ON THE ORIGIN OF THE HIERARCHY OF THE SCIENCES

NIKITA KALININ

ABSTRACT. We propose a simple “evolutionary” sandpile model exhibiting self-organised criticality and exactly $1/f$-noise (i.e. the critical exponent is equal to $-1$) and observe emergent phenomena of the same type self-organised criticality on the “next level” sandpile.

In this way we try to model climbing by the so-called hierarchy of sciences, where processes on a higher level can, in principle, be derived by laws of a lower level but this derivation is computationally unfeasible and useless from the explanatory point of view.

1. Hierarchy of sciences

According to Auguste Compte (1798–1857), each Science came into being to seek the “Laws” of a particular level of facts which man experience in the world. Next, each Science depends on the developments of its predecessor in a “hierarchy” or, better to say, in an “ordering” of Sciences by increasing complexity and decreasing generality.

One can add a reductionist flavour to this hierarchy, e.g. “The underlying physical laws necessary for the mathematical theory of a large part of physics and the whole of chemistry are thus completely known, and the difficulty is only that the exact application of these laws leads to equations much too complicated to be soluble” as Paul Dirac wrote in 1929 [9].

However, each level of hierarchy of sciences manifests emergent properties such as Darwinian evolution [5, 11, 14] or Mendeleevian periodic table [17], whose reduction to physics could be cumbersome and obscure if at all possible. Hence it is an interesting question: how emergence is possible, what are possible mechanisms of emergence? Frequently emergence phenomena are associated to so-called self-organised criticality [7].

One can compare two neighbour levels in the hierarchy through an artistic metaphor: earlier science is to later science as the frame, canvas, and paint are to a picture [3].
The goal of this paper is to present a model, in which we cook up self-organised criticality of a “higher” level using “materials” of self-organised criticality of a lower level. The main idea is as follows: in the artistic metaphor, a picture (or the art in general) is perceived if we don’t pay too much attention to all the small details but select certain more important or salient parts of the information and try to find relations between them. We describe this idea in a mathematical form and present results of simulations.

One pleasant features of our model is that it exhibits a $1/f$-noise. Research is supported by the Russian Science Foundation grant 20-71-00007.

2. Cellular automaton as a metaphor of matter and energy

We cannot resist to start this section by quoting Dirac one more time: “It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical theory of great beauty and power, needing quite a high standard of mathematics for one to understand it. You may wonder: Why is nature constructed along these lines? One can only answer that our present knowledge seems to show that nature is so constructed. We simply have to accept it. One could perhaps describe the situation by saying that God is a mathematician of a very high order, and He used very advanced mathematics in constructing the universe.” [10]

Our position is that the most primordial matter in the World is mathematics (in the same sense that certain particles correspond to irreducible unitary representations).

It has been argued that if we would like the classical spacetime to emerge from a background-independent theory of gravity, then some self-organised critical behaviour is required [19]. Loop gravity models [19,16,15], Penrose spin networks [13,12], quantum foam and tiling [6], and, more generally, cellular automata [20] in general can be thought of as discretizations of the Universe or, at least, as reasonable metaphors of the Universe.

Next, self-organised criticality for us is synonymous to the fact that certain observable variable $y$ in the system depends on another one variable $x$ as in the power law, namely, $y = cx^\alpha$. When $\alpha = -1$ we call it $1/f$-noise.
In this article we consider a cellular automaton (evolutionary sandpile model) on a part of the standard grid $\mathbb{Z}^2$, and the sign of emergence for us is a power law relation in this model between frequency of avalanches and its sizes.

3. The model

3.1. Standard sandpile model. Consider $G = [0, n]^2 \cap \mathbb{Z}^2$, a subgraph of the grid $\mathbb{Z}^2$. Each lattice point $(i, j) \in \mathbb{Z}^2, 0 \leq i, j, \leq n$ is thus a vertex of $G$ of valency 4 as we connect each two vertices $v, w$ of distance one by a link, and denote this fact as $v \sim w$. A state of sandpile model is a function $\phi : G \rightarrow \mathbb{Z}_{\geq 0}$ from the vertices of $G$ to non-negative integers. One can think that each vertex $v$ of $G$ contains a non-negative amount $\phi(v)$ of non-divisible grains of sand.

A vertex $v \in G$ with $\phi(v) \geq 4$ can topple by sending one grain to each of its neighbors, grains falling outside $G$ disappear. Formally, if $\phi(v) \geq 4$ then the toppling at $v$ produces another state $\psi$ where $\psi(v) = \phi(v) - 4$, $\psi(w) = \phi(w) + 1$ if $w \sim v$ and $\psi(w) = \phi(v)$ otherwise.

Starting with an arbitrary state $\phi$ and doing topplings while it is possible (it is called a relaxation of $\phi$) produces a state denoted by $\phi^0$ where toppling are no more possible, i.e. $\phi^0(v) \leq 3$ for each $v \in G$. One can prove that the result of relaxation does not depend on the order of topplings $\mathbb{N}$.

Let $\phi_0$ be the state with three grains at every vertex. It is the pointwise maximal stable state, i.e. a state where topplings are not possible.

Let us uniformly randomly choose a vertex $v_0 \in G$ and add one grain of sand to it, this gives a state which we denote by $\phi_0 + \delta_{v_0}$.

This state has four grains at $v_0$, so we can topple $v_0$, it produces a state where neighbors of $v_0$ have four grains, we should topple them, etc. The set of vertices which topple during the relaxation of $\phi_0 + \delta_{v_0}$ is denoted by $A_1$, and the result of the relaxation is denoted by $\phi_1$. Again, choose randomly $v_1 \in G$, add one grain to $v_1$ in $\phi_1$, relax. This way we obtain a chain of states

$$\phi_{n+1} = (\phi_n + \delta_{v_n})^0,$$

and a sequence of sets $A_n$. The number $a_n = |A_n|$ of vertices in $A_n$ is called the area of the avalanche provoked by adding a grain of sand to $v_n$. It can be observed in simulations that the histogram of the numbers $a_n$ (when $n \to \infty$) on the double log-scale looks like a line, hence the
probability $p$ that an avalanche in the above dynamics has area $x$ is

$$p = cx^\alpha.$$  

This power law statistics were observed in [2, 1] where the term “self-organised criticality” we coined. This was rigorously proven only recently [4].

3.2. **Evolutionary sandpile model.** We suppose that the graph $G$ and the above sandpile dynamics on it is the basic reality level.

Following artistic metaphor, to see a picture and not just paints, we need to distinguish some vertices among others and claim that the set of distinguished vertices is the next observed level of reality. Previously, on each step we added a grain of sand to uniformly chosen vertex. It may not be fair, let us add more sand to the vertices which produce bigger avalanches! Think of the size of an avalanche starting from a vertex $v$ as a proxy to the fitness of $v$.

Formally, let us redefine the probability $p(v)$ of adding a grain to a vertex $v$ to be proportional to

$$f(v) = 1 + \sum a \log(Area(a) + 1)$$

where $a$ runs by all avalanches starting from $v$. Call $f(v)$ the fitness of $v$. Note that the area can be zero (no one toppled after adding a grain to $v$) so we write $Area(a) + 1$ in the logarithm.

The dynamic of the evolutionary sandpile model is exactly as in the standard sandpile model except that the fitness of each vertex $v$ is updated when $v_n = v$ and on each step of the dynamic each vertex is chosen with probability proportional to its current fitness. Then, vertices with bigger fitness receive additional sand grains more frequently.

The question of emergence can be stated now as follows. Run a simulation of the evolutionary sandpile model and then, on each step $k$, distinguish vertices whose fitness is at least ten times the average fitness of all vertices. Note that the set $S_k$ of distinguished vertices on step $k$ may change over time. For the avalanche after adding a grain to $v_k$ let us count the number $s_k = |S_k \cap A_k|$ of distinguished vertices touched by the avalanche on the step $k$. Note that the vertices in $S_k$ “do not know” much about avalanches, how they propagate, these avalanches can even start at a vertex not in $S_k$.

We say that we observe an emergent phenomena if the numbers $s_k$ also satisfy a power law.
4. Results

We run evolutionary sandpile on the grid in the square $[0,200]^2$ initially filled with three grains at every point and performed $N = 40000000$ steps of dynamics, thus we added $N = 1000 \cdot 200^2$ grains of sand to the system, i.e. 1000 grains in each vertex in average. The plots and histograms that we present below stabilise much earlier, and there is no difference if we make, for example, twice less steps.

![Figure 1](image.png)

**Figure 1.** Big red squares indicate vertices whose fitness (probability to be chosen) is at least 40 times the average. Medium blue squares indicate vertices whose fitness is between 20 and 40 times the average. Small green squares indicate vertices whose fitness is between 10 and 20 times the average.

Figure 1 representing the vertices with the biggest fitness resembles a geographic map with megapolis’s and smaller towns along roads. We cannot provide any even hypothetical explanation of this observation.
Let us draw two types of histograms on the double log scale, see Figure 2. On the left we draw 400 bins of equal width and then take the logarithm for both axes. We see a line of slope $-1$. On the right we choose the widths of bins such that the next bin is $1.1$ times bigger than the previous one.

Then, on the double logarithm scale they all these bins have the same heights. This means that the density of the probability at $x$ is $c/x$ for

$$\int_{x_i}^{1.1x_i} \frac{c}{x} dx = c \log(1.1)$$

does not depend on $x_i$. Hence evolutionary sandpile model generates $1/f$-noise (a power law $y = x^\alpha$ with the exponent $\alpha = -1$). Recently $1/f$-noise was obtained for the first time [18] using much more complicated (than ours) modification of the original sandpile model.

Figure 2. Standard bins of equal width are on the left, bins $[50 \cdot 1.1^k, 50 \cdot 1.1^{k+1}]$, $k = 0 \ldots 70$ are on the right. Note that the slope of the evident line on the right hand side histogram is zero, thus we obtain the power law frequency $= c$(size)$^{-1}$.

Figure 3 shows the histograms of $s_k$ on the double logarithmic scale. Indeed, we observe an emergence phenomenon as the frequency of the size $s_k$ of the intersection of avalanche sets $A_k$ with the sets $S_k$ of distinguished vertices is propositional to $s_k^{-1}$. A histogram for bins of equal widths is presented on the left hand side, and the histogram for
Figure 3. Histograms of the frequency/size relation of the intersection size $s_k$ of avalanche with the set of distinguished vertices.

Figure 4. Histogram of fitness of the vertices on the double logarithmic scale.

Bins of type $[2 \cdot 1.1^k, 2 \cdot 1.1^{k+1}], k = 0, \ldots, 70$ is presented on the right hand side.

In Figure 4 we present the distribution of the fitness of all 40000 vertices in our graph.
ON THE ORIGIN OF THE HIERARCHY OF THE SCIENCES

5. Discussion

One can think about the evolutionary sandpile model using the following metaphor: instead of sand grains we excite (with ideas of songs) people sitting in the vertices of $G$, and the area $a_k$ of the avalanche starting from $v_k$ is the number of people who liked song produced by $v_k$. If $a_k = 0$, then no one like a song produced by $v_k$, even $v_k$ itself. Then it is natural that new song ideas excites $v$ with probability of the previous success of songs already composed by $v$.

In this metaphor, the set $S_k$ of distinguished vertices is the set of established song writers, those, for whom the total logarithmic success of their songs is at least ten times bigger than the average success. Then $s_k$ is the number of established song writers excited by $k$-th song. Note that songs in our model are spread from a vertex to its neighbors. When we restrict out attention to the set $S_k$ this topology is lost, thought some idea of closeness is preserved. Some songs are produced by not established song writers, and for them this song comes from "nowhere". Nevertheless the power law is presented and we say that we observe an emergent phenomena.

On the other hand it seems impossible to prove this power law for the distinguished sets directly from the rules of the evolutionary sandpile modes, as it seems impossible to reduce Darwinian evolution to quantum physics.

Thus our evolutionary sandpile model has two nice properties:
1) it is very simple and it shows $1/f$-noise
2) in the frame of our metaphor it demonstrates a phenomenon of emergence.

References

[1] P. Bak, C. Tang, and K. Wiesenfeld. Self-organized criticality: An explanation of the $1/f$ noise. Physical review letters, 59(4):381–384, 1987.
[2] P. Bak, C. Tang, and K. Wiesenfeld. Self-organized criticality. Physical review A, 38(1):364, 1988.
[3] A. T. Balaban and D. J. Klein. Is chemistry the central science? how are different sciences related? co-citations, reductionism, emergence, and posets. Scientometrics, 69(3):615–637, 2006.
[4] S. Bhupatiraju, J. Hanson, A. A. Járai, et al. Inequalities for critical exponents in $d$-dimensional sandpiles. Electronic Journal of Probability, 22, 2017.
[5] D. Blitz. Emergent evolution: qualitative novelty and the levels of reality, volume 19. Springer Science & Business Media, 2013.
[6] T. D. Brennan, F. Carta, and C. Vafa. The string landscape, the swampland, and the missing corner. arXiv preprint arXiv:1711.00864, 2017.
[7] V. Darley. Emergent phenomena and complexity. Artificial Life, 4:411–416, 1994.
[8] D. Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.*, 64(14):1613–1616, 1990.

[9] P. A. M. Dirac. Quantum mechanics of many-electron systems. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 123(792):714–733, 1929.

[10] P. A. M. Dirac. The evolution of the physicist’s picture of nature. *Scientific American*, 208(5):45–53, 1963.

[11] P. T. Macklem. Emergent phenomena and the secrets of life. *Journal of applied physiology*, 104(6):1844–1846, 2008.

[12] R. Penrose. Angular momentum: an approach to combinatorial space-time. *Quantum theory and beyond*, pages 151–180, 1971.

[13] R. Penrose. On the nature of quantum geometry. *Magic without magic*, pages 333–354, 1972.

[14] A. Pross. Toward a general theory of evolution: Extending darwinian theory to inanimate matter. *Journal of Systems Chemistry*, 2(1):1–14, 2011.

[15] C. Rovelli. Loop quantum gravity. *Living reviews in relativity*, 11(1):1–69, 2008.

[16] C. Rovelli. Loop quantum gravity: the first 25 years. *Classical and Quantum Gravity*, 28(15):153002, 2011.

[17] E. R. Scerri. How good is the quantum mechanical explanation of the periodic system? *Journal of Chemical Education*, 75(11):1384, 1998.

[18] A. Shapoval, B. Shapoval, and M. Shnirman. 1/x power-law in a close proximity of the bak–tang–wiesenfeld sandpile. *Scientific Reports*, 11(1):1–6, 2021.

[19] L. Smolin. An invitation to loop quantum gravity. In *Quantum Theory and Symmetries*, pages 655–682. World Scientific, 2004.

[20] S. Wolfram. Cellular automata as models of complexity. *Nature*, 311(5985):419–424, 1984.