Cartesian modules in small categories

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Abstract
In this note we extend the main results of [2] to the category of cartesian modules over a flat
presheaf of rings $R$ and on an arbitrary small category. This provides with new applications of that
paper to the categories of quasi–coherent sheaves on an Artin stack or on a Deligne-Mumford stack.

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1 Introduction

In [2] we develop a method for finding a family of generators of the so-called category of quasi-coherent $R$-
modules on an arbitrary quiver (cf. [2, Corollary 3.5]) and we prove that the class of flat quasi–coherent
$R$-modules is covering (cf. [2 Theorem 4.1]). The present note is devoted to showing that the same
arguments of [2] can also be used in a much more general setup, that is that of cartesian $R$-modules on a
flat presheaf of rings $R$ over a small category $\mathcal{C}$. This extends the main application in [2] to the category
\(\mathcal{QCoh}(X)\) of quasi–coherent sheaves on a scheme \(X\) and also to the category \(\mathcal{QCoh}(X')\) of quasi–coherent sheaves on an Artin stack or on a Deligne-Mumford stack.

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2 Cartesian modules on quivers

A quiver \(Q\) is a directed graph. An edge of a quiver from a vertex \(v_1\) to a vertex \(v_2\) is denoted by \(a : v_1 \to v_2\) or \(v_1 \xrightarrow{a} v_2\), the symbol \(E\) will denote the set of edges. A quiver \(Q\) may be thought as a category in which the objects are the vertices of \(Q\) and the morphisms are the paths (a path is a sequence of edges) of \(Q\). The set of all vertices will be denoted by \(V\).

Let \(Q = (V, E)\) be a quiver and let \(R\) be a presheaf from \(Q\) in the category of commutative rings, that is, for each vertex \(v \in V\) we have a ring \(R(v)\) and for an edge \(a : v \to w\) we have a ring homomorphism \(R(a^{op}) : R(w) \to R(v)\).

We shall say that we have an \(R\)-module \(M\) when we have an \(R(v)\)-module \(M(v)\) and a morphism \(M(a^{op}) : M(w) \to M(v)\) for each edge \(a : v \to w\) that is \(R(v)\)-linear. The \(R\)-module \(M\) is said to be a cartesian \(Q\)-module if for each edge \(a : v \to w\) as above the morphism

\[R(v) \otimes_{R(w)} M(w) \to M(v)\]

given by \(r_v \otimes m_w \mapsto r_v M(a^{op})(m_w), \ r_v \in R(v), \ m_w \in M(w)\) is an \(R(v)\)-isomorphism.

The category of cartesian \(Q\)-modules is abelian when \(R\) is such that for an edge \(v \to w\), \(R(v)\) is a flat \(R(w)\)-module (so the kernel of a morphism between two cartesian \(Q\)-modules is also cartesian). In this case we say \(R\) is flat. Coproducts and colimits may be computed componentwise so direct limits are exact and, as a result of Proposition 4.2, we can find a system of generators in the category. Therefore, the category of cartesian \(Q\)-modules is indeed a Grothendieck category when \(R\) is flat.

By the tensor product, \(M \otimes_R N\), where \(M\) is a right \(R\)-module and \(N\) a left \(R\)-module, we mean the
\(\mathbb{Z}\)-module \((\mathbb{Z}(v) = \mathbb{Z}\), for all \(v \in V\) and \(\mathbb{Z}(a) = id_\mathbb{Z}\) for all \(a \in E\)) such that

\[(M \otimes_R N)(v) = M(v) \otimes_R N(v),\]

with \((M \otimes_R N)(a)\) the obvious map. We then get the notion of a flat \(R\)-module and a flat cartesian \(Q\)-module.

Given an arbitrary quiver \(Q\) and a flat presheaf of rings \(R\) over \(Q\), we will denote by \(Q\text{Mod}_{\text{cart}}(R)\) the category of cartesian \(Q\)-modules over \(R\).

### 3 Cartesian modules on small categories

Now let \(\mathcal{C}\) be any small category, and let \(R\) be a flat presheaf of rings on \(\mathcal{C}\). We will consider the category \(\text{Mod}_{\text{cart}}(R)\) of cartesian \(R\)-modules. This is an abelian category and, as a consequence of Proposition 4.2, it will be a Grothendieck category. There is a notion of flat cartesian module as before. Let \(Q\) be the quiver whose vertices are the objects of \(\mathcal{C}\), and whose edges are the morphisms of \(\mathcal{C}\). It is then clear that the category \(\text{Mod}_{\text{cart}}(R)\) is a full subcategory of the category \(Q\text{Mod}_{\text{cart}}(R)\). Furthermore it is also clear that if \(M \subseteq N\) in \(Q\text{Mod}_{\text{cart}}(R)\) and \(N\) is a cartesian \(R\)-module, then \(M\) will be automatically a cartesian \(R\)-module as well. This easy observation is crucial in proving our main result and giving our main applications.

### 4 Generators and flat covers in \(\text{Mod}_{\text{cart}}(R)\)

With the observations made in the previous sections, we can use both Proposition 3.3 and Theorem 4.1 of \([2]\) to infer that \(\text{Mod}_{\text{cart}}(R)\) is a Grothendieck category admitting flat covers. Throughout this section we will assume that \(\mathcal{C}\) is a small category and \(R\) is a flat presheaf of rings on it. We shall denote by \(Q\) the quiver associated to \(\mathcal{C}\).

**Definition 4.1.** Let \(M\) be a cartesian \(Q\)-module. The cardinality of \(M\) is defined as the cardinality of the coproduct (in the category of sets) of all modules associated to the vertices \(v \in V\), that is

\[|M| = \big| \sqcup_{v \in V} M(v) \big|\]
Proposition 4.2. Let \( \mathcal{C} \) be any small category with associated quiver \( Q_\mathcal{C} = (V, E) \) and \( M \) a cartesian \( R \)-module. Let \( \kappa \) be an infinite cardinal such that \( \kappa \geq |R(v)| \) for all \( v \) and such that \( \kappa \geq \max\{|E|, |V|\} \). Let \( X_v \subseteq M(v) \) be subsets with \( |X_v| \leq \kappa \) for all \( v \). Then there is cartesian \( R \)-submodule \( M' \subseteq M \) with \( M'(v) \) pure for all \( v \), with \( X_v \subseteq M'(v) \) for all \( v \) and such that \( |M'| \leq \kappa \).

Proof. The proof of [2, Proposition 3.3] gives a cartesian \( Q \)-submodule \( M' \) of \( M \) satisfying the desired properties. But then by the previous comment, as \( M \) is cartesian, \( M' \) will be also cartesian \( R \)-module. \( \square \).

Definition 4.3. A cartesian \( R \)-submodule \( M' \) of an \( R \)-module \( M \) is said to be pure whenever \( M'(v) \) is a pure \( R(v) \)-submodule of \( M(v) \), for every vertex \( v \in V \).

Corollary 4.4. There exists an infinite cardinal \( \kappa \) such that every cartesian \( R \)-module \( M \) is the sum of its quasi-coherent \( R \)-submodules of type \( \kappa \).

Proof. Let \( M \) be any cartesian \( R \)-module and take an element \( x \in M \). Then, by Proposition 4.2 we find a (pure) cartesian \( R \)-submodule \( S_x \) of \( M \) with \( |S_x| \leq \kappa \) and \( x \in S_x \). Thus \( M = \sum x \in M S_x \). \( \square \)

As a consequence of this we have that \( \text{Mod}_{\text{cart}}(R) \) is a Grothendieck category whenever \( R \) is a flat presheaf of rings, for if we take a set \( Z \) of representatives of cartesian modules with cardinality bounded by \( \kappa \), it is immediate that the single cartesian \( R \)-module \( \oplus_{S \in Z} S \) generates the category of quasi-coherent \( R \)-modules.

Now if we focus on particular instances of small categories we have the following significant consequences.

Corollary 4.5. Let \((X, \mathcal{O}_X)\) be any arbitrary scheme. Each quasi-coherent sheaf can be written as a continuous chain of pure quasi-coherent subsheaves of type \( \kappa \). Thus \( \mathcal{Qcoh}(X) \) is a Grothendieck category.

Proof. We let \( \mathcal{C} \) consisting of all the affine open \( U \subseteq X \). Then the inclusion between affine open subsets defines a canonical structure of a partially ordered category on \( \mathcal{C} \). Now we let \( R \) be the structure
sheaf $\mathcal{O}_X$. Then it is standard that $\text{Mod}_{\text{cart}}(R)$ and $\mathfrak{Qcoh}(X)$ are equivalent categories. So the result will follow from Corollary 4.4. 

\textbf{Remark.} The previous proof also clarifies a possible misunderstanding on [2, Section 2]. There, the reader may wrongly think that we are considering the free category on the affine open subsets of the scheme $X$ to establish our equivalent category $\mathcal{C}$. This is obviously not true, and the gap is easily fixed by saying that we were assuming the compatibility condition on our representations there to get the desired equivalence. To be precise we are just claiming that $\mathfrak{Qcoh}(X)$ and $\text{Mod}_{\text{cart}}(R)$ (or $\mathcal{C}$ in that section) are equivalent.

Our second application goes back to Artin stacks (cf. [3]) and Deligne-Mumford stacks.

\textbf{Corollary 4.6.} Let $\mathcal{X}$ be a Deligne-Mumford stack. Then the category $\mathfrak{Qcoh}(\mathcal{X})$ is a Grothendieck category. In particular it is locally presentable and has arbitrary products.

\textbf{Proof.} We take $\mathcal{C}$ as the small subcategory of the iso classes of the category of affine schemes that are étale over $\mathcal{X}$ (such small subcategory must exist as the iso classes of such schemes form a set, as étale morphisms are of finite type). Then $\text{Mod}_{\text{cart}}(R)$ is equivalent to $\mathfrak{Qcoh}(\mathcal{X})$. 

\textbf{Corollary 4.7.} Let $\mathcal{X}$ be an algebraic stack with a flat sheaf of rings $A$. Then the category $\mathfrak{Qcoh}(\mathcal{X})$ is a Grothendieck category. In particular it is locally presentable and has arbitrary products.

\textbf{Proof.} In this case we consider $\mathcal{C}$ to be the category of affine schemes smooth over $\mathcal{X}$ and $R$ as the sheaf of rings $A$. Then $\text{Mod}_{\text{cart}}(R)$ is equivalent to the category of quasi-coherent sheaves on $\mathcal{X}$. 

\section{Flat covers and cotorsion envelopes}

\textbf{Theorem 5.1.} Let $\mathcal{C}$ be any small category and let $R$ be a flat presheaf over $\mathcal{C}$. Then $\text{Mod}_{\text{cart}}(R)$ admits flat covers and cotorsion envelopes.

\textbf{Proof.} Use the same proof as in Theorem 4.1 of [2]. 

\textbf{Corollary 5.2.} If $X$ is any scheme, the category $\mathfrak{Qcoh}(X)$ admits flat covers and cotorsion envelopes.
**Proof.** Let $M$ be a quasi-coherent sheaf over $X$. We have $\mathcal{Qcoh}(X)$ equivalent to the category $\text{Mod}_{\text{cart}}(R)$ of cartesian $R$-modules where $\mathcal{C}$ comes from all open affine subsets of $X$ and where $R$ comes from the structure sheaf $\mathcal{O}_X$. From the definition of the flat objects in the two categories we see that the equivalence functors (in both directions) preserve flatness. Then also since the functors are clearly additive and exact we get that the property of being cotorsion is also preserved (since cotorsion is defined in terms of the splitting of certain short exact sequences). So then thinking of $M$ as a cartesian $R$-module, $M$ has a flat cover and a cotorsion envelope in the category of cartesian $R$-modules. The equivalence then gives the desired flat cover and cotorsion envelope of $M$ in $\mathcal{Qcoh}(X)$. $\square$

Similarly we have the following consequence:

**Corollary 5.3.** Let $X$ be a Deligne-Mumford stack (or an Artin stack). The category $\mathcal{Qcoh}(X)$ admits flat covers and cotorsion envelopes.

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