Critical behavior of 3D $SU(2)$ gauge theory at finite temperature: exact results from universality

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Abstract

We show that universality arguments, namely the Svetitsky–Yaffe conjecture, allow one to obtain exact results on the critical behavior of 3D $SU(2)$ gauge theory at the finite temperature deconfinement transition, through a mapping into the 2D Ising model. In particular, we consider the finite-size scaling behavior of the plaquette operator, which can be mapped into the energy operator of the 2D Ising model. We obtain exact predictions for the dependence of the plaquette expectation value on the size and shape of the lattice and we compare them to Monte Carlo results, finding complete agreement. We discuss the application of this method to the computation of more general correlators of the plaquette operator at criticality, and its relevance to the study of the color flux tube structure.

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1 Introduction

The idea of universality plays a major role in the modern understanding of critical phenomena. All the physical systems undergoing a continuous phase transition are believed to fall in a certain number of universality classes, depending on the dimensionality of the system and its symmetry group, but not on the details of the microscopic interactions (as long as these are short–ranged). All the systems in a given universality class display the same critical behavior, meaning that certain dimensionless quantities have the same value for all the physical systems in a given class. Critical indices and universal amplitude ratios are examples of these universal quantities.

For gauge theories with a high temperature deconfinement transition, the universality hypothesis takes the form of the Svetitsky–Yaffe conjecture, which can be formulated as follows: suppose a $d+1$–dimensional gauge theory with gauge group $G$ has a second–order deconfinement transition at a certain temperature $T_c$; consider the $d$–dimensional statistical model with global symmetry group coinciding with the center of the gauge group: if also this model displays a second–order phase transition, then the two models belong to the same universality class. The validity of the conjecture has been well established in several Monte Carlo analyses (see e.g. and references therein). For the case we are interested in here, namely $G = SU(2)$ and $d = 2$, a precise numerical test of the Svetitsky–Yaffe conjecture can be found in Ref.[3].

The most obvious application of universality arguments, and in particular of the Svetitsky–Yaffe conjecture, is the prediction of the critical indices. For example, consider $SU(2)$ gauge theory; it undergoes a high–temperature deconfinement transition which is known to be second–order in both three and four space–time dimensions. The center of $SU(2)$ is $Z_2$, therefore the dimensionally reduced statistical model is the Ising model, which has a second–order phase transition both in $d = 2$ and $d = 3$. Hence the Svetitsky–Yaffe conjecture applies, and we can predict the critical indices of the $SU(2)$ deconfinement transition in $d+1$ dimensions to coincide with the ones of the Ising model in $d$ dimensions.

However, the predictive power of universality is certainly not limited to the values of the critical indices. In Ref.[4, 5, 6] a program has been initiated of systematic exploitation of universality arguments in studying the non–perturbative physics of gauge theories. For example, it was shown that non–trivial results on finite–size effects and correlation functions at the deconfinement point can be obtained from universality arguments. In this
way it has been possible to evaluate exactly the expectation value of the plaquette operator in presence of static sources, giving some new insight into the structure of the color flux tube for mesons and baryons.

In this paper we continue the program by analysing the finite–size scaling behavior of the plaquette operator in 3D $SU(2)$ gauge theory at the deconfinement temperature. Since the 2D Ising model is exactly solved, the Svetitsky–Yaffe conjecture gives in this case exact predictions on finite–size scaling effects. We write down these predictions for the expectation value of the plaquette operator and we compare them with Monte Carlo results. The same analysis was performed in Ref.[6] for $Z_2$ gauge theory.

2 Finite–size behavior of the plaquette expectation value

The Svetitsky–Yaffe conjecture can be seen as a mapping between observables of the 3D $SU(2)$ gauge theory at finite temperature and operators of the 2D Ising model. The Polyakov loop is mapped into the magnetization, while the plaquette operator is mapped into a linear combination of the identity and the energy operators of the statistical model [6]:

$$\langle \Box \rangle = c_1 \langle 1 \rangle + c_\epsilon \langle \epsilon \rangle + \ldots$$

where the expectation value in the l.h.s. is taken in the gauge theory, while the ones in the r.h.s. refer to the two–dimensional Ising model. The dots represent contributions from secondary fields in the conformal families of the identity and energy operators, whose contributions are subleading for asymptotically large lattices.

The finite–size dependence of the energy expectation value in the two–dimensional Ising model on a torus is [7, 8, 9]

$$\langle \epsilon \rangle = \frac{\pi \sqrt{3 m} |\eta(\tau)|^2}{\sqrt{A Z_{1/2}(\tau)}}$$

where $A \equiv L_1 L_2$ and $\tau \equiv iL_1/L_2$ are respectively the area and the modular parameter of the torus. $Z_{1/2}$ is the partition function of the Ising model at the critical point:

$$Z_{1/2} = \frac{1}{2} \sum_{\nu=2}^{4} \left| \frac{\theta_\nu(0, \tau)}{\eta(\tau)} \right|$$

2
We follow the notations of Ref. [9] for the Jacobi theta functions $\theta_\nu$ and the Dedekind function $\eta$.

Consider now 3D $SU(2)$ lattice gauge theory regularized on a $L_1 \times L_2 \times N_t$ lattice, with $L_1, L_2 \gg N_t$. For a given $N_t$ the gauge coupling $\beta$ can be tuned to a critical value $\beta_c(N_t)$ to simulate the theory at the finite temperature deconfinement phase transition. Precise evaluations of $\beta_c(N_t)$ for various values of $N_t$ are available in the literature [3]. The universality argument gives us the following prediction for the finite–size scaling behavior of the plaquette operator at the deconfinement point

$$\langle \square \rangle_{L_1 L_2} = c_1 + c_\epsilon \frac{F(\tau)}{\sqrt{L_1 L_2}} + O(1/L_1 L_2) \quad (4)$$

where $F$ is a function of the modular parameter $\tau \equiv i L_1 / L_2$ only:

$$F(\tau) = \frac{\pi \sqrt{3 m \tau} |\eta(\tau)|^2}{Z_{1/2}(\tau)}. \quad (5)$$

Here $c_1$ and $c_\epsilon$ are non–universal constants which depend on $N_t$ and must be determined numerically. Once these have been determined, Eq. (4) predicts the expectation value of the plaquette operator for all sizes and shapes of the lattice, i.e. for all values of $L_1$ and $L_2$. The $O(1/L_1 L_2)$ corrections represent the contribution of secondary fields. Therefore Eq. (4) is valid asymptotically for large lattices.

### 3 Plaquette correlators

Once the constants $c_1$ and $c_\epsilon$ have been determined at a given value of $N_t$, for example through the finite–size scaling analysis presented here, all the correlation functions of the plaquette operator at criticality are in principle exactly computable, since they can be readily derived from the corresponding correlators of the energy operator in the 2d Ising model.

Consider for example the plaquette expectation value in presence of static sources, i.e. the correlation function of the plaquette with $n$ Polyakov loops:

$$G(x; y_1, \ldots, y_n) = \langle \square(x) P(y_1) \ldots P(y_n) \rangle \quad (6)$$

This is the typical quantity to study if one is interested in the structure of the color flux tube, since it represents the density of action in presence
of external sources. Universality then tells us that the connected part of 

\[ G(x; y_1, \ldots, y_n) \] 

is given by 

\[ G_c(x; y_1, \ldots, y_n) = c_\sigma^n c_\epsilon \langle \epsilon(x) \sigma(y_1) \ldots \sigma(y_n) \rangle \tag{7} \]

where the correlator on the r.h.s. is computed in the critical 2D Ising model, \( \sigma \) is the spin operator and \( c_\sigma \) is the constant which relates it to the Polyakov loop; \( c_\sigma \) can be determined with methods similar to the ones we used here to determine \( c_\epsilon \).

Therefore the Svetitsky–Yaffe conjecture, together with the exact results of 2D conformal field theory, gives us complete control over the critical behavior of 3D SU(2) gauge theory at the finite temperature deconfinement transition. More generally, this holds for every 3D gauge theory whose deconfinement transition is second–order, including SU(3). Some studies of the flux–tube structure following this line were presented in Ref. [6].

4 Comparison with Monte Carlo results for SU(2) gauge theory

Three–dimensional finite temperature gauge theories are simulated by using \( L_1 \times L_2 \times N_t \) lattices with \( N_t \ll L_1, L_2 \). We have performed our simulations of SU(2) pure gauge theory with \( N_t = 2, 4 \) and \( L_1, L_2 \) varying between 8 and 30, and modular parameter \( \Im \tau \) between 1 and 3. The boundary conditions are periodic in all three directions. For each value of \( N_t \) we have chosen the coupling \( \beta_c(N_t) \) corresponding to the deconfinement transition point: from Ref. [3] we have

\[ \beta_c(2) = 3.469 \tag{8} \]
\[ \beta_c(4) = 6.588 \tag{9} \]

The only free parameters in Eq. (3) are the non–universal constants \( c_1 \) and \( c_\epsilon \), which depend on the value on \( N_t \) and on the kind of plaquette we are considering (time–like or space–like). Therefore we have a total of four sets of data to be compared with the theoretical prediction: for each of these sets we can perform a two–parameter fit with Eq. (4). It turns out however that finite–size effects on space–like plaquettes are of the same order of magnitude than the typical statistical uncertainties of our simulations. Therefore from now on we will consider time–like plaquettes only.
It is important to stress that for each value of \( N_t \) all the data, corresponding to different values of \( \Im \tau \), are included in the same two-parameter fit. Specifically, for each lattice with sides \( (L_1, L_2) \) we define an "effective area"

\[
\alpha(L_1, L_2) = \frac{L_1 L_2}{L_1^2 \left( \frac{i L_1}{L_2} \right)}
\]

so that Eq. (4) becomes

\[
\langle \Box \rangle_{L_1L_2} = c_1 + \frac{c_\epsilon}{\sqrt{\alpha(L_1, L_2)}}
\]

and we can fit the plaquette expectation value to a linear function of \( 1/\sqrt{\alpha} \).

The Monte Carlo results for the plaquette expectation values are reported in Tab.1 and Tab.2 while the results of the fits for \( N_t = 2 \) and \( N_t = 4 \) are reported in Tab.3. The values of the reduced \( \chi^2 \) show that the agreement is very satisfactory. In Figs. 1 and 2 the plaquette expectation values are plotted against \( 1/\sqrt{\alpha} \) together with the best fit line. The same data are plotted against \( 1/\sqrt{L_1L_2} \) in Fig. 3 for \( N_t = 2 \). This shows the crucial importance of including the non-trivial part of Eq. (4), namely its \( \tau \)-dependence.

The same analysis was performed in Ref.[6] for 3D \( \mathbb{Z}_2 \) gauge theory, whose finite temperature deconfinement transition is also in the universality class of the 2D Ising model.

5 Conclusions

We have shown that the Svetitsky–Yaffe conjecture, i.e. universality applied to the deconfinement transition, provides an exact description of finite–size effects for 3D \( SU(2) \) gauge theory at the deconfinement point. This description is obtained by mapping the gauge theory into the 2D Ising model, which is in the same universality class, and is exactly solved. We have compared the predictions obtained from this mapping with Monte Carlo results, finding complete agreement.

\footnote{The values of \( c_\epsilon \) reported in Ref.[6] for \( \mathbb{Z}_2 \) gauge theory are negative: this is due to the fact that the simulation was actually performed in the 3D spin Ising model, using 3D duality. As a consequence, the Svetitsky-Yaffe mapping between 3D observables and 2D Ising operators at criticality includes a duality transformation, which changes the sign of \( \epsilon \).}
The main motivation for this work was not to verify the validity of the Svetitsky–Yaffe conjecture, which is by now well established. Our intent is rather to stress that universality arguments provide a powerful, analytical approach to a deeply non–perturbative region, namely the deconfinement transition and its neighborhood. This is especially true for 3D gauge theories, since critical behavior in 2D is completely understood with the techniques of conformal field theory.

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Table 1: *Monte Carlo results for the time–like plaquette expectation values at the deconfinement transition point, for \( N_t = 2 \)*

| \( L_1 \) | \( L_2 \) | \( \langle \Box \rangle \) |
|---------|---------|----------------|
| 10      | 10      | 0.690737(26)   |
| 10      | 20      | 0.689930(26)   |
| 10      | 30      | 0.689402(26)   |
| 12      | 12      | 0.690393(26)   |
| 12      | 24      | 0.689762(23)   |
| 12      | 28      | 0.689593(26)   |
| 12      | 36      | 0.689304(26)   |
| 14      | 14      | 0.690201(22)   |
| 14      | 28      | 0.689626(26)   |
| 16      | 16      | 0.690090(23)   |
| 16      | 20      | 0.689912(21)   |
| 16      | 32      | 0.689566(23)   |
| 18      | 18      | 0.689963(21)   |
| 20      | 20      | 0.689835(20)   |
| 22      | 22      | 0.689753(30)   |
| 24      | 24      | 0.689697(20)   |

Table 2: *Same as Tab. 1 for \( N_t = 4 \)*

| \( L_1 \) | \( L_2 \) | \( \langle \Box \rangle \) |
|---------|---------|----------------|
| 12      | 12      | 0.8418794(72) |
| 14      | 14      | 0.8418684(64) |
| 14      | 20      | 0.8418340(52) |
| 16      | 16      | 0.8418488(57) |
| 16      | 24      | 0.8418210(54) |
| 20      | 20      | 0.8418219(49) |
| 20      | 24      | 0.8418157(54) |
| 22      | 22      | 0.8418143(43) |
Table 3: Results of the fit of the plaquette expectation values with Eq. (4) for $N_t = 2$ and $N_t = 4$. Each fit includes data from different values of the modular parameter.

| $N_t$ | $c_1$       | $c_6$       | $\chi^2_{\text{red}}$ |
|-------|-------------|-------------|------------------------|
| 2     | 0.688923(18)| 0.01856(33) | 0.92                   |
| 4     | 0.8417328(93)| 0.00186(17) | 0.32                   |

Figure 1: Plaquette expectation value as a function of $\alpha^{-1/2}$ for $N_t = 2$. Error bars are comparable to the size of the plotting symbol. The line is the best fit to Eq. (11).
Figure 2: Same as Fig. 1 for $N_t = 4$
Figure 3: Plaquette expectation value as a function of $(L_1 L_2)^{-1/2}$ for $N_t = 2$