\( \mathcal{N} = 2 \) Supersymmetry with Central Charge: a two-fold implementation.

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Abstract

In this work, we analyze an extended \( \mathcal{N} = 2 \) supersymmetry with central charge and develop its superspace formulation under two distinct points of view. Initially, in the context of classical mechanics, we discuss the introduction of deformed supersymmetric derivatives and their consequence on the deformation of a particular one-dimensional nonlinear sigma-model. After that, considering a field-theoretical framework, we present an implementation of this superalgebra in two dimensions, such that one of the coordinates is related to the central charge. As an application, in this two-dimensional scenario, we investigate the topological configurations of a special self-coupled matter model.

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I. INTRODUCTION

Back to 1975, the paper by Haag, Lopuszanski and Sohnius [1] established the most general supersymmetric algebra in four dimensions, which respects the Poincaré invariance and requirements of $S-$matrix. In addition to the Poincaré operators and usual Majorana supercharges, its is possible to include new operators, known as central charges, which have all vanishing commutation relations.

Essentially, the investigations associated with central charges in field theories can be divided in three parts. In the first one, the main subject was the classification of the multiplets associated with massless and massive representations. In this part, the papers by Salam and Strathdee [2] initially analyzed the case without central charges and the subsequent works [3]–[13] clarified the general one. These works have shown that in the presence of central charges, some states are avoided in the massive representation and, in particular cases, central charges may be related to internal symmetries.

The second part is related to the investigations in the classical level. One of the remarkable contributions is the work by Witten and Olive [14], where the authors considered some specific models in two- and four-dimensions with topological configurations and obtained a connection between central charges and topological numbers. After that, other situations involving topological defects have been explored, such as the complex projective space $CP^{n-1}$ and others nonlinear sigma-models [15]–[18]. For a more detailed discussion of central charges and topological defects, see refs. [19]–[22].

Other interesting classical point of view to central charges showed up after the dimensional reduction of supersymmetric models. Some works have discussed that central charges can be seen as an inheritance of the dimensional reduction [8] [23]–[25]. In particular cases, it is related to the momentum operator of the extra-dimension [26]. We highlight an exotic situation presented in ref. [27], in which the dimensional reduction of super-Yang-Mills model in four-dimensions with $O(N)$ group leads to Georgi-Glashow-like model in three-dimensions, where the central charge is associated with the electrical charge of the abelian subgroup, i.e., other non-topological explanation to central charge. We also indicate some supersymmetric models with central charges in higher-dimensional theories and brane-world scenarios [7] [28]–[32].
The third part concerns to the quantum aspects. Specifically, by means of quantum effects, the central charge has appeared as a quantum anomaly in the superalgebra. From this perspective, many situations have been analyzed, such as nonlinear sigma-models, kink, monopoles, domain walls and vortices [33]−[40].

From now on, let us remind some applications in mechanical systems. Firstly, in the situation without central charge, the supersymmetric mechanics have found some remarkable applications, including Factorization Method and Shape Invariance [41]−[43], Statistical Physics [44], Nuclear Physics [45]−[46], Disordered Systems, Chaos and Integrability [47]−[48]. Also, in mechanical systems, the fermionic degrees of freedom are naturally associated with spin, so that the supersymmetry has been applied in the study of magnetic interaction of point-particle [49]−[53]. We also highlight the seminal works by Witten [54], where the author focused on supersymmetry breaking in quantum mechanics as a laboratory to the field theories.

In the context of mechanical models in presence of central charge, there are some investigations in which field-theoretical models may correspond to (quantum) mechanical systems, namely, in the study of superconducting cosmic strings, localized fermions on domain walls, gapped and superconducting graphene [55]. Moreover, in some well-known mechanical cases, such as Coulomb, Aharonov–Bohm–Coulomb and Aharonov–Casher systems, it is possible to accommodate extended supersymmetries with central charge [56], which explain the degeneracy of the energy spectra. In the context of super-particles, one can introduce the so-called tensorial central charges (see [57]−[60] and references therein), which are responsible to fix some constraints in the equations of motion and have found applications in the study of higher-spin models.

Furthermore, in the presence of an external electromagnetic background, it is possible to show that the Poincaré algebra leads to a residual symmetry algebra with a central charge, deformed translations and Lorentz generators. For the models in two and three dimensions, we indicate refs. [61] and [62]−[63], respectively.

At this point, it is worthy to emphasize that we are using the term “central charge” referred to the generators with all vanishing commutation relations. However, there are situations where new bosonic generators have been added to the superalgebra with some non-vanishing commutators. For example, we address to the cases of weak supersymmetry [64] and other generalizations known as centrally extensions involving the $su(2|1)$ and $su(2|2)$
algebras [65]-[66]. Finally, we highlight some results in supersymmetric mechanics in which the introduction of central charges can be related to duality and mirror symmetries [67]-[70].

In this work, some investigations of $\mathcal{N} = 2$ supersymmetry with one (real) central charge is carried out in mechanics and two-dimensional field theory. We consider a similar superalgebra adopted in refs. [69]-[70] and we propose an alternative implementation of this supersymmetry through superspace approach. Here, we shall present an prescription to implement this extended supersymmetry by means of deformed covariant derivatives.

The paper is organized as follows: in Section II, we discuss the $\mathcal{N} = 2$ supersymmetric mechanics with central charge. We focus on the construction of the superspace formulation, central charge transformation and propose deformed (SUSY) covariant derivatives. In this context, we revise the particular one-dimensional nonlinear sigma-model discussed in ref. [69]. Then, a comparison between the two prescriptions is also carried out. After that, in Section III we turn our attention to supersymmetric field theories. Here, we develop a possible implementation of this superalgebra in two-dimensions, namely, we introduce a new coordinate related to central charge operator. In this scenario, we present a model in $(1+1)$-dimensions with topological configurations. Finally, in Section IV we display our Concluding Comments.

II. SUPERSYMMETRIC MECHANICS WITH CENTRAL CHARGE

In $\mathcal{N} = 2$ supersymmetric one-dimensional systems (mechanical systems), we have two (real) supercharges, $Q_1$ and $Q_2$, which satisfy the following superalgebra with the Hamiltonian ($H$): $Q_1^2 = Q_2^2 = H$ and $[Q_1, H] = [Q_2, H] = 0$. In this work, we are going to deal with a possible extension of this case by including one real central charge ($Z$) in the anti-commutator $\{Q_1, Q_2\} = 2Z$. We shall adopt a complex notation for the supercharges, namely, $Q = \frac{1}{\sqrt{2}} (Q_1 + iQ_2)$ and $\bar{Q} = \frac{1}{\sqrt{2}} (Q_1 - iQ_2)$, which leads to

$$\{Q, Q\} = 2H,$$

$$[Q, H] = [\bar{Q}, H] = 0,$$

$$Q^2 = iZ, \quad \bar{Q}^2 = -iZ.$$

With the aforementioned relations, one can check that $[Q, Z] = [\bar{Q}, Z] = [H, Z] = 0$, i.e.,
Z has the properties of a central charge. We also highlight that a similar superalgebra was considered in refs. [69] [70], with only different conventions in eq. (3).

In order to implement this extended supersymmetry through a superfield approach, we first introduce two (complex) Grassmann parameters, θ and ¯θ, such that the superspace can be described by \((t; \theta, \bar{\theta})\), where \(t\) is the time parameter. Throughout this work, the Grassmann derivatives are understood in the sense of acting from left to the right \([71] [72]\), such as \(\frac{\partial}{\partial \theta}(\bar{\theta}\theta) = -\bar{\theta}\). Furthermore, we draw the attention to the procedure we adopt to realize the representation of central charge. Our proposal consists in implementing the central charge by means of a deformation of the supersymmetric generators. Now, to read the explicit form of this deformation, we act with the supersymmetric transformation on a supermultiplet and, by imposing a number of conditions we are going later on to present, we get the final form of the deformation.

The superalgebra above can be realized in a differential representation. To achieve this, we define \(\delta^H = i\partial_t\) and the following (deformed) supercharge operators:

\[\delta^Q = \partial_\theta + i\bar{\theta}\partial_t + i\theta\delta^z,\]  
\[\delta^{\bar{Q}} = \partial_{\bar{\theta}} + i\theta\partial_t - i\bar{\theta}\delta^z,\]  

where we have used the notation \(\partial_t = \partial/\partial t\), \(\partial_\theta = \partial/\partial \theta\) and \(\partial_{\bar{\theta}} = \partial/\partial \bar{\theta}\).

Moreover, we point out an important comment: it is worthy to introduce a deformation \(\delta^z\) in the (supersymmetric) covariant derivatives. Similarly to the case of the supercharges, eqs. (4) and (5), we define

\[\mathcal{D} = \partial_\theta - i\bar{\theta}\partial_t - i\theta\delta^z,\]  
\[\bar{\mathcal{D}} = \partial_{\bar{\theta}} - i\theta\partial_t + i\bar{\theta}\delta^z,\]  

which satisfy \(\{\mathcal{D}, \bar{\mathcal{D}}\} = -2i\partial_t\) and have vanishing anti-commutation relations with the supercharges,

\[\{\delta^Q, \mathcal{D}\} = \{\delta^{\bar{Q}}, \bar{\mathcal{D}}\} = \{\delta^Q, \bar{\mathcal{D}}\} = \{\delta^{\bar{Q}}, \mathcal{D}\} = 0.\]  

By using these (deformed) covariant derivatives instead of the usual \(D = \partial_\theta - i\bar{\theta}\partial_t\) and \(\bar{D} = \partial_{\bar{\theta}} - i\theta\partial_t\), we automatically take into account the contribution of central charge and assure the extended supersymmetric invariance of the action formulated in terms of the covariant derivatives and superfields. In the case of the trivial central charge transformation,
we recover the usual $\mathcal{N} = 2$ supersymmetry. On the other hand, one could use the derivatives $D$ and $\bar{D}$, as done in refs. [69] [70]. However, in this situation one should perform a carefully analysis of the supersymmetric transformation of the Lagrangian (in components) and add some counter-terms in order to maintain the invariance. Here, we claim that this additional terms are exactly the ones generated by the derivatives $\mathcal{D}$ and $\bar{\mathcal{D}}$. We shall return to this point in more details in the next subsection II A, where a particular case will be analyzed.

At this moment, it is advisable to point out that, in this Section, the introduction of central charge is not associated with an extra coordinate in super space. In other words, the superspace is parametrized by $(t, \theta, \bar{\theta})$ and the central charge is implemented through deformation of supercharges and covariant derivatives.

The introduction of supercharges and deformed derivatives in connection with central charge is not exclusive to this work. In ref. [73], the authors investigated a central charge $(\Sigma)$ in the relation $\{Q, \bar{Q}\} = 2 (H - \Sigma)$ and the invariance of the action is also implemented in a more simple way by using the superfields and the correspondent deformed derivatives.

As already mentioned, one of the goals in this work is to study in what multiplet is possible to introduce a nontrivial central charge transformation related to the superalgebra (1)-(3). Here, we consider the multiplet $(1, 2, 1)$, described by one bosonic, two fermionic (Grassmann) and one auxiliary bosonic coordinates. The correspondent superfield is given by

$$X \equiv X[t; \theta, \bar{\theta}] = x(t) + i\theta \xi(t) + i\bar{\theta} \bar{\xi}(t) + \theta \bar{\theta} W(t).$$  \hspace{1cm} (9)

Let us first establish the supersymmetric transformation of these components. By using $\delta = \epsilon \delta^Q + \bar{\epsilon} \delta^\bar{Q}$, with complex Grassmann parameters $\epsilon$ and $\bar{\epsilon}$, one can show that

$$\delta x = i\epsilon \xi + i\bar{\epsilon} \bar{\xi},$$
$$\delta \xi = -\bar{\epsilon} \dot{x} - i\bar{\epsilon} W - \epsilon \delta^x x,$$
$$\delta \bar{\xi} = -\epsilon \dot{x} + i\epsilon W + \bar{\epsilon} \delta^\bar{x} x,$$
$$\delta W = \frac{d}{dt} (\epsilon \xi - \bar{\epsilon} \bar{\xi}) - \epsilon \delta^x \bar{\xi} - \bar{\epsilon} \delta^\bar{x} \xi. \hspace{1cm} (10)$$

Now we can proceed to fix the central charge transformation. In the case of real superfield, we note that $\delta \xi$ is the complex conjugation of $\delta \bar{\xi}$, thereby $\delta^x x$ must be bosonic and pure imaginary. Moreover, $\delta W$ is bosonic, then $\delta^\bar{x} \xi$ and $\delta^x \bar{\xi}$ should be fermionic. At this point, we suggest that $\delta^x x = i\mu$, where $\mu$ is a constant (real) parameter. Bearing this in mind
and applying the relation $[\delta^z, \delta] = 0$ in all superfield components, one can arrive at the conditions $\delta^z \xi = 0 = \delta^z \bar{\xi}$ and $\delta^z \mathcal{W} = 0$. Actually, the main reason to fix this particular transformation is that $\delta \mathcal{W} = \frac{d}{dt}(\ldots)$, which guarantees the invariance of the action in the superfield approach. For example, if we do not consider a constant, namely, $\delta^z x = if(x)$, with $f(x)$ being an arbitrary function of $x$, we do not obtain a total derivative in $\delta \mathcal{W}$.

Finally, we remember that in $\mathcal{N} = 2$ supersymmetry (without $Z$), the quiral ($\phi$) and anti-quiral ($\bar{\phi}$) superfields have been defined in a manner that satisfy the conditions $\bar{D} \phi = 0$ and $D \bar{\phi} = 0$. By using a Kähler (pre)potential, $K(\phi, \bar{\phi})$, these superfields were applied in the study of supersymmetric models associated with electromagnetic interaction of a point particle $[74]$ $[75]$. Here, we emphasize that the imposition of the aforementioned conditions with the deformed derivatives leads to a trivial central charge transformations.

### A. A one-dimensional Nonlinear Sigma-model

The nonlinear sigma-models in connection with extended supersymmetric mechanics has been a subject of intense investigation. In refs. $[76]$-$[81]$, the authors discussed some case of $\mathcal{N} > 2$ supersymmetries. The inclusion of central charges was done in $[69]$ $[70]$ and $[82]$ for $\mathcal{N} = 2$ and $\mathcal{N} = 4$, respectively. We also highlight some generalizations involving torsion contributions $[83]$ and applications to Black-Hole $[84]$. For a review of one-dimensional nonlinear sigma model and some aspects of geometry and topology, we point out the works $[85]$ $[86]$.

Having established the superspace formulation and the introduction of deformed supersymmetric derivatives. In this subsection, as an example, we revise the particular one-dimensional nonlinear sigma-model described in ref. $[69]$. The model consists of a point-particle restrict to the cylinder-like topology with a variable radius. More specifically, we consider a euclidean metric $ds^2 = G_{ij}(x)dx^i dx^j$ ($i, j = 1, 2$), where $G_{ij} = \text{diag}(1, h(x^1))$ and $h(x^1)$ denotes an arbitrary non-negative function. Hence, $x^1$ is the axial coordinate with $\sqrt{h(x^1)}$ being the cylinder radio and $x^2$ is the angular coordinate.

In a supersymmetric description of this model, we have two real superfields $X^1$ and $X^2$.
with the following components transformations:

\[
\begin{align*}
\delta x^1 &= i\epsilon \xi^1 + i\bar{\epsilon} \bar{\xi}^1, \\
\delta x^2 &= i\epsilon \xi^2 + i\bar{\epsilon} \bar{\xi}^2, \\
\delta \xi^1 &= -\bar{\epsilon} \dot{x}^1 - i\epsilon \mathcal{W}^1, \\
\delta \xi^2 &= -i\epsilon \mu - \bar{\epsilon} \dot{x}^2 - i\epsilon \mathcal{W}^2, \\
\delta \bar{\xi}^1 &= -\epsilon \dot{x}^1 + i\epsilon \mathcal{W}^1, \\
\delta \bar{\xi}^2 &= i\bar{\epsilon} \mu - \epsilon \dot{x}^2 + i\epsilon \mathcal{W}^2, \\
\delta \mathcal{W}^1 &= \epsilon \dot{\xi}^1 - \bar{\epsilon} \bar{\dot{\xi}}^1, \\
\delta \mathcal{W}^2 &= \epsilon \dot{\xi}^2 - \bar{\epsilon} \bar{\dot{\xi}}^2.
\end{align*}
\]

(11)

i.e., only the second superfield has a nontrivial central charge (parameter \(\mu\)).

Here, we propose what we refer to as our deformed nonlinear sigma-model in terms of the deformed derivatives:

\[
S = \int dt d\bar{\theta} d\theta \left[ \frac{1}{2} G_{ij}(X) \bar{\mathcal{D}} X^i \mathcal{D} X^j \right].
\]

(12)

After some algebraic calculations, one arrives at the corresponding off-shell Lagrangian

\[
L = \frac{1}{2} \left[ (\dot{x}^1)^2 + i \left( \xi^1 \dot{\xi}^1 - \bar{\xi}^1 \bar{\dot{\xi}}^1 \right) + (\mathcal{W}^1)^2 \right] \\
+ \frac{1}{2} h \left[ (\dot{x}^2)^2 + i \left( \xi^2 \dot{\xi}^2 - \bar{\xi}^2 \bar{\dot{\xi}}^2 \right) + (\mathcal{W}^2)^2 \right] \\
+ \frac{1}{2} h' \left[ -i\dot{x}^2 \left( \xi^1 \xi^2 + \bar{\xi}^1 \bar{\xi}^2 \right) + \mathcal{W}^2 \left( \xi^1 \tilde{\xi}^2 - \bar{\xi}^1 \bar{\tilde{\xi}}^2 \right) - \mathcal{W}^1 \xi^2 \bar{\tilde{\xi}}^2 \right] \\
- \frac{1}{2} h'' \xi^1 \bar{\xi}^1 \xi^2 \bar{\xi}^2 + \frac{1}{2} \mu h' \left( \xi^1 \xi^2 - \bar{\xi}^1 \bar{\tilde{\xi}}^2 \right) - \frac{1}{2} \mu^2 h,
\]

(13)

where \(h'\) and \(h''\) denote the first and second derivatives of \(h(x^1)\), respectively. Now, after eliminating the auxiliary coordinates \((\mathcal{W}^1, \mathcal{W}^2)\), we obtain the following on-shell Lagrangian

\[
L_{on-shell} = \frac{1}{2} \left[ (\dot{x}^1)^2 + i \left( \xi^1 \dot{\xi}^1 - \bar{\xi}^1 \bar{\dot{\xi}}^1 \right) \right] \\
+ \frac{1}{2} h \left[ (\dot{x}^2)^2 + i \left( \xi^2 \dot{\xi}^2 - \bar{\xi}^2 \bar{\dot{\xi}}^2 \right) + \frac{1}{2} \left( \frac{h'}{h} \right) \left( \xi^1 \xi^2 \bar{\tilde{\xi}}^1 \bar{\tilde{\xi}}^2 \right) \right] \\
+ \frac{1}{2} h' \left[ -i\dot{x}^2 \left( \xi^1 \xi^2 + \bar{\xi}^1 \bar{\xi}^2 \right) \right] - \frac{1}{2} h'' \xi^1 \bar{\xi}^1 \xi^2 \bar{\xi}^2 \\
+ \frac{1}{2} \mu h' \left( \xi^1 \xi^2 - \bar{\xi}^1 \bar{\tilde{\xi}}^2 \right) - \frac{1}{2} \mu^2 h.
\]

(14)

As expected in the case of nonlinear sigma-models, the elimination of auxiliary coordinates yield quartic terms on the Grassmann coordinates \((\xi, \bar{\xi})\).

Finally, we notice that the Lagrangian \(L_{on-shell}\) is equivalent to the one obtained in ref. \[69\]. It only differs from some signs and \(i\)-factors which are due our different conventions. At
this point, it is interesting to compare both methodologies. In ref. [69], the authors initially consider the following action

\[
S_0 = \int dt d\theta d\bar{\theta} \left[ \frac{1}{2} G_{ij}(X) DX^i \bar{D} X^j \right],
\]  

(15)

where \( D = \partial_\theta - i \bar{\theta} \partial_t \) and \( \bar{D} = \partial_\theta - i \theta \partial_t \) correspond to the derivatives without central charge. Then eq. (15) is decomposed in components and a analysis of Susy transformation (with central charge), eqs. (11), leads to a non-invariant action. In order to restore the supersymmetric invariance, some corrections terms are needed, i.e., a new contribution \( S_{c-t} \) is added to \( S_0 \) such that the complete action \( S = S_0 + S_{c-t} \) remains invariant. Here, we claim that this complete action coincides with eq. (12). Therefore, the deformed derivatives automatically includes the correspondent \( L_{c-t} \), namely, the last two terms in eq. (13) with parameters \( \mu \) and \( \mu^2 \).

III. SUPERSYMMETRIC FIELD THEORY WITH CENTRAL CHARGE

In this section, we present another point of view to investigate this extended supersymmetry. Our proposal is to introduce a new coordinate \( v \) related to central charge \( Z \). It is worthy to comment that the introduction of extra bosonic coordinates — and, consequently, an extended superspace — is not exclusive to this work. We highlight a series of papers [5] [10] [19] [87] [88] in which this point of view has been adopted in other contexts. For this purpose, we shall generalize the procedure described in ref. [74] for \( \mathcal{N} = 2 \) supersymmetric mechanics (without \( Z \)). Initially, let us define a group element

\[
G(t, v, \theta, \bar{\theta}) = e^{itH + ivZ + i\theta Q + i\bar{\theta} \bar{Q}}. 
\]  

(16)

By using the superalgebra, eqs. (11) - (3), and the well-known Baker-Campbell-Hausdorff identity,

\[
e^A e^B = e^{A+B} e^{(1/2)[A,B]},
\]

\[
[[A, B], A] = 0 = [[A, B], B],
\]  

(17)

one may verify the group structure with the multiplicative rule

\[
G(t', v', \theta', \bar{\theta}') G(t, v, \theta, \bar{\theta}) = 
\]
\[ G \left( t + t' + i(\theta' \bar{\theta} + \bar{\theta}' \theta), v + v' + (\theta' \theta - \bar{\theta}' \bar{\theta}) , \theta + \theta', \bar{\theta} + \bar{\theta}' \right) , \]  

which leads to the following translation in this new superspace

\[ (t, v, \theta, \bar{\theta}) \rightarrow (t + t' + i(\theta' \bar{\theta} + \bar{\theta}' \theta), v + v' + (\theta' \theta - \bar{\theta}' \bar{\theta}) , \theta + \theta', \bar{\theta} + \bar{\theta}') . \]  

Notice that a real translation acts on \( v \) in a similar way with the time, namely, in both cases appear bilinear of the Grassmann parameters \((\theta, \bar{\theta})\).

Starting with the group element \( G(t, v, \theta, \bar{\theta}) \) described previously, we establish the superfield as

\[ X(t, v, \theta, \bar{\theta}) = G(t, v, \theta, \bar{\theta}) X(0, 0, 0, 0) G^{-1}(t, v, \theta, \bar{\theta}) . \]  

Now we consider an infinitesimal transformation and obtain the differential representation to supercharges and central charge. By using infinitesimal parameters \((t', v', \theta', \bar{\theta}') \rightarrow (\varepsilon, \zeta, \epsilon, \bar{\epsilon})\) and the multiplicative rule (18), we have

\[ G(\varepsilon, \zeta, \epsilon, \bar{\epsilon}) X(t, v, \theta, \bar{\theta}) G^{-1}(\varepsilon, \zeta, \epsilon, \bar{\epsilon}) = X(t + \varepsilon + i(\epsilon \bar{\theta} + \bar{\epsilon} \theta), v + \zeta + (\epsilon \theta - \bar{\epsilon} \bar{\theta}), \theta + \epsilon, \bar{\theta} + \bar{\epsilon}) . \]  

A general operator \( \mathcal{O} \) satisfies \( i[\mathcal{O}, X] \sim \delta^\mathcal{O} X \). Thus, from eq. (21), one may expand both sides and obtain the following differential representation:

\[ \delta^H = i\partial_t , \quad \delta^Z = -i\partial_v , \]  

\[ \delta^Q = \partial_\theta + i\bar{\theta}\partial_t + \theta\partial_v , \quad \delta^{\bar{Q}} = \partial_{\bar{\theta}} + i\theta\partial_t - \bar{\theta}\partial_v . \]  

At this point, we realize that the central charge behaves like a momentum operator of the "extra-dimension" \( v \). In comparison with the supercharges in mechanical case, eqs. (4) and (5), we have an analogous structure, but now the central charge is completely fixed, \( \delta^Z = -i\partial_v \), independently of the superfield components.

These differential operators satisfy the same superalgebra,

\[ \left\{ \delta^Q, \delta^{\bar{Q}} \right\} = 2 \delta^H , \]  

\[ (\delta^Q)^2 = i\delta^Z , \quad (\delta^{\bar{Q}})^2 = -i\delta^Z , \]  

where \( \delta^H \) and \( \delta^Z \) commute with all operators.
In order to obtain the covariant derivatives, we consider an alternative multiplication to the right of the group elements, \( G(t, v, \theta, \bar{\theta}) G(t', v', \theta', \bar{\theta}') \), and the definition
\[
\varphi(t, v, \theta, \bar{\theta}) = G^{-1}(t, v, \theta, \bar{\theta}) \, X(0, 0, 0, 0) \, G(t, v, \theta, \bar{\theta}) ,
\]
which leads to
\[
G^{-1}(\varepsilon, \zeta, \epsilon, \bar{\epsilon}) \, \varphi(t, v, \theta, \bar{\theta}) \, G(\varepsilon, \zeta, \epsilon, \bar{\epsilon})
= \varphi(t + \varepsilon + i(\bar{\theta} \epsilon + \theta \bar{\epsilon}), v + \zeta + (\theta \epsilon - \bar{\theta} \bar{\epsilon}), \theta + \epsilon, \bar{\theta} + \bar{\epsilon}) .
\]
From the expansion and comparison of both sides, one can conclude that
\[
\mathcal{D} = \partial_{\theta} - i \bar{\theta} \partial_{t} - \theta \partial_{v} , \quad \bar{\mathcal{D}} = \partial_{\bar{\theta}} - i \theta \partial_{t} + \bar{\theta} \partial_{v} .
\]

These covariant derivatives exhibit the same representation of the deformed derivatives in the mechanical case (with \( \delta \bar{Z} = -i \partial_{v} \)) and they also anti-commute with the supercharges \( \delta Q \) and \( \delta \bar{Q} \).

Having established the supercharges and covariant (deformed) derivatives, we turn our attention to the discussion of superfield and supersymmetric transformation. We define a real (bosonic) superfield as
\[
X(t, v, \theta, \bar{\theta}) = f_{1}(t, v) + i \theta \psi(t, v) + i \bar{\theta} \bar{\psi}(t, v) + f_{2}(t, v) \, \theta \bar{\theta} .
\]

It is important to mention that, in this formulation, we do not have a classical mechanics description, because now we deal with the component fields \( f_{1}, f_{2}, \psi \) and \( \bar{\psi} \), which in general depend on \( (t, v) \). However, by taking a dimensional reduction, one can arrive at mechanical case. For example, in the trivial reduction à la Scherk-Schwarz, \( \partial_{v} \) (all fields) = 0, we recover the usual \( \mathcal{N} = 2 \) supersymmetry (without central charge) with the identification \( (f_{1}, \psi, \bar{\psi}, f_{2}) \rightarrow (x, \xi, \bar{\xi}, W) \).

Let us obtain the field components transformations with this kind of supersymmetry. By taking the Taylor expansion of the right-hand side of eq. (21) and comparing with
\[
\delta X \equiv \delta f_{1} + i \theta \delta \psi + i \bar{\theta} \delta \bar{\psi} + \theta \bar{\theta} \delta f_{2} ,
\]
we have the following variations
\[
\delta f_{1} = i \epsilon \psi + i \epsilon \bar{\psi} + \varepsilon \dot{f}_{1} + \zeta \partial_{v} f_{1} ,
\]

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\[
\delta \psi = i \epsilon \partial_v f_1 \epsilon = i \epsilon f_2 - \bar{\epsilon} \dot{f}_1 + \epsilon \dot{\psi} + \zeta \partial_v \psi,
\]
\[
\delta f_2 = \frac{d}{dt} \left( \epsilon \psi - \bar{\epsilon} \bar{\psi} + \epsilon f_2 \right) + \frac{d}{dv} \left( i \epsilon \bar{\psi} + i \bar{\epsilon} \psi + \zeta f_2 \right).
\]

Once \( f_2 \) transforms as a total derivative of \( v \) and \( t \), one may describe an invariant action in terms of real superfield and covariant derivatives. For susy transformations, we just need to fix \( \zeta = \epsilon = 0 \) in the last variations, such that \( \delta^{SUSY} = \epsilon \delta^Q + \bar{\epsilon} \bar{\delta}^Q \), where \( \delta^Q \) and \( \bar{\delta}^Q \) are given by eq. (23). In this case, the component transformations are very similar to the mechanical case, eqs. (10).

Finally, we emphasize that in our formalism the chiral and anti-chiral superfields do not depend on \( v \) and have the same form given by the supersymmetric model without central charge.

A. Topological configurations in (1+1)D

In this subsection, we discuss an application of the previous formalism. We investigate a particular model in two-dimensions. Using the covariant derivatives (28) and real superfield (29), we propose the following action

\[
S = \int dt \, dv \, d\bar{\theta} \, d\theta \left[ \frac{1}{2} \mathcal{D}X \mathcal{\bar{D}}X + U(X) \right],
\]

where \( U(X) \) denotes an arbitrary superpotential.

This action in components leads to the following Lagrangian density

\[
\mathcal{L}_{\text{off-shell}} = \frac{1}{2} (\dot{f}_1)^2 - \frac{1}{2} (\partial_v f_1)^2 + \frac{(f_2)^2}{2} + f_2 \frac{\partial U}{\partial f_1} + \bar{\psi} \dot{\psi} \frac{\partial^2 U}{\partial f_1^2} + \bar{\psi} \dot{\psi} \frac{\partial^2 U}{\partial f_1^2} + \left( \partial_v \bar{\psi} \right) \bar{\psi} + \frac{1}{2} \left( \partial_v \bar{\psi} \right) \bar{\psi} + \psi \partial_v \bar{\psi} \psi \partial_v \psi \right).
\]

Through the Klein-Gordon profile, we again note that \( v \) can be seen as spacial coordinate and the central charge \( Z \), in the differential representation, is the momentum operator \( (\delta Z = -i \partial_v) \). In this case, we concluded that the role of central charge is to accommodate the field structure in \((1 + 1)D\).

Since the auxiliary field \( f_2(t, v) \) does not have dynamics, it can be eliminated by its equation of motion, \( f_2(t, v) = -\partial U/\partial f_1 \), resulting on the on-shell Lagrangian density

\[
\mathcal{L} = \frac{1}{2} (\dot{f}_1)^2 - \frac{1}{2} (\partial_v f_1)^2 - \frac{1}{2} \left( \frac{\partial U}{\partial f_1} \right)^2 + \psi \bar{\psi} \frac{\partial^2 U}{\partial f_1^2} +
\]
\[ + \frac{i}{2}(\psi \dot{\bar{\psi}} - \dot{\psi} \bar{\psi}) + \frac{1}{2} \left( (\partial_v \bar{\psi}) \dot{\bar{\psi}} + \psi (\partial_v \psi) \right). \tag{36} \]

The equations of motion are given by
\[ \ddot{f}_1 - \partial_v^2 f_1 + U' U''' - \psi \bar{\psi} U''' = 0, \tag{37} \]
\[ i \dot{\psi} - \partial_v \bar{\psi} - \psi U'' = 0, \tag{38} \]
where we used \( U' \equiv \partial U / \partial f_1 \).

Here, we notice that these equations of motion can accommodate topological configurations. Let us initially consider a trivial fermionic sector, \( \psi(t, v) = \psi^{(1)}(t, v) = 0 \). Then, for a particular case \( f_1(t, v) = f_1^{(1)}(v) \), satisfying
\[ \partial_v f_1^{(1)} = \pm U'(f_1^{(1)}), \tag{39} \]
we obtain a possible solution. This first order equation can describe some topological configurations. We only need to specify the function \( U(f_1) \). Before do that, we would like to point out some comments about the fermionic sector.

It is interesting to highlight that we can take advantage of the supersymmetric structure to get a set of solutions with nontrivial fermionic sector. We shall adopt a similar procedure discussed in ref. [90], where oscillinos (fermionic) solutions was obtained by means of supersymmetric transformations in oscillons (bosonic configurations).

First, one may verify that, by applying supersymmetric transformations, eqs. (37) and (38) are mapped into each other. That is exactly the sense of supersymmetry. In order to obtain a nontrivial fermionic solution with topological structure in the bosonic sector, we use the on-shell supersymmetric transformations, namely, eqs. (31) and (32) with \( \varepsilon = \zeta = 0 \) and \( f_2 = -U' \). In other words, we need to consider a supersymmetric perturbation of the previous solution,
\[ f_1^{(2)} = f_1^{(1)} + \delta f_1^{(1)} \equiv f_1^{(1)}, \tag{40} \]
\[ \psi^{(2)} = \psi^{(1)} + \delta \psi^{(1)} \equiv \delta \psi^{(1)} = i \varepsilon \partial_v f_1^{(1)} + i \varepsilon U', \tag{41} \]
which are also solutions of the equations of motion (37) and (38).

Finally, let us present a particular case. If we fix the arbitrary function \( U(f_1) \) such that
\[ \frac{1}{2} [U'(f_1)]^2 = A \left[ 1 - \cos \left( \frac{2\pi}{F} f_1 \right) \right], \tag{42} \]
with $A$ and $F$ being constant parameters, then we obtain a new supersymmetric version of
the sine-Gordon model \[91\]. In this case, the bosonic solution is given by

$$f_1^{(2)}(v) = \frac{2F}{\pi} \arctan \left[ e^{m(v-v_0)} \right],$$

(43)

where $v_0$ is an arbitrary constant and $m \equiv 2\pi \sqrt{A/F}$.

By using the supersymmetric perturbation, eq. \[11\], one can arrive at the following
fermionic (static) solution

$$\psi^{(2)}(v) = i \epsilon 4 \sqrt{A} \frac{e^{m(v-v_0)}}{1 + e^{2m(v-v_0)}} \pm i \epsilon \left\{ 2A \left[ 1 - \cos \left( \frac{2\pi}{F} f_1^{(2)}(v) \right) \right] \right\}^{1/2},$$

(44)

Finally, it is worthy to comment that, by construction, the fermionic solution has a trivial
condensate, $\bar{\psi}^{(2)} \psi^{(2)} = 0$. This could indicate a possible relation between this superalgebra
(with central charge and coordinate $v$) and other two-dimensional supersymmetries. We
shall return to this point in our conclusions.

**IV. CONCLUDING COMMENTS**

Initially, we have discussed the $\mathcal{N} = 2$ supersymmetric mechanics with one (real) central
charge for the multiplet $(1, 2, 1)$. A prescription to obtain deformed $\mathcal{N} = 2$ models by central
charge was developed. To establish this in a superfield approach, we have introduced
deformed covariant derivatives, eqs. \[6\] and \[7\], which take into account the new terms
related to the central charge. As an example, we have recast the particular one-dimensional
nonlinear sigma-model of the ref. \[69\] and shown an equivalence between the two prescrip-
tions for the specific transformations given by eqs. \[10\]. However, we have noticed that an
introduction of deformed derivatives allows us to implement this extended supersymmetry
in a more simple way, once we maintain the superfields and it is not necessary to decom-
pose the Lagrangian in components and add counter-terms to recover the supersymmetry.
Bearing this in mind, we would like to point out some possible subjects of investigations. In
the context of general nonlinear sigma-model (with Riemann curvature), one may add the
torsion and generalized torsion terms \[83\], namely, some couplings involving 2− and 4−form
with $DX^i DX^j$, $DX^i DX^j DX^k DX^l$ and its complex conjugations. A deformation of these
models by central charge could be obtained by using the following prescription proposed
here: $D \rightarrow \mathcal{D}$ and $D \rightarrow \bar{\mathcal{D}}$. It would be interesting to study the quantization as well as the role of central charge in these mechanical systems, once these new terms could generate new dualities or other symmetries.

In the second part, we have proposed an implementation of the superalgebra (1)-(3) in two-dimensional field theory. Along the lines discussed in the Introduction, one possible scenario to obtain central charge is to consider a dimensional reduction of the higher-dimensional theory. In this case, the central charge is related to a momentum operator in the extra-dimension. Here, we have a similar situation, namely, we interpreted the central charge as a momentum operator of the spatial-dimension $v$. In this assumption, the supersymmetric transformations (with central charge) is fully fixed, given by eqs. (31)-(33), where $\delta Z = -i\partial_v$. As an application, we discussed a supersymmetric model which exhibits topological configurations in the bosonic sector and nontrivial fermionic solution.

Finally, we point out some possible investigations related to this supersymmetry in two dimensions. One could analyze the introduction of other superfields, such as complex bosonic/fermionic scalar and vector, in order to accommodate charged matter and gauge fields. Furthermore, the connection of this superalgebra with other two-dimensional (Poincaré) supersymmetries also remains as a subject for future investigation. One may study the usual supersymmetry in two dimensions and redefine or drop out some Lorentz (boost) generators. Remembering that in two dimensions, it is also possible to have a Majorana-Weyl fermion and implement the heterotic ($p,q$)–supersymmetries. For instance, the Majorana-Weyl condition in two dimensions implies one degree-of-freedom which exhibits a trivial condensate. In our context, we have also obtained a similar situation for the fermionic solution $\psi^{(2)}(v)$.

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