MHD waves within noncommutative Maxwell theory

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In the presence of a strong uniform magnetic field, we study the influence of space noncommutativity on the electromagnetic waves propagating through a quasi-static homogeneous plasma. In this treatment, we have adopted a physical model which considers plasma as quasi-neutral single fluid. By using noncommutative Maxwell theory, the ideal magnetohydrodynamics (MHD) equations are established, in which new equilibrium conditions are extracted. As an empirical study, some attractive features of MHD waves behavior are investigated. Furthermore, it is shown that the presence of space noncommutativity enhances slightly the phase velocity of the incompressive shear Alfvén waves. In a compressible plasma, the noncommutativity plays the role of an additional compression on the medium, in which its relevant effect on the fast modes occurs for highly oblique branches, while the low effect appears when the propagations are nearly parallel or anti-parallel. In addition, it turned out that the influence of space deformation on the slow modes is \( \sim 10^3 \) times smaller than that on the fast modes. The space noncommutativity effect on the slow waves is negligible in low plasma \( \beta \) value, and could appear when \( \beta \) is higher than 0.1, thus the extreme modification occurs for oblique slow waves propagating with angles between 30\(^\circ\) and 60\(^\circ\). Finally, we comment on the possible effect of such waves on CMB spectrum in photon-baryon plasma.

I. INTRODUCTION:

It is well known that magnetohydrodynamics (MHD) waves [1 - 3] play an important role in the field of space plasma. In astrophysics, it remained for a long time the preferred theory in the description of the dynamics of various astrophysical plasma systems such as the formation of the solar corona which is associated with the problem of the plasma heating and the solar wind acceleration [4 - 6]. Indeed, many examples could be given in the application of the MHD theory namely in space plasma, like as the study of the linear properties of the fast magnetosonic propagating in inhomogeneous plasma which is done by several authors [7 - 9] in order to model these waves in coronal loop. T. K. Suzuki et al. [10] proposed a collisionless plasma in which the damped fast MHD waves are responsible of the heating and acceleration of winds from rotating stars due to the observational evidence for locally strong magnetic fields in stellar atmospheres.

In cosmology, after the observation of the cosmic microwaves background (CMB) radiations in 1965, it is believed that MHD played a major role in shaping the radiation spectrum during the so-called plasma epoch. In fact, the evidence that the plasma was magnetized has been confirmed after the measurements of background magnetic fields which are of the order of \( \mu \text{Gauss} \). Moreover, most of the theories predict that these magnetic fields are the amplified remnants of a seed cosmological magnetic field generated in the early Universe [11]. The presence of such field in the primordial plasma influences the acoustic waves pattern of the CMB anisotropy power spectrum [12 - 14], e.g. Adams et al. [13] argued that the primordial density fluctuations that are generated in inflationary Universe enter the horizon before the last photon scattering, and initiate magneto-acoustic oscillations in the photon-baryon plasma due to the presence of primordial magnetic fields. These oscillations distort the primordial spectrum of fluctuations and affect CMB anisotropy. Therefore, dealing with the features of MHD wave dynamics is a significant part of cosmological plasma, and because of the magnetic forces, the theory of MHD is more complicated and fascinating than hydrodynamics itself due to the influence of the magnetic field on the traditional sound waves which consequently transform to slow and fast magneto-acoustic waves. Furthermore, a new kind of wave appears, called Alfvén wave, which arises from magnetic tension and propagates along the field lines without disturbing the thermal pressure or density of the plasma.

On the other hand, there has been a large interest in the study of physical phenomena on a non commutative (NC) spacetime. The idea of spacetime noncommutativity is not new and was first discussed by Snyder in 1947 [15]. At this time the theory of renormalization was not yet well established and the goal was to introduce a natural cut-off to deal with infinities in quantum field theory. However this theory was plagued with several problems such as the violation of unitarity and causality which make people abandoning it. The appearance of such theory, baptized noncommutative geometry, as a limit of string theory has generated a revival of interest for this theory [16, 17].

In the framework of noncommutative geometry the po-
sition vector $x^\mu$ is promoted to an operator $\hat{x}^\mu$ satisfying the relation
\[ [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (1) \]

where $\theta^{\mu\nu}$ is a real, antisymmetric constant matrix which has the dimension of area with elements of order $(\Lambda_{NC})^{-2}$ in system unit ($\hbar = c = 1$). $\Lambda_{NC}$ is the energy scale where the effects on the noncommutativity of spacetime will be relevant.

The role of $\theta^{\mu\nu}$ can be compared to that of the Planck constant $\hbar$ which quantifies in quantum mechanics the level of noncommutativity between space and momentum. In Moyal algebra, the product of two arbitrary fields is defined by $*$ product (the star or Moyal product) [18, 19]:
\[ (f * g)(x) = \left[ \exp\left(\frac{i}{2} \theta^{\mu\nu} \partial_\mu \partial_\nu \right) f(x) g(y) \right]_{x=y}, \quad (2) \]

Some cosmological effects of noncommutativity have been studied. When spacetime is noncommutative on short distance scales, this may have an imprint on early Universe physics, and leads to an interesting consequence at a microscopic level. Indeed, it could be one of possible scenarios that may cause the generation of the density perturbations and primordial magnetic fields in the inflationary Universe [20]. Such studies allow to predict some bounds on noncommutativity scale $\Lambda_{NC}$ which may have a temperature dependence [21].

Our work is devoted to study the MHD waves by taking into account the space modification, and focusing on the description of a homogeneous plasma in a non-smooth space. The aim of this work is to make a theoretical background in the field of NCMHD waves and seek some future works to study this topic. Opening this new window inevitably leads to deal with the interactions of space deformation and plasma waves, which could be considered as a new experimental area for testing space noncommutativity contribution.

This paper is organized as follows. In the second section we start with a brief review of noncommutative classical electrodynamics, from which we derive the NC-Maxwell equations. Then in the third section, and by assuming a small $\theta$ matrix, we establish the modified MHD equations to first $\theta$-parameter for a single conductor medium. As an application, in the fourth section we deal with a particularly interesting case of a high conductor ($\sigma \to \infty$) fluid which is considered as plasma medium. New equilibrium conditions are deduced as well as the main attractive features related to the NCMHD waves are studied for a homogeneous plasma around an equilibrium state. Finally, the obtained results are discussed and compared with those known in usual space.

II. NONCOMMUTATIVE MAXWELL EQUATIONS:

It is understood that NC gauge field theories cause the violation of Lorentz invariance when $\theta$ is considered as a constant matrix, except if this matrix is promoted to a tensor related to the contracted Snyder’s Lie algebra [22]. The problem of unitarity appears also with time-space noncommutativities ($\theta^{0i} \neq 0$) [9, 10]. In particular, NC Maxwell theory loses the causality due to the appearance of derivative couplings in the Lagrangian with the Lorentz invariance exhibited by plane wave solutions [23].

The free Maxwell action on noncommutative space is given by
\[ S = -\frac{1}{4} \int dx \hat{F}_{\mu\nu} \ast \hat{F}^{\mu\nu} \quad (3) \]
where $\hat{F}_{\mu\nu}$ is the noncommutative strength field
\[ \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i\epsilon [\hat{A}_\mu, \hat{A}_\nu], \quad (4) \]
where $\epsilon$ is the electric charge, and $[\hat{A}_\mu, \hat{A}_\nu]$ is the Moyal bracket defined as
\[ [\hat{A}_\mu, \hat{A}_\nu] = \hat{A}_\mu \ast \hat{A}_\nu - \hat{A}_\nu \ast \hat{A}_\mu. \]

According to the Seiberg-Witten map to the first $\theta$ order of the NC gauge and strength fields [17], we get
\[ \hat{A}_\mu = A_\mu - \frac{\epsilon}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}) \]
\[ \hat{F}_{\mu\nu} = F_{\mu\nu} + \epsilon \theta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \epsilon \theta^{\alpha\beta} A_\alpha \partial_\beta F_{\mu\nu}, \quad (5) \]
with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual strength electromagnetic field and $A_\mu$ is the vector-potential.

Hence, from action (3) the Lagrangian in four-dimensional spacetime is
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{\epsilon}{8} \theta^{\alpha\beta} F_{\alpha\beta} F_{\mu\nu}^2 - \frac{\epsilon}{2} \theta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} F_{\mu\nu} + O(\theta^4) + A_\mu J^\mu, \quad (6) \]
where we have added the external vector-current $J^\mu = 4\pi (\rho_\mu, \mathbf{j})$ and taken into consideration the integration of the term $A_\mu J^\mu$ over the whole spacetime which leads to $A_\mu J^\mu$ due to the integral property of star product [18].

By using the expressions of the electric field $E^i = F^{0i}$ and the magnetic induction field $B_k = \frac{\epsilon}{2} \epsilon_{ijk} F^{ij}$ ($e^{123} = 1$), and considering $\epsilon e^{ij} = e^{ijk}$ with $\theta^{0i} = 0$ (space-space noncommutativity), we can extract a non-linear equation, one of the most important NC Maxwell equations in Gaussian units as follows [24]
\[ \nabla \cdot \mathbf{E} = 4\pi \rho_\mu \quad (7) \]
\[ \frac{\partial}{\partial t} \mathbf{E} - \nabla \times \mathbf{H} = -4\pi \mathbf{j}, \quad (8) \]
where \( j \) is the current and \( \rho_q \) is the charge density. Eq. (8) represents the modified Ampere’s law. The displacement \( \mathcal{E} \) and magnetic \( \mathcal{H} \) fields are also given by

\[
\mathcal{E} = E + d, \\
\mathcal{H} = B + h,
\]

with

\[
d = (\theta.B)E - (\theta.E)B - (E.B)\theta \\
h = (\theta.B)B + (\theta.E)E - \frac{1}{2} (E^2 - B^2). \tag{9}
\]

The dual tensor \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \) (\( \epsilon_{\mu\nu\alpha\beta} \) is an antisymmetric tensor Levi-Civita) always satisfies the equation \( \partial_\mu \tilde{F}^{\mu\nu} = 0 \) which implies that

\[
\frac{\partial}{\partial t} B + \nabla \wedge E = 0 \tag{10} \\
\nabla \cdot B = 0, \tag{11}
\]

where the symbol \( \wedge \) denotes the vector product.

The choice of the matrix \( \theta^{\mu\nu} \) means that we are dealing with a preferred frame in which the background electromagnetic field related to \( \theta \)-matrix is only reduced to a constant background magnetic field. As it was shown in several works \([25-27]\), \( \theta \) space-space noncommutativity preserves the unitarity and is compatible with most works done on NC theories.

It turned out from the paper of Kruglov \([24]\) that the electromagnetic waves solutions of the linear equations of the classical electrodynamics are the solutions of the nonlinear wave propagation equations of the electromagnetic fields derived from NC Maxwell theory at \( j = 0 = \rho \). Also, more features of the classical waves propagating have been discussed by Z. Guralnik et al. \([28]\). The authors deduced that the phase speed of these waves is different from \( c \) (with small modification) in case of a transverse propagation with respect to the background magnetic field induction, while the parallel propagation propagation is unchanged. Furthermore, Y. Abe et al. \([29]\) studied a more general case of the electric-magnetic duality symmetry within noncommutative Maxwell theory, in which the polarizations of the propagating waves have been discussed. T. Mariz et al. \([30]\) gave a detailed study on the dispersion relation for plane waves in the presence of a constant background electromagnetic field, the authors did not find any restriction on the plane waves solution in the Seiberg-Witten approach of noncommutative gauge theory which is not the case in strictly Moyal approach, where they deduced that no plane waves are allowed when time is noncommutative.

In our study, we focus on the classical behavior of the electromagnetic waves propagating through a plasma with high conductivity in the presence of both, a magnetic field, and NC space. This treatment is based on the classical MHD theory which is worked out in the framework of NC Maxwell theory.

### III. NONCOMMUTATIVE MHD EQUATIONS:

Let us start with the continuity equation that describes the flow motion

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{12}
\]

with \( \rho \) is the mass density of the medium, \( \mathbf{V} \) its velocity. If this fluid has the ability to carry a density current \( \mathbf{J} \) (conductor), then the rising Ampere force is \( \mathbf{J} \times \mathbf{B} \), once the magnetic field \( \mathbf{B} \) is present, and the plasma is a quasi-neutral in large scale greater than Debye length, consequently the electric volume force \( \rho_q \mathbf{E} \) vanishes, and the moment fluid equation becomes

\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = -\nabla p + \mathbf{j} \wedge \mathbf{B}, \tag{13}
\]

with \( p \) is the pressure which acts on the boundaries of the infinitesimal fluid volume.

In case of a medium with high conductivity (\( \sigma \rightarrow \infty \)) (ideal plasma), the known Ohm’s law for a moving conductor which describes the current \( \mathbf{J} \) in terms of magnetic and electric fields is reduced to the following simple relationship between these fields

\[
(\mathbf{E} + \mathbf{V} \wedge \mathbf{B}) = 0. \tag{14}
\]

Since the quasi-neutrality is assumed, the equation \( \nabla \cdot \mathcal{E} \approx 0 \) does not constitute a dynamical evolution equation. Also the equation \( \nabla \cdot \mathcal{B} = 0 \) is only a constraint, not an evolution equation since it does not include time derivative. In case of non-relativistic MHD approximations where the conducting fluid moves very slowly, this term is absent (\( \frac{\partial \rho}{\partial t} \sim 0 \)) which is the limit of slow motion and large scale spatial derivatives, the displacement current is always negligible, and from Eq. (8) we get

\[
\mathbf{j} = \frac{1}{4\pi} \nabla \wedge \mathcal{H} - \frac{1}{4\pi} \frac{\partial \mathbf{d}}{\partial t}. \tag{15}
\]

In order to express the relations of \( \mathbf{d} \) and \( \mathbf{h} \) given in Eq. (9) in terms of \( \theta, \mathbf{B} \) and \( \mathbf{V} \), we use Eq. (14) to extract the following relationships up to the first order \( \theta \)

\[
\mathbf{d} = (- (\theta \cdot \mathbf{B}) (\mathbf{V} \wedge \mathbf{B}) + ((\mathbf{V} \wedge \mathbf{B}) \cdot \theta) \mathbf{B}) \\
\mathbf{h} = (\theta \cdot \mathbf{B}) \mathbf{B} + (((\mathbf{V} \wedge \mathbf{B}) \cdot \theta) (\mathbf{V} \wedge \mathbf{B}) \\
- \frac{1}{2} ((\mathbf{V} \wedge \mathbf{B})^2 - \mathbf{B}^2) \theta). \tag{16}
\]

To establish the NCMHD equations, it is useful to express the current \( \mathbf{j} \) as functions of \( \mathbf{B} \) and \( \mathbf{V} \). By using Eqs. (14) and (10), we then obtain

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{V} \wedge \mathbf{B}). \tag{17}
\]
When $\theta$ goes to zero, Eq. (13) tends to the usual known momentum equation.

Based on the NCMHD equations, we study the electromagnetic waves behavior using a linear mode analysis around the equilibrium. First, we should determine the equilibrium condition of the considered ideal plasma in the presence of noncommutativity. Plasma is said to be in equilibrium if $V = 0$ and none of the variables depend on time, so from Eq. (13), we get

$$\nabla p = j \wedge B.$$

Using the relationships (15) and (16), we deduce that

$$\nabla p = -\nabla \left( \frac{1}{4\pi} (\frac{1}{2}(\theta \cdot B) + 1) B^2 \right) + \frac{1}{4\pi} B(1 + (\theta \cdot B)) \nabla B + \frac{B^2}{4\pi} \nabla (\theta \cdot B) + \frac{1}{4\pi} \left( B + (\theta \cdot B) B + \frac{B^2}{2} \right),$$

where the first term represents the modified magnetic pressure and the rest terms are the modified magnetic tension. In the equilibrium conditions, the evolution equation of the magnetic field in the noncommutative space is different from that given in usual space. Certainly, the homogeneous case of constant $B$ and $p$ are one of the possible solutions of Eqs. (19). In the next treatment of the electromagnetic waves propagation through a plasma around the equilibrium, we will assume that the component of the background field along the direction of the magnetic field, which means that $\theta$ is parallel to $B$.

IV. NONCOMMUTATIVE MHD WAVES

It is well known that the waves play an important role in the propagative phenomena related to the plasma physics. The following treatment is mainly devoted to the description of the wave properties of plasma in noncommutative space within MHD approximation by a linear mode analysis. Notice that, the equations of ideal NCMHD (12), (13) and (17) are highly nonlinear and self-consistent.

Let us study an homogeneous plasma (in equilibrium state) under a constant magnetic field $B_0$, in which all the parameters do not depend on coordinates, i.e. $\rho_0$ and $p_0$ are constant with $V_0 = 0$. Hence, all quantities $Q(\rho, p, V, B)$ can thus be written as a sum of an equilibrium term and a small first-order perturbation

$$Q(r, t) = Q_0(r) + \epsilon Q_1(r, t).$$

(20)

Higher order terms describing the perturbations are neglected. We linearize the NCMHD equations by inserting the Fourier transformation of all quantities

$$Q_1(r, t) = \int dk \tilde{Q}_1 \exp i (k \cdot r - \omega t)$$

(21)

into eqs. (12), (13) and (17). Then, we extract the terms of first order $\epsilon$ with neglecting all remaining terms of order $\epsilon^2$ and higher as follows

$$\tilde{\rho}_1 = \frac{\rho_0}{\omega} k \cdot \tilde{V}_1,$$

$$\omega p_0 \tilde{V}_1 = \frac{\nu^2 p_0}{\omega} k (k \cdot \tilde{V}_1) - \frac{e \theta B_0}{4\pi} C \wedge B_0 - \frac{B_0 \cdot \tilde{B}_1}{4\pi} (k \wedge \theta) \wedge B_0,$$

$$\omega C = - (k \cdot \tilde{V}_1) (k \cdot B_0) + (k \wedge B_0) (k \cdot \tilde{V}_1).$$

(22)

(23)

(24)

with $1 + \theta B_0 \approx e^{\theta B_0}$ and $C = k \wedge B_1$.

Notice that we have considered the static magnetic field $B_0$ is parallel to $\theta$ which is proportional to the background magnetic field. A polytropic adiabatic law which gives the relationship between the pressure $p$ and the density $\rho$ has been taken into account

$$p = p_0 \left( \frac{\rho}{\rho_0} \right)^\gamma.$$

(25)

As a consequence, $p_1 = v_s^2 \rho_1$, from which, we derive the expression of the square of sound speed $v_s^2 = \frac{2\rho_0}{\rho_0}$, with $\gamma$ is the adiabatic index.

The equations (22), (23) and (24) are a homogeneous set of 6 independent equations for 6 variables since on the other hand, we have the constraint $k \cdot \tilde{B}_1 = 0$ which indicates a transverse propagation of electromagnetic waves. From Eq. (22), we deduce that the density variations are only related to the velocity components along the wave vector, and similarly to the commutative case, the study of the waves in an incompressible plasma ($\rho_1 = p_1 = 0$) shows that the velocity motion is transversal with respect to the wave vector $k$ and consequently, the fluctuating magnetic field $\tilde{B}_1$ is perpendicular to $B_0$.

In order to analyze the various modes arising from the perturbed homogeneous plasma, it is worth focusing on the dispersion relations which connect the wave phase velocity $v_{ph} = \frac{\omega}{k}$ with the measured quantities related to the plasma features themselves. By substituting the vector product of eq. (23) with $k$ into Eq. (24), we get

$$\left( 1 - \frac{e \theta B_0}{4\pi \rho_0 \omega^2} \right) C - \frac{1}{\omega} (k \wedge B_0) (k \cdot \tilde{V}_1)$$

$$= \frac{e}{4\pi \rho_0 \omega^2} B_0 \cdot (k \wedge B_1) (k \wedge \theta)$$

(26)

At this stage, it is worth analyzing the Alfvén and magnetosonic waves propagating in incompressible and compressible plasmas, and compare them with those obtained in commutative space. Let us separately treat the two different states of plasma when the propagation is not purely sonic ($v_{ph} \neq v_s$). Firstly, we consider the incompressible case of plasma, i.e. $B_0 \cdot B_1 = 0$ which leads to

$$C \wedge (k \wedge B_0) \neq 0.$$
From the vector product of Eq. (26) with (k ∧ B₀), we obtain the first dispersion relation for an incompressible plasma

\[
\left( 1 - \frac{e^{\theta B_0} (k \cdot B_0)^2}{4\pi \rho_0 \omega^2} \right) = 0,
\]

(27)

where we can get the norm of wave phase velocity in the presence of noncommutativity

\[
v_{\text{ph}}^2 = (v_{A}^{nc})^2 \cos^2 \alpha
\]

(28)

with \((v_{A}^{nc})^2 = e^{\theta B_0} v_A^2\) is the modified Alfvén velocity and \(v_A^2 = B_0^2 / 4\pi \rho_0\) is the usual Alfvén velocity. Indeed, in case of a parallel propagation, the modified incompressible Alfvén waves can be deduced,

\[
v_{\text{ph}}^2 = (v_{A}^{nc})^2.
\]

(29)

Note that the Alfvén mode undergoes a modification coming from space deformation. This modification plays a role of another velocity which is added to the usual Alfvén velocity, consequently, enhances the value of this latter. In fact, the influence of the noncommutativity is considerable when we deal with an incompressible plasma emerged in a strong magnetic field \(B_0\).

Secondly, if the plasma is compressible, this means that \(B_0B_1 \neq 0\), therefore, by performing the scalar product of Eq. (26) with \((k \cdot B_0)\), we obtain the second dispersion relation for a compressible plasma

\[
\left( 1 - \frac{e^{\theta B_0} (k \cdot B_0)^2}{4\pi \rho_0 \omega^2} \right) \left( 1 - \frac{k^2 v_s^2}{\omega^2} \right) =
\]

\[
\frac{1}{4\pi \rho_0 \omega^2} (k \cdot B_0)^2 \left[ e^{\theta B_0} + \frac{2}{k^2} (k \cdot B_0)^2 \right]
\]

\[
+ \frac{(k \cdot B_0)^2}{4\pi \rho_0 \omega^2} [i(k \cdot B_0)](1 - \frac{k^2 v_s^2}{\omega^2}),
\]

(30)

from which, we extract the following fast and the slow modes

\[
(v_{A}^{nc})_{\text{ph}(\pm)}^2 = \frac{1}{2} \left[ \pm \left( (v_{A}^{nc})^2 \exp (\theta B_0 \sin^2 \alpha) + v_s^2 \right)^2 - 4 (v_{A}^{nc})^2 v_s^2 \exp (\theta B_0 \sin^2 \alpha) \cos^2 \alpha \right]^{1/2}
\]

\[
+ \left( (v_{A}^{nc})^2 \exp (\theta B_0 \sin^2 \alpha) + v_s^2 \right],
\]

(31)

The signs (+) and (−) in eq. (31) indicate the fast and slow magnetosonic (or magneto-acoustic) waves which arise from the coupling between magnetic compression (Alfvénic) and medium compression (sonic). We note that the influence of noncommutativity on the magneto-acoustic waves involves again the modified Alfvén velocity (29).

Let us define the quantity \(\Delta_\pm\)

\[
\Delta_\pm = \frac{(v_{A}^{nc})_{\text{ph}(\pm)}^2 - v_{\text{ph}(\pm)}^2}{\theta B_0 v_A^2}
\]

(32)

that corresponds to the degeneracy rate of the fast and slow modes \((v_{A}^{nc})_{\text{ph}(\pm)}^2\) in the presence of space noncommutativity from the usual modes \(v_{\text{ph}(\pm)}^2\) normalized to

FIG. 1: The variation of the square phase speed of fast mode normalized to the square Alfvén velocity as a function of the propagation angle \(\alpha\) in the absence of noncommutativity \((\theta = 0)\) for \(\beta = 0.1\) (dashed line), \(\beta = 0.04\) (dotted line) and \(\beta = 0\) (line).

FIG. 2: The variation of the square phase speed of slow mode normalized to the square Alfvén velocity as a function of the propagation angle \(\alpha\) in the absence of noncommutativity \((\theta = 0)\) for \(\beta = 0.1\) (dashed line), \(\beta = 0.04\) (dotted line) and \(\beta = 0\) (line).
\( \theta B_0 \) quantity and the square of the usual Alfvén velocity \( v_A^2 \). Notice that \( v_A^2 \) and \( v_{ph(\pm)} \) are respectively \((\gamma A)^2\) and \((\gamma nc)_{ph(\pm)} \) in the absence of noncommutativity (\( \theta = 0 \)).

By considering a small value of \( \theta B_0 = 10^{-5} \) which is a consequence of a strong value of the mean magnetic field, we plot \( \Delta_+ \) and \( \Delta_- \) respectively in fig. 3 and fig. 4 in function of the propagation angle \( \alpha \) for a different values of plasma \( \beta = \frac{v}{v_A} \).

It is well known that the strength of the value of the magnetic field plays the major role in making the effect of the space deformation more relevant on the waves in plasma medium when the noncommutativity scale is relatively high.

The plasma \( \beta \) variation in fixed mean magnetic field \( B_0 \) becomes proportional to the pressure \( p_0 \) by the adiabatic index \( \gamma \), then \( \beta \) increases when the plasma is initially strongly compressed. However in our study, we consider that the magnetic pressure is dominant in such way that \( \beta < 1 \).

According to the fig. 3, it turns out that in compressible plasma, the phase velocity related to the fast waves is slightly enhanced due to space noncommutativity. This modification which is represented by the rate \( \Delta_+ \), has a very slow dependence on plasma \( \beta \) variation. This extreme modification on the fast which is around \( \sim 2 \) which occurs when the propagation is perpendicular (\( \alpha = 90^\circ \)) with respect to the mean magnetic field, and becomes smaller with \( \Delta_- \sim 1 \) when the fast waves is nearly Alfvénic in parallel or anti-parallel. Also, the noncommutativity effect on the fast mode is proportional to the phase speed of the wave which varies as a function of the propagation angle. In contrast, from fig. 4, the quantity \( \Delta_- \) that corresponds to modification rate of the slow mode has a high dependence on plasma \( \beta \) variation. Although this rate is much smaller comparing with \( \Delta_+ \), the noncommutative effect could slightly appear when plasma \( \beta \) is higher than 0.04, which means that the noncommutativity affects the slow mode as long as the plasma is strongly compressed (high \( p_0 \)). The extreme modification on slow mode occurs mainly in oblique propagations between 30° and 60°, and as it is expected from fig.2, the noncommutativity effect vanishes for \( \alpha = 90^\circ \) at which originally there is no perpendicular propagation of the slow mode. In addition, no modification appears on parallel and anti-parallel slow waves which correspond to the maximum phase velocity of this mode. This means that no proportionality between noncommutativity effect and the phase velocity of the slow wave.

It is important to discuss the interesting physical case when \( \beta \to 0 \), which means a total domination of the magnetic pressure on compressible plasma due to the strength of the mean constant magnetic field \( B_0 \) or the plasma is not relatively enough compressed. In fact, according to fig. 1, in the absence of noncommutativity (\( \theta = 0 \)), the fast mode converges to the usual Alfvén mode \( (v_+ \approx v_A) \) for any propagation angle, but when \( \theta \neq 0 \), it is clear from fig.3 that the effect of space noncommutativity does not change and is nearly the same one for \( \beta \neq 0 \). Therefore, this effect plays a role of medium compression which rises the fast mode from the Alfvénic one. While in fig.2, as it is expected, the slow mode vanishes in case of \( \beta = 0 \) which is equivalent to the incompressible plasma case.
V. CONCLUSION AND DISCUSSION

In this letter, we have studied the propagative phenomenon of the electromagnetic waves in a homogeneous plasma under the effect of a magnetic field and space-space uncertainty. By using a physical model which considers plasma as a high conductor single fluid medium, the ideal NCMHD equations are established with the help of NC Maxwell theory. Because of a quasi-static motion of the fluid, we neglected the relativistic effect which is very small and it is below the order of $(\theta B_0)^2$. In this treatment, the behavior of the electromagnetic waves propagating in this medium are studied in both cases, incompressible and compressible plasmas. It turned out that in an incompressible plasma, the Alfvén mode undergoes a modification which appears as a small additional velocity which enhances the value of the usual Alfvén velocity in commutative space.

On the other hand, it is deduced that the influence of space noncommutativity on the fast waves in a compressible plasma is proportional to the phase velocity. Moreover, the high oblique fast waves undergo a strong modification which is about 2 times higher than the perturbed parameter $\theta B_0$, and a weak modification occurs in case of nearly parallel and anti-parallel propagation. For the slow mode, the influence of space noncommutativity is very small especially when $\beta$ is low than 0.04. This effect could slightly appear when $\beta$ is beyond the value of 0.1, at which, the extreme modification on the slow mode occurs in oblique propagations between 30° and 60°, and vanishes in the parallel one. In addition, for $\beta \rightarrow 0$ there is no influence of space noncommutativity on slow mode while this influence on the fast mode is nearly identical to that for $\beta \neq 0$ which is higher in perpendicular propagation.

The impact of space noncommutativity on cosmological MHD waves properties may lead to new consequences on the study of the influence of such waves on CMB temperature spectrum. Although the effect of the space noncommutativity may decrease after the inflationary universe due to the decreasing in energy scale, the presence of a primordial magnetic field excites any possible effect of space-space uncertainty on the primordial photon-baryon plasma before the last scattering. Any role of space noncommutativity at that time depends on its scale $\Lambda_{NC}$. Several scenarios based on space noncommutativity, aimed to explain the mechanism behind the generation of the primordial magnetic field, hence, possible constraints on $\Lambda_{NC}$ parameter have been involved. In fact, the possibility that $\Lambda_{NC}$ has a temperature dependence has been discussed in [20, 21], where the authors mentioned that the world is commutative at low temperature but becomes more noncommutative once the temperature is higher than a certain threshold temperature $T_0$. In ref. [21], an intensive discussion on the temperature dependence of $\theta$-parameter based on the constraints on the primordial magnetic field which is $B(T = 10 Mev) = 10^{-8} Gev^2$ at the beginning of nucleosynthesis (see ref. [31]), the authors argued that the presence of noncommutativity may be not so efficient beyond nucleosynthesis scale. However, its effect cannot be omitted due to high temperature especially when the radiations dominate the plasma. Hence, this drives our attention, that this effect can be treated as a perturbative correction in the study of the distortion of the primordial spectrum of fluctuations by MHD waves before the last scattering. As we have seen in this letter, the space noncommutativity influences the compressible oscillations depending on their propagation angle, on the other hand, the fact that the velocity of the fast waves depends on the propagation angle between the wavenumber and the magnetic field, the CMB anisotropy would be affected [13]. This could reduce the impact of noncommutativity on some branches of the MHD waves, therefore, this leaves imprints on the primordial spectrum of the fluctuations anisotropy. Furthermore, such an influence may also correct the magnitude of the primordial magnetic field, this is because the noncommutativity can play a role of an additional magnetic field applied on the medium. In addition, dealing with the influence of the NCMHD waves on CMB radiations may provide a better estimation of the scale of $\theta$ parameter during plasma epoch. More details will be given in our future works.

[1] E. R. Priest, Solar Magnetohydrodynamics (D. Reidel Publishing Company, 1982).
[2] M. Goossens, An introduction to Plasma Astrophysics and Magnetohydrodynamics (Kluwer Academic Publishers, 2003).
[3] V. M. Nakariakov, L. Ofman, E. E. Deluca, B. Roberts, and J. M. Davila, Science 285 (1999) 862.
[4] J. A. McLaughlin and A. W. Hood, Astron. Astrophys. 420 (2004) 1129.
[5] U. Narain, P. Ulmschneider, Space. Sci. Rev. 75 (1996) 453.
[6] G. Haerendel, Nature. 360 (1992) 241.
[7] P. S. Cally, Solar. Phys. 108 (1986)183.
[8] E M. Edwin and B. Roberts, Astron. Astrophys. 192 (1988) 343.
[9] V. M. Nakariakov and B. Roberts, Solar. Phys. 159 (1995) 399.
[10] T. K. Suzuki, H. Yan, A. Lazarian and J. P. Cassinelli, Astrophys.J. 640 (2006) 1005.
[11] L. M. Widrow, Rev. Mod. Phys. 74 (2002) 775; M. Giovannini, Int. J. Mod. Phys. D 13 , (2004) 391.
[12] T. Kahniahvili and B. Ratra, Phys. Rev. D 75, (2007) 023002; M. Giovannini, Classical Quantum Gravity 23 , (2006) R1.
[13] J. Adams, U. H. Danielsson, D, Grasso, and H. Rubinstein, Phys. Lett. B388, (1996) 253; D. Grasso and H.
Rubinstein, Phys. Rep. 348, (2001) 163.
[14] D. G. Yamasaki, K. Ichiki, T. Kajino, and G. J. Mathews, Astrphys. J. 646, (2006) 719; S. Koh and C. H. Lee, Phys. Rev. D 62, (2000) 083509.
[15] H. S. Synder, Phys. Rev. 71 (1947) 38.
[16] A. Connes, M. R. Douglas and A. Schwarz, JHEP 9802 (1998) 003.
[17] N. Seiberg and E. Witten, JHEP 9909 (1999) 032; arXiv: hep-th/9908142
[18] R. J. Szabo, Phys. Rep. 378 (2003) 207; arXiv: hepth/0109162 M. R. Douglas and N. Nekrasov, Rev. Mod. Phys. 73 (2001) 977; arXiv: hep-th/0106048.
[19] S. Bourouaine and A. Benslama, Mod. Phys. Lett. A20 (2005) 1997; arXiv: hep-th/0507060 S. Bourouaine and A. Benslama, J. Phys. A: Math. Gen. 38 (2005) 7389.
[20] C. Chu, B. Greene, and G. Shiu, Mod. Phys. Lett. A16 (2001) 38; arXiv: hep-th/0011241 K. Bamba and J. Yokoyama, Phys. Rev. D 70 (2004) 083508; S. Tsujikawa and R. Maartens, Phys. Lett. B574 (2003) 141.
[21] A. Mazumdar and M. Sheikh-Jabbari, Phys. Rev. Lett. 87 (2001) 011301.
[22] C. E. Carlson, C. D. Carone and N. Zobin, Phys. Rev. D66 (2002) 075001; arXiv: hep-th/0206035
[23] H. Bozkaya, P. Fischer, H. Grosse, M. Pitschmann, V. Putz, M. Schweda and R. Wulkenhaar, Eur. Phys. J. C 26 (2002) 139; arXiv: hep-th/0205153
[24] S. I Kruglov, Annales de la foundation Louis de Broglie, 27 (2002) 343; arXiv: hep-th/0110059.
[25] J. Gomis and T. Mehen, Nucl. Phys. B591 (2000) 265; arXiv: hep-th/0005129
[26] O. Aharony, J. Gomis and T. Mehen, JHEP 0009 (2000) 23; arXiv: hep-th/0006236
[27] C. Rim and J. H. Yee, Phys. Lett. B574 (2003) 111; arXiv: hep-th/0205193
[28] Z. Gurău, R. Jackiw, S.Y. Pi and A.P. Polychronakos, Phys. Lett. B517 (2001) 450; arXiv: hep-th/0106044
[29] Y. Abe, R. Banerjee and I. Tsutsui, Phys. Lett. B517 (2001) 450; arXiv: hep-th/0009127
[30] T. Mariz, J. R. Nascimento and V. O. Rivelles, Phys. Lett. B517 (2001) 450; arXiv: hep-th/0609132
[31] D. Grasso and H. R. Rubinstein, Phys. Lett. B379 (1996) 73.