ON THE PUSH-OUT SPACES

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Abstract. Let \( f : M^m \rightarrow \mathbb{R}^{m+k} \) be an immersion where \( M \) is a smooth connected \( m \)-dimensional manifold without boundary. Then we construct a subspace \( \Omega(f) \) of \( \mathbb{R}^k \), namely push-out space, which corresponds to a set of embedded manifolds which are either parallel to \( f \), tubes around \( f \) or, in general, partial tubes around \( f \). This space is invariant under the action of the normal holonomy group, \( \text{Hol}(f) \). Moreover, we construct geometrically some examples for normal holonomy group and push-out space in \( \mathbb{R}^3 \). These examples will show that properties of push-out space that are proved in the case \( \text{Hol}(f) \) is trivial, is not true in general.

1. Introduction

In this paper we introduce push-out space for an immersion \( f : M^m \rightarrow \mathbb{R}^{m+k} \), where \( M \) is a smooth connected \( m \)-dimensional manifold without boundary. To do this, we give some examples in 3-dimensional Euclidean space, \( \mathbb{R}^3 \), in fact, in these examples we calculate normal holonomy group and push-out space geometrically. We consider the case when \( \text{Hol}(f) \) is non-trivial. This extends the work of Carter and Senturk [2], who obtained results about the case when \( \text{Hol}(f) \) is trivial. In these examples we show that some of the properties of push-out space which they obtained is not true for the case when \( \text{Hol}(f) \) is non-trivial.

2. Basic definitions

Definition 2.1 [2]. Let \( f : M^m \rightarrow \mathbb{R}^{m+k} \) be a smooth immersion where \( M \) is a smooth connected \( m \)-dimensional manifold without boundary. The total space of the normal bundle of \( f \) is defined by
\[
N(f) = \{(p,x) \in M \times \mathbb{R}^{m+k} : <x,v> = 0 \quad \forall v \in f^*T_p(M)\}
\]
The endpoint map \( \eta : N(f) \rightarrow \mathbb{R}^{m+k} \) is defined by \( \eta(p,x) = f(p) + x \) and, the set of singular points of \( \eta \) is subset \( \Sigma(f) \subset N(f) \) called the set of critical normals of \( f \) and the set of focal points of \( \eta \) is a subset \( \eta(\Sigma(f)) \subset \mathbb{R}^{m+k} \).

For \( p \in M \), we put \( N_p(f) = \{x : (p,x) \in N(f)\} \) and \( \Sigma_p(f) = \{x : (p,x) \in \Sigma(f)\} \) respectively, normal space at \( p \) and the set can be thought of as focal points with base \( p \).

Definition 2.2 [1]. For \( p_0 \in M \) and \( p \in M \) and path \( \gamma : [0,1] \rightarrow M \) from \( p_0 \) to \( p \) define \( \varphi_{p,\gamma} : N_{p_0}(f) \rightarrow N_p(f) \) by parallel transport along \( \gamma \). The \( \varphi_{p,\gamma} \)'s are isometries. The normal holonomy group on \( N_{p_0}(f) \), is
\[
\text{Hol}(f) = \{\varphi_{p_0,\gamma} : \gamma : [0,1] \rightarrow M, \quad \gamma(0) = \gamma(1) = p_0\}
\]

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If the closed path $\gamma$ at $p_0$ is homotopically trivial then $\varphi_{p_0,\gamma}$ is an element of the restricted normal holonomy group $\mathcal{Hol}_0(f)$.

**Definition 2.3 ([3]).** For a fix $p_0 \in M$ the push-out space for an immersion $f : M^m \rightarrow \mathbb{R}^{m+k}$ is defined by

$$\Omega(f) = \{ x \in N_{p_0}(f) : \forall p \in M, \forall \gamma \text{ s.t. } \gamma(0) = p_0, \gamma(1) = p \text{ then } \varphi_{p,\gamma}x \notin \Sigma_p(f) \}$$

(i.e. $\forall p \in M$, $f(p) + \varphi_{p,\gamma}(x)$ is not a focal point with base $p$ when $x$ belongs to $\Omega(f)$). Therefore $\Omega(f)$ is the set of normals at $p_0$, where transported parallely along all curves, do not meet focal points. So $\Omega(f)$ is invariant under the action of $\mathcal{Hol}(f)$.

**Definition 2.4 ([4]).** Let $B \subset N(f)$ be a smooth subbundle with type fiber $S$

1) $S$ is a smooth submanifold of $\mathbb{R}^k$

2) $B \cap \Sigma(f) = \emptyset$

3) $B$ is invariant under parallel transport (along any curve in $M$). Then $B$ is a smooth manifold and $g \equiv \eta|_B : B \rightarrow \mathbb{R}^{m+k}$ is a smooth immersion called a partial tube about $f$.

**Theorem 2.5 ([2]).** Let $\mathcal{Hol}(f)$ is trivial and $M$ be a compact manifold. Then each path-connected component of $\Omega(f)$ is open in $\mathbb{R}^k$.

**Theorem 2.6 ([2]).** Let $\mathcal{Hol}(f)$ is trivial then Each path-connected component of $\Omega(f)$ is convex.

**Remark 2.7.** In Example 3.2 if $\frac{\pi}{2}$ is irrational then $\Omega(\bar{f})$ is not open in $\mathbb{R}^2$ but $M = S^1$ is compact. Also, in Example 3.5 $\Omega(f) = \{O\}$ hence $\Omega(f)$ is closed in $\mathbb{R}^2$ but $\mathcal{Hol}(f)$ is trivial. This shows that Theorem 2.5 is false when $M$ is not compact or $\mathcal{Hol}(f)$ is non-trivial.

**Remark 2.8.** In Example 3.6 one of path-connected components of $\Omega(f)$, which is the complement space of cone and two other components in $\mathbb{R}^3$, is not convex. This shows that Theorem 2.6 is false when $\mathcal{Hol}(f)$ is non-trivial.

we conclude that the properties of push-out space that are proved in the case $\mathcal{Hol}(f)$ is trivial, is not true in general.

3. Examples of Normal Holonomy Groups and Push-Out Spaces

**Example 3.1.** We start with a curve as below

![Diagram](image)

suppose this curve is given by $s \mapsto (\xi(s),\eta(s))$ where $s \in [0, 1]$ and at $(1, 0, 0): s = 0$ $\frac{\partial \xi}{\partial s} = 1, \frac{\partial \eta}{\partial s} = 0$ for all $r \geq 0$ and at $(0, 1, 0): s = 1, \frac{\partial \eta}{\partial s} = 1, \frac{\partial ^2 \xi}{\partial s^2} = 0$ for all $r \geq 0$. Now, we take this curve in $\mathbb{R}^3$ and consider the same curves in $yz$-plane and
xz-plane and fit together to make a smooth closed curve in \( \mathbb{R}^3 \). Now by identifying \( S^1 \) with \( \mathbb{R}_/\mathbb{Z} \), the curve in \( \mathbb{R}^3 \) can be redefined as \( f : S^1 \to \mathbb{R}^3 \) where:

\[
f(s) = \begin{cases} 
(\xi(s), \eta(s), 0) & 0 \leq s \leq 1 \\
(0, \xi(s-1), \eta(s-1)) & 1 \leq s \leq 2 \\
(\eta(s-2), 0, \xi(s-2)) & 2 \leq s \leq 3
\end{cases}
\]

To find the normal holonomy group of the above curve, we will consider normal vector to the curve under parallel transport. As each part of the curve lies in a 2-plane, the normal plane at a point of the curve is spanned by the perpendicular direction to the 2-planes.

**Step 1.** Start with the normal vector at \((1,0,0)\), in the diagram, it stays in the xy-plane under parallel transport.

The normal vector \((0,1,0)\) at \((1,0,0)\) goes to normal vector \((-1,0,0)\) at \((0,1,0)\).

**Step 2.** At \((0,1,0)\) the normal vector \((-1,0,0)\) is perpendicular to the yz-plane, it stays perpendicular to the yz-plane under parallel transport form \((0,1,0)\) to \((0,0,1)\).
The normal vector (-1,0,0) at (0,1,0) goes to normal vector (-1,0,0) at (0,0,1).

**Step 3.** The normal vector (-1,0,0) is in the xz-plane at (0,0,1) and stays in the xz-plane from (0,0,1) to (1,0,0).

The normal vector (-1,0,0) at (1,0,0) by going once around the curve the normal vector will turn about $\frac{\pi}{2}$.

Going around of curve again, the normal vector moves through another $\frac{\pi}{2}$ and after four times around the curve back to its original position. This shows that $Hol(f)$ is generated by a rotation through $\frac{\pi}{2}$.

Now we find the push-out space of $f$. Except at end-points of three areas, locally the curve lies in a 2-plane so the focal points with base $s, f(s) + \Sigma_s(f)$, consists of a straight line through the center of curvature, $c(s)$, of the curve at $s$, perpendicular to the line joining $c(s)$ and $f(s)$. 

At end-points of three areas, and possibly some other points, the focal set is empty as the center of curvature "at infinity".

so $\Sigma_s(f)$ is a line in $N_s(f)$. The image of $\Sigma_s(f)$ under normal holonomy group is obtained by rotating it through $\frac{\pi}{2}$ until it returns to the original position.

Now, fix the normal plane $N_{s_0}(f)$ at $f(S_0) = (1, 0, 0)$ where $s_0 = 0$ and use parallel transport to identify all the normal planes with the normal plane $N_{s_0}(f)$. The push-out space is complement of all the $\Sigma_s(f)$ and their images under normal holonomy group.
Therefore the push-out space of $f: S^1 \to \mathbb{R}^3$ is an open square $Q$ with sides of length $2\rho$ where $\rho$ is the minimum absolute value of the radius of curvature of the original curve in the $xy$-plane. (i.e. $\Omega(f)$ is the interior of the smallest square on $N_{s_0}(f)$.)

**Example 3.2.** We consider the immersion $\bar{f}$ as in Example 3.1 except that the $xz$-plane is tilted through an angle $\alpha$.

In other words, $\bar{f} = Lof$ where $f$ is the immersion in example 3.1 and $L$ is the linear transformation given by

$$L = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \tan \alpha \\
0 & 0 & 1
\end{pmatrix}$$

where $0 < \alpha < \frac{\pi}{2}$. So, we have

$$\bar{f}(s) = \begin{cases}
(\xi(s), \eta(s), 0) & 0 \leq s \leq 1 \\
(0, \xi(s-1) + \eta(s-1) \tan \alpha, \eta(s-1)) & 1 \leq s \leq 2 \\
(\eta(s-2), \xi(s-2) \tan \alpha, \xi(s-2)) & 2 \leq s \leq 3
\end{cases}$$

The end-points of three areas of this immersion are $(1,0,0), (0,1,0)$ and $(0,\tan \alpha,1)$. Note that at these points the tangent to the curve is the radial line form $(0,0,0)$ so unit tangent at $(1,0,0)$ is $(1,0,0)$ and unit tangent at $(0,\tan \alpha,1)$ is $(0,\tan \alpha,1)$ etc.

As in Example 3.1 under parallel transport, the normal vector $(0,1,0)$ at $(1,0,0)$ goes to the normal vector $(-1,0,0)$ at $(0,1,0)$, which goes to the normal vector $(-1,0,0)$ at $(0,\tan \alpha,1)$, which goes to the normal vector $(0,\tan \alpha,1)$ at $(1,0,0)$. So going once around the curve the normal vector has moved through $\frac{\pi}{2} - \alpha$.

This shows that $\mathcal{H}ol(\bar{f})$ is generated by a rotation through $\frac{\pi}{2} - \alpha$. As in Example 3.1, the image of $\Sigma_s(\bar{f})$ under normal holonomy group is obtained by rotating the line $\Sigma_s(\bar{f})$ through $\frac{\pi}{2} - \alpha$. It depends on $\alpha$ and is obtained as:

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Start

Once around the curve
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If $R$ is the rotating through an angle $\frac{\pi}{2} - \alpha$ then $\Omega(\bar{f}) = \bigcap\{(R)^nQ : n \in \mathbb{Z}\}$ where $Q$ is a square as in Example 3.1 thus if $\frac{\pi}{2}$ is rational then $(R)^n(\Sigma_s(\bar{f})) = \Sigma_s(\bar{f})$ for some $n \in \mathbb{Z}$ and so, $\Omega(\bar{f})$ is the interior of the smallest polygon. If $\frac{\pi}{2}$ is irrational then $(R)^n(\Sigma_s(\bar{f})) \neq \Sigma_s(\bar{f})$ for any $n \in \mathbb{Z}$ and so, $\Omega(\bar{f})$ is an open disk of radius $\rho$ together with a dense set of points on the boundary circle where $\rho$ is minimum absolute value of the radius of curvature of the immersed curve by $\bar{f}$.

Example 3.3. In Example 3.2, we replace the immersion $\bar{f}$ with the immersion $\bar{f}oh$ where $h : \mathbb{R} \rightarrow S^1 \equiv \mathbb{R}^2/\mathbb{Z}$ is covering projection. Since $\mathbb{R}$ is simply connected, for any arbitrary point $s \in \mathbb{R}$, any closed path at $s$ is nullhomotopic with constant path at $s$, hence definition 2.2 shows that, the normal holonomy group of $\bar{f}oh$ is trivial (i.e. $\text{Hol}(oh) = H_{\text{hol}}(f)$). To calculate $\Omega(foh)$, we prove the theorem 3.4 in general. It will show that $\Omega(\bar{f}oh) = \Omega(\bar{f})$.

**Theorem 3.4.** Let $f : M^m \rightarrow \mathbb{R}^{m+k}$ be an immersion and $\hat{M}$ be any covering space with covering projection $h : \hat{M} \rightarrow M$. If $\hat{f} = foh$, then $\Omega(\hat{f}) = \Omega(f)$.

**Proof.** Let $x \in \Omega(f)$ and fix $p_0 \in M$. Then definition 2.3 implies that, $\forall p \in M, \forall \gamma$ s.t. $\gamma(0) = p_0, \gamma(1) = p; \varphi_{p, \gamma}x \neq \Sigma_p(f)$. we define the total space of the normal bundle of $\hat{f}$ by

$$N(\hat{f}) = \{(\hat{p}, x) \in \hat{M} \times \mathbb{R}^{m+k} : <x, v> = 0 \quad \forall v \in \hat{f}_*T_p(\hat{M})\}$$

Also, for any $\hat{p} \in h^{-1}(p)$ we have

$$\hat{f}_*T_p(\hat{M}) = (foh)_*T_p(\hat{M})$$

$$= (f_*oh_*)T_p(\hat{M})$$

$$= f_*T_p(M)$$

this shows that, for any $\hat{p} \in h^{-1}(p)$ we have $N_{\hat{p}}(\hat{f}) = N_p(f)$ and so $\Sigma_{\hat{p}}(\hat{f}) = \Sigma_p(f)$. Further, we fix $\hat{p} \in h^{-1}(p_0)$ then $\hat{\varphi}_{\hat{p}, \hat{\gamma}} = \varphi_{p, \gamma}$ where $\hat{\gamma} : [0, 1] \rightarrow \hat{M}$ s.t. $\hat{\gamma}(0) = \hat{p}_0, \hat{\gamma}(1) = \hat{p}$. Therefore, $\hat{p}$ is $\hat{f}$ in $\hat{M}$, $\forall \hat{\gamma}; \varphi_{\hat{p}, \hat{\gamma}}x \neq \Sigma_p(f)$. Now using definition 2.3 again, follows that, $x \in \Omega(\hat{f})$. By the same way proves that $\Omega(\hat{f}) \subseteq \Omega(f)$. \hfill $\square$

**Example 3.5.** If $\frac{\pi}{2} - \alpha = \frac{2\pi}{n}$, then Example 3.3 can be modified by replacing $h$ by the n-fold covering $\bar{h} : S^1 \rightarrow S^1$. 
Going once around the first $S^1$ in $\tilde{h} : S^1 \rightarrow S^1$ corresponds to moving $n$ times around the second $S^1$ so parallely transporting a normal $n$ times around the second $S^1$ which gives a rotation of
\[ n\left(\frac{\pi}{2} - \alpha\right) = n\left(\frac{2\pi}{n}\right) = 2\pi \]
i.e. the identity, so $Hol(\tilde{f} \tilde{h}) = Hol_0(\tilde{f} \tilde{h})$. Since, the immersed curve by $\tilde{f} \tilde{h}$ and the immersed curve by $\tilde{f}$ have same figure in $\mathbb{R}^3$ and $Hol(\tilde{f} \tilde{h})$ is trivial so the singular sets of them also the same (i.e. $\Sigma(\tilde{f} \tilde{h}) = \Sigma(\tilde{f})$). This implies that $\Omega(\tilde{f} \tilde{h}) = \Omega(\tilde{f})$.

**Example 3.6.** We consider a sequence of curves $f_n$ in $\mathbb{R}^3$ defined as in Example 3.1 except that $||f_n(s)||$ and the curvature tends to infinity with $n$ when $s = \frac{1}{2}, \frac{3}{2}$ or $\frac{5}{2}$ but is bounded otherwise.

Now, we define the immersion $f : \mathbb{R} \rightarrow \mathbb{R}^3$ by $f(s \pm 3n) = f_n(s)$. When $n$ tends to infinity, the immersion $f : \mathbb{R} \rightarrow \mathbb{R}^3$ has a sequence of points where the curvature tends to infinity and the radius of curvature at these points can be arbitrary small; in other words, $\exists s$ where $\Sigma_s(f)$ is arbitrary close to "O" in $N_s(f)$. So $\{O\}$ is the only point not in the image of $\Sigma_s(f)$ under normal holonomy group for all $s \in \mathbb{R}$. Then $\Omega(f) = \{O\}$. In this case because $\mathbb{R}$ is simply connected then $Hol(f)$ is trivial.

The following results have been proved in [2], when $Hol(f)$ is trivial.

**Theorem 3.7.** Let $M$ be a compact manifold, then each path-connected component of $\Omega(f)$ is open in $\mathbb{R}^k$.

**Theorem 3.8.** Each path-connected component of $\Omega(f)$ is convex.

**Remark 3.9.** In Example 3.2 if $\alpha$ is irrational then $\Omega(\tilde{f})$ is not open in $\mathbb{R}^2$ but $M = S^1$ is compact. Also, in Example 3.5 $\Omega(f) = \{O\}$ so $\Omega(f)$ is closed in $\mathbb{R}^2$ but $Hol(f)$ is trivial. This shows that Theorem 3.7 is false when $M$ is not compact or $Hol(f)$ is non-trivial.
Remark 3.10. In Example 3.6 one of the path-connected components of $\Omega(f)$, which is the complement space of cone and two other components in $\mathbb{R}^3$, is not convex. This shows that Theorem 3.8 is false when $\text{Hol}(f)$ is non-trivial.

Thus the properties of push-out space that are proved in the case $\text{Hol}(f)$ is trivial, is not true in general.

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