Quantum Two-Stream Instability with Exchange Interaction

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H.-J. Kull
Institute for Theory of Statistical Physics, Laser Physics Group, RWTH Aachen University, 52056 Aachen, Germany
E-mail: kull@ilt-extern.fraunhofer.de

Abstract. Quantum plasmas can be described in the framework of the self-consistent field approach by a set of time-dependent Schrödinger equations (TDSE) for the electrons in a self-consistent potential. This approach is also well-known as the multistream model of quantum plasmas. The self-consistent field is commonly calculated in the Vlasov approximation from the average charge density of all electrons thereby neglecting the quantum-mechanical exchange interaction. In the present work, we consider a linear Hartree-Fock model with screened Coulomb interactions to examine the role of the exchange interaction. The two-stream instability is analysed and results for the Vlasov, Hartree and Hartree-Fock approximations are compared in detail. It is found that the exchange interaction considerably modifies the stability behavior for weakly screened Coulomb interactions. In particular, crossings of the two branches of the dispersion relation lead to oscillating unstable solutions that are not present in the absence of the exchange interaction.

1. Introduction
The study of strong-field laser-atom interactions has led to increased interest in the solution of the time-dependent Schrödinger equation (TDSE) for single-electron and multi-electron atoms. For many-electron problems, on the other hand, direct solution methods become computationally very expensive, although they are presently explored in the fields of warm dense matter and high-energy density plasmas. Another standard approach to many-electron systems is the self-consistent field method. Here the electron-electron interaction is replaced by an effective interaction of each electron with the medium. Variants of this approach are the Vlasov-theory, the Hartree-theory and the density functional theory (DFT). An approximate formulation of the quantum Vlasov theory is provided by the multistream model [1–5], which is a quantum-mechanical version of the classical beam model by Dawson [6]. Here the single-particle statistical operator is approximated by an ensemble of representative quantum states. The numerical solution of the TDSE for representative states with an effective self-consistent potential is feasible in many cases. Using this method, we have recently studied the linear [7] and nonlinear dynamics [5] of quantum plasmas in the weak-coupling regime. The reduced single- and two-stream models have also attracted considerable attention in the past [3; 8–11].

In this work, we wish to discuss the role of the exchange interaction in this context, which has mostly been ignored in previous analyses. General discussions on the exchange corrections in degenerate electron gases can be found in [12] and [13]. It is well-known that the wavefunction of identical particles has to satisfy the postulate of symmetry for bosons and of antisymmetry for
fermions. In the simplest case the electronic wavefunction can be chosen as a Slater determinant and by minimizing a variational principle for the TDSE, the familiar Hartree-Fock equations are obtained [14; 15]. These equations include the exchange interaction, which is not present in the Vlasov or the Hartree approximations and is only treated on the average in most DFT formulations. To explore the role of the exchange term, we study a simple analytically solvable model of the two-stream instability. It is our goal, to compare for this particular case in detail the Vlasov, the Hartree and the Hartree-Fock approaches to gain more insight in the role of the exchange interaction in multistream plasma treatments.

2. Two-Stream Model
In this section, we briefly summarize the Hartree-Fock equations of the multistream-model of quantum plasmas. We then restrict attention to the two-stream model and derive the basic equations for small amplitude perturbations. These equations will be further considered in the subsequent section to discuss the two-stream instability under various conditions.

2.1. Multi-Stream Equations
We consider an infinitely extended plasma with a homogeneous ion density. The multistream model of the electrons consists of an ensemble of \( N \) carrier-envelope waves with envelopes \( A_s(r, t) \), carrier momenta \( p_s \) and carrier energies \( E_s \),

\[
\psi_s(r, t) = A_s(r, t)e^{i\phi_s}, \quad \varphi_s = \frac{1}{\hbar}(p_s \cdot r - E_s t), \quad s = 1, 2, 3, \ldots, N,
\]

where \( \hbar \) denotes the reduced Planck constant. These wavefunctions are assumed to be mutually orthonormal. The spin dependence of the streams is omitted for simplicity, since we will only treat an electron-pair with parallel spins and neglect any spin interactions. Using a Slater determinant of single-electron wavefunctions as a trial-function in the variational formulation of the Schrödinger equation, one arrives at the set of time-dependent Hartree-Fock equations,

\[
i\hbar \partial_t \psi_s = -\frac{\hbar^2}{2m} \Delta \psi_s + q\phi_i \psi_s + q \sum_r (\phi_{rr} \psi_s - \phi_{rs} \psi_r),
\]

where \( \phi_i \) denotes the potential produced by the ions, \( q \) is the charge, \( m \) the mass of the electron, and the electron-electron interaction is described by a set of effective potentials \( \phi_{rs} \). The diagonal terms \( \phi_{rr} \) describe the classical direct interaction, while the off-diagonal terms \( \phi_{rs} \), \( r \neq s \) correspond to the quantum exchange interaction. It can be seen that the self-interaction for \( r = s \) is excluded since the direct term is canceled by the exchange term. We also mention that the Hartree-Fock theory is a generalization of simpler approaches. In the Vlasov theory the exchange interaction is absent and the direct interaction commonly includes the self-interaction. In the Hartree theory the direct interaction is calculated by omitting the self-interaction. In this sense, the Hartree theory accounts for the \( r = s \) exchange term, while all other exchange terms are omitted.

The effective potentials can be expressed by well-known integral representations or, equivalently, by a set of Poisson equations,

\[
\Delta \phi_{rs} - k_0^2(1 - \delta_{rs})\phi_{rs} = -4\pi q n_{rs}, \quad n_{rs} = \psi^*_r \psi_s.
\]

For \( r \neq s \), we have added to the standard Poisson equation a second term on the l.h.s, which corresponds to the screening of the Coulomb interaction in the Thomas-Fermi approximation. Screening is an important mechanism that competes with the exchange-interaction in plasmas. We can take the Thomas-Fermi wavenumber \( k_0 \) just as a parameter to control the strength of the
A self-consistent treatment of screening within the Hartree-Fock theory is beyond the scope of this work.

In the following, we will use dimensionless quantities. With the density \( n_0 \) of the positive elementary charges and the plasma frequency \( \omega_p = \sqrt{4\pi q^2 n_0 / m} \) in CGS-Gaussian units, the dimensionless variables are defined by

\[
\tilde{\psi}_s = \frac{1}{\sqrt{n_0}} \psi_s, \quad \tilde{\phi}_{rs,i} = \frac{|q|}{\hbar \omega_p} \phi_{rs,i}, \quad \tilde{r} = \sqrt{\frac{m \omega_p}{\hbar}} r, \quad \tilde{k}_0 = k_0 \sqrt{\frac{\hbar}{m \omega_p}}, \quad \tilde{t} = \omega_p t.
\]

(4)

Omitting the tilde-symbol, the dimensionless Hartree-Fock and Poisson equations become

\[
i\partial_t \psi_s = -\frac{1}{2} \Delta \psi_s - \phi_s \psi_s - \sum_r \phi_{rr} \psi_s - \phi_{rs} \psi_r,
\]

\[
\Delta \phi_{rs} - k_0^2 (1 - \delta_{rs}) \phi_{rs} = n_{rs}, \quad n_{rs} = \psi_r^* \psi_s.
\]

(5a) \hspace{1cm} (5b)

In the present units, the wavefunctions are normalized by

\[
\langle \psi_r | \psi_s \rangle = \int d^3r \ n_{rs} = \frac{1}{n_0} \delta_{rs}.
\]

(5c)

### 2.2. Two-Stream Equilibrium

In this work, we will restrict attention to an analytically solvable idealized two-electron system. Let us consider two electron streams, occupied by just a pair of electrons of equal spin such that the wavefunction of both electrons has to be antisymmetric. The Hartree-Fock equations for the two streams then become a system of two coupled Schrödinger equations,

\[
i\partial_t \psi_1 = -\frac{1}{2} \Delta \psi_1 - (\phi_1 + \phi_{22}) \psi_1 + \phi_{21} \psi_2,
\]

\[
i\partial_t \psi_2 = -\frac{1}{2} \Delta \psi_2 - (\phi_1 + \phi_{11}) \psi_2 + \phi_{12} \psi_1.
\]

(6a) \hspace{1cm} (6b)

We first consider the equilibrium state of this system and specifically assume plane-wave solutions as given in (1) with equal and opposite momenta

\[
\psi_1^{(0)} = e^{i \varphi_1}, \quad \psi_2^{(0)} = e^{i \varphi_2}, \quad \mathbf{p}_{1,2} = \pm \mathbf{p}.
\]

(7)

This symmetrically counterstreaming configuration is time-reversible and as a consequence the 4th-order dispersion relation for the excitation frequency \( \omega \) will become a simpler quadratic equation for \( \omega^2 \). The charge density of each electron, \(-|\psi_{1,2}^{(0)}|^2 = -1\), is homogeneously distributed and is taken equal and opposite to the charge density +1 of the ion. The direct interaction of each electron with the other one is thereby canceled by the ion interaction,

\[
\phi_{11}^{(0)} = \phi_{22}^{(0)} = -\phi_1.
\]

(8)

The whole plasma system has one excess negative charge. This excess charge will become of minor importance for multistream models with many electrons. In a strictly neutral equilibrium, the equilibrium wavefunctions would be more complicated due to the presence of the binding potential of the ion background. Inserting (7) into the Poisson equation, one obtains the exchange potentials

\[
\phi_{21}^{(0)} = \phi_{12}^{(0)*} = -\frac{1}{|\mathbf{p}_1 - \mathbf{p}_2|^2 + k_0^2} \psi_2^{(0)*} \psi_1^{(0)}.
\]

(9)
Inserting (7) and (9) into the Hartree-Fock equations, one arrives at the energy-momentum relations

\[ E_{1,2} = \frac{p_{1,2}^2}{2} - \frac{1}{|p_1 - p_2|^2 + k_0^2}. \]  

The kinetic energy is reduced by the exchange energy. For \( k_0 = 0 \), the latter one becomes singular for \( p_1 \rightarrow p_2 \). To avoid such singularities the screening constant \( k_0 \) is assumed to be finite. It is noted that these single-electron energies can be summed up in the continuum limit and then lead to the well-known Hartree-Fock result for the energy of the free-electron gas.

2.3. Two-Stream Perturbations

Perturbations of the equilibrium state will be treated in linear-order. For this purpose the carrier-envelope waves (1) are written in the form

\[ \psi_s = (1 + \chi_s(r, t))\psi_s^{(0)}, \quad \chi_s(r, t) = h_s(r)e^{-i\omega t}. \]  

The envelope perturbation \( \chi_s(r, t) \) may be viewed as the perturbation in the interaction picture, since it evolves relative to the unperturbed stationary state. The set of linearized equations can be Fourier-transformed with respect to the spatial coordinates. The Fourier-transforms of \( \chi_s \) and \( \chi_s^* \) within the volume \( V \) are defined by

\[ \hat{\chi}_s = \frac{1}{V} \int d^3r \chi_s(r)e^{-ikr}, \quad \hat{\chi}_s^* = \frac{1}{V} \int d^3r \chi_s^*(r)e^{-ikr}. \]  

Linearizing the Hartree-Fock equations (6) with these perturbations, one obtains the system of equations

\[ (\omega - \Omega_1)\hat{\chi}_1 = -G(k)(\hat{\chi}_2 + \hat{\chi}_2) + G(k + 2p)(\hat{\chi}_1 + \hat{\chi}_2) - G(2p)(\hat{\chi}_1 - \hat{\chi}_2), \]  

\[ (\omega + \Omega_2)\hat{\chi}_2 = -G(k)(\hat{\chi}_1 + \hat{\chi}_1) + G(k - 2p)(\hat{\chi}_2 + \hat{\chi}_1) - G(2p)(\hat{\chi}_2 - \hat{\chi}_1), \]  

\[ (\omega - \Omega_2)\hat{\chi}_1 = G(k)(\hat{\chi}_2 + \hat{\chi}_2) - G(k - 2p)(\hat{\chi}_2 + \hat{\chi}_1) + G(2p)(\hat{\chi}_1 - \hat{\chi}_2), \]  

\[ (\omega + \Omega_1)\hat{\chi}_2 = G(k)(\hat{\chi}_1 + \hat{\chi}_1) - G(k + 2p)(\hat{\chi}_1 + \hat{\chi}_2) + G(2p)(\hat{\chi}_2 - \hat{\chi}_1), \]

with

\[ \Omega_{1,2} = k(p \pm \frac{k}{2}), \quad G(x) = -\frac{1}{x^2 + k_0^2}. \]

The frequencies \( \Omega_{1,2} \) are known as the frequencies of single-particle excitations, since they are asymptotic solutions for \( G \rightarrow 0 \). In general, the single-particle excitations are coupled by the direct interaction \( G(k) \) and by the exchange interactions \( G(k \pm 2p) \) and \( G(2p) \).

For comparison with the Hartree and Vlasov theories, we comment on the corresponding equations. In the Hartree approximation one has only a direct interaction of each electron with the other one. This interaction is given by the terms

\[ U_{H1,2} = \mp G(k)(\hat{\chi}_{2,1} + \hat{\chi}_{2,1}). \]  

The corresponding set of equations is obtained from (13) by omitting there the exchange terms, setting \( G(k \pm 2p) = 0 \) and \( G(2p) = 0 \).

In the Vlasov approximation, the same terms are omitted, however, the self-interaction is not excluded. The equilibrium charge density is exactly zero, since both electrons contribute to the effective electrostatic potential. Setting the corresponding ion charge to +2, the effective interaction for both electrons is the average of the two effective interactions in the Hartree model,

\[ U_V = \frac{1}{2}[U_{H1} + U_{H2}]. \]

In the following, we will discuss results from the solution of the Hartree-Fock equations (13) and make comparison with the corresponding Hartree and Vlasov results derived with the reduced interactions (14) and (15), respectively.
3. Results
The basic equations (13) for small amplitude perturbations will now be analysed in turn for the Vlasov, the Hartree and the Hartree-Fock approximations.

3.1. Vlasov Model
The two-stream instability has been analysed in the past for the Vlasov approach [3] and we will briefly summarize these results. Using the perturbation equations (13) with the effective interactions given by (15), one can easily obtain a single equation

\[ D_V \Sigma_V = 0, \]  

(16)

with

\[ \Sigma_V = \hat{\chi}_1 + \tilde{\chi}_1 + \hat{\chi}_2 + \tilde{\chi}_2, \]  

(17a)

\[ D_V = 1 - \frac{1}{k^2} \left( \frac{\Omega_1}{\omega^2 - \Omega_1^2} - \frac{\Omega_2}{\omega^2 - \Omega_2^2} \right). \]  

(17b)

The variable \( \Sigma_V \) represents the Fourier-transformed density perturbation, which is also proportional to the Fourier-transformed electrostatic potential. The mode frequencies are obtained from the solubility condition \( D_V = 0 \), which can be written as a quadratic equation in \( \omega^2 \),

\[ \omega^4 - P \omega^2 + Q = 0. \]  

(18)

Its zeros are given by

\[ \omega_{1,2} = \frac{P}{2} \pm \sqrt{\frac{P^2}{4} - Q}, \]  

(19a)

\[ P = \Omega_1^2 + \Omega_2^2 + 1 = 1 + 2k^2 \left( p^2 + \frac{k^4}{4} \right), \]  

(19b)

\[ Q = \Omega_1 \Omega_2 (\Omega_1 \Omega_2 - 1) = k^2 \left( p^2 - \frac{k^2}{4} \right) \left[ k^2 \left( p^2 - \frac{k^2}{4} \right) - 1 \right]. \]  

(19c)

The parameter region of the two-stream instability can be readily determined by looking at (19) and convincing that \( P > 0 \) and \( P^2 > 4Q \). The upper branch \( \omega_1^2 \) is always positive, while the lower branch \( \omega_2^2 \) becomes negative for \( Q < 0 \). The instability criterion is therefore

\[ Q < 0 \iff 0 < \Omega_1 \Omega_2 < 1, \iff p^2 < \frac{1}{k^2} + \frac{k^2}{4} \quad \text{and} \quad p > \frac{k}{2}. \]  

(20)

The quantum Vlasov result (19) reduces to the classical one by the substitution \( \Omega_{1,2} = kp \), thereby omitting all the terms with the factor 1/4 in the definitions of \( P \) and \( Q \). In the classical limit the instability criterion becomes \( p < \frac{1}{k} \). This classical instability boundary is the asymptote to the upper quantum-mechanical boundary for small wave numbers. The quantum version has another lower boundary \( p = k/2 \) that is related to the zero of the single-particle excitation energy \( \Omega_2 = 0 \), when the wave with momentum \( k \) can excite transitions \( p \leftrightarrow -p \) between counterpropagating beams of equal energy. These instability boundaries in the \( k p \)-plane are shown in Figure 1. One can recognize, that the quantum treatment leads to a second unstable \( k \)-interval above the minimum \( k = \sqrt{2} \), \( p = 1 \) of its upper boundary.

For small wavenumbers, the two solution branches can be classified as a stable plasma mode and as the two-stream instability

\[ \omega_1^2 \to 1 + 3 k^2 p^2, \quad \omega_2^2 \to -k^2 p^2, \quad \text{for} \quad k \to 0. \]  

(21)
Figure 1. Stability boundaries of the two-stream instability with beam momenta $\pm p$ and wave momentum $k$. Classically the unstable region is the area below the boundary $p = 1/k$ (solid). Quantum-mechanically, there is another instability region near the line $p = k/2$ where the single-particle excitation frequency $\Omega_2$ vanishes. The instability regions of the quantum Vlasov and quantum Hartree models have the same upper boundary (dashed) but different lower boundaries (dotted and dotted-dashed).

For large wavenumbers, outside the instability region, both branches approach stable single-particle excitations

$$\omega^2_{1,2} \rightarrow \Omega^2_{1,2}, \quad \text{for} \quad k \rightarrow \infty.$$  \hfill (22)

For large wavenumbers, the instability region near $\Omega_2 = 0$ can be defined by $p - \frac{k}{2} = \frac{c}{k}$, with the parameter $c$ in the interval $0 < c < 1$. The parameters $P$ and $Q$ then can be expanded as $P = k^4 + O(1)$, $Q = c(c - 1) + O(k^{-4})$. The unstable branch for large wavenumbers then can be found to be

$$\omega^2_2 = \frac{Q}{P} = -\frac{c(1 - c)}{k^4}, \quad \text{for} \quad k \rightarrow \infty, \quad 0 < c < 1.$$  \hfill (23)

### 3.2. Hartree Model

Next, we discuss results of the Hartree model. In this model each electron experiences only the average field of the other electron as given by (14). The analysis of this model is instructive, since it represents the limit of the Hartree-Fock model for strong screening of the exchange interaction.

Eliminating the variables of one electron from (13), one obtains for the other electron the condition

$$D_H \Sigma_H = 0, \quad \Sigma_H = \hat{\chi}_1 + \check{\chi}_1,$$  \hfill (24a)

$$D_H = 1 - \frac{1}{(\omega^2 - \Omega^2_1)(\omega^2 - \Omega^2_2)}.$$  \hfill (24b)

The solubility condition $D_H = 0$ leads again to a quadratic equation of the form of (18). Its
coefficients are given by
\[ P = \Omega_1^2 + \Omega_2^2 = 2k^2p^2 + \frac{1}{2}k^4, \]  
\[ Q = \Omega_1^2\Omega_2^2 - 1 = \left[k^2 \left(p^2 - \frac{k^2}{4}\right)^2\right]^2 - 1. \]  
(25)

The instability criterion is again given by \( Q < 0 \) with the corresponding boundaries
\[ p^2 = \pm \frac{1}{k^2} + \frac{k^4}{4}. \]  
(26)

The upper boundary is the same as in the Vlasov case. The lower boundary is now shifted below the line \( k = 2p \) and a finite unstable k-interval is obtained for \( p \rightarrow 0 \) (Figure 1). The two frequency branches can be obtained as
\[ \omega_{1,2}^2 = k^2p^2 + \frac{k^4}{4} \pm \sqrt{1 + p^2k^6}. \]  
(27)

The asymptotes for small and large wavenumbers become
\[ \omega_{1,2}^2 \rightarrow \pm 1 + k^2p^2, \quad \text{for} \quad k \rightarrow 0, \]  
(28a)

\[ \omega_{1,2}^2 \rightarrow \Omega_{1,2}^2, \quad \text{for} \quad k \rightarrow \infty. \]  
(28b)

The instability has now a finite growth rate equal to the plasma frequency for \( k \rightarrow 0 \).

3.3. Hartree-Fock Model
The analysis of the full Hartree-Fock system (13) is more cumbersome. We have calculated the determinant of the coefficient matrix and obtain again a polynomial of second order in \( \omega^2 \) as given by (18). Its coefficients are defined by
\[ P = \Omega_{HF,1}^2 + \Omega_{HF,2}^2 + 2b^2 - d^2 - c^2, \]  
\[ Q = (\Omega_{HF,1}^2 - c^2)(\Omega_{HF,2}^2 - d^2) + 2b^2(\Omega_{HF,1}\Omega_{HF,2} - cd) + b^4, \]  
(30a)

(30b)

with the Hartree-Fock frequencies of the single-particle excitations
\[ \Omega_{HF,1} = \Omega_1 + c - b, \quad \Omega_{HF,2} = \Omega_2 + b - d \]

and some effective-potential parameters
\[ b = G(2p) - G(k), \quad c = G(2p + k) - G(k), \quad d = G(2p - k) - G(k). \]

We have investigated the instability region in the \( kp \)-plane by evaluating the coefficients \( P \) and \( Q \) numerically. The resulting instability diagram is shown in Figure 2 for different values of the screening constant \( k_0 \). The criterion for stability is
\[ P > 0 \quad \text{and} \quad 0 \leq Q \leq \left(\frac{P}{2}\right)^2. \]  
(31)

These stable regions are colored in blue. Unstable regions are found as before for \( P > 0 \) and \( Q < 0 \). These areas are colored in red. Furthermore there is an instability region for \( P > 0 \) and \( Q > (P/2)^2 \), which is colored in yellow. Within the red region, the unstable frequency is
purely imaginary and therefore the perturbation amplitude shows monotonic exponential growth. Within the yellow region, the unstable frequencies have both nonvanishing real and imaginary parts. The perturbation amplitudes show oscillatory growth. For sufficiently large screening, the Hartree-Fock result converges to the Hartree result, as can be seen in a) for $k_0 = 2.4$. For smaller screening constants ($k_0 = 1.5, 1.0, 0.0$), the instability region for large $k$ is completely shifted below the line $k = 2p$. In addition, one can observe that the two unstable areas around the lines $k = 0$ and $k = 2p$ become disconnected. In between these disconnected areas there appears an island of solutions with oscillatory growth. This island is a unique feature of the exchange interaction in the Hartree-Fock model.

![Figure 2](image_url)

**Figure 2.** Instability regions in the Hartree-Fock model as a function of the screening constant $k_0$. Black lines indicate the Vlasov and Hartree boundaries as shown in Figure 1. The colors indicate stable regions (blue), non-oscillatory unstable regions (red) and oscillatory unstable regions (yellow). a) For $k_0 = 2.4$ exchange effects are marginal. b) With decreasing screening, the exchange interaction leads to two disconnected read areas. c) The red areas become separated by a yellow island. d) Below this island, the instability is stabilized for small wavenumbers by the exchange interaction.

An evaluation of the two branches $\omega^2_{1,2}(k)$ is shown in Figure 3 for $p = 0.5$ and different screening constants $k_0 = 2.4, 1.1, 0.6, 0.2$. For $k_0 = 2.4$, one can observe the stable plasmon branch and the unstable two-stream branch of the Hartree model. For $k_0 = 1.1$, one can see that the two branches have merged. There is now an interval, where both branches have the same real part and different non-zero imaginary parts. For $k_0 = 0.6$, the two branches to the left of the island are both positive and therefore stable. Note that they have interchanged their physical significance. The plasmon branch with the asymptote $\omega^2 \to 1$ for $k \to 0$ is now the lower branch. Finally, for $k_0 = 0.2$, this interchange is nearly complete. The plasmon branch now starts as the lower branch on the left side and continues as the upper branch on the right side of the island. The other branch approaches a first-order pole at $k = 2p$. Thereby the whole
region $k < 2p_0$ has become stable and the instability has become shifted to a region above the pole $k = 2p$.

![Figure 3](image)

**Figure 3.** Real and imaginary parts of the branches $\omega^2(k)$. a) For sufficiently strong screening of the exchange interaction one has two disconnected branches for the plasmon excitation (upper branch) and the two-stream instability (lower branch). b) With decreasing screening, the two branches merge and thereby non-zero imaginary parts appear. c) For still weaker screening, the branches on the left side have become interchanged, the plasmon branch being the lower one. d) Finally, the upper branch approaches a pole singularity at $k = 2p = 1$, that changes strongly the instability behavior.

4. Conclusions
In this work, we have investigated the role of the exchange interaction on the quantum two-stream instability. As the exchange interaction is mostly neglected in self-consistent field treatments of plasmas, some validation for such a procedure appeared appropriate. Looking at the two-stream instability of the unscreened Coulomb interaction, it became evident that the exchange contribution is quite substantial and leads to completely different modes and instability regions in comparison to the more familiar Vlasov results. Although the two-stream model is highly idealized, this conclusion will also hold for multistream models and their thermodynamic limit. The simple reason is that the effective exchange potentials of the long-range Coulomb interaction are quite singular and become dominant in various parameter regions. However, with increasing screening, these differences become marginal and then the results of the simpler Hartree model have been approached. The present differences between the Hartree and the Vlasov models will also become negligible for many-electron systems. The screening constant of metals under normal conditions is such that $k_0$ varies between about 2 and 2.5. Under these conditions the two-stream model indicates, that the exchange interaction is actually a small correction.
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