Queue Layouts, Tree-Width, and Three-Dimensional Graph Drawing

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Abstract. A three-dimensional (straight-line grid) drawing of a graph represents the vertices by points in $\mathbb{Z}^3$ and the edges by non-crossing line segments. This research is motivated by the following open problem due to Felsner, Liotta, and Wismath [Graph Drawing '01, Lecture Notes in Comput. Sci., 2002]: does every $n$-vertex planar graph have a three-dimensional drawing with $O(n)$ volume? We prove that this question is almost equivalent to an existing one-dimensional graph layout problem. A queue layout consists of a linear order $\sigma$ of the vertices of a graph, and a partition of the edges into queues, such that no two edges in the same queue are nested with respect to $\sigma$. The minimum number of queues in a queue layout of a graph is its queue-number. Let $G$ be an $n$-vertex member of a proper minor-closed family of graphs (such as a planar graph). We prove that $G$ has a $O(1) \times O(1) \times O(n)$ drawing if and only if $G$ has $O(1)$ queue-number. Thus the above question is almost equivalent to an open problem of Heath, Leighton, and Rosenberg [SIAM J. Discrete Math., 1992], who ask whether every planar graph has $O(1)$ queue-number? We also present partial solutions to an open problem of Ganley and Heath [Discrete Appl. Math., 2001], who ask whether graphs of bounded tree-width have bounded queue-number? We prove that graphs with bounded path-width, or both bounded tree-width and bounded maximum degree, have bounded queue-number. As a corollary we obtain three-dimensional drawings with optimal $O(n)$ volume, for series-parallel graphs, and graphs with both bounded tree-width and bounded maximum degree.

1 Introduction

A celebrated result independently due to de Fraysseix, Pach, and Pollack [5 and Schnyder [27] states that every $n$-vertex planar graph has a (two-dimensional) straight-line grid drawing with $O(n^2)$ area. Motivated by applications in information visualisation, VLSI circuit design and software engineering, there is a growing body of research in three-dimensional graph drawing (see [12 for example). One might expect that in three dimensions, planar graphs would admit straight-line grid drawings with $o(n^2)$ volume. However, this question has remained an

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elusive open problem. The main contribution of this paper is to prove that this question of three-dimensional graph drawing is almost equivalent to an existing one-dimensional graph layout problem regarding queue layouts. Furthermore, we establish new relationships between queue-number, tree-width and path-width; and obtain $O(n)$ volume three-dimensional drawings of series-parallel graphs, and graphs with both bounded tree-width and bounded degree.

1.1 Definitions and Notation

Throughout this paper, all graphs $G$ are undirected, simple, connected, and finite with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and maximum degree of $G$ are respectively denoted by $n = |V(G)|$ and $\Delta(G)$. For all disjoint subsets $A, B \subseteq V(G)$, the bipartite subgraph of $G$ with vertex set $A \cup B$ and edge set $\{vw \in E(G) : v \in A, w \in B\}$ is denoted by $G[A, B]$.

A tree-decomposition of a graph $G$ consists of a tree $T$ and a collection $\{T_x : x \in V(T)\}$ of subsets $T_x$ (called bags) of $V(G)$ indexed by the nodes of $T$ such that:

- $\bigcup_{x \in V(T)} T_x = V(G),$
- $\forall$ edges $vw \in E(G)$, $\exists$ node $x \in V(T)$ such that $\{v, w\} \subseteq T_x$, and
- $\forall$ nodes $x, y, z \in V(T)$, if $y$ is on the $xz$-path in $T$, then $T_x \cap T_z \subseteq T_y$.

The width of a tree-decomposition is one less than the maximum size of a bag. A path-decomposition is a tree-decomposition where the tree $T$ is a path. The path-width (respectively, tree-width) of a graph $G$, denoted by $pw(G)$ ($tw(G)$), is the minimum width of a path- (tree-) decomposition of $G$.

1.2 Three-Dimensional Straight-Line Grid Drawing

A three-dimensional straight-line grid drawing of a graph, henceforth called a three-dimensional drawing, represents the vertices by distinct points in $\mathbb{Z}^3$, and represents each edge as a line-segment between its end-vertices, such that edges only intersect at common end-vertices. In contrast to the case in the plane, it is well known that every graph has a three-dimensional drawing. We therefore are interested in optimising certain measures of the aesthetic quality of a drawing. If a three-dimensional drawing is contained in an axis-aligned box with side lengths $X - 1$, $Y - 1$ and $Z - 1$, then we speak of an $X \times Y \times Z$ drawing with volume $X \cdot Y \cdot Z$. We study three-dimensional drawings with small volume.

Cohen, Eades, Lin, and Ruskey [5] proved that every graph has a three-dimensional drawing with $O(n^3)$ volume, and this bound is asymptotically tight for the complete graph $K_n$. Calamoneri and Sterbini [4] proved that every 4-colourable graph has a three-dimensional drawing with $O(n^2)$ volume. Generalising this result, Pach, Thiele, and Tóth [23] proved that every $k$-colourable graph, for fixed $k \geq 2$, has a three-dimensional drawing with $O(n^2)$ volume, and that this bound is asymptotically optimal for the complete bipartite graph with equal
sized bipartitions. The first linear volume bound was established by Felsner, Wismath, and Liotta [14], who proved that every outerplanar graph has a drawing with $O(n)$ volume. Poranen [25] proved that series-parallel digraphs have upward three-dimensional drawings with $O(n^3)$ volume, and that this bound can be improved to $O(n^2)$ and $O(n)$ in certain special cases. di Giacomo, Liotta, and Wismath [7] proved that series-parallel graphs with maximum degree three have three-dimensional drawings with $O(n)$ volume. Dujmović, Morin, and Wood [12] proved that every graph $G$ has a three-dimensional drawing with $O(n \cdot \text{pw}(G)^2)$ volume. This implies $O(n \log^2 n)$ volume drawings for graphs of bounded tree-width, such as series-parallel graphs.

Since a planar graph $G$ is 4-colourable and has $\text{pw}(G) \in O(\sqrt{n})$, by the above results of Calamoneri and Sterbini [4], Pach et al. [23], and Dujmović et al. [12], every planar graph has a three-dimensional drawing with $O(n^2)$ volume. This result also follows from the classical algorithms of de Fraysseix et al. [6] and Schnyder [27] for producing plane grid drawings. This paper is motivated by the following open problem due to Felsner et al. [14].

**Open Problem 1 ([14]).** Does every planar graph have a three-dimensional drawing with $O(n)$ volume? In fact, any $o(n^2)$ bound would be of interest.

In this paper we prove that Open Problem 1 is almost equivalent to an existing open problem in the theory of queue layouts.

### 1.3 Queue Layouts

For a graph $G$, a linear order of $V(G)$ is called a vertex-ordering of $G$. A queue layout of $G$ consists of a vertex-ordering $\sigma$ of $G$, and a partition of $E(G)$ into queues, such that no two edges in the same queue are nested with respect to $\sigma$. That is, there are no edges $vw$ and $xy$ in a single queue with $v <_\sigma x <_\sigma y <_\sigma w$. The minimum number of queues in a queue layout of $G$ is called the queue-number of $G$, and is denoted by $qn(G)$. A similar concept is that of a stack layout (or book embedding), which consists of a vertex-ordering of $G$, and a partition of $E(G)$ into stacks (or pages) such that there are no edges $vw$ and $xy$ in a single stack with $v <_\sigma x <_\sigma w <_\sigma y$. The minimum number of stacks in a stack layout of $G$ is called the stack-number (or page-number) of $G$, and is denoted by $sn(G)$.

Motivated by applications in VLSI layout, fault-tolerant processing, parallel processing, matrix computations, and sorting networks, queue layouts have been extensively studied [19, 20, 24, 26, 29]. Heath and Rosenberg [20] characterised graphs admitting 1-queue layouts as the ‘arched leveled planar’ graphs, and proved that it is NP-complete to recognise such graphs. This result is in contrast to the situation for stack layouts — the graphs admitting 1-stack layouts are precisely the outerplanar graphs, which can be recognised in polynomial time. On the other hand, it is NP-hard to minimise the number of stacks in a stack layout which respects a given vertex-ordering [17]. However the analogous problem for queue layouts can be solved as follows. As illustrated in Fig. 1, a $k$-rainbow
Queue Layouts, Tree-Width, and Three-Dimensional Graph Drawing

Fig. 1. A rainbow of five edges in a vertex-ordering.

in a vertex-ordering $\sigma$ consists of a matching $\{v_i w_i : 1 \leq i \leq k\}$ such that $v_1 \prec_\sigma v_2 \prec_\sigma \cdots \prec_\sigma v_k \prec_\sigma w_k \prec_\sigma w_{k-1} \prec_\sigma \cdots \prec_\sigma w_1$.

A vertex-ordering containing a $k$-rainbow needs at least $k$ queues. A straightforward application of Dilworth’s Theorem \cite{9} proves the converse. That is, a fixed vertex-ordering admits a $k$-queue layout where $k$ is the size of the largest rainbow. (Heath and Rosenberg \cite{20} describe an $O(m \log \log n)$ time algorithm to compute the queue assignment.) Thus determining $q_n(G)$ can be viewed as the following vertex layout problem.

**Lemma 1 \cite{20}**. The queue-number $q_n(G)$ of a graph $G$ is the minimum, taken over all vertex-orderings $\sigma$ of $G$, of the maximum size of a rainbow in $\sigma$.

The relationship between tree-width and stack and queue layouts has previously been studied in \cite{16, 26}. Rengarajan and Veni Madhavan \cite{26} prove that a graph of tree-width at most two (that is, a graph with series-parallel biconnected components \cite{2}) has a 2-stack layout and a 3-queue layout. In the special case of an outerplanar graph a 2-queue layout is constructed. More generally, Ganley and Heath \cite{16} prove that the stack-number $s_n(G) \leq \tw(G) + 1$, and ask whether a similar relationship holds for the queue-number.

**Open Problem 2 \cite{16}**. Does every graph of bounded tree-width have bounded queue-number?

2 Our Results

This paper contributes the following two theorems. The first, proved in Section 3, provides a partial answer to Open Problem 2.

**Theorem 1**. The following classes of graphs have bounded queue-number:

1. graphs of bounded path-width, and
2. graphs of bounded tree-width and bounded maximum degree.

In particular, $q_n(G) \leq pw(G)$ and $q_n(G) \leq 36 \tw(G) \Delta(G)$ for every graph $G$.

A similar upper bound to (1) is obtained by Heath and Rosenberg \cite{20}, who show that every graph $G$ has $q_n(G) \leq \lfloor \frac{1}{2} \bw(G) \rfloor$, where $\bw(G)$ is the bandwidth of $G$. In many cases this result is weaker than (1) since $pw(G) \leq \bw(G)$ (see
Note that since $pw(G) \in O(tw(G) \cdot \log n)$ \cite{2}, the queue-number $qn(G) \in O(tw(G) \cdot \log n)$.

Theorem 2 below relates the volume of a three-dimensional drawing of a graph to its queue-number, and is proved in Section 4. While our motivation is for three-dimensional drawings of planar graphs, the theorem applies to any proper minor-closed family of graphs; that is, a graph family which is not the class of all graphs, and is closed under edge-contraction, edge-deletion, and deleting isolated vertices.

**Theorem 2.** Let $G$ be a proper minor-closed family of graphs, and let $F(n)$ be a set of functions closed under taking polynomials (for example, $O(1)$ or $O(polylog n)$). For every graph $G \in G$, $G$ has a $F(n) \times F(n) \times O(n)$ drawing if and only if $G$ has queue number $qn(G) \in F(n)$.

Graphs with constant queue-number include de Bruijn graphs, FFT and Beneš network graphs \cite{20}. By the above-mentioned result of Rengarajan and Veni Madhavan \cite{26}, and since graphs with tree-width at most some constant form a proper minor-closed family, Theorems 1 and 2 together imply the following. Part (2) is proved without using queue layouts in \cite{12}.

**Corollary 1.** The following graphs have three-dimensional drawings with $O(n)$ volume:

1. de Bruijn graphs, FFT and Beneš network graphs,
2. graphs of bounded path-width \cite{12},
3. graphs of tree-width at most two (series-parallel graphs), and
4. graphs of bounded tree-width and bounded maximum degree.

Corollary 1 improves and/or generalises the above-mentioned results for three-dimensional drawings of outerplanar graphs, series-parallel graphs, and graphs of bounded tree-width in \cite{6, 7, 12, 14, 25}. Note that the algorithm by Felser et al. \cite{14} closely parallels the construction of 2-queue layouts of outerplanar graphs due to Rengarajan and Veni Madhavan \cite{26}, both of which are based on breadth-first search, as is one of our proofs to follows.

## 3 Queue Layouts and Tree-Width

In this section we prove Theorem 1. Consider a vertex-ordering $\sigma$ of a graph $G$.

The vertex cut in $\sigma$ at a vertex $v \in V(G)$ is defined to be $\{x \in V(G) : \exists xy \in E(G), x \leq_\sigma v <_\sigma y\}$. The vertex separation number of $G$ is the minimum, taken over all vertex-orderings $\sigma$ of $G$, of a maximum vertex cut in $\sigma$. A $k$-rainbow in $\sigma$ implies $\sigma$ has a vertex cut of size $k$. Thus the queue-number of a graph is at most its vertex separation number by Lemma 1. The next result immediately follows, since the vertex separation number of a graph equals its path-width (see \cite{5}).

**Lemma 2.** Graphs of bounded path-width have bounded queue-number. In particular, $qn(G) \leq pw(G)$ for every graph $G$. 
To establish our next result we employ a structure called a tree-partition.

Let \( G \) be a graph, let \( T \) be a tree, and let \( \{ T_x : x \in V(T) \} \) be a partition of \( V(G) \) into sets (called bags) indexed by the nodes of \( T \). We denote the bag containing a vertex \( v \in V(G) \) by \( T_{\alpha(v)} \). The pair \( (T, \{ T_x \}) \) is a tree-partition of \( G \) if for every edge \( vw \in E(G) \), either \( \alpha(v) = \alpha(w) \) or \( \alpha(v) \alpha(w) \in E(T) \). We call \( vw \) an intra-bag edge if \( \alpha(v) = \alpha(w) \) and an inter-bag edge otherwise. The width of the tree-partition is the maximum size of a bag \( T_x \). The tree-partition-width of a graph \( G \), denoted by \( \text{tpw}(G) \), is the minimum width of a tree-partition of \( G \). Note that tree-partition-width has also been called strong tree-width \([3, 28]\).

**Lemma 3.** Graphs of bounded tree-partition-width, which includes graphs of bounded tree-width and bounded maximum degree, have bounded queue-number. In particular, \( \text{qn}(G) \leq \frac{3}{2} \text{tpw}(G) \leq 36 \text{tw}(G) \Delta(G) \) for every graph \( G \).

**Proof.** Let \( (T, \{ T_x \}) \) be a tree-partition of \( G \) with width \( \text{tpw}(G) \). Let \( \pi \) be a vertex-ordering of \( T \) determined by a lexicographical breadth-first-search of \( T \) starting from an arbitrary root node. Then no two edges of \( T \) are nested in \( \pi \). (This is why trees have queue-number one.) Also observe that each node \( x \in V(T) \) has at most one incident edge \( xy \) with \( y <_\pi x \).

Let \( \sigma \) be a vertex-ordering of \( G \) such that \( v <_\sigma w \) implies \( \alpha(v) \leq_\pi \alpha(w) \). Suppose \( e_1 \) and \( e_2 \) of \( G \) are nested in \( \sigma \). If \( e_1 \) and \( e_2 \) are both intra-bag edges then their end-vertices are all in a common bag. Thus there are at most \( \frac{3}{2} \text{tpw}(G) \) intra-bag edges in a rainbow of \( \sigma \). If \( e_1 \) and \( e_2 \) are both inter-bag edges then the left end-vertex of \( e_1 \) and the left end-vertex of \( e_2 \) are in a common bag. Thus there are at most \( \text{tpw}(G) \) inter-bag edges in a rainbow of \( \sigma \). Therefore a rainbow in \( \sigma \) can have at most \( \frac{3}{2} \text{tpw}(G) \) edges.

The result follows from Lemma 1 and since Ding and Oporowski \([10]\) proved that \( \text{tpw}(G) \leq 24 \text{tw}(G) \Delta(G) \) for every graph \( G \).

\( \square \)

Lemmata 2 and 3 establish Theorem 1.

## 4 Queue Layouts and Three-Dimensional Drawings

In this section we prove Theorem 2. Our proof depends on the following structure introduced by Dujmović et al. \([12]\). An ordered \( k \)-layering of a graph \( G \) consists of a partition \( V_1, V_2, \ldots, V_k \) of \( V(G) \) into layers, and a total order \( <_i \) of each \( V_i \), such that for every edge \( vw \), if \( v <_i w \) then there is no vertex \( x \) with \( v <_i x <_i w \). The span of an edge \( vw \) is \( |i - j| \) where \( v \in V_i \) and \( w \in V_j \). An intra-layer edge is an edge with zero span. An \( X \)-crossing consists of two edges \( vw \) and \( xy \) such that for distinct layers \( i \) and \( j \), \( v <_i x \) and \( y <_j w \). Dujmović et al. \([12]\) proved the following (see Fig. 2).

**Lemma 4** \([12]\). Let \( F(n) \) be a set of functions closed under taking polynomials. Then a graph \( G \) has a \( F(n) \times F(n) \times O(n) \) drawing if and only if \( G \) has an ordered \( k \)-layering with no \( X \)-crossing, for some \( k \in F(n) \). Furthermore, if \( G \) has an ordered layering with no \( X \)-crossing and maximum edge span \( s \) then \( G \) has a \( O(s) \times O(s) \times O(n) \) drawing.
Dujmović et al. [12] proved that a graph $G$ has an ordered $(\text{pw}(G)+1)$-layering with no X-crossing. That $G$ has a three-dimensional drawing with $O(n \cdot \text{pw}(G)^2)$ volume follows from Lemma 4. A result of Felsner et al. [14] also fits into this framework. To construct three-dimensional drawings of outerplanar graphs with $O(n)$ volume, they proved that such a graph has an ordered layering with no X-crossing and maximum edge span at most one. Note that the plane grid graph, which has $\Theta(\sqrt{n})$ path-width and tree-width, has an obvious ordered layering with no X-crossing and maximum edge span one. The ‘nested triangles’ graph which provides an $\Omega(n^2)$ lower bound on the area of plane grid drawings [6], has an ordered 3-layering with no X-crossing. Thus both of these important examples of planar graphs have three-dimensional drawings with $O(n)$ volume.

Lemma 4 implies that Theorem 2 can be proved if we show that $qn(G) \in F(n)$ if and only if $G$ has an ordered $k$-layering with no X-crossing, for some $k \in F(n)$. The next lemma highlights the inherent relationship between ordered layerings and queue layouts. Its proof follows immediately from the definitions (see Fig. 3).

Lemma 5. A bipartite graph $G = (A, B; E)$ has an ordered 2-layering with no X-crossing and no intra-layer edges if and only if $G$ has a 1-queue layout such that in the corresponding vertex-ordering, the vertices in $A$ appear before the vertices in $B$.

We now show that a queue layout can be obtained from an ordered layering with no X-crossing. This result can be viewed as a generalisation of the
construction of a 2-queue layout of an outerplanar graph by Rengarajan and Veni Madhavan [26] (with \( s = 1 \)).

**Lemma 6.** Let \( G \) be a graph with an ordered \( k \)-layering \( \{(V_i, <_i) : 1 \leq i \leq k\} \) with no X-crossing and maximum edge span \( s \). Then \( qn(G) \leq s + 1 \), and if there are no intra-layer edges then \( qn(G) \leq s \).

**Proof.** Let \( \sigma = V_1, \ldots, V_k \) with each \( V_i \) ordered by \( <_i \). Let \( R \) be the largest rainbow in \( \sigma \). By Lemma 5 between each pair of layers there is at most one edge in \( R \). A simple inductive argument shows that there is at most \( s \) non-intra-layer edges in \( R \); see Fig. 4. No two intra-layer edges are nested in \( \sigma \). Thus \( R \) has at most \( s + 1 \) edges. By Lemma 1, \( qn(G) \leq s + 1 \). If there are no intra-layer edges then \( R \) has at most \( s \) edges and \( qn(G) \leq s \). \( \square \)

We now prove a converse result to Lemma 6. Consider an ordered \( k \)-layering with no X-crossing and no intra-layer edges. It is easily seen that the subgraph induced by two layers is a forest of caterpillars. A slightly smaller family of graphs is a forest of stars. A proper vertex-colouring of a graph is called a star colouring if each bichromatic subgraph is a forest of stars; that is, every path on four vertices receives at least three distinct colours. The minimum number of colours in a star colouring of a graph \( G \) is called the star chromatic number of \( G \), and is denoted by \( \chi_{st}(G) \). Neˇsetˇril and Ossona de Mendez [22] proved that every planar graph \( G \) has \( \chi_{st}(G) \leq 30 \). Many other graph families have bounded star chromatic number, including graphs with bounded maximum degree [1], and graphs with bounded tree-width [15]. In particular, Fertin et al. [15] proved that \( \chi_{st}(G) \leq \frac{1}{2}tw(G)(tw(G) + 3) + 1 \). More generally, Neˇsetˇril and Ossona de Mendez [22] proved that \( G \) has bounded star chromatic number if and only if \( G \) is a member of a proper minor-closed family of graphs. In this case, \( \chi_{st}(G) \) is at most a quadratic function of the maximum chromatic number of a minor of \( G \).

**Lemma 7.** Let \( G \) be a graph with star chromatic number \( \chi_{st}(G) \leq c \), and queue-number \( qn(G) \leq q \). Then \( G \) has an ordered \( t \)-layering with no X-crossing where

\[
t \leq c(2(c - 1)q + 1)^{c - 1}.
\]

**Proof.** Let \( V_1, \ldots, V_c \) be the colour classes of a star colouring of \( G \). Pemmaraju [24] proved that a \( q \)-queue graph layout can be ‘separated’ by a vertex \( c \)-colouring to produce a \( 2(c - 1)q \)-queue layout with the vertices in each colour class consecutive in the vertex-ordering. (The proof is a straightforward application of
Lemma 3 implies that a planar graph has a three-dimensional drawing with vertex-ordering $\sigma = V_1, \ldots, V_c$, where $q' = 2(c-1)q$.

For every vertex $v \in V_i$, $1 \leq i \leq c$, and $j \in \{1, \ldots, c\} \setminus \{i\}$, let $d_j(v)$ be the degree of $v$ in $G[V_i, V_j]$. Define the $j$th label of $v$, denoted by $\phi_j(v)$, as follows. If $d_j(v) \geq 2$ then let $\phi_j(v) = 'r' (v$ is the root of a star in $G[V_i, V_j])$. If $d_j(v) = 1$ then let $\phi_j(v)$ be the degree containing the edge in $G[V_i, V_j]$ incident to $v$. If $d_j(v) = 0$ then let $\phi_j(v)$ be some arbitrary queue. Let the label of $v \in V_i$ be $\phi(v) = (\phi_1(v), \ldots, \phi_{i-1}(v), \phi_{i+1}(v), \ldots, \phi_c(v))$. Let $S_i$ be the set of possible labels for a vertex in $V_i$. Then $|S_i| = (q' + 1)^{c-1}$. 

Now group the vertices with the same colour and the same label. Let $V_{i,L} = \{v \in V_i : \phi(v) = L\}$ for all labels $L \in S_i$ and $1 \leq i \leq c$, and consider each $V_{i,L}$ to be ordered by $\sigma$. Thus $\{V_{i,L} : 1 \leq i \leq c, L \in S_i\}$ is an ordered layering of $G$. We denote the $j$th label of $L \in S_i$ by $L[j]$.

Consider a subgraph $G[V_{i,P}, V_{j,Q}]$ for some $1 \leq i < j \leq c$ and labels $P \in S_i$ and $Q \in S_j$. We claim that all edges in $G[V_{i,P}, V_{j,Q}]$ are in a single queue. If $P[j] = 'r$ and $Q[i] = 'r$ then $G[V_{i,P}, V_{j,Q}]$ has no edges. If $P[j] = 'r$ and $Q[i] = q_a$ for some queue $q_a$, then all edges in $G[V_{i,P}, V_{j,Q}]$ are in $q_a$. Similarly, if $Q[i] = 'r$ and $P[j] = q_a$ for some queue $q_a$, then all edges in $G[V_{i,P}, V_{j,Q}]$ are in $q_a$. Finally, consider the case in which $P[j] = q_a$ and $Q[i] = q_b$ for some queues $q_a$ and $q_b$. If $a \neq b$ then there are no edges in $G[V_{i,P}, V_{j,Q}]$, and if $a = b$ then all edges in $G[V_{i,P}, V_{j,Q}]$ are in queue $q_a (= q_b)$. In each case, all edges in $G[V_{i,P}, V_{j,Q}]$ are in a single queue. By Lemma 5, $V_{i,P}$ and $V_{j,Q}$ form an ordered 2-layering of $G[V_{i,P}, V_{j,Q}]$ with no X-crossing. In general, $\{V_{i,L} : 1 \leq i \leq c, L \in S_i\}$ is an ordered layering of $G$ with no X-crossing. The number of layers is $c(q' + 1)^{c-1} = c(2(c-1)q + 1)^{c-1}$. $\square$

Lemmata 4, 6 and 7 together with the result of Nešetril and Ossona de Mendez [22] establish Theorem 2.

5 Conclusion

Theorem 2 implies that a planar graph has a three-dimensional drawing with $O(n)$ volume if it has $O(1)$ queue-number. Thus an affirmative answer to the following open problem due to Heath et al. [19] would solve Open Problem 3. In fact, the two problems are almost equivalent. It is possible, however, that a planar graph has non-constant queue-number, yet has say a $O(n^{1/3}) \times O(n^{1/3}) \times O(n^{1/3})$ drawing.

Open Problem 3 ([19, 20]). Does every planar graph have $O(1)$ queue-number?

In 1992, Heath and Rosenberg [20] and Heath et al. [19] conjectured that every planar graph does have $O(1)$ queue-number. More recently, Pemmaraju [24] provided ‘evidence’ that the planar graph obtained by repeated stellation of $K_3$ (that is, by adding a degree three vertex to every face) has non-constant
Queue Layouts, Tree-Width, and Three-Dimensional Graph Drawing

This graph does have \(O(\log n)\) queue-number \[24\]. Pemmaraju \[24\] and Heath [private communication, 2002] conjecture that every planar graph has \(O(\log n)\) queue-number. By Theorem \[2\], this would imply that every planar graph has a three-dimensional drawing with \(O(n \text{polylog } n)\) volume. Note that if the stellated \(K_3\) graph, which has tree-width three, has non-constant queue-number then Open Problem \[2\] would also have a negative answer \[10\].

The best known upper bound on the queue-number of a planar graph is \(O(\sqrt{n})\), which follows from Lemma \[2\] and the fact that the path-width of a planar graph is \(O(\sqrt{n})\) (see \[2\]). This result can also be proved using a variant of the randomised algorithm of Malitz \[21\] (see \[19\]), or the derandomised algorithm of Shahrokhi and Shi \[29\].

As a final word, we estimate the constants in the \(O(n)\) volume bound of Corollary \[1\]. Take a graph \(G\) with bounded tree-width \(\text{tw}(G) \leq k\) and bounded maximum degree \(\Delta(G) \leq d\). Then \(\chi_{st}(G) \leq \frac{1}{2}k^2 + o(k^2)\) \[15\] and \(\text{qn}(G) \leq 36kd\) by Lemma \[3\]. By Lemma \[4\] \(G\) has an ordered layering with no X-crossing and approximately \(k^2(36k^3d)k^{k/2}\) layers. By Lemma \[4\] \(G\) has a three-dimensional drawing with approximately \(O(k^4(36k^3d)k^2 \cdot n)\) volume. As another example, a series-parallel graph \(G\) has \(\text{tw}(G) \leq 2\) \[2\], \(\text{qn}(G) \leq 3\) \[26\], and \(\chi_{st}(G) \leq 6\) \[13\]. By Lemma \[7\] \(G\) has an ordered layering with no X-crossing and at most 6 \(\cdot 31^5\) layers. By Lemma \[8\] the constant in the \(O(n)\) volume bound of Corollary \[8\] for series-parallel graphs is at least \(36 \cdot 31^{10} \approx 2.9 \times 10^{16}\). It is an interesting open problem to construct linear volume three-dimensional drawings with a smaller constant in the \(O(n)\) volume bound.

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Note added in proof: Dujmović and Wood \[13\] recently solved Open Problem \[2\]. That is, graphs of bounded tree-width have bounded queue-number, and hence have three-dimensional drawings with linear volume.

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