Calculation of the quark condensate through Schwinger-Dyson equation

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In this letter, we clarify the algebra expression for calculating the quark condensate based on the non-perturbative quark propagator calculated through Schwinger-Dyson equation. The quark condensates, which characterize the low energy QCD vacuum, should not get a divergent quantity at large energy scale; the re-normalization group evolution behaviour at large energy scale therefore should be interpreted as "smeared collective effects" for it contains both perturbative and non-perturbative parts. We prefer the integral expression and get a quantity which is both convergent and scale dependent.

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I. INTRODUCTION

Quark condensates, the non-vanishing vacuum expectation values of normal product $\bar{q}q$, which characterize the non-perturbative quantum chromodynamics (QCD) vacuum, correspond to the breaking of Chiral invariance. Their phenomenological importance to hardronic physics properties exhibits itself by the initial work of Shifman, Vainshtein and Zakharov [1], which is now known as QCD sum rule. The QCD sum rule approach tries to incorporate non-perturbative elements of full QCD. As the starting point, operator product expansion method was used to expand the time ordered currents into a series of quark and gluon condensates which parameterize the long distance properties, while the short distance effects are incorporated in the Wilson coefficients. However, these elements (condensates) can not yet be rigorous in QCD. In this letter, we clarify the algebra expression for calculating the quark condensate based on the non-perturbative quark propagator calculated through Schwinger-Dyson equation. The letter is arranged as: in section 2, re-normalization group evolution of the quark condensate; in section 3, calculation of the quark propagator and algebra expression for the quark condensate; in section 4, conclusion.

II. RE-NORMALIZATION GROUP EVOLUTION OF THE QUARK CONDENSATE

Let us begin our work by the Gell-Mann-Oakes-Renner (GMOR) relation [2], which announces the existence of non-zero quark condensate for the first time.

$$f_{\pi}^2 m_{\pi}^2 = (m_u + m_d) \mu \{ \langle \bar{u}u \rangle_\mu + \langle \bar{d}d \rangle_\mu \}. \quad (1)$$

This relation can be easily obtained by a few algebra manipulation. The left hand side of Eq.(1) is scale independent, while the components of the right hand side is scale dependent. To make a scale independent quantity, the $m_\mu$ and $\langle \bar{q}q \rangle_\mu$ must evolve in opposite direction:

$$m_\mu = \left( \frac{\log(\Lambda^2 / \Lambda^2_{QCD})}{\log(\mu^2 / \Lambda^2_{QCD})} \right)^d m_\Lambda, \quad \langle \bar{q}q \rangle_\mu = \left( \frac{\log(\mu^2 / \Lambda^2_{QCD})}{\log(\Lambda^2 / \Lambda^2_{QCD})} \right)^d \langle \bar{q}q \rangle_\Lambda = \{ \log(\mu^2 / \Lambda^2_{QCD}) \}^d \text{ constant}, \quad (2)$$

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where $\Lambda_{QCD}$ is the QCD scale parameter, $\Lambda$ serves as boundary condition, $d$ is the anomaly dimension. The $\langle \bar{q}q \rangle_\mu$, which characterizes the low energy non-perturbative QCD vacuum, acquires a divergent quantity at large energy scale, while the perturbative effects are dominating. It is obvious an artifact of the re-normalization group equation. We can borrow some idea from quantum electrodynamics (QED), if we take QED as self consistent theory, to avoid the Landau singularity and triviality [3], the perturbative expansion of coupling must break down at some energy scale far below Landau singularity and the QED emerges into grand unified gauge theory which is an asymptotic gauge theory, through not very successful [3]. It is an indication there must be some constrains for the application of the re-normalization group equation.

Considering the particular scale behaviour of non-abelian gauge theory, we can divide the QCD vacuum into both non-perturbative and perturbative parts. The full two point quark Green function can then be expressed into the following form:

$$S(x) = S_{PT}(x)_\mu + S_{NP}(x)_\mu.$$  (3)

The scale $\mu$ is the Chiral phase transition point, the effects above which can be calculated by the usual perturbative expansion method, while the effects below is embodied in the quark vacuum condensate. For example, for $\mu > m_c$, the values of $\alpha_s(\mu)$ are sufficiently small that the effects of strong interaction can be treated in perturbation theory.

When one moves to low energy scale, $\alpha_s$ increases. At $\mu \approx 1\,\text{GeV}$ and $\Lambda_{QCD}^{(3)}(\text{MS})$ (Here 3 is the number of quark flavor), $\alpha_s > 0.5$. It signals the breakdown of perturbation expansion. We can also get support from renormalon theory of QCD, which serves as a bridge between perturbative and non-perturbative QCD. In general, ultraviolet renormalon can be canceled by introducing counter terms while the infrared renormalon is absorbed into vacuum condensate due to the strong coupling at low energy scale [4]. If the vacuum condensate be pure collective non-perturbative objects, the $\mu$ can not taken to be very large where the perturbative effects are dominating.

III. CALCULATION OF THE QUARK PROPAGATOR AND ALGEBRA EXPRESSION FOR THE QUARK CONDENSATE

If the full two point quark Green function is known, the calculation of the quark condensate can be greatly facilitated. To obtain the full quark Green function, let us first examine the status of two main non-perturbative method, the Schwinger-Dyson(SD) equation and lattice simulation.

Quantum chromodynamics, as a non-breaking $SU(3)$ gauge theory, merits asymptotic freedom when the energy scale is large, which protects the perturbative calculation doing the work, so we can get the full two point quark Green function by loops calculation. However, at small energy scale, the expansion of large coupling constant leads to a divergent series. On the other hand, one of the main non-perturbative method, lattice calculation can not give reliable results below 1 GeV, where the most novel physics are supposed to lie. Furthermore, lattice calculation suffers its own shortcomings, such as Gribov copies, boundary conditions, extrapolation and so on. The most outstanding may be that the lattice simulation can not include the contribution from the quark loops. Neglecting the quark loops may lead to very different results comparing with the corresponding full ones. For example, although the ghost contributions are important for the protection of unitary condition for the transition amplitude, they can be safely neglected for its small numerical values at high energy scales. However, at low energy scale, including ghost greatly changes the infrared structure of gluons two point functions. That is, one get an infrared vanished instead an infrared enhanced gluon propagator [7,8] (Ref. [7,8] are comprehensive review).

We can not solve the Euler-Lagrange equation for quarks, gluons and ghost, and get the analytic expression for those fields. At the present time SD equation is the most reliable tool for studying the infrared behaviour of the
quarks and gluons propagators in the continuum limit, while it has its own shortcomings. One can get a series of coupled re-normalized integral equations for those Green functions through functional integral method by including the counter terms. However, it is a divergent series and difficult to deal with in practice, one have to make some truncation. In fact, the long part of dressed vertex can be approximated through the solution of the corresponding Ward-Takahashi or Slavnov-Taylor identity \[11\] while the transverse part of vertex can be determined with the guide of high energy behaviour, for example, Ball-Chiu vertex \[12\] and Curtis-Pennington vertex \[13\]. The re-normalized SD equation for quark propagator can be expressed in the following form:

$$S^{-1}(p, \mu) = Z_2(\mu, \Lambda)[\gamma \cdot p - m_0(\Lambda)] + \frac{16\pi i}{3}Z_1(\mu, \Lambda) \int^{\Lambda}_0 \frac{d^4k}{(2\pi)^4} \gamma_\mu S(k, \mu) \Gamma_\nu G^{\mu\nu}(k - p), \quad (4)$$

where

$$S^{-1}(p, \mu) = A(p, \mu)\gamma \cdot p - B(p, \mu) \equiv A(p, \mu)[\gamma \cdot p - m(p)], \quad (5)$$

$$G^{\mu\nu}(k) = -(g^{\mu\nu} - k^\mu k^\nu/k^2)G(k^2), \quad (6)$$

and \(m_0\) stands for an explicit quark mass-breaking term. With the explicit small mass term, we can preclude the zero solution for \(B(p)\) and in fact there indeed exist a bare current quark mass which is given by the Yukawa coupling of Higgs due to the spontaneous breaking of SU(2) Chiral symmetry at an energy scale about 246 GeV. The full vertex \(\Gamma_\mu\) can be approximated by the Ball-Chiu vertex and Curtis-Pennington vertex. This dressing comprises the notation of constituent quarks by providing a mass \(m(p) = B(p, \mu)/A(p, \mu)\), which is corresponding to dynamical symmetry breaking. Because the form of the gluon propagator \(g^2G(p)\) in the infrared region is unknown, one often uses model forms as input in the previous studies of the rainbow SD equation \[9,8\]. Although the loops integration is in general divergent, the full quark gluon loop integration maybe non divergent. There are some successful model dependent formulations for the coupling constant which insure a natural integral cut off at large energy scale, for example, confining \(\delta^4(k)\) model, gauss distribution model, flat bottom model \[9,8,14\]. There is no need for the nomenclature subtraction and hence re-normalization, if there is no divergent in the loops integration. The re-normalization coefficients can be set to 1.

The quark condensate, as isolated closed non-perturbative quark loop integral, if dependent on any energy scale, the scale is the momentum cut off in the quark loop integral,

$$\langle \bar{q}(x)q(0)\rangle_\mu = (-4N_c)\int_0^\mu \frac{d^4p}{(2\pi)^4} \frac{B(p^2)}{p^2A^2(p^2) + B^2(p^2)}e^{ipx}. \quad (7)$$

At \(x=0\) the expression for the local condensate \(\langle \bar{q}(0)q(0)\rangle\) is recovered,

$$\langle \bar{q}(0)q(0)\rangle_\mu = -\frac{12}{16\pi^2} \int_0^\mu ds A^2(s) + B^2(s). \quad (8)$$

In fact, the energy scale \(\mu\) is implicitly determined by the effective gluon propagator, we can use above expression as the definition for the quark condensate. However, presently, the non-perturbative technique can not prove the relation

$$\int_0^\mu ds \frac{B(s)}{sA^2(s) + B^2(s)} \sim (\log(\mu^2/\Lambda^2_{QCD}))^d, \quad (9)$$

at low energy scale. Here \(d\) is the anomalous dimension and takes the value \(d = \frac{12}{33 - 2n_f}\).

Operator product expansion has proven that at large Euclidean momentum, the effective quark mass evolves as

$$m(q^2)_{q^2 \rightarrow \infty} = \frac{c}{q^2} \left(\log(q^2/\Lambda^2_{QCD})\right)^{d-1} + m(\mu) \left(\frac{\log(\mu^2/\Lambda^2_{QCD})}{\log(q^2/\Lambda^2_{QCD})}\right)^d, \quad (10)$$
here \( c = -\frac{4\pi d}{\log(\mu^2/\Lambda_{QCD}^2)} \) is some constant independent of \( \mu \) \[15,16\], this implies that \( \langle \bar{q}q \rangle_\mu \sim [\log(\mu^2/\Lambda_{QCD}^2)]^d \), but not the definition of Eq.(8).

If we take the limit \( \mu \to \infty \) and assume ultraviolet dominating, the above relation \( \langle \bar{q}q \rangle_\mu \sim (\log(\mu^2/\Lambda_{QCD}^2))^d \) is indeed the case, although the quark condensate is only defined at low energy scale.

When the energy scale \( \mu \) increase, one obtain an increasing quantity. It is obvious a great mount of perturbative effects come in the integral, then \( \langle \bar{q}q \rangle_\mu \) contains both perturbative and non-perturbative parts and can be best interpreted as ”smeared collective effects”, due to the falling attraction force can hardly bind the quarks together to form vacuum condensate. There is another description of quark confinement, the propagator of a colored state should have no singularities on the real time-like \( p^2 \) axial, the absence of Kallen-Lehmann spectral representation precludes the existence of free quarks \[6,11\]. The interaction between dressed quarks may diminish at separation beyond the characteristic length scale at which point the quark Green’s function approaches its vacuum values, but the vacuum’s repulsive force due to the absence of a mass pole in the dressed quark propagator again compresses the quarks together. However, there is a long way to go before the repulsive quark-vacuum interaction can be quantified.

Here we borrow some idea from studies of strong coupling quantum electrodynamics, which is often used as an abelian model for QCD. In those studies, the effective coupling constant is often taken as constant, and there does exist divergent in the loops integral, so re-normalization is necessary \[18\]. We have to change above algebra formulation for the quark condensate through using re-normalized quark propagator,

\[
\langle \bar{q}q \rangle_\mu = -4N_c \int_0^\infty \frac{d^4p}{(2\pi)^4} \frac{m(p^2)}{A(p,\mu)(p^2 + m^2(p))}, \tag{11}
\]

where

\[
A(p,\mu)|_{p^2=\mu^2} = 1. \tag{12}
\]

The effects from hard gluonic radiative corrections to the quark propagator are connected to a possible change of the re-normalization scale \( \mu \) at which the condensates are defined. Those effects are of minor importance for the non-perturbative effects in the low and medium energy regions, and should be neglected as it is perturbative contribution if we take quark condensate as pure collective objects. In fact, the separation is hard to perform.

On the other hand, if we take the GMOR relation as the starting point, one should not take the scale be very large due to the artifact of the re-normalization group for \( \langle \bar{q}q \rangle_\mu \), the evolution of quark condensate with \( \mu \) can be determined from GMOR relation and re-normalization group equation,

\[
\langle \bar{q}q \rangle_\mu = \left\{ \frac{\log(\mu^2/\Lambda_{QCD}^2)}{\log(\Lambda^2/\Lambda_{QCD}^2)} \right\}^d \langle \bar{q}q \rangle_\Lambda. \tag{13}
\]

In the following, we write the expressions more explicitly:

\[
\langle \bar{q}q \rangle_\mu = -\left\{ \frac{\log(\mu^2/\Lambda_{QCD}^2)}{\log(\Lambda^2/\Lambda_{QCD}^2)} \right\}^d \frac{12}{16\pi^4} \int_0^\Lambda d^4p \frac{B(p^2)}{A(p^2)p^2 + B(p^2)^2}, \tag{14}
\]

and

\[
\langle \bar{q}q \rangle_\mu = -\left\{ \frac{\log(\mu^2/\Lambda_{QCD}^2)}{\log(\Lambda^2/\Lambda_{QCD}^2)} \right\}^d \frac{12}{16\pi^4} \int_0^\infty d^4p \frac{m(p^2)}{A(p,\Lambda)(p^2 + m^2(p))}. \tag{15}
\]

here one should not take \( \Lambda \) a large value. Equation (15) and Eq.(14) are expressions for quark condensate with and without hard gluonic radiative corrections respectively.

In Ref. \[17\], the authors got a scale dependent of \( \langle \bar{q}q \rangle_\mu \) through analyzing the Ward-Takahashi identity and the Beth-Salpeter equation with the tacit acceptance of the GMOR relation.
\[ 
\langle \bar{q}q \rangle_\mu = \begin{cases} 
\frac{\log(\mu^2/\Lambda_{QCD}^2)}{\log(\Lambda^2/\Lambda_{QCD}^2)} \int d^d(\bar{q}q)_{\Lambda}, \\
\frac{\log(\mu^2/\Lambda_{QCD}^2)}{\log(\Lambda^2/\Lambda_{QCD}^2)} \int d^4(-4N_c) \int_0^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{m(p^2)}{A(p, \mu)(p^2 + m^2(p))}, 
\end{cases}
\]

In fact, although the expression has explicit scale dependence, it does not satisfy the re-normalization group equation due to the scale dependence of \( A(p, \mu) \).

IV. CONCLUSION

In this letter, we examine the algebra definition of quark condensate. To obtain the quark condensate, we have to get the non-perturbative quark propagator through quark SD equation. If there is no divergent in the loops integration, re-normalization is unnecessary. The quark condensate, as isolated closed non-perturbative quark loop integral, if dependent on any energy scale, the scale is the momentum cut off in the quark loop integral. Otherwise, for a re-normalized quark propagator, the subtraction point provided a natural energy scale, we should use the definition Eq.(11). As the critical point for perturbative-nonperturbative phase transition is about 1 GeV, one should not get a quantity at large energy scale \( \langle \bar{q}q \rangle_\Lambda \) and the then evolve to lower energy scale through re-normalization group equation. To get explicitly scale dependent expressions for quark condensate, one can use Eq.(14) and Eq.(15) with \( \Lambda \) taken not far above 1 GeV. At large energy scale, the falling attraction force can hardly bind the quark together to form vacuum condensate, meanwhile, a great mount of perturbative effects come in the integral, the quark condensate \( \langle \bar{q}q \rangle_\Lambda \) contains both perturbative and non-perturbative parts and can be best interpreted as "smeared collective effects". If one take the quark condensate as pure collective objects, the \( \mu \) should not taken to be large, one can also get some support from renormalon theory of QCD.

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