Pryce’s spin operator and the isometry generators of the massive Dirac fermions

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Abstract

It is shown that the components of Pryce’s spin operator of Dirac’s theory are generators of the $SU(2)$ representation carried by the space of Pauli’s spinors defining the polarization of the plane wave solutions of Dirac’s equation. At the level of quantum field theory we find the one-particle operator corresponding to Pryce’s spin, we define the polarization operator and derive the isometry generators of Dirac’s field of any polarization, including the momentum-dependent ones. In this manner, the problem of separating conserved spin and orbital angular momentum operators is solved naturally. As an example, the spin and polarization operators as well the isometry generators are derived for the first time in momentum-helicity basis.

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1 Introduction

The historical problem of finding a good spin operator of Dirac’s theory comes from the fact that the spin part of the total angular momentum is not conserved. This problem was studied by many authors, giving rise to a rich literature, but without arriving to a commonly accepted solution (see for instance Ref. [1] and the literature indicated therein).

Remaining outside this stream we tried to study how the polarization of Dirac’s field can be changed by applying suitable operators in configuration representation (rep.). The polarization is determined by Pauli spinors, entering in the structure of the plane wave solutions of the Dirac equation, which offer us supplemental $SU(2)$ degrees of freedom that are less studied so far. This is because the usual differential or multiplicative operators are not appropriate for studying this symmetry. Consequently, we were forced to consider a class of integral operators whose kernels have a convenient spectral expansion. We obtained thus suitable transformations (transfs.) of Pauli’s spinors whose generators close an $SU(2)$ algebra and, in addition, are conserved as components (comps.) of the spin operator one looks for. As these operators can be calculated in momentum rep. we arrived at a surprising result: the $SU(2)$ generators

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acting on Pauli’s spinors which define the polarization are just the comps. of
the spin operator proposed by Pryce in momentum rep. long time ago [2].

Our aim is to present here this result as an argument in favour of Pryce’s
spin operator which becomes after quantization the one-particle spin operator
of Dirac’s fermions in quantum field theory (QFT). In this framework we derive
the one-particle operators representing the isometry generators of the Dirac
quantum field in any type of polarization, pointing out that the operator of
total angular momentum is split into a new orbital angular momentum and
Pryce’s spin operator, each one being conserved separately. Moreover, we derive
for the first time the closed forms of isometry generators, spin and polarization
operators in momentum-helicity basis of QFT.

Working explicitly with integral operators we consider simultaneously the
covariant transfs. in configuration rep. and the equivalent Wigner’s [3] induced
unitary irreducible reps. (irreps.) [4] in momentum rep., as the framework of our
investigation. For this reason we start presenting in the next section the Dirac
field in Cartesian coordinates, pointing out the operators of RQM conserved via
Noether’s theorem and the relativistic scalar product with respect to which the
covariant rep. behaves as a unitary one. The section 3, devoted to Wigner’s
theory, starts presenting the plane wave solutions of the Dirac equation which
form the momentum basis, separating the positive and negative frequences
[5, 6, 7]. We define these solutions up to polarization, keeping the degrees of
freedom of Pauli’s spinors that form the principal object of this study. In the
second part of this section we present the tools offered by Wigner’s theory in
determining the structure of Dirac’s spinors in momentum rep. [8] but without
affecting the Pauli ones which remain arbitrary.

With these preparations and inspired by the completeness condition of the
mode spinors, we propose in section 4 the general form of the kernels of inte-
gral operators able to act on the space of Pauli spinors, changing polarization.
These operators have well-defined actions in momentum rep. where they form
an algebra from which we may extract the operators of a SU(2) rep. whose
generators, calculated in momentum rep., are just the comps. of Pryce’s spin
operator [2]. After giving the definition of these operators with the help of a pair
of projection operators introduced here, we define the polarization one which, in
general, is more complicated than a simple linear combination of spin operator
components.

These results are obtained at the level of configuration and momentum reps.
of the relativistic quantum mechanics (RQM) where there are problems in in-
terpreting the anti-particle solutions. Fortunately, the framework adopted here
allows us to apply easily the Bogolyubov method of quantization transforming
the expectation values of symmetry generators of RQM into the one-particle
operators of QFT. In section 5 we show that the Pryce spin operator is related
to the conserved one-particle spin operator whose comps. are the generators of
the unitary operators changing the polarizations of Dirac’s field operators in
configuration rep.. Moreover, we derive the one-particle isometry generators in
momentum bases with any polarization showing how the one-particle operator
of total angular momentum is split naturally into conserved spin and orbital
angular momentum.

Section 6 is devoted to the momentum-helicity basis for which we derive
for the first time the isometry generators as one-particle operators of QFT.
Moreover, we show that on antiparticle sector our polarization operator behaves
2 Massive Dirac field

Let us start with the Minkowski space-time, \((M, \eta)\), having the metric \(\eta = \text{diag}(1, -1, -1, -1)\) and Cartesian coordinates \(x^\mu (\alpha, \beta, \ldots, \nu) = 0, 1, 2, 3\). The isometries of \(M\), are transfs. of the Poincaré group \(P_+ = T(4) \otimes L_+^{\mathbb{R}}\), \((\Lambda, a) : x \rightarrow x' = \Lambda x - a\), formed by transfs. \(\Lambda \in L_+^\mathbb{R}\) of the orthochronous proper Lorentz group, preserving the metric \(\eta\), and four dimensional translations \(a \in T(4)\). The universal covering group of the Poincaré one, \(\tilde{P}_+ = T(4) \otimes SL(2, \mathbb{C})\), includes transfs. \(\lambda \in SL(2, \mathbb{C})\) related to those of the Lorentz group through the canonical homomorphism, \(\lambda \rightarrow \Lambda(\lambda)\) [8].

The covariant Dirac field, \(\psi : M \rightarrow \mathcal{V}_D\), is locally defined over \(M\) with values in the vector spaces \(\mathcal{V}_D\) carrying the finite-dimensional rep. \(\rho_D = (\frac{3}{2}, 0) \oplus (0, \frac{1}{2})\) of the \(SL(2, \mathbb{C})\) group where one can define the Dirac \(\gamma\)-matrices that satisfy \(\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\), giving rise to \(SL(2, \mathbb{C})\) generators. Here we consider exclusively the chiral representation (with diagonal \(\gamma^0\)) in which the transfs.

\[
\lambda(\omega) = \exp \left( -\frac{i}{2} \omega^{\alpha\beta} s_{\alpha\beta} \right) \in \rho_D, \quad s^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu],
\]

with real-valued parameters, \(\omega^{\alpha\beta} = -\omega^{\beta\alpha}\), are reducible to the subspaces of irreps. \((\frac{3}{2}, 0)\) and \((0, \frac{1}{2})\) of \(\rho_D\), [8, 9]. We denote by \(r = \text{diag}(\hat{r}, \hat{r}) \in \rho_D \otimes SU(2)\) the transfs. we call here simply rotations, for which we use Cayley-Klein parameters \(\theta^i = \frac{1}{2} \epsilon_{ijk} \omega^{jk}\) and generators \(s_i = \frac{1}{2} \epsilon_{ijk}s^{jk}\) = diag(\(\hat{s}_i, \hat{s}_i\)), where \(\hat{s}_i = \frac{1}{2} \sigma_i\) depend on the Pauli matrices \(\sigma_i (i, j, k\ldots = 1, 2, 3)\). Similarly, we say that the transfs. \(l = \text{diag}(\hat{l}, \hat{l}^{-1}) \in \rho_D \otimes [SL(2, \mathbb{C})/SU(2)]\) are Lorentz boosts generated by the matrices \(s_{\alpha 0} = \text{diag}(-i\hat{s}_i, i\hat{s}_i)\) whose parameters are denoted by \(\tau^i = \omega^{0i}\).

Summarizing we may write

\[
\begin{align*}
 r(\theta) &= \text{diag}(\hat{r}(\theta), \hat{r}(\theta)), \quad \hat{r}(\theta) = e^{-i\theta^i \hat{s}_i} = e^{-\frac{i}{2} \theta^i \sigma_i}, \\
 l(\tau) &= \text{diag}(\hat{l}(\tau), \hat{l}^{-1}(\tau)), \quad \hat{l}(\tau) = e^{-\tau^i \hat{s}_i} = e^{-\frac{1}{2} \tau^i \sigma_i}.
\end{align*}
\]

In the covariant parametrization the associated Lorentz transfs. may be expanded as

\[
\Lambda^\alpha_{\mu'\nu}[\lambda(\omega)] = \delta^\alpha_{\mu'} + \omega^\mu_{\mu'} + \omega^\mu_{\alpha'\nu'} + \cdots,
\]

as it results from the canonical homomorphism [8].

The massive Dirac field \(\psi\) of mass \(m\) and its Dirac adjoint, \(\bar{\psi}\), are canonical variables of the action

\[
S[\psi] = \int d^4 x L_D(\psi, \bar{\psi}),
\]

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defined by the Lagrangian density,

$$\mathcal{L}_D(\psi) = \frac{i}{2} \left[ \overline{\psi} \gamma^\alpha \partial_\alpha \psi - (\partial_\alpha \psi) \gamma^\alpha \psi \right] - m \overline{\psi} \psi. \quad (6)$$

This action gives rise to the Dirac equation,

$$E_D \psi = (i \gamma^\mu \partial_\mu - m) \psi = 0, \quad (7)$$

and the form of the relativistic scalar product

$$\langle \psi, \psi' \rangle_D = \int d^3x \overline{\psi}(x) \gamma^0 \psi'(x) = \int d^3x \psi^+ (x) \psi'(x), \quad (8)$$

related to the conserved electric charge.

The Dirac field transforms under isometries according to the covariant rep. $T$ : $(\lambda, a) \rightarrow T_{\lambda,a}$ of the group $\tilde{P}^1_+$, as [8]

$$\langle T_{\lambda,a} \psi \rangle(x) = \lambda \psi (\Lambda(\lambda)^{-1}(x + a)). \quad (9)$$

The well-known basis-generators of this rep.,

$$P_\mu = \left. \frac{\partial T_{1,a}}{\partial a^\mu} \right|_{a=0}, \quad J_{\mu\nu} = \left. \frac{i \partial T_{\omega,0}}{\partial \omega^{\mu\nu}} \right|_{\omega=0}, \quad (10)$$

may be rewritten in vector notation, separating the momentum comp. and energy operator, $P^i = -i \partial_i$ and $H = P_0 = i \partial_t$, and denoting the $SL(2, \mathbb{C})$ generators as,

$$J_i = \frac{1}{2} \varepsilon_{ijk} J_{jk} = -i \varepsilon_{ijk} x^j \partial_k + s_i, \quad (11)$$

$$K_i = J_0 = i(x^i \partial_i + t \partial_t) + s_0, \quad (12)$$

obtaining the usual basis $\{H, P^i, J_i, K_i\}$ of the Lie algebra $\mathcal{L}(T)$ of rep. $T$ [8].

The action [8] is invariant under isometries such that the scalar product [8] is invariant,

$$\langle T_{A,a} \psi, T_{A,a} \psi' \rangle_D = \langle \psi, \psi' \rangle_D, \quad (13)$$

because of the generators $X \in \mathcal{L}(T)$ that are self-adjoint, obeying $\langle X \psi, \psi' \rangle_D = \langle \psi, X \psi' \rangle_D$, as the $SL(2, \mathbb{C})$ generators of the rep. $\rho_D$ are Dirac self-adjoint, $\overline{s}_{\mu\nu} = s_{\mu\nu}$. All these generators are conserved via Noether’s theorem in the sense that their expectation values $\langle \psi, X \psi \rangle_D$ are independent on time. Therefore, we may conclude that in this framework the covariant rep. $T$ behaves as an unitary one with respect to the relativistic scalar product [8].

The invariants of the Dirac field are the eigenvalues of Casimir operators of the rep. $T$ that read [8]

$$C_1 = P_\mu P^\mu \sim m^2, \quad C_2 = -\eta_{\mu\nu} W^\mu W^\nu \sim m^2 s(s + 1), \quad s = \frac{1}{2}, \quad (14, 15)$$

where the Pauli-Lubanski operator [8],

$$W^\mu = -\frac{1}{2} \varepsilon^{\mu\alpha\beta} P_\nu J_{\alpha\beta}, \quad (16)$$

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has the components
\[ W_0 = J_i P^i = s_i P^i, \quad W_i = H J_i + \varepsilon_{ijk} P^j K^k, \]  \tag{17}
as we use \( \varepsilon^{0123} = -\varepsilon_{0123} = -1 \). These operators play the role of components of a covariant spin of RQM as long as \( W_0 \) is just the helicity operator. Note that \( W_0 \) is the only differential operator able to define polarization but it leads to some difficulties at the level of QFT as we shall see later after discussing the problem of finding a suitable spin operator. We remind the reader that the matrices \( s_i \) cannot play this role as these are not conserved separately even the total angular momentum operator \( (11) \) is conserved.

3 Wigner’s theory in momentum representation

The field equation and \( \mathcal{L}(T) \) algebra offer the complete system of commuting operators \( \{ H, P^1, P^2, P^3, W_0 \} \) defining the momentum-helicity basis. Unfortunately, this rep. is not defined in rest frames (for \( p = 0 \)) which are the starting point of the Wigner theory of induced reps., For this reason we restrict ourselves to start with the incomplete set \( \{ H, P^1, P^2, P^3 \} \) determining the momentum rep. up to polarization which will be studied later.

3.1 Frequencies separation

For writing down the general solution of the Dirac equation we consider the usual sets of mode spinors, \( U_{p,\sigma} \) and \( V_{p,\sigma} = C U_{p,\sigma}^* \), of positive and respectively negative frequencies, related through the charge conjugation defined by the matrix \( C = C^{-1} = i\gamma^2 \) acting as [7]
\[ \gamma^{\mu*} = -C\gamma^\mu C \rightarrow s^*_{\mu\nu} = -Cs_{\mu\nu}C \rightarrow \lambda^* = C\lambda C. \]  \tag{18}
The mode spinors (or fundamental spinors) satisfy the Dirac equation and the eigenvalues problems,
\[ HU_{p,\sigma} = E(p)U_{p,\sigma}, \quad HV_{p,\sigma} = -E(p)V_{p,\sigma}, \]  \tag{19}\[ P^i U_{p,\sigma} = p^i U_{p,\sigma}, \quad P^i V_{p,\sigma} = -p^i V_{p,\sigma}, \]  \tag{20}
where \( E(p) = \sqrt{m^2 + p^2} \) (\( p = |p| \)) is the relativistic energy. The free Dirac field can be expanded as [6, 7]
\[ \psi(x) = \psi^+(x) + \psi^-(x) = \int d^3p \sum_{\sigma} [U_{p,\sigma}(x)\alpha_\sigma(p) + V_{p,\sigma}(x)\beta_\sigma^*(p)], \]  \tag{21}
in terms of mode functions in momentum rep., \( \alpha_\sigma \) and \( \beta_\sigma \), of particles and respectively antiparticles of arbitrary polarization \( \sigma = \pm \frac{1}{2} \) that will be defined after studying the spin operators. In this manner, the space of Dirac’s mode spinors, \( \mathcal{F}_D = \mathcal{F}_D^+ \oplus \mathcal{F}_D^- \), is split in two orthogonal subspaces of fundamental solutions of positive and respectively negative frequencies.

The mode spinors have the general form
\[ U_{p,\sigma}(x) = u_\sigma(p) \frac{1}{(2\pi)^{3/2}} e^{-iE(p)t + ip \cdot x}, \]  \tag{22}\[ V_{p,\sigma}(x) = v_\sigma(p) \frac{1}{(2\pi)^{3/2}} e^{iE(p)t - ip \cdot x}, \]  \tag{23}
where the spinors $u_{\sigma}(p)$ and $v_{\sigma}(p) = C u_{\sigma}(p)^*$ must be normalized in order to obtain the orthonormalization relations

$$\langle U_{p,\sigma}, U_{p',\sigma}' \rangle_D = \langle V_{p,\sigma}, V_{p',\sigma}' \rangle_D = \delta_{\sigma\sigma'} \delta^3(p-p') \quad (24)$$

$$\langle U_{p,\sigma}, U_{p',\sigma}' \rangle_D = \langle V_{p,\sigma}, U_{p',\sigma}' \rangle_D = 0, \quad (25)$$

and the completeness condition,

$$\int d^3p \sum_{\sigma} \left[ U_{p,\sigma}(t,x) U_{p',\sigma}^+(t,x') + V_{p,\sigma}(t,x) V_{p',\sigma}^+(t,x') \right] = \delta^3(x-x') \quad (26)$$

Eqs. (24) and (25) help us to write the inversion formulas

$$\alpha_{\sigma}(p) = \langle U_{p,\sigma}, \psi \rangle_D, \quad \beta_{\sigma}(p) = \langle \psi, V_{p,\sigma} \rangle_D, \quad (27)$$

we need in applications.

In RQM the physical meaning of the field $\psi$ is encapsulated in the wave functions in momentum rep., $\alpha_\sigma$ and $\beta_\sigma$, that form the Pauli spinors,

$$\alpha = \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} \in \mathcal{F}_\alpha, \quad \beta = \begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} \in \mathcal{F}_\beta, \quad (28)$$

of the unitary spaces $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ equipped with scalar products,

$$\langle \alpha, \alpha' \rangle = \int d^3p \sum_{\sigma} \alpha_\sigma^*(p) \alpha_{\sigma'}(p), \quad (29)$$

and similarly for the spinors $\beta$, such that we can write

$$\langle \psi, \psi' \rangle_D = \langle \alpha, \alpha' \rangle + \langle \beta, \beta' \rangle, \quad (30)$$

after using Eqs. (24) and (25).

We remind the reader that the differential operators in configuration rep., $F(i \partial_{\mu}) : \mathcal{F}_D^+ \rightarrow \mathcal{F}_D^+$, give rise to multiplicative operators in momentum rep., $\tilde{F}(p^\mu)$, acting differently on the mode spinors,

$$[F(i \partial_{\mu}) \psi](x) = \int d^3p \sum_{\sigma} \left[ \tilde{F}(p^\mu) U_{p,\sigma}(x) \alpha_{\sigma}(p) \right. \right.$$

$$\left. \left. + \tilde{F}(-p^\mu) V_{p,\sigma}(x) \beta_{\sigma}^*(p) \right] \right. \quad (31)$$

We say here that these operators are diagonal on $\mathcal{F}_D$ as they do not mix the mode spinors of different frequencies. For example, the Hamiltonian operator $H_D = -i \gamma^0 \gamma^i \partial_i + m \gamma^0$ acts as

$$\left( H_D U_{p,\sigma}(x) \right) = \tilde{F}(p) U_{p,\sigma}(x) = E(p) U_{p,\sigma}(x), \quad (32)$$

$$\left( H_D V_{p,\sigma}(x) \right) = \tilde{H}_D(-p) V_{p,\sigma}(x) = -E(p) V_{p,\sigma}(x), \quad (33)$$

where

$$\tilde{H}_D(p) = m \gamma^0 + \gamma^0 \gamma^i p^i, \quad (34)$$

is the Hamiltonian operator in momentum rep.
3.2 Wigner’s induced representations

The Wigner theory [3] focuses on functions defined on orbits, \( \Omega_p = \{ p | p = \Lambda \hat{p}, \Lambda \in L^+_{\text{SO}(3)} \} \), in momentum space that may be built by applying the Lorentz transfs. on a representative momentum \( \hat{p} \). In the case of massive particles, we discuss here, the representative momentum is just the rest frame one, \( \hat{p} = (m, 0, 0, 0) \). The rotations that leave \( \hat{p} \) invariant, \( \hat{R} \hat{p} = \hat{p} \), form the stable group \( SO(3) \subset L^+_{\text{SO}(3)} \) of \( \hat{p} \) whose universal covering group \( SU(2) \) is called the little group associated to the representative momentum \( \hat{p} \).

For each momentum \( p \) there exist a set of transfs. \( L_p \) generating it as \( p = L_p \hat{p} \). These transfs. are defined up to arbitrary rotations \( \hat{R}(p) \) which may depend on \( p \) as these do not change the representative momentum, \( L_p \hat{R}(p) \hat{p} = L_p \hat{p} \). This means that the orbit \( \Omega_p \) is in fact an homogeneous space \( L^+_{\text{SO}(3)}/SO(3) \). Consequently, the corresponding transfs. \( \lambda_p \in \rho_D \) which satisfy \( \Lambda(\lambda_p) = L_p \) and \( \lambda_{p=0} = 1 \) in \( \rho_D \) have the general form \( \lambda_p = l_p \hat{r}(p) \), being constituted by genuine Lorentz boosts \( l_p \in \rho_D [SL(2, C)/SU(2)] \) defined by Eq. (35) and arbitrary rotations \( r(p) \in \rho_D [SU(2)] \) satisfying \( r(p = 0) = 1 \in \rho_D \).

For investigating the effect of isometries in momentum rep. one assumes that there exist a rep. \( \hat{T} : (\lambda, a) \rightarrow \hat{T}_{\lambda,a} \), carried by the spaces \( \mathcal{F}_\alpha \) and \( \mathcal{F}_\beta \), associated to \( T \) as \( \mathbf{3, 5, 9} \)

\[
(T_{\lambda,a} \psi)(x) = \int d^3 \hat{p} \sum_{\sigma} \left[ U_{p,\sigma}(x)(\hat{T}_{\lambda,a} \alpha \sigma)(p) + V_{p,\sigma}(x)(\hat{T}_{\lambda,a} \beta \sigma)(p) \right].
\]

Taking into account that the covariant rep. \( T \) is defined by Eq. (38) and using the Lorentz invariant measure \( d^3 p E(p)^{-1} = d^3 p' E(p')^{-1} \) we find the action of rep. \( \hat{T} \) \[8\],

\[
\sum_{\sigma} u_{\sigma}(p)(\hat{T}_{\lambda,a} \alpha \sigma)(p) = \frac{E(p')}{E(p)} \sum_{\sigma} \lambda_{u,\sigma}(p') \alpha_{\sigma}(p') e^{-ia \cdot p'}, \tag{36}
\]

\[
\sum_{\sigma} v_{\sigma}(p)(\hat{T}_{\lambda,a} \beta \sigma)(p) = \frac{E(p')}{E(p)} \sum_{\sigma} \lambda_{v,\sigma}(p') \beta_{\sigma}(p') e^{ia \cdot p'}, \tag{37}
\]

where \( a \cdot p = a \mu p^\mu = E(p)a^0 - p \cdot a \) and

\[
p' = \Lambda(\lambda)^{-1} p. \tag{38}
\]

Furthermore, according to Wigner’s general method, we introduce the mode spinors \[9\],

\[
u_\sigma(p) = N(p) \lambda_p \hat{u}_\sigma = N(p) \hat{l}_p r(p) \hat{u}_\sigma, \tag{39}
\]

\[
u_\sigma(p) = C u_\sigma^*(p) = N(p) \lambda_p \hat{v}_\sigma = N(p) \hat{l}_p r(p) \hat{v}_\sigma, \tag{40}
\]

where \( N(p) \in \mathbb{R} \) satisfying \( N(0) = 1 \) is a normalization factor. The rest frame spinors \( \hat{u}_\sigma = u_\sigma(0) \) and \( \hat{v}_\sigma = v_\sigma(0) = C \hat{u}_\sigma^* \) are solutions of the Dirac equation in the rest frame where this equation reduces to the eigenvalues problems of the matrix \( \gamma^0 \),

\[
\gamma^0 \hat{u}_\sigma = \hat{u}_\sigma, \quad \gamma^0 \hat{v}_\sigma = -\hat{v}_\sigma. \tag{41}
\]

Then the Wigner spinors \[39\] and \[40\] are solutions of the Dirac equation in any frame of momentum rep.. Indeed, observing that the matrix \( \gamma p = E(p) \gamma^0 = \gamma^0 p \)

\[7\]
satisfies $\gamma p \lambda_p = m \lambda_p \gamma^0$ we obtain the Dirac equations in momentum rep.,

\[
(\gamma p - m)u_\sigma(p) = 0, \quad (\gamma p + m)v_\sigma(p) = 0, \quad (42)
\]

after exploiting Eqs. (11).

The matrices $\frac{1+\gamma^0}{2}$ form an orthogonal system of projection matrices such that the spinor subspaces $\frac{1+\gamma^0}{2} \mathcal{V}_D$ and $\frac{1-\gamma^0}{2} \mathcal{V}_D$ are orthogonal. Moreover we assume that all these spinors are normalized, $\bar{u}_\sigma \bar{u}_\sigma = \bar{v}_\sigma \bar{v}_\sigma = \delta_{\sigma\sigma'}$, forming complete systems on their subspaces,

\[
\sum_\sigma \bar{u}_\sigma \bar{u}_\sigma = \frac{1 + \gamma^0}{2}, \quad \sum_\sigma \bar{v}_\sigma \bar{v}_\sigma = \frac{1 - \gamma^0}{2}. \quad (43)
\]

We have now the opportunity of introducing the Pauli spinors we need for studying the polarization assuming that in the chiral rep. of the Dirac matrices (with diagonal $\gamma^5$) we may express the momentum-dependent spinors as

\[
r(p) \bar{u}_\sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_\sigma(p) \\ \xi_\sigma(p) \end{pmatrix}, \quad r(p) \bar{v}_\sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta_\sigma(p) \\ -\eta_\sigma(p) \end{pmatrix}, \quad (44)
\]

in terms of Pauli spinors $\xi_\sigma(p)$ and $\eta_\sigma(p) = i\sigma_2 \xi_\sigma(p)^*$ that form related bases, i.e. orthonormal,

\[
\xi_\sigma^+ (p) \xi_{\sigma'}(p) = \eta_\sigma^+ (p) \eta_{\sigma'}(p) = \delta_{\sigma\sigma'}, \quad (45)
\]

and complete systems,

\[
\sum_\sigma \xi_\sigma(p) \xi_{\sigma'}(p) = \sum_\sigma \eta_\sigma(p) \eta_{\sigma'}(p) = 1_{2\times 2}. \quad (46)
\]

The functions $\xi_\sigma : \mathbb{R}^3_p \rightarrow \mathcal{V}_D$ remain arbitrary representing the polarization degrees of freedom which will be determined after defining the polarization operators. In general, when these spinors depend explicitly on $p$ we say that we have a peculiar polarization while a polarization independent on $p$ will be referred as common polarization. Finally, by setting the normalization factor as

\[
N(p) = \sqrt{\frac{m}{E(p)}}, \quad (47)
\]

we obtain standard orthonormalization relations,

\[
\overline{\tau}_\sigma(p) u_{\sigma'}(p) = \overline{\tau}_\sigma(p) v_{\sigma'}(p) = \delta_{\sigma\sigma'}, \quad (48)
\]

\[
\overline{\tau}_\sigma(p) v_{\sigma'}(-p) = \overline{\tau}_\sigma(-p) u_{\sigma'}(p) = 0, \quad (49)
\]

which are independent on the concrete form of the spinors $\xi_\sigma$.

With these preparations we arrive at the final result of Wigner’s approach showing that the spinors $\alpha$ and $\beta$ transform alike under Wigner’s rep. $T$ acting as \[3, 8, 9\]

\[
(\tilde{T}_{\lambda, \alpha})_\sigma(p) = \sqrt{\frac{E(p')}{E(p)}} \sum_{\sigma'} D_{\sigma\sigma'}(\lambda, p) \alpha_{\sigma'}(p') e^{-iap}, \quad (50)
\]
where the matrix elements

\[ D_{\sigma\sigma'}(\lambda, p) = \hat{u}_\sigma^* \hat{w}(\lambda, p) \hat{u}_{\sigma'} , \tag{51} \]

depend on the Wigner tranf. 

\[ w(\lambda, p) = \lambda_p^{-1} \lambda \lambda_{p'} = r(p)^{-1} \lambda l_p r(p') , \tag{52} \]

giving the Lorentz one

\[ \Lambda[w(\lambda, p)] = R(p)^{-1} L_p^{-1} \Lambda(\lambda) L_{p'} R(p') \], \tag{53} \]

that leave invariant the representative momentum,

\[ \Lambda[w(\lambda, p)] \hat{p} = R(p)^{-1} L_p^{-1} \Lambda(\lambda) \hat{p} = R(p)^{-1} L_p^{-1} p = \hat{p}. \tag{54} \]

Therefore, \( \Lambda[w(\lambda, p)] \in SO(3) \rightarrow w(\lambda, p) \in SU(2) \) which means that we may write the definitive form of the matrix elements (51) as

\[ D_{\sigma\sigma'}(\lambda, p) = \xi_\sigma^+ (p) \hat{I}_p^{-1} \hat{I}_{p'} \xi_{\sigma'}(p') , \tag{55} \]

after exploiting Eq. (52) and our previous definitions (44). Obviouisly, the matrices \( D(\lambda, p) \) form the irrep. of spin \( s = \frac{1}{2} \) of the \( SU(2) \) group. For this reason one says the Wigner irreps. \( \tilde{T} \) are induced by the subgroup \( T(4) \subseteq SU(2) \) \cite{3,8,9}. For the antiparticle spinors \( \beta^* \) we obtain the matrix elements

\[ \hat{v}_\sigma^* w(\lambda, p) \hat{v}_{\sigma'} = (\hat{u}_\sigma^* w(\lambda, p) \hat{u}_{\sigma'})^* = [D_{\sigma\sigma'}(\lambda, p)]^* , \tag{56} \]

showing that the spinors \( \alpha \) and \( \beta \) transform alike under isometries. Note that the form of the matrix elements (55) we use here points out explicitly the Pauli spinors helping us to study the polarization.

The Wigner rep. \( \tilde{T} \) is irreducible as the matrices \( D \) are irreducible. Moreover, these are unitary with respect to the scalar product (29)

\[ \langle \tilde{T}_{\lambda,a} \alpha | \tilde{T}_{\lambda,a} \alpha' \rangle = \langle \alpha | \alpha' \rangle , \tag{57} \]

and similarly for \( \beta \). As the covariant reps. are unitary with respect to the scalar product (3) which can be decomposed as in Eq. (30) we conclude that the expansion (21) establishes the unitary equivalence, \( T = \tilde{T} \oplus \tilde{T} \), of the covariant rep. with the orthogonal sum of Wigner’s unitary irreps. \cite{4}. This means that the generators \( \tilde{X} \in L(\tilde{T}) \) of the Lie algebra of the irrep \( \tilde{T} \) defined as

\[ \tilde{P}_\mu = i \frac{\partial \tilde{T}_{1,a}}{\partial \omega^\mu} \bigg|_{a=0} , \quad \tilde{J}_{\mu
u} = i \frac{\partial \tilde{T}(\omega,0)}{\partial \omega^{\mu
u}} \bigg|_{\omega=0} , \tag{58} \]

are related to the corresponding generators \( X \in L(T) \) such that

\[ (X\psi)(x) = \int d^3 p \sum_\sigma \left[ U_{p,\sigma}(x)(\tilde{X} \alpha)_\sigma(p) - V_{p,\sigma}(x)(\tilde{X} \beta)^*_{\sigma}(p) \right] , \tag{59} \]

as we deduce deriving Eq. (35) with respect to a group parameter \( \zeta \in (\omega, a) \) in \( \zeta = 0 \). The problem here is the wrong sign of the antiparticle term which has to be corrected under quantization.
4 Looking for spin operators

The next step is to define the polarization looking for operators acting on the space of Pauli spinors. Bearing in mind that the representative momentum corresponds to a set of rest frames related among themselves through the $SO(3)$ rotations of stable group we observe that the space of Pauli’s spinors has similar degrees of freedom governed by the $SU(2)$ little group. These degrees of freedom deserve to be investigated as the last symmetry we did not studied so far.

Let us start with an arbitrary orthonormal basis $\xi \subset \mathcal{V}_P$, satisfying Eqs. (45) and (46), whose spinors may depend on $\mathbf{p}$ but without denoting this explicitly. The rotations $\hat{r} \in SU(2)$ of the little group transform this basis as

$$\hat{r} \xi_\sigma = \sum_{\sigma'} \xi_{\sigma'} D_{\sigma'\sigma}(\hat{r}) \xi_\sigma, \quad (60)$$

$$\hat{r} \eta_\sigma = \sum_{\sigma'} \eta_{\sigma'} D_{\sigma'\sigma}^*(\hat{r}) \eta_\sigma, \quad (61)$$

where $r = \text{diag}(\hat{r}, \hat{r}) \in \rho_D$ is an arbitrary rotation corresponding to $\hat{r}$ for which we use the traditional notation $D_{\sigma'\sigma}(\hat{r}) = \xi_{\sigma'}^* \hat{r} \xi_\sigma$.

For studying these transfs. it is convenient to re-denote the Dirac field (21) by $\psi_{\xi}(x)$ and the mode spinors by $U_{p,\xi_\sigma}$ and $V_{p,\eta_\sigma}$, pointing out explicitly their dependence on Pauli’s spinors. We obtain thus the transfs. of mode spinors,

$$U_{p,\hat{r}\xi_\sigma}(x) = \sum_{\sigma'} U_{p,\xi_{\sigma'}}(x) D_{\sigma'\sigma}(\hat{r}) , \quad (63)$$

$$V_{p,\hat{r}\eta_\sigma}(x) = \sum_{\sigma'} V_{p,\eta_{\sigma'}}(x) D_{\sigma'\sigma}^*(\hat{r}) , \quad (64)$$

which give the transformed field $\psi_{\hat{r}\xi}$ according to the expansion (21).

Under such circumstances, it would be interesting to look for a rep., $V : \hat{r} \rightarrow V(\hat{r})$, of the little group $SU(2)$ with values in a set of operators $\mathfrak{V} = \{ V(\hat{r}) | \hat{r} \in SU(2) \}$ able to rotate the Pauli spinors,

$$V(\hat{r})\psi_{\xi} = \psi_{\hat{r}\xi}, \quad \forall \hat{r} \in SU(2) , \quad (65)$$

but without depending on the spinor basis $\xi$ or affecting other quantities. We shall show that this is not a mission impossible constructing a such rep. with values in a set of integral operators.

4.1 Integral operators

We focus on the integral operators, $Z : \mathcal{F}_D \rightarrow \mathcal{F}_D$ whose action,

$$(Z\psi)(x) = \int d^4x' K_Z(x - x')\psi(x') , \quad (66)$$

is determined by the kernels $K_Z$ which are $4 \times 4$-matrices depending on $x - x'$. These operators are linear forming an algebra in which the multiplication is defined by the convolution,

$$K_{Z_1Z_2}(x - x') = \int d^4x'' K_{Z_1}(x - x'')K_{Z_2}(x'' - x') , \quad (67)$$
denoted as \( K_Z Z_\xi = K_Z \ast K_{Z_\xi} \). The identity operator has the kernel \( K_1(x - x') = \delta^4(x - x') \). An operators \( Z \) is invertible if there exists the operator \( Z^{-1} \) such that \( K_Z \ast K_{Z^{-1}} = K_{Z^{-1}} \ast K_Z = K_1 \). For any integral operator \( Z \) we may write the bracket
\[
\langle \psi, Z \psi' \rangle_D = \int d^4x \, d^4x' \psi^+(x) K_Z(x, x') \psi(x')
\] (68)
observing that \( Z \) is self-adjoint with respect to this scalar product only if \( K_Z(x) = K_Z^*(x) \). Note that the multiplicative or differential operators are particular cases of integral ones. For example, the operators \( \partial_t \) can be seen as integral operators having the kernels \( K_{\partial_t}(x) = \partial_t \delta^4(x) \).

Of a special interest are the equal-time operators \( Y \) whose kernels have the form
\[
K_Y(x - x') = \delta(t - t') K_Y(x - x'),
\] (69)
acting as
\[
(Y \psi)(t, x) = \int d^3x K_Y(x - x') \psi(t, x'),
\] (70)
without involving the time. In this case the kernels allow the 3-dimensional Fourier rep.,
\[
K_Y(x) = \int d^3p \, \frac{e^{ip \cdot x}}{(2\pi)^3} \tilde{Y}(p),
\] (71)
where \( \tilde{Y}(p) \) is the operator in momentum rep. of RQM corresponding to \( Y \). Then the action (70) on a field (21) can be written as
\[
(Y \psi)(x) = \int d^3p \sum_{\sigma} \left[ \tilde{Y}(p) U_{p,\xi}(x) \alpha_\sigma(p) + \tilde{Y}(-p) V_{p,\eta}(x) \beta^*_\sigma(p) \right].
\] (72)
We see thus that all these integral operators are diagonal on \( F_D \), acting separately on \( F^\pm_D \) without mixing mode spinors of different frequencies.

In what follows we consider a special type of such equal-time operators, denoted by \( Y_\xi \), acting alike on the spinors \( \alpha \) and \( \beta \), defined by the kernels
\[
K_{Y_\xi}(x - x') = \int d^3p \sum_{\sigma, \sigma'} \left[ U_{p,\xi}(t, x) \tilde{g}_{\sigma\sigma'}^* U_{p,\xi'}(t, x') \right. \\
\left. + V_{p,\eta}(t, x) \tilde{g}_{\sigma\sigma'}^* V_{p,\eta'}(t, x') \right],
\] (73)
in an arbitrary basis \( \xi \subset V_p \) that can depend on \( p \). The action of these operators can be calculated easily in momentum rep. by using the orthogonality properties (21) and (25). We obtain thus
\[
(Y_\xi \psi)(x) = \int d^3x' K_{Y_\xi}(x - x') \psi(x')
\]
\[
= \int d^3p \sum_{\sigma, \sigma'} \left[ U_{p,\xi}(t, x) \tilde{g}_{\sigma\sigma'}^* \alpha_\sigma(p) + V_{p,\eta}(t, x) \tilde{g}_{\sigma\sigma'}^* \beta^*_\sigma(p) \right],
\] (74)
defining the action of associated matrix-operator \( \tilde{g}(p) \) transforming alike the spinors \( \alpha \) and \( \beta \),
\[
\langle \tilde{g}\alpha \rangle_\sigma(p) = \langle U_{p,\xi}, Y_\xi \psi \rangle_D = \tilde{g}_{\sigma\sigma'}^* \alpha_\sigma'(p),
\] (75)
\[
\langle \tilde{g}\beta \rangle_\sigma(p) = \langle Y_\xi \psi, V_{p,\eta} \rangle_D = \tilde{g}_{\sigma\sigma'}^* \beta^*_\sigma(p).
\] (76)
The special form of these operators allow us to derive their expectation values, \[ \langle \psi_\xi, Y_\xi \psi_\xi \rangle_D = \langle \alpha, \bar{y} \alpha \rangle + \langle \beta, \bar{y}^\dagger \beta \rangle, \] exploiting Eqs. (24) and (25). Hereby we see that an operator \( Y_\xi \) is self-adjoint if the matrix \( \bar{y} \) is Hermitian, \( \bar{y}_{\sigma\sigma'} = \bar{y}_{\sigma'\sigma}^* \).

We must stress that the dependence on \( \xi \) of the operator \( Y_\xi \) is not an impediment as we know what happens when we change the spinor basis. Thus, by using Eqs. (63) and (64) we find that \( Y_\xi \to Y_\xi \Rightarrow \bar{y} \to D(\hat{r})\bar{y} \) if we keep unchanged the spinors \( \alpha \) and \( \beta \) encapsulating the physical meaning.

### 4.2 Spin and polarization

The operators \( V(\hat{r}) \in \mathfrak{U} \) we need for rotating the spinor basis \( \xi \) of the field \( \psi_\xi \) can be constructed as integral operators with kernels of the form (71) where

\[ Y = \int d^3x' \mathcal{K}_{V(\hat{r})}(x - x')\psi_\xi(t, x') = \psi_{\xi}(t, x), \]

By substituting then these matrices in Eq. (71) and applying the identities (63) and (64) we find the desired action

\[ [V(\hat{r})\psi_\xi](t, x) = \int d^3x' \mathcal{K}_{V(\hat{r})}(x - x')\psi_\xi(t, x') = \psi_{\xi}(t, x), \]

upon the basis of Pauli spinors. In addition, the general rule (77) allows us to derive the expectation values of these operators

\[ \langle \psi_\xi, V(\hat{r})\psi_\xi \rangle_D = \langle \alpha, D(\hat{r})\alpha \rangle + \langle \beta, D^+(\hat{r})\beta \rangle, \]

which depend explicitly on \( \xi \) through the matrix \( D(\hat{r}) \).

Furthermore, we consider the mode spinors (22) and (23) whose Wigner spinors are defined in Eqs. (39) and (40), the action of SU(2) rotations (60) and (61) as well the identities (A.3) and (A.4), deducing that the integral operators \( V(\hat{r}) \in \mathfrak{U} \) have kernels of the form (71) whose Fourier transforms read

\[ \tilde{V}(\hat{r}, \mathbf{p}) = \frac{m}{E(\mathbf{p})} \left[ l_\mathbf{p}^r + \frac{1 + \gamma^0}{2} l_\mathbf{p}^{-1} l_\mathbf{p}^r \gamma^0 l_\mathbf{p}^{-1} \right], \]

where \( r = \text{diag}(\hat{r}, \hat{r}) \in \rho_D \) while \( \tilde{\Pi}_\pm(\mathbf{p}) \) are the projection operators (A.5) and (A.6). The operator \( V(\hat{r}) \) is independent on the spinor bases under consideration as in momentum rep. the Dirac spinors \( u \) and \( \hat{v} \) give rise, as in Eqs. (43), to the projectors \( \frac{1 + r_\mathbf{p}}{2} \) which commute with \( r(\mathbf{p}) \) such that \( r(\mathbf{p})\frac{1 + r_\mathbf{p}}{2} r(\mathbf{p})^* = \frac{1 + r_\mathbf{p}}{2} \).

We defined thus the set \( \mathfrak{U} \) of operators whose properties can be studied in momentum rep. as their Fourier transforms obey the same algebra,

\[ V = V_1 V_2 \to \mathcal{K}_V = \mathcal{K}_{V_1} * \mathcal{K}_{V_2} \to \tilde{V}(\mathbf{p}) = \tilde{V}_1(\mathbf{p})\tilde{V}_2(\mathbf{p}). \]

By using then the identities (A.3) and (A.4), after a little calculation, we verify that

\[ \tilde{V}(\hat{r}, \mathbf{p})\tilde{V}(\hat{r}', \mathbf{p}) = \tilde{V}(\hat{r}' \hat{r}, \mathbf{p}), \]
observing that for \( \mathbf{r} = 1_{2 \times 2} \) we have

\[
\tilde{V}(1_{2 \times 2}, \mathbf{p}) = \tilde{\Pi}^+(\mathbf{p}) + \tilde{\Pi}^-(\mathbf{p}) = 1 \in \rho_D .
\]

We conclude that the set \( \mathcal{G} \) form just the \( SU(2) \) rep. we are looking for. We arrive thus at our principal objective, namely the definition of spin operator.

**Definition 1** The spin operator is the vector-operator \( \mathbf{S} \) whose components are generators of the representation \( V : SU(2) \to \mathcal{G} \).

Starting with the transf. \( V(\mathbf{r}(\theta)) \) depending on the rotation \( \mathbf{r} \) we derive the spin comps.,

\[
S_i = i \frac{\partial V(\mathbf{r}(\theta))}{\partial \theta^i} \bigg|_{\theta^i = 0} ,
\]

finding that they are integral operators acting as

\[
[S_i \psi_\xi](t, \mathbf{x}) = \int d^3x' K_{\xi', \xi}(\mathbf{x} - \mathbf{x}') \psi_\xi(t, \mathbf{x}') \psi\xi_{\xi'}(t, \mathbf{x}) ,
\]

through kernels having as Fourier transforms the spin comps. in momentum rep.,

\[
\tilde{S}_i(\mathbf{p}) = \frac{m}{E(p)} \left[ l_p s_i \frac{1 + \gamma^0 l_p}{2} - l_p^{-1} s_i \frac{1 - \gamma^0 l_p}{2} \right]
+ l_p s_i l_p^{-1} \tilde{\Pi}^+(\mathbf{p}) + l_p^{-1} s_i l_p \tilde{\Pi}^-(\mathbf{p}) ,
\]

where, as mentioned before, \( s_i = \text{diag}(\hat{s}_i, \hat{s}_i) \) are the \( SU(2) \) generators of rep. \( \rho_D \). Surprisingly, after a little calculation, we find that the operators defined above are just the comps. of Pryce’s spin operator,

\[
\tilde{S}_i(\mathbf{p}) = \frac{m}{E(p)} s_i + \frac{\mathbf{p}^i(s \cdot \mathbf{p})}{E(p)(E(p) + m)} + \frac{i}{2E(p)} \epsilon_{ijk} p^j \gamma^k ,
\]

proposed long time ago (see the third of Eqs. (6.7) of Ref. [2]). These operators generate the \( su(2) \) algebra,

\[
[S_i, \tilde{S}_j(\mathbf{p})] = i \epsilon_{ijk} \tilde{S}_k(\mathbf{p}) \to [S_i, S_j] = i \epsilon_{ijk} S_k ,
\]

are self-adjoint and conserved commuting with the Dirac Hamiltonian in momentum rep. [34]. The action of these operators on the mode spinors can be derived as in Eq. (72) obtaining

\[
(S_i U_{p, \xi}(x)) = \tilde{S}_i(p) U_{p, \xi}(x) = U_{p, \hat{s}_i \xi}(x) ,
\]

\[
(S_i V_{p, \eta}(x)) = \tilde{S}_i(-p) V_{p, \eta}(x) = V_{p, \hat{s}_i \eta}(x) ,
\]

after using the form [87], expressions of mode spinors and the identities [A.3] and [A.4].

In other respects, the polarization is given by the related spinors \( \xi_{p}(\mathbf{p}) \) and \( \eta_{p}(\mathbf{p}) \) assumed to satisfy the eigenvalues problems

\[
\hat{s}_i n^i \xi_{p}(\mathbf{p}) = \sigma \xi_{p}(\mathbf{p}) \to \hat{s}_i n^i \eta_{p}(\mathbf{p}) = -\sigma \eta_{p}(\mathbf{p}) ,
\]

where the unit vector \( \mathbf{n}(\mathbf{p}) \) give the peculiar direction with respect to which the peculiar polarization is measured. Under such circumstances we may define a convenient polarization operator proper to the present framework.
**Definition 2** The polarization operator is the integral operator $W$ whose kernel has the Fourier transform

$$W(p) = \frac{m}{E(p)} \left[ l_p s_i n_i(p) \frac{1 + \gamma^0}{2} - l_p^{-1} s_i n_i(-p) \frac{1 - \gamma^0}{2} \right]$$

This operator commutes with $H$ and $P^i$ acting on the mode spinors constructed with the spinors of Eqs. (92) as

$$(WU_{p,\xi}(p))(x) = \tilde{W}(p)U_{p,\xi}(p)(x) = U_{p,\xi[n^i(p)]}(p)(x)$$

$$(WV_{p,\eta}(p))(x) = \tilde{W}(-p)V_{p,\eta}(p)(x) = V_{p,\eta[n^i(p)]}(p)(x)$$

These eigenvalues problems convince us that $W$ is just the operator we need for completing the system of commuting operators as $\{H, P^1, P^2, P^3, W\}$ for defining properly the momentum reps. of RQM.

As the comps. of spin operator are integral operators we understand that $W$ remains an operator of this type even in the case of common polarization when $n$ and $\xi$ are independent on $p$ and, consequently, we may write $W = S \cdot n$.

The well-known example is the momentum-spin basis [7] where $n = e_3$ and the operator $W = S_3$ defines the basis spinors

$$\xi_+ = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad \xi_- = \left( \begin{array}{c} 0 \\ 1 \end{array} \right).$$

used in various applications [5, 7, 6]. Therefore, we may say that there are no situations in which the polarization might be defined properly by a differential polarization operator as, for example, the helicity one, $W_0$. We shall discuss later the difficulties related to this operator when we shall study the polarization and spin integral operators of momentum-helicity basis.

## 5 Quantization

The principal benefit of our approach is the opportunity of writing down the one-particle spin operators of QFT corresponding to Pryce’s ones, studying how these operators are related to the isometry generators of Dirac’s theory.

We adopt here the Bogolyubov method of quantization [10] replacing first the functions in momentum rep. with field operators, $(\alpha, \alpha^*) \rightarrow (a, a^\dagger)$ and $(\beta, \beta^*) \rightarrow (b, b^\dagger)$, satisfying canonical anti-commutation relations with the non-vanishing ones,

$$\{a_{\sigma}(p), a_{\sigma'}^\dagger(p')\} = \{b_{\sigma}(p), b_{\sigma'}^\dagger(p')\} = \delta_{\sigma \sigma'}\delta^3(p - p').$$

The Dirac field becomes thus a field operator while all the time-independent expectation values of isometry generators become one-particle operators, $X =: \langle \psi, X \psi \rangle_D :$, calculated respecting the normal ordering of the operator products.
We obtain thus a basis of operator algebra formed by field and one-particle operators which have the obvious properties

\[ [X, \psi(x)] = -(X \psi)(x), \quad [X, Y] = \langle \psi, ([X, Y] \psi) \rangle_D : \]

that preserve the linear structures. This basis will generate freely all the operators of Dirac’s QFT. We must specify that the quantization leads to Heisenberg’s picture in which an operator is conserved only if it commutes with the energy operator.

5.1 Charge, polarization and spin operators

Let us consider the basis of mode spinors generated by the complete system \{\mathcal{H}, P_1, P_2, P_3, W\} which are diagonal in this basis. In addition, the identity \(1 \in \rho_D\), which is the generator of the gauge group \(U(1)_{em}\), leads through quantization to the charge operator,

\[ Q := \langle \psi_\xi, \Pi \psi_\xi \rangle_D : = Q_+ + Q_- = \int d^3p \sum_\sigma [a_\sigma^+(p)a_\sigma(p) - b_\sigma^+(p)b_\sigma(p)] , \]

where the particle and antiparticle charge operators are given now just by our projection operators \(\Xi^5\) and \(\Xi^6\) as

\[ Q_{\pm} := \langle \psi_\xi, \Pi_{\pm} \psi_\xi \rangle_D : . \]

Then the operator of number of particles,

\[ N = Q_+ - Q_- := \langle \psi_\xi, (\Pi_+ - \Pi_-) \psi_\xi \rangle_D : , \]

is related to the operator whose kernel has the Fourier transform \(E(p)^{-1} \tilde{H}_D(p)\).

The polarization operator, supposed to be diagonal in this basis, has an almost trivial expectation value leading to the one-particle operator

\[ W := \langle \psi_\xi, \mathcal{W} \psi_\xi \rangle_D := \frac{1}{2} \int d^3p \sum_\sigma \sum_{\sigma'} \Sigma_{\sigma \sigma'}(p) \left[ a_{\sigma'}^+(p)a_\sigma(p) + b_{\sigma'}^+(p)b_\sigma(p) \right] , \]

which hides the dependence on \(\xi\).

For applying the same method to Pryce’s spin operator we look for expectation values of its comps. \(S_i\) at the level of RQM. These can be found deriving with respect to Cayley-Klein parameters \(\theta\) the expectation values \(\mathcal{S}_0\) where we substitute \(D(\hat{r}) \rightarrow D[\hat{r}(\theta)]\) with \(\hat{r}(\theta)\) given by Eq. \(\mathcal{S}_0\). As the quantization changes the wrong sign of the anti-particle term we obtain the one-particle operators

\[ S_i(\xi) = \langle \psi_\xi, S_i \psi_\xi \rangle_D : \]

\[ = \frac{1}{2} \int d^3p \sum_{\sigma, \sigma'} \Sigma_{i, \sigma \sigma'}(p) \left[ a_{\sigma'}^+(p)a_{\sigma}(p) + b_{\sigma'}^+(p)b_{\sigma}(p) \right] , \]

representing the comps. of the spin one. Here we denoted by

\[ \Sigma_{i, \sigma \sigma'}(p) = \xi^+_{\sigma'}(p) \sigma_i \xi_{\sigma}(p) , \]
the matrix elements of the Pauli matrices in the basis $\xi$ in which the quantum field $\psi_\xi$ was defined. Remarkably, for peculiar polarization the quantization lays out explicitly the dependence on $p$ of the spin comp. For common polarization the matrices (104) become independent on $p$ as, for example, in the traditional momentum-spin basis (96) where $\Sigma_i = \sigma_i$.

The operators $S_i(\xi)$ are self-adjoint and form an operator valued rep. of the $su(2) \sim so(3)$ algebra. They give rise to unitary operators, $V[\hat{r}(\theta)] \rightarrow V(\theta, \xi) = e^{i\theta^i S_i(\xi)}$, forming an unitary rep. of the $SU(2)$ little group able to transform the spinor bases. Indeed, by using exponential expansions we can show that $V(\theta, \xi)\psi_\xi(p) V^\dagger(\theta, \xi) = \psi_{\hat{r}(\theta)\xi}(p)$, helping us to change the spinor basis or to relate two different bases.

### 5.2 Isometry generators

Let us see now how the spin operator defined above is involved in the structure of the isometry generators. We start with the expectation values, $\langle \psi, X \psi \rangle_D = \langle \alpha, \tilde{X} \alpha \rangle - \langle \beta, \tilde{X} \beta \rangle$, derived from Eq. (59) for any pair of related generators $X \in L(T)$ and $\tilde{X} \in L(\tilde{T})$ corresponding to the same parameter. Applying then the quantization we change the wrong relative sign ($-$) in Eq. (109) after restoring the normal ordering of operator products, obtaining correct forms of one-particle operators.

In the basis under consideration the translation generators are diagonal and independent on $\xi$ such that the corresponding one-particle operators read

\begin{align}
P^i &= \langle \psi_\xi, P^i \psi_\xi \rangle_D := \int d^3p p^i \sum_\sigma \left[ a^\dagger_\sigma(p) a_\sigma(p) + b^\dagger_\sigma(p) b_\sigma(p) \right], \\
H &= \langle \psi_\xi, H \psi_\xi \rangle_D := \int d^3p E(p) \sum_\sigma \left[ a^\dagger_\sigma(p) a_\sigma(p) + b^\dagger_\sigma(p) b_\sigma(p) \right].
\end{align}

These operators commute among themselves and with $Q$ and the polarization operator (102) forming the complete system $\{H, P^1, P^2, P^3, W, Q\}$ determining the bases of the Fock state space.

The $SL(2, \mathbb{C})$ generators can be derived by using our parametrizations (2) and (3). For obtaining the rotation generators we take $\hat{\lambda} = \hat{r}(\theta)$ observing that the transformed momentum (38) can be expanded now as $p'^i = p^i + \epsilon_{ijk}p^j\theta^k + \cdots$, according to the general rule (4). Introducing these quantities in Eq. (109)
and deriving the transf. \( \text{[50]} \) with respect to the Cayley-Klein parameters \( \theta^i \) in \( \theta^i = 0 \) we obtain

\[
 J_i(\xi) =: \langle \psi_\xi, J_i \psi_\xi \rangle_D := L_i(\xi) + S_i(\xi), \quad (112)
\]

where the comps. \( S_i(\xi) \) of the spin operator are defined by Eq. \( \text{[108]} \) while the associated orbital angular momentum operator has the comps.

\[
 L_i(\xi) = -\frac{i}{2} \int d^3p \epsilon_{ijk} p^j \sum_{\sigma} \left[ a^\dagger_\sigma(p) \partial_{p^k} a_\sigma(p) + b^\dagger_\sigma(p) \partial_{p^k} b_\sigma(p) \right] - i \int d^3p \epsilon_{ijk} p^j \sum_{\sigma, \sigma'} \Omega_{\sigma, \sigma'}(p) \left[ a^\dagger_\sigma(p) a_{\sigma'}(p) + b^\dagger_\sigma(p) b_{\sigma'}(p) \right], \quad (113)
\]

where we denote

\[
 \Omega_{\sigma, \sigma'}(p) = -\Omega^*_{\sigma', \sigma}(p) = \xi^+_{\sigma'}(p) \left[ \partial_{p^i} \xi^+_{\sigma'}(p) \right], \quad (114)
\]

observing that the matrices \( i\Omega_i \) are Hermitian. The first integral gives genuine orbital operators, independent on \( \xi \), written with the notation \( f \stackrel{\hat{\lambda}}{\leftrightarrow} g = p^i \partial_{p^i} f - \frac{1}{2} \partial f \) which helps us to understand that these are self-adjoint operators. The second integral gives self-adjoint but non-diagonal terms which contribute only in the case of peculiar polarization. For common polarization we have \( \Omega_i = 0 \) and the orbital angular momentum becomes diagonal, independent on \( \xi \).

The commutation relations of these operators can be derived easily by using Eqs. \( \text{[15]} \) and \( \text{[16]} \) as well the obvious identities

\[
 \xi^+_{\sigma'}(p) \partial_{p^i} \xi^\dagger_{\sigma}(p) = \delta_{\sigma, \sigma'} \partial_{p^i} + \Omega_{i, \sigma, \sigma'} . \quad (115)
\]

We obtain first that the operators \( L_i(\xi) \) and \( S_i(\xi) \) form the bases of two independent \( \text{so}(2) \sim \text{su}(3) \) algebras commuting each other, \( [L_i(\xi), S_j(\xi)] = 0 \).

Moreover, these operators are conserved separately, each one commuting with the Hamiltonian operator,

\[
 [H, L_i(\xi)] = 0, \quad [H, S_i(\xi)] = 0 . \quad (116)
\]

Therefore, we may conclude that the Pryce spin operator of RQM gives just the conserved one-particles spin operator we need in QFT.

The generators of the Lorentz boosts can be found by choosing \( \hat{\lambda} = \hat{l}(\tau) \) as in Eq. \( \text{[3]} \), observing that now \( p^i = p^* - \tau^i E(p) + \cdots \) and deriving Eq. \( \text{[50]} \) with respect to \( \tau^i \) in \( \tau = 0 \). After a few manipulation we obtain

\[
 K_i(\xi) =: \langle \psi_\xi, K_i \psi_\xi \rangle_D : = \int d^3p \sum_{\sigma, \sigma'} K_{i, \sigma, \sigma'}(p) \left[ a^\dagger_{\sigma'}(p) a_{\sigma}(p) + b^\dagger_{\sigma'}(p) b_{\sigma}(p) \right] + \frac{i}{2} \int d^3p E(p) \sum_{\sigma} \left[ a^\dagger_{\sigma}(p) \partial_{p^i} a_{\sigma}(p) + b^\dagger_{\sigma}(p) \partial_{p^i} b_{\sigma}(p) \right], \quad (117)
\]

where we denoted

\[
 K_i(p) = iE(p)\Omega_i(p) + \frac{1}{2(E(p) + m)} \epsilon_{ijk} p^j \Sigma_\ell(p). \quad (118)
\]
These are self-adjoint operators but they are not conserved satisfying the usual commutation relation

$$[H, K_i(\xi)] = -iP^i,$$  \hspace{1cm} (119)

of the $\text{sl}(2, \mathbb{C})$ algebra. In addition, we must specify that the operators $K_i(\xi)$ cannot be split as the total angular momentum, satisfying canonical commutation relations,

$$[J_i(\xi), K_j(\xi)] = i\epsilon_{ijk}K^k(\xi),$$  \hspace{1cm} (120)

$$[K_i(\xi), K_j(\xi)] = -i\epsilon_{ijk}J^k(\xi),$$  \hspace{1cm} (121)

but without giving relevant commutators with $L_i(\xi)$ or $S_i(\xi)$. This means that the splitting (112) cannot be extended to the entire $\text{sl}(2, \mathbb{C})$ algebra.

We derived thus the self-adjoint basis-generators of a family of unitary reps. of the group $\tilde{P}^{\uparrow}_+$ with values in one-particle operators which are determined by the bases $\xi$ we chose for describing polarization.

6 Example: momentum-helicity basis

The most used peculiar polarization is the helicity giving rise to the momentum-helicity basis in which the spinors $\xi_{\sigma}(p)$ and $\eta_{\sigma}(p) = i\sigma_2\xi_{\sigma}^*(p)$ satisfy the related eigenvalues problems

$$\hat{s}_i n^i p \xi_{\sigma}(p) = \sigma \xi_{\sigma}(p) \rightarrow \hat{s}_i n^i p \eta_{\sigma}(p) = -\sigma \eta_{\sigma}(p),$$  \hspace{1cm} (122)

where $n = \frac{p}{|p|}$ is the unit vector of $p$. These spinors can be obtained transforming the spin basis (96) as

$$\xi_{\sigma}(p) = \hat{r}_h(p)\xi_{\sigma} \rightarrow \eta_{\sigma}(p) = \hat{r}_h(p)\eta_{\sigma},$$  \hspace{1cm} (123)

with the help of the $SU(2)$ rotation

$$\hat{r}_h(p) = \sqrt{\frac{p+p^3}{2p}} \left[ \begin{array}{cc} 1 & -i\hat{r}_h(p) \\ i\hat{r}_h(p) & 1 \end{array} \right].$$  \hspace{1cm} (124)

The corresponding $SO(3)$ rotation, $R(\hat{r}_h(p))$, transforms the polarization direction $e_3$ of the spin basis into the helicity one, $n_p$. Our approach offers now the opportunity of finding the corresponding transf. between the mode spinors. This is performed by the integral operator $V_h$ whose kernel has the Fourier transform

$$\tilde{V}_h(p) = l_p r_h(p) l^{-1}_p \tilde{\Pi}_+(p) + l^{-1}_p r_h(-p) l_p \tilde{\Pi}_-(p),$$  \hspace{1cm} (125)

where $r_h(p) = \text{diag}(\hat{r}_h(p), -\hat{r}_h(p)) \in \rho_D$.

Furthermore, we focus on the polarization observing that in this case the polarization operator (93) with $n = n_p$ can be put in the form

$$\tilde{W}(p) = s_i n^i_p \left[ \tilde{\Pi}_+(p) - \tilde{\Pi}_-(p) \right] = s_i n^i_p \hat{H}_D(p) E(p),$$  \hspace{1cm} (126)

as $n(-p) = -n_p$ and $s_i n^i_p$ commutes with $l_p$. Then the mode spinors constructed with the helicity spinors satisfy the eigenvalues problems (124) and (125).
which guarantee the correct quantization [102]. On the other hand, here we may use the components of the Pauli-Lubanski operator interpreted as covariant a four-vector spin operator. Its 0-th component is the helicity operator

\[ W_0 = J_i P^i = s^i P^i \to \tilde{W}_0(p) = \tilde{S}_i(p)p^i = s_i p^i, \tag{127} \]

whose action on the mode spinors,

\[
(W_0 U_{p,\xi^e(p)}(x) = \tilde{W}_0(p) U_{p,\xi^e(p)}(x),
\]

\[
(W_0 V_{p,\eta^e(p)}(x) = \tilde{W}_0(-p) V_{p,\eta^e(p)}(x)
\]

\[ = \sigma p U_{p,\xi^e(p)}(x), \tag{128} \]

\[ = \sigma p V_{p,\eta^e(p)}(x), \tag{129} \]

calculated as in the previous case, is different from that of \( W \) as the eigenvalue of Eq. (129) is \( \sigma p \) instead of \( -\sigma p \) needed for suitable quantization. This difference comes from the term \( E(p)^{-1} \tilde{H}_D(p) \) of Eq. (126) which assures correct eigenvalues of the operator \( W \). Therefore, at the level of QFT it is convenient to quantize the operator \( W \rightarrow W_0 \) as in Eq. (102) instead of \( W \).

For writing down the isometry generators of QFT we derive the matrices (104) in this basis,

\[
\Sigma_1(p) = \frac{p^1}{p} p^1 \sigma_1 + \frac{p^2}{p(p + p^3)} p^2 \sigma_2 + \sigma_1 ,
\]

\[
\Sigma_2(p) = \frac{p^2}{p} p^2 \sigma_3 - \frac{p^1}{p(p + p^3)} p^1 \sigma_1 + \sigma_2 ,
\]

\[
\Sigma_3(p) = \frac{p^3}{p} p^3 \sigma_2 + \frac{p^2}{p} p^2 \sigma_1 - \sigma_2 , \tag{130} \]

finding that the matrices (114) read

\[
\Omega_1(p) = \frac{-i}{2p^2(p + p^3)} \left[ p^1 p^2 \sigma_1 + pp^2 \sigma_2 + \sigma_1 \right],
\]

\[
\Omega_2(p) = \frac{i}{2p^2(p + p^3)} \left[ p^1 p^2 \sigma_2 + pp^1 \sigma_3 + \sigma_1 \right], \tag{131} \]

\[
\Omega_3(p) = \frac{i}{2p^2} (p^1 \sigma_2 - p^2 \sigma_1) .
\]

With their help we can derive for the first time the isometry generators in momentum-helicity basis as one-particle operators of the Dirac QFT.

In applications we may turn back to RQM but considered now as the one-particle restriction of QFT. For example, in the normalized state

\[ |\alpha\rangle = \int d^3p \sum_\sigma \alpha_\sigma(p) a_\sigma^\dagger(p)|0\rangle, \quad \langle \alpha|\alpha\rangle = 1 , \tag{132} \]

defined by the wave functions \( \alpha_\sigma(p) \) that form the normalized Pauli spinor \( \alpha \in \mathcal{F}_\alpha \) as in Eq. (28), we may calculate the expectation value of any generator \( X \) as

\[ \langle \alpha|X|\alpha\rangle = \langle \alpha, \tilde{X} \alpha \rangle , \tag{133} \]
where $\tilde{X} \in \mathcal{L}(\tilde{T})$ is the generator of RQM corresponding to $X$. The list of these generators can be written down, according to Eqs. (110-118), but omitting the explicit dependence on $p$,

$$\tilde{P}_i = p_i, \quad \tilde{H} = E, \quad \tilde{W} = \frac{1}{2} \sigma_3,$$

(134)

$$\tilde{J}_i = \tilde{L}_i + \tilde{S}_i, \quad \tilde{S}_i = \frac{1}{2} \Sigma_i, \quad \tilde{L}_i = -i \epsilon_{ijk} p^j (\partial_{p^k} + \Omega_k),$$

(135)

$$\tilde{K}_i = i E (\partial_{p^i} + \Omega_i) + \frac{1}{2(E + m)} \epsilon_{ijk} p^j \Sigma_k.$$

(136)

We obtain thus intuitive expressions which apparently are complicated as the matrices $\Sigma_i$ and $\Omega_i$ are given by Eqs. (130) and respectively (131). However, the algebra of these operators is the same as in momentum-spin basis where $\Omega_i = 0$ and $\Sigma_i = \sigma_i$. For example, the identities (115) help us to show that the operators $\Sigma_i$ and $\partial_{p^i} + \Omega_i$ satisfy the same commutation relations as $\sigma_i$ and respectively $\partial_{p^i}$. Remarkably, the polarization operator defined here becomes now just the Pauli one.

Other simple identities, $p^i \Omega_i = 0$ and $p^i \Sigma_i = p \sigma_3$, allow us to write

$$\tilde{P}^i \tilde{L}_i = 0 \rightarrow \tilde{W}_0 = \tilde{P}^i \tilde{J}_i = \tilde{P}^i \tilde{S}_i = \frac{1}{2} p \sigma_3 = p \tilde{W},$$

(137)

at least in the particle sector. In the anti-particle sector all the above operators keep their forms apart $\tilde{W}_0$ which changes its sign as we deduce observing that the eigenvalues of Eqs. (95) and (129) have opposite signs.

More algebra may be performed resorting to algebraic codes on computer. We obtain thus the space comps. of the Pauli-Lubanski operator,

$$\tilde{W}_i = E \tilde{J}_i + \epsilon_{ijk} p^j \tilde{K}_k = m \tilde{S}_i + \frac{p^i}{E + m} \tilde{W}_0,$$

(138)

and verify that the second Casimir operator (15) in this rep., gives the expected invariant $\tilde{C}_2 = -\eta^{\mu\nu} \tilde{W}_\mu \tilde{W}_\nu = \frac{3}{4} m^2$. Note that in this rep. the first invariant (14) is implicit as $E$ is just the relativistic energy.

This example shows that the structure and properties of the spin and polarization operators defined here are close to the original non-relativistic Pauli’s spin theory.

### 7 Concluding remarks

We have shown that the comps. of spin operator proposed by Pryce in momentum rep. are Fourier transforms of the kernels of integral operators generating a rep. of the little group carried by the space of Pauli’s spinors defining the polarization. After quantization they become conserved one-particle operators $\tilde{S}_i$ representing the comps. of the desired spin operator splitting naturally the total angular momentum, in two conserved parts, i. e. this spin operator and the associated conserved angular momentum of comps. $L_i$. Therefore, we must accept that this is the correct spin operator we need for defining and controlling the polarization in special relativistic QFT.

Turning back to RQM, it is obvious that a new spin operator corresponds to a new coordinate operator giving the associated orbital angular momentum if the
total angular momentum remains unchanged. For this reason Pryce proposed simultaneously the spin operator $S$ we studied here and a related coordinate operator, $X = x + \delta X$, formed by the position vector $x$ and $\delta X$ that is an integral operator defined in momentum rep. [2] such that $P \wedge \delta X = S - s$. As this operator was less studied until now we believe that it deserves a careful investigation in RQM and especially at the level of QFT. In this manner we may understand the original paradigm in which these operators were found completing thus our image about the merit of the prominent physicist which was M. H. L. Pryce.

Appendix: Boosts and projection operators

The standard boosts of $\rho_D$ have the form [3] with parameters $\tau^i = m \tanh^{-1} \frac{p^i}{E(p)}$ such that the matrix [9]

$$l_p = \frac{E(p) + m + \gamma^0 \gamma^i p^i}{\sqrt{2m(E(p) + m)}} \in \rho_D,$$  \hspace{1cm} (A.1)

satisfies $l_p = l^+_p$ and $l^{-1}_p = l^-_p = \gamma^0 l^0_p \gamma^0$. The matrices

$$l^2_p = \frac{E(p) + \gamma^0 \gamma^i p^i}{m}, \quad l^2_p = \frac{E(p) - \gamma^0 \gamma^i p^i}{m},$$  \hspace{1cm} (A.2)

give rise to the following identities

$$\frac{1 + \gamma^0}{2} l^2_p \frac{1 + \gamma^0}{2} = \frac{E(p) \frac{1 + \gamma^0}{2}}{m},$$  \hspace{1cm} (A.3)

$$\frac{1 - \gamma^0}{2} l^2_p \frac{1 - \gamma^0}{2} = \frac{E(p) \frac{1 - \gamma^0}{2}}{m},$$  \hspace{1cm} (A.4)

which help us to define the integral operators $\Pi_+$ and $\Pi_-$ whose kernels have the Fourier transforms

$$\Pi_+(p) = \frac{m}{E(p)} l^+_p \frac{1 + \gamma^0}{2} l_p = \frac{1}{2} \left(1 + \frac{\hat{H}_D(p)}{E(p)}\right),$$  \hspace{1cm} (A.5)

$$\Pi_-(p) = \frac{m}{E(p)} l^-_p \frac{1 - \gamma^0}{2} l^-_p = \frac{1}{2} \left(1 - \frac{\hat{H}_D(p)}{E(p)}\right),$$  \hspace{1cm} (A.6)

where $\hat{H}_D(p)$ is given by Eq. [34]. We can verify now that $\Pi_+$ and $\Pi_-$ form a complete system of orthogonal projection operators satisfying $\Pi^2_+ = \Pi_+$, $\Pi^2_- = \Pi_-$, $\Pi_+ \Pi_- = \Pi_- \Pi_+ = 0$ and $\Pi_+ + \Pi_- = 1 \in \rho_D$. According to Eqs. [32] and [33] we find that these operators separate the mode spinors of positive and negative frequencies as $\Pi_+ F_D = F^+_D$ and $\Pi_- F_D = F^-_D$.

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