Generalized Backprojection Operator: Fast Calculation

Eduardo X Miqueles$^1$, Elias S Helou$^2$ and Alvaro R De Pierro$^2$

$^1$ Brazilian Synchrotron Light Source/CNPEM - Campinas, SP/Brazil
$^2$ Universidade de S\~ao Paulo/ICMC - S\~ao Carlos, SP/Brazil

E-mail: edu.miqueles@gmail.com

Abstract. The inverse Radon transform and its straightforward implementation, known as filtered backprojection (also known as FBP), has become a powerful algorithm for solving a tomographic inverse problem. It has a wide range of applications, including geophysics, medicine and synchrotrons, and from kilo to centi to micro scale respectively. Such a classical inversion has a major computational disadvantage: increasing slowness proportionally to the data size. An ordinary implementation of this algorithm relies on a simple integral that has to be done pixelwise. Many accelerating techniques were proposed in the literature so as to make this part of the inversion as fast as possible. One the most promising strategies is converting the backprojection as a convolution operator (at log-polar coordinates). The generalized backprojector has many applications, for instance in the analytical inversion of single-photon emission tomography or x-ray fluorescence tomography. Our aim in this paper is to show how these ideas can be used for other inversion methods, the iterative ones; which deal much better with noise.

1. Introduction

Most of the hot-spot tomographic problems, viz.: CT (computed tomography), SPECT (single-photon emission computed tomography), PET (positron emission tomography), ERT (exponential radon transform) and XFCT (x-rays fluorescence computed tomography), are related to the inversion of following integral operator:

$$ g(t, \theta) = \int_{\Omega(t, \theta)} f(x) \omega(x, \theta) dx = \mathcal{R}_{\omega} f(t, \theta) $$

with $\Omega(t, \theta) = \{x \in \mathbb{R}^2 : x \cdot \xi_\theta = t\}$ a straight line parameterized by $(t, \theta) \in [-1, 1] \times [0, 2\pi]$. Here $\xi_\theta = (\cos \theta, \sin \theta)^T$ is the normal vector to $\Omega$ and $x$ lies in the unit ball $|x|_2 \leq \frac{\sqrt{2}}{2}$. The above integral operator is called the generalized Radon transform and the common task is: find $f$ given the function $g$ over the set $\{\Omega(t, \theta)\}$. Let us denote $U$ as the feature space, lying all the map functions of the form $z = z(x)$, and denote $V$ as the Radon space, gathering all the sinogram functions $g = g(t, \theta)$. Some typical weight functions $\omega$ are summarized on table 1. Operator $\mathcal{D}$ is the divergent beam transform, meaning that $\mathcal{D} h(x, \theta)$ (for $h \in U$) is the integral of $h$ over the line $\text{span}\{x + s\xi_\theta : s \in \mathbb{R}_+\}$. Analytical inversion formulas for $\mathcal{R}_{\omega}$ at each case of table 1 are presented in $[4, 6, 7, 8, 9]$. Note that, for a fixed angle $\theta$, $\mathcal{R}_{\omega} f(t, \theta)$ is the classical Radon transform of the function $f(\cdot) \omega(\cdot, \theta) \in U$, i.e., $\mathcal{R}_{\omega} f(t, \theta) = \mathcal{D}[f \omega(\cdot, \theta)](t, \theta)$.

(Discretization) When dealing with a discretized version of $z = z(x) \in U$, we consider that $x$ belongs to a pixel grid $\{x_0, \ldots, x_{P-1}\}$; usually with $P = s^2$. Therefore, we have $z$ as a
Table 1. Some typical weight functions for the generalized Radon transform (1).

| Name | Weight function $\omega(x, \theta)$ | Parameters |
|------|----------------------------------|------------|
| CT   | $e^{-(x \cdot \xi^{+})^{\mu}}$   | $\mu \in \mathbb{R}$ |
| ERT  | $e^{-(x \cdot \xi^{+})^{\mu}}$   | $\mu \in \mathbb{U}$ |
| PET  | $e^{-D\mu(x, \theta)}$          | $\mu \in \mathbb{U}$ |
| SPECT| $e^{-D\mu(x, \theta)}$          | $\lambda, \mu \in \mathbb{U}$, $\Gamma \subset [0, \pi]$ |
| XFCT | $\int_{\mathbb{F}} d\gamma e^{-D\mu(x, \theta)}$ | $\lambda, \mu \in \mathbb{U}$, $\Gamma \subset [0, \pi]$ |

linear combination of basis functions $\{\phi_{0}(x), \ldots, \phi_{P-1}(x)\}$ (each per pixel); hence it suffices to represent $z$ as a vector $z \in \mathbb{R}^{P}$ with coefficients $z_{i} = z(x_{i})$. The same applies to $g = g(t, \theta) \in \mathbb{V}$ with $(t, \theta)$ lying in the grid $\{-1 + \frac{jT}{2}, \ldots, \frac{k2\pi}{\gamma} \} \times \{0, \ldots, k \}$ yielding a matrix $g \in \mathbb{R}^{nv}$ with entries given by $g_{n} = g(t(n), \theta(n))$. Here, $n = j + kr \in \{0, \ldots, D - 1\}$ (therefore $k_{n} = [n/r]$). We denote $D = rv$.

2. The backprojection operator

It was shown in [10] that the adjoint operator of $\mathcal{R}_{\omega}$ in (1), is the one acting as $\mathcal{B}_{\omega}: g \in \mathbb{V} \mapsto b \in \mathbb{U}$, defined through

$$b(x) = \mathcal{B}_{\omega}g(x) = \int_{[0, 2\pi]} g(x \cdot \xi, \theta)\omega(x, \theta)d\theta \approx \frac{2\pi}{v} \sum_{k=0}^{v} g(x \cdot \xi_{k}, \theta)\omega(x, \theta_{k})$$

The approximation above comes from a finite set of measurements $\{\theta_{0}, \ldots, \theta_{v}\}$. The following fact, is a well-known procedure for discretized computed tomography, i.e. $\omega = 1$. We generalize it for $\omega \neq 1$.

Lemma 2.1 Equations $g = \mathcal{R}_{\omega}f$ are equivalent to $g = A_{\omega}f$ in the grid $\{t_{j}\} \times \{\theta_{k}\} \times \{x_{i}\}$

Proof: Indeed, writing $f(x)$ as a linear combination of the basis functions $\{\phi_{0}, \ldots, \phi_{P-1}\}$ and taking the Radon transform we get

$$g_{j+k} = \mathcal{R} \left( \sum_{i=0}^{P-1} f_{i} \phi_{i}(x) \right) = \sum_{i=0}^{P-1} f_{i} \mathcal{R}\left[\omega(x, \theta)\phi_{i}(x)\right] = \sum_{i=0}^{P-1} f_{i} w_{n,i} a_{n,i}$$

with $n = j + kr$, $a_{n,i} = \mathcal{R}\phi_{i}(t, \theta_{k})$ and $w_{n,i} = \mathcal{R}(\omega(x, \theta_{k})\phi_{i}(x))$ if $\mathcal{R}(\omega(x, \theta_{k})\phi_{i}(x)) (t, \theta_{k}) \neq 0$ or $w_{n,i} = 0$ otherwise. Notice that this is well defined if, say, $\phi_{i}(x) \geq 0$. It follows from (3) that $g = A_{\omega}f$, being $a_{n,i}w_{n,i}$ the entries of $A_{\omega} \in \mathbb{R}^{D \times P}$.

Matrix $A_{\omega}$ of Lemma 2.1 is a Hadamard product of matrices $(a_{ni})$ and $(w_{kni})$ (i.e., elementwise matrix product, see [11, 12]). Here, matrix $A = (a_{ni})$ is the one giving the intersection length of a ray $\Omega_{n}$ and a pixel $x_{i}$. As a Hadamard product, denoted by symbol $\odot$, we can see $A_{\omega}$ as

$$A_{\omega} = A \odot \left[ \begin{array}{c} ew_{1}^{T} \\ \vdots \\ ew_{v}^{T} \end{array} \right] \iff A_{\omega} = A \odot M\ ,\ M = (W \odot e)$$

with $e \in \mathbb{R}^{v}$ a vector of ones and $w_{k}^{T} \in \mathbb{R}^{1 \times P}$ the kth row of matrix $W = (w_{k,i}) \in \mathbb{R}^{v \times P}$.

Symbol $\odot$ stands for Kronecker operation $\otimes$. The product $\odot$ satisfies the property (see [12])
\[(A \otimes M)^T = (A^T \otimes M^T),\] giving us, for all \(g \in U\)

\[
[A^T g]_i = [(A^T \otimes M^T)g]_i = [(a_1 \otimes m_1)g_1 + \ldots + (a_D \otimes m_D)g_D]_i \tag{5}
\]

\[
= (a_{1,i} \otimes m_{1,i})g_1 + \ldots + (a_{D,i} \otimes m_{D,i})g_D \tag{6}
\]

\[
= (a_{1,i} \ldots a_{D,i})^T \text{diag}(m_{1,i}, \ldots, m_{D,i}) g \tag{7}
\]

Equation (7) is the discrete analogous of (2), i.e., the generalized backprojection of sinogram \(g\), evaluated at pixel \(x_i\). Note that if \(\omega \equiv 1\), matrix \(M\) becomes a matrix of ones and the above diagonal matrix becomes the identity, where we retrieve the classical backprojection algorithm. For \(\omega \neq 1\) the generalized backprojection of sinogram \(g \in U\) becomes the usual backprojection of a weighted projection. Such a weighting step could be done by means of a Kronecker product. Denoting \(c_i \in \mathbb{R}^D\) as the \(i\)th canonical vector, we get from (4):

\[
\text{diag}(m_{1,i}, \ldots, m_{D,i}) g = g \otimes \text{vec}(Wc_i \otimes e \otimes e^T), \quad e \in \mathbb{R}^r. \tag{8}
\]

### 3. Fast computation

From (7) and (8) we realize that, for computing the generalized backprojection operator \(B_\omega\), we only need to compute the classical backprojection \(B\) preceded by a weighting step of the sinogram. The same happens in a continuous setting; in fact, we just have to observe that \(g(x \cdot \xi, \theta)\omega(x, \theta)\) is a pointwise product of two sinograms, the first being the input sinogram \(g\) and the second being \(\omega\); which is dependent only on \(\theta\). In the discrete angle set \(\{\theta_0, \ldots, \theta_v\}\), this pointwise action is just a Hadamard product of both sinograms, plus an interpolation from \(x \cdot \xi\) to \(t\).

Let \(B\) denote the algorithm to compute the classical backprojection operator \(B\), i.e., \(Bg(x) \approx B[g](x)\). In a generalized/vectorial notation, we are able to write \(b(x_i) = B_\omega(x_i) \approx B[g \otimes \text{vec}(Wc_i \otimes e \otimes e^T)]\). Or, using matrices:

\[
b(x_i) \approx B \begin{bmatrix} g(t_1, \theta_1) & \ldots & g(t_r, \theta_1) \\ g(t_1, \theta_2) & \ldots & g(t_r, \theta_2) \\ \vdots & \ddots & \vdots \\ g(t_1, \theta_v) & \ldots & g(t_r, \theta_v) \end{bmatrix} \begin{bmatrix} \omega(x_i, \theta_1) \\ \omega(x_i, \theta_2) \\ \vdots \\ \omega(x_i, \theta_v) \end{bmatrix}
\]

with \(\square\) a row-wise hadamard product between matrix/vector with the same number of rows. It is easy to verify that \(\square\) is closely related to the \(\{\otimes, \otimes\}\) operations. The fast computation of \(B_\omega\) is dependent of a fast computation of \(B\), represented by algorithm \(B\). The following statement give us a powerful tool for calculating \(B\). The proof can be found in [3].

**Theorem 3.1** \(Bg\) can be computed as the convolution of \(g\) with an appropriate point-spread function, both in log-polar coordinates.

The above Theorem allow us to compute the backprojection with \(O(s^2 \log s)\) complexity using the Fast Fourier transform, in log-polar coordinates, for each pixel. This is in contrast with a naïve implementation of the integral in (2) which requires approximately \(O(s^3)\) computations using a quadrature rule for each pixel \(x\). It is not the only existing method with low complexity \(O(s^2 \log s)\), see [1, 2].
4. Application: iterative methods
We propose an immediate application of Sections 2 and 3 on iterative methods for solving (1).
From Lemma 2.1 we have to find the solution $f$ of the linear system $A_x f = g$. Iterative
techniques can be used to solve this inverse problem [5, 10], and the backprojection operator$
B$ plays an important role. For instance,

$$f^{k+1} = f^k + \frac{1}{|Wc_i|} \mathbb{B}_i \{ g - (A \circ M) f^k \} , \quad f^{k+1} = \left[ f^k \circ \mathbb{B}_i \{ g \circ (A \circ M) f^k \} \right] \circ \mathbb{e}$$

are the continuous iterative method [10] and the expectation maximization method [5]. It should
be noted that $F$ is a low-pass filtering operator while $\mathbb{BF}$ can be regarded as the FBP inversion
method. Symbol $\circ$ stands for the element-wise division of vectors and $e = (1) \in \mathbb{R}^D$.

Figure 1 shows an application with data from a micro-tomography beam line at the Brazilian
Synchrotron Light Source. The reconstruction using the filtered backprojection algorithm is
shown in (c). For a noisy data $g \in \mathbb{R}^{6184960}$ (since 6184960 = 3020 angles $\times$ 2048 rays) depicted
in (a), it is clear that an analytic method is not the best choice. To use an iterative technique
(EM or RAMLA), one has to implement a very fast backprojection operator, whose action is shown
in (b). Therefore, Theorem 3.1 seems to be the best option for a fast implementation of $\mathbb{B}$ and a
straightforward fast iterative technique. Fast generalized backprojection techniques could also
be easily implemented for XFCT measurements.

Figure 1. Left: Input sinogram $g$. Center: Action of algorithm $\mathbb{B}[g]$. Right: Filtering of $\mathbb{B}[g]$

5. Conclusion
Fast backprojection can be implemented efficiently using log-polar coordinates. In this
manuscript, we demonstrate that the generalized backprojection operator can also be made fast,
using appropriate matrix products. Such an strategy has great impact at x-rays fluorescence
tomography and single-photon emission tomography.

Acknowledgments
We thank Francis P. O’Dowd and Hugo H. Slepicka, from the Brazilian Synchrotron Light Source for providing the data. Work by Alvaro R. De Pierro was supported by Cnpq grant no. 304820/2006-7.

References
[1] Brandt A, Mann J, Brodski M and Galun M, 1999 A fast and accurate multilevel inversion of the radon transform,
Siam, J. Appl. Math, 60 (2), 437-62
[2] Basu S and Bresler Y, 2000, $O(N^2 \log_2 N)$ filtered backprojection reconstruction algorithm for tomography, IEEE,
Trans. Med. Imaging, 9 (10), 1760-73.
[3] Anderson F, 2005 Fast inversion of the radon transform using log-polar coordinates and partial back-projections, SIAM,
J. Appl. Math, 65 (3), 818.
[4] Natterer F and Wubbeling F, 2001 Mathematical methods in image reconstruction SIAM.
[5] Helou E S and De Pierro A R, 2005 Convergence results for scaled gradient algorithms in positron emission tomography, *Inverse Problems*, 21(6), 1905-14.

[6] Tretiak O and Metz C E, 1980 The exponential radon transform, *SIAM J. Appl. Math.*, 39, 341.

[7] Metz C E and Pan X, 1995, A unified analysis of exact methods of inverting the 2-D exponential radon transform, with implications for noise control in spect, *IEEE Trans. Medical Imaging*, 14 (4), 643.

[8] Fokas A S, 2008 A unified approach to boundary value problems, *SIAM, CBMS-NSF Regional Conference Series in Applied Mathematics*.

[9] Miqueles E X and De Pierro A R, 2011 On the inversion of the xfct radon transform, *Studies in Applied Mathematics*, 127(4), 394.

[10] Miqueles E X, De Pierro A R, 2013 On the iterative inversion of generalized attenuated radon transforms, *Journal of Inverse and Ill-posed Problems*, 21(5), 695.

[11] Horn R A, Johnson C R, 1985 Matrix analysis, *Cambridge University Press*, New York.

[12] Van Loan C F, 2000 The ubiquitous kroncker product, *Journal of Computational and Applied Mathematics*, 123, 85.