Numerical studies on chaoticity of a classical hard-wall billiard with openings

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Using fractal analysis, we investigate how the size of openings affects the chaotic behavior of a classical closed billiard when two openings are made on the boundary of the billiard. This kind of open billiards retains chaotic properties of original closed billiards when openings are small compared to the size of the billiard. We calculate the fractal dimension using an one-dimensional subset of all possible initial conditions that will produce trajectories of a particle injected from an opening for the billiard, and then observe how the opening size can change the classical chaotic properties of this open billiard.

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Two-dimensional (2D) hard-wall billiard systems have been a popular subject for studying the dynamics of chaotic systems[1]. Theoretically, 2D billiard systems can be used to study manifestations of classical chaos in semiclassical and quantum mechanics. The classical dynamics of a billiard system shows three distinct types of behavior: the system is either integrable (regular behavior) or non-integrable (either soft chaos, characterized by mixed phase spaces that have both regular and chaotic regions, or hard chaos, characterized by ergodicity and mixing)[2]. In experiments, on the other hand, the billiard systems can be used as models to explain fluctuating behavior of magnetoconductance through 2D semiconductor heterostructures in the ballistic regime[3, 4]. In such an experiment, the structure have the shape of a 2D billiard, of which the dynamic properties of the billiard are already known (for example, the circle as a regular system, or the stadium as a system with hard chaos). To measure the conductance through the structure, leads (from here on, we will call them openings) are attached to the structure; hence the whole structure becomes an open billiard. One usually studies transmission properties through these open billiards, and links results to the classical chaotic properties of original closed billiards, because the properties of the closed billiard are still observed indirectly when the size of the openings is small compared to the size of the billiard (see Refs. [5, 6, 7]). Therefore, to better relate the open billiard to the original closed billiard, quantitative analysis on effects of openings in classical dynamics is necessary.

In this paper, we numerically calculate a quantity that represents the chaoticity of classical open billiards, and see how the size of the openings affects the overall classical dynamics. In Ref. [8], the fractal dimension was obtained by looking at a two-dimensional set of initial conditions that will produce trajectories injected from an opening of the Sinai billiard with two openings. [A set of initial conditions in four-dimensional phase space can be reduced to a two-dimensional set, because (1) energy of the particle does not affect the trajectory of a particle, and (2) initial locations of the particle can be restricted to an one-dimensional (1D) curve.] Here, however, we will use an 1D subset of all possible initial conditions for calculations of the fractal dimension to make numerical calculations simple. The incident angle of the particle injected from the center of one opening represents an 1D subset of all possible initial conditions, which give rise to some possible trajectories of injected particles. In Ref. [8], graphs of the “exit opening” (an opening from which the particle exits) vs the incident angle from the center of an opening were used. There are “geometrical channels” inside a range of incident angles. When there are two openings, we can define transmission windows that represent initial angles for transmitting particles. The boundaries of these windows will form a fractal[8, 9]. In this paper, instead of calculating the fractal dimensions of boundaries of these transmission windows, we use the graph of the number of collisions vs the incident angle, and calculate the fractal dimension of the set of singularities, at which the number of collisions changes abruptly, inside the set of all possible incident angles (see Ref. [12]). The fractal dimensions from these two graphs will be eventually the same.

The billiard used here for numerical calculations is the circular billiard with a straight cut (the cut-circle billiard) with two openings (see Fig. 1). There are five parameters: (1) the width $W$ measured in the direction perpendicular to the cut, (2) the radius $R$, (3) the angular difference of the openings, which represents the opening size, (4) the orientation angle $\Omega$ of the cut relative to the first opening, and (5) the position of the second opening relative to the first opening as measured by the angle $\gamma$. We scale the width $W$ by $R$ so $w = W/R$, and thereby reduce the number of independent parameters to four: $w$, $\Delta$, $\Omega$, and $\gamma$. For all subsequent discussions, we set $\Omega = 135^{\circ}$ and $\gamma = 270^{\circ}$. For these values, $w$ cannot be less than $[1 - \cos(45^{\circ})] \approx 0.293$, and $\Delta$ has an upper limit.

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The size of two openings is $\Delta$, and the position of the cut and the opening II, $\Omega$ and $\gamma$, are measured from the position of the opening I. A particle is injected from the center of the opening I with an angle $\phi$.

\[ \Delta_{\text{max}}, \]
\[ \Delta_{\text{max}} = \begin{cases} 2 \left[ \cos^{-1}(1 - w) - \pi/4 \right] & \text{when } 0.293 < w < 1 \\ \pi/2 & \text{when } 1 \leq w \leq 2 \end{cases} \] (1)

in radian. When the billiard is closed ($\Delta = 0$), it has been proven that the phase space is mixed (soft chaos) when $0 < w < 1$, and that the phase space is fully chaotic (hard chaos) when $1 < w < 2$ \cite{13}. The system is integrable when $w = 1$ and 2. This billiard has a property of showing all three types of dynamic behaviors just by changing a parameter $w$, and this is the reason why the cut-circle billiard is a good example for analysis like this.

The particle is injected with an incident angle $\phi$ ($-\pi/2 < \phi < \pi/2$). One can easily calculate several quantities by numerically following these trajectories, but we focus only on the number of collisions with the wall before the exit with respect to $\phi$. In other words, we find a mapping from the incident angle in opening I to the number of collisions before the exit. In Fig. 2, we show how the number of collisions changes with $\phi$ for five different $w$ values ($w = 0.5$, $w = 0.75$, $w = 1.04$, $w = 1.5$, and $w = 1.71$) when $\Delta = 30^\circ$. In these calculations, there are finite number of singularities when $w = 0.5$, but, in other cases, there are regions where singular points are closely packed together showing infinitely fine structures. We observe that there tends to be more singularities as $w$ increases. That is because (1) when $w$ is bigger than one, the original closed billiard shows hard chaos, which means global chaos, and (2) the opening size is fixed here for all $w$’s even though the overall size of the billiard increases as $w$ increases, as a result, making the relative size of the openings smaller.

We calculate the fractal dimension $d_f$ for sets of singularities for various cases, using a simplified box-counting algorithm. For $N_P$ uniformly distributed incident angles, we numerically find the number of collisions for each incident angle representing an initial condition. For several $N_P$ values, we find the number of singularities $N_S$ (assuming that there only exists a singularity when the number of collisions changes for two successive incident angles). Then the fractal dimension $d_f$ is defined by

\[ d_f \equiv \lim_{N_P \to \infty} \frac{\log_{10} N_S}{\log_{10} N_P} \] (2)

which is the slope of the graph of $\log_{10} N_S$ vs $\log_{10} N_P$. We use the ordinary least-square fit to find the slope using points in the graph with limited $N_P$. Because the numerical accuracy for numbers is limited, $N_P$ have an upper limit. Moreover the numerically given error of an initial condition will amplify as the particle travels through the billiard and hits the boundary while traveling. Here we use the double-precision numbers with 15 significant digits, and then $N_P$ should be less than $10^{15}$, especially when the number of collisions is big. In most cases in our example, $N_P$ up to $10^7$ were used to obtain $d_f$’s with two significant digits. In some cases (for example, when $w = 1.71$ and $\Delta > 59^\circ$), however, $N_P$’s up to $10^8$ were used to find $d_f$’s, because the slopes didn’t converge sufficiently when $N_P$ is less than $10^7$.

In Fig. 2, we fix the width $w$ at five different values used in Fig. 2 and vary the opening size $\Delta$ up to $\Delta_{\text{max}}$ for
FIG. 3: Graphs of the fractal dimension \( d_f \) vs the opening size \( \Delta \). For each \( w \) (\( w = 0.5, w = 0.75, w = 1.04, w = 1.5, \) and \( w = 1.71 \)), \( \Delta \) is varied up to a maximum value \( \Delta_{\text{max}} \). We can observe that \( d_f \) does not always decreases monotonically as the opening size \( \Delta \) increases.

An interesting feature of these curves is that they are not monotonically decreasing as \( \Delta \) increases. Small peaks appear even after \( d_f \) reaches zero at a certain \( \Delta \) value. For example, the \( d_f \)-curve for \( w = 1.04 \) reaches zero near \( \Delta \sim 35^\circ \), but there is a small peak near \( \Delta \sim 79^\circ \). The reason for this phenomenon is that there are more possible trajectories when the opening is bigger. In our method for the circular billiard, the launching point of the particle gets closer to the center of the billiard as the opening size increases.

In Fig. 3 two trajectories that doesn’t exist when openings are smaller are shown. Since these trajectories have the higher numbers of collisions and there is sensitive dependence on initial conditions near them, \( d_f \) becomes non-zero. In Fig. 3(a), the billiard (\( w = 1.04, \Delta = 79^\circ \)) lost most of the original shape of the closed billiard, but trajectories near the one shown (\( \phi = 84.44^\circ \)) still make up an infinitely fine structure. In Fig. 3(b), the billiard (\( w = 1.71, \Delta = 59^\circ \)) has a trajectory (\( \phi = 88.40^\circ \)) that gets close to the 3-bounce closed orbit in the circular billiard. (The closed orbits in the circular billiard are stable, but only marginally.) Even though only few bounces are shown, this trajectory actually bounces more than two thousand times as the triangular shape rotates clockwise little by little for each rotation, before the particle finally exits through either one opening. A high peak near \( \phi = 59^\circ \) for \( w = 1.71 \) in Fig. 3 is due to trajectories near this one.

In Fig. 3, we calculate \( d_f \)'s as the width \( w \) varies from 0.5 to 2 with the step size 0.01, for several opening sizes (\( \Delta = 0.5^\circ, \Delta = 1^\circ, \Delta = 5^\circ, \Delta = 10^\circ, \Delta = 20^\circ, \Delta = 30^\circ, \Delta = 40^\circ, \) and \( \Delta = 50^\circ \)). We can compare graphs in two different regions (found in the closed cut-circle billiard): \( 0 < w < 1 \) (soft chaos) and \( 1 < w < 2 \) (hard chaos). When \( w = 1 \) and \( w = 2 \) (integrable cases), \( d_f \) is zero in our calculation. As the opening size \( \Delta \) approaches zero (see graphs for \( \Delta = 0.5^\circ \) and \( \Delta = 1^\circ \)), we observe that the fractal dimension \( d_f \) approaches one in both regions in our calculation. When the opening size \( \Delta \) is not big (see graphs for \( \Delta = 5^\circ, \Delta = 10^\circ, \) and \( \Delta = 20^\circ \)), we observe that behavior in two regions is clearly distinct. In the region \( 0 < w < 1 \), there are fluctuations, which comes from the mixed phase space structures of the billiard, and in the region \( 1 < w < 2 \), graphs are smooth because the phase spaces of the billiard have no structure due to ergodicity. On the other hand, when the opening size gets bigger (see graphs for \( \Delta = 30^\circ, \Delta = 40^\circ, \) and \( \Delta = 50^\circ \)), the distinction between two regions, observed in cases with smaller openings, starts to disappear; there are fluctuations in both regions. Ergodicity in closed cut-circle billiards with hard chaos is no longer an important factor in the dynamics of open billiards when openings are big, as expected.
Fractal dimension $d_f$ vs $w$ for 8 different opening sizes, $\Delta = 0.5^\circ$ ($\circ$), $\Delta = 1^\circ$ ($\Box$), $\Delta = 5^\circ$ ($\odot$), $\Delta = 10^\circ$ ($\triangle$), $\Delta = 20^\circ$ ($\triangledown$), $\Delta = 30^\circ$ ($+$), $\Delta = 40^\circ$ ($\circ$), and $\Delta = 50^\circ$ ($\Box$). When $\Delta = 5^\circ$ and $\Delta = 10^\circ$, the behavior of the graph is clearly distinct for two regions: hard chaos ($1 < w < 2$) and soft chaos ($0 < w < 1$). But when $\Delta > 20^\circ$, the distinction between two regions disappears.

We have so far observed numerical results for the cut-circle billiard with two openings. A method that calculates the fractal dimension using an 1D subset of all possible initial conditions has been used to determine effects of openings on chaotic closed billiard quantitatively. From numerical results, we have found that the fractal dimension as a function of the opening size is not a monotonically decreasing function, because there are more trajectories, which can be sensitive to initial conditions, for bigger openings. These new trajectories can make up infinitely fine structures, and subsequently can cause the fractal dimensions to increase with $\Delta$. We also have found that, as the opening size gets bigger, the distinction caused by soft chaos and hard chaos of the original closed billiard starts to disappear, because the open billiard that are constructed by introducing openings to the closed billiard starts to lose the original shape of the closed billiard as the size of openings gets bigger.

There are several points worth mentioning. First, if we use the 2D full set of initial conditions to calculate the fractal dimensions as was done in Ref. [8], $d_f$ will be in the range of $1 \leq d_f \leq 2$, but the overall behavior of $d_f$ is expected to be similar to that of the results obtained here. Second, we can use a mapping from initial conditions to other quantities like dwell times (or travel distances) instead of the exit openings and the number of collisions. Even though the dwell time will be more expensive to calculate numerically, the set of singularities from the dwell time will be almost the same as those from the number of collisions. Finally, it has been found to be important and necessary to understand relations between classical dynamics and quantum (or semiclassical) dynamics of billiards [16, 17, 18], because the ballistic scattering is usually realized in microscopic scales. Then it will be interesting to ask how this kind of fractal dimensions can be related to the quantum and semiclassical dynamics of the same kinds of open billiards.

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