On Affine Motions and Bar Frameworks in General Position

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Abstract

A configuration $p$ in $r$-dimensional Euclidean space is a finite collection of points $(p_1, \ldots, p_n)$ that affinely span $\mathbb{R}^r$. A bar framework, denoted by $G(p)$, in $\mathbb{R}^r$ is a simple graph $G$ on $n$ vertices together with a configuration $p$ in $\mathbb{R}^r$. A given bar framework $G(p)$ is said to be universally rigid if there does not exist another configuration $q$ in any Euclidean space, not obtained from $p$ by a rigid motion, such that $||q^i - q^j|| = ||p^i - p^j||$ for each edge $(i, j)$ of $G$.

It is known [2, 6] that if configuration $p$ is generic and bar framework $G(p)$ in $\mathbb{R}^r$ admits a positive semidefinite stress matrix $S$ of rank $(n - r - 1)$, then $G(p)$ is universally rigid. Connelly asked [8] whether the same result holds true if the genericity assumption of $p$ is replaced by the weaker assumption of general position. We answer this question in the affirmative in this paper.

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1 Introduction

A configuration $p$ in $r$-dimensional Euclidean space is a finite collection of points $(p^1, \ldots, p^n)$ in $\mathbb{R}^r$ that affinely span $\mathbb{R}^r$. A bar framework (or framework for short) in $\mathbb{R}^r$, denoted by $G(p)$, is a configuration $p$ in $\mathbb{R}^r$ together with a simple graph $G$ on the vertices $1, 2, \ldots, n$. For a simple graph $G$, we denote its node set by $V(G)$ and its edge set by $E(G)$. To avoid trivialities, we assume throughout this paper that graph $G$ is connected and not complete.

Framework $G(q)$ in $\mathbb{R}^r$ is said to be congruent to framework $G(p)$ in $\mathbb{R}^r$ if configuration $q$ is obtained from configuration $p$ by a rigid motion. That is, if $|q^i - q^j| = |p^i - p^j|$ for all $i, j = 1, \ldots, n$, where $|.|$ denotes the Euclidean norm. We say that framework $G(q)$ in $\mathbb{R}^r$ is equivalent to framework $G(p)$ in $\mathbb{R}^r$ if $|q^i - q^j| = |p^i - p^j|$ for all $(i, j) \in E(G)$. Furthermore, we say that framework $G(q)$ in $\mathbb{R}^r$ is affinely-equivalent to framework $G(p)$ in $\mathbb{R}^r$ if configuration $q$ is obtained from configuration $p$ by an affine motion; i.e., $q^i = Ap^i + b$, for all $i = 1, \ldots, n$, for some $r \times r$ matrix $A$ and an $r$-vector $b$.

A framework $G(p)$ in $\mathbb{R}^r$ is said to be universally rigid if there does exist a framework $G(q)$ in any Euclidean space that is equivalent, but not congruent, to $G(p)$. The notion of a stress matrix $S$ of a framework $G(p)$ plays a key role in the problem of universal rigidity of $G(p)$.

1.1 Stress Matrices and Universal Rigidity

Let $G(p)$ be a framework on $n$ vertices in $\mathbb{R}^r$. An equilibrium stress of $G(p)$ is a real valued function $\omega$ on $E(G)$ such that

$$\sum_{j: (i,j) \in E(G)} \omega_{ij}(p^i - p^j) = 0 \text{ for all } i = 1, \ldots, n. \quad (1)$$

Let $\omega$ be an equilibrium stress of $G(p)$. Then the $n \times n$ symmetric matrix $S = (s_{ij})$ where

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if } (i, j) \in E(G), \\ 0 & \text{if } i \neq j \text{ and } (i, j) \notin E(G), \\ \sum_{k: (i,k) \in E(G)} \omega_{ik} & \text{if } i = j, \end{cases} \quad (2)$$

is called the stress matrix associated with $\omega$, or a stress matrix of $G(p)$. The following result provides a sufficient condition for the universal rigidity of a given framework.

**Theorem 1.1 (Connelly [5, 6], Alfakih [1])** Let $G(p)$ be a bar framework in $\mathbb{R}^r$, for some $r \leq n - 2$. If the following two conditions hold:
1. There exists a positive semidefinite stress matrix $S$ of $G(p)$ of rank $(n - r - 1)$.

2. There does not exist a bar framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$.

Then $G(p)$ is universally rigid.

Note that $(n - r - 1)$ is the maximum possible value for the rank of the stress matrix $S$. In connection with Theorem 1.1, we mention the following result obtained in So and Ye [11] and Biswas et al. [4]: Given a framework $G(p)$ in $\mathbb{R}^r$, if there does not exist a framework $G(q)$ in $\mathbb{R}^s$ $(s \neq r)$ that is equivalent to $G(p)$, then $G(p)$ is universally rigid. Moreover, if $G(p)$ contains a clique of $r + 1$ points in general position, then the existence of a rank-$(n - r - 1)$ positive semidefinite stress matrix implies that framework $G(p)$ is universally rigid, regardless whether the other non-clique points are in general position or not.

Condition 2 of Theorem 1.1 is satisfied if configuration $p$ is assumed to be generic (see Lemma 2.2 below). A configuration $p$ (or a framework $G(p)$) is said to be generic if all the coordinates of $p^1, \ldots, p^n$ are algebraically independent over the integers. That is, if there does not exist a non-zero polynomial $f$ with integer coefficients such that $f(p^1, \ldots, p^n) = 0$. Thus

**Theorem 1.2 (Connelly [6], Alfakih [2])** Let $G(p)$ be a generic bar framework on $n$ nodes in $\mathbb{R}^r$, for some $r \leq n - 2$. If there exists a positive semidefinite stress matrix $S$ of $G(p)$ of rank $(n - r - 1)$. Then $G(p)$ is universally rigid.

The converse of Theorem 1.2 is also true.

**Theorem 1.3 (Gortler and Thurston [10])** Let $G(p)$ be a generic bar framework on $n$ nodes in $\mathbb{R}^r$, for some $r \leq n - 2$. If $G(p)$ is universally rigid, then there exists a positive semidefinite stress matrix $S$ of $G(p)$ of rank $(n - r - 1)$.

Connelly [8] asked whether a result similar to Theorem 1.2 holds if the genericity assumption of $G(p)$ is replaced by the weaker assumption of general position. A configuration $p$ (or a framework $G(p)$) in $\mathbb{R}^r$ is said to be in general position if no subset of the points $p^1, \ldots, p^n$ of cardinality $r + 1$ is affinely dependent. For example, a set of points in the plane are in general position if no 3 of them lie on a straight line.

In this paper we answer Connelly’s question in the affirmative. Thus the following theorem is the main result of this paper.

**Theorem 1.4** Let $G(p)$ be a bar framework on $n$ nodes in general position in $\mathbb{R}^r$, for some $r \leq n - 2$. If there exists a positive semidefinite stress matrix $S$ of $G(p)$ of rank $(n - r - 1)$. Then $G(p)$ is universally rigid.
The proof of Theorem 1.4 will be given in Section 3. This paper and [3] are first steps toward the study of universal rigidity under the general position assumption.

In [3], it was shown that the framework $G(p)$ on $n$ nodes in general position in $\mathbb{R}^r$ for some $r \leq n - 2$, where $G$ is the $(r+1)$-lateration graph, admits a rank $(n-r-1)$ positive semi-definite stress matrix.

2 Preliminaries

To develop the ingredients needed for the proof of our main result, we review the necessary background on affine motions, stress matrices, and Gale matrices.

An affine motion in $\mathbb{R}^r$ is a map $f : \mathbb{R}^r \to \mathbb{R}^r$ of the form

$$f(p^i) = Ap^i + b,$$

for all $p^i$ in $\mathbb{R}^r$, where $A$ is an $r \times r$ matrix and $b$ is an $r$-vector. A rigid motion is an affine motion where matrix $A$ is orthogonal.

Vectors $v^1, \ldots, v^m$ in $\mathbb{R}^r$ are said to lie on a quadratic at infinity if there exists a non-zero symmetric $r \times r$ matrix $\Phi$ such that

$$(v^i)^T \Phi v^i = 0,$$

for all $i = 1, \ldots, m$. (3)

Lemma 2.1 (Connelly [7]) Let $G(p)$ be a bar framework on $n$ vertices in $\mathbb{R}^r$. Then the following two conditions are equivalent:

1. There exists a framework $G(q)$ in $\mathbb{R}^r$ that is equivalent, but not congruent, to $G(p)$ such that $q^i = Ap^i + b$ for all $i = 1, \ldots, n$,

2. The vectors $p^i - p^j$ for all $(i, j) \in E(G)$ lie on a quadratic at infinity.

Lemma 2.2 (Connelly [7]) Let $G(p)$ be a generic bar framework on $n$ vertices in $\mathbb{R}^r$. Assume that each node of $G$ has degree at least $r$. Then the vectors $p^i - p^j$ for all $(i, j) \in E(G)$ do not lie on a quadratic at infinity.

Therefore, under the genericity assumption, Condition 2 in Lemma 2.1 does not hold. Consequently, Theorem 1.2 follows as a simple corollary of Theorem 1.1.

Note that Condition 2 in Lemma 2.1 is expressed in terms of the edges of $G$. An equivalent condition in terms of the missing edges of $G$ can also be obtained using Gale matrices. This equivalent condition turns out to be crucial for our proof of Theorem 1.4.

To this end, let $G(p)$ be a framework on $n$ vertices in $\mathbb{R}^r$. Then the following $(r + 1) \times n$ matrix

$$A := \begin{bmatrix} p^1 & p^2 & \cdots & p^n \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

(4)
has full row rank since $p^1, \ldots, p^n$ affinely span $\mathbb{R}^r$. Note that $r \leq n - 1$. Let

$$r = \text{the dimension of the null space of } A; \ i.e., \ r = n - 1 - \bar{r}. \quad (5)$$

**Definition 2.1** Suppose that the null space of $A$ is nontrivial, i.e., $\bar{r} \geq 1$. Any $n \times \bar{r}$ matrix $Z$ whose columns form a basis of the null space of $A$ is called a Gale matrix of configuration $p$. Furthermore, the $i$th row of $Z$, considered as a vector in $\mathbb{R}^\bar{r}$, is called a Gale transform of $p^i$.\[9\]

Let $S$ be a stress matrix of $G(p)$ then it follows from (2) and (4) that

$$AS = 0. \quad (6)$$

Thus

$$S = Z\Psi Z^T, \quad (7)$$

for some $\bar{r} \times \bar{r}$ symmetric matrix $\Psi$, where $Z$ is a Gale matrix of $p$. It immediately follows from (7) that rank $S = \text{rank } \Psi$. Thus, $S$ attains its maximum rank of $\bar{r} = (n - 1 - r)$ if and only if $\Psi$ is nonsingular, i.e., rank $\Psi = \bar{r}$.

Let $e$ denote the vector of all 1’s in $\mathbb{R}^n$, and let $V$ be an $n \times (n - 1)$ matrix that satisfies:

$$V^T e = 0, \ V^T V = I_{n-1}, \quad (8)$$

where $I_{n-1}$ is the identity matrix of order $(n - 1)$. Further, let $E^{ij}, i \neq j$, denote the $n \times n$ symmetric matrix with 1 in the $(i, j)$th and $(j, i)$th entries and zeros elsewhere, and let $E(y) = \sum_{(i,j) \not\in E(G)} y_{ij} E^{ij}$ where $y_{ij} = y_{ji}$. In other words, the $(k, l)$ entry of matrix $E(y)$ is given by

$$E(y)_{kl} = \begin{cases} 0 & \text{if } (k, l) \in E(G), \\ 0 & \text{if } k = l, \\ y_{kl} & \text{if } k \neq l \text{ and } (k, l) \not\in E(G). \end{cases} \quad (9)$$

Then we have the following result.

**Lemma 2.3** (Alfakih [2]) Let $G(p)$ be a bar framework on $n$ vertices in $\mathbb{R}^r$ and let $Z$ be any Gale matrix of $p$. Then the following two conditions are equivalent:

1. The vectors $p^i - p^j$ for all $(i, j) \in E(G)$ lie on a quadratic at infinity.

2. There exists a non-zero $y = (y_{ij}) \in \mathbb{R}^{\bar{m}}$ such that:

$$V^T E(y) Z = 0, \quad (10)$$

where $\bar{m}$ is the number of missing edges of $G$, $V$ is defined in (8), and $E(y)$ is defined in (9). $0$ here is the zero matrix of dimension $(n - 1) \times \bar{r}$.

Condition 2 of Lemma 2.3 can be easily understood if a projected Gram matrix approach is used for the universal rigidity of bar frameworks (see [2] for details).
3 Proof of Theorem 1.4

The main idea of the proof is to show that Condition 2 of Lemma 2.3 does not hold under the general position assumption, and under the assumption that $G(p)$ admits a positive semidefinite stress matrix of rank $(n - r - 1)$. The choice of the particular Gale matrix to be used in equation (10) is critical in this regard. We begin with a few necessary lemmas.

Lemma 3.1 Let $G(p)$ be a framework on $n$ nodes in general position in $\mathbb{R}^r$ and let $Z$ be any Gale matrix of configuration $p$. Then any $\bar{r} \times \bar{r}$ submatrix of $Z$ is nonsingular.

Proof. For a proof see e.g., [1]. \hfill $\Box$

Let $\bar{N}(i)$ denote the set of nodes of graph $G$ that are non-adjacent to node $i$; i.e.,
\begin{equation}
\bar{N}(i) = \{j \in V(G) : j \neq i \text{ and } (i, j) \notin E(G)\},
\end{equation}

Lemma 3.2 Let $G(p)$ be a framework on $n$ nodes in general position in $\mathbb{R}^r$. Assume that $G(p)$ has a stress matrix $S$ of rank $(n - 1 - r)$. Then there exists a Gale matrix $\hat{Z}$ of $G(p)$ such that $\hat{z}_{ij} = 0$ for all $j = 1, \ldots, \bar{r}$ and $i \in \bar{N}(j + r + 1)$.

Proof. Let $G(p)$ be in general position in $\mathbb{R}^r$ and assume that it has a stress matrix $S$ of rank $\bar{r} = (n - 1 - r)$. Let $Z$ be any Gale matrix of $G(p)$, then $S = Z\Psi Z^T$ for some non-singular symmetric $\bar{r} \times \bar{r}$ matrix $\Psi$. Let us write $Z$ as:
\begin{equation}
Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},
\end{equation}

where $Z_2$ is $\bar{r} \times \bar{r}$. By Lemma 3.1 $Z_2$ is non-singular. Now let
\begin{equation}
\hat{Z} = (\hat{z}_{ij}) = Z\Psi Z_2^T.
\end{equation}

Then $\hat{Z}$ is a Gale matrix of $G(p)$. This simply follows from the fact that the matrix obtained by multiplying any Gale matrix of $G(p)$ from the right by a non-singular $\bar{r} \times \bar{r}$ matrix, is also a Gale matrix of $G(p)$. Furthermore,

\[ S = Z\Psi Z^T = Z\Psi [Z_1^T \ Z_2^T] = [Z\Psi Z_1^T \ \hat{Z}]. \]

In other words, $\hat{Z}$ consists of the last $\bar{r}$ columns of $S$. Thus $\hat{z}_{ij} = s_{i,j+r+1}$. By the definition of $S$ we have $s_{i,j+r+1} = 0$ for all $i \neq j + r + 1$ and $(i, j + r + 1) \notin E(G)$. Therefore, $\hat{z}_{ij} = 0$ for all $j = 1, \ldots, \bar{r}$ and $i \in \bar{N}(j + r + 1)$. \hfill $\Box$
Lemma 3.3 Let the Gale matrix in (10) be $\hat{Z}$ as defined in (13). Then the system of equations (10) is equivalent to the system of equations

$$\mathcal{E}(y)\hat{Z} = 0.$$  

(14)

0 here is the zero matrix of dimension $n \times \bar{r}$.

Proof. System of equations (10) is equivalent to the following system of equations in the unknowns, $y_{ij}$ ($i \neq j$ and $(i, j) \notin E(G)$) and $\xi \in \mathbb{R}^r$:

$$\mathcal{E}(y)\hat{Z} = e\xi^T,$$

(15)

Now for $j = 1, \ldots, \bar{r}$, we have that the $(j + r + 1, j)$th entry of $\mathcal{E}(y)\hat{Z}$ is equal to $\xi_j$. But using (9) and Lemma 3.2 we have

$$(\mathcal{E}(y)\hat{Z})_{j+r+1,j} = \sum_{i=1}^{n} \mathcal{E}(y)_{j+r+1,i} \hat{z}_{ij} = \sum_{i:i \in \overline{N}(j+r+1)} y_{j+r+1,i} \hat{z}_{ij} = 0.$$  

Thus, $\xi = 0$ and the result follows.

Now we are ready to prove our main theorem.

Proof of Theorem 1.4

Let $G(p)$ be a framework on $n$ nodes in general position in $\mathbb{R}^r$. Assume that $G(p)$ has a positive semidefinite stress matrix $S$ of rank $\bar{r} = n - 1 - r$. Then $\deg(i) \geq r + 1$ for all $i \in V(G)$, i.e., every node of $G$ is adjacent to at least $r + 1$ nodes (for a proof see [1, Theorem 3.2]). Thus

$$|\overline{N}(i)| \leq n - r - 2 = \bar{r} - 1.$$  

(16)

Furthermore, it follows from Lemmas 3.2, 3.3 and 2.3 that the vectors $p_i - p_j$ for all $(i, j) \in E(G)$ lie on a quadratic at infinity if and only if system of equations (14) has a non-zero solution $y$. But (14) can be written as

$$\sum_{j: \in N(i)} y_{ij} \hat{z}^j = 0, \text{ for } i = 1, \ldots, n,$$

where $(\hat{z}^i)^T$ is the $i$th row of $\hat{Z}$. Now it follows from (16) that $y_{ij} = 0$ for all $(i, j) \notin E(G)$ since by Lemma 3.1 any subset of $\{\hat{z}^1, \ldots, \hat{z}^n\}$ of cardinality $\leq \bar{r} - 1$ is linearly independent.

Thus system (14) does not have a non-zero solution $y$. Hence the vectors $p_i - p_j$, for all $(i, j) \in E(G)$, do not lie on a quadratic at infinity. Therefore, by Lemma 2.1 there does not exist a framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$. Thus by Theorem 1.1 $G(p)$ is universally rigid.
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