Short-range interactions in an effective field theory approach for nucleon-nucleon scattering

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We investigate in detail the effect of making the range of the “contact” interaction used in effective field theory (EFT) calculations of NN scattering finite. This is done in both an effective field theory with explicit pions, and one where the pions have been integrated out. In both cases we calculate NN scattering in the $^1S_0$ channel using potentials which are second-order in the EFT expansion. The contact interactions present in the EFT Lagrangian are made finite by use of a square-well regulator. We find that there is an optimal radius for this regulator, at which second-order corrections to the EFT are identically zero; for radii near optimal these second-order corrections are small. The cutoff EFTs which result from this procedure appear to be valid for momenta up to about 100 MeV/c. We also find that the radius of the square well cannot be reduced to zero if the theory is to reproduce both the experimental scattering length and effective range. Indeed, we show that, if the NN potential is the sum of a one-pion exchange piece and a short-range interaction, then the short-range piece must extend out beyond 1.1 fm, regardless of its particular form.

I. INTRODUCTION

Effective field theory (EFT) techniques have been used successfully for many years to study problems in particle physics where a well-defined hierarchy of mass scales exists [1]. In such problems one can, in principle, integrate out the short-range degrees of freedom, i.e. those corresponding to momenta above some separation scale, thereby obtaining a non-local effective action which will be applicable at energies well below this scale. The assumption is then made that all momenta in the problem of interest are small compared to the masses of the degrees of freedom which were integrated out. It follows that a momentum expansion of the action may be made. This leads to calculations which are organized in terms of the power of momentum which a given diagram contributes—the so-called EFT expansion.

This is possible because power-counting arguments indicate that a given loop diagram only contributes at a definite order in the EFT expansion. Naturally, these loop diagrams are divergent, and must be regulated if they are to yield finite results. These results must then be renormalized. The theory as a whole is not renormalizable, but at a given order in the EFT expansion there are only a finite number of possible counterterms and so the theory does have predictive power. Hence, once the order of an EFT calculation is fixed, the predictions of that theory are independent of the regulation scheme chosen.

Several years ago, Weinberg suggested that the techniques of EFT could be modified to study low-energy problems in nuclear physics, including nucleon-nucleon scattering [2,3]. He pointed out that the fundamental difference in the NN scattering case is that the EFT expansion must be applied to the potential—not the amplitude, as in most other problems. If this is done then the potential which is to be used in the Schrödinger equation contains a delta function interaction. Direct calculation with such an equation is impossible. Regulation and renormalization of the results must be performed before physical predictions can be extracted. The question is how this is to be done.

So far two approaches exist in the literature. The first involves formally iterating the divergent interaction, and then renormalizing the resultant amplitude. This was the approach used by Weinberg [3]. More recently, it has been applied by Kaplan et al., who iterated the contact interaction formally and then used dimensional regularization and the $\overline{\text{MS}}$ renormalization scheme to calculate NN scattering in the $^1S_0$ channel [4]. The second approach is to introduce a regulator into the equation before iterating the potential to generate the t-matrix. This was the method adopted by van Kolck et al., who cut off all integrals in momentum space in studies of NN scattering [5]. In such an approach renormalization is performed by adjusting the coefficients of the potential to fit the NN scattering data. This can be done for several different values of the regulator parameter. The sensitivity of unfit physical observables to the regulator parameter may be regarded as a measure of the validity of the regulation. If an EFT solution to the NN problem can be constructed by one of these two, or some other, means, it may then be used as input in nuclear physics applications of EFT, e.g. to pion-deuteron scattering [6], the three-nucleon system [7], the $pp \rightarrow pp\pi^0$ process [8], and pion photoproduction on the deuteron [9].

One might think that in order to establish the connection with the original field theory the regulator in the second approach must ultimately be removed from the problem, i.e., that the range of the regulated contact interaction must eventually be taken to zero. However, it has recently been pointed out that the use of zero-range interactions which are defined as the limits of short-range ones as the range is taken to zero leads to peculiar results. In particular it has been shown that EFTs either with or without pions cannot contain a repulsive zero-range interaction (defined in
this sense) which has any effect on the observables in the renormalized theory \[1,2\]. Thus all non-trivial zero-range interactions must be attractive. Furthermore, in theories without explicit pions (i.e. where the pions are integrated out) it was shown that if the range of the interaction is taken to zero then the phase shifts obey:

\[
\frac{d}{dk^2}(k \cot(\delta(k))) \leq 0. \tag{1.1}
\]

A corollary is that potentials of sufficiently small radii cannot reproduce the NN \(^1S_0\) scattering length and effective range. In this paper we show that similar constraints apply when pionic degrees of freedom are explicitly included.

While these results do not invalidate the conclusions of Ref. \[3\] they do show that the approach used there is not equivalent to solving a Schrödinger equation with an interaction of range \(R\), and renormalizing the coefficients as \(R\) is adjusted. In other words, for this problem, regulating the potential and then iterating it is not equivalent to iterating it formally and then using dimensional regularization and \(\overline{\text{MS}}\) renormalization.

This leads us to investigate in detail what happens if the range of the “contact” interaction is made finite before iteration, as was done in practice in Ref. \[3\]. (The use of similar “cutoff” field theories has been advocated for some time by, for example, Lepage \[13\].) While such an approach apparently violates some of the assumptions of Weinberg’s power-counting arguments \[1\], one trusts that for a wide range of regulator parameter values the results of the renormalized theory are not particularly sensitive to the exact value of that parameter. Indeed, a virtue may be made of the necessity of keeping this parameter finite. The freedom to choose it may be exploited so as to minimize the error in truncating the EFT expansion. That is, a range may be chosen for our short-range interaction which results in the renormalized coefficient of the second-order term in the EFT expansion being zero. For other values of the short-range interaction range the value of this renormalized second-order coefficient will, of course, be non-zero. The viability of such an approach must be judged by testing how sensitive observables are to the range chosen for the regulated contact interaction.

In this paper we explore this issue for two EFT-motivated treatments of nucleon-nucleon scattering in the \(^1S_0\) channel \[4\]. In Section \[II\] we investigate a Lagrangian with only contact interactions (i.e. where the pions have been integrated out), while in Section \[III\] we look at an EFT with contact interactions and explicit pions. In both cases we never use a true contact interaction but instead always solve the Schrödinger equation keeping the regulator parameter for the contact interaction, \(R\), finite. The particular form of the regulator chosen here is a square well. In both EFT treatments we begin by calculating at zeroth order in the EFT expansion. At this order, the potential in the Schrödinger equation is either a square well alone, or the sum of a square well and one-pion exchange. For any given radius \(R\) the strength of the well can be adjusted to match the \(^1S_0\) NN scattering length. However, by exploiting the freedom to choose \(R\), we can minimize the second-order corrections to the EFT by choosing a radius which results in the total zeroth-order EFT potential reproducing both the scattering length and the effective range. The resulting value of the well radius, \(R\), is the “optimal” one, since it minimizes the second-order EFT corrections to the potential. For an arbitrary regulator radius \(R\), going to second order in the EFT means adding terms to the potential which correspond to derivatives of our “contact” interaction. The coefficients of these new terms are fixed by demanding that at different well radii the scattering length and effective range are still reproduced. At the optimal radius these coefficients are, by definition, zero. For radii close to optimal, it is found that the coefficients can be renormalized successfully. However, we will show that there is a lower bound on the radius for which this renormalization can be done. Below a certain value of \(R\) it is impossible to adjust the coefficients to reproduce low-energy \(^1S_0\) NN scattering data. This explicitly demonstrates, for the specific case of a square-well regulator, that the theorem of Ref. \[14\] prevents one from taking the limit \(R \to 0\). More generally, the result of Ref. \[12\] leads to an absolute lower bound on the radius \(R\) that may be used if the effective field theory is to correctly predict the effective range. This lower bound is completely independent of the particular regulator chosen. In the EFT without explicit pions it is a simple matter to show that the absolute lower bound is \(R = 1.3\) fm. In Section \[IV\] we use a modified version of the argument presented in Ref. \[12\] to show that the smallest possible \(R\), even when explicit one-pion exchange is included, is larger than 1.1 fm.

The sensitivity of the results to the range of the short-range potential is explored in two ways. First, we examine the phase shifts for different values of \(R\). Agreement is expected at low momenta since we have fit the scattering length and effective range. Thus the phase shifts should be identical until the point where the fourth-order terms in the effective range expansion become significant. The question is at what momentum these fourth (and higher) order terms become important for different well radii. Secondly, we examine the magnitude of the second-order terms in the t-matrix for potentials of various radii. The hope is that for radii close to optimal the second-order corrections are small for moderate momenta. This would indicate that if short-range potentials of these radii are used then the EFT expansion stays under control.
II. AN EFFECTIVE FIELD THEORY WITHOUT PIONS: THE SQUARE WELL AND DERIVATIVES

As observed in Refs. [2,3], if one integrates out all the exchanged mesons in the NN interaction then an EFT is obtained which consists solely of contact interactions and derivatives thereof. In the $^1S_0$ channel the only pieces of this Lagrangian which contribute (to second order in the EFT expansion) are:

$$
\mathcal{L} = N^\dagger \partial_t N - N^\dagger \frac{\nabla^2}{2M} N - \frac{1}{2} C_S (N^\dagger N)^2 - \frac{1}{2} C_T (N^\dagger \sigma N)^2 - \frac{1}{4} C_2 (N^\dagger \nabla^2 N) (N^\dagger N) - \frac{1}{4} C_2 ((\nabla^2 N^\dagger) N) (N^\dagger N). \tag{2.1}
$$

If the interaction terms here are used, as advocated by Weinberg, to generate a potential for use in the Schrödinger equation, then at second order in the EFT expansion the result is a Schrödinger equation which we will write with a general non-local potential,

$$
- \frac{\nabla^2}{M} \psi(x) + \int V_R(x, x') \psi(x') d^3x' = E \psi(x), \tag{2.2}
$$

where the subscript $R$ indicates that $V$ must be regulated in order to make the Schrödinger equation meaningful. The regulated non-local potential is:

$$
V_R(x, x') = (C_0 + C_2 (-\nabla^2 - \nabla'^2)) \delta^{(3)}(x - x') \delta^{(3)}_R(x). \tag{2.3}
$$

Here $C_0 = C_S - 3C_T$, and $\delta^{(3)}_R$ is a function which tends to a delta function as $R \to 0$. In this work we use a square well of width $R$ for the function $\delta^{(3)}_R$; i.e.,

$$
\delta^{(3)}_R(x) = \frac{3\Theta(R - |x|)}{4\pi R^3}. \tag{2.4}
$$

In the $l = 0$ partial wave, the radial equation obtained from the Schrödinger equation (2.2) is then

$$
\left( - \frac{1}{M} - 2C_2 \delta_R(r) \right) \frac{d^2U}{dr^2} - 2C_2 \left( \frac{dU}{dr} - \frac{U(r)}{r} \right) \frac{d\delta_R}{dr} + \left( C_0 \delta_R(r) - C_2 \frac{d^2\delta_R}{dr^2} \right) U(r) = EU(r). \tag{2.5}
$$

The solutions to this equation in the two regions $r < R$ and $r > R$ must be matched at $r = R$. By integrating Eq. (2.3), we see that the boundary condition at $r = R$ involves a discontinuity of the derivative of the wavefunction, due to the derivative terms in the potential:

$$
\left( 1 + \frac{3C_2 M}{4\pi R^3} \right) \frac{dU}{dr} \bigg|_+ = \frac{dU}{dr} \bigg|_-, \tag{2.6}
$$

where $dU/dr|_+$ denote the one-sided derivatives at $r = R$.

From Eq. (2.4) an expression for the phase-shift $\delta$ at momentum $k = (ME)^{1/2}$ may be derived. It is:

$$
k \cot \delta = \frac{B_k \cot(kR) + k \tan(kR)}{1 - B_k \cot(kR) \tan(kR)}, \tag{2.7}
$$

where

$$
B = 1 + \frac{3C_2 M}{4\pi R^3}, \quad \kappa = \left( \frac{k^2 - \frac{3C_2 M}{4\pi R^3}}{1 + \frac{3C_2 M}{4\pi R^3}} \right)^{1/2}. \tag{2.8}
$$

Expanding Eq. (2.7) in powers of $k^2$, we find that the scattering length, $a$, and effective range, $r_e$, of this potential are given by

$$
a = R - \frac{R}{Bx \cot x}, \tag{2.9}
$$

$$
r_e = \frac{R(R - a)}{a} \left[ \frac{2\pi R^4(1 - x \tan x - x \cot x)}{(2\pi R^3 + 6C_2 \mu)x^2a} - 2 \right], \tag{2.10}
$$

3
where

\[ x = R \left( \frac{-3C_0M}{4\pi R^3 + 6C_2M} \right)^{1/2}. \]  

(2.11)

As we can see from Eqs. (2.9) and (2.10) above, \( x \) is a more natural variable for this problem than \( C_0 \). Consequently, from now on we will generally use \( x \) and \( C_2 \) as variables.

The values adopted for the NN \( ^1S_0 \) scattering length and effective range here are:

\[ a = -24 \text{ fm}; \quad r_e = 2.7 \text{ fm}. \]  

(2.12)

We will examine how \( x \) and \( C_2 \) must be renormalized as \( R \) is varied in order to reproduce these values. We will begin by setting \( C_2 = 0 \) and solving Eqs. (2.9) and (2.10) numerically for \( R \) and \( x \). We will refer to the result as the “optimal” well since it is the well which minimizes the second-order corrections in the EFT expansion. We will then vary \( R \) from the optimal value and solve for \( x \) and \( C_2 \) to determine over what range of \( R \) we can successfully obtain the actual scattering length and effective range using this potential. Finally, we will examine the phase shifts and the second-order corrections to the t-matrices for these renormalized potentials and compare our results with the phase shifts extracted from the experimental data.

Before we begin this procedure note that Phillips and Cohen [12] have recently shown that a bound for \( d\delta/dk \) derived by Wigner [15] yields the following constraint on \( r_e \):

\[ r_e \leq 2 \left( R - \frac{R^2}{a} + \frac{R^3}{3a^2} \right)^{1/2}. \]  

(2.13)

Eq. (2.13) provides an absolute lower bound on the size of a potential which can reproduce a given scattering length and effective range. In the \( ^1S_0 \) channel for NN scattering, this lower bound is 1.3 fm. In fact, for this potential, we will find that the lowest value of \( R \) for which Eqs. (2.9) and (2.10) can be solved is considerably larger than this.

To determine the optimal width of the square well, Eqs. (2.9) and (2.10) are solved with \( C_2 = 0 \), with the result that the optimal width is 2.6 fm. The corresponding value of \( x \) is 1.5. These values of \( x, C_2, \) and \( R \) are equivalent to \( C_0 = -5.1 \text{ fm}^2 \). A plot of the phase shift as a function of \( k \) for these values of \( R, x \) and \( C_2 \) is shown in Figure 1. We confirm that, as we expect from Levinson’s Theorem for a potential with no bound states, the phase shifts at zero and very large energy are equal.

![FIG. 1. A plot of phase shifts in radians versus momentum for the optimal well. The phase shifts are calculated from Eq. (2.7) with \( R = 2.6 \text{ fm}, x = 1.5 \) and \( C_2 = 0 \).](image)

We now proceed to examine the behavior for other values of \( R \). When Eqs. (2.9) and (2.10) are solved numerically for \( x \) and \( C_2 \) as \( R \) is varied from the optimal value, we find the behavior shown in Figures 2 and 3. As the radius is
reduced, we find that $x$ tends to $\pi/2$ and $C_2$ diverges as we approach a radius near 1.7 fm. The fact that $x$ is tending to a finite quantity while $C_2$ diverges suggests that $C_0$ is also diverging. For large $R$, $C_2$ becomes large and negative.

Armed with renormalized values of $x$ and $C_2$ for several different radii, we can use Eq. (2.7) to examine how the phase shifts are affected by changing $R$. Figure 4 compares the phase shifts for several values of $R$ with those obtained at the optimal value of $R$. Since we have fit the scattering length and effective range, the phase shifts will agree for all radii until the fourth order terms in the effective range expansion become important. While this occurs at quite low momentum for the 5 fm well, when the radii are close to optimal the agreement persists to a much higher momentum, $k \approx 0.5$ fm$^{-1}$. This is roughly what we expect since the radii of the relevant wells are themselves around 2 fm. Since this is a theory without explicit pions this radius can be thought of as being intrinsically that of one-pion exchange. Thus one might hope that when pions are explicitly included in the theory the radius of the short-range potential will

FIG. 2. A plot of the renormalized values of $x$, defined as in Eq. (2.11), versus the radius, $R$, of a square well potential.

FIG. 3. A plot of the renormalized values of the second-order coefficient in the EFT expansion, $C_2$, versus the radius, $R$, of a square well potential. The value of $C_2$ diverges near $R = 1.7$ fm.
decrease significantly.

For non-optimal wells the phase shifts always go to positive or negative infinity as the momentum gets large. However, this does not contradict Levinson’s Theorem since these potentials are non-local.

![Graph of phase shifts](image)

**FIG. 4.** A plot of the phase shifts in radians versus momentum calculated from Eq. (2.7) for square wells with different radii. The phase shifts for the optimal well, shown in more detail in Figure 1, are shown here in bold for comparison.

Next we compare these phase shifts with the experimentally determined values. Figure 5 compares the data from the Nijmegen partial wave analysis [16] with the phase shifts from the optimal well and the phase shifts from the effective range expansion. Unfortunately, we find rather poor agreement with the data for our optimal square well. Contrastingly, the effective range expansion matches the data surprisingly well, indicating that the shape parameter for NN scattering in this channel is quite small. Of course, we could adjust all three parameters $R$, $C_0$, and $C_2$ in order to improve the agreement of our result with the data, but this would not be in the spirit of the calculations we are pursuing here. While the result displayed in Fig. 5 is not particularly promising for the approach espoused in this paper, we will see that the explicit inclusion of pion exchange will lead to a considerable improvement in the fit to the experimental data.
FIG. 5. A comparison of calculated and experimentally measured phase shifts. The solid line in this plot is the phase shift as a function of momentum for an optimal square well. The dashed line is the effective range expansion to second order. The dots are data points from the Nijmegen np partial wave analysis in the $^1S_0$ channel [16].

Having examined the behavior of this system to second order in the EFT expansion, we now wish to examine just how significant the second order, i.e. derivative, terms are. For a given value of $R$, we may examine the fractional difference between the on-shell t-matrices at zeroth and second order in the EFT expansion,

$$\Delta T_R(k) \equiv \frac{T_R^{(0)}(k) - T_R^{(2)}(k)}{T_R^{(0)}(k)},$$

(2.14)

Here, the second-order t-matrix, $T_R^{(2)}$ has already been calculated:

$$T_R^{(2)}(k) = k \cot \delta - ik,$$

(2.15)

where $k \cot \delta$ is given in Eq. (2.7). On the other hand, $T_R^{(0)}$ is the zeroth-order EFT amplitude for a square well of radius $R$. Hence, at a given $R$ we have $C_2 = 0$ and the only free parameter is $C_0$. Consequently, we must renormalize differently. In this case we use Eq. (2.4) to fix $x$ by demanding that we reproduce the $^1S_0$ NN scattering length. This gives the renormalization condition

$$x_0 \cot x_0 = \frac{R}{R - a},$$

(2.16)

$T_R^{(0)}$ is then found by using the values $x = x_0$ and $C_2 = 0$ in Eq. (2.7) for $k \cot \delta$.

For the optimal width well, $\Delta T$ is zero, by construction. For non-optimal widths, Figure 6 shows $\Delta T_R(k)$ for several values of $R$. We note that, as we would expect, in the zero-energy limit the difference is always zero. The rate at which $\Delta T_R$ grows with energy is dependent on how much the radius of the well differs from that of the optimal well.

FIG. 6. A plot of the fractional difference between the zeroth- and second-order t-matrices versus momentum, calculated from Eq. (2.14), for several values of $R$, the radius of the square well. For the optimal well with $R = 2.6$ fm, this quantity is identically zero.

To summarize, we have shown how the coefficients in the second-order EFT expansion must be renormalized if we wish to reproduce the $^1S_0$ NN scattering length and effective range with a square well potential of a given radius. In particular, we have demonstrated that there is a definite lower bound on the radius for which this renormalization can be done. For low momenta, we find that the phase shifts are insensitive to the choice of regulator parameter, as one would expect since we have fit the effective range expansion up to second order. For radii close to the optimal
value, we observe that the agreement persists to higher momenta. Furthermore, for these nearly-optimal radii the quantity $\Delta T$ is small for momenta up to roughly 0.5 fm$^{-1}$, thus indicating that the EFT expansion is under control for momenta below this scale. Again, we would not expect our EFT to be valid above this scale since it does not explicitly contain pions.

We do find, somewhat disappointingly, that a straightforward effective range expansion of the data fits the actual phase shifts to considerably higher momenta than our optimal well. This indicates that the shape parameter for our optimal square well is considerably larger than the experimental shape parameter, which appears to be very small. While this situation could be remedied by adjusting our radius to obtain a better agreement with the data such an approach would not be systematic, as it would rely on the sensitivity of the phase shifts to the cutoff parameter $R$.

### III. AN EFFECTIVE FIELD THEORY WITH EXPLICIT PIONS: ONE-PION EXCHANGE AND A SQUARE WELL

Naturally, the pionless EFT of Section II is only valid for momenta considerably less than $m_\pi$. In order to construct a better EFT and so draw nearer to cases of interest in nuclear physics in this section we include pions in our EFT.

At zeroth order the $NN$ potential resulting from such an EFT Lagrangian is:

$$V_R(x) = C_0 \delta^{(3)}(x) + V_\pi(x),$$  \tag{3.1}

where $V_\pi(x)$ is the $^1S_0$ OPE potential, which, if we absorb the delta function piece into the contact interaction takes the form

$$V_\pi(r) = -\alpha \pi e^{-m_\pi r},$$  \tag{3.2}

where

$$\alpha \pi \equiv \frac{g_A^2 m_\pi^2}{16 \pi f_\pi^2}.$$  \tag{3.3}

Strictly speaking at second order in the EFT we should calculate pionic corrections to this potential as well as including derivatives of the regulated contact interaction. However, as a first attempt we include only these derivative terms, using as our second-order EFT interaction a sum of the interaction of Eqs. (2.3) and (2.4) and the OPEP (3.2).

The radial Schrödinger equation in the $l = 0$ partial wave is then:

$$\left( -\frac{1}{M} - 2C_2 \delta R(r) \right) \frac{d^2 U}{dr^2} - 2C_2 \left( \frac{dU}{dr} + \frac{U(r)}{r} \right) \frac{d\delta R}{dr} + \left( V_\pi(r) + C_0 \delta R(r) - C_2 \frac{d^2 \delta R}{dr^2} \right) U(r) = EU(r).$$  \tag{3.4}

The boundary condition at $r = R$ is identical to the case with only a square well given in Eq. (2.6).

The equation (3.4) cannot be solved analytically. Therefore, we must solve the differential equation numerically and impose a matching condition far from the origin in order to find the phase shifts. Once this is done, the analysis proceeds exactly as in Section II. We note that because the pion potential is not compactly supported, it is no longer clear that the results of Ref. [12] demand a lower bound on $R$. We will, nevertheless, see in Section IV that such a lower bound does exist in this case.

When we set $C_2 = 0$ and solve for the values of $R$ and $C_0$ which reproduce the $^1S_0$ NN scattering length and effective range, we obtain an optimal well width of 2.3 fm with $C_0 = -3.3$ fm$^2$. A plot of the phase shifts as a function of $k$ for this potential is shown in Figure 7.
FIG. 7. A plot of phase shifts in radians versus momentum for the pion potential plus an optimal well. These are calculated from the numerical solution of Eq. (3.4) with $R = 2.3$ fm, $C_0 = -3.3$ fm$^2$ and $C_2 = 0$.

Next we renormalize the coefficients $C_0$ and $C_2$ for other values of $R$. Figures 8 and 9 plot the renormalized values. We observe that both $C_0$ and $C_2$ diverge near $R = 1.4$ fm, indicating that this is the smallest well that can be used to parameterize the NN interaction in the $^1S_0$ channel when a one-pion exchange potential is included.

FIG. 8. A plot of the renormalized values of the zeroth-order coefficient in the EFT expansion, $C_0$, from the numerical solution of Eq. (3.4) that reproduces the low-energy $^1S_0$ NN scattering data, versus the radius, $R$, of the square well piece of the potential.
FIG. 9. A plot of the renormalized values of the second-order coefficient in the EFT expansion, $C_2$, versus the radius, $R$, of the square well piece of the potential.

Now that we have renormalized $C_0$ and $C_2$, these values can be used to calculate the phase shifts for our second-order EFT potential. A comparison of the phase shifts for several values of $R$ is shown in Figure 10. We observe the same qualitative behavior as was seen in Figure 4. The phase shifts for all radii agree up to a point—as must be, since we have fit the first two terms in the effective range expansion. At this point, the results for potentials with radii far from optimal diverge rapidly. However, for values of $R$ reasonably close to the optimal value, the agreement persists to a considerably higher momentum.

FIG. 10. A plot of the phase shifts in radians versus momentum for various radii from the numerical solution of Eq. (3.4). The phase shifts for the optimal well are shown in bold.

Nevertheless, one sees that sensitivity to the particular regulator chosen creeps into the observables at a momentum of $k \approx 0.5 \text{ fm}^{-1}$ (earlier if the well is ridiculously large). This is not surprising since this is roughly the momentum at which details of the well structure begin to be probed. One might think that this is an argument for removing the regulator from the problem by taking $R$ to zero. However, in this work we have shown that $R$ cannot be decreased
below about 1.4 fm. Thus, we appear to be forced to use a regulator which limits the applicability of the EFT to momenta below some maximum, \( \Lambda \). This maximum lies considerably below the expected range of validity of the EFT. This is not a desirable situation. Similar comments apply to the results of Fig. 3, but there the pion is integrated out, and so \( \Lambda \) is of roughly the same magnitude as the expected maximum momentum for the EFT. What we see in Fig. 11 is that including the pion explicitly does not greatly decrease the sensitivity of the results to the radius of the short-range potential. This occurs because the large effective range of this problem means that the “short-range” potential cannot really be made short range at all, but instead must extend out to at least 1.4 fm. This suggests the possibility of a breakdown in the scale separation which is essential for power-counting arguments to be applicable.

Figure 11 compares the phase shifts for the OPE potential plus optimal well with the Nijmegen np partial wave analysis for the \( ^1S_0 \) channel [16]. This plot shows good agreement up to the pion production threshold. The divergence at higher momenta is sharp but not unexpected since calculated phase shifts are positive for all momenta while the experimental phase shift becomes negative just above pion threshold. This behavior could be reproduced if we tuned the potential specifically to do so.

However, we observe that there is really no reason to compare the Nijmegen phase shifts with the results from the optimal well. The phase shifts from any of the other wells whose phase shifts are plotted in Fig. 11 can be compared to the data too. Upon doing this it is clear that the optimal well happens to be a potential whose structure affects the phase shifts in a way which brings them closer to to the experimental data. In other words, the good agreement seen in Fig. 11 is fortuitous, and is dependent upon details of the particular regulator used. We cannot expect such good agreement in general.

Finally, we once again compare the the zeroth- and second-order t-matrices for different values of \( R \). We will examine the fractional difference \( \Delta T_R(k) \) as defined in Eq. (2.14). The second-order t-matrix, \( T_R^{(2)}(k) \), is calculated using the pion potential and a square well with second derivative terms. As in Section 11 the zeroth-order t-matrix is calculated with only the zeroth-order potential (3.3), fitting the scattering length to match that of the NN interaction in the \( ^1S_0 \) channel. The results are plotted in Figure 12. Again the behavior is very similar to the case where the pion was integrated out. The inclusion of the pion has not decreased the size of this quantity significantly, because the “short-range” square well is itself of roughly one-pion range.

![Figure 11](image-url)
IV. A LOWER-BOUND ON THE RANGE OF THE SHORT-RANGE INTERACTION

In the previous section we saw that the radius of the square well potential could not be reduced below 1.4 fm if the scattering length and effective range generated by the sum of a short-range and one-pion exchange potential were to agree with the experimental results. In fact, there exists a lower bound on this radius which is independent of the choice of regulating potential. In this section we extend the arguments of Ref. [12] in order to show that, if the theory is to fit the scattering length and effective range, then there is a general absolute lower bound on the range of the non-one-pion-exchange piece of the interaction.

Consider the radial Schrödinger equation for S-wave scattering in the case where both a one-pion exchange potential and an arbitrary (possibly non-local) short-range potential, $V_R$ are present:

$$-\frac{1}{M} \frac{d^2 u_E(r)}{dr^2} + \int dr' V_R(r, r') u_E(r') - \alpha \frac{e^{-m+r}}{r} u_E(r) = E u_E(r),$$

with $u_E(0) = 0$. Here $V_R(r, r') = 0$ for $r > R$ or $r' > R$. This, of course, was the case discussed in the previous section for certain specific $V_R$s. However, in this section, instead of enquiring as to the exact nature of $V_R$, we now consider a solution, $v_E(r)$ of the equation

$$-\frac{1}{M} \frac{d^2 v_E(r)}{dr^2} - \alpha \frac{e^{-m+r}}{r} v_E(r) = E v_E(r).$$

$v_E$ is chosen so as to match onto the asymptotic wave function, $u_E(r) = \sin(kr + \delta_E)$, with $k = \sqrt{ME}$ and $\delta_E$ the experimental phase shift, as $r \to \infty$, and is normalized so that $v_E(0) = 1$. Suppose now that $\tilde{u}_E(r)$ is a solution of Eq. (1.1), with the parameters of $V_R$ adjusted so that $\tilde{u}_E$ has the experimentally observed asymptotic behavior. Suppose also that $\tilde{u}_E$ is normalized so that it agrees with $v_E(r)$ at $r = R$. Given this normalization the two wave functions agree on $[R, \infty)$. However, they differ on $[0, R]$, in that $\tilde{u}_E(0) = 0$, while $v_E(0) = 1$.

Now, going through the arguments displayed in [12] yields:

$$\frac{dv_2}{dr} \bigg|_{r=0} - \frac{dv_1}{dr} \bigg|_{r=0} = \left( k_2^2 - k_1^2 \right) \int_0^{\infty} dr \left[ v_2(r)v_1(r) - \tilde{u}_2(r)\tilde{u}_1(r) \right],$$

where $v_2$ and $v_1$ are obtained by solving (1.2), with the appropriate boundary conditions at $r = R$, at two different energies $E_2$ and $E_1$. Similarly, $\tilde{u}_2$ and $\tilde{u}_1$ are solutions to (1.1) at the same two energies. From Eq. (1.3) we find

$$\frac{d}{dE} \left( \frac{dv_E}{dr} \bigg|_{r=0} \right) = M \int_0^{\infty} dr \left[ v_2^2(r) - \tilde{u}_2^2(r) \right].$$

FIG. 12. A plot of the fractional difference between the zeroth- and second-order t-matrices versus momentum for several values of $R$. For an optimal well, with $R=2.3$ fm, this quantity is identically zero.
Since \( u_E(r) \) may be chosen to be real, and \( u_E \) and \( v_E \) agree for \( r \geq R \), it follows that
\[
\frac{d}{dE} \left( \frac{dv_E}{dr} \right)_{r=0} \leq M \int_0^R dr \ v_E^2(r). \tag{4.5}
\]

Once the asymptotic behavior of \( v_E \) is specified the function \( v_E(r) \) is independent of \( R \), depending only on the experimental phase shift \( \delta_E \). By contrast, the right-hand side is a function of \( R \), but only through the integral’s upper bound. Thus, if \( \frac{d}{dE} \left( \frac{dv_E}{dr} \right)_{r=0} \) is positive, then as \( R \) is decreased towards zero a value of \( R \) will be reached for which (4.5) will be violated.

In the case where there is no one-pion exchange interaction the wavefunction \( v_E(r) \) is
\[
v_E(r) = \frac{\sin(kr + \delta_E)}{\sin(\delta_E)}, \tag{4.6}
\]
and so, Eq. (4.5) becomes:
\[
\frac{d}{dE} (k \cot \delta_E) \leq 0, \tag{4.7}
\]
as claimed in the Introduction. As mentioned in Section [I], this may be derived from an old result of Wigner [15].

In the case where one-pion exchange is included the function \( v_E(r) \) may be calculated numerically given experimental phase shift data. When this is done using the experimental values for \( a \) and \( r_e \) given in Eq. (2.12) the result:
\[
\frac{d}{dE} \left( \frac{dv_E}{dr} \right)_{r=0} \bigg|_{E=0} = 10.0, \tag{4.8}
\]
accurate to two significant figures, is obtained. (In practice this is best done by fixing the logarithmic derivative at some small, but finite, distance, integrating out to large distances and matching to the experimental phase shifts, and only then integrating in to the origin.) We may now check whether Eq. (4.5) is violated at \( E = 0 \). In Fig. 13 we plot the right-hand side of Eq. (4.5), taken at \( E = 0 \),
\[
f(R) \equiv M \int_0^R dr \ v_E^2(r), \tag{4.9}
\]
as a function of \( R \). The constraint (4.5) is violated once \( R < 1.1 \text{ fm} \). It follows that any short-range potential of range less than 1.1 fm which is used in the Schrödinger equation (4.1) will not be able to fit the experimental scattering length and effective range.

FIG. 13. A comparison of the two sides of Eq. (4.5), evaluated at \( E = 0 \) for NN scattering in the \(^1S_0\) channel. Observe that the function \( f(R) \) is less than the numerical bound (4.8) for radii less than 1.1 fm.
V. CONCLUSIONS AND DISCUSSION

In this paper we have shown how a finite-range potential can parameterize the short-range physics in an effective field theory (EFT) approach to the nucleon-nucleon interaction. We have chosen a square-well potential since it allows a simple and clear analysis using elementary quantum mechanics. Other forms of potentials could be used and should have little effect on the results. We have considered both the case where all exchanged particles are integrated out, and the case where a one-pion exchange potential is retained.

We have done EFT calculations at both zeroth and second order. In doing so we have adopted the approach of Weinberg, and done the power counting in our EFT in the potential, rather than in the amplitude. The zeroth-order calculation then involves a potential which is the sum of a square well and one-pion exchange (if explicit pions are present), while our second-order EFT calculation adds derivatives of the regulated contact interaction. (It should be noted that our “second-order” EFT calculation in the theory with pions ignores some two-pion exchange graphs which are, strictly speaking, of the same EFT order as graphs we have included in our potential. However, we believe the inclusion of these graphs in the calculation will not qualitatively alter our conclusions.) For a given regulator parameter the coefficients in the short-range interaction are fitted to the $^1S_0$ scattering length and effective range.

An optimal choice for the well radius was found in both the theory with explicit pions and that without. At that well radius the second-order piece of the potential is identically zero and yet the $^1S_0$ scattering length and effective range are reproduced. When one attempts to renormalize the coefficients in the EFT expansion so as to fit the experimental data at other well radii, there turns out to be a lower bound on the well radii for which this can be done successfully. This suggests that this method of regulating the theory does not allow one to reach the limit of truly zero-range interactions. However, it should be noted that other, non-equivalent, regulation schemes may allow one to define a Schrödinger equation containing contact interactions.

For potentials with radii close to the optimal value the second-order corrections to the $t$-matrix are small, and the phase shifts are similar to those produced by the optimal well. This lack of sensitivity to the regulator parameter indicates that it might be possible to develop a systematic power-counting scheme along the lines of the calculations performed in this paper. However, our results show that such a power-counting scheme can only hope to be successful for momenta up to about 100–150 MeV. This is a range of validity which excludes many interesting nuclear physics applications. It is somewhat surprising that the cutoff cannot be reduced beyond 1.4 fm for the case of a square-well regulator. Indeed, the results of Section [IV] show that, no matter what regulator is used, it must extend beyond $r = 1.1$ fm. This means that the physics which, together with one-pion exchange, explains the effective range in the $^1S_0$ channel, is not particularly short-ranged.

In the case where the OPE potential was included, we found that the optimal well fits the experimental phase shifts well. However, if other regulator parameters are chosen the data is not fit particularly successfully beyond about 0.5 fm$^{-1}$.

There are a number of points to observe in comparing the approach of this paper with that recently advocated by Kaplan et al. [4]. First, we note that the regulation of the delta function interactions used in that work was completely different to that employed here. Whereas we have defined a contact interaction as the limit of a sequence of square wells of decreasing radii, Kaplan et al. have calculated sets of loop diagrams using true contact interactions and then renormalized the resulting infinities using dimensional regularization and the $\overline{\text{MS}}$ scheme. The two approaches to regulating the delta function potential are not equivalent. Specifically, in our approach the range of the short-range potential cannot be taken to zero if the renormalization conditions on its coefficients are to be satisfied.

Second, upon keeping the short-range potential of finite range, we see that using the Weinberg approach to power counting allows the explanation of both the scattering length and the effective range in an EFT which is valid to momenta $k \approx 0.5$ fm$^{-1}$. This is in contrast to the results of Kaplan et al. who found that, given their method of regulation, if the scattering length and effective range were to be explained in a Weinberg power-counting approach then the domain of validity of the resulting EFT was very small. It was this that led Kaplan et al. to define a new power-counting scheme in which power counting was employed for the inverse scattering amplitude. With the form of regulation used in this paper power counting may always be applied to the potential without a poor radius of convergence for the EFT resulting. Of course, our effective field theory expansion is open to question since our regulator was never, and indeed can never be, removed from the theory. However, we have demonstrated that when the regulator parameter is kept finite and within sensible bounds, there is not great sensitivity to it for processes involving momenta up to about 100 MeV. A formal power-counting scheme for an EFT approach where the range of the short-range interaction is always kept finite remains to be worked out.

Finally, from the standpoint of correctly reproducing the experimental data, in the case without pion exchange the approach of Ref. [4] (which is identical to effective range theory) does considerably better than our “optimal well”. When the pion is included, the two approaches appear to yield nearly identical results, although the lack of an a priori reason for choosing the optimal well over wells of other radii means that we cannot unambiguously say that
the phase shifts predicted by our approach are those shown in Fig. 11. Above 100 MeV the predictions from potentials in which different regulator parameters were used differ considerably. This is a troubling result, and casts doubt on the efficacy of the approach discussed here. However, strictly speaking, the momenta to which the EFT of Kaplan et al. applies is of roughly this size, although a comparison of their results with experimental data indicates that good agreement is obtained to much higher momentum than 100 MeV. Thus, it remains to be seen if either approach can be successfully used in describing other nuclear physics problems.

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