Classical data compression with quantum side information

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March 31, 2022

Abstract

The problem of classical data compression when the decoder has quantum side information
at his disposal is considered. This is a quantum generalization of the classical Slepian-Wolf
theorem. The optimal compression rate is found to be reduced from the Shannon entropy of
the source by the Holevo information between the source and side information.

Generalizing classical information theory to the quantum setting has had varying success de-
pending on the type of problem considered. Quantum problems hitherto solved (in the asymptotic
sense of Shannon theory) may be divided into three classes. The first comprises pure bipartite
entanglement manipulation, such as Schumacher compression [1] and entanglement concentra-
tion/dilution [2, 3, 4]. Their tractability is due to the formal similarities between a pair of perfectly
correlated random variables and the Schmidt decomposition of bipartite quantum states.

The second, and largest, is the class of “hybrid” classical-quantum problems, where only a
subset (usually of size one) of the terminals in the problem is quantum and the others are classical.
The simplest example is the Holevo-Schumacher-Westmoreland (HSW) theorem [5], which deals
with the capacity of a classical → quantum channel (abbreviated {c → q}; see [6]). This carries
over to the multiterminal case involving many classical senders and one quantum receiver [7]. Then
we have Winter’s measurement compression theorem [8], and remote state preparation [9, 11, 10].
These two may be thought of as simulating quantum → classical ({q → c}) and {c → q} channels,
respectively. Another recent discovery has been quantum data compression with classical side-
information available to both the encoder and decoder [12], generalizing the rate-entropy curve of
[11] to arbitrary pure state ensembles.

The third class is that of fully quantum communication problems, such as the entanglement-
assisted capacity theorem [13] and its reverse – that of simulating quantum channels in the presence
of entanglement [14]. These rely on methods of {c → q} channel coding combined with super-dense
coding [15] and {q → c} channel simulation combined with quantum teleportation [16], respectively.
A recent addition to this class has been the long awaited proof of the channel capacity theorem
[17, 18], which also relies on classical-quantum methods.

The problem addressed here belongs to the second class and concerns classical data compression
when the decoder has quantum side information at his disposal. We shall refer to it as the classical-
quantum Slepian-Wolf (CQSW) problem in analogy to its classical counterpart [19]. We begin
by introducing the notion of a bipartite classical-quantum system. The fully classical and fully
quantum analogues are familiar concepts. The former is is embodied in a pair of correlated random

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variables \(XY\), associated with the product set \(\mathcal{X} \times \mathcal{Y}\) and a probability distribution \(p(x, y) = \Pr\{X = x, Y = y\}\) defined on \(\mathcal{X} \times \mathcal{Y}\). The latter is a bipartite quantum system \(AB\), associated with a product Hilbert space \(\mathcal{H}_A \otimes \mathcal{H}_B\) and a density operator \(\rho_{AB}\), the “quantum state” of the system \(AB\), defined on \(\mathcal{H}_A \otimes \mathcal{H}_B\). The state of a classical-quantum system \(XQ\) is now described by an ensemble \(E = \{\rho_x, p(x)\}\), with \(p(x)\) defined on \(\mathcal{X}\) and the \(\rho_x\) being density operators on the Hilbert space \(\mathcal{H}_Q\) of \(Q\). Thus, with probability \(p(x)\) the classical index and quantum state take on values \(x\) and \(\rho_x\), respectively. Such correlations may come about, for example, when Bob holds the purification of a state Alice is measuring. Indeed, let Alice and Bob initially share the quantum system \(AB\) with local density matrix \(\rho\) and \(\ast\) denotes complex conjugation in the \(\{|i\}\) basis.

A useful representation of classical-quantum systems, which we refer to as the “enlarged Hilbert space” (EHS) representation, is obtained by embedding the random variable \(X\) in some quantum system \(A\). Then our ensemble \(\{\rho_x, p(x)\}\) corresponds to the density operator

\[
\rho^A = \sum_x p(x)|x\rangle\langle x| \otimes \rho_x^Q,
\]

where \(\{|x\rangle: x \in \mathcal{X}\}\) is an orthonormal basis for the Hilbert space \(\mathcal{H}_A\) of \(A\). A classical-quantum system may, therefore, be viewed as a special case of a quantum one. The EHS representation is convenient for defining various information theoretical quantities for classical-quantum systems. The von Neumann entropy of a quantum system \(A\) with density operator \(\rho^A\) is defined as \(H(A) = -\text{Tr} \rho^A \log \rho^A\). For a bipartite quantum system \(AB\) define the conditional von Neumann entropy

\[
H(B|A) = H(AB) - H(A),
\]

and quantum mutual information

\[
I(A;B) = H(A) + H(B) - H(AB) = H(B) - H(B|A),
\]

in formal analogy with the classical definitions. Notice that for classical-quantum correlations \(E\) the von Neumann entropy \(H(A)\) is just the Shannon entropy \(H(X) = -\sum_x p(x) \log p(x)\) of \(X\). The conditional entropy \(H(Q|X)\) is defined as \(H(Q|A)\) and equals \(\sum_x p(x)H(\rho_x)\). Similarly, the mutual information of \(XQ\) is defined as \(I(X; Q) = I(A; Q)\). Notice that this is precisely the familiar Holevo information \(\chi\) of the ensemble \(E\):

\[
\chi(E) = H \left( \sum_x p(x)\rho_x \right) - \sum_x p(x)H(\rho_x).
\]

Returning to the formulation of the CQSW problem, suppose Alice and Bob share a large number \(n\) copies of the classical-quantum system \(XQ\). Alice possesses knowledge of the index \(x^n = x_1x_2 \ldots x_n\), but not the quantum system locally described by \(\rho_{x^n} = \rho_{x_1} \otimes \rho_{x_2} \cdots \otimes \rho_{x_n}\); Bob has the quantum system at his disposal but not the classical index. Note that this does
with respect to $\rho$ based on $I$ probability is required to be bounded. A rate (CQSW Theorem) converse. The “if” part of the proof is called the direct coding theorem, and the “only if” part is called the random variables $X_j$ of a large number of copies of $X$ that achieves this. We proceed to formally define the coding procedure. An $(n, \epsilon)$ CQSW code consists of

- a mapping $f : X^n \rightarrow [M]$, $[M] = \{1, 2, \ldots, M\}$, $M = 2^{nR}$, by which Alice encodes her classical message $X^n$ into the index $I = f(X^n)$;
- a set $\{\Lambda^{(1)}, \Lambda^{(2)}, \ldots, \Lambda^{(M)}\}$, where each $\Lambda^{(i)} = \{\Lambda_j^{(i)}\}$ is a POVM acting on $\mathcal{H}^{\otimes n}$ and taking values $j \in [N]$;
- a decoding map $g : [M] \times [N] \rightarrow X^n$ that provides Bob with an estimate $\hat{X}^n = g(I, J)$ of $X^n$ based on $I$ and the outcome $J$ of the POVM $\Lambda^{(I)}$ applied to Bob’s quantum system $Q^n$.

The rate $R$ signifies the number of bits per copy needed to encode the index $I$. The error probability is required to be bounded

$$P_e = \Pr\{\hat{X}^n \neq X^n\} \leq \epsilon.$$  

Denoting Bob’s residual state after the extraction of the classical information by $\hat{\rho}_{x^n}$, its disturbance with respect to $\rho_{x^n}$ must also be small on average

$$\Delta = \sum_{x^n} p(x^n) \|\hat{\rho}_{x^n} - \rho_{x^n}\|_1 \leq \epsilon.$$  

A rate $R$ is said to be achievable if for any $\epsilon, \delta > 0$ and all sufficiently large $n$ there exists an $(n, \epsilon)$ code of rate $R + \delta$. Our main result (which first appeared in [26]) is the following theorem.

**Theorem 1** (CQSW Theorem) *Given a classical-quantum system $X Q$, a rate $R$ is achievable iff*

$$R \geq H(X) - I(X; Q) = H(X|Q).$$

The “if” part of the proof is called the direct coding theorem, and the “only if” part is called the converse.

Let us first compare our result to the classical Slepian-Wolf problem. The latter is usually formulated as a three terminal problem. We are given two correlated sources described by the random variables $X$ and $Y$, known to Alice and Bob, respectively. They encode their sources separately and send them to Charlie at rates $R_1$ and $R_2$, respectively, who decodes them jointly with the aim to faithfully reconstruct $X$ and $Y$. One may now ask about the achievable rate **region**
Figure 1: The achievable rate region for the classical Slepian-Wolf problem.

(R₁, R₂). The answer is given by

\begin{align*}
R₁ &\geq H(X|Y) \\
R₂ &\geq H(Y|X) \\
R₁ + R₂ &\geq H(XY),
\end{align*}

as shown in figure 1. It suffices to show the achievability of the points (H(X), H(Y|X)) and (H(X|Y), H(Y)), since the rest of the region follows by time sharing (merging codes of length νn and (1 − ν)n, ν ∈ [0, 1], corresponding the two points, respectively). The obvious classical-quantum generalization of this result would be to replace Y by a quantum system Q, and the joint distribution of XY by the ensemble state

\[ \rho^{AQ} = \sum_{x,y} p(x,y)|x\rangle\langle x| \otimes \pi_y, \]

(4)

where π_y are density operators on Q, which for the sake of this discussion we assume to be pure. Observe that the state written here has the same form as in (2), with

\[ p(x) = \sum_y p(x,y), \]

\[ \rho_x = \frac{1}{p(x)} \sum_y p(x,y)\pi_y. \]

But here the description also contains the decomposition of ρ_x into pure states, i.e., a chosen ensemble.

The task of coding is, analogously to Schumacher’s theorem, to enable Charlie to reconstruct |x^n⟩⟨x^n| ⊗ π_y^n with high average fidelity, in a situation of many independent realizations of ρ^{AQ}. Indeed, Theorem 1 implies the achievability of the point (H(X,Q), H(Q)). Bob may Schumacher compress his quantum system and send it to Charlie at a qubit rate of R₂ = H(Q). The latter uses it as quantum side information, and Alice needs to send classical information to Charlie at a
bit rate of \( R_1 = H(X|Q) \). Furthermore, after having used the quantum system for this purpose, according to [13] it will remain basically intact. (Note that our proof of the direct coding theorem below actually shows that even the average disturbance of the \( \pi_y^n \) is small – in fact the decoder is such that it causes little disturbance to purifications of the \( \rho_x^n \).

As for the other point \( (H(X), H(Q|X)) \), we do not know to which extent the classical result carries over. There are, as in the above discussion, trivial examples where it is achievable. One example is perfect correlation, when \( p(x,y) \neq 0 \) if \( x = y \): then, knowing \( x \) one can perfectly reconstruct the pure state of \( Q \) because it has to be \( \pi_x \). So, \( R_1 = H(X) \), \( R_2 = 0 \) is achievable. Another is when \( X \) can be read off \( Q \), i.e. when the states \( \pi_y \) fall into mutually orthogonal classes \( \mathcal{Y}_x \) such that \( p(x,y) \neq 0 \) implies \( \pi_y \in \mathcal{Y}_x \). Then Alice can Shannon compress her \( x^n \), and Bob, since he can read \( x^n \) on his system, can Schumacher compress to a rate \( H(Q|X) \) (compare [11] and [12]).

Notice that there are two variants to the coding problem here: blind (where Bob has to operate on the \( \pi_y \)), and visible (where he is told \( y \)). Note that the labeling of the different ensembles for the \( \rho_x \) by the same set \( \mathcal{Y} \) is purely artificial – this is why there is more than one visible coding problem associated to the same ensemble. In particular, we cannot expect the answers to the visible and to the blind problem to be the same. Both however are open problems.

**Proof of Theorem 1 (converse)** We need to prove that, for any \( \delta, \epsilon > 0 \) and sufficiently large \( n \), if an \((n,\epsilon)\) code has rate \( R \) then \( R \geq H(X|Q) - \delta \). Without loss of generality, \( \epsilon \leq \frac{2}{nX} \) and \( n \geq 2/\delta \). We shall make use of two inequalities. The first is the Holevo bound [24], according to which the amount of information about \( X^n \) extractable from the quantum system \( Q^n \) is bounded from above by \( I(X^n;Q^n) = nI(X;Q) \). Recall that Bob makes an estimate \( \hat{X}^n = g(I,J) \) of \( X^n \) based on \( I = f(X^n) \) and the measurement outcome \( J \). Our second ingredient is Fano’s inequality [25]:

\[
H(X^n|IJ) \leq h_2(p) + P_x \log(|X|^n - 1).
\]

Here \( h_2(p) = -p \log p - (1-p) \log(1-p) \) is the binary entropy. This inequality is interpreted as: Given \( IJ \) one can specify \( X^n \) by saying whether or not it is equal to \( g(I,J) \) and, conditionally upon a negative answer, specifying which of the remaining \(|X|^n - 1\) values it has taken. We have

\[
nR + nI(X;Q) \\
\geq H(I) + I(X^n;J) \\
= H(X^n) + H(I|X^nJ) + I(I;J) - H(X^n|IJ) \\
\geq nH(X) - H(X^n|IJ) \\
\geq n \left( H(X) - \frac{1}{n} - \epsilon \log |X| \right).
\]

The first inequality follows trivially from \( I \in [2^nR] \) and Holevo’s theorem. The second comes from the non-negativity of mutual information and conditional entropy. The final one is a consequence of Fano’s inequality. Thus \( R \geq H(X) - I(X;Q) - \delta \), as claimed.

**Remark** An alternative way to demonstrate the converse uses a recent result on remote state preparation [10], according to which Alice and Bob may establish classical-quantum correlations \( XQ \) with asymptotically perfect fidelity using shared entanglement and forward classical communication at rates of \( H(Q) \) ebits and \( I(X;Q) \) bits, respectively. Let us assume that the converse fails, i.e. that it is possible to achieve a CQSW rate \( R < H(X|Q) \). Then with the help of shared entanglement she would be able to convey \( X \) at a classical rate strictly less than \( H(X) \), by first remotely preparing the quantum information then performing the CQSW protocol. We know, however, that entanglement can in no way increase the capacity of a classical channel, e.g. by [13].

**Remark** Note that the lower bound in Theorem 1 holds true even for CQSW codes which disregard condition [25]. We invite the reader to confirm that in the proof of the converse it was never used.
Before launching into the proof of achievability we give a heuristic argument. Let us recall typical sequences (see [23] for an extensive discussion) and subspaces [1] and their properties. The theorem of typical sequences states that given random variable $X$ defined on a set $\mathcal{X}$ and with probability distribution $p(x)$, for any $\epsilon, \delta > 0$ and sufficiently large $n > n_0(|\mathcal{X}|, \epsilon, \delta)$ there exists a typical set $T_{X, \delta} \subset \mathcal{X}^n$ of sequences $x^n$ such that

$$2^{n[H(X) - \delta]} \leq |T_{X, \delta}| \leq 2^{n[H(X) + \delta]},$$

and $\Pr\{X^n \in T_{X, \delta}\} \geq 1 - \epsilon$. Typical sequences are those in which the fraction of a given letter $x$ is approximated by its probability $p(x)$, and the law of large numbers guarantees that such sequences will occur with high probability. Thus one need worry only about encoding typical sequences. The quantum analogue of the typical set is the typical subspace $T_{Q, \delta}$ of $\mathcal{H}^\otimes n$, defined for a quantum system $Q$ with $d$-dimensional Hilbert space $\mathcal{H}$ and in a quantum state $\rho$. It satisfies

$$2^{n[H(Q) - \delta]} \leq \dim T_{Q, \delta} \leq 2^{n[H(Q) + \delta]},$$

and $\Tr(\rho^\otimes n \Pi_{Q, \delta}) \geq 1 - \epsilon$, where $\Pi_{Q, \delta}$ is the projector onto $T_{Q, \delta}$. Finally, for a classical quantum system $XQ$ and a particular sequence $x^n \in T_{X, \delta}$ we define the conditionally typical subspace $T_{Q|X, \delta}(x^n)$ in the following way. The Hilbert space $\mathcal{H}^\otimes n$ can be decomposed into a tensor product $\bigotimes_{x} \mathcal{H}_x$ with $\mathcal{H}_x$ collecting all the factors $k$ such that $x_k = x$. Then the conditionally typical subspace is the tensor product of the typical subspaces of the $\mathcal{H}_x$ with respect to $\rho_x$. It follows that

$$2^{n[H(Q|X) - \delta]} \leq \dim T_{Q|X, \delta}(x^n) \leq 2^{n[H(Q|X) + \delta]},$$

for some constant $K$. At the same time $\Tr(\rho_x \Pi_{Q|X, \delta}(x^n)) > 1 - |X|\epsilon$, where $\Pi_{Q|X, \delta}(x^n)$ is the projector onto $T_{Q|X, \delta}(x^n)$. The latter means that the trace decreasing measurement given by $\Pi_{Q|X, \delta}(x^n)$ will succeed with high probability when applied to the state $\rho_x^n$. One would like to construct a POVM out of such conditionally typical projectors for different $x^n$ belonging to some set $\mathcal{C}$, in order to distinguish between them. Since the $T_{Q|X, \delta}(x^n)$ are approximately contained in $T_{Q, \delta}$, the task is, roughly speaking, to “pack” the $T_{Q|X, \delta}(x^n), x^n \in \mathcal{C}$ into the typical subspace $T_{Q, \delta}$. The former have dimension $\approx 2^{n[H(Q|X)]}$ and the latter has dimension $\approx 2^n H(Q)$, hence one expects $|\mathcal{C}|$ to be at most $\approx 2^{n[H(Q|X)]} - H(Q|X)} = 2^n I(X:Q)$. This is the basic content of the HSW, or $\{c \rightarrow q\}$ channel coding, theorem [5], although the actual POVM construction is rather more subtle. Accordingly, $\mathcal{C}$ is called a channel code. Here we take one step further and ask about the minimal number of disjoint channel codes that “cover” the typical input set $T_{X, \delta}$. The size of $T_{X, \delta}$ is $\approx 2^n H(X)$, so the number of codes needed should be $\approx 2^n [H(X) - I(X:Q)]$. Now Alice need only send information about which code her source sequence $x^n$ belongs to, and Bob can perform the appropriate measurement to distinguish it from the other sequences in that code, as in the one-shot BB84 example. The described construction is depicted in figure 2.

To prove Theorem 1 we shall need some background on channel codes. For a classical-quantum system $XQ$, a channel code $\mathcal{C}$ is a subset of $\mathcal{X}^n$, associated with a POVM $\Lambda = \{\Lambda_{x^n} : x^n \in \mathcal{C}\}$ acting on $\mathcal{H}^\otimes n$. The rate of the channel code is defined as $r = \frac{1}{n} \log |\mathcal{C}|$. The error probability of a given $x^n \in \mathcal{C}$ is $p_e(x^n) = 1 - \Tr(\rho_x \Lambda_{x^n})$. $\mathcal{C}$ is called an $(n, \epsilon)$ code if $\max_{x^n \in \mathcal{C}} p_e(x^n) \leq \epsilon$. We shall need the following version of the $\{c \rightarrow q\}$ channel coding theorem [22]:

**Theorem 2** (Winter [22], Theorem 10) For all $\eta, \epsilon, \delta \in (0, 1)$, sufficiently large $n \geq n_1(|\mathcal{X}|, d, \eta, \epsilon, \delta)$ and every subset $A \in \mathcal{X}^n$ with $\Pr\{x^n \in A\} \geq \eta$, there exists an $(n, \epsilon)$ channel code $\mathcal{C}$ of rate $r \geq I(X;Q) - \delta$ satisfying $\mathcal{C} \subset A$.

The $\mathcal{C} \subset A$ condition is sufficiently strong to easily yield the achievability part of the CQSW theorem, following a standard classical argument of Csiszár and Körner [23].
Proof of Theorem 1 (coding) Fixing $0 < \epsilon < \frac{1}{2}$ and $\delta > 0$ we shall first show that for sufficiently large $n$ there exists a family of disjoint channel codes $\{C_1, C_2, \ldots, C_{M-1}\}$ such that

$$\Pr\{x^n / \not\in \bigcup_{i=1}^{M-1} C_i\} \leq 2\epsilon$$

and $\frac{1}{n} \log M \leq H(X|Q) + 2\delta$, thus upper bounding the number of channel codes needed to cover most of the high probability sequences. Recall that for $n \geq n_0(|X|, \epsilon, \delta)$ we have $\Pr(X^n \in T_\delta(X)) > 1 - \epsilon$. By Theorem 2 we also have that for $n \geq n_1(|X|, d, \eta, \epsilon, \delta)$ and every subset $A \in X^n$ with $\Pr\{x^n \in A\} \geq \epsilon$ there exists an $(n, \epsilon)$ code of rate $r \geq I(X; Q) - \delta$ satisfying $C \subset A$. We choose $n \geq \max\{n_0, n_1\}$ so that both conditions are satisfied. The idea is to keep constructing disjoint codes from $T_{X,\delta}$ for as long as Theorem 2 allows. Define $A_1 = T_{X,\delta}$, and let $C_1 \subset A_1$ be an $(n, \epsilon)$ code as specified by Theorem 2. Recursively construct in a similar manner $C_i \subset A_i$ where $A_i = T_{X,\delta} - \bigcup_{j=1}^{i-1} C_j$, which will also satisfy the conditions of the theorem as long as $\Pr\{x^n \in A_i\} \geq \epsilon$. Suppose the construction stops at $i = M$, i.e. $\Pr\{x^n \in A_M\} \leq \epsilon$. Then we have

$$\Pr\{x^n / \not\in \bigcup_{i=1}^{M-1} C_i\} = \Pr\{X^n / \not\in T_{X,\delta}\} + \Pr\{x^n \in A_M\} \leq 2\epsilon. \quad (5)$$

On the other hand

$$2^{n[H(X) + \delta]} \geq |T_{X,\delta}| \geq \sum_{i=1}^{M-1} |C_i| \geq (M - 1) 2^{n[H(X; Q) - \delta]},$$
which implies

\[ R = \frac{1}{n} \log M \leq H(X) - I(X; Q) + 2\delta. \]

The mapping \( f \) is now defined as

\[ f(x^n) = \begin{cases} i & x^n \in C_i \\ M & \text{otherwise} \end{cases} \]

The latter case, which signifies an encoding error, happens with probability \( \leq 2\epsilon \) by \( \[8\] \). Otherwise, Bob performs the POVM corresponding to the code \( C_{f(x^n)} \), which fails to correctly identify \( x^n \) with probability \( \leq \epsilon \). Therefore the total error probability is bounded \( P_e \leq 3\epsilon \). Finally, Winter’s “gentle measurement” lemma \( \[7\] \), which states that a POVM with a highly predictable outcome on a given state cannot disturb it much, guarantees that the average disturbance \( \Delta \) is bounded by \( \sqrt{8\epsilon + \epsilon} \). The direct coding theorem follows. \( \blacksquare \)

**Remark** The “gentle measurement” lemma invoked at the end of the proof actually applies equally if the measurement acts on one half of a purification of the state \( \rho_{x^n} \) – a fact we needed in the discussion of the Slepian-Wolf theorem after the statement of Theorem 1.

Finally, we would like to comment on a connection to Winter’s measurement compression theorem \( \[8\] \). Suppose Bob needs to perform a “BB84” measurement given by the operation elements \( \{\frac{1}{\sqrt{2}}|0\rangle, \frac{1}{\sqrt{2}}|1\rangle, \frac{1}{\sqrt{2}}|+\rangle, \frac{1}{\sqrt{2}}|\rangle\} \) on a quantum system described by the uniform density matrix. He would then need 2 classical bits of communication to convey the outcome to Alice. Equivalently, he can use 1 bit of shared randomness between him and Alice to decide which of the two measurements \( \{0\rangle \langle 0|, 1\rangle \langle 1|\} \) or \( \{|+\rangle \langle +|, |\rangle \langle \|\} \) he should perform, and send her only 1 bit describing the outcome. He has thus perfectly *simulated* the measurement, replacing 1 bit of communication with the weaker resource of 1 bit of shared randomness.

For a general source-POVM pair \( (\rho, \Lambda) = \{\Lambda_x\} \), define the classical system \( XQ \) by the ensemble \( \{\rho_x, p(x)\} \), given by \( \[1\] \); in other words \( XQ \) embodies the correlations between the measurement outcome \( X \) (to be sent to Alice) and the reference system \( Q \) that purifies the system to be measured. In an asymptotic and approximate setting, the measurement \( \Lambda^{\otimes n} \) is considered well simulated on \( \rho^{\otimes n} \) if the classical-quantum correlations established between Alice and Bob’s reference system closely resemble \( n \) copies of \( XQ \). It was shown in \( \[5\] \) that the optimal classical communication and shared randomness rates become \( I(X; Q) \) and \( H(X|Q) \) respectively. It is not surprising that the minimal amount of classical communication required to establish a remote classical-quantum correlation is given by the corresponding Holevo information. Achievability of this bound may be described by a diagram similar to the one depicted in figure 2, with the difference that “PACKING” should be replaced by “COVERING”. The idea is to divide the set of typical outcome sequences \( T_{X, \delta} \) into codes \( C_i, i \in [M] \), such that \( \{\rho_{x^n} : x^n \in C_i\} \) mimics the set of residual states of the reference system after performing some measurement \( \Lambda^{(i)} \). Thus \( |C_i| \) must be sufficiently large to allow

\[ \sum_{x^n \in C_i} p(x^n)\rho_{x^n} \approx \text{const.} \times \rho^{\otimes n}, \quad \forall i \in [M]. \]

Since \( \rho^{\otimes n} \) and \( \rho_{x^n} \) are “almost” uniformly supported on \( T_{Q, \delta} \) and \( T_{Q|X, \delta}(x^n) \), respectively, \( \[22\] \), dimension counting arguments again suggest \( |C_i| \geq 2^{nI(X; Q)} \); moreover \( M \geq 2^{nI(X|Q)} \) as before. \( \Lambda^{\otimes n} \) is then simulated by randomly choosing one of the \( \Lambda^{(i)} \), as in the BB84 example.

Coding with side information is a relatively unexplored and potentially rich area of quantum information theory. We have presented here an important member of this class of problems, providing yet another example of classical Shannon theory generalizing to the quantum domain.

Our main result can be understood as the translation of one of the two extremal rate points of the classical Slepian-Wolf region to a classical-quantum scenario. The main question left open is whether one can also translate the other rate point; it might actually be that the rate region in the quantum case does not look like figure \( \[11\] \).
Acknowledgments

We are indebted to Debbie Leung for comments on an earlier version of the manuscript. Thanks also go to Charles Bennett, Toby Berger, Andrew Childs, Patrick Hayden, Aram Harrow, Luis Lastras, Anthony Ndirango and John Smolin for useful discussions.

ID is supported in part by the NSA under the US Army Research Office (ARO), grant numbers DAAG55-98-C-0041 and DAAD19-01-1-06. AW is supported by the U.K. Engineering and Physical Sciences Research Council.

References

[1] B. Schumacher, Phys. Rev. A 51, 2738 (1995); R. Jozsa and B. Schumacher, J. Mod. Opt. 41, 2343 (1994)
[2] C. H. Bennett, H. J. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. A 53, 2046 (1996)
[3] H.-K. Lo and S. Popescu, Phys. Rev. Lett. 83, 1459 (1999)
[4] P. Hayden and A. Winter, quant-ph/0204092 (2002); A. W. Harrow and H.-K. Lo, quant-ph/0204096 (2002)
[5] A. S. Holevo, IEEE Trans. Inf. Theory 44, 269 (1998); B. Schumacher and M. D. Westmoreland, Phys. Rev. A 56, 131 (1997)
[6] I. Devetak and A. Winter, quant-ph/0304196 (2002)
[7] A. Winter, IEEE Trans. Inf. Theory 47, 3059 (2001)
[8] A. Winter, quant-ph/0109050 (2001)
[9] A. K. Pati, Phys. Rev. A 63, 14302 (2001); H.-K. Lo, quant-ph/9912009 (1999); C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, P. W. Shor, B. M. Terhal and W. K. Wootters, Phys. Rev. Lett. 87, 77902 (2001); D. W. Leung and P. W. Shor, quant-ph/0201008 (2002)
[10] C. H. Bennett, P. Hayden, D. W. Leung, P. W. Shor and A. Winter, quant-ph/0307100 (2003)
[11] I. Devetak and T. Berger, Phys. Rev. Lett. 87, 197901 (2001)
[12] P. Hayden, R. Jozsa and A. Winter, J. Math. Phys. 43, 4404 (2002)
[13] C. H. Bennett, P. W. Shor, J. A. Smolin and A. V. Thapliyal, IEEE Trans. Inf. Theory 48, 2637 (2002)
[14] C. H. Bennett, I. Devetak, A. W. Harrow, P. W. Shor and A. Winter, “The Quantum Reverse Shannon Theorem”, in preparation
[15] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992)
[16] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. 70, 1885 (1993)
[17] P. W. Shor, “The quantum channel capacity and coherent information ”, lecture notes, MSRI Workshop on Quantum Computation, (2002). Available at http://www.msri.org/publications/in/msri/2002/quantumcrypto/shor/1/
[18] I. Devetak, quant-ph/0304127 (2003)
[19] D. Slepian and J. K. Wolf, IEEE Trans. Inf. Theory 19, 471 (1973)
[20] L. P. Hughston, R. Jozsa and W. K. Wootters, Phys. Lett. A 183, 14 (1993)

[21] C. H. Bennett and G. Brassard, Proc. IEEE Int. Conf. Computers, Systems and Signal Processing (Bangalore, India), 175 (1984)

[22] A. Winter, IEEE Trans. Inf. Theory 45, 2481 (1999)

[23] I. Csiszár and J. Körner, Information Theory: Coding Theorems for Discrete Memoryless Systems, Academic Press, New York (1981)

[24] A. S. Holevo, Probl. Inf. Transm. 9, 177 (1973)

[25] T. M. Cover and J. A. Thomas, Elements of Information Theory, Wiley and Sons, New York (1991)

[26] A. Winter, Ph.D. thesis, quant-ph/9907077 (1999)