A note on clique-width and tree-width for structures

Hans Adler and Isolde Adler

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Abstract

We give a simple proof that the straightforward generalisation of clique-width to arbitrary structures can be unbounded on structures of bounded tree-width. This can be corrected by allowing fusion of elements.

1 Introduction

Clique-width, introduced by Courcelle and Olariu, is a good measure for the complexity of a graph in the sense that many problems that are intractable in general become tractable when restricted to graphs of bounded clique-width. Moreover, clique-width of a graph is bounded by a function of tree-width, while the converse is not true [4]. However, the clique-width of a clique is two, so trying to measure the complexity of a general structure by the clique-width of its Gaifman graph makes no sense. Therefore we are looking for a notion similar to clique-width of graphs, which should ideally have all of the following properties:

1. It is defined for arbitrary structures.
2. It specialises (essentially) to clique-width, in the case of graphs.
3. It is bounded by a function of the tree-width of the Gaifman graph.
4. Computationally hard problems should become tractable on instances of bounded width.
5. It does not increase on induced substructures.
6. The value mapping is an MS transduction.
7. Every set of structures that is the image of an MS transduction from trees has bounded width.
8. For fixed $k$ a decomposition of width $k$ (or width $f(k)$) can be computed in polynomial time.

This paper mainly addresses the third criterion. We consider the generalisation of clique-width to structures as proposed by Grohe and Turán [8]. (A similar notion was introduced by Fischer and Makowsky, resulting in values that are smaller by one because elements can also be uncoloured [6].) We give a simpler proof for a result of Courcelle, Engelfriet and Rozenberg [2, Theorem 7.5], showing that the clique-width of a class of structures is not bounded in terms of its tree-width. More specifically we show that there is a class of structures, with one ternary relation symbol, that has unbounded clique-width while the tree-width of the class is bounded by 2. It is known that there can be no such examples with structures that have only binary and unary relation symbols [4, 10].
We then consider a modified definition where we additionally allow fusion of elements. We show that the corresponding modified width is bounded from above by tree-width plus 2. In an earlier version of this paper we referred to clique-width with fusion as ‘reduced clique-width’, since we were not aware that the fusion operation had already been introduced by Courcelle and Makowsky \cite{Courcelle1990}. We have now corrected this to avoid unnecessary proliferation of technical terms.

2 Structures, decompositions, and unary clique-width

A signature $\sigma = \{R_1, \ldots, R_n\}$ is a finite set of relation symbols $R_i$, $1 \leq i \leq n$. As usual, every relation symbol $R \in \sigma$ has an associated arity $\text{ar}(R)$. A $\sigma$-structure is a tuple $A = (A, R_1^A, \ldots, R_n^A)$ where $A$ is a set, the universe of $A$, and $R_i^A \subseteq A^{\text{ar}(R_i)}$ for $1 \leq i \leq n$. All structures in this paper are finite (i.e. they have finite universes).

Given a structure $A$, we write $G_A$ for the underlying graph (also called Gaifman graph) of $A$: The vertices are the elements of the universe, and two different vertices are joined by an edge if, and only if, they appear together in some tuple that is in a relation of $A$.

A tree decomposition of a graph $G = (V, E)$ is a pair $(T, B)$, consisting of a rooted tree $T$ and a family $B = (B_t)_{t \in T}$ of subsets of $V$, the pieces of $T$, satisfying:

- For each $v \in V$ there exists $t \in T$, such that $v \in B_t$.
- For each edge $e \in E$ there exists $t \in T$, such that $e \subseteq B_t$.
- For each $v \in V$ the set $\{ t \in T \mid v \in B_t \}$ is connected in $T$.

The width of $(T, B)$ is defined as $w(T, B) := \max \{|B_t| - 1 \mid t \in T\}$. The tree-width of $G$ is defined as $\text{tw}(G) := \min \{ w(T, B) \mid (T, B) \text{ is a tree decomposition of } G \}$.

Path decompositions and path-width of $G$, $\text{pw}(G)$, are defined analogously, with the additional restriction that $T$ be a path.

Let us call a pair $(A, \gamma)$ consisting of a structure $A$ and a map $\gamma : A \to \omega$ a coloured structure. It is $k$-coloured if $\gamma(A) \subseteq \{0, 1, \ldots, k-1\}$. We will usually think of elements of colour 0 as ‘uncoloured’, and identify structures and 1-coloured structures. We call an element $a$ of a coloured structure $(A, \gamma)$ isolated if $a$ is the only element of colour $\gamma(a)$. For a signature $\sigma$ and a non-negative integer $k$, we define $\text{UCW}_k[\sigma]$ as the smallest class of $k$-coloured $\sigma$-structures such that:

1. Every (1-coloured) empty $\sigma$-structure is in $\text{UCW}_k[\sigma]$.
2. Every (1-coloured) 1-element $\sigma$-structure whose relations are all empty is in $\text{UCW}_k[\sigma]$.
3. The disjoint union $(\mathcal{A} \uplus \mathcal{B}, \gamma_A \uplus \gamma_B)$ of two coloured structures $(\mathcal{A}, \gamma_A), (\mathcal{B}, \gamma_B) \in \text{UCW}_k[\sigma]$ is again in $\text{UCW}_k[\sigma]$.
4. If $(\mathcal{A}, \gamma) \in \text{UCW}_k[\sigma]$ and $f : \{0, \ldots, k-1\} \to \{0, \ldots, k-1\}$ is any function, then $(\mathcal{A}, f \circ \gamma) \in \text{UCW}_k[\sigma]$.
5. If $(\mathcal{A}, \gamma) \in \text{UCW}_k[\sigma]$, $R \in \sigma$ is an $n$-ary relation symbol, and $c_0, \ldots, c_{n-1} \in \{0, \ldots, k-1\}$ is an $n$-tuple of colours, then for the structure $\mathcal{B}$ which is like $\mathcal{A}$ except that $\mathcal{B} \models R(a_0, \ldots, a_{n-1})$ holds iff $\mathcal{A} \models R(a_0, \ldots, a_{n-1})$ or $\gamma(a_i) = c_i$ for $i = 0, \ldots, n-1$, we have $(\mathcal{B}, \gamma) \in \text{UCW}_k[\sigma]$.

For a $\sigma$-structure $A$ (possibly coloured) let $\text{ucw}(A)$, the unary clique-width of $A$, be the smallest number $k$ such that $A \in \text{UCW}_k[\sigma]$. ‘Unary’ because only single elements are coloured. It is easy to see that our notion is equivalent to that of Grohe and Turán, even though they use a more restrictive colouring operation \cite{Grohe2008}.

Every $k$-coloured $\sigma$-structure $A$ with $|A| \leq k$ elements has unary clique-width $\text{ucw}(A) \leq k$: Take the disjoint union of $|A|$ differently coloured 1-element structures, introduce the necessary relations, and recolour all elements with their final colour.
3 Unary clique-width and tree-width

**Proposition 1.** Every (1-coloured) structure $A$ satisfies 

$$ucw(A) \leq pw(G_A) + 1.$$ 

**Proof.** Let $k = pw(G_A)+1$. We will show by induction on $n$ the stronger statement that if $G_A$ has a path decomposition $B_0, B_1, \ldots, B_{n-1}$ of width $\leq k+1$ and $\gamma : A \to \{0, \ldots, k\}$ is a $k+1$-colouring of $A$ such that all elements of $A \setminus B_{n-1}$ have colour 0, then $(A, \gamma) \in UCW_{k+1}[\sigma]$. The case $n = 0$ is trivial because then $A$ is the empty $\sigma$-structure.

For $n \geq 1$, let $A' = A \upharpoonright (B_0 \cup \cdots \cup B_{n-2})$ be the induced substructure of $A$ with domain $B_0 \cup \cdots \cup B_{n-2}$, and let $\gamma' : B_0 \cup \cdots \cup B_{n-2} \to \{0, \ldots, k+1\}$ be a colouring such that every element of $B_{n-2} \cap B_{n-1}$ is isolated and has a non-zero colour, while all other elements have colour 0. Then $(A', \gamma') \in UCW_{k+1}[\sigma]$ by the induction hypothesis. Let $A''$ be the $\sigma$-structure with domain $A$ whose relations are precisely those of $A'$. Let $\gamma'' : A \to \{0, \ldots, k+1\}$ extend $\gamma'$ so that any two elements of $B_{n-1}$ have distinct non-zero colours. $(A'', \gamma'')$ can be obtained from $(A', \gamma')$ by disjoint union with one-element structures, so clearly $(A'', \gamma'') \in UCW_{k+1}[\sigma]$. All relations of $A$ that are not also relations of $A'$ must be between elements of $B_{n-1}$. Since every element of $B_{n-1}$ has a unique colour we can introduce all these relations without introducing any unwanted relations. Similarly, for $f : \{0, \ldots, k+1\} \to \{0, \ldots, k+1\}$ such that $f(0) = 0$ and $f(\gamma''(a)) = \gamma(a)$ we get $\gamma = f \circ \gamma''$. Hence $(A, \gamma) \in UCW_{k+1}[\sigma]$.

**Lemma 2.** For every structure $A$ in the signature $\sigma = \{E\}$ of graphs, there is a structure $A'$ with universe $A' = A \cup \{t\}$ in the signature $\sigma' = \{R\}$, $R$ a ternary relation symbol, which satisfies:

1. $tw(G_{A'}) = tw(G_A) + 1$.
2. $pw(G_{A'}) = pw(G_A) + 1$.
3. $ucw(A') \geq pw(G_{A'}) + 1$.

(Note: The +1 in clauses 1 and 2 is due to the fact that $G_{A'}$ is an apex graph over $G_A$, whereas the +1 in 3 merely corrects the conventional −1 in the definition of path-width.)

**Proof.** We interpret the relation $R$ as follows:

$$A' \models Racb \iff a \neq b \text{ and } c \in \{a, b\} \text{ and } (A \models Eab \text{ or } a = t).$$

Then 1 and 2 hold because $G_{A'}$ is an apex graph over $G_A$. (I.e., $G_A$ is an induced subgraph of $G_{A'}$, and $G_{A'}$ has exactly one additional vertex, say $t$, and $t$ has an edge with every vertex of $G_A$.) Towards a proof of 3 we observe:

If $A' \in UCW_k[\sigma']$, then there is a tree of $k$-coloured $\sigma'$-structures such that the leaves are singletons with empty relations, and every inner node is either binary and the disjoint union of its two children, or unary and obtained from its only child by recolouring or by introducing a new relation. Due to the definition of $R$, for every element $a \neq t$ there must be a node containing both $a$ and $t$ as isolated elements, and for any two distinct elements $a, b \neq t$ there must be a node containing $a, b, t$ as three isolated elements.

The branch $(A_0, \gamma_0), \ldots, (A_{m-1}, \gamma_{m-1})$ of the tree which begins with the root and ends in the leaf consisting of the single element $t$ has the following properties:

1. For every element $a \in A$ the branch contains a node that has $a$ and $t$ as isolated elements.
2. For every edge $A \models Eab$ of $A$ there is a node that has $a, b, t$ as isolated elements.
3. For every element $a \in A$ there is a greatest index $j$ such that $a \in A_j$, and a smallest index $i \leq j$ such that $a$ is the only element of colour $\gamma_i^t(a)$.
Let \( \ell \) be the greatest index such that \( t \) is isolated in \( A_\ell^t \). For \( i = 0, \ldots, \ell \) let \( B_i \) consist of the isolated elements of \( (A_i^t, \gamma_i^t) \). Then \( B_0 \setminus \{t\}, \ldots, B_\ell \setminus \{t\} \) is clearly a path decomposition of \( A \). Each bag \( B_i \setminus \{t\} \) has at most \( k \) elements, so the width of the path decomposition is at most \( k - 1 \). Thus we have shown that \( ucw(A) \leq k \) implies \( pw(A) \leq k - 2 \).

**Corollary 3.** For every non-negative integer \( n \) there is a structure \( A \) with only one, ternary, relation symbol, such that

1. \( tw(G_A) = 2 \) and
2. \( ucw(A) > n \).

**Proof.** Let \( G \) be an undirected tree in the signature \( \{E\} \) of graphs, satisfying \( pw(G_G) \geq n \). Let \( A' = G' \) be as in the Lemma. Then \( tw(G_G) = tw(G_G') + 1 = 2 \) and \( ucw(G_G') \geq pw(G_G') + 1 = pw(G_G) + 2 > n \).

As Johann Makowsky pointed out to us, a result of Glikson and Makowsky [7] is cited incorrectly in the paper by Fischer and Makowsky [6] (as Theorem 3.7 (ii)), resulting in an apparent contradiction to Corollary [3]. Our result is optimal in the sense that, as shown by Courcelle and Olariu, clique-width of structures with at most binary relations is bounded by a function of tree-width of the Gaifman graph [4]. (The details for an arbitrary number of binary relations were checked by Till Scheffzik in his diplomarbeit [10]).

### 4 Clique-width with fusion

The reason why the proof of Proposition [1] works with path-width but not with tree-width is that for unary clique-width there is no way to glue together two substructures that intersect in a small bag of elements. In other words, while we can introduce new relations in the signature depending only on the colours of the elements, we cannot do this for equality, which can also be seen as a binary relation. The definition below fixes this, using the fusion operation of Courcelle and Makowsky [3, 9].

We will need a rarely used universal construction for structures: The quotient by an equivalence relation which need not be a congruence relation. Let \( A \) be a \( \sigma \)-structure, and let \( \sim \) be an equivalence relation on its domain \( A \). Let \( B = A/\sim \) be the set of equivalence classes of \( \sim \). Let \( \mathcal{B} \) be the \( \sigma \)-structure with domain \( B \) which satisfies \( \mathcal{B} \models R(b_1, \ldots, b_{n(R)}) \) if and only if there are \( a_1 \in b_1, \ldots, a_{n(R)} \in b_{n(R)} \) such that \( A \models R(a_1, \ldots, a_{n(R)}) \). We denote this quotient structure by \( \mathcal{B} = A/\sim \). \( \mathcal{B} \) is universal in the sense that the projection map \( p : A \to B \) is a homomorphism, and for every other structure \( \mathcal{B}' \) with the same domain \( B \), if the projection map \( p' : A \to B' \) is also a homomorphism then it factors through \( p \). For a coloured structure \( (A, \gamma) \) and a set \( C \subseteq \omega \) of colours we define \( a \sim_C a' \) to mean \( a = a' \) or \( \gamma(a) = \gamma(a') \in C \). The obvious colouring \( \gamma/\sim_C \) induced on the quotient structure \( A/\sim_C \) satisfies \( \gamma/\sim_C(b) = \gamma(a) \) for some/any \( a \in b \).

For a signature \( \sigma \) and a non-negative integer \( k \), the class \( UCWF_k[\sigma] \) is the smallest class of \( k \)-coloured \( \sigma \)-structures such that:

1. Every empty \( \sigma \)-structure is in \( UCWF_k[\sigma] \).
2. Every 1-element \( \sigma \)-structure whose relations are all empty is in \( UCWF_k[\sigma] \).
3. The disjoint union \( (A \sqcup \mathcal{B}, \gamma_A \sqcup \gamma_B) \) of two coloured structures \( (A, \gamma_A), (\mathcal{B}, \gamma_B) \in UCWF_k[\sigma] \) is again in \( UCWF_k[\sigma] \).
4. If \( (A, \gamma) \in UCWF_k[\sigma] \) and \( f : \{0, \ldots, k-1\} \to \{0, \ldots, k-1\} \) is any function, then \( (A, f \circ \gamma) \in UCWF_k[\sigma] \).
5. If \( (A, \gamma) \in UCWF_k[\sigma] \), \( R \in \sigma \) is an \( n \)-ary relation symbol, and \( c_0, \ldots, c_{n-1} \in \{0, \ldots, k-1\} \) is an \( n \)-tuple of colours, then for the structure \( \mathcal{B} \) which is like \( A \) except that \( \mathcal{B} \models R(a_0, \ldots, a_{n-1}) \) holds if \( A \models R(a_0, \ldots, a_{n-1}) \) or \( \gamma(a_i) = c_i \) for \( i = 0, \ldots, n-1 \), we have \( (\mathcal{B}, \gamma) \in UCWF_k[\sigma] \).
6. If \((A, \gamma_A) \in \text{UCWF}_k^{\sigma}\) and \(c \in \{0, \ldots, k-1\}\), then also \((A/\sim_{\{c\}}, \gamma/\sim_{\{c\}}) \in \text{UCWF}_k^{\sigma}\).

The only difference to the definition of \(\text{UCW}_k^{\sigma}\) is that we have added the fusion operation at the end. For a \(\sigma\)-structure \(A\) let \(\text{ucwf}(A)\), the unary clique-width with fusions of \(A\), be the smallest number \(k\) such that \(A \in \text{UCWF}_k^{\sigma}\).

Remark 4. Every \(k\)-coloured \(\sigma\)-structure \((A, \gamma)\) with at most \(k\) elements satisfies \((A, \gamma) \in \text{UCWF}_k^{\sigma}\).

Proof. Clearly for every \(\sigma\)-structure \(A'\) with no relations and every \(k\)-colouring \(\gamma'\) of \(A'\) we have \((A', \gamma') \in \text{UCWF}_k^{\sigma}\). Now for \(A\) as in the remark let \(A'\) be the \(\sigma\)-structure which has the same domain as \(A\) but no relations, and let \(\gamma'\) be a colouring of \(A'\) such that every element has a different colour. Since \((A', \gamma') \in \text{UCWF}_k^{\sigma}\), clearly also \((A, \gamma) \in \text{UCWF}_k^{\sigma}\).

Lemma 5. Suppose \(A\) has a tree decomposition \((T, B)\) of width \(\leq k\), and \(B_t\) is the bag at some tree node \(t \in T\). Let \(\gamma\) be a \((k+2)\)-colouring of \(A\) such that all elements \(a \in A\) of the domain of \(A\) have colour \(\gamma(a) = 0\). Then \((A, \gamma) \in \text{UCWF}_k^{\sigma}\).

Proof. By induction on the minimal number of nodes in a tree decomposition of \(A\) in which the bag \(B_t\) occurs, using the remark as induction base.

Corollary 6. Every relational structure \(A\) satisfies

\[
\text{ucwf}(A) \leq \text{tw}(A) + 2.
\]

The term +2 consists of +1 correcting for the conventional −1 in the definition of tree-width, and +1 for the colour 0, which we have put to special use.

5 Conclusion

We have shown that clique-width for structures fails even the basic requirement that it be bounded in terms of tree-width. We have also shown that this can be corrected by admitting the fusion operation in addition to the other operations, although we have not explored the (presumably negative) impact of this on the other desirable conditions.

Another approach would be to colour tuples rather than elements. Examples for definitions that go in that direction are Blumensath’s partition-width \(^{11}\), and patch-width, defined by Fischer and Makowsky and examined further by Shelah and Doron \(^{0,5}\). For instance, we could assign colours to tuples of length at most some fixed \(n\). But then it should still be possible to adapt our example to show that we do not obtain a bound in terms of tree-width.

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