Discussion on Experimental Teaching of Mechanics of Materials

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Abstract: Based on the teaching practice of both the theory and experiment of mechanics of materials by the author, discussion is attempted to add into experimental teaching part. In the discussion, some hidden correlations, perceptual cognitions and interesting problems are illustrated. This attempt is confirmed to be effective in improving the understanding of mechanics knowledge of the undergraduate students and in promoting the combination of theoretical teaching and experimental teaching of mechanics of materials.

1. Introduction

Mechanics of materials comes from the engineering practice and research the strength, deformation and stability of materials\[(1)\]. The course is closely related to engineering practice, so that many equatione and principles in engineering design are from this course\[(2)\]. Therefore, the experimental teaching of material mechanics is very important in the teaching of mechanics of materials. In recent years, many scholars have thought and innovated the teaching of mechanics of materials. Zhou Shuangxi et al. introduced the concept of “conception design implementation operation” namely CDIO engineering education concept, and carried out innovation and reform on the experimental teaching mode of mechanics of materials \[(3)\]. Zhang Zhizhen suggested that the experimental materials should be selected from the practical engineering directly\[(4)\].

The author teaches both of the theoretical courses and experiments of mechanics of materials for undergraduates. From the teaching practice, the author realized the importance of the integration of experimental teaching and theoretical teaching. Therefore, the author tries to integrate the theoretical teaching and experimental teaching by increasing the expanding discussion.

2. Experiments in mechanics of materials

The basic experiments of mechanics of materials include: (1) tensile and compression experiment of metal materials; (2) torsion experiment of metal materials; (3) measuring normal stresses on cross section of pure bending beam; (4) measuring principal stresses of beam under bending and torsion combined loading. The former two experiments mainly focus on the strength characteristics of materials, and the other two experiments mainly focus on the deformation characteristics of materials.

2.1 Strength characteristics experiments

Determination of strength characteristics of mild steel and cast iron by tensile and compression experiment. Both of the tension specimen and torque specimen are a bar with length \(L\) and section radius \(R\). The compression specimen is a cylinder with height \(H\) and section radius \(R\). As shown in...
Figure 1. In the experiment, the tensile (compressive) force $F$ and the deformation $\Delta L$ were measured. According to the yield loading $F_y$ of mild steel, the yield stress $\sigma_y$ is calculated. According to the tensile and compressive failure loading $F_{bt}$ and $F_{bc}$ of cast iron, the tensile and compressive failure stress $\sigma_{bt}$ and $\sigma_{bc}$ are calculated.

\[
\begin{align*}
\sigma_y &= F_y / (\pi R^2) \\
\sigma_{bt} &= F_{bt} / (\pi R^2) \\
\sigma_{bc} &= F_{bc} / (\pi R^2)
\end{align*}
\] (1) (2) (3)

Figure 1. Sketches of uniaxial tensile and compression experiment.

In the torsion experiment of metal materials, the microscopic shear stress $\tau$ is calculated by measuring the macroscopic torque $T$. As shown in Figure 2., the yield torque $T_y$ of mild steel bar and the failure torque $T_b$ of cast iron bar are measured, then the yield shear stress $\tau_y$ for mild steel and the failure shear stress $\tau_b$ for cast iron are calculated accordingly:

\[
\begin{align*}
T_y &= \frac{2}{3} \pi R^3 \tau_y \\
T_b &= \frac{1}{2} \pi R^3 \tau_b
\end{align*}
\] (4) (5)

Figure 2. Sketch of torsion experiment

2.2 Deformation characteristic experiment
Applying bending moment $M$ on both ends of rectangular section beam with width $b$ and height $h$ makes the beam in pure bending state. As shown in Figure 3, the downward is the positive direction of $y$ coordinate, and the origin is at the half height of beam.

Then the normal stress distribution $\sigma(y)$ of the section is described as follows:

\[
\sigma(y) = \frac{M}{I_y} y
\] (6)

The experiment is to verify the rationality of equation (6).
The device for measuring principal stresses is shown in Figure 4. With the loading $F$ and torque $T$ applying on the thin walled bar, the whole bar is in bending, shearing and torsion state. Point A locates at the surface of the bar, therefore, the shear stress at point a in z direction is 0, then point A can be considered as plane stress state. Three strain gauges $\text{①}$, $\text{②}$ and $\text{③}$ are pasted at point A, strain gauge $\text{①}$ is perpendicular to strain gauge $\text{③}$, and strain gauge $\text{②}$ is along the $x$ axis and on the angular bisector of strain gauge $\text{①}$ and $\text{③}$. By measuring the direct strains of three strain gauges $\varepsilon_{\text{①}}$, $\varepsilon_{\text{②}}$ and $\varepsilon_{\text{③}}$, combining the elastic modulus $E$ and Poisson’s ratio $\nu$, the magnitude and direction of principal stresses at point A are calculated according to equations (7) and (8).

\[
\sigma_1 = \frac{E\left(\varepsilon_{\text{①}} + \varepsilon_{\text{③}}\right)}{2(1-\nu)} + \frac{E\left[\left(\varepsilon_{\text{①}} - \varepsilon_{\text{③}}\right)^2 + \left(\varepsilon_{\text{③}} - \varepsilon_{\text{②}}\right)^2\right]^{\frac{1}{2}}}{\sqrt{2(1+\nu)}}
\]
\[
\sigma_3 = \frac{E\left(\varepsilon_{\text{①}} + \varepsilon_{\text{③}}\right)}{2(1-\nu)} - \frac{E\left[\left(\varepsilon_{\text{①}} - \varepsilon_{\text{③}}\right)^2 + \left(\varepsilon_{\text{③}} - \varepsilon_{\text{②}}\right)^2\right]^{\frac{1}{2}}}{\sqrt{2(1+\nu)}}
\]
\[
\alpha = \frac{1}{2}\arctan\left(\frac{\varepsilon_{\text{③}} - \varepsilon_{\text{①}}}{2\varepsilon_{\text{②}} - \varepsilon_{\text{③}} - \varepsilon_{\text{①}}}\right)
\]

3. Expanding discussion on the experiment of mechanics of materials

According to students’ questions and communication with students in teaching practice, the author tries to expand the following discussions based on the above four basic experimental teaching contents.

**Discussion 1: Correlation between allowable stresses**

The yield characteristics of mild steel are described by the fourth strength theory (Mises criterion). According to the yield stress state under uniaxial tension $(\sigma_1, \sigma_2, \sigma_3) = (\sigma_s, 0, 0)$, the expression of yield criterion is determined:

\[
\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} = \sqrt{2}\sigma_s
\]

When the mild steel bar is twisted to yielding, shear stress $\tau$ of all points on cross section reach the yield shear stress $\tau_s$, and the principle stress state is $(\sigma_1, \sigma_2, \sigma_3) = (\tau_s, 0, -\tau_s)$. According to Mises criterion, yield shear stress of mild steel $\tau_s$ can be obtained through equation (9):

\[
\tau_s = \sigma_s/\sqrt{3} \approx 0.6\sigma_s
\]
If the third strength theory (Tresca criterion) is roughly applied to describe the yield of mild steel, yield shear stress is calculated as 
\[ \tau_\text{yield} = 0.5 \sigma_\text{yield} \]. Here, students are reminded to recall the relationship between allowable tensile stress \( \sigma_\text{allow} \) and allowable shear stress \( \tau_\text{allow} \) of mild steel \( \tau_\text{allow} = (0.5\sim0.6)\sigma_\text{allow} \).

The failure condition of the compressed cast iron specimen sliding along a inclined plane can be described by Mohr-Coulomb criterion \([2]\). According to the tensile and compressive failure stress \( \sigma_\text{fail} \) and \( \sigma_\text{fail comp} \), the failure stress state of cast iron conforms to:

\[ \sigma_\text{fail} = \frac{\sigma_\text{allow}}{\alpha_\text{allow} + \sigma_\text{allow}} \]

(11)

For torsion of the cast iron bar, when shear stress of the outermost layer reaches failure shear stress \( \tau_\text{fail} \), the specimen is broken by torsion, when the stress state \( (\sigma_\text{allow}, \sigma_\text{allow comp}, \sigma_\text{field}) = (\tau_\text{fail}, 0, -\tau_\text{fail}) \). The failure shear stress \( \tau_\text{fail} \) of cast iron can be obtained by substituting it into equation (11).

\[ \tau_\text{fail} = \frac{\sigma_\text{allow}}{\alpha_\text{allow} + \sigma_\text{allow}} \sigma_\text{fail} \]

(12)

According to equation (12) and considering \( \sigma_\text{allow} = (2\sim7)\sigma_\text{allow comp} \) \([1]\), the relationship between failure shear stress \( \tau_\text{fail} \) and failure tensile stress \( \sigma_\text{allow} \) of cast iron \( \tau_\text{fail} = (0.7\sim0.9)\sigma_\text{allow} \) can be obtained. If the torsion failure of cast iron is taken into account by the first strength theory (maximum tensile strain theory), it is similar to obtain \( \tau_\text{fail} = \sigma_\text{allow} \). Considering the torsion failure of cast iron and taking Poisson's ratio \( \nu \approx 0.25 \), we can obtain \( \tau_\text{fail} = 0.8\sigma_\text{allow} \). Then students are reminded to recall the relationship between allowable tensile stress \( \sigma_\text{allow} \) and allowable shear stress \( \tau_\text{allow} \) of cast iron \( \tau_\text{allow} = (0.8\sim1.0)\sigma_\text{allow} \).

Discussion 2: Macroscopic load and microscopic stress

When the material unit is subjected to pure shear stress \( \tau \); When the beam is bent, the material unit is subjected to a unidirectional normal stress \( \sigma \). According to Hooke's law, shear stress \( \tau \) and normal stress \( \sigma \) are expressed as follows:

\[ \tau = G \gamma \]

(13a)

\[ \sigma = E \varepsilon \]

(13b)

In equation (13), \( G \) is the shear modulus and \( E \) is Young’s modulus. According to the geometric conditions: The shear strain \( \gamma \) of the torsion bar is determined by position quantity \( r \) and macroscopic torsion degree \( (d\phi / dx) \). The normal strain \( \varepsilon \) of pure bending beam is determined by position quantity \( y \) and macroscopic bending degree \( (1/\rho) \).

\[ \gamma = r \cdot (d\phi / dx) \]

(14a)

\[ \varepsilon = y \cdot (1/\rho) \]

(14b)

In equation (14a), the factor \( r \) is the distance between the micro area \( dA \) and the axis of the bar, as shown in Figure 2. \( (d\phi / dx) \) is the relative torsion angle of two sections per unit length along the axial direction \( x \) of the bar. In equation (14b), \( y \) is shown in Figure 3, \( \rho \) is the radius of curvature of the pure bending beam. According to equation (14), restricted to elastic deformation, \( y \) is linearly distributed along \( r \) direction and \( \varepsilon \) is linearly distributed along \( y \) direction, as shown in Figure 2 and 3. Then students are guided to establish the balance equations from equatione (13) and (14).

\[ T = \int_A r \cdot \tau (r) dA \]

(15a)

\[ M = \int_A y \cdot \sigma (y) dA \]

(15b)

Substituting equations (13) and (14) into (15), macroscopic relationships \( T \sim (d\phi / dx) \) and \( M \sim (1/\rho) \) are deducted:

\[ T = \frac{d\phi}{dx} G \int_A r^2 dA = \frac{d\phi}{dx} GI_p \]

(16a)

\[ M = \frac{E}{\rho} \int_A y^2 dA = \frac{E}{\rho} I_z \]

(16b)

In equation (16): \( I_p \) is the polar moment of inertia; \( I_z \) is the axial moment of inertia. For circular section with radius \( R \), \( I_p = 2I_z = \pi R^4 / 2 \), and for rectangular section with width \( b \) and height \( h \), \( I_z = bh^3 / 12 \). Considering equations (13), (14) and (16), the relationships between macroscopic load and microscopic stress are obtained:
\[
\tau = r \cdot \left( \frac{T}{I_p} \right) \\
\sigma = y \cdot \left( \frac{M}{I_z} \right)
\]  
(17a)  
(17b)

As shown in Table 1, there is a graceful symmetry between the bending and torsion equations.

| logical relationship | bending | torsion |
|----------------------|---------|---------|
| stress \( \rightarrow \) strain | I \( \sigma = E \varepsilon \) | \( \tau = G \gamma \) |
| load \( \rightarrow \) deformation | II \( \varepsilon = \frac{y}{\rho} \) | \( \gamma = \frac{\tau}{G} \frac{d\phi}{dx} \) |
| equilibrium equation \( \rightarrow \) I + II | III \( \frac{1}{\rho} = \frac{M}{E I_z} \) | \( \frac{d\phi}{dx} = \frac{T}{G I_p} \) |
| I + II + III | IV \( \sigma = \frac{M}{I_z} y \) | \( \tau = \frac{T}{I_p} r \) |

Table 1. Comparison of bending and torsion

In Table 1, applying equilibrium condition, constitutive relation I and geometric relations II, the load–deformation relation III is firstly deduced. And then the “stress–load relation IV” is derived from relationships I, II, and III.

**Discussion 3: Derivation of the relationship between elastic modulus and shear modulus**

The relationship between elastic modulus \( E \) and shear modulus \( G \) is described as follows:

\[
E = 2(1 + \nu)G
\]  
(18)

The author hopes to guide the students to understand the \( E \sim G \) relation more vividly. As shown in Figure 5, the unit ABCD with side length \( a \) is subjected to pure shear stress \( \tau \), and its deformation is AB’C’D’. The unit is in plane stress state \((\sigma_1, \sigma_2, \sigma_3) = (\tau, 0, -\tau)\).

![Figure 5. Stress and strain of pure shear unit](image)

In the direction of major principal stress \( \sigma_1 \), the diagonal length of the unit changes from \(|AC|\) before loading to \(|AC'|\) after loading. Therefore, the major principal strain \( \varepsilon_1 \) is:

\[
\varepsilon_1 = \frac{2a \cos \left( \frac{\pi}{4} - \gamma \right) - \sqrt{2a}}{\sqrt{2a}} \approx \frac{1}{2} \gamma
\]  
(19)

It can be concluded that, restricted to small strain, the pure shear strain \( \gamma \) is two times of the major principal strain \( \varepsilon_1 \). According to Hooke’s law, \( \varepsilon_1 \) is expressed as:

\[
\varepsilon_1 = \frac{\sigma_1}{E} - \nu \frac{\sigma_3}{E}
\]  
(20)

By substituting equations (20) and (13a) into equation (19), the \( E \sim G \) relation (18) is obtained.

**Discussion 4: Loading rate relationship between tension and torsion.**

The mechanical properties of materials are related to the loading rate. In general, the greater the loading rate, the greater the resistance force of the material \(^5\). Consequently, in order to compare test data from different experiments of one kind of material, it is critical to control the test loading rate.
According to reference [6], the loading rate of mild steel in tension compression experiment \( \sigma_s \) is \( 6 \text{MPa/s} \text{--} 10 \text{MPa/s} \) and that in torsion experiment \( \phi \) is \( 6^\circ/\text{min} \text{--} 30^\circ/\text{min} \). These two loading rates are relative, which is present to the students to think. From equation (19), if the surface principal strain rate of the torsion specimen is the same as that of the uniaxial tensile specimen, the relationship between the tensile and torsional loading rates \( \sigma_s \) and \( \phi \) can be derived from equation (21).

\[
\dot{\sigma} = E \dot{e} = E \frac{\dot{x}}{2} = ER \cdot \left( \frac{\dot{\phi}}{L} \right) / 2 = \frac{ER}{2L} \dot{\phi}
\]

Substituting \( E = 200 \text{GPa} \), \( R = 5 \text{mm} \) and \( L = 100 \text{mm} \) into equation (21), the relationship of \( \sigma_s \) and \( \phi \) is approximately \( \sigma_s \approx 5000 \phi \), which is consistence with the reference.

Discussion 5: load correlation and deformation correlation

The bar-shaped specimen with length \( L \) radius \( R \) is taken as an example for discussion. Restricted to elastic deformation, tension \( F \), torque \( T \) and bending moment \( M \) act on the specimen respectively, the corresponding tensile deformation \( \Delta L \), relative torsion angle \( \phi \) and relative bending angle \( \theta \) are:

\[
\Delta L = \frac{(FL)}{(EA)}
\]

\[
\phi = \frac{(TL)}{(GL_p)}
\]

\[
\theta = \frac{(ML)}{(EI_p)}
\]

According to equation (23), (24) and equation (18), when the same magnitude of torque and bending moment act respectively, the torsion angle \( \phi \) and bending angle \( \theta \) satisfy the following requirements:

\[
\phi = (1 + \nu) \theta
\]

Neglecting the Poisson’s ratio, \( \phi \) and \( \theta \) are approximately equal.

The tensile yield load \( F_s \) and tensile yield deformation \( \Delta L_s \) of mild steel sample are:

\[
F_s = \pi R^2 \sigma_s
\]

\[
\Delta L_s = L \cdot \sigma_s / E
\]

When mild steel specimen is torsional and the surface reaches the yield shear stress \( \tau_s \), the specimen will begin to enter the plastic deformation stage. According to equations (17a) and (23), the torque \( T_{se} \) and the relative torsion angle \( \phi_{se} \) are:

\[
T_{se} = \frac{\pi R^3 \tau_s}{2}
\]

\[
\phi_{se} = (\tau_s L)/(GR)
\]

For the pure bending mild steel specimen, when the upper and lower fibers reach the yield stress \( \sigma_s \), the whole bar reaches the maximum elastic bending deformation. According to equations (17b) and (24), the bending moment \( M_{se} \) and the relative torsion angle \( \theta_{se} \) are:

\[
M_{se} = \pi R^3 \sigma_s / 4
\]

\[
\theta_{se} = (\sigma_s L)/(ER)
\]

When torsion of mild steel reaches full section yield, torque \( T_s \) is shown in equation (4), and when pure bending reaches full section yield, the bending moment \( M_s \) is:

\[
M_s = 4 \sigma_s R^3 / 3
\]

The tensile failure load \( F_{bt} \) and tensile failure deformation \( \Delta L_{bt} \) of cast iron are as follows:

\[
F_{bt} = \pi R^2 \sigma_{bt}
\]

\[
\Delta L_{bt} = L \cdot \sigma_{bt} / E
\]

The failure torque of cast iron is shown in equation (5), and the failure torsion angle \( \phi_b \) is:

\[
\phi_b = (\tau_b L)/(GR)
\]

For the pure bending cast iron specimen, when the fiber at the lower end reaches the failure tensile stress \( \sigma_{bt} \), the whole bar reaches the maximum bending deformation. According to equation (17b) and (24), the bending moment \( M_b \) and the relative bending angle \( \theta_b \) are:

\[
M_b = \pi R^3 \sigma_{bt} / 4
\]

\[
\theta_b = (\sigma_{bt} L)/(ER)
\]

By comparing equation (4) with (28), and (30) with (32), it can be obtained that:

\[
T_{se} = (3/4)T_s
\]
Therefore, for mild steel, the safe stock under bending is greater than that under torsion. Combining equations (4), (10), (26), and (5), (12), (33), $T_s \sim F_s$ and $T_b \sim F_{bt}$ relationships are obtained:

$$T_s = \frac{2R}{\sqrt[3]{3}}F_s$$  

$$T_b = \frac{R}{2\sigma_{bc}+\sigma_{bt}}F_{bt}$$  

Take specimen $R=5$ mm for example, according to equations (40) and (41), $T_s \approx \frac{(1500)}{F_s}$ and $T_b \approx \frac{(1500)}{F_{bt}}$ in numerical value. Therefore, for the same specimen, the tensile force required for tensile yield or failure is several ‘tons’, while the torque for torsion yield or failure is only tens of ‘Newton \cdot meters’.

By combining equations (10), (18), (27) and (29), the $\varphi_{se} \sim \Delta L_s$ relation of the mild steel is:

$$\varphi_{se} \cdot R = \frac{2(1+\nu)\Delta L_s}{\sqrt[3]{3}} \approx 1.5\Delta L_s$$  

At this time, the major principal strain $\varepsilon_1$ occurred on the surface of the mild steel specimen is:

$$\varepsilon_1 = \frac{\varphi_{se} R}{\sqrt[3]{3}} = \frac{1+\nu}{\sqrt[3]{3}} \frac{\Delta L_s}{L} \neq \frac{\Delta l}{l}$$  

The $\varphi_b \sim \Delta L_{bt}$ relation of cast iron is obtained by combining equations (12), (18), (34) and (35).

$$\varphi_b \cdot R = \frac{2(1+\nu)\sigma_{bt}}{\sigma_{bc}+\sigma_{bt}}\Delta L_{bt} \approx 2\Delta L_b$$  

At this time, the major principal strain $\varepsilon_1$ occurred on the surface of the cast iron specimen is:

$$\varepsilon_1 = \frac{\varphi_{bt} R}{\sqrt[3]{3}} = \frac{1+\nu}{\sigma_{bc}+\sigma_{bt}} \frac{\Delta L_b}{L} \approx \frac{\Delta l}{l}$$  

For cast iron, Poisson’s ratio $\nu \approx 0.25 \approx \frac{\sigma_{bt}}{\sigma_{bc}}$, which makes ‘$\approx$’ in equation (45) true. Comparison equation (43) and (45): the major principal strains of mild steel in pure shear yield and in uniaxial tensile yield are different, while the major principal strain of cast iron in pure shear yield and in uniaxial tensile yield are close. This fact confirms that the second strength theory (maximum tensile strain theory) is applicable to describe the failure of cast iron, but not to the yield of mild steel.

Here, students can also be guided to discuss the description of the failure stress state of cast iron by the first strength theory (maximum tensile stress theory) and the second strength theory (maximum tensile strain theory). The first strength theory takes the maximum tensile stress as the failure condition, which implies that the material can withstand infinite compressive stress. Obviously, this is contrary to the objective fact of compressive strength of cast iron. In addition, Poisson's ratio for cast iron is $\nu \approx 0.25$, which is close to the ratio of tensile and compressive strength ($\sigma_{bt}/\sigma_{bc}$). Therefore, the second strength theory is consistent with Mohr-Coulomb strength theory in describing uniaxial tension, uniaxial compression and pure shear. When $\nu \approx (\sigma_{bt}/\sigma_{bc})$ is no longer satisfied (e.g. for concrete $\nu \approx 0.2$, $(\sigma_{bt}/\sigma_{bc}) < 0.1$) [7], the second strength theory is no longer applicable.

**Discussion 6: Magnitude of elastic deformation**

For mild steel $\sigma_s = 240$MPa, $E \approx 200$GPa, hence the maximum elastic strain $\varepsilon$ is about $(\sigma_s/E) \approx 1/1000$. For gray cast iron $\sigma_{bt} \approx 190$MPa, $\sigma_{bc} \approx 650$MPa, $E \approx 140$GPa, hence the maximum elastic tensile strain $\varepsilon$ is about $(\sigma_{bt}/E) \approx 1/1000$; the maximum elastic compressive strain $\varepsilon$ is about $(\sigma_{bc}/E) \approx 1/200$. In practical engineering, the working stress of metal bar must be in the range of elastic deformation. Therefore, such a small elastic deformation hardly changes the bar directions in skeletal structure before and after the loading, and the internal force of each bar can be calculated according to the bar directions before loading[8]. Considering that the maximum axial strain is less than (1/200) and Poisson's ratio $\nu \approx 0.3$, the radial strain of the bar does not exceed 3/2000, and the change rate of cross-sectional area does not exceed 3/1000. In this case, when calculating $\sigma_s$, $\sigma_{bt}$ or $\sigma_{bc}$, the initial cross section area $(\pi R^2)$ can be approximately adopted.

Considering the $\sigma_s \sim \tau_s$ relationship for mild steel in equation (10) and the $\sigma_{bt} \sim \tau_b$ relationship for gray cast iron in equation (12), the elastic tensile strain $\varepsilon$ range of mild steel and cast iron are about
(1/1000) and \( \nu \approx (\sigma_{bt}/\sigma_{bc}) \approx 0.3 \), the maximum elastic shear strains \( \gamma \) of the two materials are not more than 0.1\(^\circ\), as expressed in equation (46).

\[
\gamma_{\text{steel max}} = \frac{\tau_s}{G} = \frac{2(1+\nu)}{\sqrt{3}} \frac{\sigma_s}{E} (46a)
\]

\[
\gamma_{\text{iron max}} = \frac{\tau_b}{G} = \frac{2(1+\nu)\sigma_{bc}}{\sigma_{bc}+\sigma_{bt}} \frac{\sigma_{bt}}{E} (46b)
\]

According to equations (6) and (24), for rectangular section steel beams with width \( b \) and height \( h \) subjected to pure bending, the maximum elastic bending moment \( M_{se} \) and torsion angle \( \theta_{se} \) that the rectangular beam can withstand are:

\[
M_{se} = \sigma_s bh^2/6 (47)
\]

\[
\theta_{se} = \frac{(2\sigma_s L)}{(Eh)} (48)
\]

From equation (48), for a rectangular section bending steel beam with width \( b \) and height \( h \), the minimum radius of curvature in elastic deformation range is \( \rho = (l/\theta_{se}) \approx 500h \).

**Discussion 7: Deformations of laminated beams**

The purpose of bending normal stress measurement of pure bending beam is to verify the rationality of equation (6) under plane section assumption. And then, laminated beams problem is presented to discuss the with the students.

As shown in Figure 6, the double layer laminated beams with the same material is used, the width of the beams are \( b \), and the heights of the beams are \( h_1 \) and \( h_2 \), and there is no friction between the beams contact surface. When the two beams are subjected to bending moment \( M_1 \) and \( M_2 \) respectively and reach the same curvature radius \( (1/\rho) \), the bending moments of the two beams are:

\[
12M_1 \left( \frac{Eh_1^3}{Eh_2^3} \right) = 1/\rho = 12M_2 \left( \frac{Eh_1^3}{Eh_2^3} \right) (49)
\]

Namely, the bending moment of each beam is in positive proportion to the cube of beam height. Then, according to equation (17b), the gradient of bending normal stress \( \sigma \) along the beam height \( y \) is:

\[
d\sigma/dy = 12M/(bh^3) (50)
\]

By combining equations (49) and (50), it can be seen that the normal stress gradients \((d\sigma/dy)\) of cross section for each layer beam are the same. When a whole beam of the same material with width \( b \) and height \( (h_1 + h_2) \) reaches the same curvature \( (1/\rho) \), the bending moment \( M \) is:

\[
1/\rho = \frac{12M}{Eh(b(h_1+h_2))^3} (51)
\]

By comparing equations (49) and (51), it is easy to obtain that:

\[
\frac{M}{M_1 + M_2} = \frac{(h_1+h_2)^3}{h_1^3 + h_2^3} > 1 (52)
\]

Inequality relation \( M > (M_1 + M_2) \) shows that after splitting a whole beam to form laminated beams, the bending moment required for the same deformation decreases, and also the load capacity of the beam decreases. According to the comprehension “relaxation of constraints leads to the decrease of stiffness”, it is easy for students to infer as follows: compared with the laminated beams, the bending whole beam has some internal constraints at the position of contact surface of laminated beams. However, from a macroscopic point of view, it is impossible to have an antisymmetric shear force in a symmetrical structure with symmetrical loads.

From the knowledge of *mechanics of materials*, it can be clearly known that there is no shear stress in the pure bending beam. The key to these problems lies in whether or not “the same macroscopic
“deformation” occurs in the whole beam and laminated beams. In fact, “curvature $1/\rho$ for each laminated beam” and “curvature $1/\rho$ for a whole beam” are not ‘the same macroscopic deformation’. Therefore, the corresponding load $(M_1 + M_2)$ is different from $M$ surely.

Discussion 8: Stress circle and strain circle

Measuring principal stresses of beam under bending and torsion combined loading need the application of strain circle and stress circle. Some scholars have made a comparative discussion on both of them specially \[9\]. Considering that the part of strain circle is usually omitted in mechanics of materials, students can be guided to understand it in this experiment.

The stress circle equation expressed by principal stress and the strain circle equation expressed by principal strain are shown in equations (53) and (54) respectively:

\[
\begin{align*}
\sigma - \frac{\sigma_1 + \sigma_2}{2} + \tau \frac{\sqrt{2}}{2} &= \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 \\
\varepsilon - \frac{\varepsilon_1 + \varepsilon_2}{2} + \gamma \frac{\sqrt{2}}{2} &= \left(\frac{\varepsilon_1 - \varepsilon_2}{2}\right)^2
\end{align*}
\]

(53)

(54)

The stress and strain are connected by Hooke's law, which is described as:

\[
\begin{align*}
\sigma_1 &= \frac{E}{1-\nu^2}\left(\varepsilon_1 + \nu \varepsilon_3\right) \\
\sigma_3 &= \frac{E}{1-\nu^2}\left(\nu \varepsilon_1 + \varepsilon_3\right)
\end{align*}
\]

(55)

According to equations (54) and (55), the relationship of abscissa of circle center and radius between the strain circle and the stress circle are:

\[
\begin{align*}
\varepsilon_1 + \varepsilon_3 &= \frac{1-\nu}{E} \cdot \frac{\sigma_1 + \sigma_2}{2} \\
\varepsilon_1 - \varepsilon_3 &= \frac{1+\nu}{E} \cdot \frac{\sigma_1 - \sigma_2}{2}
\end{align*}
\]

(56)

From equation (56), if the strain and stress circles are plot on paper in the same size, namely $(1 + \nu) = E$ as shown in Figure 7, then the abscissa of strain circle center $|OO_\varepsilon|$ and stress circle center $|OO_\sigma|$ are satisfied the following requirements:

\[
|OO_\varepsilon| = \frac{1-\nu}{1+\nu} |OO_\sigma|
\]

(57)

Figure 7. Relationship of stress circle and strain circle
According to the stress and strain circle, the principal stresses $\sigma_1$ and $\sigma_3$ can be obtained from three known stresses $\sigma_6$, $\tau_{xy}$ and $\sigma_7$; and the principal strains $\varepsilon_1$ and $\varepsilon_3$ can be obtained from three known strains $\varepsilon_6$, $\gamma_{xy}$ and $\varepsilon_7$. However, strain gauges can only measure the normal strain. Therefore, in order to determine the principal strains $\varepsilon_1$ and $\varepsilon_3$, it is necessary to measure three normal strains $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ at the same point in different directions by three strain gauges. Then the principal stress $\sigma_1$ and $\sigma_3$ is calculated according to Hooke's Law (55). As shown in Figure 8, the three direct strains $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$, and the principal strains $\varepsilon_1$ and $\varepsilon_3$, satisfied equation (58).

\[
\frac{\varepsilon_1 + \varepsilon_3}{2} + \left(\frac{\varepsilon_1 - \varepsilon_3}{2}\right)^2 = \left(\frac{\varepsilon_2}{2}\right)^2 \quad (58)
\]

From equations (55) and (58), the principal stresses $\sigma_1$ and $\sigma_3$ can be obtained. Furthermore, there are 2 value difference terms, $(\varepsilon_1 - \varepsilon_2)^2$ and $(\varepsilon_3 - \varepsilon_2)^2$, in equations (7) and (8), which increase the experimental error. Thus it is beneficial to properly set $M$ and $T$ to make one of the value difference terms, say $(\varepsilon_3 - \varepsilon_2)^2$, to be zero. According to equation (59), $M = T$ leads to $(\varepsilon_3 - \varepsilon_2)^2 = 0$.

\[
\frac{M}{t_y} R = \sigma_x = 2 \tau_{xy} = 2 \frac{T}{t_p} R \quad (59)
\]

The following is an example of the expression of $\sigma_1$ in equation (7) to analyze the benefits to do so. The absolute error limits of measured values $(\varepsilon_1 + \varepsilon_3)$, $(\varepsilon_1 - \varepsilon_2)$ and $(\varepsilon_3 - \varepsilon_2)$ are $\delta$, then the absolute error limits $\Delta$ of $\sigma_1$ are expressed as:\

\[
\Delta = E \left[ \frac{1}{2(1-\nu)} + \frac{1}{\sqrt{2(1+\nu)}} (a + b) \right] \cdot \delta \quad (60)
\]

In equation (60), $a = \frac{1}{2\nu} \left( (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_3 - \varepsilon_2)^2 \right)^{1/2}$ and $b = \frac{\varepsilon_3 - \varepsilon_2}{\sqrt{\varepsilon_3 - \varepsilon_2}}$, $a^2 + b^2 = 1$, and in general $a + b > 1$. When $(\varepsilon_3 - \varepsilon_2) = 0$, $a + b = 1$. At this time, the absolute error limit $\Delta$ of $\sigma_1$ in equation (60) is the minimum.

4. Conclusion

According to the experience in the teaching practice of theoretical and experimental courses of mechanics of materials, the author supposes to add a part of discussion in experimental teaching. This discussion will not only increase the relationship between theory and experiment, but also guide the students to understand and master mechanics of materials with strong engineering flavor. 

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