Enumeration Problems for Regular Path Queries

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\section*{Abstract}

Evaluation of regular path queries (RPQs) is a central problem in graph databases. We investigate the corresponding enumeration problem, that is, given a graph and an RPQ, enumerate all paths in the graph that match the RPQ. We consider several versions of this problem, corresponding to different semantics of RPQs that have recently been considered: arbitrary paths, shortest paths, simple paths, and trails.

Whereas arbitrary and shortest paths can be enumerated in polynomial delay, the situation is much more intricate for simple paths and trails. For instance, already the question if a given graph contains a simple path or trail of a certain length has cases with highly non-trivial solutions and cases that are long-standing open problems. In this setting, we study RPQ evaluation from a parameterized complexity perspective. We define a class of simple transitive expressions that is prominent in practice and for which we can prove two dichotomy-like results: one for simple paths and one for trails paths. We observe that, even though simple path semantics and trail semantics are intractable for RPQs in general, they are feasible for the vast majority of the kinds of RPQs that users use in practice. At the heart of this study is a result of independent interest on the parameterized complexity of finding disjoint paths in graphs: the two disjoint paths problem is W[1]-hard if parameterized by the length of one of the two paths.

\section{Introduction}

Regular path queries (RPQs) are a crucial feature of graph database query languages, since they allow us to pose queries about arbitrarily long paths in the graph. Essentially, RPQs are regular expressions that are matched against labeled directed paths in the graph database. Currently, the World Wide Web Consortium \cite{w3c} and the openCypher project \cite{opencypher} are considering how RPQ evaluation can be formally defined for the development of SPARQL 1.1 \cite{sparql11} and Neo4J Cypher \cite{cypher} \cite{cypher}, respectively. Several popular candidates that have been considered are arbitrary paths, shortest paths, simple paths, and trails (cfr. \cite[Section 4.5]{deshpande2014regular}, \cite{cypher}).

We briefly explain these semantics. Given a graph, an RPQ $r$ considers directed paths for which the labels on the edges form a word in the language of $r$. We call such paths candidate matches. The different semantics restrict the kind of paths that match the RPQ, i.e., can be returned as answers. Arbitrary paths imposes no restriction and returns every candidate match. Shortest paths on the other hand, only returns the shortest candidate matches. Simple paths, resp., trails, only return candidate matches that do not have duplicate nodes, resp., edges.

Under arbitrary paths, the number of matches may be infinite if the graph is cyclic. This may pose a challenge for designing the query language, even if one does not choose to return all matching paths. Indeed, a popular alternative semantics of RPQs is to return node pairs

\footnote{Simple paths and trails are called no-repeated-node paths and no-repeated-edge paths in \cite[Section 4.5]{deshpande2014regular}, respectively.}
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\((x, y)\) such that there exists a matching path from \(x\) to \(y\). If one wants to consider a bag semantics version for node pairs, where each \((x, y)\) is returned as often as the number of matches from \(x\) to \(y\), one needs to deal with the case where this number is infinite.

Under shortest paths, simple paths, and trails, the number of matching paths is always finite, which simplifies the aforementioned design challenge. However, these versions face other challenges. Simple paths may present complexity issues. Two fundamental problems are that

- counting the number of simple paths between two nodes is \#P-complete \[43]\ and
- deciding if there exists a simple path of even length between two given nodes is NP-complete \[41].

Indeed, the first problem implies that evaluating the RPQ \(a^*\) under bag semantics is \#P-complete and the second one implies that one needs to solve an NP-complete problem to evaluate the RPQ \((aa)^*\). Trails faces similar challenges as simple paths, due to the similar no-repetition constraint. Shortest paths does not have these complexity issues, but it is unclear if its semantics is very natural. For instance, under shortest paths semantics, if we ask how many paths there are from \(x\) to \(y\), then this number may decrease if a new, shorter, path is added. This may seem counter-intuitive to users.

Since it seems that there is no one-size-fits-all solution, the openCypher project team recently proposed to support several kinds of semantics for Neo4J Cypher \[40]. This situation motivated us to shed more light on RPQ evaluation, focusing on the following aspects:

- We focus on returning paths as answers and on enumeration versions of evaluation. That is, we study problems where the task is to enumerate all matching paths, without duplicates. We are interested in which situations it is possible to answer queries in polynomial delay, i.e., such that the time between consecutive answers is at most polynomial.
- We take into account a recent study that investigated the structure of about 250K RPQs in a wide range of SPARQL query logs \[10]. It turns out that these RPQs have a relatively simple structure, which is remarkable because their syntax is not restricted by the SPARQL recommendation.

Our contributions are the following.

1. After observing that enumeration of arbitrary or shortest paths that match an RPQ can be done in polynomial delay (Section 3), we turn to enumeration for simple paths and trails. For downward-closed languages (i.e., languages that are closed under taking subsequences), this is an easy consequence of Yen’s algorithm \[47] (Section 4.1).

2. We show that Bagan et al.’s dichotomy for deciding the existence of a simple path that matches an RPQ \[7] carries over to enumeration problems (Section 4.2). Furthermore, we show that Bagan’s dichotomy carries over from simple paths to trails. Since Bagan et al.’s dichotomy is about the data complexity of RPQ evaluation, this gives us some understanding about the data complexity of enumeration.

3. However, our goal is to get a better understanding of the combined complexity of enumerating simple paths or trails. This is a challenging task because it contains subproblems that are highly non-trivial. One such subproblem is testing if there exists a directed simple path of length \(\log n\) between two given nodes in graph \(G\) with \(n\) nodes. This problem was shown to be in PTIME by Alon et al., using their color coding technique \[3]. It is open for over two decades if it can be decided in PTIME if there is a simple path.

\[2\] It is also known that answering the RPQ \(a^*ba^*\) under simple path semantics is at least as difficult as the Two Disjoint Paths problem \[50].

\[3\] Notice that each semantics only returns or counts the number of matching paths.
of length \( \log^2 n \). Notice that these two problems are special cases of RPQ evaluation under simple path semantics (i.e., evaluate the RPQs \( a^{\log n} \) and \( a^{\log^2 n} \) in a graph where every edge has label \( a \)).

We therefore investigate RPQ evaluation from the angle of parameterized complexity. We introduce the class of \textit{simple transitive expressions (STEs)} that capture over 99% of the RPQs that were found in SPARQL query logs in a recent study \[10\]. We identify a property of STEs that we call \textit{cuttability} and prove that the combined parameterized complexity for evaluating STEs \( R \) is in \( \text{FPT} \) if \( R \) is cuttable and \( \text{W}[1]\)-hard otherwise.

Examples of cuttable classes of expressions are \( a^k a^* \) and \( (a+b)^k a^* \) (for \( k \in \mathbb{N} \)). Examples of non-cuttable classes are \( a^{k} b^* \), \( a^{k} b a^* \), and \( a^{k} (a + b)^* \). For trails, we also show a dichotomy, but here the \( \text{FPT} \) fragment is larger. That is, if the class \( R \) is not cuttable, evaluation is still \( \text{FPT} \) if \( R \) is almost conflict-free. We show that these dichotomies carry over to enumeration problems (Section 6).

4. At the core of these results are two results of independent interest (Section 5). The first shows that the \textit{Two Disjoint Paths} problem is \( \text{W}[1]\)-hard when parameterized by the length of one of the two paths (Theorem 18). The second is by the authors of \[25\], who showed that it can be decided in \( \text{FPT} \) if there is a simple path of length \( \text{at least} \ k \) between two nodes in a graph (Theorem 14).

Putting everything together, we see that, although simple path and trail semantics lead to high complexity in general, their complexity for RPQs that have been found in SPARQL query logs is reasonable. We discuss this in the conclusions.

Related Work

RPQs on graph databases have been researched in the literature since the end of the 80’s \[16, 17, 46\] and many problems have been investigated, such as optimization \[11\], query rewriting and query answering using views \[13, 15\], and containment \[14, 20, 23\]. RPQ evaluation is therefore a fundamental problem in the field. We refer to \[8\] for an excellent overview on RPQs and queries for graph databases in general.

Mendelzon and Wood \[36\] were the first to consider simple paths for answering regular path queries. They proved that testing if there exist simple paths matching \( a^* ba^* \) or \( (aa)^* \) is \( \text{NP}\)-complete and studied classes of graphs for which evaluation becomes tractable. Arenas et al. \[5\] and Losemann and Martens \[34\] studied counting problems related to RPQs in SPARQL 1.1 (which are called \textit{property paths} in the specification). They showed that, under the definition of SPARQL at that time, query evaluation was highly complex. They made proposals on how to amend this, which were largely taken into account by the W3C. Extensions of RPQ-like queries with data value comparisons and branching navigation were studied by Libkin et al. \[33\].

Bagan et al. \[7\] studied the \textit{data complexity} of RPQ evaluation under simple path semantics (i.e., the regular path query is considered to be constant). They proved that there is a trichotomy for the evaluation problem: the data complexity of RPQ evaluation is \( \text{NP}\)-complete for languages outside a class they call \( C_{\text{tract}} \), it is \( \text{NL}\)-complete for infinite languages in \( C_{\text{tract}} \), and in \( \text{AC}^0 \) for finite languages. (Since the results are on data complexity, the representation of the languages does not matter.)

We also consider problems where the task is to enumerate paths in graphs. In this context we will use Yen’s algorithm \[17\] which is a procedure for enumerating simple paths in graphs. Yen’s algorithm was generalized by Lawler \[32\] and Murty \[37\] to a tool for designing general algorithms for enumeration problems. Lawler-Murty’s procedure has been used for solving enumeration problems in databases in various contexts \[27, 29, 30\].
2 Preliminaries

By $\Sigma$ we always denote an alphabet, that is, a finite set. A ($\Sigma$-)symbol is an element of $\Sigma$. A word (over $\Sigma$) is a finite sequence $w = a_1 \ldots a_n$ of $\Sigma$-symbols. The length of $w$, denoted by $|w|$, is its number of symbols $n$. We denote the empty word by $\varepsilon$. We denote the concatenation of words $w_1$ and $w_2$ as $w_1 \cdot w_2$ or simply as $w_1 w_2$. We assume familiarity with regular expressions and finite automata. The regular expressions (RE) we use in this paper are defined as follows: $\emptyset$, $\varepsilon$ and every $\Sigma$-symbol is a regular expression; and when $r$ and $s$ are regular expressions, then $(rs)$, $(r+s)$, $(r?)$, $(r^*)$, and $(r^+)$ are also regular expressions. From now on, we use the usual precedence rules to omit braces. The size $|r|$ of a regular expression is the number of occurrences of $\Sigma$-symbols in $r$. For example, $[aba] = 3$. We define the language $L(r)$ of $r$ as usual. Since it is easy to test if $L(r) = \emptyset$ for a given expression $r$, we assume in this paper that $L(r) \neq \emptyset$ for all expressions, unless mentioned otherwise. For $n \in \mathbb{N}$, we use $r^n$ to abbreviate the $n$-fold concatenation $r \cdots r$ of $r$. We abbreviate $(r?)^n$ by $r^*$. A non-deterministic finite automaton (NFA) $N$ over $\Sigma$ is a tuple $(Q, \Sigma, \Delta, Q_I, Q_F)$, where $Q$ is a finite set of states, $\Sigma$ is a finite alphabet, $\Delta : Q \times \Sigma \times Q$ is the transition relation, $Q_I \subseteq Q$ is the set of initial states, and $Q_F$ is the set of final states. By $\delta^*(w)$ we denote the set of states reachable by $N$ after reading $w$, that is, $\delta^*(w) = Q_I$ and, for every word $w$ and symbol $a$, we define $\delta^*(wa) = \{\delta(q,a) \mid q \in \delta^*(w)\}$. The size of an NFA is $|Q|$, i.e., its number of states. We define the language $L(N)$ of $N$ as usual.

2.1 Graph Databases, Paths, and Trails

We use edge-labeled directed graphs as abstractions for graph databases. A graph $G$ (with labels in $\Sigma$) will be denoted as $G = (V, E)$, where $V$ is the finite set of nodes of $G$ and $E \subseteq V \times \Sigma \times V$ is the set of edges. We say that edge $e = (u, a, v)$ goes from $u$ to node $v$ and has the label $a$. Sometimes we write an edge as $(u, v) \in V \times V$ if the label does not matter. In this paper, we assume that graphs are directed, unless mentioned otherwise. The size of a graph $G$, denoted by $|G|$ is $|V| + |E|$.

We assume familiarity with basic terminology on graphs. A path from node $u$ to node $v$ in $G$ is a sequence $p = (v_0, a_1, v_1)(v_1, a_2, v_2)\cdots(v_{n-1}, a_n, v_n)$ of edges in $G$ such that $u = v_0$ and $v = v_n$. For $0 \leq i \leq n$, we denote by $p[i, i]$ (or $p[i]$) the node $v_i$ and, for $0 \leq i < j \leq n$, we denote by $p[i, j]$ the subpath $(v_i, a_{i+1}, v_{i+1})\cdots(v_{j-1}, a_j, v_j)$. A path $p$ is a simple path if it has no repeated nodes, that is, all nodes $v_0, \ldots, v_n$ are pairwise different. It is a trail if it has no repeated edges, that is, all triples $(v_i, a_{i+1}, v_{i+1})$ are pairwise different. The length of $p$, denoted $|p|$, is the number $n$ of edges in $p$. By definition of paths, we consider two paths to be different if they are different sequences of edges. In particular, two paths going through the same nodes in the same order, but using different edge labels are different.

The set of nodes of path $p$ is $V(p) = \{v_0, \ldots, v_n\}$. The word of $p$ is $a_1 \cdots a_n$ and is denoted by lab$(G)(p)$. We omit $G$ if it is clear from the context. Path $p$ matches a regular expression $r$ (resp., NFA $N$) if lab$(p) \in L(r)$ (resp., lab$(p) \in L(N)$). The concatenation of paths $p_1 = (v_0, a_1, v_1)\cdots(v_{n-1}, a_n, v_n)$ and $p_2 = (v_n, a_{n+1}, v_{n+1})\cdots(v_{n+m-1}, a_{n+m}, v_{n+m})$ is simply the concatenation $p_1p_2$ of the two sequences.

We will often consider a graph $G = (V, E)$ together with a source node $s$ and a target node $t$, for example, when considering paths from $s$ to $t$. We denote such a graph with source $s$ and target $t$ as $(G, s, t)$ and define their size $|(G, s, t)|$ as $|G|$.

The product of graph $(G, s, t)$ and NFA $N = (Q, \Sigma, \Delta, Q_I, Q_F)$ is a graph $(G, s, t) \times N = (V', E')$ with $V' = (V \times Q)$ and $E' = \{(u_1, q_1), (u_2, q_2) \mid (u_1, a, u_2) \in E \text{ and } q_1 \in \delta^*(u_1, a, q_2)\}$.
2.2 Decision and Enumeration Problems

An enumeration problem $P$ is a set of pairs $(i, O)$ where $i$ is an input and $O$ is a finite or countably infinite set of outputs for $i$, denoted by $P(i)$. Terminologically, we say that the task is to enumerate $O$, given $i$.

We consider the following problems, where $G$ is always a graph, $s$ and $t$ are nodes in $G$, and $r$ is a regular expression (also called regular path query (RPQ)).

- **Path**: Given $(G, s, t)$ and $r$, is there a path from $s$ to $t$ that matches $r$?
- **SimPath**: Given $(G, s, t)$ and $r$, is there a simple path from $s$ to $t$ that matches $r$?
- **Trail**: Given $(G, s, t)$ and $r$, is there a trail from $s$ to $t$ that matches $r$?
- **EnumPaths**: Given $(G, s, t)$ and $r$, enumerate the paths in $G$ from $s$ to $t$ that match $r$.
- **EnumShortPaths**: Given $(G, s, t)$ and $r$, enumerate the shortest paths in $G$ from $s$ to $t$ that match $r$.
- **EnumSimPaths**: Given $(G, s, t)$ and $r$, enumerate the simple paths in $G$ from $s$ to $t$ that match $r$.
- **EnumTrails**: Given $(G, s, t)$ and $r$, enumerate the trails in $G$ from $s$ to $t$ that match $r$.

An enumeration algorithm for $P$ is an algorithm that, given input $i$, writes a sequence of answers to the output such that every answer in $P(i)$ is written precisely once. If $A$ is an enumeration algorithm for enumeration problem $P$, we say that $A$ runs in polynomial delay, if the time before writing the first answer and the time between writing every two consecutive answers is polynomial in $|i|$.

For a class $R$ of regular expressions, we denote by $\text{Path}(R)$ the problem $\text{Path}$ where we always assume that $r \in R$. If $R$ consists of a single expression $r$, we simplify the notation to $\text{Path}(r)$. We use the same convention for all other decision- and enumeration problems. We assume familiarity with the notions combined- and data complexity. In our decision problems, $(G, s, t)$ is the data and $r$ is the query.

2.3 Reducing Between Trails and Simple Paths

Lapaugh and Rivest [31] showed that there is a strong correspondence between trail and simple path problems that we will use extensively and therefore revisit here. Unfortunately, Lapaugh and Rivest’s Lemmas 1 and 2 do not precisely capture what we need, so we have to be a bit more precise.

The following construction is from [31] Proof of Lemma 1]. Let $(G, s_1, t_1, \ldots, s_k, t_k)$ be a graph $G$ together with nodes $s_1, t_1, \ldots, s_k, t_k$. We denote by split$(G, s_1, t_1, \ldots, s_k, t_k)$ the tuple $(G', s'_1, t'_1, \ldots, s'_k, t'_k)$ obtained as follows. The graph $G'$ is obtained from $G$ by replacing each node $v$ by two nodes head$(v)$ and tail$(v)$. A directed edge is added from head$(v)$ to tail$(v)$. All incoming edges of $v$ become incoming edges of head$(v)$ and all outgoing edges of $v$ become outgoing edges of tail$(v)$. For every $s_i$ and $t_i$, we define $s'_i = \text{head}(s_i)$ and $t'_i = \text{tail}(t_i)$.

**Lemma 1.** Let $(G', s'_1, t'_1, \ldots, s'_k, t'_k) = \text{split}(G, s_1, t_1, \ldots, s_k, t_k)$. Then there exists pairwise node disjoint simple paths of length $k_i$ from $s_i$ to $t_i$ in $G$ iff there exist pairwise edge disjoint trails of length $2k_i + 1$ from $s'_i$ to $t'_i$ in $G'$.

We denote by line$(G, s_1, t_1, \ldots, s_k, t_k)$ a variation on the line graph of $G$. More precisely, line$(G, s_1, t_1, \ldots, s_k, t_k)$ is the tuple $(G', s_1, t_1, \ldots, s_k, t_k)$ obtained as follows. Let $G = (q_1, a, q_2) \in \Delta$. 

((q_1, a, q_2) \in \Delta).
The nodes of $G'$ are $E \cup \{s_1, t_1, \ldots, s_k, t_k\}$. The edges of $G'$ are the disjoint union of

- $\{(u, v), (v, w) \mid (u, v) \text{ and } (v, w) \in E\}$,
- $\{(s_i, (s_i, v)) \mid i = 1, \ldots, k \text{ and } (s_i, v) \in E\}$, and
- $\{((v, t_i)) \mid i = 1, \ldots, k \text{ and } (v, t_i) \in E\}$.

It is well known that the line graph of $G$ is useful for reducing trail problems to simple path problems \[31\]

\[\text{Proof of Lemma 2.}\]

\[\text{Lemma 2. Let } (G', s_1', t_1', \ldots, s_k', t_k') = \text{line}(G, s_1, \ldots, s_k, t_k). \text{ Then there exist pairwise edge disjoint trails of length } k_i \text{ from } s_i \text{ to } t_i \text{ in } G \text{ iff there exist pairwise node disjoint simple paths of length } k_i + 1 \text{ from } s_i' \text{ to } t_i' \text{ in } G'.\]

\section{Enumerating All Regular Paths and Shortest Regular Paths}

The following result is due to Ackermann and Shallit.

\[\text{Theorem 3 (Theorem 3 in \[2\]). Given an NFA } N, \text{ enumerating the words in } L(N) \text{ can be done in polynomial delay.}\]

This result generalizes a result of Mäkinen \[35\], who proved that the words in $L(N)$ can be enumerated in polynomial delay if $N$ is deterministic. Ackermann and Shallit generalized his algorithm and proved that, for a given length $n$ (which they call cross-section), the lexicographically smallest word in $L(N)$ can be found in time $O(|Q|^2 n^2)$ \[2\], Theorem 1.

They then prove that the set of all words of length $n$ can be computed in time $O(|Q|^2 n^2 + |\Sigma||Q|^2 x)$, where $x$ is the sum of lengths of outputted words \[2\], Theorem 2. A closer inspection of their algorithm actually shows that it has delay $O(|\Sigma||Q|^2 |w|)$ where $|w|$ is the size of the next output. In fact, Ackermann and Shallit prove that the words in $L(N)$ can be enumerated in radix order.\[4\]

It is easy to extend the algorithm of Ackermann and Shallit to solve ExactPaths in polynomial delay as follows. We construct an NFA $N_r$ for $r$ and take the product with $(V, E, s, t)$. The product automaton therefore has states $(q, u)$ where $q$ is a state from $N_r$ and $u$ a node from $G$. In the resulting automaton, replace every transition $[(q_1, u_1), a, (q_2, u_2)]$ with $[(q_1, u_1), (u_1, a, u_2), (q_2, u_2)]$. Enumerating the words from the resulting automaton in radix order corresponds to enumerating the paths from $s$ to $t$ that match $r$ in radix order in polynomial delay. We therefore have the following corollary.

\[\text{Corollary 4. ExactPaths and EnumShortPaths can be solved in polynomial delay.}\]

For completeness, we note that counting the number of paths from $s$ to $t$ that match a given regular expression $r$ is $\#P$-complete in general, even if $G$ is acyclic, see \[34\] Theorem 4.8(1)] and \[6\] Theorem 6.1.\[5\] The same holds for counting the number of shortest paths, since all paths in the proof of \[31\] Theorem 4.8(1)] have equal length.

\section{Enumerating Simple Regular Paths}

We now turn to the question of enumerating simple paths with polynomial delay. A starting point is Yen’s algorithm \[17\] for finding simple paths from a source $s$ to target $t$. Yen’s
Algorithm 1 Yen’s algorithm

**Input:** Graph $G = (V,E)$, nodes $s, t$

**Output:** The simple paths from $s$ to $t$ in $G$

1: $A ← \emptyset$ \hfill $\triangleright A$ is the set of paths already written to output
2: $B ← \emptyset$ \hfill $\triangleright B$ is a set of paths from $s$ to $t$
3: $p ←$ a shortest path from $s$ to $t$ in $G$
4: while $p \neq \text{null}$ do \hfill $\triangleright$ As long as we find a path $p$
5: output $p$
6: Add $p$ to $A$
7: for $i = 1$ to $|p| - 1$ do
8: $G' ← (V', E')$, where $V' = V \setminus V(p[i-1])$ and $E' = E \cap (V' \times V')$
9: for every path $p_1$ in $A$ with $p_1[i-1] = p[i-1]$ do
10: Delete the edge $p_1[i-1,i]$ in $G'$
11: end for \hfill $\triangleright G'$ now no longer has paths already in $A$
12: Find a shortest path $p_2$ from $p[i,i]$ to $t$ in $G'$
13: Add $p_1[i,i] \cdot p_2$ to $B$
14: end for
15: $p ←$ a shortest path in $B$ \hfill $\triangleright p ← \text{null}$ if $B = \emptyset$
16: Remove $p$ from $B$
17: end while

algorithm usually takes another parameter $K$ and returns the $K$ shortest simple paths, but we present a version here for enumerating all simple paths.

**Theorem 5** (Implicit in [47]). Given a graph $G$ and nodes $s, t$, Algorithm 1 enumerates all simple paths from $s$ to $t$ in polynomial delay.

**Proof sketch.** The original algorithm of Yen [47] finds, for a given $G$, $s$, $t$, and $K \in \mathbb{N}$ the $K$ shortest simple paths from $s$ to $t$ in $G$. Its only difference to Algorithm 1 is that it stops when $K$ paths are returned.

Yen does not prove that the algorithm has polynomial delay, but instead shows that the delay is $O(KN + N^3)$, where $N$ is the number of nodes in $G$. Unfortunately, $K$ can be exponential in $|G|$ in general. However, the reason why the algorithm has $K$ in the complexity is line 9 which iterates over all paths in $A$. If we do not store $A$ as a linked list as in [47] but as a prefix tree of paths instead, the algorithm only needs $O(N^2)$ steps to complete the entire for-loop on line 9 (without any optimizations). We therefore obtain delay $O(N^3)$ from Yen’s analysis.

### 4.1 Downward Closed Languages

Yen’s algorithm immediately shows that EnumSimPaths can be solved in polynomial delay for languages that are closed under taking subsequences. Formally, we say that a language $L$ is downward closed if, for every word $w = a_1 \cdots a_n \in L$ and every sequence $0 < i_1 < \cdots < i_k < n + 1$, we have that $a_{i_1} \cdots a_{i_k} \in L$. A regular expression is downward closed if it defines a downward closed language.

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6 In [47], Section 5, he notes that computing path number $k$ in the output costs, in his terminology, $O(KN)$ time in Step I(a) and $O(N^3)$ in Step I(b).
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**Proposition 6.** $\text{EnumSimPaths}(R)$ is in polynomial delay for the class $R$ of downward closed regular expressions, even when the paths need to be output in radix order.

**Proof sketch.** Assume that $(G,s,t)$ and $r$ is an input for $\text{EnumSimPaths}$ such that $L(r)$ is downward closed. Let $N = (Q,\Sigma,\delta,Q_I,Q_F)$ be an NFA for $r$. We change Algorithm 1 as follows:

- In line 3 instead of finding a shortest path $p$ in $G$, we first find a shortest path $p$ in $(G,s,t) \times N$. We then replace every node of the form $(u,q) \in V \times Q$ in $p$ by $u$.
- In line 12 we need to find a shortest path in a product between $(G',p[i,i],t)$ and $N$. More precisely, let $J = \delta^*((\text{lab}(p[0,i])))$ and denote by $N_J$ the NFA with initial state set $J$, that is, $(Q,\Sigma,\delta,J,Q_F)$. Then, in line 12 we first find a shortest path $p_2$ from any node in $(\{(p[i,i],q_i) \mid q_i \in \delta^*(\text{lab}(p[0,i]))\})$ to any node in $\{(t,q_F) \mid q_F \in Q_F\}$ in $(G',p[i,i],t) \times N_J$. Then, we replace every node of the form $(u,q) \in V \times Q$ in $p_2$ by $u$.

We prove in Appendix B that the adapted algorithm is correct. By using Ackermann and Shallit’s algorithm [2] from Theorem 3, we can even find a smallest path in $\text{EnumSimPaths}(R)$ in polynomial delay in radix order.

Now we prove that upper bounds transfer from simple path problems to trail problems. This is not immediate from Lemma 7, since it only deals with unlabeled graphs. Furthermore, there cannot be a polynomial time reduction from $\text{SimPath}$ to $\text{edgespath}(a^*b^*)$ since $\text{Trail}(a^*b^*)$ is in FPT while $\text{SimPath}(a^*b^*)$ is W[1]-hard. We will prove this later in Theorem 26 and Theorem 27.

**Lemma 7.** Let $r$ be a regular expression and $(G,s,t)$ a graph. Then there exist graphs $(H_1,s_1,t_1),\ldots,(H_n,s_n,t_n)$ with $n \leq |G|$ such that there exists a trail from $s$ to $t$ in $G$ that matches $r$ if and only if there exists an $i$ such that there exists a simple path from $s_i$ to $t_i$ in $H_i$ that matches $r$. Furthermore, each $H_i$ is computable in polynomial time.

Using this Lemma, we can immediately show that the upper bound from Lemma 5 also holds for edge-disjoint problems.

**Corollary 8.** $\text{EnumTrails}(R)$ is in polynomial delay for the class $R$ of downward closed regular expressions, even when the paths need to be output in radix order.

**Proof.** Given $r \in R$ and a graph $G$. We use Lemma 7 to construct the graphs $(H_1,s_1,t_1),\ldots,(H_n,s_n,t_n)$. The algorithm in Lemma 6 allows us to enumerate all simple paths from $s_i$ to $t_i$ in $H_i$ in radix order. Therefore, we use $n$ parallel instances of this algorithm to enumerate, for all $i$, all simple paths from $s_i$ to $t_i$ in $H_i$ in radix order. Since each simple path in each $H_i$ corresponds to a trail in $G$, we can also output the corresponding paths in polynomial delay with radix order.

### 4.2 Beyond Downward Closed Languages, Data Complexity

Once we go beyond downward-closed languages, simple paths or trails can not always be enumerated in polynomial delay (if $P \neq NP$). For instance, the problems $\text{SimPath}(a^*ba^*)$ and $\text{SimPath}(aa^*)^*$ are well known to be NP-complete [34] and it is easy to see that the corresponding problems for trails are NP-complete too.

Bagan et al. [7] studied the data complexity of $\text{SimPath}$ and discovered a dichotomy
w.r.t. a class \( C_{\text{tract}} \) of regular languages. More precisely, although \( \text{SimPath}(r) \) can be NP-complete in general, it is in PTIME if \( L(r) \in C_{\text{tract}} \) and NP-complete otherwise [7, Theorem 2]. Here, \( C_{\text{tract}} \) is defined as follows.

\begin{definition} \text{(Similar to [7], Theorem 4)} \end{definition}

For \( i \in \mathbb{N} \), we say that a regular language can be \( i \)-loop abbreviated if, for all \( w_l, w, w_r \in \Sigma^*, w_1, w_2 \in \Sigma^+ \), we have that, if \( w_l w_1^i w w_2^i w r \in L \), then \( w_l w_1^i w r \in L \). We define \( C_{\text{tract}} \) as the set of regular languages \( L \) such that there exists an \( i \in \mathbb{N} \) for which \( L \) can be \( i \)-loop abbreviated.

We show that Bagan et al.’s classification also leads to a dichotomy w.r.t. polynomial delay enumeration in terms of data complexity.

\begin{theorem} \text{In terms of data complexity,} \end{theorem}

(a) \( \text{EnumSimPaths}(r) \) can be solved in polynomial delay if \( L(r) \in C_{\text{tract}} \) and

(b) \( \text{SimPath}(r) \) is NP-complete otherwise.

\begin{proofsketch} \text{Part (b) is immediate from [7, Theorem 1]. For (a), our plan is to use Bagan et al.’s algorithm for simple paths (which we call BBG algorithm) as a subroutine in Yen’s algorithm. We call BBG in lines 3 and 12, so that the algorithm receives}

(i) a simple path from \( s \) to \( t \) that matches \( r \) in line 3 and

(ii) a simple path \( p_2 \) from \( p[i, i] \) to \( t \) such that \( p[0, i] \cdot p_2 \) matches \( r \) in line 12, respectively. Change (i) to Yen’s algorithm is trivial. Change (ii) can be done by calling BBG with \( G' \) for the language of the automaton \( N_J \) in the proof of Proposition 6. We show that the adapted algorithm is correct in Appendix B.} \end{proofsketch}

As we argue in Appendix B, the algorithm for Theorem 10(a) can even be adapted to output paths in increasing length (even radix order).

In fact, Bagan et al.’s dichotomy can also be extended to \( \text{Trail}(r) \). We note that the NP hardness of \( \text{SimPath}(r) \) does not carry over to \( \text{Trail}(r) \) with the reductions introduced in Lemmas 1 or 7. Lemma 1 only applies to unlabeled graphs and, when adjusting it to labeled graphs, one would only obtain hardness for a very restricted class of expressions instead of all expressions in \( C_{\text{tract}} \). Lemma 7 on the other hand only allows to transfer the upper bound. We therefore need to revisit some of Bagan et al.’s methods.

\begin{theorem} Let \( r \) be a regular expression.

(a) If \( L(r) \) belongs to \( C_{\text{tract}} \), \( \text{Trail}(r) \) is in PTIME.

(b) Otherwise, \( \text{Trail}(r) \) is NP-complete.\end{theorem}

\begin{proof} Part (a) follows directly from Lemma 7 and the upper bound of Bagan et al. [7, Theorem 2]. It remains to show (b). The upper bound again follows from Lemma 7. The hardness is similar to [7, Lemma 2]. We prove it in the Appendix. \end{proof}

4.3 Beyond Downward Closed Languages, Combined Complexity

Unfortunately, Bagan et al.’s classification does not go through when we consider combined complexity. Indeed, if \( G \) is a graph with \( n \) nodes and only \( a \)-labeled edges, then asking if there is a simple path that matches the expression \( a^n \) (which is finite and therefore in \( C_{\text{tract}} \)) is the NP-complete Hamilton Path problem.

\footnote{They actually proved that there is a trichotomy: the third characterization is that \( \text{SimPath} \) is in \( AC^0 \) if \( L(r) \) is finite.}
On the other hand, Alon et al. \[3\] proved that SimPath for graphs with \(n\) nodes is in PTIME for the language \(a^{\log^2 n}\), which is also in \(C_{\text{tract}}\). It is open since 1995 whether SimPath is in PTIME for \(a^{\log^2 n}\) [3]. Recently, Björklund et al. [9] showed that, under the Exponential Time Hypothesis, there is no PTIME algorithm that can decide if there exists a simple path of length \(\Omega(f(n)\log^2 n)\) between two nodes in a graph of size \(n\) for any nondecreasing polynomial time computable function \(f\) that tends to infinity. The same holds if we consider trails instead of simple paths.

So, first of all, we see that all these languages are in \(C_{\text{tract}}\) and behave very differently in terms of combined complexity. Second, the parameter \(k\) of \(a^k\) plays a great role, which motivates us to study the problem from the angle of parameterized complexity next.

## 5 Simple Paths With Length Constraints

In this section we investigate the parameterized complexity of problems that involve simple paths with length constraints. The problems we consider here are the core of the RPQ evaluation problems in Section 6. We first give a quick overview of some notions in parameterized complexity. We follow the exposition of Cygan et al. [18] and refer to their work for further details. A parameterized problem is a language \(L_k \subseteq \Sigma^* \times \mathbb{N}\) where, as before, \(\Sigma\) is a fixed, finite alphabet. For an instance \((x, k) \in \Sigma^* \times \mathbb{N}\), we call \(k\) the parameter. The size \(|(x, k)|\) of an instance \((x, k)\) is defined as \(|x| + k\). A parameterized problem \(L_k\) is called fixed-parameter tractable if there exists an algorithm \(A\), a computable function \(f : \mathbb{N} \rightarrow \mathbb{N}\), and a constant \(c\) such that, given \((x, k) \in \Sigma^* \times \mathbb{N}\), the algorithm \(A\) correctly decides whether \((x, k) \in L_k\) in time bounded by \(f(k) \cdot |(x, k)|^c\), where \(c\) is a constant. The complexity class containing all fixed-parameter tractable problems is called FPT.

Let \(L_k\) and \(L'_k\) be two parameterized problems. A parameterized reduction from \(L_k\) to \(L'_k\) is an algorithm \(R\) that, given an instance \((x, k)\) of \(L_k\), outputs an instance \((x', k')\) of \(L'_k\) such that
- \((x, k)\) is a yes-instance of \(L_k\) if and only if \((x', k')\) is a yes-instance of \(L'_k\),
- \(k' \leq g(k)\) for some computable function \(g\), and
- the running time of \(R\) is \(f(k) \cdot |(x, k)|^{O(1)}\) for some computable function \(f\).

Downey and Fellows introduced the W-hierarchy \[21\]. The \(k\)-Clique problem is \(W[1]\)-complete, that is, complete for the first level of the W-hierarchy \[22\]. Therefore, \(k\)-Clique not being fixed-parameter tractable is equivalent to \(\text{FPT} \neq \text{W}[1]\), which is a standard assumption in parameterized complexity.

### 5.1 One Simple Path

We consider the following parameterized problems.
- **SimPath\(_k\)**: Given an instance \(((G, s, t), k)\) with \(k \in \mathbb{N}\), is there a simple path from \(s\) to \(t\) of length exactly \(k\) in \(G\)?
- **SimPath\(_{\leq k}\)** and **SimPath\(_{\geq k}\)**: these are defined analogously to SimPath\(_k\) but ask if there is a simple path of length \(\leq k\) and \(\geq k\), respectively.

The problems Trail\(_k\), Trail\(_{\leq k}\), and Trail\(_{\geq k}\) are defined analogously but consider trails instead of simple paths.

These three problems are in FPT, but the techniques to prove it are quite different. For SimPath\(_k\), membership in FPT follows from the famous color coding technique \[3\].

\textbf{Theorem 12} (Alon et al. \[3\]). SimPath\(_k\) is in FPT.

SimPath\(_{\leq k}\) is trivially in FPT because the shortest path problem is in PTIME.
Finally, SimPath$_{\leq k}$ can be shown to be in FPT by adapting methods from Fomin et al. [25]. They proved that finding simple cycles of length at least $k$ is in FPT for cycles and discovered that their technique also works for paths [19]. The following theorem is therefore due to the authors of [25]. We present a proof in Appendix C because we need it to prove Theorems 23 and 26 (We note that Fomin et al. 25 did already consider SimPath$_{\geq k}$ on undirected graphs, but the techniques needed on directed graphs are quite different.)

**Theorem 13.** SimPath$_{\leq k}$ is in PTIME (and therefore in FPT).

By Lemma 7 the complexities of Theorems 12, 13, and 14 carry over from simple paths to trails.

**Theorem 15.** Trail, Trail$_{\leq k}$, and Trail$_{\geq k}$ are in FPT

### 5.2 Two Node-Disjoint Paths

We consider variants of the TwoDisjointPaths problem [25]. A two-colored graph is a directed graph in which every edge is given one of two colors, say $a$ or $b$. An $a$-colored path is a path consisting of only $a$-colored edges. We will denote an $a$-colored edge from $u$ to $v$ with $u \xrightarrow{a} v$ (similar for $b$-colored edges). In the remainder we abbreviate $a$-colored edge and $a$-colored path by $a$-edge and $a$-path, respectively. We consider the following parameterized problems.

- **TwoNodeDisjointPaths$_k$:** Given a graph $G$, nodes $s_1, t_1, s_2, t_2$, and parameter $k \in \mathbb{N}$, are there simple paths $p_1$ from $s_1$ to $t_1$ and $p_2$ from $s_2$ to $t_2$ such that $p_1$ and $p_2$ are node-disjoint and $p_1$ has length $k$?

- **TwoColorNodeDisjointPaths$_k$:** Given a two-colored graph $G$ and nodes $s_a, t_a, s_b, t_b$, is there a simple $a$-path $p_a$ from $s_a$ to $t_a$ and a simple $b$-path $p_b$ from $s_b$ to $t_b$ such that $p_a$ and $p_b$ are node-disjoint and $p_a$ has length $k$?

It is well-known that TwoDisjointPaths, the non-parameterized version of TwoNodeDisjointPaths$_k$, is NP-complete [25]. Cai and Ye [12] proved that TwoNodeDisjointPaths$_k$ is in FPT for undirected graphs, both for the cases where one wants simple paths or trails. They left the cases for directed graphs as open problems [12] Problem 2). We solve one of the cases by showing in Theorem 16 that TwoNodeDisjointPaths$_k$ is $W[1]$-hard. We start by proving that TwoColorNodeDisjointPaths$_k$ is $W[1]$-hard, because the proof for TwoNodeDisjointPaths$_k$ relies on it.

**Theorem 16.** TwoColorNodeDisjointPaths$_k$ is $W[1]$-hard.

**Proof.** The proof is inspired by an adaptation of Grohe and Grüber [28 Lemma 16] of a proof by Slivkins [12 Theorem 2.1]. Slivkins proved that $k$ Disjoint Paths is $W[1]$-hard in acyclic graphs, that is, he showed that it $W[1]$-hard to decide, given a DAG $G$ and nodes $s_1, t_1, \ldots, s_k, t_k$ (with parameter $k$), if there are pairwise edge-disjoint simple paths from $s_i$ to $t_i$ for each $i = 1, \ldots, k$.

We reduce from $k$-Clique, which is well known to be $W[1]$-complete [22 Corollary 3.2]. Let $G = (V, E)$ be an undirected graph and assume w.l.o.g. that $V = \{1, \ldots, n\}$. We will construct a two-colored graph $G'$ with $kn \cdot 2(k + 1) + k(k - 1)/2 + 2(k + 1)$ nodes such that $G$ has a $k$-Clique if and only if $G'$ has node-disjoint simple paths $p_1$ from $s_1$ to $t_1$ and $p_2$ from $s_2$ to $t_2$ such that $p_1$ is $a$-colored and has length $k' \in \Theta(k^2)$ while $p_2$ is $b$-colored. The graph $G'$ contains $kn$ gadgets $G_{i,j}$ with $i = 1, \ldots, k$ and $j = 1, \ldots, n$, each consisting of $2(k + 1)$ nodes. Gadgets will be ordered in $k$ rows, where row $i$ has gadgets $G_{i,1}^{1,1}, \ldots, G_{i,n}^{1,1}$.
We note that Theorem 16 can be proved without using control nodes, but we need them to for Theorem 18 where we then only require a small change to the construction.

8 We note that Theorem can be proved without using control nodes, but we need them to for Theorem 18 where we then only require a small change to the construction.
Corollary 17. \textbf{TwoColorDisjointPaths}_k is W[1]-hard even if \( t_1 = s_2 \).

The two colors in the proof of Theorem 16 play a central role: since the \( a \)-path cannot use any \( b \)-edges and vice versa, we have much control over where the two paths can be. The following Theorem shows that the construction in Theorem 16 can be strengthened so that we do not need the two colors.

Theorem 18. \textbf{TwoNodeDisjointPaths}_k is W[1]-hard.

Proof. We adapt the reduction from Theorem 17. The only change we make is that we replace each \( b \)-edge by a directed path of \( k' \) edges (introducing \( k' - 1 \) new nodes for each such edge). We prove in Appendix C that the reduction is correct.

For completeness, we mention the complexity of other variants of \textbf{TwoNodeDisjointPaths}_k, some of which can be shown by extending the technique from Theorem 18. We define \textbf{TwoNodeDisjointPaths}_{\leq k} and \textbf{TwoNodeDisjointPaths}_{\geq k} analogously to \textbf{TwoNodeDisjointPaths}_k by requiring that \( p_1 \) has length \( \leq k \) and \( \geq k \), respectively.

Theorem 19. \textbf{TwoNodeDisjointPaths}_{\leq k} is W[1]-hard.

\textbf{TwoNodeDisjointPaths}_{\leq k} is NP-complete for every constant \( k \in \mathbb{N} \) \cite{20}.
Two Edge-Disjoint Trails

Here, we study the trail versions of the disjoint paths problem where we require the trails to be edge-disjoint.

**Theorem 20.** TwoEdgeDisjointTrails_k is W[1]-hard.

The W[1] hardness follows directly from Theorem 18 and the reduction in Lemma 1. Nonetheless, we give a proof analogous to Theorem 18 in Appendix C, since we feel that this might help to better understand our W[1] hardness proof of Theorem 27.

6 Parameterized Complexity of Simple Regular Paths

We now return to regular path query evaluation and consider parameterized versions of SimPath and Trail. In contrast to Section 5, the parameter k will not be a constraint on the length of the paths. Instead, we will search for paths of arbitrary length (as in Sections 3 and 4) and the parameter k will be determined by the regular expression. That is, the parameter k is implicitly determined by r.

**Some Concrete Languages.** We first consider a few simple examples of such problems and generalize the approach later in this section. For k ∈ N, we define the regular expressions 
\[ a^k, (a?)^k, a^k a^*, a^k b^*, \text{ and } a^{k-1}ba^* \]
to have parameter k. By abusing notation, we denote by \( a^k b^* \) the class of regular expressions \( \{a^k b^* | k \in \mathbb{N}\} \) (similar for the other abovementioned expressions). As such, the parameterized problem SimPath(\( a^k b^* \)) asks, given \( (G, s, t) \) and a regular expression r of the form \( a^k b^* \), if there exists a simple path from s to t that matches r. It is in FPT if it can be decided by an algorithm that runs in time \( f(k) \cdot (|G| + |r|)^e \) for a computable function f and a constant c. The following can now be easily deduced from Section 5.

**Theorem 22.**
(a) SimPath(\( a^k \)), SimPath((a?)^k), and SimPath(\( a^k a^* \)) are in FPT.
(b) SimPath(\( a^k b^* \)) and SimPath(\( a^{k-1}ba^* \)) are W[1]-hard.

**Proof.** Part (a) is immediate from Theorems 12, 13, and 14, respectively. For part (b), the hardness of SimPath(\( a^k b^* \)) is immediate from Corollary 17. The hardness of SimPath(\( a^{k-1}ba^* \))
Table 1 Structure of the 250K property paths in the corpus of Bonifati et al. [10]

| Expression Type | Relative | \( \ell \) | STE? | Expression Type | Relative | \( \ell \) | STE? |
|-----------------|----------|------------|------|-----------------|----------|------------|------|
| \((a_1 + \cdots + a_\ell)^*\) | 39.12% | 2–4 | yes | \(a_1a_2\cdots a_\ell?\) | 0.02% | 1–3 | yes |
| \(a^*\) | 26.42% | yes | (\(ab^*\) + c) | 0.01% | no |
| \(a_1 \cdots a_\ell\) | 11.65% | 2–6 | yes | \(a^*b?\) | 0.01% | yes |
| \(a^*b\) | 10.39% | yes | \(abc^*\) | 0.01% | yes |
| \(a_1 + \cdots + a_\ell\) | 8.72% | 2–6 | yes | \((a + b)^*\) | 0.01% | 2 | yes |
| \(a^+\) | 2.07% | yes | \((a^* + \cdots + a_\ell)^*\) | 0.01% | 2 | yes |
| \(a_1?a_2?\cdots a_\ell?\) | 1.55% | 1–5 | yes | \(A_1A_2\) | < 0.01% | yes |
| \(a(b_1 + \cdots + b_\ell)\) | 0.02% | 2 | yes | other | 0.01% | mixed |

is obtained from Theorem 18 by applying a simple proof of Mendelzon and Wood [36, Theorem 1 (2)], reducing TwoDisjointPaths to SimPath\((a^*ba^*)\). The idea is to add \(t_1 \rightarrow s_2\) (and label all other edges \(a\)). Then, every path from \(s_1\) to \(t_2\) matching \(a_1^*ba_2^*\) must contain the \(b\)-edge. This implies that such a path exists if and only if there exist two node-disjoint paths, one from \(s_1\) to \(t_1\) and the other from \(s_2\) to \(t_2\). ◀

We can even slightly generalize the proof of Theorem 22(a) to deal with more complex languages. Notice that the following result implies that SimPath\((a^kba^*)\) is in FPT, whereas SimPath\((a^*ba^*)\) is \(W[1]\)-hard by Theorem 22(b).

▶ Theorem 23. For every constant \(c\) and word \(w\) with \(|w| = c\), the problem SimPath \((a^kba^*)\) with parameter \(k\) is in FPT.

Proof sketch. First we use the algorithm from Theorem 14 to decide SimPath \((a^*a^*)\). If the answer is no, we enumerate all possible paths \(p\) that match \(w\) and change the algorithm from Theorem 14 to find two disjoint simple paths, not intersecting \(p\): one from \(s_1\) to \(p[0]\) matching \(a^k\) and one from \(p[c]\) to \(t\) matching \(a^*\). (Recall that \(p[i]\) denotes the \(i\)th node in \(p\).) ◀

If FPT \(\not= W[1]\), then Theorem 23 cannot be generalized to arbitrary words \(w\), since SimPath \((b^ka^*b^*)\) with parameter \(k\) is \(W[1]\)-hard. This can be shown by adapting the proof of Theorem 16. We add a path matching \(b^k\) from a new node \(s_1'\) to \(s_1\) and define \(t_1 = s_2\) as in Corollary 17. Then, the problem whether there is a simple path matching \(b^ka^*b^*\) from \(s_1'\) to \(t_2\) is the same as asking whether there is a simple path matching \(a^*b^*\) from \(s_1\) to \(t_2\).

6.1 Simple Transitive Expressions

We now aim at generalizing the previous results to more general (but still very restricted) regular expressions. However, we feel that these expressions are relevant and important from a practical perspective since they constitute more than 99% of the property paths found in SPARQL query logs in an extensive recent study [10]. Notice that SPARQL property paths are RPQs with added syntactic sugar, so the syntax of the expressions is not restricted as, e.g., in Cypher.

In the following definition, we use \(A \subseteq \Sigma\) to abbreviate \((a_1 + \cdots + a_n)\) so that \(A = \{a_1, \ldots, a_n\}\). We allow \(A = \emptyset\).
Definition 24. An atomic expression is of the form \( A \subseteq \Sigma \). A \( k \)-bounded expression is a regular expression of the form \( A_1 \cdots A_k \) or \( A_1^? \cdots A_k^? \), where \( k \geq 0 \) and each \( A_i \) is an atomic expression. Finally, a simple transitive expression (STE) is a regular expression \( B_{\text{pre}} A^* B_{\text{suff}} \), where \( B_{\text{pre}} \) and \( B_{\text{suff}} \) are bounded expressions and \( A \) is an atomic expression. For an STE \( r = B_{\text{pre}} A^* B_{\text{suff}} \), we define the parameter \( k_r = k_1 + k_2 \), where \( B_{\text{pre}} \) is \( k_1 \)-bounded and \( B_{\text{suff}} \) is \( k_2 \)-bounded.

Notice that about 99.7% of the property paths in Table 1 are STEs or trivially equivalent to an STE (by taking \( A = \emptyset \), for example).

6.2 Two Dichotomies

Dichotomy for Simple Paths

Definition 25. Let \( r = B_{\text{pre}} A^* B_{\text{suff}} \) be an STE with \( L(r) \neq \emptyset \). If \( B_{\text{pre}} = A_1 \cdots A_{k_1} \), then its left cut border \( c_1 \) is the largest value such that \( A_i \not\subseteq A_{c_1} \) if it exists and zero otherwise. If \( B_{\text{pre}} = A_1^? \cdots A_{k_1}^? \), then its left cut border is zero. We define right cut borders symmetrically (e.g., for \( B_{\text{suff}} = A'_1 \cdots A'_{k_2} \), it is the largest \( c_2 \) such that \( A_i \not\subseteq A_{c_2} \)).

We explain the intuition behind cut borders in Figure 5.

For \( c \in \mathbb{N} \), an expression is \( c \)-bordered if the maximum of its left and right cut borders is \( c \). We call a class \( R \) of STEs cuttable if there exists a constant \( c \in \mathbb{N} \) such that each expression in \( R \) is \( c' \)-bordered for some \( c' \leq c \). We can now prove a dichotomy on the complexity of \( \text{SimPath}(R) \) for classes of STEs \( R \), if \( R \) satisfies the following mild condition.

We say that \( R \) can be sampled if there exists an algorithm that, given \( k \in \mathbb{N} \), returns an expression in \( R \) that is \( k' \)-bordered with \( k' \geq k \), and “no” otherwise.

Theorem 26. Let \( R \) be a class of STEs that can be sampled. Then,

(a) if \( R \) is cuttable, then \( \text{SimPath}(R) \) is in FPT with parameter \( k_r \) and
(b) otherwise, \( \text{SimPath}(R) \) is W[1]-hard with parameter \( k_r \).

Proof idea. The main idea of the proof is to attack case (a) using the techniques for proving Theorem 14. If \( R \) is cuttable, we can use exhaustive search to enumerate all possible pre- and suffixes of length at most \( c \). We then use a variation of the representative sets technique [25] to obtain an FPT algorithm. In case (b), we show that it is possible to adapt the reduction in the proof of Theorem 18.

Notice that the difference between cuttable and non-cuttable classes of STEs can be quite subtle. For instance, \( b^k a^* \) and \( a^k (a + b)^* \) are non-cuttable, but \( (a + b)^k a^* \) is cuttable.

Dichotomy for Trails

We now present the dichotomy for trails. Perhaps surprisingly, this dichotomy is slightly different. The underlying reason is that \( \text{Trail}(a^k b^*) \) is in FPT because the \( a \)-path and the \( b \)-path can be evaluated independent of each other (no \( a \)-edge will be equal to a \( b \)-edge). On the other hand we have that \( \text{Trail}(a^k b a^*) \) is W[1]-hard.

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9 In fact, all expressions in the full version of Table 1 in [10] except for one can be handled with the techniques we present here. They just don’t fit the definition of STEs.
Let $r = A_1 \cdots A_k A^* A'_k \cdots A'_1$ be an STE with left cut border $c_1$ and right cut border $c_2$. We say that $A_i$ with $i \leq c_1$ (resp., $A'_j$ with $j \leq c_2$) is a conflict position if there exists a symbol $\sigma \in A_i \cap A$ (resp., $\sigma \in A_j \cap A$). We say that $R$ is almost conflict free if there exists a constant $c$ such that each $r \in R$ has at most $c$ conflict positions.

We say that $R$ can be conflict-sampled if there exists an algorithm that, given $k \in \mathbb{N}$, returns an expression in $R$ that has $k'$ conflict labels with $k' \geq k$, and "no" otherwise.

\begin{itemize}
  \item \textbf{Theorem 27.} Let $R$ be a class of STEs that can be conflict-sampled. Then, (a) if $R$ is almost conflict free, then $\text{Trail}(R)$ is in FPT with parameter $k_r$ and (b) otherwise, $\text{Trail}(R)$ is $W[1]$-hard with parameter $k_r$.
\end{itemize}

\section{Enumeration Problems for Simple Transitive Expressions}

We now observe that our tractability results can be carried over to the enumeration setting. To this end, a parameterized enumeration problem is defined analogously as an enumeration problem, but its input is of the form $(x,k) \in \Sigma^* \times \mathbb{N}$. It is in FPT delay if there exists an algorithm that enumerates the output such that the time between two consecutive outputs is bounded by $f(k) \cdot |x|^c$ for a constant $c$. Notice that each problem in polynomial delay is also in FPT delay.

The problems in the following theorems are straightforward enumeration versions of problems we already considered.

\begin{itemize}
  \item \textbf{Theorem 28.} $\text{EnumSimPaths}\geq k$ is in FPT delay.
  \item \textbf{Theorem 29.} For each constant $c$ and each word $w$ with length $|w| = c$, the problem $\text{EnumSimPaths}(a^k w? a^*)$ is in FPT delay.
  \item \textbf{Theorem 30.} Let $R$ be a cuttable class of STEs. Then $\text{EnumSimPaths}(R)$ is in FPT delay.
\end{itemize}

The proofs of these theorems are all along the same lines. In the proof of Theorem 10 we adapted Yen’s algorithm to work with simple instead of shortest paths. We already showed that the problems $\text{SimPath}\geq k$, $\text{SimPath}(a^k w? a^*)$, and $\text{SimPath}(R)$ are in FPT. Furthermore, these FPT algorithms can trivially be adjusted to also return a matching path if it exists. We also need to show that we can find simple paths matching suffixes in the language (for the adapted line 12 of Yen’s algorithm in the proof of Theorem 10). This can also be done for each of these theorems, essentially because the suffixes of the languages we need to

\footnote{More precisely, we need language derivatives, see Appendix B.1.}
consider again can be solved with our FPT algorithms. In Appendix \[\text{Appendix F}\] we prove that this approach works.

Furthermore, we can also show that the FPT result from Theorem 27 carries over to enumeration problems.

\[\text{Theorem 31.}\] Let \(R\) be a class of STE that is almost conflict-free. Then, \(\text{EnumTrails}(R)\) is in FPT delay.

8 Conclusions

Our main results are two dichotomies on the parameterized complexity of evaluating simple transitive expressions (STEs), which are a class of regular expressions powerful enough to capture over 99% of the RPQs occurring in a recent practical study [10]. These dichotomies are for simple path semantics and trail semantics, respectively.

For simple path semantics, the central property that we require for a class of expressions so that evaluation is in FPT is cuttability, i.e., constant-size cut borders (also see Figure 5).

For trail semantics, the dichotomy is such that the FPT fragment is slightly larger. Even if the cut borders of a class of expressions is not bounded by a constant, it can be evaluated in FPT if the number of conflict positions are bounded by a constant. An example of a non-cuttable class of expressions with a constant number of conflict positions is \(\{a^k b^* \mid k \in \mathbb{N}\}\).

For this class, evaluation over trail semantics is in FPT (with parameter \(k\)) but \(W[1]\)-hard over simple path semantics.

Looking at Table 1, we see that the cut borders for expressions in practice are indeed very small: it is one for \(a^* b\), two for \(a b c^*\), and zero in all other cases. All these expressions have FPT evaluation for simple path and trail semantics. Therefore, although the simple path and trail semantics of RPQs are known to be hard in general, it seems that the RPQs that users actually ask are much less harmful. In fact, since the vast majority of expressions in Table 1 has cut borders of at most two, our FPT result in Theorem 26 implies that evaluation for this majority of expressions is in polynomial time combined complexity. Furthermore, matching paths can be enumerated in polynomial delay. (Recall that, if \(P \neq \text{NP}\), this is impossible even for fixed expressions: evaluation for \(a^* ba^*\) or \((aa)^*\) under simple path semantics is NP-complete.)

For the expressions in Table 1, the parameter \(k_r\) is at most six. Since the function \(f\) in our FPT algorithms is only single exponential, we believe that these expressions can be dealt with in practical scenarios, in principle. The data complexity of our FPT algorithms is currently \(O(m n \log n + n^2 + mn)\) with \(m = |E|\) and \(n = |V|\). This bound comes from Fomin et al.’s representative set technique [25] and we did not yet investigate yet if this can be improved. We believe that this would be an interesting future direction.

Acknowledgments

We are grateful to Phokion Kolaitis for suggesting us to study enumeration problems on simple paths matching RPQs. We are also grateful to Holger Dell for pointing us to Theorem 14 and providing us with a proof sketch.

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In the Appendix we provide proofs for which there was no space in the body of the paper. In some proofs, we indicate by \( \cdots \) where we continue a proof that was partly presented in the body.

### A Proofs for Section 2

**Lemma 32.** Let \((G', s_i', t_1', \ldots, s_k', t_k') = \text{split}(G, s_1, t_1, \ldots, s_k, t_k)\). Then there exists pairwise node disjoint simple paths of length \( k_i \) from \( s_i \) to \( t_i \) in \( G \) iff there exist pairwise edge disjoint trails of length \( 2k_i + 1 \) from \( s_i' \) to \( t_i' \) in \( G' \).

**Proof.** This is an easy consequence of the construction. \( \blacksquare \)

**Lemma 33.** Let \((G', s_i', t_1', \ldots, s_k', t_k') = \text{line}(G, s_1, t_1, \ldots, s_k, t_k)\). Then there exist pairwise edge disjoint trails of length \( k_i \) from \( s_i \) to \( t_i \) in \( G \) iff there exist pairwise node disjoint simple paths of length \( k_i + 1 \) from \( s_i' \) to \( t_i' \) in \( G' \).

**Proof.** This is an easy consequence of the construction. \( \blacksquare \)

**Notation.** In the appendix, we sometimes use u-v-path to refer to a path from \( u \) to \( v \).

### B Proofs for Section 4

**Proposition 6.** \( \text{EnumSimPaths}(\mathcal{R}) \) is in polynomial delay for the class \( \mathcal{R} \) of downward closed regular expressions, even when the paths need to be output in radix order.

**Proof.** Assume that \((G, s, t)\) and \( r \) is an input for \( \text{EnumSimPaths} \) such that \( L(r) \) is downward closed. Let \( N = (Q, \Sigma, \delta, \mathcal{Q}_I, \mathcal{Q}_F) \) be an NFA for \( r \). We change Algorithm 1 as follows:

1. In line 3 instead of finding a shortest path \( p \) in \( G \), we first find a shortest path \( p \) in \((G', s, t) \times N\). We then replace every node of the form \((u, q) \in V \times Q\) in \( p \) by \( u \).
2. In line 12 we need to find a shortest path in a product between \((G', p[i, i], t) \times N\). More precisely, let \( J = \delta^p(\text{lab}(p[0, i])) \) and denote by \( N_J \) the NFA with initial state set \( J \), that is, \((Q, \Sigma, \delta, J, \mathcal{Q}_F)\). Then, in line 12 we first find a shortest path \( p_2 \) from any node in \{\( p[i, j] \), \( q_i \in \delta^p(\text{lab}(p[0, i])) \} \) to any node in \{\( t, q_F \) \} \( q_F \in \mathcal{Q}_F \) in \((G', p[i, i], t) \times N_J\). We then replace every node of the form \((u, q) \in V \times Q \) in \( p_2 \) by \( u \).

\( \cdots \)

We now prove that this leads to a polynomial delay algorithm for \( \text{EnumSimPaths} \). As the product can be constructed in time \( O(|G||N|) \), the algorithm still runs in polynomial delay.

To prove that this algorithm is correct, we first show that no path is written to the output more than once: Each such path is stored in \( A \) and cannot be found again, because the prefix \( p[0, i] \) differs or at least one edge will be deleted in line 9.

We now prove that the algorithm only writes simple paths that match \( r \) to the output. Each shortest path \( p_2 \) considered in the product \((G', p[i, i], t) \times N\) in line 12 is, after replacing nodes \((u, q) \) with \( u \), a simple path in \( G \), because \( L \) is downward closed. Therefore, since \( p_2 \) is disjoint from \( V(p[0, i - 1]) \) due to line 8 of the algorithm, \( p[0, i] \cdot p_2 \) is also a simple path that matches \( r \).

Finally, we prove that the algorithm finds all such simple paths. If a simple path \( p \) in \((G, s, t) \) matches \( r \), then this path is also a simple path in \((G, s, t) \times N\). So, we can find this path using the changed algorithm if and only if we do not delete any edge from \( p \) in \( G \), which is only done in Line 9. But we did not change this line, so it follows from the correctness of Yen’s algorithm that \( p \) can be found. \( \blacksquare \)
Figure 6 Example of a part of the reduction in Lemma 34. There exists a trail from $s$ to $t$ matching $r$ in the left graph if and only if there exists a simple path from $s'$ to $t'$ matching $a \cdot r$ in the right graph.

Lemma 34. Let $r$ be a regular expression and $(G,s,t)$ a graph. Then there exist graphs $(H_1,s_1,t_1), \ldots, (H_n,s_n,t_n)$ with $n \leq |G|$ such that there exists a trail from $s$ to $t$ in $G$ that matches $r$ if and only if there exists an $i$ such that there exists a simple path from $s_i$ to $t_i$ in $H_i$ that matches $r$. Furthermore, each $H_i$ is computable in polynomial time.

Proof. Given a graph $(G,s,t)$, we will construct a graph $(H,s',t')$ such that there exists a simple path from $s'$ to $t'$ matching $ar$ in $H$ if and only if there exists a trail from $s$ to $t$ matching $r$ in $G$, where $a$ is an arbitrary symbol. Excluding $s'$ and $t'$, the graph $H$ is the line graph of $G$. We can then enumerate all possible $a$-edges that start in $s'$ to obtain up to $n$ new instances $(H_1,s'_1,t'), \ldots, (H_n,s'_n,t')$, such that there exists a trail from $s$ to $t$ matching $r$ in $G$ if and only if there exists an $i$ such that there is a simple path from $s'_i$ to $t'$ in $H_i$ that matches $r$.

So it remains to give the construction of $H$ and prove the correctness of the reduction. Let $a \in \Sigma$ be fixed. Let $H = (V', E')$ with $V' = \{v_e \mid e \in E\} \cup \{s',t'\}$ and $E' = \{(u_1,a,v_1),v_1,v_2,a,v_3) \mid u_1,u_2,v_3 \in V\} \cup \{(s',a,v_{(s,a,t)}),(v_{(u,a,t)}),\sigma,t')\}$. An example of this reduction can be seen in Figure 6. We will now show the correctness of the reduction. Assume there exists a path

$$p = (s,a_0,v_1)(v_1,a_1,v_2)\cdots(v_k,a_k,t)$$

from $s$ to $t$ in $G$ that matches $r$ and has pairwise disjoint edges. Then the path

$$p' = (s',a,v_{(s,a_0,v_1)})(v_{(s,a_0,v_1)},a_0,v_{(v_1,a,v_2)})(v_{(v_1,a_1,v_2)},a_1,v_{v_2,a_2,v_3})\cdots(v_{v_k,a_k,t},a_k,t')$$

is a simple path from $s'$ to $t'$ in $H$ that matches $ar$. The other direction follows analogous since each path from $s'$ to $t'$ in $H$ that matches $ar$ has this form and we can therefore find the corresponding path from $s$ to $t$ in $G$.

We note that, in the above proof there is a clear correspondence between nodes in $H_i$ and edges in $G$.

Corollary 35. Furthermore, each node in $H_i$, except for $s_i$ and $t_i$, corresponds to exactly one edge in $G$. 

B.1 Proofs for Section 4.2

In the following proof, we need language derivatives. For a language $L$ and word $w$, the left derivative\(^{11}\) w.r.t. $w$, denoted $w^{-1}L$, is defined as $\{v \mid vw \in L\}$.

\textbf{Theorem 10.} In terms of data complexity, 
(a) EnumSimPaths$(r)$ can be solved in polynomial delay if $L(r) \in C_{\text{tract}}$ and 
(b) SimPath$(r)$ is NP-complete otherwise.

\textbf{Proof.} Part (b) immediately follows from [7, Theorem 1]. It therefore only remains to prove (a). Our plan is to use Bagan et al.’s algorithm for simple paths as a subroutine in Yen’s algorithm. We refer to Bagan et al.’s algorithm as the BBG algorithm.

Second, we show that the algorithm writes all simple paths matching $r$ into the output. Therefore, let $\pi$ be a simple path from $s$ to $t$ in $G$ that matches $r$. Due to the correctness of the BBG algorithm, YenSimple has found a simple path matching $r$ in line \[\text{12}\] and therefore $A \neq \emptyset$. We now consider an arbitrary iteration of the while-loop and prove that either $\pi$ must have been found already or YenSimple will find a path $\tilde{\pi}$ that shares a longer prefix with $\pi$ than all paths in $A$. Clearly, this shows that $\pi$ will eventually be found.

Assume that we are at the beginning of the while-loop and let $S$ be the set of paths in $A$ that share the longest prefix with $\pi$, that is, $S = \{\pi' \in A \mid 30 \leq i \leq |V| : \pi'[0,i] = \pi[0,i]\}$ and there exists no path $\tilde{\pi} \neq \pi' \in A$ with $\tilde{\pi}[0,i+1] = \pi[0,i+1]$. Since $A \neq \emptyset$ and $A$ only contains paths from $s$ to $t$ and $\pi[0,0] = s$, i.e., they share at least the first node, $S$ is not empty. Take $\pi' \in S$ such that $\pi'$ was the last element in $S$ that was added to $A$ (and therefore written to the output). As $\pi'$ and $\pi$ are both simple paths from $s$ to $t$ and $\pi$ was not yet written to the output, $\pi'$ must have at least one edge that is not in $\pi$.

After having added $\pi'$ to $A$, YenSimple searched, for each $i$ from 1 to $|\pi'|$, a simple path having $\pi'[0,i-1]$ as prefix, but not having the edge $(\pi'[i-1,i])$. Let $i'$ be maximal with $\pi'[0,i'-1] = \pi[0,i'-1]$. Notice that the edge $(\pi'[i'-1,i'])$ was not deleted in line \[\text{9}\] because otherwise there would have been a path $\tilde{\pi} \in A$ with $\tilde{\pi}[0,i'] = \pi[0,i']$, contradicting $\pi' \in S$. So, if $\pi$ is a simple path from $s$ to $t$, $\pi$ must have been found already or the algorithm will find a simple path $\tilde{\pi}$ with $\tilde{\pi}[0,i] = \pi[0,i]$ that is not in $A$. This concludes the proof that all simple paths from $s$ to $t$ that match $r$ will be found by YenSimple.

Before we turn to complexity, we need a simple observation.

\textbf{Observation 36.} $C_{\text{tract}}$ is closed under taking left derivatives, that is, if $L \in C_{\text{tract}}$ and $w$ is a word, then $w^{-1}L \in C_{\text{tract}}$.

\(^{11}\)These are sometimes also called Brzozowski derivatives of $L$ [11].
Proof. The observation immediately follows from the definition of $C_{\text{tract}}$. Indeed, if $w^{-1}L \notin C_{\text{tract}}$, then, for every $i \in \mathbb{N}$ there exist words $w_i, w_1, w_2 \in \Sigma^*$, such that $wiw_1w_2w_i \in w^{-1}L$ but $wiw_1w_2w_i \notin w^{-1}L$. However, then we also have for every $i \in \mathbb{N}$ that $wwiw_1w_2w_i \in L$ and $wwiw_1w_2w_i \notin L$, which contradicts that $L \in C_{\text{tract}}$. ◀

We now turn to complexity. In terms of data complexity, the time bound of Yen’s algorithm (i.e., polynomial delay) is not affected by searching simple instead of shortest paths, since BBG operates in polynomial time for $L(r)$, which is in $C_{\text{tract}}$ [7, Lemma 16]. The same holds for $L(NJ)$, which is in $C_{\text{tract}}$ due to Observation 36. This concludes the proof. ◀

The algorithm for Theorem 10(a) does not yield any order on the paths. But an order from shortest to longest paths can easily be obtained by changing BBG to output a shortest path. As Bagan et al. already note [7, Section 3.2], this is indeed a simple change in their algorithm. Moreover, it is also possible to use Ackerman and Shallit’s algorithm [2] for finding shortest and lexicographically smallest paths in the BBG algorithm. It is therefore also possible to enumerate the paths in radix order in polynomial delay.

Lemma 37. If $L(r) \in C_{\text{tract}}$, then in terms of data complexity, $\text{EnumSimPaths}(r)$ can be solved in polynomial delay and with all paths enumerated in radix order.

Proof. Instead of showing that Yen’s algorithm also works with simple paths, as we did in the proof of Theorem 10 part a), we slightly change the algorithm in Bagan et al.’s paper [7] to find a shortest and lexicographically smallest simple path. Then we can use this algorithm as subroutine in Yen’s algorithm.

To this end, we change the algorithm introduced in [7, Lemma 15]. There, in the second step, we replace $(\text{left}_i, \text{cut}_{C_i}, \text{right}_i)$ with a smallest simple $\Sigma_{C_i}$-restricted path in radix order from $\text{left}_i$ to $\text{right}_i$ (i.e., we require additionally that it is a lexicographically smallest path in radix order). We can find such a path using the algorithm of Ackerman and Shallit [2], see also Theorem 5 (We view the subgraph of $G$ that contains only $\Sigma_{C_i}$-restricted nodes (and right$_i$) and $\Sigma_{C_i}$-labeled edges between these nodes as NFA with start state left$_i$ and final state right$_i$.) Since shortest simple paths are always admissible [7, Lemma 13], we are indeed able to find a smallest simple path in radix order in the following way: Just like Bagan et al., we enumerate all possible candidate summaries $S$ w.r.t. $(L(r), G, s, t)$ and then apply to each the adapted algorithm from [7, Lemma 15]. If we find a solution, we do not return ‘yes’ immediately, but continue the enumeration while holding the smallest solution w.r.t. radix order in our storage and update it when necessary.

As the algorithm of Ackerman and Shallit [2] runs in polynomial time, the adapted algorithm [7, Lemma 15] still runs in polynomial time. Since $w^{-1}L(r) \in C_{\text{tract}}$ (see Observation 36), we can use this algorithm as subroutine in Yen’s algorithm in the lines 3 and 12 to obtain a polynomial delay algorithm that enumerates all paths in radix order. ◀

Theorem 11. Let $r$ be a regular expression.

(a) If $L(r)$ belongs to $C_{\text{tract}}$, $\text{Trail}(r)$ is in PTIME.

(b) Otherwise, $\text{Trail}(r)$ is NP-complete.

Proof. Part (a) follows directly from Lemma 7 and the upper bound of Bagan et al. [7, Theorem 2].
It remains to show (b). The upper bound again follows from Lemma 7. The hardness is similar to [7, Lemma 2], as we show next.

Let $L(r) \notin C_{\text{tract.}}$. We exhibit a reduction from the TwoDisjointPaths problem, that is: Given a graph $G = (V, E)$ and nodes $s_1, t_1, s_2, t_2$. Is there a pair of node-disjoint simple paths in $G$, one from $s_1$ to $t_2$ and the other from $s_2$ to $t_2$? This problem is NP-complete [26] and can be transformed into the corresponding trail problem using [21] Lemma 1. Let $N = (Q, \Sigma, \Delta, Q_I, Q_F)$ be an NFA with $Q_I = s_N$, and $L(N) = L(r)$. For this proof we need a different definition of $C_{\text{tract.}}$. A language $L(r) \notin C_{\text{tract.}}$ if and only if there exists a witness for hardness $(q, w_m, w_t, w_1, w_2)$ where $q \in Q, w_t \in \Sigma^*$, $w_1 \in \text{Loop}(w_1)$, and $w_2, w_m \in \Sigma^+$ satisfying $w_m w_2 w_1 \subseteq L_q$ and $(w_1 + w_2)^* w_t \cap L_q = \emptyset$. This definition is equivalent to Definition 9 , see [7, Definition 1, Theorem 4].

Since $L(r) \notin C_{\text{tract.}}$, there exists a witness for hardness $(q, w_1, w_m, w_1, w_2)$ [7, Lemma 1]. Let $w_t$ be a word such that $Q^*(w_t) = q$. By definition we have $w_t(w_1 + w_2)^* w_t \cap L = \emptyset$ and $w_t w_1^* w_m w_2^* w_t \subseteq L(r)$. We build from $G$ a generalized graph $G'$ whose edges are labeled by non-empty words. The generalized graph $G'$ can easily be turned into a graph by adding intermediate nodes, replacing an edge labeled by word $w$ by a path whose edges form the word $w$. The graph $G'$ is constructed as follows. The nodes of $G'$ are the same as the nodes of $G$. For each edge $(v_1, v_2)$ in $G$, we add two edges $(v_1, w_1, v_2)$ and $(v_1, v_2, w_2)$. Moreover, we add two new nodes $x, y$ and three edges $(x, w_t, x_1), (y_1, w_m, x_2)$, and $(y_2, w_r, y)$. By construction, for every trail $p$ from $x$ to $y$ in $G'$ that contains the edge $(y_1, w_m, x_2)$, we can obtain a similar path that matches a word in $w_t w_1^* w_m w_2^* w_t$ by switching $w_1$ and $w_2$ edges, keeping the same nodes. Every trail $p$ from $x$ to $y$ in $G$ that does not contain the edge $(y_1, w_m, x_2)$ matches a word in $w_t(w_1 + w_2)^* w_t$. By definition of $q \in Q, w_t \in \Sigma^*$, $w_m, w_1, w_2 \in \Sigma^+$, no path of the form $w_t(w_1 + w_2)^* w_t$ matches $r$, whereas every path matching $w_t w_1^* w_m w_2^* w_t$ automatically matches $r$. Thus, Trail($r$) returns “yes” for $(G', x, y)$ iff there is a trail from $x$ to $y$ in $G'$ that contains the edge $(y_1, w_m, x_2)$ that is, iff Two-Edge-Disjoint-Path returns “yes” for $(G, x_1, y_1, x_2, y_2)$.

**C Proofs for Section 5**

**Proofs for Section 5.1**

We present how Theorem 14 can be proved, following the explanation we received from Holger Dell [19]. (To the best of our knowledge, the result and proof should be attributed to the authors of [25].) We first need some terminology. The following is Definition 3.1 from [19]. We rephrase it from matroids to graphs to simplify presentation.

**Definition 38 (k-representative family [25]).** Given a graph $G = (V, E)$ and a family $S$ of subsets of $V$, and $k \in \mathbb{N}$, we say that a subfamily $\bar{S} \subseteq S$ is $k$-representative for $S$ if the following holds: for every set $Y \subseteq V$ of size at most $k$, if there is a set $X \in S$ disjoint from $Y$ with $|X \cup Y| \leq 2k$, then there is a set $\bar{X} \in \bar{S}$ disjoint from $Y$ with $|\bar{X} \cup Y| \leq 2k$. We abbreviate this by $\bar{S} \subseteq_{\text{rep}} S$.

In the following we define $P_{sv}^k = \{ X \mid X \subseteq V \text{ such that } s, v \in X, |X| = k, \text{ and there is a path from } s \text{ to } v \text{ in } G \text{ of length } k - 1 \text{ containing exactly the nodes in } X \}$

The following Lemma shows that, in order to find a solution for $\text{SimPath}_{>k}$, it suffices to consider paths where the first $k + 1$ nodes belong to a set in $\bar{P}_{sv}^{k+1} \subseteq_{\text{rep}} P_{sv}^{k+1}$. This lemma
and its proof are analogous to Lemma 5.1 in [25]. We state it here because it is bordered for the correctness of Algorithm 2 and we need to adapt this proof to show the correctness of Algorithms 3 and 4 which build upon Algorithm 2.

**Lemma 39.** Assume that $\hat{P}_{sv}^{k+1} \subseteq_{rep} P_{sv}^{k+1}$. Then a graph $G = (V, E)$ has a simple $s$-$t$-path of length at least $k$ if and only if there exists a node $v \in V$ and $X \in P_{sv}^{k+1} \subseteq_{rep} P_{sv}^{k+1}$ such that $G$ has a simple $s$-$t$-path of length at least $k$ with the first $k+1$ nodes belonging to $X$.

**Proof.** The only-if direction is straightforward: if $G$ has a simple path $p$ whose first $k+1$ nodes belong to $X$, then its length is at least $k$. For the other direction, let $p = (v_0, v_1) \cdots (v_{r-1}, v_r)$ be a shortest $s$-$t$-path of length at least $k$, i.e., such that $r \geq k$.

If $|p| = r \leq 2k+1$, we define $P = (v_0, v_1) \cdots (v_{k-1}, v_k)$ and $Q = (v_{k+1}, v_{k+2}) \cdots (v_r, v_r)$.

Because $|V(Q)| \leq k+1$, $V(P) \in P_{sv}^{k+1}$ and $V(P) \cap V(Q) = \emptyset$, the definition of $P_{sv}^{k+1} \subseteq_{rep} P_{sv}^{k+1}$ guarantees the existence of an $s$-$v_k$-path $P'$ with $V(P') \in P_{sv}^{k+1}$ and $V(P') \cap V(Q) = \emptyset$.

By replacing $P$ with $P'$ in $p$, we obtain a simple $s$-$t$-path of length $|p|$. Otherwise, we have that $|p| = r > 2k+1$. Then we define $P = (v_0, v_1) \cdots (v_{k-1}, v_k)$ and $R = (v_{k+1}, v_{k+2}) \cdots (v_r, v_r)$, so we have $p = P \cdot (v_k, v_{k+1}) \cdots (v_r-1, v_r)$. Since $|V(Q)| = k+1$, $V(P) \in P_{sv}^{k+1}$ and $V(P) \cap V(Q) = \emptyset$, the definition of $P_{sv}^{k+1} \subseteq_{rep} P_{sv}^{k+1}$ guarantees the existence of an $s$-$v_k$-path $P' = (v_0, v_1) \cdots (v_{k-1}, v_k)$ with $V(P') \in P_{sv}^{k+1}$ and $V(P') \cap V(Q) = \emptyset$. If $P'$ is disjoint from $R$, the path $p' = (v_0, v_1) \cdots (v_{k-1}, v_k)(v_k, v_{k+1}) \cdots (v_r, v_r)$ is a simple path of length $r$, so we are done.

We show that $P'$ must be disjoint from $R$. Towards a contradiction, assume that there is an $i \in \{1, \ldots, k-1\}$ such that $v'_i = v_i \in R$. We choose $i$ minimal and build a new simple path $p'' = (v_0, v'_1) \cdots (v'_i, v'_{j+1}) \cdots (v_r, v_r)$ with $|p''| \geq k$, because it contains $Q$. But $V(p'')$ does not contain $v_k$, so $p''$ is shorter than $p$, which contradicts that $p$ was a shortest $s$-$t$-path of length at least $k$. So $P'$ must be disjoint from $R$.

We still need to show that $P_{sv}^{k+1} \subseteq_{rep} P_{sv}^{k+1}$. This is done by Lemma 5.2 in [25], which also restricts the size of $P_{sv}^{k}$ and gives an upper bound for its computation time. Since their Lemma 5.2 in [25] is more general than we need, we give more concrete bounds here that come from the proof of Theorem 5.3 in [25].

**Lemma 40.** A collection of families $P_{sv}^{k} \subseteq_{rep} P_{sv}^{k}, v \in V \setminus \{s\}$ of size at most $\left(\frac{2^{k}}{k}\right) \cdot 2^{o(2k)}$ each can be found in time

$$O\left(8^{k+o(k)}m \log n\right),$$

where $n = |V|$ and $m = |E|$.

From Lemmas 39 and 40, we can infer that Algorithm 2 correctly solves SimPath$_{\geq k}$ in FPT.

**Theorem 14.** (Similar to Theorem 5.3 in [25]) SimPath$_{\geq k}$ is in FPT.

**Proof.** The problem can be solved using Algorithm 2. Its correctness follows directly from Lemma 39. Using Lemma 40, we now show that the algorithm is indeed a FPT algorithm.

Let $n = |V|$ and $m = |E|$. We obtain from Lemma 40 that line 2 of the Algorithm 2 takes $O\left(8^{k+o(k)}m \log n\right)$ time for each $v \in V$. Since we need to consider at most $n \cdot \left(\frac{2^{(k+1)}}{k+1}\right)$.
Algorithm 2 FLPS Algorithm

Input: Graph $G = (V, E)$, nodes $s, t$ in $G$, parameter $k$
Output: Decide if there exists a simple path from $s$ to $t$ with length at least $k$

1: for every $v ∈ V$ do
2: Compute $\mathcal{P}^{k+1}_{sv} ≜ k+1 \mathcal{P}^{k+1}_{sv}$
3: for every $X ∈ \mathcal{P}^{k+1}_{sv}$ do
4: $V′ ← (V \setminus X) ∪ \{v\}$
5: $E′ ← E \cap (V′ × V′)$
6: if there exists a path from $v$ to $t$ in $(V′, E′)$ then
7: return YES
8: end if
9: end for
10: end for
11: return NO

$2^{o(n(2k+1))}$ sets $X$ in line 3 the number of such sets we need to consider throughout the entire algorithm is at most $O(n^{4k+o(k)})$. Finally, line 6 can be checked by a reachability test (say, depth-first search) in time $O(m + n)$, so the overall running time is bounded by

$$O \left( 8^{k+o(k)} mn \log n + 4^{k+o(k)} \cdot (n^2 + mn) \right),$$

which is clearly in FPT for the parameter $k$.

Proofs for Section 5.2

Theorem 16. TwoColorNodeDisjointPaths$_k$ is $W[1]$-hard.

Proof. We now prove that the reduction is correct, that is, $G$ has a $k$-clique iff there are simple paths $p_1$ from $c_1$ to $c_{k+1}$ and $p_2$ from $r_1$ to $r_{k+1}$ such that $p_1$ and $p_2$ are node-disjoint, $p_1$ is colored $a$, and $p_2$ is colored $b$. We need the following Lemma. Let $G'_a$ denote the subgraph obtained from $G'$ by removing the $b$-edges and the nodes $r_1, \ldots, r_{k+1}$ (which have no adjacent $a$-edges). Then $G'_a$ has the following properties.

Lemma 41. $G'_a$ has the following properties:

(a) Each path in $G'_a$ has length exactly $k'$ if and only if it is from $c_1$ to $c_{k+1}$.
(b) Each path in $G'_a$ of length $k'$ visits all control nodes, i.e., it contains all $c_i$ and $c_{i+1}$, with $i ∈ \{1, \ldots, k+1\}$ and $1 ≤ i_1 < i_2 ≤ k$.
(c) Each path in $G'_a$ of length $k'$ has at least one edge $u_i \sim v_k$ in every row of $G'_a$.

We prove the lemma after the present proof.

Let us first assume that the undirected graph $G$ has a $k$-clique with nodes $\{n_1, \ldots, n_k\}$. Then an $a$-path can go from $c_1$ to $c_{k+1}$ using only the gadgets $G_{i,n_i}$ with $i = 1, \ldots, k$. The reason is that, since $(n_1, n_2) ∈ E$, the edges $G_{i_1,n_{i_2}}[v_{i_1}] \xrightarrow{a} G_{i_1,n_{i_1}}[u_{i_2}+1]$ exist for all $i_1 ≤ i_2$. Due to Lemma 41(b), this path has exactly $k'$ edges. The $b$-path, on the other hand, can go from $r_1$ to $r_{k+1}$ and skip exactly $G_{i,n_i}$ for all $i = 1, \ldots, k$ (using the diagonal edges in Figure 2). Since it skips these $G_{i,n_i}$, it is node-disjoint from the $a$-path and therefore we have a solution for TwoColorDisjointPaths$_k$.

\footnote{We only need part (a) in this proof. Parts (b) and (c) are used to prove Theorem 18.}
For the other direction let us assume that there exist a simple $a$-path $p_a$ from $c_1$ to $c_{k+1}$ and a simple $b$-path $p_b$ from $r_1$ to $r_{k+1}$ in $G'$ such that $p_a$ and $p_b$ are node-disjoint and $p_a$ has length $k'$. We show that $G$ has a $k$-clique. Since every $b$-path from $r_1$ to $r_{k+1}$ goes through each row, that is, from $r_i$ to $r_{i+1}$ for all $i = 1, \ldots, k$, this is also the case for $p_b$. By construction $p_b$ must also skip exactly one gadget in each row, using the diagonal edges in Figure 2. Furthermore, for each gadget $G_{i,j}$ that $p_b$ visits, it must be the case that it either visits all nodes $u_{11}, \ldots, u_{k+1}$ or all nodes $v_{11}, \ldots, v_{k+1}$. (This is immediate from Figure 1 showing all internal edges of a gadget.) Therefore, since $p_a$ and $p_b$ are node-disjoint, the $p_a$ cannot visit any gadget $G_{i,j}$ already visited by $p_b$. Therefore, $p_a$, which goes from $c_1$ to $c_{k+1}$ by Lemma 41, can only do so through the $k$ skipped gadgets, call them $G_{i,n_i}$ for $i = 1, \ldots, k$. Recall that the edges between the gadgets $G_{i_2,n_{i_2}}$ and $G_{i_1,n_{i_1}}$ only exist if $(n_{i_1}, n_{i_2}) \in E$. As these edges are necessary for the existence of the $a$-path from $c_1$ to $c_{k+1}$, all $n_i$ must be pairwise adjacent in $G$. That is, they form a clique of size $k$ in $G$. 

\textbf{Proof of Lemma 41} First observe that $G'_a$ contains a fixed part that only depends on $n$ and $k$, plus a set of edges that represent edges in $G$, i.e., edges that are present in $G'$ if and only if there exists a corresponding edge in $G$. Therefore, every possible graph $G'$ that the reduction produces is a subgraph of the case where $G$ is a complete graph (i.e., if $G$ has $n$ nodes, it is the $n$-clique). Let $C'$ denote the graph $G'$ in the case where $G$ is the $n$-clique. We prove the following points, which imply the Lemma:

1. The subgraph $C'_a$ of $C'$ consisting of the $a$-colored edges is a DAG.
2. Each path $C'_a$ from $c_1$ to $c_{k+1}$ has length exactly $k'$.
3. Each path in $C'_a$ has length exactly $k'$ if and only if it is from $c_1$ to $c_{k+1}$.
4. Each path in $C'_a$ of length $k'$ visits all control nodes, i.e., it contains all $c_i$ and $c_{1_2}$, with $i \in \{1, \ldots, k+1\}$ and $1 \leq i_1 < i_2 \leq k$.
5. Each path in $C'_a$ of length $k'$ has at least one edge $u_i \xrightarrow{a} v_i$ in every row of $C'_a$.

We first prove part (1). We first show that, if $C'_a$ has a cycle, then this cycle must contain a control node. Indeed, within the same row, the graph $C'_a$ only has the edges from $u_i$ to $v_i$ in all the gadgets. So, if there cannot be a cycle that only contains nodes from a single row. Therefore, the cycle must contain a path from some node in a row $i_1$ to a node in row $i_2$, for $i_1 < i_2$. Since every path in $C'_a$ from row $i_1$ to $i_2$ with $i_1 < i_2$ contains, by construction, at least one control node, we have that every cycle in $C'_a$ must contain a control node.

It therefore remains to show that $C'_a$ contains no cycle that uses a control node. To this end, observe that the relation $\prec$ where $n_1 \prec n_2$ iff $n_1 \neq n_2$ and $n_2$ is reachable from $n_1$ is a strict total order

$$c_1, c_{12}, c_{13}, \ldots, c_{1k}, c_2, c_{23}, \ldots, c_{k-2k}, c_{k-1k}, c_k, c_{k+1}$$

on the control nodes. That is, the order is such that control nodes are reachable in $C'_a$ from all control nodes to their left and none to their right.

We now prove part (2). First we prove that, between two consecutive control nodes in $C'_a$, each path has a fixed length that depends only on the kind of control nodes. Then, since $C'_a$ is a DAG by part (1), we can simply concatenate paths to obtain the length of paths from $c_1$ to $c_{k+1}$, showing (2). In this proof, when we consider a path that visits nodes in row $i$ in $C'_a$, then by construction of $C'$, the length of this path is independent of the gadget $G_{i,j}$ that the path visits. That is, the path’s length is the same for every $j = 1, \ldots, n$. To simplify notation, we therefore omit the $j$ in $G_{i,j}[u]$ and write $G_i[u]$ instead.

We first consider the length of paths between consecutive control nodes in the ordering \footnote{\text{\footnotesize{}}}. Therefore, fix two such consecutive control nodes $n_1$ and $n_2$. We make a case distinction:
We adapt the reduction from Theorem 17. The only change we make is that we also have (5).

Since \( \prec \) is a strict total order, this means that each path from \( c_1 \) to \( c_{k+1} \) in \( C_a' \) has the same length. We show that this length is exactly \( 5(k(k-1)/2) \cdot 3k = k' \). The paths \( c_i \) to \( c_{i+1} (i = 1, \ldots, k-1) \) and \( c_k \) to \( c_{k+1} \) sum up to length \( 3k \). For a fixed \( i \) we have \( 5 \cdot (k-i-1) \) paths from \( c_{i+1} \) to \( c_k \), which sum up to length \( 5(k(k-1)/2) - 5k + 5 \) for \( i = 1, \ldots, k-2 \). Finally, we need to consider the paths from \( c_{k} \) to \( c_{i+1} \), which, for \( i = 1, \ldots, k-1 \), sum up to length \( 5k - 5 \). This shows (2).

Since \( C_a' \) is a DAG, every node in \( C_a' \) is reachable from \( c_1 \), since \( c_{k+1} \) does not have outgoing edges in \( C_a' \), and since each path of length \( k' \) starting from \( c_1 \) ends in \( c_{k+1} \), we also have (3). Since \( \prec \) is a strict total order on the control nodes, we also have (4).

Due to (3) and (4) each path of length \( k' \) in \( C_a' \) contains \( c_i \) for \( i = 1, \ldots, k+1 \). Since each path from \( c_1 \) to the next control node contains \( (G_{i,j}[u_{i+1}], G_{i,j}[u_{i+1}]) \), for \( j \in \{1, \ldots, n\} \) we also have (5).

\[ \textbf{Theorem 18.} \ TwoNodeDisjointPaths_{k} \text{ is W[1]-hard.} \]

\[ \textbf{Proof.} \ We \ adapt \ the \ reduction \ from \ Theorem 17. \ The \ only \ change \ we \ make \ is \ that \ we \ replace \ each \ \textit{b}\text{-edge} \ by \ a \ directed \ path \ of \ \textit{k'} \ edges \ (introducing \ \textit{k'} - 1 \ new \ nodes \ for \ each \ such \ edge). \]

\[ \cdots \] Call the resulting graph \( G'' \). We make the following observation:

\[ \textbf{Observation 42.} \ \text{In} \ G'', \ \text{we \ have \ that} \]

(a) every path from \( c_1 \) to \( c_{k+1} \) has length \( \geq k' \) and 
(b) every path from \( c_1 \) to \( c_{k+1} \) has length \( \geq k' \) if and only if it only uses \( a\)-edges.

We prove the observation using Lemma 11(a). For part (a) we have two cases. If a path from \( c_1 \) to \( c_{k+1} \) uses \( a\)-edges only, the result is immediate from Lemma 11(a). If it uses at least one \( b\)-edge, then it uses at least \( k' \) \( b\)-edges by construction.

For part (b), if a path from \( c_1 \) to \( c_{k+1} \) has length \( \geq k' \), it uses at least one \( a\)-edge since \( c_{k+1} \) only has incoming \( a\)-edges. If it would use at least one \( b\)-edge, it uses at least \( k' \) \( b\)-edges by construction, which contradicts that the length is \( k' \). The converse direction is immediate from Lemma 11(a). This concludes the proof of Observation 12.

We show that \( G'' \) and \( k' \) are in TwoColorNodeDisjointPaths_{k}, if and only if \( G'' \) and \( k' \) are in TwoNodeDisjointPaths_{k}. That is, \( G'' \) has a simple \( a\)-path \( p_a \) from \( s_1 \) to \( c_{k+1} \) (of length \( k' \)) and simple \( b\)-path \( p_b \) from \( r_1 \) to \( r_{k+1} \) such that \( p_b \) has length \( k' \) and is node-disjoint from \( p_a \) if and only if \( G'' \) has simple paths \( p_1 \) from \( c_1 \) to \( c_{k+1} \) and \( p_2 \) from \( r_1 \) to \( r_{k+1} \), where \( p_1 \) has length \( k' \) and \( p_1 \) and \( p_2 \) are node-disjoint.

If \( G'' \) and \( k' \) are in TwoColorNodeDisjointPaths_{k}, then we can use the corresponding paths in \( G'' \) (where we follow \( b\)-paths in \( G'' \) instead of \( b\)-edges in \( G'' \)). Conversely, if \( G'' \) and \( k' \) are in TwoNodeDisjointPaths_{k}, it follows from Observation 12 that \( p_1 \) can only use \( a\)-edges. We now show that the path \( p_2 \) from \( r_1 \) to \( r_{k+1} \) can only use \( b\)-edges, and we show that it cannot use \( a\)-edges. There are 3 types of \( a\)-edges: (i) the ones from and to control-nodes,
(ii) “upward” edges that connect row \( j \) to row \( i \) with \( j > i \), and (iii) edges from \( u_t \) to \( v_t \) in one gadget.

Notice that, by construction, \( p_2 \) must visit nodes in row 1 and later also nodes in row \( k \).
To do so, \( p_2 \) cannot use edges from or to control nodes (type (i)), since, due to Lemma 41, \( p_1 \) already visits all of them. So \( p_2 \) cannot go from row \( i \) to a row \( j \) with \( i < j \) via a-edges. This means that, if \( i < j \), then \( p_2 \) can only go from row \( i \) to row \( j \) by going through \( r_{i+1} \) (and through nodes in row \( i+1 \)), since every remaining path from row \( i \) to a larger row goes through \( r_{i+1} \). So, in order to go from row 1 to row \( k \), path \( p_2 \) needs to visit all nodes \( r_1, \ldots, r_k \), in that order. This means that it is also impossible for \( p_2 \) to use edges of type (ii). Indeed, if \( p_2 \) were to use an edge from row \( j \) to row \( i \) with \( j > i \), then it would need to visit \( r_{i+1} \) a second time to arrive back in row \( j \). Finally, if \( p_2 \) used an a-edge of type (iii) in row \( i \), then, by construction, it would have to visit every gadget in this row. But since \( p_1 \) already uses at least one edge in each row, see Lemma 41, this means that \( p_2 \) cannot be node-disjoint with \( p_1 \). This shows that \( G' \) and \( k' \) are in \( \text{TwoColorNodeDisjointPaths}_k \).

\[ \text{Theorem 19.} \quad \text{TwoNodeDisjointPaths}_{\leq k} \text{ is W[1]-hard.} \]

\[ \text{TwoNodeDisjointPaths}_{\leq k} \text{ is NP-complete for every constant } k \in \mathbb{N}. \]

\[ \text{TwoColorNodeDisjointPaths}_k, \text{ TwoNodeDisjointPaths}_k, \text{ and TwoNodeDisjointPaths}_{\leq k} \text{ are in W[P].} \]

\[ \text{Proof.} \quad \text{The theorem follows immediately from Lemmas 43 and 45.} \]

\[ \text{Lemma 43.} \quad \text{TwoNodeDisjointPaths}_{\leq k} \text{ is W[1]-hard.} \]

\[ \text{Proof.} \quad \text{We start from the same graph as in the proof of Theorem 18. Then, from Observation 12 we know that there exist no path from } c_1 \text{ to } c_{k+1} \text{ that has length smaller than } k. \text{ So the answer to our problem on this instance is the same as for TwoNodeDisjointPaths}_k, \text{ which completes the reduction.} \]

For completeness, we observe that TwoNodeDisjointPaths with \( k = 0 \) is simply the TwoNodeDisjointPaths problem.

\[ \text{Lemma 44 (26).} \quad \text{TwoNodeDisjointPaths}_{\leq k} \text{ is NP-complete for every constant } k \in \mathbb{N}. \]

\[ \text{Lemma 45.} \quad \text{TwoColorNodeDisjointPaths}_k, \text{ TwoNodeDisjointPaths}_k, \text{ and TwoNodeDisjointPaths}_{\leq k} \text{ are in W[P].} \]

\[ \text{Proof.} \quad \text{We show membership in W[P] by using Definition 3.1 in Flum and Grohe. They say that W[P] is the class of parameterized problems that can be decided by a nondeterministic Turing machine (NTM) in time } f(k) \cdot |x|^{O(1)} \text{ and such that it makes at most } O(h(k) \cdot \log n) \text{ nondeterministic choices in the computation of any input } (x, k). \]

The problem TwoNodeDisjointPaths can be decided by such an nondeterministic Turing machine as follows. Given a graph \( G = (V, E) \) and a parameter \( k \). The NTM first uses \((k - 1) \cdot \log n\) steps to guess \( k - 1 \) nodes \( v_1, \ldots, v_{k-1} \) in the right order. Then we can verify in \( O(k \log n) \) steps that these nodes form a simple path \( p_1 = (s_1, v_1, v_2, \ldots, v_{k-1}, t_1) \) from \( s_1 \) to \( t_1 \). After this, the NTM tests deterministically that \( t_2 \) is reachable from \( s_2 \) while avoiding the nodes \( s_1, v_2, \ldots, v_{k-1}, t_1 \). This can be done in time polynomial in \(|G|\).

For TwoColorNodeDisjointPaths and TwoNodeDisjointPaths the proof is analogous.
C.1 Proofs for Section 5.3

► Theorem 20. TwoEdgeDisjointTrails\(k\) is \(W[1]\)-hard.

Proof. Just like the reduction in Theorem 16, this proof is inspired by the adaption of Grohe and Grüber [28, Lemma 16] of the main reduction of Slivkins [42]. In fact, since the reduction in Slivkins also considers trails, we use exactly the same gadgets here.

We give a reduction from \(k\)-Clique. Let \((G, k)\) be an instance of \(k\)-Clique. We construct the graph \(G’\) from Theorem 16, and make the following changes to obtain our final graph \(H\):

- In each gadget \(G_{i,j}\), we split each \(u_i\) in two nodes, that is \(u_i^{in}\) and \(u_i^{out}\). We call the two nodes that resulted from the same node a node pair. We redirect all incoming edges from \(u_i\) to \(u_i^{in}\) and let all outgoing edges from \(u_i\) begin in \(u_i^{out}\). Finally, we add an edge \(u_i^{in}\) to \(u_i^{out}\). We depict this in Figure 7. We make exactly the same change to all \(v_\ell, c_1, c_{i_1, i_2}\), and \(r_1\).

- We replace each \(b\)-edge by a \(b\)-path of length \(k_{\text{new}} = k(k - 1)/2 \cdot 5 + 3k + k + 1 + k(k - 1)/2 + k \cdot 2(k + 1) = 5k^2 + 3k + 1\). Notice that this \(k_{\text{new}}\) is longer than \(k’ = k(k - 1)/2 \cdot 5 + 3k\) in Theorem 15 because we split some nodes and added new edges between them. To be precise, the length of the \(a\)-path from \(c_i^{in}\) to \(c_{k+1}^{out}\) became longer because we split all \(k + 1 + k(k - 1)/2\) control nodes and it has to pass in total \(k \cdot 2(k + 1)\) new edges between \(u_i^{in}\) and \(u_i^{out}\) and between \(u_i^{in}\) and \(v_\ell^{out}\).

The correctness proof now follows the lines of the proof of Theorem 18. We show that there exist paths \(p_a\) and \(p_b\) in \(G’\) such that \(p_a\) is an \(a\)-path of length \(k’\) from \(c_1\) to \(c_{k+1}\) and \(p_b\) is a \(b\)-path from \(r_1\) to \(r_{k+1}\) if and only if there exist two edge-disjoint \(p_1\) and \(p_2\), where \(p_1\) has length exactly \(k_{\text{new}}\) and is from \(c_i^{in}\) to \(c_{k+1}^{out}\) and \(p_2\) is from \(r_1^{out}\) to \(r_{k+1}^{out}\).

We will now show that \(p_1\) corresponds to \(p_a\) and \(p_2\) corresponds to \(p_b\). This then proves the lemma. If \(p_a\) and \(p_b\) exist, then we can use their nodes (or node pairs) to build edge disjoint paths \(p_1\) and \(p_2\). (We use the same nodes or node pairs thereof and do not change the order.)

For the other direction, let us assume that \(p_1\) and \(p_2\) exist in \(H\). We first show that \(p_1\) corresponds to an \(a\)-path in \(G’\). We have constructed our graph \(H\) such that each path from \(c_i^{in}\) to \(c_{k+1}^{out}\) has length at least \(k_{\text{new}}\) and length exactly \(k_{\text{new}}\) if and only if it chooses a path corresponding to an \(a\)-path in \(G’\), see also Observation 42. We now show that \(p_2\) cannot correspond to any path in \(G’\) that uses \(a\)-edges. Recall that there are 3 types of \(a\)-edges in \(G’\): (i) the ones from and to control nodes, (ii) “upward” edges that connect row \(j\) to row \(i\) with \(j > i\), and (iii) edges from \(u_i\) to \(v_j\) in one gadget. Since \(p_1\) is a path from \(c_i^{in}\) to \(c_{k+1}^{out}\) of length exactly \(k_{\text{new}}\), the corresponding \(a\)-path from \(c_1\) to \(c_{k+1}\) in \(G’\) uses all control nodes, see Lemma 14. Therefore, \(p_1\) must do the same. Since we did split all control nodes, \(p_1\) especially contains the edge between each node pair of control nodes. This implies that \(p_2\) cannot use the edge between any node pair of control nodes and its corresponding path in \(G’\) cannot contain any \(a\)-edge from or to an control node, that is type (i). So \(p_2\) cannot go from row \(i\) to a row \(j\) with \(i < j\) via control nodes. This means that, if \(i < j\), then \(p_2\) can only go from row \(i\) to row \(j\) by going through \(r_i^{in}\) and \(r_j^{out}\) since every remaining path from row \(i\) to a larger row goes through \(r_i^{in}\) and \(r_j^{out}\). So, in order to go from row \(1\) to row \(k\), path \(p_2\) needs to visit all nodes \(r_1^{in}, r_2^{out}, \ldots, r_k^{in}, r_k^{out}\), in that order. This means that it is also impossible for \(p_2\) to use “upward” edges. (Otherwise there would be an \(i\), such that \(p_2\) would use the edge between \(r_i^{in}\) and \(r_{i+1}^{out}\) twice.) So the corresponding path in \(G’\) must not use \(a\)-edges of type (ii). Finally, if \(p_2\) used an edge between \(u_\ell^{in}\) and \(v_\ell^{out}\) in any gadget in row \(i\), then it would have to visit every gadget in this row by construction, i.e., for all \(j \in \{1, \ldots, n\}\) and all \(\ell \in \{1, \ldots, k+1\}\): \((G_{i,j}[u_\ell^{in}], a, G_{i,j}[v_\ell^{out}]) \in p_2 \lor (G_{i,j}[v_\ell^{in}], a, G_{i,j}[r_\ell^{in}]) \in p_2\). But we know...
we have an FPT-reduction.

\[ \text{Algorithm 3} \]

**Input:** Graph \( G = (V, E) \), nodes \( s, t \) in \( G \), parameter \( k \), RE \( a^k w?a^* \)

**Output:** Decide if there exists a simple path from \( s \) to \( t \) matching the given RE

1. if FLPS((\( V, E \cap (V \times (a) \times V) \)), \( s, t, k \)) then return YES \( \triangleright \) Call Algorithm [2]
2. end if
3. \( S = \{p_c | p_c = (u_0, u_1) \cdots (u_{c-1}, u_c) \text{ is a simple path that matches } w\} \)
4. for each \( p_c \in S \) do
5. \( V'_c \leftarrow V \setminus V(p_c[1, c - 1]) \) \( \triangleright \) Delete all but the first and last node of \( p_c \)
6. \( E'_c = E \cap (V'_c \times (a) \times V'_c) \) \( \triangleright \) Consider only \( a\)-edges
7. Compute \( P_{s_{t_c}}^{k+1} \leq \leq_{\text{REP}} P_{s_{t_0}}^{k+1} \text{ in } (V'_c, E'_c) \)
8. for all sets \( X \in P_{s_{t_0}}^{k+1} \) do
9. \( V' \leftarrow (V'_c \setminus X) \)
10. \( E' \leftarrow (E'_c \cap (V' \times V')) \)
11. if there exists a path from \( u_c \) to \( t \) in \( (V', E') \) then
12. return YES
13. end if
14. end for
15. end for
16. return NO

from Lemma [41](c) that the path corresponding to \( p_1 \) in \( G' \) uses at least one edge in each row. This means that in each row there exists a gadget \( G_{i,j} \) and an \( \ell \) such that \( p_1 \) uses the edges \( (G_{i,j}[u_{i,0}^{\ell}], a, G_{i,j}[v_{i,0}^{\ell}]), (G_{i,j}[u_{i,\ell}^{\ell}], a, G_{i,j}[v_{i,\ell}^{\ell}]), \) and \( (G_{i,j}[v_{i,\ell}^{\ell}], a, G_{i,j}[u_{i,0}^{\ell}]) \). So \( p_2 \) cannot be edge-disjoint with \( p_1 \) if it uses such an edge. This implies that the path corresponding to \( p_2 \) in \( G' \) cannot use edges of type (iii), so we finally know that the path corresponding to \( p_2 \) in \( G' \) only contains \( b\)-edges. So \( G' \) and \( k' \) are indeed in TwoColorNodeDisjointPaths \( k \) and we have an FPT-reduction.

\[ \triangleright \text{Theorem 23. For every constant } c \text{ and word } w \text{ with } |w| = c, \text{ the problem } \text{SimPath} \ (a^k w?a^*) \text{ with parameter } k \text{ is in FPT.} \]

**Proof.** We give an FPT algorithm that solves this problem, see Algorithm [3]. We first prove that Algorithm [3] is correct. If there exists a simple path matching \( a^k a^* \), we find it using Algorithm [2] in line [1] and return ‘yes’.

So let us assume that there exists no such path. We then use brute force to find all simple paths \( p_c = (u_0, u_1) \cdots (u_{c-1}, u_c) \) matching \( w \) and store them in a set \( S \). For each
path $p_c \in S$, we compute the graph $(V'_c, E'_c)$ that does not contain the inner nodes of $p_c$ and only contains $a$-edges. Then we compute a set $\hat{P}^k_{s_{u_0}} \subseteq_{\text{rep}} P^k_{s_{u_0}}$ in $(V'_c, E'_c)$. We will argue using Lemma 59 that it suffices to consider paths in which the first $k + 1$ nodes belong to a set $X \in \hat{P}^k_{s_{u_0}}$. To this end, assume that there is a simple path

$$p = (v_0, v_1) \cdots (v_{k-1}, u_0) \cdot p_c \cdot (u_c, v_{k+1}) \cdots (v_r, v_r)$$

matching $a^kwa^*$. (We wrote some of the concatenation operators $\cdot$ explicitly to improve readability.)

We consider two cases. If $|p| \leq 2k + c$, we define $P = (v_0, v_1) \cdots (v_{k-1}, u_0)$ and $Q = (u_c, v_{k+1}) \cdots (v_r, v_r)$. Notice that $|V(Q)| \leq k + 1$, $V(P) \cap V(Q) = \emptyset$, and $P \in \hat{P}^k_{s_{u_0}}$. Therefore, since $\hat{P}^k_{s_{u_0}} \subseteq_{\text{rep}} P^k_{s_{u_0}}$, there exists at least one path $P'$ with $V(P') \in \hat{P}^k_{s_{u_0}}$ and $V(P') \cap V(Q) = \emptyset$. Since $(V'_c, E'_c)$ does not contain any nodes of $p_c[1, c - 1]$ by definition, we also know that $V(P') \cap V(p_c) = \{u_0\}$. (Notice that $u_c$ cannot be in the intersection, because it is in $V(Q)$.) This means that $P' \cdot p_c \cdot Q$ is indeed a simple path that matches $a^kwa^*$.

Otherwise we have that $|p| > 2k + c$, in which case we define $P = (v_0, v_1) \cdots (v_{k-1}, u_0)$ and $Q = (v_{r-k}, v_{r-k+1}) \cdots (v_r, v_r)$ and $R = (u_c, v_{k+1}) \cdots (v_{r-k-2}, v_{r-k-1})$. So we have that

$$p = P \cdot p_c \cdot R \cdot (v_{r-k-1}, v_r \cdot Q).$$

We also know that $P \in \hat{P}^k_{s_{u_0}}$ and $|V(Q)| = k + 1$. Therefore, by definition of $\hat{P}^k_{s_{u_0}} \subseteq_{\text{rep}} P^k_{s_{u_0}}$, there must be a path $P' = (v_0, v'_1) \cdots (v'_{k-1}, u_0)$ with $V(P') \in \hat{P}^k_{s_{u_0}}$ such that $V(P') \cap V(Q) = \emptyset$. The path $P'$ is also does not contain any of the inner nodes of $p_c$, because $G'_c$ does not contain nodes of $V(p_c[1, c - 1])$.

There are again two possibilities: $P'$ intersects with $R$ or not. In the first case, there exists a node $v'_j \in V(P')$ with minimal $j$ such that $v'_j = v_j \in V(R)$. Then we replace the path $p$ by a new simple path $p' = (v_0, v'_1) \cdots (v'_{j-1}, v'_{j+1}) \cdots (v_{r-1}, v_r)$. But then $p'$ matches $a^k a^*$, because it does not contain $p_c$, whereas it still contains $Q$. This contradicts that no such path $p'$ exists (we would have found this path with Algorithm 2 in line 1).

Therefore, $P'$ does not intersect with $R$ and we have found our path.

Finally we note that this algorithm is indeed an FPT algorithm. Line 1 works in FPT due to Theorem 11. The set $S$ in line 4 can contain at most $O(n^2)$ different paths, so enumerating all of them is in PTIME. The rest is analogous to Algorithm 2 and therefore in FPT.

## E Proofs for Section 6.1

### E.1 Dichotomy for Node-Disjoint Paths

**Lemma 46.** Let $\mathcal{R}$ be the class of 0-bordered STEs. Then SimPath($\mathcal{R}$) is in FPT with parameter $k_f$.

**Proof.** We prove the lemma by case distinction on the form of

$$r = B_{\text{pre}} A^* B_{\text{suff}} \in \mathcal{R}.$$ 

There are two cases for $A$: either $A = \emptyset$ or $A \neq \emptyset$. In the former case, $L(r)$ is finite. Then we can use an algorithm obtained by Bagan et al. [2, Theorem 6, Corollary 2], to solve it in FPT. (Bagan et al. use the size of the NFA as parameter for their algorithm.)
Algorithm 4

Input: Graph $G = (V,E)$, nodes $s,t$ in $G$, and $0$-bordered $r = A_1 \cdots A_k A^* A'_{k_2} \cdots A'_1$ (assuming $A,A_1,\ldots,A_{k_1},A'_1,\ldots,A'_{k_2} \neq \emptyset$)

Output: Does there exist a simple path from $s$ to $t$ matching $r$?

1: $k \leftarrow k_1 + k_2$
2: if there exists a simple path from $s$ to $t$ matching $A_1 \cdots A_k A^* A'_{k_2} \cdots A'_1$ then
3: return YES
4: end if
5: for all $v \in V$ do
6: Compute $\hat{P}^k_{sv,1} \subseteq_{\text{rep}} P^k_{sv,1}$ in $G$ with $r_1 = A_1 \cdots A_k A^* A'_{k_2}$. 
7: for all sets $X \in \hat{P}^k_{sv,1}$ do
8: $V' \leftarrow (V \setminus X) \cup \{v\}$
9: $E' \leftarrow E \cap (V' \times \Sigma \times V')$
10: for all $u \in V$ do
11: Compute $\hat{P}^k_{tu,2} \subseteq_{\text{rep}} P^k_{tu,2}$ in $(V',E')$ with $r_2 = A'_{k_2} \cdots A'_1$. 
12: for all sets $X' \in \hat{P}^k_{tu,2}$ do
13: $V'' \leftarrow (V' \setminus X') \cup \{u\}$
14: $E'' \leftarrow E' \cap (V'' \times A \times V'')$ 
15: if there exists a path from $v$ to $u$ in $(V'',E'')$ then
16: return YES
17: end if
18: end for
19: end for
20: end for
21: end for
22: return NO

Otherwise, we know that $A \neq \emptyset$. Furthermore, we can assume w.l.o.g. that all $A_1,\ldots, A_{k_1},A'_1,\ldots,A'_{k_2}$ are non-empty. Indeed, if this would not be the case, then the expression can be simplified ($L(r) = \emptyset$ is easy to test and $\emptyset$ can be simplified to $\emptyset$). We now differentiate between the forms of $B_{\text{pre}}$ and $B_{\text{suff}}$. There are two possible forms, that is (1) $B_1 ? \cdots B_{\ell} ?$ with $\ell \geq 0$ or (2) $B_1 \cdots B_{\ell}$ with $\ell \geq 1$. If $B_{\text{pre}}$ and $B_{\text{suff}}$ are of form (1), the language is downward closed. Therefore we can evaluate the answer in PTIME, see Proposition 6. If $B_{\text{pre}}$ and $B_{\text{suff}}$ are both of form (2), we will show that we can use Algorithm 4. We show the correctness of this algorithm next and we explain later how to change it if $B_{\text{pre}}$ has form (2) (resp., (1)) and $B_{\text{suff}}$ has form (1) (resp., (2)).

We first give the idea of the algorithm. Let $k = k_1 + k_2$ (that is, $k$ is the parameter $k_r$ from Definition 24). First the algorithm tests if there is a simple path that matches $A_1 \cdots A_k A^* A'_{k_2} \cdots A'_1$. Dealing with this case separately simplifies the cases we need to treat in lines 5-21. In lines 5-21 the algorithm essentially performs two nestings of Algorithm 2. Since neither the language of $B_{\text{pre}}$ nor the language of $B_{\text{suff}}$ is downward closed, we need to execute Algorithm 2 once to find the prefix and once to find the suffix. Furthermore, we use a variant of $\hat{P}^k_{sv}$, namely $\hat{P}^k_{sv,1,2}$, that allows us to make sure that the prefix (suffix, resp.) of the path matches $B_{\text{pre}}$ ($B_{\text{suff}}$, resp.). More precisely,

$P^k_{sv,1} := \{X \mid X \subseteq V \text{ such that } s,v \in X, |X| = k, \text{ and there is a path from } s \to v \text{ in } G \}$
We now show the correctness. If we have a simple $s$-$t$-path matching $r$ of length up to $2k$, it will be found in line 2. So it remains to test whether there exists a simple $s$-$t$-path matching $r$ of length at least $2k+1$. In each such path the first $k+1$ nodes must match $r_1 = A_1 \cdots A_{k_1} A_{k_2}$, while the rest of the path matches $A^* A_{k_2}' \cdots A_1'$. We now prove analogously to Lemma 39 that it suffices to consider paths in which the first $k+1$ nodes belong to $X \in \hat{P}_{sv,r_1}$ of size $r$. In fact, the proof of Lemma 39 needs no adaption. It still works since $r$ is 0-bordered. To be more precise: if we start with a shortest simple path $p = P \cdot (v_k, v_{k+1}) \cdot R \cdot (v_{|p|-k-1}, v_{|p|-k}) \cdot Q$ of length at least $2k+2$ (or $p = P \cdot (v_k, v_{k+1}) \cdot Q$ with $R = \emptyset$ if $|p| = 2k+1$) that matches $r$, we also find a path $P'$ with $V(P') \in \hat{P}_{sv,r_1}$ that is disjoint from $V(Q)$. If $P'$ and $R$ intersect, we obtain a shorter simple path that still matches $r$ because $r$ is 0-bordered and the resulting path is still longer than $k$ (it contains $Q$). Notice that this is the reason why we need to consider paths of length $k$ in line 6 of the algorithm, instead of length $k_1$.

We now obtained that if there is a simple $s$-$t$-path matching $r$ of length at least $2k+1$, then there exists a $v \in V$ and a set $X$ in $\hat{P}_{sv,r_1}$, such that its first $k+1$ nodes belong to $X$. Then we need to find the rest of the path, that is, a simple $v$-$t$-path matching $A^* A_{k_2}' \cdots A_1'$ in the graph without $X \setminus \{v\}$.

Due to line 2 we know that the $v$-$t$-path must have length at least $k_2+1$. Symmetrically to before we can show that, if such a path exists, then there exists a $u$ such that its last $k_2+1$ nodes belong to a set $X'$ in $\hat{P}_{ut,r_2}$ of size $r$ that intersects $X$. In fact, the proof of Lemma 39 needs no adaption. It remains to test if there is a path from $v$ to $u$ that matches $A^*$ which is done in line 15. This concludes the correctness proof.

We next show that the algorithm is indeed in FPT. Bagan et al. [7, Theorem 6] showed that the test in line 2 is in FPT, to be precise $O(2^{O(2k)} \cdot |N| \cdot |G| \cdot \log |G|)$ where $|N|$ is the size of the NFA used. Here we use an NFA of size $2k$. It remains to show that the $\hat{P}_{sv,r_1}$ in line 6 (and $\hat{P}_{ut,r_2}$ in line 11, resp.) can be computed in FPT time and its size depends only on $k$. This is guaranteed by Lemma 49 (Notice that we can efficiently compute $\hat{P}_{sv,r_1}$ for all $u$ by using the construction in Lemma 49 to compute $\hat{P}_{sv,r_1}$ on the graph with reversed edges.)

So Algorithm 4 is indeed an FPT algorithm if $B_{pre}$ and $B_{suff}$ are of form (2). We now explain how it can be changed to work if $B_{pre}$ is of form (2) and $B_{suff}$ of form (1), that is: Assume we have $r = A_1 \cdots A_{k_1} A^* A_{k_2}' \cdots A_1'$. Then we make the following changes:

- in line 2 the path should match $A_1 \cdots A_{k_1} A^* A_{k_2}' \cdots A_1'$.
- instead of line 10 to 19 with a test for an $v$-$t$-path matching $r' = A^* A_{k_2}' \cdots A_1'$. Notice that this implies that there exists a simple $v$-$t$-path matching $r'$ since $L(r')$ is downward closed.

The case $r = A_1 ? \cdots A_{k_1} ? A^* A_{k_2}' \cdots A_1'$ is symmetric. It has indeed the same answer as $A_1' \cdots A_{k_1}' A^* A_{k_2} ? \cdots A_1'$ on the graph with reversed edges.

So, to complete the proof of Lemma 46 it remains to prove Lemma 49. We need to introduce some terminology and notation. For $p \in \mathbb{N}$, a $p$-family $\mathcal{A}$ is a set containing sets of size $p$. By $|A|$ we denote the number of sets in $\mathcal{A}$.

We also restate Lemma 3.3 and Corollary 4.16 from [25] since we need them in the proof. Lemma 47 states that the relation “is a $q$-representative set for” is transitive.

**Lemma 47** (Lemma 3.3 in [25] for directed graphs). Given a graph $G = (V,E)$ and $\mathcal{S}$ a family of subsets of $V$. If $S' \subseteq_{\mathbb{R}} S$ and $\tilde{S} \subseteq_{\mathbb{R}} S'$, then $\tilde{S} \subseteq_{\mathbb{R}} S$. 

\[ \]
Corollary 48 (Corollary 4.16 in [25], without weight function). There is an algorithm that, given a \( p \)-family \( \mathcal{A} \) of sets over a universe \( U \) of size \( n \) and an integer \( q \), computes in time \( O\left( |\mathcal{A}| \cdot \left( \frac{p + q}{q} \right)^q \cdot 2^{o(p+q)} \cdot \log n \right) \) a subfamily \( \tilde{\mathcal{A}} \subseteq_{\text{rep}} \mathcal{A} \) such that \( |\tilde{\mathcal{A}}| \leq \left( \frac{p+q}{p} \right) \cdot 2^{o(p+q)} \).

We now adapt Lemma 5.2 in Fomin et al. [25] to show a time and space bound for the sets \( \tilde{P}_{sv,r}^{k+1} \) and \( \tilde{P}_{sv,r}^{k+1} \) in the proof of Lemma 46 (where \( r_1 = A_1 \cdots A_k \tilde{A}_{k+2} \) and \( r_2 = A_k^1 \cdots A_1^1 \)). We think that this lemma might be of interest for others that try to find more languages that can be evaluated in FPT.

Recall that \( P_{sv,r}^k := \{X | X \subseteq V \text{ such that } s, v \in X, |X| = k, \text{ and there is a path from } s \text{ to } v \text{ in } G \text{ of length } k - 1, \text{ matching } r, \text{ and containing exactly the nodes in } X \}. \)

Lemma 49. For each regular expression \( r = A_1 \cdots A_{k-1} \), a collection of families \( \tilde{P}_{sv,r}^k \subseteq_{\text{rep}} P_{sv,r}^k \), \( v \in V \setminus \{s\} \) of size at most \( \left( \frac{2k}{k} \right)^k \cdot 2^{o(2k)} \) each can be found in time \( O\left( \log n \right) \), where \( n = |V| \) and \( m = |E| \).

Proof. We describe a dynamic programming-based algorithm. Let \( V = \{s, v_1, \ldots, v_{n-1}\} \) and \( D \) be a \((k-1) \times (n-1)\) matrix where the rows are indexed with integers in \( 2, \ldots, k \) and the columns are indexed with nodes in \( \{v_1, \ldots, v_{n-1}\} \). The entry \( D[i, v] \) will store the family \( \tilde{P}_{sv,r}^{i-1} \subseteq_{\text{rep}} P_{sv,r}^{i-1} \), where \( r' \) denotes the prefix \( A_1 \cdots A_i \) of \( r \). We fill the entries in the matrix \( D \) in the increasing order of rows. For \( i = 2 \), we set \( D[2, v] = \{\{s, v\}\} \) if \( G \) has an edge of the form \( s \to v \) with \( a \neq A_1 \) (otherwise \( D[2, v] = \{\} \)). Assume that we have filled all the entries until row \( i \). For two families of sets \( \mathcal{A} \) and \( \mathcal{B} \), we define

\[
\mathcal{A} \cdot \mathcal{B} = \{X \cup Y | X \in \mathcal{A}, Y \in \mathcal{B}, \text{ and } X \cap Y = \emptyset\}.
\]

We denote by \( u \xrightarrow{A_i} v \) that there exists an edge \( u \leadsto v \) with \( a \in A_i \). Let

\[
N_{sv,r}^{i+1} = \bigcup_{u \xrightarrow{A_i} v} \tilde{P}_{sv,r}^{i-1} \cdot \{v\}.
\]

We next adapt Claim 5.1 in [25] such that it takes \( r \) into account, that is:

Claim 50. \( N_{sv,r}^{i+1} \subseteq_{\text{rep}} \left( P_{sv,r}^{i+1} \right)^{2k-(i+1)} \)

Proof. Let \( S \in P_{sv,r}^{i+1} \) and \( Y \) be a set of size \( 2k - (i + 1) \) such that \( S \cap Y = \emptyset \). We will show that there exists a set \( S' \in N_{sv,r}^{i+1} \) such that \( S' \cap Y = \emptyset \). This will imply the desired result.

Since \( S \in P_{sv,r}^{i+1} \), there exists a path \( P = (s, u_1) \cdots (u_{i-1}, v) \) in \( G \) such that \( S = V(P) \) and \( u_{i-1} \xrightarrow{A_i} v \). The existence of path \( P[0, i - 1] \), the subpath of \( P \) between \( s \) and \( u_{i-1} \), implies that \( X^* = S \setminus \{v\} \in P_{sv,r}^{i-1} \). Take \( Y^* = Y \cup \{v\} \). Observe that \( X^* \cap Y^* = \emptyset \) and \( |Y^*| = 2k - i \). Since \( P_{sv,r}^{i-1} \subseteq_{\text{rep}} P_{sv,r}^{i-1} \), there exists a set \( \hat{X}^* \in P_{sv,r}^{i-1} \) such that \( \hat{X}^* \cap Y^* = \emptyset \). However, since \( u_{i-1} \xrightarrow{A_i} v \) and \( \hat{X}^* \cap \{v\} = \emptyset \), we have \( \hat{X}^* \cdot \{v\} = \hat{X}^* \cup \{v\} \) and \( \hat{X}^* \cup \{v\} \in N_{sv,r}^{i+1} \). Taking \( S' = \hat{X}^* \cup \{v\} \) suffices for our purpose. This completes the proof of the claim.

We fill the entry for \( D[i+1, v] \) as follows. Observe that

\[
N_{sv,r}^{i+1} = \bigcup_{u \xrightarrow{A_i} v} D[i, u] \cdot \{v\}.
\]
We already have computed the family corresponding to $D[i, u]$ for all $u$. By Corollary \footnote{48} we have $|P_{sv, r}^{\text{pre}}| \leq \left(2k^{2}\right)2^{o(2k)}$ and thus also $|N_{sv, r}^{i+1}| \leq d^-(v)\left(\frac{2k}{2k - i - 1}\right)^{2k-i-1}2^{o(2k)}$. Furthermore, we can compute $N_{sv, r}^{i+1}$ in time $O\left(d^-(v)\left(\frac{2k}{2k - i - 1}\right)^{2k-i-1}2^{o(2k)}\cdot \log n\right)$.

By Claim 5.1, we know that $N_{sv, r}^{i+1} \subseteq P_{sv, r}^{i+1}$. Thus, Lemma \footnote{47} implies that $N_{sv, r}^{i+1} = P_{sv, r}^{i+1}$. We assign this family to $D[i + 1, v]$. This completes the description and the correctness of the algorithm.

Notice that, if we keep the elements in the sets in the order in which they were built using the $\bullet$ operation, then they directly correspond to paths. As such, every ordered set in our family represents a path in the graph.

Since our only change was that we test $u \rightarrow v$ instead of $u \rightarrow v$, the time bound of $O\left(8^{k+o(k)}m\log n\right)$ \footnote{25} Lemma 5.2 carries over. The size bound is still guaranteed by Corollary \footnote{48}.

\begin{theorem}
Let $\mathcal{R}$ be a class of STEs that can be sampled. Then,
\begin{enumerate}[(a)]
  \item if $\mathcal{R}$ is cuttable, then $\text{SimPath}(\mathcal{R})$ is in FPT with parameter $k_r$ and
  \item otherwise, $\text{SimPath}(\mathcal{R})$ is $W[1]$-hard with parameter $k_r$.
\end{enumerate}
\end{theorem}

\begin{proof}
We first prove part (a). To this end, let $c$ be a constant such that every $r \in \mathcal{R}$ is at most $c$-bordered. Let $r \in \mathcal{R}$. Then we know that the maximum of its left cut border $c_1$ and its right cut borders $c_2$ is at most $c$. So we can enumerate, for all $u, v \in V$, all simple $s$-$u$-paths $p_1$ matching $A_1 \cdots A_c$, and all simple $v$-$t$-paths $p_2$ matching $A'_1 \cdots A'_c$ in time $O(n^c)$. For all such node-disjoint paths $p_1$ and $p_2$, delete in $G$ all nodes in $(V(p_1) \setminus \{u\}) \cup (V(p_2) \setminus \{v\})$. In the remaining graph, we search a $u$-$v$-path that matches

\begin{align*}
  r' &= A_{c_1 + 1} \cdots A_k A'_{c_2} \cdots A'_{c_2 + 1}, \\
  r' &= A_1 \cdots A_k A'_{s_2} \cdots A'_{c_2 + 1}, \\
  r' &= A_1 \cdots A_k A'_{t_2} \cdots A'_{c_2 + 1}, \\
  r' &= A_1 \cdots A_k A'_{s_2} \cdots A'_{c_2 + 1}, \text{ or}
\end{align*}

This are the only possibilities. Remember that the left cut border of $B_{\text{pre}} = A_1 \cdots A_k$, is 0. Since $r'$ is 0-bordered, Lemma \footnote{46} allows us to solve $\text{SimPath}(r')$ in time $f(k_r) \cdot |G|^d$ for a constant $d$ and a computable function $f$. Since $k_r + c_1 + c_2 = k_r$, this shows that $\text{SimPath}(\mathcal{R})$ is in FPT with parameter $k_r$.

We now prove part (b). Let $\mathcal{R}$ be an arbitrary but fixed class of STEs that can be sampled. We show that $\text{SimPath}(\mathcal{R})$ is $W[1]$-hard by giving an FPT reduction from $k$-Clique, which is known to be $W[1]$-hard (with parameter $k$). Let $(G, k)$ be an input to $k$-Clique. We will construct a graph $(H, s, t)$ and an expression $r \in \mathcal{R}$ such that $(G, k) \in k$-Clique if and only if $H$ has a simple $s$-$t$-path that matches $r$. Let $k' = k(k - 1)/2 \cdot 5 + 3k$. Since $\mathcal{R}$ can be sampled, a $k'$-bordered expression $r \in \mathcal{R}$ with $k' \geq k + 1$ can be computed within time $f(k)$, for some computable function $f$. Therefore, we also know $k_r \leq f(k)$, else $r$ could not be computed in this time. Since $r$ is $k'$-bordered, we have that its left cut border is $k'$ or its right cut border is $k''$ (or both).
Here we only consider the case that the left cut border is \( k'' \), i.e., \( A \not\subseteq A_{k''} \), the other is symmetric. For \( r \) to have a left cut border of \( k'' \), it must be of the form

\[
r = A_1 \cdots A_{k'} \cdots A_{k_1} A_{k_2}' \cdots A_1' \quad \text{or} \quad r = A_1 \cdots A_{k'} \cdots A_{k_1} A_{k_2}' \cdots A_1'
\]

with \( A, A_1, \ldots, A_{k_1}, A_{k_2}', \ldots, A_1' \not= \emptyset \). (Remember that we assume \( L(r) \not= \emptyset \) and the left cut border of \( B_{pre} = A_1' \cdots A_{k_2}' \) is 0. Furthermore, \( A \not\subseteq A_{k''} \) implies \( A \not= \emptyset \).) The following construction holds for both forms that \( r \) can have.

We now construct \((H,s,t)\). The main idea is to have at most one edge with a label in \( A_{k''} \) that is reachable from \( s \) by a path of length \( k'' - 1 \). Then each path matching \( r \) must route through it and we can do a similar proof as for \( \text{SimPath}(a^{k-1}ba^*) \) in Corollary 22.

More formally, fix an \( x \in (A \setminus A_{k''}) \). Fix three words \( w_1, w_2, \) and \( w_3 \) such that

1. \( w_1 \in L(A_1 \cdots A_{k'}) \),
2. \( w_2 \in L(A_{k'+1} \cdots A_{k'} \cdots A_{k_1}) \), and
3. \( w_3 \in L(A_{k_2}' \cdots A_1') \)^{14}

Notice that such words indeed exist. For the construction of \( H \), we start with the graph \( G' \) used in the proof of Theorem 16 and make the following changes:

- We replace each \( b \)-edge in \( G' \) with an \( x \)-path of length \( k'' \) (using \( k'' - 1 \) new nodes for each replacement).
- We change the labels of the \( a \)-edges in \( G' \) such that each \( c_1\)-\( c_{k+1} \)-path is labeled \( w_1 \).
- Notice that the label for each such edge is well-defined. Indeed, by Lemma 11(5) we have that each \( a \)-path from \( c_1 \) to \( c_{k+1} \) has length exactly \( k' \). If there would be an edge \( e \) on an \( a \)-labeled \( c_1\)-\( c_{k+1} \)-path that is reachable from \( c_1 \) through \( n_1 \) edges and also through \( n_2 \) edges, with \( n_1 \not= n_2 \), then, since \( c_{k+1} \) is reachable from \( e \), it means that there would be paths of different lengths from \( c_1 \) to \( r_1 \). We relabel all other edges with \( x \).
- We add a path labeled \( w_2 \) from \( c_{k+1} \) to \( r_1 \). We refer to this path as the \( w_2 \)-labeled path in the remainder of the proof.
- We add a path labeled \( w_3 \) from \( r_{k+1} \) to a new node \( t_{new} \), to which we will refer as the \( w_3 \)-labeled path in the remainder of the proof.

The resulting graph \((H,c_1,t_{new})\) together with the expression \( r \in R \) serves as input for \( \text{SimPath}(R) \). This concludes the reduction.

We show that the reduction is correct. This can be proved analogously to the proof of Theorem 18 that is, we show that \( G' \) and \( k' \) are in \( \text{TwoColorNodeDisjointPaths}_k \) if and only if \((H,c_1,t_{new})\) and \( r \) are in \( \text{SimPath}(R) \).

If \( G' \) and \( k' \) are in \( \text{TwoColorNodeDisjointPaths}_k \) with solution \( p_a \) and \( p_b \), then there exists a (unique) simple path from \( c_1 \) to \( t_{new} \) in \( H \) that contains the nodes \( V(p_a) \cup V(p_b) \) and matches \( r \).

Conversely, if \((H,c_1,t_{new})\) and \( r \) are in \( \text{SimPath}(R) \), there exists a simple path \( p \) from \( c_1 \) to \( t_{new} \) in \( H \) that matches \( r \). We will prove the following:

(i) The prefix of \( p \) of length \( k' \) corresponds to a simple path from \( c_1 \) to \( c_{k+1} \) in \( G' \) from the proof of Theorem 16 in Appendix C^{15}(That is, \( p[0,k'] \) is a path from \( c_1 \) to \( c_{k+1} \)-path in \( G' \)).

(ii) The prefix of \( p \) of length \( k_1 \) ends in \( r_1 \). Its prefix is labeled \( w_1 \) and its suffix is the \( w_2 \)-labeled path.

(iii) We show that \( \text{lab}(p) = w_1w_2w'w_3 \) with \( w' \in L(A^*) \).

---

^{14} We use \( w_3 \in L(A_{k_2}' \cdots A_1') \) in case that \( r \) ends with \( A_{k_2}' \cdots A_1' \) but also if it ends with \( A_{k'_2}' \cdots A_1' \).

^{15} \( G'_n \) is the graph obtained from \( G' \) by deleting all \( b \)-edges and nodes that have no adjacent \( a \)-edges.
We prove (i). By definition of $r$, the edge $p[k'' - 1, k'']$ is labeled by some symbol in $A_{k''}$. Therefore, this symbol cannot be $x$. By construction of $H$, this edge is either an edge that was labeled $a$ in $G'$, an edge on the $w_2$-labeled path, or an edge on the $w_3$-labeled path (since all other edges are labeled $x$).

Since the $w_3$-labeled path is not reachable with a path of length smaller than $k''$ and the $w_2$-labeled path starts in $c_{k+1}$ and is therefore only reachable with a path of length at least $k'$, see Observation 42, the first $k' + 1$ nodes must form an $a$-path. This implies that $p[0, k']$ is entirely in $G''_a$. From Lemma 24, we know that each path in $G''_a$ of length $k'$ goes from $c_1$ to $c_{k+1}$ which implies (i). Since all nodes (except $r_1$) that belong to the $w_2$-labeled path of length $k_1 - k'$ have only one outgoing edge, we have that $p[0, k_1]$ ends in $r_1$ and must match $w_1w_2$. This shows (ii).

Since $p$ matches $r = A_1 \cdots A_{k_1} A^* A_{k_2}' \cdots A_1'$ or $r = A_1 \cdots A_{k_1} A^* A_{k_2}'' \cdots A_1''$, and since each word in $A_1 \cdots A_{k_1}$ has length $k_1$, it follows that $\text{lab}(p) = w_1w_2w'$ with $w' \in L(A^* A_{k_2} \cdots A_1') \cup L(A^* A_{k_2}'' \cdots A_1'')$.

By construction of $H$, the $w_3$-labeled path is the unique path of length $|w_3|$ leading to $t_{new}$. Therefore, each $c_1$-$t_{new}$-path in $H$ must end with the $w_3$-labeled path. Since $w_3 \in L(A_{k_2}' \cdots A_1')$ and $|w_3|$ is the length of every word in $L(A_{k_2}' \cdots A_1')$, we have that $\text{lab}(p) = w_1w_2w'$ where $w \in L(A^*)$. So we have (iii). Let $p'$ be the part of $p$ labeled $w'$. We now show that $p'$ can only consist of edges labeled $x$. Since $p$ is a simple path, it must be node-disjoint with its prefix $p[0, k']$. We showed in (i) that $p[0, k']$ corresponds to a $c_1$-$c_{k+1}$-path in $G''_a$, so we know from Lemma 21 that it uses all control-nodes and at least one edge in each row. Therefore, it follows as in the proof of Theorem 18 that $p'$ cannot use edges that correspond to ones in $G''_a$. Therefore, $p'$ only consists of edges labeled $x$. This shows that $G'$ and $k'$ are in TwoColorNodeDisjointPaths, because $p[0, k']$ corresponds to a path $p_0$ and $p'$ to $p_0$, which are solutions to TwoColorNodeDisjointPaths.

Finally, we note that the construction can indeed be done in FPT since the expression $r \in \mathcal{R}$ can be determined in time $f(k)$ for a computable function $f$, the graph from the proof of Theorem 19 was constructed in FPT, and all changes we made to the graph are in time $h(k) \cdot |G''|^c$, for a constant $c$ and a computable function $h$, which is FPT. Indeed, we only relabeled all edges, replaced each edge at most once with $k''$ new edges and added other paths of length at most $|r| \leq f(k)$. Since $|r| \leq f(k)$, we also have $k_r \leq f(k)$, so we have indeed an FPT reduction.

E.2 Dichotomy for Edge-Disjoint Paths

Next we will prove the dichotomy on STEs for trails.

**Theorem 27.** Let $\mathcal{R}$ be a class of STEs that can be conflict-sampled. Then, (a) if $\mathcal{R}$ is almost conflict free, then $\text{Trail}(\mathcal{R})$ is in FPT with parameter $k_r$ and (b) otherwise, $\text{Trail}(\mathcal{R})$ is W[1]-hard with parameter $k_r$.

**Proof.** We first prove part (a). Since $\mathcal{R}$ is almost conflict free, there exists a constant $c$ such that each $r \in \mathcal{R}$ has at most $c$ conflict labels. Let $r \in \mathcal{R}$ an STE with left cut border $c_1$ and right cut border $c_2$. We will show how to decide whether there exists a path from $s$ to $t$ in $G$ matching $r$ in FPT.

We use the reduction from Lemma 7 to convert this problem into at most $n$ instances of the corresponding problem SimPath $(r)$. We now show how to decide SimPath $(r)$ on an instance $(H_i, s', t')$ in time $f(k_r) \cdot |H_i|^{O(1)}$. We observe that each node in $H_i$ (except $s'$ and $t'$) corresponds to exactly one edge in $G$. ▷
If \( r \) is \( c' \)-bordered for a \( c' \leq c \), then we can immediately use the methods of Theorem 26(a). If this is not the case, we know that \( A, A_1, \ldots, A_{k_1}, A'_1, \ldots, A_{k_2} \neq \emptyset \) and it remains to consider \( r = A_1 \cdots A_{k_1} A^* A'_{k_2} \cdots A'_1 \), \( r = A_1 \cdots A_{k_1} A^* A'_{k_2} \cdots A'_1 \), \( r = A_1 \cdots A_{k_1} A^* A'_{k_2} \cdots A'_1 \). We will show how to solve this problem for

\[
r = A_1 \cdots A_{k_1} A^* A'_{k_2} \cdots A'_1,
\]

by adapting Algorithm 4. The other cases then follow as in Theorem 25(a).

We will first explain what we change in Algorithm 4 and show its correctness afterwards.

1. We also change \( r_1 \) (line 6) and \( r_2 \) (line 13) by relabeling the conflict labels. Assume \( A_\ell \) is an conflict label. Then we define \( A_\ell \) in \( r_1 \) as \( A_\ell \setminus A \cup \{ a' \mid a \in A \cap A_\ell \} \). We proceed analogously for conflict labels \( A'_j \) in \( r_2 \). easier understandable if we do this generally for \( A_\ell \) with \( j \leq c_1 \) or \( A'_j \) with \( j \leq c_2 \).

2. We enumerate all subsets of up to \( c \) nodes \( v_{u_1, a, u_2} \) with \( a \in A \) in \( H_i \). For each possible subset \( S \), we generate a graph \( H_S \) by changing the nodes \( v_{u_1, a, u_2} \in S \) to \( v_{u_1, a', u_2} \in S \) and relabel the outgoing edges with \( \sigma' \) (where \( \sigma' \) is a new symbol, i.e., we add at most \(|A|\) different symbols in total).

This completes the changes we do. It is obvious that these changes are possible in FPT, since enumerating all subsets of size at most \( c \) is in \( O(|H_i|^c) \). Since the original algorithm was in FPT, the adapted one is as well. We now show the correctness. We first show that it indeed suffices to consider subsets of up to \( c \) nodes. Let \( p_{\text{pref}} \) be an arbitrary path matching \( A_1 \cdots A_{c_1} \) and \( p_{\text{aff}} \) be an arbitrary path matching \( A'_{c_2} \cdots A'_1 \) that is edge-disjoint from \( p_{\text{pref}} \). Since \( r \) only has \( c \) conflict labels, we know that there are at most \( c \) edges that can be shared between \( p_{\text{pref}} \) and \( p_{\text{aff}} \) with an arbitrary path matching \( A^* \). Since we constructed \( H_i \) such that every edge in \( G \) corresponds to exactly one node in \( H_i \), see Lemma 7 this means that simple paths matching \( A_1 \cdots A_{c_1} \) and \( A'_{c_2} \cdots A'_1 \) can share at most \( c \) nodes with an arbitrary path matching \( A^* \). So, in order to assure node disjointness between those paths, it indeed suffices to consider subsets of up to \( c \) nodes and force the paths matching \( A_1 \cdots A_{c_1} \) and \( A'_{c_2} \cdots A'_1 \) to only use those while the \( A \)-path may only choose other nodes. We enforce this by changing the labels.

If there exists a simple path from \( s' \) to \( t' \) in \( H_i \) of length at most \( 2k \), where \( k = k_1 + k_2 \), it will be found in line 2. We now show that, if there exists a path from \( s' \) to \( t' \) in \( H_i \) of length at least \( 2k + 1 \), then it will be found in the adapted algorithm between line 3 and 21. Like in Lemma 16 we have that it suffices to consider paths in which the \( k + 1 \) first nodes belong to \( X \in P_{\text{st}}^{k+1} \). The proof is again analogous to Lemma 19. The paths \( P' \) and \( R \) it cannot intersect in the first \( c_1 + 1 \) nodes of \( P' \) since those nodes only have outgoing edges that have labels not in \( A \). Since \( R \) matches \( A^* \), it cannot use them. And if \( P' \) and \( R \) intersect after the first \( c_1 + 1 \), the obtained simple path still matches (if we replace the new symbols with their usual ones) \( r \), since we have that \( A \subseteq A_{c_1+1}, \ldots, A_{k_1} \), due to definition of \( c_1 \), and the path is long enough because it still contains \( Q \). From line 2 we know that the remaining path from \( r \) to \( t \) must have length at least \( k_2 + 1 \). So we can prove again analogous to Lemma 19 that it suffices to consider paths in which the last \( k_2 + 1 \) nodes belong to \( X' \in P_{\text{st}}^{k+1} \), for some \( u \). So the adapted algorithm is indeed correct.

It remains to consider case (b). Notice that \( \mathcal{R} \) is not cuttable, as this would imply that it is almost conflict free. The proof follows the lines of Theorem 20 part (b), i.e., we give an reduction from \( k \)-Clique. Given an instance \((G,k)\) from \( k \)-Clique, we find an \( r \in \mathcal{R} \) that has at least \( 2k_{\text{new}} \) conflict labels where \( k_{\text{new}} = 5k^2 + 3k + 1 \) (this comes from Theorem 20). Notice that we need so many conflict labels to ensure that they are on the right position). Let us assume that we have at least \( k''_{\text{new}} \) conflict labels in \( A_1 \cdots A_{k''} \), where \( k'' = c_1 \) is the left cut.
border of \( r \). The case that we have at least \( k_{\text{new}} \) conflict labels in \( A'_c \cdots A'_1 \) is symmetric. Notice that we use \( k'' \) instead of \( c_1 \) to avoid confusion with the node \( c_1 \). Furthermore, \( A_1 \) has the same property as \( A_{k''} \) in Theorem \ref{thm:construction} part (b). Due to definition of cut border, we have that \( A \not\subseteq A_{k''} \).

We will now basically use the graph from Theorem \ref{thm:construction} part (b) except that we label the \( a \)-path from \( c_1 \) to \( r_1 \) differently and split the nodes like in Theorem \ref{thm:construction}. We will now explain the changes in detail.

First we fix an \( x \in (A \setminus A_{k''}) \). Fix three words \( w_1, w_2, \) and \( w_3 \) such that
\[
\begin{align*}
&\text{\( w_1 \in L(A_1 \cdots A_{k''}) \), such that \( w_1 \) contains as many symbols in \( A \) as possible} \\
&\text{\( w_2 \in L(A_{k''+1} \cdots A_{k_1}) \), and} \\
&\text{\( w_3 \in L(A'_{k_2} \cdots A'_1) \).}
\end{align*}
\]
We start with the graph \( G' \) from Theorem \ref{thm:construction}.

In each gadget \( G_{i,j} \), we split each \( u_{\ell} \) in two nodes, that is \( u_{\ell}^{\text{in}} \) and \( u_{\ell}^{\text{out}} \). We call the two nodes that resulted from the same node a \textit{node-pair}. We redirect all incoming edges from \( u_{\ell} \) to \( u_{\ell}^{\text{in}} \), while all outgoing edges begin in \( u_{\ell}^{\text{out}} \). We depict this in Figure \ref{fig:construction}. Finally, we add an edge \( u_{\ell}^{\text{in}} \to u_{\ell}^{\text{out}} \). We make exactly the same change to all \( c_{\ell}, c_{\ell_1}, c_{\ell_2}, \) and \( r_1 \).

We replace each \( b \)-edge by a \( x \)-path of length \( k'' \) and label the edge between \( r_{\ell}^{\text{in}} \) and \( r_{\ell}^{\text{out}} \) with \( x \) for all \( i \).

We will now adapt the graph so that each path from \( c_{\ell}^{\text{in}} \) to \( c_{\ell+1}^{\text{out}} \) has length at least \( k'' \) and exactly \( k'' \) if and only if it uses edges corresponding to an \( a \)-path from \( c_1 \) to \( c_{k+1} \) in \( G' \). Since we have at least \( k_{\text{new}} \) symbols from \( A \) in \( w_1 \) and \( k_{\text{new}} \leq k'' = |w_1| \), we can indeed label each edge with a symbol from \( A \). We choose the first \( k_{\text{new}} \) symbols from \( c_{\ell}^{\text{in}} \) to \( c_{\ell+1}^{\text{out}} \) in the right order. We label \( c_{\ell+1}^{\text{in}} \xrightarrow{A''} c_{\ell+1}^{\text{out}} \). To obtain a path matching \( w_1 \), we then insert paths matching subwords of \( w_1 \) that contain no symbol in \( A \) between two such edges, or containing arbitrary symbols, right before \( c_{\ell+1}^{\text{out}} \). (Here we just need to make sure that all paths have the same length.) Sounds awful, I know, but we somehow need to make sure that all paths have the same length.

We add a path matching \( w_2 \) from \( c_{k+1}^{\text{out}} \) to \( r_{1}^{\text{in}} \), which we will call \( w_2 \)-\textit{labeled path}, and a path matching \( w_3 \) from \( r_{k+1}^{\text{out}} \) to a new node \( t_{\text{new}} \), which we will call \( w_3 \)-\textit{labeled path}.

This completes the construction of our graph \( H \). We can now prove the correctness analogously to Theorem \ref{thm:construction} part (b) and Theorem \ref{thm:construction}. For the direction, let \( p_k \) be a simple \( a \)-path from \( c_1 \) to \( c_{k+1} \) and \( p_b \) a simple \( b \)-path from \( r_1 \) to \( r_{k+1} \) in \( G' \), such that \( p_b \) has length \( k' \) and is \textit{node-disjoint from} \( p_b \). By construction, we can use the same nodes (or node-pairs) as \( p_a \) to obtain a trail \( p_1 \) from \( c_{k+1}^{\text{in}} \) to \( c_{k+1}^{\text{out}} \) in \( H \) that matches \( w_1 \). And we can use the same nodes (or node-pairs) as \( p_b \) to obtain a trail \( p_2 \) from \( r_{k+1}^{\text{in}} \) to \( r_{k+1}^{\text{out}} \) in \( H \) that matches \( x' \), i.e., \( A'' \). We can complete it to a trail from \( c_1^{\text{in}} \) to \( t_{\text{new}} \) that matches \( r \) by adding the \( w_2 \)-labeled and \( w_3 \)-labeled path. For the other direction, let \( p \) be a trail from \( c_1^{\text{in}} \) to \( t_{\text{new}} \) in \( H \) that matches \( r \). We will prove the following:

(i) The prefix of \( p \) of length \( k' \) corresponds to a simple path from \( c_1 \) to \( c_{k+1} \) in \( G_a' \) from the proof of Theorem \ref{thm:construction} in Appendix \ref{app:construction} (That is, \( p[0,k'] \) is a path from \( c_1 \) to \( c_{k+1} \)-path in \( G_a' \)).

(ii) The prefix of \( p \) of length \( k_1 \) ends in \( r_1 \). Its prefix is labeled \( w_1 \) and its suffix is the \( w_2 \)-labeled path.

---

\footnote{We use \( w_3 \in L(A'_{k_2} \cdots A'_1) \) in case that \( r \) ends with \( A'_{k_2} \cdots A'_1 \) but also if it ends with \( A'_2 \cdots A'_1 \).}

\footnote{It would suffice to label each edge between each \( u_{\ell}^{\text{in}} \to u_{\ell}^{\text{out}} \) and each \( v_{\ell}^{\text{in}} \to v_{\ell}^{\text{out}} \) with a label in \( A \).}

\footnote{\( G_a' \) is the graph obtained from \( G' \) by deleting all \( b \)-edges and nodes that have no adjacent \( a \)-edges.
(iii) We show that \( \text{lab}(p) = w_1w_2w_3w_4 \) with \( w_4 \in L(A^*) \).

We prove (i). By definition of \( r \), the edge \( p[k'' - 1, k''] \) is labeled by some symbol in \( A_k \).

Therefore, this symbol cannot be \( x \). By construction of \( H \), this edge is either an edge that was labeled \( a \) in \( G' \), an edge on the \( w_2 \)-labeled path, or an edge on the \( w_3 \)-labeled path (since all other edges are labeled \( x \)).

Since the \( w_2 \)-labeled path is not reachable with a path of length at most \( k'' \) and the \( w_2 \)-labeled path starts in \( c_{k+1} \) and is therefore only reachable with a path of length at least \( k'' \) (due to construction), the first \( k'' + 1 \) nodes must form an \( a \)-path. This implies that \( p[0, k''] \) is entirely in \( G' \). From Lemma 11(a), we know that each path in \( G' \) of length \( k'' \) goes from \( c_1 \) to \( c_{k+1} \) which implies (i). Since all nodes that belong to the \( w_2 \)-labeled path of length \( k_1 - k'' \) have only one outgoing edge, we have that \( p[0, k_1] \) ends in \( r_1 \) and must match \( w_1w_2 \). This shows (ii).

Since \( p \) matches \( r = A_1 \cdots A_k A_1 A_2' \cdots A_1' \) (the case \( r = A_1 \cdots A_k A_1 A_2' \cdots A_1' \) is analogous) and each word in \( A_1 \cdots A_k \) has length \( k_1 \), it follows that \( \text{lab}(p) = w_1w_2w_4' \) with \( w' \in L(A_1 A_2' \cdots A_1') \).

By construction of \( H \), the \( w_3 \)-labeled path is the unique path of length \( |w_3| \) leading to \( t_{\text{new}} \). Therefore, each \( c_i^a-t_{\text{new}} \)-path in \( H \) must end with the \( w_3 \)-labeled path. Since \( w_2 = L(A_1 A_2' \cdots A_1') \) and \( |w_3| \) is the length of every word in \( L(A_1 A_2' \cdots A_1') \), we have that \( \text{lab}(p) = w_1w_2w_3w_4' \) where \( w \in L(A^*) \). So we have (iii). Let \( p' \) be the part of \( p \) labeled \( w' \).

We now show that \( p' \) can only consist of edges labeled \( x \).

Since \( p \) is a simple path, it must be node-disjoint with its prefix \( p[0, k'' \). We showed in (i) that \( p[0, k''] \) corresponds to a \( c_1-c_{k+1} \)-path in \( G' \) and (iii) that it uses all control nodes and at least one edge in each row. Therefore, it follows as in the proof of Theorem 20 that \( p' \) cannot use edges that correspond to ones in \( G' \). Therefore, \( p' \) only consists of edges labeled \( x \). This shows that \( G' \) and \( k' \) are in \textbf{TwoColorNodeDisjointPaths}_k, because \( p[0, k'' \) corresponds to a path \( p_a \) and \( p' \) to \( p_b \), which are solutions to \textbf{TwoColorNodeDisjointPaths}_k.

Finally, we note that the construction can indeed be done in FPT since the expression \( r \in R \) can be determined in time \( f(k) \) for a computable function \( f \), the graph from the proof of Theorem 16 was constructed in FPT, and all changes we made to the graph are in time \( h(k) \cdot |G'| \), for a constant \( c \) and a computable function \( h \), which is FPT. Indeed, we only relabeled all edges, replaced each edge at most once with \( c_0 \) new edges, split each node at most once in two new ones, and added other paths of length at most \( |r| \leq f(k) \). Since \( |r| \leq f(k) \), we also have \( k_r \leq f(k) \), so we have indeed an FPT reduction.

**F** Proofs for Section 7

**Theorem 28.** \textbf{EnumSimPaths}$_{\geq k}$ is in FPT delay.

**Proof.** In the proof of Theorem 10 we adapted Yen’s algorithm to work with simple instead of shortest paths. We already showed that the problem \textbf{SimPath}$_{\geq k}$ is in FPT. Furthermore, the FPT algorithm can be adjusted to also return a matching path. This is because, due to definition of \( P_{sv}^{k+1} \), the nodes in \( \hat{P}_{sv}^{k+1} \) form a path from \( s \) to \( v \) of length \( k \). Given those nodes, we can easily build such a path. (In fact, the construction of \( \hat{P}_{sv}^{k+1} \) allows to order the elements in the sets in \( \hat{P}_{sv}^{k+1} \) so, that they directly correspond to such a path, see Lemma 5.2.) If Algorithm 1 returns ‘yes’, then there exists a \( v \) and a \( X \in \hat{P}_{sv}^{k+1} \) such that there exists a path from \( v \) to \( t \) in the graph without \( X \\setminus \{v\} \). As explained before we can construct a path from \( s \) to \( v \) that uses only nodes in \( X \). We concatenate this path with a simple path from \( v \) to \( t \) that does not use nodes in \( X \) except \( v \) to obtain a simple path from
s to t that has length at least k. Therefore, we can use this algorithm as a subroutine of YangSimple to obtain an FPT delay algorithm. For proving the correctness of this approach, we need to note that we can also deal with derivatives of the language, i.e., \( \text{SimPath} \geq j \), with \( i \leq k \), which is needed in line 12 of the YangSimple algorithm. However, in this case, we can simply solve \( \text{SimPath} \geq j \) with \( j = \max\{k-i,0\} \) with the same technique as \( \text{SimPath} \geq k \).

\[\textbf{Theorem 29.} \] For each constant \( c \) and each word \( w \) with length \(|w| = c\), the problem \( \text{EnumSimPaths}(a^k w^i a^* \text{ is in FPT delay.} \)

**Proof.** We adapt the proof of Theorem 28 for this case. Therefore, we first show that the Algorithm 3 can indeed output simple paths. If there exists a path matching \( \text{SimPath} (a^k w^i a^*) \), Algorithm 2 finds it and can output it (see Theorem 28). Otherwise the algorithm only returns ‘yes’ if there exists a path \( p_c \in S \), a set \( X \in \hat{P}_{\text{suff}} \) and a path from \( u_c \) to \( t \) that are all node-disjoint except for \( u_c \) and \( u_c \). As explained before we can built a simple path \( p_1 \) from \( s \) to \( u_0 \) from the nodes in \( X \) and find a simple path \( p_2 \) from \( u_c \) to \( t \) that does not use nodes in \( X \). We then return \( p_1 p_2 \), which is indeed a simple path matching \( a^k w^i a^* \).

For the derivatives, we need to deal with \( \text{SimPath} (a^k w^i a^*) \), \( \text{SimPath} (w^i a^*) \) for suffixes of \( w \), and \( \text{SimPath} (a^*) \). The first, \( \text{SimPath} (a^k w^i a^*) \) with \( i \leq k \) can be solved with the same technique as \( \text{SimPath} (a^k w^i a^*) \). For \( \text{SimPath} (w^i a^*) \), where \( w' \) is a suffix of \( w \), we enumerate all possible paths \( p' \) that match \( w' \), which are again at most \( O(n') \) many since \(|w'| \leq |w| = c\). Then, we search for a simple path \( p_2 \) from the last node of \( p' \) to \( t \) that does not use other nodes from \( p' \). If we found one (which must be the case if we return ‘yes’), we return \( p' p_2 \). We can use the same technique to deal with \( \text{SimPath} (a^*) \), that is, we choose \( w' = e \).

\[\textbf{Lemma 51.} \] Let \( w \in \Sigma^* \) and \( r \) be a \( c \)-bordered STE of size \( n \). Then \( w^{-1} L(r) \) is a union of \( \text{STEs} \{r_1, \ldots, r_m\} \) such that

\[m \leq n \text{ and } \forall r_i \text{ is } c' \text{-bordered for some } c' \leq c.\]

**Proof.** Let \( r = B_1 \cdots B_n \) be a \( c \)-bordered STE such that each \( B_i \) is either of the form \( A, A? \) or \( A^* \) as in Definition 24. Let \( w \in \Sigma^* \) and \( J = \{ j \mid w \in L(B_1 \cdots B_j) \} \). Then \( w^{-1} L(r) = L(\Sigma_{j \in J} B_{j+1} \cdots B_n) \). Since \(|J| \leq n \) and each expression \( B_{j+1} \cdots B_n \) is \( c' \)-bordered for some \( c' \leq c \) by definition, the result follows.

\[\textbf{Theorem 30.} \] Let \( R \) be a cuttable class of \( \text{STEs} \). Then \( \text{EnumSimPaths}(R) \) is in FPT delay.

**Proof.** We adapt the proof of Theorem 28 for this case. So let \( R \) be a cuttable class of \( \text{STEs} \) and let \( r \in R \) with left cut border \( c_1 \) and right cut border \( c_2 \). First we enumerate all possible paths \( p_{c_1} \) that can match \( A_1 \cdots A_{c_1} \) and \( p_{c_2} \) that match \( A'_{c_2} \cdots A'_{c_1} \). (These paths can also be empty if \( c_1 = 0 \) or \( c_2 = 0 \).) We know now that the remaining regular expression, that is \( r' = B_{p_{c_1}} A^* B_{p_{c_2}} \text{ is 0-bordered.} \) So we now search for a path matching \( r' \) from the last node of \( p_1 \) to the first of \( p_2 \) in the graph without the other nodes of \( p_1 \) and \( p_2 \).

Now we do case distinctions depending on its actual form. If \( A = \emptyset \), we can use the algorithm of Bagan et al. [7, Theorem 6]. In the appendix of the arXiv-version they give the corresponding algorithm which is based on color coding and dynamic programming. This algorithm can easily be adapted to generate a witness \( p \) in case the decision algorithm returns ‘yes’. Indeed, while running the dynamic algorithm, we can always store the witnessing information for each newly computed entry, from which the witnessing path can be computed at the end of the algorithm. We then return \( p_{c_1} p p_{c_2} \). Otherwise we know that
Adding each simple path longer than length at least $k$, we show that Algorithm 3 can indeed output simple paths. As explained before, we can obtain a simple path matching $v'$ of length at most $2k$ in line 2 by using the algorithm of Bagan et al. [7, Theorem 6]. So, if Algorithm 4 returned ‘yes’, but we did not find a path in line 2, there exists nodes $u, v \in V$, sets $X \in \bar{P}^{k+1}_{sv,r}$ and $X' \in \bar{P}^{k+1}_{ut,r}$, and a simple path $p$ from $v$ to $u$ that matches $A^*$ and is node disjoint from $X$ and $X'$ except for $v$ and $u$. Due to definition of $P^{k+1}_{sv,r}$, the nodes in $X \in \bar{P}^{k+1}_{sv,r}$ form a path from $s$ to $v$ that matches $r_1$ and has length $k$. Since we built the sets in $\bar{P}^{k+1}_{sv,r}$ analogous to [25, Lemma 5.2], we can easily build such a path. (In fact, the construction of $\bar{P}^{k+1}_{sv,r}$ allows to order the elements in the sets so, that they directly correspond to such a path). So, we can construct a path $p_1$ from $s$ to $v$ that uses only nodes in $X$ and matches $r_1$ and a path $p_2$ from $u$ to $t$ that uses only nodes in $X'$ and matches $r_2$. So Algorithm 4 can indeed output the path $p_1 p_2$. We then obtain our solution for $r$ by adding $p_1$ and $p_2$, if necessary.

Since, we can use an easier variant of Algorithm 4 if $r' = A_1 \cdots A_k ? A^* A'_{k_2} \cdots A'_{k_2} ? \cdots A'_1 ?$ or $r' = A_1 \cdots A_k ? A^* A'_{k_2} \cdots A'_{k_2} ? \cdots A'_1$ in line 2, we can also output paths in this cases.

It remains to show that we can handle all possible derivatives of STEs. According to Lemma 51, we only need to consider $k_1 + k_2 + 1$ STEs that are $c'$-bordered for some $c' \leq c$. Indeed, according to Lemma 51, each possible derivative is a union of at most $k_1 + k_2 + 1$ STEs. Since each such STE is $c'$-bordered for some $c' \leq c$, we can solve SimPath for each of them in FPT. And, since it are at most $k_1 + k_2 + 1$ many, solving it for each of them is still in FPT. We can obviously use the same case distinctions as above and return a path if SimPath answers ‘yes’.

We will now show that we can even output paths in FPT delay with radix order. Therefore, we will use Yen’s algorithm, so we need algorithms that output shortest and lexicographically smallest paths. We will show how to achieve this, also for the derivatives needed in line 12 of Yen’s algorithm.

**Lemma 52.** EnumSimPaths$_{\geq k}$ is in FPT delay with radix order.

**Proof.** We have seen in Theorem 28 that Algorithm 3 can indeed output paths. We now explain how to change this algorithm to output a shortest simple path from $s$ to $t$ that has length at least $k$. Therefore, we first observe that the proof of Lemma 29 works with the shortest simple path longer than $k$. So this lemma also implies that there exists a shortest simple path longer than $k$ such that its first $k + 1$ nodes belong to a $X \in \bar{P}^{k+1}_{sv}$. So we can find a shortest simple path longer than $k$ by running the algorithm for each $v \in V$ and each $X \in \bar{P}^{k+1}_{sv}$ and searching a shortest $v$-$t$-path in line 6. We always store the actual shortest simple path that is still longer than $k$. Therefore, we can use this algorithm as a subroutine of Yen’s algorithm to obtain an FPT delay algorithm that enumerates the paths from shortest to longest. Notice that this algorithm can also deal with derivatives of the language, i.e., SimPath$_{\geq j}$ with $j = \max\{k - i, 0\}$, which is needed in line 12 of Yen’s algorithm.

**Lemma 53.** For each constant $c$ and each word $w$ with length $|w| = c$, the problem EnumSimPaths$(a^k w ? a^*)$ is in FPT delay with radix order.

**Proof.** We have seen in Theorem 29 that Algorithm 3 can indeed output paths. We show how to change this algorithm to output a shortest and lexicographically smallest simple path
that matches $a^k w? a^*$. If there exists a shortest simple path matching $a^k a^*$, we will find it in line 4 and store it in $p_1$. We explain in Lemma 52 how this can be done. In addition to this, we find the overall shortest and lexicographically smallest path matching $a^k w a^*$ (if it exists) by enumerating all paths $p_c \in S$ and again using the algorithm obtained in Lemma 52 to find the shortest paths that completes $p_c$ to a path matching $a^k w a^*$. We store the actual shortest and lexicographically smallest path that matches $a^k w a^*$ in $p_2$. If we only have one path in $p_1$ or $p_2$, we will output this one. Otherwise, we compare the length of $p_1$ with the length of $p_2$. If $|p_1| = |p_2|$, we output the lexicographically smaller one, else the shorter one. This completes the changes. Notice that this algorithm also works if $a$ is not the lexicographically smallest symbol. Since we output the shorter path, the path must indeed be simple. Therefore, the algorithm still works correctly.

We again need to show how to handle the derivatives. This algorithm can also be used for $\text{SimPath}(a^{k-1} w? a^*)$ with $i \leq k$. For $\text{SimPath}(w' a^*)$, where $w'$ is a suffix of $w$, we can enumerate all possible simple paths $p'$ that match $w'$, which are again at most $O(n')$, many, since $|w'| \leq |w| = c$. Then, we search for a shortest path $p_2$ from the last node of $p'$ to $t$ matching $a^*$ that does not use other nodes from $p'$. We then choose the shortest and lexicographically smallest path $p' p_2$ and output it.

\textbf{Lemma 54.} Let $\mathcal{R}$ be a cuttable class of STEs. Then $\text{EnumSimPaths}(\mathcal{R})$ is in FPT delay with radix order if $|A_i| \leq c$ and $|A'_i| \leq c$ for a constant $c$.

\textbf{Proof.} We have seen in Theorem 30 that we can enumerate paths for all kinds of STEs. Our goal is now to show that it is also possible to enumerate them in radix order. Let $r \in \mathcal{R}$ with left cut border $c_1$ and right cut border $c_2$. We enumerate all possible paths $p_c$ that can match $A_1 \cdots A_{c_1}$ and $p_{c_2}$ that match $A'_{c_2} \cdots A'_{c_1}$. (These paths can also be empty if $c_1 = 0$ or $c_2 = 0$.) We know now that the remaining regular expression, that is $r' = B_{\text{pre}}' A^* B_{\text{suff}}'$, is 0-bordered. So we now search for a shortest and lexicographically smallest simple path matching $r'$ from the last node of $p_1$ to the first of $p_2$ in the graph without the other nodes of $p_1$ and $p_2$.

We do a case distinctions depending on the form of $r'$. If $A = \emptyset$, we can use the algorithm of Bagan et al. [27, Theorem 6], please write shorter: This algorithm can also be used to obtain a smallest path in radix order. Since the witnessing path is computed from the back, we cannot directly determine the lexicographically smallest path, but we can compute the smallest path in reversed radix order, that is: $w_1 \leq^R w_2$ in reversed radix order if $|w_1| < |w_2|$ or $|w_1| = |w_2|$ and $w_1^R$ is lexicographically smaller than $w_2^R$, where $R$ denotes the symbol-wise reverse order of a word. We use the algorithm on the graph with reversed edges and with the “reversed” regular expression, i.e., instead of $r' = A_{c_1+1} \cdots A_{c_k} A'_{c_k} \cdots A'_{c_1+1}$, we use $r'^R = A'_{c_1+1} \cdots A'_{c_k} A_{c_k} \cdots A_{c_1+1}$. In the end, we reverse the path to obtain the smallest path in radix order in the original graph that matches $r$ and is simple. We enumerate all possible color codings and compare for each color coding the smallest path in radix order to obtain the overall smallest path in radix order, which we call $p$. We then return $p_{c_1} p p_{c_2}$.

Otherwise we know that $A \neq \emptyset$. If $r' = A_1 ? \cdots A_{k_1} ? A^* A'_{k_2} ? \cdots A'_{k_1} ?$, its language $L(r')$ is downward closed, so we can find a simple path $p$ matching $r'$ that is smallest in radix order,

\textbf{19} We used in the correctness proof of this algorithm that there is no path matching $a^k a^*$ if we search for a simple path matching $a^k w a^*$. Indeed the much weaker version suffices: there exists no simple path matching $a^k a^*$ that is strictly shorter than the shortest path matching $a^k w a^*$. This suffices since, if the path matching $a^k a^*$ is not simple and the repeated node is neither in the subpath matching $w$ nor in the subpath matching the last $k$ nodes, then the resulting simple path matches $a^k a^*$ and is really shorter.
Since to a simple path from be a smallest path in radix order from as explained above, we can use the algorithm of Bagan et al. [7, Theorem 6] to output smallest paths in line 2 if SimPath(r') has a solution of length at most 2k, where \( k = k_1 - c_1 + k_2 - c_2 \).

The lines 5 to 10 highly resemble Algorithm 3 and can therefore be changed to output shortest paths, see Lemma 52. The same holds for lines 10 to 19. So shortest paths are no problem.

We can guarantee that the results are in radix order if there exists a constant \( c \) with \( |A_i| \leq c \) and \( |A'_i| \leq c \) for all \( i \). This is because we can then enumerate in line 6 all up to \( c^k \) words \( w_1 \in L(r_1) \) and compute \( \hat{P}_{sv,w_1}^{k+1} = \hat{P}_{sv,w_1}^{k+1} \) for each such word. This way we can ensure that we really considered each lexicographically smallest word. We proceed analogous in line 11 for all \( w_2 \in L(r_2) \).

We will now show that we can indeed obtain the smallest path in radix order this way. We will therefore use a variant of Lemma 39. Let \( p = (s, a_0, v_1)(v_1, a_1, v_2) \cdots (v_{t-1}, a_{t-1}, t) \) be a smallest path from \( s \) to \( t \) in radix order that is simple and matches \( r' \). Then \( w_1 = a_0 \cdots a_{t-1} \in L(r_1) \). From line 2 we know that the solution now must have length longer than 2k. We now define

\[
P = (s, a_0, v_1)(v_1, a_1, v_2) \cdots (v_{k-1}, a_{k-1}, v_k),
\]
\[
R = (v_{k+1}, a_{k+1}, v_{k+2}) \cdots (v_{t-k-2}, a_{t-k-2}, v_{t-k-1}),
\]
\[
Q = (v_{t-k}, a_{t-k}, v_{t-k+1}) \cdots (v_{t-1}, a_{t-1}, t).
\]

As usual we write

\[
p = P \cdot (v_k, a_k, v_{k+1}) \cdot R \cdot (v_{t-k-1}, a_{t-k-1}, v_{t-k}) \cdot Q.
\]

If \( k = \ell - k - 1 \), i.e., \( \ell = 2k + 1 \), we write \( p = P \cdot (v_k, a_k, v_{k+1}) \cdot Q \) instead with \( R = \varepsilon \).

Since \( |V(Q)| = k + 1 \) and \( V(P) \cap V(Q) = \emptyset \), we can find a simple path \( P' \) from \( s \) to \( v_k \) that matches \( w_1 \) and consists of nodes in \( \hat{P}_{sv,w_1}^{k+1} \). Furthermore, \( V(P') \cap V(Q) = \emptyset \). If \( P' \) and \( R \) intersect, the resulting path would contradict our choice of \( p \) as smallest path from \( s \) to \( t \) that is simple and matches \( r' \), since the resulting path is shorter. If \( P' \) and \( R \) do not intersect, the path \( p' = P'(v_k, a_k, v_{k+1})Q(v_{t-k-1}, a_{t-k-1}, v_{t-k})R \) is still a smallest path from \( s \) to \( t \) in radix order that is simple and matches \( r' \) and its first \( k+1 \) nodes indeed belong to a set of nodes in \( \hat{P}_{sv,w_1}^{k+1} \).

It remains to show that its last \( k+1 \) nodes belong to \( \hat{P}_{v_{t-k},w_2}^{k+1} \), where \( w_2 = a_{t-k} \cdots a_{t-1} \).

We assume that our prefix \( P' \) is fix, i.e., let \( p' = P'(v_k, a_k, v_{k+1})Q(v_{t-k-1}, a_{t-k-1}, v_{t-k})R \) be a smallest path in radix order from \( s \) to \( t \) that matches \( r' \) and is simple. If the length of \( Q(v_{t-k-1}, a_{t-k-1}, v_{t-k})R \) is smaller than 2k, i.e., \( \ell \leq k + 2k + 1 \), we have

\[
p' = P' \cdot Q_2 \cdot (v_{t-k-1}, a_{t-k-1}, v_{t-k}) \cdot P_2
\]

with

\[
Q_2 = (v_k, a_k, v_{k+1}) \cdots (v_{t-k-2}, a_{t-k-2}, v_{t-k-1})
\] and
\[
P_2 = (v_{t-k}, a_{t-k}, v_{t-k+1}) \cdots (v_{t-1}, a_{t-1}, t).
\]

Since \( V(Q_2) \leq k+1 \) and \( V(Q_2) \cap V(P_2) = \emptyset \), we find a set \( X' \in \hat{P}_{v_{t-k},w_2}^{k+1} \) that corresponds to a simple path from \( v_{t-k} \) to \( t \) that matches \( w_2 \) and does not intersect with \( Q_2 \).
Let us now assume that \( \ell > k + 2k_2 + 1 \). In this case we have

\[
p' = P' \cdot Q_2 \cdot (v_{k+k_2}, a_{k+k_2}, v_{k+k_2+1}) \cdot R_2 \cdot (v_{\ell-k_2-1}, a_{\ell-k_2-1}, v_{\ell-k_2}) \cdot P_2
\]

with

\[
Q_2 = (v_{k}, a_{k}, v_{k+1}) \cdots (v_{k+k_2-1}, a_{k+k_2-1}, v_{k+k_2}).
\]

\[
R_2 = (v_{k+k_2+1}, a_{k+k_2+1}, v_{k+k_2+2}) \cdots (v_{\ell-k_2-2}, a_{\ell-k_2-2}, v_{\ell-k_2-1})
\]

\[
P_2 = (v_{\ell-k_2}, a_{\ell-k_2}, v_{\ell-k_2+1}) \cdots (v_{\ell-1}, a_{\ell-1}, t).
\]

Since \( V(Q_2) = k_2 + 1 \) and \( V(Q_2) \cap V(P_2) = \emptyset \), we find a set \( X' \in \hat{P}_{k_2+1}^{k_2} v_{\ell-k_2} t, w_2 \) that corresponds to a simple path \( P_2' \) from \( v_{\ell-k_2} \) to \( t \) that matches \( w_2 \) and does not intersect with \( Q_2 \). If \( P_2' \) and \( R_2 \) intersect, the resulting path would contradict our choice of \( p' \) as smallest path from \( s \) to \( t \) that is simple and matches \( r' \), since the resulting path is shorter. If \( P_2' \) and \( R_2 \) do not intersect, the path \( p' = P' \cdot Q_2 \cdot (v_{k+k_2}, a_{k+k_2}, v_{k+k_2+1}) \cdot R_2 \cdot (v_{\ell-k_2-1}, a_{\ell-k_2-1}, v_{\ell-k_2}) \cdot P_2' \) is still a smallest path from \( s \) to \( t \) that is simple and matches \( r' \) and its last \( k_2 + 1 \) nodes indeed belong to a set of nodes in \( \hat{P}_{k_2+1}^{k_2} v_{\ell-k_2} t, w_2 \).

In line 15 we can use the algorithm of Ackermann and Shallit [2] to find a shortest and lexicographically smallest \( v-u \)-path matching \( A' \), see Theorem 3.

Since, we can use an easier variant of Algorithm 4 if \( r' = A_{q_1+1} \cdots A_{k_2} A'_{k_2} \cdots A'_{\ell-1} \) or \( r' = A_1 \cdots A_{k_1} A'_{k_1} \cdots A'_{\ell-1} \), we can also output paths in radix order these cases. Since for each expression \( r \in \mathcal{R} \), \( w^{-1} L(r) \) is a STE, we can also use this obtained algorithm as subroutine in Yen’s algorithm in line 12.

For expressions of the form \( A_1 \cdots A_{k_1} A'_{k_1} \cdots A'_{\ell-1} \), it follows analogous. Since the derivative of an STE is again an STE with at most the same cut border (see Lemma 5.1), we can use the same methods for them.

\[\mathbf{\triangleright Theorem 31.}\ \text{Let } \mathcal{R} \text{ be a class of STEs that is almost conflict-free. Then, EnumTrails}(\mathcal{R}) \text{ is in } \text{FPT delay.}\]

\[\textbf{Proof.}\ \text{We will use the reduction from Lemma 7 to reduce } \text{SimPath}(r) \text{ to } \text{SimPath}(r). \text{Since the paths have a one-to-one correspondence, we can use the path in the output of } \text{SimPath}(r) \text{ to obtain a path for } \text{SimPath}(r). \text{So, if we can show that } \text{SimPath}(r) \text{ in } \text{FPT delay for the graphs that can be constructed in the reduction, we can use this correspondence to also output the corresponding paths. Notice that we have already covered many cases of } r \text{ in Theorem 30 but since } \mathcal{R} \text{ does not need to be cuttable, we still have to show that Algorithm 4 can also output paths after the changes in Theorem 27 part (a). Since we only relabeled some edges and labels in the regular expression, the adapted Algorithm 4 can still output a witness. In this output, we reverse our label-changes again to obtain a path in the original graph without label-changes and therefore corresponding to a path in } \text{SimPath}(r). \text{Since derivatives of } r \text{ have at most the same number of conflict labels and the graphs used in Yen’s algorithm still have the property that each node corresponds to at most one edge, we can again use the same strategy in Yen’s algorithm in line 12 and therefore solve } \text{EnumSimPaths}(\mathcal{R}).\]
that can be constructed in the reduction, we can use this correspondence to also output the corresponding paths in $\text{Trail}(r)$ with radix order. Notice that we have already covered many cases of the form of $r$ in Theorem 30 and can in this cases find the smallest in radix order, see Lemma 54. But since $\mathcal{R}$ does not need to be cuttable, we still have to show that Algorithm 4 can also output smallest paths in radix order after the changes we did in Theorem 27 part (a). Recall that our changes were to enumerate sets of at most $c$ nodes, relabel the outgoing edges of those nodes and changed some labels in the regular expression. Obviously, the adapted Algorithm 4 can still output a witness and, since we obtain our original instance by removing all single quotes, i.e., we relabel $a'$ to our original symbol $a$, we can even compare each candidate and only output the smallest in radix order. Since the reduction in Lemma 7 preserves the labels, we therefore have a smallest path in radix order in $\text{Trail}(r)$.

Since derivatives of $r$ have at most the same number of conflict labels and the graphs used in Yen’s algorithm still have the property that each node corresponds to at most one edge, we can again use the same strategy in Yen’s algorithm in line 12 and therefore solve $\text{EnumSimPaths}(r)$ by solving $\text{EnumSimPaths}(r)$ in FPT delay with radix order and output the corresponding paths.