Advances in the studies of anomalous diffusion in velocity space

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Abstract

A generalized Fokker-Planck equation is derived to describe particle kinetics in specific situations when the probability transition function (PTF) has a long tail in momentum space. The equation is valid for an arbitrary value of the transferred in a collision act momentum and for the arbitrary mass ratio of the interacting particles. On the basis of the generalized Fokker-Planck equation anomalous diffusion in velocity space is considered for hard sphere model of particle interactions, Coulomb collisions and interactions typical for dusty plasmas. The example of dusty plasma interaction is peculiar in way that it leads to a new term in the obtained Fokker-Planck-like equation due to the dependence of the differential cross-section on the relative velocity. The theory is also applied to diffusion of heavy particles in the ambience of light particles with a prescribed power-type velocity distribution function. In general, the theory is applicable to consideration of anomalous diffusion in velocity space if the typical velocity of one sort of particles undergoing diffusion is small compared to the typical velocity of the background particles.

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Introduction

Interest to anomalous diffusion in coordinate space is explained by its great variety of applications, i.e., semiconductors, polymers, some granular systems, plasmas under specific conditions, various objects in biological systems, physical-chemical systems, and others.

Non-linear time dependence $< r^2(t) >$ of the mean-squared displacement has been experimentally observed, in particular, under essentially non-equilibrium conditions and in some disordered systems [1, 2]. As is well known there are two types of anomalous diffusion in coordinate space which are referred to as superdiffusion $< r^2(t) > \sim t^\alpha$ ($\alpha > 1$) and subdiffusion ($\alpha < 1$) [3]. For description of these two anomalous diffusion regimes a number of efficient models have been proposed. The continuous time random walk (CTRW) model by Scher and Montroll [4] for the process of subdiffusion is a basis for understanding photoconductivity in strongly disordered and glassy semiconductors. The Levy-flight superdiffusion model [5] describes such phenomena as self-diffusion in micelle systems [6] as well as reaction and transport in polymer systems [8]. Another application of the model is studying stochastic behavior of financial market indices [8]. The method
of so-called fractional differential equations in coordinate and time spaces has been successfully
developed to treat both cases of anomalous diffusion [9].

Recently a more general approach has been proposed in [10] which not only reproduces the
results of the standard fractional differentiation method (whenever it is applicable) but also allows
to tackle more complicated cases of anomalous diffusion. In [11], for example, this approach is used
for studying diffusion in a time-dependent external field.

Some aspects of anomalous diffusion in velocity space have been considered in papers [13]-[18].
However, compared to anomalous diffusion in coordinate space anomalous diffusion in velocity space
is still poorly studied but it is actively attracting more and more attention [19]-[21].

In this paper the problem of anomalous diffusion in momentum (velocity) space is considered
detail in the spirit of the approach proposed in [10] for the diffusion in coordinate space. Extra
terms (cp. [19]-[21]) of the generalized Fokker-Planck equation are found. The results are applied to
hard sphere model of collision, Coulomb systems and dusty plasma. In dusty plasma the additional
terms are numerically small but not negligible due to to velocity dependence of the cross-section.

The paper is organized as follows. Diffusion in velocity space for the cases of normal and
anomalous behavior of the PTF is presented in Section 1. Starting from the Boltzmann-type PTF
we arrive at explicit expansion PTF into a series which is applicable not only to the Boltzmann-type
processes but also to a wide class of processes with other types of PT-functions. The generalized
Fokker-Planck equation is presented in Section 2. Special cases of anomalous diffusion with the
specific power-type distribution function of ambient light particles are analysed in Section 3. In this
section the cases of hard sphere and Coulomb models of diffusion are considered. Subsection 3.3 is
devoted to the example of anomalous diffusion in dusty plasma where the differential cross-section
depends on velocity of the colliding particles. In this case the additional term in the generalized
kinetic equation is not negligible.

1 Diffusion in velocity space on the basis of the master-type
equation

Let us consider diffusion in velocity space using an approach based on the corresponding master
equation for the distribution function \( f(p, t) \) which describes the balance of particles coming at and
from the point \( p \) at the instant \( t \). The master equation written in velocity space reads (see, e.g.,
[22])

\[
\frac{df(p, t)}{dt} = \int dq \left\{ W(q, p + q)f(p + q, t) - W(q, p)f(p, t) \right\}.
\] (1)

The probability transition function \( W(q, p') \) describes the probability that a particle with momentum
\( p' \) passes from the point \( p' \) in the velocity phase space to the point \( p \) per unit time by transferring
momentum \( q = p' - p \) to the surrounding medium. Under the assumption that the characteristic
transferred momentum \( q \) is much smaller than \( p \) equation (1) can be expanded into a series with
respect to \( q \) to the second order. The expansion leads to the usual form of the Fokker-Planck
equation for the density distribution function \( f(p, t) \)
\[
\frac{df(p,t)}{dt} = \frac{\partial}{\partial p} \left[ A_\alpha(p) f(p,t) \right] + \frac{\partial}{\partial p} \left( B_{\alpha\beta}(p) f(p,t) \right),
\]

(2)

where
\[
A_\alpha(p) = \int q_\alpha W(q,p) dq,
\]

(3)

\[
B_{\alpha\beta}(p) = \frac{1}{2} \int q_\alpha q_\beta W(q,p) dq.
\]

(4)

The coefficients \(A_\alpha\) and \(B_{\alpha\beta}\) are responsible for the processes of friction and diffusion, respectively. The subscripts \(\alpha\) and \(\beta\) correspond to the coordinate axes.

For the Boltzmann elastic collisions the PTF-function \(W(q,p)\) has been found in [23, 24] and can be represented in the form [10]
\[
W(p,q) = \frac{1}{\mu^2} \int du \delta \left( u \cdot q + \frac{q^2}{2\mu} \right)
\]

\[
\times \frac{d\sigma}{d\Omega} \left[ \arccos \left( 1 - \frac{q^2}{2\mu^2 u^2} \right), u \right] f_b(u + v),
\]

(5)

Here \(p = Mv\) is the momentum of a scattered particle, \(\mu = Mm/(m + M)\) is reduced mass and \(f_b\) is the distribution function for the background scattering particles. The relative collision velocity is denoted with \(u\) and \(d\sigma(\chi,u)/d\Omega\) is the scattering differential cross-section calculated in the coordinate system in which a scattering particles with mass \(m\) is fixed throughout the whole act of collision [25, 26]. The scattering angle \(\chi = \arccos[1 - q^2/(2\mu^2 u^2)]\) stands for the rotation of the relative velocity \(u\). Assuming that the vector \(q\) is directed along the axis \(z\) and performing integration with respect to the projection of \(u\) on the axis \(z\) one arrives at the more explicit form of the equation for the PTF-function
\[
W(q,p) = \frac{\pi}{\mu^2 q} \int du \frac{d\sigma}{d\Omega} \left[ \arccos(1 - \frac{q^2}{2\mu^2 u^2}), u \right] f_b(u + v),
\]

(6)

where \(u_z = -q/2\mu\), \(u_\perp\) and \(v_\perp\) mean projections of the vectors \(u\) and \(v\), respectively, on the plane \(xy\) perpendicular to the axis \(z\).

For the scattering cross-sections independent on \(u\) (e.g., for hard sphere or Coulomb interactions) equation (6) is reduced to the following equation by changing the variable of integration \(u = u_\perp + v_\perp\)
\[
W(q,p) = \frac{\pi}{\mu^2 q} \int du \frac{d\sigma}{d\Omega} \left[ \arccos \left( 1 - \frac{q^2}{4\mu^2 u^2} \right), \frac{q \cdot v}{\mu} \right] f_b(u + v),
\]

(7)

Now let us consider the case when the characteristic velocity of particles diffusing in velocity space is small compared to the characteristic velocities in the surrounding medium (e.g., [22]).
For the Boltzmann-type collisions it means that there are two types of particles – heavy particles undergoing diffusion with the characteristic velocity \( v \sim p/M \) and light particles characterized by velocity \( u \) and \( u \gg v \). In the zero order approximation with respect to \( p \) when we omit any dependence of the PTF on \( p \) (i.e., \( W(q, p) = W(q, p = 0) = W(q) \)) the coefficients \( A_\alpha \) and \( B_{\alpha\beta} \) yield the equations

\[
A_\alpha = 0, \\
B_{\alpha\beta} = \delta_{\alpha\beta} B_{\alpha} = \delta_{\alpha\beta} \frac{1}{6} \int q^2 W(q) dq,
\]

where \( \delta_{\alpha\beta} \) is the Kronecker delta. The neglect of \( p \)-dependence is not correct. Usually the coefficient \( A_\alpha \) for the Fokker-Planck equation is found assuming that the stationary distribution function is Maxwellian. This assumption leads to the standard relation for the coefficients \( MT A_\alpha(p) = p_\alpha B \) where \( M \) is mass of the diffusing particle and \( T \) is temperature of the particles in equilibrium. The relation is analogous to the Einstein relation in coordinate space.

For the systems far from equilibrium, for example, for slowly decreasing PTF, the approach above is not applicable. In this case rigorous calculation of the coefficients \( A_\alpha \) and \( B_{\alpha\beta} \) requires the next order of approximation with respect to the smallness of \( p \) [19]. As the function \( W(q, p) \) is a scalar it can be regarded as a function depending only on variables \( q, q \cdot p, p^2 \). We imply that the PTF is such a function of \( q \cdot p \) and \( p^2 \) that it can be analytically expanded into a series in the vicinity of the point \( q \cdot p = 0, p^2 = 0 \). Note that in the expansion of the PTF we leave only those terms which have \( p \) to the power no higher than 2. Expansion of \( W(q, p) \) into a series to the second order is given by

\[
W(q, p) \simeq W(q) + \tilde{W}'(q)(q \cdot p) + \frac{1}{2} \tilde{W}''(q)(q \cdot p)^2 + \tilde{W}'(q)p^2,
\]

where

\[
\tilde{W}'(q) = \frac{\partial W(q, q \cdot p, p^2)}{\partial (q \cdot p)} \bigg|_{q \cdot p = 0, p^2 = 0}, \\
\tilde{W}''(q) = \frac{\partial^2 W(q, q \cdot p, p^2)}{\partial (q \cdot p)^2} \bigg|_{q \cdot p = 0, p^2 = 0}, \\
\tilde{W}'(q) = \frac{\partial W(q, q \cdot p, p^2)}{\partial p^2} \bigg|_{q \cdot p = 0, p^2 = 0}.
\]

Basically, the developed approximation is true if the typical velocity of one sort of particles undergoing diffusion is small compared to the typical velocity of the background particles. Let us note that for the differential cross-sections independent on the relative velocity \( u \) the coefficient \( \tilde{W}'(q) \equiv 0 \) as it can be seen from equation (7). In general, this term is not negligible and it should be taken into account (it was omitted in [19, 21]).
Then with the necessary accuracy we find the coefficient $A_\alpha$

$$A_\alpha(p) = \int q_\alpha q_\beta p_\beta \tilde{W}'(q) dq$$

$$= p_\alpha \int q_\alpha q_\alpha \tilde{W}'(q) dq$$

$$= \frac{p_\alpha}{3} \int q^2 \tilde{W}'(q) dq.$$  

(11)

Remarkably, if the equality $\tilde{W}'(q) = \frac{mW(q)}{(2M\mu T)}$ is fulfilled for the function $W(q,p)$ the following relation for the coefficients $A_\alpha$ and $B$ takes place

$$\frac{\mu}{m}MA_\alpha(p) = p_\alpha B$$  

(12)

which turns into the usual Einstein relation in the limit $m \ll M$

$$MTA_\alpha(p) = p_\alpha B.$$  

(13)

Straightforward differentiation of $W(p,q)$ in Eq. (5) with respect to the variable $p \cdot q$ which is assumed independent in the case of the equilibrium Maxwellian distribution leads to equality (12) and we arrive at the ordinary Fokker-Planck equation in velocity space with the constant diffusion coefficient $D = B/M^2$ and the constant friction coefficient $\beta = mB/\mu T = D(m + M)/T$ which satisfy the Einstein relation in the limit $m \ll M$.

For systems close to equilibrium the PT-function is calculated and the appropriate Fokker-Planck equation is discussed in detail in review [23]. In case of hard-sphere interactions and the equilibrium Maxwellian distribution $f_b$ equation (5) is reduced to the equation which is consistent with the result given in papers [23, 24]. However, even for quasi-equilibrium regimes when long tails of the PTF are absent the consideration in [23] is restricted to hard sphere interactions. Non-equilibrium forms of the PTF and changes in the structure of the Fokker-Planck equation for such systems are considered in [19, 20]. In the following section this consideration is extended. It brings up the importance of of the additional term in the expansion of the PTF with respect to the variable $p^2$ (see Eq. (9)). The chapter also gives correct coefficients in the generalised Fokker-Planck equation.

2 Generalized Fokker-Planck equation

In some non-equilibrium (stationary or non-stationary) systems PTF can have a long tail as a function of $q$. For such systems the ordinary Fokker-Planck equation (2) is not valid as the kinetic coefficients diverge at large values of $q$.

Now let us refer to the expansion of the PTF given by equation (9) making no assumptions as for the dependence of the differential cross-section on the relative velocity $u$. It results in an additional (cp. [19, 20]) term proportional to the first derivative of the PTF with respect to $p^2$. Substituting this expansion into Eq. (1) and making use of the Fourier transformation we arrive at
the following equation

\[
\frac{df(s)}{dt} = \int W(q) \left[ f(p+q) - f(p) \right] e^{ip\cdot s}(2\pi)^3 dp dq
\]

\[
+ \int \tilde{W}'(q) \left[ (q\cdot(p+q)) f(p+q) - (q\cdot p) f(p) \right] e^{ip\cdot s}(2\pi)^3 dp dq
\]

\[
+ \int \tilde{W}''(q) \left[ (q\cdot(p+q))^2 f(p+q) - (q\cdot p)^2 f(p) \right] e^{ip\cdot s}(2\pi)^3 dp dq.
\]

(14)

where \(\hat{f}(s)\) is the Fourier image of the function \(f(p)\). We omitted the arguments \(t\) and \(q\) of the distribution function to make the notation brief. Introducing the coefficients

\[
A(s) = \int \left[ e^{-i(q\cdot s)} - 1 \right] W(q) dq,
\]

\[
B_\alpha(s) = -\frac{i}{s^2} \int (q\cdot s) \left[ e^{-i(q\cdot s)} - 1 \right] \tilde{W}'(q) dq
\]

\[
C_{\alpha\beta}(s) = -\frac{1}{2} \int q_\alpha q_\beta \left[ e^{-i(q\cdot s)} - 1 \right] \tilde{W}''(q) dq,
\]

\[
E(s) = -\int \left[ e^{-i(q\cdot s)} - 1 \right] \tilde{W}'(q) dq
\]

we come to the Fokker-Planck-like equation

\[
\frac{d\hat{f}(s)}{dt} = A(s)\hat{f}(s) + B_\alpha(s) \frac{\partial \hat{f}(s)}{\partial s_\alpha}
\]

\[
+ C_{\alpha\beta}(s) \frac{\partial^2 \hat{f}(s)}{\partial s_\alpha \partial s_\beta} + E(s) \Delta \hat{f}(s).
\]

(16)

After a number of calculations it can be shown that

\[
A(s) = A(s) = 4\pi \int_0^\infty q^2 \left[ \sin(qs) - \frac{qs}{qs} - 1 \right] W(q) dq,
\]

(17)

\[
B_\alpha(s) = s_\alpha B(s),
\]

\[
B(s) = \frac{4\pi}{s^2} \int_0^\infty q^2 \left[ \cos(qs) - \frac{\sin(qs)}{qs} \right] \tilde{W}'(q) dq,
\]

(18)
\[ C_{\alpha \beta}(s) = s_{\alpha \beta}C_1(s) + s^2 \delta_{\alpha \beta}C_2(s), \]

\[ C_1(s) = \frac{12\pi}{s^2} \int_0^\infty q^4 \left[ \frac{\sin(qs)}{3qs} + \frac{\cos(qs)}{q^3 s^2} - \frac{\sin(qs)}{q^5 s^4} \right] \tilde{W}''(q)dq, \]

\[ C_2(s) = \frac{4\pi}{s^2} \int_0^\infty q^4 \left[ -\frac{1}{3} - \frac{\cos(qs)}{q^3 s^2} + \frac{\sin(qs)}{q^5 s^4} \right] \tilde{W}''(q)dq, \]

\[ E(s) = s^2 C_3(s), \]

\[ C_3(s) = \frac{4\pi}{s^2} \int_0^\infty q^2 \left[ 1 - \frac{\sin(qs)}{qs} \right] W'(q)dq. \]

For an isotropic function \( f(s) \equiv f(s) \) the Fokker-Planck-like equation (19) is reduced to equation

\[
\frac{df(s)}{dt} = A(s) \hat{f}(s) + [B(s) + 2C_2(s) + 2C_3(s)] s \frac{\partial \hat{f}(s)}{\partial s} + [C_1(s) + C_2(s) + C_3(s)] \frac{\partial^2 \hat{f}(s)}{\partial s^2}.
\]

This equation is a generalisation of the usual Fokker-Planck equation which implies certain smallness of the transferred momentum \( q \). The advantage of our equation is that it is valid for an arbitrary value of \( q \). This virtue lets us go far beyond the scope of the phenomena described by the ordinary Fokker-Planck equation.

However, it is instructive to match Eq. (24) and the usual Fokker-Planck equation. The latter tackles the problem of diffusion of heavy particle in a gas of light particles. Basically, Eq. (24) should be reduced to the Fokker-Planck equation in the limit \( m/M \to 0 \) and for small values of \( q \). First of all, let us note that in the case when the PTF and the functions \( W(q), \tilde{W}'(q), \tilde{W}''(q) \) strongly decrease for large values of \( q \) the exponents in the integrals in the functions \( A(s), B(s), C_1(s), C_2(s) \) and \( C_3(s) \) can be expanded into a series as follows

\[
A(s) \simeq -\frac{s^2}{6} \int q^2 W(q)dq,
\]

\[
B(s) \simeq -\frac{1}{3} \int q^2 \tilde{W}'(q)dq,
\]

\[
C_1(s) \simeq -\frac{1}{15} \int q^4 \tilde{W}''(q)dq,
\]

\[
C_2(s) \simeq -\frac{1}{30} \int q^4 \tilde{W}''(q)dq,
\]

\[
C_3(s) \simeq \frac{1}{6} \int q^2 \tilde{W}'(q)dq.
\]
Then we can set the coefficients $C_1$, $C_2$ and $C_3$ equal to zero as they are of next order of smallness compared to the coefficients $A$ and $B$ with respect to the small parameter $m/M$. The fact can be deduced from the expression for the PTF (3). And the simplified kinetic equation in velocity space based on the PTF (which is non-equilibrium in the general case) yields

$$
\frac{df(s,t)}{dt} = A_0 s^2 f(s) + Bs \frac{\partial f(s)}{\partial s},
$$

(26)

where $A_0 = -1/6 \int q^2 W(q) dq$.

The stationary solution of this equation is given by

$$
f(s) = \text{Const} \cdot \exp \left[ -\frac{A_0 s^2}{2B} \right].
$$

(27)

The respective normalized stationary momentum distribution reads

$$
f(p) = \frac{NB^{3/2}}{(2\pi A_0)^{3/2}} \exp \left[ -\frac{Bp^2}{2A_0} \right].
$$

(28)

3 Special cases of anomalous diffusion in velocity space

Let us calculate the kinetic coefficients in the generalized Fokker-Planck equation for the special cases of anomalous diffusion. In our model diffusing heavy particles with mass $M$ interact with light particles of the surrounding medium with mass $m \ll M$ and with power-type velocity distribution

$$f_b(u + v) = \frac{n_b}{u_0^{2-2\gamma}(u + v)^{2\gamma}},
$$

(29)

where $u_0$ means the characteristic velocity, $n_b$ is the constant which has the dimension of the density in the coordinate space but since the function $f_b$ can not be normalized we can not think of $n_b$ as of a density. Impossibility to normalize the distribution function $f_b$ is explained with the fact that the normalizing integral diverges either at small or at large values of the argument of $f_b$. In reality normalization can always be fulfilled. At small values of the argument the power-law (29) of the distribution function is violated. But it is highly unlikely to obtain an analytical expression for $W(q,p)$ for realistic distribution functions.

3.1 Hard-sphere interactions

Interactions in systems of hard-sphere particles of radius $a$ are described by the differential cross-section $d\sigma/d\omega = a^2/4$. Substitution of the specified distribution function and cross-section into equation (3) gives

$$W(q, p) = \frac{n_b a^2}{4\mu^2 q u_0^{2-2\gamma}} \int \frac{\delta \left( u_z + \frac{p}{2\mu} \right)}{(u + v)^{2\gamma}} d^3 u.
$$

(30)

For $\gamma > 1$ the integral (30) converges and the result of integration reads

$$W(p, q) = \frac{\pi n_b a^2 (\xi - 1)^{2-2\gamma}}{(2\mu)^{4-2\gamma}(\gamma - 1)q^{2\gamma-1}u_0^{3-2\gamma}}.
$$

(31)
where $\xi = 2\mu(q \cdot v)/q^2$. Expanding equation (31) into a series according to (9) we leave only the first three terms

$$W(q) = \frac{\pi n_0 a^2}{(2\mu)^4 - 2^4} = \frac{R_0}{q^{2\gamma - 1}}, \quad (32)$$

$$\hat{W}(q) = \frac{2\pi n_0 a^2}{M(2\mu)^3 - 2^3} = \frac{R_1}{q^{2\gamma + 1}}, \quad (33)$$

$$R_1 = \frac{2(2\gamma - 2)}{M} R_0,$$

$$\hat{W}'(q) = \frac{2\pi n_0 a^2(2\gamma - 1)}{M^2(2\mu)^2 - 2^2} = \frac{R_2}{q^{2\gamma + 3}}, \quad (34)$$

$$R_2 = \frac{(2\mu)^2(2\gamma - 2)(2\gamma - 1)}{M^2} R_0,$$

$$\hat{W}'(q) = 0. \quad (35)$$

Now let us substitute $W(q)$ into equation (17) which introduces the coefficient $A(s)$ and discuss the convergence criteria.

$$A(s) = 4\pi \int_0^\infty q^2 \left[ \frac{\sin (qs)}{qs} - 1 \right] W(q) dq$$

$$= 4\pi R_0 \int_0^\infty \frac{dq}{q^{2\gamma - 3}} \left[ \frac{\sin (qs)}{qs} - 1 \right]. \quad (36)$$

The convergence of the integral in equation (36) is guaranteed when the inequality $2 < \gamma < 3$ takes place. Whereas $\gamma < 3$ gives the convergence for small values of $q$ ($q \to 0$) the convergence for $q \to \infty$ is provided at $\gamma > 2$.

The values of the parameter $\gamma$ at which the coefficient $B(s)$ remains finite can be found directly from the definition of $B(s)$ (Eq. (15)). It is clear to see that condition for convergence of the integral in equation for the coefficient $B(s)$ for small values of $q$ is $\gamma < 2$ and for large values of $q$ is $\gamma > 1/2$. Finally, finiteness of the coefficients $C_1(s)$ and $C_2(s)$ follows from the inequality $\gamma < 2$ for small values of $q$. For large values of $q$ the coefficient $C_1$ is finite at $\gamma > 1$ and the coefficient $C_2$ is finite at $\gamma > 0$ (Eq. (16) (21)).

To sum up, in the case under consideration, the convergence of the coefficients $A$, $B$, and $C_{1,2}$ for large values of $q$ is defined only by the convergence of the coefficient $A$, which means $\gamma > 2$. For small values of $q$ it is sufficient to guarantee convergence for the coefficient $B(s)$ with $\gamma < 2$. In other words, for purely power-type behavior of the function $f_0(\xi)$ the simultaneous convergence of the coefficients $A$, $B$ and $C_{1,2}$ does not exist. However, in real physical models, the convergence of the coefficients at $q \to 0$ can be achieved, for example, due to the non-power-type behavior of the PTF $W$ for small values of $q$ (compare with examples of anomalous diffusion in coordinate space [12]). Therefore, in the system of hard-sphere heavy particles undergoing anomalous diffusion in the medium of light particles with power-type distribution at large values of the variable $q$ the kinetic coefficients exist at $\gamma > 2$. This result approves of the qualitative conclusion in paper [19] but the numbers do not coincide as we started from another more explicit expression for the PTF [20].
3.2 Coulomb collisions

By analogy, we can consider a system of particles characterized by Coulomb interactions. Formally, the Coulomb interactions are described by a differential cross-section

\[
\frac{d\sigma_{\text{Coul}}}{d\omega} = \left( \frac{Ze^2}{2\mu u^2} \right)^2 \frac{1}{\sin^4 \frac{\chi}{2}} = \frac{4Ze^4\mu^2}{q^4},
\]

(37)

where \(Z\) is the charge number and the scattering angle \(\chi = \arccos(1 - q^2/2\mu u^2)\) (see Eq. (5)). From the mathematical point of view analysis of the kinetic coefficients in the case of Coulomb collisions is no more difficult than the case of hard-sphere collisions as the Coulomb cross-section is also independent on the relative velocity \(u\). The only difference is the power in the dependence of the PTF on the transfer momentum \(q\)

\[
W_{\text{Coul}}(q, p) = \frac{4\pi Z^2e^4n_b}{q^3u_0^{2\gamma}} \int \frac{\delta(u_x + \frac{q}{2p})}{(u + v)^{2\gamma}} q^3 u.
\]

(38)

Performing integration we arrive at the equation

\[
W_{\text{Coul}}(p, q) = \frac{4\pi Z^2e^4n_b(\xi - 1)^{2-2\gamma}}{(2\mu)^{2-2\gamma}(\gamma - 1)q^{2\gamma+3}u_0^{3-2\gamma}},
\]

(39)

Expanding the PTF into a series we obtain the functions \(W_{\text{Coul}}(q), \tilde{W}_{\text{Coul}}'(q)\) and \(\tilde{W}_{\text{Coul}}''(q)\)

\[
W_{\text{Coul}}(q) = \frac{4\pi Z^2e^4n_bu_0^{2\gamma-3}}{(2\mu)^{2-2\gamma}(\gamma - 1)q^{2\gamma+3}} = \frac{K_0}{q^{2\gamma+3}},
\]

(40)

\[
\tilde{W}_{\text{Coul}}'(q) = \frac{8\pi Z^2e^4n_bu_0^{2\gamma-3}}{M(2\mu)^{1-2\gamma}q^{2\gamma+5}} = \frac{K_1}{q^{2\gamma+5}},
\]

(41)

\[
K_1 = \frac{2\mu(2\gamma - 2)}{M} K_0,
\]

\[
\tilde{W}_{\text{Coul}}''(q) = \frac{8\pi Z^2e^4n_b(2\gamma - 1)u_0^{2\gamma-3}}{M^2(2\mu)^{-2\gamma}q^{2\gamma+7}} = \frac{K_2}{q^{2\gamma+7}},
\]

(42)

\[
K_2 = \frac{(2\mu)^2(2\gamma - 2)(2\gamma - 1)}{M^2} K_0,
\]

(43)

\[
\tilde{W}_{\text{Coul}}'(q) = 0.
\]

(44)

Reasoning by analogy with the previous section concerning hard-sphere interactions we can draw a conclusion that the coefficients \(A(s), B(s)\) and \(C_{1,2}(s)\) are finite at large values of \(q\) provided the inequality \(\gamma > 0\) is fulfilled. We assume that the convergence of the integrals in these coefficients at small values of \(q\) is due to various neglected in our model factors, such as screening effect, etc. The asymptotic behavior of the PTF and the respective derivatives of the PTF for large \(q\) is similar to one obtained in [21]. However, the coefficients are calculated on the basis of the explicit representation for the PTF (6).
3.3 Interactions typical for dusty plasmas

A more complicated and interesting case is interaction of dusty particles with electrons and ions. The differential cross-section for this type of interaction depends on the relative velocity of the particles unlike the cases of Coulomb and hard-sphere models of collisions. This fact leads to the dependence of the PTF on $p^2$. To demonstrate this let us refer to the differential cross-section for grains-electrons interactions [27]. The respective differential cross-section yields

$$
\sigma_e(u) = \begin{cases} 
\pi \rho^2 \left( 1 - \frac{2e^2}{\rho m_e u^2} \right), & \frac{2e^2}{\rho m_e u^2} < 1, \\
0, & \frac{2e^2}{\rho m_e u^2} > 1,
\end{cases}
$$

(44)

$$
\sigma_i(u) = \pi \rho^2 \left( 1 + \frac{2e^2}{\rho m_i u^2} \right),
$$

(45)

where $m_{e(i)}$ is electron (ion) mass, $\rho$ is radius of a grain, $e$ is the elementary charge. The ions are assumed singly charged.

Let us consider electron-grain collisions and rewrite Eq. (44) in a more convenient way

$$
\sigma_e(u) = \begin{cases} 
\frac{\rho^2}{4} \left( 1 - \frac{\Delta^2}{u^2} \right), & u > \Delta, \\
0, & u < \Delta,
\end{cases}
$$

(46)

where $\Delta^2 = 2e^2/\rho m_e$. Along with the previous special cases we assumed scattering particles (electrons) have a power-type velocity distribution [49]. Substituting the announced cross-section and the distribution function into the PTF (5) we arrive at

$$
W(q, v) = \frac{\pi \rho^2 n_b}{6\mu^2} q \int_{u>\Delta} \left( 1 - \frac{\Delta^2}{u^2} \right) \frac{dn}{u^2 (u + v)^{2\gamma + 1}}.
$$

(47)

Analyzing the integral we can conclude that for large values of $q/2\mu > \Delta$ the inequality $u > \Delta$ is guaranteed. In our paper we restrict our consideration only with large values of $q$. The point is that the power-type distribution function is realistic only for large values of $q$. For small values it diverges and the processes which suppress its growth should be taken into account.

But even for $q/2\mu > \Delta$ when the integration should be performed over the whole velocity space it is not possible to obtain an analytical expression for the PTF in terms of elementary functions. However, we can calculate the coefficients $W(q), \tilde{W}(q), \tilde{W}''(q)$ and $\tilde{W}''(q)$ expanding the integral into a series in the vicinity of the point $(q \cdot p) = 0$ a $p^2 = 0$ when it is necessary.

$$
W(q) = \frac{\pi \rho^2 n_b \gamma}{4\mu^2 q} q^{2\gamma - 3} \times \left[ \frac{1}{\gamma + 1} - \frac{\Delta^2}{q^2 \gamma + 2} \right],
$$

(48)
where $\bar{q} = q/2\mu$.

\[
\tilde{W}'(q) = \frac{\pi \rho^2}{4M^2\mu^2q} \frac{\gamma}{q^{2(\gamma+2)}} \frac{n_{b}u_{0}^{2\gamma-3}}{q^{2(\gamma+2)}} \times \left[ \frac{\Delta^2}{q^2} \frac{3}{\gamma + 4} - \frac{2}{\gamma + 2} \right]. 
\]

(49)

\[
\tilde{W}'(q) = \frac{\pi \rho^2}{M\mu q} \frac{n_{b}u_{0}^{2\gamma-3}}{q^{2(\gamma+2)}} \times \left[ \frac{1}{\gamma + 2} - \frac{\Delta^2}{q^2} \frac{1}{\gamma + 3} \right]. 
\]

(50)

\[
\tilde{W}''(q) = \frac{2\pi \rho^2}{M^2\mu^2q} \frac{\gamma(\gamma + 1)}{q^{2(\gamma+3)}} \frac{n_{b}u_{0}^{2\gamma-3}}{q^{2(\gamma+3)}} \times \left[ \frac{\Delta^2}{q^2} \frac{1}{\gamma + 4} - \frac{1}{\gamma + 2} \right]. 
\]

(51)

The convergences of all coefficients above at large values of $q$ fulfills when $\gamma > 0$. Remarkably for the dusty particles interacting with electrons the coefficient $\tilde{W}'(q)$ is not equal to zero although it is small due to the mass ratio of electrons and ions.

Similar analysis can be done for the dusty particles interacting with ions.

4 Conclusion

A more general Fokker-Planck-like equation is derived on the basis of the master equation. The equation is applicable for any values of the transferred momentum in a collision act unlike the usual Fokker-Planck equation which is valid for relatively small values of the transferred momentum. The mass ratio of the interacting particles is not important for the derived Fokker-Planck-like equation. It is based on the mere assumption that there are two species of the colliding particles, namely, with large and small velocity values.

The coefficients for the general Fokker-Planck-like equation are calculated using the general expression for the probability transition function describing elastic collisions. The scattering particles were assumed to have a velocity distribution function with power-type tail. It leads to the power-type dependence of the PTF on the transferred momentum and the corresponding process of diffusion in velocity space has anomalous character. Three different examples of the differential cross-section have been considered. The hard-sphere model of collisions as well Coulomb model demonstrate similar behavior of the PTF. However, for Coulomb interactions PTF has a more strongly decreasing tail. The case of dusty particles scattering on plasma particles (electrons or ions) is more complicated leads to the less trivial dependence of the PTF on the transferred momentum.

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