ON THE SPECTRAL WINDOW OF THOMSON’S ESTIMATOR

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Abstract. Thomson’s multi-window estimator is one of the most widely used techniques in the estimation of the spectrum of a time series with bandwidth $W$ from $N$ observations in the time domain. The success of the method is due to variance reduction and low bias (low spectral leakage resulting from convolving with the spectral window). While the reduction of variance is achieved by an averaging process, the control on the bias is based on the fact that the average of the square of the first $K = \lfloor 2NW \rfloor$ discrete prolate wave functions becomes closer, as $K$ grows, to the ideal averaging kernel $\frac{1}{2W}[−W,W]$, given by the normalized characteristic function of the target bandwidth region. In this technical report, we derive an analytic estimate supporting and quantifying this fact: we bound the $L^1$ distance between the spectral window and the ideal averaging kernel.

1. Introduction

Let $I = [−1/2,1/2]$. Any stationary, real, ergodic, zero-mean, Gaussian stochastic process has a Cramér spectral representation

$$x(t) = \int_I e^{2\pi i \xi t} dZ(\xi).$$

The spectrum $S(\xi)$, sometimes called the power spectral density of the process, is defined by

$$S(\xi) d\xi = \mathbb{E}\{|dZ(\xi)|^2\}.$$ 

The purpose of spectral estimation is solving the highly undetermined problem of estimating $S(\xi)$ from a sample of $N$ contiguous observations $x(0), ..., x(N−1)$. This can be done by forming a direct spectrum estimate by averaging the periodogram with a data window $\{D(t)\}_{t=0}^{N-1}$, usually called a taper:

$$\hat{S}_D(\xi) = \left| \sum_{t=0}^{N-1} x(t) D(t) e^{-2\pi i \xi t} \right|^2.$$

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In the frequency domain, the properties of the taper are deduced from its Fourier transform $G_D(\xi)$ which is called the spectral window:

$$G_D(\xi) = \sum_{t=0}^{N-1} D(t)e^{-2\pi i \xi t}.$$ 

The choice of the taper $\{D(t)\}_{t=0}^{N-1}$ can have a significant effect on the resulting spectrum estimate. This is apparent by observing that its expectation is the convolution of the true spectrum $S(\xi)$ with the spectral window $G_D(\xi)$

$$\mathbb{E}\left[\hat{S}_D(\xi)\right] = S(\xi) * G_D(\xi).$$

Thus, $\{D(t)\}_{t=0}^{N-1}$ has a smoothing effect on the true spectrum. In [2], Thomson introduced a unified algorithm, where the number $N$ of observations enters explicitly in the methods and performance bounds. It uses the solutions of the integral equation

$$\int_{-W}^{W} \frac{\sin N\pi(\xi - \xi')}{\sin \pi(\xi - \xi')} U_k(N, W; \xi')d\xi' = \lambda_k(N, W)U_k(N, W; \xi),$$

where $0 < W < \frac{1}{2}$. The functions $U_j(N, W, \xi)$ are called discrete prolate spheroidal functions. The discrete prolate spheroidal sequences $v^{(k)}(N, W)$ are defined as

$$v_t^{(k)}(N, W) = \frac{1}{i^k\lambda_k(N, W)} \int_{-W}^{W} U_k(N, W; \xi)e^{-2\pi i [t-(N-1)/2]}d\xi.$$ 

The core of Thomson’s method consists in using a number $K = \lfloor 2NW \rfloor$ of discrete prolate spheroidal sequences $\{v_t^{(k)}(N, W)\}_{t=0}^{N-1}$ as the tapers in (1.1) and then averaging the resulting estimates. Each estimate is thus defined as

$$\hat{S}_k(\xi) = \sum_{t=0}^{N-1} x(t)v_t^{(k)}(N, W)e^{-2\pi i \xi t}.$$

The expected value of such an estimator is given by

$$\mathbb{E}\{\hat{S}_k(\xi)\} = |U_k(N, W; \xi)|^2 * S(\xi),$$

where $S(\xi)$ is the unobservable spectrum. Thus, $\mathbb{E}\{\hat{S}_k(\xi)\}$ is a smoothing of the unobservable spectrum and $|U_k(N, W; \xi)|^2$ replaces the spectral window in (1.2). While this provides a good estimate for $k = 0$, because of the optimal concentration of the first prolate function on the interval $[-W, W]$, the bias of the estimate increases with $k$, because the amount of energy of $U_k(N, W; \xi)$ on $[-W, W]$ decreases with $k$. Thomson suggested using an average

$$\hat{S}_K(\xi) = \frac{1}{K} \sum_{k=0}^{K-1} \hat{S}_k(\xi).$$
of the best $K = \lfloor 2NW \rfloor$ estimates. The averaging process has the obvious advantage of reducing the variance in the spectral estimate. But it has another less obvious advantage: the bias of the averaged estimator (spectral leakage),

$$\text{Bias} (N, K) = \mathbb{E} \{ \hat{S}_K (\xi_0) \} - S(\xi_0)$$

defined as the difference between the expectation of the estimator and the true nonobservable spectrum, decreases with the bandwidth product $2NW$. From equation (1.4), defining the single estimates associated with $k$, it is clear that the bias efficiency of the estimator is better when the spectral window is, in some sense, close to an ideal averaging kernel $\frac{1}{2W} 1_{[-W,W]}$. This scenario becomes less accurate if we increase $k$. It is a remarkable fact that, however, if we take the average (1.5), the spectral window of the estimator becomes again similar to an averaging kernel localized on $[-W,W]$. The reasons behind this behavior are intriguing.

Define

$$\rho_K(N, W, \xi) = \sum_{j=0}^{K-1} |U_j(N, W, \xi)|^2$$

and observe that the expected value of the estimator (1.5) is

$$\mathbb{E} \{ \hat{S}_K (\xi) \} = \frac{1}{K} \rho_K(N, W, \xi) \ast S(\xi).$$

The fact that the spectral window in (1.8) approaches a flat function concentrated on $[-W,W]$ means that the functions in the sequence $\{|U_j(N, W, \cdot)|^2 : k = 0, \ldots K - 1\}$ are organized inside the interval $[-W,W]$ in such a way that each function tends to fill in the empty energy spots left by the sum of the previous ones, much in analogy to the Pythagorean relation for pure frequencies: $\sin^2(t) + \cos^2(t) = 1$. More precisely, the claim that the spectral window in Thomson’s method approximates an ideal averaging kernel means that the two functions

$$\frac{1}{K} \rho_K(N, W, \cdot) \quad \text{and} \quad \frac{1}{2W} 1_{[-W,W]},$$

become more and more similar as $K$ increases. In this note we provide an analytic bound supporting and quantifying this fact. The main result is Theorem (1) below that provides the following estimate for the $L^1$-distance between the two functions in (1.9):

$$\left\| \frac{1}{K} \rho_K(N, W, \cdot) - \frac{1}{2W} 1_{[-W,W]} \right\|_{L^1(I)} \lesssim \sqrt{\frac{\log N}{K}}, \quad N \geq 2.$$
2. Proof of the main result

Let \( I := [-1/2, 1/2] \) and let us denote the exponentials by \( e_\omega(x) := e^{2\pi i x \omega} \). By a slight abuse of notation we use \( x, y, z \) to denote variables in \( I \), although later we will use \( \xi \) to the same end. We will always let \( N \geq 2 \) be an integer and \( W \in (-1/2, 1/2) \), while \( K := \lfloor 2NW \rfloor \) denotes the smallest integer not greater than \( 2NW \). For two non-negative functions \( f, g \), the notation \( f \lesssim g \) means that there exists a constant \( C > 0 \) such that \( f \leq Cg \). (The constant \( C \), of course, does not depend on the parameters \( N, W \).)

2.1. Trigonometric polynomials. For notational convenience, we use a temporal normalization that is slightly different to the one in the Introduction (this has no impact on the announced estimates). We consider the space of trigonometric polynomials

\[
P_N = \text{Span} \left\{ e_{\frac{-N-1}{2} + j} : 0 \leq j \leq N - 1 \right\} \subseteq L^2(I)
\]

and the associated kernel:

\[
K^N(x, y) = \frac{\sin(\pi N(x - y))}{\sin(\pi(x - y))}, \quad x, y \in I, \quad N \in \mathbb{N}.
\]

Let us also define

\[
\Phi^N(x) = \left| \frac{\sin(\pi Nx)}{\sin(\pi x)} \right|^2,
\]

so that

\[
|K^N(x, y)|^2 = \Phi^N(x - y),
\]

for all \( x, y \in I \). Note that \( \int_I \Phi^N = N \).

2.2. Toeplitz operators. Given a closed subspace \( S \subseteq L^2(I) \), \( P_S \) stands for the orthogonal projection onto \( S \). For \( W \in (-1/2, 1/2) \) the Toeplitz operator \( H_N^W \) is

\[
H_N^W f := P_{P_N} \left( (P_{P_N} f) \cdot 1_{[-W,W]} \right), \quad f \in L^2(I).
\]

When \( f \in \mathcal{P}_N \), \( H_N^W f \) is simply the projection of \( f \cdot 1_{[-W,W]} \) into \( \mathcal{P}_N \). \( H_N^W \) can be explicitly described by the formula

\[
H_N^W f(x) = \int_I f(y) K_N^W(x, y) dy,
\]

where the kernel \( K_N^W(x, y) \) is

\[
K_N^W(x, y) = \int_{[-W,W]} K^N(x, z) \overline{K^N(y, z)} dz.
\]

(The conjugation bars are not necessary since \( K^N \) is real-valued, but we prefer keeping them for symmetry reasons.)
The operator $H_N^W$ can be diagonalized as:

$$H_N^W f = \sum_{j=0}^{N-1} \lambda_j(N, W) \langle f, U_j(N, W) \rangle U_j(N, W),$$

where $\lambda_0 \geq \ldots \geq \lambda_{N-1}$, and $U_j(N, W)$ are the discrete prolate spheroidal functions - cf. (1.3) - normalized by $\|U_j(N, W)\|_{L^2(I)} = 1$. The function $\rho_K(N, W, \xi)$ is defined by (1.7).

The diagonalization in (2.3) means that the integral kernel in (2.2) can be written as

$$K_N^W(x, y) = \sum_{j=0}^{N-1} \lambda_j(N, W) U_j(N, W)(x) \overline{U_j(N, W)(y)}, \quad x, y \in I.$$

2.3. Some preliminaries. We now collect some results whose proof is postponed to the Appendix.

Lemma 1. Let $f : I \to \mathbb{C}$ an integrable function, of bounded variation, and supported on $I^o = (-1/2, 1/2)$. For $N \geq 2$, let

$$f * \Phi^N(x) = \int_I f(y) \Phi^N(x - y) \, dy, \quad x \in I.$$

Then

$$\left\| f - \frac{1}{N} f * \Phi^N \right\|_{L^1(I)} \lesssim \text{Var}(f, I) \frac{\log N}{N}.$$  

Remark 1. In the above estimate, \text{Var}(f, I) denotes the total variation of $f$ on $I$. If $f = 1_{[-W, W]}$, with $W \in (-1/2, 1/2)$, then $\text{Var}(f, I) = 2$ and the estimate reads

$$\left\| 1_{[-W, W]} - \frac{1}{N} 1_{[-W, W]} * \Phi^N \right\|_{L^1(I)} \lesssim \frac{\log N}{N}.$$

We will also need a description of the profile of the eigenvalues of $H_N^W$.

Proposition 1. For each $\delta \in (0, 1)$,

$$|\# \{k : \lambda_k(N, W) > 1 - \delta\} - 2NW| \lesssim \max \left\{ \frac{1}{\delta}, \frac{1}{1 - \delta} \right\} \log N.$$

The estimate in Proposition 2 can be substantially refined, but the present form will be sufficient for the purpose of this note. Indeed, from the profile of the eigenvalues we can obtain our first estimate on $\rho_K(N, W, \xi)$.

Proposition 2. For $N \geq 2$ and $K \geq \log N$:

$$\|\rho_K(N, W, \cdot) - 1_{[-W, W]} * \Phi^N\|_{L^1(I)} \lesssim \sqrt{2NW} \sqrt{\log N}.$$
Proof. Set

\[ E(N, W) = 1 - \frac{1}{2NW} \sum_{j=0}^{K-1} \lambda_j(N, W). \]

A calculation using the eigenvalue estimates from Proposition 1 leads to the bound

(2.6) \[ E(N, W) \lesssim \sqrt{\frac{\log N}{NW}}, \]

see details in the Appendix. Comparing (2.2) and (2.4) we see that

\[ \int_{[-W,W]} K_N(x, z)K_N(y, z)dz = \sum_{j=0}^{N-1} \lambda_j(N, W)U_j(N, W)(x)U_j(N, W)(y), \quad x, y \in I. \]

In particular, taking \( x = y = \xi \in I \) yields,

\[ (1_{[-W,W]} \ast \Phi^N)(\xi) = \sum_{j=0}^{N-1} \lambda_j(N, W) |U_j(N, W, \xi)|^2, \]

Let us set \( L_j = 1 \) for \( 0 \leq j \leq K - 1 \) and \( L_j = 0 \) for \( K \leq j \leq N - 1 \). With this notation,

\[ \rho_K(N, W, \xi) - (1_{[-W,W]} \ast \Phi^N)(\xi) = \sum_{j=0}^{N-1} (L_j - \lambda_j(N, W)) |U_j(N, W, \xi)|^2. \]

Therefore we can estimate,

\[
\left\| \rho_K(N, W, \cdot) - 1_{[-W,W]} \ast \Phi^N \right\|_{L^1(I)} \\
\leq \sum_{j=0}^{N-1} |L_j - \lambda_j(N, W)| = \sum_{j=0}^{K-1} (1 - \lambda_j(N, W)) + \sum_{j=K}^{N-1} \lambda_j(N, W) \\
= K - 2 \sum_{j=0}^{K-1} \lambda_j(N, W) + \text{trace}(H^N_W) \\
= K - 2 \sum_{j=0}^{K-1} \lambda_j(N, W) + 2NW = (K - 2NW) - 2 \sum_{j=0}^{K-1} \lambda_j(N, W) + 4NW \\
\leq 2 \left[ 2NW - \sum_{j=0}^{K-1} \lambda_j(N, W) \right] = 4NW \left[ 1 - \frac{1}{2NW} \sum_{j=0}^{K-1} \lambda_j(N, W) \right] \\
= 4NW \cdot E(N, W) \lesssim \sqrt{2NW} \sqrt{\log N},
\]

where in the last step we used (2.6). \[ \square \]

Finally we derive our main result.

**Theorem 1.** Let \( N \geq 2 \) be an integer, \( W \in (-1/2, 1/2) \) and set \( K := \lfloor 2NW \rfloor \). Then

(2.7) \[ \left\| \frac{1}{K} \rho_K(N, W, \cdot) - \frac{1}{2W} 1_{[-W,W]} \right\|_{L^1(I)} \lesssim \sqrt{\frac{\log N}{K}}. \]
Proof. First note that

\[ \left\| \frac{1}{K} \rho_K(N, W, \cdot) \right\|_{L^1(I)} = \left\| \frac{1}{2W} 1_{[-W,W]} \right\|_{L^1(I)} = 1, \]

so the bound in (2.7) is trivial if \( K \leq \log N \). Hence we assume that \( K \geq \log N \) and estimate:

\[
\left\| \frac{1}{K} \rho_K(N, W, \cdot) - \frac{1}{2W} 1_{[-W,W]} \right\|_{L^1(I)} \leq \left\| \frac{1}{K} \rho_K(N, W, \cdot) - \frac{1}{K} 1_{[-W,W]} * \Phi^N \right\|_{L^1(I)} + \left\| \frac{1}{K} 1_{[-W,W]} * \Phi^N - \frac{N}{K} 1_{[-W,W]} \right\|_{L^1(I)} = A + B + C.
\]

\( A \) can be bounded immediately by using Proposition \([2]\) (and noting that \( K \approx 2NW \)). In order to bound \( B \) we use Lemma \([1]\) with \( f = 1_{[-W,W]} \) to obtain

\[
\left\| \frac{1}{K} 1_{[-W,W]} * \Phi^N - \frac{N}{K} 1_{[-W,W]} \right\|_{L^1(I)} = \frac{N}{K} \left\| 1_{[-W,W]} * \Phi^N - 1_{[-W,W]} \right\|_{L^1(I)} \lesssim \frac{N \log N}{K} = \frac{\log N}{K}.
\]

Since \( K \geq \log N, \frac{\log N}{K} \leq \sqrt{\frac{\log N}{K}} \), and the bound on \( B \) follows. Finally, we bound \( C \) as

\[
\left\| \frac{N}{K} 1_{[-W,W]} - \frac{1}{2W} 1_{[-W,W]} \right\|_{L^1(I)} = \left| \frac{2WN}{K} - 1 \right| = \left| \frac{2WN - K}{K} \right| \leq \frac{1}{K}.
\]

Since \( N \geq 2, \frac{1}{K} \lesssim \frac{\log N}{K} \leq \sqrt{\frac{\log N}{K}} \). This completes the proof.

\[ \square \]

3. Appendix

3.1. Proof of Lemma \([1]\). By an approximation argument, we assume without loss of generality that \( f \) is smooth (cf. \([1]\) Lemma 3.2]). We also extend \( f \) periodically to \( \mathbb{R} \). Note that this extension is still smooth because \( f|I \) is supported on \( I^0 \).

Step 1. We note the following estimate:

\[ (3.1) \quad \| f(\cdot + h) - f \|_{L^1(I)} \lesssim \text{Var}(f, I) |\sin(\pi h)|, \quad h \in \mathbb{R}. \]

Since \( f(x + h) - f(x) = \int_0^1 f'(th + x) h \, dt \), we can estimate

\[
\| f(\cdot + h) - f \|_{L^1(I)} \leq \int_0^1 \int_{-1/2}^{1/2} |f'(th + x)| \, dx \, |h| \, dt = \int_0^1 \int_{-1/2+th}^{1/2+th} |f'(x)| \, dx \, |h| \, dt = \int_0^1 \int_{-1/2}^{1/2} |f'(x)| \, dx \, |h| \, dt = \text{Var}(f, I) |h|,
\]
where the last step uses the periodicity of $f$. Given $h \in \mathbb{R}$, there exists $k \in \mathbb{Z}$ such that $|h - k| \leq 1/2$. Using again the periodicity of $f$,
\[
\|f(\cdot + h) - f\|_{L^1(I)} = \|f(\cdot + h - k) - f\|_{L^1(I)} \leq \text{Var}(f) |h - k|.
\]
Finally, since $|h - k| \leq 1/2$, it follows that $|h - k| \lesssim |\sin(\pi(h - k))| = |\sin(\pi h)|$.

**Step 2.** With the notation $f^N := f * \frac{1}{N} \Phi^N$,
\[
\begin{align*}
 f(x) - f^N(x) &= \frac{1}{N} \int_{-1/2}^{1/2} (f(x) - f(y)) \Phi^N(x - y)dy \\
 &= \frac{1}{N} \int_{-1/2}^{1/2} (f(x) - f(y + x)) \Phi^N(-y)dy \\
 &= \frac{1}{N} \int_{-1/2}^{1/2} (f(x) - f(y + x)) \Phi^N(y)dy,
\end{align*}
\]
where the last equality follows by periodicity. Using (3.1) we derive the bound:
\[
\|f - f^N\|_{L^1(I)} \leq \frac{1}{N} \int_{-1/2}^{1/2} \|f - f(\cdot + y)\|_{L^1(I)} \Phi^N(-y)dy \\
\lesssim \text{Var}(f, I) \frac{1}{N} \int_{-1/2}^{1/2} |\sin(\pi h)| \Phi^N(y)dy
\]

**Step 3.** We show that
\[
\int_{-1/2}^{1/2} |\sin(\pi x)| \Phi^N(x) \ dx \lesssim \log(N).
\]
We estimate
\[
\int_{-1/2}^{1/2} |\sin(\pi x)| \Phi^N(x) \ dx = \int_{-1/2}^{1/2} \frac{|\sin(\pi x)|^2}{|\sin(\pi x)|} \ dx \\
\approx \int_0^{1/2} \frac{|\sin(\pi x)|^2}{|\sin(\pi x)|} \ dx \lesssim \int_0^{1/2} \frac{|\sin(\pi x)|^2}{|x|} \ dx \\
= \int_0^{N/2} \frac{|\sin(\pi x)|^2}{|x|} \ dx \leq C + \int_1^{N/2} \frac{1}{|x|} \ dx \lesssim \log N,
\]
for $N \geq 2$.

### 3.2. Proof of Proposition

The proof uses the following classic eigenvalue estimate for Toeplitz operators:

\[\#\{k : \lambda_k(N, W) > 1 - \delta\} - \text{trace}(H^N_W) \leq \max \left\{ \frac{1}{\delta}, \frac{1}{1 - \delta} \right\} \text{trace} \left[ (H^N_W) - (H^N_W)^2 \right].\]

(See for example [1] Lemma 3.3.) To use (3.2) we first note that
\[\text{trace} (H^N_W) = \int_I K^N_W(x, x)dx = \int_{[-W,W]} \int_I \Phi^N(x - y)dydx = 2NW,\]
because \( \int \Phi^N = N \), cf. (2.1). Moreover a similar calculation gives

\[
\text{trace} \left( H^N_W \right)^2 = \int_{[-W,W]} \int_I 1_{[-W,W]}(y) \Phi^N(x-y) dy dx.
\]

(See for example [1] Lemma 2.1 for a detailed argument in a slightly different context.) Hence we have the following formula:

\[
(3.4) \quad \text{trace} \left[ (H^N_W) - (H^N_W)^2 \right] = \int_I \left[ N 1_{[-W,W]}(x) - \left( 1_{[-W,W]}(y) * \Phi^N \right)(x) \right] dx.
\]

Using (3.2), together with (3.3), (3.4), and Lemma 1 we obtain

\[
\left| \#\{ k : \lambda_k(N,W) > 1 - \delta \} - 2NW \right| 
\leq \left\{ \frac{1}{\delta}, \frac{1}{1-\delta} \right\} \int_{[-W,W]} \left[ N 1_{[-W,W]}(x) - \left( 1_{[-W,W]}(y) * \Phi^N \right)(x) \right] dx 
\leq \max \left\{ \frac{1}{\delta}, \frac{1}{1-\delta} \right\} \left\| N 1_{[-W,W]} - N 1_{[-W,W]} * \frac{\Phi^N}{N} \right\|_{L^1(I)} 
\lesssim \max \left\{ \frac{1}{\delta}, \frac{1}{1-\delta} \right\} \log N.
\]

3.3. Proof of (2.6). Let \( \delta \in (0,1) \) and let us set \( n_\delta := \#\{ k : \lambda_k(N,W) > 1 - \delta \} \) and \( C_\delta := \max \left\{ \frac{1}{\delta}, \frac{1}{1-\delta} \right\} \). Let us also consider the number \( l_\delta := \min\{K, n_\delta\} \).

According to Proposition 1 there is a constant \( C > 0 \) such that

\[
n_\delta \geq 2NW - CC_\delta \log N.
\]

In addition, since \( N \geq 2 \), \( K = \lfloor 2NW \rfloor \geq 2NW - 1 \geq 2NW - CC_\delta \log N \), with a possibly smaller constant \( C \). Hence,

\[
(3.5) \quad l_\delta \geq 2NW - CC_\delta \log N.
\]

Since \( l_\delta \leq L \),

\[
\sum_{j=0}^{K-1} \lambda_j(N,W) \geq \sum_{j=0}^{l_\delta-1} \lambda_j(N,W) \geq (1-\delta)l_\delta.
\]

Therefore, using (3.5),

\[
E(N,W) \leq 1 - (1-\delta)l_\delta 2NW \leq 1 - (1-\delta) \left( 1 - CC_\delta \frac{\log N}{2NW} \right) 
\leq \delta + \frac{1}{\delta} C \frac{\log N}{2NW}.
\]

Since \( 2NW \geq \lfloor 2NW \rfloor = K \geq \log N \), we can choose \( \delta := \sqrt{\frac{\log N}{2NW}} \), giving \( E(N,W) \lesssim \sqrt{\frac{\log N}{2NW}} \), as desired.


 References

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