Algebraicity of the central critical values of twisted triple product $L$-functions

Shih-Yu Chen

Received: 23 January 2021 / Accepted: 1 June 2021 / Published online: 22 June 2021
© Fondation Carl-Herz and Springer Nature Switzerland AG 2021

Abstract
We study the algebraicity of the central critical values of twisted triple product $L$-functions associated to motivic Hilbert cusp forms over a totally real étale cubic algebra in the totally unbalanced case. The algebraicity is expressed in terms of the cohomological period constructed via the theory of coherent cohomology on quaternionic Shimura varieties developed by Harris. As an application, we generalize our previous result with Cheng on Deligne’s conjecture for certain automorphic $L$-functions for $GL_3 \times GL_2$.

Keywords
Critical values · Triple product $L$-functions · Deligne’s conjecture

Mathematics Subject Classification
Primary 11F67; Secondary 11F70 · 11F75

1 Introduction and main results

1.1 Introduction

Let $E$ be a totally real cubic extension over $\mathbb{Q}$ and $\Sigma_E = \{\infty_1, \infty_2, \infty_3\}$ the set of embeddings of $E$ into $\mathbb{R}$. Let $\varPi = \bigotimes_v \varPi_v$ be an irreducible unitary cuspidal automorphic representation of $(R_{E/Q} GL_2/E)(\hat{A}_{\mathbb{Q}}) = GL_2(\hat{A}_{E})$ with central character $\omega_{\varPi}$, where $v$ runs through the places of $\mathbb{Q}$ and $R_{E/Q}$ denotes Weil’s restriction of scalars. We assume $\varPi$ is motivic of weight
Remark 1.3. When \( \kappa_1, \kappa_2, \kappa_3 \in \mathbb{Z}_{\geq 2} \), that is, \( \Pi_{\infty} \) is a discrete series representation of weight \((\kappa_1, \kappa_2, \kappa_3)\) with \( \kappa_i \) corresponding to \( \infty_i \) and \( \kappa_i \equiv \kappa_j \pmod{2} \) for \( 1 \leq i, j \leq 3 \). We assume further that \( \kappa_1 + \kappa_2 + \kappa_3 \equiv 0 \pmod{2} \) and \( \kappa_1 \geq \kappa_2 \geq \kappa_3 \). Consider the (finite) Asai cube \( L \)-function

\[
L^{(\infty)}(s, \Pi, As) = \prod_p L(s, \Pi_p, As)
\]

of \( \Pi \). A critical point for \( L(s, \Pi, As) \) is a half-integer \( m + \frac{1}{2} \) which is not a pole of the archimedean local factors \( L(s, \Pi_{\infty}, As) \) and \( L(1-s, \Pi_{\infty}, As) \). Explicating Yoshida’s calculation \([52, (5.11)]\) of the (conjectural) motivic Asai periods in this case, we have the following conjecture on the algebraicity of the critical values of \( L(s, \Pi, As) \) in terms of Shimura’s \( Q \)-invariant \([46, \text{Sect. 7}]\) which we shall now recall. Let \( I \subseteq \Sigma_E \). Suppose there exists a quaternion algebra \( B \) over \( \mathbb{E} \) such that

- \( B \) is unramified at places in \( I \) and ramified at places in \( \Sigma_E \setminus I \);
- there exists an irreducible cuspidal automorphic representation \( \Pi_B \) of \( B^\times (\mathbb{A}_E) \) associated with \( \Pi \) by the Jacquet–Langlands correspondence.

Then the period invariant \( \mathcal{Q}(\Pi, I) \in \mathbb{C}^x / \overline{\mathbb{Q}}^x \) is defined to be the class represented by the Petersson norm of a non-zero \( \overline{\mathbb{Q}} \)-rational vector-valued cusp form on \( B^\times (\mathbb{A}_E) \) associated to \( \Pi_B \) (we refer to \([45, \text{Sect. 2}]\) for the notion of \( \overline{\mathbb{Q}} \)-rational automorphic forms). Note that \( \mathcal{Q}(\Pi, I) \) does not depend on the choice of \( B \) and \( \overline{\mathbb{Q}} \)-rational automorphic forms (cf. \([52, \text{Theorem 6.6}]\)) and is always defined when \( I = 1, 3 \).

**Conjecture 1.1** Let \( m + \frac{1}{2} \) be critical for \( L(s, \Pi, As) \). We have

\[
\frac{L^{(\infty)}(m + \frac{1}{2}, \Pi, As)}{(2\pi \sqrt{-1})^{4m} \cdot q(\Pi, As)} \in \overline{\mathbb{Q}},
\]

where

\[
q(\Pi, As) = \begin{cases} 
\pi^{\kappa_1+\kappa_2+\kappa_3+2} \mathcal{Q}(\Pi, \Sigma_E) & \text{if } \kappa_1 < \kappa_2 + \kappa_3, \\
\pi^{2\kappa_2+2} \mathcal{Q}(\Pi, \{\infty_1\})^2 & \text{if } \kappa_1 \geq \kappa_2 + \kappa_3.
\end{cases}
\]

A similar conjecture was proposed by Blasius \([2]\) if we replace \( \mathbb{E} \) by \( \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \). One can also propose a conjecture if we replace \( \mathbb{E} \) by \( \mathbb{K} \times \mathbb{Q} \) for some real quadratic extension \( \mathbb{K} \) over \( \mathbb{Q} \). We remark that the conjecture is compatible with Deligne’s conjecture \([9]\) (see Sect. 4.5). When \( \kappa_1 < \kappa_2 + \kappa_3 \), the conjecture was proved by Garrett and Harris in \([11, \text{Theorem 4.6}]\) for \( |m| > 1 \) and by the author and Cheng in \([5, \text{Corollary 6.4}]\) for \( m = 0 \). When \( \kappa_1 \geq \kappa_2 + \kappa_3 \), we have the following result which is a special case of our main result Theorem 1.4 (see Remark 1.5).

**Theorem 1.2** Assume \( \kappa_1 \geq \kappa_2 + \kappa_3 \), \( \omega_{\Pi}|_{\mathbb{A}_E} \) is trivial, and \( \text{Hom}_{\text{GL}_2(\mathbb{Q}_p)}(\Pi_p, \mathbb{C}) \neq 0 \) for all rational primes \( p \). Then Conjecture 1.1 holds for \( m = 0 \).

**Remark 1.3** By the results of Prasad \([37]\) and \([38]\), \( \text{Hom}_{\text{GL}_2(\mathbb{Q}_p)}(\Pi_p, \mathbb{C}) \neq 0 \) whenever \( \Pi_p \) is a principal series representation. Note that the result of Prasad holds for totally real étale cubic algebra \( \mathbb{E} \) over \( \mathbb{Q} \).
1.2 Main results

Let $E$ be a totally real étale cubic algebra over a totally real number field $F$. Let $\Pi = \bigotimes_v \Pi_v$ be an irreducible cuspidal automorphic representation of $(\mathbb{R}_E/F \text{GL}_2(E))(\mathbb{A}_E) = \text{GL}_2(\mathbb{A}_E)$ with central character $\omega_{\Pi}$, where $v$ runs through the places of $F$. We have the Asai cube representation

$$\text{As} : L(\mathbb{R}_E/F \text{GL}_2(E)) \longrightarrow \text{GL}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$$

of the $L$-group $L(\mathbb{R}_E/F \text{GL}_2(E))$ of $\mathbb{R}_E/F \text{GL}_2(E)$. The associated automorphic $L$-function

$$L(s, \Pi, \text{As}) = \prod_v L(s, \Pi_v, \text{As})$$

is called the twisted triple product $L$-function of $\Pi$. We denote by $L(\infty)(s, \Pi, \text{As})$ the $L$-function obtained by excluding the archimedean $L$-factors. Let $\Sigma_E$ (resp. $\Sigma_F$) be the set of non-zero algebra homomorphisms from $E$ (resp. $F$) into $\mathbb{R}$. We assume that $\omega_{\Pi}|_{\mathbb{A}_F}$ is trivial and $\Pi$ is motivic (cf. Sect. 2.2) of weight

$$\kappa = \sum_{w \in \Sigma_E} \kappa_w w \in \mathbb{Z}_{\geq 1}[\Sigma_E].$$

We say $\Pi$ is totally unbalanced (resp. totally balanced) if for all $v \in \Sigma_F$ we have

$$2 \max_{w|v} \{\kappa_w\} - \sum_{w|v} \kappa_w \geq 0 \quad (\text{resp.} \ < 0).$$

In the totally balanced case, the algebraicity of $L(s, \Pi, \text{As})$ at the critical points (except for $s = -\frac{1}{2}, \frac{3}{2}$) were proved by Garrett–Harris [11] and C.–Cheng [5] in terms of the Petersson norm of the normalized newform of $\Pi$ and the result is compatible with Deligne’s conjecture [9]. The aim of this paper is to prove, in the totally unbalanced case, the algebraicity of the central critical value $L(\frac{1}{2}, \Pi, \text{As})$ in terms of Harris’ cohomological period which we shall now describe. Our result is compatible with Deligne’s conjecture which predicts that the algebraicity can be expressed in terms of the (conjectural) motivic periods in [52] (cf. Sect. 4.5). Suppose the global root number $\varepsilon(\Pi, As)$ of $\Pi$ with respect to the Asai cube representation is equal to 1. By the results of Prasad [37] and [38] and Loke [33], there exists a unique quaternion algebra $D$ over $F$ so that there exists an irreducible cuspidal automorphic representation $\Pi_D = \bigotimes_v \Pi_D^v$ of $D^\times(\mathbb{A}_E)$ associated to $\Pi$ by the Jacquet–Langlands correspondance and such that

$$\text{Hom}_{D^\times(\mathbb{A}_F)}(\Pi_D^\#, \mathbb{C}) \neq 0$$

for all places $v$ of $F$. Note that $D$ is totally indefinite if and only if $\Pi$ is totally unbalanced. In this case, for each subset $I$ of $\Sigma_E$, we denote by $\Omega^I(\Pi_D^\#) \in \mathbb{C}^\times$ (resp. $\Omega^I((\Pi_D^\#)^\vee) \in \mathbb{C}^\times$) the period obtained by comparing the rational structures on $\Pi_D^\# = \bigotimes_{v|\infty} \Pi_D^v$ (resp. $(\Pi_D^\#)^\vee = \bigotimes_{v|\infty} (\Pi_D^v)^\vee$) via the zeroth and $I$-th coherent cohomology of certain automorphic line bundles on toroidal compactification of the quaternionic Shimura variety associated to $D^\times$ (see Sect. 2 for the precise definition).

For $\sigma \in \text{Aut}(\mathbb{C})$, there exists a unique motivic irreducible cuspidal automorphic representation $^\sigma\Pi$ of $\text{GL}_2(\mathbb{A}_E)$ such that its finite part is isomorphic to the $\sigma$-conjugate of the finite part of $\Pi$. We denote by $^\sigma\kappa$ the weight of $^\sigma\Pi$. The rationality field $\mathbb{Q}(5)$ of $5$ is defined to be the fixed field of $\{\sigma \in \text{Aut}(\mathbb{C}) | ^\sigma\Pi = \Pi\}$ and is a number field. For each $v \in \Sigma_F$, let $\bar{v}(k) \in \Sigma_E$.

\(\text{Springer}\)
be the homomorphism such that \( \max_{w|\nu} \{ \kappa_w \} = \kappa(\kappa) \). Note that \( \tilde{v}(\kappa) \) is well-defined by the totally unbalanced assumption. Put

\[
I_\kappa = \{ \tilde{v}(\kappa) \ | \ v \in \Sigma_E \} \subset \Sigma_E.
\]

Following is our main result for totally unbalanced \( \Pi \). When \( \mathbb{E} = \mathbb{F} \times \mathbb{F} \times \mathbb{F} \) and \( D \) is the matrix algebra, the theorem was proposed and proved by Harris [16]. Following the ideas in [16], we generalize the result of Harris to arbitrary totally real étale cubic algebra \( \mathbb{E} \) over \( \mathbb{F} \).

**Theorem 1.4** Let \( \Pi \) be a motivic irreducible cuspidal automorphic representation of \( \text{GL}_2(\mathbb{A}_\mathbb{E}) \). Assume that \( \sigma \Pi \big|_{\mathbb{A}_\mathbb{E}} \) is trivial and \( \Pi \) is totally unbalanced.

1. If \( \varepsilon(\Pi, As) = -1 \), then
   \[
   L \left( \frac{1}{2}, \sigma \Pi, As \right) = 0
   \]
   for all \( \sigma \in \text{Aut}(\mathbb{C}) \).

2. Suppose \( \varepsilon(\Pi, As) = 1 \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), we have
   \[
   \sigma \left( L(\infty) \left( \frac{1}{2}, \Pi, As \right) \right) = \frac{L(\infty) \left( \frac{1}{2}, \sigma \Pi, As \right)}{(D_\mathbb{E}/D_\mathbb{F})^{1/2}(2\pi \sqrt{-1})^{2[\mathbb{F}:\mathbb{Q}]} \cdot \Omega^L_1(\Pi D) \cdot \Omega^L_2((\Pi D)\gamma)}.
   \]
   \[
   \frac{(D_\mathbb{E}/D_\mathbb{F})^{1/2}(2\pi \sqrt{-1})^{2[\mathbb{F}:\mathbb{Q}]} \cdot \Omega^L_1(\Pi D) \cdot \Omega^L_2((\Pi D)\gamma)}{(D_\mathbb{E}/D_\mathbb{F})^{1/2}(2\pi \sqrt{-1})^{2[\mathbb{F}:\mathbb{Q}]} \cdot \Omega^L_1(\Pi D) \cdot \Omega^L_2((\Pi D)\gamma)}
   \]
   Here \( D_\mathbb{E} \) and \( D_\mathbb{F} \) are the absolute discriminant of \( \mathbb{F}/\mathbb{Q} \) and \( \mathbb{E}/\mathbb{Q} \), respectively, and \( D \) is the unique quaternion algebra over \( \mathbb{F} \) such that \( \text{Hom}_{D \times (\mathbb{F})}(\Pi D, \mathbb{C}) \neq 0 \) for all places \( v \) of \( \mathbb{F} \). In particular, we have
   \[
   L(\infty) \left( \frac{1}{2}, \Pi, As \right) \frac{\Omega^L_1(\Pi D) \cdot \Omega^L_2((\Pi D)\gamma)}{(D_\mathbb{E}/D_\mathbb{F})^{1/2}(2\pi \sqrt{-1})^{2[\mathbb{F}:\mathbb{Q}]} \cdot \Omega^L_1(\Pi D) \cdot \Omega^L_2((\Pi D)\gamma)} \in \mathbb{Q}(\Pi).
   \]
   Moreover, when \( D \) is the matrix algebra, we can replace \( \Omega^L_1((\Pi D)\gamma) \) by \( \Omega^L_1(\Pi) \).

**Remark 1.5** Assume \( \kappa \in \mathbb{Z} \geq 2[\Sigma\mathbb{E}] \). Let \( I \subseteq \Sigma\mathbb{E} \). By the result of Harris [19, Theorem 1] and the period relation in Lemma 2.8 below, suppose that Shimura’s period invariant \( Q(\Pi, I) \in \mathbb{C}^x/\overline{\mathbb{Q}}^x \) is defined, then we have

\[
Q(\Pi, I) = (2\pi \sqrt{-1})^{-\sum_{w \in I} \kappa_w} \cdot \Omega^L_1(\Pi) \mod \overline{\mathbb{Q}}^x.
\]

Therefore, if \( D \) is the matrix algebra and \( Q(\Pi, I_\kappa) \) is defined, then we can express the algebraicity of \( L(\infty) \left( \frac{1}{2}, \Pi, As \right) \) in terms of \( Q(\Pi, I_\kappa) \). In particular, Theorem 1.2 holds.

As an application of Theorem 1.4, we generalize our previous result [5], which is compatible with Deligne’s conjecture, on the algebraicity of the central critical value of certain automorphic \( L \)-functions for \( \text{GL}_3 \times \text{GL}_2 \). Let \( \Pi = \bigotimes_v \Pi_v, \Pi' = \bigotimes_v \Pi'_v \) be motivic irreducible cuspidal automorphic representations of \( \text{GL}_2(\mathbb{A}_\mathbb{F}) \) with central characters \( \omega_{\Pi}, \omega_{\Pi'} \) and of weights \( \ell, \ell' \in \mathbb{Z}_{\geq 1}[\Sigma\mathbb{F}] \), respectively. For each subset \( I \) of \( \Sigma\mathbb{F} \) which is admissible with respect to \( \ell' \), let \( \Omega^L_1(\Pi') \subseteq \mathbb{C}^x \) be the Harris’ period of \( \Pi \) recalled in Sect. 2.3. Let \( \text{Sym}^2(\Pi) \) be the Gelbart–Jacquet lift [12] of \( \Pi \), which is an isobaric automorphic representation of \( \text{GL}_3(\mathbb{A}_\mathbb{F}) \) and is the functorial lift of the symmetric square representation of \( \text{GL}_2 \) associated to \( \Pi \). Let

\[
L(s, \text{Sym}^2(\Pi) \times \Pi') = \prod_v L(s, \text{Sym}^2(\Pi_v) \times \Pi'_v)
\]
the Rankin–Selberg automorphic $L$-function for $GL_3(\mathbb{A}_F) \times GL_2(\mathbb{A}_F)$ associated to $\text{Sym}^2(\Pi) \times \Pi'$. We denote by $L^{(\infty)}(\iota, \text{Sym}^2(\Pi) \times \Pi')$ the $L$-function obtained by excluding the archimedean $L$-factors.

**Theorem 1.6** Suppose the following conditions hold:

(i) $\omega^2 \Pi \omega_{\Pi'}$ is trivial;
(ii) $\ell - 2\ell \in \mathbb{Z}_{\geq 0}[\Sigma_F]$;
(iii) there exists a totally real quadratic extension $\mathbb{K}$ over $\mathbb{F}$ such that

$$\varepsilon(\Pi' \otimes \omega_{\Pi' \mathbb{K}/\mathbb{F}}) = \varepsilon(\text{Sym}^2(\Pi) \times \Pi'),$$

where $\varepsilon(*)$ is the global root number of $*$ and $\omega_{\mathbb{K}/\mathbb{F}}$ is the quadratic Hecke character of $\mathbb{A}_\mathbb{K}$ associated to $\mathbb{K}/\mathbb{F}$ by class field theory.

Then we have

$$\sigma \left( \frac{L^{(\infty)}(\frac{1}{2}, \text{Sym}^2(\Pi) \times \Pi')}{(2\pi \sqrt{-1})^{[\mathbb{F} : \mathbb{Q}](1+r)} \cdot G(\omega_{\Pi}) \cdot \Omega^2(\Pi') \cdot p(((-1)^{1+r} \text{sgn}(\omega_{\Pi}), \Pi'))} \right)$$

$$= \frac{L^{(\infty)}(\frac{1}{2}, \text{Sym}^2(\sigma \Pi) \times \sigma \Pi')}{(2\pi \sqrt{-1})^{[\mathbb{F} : \mathbb{Q}](1+r)} \cdot G(\sigma \omega_{\Pi}) \cdot \Omega^2(\sigma \Pi') \cdot p(((-1)^{1+r} \text{sgn}(\omega_{\Pi}), \sigma \Pi'))}$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. Here $r \in \mathbb{Z}$ is defined so that $|\omega_{\Pi}| = |\ell|_{\mathbb{A}_\mathbb{F}}$, $G(\omega_{\Pi})$ is the Gauss sum of $\omega_{\Pi}$, $\text{sgn}(\omega_{\Pi}) \in \{\pm 1\}^\Sigma_F$ is the signature of $\omega_{\Pi}$, and $p(\varepsilon, \Pi')$ are the periods for $\Pi'$ defined for $\varepsilon \in \{\pm 1\}^\Sigma_F$ in [44].

**Remark 1.7** Condition (iii) is satisfied when either $\Pi'_v$ is not a discrete series representation for any finite place $v$ or the conductors of $\Pi$ and $\Pi'$ are square-free. We also refer to Lemma 2.9 for the period relation between Petersson norm and $\Omega^2(\Pi')$.

**Remark 1.8** When $\ell - 2\ell \in \mathbb{Z}_{< 0}[\Sigma_F]$, the algebraicity of $L^{(\infty)}(\frac{1}{2}, \text{Sym}^2(\sigma \Pi) \times \sigma \Pi')$ was proved by several authors. Suppose that $\mathbb{F} = \mathbb{Q}$ and the conductors of $\Pi$ and $\Pi'$ are square-free, it is proved in [24, Corollary 2.6], [51, Theorem 1.1], [36, Corollary 1.4], [5, Theorem A], and [7, Corollary 1.2] in terms of Petersson norm and Shimura’s period [43] and the result is compatible with Deligne’s conjecture. The algebraicity is also proved in [40] and [29] in terms of certain cohomological period.

### 1.3 An outline of the proof

There are two main ingredients for the proof of Theorem 1.4:

- Ichino’s central value formula for $L(\frac{1}{2}, \Pi, As)$;
- cohomological interpretation of the global trilinear period integral.

We have the global trilinear period integral $I^D \in \text{Hom}_{\mathbb{C}}(\Pi^D \otimes (\Pi^D)^\vee, \mathbb{C})$ in (4.1) defined by integration on $D^\times(\mathbb{A}_F) \times D^\times(\mathbb{A}_F)$ of cusp forms in $\Pi^D \otimes (\Pi^D)^\vee$. In [25], Ichino established a formula which decomposes the global trilinear period integral $I^D$ into a product of $L(\frac{1}{2}, \Pi, As)$ and the local trilinear period integrals $I^D_v$ defined in (4.2). Note that our choice of the quaternion algebra $D$ guarantees the non-vanishing of $I^D_v$ (cf. Lemma 4.5). On the other hand, consider the automorphic line bundles $\mathcal{L}_{(\mathbb{A}, \mathcal{L})}$ and $\mathcal{L}_{(\mathbb{A}, \mathcal{L})}$ associated to the algebraic characters $\rho_{(\mathbb{A}, \mathcal{L})}$ and $\rho_{(\mathbb{A}, \mathcal{L})}$ of $(\mathbb{R}^\times \cdot SO(2))^{\Sigma_F}$ in (2.4) on the quaternion Shimura variety for the Shimura datum $(\mathbb{R}^\times / \mathbb{Q}(\mathbb{F} \otimes \mathbb{E})^\times, (\mathbb{F}^\times \otimes \mathbb{E})^{\Sigma_F})$. The period $\Omega^D(\Pi^D) \in \mathbb{C}^\times$ is obtained.
by comparing the \( \mathbb{Q}(\Pi) \)-rational structures on the \( \Pi_f \)-isotypic components of the cuspidal cohomology \( H^0_{\text{cusp}}(\mathcal{L}(\kappa, \ell)) \) and \( H^{[\mathbb{Q}]}_{\text{cusp}}(\mathcal{L}(\kappa, \ell)) \). Similarly for \( \Omega^f((\Pi^D)^{\vee}) \). Under the totally unbalanced condition, we constructed in Sect. 3 certain trilinear differential operator \([\delta(\kappa)]\) rational \( \mathbb{Q}(\kappa) \) from \( \mathcal{L}(\kappa, \ell) \) to \( \mathcal{L}(2, 0) \), the automorphic line bundle associated to the algebraic character \( \rho(2, 0) \) of \( (\mathbb{R}^\times \otimes \mathbb{SO}(2))^{\Sigma_\mathbb{F}} \) on the quaternion Shimura variety for the Shimura datum \((\mathbb{R}_F/\mathbb{Q}) D^\times, (\delta^\pm)^{\Sigma_\mathbb{F}} \). It induces a \( D^\times (A_{\mathbb{F}, f}) \)-module homomorphism

\[
[\delta(\kappa)]: H^{[\mathbb{F}; \mathbb{Q}]}(\mathcal{L}_{\kappa}(\ell, \ell)) \longrightarrow H^{[\mathbb{F}; \mathbb{Q}]}((\mathcal{L}(2, 0))^{\text{sub}})
\]

which is rational \( \mathbb{Q}(\kappa) \). Comparing the trilinear differential operator with the \( \mathbb{Q} \)-rational trace map

\[
H^{[\mathbb{F}; \mathbb{Q}]}((\mathcal{L}(2, 0))^{\text{sub}}) \longrightarrow \mathbb{C}
\]

in Lemma 2.2, we then deduce the cohomological interpretation of the global trilinear period integral \( I^D \). Combining with Ichino’s formula, we obtain the algebraicity of \( L(1, \kappa, \text{Ad}) \). We mention one subtlety in the proof. In order to apply Ichino’s formula, it is necessary to compare \( L(1, \mathbb{F}, \mathbb{A}) \), which is the special value of the adjoint \( L \)-function of \( \Pi \) at \( s = 1 \), with the Petersson bilinear pairing \( (2.18) \) on \( \Pi^D \times (\Pi^D)^{\vee} \). It is known that \( L(1, \mathbb{F}, \text{Ad}) \) is essentially equal to the Petersson pairing of \( \mathbb{Q}(\Pi) \)-rational holomorphic cusp form in \( \Pi^D \times (\Pi^D)^{\vee} \) (see Lemma 2.9 and Corollary 2.10). On the other hand, the rationality of the global trilinear period integral \( I^D \) is related to the rational structure of \( H^{[\mathbb{F}; \mathbb{Q}]}(\mathcal{L}(\kappa, \ell, \ell))[(\Pi^D)_{\mathbb{Q}}] \) as we have explained above. This is the key reason why we need to compare the rational structures on \( H^{0}_{\text{cusp}}(\mathcal{L}(\kappa, \ell, \ell))[(\Pi^D)_{\mathbb{Q}}] \) and \( H^{[\mathbb{F}; \mathbb{Q}]}(\mathcal{L}(\kappa, \ell, \ell))[(\Pi^D)_{\mathbb{Q}}] \).

This paper is organized as follows. In Sect. 2, we recall the theory of coherent cohomology on quaternion Shimura varieties based on the general results of Harris [18]. The cohomological periods \( \Omega^f(\Pi) \) for admissible \( I \subseteq \Sigma_\mathbb{E} \) are defined in Proposition 2.6. In Sect. 3, we construct the trilinear differential operator under the totally unbalanced condition. We specialize the results of Harris in [13, Sect. 3] and [14, Sect. 7] to the natural inclusion

\[
(\mathbb{R}_F/\mathbb{Q}) D^\times, (\delta^\pm)^{\Sigma_\mathbb{F}} \subset (\mathbb{R}_F/\mathbb{Q}) (D \otimes_{\mathbb{F}} E)^\times, (\delta^\pm)^{\Sigma_\mathbb{E}}
\]

of Shimura data and the automorphic line bundles \( \mathcal{L}(\kappa, \ell, \ell) \) and \( \mathcal{L}(2, 0) \). In Sect. 4, we prove our main results Theorems 1.4 and 1.6. We also compare Theorem 1.4 with Deligne’s conjecture in Sect. 4.5.

### 1.4 Notation

Fix a totally real number field \( \mathbb{F} \). Let \( A_{\mathbb{F}} \) (resp. \( A \)) be the ring of adeles of \( \mathbb{F} \) (resp. \( \mathbb{Q} \)) and \( A_{\mathbb{F}, f} \) (resp. \( A_{f} \)) be its finite part. Let \( \mathfrak{o}_F \) be the ring of integers of \( \mathbb{F} \) and \( D_F \) the absolute discriminant of \( \mathbb{F}/\mathbb{Q} \). We denote by \( \delta_F \) the closure of \( \mathfrak{o}_F \) in \( A_{\mathbb{F}, f} \). Let \( \psi_{\mathbb{Q}} = \otimes_v \psi_{\mathbb{Q}_v} \), be the standard additive character of \( \mathbb{Q} \backslash A \) defined so that

\[
\psi_{\mathbb{Q}_p}(x) = e^{-2\pi \sqrt{-1} x} \text{ for } x \in \mathbb{Z}[p^{-1}], \\
\psi_{\mathbb{R}}(x) = e^{2\pi \sqrt{-1} x} \text{ for } x \in \mathbb{R}.
\]

The standard additive character \( \psi_{\mathbb{F}} \) of \( \mathbb{F} \backslash A_{\mathbb{F}} \) is defined by \( \psi_{\mathbb{F}} = \psi_{\mathbb{Q}} \circ \text{tr}_{\mathbb{F}/\mathbb{Q}} \). For \( \alpha \in \mathbb{F} \), let \( \psi_{\mathbb{F}_v}^\alpha \) be the additive character defined by \( \psi_{\mathbb{F}_v}^\alpha(x) = \psi_{\mathbb{F}_v}(\alpha x) \). Similarly we define \( \psi_{\mathbb{F}_v}^\alpha \) for \( \alpha \in \mathbb{F}_v \).
Let $E$ be a totally real étale algebra over $F$. Let $\Sigma_E$ be the set of non-zero algebra homomorphisms from $E$ to $R$. We identify $E_\infty = E \otimes Q \otimes R$ with $R^{\Sigma_E}$ so that the $w$-th coordinate of $R^{\Sigma_E}$ corresponds to the completion of $E$ at $w$. Let $\kappa = \Sigma_{w \in E} \kappa_w w \in Z[\Sigma_E]$. For $\sigma \in \text{Aut}(C)$, define $\sigma^w = \Sigma_{w \in E} \sigma^w w \in Z[\Sigma_E]$ with $\sigma^w = \kappa^{-1} w$. Let $Q(\kappa)$ be the fixed field of $\{ \sigma \in \text{Aut}(C) \mid \sigma^w = \kappa \}$. For each subset $I$ of $\Sigma_E$, let $\kappa_I = \Sigma_{w \in E} \kappa(I)_w w \in Z[\Sigma_E]$ defined by

$$\kappa(I)_w = \begin{cases} 2 - \kappa_w & \text{if } w \in I, \\ \kappa_w & \text{if } w \notin I, \end{cases}$$

and let $Q(I)$ be the fixed field of $\{ \sigma \in \text{Aut}(C) \mid \sigma^w = \kappa \}$. Note that $Q(\kappa(I)) \subseteq Q(\kappa) \cdot Q(I)$. Consider the map

$$I \mapsto \{^I, \kappa(I)\}.$$  

We say a subset $I$ of $\Sigma_E$ is admissible with respect to $\kappa$ if the fiber of $\{^I, \kappa(I)\}$ under the above map contains only $I$. In this case, we have $Q(\kappa(I)) = Q(\kappa) \cdot Q(I)$. It is clear that the empty set and $\Sigma_E$ are admissible. We will use the notion of admissibility only when $\kappa \in \mathbb{Z}_{\geq 1}(\Sigma_E)$. In this case, any subset $I$ such that $\kappa_w \geq 2$ for all $w \in I$ is admissible. We assume

$$E = F_1 \times \cdots \times F_n$$

for some totally real number fields $F_1, \ldots, F_n$ over $F$. Let $\tilde{E}$ be the composite of the Galois closure of $F_i$ over $Q$ for $1 \leq i \leq n$. We identify $\Sigma_E$ with the disjoint union of $\tilde{E}$, for $1 \leq i \leq n$ in a natural way. Let $(\kappa, r) \in Z[\Sigma_E] \times Z[\Sigma_E]$. We say $(\kappa, r)$ is motivic if $\kappa_w \equiv r_w (\text{mod } 2)$ for $w \in E$ and $r_w = r_{w'}$ whenever $w, w' \in E_i$ for some $1 \leq i \leq n$.

In $\text{GL}_2$, let $B$ be the Borel subgroup consisting of upper triangular matrices and $N$ be its unipotent radical, and put

$$a(v) = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \quad d(v) = \begin{pmatrix} 0 & 1 \\ 0 & v \end{pmatrix}, \quad m(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

for $v, t \in \text{GL}_1$ and $x \in \mathbb{G}_a$. Let $\mathfrak{g}_2$ be the Lie algebra of $\text{GL}_2(R)$ and $\mathfrak{g}_{2, \mathbb{C}}$ be its complexification. We have

$$\mathfrak{g}_{2, \mathbb{C}} = \mathbb{C} \cdot Z \oplus \mathbb{C} \cdot H + \mathbb{C} \cdot X_+ \oplus \mathbb{C} \cdot X_-,$$

where

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad X_+ = \begin{pmatrix} \sqrt{-1} & -1 \\ -1 & \sqrt{-1} \end{pmatrix}, \quad X_- = \begin{pmatrix} \sqrt{-1} & 1 \\ 1 & -\sqrt{-1} \end{pmatrix}.$$  

We denote by $U(\mathfrak{g}_{2, \mathbb{C}})$ the universal enveloping algebra of $\mathfrak{g}_{2, \mathbb{C}}$. Let

$$SO(2) = \left\{ k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \left| \theta \in \mathbb{R}/2\pi \mathbb{Z} \right. \right\}.$$  

For $\kappa \in \mathbb{Z}_{\geq 1}$ and $\varepsilon \in \{ \pm 1 \}$, let $D(\kappa)^\varepsilon$ be the irreducible admissible $(\mathfrak{g}_2, SO(2))$-module characterized so that there exists a non-zero $v \in D(\kappa)^\varepsilon$ such that

$$Z \cdot v = 0, \quad H \cdot v = \varepsilon \kappa \cdot v, \quad X_{-\varepsilon} \cdot v = 0.$$  

\textit{Springer}
Let $D(\kappa) = D(\kappa)^+ \oplus D(\kappa)^-$, which is an irreducible admissible $(\mathfrak{gl}_2, \mathfrak{o}(2))$-module. Note that $D(\kappa)$ is the $\mathfrak{o}(2)$-finite part of the (limit of) discrete series representation of $GL_2(\mathbb{R})$ with weight $\kappa$. For $r \in \mathbb{Z}$ such that $\kappa \equiv r \pmod{2}$ and $\alpha \in \mathbb{R}^\times$, let $W_{(\kappa, r), \psi^\alpha_R}$ be the Whittaker function for $D(\kappa) \otimes |t|^{r/2}$ with respect to $\psi^\alpha_R$ of weight $\pm \kappa$ normalized so that $W_{(\kappa, r), \psi^\alpha_R}(1) = e^{-2\pi \alpha}$. The explicit formula is given by

$$W_{(\kappa, r), \psi^\alpha_R}(z(x) \ell(y)) = \zeta^r(\pm \alpha x)^{(\kappa+r)/2} e^{2\pi \sqrt{-1}(\alpha x \pm \sqrt{-1} y) \pm \sqrt{-1} \kappa \theta} \cdot \|_{\mathbb{R}^\times}(\pm \alpha y) \quad (1.1)$$

for $x, y \in \mathbb{R}$, $z \in \mathbb{R}^\times$, and $k_\theta \in SO(2)$.

Let $\sigma \in \text{Aut}(\mathbb{C})$. Define the $\sigma$-linear action on $\mathbb{C}(X)$, which is the field of formal Laurent series in variable $X$ over $\mathbb{C}$, as follows:

$$\sigma P(X) = \sum_{n \gg -\infty} \sigma(a_n) X^n$$

for $P(X) = \sum_{n \gg -\infty} a_n X^n \in \mathbb{C}(X)$. For a complex representation $\Pi$ of a group $G$ on the space $V_{\Pi}$ of $\Pi$, let $\sigma \Pi$ be the representation of $G$ on $V_{\Pi}$ defined

$$\sigma \Pi(g) = t \circ \Pi(g) \circ t^{-1},$$

where $t : V_{\Pi} \to V_{\Pi}$ is a $\sigma$-linear isomorphism. Note that the isomorphism class of $\sigma \Pi$ is independent of the choice of $t$. We call $\sigma \Pi$ the $\sigma$-conjugate of $\Pi$. When $v$ is a finite place of $\mathbb{F}$ and $f$ is a complex-valued function on $\mathbb{F}_v^m$ or $(\mathbb{F}_v^\times)^m$ for some $m \in \mathbb{Z}_{\geq 1}$, we define $\sigma f$ by $\sigma f(x) = \sigma(f(x))$ for $x \in \mathbb{F}_v^m$ or $x \in (\mathbb{F}_v^\times)^m$.

For an algebraic Hecke character $\chi$ of $\mathbb{A}_{\mathbb{F}}^\times$, the Gauss sum $G(\chi)$ of $\chi$ is defined by

$$G(\chi) = D_{\mathbb{F}}^{-1/2} \prod_{v \mid \infty} \varepsilon(0, \chi_v, \psi_{\mathbb{F}_v}),$$

where $\varepsilon(s, \chi_v, \psi_{\mathbb{F}_v})$ is the $\varepsilon$-factor of $\chi_v$ with respect to $\psi_{\mathbb{F}_v}$ defined in [49]. For $\sigma \in \text{Aut}(\mathbb{C})$, let $\sigma \chi$ be the unique algebraic Hecke character of $\mathbb{A}_{\mathbb{F}}^\times$ such that $\sigma \chi(x) = \sigma(\chi(x))$ for $x \in \mathbb{A}_{\mathbb{F}, f}^\times$. It is easy to verify that

$$\sigma(G(\chi)) = \sigma(\chi) G(\sigma \chi),$$

$$\sigma \left( \frac{G(\chi \chi')}{G(\chi) G(\chi')} \right) = \frac{G(\sigma \chi \sigma \chi')}{G(\sigma \chi) G(\sigma \chi')} \quad (1.2)$$

for algebraic Hecke characters $\chi, \chi'$ of $\mathbb{A}_{\mathbb{F}}^\times$, where $u \in \hat{\mathbb{Z}}^\times$ is the unique element such that $\sigma(\psi_{\mathbb{F}}(x)) = \psi_{\mathbb{F}}(ux)$ for $x \in \mathbb{A}_{\mathbb{F}, f}$. Let $\text{sgn}(\chi) \in \{\pm 1\} \Sigma_{\mathbb{F}}$ be the signature of $\chi$ defined by $\text{sgn}(\chi)_v = \chi_v(-1)$ for $v \in \Sigma_{\mathbb{F}}$. Note that $\text{sgn}(\sigma \chi) = \text{sgn}(\chi)$ for all $\sigma \in \text{Aut}(\mathbb{C})$.

## 2 Periods of motivic quaternionic modular forms

### 2.1 Coherent cohomology groups on the quaternionic Shimura variety

Let

$$\mathbb{E} = \mathbb{F}_1 \times \cdots \times \mathbb{F}_n$$

Springer
be a totally real étale algebra over $\mathbb{F}$, where $\mathbb{F}_1, \cdots, \mathbb{F}_n$ are totally real number fields over $\mathbb{F}$. For $1 \leq i \leq n$, let $D_i$ be a totally indefinite quaternion algebra over $\mathbb{F}_i$. Put $D = D_1 \times \cdots \times D_n$. Let

$$G = \mathbb{R}_{E/Q} D^x = \mathbb{R}_{F_1/Q} D_1^x \times \cdots \times \mathbb{R}_{F_n/Q} D_n^x.$$ 

It is a connected reductive linear algebraic group over $\mathbb{Q}$. We identify $G(\mathbb{R})$ with $\text{GL}_2(\mathbb{R}) \Sigma_{E}$ via the identification of $\mathbb{E}_\infty$ with $\mathbb{R} \Sigma_{E}$. Let $h : \mathbb{R}_{C/R} G_m \to G_{\mathbb{R}}$ be the homomorphism defined by

$$h(x + \sqrt{-1} y) = \left( \begin{array}{cc} x & y \\ -y & x \end{array} \right), \cdots, \left( \begin{array}{cc} x & y \\ -y & x \end{array} \right)$$

(2.1)
on $\mathbb{R}$-points. Let $X$ be the $G(\mathbb{R})$-conjugacy class containing $h$. Then $(G, X)$ is a Shimura datum and the associated Shimura variety

$$\text{Sh}(G, X) = \lim_{\overline{K}} \text{Sh}_{\overline{K}}(G, X) = \lim_{\overline{K}} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/\overline{K}$$

is called the quaternionic Shimura variety associated to $(G, X)$, where $\overline{K}$ runs through neat open compact subgroups of $G(\mathbb{A}_f)$. It is a pro-algebraic variety over $\mathbb{C}$ with continuous $G(\mathbb{A}_f)$-action and admits canonical model over $\mathbb{Q}$. Let $K_\infty$ be the stabilizer of $h$ in $G(\mathbb{R})$. Note that

$$K_\infty = Z_G(\mathbb{R}) \cdot SO(2) \Sigma_{E}$$

and we have isomorphisms

$$G(\mathbb{R})/K_\infty \longrightarrow X \longrightarrow (\mathfrak{h}^\pm)^{\Sigma_{E}}, \quad gK_\infty \longmapsto ghg^{-1} \longmapsto g \cdot (\sqrt{-1}, \cdots, \sqrt{-1}).$$

(2.2)

Here $\mathfrak{h}^\pm = \mathfrak{c} \setminus \mathbb{R}$ is the union of the upper and lower half-planes and $G(\mathbb{R})$ acts on $(\mathfrak{h}^\pm)^{\Sigma_{E}}$ by the linear fractional transformation. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G(\mathbb{R})$ and $K_\infty$, respectively. The Hodge decomposition on $\mathfrak{g}_C$ induced by $\text{Ad} \circ h$ is given by

$$\mathfrak{g}_C = p^+ \oplus \mathfrak{t}_C \oplus p^-$$

(2.3)

with $p^+ = \mathfrak{g}_C^{(-1,1)}$, $p^- = \mathfrak{g}_C^{(1,-1)}$, and $\mathfrak{t}_C = \mathfrak{g}_C^{(0,0)}$. Here

$$\mathfrak{g}_C^{(p,q)} = \{X \in \mathfrak{g}_C \mid h(z)^{-1} X h(z) = z^{-p} z_q X \text{ for } z \in \mathbb{C} \}.$$ 

We identify $\mathfrak{g}$ with $\mathfrak{gl}_2^{\Sigma_{E}}$ via the identification of $\mathbb{E}_\infty$ with $\mathbb{R} \Sigma_{E}$. Let $(\kappa, r) \in \mathbb{Z}[\Sigma_{E}] \times \mathbb{Z}[\Sigma_{E}]$ such that $\kappa w \equiv r w \pmod{2}$ for $w \in \Sigma_{E}$. We denote by $\mathbb{C}(\kappa, r)$ the complex field $\mathbb{C}$ equipped with the action $\rho_{(\kappa, r)}$ of $K_\infty$ given by

$$\rho_{(\kappa, r)}(a \cdot k_\theta) z = \prod_{w \in \Sigma_{E}} a_w r w e^{-\sqrt{-1} \kappa w \theta w} \cdot z$$

(2.4)

for $a = (a_w)_{w \in \Sigma_{E}} \in (\mathbb{R}^\times)^{\Sigma_{E}}$ and $k_\theta = (k_{\theta w})_{w \in \Sigma_{E}} \in \text{SO}(2)^{\Sigma_{E}}$, and $z \in \mathbb{C}$. Conversely, any one-dimensional algebraic representation of $K_\infty$ over $\mathbb{C}$ is of this form. We say $\rho_{(\kappa, r)}$ is motivic if $\rho_{(\kappa, r)}|Z_G(\mathbb{R})$ is the base change of a $\mathbb{Q}$-rational character of $Z_G$. It is easy to see that $\rho_{(\kappa, r)}$ is motivic if and only if $(\kappa, r)$ is motivic (cf. Sect. 1.4). Let $A_{\infty}(G(\mathbb{A}))$ (resp. $A_0(G(\mathbb{A}))$) be the space of essentially square-integrable automorphic forms (resp. cusp forms) on $G(\mathbb{A})$. Let

$$C_{s\alpha}(G(\mathbb{A})) = \left\{ \varphi \in C^\infty(G(\mathbb{Q}) \backslash G(\mathbb{A})) \mid X \cdot \varphi \text{ is slowly increasing for all } X \in U(\mathfrak{g}_C) \right\},$$

$$C_{r\alpha}(G(\mathbb{A})) = \left\{ \varphi \in C^\infty(G(\mathbb{Q}) \backslash G(\mathbb{A})) \mid X \cdot \varphi \text{ is rapidly decreasing for all } X \in U(\mathfrak{g}_C) \right\}.$$
Let $\mathfrak{P} = p^- \oplus \mathfrak{t}_C$ be a parabolic subalgebra of $\mathfrak{g}_C$. We have the $(\mathfrak{P}, K_\infty)$-modules
\[
A_\ast (G(\mathbb{A})) \otimes_C C_{(k, \xi)}\mathfrak{L} , \quad C_{q}^\infty (G(\mathbb{A})) \otimes_C C_{(k, \xi)}\mathfrak{L}
\]
for $\ast \in \{(2), 0\}$ and $\ast' \in \{\text{sia}, \text{rda}\}$, where the action of $\mathfrak{P}$ on $C_{(k, \xi)}\mathfrak{L}$ factors through $\mathfrak{t}_C$. Consider the complexes with respect to the Lie algebra differential operator (cf. [4, Chapter I]):
\[
C_{(2), (k, \xi)}^q = \left( A_0 (G(\mathbb{A})) \otimes_C T^+ \otimes_C C_{(k, \xi)}\mathfrak{L} \right)_{K_\infty},
\]
\[
C_{\text{cusp}, (k, \xi)}^q = \left( A_0 (G(\mathbb{A})) \otimes_C T^+ \otimes_C C_{(k, \xi)}\mathfrak{L} \right)_{K_\infty},
\]
\[
C_{\text{sia}, (k, \xi)}^q = \left( C_{\text{sia}}^\infty (G(\mathbb{A})) \otimes_C T^+ \otimes_C C_{(k, \xi)}\mathfrak{L} \right)_{K_\infty},
\]
\[
C_{\text{rda}, (k, \xi)}^q = \left( C_{\text{rda}}^\infty (G(\mathbb{A})) \otimes_C T^+ \otimes_C C_{(k, \xi)}\mathfrak{L} \right)_{K_\infty}
\]
for $q \in \mathbb{Z}_{\geq 0}$. The corresponding $q$-th cohomology groups of the above complexes are denoted respectively by
\[
H_{(2)}^q (\mathfrak{L}_{(k, \xi)}), \quad H_{\text{cusp}}^q (\mathfrak{L}_{(k, \xi)}), \quad H^q (\mathfrak{L}_{\text{can}}), \quad H^q (\mathfrak{L}_{\text{sub}}).
\]
Note that $G(\mathbb{A}_f)$ acts on the above complexes by right translation. This in turn defines $G(\mathbb{A}_f)$-module structures on the cohomology groups. It is clear that $H_{\text{cusp}}^q (\mathfrak{L}_{(k, \xi)})$ and $H_{(2)}^q (\mathfrak{L}_{(k, \xi)})$ are semisimple $G(\mathbb{A}_f)$-modules. By the results of Harris [17] and [18], the relative Lie algebra cohomology groups $H^q (\mathfrak{L}_{\text{can}})$ and $H^q (\mathfrak{L}_{\text{sub}})$ are isomorphic to the $q$-th coherent cohomology groups of the canonical and subcanonical extension, respectively, of certain automorphic line bundle $\mathcal{L}_{(k, \xi)}$ on $\text{Sh}(G, X)$ to a good toroidal compactification. Thus the terminology is justified. The natural inclusions
\[
\begin{array}{c}
A_0 (G(\mathbb{A})) \longrightarrow A_0 (G(\mathbb{A})) \\
C_{\text{rda}}^\infty (G(\mathbb{A})) \longrightarrow C_{\text{sia}}^\infty (G(\mathbb{A}))
\end{array}
\]
induce the following commutative diagram for $G(\mathbb{A}_f)$-module homomorphisms:
\[
\begin{array}{c}
H_{\text{cusp}}^q (\mathfrak{L}_{(k, \xi)}) \longrightarrow H_{(2)}^q (\mathfrak{L}_{(k, \xi)}) \\
H(q (\mathfrak{L}_{\text{sub}})_{(k, \xi)}) \longrightarrow H(q (\mathfrak{L}_{\text{can}})_{(k, \xi)}).
\end{array}
\]
Let $H_{1}^q (\mathfrak{L}_{(k, \xi)})$ be the image of the homomorphism $H^q (\mathfrak{L}_{\text{sub}})_{(k, \xi)} \rightarrow H^q (\mathfrak{L}_{\text{can}})_{(k, \xi)}$. By Theorem 2.1-(3) below, the homomorphism $H_{\text{cusp}}^q (\mathfrak{L}_{(k, \xi)}) \rightarrow H_{1}^q (\mathfrak{L}_{(k, \xi)})$ is injective. We identify $H_{\text{cusp}}^q (\mathfrak{L}_{(k, \xi)})$ with a $G(\mathbb{A}_f)$-submodule of $H^q (\mathfrak{L}_{\text{can}})_{(k, \xi)}$ by this injection.

We recall in the following theorem some results of Harris [13], [18] and Milne [35] specialized to the Shimura datum $(G, X)$. Let $A_0 (G(\mathbb{A}_f), (k, \xi))$ be the space of essentially square-integrable automorphic forms $\varphi$ on $G(\mathbb{A})$ such that

\[\text{(2.6)}\]

\[\text{Springer}\]
\(\varphi(g) = \prod_{w \in \Sigma_E} a_w^g \cdot \varphi(g)\) for \(g = (a_w)_{w \in \Sigma_E} \in ZG(\mathbb{R}) = (\mathbb{R}^\times)^{\Sigma_E}\) and \(g \in G(\mathbb{A})\);

- \(\varphi\) is an eigenfunction of the Casimir operator of \(g_0\) with eigenvalue \(\prod_{w \in \Sigma_E} (\frac{1}{2} k_w^2 - \kappa_w)\).

Let \(\mathcal{A}_0(G(\mathbb{A}), (\kappa, r))\) be the subspace of \(\mathcal{A}_2(G(\mathbb{A}), (\kappa, r))\) consisting of cusp forms on \(G(\mathbb{A})\). We fix a \(\mathbb{Q}\)-rational structure on \(C_{(\kappa, \ell)}\) spanned by a non-zero vector \(v_{(\kappa, \ell)} \in C_{(\kappa, \ell)}\). We also fix a \(\mathbb{E}\)-rational structure on \(p^\pm\) with \(\mathbb{E}\)-basis given by

\[
\{ X_{\pm, w} \mid w \in \Sigma_E \},
\]

where \(X_{\pm, w}\) is defined so that its \(w\)-component is equal to \(X_{\pm}\) and zero otherwise. For \(\sigma \in \text{Aut}(\mathbb{C})\), we have the \(\sigma\)-linear isomorphisms

\[
\begin{align*}
\mathbb{C}_{(\kappa, \ell)} & \longrightarrow \mathbb{C}_{(\kappa, \ell)}^\sigma, \quad z \cdot v_{(\kappa, \ell)} \longmapsto \sigma(z) \cdot v_{(\kappa, \ell)}; \\
p^\pm & \longrightarrow p^\pm, \quad \sum_{w \in \Sigma_E} z_w \cdot X_{\pm, w} \longmapsto \sum_{w \in \Sigma_E} \sigma(z_w) \cdot X_{\pm, \sigma w}.
\end{align*}
\]

(2.7)

Note that we obtain a \(\mathbb{Q}\)-rational structure on \(p^\pm\) by taking the \(\text{Aut}(\mathbb{C})\)-invariants with respect to the above \(\sigma\)-linear isomorphism.

**Theorem 2.1** Assume \((\kappa, \ell) \in \mathbb{Z}[\Sigma_E] \times \mathbb{Z}[\Sigma_E]\) is motivic.

1. For \(\sigma \in \text{Aut}(\mathbb{C})\), with respect to the \(\sigma\)-linear isomorphisms in (2.7), conjugation by \(\sigma\) induces natural \(\sigma\)-linear \(G(\mathbb{A}_f)\)-module isomorphisms

\[
T_\sigma : H^q(L^\text{sub}_{(\kappa, \ell)}) \longrightarrow H^q(L^\text{sub}_{(\kappa, \ell)}^\sigma), \quad T_\sigma : H^q(L^\text{can}_{(\kappa, \ell)}) \longrightarrow H^q(L^\text{can}_{(\kappa, \ell)}^\sigma),
\]

and such that the diagram

\[
\begin{array}{ccc}
H^q(L^\text{sub}_{(\kappa, \ell)}) & \xrightarrow{T_\sigma} & H^q(L^\text{sub}_{(\kappa, \ell)}^\sigma) \\
\downarrow & & \downarrow \\
H^q(L^\text{can}_{(\kappa, \ell)}) & \xrightarrow{T_\sigma} & H^q(L^\text{can}_{(\kappa, \ell)}^\sigma)
\end{array}
\]

commutes. Moreover, \(H^q(L^\text{sub}_{(\kappa, \ell)})\) and \(H^q(L^\text{can}_{(\kappa, \ell)})\) are admissible \(G(\mathbb{A}_f)\)-modules and have canonical rational structures over \(\mathbb{Q}(\kappa)\) given by taking the Galois invariants with respect to \(T_\sigma\) for \(\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\kappa))\).

2. We have

\[
H^q_{\text{cusp}}(L_{(\kappa, \ell)}) = \left( \mathcal{A}_0(G(\mathbb{A}), (\kappa, r)) \otimes_{\mathbb{C}} \bigwedge^q p^+ \otimes_{\mathbb{C}} \mathbb{C}_{(\kappa, \ell)} \right)^{K_{\infty}},
\]

\[
H^q_{(2)}(L_{(\kappa, \ell)}) = \left( \mathcal{A}_2(G(\mathbb{A}), (\kappa, r)) \otimes_{\mathbb{C}} \bigwedge^q p^+ \otimes_{\mathbb{C}} \mathbb{C}_{(\kappa, \ell)} \right)^{K_{\infty}}.
\]

3. The composite of the left vertical and lower horizontal homomorphisms in (2.6) is an injective \(G(\mathbb{A}_f)\)-module homomorphism \(H^q_{\text{cusp}}(L_{(\kappa, \ell)}) \rightarrow H^q(L_{(\kappa, \ell)})\).

4. The image of the homomorphism \(H^q_{(2)}(L_{(\kappa, \ell)}) \rightarrow H^q(L^\text{can}_{(\kappa, \ell)})\) in (2.6) contains \(H^q(L_{(\kappa, \ell)})\). In particular, \(H^q_{(2)}(L_{(\kappa, \ell)})\) is a semisimple \(G(\mathbb{A}_f)\)-module.

Let \(\mathcal{K}\) be a neat open compact subgroup of \(G(\mathbb{A}_f)\). For \(\star = \text{sia or rda}\), we have the complexe \(\mathcal{K}^\star C^\infty_{(\kappa, \ell)}\) analogous to (2.5) with \(C^\infty_{(\kappa, \ell)}\) replacing by \(C^\infty_{(\kappa, \ell)}\). We denote by \(H^q_{\mathcal{K}}(L^\text{can}_{(\kappa, \ell)})\) (resp. \(H^q_{\mathcal{K}}(L^\text{sub}_{(\kappa, \ell)})\)) the corresponding \(q\)-th cohomology group if \(\star = \text{sia} \)
(resp. \( \star = \mathrm{rda} \)). Note that \( H^q_{\mathcal{K}}(\mathcal{L}^\text{can}_{(\mathcal{L},\mathcal{L})}) \) and \( H^q_{\mathcal{K}}(\mathcal{L}^\text{sub}_{(\mathcal{L},\mathcal{L})}) \) are finite dimensional vector spaces over \( \mathbb{C} \) and isomorphic to the \( q \)-th coherent cohomology groups of the canonical and subcanonical extension, respectively, of certain automorphic line bundle \( \mathcal{K} \mathcal{L}_{(\mathcal{L},\mathcal{L})} \) associated to the representation \( \rho_{(\mathcal{L},\mathcal{L})} \) on \( \text{Sh}_{\mathcal{K}}(G, X) \) to a good toroidal compactification. For each \( g \in G(\mathbb{A}_f) \), let \( \mathcal{K}^g = g^{-1} \mathcal{K} g \). The natural isomorphism \( \text{Sh}_{\mathcal{K}^g}(G, X) \to \text{Sh}_{\mathcal{K}}(G, X) \) induces isomorphisms

\[
H^q_{\mathcal{K}}(\mathcal{L}^\text{can}_{(\mathcal{L},\mathcal{L})}) \to H^q_{\mathcal{K}^g}(\mathcal{L}^\text{can}_{(\mathcal{L},\mathcal{L})}), \quad H^q_{\mathcal{K}}(\mathcal{L}^\text{sub}_{(\mathcal{L},\mathcal{L})}) \to H^q_{\mathcal{K}^g}(\mathcal{L}^\text{sub}_{(\mathcal{L},\mathcal{L})}). \tag{2.8}
\]

Then similar assertions as in Theorem 2.1-(1) hold for \( H^q_{\mathcal{K}}(\mathcal{L}^\text{can}_{(\mathcal{L},\mathcal{L})}) \) and \( H^q_{\mathcal{K}}(\mathcal{L}^\text{sub}_{(\mathcal{L},\mathcal{L})}) \) so that the corresponding \( \sigma \)-linear isomorphisms \( T_{\sigma} \) in (1) are compatible with (2.8). Moreover, the natural morphisms of complexes \( \mathcal{K} C^*_{\mathcal{L},(\mathcal{L},\mathcal{L})} \to C^*_{\mathcal{L},(\mathcal{L},\mathcal{L})} \) induce \( G(\mathbb{A}_f) \)-module isomorphisms

\[
\lim_{\mathcal{K} \to \mathcal{K}^g} H^q_{\mathcal{K}}(\mathcal{L}^\text{can}_{(\mathcal{L},\mathcal{L})}) \to H^q(\mathcal{L}^\text{can}_{(\mathcal{L},\mathcal{L})}), \quad \lim_{\mathcal{K} \to \mathcal{K}^g} H^q_{\mathcal{K}}(\mathcal{L}^\text{sub}_{(\mathcal{L},\mathcal{L})}) \to H^q(\mathcal{L}^\text{sub}_{(\mathcal{L},\mathcal{L})})
\]

which are compatible with \( T_{\sigma} \) for all \( \sigma \in \text{Aut}(\mathbb{C}) \). Here the \( G(\mathbb{A}_f) \)-module structure on the direct limits are defined by the isomorphisms (2.8).

Let \( 2 = (2, \cdots, 2) \) and \( \underline{0} = (0, \cdots, 0) \). The automorphic line bundle \( \mathcal{K} \mathcal{L}^\text{sub}_{(\underline{2},\underline{0})} \) is isomorphic to the dualizing sheaf for toroidal compactification of \( \text{Sh}_{\mathcal{K}}(G, X) \) (cf. [18, Proposition 2.2.6]) with trace map

\[
H^{|[\mathbb{E}:\mathbb{Q}|_{(\underline{2},\underline{0})}}_{\mathcal{K}}(\mathcal{L}^\text{sub}_{(\underline{2},\underline{0})}) \to \mathbb{C}, \quad \omega \mapsto \int_{\text{Sh}_{\mathcal{K}}(G, X)} \omega,
\]

where

\[
\int_{\text{Sh}_{\mathcal{K}}(G, X)} \omega = \int_{G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_f)/\mathcal{K}} \varphi \circ t_{\mathcal{K}}(x) \, d\mu_{\mathcal{K}}
\]

if \( \omega \) is represented by \( \varphi \otimes \bigwedge_{w \in \Sigma_E} X_+ , w \otimes v_{(\underline{2},\underline{0})} \in \mathcal{K} C^{|[\mathbb{E}:\mathbb{Q}|_{(\underline{2},\underline{0})}}_{\mathcal{K}} \rightleftarrows \mathbb{C} \rightleftarrows \Sigma_E \). Here

\[
t_{\mathcal{K}} : G(\mathbb{Q}) \times G(\mathbb{A}_f)/\mathcal{K} \to G(\mathbb{Q}) \setminus G(\mathbb{A})/K_{\infty} \mathcal{K}
\]

is the natural isomorphism, and the measure on \( G(\mathbb{Q}) \times G(\mathbb{A}_f)/\mathcal{K} \) is defined as follows:

- on \( G(\mathbb{A}_f)/\mathcal{K} \), we take the counting measure;
- on \( X \), we take the measure \( \left( \frac{dz \wedge d\bar{z}}{2\pi \sqrt{-1}} \right)^{|[\mathbb{E}:\mathbb{Q}|_{(\underline{2},\underline{0})}}_{\mathbb{C}} \rightleftarrows (\Sigma_{E})^{-1} \Sigma_E \) via the isomorphism (2.2).

Under this normalization of measure, we have the following Galois equivariant property:

\[
\sigma \left( \int_{\text{Sh}_{\mathcal{K}}(G, X)} \omega \right) = \int_{\text{Sh}_{\mathcal{K}}(G, X)} T_{\sigma} \omega \tag{2.9}
\]

for \( \omega \in H^{|[\mathbb{E}:\mathbb{Q}|_{(\underline{2},\underline{0})}}_{\mathcal{K}}(\mathcal{L}^\text{sub}_{(\underline{2},\underline{0})}) \) and \( \sigma \in \text{Aut}(\mathbb{C}) \) (cf. [18, (3.8.4)])). In the following lemma, we show that the family of trace maps as \( \mathcal{K} \) varies can be normalized to define a trace map

\[
H^{|[\mathbb{E}:\mathbb{Q}|_{(\underline{2},\underline{0})}}_{\mathcal{K}}(\mathcal{L}^\text{sub}_{(\underline{2},\underline{0})}) \to \mathbb{C}.
\]

**Lemma 2.2** We have a well-defined \( G(\mathbb{A}_f) \)-equivariant trace map

\[
H^{|[\mathbb{E}:\mathbb{Q}|_{(\underline{2},\underline{0})}}_{\mathcal{K}}(\mathcal{L}^\text{sub}_{(\underline{2},\underline{0})}) \to \mathbb{C}, \quad \omega \mapsto \int_{\text{Sh}(G, X)} \omega,
\]

\( \Theta \) Springer
where
\[
\int_{\text{Sh}(G,X)} \omega = \left[ \hat{\phi}_E^\times : \sigma_E^\times \cdot \mathcal{U} \right]^{-1} \int_{Z_G(\hat{\lambda})G(\mathbb{Q})/G(\lambda)} \sum_{a \in E^\times \setminus A_E^\times / E_X^\times \mathcal{U}} \varphi(ag) \, dg_{\text{Tam}}
\]
if \( \omega \) is represented by \( \varphi \otimes \bigwedge_{w \in \Sigma_E} X^+,w \otimes \nu(2,0) \in C^{[\overline{E,Q}]}_{rda,(2,0)} \). Here \( dg_{\text{Tam}} \) is the Tamagawa measure on \( Z_G(\hat{\lambda}) \backslash G(\lambda) \) and \( \mathcal{U} \) is any open compact subgroup of \( \mathbb{A}_{E,f}^\times \) such that \( \varphi \) is right \( \mathcal{U} \)-invariant. Moreover, the trace map satisfies the Galois equivariant property:
\[
\sigma \left( \int_{\text{Sh}(G,X)} \omega \right) = \int_{\text{Sh}(G,X)} T_{\sigma} \omega
\]
for \( \omega \in H^{[\overline{E,Q}]}(L_{\text{sub}}^{\text{sub}}(2,0)) \) and \( \sigma \in \text{Aut}(\mathbb{C}) \).

**Proof**
We identify \( Z_G(\hat{\lambda}) \) with \( \mathbb{A}_{E,f}^\times \). Let \( \omega \in H^{[\overline{E,Q}]}(L_{\text{sub}}^{\text{sub}}(2,0)) \) be a class represented by \( \varphi \otimes \bigwedge_{w \in \Sigma_E} X^+,w \otimes \nu(2,0) \in C^{[\overline{E,Q}]}_{rda,(2,0)} \). For any neat open compact subgroup \( \mathcal{K} \) of \( G(\mathbb{A}_f) \) such that \( \varphi \) is right \( \mathcal{K} \)-invariant, we let \( \mathcal{U}_K = Z_G(\hat{\lambda}_f) \cap \mathcal{K} \) and \( \omega_K \in H^{[\overline{E,Q}]}_{\mathcal{K}}(L_{\text{sub}}^{\text{sub}}(2,0)) \) be the class represented by \( \varphi \otimes \bigwedge_{w \in \Sigma_E} X^+,w \otimes \nu(2,0) \) considered as an element in \( \mathcal{K}^+ C^{[\overline{E,Q}]}_{rda,(2,0)} \). By [28, Lemmas 6.1 and 6.3], there exists a non-zero rational number \( C \) depending only on \( G \) such that
\[
\int_{\text{Sh}_K(G,X)} \omega_K = \int_{G(\mathbb{Q}) \backslash X \times G(\hat{\lambda}_f)/\mathcal{K}} \varphi \circ \iota_K(x) \, d\mu_K
\]
\[
= C \cdot [K_0 : K] \cdot [\hat{\phi}_E^\times : \mathcal{U}_K]^{-1} \int_{G(\mathbb{Q}) \backslash G(\hat{\lambda})/Z_{G(\mathbb{R})}^{\mathcal{U}_K}} \varphi(g) \, dg_{\text{Tam}}.
\]
Here \( K_0 \) is any maximal open compact subgroup of \( G(\mathbb{A}_f) \) containing \( \mathcal{K} \). We remark that the above formula was proved in [28, Lemmas 6.3] for \( \mathcal{U}_K = \hat{\phi}_E^\times \). The general case can be proved in a similar way. We rewrite the formula as
\[
C^{-1} \cdot [K_0 : K]^{-1} \cdot [\hat{\phi}_E^\times : \mathcal{U}_K] \int_{\text{Sh}_K(G,X)} \omega_K
\]
\[
= [\hat{\phi}_E^\times : \sigma_E^\times \cdot \mathcal{U}_K]^{-1} \int_{G(\mathbb{Q}) \backslash G(\hat{\lambda})/Z_{G(\mathbb{R})}^{\mathcal{U}_K}} \varphi(g) \, dg_{\text{Tam}}
\]
\[
= [\hat{\phi}_E^\times : \sigma_E^\times \cdot \mathcal{U}_K]^{-1} \int_{Z_G(\hat{\lambda})G(\mathbb{Q}) \backslash G(\lambda)} \sum_{a \in E^\times \setminus A_E^\times / E_X^\times \mathcal{U}_K} \varphi(ag) \, dg_{\text{Tam}}
\]
Note that the constant \( C^{-1} \cdot [K_0 : K]^{-1} \cdot [\hat{\phi}_E^\times : \mathcal{U}_K] \) depends only on \( G \) and \( \mathcal{K} \). Let \( \mathcal{U} \) be any open compact subgroup of \( \mathbb{A}_{E,f}^\times \) such that \( \varphi \) is right \( \mathcal{U} \)-invariant. Then we have
\[
[\hat{\phi}_E^\times : \sigma_E^\times \cdot (\mathcal{U} \cap \mathcal{U}_K)]^{-1} \int_{Z_G(\hat{\lambda})G(\mathbb{Q}) \backslash G(\lambda)} \sum_{a \in E^\times \setminus A_E^\times / E_X^\times (\mathcal{U} \cap \mathcal{U}_K)} \varphi(ag) \, dg_{\text{Tam}}
\]
\[
= [\hat{\phi}_E^\times : \sigma_E^\times \cdot (\mathcal{U} \cap \mathcal{U}_K)]^{-1} \cdot [E^\times E^\times \mathcal{U} : E^\times E^\times (\mathcal{U} \cap \mathcal{U}_K)]
\]
\[
\int_{Z_G(\hat{\lambda})G(\mathbb{Q}) \backslash G(\lambda)} \sum_{a \in E^\times \setminus A_E^\times / E_X^\times \mathcal{U}} \varphi(ag) \, dg_{\text{Tam}}
\]
\[
= [\hat{\phi}_E^\times : \sigma_E^\times \cdot \mathcal{U}]^{-1} \int_{Z_G(\hat{\lambda})G(\mathbb{Q}) \backslash G(\lambda)} \sum_{a \in E^\times \setminus A_E^\times / E_X^\times \mathcal{U}} \varphi(ag) \, dg_{\text{Tam}}.
\]
We conclude that the trace map \( \omega \mapsto \int_{\text{Sh}(\mathcal{G}, \mathcal{X})} \omega \) is well-defined. Finally, the Galois equivariance property follows from \( C \in \mathbb{Q}^\times \) and (2.9). This completes the proof.

### 2.2 Rational structures via the coherent cohomology

Let \( \Pi = \bigotimes_v \Pi_v \) be an irreducible admissible automorphic representation of \( G(\mathbb{A}) \), where \( v \) runs through the places of \( \mathbb{Q} \). Let \( \Pi_f = \bigotimes_p \Pi_p \) be the finite part of \( \Pi \). We assume that \( \Pi \) is motivic, that is, there exists motivic \((\kappa, \mathcal{R}) \in \mathbb{Z}_{\geq 1}[\Sigma_\mathbb{E}] \times \mathbb{Z}[\Sigma_\mathbb{E}] \) such that

\[
\Pi_\infty = \bigotimes_{w \in \Sigma_\mathbb{E}} (D(\kappa_w) \otimes | -r_w/2 |).
\]

Here \( D(\kappa_w) \) is the (limit of) discrete series representation of \( GL_2(\mathbb{R}) \) with weight \( \kappa_w \) (cf. Sect. 1.4). We call \( \kappa \) (resp. \( (\kappa, \mathcal{R}) \)) the weight (resp. motivic weight) of \( \Pi \). When \( \mathbb{E} \) is a field, we necessarily have \( \mathcal{R} = (r, \cdots, r) \) for some \( r \in \mathbb{Z} \) and also call \( (\kappa, \mathcal{R}) \in \mathbb{Z}_{\geq 1}[\Sigma_\mathbb{E}] \times \mathbb{Z} \) the motivic weight of \( \Pi \). Note that \( \Pi \) occurs in \( A_0(G(\mathbb{A}), (\kappa, \mathcal{R})) \). For each motivic \((\kappa', \mathcal{R}') \in \mathbb{Z}[\Sigma_\mathbb{E}] \times \mathbb{Z}[\Sigma_\mathbb{E}] \) and \( \star \in \{{\text{cusp}}, (2), !\} \), we denote by \( H^q_\mathrm{cusp}(\mathcal{L}(\kappa', \mathcal{R}'))[\Pi_f] \) the \( \Pi_f \)-isotypic component of \( \Pi_f \) in \( H^q(\mathcal{L}(\kappa', \mathcal{R}')) \).

**Lemma 2.3** Let \((\kappa', \mathcal{R}') \in \mathbb{Z}[\Sigma_\mathbb{E}] \times \mathbb{Z}[\Sigma_\mathbb{E}]\) be motivic.

1. If \( \mathcal{R} \neq \mathcal{R}' \) or \((\kappa_w - \kappa'_w)(\kappa_w + \kappa'_w - 2) \neq 0 \) for some \( w \in \Sigma_\mathbb{E} \), then \( H^q_\mathrm{cusp}(\mathcal{L}(\kappa', \mathcal{R}'))[\Pi_f] = 0 \) for all \( q \) and \( \star \in \{{\text{cusp}}, (2), !\} \).
2. If \( \mathcal{R} = \mathcal{R}' \) and \((\kappa_w - \kappa'_w)(\kappa_w + \kappa'_w - 2) = 0 \) for all \( w \in \Sigma_\mathbb{E} \), then the homomorphism in Theorem 2.1-(3) induces an isomorphism of \( G(\mathbb{A}_f) \)-modules \( H^q_\mathrm{cusp}(\mathcal{L}(\kappa', \mathcal{R}'))[\Pi_f] \to H^q(\mathcal{L}(\kappa', \mathcal{R}'))[\Pi_f] \). Moreover, the multiplicity of \( \Pi_f \) in \( H^q_\mathrm{cusp}(\mathcal{L}(\kappa', \mathcal{R}'))[\Pi_f] \) is equal to the number of subsets \( I \) of \( \Sigma_\mathbb{E} \) such that

\[
\kappa' = \kappa(I), \quad q = \sharp I.
\]

**Proof** For each subset \( I \) of \( \Sigma_\mathbb{E} \), let

\[
\varepsilon(I)_w = \begin{cases} -1 & \text{if } w \in I, \\ +1 & \text{if } w \notin I \end{cases}
\]

for \( w \in \Sigma_\mathbb{E} \) and

\[
\Pi_{\infty, I} = \bigotimes_{w \in \Sigma_\mathbb{E}} (D(\kappa_w)^{\varepsilon(I)_w} \otimes | -r_w/2 |)
\]

be an irreducible admissible \((\mathfrak{g}, K_\infty)\)-module. Note that we have a \((\mathfrak{g}, K_\infty)\)-module isomorphism

\[
\Pi_\infty \simeq \bigoplus_{I \subseteq \Sigma_\mathbb{E}} \Pi_{\infty, I}.
\]

Specializing [18, Theorem 4.6.2] to the (limit of) discrete series representation \( \Pi_{\infty, I} \), we have

\[
\dim \left( \Pi_{\infty, I} \otimes_{\mathbb{C}} \bigoplus_{q} \mathbb{P}^+ \otimes_{\mathbb{C}} \mathbb{C}(\kappa', \mathcal{R}') \right)_{K_\infty} = \begin{cases} 0 & \text{if } (\kappa', \mathcal{R}') \neq (\kappa(I), \mathcal{R}) \text{ or } q \neq \sharp I, \\ 1 & \text{if } (\kappa', \mathcal{R}') = (\kappa(I), \mathcal{R}) \text{ and } q = \sharp I. \end{cases}
\]

(2.10)
On the other hand, by the strong multiplicity one theorem, we have
\[
\mathcal{A}_0(G(\mathbb{A}), (\mathfrak{r}, \varrho)) [\Pi_f] = \mathcal{A}_0(G(\mathbb{A}), (\mathfrak{r}, \varrho)) [\Pi_f] = \Pi.
\]
The assertions then follow from Theorem 2.1-(2)-(4) and (2.10). This completes the proof. 

For \( \sigma \in \text{Aut}(\mathbb{C}) \), let \( \sigma \Pi \) be the irreducible admissible representation of \( G(\mathbb{A}) \) defined by
\[
\sigma \Pi = \sigma \Pi_\infty \otimes \sigma \Pi_f,
\]
where \( \sigma \Pi_f \) is the \( \sigma \)-conjugate of \( \Pi_f \) and \( \sigma \Pi_\infty \) is the representation of \( G(\mathbb{R}) = \text{GL}_2(\mathbb{R})^\Sigma_{\mathbb{E}} \) so that its \( w \)-component is equal to the \( \sigma^{-1} \circ w \)-component of \( \Pi_\infty \). The following lemma is well-known and can be deduced from the result of Shimura [44, Proposition 1.6]. When \( \kappa_w \geq 2 \) for all \( w \in \Sigma_{\mathbb{E}} \), the lemma was also proved by Waldspurger [50] and Harder [15] and is based on the study of \( (g, K_\infty) \)-cohomology. We provide another proof based on the results of Harris [18], which is \( (\mathfrak{g}, K_\infty) \)-cohomological in natural.

**Lemma 2.4** For \( \sigma \in \text{Aut}(\mathbb{C}) \), the representation \( \sigma \Pi \) is a motivic irreducible cuspidal automorphic representation of \( G(\mathbb{A}) \) of motivic weight \( (\sigma \kappa_r, r) \). Moreover, the rationality field \( \mathbb{Q}(\Pi) \) of \( \Pi \) is equal to the fixed field of \( \{ \sigma \in \text{Aut}(\mathbb{C}) \} | \sigma \Pi = \Pi \} \).

**Proof** Fix \( \sigma \in \text{Aut}(\mathbb{C}) \). Since \( H^0_!([\mathcal{L}_{(\mathfrak{r}, r)}]) [\sigma \Pi_f] \simeq \Pi_f \), we have
\[
H^0_!([\mathcal{L}_{(\mathfrak{r}, r)}]) [\sigma \Pi_f] = T_\sigma(H^0_!([\mathcal{L}_{(\mathfrak{r}, r)}]) [\Pi_f]) \simeq \sigma \Pi_f.
\]
On the other hand, the homomorphism in Theorem 2.1-(3) is an isomorphism by [18, Proposition 5.4.2]. It follows that
\[
H^0_{\text{cusp}}([\mathcal{L}_{(\mathfrak{r}, r)}]) [\sigma \Pi_f] = (A_0(G(\mathbb{A}), (\mathfrak{r}, \varrho)) [\sigma \Pi_f] \otimes_{\mathbb{C}} \mathcal{C}_{(\mathfrak{r}, r)}) \mathbb{K}_\infty \simeq \sigma \Pi_f.
\]
We conclude that \( A_0(G(\mathbb{A}), (\mathfrak{r}, \varrho)) [\sigma \Pi_f] = \Pi' \) for some irreducible cuspidal automorphic representation \( \Pi' = \Pi_\infty \otimes \sigma \Pi_f \) of \( G(\mathbb{A}) \) such that
\[
(\Pi'_\infty \otimes_{\mathbb{C}} \mathcal{C}_{(\mathfrak{r}, r)}) \mathbb{K}_\infty \neq 0.
\]
This implies that there exists a non-zero vector \( v \in \Pi'_\infty \) such that
\[
\Pi'_\infty (a \cdot \mathbf{k}_a) v = \prod_{w \in \Sigma_{\mathbb{E}}} a^w \mathbf{e}^\mathbf{r} \mathbf{k}_w \mathbf{v} \quad (2.11)
\]
for \( a = (a_w)_{w \in \Sigma_{\mathbb{E}}} \in (\mathbb{R}^\times)^{\Sigma_{\mathbb{E}}} \) and \( \mathbf{k}_a = (k_{\theta_w})_{w \in \Sigma_{\mathbb{E}}} \in \text{SO}(2)^{\Sigma_{\mathbb{E}}} \). In particular, we have
\[
(H, \cdots, H) \cdot v = \prod_{w \in \Sigma_{\mathbb{E}}} \sigma \mathbf{k}_w \cdot v = \prod_{w \in \Sigma_{\mathbb{E}}} \mathbf{k}_w \cdot v.
\]
Note that the Casimir operator of \( \mathfrak{g}(2, \mathbb{C}) \) is given by
\[
\frac{1}{2} H^2 - H - \frac{1}{2} X_+ X_-.
\]
We deduce from the second condition defining the space \( A_0(G(\mathbb{A}), (\mathfrak{r}, r)) \) that
\[
(X_+ X_-, \cdots, X_+ X_-) \cdot v = 0. \quad (2.12)
\]
By [30, Lemma 5.6], we see that (2.11) and (2.12) imply that

$$\Pi'_\infty = \bigotimes_{w \in \Sigma_E} (D(\sigma_{k_w}) \otimes |r_w/2|).$$

Therefore $\Pi'$ is isomorphic to $\sigma\Pi$. For the second assertion, assume $\sigma\Pi_f = \Pi_f$. Then it follows from the strong multiplicity one theorem that $\sigma\Pi = \Pi$. This completes the proof. □

**Lemma 2.5** Let $I \subseteq \Sigma_E$. For any field extension $\mathbb{Q}(\Pi, I) \subseteq F \subseteq \mathbb{C}$, we have the $F$-rational structure on $H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})[\Pi_f]$ by taking the $\text{Aut}(\mathbb{C}/F)$-invariants

$$H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})[\Pi_f]^{\text{Aut}(\mathbb{C}/F)} = \left\{ c \in H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})[\Pi_f] \mid T_{\sigma} c = c \text{ for all } \sigma \in \text{Aut}(\mathbb{C}/F) \right\}. $$

Here $\mathbb{Q}(\Pi, I) = \mathbb{Q}(\Pi) \cdot \mathbb{Q}(I)$. Moreover, suppose that $I$ is admissible with respect to $k$, then the $\mathbb{G}(\mathbb{A}_f)$-module $H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})[\Pi_f]$ is isomorphic to $\Pi_f$ and the $F$-rational structures are unique up to homotheties.

**Proof** By the commutativity of the diagram in Theorem 2.1-(1) and Lemma 2.3-(2), we have

$$H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})[\Pi_f] = T_{\sigma} (H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})[\Pi_f])$$

(2.13)

for all $\sigma \in \text{Aut}(\mathbb{C})$. Let $\mathbb{Q}(\Pi) \cdot \mathbb{Q}(I) \subseteq F \subseteq \mathbb{C}$ be a field extension. By Theorem 2.1-(1), $H^2(\mathcal{L}_{(k, I, \underline{\xi})})$ admits a $F$-rational structure given by

$$H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})^{\text{Aut}(\mathbb{C}/F)} = \left\{ c \in H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})}) \mid T_{\sigma} c = c \text{ for all } \sigma \in \text{Aut}(\mathbb{C}/F) \right\}. $$

By (2.13), $H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})[\Pi_f]$ is invariant by $T_{\sigma}$ for all $\sigma \in \text{Aut}(\mathbb{C}/F)$. Therefore, by [8, Lemme 3.2.1], we have a $F$-rational structure on $H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})[\Pi_f]$ given by

$$H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})[\Pi_f] \cap H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})^{\text{Aut}(\mathbb{C}/F)} = H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})[\Pi_f]^{\text{Aut}(\mathbb{C}/F)}.$$

Finally, suppose that $I$ is admissible with respect to $k$. Hence $H^2_{\text{cusp}}(\mathcal{L}_{(k, I, \underline{\xi})})[\Pi_f] \simeq \Pi_f$ is irreducible by Lemma 2.3-(2) and the uniqueness up to homotheties then follows from Schur’s lemma. This completes the proof. □

### 2.3 Harris’ periods

Let $\Pi$ be a motivic irreducible cuspidal automorphic representation of $G(\mathbb{A})$ of motivic weight $(k, r) \in \mathbb{Z}_{\geq 1}[\Sigma_E] \times \mathbb{Z}[\Sigma_E]$. Let $\Pi_{\text{hol}}$ be the subspace of $\Pi$ consisting of holomorphic cusp forms $\varphi \in \Pi$, that is,

$$\varphi(k_{\theta_w}) = e^{r_{\Gamma} \sum_{w \in \Sigma_E} x_w \theta_w} \varphi(g)$$

for $k_{\theta_w} = (k_{\theta_w})_{w \in \Sigma_E} \in \text{SO}(2)_{\Sigma_E}$ and $g \in G(\mathbb{A})$. Let $I$ be a subset of $\Sigma_E$. Define $\tau_I \in G(\mathbb{R}) = \text{GL}_2(\mathbb{R})_{\Sigma_E}$ by

$$\tau_I = (a(\varphi(I)_w))_{w \in \Sigma_E},$$

(2.14)

where

$$\varphi(I)_w = \begin{cases} -1 & \text{if } w \in I, \\ 1 & \text{if } w \notin I. \end{cases}$$
For \( \varphi \in \Pi_{\text{hol}} \), let \( \varphi^I \in \Pi \) defined by

\[
\varphi^I(g) = \varphi(g \cdot \tau_I)
\]

for \( g \in G(\mathbb{A}) \). We then have the homomorphism of \( G(\mathbb{A}_f) \)-modules:

\[
\xi_I : \Pi_{\text{hol}} \to H_{\text{cusp}}^I(\mathcal{L}(\kappa(I), \xi))[\Pi_f], \quad \varphi \mapsto \varphi^I \otimes \bigwedge_{w \in I} X_{+,w} \otimes v(\kappa(I), \xi).
\]

Here we fix an ordering of the wedge \( \bigwedge_{w \in I} X_{+,w} \) once and for all such that \( \bigwedge_{w \in \sigma I} X_{+,w} \) under the \( \sigma \)-linear isomorphism in (2.7) for all \( \sigma \in \text{Aut}(\mathbb{C}) \). Note that \( \xi_I \) is an isomorphism if and only if \( I \) is admissible with respect to \( \kappa \) by Lemma 2.3-(2). By taking \( I = \emptyset \) to be the empty set, we identify \( \Pi_{\text{hol}} \) with \( H_{\text{cusp}}^0(\mathcal{L}(\kappa(I), \xi))[\Pi_f] \) by the isomorphism \( \xi_\emptyset \). Moreover, for \( \sigma \in \text{Aut}(\mathbb{C}) \) we have the \( \sigma \)-linear isomorphism

\[
\Pi_{\text{hol}} \to \sigma \Pi_{\text{hol}}, \quad \varphi \mapsto \sigma \varphi
\]

defined so that the diagram

\[
\begin{array}{ccc}
\Pi_{\text{hol}} & \xrightarrow{\xi_\emptyset} & \sigma \Pi_{\text{hol}} \\
\downarrow & & \downarrow \\
H_{\text{cusp}}^0(\mathcal{L}(\kappa(I), \xi))[\Pi_f] & \xrightarrow{T_\sigma} & H_{\text{cusp}}^0(\mathcal{L}(\kappa(I), \xi))[\sigma \Pi_f]
\end{array}
\]

is commute. Comparing the \( \mathbb{Q}(\Pi, I) \)-rational structures on \( H_{\text{cusp}}^0(\mathcal{L}(\kappa(I), \xi))[\Pi_f] \) and \( H_{\text{cusp}}^I(\mathcal{L}(\kappa(I), \xi))[\Pi_f] \) defined in Lemma 2.5, we obtain the following result.

**Proposition 2.6** Let \( I \subseteq \Sigma_E \) be admissible with respect to \( \kappa \). There exists \( \Omega^I(\Pi) \in \mathbb{C}^\times \), unique up to \( \mathbb{Q}(\Pi, I)^\times \), such that

\[
\xi_I \left( \Pi_{\text{hol}}^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi, I))} \right) = H_{\text{cusp}}^I(\mathcal{L}(\kappa(I), \xi))[\Pi_f]^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\cdot, I))}.
\]

Moreover, we can normalize the periods so that

\[
T_\sigma \left( \frac{\xi_I(\varphi)}{\Omega^I(\Pi)} \right) = \frac{\xi_{\sigma I}(\sigma \varphi)}{\Omega^{\sigma I}(\sigma \Pi)}
\]

for \( \varphi \in \Pi_{\text{hol}} \) and \( \sigma \in \text{Aut}(\mathbb{C}) \).

**Remark 2.7** For non-admissible \( I \subseteq \Sigma_E \), the multiplicity of \( \Pi_f \) in \( H_{\text{cusp}}^I(\mathcal{L}(\kappa(I), \xi))[\Pi_f] \) is greater than one and it is not known whether the equality \( T_\sigma \circ \xi_I(\Pi_{\text{hol}}) = \xi_{\sigma I}(\sigma \Pi_{\text{hol}}) \) holds for any \( \sigma \in \text{Aut}(\mathbb{C}) \). Therefore, we do not know whether \( \xi_I(\Pi_{\text{hol}}) \) is defined over \( \mathbb{Q}(\Pi, I) \). However, since \( H_{\text{cusp}}^I(\mathcal{L}(\kappa(I), \xi))[\Pi_f] \) is defined over \( \mathbb{Q}(\Pi) \cdot \mathbb{Q}(\kappa(I)) \), it follows that \( \xi_I(\Pi_{\text{hol}}) \) is defined over some finite extension \( F \) over \( \mathbb{Q}(\Pi) \cdot \mathbb{Q}(\kappa(I)) \). Hence there exists \( \Omega^I(\Pi) \in \mathbb{C}^\times \), unique up to \( F^\times \), such that

\[
\xi_I \left( \Pi_{\text{hol}}^{\text{Aut}(\mathbb{C}/F)} \right) = \xi_I(\Pi_{\text{hol}}) \cap H_{\text{cusp}}^I(\mathcal{L}(\kappa(I), \xi))[\Pi_f]^{\text{Aut}(\mathbb{C}/F)}.
\]

We have assumed that \( E = F_1 \times \cdots \times F_n \) and \( D = D_1 \times \cdots \times D_n \) for some totally real number fields \( F_i \) and some totally indefinite quaternion algebra \( D_i \) over \( F_i \). Then

\[
\Pi = \Pi_1 \times \cdots \times \Pi_n
\]
for some motivic irreducible cuspidal automorphic representation \( \Pi_i \) of \( D_i^\times(\mathbb{A}_{\mathbb{F}_i}) \) for \( 1 \leq i \leq n \). We identify \( \Sigma_E \) with the disjoint union of \( \Sigma_{\mathbb{F}_i} \) for \( 1 \leq i \leq n \) in a natural way. Then

\[
I = \bigcap_{i=1}^{n} I \cap \Sigma_{\mathbb{F}_i}.
\]

We have the following period relation.

**Lemma 2.8** Let \( I \subseteq \Sigma_E \) be admissible with respect to \( \kappa \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), we have

\[
\sigma \left( \prod_{i=1}^{n} \Omega^I(\Pi_{\Pi_i}) \right) = \prod_{i=1}^{n} \Omega^I(\sigma \Pi_i).
\]

**Proof** Indeed, we have

\[
\text{Sh}(G, X) = \text{Sh}(G_1, X_1) \times \cdots \times \text{Sh}(G_n, X_n),
\]

where \( (G_i, X_i) = (R_{\mathbb{F}_i/\mathbb{Q}} D_i^\times, (\mathfrak{g}^\pm)_{\Sigma_{\mathbb{F}_i}}) \). Write

\[
(\kappa, r) = (\kappa_1, r_1) \times \cdots \times (\kappa_n, r_n)
\]

under the identification

\[
\mathbb{Z}_{\geq 1}[\Sigma_E] \times \mathbb{Z}[\Sigma_E] = (\mathbb{Z}_{\geq 1}[\Sigma_{\mathbb{F}_1}] \times \mathbb{Z}[\Sigma_{\mathbb{F}_1}]) \times \cdots \times (\mathbb{Z}_{\geq 1}[\Sigma_{\mathbb{F}_n}] \times \mathbb{Z}[\Sigma_{\mathbb{F}_n}]).
\]

Then we have

\[
\mathcal{L}_{(\kappa, r)} = \prod_{i=1}^{n} \mathcal{L}_{(\kappa_i, r_i)}.
\]

Then it follows from the Künneth formula that we have a canonical \( G(\mathbb{A}_f) \)-module isomorphism

\[
H^1(\mathcal{L}^*_{(\kappa, r)}) \simeq \bigoplus_{q_1 \cdots q_n = q} \bigotimes_{i=1}^{n} H^q_i(\mathcal{L}^*_{(\kappa_i, r_i)}).
\]

for \( \star \in \{\text{sub}, \text{can}\} \). Taking the \( \Pi_f \)-isotypic parts and note that \( H^q_i(\mathcal{L}_{(\kappa_i, r_i)})[\pi_i, f] \) is zero unless \( q_i = \sharp I \cap \Sigma_{\mathbb{F}_i} \) by Lemma 2.3 and the admissibility of \( I \), we thus obtain an isomorphism

\[
H^1(\mathcal{L}_{(\kappa, r)})[\Pi_f] \simeq \bigotimes_{i=1}^{n} H^1(\mathcal{L}_{(\kappa_i, r_i)})[\pi_i, f].
\]

We deduce from Lemma 2.3-(2) that

\[
H^1_{\text{cusp}}(\mathcal{L}_{(\kappa, r)})[\Pi_f] \simeq \bigotimes_{i=1}^{n} H^1_{\text{cusp}}(\mathcal{L}_{(\kappa_i, r_i)})[\pi_i, f].
\]

In the above isomorphism, we normalize the \( \mathbb{Q}(\kappa_i) \)-rational structure on \( \mathbb{C}_{(\kappa_i, r_i)} \) for \( 1 \leq i \leq n \) such that

\[
\mathbb{V}_{(\kappa_i, r_i)} = \bigotimes_{i=1}^{n} \mathbb{V}_{(\kappa_i, r_i)}
\]
under the isomorphism

$$(\rho_{(\xi,\mathcal{L})}, C_{(\xi,\mathcal{L})}) \simeq \bigotimes_{i=1}^{n} (\rho_{(\xi_{i},\mathcal{L}_{i})}, C_{(\xi_{i},\mathcal{L}_{i})}).$$

Then for $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$T_\sigma = T_\sigma^{(1)} \otimes \cdots \otimes T_\sigma^{(n)}.$$

Here

$$T_\sigma^{(i)} : H^{i}I^{1} \otimes \Sigma_{E} \left( \mathcal{L}_{(\xi, I^{1} \otimes \Sigma_{E})}^{\text{sub}} \right) \to H^{i}I^{1} \otimes \Sigma_{E} \left( \mathcal{L}_{(\xi, I^{1} \otimes \Sigma_{E})}^{\text{sub}} \right)$$

is the $\sigma$-linear isomorphism in Theorem 2.1 -(1). The assertion then follows at once. □

For $I = \Sigma_{E}$, the period $\Omega_{E}^{\Sigma_{E}}(\Pi)$ can be expressed in terms of the Petersson pairing of holomorphic cusp forms. Let $\langle \ , \ \rangle : \Pi_{\text{hol}} \times \Pi_{\text{hol}}^{\vee} \to \mathbb{C}$ be the $G(\mathbb{A}_{f})$-equivariant Petersson bilinear pairing defined by

$$\langle \varphi_{1}, \varphi_{2} \rangle = \int_{Z_{G}(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi_{1}^{\Sigma_{E}}(g)\varphi_{2}(g) \, dg_{\text{Tam}}^{\text{Tam}}. \tag{2.18}$$

Here $dg_{\text{Tam}}^{\text{Tam}}$ is the Tamagawa measure on $Z_{G}(\mathbb{A})\backslash G(\mathbb{A})$.

**Lemma 2.9** We have

$$\sigma \left( \langle \varphi_{1}, \varphi_{2} \rangle \right) = \langle \sigma \varphi_{1}, \sigma \varphi_{2} \rangle \frac{\Omega_{E}^{\Sigma_{E}}(\Pi)}{\Omega_{E}^{\Sigma_{E}}(\sigma \Pi)}$$

for $\varphi_{1} \in \Pi_{\text{hol}}, \varphi_{2} \in \Pi_{\text{hol}}^{\vee}$, and $\sigma \in \text{Aut}(\mathbb{C})$.

**Proof** We have the morphism for complexes

$$C_{(\xi, \mathcal{L})}^{[\mathbb{E} \setminus \mathcal{Q}]} \times C_{\text{sil}, (\xi_{i}, -\mathcal{L}_{i})}^{0} \to C_{\text{sil}, (\xi, -\mathcal{L})}^{[\mathbb{E} \setminus \mathcal{Q}]}$$

$$\left( \varphi_{1} \otimes \bigwedge_{w \in \Sigma_{E}} X_{+, w} \otimes v_{(2-\mathcal{L}, \mathcal{L})}, \varphi_{2} \otimes v_{(\xi, -\mathcal{L})} \right) \mapsto \varphi_{1} \varphi_{2} \otimes \bigwedge_{w \in \Sigma_{E}} X_{+, w} \otimes v_{(2-\mathcal{L}, \mathcal{L})}.$$

This induces $G(\mathbb{A}_{f})$-module homorphism of cohomology groups

$$H^{[\mathbb{E} \setminus \mathcal{Q}]}(\mathcal{L}_{(2-\mathcal{L}, -\mathcal{L})}^{\text{can}}) \times H^{0}(\mathcal{L}_{(2-\mathcal{L}, -\mathcal{L})}^{\text{can}}) \to H^{[\mathbb{E} \setminus \mathcal{Q}]}(\mathcal{L}_{(2-\mathcal{L}, -\mathcal{L})}^{\text{can}}), \quad (c_{1}, c_{2}) \mapsto c_{1} \wedge c_{2}.$$

The homomorphism satisfies the Galois equivariant property

$$T_{\sigma}(c_{1} \wedge c_{2}) = T_{\sigma}c_{1} \wedge T_{\sigma}c_{2}$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. Composing with the trace map in Lemma 2.2, we obtain the $G(\mathbb{A}_{f})$-equivariant homomorphism

$$\Pi_{\text{hol}} \times \Pi_{\text{hol}}^{\vee} \to \mathbb{C}, \quad (\varphi_{1}, \varphi_{2}) \mapsto \int_{\text{Sh}(G, X)} \xi_{E}(\varphi_{1}) \wedge \xi_{\phi}(\varphi_{2})$$

which satisfies

$$\sigma \left( \int_{\text{Sh}(G, X)} \xi_{E}(\varphi_{1}) \wedge \xi_{\phi}(\varphi_{2}) \right) = \int_{\text{Sh}(G, X)} \xi_{E}^{\sigma}(\varphi_{1}) \wedge \xi_{\phi}^{\sigma}(\varphi_{2})$$
for all $\sigma \in \text{Aut}(\mathbb{C})$. Note that we may take $\Omega^\vee(\Pi') = 1$ by definition. Since the class $\xi_{\Sigma E}(\varphi_1) \wedge \xi_{\varphi_2}$ in $H^{1|E:Q}(L_{\Sigma E}(2,0))$ is represented by $\varphi_1^{\Sigma E} \cdot \varphi_2 \otimes \bigwedge_{w \in \Sigma E} \chi_{+}w \otimes \nu(2,0)$, by Lemma 2.2 we have
\[
\int_{\text{Sh}(G,X)} \xi_{\Sigma E}(\varphi_1) \wedge \xi_{\varphi_2} = [\hat{\Delta}_{\Sigma E}^{\infty} : \hat{E}_{\Sigma E}^{\infty} \hat{\Delta}_{\Sigma E}^{\infty}] \cdot \langle \varphi_1, \varphi_2 \rangle.
\]
This completes the proof. \hfill \Box

Let
\[
L(s, \Pi, \text{Ad}) = \prod_v L(s, \Pi_v, \text{Ad})
\]
be the adjoint $L$-function of $\Pi$, where $\text{Ad}$ is the adjoint representation of $L$ $G$ on $pgl_2(\mathbb{C})^{[E:Q]}$. Note that $L(s, \Pi, \text{Ad})$ is holomorphic and non-zero at $s = 1$. Combining with the results of Shimura [45] and Takase [48], we obtain the following corollary.

**Corollary 2.10** We have
\[
\sigma \left( \frac{L(1, \Pi, \text{Ad})}{(2\pi \sqrt{-1})^{-\sum_{w \in \Sigma E} k_w \cdot \left[ E:Q \right]} \cdot \Omega_{\Sigma E}(\Pi)} \right) = \frac{L(1, \sigma \Pi, \text{Ad})}{(2\pi \sqrt{-1})^{-\sum_{w \in \Sigma E} k_w \cdot \left[ E:Q \right]} \cdot \Omega_{\Sigma E}(\sigma \Pi)}
\]
for all $\sigma \in \text{Aut}(\mathbb{C})$.

**Proof** By Lemma 2.8, it suffices to consider the case when $E$ is a field. Let $\Pi'$ be the Jacquet-Langlands transfer of $\Pi$ to $\text{GL}_2(\hat{A}_E)$. By a variant of the result [22, Theorem 12.3] (see also [45, Theorem 3.8]), we have
\[
\sigma \left( \frac{\langle \varphi_1, \varphi_2 \rangle}{\langle \varphi_3, \varphi_4 \rangle} \right) = \frac{\langle \sigma \varphi_1, \sigma \varphi_2 \rangle}{\langle \sigma \varphi_3, \sigma \varphi_4 \rangle}
\]
(2.19)
for $\varphi_1 \in \Pi_{\text{hol}}, \varphi_2 \in \Pi'_{\text{hol}}, \varphi_3 \in \Pi'_{\text{hol}}, \varphi_4 \in (\Pi')_{\text{hol}}$ with $\langle \varphi_3, \varphi_4 \rangle \neq 0$ and $\sigma \in \text{Aut}(\mathbb{C})$. We remark that although the assertion in [22, Theorem 12.3] is stated for $E = Q$, one can prove (2.19) for general totally real number fields following the same argument in [22, Sect. 15.2]. On the other hand, it follows from the result of Takase [48, Proposition 1] that
\[
\sigma \left( \frac{L(1, \Pi, \text{Ad})}{(2\pi \sqrt{-1})^{-\sum_{w \in \Sigma E} k_w \cdot \left[ E:Q \right]} \cdot \langle \varphi_1, \varphi_2 \rangle} \right) = \frac{L(1, \sigma \Pi, \text{Ad})}{(2\pi \sqrt{-1})^{-\sum_{w \in \Sigma E} k_w \cdot \left[ E:Q \right]} \cdot \langle \sigma \varphi_1, \sigma \varphi_2 \rangle}
\]
for $\varphi_1 \in \Pi'_{\text{hol}}, \varphi_2 \in (\Pi')_{\text{hol}}$ with $\langle \varphi_1, \varphi_2 \rangle \neq 0$, and $\sigma \in \text{Aut}(\mathbb{C})$. We remark that the factor $(2\pi \sqrt{-1})^{-\sum_{w \in \Sigma E} k_w}$ is obtained from the comparison between rational structures on $\Pi'_{\text{hol}}$ given by the zeroth coherent cohomology and by the Whittaker model (cf. [11, (A.4.6)]). The assertion then follows immediately from Lemma 2.9. \hfill \Box

3 Trilinear differential operators

Let $E$ be a totally real étale cubic algebra over a totally real number field $F$. Let $D$ be a totally indefinite quaternion algebra over $F$. Let
\[
G' = R_{F/Q} D^\times, \quad G = R_{E/Q} (D \otimes_F E)^\times
\]
be connected reductive linear algebraic groups over \( \mathbb{Q} \). We identify \( G'(\mathbb{R}) \) and \( G(\mathbb{R}) \) with \( \text{GL}_2(\mathbb{R})^{\Sigma_F} \) and \( \text{GL}_3(\mathbb{R})^{\Sigma_E} \) via the identifications \( \mathbb{F}_\infty = \mathbb{R}^{\Sigma_F} \) and \( \mathbb{E}_\infty = \mathbb{R}^{\Sigma_E} \), respectively. Let \( X' \) (resp. \( X \)) be the \( G'(\mathbb{R}) \)-conjugacy class (resp. \( G(\mathbb{R}) \)-conjugacy class) containing \( h' \) (resp. \( h \)) defined as in (2.1). The natural diagonal embedding \( \mathbb{F} \rightarrow \mathbb{E} \) induces the natural injective morphism \( G' \rightarrow G \), which defines the inclusion of Shimura data

\[
(G', X') \subset (G, X).
\]

We begin with a well-known lemma.

**Lemma 3.1** Let \( (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}_{\geq 1}^3 \) such that \( \ell_1 \geq \ell_2 + \ell_3 \) and \( \ell_1 + \ell_2 + \ell_3 \equiv 0 \pmod{2} \). Let \( v_{\ell_2} \) and \( v_{\ell_3} \) be non-zero vectors in \( D(\ell_2)^+ \) and \( D(\ell_3)^+ \) of weights \( \ell_2 \) and \( \ell_3 \), respectively. Then there exists a non-zero element in \( U(\mathfrak{gl}_2^2, \mathbb{C}) \) of the form

\[
\sum_{2m_1+2m_2=\ell_1-\ell_2-\ell_3} c_{m_1,m_2}(\ell_1, \ell_2, \ell_3)(X_{m_1}^+ \otimes X_{m_2}^+),
\]

unique up to scalars, such that

\[
H \cdot \sum_{2m_1+2m_2=\ell_1-\ell_2-\ell_3} c_{m_1,m_2}(\ell_1, \ell_2, \ell_3)(X_{m_1}^+ v_{\ell_2} \otimes X_{m_2}^+ v_{\ell_3}) = \ell_1 \cdot \sum_{2m_1+2m_2=\ell_1-\ell_2-\ell_3} c_{m_1,m_2}(\ell_1, \ell_2, \ell_3)(X_{m_1}^+ v_{\ell_2} \otimes X_{m_2}^+ v_{\ell_3}),
\]

(3.1)

and

\[
X_- \cdot \sum_{2m_1+2m_2=\ell_1-\ell_2-\ell_3} c_{m_1,m_2}(\ell_1, \ell_2, \ell_3)(X_{m_1}^+ v_{\ell_2} \otimes X_{m_2}^+ v_{\ell_3}) = 0.
\]

(3.2)

Here the action of \( \mathfrak{gl}_2^2 \) on \( D(\ell_2)^+ \otimes D(\ell_3)^+ \) is given by

\[
X \cdot (v \otimes v') = X \cdot v \otimes v' + v \otimes X \cdot v'.
\]

Moreover, the coefficients can be normalized so that \( c_{m_1,m_2}(\ell_1, \ell_2, \ell_3) \in \mathbb{Q} \).

**Proof** The existence and uniqueness follow directly from the decomposition (cf. [41])

\[
D(\ell_2)^+ \otimes D(\ell_3)^+|_{\mathfrak{gl}_2^2, \mathbb{C}} = \bigoplus_{j=0}^{\infty} D(\ell_2 + \ell_3 + 2j)^+.
\]

For the rationality of the differential operator, note that

\[
H \cdot v_{\ell_2} = \ell_2 \cdot v_{\ell_2}, \quad H \cdot v_{\ell_3} = \ell_3 \cdot v_{\ell_3}, \quad X_+ X_- - X_- X_+ = -4H.
\]

Therefore, by a simple induction argument, we see that the linear equations defined by (3.2) have coefficients in \( \mathbb{Q} \). In particular, the solutions can be chosen to be rational numbers. \( \square \)

For a triplet \( (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}_{\geq 1}^3 \) such that \( \ell_1 + \ell_2 + \ell_3 \equiv 0 \pmod{2} \) and satisfying the unbalanced condition

\[
2 \max\{\ell_1, \ell_2, \ell_3\} \geq \ell_1 + \ell_2 + \ell_3,
\]

we fix a choice of \( c_{m_1,m_2}(\ell_1, \ell_2, \ell_3) \in \mathbb{Q} \) for each pair of non-negative integers \( m_1, m_2 \) with \( 2m_1 + 2m_2 = 2 \max\{\ell_1, \ell_2, \ell_3\} - \ell_1 + \ell_2 + \ell_3 \) such that the similar assertion in Lemma 3.1...
holds. Define $X(\ell_1, \ell_2, \ell_3) \in U(\mathfrak{gl}_2^3, \mathbb{C})$ by
\[
X(\ell_1, \ell_2, \ell_3) = \begin{cases} 
\sum_{2m_1+2m_2=\ell_1-\ell_2-\ell_3} c_{m_1,m_2} \sum_{\ell_1, \ell_2, \ell_3} (1 \otimes X^m_+ \otimes X^m_+) & \text{if } \ell_1 \geq \ell_2 + \ell_3, \\
\sum_{2m_1+2m_2=\ell_2-\ell_1-\ell_3} c_{m_1,m_2} \sum_{\ell_1, \ell_2, \ell_3} (X^m_+ \otimes 1 \otimes X^m_+) & \text{if } \ell_2 \geq \ell_1 + \ell_3, \\
\sum_{2m_1+2m_2=3-\ell_1-\ell_2} c_{m_1,m_2} \sum_{\ell_1, \ell_2, \ell_3} (X^m_+ \otimes X^m_+ \otimes 1) & \text{if } \ell_3 \geq \ell_1 + \ell_2.
\end{cases}
\]

Let $(\kappa, r) \in \mathbb{Z}_{\geq 1}[\Sigma_{\mathbb{E}}] \times \mathbb{Z}[\Sigma_{\mathbb{E}}]$ be motivic. We assume $\kappa$ satisfies the totally unbalanced condition
\[
2 \max_{w|\nu} \{\kappa_w\} - \sum_{w|\nu} \kappa_w \geq 0 \quad (3.3)
\]
for all $\nu \in \Sigma_{\mathbb{E}}$ and
\[
\sum_{w \in \Sigma_{\mathbb{E}}} r_w = 0.
\]

For each $\nu \in \Sigma_{\mathbb{E}}$, let $v^{(1)}, v^{(2)}, v^{(3)} \in \Sigma_{\mathbb{E}}$ be the extensions of $v$ and $\tilde{v}(\kappa) \in \Sigma_{\mathbb{E}}$ the homomorphism such that $\max_{w|\nu} \{\kappa_w\} = \kappa_{\tilde{v}(\kappa)}$. Put
\[
I_{\kappa} = \{\tilde{v}(\kappa) \mid \nu \in \Sigma_{\mathbb{E}}\} \subset \Sigma_{\mathbb{E}}.
\]

Let $\mathcal{L}'(2, 0)$ and $\mathcal{L}'(\kappa, I_\kappa)$ be the automorphic line bundles on $\text{Sh}(G', X')$ and $\text{Sh}(G, X)$ defined by the motivic algebraic representations $\rho'_\mathbb{Q}(\kappa)$ of $K'_\infty$ and $\rho_{\kappa, I_\kappa}$ of $K_\infty$ in (2.4), respectively. Here
\[
(2, 0) = ((2, \cdots, 2), (0, \cdots, 0)) \in \mathbb{Z}[\Sigma_{\mathbb{E}}] \times \mathbb{Z}[\Sigma_{\mathbb{E}}].
\]

**Proposition 3.2** There exists a homogeneous $\mathbb{Q}(\kappa)$-rational differential operator $[\delta(\kappa)]$ from $\mathcal{L}'(\kappa, I_\kappa)$ to $\mathcal{L}'(2, 0)$ satisfying the following conditions:

1. Let
\[
[\delta(\kappa)] : H^{[\mathbb{F}, \mathbb{Q}]}(\mathcal{L}'(\kappa, I_\kappa)) \longrightarrow H^{[\mathbb{F}, \mathbb{Q}]}((\mathcal{L}'(2, 0))_{\text{sub}})
\]
be the induced $G'(\mathbb{A}_f)$-module homomorphism. Then we have
\[
T_{\sigma} \circ [\delta(\kappa)] = [\delta(\sigma \kappa)] \circ T_{\sigma}
\]
for all $\sigma \in \text{Aut}(\mathbb{C})$.

2. If a class in $H^{[\mathbb{F}, \mathbb{Q}]}(\mathcal{L}'(\kappa, I_\kappa))$ is represented by $\varphi \otimes \bigwedge_{w \in I_\kappa} X_{+, w} \otimes v(\kappa, I_\kappa)$, then its image under $[\delta(\kappa)]$ in $H^{[\mathbb{F}, \mathbb{Q}]}((\mathcal{L}'(2, 0))_{\text{sub}})$ is represented by
\[
(X(\kappa) \cdot \varphi)|_{G'(\mathbb{A})} \otimes \bigwedge_{v \in \Sigma_{\mathbb{E}}} X_{+, v} \otimes v(2, 0).
\]

Here $X(\kappa) \in U(\mathfrak{gl}^3_{\mathbb{E}, \mathbb{C}})$ is defined by
\[
X(\kappa) = \bigotimes_{v \in \Sigma_{\mathbb{E}}} X(\kappa_{v(1)}, \kappa_{v(2)}, \kappa_{v(3)}).
\]
Algebraicity of the central critical values...

Proof  Recall \( \mathfrak{g} \) and \( \mathfrak{t} \) (resp. \( \mathfrak{g}' \) and \( \mathfrak{t}' \)) are the Lie algebras of \( G(\mathbb{R}) \) and \( K_\infty = Z_G(\mathbb{R}) \cdot \text{SO}(2)_{\Sigma_E} \) (resp. \( G'(\mathbb{R}) \) and \( K'_\infty = Z_{G'}(\mathbb{R}) \cdot \text{SO}(2)_{\Sigma_F} \)), respectively, and we have the Hodge decompositions for \( \mathfrak{g} \) and \( \mathfrak{g}' \) as in (2.3). We identify \( \mathfrak{g} \) and \( \mathfrak{g}' \) with \( \mathfrak{gl}_2_{\Sigma_E} \) and \( \mathfrak{gl}_2_{\Sigma_F} \), respectively. For each \( w \in \Sigma_E \) (resp. \( v \in \Sigma_F \)), let \( \mathfrak{g}_w \) (resp. \( \mathfrak{g}'_v \)) be the \( w \)-component of \( \mathfrak{g} \) (resp. \( v \)-component of \( \mathfrak{g}' \)). Let

\[
\mathfrak{P} = p^{-} \oplus \mathfrak{t}_{C}, \quad \mathfrak{P}' = (p')^{-} \oplus \mathfrak{t}'_{C}.
\]

By [13, Theorem 4.8] and [14, Lemma 7.2], each element

\[
\delta^* \in \text{Hom}_{U(\mathfrak{P}')}((C_{(2,0)}^{\vee}, U(\mathfrak{g}_C) \otimes U(\mathfrak{p}) \subset \mathfrak{C}_{(\kappa,2)}, \mathfrak{L})) = \text{Hom}_{U(\mathfrak{p}')}((C_{(2,0)}^{\vee}, U(\mathfrak{g}_C) \otimes U(\mathfrak{p}) \subset \mathfrak{C}_{(-\kappa,2), -\mathfrak{L}})
\]

gives rise to a homogeneous differential operator \([\delta] \) from \( \mathcal{L}(\kappa,2), \mathfrak{L}) \) to \( \mathcal{L}'(2,0), \mathfrak{L}) \). Here the action of \( \mathfrak{P} \) and \( \mathfrak{P}' \) on \( C_{(2,0)}^{\vee}, \mathfrak{L}) \) and \( C_{(2,0)}^{\vee}, \mathfrak{L}) \) factor through \( \mathfrak{t}_{C} \) and \( \mathfrak{t}'_{C} \), respectively. We define \([\delta(\kappa)] \) to be the homogeneous differential operator corresponding to the element

\[
\delta(\kappa)^* \in \text{Hom}_{U(\mathfrak{P}')}((C_{(2,0)}^{\vee}, U(\mathfrak{g}_C) \otimes U(\mathfrak{p}) \subset \mathfrak{C}_{(-\kappa,2), -\mathfrak{L}})
\]

defined by

\[
\delta(\kappa)^*(v(-2,0)) = X(\kappa) \cdot (1 \otimes v(-\kappa,2), -\mathfrak{L}),
\]

where \( X(\kappa) \in U(\mathfrak{g}_{C}) = U(\mathfrak{gl}_{2_{\Sigma_E}}) \) is in (3.4). We shall show that \( \delta(\kappa)^* \) is indeed \( U(\mathfrak{P}') \)-equivariant. For \( (\ell, t) \in \mathbb{Z} \times \mathbb{Z} \) with \( \ell \equiv t \) (mod 2), let \( C_{(\ell,t)} \) be the complex field equipped with the action of \( \mathbb{R}^\times \cdot \text{SO}(2) \) given by \( ak_\theta \cdot z = a^{-t}e^{-\sqrt{-1}\ell \theta} \cdot z \) for \( a \in \mathbb{R}^\times \) and \( k_\theta \in \text{SO}(2) \). Thus we have

\[
C_{(-\kappa,2), -\mathfrak{L}} = \bigotimes_{w \in \Sigma_E} C_{(-\kappa,2), -r_w}
\]

as algebraic characters of \( K_\infty = (\mathbb{R}^\times \cdot \text{SO}(2))_{\Sigma_E} \). In the above isomorphism, we fix \( v_w \in C_{(-\kappa,2), -r_w} \) for each \( w \in \Sigma_E \) such that

\[
v(-\kappa,2), -\mathfrak{L}) = \bigotimes_{w \in \Sigma_E} v_w.
\]

Note that we also have

\[
U(\mathfrak{g}_{C}) \otimes U(\mathfrak{p}) \subset \mathfrak{C}_{(-\kappa,2), -\mathfrak{L}} = \bigotimes_{w \in \Sigma_E} U(\mathfrak{g}_{C,w}) \otimes U(\mathfrak{p}_{w}) \subset \mathfrak{C}_{(-\kappa,2), -r_w}.
\]

We write \( \mathcal{D}_w = U(\mathfrak{g}_{C,w}) \otimes U(\mathfrak{p}_{w}) \subset \mathfrak{C}_{(-\kappa,2), -r_w} \) for each \( w \in \Sigma_E \). Let \( v \in \Sigma_F \). The action of \( \mathfrak{g}'_v \) on \( \mathcal{D}_w \otimes \mathcal{D}_{v(1)} \otimes \mathcal{D}_{v(2)} \otimes \mathcal{D}_{v(3)} \) is given by

\[
X \cdot (v_1 \otimes v_2 \otimes v_3) = X \cdot v_1 \otimes v_2 \otimes v_3 + v_1 \otimes X \cdot v_2 \otimes v_3 + v_1 \otimes v_2 \otimes X \cdot v_3
\]

Springer
for $X \in \mathfrak{g}'_\nu$ and $v_i \in D_{v(i)}$ for $i = 1, 2, 3$. Then we have

$$Z \cdot \left( X(\kappa_{v(1)}, \kappa_{v(2)}, \kappa_{v(3)}) \cdot \bigotimes_{i=1}^{3} (1 \otimes v_{v(i)}) \right) = 0,$$

$$H \cdot \left( X(\kappa_{v(1)}, \kappa_{v(2)}, \kappa_{v(3)}) \cdot \bigotimes_{i=1}^{3} (1 \otimes v_{v(i)}) \right) = 2 \cdot \left( X(\kappa_{v(1)}, \kappa_{v(2)}, \kappa_{v(3)}) \cdot \bigotimes_{i=1}^{3} (1 \otimes v_{v(i)}) \right),$$

$$X_- \cdot \left( X(\kappa_{v(1)}, \kappa_{v(2)}, \kappa_{v(3)}) \cdot \bigotimes_{i=1}^{3} (1 \otimes v_{v(i)}) \right) = 0.$$

(3.5)

Indeed, suppose $v^{(1)} = \tilde{v}(\kappa)$, then $\kappa(I_{\xi})_{v^{(1)}} = 2 - \kappa_{v(1)}, \kappa(I_{\xi})_{v^{(2)}} = \kappa_{v(2)}$, and $\kappa(I_{\xi})_{v^{(3)}} = \kappa_{v(3)}$. Thus we have

$$D_{v^{(2)}} = D(\kappa_{v(2)})^{+} \otimes | r_{v(2)}/2 |, \quad D_{v^{(3)}} = D(\kappa_{v(3)})^{+} \otimes | r_{v(3)}/2 |.$$

Also note that

$$H \cdot (1 \otimes v_{v(1)}) = (2 - \kappa_{v(1)}) \cdot (1 \otimes v_{v(1)}), \quad X_- \cdot (1 \otimes v_{v(1)}) = 0$$

by definition. Thus (3.5) follows from the condition $r_{v^{(1)}} + r_{v^{(2)}} + r_{v^{(3)}} = 0$, (3.1), and (3.2). Since

$$X(\kappa) \cdot (1 \otimes v_{(-\kappa(I_{\xi}), -\xi)}) = \bigotimes_{v \in \Sigma_{F}} \left( X(\kappa_{v(1)}, \kappa_{v(2)}, \kappa_{v(3)}) \cdot \bigotimes_{i=1}^{3} (1 \otimes v_{v(i)}) \right),$$

we deduce from (3.5) that $\delta(\kappa)^*$ is $U(\mathfrak{g}_v')$-equivariant. Moreover, it is clear that the diagram

$$\begin{array}{ccc}
\mathbb{C}_{(-2, 0)} & \xrightarrow{\delta(\kappa)^*} & \mathbb{C}_{(-2, 0)} \\
\downarrow & & \downarrow \\
\mathbb{C}_{(-2, 0)} & \xrightarrow{\delta^{(\sigma)})^*} & \mathbb{C}_{(-2, 0)}
\end{array}$$

is commutative for all $\sigma \in \text{Aut}(\mathbb{C})$, where the vertical homomorphisms are induced by the $\sigma$-linear isomorphisms in (2.7). The assertions (1) and (2) then follow from the construction of $\delta(\kappa)^*$. This completes the proof.

4 Proof of main results

In this section, we prove the main results Theorems 1.4 and 1.6 of this paper. In Sects. 4.1–4.3, we keep the notation of Sect. 3 and let $\Pi = \bigotimes_v \Pi_v$ be an irreducible cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_E)$ with central character $\omega_{\Pi}$, where $v$ runs through the places of $\mathbb{F}$. We assume the following conditions hold:

- $\omega_{\Pi}|_{A_{\kappa}^{\times}}$ is trivial;
- $\Pi$ is motivic of weight $(\kappa, r) \in \mathbb{Z}_{\geq 1}[\Sigma_{E}] \times \mathbb{Z}[\Sigma_{E}]$;
- $\kappa$ satisfies the totally unbalanced condition.

\(\square\) Springer
Lemma 2.1. When \( D \otimes \langle I \rangle \) trilinear period integrals \( \phi \) for cusp forms fix a non-zero \( D \). Let \( \zeta \) here. Here \( v \) be the Haar measure on \( \mathbb{F} \) of \( A \). As \( 1 < 2 \) and \( \varphi_1, \varphi_2 \) of \( \mathbb{F} \) such that there exists an irreducible cuspidal automorphic representation \( \Pi^D = \otimes_v \Pi_v^D \) of \( D^\times \langle \mathbb{A}_F \rangle \) associated to \( \Pi \) by the Jacquet–Langlands correspondence. Define the functional \( I^D \in \text{Hom}_{D^\times \langle \mathbb{A}_F \rangle} \otimes D^\times (\mathbb{A}_F) \otimes (\Pi^D)^\vee, \mathbb{C}) \) by the global trilinear period integral

\[
I^D(\varphi_1 \otimes \varphi_2) = \int_{\mathbb{A}_F^\times(\mathbb{F})} \int_{\mathbb{A}_F^\times(\mathbb{F})} \varphi_1(g_1)\varphi_2(g_2) dg_1^{\text{Tam}} dg_2^{\text{Tam}}
\]

for cusp forms \( \varphi_1 \) and \( \varphi_2 \) in the representation space of \( \Pi^D \) and \( (\Pi^D)^\vee \), respectively. Here \( dg_1^{\text{Tam}} \) and \( dg_2^{\text{Tam}} \) are the Tamagawa measures on \( A_{\mathbb{F}}^\times \langle \mathbb{A}_F \rangle \). For each place \( v \) of \( \mathbb{F} \), we fix a non-zero \( D^\times (\mathbb{E}_v) \)-equivariant bilinear pairing

\[
\langle , \rangle_v : \Pi^D_v \times (\Pi^D_v)^\vee \longrightarrow \mathbb{C}
\]

Let \( dg_v \) be the Haar measure on \( \mathbb{F}_v^\times(\mathbb{F}_v) \) defined as follows:

- If \( v \) is a finite place, let \( dg_v \) be the Haar measure normalized so that \( \text{vol}(\sigma_{\mathbb{F}}^\times \langle \sigma_{\mathbb{F}} \rangle, dg_v) = 1 \). Here \( \sigma_{D_v} \) is a maximal order of \( D(\mathbb{F}_v) \).
- If \( v \) is a real place, let

\[
dg_v = \frac{dx_v dy_v}{|y_v|^2} dk_v
\]

for \( g_v = n(x_v)a(y_v)k_v \) with \( x_v \in \mathbb{R}, y_v \in \mathbb{R}^\times \), and \( k_v \in \text{SO}(2) \). Here \( dx_v \) and \( dy_v \) are the Lebesgue measures and \( dk_v \) is the Haar measure on \( \text{SO}(2) \) such that \( \text{vol}(\text{SO}(2), dk_v) = 2 \).

Let \( I_v^D \in \text{Hom}_{D^\times (\mathbb{F}_v)} \otimes D^\times (\mathbb{F}_v) \otimes (\Pi^D_v \otimes (\Pi^D_v)^\vee, \mathbb{C}) \) be the functional defined by the local trilinear period integrals

\[
I_v^D(\varphi_{1,v} \otimes \varphi_{2,v}) = \frac{\zeta_{\mathbb{E}_v}(2)}{\zeta_{\mathbb{E}_v}(2)} \cdot \frac{L(1, \Pi_v, \text{Ad})}{L(1, \Pi_v, \text{As})} \cdot \int_{\mathbb{F}_v^\times(\mathbb{F}_v)} (\Pi^D_v(g_v)\varphi_{1,v}, \varphi_{2,v})_v dg_v.
\]

Here \( \zeta_{\mathbb{E}_v}(s) \) and \( \zeta_{\mathbb{E}_v}(s) \) are the local zeta functions of \( \mathbb{E}_v \) and \( \mathbb{F}_v \), respectively. Note that \( L(1, \Pi_v, \text{As}) \neq 0 \) (cf. [6, Lemma 3.1]) and the integral is absolutely convergent by [25, Lemma 2.1]. When \( D \) is unramified at \( v \), we also write \( I_v = I_v^D \). We normalize the pairings \( \langle , \rangle_v \) so that if \( \varphi_1 = \bigotimes_v \varphi_{1,v} \in \Pi^D \) and \( \varphi_2 = \bigotimes_v \varphi_{2,v} \in (\Pi^D)^\vee \), then \( \langle \varphi_{1,v}, \varphi_{2,v} \rangle_v = 1 \) for almost all \( v \) and

\[
\int_{\mathbb{A}_F^\times(\mathbb{E})} \varphi_1(g)\varphi_2(g) dg_v^{\text{Tam}} = \prod_v \langle \varphi_{1,v}, \varphi_{2,v} \rangle_v.
\]

Here \( dg_v^{\text{Tam}} \) is the Tamagawa measure on \( A_{\mathbb{E}}^\times \langle \mathbb{A}_E \rangle \). Let \( C_D \) be the constant such that

\[
C_D \cdot \prod_v dg_v
\]
is the Tamagawa measure on \( A_F^\times \setminus D^\times (A_F) \). Then we have (cf. [28, Lemma 6.1])

\[
C_D = \prod_{v \in \Sigma_D} (q_v - 1)^{-1} \cdot D_{\mathbb{F}}^{-3/2} \cdot \zeta_{\mathbb{F}}(2)^{-1},
\]

where \( \Sigma_D \) is the set of places of \( \mathbb{F} \) at which \( D \) is ramified, \( q_v \) is the cardinality of the residue field of \( \mathbb{F}_v \), and \( \zeta_{\mathbb{F}} \) is the completed Dedekind zeta function of \( \mathbb{F} \). We have the following central value formula of Ichino [25].

**Theorem 4.1** (Ichino) As functionals in \( \text{Hom}_{D^\times (A_F)^\times \times D^\times (A_F)}(\Pi_D \otimes (\Pi_D)^\vee, \mathbb{C}) \), we have

\[
P_D = \frac{C_D}{2^n} \cdot \frac{\zeta_{\mathbb{F}}(2)}{\zeta_{\mathbb{F}}(2)} \cdot \frac{L(\frac{1}{2}, \Pi, \text{As})}{L(1, \Pi, \text{Ad})} \cdot \prod_v I_{v}^D.
\]

Here \( \zeta_{\mathbb{E}}(s) \) and \( \zeta_{\mathbb{F}}(s) \) are the completed Dedekind zeta functions of \( \mathbb{E} \) and \( \mathbb{F} \), respectively, and

\[
c = \begin{cases} 
3 & \text{if } \mathbb{E} = \mathbb{F} \times \mathbb{F} \times \mathbb{F}, \\
2 & \text{if } \mathbb{E} = \mathbb{K} \times \mathbb{F} \text{ for some totally real quadratic extension } \mathbb{K} \text{ of } \mathbb{F}, \\
1 & \text{if } \mathbb{E} \text{ is a field}.
\end{cases}
\]

### 4.2 Local trilinear period integrals

For each finite place \( v \) of \( \mathbb{F} \) and \( \sigma \in \text{Aut}(\mathbb{C}) \), we fix \( \sigma \)-linear isomorphisms

\[
t_{\sigma, v} : \Pi_v^D \longrightarrow \sigma \Pi_v^D, \quad t_{\sigma, v}^\vee : (\Pi_v^D)^\vee \longrightarrow \sigma (\Pi_v^D)^\vee.
\]

By abuse of notation, we denote by the same notation \( I_v^D \in \text{Hom}_{D^\times (\mathbb{E}_v)^\times \times D^\times (\mathbb{F}_v)}(\sigma \Pi_v^D \otimes (\Pi_v^D)^\vee, \mathbb{C}) \) the functional defined as in (4.2) with respect to the \( D^\times (\mathbb{E}_v) \)-equivariant bilinear pairing

\[
\langle \cdot, \cdot \rangle_v : \sigma \Pi_v^D \times (\Pi_v^D)^\vee \longrightarrow \mathbb{C}
\]

defined by

\[
\langle t_{\sigma, v} \varphi_{1, v}, t_{\sigma, v}^\vee \varphi_{2, v} \rangle_v = \sigma(\varphi_{1, v}, \varphi_{2, v})_v
\]

for \( \varphi_{1, v} \in \Pi_v^D \) and \( \varphi_{2, v} \in (\Pi_v^D)^\vee \). In the following lemmas, we show that the local \( L \)-factors and the local trilinear period integrals satisfy the Galois equivariant property.

**Lemma 4.2** Let \( v \) be a finite place of \( \mathbb{F} \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), we have

\[
\sigma L(1, \Pi_v, \text{Ad}) = L(1, \sigma \Pi_v, \text{Ad}), \quad \sigma L(\frac{1}{2}, \Pi_v, \text{As}) = L(\frac{1}{2}, \sigma \Pi_v, \text{As}.
\]

**Proof** We prove the assertion for \( L(\frac{1}{2}, \Pi_v, \text{As}) \) in the case when \( \mathbb{E}_v \) is a field. The assertion for \( L(1, \Pi_v, \text{Ad}) \) or arbitrary \( \mathbb{E}_v \) can be proved in a similar way and we omit it. Let \( W_{\mathbb{E}_v} \) and \( W_{F_v} \) be the Weil–Deligne groups of \( \mathbb{E}_v \) and \( F_v \), respectively. Fix \( \sigma \in \text{Aut}(\mathbb{C}) \). Let \( \chi_{\mathbb{E}_v, \sigma} \) and \( \chi_{F_v, \sigma} \) be the quadratic characters of \( \mathbb{E}_v^\times \) and \( F_v^\times \), respectively, defined by

\[
\chi_{\mathbb{E}_v, \sigma} = \sigma(1 |_{\mathbb{E}_v}) \cdot |_{\mathbb{E}_v}^{-1/2}, \quad \chi_{F_v, \sigma} = \sigma(1 |_{F_v}) \cdot |_{F_v}^{-1/2}.
\]

For \( n \geq 1 \), we identify the Langlands dual group \( L(\mathbb{R}_{\mathbb{E}_v/F_v}, \text{GL}_n) \) of \( \mathbb{R}_{\mathbb{E}_v/F_v} \text{GL}_n \) with \( \text{GL}_n(\mathbb{C})^3 \times \text{Gal}(\mathbb{F}_v/F_v) \) (cf. [3, Sect.5]), where the action of \( \text{Gal}(\mathbb{F}_v/F_v) \) on \( \text{GL}_n(\mathbb{C})^3 \)
Lemma 4.3

Let \( s \) be a finite place of \( \mathbb{F} \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), we have

\[
\sigma I^D_v(\psi_{1,v} \otimes \psi_{2,v}) = I^D_v(t_{\sigma,v} \psi_{1,v} \otimes t^\vee_{\sigma,v} \psi_{2,v})
\]

for \( \psi_{1,v} \in \Pi^D_v \) and \( \psi_{2,v} \in (\Pi^D_v)^\vee \).

**Proof** Let \( \psi_{1,v} \in \Pi^D_v \) and \( \psi_{2,v} \in (\Pi^D_v)^\vee \). Note that by definition (4.4) we have

\[
\sigma(\Pi^D_v(g_v)\psi_{1,v}, \psi_{2,v}) = (\Pi^D_v(g_v)t_{\sigma,v} \psi_{1,v}, t^\vee_{\sigma,v} \psi_{2,v})
\]

for \( g_v \in D^\times(\mathbb{F}) \) and \( \sigma \in \text{Aut}(\mathbb{C}) \). Together with Lemma 4.2, it suffices to show that

\[
\sigma \left( \int_{\mathbb{F}^\times_v \setminus D^\times(\mathbb{F})} (\Pi^D_v(g_v)\psi_{1,v}, \psi_{2,v}) dg_v \right) = \int_{\mathbb{F}^\times_v \setminus D^\times(\mathbb{F})} \sigma(\Pi^D_v(g_v)\psi_{1,v}, \psi_{2,v}) dg_v \quad (4.5)
\]
for all $\sigma \in \text{Aut}(\mathbb{C})$. If $D$ is ramified at $v$, then $\mathbb{F}_v \setminus D^\times(\mathbb{F}_v)$ is compact. Therefore the local period integral is a finite sum and equality (4.5) holds trivially. Suppose $D$ is unramified at $v$ and identify $D^\times$ with $\text{GL}_2$. Let $K_v = \sigma^\times_v \setminus \text{GL}_2(\sigma_{\mathbb{F}_v})$. By the Cartan decomposition, we have

$$
\int_{\mathbb{F}_v^\times \setminus \text{GL}_2(\mathbb{F}_v)} \langle \Pi_v(g_v)\varphi_{1,v}, \varphi_{2,v} \rangle_v \, dg_v
$$

$$
= \int_{K_v} \int_{K_v} \int_{\mathbb{F}_v^\times} \langle \Pi_v(k_1,v a(t_v)k_2,v)\varphi_{1,v}, \varphi_{2,v} \rangle_v \, dg_v \, d^\times t_v \, dk_{1,v} \, dk_{2,v},
$$

where $dk_{1,v}, dk_{2,v},$ and $d^\times t_v$ are Haar measures normalized so that

$$\text{vol}(K_v, dk_{1,v}) = \text{vol}(K_v, dk_{2,v}) = \text{vol}(\sigma_v^\times, d^\times t_v) = 1.\text{vol}(K_v, dk_{1,v}) = \text{vol}(K_v, dk_{2,v}) = \text{vol}(\sigma_v^\times, d^\times t_v) = 1.$$

Note that

$$\text{vol}(K_v a(t_v)K_v, dg_v) = \begin{cases} 1 & \text{if } |t_v|_{\mathbb{F}_v} = 1, \\ |t_v|_{\mathbb{F}_v}^{-1}(1 + q_v^{-1}) & \text{if } |t_v|_{\mathbb{F}_v} < 1 \end{cases}$$

for $t_v \in \sigma_{\mathbb{F}_v} \setminus \{0\}$. It is well-known that there exist characters $\chi_{1,v}$ and $\chi_{2,v}$ of $\mathbb{E}_v^\times$ depending only on $\Pi_v$ and locally constant functions $f_{1,v}$ and $f_{2,v}$ on $\mathbb{E}_v \times \text{GL}_2(\sigma_{\mathbb{E}_v}) \times \text{GL}_2(\sigma_{\mathbb{E}_v})$ such that

$$\langle \Pi_v(k_1,v a(t_v)k_2,v)\varphi_{1,v}, \varphi_{2,v} \rangle_v = \chi_{1,v}(t_v) f_{1,v}(t_v, k_1,v, k_2,v) + \chi_{2,v}(t_v) f_{2,v}(t_v, k_1,v, k_2,v)$$

for $(t_v, k_1,v, k_2,v) \in (\sigma_{\mathbb{E}_v} \setminus \{0\}) \times \text{GL}_2(\sigma_{\mathbb{E}_v}) \times \text{GL}_2(\sigma_{\mathbb{E}_v})$. For a character $\chi$ of $\mathbb{E}_v^\times$ and a locally constant function $f: \mathbb{F}_v \to \mathbb{C}$ with compact support, let $Z(\chi, f)$ be the Tate integral defined by

$$Z(\chi, f) = \int_{\mathbb{F}_v^\times} \chi(t_v) f(t_v) \, d^\times t_v.$$

It is easy to show that the integral converges absolutely when $|\chi| = |\frac{\lambda}{\mathbb{F}_v}|$ for some $\lambda > 0$ and we have

$$\sigma Z(\chi, f) = Z(\sigma\chi, \sigma f)$$

for all $\sigma \in \text{Aut}(\mathbb{C})$ when both sides are absolutely convergent. We have

$$\int_{\mathbb{F}_v^\times \setminus \text{GL}_2(\mathbb{F}_v)} \langle \Pi_v(g_v)\varphi_{1,v}, \varphi_{2,v} \rangle_v \, dg_v$$

$$= [K_v : U_v]^2 \sum_{k_1,v \in K_v/U_v} \sum_{k_2,v \in K_v/U_v} \left[ Z(\chi_{1,v}|\mathbb{F}_v^\times, f_{1,v}^{(1)}(k_1,v,k_2,v)) + Z(\chi_{2,v}|\mathbb{F}_v^\times, f_{1,v}^{(2)}(k_1,v,k_2,v)) \right].$$

Here $U_v$ is an open compact normal subgroup of $K_v$ such that both $\varphi_{1,v}$ and $\varphi_{2,v}$ are $U_v$-invariant and

$$f_{1,v}^{(i)}(t_v) = \begin{cases} 0 & \text{if } t_v \notin \sigma_{\mathbb{F}_v}, \\ f_{i,v}(t_v, k_1,v, k_2,v) & \text{if } |t_v|_{\mathbb{F}_v} = 1, \\ (1 + q_v^{-1}) f_{i,v}(t_v, k_1,v, k_2,v) & \text{if } |t_v|_{\mathbb{F}_v} < 1 \end{cases}$$
for \( i = 1, 2 \) and \( t_v \in \mathbb{F}_p \). We remark that \( |\chi_{t_v}|_{\mathbb{F}_p} = |\chi_{\lambda_i}|_{\mathbb{F}_p} \) for some \( \lambda_i > 1 \) by the result of Kim–Shahidi [32]. Hence the above Tate integrals are absolutely convergent. Equality (4.5) then follows immediately from (4.6) and (4.7). This completes the proof. \( \square \)

Now we consider the archimedean local trilinear period integrals. The totally unbalanced condition implies that \( D \) is unramified at \( v \) and \( \Pi_v^D = \Pi_v \) for each \( v \in \Sigma_F \). Let \( \Pi_\infty = \bigotimes_{v \in \Sigma_F} \Pi_v \) be a representation of \( D^\times(\mathbb{F}_\infty) = \text{GL}_2(\mathbb{R})^\Sigma_F \). Define \( I_\infty \in \text{Hom}_{\text{GL}_2(\mathbb{R})^\Sigma_F}((\Pi_\infty \otimes \Pi_\infty^\vee, \mathbb{C})) \) and \((, \, )_\infty: \Pi_\infty \times \Pi_\infty^\vee \to \mathbb{C}\) by

\[
I_\infty = \bigotimes_{v \in \Sigma_F} I_v, \quad (, \, )_\infty = \prod_{v \in \Sigma_F} (, \, )_v.
\]

We recall in the following lemma our previous calculation of local trilinear period integral.

**Lemma 4.4** Let \( \varphi(\kappa, \ell) \in \Pi_\infty, \varphi(\kappa, -\ell) \in \Pi_\infty^\vee \) be non-zero vectors of weight \( \kappa \) and \( X(\kappa) \in U(\text{gl}_2^\vee, \mathbb{C}) \) be the differential operator defined in (3.4). We have

\[
I_\infty(X(\kappa) \cdot \Pi_\infty(\tau_\kappa) \varphi(\kappa, \ell) \otimes X(\kappa) \cdot \Pi_\infty^\vee(\tau_\kappa) \varphi(\kappa, -\ell))
\]

\[
\langle \varphi(\kappa, \ell), \Pi_\infty(\tau_\kappa) \varphi(\kappa, -\ell) \rangle_\infty
\]

\[\in (2\pi \sqrt{-1}) \sum_{v \in \Sigma_F} (2 \text{max}_{w \mid v}(w \kappa) - \text{sum}_{w \mid v}(w \kappa)) \cdot \mathbb{Q}^\times.\]

Here \( \tau_\kappa \in \text{GL}_2(\mathbb{R})^\Sigma_F \) are defined in (2.14).

**Proof** Let \( v \in \Sigma_F \) and \( v^{(1)}, v^{(2)}, v^{(3)} \in \Sigma_E \) be the extensions of \( v \). Note that

\[
\Pi_v = \bigotimes_{i=1}^3 (D(\kappa_v)) \otimes |r_v^{(i)/2}/2\rangle, \quad \Pi_v^\vee = \bigotimes_{i=1}^3 (D(\kappa_v)) \otimes |r_v^{-(i)/2}/2\rangle.
\]

Let \( v^{(i)} \in (D(\kappa_v)) \otimes |r_v^{(i)/2}/2\rangle \) and \( v^{\vee (i)} \in (D(\kappa_v)) \otimes |r_v^{-(i)/2}/2\rangle \) be non-zero vectors of weight \( \kappa_v^{(i)} \) for \( i = 1, 2, 3 \). Put \( v_v = v_v^{(1)} \otimes v_v^{(2)} \otimes v_v^{(3)} \) and \( v_v^\vee = v_v^{(1)} \otimes v_v^{(2)} \otimes v_v^{(3)} \). We may assume \( v_v^{(1)} = v^{(1)}_\kappa \). Then

\[
I_v(X(\kappa_v^{(1)}, \kappa_v^{(2)}, \kappa_v^{(3)}) \cdot \Pi_v((\tau_\kappa)_v) v_v \otimes X(\kappa_v^{(1)}, \kappa_v^{(2)}, \kappa_v^{(3)}) \cdot \Pi_v^\vee((\tau_\kappa)_v)v_v^\vee)
\]

\[
= \zeta_\mathbb{R}(2)^{-2} \frac{L(1, \Pi_v, \text{Ad})}{L(1, \Pi_v, \text{As})} \sum_{m_1, m_2, m_1', m_2'} c_{m_1, m_2}(\kappa_v^{(1)}, \kappa_v^{(2)}, \kappa_v^{(3)}) c_{m_1', m_2'}(\kappa_v^{(1)}, \kappa_v^{(2)}, \kappa_v^{(3)})
\]

\[
\times \int_{\mathbb{R}^n \setminus \text{GL}_2(\mathbb{R})} \left( \Pi_v(g(\tau_\kappa)) v_v^{(1)} \otimes X^{m_1}_+ v_v^{(2)} \otimes X^{m_2}_+ v_v^{(3)} \otimes \Pi_v^\vee((\tau_\kappa)_v) v_v^{(1)} \otimes X^{m_1'}_+ v_v^{(2)} \otimes X^{m_2'}_+ v_v^{(3)} \right)
\]

\[
\langle v_v, \Pi_v((a(-1)) v_v^\vee) \rangle_v,
\]

Here \( m_1, m_2, m_1', m_2' \) runs through non-negative integers such that

\[
2m_1 + 2m_2 = 2m_1' + 2m_2' = \kappa_v^{(1)} - \kappa_v^{(2)} - \kappa_v^{(3)}.
\]

By Lemma 3.1, the vector

\[
\sum c_{m_1, m_2}(\kappa_v^{(1)}, \kappa_v^{(2)}, \kappa_v^{(3)}) (X^{m_1}_+ v_v^{(2)} \otimes X^{m_2}_+ v_v^{(3)}
\]

has weight \( \kappa_v^{(1)} \) and the \((\text{gl}_2, \text{O}(2))-\text{module generated by it under the diagonal action is isomorphic to } D(\kappa_v^{(1)}) \otimes |r_v^{(1)/2}/2\rangle\). It then follows from the Schur orthogonality relations that
the above integral is non-zero. On the other hand, by [5, Proposition 4.1 and Corollary 4.4],
the above integral is equal to
\[
(2\pi \sqrt{-1})^{\kappa_v(1) - \kappa_v(2) - \kappa_v(3)} \cdot 2^{\kappa_v(1) - \kappa_v(2) - \kappa_v(3) + 1} \cdot \sum_{m_1 + 2m_2 = \kappa_v(1) - \kappa_v(2) - \kappa_v(3)} (-1)^{m_1} c_{m_1, m_2} (\kappa_v(1), \kappa_v(2), \kappa_v(3)) \cdot 2^{\kappa_v(1)} - \kappa_v(2) - \kappa_v(3) + 1 \cdot 2^{\kappa_v(1)} - \kappa_v(2) - \kappa_v(3) + 1 \cdot \sum_{v \in \Sigma_{\mathbb{F}}} 2^{\max_{w \mid \kappa_w} (\kappa_w - \sum_{w \mid \kappa_w} \kappa_w)}. \]

We conclude that
\[
I_\infty(X(\kappa) \cdot \Pi_\infty(\tau L_0) \psi_{(\kappa, \mathbb{E})} \otimes X(\kappa) \cdot \Pi_\infty^\vee(\tau L_0) \psi_{(\kappa, -\mathbb{E})})
\]
\[
= \prod_{v \in \Sigma_{\mathbb{F}}} \frac{I_v(X(\kappa_v(1), \kappa_v(2), \kappa_v(3)) \cdot \Pi_v((\tau L_0)_v) \psi_v \otimes X(\kappa_v(1), \kappa_v(2), \kappa_v(3)) \cdot \Pi_v^\vee((\tau L_0)_v) \psi_v^\vee)}{(\psi_v, \Pi_v(a(-1)) \psi_v^\vee)}
\]
\[
\in (2\pi \sqrt{-1}) \sum_{v \in \Sigma_{\mathbb{F}}} 2^{\max_{w \mid \kappa_w} (\kappa_w - \sum_{w \mid \kappa_w} \kappa_w)} \cdot \mathbb{Q}^\times.
\]
This completes the proof. \[\Box\]

Let \(v\) be a place of \(\mathbb{F}\). By the results of Prasad [37], [38] and Loke [33]. We have
\[
\dim \text{Hom}_{D}(\Pi_v^D, \mathbb{C}) \leq 1.
\]

**Lemma 4.5** Let \(v\) be a place of \(\mathbb{F}\). Then \(\text{Hom}_{D}(\Pi_v^D, \mathbb{C}) \neq 0\) if and only if \(I_v^D \neq 0\).

**Proof** When \(v \in \Sigma_{\mathbb{F}}\), the assertion follows from Lemma 4.4. Assume \(v\) is a finite place. The
assertion can be proved word by word following the proof of [7, Corollary 9.6] except we
replace \(\mathbb{R}\) and [34, Theorem 4.12] therein by \(\mathbb{F}_v\) and [31, Corollary 3.7], respectively. \[\Box\]

### 4.3 Proof of Theorem 1.4

Let
\[
\varepsilon(s, \Pi, As) = \prod_v \varepsilon(s, \Pi_v, As, \psi_v)
\]
be the global Asai \(\varepsilon\)-factor of \(\Pi\), where \(\bigotimes_v \psi_v\) is a non-trivial additive character of \(\mathbb{F}\setminus\mathbb{A}_\mathbb{F}\)
and \(\varepsilon(s, \Pi_v, As, \psi_v)\) is the local Asai \(\varepsilon\)-factor of \(\Pi_v\) with respect to \(\psi_v\) defined via the Weil–
Deligne representation. Note that \(\varepsilon(s, \Pi_v, As, \psi_v) = 1\) for all finite places \(v\) of \(\mathbb{F}\) such that
\(\mathbb{E}_v\) is unramified over \(\mathbb{F}_v\), \(\Pi_v\) is unramified, and \(\psi_v\) is of conductor \(\sigma_{\mathbb{F}_v}\). By our assumption
that \(\omega \Pi \mid_{\mathbb{A}_{\mathbb{F}}}\) is trivial, we have \(\varepsilon(s, \Pi_v, As, \psi_v) \in \{\pm 1\}\) and is independent of the choice of
\(\psi_v\). We write
\[
\varepsilon(s, \Pi_v, As) = \varepsilon(s, \Pi_v, As, \psi_v).
\]
Recall the global root number \(\varepsilon(\Pi, As)\) is defined by
\[
\varepsilon(\Pi, As) = \varepsilon(\frac{1}{2}, \Pi, As) \in \{\pm 1\}.
\]
On the other hand, analogous to the proof of Lemma 4.2, we can show that
\[
\sigma \varepsilon \left(\frac{1}{2}, \Pi_v, As\right) = \varepsilon \left(\frac{1}{2}, \sigma \Pi_v, As\right)
\]
\[\Box\]
for all finite places \( v \) of \( \mathbb{F} \) and \( \sigma \in \text{Aut}(\mathbb{C}) \). In particular, we have

\[
\varepsilon(\Pi, \text{As}) = \varepsilon(\sigma \Pi, \text{As})
\]

for all \( \sigma \in \text{Aut}(\mathbb{C}) \). We see that assertion (1) of Theorem 1.4 follows immediately from the expected functional equation

\[
L(s, \Pi, \text{As}) = \varepsilon(s, \Pi, \text{As})L(1 - s, \Pi, \text{As})
\]  

(4.8)

which we now prove. Let

\[
L_{\text{PSR}}(s, \Pi, \text{As}) = \prod_v L_{\text{PSR}}(s, \Pi_v, \text{As}), \quad \varepsilon_{\text{PSR}}(s, \Pi, \text{As}) = \prod_v \varepsilon_{\text{PSR}}(s, \Pi_v, \text{As}, \psi_v)
\]

be the global Asai \( L \)-function and \( \varepsilon \)-factor of \( \Pi \) defined by the Rankin–Selberg method as well as the local zeta integrals developed by Piatetski-Shapiro–Rallis [39] and Ikeda [27]. In the Rankin–Selberg context, the functional equation

\[
L_{\text{PSR}}(s, \Pi, \text{As}) = \varepsilon_{\text{PSR}}(s, \Pi, \text{As})L_{\text{PSR}}(1 - s, \Pi, \text{As})
\]

holds and is a direct consequence of the functional equation of Siegel Eisenstein series. On the other hand, by [32] and [6, Corollary 1.4], we have

\[
L(s, \Pi_v, \text{As}) = L_{\text{PSR}}(s, \Pi_v, \text{As}), \quad \varepsilon(s, \Pi_v, \text{As}, \psi_v) = \varepsilon_{\text{PSR}}(s, \Pi_v, \text{As}, \psi_v)
\]

for all places \( v \) of \( \mathbb{F} \). Hence the functional Eq. (4.8) holds.

Now we assume \( \varepsilon(\Pi, \text{As}) = 1 \). Let \( \mathbb{K} \) be the quadratic discriminant algebra of \( \mathbb{E}/\mathbb{F} \) and \( \omega_{\mathbb{K}/\mathbb{F}} = \prod_v \omega_{\mathbb{K}_v/\mathbb{F}_v} \) the quadratic character of \( \mathbb{F}^\times \backslash \mathbb{A}_\mathbb{F}^\times \) associated to \( \mathbb{K}/\mathbb{F} \) by class field theory. By the assumption \( \varepsilon(\Pi, \text{As}) = 1 \), there exists a unique quaternion algebra \( D \) over \( \mathbb{F} \) such that \( D \) is ramified at \( v \) if and only if

\[
\varepsilon\left(\frac{1}{2}, \Pi_v, \text{As}\right) \cdot \omega_{\mathbb{K}_v/\mathbb{F}_v}(-1) = -1.
\]

In particular, it follows from the totally unbalanced condition that \( D \) is totally indefinite. By the results of Prasad [37], [38] and Loke [33], the above sign condition implies that there exists an irreducible cuspidal automorphic representation \( \Pi^D = \bigotimes_v \Pi^D_v \) of \( D^\times(\mathbb{A}_\mathbb{F}) \) associated to \( \Pi \) by the Jacquet–Langlands correspondence such that

\[
\text{Hom}_{D^\times(\mathbb{F}_v)}(\Pi^D_v, \mathbb{C}) \neq 0
\]

for all places \( v \) of \( \mathbb{F} \). Fix non-zero vectors \( \varphi_{(\mathbb{k}, \ell)} \in \Pi_\infty = \bigotimes_{v \in \Sigma_{\mathbb{E}}} \Pi_v \) and \( \varphi_{(\mathbb{k}, -\ell)} \in \Pi^\vee_\infty = \bigotimes_{v \in \Sigma_{\mathbb{E}}} \Pi^\vee_v \) of weight \( \mathbb{k} \), that is,

\[
\Pi_\infty(k_\ell)\varphi_{(\mathbb{k}, \ell)} = \prod_{w \in \Sigma_{\mathbb{E}}} e^{\sqrt{-1} k_{\ell w} \theta_w} \cdot \varphi_{(\mathbb{k}, \ell)},
\]

\[
\Pi^\vee_\infty(k_\ell)\varphi_{(\mathbb{k}, -\ell)} = \prod_{w \in \Sigma_{\mathbb{E}}} e^{\sqrt{-1} k_{\ell w} \theta_w} \cdot \varphi_{(\mathbb{k}, -\ell)}
\]

for \( k_\ell = (k_{\ell w})_{w \in \Sigma_{\mathbb{E}}} \in \text{SO}(2\Sigma_{\mathbb{E}}) \). Fix \( \sigma \in \text{Aut}(\mathbb{C}) \). Let \( \varphi_{(\sigma, \mathbb{k}, \ell)} \in \sigma\Pi_\infty \) and \( \varphi_{(\sigma, \mathbb{k}, -\ell)} \in \sigma\Pi^\vee_\infty \) be the vectors of weight \( \sigma \mathbb{k} \) such that \( \varphi_1 = \varphi_{(\mathbb{k}, \ell)} \otimes \left( \bigotimes_{v \mid \infty} \varphi_{1, v} \right) \in \Pi^D_{\text{hol}} \) and \( \varphi_2 = \varphi_{(\mathbb{k}, -\ell)} \otimes \left( \bigotimes_{v \mid \infty} \varphi_{2, v} \right) \in \sigma(\Pi^D_{\text{hol}})^\vee \), then

\[
\sigma \varphi_1 = \varphi_{(\sigma, \mathbb{k}, \ell)} \otimes \left( \bigotimes_{v \mid \infty} t_{\sigma, v} \varphi_{1, v} \right) \in \sigma\Pi^D_{\text{hol}}, \quad \sigma \varphi_2 = \varphi_{(\sigma, \mathbb{k}, -\ell)} \otimes \left( \bigotimes_{v \mid \infty} t_{\sigma, v}^\vee \varphi_{2, v} \right) \in \sigma(\Pi^D_{\text{hol}})^\vee.
\]
Here the $\sigma$-linear isomorphisms $t_{\sigma,v}, t_{\sigma,v}^\vee$ for finite places $v$ of $F$ are fixed in Sect. 4.2 and
\[
\begin{align*}
\Pi^D_{\text{hol}} & \longrightarrow \sigma \Pi^D_{\text{hol}}, \quad \varphi \longmapsto \sigma \varphi, \\
(\Pi^D_{\text{hol}})^\vee & \longrightarrow \sigma(\Pi^D_{\text{hol}})^\vee, \quad \varphi^\vee \longmapsto \sigma \varphi^\vee
\end{align*}
\]
are defined in (2.17). By Lemma 4.5, for each finite place $v$ of $F$, there exist $\varphi_{1,v} \in \Pi^D_v$ and $\varphi_{2,v} \in (\Pi^D_v)^\vee$ such that
\[
I^D_v(\varphi_{1,v} \otimes \varphi_{2,v}) \neq 0.
\]
Moreover, when $v$ is a finite place such that $E_v$ is unramified over $F_v$ and $\Pi_v$ is unramified, then $D$ is unramified at $v$ and we can choose $\varphi_{1,v}$ and $\varphi_{2,v}$ to be non-zero $\text{GL}_2(\sigma_{E_v})$-invariant vectors. In this case, by [25, Lemma 2.2], we have
\[
I^D_v(\varphi_{1,v} \otimes \varphi_{2,v}) = \langle \varphi_{1,v}, \varphi_{2,v} \rangle_v.
\]

With this local choice, we put $\varphi_1 = \varphi(e,\ell) \otimes \left( \bigotimes_{v \mid \infty} \varphi_{1,v} \right) \in \Pi^D_{\text{hol}}$ and $\varphi_2 = \varphi(e,-\ell) \otimes \left( \bigotimes_{v \mid \infty} \varphi_{2,v} \right) \in (\Pi^D_{\text{hol}})^\vee$. We also fix $\varphi_3 = \varphi(e,\ell) \otimes \left( \bigotimes_{v \mid \infty} \varphi_{3,v} \right) \in \Pi^D_{\text{hol}}$ and $\varphi_4 = \varphi(e,-\ell) \otimes \left( \bigotimes_{v \mid \infty} \varphi_{4,v} \right) \in (\Pi^D_{\text{hol}})^\vee$ such that
\[
\langle \varphi_3, \varphi_4 \rangle = \int_{G(\mathbb{A})} \varphi_3(g) \varphi_4(g \cdot t_{\Sigma_E}) \, dg_{\text{Tam}} \neq 0.
\]

Let $X(e) \in U(\text{GL}_2(\mathbb{C}_E))$ be the differential operator defined in (3.4). By Ichino’s formula Theorem 4.1, we have
\[
\begin{align*}
I^D(\mathbf{X}(e) \cdot \varphi_1^l_{\ell} \otimes \mathbf{X}(e) \cdot \varphi_2^l_{\ell}) \\
& \quad \langle \varphi_3, \varphi_4 \rangle \\
& = \frac{C_D}{2^c} \cdot \frac{\zeta(2)}{\zeta(2)} \cdot \frac{L(1, \Pi, \text{Ad})}{L(1, \Pi, \text{Ad})} \cdot \prod_{v \mid \infty} \frac{I^D_v(\varphi_{1,v} \otimes \varphi_{2,v})}{\langle \varphi_{3,v}, \varphi_{4,v} \rangle_v} \\
& \quad \cdot \frac{I^\infty(X(e) \cdot \Pi^\infty(\tau_{\Sigma_E}) \varphi(e,\ell) \otimes X(e) \cdot \Pi^\vee(\tau_{\Sigma_E}) \varphi(e,-\ell))}{\langle \varphi(e,\ell), \Pi^\infty(\tau_{\Sigma_E}) \varphi(e,-\ell) \rangle_\infty} \\
& \quad \cdot \frac{I^D(\mathbf{X}(e) \cdot \varphi_1^l_{\ell} \otimes \mathbf{X}(e) \cdot \varphi_2^l_{\ell})}{\langle \varphi_3, \varphi_4 \rangle} \\
& = \frac{C_D}{2^c} \cdot \frac{\zeta(2)}{\zeta(2)} \cdot \frac{L(1, \Pi, \text{Ad})}{L(1, \Pi, \text{Ad})} \cdot \prod_{v \mid \infty} \frac{I^D_v(t_{\sigma,v} \varphi_{1,v} \otimes \varphi_{2,v})}{\langle \varphi_{3,v}, \varphi_{4,v} \rangle_v} \\
& \quad \times \frac{I^\infty(X(e) \cdot \Pi^\infty(\tau_{\Sigma_E}) \varphi(e,\ell) \otimes X(e) \cdot \Pi^\vee(\tau_{\Sigma_E}) \varphi(e,-\ell))}{\langle \varphi(e,\ell), \Pi^\infty(\tau_{\Sigma_E}) \varphi(e,-\ell) \rangle_\infty}.
\end{align*}
\]

Here $\tau_{\Sigma_E}$ and $\varphi_1^l_{\ell}, \varphi_2^l_{\ell}$ are defined in (2.14) and (2.15), respectively. By Lemma 2.2 and Proposition 3.2, we have the Galois equivariant property of the global trilinear period integral that
\[
\sigma \left( \frac{I^D(\mathbf{X}(e) \cdot \varphi_1^l_{\ell} \otimes \mathbf{X}(e) \cdot \varphi_2^l_{\ell})}{\Omega^l_{\mathbf{X}(e)(\Pi^D)} \cdot \Omega^l_{\mathbf{X}(e)(\Pi^D)^\vee}} \right) = \frac{I^D(\mathbf{X}(e) \cdot \varphi_1^l_{\ell} \otimes \mathbf{X}(e) \cdot \varphi_2^l_{\ell})}{\Omega^l_{\mathbf{X}(e)(\sigma \Pi^D)} \cdot \Omega^l_{\mathbf{X}(e)(\sigma \Pi^D)^\vee}}. (4.9)
\]
On the other hand, by Lemma 2.9 and Corollary 2.10, we have

$$\sigma \left( \frac{L(1, \Pi, \text{Ad})}{(2\pi \sqrt{-1})^{\sum_{w \in \Sigma_F} \kappa_w} \cdot \pi^{3[F:Q]} \cdot \langle \varphi_3, \varphi_4 \rangle} \right) = \frac{L(1, \sigma \Pi, \text{Ad})}{(2\pi \sqrt{-1})^{\sum_{w \in \Sigma_F} \kappa_w} \cdot \pi^{3[F:Q]} \cdot \langle \sigma \varphi_3, \sigma \varphi_4 \rangle}.$$  (4.10)

By Lemmas 4.3 and 4.4, we have

$$\sigma \left( \prod_{v|\infty} \frac{I^D_v (\varphi_{1,v} \otimes \varphi_{2,v})}{\langle \varphi_{3,v}, \varphi_{4,v} \rangle} \right) = \prod_{v|\infty} \frac{I^D_v \left( t_{\sigma,v} \varphi_{1,v} \otimes t_{\sigma,v} \varphi_{2,v} \right)}{\langle t_{\sigma,v} \varphi_{3,v}, t_{\sigma,v} \varphi_{4,v} \rangle},$$  (4.11)

and

$$\frac{I_\infty (X(\kappa) \cdot \Pi_\infty (\tau_{I_\kappa}) \varphi(\kappa, \ell) \otimes \Pi_\infty^\vee (\tau_{I_\kappa}) \varphi(\kappa, -\ell))}{\langle \varphi(\kappa, \ell), \Pi_\infty^\vee (\tau_{\Sigma E}) \varphi(\kappa, -\ell) \rangle} = \frac{I_\infty (X(\kappa) \cdot \sigma \Pi_\infty^\vee (\tau_{I_{\kappa}}) \varphi(\sigma \kappa, \ell) \otimes \Pi_\infty^\vee (\tau_{I_{\kappa}}) \varphi(\sigma \kappa, -\ell))}{\langle \varphi(\sigma \kappa, \ell), \sigma \Pi_\infty^\vee (\tau_{\Sigma E}) \varphi(\sigma \kappa, -\ell) \rangle}.$$  (4.12)

By (4.3) and the result of Siegel [47], we have

$$C_D \in D_F^{1/2} \cdot \zeta_F (2)^{-1} \cdot \mathbb{Q}^\times, \quad \zeta_F (2) \in D_F^{1/2} \cdot \pi^{3[F:Q]} \cdot \mathbb{Q}^\times, \quad \zeta_E (2) \in D_E^{1/2} \cdot \pi^{3[F:Q]} \cdot \mathbb{Q}^\times.$$

Also note that

$$\prod_{v \in \Sigma_F} L \left( \frac{1}{2}, \Pi_v, \text{As} \right) \in \pi^{-2 \sum_{w \in \Sigma_F} \max_{w|v} \{ \kappa_w \}} \cdot \mathbb{Q}^\times.$$

The algebraicity for $L \left( \frac{1}{2}, \Pi, \text{As} \right)$ then follows from (4.9)–(4.12). Finally, assume $D$ is the matrix algebra, we show that $\Omega^D (\Pi^D (\nu)) = \Omega^D (\Pi (\nu))$ can be replaced by $\Omega^D (\Pi)$. When $L \left( \frac{1}{2}, \Pi, \text{As} \right) = 0$, the assertion holds by assertion (1). Thus we may assume $L \left( \frac{1}{2}, \Pi, \text{As} \right) \neq 0$. In this case, by Ichino’s formula Theorem 4.1, there exists $\varphi \in \Pi_{\text{hot}}$ such that

$$\int_{A_F \backslash \text{GL}_2 (F) \backslash \text{GL}_2 (A_F)} \frac{X(\kappa) \cdot \varphi (\kappa, g) d \gamma}{\Omega^D (\Pi)} \neq 0.$$

By Lemma 2.2 and Proposition 3.2, we have

$$\sigma \left( \int_{A_F \backslash \text{GL}_2 (F) \backslash \text{GL}_2 (A_F)} \frac{X(\kappa) \cdot \varphi (\kappa, g) d \gamma}{\Omega^D (\Pi)} \right) = \int_{A_F \backslash \text{GL}_2 (F) \backslash \text{GL}_2 (A_F)} \frac{X(\sigma \kappa) \cdot \sigma \varphi (\kappa, g) d \gamma}{\Omega^D (\sigma \Pi)}.$$
On the other hand, since \( \omega^\Pi \big|_{\Lambda^x_{\Sigma}} \) is trivial, we have \( \varphi|_{GL_2(\Lambda_{\Sigma})} = (\varphi \otimes \omega^{-1}_\Pi)|_{GL_2(\Lambda_{\Sigma})} \). By Lemma 2.2 and Proposition 3.2 again, we have

\[
\sigma \left( \int_{\Lambda^x_{\Sigma}GL_2(F) \setminus GL_2(\Lambda_{\Sigma})} \frac{X(\kappa) \cdot \varphi^\infty(g)}{\Omega^\kappa_{\omega}(\Pi)} dg_{\text{Tam}} \right) = \omega^\Pi (\det(\tau_{\kappa})) \cdot \sigma \left( \int_{\Lambda^x_{\Sigma}GL_2(F) \setminus GL_2(\Lambda_{\Sigma})} \frac{X(\kappa) \cdot (\varphi \otimes \omega^{-1}_\Pi)^\infty(g)}{\Omega^\kappa_{\omega}(\Pi)} dg_{\text{Tam}} \right) = \omega^\Pi (\det(\tau_{\kappa})) \cdot \sigma \left( \int_{\Lambda^x_{\Sigma}GL_2(F) \setminus GL_2(\Lambda_{\Sigma})} \frac{X^\infty(\kappa) \cdot (\varphi \otimes \omega^{-1}_\Pi)^\infty(g)}{\Omega^\kappa_{\omega}(\sigma\Pi)} dg_{\text{Tam}} \right).
\]

Here \( (\varphi \otimes \omega^{-1}_\Pi) \) is defined in (2.17) with \( \Pi \) replaced by \( \Pi^\vee \). By the \( q \)-expansion principle [14, Theorem 6.4] and [1, (1.1.13)] (see also [11, (A.4.6) and (A.4.7)]) for Hilbert modular forms, we have

\[
\sigma (\varphi \otimes \omega^{-1}_\Pi) = \sigma \varphi \otimes \omega^{-1}_\Pi.
\]

Therefore, we have

\[
\omega^\Pi (\det(\tau_{\kappa})) \cdot \sigma \left( \int_{\Lambda^x_{\Sigma}GL_2(F) \setminus GL_2(\Lambda_{\Sigma})} X^\infty(\kappa) \cdot (\varphi \otimes \omega^{-1}_\Pi)^\infty(g) dg_{\text{Tam}} \right) = \int_{\Lambda^x_{\Sigma}GL_2(F) \setminus GL_2(\Lambda_{\Sigma})} X^\infty(\kappa) \cdot (\varphi \otimes \omega^{-1}_\Pi)^\infty(g) \cdot \omega^{-1}_\Pi (\det(g)) dg_{\text{Tam}} = \int_{\Lambda^x_{\Sigma}GL_2(F) \setminus GL_2(\Lambda_{\Sigma})} X^\infty(\kappa) \cdot (\varphi \otimes \omega^{-1}_\Pi)^\infty(g) dg_{\text{Tam}}.
\]

We conclude that

\[
\sigma \left( \frac{\Omega^\kappa_{\omega}(\Pi^\vee)}{\Omega^\kappa_{\omega}(\Pi)} \right) = \frac{\Omega^\kappa_{\omega}(\sigma\Pi^\vee)}{\Omega^\kappa_{\omega}(\sigma\Pi)}.
\]

This completes the proof.

### 4.4 Proof of Theorem 1.6

First we recall the result of Shimura [44] on the algebraicity of special values of the twisted standard \( L \)-functions for motivic irreducible cuspidal automorphic representations of \( GL_2(\Lambda_{\Sigma}) \). We also refer to [42] for a different proof.

**Theorem 4.6 (Shimura)** Let \( \Pi \) be a motivic irreducible cuspidal automorphic representation of \( GL_2(\Lambda_{\Sigma}) \) of motivic weight \((\ell, r) \in \mathbb{Z}_{\geq 2}[\Sigma_{\Sigma}] \times \mathbb{Z}\) and central character \( \omega_{\Pi} \). There exist complex numbers \( p(\varepsilon, \sigma\Pi) \in \mathbb{C}^\times \) defined for all \( \sigma \in \text{Aut}(\mathbb{C}) \) and \( \varepsilon \in \{ \pm 1 \}^{\Sigma_{\Sigma}} \) satisfying the following assertions:

1. We have

\[
\sigma \left( \frac{L(\infty)(m + \frac{1}{2}, \Pi \otimes \chi)}{(2\pi \sqrt{-1})^{[F:Q]m} \cdot G(\chi) \cdot p((-1)^m \text{sgn} (\chi), \Pi)} \right) = L(\infty)(m + \frac{1}{2}, \sigma\Pi \otimes \sigma\chi) = \frac{(2\pi \sqrt{-1})^{[F:Q]m} \cdot G(\sigma\chi) \cdot p((-1)^m \text{sgn} (\chi), \sigma\Pi)}{(2\pi \sqrt{-1})^{[F:Q]m} \cdot G(\chi) \cdot p((-1)^m \text{sgn} (\chi), \Pi)}.
\]
for any finite order Hecke character $\chi$ of $\mathbb{A}_F^\times$, $\sigma \in \text{Aut}(\mathbb{C})$, and $m \in \mathbb{Z}$ such that
\[-\frac{\min_{v \in \Sigma_F} \{ \ell_v \}}{2} - \frac{r}{2} < m < \frac{\min_{v \in \Sigma_F} \{ \ell_v \}}{2} - \frac{r}{2}.\]

(2) We have
\[\sigma \left( \frac{p(\varepsilon, \Pi) \cdot p(-\varepsilon, \Pi)}{(2\pi \sqrt{-1})^{[F:Q](1+r)}(\sqrt{-1})^{\sum_{v \in \Sigma_F} \ell_v} \cdot G(\omega_{\Pi}) \cdot \Omega^{\Sigma_F}(\Pi)} \right) = \frac{p(\varepsilon, \sigma \Pi) \cdot p(-\varepsilon, \sigma \Pi)}{(2\pi \sqrt{-1})^{[F:Q](1+r)}(\sqrt{-1})^{\sum_{v \in \Sigma_F} \ell_v} \cdot G(\sigma \omega_{\Pi}) \cdot \Omega^{\Sigma_F}(\sigma \Pi)}\]

for any $\varepsilon \in \{\pm 1\}^{\Sigma_F}$ and $\sigma \in \text{Aut}(\mathbb{C})$.

**Remark 4.7** In [44, Theorem 4.3], the theorem was stated for motivic $(\ell, r) \in \mathbb{Z}_{\geq 3}[\Sigma_F] \times \mathbb{Z}$. It is straightforward to extend the results to $(\ell, r) \in \mathbb{Z}_{\geq 2}[\Sigma_F] \times \mathbb{Z}$ by [44, (4.16)] and the non-vanishing theorem of Friedberg–Hoffstein [10]. We also refer to Lemma 2.9 for the period relation between Petersson norm and $\Omega^{\Sigma_F}(\Pi)$.

Now we begin the proof of Theorem 1.6. Let $\Pi = \bigotimes_v \Pi_v$, $\Pi' = \bigotimes_v \Pi'_v$ be motivic irreducible cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$ with central characters $\omega_{\Pi}$, $\omega_{\Pi'}$ and of weights $\ell$, $\ell' \in \mathbb{Z}_{\geq 1}[\Sigma_F]$, respectively. By assumption (i), the automorphic representation $\text{Sym}^2(\Pi) \times \Pi'$ of $GL_3(\mathbb{A}_F) \times GL_2(\mathbb{A}_F)$ is self-dual. Since $\text{Sym}^2(\Pi) \times \Pi'$ is isobaric, we have the global functional equation
\[L(s, \text{Sym}^2(\Pi) \times \Pi') = \varepsilon(s, \text{Sym}^2(\Pi) \times \Pi') L(1-s, \text{Sym}^2(\Pi) \times \Pi').\]
Recall the global root number $\varepsilon(\text{Sym}^2(\Pi) \times \Pi')$ is defined by
\[\varepsilon(\text{Sym}^2(\Pi) \times \Pi') = \varepsilon(\frac{1}{2}, \text{Sym}^2(\Pi) \times \Pi') \in \{\pm 1\}.
\]
Analogous to the proof of Lemma 4.2, we can show that
\[\varepsilon(\text{Sym}^2(\Pi) \times \Pi') = \varepsilon(\text{Sym}^2(\sigma \Pi) \times \sigma \Pi')\]
for all $\sigma \in \text{Aut}(\mathbb{C})$. In particular, if $\varepsilon(\text{Sym}^2(\Pi) \times \Pi') = -1$, then it follows from the global functional equation that
\[L\left(\frac{1}{2}, \text{Sym}^2(\sigma \Pi) \times \sigma \Pi') = 0\]
for all $\sigma \in \text{Aut}(\mathbb{C})$. Therefore, we may assume that $\varepsilon(\text{Sym}^2(\Pi) \times \Pi') = 1$. By assumption (iii) of Theorem 1.6 and the non-vanishing theorem of Friedberg–Hoffstein [10], there exists a totally real quadratic extension $K$ over $F$ such that the base change lift $\Pi_K = \bigotimes_v \Pi_{K,v}$ of $\Pi$ to $GL_2(\mathbb{A}_K)$ is cuspidal and
\[L\left(\frac{1}{2}, \Pi' \otimes \omega_{\Pi} \omega_{\Pi}/F\right) \neq 0.\]
Consider the totally real étale cubic algebra $E = K \times F$ over $F$ and the motivic irreducible cuspidal automorphic representation $\Pi_K \times \Pi'$ of $GL_2(\mathbb{A}_E)$. We identify $\Sigma_E$ with $\Sigma_K \cup \Sigma_F$ in a natural way. Note that the weight $K \in \mathbb{Z}_{\geq 1}[\Sigma_E]$ of $\Pi_K \times \Pi'$ is given as follows: for $w \in \Sigma_E$ lying over $v \in \Sigma_F$, we have
\[\kappa_w = \begin{cases} \ell_v & \text{if } w \in \Sigma_K, \\ \ell'_v & \text{if } w \in \Sigma_F. \end{cases}\]
In particular, $\Pi_K \times \Pi'$ is totally unbalanced and $I_\Sigma = \Sigma_F$. Let $D$ be the unique totally indefinite quaternion algebra over $F$ so that there exists an irreducible cuspidal automorphic representation

$$\Pi_K^D \times (\Pi')^D = \bigotimes_v \left( \Pi_{K,v}^D \times (\Pi_{v}')^D \right)$$

of $D^\times(\mathbb{A}_{\Sigma})$ associated to $\Pi_K \times \Pi'$ by the Jacquet–Langlands correspondence such that

$$\text{Hom}_{D^\times(\mathbb{A}_{\Sigma})}(\Pi_K^D, (\Pi_{v}')^D, \mathbb{C}) \neq 0$$

for all places $v$ of $\mathbb{F}$. By Lemma 2.8, we have the period relation

$$\sigma \left( \frac{\Omega_{\Sigma^F}(\Pi_K^D \times (\Pi')^D)}{\Omega^\phi(\Pi_K^D) \cdot \Omega_{\Sigma^F}(\Pi')^D} \right) = \frac{\Omega_{\Sigma^F}(\sigma \Pi_K^D \times \sigma(\Pi')^D)}{\Omega^\phi(\sigma \Pi_K^D) \cdot \Omega_{\Sigma^F}(\sigma(\Pi')^D)}$$

(4.13)

for all $\sigma \in \text{Aut}(\mathbb{C})$. Note that by definition we may take $\Omega^\phi(\Pi_K^D) = 1$ and by Corollary 2.10 we have

$$\sigma \left( \frac{\Omega_{\Sigma^F}(\Pi')^2}{\Omega_{\Sigma^F}(\Pi)^D \cdot \Omega_{\Sigma^F}(\Pi')^D} \right) = \frac{\Omega_{\Sigma^F}(\sigma \Pi')^2}{\Omega_{\Sigma^F}(\sigma \Pi^D) \cdot \Omega_{\Sigma^F}(\sigma(\Pi')^D)}$$

(4.14)

for all $\sigma \in \text{Aut}(\mathbb{C})$. Also we have the well-known fact for the rationality of the quadratic Gauss sum that

$$D_{\Sigma}^{1/2} G(\omega_{\mathbb{K}/\mathbb{F}}) \in \mathbb{Q}^\times.$$ 

(4.15)

Therefore, by (4.13)–(4.15) and Theorem 1.4, we have

$$\sigma \left( \frac{L^{(\infty)}(\frac{1}{2}, \Pi_K \times \Pi', \text{As})}{(2\pi \sqrt{-1})^{2[\mathbb{F} : \mathbb{Q}]} \cdot G(\omega_{\mathbb{K}/\mathbb{F}}) \cdot \Omega_{\Sigma^F}(\Pi')^2} \right) = \frac{L^{(\infty)}(\frac{1}{2}, \sigma \Pi_K \times \sigma \Pi', \text{As})}{(2\pi \sqrt{-1})^{2[\mathbb{F} : \mathbb{Q}]} \cdot G(\omega_{\mathbb{K}/\mathbb{F}}) \cdot \Omega_{\Sigma^F}(\sigma \Pi')^2}$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. Finally, we have the factorization of $L$-functions:

$$L(s, \Pi_K \times \Pi', \text{As}) = L(s, \text{Sym}^2(\Pi) \times \Pi') L(s, \Pi' \otimes \omega_H \omega_{\mathbb{K}/\mathbb{F}}).$$

Theorem 1.6 then follows immediately from Theorem 4.6 of Shimura for the central critical value

$$L^{(\infty)}(\frac{1}{2}, \Pi' \otimes \omega_H \omega_{\mathbb{K}/\mathbb{F}}) = L^{(\infty)}(r + \frac{1}{2}, \Pi' \otimes |_{\mathbb{A}_{\Sigma}}^r \omega_H \omega_{\mathbb{K}/\mathbb{F}})$$

and the condition that $L(\frac{1}{2}, \Pi' \otimes \omega_H \omega_{\mathbb{K}/\mathbb{F}}) \neq 0$. This completes the proof.

### 4.5 Compatibility with Deligne’s conjecture

In this section, we interpret Theorem 1.4 in terms of Deligne’s conjecture [9]. We assume $E$ is a field. The other cases can be treated in a similar way.

Suppose $\kappa_w \geq 2$ for all $w \in \Sigma_\Sigma$. Let

$$M(\Pi)$$

be the (conjectural) motive attached to $\Pi$ of rank 2 over $\mathbb{E}$ with coefficients in $\mathbb{Q}(\Pi)$ satisfying the conditions in [52, Conjecture 4.1] with $k_0$ therein replaced by $-r$, where $|\omega_H| = |\gamma_{H_\Sigma}^r|$ for some $r \in \mathbb{Z}$. Let $\Sigma_{\mathbb{Q}(\Pi)}$ be the set of embeddings from $\mathbb{Q}(\Pi)$ to $\mathbb{C}$ and identify $\mathbb{Q}(\Pi) \otimes_{\mathbb{Q}} \mathbb{C}$ with $\mathbb{C}^{\Sigma_{\mathbb{Q}(\Pi)}}$ in a natural way. We also identify $\mathbb{Q}(\Pi)$ as a subfield of $\mathbb{Q}(\Pi) \otimes_{\mathbb{Q}} \mathbb{C}$ by the

[S. Y. Chen]
diagonal embedding. For each subset $I$ of $\Sigma_G$ and $\sigma \in \text{Aut}(\mathbb{C})$, denote by $\Omega^I(\sigma\Pi) \in \mathbb{C}^*$ the Harris’ period of $\sigma\Pi$. For $w \in \Sigma_E$, let

$$c_w^\pm(M(\Pi)) = (c_w^\pm(\sigma, M(\Pi)))_\sigma \in (\mathbb{Q}(\Pi) \otimes_{\mathbb{Q}} \mathbb{C})^*,$$

$$\delta_w(\text{Art}_{\omega_1^{-1}}) = (\delta_w(\sigma, \text{Art}_{\omega_1^{-1}}))_\sigma \in (\mathbb{Q}(\Pi) \otimes_{\mathbb{Q}} \mathbb{C})^*$$

be the covariantly defined $w$-periods in [52, Sect.2.4]. Here $\text{Art}_{\omega_1^{-1}}$ is the Artin motive attached to $\omega_1^{-1}$ defined as in [9, Sect.6] and $\sigma \in \Sigma_{Q(\Pi)}$. Comparing [52, (4.14)] with [16, Theorem 3.5.1] on the algebraicity of Rankin–Selberg $L$-functions for $GL_2 \times GL_2$, it is natural to expect that

$$\left(\prod_{w \in \sigma I} c_w^\pm(\sigma, M(\Pi))c_w^-(\sigma, M(\Pi))\delta_w(\sigma, \text{Art}_{\omega_1^{-1}})^{-1}\right)_{\sigma} \equiv (2\pi \sqrt{-1})^{-r_1} (\sqrt{-1})^{r_2} \left(\prod_{w \in \sigma I} \mathcal{C}_w(\sigma, M(\Pi))\delta_w(\sigma, \text{Art}_{\omega_1^{-1}})^{-1}\right)_{\sigma} \mod \mathbb{Q}(\Pi, I)^\times. \tag{4.16}$$

Here we have enlarged the coefficients to $\mathbb{Q}(\Pi, I) = \mathbb{Q}(\Pi) \cdot \mathbb{Q}(I)$. On the other hand, for a totally indefinite quaternion algebra $D$ over $\mathbb{F}$, it was conjectured in [17, Conjecture 7.1.6] that

$$\sigma\left(\frac{\Omega^I(\Pi D)}{\Omega^I(\Pi)}\right) = \frac{\Omega^I(\sigma\Pi D)}{\Omega^I(\sigma\Pi)} \tag{4.17}$$

for all $\sigma \in \text{Aut}(\mathbb{C})$ and $I \subseteq \Sigma_E$. Similarly for $\Pi^\vee$. This conjecture holds trivially for $I = \emptyset$ and is known for $I = \Sigma_E$ (cf. Corollary 2.10). We also expect that

$$\sigma\left(\frac{\Omega^I(\Pi^\vee)}{\Omega^I(\Pi)}\right) = \frac{\Omega^I(\sigma\Pi^\vee)}{\Omega^I(\sigma\Pi)}.$$

Indeed, let $\chi$ be an algebraic Hecke character of $A^\times_E$. Enlarging the coefficients to $\mathbb{Q}(\Pi, I, \chi) = \mathbb{Q}(\Pi) \cdot \mathbb{Q}(I) \cdot \mathbb{Q}(\chi)$ and applying [52, Proposition 3.1] to $M \otimes N = M(\Pi) \otimes \text{Art}_{\chi^{-1}} = M(\Pi \otimes \chi)$, we have

$$c_w^+(M(\Pi \otimes \chi))c_w^-(M(\Pi \otimes \chi))\delta_w(\text{Art}_{\omega_1^{-1}\chi^{-1}})^{-1} = c_w^+(M(\Pi))c_w^-(M(\Pi))\delta_w(\text{Art}_{\omega_1^{-1}})^{-1}$$

for each $w \in \Sigma_E$. Therefore, it is natural to expect that

$$\left(\Omega^I(\sigma\Pi \otimes \sigma\chi)\right)_{\sigma} \equiv \left(\Omega^I(\sigma\Pi)\right)_{\sigma} \mod \mathbb{Q}(\Pi, I, \chi)^\times.$$

Let $\text{As}(M(\Pi))$ be the (conjectural) Asai motive associated to $M(\Pi)$, that is,

$$\text{As}(M(\Pi)) = \bigotimes_{\Omega} \text{Res}_{E/F} M(\Pi)$$

in the notation of [52, Conjecture 1.8] with $\Omega = \text{Gal}(\overline{Q}/F)/\text{Gal}(\overline{Q}/E)$. Then we have

$$L(\text{Res}_{F/Q} \text{As}(M(\Pi)), s) = \left(L^*(s + \frac{3}{2}, \sigma\Pi, \text{As})\right)_{\sigma}$$

and Deligne’s conjecture predicts that

$$\left(\frac{\Omega^{(\infty)}(\sigma\Pi, m)}{\Omega^{(\infty)}} \cdot \chi^{-1}(\sigma\Pi, m) \cdot c^*(\sigma\Pi, m)\right)_{\sigma} \in \mathbb{Q}(\Pi)$$

for all critical points $m \in \mathbb{Z}$ for $\text{Res}_{F/Q} \text{As}(M(\Pi))$. Here $c^*(\sigma\Pi, m)$ are Deligne’s periods attached to $\text{Res}_{F/Q} \text{As}(M(\Pi))$. 

\text{Springer}
Lemma 4.8  Subject to the period relations (4.16) and (4.17), Theorem 1.4 is compatible with Deligne’s conjecture for $m = -1$ modulo $\widetilde{E}^\times$.

Proof  Explicate Yoshida’s calculation [52, (5.9)] of the Deligne’s period $c^\pm(R_{F/Q} \text{As}(M(\Pi)))$ in our case, we then deduce from the totally unbalanced condition that

$$c^\pm(R_{F/Q} \text{As}(M(\Pi))) \in (2\pi \sqrt{-1})^4 \mathbb{F}[\overline{\mathbb{F}}] \left( G(\sigma \omega_{\Pi})^2 \prod_{w \in I_{\omega}} c^+_w(\sigma, M(\Pi))^2 c^-_w(\sigma, M(\Pi))^2 \delta_w(\sigma, \text{Art}_{\omega_{\Pi}})^{-2} \right)_{\sigma} \cdot (\overline{\mathbb{E}}^\times)^{\Sigma(\Pi)}$$

where $G(\sigma \omega_{\Pi})$ is the Gauss sum of $\sigma \omega_{\Pi}$ and $\widetilde{E}$ is the Galois closure of $E$ over $\mathbb{Q}$. Note that the assumption $\omega_{\Pi} |_{k^\times}$ is trivial and (1.2) imply that $(G(\sigma \omega_{\Pi}))_{\sigma} \in \mathbb{Q}(\Pi)^\times$. Therefore, when $D$ is the matrix algebra, our main result Theorem 1.4 is compatible with Deligne’s conjecture for $m = -1$, at least modulo $\widetilde{E}^\times$, subject to (4.16). For general $D$, we need to assume further the period relation (4.17).

Acknowledgements  This work was initiated during the author’s visit to Kyoto University supported by the Graduate Students Study Abroad Program sponsored by Ministry of Science and Technology. The author would like to thank Atsushi Ichino for suggesting the problem and his advice and encouragement. Finally, the author is grateful to the referees for the suggestions and comments on the improvement of the manuscript.

References

1. D. Blasius, M. Harris, and D. Ramakrishnan. Coherent cohomology, limits of discrete series, and Galois conjugation. *Duke Math. J.*, 73(3):647–685, 1994.
2. D. Blasius. Appendix to Orloff, Critical values of certain tensor product $L$-functions. *Invent. Math.*, 90:181–188, 1987.
3. A. Borel. Automorphic $L$-functions. In *Automorphic forms, representations, and $L$-functions*, volume 33, Part 2, pages 27–61. Proceedings of Symposia in Pure Mathematics, 1979.
4. A. Borel and N. Wallach. Continuous cohomology, discrete subgroups, and representations of reductive groups, second edition, volume 67 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., 2000.
5. S.-Y. Chen and Y. Cheng. On Deligne’s conjecture for certain automorphic $L$-functions for $GL(3) \times GL(2)$ and $GL(4)$. *Doc. Math.*, 24:2241–2297, 2019.
6. S.-Y. Chen. Gamma factors for the Asai cube representation. Math. Z., 297:747–773, 2021.
7. S.-Y. Chen. Pullback formulas for nearly holomorphic Saito-Kurokawa lifts. *Manuscripta Math.*, 161:501–561, 2020.
8. L. Clozel. Motifs et Formes Automorphes: Applications du Principe de Fonctorialitè. In *Automorphic Forms, Shimura Varieties, and $L$-functions, Vol. I*, Perspectives in Mathematics, pages 77–159, 1990.
9. P. Deligne. Valeurs de fonctions $L$ et périodes d’intégrales. In *Automorphic Forms, Representations and $L$-Functions*, volume 33, pages 313–346. Proceedings of Symposia in Pure Mathematics, 1979. Part 2.
10. S. Friedberg and J. Hoffstein. Nonvanishing Theorems for Automorphic $L$-Functions on $GL(2)$. *Ann. of Math.*, 142(2):385–423, 1995.
11. P. Garrett and M. Harris. Special values of triple product $L$-functions. *Amer. J. Math.*, 1993.
12. S. Gelbart and H. Jacquet. A relation between automorphic representations of $GL(2)$ and $GL(3)$. *Ann. Sci. Éc. Norm. Supér.*, 11:471–542, 1978.
13. M. Harris. Arithmetic vector bundles and automorphic forms on Shimura varieties. I. *Invent. Math.*, 82:151–189, 1985.
14. M. Harris. Arithmetic vector bundles and automorphic forms on Shimura varieties, II. *Compos. Math.*, 60(3):323–378, 1986.
15. G. Harder. Eisenstein cohomology of arithmetic groups The case $GL_2$. *Invent. Math.*, 89:37–118, 1987.
16. M. Harris. Period invariants of Hilbert modular forms, I: Trilinear differential operators and $L$-functions. In *Cohomology of Arithmetic Groups and Automorphic Forms*, volume 1447 of *Lecture Notes in Mathematics*, pages 155–202. Springer-Verlag, 1989.
17. M. Harris. Automorphic forms and the cohomology of vector bundles on shimura varieties. In Automorphic Forms, Shimura Varieties, and L-functions, Vol. II, Perspectives in Mathematics, pages 41–91, 1990.

18. M. Harris. Automorphic forms of $\delta$-cohomology type as coherent cohomology classes. J. Differential Geom., 32:1–63, 1990.

19. M. Harris. Periods invariants of Hilbert modular forms, II. Compos. Math., 94:201–226, 1994.

20. G. Henniart. Sur la conjecture de Langlands locale pour GL$_n$. J. Théor. Nombres Bordeaux, 13(1):167–187, 2001.

21. M. Harris and R. Taylor. On the geometry and cohomology of some simple Shimura varieties. Princeton University Press, 2001.

22. A. Ichino. Pullbacks of Saito-Kurokawa lifts. Invent. Math., 162:551–647, 2005.

23. A. Ichino. Trilinear forms and the central values of triple product L-functions. Duke Math. J., 145(2):281–307, 2008.

24. A. Ichino and T. Ikeda. On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture. Geom. Funct. Anal., 19:1378–1425, 2010.

25. T. Ikeda. On the functional equations of the triple L-functions. Kyoto J. Math., 29:175–219, 1989.

26. A. Ichino and K. Prasanna. Period of quaternionic Simura varieties. Mem. Amer. Math. Soc., 2018. to appear.

27. F. Januszewski. On Period Relations for Automorphic L-functions I. Trans. Amer. Math. Soc., 371:6547–6580, 2019.

28. H. Jacquet and R. Langlands. Automorphic Forms on GL(2), volume 114 of Lecture Notes in Mathematics. Springer-Verlag, Berlin and New York, 1970.

29. S. S. Kudla and S. Rallis. Ramified degenerate principal series representations for Sp$(n)$. Israel J. Math., 78:209–256, 1992.

30. H. H. Kim and F. Shahidi. Functorial products for GL$_2 \times$ GL$_3$ and the symmetric cube for GL$_2$. Ann. of Math., 155(2):837–893, 2002.

31. H. Y. Loke. Trilinear forms of gl$_2$. Pacific J. Math., 197(1):119–144, 2001.

32. S. T. Lee and C. B. Zhu. Degenerate principal series and local theta correspondence II. Israel J. Math., 100:29–59, 1997.

33. J. S. Milne. The action of an automorphism of $\mathbb{C}$ on a Shimura variety and its special points. In Arithmetic and Geometry, Volume I, volume 35 of Progress in Mathematics, pages 239–265. Birkhauser, 1983.

34. A. Pal and C. de Vera-Piquero. Pullbacks of Saito-Kurokawa lifts and a central value formula for degree 6 L-series. Doc. Math., 24:1935–2036, 2019.

35. D. Prasad. Trilinear forms for representations of GL$(2)$ and local $\epsilon$-factors. Compos. Math., 75(1):1–46, 1990.

36. D. Prasad. Invariant linear forms for representations of GL$(2)$ over a local field. Amer. J. Math., 114:1317–1363, 1992.

37. I.I. Piatetskii-Shapiro and S. Rallis. Rankin triple L functions. Compos. Math., 64:31–115, 1987.

38. A. Raghuram. Critical values for Rankin-Selberg L-functions for GL$_n \times$ GL$_{n-1}$ and the symmetric cube L-functions for GL$_2$. Forum Math., 28:457–489, 2016.

39. J. Repka. Tensor products of unitary representations of SL$_2(\mathbb{R})$. Bull. Amer. Math. Soc., 82(6):930–932, 1976.

40. A. Raghuram and N. Tanabe. Note on the arithmetic of Hilbert modular forms. J. Ramanujan Math. Soc., 26(3):261–319, 2011.

41. G. Shimura. On the periods of modular forms. Math. Ann., 229:211–221, 1977.

42. G. Shimura. The special values of the zeta functions associated with Hilbert modular forms. Duke Math. J., 45(3):637–679, 1978.

43. G. Shimura. On certain zeta functions attached to two Hilbert modular forms: II. The case of automorphic forms on a quaternion algebra. Ann. of Math., 114:569–607, 1981.

44. G. Shimura. On the critical values of certain Dirichlet series and the periods of automorphic forms. Invent. Math., 94:245–305, 1988.

45. C.L. Siegel. Berechnung von Zetafunktionen an ganzzahligen Stellen. Nachr. Akad. Wiss. Göttingen, pages 87–102, 1969.

46. K. Takase. On the trace formula of the Hecke operators and the special values of the second L-functions attached to the Hilbert modular forms. Manuscripta Math., 55:137–170, 1986.

47. J. Tate. Number theoretic background. In Automorphic forms, representations, and L-functions, volume 33, pages 3–26. Proceedings of Symposia in Pure Mathematics, 1979. Part 2.
50. J.-L. Waldspurger. Quelques propriétés arithmétiques de certaines formes automorphes sur GL(2). *Compos. Math.*, 54:121–171, 1985.
51. H. Xue. Central values of degree six $L$-functions. *J. Number Theory*, 203:350–359, 2019.
52. H. Yoshida. On the zeta functions of Shimura varieties and periods of Hilbert modular forms. *Duke Math. J.*, 75(1):121–191, 1994.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.