A new representation for the partition function of the six-vertex model with domain wall boundaries

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Abstract. We obtain a new representation for the partition function of the six-vertex model with domain wall boundaries using a functional equation recently derived by the author. This new representation is given in terms of a sum over the permutation group where the partial homogeneous limit can be taken trivially. We also show by construction that this partition function satisfies a linear partial differential equation.

Keywords: correlation functions, integrable spin chains (vertex models), solvable lattice models
1. Introduction

In the past few decades the mathematical structure underlying integrable systems has been intensively studied and even after many years it seems that its richness has not yet been completely exhausted. Integrable systems can be solved by a variety of methods, ranging from functional to algebraic methods, whose solution is usually expressed in terms of solutions of Bethe ansatz equations [1]–[4]. By way of contrast, the six-vertex model with domain wall boundary conditions (DWBC) is one exception to this general behaviour, and in fact the computation of its partition function and free energy does not rely on Bethe ansatz like solutions [5]–[7].

In the pioneer work [5] it was demonstrated that this model obeys a recurrence relation relating the partition function of the system on a square lattice of size $L \times L$ to the one on a lattice of size $(L-1) \times (L-1)$. Five years later Izergin proposed a determinant solution for this recurrence relation which, together with extra properties, determines uniquely the partition function of the system [6].

On the other hand the fundamental role of the Yang–Baxter equation and the Yang–Baxter algebra in the construction and solution of integrable systems is nowadays well understood, though the Yang–Baxter algebra does not seem to play any explicit role in the Izergin–Korepin solution of the six-vertex model with DWBC. Recently an alternative approach for computing this partition function was proposed, where the Yang–Baxter algebra is the main ingredient [8]. The approach of [8] makes explicit use of the Yang–Baxter algebra in order to derive a functional equation determining the partition function of the six-vertex model with DWBC. Though lacking a rigorous proof, in the framework of [8] the partition function is uniquely determined by three conditions:

(i) functional equation;
(ii) polynomial structure;
(iii) asymptotic behaviour;

and here we aim to demonstrate that Korepin’s recurrence relation can be suitably introduced in this framework, removing the need for polynomial solutions. This is of particular interest for further applications since there exist relevant models, such as the
Hubbard model, whose $R$-matrix contains non-polynomial elements [9]. Moreover, the introduction of the recurrence relation in the functional equation derived in [8] yields naturally an explicit representation for the partition function of the six-vertex model with DWBC whose homogeneous limit for the vertical degrees of freedom can be obtained trivially. In a second analysis we also complement the results of [8] by showing that the functional equation previously obtained for the partition function of the six-vertex model with DWBC can be converted into a linear partial differential equation.

This paper is organized as follows. In section 2 we recall some basic definitions and results associated with the derivation of a functional equation determining the partition function of the six-vertex model with DWBC. In section 4 we recast the functional equation previously obtained for the partition function as an operator equation and this is followed by its differential representation. Final comments and concluding remarks are discussed in section 5.

2. Functional relations and domain wall boundaries

In this section we recall some previous results and definitions associated with the derivation of a functional equation determining the partition function of the six-vertex model with domain wall boundaries. Let us consider the matrix $L$ of a functional equation determining the partition function of the six-vertex model with

$$L = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}$$

(1)

containing the statistical weights of the six-vertex model. The non-null entries are given by $a(\lambda) = \sinh(\lambda + \gamma)$, $b(\lambda) = \sinh(\lambda)$ and $c(\lambda) = \sinh(\gamma)$, where $\lambda$ and $\gamma$ are complex variables. The matrix $L(\lambda)$ satisfies the Yang–Baxter equation, namely

$$L_{12}(\lambda - \mu) L_{13}(\lambda - \nu) L_{23}(\mu - \nu) = L_{23}(\mu - \nu) L_{13}(\lambda - \nu) L_{12}(\lambda - \mu)$$

(2)

where $L_{ij} \in \text{End}(V_i \otimes V_j)$ and $V_i \cong \mathbb{C}^2$, and in this way the monodromy matrix $T$ defined by

$$T(\lambda, \{\mu_k\}) = L_{A1}(\lambda - \mu_1) L_{A2}(\lambda - \mu_2) \ldots L_{AL}(\lambda - \mu_L)$$

(3)

satisfies the following quadratic relation usually referred to as the Yang–Baxter algebra:

$$R(\lambda - \nu) T(\lambda, \{\mu_k\}) \otimes T(\nu, \{\mu_k\}) = T(\nu, \{\mu_k\}) \otimes T(\lambda, \{\mu_k\}) R(\lambda - \nu).$$

(4)

Here $R(\lambda) = P L(\lambda)$ and $P$ denotes the standard permutation matrix.

The monodromy matrix $T(\lambda, \{\mu_k\})$ can be conveniently written in terms of operators $A(\lambda, \{\mu_k\})$, $B(\lambda, \{\mu_k\})$, $C(\lambda, \{\mu_k\})$ and $D(\lambda, \{\mu_k\})$, i.e.

$$T(\lambda, \{\mu_k\}) = \begin{pmatrix} A(\lambda, \{\mu_k\}) & B(\lambda, \{\mu_k\}) \\ C(\lambda, \{\mu_k\}) & D(\lambda, \{\mu_k\}) \end{pmatrix},$$

(5)

and as demonstrated in [5] the partition function $Z$ of the six-vertex model with domain wall boundaries in a lattice of size $L \times L$ can be expressed as

$$Z = \langle 0 | \prod_{j=1}^L B(\lambda_j, \{\mu_k\}) | 0 \rangle$$

(6)

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where the states $|0\rangle$ and $|\bar{0}\rangle$ consist of the ferromagnetic states

$$|0\rangle = \bigotimes_{i=1}^{L} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |\bar{0}\rangle = \bigotimes_{i=1}^{L} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  \hfill (7)

Among the three conditions given in [8] determining the partition function $Z$, the main one consists of the functional equation

$$\sum_{i=1}^{L+1} M_i \bar{Z}(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{L+1})$$

$$+ \sum_{1 \leq i < j \leq L+1} N_{ji} \bar{Z}(\lambda_0, \lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{L+1}) = 0 \hfill (8)$$

where we write $Z(\lambda_1, \ldots, \lambda_L)$ for the partition function (6), omitting the dependence on the variables $\{\mu_j\}$. In their turn, the coefficients $M_i$ and $N_{ji}$ are given by

$$M_i = c(\lambda_i - \lambda_0) b(\lambda_i - \lambda_0) \prod_{l=1}^{L} a(\lambda_0 - \mu_l) b(\lambda_i - \mu_l) \prod_{k=1}^{L+1} a(\lambda_i - \lambda_k) a(\lambda_k - \lambda_0) b(\lambda_i - \lambda_k) b(\lambda_k - \lambda_0)$$

$$+ c(\lambda_0 - \lambda_i) \prod_{l=1}^{L} a(\lambda_i - \mu_l) b(\lambda_0 - \mu_l) \prod_{k=1}^{L+1} a(\lambda_0 - \lambda_k) a(\lambda_k - \lambda_i) b(\lambda_0 - \lambda_k) b(\lambda_k - \lambda_i) \hfill (9)$$

$$N_{ji} = \frac{c(\lambda_0 - \lambda_j) c(\lambda_i - \lambda_0) a(\lambda_j - \lambda_i)}{b(\lambda_0 - \lambda_j) b(\lambda_i - \lambda_0) b(\lambda_j - \lambda_i)} \prod_{l=1}^{L} a(\lambda_i - \mu_l) b(\lambda_j - \mu_l) \prod_{m=1}^{L+1} a(\lambda_j - \lambda_m) a(\lambda_m - \lambda_i) b(\lambda_j - \lambda_m) b(\lambda_m - \lambda_i)$$

$$+ \frac{c(\lambda_0 - \lambda_j) c(\lambda_j - \lambda_0) a(\lambda_j - \lambda_i)}{b(\lambda_0 - \lambda_j) b(\lambda_j - \lambda_0) b(\lambda_j - \lambda_i)} \prod_{l=1}^{L} a(\lambda_j - \mu_l) b(\lambda_i - \mu_l) \times \prod_{m=1}^{L+1} a(\lambda_i - \lambda_m) a(\lambda_m - \lambda_j) b(\lambda_i - \lambda_m) b(\lambda_m - \lambda_j). \hfill (10)$$

Here we shall also employ the variables $x_i = e^{2(\lambda_i - \mu_i)}$ in order to characterize the polynomial structure of $Z$. More precisely, the partition function (6) exhibits the following polynomial structure:

$$Z(\lambda_1, \ldots, \lambda_L) = \tilde{Z}(x_1, \ldots, x_L) \prod_{i=1}^{L} x_i^{(L-1)/2} \hfill (11)$$

where $\tilde{Z}(x_1, \ldots, x_L)$ is a polynomial of degree $L - 1$ in each variable $x_i$ separately.

Besides the two conditions discussed above, the full determination of the partition function also makes use of the asymptotic behaviour $\tilde{Z}(x_1, \ldots, x_L) \sim ((q - q^{-1})^L/2^L)[L]_q^2! (x_1 \ldots x_L)^{L-1}$ as $x_i \rightarrow \infty$, where $[L]_q^2!$ denotes the $q$-factorial function defined as

$$[L]_q^2! = 1(1 + q^2)(1 + q^2 + q^4) \ldots (1 + q^2 + \ldots + q^{2(L-1)}). \hfill (12)$$

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The partition function (6) also exhibits some extra properties besides the three conditions discussed above. For instance, \( Z \) is a symmetric function under the exchange of variables \( \lambda_i \leftrightarrow \lambda_j \), i.e.

\[
Z(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_L) = Z(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_L),
\]

which follow directly from the commutation relation

\[
B(\lambda, \{\mu_k\})B(\nu, \{\mu_k\}) = B(\nu, \{\mu_k\})B(\lambda, \{\mu_k\})
\]

encoded in the relation (4).

We close this section by remarking that the partition function (6) is also symmetric under the exchange of variables \( \mu_i \leftrightarrow \mu_j \) as discussed in [5]; however this property will not be required for our forthcoming analysis.

3. A new representation for \( Z(\lambda_1, \ldots, \lambda_L) \)

In this section we aim to demonstrate how we can combine the functional equation (8) and Korepin’s recurrence relation [5] in order to produce a new representation for the partition function \( Z \). As a matter of fact it turns out that the functional equation and the recurrence relation together are enough to completely determine the partition function.

In order to construct a representation for the partition function \( Z \) we firstly recall the recurrence relation derived in [5] adjusted to the notation that we are considering here. Since this recurrence relation has already been discussed in many works [4, 5, 11] we shall omit its derivation and here we only need the relation

\[
Z(\lambda_1, \ldots, \lambda_L | \mu_1, \ldots, \mu_L) |_{\lambda_1 = \mu_1} = c \prod_{j=2}^{L} a(\lambda_j - \mu_1)a(\mu_1 - \mu_j)Z(\lambda_2, \ldots, \lambda_L | \mu_2, \ldots, \mu_L) \tag{15}
\]

with \( Z(\lambda_1 | \mu_1) = c(\lambda_1 - \mu_1) \). In the relation (15) we have included explicitly the dependence of \( Z \) on the variables \( \{\mu_j\} \) and also have assumed that \( Z(\lambda_1, \ldots, \lambda_n | \mu_1, \ldots, \mu_n) \) denotes the partition function \( Z \) on a lattice of size \( n \times n \). In this way, for instance, equation (15) is a first-order recurrence relation connecting the partition function on an \((L-1) \times (L-1)\) lattice to the one on an \( L \times L \) lattice for a particular value of the variable \( \lambda_1 \).

For illustrative purposes we shall first consider the functional equation (8) with \( L = 2 \) which is then given by

\[
M_1 Z(\lambda_2, \lambda_3 | \mu_1, \mu_2) + M_2 Z(\lambda_1, \lambda_3 | \mu_1, \mu_2) + M_3 Z(\lambda_1, \lambda_2 | \mu_1, \mu_2) + N_{21} Z(\lambda_0, \lambda_3 | \mu_1, \mu_2) + N_{31} Z(\lambda_0, \lambda_2 | \mu_1, \mu_2) + N_{32} Z(\lambda_0, \lambda_1 | \mu_1, \mu_2) = 0 \tag{16}
\]

where the coefficients \( M_i \) and \( N_{ji} \) follow from (9) and (10) with the appropriate value of \( L \).

By setting \( \lambda_0 = \mu_1 \) and \( \lambda_3 = \mu_1 - \gamma \) we obtain

\[
M_1 |_{\lambda_0 = \mu_1, \lambda_3 = \mu_1 - \gamma} = M_2 |_{\lambda_0 = \mu_1, \lambda_3 = \mu_1 - \gamma} = 0 \tag{17}
\]
while $M_3$, $N_{21}$, $N_{31}$ and $N_{32}$ remain finite. More precisely we have

$$M_3|_{\lambda_0=\mu_1,\lambda_3=\mu_1-\gamma} = -c^2 a(\mu_1 - \mu_2) a(\mu_2 - \mu_1)$$

$$N_{21}|_{\lambda_0=\mu_1,\lambda_3=\mu_1-\gamma} = c^2 a(\mu_2 - \mu_1) a(\lambda_1 - \mu_2) \frac{b(\lambda_2 - \mu_1) a(\lambda_2 - \lambda_1)}{a(\lambda_2 - \mu_1) b(\lambda_2 - \lambda_1)}$$

(18)

$$N_{32}|_{\lambda_0=\mu_1,\lambda_3=\mu_1-\gamma} = c^2 a(\mu_2 - \mu_1) a(\lambda_2 - \mu_2) \frac{b(\lambda_1 - \mu_1) a(\lambda_1 - \lambda_2)}{a(\lambda_1 - \mu_1) b(\lambda_1 - \lambda_2)}$$

and we omit the explicit form of $N_{21}|_{\lambda_0=\mu_1,\lambda_3=\mu_1-\gamma}$ since it will not be required.

For this particular choice of variables $\lambda_0$ and $\lambda_3$ equation (16) is then reduced to

$$Z(\lambda_1, \lambda_2|\mu_1, \mu_2) = \frac{a(\lambda_1 - \mu_2) b(\lambda_2 - \mu_1) a(\lambda_2 - \lambda_1)}{a(\mu_1 - \mu_2) a(\lambda_2 - \mu_1) b(\lambda_2 - \lambda_1)} Z(\mu_1, \lambda_2|\mu_1, \mu_2)$$

$$+ \frac{a(\lambda_2 - \mu_2) b(\lambda_1 - \mu_1) a(\lambda_1 - \lambda_2)}{a(\mu_1 - \mu_2) a(\lambda_1 - \mu_1) b(\lambda_1 - \lambda_2)} Z(\mu_1, \lambda_1|\mu_1, \mu_2)$$

$$- Z(\mu_1, \mu_1 - \gamma|\mu_1, \mu_2) \cdot \frac{N_{21}}{M_3} |_{\lambda_0=\mu_1,\lambda_3=\mu_1-\gamma},$$

(19)

and if we set $\lambda_2 = \mu_1$ in (19) and consider that the partition function $Z$ is symmetric under the exchange of variables $\lambda_i \leftrightarrow \lambda_j$, we are left with the following identity:

$$\left\{ \frac{N_{21}}{M_3} |_{\lambda_0=\mu_1,\lambda_3=\mu_1-\gamma} \right\} : Z(\mu_1, \mu_1 - \gamma|\mu_1, \mu_2) = 0.$$

(20)

Since the quantity inside the brackets in (20) is finite, we can conclude that $Z(\mu_1, \mu_1 - \gamma|\mu_1, \mu_2) = 0$. Thus the relation (19) simplifies to

$$Z(\lambda_1, \lambda_2|\mu_1, \mu_2) = \frac{a(\lambda_1 - \mu_2) b(\lambda_2 - \mu_1) a(\lambda_2 - \lambda_1)}{a(\mu_1 - \mu_2) a(\lambda_2 - \mu_1) b(\lambda_2 - \lambda_1)} Z(\mu_1, \lambda_2|\mu_1, \mu_2)$$

$$+ \frac{a(\lambda_2 - \mu_2) b(\lambda_1 - \mu_1) a(\lambda_1 - \lambda_2)}{a(\mu_1 - \mu_2) a(\lambda_1 - \mu_1) b(\lambda_1 - \lambda_2)} Z(\mu_1, \lambda_1|\mu_1, \mu_2).$$

(21)

Notice however that the property $Z(\mu_1, \mu_1 - \gamma|\mu_1, \mu_2) = 0$ could also have been inferred from the recurrence relation (15).

Now we can simply insert the recurrence relation (15) into the relation (19), and by doing so we automatically obtain

$$Z(\lambda_1, \lambda_2|\mu_1, \mu_2) = F_{12} + F_{21}$$

(22)

where

$$F_{ij} = c^2 a(\lambda_i - \mu_2) b(\lambda_j - \mu_1) \frac{a(\lambda_j - \lambda_i)}{b(\lambda_j - \lambda_i)}.$$

(23)

This procedure can be straightforwardly extended to the case $L = 3$. In that case the functional equation (8) reads

$$M_1 Z(\lambda_2, \lambda_3, \lambda_4|\mu_1, \mu_2, \mu_3) + M_2 Z(\lambda_1, \lambda_3, \lambda_4|\mu_1, \mu_2, \mu_3) + M_3 Z(\lambda_1, \lambda_2, \lambda_4|\mu_1, \mu_2, \mu_3)$$

$$+ M_4 Z(\lambda_1, \lambda_2, \lambda_3|\mu_1, \mu_2, \mu_3) + N_{21} Z(\lambda_0, \lambda_3, \lambda_4|\mu_1, \mu_2, \mu_3)$$

$$+ N_{31} Z(\lambda_0, \lambda_2, \lambda_4|\mu_1, \mu_2, \mu_3) + N_{41} Z(\lambda_0, \lambda_2, \lambda_3|\mu_1, \mu_2, \mu_3)$$

$$+ N_{32} Z(\lambda_0, \lambda_1, \lambda_4|\mu_1, \mu_2, \mu_3) + N_{42} Z(\lambda_0, \lambda_1, \lambda_3|\mu_1, \mu_2, \mu_3)$$

$$+ N_{43} Z(\lambda_0, \lambda_1, \lambda_2|\mu_1, \mu_2, \mu_3) = 0$$

(24)

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and now we set \( \lambda_0 = \mu_1 \) and \( \lambda_1 = \mu_1 - \gamma \). On doing so we find that
\[
M_1|_{\lambda_0=\mu_1, \lambda_4=\mu_1-\gamma} = M_2|_{\lambda_0=\mu_1, \lambda_4=\mu_1-\gamma} = M_3|_{\lambda_0=\mu_1, \lambda_4=\mu_1-\gamma} = 0.
\] (25)

The remaining coefficients do not vanish, but from the recurrence relation (15) it is easy to see that
\[
Z(\mu_1, \mu_1 - \gamma, \lambda|\mu_1, \mu_2, \mu_3) = 0,
\] (26)
and we only need to consider the terms \( M_4, N_{41}, N_{42} \) and \( N_{43} \) in equation (24). When \( \lambda_0 = \mu_1 \) and \( \lambda_4 = \mu_1 - \gamma \) those terms simplify to
\[
m_3 = M_4|_{\lambda_0=\mu_1, \lambda_4=\mu_1-\gamma} = c^2 a(\mu_1 - \mu_2)a(\mu_2 - \mu_1) a(\mu_1 - \mu_3) a(\mu_3 - \mu_1)
\]
\[
\bar{m}_1 = N_{41}|_{\lambda_0=\mu_1, \lambda_4=\mu_1-\gamma} = -c^2 a(\mu_2 - \mu_1)a(\mu_3 - \mu_1)a(\lambda_1 - \mu_2)a(\lambda_1 - \mu_3)
\times \frac{b(\lambda_2 - \mu_1) b(\lambda_3 - \mu_1) a(\lambda_2 - \lambda_1) a(\lambda_3 - \lambda_1)}{a(\lambda_2 - \mu_1) a(\lambda_3 - \mu_1) b(\lambda_2 - \lambda_1) b(\lambda_3 - \lambda_1)}
\]
\[
\bar{m}_2 = N_{42}|_{\lambda_0=\mu_1, \lambda_4=\mu_1-\gamma} = -c^2 a(\mu_2 - \mu_1)a(\mu_3 - \mu_1)a(\lambda_2 - \mu_2)a(\lambda_2 - \mu_3)
\times \frac{b(\lambda_1 - \mu_1) b(\lambda_3 - \mu_1) a(\lambda_1 - \lambda_2) a(\lambda_3 - \lambda_2)}{a(\lambda_1 - \mu_1) a(\lambda_3 - \mu_1) b(\lambda_1 - \lambda_2) b(\lambda_3 - \lambda_2)}
\]
\[
\bar{m}_3 = N_{43}|_{\lambda_0=\mu_1, \lambda_4=\mu_1-\gamma} = -c^2 a(\mu_2 - \mu_1)a(\mu_3 - \mu_1)a(\lambda_3 - \mu_2)a(\lambda_3 - \mu_3)
\times \frac{b(\lambda_1 - \mu_1) b(\lambda_2 - \mu_1) a(\lambda_1 - \lambda_3) a(\lambda_2 - \lambda_3)}{a(\lambda_1 - \mu_1) a(\lambda_2 - \mu_1) b(\lambda_1 - \lambda_3) b(\lambda_2 - \lambda_3)}
\]
and we are left with the equation
\[
Z(\lambda_1, \lambda_2, \lambda_3|\mu_1, \mu_2, \mu_3) = -c a(\lambda_2 - \mu_1)a(\lambda_3 - \mu_1)a(\mu_1 - \mu_2)a(\mu_1 - \mu_3) \bar{m}_1 Z(\lambda_2, \lambda_3|\mu_2, \mu_3)
\]
\[
\quad - c a(\lambda_1 - \mu_1)a(\lambda_3 - \mu_1)a(\mu_1 - \mu_2)a(\mu_1 - \mu_3) \bar{m}_2 Z(\lambda_1, \lambda_3|\mu_2, \mu_3)
\]
\[
\quad - c a(\lambda_1 - \mu_1)a(\lambda_2 - \mu_1)a(\mu_1 - \mu_2)a(\mu_1 - \mu_3) \bar{m}_3 Z(\lambda_1, \lambda_2|\mu_2, \mu_3)
\] (28)
where we have already considered the recurrence relation (15). Now we only need to substitute the expressions (22) and (23) in (28) in order to obtain the partition function
\[
Z(\lambda_1, \lambda_2, \lambda_3|\mu_1, \mu_2, \mu_3).
\]
With this procedure we automatically obtain
\[
Z(\lambda_1, \lambda_2, \lambda_3|\mu_1, \mu_2, \mu_3) = F_{123} + F_{132} + F_{213} + F_{231} + F_{321} + F_{312}
\] (29)
where
\[
F_{ijk} = c^2 a(\lambda_i - \mu_2)a(\lambda_i - \mu_3)a(\lambda_j - \mu_2)b(\lambda_j - \mu_1)b(\lambda_k - \mu_1)b(\lambda_k - \mu_2)
\times \frac{a(\lambda_j - \lambda_i) a(\lambda_k - \lambda_j) a(\lambda_k - \lambda_i)}{b(\lambda_j - \lambda_i) b(\lambda_k - \lambda_i) b(\lambda_k - \lambda_j)}.
\] (30)
The expressions (22) and (29) suggest that for general \( L \) we can write the partition function as a sum over the permutation group of rather simple elements. In fact we can extend this analysis to general \( L \) by considering \( \lambda_0 = \mu_1 \) and \( \lambda_{L+1} = \mu_1 - \gamma \) in the relation (8). From the definition (9) we can immediately see that
\[
M_j|_{\lambda_0=\mu_1, \lambda_{L+1}=\mu_1-\gamma} = 0 \quad j = 1, \ldots , L
\] (31)
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and from the recurrence relation (15) we obtain the property

$$Z(\mu_1, \mu_1 - \gamma, \lambda_1, \ldots, \lambda_{L-2}|\mu_1, \ldots, \mu_L) = 0. \quad (32)$$

In this way, for this particular choice of $\lambda$ and by doing so we find previously in [10]. The relation (33) can now be iterated using the results (22) and (23),

Here we remark that a similar, though not equivalent, recurrence relation has appeared difficult to see that the invariance of $Z$ it contains the same number of terms as the determinant representation [6]. Furthermore, the computation of the homogeneous limit $\mu \rightarrow \mu$ of an $S$ matrix [6], it is not clear whether the relations (35) and (36) can be converted into a determinant. However, since (35) consists of a sum over the permutation group $S_L$, it contains the same number of terms as the determinant representation [6]. Furthermore, the computation of the homogeneous limit $\mu_k \rightarrow \mu$ from the expressions (35) and (36) is trivial, in contrast to the case for the Izergin–Korepin determinant representation, where the evaluation of the homogeneous limit is rather intricate [11].

$$Z(\lambda_1, \ldots, \lambda_L) = -c \sum_{j=1}^L \prod_{k \neq j} a(\lambda_k - \mu_1) \prod_{k=2}^L a(\mu_1 - \mu_k) \frac{\bar{m}_j}{m_L} Z(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_L) \quad (33)$$

where again we omit the dependence on the variables $\{\mu_j\}$. In their turn the coefficients $m_L$ and $\bar{m}_j$ are given by

$$m_L = M_{L+1} |_{\lambda_0=\mu_1; \lambda_{L+1}=\mu_1-\gamma} = (-1)^{L+1} c^2 \prod_{j=2}^L a(\mu_1 - \mu_j) a(\mu_j - \mu_1)$$

$$\bar{m}_j = N_{L+1,j} |_{\lambda_0=\mu_1; \lambda_{L+1}=\mu_1-\gamma} = (-1)^L c^2 \prod_{k=2}^L a(\mu_k - \mu_1) a(\mu_1 - \mu_k) \prod_{k=1}^L b(\lambda_k - \mu_1) a(\lambda_k - \mu_1) \prod_{k=1}^L b(\lambda_k - \lambda_j) a(\lambda_k - \lambda_j) \quad (34)$$

Here we remark that a similar, though not equivalent, recurrence relation has appeared previously in [10]. The relation (33) can now be iterated using the results (22) and (23), and by doing so we find

$$Z(\lambda_1, \ldots, \lambda_L) = \sum_{\{i_1, \ldots, i_L\} \in S_L} F_{i_1 \ldots i_L} \quad (35)$$

where $S_L$ denotes the permutation group of order $L$ and

$$F_{i_1 \ldots i_L} = c^L \prod_{n=1}^L \prod_{j=1}^L a(\lambda_{i_n} - \mu_j) \prod_{j=1}^L b(\lambda_{i_n} - \mu_j) \prod_{n=1}^{L-1} \prod_{m>n}^L a(\lambda_{i_m} - \lambda_{i_n}) \prod_{n=1}^{L-1} \prod_{m>n}^L b(\lambda_{i_m} - \lambda_{i_n}) \quad (36)$$

Since the partition function (35) is given by a sum over permutations, it is not difficult to see that the invariance of $Z$ under the exchange of variables $\lambda_i \leftrightarrow \lambda_j$ is explicitly manifested in the representation given by (35) and (36). On the other hand, the symmetry of $Z$ under the exchange of variables $\mu_i \leftrightarrow \mu_j$ is not apparent, though the explicit evaluation of (35) for small values of $L$ does indeed corroborate this property.

Though it is well known that the partition function $Z$ can be written as a determinant of an $L \times L$ matrix [6], it is not clear whether the relations (35) and (36) can be converted into a determinant. However, since (35) consists of a sum over the permutation group $S_L$, it contains the same number of terms as the determinant representation [6]. Furthermore, the computation of the homogeneous limit $\mu_k \rightarrow \mu$ from the expressions (35) and (36) is trivial, in contrast to the case for the Izergin–Korepin determinant representation, where the evaluation of the homogeneous limit is rather intricate [11].

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4. From functional relations to partial differential equations

In the course of the investigation of integrable systems many connections between previously unrelated topics have emerged. For example, the Knizhnik–Zamolodchikov (KZ) equation is a fundamental differential equation in conformal field theories and its rich mathematical structure is manifested in the variety of topics in which the KZ and its quantized version appears [12,13]. Interestingly, the KZ equation establishes a connection between two representation theories, one associated with Lie algebras and the other one associated with quantum groups [13,14]. Though not in the same way as the KZ equation connects the representation theory of Lie algebras and quantum groups, the derivation of the functional equation (8) explores a connection between the highest weight representation theories of the Yang–Baxter algebra and the \( \mathfrak{su}(2) \) Lie algebra [8].

Moreover, in conformal field theory the KZ equation is a differential equation for the matrix coefficients of the product of intertwining operators for an affine Lie algebra \( \hat{\mathfrak{g}} \), while in the case of the six-vertex model with DWBC we have a functional equation for a coefficient of the Bethe vectors.

A priori it is not clear whether there exists some relation between the aforementioned functional equation and the KZ equation or its quantized version. In order to shed some light on possible connections, we aim in this section to complement the results of [8] by showing that the functional equation previously obtained for the partition function of the six-vertex model with DWBC can be converted into a linear partial differential equation.

Let \( f \) be a complex valued function \( f(z) \in \mathbb{C}[z] \) with \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \). For \( \alpha \notin [1, n] \) we define the operator \( D^\alpha_i \) as

\[
D^\alpha_i : f(z_1, \ldots, z_i, \ldots, z_n) \mapsto f(z_1, \ldots, z_\alpha, \ldots, z_n)
\]

which basically replaces the variable \( z_i \) with \( z_\alpha \).

Now we shall make use of the property (13) and in terms of operators \( D^\alpha_i \) the functional equation (8) simply reads

\[
\left\{ \sum_{1 \leq i < j \leq L} N_{ji} D^0_i D^{L+1}_j + \sum_{i=1}^L \left[ M_i D^{L+1}_i + N_{L+1,i} D^0_i \right] + M_{L+1} \right\} Z(\lambda_1, \ldots, \lambda_L) = 0 \tag{38}
\]

which consists of a second-order equation in terms of the operator \( D^\alpha_i \). Moreover, as we shall demonstrate, the operator \( D^\alpha_i \) possesses a differential representation when it is restricted to the ring of polynomials.

In order to proceed it is convenient to define the functions

\[
\bar{M}_i = \prod_{j=1, j \neq i}^{L+1} x_j^{(1-L)/2} M_i \quad \text{and} \quad \bar{N}_{ji} = \prod_{k=0, k \neq i,j}^{L+1} x_k^{(1-L)/2} N_{ji}
\]

such that equation (38) becomes

\[
\left\{ \sum_{1 \leq i < j \leq L} \bar{N}_{ji} D^0_i D^{L+1}_j + \sum_{i=1}^L \left[ \bar{M}_i D^{L+1}_i + \bar{N}_{L+1,i} D^0_i \right] + \bar{M}_{L+1} \right\} \bar{Z}(x_1, \ldots, x_L) = 0, \tag{40}
\]

where the function \( \bar{Z} \) defined by equation (11) consists of a polynomial of order \( L - 1 \) in each variable \( x_i \) separately.

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In this way we can restrict the action of the operator $D_i^\alpha$ to the space of polynomials of order $m$, and this restriction is manifested in the property

$$\frac{\partial^k f}{\partial z_i^k} = 0 \quad \text{for } k > m.$$  \hfill (41)

Consequently the Taylor expansion of $f$ is also truncated and convergent.

Now considering that $f = f(z_1, \ldots, z_n)$ and also taking into account equation (41) we have

$$f = f(z_1, \ldots, z_{i-1}, z_\alpha, z_{i+1}, \ldots, z_n) + \frac{\partial f}{\partial z_i} (z_i - z_\alpha)$$

$$+ \frac{1}{2!} \frac{\partial^2 f}{\partial z_i^2} (z_i - z_\alpha)^2 + \cdots + \frac{1}{m!} \frac{\partial^m f}{\partial z_i^m} (z_i - z_\alpha)^m$$  \hfill (42)

for any $i \in [1, n]$. On the other hand, since $\alpha \notin [1, n]$ we can write

$$\frac{\partial^k f}{\partial z_i^k} = \frac{\partial^k f(z_1, \ldots, z_{i-1}, z_\alpha, z_{i+1}, \ldots, z_n)}{\partial z_\alpha^k}$$  \hfill (43)

and the Taylor expansion (42) can be rewritten as

$$f(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n) = \left[ \sum_{k=0}^{m} \frac{(z_i - z_\alpha)^k}{k!} \frac{\partial^k f}{\partial z_\alpha^k} \right] f(z_1, \ldots, z_{i-1}, z_\alpha, z_{i+1}, \ldots, z_n).$$  \hfill (44)

The term inside the brackets performs the operation (37) and we thus obtain

$$D_i^\alpha = \sum_{k=0}^{m} (z_\alpha - z_i)^k \frac{\partial^k f}{\partial z_\alpha^k}. \hfill (45)$$

At this stage we have already gathered all the ingredients required to convert the functional equation (8) into a partial differential equation. As discussed in section 1, the partition function (6) is completely determined by the conditions (i), (ii) and (iii). Among the three conditions, the condition (iii) is the weakest one since it just determines the leading order coefficient of the polynomial $\bar{Z}$. On the other hand the conditions (i) and (ii) play a major role in the determination of the partition function (6) and their combination is what allows us to express equation (8) as a linear partial differential equation.

Taking into account that the function $\bar{Z}(x_1, \ldots, x_{L})$ is a polynomial of degree $L - 1$ in each variable $x_i$, we can substitute the representation (45) with $m = L - 1$ in the equation (40) and we are left with

$$\left\{ \sum_{1 \leq i < j \leq L} \sum_{k,l=0}^{L-1} \frac{\bar{N}_{ij}}{k!l!} (x_0 - x_i)^k (x_{L+1} - x_j)^l \frac{\partial^{k+l}}{\partial x_i^k \partial x_j^l}$$

$$+ \sum_{i=1}^{L} \sum_{k=0}^{L-1} \frac{1}{k!} \left[ \bar{M}_i (x_{L+1} - x_i)^k + \bar{\bar{M}}_{L+1,i} (x_0 - x_i)^k \right] \frac{\partial^k}{\partial x_i^k} + \bar{\bar{M}}_{L+1} \right\} \times \bar{Z}(x_1, \ldots, x_L) = 0$$  \hfill (46)

which consists of a linear partial differential equation of order $2(L - 1)$.

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5. Concluding remarks

In this work we have derived a new representation for the partition function of the six-vertex model with DWBC. Since Izergin’s proposal of the determinant representation, this partition function has been rewritten in different ways over the years [15]–[19] and many connections between this partition function and the theory of polynomials have also emerged [20,21]. In particular, it was shown in [21] that this partition function consists of the Schubert polynomial.

Here we have obtained a representation for this partition function in terms of a sum over the permutation group whose possible interpretation as a determinant has not been clear so far. On the other hand, the representation (35) and (36) has the advantage of allowing the evaluation of the partial homogeneous limit $\mu_k \rightarrow \mu$, in a trivial way.

Furthermore, this new representation is a direct consequence of the functional equation (8) and Korepin’s recurrence relation (15), removing all other requirements considered in [8] and [5] concerning the nature of the solutions. It is also worthwhile to stress here that, though we have obtained a representation for the partition function $Z$ from the functional equation (8), we have not solved equation (8) strictly speaking since we have only considered the equation (8) at the special points $\lambda_0 = \mu_1$ and $\lambda_{L+1} = \mu_1 - \gamma$. However, a rigorous proof of the uniqueness of the solution for the system of equations formed by (8) and (15) seems to imply that the representation given by (35) and (36) does indeed solve equation (8) for general values of $\lambda_0$ and $\lambda_{L+1}$.

As regards a second analysis of equation (8), we have also demonstrated in section 4 that the requirement of polynomial solutions allows us to rewrite the functional relation (8) as a linear partial differential equation. The partition function of the six-vertex model with DWBC is known to correspond to a KP $\tau$ function [22] and also to satisfy a Toda lattice differential equation in the homogeneous limit [11]. In this way we hope that the linear partial differential equation (46) will help shed some light on possible connections between this partition function and the theory of differential equations.

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