Accuracy and Numerical Stability Analysis of Lattice Boltzmann Method with Multiple Relaxation Time for Incompressible Flows

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Abstract. Lattice Boltzmann Method (LBM) is the novel method for simulating fluid dynamics. Nowadays, the application of LBM ranges from the incompressible flow, flow in the porous medium, until microflows. The common collision model of LBM is the BGK with a constant single relaxation time τ. However, BGK suffers from numerical instabilities. These instabilities could be eliminated by implementing LBM with multiple relaxation time. Both of those scheme have implemented for incompressible 2 dimensions lid-driven cavity. The stability analysis has done by finding the maximum Reynolds number and velocity for converged simulations. The accuracy analysis is done by comparing the velocity profile with the benchmark results from Ghia, et al and calculating the net velocity flux. The tests concluded that LBM with MRT are more stable than BGK, and have a similar accuracy. The maximum Reynolds number that converges for BGK is 3200 and 7500 for MRT respectively.

1. Introduction
The Lattice Boltzmann Method (LBM) was an extension from the Lattice Gas Automata (LGA) [1]. LBM has developed in significant way and already applied to several kinds of fluid dynamics such as flow in porous media and multiphase flow [2, 3]. The common collision process is the Bhatnagar-Gross-Krook (BGK) with a single relaxation time τ. Despite its simplicity, BGK suffers numerical instability. This problem has already solved by implementing the multiple relaxation time (MRT)[7, 8]. This generalized view has an agreement with the kinetic theory of gases since the collision according to the kinetic theory of gases has various relaxation time and is related to several physical parameters [6]. Beside the numerical stability, another important aspect to analyze from Computational Fluid Dynamics(CFD) is the accuracy. The test-case for the accuracy and numerical stability analysis is the incompressible 2 dimensional Lid-Driven Cavity. Numerical stability and accuracy analysis in 2 dimensional lid-driven cavity has been done for low Reynolds number by Luo, et. al.[14]. This research will continue the works by Luo, et. al [14] for higher Reynolds number. Moreover, we will do the continuity analysis by Aydin [13] and further accuracy analysis by comparing the benchmark results from Ghia, et. al. In general, this paper begins with the discussion about LBM with BGK and MRT for D2Q9 lattices, the initial and boundary condition, and the lid-driven cavity itself. Last, we will discuss the accuracy and numerical stability.
2. The Lattice Boltzmann Method.

2.1. Bhatnagar-Gross-Krook Approximation,

The evolving equation for the distribution function with respect to canonical coordinate \( f(\mathbf{r}, \mathbf{p}, t) \) in phase space \( d^3r d^3p \) is the Boltzmann’s equation,

\[
\left( \frac{\partial}{\partial t} + \frac{\mathbf{u}}{m} \cdot \nabla \mathbf{r} + \mathbf{F} \cdot \nabla \mathbf{p} \right) f(\mathbf{r}, \mathbf{p}, t) = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}
\]  

(1)

which stated that the spatial and temporal evolution depend on the changes from collision \( \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} \). The macroscopic parameters can be calculated from the moment integral.

\[
\rho = \int f(\mathbf{r}, \mathbf{p}, t)d\mathbf{u} \quad \rho \mathbf{v} = \int \mathbf{u} f(\mathbf{r}, \mathbf{p}, t)d\mathbf{u}
\]

(2)

With Bhatnagar-Gross-Krook approximation, \( \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} \) is affected by single relaxation time \( \tau \),

\[
\frac{\partial f(\mathbf{r}, \mathbf{p}, t)}{\partial t} + \frac{\mathbf{u}}{m} \cdot \nabla f(\mathbf{r}, \mathbf{p}, t) = -\frac{1}{\tau} \left[ f(\mathbf{r}, \mathbf{p}, t) - f^{eq}(\mathbf{r}, \mathbf{p}, t) \right]
\]

(3)

Furthermore, the integral on equation (2) is discretized with quadrature,

\[
\rho = \sum_i f_i \quad \rho \mathbf{v} = \sum_i f_i \mathbf{c}_i
\]

(4)

\( f_i \) is discrete distribution function and \( \mathbf{c}_i \) is discrete velocity. Hence we have the discrete form of Boltzmann’s equation,

\[
f_i(\mathbf{r} + \mathbf{c}_i \Delta t, t + \Delta t) - f_i(\mathbf{r}, t) = -\frac{1}{\tau} \left[ f_i(\mathbf{r}, t) - f_i^{eq}(\mathbf{r}, t) \right]
\]

(5)

The l.h.s is the streaming process and the r.h.s is the collision process. With Chapman-Enskog expansion [2, 3], we obtained the exact relation between LBM and Navier-Stokes for fluid dynamics namely the kinematic viscosity,

\[
\nu = c_s^2 \left( \tau - \frac{1}{2} \right)
\]

(6)

The solution of equation (5) is executed in lattice domain. We will use D2Q9 lattice with 9 discrete velocities,

\[
\mathbf{c}_i = \begin{cases} (0,0), & i = 0 \\ (0,\pm 1)(\pm 1,0), & i = 1, 2, 3, 4 \\ (\pm 1, \pm 1), & i = 5, 6, 7, 8 \end{cases}
\]

(7)

In D2Q9 the lattice speed will be \( c = \frac{\sqrt{3}}{\sqrt{2}} = 1 \) and sound speed \( c_s^2 = RT = c^2/3 \). As a result, the equilibrium distribution function \( f^{eq} \),

\[
f_i^{(eq)} = w_i \rho \left[ 1 + \frac{3(c_i \cdot \mathbf{v})}{c^2} + \frac{9(c_i \cdot \mathbf{v})^2}{2c^4} - \frac{3v^2}{2c^2} \right]
\]

(8)

\( f^{eq} \) is obtained by second order Taylor expansion for velocity from the equilibrium Maxwell-Boltzmann distribution. Furthermore, the moment integral is evaluated with third-order Gauss-Hermite quadrature [5], the value of the weight \( w_i \) then become,

\[
w_i = \begin{cases} 4/9, & i = 0 \\ 1/9, & i = 1, 2, 3, 4 \\ 1/36, & i = 5, 6, 7, 8 \end{cases}
\]

(9)
2.2. Multiple Relaxation Time

The collision process on MRT is solved on moment space $M$, while the streaming process still on the velocity space $V$, the transformation to moment space is done with a group of base vectors $\{|\phi_k| k = 0, 1, \ldots, N\}$ which are the results of the Gram-Schmidt orthogonalisation for polynomials of the velocity. Hence, the base vectors for 9 discrete velocities [8],

$$
\begin{align*}
|\phi_1| &= |c_i|^0 \\
|\phi_2| &= -4|c_i|^0 + 3(c_{ix}^2 + c_{iy}^2) \\
|\phi_3| &= 4|c_i|^0 - \frac{21}{2}(c_{ix}^2 + c_{iy}^2) + \frac{9}{2}(c_{ix}^2 + c_{iy}^2)^2 \\
|\phi_4| &= c_{ix} \\
|\phi_5| &= [-5|c_i|^0 + 3(c_{ix}^2 + c_{iy}^2)]c_{ix} \\
|\phi_6| &= c_{iy} \\
|\phi_7| &= [-5|c_i|^0 + 3(c_{ix}^2 + c_{iy}^2)]c_{iy} \\
|\phi_8| &= c_{ix}^2 + c_{iy}^2 \\
|\phi_9| &= c_{ix}c_{iy}
\end{align*}
$$

One can state the base vectors with the transformation matrix $M$,

$$
M = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-4 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\
4 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & -1 & 0 & 1 & -1 & -1 & 1 \\
0 & -2 & 0 & 2 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & -1 \\
0 & 0 & -2 & 0 & 2 & 1 & 1 & -1 & -1 \\
0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1
\end{pmatrix}
$$

Hence, the moment distribution function,

$$
\mathbf{m} = \langle f | \phi_k \rangle = (\rho, e, c, \mathbf{j}_x, \mathbf{q}_x, \mathbf{j}_y, \mathbf{q}_y, \mathbf{p}_{xx}, \mathbf{p}_{xy})^T
$$

The first three elements $(\rho, e, c)$ are rank-0 tensor or scalar, $\rho$ is density, $e$ stands for kinetic energy and $c$ is the energy square. The next 4 elements $(\mathbf{j}_x, \mathbf{q}_x, \mathbf{j}_y, \mathbf{q}_y)$ are rank-1 tensor which denotes the x-axes momentum, energy flux on x-axes, y-axes momentum, and energy flux on y-axes, respectively. The last 2 elements $(\mathbf{p}_{xx}, \mathbf{p}_{xy})$ are rank-2 tensors that denotes the diagonal and off-diagonal off stress tensor.

Then, the equilibrium distribution function on moment space [3],

$$
\mathbf{m}^{eq} = \rho(1, -2 + 3v_x^2, 1 - 3v_x^2, v_x, -v_x, v_y, -v_y, v_x^2 - v_y^2, v_xv_y)^T
$$

The relaxation rate can be set to each physical process related to equation (20). As a result, there are 9 relaxation rate for different processes.

$$
S = (0, s_c, s_e, 0, s_q, 0, s_q, s_p, s_p)
$$

Hence, the evolving equation for LB-MRT,

$$
|f(r + c_i\Delta t, t + \Delta t)| - |f(r, t)| = -M^{-1}S[|m(r, t)| - |m^{eq}(r, t)|]
$$

Furthermore, the relaxation rate for density and momentum is equal to zero because they conserved against collision. Other relaxation parameters are set with such values,

$$
\begin{align*}
s_c &= 1.64 & s_e &= 1.54 \\
s_q &= 1.92 & s_p &= 1/\tau
\end{align*}
$$
3. Lid-Driven Cavity

In 2 dimensional lid-driven cavity, the fluid is in a $L_x \times L_y$ square cavity. The top wall of the cavity is moving with constant velocity $v_0$. The benchmark velocity profile for 2 dimensional lid-driven cavity from Ghia, et. al [12] is used for accuracy comparison. The characterization parameter for incompressible flow in lid-driven cavity is the Reynolds number (Re),

$$Re = \frac{Lv_0}{\nu}$$  

(24)

The total horizontal and vertical velocity flux must equal to zero since the flow is closed. One can determine the total flux by calculating the volumetric integrals [13],

$$Q_{\text{hor}} = \int_{0}^{L_x} v_y dx$$  

(25)

$$Q_{\text{ver}} = \int_{0}^{L_y} v_x dy$$  

(26)

The first integral is evaluated at $L_y/2$ and the latter is evaluated at $L_x/2$.

4. Accuracy and Numerical Stability Analysis

The simulation is done with various Reynolds number from 100 to 7500, wall velocity $v_0 = 0.1$, and two lattice sizes $128 \times 128$ and $256 \times 256$. First, the evaluation of streamline plot for $N = 128 \times 128$

There are differences between the streamline plot of the BGK and MRT implementation. BGK cannot capture the vortex in the bottom left of the cavity on Re=400 and Re=1000. We get different outcome when using $N = 256 \times 256$ where the results are identical. Here, we could indicate that the implementation of MRT have greater accuracy for smaller lattice size and the MRT implementation is more stable against the lattice size. Further accuracy test is comparing the velocity profiles with the benchmark results from Ghia, et. al.[12].

The velocity profile is slightly deviates from the references for N=128. Most of the errors are occured near the boundary. Surprisingly, the results for BGK and MRT are again identical with N=256. Here it indicates once more that implementation of MRT is more stable against the variation of lattice size.

Moreover, the pressure contour will also be evaluated. The value of pressure is obtained from the equation of ideal gas $p = \rho c_s^2$ and normalized with $v_0^2$. Here we found that the pressure contour plot is oscillating at top corner of the cavity and increased as the Reynolds number increased. The MRT implementation cannot eliminate this effect but it can damp the oscillation. This oscillation is caused by the singular velocity on top corner of the cavity since the top wall is...
moving and the side wall is stationary. This is called the compressibility effect which is the main source of the density fluctuation in the corner that caused the oscillation. This effect is intrinsic within LBM since it is used to simulate small velocity but cannot fully incompressible. These effects can be reduced by setting the dissipation of the density fluctuation. This can be done only if we set the relaxation rate with respect to the energy $s_e$. Hence, it is obvious that only the MRT implementation could do that since the relaxation rate for BGK only depends on $\tau$.

In order to analyze the numerical stability, we tried to find the maximum value of Reynolds number that still converges with each implementation. The BGK can converge until $Re=3200$ and the MRT can converge until $Re=5000$ with $N=128$ and $Re=7500$ with $N=256$.

The final test for the numerical stability analysis is to determine the maximum wall velocity that still converges within 1000 iterations.

The test showed the MRT implementation is more stable than BGK. On BGK, when $1/\tau$ approach 2, then the viscosity $\nu$ approach 0, and when $1/\tau > 2$, we have a negative viscosity which is certainly unphysical. From the graph we can see that the highest velocity is reached when $1/\tau = 1.92$ which is the exact value of $s_q$. Furthermore, the numerical stability could be

Figure 2: Streamline for $Re=400$ with $N=128$: (a)BGK, (b)LB-MRT, Streamline for $Re=1000$ with $N=128$: (c)BGK, (d)LB-MRT
enhanced if we set the value of $s_q$ equal to $s_{\nu}$, to realized the concept of no-slip at bounce-back boundary condition [14].
5. Conclusions
The implementation of multiple relaxation time (MRT) for collision process of the lattice Boltzmann method for 128 × 128 lattices has better accuracy compared to the implementation
of Bhatnagar-Gross-Krook (BGK) approximation, both of them have identical accuracy on 256 × 256 lattices. The maximum Reynolds number that still converges is 3200 for BGK and 5000 and 7500 for MRT with N=128 and N=256 respectively. Hence, we can conclude that the implementation of MRT is more stable against variation of Reynolds number and lattice size. Moreover, the MRT implementation can reduce the compressibility effect by dissipating the density fluctuation, this can be done by setting the relaxation rate of energy $s_e$. The numerical stability test also showed that MRT is more stable than BGK, and can enhance the numerical stability by setting $s_q$ equals to $s_{\nu}$. The test showed that it reach its maximum velocity when $1/\tau = 1.92$ which is the value of $s_q$.  

Figure 7: Velocity profiles for Re=5000 and Re=7500.

Figure 8: Numerical stability test.
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