ON DISTRIBUTION OF THE DEPTH INDEX ON PERFECT MATCHINGS

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Abstract. We study the restriction of depth index statistic on the set of perfect matchings. In particular, we provide additional combinatorial description of the statistic for perfect matchings and calculate the generating polynomial. The main result of the present short paper is that the depth index on perfect matchings is equidistributed with the rank function of the Bruhat order.

1. Introduction

This paper studies the distributions of the depth index statistics on the set of perfect matchings in $PM_{2n}$. The depth-index of a set partition, was introduced and studied in [2] by the first author together with Can. In [3] together with Rubey they established connection to another statistic of set partitions and provided a combinatorial interpretation of these two statistics. As was shown in [2], the depth index statistic is equal to the rank function of a certain graded EL-shellable order on $\Pi_n$, the set partitions $[n]$. That order is isomorphic to the Bruhat-Chevalley-Renner order on the upper-triangular matrices, and thus, the depth index appears to be equal to the dimension of a certain matrix variety. The present paper studies the restriction of the depth index to the set of partitions with blocks of size 2, which can be interpreted as perfect matchings on the set $[2n]$.

The set of perfect matchings is a classical combinatorial object that was studied in various contexts. Perfect matchings, when viewed as fixed-point free involutions on $[2n]$ have a natural poset structure – namely, the restriction of the strong Bruhat order on $S_{2n}$. This poset was studied in detail in [1]. It was shown that the length function (let us denote it $\ell$) of Bruhat order on $PM_{2n}$ is $[2n - 1]_q!!$, and it equals to the length function of the poset that was studied by Deodhar and Srinivasan in [4].

It can be easily checked (for example, for the case $n = 3$), that the classical strong Bruhat order on $PM_{2n}$ and the order on $PM_{2n}$ induced from the order on set partitions studied
in [2], are different. However, we prove in this paper that their generating polynomials satisfy the following identity.

\[ \sum_{\pi \in PM_{2n}} q^{l(\pi)} = q^{\frac{(n+1)}{2}}[2n - 1]q!! \]

2. Background

Definition 2.1. A collection of subsets \( \{S_i\}_{i \in I} \) of a set \( S \) is said to be a set partition of \( S \) if the sets \( S_i \) are mutually disjoint and we have \( \bigcup_{i \in I} S_i = S \). The sets \( S_i \) are called the blocks of the partition. For \( n > 0 \), the set of all set partitions of \( [n] \) is denoted by \( \Pi_n \).

We will often drop set parentheses and commas and just put vertical bars between blocks. If \( B_1, \ldots, B_k \) are the blocks of the set partition \( \pi \in \Pi_n \), then the standard form of \( \pi \) is defined as \( B_1|\ldots|B_k \) where we assume that \( \min B_i < \min B_{i+1} \) for every \( 1 \leq i \leq k \), and the elements of every block are listed in increasing order. For example, \( \pi = 1378|26|45 \) is a set partition in \( \Pi_8 \).

Definition 2.2. Let \( \pi \in \Pi_n \) be a set partition. The minimal elements of the blocks are the openers, and the maximal elements of are the closers of \( \pi \).

Example 2.3. The partition \( \pi = 1378|26|45 \) in \( \Pi_8 \) has openers 1, 4, 7 and closers 5, 6, 8.

Definition 2.4. Let \( \pi \in \Pi_n \) be a set partition with the standard form \( B_1|\ldots|B_k \). The arc diagram of \( \pi \) is obtained by placing labels 1, \ldots, \( n \) on horizontal line and connecting consecutive elements of each block by arcs, such the arcs do not cross each other if it is possible to draw them without a crossing as in Figure 2.1.

Definition 2.5. Let \( \pi \in \Pi_n \) be a set partition of \( [n] \). The extended arc diagram of \( \pi \) is obtained from the arc diagram of \( \pi \) by adding a half-arc \( (-\infty, i) \) from far left to each opener \( i \) and a half-arc \( (j, \infty) \) from each closer \( j \) to far right. These arcs are drawn in such a way that the half-arcs to the left do not cross, and half-arcs to the right do not cross either. See Figure 2.2 for an example.

Remark 2.6. The correspondence between set partitions of \( \pi \) and their arc diagrams is a bijection, and in the sequel we will identify the partition \( \pi \) of \( [n] \) with its arc diagram \( A(\pi) \).
The intertwining number was introduced in [5]. Using the extended diagram representations of set partitions, it can be defined equivalently as in [3].

Definition 2.7. Let $A \in \Pi_n$ be a set partition of $[n]$. Two (generalized) arcs $(i, j)$ and $(k, l)$ in the extended arc diagram of $A$ are said to cross in $\pi$ if $i < k < j < l$. The number of crossings in the extended arc diagram of $A$ is called the intertwining number of $A$ and denoted $i(A)$. (See [3] for further details and references).

A closely related statistic to intertwining number is the depth-index, defined as follows.

Definition 2.8. Let $A \in \Pi_n$ be a set partition of $[n]$. Denote by $\text{Arcs}(A)$ the set of arcs of $A$. For every $1 \leq v \leq n$, the depth of vertex $v$, denoted by $\text{depth}(v)$ is the number of arcs $(i, j) \in \text{Arcs}(A)$ with $1 \leq i < v < j \leq n$. For every $\alpha = (u, v) \in \text{Arcs}(A)$, the depth of the arc $\alpha$ is the number of arcs $(i, j) \in \text{Arcs}(A)$ with $1 \leq i < u < v < j \leq n$.

The depth index $t(A)$ of $A$ is

$$t(A) = \sum_{i=1}^{\lfloor \text{Arcs}(A) \rfloor} (n - i) - \sum_{v=1}^{n} \text{depth}(v) + \sum_{\alpha \in \text{Arcs}(A)} \text{depth}(\alpha).$$

Remark 2.9. Intuitively, the depth of a vertex or an arc is the number of arcs above it in the arc diagram.

Intertwining number and depth-index satisfy the following identity.

Theorem 2.10. [3, Theorem 1] For any set partition $A \in \Pi_n$, we have

$$t(A) + i(A) = \binom{n}{2}.$$

We are interested in the distribution of the depth-index on perfect matchings on $[2n]$.

Example 2.11. For partition $\pi = \{3, 7\} \{2\} \{6\} \{4\}$ as in Figure 2.2, $i(\pi)$ is the number of crossings in the extended arc diagram, thus we have $i(\pi) = 8$. The number of arcs in $\pi$ is 5, the only arc of nonzero depth is $(4, 5)$ with depth($(4, 5)) = 2$, and we have

- $\text{depth}(2) = \text{depth}(3) = \text{depth}(6) = 1$,
- $\text{depth}(4) = \text{depth}(5) = 2$,
- $\text{depth}(1) = \text{depth}(8) = 0$. 

![Figure 2.2. The extended arc diagram of the set partition $\pi = \{3, 7\} \{2\} \{6\} \{4\}$.](image)
Hence, by definition we have
\[ t(\pi) = \sum_{i=1}^{8} (8 - i) - \sum_{v=1}^{8} \text{depth}(v) + \sum_{\alpha \in \text{Arcs}(\pi)} \text{depth}(\alpha) = (7 + 6 + 5 + 4 + 3) - (1 + 2 + 1 + 1) + 2 = 20. \]

It is easily verified that \( i(\pi) + t(\pi) = 28 = \binom{8}{2} \).

**Definition 2.12.** A perfect matching on \([2n]\) is a set partition of \([2n]\) in which every block has size 2, or equivalently an arc diagram in which every vertex touches exactly one arc. The set of perfect matchings on \([2n]\) will be denoted by \( PM_{2n} \).

**Remark 2.13.** By interpreting the arcs in the arc diagram of a perfect matching \( \pi \in PM_{2n} \) as transpositions in \( S_{2n} \), we obtain a fixed point free involution in \( S_{2n} \). In fact, this is a bijection from perfect matchings to fixed point free involutions in \( S_{2n} \). In the rest of the text, every perfect matching \( \pi \in PM_{2n} \) will be identified with its corresponding fixed point free involution.

We proceed to describe this function in terms of \([4]\).

**Definition 2.14.** Let \( \pi \in PM_{2n} \) be a perfect matching on \([2n]\). The crossing number \( c(\pi) \) of \( \pi \) is the number of arcs \((i,j),(k,l) \in \text{Arcs}(\pi)\) that cross. The span of an arc \( \alpha = (i,j) \in \text{Arcs}(\pi) \) is \( \text{span}(\alpha) = j - i - 1 \), which is the number of dots below the arc.

**Theorem 2.15.** \([4]\) Theorem 1.3
1. For every \( \pi \in PM_{2n} \),
\[ \ell(\pi) = \sum_{\alpha \in \text{Arcs}(\pi)} \text{span}(\alpha) - c(\pi) \]
2. The generating polynomial of \( \ell \) in \( PM_{2n} \) is
\[ L_{PM_{2n}}(q) = \sum_{\pi \in PM_{2n}} q^{\ell(\pi)} = [2n - 1]_{q!!}. \]

Recall, that \([2n - 1]_{q!!}\) is the \( q \)-analogue of the double factorial, that is defined as follows
\[ [2n - 1]_{q!!} = \prod_{i=1}^{n} \frac{1 - q^{2i-1}}{1 - q}. \]

**3. Main result**

We prove the following relation between the depth-index and the length function of Bruhat-Chevalley order on fixed point free involutions.
Theorem 3.1. The depth index on $PM_{2n}$ is distributed as follows:

$$T_{PM_{2n}}(q) = \sum_{\pi \in PM_{2n}} q^{e(\pi)} = q^{(n+1)}[2n-1]!!.$$ 

We examine the length function and depth index restricted to perfect matchings first. The definitions that follow are valid for arcs in the regular arc diagrams and not extended ones.

Definition 3.2. Let $\pi \in PM_{2n}$. We say that two arcs $e = (i,j), f = (k,l) \in Arcs(\pi)$ form:

(i) a crossing with $e$ as initial edge if $i < k < j < l$;

(ii) a nesting with $e$ as initial edge if $i < k < l < j$;

(iii) a alignment with $e$ as initial edge if $i < j < k < l$.

The number of crossings, nestings and alignments will be denoted by $cr, ne$ and $al$ respectively.

Observation 3.3. For every perfect matching $\pi$ in $PM_{2n}$ there are exactly $\binom{n}{2}$ pairs $(e, f) \in Arcs(\pi) \times Arcs(\pi)$ with $e = (i, j), f = (k, l)$ and $i < k$. Therefore, we have

$$cr(\pi) + ne(\pi) + al(\pi) = \binom{n}{2}$$

for every $\pi \in PM_{2n}$.

By changing the summation order, we obtain the following identities:

Observation 3.4. For every $\pi \in PM_{2n}$, we have:

1. $\sum_{v=1}^{2n} depth(v) = \sum_{\alpha \in Arcs(\pi)} span(\alpha)$;
2. ne(\pi) = \sum_{\alpha \in Arcs(\pi)} depth(\alpha)$.

Definition 3.5. Let $\pi \in PM_{2n}$. The sum $\sum_{v=1}^{2n} depth(v) = \sum_{\alpha \in Arcs(\pi)} span(\alpha)$ is called the total vertex depth of $\pi$ and denoted by $tvd(\pi)$.

By combining Observations 3.3 and 3.4 we obtain the following equalities for the depth index.
**Observation 3.6.** For every $\pi \in PM_{2n}$ we have the following equality:

$$t(\pi) = \sum_{i=1}^{\lfloor \text{Arcs}(\pi) \rfloor} (n - i) - \text{tvd}(\pi) + \text{ne}(\pi)$$

$$= n^2 + \binom{n}{2} - \text{tvd}(\pi) + \binom{n}{2} - \text{cr}(\pi) - \text{al}(\pi)$$

$$= n^2 + 2\binom{n}{2} - \text{tvd}(\pi) - \text{cr}(\pi) - \text{al}(\pi).$$

We are interested in expressing the term $\text{tvd}(\pi)$ using $\text{cr}(\pi), \text{ne}(\pi)$ and $\text{al}(\pi)$.

Recall the definition of the intertwining number (Definition 2.7) - the number of crossings in the extended arc diagram.

**Lemma 3.7.** Let $\pi \in PM_{2n}$. Then $i(\pi) = 3\text{cr}(\pi) + 2\text{ne}(\pi) + \text{al}(\pi)$.

**Proof.** Let $e = (i,j)$ and $f = (k,l)$ with $i < k$. If the pair $(e,f)$ is a nesting - it adds exactly two crossings to the extended arc diagram: $(-\infty, k)$ with $(i,j)$ and $(i,j)$ with $(l, \infty)$. If the pair is a crossing, it adds 3 crossings in the extended arc diagram: $e$ with $f$, $(-\infty, k)$ with $e$, and $f$ with $(j, \infty)$. Finally, if $(e,f)$ is an alignment, we get a single crossing $(j, \infty)$ with $(k, -\infty)$. By summing everything, we get the desired result. \(\square\)

By using Theorem 2.10 one can express $\text{tvd}(\pi)$ in terms of $\text{ne}(\pi), \text{al}(\pi)$ and $\text{cr}(\pi)$.

**Lemma 3.8.** For every $\pi \in PM_{2n}$ we have:

$$\text{tvd}(\pi) = 2\binom{n}{2} - 2\text{al}(\pi) = 2(\text{cr}(\pi) + \text{ne}(\pi)).$$

**Proof.** By Theorem 2.10, for $\pi \in PM_{2n}$, and using Observation 3.6 and Lemma 3.7 we obtain

$$\binom{2n}{2} = t(\pi) + i(\pi)$$

$$= n^2 + 2\binom{n}{2} - \text{tvd}(\pi) - \text{cr}(\pi) - \text{al}(\pi) + 3\text{cr}(\pi) + 2\text{ne}(\pi) + \text{al}(\pi)$$

$$= n^2 + 2\binom{n}{2} - \text{tvd}(\pi) + 2\text{cr}(\pi) + 2\text{ne}(\pi).$$

By rearranging the equation we obtain:

$$\text{tvd}(\pi) = n^2 + 2\binom{n}{2} - \binom{2n}{2} + 2\text{cr}(\pi) + 2\text{ne}(\pi)$$

$$= 2\text{cr}(\pi) + 2\text{ne}(\pi)$$

$$= 2\binom{n}{2} - 2\text{al}(\pi),$$

as desired. \(\square\)
We can now express $\ell(\pi)$ and $t(\pi)$ using $ne(\pi)$ and $cr(\pi)$.

**Corollary 3.9.** For every $\pi \in PM_{2n}$ we have:

$$\ell(\pi) = cr(\pi) + 2ne(\pi)$$

and

$$t(\pi) = n^2 + \binom{n}{2} - 2cr(\pi) - ne(\pi).$$

**Proof.** Substitute $tvd(\pi)$ with $2ne(\pi) + cr(\pi)$ in the expressions $\ell(\pi) = tvd(\pi) - cr(\pi)$ and $t(\pi) = n^2 + \binom{n}{2} - tvd(\pi) + ne(\pi)$. ☐

We are ready to prove the main theorem. In order to do it, we rely on the following facts.

**Theorem 3.10** (Theorem 1.2). There exists an involution $\phi : PM_{2n} \to PM_{2n}$ that preserves the number of alignments and exchanges the number of crossings and nestings. In other words, for each $\pi \in PM_{2n}$ we have

$$al(\phi(\pi)) = al(\pi), cr(\phi(\pi)) = ne(\pi), ne(\phi(\pi)) = cr(\pi).$$

**Fact 3.11.** The polynomial $[2n - 1]_{q!!}$ has degree $n^2 - n$ and is palindromic. Namely, the coefficients of $q^r$ and $q^{n^2 - n - r}$ in $[2n - 1]_{q!!}$ are equal for each $0 \leq r \leq n^2 - n$.

**Corollary 3.12.** There exists a bijection $\psi : PM_{2n} \to PM_{2n}$ such that

$$\ell(\psi(\pi)) = n^2 - n - \ell(\pi)$$

for every $\pi \in PM_{2n}$.

**Proof.** By Theorem 2.15 the generating polynomial on $PM_{2n}$ of the length function is $[2n - 1]_{q!!}$. Together with Fact 3.11 it implies that for every $0 \leq r \leq n^2 - n$, the number of elements of $PM_{2n}$ with length $r$ is equal to the number of elements with length $n^2 - n - r$, as desired. ☐

We proceed to prove the main theorem.

**Proof of Theorem 3.1.** By Corollary 3.9 and Theorem 3.10 we have:

$$\ell(\phi(\pi)) = 2ne(\phi(\pi)) + cr(\phi(\pi)) = 2cr(\pi) + ne(\pi).$$

By applying Corollary 3.9 again we obtain:

$$t(\pi) + \ell(\phi(\pi)) = n^2 + \binom{n}{2}.$$

Using $\psi$ from Corollary 3.12 and the relation $\ell(\phi(\pi)) = n^2 - n - \ell(\pi)$ we obtain:

$$t(\pi) + n^2 - n - \ell(\psi(\phi(\pi))) = n^2 + \binom{n}{2}.$$
Finally, by rearranging the equation, we obtain \( t(\pi) = \binom{n+1}{2} + \ell(\psi(\phi(\pi))) \). Since, \( \psi \circ \phi \) is a bijection, the statistics \( t \) and \( \binom{n+1}{2} + \ell \) are equi-distributed, and the main theorem follows.

By using the identity \( i(\pi) + t(\pi) = \binom{2n}{2} \) for every \( \pi \in \Pi_{2n} \) we obtain a similar formula for the generating polynomial of the intertwining number.

We rely on the following observation.

**Fact 3.13.** If \( p(q) \) is a palindromic polynomial in \( q \) of degree \( m \), then
\[
q^mp\left(\frac{1}{q}\right) = p(q)
\]

**Corollary 3.14.** The generating polynomial of the intertwining number \( I_{PM_{2n}}(q) \) is
\[
I_{PM_{2n}}(q) = \sum_{\pi \in PM_{2n}} q^{i(\pi)} = q^{\binom{2n}{2}} [2n - 1]_q!!.
\]

**Proof.** By Theorem 2.10 we have
\[
\sum_{\pi \in PM_{2n}} q^{i(\pi)} = \sum_{\pi \in PM_{2n}} q^{\binom{2n}{2} - t(\pi)} = q^{\binom{2n}{2}} \sum_{\pi \in PM_{2n}} \frac{1}{q^{-t(\pi)}}.
\]

By applying Theorem 3.11 we obtain
\[
q^{\binom{2n}{2}} \sum_{\pi \in PM_{2n}} \frac{1}{q^{-t(\pi)}} = q^{\binom{2n}{2}} \frac{1}{q^{\binom{n+1}{2}}} [2n - 1]_q!!.
\]

By Facts 3.13 and Fact 3.11 \( q^{n^2 - n[2n - 1]_q!!} = [2n - 1]_q!! \), and therefore
\[
q^{\binom{2n}{2}} \frac{1}{q^{\binom{n+1}{2}}} [2n - 1]_q!! = q^{\binom{2n}{2}} \frac{1}{q^{\binom{n+1}{2}}} [2n - 1]_q!!
\]
\[= q^{\binom{2n}{2} - \binom{n+1}{2} - n^2 - n} [2n - 1]_q!! = q^{\binom{n}{2}} [2n - 1]_q!!,\]
as desired.

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