Dynamics of the nonlinear Klein-Gordon equation in the nonrelativistic limit, II

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October 15, 2018

Abstract
We study the nonlinear Klein-Gordon (NLKG) equation on a manifold $M$ in the nonrelativistic limit, namely as the speed of light $c$ tends to infinity. In particular, we consider an order-$r$ normalized approximation of NLKG (which corresponds to the NLS at order $r = 1$), and prove that when $M = \mathbb{R}^d$, $d \geq 2$, small radiation solutions of the order-$r$ normalized equation approximate solutions of the NLKG up to times of order $O(c^{2(r-1)})$.

Keywords: nonrelativistic limit, nonlinear Klein-Gordon equation

MSC2010: 37K55, 70H08, 70K45, 81Q05

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1 Introduction

This paper is a continuation of [Pas17]. In these two papers the nonlinear Klein-Gordon (NLKG) equation in the nonrelativistic limit, namely as the speed of light $c$ tends to infinity, is studied.

The nonrelativistic limit for the Klein-Gordon equation on $\mathbb{R}^d$ has been extensively studied over more than 30 years, and essentially all the known results only show convergence of the solutions of NLKG to the solutions of the approximate equation for times of order $O(1)$. The typical statement ensures convergence locally uniformly in time. In a first series of results (see [Tsu84], [Naj90] and [Mac01]) it was shown that, if the initial data are in a certain smoothness class, then the solutions converge in a weaker topology to the solutions of the approximating equation. These are informally called “results with loss of smoothness”. Although in this paper a longer time convergence is proved, this result also fills in this group.

Recently, Lu and Zhang in [LZ16] proved a result which concerns the NLKG with a quadratic nonlinearity. Here the problem is that the typical scale over which the standard approach allows to control the dynamics is $O(c^{-1})$, while the dynamics of the approximating equation takes place over time scales of order $O(1)$. In that work the authors are able to use a normal form transformation (in a spirit quite different from ours) in order to extend the time of validity of the approximation over the $O(1)$ time scale. We did not try to reproduce or extend that result.

In [Pas17] Birkhoff normal form methods were used in order to extend the approximation up to order $O(1)$ to the NLKG equation on $M$, $M$ being a compact smooth manifold or $\mathbb{R}^d$; when $M = \mathbb{R}^d$, $d \geq 2$, the approximation of solutions of the linear KG equation with solutions of the linearized order-$r$ normalized equation up to times of order $O(c^{2(r-1)})$ is proved.

In this paper we prove a long-time approximation result for the dynamics of the NLKG: we consider the NLKG equation on $\mathbb{R}^d$, $d \geq 2$, and we prove that for $r > 1$ solutions of the order-$r$ normalized equation approximate solutions of the NLKG equation up to times of order $O(c^{2(r-1)})$.

The present paper and [Pas17] can be thought as examples in which techniques from canonical perturbation theory are used together with results from the theory of dispersive equations in order to understand the singular limit of some Hamiltonian PDEs. In this context, the nonrelativistic limit of the NLKG is a relevant example.

The issue of nonrelativistic limit has been studied also in the more general Maxwell-Klein-Gordon system ([BMS04], [MN03]), in the Klein-Gordon-Zakharov system ([MN08], [MN10]), in the Hartree equation ([CO06]) and in the pseudo-relativistic NLS ([CS16]). However, all these results proved the convergence of the solutions locally uniformly in time; no information could be obtained about the convergence of solutions for longer (in the case of NLKG, that means $c$-dependent) timescales. On the other hand, in the recent [HKNR18], which studies the non-relativistic limit of the Vlasov-Maxwell system, the authors were able to prove a stability result valid for times which are polynomial in terms of the speed of light for solutions which lie in a neighbourhoo of stable equilibria of the system.

Another example of singular perturbation problem that has been studied with canonical perturbation theory is the problem of the continuous approximation of lattice dynamics (see e.g. [BP06]). In the framework of lattice dynamics, the approximation has been justified only for the typical time scale of averaging theorems, which corresponds to our $O(1)$ time scale. Hopefully the methods developed in [Pas17] and in the present paper could allow to extend the time of validity of those results.
The paper is organized as follows. In sect. 2 we state the results of the paper, together with some examples and comments. In sect. 3 we show Strichartz estimates for the linear KG equation on $\mathbb{R}^d$. In sect. 4 we recall an abstract result from [Pas17]; next, in sect. 4.1 we apply the abstract theorem to the real NLKG equation, making some explicit computations of the normal form at the first and at the second step. In sect. 5 we study the properties of the normalized equation, namely its dispersive properties in the linear case and its well-posedness for solutions with small initial data in the nonlinear case. In sect. 6 we discuss the approximation for longer timescales: in particular, to deduce the latter we will exploit some dispersive properties of the KG equation reported in sect. 3.

Acknowledgments. This work is a revised and extended version of a part of the author’s PhD thesis. The author would like to thank his supervisor for the PhD thesis Professor Dario Bambusi.

The author is supported by the ERC grant “HamPDEs”.

2 Statement of the Main Results

The NLKG equation describes the motion of a spinless particle with mass $m > 0$. Consider first the real NLKG

$$
\frac{\hbar^2}{2mc^2} u_{tt} - \frac{\hbar^2}{2m} \Delta u + \frac{mc^2}{2} u + \lambda |u|^{2(l-1)} u = 0,
$$

(2.1)

where $c > 0$ is the speed of light, $\hbar > 0$ is the Planck constant, $\lambda \in \mathbb{R}$, $l \geq 2$, $c > 0$.

In the following $m = 1$, $\hbar = 1$. As anticipated above, one is interested in the behaviour of solutions as $c \to \infty$.

First it is convenient to reduce equation (2.1) to a first order system, by making the following symplectic change variables

$$
\psi := \frac{1}{\sqrt{2}} \left[ \left( \langle \nabla \rangle_c \right)^{1/2} u - i \left( \frac{c}{\langle \nabla \rangle_c} \right)^{1/2} v \right], \quad v = u_t/c^2,
$$

where

$$
\langle \nabla \rangle_c := (c^2 - \Delta)^{1/2},
$$

(2.2)

which reduces (2.1) to the form

$$
-iv_t = c \langle \nabla \rangle_c \psi + \lambda \left( \frac{c}{\langle \nabla \rangle_c} \right)^{1/2} \left( \left( \frac{c}{\langle \nabla \rangle_c} \right)^{1/2} \left( \psi + \bar{\psi} \right) \right)^{2l-1},
$$

(2.3)

which is hamiltonian with Hamiltonian function given by

$$
H(\bar{\psi}, \psi) = \langle \bar{\psi}, c \langle \nabla \rangle_c \psi \rangle + \lambda \frac{1}{2l} \int \left[ \left( \frac{c}{\langle \nabla \rangle_c} \right)^{1/2} \frac{\psi + \bar{\psi}}{\sqrt{2}} \right]^{2l} \, dx.
$$

(2.4)

In the following the notation $a \lesssim b$ is used to mean: there exists a positive constant $K$ that does not depend on $c$ such that $a \leq Kb$.

Before discussing the approximation of the solutions of NLKG with NLS-type equations, we describe the general strategy we use to get them.
Remark that Eq. (2.1) is Hamiltonian with Hamiltonian function (2.4). If one divides the Hamiltonian by a factor $c^2$ (which corresponds to a rescaling of time) and expands in powers of $c^{-2}$ it takes the form

$$
\langle \psi, \hat{\psi} \rangle + \frac{1}{c^2} P_c(\psi, \hat{\psi})
$$

(2.5)

with a suitable function $P_c$. One can notice that this Hamiltonian is a perturbation of $h_0 := \langle \psi, \hat{\psi} \rangle$, which is the generator of the standard Gauge transform, and which in particular admits a flow that is periodic in time. Thus the idea is to exploit canonical perturbation theory in order to conjugate such a Hamiltonian system to a system in normal form, up to remainders of order $O(c^{-2r})$, for any given $r \geq 1$.

The problem is that the perturbation $P_c$ has a vector field which is small only as an operator extracting derivatives: hence, if one Taylor expands $P_c$ and its vector field, the number of derivatives extracted at each order increases. This situation is typical in singular perturbation problems, and the price to pay to get a normal form is that the remainder of the perturbation turns out to be an operator that extracts a large number of derivatives.

In Sect. 4.1 the normal form equation is explicitly computed in the case $r = 2, l = 2$:

$$
i \psi_t = c^2 \psi - \frac{1}{2} \Delta \psi + \frac{3}{4} \lambda |\psi|^2 \psi
$$

$$
+ \frac{1}{c^2} \left[ \frac{51}{8} \lambda^2 |\psi|^4 \psi + \frac{3}{16} \lambda \left( 2|\psi|^2 \Delta \psi + \psi^2 \Delta \psi + \Delta(|\psi|^2 \psi) \right) - \frac{1}{8} \Delta^2 \psi \right],
$$

(2.6)

namely a singular perturbation of a Gauge-transformed NLS equation. If one, after a gauge transformation, only considers the first order terms, one has the NLS.

The standard way to exploit such a “singular” normal form is to use it just to construct some approximate solution of the original system, and then to apply Gronwall Lemma in order to estimate the difference with a true solution with the same initial datum (see for example [BCP02]).

This strategy works also here, but it only leads to a control of the solutions over times of order $O(c^2)$. When scaled back to the physical time, this allows to justify the approximation of the solutions of NLKG by solutions of the NLS over time scales of order $O(1)$, on any manifold admitting a Littlewood-Paley decomposition (such as Riemannian smooth compact manifolds, or $\mathbb{R}^d$; see the introduction of [Bou10] and section 2.1 of [BGT04] for the construction of Littlewood-Paley decomposition on compact manifolds).

A similar result has been obtained for the case $M = \mathbb{T}^d$ by Faou and Schratz [FS14], who aimed to construct numerical schemes which are robust in the nonrelativistic limit.

The idea one uses here in order to improve the time scale of the result is that of substituting Gronwall Lemma with a more sophisticated tool, namely dispersive estimates and the retarded Strichartz estimate. This can be done each time one can prove a dispersive or a Strichartz estimate for the linearization of equation (2.3) on the approximate solution, uniformly in $c$. Now we state our result for the approximation of small radiation solutions of the NLKG equation.

**Theorem 2.1.** Consider (2.3) on $\mathbb{R}^d$, $d \geq 2$. Let $r > 1$, and fix $k_1 \gg 1$. Assume that $l \geq 2$ and $r < \frac{d}{2}(l - 1)$. Then $\exists k_0 = k_0(r) > 0$ such that for any $k \geq k_0$ and for any $\sigma > 0$ the following holds: consider the solution $\psi_\tau$ of the normalized equation (6.1), with initial datum $\psi_{\tau,0} \in H^{k+k_0+\sigma+d/2}$. Then there exist $\alpha^* := \alpha^*(d, l, r) > 0$ and there exists $c^* := c^*(r, k) > 1$, such that for any $\alpha > \alpha^*$ and for any $c > c^*$, if $\psi_{\tau,0}$ satisfies

$$
\|\psi_{\tau,0}\|_{H^{k+k_0+\sigma+d/2}} \lesssim c^{-\alpha},
$$

then
then
\[ \sup_{t \in [0,T]} \| \psi(t) - \psi_r(t) \|_{H^k} \lesssim \frac{1}{c^2}, \quad T \lesssim c^{2(r-1)}, \]
where \( \psi(t) \) is the solution of (4.13) with initial datum \( \psi_{r,0} \).

**Remark 2.2.** The assumption of existence of \( \psi_r \) up to times of order \( \mathcal{O}(c^{2(r-1)}) \) is actually a delicate matter. Equation (2.6), for example, is a quasilinear perturbation of a fourth-order Schrödinger equation (4NLS). Even if we restrict to the case \( r = 2 \), the issues of global well-posedness and scattering for solutions with large initial data for Eq. (2.6) have not been solved. For solutions with small initial data, on the other hand, there are some papers dealing with the local well-posedness of 4NLS (see for example [HHW07]), and with global well-posedness and scattering of 4NLS (see [RWZ16]). In Sec. 5.2 we prove the local well-posedness for times of order \( \mathcal{O}(c^{2(r-1)}) \) for solutions of the order-\( r \) normalized equation with small initial data under the assumptions that \( l \geq 2 \) and \( r < \frac{d}{2}(l-1) \).

**Remark 2.3.** Just to be explicit, we make some examples of Theorem 2.1. For \( M = \mathbb{R}^2 \) and a nonlinearity of order \( 2l \), we can justify the approximation of small radiation solutions up to times of order \( \mathcal{O}(c^{2(r-1)}) \), for \( r < \frac{1}{2}(l-1) \). For \( M = \mathbb{R}^3 \) and a nonlinearity of order \( 2l \), we can justify the approximation of small radiation solutions up to times of order \( \mathcal{O}(c^{2(r-1)}) \), for \( r < \frac{1}{2}(l-1) \).

On the other hand, when \( \frac{d}{2}(l-1) \leq 2 \), we cannot justify the approximation over long time scales: examples of such cases are the cubic NLKG in 2, 3 and 4 dimensions, or the quintic NLKG in 2 dimensions.

Before closing the subsection, we remark that the condition on \( r \) in Theorem 2.1 depends on the assumption under which we were able to prove a well-posedness result for the normalized equation, which in turn depends on the approach presented recently in [RWZ16]; we do not exclude that this technical condition could be improved.

### 3 Dispersive properties of the Klein-Gordon equation

We briefly recall some classical notion of Fourier analysis on \( \mathbb{R}^d \). Recall the definition of the space of Schwartz (or rapidly decreasing) functions,

\[ \mathcal{S} := \{ f \in C^{\infty}(\mathbb{R}^d, \mathbb{R}) \mid \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^{\alpha/2} |\partial^\beta f(x)| < +\infty, \forall \alpha \in \mathbb{N}^d, \forall \beta \in \mathbb{N}^d \}. \]

In the following \( \langle x \rangle := (1 + |x|^2)^{1/2} \).

Now, for any \( f \in \mathcal{S} \) the *Fourier transform* of \( f \), \( \mathcal{F}f : \mathbb{R}^d \to \mathbb{R} \), is defined by the following formula

\[ \mathcal{F}f(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-i\langle x,\xi \rangle} dx, \quad \forall \xi \in \mathbb{R}^d, \]

where \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^d \).

At the beginning we will obtain Strichartz estimates for the linear equation

\[ -i \psi_t = c(\nabla)_x \psi, \quad x \in \mathbb{R}^d. \quad (3.1) \]
Proposition 3.1. Let \( d \geq 2 \). For any Schrödinger admissible couples \((p,q)\) and \((r,s)\), namely such that

\[
2 \leq p, r \leq \infty, \\
2 \leq q, s \leq \frac{2d}{d-2}, \\
\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad \frac{2}{r} + \frac{d}{s} = \frac{d}{2}, \\
(p,q,d), (r,s,d) \neq (2, +\infty, 2),
\]

one has

\[
\|\langle \nabla \rangle_{c}^{\frac{1}{2} - \frac{1}{p} - \frac{1}{q}} e^{it \langle \nabla \rangle_{c}} \psi_{0}\|_{L_{t}^{p}L_{x}^{q}} \lesssim c_{1}^{\frac{1}{2} - \frac{1}{p} - \frac{1}{2}} \|\langle \nabla \rangle_{c}^{1/2} \psi_{0}\|_{L_{x}^{2}}, 
\]

(3.2)

\[
\left\| \langle \nabla \rangle_{c}^{\frac{1}{2} - \frac{1}{r} - \frac{1}{s}} \int_{0}^{t} e^{i(t-s) \langle \nabla \rangle_{c}} F(s) \, ds \right\|_{L_{t}^{p}L_{x}^{q}} \lesssim c_{1}^{\frac{1}{2} - \frac{1}{r} - \frac{1}{s} - 1} \|\langle \nabla \rangle_{c}^{\frac{1}{2} - \frac{1}{r} - \frac{1}{s} + 1} F\|_{L_{t}^{r'}L_{x}^{s'}}.
\]

(3.3)

Proof. By a simple scaling argument, from the following result reported by D’Ancona-Fanelli in [DF08] for the operator \( \langle \nabla \rangle \defeq \langle \nabla \rangle_{c} \) (for more details see the proof of Proposition 3.1 in [Pas17]).

Lemma 3.2. For all \((p,q)\) Schrödinger-admissible exponents

\[
\|e^{it \langle \nabla \rangle_{c}} \phi_{0}\|_{L_{t}^{p}W_{x}^{\frac{1}{2} - \frac{1}{p} - \frac{1}{q}}} = \|\langle \nabla \rangle_{c}^{\frac{1}{2} - \frac{1}{p} - \frac{1}{q}} e^{it \langle \nabla \rangle_{c}} \phi_{0}\|_{L_{t}^{p}L_{x}^{q}} \leq \|\phi_{0}\|_{L_{x}^{2}}.
\]

\(\square\)

Remark 3.3. By choosing \( p = +\infty \) and \( q = 2 \), we get the following a priori estimate for finite energy solutions of (3.1),

\[
\|e^{it/2 \langle \nabla \rangle_{c}^{1/2} e^{it \langle \nabla \rangle_{c}} \psi_{0}\|_{L_{t}^{p}L_{x}^{q}} \lesssim \|e^{it/2 \langle \nabla \rangle_{c}^{1/2} \psi_{0}\|_{L_{x}^{2}}.
\]

We also point out that, since the operators \( \langle \nabla \rangle \) and \( \langle \nabla \rangle_{c} \) commute, the above estimates in the spaces \( L_{t}^{p}L_{x}^{q} \) extend to estimates in \( L_{t}^{p}W_{x}^{k,q} \) for any \( k \geq 0 \).

4 A Birkhoff Normal Form result

Consider the scale of Banach spaces \( W^{k,p}(M, \mathbb{C}^{n} \times \mathbb{C}^{n}) \ni (\psi, \bar{\psi}) \) \((k \geq 1, 1 < p < +\infty, n \in \mathbb{N}_{0})\) endowed by the standard symplectic form. Having fixed \( k \) and \( p \), and \( U_{k,p} \subset W^{k,p} \) open, we define the gradient of \( H \in C^{\infty}(U_{k,p}, \mathbb{R}) \) w.r.t. \( \psi \) as the unique function s.t.

\[
\langle \nabla_{\psi} H, \bar{h}\rangle = d_{\psi}Hh, \quad \forall h \in W^{k,p},
\]

so that the Hamiltonian vector field of a Hamiltonian function \( H \) is given by

\[
X_{H}(\psi, \bar{\psi}) = (i \nabla_{\psi} H, -i \nabla_{\psi} H).
\]

The open ball of radius \( R \) and center \( 0 \) in \( W^{k,p} \) will be denoted by \( B_{k,p}(R) \).
Remark 4.1. Let \( k \geq 0, 1 < p < +\infty \), we now introduce the Littlewood-Paley decomposition on the Sobolev space \( W^{k,p} = W^{k,p}(\mathbb{R}^d) \) (see [Tay11], Ch. 13.5).

In order to do this, define the cutoff operators in \( W^{k,p} \) in the following way: start with a smooth, radial nonnegative function \( \phi_0 : \mathbb{R}^d \to \mathbb{R} \) such that \( \phi_0(\xi) = 1 \) for \( |\xi| \leq 1/2 \), and \( \phi_0(\xi) = 0 \) for \( |\xi| \geq 1 \); then define \( \phi_1(\xi) := \phi_0(\xi/2) - \phi_0(\xi) \), and set

\[
\phi_j(\xi) := \phi_1(2^{1-j}\xi), \quad j \geq 2. \tag{4.1}
\]

Then \( (\phi_j)_{j \geq 0} \) is a partition of unity,

\[
\sum_{j \geq 0} \phi_j(\xi) = 1.
\]

Now, for each \( j \in \mathbb{N} \) and each \( f \in W^{k,2} \), we can define \( \phi_j(D)f \) by

\[
F(\phi_j(D)f)(\xi) := \phi_j(\xi)F(f)(\xi).
\]

It is well known that for \( p \in (1, +\infty) \) the map \( \Phi : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d, l^2) \),

\[
\Phi(f) := (\phi_j(D)f)_{j \in \mathbb{N}},
\]

maps \( L^p(\mathbb{R}^d) \) isomorphically onto a closed subspace of \( L^p(\mathbb{R}^d, l^2) \), and we have compatibility of norms ([Tay11], Ch. 13.5, (5.45)-(5.46)),

\[
K'_p \|f\|_{L^p} \leq \|\Phi(f)\|_{L^p(\mathbb{R}^d, l^2)} := \left\| \sum_{j \in \mathbb{N}} |\phi_j(D)f|^2 \right\|_{l^2}^{1/2} \leq K_p \|f\|_{L^p},
\]

and similarly for the \( W^{k,p} \)-norm, i.e. for any \( k > 0 \) and \( p \in (1, +\infty) \)

\[
K'_{k,p} \|f\|_{W^{k,p}} \leq \left\| \sum_{j \in \mathbb{N}} 2^{jk} |\phi_j(D)f|^2 \right\|_{l^2}^{1/2} \leq K_{k,p} \|f\|_{W^{k,p}}. \tag{4.2}
\]

We then define the cutoff operator \( \Pi_N \) by

\[
\Pi_N \psi := \sum_{j \leq N} \phi_j(D)\psi. \tag{4.3}
\]

We point out that the Littlewood-Paley decomposition, along with equality (4.2), can be extended to compact manifolds (see [BGT04]), as well as to some particular non-compact manifolds (see [Bou10]).

Now we consider a Hamiltonian system of the form

\[
H = h_0 + \epsilon h + \epsilon F, \tag{4.4}
\]

where \( \epsilon > 0 \) is a parameter. We assume that

PER \( h_0 \) generates a linear periodic flow \( \Phi^t \) with period \( 2\pi \),

\[
\Phi^{t+2\pi} = \Phi^t \quad \forall t.
\]

We also assume that \( \Phi^t \) is analytic from \( W^{k,p} \) to itself for any \( k \geq 1 \), and for any \( p \in (1, +\infty) \);
INV for any \( k \geq 1 \), for any \( p \in (1, +\infty) \), \( \Phi^t \) leaves invariant the space \( \Pi_j W^{k,p} \) for any \( j \geq 0 \). Furthermore, for any \( j \geq 0 \)

\[
\pi_j(D) \circ \Phi^t = \Phi^t \circ \pi_j(D);
\]

NF \( h \) is in normal form, namely

\[
h \circ \Phi^t = h.
\]

Next we assume that both the Hamiltonian and the vector field of both \( h \) and \( F \) admit an asymptotic expansion in \( \epsilon \) of the form

\[
h \sim \sum_{j \geq 1} \epsilon^{j-1} h_j, \quad F \sim \sum_{j \geq 1} \epsilon^{j-1} F_j;
\]

and that the following properties are satisfied

HVF There exists \( R^* > 0 \) such that for any \( j \geq 1 \)

\[
\cdot \ X_{h_j} \text{ is analytic from } B_{k+2j,p}(R^*) \text{ to } W^{k,p};
\]

\[
\cdot \ X_{F_j} \text{ is analytic from } B_{k+2(j-1),p}(R^*) \text{ to } W^{k,p}.
\]

Moreover, for any \( r \geq 1 \) we have that

\[
\cdot \ X_{h - \sum_{j=1}^{r} \epsilon^{j-1} h_j} \text{ is analytic from } B_{k+2(r+1),p}(R^*) \text{ to } W^{k,p};
\]

\[
\cdot \ X_{F - \sum_{j=1}^{r} \epsilon^{j-1} F_j} \text{ is analytic from } B_{k+2r,p}(R^*) \text{ to } W^{k,p}.
\]

In [Pas17] we proved the following theorem.

**Theorem 4.2** (see Theorem 4.3 in [Pas17]). Fix \( r \geq 1 \), \( R > 0 \), \( k_1 \gg 1 \), \( 1 < p < +\infty \). Consider (4.4), and assume PER, INV, NF and HVF. Then \( \exists k_0 = k_0(r) > 0 \) with the following properties: for any \( k \geq k_1 \) there exists \( \epsilon_{r,k,p} \ll 1 \) such that for any \( \epsilon < \epsilon_{r,k,p} \) there exists \( T_\epsilon^{(r)} : B_{k,p}(R) \to B_{k,p}(2R) \) analytic canonical transformation such that

\[
H_\epsilon := H \circ T_\epsilon^{(r)} = h_0 + \sum_{j=1}^{r} \epsilon^j Z_j + \epsilon^{r+1} R_\epsilon^{(r)},
\]

where \( Z_j \) are in normal form, namely

\[
\{ Z_j, h_0 \} = 0,
\]

and

\[
\sup_{B_{k+k_0,p}(R)} \| X_{Z_j} \|_{W^{k,p}} \leq C_{k,p},
\]

\[
\sup_{B_{k+k_0,p}(R)} \| X_{R^{(\epsilon)}} \|_{W^{k,p}} \leq C_{k,p},
\]

\[
\sup_{B_{k,p}(R)} \| T_\epsilon^{(r)} - id \|_{W^{k,p}} \leq C_{k,p} \epsilon.
\]
In particular, we have that
\[ Z_1(\psi, \bar{\psi}) = h_1(\psi, \bar{\psi}) + \langle F_1 \rangle (\psi, \bar{\psi}), \]
where \( (F_1) (\psi, \bar{\psi}) := \int_0^{2\pi} F_1 \circ \Phi^t(\psi, \bar{\psi}) \frac{dt}{2\pi}. \)

### 4.1 The real nonlinear Klein-Gordon equation

We first consider the Hamiltonian of the real non-linear Klein-Gordon equation with power-type nonlinearity on a smooth manifold \( M \) (\( M \) is such the Littlewood-Paley decomposition is well-defined; take, for example, a smooth compact manifold, or \( \mathbb{R}^d \)). The Hamiltonian is of the form
\[ H(u, v) = \frac{c^2}{2} \langle v, v \rangle + \frac{1}{2} \langle u, \langle \nabla \rangle_c^2 u \rangle + \lambda \int \frac{u^{2l}}{2l}, \tag{4.10} \]
where \( \langle \nabla \rangle_c := (c^2 - \Delta)^{1/2} \). \( \lambda \in \mathbb{R}, \ l \geq 2. \)

If we introduce the complex-valued variable
\[ \psi := \frac{1}{\sqrt{2}} \left[ \left( \frac{\langle \nabla \rangle_c}{c} \right)^{1/2} u - i \left( \frac{c}{\langle \nabla \rangle_c} \right)^{1/2} v \right], \tag{4.11} \]
the corresponding symplectic 2-form becomes \( \iota d\psi \wedge d\bar{\psi} \), the Hamiltonian (4.10) in the coordinates \((\psi, \bar{\psi})\) is
\[ H(\bar{\psi}, \psi) = \langle \bar{\psi}, c(\nabla)_c \psi \rangle + \frac{\lambda}{2l} \int \left[ \left( \frac{c}{\langle \nabla \rangle_c} \right)^{1/2} \psi + \bar{\psi} \right]^{2l} \dx. \tag{4.12} \]

If we rescale the time by a factor \( c^2 \), the Hamiltonian takes the form (4.4), with \( \epsilon = \frac{1}{c^2} \), and
\[ H(\bar{\psi}, \psi) = h_0(\psi, \bar{\psi}) + \epsilon h(\psi, \bar{\psi}) + \epsilon F(\psi, \bar{\psi}), \tag{4.13} \]
where
\[ h_0(\psi, \bar{\psi}) = \langle \bar{\psi}, \psi \rangle, \tag{4.14} \]
\[ h(\psi, \bar{\psi}) = \langle \bar{\psi}, (c(\nabla)_c - c^2) \psi \rangle \sim \sum_{j \geq 1} \epsilon^{j-1} \langle \bar{\psi}, a_j \Delta^j \psi \rangle =: \sum_{j \geq 1} \epsilon^{j-1} h_j(\psi, \bar{\psi}), \tag{4.15} \]
\[ F(\psi, \bar{\psi}) = \frac{\lambda}{2^{l+1}} \int \left[ \left( \frac{c}{\langle \nabla \rangle_c} \right)^{1/2} (\psi + \bar{\psi}) \right]^{2l} \dx \tag{4.16} \]
\[ \sim \frac{\lambda}{2^{l+1}} \int (\psi + \bar{\psi})^{2l} \dx \]
\[ + \epsilon b_2 \int [\psi + \bar{\psi}]^{2l-1} \Delta (\psi + \bar{\psi}) + \ldots + (\psi + \bar{\psi}) \Delta ((\psi + \bar{\psi})^{2l-1}) \] \[ + \mathcal{O}(\epsilon^2) \]
\[ =: \sum_{j \geq 1} \epsilon^{j-1} F_j(\psi, \bar{\psi}), \tag{4.17} \]
where \((a_j)_{j \geq 1}\) and \((b_j)_{j \geq 1}\) are real coefficients, and \(F_j(\psi, \bar{\psi})\) is a polynomial function of the variables \(\psi\) and \(\bar{\psi}\) (along with their derivatives) and which admits a bounded vector field from a neighborhood of the origin in \(W^{k+2(j-1),p}\) to \(W^{k,p}\) for any \(1 < p < +\infty\).

This description clearly fits the scheme treated in the previous section, and one can easily check that assumptions PER, NF and HVF are satisfied. Therefore we can apply Theorem 4.2 to the Hamiltonian (4.13).

**Remark 4.3.** About the normal forms obtained by applying Theorem 4.2, we remark that in the first step (case \(r = 1\) in the statement of the Theorem) the homological equation we get is of the form
\[
\{\chi_1, h_0\} + F_1 = \langle F_1 \rangle, \tag{4.18}
\]
where \(F_1(\psi, \bar{\psi}) = \frac{\lambda}{2^{l+1}} \int (\psi + \bar{\psi})^2 \, dx\). Hence the transformed Hamiltonian is of the form
\[
H_1(\psi, \bar{\psi}) = h_0(\psi, \bar{\psi}) + \frac{1}{c^2} \left[ -\frac{1}{2} \langle \bar{\psi}, \Delta \psi \rangle + \langle F_1 \rangle (\psi, \bar{\psi}) \right] + \frac{1}{c^4} R_1^{(1)}(\psi, \bar{\psi}), \tag{4.19}
\]
where
\[
\langle F_1 \rangle (\psi, \bar{\psi}) = \frac{\lambda}{2^{l+1}} \int |\psi|^{2l} \, dx. \tag{4.20}
\]
If we neglect the remainder and we derive the corresponding equation of motion for the system, we get
\[
-i \dot{\psi}_t = \psi + \frac{1}{c^2} \left[ -\frac{1}{2} \Delta \psi + \frac{\lambda}{2^{l+1}} \left( \frac{2l}{l} \right) \langle \psi \rangle^{2(l-1)} \psi \right], \tag{4.21}
\]
which is the NLS, and the Hamiltonian which generates the canonical transformation is given by
\[
\chi_1(\psi, \bar{\psi}) = \frac{\lambda}{2^{l+1}} \sum_{j=0, \ldots, 2l \neq l} \frac{1}{i \cdot 2(l-j)} \left( \frac{2l}{j} \right) \int |\psi|^{2l-j} \bar{\psi}^j \, dx. \tag{4.22}
\]
Such computations already appeared in [Pas17].

**Remark 4.4.** Now we iterate the construction by passing to the case \(r = 2\).

If we neglect the remainder of order \(c^{-6}\), we have that
\[
H \circ T^{(1)} = h_0 + \frac{1}{c^2} h_1 + \frac{1}{c^4} \{\chi_1, h_1\} + \frac{1}{c^4} h_2 + \\
+ \frac{1}{c^2} \langle F_1 \rangle + \frac{1}{c^4} \{\chi_1, F_1\} + \frac{1}{2c^4} \left\{ \langle \chi_1, h_0 \rangle + \{\chi_1, h_1\} \right\} + \frac{1}{c^4} F_2 \tag{4.23}
\]
\[
= h_0 + \frac{1}{c^2} [h_1 + \langle F_1 \rangle] + \frac{1}{c^4} \left\{ \langle \chi_1, h_1 \rangle + h_2 + \{\chi_1, F_1\} + \frac{1}{2} \{\chi_1, \langle F_1 \rangle - F_1\} \right\} + F_2, \tag{4.24}
\]
where \(h_1(\psi, \bar{\psi}) = -\frac{1}{2} \langle \bar{\psi}, \Delta \psi \rangle\), and \(\chi_1\) is of the form (4.22).
Now we compute the terms of order $\frac{1}{c^4}$.

\[
\{\chi_1, h_1\} = d\chi_1 X_{h_1} = \frac{\partial \chi_1}{\partial \psi} \frac{\partial h_1}{\partial \psi} - i \frac{\partial \chi_1}{\partial \bar{\psi}} \frac{\partial h_1}{\partial \bar{\psi}}
\]

\[
\begin{align*}
&= -\frac{\lambda}{2^{l+3l}} \int \left[ \sum_{j=0,\ldots,2l-1, j \neq l} \frac{1}{l-j} \binom{2l}{j} (2l-j) \psi^{2l-j-1} \bar{\psi}^j \right] \Delta \psi \, dx \\
&\quad + \frac{\lambda}{2^{l+3l}} \int \left[ \sum_{j=1,\ldots,2l, j \neq l} \frac{1}{l-j} \binom{2l}{j} j \psi^{2l-j-1} \bar{\psi}^j \right] \Delta \bar{\psi} \, dx \\
&\quad - \frac{\lambda}{2^{l+3l}} \int \Delta \psi \psi^{2l-1} + \Delta \bar{\psi} \bar{\psi}^{2l-1} \, dx \\
&\quad - \frac{\lambda}{2^{l+3l}} \int \sum_{j=1,\ldots,2l, j \neq l} \frac{1}{l-j} \binom{2l}{j} \int (2l-j) \psi^{2l-j-1} \bar{\psi}^j \Delta \psi - j \psi^{2l-j} \bar{\psi}^j \Delta \bar{\psi} \, dx,
\end{align*}
\]

(4.25)

and since $j \neq l$ in the sum we have that

\[
\langle \{\chi_1, h_1\} \rangle = 0.
\]

(4.26)

Next,

\[
h_2 = -\frac{1}{8} \langle \bar{\psi}, \Delta^2 \psi \rangle,
\]

(4.27)

\[
\begin{align*}
\{\chi_1, F_1\} &= \frac{\lambda^2}{2^{2l+3l}} \int \left[ \sum_{j=0,\ldots,2l-1, j \neq l} \frac{1}{l-j} \binom{2l}{j} (2l-j) \psi^{2l-j-1} \bar{\psi}^j \right] \left[ \sum_{h=1}^{2l} \binom{2l}{h} \psi^{2l-h} \bar{\psi}^{h-1} \right] \, dx \\
&\quad - \frac{\lambda^2}{2^{2l+3l}} \int \left[ \sum_{j=1,\ldots,2l, j \neq l} \frac{1}{l-j} \binom{2l}{j} j \psi^{2l-j-1} \bar{\psi}^j \right] \left[ \sum_{h=0}^{2l-1} \binom{2l}{h} (2l-h) \psi^{2l-h-1} \bar{\psi}^h \right] \, dx
\end{align*}
\]

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\[
\langle \{ \chi_1, F_1 \} \rangle = \lambda^2 K(l) \int |\psi|^{2(2l-1)} \, dx,
\]

\[
K(l) := \frac{1}{2^{2l+3l^2}} \left\{ \sum_{j, h = 1, \ldots, 2l-1} \frac{1}{l-j} \binom{2l}{j} \binom{2l}{h} [(2l - j)h - j(2l - h)] + 16l \right\},
\]

where \( K(l) > 0 \) by the conditions on \( j \) and \( h \) in the sum.

Then,

\[
\{ \chi_1, \langle F_1 \rangle \} = \frac{\lambda^2}{2^{2l+3l^2}} \left\{ \sum_{j, h = 1, \ldots, 2l-1} \frac{1}{l-j} \binom{2l}{j} \binom{2l}{h} [(2l - j)h - j(2l - h)] \right\} + \sum_{j = 1, \ldots, 2l-1} 2 \binom{2l}{j} \int \psi^{2l-1-j} \bar{\psi}^{j+1-1} \, dx,
\]

and since \( j \neq l \) in the sum we have that

\[
\langle \{ \chi_1, \langle F_1 \rangle \} \rangle = 0.
\]
Furthermore,

\[
F_2 = \frac{\lambda}{2^{l+3l}} \cdot 2l \int (\psi + \bar{\psi})^{2l-1} \Delta(\psi + \bar{\psi}) \, dx
\]

\[
= \frac{\lambda}{2^{l+2}} \sum_{j=0}^{2l-1} \left( \frac{2l-1}{j} \right) \int \psi^{2l-1-j} \bar{\psi}^j (\Delta \psi + \Delta \bar{\psi}) \, dx,
\]

(4.33)

\[
\langle F_2 \rangle = \frac{\lambda}{2^{l+2}} \int \left( \frac{2l-1}{l} \right) \psi^{2l-1} \bar{\psi}^l \Delta \psi + \left( \frac{2l-1}{l-1} \right) \psi^{2l-1} \bar{\psi}^{l-1} \Delta \bar{\psi} \, dx
\]

\[
= \frac{\lambda}{2^{l+2}} \left( \frac{2l-1}{l} \right) \int |\psi|^{2(l-1)} (\bar{\psi} \Delta \psi + \psi \Delta \bar{\psi}) \, dx
\]

(4.34)

Hence, up to a remainder of order \( O \left( \frac{1}{c}^6 \right) \), we have that

\[
H_2 = h_0 + \frac{1}{c^2} \int \left[ -\frac{1}{2} \langle \bar{\psi}, \Delta \psi \rangle + \frac{\lambda}{2^{l+1}} \left( \frac{2l}{l} \right) |\psi|^2 \right] \, dx
\]

\[+
\frac{1}{c^2} \int \left[ \lambda^2 K(l) |\psi|^{2(l-1)} + \frac{\lambda}{2^{l+2}} \left( \frac{2l-1}{l} \right) |\psi|^{2(l-1)} (\bar{\psi} \Delta \psi + \psi \Delta \bar{\psi}) \right] \, dx,
\]

(4.35)

which, by neglecting \( h_0 \) (that yields only a gauge factor) and by rescaling the time, leads to the following equations of motion

\[
-i \dot{\psi}_t = -\frac{1}{2} \Delta \psi + \frac{\lambda}{2^{l+1}} \left( \frac{2l}{l} \right) |\psi|^{2(l-1)} \psi + \frac{1}{c^2} \left[ -\frac{1}{8} \Delta^2 \psi + \lambda^2 K(l) (2l-1) |\psi|^{4(l-1)} \psi \right]
\]

\[+
\frac{1}{c^2} \left[ \lambda^2 K(l) |\psi|^{2(l-1)} \Delta \psi + (l-1) |\psi|^{2(l-2)} \psi^2 \Delta \bar{\psi} + \Delta (|\psi|^{2(l-1)} \bar{\psi}) \right],
\]

(4.36)

which for example in the case of a cubic nonlinearity \( (l = 2) \) reads

\[
-i \dot{\psi}_t = -\frac{1}{2} \Delta \psi + \frac{3}{4} \lambda |\psi|^2 \psi
\]

\[+
\frac{1}{c^2} \left[ \frac{51}{8} \lambda^2 |\psi|^4 \psi + \frac{3}{16} \lambda (2 |\psi|^2 \Delta \psi + \psi^2 \Delta \bar{\psi} + \Delta (|\psi|^2 \bar{\psi})) \right] - \frac{1}{8} \Delta^2 \psi \]

(4.37)

Eq. (4.37) is the nonlinear analogue of a linear higher-order Schrödinger equation that appears in [CM12] and [CLM15] in the context of semi-relativistic equations.

5 Properties of the normal form equation

5.1 Linear case

Now let \( r \geq 1, d \geq 2 \). In [CM12] and [CLM15] the authors proved that the linearized normal form system, namely the one that corresponds (up to a rescaling of time by a factor \( c^2 \)) to

\[
-i \dot{\psi}_t = X_{h_0} + \sum_{j=1}^{r} \lambda h_j (\psi_r),
\]

\( \psi_r(0) = \psi_0 \),

(5.1)
admits a unique solution in $L^\infty_t H^{k+k_0} (\mathbb{R}^d)$ (this is a simple application of the properties of the Fourier transform), and by a perturbative argument they also proved the global existence also for the higher order Schrödinger equation with a bounded time-independent potential.

Moreover, by following the arguments of Theorem 4.1 in [KAY12] and Lemma 4.3 in [CLM15] one obtains the following dispersive estimates and local-in-time Strichartz estimates for solutions of the linearized normal form equation (5.1).

**Proposition 5.1.** Let $r \geq 1$ and $d \geq 2$, and denote by $U_r(t)$ the evolution operator of (5.1) at the time $c^2 t$ ($c \geq 1$, $t > 0$). Then one has the following local-in-time dispersive estimate
\[
\|U_r(t)\|_{L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} \lesssim c^{d(1 - \frac{1}{r})} |t|^{-d/(2r)} , \quad 0 < |t| \lesssim c^{2(r-1)} .
\]

On the other hand, $U_r(t)$ is unitary on $L^2(\mathbb{R}^d)$. Now introduce the following set of admissible exponent pairs:
\[
\Delta_r := \{(p, q) : (1/p, 1/q) \text{ lies in the closed quadrilateral } ABCD\} ,
\]
where
\[
A = \left(1, \frac{1}{r} \right) , \quad B = \left(1, \frac{1}{r'} \right) , \quad C = (1, 0) , \quad D = \left(\frac{1}{r'}, 0\right) , \quad \tau_r = \frac{2r - 1}{r - 1} , \quad \frac{1}{r} + \frac{1}{r'} = 1 .
\]
Then for any $(p, q) \in \Delta_r \setminus \{(2, 2), (1, \tau_r), (\tau_r', \infty)\}$
\[
\|U_r(t)\|_{L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \lesssim c^{d(1 - \frac{1}{r})(\frac{1}{r} - \frac{1}{r'})|t|^{-d/(2r)}} , \quad 0 < |t| \lesssim c^{2(r-1)} .
\]

Let $r \geq 1$ and $d \geq 2$: in the following lemma $(p, q)$ is called an order-$r$ admissible pair when $2 \leq p, q \leq +\infty$ for $r \geq 2$ ($2 \leq q \leq 2d/(d-2)$ for $r = 1$), and
\[
\frac{2}{p} + \frac{d}{rq} = \frac{d}{2r} .
\]

**Proposition 5.2.** Let $r \geq 1$ and $d \geq 2$, and denote by $U_r(t)$ the evolution operator of (5.1) at the time $c^2 t$ ($c \geq 1$, $t > 0$). Let $(p, q)$ and $(a, b)$ be order-$r$ admissible pairs, then for any $T \leq c^{2(r-1)}$
\[
\|U_r(t)\|_{L^p([0, T])L^q(\mathbb{R}^d)} \lesssim c^{d(\frac{1}{r'} - \frac{1}{r})|t|^{-d/(2r)}} \|\phi_0\|_{L^2(\mathbb{R}^d)} = c^{(1 - \frac{1}{r'})\frac{d}{2r}} \|\phi_0\|_{L^2(\mathbb{R}^d)} ,
\]
\[
\left\|\int_0^t U_r(t - \tau)\phi(\tau) d\tau\right\|_{L^p([0, T])L^q(\mathbb{R}^d)} \lesssim c^{(\frac{1}{r'} - \frac{1}{r})2r(\frac{1}{r} + \frac{1}{r'})} \|\phi_l(0, T)\|_{L^{r'}(\mathbb{R}^d)} ,
\]

### 5.2 Well-posedness of higher order nonlinear Schrödinger equations with small data

Here we discuss the local well-posedness of
\[
- \imath \psi_t = A_{c, r} \psi + P((\partial_x^2 \psi)|_{|\alpha| \leq 2(r-1)}, (\partial_x^2 \psi)|_{|\alpha| \leq 2(r-1)}), \quad t \in I, \quad x \in \mathbb{R}^d ,
\]
\[
\psi(0, x) = \psi_0(x) ,
\]
where $r \geq 2$, $I := [0, T]$, $T > 0$,
\[
A_{c, r} = c^2 \sum_{j=1}^r \frac{\Delta_j}{c^{2(r-1)}} , \quad c \geq 1 ,
\]
\[
A_{c, r} = c^2 \sum_{j=1}^r \frac{\Delta_j}{c^{2(r-1)}} , \quad c \geq 1 ,
\]
and $P$ is an analytic function at the origin of the form
\[ P(z) = \sum_{m+1 \leq |\beta| < M} a_{\beta} z^\beta, \quad |a_{\beta}| \leq K |\beta|, \quad |z| \ll 1, \tag{5.10} \]
where $M > m \geq 2$, $m, M \in \mathbb{N}$.

We will exploit this result during the proof of Theorem 2.1. We will adapt an argument of [RWZ16] in order to show the local well-posedness of Eq. for data with small norm in the so-called modulation spaces.

Modulation spaces $M^{s}_{p,q} (s \in \mathbb{R}, 0 < p, q < +\infty)$ were introduced by Feichtinger, and they can be seen as a variant of Besov spaces, in the sense that they allow to perform a frequency decomposition of operators, and to study their properties with respect to lower and higher frequencies. This spaces were recently used in order to prove global well-posedness and scattering for small data for nonlinear dispersive PDEs, especially in the case of derivative nonlinearities (see for example [WH07], [WHH09] and [RWZ16]). We refer to [RSW12] for a survey about modulation spaces and nonlinear evolution equations.

We define the norm on modulation spaces via the following decomposition: let $\sigma : \mathbb{R}^{d} \to \mathbb{R}$ be a function such that
\[ \text{supp}(\sigma) \subset [-3/4, 3/4]^d, \]
and consider a function sequence $(\sigma_k)_{k \in \mathbb{Z}^d}$ satisfying
\[ \sigma_k(\cdot) = \sigma(\cdot - k), \quad \sum_{k \in \mathbb{Z}^d} \sigma_k(\xi) = 1, \quad \forall \xi \in \mathbb{R}^d. \tag{5.11} \]
Denote by
\[ \mathcal{Y}_d := \{ (\sigma_k)_{k \in \mathbb{Z}^d} : (\sigma_k)_{k \in \mathbb{Z}^d} \text{satisfies (5.11) - (5.12)} \}. \]
Let $(\sigma_k)_{k \in \mathbb{Z}^d} \in \mathcal{Y}_d$, and define the frequency-uniform decomposition operators
\[ \Box_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}, \tag{5.13} \]
where by $\mathcal{F}$ we denote the Fourier transform on $\mathbb{R}^d$, then we define the modulation spaces $M^{s}_{p,q}(\mathbb{R}^d)$ via the following norm,
\[ \|f\|_{M^{s}_{p,q}(\mathbb{R}^d)} := \left( \sum_{k \in \mathbb{Z}^d} (k)^{sq} \|\Box_k f\|_p^q \right)^{1/q}, \quad s \in \mathbb{R}, 0 < p, q < +\infty. \tag{5.14} \]
Actually, in our application we will always be interested in the spaces $M^{s}_{p,1}(\mathbb{R}^d)$ with $s \in \mathbb{R}$ and $p > 1$. We just mention some properties of modulation spaces.

**Proposition 5.3.** Let $s, s_1, s_2 \in \mathbb{R}$ and $1 < p, p_1, p_2 < +\infty$.

1. $M^{s}_{p,1}(\mathbb{R}^d)$ is a Banach space;

2. $S(\mathbb{R}^d) \subset M^{s}_{p,1}(\mathbb{R}^d) \subset S'(\mathbb{R}^d)$.
3. \( S(\mathbb{R}^d) \) is dense in \( M^a_{p,1}(\mathbb{R}^d) \);

4. if \( s_2 \leq s_1 \) and \( p_1 \leq p_2 \), then \( M^a_{p_1,1} \subseteq M^a_{p_2,1} \);

5. \( M^a_{p_1,1}(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \);

6. let \( \tau(p) = \max(0, d(1 - 1/p), d/p) \) and \( s_1 > s_2 + \tau(p) \), then \( W^{s_1,p}(\mathbb{R}^d) \subset M^a_{p,1}(\mathbb{R}^d) \);

7. let \( s_1 \geq s_2 \), then \( M^a_{p,1}(\mathbb{R}^d) \subset W^{s_2,p}(\mathbb{R}^d) \).

The last two properties are not trivial, and have been proved in [KS11].

We also introduce other spaces which are often used in this context: the anisotropic Lebesgue space \( L^{p_1,p_2}_{x_1;(x_j)}_{\mathbb{R}^d,t} \):

\[
\|f\|_{L^{p_1,p_2}_{x_1;(x_j)}_{\mathbb{R}^d,t}} := \left\| \left\| f \right\|_{L^{p_2}_{x_1-1,x_{i+1}+1,\ldots,x_d,t}(\mathbb{R}^d \times I)} \right\|_{L^{p_1}_{x_1}(\mathbb{R})},
\]

and, for any Banach space \( X \), the spaces \( l^{1,s}_\Box(X) \) and \( l^{1,s+i}_\Box(X) \),

\[
\|f\|_{l^{1,s}_\Box(X)} := \sum_{k \in \mathbb{Z}^d} \langle k \rangle^s \| \square_k f \|_X, \quad (5.15)
\]

\[
\|f\|_{l^{1,s+i}_\Box(X)} := \sum_{k \in \mathbb{Z}^d} \langle k \rangle^s \| \square_k f \|_X, \quad i \in \mathbb{Z} := \{ k \in \mathbb{Z}^d : |k| = \max_{1 \leq j \leq d} |k_j|, |k_i| > c \}. \quad (5.16)
\]

For simplicity, we write \( l^{1}_\Box(X) = l^{1,0}_\Box(X) \) and \( M^a_{p,1} = M^a_{p,1}(\mathbb{R}^d) \).

**Proposition 5.4.** Let \( d \geq 2, m \geq 2, m > 4r/d \) and \( s > 2(r - 1) + 1/m \).

(i) There exist \( c_0 > 1 \) and \( \delta_0 = \delta_0(d, m, r) > 0 \) such that for any \( c \geq c_0 \), for any \( \delta > \delta_0 \) and for any \( \psi \in M^a_{2,1} \) with \( \| \psi \|_{M^a_{2,1}} \leq c^{-\delta} \) the equation \( (5.8) \) admits a unique solution \( \psi \in C(I, M^a_{2,1}) \cap D \), where \( T = T(\| \psi_0 \|_{M^a_{2,1}}) = \mathcal{O}(c^{2(r-1)}) \), and

\[
\| \psi \|_D = \sum_{\alpha = 0}^{2(r-1)} \sum_{i,j=1}^d \| \partial^{r-\alpha}_x \psi \|_{l^{1,(r-z)z,+(r-z)z}_{\Box,i,j,\Box}} \| \psi \|_{L^\infty_\Box(L^\infty_{x_1,\ldots,x_\alpha,\Box}) \cap L^\infty_{x_1,\ldots,x_\alpha,\Box}(L^\infty_{x_{\alpha+1},\ldots,x_d,\Box}) \cap L^{2,2}_{x_1,\ldots,x_d,\Box} \cap L^{2,2}_{x_1,\ldots,x_d,\Box} \} \lesssim c^{-\delta}. \quad (5.17)
\]

(ii) Moreover, if \( s \geq s_0(\alpha) := d + 2 + \frac{1}{r} \), then there exists \( \delta_1 = \delta_1(d, m, r) > 0 \) such that for any \( c \geq c_0 \), for any \( \delta > \delta_1 \) and for any \( \psi \in M^a_{2,1} \) with \( \| \psi \|_{M^a_{2,1}} \leq c^{-\delta} \) the equation \( (5.8) \) admits a unique solution \( \psi \in C(I, H^s) \), where \( T = T(\| \psi_0 \|_{M^a_{2,1}}) = \mathcal{O}(c^{2(r-1)}) \), and

\[
\| \psi(t) \|_{H^s} \lesssim c^{-\delta}, \quad |t| \lesssim c^{2(r-1)}. \quad (5.18)
\]

From the above Proposition and from the embedding \( H^{s+\sigma+d/2} \subset M^a_{2,1} \) for any \( \sigma > 0 \) we can deduce
5.2.1 Time decay and one can check that \( r \) separately. For simplicity we discuss the case its frequency-localized version, and we want to consider lower, medium and higher frequency

Lemma 5.6. then \(( \psi )\) estimates in the framework of frequency-uniform localization. its local smoothing property, Strichartz estimates with □

The rest of this subsection is devoted to the proof of Proposition 5.4. For convenience, we also write The rest of this subsection is devoted to the proof of Proposition 5.4. For convenience, we also write \( A \)

\[ \sigma_k(\xi) := \eta_k(\xi_1) \cdots \eta_k(\xi_d), \quad k = (k_1, \ldots, k_d) \in \mathbb{Z}^d, \]

(5.20)

For convenience, we also write

\[ \tilde{\sigma}_k = \sum_{\|\ell\|_{\infty} \leq 1} \sigma_{k+\ell}, \quad \tilde{\Box}_k = \sum_{\|\ell\|_{\infty} \leq 1} \Box_{k+\ell}, \quad k \in \mathbb{Z}^d, \]

(5.21)

and one can check that

\[ \tilde{\sigma}_k \sigma_k = \sigma_k, \quad \tilde{\Box}_k \circ \Box_k = \Box_k, \quad k \in \mathbb{Z}^d. \]

(5.22)

We also write \( A_c f(t, x) := \int_0^t \mathcal{U}_c(t - \tau) f(\tau, x) d\tau \).

5.2.1 Time decay

Now, the time-decay of the operator \( \mathcal{U}_c(t) \) is known (see (5.2)), but now we are interested in its frequency-localized version, and we want to consider lower, medium and higher frequency separately. For simplicity we discuss the case \( r = 2 \), and we defer to the the end of this section a remark about the case \( r > 2 \). So, consider

\[ \mathcal{U}_c(t) = e^{itA_c - r} = e^{it\xi^2} \mathcal{F}^{-1} e^{it(|\xi|^2 - \frac{1}{2}k^2)} \mathcal{F}, \]

and write \( \epsilon = e^{-2} \). It is known that the time decay of \( \mathcal{U}_c(t) \) is determined by the critical points of \( P_2(|\xi|) = |\xi|^2 - \epsilon |\xi|^4 \). Notice that \( P_2'(R) = 4R(e^{1/2}R + \frac{1}{\sqrt{2}})(e^{1/2}R - \frac{1}{\sqrt{2}}) \), the singular points of \( P_2 \) are \( \xi = 0 \) and the points of the sphere \( \xi = (2\epsilon)^{-1/2} \). To handle these points, we exploit Littlewood-Paley decomposition, Van der Corput lemma and some properties of the Fourier transform of radial functions.

Indeed, it is known that the Fourier transform of a radial function \( f \) is radial,

\[ \mathcal{F}f(\xi) = 2\pi \int_0^{\infty} f(R) R^{d-1} (R|\xi|)^{-(d-2)/2} J_{\frac{d-2}{2}}(R|\xi|) dR, \]
where $J_m$ is the order $m$ Bessel function,

$$J_m(R) = \frac{(R/2)^m}{\Gamma(m + 1/2)\pi^{1/2}} \int_{-1}^{1} e^{iRT(1 - t^2)^{m-1/2}} dt, \ m > -1/2.$$ 

By following the computations in [RZ16] we obtain that

$$Ff(s) = K_d\pi \int_{0}^{\infty} f(R)R^{d-1}e^{-iRs}h(Rs)dR + K_d\pi \int_{0}^{\infty} f(R)R^{d-1}e^{iRs}h(Rs)dR, \ K_d > 0,$$

$$|h^{(k)}(R)| \leq K_d(1 + R)^{-\frac{d+1}{2}}k, \ \forall k \geq 0.$$ (5.23)

Now we make a Littlewood-Paley decomposition of the frequencies: choose $\rho$ a smooth cut-off function equal to 1 in the unit ball and equal to 0 outside the ball of radius 2, write $\phi_0 = \rho(\cdot) - \rho(2\cdot), \ \phi_j(\cdot) = F^{-1}\phi_0(2^{-j}\cdot)F, \ j \in \mathbb{Z},$ and consider

$$U_2(t)\psi_0 = \sum_{|j| \leq K} \phi_j(D)U_2(t)\psi_0 + \sum_{j < -K} \phi_j(D)U_2(t)\psi_0 + \sum_{j > K} \phi_j(D)U_2(t)\psi_0$$

$$=: P_-U_2(t)\psi_0 + P_UU_2(t)\psi_0 + P_UU_2(t)\psi_0,$$ (5.25)

where

$$K := K(\epsilon) = 10 - \frac{1}{2}[\log_2 \epsilon].$$ (5.26)

Notice that the singular point $R = 0$ is in the support set of $F(P_-U_2(t)\psi_0)$. Roughly speaking, if $j < -K$, the dominant term in $P_2(R)$ is $R^2$, while if $j > K$ the dominant term in $P_2(R)$ is $\epsilon R^k$; hence, by (5.2)

$$\|P_-U_2(t)\psi_0\|_{L^\infty} \lesssim |t|^{-d/2}\|\psi_0\|_{L^1},$$ (5.27)

$$\|P_UU_2(t)\psi_0\|_{L^\infty} \lesssim \epsilon^{d/2}|t|^{-d/4}\|\psi_0\|_{L^1}, \ 0 < |t| \leq \epsilon^2.$$ (5.28)

The time decay estimate for $P_-U_2(t)\psi_0$ is more difficult, since $P_2(R)$ has a singular point in $R = R_1 := (2\epsilon)^{-1/2}$, which corresponds to the sphere $|\xi| = R_1$ in the support set of $F(P_-U_2(t)\psi_0)$. We notice that also the point that satisfies $P_2(R) = 0, \ R = (6\epsilon)^{-1/2}$, corresponds to a sphere $\xi = R_2$ contained in the support set of $F(P_-U_2(t)\psi_0)$; we shall use this fact later.

In order to handle the singular point $R_1$, we perform another decomposition around the sphere $|\xi| = R_1$. Denote $\bar{\rho}(\cdot) = \rho(2^{-K}\cdot) - \rho(2^{(K+1)}\cdot),$ then $P_\bar{\rho} = F^{-1}\bar{\rho}F,$ write $P_k = F^{-1}\phi_k(|\xi| - R_1)F,$ we get

$$\sum_{|j| \leq K} \phi_j(D)U_2(t)\psi_0 = \sum_{k \in \mathbb{Z}} P_kU_2(t)\psi_0$$ (5.29)

By Young’s inequality

$$\|P_kU_2(t)\psi_0\|_{L^\infty} \lesssim \|F^{-1}(\bar{\rho}\phi_k(|\xi| - R_1)e^{-itP_2(|\xi|)})\|_{L^\infty}\|\psi_0\|_{L^1},$$ (5.30)
Moreover,
\[
\mathcal{F}^{-1}\left(\tilde{\rho}\phi_k(|\xi| - R_1)e^{-itP_2(|\xi|)}\right)
\]
\overset{(5.23)}{=} K_d\pi \int_0^\infty R^{d-1}\tilde{\rho}(R)|\phi_k(R - R_1)e^{-itP_2(R) - itR|x|}\tilde{h}(R|x|)dR
\]
\[+ K_d\pi \int_0^\infty R^{d-1}\tilde{\rho}(R)|\phi_k(R - R_1)e^{-itP_2(R) + iR|x|}\tilde{h}(R|x|)dR
\]
\[=: A_k(|x|) + B_k(|x|).
\]

In order to estimate \(A_k(s)\) we rewrite it as
\[
A_k(s) = K_d\pi \left( \int_{R_1}^\infty + \int_0^{R_1} \right) R^{d-1}\tilde{\rho}(R)|\phi_k(R - R_1)e^{-itP_2(R) - iRs}\tilde{h}(Rs)dR
\]
\[=: A_k^{(1)}(s) + A_k^{(2)}(s).
\]

We begin by estimating \(A_k^{(1)}\): notice that \(A_k^{(1)}(s)\) for \(k > K + 2\), hence we can assume that \(k \leq K + 2\). By a change of variables we obtain
\[
A_k^{(1)}(s) = \frac{R_1 + 2^k s}{2^k K_d\pi e^{-iR_1 s}} \int_{1/2}^{2} F(\sigma)e^{it2^k\tilde{P}_2(\sigma)}d\sigma,
\]
\[
F(\sigma) := (R_1 + 2^k \sigma)^{d-1}\tilde{\rho}(R_1 + 2^k \sigma)|\phi_0(\sigma)|\tilde{h}((R_1 + 2^k \sigma)s),
\]
\[
\tilde{P}_2(\sigma) := (2^k t)^{-1}(tp_2(R_1 + 2^k \sigma) - 2^k \sigma s).
\]

One can check that
\[
|\tilde{P}_2'(\sigma)| = \left| 4(R_1 + 2^k \sigma)(2R_1 + 2^k \sigma)\sigma e - \frac{s}{t2^k} \right|.
\]

Let \(s \gg 1\) if \(s \ll 2^k t/e\), then
\[
|F^{(m)}(\sigma)| \lesssim 1, \quad \forall m \geq 1, \quad |\tilde{P}_2'(\sigma)| \lesssim \epsilon, \quad |\tilde{P}_2''(\sigma)| \lesssim \epsilon^{1/2}, \quad |\tilde{P}_2'''(\sigma)| \lesssim \epsilon, \quad |\tilde{P}_2^{(m)}(\sigma)| \lesssim \epsilon^{1}, \quad \forall m \geq 4
\]
while for \(s \gg 2^k t/e\)
\[
|F^{(m)}(\sigma)| \ll 1, \quad \forall m \geq 1, \quad |\tilde{P}_2^{(m)}(\sigma)| \lesssim \epsilon^{1}, \quad \forall m \geq 1.
\]

Integrating by parts we get
\[
A_k^{(1)}(s) = 2^k(2^k t)^{-N}K_d\pi e^{iR_1 s} \int_{1/2}^{2} e^{it2^k\tilde{P}_2(\sigma)} \frac{d}{d\sigma} \left( \frac{1}{\tilde{P}_2'(\sigma)} \ldots \frac{1}{\tilde{P}_2'(\sigma)} \frac{d}{d\sigma} \left( \frac{F(\sigma)}{\tilde{P}_2(\sigma)} \right) \right) d\sigma.
\]

Therefore
\[
|A_k^{(1)}(s)| \lesssim 2^k(2^k t)^{-N}.
\]

If \(s \sim 2^k t/e\), we apply Van der Corput Lemma,
\[
|A_k^{(1)}(s)| \lesssim 2^k(2^k t)^{-1/2} \int_{1/2}^{2} |\partial_\sigma F(\sigma)|d\sigma
\]
\[\overset{(5.24)}{\lesssim} 2^k(2^k t)^{-1/2} s^{-(d-1)/2} \lesssim 2^k(2^k t)^{-d/2} s^{(d-1)/2}.
\]
Moreover, we can check that $|A_k^{(1)}(s)| \lesssim 2^k$; hence, for $s \gg 1$

$$|A_k^{(1)}(s)| \lesssim 2^k \min(1, (2^k t)^{-d/2}). \quad (5.34)$$

If $s \ll 1$, we rewrite $A_k^{(1)}$ in the following form

$$A_k^{(1)}(s) = 2^k K_2 \pi e^{-i R_1 s} \int_{1/2}^2 F_1(\sigma)e^{itP_2(R_1+2^k \sigma)}d\sigma,$$

$$F_1(\sigma) := (R_1 + 2^k \sigma)^{d-1-\bar{\rho}}(R_1 + 2^k \sigma)\bar{h}(R_1 + 2^k \sigma)se^{-i2^k \sigma}.$$

Again integrating by parts, we obtain

$$|A_k^{(1)}(s)| \lesssim 2^k \min(1, (2^k t)^{-d/2}). \quad (5.35)$$

Now we estimate $A_k^{(2)}$. We notice that $R_2 \in \text{supp}(\phi_k(R_1 - \cdot))$ if and only if $k \in \{-2, -1\}$; when $k \notin \{-2, -1\}$ one can repeat the above argument and show that

$$|A_k^{(2)}(s)| \lesssim 2^k \min(1, (2^k t)^{-d/2}). \quad (5.36)$$

Let $k \in \{-2, -1\}$. If $s \ll t$ or $s \gg t$ we have by integration by parts that

$$|A_k^{(2)}(s)| \lesssim \min(1, t^{-N}), \forall N \in \mathbb{N}.$$

On the other hand, if $s \sim t$ we can use Van der Corput Lemma and obtain

$$|A_k^{(2)}(s)| \lesssim t^{-1/3} s^{-(d-1)/2} \lesssim t^{-\frac{d}{2}+\frac{1}{6}}.$$

Therefore, for $k \in \{-2, -1\}$ we have

$$|A_k^{(2)}(s)| \lesssim \min(1, t^{-\frac{d}{2}+\frac{1}{6}}). \quad (5.38)$$

Combining (5.37) and (5.38) we can deduce that

$$|A_k^{(2)}(s)| \lesssim 2^k \min(1, (2^k t)^{-\frac{d}{2}+\frac{1}{6}}). \quad (5.39)$$

If we sum up all the $A_k$ for $k \leq K + 2$ we finally conclude that for any $d \geq 2$

$$\|P_u U_2(t)\psi_0\|_{L^2} \lesssim c \min(|t|^{-d/2}, |t|^{-d/2+1/6})\|\psi_0\|_{L^1}. \quad (5.40)$$

**Remark 5.7.** In the general case $r > 2$, we have to determine critical points for the polynomial

$$P_r(R) = \sum_{j=1}^r (-1)^{j+1} e^{j-1} R^{2j}, \quad (5.41)$$

namely the roots of the polynomial

$$P'_r(R) = \sum_{j=1}^r (-1)^{j+1} e^{j-1} j R^{2j-1} = R \left( \sum_{j=1}^r (-1)^{j+1} e^{j-1} 2j R^{2j-1} \right). \quad (5.42)$$
Besides the trivial value $R = 0$, which we deal as in the case $r = 2$, one should rely on lower and upper bounds to determine the other (if any) real roots. For a lower bound, we rely on a well-known corollary of Rouché theorem from complex analysis, and we obtain that the other roots satisfy
\[
R \geq \frac{2}{\max \left(2, \sum_{j=0}^{r} 2j\epsilon^{-1} \right)} \geq \frac{2}{\max \left(2, 2r \sum_{j=0}^{r-1} \epsilon^j \right)}
\]
for $r \leq 1/2$ and
\[
R \geq \frac{2}{\max(2, 4r)} \geq 1.
\]
For what concerns an upper bound, we exploit an old result by Fujiwara ([Fuj16]), and we get that the roots satisfy
\[
R \leq \max_{1 \leq j \leq r-1} \left( 2(r-1) \frac{2j\epsilon^{-1}}{2r\epsilon^{-1}} \right)^{\frac{1}{r-1}} \leq 2(r-1) \max_{1 \leq j \leq r-1} \left( \frac{j}{r} \right)^{\frac{1}{r-1}} \epsilon^{\frac{r}{2(r-1)}} \leq K_r \epsilon^{-1/2}
\]
for some $K_r > 0$.

Hence, in the case $r > 2$, if $\epsilon$ sufficiently small (depending on $r$), then the polynomial $P^r_\epsilon$ has critical points (apart from 0) which have modulus between 1 and $O(\epsilon^{-1/2})$ (a similar argument works also for the polynomial $P^n_\epsilon$), and this affects the medium-frequency decay of $U_\epsilon(t)$. In any case, we can deal with this problem as in the case $r = 2$, and we get
\[
\|P_\epsilon U_\epsilon(t)\psi\|_{L^1} \lesssim |t|^{-d/2} \|\psi_0\|_{L^1}, \quad (5.43)
\]
\[
\|P_\epsilon U_\epsilon(t)\psi\|_{L^\infty} \lesssim c \min(|t|^{-d/2}, |t|^{-d/2 + 1/6}) \|\psi_0\|_{L^1}, \quad (5.44)
\]
\[
\|P_\epsilon U_\epsilon(t)\psi\|_{L^\infty} \lesssim e^{d/2} |t|^{-d/4} \|\psi_0\|_{L^1}, \quad 0 < |t| \leq c^2(r-1). \quad (5.45)
\]

### 5.2.2 Smoothing estimates

As already pointed out, one needs smoothing estimates to ensure the well-posedness of Eq. (5.8) because of the presence of derivatives in the nonlinearity. Again, we first consider the case $r = 2$, and then we mention the results for $r > 2$.

**Proposition 5.8.** For any $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ with $|k_i| = |k|_\infty$ and $|k_i| \gtrsim c$
\[
\left\| \square_k D^{3/2}_{z_1} U_\epsilon(t)\psi \right\|_{L^\infty_{z_1}(x_1)} \lesssim c \|\square_k \psi_0\|_{L^2_{z_1}}. \quad (5.46)
\]

**Proof.** It suffices to consider the case $i = 1$. For convenience, we write $\bar{z} = (z_1, \ldots, z_d)$. Then,
\[
\left\| \square_k D^3_{z_1} U_\epsilon(t)\psi \right\|_{L^\infty_{z_1}(x_1)} \lesssim \int \sigma_k(\xi)|\xi_1|^{3/2} e^{itP_\epsilon(\xi)} F(\psi_0)(\xi)e^{ix_1 \xi_1} d\xi_1 \left\| L^\infty_{L^2_{\xi_1}} \right. \leq L.
\]

\[
\begin{align*}
\left\| \square_k D^3_{z_1} U_\epsilon(t)\psi \right\|_{L^\infty_{z_1}(x_1)} &\lesssim \int \eta_k(\xi_1)|\xi_1|^{3/2} e^{itP_\epsilon(\xi)} F(\psi_0)(\xi)e^{ix_1 \xi_1} d\xi_1 \left\| L^\infty_{L^2_{\xi_1}} \right. \\
&= L.
\end{align*}
\]
Now, we estimate $L$: if $k_1 \gtrsim c$, then $\xi_1 > 0$ for $\xi \in \text{supp}(\eta_{k_1})$. Hence, by changing variable, $\theta = P_2(\xi)$, we get

$$L \lesssim \left\| \int \eta_{k_1}(\xi(\theta))\xi_1(\theta)^{3/2}e^{it\theta}F(\psi_0(\xi(\theta))e^{ix_1\xi_1(\theta)}\frac{1}{2}\xi_1^{-1}(\theta) \left( \frac{2|\xi|^2}{c^2} - 1 \right)^{-1} \right\|_{L^\infty_{\tau}L^2_{\xi, t}},$$

$$\lesssim \left\| \eta_{k_1}(\xi(\theta))\xi_1(\theta)^{1/2}F(\psi_0(\xi(\theta)) \left( \frac{2|\xi|^2}{c^2} - 1 \right)^{-1/2} \xi_1^{1/2} \right\|_{L^2_{\xi}},$$

$$\lesssim \left\| \eta_{k_1}(\xi_1)\xi_1F(\psi_0) \xi_1 \left( \frac{2|\xi|^2}{c^2} - 1 \right)^{-1/2} \right\|_{L^2_{\xi}} \lesssim c\|\psi_0\|_{L^2}.$$

The proof for the case $k_1 \lesssim -c$ is similar.

By duality we have the following

**Proposition 5.9.** For any $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ with $|k_1| = |k|_{\infty}$ and $|k_1| \gtrsim c$

$$\|\Box_k \partial^2_{x_i}A_{22}f\|_{L^p_tL^q_x} \lesssim c\|\Box_k D^2_t f\|_{L^1_tL^{2,p}_{x, x}},$$

(5.47)

Now consider the inhomogeneous Cauchy problem

$$-i\psi_t = A_{c, 2}\psi + f(t, x), \quad \psi(0, x) = 0.$$  

(5.48)

**Proposition 5.10.** For any $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ with $|k_1| = |k|_{\infty}$ and $|k_1| \gtrsim c$

$$\|\Box_k \partial^2_{x_i} \psi\|_{L^p_tL^q_x} \lesssim \|\Box_k f\|_{L^1_tL^{2,p}_{x, x}},$$

(5.49)

**Proof.** It suffices to consider $i = 1$. We write

$$\psi = \mathcal{F}^{-1}_{\tau, \xi} \frac{1}{\tau - c^2 - P_2(|\xi|)}(\mathcal{F}_{t, x}f)(\tau, \xi).$$

We have

$$\partial^2_{x_i} \psi = \mathcal{F}^{-1}_{\tau, \xi} \frac{\xi_1^2}{\tau^2 - P_2(|\xi|) + c^2 - \tau} \mathcal{F}_{t, x}f.$$  

(5.50)

We want to show that

$$\left\| \mathcal{F}^{-1}_{\tau, \xi} \frac{\eta_{k_1}(\xi_1)\xi_1^2}{\tau - P_2(|\xi|) + c^2 - \tau} \mathcal{F}_{t, x}f \right\|_{L^\infty_{\tau}L^2_{\xi, t}} \lesssim \left\| \mathcal{F}^{-1}_{\xi_1} \eta_{k_1}(\xi_1) \mathcal{F}_{x_1}f \right\|_{L^1_{x_1}L^2_{\xi_1, t}},$$

which, by Young's inequality, is equivalent to show that

$$\sup_{x_1, \tau, \xi_1, (j \neq 1)} \left| \mathcal{F}^{-1}_{\xi_1} \frac{\sigma_k(\xi_1)\xi_1^2}{\tau - P_2(|\xi|) + c^2 - \tau} \right| \lesssim 1.$$  

(5.51)
We prove (5.51): first, notice that when $|k_1| = |k|_\infty$, then $|\xi_1| \sim |\xi|_\infty$ for $\xi \in \text{supp}(\sigma_k)$. We split the argument according to the cases $\tau - c^2 > 0$ and $\tau - c^2 \leq 0$. In the case $\tau - c^2 > 0$

\[
\sup_{x, \tau, \xi, (j \neq 1)} \left| \mathcal{F}_{\xi_1}^{-1} \frac{\sigma_{k}(\xi)\xi_1^2}{P_2(|\xi|) + c^2 - \tau} \right| \lesssim \left| \int_{k_1 - 3/4}^{k_1 + 3/4} \frac{c^2}{\xi_1^2} \, d\xi_1 \right| \lesssim 1.
\]

When $\tau - c^2 \leq 0$ we set $\tau_2 := \tau_2(c) = c \left( \sqrt{\frac{\tau}{c^2}} - \frac{1}{2} + \frac{1}{2} \right) > 0$, in order to write

\[
P_2(|\xi|) + c^2 - \tau = \left( \frac{|\xi|^2}{c} + \tau_2 \right) \left( - \frac{|\xi|^2}{c} + \tau_2 + c \right).
\]

Hence

\[
\mathcal{F}_{\xi_1}^{-1} \frac{\sigma_{k}(\xi)\xi_1^2}{P_2(|\xi|) + c^2 - \tau} = \mathcal{F}_{\xi_1}^{-1} \frac{\sigma_{k}(\xi)\xi_1^2}{\left( \frac{|\xi|^2}{c} + \tau_2 \right) \left( - \frac{|\xi|^2}{c} + \tau_2 + c \right)}
\]

\[
= \mathcal{F}_{\xi_1}^{-1} \frac{\eta_{k_1}(\xi_1)\xi_1^2}{\xi_1 \left( \frac{|\xi|^2}{c} + \tau_2 \right) \left( - \frac{|\xi|^2}{c} + \tau_2 + c \right) \eta_{k_1}(\xi_1)}
\]

\[
\lesssim c_{1/2} \mathcal{F}_{\xi_1}^{-1} \frac{\eta_{k_1}(\xi_1)\xi_1}{\xi_1 + \frac{A^2}{2} B^2 - \frac{\xi_1}{c}} \left( B - \frac{\xi_1}{c} + \frac{1}{2} \right) \eta_{k_1}(\xi_1)
\]

\[
=: I + II.
\]

We estimate only $I$, as the argument of $II$ is similar. First we write

\[
I = \frac{c}{2} \mathcal{F}_{\xi_1}^{-1} \frac{\eta_{k_1}(\xi_1)\xi_1}{\xi_1 + \frac{A^2}{2}} + \frac{c}{2} \mathcal{F}_{\xi_1}^{-1} \frac{\eta_{k_1}(\xi_1)B}{(B - \frac{\xi_1}{c} + \frac{1}{2}) (\xi_1 + \frac{A^2}{2})} := I_1 + I_2.
\]

Since $\mathcal{F}_{\xi_1}^{-1}(1/\xi_1)$ is the function $\text{sgn}(\xi_1)$, we have that $I_1$ is bounded uniformly with respect to $c$. For $I_2$, it suffices to show

\[
cB \sup_{x_1} \left| \mathcal{F}_{\xi_1}^{-1} \frac{1}{(B - \frac{\xi_1}{c} + \frac{1}{2}) (\xi_1 + \frac{A^2}{2})} \right| \lesssim 1.
\]

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Since $|\mathcal{F}(e^{-i|\cdot|})(\xi)| \lesssim \frac{1}{1 + |\xi|}$,

$$cB \left\| \mathcal{F}^{-1}_{\xi_1} \left( \frac{1}{B - \frac{\xi_1^2}{c^2} + A^2} \right) \right\|_{L^\infty_{\xi_1}} \lesssim cB \left\| \mathcal{F}^{-1}_{\xi_1} \left( \frac{1}{B - \frac{\xi_1^2}{c^2} + A^2} \right) \right\|_{L^1_{\xi_1}} \lesssim \frac{c^2B}{A^2} \left\| \mathcal{F}^{-1}_{\xi_1} \left( \frac{1}{B - \frac{\xi_1^2}{c^2} + A^2} \right) \right\|_{L^1_{\xi_1}} \lesssim B \left\| \mathcal{F}^{-1}_{\xi_1} \left( \frac{1}{B - \frac{\xi_1^2}{c^2} + A^2} \right) \right\|_{L^1_{\xi_1}} \lesssim 1.$$

Finally, we observe that in general the solution $\psi$ of (5.48) may not vanish at $t = 0$. However, by Parseval identity

$$\psi(0, x) = \psi(t, x)|_{t=0} = K \int_I U_2(s)f(s, x) ds,$$

for some $K > 0$, and if we combine it with (5.47), we have that $\Box U_2(t)\partial^2_{t} \psi(0, x) \in L^2$. Hence, by (5.46)

$$\tilde{\psi}(t) := \psi(t) - U_2(t)\psi(0, \cdot) = i \int_I U_2(t - \tau)f(\tau) d\tau$$

is the solution of (5.48), and it satisfies (5.49).

**Lemma 5.11.** For any $\sigma \in \mathbb{R}$ and $k \in \mathbb{Z}^d$ with $|k| \geq 4$,

$$\left\| \Box_k D^\sigma_{\xi_1} \psi \right\|_{L^{p_1}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)} \lesssim \langle k \rangle^\sigma \left\| \Box_k \psi \right\|_{L^{p_1}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)}.$$  

(5.54)

If we replace $D^\sigma_{\xi_1}$ by $\partial^\sigma_{\xi_1}$, the above inequality holds for all $k \in \mathbb{Z}^d$.

**Proof.** See the proof of Lemma 3.4 in [WHH99]. One can check that both sides of (5.54) are equivalent for $|k| \geq 4$. \qed

By combining (5.49), (5.47) and (5.54) we obtain

**Proposition 5.12.** For any $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ with $|k| = |k|_{\infty} \gtrsim c$ we have

$$\left\| \Box_k \partial^2_{\xi_1} \mathcal{A}_f \right\|_{L^{p_1}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)} \lesssim \left\| \Box_k f \right\|_{L^{1,2}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)},$$

(5.55)

and

$$\left\| \Box_k \partial^2_{\xi_1} \mathcal{A}_f \right\|_{L^{\infty}_{x_1}L^2} \lesssim c \left\| \langle k \rangle \right\|^{1/2} \left\| \Box_k f \right\|_{L^{1,2}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)}.$$  

(5.56)

**Remark 5.13.** For the case $r > 2$ we replace (5.46), (5.47), (5.49), (5.55) and (5.56) with

$$\left\| \Box_k D^{r-1/2}_{\xi_1} \mathcal{U}_t(\psi) \right\|_{L^{p_1}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)} \lesssim c^{-1} \left\| \Box_k \psi \right\|_{L^{2}},$$

(5.57)

$$\left\| \Box_k \partial^r_{\xi_1} \mathcal{A}_f \right\|_{L^{\infty}_{x_1}L^2} \lesssim c^{-1} \left\| \Box_k D^{1/2}_{\xi_1} f \right\|_{L^{1,2}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)},$$

(5.58)

$$\left\| \Box_k \partial^{2(r-1)}_{\xi_1} \psi \right\|_{L^{\infty}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)} \lesssim \left\| \Box_k f \right\|_{L^{1,2}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)},$$

(5.59)

$$\left\| \Box_k \partial^{2(r-1)}_{\xi_1} \mathcal{A}_f \right\|_{L^{\infty}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)} \lesssim \left\| \Box_k f \right\|_{L^{1,2}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)},$$

(5.60)

$$\left\| \Box_k \partial^{2(r-1)}_{\xi_1} \mathcal{A}_f \right\|_{L^{\infty}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)} \lesssim c^{-1} \left\| \langle k \rangle \right\|^{-3/2} \left\| \Box_k f \right\|_{L^{1,2}_{x_1}(\mathbb{R}^j \times \mathbb{R}^d)}.$$  

(5.61)
Remark 5.14. We point out the fact that we have worked out smoothing estimates only in the higher frequencies. As in [RWZ16], only these smoothing estimates are needed in order to discuss the well-posedness of (5.8).

5.2.3 Strichartz estimates

By exploiting (5.6) we can deduce Strichartz estimates for solutions of (5.8) combined with \(\square_k\)-decomposition operators.

Proposition 5.15. Let \(r \geq 1, d \geq 2, c \geq 1, t > 0\). Let \((p,q)\) and \((a,b)\) be order-\(r\) admissible pairs. Then for any \(0 < T \lesssim c^{2(r-1)}\) and for any \(k \in \mathbb{Z}^d\) with \(|k| \gtrsim K\) (\(K = K(c)\) is defined in (5.26))

\[
\|\square_k \mathcal{U}_t(t)\phi_0\|_{L^p((0,T)]L^r(\mathbb{R}^d)} \lesssim c^{d(\frac{1}{p} - \frac{1}{r})} \|\square_k \phi_0\|_{L^2(\mathbb{R}^d)},
\]

(5.62)

\[
\|\square_k \int_0^T \mathcal{U}_t(t-\tau)\phi(\tau)d\tau\|_{L^p((0,T)]L^r(\mathbb{R}^d)} \lesssim c^{(\frac{1}{r} - \frac{1}{p})}\|\square_k \phi\|_{L^a(\mathbb{R}^d)}\|\mathcal{U}_t(t)\|_{L^b(\mathbb{R}^d)}\|
\]

(5.63)

Furthermore, by (5.2) we have that

\[
\|\square_k \mathcal{U}_t(t)\|_{L^1(\mathbb{R}^d)} \lesssim c^{(\frac{1}{r} - \frac{1}{p})}(t)^{-d/(2r)},\]

and by following closely the argument in Section 5 of [WH07] we can deduce

Proposition 5.16. Let \(r \geq 1, d \geq 2, c \geq 1\). Let \((p,q)\) be a Schrödinger admissible pair, then

\[
\|\mathcal{U}_t(t)\psi_0\|_{L^p_t((0,T])L^r_x(\mathbb{R}^d)} \lesssim c^{(\frac{1}{r} - \frac{1}{p})}\|\psi_0\|_{M^r_{p,2}(\mathbb{R}^d)},\]

(5.64)

\[
\|A_t f\|_{L^p_t((0,T])L^r_x(\mathbb{R}^d)} \lesssim c^{(\frac{1}{r} - \frac{1}{p})}\|f\|_{L^a_t((0,T])L^b_x(\mathbb{R}^d)}.
\]

(5.65)

5.2.4 Maximal function estimates

In this subsection we study the maximal function estimates for the semigroup \(\mathcal{U}_t(t)\) and the integral operator \(\int_0^T \mathcal{U}_t(t-\tau)\cdot d\tau\) in anisotropic Lebesgue spaces. To do this, we will need the time decay properties proved in Sec. 5.2.1. As always, we first prove results for the case \(r = 2\), and then we write the modification for the general case.

Lemma 5.17. 1. Let \(q \geq 2, \frac{d}{q} < q \leq +\infty\) and \(k \in \mathbb{Z}^d\) with \(|k| \gtrsim K\), then

\[
\|\square_k \mathcal{U}_t(t)\psi_0\|_{L^{q,\infty}_{x_i(x_j)j \neq i}(\mathbb{R}^d)} \lesssim c^{d/2}\langle k \rangle^{1/q}\|\square_k \psi_0\|_{L^2(\mathbb{R}^d)},\]

(5.66)

2. Let \(q \geq 2, \frac{d}{q} < q \leq +\infty\) and \(k \in \mathbb{Z}^d\) with \(|k| \lesssim K\), then

\[
\|\square_k \mathcal{U}_t(t)\psi_0\|_{L^{q,\infty}_{x_i(x_j)j \neq i}(\mathbb{R}^d)} \lesssim c^{\langle k \rangle^{1/q}}\|\square_k \psi_0\|_{L^2(\mathbb{R}^d)}, \forall i = 1,\ldots,d.
\]

(5.67)

Proof. Clearly it suffices to show the thesis for \(i = 1\); recall that for any \(x = (x_1,\ldots,x_d) \in \mathbb{R}^d\), we denote \(\bar{x} = (x_2,\ldots,x_d)\). By a standard \(TT^*\) argument, (5.66) is equivalent to

\[
\left\|\int_{\mathbb{R}^d} e^{i(x,\xi)}e^{it(x^2 + P(x))}a_k(\xi)d\xi\right\|_{L^{q/2,\infty}_{x_i(x_j)j \neq i}(\mathbb{R}^d)} \lesssim \langle k \rangle^{2/q}.
\]

(5.68)
If $|k| \gtrsim K(c)$, then
\[
\| \mathcal{F}^{-1} e^{it(c^2 + P_2(|\xi|))} \sigma_k(\xi) \|_{L^\infty_x} \overset{(5.28)}{\lesssim} c^{d/2} \langle k \rangle^{-d} |t|^{-d/4}, \quad 0 < |t| \lesssim c^2; \tag{5.69}
\]
on the other hand
\[
\| \Box \mathcal{U}_2(t) \mathcal{F}^{-1} \sigma_k \|_{L^\infty_x} \lesssim \| \Box \mathcal{U}_2(t) \mathcal{F}^{-1} \sigma_k \|_{L^\infty_x L^2_x} \lesssim 1. \tag{5.70}
\]
If we combine (5.69) and (5.70), we obtain
\[
\| \Box \mathcal{U}_2(t) \mathcal{F}^{-1} \sigma_k \|_{L^\infty_x} \lesssim c^{d/2}(1 + \langle k \rangle^4 |t|)^{-d/4}, \quad 0 < |t| \lesssim c^2. \tag{5.71}
\]
Now, if $|x_1| \gtrsim 1 + |t| \langle k \rangle^5$, by integrating by parts we get
\[
\| \Box \mathcal{U}_2(t) \mathcal{F}^{-1} \sigma_k \| \lesssim c^{d/2} \langle x_1 \rangle^{-2}. \tag{5.72}
\]
If $|x_1| \lesssim 1 + |t| \langle k \rangle^5$, by (5.71) we can deduce
\[
\| \Box \mathcal{U}_2(t) \mathcal{F}^{-1} \sigma_k \| \lesssim c^{d/2}(1 + |x_1| \langle k \rangle^{-1})^{-d/4}. \tag{5.73}
\]
Combining (5.72) and (5.73) we have
\[
\sup_{\bar{x}, \bar{t}} \| \Box \mathcal{U}_2(t) \mathcal{F}^{-1} \sigma_k \| \lesssim c^{d/2} \langle x_1 \rangle^{-2} + c^{d/2}(1 + |x_1| \langle k \rangle^{-1})^{-d/4}, \tag{5.74}
\]
from which, by taking the $L^{3/2}_x$ norm on both sides, we obtain (5.68). The proof for the case $|k| \lesssim K(c)$ is similar. \qed

Lemma 5.18. Let $q \geq 2$, $\frac{4}{d} < q \leq +\infty$ and $k \in \mathbb{Z}^d$ with $|k| \gtrsim K(c)^2$, then
\[
\| \Box_k A_2 f \|_{L^q_{x_i(t), x_j(t)}} \lesssim c^{d/2} \langle k \rangle^{-3/2 + 1/q} \| \Box_k f \|_{L^1_{x_i(t), x_j(t)}}, \quad 0 < |t| \lesssim c^2, \quad \forall i = 1, \ldots, d. \tag{5.75}
\]

Proof. It suffices to prove the case $i = 1$. Recall that the solution of (5.48) is of the form
\[
\psi = \mathcal{F}^{-1}_{\tau, k} \frac{1}{c^2 + P_2(|\xi|) - \tau} \mathcal{F}_{\tau, x} f,
\]
hence its frequency localization can be written as
\[
\Box_k \psi = \mathcal{F}^{-1}_{\tau, k} \frac{1}{c^2 + P_2(|\xi|) - \tau} (\mathcal{F}_{\tau, x} \Box_k f)(\tau, \xi).
\]
For convenience, we introduce the following regions
\[
\mathcal{E}_1 = \{ \tau - c^2 \leq -c^2/4 \}, \quad \mathcal{E}_2 = \left\{ -c^2/4 \leq \tau - c^2 \leq |\xi|^2 \left( -\frac{|\xi|^2}{c^2} + 1 \right) \right\}, \quad \mathcal{E}_3 = \left\{ \tau - c^2 \geq |\xi|^2 \left( -\frac{|\xi|^2}{c^2} + 1 \right) \right\},
\]
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and we make the following decomposition

\[
c^2 + P_2(|\xi|) - \tau = \begin{cases} 
\left( \frac{|\xi|^2}{c^2} + \tau_2(c, \tau) \right) \left( -\frac{\xi \cdot \tau}{c^2} + a \right) + \left( \frac{\xi \cdot \tau}{c^2} + a \right), & (\xi, \tau) \in \mathbb{E}_1, \\
\left( \frac{|\xi|^2}{c^2} + \tau_2(c, \tau) \right) \left( -\frac{\xi \cdot \tau}{c^2} + \tau_2(c, \tau) + c \right), & (\xi, \tau) \in \mathbb{E}_2, \\
\left( \frac{|\xi|^2}{c^2} - \tau \right)^2 + \left( \frac{\xi \cdot \tau}{c^2} - \tau \right), & (\xi, \tau) \in \mathbb{E}_3,
\end{cases}
\]

(5.76)

where \(a = a(c, \xi, \tau) := (\tau_2(c, \tau) - |\xi|^2/c + c)^{1/2} \). We denote

\[
\Box_k \psi_1 = \mathcal{F}^{-1}_{\tau, \xi} \frac{\chi_{E_1}(\tilde{\xi}, \tau)}{c^2 + P_2(|\xi|) - \tau} (\mathcal{F}_{t,x} \Box_k f)(\tau, \xi), \quad i = 1, 2, 3.
\]

First, we estimate \(\Box_k \psi_1\). Set \(\tilde{\eta}_k(\xi_1) = \sum_{|\xi| \leq 10} \tilde{\eta}_{1+i}(\xi_1)\). First we notice that

\[
= \sum_{j=1}^{3} A_j(c, \xi, \tau).
\]

According to the above decomposition, we can rewrite \(\Box_k \psi_1\) as

\[
\Box_k \psi_1 = \mathcal{F}^{-1}_{\tau, \xi} \frac{\chi_{E_1}(\tilde{\xi}, \tau)}{c^2 + P_2(|\xi|) - \tau} (\mathcal{F}_{t,x} \Box_k f)(\tau, \xi) + \mathcal{F}^{-1}_{\tau, \xi} \frac{\chi_{E_1}(\tilde{\xi}, \tau)(1 - \tilde{\eta}_{k_1}(ac^{1/2}))}{c^2 + P_2(|\xi|) - \tau} (\mathcal{F}_{t,x} \Box_k f)(\tau, \xi)
\]

\[
= \sum_{j=1}^{3} \mathcal{F}^{-1}_{\tau, \xi} \chi_{E_1}(\tilde{\xi}, \tau) A_j(c, \xi, \tau) \tilde{\eta}_{k_1}(ac^{1/2}) (\mathcal{F}_{t,x} \Box_k f)(\tau, \xi)
\]

\[
+ \mathcal{F}^{-1}_{\tau, \xi} \chi_{E_1}(\tilde{\xi}, \tau)(1 - \tilde{\eta}_{k_1}(ac^{1/2})) (\mathcal{F}_{t,x} \Box_k f)(\tau, \xi)
\]

\[
=: I + II + III + IV.
\]

Case \(k_1 \geq K(c)^2\): first, we estimate II. Let \(\tilde{\sigma}_k\) be as in (5.21), then

\[
II = \int_{\mathbb{R}^d} \frac{e^{it\tau + i(x, \xi)} \chi_{E_1}(\tilde{\xi}, \tau)}{2a(2\tau_2(c, \tau) + c)} \tilde{\sigma}_k(\tilde{\xi}) \tilde{\eta}_{k_1}(ac^{1/2}) \int_{\mathbb{R}^d} \tilde{f}(y_1, \cdot)(\xi, \tau)e^{i(x_1 - y_1)ac^{1/2}} sgn(x_1 - y_1) d\xi dy_1 d\tau.
\]

By changing variable, \(\xi_1 = c^{1/2}a(c, \xi, \tau)\), and by setting \(\tilde{\rho}_k(\xi) = \tilde{\sigma}_k(\xi) \tilde{\eta}_{k_1}(\xi_1)\), we obtain

\[
|II| \leq \int d\xi \ sgn(x_1 - y_1) \int e^{itc^2 + P_2(|\xi|)} e^{i(x_1 - y_1)\xi + i(x, \xi)} \tilde{\rho}_k(\xi) \tilde{f}(y_1, \cdot) (e^{2 + P_2(|\xi|)}, \tau) d\xi,
\]

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and by applying (5.66) we get
\[
\|II\|_{L^\infty_t L^2_x} \lesssim \int \left\| \int e^{it(c^2 + P_2(|\xi|))} e^{i(x_1 - y_1)\xi + i(x, \xi) \rho_k(\xi) \int_k f(y_1, \cdot)} (c^2 + P_2(|\xi|), \tau) d\xi \right\|_{L^\infty_t L^2_x} dy_1
\]
\[
\lesssim c^{d/2} (k_1)^{1/4} \int \|\hat{\rho}_k(\xi)\|_{L^{\infty_t} L^{2}_x} \|e^{it(\xi, \tau)}(c^2 + P_2(|\xi|), \tau)\|_{L^2_x} dy_1
\]
(5.74), (5.54) \( c^{d/2} (k_1)^{1/4 - 3/2} \|\hat{k} f\|_{L^1_t L^{2}_x} \). (5.77)

Since \( k_1 > 0 \), \( III \) has the same upper bound as in (5.77).

Now we estimate \( IV \): first notice that
\[
IV = \int \left\| 1 \right\|_{L^\infty_t L^2_x} \lesssim \left\| 1 \right\|_{L^\infty_t L^2_x}
\]
(5.54) \( c^{d/2} (k_1)^{1/4 - 3/2} \|\hat{k} f\|_{L^1_t L^{2}_x} \). (5.77)

By Young’s inequality for convolutions, Hölder’s inequality and Minkowski’s inequality we have
\[
\|IV\|_{L^\infty_t L^2_x} \lesssim \left\| \int \|\hat{\sigma}_k(\xi)\|_{L^{\infty_t} L^{2}_x} \left\| K(x_1, a, \xi) \right\|_{L^1_{t, \tau}} dy_1 \right\|_{L^2_{\xi, \tau}}
\]
\[
\lesssim \left\| \right\|_{L^1_{t, \tau}} \left\| \hat{\sigma}_k(\xi) K(x_1, a, \xi) \right\|_{L^1_{t, \tau}}
\]
\[
\lesssim \left\| \right\|_{L^1_{t, \tau}} \left\| \hat{\sigma}_k(\xi) K(x_1, a, \xi) \right\|_{L^1_{t, \tau}} \|L^\infty_t L^2_x \|L^1_{t, \tau}
\]
Integrating by parts it follows that
\[
\|\hat{\sigma}_k(\xi) K(x_1, a, \xi)\|_{L^\infty_t L^2_x L^1_{t, \tau}} \lesssim \sup_{|\xi - k| \leq 3} \sum_{j=0}^1 \left\| 1 \right\|_{L^\infty_t L^2_x} \left\| (1 - \bar{\eta}_k(\xi) D_{\xi_1} (c^2 + P_2(|\xi|) - \tau)^{-1} \right\|_{L^2_{\xi}}.
\]
(5.78)

Noticing that \(|\xi - ac^{1/2}| \geq c^{1/2} \geq 1 \) in the support set of \((1 - \bar{\eta}_k(\xi) D_{\xi_1} (c^2 + P_2(|\xi|) - \tau)^{-1} \) we can deduce from (5.76) that there is no singularity if we integrate (5.78), and this gives
\[
\|\hat{\sigma}_k(\xi) K(x_1, a, \xi)\|_{L^\infty_t L^2_x L^1_{t, \tau}} \lesssim c^{1/2} |k_1|^{-3/2}.
\]

Now we estimate \( I \): we begin by setting
\[
J(x_1, a, \xi) = \chi_{E_1}(\xi, \tau) \bar{\eta}_k(\xi) \sum_{|\xi_1| \leq \eta_0 f(\xi_1 \xi + i\xi_1) \left( \frac{|\xi_1|^2}{c} + \tau(x, \tau) \right) d\xi.
\]
(5.79)

One can check that
\[
I = \int \left\| \right\|_{L^1_{t, \tau}} \left\| (1 - \bar{\eta}_k(\xi) D_{\xi_1} (c^2 + P_2(|\xi|) - \tau)^{-1} \right\|_{L^2_{\xi}}.
\]

Similar to the estimate of \( IV \), by Young’s, Hölder’s and Minkowski’s inequalities we obtain
\[
\|I\|_{L^\infty_t L^2_x} \lesssim c^{1/2} \|\hat{\sigma}_k(\xi) J(x_1, a, \xi)\|_{L^\infty_t L^2_x L^1_{t, \tau}}.
\]

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By integration by parts we get
\[ |J(x_1, a, \xi)| \lesssim \chi_{E_1}(\xi, \tau) \eta_k_1(\alpha c^{1/2}) \left( \frac{|x_1|^2}{c} + \tau_2(c, \tau) \right)^{-1} \left( 1 + |x_1| \right) \sum_{j=0}^{1} \left| \partial^j_{\xi_1} \left( \frac{|x_1|^2}{c} + \tau_2(c, \tau) \right)^{-1} \right| d\xi_1. \]

Therefore
\[
\| \tilde{\delta}_k(\xi) J(x_1, a, \xi) \|_{L^\infty_k E_1 L^2_x L^4_t} \lesssim \sup_{|\xi-k|_\infty \leq 3} \left\| \chi_{E_1}(\xi, \tau) \eta_k_1(\alpha c^{1/2}) \left( \frac{|x_1|^2}{c} + \tau_2(c, \tau) \right)^{-1} \right\|_{L^2_t},
\]
and noticing that $|\alpha c^{1/2} - k| \leq 20$ in the support set of $\eta_k_1(\alpha c^{1/2})$, we can deduce that $2\tau_2(c, \tau) + c \gtrsim k^2$, and finally we obtain
\[
\| \tilde{\delta}_k(\xi) J(x_1, a, \xi) \|_{L^\infty_k E_1 L^2_x L^4_t} \lesssim |k_1|^{-2}.
\]

The proof for the case $k \lesssim -K(c)^2$ is similar. Furthermore, in the estimate of $\Box_k \psi_2$ and $\Box_k \psi_3$ we can check that there is no singularity in $(c^2 + P_2(|\xi|) - \tau)^{-1}$ for $|\xi| \geq c^{1/2}$ and $(\xi, \tau) \in E_2 \cup E_3$. Hence, one can argue as in (5.79)-(5.81) and conclude. \(\Box\)

In the last Lemma we proved that $\Box_k A_2 : L^{1,2}_{x_1, (x_j), x_2,t} \to L^{2,\infty}_{x_1, (x_j), x_2,t}$. In the next Lemma we show that $\Box_k A_2 : L^{1,2}_{x_2, (x_j), x_1,t} \to L^{2,\infty}_{x_1, (x_j), x_2,t}$.

**Lemma 5.19.** Let $q \geq 2, \frac{8}{d} < q \leq +\infty, k \in \mathbb{Z}^d$ with $|k_1| \gtrsim c$ and $h, i \in \{1, \ldots, d\}$ with $h \neq i$, then
\[
\| \Box_k A_2 f \|_{L^{q,\infty}_{x_2, (x_j), x_1,t}} \lesssim c^{1+d/2} \| k \|_{L^{1,2}_{x_1, (x_j), x_2,t}}^{3/2 + 1/q} \left\| \Box_k f \right\|_{L^{1,2}_{x_1, (x_j), x_2,t}}, \quad 0 < |t| \lesssim c^2.
\]

**Proof.** It clearly suffices to consider the case $h = 1, i = 2$ and $k_2 \gtrsim c$. The proof goes along the same line of that of (5.75), and we will only prove in detail the parts that are different. For convenience, we denote $\xi = (\xi_1, \xi_3, \ldots, \xi_d)$. We introduce the following regions
\[
F_1 = \{ \tau - c^2 \leq -c^2/4 \},
F_2 = \left\{ -c^2/4 \leq \tau - c^2 \leq \frac{|\xi|^2}{c^2} \right\},
F_3 = \{ \tau - c^2 \geq \frac{|\xi|^2}{c^2} + 1 \},
\]
and we make the following decomposition
\[
c^2 + P_2(|\xi|) - \tau = \begin{cases} 
\left( \frac{|\xi|^2}{c} + \tau_2(c, \tau) - \frac{\xi_1^2}{c} + a \right) \left( \frac{\xi_1^2}{c} + a \right), & (\xi, \tau) \in F_1, \\
\left( \frac{|\xi|^2}{c} + \tau_2(c, \tau) - \frac{|\xi|^2}{c} + \tau_2(c, \tau) + c \right), & (\xi, \tau) \in F_2, \\
\left( \frac{|\xi|^2}{c} - \frac{\xi_1^2}{c} \right)^2 + \left( \frac{\xi_1^2}{c} - \frac{\xi_1^2}{a} \right), & (\xi, \tau) \in F_3,
\end{cases}
\]
where $b = b(c, \xi, \tau) := (\tau_2(c, \tau) - \frac{|\xi|^2}{c} + c)^{1/2}, \tau_2(c, \tau) = c \left( \sqrt{\frac{c}{2}} - \frac{\xi_1^2}{c} - \frac{1}{2} \right)$. We denote
\[
\Box_k \psi_i = F^{-1}_{\tau, c^2 + P_2(|\xi|)} \chi_{F_i} \chi_{\mathbb{R}^d} \left( c \left( \sqrt{\frac{c}{2}} - \frac{\xi_1^2}{c} - \frac{1}{2} \right) \right), \quad i = 1, 2, 3.
\]
We estimate $\square_k \tilde{\psi}_1$, since by definition of the regions $F_i$, the estimate of the other terms follow more easily, like in the last Lemma.

Set $\tilde{\eta}_k(\xi_2) = \sum_{|i| \leq 10} \eta_k x_2 + i(\xi_2)$. First we notice that

\[
\frac{\chi_{F_i}(\xi, \tau)}{c^2 + P_2(|\xi|) - \tau} = \frac{\chi_{F_i}(\xi, \tau)}{(2\tau_2(c, \tau) + c\left(\frac{|\xi|}{c} + \tau_2(c, \tau)\right) + \frac{1}{c^{2/\tau_2} + b} + \frac{1}{c^{2/\tau_2} + b})^3} B_j(c, \xi, \tau).
\]

According to the above decomposition, we can rewrite $\square_k \tilde{\psi}_1$ as

\[
\square_k \tilde{\psi}_1 = \sum_{j=1}^3 F_{-j}^{-1} x_2 \chi_{F_i}(\xi, \tau) B_j(c, \xi, \tau) \tilde{\eta}_k(b c^{1/2}) (F_{t,x} \square_k f)(\tau, \xi)
\]

and notice that $2\tau_2(c, \tau) + c \gtrsim k^2$ in the support set of $m$; hence, for sufficiently large $c$, we have

\[
m(\xi, \tau) \lesssim \frac{\chi_{F_i}(\xi, \tau)}{k^2},
\]

and therefore

\[
\|m\|_{L^2_{x_j} L^2_{x_2} \ldots \xi_d, \tau^2} \lesssim |k_2|^{-2}.
\]

Now, since by Young’s, H"older’s and Minkowski’s inequalities we have

\[
\|F_{-j}^{-1} m(\xi, \tau) (F_{t,x} \square_k f)\|_{L^{q/2}_{x_j} L^2_{x_2} \ldots \xi_d, \tau^{2/\xi}} \lesssim \|F_{-j}^{-1} m(\xi, \tau) (F_{t,x} \square_k f)\|_{L^2_{x_j} L^2_{x_2} \ldots \xi_d, \tau^{2/\xi}} \lesssim \|m(\xi, \tau) (F_{t,x} \square_k f)\|_{L^2_{x_j} L^2_{x_2} \ldots \xi_d, \tau^{2/\xi}} \lesssim \|m\|_{L^2_{x_j} L^2_{x_2} \ldots \xi_d, \tau^{2/\xi}} \|F_{t,x} \square_k f\|_{L^2_{x_j} L^2_{x_2} \ldots \xi_d, \tau^{2/\xi}}
\]

we can deduce that

\[
\|I\|_{L^{q/2}_{x_j} L^2_{x_2} \ldots \xi_d, \tau} \lesssim |k_2|^{-2} \|F_{t,x} \square_k f\|_{L^2_{x_j} L^2_{x_2} \ldots \xi_d, \tau^{2/\xi}}.
\]
Now we estimate $IV$: set
\[ m_k(\xi, \tau) := \frac{\chi_{\mathbb{F}_3}(\xi, \tau) \tilde{\sigma}_k(\xi)(1 - \tilde{\eta}_k(b))}{c^2 + P_2(|\xi|) - \tau}, \quad M_k(f) := \mathcal{F}^{-1}_{\tau, \xi} m_k(\xi, \tau)(\mathcal{F}_{t,x} f), \] (5.88)
and notice that $M_k(f)$ is the solution of the inhomogeneous equation
\[ -i\psi_t = A_{c, 2}\psi - \mathcal{F}^{-1}_{\tau, \xi} m_k(\xi, \tau)(c^2 + P_2(|\xi|) - \tau)(\mathcal{F}_{t,x} f). \]

Applying (5.56) (recall that $k_2 \geq c$), we have
\[
\|M_k(f)\|_{L^\infty_t L^2_x} \lesssim c^{3/2} \|M_k(f)\|_{L^\infty_t L^2_x} \\
\lesssim c^{d/2} |k_2|^{-3/2} \|f\|_{L^2_{t,x} L^2_{(\sigma, \tau)}}, \\
= c^{d+1} |k_2|^{-3/2} \|f\|_{L^1_{t,x} L^2_{(\sigma, \tau)}}. \] (5.90)

Next, for $(\xi, \tau) \in \text{supp}(m_k)$,
\[
|c^2 + P_2(|\xi|) - \tau| \geq c^{-1} (k)^2 |k_2|. \] (5.91)

By the definition of $b$ we have that for \( \frac{|\tilde{\xi}|}{\tilde{\tau}} \lesssim \frac{1}{d}(c + \tau_2(c, \tau)) \)
\[
|c^2 + P_2(|\xi|) - \tau| \gtrsim (c + \tau_2(c, \tau))^{3/2}, \quad (\xi, \tau) \in \text{supp}(m_k), \] (5.92)

while for \( \frac{|\tilde{\xi}|}{\tilde{\tau}} \gtrsim \frac{1}{2}(c + \tau_2(c, \tau)) \) we can exploit the fact that $|k|_\infty = |k_2| \gtrsim c$ to obtain again that
\[
|c^2 + P_2(|\xi|) - \tau| \gtrsim (c + \tau_2(c, \tau))^{3/2}, \quad (\xi, \tau) \in \text{supp}(m_k), \] (5.93)

and by combining (5.88) with (5.92)-(5.93) we obtain
\[
m_k(\xi, \tau) \lesssim c \frac{\chi_{\mathbb{F}_3}(\xi, \sigma) \tilde{\sigma}_k(\xi)}{(|k_2|^2 + c + \tau_2(c, \tau))^{3/2}}, \] (5.94)

which gives
\[
\|m_k\|_{L^1_{t,x} L^2_{\mathbb{F}_3} \ldots \mathbb{F}_d, L^\infty_{(\xi, t)}} \lesssim c |k_2|^{-1}. \] (5.95)

Therefore, from (5.88) and (5.86) we can deduce
\[
\|M_k(f)\|_{L^2_{t,x} L^\infty_{(\sigma, \tau)}} \lesssim c |k_2|^{-1} \|f\|_{L^2_{t,x} L^2_{(\sigma, \tau)}}, \] (5.96)

For any $q \geq 2$ we obtain by interpolation between (5.90) and (5.96)
\[
\|M_k(f)\|_{L^2_{t,x} L^\infty_{(\sigma, \tau)}} \lesssim c^{1 + d(\frac{1}{q} - \frac{1}{2})} |k_2|^{-3/2 + 1/q} \|f\|_{L^1_{t,x} L^2_{(\sigma, \tau)}}, \] (5.97)

and replacing $f$ by $\Box_k f$ in (5.97), we finally obtain
\[
\|IV\|_{L^2_{t,x} L^\infty_{(\sigma, \tau)}} \lesssim c^{1 + d(\frac{1}{q} - \frac{1}{2})} |k_2|^{-3/2 + 1/q} \|f\|_{L^1_{t,x} L^2_{(\sigma, \tau)}}, \]
\[
\square \]
If we collect (5.75) and (5.82), we can deduce

Lemma 5.20. Let \( q \geq 2, \frac{4}{q'} < q < +\infty, k \in \mathbb{Z}^d \) with \( |k| \geq c \) and \( h, i \in \{1, \ldots, d\} \), then

\[
\| \square_k \partial_{x_i}^2 A_k f \|_{L^{q,\infty}_{t,x_i} \times \mathbb{R}^+} \lesssim c^{1+d/2} \langle k \rangle^{1/2+1/q} \| \square_k f \|_{L^{1,2}_{t,x_i} \times \mathbb{R}^+}, \quad 0 < |t| \lesssim c^2.
\]  

(5.98)

Remark 5.21. In the general case \( r > 2 \) we have

1. Let \( q \geq 2, \frac{4}{q'} < q < +\infty \) and \( k \in \mathbb{Z}^d \) with \( |k| \geq K(c) \), then

\[
\| \square_k U_r(t) \psi_0 \|_{L^{q,\infty}_{t,x_i} \times \mathbb{R}^+} \lesssim c \langle k \rangle^{1/q} \| \square_k \psi_0 \|_{L^2}, \quad 0 < |t| \lesssim c^{2(r-1)}, \quad \forall i = 1, \ldots, d.
\]  

(5.99)

2. Let \( q \geq 2, \frac{4}{q'} < q < +\infty \) and \( k \in \mathbb{Z}^d \) with \( |k| \leq K(c) \), then

\[
\| \square_k U_r(t) \psi_0 \|_{L^{q,\infty}_{t,x_i} \times \mathbb{R}^+} \lesssim c \langle k \rangle^{1/q} \| \square_k \psi_0 \|_{L^2}, \quad \forall i = 1, \ldots, d.
\]  

(5.100)

3. Let \( q \geq 2, \frac{4}{q'} < q < +\infty \) and \( k \in \mathbb{Z}^d \) with \( |k| \geq K(c)^2 \) and \( i \in \{1, \ldots, d\} \), then

\[
\| \square_k A_r f \|_{L^{q,\infty}_{t,x_i} \times \mathbb{R}^+} \lesssim c^{r-1+d(1-\frac{2}{q})} \langle k \rangle^{-r+1/2+1/q} \| \square_k f \|_{L^{1,2}_{t,x_i} \times \mathbb{R}^+}, \quad 0 < |t| \lesssim c^{2(r-1)}.
\]  

(5.101)

4. Let \( q \geq 2, \frac{4}{q'} < q < +\infty \), \( k \in \mathbb{Z}^d \) with \( |k| \geq c \) and \( h, i \in \{1, \ldots, d\} \) with \( h \neq i \), then

\[
\| \square_k A_r f \|_{L^{q,\infty}_{t,x_i} \times \mathbb{R}^+} \lesssim c^{r-1+d(1-\frac{2}{q})} \langle k \rangle^{-r+1/2+1/q} \| \square_k f \|_{L^{1,2}_{t,x_i} \times \mathbb{R}^+}, \quad 0 < |t| \lesssim c^{2(r-1)}.
\]  

(5.102)

5. Let \( q \geq 2, \frac{4}{q'} < q < +\infty \), \( k \in \mathbb{Z}^d \) with \( |k| \geq c \) and \( h, i \in \{1, \ldots, d\} \), then

\[
\| \square_k \partial_{x_i}^{2(r-1)} A_r f \|_{L^{q,\infty}_{t,x_i} \times \mathbb{R}^+} \lesssim c^{r-1+d(1-\frac{2}{q})} \langle k \rangle^{-3/2+1/q} \| \square_k f \|_{L^{1,2}_{t,x_i} \times \mathbb{R}^+}, \quad 0 < |t| \lesssim c^{2(r-1)}.
\]  

(5.103)

5.2.5 Proof of the local well-posedness

In this subsection we use smoothing estimates, Strichartz estimates and maximal function estimates in order to prove Proposition 5.4. In order to do so, it seems necessary to estimate norms in which partial derivatives and anisotropic Lebesgue spaces have different directions, for example \( \| \partial_{x_i}^2 \square_k A_f \|_{L^{2,\infty}_{t,x_i} \times \mathbb{R}^+} \) with \( |k|_{\infty} = |k|_3 \). As usual, we show results for the case \( r = 2 \), and then we point out the modifications for the case \( r > 2 \).

Lemma 5.22. Let \( i, l, m \in \{1, \ldots, d\}, 1 \leq p, q, \leq +\infty \). Assume that \( k = (k_1, \ldots, k_d) \) with \( |k|_{\infty} = |k_m| \geq c \), then

\[
\| \square_k \square_{x_i}^{2} f \|_{L^{p,q}_{t,x_i} \times \mathbb{R}^+} \lesssim \| \square_k \partial_{x_i}^{2} f \|_{L^{p,q}_{t,x_i} \times \mathbb{R}^+}.
\]  

(5.104)
Proof.

\[ \| \Box_k \partial_x^2 f \|_{L^{p,q}_{x,v(t),r(t)}} \lesssim \sum_{|l| \leq |k|} \left\| F_{l}^{-1} (\xi_t) \left( \left( \frac{\xi_t}{\xi_m} \right)^2 \eta_{k_l} (\xi_t) \eta_{k_m} (\xi_t) \right) \right\|_{L^1(\mathbb{R}^2)} \times \left\| \Box_k \partial_{x,v(t)}^2 f \|_{L^{p,q}_{x,v(t),r(t)}} \right\| \]
\[ \lesssim \| \Box_k \psi \|_{L^{p,q}_{x,v(t)}}. \]

\[ \square \]

Lemma 5.23. 1. Let \((a, b)\) be order-2 admissible, \(i \in \{1, \ldots, d\}\), \(q \geq 2\), \(\frac{n}{2} < q < +\infty\) and \(k \in \mathbb{Z}^d\) with \(|k|_\infty \geq K(c)\), then

\[ \| \Box_k \partial_x^2 A_2 f \|_{L^{p,q(\xi)}_{x,v(t),r(t)}} \lesssim c^{4+\frac{2}{p}} \langle |k|_\infty \rangle^{n+1/q} \| \Box_k f \|_{L^2_x L^q_v}, \quad 0 < |t| \lesssim c^2. \]  \hspace{1cm} (5.105)

2. Let \((a, b)\) be Schrödinger admissible, \(i \in \{1, \ldots, d\}\), then

\[ \| \Box_k \partial_x^2 A_2 f \|_{L^{p,q(\xi)}_{x,v(t),r(t)}} \lesssim c^{4+\frac{2}{p}} \langle |k|_\infty \rangle^{n+1/q} \| \Box_k f \|_{L^2_x L^q_v}, \quad 0 < |t| \lesssim c^2. \]  \hspace{1cm} (5.106)

Proof. Denote

\[ \mathcal{L}_k(f, \psi) = \int \left( \Box_k \int \mathcal{U}_2(t - \tau) f(\tau) d\tau, \psi(t) \right) dt. \]

By duality and the maximal function estimate (5.66)

\[ |\mathcal{L}_k(f, \psi)| \leq \| \Box_k f \|_{L^q_x L^2_t} \sum_{|l| \leq 1} \left\| \Box_k \int \mathcal{U}_2(t - \tau) \psi(t) dt \right\|_{L^q_x L^2_t} \]
\[ \leq \| \Box_k f \|_{L^q_x L^2_t} \sum_{|l| \leq 1} \left\| \Box_k \mathcal{U}_2(t - \tau) \psi(t) dt \right\|_{L^2_x L^\infty_t} \]
\[ \leq c^{d/2} \langle k \rangle^{1/q} \| \Box_k f \|_{L^q_x L^2_t} \| \psi \|_{L^1_x L^2_t}, \]  \hspace{1cm} (5.66)

so by duality we obtain

\[ \left\| \Box_k \int \mathcal{U}_2(t - \tau) f(\tau) d\tau \right\|_{L^p_x L^q_t} \lesssim c^{d/2} \langle k \rangle^{1/q} \| \Box_k f \|_{L^q_x L^2_t}. \]  \hspace{1cm} (5.107)

Therefore, by duality, Strichartz estimates (5.63) and (5.108)

\[ |\mathcal{L}_k(f, \psi)| \leq \left\| \Box_k \int \mathcal{U}_2(-\tau) f(\tau) d\tau \right\|_{L^2_x} \left\| \Box_k \int \mathcal{U}_2(-t) \psi(t) dt \right\|_{L^2_x} \]
\[ \leq c^{d/2} \langle k \rangle^{1/q} \| f \|_{L^q_x L^2_t} c^{(1-1/r)2/q} \| \Box_k \psi \|_{L^q_x L^2_v}, \]  \hspace{1cm} (5.109)

which implies (5.105) for \(q > 2\) or \(a > 2\). In the case \(a = q = 2\), (5.105) can be directly deduced from (5.66). Furthermore, by (5.65), (5.54) and (5.56) we get

\[ \mathcal{L}_k(\partial_x^2 f, \psi) \lesssim c^{1+4/p} \langle |k|_\infty \rangle^{1/2} \| \Box_k f \|_{L^{1,2}_{x,v(t)}} c^{4/p} \| \psi \|_{L^q_x L^2_v}, \]  \hspace{1cm} (5.110)

and we can deduce (5.106); by exchanging \(f\) and \(\psi\), we get (5.107).  \hspace{1cm} \(\square\)
We now summarize the results we will use in order to prove the local well-posedness of (5.8): we omit the proof, it follows from the results of the previous subsections, together with (5.104).

**Proposition 5.24.** Let \( d \geq 2, 8/d \leq p < +\infty, 2 \leq q < +\infty, q > 8/d, k \in \mathbb{Z}^d \) with \( |k|_{\infty} = |k| > c, h, i, l \in \{1, \ldots, d\} \). Then

\[
\left\| \Box_k D_k^{1/2} U_2(t) \psi_0 \right\|_{L^2_{k, t}} \lesssim c \left\| \Box_k \psi_0 \right\|_{L^2},
\]

\[
\left\| \Box_k U_2(t) \psi_0 \right\|_{L^\infty_{k, t}} \lesssim c^{d/2} (k)^{1/q} \left\| \Box_k \psi_0 \right\|_{L^2}, \quad 0 < |t| \lesssim c^2,
\]

\[
\left\| \Box_k U_2(t) \phi_0 \right\|_{L^\infty_{k, t}} \lesssim c \left\| \Box_k \phi_0 \right\|_{L^2}, \quad 0 < |t| \lesssim c^2
\]

\[
\left\| \Box_k \partial_{x_l}^2 A_2 \right\|_{L^2_{k, t}} \lesssim \left\| \Box_k f \right\|_{L^{1/2}_{k, t}},
\]

\[
\left\| \Box_k \partial_{x_l} A_2 \right\|_{L^t_{k, t}} \lesssim c^{1+d/2} (k) \left\| \Box_k f \right\|_{L^{1/2}_{k, t}}, \quad 0 < |t| \lesssim c^2,
\]

\[
\left\| \Box_k \partial_{x_l}^2 A_2 \right\|_{L^t_{k, t}} \lesssim c^{1+ \frac{1}{1+q}} (k) \left\| \Box_k f \right\|_{L^{1/2}_{k, t}}, \quad 0 < |t| \lesssim c^2
\]

\[
\left\| \Box_k \partial_{x_l} A_2 \right\|_{L^t_{k, t}} \lesssim c^{\frac{3}{2} + \frac{1}{1+q}} (k) \left\| \Box_k f \right\|_{L^{1/2}_{k, t}}(t, p), \quad 0 < |t| \lesssim c^2.
\]

For the case \( r > 2 \) we have the following results

**Remark 5.25.**

1. Let \( (a, b) \) be order-\( r \) admissible, \( i \in \{1, \ldots, d\}, q \geq 2, \frac{4}{\alpha} < q < +\infty \) and \( k \in \mathbb{Z}^d \) with \( |k|_{\infty} > c \), then

\[
\left\| \Box_k \partial_{x_l}^r A_r \right\|_{L^t_{k, t}} \lesssim c 2^{(r-1)} \left\| \Box_k f \right\|_{L^{r/2}_{k, t}}, \quad 0 < |t| \lesssim c^{2(r-1)}.
\]

2. Let \( (a, b) \) be Schrödinger admissible, \( i \in \{1, \ldots, d\}, \) then

\[
\left\| \Box_k \partial_{x_l}^r A_r \right\|_{L^{t, 2}_{k, t}} \lesssim c 2^{r-1+2r/a} (k) \left\| \Box_k f \right\|_{L^{1/2}_{k, t}} \quad 0 < |t| \lesssim c^{2(r-1)}
\]

\[
\left\| \Box_k \partial_{x_l}^r A_r \right\|_{L^{t, 2}_{k, t}} \lesssim c 2^{r-1+2r/a} (k) \left\| \Box_k f \right\|_{L^{1/2}_{k, t}} \quad 0 < |t| \lesssim c^{2(r-1)}.
\]

**Proposition 5.26.** Let \( d \geq 2, 4r/d \leq p < +\infty, 2 \leq q < +\infty, q > 4r/d, k \in \mathbb{Z}^d \) with
Lemma 5.27. Let $s \geq 0$, $N \geq 3$, $i \in \{1, \ldots, d\}$, then

$$\sum_{\alpha=1}^{N} \left\| \sum_{1 \leq 1 \leq \ldots \leq N} \sum_{\beta=1}^{N} \sum_{\alpha=1}^{N} \left\| \psi_{\alpha} \right\|_{r_{i}^{1, \ldots, 1, (k_{t}^{1, \ldots, 1})}} \prod_{\beta=1}^{N} \left\| \psi_{\beta} \right\|_{r_{i}^{1, \ldots, 1, (k_{t}^{1, \ldots, 1})}} \right\|_{t, i, c, \left(\begin{array}{c}1^{1, \ldots, 1} \end{array}\right)_{x_{i}(t)}}.$$ (5.132)

Proof. See proof of Lemma 3.1 in [RWZ16].

Lemma 5.28. Let $N \geq 1$ and $i \in \{1, \ldots, d\}$, and assume that $1 \leq p, q, p_{1}, q_{1}, \ldots, p_{N}, q_{N} \leq +\infty$ satisfy

$$\frac{1}{p} = \frac{1}{p_{1}} + \ldots + \frac{1}{p_{N}}, \quad \frac{1}{q} = \frac{1}{q_{1}} + \ldots + \frac{1}{q_{N}}.$$
then
\[
\left\| \sum_{N=1}^{N} \Box_{k}^{N} \psi_{1} \cdots \Box_{k}^{N} \psi_{N} \right\|_{L^{p}_{t} L^{q}_{x}} \lesssim c^{d} N^{d} \prod_{i=1}^{N} \left\| \Box_{k}^{N} \psi_{i} \right\|_{L^{p}_{t} L^{q}_{x}}. \tag{5.133}
\]

**Proof.** See proof of Lemma 3.3 in [RWZ16].

**Lemma 5.29.** Let \( s \geq 0, N \geq 1 \) and \( i \in \{1, \ldots, d\} \), and assume that \( 1 \leq p, q, p_{1}, \ldots, p_{N}, q_{N} \leq +\infty \) satisfy
\[
\frac{1}{p} = \frac{1}{p_{1}} + \ldots + \frac{1}{p_{N}}, \quad \frac{1}{q} = \frac{1}{q_{1}} + \ldots + \frac{1}{q_{N}},
\]
then
\[
\left\| \psi_{1} \cdots \psi_{N} \right\|_{L^{p}_{t, \infty}(L^{q}_{x})} \lesssim N^{d} \prod_{i=1}^{N} \left\| \psi_{i} \right\|_{L^{p_{i}}_{t, \infty}(L^{q_{i}}_{x})}. \tag{5.134}
\]

**Proof.** See proof of Lemma 8.2 in [WH07].

**Proof (Proposition 5.4, part (i), case \( r = 2 \)).** Since the nonlinearity contains terms of the form \((\partial_{x}^{2} \psi)^{\beta}\) with \(|\alpha| \leq 2, |\beta| \geq m + 1\), we introduce the space
\[
D := \{ \psi \in \mathcal{S}': \left\| \psi \right\|_{D} := \sum_{|\alpha| \leq 2} \sum_{i=1}^{d} \rho_{1}^{(i)} (\partial_{x}^{2} \psi) \lesssim c^{-\delta_{0}} \},
\]
where
\[
\rho_{1}^{(i)} (\psi) := \left\| \psi \right\|_{L^{1,1+1/2+1/m}_{t,x}} (L^{\infty}_{x}(L^{2}_{x})), \quad \rho_{2}^{(i)} (\psi) := \left\| \psi \right\|_{L^{1,1} (L^{m}_{x}(L^{2}_{x}, \chi))}, \quad \rho_{3}^{(i)} (\psi) := \left\| \psi \right\|_{L^{1,1+1/m}_{t,x} (L^{\infty}_{x}(L^{2}_{x} \cap L^{2+m}_{x})).
\]
and for some \( \delta_{0} > 0 \) that we will choose later.

Since \( \left\| \psi \right\|_{D} = \left\| \bar{\psi} \right\|_{D} \), without loss of generality we can assume that the nonlinearity contains only terms of the form
\[
\psi^{\beta_{0}} (\partial_{x}^{2} \psi)^{\beta_{1}} (\partial_{x}^{2} \psi)^{\beta_{2}} =: \Psi_{1} \cdots \Psi_{R},
\]
where \( R := |\beta| = \beta + |\beta_{1}| + |\beta_{2}|, |\alpha_{i}| = i \) (i = 1, 2).

To prove the first part of Proposition 5.4 we will show that the map
\[
\mathcal{F} : D \to D,
\]
\[
\psi(t) \mapsto \mathcal{U}_{2}(t) \psi_{0} + i A_{2} P((\partial_{x}^{2} \psi)_{|\alpha| \leq 2}, (\partial_{x}^{2} \bar{\psi})_{|\alpha| \leq 2})
\]
is a contraction mapping.
First, we have that by Proposition 5.24
\[ \|U_2(t)\psi_0\| \lesssim c^{\frac{1}{2} + \frac{n}{m+3/m}} \|\psi_0\|_{M^{n+3/m}_2}. \]

Now, for the estimate of \( \rho_1^{(1)}(A_2\partial_{x_j}^a F) \) \((i, j = 1, \ldots, d)\) it suffices to estimate \( \rho_1^{(1)}(A_2\partial_{x_1}^a F) \): indeed, by (5.104)
\[ \rho_1^{(1)}(A_2\partial_{x_2}^a F) \lesssim \rho_1^{(1)}(A_2\partial_{x_1}^a F). \]

Using frequency-uniform decomposition, we write
\[ \Box_k(\Psi_1 \cdots \Psi_R) = \sum_{B^{(R)}_1, 1} \Box_k(\Box_{k(1)} \Psi_1 \cdots \Box_{k(n)} \Psi_R) + \sum_{B^{(R)}_2, 2} \Box_k(\Box_{k(1)} \Psi_1 \cdots \Box_{k(n)} \Psi_R). \]

By exploiting (5.114) and (5.32) for the first sum and (5.117) and (5.134) for the second sum we obtain
\[ \rho_1^{(1)}(A_2 \partial_{x_2}^a (\Psi_1 \cdots \Psi_R)) \lesssim \left\| \sum_{B^{(R)}_1} \Box_k(\Box_{k(1)} \Psi_1 \cdots \Box_{k(n)} \Psi_R) \right\|_{L^{1, s-r+1/2+1/m,(L^{1, 2}(x, y))}_{1, c}} \]
\[ + c^{\frac{1}{2} + \frac{n}{m+2}} \left\| \sum_{B^{(R)}_2} \Box_k(\Box_{k(1)} \Psi_1 \cdots \Box_{k(n)} \Psi_R) \right\|_{L^{1, s-r+1/2+1/m,(L^{1, 2}(x, y))}_{1, c}} \]
\[ \lesssim c^{\frac{1}{2} + \frac{n}{m+2}} R^d \|\psi\|_{D}. \]

Next, we estimate \( \rho_2^{(1)}(A_2(\Psi_1 \cdots \Psi_R)) \) and \( \rho_3^{(1)}(A_2(\Psi_1 \cdots \Psi_R)) \). By (5.119) and (5.118) we have
\[ \sum_{j=2}^{3} \rho_j^{(1)}(A_2(\Psi_1 \cdots \Psi_R)) \lesssim c^{\frac{1}{2} + \frac{n}{m+2}} \|\Psi_1 \cdots \Psi_R\|_{L^{1, s+1/m,(L^{1, 2}(x, y))}_{1, c}} \]
\[ \lesssim c^{\frac{1}{2} + \frac{n}{m+2}} R^d \|\psi\|_{D}. \]

Then we consider \( \rho_2^{(1)}(A_2 \partial_{x_1}^a (\Psi_1 \cdots \Psi_R)) \): we have
\[ \rho_2^{(1)}(A_2 \partial_{x_1}^a (\Psi_1 \cdots \Psi_R)) \lesssim \left( \sum_{k\in\mathbb{Z}^d} \sum_{|k|_\infty \geq c} \left\| \Box_k A_2 \partial_{x_1}^a (\Psi_1 \cdots \Psi_R) \right\|_{L^{m, \infty}(x_1 \times \mathbb{R}^d)}^{m+1} \right) \]
\[ =: III + IV. \]

Again by (5.118) and (5.134) we obtain
\[ IV \lesssim c^{\frac{1}{2} + \frac{n}{m+2}} R^d \|\psi\|_{D}. \]
Furthermore, we have that
\[ III \lesssim \left( \sum_{k \in \mathbb{Z}^d} + \cdots + \sum_{k \in \mathbb{Z}^d} \right) \| \Box_k A_2 \partial_{x_1}^2 (\Psi_1 \cdots \Psi_R) \|_{L^{m, \infty}_{x_1, (x_j)_{j \neq 1}, t}} \]
\[ =: G_1(\psi) + \cdots + G_d(\psi). \]
Using the frequency-uniform decomposition, (5.115), (5.132) and (5.133) we have that
\[ G_i(\psi) \lesssim c_1^{1 + 3 \frac{d}{2} R^d} \| \psi \|_D, \quad i = 1, \ldots, d, \]
therefore
\[ III \lesssim c_1^{1 + 3 \frac{d}{2} R^d} \| \psi \|_D. \]
Finally, we estimate \( \rho_{(1)}^3 (A_2 \partial_{x_1}^2 (\Psi_1 \cdots \Psi_R)) \). It suffices to consider the case \( i = 1 \): by (5.65) and (5.54) we have
\[ \| \Box_k A_2 \partial_{x_1}^2 f \|_{L^\infty_{x} L^2_t L^{r+1}_{x_m}} \lesssim \frac{\| f \|_{L^\infty_{x} L^2_t L^{r+1}_{x_m}}}{m}, \]
and by (5.116) and (5.105) we obtain
\[ \rho_{(1)}^3 (A_2 \partial_{x_1}^2 (\Psi_1 \cdots \Psi_R)) \lesssim c_1^{1 + \frac{8}{m} (m + 2)} R^d \| \psi \|_D. \]
Collecting all estimates, we have
\[ \| F(\psi) \|_D \lesssim c_1^{\frac{d+1}{m+1} (m^2+1)} + c_1^{1 + \frac{4d+1}{m+1} \frac{2}{m+1}} \sum_{m+1 \leq R \leq M} R^{n+1} \| \psi \|_D. \] (5.135)
and for \( c \geq 1 \) sufficiently large we can conclude by a standard contraction mapping argument (see for example the proof of Theorem 1.1 in [CW90]), by choosing
\[ \delta > \delta_0(d, m, 2) := \max \left( \frac{d}{2}, \frac{4}{m(m+2)}, \frac{1}{m} + \frac{3d}{2m}, \frac{1}{m} + \frac{2}{m(m+2)} + \frac{8}{m^2(m+2)} + \frac{4}{m^3} \right). \] (5.136)

Remark 5.30. By arguing in the same way for the general case \( r > 2 \) we end up with the condition
\[ \delta > \delta_0(d, m, r) := \max \left( d \left( 1 - \frac{1}{r} \right) + \frac{4r}{m^2(m+2)} + \frac{r-1}{m} + \frac{3d}{2m} + \frac{2rm + 8(r-1)^2}{m^2(m+2(r-1))} + \frac{4(r-1)^2}{m^3} \right). \] (5.137)

Remark 5.31. The quantity \( \delta_0(d, l, r) \) defined in Corollary 5.5 is actually the right-hand side of (5.137) with \( m \) replaced by \( 2(l - 1) \).

In order to prove the second part of Proposition 5.4 we will exploit another contraction mapping argument, like in the proof of Theorem 1 in [HHW07] (which in turn is based on the proof of Theorem 4.1 of [KPV93]). In the following, we denote by a \( (Q_\alpha)_{\alpha \in \mathbb{Z}^d} \) a fixed family of nonoverlapping cubes of size \( R \) such that \( \mathbb{R}^d = \bigcup_\alpha Q_\alpha \).
Lemma 5.32. Let $d \geq 2$ and $r \geq 2$, then the following estimates hold.

* (Local smoothing, homogeneous case)

\[
\sup_{\alpha \in \mathbb{Z}^d} \left( \int_{Q_\alpha} \int_{\mathbb{R}} |D_x^{r-1/2} \partial_t \psi_0(x)|^2 \, dx \right)^{1/2} \lesssim e^{r-1} R^{1/2} \|\psi_0\|_{L^2},
\]

(5.138)

\[
\left\| D_x^r \int_{I} \partial_t(t-\tau) \psi(t,\cdot) \, d\tau \right\|_2 \lesssim e^{r-1} R^{1/2} \sum_{\alpha \in \mathbb{Z}^d} \left( \int_{Q_\alpha} \int_{I} |\psi(t,x)|^2 \, dx \right)^{1/2} ;
\]

(5.139)

* (Local smoothing, inhomogeneous case) the solution of the inhomogeneous Cauchy problem

\[-i\psi_t = A_{c,r} \psi + f(t,x), \quad t \in I, x \in \mathbb{R}^d,
\]

such that $\psi_0 \equiv 0$ satisfies

\[
\sup_{\alpha \in \mathbb{Z}^d} \|D_x^{2(r-1)} \psi\|_{L^2_0(Q_\alpha);L^2_0(t)} \lesssim e^{2(r-1)} R T^{1/(4d)} \sum_{\alpha \in \mathbb{Z}^d} \|f\|_{L^2_0(Q_\alpha);L^2_0(t)}
\]

(5.140)

* (Maximal function estimate) For any $s > d + \frac{1}{2}$ we have

\[
\left( \int_{\mathbb{R}^d} \sup_{|t| \leq e^{2(r-1)}} |\partial_t(t)\psi_0(x)|^2 \, dx \right)^{1/2} \lesssim e^{d(1-\frac{1}{2})} \|\psi_0\|_{H^s}.
\]

(5.141)

**Proof (sketch).** The proof in the case $r = 2$ can be obtained simply by rescaling Lemma 3, Lemma 4, Lemma 5 and Lemma 6 of [HHW07]. The proof in the case $r > 2$ can be obtained by considering the operator $\mathcal{U}_k(t)$ and $A_k(t)$ instead of $\mathcal{U}_2(t)$ and $A_2(t)$.

**Proof (Proposition 5.4, part (ii), case $r = 2$).** We will prove the result only for $s = s_0$, since the general case follows from commutator estimates. For simplicity, we only deal with the case

\[
P((\partial_x^\alpha \bar{\psi})_{|\alpha| \leq 2}, (\partial_x^\alpha \bar{\psi})_{|\alpha| \leq 2}) = \partial_{x_1}^\alpha \bar{\psi} \partial_{x_k}^\alpha \bar{\psi} \partial_{x_m}^\alpha \psi.
\]

More precisely, we fix a positive constant $\nu < 1/3$, and we define the space $Z^s_\mathbb{C}$ of all function $\phi: I \times \mathbb{R}^d \rightarrow \mathbb{C}$ such that the following three conditions hold

\[
\|\phi\|_{L_\infty(I;H^{s_0})} \leq c^{-s},
\]

(5.142)

\[
\sum_{|\beta| = s_0 + 1/2} \left( \int_{I} \sup_{\alpha \in \mathbb{Z}^d} \left( \int_{J_{Q_\alpha}} \|\partial_x^\beta \phi(t,x)\|^2 \, dx \right)^{1/2} \right. \left. \|A_2(t)\|_{H^{s_0}} \right) \leq T^\nu,
\]

(5.143)

\[
\left( \sup_{t \in I} \sup_{x \in Q_\alpha} \left| D_x^2 \phi(t,x) \right|^2 \right)^{1/2} \lesssim c^{-s}.
\]

(5.144)

We want to show that the map

\[
\mathcal{F} : Z^s_\mathbb{C} \rightarrow Z^s_\mathbb{C},
\]

\[
\psi(t) \mapsto \mathcal{U}_2(t)\psi_0 + i A_2 P((\partial_x^\alpha \bar{\psi})_{|\alpha| \leq 2}, (\partial_x^\alpha \bar{\psi})_{|\alpha| \leq 2})
\]

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is a contraction mapping.

We can observe that for any \( \beta \in \mathbb{Z}^d \) with \( |\beta| = s_0 - \frac{1}{2} \)
\[
\partial_x^\beta (\partial_x^2 \psi \partial_x^2 \psi \partial_x^2 \partial_x^2 \psi) = \partial_x^\beta \partial_x^2 \psi \partial_x^2 \psi \partial_x^2 \partial_x^2 \psi + \partial_x^\beta \psi \partial_x^2 \partial_x^2 \psi \partial_x^2 \partial_x^2 \psi + \partial_x^\beta \psi \partial_x^2 \partial_x^2 \psi \partial_x^2 \partial_x^2 \psi + R((\partial_x^2 \psi)_{2|\gamma| \leq s_0 - 1/2}).
\]

Now, for any \( \psi \in Z^d \) we have
\[
\sum_{|\beta| = s_0 + 1/2} \sup_{\alpha \in \mathbb{Z}^d} \left( \int \int_{Q_0} |\partial_x^\beta \psi(t,x)|^2 \, dx \, dt \right)^{1/2} \leq\]
\[
\lesssim \sum_{|\beta| = s_0 + 1/2} \sup_{\alpha \in \mathbb{Z}^d} \left( \int \int_{Q_0} |\mathcal{U}_2(t) \partial_x^\beta \psi_0(x)|^2 \, dx \, dt \right)^{1/2} + \sum_{|\beta| = s_0 + 1/2} \sup_{\alpha \in \mathbb{Z}^d} \left( \int \int_{Q_0} \left( \int_0^t \mathcal{U}_2(t-\tau) \partial_x^\beta (\partial_x^\beta \psi \partial_x^2 \partial_x^2 \psi \partial_x^2 \partial_x^2 \psi) \, d\tau \right)^2 \, dx \, dt \right)^{1/2} (5.138),(5.139)
\]
\[
\lesssim c T^{1/3} \| \psi_0 \|_{H^{s_0}} + c^2 T^{1/4(4d)} \sum_{|\beta_0| = s_0 - 3/2} \sum_{j,k,m=1}^d \| \partial_x^\beta \partial_x^2 \partial_x^2 \partial_x^2 \partial_x^2 \partial_x^2 \psi \|_{L^2(Q_0; L^2(I))} + c^2 \int_0^T \| D_x^{1/2} \mathcal{R}((\partial_x^2 \psi)_{2|\gamma| \leq s_0 - 1/2}) \|_{L^2} \, dt \lesssim c T^{1/3} \| \psi_0 \|_{H^{s_0}} + c^2 T^{1/4(d)} \sum_{|\beta_0| = s_0 + 1/2} \sup_{\alpha \in \mathbb{Z}^d} \left( \int \int_{Q_0} |\partial_x^\beta \psi(t,x)|^2 \, dx \, dt \right)^{1/2} \left( \sum_{\alpha \in \mathbb{Z}^d} \sup_{t \in I} \sup_{x \in Q_0} |D_x^2 \psi|^2 \right) \leq c^{1-\delta} T^{1/3} + c^2 T^{1/4(d)} T^{\nu} e^{-2\delta} + c^2 T e^{-3\delta} \leq T^{\nu}, \quad (5.145)
\]
where in the last inequality we have chosen \( \delta \gg 1 \) such that
\[
c^{1-\delta} T^{-\nu + 1/3} + c^2(1-\delta) T^{1/4(d)} + c^2 e^{-3\delta} T^{-\nu} \lesssim 1, \quad T = O(e^{2(r-1)}). \quad (5.146)
\]
Next, we have that for any $\psi \in Z_\delta^I$

$$\|\psi\|_{L^\infty(I)H^{s_0}} \leq \|\psi_0\|_{H^{s_0}} + \sup_{t \in I} \int_0^t \|U_2(t-\tau)d_{x,\tau}^2 \psi(\tau) \partial^2_{x,\tau} \psi(\tau) \|_{L^2} d\tau$$

$$+ \sup_{t \in I} \left\| D_{x,\tau}^{3/2} \int_0^t U_2(t-\tau)D_{x,\tau}^{s_0-3/2} \partial^2_{x,\tau} \psi(\tau) \partial^2_{x,\tau} \psi(\tau) d\tau \right\|_{L^2}$$

(5.139)

$$\lesssim \sup_{t \in I} \|\partial^2_{x,\tau} \psi(t) \partial^2_{x,\tau} \psi(\tau)\|_{L^2}$$

and

$$\approx \sup_{t \in I} \|\partial^2_{x,\tau} \psi(t) \partial^2_{x,\tau} \psi(\tau)\|_{L^2}$$

$$+ c^{e^{-1}} \sum_{\alpha \in \mathbb{Z}^d} \left( \int_{Q_\alpha} \left| D_{x,\tau}^{s_0-3/2} \partial^2_{x,\tau} \psi(t) \partial^2_{x,\tau} \psi(\tau) \right|^2 d\tau \right)^{1/2}$$

$$\lesssim \|\psi_0\|_{H^{s_0}} + T \sup_{t \in I} \|\psi\|_{H^4}$$

$$+ c \sum_{j,k,m=1}^d \sum_{\alpha \in \mathbb{Z}^d} \left( \int_{Q_\alpha} \left| D_{x,\tau}^{s_0-3/2} \partial^2_{x,\tau} \psi(t) \partial^2_{x,\tau} \psi(\tau) \right|^2 d\tau \right)^{1/2}$$

(5.147)

where in the last inequality we have chosen $\delta \gg 1$ such that

$$(T + cT^{1/2})e^{-35} + T^c e^{-25} \lesssim \frac{1}{2}, \quad T = \mathcal{O}(e^2(r-1)).$$

Then, we have that for any $\psi \in Z_\delta^I$

$$\left( \sum_{\alpha \in \mathbb{Z}^d} \sup_{t \in I} \sup_{x \in Q_\alpha} |D_{x,\tau}^2 \psi(t, x)|^2 \right)^{1/2}$$

(5.148)

$$\lesssim \|\psi_0\|_{H^{s_0}} + c^\delta T \|\psi\|_{L^\infty(I)H^{s_0}}$$

$$\lesssim \|\psi_0\|_{H^{s_0}} + c^\delta \|\psi\|_{L^\infty(I)H^{s_0}}$$

$$\lesssim \|\psi_0\|_{H^{s_0}} + c^\delta T e^{-35}$$

(5.149)

where in the last inequality we have chosen $\delta \gg 1$ such that

$$c^\delta T e^{-35} \lesssim 1, \quad T = \mathcal{O}(e^2(r-1)).$$

(5.150)
Finally, if for any $\phi \in Z^T_\delta$ we set $\Lambda_T(\phi)$ as the maximum between the three following quantities,

$$
\sum_{\alpha \in \mathbb{Z}^d} \sup_{t \in I} \sup_{x \in Q_{\alpha}} |D^2_x \psi(t, x)|^2,
$$

$$
\|\phi\|_{L^\infty(I)H^\alpha},
$$

$$
e^{-\delta T^{-\nu}} \sum_{|\beta| = s_0 + 1/2} \sup_{\alpha \in \mathbb{Z}^d} \left( \int \int_{Q_{\alpha}} |\partial_\beta^2 \phi(t, x)|^2 \, dx \, dt \right)^{1/2},
$$

we can observe that for any $\phi_1, \phi_2 \in Z^T_\delta$

$$
\Lambda_T(\mathcal{F}(\phi_1) - \mathcal{F}(\phi_2)) \leq KT^\nu e^{-2\delta} \Lambda_T(\phi_1 - \phi_2),
$$

where $K$ is a positive constant which does not depend on $c$. Hence, if we choose $\delta \gg 1$ such that (5.148), (5.146), (5.150) and

$$
KT^\nu e^{-2\delta} \leq \frac{1}{2}
$$

hold true, we can conclude.

\[ \square \]

6 Long time approximation of radiation solutions

Now we want to exploit the result of the previous section in order to deduce some consequences about the dynamics of the NLKG equation (2.3) on $M = \mathbb{R}^d$, $d \geq 2$, in the nonrelativistic limit.

Consider the simplified system, that is the Hamiltonian $H_r$ in the notations of Theorem 4.2, where we neglect the remainder:

$$
H_{\text{simp}} := h_0 + \epsilon (h_1 + \langle F_1 \rangle) + \sum_{j=2}^r \epsilon^j (h_j + Z_j).
$$

We recall that in the case of the NLKG the simplified system is actually the NLS (given by $h_0 + \epsilon (h_1 + \langle F_1 \rangle)$), plus higher-order normalized corrections. Now let $\psi_r$ be a solution of

$$
-i \dot{\psi}_r = X_{H_{\text{simp}}} (\psi_r),
$$

then $\psi_r(t, x) := T^{(r)}(\psi_r(c^2 t, x))$ solves

$$
\dot{\psi}_a = i c \langle \nabla \rangle_c \psi_a + \frac{\lambda}{2l} \left( \frac{c}{\langle \nabla \rangle_c} \right)^{1/2} \left[ \frac{c}{\langle \nabla \rangle_c} \right]^{1/2} \psi_a + \psi_a + \frac{\bar{\psi}_a}{\sqrt{2}} \right]^{2l-1} - \frac{1}{c^{2r}} X_{T^{(r)} R^{(r)}} (\psi_a, \bar{\psi}_a),
$$

that is, the NLKG plus a remainder of order $c^{-2r}$ (in the following we will refer to equation (6.2) as approximate equation, and to $\psi_a$ as the approximate solution of the original NLKG). We point out that the original NLKG and the approximate equation differ only by a remainder of order $c^{-2r}$, which is evaluated on the approximate solution. This fact is extremely important: indeed, if one can prove the smoothness of the approximate solution (which often is easier to check than the smoothness of the solution of the original equation), then the contribution of the remainder may be considered small in the nonrelativistic limit. This property is rather general, and has

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been already applied in the framework of normal form theory (see for example [BCP02]).

Now let \( \psi \) be a solution of the NLKG equation (2.3) with initial datum \( \psi_0 \), and let \( \delta := \psi - \psi_a \) be the error between the solution of the approximate equation and the original one. One can check that \( \delta \) fulfills

\[
\delta = ic(\nabla)\delta + [P(\psi_a + \delta, \bar{\psi}_a + \bar{\delta}) - P(\psi_a, \bar{\psi}_a)] + \frac{1}{c^2}X_{T^{-1}R^{-1}}(\psi_a(t), \bar{\psi}_a(t)),
\]

where

\[
P(\psi, \bar{\psi}) = \frac{\lambda}{2t} \left( \frac{c}{\langle \nabla \rangle} \right)^{1/2} \left[ \left( \frac{c}{\langle \nabla \rangle} \right)^{1/2} \frac{\psi + \bar{\psi}}{\sqrt{2}} \right]^{2t-1}.
\]

Thus we get

\[
\delta = ic(\nabla)\delta + dP(\psi_a(t))\delta + O(\delta^2) + O\left( \frac{1}{c^2t} \right);
\]

\[
\delta(t) = e^{ic(\nabla)t}\delta_0 + \int_0^t e^{i(t-s)c(\nabla)} dP(\psi_a(s))\delta(s) ds + O(\delta^2) + O\left( \frac{1}{c^2t} \right).
\]

By applying Gronwall inequality to (6.4) we can obtain an approximation result which is valid only locally uniformly in time, namely up to times of order \( O(1) \) (see Theorem 2.3 of [Pas17]).

Observe that the evolution of the error \( \delta \) between the approximate solution \( \psi_a \), namely the solution of (6.2), and the original solution \( \psi \) of (2.3) is described by

\[
\delta(t) = ic(\nabla)\delta + dP(\psi_a(t))\delta(t);
\]

\[
\delta(t) = e^{ic(\nabla)t}\delta_0 + \int_0^t e^{i(t-s)c(\nabla)} dP(\psi_a(s))\delta(s) ds,
\]

up to a remainder which is small, if we assume the smoothness of \( \psi_a \).

Now we study the evolution of the error for long (that means, \( c \)-dependent) time intervals.

We pursue such a program by a perturbative argument, considering a small radiation solution \( \psi_r = \eta_{rad,r} \) of the normalized system (6.1) that exists up to times of order \( O(c^{2(r-1)}) \), \( r > 1 \).

As an application of Proposition 3.1, we consider the following case. Fix \( r > 1 \), let \( \sigma > 0 \) and let \( \psi_r = \eta_{rad} \) be a radiation solution of (6.1), namely such that

\[
\eta_{rad,0} = \eta_{rad}(0) \in H^{k+k_0+\sigma+d/2}(\mathbb{R}^d),
\]

where \( k_0 > 0 \) and \( k \gg 1 \) are the ones in Theorem 4.2.

Let \( \delta(t) \) be a solution of (6.5); then by Duhamel formula

\[
\delta(t) := \mathcal{U}(t,0)\delta_0 + \int_0^t e^{ic(\nabla)(s)} dP(\psi_a(s))\mathcal{U}(s,0)\delta_0 ds.
\]

Now fix \( T \leq c^{2(r-1)} \); we want to estimate the local-in-time norm in the space \( L^\infty([0, T])H^k(\mathbb{R}^d) \) of the error \( \delta(t) \).

By (3.2) we can estimate the first term. We can estimate the second term by (3.3): hence for any \( (p, q) \) Schrödinger-admissible exponents
\[ \left\| \int_0^t e^{i(t-s)c\langle \nabla \rangle} dP(\psi_a(s))\delta(s) ds \right\|_{L^r_x([0,T])H^{\eta}_{k,q}} \]

\[ \lesssim c^{1+}\frac{1}{p} - \frac{1}{q} \left\| \langle \nabla \rangle \right\|_{L^r_x([0,T])H^{\eta}_{k,q}} \]

\[ \lesssim c^{1+}\frac{1}{p} - \frac{1}{q} \left\| \langle \nabla \rangle \right\|_{L^r_x([0,T])H^{\eta}_{k,q}} \]

\[ c^{1+}\frac{1}{p} - \frac{1}{q} \left\| \langle \nabla \rangle \right\|_{L^r_x([0,T])H^{\eta}_{k,q}} \]

\[ + c^{1+}\frac{1}{p} - \frac{1}{q} \left\| \langle \nabla \rangle \right\|_{L^r_x([0,T])H^{\eta}_{k,q}} \]

\[ =: I_p + I_{p}. \]

but recalling (6.3) one has that

\[ I_p \lesssim \frac{|\lambda|}{2^{t-1/2}(2l)(2l-1)} c^{1+}\frac{1}{p} - \frac{1}{q} \left\| \langle \nabla \rangle \right\|_{L^r_x([0,T])H^{\eta}_{k,q}} \]

\[ \left\| \left( \frac{c}{\langle \nabla \rangle} \right)^{1/2} (\eta_{\text{rad}}(c^2 t) + \tilde{\eta}_{\text{rad}}(c^2 t)) \right\|_{L^r_x([0,T])H^{\eta}_{k,q}} \]

\[ \leq \left\| \left( \frac{c}{\langle \nabla \rangle} \right)^{1/2} (\eta_{\text{rad}}(c^2 t) + \tilde{\eta}_{\text{rad}}(c^2 t)) \right\|_{L^r_x([0,T])H^{\eta}_{k,q}} \]

\[ \leq \int_0^T \left\| \left( \frac{c}{\langle \nabla \rangle} \right)^{1/2} (\eta_{\text{rad}}(c^2 t) + \tilde{\eta}_{\text{rad}}(c^2 t)) \right\|_{L^r_x([0,T])H^{\eta}_{k,q}} dt \]

by Sobolev product theorem (recall that \( l \geq 2 \), and that \( k \gg 1 \) we can deduce that

\[ \left\| \left( \frac{c}{\langle \nabla \rangle} \right)^{1/2} (\eta_{\text{rad}}(c^2 t) + \tilde{\eta}_{\text{rad}}(c^2 t)) \right\|_{L^r_x([0,T])H^{\eta}_{k,q}} \]

\[ \leq \left\| \eta_{\text{rad}}(c^2 t) + \tilde{\eta}_{\text{rad}}(c^2 t) \right\|_{L^r_x([0,T])H^{\eta}_{k,q}}. \]
but since by Proposition 5.3 we have that for any \( \sigma > 0 \)
\[
L_t^{2(l-1)(\frac{2}{l-1})}([0,T])W^{k,d(1+\sigma/2)}_x \geq L_t^{2(l-1)(\frac{2}{l-1})}([0,T])M_{d(1+\sigma/2),1,x}^k
\]
\[
\geq L_t^{2(l-1)(\frac{2}{l-1})}([0,T])M_{2,1,x}^k
\]
\[
\geq L_t^{2(l-1)(\frac{2}{l-1})}([0,T])H_x^{k+\sigma+d/2}
\]
\[
\geq L_t^{\infty}([0,T])H_x^{k+\sigma+d/2},
\]
we have that
\[
\|\eta_{rad}\|^2_{L_t^{2(l-1)(\frac{2}{l-1})}([0,T])W^{k,d(1+\rho/2)}_x} \lesssim T^{\frac{1}{l-1}} \|\eta_{rad}\|^2_{L_t^{\infty}([0,T])H_x^{k+\sigma+d/2}},
\]
but by Corollary 5.5 the right-hand side of (6.11) is finite and does not depend on \( c \geq 1 \) for
\[
\|\eta_{rad,0}\|_{H_x^{k+k_0+\sigma+d/2}} \lesssim c^{-\alpha}, \quad (6.12)
\]
\[
\alpha > \max \left( \delta_0(d,l,r), \delta_1(d,l,r), \frac{r-1}{l-1} \right) := \alpha^*(d,l,r). \quad (6.13)
\]
where \( c \geq c_0 \) is sufficiently large, and where \( \delta_0(d,l,r) \) and \( \delta_1(d,l,r) \) are defined in Corollary 5.5.

Furthermore, via (4.9) one can show that there exists \( c_{r,k} > 0 \) sufficiently large such that for
\( c \geq c_{r,k} \) the term \( I_{L_2} \) can be bounded by \( \frac{1}{c^4} \).

This means that we can estimate the \( L^{\infty}([0,T])H^k \) norm of the error only for a small (with respect to \( c \)) radiation solution, which is the statement of Proposition 6.1.

To summarize, we get the following result.

**Proposition 6.1.** Consider (4.12) on \( \mathbb{R}^d \), \( d \geq 2 \). Let \( r > 1 \), and fix \( k_1 \gg 1 \). Assume that \( l \geq 2 \) and \( r < \frac{d}{2}(l-1) \). Then \( \exists k_0 = k_0(r) > 0 \) such that for any \( k \geq k_1 \) and for any \( \sigma > 0 \) the following holds: consider the solution \( \eta_{rad} \) of (6.1) with initial datum \( \eta_{rad,0} \in H^{k+k_0+\sigma+d/2}(\mathbb{R}^d) \), and call \( \delta \) the difference between the solution of the approximate equation (6.2) and the original solution of the Hamilton equation for (4.12). Assume that \( \delta_0 := \delta(0) \) satisfies
\[
\|\delta_0\|_{H^k} \lesssim \frac{1}{c^2}.
\]
Then there exist \( \alpha^* := \alpha^*(d,l,r) > 0 \) and there exists \( c^* := c^*(r,k) > 1 \), such that for any \( \alpha > \alpha^* \) and for any \( c > c^* \), if \( \eta_{rad,0} \) satisfies
\[
\|\eta_{rad,0}\|_{H^{k+k_0+\sigma+d/2}} \lesssim c^{-\alpha},
\]
then
\[
\sup_{t \in [0,T]} \|\delta(t)\|_{H^k} \lesssim \frac{1}{c^2}, \quad T \lesssim c^{2(r-1)}.
\]

By exploiting (4.9) and Proposition 6.1, we obtain Theorem 2.1.
References

[BCP02] Dario Bambusi, Andrea Carati, and Antonio Ponno. The nonlinear Schrödinger equation as a resonant normal form. Discrete and Continuous Dynamical Systems Series B, 2(1):109–128, 2002.

[BGT04] Nicolas Burq, Pierre Gérard, and Nikolay Tzvetkov. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. American Journal of Mathematics, 126(3):569–605, 2004.

[BMS04] Philippe Bechouche, Norbert J Mauser, and Sigmund Selberg. Nonrelativistic limit of Klein-Gordon-Maxwell to Schrödinger-Poisson. American Journal of Mathematics, 126(1):31–64, 2004.

[Bou10] Jean-Marc Bouclet. Littlewood-Paley decompositions on manifolds with ends. Bull. Soc. Math. France, 138(1):1–37, 2010.

[BP06] Dario Bambusi and Antonio Ponno. On metastability in FPU. Communications in Mathematical Physics, 264(2):539–561, 2006.

[CLM15] Rémi Carles, Wolfgang Lucha, and Emmanuel Moulay. Higher-order Schrödinger and Hartree–Fock equations. Journal of Mathematical Physics, 56(12):122301, 2015.

[CM12] Rémi Carles and Emmanuel Moulay. Higher order Schrödinger equations. Journal of Physics A: Mathematical and Theoretical, 45(39):395304, 2012.

[CO06] Yonggeun Cho and Tohru Ozawa. On the semirelativistic Hartree-type equation. SIAM Journal on Mathematical Analysis, 38(4):1060–1074, 2006.

[CS16] Woocheol Choi and Jinmyoung Seok. Nonrelativistic limit of standing waves for pseudo-relativistic nonlinear Schrödinger equations. Journal of Mathematical Physics, 57(2):021510, 2016.

[CF90] Thierry Cazenave and Fred B Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in $H^s$. Nonlinear Analysis: Theory, Methods & Applications, 14(10):807–836, 1990.

[DF08] Piero D’Ancona and Luca Fanelli. Strichartz and smoothing estimates for dispersive equations with magnetic potentials. Communications in Partial Differential Equations, 33(6):1082–1112, 2008.

[FS14] Erwan Faou and Katharina Schratz. Asymptotic preserving schemes for the Klein–Gordon equation in the non-relativistic limit regime. Numerische Mathematik, 126(3):441–469, 2014.

[Fuj16] Matsusaburō Fujikura. Über die obere Schranke des absoluten Betrages der Wurzeln einer algebraischen Gleichung. Tohoku Mathematical Journal, First Series, 10:167–171, 1916.

[HHW07] Chengchun Hao, Ling Hsiao, and Baoxiang Wang. Well-posedness of Cauchy problem for the fourth order nonlinear Schrödinger equations in multi-dimensional spaces. Journal of Mathematical Analysis and Applications, 328(1):58–83, 2007.
[HKNR18] Daniel Han-Kwan, Toan T Nguyen, and Frederic Rousset. Long time estimates for the Vlasov-Maxwell system in the non-relativistic limit. *Communications in mathematical physics*, 363(2):389–434, 2018.

[KAY12] JinMyong Kim, Anton Arnold, and Xiaohua Yao. Global estimates of fundamental solutions for higher-order Schrödinger equations. *Monatshefte für Mathematik*, 168(2):253–266, 2012.

[KPV93] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. Small solutions to nonlinear Schrödinger equations. *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, 10(3):255–288, 1993.

[KS11] Masaharu Kobayashi and Mitsuru Sugimoto. The inclusion relation between Sobolev and modulation spaces. *Journal of Functional Analysis*, 260(11):3189–3208, 2011.

[LZ16] Yong Lu and Zhifei Zhang. Partially strong transparency conditions and a singular localization method in geometric optics. *Archive for Rational Mechanics and Analysis*, 222(1):245–283, 2016.

[Mac01] Shuji Machihara. The nonrelativistic limit of the nonlinear Klein-Gordon equation. *FUNKCIALAJ EKVACIOJ SERIO INTERNACIA*, 44(2):243–252, 2001.

[MN03] Nader Masmoudi and Kenji Nakanishi. Nonrelativistic limit from Maxwell-Klein-Gordon and Maxwell-Dirac to Poisson-Schrödinger. *International Mathematics Research Notices*, 2003(13):697–734, 2003.

[MN08] Nader Masmoudi and Kenji Nakanishi. Energy convergence for singular limits of Zakharov type systems. *Inventiones mathematicae*, 172(3):535–583, 2008.

[MN10] Nader Masmoudi and Kenji Nakanishi. From the Klein–Gordon–Zakharov system to a singular nonlinear Schrödinger system. *Annales de l’Institut Henri Poincare - Analyse Non Lineaire*, 27(4):1073–1096, 2010.

[Naj90] Branko Najman. The nonrelativistic limit of the nonlinear Klein-Gordon equation. *Nonlinear Analysis: Theory, Methods & Applications*, 15(3):217–228, 1990.

[Pas17] Stefano Pasquali. Dynamics of the nonlinear Klein-Gordon equation in the nonrelativistic limit, I. *arXiv preprint arXiv:1703.01609*, 2017.

[RSW12] Michael Ruzhansky, Mitsuru Sugimoto, and Baoxiang Wang. Modulation spaces and nonlinear evolution equations. *Progress in Mathematics*, 301:267–283, 2012.

[RWZ16] Michael Ruzhansky, Baoxiang Wang, and Hua Zhang. Global well-posedness and scattering for the fourth order nonlinear Schrödinger equations with small data in modulation and Sobolev spaces. *Journal de Mathématiques Pures et Appliquées*, 105(1):31–65, 2016.

[Tay11] ME Taylor. Partial differential equations III. Nonlinear equations. Second. vol. 117. *Applied Mathematical Sciences. Springer, New York*, pp. xxi, 715, 2011.

[Tsu84] Masayoshi Tsutsumi. Nonrelativistic approximation of nonlinear Klein-Gordon equations in two space dimensions. *Nonlinear Analysis: Theory, Methods & Applications*, 8(6):637–643, 1984.
[WH07] Baoxiang Wang and Henryk Hudzik. The global Cauchy problem for the NLS and NLKG with small rough data. *Journal of Differential Equations*, 232(1):36–73, 2007.

[WHH09] Baoxiang Wang, Lijia Han, and Chunyan Huang. Global well-posedness and scattering for the derivative nonlinear Schrödinger equation with small rough data. *Annales de l’Institut Henri Poincaré - Analyse Non Linéaire*, 26(6):2253–2281, 2009.