Abstract. Trace diagrams are structured graphs with edges labeled by matrices. Each diagram has an interpretation as a particular multilinear function, allowing algebraic proofs to be made in the diagrammatic category. We provide a rigorous combinatorial definition of these diagrams, and prove that they may be efficiently represented in terms of matrix minors. Using this viewpoint, we provide new proofs of several standard determinant formulas and show how they may be generalized.

1. Introduction

In the early 1950s, Roger Penrose invented a diagrammatic notation to streamline calculations in multilinear algebra. In his context, indices became labels on edges between spider-like nodes, and tensor contraction meant gluing two edges together [18]. Penrose’s work marked the beginning of what Jones would later call planar algebra [10], a field which has grown out of a confluence of techniques from knot theory, mathematical physics, and quantum topology.

A trace diagram is a generalization of Penrose’s tensor diagram, in which the edges correspond to vectors and may be marked by matrices. This term was first used in [19], in which a special class of trace diagrams generalizing spin networks were shown to provide an additive basis for a certain ring of invariants. While the name is new, the idea behind trace diagrams has appeared before in several contexts [1, 2, 5, 14, 23].

The purpose of this paper is to provide a rigorous theory of trace diagrams. We start with a purely combinatorial definition. Using signed graph colorings, we define a monoidal functor between the diagrammatic category and the category of multilinear functions. Besides the

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definition, we prove two important properties of trace diagrams. Theorem 5.4 shows that trace diagram notation is strictly more powerful than the standard notation, since a single diagrammatic relation may have multiple expressions in the typical notation. Theorem 7.4 describes how any trace diagram may be understood in terms of matrix minors.

As an application, we show how trace diagrams provide insight into classical determinant calculations. Cayley, Jacobi, and other 19th century mathematicians described several methods for calculating both general determinants and special cases [17]. The calculations could often take pages to complete because of the complex notation and the need to keep track of indices. In contrast, we show that diagrammatic proofs of these classic results come very quickly, once the theory has been suitably developed. The diagrammatic proofs are also more easily generalized, as the translation from (20) to (25).

Diagrammatic notations have been in use for some time. Penrose’s work was preceded by Levinson [15], who used diagrams to study angular momentum. The physicists Stedman [23] and Cvitanovic [5] both published extensive books on the subject. In knot theory, Kauffman generalized Penrose’s diagrams and showed their relation to knot polynomials [11]. Przytycki and others placed Kauffman’s work in the context of skein modules [3]. More recently, Kuperberg introduced spiders [12] as a means of studying representation theory. Most of these fall under the general umbrella of Jones’ planar algebras [10]. What we call trace diagrams is referred to occasionally in these works, but has not yet been studied systematically.

This paper is organized as follows. Section 2 provides a short review of multilinear algebra. In Section 3 we introduce the idea of signed graph coloring, which forms the basis for the translation between trace diagrams and multilinear algebra described rigourously in sections 4 and 5. Sections 6 and 7 describe the basic properties of trace diagrams, from which the new proofs of classical determinant results in Section 8 are derived.

2. Multilinear Algebra

This section reviews multilinear algebra and tensors. A nice introductory treatment of tensors is given in Appendix B of [8].

Let $V$ be a finite-dimensional vector space over a field $F$. Informally, a 2-tensor consists of finite sums of vector pairs $(u, v) \in V \times V$ modulo the relations

$$(\lambda u, v) = \lambda(u, v) = (u, \lambda v)$$
for all $\lambda \in F$. The resulting term is denoted $u \otimes v$. In general, tensors are formed from $k$-tuples of vectors subject to the allowance that scalars may be freely moved between terms.

In what follows, we assume that $V$ has basis $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$. The space of $k$-tensors $V^\otimes k \equiv V \otimes \cdots \otimes V$ is itself a vector space with $n^k$ basis elements of the form

$$\hat{e}_{\vec{\alpha}} \equiv \hat{e}_{\alpha_1} \otimes \hat{e}_{\alpha_2} \otimes \cdots \otimes \hat{e}_{\alpha_k},$$

one for each $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in N^k$. By convention $V^\otimes 0 = F$.

Let $\langle \cdot, \cdot \rangle$ be the inner product on $V$ defined by

$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker delta. This extends to an inner product on $V^\otimes k$ with

$$\langle \hat{e}_{\vec{\alpha}}, \hat{e}_{\vec{\beta}} \rangle = \delta_{\alpha_1, \beta_1} \delta_{\alpha_2, \beta_2} \cdots \delta_{\alpha_k, \beta_k},$$

making $\hat{e}_{\vec{\alpha}}$ an orthonormal basis for $V^\otimes k$.

Denote by $\text{Fun}(V^\otimes j, V^\otimes k)$ the space of multilinear functions from $V^\otimes j$ to $V^\otimes k$. Besides composition $f \circ g$, there is another way to combine such functions. Given $f_1 \in \text{Fun}(V^\otimes j_1, V^\otimes k_1)$ and $f_2 \in \text{Fun}(V^\otimes j_2, V^\otimes k_2)$, then $f_1 \otimes f_2 \in \text{Fun}(V^\otimes (j_1+j_2), V^\otimes (k_1+k_2))$ is the multilinear function defined by letting $f_1$ operate on the first $k_1$ tensor components of $V^\otimes (j_1+j_2)$ and $f_2$ on the last $j_2$ components. This makes the space of multilinear functions a monoidal category.

A multilinear function $f \in \text{Fun}(V^\otimes r)$ is usually called a multilinear form. When both $j = k = 0$, functions $f : F \to F$ may be thought of as elements of $F$. In particular, $\text{Fun}(F, F) \cong F$ via the isomorphism $f \mapsto f(1)$.

The space of rank-$r$ tensors is isomorphic to $\text{Fun}(V^\otimes r)$. Given $f \in \text{Fun}(V^\otimes r)$, the isomorphism maps

$$f \mapsto \sum_{\vec{\alpha} \in N^r} f(\hat{e}_{\vec{\alpha}}) \hat{e}_{\vec{\alpha}} \in V^\otimes r.$$

This is the duality property of tensor algebra. Loosely speaking, multilinear functions do not distinguish between inputs and outputs; up to isomorphism all that matters is the total number of inputs and outputs.

One relevant example is the determinant, which can be written as a function $V^\otimes k \to F$ on the column vectors. This may be defined on the tensor product since a scalar multiplied on a single column may be factored outside the determinant. Determinants additionally are antisymmetric, since switching any two columns changes the sign of the determinant. Antisymmetric functions can also be considered as
functions on an exterior (wedge) product of vector spaces, which we do not define here.

3. Signed Graph Coloring

This section introduces graph theoretic principles that will be used in defining trace diagrammatic functions. Although the terminology is borrowed from that common in graph theory, to our knowledge the notion of signed graph coloring is new.

Given a vertex \( v \in V \) of a graph \( G = (V, E) \), denote by \( E(v) \) the set of edges adjacent to \( v \). We say that two edges in the same \( E(v) \) for some \( v \) are adjacent.

**Definition 3.1.** A ciliated graph \( G = (V, E, \sigma^*) \) is a graph \( (V, E) \) together with an ordering \( \sigma^*_v : \{1, 2, \ldots, \deg(v)\} \rightarrow E(v) \) of edges at each vertex \( v \in V \).

By convention, when such graphs are drawn in the plane, the ordering is specified by enumerating edges in a counter-clockwise fashion from a ciliation, as shown in Figure 3. Ciliated graphs are sometimes also called fat graphs.

![Figure 1](image_url)

**Figure 1.** By convention, a ciliation on a vertex induces a counter-clockwise ordering, shown as \((e_1, e_2, e_3, e_4)\) at left. The coloring shown at right induces the permutation \((2 4 1 3)\) at the vertex.

**Definition 3.2.** Given the set \( N = \{1, 2, \ldots, n\} \), an \( n \)-edge coloring of a graph \( G = (V, E) \) is a map \( \kappa : E \rightarrow N \) such that no two adjacent edges have the same label. We denote the set of all \( n \)-edge colorings of a graph \( G \) by \( \text{col}(G) \) when \( n \) is understood.

In graph theory, edge colorings are sometimes called Tait colorings.

Edge colorings induce permutations at the vertices of ciliated graphs. Given an edge coloring \( \kappa \) and a degree-\( n \) vertex \( v \), there is a well-defined permutation \( \pi_\kappa(v) \in S_n \) at each interior vertex \( v \in V_n \) defined by

\[
\pi_\kappa(v) : i \mapsto \kappa(\sigma_v(i)).
\]

In other words, 1 is taken to the label on the first edge adjacent to the vertex, 2 is taken to the label on the second edge, and so on.
Definition 3.3. Given an admissible coloring $\kappa$ of a ciliated graph $G = (V, E, \sigma_*)$, the signature $\text{sgn}_\kappa(G)$ is the product of permutation signatures on the degree-$n$ vertices:

$$\text{sgn}_\kappa(G) = \prod_{v \in V_n} \text{sgn}(\pi_\kappa(v)),$$

where $\text{sgn}(\pi_\kappa(v))$ is the signature of the permutation $\pi_\kappa(v)$.

The signed chromatic index $\chi(G)$ is the sum of signatures over all possible colorings:

$$\chi(G) = \sum_{\kappa \in \text{col}(G)} \text{sgn}_\kappa(G).$$

Definition 3.4. A pre-coloring of a graph $G = (E, V)$ is a coloring $\tilde{\kappa} : \tilde{E} \to N$ of a subset $\tilde{E} \subset E$ of the edges of $G$. A leaf coloring is a pre-coloring of the edges adjacent to the degree-1 vertices.

Two pre-colorings $\tilde{\kappa}_1 : \tilde{E}_1 \to N$ and $\tilde{\kappa}_2 : \tilde{E}_2 \to N$ are compatible if they agree on the intersection $\tilde{E} \equiv \tilde{E}_1 \cap \tilde{E}_2$. In this case, the union $\tilde{\kappa}_1 \cup \tilde{\kappa}_2$ with $|\tilde{\kappa}_1 \cup \tilde{\kappa}_2||_{\tilde{E}_i} = \tilde{\kappa}_i|_{\tilde{E}_i}$ is also a pre-coloring.

If $\tilde{E}_1 \subset \tilde{E}_2$, we say that $\tilde{\kappa}_2$ extends $\tilde{\kappa}_1$ and write $\tilde{\kappa}_2 \succ \tilde{\kappa}_1$. We denote the (possibly empty) set of colorings that extend $\tilde{\kappa}$ by

$$\text{col}_{\tilde{\kappa}}(G) \equiv \{ \kappa \in \text{col}(G) : \kappa \succ \tilde{\kappa} \}.$$

The signed chromatic subindex of a pre-coloring $\tilde{\kappa}$ is the sum of signatures of its extensions:

$$\chi_{\tilde{\kappa}}(G) = \sum_{\kappa \succ \tilde{\kappa}} \text{sgn}_\kappa(G).$$

Example. For $n = 3$, the pre-coloring has two colorings

(2)

In the first case, the permutations are $(\frac{1}{2} \frac{2}{3} \frac{3}{1})$ and $(\frac{3}{2} \frac{2}{1} \frac{1}{3})$, so that the signature is $\text{sgn}(\frac{1}{2} \frac{2}{3} \frac{3}{1}) \cdot \text{sgn}(\frac{3}{2} \frac{2}{1} \frac{1}{3}) = -1$. In the second case, the permutations are $(\frac{1}{2} \frac{2}{3} \frac{3}{1})$ and $(\frac{1}{2} \frac{2}{3} \frac{3}{1})$, indicating a positive signature. Hence the signed chromatic subindex of this pre-coloring is 0.

4. Trace Diagrams

Penrose was probably the first to describe how tensor algebra may be performed diagrammatically [18]. In his framework, lines in a graph represent elements of the vector field $F$, and nodes represent multilinear functions. Trace diagrams are a generalization of Penrose’s tensor
diagrams, in which the edges are directed and may be labeled by matrices and the nodes represent the determinant form.

**Definition 4.1.** A trace diagram is a directed ciliated graph $\mathcal{D} = (V_1 \sqcup V_n, E, \sigma_*)$ together with a marking of edges by linear transformations in $\text{Fun}(V, V)$. Vertices have either degree 1 (in $V_1$) or degree $n$ (in $V_n$). The diagram is closed if $V_1$ is empty. A framed trace diagram is a diagram together with a partition $E_1 = E_I \sqcup E_O$ of the leaf edges adjacent to $V_1$ into ordered inputs $E_I$ and outputs $E_O$.

By convention, framed trace diagrams are drawn with inputs at the bottom of the diagram and outputs at the top. Both are assumed to be ordered left to right.

We also permit multiple markings on the same edge, with the understanding that $\begin{aligned} \text{If there are no markings, then } \mathcal{D} \text{ is a determinant diagram, and the orientation of edges may be omitted.} 
\end{aligned}$

Trace diagrams require an expanded definition of coloring:

**Definition 4.2.** A coloring of a trace diagram $\mathcal{D}$ is a map $\kappa : E \to N \times N$ labeling the head and tail of each edge by $\kappa(e)_1$ and $\kappa(e)_2$, respectively, in such a way that all labels near an $n$-vertex are different, and unmarked edges $e$ have $\kappa(e)_1 = \kappa(e)_2$.

**Definition 4.3.** Let $\mathcal{D}$ be a trace diagram with edge $e \in E$ marked by matrix $A_e$. The coefficient $\psi_\kappa(\mathcal{D})$ of the coloring is

$$\psi_\kappa(\mathcal{D}) \equiv \prod_{e \in E} (A_e)_{\kappa(e)_2 \kappa(e)_1},$$

where $(A)_{ij}$ represents the $i, j$-matrix entry.

Thus, the coloring “picks out” the entry in the column corresponding to the incoming edge and the row corresponding to the outgoing edge:

$$\begin{aligned} (3) \quad \langle i, e_i \rangle \leftrightarrow (A)_{ij} = \langle e_i, A\hat{e}_j \rangle. \end{aligned}$$
Pre-colorings \( \tilde{\kappa} \alpha : (E_I)_i \mapsto \alpha_i \) are in a one-to-one correspondence with basis elements \( \hat{e}_\alpha \in V^\otimes|E_I| \); we will use \( \tilde{\alpha} \) as shorthand for the pre-coloring. Together with some \( \beta \in V^\otimes|E_O| \), the combination \( \alpha \cup \beta \) is a leaf coloring. The coefficients of diagram functions are the signed chromatic subindices of these leaf colorings, weighted by the coloring coefficients.

**Definition 4.4.** Given a framed trace diagram \( \mathcal{D} \), define the weight \( \chi_\gamma(\mathcal{D}) \) of a leaf coloring \( \gamma \) by

\[
\chi_\gamma(\mathcal{D}) = \sum_{\kappa \succ \gamma} \sgn_\kappa(\mathcal{D}) \psi_\kappa(\mathcal{D}).
\]

Define the *trace diagram function* of \( \mathcal{D} \) by linear extension of

\[
f_\mathcal{D} : \hat{e}_\alpha \mapsto \sum_{\tilde{\beta} \in N^{|E_O|}} \chi_{\alpha \cup \beta}(\mathcal{D}) \hat{e}_{\tilde{\beta}}.
\]

If the diagram is closed, we define its *value* to be

\[
\chi(\mathcal{D}) = \sum_{\kappa \in \text{col}(\mathcal{D})} \sgn_\kappa(\mathcal{D}) \psi_\kappa(\mathcal{D}).
\]

**Remark 4.5.** If \( n \) is odd, trace diagrams may be drawn without ciliations, since \( \sgn(\sigma) \) is invariant under cyclic re-orderings:

\[
\sgn \begin{pmatrix} a_1 & \cdots & a_n \\ 1 & \cdots & n \end{pmatrix} = \sgn \begin{pmatrix} a_2 & \cdots & a_n & a_1 \\ 1 & \cdots & n-1 & n \end{pmatrix}.
\]

We will sometimes abuse notation by using the diagram \( \mathcal{D} \) interchangeably with \( f_\mathcal{D} \). We also write formal linear sums of diagrams to indicate the corresponding sums of functions. See the next section for details on why this is permissible.

**Example.** The diagram \( \int \) has no vertices, so the signature is trivially +1, and the input and output must have the same label. This implies that \( \int : \hat{e}_1 \mapsto \hat{e}_1 \), verifying that the diagram is the identity.

The two colorings of \( \mathcal{D} \) describe the action of the underlying diagram on the input \( \hat{e}_1 \otimes \hat{e}_2 \):

\[
\int : \hat{e}_1 \otimes \hat{e}_2 \mapsto (\hat{e}_2 \otimes \hat{e}_1 - \hat{e}_1 \otimes \hat{e}_2).
\]

The input \( \hat{e}_1 \otimes \hat{e}_2 \) corresponds to the pre-coloring at the bottom of the diagram, while each term in the output corresponds to a coloring extension whose coefficient is coloring’s signature.
Example. The simple loop $\bigcirc$ has $n$ colorings. Each coloring is assigned a default signature of 1, since there are no vertices. Hence, the value of the circle is $n$. The “barbell” diagram $\bigcirc - \bigcirc$ has no colorings, since in any coloring the same color meets a vertex twice. Therefore, the diagram evaluates to zero.

Example. The simplest closed trace diagram with a matrix has edge colorings $A_i$ for $i = 1, 2, \ldots, n$. Thus, the diagram’s value is

$$ (6) \quad \bigcirc = a_{11} + \cdots + a_{nn} = \text{tr}(A). $$

5. Trace Diagram Relations and Monoidal Structure

Denote by $\mathfrak{D}(I, O)$ the free $F$-module over framed trace diagrams with $I$ inputs and $O$ outputs. One may compose elements of $\mathfrak{D}(I_1, O_1)$ with those of $\mathfrak{D}(O_1, O_2)$ by gluing outputs to inputs. Since inputs are drawn at the bottom of a diagram and outputs at the top, composition involves drawing one diagram above another.

One may also define $\mathfrak{D}_1 \otimes \mathfrak{D}_2$ as the diagram placing $\mathfrak{D}_2$ to the right of $\mathfrak{D}_1$, making the space of framed trace diagrams a monoidal category.

**Theorem 5.1.** The mapping $D \mapsto f_D$ of Definition 4.4 is a functor of monoidal categories.

**Proof.** It is clear that the mapping preserves the tensorial structure.

To see that it respects composition, let $D_1 \in \mathfrak{D}(I_1, O_1)$ and $D_2 \in \mathfrak{D}(O_1, O_2)$. Then by applying (5) twice

$$ f_{D_2} \circ f_{D_1} : \hat{e}_\alpha \mapsto \sum_{\gamma \in N^{O_2}} \left( \sum_{\beta \in N^{O_1}} \chi_{\alpha, \beta}^\gamma(D_1) \chi_{\beta, \gamma}^\gamma(D_2) \right) \hat{e}_\gamma. $$

The term in parentheses simplifies to

$$ \sum_{\beta \in N^{O_1}} \chi_{\alpha, \beta}^\gamma(D_1) \chi_{\beta, \gamma}^\gamma(D_2) = \chi_{\alpha, \gamma}^\gamma(D_2 \circ D_1). $$

Hence, $f_{D_2} \circ f_{D_1} = f_{D_2 \circ D_1}$. \qed

Intuitively, this result means that a trace diagram’s function may be understood by breaking the diagram up into little pieces and gluing them back together.

**Definition 5.2.** A trace diagram relation is a summation $\sum_D c_D D \in \mathfrak{D}(I, O)$ of framed trace diagrams for which $\sum_D c_D f_D = 0$. 
Given the monoidal structure, one can apply trace diagram relations on small pieces of larger diagrams (called local relations). Indeed, some diagrammatic structures are defined as free summations over diagrams modulo one or more local relations. [3]

The following result establishes the intrinsic meaning of edge orientations.

**Proposition 5.3** (Transpose Diagrams). Let $D$ be a trace diagram, let $D^T$ represent the same diagram in which all matrices have been replaced by their transpose, and let $D^*$ represent the same diagram in which all edges have opposite orientations. Then $f_{D^*} = f_{D^T}$.

**Proof.** Since, $A_{ji} = A_{ji} = A_{ji}$ the impact of transposing matrices on the underlying function is the same as that of reversing edge orientations. □

Trace diagram relations can be made more general than multilinear relations by relaxation of the framing. Denote by $D(m)$ the free $F$-module over tensor diagrams with $m$ ordered leaves. A leaf partition gives a mapping $D(m) \rightarrow D(I, O)$, where $I + O = m$. A (general) trace diagram relation is a summation $\sum_D c_D D \in D(m)$ that restricts under some partition to a framed trace diagram relation.

**Theorem 5.4.** Every leaf partition of a general trace diagram relation gives a framed trace diagram relation.

**Proof.** By Definition 4.4, the weights of a function depend only on the leaf labels, and not on the partition of framing of the diagram. Since the weights are the same, the relations do not depend on the framing. □

The fact that diagrammatic relations are independent of framing is very powerful. One may sometimes read off several identities of multilinear algebra from the same diagrammatic relation.

**Example.** Let $u, v, w \in \mathbb{C}^3$. One can show the cross product and inner product to be

$u \times v = \begin{array}{c} u \cr \hline v \end{array}$ and $u \cdot v = \begin{array}{c} u \cr \bowtie \cr v \end{array}$.

One can also show that

$\begin{array}{c} \times \cr \hline \end{array} = \begin{array}{c} X \cr - \cr \hline \end{array}$. (7)
Consequently,
\[
\begin{align*}
\begin{array}{c}
\text{u} \text{v} \text{w} \text{x} \\
\text{u} \text{v} \text{w} \text{x} \\
\text{u} \text{v} \text{w} \text{x}
\end{array}
\end{align*}
\]
which is the vector identity
\[
(u \times v) \cdot (w \times x) = (u \cdot w)(v \cdot x) - (u \cdot x)(v \cdot w).
\]

It is even possible for certain diagrams to be read in multiple ways, leading to algebraic identities where no diagrammatic ones even exist.

**Example.** The single diagram
\[
\begin{align*}
\begin{array}{c}
\text{u} \text{v} \text{w} \\
\text{u} \text{v} \text{w} \\
\text{u} \text{v} \text{w}
\end{array}
\end{align*}
\]
implies the vector identities
\[
(u \times v) \cdot w = u \cdot (v \times w) = (w \times u) \cdot v = \text{det}[u \; v \; w].
\]

Bullock used similar results for \( n = 2 \) to classify \( \text{SL}(2, \mathbb{C}) \) trace relations \cite{2}.

6. **Diagrammatic Building Blocks**

This section builds a library of local diagrammatic relations that are needed for reasoning about general diagrams.

**Notation 6.1.** Given an ordered set of distinct elements \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \), let \( \tilde{\alpha} \) denote \((\alpha_k, \ldots, \alpha_2, \alpha_1)\). The switch between \( \alpha \) and \( \tilde{\alpha} \) requires \( \lfloor \frac{n}{2} \rfloor \) transpositions, and so \( \text{sgn}(\tilde{\alpha}) = (-1)^{\lfloor \frac{n}{2} \rfloor} \text{sgn}(\alpha) \).

Let \( S^c_{\alpha} \) represent the set of permutations of \( N \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \). If \( \beta = (\beta_1, \beta_2, \ldots, \beta_{n-k}) \in S^c_{\alpha} \), let \( (\alpha \; \beta) \) denote the permutation
\[
(\alpha \; \beta) \equiv \begin{pmatrix}
\alpha_1 & \cdots & \alpha_k & \beta_{n-k} & \cdots & \beta_1 \\
1 & \cdots & k & k+1 & \cdots & n
\end{pmatrix}.
\]

**Proposition 6.2.** If \( \alpha \in N^k \) has no repeated elements then
\[
\begin{align*}
\begin{array}{c}
\text{n-k choose k} \\
\text{n-k choose k}
\end{array}
\end{align*}
\]

If \( \alpha \in N^k \) has repeated elements, then the diagram maps it to 0.

**Proof.** This follows immediately from Definition 4.4. □
We note the following special cases of this result:

(8) \[ \hat{e}_\alpha \mapsto \text{sgn}(\alpha) = \det(\hat{e}_{\alpha_1} \cdots \hat{e}_{\alpha_n}). \]

(9) \[ \sum_{\beta \in S_n} \text{sgn}(\beta)\hat{e}_\beta = (-1)^{|S|/2} \sum_{\beta \in S_n} \text{sgn}(\beta)\hat{e}_\beta. \]

Also, the case \( k = n-1 \) provides a generalization of the three-dimensional cross product.

For diagrams labeled by matrices, the most important result is:

**Proposition 6.3** (Matrix Action at Nodes).

(10) \[ A \hat{n} = \det(A) \hat{n}. \]

If \( A \) has inverse \( A^{-1} \) then

(11) \[ \det(A) = (-1)^{|S|/2}(n-k)! \sum_{\sigma \in S_{n-k}} \text{sgn}(\sigma)\hat{e}_\sigma. \]

**Proof.** The first statement is an immediate consequence of (8). The second follows from the first. \( \square \)

Note that repeated application of (10) implies that \( \det(AB) = \det(A)\det(B) \).

By Proposition 5.3, it also shows that \( \det(A^T) = \det(A) \).

**Proposition 6.4.** If \( \alpha \in N^k \) has no repeated elements, then

\[ \hat{e}_\alpha \mapsto (-1)^{|S|/2}(n-k)! \sum_{\sigma \in S_{n-k}} \text{sgn}(\sigma)\hat{e}_\sigma. \]

**Proof.** The previous result applied twice gives the image of \( \hat{e}_\alpha \) as

\[ (n-k)! \sum_{\sigma \in S_{n-k}} \text{sgn}(\sigma)\hat{e}_\sigma. \]

The signature term simplifies as follows:

\[ \text{sgn}(\sigma)\hat{e}_\sigma = (-1)^{|S|/2}\text{sgn}(\sigma)\hat{e}_\sigma \]

\[ = (-1)^{|S|/2}\text{sgn}(\sigma)\hat{e}_\sigma. \]

**Corollary 6.5** (Determinant Diagram).

(12) \[ = (-1)^{|S|/2}n!\det(A). \]

Proof. Proposition 6.3 gives the factor $\det(A)$, while Proposition 6.4 with $k = 0$ gives the factor $(-1)^{\lfloor \frac{n}{2} \rfloor} n!$. □

The following result will be used repeatedly in later sections.

**Lemma 6.6 (cut-and-paste lemma).** If $\alpha \in N^k$ has no repeated elements and $\beta \in S^k_\alpha$, then

\begin{equation}
\text{sgn}(\alpha \leftarrow \beta) (n-k)! \end{equation}

If $\alpha \in N^k$ has repeated elements, then the diagram maps it to 0.

Proof. As in the previous proposition, the result does not depend on a specific choice of $\beta \in S^k_\alpha$. Hence the summation $\sum_{\beta \in S^k_\alpha} \text{sgn}(\alpha \leftarrow \beta) \hat{e}_\beta$ obtained after applying Proposition 6.2 to the bottom node simplifies to $(n-k)! \text{sgn}(\alpha \leftarrow \beta) \hat{e}_\beta$, where $\beta$ may be chosen arbitrarily. □

Intuitively, this states that when multiple edges connect the same two nodes, a partial labeling may be passed to the inside with a signed factor of $(n-k)!$. And conversely, extra nodes can be added on to the diagram. We will usually choose $\beta$ to be the choice of elements in increasing order.

### 7. Matrix Minors

This section reveals the fundamental role of matrix minors in trace diagram functions. We begin with notation and a review of matrix minors. A good classical treatment of matrix minors is given in Section 2.4 of [13].

Let $A$ be an $n \times n$ matrix over a field $F$ with

$$A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}.$$

A *submatrix* of a matrix $A$ is a smaller matrix formed by “crossing out” a number of rows and columns in $A$.

Let $N \equiv \{1, 2, \ldots, n\}$. Let $I = (I_1, \ldots, I_{k_1})$ and $J = (J_1, \ldots, J_{k_2})$ be ordered subsets of $N$, so that $1 \leq I_1 < \cdots < I_{k_1} \leq n$. Let $A_{I,J}$ denote the submatrix formed from the rows in $I$ and the columns in $J$. The *complementary submatrix* $A_{I,J}^c$ is formed by crossing out the rows in $I$
and the columns in $J$. For $n \geq 3$, the interior $\text{int}(A)$ is the submatrix $A_{(1,n),(1,n)}^c$.

**Example.** Let $I = (1, 2)$ and $J = (3, 4)$. If

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix},$$

then $A_{I,J} = \begin{pmatrix} c & d \\ g & h \end{pmatrix}$, $A_{I,J}^c = \begin{pmatrix} i & j \\ m & n \end{pmatrix}$, and $\text{int}(A) = \begin{pmatrix} f & g \\ j & k \end{pmatrix}$.

**Definition 7.1.** If $I$ and $J$ have the same number of entries, the minor $[A_{I,J}]$ is the determinant of the submatrix $A_{I,J}$. The complementary minor $[A_{I,J}^c]$ is the determinant of the complemental submatrix $A_{I,J}^c$.

A direct formula for the $k \times k$ minor is

$$[A_{I,J}] = \sum_{\sigma \in S_k} \text{sgn}(\sigma) a_{I_1,J_{\sigma(1)}} a_{I_2,J_{\sigma(2)}} \cdots a_{I_k,J_{\sigma(k)}}. \tag{14}$$

In the above example, $[A_{I,J}] = ch - gd$.

**Definition 7.2.** The $(i, j)$-cofactor of $A$ is

$$C_{ij} \equiv (-1)^{i+j} [A_{i,j}^c].$$

The $(I, J)$-complementary cofactor of $A$ is

$$C_{I,J} = (-1)^{I_1+\cdots+I_k+J_1+\cdots+J_k} [A_{I,J}^c].$$

The adjugate (or adjoint) $\text{adj}(A)$ of a square matrix is the matrix comprised of entries $(\text{adj}(A))_{ij} \equiv C_{ji}$.

A student often sees cofactors first in the cofactor expansion formula useful for by-hand calculations of the determinant:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}, \tag{15}$$

where $i \in \mathbb{N}$ is an arbitrary row. Adjugates are sometimes used to compute the matrix inverse since $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

**Proposition 7.3.** Let $A$ be an $n \times n$ matrix. Then

$$[A_{I,J}] = \text{sgn}(J^c \setminus J) \overbracket{I_{J_2}}{I_1} \overbrace{I_k} = \text{sgn}(I^c \setminus I) \overbrace{J_{I_2}}{J_1} \overbracket{J_k}.$$

$$[A_{I,J}] = \text{sgn}(J^c \setminus J) \overbracket{I_{J_2}}{I_1} \overbrace{I_k} = \text{sgn}(I^c \setminus I) \overbrace{J_{I_2}}{J_1} \overbracket{J_k}.$$
Proof. By Proposition 6.4 and the minor formula (14),
\[
\begin{align*}
\sum_{\sigma \in S_k} \text{sgn}(\sigma) \cdot A_{J_\sigma(1)} I_1 A_{J_\sigma(2)} I_2 \cdots A_{J_\sigma(k)} I_k
\end{align*}
\]

Using the cut-and-paste lemma (13), the same diagram reduces to
\[
(n - k)! \text{sgn}(\overrightarrow{J J^c}) = (n - k)! (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \text{sgn}(\overrightarrow{J^c J}) .
\]

This verifies the first function. The second case is similar. □

The following result describes a way to efficiently express arbitrary trace diagrams as summations over matrix minors. In the statement, we let \( M \) be the set of matrices in the diagram and we let \( E_A \) denote the set of edges marked by a matrix \( A \in M \).

**Theorem 7.4.** Let \( D \) be a trace diagram such that each edge has at most one matrix label and vertices are either sources or sinks (meaning every edge adjacent to a common vertex has the same orientation there). Then the following properties hold.

(i) The diagram is equivalent to one that may be decomposed into pieces containing either a single node with multiple markings by the same matrix or no matrix markings.

(ii) Let \( n_A \) be the minimum size of a subset of vertices \( V' \subset V \) for which some assignment \( E_A \to V' \) sends all edges marked by \( A \) to adjacent vertices. Then the diagram’s function may be written as a sum over products of matrix minors, with each \( A \in M \) contributing \( n_A \) minors.

**Proof.** The decomposition may not be possible in general, but it does work if we introduce additional nodes using the cut-and-paste-lemma. This may introduce a constant factor, but it implies that there is an assignment of all matrices in the diagram to nodes such that no node receives two different matrices. By choosing where to add the nodes appropriately, we may assume that there are exactly \( n_A \) nodes receiving the matrix \( A \) in this assignment.

With this done, decompose the diagram by cutting around each node and keeping the matrices in this assignment together with the node. This verifies the first statement.
For the second statement, note that the decomposition allows the function to be expressed as a sum over pre-colorings along the boundaries of each piece in the decomposition. The marked pieces will have the form (16) and correspond to matrix minors; the remaining pieces evaluate to constants.

The next section requires understanding the following diagrams for the cofactor and the adjugate:

**Proposition 7.5.** Let $A$ be an $n \times n$ matrix. Then

\[
C_{I,J} = \frac{(-1)^{\frac{n}{2}}}{(n-k)!} A_{J_1 J_2 \cdots J_k} \quad \text{and} \quad \adj(A) = \frac{(-1)^{\frac{n}{2}}}{(n-1)!} A_{J_1 J_2 \cdots J_k}.
\]

**Proof.** By Proposition 7.3, the complemental minor is

\[
[A^c_{I,J}] = \sgn(J \leftarrow J^c) \frac{\sgn(I \leftarrow I^c)(n-k)!}{(n-k)!} A_{J_1 J_2 \cdots J_k} = \sgn(J \leftarrow J^c) \frac{\sgn(I \leftarrow I^c)(n-k)!}{(n-k)!} A_{J_1 J_2 \cdots J_k}.
\]

Matching this up with the cofactor $C_{I,J} = (-1)^{I_1 + \cdots + I_k + J_1 + \cdots + J_k}$ requires a little bit of work with the signs.

**Lemma 7.6.** Let $J = (J_1, \ldots, J_k)$ and $J^c = (J_1^c, \ldots, J_{n-k}^c)$ be ordered subsets of $N$ whose union is $N$. Then

\[
\sgn(J^c \leftarrow J) = (-1)^{nk + J_1 + J_2 + \cdots + J_k}.
\]

**Proof.** Move the $\{J_i\}$ one at a time to their “proper” positions among the $J^c$. The ordering implies

\[
(\ldots, J^c_{k-1}, \ldots, n-k, \ldots, J_{n-k}, J_k, \ldots) = (\ldots, J_k + 1, \ldots, n, J_k, \ldots),
\]

so $n - J_k$ transpositions are required to return $J_k$ to its proper place. Repeating this for each other $J_i$ gives the identity after a total of $nk - (J_1 + \cdots + J_k)$ transpositions.

Thus $\sgn(J^c \leftarrow J) \sgn(I \leftarrow I^c) = (-1)^{\frac{n}{2}} (-1)^{I_1 + \cdots + I_k + J_1 + \cdots + J_k}$, verifying the diagram for the general cofactor is as stated, and the first result is proven.

The adjugate diagram is the case $k = 1$ with edges reversed.
8. Three Short Determinant Proofs

There are several standard methods for computing the determinant. The Leibniz rule is the common definition using permutations. Cofactor expansion provides a recursive technique that lends itself well to by-hand calculations. Laplace expansion is similar but uses generalized cofactors. A lesser known technique is Dodgson condensation [6], which involves recursive computations using $2 \times 2$ determinants.

Theorem 7.4 implies a straightforward diagrammatic approach to finding determinant identities: decompose the diagram for the determinant into pieces containing at most one node, and express the result as a summation over matrix minors. This approach gives the cofactor and Laplace formulae.

**Proposition 8.1** (Cofactor Expansion). For an $n \times n$ matrix $A$ and $j \in \{1, 2, \ldots, n\}$,

$$\text{det}(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ji} C_{ji}. \quad (19)$$

**Proof.** Proposition 6.5 states that

$$\begin{array}{c}
\text{det}(A) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} n! \text{det}(A) \n
Using the cut-and-paste lemma [13], we may choose one strand label arbitrarily, so that

$$\begin{array}{c}
\text{det}(A) = n \sum_{j=1}^{n} C_{ij} a_{ij} = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} n(n-1)! \sum_{j=1}^{n} C_{ij} a_{ij}.
\end{array}$$

Canceling the common $(-1)^{\left\lfloor \frac{n}{2} \right\rfloor} n!$ factor proves the first equality. The second equality follows by transposing the diagrams. \[\square\]

This result is easily generalized by labeling several strands instead of just one (cf. Theorem 1 in Section 2.4 of [13]).

**Proposition 8.2** (Laplace’s Theorem).

$$\text{det}(A) = \sum_{1 \leq J_1 < \cdots < J_k \leq n} C_{I,J} [A_{I,J}] = \sum_{1 \leq J_1 < \cdots < J_k \leq n} C_{J,I} [A_{J,I}].$$
**Proposition 8.3.** Let which is used to derive Dodgson condensation \[21\]. In contrast to the first diagram in the last term is \((-1)^{\frac{k}{2}}(n-k)!C_{I,J}\), and the second (including the sign term) is \(A_{I,J}\).

We now turn to the Jacobi determinant theorem, first stated in [9], which is used to derive Dodgson condensation [21]. In contrast to the previous proofs, we state first the diagrammatic theorem, and show Jacobi’s result as a corollary. This proof was first given in [10].

**Proposition 8.3.** Let \(A\) be an invertible \(n \times n\) matrix, and let \(I\) and \(J\) be ordered subsets of \(N\). Then

\[
\begin{align*}
\text{(20)} & \quad = c_1c_2\det(A)^{k-1} \quad \text{(20)} & \quad = c_1c_2\det(A)^{k-1} \quad \text{(20)} & \quad = c_1c_2\det(A)^{k-1} \\
& \quad c_1 = \frac{(n-1)!}{k!(n-k)!} \quad c_1 = \frac{(n-1)!}{k!(n-k)!} \quad c_1 = \frac{(n-1)!}{k!(n-k)!} \\
& \quad \text{where } c_1c_2 = \left((-1)^{\frac{k}{2}}(n-1)!\right)^k \text{sgn}(J^c) \text{sgn}(I^c) \frac{k!}{(n-k)!}.
\end{align*}
\]

**Proof.** Use Proposition [6.3] to move each group of \(n-1\) matrices in the lefthand diagram of (20) onto a single edge labeled by \(A\), then use Proposition [6.4] with \(k = 1\) to eliminate the “bubbles” in the graph. This reduces the diagram to

\[
\begin{align*}
& \quad c_1\det(A)^k \quad c_1 = \frac{k!}{n!} \quad c_1 = \frac{k!}{n!} \\
& \quad = c_1\det(A)^{k-1} \quad = \frac{k!}{n!} \quad = \frac{k!}{n!} \\
& \quad c_1 = c_2\det(A)^{k-1} \quad c_1 = c_2\det(A)^{k-1} \quad c_1 = c_2\det(A)^{k-1}.
\end{align*}
\]
The second step is also a consequence of Proposition 6.3. The third step uses the cut-and-paste lemma (13) twice. The constants are 
\[ c_1 = \left( (-1)^{\frac{n}{2}} (n-1)! \right)^k \] and 
\[ c_2 = \text{sgn}(J^c \overleftarrow{J}) \text{sgn}(I^c \overleftarrow{I}) \frac{k!}{(n-k)!}. \]
\[ \square \]

Corollary 8.4 (Jacobi Determinant Theorem). Let \( A \) be an \( n \times n \) invertible matrix, and let \( A_{I,J} \) be a \( k \times k \) submatrix of \( A \). Then
\[ [\mathbf{adj}(A)]_{I,J} = C_{J,I} \text{det}(A)^{k-1}, \]
where \([\mathbf{adj}(A)]_{I,J}\) is the corresponding minor in the adjugate of \( A \).

Proof. Rewrite (20) as \( D_1 = c_1 c_2 \text{det}(A)^{k-1} D_2 \). By (17),
\[ D_2 = (-1)^{\frac{n}{2}} (n-k)! C_{J,I} \equiv c_3 C_{J,I}. \]

To see the meaning of \( D_1 \), consider the following revision of (18):
\[ [A_{I,J}] = \frac{\text{sgn}(J^c \overleftarrow{J}) \text{sgn}(I^c \overleftarrow{I})}{k!} \]
\[ \frac{J^c_1 \ldots J^c_{n-k}}{I^c_1 \ldots I^c_{n-k}}. \]

From this, one obtains a diagram for \([\mathbf{adj}(A)]_{I,J}\) by replacing each \( A \) with the adjugate diagram (17). The result is a multiple of \( D_1 \):
\[ \mathbf{adj}(A)_{I,J} = \frac{\text{sgn}(J^c \overleftarrow{J}) \text{sgn}(I^c \overleftarrow{I}) \left( (-1)^{\frac{n}{2}} \right)^k}{k!\left( (n-1)! \right)^k} \]
\[ \mathbf{D} \equiv c_4 D_1. \]

Combining (20), (22), and (23) gives
\[ \mathbf{adj}(A)_{I,J} = c_4 \mathbf{D}_1 = c_1 c_2 c_3 \mathbf{det}(A)^{k-1} \mathbf{D}_2 = c_1 c_2 c_3 c_4 \mathbf{det}(A)^{k-1} C_{J,I}. \]

It is straightforward to check that \( c_1 c_2 c_3 c_4 = 1 \). \[ \square \]

The first proofs of this theorem took several pages to complete, and required careful attention to indices and matrix elements. A modern proof is given in [21] that also takes several pages, and relies on expressing the minor as the determinant of an \( n \times n \) matrix derived from \( A \). By contrast, the diagrammatic portion of the proof (Proposition 8.3) contains the essence of the result and was relatively easy. The more difficult part was showing that the diagrammatic relation corresponded to the correct algebraic statement.

Many identities in linear algebra are simply special cases of this theorem. For example, when \( I = J = N \), then \([A^c_{I,J}] = 1\) trivially and so
\[ \text{det}(\mathbf{adj}(A)) = \text{det}(A)^{n-1}. \]
Charles Dodgson’ condensation method [6] also depends on this result. The following example shows the condensation method at work on a $4 \times 4$ determinant.

$$
\begin{vmatrix}
-2 & -1 & 1 & 4 \\
-1 & -2 & -1 & 6 \\
-1 & -1 & 2 & 4 \\
2 & 1 & -3 & -8 \\
\end{vmatrix} \rightarrow \begin{vmatrix}
3 & -1 & 2 \\
-1 & -5 & 8 \\
1 & 1 & -4 \\
\end{vmatrix} \rightarrow \begin{vmatrix}
8 & -2 \\
-4 & 6 \\
\end{vmatrix} \rightarrow -8,
$$

where -8 is the determinant of the original matrix. Each step involves taking $2 \times 2$ determinants, making the process easy to do by hand. However, the technique fails for some matrices since it involves division.

The method relies on the particular case $I = J = \{1, n\}$. Then $[A_{i,j}]$ is the determinant of the interior entries, and $[\text{adj}(A)_{i,j}] = C_{11}C_{nn} - C_{1n}C_{n1}$, where $C_{ij}$ is the cofactor, so (21) becomes

$$
\text{det}(A) = \frac{C_{11}C_{nn} - C_{1n}C_{n1}}{\text{det} \text{int}(A)}.
$$

For $3 \times 3$ matrices, this is precisely Dodgson’s method. Larger determinants are computed using several iterations of this formula.

9. Conclusion

The main purpose of this paper has been to develop the theory of trace diagrams. A secondary purpose is to provide a lexicon for their translation into linear algebra. The advantage in this approach to linear algebra lies in the ability to generalize results. For example, the proof of Proposition 8.3 is easily generalized when $ik \leq n$ to

$$
C \cdots C \cdots C \cdots C = C \text{det}(A)^{k-1} \cdots C
$$

for some $C$ independent of $A$. The diagram on the left places a generalization of $\text{adj}(A)$ within a minor diagram.

There is much more to be said about trace diagrams. The case $n = 2$ was the starting point of the theory [15] and has been studied extensively, most notably providing the basis for spin networks [4] [11] and the Kauffman bracket skein module [3]. In the general case, the
coefficients of the characteristic equation of a matrix can be understood as the \( n + 1 \) “simplest” closed trace diagrams \([20]\).

The diagrammatic language also proves to be extremely useful in invariant theory. It allows for easy expression of the “linearization” of the characteristic equation \([20]\), from which several classical results of invariant theory are derived \([7]\). Diagrams have already given new insights in the theory of character varieties and invariant theory \([2, 22, 14]\), and it is likely that more will follow.

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