Critical threshold and dynamics of a general rumor model on complex networks

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We derive mean-field equations that describe the dynamics of a general model of rumor spreading on complex networks, and use analytical and numerical solutions of these equations to examine the threshold behavior and dynamics of the model on random graphs, uncorrelated scale-free networks and scale-free networks with assortative degree correlations. We show that in both homogeneous networks and random graphs the model exhibits a critical threshold in the rumor spreading rate below which a rumor cannot propagate in the system. In the case of scale-free networks, on the other hand, this threshold becomes vanishingly small in the limit of infinite system size. We find that the initial rate at which a rumor spreads is much higher in scale-free networks than in random graphs, and that the rate at which the spreading proceeds on scale-free networks is further increased when assortative degree correlations are introduced. The impact of degree correlations on the final fraction of nodes that ever hears a rumor, however, depends on the interplay between network topology and the rumor spreading rate. Our results show that scale-free networks are prone to the spreading of rumors, just as they are to the spreading of infections. They are relevant to the spreading dynamics of chain emails, viral advertising and large-scale data dissemination algorithms on the Internet, the so-called gossip protocols.

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I. INTRODUCTION

Many real-life technological, social and biological systems have complex network-like structures, where vertices represent entities comprising the systems and edges the presence of some form of interaction or relationship between them. Some important examples include the Internet [1, 2], the World Wide Web [3] and social interaction networks [4, 5, 6, 7]. In recent years a significant amount of research has been devoted to empirical and theoretical studies of such networks [8, 9, 10].

A large body of this work has been focused on finding statistical properties, such as degree distributions, average shortest paths, and clustering coefficients that characterize the structure of complex networks. Another active area of research is on creating realistic models of such networks which can help us better understand the underlying organization mechanisms behind their formation and evolution. A third area of research, which is still at its early stages, investigates dynamical processes on complex networks with the aim of understanding the impact of network topology on the dynamics [11, 12, 13, 14, 15].

A prominent example of such process is the spreading of rumors in social networks [12, 13, 15]. Rumors bear an immediate resemblance with epidemics and their spreading plays an important role in a variety of human affairs, such as stock markets [16], politics and warfare. Rumor-like mechanisms also form the basis for the phenomena of viral marketing, where companies exploit social networks of their customers on the Internet in order to promote their products via the so-called ‘word-of-email’ and ‘word-of-web’ [20, 21]. Finally, rumor-mongering forms the basis for an important class of communication protocols, called gossip algorithms, which are used for large-scale information dissemination on the Internet, and in peer-to-peer file sharing applications [22, 23].

Unlike epidemic spreading which has been extensively studied, quantitative models and investigations of rumor dynamics have been rather limited. Furthermore, most existing models either do not take into account the topology of the underlying social interaction networks along which rumors spread, or use highly simplified models of the topology. While such simple models may adequately describe the spreading process in small-scale social networks, via the word-of-mouth, they become highly inadequate when applied to the spreading of rumors in large social interaction networks, in particular those which are formed on the Web. Such networks, which include email networks [24], and social networking sites [7], typically number in tens of thousands to millions of nodes and their topology can show highly complex connectivity patterns [5, 24].

In two previous papers [15, 25] we investigated the dynamics of a classical model of rumor spreading, the so-called Daley-Kendall model [17], on such networks. In this paper we describe a more general model of ru-
mor dynamics on networks, which unifies our previous model with the Susceptible-Infected-Removed (SIR) model of epidemics. We formulate the dynamics of this model within the framework of Interacting Markov Chains (IMC) \[22\], and use this framework to derive a set of equations that describe on a mean-field level the dynamics of the model on complex Markovian networks. Using analytical and numerical solutions of these equations we examine both the steady-state and the time-dependent behavior of the model on Erdős-Rényi (ER) random graphs, as a prototypical example of homogeneous networks, and strongly heterogeneous scale-free (SF) networks, both in the absence and presence of assortative degree correlations.

Our results point out to several important differences in the dynamics of rumor spreading on the above networks. In particular, we find the presence of a critical value for the rumor spreading rate (a rumor threshold) below which rumors cannot spread in random graphs, and its absence in scale-free networks with unbounded fluctuations in node degree distribution. We find that the initial spreading rate of a rumor is much higher on scale-free networks as compared to random graphs, and that this spreading rate is further increased when assortative degree-degree correlations are introduced. The final fraction of nodes which ever hears a rumor (we call this the final size of a rumor), however, depends on the interplay between the model parameters and the underlying topology. Our findings provide a better quantitative understanding of the complex interplay between the characteristics of a rumor, i.e. its spreading and cessation rates, and the topology of the underlying interaction network that supports its propagation. They are relevant to a number of rumor-like processes such as chain emails, viral advertising and large-scale data dissemination in computer and communication networks via gossip protocols.

The rest of this paper is organized as follows. In Section II we describe our general rumor model. In section III a formulation of the model within the framework of Interactive Markov Chains is given, and the corresponding mean-field equations are derived. In section IV analytical results are presented for the case of homogeneous networks, characterized by a constant degree and no degree correlations. This is followed by a numerical study of the model on ER random graphs and both uncorrelated and assortatively correlated SF networks. We close this paper in section V with conclusions.

II. A GENERAL RUMOR MODEL ON NETWORKS

The model we shall consider is defined in the following way. A closed population consisting of \( N \) members is subdivided into three groups, those who have not heard the rumor, those who have heard it and wish to spread it, and those who have heard it but have ceased communicating it. We call these groups ignorants, spreaders and stiflers, respectively. The rumor spreads by directed contact of the spreaders with others in the population, which takes place along the links of an undirected social interaction network \( G = (V, E) \), where \( V \) and \( E \) denote the vertices and the edges of the network, respectively. The outcome of contact of a specified spreader with (a) an ignorant is that the ignorant becomes an spreader at a rate \( \lambda \), and (b) another spreader or a stifler is that the initiating spreader becomes stifler at a rate \( \alpha \). The spreading process starts with one (or more) element(s) becoming informed of a rumor and terminates when no spreaders are left in the population. In the Daley-Kendall rumor model and its variants stifling is the only mechanism that results in cessation of rumor spreading. In reality, however, cessation can occur also purely as a result of spreaders forgetting to tell the rumor, or their disinclination to spread the rumor anymore. Following a suggestion in \[16\], we take this important mechanism into account by assuming that individuals may also cease spreading a rumor spontaneously (i.e. without the need for any contact) at a rate \( \delta \).

III. INTERACTIVE MARKOV CHAIN MEAN-FIELD EQUATIONS

We can describe the dynamics of the above model on a network within the framework of the Interactive Markov Chains (IMC). The IMC was originally introduced in mathematical sociology, as a means for modeling social processes involving many interacting actors (or agents) \[20\]. More recently they have been applied to the dynamics of cascading failures in electrical power networks \[27\], and the spread of malicious software (malware) on the Internet \[28\]. An IMC consists of \( N \) interacting nodes, with each node having a state that evolves in time according to an internal Markov chain. Unlike conventional Markov chains, however, the corresponding internal transition probabilities depend not only on the current state of a node itself, but also on the states of all the nodes to which it is connected. The overall system evolves according to a global Markov Chain whose state space dimension is the product of states describing each node. When dealing with large networks, the exponential growth in the state space renders direct numerical solution of the IMCs extremely difficult, and one has to resort to either Monte Carlo simulations or approximate solutions. In the case of rumor model, each internal Markov chain can be in one of the three states: ignorant, spreader or stifler. For this case we derive below a set of coupled rate equations which describe on a mean-field level the dynamics of the IMC. We note that a similar mean-field approach may also be applicable to other dynamical processes on networks which can be described within the IMC framework.

Consider now a node \( j \) which is in the ignorant state at time \( t \). We denote with \( p^t_{ij} \) the probability that this node
stays in the ignorant state in the time interval \([t, t + \Delta t]\), and with \(p^i_{ij} = 1 - p^s_{ij}\) the probability that it makes a transition to the spreader state. It then follows that

\[
p^i_{ij} = (1 - \Delta t \lambda)^g, \tag{1}
\]

where \(g = g(t)\) denotes the number of neighbors of node \(j\) which are in the spreader state at time \(t\). In order to progress, we shall coarse-grain the micro-dynamics of the system by classifying nodes in our network according to their degree and taking statistical average of the above transition probability over degree classes.

Assuming that a node \(j\) has \(k\) links, \(g\) can be considered as an stochastic variable which has the following binomial distribution:

\[
\Pi(g, t) = \binom{k}{g} \theta(k, t)^g (1 - \theta(k, t))^{k - g}, \tag{2}
\]

where \(\theta(k, t)\) is the probability at time \(t\) that an edge emanating from an ignorant node with \(k\) links points to a spreader node. This quantity can be written as

\[
\theta(k, t) = \sum_{k'} P(k'|k)P(s_{k'|ik}) \approx \sum_{k'} P(k'|k)\rho^s(k', t). \tag{3}
\]

In this equation \(P(k'|k)\) is the degree-degree correlation function, \(P(s_{k'|ik})\) the conditional probability that a node with \(k'\) links is in the spreader state, given that it is connected to an ignorant node with degree \(k\), and \(\rho^s(k', t)\) is the density of spreader nodes at time \(t\) which belong to connectivity class \(k\). In the above equation the final approximation is obtained by ignoring dynamic correlations between the states of neighboring nodes.

The transition probability \(\tilde{p}_{ii}(k, t)\) averaged over all possible values of \(g\) is then given by:

\[
\tilde{p}_{ii}(k, t) = \sum_{g=0}^{k} \binom{k}{g} (1 - \lambda \Delta t)^g \theta(k, t)^g (1 - \theta(k, t))^{k - g} = \left(1 - \lambda \Delta t \sum_{k'} P(k'|k)\rho^s(k', t)\right)^k. \tag{4}
\]

In a similar fashion we can derive an expression for the probability \(\tilde{p}_{ss}(k, t)\) that a spreader node which has \(k\) links stays in this state in the interval \([t, t + \Delta t]\). In this case, however, we also need to compute the expected value of the number of stifler neighbors of the node at time \(t\). Following steps similar to the previous paragraphs we obtain

\[
\tilde{p}_{ss}(k, t) = \left(1 - \alpha \Delta t \sum_{k'} P(k'|k)(\rho^s(k', t) + \rho^r(k', t))\right)^k \times (1 - \delta \Delta t). \tag{5}
\]

The corresponding probability for a transition from the spreader to the stifler state, \(\tilde{p}_{sr}(k, t)\) is given by \(\tilde{p}_{sr}(k, t) = 1 - \tilde{p}_{ss}(k, t)\).

The above transition probabilities can be used to set up a system of coupled stochastic equations for change in time of the population of ignorants, spreaders and stiflers within each connectivity class. However, ignoring fluctuations around expected values we can also obtain a set of deterministic rate equations for the expected values of these quantities. In the limit \(\Delta t \to 0\) these equations are given by

\[
\frac{\partial \rho^i(k, t)}{\partial t} = -k \lambda \rho^i(k, t) \sum_{k'} \rho^s(k', t) P(k'|k) \tag{6}
\]

\[
\frac{\partial \rho^s(k, t)}{\partial t} = k \lambda \rho^i(k, t) \sum_{k'} \rho^s(k', t) P(k'|k) - k \alpha \rho^s(k, t) \sum_{k'} (\rho^s(k', t) + \rho^r(k', t)) P(k'|k) - \delta \rho^s(k, t) \tag{7}
\]

\[
\frac{\partial \rho^r(k, t)}{\partial t} = k \alpha \rho^s(k, t) \sum_{k'} (\rho^s(k', t) + \rho^r(k', t)) P(k'|k) + \delta \rho^s(k, t). \tag{8}
\]

In the above equations \(\rho^i(k, t), \rho^s(k, t),\) and \(\rho^r(k, t)\) are the densities at time \(t\) of, respectively, ignorants, spreaders and stiflers in class \(k\). These quantities satisfy the normalization condition \(\rho^i(k, t) + \rho^s(k, t) + \rho^r(k, t) = 1\).

For future reference we note here that information on the underlying network is incorporated in the above equations solely via the degree-degree correlation function. Thus in our analytical and numerical studies reported in the next section we do not need to generate any actual network. All that is required is either an analytical expression for \(P(k'|k)\) or a numerical representation of this quantity.

IV. RESULTS AND DISCUSSIONS

A. Homogeneous networks

In order to understand some basic features of our rumor model we first consider the case of homogeneous networks, in which degree fluctuations are very small and there are no degree correlations. In this case the rumor equations become:

\[
\frac{d\rho^i(t)}{dt} = -\lambda \bar{k} \rho^i(t) \rho^s(t) \tag{9}
\]

\[
\frac{d\rho^s(t)}{dt} = \lambda \bar{k} \rho^i(t) \rho^s(t) - \alpha \bar{k} \rho^s(t)(\rho^s(t) + \rho^r(t)) - \delta \rho^s(t) \tag{10}
\]

\[
\frac{d\rho^r(t)}{dt} = \alpha \bar{k} \rho^s(t)(\rho^s(t) + \rho^r(t)) + \delta \rho^s(t), \tag{11}
\]

where \(\bar{k}\) denotes the constant degree distribution of the network (or the average value for networks in which the
probability of finding a node with a different connectivity decays exponentially fast).

The above system of equations can be integrated analytically using an standard approach. In the infinite time limit, when \( s(\infty) = 0 \), we obtain the following transcendental equation for \( R = r(\infty) \), the final fraction of nodes which ever hear the rumor (we call this the final rumor size)

\[
R = 1 - e^{-\epsilon R}
\]

(12)

where

\[
\epsilon = \frac{(\alpha + \lambda)\bar{k}}{\delta + \alpha k},
\]

(13)

Eq. (12) admits a non-zero solution only if \( \epsilon > 1 \). For \( \delta \neq 0 \) this condition is fulfilled provided

\[
\frac{\lambda}{\delta} \bar{k} > 1,
\]

(14)

which is precisely the same threshold condition as found in the SIR model of epidemic spreading on homogeneous networks \( ^{12} ^{26} \). On the other hand, in the special case \( \delta = 0 \) (i.e when the forgetting mechanism is absent) \( \epsilon = 1 + \lambda/\alpha > 1 \), and so Eq. (14) always admits a non-zero solution, in agreement with the result in \( ^{27} \).

The above result shows, however, that the presence of a forgetting mechanism results in the appearance of a finite threshold in the rumor spreading rate below which rumors cannot spread in homogeneous networks. Furthermore, the value of the threshold is independent of \( \alpha \) (i.e. the stifling mechanism), and is the same as that for the SIR model of epidemics on such networks. This result can be understood by noting that in the above equations the terms corresponding to the stifling mechanism are of second order in \( \rho^* \), while the terms corresponding to the forgetting mechanism are only of first order in this quantity. Thus in the initial state of spreading process, where \( \rho^* \approx 0 \) and \( \rho^f \approx 0 \), the effect of stifling is negligible relative to that of forgetting, and the dynamics of the model reduces to that of the SIR model.

### B. Random graphs and uncorrelated scale-free networks

Next we numerically investigate the dynamics of rumor spreading on complex networks. We consider first uncorrelated networks, for which the degree-degree correlations can be written as

\[
P(k'|k) = q(k') = \frac{k' P(k')}{\langle k \rangle},
\]

(15)

where \( P(k') \) is the degree distribution and \( \langle k \rangle \) is the average degree. We consider here two classes of such networks. The first class is the Erdős-Rényi random networks, which for large \( N \) have a Poisson degree distribution:

\[
P(k) = e^{-k} \frac{(k)^k}{k!}.
\]

(16)

The above degree distribution peaks at an average value \( \langle k \rangle \) and show small fluctuations around this value. The second class we consider are scale-free networks which are characterized by a highly right-skewed power law degree distribution:

\[
P(k) = \begin{cases} A k^{-\gamma} & k_{\min} \leq k \\ 0 & \text{otherwise}. \end{cases}
\]

(17)

In the above equation \( k_{\min} \) is the minimum degree of the networks and \( A \) is a normalization constant. For \( 2 \leq \gamma \leq 3 \) the variance of the above degree distribution becomes infinite in the limit of infinite system size while the average degree distribution remains finite. We shall consider henceforth SF networks with \( \gamma = 3 \).

Our studies of uncorrelated networks were performed using the above forms of \( P(k) \) to represent ER and SF networks, respectively. The size of the networks considered was \( N = 10^5 \), and the average degree was fixed at \( \langle k \rangle = 7 \). For each network considered we generated a sequence of \( N \) random integers distributed according to its degree distribution. The coupled set of differential equation (6-8) were then solved numerically using an standard finite difference scheme, and numerical convergence with respect to the step size was checked numerically. In the following and throughout the paper all calculations reported are performed by starting the rumor from a randomly chosen initial spreader, and averaging the results over 300 runs with different initial spreaders. The calculations reported below were performed for networks consisting of \( N = 10^6 \) nodes.

In our first set of calculations we set \( \delta = 1 \) and investigate the dynamics as a function of the rumor spreading rate \( \lambda \) and the stifling rate \( \alpha \). First we focus on the impact of network topology on the final size of a rumor, \( R \), which for inhomogeneous networks is obtained from

\[
R = \sum_k \rho^f (k, t_{\infty}),
\]

(18)

where \( t_{\infty} \) denotes a sufficiently long time at which the spreading process has reached its steady state (i.e. no spreaders are left in the network). In Fig. 1 \( R \) corresponding to the ER network is plotted as a function of \( \lambda \), and for several different values of the stifling parameter \( \alpha \). It can be seen that in this network \( R \) exhibits a critical threshold \( \lambda_c \) below which a rumor cannot spread in the network. Furthermore, just as in the case of homogeneous networks, the value of the threshold does not depend on \( \alpha \), and is at \( \lambda_c = 0.12507 \). This value is in excellent agreement with the analytical results for the SIR model on an infinite size ER network, which is given by \( \lambda_c = \langle k \rangle / \langle k^2 \rangle = 0.125 \). We also verified numerically that the behavior of \( R \) in the vicinity of the critical point can be described with the form

\[
R \sim A(\lambda - \lambda_c)^\beta,
\]

(19)

where \( \beta = 1 \), and \( A = A(\alpha) \) is a smooth and monotonically decreasing function of \( \alpha \). The results are shown in
The final size of the rumor, $R$, is shown as a function of the spreading rate $\lambda$ for the ER network of size $10^6$. The results are shown for several values of the stifling parameter $\alpha$.

We further examined the above numerical findings by analytically solving Eqs. (6-8) in the vicinity of the critical rumor threshold, to first order in $\alpha$. Details of the calculations are given in the Appendix, and they confirm the above numerical results for $\lambda_c$, and the behavior of $R$ in the vicinity of the critical rumor threshold.

Next we turn to our results for the SF network. In Fig. 3 results for $R$ in this network are shown. In this case we also observe the presence of an $\alpha$-independent rumor threshold, albeit for much smaller spreading rates than for the ER network. We have verified numerically that in this case the threshold is approached with zero slope, as can also be gleaned from Fig. 3. Since the value of the threshold is independent of $\alpha$, we can use the well-known result for the SIR model (the $\alpha = 0$ case) to conclude that in the limit of infinite system size the threshold seen in the SF network will approach zero. It is therefore not an intrinsic property of rumor spreading on this network.

In order to further analyze the behavior of $R$ in SF networks, we have numerically fitted our results to the stretched exponential form,

$$R \sim \exp(-C/\lambda),$$

with $C$ depending only weakly on $\alpha$. This form was found to describe the epidemic prevalence in both the SIS and the SIR model of epidemics [11, 12]. The results are displayed in Fig. 4, and they clearly indicate that the stretched exponential form also nicely describes the behavior of $R$ in our rumor model. This result provides further support for our conjecture that the general rumor model does not exhibit any threshold behavior on SF networks (at least in the limit of infinite systems size).

In addition to investigating the impact of network topology on the steady-state properties of the model, it is of great interest to understand how the time-dependent behavior of the model is affected by topology. In Figs. 5 and 6 we display, as representative examples, time evolution of, respectively, the total fractions of stiflers and spreaders, in both networks for $\lambda = 1$ and two sets of values of the cessation parameters: $\{\delta = 1, \alpha = 0\}$, and $\{\delta = 0, \alpha = 1\}$. The first set of parameters corresponds to a spreading process in which cessation results purely from spontaneous forgetting of a rumor by spreaders, or their disinclination to spread the rumor any further. The second set corresponds to a scenario where individuals keep spreading the rumor until they become stiflers due to their contacts with other spreaders or stiflers in the network. As can be seen in Fig. 5, in the first scenario the initial spreading rate of a rumor on the SF network is much faster than on the ER network. In fact, we find
that the time required for the rumor to reach 50% of nodes in the ER random graph is nearly twice as long as the corresponding time on the SF networks. This large difference in the spreading rate is due to the presence of highly connected nodes (hubs) in the SF network, whose presence greatly speeds up the spreading of a rumor. We note that in this scenario not only a rumor spreads initially faster on SF networks, but it also reaches a higher fraction of nodes at the end of the spreading process.

It can be seen from Figs. 5 and 6 that in the second spreading scenario (i.e. when stifling is the only mechanism for cessation) the initial spreading rate on the SF network is, once again, higher than on the ER network. However, unlike the previous situation, the ultimate size of the rumor is higher on the ER network. This behavior is due to the conflicting roles that hubs play when the stifling mechanism is switched on. Initially the presence of hubs speeds up the spreading but once they turn into stiflers they also effectively impede further spreading of the rumor.

C. Assortatively correlated scale-free networks

Recent studies have revealed that social networks display assortative degree correlations, implying that highly connected vertices preferentially connect to vertices which are also highly connected [10]. In order to study the impact of such correlations on the dynamics of our model, we make use of the following “local” ansatz for the degree-degree correlation function

$$P(k'|k) = (1 - \beta)g(k') + \beta\delta_{kk'}; \quad (0 \leq \beta < 1). \quad (21)$$

This form has been used previously in recent studies of the SIR dynamics on correlated scale-free networks [31, 32], and allows us to study in a controlled way the impact of degree correlations on the spreading of rumor.

Using the above degree-degree correlation function we numerically solved Eqs. (6-8) for a SF network characterized by $\gamma = 3$ and $< k > = 7$. The network size was fixed at $N = 100,000$, and we used two values for the correlation parameter: $\beta = 0.2$ and $\beta = 0.4$. Fig. 7 displays $R$ as a function of $\lambda$, and for $\alpha = 0.5, 0.75, 1$ (the value of $\delta$ was fixed at 1).

It can be seen that below $\lambda \approx 0.5$ a rumor will reach a somewhat smaller fraction of nodes on the correlated networks than on the uncorrelated ones. However for larger values of $\lambda$ this behavior reverses, and the final size of the rumor in assortatively correlated networks shows a
higher value than in the uncorrelated network. We thus conclude that the qualitative impact of degree correlations on the final size of a rumor depends very much on the rumor spreading rate. We also investigated the effect of assortative correlations on the dynamics (temporal behavior) of rumor spreading and found that such correlations slightly increase the initial rate of spreading.

V. CONCLUSIONS

In this paper we introduced a general model of rumor spreading on complex networks. Unlike previous rumor models, our model incorporates two distinct mechanisms that cause cessation of a rumor, stifling and forgetting. We used an Interactive Markov Chain formulation of the model to derive deterministic mean-field equations for the dynamics of the model on Markovian complex networks. Using these equations we investigated analytically and numerically the behavior of the model on Erdős-Rényi random graphs and scale-free networks with exponent \( \gamma = 3 \).

Our results show the presence of a critical threshold in the rumor spreading rate below which a rumor cannot spread in ER networks. The value of this threshold was found to be independent of the stifling mechanism, and to be the same as the critical infection rate of the SIR epidemic model. Such a threshold is also present in the finite-size SF networks we studied, albeit at a much smaller value. However in SF networks this threshold is reached with a zero slope and its value becomes vanishingly small in the limit of infinite network size. We also found the initial rate of spreading of a rumor to be much higher on scale-free networks than on ER random graphs. An effect which is caused by the presence of hubs in these networks, which efficiently disseminate a rumor once they become informed. Our results show that SF networks are prone to the spreading of rumors, just as they are to the spreading of infections.

Finally we used a local ansatz for the degree-degree correlation function in order to numerically investigate the impact of assortative degree correlations on the dynamics of our model on SF networks. These correlations were found to increase slightly the initial rate of spreading in SF networks. However, their impact on the final fraction of nodes which hear a rumor depends very much on the rate of rumor spreading.

The basic assumption underlying the mean-field equations derived in this paper is that all vertices within a given degree class can be considered statistically equivalent. Therefore our results are not directly applicable to structured networks in which a distance or time ordering can be introduced, or there is a high level of clustering. We are currently working on more elaborate approximations of our model which could take into account such properties, and in particular the presence of clustering, which is known to be an important feature of social networks.

Furthermore, in the present work we assumed the underlying network to be static, i.e. a time-independent network topology. In reality, however, many social and communication networks are highly dynamic. An example of such time-dependent social networks is Internet chatrooms, where individuals continuously make new social contacts and break old ones. Modeling the dynamics of rumor spreading on such dynamic networks is highly challenging, in particular when the time scale at which network topology changes becomes comparable with the time scale of the process dynamics. We also aim to tackle this highly interesting problem in future work.

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APPENDIX A: CRITICAL RUMOR THRESHOLD IN ER RANDOM GRAPHS

Eq. (6) can be integrated exactly to yield:

\[
\rho'(k, t) = \rho'(k, 0) e^{-k \lambda \phi(t)},
\]  

where \( \rho'(k, 0) \) is the initial density of ignorant nodes with connectivity \( k \), and we have introduced the auxi-
where, without loss of generality, we have also put \( \rho \).

\[
\phi(t) = \sum_k q(k) \int_0^t \rho^s(k, t') dt' \equiv \int_0^t \langle \rho^s(k, t') \rangle dt'.
\]

(A2)

In the above equation and hereafter we use the shorthand notation

\[
\langle O(k) \rangle = \sum_k q(k) O(k)
\]

(A3)

with

\[
q(k) = \frac{kP(k)}{\langle k \rangle}
\]

(A4)

In order to obtain an expression for the final size of the rumor, \( R \), it is more convenient to work with \( \phi \). Assuming an homogeneous initial distribution of ignorants, \( \rho^s(k, 0) = \rho_0^k \), we can obtain a differential equation for this quantity by multiplying Eq. (7) with \( q(k) \) and summing over \( k \). This yields after some elementary manipulations:

\[
\frac{d\phi}{dt} = 1 - \langle e^{-\lambda k \phi} \rangle - \delta \phi
\]

\[
- \alpha \int_0^t \left[ 1 - \langle e^{-\lambda k \phi(t')} \rangle \right] \langle \rho^s(k, t') \rangle dt'.
\]

(A5)

where, without loss of generality, we have also put \( \rho_0^0 = 1 \).

In the limit \( t \to \infty \) we have \( \frac{d\phi}{dt} = 0 \), and Eq. (A5) becomes:

\[
0 = 1 - \langle e^{-\lambda k \phi_\infty} \rangle - \delta \phi_\infty
\]

\[
- \alpha \int_0^\infty \left[ 1 - \langle e^{-\lambda k \phi(t')} \rangle \right] \langle \rho^s(k, t') \rangle dt'.
\]

(A6)

where \( \phi_\infty = \lim_{t \to \infty} \phi(t) \).

For \( \alpha = 0 \), Eq. (A5) can be solved explicitly to obtain \( \Phi_\infty \) [12]. For \( \alpha \neq 0 \), we solve (A5) to leading order in \( \alpha \). Integrating Eq. (7) to zero order in \( \alpha \) we obtain

\[
\rho^s(k, t) = 1 - e^{-\lambda k \phi} - \delta \int_0^t e^{\delta(t-t')} \left[ 1 - e^{-\lambda k \phi(t')} \right] dt' + O(\alpha).
\]

(A7)

Close to the critical threshold both \( \phi(t) \) and \( \phi_\infty \) are small. Writing \( \phi(t) = \phi_\infty f(t) \), where \( f(t) \) is a finite function, and working to leading order in \( \phi_\infty \), we obtain

\[
\rho^s(k, t) \approx -\delta \lambda k \phi_\infty \int_0^t e^{\delta(t-t')} f(t') dt' + O(\phi^2_\infty) + O(\alpha)
\]

(A8)

Inserting this in Eq. (A6) and expanding the exponential to the relevant order in \( \phi_\infty \) we find

\[
0 = \phi_\infty \left[ \lambda \langle k \rangle - \delta - \lambda^2 \langle k^2 \rangle (1/2 + \alpha \langle k \rangle) I \phi_\infty \right]
\]

\[
+ O(\alpha^2) + O(\phi_\infty^3)
\]

(A9)

where \( I \) is a finite and positive-defined integral. The non-trivial solution of this equation is given by:

\[
\phi_\infty = \frac{\lambda \langle k \rangle - \delta}{\lambda^2 \langle k^2 \rangle (1/2 + \alpha I \langle k \rangle)}.
\]

(A10)

Noting that \( \langle k \rangle = \langle k^2 \rangle / \langle k \rangle \) and \( \langle k^2 \rangle = \langle k^3 \rangle / \langle k \rangle \) we obtain:

\[
\phi_\infty = \frac{2\langle k \rangle (\lambda / \langle k \rangle) \lambda - \delta}{\lambda^2 \langle k^3 \rangle (1 + 2\alpha I \langle k^2 \rangle / \langle k \rangle)}.
\]

(A11)

This yields a positive value for \( \phi_\infty \) provided that

\[
\frac{\lambda}{\delta} \geq \frac{\langle k \rangle}{\langle k^2 \rangle}.
\]

(A12)

Thus to leading order in \( \alpha \) the critical rumor threshold is independent of this quantity, and is the same as for the SIR model. In particular, for \( \delta = 1 \) the critical rumor spreading threshold is given by \( \lambda_c = \langle k \rangle / \langle k^2 \rangle \), and Eq. (A12) simplifies to:

\[
\phi_\infty = \frac{2\langle k \rangle (\lambda - \lambda_c)}{\lambda^2 \langle k^4 \rangle (\lambda_c + 2\alpha I)}.
\]

(A13)

Finally, \( R \) is given by

\[
R = \sum_k P(k)(1 - e^{-\lambda k \phi_\infty}),
\]

(A14)

and expanding the exponential in Eq. (A6) we obtain

\[
R \approx \sum_k P(k)\lambda k \phi_\infty = \frac{2\langle k \rangle^2 (\lambda - \lambda_c)}{\lambda \langle k^3 \rangle (\lambda_c + 2\alpha I)},
\]

(A15)

which shows that \( R \sim (\lambda - \lambda_c) \) in the vicinity of the rumor threshold.

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