Thermodynamic curvature and isoperimetric inequality for the charged BTZ black hole

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In this paper, we investigate the thermodynamic curvature of the charged Banados-Teitelboim-Zanelli (BTZ) black hole and the relationship between thermodynamic curvature and reverse isoperimetric inequality. There are two schemes for analyzing the thermodynamic behavior of the charged BTZ black hole. In one scheme, the charged BTZ black hole is super-entropic, while in the other, it is not (the reverse isoperimetric inequality is saturated). In both schemes, the thermodynamic curvature is always positive, which may be related to the information of repulsive interaction between black hole molecules for the charged BTZ black hole. Furthermore, we find that for different phase spaces, the obtained thermodynamic curvatures in both schemes are equivalent respectively, which provides a clue to establish the coordinate-free definition of the thermodynamic geometry. Meanwhile, we give a conjecture that when the reverse isoperimetric inequality is saturated, the thermodynamic curvature of extreme black hole tends to be infinity and for super-entropic black holes, the thermodynamic curvature of extreme black hole will go to a finite value.

I. INTRODUCTION

At present, black hole physics is generally considered as one of the best and effective ways to explore quantum gravity. Especially with the pioneering discovery of Hawking and Bekenstein about the temperature and entropy in black holes [1–4], the general relativity, quantum mechanics and statistical physics are closely linked together to make it possible for us to glimpse the tip of the iceberg of quantum gravity. Thermodynamics theory, which has been rested on general principles spanning a wide range of pure fluids physical systems, is now applied to black hole systems extensively

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and successfully [5]. One of the most prominent is the introduction of extended phase space [6–8], which makes the charged AdS black hole and the van der Waals fluid have a close relationship [9, 10]. Black holes exhibit abundant phase transitions and critical behaviors in such a extended phase space [11–13].

Recently, the theory of thermodynamics geometry [14–16] is widely applied to the thermodynamics system of black hole, which provides a new and more traceable perspective for studying the micro-mechanism of black holes from the axioms of thermodynamics phenomenologically. This new scheme is mainly to use the Hessian matrix structure to represent the thermodynamic fluctuation theory [14]. Thermodynamic curvature is the most important physical quantity in the theory. Taking advantage of the new scheme, the primary microscopic information of the Banados-Teitelboim-Zanelli (BTZ) black hole, Schwarzschild (-AdS) black hole, Reissner-Nordström (-AdS) black hole, Gauss-Bonnet (-AdS) black hole, higher dimensional black holes and other black holes are explored [17–41].

In this paper, we want to explore the thermodynamic curvature of (2 + 1)-dimensional charged BTZ black hole in different phase spaces and study the relation between thermodynamic curvature of black hole and reverse isoperimetric inequality. The mainly motivations are: i) in the early researches [30, 42–46] on the thermodynamic geometry of BTZ black holes, the cosmological constant was regarded as a fixed parameter or fluctuation parameter. Such a system has no pressure and volume terms, that is, no extended phase space. Now the notion of extended phase space has been widely recognized and a good thermodynamic system must have pressure and volume terms. Therefore, it is necessary to discuss the thermodynamic geometry of BTZ black hole again in the extended phase space. ii) the lower-dimensional theories of gravity have gained renewed interest since on the one hand the physics of quantum black holes may be effectively lower-dimensional and on the other hand the lower-dimensional gravity can provide a guidance to the understanding of the higher dimensional counterparts. Therefore the investigation of the BTZ black hole is interesting and necessary in its own right. iii) due to the asymptotic structure of the black hole solution, the computation of the mass of charged BTZ black hole is quite problematic. Ref. [47] provides two schemes for analyzing the thermodynamic behavior of the charged BTZ black hole. In one case, the charged BTZ black hole is super-entropic [48, 49], while in the another, it is not. Through the calculation of the thermodynamic curvature, we provide a diagnosis for the discrimination of the two schemes. Meanwhile we give a conjecture about the relationship between thermodynamic curvature and reverse isoperimetric inequality of black holes. iiiii) the analytical calculation of the thermodynamic curvature in the different phase space of the (2 + 1)-dimensional black hole provides an attempt to establish the coordinate-free definition of the thermodynamic geometry.

Note: when organizing this work, we notice a similar study [50], where the thermodynamic curvature of the charged rotating BTZ black hole in \{S, P\} phase space and the behavior of the thermodynamic curvature with respect to entropy and volume are investigated. In this paper, we
discuss the thermodynamic curvature of charged BTZ black hole in different phase spaces under two different thermodynamic schemes, and give an empirical conjecture between the thermodynamic curvature and isoperimetric inequality of black holes.

II. RUPPEINER THERMODYNAMIC GEOMETRY

Completely from a thermodynamic point of view, now the Ruppeiner thermodynamic geometry is dealt with as a new attempt to extract the microscopic interaction information from the axioms of thermodynamics phenomenologically or qualitatively. Its line element can be written as in terms of the Hessian matrix about the black hole entropy [14]
\[
\Delta l^2 = -\frac{\partial^2 S}{\partial X^\mu \partial X^\nu} \Delta X^\mu \Delta X^\nu,
\]
where \( X^\mu \) represents some independent thermodynamic quantities. The line element \( \Delta l^2 \) measures the distance between two neighbouring fluctuation states in the state space. For black holes with AdS background, the most basic thermodynamic differential relation is \( dM = T dS + V dP + \) others works, where enthalpy \( M \), Hawking temperature \( T \), entropy \( S \), thermodynamic pressure \( P \), thermodynamic volume \( V \), and others works like as charge \( Q \) and electrostatic potential \( \Phi \), or angular momentum \( J \) and angular velocity \( \Omega \), or some coupling constants and their conjugations. Now we consider the situation with fixed other work terms, so there are four situations.

(i) When the phase space is \( \{ S, P \} \), the line element takes the form [34]
\[
\Delta l^2 = \frac{1}{T} \left( \frac{\partial T}{\partial S} \right)_P \Delta S^2 + \frac{2}{T} \left( \frac{\partial T}{\partial P} \right)_S \Delta S \Delta P + \frac{1}{T} \left( \frac{\partial V}{\partial P} \right)_S \Delta P^2.
\]

(ii) When the phase space is \( \{ T, V \} \), the line element reads as [34]
\[
\Delta l^2 = \frac{1}{T} \left( \frac{\partial S}{\partial T} \right)_V \Delta T^2 + \frac{2}{T} \left( \frac{\partial S}{\partial V} \right)_T \Delta T \Delta V + \frac{1}{T} \left( \frac{\partial P}{\partial V} \right)_T \Delta V^2.
\]

(iii) When the phase space is \( \{ S, V \} \), the line element becomes
\[
\Delta l^2 = \frac{1}{T} \left( \frac{\partial T}{\partial S} \right)_V \Delta S^2 + \frac{1}{T} \left( \frac{\partial P}{\partial V} \right)_S \Delta V^2.
\]

(iv) When the phase space is \( \{ T, P \} \), the line element changes into
\[
\Delta l^2 = \frac{1}{T} \left( \frac{\partial S}{\partial T} \right)_P \Delta T^2 + \frac{1}{T} \left( \frac{\partial V}{\partial P} \right)_T \Delta P^2.
\]

Next we will use the above four different phase spaces to calculate the thermodynamic curvature of the charged BTZ black hole respectively, so as to explore some micro-information that the black hole may have.
III. THERMODYNAMIC PROPERTIES OF CHARGED BTZ BLACK HOLE

For the (2 + 1)-dimensional charged BTZ black hole, its metric and the gauge field are \[47\,\text{[51]}\]

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\varphi^2,
\]

\[
F = dA, \quad A = -Q \ln \left(\frac{r}{l}\right) dt,
\]

where \(f(r)\) is

\[
f(r) = -2m - \frac{Q^2}{2} \ln \left(\frac{r}{l}\right) + \frac{r^2}{l^2}.
\]

The function \(f(r)\) is

\[
\text{here the function } f(r) \text{ is}
\]

where \(m\) is related to black hole mass, \(l\) is the AdS radius which is connected with the negative cosmological constant \(\Lambda\) via \(\Lambda = -1/l^2\) and \(Q\) is total charge of black hole. About the basic thermodynamic properties in terms of the event horizon radius \(r_h\) which is determined by the largest root of the equation \(f(r_h) = 0\), there are generally two different forms of these quantities.

A. Case I

By taking advantage of the Komar formula to determine the mass of the black hole, authors of Ref. \[47\] write the first law of thermodynamics of the charged BTZ black hole as \(dM = TdS + VdP + \Phi dQ\), where the relevant quantities are

\[
M = \frac{m}{4} = \frac{r_h^2}{8l^2} - \frac{Q^2}{16} \ln \left(\frac{r_h}{l}\right),
\]

\[
T = \frac{r_h}{2\pi l^2} - \frac{Q^2}{8\pi r_h},
\]

\[
S = \frac{1}{2}\pi r_h,
\]

\[
P = -\frac{\Lambda}{8\pi} = \frac{1}{8\pi l^2},
\]

\[
V = \pi r_h^2 - \frac{1}{4}\pi Q^2 l^2,
\]

\[
\Phi = -\frac{1}{8}Q \ln \left(\frac{r_h}{l}\right).
\]

B. Case II

An alternative scheme is renormalization procedure by enclosing the system in a circle of radius \(r_0\) and taking the limit \(r_0 \to \infty\) whilst keeping the ratio \(r/r_0 = 1 \text{[52]}\). Then the black hole mass is interpreted as the total energy inside the circle of radius \(r_0\). Based on this fact, Ref. \[47\] introduces a new thermodynamic parameter associated with the renormalization length scale \(R = r_0\) via writing

\[
f(r) = -2m_0 - \frac{Q^2}{2} \ln \left(\frac{r}{R}\right) + \frac{r^2}{l^2}.
\]
The first law of thermodynamics of the charged BTZ black hole becomes \( d\tilde{M} = TdS + \tilde{V}dP + \tilde{\Phi}dQ + KdR \), where the relevant quantities are

\[
\tilde{M} = \frac{m_0}{4} = \frac{r_h^2}{8l^2} - \frac{Q^2}{16} \ln \left( \frac{r_h}{R} \right),
\]

\[
\tilde{V} = \pi r_h^2,
\]

\[
\tilde{\Phi} = -\frac{1}{8}Q \ln \left( \frac{r_h}{R} \right),
\]

\[
K = \frac{Q^2}{16R},
\]

and the other thermodynamic quantities \( T, S, P \) are still expressions (9), (10) and (11).

IV. THERMODYNAMIC CURVATURE OF CHARGED BTZ BLACK HOLE

A. Thermodynamic curvature for Case I

In principle, there should be four ways, i.e., in phase spaces \( \{S, P\}, \{S, V\}, \{T, V\} \) and \( \{T, P\} \). For the phase space \( \{T, V\} \), we need to write the entropy \( S \) and thermodynamic pressure \( P \) as a function of temperature \( T \) and thermodynamic volume \( V \), respectively. The analytical forms are very complex. For the sake of simplicity, we will not consider this case.

- In the phase space \( \{S, P\} \), we need to write the temperature \( T \) and thermodynamic volume \( V \) as functions of entropy \( S \) and pressure \( P \), respectively

\[
T = \frac{128PS^2 - \pi Q^2}{16\pi S}, \quad V = \frac{128PS^2 - \pi Q^2}{32\pi P}.
\]

Hence according to Eq. (2), we can directly calculate the expression of thermodynamic curvature

\[
R_{SP} = \frac{384\pi Q^2PS}{(\pi Q^2 + 256PS^2)^2}.
\]

- In the phase space \( \{S, V\} \), the temperature \( T \) and thermodynamic pressure \( P \) can be written as functions of entropy \( S \) and volume \( V \), respectively

\[
T = \frac{\pi Q^2V}{16S(4S^2 - \pi V)}, \quad P = \frac{\pi Q^2}{128S^2 - 32\pi V}.
\]

Hence according to Eq. (4), we can obtain the expression of thermodynamic curvature

\[
R_{SV} = \frac{12S(4S^2 - \pi V)}{(\pi V - 12S^2)^2}.
\]
In the phase space \( \{ T, P \} \), we must have the expressions of entropy \( S \) and thermodynamic volume \( V \) in terms of temperature \( T \) and pressure \( P \), respectively

\[
S = \frac{\pi T + \sqrt{\pi^2 T^2 + 2\pi Q^2 P}}{16 P}, \quad V = \frac{T}{32 P^2} \left( \pi T + \sqrt{\pi^2 T^2 + 2\pi Q^2 P} \right).
\]  

(22)

Hence according to Eq. (5), we can obtain the expression of thermodynamic curvature

\[
R_{TP} = \frac{24P \left[ 8\pi T^2 \left(-\sqrt{\pi T} + \sqrt{2Q^2 P + \pi T^2} \right) + 3Q^2 P \left(-5\sqrt{\pi T} + 3\sqrt{2Q^2 P + \pi T^2} \right) \right]}{\sqrt{\pi} \left(9Q^2 P + 4\pi T^2 \right)^2}.
\]  

(23)

Meanwhile we can easily test the identity

\[
R_{SP} = R_{SV} = R_{TP} > 0.
\]  

(24)

Furthermore, for the extreme black hole \( T = 0 \), we can observe clearly that thermodynamic curvature is finite positive value. Based on an empirical conclusion under the framework of thermodynamic geometry theory, i.e., the negative (positive) thermodynamic curvature is associated with attractive (repulsive) microscopic interactions for a thermodynamic system [17, 19–26, 34, 35], we can speculate that the charged BTZ black hole is likely to present a repulsive between its molecules phenomenologically or qualitatively.

**B. Thermodynamic curvature for Case II**

In principle, there should also be four ways, i.e., in phase spaces \( \{ S, P \} \), \( \{ T, \tilde{V} \} \), \( \{ S, \tilde{V} \} \) and \( \{ T, P \} \). Because entropy \( S \) and thermodynamic volume \( \tilde{V} \) are not independent of each other, the phase space \( \{ S, \tilde{V} \} \) is invalid.

- In the phase space \( \{ S, P \} \), we need to write the temperature \( T \) and thermodynamic volume \( \tilde{V} \) as functions of entropy \( S \) and pressure \( P \), respectively

\[
T = \frac{128PS^2 - \pi Q^2}{16\pi S}, \quad \tilde{V} = \frac{4S^2}{\pi}.
\]  

(25)

Hence according to Eq. (2), we can directly calculate the expression of thermodynamic curvature

\[
\dot{R}_{SP} = \frac{2\pi Q^2}{S(128PS^2 - \pi Q^2)}.
\]  

(26)

- In the phase space \( \{ T, \tilde{V} \} \), the entropy \( S \) and thermodynamic pressure \( P \) can be written as functions of temperature \( T \) and volume \( \tilde{V} \), respectively

\[
S = \left( \frac{\pi \tilde{V}}{4} \right)^{1/2}, \quad P = \frac{\sqrt{\pi T}}{4\sqrt{\tilde{V}}} + \frac{Q^2}{32\tilde{V}}.
\]  

(27)
Hence according to Eq. (3), we can obtain the expression of thermodynamic curvature

$$\tilde{R}_{T\tilde{V}} = \frac{Q^2}{2\pi TV}.$$  \hspace{1cm} (28)

- In the phase space \{T, P\}, we must have the expressions of entropy \(S\) and thermodynamic volume \(\tilde{V}\) in terms of temperature \(T\) and pressure \(P\), respectively

\[
S = \pi T + \sqrt{\pi^2 T^2 + 2\pi Q^2 P} \quad \frac{16 P}{16 P}, \quad \tilde{V} = \frac{\pi T^2 + Q^2 P + T \sqrt{\pi^2 T^2 + 2\pi Q^2 P}}{32 P^2}. \hspace{1cm} (29)
\]

Hence according to Eq. (5), we can obtain the expression of thermodynamic curvature

$$\tilde{R}_{TP} = \frac{16Q^2P + 16T\left(\pi T - \sqrt{\pi^2 T^2 + 2\pi Q^2 P}\right)}{\pi Q^2 T}.$$  \hspace{1cm} (30)

Meanwhile we also have the identity

$$\tilde{R}_{SP} = \tilde{R}_{T\tilde{V}} = \tilde{R}_{TP} > 0.$$  \hspace{1cm} (31)

Moreover, for the extreme black hole \(T = 0\), we can observe clearly that thermodynamic curvature tends to be positive infinity. Hence we can conjecture that the charged BTZ black hole is likely to present a repulsive between its molecules phenomenologically or qualitatively.

V. CONCLUSION AND DISCUSSION

Comparing the results of the Case I and Case II, we can observe clearly the following four very interesting conclusions.

- Both of Case I and Case II, the obtained thermodynamic curvature is always positive, which may be related to the information of repulsive interaction between black hole molecules for the charged BTZ black hole.

- When \(Q = 0\), in both cases, we can see that the thermodynamic curvature degenerates to zero, which show that the neutral BTZ black hole is Ruppeiner flat [30, 42, 45].

- For the extreme black hole, i.e., \(T = 0\), the thermodynamic curvature is finite positive value in Case I, while in Case II, it tends to be positive infinity. In the previous thermodynamic geometric analysis of AdS black holes [19–26, 34, 38], we usually see that in extreme black holes, the thermodynamic curvature tends to be positive or negative infinity. In Case I of the thermodynamic analysis about charged BTZ black hole, the thermodynamic curvature of extreme black hole is a positive finite value, which is not consistent with the previous discussion. From this point of view, thermodynamic curvature of extreme black hole discussed in present...
paper may serve as a criterion to discriminate the two thermodynamic approaches introduced in Ref. [47] and our result seems to support the Case II of the thermodynamic analysis about charged BTZ black hole.

- In both cases, we find in different phase spaces, the obtained thermodynamic curvatures are equivalent. Thus the result provides a clue to establish the coordinate-free definition of the thermodynamic geometry. Similarly, this phenomenon has also been observed in Schwarzschild AdS black hole [34] and Reissner-Nordström AdS black hole [53].

We have known that for Case I, the charged BTZ solution is proved to be a super-entropic black hole [48, 49], which violates the reverse isoperimetric inequality. Generally, for a $d$-dimensional black hole, its thermodynamic volume $V$ and entropy $S$ satisfy the reverse isoperimetric inequality [54]

$$\mathcal{R} = \left( \frac{(d - 1)V}{\omega_{d-2}} \right)^{\frac{1}{d-1}} \left( \frac{\omega_{d-2}}{4S} \right)^{\frac{1}{d-2}} \geq 1,$$

where $\omega_n = 2\pi^{(n+1)/2}/\Gamma[(n + 1)/2]$ is the standard volume of the round unit sphere. In $d = 3$, the charged BTZ black hole has $\mathcal{R} < 1$ in Case I. While in Case II, because the thermodynamic volume $\tilde{V}$ and entropy $\tilde{S}$ are not independent of each other, then the above reverse isoperimetric inequality is saturated by the charged BTZ black hole, i.e., $\mathcal{R} = 1$. The underlying reason for the satisfaction, violation or saturation of the reverse isoperimetric inequality lies in the definition of the thermodynamic volume of a black hole, which is also a very significant research issue in black hole thermodynamics. If the thermodynamic volume is consistent with the expression of the geometric volume, often the reverse isoperimetric inequality is saturated ($\mathcal{R} = 1$). While the thermodynamic volume of black hole does not look like any geometric volume, it generally corresponds to a super-entropic black hole ($\mathcal{R} < 1$) or sub-entropic black hole ($\mathcal{R} > 1$).

When the reverse isoperimetric inequality is saturated, the thermodynamic curvature of extreme black hole tends to be (positive or negative) infinity. This conjecture is verified by many examples, like Schwarzschild AdS black hole, Reissner-Nordström AdS black hole and Gauss-Bonnet AdS black hole (and various other simple static black hole solutions of the pure Einstein gravity or higher-derivative generalizations thereof), and Case II of our current discussing. Moreover, according to above calculation and analysis of Case I of the analysis about charged BTZ black hole, it is conjectured that for super-entropic black holes, the thermodynamic curvature of extreme black hole will go to a finite (positive or negative) value. In the future works, we will further test the conjecture in other super-entropy black holes, like ultra-spinning limit of Kerr-AdS black holes [55, 56]. Meanwhile, does this conjecture hold for the sub-entropic black hole, such as the Kerr-AdS black hole [54, 57], STU black holes [57, 58], Taub-NUT/Bolt black hole [59], generalized exotic BTZ black hole [49] and noncommutative black hole [60]? These are also very interesting topics to be discussed in the future.
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