Complexity Study of a Single Particle Under q-Deformed Potentials

Ferhat Nutku\textsuperscript{1}, K. D. Sen\textsuperscript{1,2} and Ekrem Aydiner	extsuperscript{1}

\textsuperscript{1}Department of Physics, Faculty of Science, Istanbul University, Vezneciler, Istanbul, 34134, Turkey
\textsuperscript{2}School of Chemistry, University of Hyderabad, Hyderabad 500 046, India

(Dated: June 26, 2018)

We have studied the variation of the position space statistical complexity measure defined by López-Ruiz, Mancini, and Calbet such as the product of exponential of the Shannon information entropy and the disequilibrium by using the 1D-normalized probability densities derived from solutions of the Schrödinger equation corresponding to the $q$-deformed harmonic oscillator and $q$-deformed Morse potentials. An analysis of the numerical results in terms of Shannon information entropy, disequilibrium and complexity measure are presented. In $q$-deformed harmonic oscillator, $q$-dependence of the complexity shows a minimum point for all excited energy levels. In the case of $q$-deformed Morse Potential, complexity decreases with increasing $q$ for the investigated diatomic molecules.

PACS numbers: 89.75.-k, 89.70.-a, 89.70.Cf

\textsuperscript{*}ekrem.aydiner@istanbul.edu.tr
I. INTRODUCTION

Complexity measures are being increasingly employed in order to understand the behavior of systems encountered in several disciplines of scientific inquiry. Everybody knows that it is very difficult to define a universal definition of complexity for all systems. Therefore, it can be seen that different measures for complexity have been proposed in the literature [1]. A few examples of them are (i) algorithmic complexity [2, 3], (ii) a measure of the self-organization capacity of a system [4], (iii) Crutchfield and Young’s complexity [5] etc. For a finite many-particle system, complexity may be regarded as a measure of it’s internal order/disorder which can be represented by the corresponding information entropy and the distance from equilibrium which is called as disequilibrium. In the context of electronic structure of atoms and molecules, a suitable measure which is so called as $C_{LMC}$ has been proposed by López-Ruiz, Mancini and Calbet (LMC) [6, 7] to analysis the statistical complexity. This measure has been widely employed to many problems in the literature [7–9]. Indeed, $C_{LMC}$ allocates a multiplicative role to the measure of distance from equilibrium in conjunction with the information entropy to define a measure for the complexity of a finite system.

In most of the previous studies focused on quantum systems, the wave function of a particle has been used to obtain quantities such as information entropy, disequilibrium and complexity. It is known that a Gaussian type potential leads to a Gaussian wave function, which strongly satisfies the Heisenberg inequality. However a non-Gaussian type potential leads to a more non-exponential wave function which does not strongly satisfy Heisenberg inequality. In this case, instead of Heisenberg inequality, a different one such as Bialynicki-Birula and Mycielski (BBM) inequality [10] can be often preferred. BBM inequality is a theoretic-informatic inequality which can be applied to many different problems. In the complexity discussion of quantum systems, the form of the wave function is very important, which may present different aspects of the complex behavior of these systems. One of the source of the non-exponential form is q-deformation over the wave function, which can emerge from non-linear interactions and non-Markovian memorial effects within the systems or from the interactions between the system and its surrounding environment. True understanding of the effect of deformed potential can shed light on the deep physics of complexity of several real systems. q-deformed potentials have been considered in physics to discuss many problems. For example; the q-deformed hyperbolic potentials have been proposed firstly by Arai [11, 12] and they have found several applications in various fields of physics and chemistry. They are used for modeling and describing electronic conductance in disordered metals and doped semiconductors [13], phonon spectrum in $^4$He [14], oscillatory-rotational spectra of diatomic [15] and multi-atomic molecules [16]. Furthermore, q-deformation of the Morse potential has been investigated by [17–19]. Some recent works are the investigation and comparison of the energy spectrum of q-oscillator and Morse-like anharmonic potential in Ref. [20] and derivation of the exact normalized wave functions for the q-deformed screened Coulomb Hulthen potential in Ref. [21] etc.

The aim of this work is to relate complexity of a quantum system with its energy level $n$ and potential deformation parameter $q$. Therefore, in this letter we consider two simple q-deformed potentials and analyze complexity of the system depending on these potentials. The outline of article is like the following: a definition and meaning of López-Ruiz, Mancini and Calbet complexity measure is presented in Sec. 2. Afterwards in Sec. 3, $C_{LMC}$ is applied to the probability distribution of a single particle under q-deformed harmonic oscillator and q-deformed Morse potentials. As a summary, significant results and remarks are presented in Sec. 4.

II. COMPLEXITY MEASURE OF LÓPEZ-RUIZ, MANCINI AND CALBET

The LMC complexity measure is defined for continuous systems as the following [7],

$$C_{LMC} = e^{S_x} D_x$$

(1)

where $S_x$ and $D_x$ are Shannon information entropy and disequilibrium, respectively. These are defined in one dimension as

$$S_x = -\int_{-\infty}^{\infty} \rho(x) \ln \rho(x) \, dx$$

(2)

$$D_x = \int_{-\infty}^{\infty} \rho^2(x) \, dx$$

(3)

where $\rho(x)$ is the probability density given in terms of the wavefunction as $\rho(x) = |\psi(x)|^2$. Above expressions are obtained by taking the continuous limit of the following expressions which are expressed in terms of a discrete
probability distribution.

\[ S_x = -k_B \sum_{i=1}^{N} p_i(x) \ln p_i(x) \]  
\[ D_x = \sum_{i=1}^{N} \left( p_i - \frac{1}{N} \right)^2 \]

where \( p_i \) is the probability of occupying the state \( i \), and \( N \) is the total number of accessible states in position space. For a reminder, this definition of complexity is valid only for position space. However, a complexity measure for momentum space can be also obtained by applying a Fourier transformation to probability distribution expressed in the position space.

On the other hand, according to the LMC definition one can deduce that complexity is like predictability \( P \). LMC can be used to predict whether a system is fully ordered or not by using \( P = 1 - C_{LMC} \). For instance, at the statistical limit case \( C_{LMC} \) becomes zero for both totally ordered (perfect crystal) and totally disordered (random gas) systems. Ideal gas is totally random and one can predict its randomness. On the other hand, perfect crystal is also totally predictable, because it can be constructed by a unit cell and a symmetry operation transformation.

It is generally assumed that \( C_{LMC} \) in Eq. 1 is a well known statistical measure of complexity for ergodic systems. On the other hand, we show in this study that \( C_{LMC} \) is also a novel candidate measure to identify complexity for q-deformed quantum systems. Therefore, we discuss the complexity in the whole paper caused from q-deformed harmonic and Morse potentials by using Eqs. 13.

III. COMPLEXITY STUDY OF Q-DEFORMED POTENTIALS

A. q-Deformed harmonic oscillator

Now, we consider, as a first example, q-deformed quantum harmonic oscillator. The q-oscillator is described by the Hamiltonian

\[ H = \frac{\hbar \omega}{2} (a - a^+ + a^+ a^-) \]  

where \( \omega \) is the oscillator frequency. In the complexity study of q-oscillator \( \hbar \) is set to 1. \( a^+ \) and \( a^- \) are creation and annihilation operators satisfying the commutation relation

\[ [a^-, a^+]_q = a^- a^+ - qa^+ a^- = 1, \]

with the deformation parameter \( q \) taking values in the interval (0, 1). The effect of annihilation and creation operators to the states are given as below,

\[ a^- |n\rangle = \sqrt{|n|} |n-1\rangle, \quad a^+ |n\rangle = \sqrt{|n+1|} |n+1\rangle, \]

where \( |n\rangle = \frac{1-q^n}{1-q} \) which satisfies the limits, \( \lim_{q \to 1} |n\rangle = n \) and \( \lim_{q \to 0} |n\rangle = 1 \).

The wave function of q-oscillator is more complex than that of simple quantum oscillator. However, if the calculation steps in Ref. 22 are repeated, the normalized wave functions can be obtained as

\[ |n\rangle = \Psi_n(x) = \frac{\exp(-x^2/2 + \frac{3}{2}i\alpha x)}{\pi^{1/4}i^n(1 - \exp(-2\alpha^2))^{7/4} \sqrt{|n\rangle q!}} \sum_{k=0}^{n} (-1)^k |n\rangle_q! \exp(2i\alpha(n-k)x - k\alpha^2) \]

where \( \alpha = i\sqrt{(\log q)/2} \). In the above formula \( |n\rangle_q! \) is the q-factorial and defined as

\[ |n\rangle_q! = \prod_{k=1}^{n} [k]_q = \frac{1-q^n}{1-q} \frac{1-q^{n-1}}{1-q} \cdots \frac{1-q^2}{1-q} \frac{1-q}{1-q} = (q; q)_n. \]

where \( (q; q)_n \) is the q-Pochhammer symbol. On the other hand, energy eigenvalues for q-oscillator is given by 22.

\[ E_n = \omega \left( |n\rangle_q + \frac{q^n}{2} \right) = \omega \left( |n\rangle_q + \frac{|n+1\rangle_q}{2} \right). \]
We note that, in the limit of $q \to 1$, energy eigenvalues reduce to $E_n = \omega \left( n + \frac{1}{2} \right)$ of the ordinary quantum harmonic oscillator. However, under a small perturbation from unity ($q = 1 - \varepsilon$), the energy spectrum becomes quadratic and energy eigenvalues can be approximated as the following,

$$E_n = \omega \left( n + 1 - \frac{n^2}{2} \varepsilon + O(\varepsilon^2) \right).$$

(12)

For $\varepsilon = 0$, Eq. $12$ reduces to eigenvalue of the harmonic oscillator. We note that the q-oscillator has a nonlinear spectrum and it satisfies the Heisenberg uncertainty as

$$\Delta x \Delta p = \frac{E_n}{\omega} = \left( \frac{2 - (1 + q)q^n}{2(1 - q)} \right) \leq \left( n + \frac{1}{2} \right)$$

(13)

for all states where $q$ changes in the interval of $(0, 1)$. Therefore, uncertainty of q-deformed quantum harmonic oscillator is less than the uncertainty of quantum harmonic oscillator.

**FIG. 1:** $q$ dependence of the probability distribution under q-deformed harmonic oscillator for levels $n = 1, 2, 5, 6$. The arrow indicates increment direction of $q$.

Applying a simple numerical procedure to the related equations, with help of the Eq. 9, probability distribution $\rho(x)$ of q-oscillator for different energy levels can be computed. Here we present the results for $n = 1, 2, 5, 6$ to quantify discussing. Related numerical results for these excited states for different q values are given in Figs. 1a-d. As it can be seen from figures, q-oscillator show very interesting behavior depend on both of exited states and q-parameter. The shape of the probability distribution dependence upon $n$ is generally known which corresponds to solution of the quantum harmonic oscillator i.e, q=1. Here we note that the number of peaks in the probability distribution
increases with \( n \). In the case of \( q = 1 \), the most probable value of position for the lower states is very different from the classical harmonic oscillator where it spends more time near the boundaries of its motion. But as the quantum number increases, the probability distribution becomes more like that of the classical oscillator - this tendency to approach the classical behavior for high quantum numbers is called the correspondence principle. This is the first important point to understand of the solution of the quantum harmonic oscillator. We will remind this point in below discussing. On the other hand, in the numerical study, we found that the \( q \)-dependence of \( q \)-oscillator is very remarkable. For \( q = 1 \) the solution gives the standard \( \rho(x) \) of the quantum harmonic oscillator. This is a known case as mentioned above. However, while \( q \) parameter decreases from 1 to 0 which indicates that deformation increases and \( \rho(x) \) becomes more localized in both figures. This behavior is the second important point.

If we turn back to figures, we can see from Figs. 1a and 1b that the shape of the probability function has similar characteristic form for some \( q \) values. However, for very small \( q \) values this behavior of the \( \rho(x) \) dramatically changes. For instance; at \( q = 0.001 \), probability surprisingly becomes localized at the \( x = 0 \). We say that \( q \) dominates the oscillator from quantum mechanical behavior to the classical Gaussian distribution one. For \( q = 1 \), there is no memory effects in the system which behaves completely quantum mechanical. However, when \( q \) decreases, indicating that memory effects in the system emerges. We can note that source of the memory effect is the correlations between peaks of the wave functions which might be arisen from the broken symmetry of the harmonic oscillator potential with increasing \( q \) deformation.

It is possible to see similar behavior in the case of higher excited states for example \( n = 5 \) and 6. The oscillating form of the \( \rho(x) \) clearly changes depending on \( q \) values. When \( q \) decreases from 1 to 0.1, the form gradually becomes more narrow and the oscillating behavior of the \( \rho(x) \) between two edges decreases and vanishes so that oscillating form evolves into a straight oscillating plateau. In the case of \( n = 1, 2, 5, 6 \), for a small \( q = 0.001 \) value, as it can be seen from Fig. 1-d localization behavior in the the probability appears i.e., probability distribution becomes more Gaussian. We note that \( q \) dependence behavior of the \( \rho(x) \) is a very interesting result which probably emerges from deformation of the external potential. These results can be concluded that when \( q \) approaches zero, probability of the corresponding wave function of the quantum \( q \)-oscillator becomes localized at the origin which might be a result of memorial or non-Markovian effects.

So far, we have discussed the probability distribution and eigenenergies of \( q \)-oscillator. Now we can analyze disequilibrium, information entropy and complexity properties of \( q \)-oscillator. Based on Eqs. 1-3, these quantities can be numerically obtained for ground and several exited states depending on \( q \). All numerical results are given in Fig. 2. As it can be seen from Fig. 2a, that the ground state information entropy is constant and independent from \( q \). However, it has different shapes for all excited states. For example the information entropy increases and smoothly reach up to a saturation for small \( n \) values while \( q \) increase from 0 to 1. However, this behavior rapidly changes for higher \( n \) values. Information entropy increases while \( q \) increases and after a critical saturation it decreases again between \( q = 0.8 - 1.0 \). This behavior appears depending on \( n \) values. Therefore, we see a smooth transition from quantum to classical behavior in information entropy. This is a novel result. This important result shows that \( q \) plays an important role as a memory or non-Markovian effects on the system, which drives the system from quantum to classical one.

In Fig. 2b, \( q \) dependence of the disequilibrium is given for different excited states. As it can be seen from Fig. 2b, disequilibrium of ground state is independent from \( q \) and disequilibrium has a large value. However, disequilibrium of \( q \)-oscillator decreases with increasing \( q \) and \( n \) values, which means that the system passes from quantum to classical behavior with increasing \( q \) and \( n \) values. On the other hand, after attaining a certain minimum value in the region \( q = 0.7 - 0.9 \), the disequilibrium is found to increase as \( q \) decreases below 0.8. For a given value of \( q \), the disequilibrium is found to decrease with the increasing quantum number of the excited state. The amount of decrements increases near the minima of disequilibrium versus \( q \) plot. According to Fig. 2b, with reference to the ground state, for a given excited state, complexity of the \( q \)-oscillator has a very interesting and very complicated behavior. For all excited states complexity decrease depending on the increment in \( q \), however, it starts to increase from above a critical minimum \( q \) value. This complication of the complexity is caused by the conflict between disequilibrium and information entropy. We can conclude that system has a critical transition in complexity. Around the value of \( q \) in the neighbourhood of 0.9 where information entropy exhibits a maximum and the disequilibrium exhibits a minimum, the statistical complexity attains an approximately similar value for all excited states. Below this bunching point of statistical complexity for the excited states, the \( C_{LMC} \) vs \( q \) curves are found to cross over with the inversion of relative ordering of the quantum number \( n \) of the excited states. Beyond the bunching region, as \( q \) decreases, for each excited state the statistical complexity goes through a minimum at a certain \( q \). The depth of the minimum value is found to decrease with decreasing quantum number of the excited state. With further decrease in \( q \) beyond the minimum value statistical complexity, the \( C_{LMC} \) values are found to increase, finally, culminating at a common value at the lowest \( q \).

Another interesting result that appears in \( q \)-oscillator is the following, for low \( n \) values, entropy takes low values, however, disequilibrium is also large. Then complexity also takes large values for low \( n \) values.
Finally, in order to comprehend the numerical changes, Shannon information entropy, disequilibrium and complexity values obtained by using Eqs. [13] are given in Table I.

![Graphs showing q dependence of Shannon information entropy, disequilibrium, and complexity for q-deformed harmonic oscillator](image)

**Fig. 2:** q dependence of the Shannon information entropy (a), disequilibrium (b) and complexity (c) for q-deformed harmonic oscillator for energy levels changing in [0,10] where n=7,9 have been omitted for preventing messiness.

| n   | q    | S    | D    | C = e^qD |
|-----|------|------|------|---------|
| 0   | 0.001| 1.07236 | 0.39894 | 1.16582 |
| 0.4 | 1.07236 | 0.39894 | 1.16582 |
| 1   | 1.07236 | 0.39894 | 1.16582 |
| 5   | 0.001| 1.07829 | 0.39232 | 1.15329 |
| 0.4 | 1.59322 | 0.21132 | 1.03962 |
| 1   | 1.76806 | 0.19666 | 1.15235 |
| 10  | 0.001| 1.07829 | 0.39232 | 1.15329 |
| 0.4 | 1.61096 | 0.20561 | 1.02962 |
| 1   | 2.01018 | 0.15668 | 1.16957 |

**Table I**: Energy level and q-deformation dependence of Shannon information entropy, disequilibrium and complexity for q-deformed harmonic oscillator.

B. q-Deformed Morse potential

After quantifying the complexity of q-oscillator, here we consider, as an second example, q-deformed Morse potential [17–20]. The general form of the q-deformed Morse potential is expressed in the Ref. [23] as the following,

\[ V_{MP}(x) = V_1 e^{-2\alpha x} - V_2 e^{-\alpha x}, \quad V_1 = D_e, \quad V_2 = 2qD_e \]  \(14\)

where \(\alpha = ar_e, \quad x = (r - r_e)/r_e, \quad V_1 \) and \(V_2 \) are the repulsive and attractive terms, respectively. \(r_e \) is the equilibrium position of the nuclei, \(\alpha \) is a constant related with the range of potential and \(D_e \) is a measure of the depth of potential well at equilibrium distance. This form of the q-deformed Morse potential is the special case of the well-known diatomic Morse potential.

Wave function of a particle for q-deformed Morse potential can be obtained after several steps as in Ref. [18]

\[ \Psi_n(x) = N_n \exp \left\{ -\alpha \left( \lambda q - n - \frac{1}{2} \right) x - \lambda e^{-\alpha x} \right\} L_n^{2\left(\lambda q - n - \frac{1}{2}\right)}(2\lambda e^{-\alpha x}) \]  \(15\)

where \(N_n \) is the normalizing factor. \(L_n^{\alpha}(x) \) are called generalized Laguerre polynomials or associated Laguerre polynomials. In this work, we numerically computed the normalization constant \(N_n \) by using Mathematica. On the other hand, the energy eigenvalues of q-Morse potential are given as

\[ E_n = -\alpha^2 E_0 \left( \lambda q - n - \frac{1}{2} \right)^2, \quad n = 0, 1, 2, \ldots, n_{\text{max}} \]  \(16\)
where $\lambda$ and $E_0$ are given as the following form,

$$\lambda = \left( \frac{D_e}{\alpha^2 E_0} \right)^{1/2}, \quad E_0 = \frac{\hbar^2}{2\mu r_e^2}$$

(17)

here $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the diatomic molecule. As it can be seen from Eq. 16, the form of the energy is clearly different from that of the classical Morse potential one. For example, energy levels corresponding to q-deformed Morse potential are upper bounded by $q$ and maximum level number is restricted by the inequality $n_{max} \leq (\lambda q - 1/2)$.

Here, we also investigate $q$ dependence of the complexity, disequilibrium and information entropy for q-Morse potential. However, in the present case, we compute and discuss these quantities by using well known experimental parameters of HCl and H$_2$ molecules, which are given in Table II. Here we note that, in numerical procedure, we defined $q = 0.35$ as a lower bound. Under this lower bound of $q$, numerical error appears in computation. Therefore, we study $q$ dependence between 0.35 and 1.0. However, on the other hand, the lower bound of the $q$ leads to an upper bound which covers the number of possible excited states. For example, by using these values, in our work for $q = 0.35$, $n_{max}$ value is found as 8 and 5 for HCl and H$_2$ molecules, respectively.

TABLE II: Molecular specific parameters for HCl and H$_2$.

| Parameter   | HCl [24] | H$_2$ [25] | Unit       |
|-------------|----------|------------|------------|
| $a_1$       |   1.868  |   1.944    | Å$^{-1}$   |
| $r_e$       |   1.275  |   0.742    | Å          |
| $D_e$       |  37255   |  38266     | cm$^{-1}$  |
| $\mu$       |   0.980  |   0.504    | amu        |

The spatial probability distribution change with the q-deformation parameter is presented for HCl and H$_2$ molecules in Figs. 3 and 4, respectively. As it can be seen from figures that probability distribution of Morse potential has an asymmetric shape. For $q = 1$ peaks take large values however, when the $q$ parameter decreases $\rho(x)$ becomes much more spreading over position space. This change can be more easily seen in excited states such as shown in Figs. 3b and 4b for state $n = 4$.

FIG. 3: $q$ dependence of the probability distribution under q-deformed Morse potential for levels $n = 1$ and $n = 4$ of HCl molecule. The arrow indicates increment direction of $q$.

Here we compute Shanon information entropy, disequilibrium and complexity by using solutions of q-deformed Morse potential for HCl and H$_2$ diatomic molecules. By using parameters in Table II, we compute all quantities. The numerical results are given in Figs. 5 and 6 for several excited states for HCl and H$_2$ molecules, respectively. As it can be seen from figures that there is no any complicated behavior in entropy, disequilibrium and complexity as in q-harmonic oscillator. For both molecules, entropy $S$ and complexity $C$ smoothly decreases with $q$, however, disequilibrium $D$ smoothly increases with $q$. 
FIG. 4: $q$ dependence of the probability distribution in q-deformed harmonic oscillator for levels $n = 1$ and $n = 4$ of H$_2$ molecule. The arrow indicates increment direction of $q$.

FIG. 5: Shannon information entropy (a), disequilibrium (b) and complexity (c) for HCl molecule at energy levels $n$ in $[1,7]$.

FIG. 6: Shannon information entropy (a), disequilibrium (b) and complexity (c) for H$_2$ molecule at energy levels $n$ in $[1,5]$.

Now we can compare some result of the q-deformed Morse potential with the q-oscillator. When comparing results we can see that for low $n$ values entropy takes low values, however disequilibrium is also large. These results are compatible with the results of q-oscillator. However, the complexity of q-Morse takes low values for low $n$ values unlike q-oscillator.
When q-deformation applied to a quantum harmonic oscillator, it is found that probability distribution becomes more classical rather than quantum mechanical as q approaches to zero. In q-deformed harmonic oscillator, as q decreases ρ(x) becomes more localized. q deformation enforces q-oscillator to behave like a classical oscillator without spreading the probability distribution but preserving the discreteness of energy. In other words, in the limit case in which deformation increases, q approaches to 0, probability distribution of a single particle collapses in to a Gaussian form. q-deformation phenomena reduces the Heisenberg uncertainty ΔxΔp product for the sake of gradual lose of quantumness. Entropy, disequilibrium and complexity values are independent of q for the ground state of the q-oscillator. However, q-deformation causes a q dependent change in entropy, disequilibrium and complexity calculated for excited states. q-deformation affects the oscillator differently depending on its energy level n. For instance, minima of the complexity shift to higher q values as the quantum number n increases. q-oscillator takes the same complexity value, which is calculated for position space, independent of its energy when q is set to 0.9.

Complexity measure of probability distribution of an electron belonging to a q-deformed diatomic Morse potential decreases with increasing q. Meanwhile, when q is increased, entropy decreases and disequilibrium increases. When q-dependence of the probability distributions of position space for q-harmonic and q-Morse potentials are compared it is seen that q-deformation (as q approaches zero) causes localization in q-oscillator by squeezing ρ(x), whereas it causes a spreading of ρ(x) under q-Morse potential.

Complexity measure of probability distribution of an electron belonging to a q-deformed diatomic Morse potential decreases with increasing q. Meanwhile, when q is increased, entropy decreases and disequilibrium increases. When q-dependence of the probability distributions of position space for q-harmonic and q-Morse potentials are compared it is seen that q-deformation (as q approaches zero) causes localization in q-oscillator by squeezing ρ(x), whereas it causes a spreading of ρ(x) under q-Morse potential.

V. ACKNOWLEDGMENTS

KDS is grateful to The Scientific and Technological Research Council of Turkey (TÜBİTAK) for a visiting scientist award under its 2221-program, grant number 1059B211601794 and acknowledges with thanks the support received under the Emeritus Scientist scheme , C.S.I.R. New Delhi.

[1] S. Lloyd, IEEE Control Systems Magazine 21, 7 (2001).
[2] G. J. Chaitin, Journal of the ACM 13, 547 (1966).
[3] D. W. Miller, Selected Translations in Mathematical Statistics and Probability, Vol. 7 (JSTOR, 1969).
[4] G. Y. Georgiev, K. Henry, T. Bates, E. Gombos, A. Casey, M. Daly, A. Vinod, and H. Lee, Complexity 21, 18 (2015).
[5] J. P. Crutchfield and K. Young, Physical Review Letters 63, 105 (1989).
[6] R. López-Ruiz, M. C. S. Rodrigues, and S. Wulck, Phys. Rev. Lett. 76, 1098 (1996).
[7] J. L. Cooper and R. K. Gupta, International Journal of Modern Physics E 7, 553 (1998).
[8] D. Bonatsos, C. Daskaloyannis, and P. Kolokotronis, The Journal of chemical physics 106, 605 (1997).
[9] K. Sen, ed., Statistical Complexity: Applications in Electronic Structure (Springer, New York, 2012).
[10] J. Sañudo and R. López-Ruiz, Physics Letters, Section A: General, Atomic and Solid State Physics 375, 1674 (2011).
[11] I. Bialynicki-Birula and J. Mycielski, Communications in Mathematical Physics 44, 129 (1975).
[12] A. Arai, Journal of Mathematical Analysis and Applications 158, 63 (1991).
[13] A. Arai, Journal of Physics A: Mathematical and General 34, 4281 (2001).
[14] M. R-Monteiro, L. M. C. S. Rodrigues, and S. Walck, Phys. Rev. Lett. 1098 (1996).
[15] R. S. Johal and R. K. Gupta, International Journal of Modern Physics E 7, 553 (1998).
[16] D. Bonatsos, C. Daskaloyannis, and P. Kolokotronis, The Journal of chemical physics 106, 605 (1997).
[17] I. L. Cooper and R. K. Gupta, Physical Review A 52, 941 (1995).
[18] S. Ikhdair, Chemical Physics Letters 361, 9 (2000).
[19] A. Dobrogowska, Applied Mathematics Letters 26, 769 (2013).
[20] N. G. Vinh, M. Van Ngu, N. T. Lan, L. T. K. Thanh, and N. A. Viet, Journal of Physics: Conference Series 627, 012022 (2015).
[21] N. S. R. Hall and K. Sen, J. Math. Phys. (Submitted).
[22] V. V. Eremin and A. A. Meldianov, Theoretical and Mathematical Physics 147, 709 (2006).
[23] M. Znojil, Physics Letters A 264, 108 (1999).
[24] W.-C. Qiang and S.-H. Dong, Physics Letters A 363, 169 (2007).
[25] C. Flavio, C. Enrique, M. Pablo, and C.-V. Luis, Journal of Physics: Conference Series 574, 012105 (2015).