Lipschitzian solutions to linear iterative equations revisited

KAROL BARON AND JANUSZ MORAWIEC

Abstract. We study the problems of the existence, uniqueness and continuous dependence of Lipschitzian solutions \( \varphi \) of equations of the form

\[
\varphi(x) = \int_{\Omega} g(\omega) \varphi(f(x, \omega)) \mu(d\omega) + F(x),
\]

where \( \mu \) is a measure on a \( \sigma \)-algebra of subsets of \( \Omega \) and \( \int_{\Omega} g(\omega) \mu(d\omega) = 1 \).

Mathematics Subject Classification. Primary 39B12.

Keywords. Iterative equations, Lipschitzian solutions, continuous dependence of solutions, Bochner integral.

1. Introduction

Fix a measure space \((\Omega, A, \mu)\) and a separable metric space \((X, \rho)\).

We continue a study of Lipschitzian solutions \( \varphi \) of equations of the form

\[
\varphi(x) = \int_{\Omega} g(\omega) \varphi(f(x, \omega)) \mu(d\omega) + F(x)
\]

(1)

assuming now, contrary to [2], that

\[
\int_{\Omega} g(\omega) \mu(d\omega) = 1.
\]

(2)

Concerning the given functions \( f, g \) and \( F \) we assume the following hypotheses in which \( B \) stands for the \( \sigma \)-algebra of all Borel subsets of \( X \) and \( \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} \).

(H1) The function \( f \) maps \( X \times \Omega \) into \( X \) and for every \( x \in X \) the function \( f(x, \cdot) \) is \( A \)-measurable, i.e.

\[
\{\omega \in \Omega : f(x, \omega) \in B\} \in A \quad \text{for all } x \in X \text{ and } B \in B.
\]
(H2) The function \( g: \Omega \to \mathbb{K} \) is integrable, satisfies (2),
\[
\int_{\Omega} |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega) < \infty \quad \text{for every } x \in X,
\]
and
\[
\int_{\Omega} |g(\omega)| \rho(f(x, \omega), f(z, \omega)) \mu(d\omega) \leq \lambda \rho(x, z) \quad \text{for all } x, z \in X
\]
with \( \lambda \in [0, 1) \).

(H3) The function \( F \) maps \( X \) into a separable Banach space \( Y \) over \( \mathbb{K} \) and
\[
\|F(x) - F(z)\| \leq L \rho(x, z) \quad \text{for all } x, z \in X
\]
with \( L \in [0, +\infty) \).

As emphasized in [3, Section 0.3] iteration is the fundamental technique for solving functional equations in a single variable, and iterates usually appear in the formulae for solutions. We iterate, as in [2], the operator which transforms a Lipschitzian \( F: X \to Y \) into \( \int_{\Omega} g(\omega) F(f(x, \omega)) \mu(d\omega) \), contrary to [3, Section 7.2D] where the Schauder fixed point theorem is used. The special case where \( g(\omega) = 1 \) for all \( \omega \in \Omega \) and \( \mu(\Omega) = 1 \) was considered in [1, Section 4] on a base of iteration of random-valued functions.

For integrating vector functions we use the Bochner integral.

2. Existence and uniqueness

Assuming (H1)–(H3) and making use of [2, Lemma 2.2] we define
\[
F_0(x) = F(x), \quad F_n(x) = \int_{\Omega} g(\omega) F_{n-1}(f(x, \omega)) \mu(d\omega)
\]
for all \( x \in X \) and \( n \in \mathbb{N} \), and note that
\[
\|F_n(x) - F_n(z)\| \leq L \lambda^n \rho(x, z) \quad \text{for all } x, z \in X \text{ and } n \in \mathbb{N}.
\]

Our main result reads as follows.

**Theorem 2.1.** Assume (H1)–(H3). Then:

(i) There is a \( y_0 \in Y \) such that \( \lim_{n \to \infty} F_n(x) = y_0 \) for every \( x \in X \);

(ii) Equation (1) has a Lipschitzian solution \( \varphi: X \to Y \) if and only if
\[
\lim_{n \to \infty} F_n(x_0) = 0 \quad \text{for an } x_0 \in X.
\]

(iii) Any Lipschitzian solution \( \varphi: X \to Y \) of (1) has the form
\[
\varphi(x) = c + \sum_{n=0}^{\infty} F_n(x) \quad \text{for every } x \in X,
\]
where \( c \) is a constant from \( Y \).
(iv) If $\varphi : X \to Y$ is a Lipschitzian solution of (1), then
\[
\| \varphi(x) - \varphi(z) \| \leq \frac{L}{1 - \lambda} \rho(x, z) \quad \text{for all } x, z \in X.
\] (6)

Proof. It follows from (3), (2) and (4) that
\[
\| F_n(x) - F_{n-1}(x) \| \leq L \lambda^{n-1} \int_{\Omega} |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega)
\]
for all $x \in X$ and $n \in \mathbb{N}$. Consequently, for every $x \in X$ the series
\[
\sum_{n=1}^{\infty} (F_n(x) - F_{n-1}(x))
\]
converges, i.e., $(F_n(x))_{n \in \mathbb{N}}$ converges in $Y$. Hence and from (4) assertion (i) follows.

Passing to the proof of assertion (ii) assume that $\varphi : X \to Y$ is a Lipschitzian solution of (1) with a Lipschitz constant $L_\varphi$ and define
\[
\varphi_0(x) = \varphi(x), \quad \varphi_n(x) = \int_{\Omega} g(\omega) \varphi_{n-1}(f(x, \omega)) \mu(d\omega)
\] (7)
for all $x \in X$ and $n \in \mathbb{N}$. Since
\[
\varphi_{n-1} = \varphi_n + F_{n-1}
\]
for every $n \in \mathbb{N}$, we have
\[
\varphi = \varphi_n + \sum_{k=0}^{n-1} F_k
\] (8)
and by (i) there is a $c \in Y$ such that
\[
\lim_{n \to \infty} \varphi_n(x) = c \quad \text{for every } x \in X.
\]
Taking this and (8) into account we see that for every $x \in X$ the series occurring in (5) converges and (5) holds. In particular, $\lim_{n \to \infty} F_n(x) = 0$ for every $x \in X$. Applying (5) and (4) we obtain (6).

For the proof of the existence assume that $\lim_{n \to \infty} F_n(x_0) = 0$ for an $x_0 \in X$ and making use of (4) define $\Phi : X \to Y$ by
\[
\Phi(x) = \sum_{n=0}^{\infty} (F_n(x) - F_n(x_0)).
\] (9)
Clearly,
\[
\| \Phi(x) - \Phi(z) \| \leq \frac{L}{1 - \lambda} \rho(x, z) \quad \text{for all } x, z \in X.
\] (10)
We shall show that $\Phi$ solves (1). To this end fix $x \in X$. According to [2, Lemma 2.2] the function
\[
\omega \mapsto g(\omega) \Phi(f(x, \omega)), \quad \omega \in \Omega,
\]
is Bochner integrable and by (4) for all \( n \in \mathbb{N} \) and \( \omega \in \Omega \) we have
\[
\left\| g(\omega) \left( F_n(f(x, \omega)) - F_n(x_0) \right) \right\| \leq L \lambda^n |g(\omega)| (\rho(f(x, \omega), x) + \rho(x, x_0)).
\]
Hence, applying the dominated convergence theorem, (9), (3) and (2) we see that
\[
\int_{\Omega} g(\omega) \Phi(f(x, \omega)) \mu(\mathrm{d}\omega) = \sum_{n=0}^{\infty} \int_{\Omega} g(\omega) \left( F_n(f(x, \omega)) - F_n(x_0) \right) \mu(\mathrm{d}\omega)
\]
\[
= \sum_{n=0}^{\infty} (F_{n+1}(x) - F_n(x_0))
\]
\[
= \sum_{n=0}^{\infty} (F_{n+1}(x) - F_{n+1}(x_0))
\]
\[
+ \sum_{n=0}^{\infty} (F_{n+1}(x_0) - F_n(x_0))
\]
\[
= \sum_{n=1}^{\infty} (F_n(x) - F_n(x_0))
\]
\[
+ \lim_{n \to \infty} (F_n(x_0) - F_0(x_0))
\]
\[
= \sum_{n=0}^{\infty} (F_n(x) - F_n(x_0)) - F_0(x)
\]
\[
= \Phi(x) - F(x),
\]
which ends the proof. \( \square \)

The following example shows that sometimes the limit of the sequence \( (F_n)_{n \in \mathbb{N}} \) can be easily calculated, but its value may be a surprise.

**Example 2.2.** Given a Lipschitzian \( F : [0, 1] \to \mathbb{R} \) consider the equation
\[
\varphi(x) = \frac{1}{2} \varphi \left( \frac{1}{2} x \right) + \frac{1}{2} \varphi \left( \frac{1}{2} x + \frac{1}{2} \right) + F(x).
\] (11)
In this case \( f(x, \omega) = \frac{1}{2} x + \omega \) and \( g(\omega) = 1 \) for all \( x \in [0, 1] \) and \( \omega \in \Omega = \{0, \frac{1}{2}\} \), \( \mu(\{0\}) = \mu(\{\frac{1}{2}\}) = \frac{1}{2} \), and
\[
F_n(x) = \frac{1}{2} F_{n-1} \left( \frac{1}{2} x \right) + \frac{1}{2} F_{n-1} \left( \frac{1}{2} x + \frac{1}{2} \right) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} F \left( \frac{1}{2^n} x + \frac{k}{2^n} \right)
\]
for all \( x \in [0, 1] \) and \( n \in \mathbb{N} \), whence
\[
\lim_{n \to \infty} F_n(x) = \int_{0}^{1} F(y) \mathrm{d}y \quad \text{for every } x \in [0, 1].
According to Theorem 2.1 Eq. (11) has a Lipschitzian solution $\varphi: [0,1] \to \mathbb{R}$ if and only if
$$\int_0^1 F(x)dx = 0.$$

Remark 2.3. Note that [2, Example 3.3] shows that assumptions (H1)–(H3) do not guarantee the existence of a continuous solution $\varphi: X \to Y$ of (1).

Remark 2.4. As [1, Example 4.2] shows, under the assumptions of Theorem 2.1 besides a Lipschitzian solution Eq. (1) may also have a continuous one which is not Lipschitzian.

Remark 2.5. For every $\lambda \in (0,1)$ the logarithmic function restricted to $[1,\infty)$ is a Lipschitzian solution to the equation
$$\varphi(x) = \varphi(\lambda x + 1 - \lambda) + \log \frac{x}{\lambda x + 1 - \lambda};$$
here $f(x, \omega) = \lambda x + 1 - \lambda$, $g(\omega) = 1$, $F(x) = \log \frac{x}{\lambda x + 1 - \lambda}$ for all $x \in [1,\infty)$ and $\omega \in \Omega$, $\mu(\Omega) = 1$. According to Theorem 2.1 it is the only up to an additive constant Lipschitzian solution $\varphi: [1,\infty) \to \mathbb{R}$ to this equation, and it is unbounded in spite of the fact that $F$ is bounded.

3. Continuous dependence

Assume (H1) and (H2), fix $x_0 \in X$ and let $(Y, \| \cdot \|)$ be a separable Banach space over $\mathbb{K}$. In what follows we consider the linear space $\text{Lip}(X,Y)$ of all Lipschitzian functions mapping $X$ into $Y$ with the norm
$$\| F \|_{\text{Lip}} = \| F(x_0) \| + \| F \|_L,$$
where $\| F \|_L$ stands for the smallest Lipschitz constant for $F$, and the linear subspace $\mathcal{F}$ of $\text{Lip}(X,Y)$ consists of all $F \in \text{Lip}(X,Y)$ such that for the sequence $(F_n)_{n \in \mathbb{N}}$ defined by (3) we have $\lim_{n \to \infty} F_n(x_0) = 0$. It is clear that the norm $\| \cdot \|_{\text{Lip}}$ depends on the fixed point $x_0$, but for different points such norms are equivalent, and it follows from (4) that $\mathcal{F}$ does not depend on $x_0$. Putting
$$\mathcal{F}_0 = \{ F \in \text{Lip}(X,Y) : F(x_0) = 0 \}$$
we shall prove the following theorem.

**Theorem 3.1.** If (H1), (H2) hold and $Y$ is a separable Banach space over $\mathbb{K}$, then for every $F \in \mathcal{F}$ the formula
$$\varphi^F(x) = \sum_{n=0}^{\infty} (F_n(x) - F_n(x_0))$$
for every $x \in X$. 
defines the only Lipschitzian and vanishing at \( x_0 \) solution \( \varphi^F: X \to Y \) of (1), the operator

\[
F \mapsto \varphi^F, \quad F \in \mathcal{F},
\]

is a linear homeomorphism of \( \mathcal{F} \) onto \( \mathcal{F}_0 \) and

\[
\frac{1}{1 + \lambda + \int_{\Omega} |g(\omega)| \rho(f(x_0, \omega), x_0) \mu(d\omega)} \| F \|_{Lip} \leq \| \varphi^F \|_{Lip} \leq \frac{1}{1 - \lambda} \| F \|_{Lip}
\]

for every \( F \in \mathcal{F} \).

**Proof.** In view of Theorem 2.1 we have to show that operator (12) maps \( \mathcal{F} \) onto \( \mathcal{F}_0 \) and for every \( F \in \mathcal{F} \) the first inequality in (13) holds.

Fix \( \psi \in \mathcal{F}_0 \). According to [2, Lemma 2.2] the formula

\[
F(x) = \psi(x) - \int_{\Omega} g(\omega) \psi(f(x, \omega)) \mu(d\omega)
\]

defines an \( F \in Lip(X,Y) \). Since \( \psi \) is a Lipschitzian solution of (1), by Theorem 2.1 the function \( F \) is in \( \mathcal{F} \) with

\[
\| F \|_{L} \leq \| \psi \|_{L} + \lambda \| \psi \|_{L}.
\]

Moreover, as \( \psi(x_0) = 0 \), \( \psi = \varphi^F \) and

\[
\| F(x_0) \| \leq \int_{\Omega} |g(\omega)| \| \psi(f(x_0, \omega)) \| - \psi(x_0) \| \mu(d\omega) \\
\leq \| \psi \|_{L} \int_{\Omega} |g(\omega)| \rho(f(x_0, \omega), x_0) \mu(d\omega).
\]

Consequently,

\[
\| F \|_{Lip} \leq \| \varphi^F \|_{Lip} \left( \int_{\Omega} |g(\omega)| \rho(f(x_0, \omega), x_0) \mu(d\omega) + 1 + \lambda \right),
\]

which ends the proof. \( \square \)

**Remark 3.2.** Since \( \mathcal{F}_0 \) is closed in the Banach space \( (Lip(X,Y), \| \cdot \|_{Lip}) \), it follows from Theorem 3.1 that also \( \mathcal{F} \) is a closed subspace of \( (Lip(X,Y), \| \cdot \|_{Lip}) \).

**Acknowledgements**

This research was supported by the University of Silesia Mathematics Department (Iterative Functional Equations and Real Analysis program).

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.
References

[1] Baron, K.: On the convergence in law of iterates of random-valued functions. Aust. J. Math. Anal. Appl. 6(1), 9 (2009) (Art. 3)
[2] Baron, K., Morawiec, J.: Lipschitzian solutions to linear iterative equations. Publ. Math. Debr. 89(3), 277–285 (2016)
[3] Kuczma, M., Choczewski, B., Ger, R.: Iterative functional equations, Encyclopedia of Mathematics and its Applications, vol. 32. Cambridge University Press, Cambridge (1990)

Karol Baron and Janusz Morawiec
Instytut Matematyki
Uniwersytet Śląski
Bankowa 14
40-007 Katowice
Poland
e-mail: baron@us.edu.pl

Janusz Morawiec
e-mail: morawiec@math.us.edu.pl

Received: July 14, 2016