partial cross section for the transition $1 \rightarrow 1$ of the processes (1) in the state with $J^\pi = 0^+$. The dots are the “experimental” points obtained from the $R$-matrix taken from Ref. [3]. The curve is our fit with the Jost matrix parameters given in Table [1] The collision energy is counted from the $p^7\text{Be}$ threshold.
Figure 6: Partial cross section for the transition $1 \rightarrow 2$ of the processes (1) in the state with $J^\pi = 0^+$. The dots are the "experimental" points obtained from the $R$-matrix taken from Ref. [3]. The curve is our fit with the Jost matrix parameters given in Table 1. The collision energy is counted from the $p^7\text{Be}$ threshold.

\begin{align*}
\sigma^J (\text{mb})
\end{align*}

\begin{align*}
7\text{Be} \left[ ^{3/2} - \right] (p, p) 7\text{Be}^* \left[ ^{1/2} - \right] \\
J^\pi = 0^+ \\
(\ell, s)_{\text{in}} = (1, 1) \\
(\ell, s)_{\text{out}} = (1, 1)
\end{align*}

dots: $R$-matrix curve: Jost-matrix fit

\begin{align*}
E &= E_{\text{cm}} - E_{p^7\text{Li}}^{\text{th}}
\end{align*}
Figure 7: Partial cross section for the transition $2 \rightarrow 1$ of the processes (1) in the state with $J^\pi = 0^+$. The dots are the "experimental" points obtained from the $R$-matrix taken from Ref. [3]. The curve is our fit with the Jost matrix parameters given in Table 1. The collision energy is counted from the $p^7\text{Be}$ threshold.
Figure 8: Partial cross section for the transition $2 \rightarrow 2$ of the processes (1) in the state with $J^\pi = 0^+$. The dots are the "experimental" points obtained from the $R$-matrix taken from Ref. [3]. The curve is our fit with the Jost matrix parameters given in Table [1]. The collision energy is counted from the $p^7\text{Be}$ threshold.
energy. We therefore can approximate them by the $(M + 1)$ Taylor terms,

$$A_{n'n}(E) \approx \sum_{m=0}^{M} a_{n'n}^{(m)}(E - E_0)^m, \quad (25)$$

$$B_{n'n}(E) \approx \sum_{m=0}^{M} b_{n'n}^{(m)}(E - E_0)^m, \quad n', n = 1, 2, \ldots, N, \quad (26)$$

with $E_0$ taken somewhere in the middle of the interval covered by the experimental points. The unknown expansion coefficients $a_{n'n}^{(m)}$ and $b_{n'n}^{(m)}$ serve as the fitting parameters and $N = 2$ is the number of the coupled channels. These matrices $A$ and $B$, when substituted in the semi-analytic expressions (22), give us the approximate Jost matrices and the corresponding $S$-matrix (9), which is used to calculate the approximate partial cross sections (10), $\tilde{\sigma}_{n'\leftarrow n}$, depending on the fitting parameters.

The optimal values of the fitting parameters are found by minimizing the following function:

$$\chi^2 = W_{11} \sum_{i=1}^{K} |\tilde{\sigma}_{1\leftarrow 1}(E_i) - \sigma_{1\leftarrow 1}(E_i)|^2 +$$

$$+ W_{21} \sum_{i=1}^{K} |\tilde{\sigma}_{2\leftarrow 1}(E_i) - \sigma_{2\leftarrow 1}(E_i)|^2 +$$

$$+ W_{12} \sum_{i=1}^{K} |\tilde{\sigma}_{1\leftarrow 2}(E_i) - \sigma_{1\leftarrow 2}(E_i)|^2 +$$

$$+ W_{22} \sum_{i=1}^{K} |\tilde{\sigma}_{2\leftarrow 2}(E_i) - \sigma_{2\leftarrow 2}(E_i)|^2,$$  

where $K$ is the number of experimental points, and $\sigma_{n'\leftarrow n}(E_i)$ is the experimental cross section at the energy $E_i$. The experimental errors are not defined because the data are taken from the $R$-matrix analysis. We therefore put all of them to unity in the $\chi^2$-function (27). Since the experimental errors are absent, each point is equally important in this function. However the magnitudes of the cross sections in different channels are significantly different (compare, for example, Figs. 5 and 8). As a result of such a difference, the minimization tends to give preference to the curves with larger values of $\sigma_{n'\leftarrow n}$, while the quality of the fitting of the smaller cross sections remains poor. To avoid this tendency, we introduce the weight factors
| $E_0$ | 1.6 MeV | 1.8 MeV | 2.0 MeV |
|-------|---------|---------|---------|
| $m$ | $n'$ | $n$ | $a_{n'n'}^{(m)}$, $b_{n'n'}^{(m)}$ [MeV$^{-m}$] | $a_{n'n'}^{(m)}$, $b_{n'n'}^{(m)}$ [MeV$^{-m}$] | $a_{n'n'}^{(m)}$, $b_{n'n'}^{(m)}$ [MeV$^{-m}$] |
| 0 | 1 | 1 | $-3.8980$, $260.02$ | $-2.5158$, $132.16$ | $-1.6501$, $51.674$ |
| | 2 | $-0.076997$, $19.287$ | $-0.85222$, $61.581$ | $-1.3865$, $66.164$ |
| | 1 | $0.0071846$, $142.56$ | $0.28927$, $113.93$ | $0.46568$, $71.761$ |
| | 2 | $-0.19860$, $-19.181$ | $-0.16232$, $13.827$ | $-0.10506$, $29.145$ |
| 1 | 1 | $3.5822$, $-537.13$ | $-2.4338$, $-67.781$ | $-5.7215$, $37.455$ |
| | 2 | $0.18923$, $-15.205$ | $-0.74242$, $-1.6230$ | $-4.5979$, $69.555$ |
| | 1 | $1.2508$, $41.701$ | $2.0021$, $149.42$ | $2.0154$, $131.71$ |
| | 2 | $-0.036673$, $-13.918$ | $-0.028947$, $22.033$ | $-0.37030$, $42.200$ |
| 2 | 1 | $-7.7575$, $1313.9$ | $-19.442$, $581.08$ | $-16.051$, $118.57$ |
| | 2 | $-0.46898$, $111.02$ | $-6.3748$, $250.95$ | $-12.680$, $135.59$ |
| | 1 | $2.8715$, $353.10$ | $3.7550$, $330.22$ | $3.4106$, $192.31$ |
| | 2 | $-0.77670$, $10.873$ | $-1.3155$, $33.713$ | $-1.8707$, $25.314$ |
| 3 | 1 | $-29.889$, $-698.57$ | $-27.627$, $-462.76$ | $-14.980$, $-195.76$ |
| | 2 | $0.032721$, $-78.511$ | $-8.0587$, $-198.45$ | $-10.889$, $-190.90$ |
| | 1 | $2.8100$, $220.84$ | $3.0660$, $109.69$ | $2.6138$, $57.036$ |
| | 2 | $-2.4411$, $-151.14$ | $-2.5425$, $-92.968$ | $-2.2098$, $-51.518$ |
| 4 | 1 | $-13.780$, $-18.493$ | $-9.4436$, $-19.098$ | $-4.1096$, $-12.805$ |
| | 2 | $0.86303$, $27.495$ | $-2.1777$, $19.390$ | $-2.5535$, $14.865$ |
| | 1 | $1.2247$, $-79.035$ | $0.98110$, $-50.869$ | $0.74022$, $-30.866$ |
| | 2 | $-1.4628$, $26.744$ | $-1.1201$, $10.597$ | $-0.72228$, $-1.1996$ |

Table 1: Parameters of the expansions (25,26) with three choices of the central point $E_0$. These parameters for $E_0 = 1.8$ MeV were used to generate the curves shown in Figs. 5, 6, 7 and 8.
$W_{n'n}$ in the $\chi^2$-function (27). These factors are chosen in such a way that the contributions from the four terms are more or less the same.

For the minimization, we use the MINUIT program developed in CERN [11]. The function (27) has many local minima. The search for the best of them can be based on the following strategy. First of all, the minimization procedure should be repeated many times (we did it $\sim 1000$ times) with randomly chosen starting values of the parameters. Then, after a good minimum is found, it can be refined by choosing random starting point around the best point found in the parameter space. After each improvement, the new starting parameters are chosen by random variations of the parameters around the new best point.

The cross sections (as well as any other observables) are expressed via the elements of the $S$-matrix (9), i.e. via the ratio of the Jost matrices. In such a ratio, any common factor in $f^{(in)}$ and $f^{(out)}$ cancels out. This means that the set of the parameters $a^{(m)}_{n'n}$ and $b^{(m)}_{n'n}$ can be scaled by any convenient factor. Such a scaling does not affect any results.

4 Results

The experimental data (obtained from the $R$-matrix given in Ref. [3]) for the four processes (1), were fitted as is described in Sec. 3, with $M = 4, W_{11} = 3, W_{12} = 15, W_{21} = 0.04$, and $W_{22} = 0.002$. We repeated the fit with five different values of the central energy $E_0$, namely, with $E_0 = 1.6$ MeV, $E_0 = 1.7$ MeV, $E_0 = 1.8$ MeV, $E_0 = 1.9$ MeV, and $E_0 = 2.0$ MeV (the energy is counted from the $p^7$Be-threshold). Formally, the results should not depend on the choice of $E_0$. However, the Taylor expansions (25, 26) are truncated and the minimization procedure is always an approximate one. The calculations with several different $E_0$ allow us to see how stable the results are, to find the average values of the resonance parameters and their standard deviations, and to exclude any possible spurious poles of the $S$-matrix (that should be unstable).

The results of the fit with $E_0 = 1.8$ MeV are graphically shown in Figs. 5, 6, 7, and 8. For the other choices of $E_0$, the quality of the fit is the same and it would be impossible to distinguish the corresponding curves. The optimal parameters for the three (out of five) choices of $E_0$ are given in Table 1. The units for the parameters are chosen in such a way that the Jost matrices are dimensionless.

The resonances were sought as zeros of $\det f^{(in)}(E)$ on the principal sheet $(-\ldots)^{00}$ of the Riemann surface. This was done using the Newton’s method [12]. In this way, we found two resonances that are close to each other. For each of the five choices of $E_0$, their parameters are given in Tables 2 and 3. It is seen that our procedure gives at least three stable digits.

The resonance energies obtained with different $E_0$, are statistically independent. We assume that they have the normal distribution and calculate the corresponding average values.
Table 2: Parameters of the first $0^+$ resonance found with five different choices of the expansion parameter $E_0$. The energy $E_r$ is counted from the $p^7\text{Be}$ threshold. The partial widths $\Gamma_1$ and $\Gamma_2$ correspond to the elastic and inelastic channels, respectively.

| $E_0$ (MeV) | $E_r$ (MeV) | $\Gamma$ (MeV) | $\Gamma_1$ (MeV) | $\Gamma_2$ (MeV) |
|------------|-------------|----------------|------------------|------------------|
| 1.6        | 1.65255     | 0.44772        | 0.13420          | 0.31352          |
| 1.7        | 1.65198     | 0.44653        | 0.13357          | 0.31295          |
| 1.8        | 1.65283     | 0.44908        | 0.13486          | 0.31422          |
| 1.9        | 1.65230     | 0.44713        | 0.13404          | 0.31309          |
| 2.0        | 1.65223     | 0.44745        | 0.13412          | 0.31333          |

Table 3: Parameters of the second $0^+$ resonance found with five different choices of the expansion parameter $E_0$. The energy $E_r$ is counted from the $p^7\text{Be}$ threshold. The partial widths $\Gamma_1$ and $\Gamma_2$ correspond to the elastic and inelastic channels, respectively.

| $E_0$ (MeV) | $E_r$ (MeV) | $\Gamma$ (MeV) | $\Gamma_1$ (MeV) | $\Gamma_2$ (MeV) |
|------------|-------------|----------------|------------------|------------------|
| 1.6        | 1.81879     | 0.83658        | 0.54461          | 0.29196          |
| 1.7        | 1.82066     | 0.83932        | 0.53713          | 0.30219          |
| 1.8        | 1.81891     | 0.84133        | 0.54164          | 0.29968          |
| 1.9        | 1.81947     | 0.86394        | 0.51899          | 0.34495          |
| 2.0        | 1.82101     | 0.83255        | 0.55292          | 0.27963          |

Table 4: Statistically averaged parameters of the two $0^+$ resonances (the first two lines) and the single $0^+$ resonance reported in Ref. [3]. The energy $E_{ex}$ is counted from the ground state of $^8\text{B}$ nucleus.

| $E_{ex}$ (MeV) | $\Gamma$ (MeV) | $\Gamma_1$ (MeV) | $\Gamma_2$ (MeV) | Ref.      |
|---------------|----------------|------------------|------------------|-----------|
| 1.7899 ± 0.0003 | 0.4476 ± 0.0009 | 0.1342 ± 0.0005 | 0.3134 ± 0.0005 | this work |
| 1.9573 ± 0.0010 | 0.8427 ± 0.0123 | 0.5391 ± 0.0126 | 0.3037 ± 0.0247 | this work |
| 1.9 ± 0.1     | 0.53 ± 0.60    | 0.06 ± 0.30      | 0.47 ± 0.40      | [3]       |
| S-matrix poles (MeV) | \((++)_0\) | \((-+)_0\) | \((+-)_0\) | \((-)_0\) |
|---------------------|----------|----------|----------|---------|
| \(1.8042 - i0.3971\) | \(1.8013 - i0.2575\) | \(1.6693 - i0.2178\) | \(1.6528 - i0.2245\) |
| \(1.8068 - i0.2664\) | \(1.8325 - i0.4072\) | \(1.7904 - i0.4035\) | \(1.8189 - i0.4207\) |
| \(1.8042 + i0.3971\) | \(1.7648 + i0.5187\) | \(1.7697 + i0.3874\) | \(1.7178 + i0.4407\) |
| \(1.8068 + i0.2664\) | \(1.8251 + i0.2468\) | \(1.8415 + i0.5646\) | \(1.7793 + i0.6420\) |

Table 5: Poles of the two-channel \(S\)-matrix on all the principal sheets of the Riemann surface within a distance of \(\sim 1\) MeV from the central point, \(E_0 = 1.8\) MeV, of the expansions \((25)\ (26)\). The energy is counted from the \(p^7\)Be threshold.

as well as the standard deviations. The results of these calculations (statistical averaging) are given in Table 4, where for the purpose of comparison, we also put the parameters of the \(0^+\) resonance obtained in Ref. [3].

By scanning all four principal sheets of the Riemann surface within a distance of \(\sim 1\) MeV around the central energy \(E_0\), we found several \(S\)-matrix poles on each of the sheets. These calculations were done with \(E_0 = 1.8\) MeV. Thus found poles are listed in Table 5.

Among all the poles, only those that are adjacent to the physical scattering energies, may influence the physical observables. They are those given in the left bottom and right top blocks of Table 5. There are four of them: two resonances on the sheet \((-\))_0 and two poles on the physical sheet \((++)_0\). They are depicted in Fig. 9.

The sheets \((-\))_0 and \((++)_0\) are cut along the real axis. At the energies above the second threshold, the upper rim of the \((-\))_0-cut is connected to the lower rim of the \((++)_0\)-cut. The connecting line is the real axis of the physical scattering energies. Thanks to the connection, it is possible to continuously move from the sheet \((-\))_0 to \((++)_0\) and back, for example, along the rectangular contour shown in Fig. 9.

In contrast to the resonances, the solutions of the Schrödinger equation, corresponding to the complex poles on the sheet \((++)_0\), have an unphysical behaviour (in particular, they have unphysical time dependence). The physical system cannot be in such a state, but mathematically the poles exist anyway and may influence the behaviour of the \(S\)-matrix on the real axis. Such poles are sometimes called shadow ones. The influence of these poles on the scattering cross section is explored in the next section.

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4.1 Contributions from individual poles

The $S$-matrix has many poles on the Riemann surface. Even just on the principal sheets and only around the resonance energy ($E \sim 1.8$ MeV), it has eight poles given in Table 5. Of course, their influence on the scattering cross sections are different. Apparently, the poles that are far away from the axis of the real scattering energies, contribute very little, if any. This axis passes through the connection of the sheets $(++)_{00}$ and $(- -)_{00}$ (see Fig. 2). Therefore the only noticeable influence can be expected from the four poles shown in Fig. 9, which are near that axis.

It is always useful to know how important each individual pole is. A reasonable answer to such a question can be obtained by decomposing the $S$-matrix in a sum of the pole terms and the background integral. Such a decomposition is possible thanks to the Mittag-Leffler theorem known in the complex analysis. In fact, for our purpose, it is sufficient to apply a more simple residue theorem, which leads to the Mittag-Leffler decomposition (see Refs. [4] and [7]).

Consider the rectangular contour shown in Fig. 9 which encloses the four chosen poles. If $E$ is a point inside this contour (for calculating the cross section, we choose it on the real
axis), then according to the residue theorem, we have
\[ \oint S(\zeta) \frac{d\zeta}{\zeta - E} = 2\pi i S(E) + 2\pi i \sum_{j=1}^{L} \frac{\text{Res}[S, E_j]}{E_j - E}, \]  
(28)
where \( E_j \) are the poles \((L = 4)\). This gives
\[ S(E) = \sum_{j=1}^{L} \frac{\text{Res}[S, E_j]}{E - E_j} + \frac{1}{2\pi i} \oint \frac{S(\zeta)}{\zeta - E} d\zeta, \]  
(29)
which is a particular form of the Mittag-Leffler decomposition, where the matrix \( S(E) \) is written as a sum of the individual pole contributions and a background integral.

For any given scattering energy \( E \), the background integral can be found by numerical integration of the \( S \)-matrix, which we obtained after fitting the experimental data. We assume that all the poles of the \( S \)-matrix (9) are simple, i.e.
\[ \det f^{(in)}(E) \longrightarrow \text{const} \cdot (E - E_j). \]  
(30)
Therefore the residues of the \( S \)-matrix at the poles can be found by numerical differentiation of the determinant of the Jost matrix,
\[ \text{Res}[S, E] = f^{(out)}(E) \left( \begin{array}{cc} f_{22}^{(in)}(E) & -f_{12}^{(in)}(E) \\ -f_{21}^{(in)}(E) & f_{11}^{(in)}(E) \end{array} \right) \left[ \frac{d}{dE} \det f^{(in)}(E) \right]^{-1}. \]  
(31)
Thus calculated residues for the four poles are given in Table 6. Using these residues and the numerically calculated background integral, we obtained (as it should be) exactly the same cross sections that are shown in Figs. 5, 6, 7, and 8. This is a kind of cross-check of our calculations.

Now, in order to get an idea of the role of each pole, we can omit them one by one from the sum (29) and see how this affects the partial cross sections. The results of such pole exclusions are shown in Fig. 10. The curves show the cross sections when one pole is excluded. The dots are the experimental data (i.e. the \( R \)-matrix cross sections).

It is seen that the second resonance (pole number 2) contributes very little and mainly to the elastic cross section in the first channel. The influences of the other three poles are noticeable in various channels. It is also interesting to know what happens if we only leave the background integral and exclude all the pole terms from the Mittag-Leffler expansion (29). The result of such an exclusion can be seen in Fig. 11. The background term describes the
Table 6: Poles of the two-channel \( S \)-matrix and the corresponding residues of its elements in the domains of the Riemann sheets \((-\cdots)_{00}\) and \((++\cdots)_{00}\) adjacent to the axis of the real scattering energies (see Fig. 9). The energy is counted from the \( p^7\text{Be} \)-threshold.
Figure 10: The dots represent the experimental (i.e., the $R$-matrix) cross sections for the inter-channel transitions $n \rightarrow n'$, where the channels are labeled as in Eq. (1). The curves show the corresponding cross sections obtained when a single pole is excluded from the Mittag-Leffler sum \( (29) \).

| shadow poles | resonances | pole excluded |
|--------------|------------|---------------|
| 4            | 2          | 1 $\rightarrow$ 1 |
|              | 1          | 1 $\rightarrow$ 2 |
| 3            | 2          | 2 $\rightarrow$ 1 |
|              | 1          | 2 $\rightarrow$ 2 |
general behaviour of the cross sections and gives a reasonable approximation for them to the left and to the right of the resonance energies. However, inside the resonance energy-interval, without the resonant and the shadow poles all the cross sections are far from the experimental points.

5 Summary and conclusion

As was stated in the Introduction, the main task of the present work was to confirm the existence and to accurately determine the parameters of the lowest resonance level with the quantum numbers \( J^\pi = 0^+ \) in the spectrum of the eight-nucleon system \(^8\text{B}\). For this purpose, we constructed the two-channel Jost-matrices that have proper analytic structure and are defined on the Riemann surface with the proper topology (with both the square-root and logarithmic branching). The free parameters of these Jost matrices were fixed using an available \( R \)-matrix fit \([3]\) of experimental data on \( p\text{\ }^7\text{Be} \) scattering.

Exploring the behaviour of these Jost matrices on the principal sheets of the Riemann surface, we located 16 poles of the \( S \)-matrix (see Table 5). Among them, only four poles (two resonances and two shadow poles) are located close enough to the axis of the real scattering energies and therefore can influence the observable cross sections (see Fig. 9).

Therefore, we found that instead of a single \( 0^+ \) resonance, there are two overlapping resonances with almost the same parameters as were reported in Ref. \([3]\) (see Table 4). In addition to them, there are also two overlapping shadow poles on the opposite side of the real axis.

In order to isolate the individual contributions to the \( S \)-matrix from the resonances and the
shadow poles, we used the Mittag-Leffler decomposition. In this way it was established that the second resonance has a rather weak influence on the energy dependencies of the partial cross sections. The roles of the other three poles are noticeable.

As is seen from Fig. [10], the first resonance and the second shadow pole significantly change the inelastic cross sections and the elastic scattering in the second channel. In principle, such changes could be detected experimentally, if the $^7$Be target is exposed to $\gamma$-rays of the energy $\sim 0.5$ MeV, when the cross section of $p^7$Be collision is being measured. In such a case, the electromagnetic radiation could cause part of the target nuclei to transit from the ground to the first excited state, $^7$Be($\frac{3}{2}^-$) + $\gamma$ $\rightarrow$ $^7$Be$^*$($\frac{1}{2}^-$).

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