Deductive Inference for the Interiors and Exteriors of Horn Theories

Kazuhisa Makino
Department of Mathematical Informatics,
University of Tokyo,
Tokyo, 113-8656, Japan.
makino@mist.i.u-tokyo.ac.jp

Hirotaka Ono
Department of Computer Science and Communication Engineering,
Kyushu University,
Fukuoka 812-8581, Japan.
ono@csce.kyushu-u.ac.jp

Abstract

In this paper, we investigate the deductive inference for the interiors and exteriors of Horn knowledge bases, where the interiors and exteriors were introduced by Makino and Ibaraki [11] to study stability properties of knowledge bases. We present a linear time algorithm for the deduction for the interiors and show that it is co-NP-complete for the deduction for the exteriors. Under model-based representation, we show that the deduction problem for interiors is NP-complete while the one for exteriors is co-NP-complete. As for Horn envelopes of the exteriors, we show that it is linearly solvable under model-based representation, while it is co-NP-complete under formula-based representation. We also discuss the polynomially solvable cases for all the intractable problems.

1 Introduction

Knowledge-based systems are commonly used to store the sentences as our knowledge for the purpose of having automated reasoning such as deduction applied to them (see e.g., [1]). Deductive inference is a fundamental mode of reasoning, and usually abstracted as follows: Given the knowledge base $KB$, assumed to capture our knowledge about the domain in question, and a query $\chi$ that is assumed to capture the situation at hand, decide whether $KB \models \chi$. 

*An extended abstract of this article was presented in Proceedings of Algorithms and Computation, 19th International Symposium (ISAAC 2008), Lecture Notes in Computer Science, Vol. 5369, pp. 390–401, Springer-Verlag Berlin Heidelberg, 2008.*
which can be understood as the question: “Is $\chi$ consistent with the current state of knowledge?”

In this paper, we consider the interiors and exteriors of knowledge base. Formally, for a given positive integer $\alpha$, the $\alpha$-interior of $KB$, denoted by $\sigma_{-\alpha}(KB)$, is a knowledge that consists of the models (or assignments) $v$ satisfying that the $\alpha$-neighbors of $v$ are all models of $KB$, and the $\alpha$-exterior of $KB$, denoted by $\sigma_{\alpha}(KB)$, is a knowledge that consists of the models $v$ satisfying that at least one of the $\alpha$-neighbors of $v$ is a model of $KB$ [11]. Intuitively, the interior consists of the models $v$ that strongly satisfy $KB$, since all neighbors of $v$ are models of $KB$, while the exterior consists of the models $v$ that weakly satisfy $KB$, since at least one of the $\alpha$-neighbors of $v$ is a model of $KB$. Here we note that $v$ might not satisfy $KB$, even if we say that it weakly satisfies $KB$. As mentioned in [11], the interiors and exteriors of knowledge base merit study in their own right, since they shed light on the structure of knowledge base. Moreover, let us consider the situation in which knowledge base $KB$ is not perfect in the sense that some sentences in $KB$ are wrong and/or some are missing in $KB$ (see also [11]).

Suppose that we use $KB$ as a knowledge base for automated reasoning, say, deductive inference $KB \models \chi$. Since $KB$ does not represent real knowledge $KB^*$, the reasoning result is no longer true. However, if we use the interior $\sigma_{-\alpha}(KB)$ of $KB$ as a knowledge base and have $\sigma_{-\alpha}(KB) \not\models \chi$, then we can expect that the result is true for real knowledge $KB^*$, since $\sigma_{-\alpha}(KB)$ consists of models which strongly satisfy $KB$. On the other hand, if we use the exterior $\sigma_{\alpha}(KB)$ of $KB$ as a knowledge base and have $\sigma_{\alpha}(KB) \models \chi$, then we can expect that the result is true for real knowledge $KB^*$, since $\sigma_{\alpha}(KB)$ consists of models which weakly satisfy $KB$. In this sense, the interiors and exteriors help to have safe reasoning.

**Main problems considered.** In this paper, we study the deductive inference for the interiors and exteriors of propositional Horn theories, where Horn theories are ubiquitous in Computer Science, cf. [13], and are of particular relevance in Artificial Intelligence and Databases. It is known that important reasoning problems like deductive inference and satisfiability checking, which are intractable for arbitrary propositional theories, are solvable in linear time for Horn theories (cf. [3]).

More precisely, we address the following problems:

- Given a Horn theory $\Sigma$, a clause $c$, and nonnegative integer $\alpha$, we consider the problems of deciding if deductive queries hold for the $\alpha$-interior and exterior of $\Sigma$, i.e., $\sigma_{-\alpha}(\Sigma) \models c$ and $\sigma_{\alpha}(\Sigma) \models c$. It is well-known [3] that a deductive query for a Horn theory can be answered in linear time. Note that it is intractable to construct the interior and exterior for a Horn theory [11, 12], and hence a direct method (i.e., first construct the interior (or exterior) and then check a deductive query) is not possible efficiently.
• We contrast traditional formula-based (syntactic) with model-based (semantic) representation of Horn theories. The latter form of representation has been proposed as an alternative form of representing and accessing a logical knowledge base, cf. [2, 4, 5, 7, 8, 6, 9, 10]. In model-based reasoning, $\Sigma$ is represented by a subset of its models $M$, which are commonly called characteristic models. As shown by Kautz et al. [7], the deductive inference can be done in polynomial time, given its characteristic models.

• Finally, we consider Horn approximations for the exteriors of Horn theories. Note that the interiors of Horn theories are Horn, while the exteriors might not be Horn. We deal with the least upper bounds, called the Horn envelopes [15], for the exteriors of Horn theories.

Main results. We investigate the problems mentioned above from an algorithmical viewpoint. For all the problems, we provide either polynomial time algorithms or proofs of the intractability; thus, our work gives a complete picture of the tractability/intractability frontier of deduction for interiors and exteriors of Horn theories. Our main results can be summarized as follows (see Figure 1).

• We present a linear time algorithm for the deduction for the interiors of a given Horn theory, and show that it is co-NP-complete for the deduction for the exteriors. Thus, the positive result for ordinary deduction for Horn theories extends to the interiors, but does not to the exteriors. We also show that the deduction for the exteriors is possible in polynomial time, if $\alpha$ is bounded by a constant or if $|N(c)|$ is bounded by a logarithm of the input size, where $N(c)$ corresponds to the set of negative literals in $c$.

• Under model-based representation, we show that the consistency problem and the deduction for the interiors of Horn theories are both co-NP-complete. As for the exteriors, we show that the deduction is co-NP-complete. We also show that the deduction for the interiors is possible in polynomial time if $\alpha$ is bounded by a constant, and so is for the exteriors, if $\alpha$ or $|P(c)|$ is bounded by a constant, or if $|N(c)|$ is bounded by a logarithm of the input size, where $P(c)$ corresponds to the set of positive literals in $c$.

• As for Horn envelopes of the exteriors of Horn theories, we show that it is linearly solvable under model-based representation, while it is co-NP-complete under formula-based representation. The former contrasts to the negative result for the exteriors. We also present a polynomial algorithm for formula-based representation, if $\alpha$ is bounded by a constant or if $|N(c)|$ is bounded by a logarithm of the input size.
The rest of the paper is organized as follows. In the next section, we review the basic concepts and fix notations. Sections 3 and 4 investigate the deductive inference for the interiors and exteriors of Horn theories. Section 5 considers the deductive inference for the envelopes of the exteriors of Horn theories.

## 2 Preliminaries

### Horn Theories

We assume a standard propositional language with atoms \( At = \{x_1, x_2, \ldots, x_n\} \), where each \( x_i \) takes either value 1 (true) or 0 (false). A literal is either an atom \( x_i \) or its negation, which we denote by \( \overline{x_i} \). The opposite of a literal \( \ell \) is denoted by \( \overline{\ell} \), and the opposite of a set of literals \( L \) by \( \overline{L} = \{\overline{\ell} \mid \ell \in L\} \). Furthermore, \( Lit = At \cup \overline{At} \) denotes the set of all literals.

A clause is a disjunction \( c = \bigvee_{i \in P(c)} x_i \lor \bigvee_{i \in N(c)} \overline{x_i} \) of literals, where \( P(c) \) and \( N(c) \) are the sets of indices whose corresponding variables occur positively and negatively in \( c \) and \( P(c) \cap N(c) = \emptyset \). Dually, a term is conjunction \( t = \bigwedge_{i \in P(t)} x_i \land \bigwedge_{i \in N(t)} \overline{x_i} \) of literals, where \( P(t) \) and \( N(t) \) are similarly defined. We also view clauses and terms as sets of literals. A conjunctive normal form (CNF) is a conjunction of clauses. A clause \( c \) is Horn, if \( |P(c)| \leq 1 \). A theory \( \Sigma \) is any set of formulas; it is Horn, if it is a set of Horn clauses. As usual, we identify \( \Sigma \) with \( \varphi = \bigwedge_{c \in \Sigma} c \), and write \( c \in \varphi \) etc. It is known [3] that the deductive query for a Horn theory, i.e., deciding if \( \Sigma \models c \) for a clause \( c \) is possible in linear time.

We recall that Horn theories have a well-known semantic characterization. A model is a vector \( v \in \{0, 1\}^n \), whose \( i \)-th component is denoted by \( v_i \). For a model \( v \), let \( \text{ON}(v) = \{i \mid v_i = 1\} \) and \( \text{OFF}(v) = \{i \mid v_i = 0\} \). The value of a formula \( \varphi \) on a model \( v \), denoted \( \varphi(v) \), is inductively defined as usual; satisfaction of \( \varphi \) in \( v \),
Example 2. Let us consider a Horn theory \( \Sigma = \sigma \). By definition, \( \Sigma \) is Horn representable if and only if

\[
\psi \in \{ v, w \} \ \forall \ \alpha \leq n \text{ such that } \alpha \in \text{od}(\Sigma) \text{ and } \sigma(\alpha) \text{ holds for any integer } \alpha \text{.}
\]

Denote by \( \psi \cup \phi \) the closure of \( \phi \). For two theories \( \Sigma \), \( \sigma \leq n \text{ respectively, are theories defined by}

\[
\text{mod}(\sigma(\Sigma)) = \{ v \in \{0,1\}^n \mid \psi \cup \phi \}
\]

and \( \sigma(\Sigma) \) respectively, are theories defined by

\[
\text{mod}(\sigma(\Sigma)) = \{ v \in \{0,1\}^n \mid \psi \cup \phi \}
\]

By definition, \( \sigma(\Sigma) = \sigma, \sigma(\Sigma) \models \sigma(\Sigma) \) for integers \( \alpha \) and \( \beta \) with \( \alpha < \beta \), and \( \sigma(\Sigma_1) \models \sigma(\Sigma_2) \) holds for any integer \( \alpha \), if two theories \( \Sigma_1 \) and \( \Sigma_2 \) satisfy \( \Sigma_1 \models \Sigma_2 \).

Example 2. Let us consider a Horn theory \( \Sigma = \{ \overline{x}_1 \lor x_3, \overline{x}_2 \lor x_3, \overline{x}_2 \lor x_4 \} \) of 4 variables, where \( \text{mod}(\Sigma) \) is given by

\[
\text{mod}(\Sigma) = \{(1111), (1011), (1010), (0111), (0011), (0010), (0001), (0000)\}
\]
(See Figure 2). Then we have $\sigma_\alpha(\Sigma) = \{\emptyset\}$ for $\alpha \leq -2$, $\{\overline{x}_1, \overline{x}_2, x_3, x_4\}$ for $\alpha = -1$, $\Sigma$ for $\alpha = 0$, $\{\overline{x}_1 \lor \overline{x}_2 \lor x_3 \lor x_4\}$ for $\alpha = 1$, and $\emptyset$ for $\alpha \geq 2$. For example, (0011) is the unique model of $\text{mod}(\sigma_{-1}(\Sigma))$, since $\mathcal{N}_1(0011) \subseteq \text{mod}(\Sigma)$ and $\mathcal{N}_1(v) \not\subseteq \text{mod}(\Sigma)$ holds for all the other models $v$. For the 1-exterior, we can see that all models $v$ with $(\overline{x}_1 \lor \overline{x}_2 \lor x_3 \lor x_4)(v) = 1$ satisfy $\mathcal{N}_1(v) \cap \text{mod}(\Sigma) \neq \emptyset$, and no other such model exists. For example, (0101) is a model of $\sigma_1(\Sigma)$, since (0111) $\in \mathcal{N}_1(0101) \cap \text{mod}(\Sigma)$. On the other hand, (1100) is not a model of $\sigma_1(\Sigma)$, since $\mathcal{N}_1(1100) \cap \text{mod}(\Sigma) = \emptyset$. Notice that $\sigma_{-1}(\Sigma)$ is Horn, while $\sigma_1(\Sigma)$ is not.

Makino and Ibaraki [11] introduced the interiors and exteri ors to analyze stability of Boolean functions, and studied their basic properties and complexity issues on them (see also [12]). For example, it is known [11] that, for a theory $\Sigma$ and nonnegative integers $\alpha$ and $\beta$, $\sigma_{-\alpha}(\sigma_{-\beta}(\Sigma)) = \sigma_{-\alpha-\beta}(\Sigma)$, $\sigma_\alpha(\sigma_\beta(\Sigma)) = \sigma_{\alpha+\beta}(\Sigma)$, and

$$\sigma_\alpha(\sigma_{-\beta}(\Sigma)) \models \sigma_{-\alpha-\beta}(\Sigma) \models \sigma_{\alpha+\beta}(\sigma_\beta(\Sigma)).$$

(3)

For a nonnegative integer $\alpha$ and two theories $\Sigma_1$ and $\Sigma_2$, we have

$$\sigma_{-\alpha}(\Sigma_1 \cup \Sigma_2) = \sigma_{-\alpha}(\Sigma_1) \cup \sigma_{-\alpha}(\Sigma_2)$$

(4)

$$\sigma_{\alpha}(\Sigma_1 \cup \Sigma_2) \models \sigma_{\alpha}(\Sigma_1) \cup \sigma_{\alpha}(\Sigma_2)$$

(5)
where \( \sigma_\alpha(\Sigma_1 \cup \Sigma_2) \neq \sigma_\alpha(\Sigma_1) \cup \sigma_\alpha(\Sigma_2) \) holds in general.

As demonstrated in Example 2, it is not difficult to see that the interiors of any Horn theory are Horn, which is, for example, proved by (4) and Lemma 3, while the exteriors might be not Horn.

### 3 Deductive Inference from Horn Theories

In this section, we investigate the deductive inference for the interiors and exteriors of a given Horn theory.

#### 3.1 Interiors

Let us first consider the deduction for the \( \alpha \)-interiors of a Horn theory: Given a Horn theory \( \Sigma \), a clause \( c \), and a positive integer \( \alpha \), decide if \( \sigma_{-\alpha}(\Sigma) \models c \) holds. We show that the problem is solvable in linear time after showing a series of lemmas.

The following lemma is a basic property of the interiors.

**Lemma 3.** Let \( c \) be a clause. Then for a nonnegative integer \( \alpha \), we have

\[
\sigma_{-\alpha}(c) = \bigvee_{S \subseteq c} (\bigwedge_{|S| = \alpha+1} (\bigvee_{\ell \in S} \ell)) = \bigwedge_{S \subseteq c} (\bigvee_{|S| = |c|-\alpha} (\bigwedge_{\ell \in S} \ell)).
\]

For example, let us consider \( c = x_1 \vee x_2 \vee \overline{x}_3 \vee \overline{x}_4 \), \( \alpha = 2 \). Then we have

\[
\sigma_{-2}(c) = x_1 x_2 \overline{x}_3 \vee x_1 x_2 \overline{x}_4 \vee x_1 \overline{x}_3 \overline{x}_4 \vee x_2 \overline{x}_3 \overline{x}_4 = (x_1 \vee x_2)(x_1 \vee \overline{x}_3)(x_1 \vee \overline{x}_4)(x_2 \vee \overline{x}_3)(x_2 \vee \overline{x}_4)(\overline{x}_3 \vee \overline{x}_4).
\]

This lemma, together with (4), implies that for a CNF \( \varphi \) and a nonnegative integer \( \alpha \), we have

\[
\sigma_{-\alpha}(\varphi) = \bigwedge_{c \in \varphi} \left( \bigvee_{S \subseteq c} (\bigwedge_{|S| = \alpha+1} (\bigvee_{\ell \in S} \ell)) \right) = \bigwedge_{c \in \varphi} \left( \bigwedge_{S \subseteq c} (\bigvee_{|S| = |c|-\alpha} (\bigwedge_{\ell \in S} \ell)) \right),
\]

where we regard \( c \) as a set of literals.

**Lemma 4.** Let \( \Sigma \) be a Horn theory, and let \( c \) be a clause. For a nonnegative integer \( \alpha \), if there exists a clause \( d \in \Sigma \) such that \( |N(d) \setminus N(c)| \leq \alpha - 1 \) or \( (|N(d) \setminus N(c)| = \alpha \) and \( P(d) \subseteq P(c) \) \), then we have \( \sigma_{-\alpha}(\Sigma) \models c \).

**Proof.** If \( \Sigma \) has a clause \( d \) such that \( |N(d) \setminus N(c)| \leq \alpha - 1 \), then \( (|N(d) \setminus N(c)| \cup P(d)| \leq \alpha \) holds. Thus by Lemma 3, we have \( \sigma_{-\alpha}(d) \models \bigvee_{i \in N(c) \setminus N(d)} \overline{x}_i \models c \). Therefore, by (4), \( \sigma_{-\alpha}(\Sigma) \models c \) holds.

On the other hand, if \( \Sigma \) has a clause \( d \) such that \( |N(d) \setminus N(c)| = \alpha \) and \( P(d) \subseteq P(c) \), then by Lemma 3, we have \( \sigma_{-\alpha}(d) \models \bigvee_{i \in P(c)} x_i \models \bigvee_{i \in N(c) \setminus N(d)} \overline{x}_i \models c \). Therefore, by (4), \( \sigma_{-\alpha}(\Sigma) \models c \) holds. \( \square \)
Lemma 5. Let Σ be a Horn theory, and let c be a clause. For a nonnegative integer \( \alpha \), if (i) \(|N(d) \setminus N(c)| \geq \alpha \) holds for all \( d \in \Sigma \) and (ii) \( \emptyset \neq P(d) \subseteq N(c) \) holds for all \( d \in \Sigma \) with \(|N(d) \setminus N(c)| = \alpha \), then we have \( \sigma_{-\alpha}(\Sigma) \not\models c \).

Proof. Let \( v \) be the unique minimal model that does not satisfy \( c \), i.e., \( v_i = 1 \) if \( x_i \in c \) and 0, otherwise. We show that \( v \models \sigma_{-\alpha}(\Sigma) \), which implies \( \sigma_{-\alpha}(\Sigma) \not\models c \).

Let \( d \) be a clause in \( \Sigma \) with \(|N(d) \setminus N(c)| \geq \alpha + 1 \), and let \( t \) be a term obtained by conjuncting arbitrary \( \alpha + 1 \) literals in \( N(d) \setminus N(c) \). Then we have \( t(v) = 1 \) and \( t \models \sigma_{-\alpha}(d) \) by Lemma 3. On the other hand, for a clause \( d \) in \( \Sigma \) with \(|N(d) \setminus N(c)| = \alpha \), let \( t \) be a term obtained by conjuncting all literals in \( (N(d) \setminus N(c)) \cup P(d) \). Then we have \( |t| = \alpha + 1 \) and \( t \models \sigma_{-\alpha}(d) \) by Lemma 3. Moreover, it holds that \( t(v) = 1 \) by \( P(d) \subseteq N(c) \). Therefore, by (4), we have \( v \models \sigma_{-\alpha}(\Sigma) \).

By Lemmas 4 and 5, we can easily answer the deductive queries, if \( \Sigma \) satisfies certain conditions mentioned in them. In the remaining case, we have the following lemma.

Lemma 6. For a Horn theory \( \Sigma \) that satisfies none of the conditions in Lemmas 4 and 5, let \( d \) be a clause in \( \Sigma \) such that \(|N(d) \setminus N(c)| = \alpha \), and \( P(d) = P(d) \setminus (P(c) \cup N(c)) \). Then \( \sigma_{-\alpha}(\Sigma) \models c \lor x_j \) holds.

Proof. By Lemma 3, we have \( \sigma_{-\alpha}(d) \models \bigvee_{i \in N(c) \cap N(d)} \overline{x_i} \lor x_j \models c \lor x_j \). This implies \( \sigma_{-\alpha}(\Sigma) \models c \lor x_j \) by (4).

From this lemma, we have only to check a deductive query \( \sigma_{-\alpha}(\Sigma) \models c \lor \overline{x_j} \), instead of \( \sigma_{-\alpha}(\Sigma) \models c \). Since \(|c| < |c \lor \overline{x_j}| \leq n \), we can answer the deduction by checking the conditions in Lemmas 4 and 5 at most \( n \) times.

We can see that a straightforward implementation of the algorithm requires \( O(n(\|\Sigma\| + |c|)) \) time, where \( \|\Sigma\| \) denotes the length of \( \Sigma \), i.e., \( \|\Sigma\| = \sum_{d \in \Sigma} |d| \). However, it is not difficult to see that we have a linear time algorithm for the problem, if \( N(d) \setminus N \) for \( d \in \Sigma \) is maintained by using the proper data structure.

Theorem 7. Given a Horn theory \( \Sigma \), a clause \( c \) and a nonnegative integer \( \alpha \), a deductive query \( \sigma_{-\alpha}(\Sigma) \models c \) can be answered in linear time, i.e., \( O(\|\Sigma\| + |c|) \) time.

3.2 Exteriors

Let us next consider the deduction for the \( \alpha \)-exteriors of a Horn theory. In contrast to the interior case, we have the following negative result.

Theorem 8. Given a Horn theory \( \Sigma \), a clause \( c \) and a positive integer \( \alpha \), it is co-NP-complete to decide whether a deductive query \( \sigma_{\alpha}(\Sigma) \models c \) holds, even if \( P(c) = \emptyset \).
Algorithm 1 Deduction-Interior-from-Horn-Theory

**Input:** A Horn theory $\Sigma$, a clause $c$ and a nonnegative integer $\alpha$.

**Output:** Yes, if $\sigma_{-\alpha}(\Sigma) \models c$; Otherwise, No.

**Step 0.** Let $N := N(c)$ and $P := P(c)$.

**Step 1.** /* Check the condition in Lemma 4. */

If there exists a clause $d \in \Sigma$ such that $|N(d) \setminus N| \leq \alpha - 1$ or ($|N(d) \setminus N| = \alpha$ and $P(d) \subseteq P$), then output Yes and halt.

**Step 2.** /* Check the condition in Lemma 5. */

If $P(d) \subseteq N$ holds for all $d \in \Sigma$ with $|N(d) \setminus N| = \alpha$, then output No and halt.

**Step 3.** /* Update $N$ by Lemma 6. */

For a clause $d$ in $\Sigma$ such that $|N(d) \setminus N| = \alpha$ and $P(d) = P(d) \setminus (P \cup N) = \{j\}$, update $N := N \cup \{j\}$ and return to Step 1.

**Proof.** By definition, $\sigma = \alpha(\Sigma) \not\models c$ if and only if there exists a model $v$ of $\Sigma$ such that some model in $N_{\alpha}(v)$ does not satisfy $c$. The latter is equivalent to the condition that there exists a model $v$ of $\Sigma$ such that $|ON(v) \cap P(c)| + |OFF(v) \cap N(c)| \leq \alpha$, which can be checked in polynomial time. Thus the problem is in co-NP.

We then show the hardness by reducing a well-known NP-complete problem INDEPENDENT SET to the complement of our problem. INDEPENDENT SET is the problem of deciding if a given graph $G = (V, E)$ has an independent set $W \subseteq V$ such that $|W| \geq k$ for a given integer $k$. Here we call a subset $W \subseteq V$ is an independent set of $G$ if $|W \cap e| \leq 1$ for all edges $e \in E$. For a problem instance $G = (V = \{1, 2, \ldots, n\}, E)$ and $k$ of INDEPENDENT SET, let us define a Horn theory $\Sigma_G$ over $At = \{x_1, x_2, \ldots, x_n\}$ by

$$\Sigma_G = \{(\bar{x}_i \lor \bar{x}_j) \mid \{i, j\} \in E\}.$$  

Let $c = \bigvee_{i=1}^{n} \bar{x}_i$ and $\alpha = n - k$. Note that $(1 \cdots 1)$ is the unique model that does not satisfy $c$. Thus $\sigma_{\alpha}(\Sigma) \not\models c$ if and only if $\sigma_{\alpha}(\Sigma)(1 \cdots 1) = 1$. Since $W$ is an independent set of $G$ if and only if $\Sigma_G$ contains a model $w$ defined by $ON(w) = W$, $\sigma_{\alpha}(\Sigma_G)(1 \cdots 1) = 1$ is equivalent to the condition that $G$ has an independent set of size at least $k (= n - \alpha)$. This completes the proof.

We remark that this result can also be derived from the ones in [11].
However, by using the next lemma, a deductive query can be answered in polynomial time, if $\alpha$ or $N(c)$ is small.

**Lemma 9.** Let $\Sigma_1$ and $\Sigma_2$ be theories. For a nonnegative integer $\alpha$, then $\sigma_\alpha(\Sigma_1) \models \Sigma_2$ if and only if $\Sigma_1 \models \sigma_{-\alpha}(\Sigma_2)$.

**Proof.** For the if part, if $\Sigma_1 \models \sigma_{-\alpha}(\Sigma_2)$, then we have $\sigma_\alpha(\Sigma_1) \models \Sigma_2$ by (3). On the other hand, if $\sigma_\alpha(\Sigma_1) \models \Sigma_2$, then we have $\Sigma_1 \models \sigma_{-\alpha}(\sigma_\alpha(\Sigma_1)) \models \sigma_{-\alpha}(\Sigma_2)$ by (3).

From Lemma 9, the deductive query for the $\alpha$-interior of a theory $\Sigma$, i.e., $\sigma_\alpha(\Sigma) \models c$ for a given clause $c$ is equivalent to the condition that $\Sigma \models \sigma_{-\alpha}(\Sigma_2)$.

Since we have $\sigma_{-\alpha}(\Sigma) = \bigwedge_{S \subseteq c : |S| = |c| - \alpha} \left( \left\| \bigvee_{\ell \in S} \ell \right\| \right)$ by Lemma 3, the deductive query for the $\alpha$-interior can be done by checking $(\left\| c \right\|)$ deductions for $\Sigma$. More precisely, we have the following lemma.

**Lemma 10.** Let $\Sigma$ be a Horn theory, let $c$ be a clause, and $\alpha$ be a nonnegative integer. Then $\sigma_\alpha(\Sigma) \models c$ holds if and only if, for each subset $S$ of $N(c)$ such that $|S| \geq |N(c)| - \alpha$, at least $(\alpha - |N(c)| + |S| + 1)$'s in $P(c)$ satisfy $\Sigma \models \bigvee_{i \in S} \overline{x_i} \vee x_j$.

**Proof.** From Lemmas 3 and 9, $\sigma_\alpha(\Sigma) \models c$ if and only if $\Sigma \models \bigwedge_{S \subseteq c : |S| = |c| - \alpha} \left( \left\| \bigvee_{\ell \in S} \ell \right\| \right)$. It is known that for a Horn theory $\Sigma$ and clause $d$, $\Sigma \models d$ if and only if $\Sigma \models \bigvee_{i \in N(d)} \overline{x_i} \vee x_j$ holds for some $j \in P(d)$ (i.e., All the prime implicates of Horn theory are Horn). This proves the lemma.

This lemma implies that the deductive query can be answered by checking the number of $j$'s in $P(c)$ that satisfy $\Sigma \models \bigvee_{i \in S} \overline{x_i} \vee x_j$ for each $S$. Since we can check this condition in linear time and there are $\sum_{p=0}^{\alpha} \binom{|N(c)|}{p}$ such $S$'s, we have the following result, which complements Theorem 8 that the problem is intractable, even if $P(c) = \emptyset$.

**Theorem 11.** Let $\Sigma$ be a Horn theory, let $c$ be a clause, and let $\alpha$ be a nonnegative integer. Then a deductive query $\sigma_\alpha(\Sigma) \models c$ can be answered in $O\left( \sum_{p=0}^{\alpha} \binom{|N(c)|}{p} \| \Sigma \| + |P(c)| \right)$ time. In particular, it is polynomially solvable, if $\alpha = O(1)$ or $|N(c)| = O(\log \| \Sigma \|)$.

## 4 Deductive Inference from Characteristic Sets

In this section, we consider the case when Horn knowledge bases can be represented by characteristic sets. Different from formula-based representation, the deductions for interiors and exteriors are both intractable, unless $P=NP$. 


4.1 Interiors

We first present an algorithm to solve the deduction problem for the interiors of Horn theories. The algorithm requires exponential time in general, but it is polynomial when \( \alpha \) is small.

Let \( \Sigma \) be a Horn theory given by its characteristic set \( \text{char}(\Sigma) \), and let \( c \) be a clause. Then for a nonnegative integer \( \alpha \), we have

\[
\sigma_{-\alpha}(\Sigma) \models c \text{ if and only if } \sigma_{-\alpha}(\Sigma) \land \overline{c} \equiv 0. \tag{6}
\]

Let \( v^* \) be a unique minimal model such that \( c(v^*) = 0 \) (i.e., \( \overline{c}(v^*) = 1 \)). By the definition of interiors, \( v^* \) is a model of \( \sigma_{-\alpha}(\Sigma) \) if and only if all \( v \)'s in \( N_\alpha(v^*) \) are models of \( \Sigma \). Therefore, for each model \( v \) in \( N_\alpha(v^*) \), we check if \( v \in \text{mod}(\Sigma) \), which is equivalent to

\[
\bigwedge_{w \in \text{char}(\Sigma) \atop w \geq v} w = v. \tag{7}
\]

If (7) holds for all models \( v \) in \( N_\alpha(v^*) \), then we can immediately conclude by (6) that \( \sigma_{-\alpha}(\Sigma) \not\models c \). On the other hand, if there exists a model \( v \) in \( N_\alpha(v^*) \) such that (7) does not hold, let \( J = \text{ON}(\bigwedge_{w \in \text{char}(\Sigma) \atop w \geq v} w) \setminus \text{ON}(v) \). By definition, we have \( J \neq \emptyset \), and we can see that

\[
\sigma_{-\alpha}(\Sigma) \models \bigvee_{i \in \text{ON}(v)} \overline{x}_i \vee x_j \text{ for all } j \in J. \tag{8}
\]

If \( J \cap N(c) \neq \emptyset \), then by Lemma 3 and (8), we have \( \sigma_{-\alpha}(\Sigma) \models \bigvee_{i \in \text{ON}(v) \cap N(c)} \overline{x}_i \), since \( |\text{ON}(v) \setminus N(c)| \leq \alpha - 1 \). This implies \( \sigma_{-\alpha}(\Sigma) \models c \). On the other hand, if \( J \cap N(c) = \emptyset \), then by Lemma 3 and (8), we have \( \sigma_{-\alpha}(\Sigma) \models \bigvee_{j \in N(c)} \overline{x}_j \vee x_j \) for all \( j \in J \). Thus, if \( J \) contains an index in \( P(c) \), then we can conclude that \( \sigma_{-\alpha}(\Sigma) \models c \); otherwise, we check the condition \( \sigma_{-\alpha}(\Sigma) \models c \vee \bigvee_{j \in J} \overline{x}_j \), instead of \( \sigma_{-\alpha}(\Sigma) \models c \). Since a new clause \( d = c \vee \bigvee_{j \in J} \overline{x}_j \) is longer than \( c \), after at most \( n \) iterations, we can answer the deductive query. Formally, our algorithm can be described as Algorithm 2.

Theorem 12. Given the characteristic model \( \text{char}(\Sigma) \) of a Horn theory \( \Sigma \), a clause \( c \) and a nonnegative integer \( \alpha \), a deductive query \( \sigma_{-\alpha}(\Sigma) \models c \) can be answered in \( O(n^{\alpha+2}|\text{char}(\Sigma)|) \) time. In particular, it is polynomially solvable, if \( \alpha = O(1) \).

Proof. Since we can see algorithm \textsc{Deduction-Interior-from-Charset} correctly answers a deductive query from the discussion before the description, we only estimate the running time of the algorithm.

Steps 0, 1 and 3 require \( O(n) \) time. Step 2 requires \( O(n^{\alpha+1}|\text{char}(\Sigma)|) \) time, since (7) can be checked in \( O(n|\text{char}(\Sigma)|) \) time. Since we have at most \( n \) iterations between Steps 1 and 2, the algorithm requires \( O(n^{\alpha+2}|\text{char}(\Sigma)|) \) time.
Algorithm 2 Deduction-Interior-from-Charset

**Input:** The characteristic set $\text{char}(\Sigma)$ of a Horn theory $\Sigma$, a clause $c$ and a non-negative integer $\alpha$.

**Output:** Yes, if $\sigma_{-\alpha}(\Sigma) \models c$; Otherwise, No.

**Step 0.** Let $N := N(c)$, $d := c$ and $q := 1$.

**Step 1.** Let $v^*$ be the unique minimal model such that $d(v^*) = 0$.

**Step 2.** For each $v$ in $N_{\alpha}(v^*)$ do

- If (7) does not hold, then let $v^{(q)} := v$, $J := ON(\bigwedge_{w \in \text{char}(\Sigma)} w) \setminus ON(v)$ and $q := q + 1$
- If $J \cap (N \cup P(c)) \neq \emptyset$, then output yes and halt.
- Let $N := N \cup J$ and $d := \bigvee_{i \in N} \overline{x_i} \lor \bigvee_{i \in P(c)} x_i$.
- Go to Step 1.

**Step 3.** Output No and halt.
However, in general, the problem is intractable, which contrasts with the formula-model representation.

**Theorem 13.** Given the characteristic set \( \text{char}(\Sigma) \) of a Horn theory \( \Sigma \) and a positive integer \( \alpha \), it is co-NP-complete to decide whether \( \sigma^{-\alpha}(\Sigma) \) is consistent, i.e., \( \text{mod}(\sigma^{-\alpha}(\Sigma)) \neq \emptyset \).

**Proof.** Let us first show the co-NP-completeness of the problem. Apply Algorithm Deduction-Interior-from-Charset to the instance \((\text{char}(\Sigma), c = \emptyset, \alpha)\). If \( \sigma^{-\alpha}(\Sigma) \) is not consistent, then the algorithm constructs vectors \( v^{(1)}, \ldots, v^{(k)} \), \( k \leq n \), in Step 2. We can see that these vectors form a witness to the inconsistency of \( \sigma^{-\alpha}(\Sigma) \). In fact, if we are given these vectors, the inconsistency can be checked in polynomial time. This implies that the problem belongs to co-NP.

We show the co-NP-hardness by reducing INDEPENDENT SET to our problem. Given a problem instance \( G = (V = \{1, 2, \ldots, n\}, E) \) and \( k \) of INDEPENDENT SET, let us define a Horn theory \( \Sigma_G \) over \( A_t = \{x_1, x_2, \ldots, x_n\} \)

\[
\text{char}(\Sigma_G) = \{v^{(i,j)}, v^{(i,j,l)} \mid \{i,j\} \in E, l \in V \setminus \{i,j\} \},
\]

where \( v^{(i,j)} \) and \( v^{(i,j,l)} \) are respectively the vectors defined by \( \text{OFF}(v^{(i,j)}) = \{i,j\} \) and \( \text{OFF}(v^{(i,j,l)}) = \{i,j,l\} \). Let \( \alpha = n - k \). Note that \( \Sigma_G \) is a negative theory, and hence \( \sigma^{-\alpha}(\Sigma_G) \) is consistent if and only if \((00\cdots0)\) is a model of \( \sigma^{-\alpha}(\Sigma_G) \). Moreover, the latter condition is equivalent to the one that \( G \) has no independent set of size at least \( k (= n - \alpha) \). This completes the proof.

This result immediately implies the following corollary.

**Corollary 14.** Given the characteristic set \( \text{char}(\Sigma) \) of a Horn theory \( \Sigma \), a clause \( c \) and a positive integer \( \alpha \), it is NP-complete to decide whether a deductive query \( \sigma^{-\alpha}(\Sigma) \models c \) holds, even if \( c = \emptyset \).

Note that, different from the other hardness results, the hardness is not sensitive to the size of \( c \).

### 4.2 Exteriors

Let us consider the exteriors. Similarly to the formula-based representation, we have the following negative result.

**Theorem 15.** Given the characteristic set \( \text{char}(\Sigma) \) of a Horn theory \( \Sigma \), a clause \( c \) and a positive integer \( \alpha \), it is co-NP-complete to decide if a deductive query \( \sigma^{-\alpha}(\Sigma) \models c \) holds.
For this problem instance, we construct our problem instance. For each $\Sigma$ of Horn theory. Note that $\Sigma$ is a subclause of $\alpha$ and let $\sigma$ correspond to $\Sigma$. This means that $\sigma$ contains a model $\alpha$. By using Lemma 10, we can see that the problem can be solved in polynomial time, if $\sigma$ or $|N(c)|$ is small. Namely, for each subset $S$ of $N(c)$ such that $|S| \geq |N(c)| - \alpha$, let $\nu_{w^S}$ denotes the model such that $ON(w^S) = S$. Then $w^S = \bigwedge_{w \in \text{char}(\Sigma)} w$ is the unique minimal model of $\Sigma$ such that $ON(w^S) \supseteq S$, and hence it follows from Lemma 10 that it is enough to check if $|ON(w^S) \cap \nu_{w^S}| \geq \alpha - |N(c)| + |S| + 1$. Clearly, this can be done in $O\left(\sum_{i=0}^{\alpha} \left(\binom{|N(c)|}{i}\right)n|\text{char}(\Sigma)|\right)$ time.
Moreover, if $|P(c)|$ is small, then the problem also become tractable, which contrasts with Theorem 8.

**Lemma 16.** Let $\Sigma$ be a Horn theory, let $c$ be a clause, and $\alpha$ be a nonnegative integer. Then $\sigma_\alpha(\Sigma) \models c$ holds if and only if each $S \subseteq P(c)$ such that $|S| \geq |P(c)| - \alpha$ satisfies

$$|OFF(w) \cap N(c)| \geq \alpha - |P(c)| + |S| + 1$$

(9)

for all models $w$ of $\Sigma$ such that $OFF(w) \cap P(c) = S$.

Note that (9) is monotone in the sense that, if a model $w$ satisfies (9), then all models $v$ with $v < w$ also satisfy it. Thus it is sufficient to check if (9) holds for all maximal models $w$ of $\Sigma$ such that $OFF(w) \cap P(c) = S$. Since such maximal models $w$ can be obtained from $w^{(i)} (i \in S)$ with $i \in OFF(w^{(i)}) \cap P(c) \subseteq S$ by their intersection $w = \bigwedge_{i \in S} w^{(i)}$, we can answer the deduction problem in $O\left(n \sum_{p=|P(c)|-\alpha}^{|P(c)|} |\text{char}(\Sigma)|^p\right)$ time.

**Theorem 17.** Given the characteristic set $\text{char}(\Sigma)$ of a Horn theory, a clause $c$, and a nonnegative integer $\alpha$, a deductive query $\sigma_\alpha(\Sigma) \models c$ can be answered in $O\left(n \min\{\sum_{p=0}^{|N(c)|} |\text{char}(\Sigma)|^p, \sum_{p=|P(c)|-\alpha}^{|P(c)|} |\text{char}(\Sigma)|^p\}\right)$ time. In particular, it is polynomially solvable, if $\alpha = O(1)$, $|P(c)| = O(1)$, or $|N(c)| = O(\log n \cdot |\text{char}(\Sigma)|)$.

## 5 Deductive Inference for Envelopes of the Exteriors of Horn Theories

We have considered the deduction for the interiors and exteriors of Horn theories. As mentioned before, the interiors of Horn theories are also Horn, while this does not hold for the exteriors. This means that the exteriors of Horn theories might lose beneficial properties of Horn theories. One of the ways to overcome such a hurdle is *Horn Approximation*, that is, approximating a theory by a Horn theory [15]. There are several methods for approximation, but one of the most natural ones is to approximate a theory by its *Horn envelope*. For a theory $\Sigma$, its *Horn envelope* is the Horn theory $\Sigma_e$ such that $\text{mod}(\Sigma_e) = \text{Cl}\_\Lambda(\text{mod}(\Sigma))$. Since Horn theories are closed under intersection, Horn envelope is the least Horn upper bound for $\Sigma$, i.e., $\text{char}(\Sigma_e) \supseteq \text{char}(\Sigma)$ and there exists no Horn theory $\Sigma^*$ such that $\text{char}(\Sigma_e) \supseteq \text{char}(\Sigma^*) \supseteq \text{char}(\Sigma)$. In this section, we consider the deduction for Horn envelopes of exteriors of Horn theories, i.e., $\sigma_\alpha(\Sigma)_e \models c$. 

15
5.1 Model-Based Representations

Let us first consider the case in which knowledge bases are represented by characteristic sets.

**Proposition 18.** Let $\Sigma$ be a Horn theory, and let $\alpha$ be a nonnegative integer. Then we have

$$mod(\sigma_\alpha(\Sigma)_e) = Cl_\wedge \bigcup_{v \in \text{char}(\Sigma)} \mathcal{N}_\alpha(v).$$  (10)

**Proof.** By definition, $mod(\sigma_\alpha(\Sigma)_e) = Cl_\wedge (mod(\sigma_\alpha(\Sigma))) \supseteq Cl_\wedge (\bigcup_{v \in \text{char}(\Sigma)} \mathcal{N}_\alpha(v))$ holds. For the converse direction, let $v^*$ be a model of Horn envelope of the $\alpha$-exterior, i.e., $v^* \in mod(\sigma_\alpha(\Sigma)_e)$. Then $v^*$ can be represented by $v^* = \wedge_{w \in W} w$ for some $W \subseteq mod(\sigma_\alpha(\Sigma))$. Assume that $w \in W$ is contained in $N_\alpha(u)$ for some model $u$ of $\Sigma$. Since such a $u$ can be represented by $u = \wedge_{z \in S_w} z$ for some $S_w \subseteq \text{char}(\Sigma)$, $w$ belongs to $Cl_\wedge (\bigcup_{v \in S_w} \mathcal{N}_\alpha(v))$. This, together with $v^* = \wedge_{w \in W} w$, implies that $v^*$ also belongs to $Cl_\wedge (\bigcup_{v \in \text{char}(\Sigma)} \mathcal{N}_\alpha(v))$. \qed

For a clause $c$, let $v^*$ be the unique minimal model such that $c(v^*) = 0$. We recall that, for a Horn theory $\Phi$,

$$\Phi \models c \text{ if and only if } c(\bigwedge_{v \in \text{char}(\Phi)} v_{v \geq v^*}) = 1.$$  (11)

Therefore, Proposition 18 immediately implies an algorithm for the deduction for $\sigma_\alpha(\Sigma)_e$ from $\text{char}(\Sigma)$, since we have $\text{char}(\sigma_\alpha(\Sigma)_e) \subseteq \bigcup_{v \in \text{char}(\Sigma)} \mathcal{N}_\alpha(v)$. However, for a general $\alpha$, $\bigcup_{v \in \text{char}(\Sigma)} \mathcal{N}_\alpha(v)$ is exponentially larger than $\text{char}(\Sigma)$, and hence this direct method is not efficient. The following lemma helps developing a polynomial time algorithm.

**Lemma 19.** Let $\Sigma$ be a Horn theory, let $c$ be a clause, and let $\alpha$ be a nonnegative integer. Then $\sigma_\alpha(\Sigma)_e \models c$ holds if and only if the following two conditions are satisfied.

(i) $|OFF(v) \cap N(c)| \geq \alpha$ holds for all $v \in \text{char}(\Sigma)$.

(ii) If $S = \{v \in \text{char}(\Sigma) \mid |OFF(v) \cap N(c)| = \alpha\} \neq \emptyset$, $P(c)$ is not covered with $OFF(v)$ for models $v$ in $S$, i.e., $P(c) \not\subseteq \bigcup_{v \in \text{char}(\Sigma)} OFF(v)$.

**Proof.** To show the if part, let us first assume that (i) and (ii) in the lemma holds. Let $v$ be a model in $\text{char}(\Sigma)$ such that $|OFF(v) \cap N(c)| > \alpha$. Then all models $w$ in $\mathcal{N}_\alpha(v)$ satisfy $OFF(w) \cap N(c) \neq \emptyset$. Therefore, if all the models $v$ in $\text{char}(\Sigma)$
satisfy $|\text{OFF}(v) \cap N(c)| > \alpha$, then by Proposition 18, we have $\text{OFF}(w) \cap N(c) \neq \emptyset$ for any model $w$ of $\sigma(\Sigma)_e$. This implies $\sigma(\Sigma)_e \models c$. Therefore, let us consider the case when $S = \{v \in \text{char}(\Sigma) \mid |\text{OFF}(v) \cap N(c)| = \alpha\}$ is not empty. Let $v^*$ be the unique minimal model such that $c(v^*) = 0$. Then by Proposition 18, we have

$$\{v \in \text{char}(\sigma(\Sigma)_e) \mid v \geq v^*\} \subseteq \{w \mid ON(w) = ON(v) \cup N(c) \text{ for some } v \in S\}. \quad (12)$$

Since $P(c)$ is not covered with $\text{OFF}(v)$ for models $v$ in $S$, this, together with (11) implies $\sigma(\Sigma)_e \models c$.

Let us next show the only-if part. Assume that (i) is satisfied, but (2) is not. Then (11) and (12) imply $\sigma(\Sigma)_e \not\models c$. On the other hand, if (1) is not satisfied, i.e., there exists a $v \in \text{char}(\Sigma)$ such that $|\text{OFF}(v) \cap N(c)| < \alpha$, let $w^{(i)}$, $i \in P(c)$, be a model in $N_\alpha(v)$ such that $ON(w^{(i)}) \supseteq N(c)$ and $OFF(w^{(i)}) \supseteq \{i\}$, and let $w^* = \bigwedge_{i \in P(c)} w^{(i)}$. Then we have $c(w^*) = 0$ and $w^* \in \text{mod}(\sigma(\Sigma)_e)$ by Proposition 18. This implies $\sigma(\Sigma)_e \not\models c$. \hfill \square

The lemma immediately implies the following theorem.

**Theorem 20.** Given the characteristic set $\text{char}(\Sigma)$ of a Horn theory $\Sigma$, a clause $c$, and a nonnegative integer $\alpha$, a deductive query $\sigma(\Sigma)_e \models c$ can be answered in linear time.

We remark that this contrasts with Corollary 14. Namely, if we are given the characteristic set $\text{char}(\Sigma)$ of a Horn theory $\Sigma$, $\sigma(\Sigma)_e \models c$ is polynomially solvable, while it is co-NP-complete to decide if $\sigma(\Sigma) \models c$.

### 5.2 Formula-Based Representation

Recall that a negative theory (i.e., a theory consisting of clauses with no positive literal) is Horn and the exteriors of negative theory are also negative, and hence Horn. This means that, for a negative theory $\Sigma$, we have $\sigma(\Sigma)_e = \sigma(\Sigma)$. Therefore, we can again make use of the reduction in the proof of Theorem 8, since the reduction uses negative theories.

**Theorem 21.** Given a Horn theory $\Sigma$, a clause $c$, and a nonnegative integer $\alpha$, it is co-NP-complete to decide whether $\sigma(\Sigma)_e \models c$ holds, even if $P(c) = \emptyset$.

**Proof.** Since the hardness is proved similarly to Theorem 8, we show that the problem belongs to co-NP.
Let \( v \) be a model of \( \sigma_\alpha(\Sigma)_e \). Then \( v \) can be represented by \( v = \bigwedge_{w \in W} w \) for some \( W \subseteq \text{char}(\sigma_\alpha(\Sigma)) \). Since \( \text{char}(\sigma_\alpha(\Sigma)) \subseteq \bigcup_{w \in \text{char}(\Sigma)} N_\alpha(w) \) by Proposition 18, we have
\[
v = \bigwedge_{w \in \text{char}(\Sigma)} \left( \bigwedge_{u \in S_w} u \right)
\]
for some \( S_w \subseteq N_\alpha(w) \). We claim that there exists such a representation that \( |S_w| \leq n \) holds for all \( w \)'s in (13). Let \( w^* = \bigwedge_{u \in S_w} u \), and let \( I = \text{ON}(w^*) \cap \text{OFF}(w) \) and \( J = \text{OFF}(w^*) \cap \text{ON}(w) \). Then we have \( w^* = \bigwedge_{j \in J} \left( w - e^{(j)} + \sum_{i \in I} e^{(i)} \right) \), where \( e^{(i)} \) denotes the \( i \)th unit model. Since \( w - e^{(j)} + \sum_{i \in I} e^{(i)} \in N_\alpha(w) \) for all \( j \in J \), the claim is proved.

Note that \( \sigma_\alpha(\Sigma)_e \models c \) if and only if there exists a model \( v \) of \( \sigma_\alpha(\Sigma)_e \) such that \( c(v) = 0 \). Since any model \( v \) of \( \sigma_\alpha(\Sigma)_e \) can be represented by \( v = \bigwedge_{w \in \text{char}(\Sigma)} \left( \bigwedge_{u \in S_w} u \right) \) for some \( S_w \subseteq N_\alpha(w) \) with \( |S_w| \leq n \) by our claim, the problem is in co-NP. \( \square \)

However, if \( \alpha \) or \( N(c) \) is small, the problem becomes tractable by algorithm \textsc{Deduction-Envelope-Exterior-from-Horn-Theory} (Algorithm 3).

The algorithm is based on a necessary and sufficient condition for \( \sigma_\alpha(\Sigma)_e \models c \), which is obtained from Lemma 19 by replacing all \( \text{char}(\Sigma) \)'s with \( \text{mod}(\Sigma) \)'s. It is not difficult to see that such a condition holds from the proof of Lemma 19.

**Theorem 22.** Given a Horn theory \( \Sigma \), a clause \( c \), and a nonnegative integer \( \alpha \), a deductive query \( \sigma_\alpha(\Sigma)_e \models c \) can be answered in \( O\left( \left( \binom{|N(c)|}{\alpha} + \binom{|N(c)|}{\alpha-1} \right) \| \Sigma \| + |P(c)| \right) \) time. In particular, it is polynomially solvable, if \( \alpha = O(1) \) or \( |N(c)| = O(\log \| \Sigma \|) \).

**Proof.** The correctness of the algorithm follows from the discussion after its description. For the time complexity, it is known \cite{3} that the satisfiability problem, together with computing a unique minimal model for a Horn theory, is possible in linear time. Since the number of the iterations of for-loops in Steps 2 and 3 are bounded by \( (\binom{|N(c)|}{\alpha-1}) \) and \( (\binom{|N(c)|}{\alpha}) \), respectively, the algorithm requires \( O\left( \left( \binom{|N(c)|}{\alpha-1} + \binom{|N(c)|}{\alpha} \right) \| \Sigma \| + |P(c)| \right) \) time. \( \square \)

**References**

[1] R. J. Brachman and H. J. Levesque. *Knowledge Representation and Reasoning*. Elsevier, 2004.

[2] R. Dechter and J. Pearl. Structure identification in relational data. *Artificial Intelligence*, 58 (1992) 237–270.
Algorithm 3 Deduction-Envelope-Exterior-from-Horn-Theory

**Input:** A Horn theory $\Sigma$, a clause $c$ and a nonnegative integer $\alpha$.

**Output:** Yes, if $\sigma_\alpha(\Sigma)_e \models c$; Otherwise, No.

**Step 1.** /* Check if there exists a model $v$ of $\Sigma$ such that $|OFF(v) \cap N(c)| < \alpha$. */

For each $N \subseteq N(c)$ with $|N| = |N(c)| - \alpha + 1$ do

Check if the theory obtained from $\Sigma$ by assigning $x_i = 1$ for $i \in N$ is satisfiable.

If so, then output No and halt.

end{for}

**Step 2.** /* Check if there exists a set $S = \{v \in mod(\Sigma) \mid |OFF(v) \cap N(c)| = \alpha\}$ such that $\bigcup_{v \in S} OFF(v) \supseteq P(c)$. */

Let $J := \emptyset$.

For each $N \subseteq N(c)$ with $|N| = |N(c)| - \alpha$ do

Compute a unique minimal satisfiable model $v$ for the theory obtained from $\Sigma$ by assigning $x_i = 1$ for $i \in N$ is satisfiable.

Update $J := J \cup \{j \in P(c) \mid v_j = 0\}$.

end{for}

If $J = P(c)$, then output NO and halt.

**Step 3.** Output Yes and halt.
[3] W. Dowling and J. H. Galliear. Linear-time algorithms for testing the satisfiability of propositional Horn theories. *Journal of Logic Programming* 3 (1983) 267–284.

[4] T. Eiter, T. Ibaraki, and K. Makino. Computing intersections of Horn theories for reasoning with models. *Artificial Intelligence*, 110 (1999) 57-101.

[5] T. Eiter and K. Makino. On computing all abductive explanations. In *Proceedings AAAI-2002* (2002) 62-67.

[6] D. Kavvadias, C. Papadimitriou, and M. Sideri. On Horn Envelopes and Hypergraph Transversals. In W. Ng, editor, *Proceedings 4th International Symposium on Algorithms and Computation (ISAAC-93)*, LNCS 762, pages 399–405, Hong Kong, December 1993. Springer.

[7] H. Kautz, M. Kearns, and B. Selman. Reasoning with characteristic models. In *Proceedings AAAI-93* (1993) 34–39.

[8] H. Kautz, M. Kearns, and B. Selman. Horn approximations of empirical data. *Artificial Intelligence*, 74 (1995) 129–245.

[9] R. Khardon and D. Roth. Reasoning with models. *Artificial Intelligence*, 87 (1996) 187–213.

[10] R. Khardon and D. Roth. Defaults and relevance in model-based reasoning. *Artificial Intelligence*, 97 (1997) 169–193.

[11] K. Makino and T. Ibaraki, Interior and exterior functions of Boolean functions, *Discrete Applied Mathematics*, 69 (1996) 209–231.

[12] K. Makino, H. Ono and T. Ibaraki, Interior and exterior functions of positive Boolean functions, *Discrete Applied Mathematics*, 130 (2003) 417–436.

[13] J. Makowsky. Why Horn formulas matter for computer science: Initial structures and generic examples. *Journal of Computer and System Sciences*, 34 (1987) 266–292.

[14] J. McKinsey. The decision problem for some classes of sentences without quantifiers. *Journal of Symbolic Logic*, 8 (1943) 61–76.

[15] B. Selman and H. Kautz, Knowledge compilation using Horn approximations, *the Proceedings of the 9th National Conference on Artificial Intelligence AAAI-91*, (1991) 904–909.