AN ADAPTIVE STRONG ORDER 1 METHOD FOR SDES WITH DISCONTINUOUS DRIFT COEFFICIENT

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ABSTRACT. In recent years, an intensive study of strong approximation of stochastic differential equations (SDEs) with a drift coefficient that may have discontinuities in space has begun. In many of these results it is assumed that the drift coefficient satisfies piecewise regularity conditions and the diffusion coefficient is Lipschitz continuous and non-degenerate at the discontinuity points of the drift coefficient. For scalar SDEs of that type the best $L^p$-error rate known so far for approximation of the solution at the final time point is 3/4 in terms of the number of evaluations of the driving Brownian motion and it is achieved by the transformed equidistant quasi-Milstein scheme, see [18]. Recently in [21] it has been shown that for such SDEs the $L^p$-error rate 3/4 cannot be improved in general by no numerical method based on evaluations of the driving Brownian motion at fixed time points. In the present article we construct for the first time in the literature a method based on sequential evaluations of the driving Brownian motion, which achieves an $L^p$-error rate of at least 1 in terms of the average number of evaluations of the driving Brownian motion for such SDEs.

1. Introduction

In this article we consider a scalar autonomous stochastic differential equation (SDE)

$$dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t, \quad t \geq 0,$$

$$X_0 = x_0,$$

where $x_0 \in \mathbb{R}$ is the initial value, $\mu : \mathbb{R} \to \mathbb{R}$ is the drift coefficient, $\sigma : \mathbb{R} \to \mathbb{R}$ is the diffusion coefficient, $W = (W_t)_{t \geq 0}$ is a 1-dimensional Brownian motion and we assume that the SDE (1) has a unique strong solution $X$. Our computational task is $L^p$-approximation of $X_1$ by numerical methods that are based on finitely many evaluations of the driving Brownian motion $W$ at points in $[0, 1]$ in the case when the drift coefficient $\mu$ may have finitely many discontinuity points.

Strong approximation of SDEs with a discontinuous drift coefficient has gained a lot of interest in the literature in recent years. See [4, 5] for results on convergence in probability and almost sure convergence of the Euler-Maruyama scheme and [1, 3, 6, 13, 14, 15, 18, 20, 22, 23, 24, 25, 26, 27] for results on $L^p$-approximation. In many of these articles it is assumed that the drift coefficient satisfies piecewise regularity conditions and the diffusion coefficient is Lipschitz continuous and non-degenerate at the discontinuity points of the drift coefficient. For SDEs of that type the best $L^p$-error rate known up to now for approximation of $X_1$ is 3/4, see [18]. In the present article we construct for the first time in the literature a numerical method, which achieves an $L^p$-error rate of at least 1 for such SDEs.

To be more precise, let us consider the following assumptions on the coefficients $\mu$ and $\sigma$. 

There exist \( k \in \mathbb{N} \) and \( \xi_0, \ldots, \xi_k+1 \in [-\infty, \infty] \) with \(-\infty = \xi_0 < \xi_1 < \ldots < \xi_k < \xi_{k+1} = \infty\) such that \( \mu \) is Lipschitz continuous on the interval \((\xi_{i-1}, \xi_i)\) for all \( i \in \{1, \ldots, k+1\} \).

\[(\sigma1) \sigma \text{ is Lipschitz continuous on } \mathbb{R} \text{ and } \sigma(\xi_i) \neq 0 \text{ for all } i \in \{1, \ldots, k\}.\]

If \((\mu1)\) and \((\sigma1)\) hold then the SDE \((1)\) has a unique strong solution, see \[13\]. In \[13\], \[14\], \[15\], \[18\] the \(L_p\)-approximation of \(X_1\) under the assumptions \((\mu1)\) and \((\sigma1)\) has been analyzed. In particular, in \[13\], \[14\] the transformed equidistant Euler-Maruyama scheme has been constructed, which achieves an \(L_2\)-error rate of at least \(1/2\) in terms of the number of evaluations of the driving Brownian motion \(W\). After that, in \[23\] an adaptive Euler-Maruyama scheme has been constructed, which achieves up to a logarithmic factor an \(L_2\)-error rate of at least \(1/2\) in terms of the average number of evaluations of \(W\) used by the scheme. Finally, in \[20\] it has been proven that the classical equidistant Euler-Maruyama scheme achieves for all \(p \in [1, \infty)\) an \(L_p\)-error rate of at least \(1/2\) in terms of the number of evaluations of \(W\) as in the case of SDEs with globally Lipschitz continuous coefficients.

In \[18\] the first higher-order method has been constructed for such SDEs. This method is based on equidistant evaluations of \(W\) and achieves for all \(p \in [1, \infty)\) an \(L_p\)-error rate of at least \(3/4\) in terms of the number of evaluations of \(W\) if \(\mu\) and \(\sigma\) satisfy \((\mu1)\) and \((\sigma1)\) and additionally the following piecewise regularity assumptions

\[(\mu2) \mu \text{ has a Lipschitz continuous derivative on } (\xi_{i-1}, \xi_i) \text{ for every } i \in \{1, \ldots, k+1\}.,\]

\[(\sigma2) \sigma \text{ has a Lipschitz continuous derivative on } (\xi_{i-1}, \xi_i) \text{ for every } i \in \{1, \ldots, k+1\}.\]

Furthermore, in \[22\] it has been shown that for SDEs \((1)\) with additive noise and a bounded and piecewise \(C_b^2\) drift coefficient \(\mu\) the equidistant Euler-Maruyama scheme in fact achieves an \(L_2\)-error rate of at least \(3/4\) in terms of the number of evaluations of \(W\). Note that in this case the Euler-Maruyama scheme coincides with the Milstein scheme.

Recently in \[21\] it has been shown that an \(L_p\)-error rate better than \(3/4\) can not be achieved in general under the assumptions \((\mu1)\), \((\mu2)\), \((\sigma1)\) and \((\sigma2)\) by no numerical method based on evaluations of \(W\) at fixed time points in \([0, 1]\). More precisely, it has been proven in \[21\] that if \(\sigma = 1\) and if \(\mu\) satisfies \((\mu1)\) and \((\mu2)\), \(\mu\) is bounded, increasing and there exists \(i \in \{1, \ldots, k\}\) such that \(\mu(\xi_i+) \neq \mu(\xi_i^-)\), then there exists \(c \in (0, \infty)\) such that for all \(p \in [1, \infty)\) and all \(n \in \mathbb{N}\),

\[
\inf_{g: \mathbb{R}^n \to \mathbb{R} \text{ measurable}} \mathbb{E}[|X_1 - g(W_{t_1}, \ldots, W_{t_n})|^p]^{1/p} \geq \frac{c}{n^{3/4}}.
\]

Note that the lower bound \((2)\) does not cover adaptive methods, i.e. methods that may choose the number as well as the location of the evaluations of the Brownian motion \(W\) in a sequential way dependent on the values of \(W\) observed so far. See e.g. \[2\], \[9\], \[10\], \[12\], \[16\], \[17\], \[23\], \[29\] for examples of such methods. It is well-known that for a large class of SDEs \((1)\) with globally Lipschitz continuous coefficients the best possible \(L_p\)-error rate that can be achieved by non-adaptive methods coincides with the best possible \(L_p\)-error rate that can be achieved by adaptive methods and is equal to \(1\), see \[16\], \[17\]. Moreover, up to now there is no example of an SDE with globally Lipschitz continuous coefficients known in the literature, for which adaptive methods are superior to non-adaptive ones with respect to the \(L_p\)-error rate. However, the superiority of adaptive methods to non-adaptive ones with respect to the \(L_p\)-error rate has recently been
demonstrated in [7, 19] for some examples of SDEs with non-globally Lipschitz continuous drift or diffusion coefficients.

In view of the latter results it is natural to ask whether there exists an adaptive method that achieves under the assumptions \( (\mu_1), (\mu_2) \) and \( (\sigma_1), (\sigma_2) \) a better \( L_p \)-error rate than the rate \( 3/4 \). To the best of our knowledge the answer to this question was not known in the literature up to now. In the present article we answer this question in the positive. More precisely, we construct a family of approximations \( \hat{X}^\delta_1 \) with \( \delta \in (0, \delta_0] \) for some \( \delta_0 > 0 \) such that each approximation \( \hat{X}^\delta_1 \) is based on at most \( c \cdot \delta^{-1} \) adaptively chosen evaluations of \( W \) in the interval \([0, 1]\) on average and such that for all \( p \in [1, \infty) \) and all \( \delta \in (0, \delta_0] \),

\[
\mathbb{E}[|X_1 - \hat{X}^\delta_1|^p]^{1/p} \leq c(p) \cdot \delta,
\]

where the constants \( c, c(p) \in (0, \infty) \) do not depend on \( \delta \), see Theorem 2. Thus, the approximations \( \hat{X}^\delta_1 \) achieve an \( L_p \)-error rate of at least 1 in terms of the average number of evaluations of \( W \). The methods \( \hat{X}^\delta_1 \) are obtained by applying a suitable transformation \( G : \mathbb{R} \to \mathbb{R} \) to the strong solution \( X \) of the SDE (1) such that the transformed solution \( Z = (G(X_t))_{t \geq 0} \) is a strong solution of a new SDE with coefficients \( \tilde{\mu} \) and \( \tilde{\sigma} \) which satisfy \( (\mu_1), (\mu_2) \) and \( (\sigma_1), (\sigma_2) \), respectively, and such that \( \tilde{\mu} \) is continuous, which implies that \( \tilde{\mu} \) is Lipschitz continuous. An adaptive quasi-Milstein scheme \( \hat{Z}^\delta = (\hat{Z}^\delta_t)_{t \geq 0} \) is used to approximate \( Z \) and the approximation \( \hat{X}^\delta_1 \) is then given by \( G^{-1}(\hat{Z}^\delta_1) \). The adaptive time stepping strategy used for the adaptive quasi-Milstein scheme \( \hat{Z}^\delta \) is an appropriate modification of the adaptive time stepping strategy used for the adaptive Euler-Maruyama scheme in [23]. We add that an \( L_p \)-error rate better than 1 can not be achieved in general under the assumptions \( (\mu_1), (\mu_2) \) and \( (\sigma_1), (\sigma_2) \) by no adaptive method based on finitely many evaluations of \( W \), see [8, 16, 17] for corresponding lower error bounds.

The implementation of our method requires the ability to evaluate the functions \( G \) and \( G^{-1} \) at each step of the adaptive quasi-Milstein scheme \( \hat{Z}^\delta \). While the transformation \( G \) is known explicitly, this is so far not the case for \( G^{-1} \), and therefore a numerical inverse of \( G \) has to be used to approximate \( G^{-1} \). This makes our method rather slow in practice. We conjecture however that the transformation of the SDE (1) is actually not needed and that an adaptive quasi-Milstein scheme for the SDE (1) itself achieves under the assumptions \( (\mu_1), (\mu_2) \) and \( (\sigma_1), (\sigma_2) \) an \( L_p \)-error rate of at least 1 in terms of the average number of evaluations of \( W \). The proof of this conjecture will be the subject of future work.

We briefly describe the content of the paper. In Section 2 we introduce some notation. Section 3 contains the construction and the error and cost analysis of the adaptive quasi-Milstein scheme in the case when the coefficients of the SDE (1) satisfy the assumptions \( (\mu_1), (\mu_2) \) and \( (\sigma_1), (\sigma_2) \) and the drift coefficient is continuous, see Theorem 1. In Section 4 we introduce the bi-Lipschitz transformation \( G \) that is then used to construct a method of order 1 under the assumptions \( (\mu_1), (\mu_2) \) and \( (\sigma_1), (\sigma_2) \), see Theorem 2. Section 5 is devoted to the proof of Theorem 1.
2. Notation

For $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ we put $d(x, A) = \inf\{|x - y|: y \in A\}$. For a function $f: \mathbb{R} \to \mathbb{R}$ we define $d_f: \mathbb{R} \to \mathbb{R}$ by

$$d_f(x) = \begin{cases} f'(x), & \text{if } f \text{ is differentiable in } x, \\ 0, & \text{otherwise.} \end{cases}$$

3. An adaptive quasi-Milstein scheme for SDEs with Lipschitz continuous coefficients

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, let $W: [0, \infty) \times \Omega \to \mathbb{R}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, let $x_0 \in \mathbb{R}$ and let $\mu: \mathbb{R} \to \mathbb{R}$ and $\sigma: \mathbb{R} \to \mathbb{R}$ be functions that satisfy the assumptions $(\mu_1)$, $(\mu_2)$ and $(\sigma_1)$, $(\sigma_2)$, respectively, and assume that $\mu$ is continuous. We consider the SDE

$$dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t, \quad t \geq 0,$$

$$X_0 = x_0.$$  

Observe that in this case both $\mu$ and $\sigma$ are Lipschitz continuous on $\mathbb{R}$, and therefore the SDE (3) has a unique strong solution and for every $p \in [1, \infty)$ it holds

$$\mathbb{E}\left[\sup_{t \in [0,1]} |X_t|^p\right] < \infty.$$  

Put $\Theta = \{\xi_1, \ldots, \xi_k\}$ and for $\varepsilon > 0$ let $\Theta^\varepsilon = \{x \in \mathbb{R}: d(x, \Theta) < \varepsilon\}$. Let $\varepsilon_0 \in (0, 1]$ and assume that

$$\varepsilon_0 \leq \frac{1}{2} \min\{\xi_i - \xi_{i-1}: i = 2, \ldots, k\}$$

if $k \geq 2$. For $\delta > 0$ put

$$\varepsilon_1^\delta = \sqrt{\delta} \cdot \log^2(1/\delta), \quad \varepsilon_2^\delta = \delta \cdot \log^4(1/\delta).$$

Let $\delta_0 \in (0, 1)$ be small enough such that for all $\delta \in (0, \delta_0]$ it holds

$$\varepsilon_2^\delta \leq \varepsilon_1^\delta \leq \varepsilon_0/2.$$  

For $\delta \in (0, \delta_0]$ we define a time-continuous adaptive quasi-Milstein scheme $\hat{X}^\delta = (\hat{X}^\delta_t)_{t \geq 0}$ recursively by

$$\tau_0^\delta = 0, \quad \hat{X}^\delta_{\tau_0^\delta} = x_0$$

and

$$\tau_{i+1}^\delta = \tau_i^\delta + h^\delta(\hat{X}^\delta_{\tau_i^\delta}),$$

$$\hat{X}^\delta_t = \hat{X}^\delta_{\tau_i^\delta} + \mu(\hat{X}^\delta_{\tau_i^\delta}) \cdot (t - \tau_i^\delta) + \sigma(\hat{X}^\delta_{\tau_i^\delta}) \cdot (W_t - W_{\tau_i^\delta}) + \frac{1}{2} \sigma^2 \sigma(\hat{X}^\delta_{\tau_i^\delta}) \cdot ((W_t - W_{\tau_i^\delta})^2 - (t - \tau_i^\delta)), \quad t \in (\tau_i^\delta, \tau_{i+1}^\delta).$$
for $i \in \mathbb{N}_0$, where the step size function $h^\delta : \mathbb{R} \to (0,1)$ is defined by

$$h^\delta(x) = \begin{cases} 
\delta, & x \notin \Theta^\delta_i, \\
\left(\frac{d(x,\Theta)}{\log^2(1/\delta)}\right)^2, & x \in \Theta^\delta_i \setminus \Theta^\delta_{i+1}, \\
\delta^2 \cdot \log^4(1/\delta), & x \in \Theta^\delta_{i+1}.
\end{cases}$$

Note that the assumption implies that $\Theta^\delta_i \subseteq \Theta^\delta_{i+1}$ for all $\delta \in (0,\delta_0]$ and hence $h^\delta$ is well-defined for all $\delta \in (0,\delta_0]$. Moreover, $h^\delta$ is continuous and it holds

$$\delta^2 \cdot \log^4(1/\delta) \leq h^\delta \leq \delta$$

for all $\delta \in (0,\delta_0]$. We add that the step size function $h^\delta$ we use for the adaptive quasi-Milstein scheme $\bar{X}^\delta$ is an appropriate modification of the step size function used for the adaptive Euler-Maruyama scheme in [23].

For $\delta \in (0,\delta_0]$ let $N(\bar{X}^\delta)$ denote the number of evaluations of $W$ used to compute $\bar{X}^\delta_1$, i.e.

$$N(\bar{X}^\delta) = \min\{i \in \mathbb{N} : \tau^\delta_i \geq 1\}.$$}

Clearly, for all $\delta \in (0,\delta_0]$,

$$N(\bar{X}^\delta) \leq \lceil \delta^{-2} \log^{-4}(1/\delta) \rceil.$$}

We have the following upper bounds for the $p$-th root of the $p$-th mean of the maximum error of $\bar{X}^\delta$ on the time interval $[0,1]$ and for the average number of evaluations of $W$ used to compute $\bar{X}^\delta_1$.

**Theorem 1.** Assume $(\mu_1)$, $(\mu_2)$ and $(\sigma_1)$, $(\sigma_2)$ and assume that $\mu$ is continuous. Let $p \in [1,\infty)$. Then there exists $c_1, c_2 \in (0,\infty)$ such that for all $\delta \in (0,\delta_0]$,

$$E\left[ \sup_{t \in [0,1]} |X_t - \bar{X}^\delta_t|^{1/p} \right]^{1/p} \leq c_1 \cdot \delta$$

and

$$E[N(\bar{X}^\delta)] \leq c_2 \cdot \delta^{-1}.$$}

The proof of Theorem 1 is postponed to Section 5.

4. AN ADAPTIVE STRONG ORDER 1 METHOD FOR SDES WITH DISCONTINUOUS DRIFT COEFFICIENT

As in Section 3 we consider a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ and we assume that $W : [0,\infty) \times \Omega \to \mathbb{R}$ is a Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$. In contrast to Section 3 we now turn to SDEs with a drift coefficient $\mu$ that may have discontinuity points.

Let $x_0 \in \mathbb{R}$ and let $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ be functions that satisfy the assumptions $(\mu_1)$, $(\mu_2)$ and $(\sigma_1)$, $(\sigma_2)$, respectively. For later purposes we note that $(\mu_1)$ implies the existence of the one-sided limits $\mu(\xi^-)$ and $\mu(\xi^+)$ for all $i \in \{1,\ldots,k\}$. We consider the SDE

$$dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t, \quad t \geq 0,$$

$$X_0 = x_0,$$

which has a unique strong solution, see [13, Theorem 2.2].
We now construct an adaptive method for approximating the strong solution of the SDE (12) at the time 1. To this end we employ the transformation strategy from [18]. We use that \( X_1 \) can be obtained by applying a Lipschitz continuous transformation to the strong solution of an SDE with coefficients \( \tilde{\mu}, \tilde{\sigma} \) satisfying the assumptions \((\mu_1), (\mu_2)\) and \((\sigma_1), (\sigma_2)\), respectively, such that \( \tilde{\mu} \) is continuous, and then we employ Theorem [11].

We start by introducing the transformation procedure from [18]. For

\[
G = \{ (z_1, \ldots, z_k) \in \mathbb{R}^k : z_1 < \cdots < z_k \}
\]

and \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k \) we put

\[
\rho_{z,\alpha} = \begin{cases} 
\frac{1}{8|\alpha_1|}, & \text{if } k = 1, \\
\min\left\{ \frac{1}{8|\alpha_i|} : i \in \{1, \ldots, k\} \cup \left\{ \frac{z_{i+1}-z_i}{\alpha_i} : i \in \{2, \ldots, k\} \right\} \right\}, & \text{if } k \geq 2,
\end{cases}
\]

where we use the convention \( 1/0 = \infty \). Let \( \phi : \mathbb{R} \to \mathbb{R} \) be given by

\[
\phi(x) = (1 - x^2)^4 \cdot 1_{[-1,1]}(x).
\]

For all \( k \in \mathbb{N}, z \in T_k, \alpha \in \mathbb{R}^k \) and \( \nu \in (0, \rho_{z,\alpha}) \) we define a function \( G_{z,\alpha,\nu} : \mathbb{R} \to \mathbb{R} \) by

\[
G_{z,\alpha,\nu}(x) = x + \sum_{i=1}^{k} \alpha_i \cdot (x - z_i) \cdot |x - z_i| \cdot \phi\left( \frac{x - z_i}{\nu} \right).
\]

The following two technical lemmas provide the properties of the mappings \( G_{z,\alpha,\nu} \) that are crucial for our purposes. For the proofs of both lemmas see [18].

**Lemma 1.** Let \( k \in \mathbb{N}, z \in T_k, \alpha \in \mathbb{R}^k, \nu \in (0, \rho_{z,\alpha}) \) and put \( z_0 = -\infty \) and \( z_{k+1} = \infty \). The function \( G_{z,\alpha,\nu} \) has the following properties.

(i) \( G_{z,\alpha,\nu} \) is differentiable on \( \mathbb{R} \) with a Lipschitz continuous derivative \( G'_{z,\alpha,\nu} \) that satisfies \( \inf_{x \in \mathbb{R}} G'_{z,\alpha,\nu}(x) > 0 \). In particular, \( G_{z,\alpha,\nu} \) has an inverse \( G^{-1}_{z,\alpha,\nu} : \mathbb{R} \to \mathbb{R} \) that is Lipschitz continuous.

(ii) For every \( i \in \{1, \ldots, k+1\} \), the function \( G'_{z,\alpha,\nu} \) is differentiable on \( (z_{i-1}, z_i) \) with Lipschitz continuous derivatives \( G''_{z,\alpha,\nu} \).

(iii) For every \( i \in \{1, \ldots, k\} \) the one-sided limits \( G''_{z,\alpha,\nu}(z_i-) \) and \( G''_{z,\alpha,\nu}(z_i+) \) exist and satisfy

\[
G''_{z,\alpha,\nu}(z_i-) = -2\alpha_i, \quad G''_{z,\alpha,\nu}(z_i+) = 2\alpha_i.
\]

**Lemma 2.** Assume \((\mu_1), (\mu_2)\) and \((\sigma_1), (\sigma_2)\). Put \( \xi = (\xi_1, \ldots, \xi_k) \), define \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k \) by

\[
\alpha_i = \frac{\mu(\xi_i-) - \mu(\xi_i+)}{2\sigma^2(\xi_i)}
\]

for \( i \in \{1, \ldots, k\} \), and let \( \nu \in (0, \rho_{\xi,\alpha}) \). Consider the function \( G_{\xi,\alpha,\nu} \) and extend \( G''_{\xi,\alpha,\nu} : \cup_{i=1}^{k+1} (\xi_{i-1}, \xi_i) \to \mathbb{R} \) to the whole real line by taking

\[
G''_{\xi,\alpha,\nu}(\xi_i) = 2\alpha_i + 2 \frac{\mu(\xi_i+) - \mu(\xi_i)}{\sigma^2(\xi_i)}
\]

for \( i \in \{1, \ldots, k\} \). Then the functions

\[
\tilde{\mu} = (G'_{\xi,\alpha,\nu} \cdot \mu + \frac{1}{2} G''_{\xi,\alpha,\nu} \cdot \sigma^2) \circ G^{-1}_{\xi,\alpha,\nu} \quad \text{and} \quad \tilde{\sigma} = (G'_{\xi,\alpha,\nu} \cdot \sigma) \circ G^{-1}_{\xi,\alpha,\nu}
\]
satisfy the assumptions (μ1), (μ2) and (σ1), (σ2), respectively, and \( \tilde{\mu} \) is continuous.

We turn to the transformation of the SDE (12). Take \( \xi, \alpha, \nu \) as in Lemma 2 and define a stochastic process \( Z: [0, \infty) \times \Omega \to \mathbb{R} \) by

\[
Z_t = G_{\xi,\alpha,\nu}(X_t), \quad t \geq 0.
\]

Then the process \( Z \) is the unique strong solution of the SDE

\[
dZ_t = \tilde{\mu}(Z_t) \, dt + \tilde{\sigma}(Z_t) \, dW_t, \quad t \geq 0,
\]

\[
Z_0 = G_{\xi,\alpha,\nu}(x_0)
\]

with \( \tilde{\mu} \) and \( \tilde{\sigma} \) given by (15), see [18]. For every \( \delta \in (0, \delta_0] \) we use \( \hat{Z}^\delta = (\hat{Z}^\delta_t)_{t \geq 0} \) to denote the time-continuous adaptive quasi-Milstein scheme (6), (7) associated to the SDE (17), i.e. \( \hat{Z}^\delta \) is defined recursively by

\[
\tau^\delta_0 = 0, \quad \hat{Z}^\delta_{\tau^\delta_0} = G_{\xi,\alpha,\nu}(x_0)
\]

\[
\tau^\delta_{i+1} = \tau^\delta_i + h^\delta(\hat{Z}^\delta_{\tau^\delta_i}),
\]

\[
\hat{Z}^\delta_{\tau^\delta_i} = \hat{Z}^\delta_{\tau^\delta_i} + \tilde{\mu}(\hat{Z}^\delta_{\tau^\delta_i}) \cdot (t - \tau^\delta_i) + \tilde{\sigma}(\hat{Z}^\delta_{\tau^\delta_i}) \cdot (W_t - W_{\tau^\delta_i})
\]

\[
+ \frac{1}{2} \tilde{\sigma} d\tilde{\sigma}(\hat{Z}^\delta_{\tau^\delta_i}) \cdot ((W_t - W_{\tau^\delta_i})^2 - (t - \tau^\delta_i)), \quad t \in (\tau^\delta_i, \tau^\delta_{i+1}],
\]

for \( i \in \mathbb{N}_0 \), where the step size function \( h^\delta \) is given by (8).

We approximate \( X \) by the stochastic process \( \hat{X}^\delta = (\hat{X}^\delta_t)_{t \geq 0} \) with \( \hat{X}^\delta_t = G_{\xi,\alpha,\nu}^{-1}(\hat{Z}^\delta_t), t \geq 0 \). For \( \delta \in (0, \delta_0] \) let \( N(\hat{X}^\delta) \) denote the number of evaluations of \( W \) used to compute \( \hat{X}^\delta_t \). We have the following upper bounds for the \( p \)-th root of the \( p \)-th mean of the maximum error of \( \hat{X}^\delta \) on the time interval \([0, 1]\) and for the average number of evaluations of \( W \) used to compute \( \hat{X}^\delta_t \).

**Theorem 2.** Assume (μ1), (μ2) and (σ1), (σ2). Let \( p \in [1, \infty) \). Then there exists \( c_1, c_2 \in (0, \infty) \) such that for all \( \delta \in (0, \delta_0] \),

\[
\mathbb{E}\left[ \sup_{t \in [0, 1]} |X_t - \hat{X}^\delta_t|^p \right]^{1/p} \leq c_1 \cdot \delta
\]

and

\[
\mathbb{E}[N(\hat{X}^\delta_1)] \leq c_2 \cdot \delta^{-1}.
\]

**Proof.** Using the Lipschitz continuity of \( G_{\xi,\alpha,\nu}^{-1} \), see Lemma (1)(i), the fact that \( \tilde{\mu} \) and \( \tilde{\sigma} \) satisfy the assumptions (μ1), (μ2) and (σ1), (σ2), respectively, and that \( \tilde{\mu} \) is continuous as well as the estimate (10) in Theorem 1 we obtain that there exist \( c_1, c_2 \in (0, \infty) \) such that for all \( \delta \in (0, \delta_0] \),

\[
\mathbb{E}\left[ \sup_{t \in [0, 1]} |X_t - \hat{X}^\delta_t|^p \right]^{1/p} = \mathbb{E}\left[ \sup_{t \in [0, 1]} |X_t - G_{\xi,\alpha,\nu}^{-1}(\hat{Z}^\delta_t)|^p \right]^{1/p} \leq c_1 \cdot \mathbb{E}\left[ \sup_{t \in [0, 1]} |Z_t - \hat{Z}^\delta_t|^p \right]^{1/p} \leq c_2 \cdot \delta.
\]

Thus, (20) holds. The estimate (21) follows from the fact that \( N(\hat{X}^\delta_1) = N(\hat{Z}^\delta_1) \) and the estimate (11) in Theorem 1. \( \square \)
5. Proof of Theorem 1

Throughout this section we assume that \( \mu \) and \( \sigma \) satisfy \((\mu 1), (\mu 2)\) and \((\sigma 1), (\sigma 2)\), respectively, and that \( \mu \) is continuous. Moreover, for \( \delta \in (0, \delta_0) \) and \( t \in [0, 1] \) we put

\[
\xi_t^\delta = \max\{\tau_i^\delta : i \in \mathbb{N}_0, \tau_i^\delta \leq t\}.
\]

We first briefly describe the structure of the proof of the error estimate \([10]\) in Theorem 1 and the relation of our analysis and the error analysis of the equidistant quasi-Milstein scheme in \([18]\). Let \( \hat{X}^{\delta, eq} = (\hat{X}^{\delta, eq}_t)_{t \geq 0} \) denote the equidistant quasi-Milstein scheme with step size \( \delta \), i.e. \( \hat{X}^{\delta, eq} \) is defined in the same way as \( \hat{X}^\delta \) in (7), but with \( h^\delta = \delta \) in place of (8). For simplicity let us restrict to the case \( p = 2 \). In \([18]\) it is shown that there exists \( c \in (0, \infty) \) such that for all \( \delta \in \{1/n : n \in \mathbb{N}\} \),

\[
\mathbb{E}[\sup_{t \in [0, 1]} |X_t - \hat{X}^{\delta, eq}_t|^2]^{1/2} \leq c \cdot \delta + c \cdot \left( \int_0^1 \mathbb{E}[|\hat{X}^{\delta, eq}_t - \hat{X}^{\delta, eq}_{t^\delta}|^2 \cdot 1_S(\hat{X}^{\delta, eq}_t, \hat{X}^{\delta, eq}_{t^\delta})] dt \right)^{1/2},
\]

where

\[
S = \left( \bigcup_{i=0}^{k+1} (\xi_{i-1}, \xi_i)^2 \right) \tag{22}
\]

is the set of pairs \((x, y)\) in \( \mathbb{R}^2 \), which do not allow for a joint Lipschitz estimate of \(|d_\mu(x) - d_\mu(y)|\) or of \(|d_\sigma(x) - d_\sigma(y)|\) if \( \mu \) or \( \sigma \) is not differentiable at one of the points \( \xi_1, \ldots, \xi_k \). Transforming the condition \((\hat{X}^{\delta, eq}_t, \hat{X}^{\delta, eq}_t) \in S \) into a condition solely on the sizes of the random variables \(|\hat{X}^{\delta}_t - \hat{X}^{\delta}_{t^\delta} - \xi_i|, |\hat{X}^{\delta}_{t^\delta} - \hat{X}^{\delta}_t - \xi_i|\) and \(|\hat{X}^{\delta}_t - \hat{X}^{\delta}_t - \xi_i|\), where \( \xi_i \) lies between \( \hat{X}^{\delta, eq}_t \) and \( \hat{X}^{\delta, eq}_{t^\delta} \), and employing a Markov-type property of \( \hat{X}^{\delta, eq} \) and occupation time estimates for \( \hat{X}^{\delta, eq} \) it is shown in \([18]\) that there exists \( c \in (0, \infty) \) such that for all \( \delta \in \{1/n : n \in \mathbb{N}\} \),

\[
\int_0^1 \mathbb{E}[|\hat{X}^{\delta, eq}_t - \hat{X}^{\delta, eq}_{t^\delta}|^2 \cdot 1_S(\hat{X}^{\delta, eq}_t, \hat{X}^{\delta, eq}_{t^\delta})] dt \leq c \cdot \delta^{3/2}. \tag{23}
\]

Combining (22) and (23) yields the rate of convergence 3/4 for the root mean square of the maximum error of the equidistant quasi-Milstein scheme \( \hat{X}^{\delta, eq} \) on the time interval \([0, 1] \).

Our proof of (10) reproduces the estimate (22). Proceeding similarly to \([18] \) Subsection 5.3 we show that there exists \( c \in (0, \infty) \) such that for all \( \delta \in (0, \delta_0) \) the adaptive quasi-Milstein scheme \( \hat{X}^{\delta} \) satisfies

\[
\mathbb{E}[\sup_{t \in [0, 1]} |X_t - \hat{X}^{\delta}_t|^2]^{1/2} \leq c \cdot \delta + c \cdot \left( \int_0^1 \mathbb{E}[|\hat{X}^{\delta}_t - \hat{X}^{\delta}_{t^\delta}|^2 \cdot 1_S(\hat{X}^{\delta}_t, \hat{X}^{\delta}_{t^\delta})] dt \right)^{1/2}. \tag{24}
\]

However, we obtain a much better upper bound for the integral on the right hand side of (24) than the upper bound \( c \cdot \delta^{3/2} \) in (23) in the case of the equidistant quasi-Milstein scheme \( \hat{X}^{\delta, eq} \). More precisely, we show that there exists \( c \in (0, \infty) \) such that for all \( \delta \in (0, \delta_0) \),

\[
\int_0^1 \mathbb{E}[|\hat{X}^{\delta}_t - \hat{X}^{\delta}_{t^\delta}|^2 \cdot 1_S(\hat{X}^{\delta}_t, \hat{X}^{\delta}_{t^\delta})] dt \leq c \cdot \delta^2, \tag{25}
\]
which jointly with (24) yields the error estimate (10). For the proof of (25) we split the integral on the left hand side of (25) into four terms using the identities

\[ 1 = 1_{1}(\Theta^0_\delta)\hat{X}_t^\delta + 1_{\Theta^0_\delta \setminus \Theta^1_\delta} \hat{X}_t^\delta + 1_{\Theta^1_\delta \setminus \Theta^2_\delta} \hat{X}_t^\delta + 1_{\Theta^2_\delta} \hat{X}_t^\delta, \quad t \in [0,1], \]

and prove the upper bound \( c \cdot \delta^2 \) for each of the resulting terms employing uniform \( L_p \)-estimates of \( \hat{X}^\delta \), appropriate upper bounds for the probabilities that the increments \( |\hat{X}_t^\delta - \hat{X}_u^\delta| \) are large compared to the distance of the actual filtration generated by \( W \) and prove the upper bound \( c \) compared to the distance of \( \hat{X}_t^\delta \) from the set \( \Theta \) as well as estimates for the expected value of certain occupation time functionals of \( \hat{X}^\delta \). We add that for the proof of (25) it is crucial that the adaptive quasi-Milstein scheme \( \hat{X}^\delta \) uses smaller step sizes when it is close to the discontinuity points of \( \mu \).

For the proof of the estimate (11) we proceed similarly to the cost analysis of the adaptive Euler-Maruyama scheme in [22, Section 5].

We briefly describe the structure of this section. In Section 5.1 we provide properties of the random times \( \tau_i^\delta \) and \( t_i^\delta \) that are crucial for our proofs. In Section 5.2 we prove \( L_p \)-estimates of the adaptive quasi-Milstein scheme \( \hat{X}^\delta \). Section 5.3 contains estimates for the expected value of occupation time functionals of \( \hat{X}^\delta \) as well as estimates for the probabilities that the increments \( |\hat{X}_t^\delta - \hat{X}_u^\delta| \) of the adaptive quasi-Milstein scheme are large compared to the distance of the actual value of the scheme \( \hat{X}_t^\delta \) from the set \( \Theta \), which finally lead to the proof of the estimate (25), see Proposition 1. The results in Sections 5.2 and 5.3 are then used in Section 5.4 to derive the error estimate (10) in Theorem 1. Section 5.5 is devoted to the proof of the estimate (11) in Theorem 1.

Throughout the following we will employ the following facts, which are an immediate consequence of the assumptions (\( \mu_1 \)), (\( \mu_2 \)) and (\( \sigma_1 \)), (\( \sigma_2 \)) and the assumption that \( \mu \) is continuous. Namely, the function \( \mu \) is Lipschitz continuous on \( \mathbb{R} \), the functions \( \mu \) and \( \sigma \) satisfy a linear growth condition, i.e.

\[ \exists K \in (0,\infty) \forall x \in \mathbb{R}: \quad |\mu(x)| + |\sigma(x)| \leq K \cdot (1 + |x|), \]

the functions \( d_\mu \) and \( d_\sigma \) are bounded, i.e.

\[ \|d_\mu\|_\infty + \|d_\sigma\|_\infty < \infty, \]

and it holds

\[ \exists c \in (0,\infty) \forall f \in \{\mu,\sigma\} \forall i \in \{1,\ldots,k+1\} \forall x, y \in (\xi_{i-1},\xi_i): \quad |f(y) - f(x) - f'(x)(y-x)| \leq c \cdot |y-x|^2. \]

5.1. Properties of the random times \( \tau_i^\delta \) and \( t_i^\delta \). Let \( (\mathcal{F}_t)_{t \geq 0} \) denote the augmentation of the filtration generated by \( W \), i.e. for all \( t \geq 0 \),

\[ \mathcal{F}_t = \sigma(\{W_s: s \in [0,t]\} \cup \mathcal{N}), \]

where \( \mathcal{N} = \{N \in \mathcal{F}: \mathbb{P}(N) = 0\} \). For a stopping time \( \tau: \Omega \to [0,\infty) \) let \( \mathcal{F}_\tau \) denote the \( \sigma \)-algebra of \( \tau \)-past, i.e.

\[ \mathcal{F}_\tau = \{A \in \mathcal{F}: A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}. \]
Moreover, for a random time \( \tau : \Omega \to [0, \infty) \) define a stochastic process \( W^\tau : [0, \infty) \times \Omega \to \mathbb{R} \) by
\[
W^\tau_t = W_{\tau+t} - W_{\tau}, \quad t \geq 0.
\]
The following two lemmas provide the properties of the random times \( \tau^i \) and \( \hat{\tau}^i \) that are crucial for our proofs.

**Lemma 3.** Let \( \delta \in (0, \delta_0] \). Then for all \( i \in \mathbb{N}_0 \),
\[
\begin{align*}
(i) \quad & \tau^i \delta \text{ is a stopping time and } X^\delta_{\tau^i \delta} \text{ is } \mathcal{F}_{\tau^i \delta}/\mathcal{B}(\mathbb{R})\text{-measurable}, \\
(ii) \quad & \tau^i+1 \delta \text{ is } \mathcal{F}_{\tau^i+1 \delta}/\mathcal{B}(\mathbb{R})\text{-measurable}, \\
(iii) \quad & W^{\tau^i \delta} \text{ is a Brownian motion and independent of } \mathcal{F}_{\tau^i \delta}
\end{align*}
\]

and
\[
\begin{align*}
(iv) \quad & \tau^i \delta \wedge 1 \text{ is a stopping time and } X^\delta_{\tau^i \delta \wedge 1} \text{ is } \mathcal{F}_{\tau^i \delta \wedge 1}/\mathcal{B}(\mathbb{R})\text{-measurable}, \\
(v) \quad & \tau^i+1 \delta \wedge 1 \text{ is } \mathcal{F}_{\tau^i+1 \delta \wedge 1}/\mathcal{B}(\mathbb{R})\text{-measurable}, \\
(vi) \quad & W^{\tau^i \delta \wedge 1} \text{ is a Brownian motion and independent of } \mathcal{F}_{\tau^i \delta \wedge 1}.
\end{align*}
\]

**Proof.** We prove (i) by induction on \( i \in \mathbb{N}_0 \). Clearly, (i) holds for \( i = 0 \). Next, assume that (i) holds for some \( i \in \mathbb{N}_0 \). Then using the definition (7) of \( \tau^i+1 \delta \) we conclude that \( \tau^i+1 \delta \) is \( \mathcal{F}_{\tau^i \delta}/\mathcal{B}(\mathbb{R})\text{-measurable} \) and \( \tau^i \delta \geq \tau^i+1 \delta \). Applying [11, Exercise 1.2.14] we thus obtain that \( \tau^i+1 \delta \) is a stopping time. This in particular yields that \( W_{\tau^i+1 \delta} \) is \( \mathcal{F}_{\tau^i+1 \delta}/\mathcal{B}(\mathbb{R})\text{-measurable} \) and \( W_{\tau^i \delta} \) is \( \mathcal{F}_{\tau^i \delta}/\mathcal{B}(\mathbb{R})\text{-measurable} \). Thus, using the fact that \( \mathcal{F}_{\tau^i \delta} \subseteq \mathcal{F}_{\tau^i+1 \delta} \) as well as the induction assumption we obtain from the definition (7) of \( \hat{X}_{\tau^i \delta} \) that \( \hat{X}_{\tau^i \delta} \) is \( \mathcal{F}_{\tau^i \delta}/\mathcal{B}(\mathbb{R})\text{-measurable} \). The definition (7) of \( \tau^i \delta \) and (i) imply (ii). The strong Markov property of \( W \) yields (iii).

For the proof of (iv)-(vi) put \( s^i_0 = 0, X^\delta_{s^i_0} = x_0 \) and
\[
s^{i+1}_0 = (s^i + \mu(\theta)\cdot \delta^i_{s^i} + \sigma(\theta)\cdot (W^\delta_{s^i} - W^\delta_{s^i})) \wedge 1,
\]
\[
\hat{X}_{s^{i+1}} = X^\delta_{s^i} + \mu(\theta)\cdot (s^{i+1}_0 - s^i) + \sigma(\theta)\cdot (W^\delta_{s^{i+1}} - W^\delta_{s^i})
\]
\[
+ \frac{1}{2} \sigma d\sigma(\theta)\cdot (W^\delta_{s^{i+1}} - W^\delta_{s^i})^2 - (s^{i+1}_0 - s^i)
\]
for \( i \in \mathbb{N}_0 \) and proceed similarly to the proof of (i)-(iii). \( \square \)

**Lemma 4.** Let \( \delta \in (0, \delta_0] \) and \( t \in [0, \infty) \). Then \( W^{\hat{\tau}^i} \) is a Brownian motion and independent of \( \hat{X}^\delta_{\xi} \).

**Proof.** Clearly, \( W^{\hat{\tau}^i} \) is continuous. Employing Lemma 3(i),(ii),(iii) we obtain that for all \( A \in \mathcal{B}(C([0, \infty); \mathbb{R})) \),
\[
\mathbb{P}(W^{\hat{\tau}^i} \in A) = \sum_{i=0}^{\infty} \mathbb{P}(W^{\tau^i} \in A, \tau^i \leq t < \tau^{i+1} \delta) = \sum_{i=0}^{\infty} \mathbb{P}(W^{\tau^i} \in A) \cdot \mathbb{P}(\tau^i \leq t < \tau^{i+1} \delta) = \mathbb{P}(W \in A).
\]
Thus, $W^\delta$ is a Brownian motion. Applying the latter fact as well as Lemma 3(i),(ii),(iii) we conclude that for all $A \in \mathcal{B}(\mathbb{C}([0, \infty); \mathbb{R}))$ and all $B \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{P}(W^\delta \in A, \hat{X}_t^\delta \in B) = \sum_{i=0}^{\infty} \mathbb{P}(W^\delta_{\tau_i^\delta} \in A, \hat{X}_{\tau_i^\delta}^\delta \in B, \tau_i^\delta \leq t < \tau_{i+1}^\delta)$$

$$= \sum_{i=0}^{\infty} \mathbb{P}(W^\delta_{\tau_i^\delta} \in A) \cdot \mathbb{P}(\hat{X}_{\tau_i^\delta}^\delta \in B, \tau_i^\delta \leq t < \tau_{i+1}^\delta)$$

$$= \mathbb{P}(W^\delta \in A) \cdot \sum_{i=0}^{\infty} \mathbb{P}(\hat{X}_{\tau_i^\delta}^\delta \in B, \tau_i^\delta \leq t < \tau_{i+1}^\delta)$$

$$= \mathbb{P}(W^\delta \in A) \cdot \mathbb{P}(\hat{X}_t^\delta \in B),$$

which shows that $W^\delta$ and $\hat{X}_t^\delta$ are independent and completes the proof of the lemma. □

5.2. $L_p$ estimates of the adaptive quasi-Milstein scheme. Using Lemma 3(i) one can show in a straightforward way that for all $\delta \in (0, \delta_0]$ and all $t \in [0, \infty)$,

$$\hat{X}_t^\delta = x_0 + \int_0^t \mu(\hat{X}_s^\delta) \, ds + \int_0^t (\sigma(\hat{X}_s^\delta) + \sigma_d(\hat{X}_s^\delta) \cdot (W_s - W_s^\delta)) \, dW_s \quad \mathbb{P}\text{-a.s.}$$

Employing (29) we obtain the following uniform $L_p$-estimates for $\hat{X}^\delta$, $\delta \in (0, \delta_0]$.

Lemma 5. Let $p \in [1, \infty)$. Then there exists $c \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$,

$$\mathbb{E} \left[ \sup_{t \in [0,1]} |\hat{X}_t^\delta|^p \right]^{1/p} \leq c.$$

Moreover, there exists $c \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$, all $\Delta \in [0, 1]$ and all $t \in [0, 1 - \Delta]$,

$$\mathbb{E} \left[ \sup_{s \in [t, t+\Delta]} |\hat{X}_s^\delta - \hat{X}_t^\delta|^p \right]^{1/p} \leq c \cdot \sqrt{\Delta}.$$

Proof. We first show that for all $\delta \in (0, \delta_0]$ and all $i \in \mathbb{N}_0$,

$$\mathbb{E} \left[ |\hat{X}_{\tau_i^\delta}|^p \right] < \infty.$$

Let $\delta \in (0, \delta_0]$. We prove (32) by induction on $i \in \mathbb{N}_0$. Clearly, (32) holds for $i = 0$. Next, assume that (32) holds for some $i \in \mathbb{N}_0$. By (7), (20) and (27) there exist $c_1, c_2 \in (0, \infty)$ such that

$$|\hat{X}_{\tau_{i+1}^\delta}|^p \leq c_1 \cdot (|\hat{X}_{\tau_i^\delta}|^p + |\mu(\hat{X}_{\tau_i^\delta})|)^p \cdot \delta^p + |\sigma(\hat{X}_{\tau_i^\delta})|^p \cdot |W_{\tau_{i+1}^\delta} - W_{\tau_i^\delta}|^p$$

$$+ \frac{1}{2} |\sigma_d(\hat{X}_{\tau_i^\delta})|^p \cdot (|W_{\tau_{i+1}^\delta} - W_{\tau_i^\delta}|^{2p} + \delta^p)$$

$$\leq c_2 \cdot (1 + |\hat{X}_{\tau_i^\delta}|^p) \cdot (1 + \sup_{t \in [0, \delta]} |W_t^{\tau_i^\delta}|^{2p} + \sup_{t \in [0, \delta]} |W_t^{\tau_i^\delta}|^p).$$

Using the independence of $\hat{X}_{\tau_i^\delta}$ and $W^{\tau_i^\delta}$, the fact that $W^{\tau_i^\delta}$ is a Brownian motion as well as the induction assumption we therefore conclude that

$$\mathbb{E} \left[ |\hat{X}_{\tau_{i+1}^\delta}|^p \right] \leq c_2 \cdot (1 + \mathbb{E}[|\hat{X}_{\tau_i^\delta}|^p]) \cdot (1 + \mathbb{E}\left[ \sup_{t \in [0, \delta]} |W_t^{\tau_i^\delta}|^{2p} \right] + \mathbb{E}\left[ \sup_{t \in [0, \delta]} |W_t^{\tau_i^\delta}|^p \right]) < \infty,$$
which completes the proof of \((32)\).

For \(\delta \in (0, \delta_0]\) put

\[
n^\delta = \lceil \delta^{-2} \log^{-4}(1/\delta) \rceil.
\]

It follows from \((32)\) that for all \(\delta \in (0, \delta_0]\),

\[
\sup_{t \in [0,1]} \mathbb{E}[|\hat{X}^\delta_t|^p] = \sup_{t \in [0,1]} \sum_{i=0}^{n^\delta} \mathbb{E}[|\hat{X}^\delta_i|^p \cdot 1_{\{\tau^\delta_i = t\}}] \leq \sum_{i=0}^{n^\delta} \mathbb{E}[|\hat{X}^\delta_i|^p] < \infty.
\]

We next prove \((30)\). By \((29)\), for all \(\delta \in (0, \delta_0]\) and all \(t \in [0,1]\),

\[
\mathbb{E}\left[ \sup_{s \in [0,t]} |\hat{X}^\delta_s|^p \right] \leq 3^p \cdot |x_0|^p + 3^p \cdot \mathbb{E}\left[ \left| \int_0^t |\mu(\hat{X}^\delta_u)| \, du \right|^p \right] + 3^p \cdot \mathbb{E}\left[ \sup_{s \in [0,t]} \left( \int_s^t (\hat{X}^\delta_u + \sigma(\hat{X}^\delta_u) \cdot (W_u - W_{u^\delta}) \cdot dW_u)^p \right) \right].
\]

Using the Hölder inequality, the Burkholder-Davis-Gundy inequality, \((26)\) and \((27)\) we conclude that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0]\) and all \(t \in [0,1]\),

\[
\mathbb{E}\left[ \sup_{s \in [0,t]} |\hat{X}^\delta_s|^p \right] \leq c + c \cdot \int_0^t \mathbb{E}[|\hat{X}^\delta_u|^p] \, du + c \cdot \int_0^t \mathbb{E}[1 + |\hat{X}^\delta_u|^p] \cdot |W_u - W_{u^\delta}| \, du.
\]

Lemma 4 implies that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0]\) and all \(u \in [0,1]\),

\[
\mathbb{E}(1 + |\hat{X}^\delta_{u^\delta}|^p) \cdot |W_u - W_{u^\delta}| \leq \mathbb{E}(1 + |\hat{X}^\delta_{u^\delta}|^p) \cdot \mathbb{E}(\sup_{s \in [0,\delta]} |W_{u^\delta}|) = \mathbb{E}(1 + |\hat{X}^\delta_{u^\delta}|^p) \cdot \mathbb{E}(\sup_{s \in [0,\delta]} |W_s|) \leq c \cdot \mathbb{E}(1 + |\hat{X}^\delta_{u^\delta}|^p).
\]

Combining \((35)\) and \((36)\) we conclude that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0]\) and all \(t \in [0,1]\),

\[
\mathbb{E}\left[ \sup_{s \in [0,t]} |\hat{X}^\delta_s|^p \right] \leq c + c \cdot \int_0^t \mathbb{E}[|\hat{X}^\delta_u|^p] \, du.
\]

Employing \((34)\) we therefore obtain that for all \(\delta \in (0, \delta_0]\),

\[
\mathbb{E}\left[ \sup_{s \in [0,1]} |\hat{X}^\delta_s|^p \right] < \infty.
\]

Moreover, by \((37)\), for all \(\delta \in (0, \delta_0]\) and all \(t \in [0,1]\),

\[
\mathbb{E}\left[ \sup_{s \in [0,t]} |\hat{X}^\delta_s|^p \right] \leq c + c \cdot \int_0^t \mathbb{E}\left[ \sup_{u \in [0,s]} |\hat{X}^\delta_u|^p \right] \, ds.
\]

Applying the Gronwall inequality completes the proof of \((30)\).
For the proof of (31) observe that for all \( \delta \in (0, \delta_0] \), all \( \Delta \in [0, 1] \) and all \( t \in [0, 1 - \Delta] \),
\[
\mathbb{E} \left[ \sup_{s \in [t, t+\Delta]} |\hat{X}^\delta_s - \hat{X}^\delta_t|^p \right] \leq 2^{p} \cdot \mathbb{E} \left[ \left| \int_t^{t+\Delta} |\mu(\hat{X}^\delta_u)| \, du \right|^p \right] \\
+ 2^{p} \cdot \mathbb{E} \left[ \sup_{s \in [t, t+\Delta]} \left| \int_t^{s} \left( \sigma(\hat{X}^\delta_u) + \sigma \cdot d_{\sigma}(\hat{X}^\delta_u) \cdot (W_u - W_\Delta) \right) \, dW_u \right|^p \right]
\]
and employ the Hölder inequality, the Burkholder-Davis-Gundy inequality, (26), (27), (36) and Lemma 6.

\[ \boxed{\text{(31)}} \]

\[ \text{□} \]

5.3. Occupation time estimates for the adaptive quasi-Milstein scheme. We first provide an estimate for the expected value of occupation time functionals of \( \hat{X}^\delta \).

**Lemma 6.** Let \( f: [0, \infty) \to [0, \infty) \) be \( \mathcal{B}([0, \infty)) / \mathcal{B}([0, \infty)) \)-measurable and let \( \gamma > 0 \). Then there exists \( c \in (0, \infty) \) such that for all \( \delta \in (0, \delta_0] \) and all \( \varepsilon \in (0, \varepsilon_0] \),
\[
\mathbb{E} \left[ \int_0^1 f(d(\hat{X}^\delta_t, \Theta)) \cdot 1_{\Theta=\hat{X}^\delta_t} \, dt \right] \leq c \cdot \int_0^\varepsilon f(x) \, dx + c \cdot \sup_{x \in [0, c]} f(x) \cdot (\varepsilon^{3/2-\gamma} + \delta^{3/2-\gamma}).
\]

**Proof.** Clearly, it is enough to show that for all \( i \in \{1, \ldots, k\} \) there exists \( c \in (0, \infty) \) such that for all \( \delta \in (0, \delta_0] \) and all \( \varepsilon \in (0, \varepsilon_0] \),
\[
\mathbb{E} \left[ \int_0^1 f(|\hat{X}^\delta_t - \xi_i|) \cdot 1_{[\xi_i-\varepsilon, \xi_i+\varepsilon]}(\hat{X}^\delta_t) \, dt \right] \leq c \cdot \int_0^\varepsilon f(x) \, dx + c \cdot \sup_{x \in [0, c]} f(x) \cdot (\varepsilon^{3/2-\gamma} + \delta^{3/2-\gamma}).
\]

In the following fix \( i \in \{1, \ldots, k\} \).

Let \( \delta \in (0, \delta_0] \). For \( t \in [0, 1] \) put
\[
\Sigma_t^\delta = \sigma(\hat{X}^\delta_t) + \sigma \cdot d_{\sigma}(\hat{X}^\delta_t) \cdot (W_t - W_\Delta).
\]

Using (26), (27), (29) and Lemma 5 we conclude that \( \hat{X}^\delta \) is a continuous semi-martingale with quadratic variation
\[
\langle \hat{X}^\delta \rangle_t = \int_0^t (\Sigma_s^\delta)^2 \, ds, \quad t \in [0, 1].
\]

For \( a \in \mathbb{R} \) let \( L^a(\hat{X}^\delta) = (L^a_t(\hat{X}^\delta))_{t \in [0, 1]} \) denote the local time of \( \hat{X}^\delta \) at the point \( a \). Thus, for all \( a \in \mathbb{R} \) and all \( t \in [0, 1] \),
\[
|\hat{X}^\delta_t - a| = |x_0 - a| + \int_0^t \text{sgn}(\hat{X}^\delta_s - a) \cdot \mu(\hat{X}^\delta_s) \, ds + \int_0^t \text{sgn}(\hat{X}^\delta_s - a) \cdot \Sigma_s^\delta \, dW_s + L^a_t(\hat{X}^\delta),
\]
where \( \text{sgn}(y) = 1_{(0, \infty)}(y) - 1_{(-\infty, 0]}(y) \) for \( y \in \mathbb{R} \), see, e.g. [28] Chap. VI. Hence, for all \( a \in \mathbb{R} \) and all \( t \in [0, 1] \),
\[
L^a_t(\hat{X}^\delta) \leq |\hat{X}^\delta_t - x_0| + \int_0^t |\mu(\hat{X}^\delta_s)| \, ds + \left| \int_0^t \text{sgn}(\hat{X}^\delta_s - a) \cdot \Sigma_s^\delta \, dW_s \right|
\]
\[
\leq 2 \int_0^t |\mu(\hat{X}^\delta_s)| \, ds + \left| \int_0^t \Sigma_s^\delta \, dW_s \right| + \left| \int_0^t \text{sgn}(\hat{X}^\delta_s - a) \cdot \Sigma_s^\delta \, dW_s \right|.
\]

\[ \boxed{\text{(40)}} \]
Using \((26), (41)\), the Hölder inequality, the Burkholder-Davis-Gundy inequality and Lemma\(^5\) we obtain that there exist \(c_1, c_2 \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0)\), all \(a \in \mathbb{R}\) and all \(t \in [0, 1]\),

\[
\mathbb{E}[L_1^a(\hat{X}^\delta)] \leq c_1 \cdot \int_0^1 (1 + \mathbb{E}[|\hat{X}^\delta_{a,t}|]) \, ds + c_1 \left( \int_0^1 \mathbb{E}[|\Sigma^\delta_s|^2] \, ds \right)^{1/2} \\
\leq c_2 + c_1 \left( \int_0^1 \mathbb{E}[|\Sigma^\delta_s|^2] \, ds \right)^{1/2}.
\]

\label{eq:42}

Moreover, by \((26), (27), \text{Lemma} \ 4 \text{ and Lemma} \ 5\) there exist \(c_1, c_2 \in (0, \infty)\) such that for all \(s \in [0, 1]\) and all \(\delta \in (0, \delta_0)\),

\[
\mathbb{E}[|\Sigma^\delta_s|^2] \leq c_1 \cdot \mathbb{E}[(1 + |\hat{X}^\delta_{s,a}|)^2] \cdot (1 + |W_s - W^\delta_{a,s}|)^2 \\
\leq c_1 \cdot \mathbb{E}[(1 + |\hat{X}^\delta_{s,a}|)^2] \cdot \mathbb{E}[(1 + \sup_{u \in [0, \delta]} |W^\delta_{a,u}|)^2] \leq c_2.
\]

\label{eq:43}

Combining \eqref{eq:42} and \eqref{eq:43} we obtain that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0)\), all \(a \in \mathbb{R}\) and all \(t \in [0, 1]\),

\[
\mathbb{E}[L_1^a(\hat{X}^\delta)] \leq c.
\]

Using \((40), (44)\) and the occupation time formula it follows that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0)\) and all \(\varepsilon \in (0, \varepsilon_0)\),

\[
\mathbb{E} \left[ \int_0^1 f(|\hat{X}^\delta_t - \xi|) \cdot 1_{[\xi_t - \varepsilon, \xi_t + \varepsilon]}(\hat{X}^\delta_t) \cdot (\Sigma^\delta_t)^2 \, dt \right] \\
= \int_{\mathbb{R}} f(|a - \xi|) \cdot 1_{[\xi_t - \varepsilon, \xi_t + \varepsilon]}(a) \cdot \mathbb{E}[L_1^a(\hat{X}^\delta)] \, da \leq c \cdot \int_0^\varepsilon f(x) \, dx.
\]

\label{eq:45}

\[
\text{By} \ (26), \ (27) \ \text{and the Lipschitz continuity of} \ \sigma \ \text{we obtain that there exist} \ c_1, c_2 \in (0, \infty) \ \text{such that for all} \ \delta \in (0, \delta_0) \ \text{and all} \ t \in [0, 1],
\]

\[
|\sigma^2(\hat{X}^\delta_t) - (\Sigma^\delta_t)^2| \leq |\sigma(\hat{X}^\delta_t) - \Sigma^\delta_t| \cdot (|\sigma(\hat{X}^\delta_t)| + |\Sigma^\delta_t|) \\
\leq c_1 \cdot (|\sigma(\hat{X}^\delta_t) - \sigma(\hat{X}^\delta_{a,t})| + |\sigma(\hat{X}^\delta_t) - \Sigma^\delta_t|) \\
\cdot (1 + |\hat{X}^\delta_t| + (1 + |\hat{X}^\delta_{a,t}|) \cdot (1 + |W_t - W^\delta_{a,t}|)) \\
\leq c_2 \cdot (|\hat{X}^\delta_t| + |\hat{X}^\delta_{a,t}|) \cdot (1 + \sup_{u \in [0, \delta]} |W^\delta_{a,u}|).
\]

\[
\text{Thus, using the Hölder inequality, \text{Lemma} \ 5 \text{ and \text{Lemma} \ 4 we conclude that for all} \ q \in [1, \infty) \ \text{there exists} \ c \in (0, \infty) \ \text{such that for all} \ \delta \in (0, \delta_0) \ \text{and all} \ t \in [0, 1],}
\]

\[
\mathbb{E} \left[ |\sigma^2(\hat{X}^\delta_t) - (\Sigma^\delta_t)^2|^{q/2} \right]^{1/q} \leq c \cdot \sqrt{\delta}.
\]

\label{eq:46}

Since \(\sigma\) is continuous and \(\sigma(\xi_t) \neq 0\) there exist \(\kappa_i, \rho_i \in (0, \infty)\) such that

\[
\inf_{x \in \mathbb{R} : |x - \xi_t| \leq \rho_i} \sigma^2(x) \geq \kappa_i.
\]

\label{eq:47}
Using (45), (46), (47) and the Hölder inequality we obtain that for all $q \in (1,\infty)$ there exists $c \in (0,\infty)$ such that for all $\delta \in (0,\delta_0]$ and all $\varepsilon \in (0,\rho_i \wedge \varepsilon_0]$,

$$
\begin{align*}
E\left[\int_0^1 f(|\hat{X}_t^\delta - \xi_i|) \cdot 1_{[|\xi_i| - \varepsilon,|\xi_i| + \varepsilon]}(\hat{X}_t^\delta) \, dt\right] \\
\leq \frac{1}{\kappa} \cdot E\left[\int_0^1 f(|\hat{X}_t^\delta - \xi_i|) \cdot 1_{[|\xi_i| - \varepsilon,|\xi_i| + \varepsilon]}(\hat{X}_t^\delta) \cdot \sigma^2(\hat{X}_t^\delta) \, dt\right] \\
\leq \frac{1}{\kappa} \cdot E\left[\int_0^1 f(|\hat{X}_t^\delta - \xi_i|) \cdot 1_{[|\xi_i| - \varepsilon,|\xi_i| + \varepsilon]}(\hat{X}_t^\delta) \cdot (\Sigma_t^\delta)^2 \, dt\right] \\
+ \frac{1}{\kappa} \cdot E\left[\int_0^1 f(|\hat{X}_t^\delta - \xi_i|) \cdot 1_{[|\xi_i| - \varepsilon,|\xi_i| + \varepsilon]}(\hat{X}_t^\delta) \cdot \sigma^2(\hat{X}_t^\delta) - (\Sigma_t^\delta)^2 \right] \, dt \\
\leq c \cdot \int_0^\varepsilon f(x) \, dx + c \cdot \sup_{x \in [0,\varepsilon]} f(x) \cdot \sqrt{\delta} \cdot \left(\int_0^1 \mathbb{P}(|\hat{X}_t^\delta - \xi_i| \leq \varepsilon)\right)^{1/q} \, dt \\
\leq c \cdot \int_0^\varepsilon f(x) \, dx + c \cdot \sup_{x \in [0,\varepsilon]} f(x) \cdot \sqrt{\delta} \cdot \left(\int_0^1 \mathbb{P}(|\hat{X}_t^\delta - \xi_i| \leq \varepsilon)\right)^{1/q}.
\end{align*}
$$

(48)

Note that in the case of $f = 1$ the estimate (48) yields that for all $q \in (1,\infty)$ there exists $c \in (0,\infty)$ such that for all $\delta \in (0,\delta_0]$ and all $\varepsilon \in (0,\rho_i \wedge \varepsilon_0]$,

$$
\int_0^1 \mathbb{P}(|\hat{X}_t^\delta - \xi_i| \leq \varepsilon) \, dt \leq c \cdot \varepsilon + c \cdot \sqrt{\delta} \cdot \left(\int_0^1 \mathbb{P}(|\hat{X}_t^\delta - \xi_i| \leq \varepsilon)\right)^{1/q}.
$$

Thus, observing that $\varepsilon_0 \in (0,1]$ and $\delta_0 \in (0,1)$ and using the Young inequality we obtain that for all $q \in (1,2]$ there exist $c_1,c_2 \in (0,\infty)$ such that for all $\delta \in (0,\delta_0]$ and all $\varepsilon \in (0,\rho_i \wedge \varepsilon_0]$,

$$
\int_0^1 \mathbb{P}(|\hat{X}_t^\delta - \xi_i| \leq \varepsilon) \, dt \leq c_1 \cdot \varepsilon + c_2 \cdot \sqrt{\delta} \cdot \left(c_1 \cdot \varepsilon + c_1 \cdot \sqrt{\delta} \right)^{1/q} \leq c_2 \cdot \varepsilon + c_2 \cdot \delta^{\frac{1}{q} + \frac{1}{2q}}.
$$

(49)

It follows from (48) and (49) that for all $q \in (1,2]$ there exists $c \in (0,\infty)$ such that for all $\delta \in (0,\delta_0]$ and all $\varepsilon \in (0,\rho_i \wedge \varepsilon_0]$,

$$
E\left[\int_0^1 f(|\hat{X}_t^\delta - \xi_i|) \cdot 1_{[|\xi_i| - \varepsilon,|\xi_i| + \varepsilon]}(\hat{X}_t^\delta) \, dt\right] \leq c \cdot \int_0^\varepsilon f(x) \, dx + c \cdot \sup_{x \in [0,\varepsilon]} f(x) \cdot \sqrt{\delta} \cdot \left(\varepsilon^{\frac{1}{q}} + \delta^{\frac{1}{2q} + \frac{1}{2q^2}}\right).
$$

By the Young inequality, for all $q \in (1,2]$, all $\delta \in (0,\delta_0]$ and all $\varepsilon \in (0,\rho_i \wedge \varepsilon_0]$,

$$
\sqrt{\delta} \cdot \varepsilon^{\frac{1}{q}} \leq \frac{1}{3} \delta^{3/2} + \frac{2}{3} \varepsilon^{\frac{3}{2q}}.
$$

Combining the latter two estimates we conclude that for all $q \in (1,2]$ there exists $c \in (0,\infty)$ such that for all $\delta \in (0,\delta_0]$ and all $\varepsilon \in (0,\rho_i \wedge \varepsilon_0]$,

$$
E\left[\int_0^1 f(|\hat{X}_t^\delta - \xi_i|) \cdot 1_{[|\xi_i| - \varepsilon,|\xi_i| + \varepsilon]}(\hat{X}_t^\delta) \, dt\right] \leq c \cdot \int_0^\varepsilon f(x) \, dx + c \cdot \sup_{x \in [0,\varepsilon]} f(x) \cdot \left(\varepsilon^{\frac{3}{2q}} + \delta^{\frac{1}{2q} + \frac{1}{2q^2}}\right).
$$

This yields (39) and completes the proof of the lemma.
Lemma 7. Let \( \alpha \in (0, \infty) \) and \( q \in [1, \infty) \). Then there exist \( c_1, c_2, c_3 \in (0, \infty) \) such that for all \( \delta \in (0, \delta_0] \) and all \( t \in [0, 1] \),

(i) \( \mathbb{P}(|\hat{X}^\delta_t - \hat{X}^\delta_t| \geq \alpha \cdot \varepsilon^\delta_2, \hat{X}^\delta_t \in \Theta^\delta_2) \leq c_1 \cdot \delta^q \),

(ii) \( \mathbb{P}(|\hat{X}^\delta_t - \hat{X}^\delta_t| \geq \alpha \cdot d(\hat{X}^\delta_t, \Theta), \hat{X}^\delta_t \in \Theta^\delta_1 \setminus \Theta^\delta_2) \leq c_2 \cdot \delta^q \),

(iii) \( \mathbb{P}(|\hat{X}^\delta_t - \hat{X}^\delta_t| \geq \alpha \cdot \varepsilon^\delta_1, \hat{X}^\delta_t \in \Theta^\infty \setminus \Theta^\delta_1) \leq c_3 \cdot \delta^q \).

Proof. Define \( \Phi: \mathbb{R} \times C([0, \infty); \mathbb{R}) \to C([0, \infty); \mathbb{R}) \) by

\[
\Phi(y, u)(t) = y + \mu(y) \cdot t + \sigma(y) \cdot u(t) + \frac{1}{2} \sigma d^\alpha(y) \cdot (u^2(t) - t)
\]

for \( y \in \mathbb{R}, \ u \in C([0, \infty); \mathbb{R}) \) and \( t \in [0, \infty) \) and observe that there exists \( \kappa \in (0, \infty) \) such that for all \( y \in \Theta^\infty \), all \( u \in C([0, \infty); \mathbb{R}) \) and all \( t \in [0, \infty) \),

\[
|\Phi(y, u)(t) - y| \leq \kappa \cdot (t + |u(t)| + u^2(t)).
\]

We first proof (i). Using Lemma 4 we obtain that for all \( \delta \in (0, \delta_0] \) and all \( t \in [0, 1] \),

\[
\mathbb{P}(\hat{X}^\delta_t \in \Theta^\delta_2) = \mathbb{P}(\Phi(0, 0)(t - \delta) - \hat{X}^\delta_t \in \Theta^\delta_2)
\]

\[
\leq \mathbb{P}(\sup_{s \in [0, h^\delta(\hat{X}^\delta_t)])} \Phi(\hat{X}^\delta_t, W^\delta_\sigma)(s) - \hat{X}^\delta_t \geq \alpha \cdot \varepsilon^\delta_2, \hat{X}^\delta_t \in \Theta^\delta_2)
\]

\[
= \int_{\Theta^\delta_2} \mathbb{P}(\sup_{s \in [0, h^\delta(y)])} |\Phi(y, W)(s) - y| \geq \alpha \cdot \varepsilon^\delta_2) d \hat{X}^\delta_t(\sigma dy).
\]

By (50), for all \( \delta \in (0, \delta_0] \) and all \( y \in \Theta^\delta_2 \),

\[
\mathbb{P}(\sup_{s \in [0, h^\delta(y)])} |\Phi(y, W)(s) - y| \geq \alpha \cdot \varepsilon^\delta_2)
\]

\[
\leq \mathbb{P}(h^\delta(y) + \sup_{s \in [0, h^\delta(y)])} |W_s| + \sup_{s \in [0, h^\delta(y)])} W^2_s \geq \frac{\alpha \varepsilon^\delta_2}{\alpha \varepsilon^\delta_2}
\]

\[
\leq \mathbb{P}(h^\delta(y) + \sup_{s \in [0, h^\delta(y)])} |W_s| \geq \frac{\alpha \varepsilon^\delta_2}{\alpha \varepsilon^\delta_2}) + \mathbb{P}(\sup_{s \in [0, h^\delta(y)])} W^2_s \geq \frac{\alpha \varepsilon^\delta_2}{2\alpha \varepsilon^\delta_2}
\]

\[
= \mathbb{P}(\sup_{s \in [0, h^\delta(y)])} |W_s| \geq \frac{\alpha \varepsilon^\delta_2}{2\alpha \varepsilon^\delta_2} - h^\delta(y)) + \mathbb{P}(\sup_{s \in [0, h^\delta(y)])} |W_s| \geq \sqrt{\frac{\alpha \varepsilon^\delta_2}{2\alpha \varepsilon^\delta_2}}).
\]

Recall that for all \( \delta \in (0, \delta_0] \) and all \( y \in \Theta^\delta_2 \) we have \( h^\delta(y) = \delta^2 \log^4(1/\delta) \). Moreover, by Lemma 3.4], there exists \( c \in (0, \infty) \) such that for all \( u \in (0, \infty) \) and all \( x \in \mathbb{R} \),

\[
\mathbb{P}(\sup_{s \in [0, u]} |W_s| \geq x) \leq c \cdot e^{-\frac{x^2}{u}}.
\]
Hence, there exist \( c_1, c_2, c_3 \in (0, \infty) \) such that for all \( \delta \in (0, \delta_0) \) and all \( y \in \Theta^{e_\delta} \),
\[
\mathbb{P}\left( \sup_{s \in [0, h^\delta(y)]} |W_s| \geq \frac{\alpha x_\delta}{2\kappa} - h^\delta(y) \right) \leq c_1 \cdot e^{-\frac{\sqrt{\alpha x_\delta}}{2\kappa} \log^2 (1/\delta) + \frac{1}{2} \log^2 (1/\delta)} \leq c_2 \cdot \delta^q
\]
as well as
\[
\mathbb{P}\left( \sup_{s \in [0, h^\delta(y)]} |W_s| \geq \frac{\sqrt{\alpha x_\delta}}{2\kappa} \right) \leq c_1 \cdot e^{-\frac{\sqrt{\alpha x_\delta}}{2\kappa}} \leq c_3 \cdot \delta^q.
\]
The latter two estimates together with (51) and (52) imply (i).

We next prove (ii). Proceeding similarly to (51) and (52) we obtain that for all \( \delta \in (0, \delta_0) \) and all \( t \in [0, 1] \),
\[
\mathbb{P}(\hat{X}_t^\delta - \hat{X}_s^\delta) \geq \alpha \cdot d(\hat{X}_s^\delta, \Theta), \hat{X}_s^\delta \in \Theta^{e_\delta} \setminus \Theta^{e_0})
\]
\[
\leq \int_{\Theta^{e_\delta} \setminus \Theta^{e_0}} \mathbb{P}\left( \sup_{s \in [0, h^\delta(y)]} |\Phi(y, W)(s) - y| \geq \alpha \cdot d(y, \Theta) \right) \mathbb{P} \hat{X}_s^\delta (dy)
\]
and for all \( \delta \in (0, \delta_0) \) and all \( y \in \Theta^{e_\delta} \setminus \Theta^{e_0} \),
\[
\mathbb{P}\left( \sup_{s \in [0, h^\delta(y)]} |\Phi(y, W)(s) - y| \geq \alpha \cdot d(y, \Theta) \right)
\]
\[
\leq \mathbb{P}\left( \sup_{s \in [0, h^\delta(y)]} |W_s| \geq \frac{\alpha \delta}{2\kappa} - h^\delta(y) \right) + \mathbb{P}\left( \sup_{s \in [0, h^\delta(y)]} |W_s| \geq \frac{\sqrt{\alpha \delta}}{2\kappa} \right) \leq c_1 \cdot e^{-\frac{\sqrt{\alpha \delta}}{2\kappa} \log^2 (1/\delta) + \sqrt{h^\delta(y)}} \leq c_2 \cdot \delta^q
\]
as well as
\[
\mathbb{P}\left( \sup_{s \in [0, h^\delta(y)]} |W_s| \geq \frac{\sqrt{\alpha \delta}}{2\kappa} \right) \leq c_1 \cdot e^{-\frac{\sqrt{\alpha \delta}}{2\kappa}} \leq c_3 \cdot \delta^q.
\]
The latter two estimates together with (51) and (55) yield (ii).

We finally prove (iii). Proceeding similarly to (51) and (52) we obtain that for all \( \delta \in (0, \delta_0) \) and all \( t \in [0, 1] \),
\[
\mathbb{P}(\hat{X}_t^\delta - \hat{X}_s^\delta) \geq \alpha \cdot \varepsilon_1, \hat{X}_s^\delta \in \Theta^{e_0} \setminus \Theta^{e_\delta})
\]
\[
\leq \int_{\Theta^{e_0} \setminus \Theta^{e_\delta}} \mathbb{P}\left( \sup_{s \in [0, h^\delta(y)]} |\Phi(y, W)(s) - y| \geq \alpha \cdot \varepsilon_1 \right) \mathbb{P} \hat{X}_s^\delta (dy)
\]
and for all \( \delta \in (0, \delta_0) \) and all \( y \in \Theta^{e_0} \setminus \Theta^{e_\delta} \),
\[
\mathbb{P}\left( \sup_{s \in [0, h^\delta(y)]} |\Phi(y, W)(s) - y| \geq \alpha \cdot \varepsilon_1 \right)
\]
\[
\leq \mathbb{P}\left( \sup_{s \in [0, h^\delta(y)]} |W_s| \geq \frac{\alpha \varepsilon_1}{2\kappa} - h^\delta(y) \right) + \mathbb{P}\left( \sup_{s \in [0, h^\delta(y)]} |W_s| \geq \frac{\sqrt{\alpha \varepsilon_1}}{2\kappa} \right) \leq c_1 \cdot e^{-\frac{\sqrt{\alpha \varepsilon_1}}{2\kappa} \log^2 (1/\delta) + \sqrt{h^\delta(y)}} \leq c_2 \cdot \delta^q.
\]
Recall that for all $\delta \in (0, \delta_0]$ and all $y \in \Theta_{\varepsilon_0} \setminus \Theta_{\varepsilon_1}$ we have $h^\delta(y) = \delta$. Applying (53) we therefore obtain that there exist $c_1, c_2, c_3 \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$ and all $y \in \Theta_{\varepsilon_0} \setminus \Theta_{\varepsilon_1}$,

$$
P\left( \sup_{s \in [0, h^\delta(y)]} |W_s| \geq \frac{\alpha \varepsilon_1}{2\kappa} - h^\delta(y) \right) \leq c_1 \cdot e^{-\frac{\alpha}{2\kappa} \log^2(1/\delta) + \sqrt{\delta}} \leq c_2 \cdot \delta^q
$$

as well as

$$
P\left( \sup_{s \in [0, h^\delta(y)]} |W_s| \geq \frac{\sqrt{\alpha \varepsilon_1}}{\sqrt{2\kappa}} \right) \leq c_1 \cdot e^{-\frac{\sqrt{\alpha}}{2\kappa} \frac{\log(1/\delta)}{\delta^{1/4}}} \leq c_3 \cdot \delta^q.
$$

The latter two estimates together with (56) and (57) imply (iii) and complete the proof of the lemma.

\[ \square \]

Next, put

$$
S = \left( \bigcup_{\ell=1}^{k+1} (\xi_{\ell-1}, \xi_{\ell}) \right)^c
$$

and note that $S = \bigcup_{\ell=1}^{k} \{ (x, y) \in \mathbb{R}^2 : (x - \xi_{\ell}) \cdot (y - \xi_{\ell}) \leq 0 \}$. We are ready to establish the main result in this section, which provides a $p$-th mean estimate of the time average of $|\hat{X}^\delta_t - \hat{X}^\delta_{\ell}|^2$ subject to the condition that the pair $(\hat{X}^\delta_t, \hat{X}^\delta_{\ell})$ lies in the set $S$.

**Proposition 1.** Let $p \in [1, \infty)$. Then there exists $c \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$,

$$
\mathbb{E}\left[ \left| \int_0^1 |\hat{X}^\delta_t - \hat{X}^\delta_{\ell}|^2 \cdot 1_{S}(\hat{X}^\delta_t, \hat{X}^\delta_{\ell}) \, dt \right|^{1/p} \right] \leq c \cdot \delta^2.
$$

**Proof.** For $\delta \in (0, \delta_0]$ and $i \in \{1, 2, 3, 4\}$ let

$$
E_i^\delta = \mathbb{E}\left[ \left| \int_0^1 |\hat{X}^\delta_t - \hat{X}^\delta_{\ell}|^2 \cdot 1_{S}(\hat{X}^\delta_t, \hat{X}^\delta_{\ell}) \cdot 1_{O_i^\delta}(\hat{X}^\delta_{\ell}) \, dt \right| 
$$

where

$$
O_1^\delta = (\Theta_{\varepsilon_0})^c, \quad O_2^\delta = \Theta_{\varepsilon_0} \setminus \Theta_{\varepsilon_1}^\delta, \quad O_3^\delta = \Theta_{\varepsilon_1}^\delta \setminus \Theta_{\varepsilon_2}^\delta, \quad O_4^\delta = \Theta_{\varepsilon_2}^\delta.
$$

Then for all $\delta \in (0, \delta_0]$,

$$
\mathbb{E}\left[ \left| \int_0^1 |\hat{X}^\delta_t - \hat{X}^\delta_{\ell}|^2 \cdot 1_{S}(\hat{X}^\delta_t, \hat{X}^\delta_{\ell}) \, dt \right|^{p} \right] \leq \sum_{i=1}^{4} E_i^\delta.
$$

Below we show that for all $i \in \{1, 2, 3, 4\}$ there exists $c \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$,

$$
E_i^\delta \leq c \cdot \delta^{2p}.
$$

Clearly, (60) and (61) imply (59).

It remains to prove (61). We start with the analysis of $E_1^\delta$. For all $\delta \in (0, \delta_0]$ and all $t \in [0, 1]$,

$$
\{(\hat{X}^\delta_t, \hat{X}^\delta_{\ell}) \in S \} \cap \{\hat{X}^\delta_t \in O_1^\delta\} \subseteq \{|\hat{X}^\delta_t - \hat{X}^\delta_{\ell}| \geq \varepsilon_0\}.
$$
Thus, using the Markov inequality and Lemma 5, we obtain that there exist \(c_1, c_2 \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0]\),

\[
E_1^\delta \leq \int_0^1 \mathbb{E}[|X_s^\delta - \hat{X}_{s}\rangle | 2p \cdot \mathbb{1}_{(|X_s^\delta - \hat{X}_{s}| \geq \varepsilon_0)}] \, dt
\]

\[
\leq \int_0^1 \mathbb{E}[|X_s^\delta - \hat{X}_{s}\rangle | 4p \cdot (\mathbb{P}(|X_s^\delta - \hat{X}_{s}| \geq \varepsilon_0))^{1/2}] \, dt \leq \frac{1}{\varepsilon_0^{2p}} \int_0^1 \mathbb{E}[|X_s^\delta - \hat{X}_{s}\rangle | 4p] \, dt
\]

\[
\leq \frac{c_1}{p_2} \int_0^1 \mathbb{E}[|X_s^\delta - \hat{X}_{s}\rangle | 4p] + \mathbb{E}[|X_s^\delta - \hat{X}_{s}\rangle | 4p] \, dt \leq c_2 \cdot \delta^{2p},
\]

which shows that (61) holds for \(i = 1\).

We next estimate \(E_2^\delta\). Using Lemma 5, we obtain that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0]\),

\[
E_2^\delta \leq \int_0^1 \mathbb{E}[|X_s^\delta - \hat{X}_{s}\rangle | 4p] \cdot (\mathbb{P}((X_s^\delta, \hat{X}_{s}) \in S, \hat{X}_{s} \in O_2^\delta))^{1/2} \, dt
\]

\[
\leq c \cdot \delta^{p} \cdot \int_0^1 (\mathbb{P}((X_s^\delta, \hat{X}_{s}) \in S, \hat{X}_{s} \in \Theta_{\varepsilon_0} \setminus \Theta_{\varepsilon_1}^\delta))^{1/2} \, dt.
\]

(62)

Moreover, using Lemma 7(iii) with \(\alpha = 1\) and \(q = 2p\) we conclude that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0]\) and all \(t \in [0, 1]\),

\[
\mathbb{P}((X_s^\delta, \hat{X}_{s}) \in S, \hat{X}_{s} \in \Theta_{\varepsilon_0} \setminus \Theta_{\varepsilon_1}^\delta) \leq \mathbb{P}(|X_s^\delta - \hat{X}_{s}| \geq \varepsilon_1, \hat{X}_{s} \in \Theta_{\varepsilon_0} \setminus \Theta_{\varepsilon_1}^\delta) \leq c \cdot \delta^{2p}.
\]

The latter estimate together with (62) yields (61) for \(i = 2\).

We next estimate \(E_3^\delta\). Similarly to (62) we obtain that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0]\),

\[
E_3^\delta \leq c \cdot \delta^{p} \cdot \int_0^1 (\mathbb{P}((X_s^\delta, \hat{X}_{s}) \in S, \hat{X}_{s} \in \Theta_{\varepsilon_1} \setminus \Theta_{\varepsilon_2}^\delta))^{1/2} \, dt.
\]

(63)

Moreover, using Lemma 7(ii) with \(\alpha = 1\) and \(q = 2p\) we conclude that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0]\) and all \(t \in [0, 1]\),

\[
\mathbb{P}((X_s^\delta, \hat{X}_{s}) \in S, \hat{X}_{s} \in \Theta_{\varepsilon_1} \setminus \Theta_{\varepsilon_2}^\delta) \leq \mathbb{P}(|X_s^\delta - \hat{X}_{s}| \geq \varepsilon_1, \hat{X}_{s} \in \Theta_{\varepsilon_1} \setminus \Theta_{\varepsilon_2}^\delta) \leq c \cdot \delta^{2p}.
\]

The latter estimate together with (63) yields (61) for \(i = 3\).

We finally estimate \(E_4^\delta\). Note that for all \(\delta \in (0, \delta_0]\), all \(t \in [0, 1]\) and all \(\omega \in \{\hat{X}_{s} \in \Theta_{\varepsilon_2}^\delta\}\) we have \(t - \delta^2(\omega) \leq \delta^2 \cdot \log^2(1/\delta)\). Thus, using Lemma 5 we obtain that there exist \(c_1, c_2 \in (0, \infty)\)
such that for all $\delta \in (0, \delta_0]$,
\[
E_4^\delta \leq \int_0^1 \mathbb{E}[|\hat{X}_t^\delta - \hat{X}_t^\delta|^{2p} \cdot 1_{\Theta_2^\delta(\hat{X}_t^\delta)}] \, dt
\leq \int_0^1 \mathbb{E}[|\hat{X}_t^\delta - \hat{X}_t^\delta|^{4p} \cdot 1_{\Theta_2^\delta(\hat{X}_t^\delta)}]^{1/2} \cdot (\mathbb{P}(\hat{X}_t^\delta \in \Theta_2^\delta))^{1/2} \, dt
\]
\[
\leq c_1 \int_0^1 \mathbb{E}\left[ \sup_{s \in [0, (t - \delta^2 \log^4(1/\delta)) \cdot t]} |\hat{X}_s^\delta - \hat{X}_s^\delta|^{1/2} \cdot (\mathbb{P}(\hat{X}_s^\delta \in \Theta_2^\delta))^{1/2} \, dt
\leq c_2 \cdot \delta^{2p} \cdot \log^{4p}(1/\delta) \cdot \int_0^1 (\mathbb{P}(\hat{X}_t^\delta \in \Theta_2^\delta))^{1/2} \, dt
\leq c_2 \cdot \delta^{2p} \cdot \log^{4p}(1/\delta) \cdot \left( \int_0^1 \mathbb{P}(\hat{X}_t^\delta \in \Theta_2^\delta) \, dt \right)^{1/2}.
\]

Employing Lemma \[\text{[5]}\] with $f = 1$ and $\gamma = 1/2$ and Lemma \[\text{[7]}(i)\] with $\alpha = 1$ and $q = 1$ we obtain that there exist $c \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$,
\[
\int_0^1 \mathbb{P}(\hat{X}_t^\delta \in \Theta_2^\delta) \, dt = \int_0^1 \mathbb{P}(\hat{X}_t^\delta - \hat{X}_t^\delta < \varepsilon_2^\delta, \hat{X}_t^\delta \in \Theta_2^\delta) \, dt
\leq c \cdot \varepsilon_2^\delta + c \cdot \delta \leq 2c \cdot \delta \cdot \log^4(1/\delta).
\]

The latter estimate together with (64) implies that there exist $c_1, c_2 \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$,
\[
E_4^\delta \leq c_1 \cdot \delta^{2p+1/2} \cdot \log^{4p+2}(1/\delta) \leq c_2 \cdot \delta^{2p},
\]
which shows that (64) holds for $i = 4$ and completes the proof of the proposition.

**5.4. Convergence analysis.** In this subsection we prove the estimate (10). Clearly, it is enough to consider the case $p \in \mathbb{N} \setminus \{1\}$. For $\delta \in (0, \delta_0]$ and $t \in [0, 1]$ we put
\[
A_t = \int_0^t \mu(X_s) \, ds, \quad \tilde{A}_t^\delta = \int_0^t \mu(\hat{X}_s^\delta) \, ds
\]
and
\[
B_t = \int_0^t \sigma(X_s) \, dW_s, \quad \tilde{B}_t^\delta = \int_0^t (\sigma(\hat{X}_s^\delta) + \sigma_d(\hat{X}_s^\delta) \cdot (W_s - W_s^\delta)) \, dW_s
\]
as well as
\[
U_t^\delta = \int_0^t \sigma_d(\hat{X}_s^\delta) \cdot (W_s - W_s^\delta) \, ds
\]
and we use the decomposition
\[
X_t - \hat{X}_t^\delta = (A_t - \tilde{A}_t^\delta - U_t^\delta) + (B_t - \tilde{B}_t^\delta) + U_t^\delta.
\]
Recall the definition (58) of the set $S$. For all $\delta \in (0, \delta_0]$, all $s \in [0, 1]$ and all $f \in \{\mu, \sigma\}$ we have

$$
|f(X_s) - f(\hat{X}_s^\delta) - \sigma f(\hat{X}_s^\delta) \cdot (W_s - W_{s^\delta})| \\
\leq |f(X_s) - f(\hat{X}_s)| + |f(\hat{X}_s) - f(\hat{X}_s^\delta) - d_f(\hat{X}_s^\delta) \cdot (\hat{X}_s^\delta - \hat{X}_s)| \cdot 1_{S^c}(\hat{X}_s^\delta, \hat{X}_s) \\
+ |d_f(\hat{X}_s) - f(\hat{X}_s^\delta) - d_f(\hat{X}_s^\delta) \cdot (\hat{X}_s^\delta - \hat{X}_s)| \cdot 1_S(\hat{X}_s^\delta, \hat{X}_s) \\
+ |d_f(\hat{X}_s^\delta) \cdot (\mu(\hat{X}_s^\delta)(s - s^\delta) + \frac{1}{2} \sigma d_s(\hat{X}_s^\delta) \cdot ((W_s - W_{s^\delta})^2 - (s - s^\delta)))|.
$$

Using the Lipschitz continuity of $\mu$ and $\sigma$ as well as (26), (27) and (28) we thus obtain that there exists $c \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$, all $s \in [0, 1]$ and all $f \in \{\mu, \sigma\},$

$$
|f(X_s) - f(\hat{X}_s^\delta) - \sigma d_f(\hat{X}_s^\delta) \cdot (W_s - W_{s^\delta})| \\
\leq c \cdot |X_s - \hat{X}_s^\delta| + c \cdot |\hat{X}_s^\delta - \hat{X}_s| + c \cdot |\hat{X}_s^\delta - \hat{X}_s^\delta| \cdot 1_S(\hat{X}_s^\delta, \hat{X}_s^\delta) \\
+ c \cdot (1 + |\hat{X}_s^\delta|) \cdot (\delta + |W_s - W_{s^\delta}|^2).
$$

Employing (67), Lemma 5 and Proposition 1 we conclude that there exist $c_1, c_2, c_3 \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$ and all $t \in [0, 1],$

$$
\mathbb{E}\left[ \sup_{0 \leq s \leq t} |A_s - \tilde{A}_s^\delta - U_s^\delta|^p \right] \\
\leq \mathbb{E}\left[ \int_0^t |\mu(X_s) - \mu(\hat{X}_s^\delta) - \sigma d_\mu(\hat{X}_s^\delta) \cdot (W_s - W_{s^\delta})| \, ds \right]^p \\
\leq c_1 \cdot \int_0^t \mathbb{E}[|X_s - \hat{X}_s^\delta|^p] \, ds + c_1 \cdot \int_0^t \mathbb{E}[|\hat{X}_s^\delta - \hat{X}_s|^2] \, ds \\
+ c_1 \cdot \mathbb{E}\left[ \int_0^t |\hat{X}_s^\delta - \hat{X}_s| \cdot 1_S(\hat{X}_s^\delta, \hat{X}_s^\delta) \, ds \right]^p \\
+ c_1 \cdot \int_0^t \mathbb{E}[1 + \sup_{u \in [0, 1]} |\hat{X}_u^\delta|^{2p}]^{1/2} \cdot \mathbb{E}[\delta^{2p} + \sup_{u \in [0, \nu(s - \delta), s]} |W_u - W_u|^4]^{1/2} \, ds \\
\leq c_1 \cdot \int_0^t \mathbb{E}[|X_s - \hat{X}_s^\delta|^p] \, ds + c_1 \cdot \mathbb{E}\left[ \int_0^t |\hat{X}_s^\delta - \hat{X}_s|^4 \cdot 1_S(\hat{X}_s^\delta, \hat{X}_s^\delta) \, ds \right]^{p/2} \\
+ c_2 \cdot \int_0^t \mathbb{E}[\sup_{u \in [0, \nu(s - \delta), s]} |\hat{X}_u^\delta - \hat{X}_{0\nu(s - \delta)}|^{2p}] \, ds + c_2 \cdot \delta^p \\
\leq c_1 \cdot \int_0^t \mathbb{E}[|X_s - \hat{X}_s^\delta|^p] \, ds + c_3 \cdot \delta^p.
$$
Using the Burkholder-Davis-Gundy inequality, (67), Lemma 5 and Proposition 1 we obtain that there exist \(c_1, c_2, c_3 \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0]\) and all \(t \in [0, 1],\)

\[
E\left[\sup_{0 \leq s \leq t} |B_s - \hat{B}_s|^p\right] \\
\leq c_1 \cdot E\left[\int_0^t |\sigma(X_s) - \sigma(\hat{X}_s) - \sigma d_s (\hat{X}_s^\delta - W_s)|^2 ds\right]^{\frac{p}{2}} \\
\leq c_2 \cdot \int_0^t E[|X_s - \hat{X}_s|^p] ds + c_2 \cdot \int_0^t E[|\hat{X}_s^\delta - \hat{X}_s|^p] ds \\
+ c_2 \cdot \int_0^t E[1 + \sup_{u \in [0,1]} |\hat{X}_u^\delta|^{2p}]^{\frac{1}{2}} \cdot E[\delta^p + \sup_{u \in [0, (s-\delta), s]} |W_s - W_u|^p]^{\frac{1}{2}} ds
\]

(69)

\[
\leq c_2 \cdot \int_0^t E[|X_s - \hat{X}_s|^p] ds + c_3 \cdot \delta^p.
\]

Combining (66) with (68) and (69) we conclude that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0]\) and all \(t \in [0, 1],\)

(70) \[
E\left[\sup_{0 \leq s \leq t} |X_t - \hat{X}_t|^p\right] \\
\leq c \cdot \int_0^t E\left[\sup_{0 \leq u \leq s} |X_u - \hat{X}_u|^p\right] ds + c \cdot \delta^p + E\left[\sup_{0 \leq s \leq t} |U_s^\delta|^p\right].
\]

Note that \(E[\sup_{0 \leq u \leq 1} |X_u - \hat{X}_u|^p] < \infty\) due to (41) and Lemma 5. Below we show that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0],\)

(71) \[
E\left[\sup_{0 \leq s \leq 1} |U_s^\delta|^p\right] \leq c \cdot \delta^p.
\]

Inserting (71) into (70) and applying the Gronwall inequality then yields the error estimate (10) in Theorem 1

We turn to the proof of (71). Clearly, for all \(\delta \in (0, \delta_0], all i \in \mathbb{N}_0\) and all \(s \in [\tau_i^\delta, \tau_{i+1}^\delta],\)

(72) \[
U_{s \wedge 1}^\delta = U_{\tau_i^\delta \wedge 1}^\delta + \sigma d_{\mu}(\hat{X}_{\tau_i^\delta \wedge 1}^\delta) \cdot \int_{\tau_i^\delta \wedge 1}^{s \wedge 1} (W_u - W_{\tau^\delta_i}) du.
\]

For \(\delta \in (0, \delta_0]\) let \(n^\delta\) be given by (33). Using (26) and (27) we obtain from (72) that there exists \(c \in (0, \infty)\) such that for all \(\delta \in (0, \delta_0]\),

\[
\sup_{0 \leq s \leq 1} |U_s^\delta| = \max_{i=0, \ldots, n^\delta - 1} \sup_{0 \leq s \leq s^\delta_{i+1}} |U_s^\delta| \\
\leq \max_{i=0, \ldots, n^\delta - 1} |U_{\tau_i^\delta \wedge 1}^\delta| + \max_{i=0, \ldots, n^\delta - 1} |\sigma d_{\mu}(\hat{X}_{\tau_i^\delta \wedge 1}^\delta)| \cdot \int_{\tau_i^\delta \wedge 1}^{\tau_{i+1}^\delta \wedge 1} |W_u - W_{\tau^\delta_i}| du
\]

(73) \[
\leq \max_{i=0, \ldots, n^\delta - 1} |U_{\tau_i^\delta \wedge 1}^\delta| + c \cdot (1 + \sup_{0 \leq s \leq 1} |\hat{X}_s^\delta|) \cdot \max_{i=0, \ldots, n^\delta - 1} \int_{\tau_i^\delta \wedge 1}^{\tau_{i+1}^\delta \wedge 1} |W_u - W_{\tau^\delta_i}| du.
\]
Let \( \delta \in (0, \delta_0] \). Employing (72), (26), (27) and Lemma 5 one can show by induction on \( i \in \{0, \ldots, n^\delta - 1\} \) that \( \mathbb{E}[U_{\tau^\delta_{i+1} \wedge 1}^\delta] < \infty \) for all \( i \in \{0, \ldots, n^\delta - 1\} \). Moreover, using Lemma 3(vi),(v) one can show by induction on \( i \in \{0, \ldots, n^\delta - 1\} \) that \( U_{\tau^\delta_{i+1} \wedge 1}^\delta \) is \( \mathcal{F}_{\tau^\delta_{i+1} \wedge 1} / \mathcal{B}(\mathbb{R}) \)-measurable for all \( i \in \{0, \ldots, n^\delta - 1\} \). Finally, observe that for all \( i \in \{0, \ldots, n^\delta - 2\} \),

\[
\int_{\tau^\delta_i \wedge 1}^{\tau^\delta_{i+1} \wedge 1} (W_u - W_{\tau^\delta_i}) \, du = \int_0^{(\tau^\delta_{i+1} \wedge 1) - (\tau^\delta_i \wedge 1)} W_u^{\tau^\delta_i \wedge 1} \, du.
\]

Using Lemma 3(vi),(v),(vi) we therefore obtain that for all \( i \in \{0, \ldots, n^\delta - 2\} \),

\[
\mathbb{E}[U_{\tau^\delta_{i+1} \wedge 1}^\delta | \mathcal{F}_{\tau^\delta_i \wedge 1}] = U_{\tau^\delta_i \wedge 1}^\delta + \sigma d_\mu(\hat{X}_{\tau^\delta_i \wedge 1}^\delta) \cdot \int_0^{(\tau^\delta_{i+1} \wedge 1) - (\tau^\delta_i \wedge 1)} \mathbb{E}[W_u^{\tau^\delta_i \wedge 1} | \mathcal{F}_{\tau^\delta_i \wedge 1}] \, du = U_{\tau^\delta_i \wedge 1}^\delta.
\]

Hence, the sequence \( (U_{\tau^\delta_i \wedge 1}^\delta, \mathcal{F}_{\tau^\delta_i \wedge 1})_{i \in \{0, \ldots, n^\delta - 1\}} \) is a martingale.

Employing the Burkholder-Davis-Gundy inequality as well as (26), (27), Lemma 5 and Lemma 4 we conclude that there exist \( c_1, c_2, c_3 \in (0, \infty) \) such that for all \( \delta \in (0, \delta_0] \),

\[
\mathbb{E}\left[\max_{i=0, \ldots, n^\delta - 1} |U_{\tau^\delta_i \wedge 1}^\delta|^p\right] \\
\leq \mathbb{E}\left[\left(\sum_{i=0}^{n^\delta - 1} \left(\sigma d_\mu(\hat{X}_{\tau^\delta_i \wedge 1}^\delta) \cdot \int_{\tau^\delta_i \wedge 1}^{\tau^\delta_{i+1} \wedge 1} (W_u - W_{\tau^\delta_i}) \, du\right)^2\right)^{p/2}\right] \\
\leq c_1 \cdot \mathbb{E}\left[(1 + \sup_{0 \leq s \leq 1} |\hat{X}_s^\delta|^{2p})\right]^{1/2} \cdot \mathbb{E}\left[\left(\sum_{i=0}^{n^\delta - 1} \left(\int_{\tau^\delta_i \wedge 1}^{\tau^\delta_{i+1} \wedge 1} (W_u - W_{\tau^\delta_i}) \, du\right)^2\right)^{p/2}\right] \\
\leq c_1 \cdot \delta^{p/2} \cdot \mathbb{E}\left[\left(\int_0^1 (W_u - W_{\tau^\delta_i})^2 \, du\right)^{p/2}\right] \\
\leq c_2 \cdot \delta^{p/2} \cdot \mathbb{E}\left[\left(\int_0^1 \mathbb{E}\left[\sup_{s \in [0, \delta]} |W_s^u|^{2p}\right] \, du\right)^{1/2}\right] \\
\leq c_3 \cdot \delta^p.
\]
Furthermore, using Lemma 5 and Lemma 4 we obtain that there exists \( c_1 \in (0, \delta_0) \),

\[
\mathbb{E} \left[ \left( 1 + \sup_{0 \leq s \leq 1} |\hat{X}_s^\delta| \right) \cdot \max_{i=0, \ldots, n} \int_{\tau_i^\delta \wedge 1}^{\tau_{i+1}^\delta \wedge 1} |W_u - W_{\tau_i^\delta}| \, du \right]^{2p}
\]

\[
\leq \mathbb{E} \left[ (1 + \sup_{0 \leq s \leq 1} |\hat{X}_s^\delta|) \cdot \left( \sum_{i=0}^{n} \int_{\tau_i^\delta \wedge 1}^{\tau_{i+1}^\delta \wedge 1} |W_u - W_{\tau_i^\delta}| \, du \right)^2 \right]^{1/2}
\]

\[
\leq c_1 \cdot \mathbb{E} \left[ \sum_{i=0}^{n} \left( (\tau_i^\delta \wedge 1) - (\tau_{i+1}^\delta \wedge 1) \right)^{2p-1} \cdot \int_{\tau_i^\delta \wedge 1}^{\tau_{i+1}^\delta \wedge 1} |W_u - W_{\tau_i^\delta}| \, du \right]^{1/2}
\]

\[
\leq c_1 \cdot \delta^{2p-1} \cdot \left( \int_0^1 \mathbb{E} \left[ \sup_{0 \leq s \leq 1} |\hat{X}_s^\delta| \right] \, ds \right)^{1/2} \leq c_1 \cdot \delta^{\frac{2p-1}{2}} \leq c_1 \cdot \delta^p.
\]

Combining (73) with (74) and (75) yields (71) and completes the proof of the estimate (10) in Theorem 1.

5.5. Cost analysis. In this subsection we proof the estimate (11). Clearly, for all \( \delta \in (0, \delta_0) \) and all \( i \in \mathbb{N} \) we have

\[
1 = \int_{\tau_{i-1}^\delta}^{\tau_i^\delta} \frac{1}{\tau_i^\delta - \tau_{i-1}^\delta} \, dt = \int_{\tau_{i-1}^\delta}^{\tau_i^\delta} \frac{1}{h^\delta(\hat{X}_s^\delta)} \, dt.
\]

Thus, for all \( \delta \in (0, \delta_0) \),

\[
N(\hat{X}_1^\delta) = 1 + \sum_{i=1}^{\infty} 1_{\{\tau_i^\delta < 1\}} = 1 + \sum_{i=1}^{\infty} 1_{\{\tau_i^\delta < 1\}} \cdot \int_{\tau_{i-1}^\delta}^{\tau_i^\delta} \frac{1}{h^\delta(\hat{X}_s^\delta)} \, dt \leq 1 + \int_0^1 \frac{1}{h^\delta(\hat{X}_s^\delta)} \, ds.
\]

For \( \delta \in (0, \delta_0) \) and \( i \in \{1, 2, 3\} \) put

\[
I_i^\delta = \mathbb{E} \left[ \int_0^1 \frac{1}{h^\delta(\hat{X}_s^\delta)} \cdot \mathbb{I}_{O_i^\delta}(\hat{X}_s^\delta) \, ds \right],
\]

where

\[
O_1^\delta = (\Theta^{\xi_1})^c, \quad O_2^\delta = \Theta^{\xi_1} \setminus \Theta^{\xi_2}, \quad O_3^\delta = \Theta^{\xi_2}.
\]

Then for all \( \delta \in (0, \delta_0) \),

\[
\mathbb{E}[N(\hat{X}_1^\delta)] \leq 1 + \sum_{i=1}^{3} I_i^\delta.
\]

Clearly,

\[
I_1^\delta = \delta^{-1} \cdot \int_0^1 P(\hat{X}_s^\delta \in (\Theta^{\xi_1})^c) \, ds \leq \delta^{-1}.
\]

Moreover, observing (65) we obtain that there exists \( c \in (0, \infty) \) such that for all \( \delta \in (0, \delta_0) \),

\[
I_3^\delta = \delta^{-2} \cdot \log^{-4}(1/\delta) \cdot \int_0^1 P(\hat{X}_s^\delta \in \Theta^{\xi_2}) \, ds \leq c \cdot \delta^{-1}.
\]
Below we show that there exists $c \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$,

\begin{equation}
I^\delta_2 \leq c \cdot \delta^{-1}.
\end{equation}

Combining (76) to (79) we obtain (11).

It remains to prove (79). For $\delta \in (0, \delta_0]$ and $t \in [0, 1]$ put

$$D^\delta_t = \{|\hat{X}^\delta_t - \tilde{X}^\delta_t| \leq \frac{1}{2}d(\hat{X}^\delta_t, \Theta)\}.$$  

Clearly, for all $\delta \in (0, \delta_0]$,

\begin{equation}
I^\delta_2 = I^\delta_{2,1} + I^\delta_{2,2},
\end{equation}

where

$$I^\delta_{2,1} = \mathbb{E}\left[\int_0^1 \frac{1}{h^\delta(X^\delta_{\xi})} \cdot 1_{O^\delta_{1}}(\hat{X}^\delta_{\xi}) \cdot 1_{D^\delta_t} \, dt\right], \quad I^\delta_{2,2} = \mathbb{E}\left[\int_0^1 \frac{1}{h^\delta(X^\delta_{\xi})} \cdot 1_{O^\delta_{1}}(\hat{X}^\delta_{\xi}) \cdot 1_{(D^\delta_t)^c} \, dt\right].$$

Observing the fact that the distance function $d(\cdot, \Theta): \mathbb{R} \to [0, \infty)$ is Lipschitz continuous with Lipschitz seminorm 1, i.e. for all $x, y \in \mathbb{R}$,

$$|d(x, \Theta) - d(y, \Theta)| \leq |x - y|,$$

we obtain that for all $\delta \in (0, \delta_0]$ and all $t \in [0, 1]$,

$$\{\hat{X}^\delta_t \in O^\delta_{1}\} \cap D^\delta_t \subseteq \{\hat{X}^\delta_t \in \Theta^\delta_{\frac{3}{4}} \setminus \Theta^\delta_{\frac{1}{2}}\} \cap \{\frac{1}{2}d(\hat{X}^\delta_t, \Theta) \leq d(\hat{X}^\delta_t, \Theta) \leq \frac{3}{2}d(\hat{X}^\delta_t, \Theta)\}.$$  

Thus, for all $\delta \in (0, \delta_0]$,

\begin{equation}
I^\delta_{2,1} = \log^4(1/\delta) \cdot \mathbb{E}\left[\int_0^1 \frac{1}{d(\hat{X}^\delta_t, \Theta)^2} \cdot 1_{O^\delta_{1}}(\hat{X}^\delta_t) \cdot 1_{D^\delta_t} \, dt\right]
\end{equation}

\begin{equation}
\quad \leq \frac{9}{4} \log^4(1/\delta) \cdot \mathbb{E}\left[\int_0^1 \frac{1}{d(\hat{X}^\delta_t, \Theta)^2} \cdot 1_{\Theta^\delta_{\frac{3}{4}} \setminus \Theta^\delta_{\frac{1}{2}}} (\hat{X}^\delta_t) \, dt\right].
\end{equation}

For $\delta \in (0, \delta_0]$ put $\bar{\epsilon}^\delta = \delta^{3/4} \cdot \log^3(1/\delta)$ and observe that $\bar{\epsilon}^\delta_2 \leq \bar{\epsilon}^\delta \leq \bar{\epsilon}^\delta_1$ for all $\delta \in (0, \delta_0]$. Hence, (81) implies that for all $\delta \in (0, \delta_0]$,

$$I^\delta_{2,1} \leq \frac{9}{4} \log^4(1/\delta) \cdot \mathbb{E}\left[\int_0^1 \frac{1}{d(\hat{X}^\delta_t, \Theta)^2} \cdot 1_{\Theta^\delta_{\frac{3}{4}} \setminus \Theta^\delta_{\frac{1}{2}}} (\hat{X}^\delta_t) \, dt\right] +$$

$$+ \frac{9}{4} \log^4(1/\delta) \cdot \mathbb{E}\left[\int_0^1 \frac{1}{d(\hat{X}^\delta_t, \Theta)^2} \cdot 1_{\Theta^\delta_{\frac{3}{4}} \setminus \Theta^\delta_{\frac{3}{2}}} (\hat{X}^\delta_t) \, dt\right]$$

\begin{equation}
\quad \leq \frac{9}{4} \log^4(1/\delta) \cdot \mathbb{E}\left[\int_0^1 \frac{1}{\max(\bar{\epsilon}^\delta, d(\hat{X}^\delta_t, \Theta))^2} \cdot 1_{\Theta^\delta_{\frac{3}{4}}} (\hat{X}^\delta_t) \, dt\right] +$$

$$+ \frac{9}{4} \log^4(1/\delta) \cdot \mathbb{E}\left[\int_0^1 \frac{1}{\max(\bar{\epsilon}^\delta_2, d(\hat{X}^\delta_t, \Theta))^2} \cdot 1_{\Theta^\delta_{\frac{3}{4}}} (\hat{X}^\delta_t) \, dt\right].$$
Applying Lemma 6 with $f = 1/\max(\varepsilon^2, \gamma^2)$ and $\gamma = 1/2$ and with $f = 1/\max(\varepsilon^2, \gamma^2)$ and $\gamma = 1/6$ we therefore conclude that there exist $c_1, c_2, c_3 \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$,

$$I_{2,1}^{\delta} \leq c_1 \cdot \log^4(1/\delta) \cdot \left( \int_0^{\frac{3}{2} \varepsilon_1} \frac{1}{\max(\varepsilon^2, x^2)} dx + (\varepsilon^2)^{-2} \cdot (\varepsilon_1^2 + \delta) \right)$$

and

$$+ \int_0^{\varepsilon_2^2} \frac{1}{\max(\frac{1}{x^2}, x^2)} dx + (\varepsilon_2^2)^{-2} \cdot \left( (\varepsilon_1^2)^{\frac{1}{3}} + \delta^{\frac{1}{3}} \right) \right)$$

$$\leq c_2 \cdot \log^4(1/\delta) \cdot (\varepsilon^2)^{-1} + (\varepsilon^2)^{-2} \cdot \varepsilon_1^2 + (\varepsilon_2^2)^{-1} + (\varepsilon_2^2)^{-2} \cdot (\varepsilon_1^2)^{\frac{1}{3}} \leq c_3 \cdot \delta^{-1}.$$ 

Moreover, employing (9) and Lemma 7(ii) with $\alpha = 1/2$ and $q = 2$ we obtain that there exists $c \in (0, \infty)$ such that for all $\delta \in (0, \delta_0]$,

$$I_{2,2}^{\delta} \leq \delta^{-2} \cdot \log^4(1/\delta) \cdot \int_0^{1} \mathbb{P}(\|\tilde{X}_t^\delta - \tilde{X}_t^{\delta_2}\|_2 > \frac{1}{2} d(\tilde{X}_t^\delta, \Theta), \tilde{X}_t^{\delta_2} \in \Theta^{\delta_1} \setminus \Theta^{\delta_2}) \, dt$$

$$\leq c \cdot \log^{-4}(1/\delta).$$

Combining (80), (82) and (83) we obtain (79). This completes the proof of the estimate (11) in Theorem 1.

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REFERENCES

[1] Dareiotis, K., and Gerencsér, M. On the regularisation of the noise for the Euler-Maruyama scheme with irregular drift. Electron. J. Probab. 25 (2020), Paper No. 82, 18.
[2] Gaines, J. G., and Lyons, T. J. Variable step size control in the numerical solution of stochastic differential equations. SIAM J. Appl. Math. 57, 5 (1997), 1455–1484.
[3] Göttsch, S., Lux, K., and Neuenkirch, A. The Euler scheme for stochastic differential equations with discontinuous drift coefficient: A numerical study of the convergence rate. Adv. Difference Equ. (2019), Paper No. 429, 21 pp.
[4] Gyöngy, I. A note on Euler’s approximations. Potential Anal. 8, 3 (1998), 205–216.
[5] Gyöngy, I., and Krylov, N. Existence of strong solutions for Itô’s stochastic equations via approximations. Probab. Theory Related Fields 105, 2 (1996), 143–158.
[6] Halidias, N., and Kloeden, P. E. A note on the Euler-Maruyama scheme for stochastic differential equations with a discontinuous monotone drift coefficient. BIT 48, 1 (2008), 51–59.
[7] Hefter, M., and Herzwurm, A. Optimal strong approximation of the one-dimensional squared Bessel process. Commun. Math. Sci. 15 (2017), 2121–2141.
[8] Hefter, M., Herzwurm, A., and Müller-Gronbach, T. Lower error bounds for strong approximation of scalar sdes with non-lipschitzian coefficients. Ann. Appl. Probab. 29, 1 (2019), 178–216.
[9] Hoel, H., von Schwerin, E., Szepessy, A., and Tempone, R. Adaptive multilevel Monte Carlo simulation. In Numerical analysis of multiscale computations, vol. 82 of Lect. Notes Comput. Sci. Eng. Springer, Heidelberg, 2012, pp. 217–234.
[10] Hoel, H., von Schwerin, E., Szepessy, A., and Tempone, R. Implementation and analysis of an adaptive multilevel Monte Carlo algorithm. Monte Carlo Methods Appl. 20, 1 (2014), 1–41.
[11] Karatzas, I., and Shreve, S. E. Brownian motion and stochastic calculus, second ed., vol. 113 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1991.
Lamba, H., Mattingly, J. C., and Stuart, A. M. An adaptive Euler-Maruyama scheme for SDEs: convergence and stability. IMA J. Numer. Anal. 27, 3 (2007), 479–506.

Leobacher, G., and Szölgyenyi, M. A numerical method for SDEs with discontinuous drift. BIT 56, 1 (2016), 151–162.

Leobacher, G., and Szölgyenyi, M. A strong order 1/2 method for multidimensional SDEs with discontinuous drift. Ann. Appl. Probab. 27 (2017), 2383–2418.

Leobacher, G., and Szölgyenyi, M. Convergence of the Euler-Maruyama method for multidimensional SDEs with discontinuous drift and degenerate diffusion coefficient. Numer. Math. 138, 1 (2018), 219–239.

Müller-Gronbach, T. Strong approximation of systems of stochastic differential equations. Habilitation thesis, TU Darmstadt (2002), iv+161.

Müller-Gronbach, T. Optimal pointwise approximation of SDEs based on Brownian motion at discrete points. Ann. Appl. Probab. 14, 4 (2004), 1605–1642.

Müller-Gronbach, T., and Yaroslavtseva, L. A strong order 3/4 method for SDEs with discontinuous drift coefficient. To appear in: IMA Journal of Numerical Analysis.

Müller-Gronbach, T., and Yaroslavtseva, L. A note on strong approximation of SDEs with smooth coefficients that have at most linearly growing derivatives. J. Math. Anal. Appl. 467 (2018), 1013–1031.

Müller-Gronbach, T., and Yaroslavtseva, L. On the performance of the Euler-Maruyama scheme for SDEs with discontinuous drift coefficient. Annales de l’Institut Henri Poincaré (B) Probability and Statistics 56, 2 (2020), 1162–1178.

Müller-Gronbach, T., and Yaroslavtseva, L. Sharp lower error bounds for strong approximation of SDEs with discontinuous drift coefficient by coupling of noise. arXiv:2010.00915 (2020), 36 pages.

Neuenkirch, A., Szölgyenyi, M., and Szpruch, L. An adaptive Euler-Maruyama scheme for stochastic differential equations with discontinuous drift and its convergence analysis. SIAM J. Numer. Anal. 57 (2019), 378–403.

Ngo, H.-L., and Taguchi, D. Strong rate of convergence for the Euler-Maruyama approximation of stochastic differential equations with irregular coefficients. Math. Comp. 85, 300 (2016), 1793–1819.

Ngo, H.-L., and Taguchi, D. On the Euler-Maruyama approximation for one-dimensional stochastic differential equations with irregular coefficients. IMA J. Numer. Anal. 37, 4 (2017), 1864–1883.

Ngo, H.-L., and Taguchi, D. Strong convergence for the Euler-Maruyama approximation of stochastic differential equations with discontinuous drift coefficients. Statist. Probab. Lett. 125 (2017), 55–63.

Przybyłowicz, P., and Szölgyenyi, M. Existence, uniqueness, and approximation of solutions of jump-diffusion SDEs with discontinuous drift. arXiv:1912.04215 (2019).

Revuz, D., and Yor, M. Continuous martingales and Brownian motion, third ed. Springer-Verlag, Berlin, 1995.

Römisch, W., and Winkler, R. Stepsize control for mean-square numerical methods for stochastic differential equations with small noise. SIAM J. Sci. Comput. 28, 2 (2006), 604–625 (electronic).

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