On outer-connected domination for graph products

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Abstract

An outer-connected dominating set for an arbitrary graph $G$ is a set $\tilde{D} \subseteq V$ such that $\tilde{D}$ is a dominating set and the induced subgraph $G[V \setminus \tilde{D}]$ be connected. In this paper, we focus on the outer-connected domination number of the product of graphs. We investigate the existence of outer-connected dominating set in lexicographic product and Corona of two arbitrary graphs, and we present upper bounds for outer-connected domination number in lexicographic and Cartesian product of graphs. Also, we establish an equivalent form of the Vizing’s conjecture for outer-connected domination number in lexicographic and Cartesian product as $\tilde{\gamma}_c(G \circ K) \tilde{\gamma}_c(H \circ K) \leq \tilde{\gamma}_c(G \square H) \circ K$. Furthermore, we study the outer-connected domination number of the direct product of finitely many complete graphs.

Keywords: Outer-connected domination; Cartesian product; Lexicographic product; Corona product; Direct product; Vizing’s conjecture.

1 Introduction & preliminary

Domination and its variations in graphs are a well studied topic in the literature, e.g. \cite{6} gives a survey on the topic. The concept of outer-connected domination number, as a variant of graph domination problem, is introduced by Cyman \cite{2} and is further studied by others in \cite{15}. The outer-connected domination problem is $NP$-complete for arbitrary graphs \cite{2}. A set $\tilde{D} \subseteq V$ of a graph $G = (V, E)$ is called an outer-connected dominating set for $G$ if (1) $\tilde{D}$ is a dominating set for $G$, and (2) $G[V \setminus \tilde{D}]$, the induced subgraph of $G$ by $V \setminus \tilde{D}$, is connected. The minimum size among all outer-connected dominating sets of $G$ is called the outer-connected domination number of $G$ and is denoted by $\tilde{\gamma}_c(G)$ \cite{2}.

The problem of finding a minimum sized outer connected dominating set has applications in computer networks. For example consider a client-server architecture based network in which any client must be able to communicate to one of the servers. Since overload of the servers is a bottleneck in such a network, every client must be able to communicate to another client directly without interrupting any server. The smallest group of servers with these properties is a minimum outer-connected dominating set for the graph representing the computer network \cite{13}.

We make use the following result related to the outer-connected domination number in this paper.

\textbf{Theorem 1.1.} \cite{2} If $G$ is a connected graph of order $n$, then

$$\tilde{\gamma}_c(G) \leq n - \delta(G).$$

Graphs are basic combinatorial structures and products of structures are a fundamental construction in graph theory. Such construction is a challenging problem and has many applications. In graph theory there are three fundamental graph products, namely the Cartesian product, the direct product, and the strong product, each with its own set of applications and theoretical interpretations. Computer science is one of the many fields in which graph products are becoming commonplace. As one specific example, one can mention the load balancing problem for massively parallel computer architectures \cite{4}. In addition to large networks
such as the graph of the which has Internet several hundred million hosts, can be efficiently modeled by subgraphs of powers of small graphs with respect to the direct product. This is one of the many examples of the dichotomy between the structure of products and that of their subgraphs \[4\]. The classification also leads to a other two products worthy of special attention, the lexicographic and the Corona products \[4\].

The study of domination number in product graphs has a long history. Back in 1963, Vizing \[16\] posed a conjecture, which is main open problem in graph domination, concerning the domination number of the Cartesian product graphs

\[ \gamma(G) \leq \gamma(G \Box H) \]

For a survey of domination in Cartesian products, an interested reader can consult \[5\] for more information. Gravier and Khelladi \[3\] posed an analogous conjecture for direct product graphs, namely

\[ \gamma(G) \gamma(H) \leq \gamma(G \times H) \]

Domination number of direct products of certain graphs has exact values, for instance, the products of two paths, the product of a path and a complement of a path, the product of K2 and a tree, bipartite graph and an odd cycle \[3, 4, 10\]. In 2010, Gasper Mekis \[11\] gave a lower bound for the domination number of a direct product and proved that this bound is sharp. Also, he studied the domination number of the direct product of finitely many complete graphs.

For the lexicographic product graphs, various types of domination were investigated in the literature, including domination \[12, 14\], total domination \[18\], rainbow domination \[15\], Roman domination \[14\], and restrained domination \[18\]. Other various types of dominating sets for products of graphs were intensively investigated in \[2, 17\]. However, outer-connected dominating sets for products of graphs has not been investigated. So, we investigate the topic in this paper by studying the outer-connected dominating sets and related notions in the lexicographic product, the direct product, the Cartesian product and the Corona product graphs.

For notation and graph theory terminology, we in general follow \[4\]. Let \( G = (V, E) \) be a graph with vertex set \( V \) and edge set \( E \) of order \(|V|\) denoted by \( n \) and size \(|E|\) denoted by \( m \). We also use \( V(G) \) and \( E(G) \) to denote the vertex set and edge set for a graph \( G \). Let \( v \) be a vertex in \( V \). The open neighborhood of \( v \) is denoted by \( N_G(v) \) and is defined as \( \{u \in V : \{u, v\} \in E(G)\} \). Similarly, the closed neighborhood of \( v \) is denoted by \( N_G[v] \) and is defined as \( \{v\} \cup N_G(v) \). Whenever the graph \( G \) is clear from the context, we simply write \( N(v) \) to denote \( N_G(v) \). For a set \( S \subseteq V \), its open neighborhood is the set \( N(S) = \bigcup_{v \in S} N(v) \) and its closed neighborhood is the set \( N[S] = N(S) \cup S \). A subset \( S \subseteq V \) is a dominating set of \( G \) if every vertex not in \( S \) is adjacent to a vertex in \( S \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality among all dominating sets of \( G \). A dominating set \( S \) is called a \( \gamma \)-set of \( G \) if \( |S| = \gamma(G) \). A dominating set \( S \) in a graph with no isolated vertex is called a total dominating set if the induced subgraph \( G[S] \) has no isolated vertex. The total domination number of \( G \), denoted by \( \gamma_t(G) \), is the minimum cardinality among all total dominating sets of \( G \). A total dominating set \( S \) is called a \( \gamma_t \)-set of \( G \) if \( |S| = \gamma_t(G) \).

We in general follow the product of graphs in \[4\]. The lexicographic product of two graphs \( G \) and \( H \), denoted by \( G \Box H \), is the graph with vertex set equal to \( V(G) \times V(H) \) such that two vertices \((u_1, u_2)\) and \((v_1, v_2)\) are connected by an edge if either \( \{u_1, v_1\} \in E(G) \) or \( u_1 = v_1 \) and \( \{u_2, v_2\} \in E(H) \). The Corona product of two graphs \( G \) and \( H \), denoted by \( G \circ H \), is the graph obtained by taking one copy of \( G \) and \( n \) copies of \( H \), where \( G \) has \( n \) vertices, and joining the \( i \)-th vertex of \( G \) to every vertex in the \( i \)-th copy of \( H \). For every \( x \in V(G) \), we denote the copy of \( H \) whose vertices are attached to the vertex \( x \) in \( G \) by \( H^x \). The Cartesian product of two graphs \( G \) and \( H \), denoted by \( G \times H \), is the graph with vertex set equal to \( V(G) \times V(H) \) such that two vertices \((u_1, u_2)\) and \((v_1, v_2)\) are connected by an edge if either \( \{u_1, v_1\} \in E(G) \) and \( u_2 = v_2 \), or \( u_1 = v_1 \) and \( \{u_2, v_2\} \in E(H) \). For graphs \( G \) and \( H \), the direct product denoted by \( G \times H \) (also known as the tensor product, cross product, cardinal product and categorical product), is the graph with vertex set equal to \( V(G) \times V(H) \) such that two vertices \((u_1, u_2)\) and \((v_1, v_2)\) are connected by an edge if and only if \( \{u_1, v_1\} \in E(G) \) and \( \{u_2, v_2\} \in E(H) \).

The rest of the paper is organized as follows: In Section 2, we characterize the outer-connected domination of lexicographic products of graphs by constructing minimum sized ones. In Section 3, we investigate the
outer-connected domination of Corona products of graphs. In Section 4, an upper bound for the outer-connected domination number in the Cartesian product graphs is defined. Finally, in Section 5, we study the outer-connected domination number of the direct product of finitely many complete graphs.

2 Outer Connected Domination in the Lexicographic Product of Two Connected Graphs

In this section we must determine the outer-connected domination number in the lexicographic product of two graphs. To this aim, we first prove the following three lemmas.

**Lemma 2.1.** Let $G$ and $H$ be two graphs. Then, $\gamma(G) \leq \gamma(G \circ H)$.

*Proof.* Let $T = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$ be the dominating set for $G \circ H$. Then for every vertex $(a, b) \in G \circ H$, there exists a vertex $(x_i, y_i) \in T$ such that $\{(x_i, y_i), (a, b)\} \in E(G \circ H)$. That means either $\{x_i, a\} \in E(G)$ or $x_i = a$ and $\{y_i, b\} \in E(H)$. In both cases, the set $T' = \{x_i \mid (x_i, y_i) \in T\}$ is a dominating set for $G$ and its cardinality is less than or equal to the cardinality of the set $T$. $\square$

**Lemma 2.2.** Let $\gamma(H) \neq 1$ and $T$ is a dominating set for $G \circ H$ and $(x_i, y_i) \in T$. then there exists at least one vertex of the form $(x_j, v) \in T$ such that either $\{x_i, x_j\} \in E(G)$ or $x_i = x_j$ and $v \neq y_i$.

*Proof.* It is clear that the vertex $(x_i, y_i) \in T$ cannot cover all of the vertices $(x_i, u) \in V(G \circ H)$ because $\gamma(H) \neq 1$. So, the vertex $(x_i, u)$ is covered by the vertex $(x_j, v) \in T$ such that either $\{x_i, x_j\} \in E(G)$ or $x_i = x_j$ and $v \neq y_i$. $\square$

**Lemma 2.3.** Let $G$ and $H$ be two graphs such that $\gamma(H) \neq 1$. Then, there exists a total dominating set for $G$ with cardinality less than or equal to $\gamma(G \circ H)$.

*Proof.* According to Lemma 2.1, the set $T'$ is a dominating set for $G$. On the other hand, we have $\gamma(H) \neq 1$, so by Lemma 2.2, for all vertices $(x, v) \in T'$, there exists at least one node with the necessary conditions. Then, for all vertices $x_i \in T'$, we also add one of its neighbors to $T'$ which would be a total dominating set for $G$ of cardinality less than or equal to $|T|$. So, $\gamma_t(G) \leq \gamma(G \circ H)$. $\square$

**Theorem 2.4.** Suppose that $G$ and $H$ are two connected graphs. Then, we have

$$\gamma_{c}(G \circ H) = \begin{cases} 
1 & \text{if } \gamma(G) = 1 \text{ and } \gamma(H) = 1, \\
2 & \text{if } \gamma(G) = 1 \text{ and } \gamma(H) \neq 1, \\
\gamma(G) & \text{if } \gamma(G) \neq 1 \text{ and } \gamma(H) = 1, \\
\gamma_t(G) & \text{if } \gamma(G) \neq 1 \text{ and } \gamma(H) \neq 1.
\end{cases}$$

(1)

*Proof.* The proof is by construction:

**Case 1:** $\gamma(G) = \gamma(H) = 1$: Without loss of generality, assume that $\{x\}$ and $\{y\}$ are minimum dominating sets for $G$ and $H$, respectively. We claim that $\{(x, y)\}$ is an outer-connected dominating set for $G \circ H$. For every vertex $u \in V(G)$ and $u \neq x$, it is clear that $\{x, u\} \in E(G)$. So, by the definition of the lexicographic product, for all vertices $v \in V(H)$, we have

$$\{(x, y), (u, v)\} \in E(G \circ H).$$

(2)

On the other hand, by the definition of lexicographic product and the fact that $\{y\}$ is a dominating set for $H$, we have

$$\{(x, v), (x, y)\} \in E(G \circ H).$$

(3)
Therefore, the set \( \{(x, y)\} \) is a dominating set for \( G \circ H \). Next, we need to show that the induced graph \( G \circ H \setminus (x, y) \) is connected. Let \( V(G) = \{v_1, v_2, \ldots, v_{n-1}, x\} \) and \( V(H) = \{u_1, u_2, \ldots, u_{n-1}, y\} \). So, it suffices to show that for all \((a, b), (c, d) \in V(G \circ H) \setminus (x, y)\), there exists a path from \((a, b)\) to \((c, d)\) in \( G \circ H \) which does not pass through the vertex \((x, y)\). To this end, there are three cases to consider:

**Case 1-a:** \( a \neq x \) and \( c \neq x \): By the definition of the lexicographic product and given that the set \( \{x\} \) is a dominating set for \( G \), we have

\[
\{(x, v), (a, b)\} \in E(G \circ H),
\]

\[
\{(x, v), (c, d)\} \in E(G \circ H).
\]

So, there exists a path of length two between vertices \((a, b)\) and \((c, d)\) which passes through the vertex \((x, v)\).

**Case 1-b:** \( a = x \) and \( c \neq x \): By the definition of the lexicographic product and given that the set \( \{x\} \) is a dominating set for \( G \), it is clear that for all \( b \in H \) where \( b \neq y \), we have

\[
\{(x, b), (c, d)\} \in E(G \circ H).
\]

**Case 1-c:** \( a = c = x \), \( b \neq y \) and \( d \neq y \): For vertex \((t, u)\) where \( t \neq x \), we have

\[
\{(a, b), (t, u)\} \in E(G \circ H),
\]

\[
\{(c, d), (t, u)\} \in E(G \circ H).
\]

So, there exists a path of length two between vertices \((a, b)\) and \((c, d)\) which passes through the vertex \((t, u)\).

Finally, because the minimum possible size for any dominating set is one, this is clearly a minimum outer-connected dominating set for \( G[H] \).

**Case 2:** \( \gamma(G) = 1 \) and \( \gamma(H) \neq 1 \): Assume that the set \( \{x\} \) is a dominating set for \( G \). Similar to the previous case, for any vertex \( u \neq x \) where \( u \in V(G) \), all the vertices \((u, v)\) are adjacent to the vertex \((x, y)\) for \( y, v \in V(H) \). Also, each vertex \((u', v')\) where \( u' \neq x \) dominates all the vertices of the form \((x, v)\). So, the set

\[
\tilde{D}(G \circ H) = \{(x, y), (u', v')\},
\]

is a dominating set for \( G \circ H \). Then, we have

\[
\gamma_c(G \circ H) = 2.
\]

Finally, we need to show that the induced graph \( G \circ H \setminus \tilde{D}(G \circ H) \) is connected. To do so, we can apply the same method as in the previous case except that in case 1-c, we need to consider the constraint \((t, u) \neq (u', v')\).

**Case 3:** \( \gamma(G) \neq 1 \) and \( \gamma(H) = 1 \): Suppose that the set \( S = \{x_1, x_2, \ldots, x_m\} \) is a minimum cardinality dominating set for \( G \). By the definition of the lexicographic product and the fact that \( \{y\} \) is a dominating set of \( H \), we have

\[
S' = \{(x_1, y), (x_2, y), \ldots, (x_m, y)\},
\]

is a dominating set for \( G \circ H \) and \( \gamma_c(G \circ H) = \gamma(G) \) since \(|S'| = |S|\).

Next, we consider vertices \((a, b), (t, p) \in V(G \circ H) \setminus S'\). We know that \( G \) is a connected graph, so there exists a path from vertex \( a \) to vertex \( t \) in graph \( G \) which is denoted by

\[
a \rightarrow a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_k \rightarrow t,
\]
Case 4: 
\[ \gamma(G) \neq 1 \text{ and } \gamma(H) \neq 1 \]

Suppose that the set \( S = \{x_1, x_2, \ldots, x_t\} \) is a minimum cardinality total dominating set for \( G \). For every vertex \( x_i \in S \), the set of vertices dominated by the vertex \( x_i \) is denoted as \( S_i \). Since the set \( S \) is a total dominating set for \( G \), we have
\[
\bigcup_{i=1}^{t} S_i = V(G),
\]
and for all \( x_j \in S \), there exists a vertex \( x_i \in S \) such that \( x_j \in S_i \). So, the vertex \((x_i, v)\) dominates all the vertices of the form \((x_j, v')\) in \( G \circ H \). So, the set
\[
S' = \{(x_1, v), (x_2, v), \ldots, (x_t, v)\},
\]
dominates all the vertices \((a, b) \in V(G \circ H)\) where \( a \in \bigcup_{i=1}^{t} S_i \). Therefore, the set \( S' \) is a dominating set for \( G \circ H \).

Next, we consider the vertices \((x, y), (t, p) \in V(G \circ H) \setminus S'\). We know that \( G \) is a connected graph so, there exists a path from vertex \( x \) to vertex \( t \) in graph \( G \) which is denoted by
\[
x \to a_1 \to a_2 \to \cdots \to a_k \to t,
\]
where \( a_1, a_2, \ldots, a_k \in G \). So, there exists a path from \((x, y)\) to \((t, p)\) in \( G \circ H \setminus S' \) denoted by
\[
(x, y) \to (a_1, u'_1) \to (a_2, u'_2) \to \cdots \to (a_k, u'_k) \to (t, p).
\]
So, the induced graph \( V(G \circ H) \setminus S' \) is connected.

Finally, to complete the proof, it is necessary to show that the dominating set obtained for \( G \circ H \) is minimum. To do so, we can apply the same method as in the previous case. Let \( S' \) be an outer-connected dominating set for \( G \circ H \). Then,
\[
\gamma_d(G \circ H) \leq |S'| = \gamma_t(G).
\]

Now, suppose \( S^* \) is a minimum cardinality outer-connected dominating set for \( G \circ H \). So, by Lemma 2.3, there is a total dominating set for \( G \) with cardinality \(|S^*| \) which we call it \( T^* \). So, we have
\[
\gamma_t(G) \leq |T^*| \leq |S^*| = \gamma_d(G \circ H),
\]
which leads to \( \gamma_d(G \circ H) = \gamma_t(G) \) by Equations 6 and 7.

\[ \square \]
By using the above theorem, we can write an equivalent form of the Vizing’s conjecture as follow.

**Theorem 2.5.** Let $G$, $H$ and $K$ be graphs such that $\gamma(G) \neq 1$, $\gamma(H) \neq 1$ and $\gamma(K) = 1$. The Vizing’s conjecture is true if and only if

$$\gamma_c(G \circ K) \gamma_c(H \circ K) \leq \gamma_c(G \square H) \circ K.$$ 

**Proof.** According to Vizing’s conjecture for all graphs $G$ and $H$,

$$\gamma(G) \gamma(H) \leq \gamma(G \square H).$$

By using Theorem 2.4 we get the following inequality:

$$\gamma_c(G \circ K) \gamma_c(H \circ K) = \gamma(G) \gamma(H) \leq \gamma(G \square H) \leq \gamma_c(G \square H) = \gamma_c((G \square H) \circ K),$$

therefore

$$\gamma_c(G \circ K) \gamma_c(H \circ K) \leq \gamma_c((G \square H) \circ K).$$

Conversely, we consider that the inequality $\gamma_c(G \circ K) \gamma_c(H \circ K) \leq \gamma_c((G \square H) \circ K)$ is true. By assumptions of the theorem, $\gamma(G) \neq 1$, and that $\gamma(G \square H) \geq \gamma(G)$ we have $\gamma(G \square H) \neq 1$. So, by using Theorem 2.4 we get

$$\gamma(G) \gamma(H) = \gamma_c(G \circ K) \gamma_c(H \circ K) \leq \gamma_c((G \square H) \circ K) = \gamma(G \square H),$$

or

$$\gamma(G) \gamma(H) \leq \gamma(G \square H).$$

□

**Lemma 2.6.** If $H = K_1$, then we have $\gamma_c(G \circ H) = \gamma_c(G)$.

**Proof.** It is easy to verify, since for every $x \in V(G)$, there exists exactly one vertex $(x, y) \in V(G \circ H)$ where $y$ is the only vertex in $H$ and for every $(x, u) \in E(G)$, there exists exactly one edge connecting $(x, y)$ and $(u, y)$ in $G \circ H$.

□

**Lemma 2.7.** If $G = K_1$, then we have $\gamma_c(G \circ H) = \gamma_c(H)$.

**Proof.** It is clear since for every $y \in V(H)$, there exists exactly one vertex $(x, y) \in V(G \circ H)$ where $x$ is the only vertex in $G$ and for every $(v, y) \in E(G)$, there exists exactly one edge connecting $(x, y)$ and $(x, v)$ in $G \circ H$.

□

By Theorem 2.4 outer-connected domination number of $G \circ H$ denoted by $\gamma_c(G \circ H)$ depends on the values $\gamma(G)$ and $\gamma_c(G)$. We know that it is NP-hard to compute the domination number and the total domination number. So, We apply the following Lemma to determine the upper bound for $\gamma_c(G \circ H)$.

**Lemma 2.8.** If $G$ is a connected graph of order $n$ and $H$ is a graph of order $m$, then we have

$$\gamma_c(G \circ H) \leq mn - (\delta(G) + \delta(H)).$$

(8)

**Proof.** By the definition of the Lexicographic products, $G \circ H$ is a graph of order $mn$ and $\delta(G \circ H) = \delta(G) + \delta(H)$. Therefore we have

$$\gamma_c(G \circ H) \leq mn - (\delta(G) + \delta(H)),$$

(9)

by Theorem 5 in [2].

□
The following is a tight example for Lemma 2.8

Example 1. Let $G$ and $H$ be two graphs with $V(G) = \{p\}$, $V(H) = \{x,y\}$ and $E(H) = \{\{x,y\}\}$. Then, we have

\[
V(G \circ H) = \{(p,x),(p,y)\},
\]

\[
E(G \circ H) = \{\{(p,x),(p,y)\}\},
\]

\[
\gamma_c(G \circ H) = 1,
\]

and

\[
mn - (\delta(G) + \delta(H)) = 1 * 2 - (0 + 1) = 1.
\]

3 Outer Connected Domination in the Corona Product of Two Graphs

Lemma 3.1. Suppose that $G$ is a connected graph and $\tilde{D} \subseteq V(G)$ is an outer-connected dominating set for $G$. If $u \in \tilde{D}$ is a cut vertex, then all the vertices $v \in V \setminus \tilde{D}$ belong to exactly a single component of $V \setminus \{u\}$.

Proof. Let $u \in \tilde{D}$ be a cut vertex and $c_1, c_2, \ldots, c_m$ be components of the induced subgraph $G[V \setminus \{u\}]$. Suppose that there exist arbitrary vertices $x, y \in V \setminus \tilde{D}$ such that $x \in c_i$ and $y \in c_j$ for $i \neq j, 1 \leq i, j \leq m$. Thus, there exists no path from $x$ to $y$ in $G[V \setminus \{u\}]$, and this is a contradiction to the assumption that $\tilde{D}$ is an outer-connected dominating set. So, $x$ and $y$ are certainly in the same component of $G[V \setminus \{u\}]$. Since the vertices $x$ and $y$ are chosen arbitrarily, then the theorem is proven.

Corollary 3.2. Suppose that $G$ is a connected graph and $\tilde{D} \subseteq V(G)$ is an outer-connected dominating set for $G$. Let $u \in \tilde{D}$ be a cut vertex and $c_1, c_2, \ldots, c_m$ be components of the induced subgraph $G[V \setminus \{u\}]$. Then, all the vertices in its $m - 1$ components are included in $\tilde{D}$.

Theorem 3.3. Let $G$ be a connected graph and $H$ is an arbitrary graph. The set $\tilde{D} \subseteq V(G \circ_c H)$ is an outer-connected domination for $G \circ_c H$ if and only if

\[
\tilde{D} = \cup_{x \in V(G)}(D(H^x)),
\]

where $D$ is a minimum dominating set of graph $H^x$, and $H^x$ is the copy of graph $H$ whose vertices are attached to the vertex $x$ in $G$.

Proof. If the set $\tilde{D} \subseteq V(G \circ_c H)$ is an outer-connected domination for $G \circ_c H$ and $x \in \tilde{D}$, then $x$ is not a cut vertex. Otherwise, according to Theorem 3.1 and by the assumption that the graph $(G \circ_c H) \setminus v$ has $m$ components, all the vertices in its $m - 1$ components are included in the outer-connected dominating set. As a result, the set $\tilde{D}$ is not minimum.

Since every vertex $v \in V(G)$ is a cut vertex in $G \circ_c H$, then none of the vertices $v \in G$ are in $\tilde{D}$. Therefore, by the definition of the Corona product and the obtained fact that $v \in G$ is not in $\tilde{D}$, we have

\[
\tilde{D} = \cup_{x \in V(G)}(D(H^x)).
\]

To prove the converse of the theorem, suppose that $\tilde{D} = \cup_{x \in V(G)}(D(H^x))$. It is clear that $D(H^x)$ is a dominating set for $H^x \cup \{x\}$, then $\cup_{x \in V(G)}(D(H^x))$ is a dominating set for $G \circ_c H$. On the other hand, $G$ is a connected graph and all of its vertices $u \in H^x \setminus D(H^x)$ are connected to $x$. So, $\tilde{D}$ is an outer-connected dominating set for $G \circ_c H$. Eventually, it is clear that

\[
|D(H^x)| \leq |\tilde{D}(H^x)|.
\]

So, the set $\tilde{D}$ is minimum.
Corollary 3.4. Let $G$ be a connected graph and $H$ be an arbitrary graph. Then, we have

$$
\tilde{\gamma}_c(G \circ H) = |V(G)| \gamma(G).
$$

(17)

Proof. The proof is clear from Theorem 3.3 and the definition of the corona product.

\[ \square \]

4 Outer Connected Domination in the Cartesian Product of Two Graphs

In this section, we present an upper bound for outer-connected domination number in Cartesian product graphs.

Theorem 4.1. For any graphs $G$ and $H$, we have

$$
\tilde{\gamma}_c(G \square H) \leq \gamma_c(G) \times |V(H)|.
$$

(18)

Proof. We first show the following claim and using it, we have

$$
\tilde{\gamma}_c(G \square H) \leq |T| = |\tilde{D}| \times |V(H)| = \gamma_c(G) \times |V(H)|.
$$

(19)

Claim 4.1.1. Let $\tilde{D}$ be an outer-connected dominating set of $G$. Then, $T = \tilde{D} \times V(H)$ is an outer-connected dominating set for $G \square H$.

Proof. Let $(a, b) \in V(G \square H) \setminus T$, then we have $a \notin \tilde{D}$. Since $\tilde{D}$ is an outer-connected dominating set for $G$, then there exists a vertex $x$ in $\tilde{D}$ such that $(a, x) \in E(G)$. On the other hand, $b \in V(H)$ so we have $(a, b) \in T$. Then, according to the definition of the Cartesian product, the vertex $(a, b)$ is adjacent to vertex $(x, b)$ in $G \square H$ which means that the vertex $(a, b)$ is dominated by the vertex $(x, b) \in T$.

Now, consider two vertices $(a, b), (x, y) \in V(G \square H) \setminus T$. If $a = x$, then there exists a path from vertex $b$ to vertex $y$ in graph $H$ in the form

$$
b \to t_1 \to t_2 \to \cdots \to t_k \to y,
$$

where $t_1, t_2, \cdots, t_k \in H$ since $H$ is a connected graph. So, there exists a path from $(x, y)$ to $(a, b)$ in $(G \square H) \setminus T$ denoted by

$$(a, b) \to (a, t_1) \to (a, t_2) \to \cdots \to (a, t_k) \to (x, y).$$

If $a \neq x$ and $y = b$, since $\tilde{D}$ is an outer-connected dominating set of graph $G$ and $a, x \notin \tilde{D}$, then there exists a path from vertex $a$ to vertex $x$ in the graph $G \setminus \tilde{D}$ which is denoted by

$$a \to t_1 \to t_2 \to \cdots \to t_k \to x,$$

where $t_1, t_2, \cdots, t_k \in G \setminus \tilde{D}$. So, there exists a path from $(x, y)$ to $(a, b)$ in $(G \square H) \setminus T$ denoted by

$$(a, b) \to (t_1, b) \to (t_2, b) \to \cdots \to (t_k, b) \to (x, y).$$

If $a \neq x$ and $y \neq b$, then there exists a path from vertex $a$ to vertex $x$ in graph $G \setminus \tilde{D}$ which is denoted by

$$a \to t_1 \to t_2 \to \cdots \to t_k \to x,$$

where $t_1, t_2, \cdots, t_k \in G \setminus \tilde{D}$ since $\tilde{D}$ is an outer-connected dominating set of graph $G$ and $a, x \notin \tilde{D}$. So, there exists a path from $(a, y)$ to $(x, y)$ in $(G \square H) \setminus T$ denoted by

$$(a, y) \to (t_1, b) \to (t_2, b) \to \cdots \to (t_k, b) \to (x, y).$$
On the other hand, since $H$ is a connected graph, then there exists a path from the vertex $b$ to the vertex $y$ in graph $H$ of the form

$$b \to t_1 \to t_2 \to \cdots \to t_k \to y,$$

where $t_1, t_2, \cdots, t_k \in H$. So, there exists a path from $(a, b)$ to $(a, y)$ in $(G \Box H) \setminus T$ denoted by

$$(a, b) \to (a, t_1) \to (a, t_2) \to \cdots \to (a, t_k) \to (a, y).$$

Therefore, there exists a path from $(a, b)$ to $(x, y)$ in $(G \Box H) \setminus T$ which passes through the vertex $(a, y)$.

So, the induced graph $V(G \Box H) \setminus T$ is connected, and hence, $T$ is an outer-connected dominating set for $G \Box H$.

The following tight example shows that the bound given in Theorem 5.1 is sharp.

**Example 2.** Let $G$ and $H$ be two graphs with $V(G) = \{a, b, c\}$, $V(H) = \{x, y\}$, $E(G) = \{\{a, b\}, \{a, c\}, \{b, c\}\}$ and $E(H) = \{\{x, y\}\}$. Then, we have

$$V(G \Box H) = \{(a, x), (a, y), (b, x), (b, y), (c, x), (c, y)\},$$

$$E(G \circ H) = \{(a, x), (a, y), (b, x), (b, y), (c, x), (c, y)\}
\begin{array}{c}
\{(a, x), (b, x)\}, \{(a, x), (c, x)\}, \{(a, y), (b, y)\}
\end{array}
\begin{array}{c}
\{(a, y), (c, y)\}, \{(b, x), (c, x)\}, \{(b, y), (c, y)\}\}.
\end{array}$$

The outer-connected dominating set for $G$ is $\{c\}$ and the outer-connected dominating set for $G \Box H$ is $\{(c, x), (c, y)\}$. So,

$$\gamma_c(G \Box H) = \gamma_c(G) + |V(H)| = 1 \times 2 = 2.\quad (22)$$

## 5 Outer Connected Domination in the Direct Product of Two Graphs

Gasper Mekis [11] gave a lower bound for the domination number of a direct product and proved the following result.

**Theorem 5.1.** [11] Let $G = \times_{i=1}^{t}K_{n_i}$, where $t \geq 3$ and $n_i \geq 2$ for all $i$. Then $\gamma(G) \geq t + 1$.

From Theorem 5.1 and that $\gamma_c(G) \geq \gamma(G)$, we obtain the following result.

**Corollary 5.2.** Let $G = \times_{i=1}^{t}K_{n_i}$, where $t \geq 3$ and $n_i \geq 2$ for all $i$. Then $\gamma_c(G) \geq t + 1$.

The bound given in Theorem 5.1 is sharp [11] and it also remains sharp for the outer-connected domination number.

**Theorem 5.3.** Let $G = \times_{i=1}^{t}K_{n_i}$, where $t \geq 3$ and $n_i \geq t + 1$ for all $i$. Then $\gamma_c(G) = t + 1$.

**Proof.** Let $\hat{D} = \{(0, 0, \ldots, 0), (1, 1, \ldots, 1), \ldots, (t, t, \ldots, t)\}$ be an outer-connected dominating set for $G$. Suppose that $u = (u_1, u_2, \ldots, u_t) \in V(G) \setminus \hat{D}$ and $u$ is not adjacent to any of the vertices from $\hat{D}$, in which case $u$ must agree in at least one coordinate with every vertex from $\hat{D}$. Hence each of the $t + 1$ elements from $\{0, 1, \ldots, t\}$ must appear on some coordinate of $u$, which is not possible as $u$ has only $t$ coordinates available. Now, it suffices to show that $V(G) \setminus \hat{D}$ induces a connected graph.

Suppose that $u = (u_1, u_2, \ldots, u_t), v = (v_1, v_2, \ldots, v_t)$ are two arbitrary vertices in $V(G) \setminus \hat{D}$, we claim there exists a vertex $x = (x_1, x_2, \ldots, x_t)$ such that the path from $u$ to $v$ in $V(G) \setminus \hat{D}$ passes through it. Let $p_i$ be $i^{th}$ coordinate of the vertex $p$. The coordinate $x_i$ have to be three properties: (1) $x_i \neq u_i$, (2) $x_i \neq v_i$, and $x_i \neq x_{i-1}$ or $x_i \neq x_{i+1}$. So, if there exist at least four vertices in all $K_{n_i}$, the claim is true which completes the proof. This end is clear since $n_i \geq t + 1$ and $t \geq 3$.\qed
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