THE TATE-VOLOCH CONJECTURE IN A POWER OF A MODULAR CURVE

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Abstract. Let \( p \) be a prime. Tate and Voloch proved that a point of finite order in the algebraic torus cannot be \( p \)-adically too close to a fixed subvariety without lying on it. The current work is motivated by the analogy between torsion points on semi-abelian varieties and special or CM points on Shimura varieties. We prove the analog of Tate and Voloch’s result in a power of the modular curve \( Y(1) \) on replacing torsion points by points corresponding to a product of elliptic curves with complex multiplication and ordinary reduction. Moreover, we show that the assumption on ordinary reduction is necessary.

1. Introduction

Let \( p \) be a fixed prime. Tate and Voloch conjectured \([22]\) that a torsion point in a semi-abelian variety cannot be \( p \)-adically too close to a subvariety without actually lying on it. They proved their conjecture when the semi-abelian variety is an algebraic torus. Buium \([1]\) obtained related results in a more abstract framework and Scanlon later proved \([20, 21]\) the conjecture for semi-abelian varieties. He used work of Chatzidakis and Hrushovski on the model theory of difference fields which also enabled Hrushovski’s proof of the Manin-Mumford Conjecture.

There is a well-established analogy between torsion points on semi-abelian varieties and CM points on Shimura varieties. It is reflected in the formal similarity between the conjectures of Manin-Mumford and André-Oort. The purpose of this paper is to begin investigating the Tate-Voloch Conjecture from the modular point of view. We will confine ourselves to a power of the modular curve \( Y(1) \) which is the coarse moduli space of elliptic curves. As a variety this is the affine line.

The \( p \)-adic absolute value \( \cdot \ |

|_p \) extends uniquely from the field \( \mathbb{Q}_p \) of \( p \)-adics numbers to an algebraic closure \( \overline{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \) and then to a completion \( \mathbb{C}_p \) of \( \overline{\mathbb{Q}}_p \). The ring of integers in \( \mathbb{C}_p \) will be denoted by \( \mathcal{O}_p \). If not stated otherwise, we will consider \( Y(1)^n \) as a scheme over the spectrum of \( \mathcal{O}_p \).

Let \( Z \subset Y(1)^n \) be a Zariski closed subset with vanishing ideal \( I \subset \mathcal{O}_p[X_1, \ldots, X_n] \). We define the \( p \)-adic distance of \( x \in \mathcal{O}_p^n \) to \( Z \) as

\[
\text{dist}_p(x, Z) = \sup\{|f(x)|_p; \ f \in I \cap \mathcal{O}_p[X_1, \ldots, X_n]|.\]

Then dist\(_p(x, Z) = 0 \) if and only if \( x \in Z(\mathcal{O}_p) \). We certainly have \( |f(x)|_p \leq 1 \) and so dist\(_p(x, Z) \leq 1 \).

The \( j \)-invariant of an elliptic curve defined over \( \mathbb{C}_p \) with complex multiplication is an algebraic integer and therefore an element of \( \mathcal{O}_p \). We call a point of \( Y(1)^n(\mathcal{O}_p) \) a CM point if its coordinates are \( j \)-invariants of elliptic curves with complex multiplication. It is convenient to call CM points of \( Y(1) \) singular moduli. For example, \( 0 \in Y(1)(\mathcal{O}_p) \) is a
singular moduli since it is the $j$-invariant of the elliptic curve with complex multiplication by $\mathbb{Z}[(\sqrt{-3} + 1)/2]$.

An elliptic curve over $\mathbb{C}_p$ with complex multiplication has good reduction at the maximal ideal of $\mathcal{O}_p$. A CM point of $Y(1)^n$ is called ordinary if its coordinates correspond to elliptic curves with ordinary reduction.

If not stated otherwise, a subvariety of $Y(1)^n$ is an irreducible closed subvariety of $Y(1)^n$ defined over $\mathbb{C}_p$.

Theorem 1. Let $X$ be a subvariety of $Y(1)^n$. There exists $\epsilon > 0$ such that if $x$ is an ordinary CM point of $Y(1)^n$ with $x \notin X(\mathbb{C}_p)$, then $\text{dist}_p(x, X) \geq \epsilon$.

Pink and Roessler [19] used Hrushovski’s setup to prove the Manin-Mumford Conjecture using only algebraic geometry. In the same vein our argument avoids the model theory of difference field employed by Scanlon. But we still rely on a carefully chosen field automorphism coming from class field theory that carries the arithmetic information. Roughly speaking, the uniformity statements provided by model theory are replaced by an effective version of Hilbert’s Nullstellensatz due to Kollár [13]. As in Scanlon’s argument we reduce the proof of Theorem 1 to the case where $X$ is a special subvariety of $Y(1)^n$. In Section 3 we give a complete description of all such special subvarieties. To treat special subvarieties we will apply Serre-Tate theory for ordinary elliptic curves in characteristic $p$. It enriches the formal deformation space of an ordinary elliptic curve with the structure of a formal torus. De Jong and Noot’s [3] characterization of ordinary CM points as points of finite order will also play an important role.

Our approach retains a connection to model theory. Indeed, we need recent results of Pila on the weakly special subvarieties contained in $X$ [18] and on the Zariski closure in $Y(1)^n$ of a Hecke orbit [17]. The latter extends to varieties over $\mathbb{C}$ an earlier theorem proved together with the author [10] if the Hecke orbit consists of algebraic elements. These results rely on a strategy initially proposed by Zannier to prove the Manin-Mumford Conjecture using a theorem of Pila and Wilkie on rational points of sets definable in an o-minimal structure.

The choice of distance function (1) was in part for convenience. Another natural choice would be

$$\text{dist}_p'(x, Z) = \inf\{|x - y|_p; \ y \in Z(\mathbb{C}_p)\}$$

where $|\cdot|_p$ denotes also the $p$-adic sup-norm on $\mathbb{C}_p^n$. Using the Taylor expansion of an element $f \in I \cap \mathcal{O}_p[\mathbb{X}]$ around $y \in Z(\mathbb{C}_p)$ together with the ultrametric triangle inequality yields $|f(x)|_p \leq |x - y|_p \max\{1, |x - y|_p^{\deg f^{-1}}\}$ for $x \in \mathcal{O}_p^n$. But $|f(x)|_p \leq 1$ and therefore, $|f(x)|_p \leq |x - y|_p$. Taking first the infimum over $y \in Z(\mathbb{C}_p)$ and then the supremum over the admissible $f$ yields

$$\text{dist}_p(x, Z) \leq \text{dist}_p'(x, Z).$$

Therefore, Theorem 1 holds for the alternative distance $\text{dist}_p'(x, Z)$.

The connection of our result to the Tate-Voloch Conjecture in the semi-abelian case begs the question why we restrict ourselves to ordinary CM points. The reason, apparent by the proposition below, is that subvarieties can be approximated arbitrarily well $p$-adically by general singular moduli. More precisely, we show that already the zero-dimension variety $X = \{0\}$ is the $p$-adic limit of a sequence of singular moduli $x$ corresponding to elliptic curves with supersingular reduction at a place above $p$. We will
bound the $p$-adic distance in terms of the discriminant $\Delta(x) < 0$ of the endomorphism ring of an elliptic curve attached to $x$.

**Proposition 2.** There is a constant $c > 0$ with the following property. Let $p$ be an odd prime with $p \equiv 2 \mod 3$. There exists a sequence $x_1, x_2, \ldots$ of non-zero singular moduli with $\Delta(x_n)$ a fundamental discriminant, $\lim_{n \to \infty} \Delta(x_n) = -\infty$, and $|x_n|_p \leq c|\Delta(x_n)|^{-1/2}$.

The proof of this proposition relies on explicit computations in the endomorphism ring of the supersingular elliptic curve $y^2 = x^3 + 1$ in characteristic $p$ combined with ideas of Gross and Zagier \[8\]. We also make explicit an old result of Nagel \[15\] on square-free values of quadratic polynomials.

Although a uniform lower bound as Theorem 1 is impossible for unrestricted CM points, we ask if a weaker bound holds true. Let $X$ be as in the theorem. Does there exist positive constants $c$ and $\lambda$ such that any CM point $x = (x_1, \ldots, x_n)$ with $x \not\in X(\mathbb{C}_p)$ satisfies $\text{dist}_p(x, X) \geq c \max\{|\Delta(x_1)|, \ldots, |\Delta(x_n)|\}^{-\lambda}$? Certainly, $\lambda \geq 1/2$ if such an inequality were true.

It is natural to ask if Theorem 1 can be extended, for example, to the coarse moduli space of principally polarized abelian varieties of fixed dimension. Our approach would require a variant of Pila’s result mentioned above in this context. From this point of view, it would also be interesting to have a proof of our result that circumvents Pila’s view, it would also be interesting to have a proof of our result that circumvents Pila’s

The paper is organized as follows. In Section 2 we use class field theory to construct the field automorphism alluded to above. Section 3 uses Pila’s results to describe subvarieties that are almost invariant under Hecke orbits with sufficiently large level. Proposition 15 in Section 4 is a weak version of Theorem 1 that cannot yet account for special subvarieties. In Section 5 we review aspects of Serre-Tate theory that are required for treating special subvarieties and proving Theorem 1 in Section 6. Finally, Proposition 2 is proved in the appendix.

The author heartily thanks Thomas Scanlon for productive discussions and especially for pointing him towards Serre-Tate theory which is a crucial ingredient in the work at hand.

## 2. Finding a Good Galois Element

We let $\text{ord}_p : \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}$ denote the valuation on $\mathbb{C}_p$ normalized such that $\text{ord}_p(p) = 1$. As usual $\mathbb{Z}_p$ is the ring of $p$-adic integers. We write $\mathbb{N}$ for the set of positive integers.

**Lemma 3.** Let $\gamma \in 1 + p\mathbb{Z}_p$ and suppose $D \in \mathbb{N}$. Then $\text{ord}_p(\gamma^D - 1) = \text{ord}_p(D) + \text{ord}_p(\gamma - 1)$ if $p \geq 3$ and $\text{ord}_2(\gamma^D - 1) \leq \text{ord}_2(D) + \text{ord}_2(\gamma^2 - 1) - 1$ if $p = 2$.

**Proof.** First we suppose that $p \geq 3$. By Proposition II.5.5 \[15\] the $p$-adic logarithm $\log : 1 + p\mathbb{Z}_p \to p\mathbb{Z}_p$ is a homomorphism with $\text{ord}_p\log(1+z) = \text{ord}_p(z)$ for $z \in p\mathbb{Z}_p \setminus \{0\}$. We apply these facts to $z = \gamma^D - 1$ and $z = \gamma - 1$ and obtain

\[
\text{ord}_p(\gamma^D - 1) = \text{ord}_p\log(\gamma^D) = \text{ord}_p(D\log\gamma) = \text{ord}_p(D) + \text{ord}_p\log\gamma = \text{ord}_p(D) + \text{ord}_p(\gamma - 1).
\]

The logarithm has similar properties for $p = 2$ albeit with a smaller domain of convergence. Now $\log : 1 + 4\mathbb{Z}_2 \to 4\mathbb{Z}_2$ satisfies $\text{ord}_2\log(1+z) = \text{ord}_2(z)$ for $z \in 4\mathbb{Z}_2 \setminus \{0\}$. 

If $\gamma \in 1+2\mathbb{Z}_2$, then $\gamma^2 \in 1+4\mathbb{Z}_2$. After replacing $\gamma$ by $\gamma^2$, the same argument as for odd primes yields

$$\text{ord}_2(\gamma^{2D} - 1) = \text{ord}_2(D) + \text{ord}_2(\gamma^2 - 1).$$

But $\text{ord}_2(\gamma^D - 1) = \text{ord}_2(\gamma^{2D} - 1) - \text{ord}_2(\gamma^D + 1)$ and so $\text{ord}_2(\gamma^D + 1) \geq 1$ completes the proof.

Lemma 4. Let $k_0 \in \mathbb{N}$ and $D \in \mathbb{N}$ with $k_0 \geq 2\text{ord}_p(2D)$ and assume $A_1, \ldots, A_n \in \text{GL}_2(\mathbb{Q}_p)$. Then there exist $\alpha, \beta \in \mathbb{Q}_p^\times$ and $e \in \mathbb{Z}$ such that the matrices

$$B_i = A_i^{-1} \begin{pmatrix} p^{-e} \alpha^D & p^{-e} \beta^D \\ \end{pmatrix} A_i.$$

satisfy the following properties.

(i) For $1 \leq i \leq n$ the matrix $B_i$ has coefficients in $\mathbb{Z}_p$ and there is an $i$ with $B_i \not\in p\text{Mat}_2(\mathbb{Z}_p)$.

(ii) We have

$$k_0 \leq \text{ord}_p(p^{-2e} \alpha^D \beta^D) \leq 3Dk_0.$$

Proof. The matrix (2) is invariant under replacing $A_i$ by a scalar multiple of itself. So we may suppose that all $A_i$ lie in $\text{Mat}_2(\mathbb{Z}_p)$.

We set $\delta_i = \det A_i$ and write

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \\ \end{pmatrix}.$$

Then

$$B_i = A_i^{-1} \begin{pmatrix} p^{-e} \alpha^D & p^{-e} \beta^D \\ \end{pmatrix} A_i = \frac{1}{p^e \delta_i} \begin{pmatrix} a_i d_i (\alpha^D - \beta^D) + \delta_i \beta^D & b_i d_i (\alpha^D - \beta^D) \\ -a_i c_i (\alpha^D - \beta^D) & -a_i d_i (\alpha^D - \beta^D) + \delta_i \alpha^D \\ \end{pmatrix}$$

where the values $\alpha, \beta, e$ will be specified later on. We define integers

$$k_i = \text{ord}_p(\delta_i) - \min\{\text{ord}_p(a_i d_i), \text{ord}_p(b_i d_i), \text{ord}_p(a_i c_i)\}.$$

The proof is a case by case analysis depending on the value of

$$k = \max\{k_1, \ldots, k_n\} \in \mathbb{Z}.$$

The first case is $k \leq k_0$. Here we set

$$\alpha = 1, \quad \beta = p^k, \quad \text{and} \quad e = -\max\{0, k\}.$$

The valuation of any

$$\frac{a_i d_i}{p^e \delta_i}(1 - p^{k_0 D}), \quad \frac{b_i d_i}{p^e \delta_i}(1 - p^{k_0 D}), \quad \frac{-a_i c_i}{p^e \delta_i}(1 - p^{k_0 D}), \quad \frac{-a_i d_i}{p^e \delta_i}(1 - p^{k_0 D}),$$

is at least $-k_1 - e \geq -k - e \geq 0$. From this and since the additional terms $p^{k_0 D} p^{-e}$ and $p^{-e}$ in the upper left and lower right entries of $B_i$ are in $\mathbb{Z}_p$, we deduce $B_i \in \text{Mat}_2(\mathbb{Z}_p)$. If $k = k_i$, then one of the four elements (11) has order precisely $-k - e$. If $k \geq 1$, then $-e = k \geq 1$ and said order is 0. The corresponding entry of $B_i$ also has order 0 and the additional terms are harmless. Hence (i) holds for positive $k$. If $k \leq 0$ then $e = 0$. The lower right entry of $B_i$ is $-a_i d_i/(1 - p^{k D})/\delta_i + 1$. It is not divisible by $p$ if $\text{ord}_p(a_i d_i/\delta_i) \geq 1$. But $a_i d_i/\delta \in \mathbb{Z}_p$ and so otherwise it is a unit. But then $a_i d_i(1 - p^{k D})/\delta_i + p^{k D}$, the upper left entry of $B_i$, is not divisible by $p$ as $Dk_0 \geq 1$. So (i) also holds if $k \leq 0$. 
To prove (ii) we note that $\text{ord}_p(p^{-2e}\alpha D\beta D) = -2e + Dk_0 = 2\max\{0, k\} + Dk_0 \geq Dk_0$. But this order is at most $3Dk_0$ because of $k \leq k_0$.

The second case is $k \geq k_0 + 1$. We set values

$$\alpha = p^{k_0} + p^k, \quad \beta = p^{k_0}, \quad \text{and} \quad e = \text{ord}_p(\alpha D - \beta D) - k.$$

Note that $\text{ord}_p(\alpha) = \text{ord}_p(\beta) = k_0$ and $\text{ord}_p(\alpha D - \beta D) \geq \text{ord}_p(\alpha - \beta) = k$. So $e \geq 0$ and

$$\text{ord}_p(a_i d_i (\alpha D - \beta D)) - \text{ord}_p \delta_i - e = \text{ord}_p(a_i d_i) + k - \text{ord}_p \delta_i \geq \min\{\text{ord}_p(a_i d_i), \text{ord}_p(b_i d_i), \text{ord}_p(c_i)\} + k_i - \text{ord}_p \delta_i = 0,$$

where the final equality is (3). A simple modification of this argument shows

$$\text{ord}_p(b_i d_i (\alpha D - \beta D)) - \text{ord}_p \delta_i - e \geq 0 \quad \text{and} \quad \text{ord}_p(a_i c_i (\alpha D - \beta D)) - \text{ord}_p \delta_i - e \geq 0.$$

We apply Lemma 3 to $\gamma = \alpha/\beta = 1 + p^{k-k_0}$ and find

$$e = \text{ord}_p(\alpha D - \beta D) - k \leq Dk_0 - k + \text{ord}_p(\gamma - 1) + \text{ord}_p(D) + \begin{cases} 0 & \text{if } p \geq 3, \\ \text{ord}_2(\gamma + 1) - 1 & \text{if } p = 2. \end{cases}$$

We note that $\text{ord}_p(\gamma - 1) = k - k_0$ and $\text{ord}_2(\gamma + 1) = \text{ord}_2(2^{k-k_0} + 2) \leq 2$ if $p = 2$. All in all we obtain $e \leq Dk_0 - k_0 + \text{ord}_p(2D) \subseteq k_0$ regardless of $p$. The hypothesis $2\text{ord}_p(2D) \leq k_0$ implies

$$e \leq Dk_0 - k_0/2.$$

In particular, $Dk_0 - e \geq k_0/2 > 0$ and so the additional terms $\beta D p^{-e}$ and $\alpha^D p^{-e}$ which appear in $B_i$ have positive order. We have thus proved $B_i \in \text{Mat}_2(\mathbb{Z}_p)$.

However, one of the 3 inequalities in (5) and (6) must be an equality if $k_i = k$. The additional terms are again harmless since they have positive order. Therefore $B_i \notin \text{Mat}_2(\mathbb{Z}_p)$ for all $i$ with $k_i = k$. This conclude the proof of part (i).

The relevant order in (ii) is $-2e + \text{ord}_p(\alpha D \beta D) = -2e + 2Dk_0 \leq 2Dk_0$ since $e \geq 0$. The lower bound follows from (7). \hfill \Box

A place $v$ of a number field $F$ is a non-trivial absolute value whose restriction to $\mathbb{Q}$ is either the restricted complex absolute value or the $p$-adic absolute value. We call $v$ finite if it is an ultrametric absolute value and we call $v$ infinite otherwise. It is well-known that finite places are in natural bijection with non-zero prime ideals of the ring of integers of $F$. We let $F_v$ denote the completion of $F$ with respect to $v$.

Suppose for the moment that $L/F$ is a finite abelian extension of number fields and let $w$ be a place of $L$ that extends $v$. Let $\mathcal{A}_F^\times$ denote the ideles of $F$ and $(\cdot, L_w/F_v) : F_w^\times \rightarrow \text{Gal}(L_w/F_v)$ is the local norm residue symbol. If $s = (s_v)_v \in \mathcal{A}_F^\times$, then the global norm residue symbol

$$(s, L/F) = \prod_v (s_v, L_w/F_v) \in \text{Gal}(L/F)$$

is a product of the local norm residue symbols. The decomposition group $\text{Gal}(L_w/F_v)$ is a subgroup of $\text{Gal}(L/F)$ and depends only on $v$ since $L/F$ is abelian. For brevity we write $D(v) = \text{Gal}(L_w/F_v)$. By abuse of notation we will write $\text{Gal}(L_v/F_v)$ and $(\cdot, L_v/F_v)$ instead of $\text{Gal}(L_w/F_v)$ and $(\cdot, L_w/F_v)$. 
Suppose \( \sigma \) is an automorphism of \( L \) and let \( p \) be a prime ideal of the ring of integers of \( L \) corresponding to a finite place \( v \). Then \( \sigma v \) is a finite place of \( L \) that corresponds to the prime ideal \( \sigma(p) \).

We write \( \tau \) for complex conjugation.

**Lemma 5.** Let \( x_1, \ldots, x_n \in \mathbb{C} \) be \( j \)-invariants of elliptic curves with complex multiplication by orders in imaginary quadratic fields \( K_1, \ldots, K_n \subset \mathbb{C} \), respectively. We abbreviate \( F = K_1 \cdots K_n \) and \( L = F(x_1, \ldots, x_n) \). We suppose that \( p \) splits in all \( K_i \) and let \( v \) be a place of \( F \) above \( p \). Then \( \tau v \neq \tau v \). For \( \alpha, \beta \in \mathbb{Q}_p^* \) let \( s \) be the idèle

\[
(\ldots, 1, \underbrace{\alpha}_{v|\mathcal{O}_v}, \underbrace{\beta}_{\tau v|\mathcal{O}_v}, 1, \ldots) \in \mathcal{A}_F^\times.
\]

Then the following properties hold.

(i) The extension \( L/F \) is abelian.
(ii) For each \( 1 \leq i \leq n \) the restrictions \( v|_{K_i} \) and \( \tau v|_{K_i} \) are distinct places of \( K_i \).
(iii) For any \( 1 \leq i \leq n \) we have

\[
(s, L/F)|_{K_i(x_i)} = \left( (\ldots, 1, \underbrace{\alpha}_{v|K_i}, \underbrace{\beta}_{\tau v|K_i}, 1, \ldots), K_i(x_i)/K_i \right).
\]

(iv) We have \( (s, L/F) \in \mathcal{D}(v) \).

**Proof.** The classical theory of complex multiplication implies that each \( K_i(x_i)/K_i \) is an abelian extension, cf. Chapter 10.3 [14]. By Galois theory the extension \( L/F \) is abelian and part (i) holds true.

Statement (ii) follows since \( p \) splits in each \( K_i \). In particular, \( \tau v \neq v \).

Part (iii) follows directly from Proposition VI.5.2 [16] and (ii).

Now for part (iv). By Lemma 9.3 [2] the extension \( K_i(x_i)/\mathbb{Q} \) is Galois with group \( \text{Gal}(K_i(x_i)/K_i) \cong \text{Gal}(K_i/\mathbb{Q}) \) where complex conjugation acts as \( \tau \eta \tau|_{K_i(x_i)} = \eta^{-1} \) for \( \eta \in \text{Gal}(K_i(x_i)/K_i) \). In particular, \( \tau(K_i(x_i)) = K_i(x_i) \) and so \( \tau(L) = L \) since \( L \) is generated by the \( K_i(x_i) \). For arbitrary \( \sigma \in \text{Gal}(L/F) \) the automorphisms \( \tau \sigma \tau|_L \) and \( \sigma^{-1} \) coincide on \( K_i(x_i) \) and hence also on \( L \).

We have

\[
(s, L/F) = (\alpha, L_v/F_v)(\beta, L_{\tau v}/F_{\tau v})
\]

where \( (\alpha, L_v/F_v) \in \mathcal{D}(v) \) and \( (\beta, L_{\tau v}/F_{\tau v}) \in \mathcal{D}(\tau v) \). If \( \sigma \in \mathcal{D}(\tau v) \), then \( \sigma \tau|_LV = \tau|_LV \) and hence \( \tau \sigma^{-1}v = \tau|_LV \) since \( \sigma \tau|_L = \sigma^{-1} \). Eliminating \( \tau \) yields \( \sigma \in \mathcal{D}(v) \) and therefore \( (\beta, L_{\tau v}/F_{\tau v}) \in \mathcal{D}(v) \). This completes the proof. \( \square \)

If \( N = (N_1, \ldots, N_n) \) is a tuple of positive integers, then we write \( T_N \subset Y(1)^n \times Y(1)^n \) for the correspondence that connects the two \( i \)-th coordinates in each copy of \( Y(1)^n \) by a cyclic isogeny of degree \( N_i \). This correspondence can be expressed explicitly using modular polynomials \( \Phi_1, \Phi_2, \Phi_3, \ldots \in \mathbb{Z}[X,Y] \), cf. Chapter 5 [17] for properties. More precisely,

\[
T_N = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in Y(1)^n \times Y(1)^n; \Phi_{N_1}(x_1, y_1) = \cdots = \Phi_{N_n}(x_n, y_n) = 0\}.
\]
We call $T_N$ the Hecke correspondence of level $N$. By Theorem 3, Chapter 5.2 \cite{14} we have $\deg_Y \Phi_N = \deg_X \Phi_N = \Psi(N)$ with

$$\Psi(N) = N \prod_{\ell|N} \frac{\ell + 1}{\ell}$$

where the product runs over primes $\ell$.

Let $\pi_1, \pi_2 : Y(1)^n \times Y(1)^n \to Y(1)^n$ denote the projections to the first and last tuple of $n$ coordinates.

The modular polynomials are monic considered as polynomials in $X$ or $Y$. Thus $\pi_1|_{T_N} : T_N \to Y(1)^n$ is a finite morphism and if $Z \subset Y(1)^n$ is Zariski closed, then so is

$$T_N(Z) = \pi_1(\pi_2^{-1}(Z) \cap T_N) \subset Y(1)^n.$$ 

If $x \in Y(1)^n(C)$, we abbreviate the finite set $T_N(\{x\})$ by $T_N(x)$.

**Proposition 6.** Let $x_1, \ldots, x_n \in C$ be $j$-invariants of elliptic curves with complex multiplication by orders in imaginary quadratic number fields $K_1, \ldots, K_n$, respectively. Let $F = K_1 \cdots K_n$ and $L = F(x_1, \ldots, x_n)$, then $L/F$ is abelian by the previous lemma. Suppose $p$ splits in all $K_i$ and $v$ is a place of $F$ above $p$. Let $k_0 \in \mathbb{N}$ and $D \in \mathbb{N}$ satisfy $k_0 \geq 2\text{ord}_p(2D)$. There exist integers $k_1, \ldots, k_n \geq 0$ and $\sigma \in \text{Gal}(L/F)$ with the following properties.

(i) We have $k_0 \leq \max\{k_1, \ldots, k_n\} \leq 3Dk_0$.

(ii) We have $\sigma \in D(v)$.

(iii) We have $\sigma^D(x_1, \ldots, x_n) \in T(p^{k_1}, \ldots, p^{k_n})(x_1, \ldots, x_n)$.

**Proof.** For brevity we write $H_i = K_i(x_i)$ and $v_i = v|_{K_i}$. Recall that $H_i/K_i$ is an abelian extension.

For any $1 \leq i \leq n$ let $E_i$ be an elliptic curve over $C$ with $j$-invariant $x_i$. There is a lattice $\Omega_i$ inside $K_i$ such that $E_i(C)$ and $\mathbb{C}/\Omega_i$ are isomorphic complex tori. Since $p$ splits in each $K_i$ there are two distinct embeddings $K_i \hookrightarrow \mathbb{Q}_p$, the first determined by $v_i$ and the second one by $v_{v_i}$.

If $K$ is a field extension of $\mathbb{Q}$ and $\ell$ a prime we write $K_\ell = K \otimes \mathbb{Q}_\ell$. If $M$ is a $\mathbb{Z}$-module then we write $M_\ell = M \otimes \mathbb{Z}_\ell$.

We may identify $(K_i)_p$ with $\mathbb{Q}_p^2$ as $\mathbb{Q}_p$-algebras using the two embeddings mentioned above. Elements of $(K_i)_p$ will be represented by column vectors in $\mathbb{Q}_p^2$. We may identify $(\Omega_i)_p$ with a free $\mathbb{Z}_p$-module of rank 2 inside $\mathbb{Q}_p^2$. We fix a $\mathbb{Z}$-Basis of $\Omega_i$ and arrange its image in $\mathbb{Q}_p^2$ as the columns of a matrix $A_i \in \text{GL}_2(\mathbb{Q}_p)$.

We apply Lemma 4 to the $A_i, k_0$, and $D$ in order to obtain $\alpha, \beta \in \mathbb{Q}_p^\times$ and $e$.

Let $s$ be the idèle as in Lemma 5 and

$$s' = (\ldots, p^{-e}, p^{-e}, p^{-e}, \ldots) \in \mathbb{A}_K^\times,$$

this is a principal idèle. We set

$$\sigma = (s, L/F)^{-1}$$

and use Artin reciprocity to deduce

$$\sigma^D = (s's^D, L/F)^{-1}.$$ 

Part (ii) of the current lemma follows from Lemma 5\textit{iv}).

Say $1 \leq i \leq n$. We abbreviate $A = A_i, H = H_i, K = K_i, x = x_i$, and $v = v_i$. 

Let $E^{\sigma_D}$ be an elliptic curve over $\mathbb{C}$ with $j$-invariant $\sigma_D(x)$. We now apply the Main Theorem of Complex Multiplication, Theorem 3 in Chapter 10.2 [14], to $E^{\sigma_D}$. Indeed, $E^{\sigma_D}(\mathbb{C})$ and $\mathbb{C}/s_K\Omega$ are isomorphic complex tori where $s_K \in A^*_K$ is the idèle norm of $s/K^{s_D}$.

The lattice $\Lambda = s_K\Omega$ is determined locally by $\Lambda_\ell$ at all primes $\ell$ as follows. If $\ell \neq p$, then $\Lambda_\ell = \Omega_\ell$ since the idèle $s_K$ has entry $p^e$ at all places above $\ell$. However, things are different when $\ell = p$. Since the columns of $A$ are a $\mathbb{Z}_p$ basis of $\Omega_p$, the columns of

$$\begin{pmatrix} p^{-e}\alpha D & p^{-e}\beta D \\ - e \end{pmatrix} A$$

constitute a $\mathbb{Z}_p$-basis of $\Lambda_p$. By the first statement of (i) in Lemma 4 we have $\Lambda_p \subset \Omega_p$. The Elementary Divisor Theorem implies $[\Omega_p : \Lambda_p] = p^{\deg(p^{-e}\alpha D \beta D)}$ and that $\Omega_p/\Lambda_p \cong \mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}/p^b\mathbb{Z}$ for integers $a \geq b$. Moreover, $b = 0$ if

$$A^{-1} \begin{pmatrix} p^{-e}\alpha D & p^{-e}\beta D \end{pmatrix} \notin p\text{Mat}_2(\mathbb{Z}_p).$$

By the Main Theorem of Complex Multiplication, the natural map $\mathbb{C}/\Lambda \to \mathbb{C}/\Omega$ restricted to the respective torsion groups is given by $K/\Lambda \to K/\Omega$. We obtain a commutative diagram where the horizontal maps are the natural group isomorphisms

$$\begin{array}{ccc}
K/\Lambda & \longrightarrow & \bigoplus_\ell K_\ell/\Lambda_\ell \\
\downarrow & & \downarrow \\
K/\Omega & \longrightarrow & \bigoplus_\ell K_\ell/\Omega_\ell
\end{array}$$

Thus the kernel of $K/\Lambda \to K/\Omega$ is isomorphic to $\Omega_p/\Lambda_p$. Therefore, $\mathbb{C}/\Lambda \to \mathbb{C}/\Omega$ is an isogeny of degree $p^{a+b}$. It is not cyclic if $b > 0$. This yields part (iii) of the proposition. Of course $k$ depends on $i$. We remark that

$$k = a - b \leq a + b = \text{ord}_p(p^{-e}\alpha D \beta D) \leq 3Dk_0$$

by (ii) of Lemma 4. So the upper bound for $\max\{k_1, \ldots, k_n\}$ in (i) of the assertion holds true.

To complete the proof of the proposition we need to prove the lower bound as well. Now $\Omega_p/\Lambda_p$ is cyclic if (1) holds true. But this is the case for some $A = A_i$ by the second statement of (i) in Lemma 4. For such an $i$ we have $k_i = \text{ord}_p(p^{-e}\alpha \beta) \geq k_0$. This completes the proof of (i). \hfill \Box

3. Hecke Translates

We briefly recall the definition of weakly special, special, and strongly special subvarieties of $Y(1)^n$.

Let $S_0, \ldots, S_r$ be a partition of $\{1, \ldots, n\}$ where $S_0$ may be empty but the $S_1, \ldots, S_r$ are non-empty. Let $s_j = \#S_j$ and let us write $S_j = \{i_{j1}, \ldots, i_{jsj}\}$ with $i_{j1} < i_{j2} < \cdots < i_{jsj}$. A weakly special subvariety of $Y(1)^n$ is an irreducible Zariski closed set $S$ determined by

$$\Phi_{N_{ij}}(x_{i_{j1}}, x_{i_{jsj}}) = 0 \text{ for all } 1 \leq j \leq r, 2 \leq k \leq s_j \text{ and } x_i = c_i \text{ for all } i \in S_0,$$

where $N_{ij}$ are positive integers and $c_i$ are fixed elements of $\mathbb{C}_p$ (or $\mathbb{C}$) for all $i \in S_0$.\hfill \Box
If all \( c_i \) are singular moduli then we call \( S \) a special subvariety of \( Y(1)^n \).
If \( S_0 = \emptyset \) then we call \( S \) a strongly special subvariety of \( Y(1)^n \).
To ease notation we often omit the suffix “of \( Y(1)^n \)” when speaking of weakly special, special, or strongly special subvarieties of \( Y(1)^n \).
Edixhoven \([4]\) showed that this definition of special subvariety is equivalent to one that is more natural from the point of view of Shimura varieties.

Let \( X \subset Y(1)^n \) be a subvariety. We define \( X^{\circ} \) to be the union \( \bigcup_{Z \subset X} Z \) where \( Z \) runs over all weakly special subvarieties with \( \dim Z \geq 1 \) that are contained in \( X \).

The following proposition is a quick consequence of Pila’s Structure Theorem on weakly special subvarieties.

**Proposition 7.** Let \( X \) be an irreducible closed subvariety of \( Y(1)^n \) both taken as over \( \mathbb{C} \). There are finitely many strongly special subvarieties \( S_i \subset Y(1)^{n-n_i} \) \( (1 \leq i \leq s) \) with \( \dim S_i \geq 1 \) and \( 0 \leq n_i \leq n-1 \), subvarieties \( W_i \subset Y(1)^{n_i} \), and coordinate permutations \( \rho_i : Y(1)^n \to Y(1)^n \) such that

\[
X^{\circ} = \bigcup_{i=1}^s \rho_i(S_i \times W_i).
\]

**Proof.** Any weakly special subvariety of \( Y(1)^n \) of positive dimension is of the form \( \rho(S \times \{x'\}) \) with \( S \subset Y(1)^{n-n'} \) strongly special, \( \rho \) a coordinate permutation, and \( x' \) a point. The case \( n'=0 \) is possible.

If for some choice of \( x' \) the weakly special subvariety \( \rho(S \times \{x'\}) \) is maximal among all weakly special subvarieties contained in \( X \), then \( S \) must come from a finite set depending only on \( X \) by Proposition 13.1 \([18]\).

The theorem on the dimension of a fiber of morphism, cf. Exercise II.3.22 \([11]\), implies that the condition \( x' \in Y(1)^{n'}(\mathbb{C}) \) with \( \rho(S \times \{x'\}) \subset X \) determines a Zariski closed set for a fixed \( S \). The proposition follows as we may restrict to finitely many \( S \). \( \square \)

The full Hecke orbit of a point \( x \in Y(1)^n(\mathbb{C}) \) is

\[
T(x) = \bigcup_{N \in \mathbb{N}^n} T_N(x).
\]

**Theorem 8 (Pila).** Let \( X \) be an irreducible closed subvariety of \( Y(1)^n \) both taken as over \( \mathbb{C} \). If \( x \in X(\mathbb{C}) \) then \( (X \setminus X^{\circ}) \cap T(x) \) is finite.

**Proof.** This is an immediate consequence of Theorem 6.2 \([17]\) which holds for varieties over \( \mathbb{C} \). \( \square \)

We say that a subvariety of \( Y(1)^n \) has a special factor if, after possibly permuting coordinates, it is of the form \( S \times W \) with

\[
S \subset Y(1)^{n-n'} \quad \text{a special subvariety where } n-n' \geq 1
\]

and \( W \subset Y(1)^{n'} \) Zariski closed.

Theorem \([8]\) is used to prove the following proposition.

**Proposition 9.** Let \( X \subset Y(1)^n \) be a subvariety without a special factor. Then there exists \( N_0 \) such that

\[
(10) \quad \text{if } N \in \mathbb{N}^n \text{ and } |N| \geq N_0, \text{ then } X \not\subset T_N(X).
\]
Before we come to the proof we briefly discuss the connection to Edixhoven’s Theorem 4.1 which, under a suitable restriction, draws a similar conclusion. The restriction requires all coordinates of \( N \) to be equal. More importantly, the prime divisors of the level need to be large with respect to \( X \). In our application the coordinates are powers of the fixed prime \( p \) which is unrelated to \( X \).

Proof of Proposition 7. By the Lefschetz principle it suffices to prove the proposition for an irreducible closed subvariety \( X \) of \( Y(1)^n \) defined over \( \mathbb{C} \).

What happens if, after permuting coordinates, we have \( X = X' \times \{x'' \} \), no coordinate function is constant on \( X' \subset Y(1)^{n'} \), and \( n' \leq n - 1 \)? Then no coordinate of \( x'' \) is a singular moduli, since \( X \) would have a special factor otherwise. There cannot be two cyclic isogenies of different degree between a pair of elliptic curves without complex multiplication. So \( x'' \in T_{N''}(x'') \) only if \( N'' = (1, \ldots, 1) \). Since \( T_{N}(X) = T_{N'}(X') \times T_{N''}(x'') \) for any \( N \in \mathbb{N}^n \) it is enough to verify the proposition for \( X = X' \text{ and } \dim X \geq 1 \).

The hypothesis and Proposition 4 imply that \( Z = X \setminus X^0 \) is a Zariski closed subset of \( X \) with \( \dim Z \leq \dim X - 1 \).

We aim at a contradiction and suppose
\[
X \subset T_{N_i}(X) \quad \text{for infinitely many} \quad N_i \in \mathbb{N}^n.
\]
We define
\[
\Sigma_1 = \bigcup_{i \geq 1} X(\mathbb{C}) \cap T_{N_i}(Z)(\mathbb{C})
\]
and for good measure \( \Sigma_2 = \{ (x_1, \ldots, x_n) \in X(\mathbb{C}); \text{ some } x_i \text{ is a singular moduli} \} \).

Then \( \Sigma_1 \) and \( \Sigma_2 \) are both countable unions of Zariski closed subsets of \( X \), each of dimension \( \leq \dim X - 1 \). Indeed, \( \dim X \cap T_{N_i}(Z) \leq \dim T_{N_i}(Z) = \dim Z \leq \dim X - 1 \) proves the claim for \( \Sigma_1 \). We recall that no coordinate function is constant on \( X \) and that there are only countably many CM points. So the claim holds for \( \Sigma_2 \) too.

We find, by the Baire Category Theorem for example, that \( \Sigma_1 \cup \Sigma_2 \not\subset X \). We fix an auxiliary point \( x \in X(\mathbb{C}) \setminus (\Sigma_1 \cup \Sigma_2) \).

After permuting coordinates we may suppose that if \( N_1 \) is the first coordinate of \( N_i \), then \( N_1 \to \infty \). By (11) there is \( x^{(i)} \in X(\mathbb{C}) \) with \( (x, x^{(i)}) \in T_{N_i}(\mathbb{C}) \). Each \( x^{(i)} \) lies in the full Hecke orbit \( T(x) \). Recall that the first coordinate of \( x \) is not a singular moduli. By evoking the argument from above on cyclic isogenies between elliptic curves without complex multiplication we find that \( x^{(1)}, x^{(2)}, \ldots \) is an infinite sequence.

Some \( x^{(i)} \) must be in \( Z(\mathbb{C}) \) by Theorem 8. But then \( x \in T_{N_i}(x^{(i)}) \subset T_{N_i}(Z)(\mathbb{C}) \), contradicting \( x \not\in \Sigma_1 \).

\[ \square \]

4. Approximating Non-Special Subvarieties

By continuity we may extend any element of \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \) uniquely to an automorphism of \( \mathbb{C}_p \). So we will apply elements of \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \) to \( \mathbb{C}_p \) without further comment.

Lemma 10 (Approximation Lemma). Let \( \alpha_1, \ldots, \alpha_N \in \mathcal{O}_p \) and suppose \( \delta \in (0, 1] \). There exists \( D \in \mathbb{N} \) such that if \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \), then
\[
|\sigma^D(\alpha_i) - \alpha_i|_p \leq \delta \quad \text{for} \quad 1 \leq i \leq N.
\]
Proof. Since $\overline{\mathbb{Q}}_p$ lies dense in $\mathbb{C}_p$ there exists a finite Galois extension $K$ of $\mathbb{Q}_p$ containing $x_1, \ldots, x_N$ with $|x_i - \alpha_i|_p \leq \delta$ for $1 \leq i \leq N$. Note that all $x_i$ must be integers since $\delta \leq 1$.

Let $\mathcal{O}$ be the ring of integers of $K$ and $\pi$ a generator of its maximal ideal. We fix the smallest integer $n \geq 0$ with $|\pi^n|_p \leq \delta$. Any element of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ induces an automorphism of the finite ring $\mathcal{O}/\pi^n\mathcal{O}$. If $D = (\#\mathcal{O}/\pi^n\mathcal{O})!$ then $\sigma^D$ acts trivially on $\mathcal{O}/\pi^n\mathcal{O}$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. In other words, $\sigma^D(x_i) \in x_i + \pi^n\mathcal{O}$ or

$$|\sigma^D(x_i) - x_i|_p \leq |\pi^n|_p \leq \delta. \tag{12}$$

The ultrametric triangle inequality implies

$$|\sigma^D(\alpha_i) - \alpha_i| = |\sigma^D(x_i) - x_i + x_i - \alpha_i + \sigma^D(\alpha_i) - \sigma^D(x_i)|_p \leq \max\{|\sigma^D(x_i) - x_i|_p, |\alpha_i - x_i|_p, |\alpha_i - \sigma^D(x_i)|_p\} \leq \delta$$

where we used $|\sigma^D(\alpha_i - x_i)|_p = |\alpha_i - x_i|_p \leq \delta$ and (12). \hfill $\square$

We extend $| \cdot |_p$ from $\mathbb{C}_p$ to the Gauss norm on $\mathbb{C}_p[X_1, \ldots, X_n]$.

**Lemma 11.** Let $f_1, \ldots, f_N \in \mathbb{C}_p[X_1, \ldots, X_n]$. There exists a constant $c = c(f_1, \ldots, f_N) > 0$ such that if $f$ lies in the ideal of $\mathbb{C}_p[X_1, \ldots, X_N]$ generated by $f_1, \ldots, f_N$ then there are $a_1, \ldots, a_N \in \mathbb{C}_p[X_1, \ldots, X_n]$ with

$$f = \sum_{i=1}^{N} a_if_i$$

and

$$\max_{1 \leq i \leq N} |a_i|_p \leq c|f|_p.$$

**Proof.** Stating that $f$ is in the ideal generated by the $a_i$ is equivalent to stating that a certain inhomogeneous linear equation $Fx = f$ is solvable. Here $F$ is a matrix whose entries are coefficients of the $f_i$ and $f$ is identified with its coefficients suitably arranged as a column vector. The entries of $x$ are coefficients of polynomials $a_i$ such that $f = \sum_{i=1}^{N} a_if_i$. After transforming $F$ into reduced row echelon form it is evident how to replace the $a_i$ by new polynomials satisfying the assertion. \hfill $\square$

We collect some basic facts on the $p$-adic distance function in the next lemma.

**Lemma 12.** Let $Z$ be Zariski closed in $Y(1)^n$ and $Z'$ Zariski closed in $Y(1)^m$.

(i) If $n = m$, then

$$\text{dist}_p(x, Z)\text{dist}_p(x, Z') \leq \text{dist}_p(x, Z \cup Z') \leq \min\{\text{dist}_p(x, Z), \text{dist}_p(x, Z')\}$$

for all $x \in \mathcal{O}_p^n$.

(ii) There is a constant $c = c(Z, Z') \geq 1$ such that if $x \in \mathcal{O}_p^n$ and $y \in \mathcal{O}_p^m$, then

$$\text{dist}_p((x, y), Z \times Z') \leq c \max\{\text{dist}_p(x, Z), \text{dist}_p(y, Z')\}.$$

Moreover,

$$\max\{\text{dist}_p(x, Z), \text{dist}_p(y, Z')\} \leq \text{dist}_p((x, y), Z \times Z').$$
Proof. We recall that $I(Z \cup Z') = I(Z) \cap I(Z')$ which implies the second inequality in (i). The first one follows from $I(Z)I(Z') \subset I(Z \cup Z')$.

We come to part (ii). The second inequality is immediate as any element of $I(Z)$ or $I(Z')$ also vanishes on $Z \times Z'$ when considered as a polynomial in additional variables.

The first inequality requires some care. Let $f_1, \ldots, f_N$ and $g_1, \ldots, g_M$ be generators of the ideals $I(Z)$ and $I(Z')$, respectively. The first $N$ polynomials have variables $X_1, \ldots, X_n$ and those of the second $M$ polynomials are $Y_1, \ldots, Y_m$. Without loss of generality, we suppose that the $f_i$ and $g_i$ have coefficients in $\mathcal{O}_p$.

Let $f \in I(Z \times Z')$ have coefficients in $\mathcal{O}_p$. The ideal $I(Z \times Z')$ is generated by $f_1, \ldots, f_N$ and $g_1, \ldots, g_M$. By Lemma 11 we can find $a_i, b_i \in \mathbb{C}_p[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ with $f = \sum_{i=1}^N a_i f_i + \sum_{i=1}^M b_i g_i$ and $\max_i\{|a_i|_p, |b_i|_p| \leq c|f|_p \leq c$; the constant $c$ depends only on the $f_i$ and $g_i$ but not on $f$. The ultrametric triangle inequality implies

$$|f(x, y)|_p \leq \max_i\{|a_i|_p|f_i(x)|_p, |b_i|_p|g_i(y)|_p| \leq c \max\{|f_i(x)|_p, |g_i(y)|_p|.$$

Therefore, $|f(x, y)|_p \leq c \max\{\text{dist}_p(x, Z), \text{dist}_p(y, Z')\}$ and part (ii) follows by taking the supremum over all admissible $f$. □

From now on, let $X$ be a subvariety of $Y(1)^n$.

We now prove that an ordinary CM point that lies sufficiently close to $X$ must lie close to one of finitely many special subvarieties contained in $X$. We do this in 3 steps given by the following 3, increasingly refined, statements. Handling ordinary CM points that are close to a special subvariety requires Serre-Tate theory and will be postponed to the next section.

**Lemma 13** (Induction step - first version). We suppose that $X$ does not have a special factor. There exists a Zariski closed subset $Z \subseteq X$ with the following properties. Let $\epsilon > 0$. There is a constant $\delta = \delta_1(\epsilon, X) > 0$ such that if $x \in Y(1)^n(\mathcal{O}_p)$ is an ordinary CM point

$$\text{with dist}_p(x, X) \leq \delta \quad \text{then dist}_p(x, Z) \leq \epsilon.$$

**Proof.** We fix polynomials $f_1, \ldots, f_N \in \mathbb{C}_p[X_1, \ldots, X_n]$ that generate $I(X)$. Without loss of generality, we may suppose that the $f_i$ have coefficients in $\mathcal{O}_p$. Later on we will use Kollár’s Sharp Effective Nullstellensatz [13]. Quadratic polynomials are not allowed in this result but this can be amended by replacing $f_i$ by $f_i^2$ if necessary. Although the $f_i$ may no longer generate $I(X)$, their set of common zeros is still $X$. This is the only property we will need.

The constant $\delta = \delta_1(\epsilon, X) > 0$ will be determined in due course. It will satisfy $\delta \leq 1$.

We apply the Approximation Lemma to $\delta$ and all coefficients of all $f_i$ to obtain $D$ with

$$|\sigma^D(f_i) - f_i|_p \leq \delta$$

for all $i$ and all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

We continue by choosing the least integer $k_0$ with $k_0 \geq \max\{1, 2\text{ord}_p(2D)\}$ and $p^{k_0} \geq N_0$ where $N_0$ is as in Proposition 6. So $k_0$ satisfies the hypothesis of Proposition 6. Moreover, if $N = (p^{k_1}, \ldots, p^{k_n})$ with $(k_1, \ldots, k_n)$ is as in Proposition 6(i), then $X \nsubseteq$
Let us keep in mind that the modular polynomials vanish. We now show that the remaining terms are small for \( f \) and \( \max_i f \) and \( L \) we now take both \( f \) and \( \Phi \) of common zeros of (14) in \( \mathbb{Z} \)
It is Zariski closed and satisfies \( Z \subseteq X \).

Now suppose \( x = (x_1, \ldots, x_n) \in \mathcal{O}_p^n \) as in the assertion.

Proposition 3 gives us \( \sigma \) and \( (k_1, \ldots, k_n) \) where \( v \) is induced by restricting \( | \cdot |_p \) to \( F \), we now take both \( L \) and \( F \) of this proposition as subfields of \( \overline{\mathbb{Q}}_p \). We extend \( \sigma \) to an element of \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \).

We introduce new independent variables \( Y_1, \ldots, Y_n \) and consider the the collection of polynomials (14)
\[
f_1(X_1, \ldots, X_n), \ldots, f_n(X_1, \ldots, X_n), f_1(Y_1, \ldots, Y_n), \ldots, f_n(Y_1, \ldots, Y_n), \Phi_{p^{k_1}}(X_1, Y_1), \ldots, \Phi_{p^{k_n}}(X_n, Y_n)
\]

involving the modular polynomials. We note that the modular polynomials cannot have degree 2.

By Kollár’s Corollary 1.7 [13] there exists \( s \), bounded solely in terms of the \( \deg f_i \) and \( p^{k_i} \), with the following property. The \( s \)-th power of any polynomial vanishing on the set of common zeros of (14) in \( \mathbb{C}_p^n \) is in the ideal generated by said polynomials.

We fix \( f \in \mathcal{O}_p[X_1, \ldots, X_n] \) that vanishes on \( Z \) with
\[
|f(x)|_p \geq \text{dist}_p(x, Z) - \delta^{1/s}.
\]

Let us keep in mind that \( f \) depends on \( x \).

If \( (x'_1, \ldots, x'_n, \sigma(x'_1), \ldots, \sigma(x'_n)) \) is any common zero of (14), then \( (x'_1, \ldots, x'_n) \in Z(\mathbb{C}_p) \). So \( f^s \) lies in the ideal generated by the polynomials (14). We invoke Lemma 11 to find
\[
f^s = \left( \sum_i a_i f_i(X_1, \ldots, X_N) + b_i f_i(Y_1, \ldots, Y_N) \right) + \sum_i c_i \Phi_{p^{k_i}}(X_1, Y_1)
\]
and \( \max_i \{|a_i|, |b_i|, |c_i| \} \leq c |f^s|_p \leq c \); here \( c \) depends on the \( f_i \) and the \( p^{k_i} \) but not on \( f \) or \( x \).

Substituting \( (X_1, \ldots, X_n) \) by \( x \) and \( (Y_1, \ldots, Y_n) \) by \( \sigma^D(x) \) makes the terms involving the modular polynomials vanish. We now show that the remaining terms are small \( p \)-adically. To start off, we use the ultrametric triangle inequality to show
\[
|f(x)|_p^s \leq \max_i \{|a_i(x, \sigma^D(x))f_i(x)|_p, |b_i(x, \sigma^D(x))f_i(\sigma^D(x))|_p \}.
\]

Now \( |a_i(x, \sigma^D(x))|_p \leq |a_i|_p \) since \( x \) and \( \sigma(x) \) have integral coordinates. A similar bound holds for the \( b_i \). We also remark that \( |f_i(x)|_p \leq \text{dist}_p(x, X) \leq \delta \) by definition of the distance. So
\[
|f(x)|_p^s \leq \max_i \{|a_i|_p, |b_i|_p \} \max_i \delta, |f_i(\sigma^D(x))|_p \}.
\]

We may use (13) to get rid of the remaining \( |f_i(\sigma^D(x))|_p \). Indeed, \( |f_i(\sigma^D(x))|_p = |(\sigma^{-D} f_i)(x)|_p \), so
\[
|f_i(\sigma^D(x))|_p = |(\sigma^{-D} f_i)(x) - f_i(x) + f_i(x)|_p \leq \max\{|(\sigma^{-D} f_i - f_i)(x)|_p, |f_i(x)|_p \}
\]
and thus
\[
|f_i(\sigma^D(x))|_p \leq \max\{|(\sigma^{-D} f_i) - f_i|_p, |f_i(x)|_p \} \leq \delta.
\]
We find
\[ |f(x)|_p \leq \max\{|a_i|_p, |b_i|_p\}^{1/s} \delta^{1/s} \leq c^{1/s} \delta^{1/s} \]
and (15) yields \( \text{dist}_p(x, Z) \leq (c^{1/s} + 1)\delta^{1/s} \). The proposition follows for any \( \delta \in (0, 1] \) with \((c^{1/s} + 1)\delta^{1/s} \leq \epsilon \).

We now generalize our statement from subvarieties without special factors to subvarieties that are not special.

**Lemma 14** (Induction step - second version). **We suppose that** \( X \) **is not a special subvariety of** \( Y(1)^n \). **There exists a Zariski closed subset** \( Z \subseteq X \) **with the following property. Let** \( \epsilon > 0 \). **There is a constant** \( \delta = \delta_2(\epsilon, X) > 0 \) **such that if** \( x \in Y(1)^n(\mathbb{C}_p) \) **is an ordinary CM point**
\[ \text{with } \text{dist}_p(x, Z) \leq \delta \text{ then } \text{dist}_p(x, Z) \leq \epsilon. \]
**Proof.** If \( X \) has no special factor, then the proposition follows from Lemma 13.

After permuting coordinates we may suppose that \( X = S \times W \) with \( S \subseteq Y(1)^{n'} \) special and \( W \subseteq Y(1)^{n''} \) a subvariety that has no special factors. We remark that \( n' \geq 1 \) because \( X \) has a special factor and \( n'' \geq 1 \) because \( X \) is not special. The first version of the induction step applied to \( W \) yields a Zariski closed set \( Z' \subseteq W \). We remark that \( Z' \) is independent of \( \epsilon \).

First we apply the initial bound of Lemma 12(ii) to find \( c \geq 1 \) with
\[ \text{dist}_p((x', x''), S \times Z') \leq c \max\{\text{dist}_p(x', S), \text{dist}_p(x'', Z')\} \]
for all \( x' \in \mathcal{O}_p^{n'} \) and \( x'' \in \mathcal{O}_p^{n''} \).

We make the choice \( \delta = \delta_2(\epsilon, X) = \min\{\delta_1(\epsilon/c, W), \epsilon/c\} \) and proceed to show that the Zariski closed set \( Z = S \times Z' \) satisfies the assertion.

Say \( x = (x', x'') \) is as in the hypothesis. If \( \text{dist}_p(x, X) \leq \delta \), then \( \text{dist}_p(x'', W) \leq \delta \) and \( \text{dist}_p(x', S) \leq \delta \) by the second bound of Lemma 12(ii). The previous lemma provides \( \text{dist}_p(x'', Z') \leq \epsilon/c \). Recalling (16) yields
\[ \text{dist}_p(x, S \times Z') \leq c \max\left\{ \delta, \frac{\epsilon}{c} \right\} \leq \epsilon. \]

**Proposition 15** (Induction step - final version). **There exist** \( N \geq 0 \) **and a finite number of special subvarieties** \( S_1, \ldots, S_N \) **of** \( Y(1)^n \) **contained in** \( X \) **with the following property. Let** \( \epsilon > 0 \). **There is a constant** \( \delta = \delta_3(X, \epsilon) > 0 \) **such that if** \( x \in Y(1)^n(\mathbb{C}_p) \) **is an ordinary CM point**
\[ \text{with } \text{dist}_p(x, Z) \leq \delta \text{ then } \text{dist}_p(x, S_i) \leq \epsilon \text{ for some } 1 \leq i \leq N. \]
**Proof.** Without loss of generality \( X \) is not special. We prove the proposition by induction on \( \dim X \).

Suppose \( \dim X = 0 \). Since \( X \) is not a CM point, the previous lemma implies that \( \text{dist}_p(x, X) \) is uniformly bounded from below for any \( x \) as in the assertion. This proposition follows with \( N = 0 \) for \( \delta_3(X, \epsilon) > 0 \) sufficiently small.

Now we assume \( \dim X \geq 1 \).

Let \( \delta_2(\epsilon, X) \) be the constant from the previous lemma and let \( Z_1, \ldots, Z_M \) be the irreducible components of the Zariski closed subset of \( X \) it provides. The \( Z_i \) are independent of \( \epsilon \). By induction we get special subvarieties \( S_1, \ldots, S_N \) contained in the \( Z_i \) and constants \( \delta_3(Z_i, \epsilon) \).
We now prove that $\delta = \delta_3(X, \epsilon) = \min \{ \delta_2(\delta_3(\epsilon, Z_i)^M, X) \}$ is sufficient.

Say $x$ is as in the assertion and $\text{dist}_p(x, X) \leq \delta$. Then $\text{dist}_p(x, Z) \leq \delta_3(\epsilon, Z_i)^M$ for all $i$ by Lemma 14. Lemma 12(i) implies that some component of $Z$, say $Z_1$, satisfies $\text{dist}_p(x, Z_1)^M \leq \text{dist}_p(x, Z)$. So

$$\text{dist}_p(x, Z_1) \leq \delta_3(\epsilon, Z_1).$$

The induction hypothesis now guarantees $\text{dist}_p(x, S_i) \leq \epsilon$ for some $i$, as desired. \( \Box \)

5. A Brief Review of Serre-Tate Theory

We recall some consequences of Serre-Tate theory for an ordinary abelian variety $A$ defined over algebraically closed field $\kappa$ of characteristic $p > 0$. Our applications will however be restricted to elliptic curves.

Let $R$ be an Artinian local ring with residue field $\kappa$. An admissible pair is a tuple $(A, f)$ with $A \to \text{Spec}(R)$ an abelian scheme and $f : A \to A \otimes_R \kappa$ an isomorphism of abelian varieties. Two admissible pairs $(A, f), (A', f')$ are called equivalent if there exists an isomorphism $A \to A'$ of abelian schemes over $\text{Spec}(R)$ whose restriction to the special fiber composed with $f$ is $f'$. The equivalence classes of admissible pairs form a set $\mathcal{M}_A(R)$. The association $R \mapsto \mathcal{M}_A(R)$ is a functor from the category of Artinian local rings to the category of sets.

Let $M_R$ be the maximal ideal of $R$. We define the group

$$\widehat{G}_m(R) = 1 + M_R.$$

Note that there is an $n$ with $p^nM_R = 0$ by Nakayama’s Lemma. So the abelian group $1 + M_R$ is a $\mathbb{Z}_p$-module. Moreover, $\widehat{G}_m$ is a functor from Artinian local rings with residue field $\kappa$ to the category of $\mathbb{Z}_p$-modules.

Since $A$ is ordinary, the subgroup $A[p^n] \subset A(\kappa)$ of elements of order dividing $p^n$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^{\dim A}$ for all $n \in \mathbb{N}$. Let

$$T_p(A) = \lim_{\leftarrow} A[p^n]$$

denote the Tate module of $A$. It is a free $\mathbb{Z}_p$-module of rank $\dim A$. Let $A'\,\text{dual of }\, A$. It is a free $\mathbb{Z}_p$-module.

**Theorem 16** (Serre-Tate). There exists a natural isomorphism

$$q : \mathcal{M}_A \to \text{Hom}(T_p(A) \otimes T_p(A'), \widehat{G}_m)$$

of functors.

**Proof.** This is part of Theorem 2.1 [12]. \( \Box \)

Now assume $K$ is a finite extension of the maximal unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$. The ring of integers $\mathcal{O}$ of $K$ is a discrete valuation ring with maximal ideal $M_\mathcal{O}$. The residue field $\kappa = \mathcal{O}/M_\mathcal{O}$ is an algebraic closure of $\mathbb{F}_p$. We set

$$\mathcal{M}_A(\mathcal{O}) = \lim_{\leftarrow} \mathcal{M}_A(\mathcal{O}/M_\mathcal{O}^{n+1}) \quad \text{and} \quad \widehat{G}_m(\mathcal{O}) = \lim_{\leftarrow} \widehat{G}_m(\mathcal{O}/M_\mathcal{O}^{n+1}) = 1 + M_\mathcal{O}.$$

The former limit is to be understood as the set of formal deformations of $A$. The latter is a $\mathbb{Z}_p$-module.
Let $A \to \text{Spec}(\mathcal{O})$ be an abelian scheme and say $A \to A \otimes \mathcal{O}$ $\kappa$ is an isomorphism of abelian varieties. The direct system of schemes $A \otimes \mathcal{O} (\mathcal{O}/\mathcal{O}^{n+1})$ yields an element

$$ q(A) \in \lim \text{Hom}(T_p(A) \otimes T_p(A^\vee), \hat{\mathcal{G}}_m(\mathcal{O}/\mathcal{O}^{n+1})) = \text{Hom}(T_p(A) \otimes T_p(A^\vee), \hat{\mathcal{G}}_m(\mathcal{O})) $$

where we take $\mathbb{Z}_p$-module homomorphisms. The choice of a $\mathbb{Z}_p$-basis of $T_p(A)$ and of $T_p(A^\vee)$ induces an isomorphism

$$ \text{Hom}(T_p(A) \otimes T_p(A^\vee), \hat{\mathcal{G}}_m(\mathcal{O})) \cong \hat{\mathcal{G}}_m(\mathcal{O})^{(\dim A)^2}. $$

Serre and Tate’s Theorem enriches the set of formal deformations of $A$ with the structure of an abelian group. De Jong and Noot characterized the elements of finite order.

**Theorem 17** (de Jong-Noot). We keep the notation from above. Then $q(A)$ has finite order if and only if $A \otimes \mathcal{O} K$ has complex multiplication.

**Proof.** See Proposition 3.5 [3]. □

### 6. Approximating Special Subvarieties

We recall that $\Phi_N \in \mathbb{Z}[X,Y]$ denotes the modular polynomial of level $N \in \mathbb{N}$.

**Lemma 18.** Suppose $N \in \mathbb{N}$ and let $x_1, x_2 \in \mathcal{O}_p$ be ordinary singular moduli. If

$$ \text{ord}_p \Phi_N(x_1, x_2) > 6\frac{\Psi(N)}{p-1} $$

then $\Phi_N(x_1, x_2) = 0$.

**Proof.** Let us consider $P = \Phi_N(x_1, x_2 + T) = a_0 + \cdots + a_T T^d \in \mathcal{O}_p[T]$. The polynomial $\Phi_N$ is monic of degree $\Psi(N)$ in $Y$, cf. [3]. So $P$ is monic and of degree $\Psi(N)$. Let $t \in \mathbb{C}_p$ be a root of $P$ with maximal order. Then

$$ \Psi(N) \text{ord}_p(t) \geq \text{ord}_p(a_0) = \text{ord}_p \Phi_N(x_1, x_2) > 6 \frac{\Psi(N)}{p-1} \quad \text{hence} \quad \text{ord}_p(t) \geq \frac{6}{p-1}. $$

We have $\Phi_N(x_1, x'_2) = 0$ with $x'_2 = x_2 + t$. In particular, $x'_2$ is a singular moduli and

$$ \text{ord}_p(x'_2 - x_2) = \text{ord}_p(t) > \frac{6}{p-1} $$

implies that it is ordinary.

We first treat the case $p \neq 2$. Below $K$ will denote a finite extension of the maximal unramified extension of $\mathbb{Q}_p$ to be specified during the argument below. Let $\mathcal{O}$ be the ring of integers in $K$ and $\mathcal{M}_\mathcal{O}$ the maximal ideal of $\mathcal{O}$.

Let us fix a root $\lambda_2 \in K$ of

$$ 2^s(1 - \lambda_2(1 - \lambda_2))^3 - x_2 \lambda_2^3(1 - \lambda_2)^2 = 0. $$

So $\lambda_2 \in \mathcal{O}_p$ since it is integral over $\mathcal{O}_p$. The ultrametric triangle inequality implies $\text{ord}_p \lambda_2 = 0$ and $\text{ord}_p (1 - \lambda_2) = 0$. The equation

$$ y^2 = x(x - 1)(x - \lambda_2) $$

defines an elliptic curve with $j$-invariant $x_2$. The coefficients of this model are integers in $K$ and its discriminant is $2^4 \lambda_2^3(\lambda_2 - 1)^2$. Thus our elliptic curve has good reduction. In other words, we obtain an elliptic scheme $E_2 \to \text{Spec}(\mathcal{O})$ whose special fiber $E_2$ is an elliptic curve over the residue field of $K$. 

After possibly increasing $K$ we find $\lambda'_2 \in K$ with
\[ 2^8 (1 - \lambda'_2(1 - \lambda'_2))^3 - x'_2\lambda'_2^2(1 - \lambda'_2)^2 = 0 \]
and
\[ \text{ord}_p(\lambda'_2 - \lambda_2) \geq \frac{1}{6} \text{ord}_p(x'_2 - x_2) > \frac{1}{p - 1}. \] (18)

This element gives us a second elliptic curve in Weierstrass form
\[ y^2 = x(x - 1)(x - \lambda'_2), \]
with $j$-invariant $x'_2$. As above we find that this curve has good reduction and thus it yields an elliptic scheme $E'_2 \to \text{Spec}(\mathcal{O})$ with special fiber $E_2$.

Note that $\lambda'_2$ and $\lambda_2$ have equal reduction. Therefore, $E_2$ and $E'_2$ are the same ordinary elliptic curve over the residue field of $\mathcal{O}$. We fix a generator of $\mathcal{T}_p(E_2)$ and henceforth consider the Serre-Tate parameter as an element of $1 + M_\mathcal{O}$. Let $q_2$ and $q'_2$ be the Serre-Tate parameters of $E_2$ and $E'_2$. Then $E_2$ and $E'_2$ are isomorphic modulo $p^{\text{ord}_p(\lambda_2 - \lambda'_2)}$ and we obtain
\[ \text{ord}_p(q_2 - q'_2) \geq \text{ord}_p(\lambda_2 - \lambda'_2) > \frac{1}{p - 1} \] (19)
from (15).

The case $p = 2$ is similar, but we cannot rely on the Legendre model as it necessarily leads to a curve of bad reduction. In any case $\text{ord}_2(x_2) = 0$ since an elliptic curve with $j$-invariant 0 is supersingular in characteristic 2. This and (17) entail $\text{ord}_2(x'_2) = 0$. The $j$-invariant of the elliptic curve defined by
\[ y^2 + xy = x^3 - \frac{36}{x^2 - 1728}x - \frac{1}{x^2 - 1728} \]
is $x_2$. The coefficients involved are in $\mathcal{O}$ and the discriminant equals $x^2_2(x_2 - 1728)^{-3}$. As before we get an elliptic scheme $E_2$ with special fiber $E_2$. And again we introduce an elliptic scheme $E'_2$ with special fiber $E_2$ and determined by
\[ y^2 + xy = x^3 - \frac{36}{x^2'_2 - 1728}x - \frac{1}{x^2'_2 - 1728}. \]
The generic fiber has $j$-invariant $x'_2$. Just as in the case of odd characteristic the corresponding Serre-Tate parameters $q_2, q'_2$ satisfy
\[ \text{ord}_2(q_2 - q'_2) \geq \text{ord}_2(\lambda_2 - \lambda'_2) > \frac{1}{p - 1} = 1. \] (20)

Now we suppose again that $p$ is arbitrary. We note that (19) and (20) both lead to
\[ \text{ord}_p(q_2 - q'_2) > \frac{1}{p - 1}. \] (21)

Since the generic fibers of $E_2$ and $E'_2$ have complex multiplication, $\zeta = q'_2q_2^{-1}$ is a root of unity by de Jong and Noot’s Theorem. Any root of unity in $1 + M_\mathcal{O}$ has order $p^e$ for some integer $e \geq 0$ and it is classical that
\[ \zeta = 1 \quad \text{or} \quad \zeta \neq 1 \quad \text{and} \quad \text{ord}_p(1 - \zeta) = \frac{1}{p^e - 1(p - 1)} \leq \frac{1}{p - 1}. \]
Lemma 1. Let $x_1, x_2 \in \mathcal{O}_p$ be ordinary singular moduli. If

$$\text{ord}_p(x_1 - x_2) > \frac{6}{p - 1} \quad \text{then} \quad x_1 = x_2.$$ 

Proof. This follows from the previous lemma in the case $N = 1$ since $\Phi_1 = X - Y$. \qed

We now prove the main result of this section.

Proposition 20. Let $S \subset Y(1)^n$ be a special subvariety. There exists a constant $\epsilon > 0$ with the following property. If $x = (x_1, \ldots, x_n) \in \mathcal{O}_p^n$ is an ordinary CM point with $x \notin S(\mathbb{C}_p)$, then

$$\text{dist}_p(x, S) \geq \epsilon.$$ 

Proof. Without loss of generality, we may suppose $S \neq Y(1)^n$. After permuting coordinates we have $S = S' \times S''$ where $S' \subset Y(1)^{n'}$ consists of a CM point and no coordinate function is constant when restricted to $S'' \subset Y(1)^{n''}$.

Say $x = (x', x'')$ is as in the hypothesis with $x' \in \mathcal{O}_p^{n'}$ and $x'' \in \mathcal{O}_p^{n''}$.

We suppose first that $n' \geq 1$ and that $x'$ is not the CM point $S'$. We can use Lemma 19 to obtain $\text{dist}_p(x', S') \geq p^{-6/(p-1)}$. Here the proposition follows from Lemma 12(ii).

If $n' = 0$ or if $x'$ is the CM point $S'$, then $S''$ is not a power of $Y(1)$ and $x'' \notin S''(\mathbb{C}_p)$. By the classification of special subvarieties at the beginning of Section 3 there is $N \in \mathbb{N}$ in a finite set depending only on $S''$ and indices $n' < i < j \leq n$ with the following properties. The polynomial $\Phi_N(x_i, x_j)$ vanishes on $S''$ and $\Phi_N(x_i, x_j) \neq 0$ for the corresponding coordinates $x_i, x_j$ of $x$. We now refer directly to Lemma 18 to obtain $\text{ord}_p \Phi_N(x_i, x_j) \leq 6\Phi(N)/(p - 1)$. Since $\Phi_N(x_i, x_j)$ is in the ideal of $X$ and has coefficients in $\mathbb{Z}$, we deduce $\text{dist}_p(x, S) \geq p^{-6\Phi(N)/(p-1)}$. \qed

Proof of Theorem 4. Our theorem is now a direct consequence of the previous proposition and Proposition 15. \qed

APPENDIX A. HIGHLY DIVISIBLE SINGULAR MODULI

A.1. Warming-up. As a warming-up for the proof of Proposition 2 we discuss a variation in the case $p = 2$. This subsection is conditional on a conjecture on prime values of quadratic polynomials. Our main tool is a result of Gross and Zagier [5]. We proceed by setting up some notation.

If $K$ is a number field then $\mathcal{O}_K$ denotes its ring of integers and $\text{Cl}_K$ its class group.

Let $\ell \geq 5$ be a prime with $\ell \equiv 3 \mod 8$. Then $\left(\frac{\ell}{2}\right) = \left(\frac{2}{\ell}\right) = -1$ where $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol. The imaginary quadratic field $K = \mathbb{Q}(\sqrt{-\ell})$ has discriminant
Let \( n \) be odd and suppose that there exists \( x \in \mathbb{Z} \) with
\[
3x^2 + 2^{2+n} = \ell.
\]
We remark that \( \ell \) satisfies all conditions imposed further up.

After multiplying by 3 and rearranging we find \((3\ell - (3x)^2)/2^{2+k} = 2^{n-k}.3\) where \( k \) denotes an odd integer with \( 1 \leq k \leq n \). Equation (23) implies \( \left( \frac{3x^2}{\ell} \right) = -\left( \frac{2}{\ell} \right)^{2+n} = -( -1)^{2+n} = 1 \) since \( n \) is odd. So \( 1 = \left( \frac{2}{\ell} \right) = \left( \frac{\ell}{3} \right) \) by quadratic reciprocity and 3 splits in \( K \). This means that \( \mathcal{O}_K \) contains precisely two ideals of norm 3. By the theory of genera by Gauss and since the discriminant of \( K \) is a prime, \( \text{Cl}_K \) has no elements of order 2, cf. Proposition 3.11 [2]. So \( \mathcal{A} \mapsto \mathcal{A}^2 \) induces an automorphism of \( \text{Cl}_K \). Hence we may choose \( \mathcal{A} \) such that \( \mathcal{A}^2 \) contains an ideal of norm 3. But \( n \) and \( k \) are both odd, so \( 2^{n-k} \) is a perfect square and therefore \( r_{\mathcal{A}^2}(2^{n-k}.3) \geq 1 \). We deduce
\[
r_{\mathcal{A}^2}\left( \frac{3\ell - (3x)^2}{2^{2+k}} \right) \geq 1 \quad \text{and hence} \quad \sum_{x \in \mathbb{Z}} 2^{\omega(\gcd(2,x))} r_{\mathcal{A}^2}\left( \frac{3\ell - x^2}{2^{2+k}} \right) \geq 2.
\]
On summing over all odd \( k \) with \( 1 \leq k \leq n \) we use (22) to deduce
\[
\ord_2(\sigma(j)) \geq \frac{3}{2}(n + 1).
\]
So \( \sigma(j) \) is non-zero and converges to 0 in the 2-adic topology if \( n \) can be made arbitrarily large.

However, our construction only works for \( n \) if there exists \( x \in \mathbb{Z} \) with (23) a prime since Gross and Zagier’s result requires a prime discriminant. Schinzel’s Hypothesis predicts that there are infinitely many \( x \) with (23) a prime. This conjecture is open and seems out of reach at the moment.

A.2. Computations with Quaternions. Gross and Zagier’s work provides precise information on the \( p \)-adic valuation of differences of certain singular moduli. But cruder estimates are sufficient for the proof of Proposition 2. These will follow from very explicit calculations in a quaternion algebra and some ideas of Gross and Zagier.

Our quaternion algebras are all over \( \mathbb{Q} \) and as a reference we use mainly the book of Vigneras [23]. Let \( K \) be the number field \( \mathbb{Q}(\sqrt{-3}) \) and \( \mathcal{O}_K = \mathbb{Z}[\theta] \) its ring of integers where \( \theta = (\sqrt{-3} + 1)/2. \) Below \( \alpha \mapsto \overline{\alpha} \) denotes the non-trivial automorphism of \( K \ni \alpha \). The different \( D \) of \( K \) is the ideal \( \sqrt{-3}\mathcal{O}_K \). We define \( q = (2 + \sqrt{-3})\mathcal{O}_K \) and remark

\( -\ell \equiv 1 \mod 4 \) and the prime 2 is inert in \( K \). There is a unique singular moduli \( j \) whose associated elliptic curve has complex multiplication by \( \mathcal{O}_K = \mathbb{Z}[\sqrt{-3}] \) with \( j \in \mathbb{Q}_2 \). We note that \( j \neq 0 \) since \( \ell \neq 3 \). The Hilbert class field of \( K \) is \( K(j) \). Let \( \mathcal{A} \in \text{Cl}_K \), we will make a precise choice later on. For \( x \in \mathbb{R} \) we set
\[
r_{\mathcal{A}}(x) = \# \{ I \in \mathcal{A}; \ I \text{ has norm } x \}.
\]
that \( q \overline{q} = 7 \mathcal{O}_K \). Let us suppose that \( p \) is as in Proposition 2. Thus \( p \) is inert in \( K \) and odd.

For \( \alpha, \beta \in K \) we set

\[
[\alpha, \beta] = \left( \begin{array}{c} \alpha \\ -7p \beta \\ \alpha \end{array} \right) \quad \text{and} \quad Q = \{ [\alpha, \beta]; \alpha, \beta \in K \}.
\]

We consider \( K \) as a subset of \( Q \) by virtue of \( \alpha \mapsto \left[ \alpha, 0 \right] \). Then \( Q \) is a ring because \( u^2 = -7p \) where \( u = [0,1] \) and even a four dimension \( \mathbb{Q} \)-algebra. We have \( u\alpha = \overline{\alpha}u \) from which it is easy to deduce that \( Q \) is the center of \( Q \). Moreover, \( [\alpha, \beta][\overline{\alpha}, -\beta] \) is \( \alpha \overline{\alpha} + 7p \beta \overline{\beta} \in \mathbb{Q} \) and vanishes if and only if \( [\alpha, \beta] = 0 \). Hence \( Q \) is a quaternion algebra and a skew-field. It therefore ramifies at at least one finite place of \( Q \). We will see momentarily that \( p \) is such a place and that it is the only one.

The subset \( O = \{ [\alpha, \beta]; \alpha \in \mathcal{D}^{-1}, \beta \in q^{-1}\mathcal{D}^{-1}, \text{ and } \alpha - 7\beta \in \mathcal{O}_K \} \subset Q \)

is a free \( \mathbb{Z} \)-module of rank 4. A short calculation verifies that it is a sub-ring of \( Q \). Thus \( O \) is an order of \( Q \). It is readily checked that

\[
[1 - 2\theta, 0], [1 - \theta, 0], \left[ \frac{1 - 2\theta}{3}, \frac{4 - 5\theta}{21} \right], \left[ 1 - \theta, \frac{3 - 9\theta}{21} \right]
\]

are elements of \( O \). Let us denote them by \( b_1, \ldots, b_4 \). In a moment we will prove that these elements constitute a \( \mathbb{Z} \)-basis of \( O \), that \( O \) is a maximal order, and that \( Q \) ramifies precisely at \( p \) and \( \infty \).

Indeed, the reduced trace of any element \( [\alpha, \beta] \in Q \) is \( \text{Tr}([\alpha, \beta]) = \alpha + \overline{\alpha} \) so the \( 4 \times 4 \) matrix with entries \( \text{Tr}([b_i b_j]) \) is

\[
\begin{pmatrix}
6 & 3 & 2 & 3 \\
3 & 1 & 1 & 1 \\
2 & 1 & \frac{2p+2}{3} & p + 1 \\
3 & 1 & p + 1 & 2p + 1
\end{pmatrix}
\]

and has determinant \(-p^2\). Thus \( \{b_1, \ldots, b_4\} \) is linearly independent and the reduced discriminant \( d(O) \) of \( O \) is a divisor of \( p \). But \( d(O) \) is a multiple of the product of all finite primes where \( Q \) ramifies. We have seen above that this product is not 1. So \( d(O) = p \) and the remaining claims hold as well.

We write \( W \subset \mathbb{C}_p \) for the ring of integers in the completion of the maximal unramified algebraic extension of \( \mathbb{Q}_p \).

If \( E \) is an elliptic curve defined over a field then \( \text{End}(E) \) denotes the ring of its endomorphisms defined over an algebraic closure of the base field. Say \( E \) is the elliptic curve determined by \( y^2 = x^3 + 1 \) taken as an equation with coefficients in \( W \). It has complex multiplications by \( \mathcal{O}_K \) and good reduction \( E_0 \) modulo \( pW \) since \( p \geq 5 \). Moreover, \( E_0 \) is supersingular because \( p \) is inert in \( K \).

The following statements are well-known facts from the theory of elliptic curves. The ring \( \text{End}(E_0) \) is a maximal order in \( \text{End}(E_0) \otimes \mathbb{Q} \) and the latter is a quaternion algebra which is ramified precisely at \( p \) and \( \infty \). Reduction modulo \( pW \) induces an injective homomorphism from \( \text{End}(E) \) to \( \text{End}(E_0) \). We will thus take \( K \) as a sub-ring of \( \text{End}(E_0) \otimes \mathbb{Q} \).
Up-to isomorphism there is only one quaternion algebra that is ramified at $p$ and $\infty$. By the Skolem-Noether Theorem we may choose an isomorphism $\text{End}(E_0) \otimes \mathbb{Q} \to \mathbb{Q}$ that is compatible with the two inclusions $K \subset \text{End}(E_0) \otimes \mathbb{Q}$ and $\mathbb{K} \subset \mathbb{Q}$. Since $\mathcal{O}$ and $\text{End}(E_0)$ both contain the principal ideal domain $\mathcal{O}_K$ and all three are maximal orders, in their respective sense, Eichler’s Theorem 4, Chapter 4 [5] implies that $\mathcal{O}$ and $\text{End}(E_0)$ are conjugate by an element of $\mathcal{O}_K$. Henceforth we will identify $\text{End}(E_0) \otimes \mathbb{Q}$ with $\mathbb{Q}$ and $\text{End}(E_0)$ with $\mathcal{O}$.

This setup enables explicit calculations with endomorphisms of $E_0$.

**Lemma 21.** Let $n, x \in \mathbb{Z}$ with $n \geq 0$ and $x$ odd. Then

$$\varphi = \left[\frac{1}{2} - \frac{2\theta - 1}{2}x, \frac{3 - 2\theta}{7}p^n\right]$$

is an element of $\mathcal{O}_K + p^n\mathcal{O}$ which satisfies $\varphi^2 - \varphi + (1 + d)/4 = 0$ where $d = 3x^2 + 4p^{2n+1}$.

**Proof.** The proof is by direct verification. We abbreviate $\varphi = [\alpha, \beta]$ and remark that $\alpha = 1/2 - (2\theta - 1)x/2 = (1 - \sqrt{-3}x)/2 \in \mathcal{O}_K$ since $x$ is odd. Hence it suffices to verify $[0, (3 - 2\theta)/7] \in \mathcal{O}$. But this follows from $D_q(3 - 2\theta)/7 = \sqrt{-3}(-2 + \sqrt{-3})(3 - 2\theta)/7\mathcal{O}_K = \sqrt{-3}\mathcal{O}_K$.

To prove the second claim we remark that $\varphi^2 - T\varphi + N = 0$ where $T$ and $N$ are, respectively, the reduced trace and reduced norm of $\varphi$. Now $T = \alpha + \overline{\alpha} = 1$ and

$$N = \alpha\overline{\alpha} + 7p\beta\overline{\beta} = \frac{1 + 3x^2}{4} + p^{2n+1}\frac{(3 - 2\theta)(3 - 2\overline{\theta})}{7} = \frac{1 + 3x^2}{4} + p^{2n+1} = \frac{1 + d}{4}. \quad \square$$

We remark that $d$ as in this lemma satisfies $-d \equiv 1 \mod 4$.

Let $n \geq 0$ be an integer and let $E$ be the elliptic curve from above. Let $E_n$ denote the elliptic scheme defined by the base change to $\text{Spec}(W/p^{n+1}W)$ of the elliptic scheme over $\text{Spec}(W)$ determined by $y^2 = x^3 + 1$. The reduction operation implies that the endomorphism rings are filtered as

$$\mathcal{O}_K \subset \cdots \subset \text{End}(E_n) \subset \cdots \subset \text{End}(E_1) \subset \text{End}(E_0).$$

Let $\hat{E}$ be the formal group law attached to $E$ with respect to the model $y^2 = x^3 + 1$. It is a power series in two variables and coefficients in $W$. If we reduce this power series modulo $p$ we obtain the formal group law $\hat{E}_0$ of $E_0$.

Suppose that $\varphi \in \text{End}(E_0)$ and let $\hat{\varphi} \in W/pW[[T]]$ be the power series representing $\varphi$. Recall that multiplication-by-$p$ of $\hat{E}$ is a power series $[p] = pf(T) + g(T^p)$ where $f, g \in W[T]$ have no constant term. If $\hat{\varphi}' \in W[[T]]$ is any lift of $\hat{\varphi}$, then a simple induction shows that $[p]^n \hat{\varphi}'$ is well-defined modulo $p^{n+1}$. This implies that $p^n \varphi$ is an endomorphism of $E_n$. Therefore, $p^n \text{End}(E_0) \subset \text{End}(E_n)$. Since elements of $\mathcal{O}_K$ are endomorphisms of $E_n$ we find

$$\mathcal{O}_K + p^n\text{End}(E_0) \subset \text{End}(E_n)$$

for all $n \geq 0$.

The next lemma relies on Gross and Zagier’s [8] version of Deuring’s Lifting Theorem.

**Lemma 22.** Let $n, x$, and $d$ be as in Lemma 21. We suppose in addition that $p \nmid d$ and that $d$ is square-free. Then there exists a singular moduli $x_n \in W$ such that $(1 + \sqrt{-d})/2$ is an endomorphism of the associated elliptic curve and $|x_n|_p \leq p^{-(n+1)}$. 


Proof. Let \( \varphi \) be the endomorphism from Lemma \([21]\). It satisfies \( \varphi^2 - \varphi + (1 + d)/4 = 0 \) and lies in \( \text{End}(E_n) \) by \([24]\). Proposition 2.7 \([8]\) provides an elliptic curve \( E' \) whose endomorphism ring contains \((1 + \sqrt{-d})/2 \) and which is isomorphic to \( E \) modulo \( p^{n+1}W \). We remark that Hensel’s Lemma and \( p \nmid d \) imply that the polynomial \( x^2 - x + (1 + d)/4 \) has precisely two roots in \( W/p^{n+1}W \). The hypothesis of Proposition 2.7 requires that \( d \) is a fundamental discriminant. The \( j \)-invariant of \( E \) is zero. So the \( j \)-invariant \( x_n \) of \( E' \) satisfies \( |x_n|_p \leq p^{-(n+1)} \) by Proposition 2.3 \([8]\). \( \square \)

In general \( 3x^2 + 4p^{2n+1} \) may have a quadratic factor, eg. \( 3 + 4 \cdot 11^2 \cdot 7 + 1 \equiv 0 \mod 49 \). So we must carefully choose \( x \) in order to apply Lemma \([22]\). By an old result of Nagel \([15]\) the polynomial \( 3x^2 + 4p^{2n+1} \) attains infinitely many square-free values at integer arguments. If \( n \geq 1 \), then any such value is coprime to \( p \) and its corresponding argument is necessarily odd. So the hypotheses of Lemmas \([21]\) and \([22]\) are satisfied. We obtain a singular moduli \( x_n \in W \) with \( |x_n|_p \leq p^{-(n+1)} \). Moreover, \( x_n \neq 0 \) because \( 3 \nmid |\Delta(x_n)| = 3x^2 + 4p^{2n+1} \).

A.3. Square-free Values of a Quadratic Polynomial. To prove Proposition \([2]\) we must bound \( |x_n|_p \) from the previous section in terms of \( |\Delta(x_n)| \). To this end we need an upper bound for one square-free value of the polynomial

\[
(25) \quad f = 3x^2 + 4p^{2n+1}
\]

in terms of \( p^{n+1} \).

As Igor Shparlinski pointed out to the author, Iwaniec and Friedlander \([7]\) recently gave estimates for squarefree values of quadratic, monic polynomials. To treat \((25)\) we will make explicit a sieving method presented in Chapter 1 of the same authors’ book \([6]\).

In this section we assume \( p \geq 5 \) and that \( n \in \mathbb{N} \).

If \( y \geq 0 \) is a real number we define

\[
N(y) = \# \{ x \in \mathbb{N}; f(x) \leq y \text{ and } f(x) \text{ is square-free} \}.
\]

Let \( \mu(\cdot) \) denote the Möbius function. If \( m \) is a positive integer then \( \sum_{d \mid m} \mu(d) = 1 \) if and only if \( m \) is square-free, otherwise this sum vanishes. Thus

\[
(26) \quad N(y) = \sum_{\substack{\sigma \geq 1 \\\{ f(x) \leq y \}}} \sum_{d^2 \mid f(x)} \mu(d) = \sum_{1 \leq d \leq y^{1/2}} \mu(d) A_{d^2}(y) \quad \text{with} \quad A_{d^2}(y) = \sum_{\substack{\sigma \geq 1 \\\{ f(x) \leq y \}}} \sum_{d^2 \mid f(x)} 1.
\]

Let \( \epsilon \in (0,1) \) be a constant to be determined in course of the argument below. In the following we use Landau’s big-\( O \) notation. All implicit constants are absolute and thus independent of \( p, n, \epsilon, \) and the parameter \( y \).

Lemma 23. Let \( d \) be a positive integer and suppose \( y \geq 4p^{2n+1} \epsilon^{-1} \). Then

\[
(27) \quad A_{d^2}(y) = \sqrt{\frac{y}{3} \rho(d^2)} + O \left( \rho(d^2) + \epsilon y^{1/2} \frac{\rho(d^2)}{d^2} \right)
\]

where \( \rho(m) = \# \{ b \in \mathbb{Z}/m\mathbb{Z}; f(b) \equiv 0 \mod m \} \) for any integer \( m \geq 1 \).
The following estimates hold true.

**Lemma 24.** We have

\[ \sum_{1 \leq d \leq y^{1/3}} \rho(d^2) = O\left(y^{5/12}\right) \quad \text{and} \quad \sum_{1 \leq d \leq y^{1/3}} \frac{\rho(d^2)}{d^2} = O(1). \]

(ii) We have

\[ \sum_{d>y^{1/3}} \frac{\rho(d^2)}{d^2} = O(y^{-1/4}). \]

(iii) Let \( k \in \mathbb{N} \), then

\[ \# \{(x,d) \in \mathbb{N}^2; 3x^2 + 4p^{2n+1} = d^2k \leq y\} = O((\log y)^2). \]
Proof. We abbreviate $t = y^{1/3}$ and set $I_s = \sum_{1 \leq d \leq t} \rho(d^2) d^{-s}$ for $s \in \{0, 2\}$. Then $I_s = O(\sum_{e \geq 0} p^e \sum_{1 \leq d \leq t} d^{1/5-s})$ thus $I_s = O(\sum_{e \geq 0} p^e (1 + 1/5-s) \sum_{1 \leq k \leq \ell/p^e} k^{1/5-s})$ with $d = p^e k$ in the final sum. This final sum vanishes if $p^e > t$. If $t \geq p^e$ and $s = 0$, it is at most $(t/p^e)^{6/5}$. Using $t = y^{1/3} \geq (p^{2n+1})^{1/3} \geq p$ we get $I_0 = O((t^{6/5} (\log t) / (\log p))$. This implies the first half of (i). If $s = 2$ then said final sum is at most $\sum_{k \geq 1} k^{-9/5}$. So $I_2 = O(\sum_{e \geq 0} p^{-4e/5})$. Now $\sum_{e \geq 0} p^{-4e/5} = 1/(1 - p^{-4/5}) = O(1)$ gives $I_2 = O(1)$, the second half of (i).

For (ii) we use $\rho(p^{2n+2}) = 0$ and estimate

$$I = \sum_{d > t} \frac{\rho(d^2)}{d^2} = O \left( \sum_{e=0}^{2n+1} p^e \sum_{d > t} d^{-1/5} \right) \quad \text{so} \quad I = O \left( \sum_{e=0}^{2n+1} p^e (1/5-1) \sum_{k > t/p^e} k^{1/5-2} \right).$$

We compare the final sum with the corresponding integral and obtain $I = O(\sum_{e=0}^{2n+1} p^{-4e/5} (t/p^e)^{-4/5})$. This sum simplifies to $\sum_{e=0}^{2n+1} t^{-4/5} = (2n + 2)^{-1/5}$ and part (ii) follows since $y \geq p^n$.

Let us now assume $(x, d)$ is in the set on the left side of (23). Then $3 \mid k$ since $p \neq 3$. We write $3k = k's^2$ with $s \in \mathbb{N}$, $k' \in \mathbb{N}$ square-free, and $3|k'$. We set $d' = ds$, $x' = 3x$ and note that $x'^2 - d'^2 k' = -12p^{2n+1}$. In other words, the norm of $z' = x' - d' \sqrt{k'}$ is $-12p^{2n+1}$ as an element of the real quadratic field $F = \mathbb{Q}(\sqrt{k'})$. Say $\mathcal{O}_F$ is the ring of integers of this field. It contains $z'$ and the principal ideal $z'\mathcal{O}_F$ divides $12p^{2n+1}\mathcal{O}_F$. There are $O(n)$ possibilities for the ideal $z'\mathcal{O}_F$ as $p\mathcal{O}_F$ has at most 2 prime ideal factors. But $p^n \leq y$ and so the number of $z'\mathcal{O}_F$ is also $O((\log y)^2)$.

However, distinct elements from our original set may lead to the same ideal. Indeed, the corresponding elements $z', z''$ could be associated. In this case $z'' = \pm \eta N z'$ for some $N \in \mathbb{Z}$ where $\eta > 0$ is the fundamental unit of $F$. We must count how often this happens. Taking the exponential absolute Weil height $H(\cdot)$ and using its basic properties gives

$$H(\eta)^{|N|} = H(\pm \eta^N) = H(z''/z') \leq H(z') H(z''),$$

From the definition of $z'$ we find $H(z') \leq 2H(x') H(d' \sqrt{k'}) = 2x'd' \sqrt{k'} \leq 6y$. The same inequality holds for $H(z'')$ and thus $|N| \log H(\eta) \leq 2 \log (6y)$. But $\log H(\eta) > 0$ and since $\eta$ is in an quadratic number field, its logarithmic height is bounded from below by a positive absolute constant. We derive $|N| = O(\log y)$. The number of possibilities for the pair $(x, d)$ is thus $O((\log y)^2)$.

\begin{lemma}
We have

$$N(y) = c(p, n) \sqrt{\frac{y}{3}} + O(y^{5/12} + ey^{1/2})$$

where $c(p, n) = \sum_{d=1}^{\infty} \mu(d) \frac{\rho(d^2)}{d^2} > 1/7$.
\end{lemma}

Proof. We split $N(y)$ from (23) up into $\sum_{1 \leq d \leq y^{1/3}} \mu(d) A_d(y) + \sum_{y^{1/3} < d \leq y^{1/2}} \mu(d) A_d(y)$. The absolute value of the second sum is at most

$$\sum_{y^{1/3} < d \leq y^{1/2}} A_d(y) = \# \{(x, d) \in \mathbb{N}^2; \ 3x^2 + 4y^{2n+1} = d^2 k \leq y \text{ and } \ d > y^{1/3} \text{ for some } k \in \mathbb{N}\}.$$
We use (27) and Lemma 24(i) to handle the first sum and obtain

\[ N(y) = \sqrt{\frac{y}{3}} \sum_{1 \leq d \leq y^{1/3}} \mu(d) \frac{\rho(d^2)}{d^2} + O(y^{5/12} + \epsilon y^{1/2}). \]

Next we want to replace the sum on the right to a sum over all \( d \). Doing this introduces an error \( O(y^{1/4}) \) by Lemma 24(ii). So

\[ N(y) = c(p, n) \sqrt{\frac{y}{3}} + O(y^{5/12} + \epsilon y^{1/2}) \]

with \( c(p, n) \) as in the hypothesis.

The function \( d \mapsto \mu(d) \rho(d^2) \) is multiplicative. Hence \( c(p, n) \) is represented by the Euler product \( c(p, n) = \prod\ell (1 - \rho(\ell^2)/\ell^2) \). We already know that \( \rho(\ell^2) = \rho(\ell) \leq 2 \) if \( \ell \notin \{2, 3, p\} \) and \( \rho(3^2) = 0 \). So \( \rho(\ell^2)/\ell^2 \leq 1/\ell^{3/2} \) except possibly for \( \ell = 2 \) or \( p \). One readily checks \( \rho(4) = 2 \) and we know \( \rho(p^2) \leq p \). So

\[ c(p, n) \geq \frac{p-1}{2} \prod_{\ell \neq 2, p} \left( 1 - \frac{1}{\ell^{3/2}} \right) \geq \frac{2}{5} \prod_{\ell} \left( 1 - \frac{1}{\ell^{3/2}} \right) \]

because \( p \geq 5 \). Elementary estimates now yield \( c(p, n) > 1/7 \).

\[ \square \]

**Proof of Proposition 2.** By Lemma 25 and since \( 7\sqrt{3} < 13 \) we may fix \( \epsilon > 0 \) such that

\[ N(y) \geq \sqrt{y/13} + O(y^{5/12}). \]

So there is an integer \( x \) such that \( 3x^2 + 4p^{2n+1} \) is square-free and at most \( c'p^{2n+1} \) where \( c' > 0 \) is absolute. The argument concludes as in the final paragraph of Section A.2.

\[ \square \]

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