Minimum-error discrimination of quantum states: New bounds and comparison

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\textsuperscript{*}This work is supported by the National Natural Science Foundation (Nos. 60573006, 60873055), the Research Foundation for the Doctoral Program of Higher School of Ministry of Education (No. 20050558015), and NCET of China.
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Abstract

The minimum-error probability of ambiguous discrimination for two quantum states is the well-known Helstrom limit presented in 1976. Since then, it has been thought of as an intractable problem to obtain the minimum-error probability for ambiguously discriminating arbitrary \( m \) quantum states. In this paper, we obtain a new lower bound on the minimum-error probability for ambiguous discrimination and compare this bound with six other bounds in the literature. Moreover, we show that the bound between ambiguous and unambiguous discrimination does not extend to ensembles of more than two states. Specifically, the main technical contributions are described as follows: (1) We derive a new lower bound on the minimum-error probability for ambiguous discrimination among arbitrary \( m \) mixed quantum states with given prior probabilities, and we present a necessary and sufficient condition to show that this lower bound is attainable. (2) We compare this new lower bound with six other bounds in the literature in detail, and, in some cases, this bound is optimal. (3) It is known that if \( m = 2 \), the optimal inconclusive probability of unambiguous discrimination \( Q_U \) and the minimum-error probability of ambiguous discrimination \( Q_E \) between arbitrary given \( m \) mixed quantum states have the relationship \( Q_U \geq 2Q_E \). In this paper, we show that, however, if \( m > 2 \), the relationship \( Q_U \geq 2Q_E \) may not hold again in general, and there may be no supremum of \( Q_U/Q_E \) for more than two states, which may also reflect an essential difference between discrimination for two-states and multi-states. (4) A number of examples are constructed.

Index Terms—Quantum state discrimination, quantum state detection, ambiguous discrimination, unambiguous discrimination, quantum information theory

I. Introduction

A fundamental issue in quantum information science is that nonorthogonal quantum states cannot be perfectly discriminated, and indeed, motivated by the study of quantum communication and quantum cryptography [1], distinguishing quantum states has become a more and more important subject in quantum information theory [2–9]. This problem may be roughly described by the connection between quantum communication and quantum state discrimination in this manner [2,3,6,8,9]: Suppose that a transmitter, Alice, wants to convey classical information to a receiver, Bob, using a quantum channel, and Alice represents the message conveyed as a mixed quantum state that, with given prior probabilities, belongs to a finite set of mixed quantum states, say \( \{ \rho_1, \rho_2, \ldots, \rho_m \} \); then Bob identifies the state by a measurement.

As it is known, if the supports of mixed states \( \rho_1, \rho_2, \ldots, \rho_m \) are not mutually orthogonal, then
Bob cannot reliably identify which state Alice has sent, namely, $\rho_1, \rho_2, \ldots, \rho_m$ cannot be faithfully distinguished [2,8,9]. However, it is always possible to discriminate them in a probabilistic means. To date, there have been many interesting results concerning quantum state discrimination, we may refer to [3,4,6,10] and the references therein. It is worth mentioning that some schemes of quantum state discrimination have been experimentally realized (for example, see [11–13] and the detailed review in [6]).

Various strategies have been proposed for distinguishing quantum states. Assume that mixed states $\rho_1, \rho_2, \ldots, \rho_m$ have the a priori probabilities $p_1, p_2, \ldots, p_m$, respectively. In general, there are three fashions to discriminate them. The first approach is ambiguous discrimination (also called quantum state detection) [2,8,9] that will be further studied in this paper, in which inconclusive outcome is not allowed, and thus error may result. A measurement for discrimination consists of $m$ measurement operators (e.g., positive semidefinite operators) that form a resolution of the identity on the Hilbert space spanned by the all eigenvectors corresponding to all nonzero eigenvalues of $\rho_1, \rho_2, \ldots, \rho_m$. Much work has been devoted to devising a measurement maximizing the success probability (i.e., minimizing the error probability) for detecting the states [14–18].

The first important result is the pioneering work by Helstrom [2]—a general expression of the minimum achievable error probability for distinguishing between two mixed quantum states. For the case of more than two quantum states, some necessary and sufficient conditions have been derived for an optimum measurement maximizing the success probability of correct detection [8,9,15]. However, analytical solutions for an optimum measurement have been obtained only for some special cases (see, for example, [19–21]).

Regarding the minimum-error probability for ambiguous discrimination between arbitrary $m$ mixed quantum states with given prior probabilities, Hayashi et al. [22] gave a lower bound in terms of the individual operator norm. Recently, Qiu [10] obtained a different lower bound by means of pairwise trace distance. When $m = 2$, these two bounds are precisely the well-known Helstrom limit [2]. Afterwards, Montanaro [23] derived another lower bound by virtue of pairwise fidelity. However, when $m = 2$, the lower bound in [23] is smaller than Helstrom limit. Indeed, it is worth mentioning that, with a lemma by Nayak and Salzman [24], we can also obtain a different lower bound represented by the prior probabilities (we will review these bounds in detail in Section II). Besides this, there also exist the other lower bounds [25,26], and upper bounds [27,28].

The second approach is the so-called unambiguous discrimination [3,29–33], first suggested by Ivanovic, Dicks, and Peres [29–31] for the discrimination of two pure states. In contrast to ambiguous discrimination, unambiguous discrimination allows an inconclusive result to be returned, but no error occurs. In other words, this basic idea for distinguishing between $m$ pure states is to devise a measurement that with a certain probability returns an inconclusive result, but, if the measurement returns an answer, then the answer is fully correct. Therefore, such a measurement consists of $m+1$ measurement operators, in which a measurement operator returns an inconclusive outcome. Analytical solutions for the optimal failure probabilities have been given for distinguishing between two and three pure states [29–34]. Chefles [35] showed that a set of pure states is amendable to unambiguous discrimination if and only if they are linearly independent. The opti-
mal unambiguous discrimination between linearly independent symmetric and equiprobable pure states was solved in [36]. A semidefinite programming approach to unambiguous discrimination between pure states has been investigated in detail by Eldar [37]. Some upper bounds on the optimal success probability for unambiguous discrimination between pure states have also been presented (see, for example, [6, 38–40] and references therein).

We briefly recollect unambiguous discrimination between mixed quantum states. In [41, 42], general upper and lower bounds on the optimal failure probability for distinguishing between two and more than two mixed quantum states have been derived. The analytical results for the optimal unambiguous discrimination between two mixed quantum states have been derived in [43, 44]. For more work regarding unambiguous discrimination, we may refer to [4, 6].

The third strategy for discrimination combines the former two methods [45–47]. That is to say, under the condition that a fixed probability of inconclusive outcome is allowed to occur, one tries to determine the minimum achievable probability of errors for ambiguous discrimination. Such a scheme for discriminating pure states has been considered in [45, 46], and, for discrimination of mixed states, it was dealt with in [47]. Indeed, by allowing for an inconclusive result occurring, then one can obtain a higher probability of correct detection for getting a conclusive result, than the probability of correct detection attainable without inconclusive results appearing [45–47].

In this paper, we derive a new lower bound on the minimum-error probability for ambiguous discrimination between arbitrary $m$ mixed quantum states with given prior probabilities. We show that this bound improves, in some cases, the previous six lower bounds in the literature, and also it better the one derived in [10]. Also, we further present a necessary and sufficient condition to show how this new lower bound is attainable.

It is known that if $m = 2$, the optimal inconclusive probability of unambiguous discrimination $Q_U$ and the minimum-error probability of ambiguous discrimination $Q_E$ have the relationship $Q_U \geq 2Q_E$ [48]. For $m > 2$, it was proved in [10] that $Q_U \geq 2Q_E$ holds only under the restricted condition of the minimum-error probability attaining the bound derived in [10] (this restriction is rigorous). In this paper, we show that, however, for $m > 2$, the relationship $Q_U \geq 2Q_E$ does not hold in general, which may also reflect an essential difference between discrimination of two-states and multi-states.

The remainder of the paper is organized as follows. In Section II, we review six of the existing lower bounds on the minimum-error probability for ambiguous discrimination between arbitrary $m$ mixed states and also give the new bound in this paper that will be derived in the next section. Then, in Section III, we present the new lower bound on the minimum-error probability for ambiguous discrimination between arbitrary $m$ mixed states, and we give a necessary and sufficient condition to show how this new lower bound is attainable. Furthermore, in Section IV, we show that this new bound improves the previous one in [10]. In particular, we try to compare these seven different lower bounds reviewed in Section II with each other. Afterwards, in Section V, we show that, for $m > 2$, the relationship $Q_U \geq 2Q_E$ does not hold in general, where $Q_U$ and $Q_E$ denote the optimal inconclusive probability of unambiguous discrimination and the minimum-error probability of ambiguous discrimination between arbitrary $m$ mixed quantum
II. Reviewing the lower bounds on the minimum-error probability

In this section, we review six of the existing lower bounds on the minimum-error probability for ambiguous discrimination between arbitrary $m$ mixed states. Also, we present the new bound in this paper, but its proof is deferred to the next section.

Assume that a quantum system is described by a mixed quantum state, say $\rho$, drawn from a collection $\{\rho_1, \rho_2, \ldots, \rho_m\}$ of mixed quantum states on an $n$-dimensional complex Hilbert space $\mathcal{H}$, with the a priori probabilities $p_1, p_2, \ldots, p_m$, respectively. We assume without loss of generality that the all eigenvectors of $\rho_i$, $1 \leq i \leq m$, span $\mathcal{H}$, otherwise we consider the spanned subspace instead of $\mathcal{H}$. A mixed quantum state $\rho$ is a positive semidefinite operator with trace 1, denoted $\text{Tr}(\rho) = 1$. (Note that a positive semidefinite operator must be a Hermitian operator [49, 50].) To detect $\rho$, we need to design a measurement consisting of $m$ positive semidefinite operators, say $\Pi_i, 1 \leq i \leq m$, satisfying the resolution

$$\sum_{i=1}^{m} \Pi_i = I, \quad (1)$$

where $I$ denotes the identity operator on $\mathcal{H}$. By the measurement $\Pi_i, 1 \leq i \leq m$, if the system has been prepared by $\rho$, then $\text{Tr}(\rho \Pi_i)$ is the probability to deduce the system being state $\rho_i$. Therefore, with this measurement the average probability $P$ of correct detecting the system’s state is as follows:

$$P = \sum_{i=1}^{m} p_i \text{Tr}(\rho_i \Pi_i) \quad (2)$$

and, the average probability $Q$ of erroneous detection is then as

$$Q = 1 - P = 1 - \sum_{i=1}^{m} p_i \text{Tr}(\rho_i \Pi_i). \quad (3)$$

A main objective is to design an optimum measurement that minimizes the probability of erroneous detection. As mentioned above, for the case of $m = 2$, the optimum detection problem has been completely solved by Helstrom [4], and the minimum attainable error probability, say $Q_E$, is by the Helstrom limit [4]

$$Q_E = \frac{1}{2}(1 - \text{Tr}|p_2 \rho_2 - p_1 \rho_1|), \quad (4)$$

where $|A| = \sqrt{A^\dagger A}$ for any linear operator $A$, and $A^\dagger$ denotes the conjugate transpose of $A$.

For discriminating more than two states, some bounds have been obtained [10, 22–28], and we review six [10, 22–26] of them in the following. We first give a lower bound, and it follows from the following lemma that is referred to [24] by Nayak and Salzman.
Lemma 1 ([24]). If $0 \leq \lambda_i \leq 1$, and $\sum_{i=1}^{m} \lambda_i \leq l$, then $\sum_{i=1}^{m} p_i \lambda_i \leq Pr(\{p_i\}, l)$, where \{p_1, p_2, \ldots, p_m\} is a probability distribution, and $Pr(\{p_i\}, l)$ denotes the sum of the $l$ comparatively larger probabilities of \{p_1, p_2, \ldots, p_m\} (e.g., if $p_{i_1} \geq p_{i_2} \geq \ldots \geq p_{i_m}$ and $l \leq m$, then $Pr(\{p_i\}, l) = \sum_{k=1}^{l} p_{i_k}$).

From this lemma it follows a lower bound on the minimum-error probability for ambiguous discrimination between \{ρ_1, ρ_2, \ldots, ρ_m\} with the a priori probabilities $p_1, p_2, \ldots, p_m$. We first recall the operator norm and trace norm of operator $A$. $\|A\|$ denotes the operator norm of $A$, i.e., $\|A\| = \max\{\|A\phi\| : \psi \in S\}$, where $S$ is the set of all unit vectors, that is to say, $\|A\|$ is the largest singular value of $A$. $\|A\|_{tr} = \text{Tr}\sqrt{A^*A}$ denotes the trace norm of $A$, equivalently, $\|A\|_{tr}$ is the sum of the singular values of $A$.

Theorem 2. For any $m$ mixed quantum states $\rho_1, \rho_2, \ldots, \rho_m$ with a priori probabilities $p_1, p_2, \ldots, p_m$, respectively, then the minimum-error probability $Q_E$ satisfies $Q_E \geq L_0$, where

$$L_0 = 1 - Pr(\{p_i\}, d), \quad (5)$$

and $d$ denotes the dimension of the Hilbert space spanned by $\{\rho_i\}$.

Proof. Let $P_S$ denote the optimal correct probability, and let $\mathbb{E}_m$ denote the class of all POVM of the form $\{E_i : 1 \leq i \leq m\}$. Due to

$$\sum_{i=1}^{m} \text{Tr}(\rho_i E_i) \leq \sum_{i=1}^{m} \|\rho_i\| \cdot \|E_i\|_{tr} = \sum_{i=1}^{m} \|E_i\|_{tr} = \sum_{i=1}^{m} \text{Tr}(E_i) = \text{Tr}(I) = d, \quad (6)$$

and with Lemma 1 we have

$$\sum_{i=1}^{m} p_i \text{Tr}(\rho_i E_i) \leq Pr(\{p_i\}, d). \quad (7)$$

We get

$$P_S = \max_{\{E_i\} \in \mathbb{E}_m} \sum_{i=1}^{m} p_i \text{Tr}(\rho_i E_i) \leq Pr(\{p_i\}, d), \quad (8)$$

Thus, we have

$$Q_E = 1 - P_S \geq 1 - Pr(\{p_i\}, d). \quad (9)$$

The proof is completed. □

Another lower bound $L_1$ was given by Hayashi et al. [22] in terms of the individual operator norm. That is,

$$L_1 = 1 - d \max_{i=1, \ldots, m} \{||p_i \rho_i||\}, \quad (10)$$

where $d$, as above, is the dimension of the Hilbert space spanned by $\{\rho_i\}$. It is easily seen that $L_1$ may be negative for discriminating some states.
Recently, Qiu [10] gave a lower bound $L_2$ in terms of pairwise trace distance, i.e.,

$$L_2 = \frac{1}{2} \left( 1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}[p_j\rho_j - p_i\rho_i] \right). \quad (11)$$

Then, Montanaro [23] derived a lower bound $L_3$ in terms of pairwise fidelity, that is,

$$L_3 = \sum_{1 \leq i < j \leq m} p_ip_j F^2(\rho_i, \rho_j), \quad (12)$$

where, also in this paper, $F(\rho_i, \rho_j) = \text{Tr} \sqrt{\rho_i \rho_j \sqrt{\rho_i}}$ as usual [50].

In this paper, we will derive a new lower bound $L_4$ in terms of trace distance. More exactly,

$$L_4 = 1 - \min_{k=1, \ldots , m} \left( p_k + \sum_{j \neq k} \text{Tr}(p_j\rho_j - p_k\rho_k)_+ \right), \quad (13)$$

where $(p_j\rho_j - p_k\rho_k)_+$ denotes the positive part of a spectral decomposition of $p_j\rho_j - p_k\rho_k$. The proof for deriving $L_4$ is deferred to Section III.

Besides, Tyson [26] derived a lower bound $L_5$, that is,

$$L_5 = 1 - \text{Tr} \left( \sum_{i=1}^{m} p_i^2 \rho_i^2 \right). \quad (14)$$

Montanaro [25] derived a lower bound of pure states discrimination. For discriminating pure states $\{|\psi_i\rangle\}$ with a priori probabilities $p_i$, the minimum error probability satisfy

$$Q_E^* \geq 1 - \sqrt{\sum_{i=1}^{m} (\langle \psi_i^\prime | \rho^{-\frac{1}{2}} | \psi_i^\prime \rangle)^2}, \quad (15)$$

where $|\psi_i^\prime\rangle = \sqrt{p_i} |\psi_i\rangle$ and $\rho = \sum_{i=1}^{m} |\psi_i^\prime\rangle \langle \psi_i^\prime|$. By the following lemma that is referred to Tyson [26], a mixed state lower bound can be obtained from the pure-state lower bound.

**Lemma 3** ([26]). Take spectral decompositions $\rho_i = \sum_k \lambda_{ik} |\psi_{ik}\rangle \langle \psi_{ik}|$, and consider the pure-state ensemble $\xi^* = \{(|\psi_{ik}\rangle, p_i\lambda_{ik})\}$. Then the minimum error probability $Q_E^*$ for discriminating $\xi^*$ satisfies

$$Q_E \leq Q_E^* \leq (2 - Q_E)Q_E. \quad (16)$$

From the above lemma, we can get

$$Q_E \geq 1 - \sqrt{1 - Q_E^*}. \quad (17)$$

So, we get a lower bound for discriminating mixed state $\{\rho_i\}$, that is

$$Q_E \geq 1 - \sqrt{\sum_{i=1}^{m} \sum_{k=1}^{\text{rank}(\rho_i)} (\langle \psi_{ik}^\prime | \rho^{-\frac{1}{2}} | \psi_{ik}^\prime \rangle)^2}, \quad (18)$$

where $\rho = \sum_i p_i \rho_i$, $|\psi_{ik}^\prime\rangle = \sqrt{p_i} \lambda_{ik} |\psi_{ik}\rangle$, and $\rho_i = \sum_{k=1}^{\text{rank}(\rho_i)} \lambda_{ik} |\psi_{ik}\rangle \langle \psi_{ik}|$. We denote this lower bound as

$$L_6 = 1 - \sqrt{\sum_{i=1}^{m} \sum_{k=1}^{\text{rank}(\rho_i)} (\langle \psi_{ik}^\prime | \rho^{-\frac{1}{2}} | \psi_{ik}^\prime \rangle)^2}. \quad (19)$$
III. A new lower bound and its attainability

In this section, we derive the new lower bound $L_4$ on the minimum-error discrimination between arbitrary $m$ mixed quantum states, and then we give a sufficient and necessary condition to achieve this bound.

The measures (e.g., various trace distances and fidelities) between quantum states are of importance in quantum information [50–53]. Here we first give three useful lemmas concerning the usual trace distance and fidelity. As indicated above, in this paper, $F(\rho, \sigma) = \text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}$.

**Lemma 4** ([50]). Let $\rho$ and $\sigma$ be two quantum states. Then

\[2(1 - F(\rho, \sigma)) \leq \text{Tr}\left|\rho - \sigma\right| \leq 2\sqrt{1 - F^2(\rho, \sigma)}. \tag{20}\]

**Lemma 5** ([10]). Let $\rho$ and $\sigma$ be two positive semidefinite operators. Then

\[\text{Tr}(\rho) + \text{Tr}(\sigma) - 2F(\rho, \sigma) \leq \text{Tr}\left|\rho - \sigma\right| \leq \text{Tr}(\rho) + \text{Tr}(\sigma). \tag{21}\]

In addition, the second equality holds if and only if $\rho \perp \sigma$.

**Definition 1.** Let $A$ be a self-adjoint matrix. Then the positive part is given by

\[A_+ = \sum_{\lambda_k > 0} \lambda_k \Pi_k, \tag{22}\]

where $A = \sum_k \lambda_k \Pi_k$ is a spectral decomposition of $A$.

**Lemma 6.** Let $E$, $\rho$ and $\sigma$ are three positive semidefinite matrices, with $E \leq I$. Then

\[\text{Tr}(E(\rho - \sigma)) \leq \text{Tr}(\rho - \sigma)_+, \tag{23}\]

with equality iff $E$ is of the form

\[E = P^+ + P_2, \tag{24}\]

where $P^+$ is the projection onto the support of $(\rho - \sigma)_+$, and $0 \leq P_2 \leq I$ is supported on the kernel of $(\rho - \sigma)$.

**Proof.** See Appendix A.

The new bound is presented by the following theorem.

**Theorem 7.** For any $m$ mixed quantum states $\rho_1$, $\rho_2$, $\cdots$, $\rho_m$ with a priori probabilities $p_1$, $p_2$, $\cdots$, $p_m$, respectively, then the minimum-error probability $Q_E$ satisfies

\[Q_E \geq L_4 = 1 - \min_{k=1,\ldots,m} \left(p_k + \sum_{j \neq k} \text{Tr}(p_j \rho_j - p_k \rho_k)_+\right). \tag{25}\]
Proof. Let $P_S$ denote the maximum probability and let $E_m$ denote the class of all POVM of the form \( \{E_i : 1 \leq i \leq m\} \). Then we have that, for any \( k \in \{1, 2, \ldots, m\} \),

\[
P_S = \max_{\{E_i\} \in E_m} \sum_{j=1}^{m} \text{Tr}(E_j p_j \rho_j) \tag{26}
\]

\[
= \max_{\{E_i\} \in E_m} \left[ p_k + \sum_{j \neq k} \text{Tr}(E_j (p_j \rho_j - p_k \rho_k)) \right] \tag{27}
\]

\[
\leq p_k + \sum_{j \neq k} \text{Tr}(p_j \rho_j - p_k \rho_k)_+, \tag{28}
\]

where the inequality (28) holds by Lemma 6.

Consequently, we get

\[
P_S \leq \min_{k=1, \ldots, m} \left( p_k + \sum_{j \neq k} \text{Tr}(p_j \rho_j - p_k \rho_k)_+ \right). \tag{29}
\]

Therefore, we conclude that inequality (28) holds by $Q_E = 1 - P_S$. \( \square \)

Remark 1. With Lemma 5, $\text{Tr}|p_j \rho_j - p_i \rho_i| \leq p_i + p_j$, and the equality holds if and only if $\rho_j \perp \rho_i$. Therefore, in Theorem 7, the upper bound on the probability of correct detection between $m$ mixed quantum states satisfies

\[
p_{k_0} + \sum_{j \neq k} \text{Tr}(p_j \rho_j - p_{k_0} \rho_{k_0})_+ \\
= \frac{1}{2} \left[ 1 + \sum_{j \neq k_0} \text{Tr}|p_j \rho_j - p_{k_0} \rho_{k_0}| - (m - 2)p_{k_0} \right] \\
\leq \frac{1}{2} \left[ 1 + \sum_{j \neq k_0} (p_j + p_{k_0}) - (m - 2)p_{k_0} \right] = 1. \tag{30}
\]

By Lemma 5 we further see that this bound is strictly smaller than 1 usually unless $\rho_1, \rho_2, \ldots, \rho_m$ are mutually orthogonal.

Remark 2. When $m = 2$, the lower bound in Theorem 7 is precisely $\frac{1}{2}(1 - \text{Tr}|p_2 \rho_2 - p_1 \rho_1|)$, which is in accord with the well-known Helstrom limit [2]; and indeed, in this case, this bound can always be attained by choosing the optimum POVM: $E_2 = P_{12}^+$ and $E_1 = I - E_2$, here $P_{12}^+$ denotes the projective operator onto the subspace spanned by the all eigenvectors corresponding to all positive eigenvalues of $p_2 \rho_2 - p_1 \rho_1$.

From the proof of Theorem 7, we can obtain a sufficient and necessary condition on the minimum-error probability $Q_E$ attaining the lower bound $L_4$, which is described by the following theorem.

Theorem 8. Equality is attained in the bound (27) iff for some fixed $k$, the operators $\{(p_j \rho_j - p_k \rho_k)_+\}_{j \neq k}$ have mutually orthogonal supports.

Proof. See Appendix B. \( \square \)
IV. Comparisons between the seven different lower bounds

In this section, we compare the seven different lower bounds ($L_i$, $i = 0, 1, 2, 3, 4, 5, 6$) on the minimum-error probability for discriminating arbitrary $m$ mixed quantum states with the a priori probabilities $p_1$, $p_2$, $\cdots$, $p_m$, respectively. Also, when discriminating two states, we consider their relation to Helstrom limit.

First, concerning the relation between $L_4$ and $L_2$, we have the following result.

**Theorem 9.** For any $m$ mixed quantum states $\rho_1$, $\rho_2$, $\cdots$, $\rho_m$ with the a priori probabilities $p_1$, $p_2$, $\cdots$, $p_m$, respectively, the two lower bounds $L_2$ and $L_4$ on the minimum-error probability for ambiguously discriminating these $m$ states have the following relationship

$$L_4 \geq L_2.$$  \hspace{1cm} (31)

**Proof.** First we recall

$$L_4 = 1 - \min_{k=1,\cdots,m} \left( p_k + \sum_{j \neq k} Tr(p_j \rho_j - p_k \rho_k) \right)$$  \hspace{1cm} (32)

$$= \frac{1}{2} \left[ 1 - \min_{k=1,\cdots,m} \left\{ \sum_{j \neq k} Tr|p_j \rho_j - p_k \rho_k| - (m-2)p_k \right\} \right]$$  \hspace{1cm} (33)

and

$$L_2 = \frac{1}{2} \left( 1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} Tr|p_j \rho_j - p_i \rho_i| \right).$$  \hspace{1cm} (34)

Let

$$\min_{k=1,\cdots,m} \left\{ \sum_{j \neq k} Tr|p_j \rho_j - p_k \rho_k| - (m-2)p_k \right\} = \sum_{j \neq k_0} Tr|p_j \rho_j - p_{k_0} \rho_{k_0}| - (m-2)p_{k_0}$$  \hspace{1cm} (35)

for some $k_0 \in \{1,2,\ldots,m\}$. Then

$$L_4 = \frac{1}{2} \left[ 1 - \left( \sum_{j \neq k_0} Tr|p_j \rho_j - p_{k_0} \rho_{k_0}| - (m-2)p_{k_0} \right) \right].$$  \hspace{1cm} (36)

We can obtain the following inequality:

$$2L_4 - 2L_2 \geq \frac{m-2}{2(m-1)} - \frac{m-2}{2(m-1)} \left( \sum_{j \neq k_0} Tr|p_j \rho_j - p_{k_0} \rho_{k_0}| - (m-2)p_{k_0} \right).$$  \hspace{1cm} (37)

The proof of inequality (37) is arranged in Appendix C.

With Lemma 5, we know that $Tr|p_j \rho_j - p_{k_0} \rho_{k_0}| \leq p_j + p_{k_0}$. Therefore, according to the inequality (37), we further have

$$2L_4 - 2L_2$$
We can calculate the eigenvalues of the above matrix as

\[
\frac{m - 2}{2(m - 1)} - \frac{m - 2}{2(m - 1)} \left( \sum_{j \neq k_0} (p_j + p_{k_0}) - (m - 2)p_{k_0} \right) \tag{38}
\]

\[
= \frac{m - 2}{2(m - 1)} - \frac{m - 2}{2(m - 1)} \left[ 1 + (m - 2)p_{k_0} - (m - 2)p_{k_0} \right] \tag{39}
\]

\[
= 0, \tag{40}
\]

Consequently, we conclude that the inequality (31) holds and the proof is completed. \qed

Example 1. Indeed, \(L_4 > L_2\) is also possible for discriminating some states. Let \(p_1 = p_2 = p_3 = \frac{1}{3}\), and \(\rho_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|, \rho_2 = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|2\rangle\langle 2|, \rho_3 = \frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|3\rangle\langle 3|\). Then we can work out directly the seven lower bounds as: \(L_0 = 0, L_1 = 0, L_2 = \frac{5}{36}, L_3 = \frac{1}{24}, L_4 = \frac{7}{36}, L_5 = \frac{13 - \sqrt{37}}{36}\) and \(L_6 = 1 - 4\sqrt{\frac{10}{13}}\). Hence, \(L_4 > L_5 > L_2 > L_6 > L_3 > L_1 = L_0\).

Indeed, the minimum-error probability \(Q_E = \frac{7}{36} = L_4\). We leave the calculation process out here, and we refer to the method of calculation by using Lemma 13 in Section V.

In the sequel, we need another useful lemma.

Lemma 10. Let \(\rho_1\) and \(\rho_2\) be two mixed states, and \(p_1 + p_2 \leq 1\) with \(p_i \geq 0, i = 1, 2\). Then

\[
p_1 + p_2 - 2\sqrt{p_1 p_2} F(\rho_1, \rho_2) \leq \text{Tr}|p_1 \rho_1 - p_2 \rho_2| \leq p_1 + p_2 - 2p_1 p_2 F^2(\rho_1, \rho_2). \tag{41}
\]

Proof. Since \(p_1 \rho_1\) and \(p_2 \rho_2\) are positive semidefinite operators and \(F(p_1 \rho_1, p_2 \rho_2) = \sqrt{p_1 p_2} F(\rho_1, \rho_2)\), we can directly get the first inequality from Lemma 3.

Now, we prove the second inequality. By Uhlmann’s theorem [51, 52], we let \(|\psi_1\rangle\) and \(|\psi_2\rangle\) be the purifications of \(\rho_1\) and \(\rho_2\), respectively, such that \(F(\rho_1, \rho_2) = |\langle \psi_1 | \psi_2 \rangle|\). Since the trace distance is non-increasing under the partial trace [50], we obtain

\[
\text{Tr}|p_1 \rho_1 - p_2 \rho_2| \leq \text{Tr}|p_1 |\psi_1\rangle\langle \psi_1| - p_2 |\psi_2\rangle\langle \psi_2||. \tag{42}
\]

Let \(\{|\psi_1\rangle, |\psi_2\rangle\}\) be an orthonormal basis in the subspace spanned by \(\{|\psi_1\rangle, |\psi_2\rangle\}\). Then \(|\psi_2\rangle\) can be represented as \(|\psi_2\rangle = \cos \theta |\psi_1\rangle + \sin \theta |\psi_1^\perp\rangle\). In addition, we have

\[
\text{Tr}|p_1 |\psi_1\rangle\langle \psi_1| - p_2 |\psi_2\rangle\langle \psi_2|| = \text{Tr} \left| \begin{pmatrix} p_1 - p_2 \cos^2 \theta & -p_2 \cos \theta \sin \theta \\ -p_2 \cos \theta \sin \theta & -p_2 \sin^2 \theta \end{pmatrix} \right|. \tag{43}
\]

We can calculate the eigenvalues of the above matrix as

\[
\frac{1}{2} \left( p_1 - p_2 \pm \sqrt{p_1^2 + p_2^2 - 2p_1 p_2 \cos(2\theta)} \right). \tag{44}
\]

Therefore, we have

\[
\text{Tr} |p_1 |\psi_1\rangle\langle \psi_1| - p_2 |\psi_2\rangle\langle \psi_2|| = \sqrt{p_1^2 + p_2^2 - 2p_1 p_2 \cos(2\theta)}. \tag{45}
\]

Since

\[
2p_1 p_2 F^2(\rho_1, \rho_2) = 2p_1 p_2 |\langle \psi_1 | \psi_2 \rangle|^2 = 2p_1 p_2 \cos^2 \theta, \tag{46}
\]
it suffices to show
\[
\sqrt{p_1^2 + p_2^2 - 2p_1p_2 \cos(2\theta)} \leq p_1 + p_2 - 2p_1p_2 \cos^2 \theta.
\] (47)
That is,
\[
p_1^2 + p_2^2 - 2p_1p_2 \cos(2\theta) \leq (p_1 + p_2 - 2p_1p_2 \cos^2 \theta)^2,
\] (48)
and equivalently,
\[
4p_1p_2 \cos^2 \theta[1 - (p_1 + p_2) + p_1p_2 \cos^2 \theta] \geq 0,
\] (49)
which is clearly true. Consequently, we complete the proof. \qed

When \( m = 2 \), we have the following relations between \( L_2, L_3, L_4 \) and the Helstrom limit \( H \).

**Proposition 11.** When \( m = 2 \),
\[
L_4 = L_2 = H \geq L_3,
\] (50)
where \( H \) is the Helstrom limit \([2]\), that is, \( H = \frac{1}{2}(1 - \text{Tr}[p_1\rho_1 - p_2\rho_2]) \).

**Proof.** It is easy to verify that, when \( m = 2 \), \( L_1 = L_2 = \frac{1}{2}(1 - \text{Tr}[p_1\rho_1 - p_2\rho_2]) = H \), and \( L_3 = p_1p_2F^2(\rho_1,\rho_2) \). As a result, to prove the inequality (50), we should show that \( \frac{1}{2}(1 - \text{Tr}[p_1\rho_1 - p_2\rho_2]) \geq p_1p_2F^2(\rho_1,\rho_2) \). Due to \( p_1 + p_2 = 1 \), according to the second inequality of Lemma 10, we easily get the conclusion, and therefore, (50) holds. \qed

**Remark 3.** From the proof of Lemma 10, we know that when \( m = 2 \), \( L_3 \) is smaller than Helstrom limit unless the mixed states are mutually orthogonal.

Moreover, if we discriminate \( m \) equiprobable mixed states, i.e., the \( m \) mixed states are chosen uniformly at random \( (p_i = \frac{1}{m}; i = 1, 2, \cdots, m) \), then \( L_3 \) and \( L_4 \) have the following relationship.

**Proposition 12.** If \( p_i = \frac{1}{m} \) \( (i = 1, 2 \cdots, m) \), then we have \( L_4 \geq L_3 \).

**Proof.** See Appendix D. \qed

Furthermore, even if the prior probabilities are not equal, under some restricted conditions, \( L_2, L_3 \) and \( L_4 \) also have certain relationships. We present a sufficient condition as follows.

**Proposition 13.** Let \( a_i = \sum_{j \neq i} p_i p_j F^2(\rho_i,\rho_j) \). Then \( L_2, L_3 \) and \( L_4 \) have the following relationship: for any \( m \geq 2 \),
\[
L_2 \geq \frac{1}{m - 1} L_3,
\] (51)
and when \( \max_{i=1,\cdots,m} \{a_i\} \geq \frac{1}{2} \sum_{i=1}^m a_i \), we have
\[
L_4 \geq L_3.
\] (52)
Proof. See Appendix E.

\textbf{Example 2.} $L_0 = L_1 > L_4 > L_3 > L_6 > L_2 > L_5$ is also possible. Let $p_1 = p_2 = p_3 = \frac{1}{3}$, and $\rho_1 = |0\rangle\langle 0|$, $\rho_2 = |+\rangle\langle +|$, $\rho_3 = |1\rangle\langle 1|$, where $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$. Then we can calculate explicitly the values of the seven lower bounds: $L_0 = \frac{1}{3}$, $L_1 = \frac{1}{3}$, $L_2 = \frac{2 - \sqrt{2}}{6}$, $L_3 = \frac{1}{9}$, $L_4 = \frac{2 - \sqrt{2}}{3}$, $L_5 = 0$, and $L_6 = 1 - 4\sqrt{\frac{5 + 2\sqrt{2}}{12}}$.

\textbf{Example 3.} $L_1 = L_4 > L_2 > L_5 > L_6 > L_3 > L_0$ is also possible. Let $p_1 = p_2 = p_3 = \frac{1}{3}$, and $\rho_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$, $\rho_2 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|2\rangle\langle 2|$, $\rho_3 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|3\rangle\langle 3|$. Then we can calculate explicitly the values of the seven lower bounds as: $L_0 = 0$, $L_1 = \frac{1}{3}$, $L_2 = \frac{1}{4}$, $L_3 = \frac{1}{12}$, $L_4 = \frac{1}{3}$, $L_5 = \frac{3 - \sqrt{3}}{6}$ and $L_6 = 1 - \sqrt{\frac{7}{3}}$.

\textbf{Example 4.} $L_4 > L_5 > L_3 > L_2 > L_6 > L_0 > L_1$ is possible. Let $p_1 = \frac{1}{10}$, $p_2 = \frac{1}{10}$, $p_3 = \frac{8}{10}$, and $\rho_1 = \frac{9}{10}|0\rangle\langle 0| + \frac{1}{10}|1\rangle\langle 1|$, $\rho_2 = \frac{9}{10}|0\rangle\langle 0| + \frac{1}{10}|2\rangle\langle 2|$, $\rho_3 = \frac{9}{10}|0\rangle\langle 0| + \frac{1}{10}|3\rangle\langle 3|$. Similarly, we can calculate explicitly the values of the seven lower bounds as: $L_0 = 0$, $L_1 = -\frac{47}{25}$, $L_2 = \frac{1350}{10000}$, $L_3 = \frac{1377}{10000}$, $L_4 = \frac{1800}{10000}$, $L_5 = \frac{90 - 9\sqrt{694}}{100}$ and $L_6 = 1 - \sqrt{\frac{694}{1000}}$.

To sum up, when $m = 2$, we have $L_4 = L_2 = H \geq L_3$ ($\geq$ can be strict for some states), and for any $m$ states, $L_4 \geq L_2$ always holds ($\geq$ can be strict for some states). For the equiprobable case (the prior probabilities are equivalent), $L_4 \geq L_3$ always holds. Besides, in general, there are no absolutely big and small relations between the other bounds, and we have provided a number of examples to verify this result.

\section*{V. Comparison between ambiguous and unambiguous discrimination}

For the sake of readability, we briefly recall the scheme of unambiguous discrimination between mixed quantum states $\{\rho_i : i = 1, 2, \ldots, m\}$ with the \textit{a priori} probabilities $\{p_i : i = 1, 2, \ldots, m\}$, respectively. To distinguish between $\rho_i$ unambiguously, we need to design a measurement consisting of $m + 1$ positive semidefinite operators, say $\Pi_i$, $0 \leq i \leq m$, satisfying the resolution

$$\sum_{i=0}^{m} \Pi_i = I,$$  \hfill (53)

and, for $1 \leq i, j \leq m$, if $i \neq j$,

$$\text{Tr}(\Pi_i \rho_j) = 0.$$  \hfill (54)

$\Pi_0$ is related to the inconclusive result and $\Pi_i$ corresponds to an identification of $\rho_i$ for $1 \leq i \leq m$. Therefore, the average probability $P$ of correctly distinguishing these states is as follows:

$$P = \sum_{i=1}^{m} p_i \text{Tr}(\rho_i \Pi_i)$$  \hfill (55)
and, the average failure (inconclusive) probability \( Q \) is then as

\[
Q = 1 - P = \sum_{i=1}^{m} p_i \text{Tr}(\rho_i \Pi_0) \tag{56}
\]

It is known that if \( m = 2 \), \( Q_U \) and \( Q_E \) have the relationship \( Q_U \geq 2Q_E \) [48]. For \( m \geq 3 \), it was proved that, under the restricted condition of the minimum-error probability attaining \( L_2 \), \( Q_U \geq 2Q_E \) still holds [10]. A natural question is that whether or not it still holds without any restricted condition. In this section, we will prove that, however, for \( m \geq 3 \), it may not hold again in general. We can reuse the states of Example 1 to show this conclusion.

**Example 5.** Suppose that \( \rho_1, \rho_2, \rho_3 \) and \( p_1, p_2, p_3 \) are the same as those in Example 1, that is, \( p_1 = p_2 = p_3 = \frac{1}{3} \), and \( \rho_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \), \( \rho_2 = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|2\rangle\langle 2| \), \( \rho_3 = \frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|3\rangle\langle 3| \). Then, for any POVM \( \{E_1, E_2, E_3\} \), by Lemma 4 we have

\[
\frac{1}{3} \times [\text{Tr}(E_1 \rho_1) + \text{Tr}(E_2 \rho_2) + \text{Tr}(E_3 \rho_3)]
= \frac{1}{3} \times [1 + \text{Tr}(E_2(\rho_2 - \rho_1)) + \text{Tr}(E_3(\rho_3 - \rho_1))]
\leq \frac{1}{3} \times [1 + \frac{1}{2}\text{Tr}|\rho_2 - \rho_1| + \frac{1}{2}\text{Tr}|\rho_3 - \rho_1| ]
= \frac{29}{36} \tag{59}
\]

In particular, when \( E_2 = |2\rangle\langle 2| \), \( E_3 = |3\rangle\langle 3| \), and \( E_1 = I - E_2 - E_3 \), the above becomes an equality. In other words, the average success probability can achieve the upper bound \( \frac{29}{36} \). Therefore, we obtain the minimum-error probability \( Q_E \) as

\[
Q_E = \frac{7}{36} \tag{60}
\]

which, as calculated in Example 1, is equal to the lower bound \( L_4 \), but not equal to the lower bound \( L_2 = \frac{5}{36} \). (In the end of the section, we will recheck that \( Q_E = \frac{7}{36} \) holds exactly.)

Next, we consider the optimal inconclusive probability of unambiguous discrimination \( Q_U \). We have known that unambiguous discrimination should satisfy the following two conditions:

\[
\text{Tr}(\Pi_i \rho_j) = \delta_{ij} p_i \tag{61}
\]

\[
\Pi_0 + \sum_{i=1}^{m} \Pi_i = I \tag{62}
\]

The condition (61) is also equivalent to

\[
\Pi_i \rho_j = 0, \tag{63}
\]

for \( i \neq j, i, j = 1, 2, \cdots, m \).

As a result, in order to unambiguously discriminate the above three states \( \rho_1, \rho_2, \rho_3 \), the POVM will be the form: \( \Pi_1 = \alpha_1|1\rangle\langle 1|, \Pi_2 = \alpha_2|2\rangle\langle 2|, \Pi_3 = \alpha_3|3\rangle\langle 3| \), and \( \Pi_0 = I - \sum_{i=1}^{m} \Pi_i \), where \( 0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1 \). Therefore,

\[
\frac{1}{3} \sum_{i=1}^{3} \text{Tr}(\Pi_i \rho_i) = \frac{1}{3} \times [\frac{1}{2}\alpha_1 + \frac{2}{3}\alpha_2 + \frac{3}{4}\alpha_3] \leq \frac{23}{36} \tag{64}
\]
When $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the above (61) will be an equality. That is to say, the optimal success probability can achieve this bound $\frac{23}{36}$. Therefore, we have the optimal inconclusive probability $Q_U$ of unambiguous discrimination between $\rho_1, \rho_2, \rho_3$ as follows:

$$Q_U = \frac{13}{36}. \quad (65)$$

Consequently, by combining (60) and (65) we have

$$Q_U = \frac{13}{36} \geq 2 \times \frac{7}{36} = 2Q_E. \quad (66)$$

To conclude, $Q_U \geq 2Q_E$ may not hold again if no condition is imposed upon the discriminated states and prior probabilities.

A natural question is what is the supremum of $Q_U/Q_E$ for 3 states or $n$ states? Indeed, motivated by the above Example 5, we can give a more general example to demonstrate that there is no supremum of $Q_U/Q_E$ for more than two states.

**Example 6.** Assume that the three mixed states $\rho_1, \rho_2, \rho_3$ have the a priori probabilities $p_1, p_2, p_3$, respectively, where, for $\alpha, \beta, \gamma \geq 0$, $\rho_1 = \alpha|0\rangle\langle 0| + (1 - \alpha)|1\rangle\langle 1|$, $\rho_2 = \beta|0\rangle\langle 0| + (1 - \beta)|2\rangle\langle 2|$, $\rho_3 = \gamma|0\rangle\langle 0| + (1 - \gamma)|3\rangle\langle 3|$. 

First, we consider the optimal inconclusive probability of unambiguous discrimination $Q_U$. Similar to Example 5, by taking $\Pi_1 = |1\rangle\langle 1|$, $\Pi_2 = |2\rangle\langle 2|$, $\Pi_3 = |3\rangle\langle 3|$, and $\Pi_0 = I - \sum_{i=1}^m \Pi_i = |0\rangle\langle 0|$, we can obtain the optimal inconclusive probability $Q_U$ as

$$Q_U = p_1\alpha + p_2\beta + p_3\gamma. \quad (67)$$

Then, we consider the minimum-error probability of ambiguous discrimination $Q_E$. Note that

$$p_2\rho_2 - p_1\rho_1 = (p_2\beta - p_1\alpha)|0\rangle\langle 0| + p_2(1 - \beta)|2\rangle\langle 2| - p_1(1 - \alpha)|1\rangle\langle 1|, \quad (68)$$

$$p_3\rho_3 - p_1\rho_1 = (p_3\gamma - p_1\alpha)|0\rangle\langle 0| + p_3(1 - \gamma)|3\rangle\langle 3| - p_1(1 - \alpha)|1\rangle\langle 1|, \quad (69)$$

$$p_3\rho_3 - p_2\rho_2 = (p_3\gamma - p_2\beta)|0\rangle\langle 0| + p_3(1 - \gamma)|3\rangle\langle 3| - p_2(1 - \beta)|2\rangle\langle 2|. \quad (70)$$

If we let $p_2\beta \geq p_1\alpha \geq p_3\gamma$, then, similar to Example 5, by taking $E_1 = |1\rangle\langle 1|$, $E_3 = |3\rangle\langle 3|$, we can obtain $E_2 = I - E_1 - E_3 = |2\rangle\langle 2| + |0\rangle\langle 0|$, and

$$Q_E = p_1\alpha + p_3\gamma. \quad (71)$$

Likewise, if $p_1\alpha \geq p_2\beta \geq p_3\gamma$, we can get

$$Q_E = p_2\beta + p_3\gamma, \quad (72)$$

and if $p_3\gamma \geq p_1\alpha \geq p_2\beta$, we have

$$Q_E = p_1\alpha + p_2\beta. \quad (73)$$
In a word, we can always get that

$$Q_E = p_1\alpha + p_2\beta + p_3\gamma - \max\{p_1\alpha, p_2\beta, p_3\gamma\}. \quad (74)$$

Consequently, with (67) we have

$$\frac{Q_U}{Q_E} = \frac{p_1\alpha + p_2\beta + p_3\gamma}{p_1\alpha + p_2\beta + p_3\gamma - \max\{p_1\alpha, p_2\beta, p_3\gamma\}}. \quad (75)$$

Therefore, if we let $p_1\alpha = a, p_2\beta$ and $p_3\gamma$ be infinite small but not zero (As we know, this can be always preserved for appropriate $p_i (i = 1, 2, 3)$ and $\alpha, \beta, \gamma$), then $Q_U/Q_E$ will be infinite large. To conclude, there is no supremum of $Q_U/Q_E$ for more than two states.

**Remark 4.** In fact, by virtue of a sufficient and necessary condition regarding the minimum-error probability of ambiguous discrimination, we can recheck the optimum measurement in Examples 5 and 6.

We recall this condition described by the following lemma, that is from [2, 6, 8, 9, 15].

**Lemma 14** ([2, 6, 8, 9, 15]). $\{E_i : i = 1, \cdots, m\}$ is an optimum measurement for achieving the minimum-error probability of ambiguously discriminating the mixed quantum states $\{\rho_i : i = 1, \cdots, m\}$ with the a priori probabilities $\{p_i : i = 1, \cdots, m\}$, respectively, if and only if

$$R - p_j\rho_j \geq 0, \quad \forall j, \quad (76)$$

where the operator

$$R = \sum_{i=1}^{m} p_i\rho_i E_i \quad (77)$$

is required to be Hermitian.

By utilizing Lemma 14 we can recheck the optimum measurements in Examples 5.

In Example 5, by using the POVM $E_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$, $E_2 = |2\rangle\langle 2|$, $E_3 = |3\rangle\langle 3|$, we obtain that $Q_E = \frac{7}{36}$ is the minimum-error probability for ambiguously discriminating $\rho_1, \rho_2, \rho_3$ with $p_1 = p_2 = p_3 = \frac{1}{3}$. Indeed, such a POVM is optimum by Lemma 14. We can verify that

$$\sum_{i=1}^{3} p_i\rho_i E_i = \frac{1}{6}|0\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1| + \frac{2}{9}|2\rangle\langle 2| + \frac{1}{4}|3\rangle\langle 3| \quad (78)$$

is Hermitian, and

$$\sum_{i=1}^{3} p_i\rho_i E_i - p_1\rho_1 = \frac{2}{9}|2\rangle\langle 2| + \frac{1}{4}|3\rangle\langle 3| \geq 0, \quad (79)$$

$$\sum_{i=1}^{3} p_i\rho_i E_i - p_2\rho_2 = \frac{1}{18}|0\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1| + \frac{1}{4}|3\rangle\langle 3| \geq 0, \quad (80)$$

$$\sum_{i=1}^{3} p_i\rho_i E_i - p_3\rho_3 = \frac{1}{12}|0\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1| + \frac{2}{9}|2\rangle\langle 2| \geq 0. \quad (81)$$
By Lemma[14] we can conclude that $E_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$, $E_2 = |2\rangle\langle 2|$, $E_3 = |3\rangle\langle 3|$ compose an optimum measurement. Therefore, we have the minimum error probability $Q_E = 1 - \sum_i Tr(p_i \rho_i E_i) = 1 - (\frac{1}{3} + \frac{2}{9} + \frac{1}{4}) = \frac{7}{36}$.

In Example 6, we consider three cases:

1) If $\max(p_1 \alpha, p_2 \beta, p_3 \gamma) = p_1 \alpha$, then let $E_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$, $E_2 = |2\rangle\langle 2|$, $E_3 = |3\rangle\langle 3|$. 

2) If $\max(p_1 \alpha, p_2 \beta, p_3 \gamma) = p_2 \beta$, then let $E_1 = |1\rangle\langle 1|$, $E_2 = |0\rangle\langle 0| + |2\rangle\langle 2|$, $E_3 = |3\rangle\langle 3|$. 

3) If $\max(p_1 \alpha, p_2 \beta, p_3 \gamma) = p_3 \gamma$, then let $E_1 = |1\rangle\langle 1|$, $E_2 = |2\rangle\langle 2|$, $E_3 = |0\rangle\langle 0| + |3\rangle\langle 3|$. 

We can verify that

$$\sum_{i=1}^{3} p_i \rho_i E_i = \max(p_1 \alpha, p_2 \beta, p_3 \gamma) |0\rangle\langle 0| + p_1 (1 - \alpha)|1\rangle\langle 1| + p_2 (1 - \beta)|2\rangle\langle 2| + p_3 (1 - \gamma)|3\rangle\langle 3|$$

is Hermitian and, for each case,

$$\sum_{i=1}^{3} p_i \rho_i E_i - p_j \rho_j \geq 0, j = 1, 2, 3. \quad (82)$$

By virtue of Lemma[14] we therefore obtain the minimum-error probability $Q_E = p_1 \alpha + p_2 \beta + p_3 \gamma - \max(p_1 \alpha, p_2 \beta, p_3 \gamma)$.

**VI. Concluding remarks**

Quantum states discrimination is an intriguing issue in quantum information processing [1–8]. In this paper, we have reviewed a number of lower bounds on the minimum-error probability for ambiguous discrimination between arbitrary $m$ quantum mixed states. In particular, we have derived a new lower bound on the minimum-error probability and presented a sufficient and necessary condition for achieving this bound. Also, we have proved that our bound improves the previous one obtained in [10]. In addition, we have compared the new bound with six of the previous bounds, by a series of propositions and examples. Finally, we have shown that, for $m > 2$, the relationship $Q_U \geq 2Q_E$ may not hold again in general, where $Q_U$ and $Q_E$ denote the optimal inconclusive probability of unambiguous discrimination and the minimum-error probability of ambiguous discrimination between arbitrary given $m$ mixed quantum states, respectively. In addition, we have demonstrated that there is no supremum of $Q_U/Q_E$ for more than two states by giving an example. As we know, for $m = 2$, $Q_U \geq 2Q_E$ always holds [48], while for $m > 2$, it holds only under a certain restricted condition [10].

A further problem worthy of consideration is how to calculate the minimum-error probability for ambiguous discrimination between arbitrary $m$ quantum mixed states with the prior probabilities, respectively, and devise an optimum measurement correspondingly. In particular, we would consider the appropriate application of these bounds presented in this paper in quantum communication [24, 54]. Indeed, it is worth mentioning that quantum state discrimination has already been applied to quantum encoding [55].
Appendix A. The proof of Lemma 6

Proof. It is obvious that

\[(\rho - \sigma) \leq (\rho - \sigma)_+\]  \hspace{1cm} (83)

It follows immediately by the positivity of E (or by Lemma 2 of Yuen-Kennedy-Lax [9]) that

\[TrE(\rho - \sigma) \leq TrE(\rho - \sigma)_+.\]  \hspace{1cm} (84)

Since \(E \leq I\), it similarly follows that

\[TrE(\rho - \sigma)_+ \leq Tr(\rho - \sigma)_+,\]  \hspace{1cm} (85)

proving (23). The equality condition is left as an exercise for the reader. \(\Box\)

Appendix B. The proof of Theorem 8

Proof. Suppose for some POVM \(\{E_k\}\), we have equality in

\[Tr \sum E_k \rho_k = Tr \left( \rho_k + \sum_{j \neq k} E_j (\rho_j - \rho_k) \right) \leq Tr \left( \rho_k + \sum_{j \neq k} (\rho_j - \rho_k)_+ \right).\]  \hspace{1cm} (86)

Then by Lemma 5

\[E_j \geq \Pi_+(\rho_j - \rho_k),\]  \hspace{1cm} (87)

where \(\Pi_+(\rho_j - \rho_k)\) is the positive projection onto the positive subspace of \(\rho_j - \rho_k\). If the unit vector \(|\psi\rangle\) is in the support of \((\rho_{j_0} - \rho_k)_+\), then one has

\[1 = |||\psi|||^2 = \sum_j \langle \psi | E_j | \psi \rangle = 1 + \sum_{j \neq j_0} \langle \psi | E_j | \psi \rangle \geq 1.\]  \hspace{1cm} (88)

It follows that \(\langle \psi | E_j | \psi \rangle = 0\) for all \(j \neq j_0\). In particular, the support of \(E_{j_0}\) is orthogonal to the supports of the other \(E_j\)'s.

Conversely, if the supports of the other \((\rho_j - \rho_k)_+\) are mutually orthogonal, then the middle term of (86) attains a maximum for the POVM

\[E_j = \Pi_+(\rho_j - \rho_k), \quad j \neq k\]  \hspace{1cm} (89)

\[E_k = I - \sum_{j \neq k} E_j.\]  \hspace{1cm} (90)

In this case, one has equality of all terms in (86). \(\Box\)
Appendix C. The proof of inequality (37)

Proof. First we recall that

\[ L_4 = 1 - \min_{k=1, \ldots, m} \left( p_k + \sum_{j \neq k} \text{Tr}(p_j \rho_j - p_k \rho_k) \right) \]  

(91)

\[ = \frac{1}{2} \left[ 1 - \left( \sum_{j \neq k_0} \text{Tr}(p_j \rho_j - p_{k_0} \rho_{k_0}) - (m - 2)p_{k_0} \right) \right], \]  

(92)

and

\[ L_2 = \frac{1}{2} \left[ 1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}(p_j \rho_j - p_i \rho_i) \right]. \]  

(93)

Therefore,

\[ 2L_4 - 2L_2 = \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}(p_j \rho_j - p_i \rho_i) - \left( \sum_{j \neq k_0} \text{Tr}(p_j \rho_j - p_{k_0} \rho_{k_0}) - (m - 2)p_{k_0} \right). \]  

(94)

Note that

\[ \sum_{1 \leq i < j \leq m} \text{Tr}(p_j \rho_j - p_i \rho_i) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j \neq i} \text{Tr}(p_j \rho_j - p_i \rho_i) \]  

(95)

and

\[ \sum_{j \neq k_0} \text{Tr}(p_j \rho_j - p_{k_0} \rho_{k_0}) - (m - 2)p_{k_0} = \frac{1}{m} \sum_{i=1}^{m} \left( \sum_{j \neq k_0} \text{Tr}(p_j \rho_j - p_{k_0} \rho_{k_0}) - (m - 2)p_{k_0} \right). \]  

(96)

By combining Eqs. (95, 96) with Eq. (94), we have

\[ 2L_4 - 2L_2 = \frac{1}{2(m-1)} \sum_{i=1}^{m} \sum_{j \neq i} \text{Tr}(p_j \rho_j - p_i \rho_i) - \frac{1}{m} \sum_{i=1}^{m} \left( \sum_{j \neq k_0} \text{Tr}(p_j \rho_j - p_{k_0} \rho_{k_0}) - (m - 2)p_{k_0} \right). \]  

(97)

Furthermore, we can equivalently rewrite Eq. (97) as follows:

\[ 2L_4 - 2L_2 = \frac{1}{2(m-1)} \sum_{i=1}^{m} \left[ \left( \sum_{j \neq i} \text{Tr}(p_j \rho_j - p_i \rho_i) - (m - 2)p_i \right) - \left( \sum_{j \neq k_0} \text{Tr}(p_j \rho_j - p_{k_0} \rho_{k_0}) - (m - 2)p_{k_0} \right) \right] \]

\[ + \frac{m-2}{2(m-1)} - \left( \frac{1}{m} - \frac{1}{2(m-1)} \right) \sum_{i=1}^{m} \left( \sum_{j \neq k_0} \text{Tr}(p_j \rho_j - p_{k_0} \rho_{k_0}) - (m - 2)p_{k_0} \right) \]  

(98)

With Eq. (85) we know that, for any \( i \in \{1, 2, \ldots, m\} \),

\[ \sum_{j \neq i} \text{Tr}(p_j \rho_j - p_i \rho_i) - (m - 2)p_i \geq \sum_{j \neq k_0} \text{Tr}(p_j \rho_j - p_{k_0} \rho_{k_0}) - (m - 2)p_{k_0}. \]  

(99)
Note that \(\frac{1}{m} - \frac{1}{2(m-1)} = \frac{m-2}{2(m-1)}\). Therefore, with Eq. (98) we have

\[
2L_4 - 2L_2 \\
\geq \frac{m-2}{2(m-1)} - \frac{m-2}{2m(m-1)} \sum_{i=1}^{m} \left( \sum_{j \neq k_0} \text{Tr} |p_j \rho_j - p_{k_0} \rho_{k_0}| - (m-2)p_{k_0} \right)
\]

which is the inequality (37) as desired. 

\[\square\]

**Appendix D. The proof of Proposition 12**

**Proof.** If \(p_i = \frac{1}{m} \) \((i = 1, 2 \cdots, m)\), we have

\[
L_3 = \frac{1}{m^2} \sum_{i < j} F^2(\rho_i, \rho_j),
\]

and for any \(k_0 \in \{1, 2, \cdots, m\}\),

\[
L_4 = 1 - \min_{k=1, \cdots, m} \left( p_k + \sum_{j \neq k} \text{Tr} (p_j \rho_j - p_k \rho_k)_+ \right)
\]

\[
= \frac{1}{2} \left[ 1 - \min_{k=1, \cdots, m} \left\{ \sum_{j \neq k} \text{Tr} |p_j \rho_j - p_k \rho_k| - (m-2)p_k \right\} \right]
\]

\[
= \frac{1}{2} \left[ \frac{2m-2}{m} - \frac{1}{m} \min_{k=1, \cdots, m} \left\{ \sum_{j \neq k} \text{Tr} |\rho_j - \rho_k| \right\} \right]
\]

\[
\geq \frac{1}{2} \left[ \frac{2m-2}{m} - \frac{1}{m} \sum_{j \neq k_0} \text{Tr} |\rho_j - \rho_{k_0}| \right]
\]

\[
\geq \frac{1}{2} \left[ \frac{2m-2}{m} - \frac{2}{m} \sum_{j \neq k_0} \sqrt{1 - F^2(\rho_j, \rho_{k_0})} \right],
\]

where the last inequality holds by Lemma 4. Thus, we get

\[
L_4 \geq \frac{1}{2} \left[ \frac{2m-2}{m} - \frac{2}{m} \min_{k=1, \cdots, m} \left\{ \sum_{j \neq k} \sqrt{1 - F^2(\rho_j, \rho_k)} \right\} \right].
\]

Therefore, we have

\[
2m^2(L_4 - L_3) \\
\geq 2m^2 - 2m - 2m \min_{k=1, \cdots, m} \left\{ \sum_{j \neq k} \sqrt{1 - F^2(\rho_j, \rho_k)} \right\} - 2 \sum_{i<j} F^2(\rho_i, \rho_j)
\]

\[
= 2m^2 - 2 \sum_{i=1}^{m} \min_{k=1, \cdots, m} \left\{ \sum_{j \neq k} \sqrt{1 - F^2(\rho_j, \rho_k)} \right\} - \sum_{i=1}^{m} \sum_{j \neq i} F^2(\rho_i, \rho_j)
\]
\[
\begin{align*}
\geq & \quad 2m^2 - 2m - 2 \sum_{i=1}^{m} \sum_{j \neq i} \sqrt{1 - F^2(\rho_j, \rho_i)} - \sum_{i=1}^{m} \sum_{j \neq i} F^2(\rho_i, \rho_j) \\
= & \quad \sum_{i=1}^{m} \sum_{j \neq i} \left( \sqrt{1 - F^2(\rho_j, \rho_i)} - 1 \right)^2 \\
\geq & \quad 0.
\end{align*}
\]

Thus, we have \(L_4 \geq L_3\). We complete the proof. \qed

**Appendix E. The proof of Proposition 13**

*Proof.* By Lemma 10 we have

\[
L_2 = \frac{1}{2} \left( 1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr} |p_{ij} - p_{ii}| \right)
\]

\[
\geq \frac{1}{2} \left( 1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} [p_i + p_j - 2p_i p_j F^2(\rho_i, \rho_j)] \right)
\]

\[
= \frac{1}{m-1} L_3.
\]

For any given \(k_0 = 1, \ldots, m\),

\[
L_4 = 1 - \min_{k=1, \ldots, m} \left( p_k + \sum_{j \neq k} \text{Tr}(p_{ij} p_{ij} - p_{ik} p_{ik}) \right)
\]

\[
= \frac{1}{2} \left[ 1 - \min_{k=1, \ldots, m} \left\{ \sum_{j \neq k} \text{Tr}|p_{ij} - p_{ik}| - (m-2)p_k \right\} \right]
\]

\[
\geq \frac{1}{2} \left[ 1 - \left( \sum_{j \neq k_0} \text{Tr}|p_{ij} - p_{ik_0}| - (m-2)p_{k_0} \right) \right]
\]

\[
= \frac{1}{2} \frac{1}{2} \sum_{j \neq k_0} \text{Tr}|p_{ij} - p_{ik_0}| + \frac{m-2}{2} p_{k_0}
\]

\[
\geq \frac{1}{2} \frac{1}{2} \sum_{j \neq k_0} [p_{k_0} + p_j - 2p_{k_0} p_j F^2(\rho_{k_0}, \rho_j)] + \frac{m-2}{2} p_{k_0}
\]

\[
= \sum_{j \neq k_0} p_{k_0} p_j F^2(\rho_{k_0}, \rho_j).
\]

So, we have

\[
L_4 \geq \max_{k=1, \ldots, m} \left\{ \sum_{j \neq k} p_k p_j F^2(\rho_k, \rho_j) \right\}.
\]

Moreover, we have

\[
L_3 = \sum_{1 \leq i < j \leq m} p_i p_j F^2(\rho_i, \rho_j) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j \neq i} p_i p_j F^2(\rho_i, \rho_j).
\]
Let $a_i = \sum_{j \neq i} p_j F^2(\rho_i, \rho_j)$. Then we get

$$L_4 - L_3 \geq \max_{i=1, \ldots, m} \{a_i\} - \frac{1}{2} \sum_{i=1}^{m} a_i.$$  \hspace{1cm} (125)

If $\max_{i=1, \ldots, m} \{a_i\} - \frac{1}{2} \sum_{i=1}^{m} a_i \geq 0$, then $L_4 \geq L_3$. We complete the proof. \hfill \Box

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