A Decomposition for Hardy Martingales. Part II.

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Abstract

This paper continues [13] on Davis and Garsia Inequalities (DGI). We prove DGI for dyadic perturbations of Hardy martingales, and apply them to estimate the $L^1$ distance of a dyadic martingale on $\mathbb{T}^N$ to the class of Hardy martingales. We revisit Bourgain’s embedding of $L^1$ into the quotient space $L^1/H_0^1$. The Appendix reviews well known estimates on cosine-martingales complementary to DGI [2].

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1 Introduction

The introduction is divided into three separate parts. We first list preliminary material, standard notations and tools employed throughout this paper. Then we survey the Davis and Garsia inequalities for Hardy martingales and their dyadic perturbations. We discuss the essential steps of the proof, and point out the role DGI are playing in Bourgain’s embedding of $L^1$ into $L^1/H_0^1$.

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Preliminaries, Notation, Conventions

Let \( T = \{ e^{i\theta} : \theta \in [0, 2\pi] \} \) be the torus equipped with the normalized angular measure. Let \( T^N = \{ (x_i)_{i=1}^\infty \} \) its countable product equipped with its product Haar measure \( \mathbb{P} \). We let \( \mathbb{E} \) denote expectation with respect to \( \mathbb{P} \).

**Martingales on \( T^N \).** Denote by \( \mathcal{F}_n \) the sigma-algebra on \( T^N \) generated by the cylinder sets \( \{(A_1, \ldots, A_n, T^N)\} \), where \( A_i, i \leq n \) are measurable subsets of \( T \). Thus \( (T^N, (\mathcal{F}_k), \mathbb{P}) \) becomes a filtered probability space. We let \( \mathbb{E}_n \) denote the conditional expectation with respect to \( \mathcal{F}_n \).

Let \( \{F_k\} \) be a bounded \( L^1(T^N) \)-martingale. Conditioned on \( \mathcal{F}_{k-1} \) the martingale difference \( \Delta F_k = F_k - F_{k-1} \) defines an element in \( L^1_0(T) \), the Lebesgue space of integrable, functions with vanishing mean.

**Dyadic martingales.** The dyadic sigma-algebra on \( T^N \) is defined with Rademacher functions. For \( x = (x_k) \in T^N \) define \( \cos_k(x) = \Re x_k \) and

\[
\sigma_k(x) = \text{sign}(\cos_k(x)).
\]

We let \( \mathcal{D} \) be sigma-algebra generated by \( \{\sigma_k, k \in \mathbb{N}\} \). Let \( D = (D_k) \) be an integrable \( (\mathcal{F}_k) \)-martingale in \( T^N \). If each \( D_k \) is measurable with respect to \( \mathcal{D} \), then we say that \( D = (D_k) \) is a dyadic martingale on \( T^N \). Define the nonnegative kernel

\[
B(w, z) = \prod_{k=1}^\infty (1 + \sigma_k(w)\sigma_k(z)), \quad w, z \in T^N.
\]

Conditional expectation \( \mathbb{E}_\mathcal{D} \) is the integral operator with kernel \( B \),

\[
\mathbb{E}_\mathcal{D} G(w) = \mathbb{E}_z(B(w, z)G(z)), \quad G \in L^1(T^N).
\]

**Hardy martingales.** Let \( H^1_0(T) \subset L^1_0(T) \) consist of those integrable functions for which the harmonic extension to the unit disk is analytic. And put \( H^\infty(T) = H^1_0(T) \cap L^\infty(T) \). See [7].

An \( L^1(T^N) \)-bounded \( (\mathcal{F}_n) \)-martingale \( F = (F_k) \) is called a Hardy martingale if conditioned on \( \mathcal{F}_{n-1} \) the martingale difference

\[
\Delta F_n = F_n - F_{n-1}
\]

defines an element in \( H^1_0(T) \). See [6], [5].

**Sine and Cosine martingales.** A cosine martingale \( U = (U_k) \) on \( T^N \) is defined by the relation

\[
\Delta U_k(x, y) = \Delta U_k(x, y), \quad x \in T^{k-1}, \quad y \in T. \tag{1.1}
\]

A sine-martingale \( V = (V_k) \) on \( T^N \) is defined by

\[
\Delta V_k(x, y) = -\Delta V_k(x, y), \quad x \in T^{k-1}, \quad y \in T. \tag{1.2}
\]
Classical martingale spaces. Let \((\Omega, (\mathcal{F}_n), \mathbb{P})\) be a filtered probability spaces. Corresponding to the fixed filtration we define the well known spaces of martingales \(H^1\), \(\mathcal{P}\) and \(\mathcal{A}\) by specifying their norms. Let \(G = (G_k)\) be an integrable \((\mathcal{F}_n)\) martingale. Define the previsible norm \(\mathcal{P}\),
\[
\|G\|_{\mathcal{P}} = \mathbb{E}\left(\sum_{k=1}^{n} \mathbb{E}_{k-1}|\Delta G_k|^2 \right)^{1/2},
\]
the \(H^1\) norm, and the absolutely summing norm \(\mathcal{A}\),
\[
\|G\|_{H^1} = \mathbb{E}\left(\sum_{k=1}^{n} |\Delta G_k|^2 \right)^{1/2} \quad \text{and} \quad \|G\|_{\mathcal{A}} = \mathbb{E}\left(\sum_{k=1}^{n} |\Delta G_k| \right).
\]
We refer to
\[
\left(\sum_{k=1}^{n} \mathbb{E}_{k-1}|\Delta G_k|^2 \right)^{1/2}
\]
as the conditional square function of \(G\). See [8] for the classical inclusions,
\[
\mathcal{A} \subseteq H^1 \subseteq L^1, \quad \text{and} \quad \mathcal{P} \subseteq H^1.
\]
The last inclusion is the content of the Burkholder-Gundy inequality [8],
\[
\|G\|_{H^1} \leq 2\|G\|_{\mathcal{P}}.
\]
The B. Davis inequality (see [8]) asserts that
\[
C^{-1}\|G\|_{H^1} \leq \mathbb{E}\sup_{k\leq n} |G_k| \leq C\|G\|_{H^1}.
\]

Martingale transforms. Let \((\Omega, (\mathcal{F}_k), \mathbb{P})\) be a filtered probability spaces and \(G = (G_k)\) be an integrable complex valued \((\mathcal{F}_k)\) martingale. Define martingale transforms
\[
T(G) = \Im \left[ \sum_{k=1}^{n} w_{k-1} \cdot \Delta G_k \right]
\]
where \(w_k\) is complex valued, \(\mathcal{F}_k\) measurable and \(|w_k| \leq 1\). Clearly the transform \(T\) satisfies
\[
\|T(G)\|_{H^1} \leq \|G\|_{H^1} \quad \text{and} \quad \|T(G)\|_{\mathcal{P}} \leq \|G\|_{\mathcal{P}}.
\]
This transform, with its unusual imaginary part, will be studied in detail in Section 2 where we prove upper \(L^1\) estimates for \(T\).

Regular martingales. Fix a filtered probability space \((\Omega, (\mathcal{F}_n), \mathbb{P})\). We say that \(D = (D_k)\) is an \(\alpha\)-regular martingale if there exist \((\mathcal{F}_k)\) adapted sequences \((\sigma_k)\) and \((d_k)\) so that
\[
\mathbb{E}_{k-1}(\sigma_k) = 0, \quad |\sigma_k| \leq 1, \quad \mathbb{E}_{k-1}(|\sigma_k|^2) > \alpha,
\]
and
\[
\Delta D_k = d_{k-1}\sigma_k, \quad k \leq n.
\]
Dyadic martingales are regular, and \(\|D\|_{\mathcal{P}} \leq \|D\|_{H^1} \).
**The Hilbert transform.** The Hilbert transform on $L^2(\mathbb{T})$ is defined as Fourier multiplier by

$$
H(e^{i\theta}) = -i \text{sign}(n) e^{i\theta}.
$$

Let $h \in H_0^2(\mathbb{T})$. Define the complex valued even part of $h$ by $u(e^{i\theta}) = \frac{1}{2}(h(e^{i\theta}) + h(e^{-i\theta}))$. The even part of $h$ is sometimes called the cosine series of $h$. The odd part of $g$ is simply $v = h - u$. Since $h$ is analytic and of vanishing mean, we recover it from its even part. Indeed

$$
h = u + iHu,
$$

where $H$ is the Hilbert transform. We also use the following identity

$$
\int \! u(e^{i\theta}) \cos(\theta) d\theta = -i \int \! v(e^{i\theta}) \sin(\theta) d\theta.
$$

Let $f \in L^2(\mathbb{T})$ be even, i.e., $f(w) = f(\overline{w})$, then $g = Hf$ is odd, i.e., $g(w) = -g(\overline{w})$.

Let $h \in H_0^2(\mathbb{T})$ and let $y = \Re h$. The Hilbert transform recovers $h$ from its imaginary part $y$, we have $h = -Hy + iy$. Since $h$ is analytic and of vanishing mean, we recover it from its even part.

Indeed

$$
h = u + iHu,
$$

and

$$
\|h\|_2 = \sqrt{2} \|\Re(w \cdot h)\|_2.
$$

**The Davis and Garsia Inequality for Hardy Martingales**

We review here results from [13]. The $L^1$ norm of a Hardy martingale $F = (F_k)_{k=1}^n$ is equivalent to the $L^1$ norm of its square function. This result of J. Bourgain [2], [3] and a strengthened version thereof [13] are the starting point for this work. Specifically we have

$$
\mathbb{E}\left(\sum_{k=1}^n |\Delta F_k|^2\right)^{1/2} \leq C \mathbb{E}|F|, \tag{1.9}
$$

and there are Hardy martingales $G = (G_k)_{k=1}^n$ and $B = (B_k)_{k=1}^n$ so that

$$
F = G + B, \tag{1.10}
$$

$$
\mathbb{E}\left(\sum_{k=1}^n \mathbb{E}_{k-1} |\Delta G_k|^2\right)^{1/2} + \mathbb{E}\sum |\Delta B_k| \leq C \mathbb{E}|F|, \tag{1.11}
$$

$$
|\Delta G_k| \leq C |F_k|. \tag{1.12}
$$

We refer to (1.9) as the square function inequality and to (1.10) — (1.12) as the Davis and Garsia inequalities for Hardy martingales. We sketch next a unified way of proving set of estimates (1.9) to (1.12). See Theorem 4.1.

**Martingale decomposition.** The joint proof of (1.9) to (1.12) is based on the following decomposition obtained in Theorem 4.2. To each Hardy martingale $F = (F_k)_{k=1}^n$ there exist Hardy martingales $G = (G_k)_{k=1}^n$ and $B = (B_k)_{k=1}^n$ so that

$$
F = G + B, \tag{1.13}
$$

$$
|F_{k-1}| + \frac{1}{4} \mathbb{E}_{k-1} |\Delta B_k| \leq \mathbb{E}_{k-1} |F_k|, \tag{1.14}
$$

$$
|\Delta G_k| \leq C_0 |F_{k-1}|. \tag{1.15}
$$

We list consequences of (1.13) — (1.15) leading to (1.9) and (1.11). See Theorem 4.1.
1. Taking expectations on both sides of (1.14) and summing the resulting telescoping series gives
\[ \mathbb{E} \sum |\Delta B_k| \leq 4\mathbb{E}|F|. \] (1.16)
Subsequently we denote by \( \|B\|_A \) the norm on the left hand side of (1.16).

2. Conditioned to \( \mathcal{F}_{k-1} \) the martingale differences \( \Delta G_k \) are analytic, square integrable, and of mean zero. Hence
\[ \mathbb{E}_{k-1}|\Delta G_k|^2 = 2\mathbb{E}_{k-1}|\Im(w_{k-1} \cdot \Delta G_k)|^2, \] (1.17)
whenever \( w_{k-1} \) is \( \mathcal{F}_{k-1} \) measurable and \( |w_{k-1}| = 1 \).

3. Theorem 2.1 combined with (1.17) and (1.16) gives
\[ \|G\|_P \leq C(\|F\|_{H^1}\|F\|_{L^1})^{1/2}. \] (1.18)

4. As \( F = G + B \) the Burkholder Gundy estimate (1.5), (1.18) and (1.16) imply
\[ \|F\|_{H^1} \leq 2\|G\|_P + \|B\|_A \leq C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2}. \] (1.19)
Canceling the factor \( \|F\|_{H^1}^{1/2} \) gives
\[ \|F\|_{H^1} \leq C\|F\|_{L^1}. \] (1.20)

5. Substitute (1.20) back into (1.18) to obtain
\[ \|G\|_P + \|B\|_A \leq C\|F\|_{L^1}. \] (1.21)

Note that (1.20) is the square function estimate for Hardy martingales and that (1.21) are the Davis and Garsia inequalities. We view the Davis and Garsia inequality as a lower bound for the \( L^1 \) norm of the Hardy martingale \( F = (F_k)_{k=1}^n \).

### Dyadic Perturbation and Stability of Davis and Garsia Inequalities

We next discuss the main results of this paper, Theorem 4.4 and Theorem 4.3. Fix a Hardy martingale \( F = (F_k)_{k=1}^n \) and a dyadic martingale \( D = (D_k)_{k=1}^n \) on \( \mathbb{T}^N \). We say that \( F - D \) is a *dyadic perturbation* of the Hardy martingale \( F = (F_k)_{k=1}^n \). The central results this paper are the Davis and Garsia inequalities for dyadic perturbations of Hardy martingales.

Let \( T \) be the martingale transform defined as
\[ T(H) = \Im \left[ \sum w_{k-1} \Delta H_k \right], \quad w_k = \frac{(F_k - D_k)}{|F_k - D_k|}. \] (1.22)
We obtain in Section 4 the Davis and Garsia inequalities for the perturbed Hardy martingale \( F - D \). That is, there exist Hardy martingales \( G \) and \( B \) so that
\[ F = G + B, \]
\[ \|B\|_A \leq C\|F - D\|_{L^1}, \] (1.23)
and
\[ \|T(G - D)\|_P \leq C\|F - D\|_{L^1}^{1/2}\|F - D\|_{H^1}^{1/2}. \] (1.24)
A simple consequence of (1.24) is
\[ \|G\|_P \leq C\|F\|_{L^1} + C\|D\|_{H^1}. \] (1.25)
See Theorem 4.4.
Martingale decomposition. The proof of (1.24) uses martingale decomposition Theorem 4.3: For each Hardy martingale $F = (F_k)_{k=1}^n$ and dyadic martingale $D = (D_k)_{k=1}^n$ there exists a Hardy martingale $G = (G_k)_{k=1}^n$ so that

$$|\Delta G_k| \leq C_0|F_{k-1} - D_{k-1}|,$$

(1.26)

$$|F_{k-1} - D_{k-1}| + \frac{1}{4}E_{k-1}|\Delta B_k| \leq E_{k-1}|F_k - D_k|,$$

(1.27)

where

$$B = F - G.$$

(1.28)

The decomposition (1.26) – (1.28) implies the Davis and Garsia inequalities (1.24) for dyadic perturbations of Hardy martingales. See Theorem 4.4.

1. Take expectations in (1.27) and sum the resulting telescoping series. This gives

$$\|B\|_A \leq C\|F - D\|_{L^1}.$$

(1.29)

2. Theorem 2.5 combined with (1.29) and (1.26) implies that the martingale transform operator $T$ defined in (1.22) satisfies

$$\|T(G - D)\|_P \leq C\|F - D\|_{L^1}^{1/2}\|F - D\|_{H^1}^{1/2}.$$

(1.30)

3. By (1.30) and the triangle inequality

$$\|T(G)\|_P \leq C\|F\|_{L^1} + C\|D\|_{H^1} + C\|T(D)\|_P.$$

(1.31)

It remains to note that $\|G\|_P = \sqrt{2}\|T(G)\|_P$, by analyticity, and that $\|T(D)\|_P \leq \|D\|_{H^1}$ by regularity.

The Specialization to $D = \mathbb{E}_D F$

We specialize the decomposition (1.26)— (1.28) and the estimates (1.23) –(1.25) to the case when the dyadic perturbation $D$ is the conditional expectation $\mathbb{E}_D(F)$ of the Hardy martingale $F$. Then, to each Hardy martingale $F = (F_k)_{k=1}^n$ there exist Hardy martingales $G = (G_k)_{k=1}^n$ and $B = (B_k)_{k=1}^n$ so that

$$F = G + B,$$

$$\|B\|_A \leq C\|F - \mathbb{E}_D F\|_{L^1},$$

(1.32)

and

$$\|T(G - \mathbb{E}_D G)\|_P \leq C\|F - \mathbb{E}_D F\|_{L^1}^{1/2}\|F\|_{L^1}^{1/2},$$

(1.33)

where the martingale transform $T$ operator is given by (1.22). Moreover (1.33) gives

$$\|G\|_{H^1} \leq 2\|G\|_P \leq C\|F\|_{L^1}.$$
We record also the following martingale inequality of independent interest. See [2]. For each Hardy martingale $G$,

$$\|\mathbb{E}_D G\|_{L^1} \leq C \|T(G - \mathbb{E}_D G)\|_{\mathcal{P}}^{1/4} \|G\|_{\mathcal{P}}^{3/4} + C \|G - \mathbb{E}_D G\|_{L^1}^{1/2} \|G\|_{L^1}^{1/2}. \tag{1.34}$$

where $T$ is the martingale transform that arose in (1.33). See [2] and the Appendix for (1.34).

The embedding Theorem revisited

The embedding theorem of J. Bourgain [2] states that $L^1(T)$ is isomorphic to a subspace of $L^1(T)/H_0^1(T)$. The construction of the $L^1$ subspace in $L^1(T)/H_0^1(T)$ exploited Hardy martingales. The $L^1$ distance of a dyadic martingale to the space of integrable Hardy martingales is the key to Bourgain’s embedding [2]:

There exists $\delta > 0$ such that for each dyadic martingale $D = (D_k)_{k=1}^n$ on $T^N$,

$$\inf \|D - F\|_{L^1} > \delta \|D\|_{L^1}, \tag{1.35}$$

where the infimum is taken over all integrable Hardy martingales $F$.

The $L^1$ distance estimate (1.35) results from the following inequality of Bourgain [2]. (See Section 4.) There exists $C > 0$ and $\alpha > 0$ so that for any Hardy martingale $F = (F_k)_{k=1}^n$

$$\|\mathbb{E}_D F\|_{L^1} \leq C \|F - \mathbb{E}_D(F)\|_{L^1}^{\alpha} \|F\|_{L^1}^{1-\alpha}. \tag{1.36}$$

Note that (1.36) is self improving. It implies that there exists $A_0 > 0$ so that for each Hardy martingale,

$$\|F\|_{L^1} \leq A_0 \|F - \mathbb{E}_D F\|_{L^1}. \tag{1.37}$$

Indeed, for a given $F$ consider separately the cases

$$\|\mathbb{E}_D F\|_{L^1} \geq \frac{1}{2} \|F\|_{L^1}, \quad \text{and} \quad \|\mathbb{E}_D F\|_{L^1} \leq \frac{1}{2} \|F\|_{L^1}$$

In the first case (1.36) gives (1.37) by arithmetic. In the second case (1.37) follows by applying triangle inequality

$$\|F\|_{L^1} \leq \|F - \mathbb{E}_D F\|_{L^1} + \|\mathbb{E}_D F\|_{L^1} \leq \|F - \mathbb{E}_D F\|_{L^1} + \frac{1}{2} \|F\|_{L^1}.$$

Proving (1.36): Davis and Garsia inequalities apply to the proof of (1.36): We use the estimates (1.32), (1.33) in combination with (1.34) to show that (1.36) holds. The martingale transform (1.22) is the crucial link between the right hand side of (1.33) and the left hand side of (1.34).

1. Let $F$ be a Hardy martingale with $\|F\|_{L^1} = 1$. Assume that

$$\|F - \mathbb{E}_D(F)\|_{L^1} = \epsilon \tag{1.38}$$

with $\epsilon << 1$, since otherwise there is nothing to prove.
2. Determine Hardy martingales $G, B$, satisfying (1.32) and (1.33) and so that
\[ F = G + B \] (1.39)

3. By (1.38), (1.39) and (1.32) we have,
\[ \|G\|_{L^1} \leq 1 + \epsilon, \quad \|G - E_D G\|_{L^1} \leq C\epsilon, \quad \|E_D B\|_{H^1} \leq C\epsilon. \] (1.40)

4. Invoke (1.34) and use (1.40) to get
\[ \|E_D F\|_{H^1} \leq \|E_D G\|_{H^1} + \|E_D B\|_{H^1} \leq C\|T(G - E_D G)\|_P^{1/4} + C\epsilon^{1/2}, \] (1.41)
where $T$ is the martingale transform operator arising in (1.33).

5. By the estimate (1.33),
\[ \|T(G - E_D G)\|_P \leq C\epsilon^{1/2}. \] (1.42)

6. Combining (1.42) and (1.41) we get
\[ \|E_D F\|_{H^1} \leq C\epsilon^{1/8}, \]

hence $\|E_D F\|_{L^1} \leq C\epsilon^{1/8}$, as claimed.

We derived the interpolatory estimate (1.36) in a straightforward manner from three basic estimates: (1.32), (1.33), and (1.34). This was the motivation for considering dyadic perturbations of Hardy martingales and their Davis Garsia decompositions.

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Organization: In Section 2 we prepare general martingale tools. In Section 3 we prepare the complex analytic tools. Section 4 contains Davis Garsia Inequalities for dyadic perturbations of Hardy martingales and their applications to the proof of the embedding theorem. The Appendix contains the estimates that relate Hardy martingales, cosine martingales and martingale transforms.

2 Martinigale Transforms

Let $(\Omega, (\mathcal{F}_n), \mathbb{P})$ be a filtered probability spaces.

The following class of martingale transforms plays a central role in the proof of Bourgain’s embedding theorem [2].

\[ T(G) = \mathcal{Z} \left[ \sum_{k=1}^{n} w_{k-1} \cdot \Delta G_k \right] \] (2.1)

where $w_k$ is complex valued, adapted, and $|w_k| \leq 1$. This section contains the point-wise estimates for $T$ as needed in Theorem 4.4 and Theorem 4.3.
The Transform Estimate.

The spaces $L^1$, $H^1$, $\mathcal{P}$ and $\mathcal{A}$ corresponding to the filtration $(\mathcal{F}_n)$ are defined in (1.3) and (1.4).

**Theorem 2.1** Let $F = (F_k)$ be a martingale. Let $A > 0$. If the martingale $G = (G_k)$ satisfies

$$|\Delta G_k| \leq A|F_{k-1}|,$$

(2.2)

then the transform $T$ defined as

$$T(G) = \Im\left[\sum_{k=1}^{n} w_{k-1} \cdot \Delta G_k\right]$$

where $w_{k-1} = \frac{F_{k-1}}{|F_{k-1}|}$, $k \leq n$ (2.3)

satisfies the point-wise estimate

$$\|T(G)\|_P \leq C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2} + C\|B\|_A,$$

(2.4)

where $C = C(A)$ and $B = F - G$.

**Comments:**

1. The hypothesis (2.2) matches property (1.15) of the martingale decompositions.

2. The sequence $(w_k)$ defining $T$ in (2.3) depends on the martingale $F$ appearing on the right hand side of (2.4).

3. The appearance of the $\|F\|_{L^1}^{1/2}$ on the right hand side of (2.4) makes Theorem 2.1 our basic tool. (Note, $T$ is obviously a contraction on $\mathcal{P}$ and on $H^1$.)

4. The unusual imaginary part in the definition of $T$ is the price to pay for having the $L^1-$ factor.

5. We will apply Theorem 2.1 only to those decompositions $F = G + B$ for which $\|B\|_A$ is properly under control.

**Iteration**

The following iteration method [2] provides the framework to produce the conditional square function estimates of Theorem 2.1.

**Theorem 2.2** Let $n \in \mathbb{N}$. Given non-negative and integrable $M_1, \ldots, M_n, V_1, \ldots, V_n,$ and integrable $w_1, \ldots, w_n$ so that the following estimates hold:

$$\mathbb{E}(M^2_{k-1} + V^2_k)^{1/2} + \mathbb{E}w_k \leq \mathbb{E}M_k \quad \text{for } 1 \leq k \leq n.$$

(2.5)

Then

$$\mathbb{E}\left(\sum_{k=1}^{n} V^2_k\right)^{1/2} + \mathbb{E}\sum_{k=1}^{n} w_k \leq 2(\mathbb{E}M_n)^{1/2}(\mathbb{E}\max_{k \leq n} M_k)^{1/2}$$

(2.6)
See [13] for a proof of Theorem 2.2. We use it here to get Theorem 2.1. Proposition 2.3 is needed to establish the hypothesis (2.5) of Theorem 2.2.

**Proposition 2.3** Fix a probability space \((\Omega, \mathbb{P})\). Let \(A > 0, z \in \mathbb{C}, w = \overline{z}/|z|\). Assume that

\[ |g| \leq A|z| \text{ and } \int_{\Omega} g d\mathbb{P} = 0. \]

Then for

\[ y = \Im(g \cdot w) \]

the following estimate holds,

\[ \left( |z|^2 + \alpha^2 \int_{\Omega} y^2 d\mathbb{P} \right)^{1/2} \leq \int_{\Omega} |z + g| d\mathbb{P}, \tag{2.7} \]

where \(\alpha = \alpha(A)\). Consequently, for any integrable \(f\)

\[ \left( |z|^2 + \alpha^2 \int_{\Omega} y^2 d\mathbb{P} \right)^{1/2} \leq \int_{\Omega} |z + f| d\mathbb{P} + \int_{\Omega} |f - g| d\mathbb{P}. \tag{2.8} \]

We first treat the case when \(z = 1\) in a separate Lemma. Thereafter we prove the estimates (2.7) and (2.8).

**Lemma 2.4** Let \(A > 0\). Assume that

\[ |g| \leq A \text{ and } \int_{\Omega} g d\mathbb{P} = 0. \]

Then \(u = \Im(g)\) satisfies

\[ \left( 1 + \alpha^2 \int_{\Omega} u^2 d\mathbb{P} \right)^{1/2} \leq \int_{\Omega} |1 + g| d\mathbb{P}, \tag{2.9} \]

where \(\alpha = \alpha(A)\).

**Proof.** The idea used below is taken from [2] p. 695. Since \(\int g = 0\),

\[ \int_{\Omega} |1 + \frac{g}{M}| d\mathbb{P} \leq \int_{\Omega} |1 + g| d\mathbb{P}. \tag{2.10} \]

for any \(M \geq 1\). Now write \(g = u + iv\) where \(u, v\) are bounded real valued. Choose \(M = M(A) > 2A\). Then rewrite

\[ |1 + \frac{g}{M}| = \left( (1 + \frac{u}{M})^2 + \frac{v^2}{M^2} \right)^{1/2}. \tag{2.11} \]

Pull out the factor \((1 + u/M)\) from the term on the right hand side of (2.11). By arithmetic the right hand side of (2.11) becomes

\[ \left( 1 + \frac{u}{M} \right) \left( 1 + \frac{v^2}{(M + u)^2} \right)^{1/2}. \tag{2.12} \]
Since \( 1 + u/M > 0 \) and
\[
\left( 1 + \frac{v^2}{(M+u)^2} \right)^{1/2} \geq 1 + \frac{v^2}{3(M+A)^2}, \tag{2.13}
\]
we multiply \( 1 + u/M \) and the right hand side of (2.13) to obtain
\[
|1 + \frac{g}{M}| \geq 1 + \frac{u}{M} + \frac{v^2}{6(M+A)^2}. \tag{2.14}
\]
Since \( g \) has vanishing mean, we have also \( \int u = 0 \), hence taking the expectation in (2.14) gives
\[
\int_\Omega |1 + \frac{g}{M}| d\mathbb{P} \geq 1 + \int_\Omega \frac{v^2}{6(M+A)^2} d\mathbb{P}. \tag{2.15}
\]
Finally, since the right hand side of (2.15) is \( \geq 1 \) we may replace it by its square-root and arrive at (2.9).

**Proof of Proposition 2.3.** Scaling and rotation reduces matters the special case of Lemma 2.4. Write \( 1/z = (1/|z|)(w) \) where \( w = \overline{z}/|z| \). Then
\[
\int_\Omega |z + g| d\mathbb{P} = |z| \int_\Omega |1 + \frac{1}{|z|}(g \cdot w)| d\mathbb{P}.
\]
Next put \( y = \Im(g \cdot w) \). Since \( |g|/|z| \leq A \), Lemma 2.4 implies
\[
|z| \left( 1 + \frac{\alpha^2}{|z|^2} \int_\Omega g^2 d\mathbb{P} \right)^{1/2} \leq \int_\Omega |z + g| d\mathbb{P}, \tag{2.16}
\]
where \( \alpha = \alpha(A) \). Note that (2.16) is just the same as (2.7). The triangle inequality gives now (2.8).

**Proof of Theorem 2.1.** Fix \( k \leq n \). Condition on \( F_{k-1} \) and put
\[
z = F_{k-1}, \quad w = \overline{F_{k-1}}/|F_{k-1}|, \quad f = \Delta F_k \quad \text{and} \quad g = \Delta G_k.
\]
Apply Proposition 2.3 with the above specification. Then (2.8) implies that
\[
Y_k = \Im(w_{k-1} \cdot \Delta G_k),
\]
satisfies the following estimate
\[
(|F_{k-1}|^2 + \alpha^2 \mathbb{E}_{k-1} Y_k^2)^{1/2} \leq \mathbb{E}_{k-1}|F_k| + \mathbb{E}_{k-1}|\Delta B_k|.
\]
Taking expectation yields
\[
\mathbb{E}(|F_{k-1}|^2 + \alpha^2 \mathbb{E}_{k-1} Y_k^2)^{1/2} \leq \mathbb{E}|F_k| + \mathbb{E}|\Delta B_k|
\]
hence Theorem 2.2 gives
\[ \mathbb{E}(\sum_{k=1}^{n} \mathbb{E}_{k-1} Y_k^2)^{1/2} \leq C(\mathbb{E}|F_n|)^{1/2} \cdot (\mathbb{E}\max_{k \leq n} |F_k|)^{1/2} + C\mathbb{E}\sum_{k=1}^{n} |\Delta B_k|. \]

Since \( T(G) = \sum Y_k \) we get
\[ \|T(G)\|_p = \mathbb{E}(\sum \mathbb{E}_{k-1} |Y_k|^2)^{1/2}. \]

The theorem of B. Davis [8] asserts that \( \mathbb{E}\max_{k \leq n} |F_k| \leq C\|F\|_{H^1}. \) This completes the proof.

\[ \blacksquare \]

Perturbation

We next perturb the martingales in Theorem 2.1 by regular martingales and obtain estimates for the resulting martingale transforms. The perturbations we consider here are not small in any sense, but rather structurally simple.

Regular martingales on a filtered probability space \((\Omega, (\mathcal{F}_n), P)\) are defined in (1.7), (1.8).

**Theorem 2.5** Let \( F = (F_k)_{k=1}^{N} \) and \( D = (D_k) \) be martingales, so that \( D = (D_k) \) is \( \alpha \)-regular. Let \( G = (G_k)_{k=1}^{N} \) be a martingale such that
\[ |\Delta G_k| \leq A|F_{k-1} - D_{k-1}|, \quad k \leq N. \] (2.17)

Then the transform \( T \) given by
\[ T(H) = \Im \left[ \sum w_{k-1} \Delta H_k \right], \quad w_k = (F_k - D_k)/|F_k - D_k|. \]

satisfies
\[ \|T(G - D)\|_p \leq C\|F - D\|_{L^1}^{1/2}\|F - D\|_{H^1}^{1/2} + C\|B\|_A, \]
where \( C = C(A, \alpha) \) and
\[ B = F - G. \]

Note that the hypothesis (2.17) in Theorem 2.5 is matched by the decompositions for Hardy martingales (1.26).

The proof of Theorem 2.5 is based on the iteration principle Theorem 2.2. We use Proposition 2.6 below to verify the assumptions (2.5). Following is an extension of Proposition 2.3.

**Proposition 2.6** Fix \( 0 < \alpha \leq 1, C \geq 1 \) and a probability space \((\Omega, \mathbb{P})\). Let \( \sigma : \Omega \rightarrow \mathbb{C} \) satisfy
\[ \int \sigma = 0, \quad |\sigma| \leq 1, \quad \int |\sigma|^2 > \alpha. \]

Let \( z \in \mathbb{C}, w = \overline{z}/|z|, \) and assume \( g : \Omega \rightarrow \mathbb{C} \) satisfy
\[ \int g = 0, \quad |g| \leq C|z|. \]
Then for any \( b \in \mathbb{C} \),
\[
y = \Im(g - b\sigma) \cdot w
\]
satisfies
\[
\left( |z|^2 + \delta^2 \int y^2 \right)^{1/2} \leq \int |z + g - b\sigma|,
\]
where \( \delta = \delta(\alpha, C) \). Consequently, for any integrable \( f \)
\[
\left( |z|^2 + \delta^2 \int y^2 \right)^{1/2} \leq \int |z + f - b\sigma| + \int |f - g|.
\]

(2.18)

**Proof.** Put \( A = 4C/\alpha \). Choose \( b \in \mathbb{C} \). Then we distinguish between two cases.

**Case 1.** Let \( |b| \leq A|z| \). In that case we have
\[
|g - b\sigma| \leq (C + A)|z|.
\]
Proposition 2.3 implies that \( y = \Im(g - b\sigma) \cdot w \) satisfies
\[
\left( |z|^2 + \delta^2 \int y^2 \right)^{1/2} \leq \int |z + g - b\sigma|,
\]
with \( \delta = \delta(\alpha, C) \)

**Case 2.** Let \( |b| \geq A|z| \). This case is straightforward since \( b \) dominates everything else. By rotation we assume that \( w = 1 \). Define the testing function \( m = -\sigma b/|b| \). Note that
\[
\int m = 0, \quad |m| \leq 1, \quad \int b\sigma m = -|b| \int |\sigma|^2.
\]
This gives,
\[
\int |z + g - b\sigma| \geq \int (z + g - b\sigma)m \geq |b|\alpha - \int |g|.
\]
Since \( |g| \leq C|z| \) and \( |b| \geq A|z| \)
\[
\int |z + g - b\sigma| \geq |z| + |b|A^{-1}(A\alpha - C - 1).
\]
Recall that we set \( A = 4C/\alpha \), and \( C > 1 \). Hence \( A\alpha - C - 1 \geq 1 \). Using again \( |g| \leq C|z| \) and \( |b| \geq A|z| \) we have, that
\[
|b| \geq \delta_0 \left( \int |g - b\sigma|^2 \right)^{1/2},
\]
where \( \delta_0 = (1 + C/A)^{-1} \). Inserting gives,
\[
\int |z + g - b\sigma| \geq |z| + \delta \left( \int |g - b\sigma|^2 \right)^{1/2},
\]
where \( \delta = \delta(\alpha, C) \).
Proof of Theorem 2.5.

Fix \( k \leq N \). Condition on \( F_{k-1} \) and put

\[
z = F_{k-1} - D_{k-1}, \quad w = \frac{z}{|z|} \quad g = \Delta G_k \quad \text{and} \quad b\sigma = \Delta D_k.
\]

Apply Proposition 2.6 with these parameters. By (2.18)

\[
Y_k = \Im(w_{k-1}(\Delta G_k - d_{k-1}\sigma_k)) \quad \text{where} \quad w_{k-1} = \frac{F_{k-1} - D_{k-1}/|F_{k-1} - D_{k-1}|}{|F_{k-1} - D_{k-1}|^2 + A_0^{-2}\mathbb{E}_{k-1}|Y_k|^2} \leq \mathbb{E}_{k-1}|F_k - D_k| + \mathbb{E}_{k-1}\Delta B_k.
\]

Taking expectations gives

\[
\mathbb{E}(|F_{k-1} - D_{k-1}|^2 + A_0^{-2}\mathbb{E}_{k-1}|Y_k|^2)^{1/2} \leq \mathbb{E}|F_k - D_k| + \mathbb{E}\Delta B_k.
\]

By Theorem 2.2, and the theorem of B. Davis [8]

\[
\mathbb{E}\left(\sum \mathbb{E}_{k-1}|Y_k|^2\right)^{1/2} \leq C\|F - D\|_{L^1}^{1/2}\|F - D\|_{H^1}^{1/2} + C\|B\|_A,
\]

where \( C = C(A, \alpha) \). Since \( T(G - D) = \sum Y_k \) we have

\[
\|T(G - D)\|_p = \mathbb{E}\left(\sum \mathbb{E}_{k-1}|Y_k|^2\right)^{1/2}.
\]

This completes the proof.

\[\blacksquare\]

3 Brownian Motion and Truncation

This section contains our complex analytic ingredients. Our aim is Theorem 3.2. The proofs rely on estimates for outer functions and stopping time decompositions for complex Brownian motion.

Outer Functions

Let \( H \) denote the Hilbert transform on \( L^2(\mathbb{T}) \). Let \( p \in L^\infty(\mathbb{T}) \) be real valued. Assume that \( \log(1-p) \) is well defined and bounded. Then

\[
q = \exp[\log(1-p) + iH\log(1-p)].
\]

defines an element in \( H^\infty(\mathbb{T}) \). See Garnett [7] for background.

Lemma 3.1 Let \( p \in L^\infty(\mathbb{T}) \) with \( 0 \leq p \leq 1/2 \). Then the outer function (3.1) satisfies the following properties

1. Then \( p + |q| = 1 \).
2. Let \( q_1 \) be the real part of \( q \) and \( q_2 \) its imaginary part so that \( q = q_1 + iq_2 \). Then
\[
\int_T q_2 dm = 0, \tag{3.2}
\]
and
\[
\int_T |1 - q_1| dm \leq C_1 \int_T p dm, \tag{3.3}
\]
where \( C_1 = 7/8 \).

3. If \( p \) is even, \( p(e^{i\theta}) = p(e^{-i\theta}) \), then \( q_2 \) is odd,
\[
q_2(e^{i\theta}) = -q_2(e^{-i\theta}). \tag{3.4}
\]

**Proof.** Since
\[
|q| = \exp[\log(1 - p)] = 1 - p,
\]
we have \( p + |q| = 1 \). Note \( \int H \log(1 - p) = 0 \), gives (3.2).

We now turn to (3.3). Note that
\[
q_1 = (1 - p) \cos H(\log(1 - p)),
\]
and
\[
1 - q_1 = (1 - \cos H(\log(1 - p))) + p \cos H(\log(1 - p)).
\]
Since \( 1 - \cos(x) \leq x^2/2 \) we obtain the point-wise estimate
\[
|1 - q_1| \leq \frac{1}{2} |H(\log(1 - p))|^2 + p.
\]

By the \( L^2 \) estimates for the Hilbert transform,
\[
\int_T |H(\log(1 - p))|^2 dm \leq 2 \int_T |\log(1 - p))|^2 dm.
\]
Invoking that \( 0 \leq p \leq 1/2 \) gives (3.3) as follows
\[
\int_T |\log(1 - p))|^2 dm \leq C_1 \int_T p^2 dm \leq C_1 \int_T p dm.
\]
If moreover \( p \) is even, then by inspection
\[
q_2 = (1 - p) \sin H \log(1 - p)
\]
is odd, hence (3.4) holds.
Brownian Motion

Let \((B_t)\) denote complex 2D-Brownian motion on Wiener space, and \(((\mathcal{F}_t), \mathbb{P})\), the associated filtered probability space. Put

\[
\tau = \inf \{ t > 0 : |B_t| > 1 \}.
\]

See Durrett [4].

The following theorem is our main complex analytic tool.

**Theorem 3.2** There exists \(C_0 \geq 1\) so that the following holds. For \(h \in H^1_0(T)\) and \(z \in \mathbb{C}\), let

\[
\rho = \inf \{ t < \tau : |h(B_t)| > C_0|z| \}, \quad \text{and} \quad g(e^{i\theta}) = \mathbb{E}(h(B_{\rho})|B_{\tau} = e^{i\theta}).
\]  

Then \(g \in H^\infty_0(T)\),

\[
|g| \leq C_0|z|,
\]

and for any \(b \in \mathbb{C}\)

\[
|z| + \frac{1}{4} \int_T |h - g| dm \leq \int_T |z + h - b\sigma| dm,
\]

where

\[
\sigma(e^{i\theta}) = \text{sign}(\cos(\theta)).
\]

**Proof.** By a result of N. Varopoulos [14], \(g\) defined by (3.5) is bounded, analytic with vanishing mean, hence in \(H^\infty_0(T)\). See also [9]. The upper bound (3.6) results from (3.5). We get lower bounds for \(\int_T |z + h - b\sigma| dm\) by integrating against testing functions. In the case \(|b| \leq 8|z|\) we take the outer functions of Lemma 3.1. The case \(|b| \geq 8|z|\) is straightforward and uses simple exponentials as testing functions.

**Case 1.** Assume \(|b| \leq 8|z|\). Define

\[
p(e^{i\theta}) = \frac{1}{4} \left[ \mathbb{E}(1_A|B_{\tau} = e^{i\theta}) + \mathbb{E}(1_A|B_{\tau} = e^{-i\theta}) \right], \quad \text{where} \quad A = \{ \rho < \infty \}.
\]

Clearly \(p\) is even, \(0 \leq p \leq 1/2\), and

\[
\int_T p dm = \frac{1}{2} \mathbb{P}(A).
\]

By definition of \(A = \{ \rho < \infty \},

\[
2 \int_T |h|p dm = \mathbb{E}|h(B_{\tau})1_A| \geq C_0|z|\mathbb{P}(A).
\]

Using that \(|b| \leq 8|z|\), in the present case, together with (3.8) and (3.9) we get by the triangle inequality

\[
\int_T |z + h - b\sigma| p dm \geq \frac{1}{2} \mathbb{E}|h(B_{\tau})1_A| - (|z| + |b|) \frac{1}{2} \mathbb{P}(A)
\]

\[
\geq (1 - \frac{9}{2C_0}) \mathbb{E}|h(B_{\tau})1_A|.
\]
Let \( q \in H^\infty(\mathbb{T}) \) be the outer function defined by (3.1). Note that \( q \in H^\infty(\mathbb{T}) \) is orthogonal to \( h \). By (3.4) \( q_2 = 3q \) is an odd function, hence orthogonal to \( \sigma \), and to constants. This gives the identities below:

\[
\int_{\mathbb{T}} |z + h - b\sigma| \cdot |q| dm \geq \left| \int_{\mathbb{T}} (z + h - b\sigma) q dm \right| = \left| \int_{\mathbb{T}} (z + b\sigma) q dm \right| = \left| \int_{\mathbb{T}} (z + b\sigma) q_1 dm \right|. \tag{3.11}
\]

Recall (3.3), that

\[
\int_{\mathbb{T}} |1 - q_1| dm \leq C_1 \mathbb{P}(A)/2.
\]

Hence writing \( q_1 = 1 + (q_1 - 1) \) and using that \(|b| \leq 8|z|\), gives

\[
\left| \int_{\mathbb{T}} (z + b\sigma) q_1 dm \right| \geq |z| - \int_{\mathbb{T}} |(z + b\sigma)(1 - q_1)| dm \geq |z| - \frac{9C_1}{2C_0}|z| \mathbb{P}(A). \tag{3.12}
\]

Hence combining (3.9) with (3.12) and (3.11) gives

\[
\int_{\mathbb{T}} |z + h - b\sigma| \cdot |q| dm \geq |z| - \frac{9C_1}{2C_0} \mathbb{E}[h(B_\tau)1_A]. \tag{3.13}
\]

Take the sum of (3.10) and (3.13). Since \( p + |q| = 1 \) we obtain with \( \alpha_0 = (1/2 - 9/2C_0 - 9C_1/2C_0) \) that

\[
\int_{\mathbb{T}} |z + h - b\sigma| dm \geq |z| + \alpha_0 \mathbb{E}[h(B_\tau)1_A].
\]

If \( C_0 > 0 \) is large enough, \( \alpha_0 > 1/4 \). Since

\[
\int_{\mathbb{T}} |h - g| dm \leq \mathbb{E}|h(B_\tau)1_A|,
\]

this gives (3.7) in the case \(|b| \leq 8|z|\).

**Case 2.** Next we turn to the case when \(|b| > 8|z|\). This case is straightforward. The testing functions involved are the simple exponentials. Note first that

\[
\int_{\mathbb{T}} |z + h - b\sigma| dm \geq \left| \int_{\mathbb{T}} (z + h - b\sigma) e^{iz} dm \right| = \frac{2|b|}{\pi}. \tag{3.14}
\]

Next by triangle inequality

\[
\int_{\mathbb{T}} |z + h - b\sigma| dm \geq \int_{\mathbb{T}} |h| dm - (|z| + |b|) \geq \int_{\mathbb{T}} |h| dm - \frac{9|b|}{8}. \tag{3.15}
\]

Take a weighted average of the equation (3.15) and (3.14), to get

\[
\int_{\mathbb{T}} |z + h - b\sigma| dm \geq \frac{|b|}{8} + \frac{1}{4} \int_{\mathbb{T}} |h| dm.
\]
Finally since $|z| \leq |b|/8$, and $\int_{T} |h-g| dm \leq \int_{T} |h| dm$, we get

$$\int_{T} |z+h-b\sigma| dm \geq |z| + \frac{1}{4} \int_{T} |h-g| dm.$$ 

\[\blacksquare\]

## 4 Davis and Garsia Inequalities

Let $\mathbb{T}^N = \{(x_i)_{i=1}^{\infty}\}$ denote the countable product of the torus $\mathbb{T}$ equipped with its product Haar measure. We return to considering martingales on $\mathbb{T}^N$.

### Davis - Garsia Inequalities for Hardy Martingales revisited

We begin, explaining how to get simultaneously the square function estimate [2] and the Davis-Garsia inequalities [13] for Hardy martingales. We use the complex analytic truncation Theorem 3.2, and the transform estimates in Theorem 2.1. This proof will be extended further on to obtain Davis-Garsia inequalities for perturbed Hardy martingales. See Theorem 4.3 below.

**Theorem 4.1** For each Hardy martingale $F = (F_k)_{k=1}^{n}$

$$\|F\|_{H^1} \leq C\|F\|_{L^1}, \quad (4.1)$$

and there exists a Hardy Martingale $G = (G_k)_{k=1}^{n}$

$$\|G\|_{P} + \|B\|_{A} \leq C\|F\|_{L^1}, \quad (4.2)$$

where

$$B = F - G$$

The Hardy martingale $G = (G_k)_{k=1}^{n}$ with the properties stated in Theorem 4.1 is obtained in the course of proving the following decomposition. This construction carries the complex analytic content of the Davies and Garsia inequalities.

**Theorem 4.2** There exists $C_0 > 0$ so that: For any Hardy martingale $F = (F_k)_{k=1}^{n}$ there is a Hardy martingale $G = (G_k)_{k=1}^{n}$ so that

$$|\Delta G_k| \leq C_0|F_{k-1}| \quad (4.3)$$

and

$$|F_{k-1}| + \frac{1}{4}\mathbb{E}_{k-1}|\Delta B_k| \leq \mathbb{E}_{k-1}|F_k|, \quad (4.4)$$

where

$$B = F - G.$$
PROOF. We first define the Hardy martingale $G$. Fix $k \leq n$. Condition to $\mathcal{F}_{k-1}$. Fix $x = (x_1, \ldots, x_{k-1}) \in \mathbb{T}^{k-1}$ and $y \in \mathbb{T}$. Put
\[ h(y) = \Delta F_k(x, y) \text{ and } z = F_{k-1}(x). \]
Let
\[ \rho = \inf \{ t < \tau : |h(B_t)| > C_0 |z| \}, \quad g = \mathbb{E}(h(B_\rho)|B_\tau = e^{i\theta}). \]
Then by [14] $g \in H_0^\infty(\mathbb{T})$ and clearly
\[ |g| \leq C_0 |z|. \quad (4.5) \]
We apply Theorem 3.2 with $b = 0$ and get
\[ |z| + \frac{1}{4} \int_{\mathbb{T}} |h - g| dm \leq \int_{\mathbb{T}} |z + h| dm. \quad (4.6) \]
Put
\[ \Delta G_k(x, y) = g(y), \quad \text{and} \quad \Delta B_k(x, y) = h(y) - g(y). \]
so that
\[ \Delta F_k = \Delta G_k + \Delta B_k. \]
Hence (4.5) gives
\[ |\Delta G_k| \leq C_0 |F_{k-1}| \]
and by (4.6)
\[ |F_{k-1}| + \frac{1}{4} \mathbb{E}_{k-1} |\Delta B_k| \leq \mathbb{E}_{k-1} |F_k|. \]

**Proof of Theorem 4.1.** Apply Theorem 4.2 to $F$. Let $F = G + B$ be the resulting decomposition into Hardy martingales satisfying (4.3) and (4.4). Define the rotation
\[ w_{k-1} = \overline{F_{k-1}}/|F_{k-1}|, \]
and the transform
\[ T(G) = \Im \left[ \sum w_{k-1} \cdot \Delta G_k \right]. \]
Since $|\Delta G_k| \leq C_0 |F_{k-1}|$, Theorem 2.1 implies that
\[ \|T(G)\|_{\mathcal{P}} \leq C \|F\|_{L_1}^{1/2} \|F\|_{H^1}^{1/2} + C \|B\|_{\mathcal{A}}, \quad (4.7) \]
where $C = C(C_0)$. Integrating (4.4) gives
\[ \mathbb{E}|F_{k-1}| + \frac{1}{4} \mathbb{E} |\Delta B_k| \leq \mathbb{E}|F_k|, \]
and by summing the telescoping estimates, one obtains
\[ \|B\|_{\mathcal{A}} \leq 4 \|F\|_{L_1}. \quad (4.8) \]
Since $G$ is a Hardy martingale and $|w_{k-1}| = 1$ we have
\[ \mathbb{E}_{k-1} |\Delta G_k|^2 = 2 \mathbb{E}_{k-1} |\Im [w_{k-1} \cdot \Delta G_k]|^2. \]
Hence
\[ \|G\|_P = \sqrt{2}\|T(G)\|_P. \]  
(4.9)

Inserting the estimates (4.8) and (4.9) into equation (4.7) gives
\[ \|G\|_P \leq C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2}. \]  
(4.10)

It remains to replace in (4.10) the right hand side by \( \|F\|_{H^1} \). To this end we use the Burkholder Gundy inequality in combination with (4.10).
\[ \|F\|_{H^1} \leq 2\|G\|_P + \|B\|_A \leq C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2}. \]

Cancellation of \( \|F\|_{H^1}^{1/2} \) gives the square function estimate, (4.1) and with (4.10) the Davis and Garsia inequality (4.2) at the same time.

Remarks:

1. The proof of Theorem 4.1 and Theorem 2.1 yield general conditions on an integrable martingale to be in \( H^1 \). Consider a martingale \( F = (F_k)_{k=1}^n \) with a decomposition into \( G = (G_k)_{k=1}^n \) and \( B = (B_k)_{k=1}^n \) so that \( F = G + B \). Assume that there are \( C > 0 \) and \( \delta > 0 \) so that the following conditions are satisfied
\[ |\Delta G_k| \leq C|F_{k-1}|, \]
\[ \mathbb{E}|F_{k-1}| + \delta\mathbb{E}|\Delta B_k| \leq \mathbb{E}|F_k|, \]
\[ \mathbb{E}|G_k|^2 \leq C\mathbb{E}|F_{k-1}|3(w_{k-1}\Delta G_k)^2, \quad w_{k-1} = \frac{|F_{k-1}|}{|F_{k-1}|}. \]
Then
\[ \|F\|_{H^1} \leq A\|F\|_{L^1}, \]
where \( A = A(C, \delta) \)

2. Note also that
\[ \mathbb{E}|F_{k-1}| + \delta\mathbb{E}|\Delta B_k| \leq \mathbb{E}|F_k|, \]
and
\[ \mathbb{E}(|F_{k-1}|^2 + \delta\mathbb{E}|\Delta G_k|^2)^{1/2} \leq \mathbb{E}|F_k| + C\mathbb{E}|\Delta B_k| \]
give
\[ \|F\|_{H^1} \leq A\|F\|_{L^1}. \]
Dyadic Perturbation and Stability

The main results of this paper are Theorem 4.4 and Theorem 4.3. These theorems determine to which extent Theorem 4.1 is stable under dyadic perturbation. Theorem 4.5 and its application to the embedding theorem [2] were impetus for considering dyadic perturbations of Hardy martingales.

Dyadic martingales. We recall the definition of the dyadic $\sigma$ algebra on $\mathbb{T}^N$. It is defined by means of the independent Rademacher functions

$$\sigma_k(x) = \text{sign}(\cos_k(x)), \quad x = (x_k) \in \mathbb{T}^N,$$

where $\cos_k(x) = \Re x_k$. Let $\mathcal{D}$ be the $\sigma-$algebra on $\mathbb{T}^N$ generated by $\sigma_1, \ldots, \sigma_k, \ldots$.

Stopping time decomposition. We next fix two martingales, $F = (F_k)_{k=1}^n$ is Hardy and $D = (D_k)_{k=1}^n$ is dyadic. Fix $k \leq n$. Condition to $\mathcal{F}_{k-1}$. That is, fix $(x_1, \ldots, x_{k-1}) \in \mathbb{T}^{k-1}$. Put

$$h(y) = \Delta F_k(x_1, \ldots, x_{k-1}, y) \quad \text{and} \quad z = F_{k-1}(x_1, \ldots, x_{k-1}) - D_{k-1}(x_1, \ldots, x_{k-1}),$$

Let

$$\rho = \inf \{ t < \tau : |h(B_t)| > C_0|z|\}, \quad g = \mathbb{E}(h(B_{\rho})|B_\tau = e^{i\theta}),$$

and put

$$\Delta G_k(x_1, \ldots, x_{k-1}, y) = g(y), \quad \text{and} \quad \Delta B_k(x_1, \ldots, x_{k-1}, y) = h(y) - g(y),$$

such that

$$\Delta F_k = \Delta G_k + \Delta B_k. \quad (4.13)$$

The title of this paper refers to the martingale decomposition defined by (4.11) — (4.13). We turn now to proving the key properties of the Hardy martingale $G = (G_k)_{k=1}^n$ defined by the stopping times (4.12).

**Theorem 4.3** For every Hardy martingale $F = (F_k)_{k=1}^n$ and dyadic martingale $D = (D_k)_{k=1}^n$ the Hardy martingale $G = (G_k)_{k=1}^n$ defined by (4.12) satisfies the following estimates

$$|\Delta G_k| \leq C_0|F_{k-1} - D_{k-1}| \quad (4.14)$$

and

$$|F_{k-1} - D_{k-1}| + \frac{1}{4}\mathbb{E}_{k-1}|\Delta B_k| \leq \mathbb{E}_{k-1}|F_k - D_k| \quad (4.15)$$

where

$$B = F - G.$$

**Proof.** Fix $k \leq n$. Condition to $\mathcal{F}_{k-1}$ and put

$$z = F_{k-1} - D_{k-1}, \quad h = \Delta F_k, \quad g = \Delta G_k, \quad b\sigma = \Delta D_k.$$

By (4.11) $|g| \leq C_0|z|$, hence (4.14) holds. Theorem 3.2 gives

$$|z| + \frac{1}{4}\int_T|h - g|dm \leq \int_T|z + h - b\sigma|dm.$$

Since

$$z + h - b\sigma = F_k - D_k \quad \text{and} \quad h - g = \Delta B_k,$$

we translate back and get (4.15).
Following are the consequences of Theorem 4.3. For a given Hardy martingale $F = (F_k)_{k=1}^n$ and dyadic martingale $D = (D_k)$ let $T$ be defined as

$$T(H) = \Im \left[ \sum_{k=1}^n w_{k-1} \Delta H_k \right], \quad w_k = \frac{(F_k - D_k)}{|F_k - D_k|}. \quad (4.16)$$

The next theorem states Davis and Garsia inequalities for a perturbed Hardy martingale. In its proof we exploit Theorem 4.3 and Theorem 2.5.

**Theorem 4.4** For every Hardy martingale $F = (F_k)_{k=1}^n$ and dyadic martingale $D = (D_k)$ the Hardy martingale $G = (G_k)_{k=1}^n$ defined by (4.12) satisfies

$$\|B\|_A \leq C \|F - D\|_{L^1}, \quad (4.17)$$

where

$$B = F - G,$$

and $T$ defined by (4.16) satisfies

$$\|T(G - D)\|_P \leq C \|F - D\|^{1/2}_{L^1} \|F - D\|^{1/2}_{H^1}, \quad (4.18)$$

and

$$\|G\|_P \leq C \|F\|_{L^1} + C \|D\|_{H^1}. \quad (4.19)$$

**Proof.** Invoke the estimates of Theorem 4.3. Taking expectations in (4.15) gives

$$E|F_{k-1} - D_{k-1}| + \frac{1}{4} E|\Delta B_k| \leq E|F_k - D_k|.$$

Summing the telescoping series gives

$$E \sum_{k=1}^n |\Delta B_k| \leq 4E|F_n - D_n|, \quad (4.20)$$

or (4.17). Next use (4.14) and apply Theorem 2.5 to $T$ (defined in (4.16)). This gives

$$\|T(G - D)\|_P \leq C \|F - D\|^{1/2}_{L^1} \|F - D\|^{1/2}_{H^1} + C \|B\|_A,$$

Invoking (4.20) we get

$$\|T(G - D)\|_P \leq C \|F - D\|^{1/2}_{L^1} \|F - D\|^{1/2}_{H^1},$$

as claimed. The remaining estimate (4.19) is a simple consequence of the above. We get first,

$$\|T(G)\|_P \leq C \|F\|_{L^1} + C \|D\|_{H^1} + C \|T(D)\|_P. \quad (4.21)$$

Since $G$ is a Hardy martingale we have

$$E_{k-1}[\Delta G_k]^2 = 2E_{k-1}[\Im (w_{k-1} \cdot \Delta G_k)]^2, \quad (4.22)$$

whenever $w_{k-1}$ is $F_{k-1}$ measurable and $|w_{k-1}| = 1$. Hence $\|G\|_P = \sqrt{2} \|T(G)\|_P$. Note also that for regular martingales

$$\|T(D)\|_P \leq \|D\|_P \leq \|D\|_{H^1}.$$

This gives (4.19) as claimed.
The special Case $D = \mathbb{E}_D F$

We next specialize Theorem 4.4. We fix a Hardy martingale $F$ and $\mathbb{E}_D F$ its conditional expectation with respect to the dyadic $\sigma$ algebra. The resulting martingale transform $T$ is then

$$T(H) = \Im \left[ \sum w_{k-1} \cdot \Delta H_k \right], \quad w_k = (F_k - \mathbb{E}_D F_k) / |F_k - \mathbb{E}_D F_k|. \quad (4.23)$$

The following theorem records the content of Theorem 4.4 in this specialized setting.

**Theorem 4.5** For any Hardy martingale $F$ there is a splitting into Hardy martingales $G$ and $B$ so that

$$\|B\|_A \leq C_1 \|F - \mathbb{E}_D F\|_{L^1}, \quad (4.24)$$

and $T$ defined in (4.23) satisfies

$$\|T(G - \mathbb{E}_D G)\|_P \leq C_1 \|F - \mathbb{E}_D F\|_{L^1}^{1/2} \|F\|_{L^1}^{1/2}. \quad (4.25)$$

and

$$\|G\|_{H^1} \leq 2 \|G\|_P \leq C \|F\|_{L^1}. \quad (4.26)$$

**Proof.** Apply Theorem 4.4 to the Hardy martingale $F$ and its conditional expectation $\mathbb{E}_D F$. Use that $\|\mathbb{E}_D (F)\|_{H^1} \leq C_1 \|F\|_{L^1}$.

---

**An upper Estimate for $\mathbb{E}_D G$.**

It remains to complement the Davis Garsia inequalities in Theorem 4.5 with an upper bound for $\mathbb{E}_D G$. For any Hardy martingale $G = (G_k)_{k=1}^n$ and any (!) adapted sequence $W = (w_k)$ satisfying $|w_k| = 1$ the following holds

$$\|\mathbb{E}_D G\|_{H^1} \leq C \|T_W (G - \mathbb{E}_D G)\|_{L^1}^{1/4} \|G\|_{L^1}^{3/4} + C \|G - \mathbb{E}_D G\|_{L^1}^{1/2} \|G\|_{L^1}^{1/2}, \quad (4.27)$$

where $T_W$ is the martingale transform operator

$$T_W (G - \mathbb{E}_D G) = \Im \left[ \sum w_{k-1} \Delta_k (G - \mathbb{E}_D G) \right]. \quad (4.28)$$

See [2] and the Appendix for (4.27).

**The Embedding Theorem revisited**

J. Bourgain [2] determines a subspace of $L^1(\mathbb{T})/H_0^1(\mathbb{T})$ isomorphic to $L^1(\mathbb{T})$. The construction of such a subspace relies on the following $L^1-$ distance estimate.

There exists $\delta > 0$ such that for each dyadic martingale $D = (D_k)_{k=1}^n$ on $\mathbb{T}^N$,

$$\inf \|D - F\|_{L^1} > \delta \|D\|_{L^1}, \quad (4.29)$$

where the infimum is taken over all integrable Hardy martingales $F$. 

The embedding theorem will be deduced from the following interpolatory estimate:
For any Hardy martingale \( F = (F_k)_{k=1}^n \)
\[
\| \mathbb{E}_D F \|_{L^1} \leq C \| F - \mathbb{E}_D(F) \|_{L^1}^{\alpha} \| F \|^{1-\alpha}_{L^1}, \tag{4.30}
\]
for some \( \alpha > 0 \). See [2]. Recall that the estimate (4.30) is self improving. It implies that there exists \( A_0 > 0 \) so that for each Hardy martingale,
\[
\| F \|_{L^1} \leq A_0 \| F - \mathbb{E}_D F \|_{L^1}. \tag{4.31}
\]

Proof that (4.31) implies (4.29). The following proof is straightforward, and included for the sake being definite. Fix a dyadic martingale \( D = (D_k)_{k=1}^n \), resolve the inf on the left hand side of (4.29), thereby select a Hardy martingale \( F_0 \) so that
\[
\inf \| D - F \|_{L^1} \geq \frac{1}{2} \| D - F_0 \|_{L^1}.
\]
If \( \| F_0 \|_{L^1} \leq \| D \|_{L^1}/2 \) we have
\[
\| D - F_0 \|_{L^1} \geq \| D \|_{L^1} - \| F_0 \|_{L^1} \geq \| D \|_{L^1}/2.
\]
If conversely \( \| F_0 \|_{L^1} \geq \| D \|_{L^1}/2 \) we proceed by treating separately these two cases:
\[
\| D \|_{L^1} \geq 4A_0 \| \mathbb{E}_D F_0 - D \|_{L^1} \quad \text{and} \quad \| D \|_{L^1} \leq 4A_0 \| \mathbb{E}_D F_0 - D \|_{L^1}. \tag{4.32}
\]
In the first case we write
\[
\| D - F_0 \|_{L^1} \geq \| F_0 - \mathbb{E}_D F_0 \|_{L^1} - \| \mathbb{E}_D F_0 - D \|_{L^1}. \tag{4.33}
\]
Invoke (4.31) and use the first case in (4.32), that is,
\[
\| F_0 - \mathbb{E}_D F_0 \|_{L^1} \geq (1/A_0) \| F_0 \|_{L^1}, \quad \| \mathbb{E}_D F_0 - D \|_{L^1} \leq (1/4A_0) \| D \|_{L^1}. \tag{4.34}
\]
Since we assumed that \( \| F_0 \|_{L^1} \geq \| D \|_{L^1}/2 \) we get from (4.33) and (4.34) that
\[
\| D - F_0 \|_{L^1} \geq (1/4A_0) \| D \|_{L^1}.
\]
In the second case of (4.32) write
\[
\| D - F_0 \|_{L^1} \geq \| \mathbb{E}_D(D - F_0) \|_{L^1} = \| D - \mathbb{E}_D F_0 \|_{L^1} \geq (1/4A_0) \| D \|_{L^1}.
\]

Proof of (4.30).

The interpolatory estimate (4.30) follows routinely from Davis and Garsia inequalities (4.25) and (4.24) of Theorem 4.5 and the martingale estimate (4.27). The details of the derivation are given in the following string of remarks.

Let \( F \) be a Hardy martingale and apply to it Theorem 4.5. Let \( F = G + B \) be the corresponding decomposition. Then
\[
\mathbb{E}_D F = \mathbb{E}_D G + \mathbb{E}_D B. \tag{4.35}
\]
Step 1. Use (4.24) directly to bound $\mathbb{E}_\mathcal{D}B$. Since $\Delta(\mathbb{E}_\mathcal{D}B)_k = \mathbb{E}_\mathcal{D}(\Delta(B)_k)$, we get with (4.24),

$$\|\mathbb{E}_\mathcal{D}B\|_{L^1} \leq \|B\|_\mathcal{A} \leq C_1\|F - \mathbb{E}_\mathcal{D}(F)\|_{L^1}. \quad (4.36)$$

Step 2. Use (4.27) to bound $\mathbb{E}_\mathcal{D}G$. This gives

$$\|\mathbb{E}_\mathcal{D}G\|_{L^1} \leq \|T(G - \mathbb{E}_\mathcal{D}G)\|_P^{1/4}\|G\|_P^{3/4} + C\|G - \mathbb{E}_\mathcal{D}G\|_{L^1}^{1/2}\|G\|_{L^1}^{1/2}, \quad (4.37)$$

where the operator $T$ is the one appearing in (4.25).

Step 3. Theorem 4.5 controls the terms on the right hand side of (4.37). Indeed (4.25) gives

$$\|T(G - \mathbb{E}_\mathcal{D}G)\|_P \leq C\|F - \mathbb{E}_\mathcal{D}F\|_{L^1}^{1/2}\|F\|_{L^1}^{1/2}. \quad (4.38)$$

Moreover by (4.26), $\|G\|_{\mathcal{H}^1} \leq 2\|G\|_P \leq C\|F\|_{L^1}$.

Step 4. The remaining factor in (4.37) is $\|G - \mathbb{E}_\mathcal{D}G\|_{L^1}$. By triangle inequality

$$\|G - \mathbb{E}_\mathcal{D}G\|_{L^1} \leq \|F - \mathbb{E}_\mathcal{D}F\|_{L^1} + \|B - \mathbb{E}_\mathcal{D}B\|_{L^1}.$$ 

By (4.36),

$$\|B - \mathbb{E}_\mathcal{D}B\|_{L^1} \leq 2\|B\|_\mathcal{A} \leq 8\|F - \mathbb{E}_\mathcal{D}F\|_{L^1}.$$ 

Hence

$$\|G - \mathbb{E}_\mathcal{D}G\|_{L^1} \leq C\|F - \mathbb{E}_\mathcal{D}F\|_{L^1}. \quad (4.39)$$

Estimating the right hand side of (4.37) using (4.38) and (4.39), gives

$$\|\mathbb{E}_\mathcal{D}G\|_{L^1} \leq C\|F - \mathbb{E}_\mathcal{D}F\|_{L^1}^{1/8}\|F\|_{L^1}^{7/8}. \quad (4.40)$$

Step 5. The identity (4.35) and (4.36), (4.40) give

$$\|\mathbb{E}_\mathcal{D}F\|_{\mathcal{H}^1} \leq C\|F - \mathbb{E}_\mathcal{D}(F)\|_{L^1}^{\alpha}\|F\|_{L^1}^{1-\alpha}, \quad \text{with} \quad \alpha = 1/8.$$

5 Appendix I. Sine and Cosine Martingales

In the course of proving (4.30) we invoked (4.27). The proof of (4.27) is done in two independent propositions concerning estimates for cosine martingales. Cosine martingales are defined in (1.1). Recall that the cosine-martingale $U = (U_k)$ of a Hardy martingale $G = (G_k)$ is defined by

$$\Delta U_k(x, y) = \frac{1}{2} [\Delta G_k(x, y) + \Delta G_k(x, \bar{y})], \quad x \in \mathbb{T}^{k-1}, y \in \mathbb{T}. \quad (5.1)$$

Proposition 5.1 and Proposition 5.2 form the link between cosine martingale, $\mathbb{E}_\mathcal{D}G$ and $G - \mathbb{E}_\mathcal{D}G$. 

\[\]
Proposition 5.1 Let $G = (G_k)$ be a Hardy martingale, and let $U = (U_k)$ be its cosine martingale defined by (5.1). Then

$$
\| E_D G \|_{H^1} \leq C \| U - E_D U \|_{H^1}^{1/2} \| G \|_{H^1}^{1/2} + C \| G - E_D G \|_{L^1}^{1/2} \| G \|_{H^1}^{1/2}
$$

(5.2)

The proof of Proposition 5.1 is in [2] pp. 700 –702.

Proposition 5.2 Assume that $W = (w_k)$ is adapted and $|w_k| = 1$, then

$$
\| U - E_D U \|_P \leq C \| T_W (G - E_D G) \|_P^{1/2} \| G \|_P^{1/2},
$$

(5.3)

where $T_W$ is defined by (4.28).

The proof of Proposition 5.2 is in [2] p. 700.

Randomizing Martingales

We give the proof of Proposition 5.1 as in [2] pp. 700 –702. The interpolatory estimates (5.8) for the averaging projection $P$ defined in (5.6) are the central ingredient.

Randomizing. Let $V = (V_k)$ be any martingale on $\mathbb{T}^N$. We define two ways of randomizing $V = (V_k)$.

1. We associate to $V = (V_k)$ a family of martingales on $\mathbb{T}^N$ parametrized by $\varepsilon \in \{-1, 1\}^N$. Put $v_k = \Delta V_k$ and define

$$
v_k(x, \varepsilon) = v_k(x_1^{\varepsilon_1}, \ldots, x_k^{\varepsilon_k}),
$$

and

$$
V(x, \varepsilon) = \sum v_k(x, \varepsilon), \quad x \in \mathbb{T}^N, \quad \varepsilon \in \{-1, 1\}^N.
$$

The partial sums of the series on the right hand side form a family of martingales on $\mathbb{T}^N$ parametrized by $\varepsilon \in \{-1, 1\}^N$ so that for each fixed $\varepsilon$,

$$
E_x |V(x, \varepsilon)| = E_x |V(x)| = \|V\|_{L^1}.
$$

(5.4)

and

$$
E_x \left( \sum_{k=1}^n |v_k(x, \varepsilon)|^2 \right)^{1/2} = E_x \left( \sum_{k=1}^n |v_k(x)|^2 \right)^{1/2} = \|V\|_{H^1}.
$$

(5.5)

2. Next we associate to the martingale $V = (V_k)$ a family of dyadic martingales parametrized by $x \in \mathbb{T}^N$. Put

$$
d_{k-1}(x, \varepsilon) = v_k(x_1^{\varepsilon_1}, \ldots, x_{k-1}^{\varepsilon_{k-1}}, x_k),
$$

and form the family of dyadic martingales

$$
D(x, \varepsilon) = \sum d_{k-1}(x, \varepsilon) \varepsilon_k,
$$

parametrized by $x \in \mathbb{T}^N$. 

Rademacher coefficients. The Rademacher coefficients of the parametrized dyadic martingales $D(x, \varepsilon)$ give rise to averaging projections for $V = (V_k)$.

1. Given the dyadic martingales $D(x, \varepsilon)$ the Rademacher coefficients are defined as

$$ (R_k(D)(x) = \mathbb{E}_\varepsilon(D(x, \varepsilon)\varepsilon_k), $$

where again $x \in \mathbb{T}_N$ is just the parameterindex of the family $D(x, \varepsilon)$. Remark that $R_k(D) = \mathbb{E}_\varepsilon(d_{k-1})$ and that $R_k(D)$ is determined by the martingale difference $v_k = \Delta V_k$. We put

$$ P(v_k) = R_k(D) $$

and form the linear extension,

$$ P(V) = \sum P(v_k) = \sum R_k(D). \tag{5.6} $$

2. Bourgain’s version of the Garnett Jones inequality (see [2], [4], [12]) implies that for the Rademacher coefficients

$$ R_k(D) = \mathbb{E}_\varepsilon(D(\varepsilon)\varepsilon_k) $$

of a dyadic martingale

$$ D(\varepsilon) = \sum d_{k-1}(\varepsilon)\varepsilon_k $$

there holds

$$ \sum |R_k(D)|^2 \leq C\mathbb{E}_\varepsilon(\sum |d_{k-1}(\varepsilon)|^2)^{1/2} \tag{5.7} $$

Sine martingales. Comparing the series representation of $V(x, \varepsilon)$ and $D(x, \varepsilon)$ it is clear that we constructed two different objects, unless we have further assumptions on the underlying martingale $V = (V_k)$. If $V = (V_k)$ is a sine martingale (see (1.2)) then

$$ v_k(x, \varepsilon) = d_{k-1}(x, \varepsilon)\varepsilon_k $$

and $D(x, \varepsilon) = V(x, \varepsilon)$.

Proposition 5.3 Let $V = (V_k)$ be a sine-martingale on $\mathbb{T}_N$.

$$ \|P(V)\|_{H^1} \leq C\|V\|_{L^1}^{1/2}\|V\|_{H^1}^{1/2}. \tag{5.8} $$

Proof. Since $V = (V_k)$ is a sine-martingale we have $v_k(x, \varepsilon) = d_{k-1}(x, \varepsilon)\varepsilon_k$ and

$$ P(v_k)(x) = \mathbb{E}_\varepsilon(d_{k-1}(x, \varepsilon)) = \mathbb{E}_\varepsilon(v_k(x, \varepsilon)\varepsilon_k). $$

Hence by (5.7),

$$ \sum |P(v_k)(x)|^2 \leq C\mathbb{E}_\varepsilon(V(x, \varepsilon))\mathbb{E}_\varepsilon(\sum |v_k(x, \varepsilon)|^2)^{1/2}. \tag{5.9} $$

Applying (5.9) shows that

$$ \|P(V)\|_{H^1} = \mathbb{E}_\varepsilon(\sum |P(v_k)(x)|^2)^{1/2} $$
is bounded by a multiple of
\[ \mathbb{E}_x \left( \mathbb{E}_\varepsilon |V(x, \varepsilon)| \mathbb{E}_\varepsilon \left( \sum |v_k(x, \varepsilon)|^2 \right)^{1/2} \right)^{1/2} \].
\[ (5.10) \]
Next apply the Cauchy Schwarz inequality so that (5.10) is bounded by
\[ (\mathbb{E}_x \mathbb{E}_\varepsilon |V(x, \varepsilon)|)^{1/2} \left( \mathbb{E}_x \mathbb{E}_\varepsilon \left( \sum |v_k(x, \varepsilon)|^2 \right)^{1/2} \right)^{1/2} \].
Apply Fubini and invoke the identities (5.4) and (5.4) to get
\[ \|P(V)\|_{H^1} \leq C \|V\|^{1/2}_{L^1} \|V\|^{1/2}_{H^1}. \]
\[ (5.11) \]

Next we record an application to Hardy martingales. Let \( G = (G_k) \) be a Hardy martingale and \( U = (U_k) \) its cosine martingale given by (5.1).

**Proposition 5.4** Let \( Z = (Z_k) \) be the \((\mathcal{F}_k)\) martingale with difference sequence
\[ \Delta Z_k = \mathbb{E}_{k-1}(\Delta U_k \cos k)\sigma_k. \]
Then,
\[ \|P(Z)\|_{H^1} \leq C \|P(G - U)\|_{H^1}, \]
and consequently
\[ \|P(Z)\|_{H^1} \leq C \|G - U\|^{1/2}_{L^1} \|G - U\|^{1/2}_{H^1}. \]
\[ (5.12) \]

**PROOF.** Note, \( V = G - U \) is a sine martingale and by the analyticity of \( \Delta G_k \),
\[ \mathbb{E}_{k-1}(\Delta U_k \cos k) = -i\mathbb{E}_{k-1}(\Delta V_k \sin k). \]
Next, (5.13) implies
\[ |P\Delta Z_k| = |P\mathbb{E}_{k-1}(\Delta V_k \sin k)|, \]
and
\[ |P\Delta Z_k| \leq \mathbb{E}_{k-1}|P\Delta V_k|. \]
\[ (5.14) \]
Hence
\[ \left( \sum |P\Delta Z_k|^2 \right)^{1/2}_{L^1} \leq C \left( \sum \mathbb{E}_{k-1}^2 |P\Delta V_k|^2 \right)^{1/2}_{L^1}. \]
\[ (5.15) \]
With Lepingle inequality [10], the right hand side of (5.15) is bounded by \( C \|PV\|_{H^1} \), that is (5.11) holds. Since \( V = G - U \) is a sine-martingale, Proposition 5.3 yields (5.12).
APPENDIX I. SINE AND COSINE MARTINGALES

Proof of Proposition 5.1.

Part 1. With the notation of Proposition 5.4 we claim that
\[ \|E_D G\|_{H^1} \leq C \|U - E_D U\|_{H^1} + \|G - U\|_{L^1}^{1/2} \|G - U\|_{H^1}^{1/2}. \] (5.16)

To this end let \( Z = (Z_k) \) be the martingale with difference sequence
\[ \Delta Z_k = E_{k-1}(\Delta U_k \cos k) \sigma_k. \]

Observe that \( E_D G = E_D U \). The key identity is
\[ E_D G = P(Z) + (E_D U - P(Z)), \] (5.17)
where \( P \) is defined in (5.6). Proposition 5.4 readily gives estimates for the first summand \( P(Z) \) in (5.17).
\[ \|P(Z)\|_{H^1} \leq C \|G - U\|_{L^1}^{1/2} \|G - U\|_{H^1}^{1/2}. \] (5.18)

Next we turn to estimating \( E_D U - P(Z) \). Since \( E_D U \) is just even, we have \( P(E_D U) = E_D U \), and
\[ P(Z) - E_D U = P(Z - E_D U). \]

By definition \( Z \) is a cosine martingale, hence the operator \( P \) acts as averaging on \( Z - E_D U \) and
\[ \|P(Z - E_D U)\|_{H^1} \leq C \|Z - E_D U\|_{H^1}. \]

This, and invoking Lepingle inequality gives
\[ \|P(Z) - E_D U\|_{H^1} \leq C \|Z - E_D U\|_{H^1} \leq C \|U - E_D U\|_{H^1}. \] (5.19)
Summing up, (5.16) follows from the identity (5.17) combined with the estimates (5.18) and (5.19).

Part 2. By (5.16)
\[ \|E_D G\|_{H^1} \leq C \|G - U\|_{L^1}^{1/2} \|G - U\|_{H^1}^{1/2} + C \|U - E_D U\|_{H^1}. \] (5.20)

Since
\[ E_D G = E_D U, \]
we have the identity
\[ G - U = G - E_D G - (U - E_D U), \] (5.21)
which gives immediately
\[ \|G - U\|_{L^1} \leq \|G - E_D G\|_{L^1} + C \|U - E_D U\|_{H^1}. \] (5.22)

Next invoke the (routine) estimates
\[ \|G - U\|_{H^1} \leq C \|G\|_{H^1} \quad \text{and} \quad \|U - E_D U\|_{H^1} \leq C \|G\|_{H^1}. \]

Thus, (5.22) and (5.21) imply that
\[ \|G - U\|_{L^1}^{1/2} \|G - U\|_{H^1}^{1/2} \leq C \|G - E_D G\|_{L^1}^{1/2} \|G\|_{L^1}^{1/2} + C \|U - E_D U\|_{H^1}^{1/2} \|G\|_{H^1}^{1/2}, \]
hence with (5.20) we obtained (5.2).
Estimating Cosine Martingales

We give the proof of Proposition 5.2 as in [2] p. 700.

Let $L^2_G(T)$ denote the space of (complex valued) even functions in $L^2(T)$, and $L^2_U(T)$ the subspace of $L^2(T)$ consisting of (complex valued) odd functions. The space $L^2(T)$ is the direct sum of the orthogonal subspaces $L^2_G(T)$ and $L^2_U(T)$,

$L^2(T) = L^2_G(T) \oplus L^2_U(T)$.

Recall that $\sigma(\theta) = \text{sign} \cos(\theta)$. Put $w_0 = 1_T$, $w_1 = \sigma$, and choose any orthonormal system \{w_k : k \geq 2\} in $L^2_G(T)$ so that \{w_k : k \geq 0\} is a complete orthonormal basis for $L^2_G(T)$.

We next observe that in $L^2(T)$ an orthonormal basis is given by the system \{(w_k, Hw_k) : k \geq 0\}.

For the Hardy space $H^2(T)$ the analytic system

\{(w_k + iHw_k) : k \geq 0\}

is an orthogonal basis with $\|w_k + iHw_k\|_2 = \sqrt{2}$, $k \geq 1$.

**Proposition 5.5** Let $h \in H^2_0(T)$, and $u(z) = (h(z) + h(\bar{z}))/2$ be the even part of $h$. Then for $w, b \in \mathbb{C}$, with $|w| = 1$,

$$\Im (w \cdot (\langle u, \sigma \rangle - b)) + \Re (w \cdot \langle u, \sigma \rangle) + \int_T |u - \langle u, \sigma \rangle \sigma|^2 d\mu = \int_T \Im (w \cdot (h - b\sigma)) d\mu$$

**Proof.** Fix $h \in H^2_0(T)$ and $w, b \in \mathbb{C}$, with $|w| = 1$. Clearly by replacing $h$ by $wh$ and $b$ by $wb$ it suffices to prove the proposition with $w = 1$. Since $\int u = 0$ we have that

$$u = \sum_{n=1}^{\infty} c_n w_n.$$  

Apply the Hilbert transform and regroup to get

$$h - b\sigma = (c_1 - b)\sigma + ic_1 H\sigma + \sum_{n=2}^{\infty} c_n (w_n + iHw_n). \quad (5.23)$$

Then, taking imaginary parts gives

$$\Im(h - b\sigma) = \Im(c_1 - b)\sigma + \Re c_1 H\sigma + \sum_{n=2}^{\infty} \Im c_n w_n + \Re c_n Hw_n. \quad (5.24)$$

By ortho-gonality the identity (5.24) yields

$$\int_T \Im^2(h - b\sigma) d\mu = \Im^2(c_1 - b) + \Re^2 c_1 + \sum_{n=2}^{\infty} |c_n|^2. \quad (5.25)$$

On the other hand, since $\int u = 0$, $c_1 = \langle u, \sigma \rangle$, and $w_1 = \sigma$ we get

$$\int_T |u - \langle u, \sigma \rangle \sigma|^2 d\mu = \sum_{n=2}^{\infty} |c_n|^2. \quad (5.26)$$

Comparing the equations (5.25) and (5.26) completes the proof.

\[\blacksquare\]
We use below some arithmetic, that we isolate first.

**Lemma 5.6** Let \( \mu, b \in \mathbb{C} \) and

\[
|\mu| + |\mu - b|^2 = a. \tag{5.27}
\]

Then for any \( w \in \mathbb{T} \),

\[
(a - |b|)^2 \leq 4(\Re^2(w \cdot (\mu - b)) + \Re^2(w \cdot \mu)). \tag{5.28}
\]

and

\[
|\mu - b|^2 \leq 2(a^2 - |\mu|^2). \tag{5.29}
\]

**Proof.** By rotation invariance it suffices to prove (5.28) for \( w = 1 \). Let \( \mu = m_1 + im_2 \) and \( b = b_1 + ib_2 \). By definition (5.27), we have

\[
a - |b| = \frac{|\mu|^2 - |b|^2 + |\mu - b|^2}{|\mu| + |b|}.
\]

Expand and regroup the numerator

\[
|\mu|^2 - |b|^2 + |\mu - b|^2 = 2m_1(m_1 - b_1) + 2m_2(m_2 - b_2). \tag{5.30}
\]

By the Cauchy Schwarz inequality, the last term in (5.30) is bounded by

\[
2(m_1^2 + (m_2 - b_2)^2)^{1/2}(m_2^2 + (m_1 - b_1)^2)^{1/2}
\]

Note that \( m_1 = \Re \mu \) and \( m_2 - b_2 = \Im(\mu - b) \). It remains to observe that

\[
(m_2^2 + (m_1 - b_1)^2)^{1/2} \leq |\mu| + |b|.
\]

or equivalently

\[
m_1^2 + m_2^2 - 2m_1b_2 + b_1^2 \leq |\mu|^2 + 2|\mu||b| + |b|^2,
\]

which is obviously true.

Next we turn to verifying (5.29). We have \( a^2 - |\mu|^2 = (a + |\mu|)(a - |\mu|) \) hence

\[
a^2 - |\mu|^2 = \left[2|\mu| + \frac{|\mu - b|^2}{|\mu| + |b|}\right] \frac{|\mu - b|^2}{|\mu| + |b|}. \tag{5.31}
\]

In view of (5.31) we get (5.29) by showing that

\[
2|\mu|^2 + 2|\mu||b| + |\mu - b|^2 \geq \frac{1}{2}(|\mu| + |b|)^2. \tag{5.32}
\]

The left hand side of (5.32) is larger than \( |\mu|^2 + |b|^2 \) while the right hand side of (5.32) is smaller \( |\mu|^2 + |b|^2 \). 

\[\blacksquare\]
We merge the inequalities of Lemma 5.6 with the identity in Proposition 5.5.

**Proposition 5.7** There exists $C_0 > 0$ so that the following holds. Let $w, b \in \mathbb{C}$, with $|w| = 1$, $h \in H^2_0(\mathbb{T})$, let $u$ be the even part of $h$ and put

$$|\langle u, \sigma \rangle| + \frac{|\langle u, \sigma \rangle - b|^2}{|b| + |\langle u, \sigma \rangle|} = a.$$ 

Then

$$\int_{\mathbb{T}} |u - b\sigma|^2 dm(y) \leq 8(a^2 - |\langle u, \sigma \rangle|^2) + \int_{\mathbb{T}} |u - \langle u, \sigma \rangle\sigma|^2 dm(y). \quad (5.33)$$

and

$$(a - |b|)^2 + \int_{\mathbb{T}} |u - \langle u, \sigma \rangle\sigma|^2 dm(y) \leq 8 \int_{\mathbb{T}} \Im^2(w \cdot (h - b\sigma)) dm(y). \quad (5.34)$$

**Proof.** Put

$$J^2 = \int_{\mathbb{T}} \Im^2(w \cdot (h - b\sigma)) dm(y). \quad (5.35)$$

The proof exploits the basic identities for the integral $J^2$ and $\int_{\mathbb{T}} |u - b\sigma|^2 dm(y)$ and intertwines them with the arithmetic (5.27) – (5.28).

**Step 1.** Use the straightforward identity,

$$\int_{\mathbb{T}} |u - b\sigma|^2 dm(y) = |\langle u, \sigma \rangle - b|^2 + \int_{\mathbb{T}} |u - \langle u, \sigma \rangle\sigma|^2 dm(y). \quad (5.36)$$

Apply (5.29), so that

$$|\langle u, \sigma \rangle - b|^2 \leq 8(a^2 - |\langle u, \sigma \rangle|^2),$$

hence by (5.36) we get (5.33),

$$\int_{\mathbb{T}} |u - b\sigma|^2 dm(y) \leq 8(a^2 - |\langle u, \sigma \rangle|^2) + \int_{\mathbb{T}} |u - \langle u, \sigma \rangle\sigma|^2 dm(y).$$

**Step 2.** Proposition 5.5 asserts that

$$\Im^2(w \cdot (\langle u, \sigma \rangle - b)) + \Re^2(w \cdot \langle u, \sigma \rangle) + \int_{\mathbb{T}} |u - \langle u, \sigma \rangle\sigma|^2 dm(y) = J^2. \quad (5.37)$$

Apply (5.28) with $\mu = \langle u, \sigma \rangle$ to the left hand side in (5.37), and get (5.34),

$$(a - |b|)^2 + \int_{\mathbb{T}} |u - \langle u, \sigma \rangle\sigma|^2 dm(y) \leq 8J^2.$$

$\blacksquare$
Proof of Proposition 5.2.

Let \( \{g_k\} \) be the martingale difference sequence of the Hardy martingale \( G = (G_k) \), and let \( \{u_k\} \) be the martingale difference sequence of the associated cosine martingale \( U = (U_k) \). Note

\[
E_D(u_k) = E_D E_{k-1}(u_k \sigma_k) \sigma_k.
\]

**Step 1.** Put \( b_k = E_D E_{k-1}(u_k \sigma_k) \), and put

\[
Y^2 = \sum_{k=1}^{\infty} |E_{k-1}(u_k \sigma_k)|^2 \quad \text{and} \quad Z^2 = \sum_{k=1}^{\infty} |b_k|^2.
\]

We have

\[
E_D \left( \sum_{k=1}^{\infty} |E_{k-1}(u_k \sigma_k)|^2 \right)^{1/2} \geq \left( \sum_{k=1}^{\infty} |E_D E_{k-1}(u_k \sigma_k)|^2 \right)^{1/2}.
\]

hence

\[
E(Y) \geq E(Z). \quad (5.38)
\]

**Step 2.** Since \( E_D(g_k) = E_D(u_k) \), the square of the conditioned square functions of \( T_W(G - E_D G) \) is

\[
\sum_{k=1}^{\infty} E_{k-1} |3(w_{k-1} \cdot (g_k - b_k \sigma_k))|^2. \quad (5.39)
\]

**Step 3.** The martingale differences of \( U - E_D(U) \) is \( \{u_k - b_k \sigma_k\} \). The square of its conditioned square functions of

\[
\sum_{k=1}^{\infty} E_{k-1} |u_k - b_k \sigma_k|^2. \quad (5.40)
\]

Following the pattern of (5.27) define

\[
a_k = |E_{k-1}(u_k \sigma_k)| + |E_{k-1}(u_k \sigma_k) - b_k|^2.
\]

and

\[
u_k = u_k - E_{k-1}(u_k \sigma_k), \quad r_k^2 = E_{k-1}|u_k|^2.
\]

By (5.33)

\[
E_{k-1}|u_k - b_k \sigma_k|^2 \leq 8(a_k^2 + r_k^2 - |E^2_{k-1}(u_k \sigma_k)|). \quad (5.41)
\]

**Step 4.** Define

\[
X^2 = \sum_{k=1}^{\infty} a_k^2 + r_k^2,
\]

then \( X \geq Y \) and

\[
\|U - E_D(U)\|_p \leq \sqrt{E(X^2 - Y^2)^{1/2}} \leq C(E(X - Y))^{1/2}(E(X + Y))^{1/2}. \quad (5.42)
\]

The second factor \( E(X + Y) \) in (5.42) is simply bounded as

\[
E(X + Y) \leq C\|U\|_p \leq C\|G\|_p. \quad (5.43)
\]
Step 5. Next we turn to estimates for $\mathbb{E}(X - Y)$. First recall $\mathbb{E}(X - Y) \leq \mathbb{E}(X - Z)$, and by triangle inequality

$$X - Z \leq \left( \sum_{k=1}^{\infty} (a_k - |b_k|)^2 + r_k^2 \right)^{1/2}.$$  

By (5.34)

$$(a_k - |b_k|)^2 + r_k^2 \leq 8 \mathbb{E}_{k-1}[\mathfrak{S}(w_{k-1} : (g_k - b_k\sigma_k))].$$  

Combining (5.42) — (5.44) gives

$$\mathbb{E}(X - Y) \leq \mathbb{E}(X - Z) \leq C\|T_W(G - \mathbb{E}_D G)\|_{\mathcal{P}}.$$  

6 Appendix II. The Transfer Operator

We transfer the the $L^1$ distance estimate from the infinite torus to $\mathbb{T}$. See [2] p. 697.

The space of integrable Hardy martingale is denoted by $H^1(\mathbb{T}^\mathbb{N})$. We showed that there exists $A_0 > 0$ so that

$$\|D\|_{L^1} \leq A_0\|F - D\|_{L^1},$$  

whenever $F \in H^1(\mathbb{T}^\mathbb{N})$ and $D$ is a dyadic martingale on $\mathbb{T}^\mathbb{N}$. We construct a diffuse sigma-algebra $\Sigma$ on $\mathbb{T}$, so that

$$\|h\|_{L^1(\mathbb{T})} \leq A\|f - h\|_{L^1(\mathbb{T})},$$  

whenever $h \in L^1(\Sigma)$ and $f \in H^1(\mathbb{T})$, or equivalently, $\|h\|_{L^1(\Sigma)} \leq A\|h\|_{L^1(\mathbb{T})/H_0^1(\mathbb{N})}$ for $h \in L^1(\Sigma)$. J. Bourgain [2] p. 697 determined a bounded linear operator $J : L^1(\mathbb{T}) \to L^1(\mathbb{T}^\mathbb{N})$ so that the restrictions to $L^1(\Sigma)$ respectively to $H^1(\mathbb{T})$ satisfy the following conditions.

1. The restriction of $J$ to $L^1(\Sigma)$ is an embedding, There exists $A_1 > 0$ so that

$$\frac{1}{A_1}\|h\|_{L^1(\mathbb{T})} \leq \|Jh\|_{L^1} \leq A_1\|h\|_{L^1(\mathbb{T})}, \quad h \in L^1(\Sigma).$$  

2. The conditional expectation operator $\mathbb{E}_D$ onto the subspace of dyadic martingales in $L^1(\mathbb{T}^\mathbb{N})$ acts as a small perturbation of the identity on $J(L^1(\Sigma))$,

$$\|H - \mathbb{E}_D H\|_{L^1} \leq \frac{1}{4A_0}\|\mathbb{E}_D H\|_{L^1} \quad H \in J(L^1(\Sigma)).$$  

3. The restriction of $J$ to $H^1(\mathbb{T})$ maps into the space of integrable Hardy martingales,

$$J(H^1(\mathbb{T})) \subseteq H^1(\mathbb{T}^\mathbb{N}).$$  

We use the bounded operator $J$ satisfying (6.2)—(6.4) to prove that $L^1(\Sigma)$ embeds as a closed linear subspace into $L^1(\mathbb{T})/H_0^1(\mathbb{T})$.

**Theorem 6.1** There exists $A > 0$ so that for each $h \in L^1(\Sigma)$

$$\|h\|_{L^1(\mathbb{T})} \leq A\|h\|_{L^1(\mathbb{T})/H_0^1(\mathbb{T})}.$$
Proof. Let \( h \in L^1(\Sigma) \) and \( f \in H^1(T) \). Put \( H = Jh, F = Jf \) and \( D = E_DH \). By (6.4), \( F \) is a Hardy martingale. Since \( D \) is dyadic by definition, (6.1) gives
\[
\|D\|_{L^1} \leq A_0\|F - D\|_{L^1}.
\] (6.5)
Write \( F - D = (F - H) + (H - D) \). Since \( D = E_DH \) we get from (6.3) and (6.5) that
\[
\|D\|_{L^1} \leq A_0\|F - H\|_{L^1} + \frac{1}{4}\|D\|_{L^1}.
\] (6.6)
Next use that \( J \) is an embedding of \( L^1(\Sigma) \). By (6.2) and (6.3) we have
\[
\|h\|_{L^1(T)} \leq 2A_1\|D\|_{L^1},
\] (6.8)
where \( A = 4A_1A_0\|J\| \).

The space \( L^1(\Sigma) \).

The Fejer kernels \( F_a, a \in \mathbb{N} \) on \( T \) are defined as
\[
F_a(z) = \sum_{|j| \leq a} \left( 1 - \frac{|j|}{a + 1} \right) z^j, \quad z \in T.
\]
We let \( \sigma(z) = \text{sign}(\Re z) \). Define inductively the sigma algebra \( \Sigma \) on \( T \).

**Step 1.** Let \( A_0 > 0 \) be the constant appearing in (6.1).
Fix \( \epsilon > 0 \) where \( \epsilon = \epsilon(A_0) \leq (100A_0)^{-1} \). Let \( n_1 = 1 \). Select \( a_1 \in \mathbb{N} \) so that
\[
\|s_1 - \sigma\|_1 < \epsilon/2,
\]
where \( s_1 = K_{a_1} \ast \sigma \). Put
\[
E_1 = \{k_1 \in (-a_1, a_1) \cap \mathbb{Z}\}.
\]
Having defined integers \( n_1 < \cdots < n_m \) and \( a_1 < \cdots < a_m \). Form
\[
E_m = \{\sum_{i=1}^{m} k_i n_i : k_i \in (-a_i, a_i) \cap \mathbb{Z}\}.
\]

**Step \( m + 1 \).** Choose \( n_{m+1} \geq n_m \) so that
\[
4^m|j| \leq n_{m+1}, \quad j \in E_m.
\] (6.9)
Select \( a_{m+1} \geq a_m \) so that
\[
\|s_{m+1} - \sigma\|_1 \leq 2^{-m}\epsilon.
\]
where \( s_{m+1} = K_{a_{m+1}} \ast \sigma \) and \( \sigma(z) = \text{sign}(\Re z) \).
**The conditional expectation** $\mathbb{E}_\Sigma$. Define $\Sigma$ to be the $\sigma-$algebra on $\mathbb{T}$ generated by the sequence of Rademacher functions

$$\sigma(z^{n_k}), \quad z \in \mathbb{T}.$$ 

Define the non-negative kernel as

$$B(z, \zeta) = \prod_{k=1}^{\infty} (1 + \sigma(z^{n_k})\sigma(\zeta^{n_k})), \quad z, \zeta \in \mathbb{T}.$$ 

Let $\mathbb{E}_\Sigma$ be the conditional expectation operator acting on $L^1(\mathbb{T})$ onto $L^1(\Sigma)$. It is an integral operator with kernel $B(z, \zeta)$,

$$\mathbb{E}_\Sigma(g)(z) = \int_{\mathbb{T}} B(z, \zeta) g(\zeta) dm(\zeta), \quad g \in L^1(\mathbb{T}). \quad (6.10)$$

**Products of Fejer kernels** Put $E = \bigcup E_m$. Let $(a_k)$ and $(n_k)$ be the sequence given in the construction of $\Sigma$. Form the pointwise products of Fejer kernels

$$K(\zeta) = \prod_{k=1}^{\infty} F_{a_k}(\zeta^{n_k}).$$

We have the following relations for the Fourier coefficients of $K$,

$$\hat{K}(0) = 1, \quad \text{and} \quad \{m \in \mathbb{Z} : \hat{K}(m) \neq 0\} \subseteq E,$$

$$\hat{K}(\sum_{k=1}^{\infty} b_k n_k) = \prod_{k=1}^{\infty} \hat{F}_{a_k}(b_k).$$

Hence the Fourier expansion of $K$ is as follows,

$$K(\zeta) = \sum_{k=1}^{\infty} \sum_{b_k=-a_k, b_k \neq 0}^{a_k} \cdots \sum_{b_1=-a_1, j=1}^{a_1} \prod_{j=1}^{k} \hat{F}_{a_j}(b_j) \zeta^{n_j b_j}.$$ 

Let $L^1_E = \{f \in L^1(\mathbb{T}) : \hat{f}(m) = 0 \text{ for } m \notin E\}$. We observed that $K \in L^1_E$.

**Embedding** $L^1(\Sigma)$ into $L^1_E$. The next proposition identifies the integral kernel of the conditional expectation operator $\mathbb{E}_\Sigma$ after its convolution with $K$.

**Proposition 6.2** Let $Rg = K * (\mathbb{E}_\Sigma g), \ g \in L^1(\mathbb{T}),$ and put

$$A(z, \zeta) = \prod_{k=1}^{\infty} (1 + s_k(z^{n_k})\sigma(\zeta^{n_k})), \quad z, \zeta \in \mathbb{T}.$$ 

Then

$$Rg(z) = \int_{\mathbb{T}} A(z, \zeta) g(\zeta) dm(\zeta), \quad z \in \mathbb{T}. \quad (6.11)$$
Proof. We show that

\[ K \ast_z B(z, \zeta) = A(z, \zeta), \]

where the convolution is taken with respect to the \( z \) variable. To this end we observe that for fixed \( \zeta \in \mathbb{T} \) the following identities hold.

\[
K \ast_z B(z, \zeta) = \sum_{k=1}^{\infty} \sum_{b_k = -a_k, b_k \neq 0}^{a_k} \cdots \sum_{b_1 = -a_1}^{a_1} \prod_{j=1}^{k} F_{a_j}(b_j)(z^{n_j b_j}) \sigma(\zeta^{n_j})
\]

\[
= \sum_{k=1}^{\infty} \sum_{b_k = -a_k, b_k \neq 0}^{a_k} \cdots \sum_{b_1 = -a_1}^{a_1} \prod_{j=1}^{k} \delta_{j}(b_j)(z^{n_j b_j}) \sigma(\zeta^{n_j})
\]

\[
= \prod_{k=1}^{\infty} (1 + s_k(z^{n_k}) \sigma(\zeta^{n_k})).
\]

Proposition 6.3 On \( L^1(\Sigma) \) convolution by \( K \) is a small perturbation of the identity,

\[
\| K \ast h - h \|_{L^1(\mathbb{T})} \leq \epsilon \| h \|_{L^1(\mathbb{T})} \quad h \in L^1(\Sigma).
\]

The operator \( Rg(z) = K \ast (\mathbb{E}_\Sigma g) \) satisfies

\[
\| Rg - \mathbb{E}_\Sigma g \|_{L^1(\mathbb{T})} \leq \epsilon \| g \|_{L^1(\mathbb{T})}, \quad g \in L^1(\mathbb{T}).
\]

Proof. In view of the integral representations (6.10) and (6.11) it suffices to prove that

\[
\sup_{\zeta \in \mathbb{T}} \int_{\mathbb{T}} |A(z, \zeta) - B(z, \zeta)| dm(z) \leq \epsilon.
\]

To this end fix \( \zeta \in \mathbb{T} \), put \( \tau_k = \sigma(\zeta^{n_k}) \). Let \( A_0 = B_0 = 1 \), and for \( j \in \mathbb{N} \) put

\[
A_j(z) = \prod_{k=1}^{j} (1 + s_k(z^{n_k}) \tau_k), \quad B_j(z) = \prod_{k=1}^{j} (1 + \sigma(z^{n_k}) \tau_k).
\]

Rewrite the difference \( A_j(z) - B_j(z) \) of the kernels as follows

\[
A_{j-1}(z) \tau_j(s_j(z^{n_j}) - \sigma(z^{n_j})) + (A_{j-1}(z) - B_{j-1}(z)) \tau_j(1 + \sigma(z^{n_j})).
\]

Next take absolute values and exploit that \( (z^{n_k}) \) is an almost independent sequence. Since \( A_{j-1} \geq 0 \),

\[
\int_{\mathbb{T}} A_{j-1}(z) dm(z) = 1 \quad \text{and} \quad \int_{\mathbb{T}} (1 \pm \sigma(z)) dm(z) = 1,
\]

and invoking (6.9) it is easy to see that

\[
\int_{\mathbb{T}} |A_j(z) - B_j(z)| dm(z)
\]

is bounded by

\[
(1 + \delta_j) \int_{\mathbb{T}} |A_{j-1}(z) - B_{j-1}(z)| dm(z) + (1 + \delta_j) \int_{\mathbb{T}} |s_j(z) - \sigma(z)| dm(z),
\]

where \( \delta_j \leq 2^{-j} \). A simple iteration proves the Lemma.
The spaces $L^1_E$ and $H^1_E$.

Recall
\[ L^1_E = \{ f \in L^1(\mathbb{T}) : \hat{f}(k) = 0 \text{ for } k \notin E \} \quad \text{and} \quad H^1_E = L^1_E \cap H^1. \]

Embedding $L^1_E$ into $L^1(\mathbb{T}^N)$. We next define an embedding
\[ T : L^1_E \to L^1(\mathbb{T}^N) \]
which maps $H^1_E = L^1_E \cap H^1$ to the space of Hardy martingales $H^1(\mathbb{T}^N)$.

We define the operator by mapping the monomials $\{z^m, m \in E\}$, into $L^1(\mathbb{T}^N)$. Recall that for $m \in E$ there exists a unique set of integers $k_j \in (-a_j, a_j)$ so that
\[ m = \sum k_j n_j. \]

Hence mapping the monomials $\{z^m, m \in E\}$, in $L^1(\mathbb{T})$ to the monomials $\{\prod w_j^{k_j}, k_j \in (-a_j, a_j)\}$, gives a well defined operator on $\text{span}\{z^m : m \in E\}$,
\[ T : z^m \to \prod w_j^{k_j}, \quad m = \sum k_j n_j. \]

Fix $f \in L^1_E$. To exhibit the martingale structure of $T(f)$ we rewrite $T$ as follows. Fix $w \in \mathbb{T}^N$ and $m \in \mathbb{N}$. Then write
\[ A(k_1, \cdots, k_{m-1}, k) = \hat{f}(k_1 n_1 + \cdots + k_{m-1} n_{m-1} + k n_m), \]
and form the Fourier coefficients
\[ a_m(k) = \sum_{k_{m-1} = -a_{m-1}}^{a_{m-1}} \cdots \sum_{k_1 = -a_1}^{a_1} A(k_1, \cdots, k_{m-1}, k) w_1^{k_1} \cdots w_{m-1}^{k_{m-1}}. \]

Thus $a_m(k) = a_m(k; w_1, \cdots, w_{m-1})$. Then put
\[ d_m(w) = \sum_{k = -a_m, k \neq 0}^{a_m} a_m(k) w_m^k. \]

We have
\[ T(f)(w) = \sum_{m=1}^{\infty} d_m(w). \]

**Proposition 6.4** For $f \in L^1_E$
\[ c \|f\|_{L^1(\mathbb{T})} \leq \|T(f)\|_{L^1} \leq C \|f\|_{L^1(\mathbb{T})}. \]

If $f \in H^1_E$, then $T(f)$ is a Hardy martingale.
PROOF. By inspection, the following properties of \( d_m(w) \) hold. First \( d_m(w) = d_m(w_1, \ldots, w_m) \), second for fixed \( w_1, \ldots, w_{m-1} \in \mathbb{T} \)

\[
\int_{\mathbb{T}} d_m(w_1, \ldots, w_{m-1}, w_m) \, dm(w_m) = 0,
\]

and third, if \( f \in H^1_E \), then

\[
w_m \to d_m(w_1, \ldots, w_{m-1}, w_m)
\]

defines an analytic polynomial, hence an element in \( H^1_0 \). In summary

\[
F_n(w) = \sum_{m=0}^{n} d_m(w), \quad n \in \mathbb{N},
\]
gives a Hardy martingale. The theorem of Meyer (see [11], [1]) asserts that for \( f \in L^1_{E_n}(\mathbb{T}) \)

\[
c\|f\|_{L^1(\mathbb{T})} \leq \|F_n\|_{L^1} \leq C\|f\|_{L^1(\mathbb{T})}.
\]

\[\blacksquare\]

**The transfer operator \( J \).**

Define the operator \( J : L^1(\mathbb{T}) \to L^1(\mathbb{T}^N) \) by putting \( Jg = T(K \ast g) \). Clearly \( J \) is bounded since convolution by \( K \) is a norm one operator on \( L^1(\mathbb{T}) \) with range on \( L^1_E \), and \( T \) is bounded on \( L^1_E \) by Proposition 6.4. By Proposition 6.3 and Proposition 6.4 for \( h \in L^1(\Sigma) \),

\[
\|h\|_{L^1(\mathbb{T})} \leq 2\|K \ast h\|_{L^1(\mathbb{T})} \leq 4\|T(K \ast h)\|_{L^1}.
\]

(6.12)

hence

\[
\frac{1}{4}\|h\|_{L^1(\mathbb{T})} \leq \|Jh\|_{L^1} \leq 4\|h\|_{L^1(\mathbb{T})}, \quad h \in L^1(\Sigma).
\]

Moreover by Proposition 6.4 we have the inclusion

\[
J(H^1(\mathbb{T})) \subseteq H^1(\mathbb{T}^N).
\]

(6.13)

Hence by (6.12) and (6.13) we proved that \( J \) satisfies (6.2) and (6.4).

**Conditional expectation \( E_D \).** Next we prove that \( E_D \) is a small perturbation of the identity on \( JL^1(\Sigma) \). Define the non-negative kernel

\[
B(w, z) = \prod_{k=1}^{\infty} (1 + \sigma_k(w)\sigma_k(z)), \quad w, z \in \mathbb{T}^N,
\]

where \( \sigma_k(w) = \sigma(w_k) \). Conditional expectation \( E_D \) is an integral operator with kernel \( B \),

\[
E_D F(w) = E_z(B(w, z)G(z)), \quad F \in L^1(\mathbb{T}^N).
\]
The kernels for \( J\mathbb{E}_\Sigma \) and \( \mathbb{E}_D J\mathbb{E}_\Sigma \). Define the kernel
\[
A(w, \zeta) = \prod_{k=1}^\infty (1 + s_k(w_k)\sigma(\zeta^{n_k})), \quad w \in \mathbb{T}^N, \quad \zeta \in \mathbb{T}.
\]
Then
\[
J\mathbb{E}_\Sigma g(w) = \int_\mathbb{T} A(w, \zeta)g(\zeta)dm(\zeta), \quad w \in \mathbb{T}^N.
\]
Let
\[
G(w, \zeta) = \prod_{k=1}^\infty \int_\mathbb{T} (1 + \sigma(z)\sigma(w_k))(1 + s_k(z)\sigma(\zeta^{n_k}))dm(z), \quad w \in \mathbb{T}^N, \quad \zeta \in \mathbb{T}.
\]
Then
\[
\mathbb{E}_D J\mathbb{E}_\Sigma g(w) = \int_\mathbb{T} G(w, \zeta)g(\zeta)dm(\zeta) \quad w \in \mathbb{T}^N.
\]
The integrals appearing in the factors of the kernel \( G(w, e^{i\psi}) \) may be evaluated as follows
\[
\int_\mathbb{T} (1 + \sigma(z)\sigma(w_k))(1 + s_k(z)\sigma(\zeta^{n_k}))dm(z) = 1 + \gamma_k\sigma(w_k)\sigma(\zeta^{n_k}),
\]
where \( \gamma_k = \int_\mathbb{T} \sigma(z)s_k(z)dm(z) \).

**Proposition 6.5** Let \( g \in L^1(\mathbb{T}) \). Put \( G = J\mathbb{E}_\Sigma g \). Then
\[
\|\mathbb{E}_D G - G\|_{L^1} \leq \epsilon \|g\|_{L^1}.
\]

**Proof.** In view of the integral representations (6.14) and (6.15) it suffices to prove that
\[
\sup_{\zeta \in \mathbb{T}} \mathbb{E}_w(|A(w, \zeta) - G(w, \zeta)|) \leq \epsilon.
\]
To this end fix \( \zeta \in \mathbb{T} \) put \( \tau_k = \sigma(\zeta^{n_k}) \). Define \( A_o = G_o = 1 \), and for \( j \in \mathbb{N} \),
\[
A_j(w) = \prod_{k=1}^j (1 + s_k(w_k)\tau_k), \quad G_j(w) = \prod_{k=1}^j (1 + \gamma_k\sigma(w_k)\tau_k), \quad w \in \mathbb{T}^N.
\]
Rewrite the difference \( A_j(w) - G_j(w) \) as
\[
A_{j-1}(w)\tau_j(s_j(w_j) - \sigma(w_j)) + (A_{j-1}(w) - G_{j-1}(w))\tau_j(1 + \gamma_j\sigma(w_j)).
\]
The second term coincides with
\[
(A_{j-1}(w) - G_{j-1}(w))\tau_j(1 + \sigma(w_j)) - (A_{j-1}(w) - G_{j-1}(w))\tau_j(1 - \gamma_j)\sigma(w_j)).
\]
Hence
\[
\mathbb{E}|A_j - G_j| \leq (1 + \epsilon_j)\mathbb{E}|A_{j-1} - G_{j-1}| + 2\epsilon_j\mathbb{E}|A_j|,
\]
where we put \( \int_\mathbb{T} |s_j - \sigma|dm = \epsilon_j \). Since \( G_j > 0, A_j > 0 \)
\[
\mathbb{E}|A_j| = 1, \quad \mathbb{E}|G_j| = 1,
\]
Hence iterating gives \( \mathbb{E}|A_n - G_n| \leq C \sum_{j=1}^n \epsilon_j \).
In Summary: We proved that the linear operator \( J : L^1(\mathbb{T}) \to L^1(\mathbb{T}^N) \) defined by \( Jg = T(K \ast g) \) is bounded and satisfies the conditions (6.2)–(6.4).

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