Jointly Modeling and Clustering Tensors in High Dimensions

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Abstract

We consider the problem of jointly modeling and clustering populations of tensors by introducing a high-dimensional tensor mixture model with heterogeneous covariances. To effectively tackle the high dimensionality of tensor objects, we employ plausible dimension reduction assumptions that exploit the intrinsic structures of tensors such as low-rankness in the mean and separability in the covariance. In estimation, we develop an efficient high-dimensional expectation-conditional-maximization (\texttt{HECM}) algorithm that breaks the intractable optimization in the M-step into a sequence of much simpler conditional optimization problems, each of which is convex, admits regularization and has closed-form updating formulas. Our theoretical analysis is challenged by both the non-convexity in the EM-type estimation and having access to only the solutions of conditional maximizations in the M-step, leading to the notion of dual non-convexity. We demonstrate that the proposed \texttt{HECM} algorithm, with an appropriate initialization, converges geometrically to a neighborhood that is within statistical precision of the true parameter. The efficacy of our proposed method is demonstrated through comparative numerical experiments and an application to a medical study, where our proposal achieves an improved clustering accuracy over existing benchmarking methods.

Keywords: expectation conditional maximization; computational and statistical errors; tensor clustering; tensor decomposition; unsupervised learning.
1 Introduction

In modern data science, tensor data, where the data take the form of a multidimensional array, are becoming ubiquitous in a wide variety of scientific and business applications. For example, in recommender systems, the data are collected as a three-way (user, item, context) tensor (Bi et al., 2018), where the context can be item features such as time, location and publisher. Due to the rapidly increasing interests in analyzing tensor data, the literature on tensor data analysis is fast growing, including topics such as tensor decomposition (Anandkumar et al., 2014; Sun et al., 2017; Zhang and Xia, 2018; Hao et al., 2020; Xia and Yuan, 2021), tensor completion (Zhang, 2019; Cai et al., 2020; Xia et al., 2021; Cai et al., 2021), and tensor regression (Li and Zhang, 2017; Zhang et al., 2018; Raskutti et al., 2019; Zhou et al., 2020). We refer to a recent survey by Bi et al. (2020) for a comprehensive review on tensor data analysis.

In this paper, we consider the problem of jointly modeling and clustering populations of tensors. When tensors are collected from heterogeneous populations, an important task to cluster the tensor samples into homogeneous groups and characterize distributions of the different populations. This is an important task that finds applications in face clustering (Cao et al., 2014), video summarization (Rabbouch et al., 2017), brain imaging segmentation (Mirzaei and Adeli, 2018), user clickstream clustering (Wang et al., 2016) and so on. An intrinsic challenge in modeling and clustering tensors is the high dimensionality of tensor objects. For example, in our real data analysis in Section 6, there are $n = 57$ tensor objects to be modeled and clustered, each of dimension $116 \times 116 \times 30$ yielding $403,680$ entries. To perform clustering, one may first vectorize the tensor objects and then apply clustering techniques developed for high-dimensional vectors (Wang et al., 2015; Hao et al., 2017; Cai et al., 2019). However, as the structures in tensors are largely ignored after vectorization, these vector based approaches can result in a loss of information, leading to reduced efficiency and accuracy. Another approach is to consider tensor subspace clustering methods, which find latent cluster structures embedded in one or more modes of a single tensor (Sun and Li, 2019; Chi et al., 2020; Luo and Zhang, 2020). When $n$ tensor samples are available, it seems sensible to stack them into one higher-order tensor, where the last mode is of dimension $n$, and then apply a tensor subspace clustering method to recover cluster labels along the last mode.
mode. However, this approach has one fundamental limit as clustering along one mode of a single tensor inevitably runs into the curse of dimensionality, in that the clustering accuracy is expected to deteriorate with $n$, the dimension of the last mode. As shown in Sun and Li (2019); Chi et al. (2020), to consistently estimate labels along one mode of a tensor, the dimension of this mode must be small compared to others. This condition seems unnatural under our setting as the clustering accuracy is expected to improve with the sample size $n$.

Recently, some progresses have been made for clustering a collection of tensors. Specifically, Tait and McNicholas (2019) considered a mixture model estimated using a standard EM algorithm. Without any dimension reduction assumption on the tensor mean, this method could not handle cases where the sample size is smaller than the total number of tensor entries. Mai et al. (2021) proposed DEEM, which clusters tensors using a carefully designed discriminant analysis and the discriminant tensors are assumed to be sparse. The main focus of DEEM was to develop a clustering rule while subpopulation distributions were not directly estimated. Characterizing subpopulation distributions can be useful, as one might wish to examine the differences in means and covariances across subpopulations. Moreover, DEEM assumed homogeneous covariances across clusters and may not perform well when covariances differ among clusters. As shown in the numerical experiments in Sections 5-6, DEEM can be numerically unstable and sensitive to potential model misspecifications.

In this paper, we introduce a flexible high-dimensional tensor mixture model with heterogeneous covariances to jointly model and cluster a collection of tensors. To facilitate estimability and interpretability, we employ effective dimension reduction assumptions that take advantages of the intrinsic structures of tensors and improve model interpretability. Specifically, we assume the tensor means to be low-rank and internally sparse (defined in Section 2.3), and the tensor covariances to be separable and conditionally sparse. These assumptions are plausible in a wide range of applications and are commonly employed in the tensor analysis literature (Anandkumar et al., 2014; Sun et al., 2017; Zhang and Xia, 2018; Pan et al., 2019; Mai et al., 2021; Zhou et al., 2020). The mixture components in our proposal are allowed to have heterogeneous covariances, which greatly relaxes the homogeneous and/or isotropic covariance assumption commonly employed in the mixture model literature (Balakrishnan et al., 2017; Cai et al., 2019; Mai et al., 2021).
In estimation, we employ a high-dimensional expectation-maximization (EM) type algorithm. One major challenge is that the M-step in the standard EM algorithm (Dempster et al., 1977) requires an optimization with respect to the low-rank tensor means and separable covariances from each mixture component. This is an intractable non-convex optimization problem. To tackle this challenge, we propose a high-dimensional expectation-conditional-maximization (HECM) algorithm that breaks the challenging optimization problem in the M-step into several simpler alternating conditional optimization problems, each of which is convex, has closed-form updating formulas and admits regularization. An attractive property of the proposed H ECM algorithm is that sparsity structures can be easily incorporated into parameter estimation by adding regularization to the smaller conditional optimizations.

While convergence to an arbitrary fixed point has been studied for ECM-type algorithms (Meng, 1994), to our knowledge, local convergence has yet to be investigated, even in the low-dimensional regime. In our theoretical analysis, we show that the H ECM algorithm converges geometrically to a neighborhood that is within statistical precision of the unknown true parameter given a suitable initial. This is a useful statistical guarantee that sheds light on when and how quickly the H ECM iterates converge to the true parameter. Our theoretical analysis is highly nontrivial, as the conditional updating scheme in the H ECM requires a delicate treatment in order to establish the contraction of the iterations. In particular, our analysis builds on a collection of conditional Q functions in the form of $Q_n(\vartheta, \tilde{\Theta}_{-\vartheta} | \Theta^{(t)})$, where $\Theta^{(t)}$ is the parameter update from the $t$-th step, $\vartheta$ is a subset of $\Theta$ to be updated in the $(t + 1)$-th step and $\tilde{\Theta}_{-\vartheta}$ collects all other parameters being conditioned on, with some taking values from the $t$-th step (i.e., those yet to be updated in $\tilde{\Theta}_{-\vartheta}$) and some from the $(t + 1)$-th step (i.e., those already updated in $\tilde{\Theta}_{-\vartheta}$). As the H ECM does not have access to $\arg \max_{\Theta} Q_n(\Theta | \Theta^{(t)})$ in the M-step, existing arguments and techniques in the population and sample-based analysis of the standard EM algorithms (Yi and Caramanis, 2015; Wang et al., 2015; Balakrishnan et al., 2017) are no longer directly applicable. Our analysis is accomplished by identifying new statistical and computational properties of the conditional Q functions and employing new proof strategies in establishing one-step contraction; see Section 4.2. Finally, besides the challenge of conditional updates in the M-step, our analysis is further complicated by considering sparsity structures in the estimation.
To recap, our work contributes to both methodology and theory. As to methodology, we propose a high-dimensional tensor mixture model with heterogeneous covariances. To take advantage of the intrinsic structures of tensors and facilitate estimability, we employ several effective dimension reduction assumptions including low-rankness, sparsity and separability. To tackle the intractable optimization in the M-step, we propose an efficient HECDM algorithm and show that, the HECDM algorithm converges geometrically to the unknown true parameter given a suitable initial. To our knowledge, this is the first statistical guarantee on the local convergence of EM-type algorithms with conditional maximizations. Thus, our theoretical results may further advance the growing literature on statistical guarantees for EM-type algorithms (Wang et al., 2015; Ho and Nguyen, 2016; Balakrishnan et al., 2017; Cai et al., 2019; Dwivedi et al., 2020; Kwon et al., 2021).

The rest of this paper is organized as follows. Section 2 introduces the high-dimensional tensor mixture model with heterogeneous covariances. Section 3 discusses details of the HECDM algorithm. Section 4 investigates the statistical properties of our proposed method. Section 5 presents numerical experiments and Section 6 illustrates with a real data analysis.

2 Model and Problem

2.1 Notation and tensor algebra

A tensor is a multidimensional array and the order of a tensor is the number of dimensions, also referred to as modes. We denote vectors using lower-case bold letters (e.g., \( \mathbf{x} \)), matrices using upper-case bold letters (e.g., \( \mathbf{X} \)), high-order tensors using upper-case bold script letters (e.g., \( \mathcal{X} \)), and let \( [n] = \{1, 2, \ldots, n\} \). Given a vector \( \mathbf{x} \in \mathbb{R}^d \), we let \( \|\mathbf{x}\|_0, \|\mathbf{x}\|_1 \) and \( \|\mathbf{x}\|_2 \) denote the vector \( \ell_0, \ell_1 \) and \( \ell_2 \) norms, respectively. We use \( \mathbf{x}_j \) or \( \mathbf{x}(j) \) to denote \( j \)-th entry of \( \mathbf{x} \). Given a matrix \( \mathbf{X} \in \mathbb{R}^{d_1 \times d_2} \), we let \( \|\mathbf{X}\|_{0,\text{off}} = \sum_{i \neq j} 1(\mathbf{X}_{ij} \neq 0) \), \( \|\mathbf{X}\|_{1,\text{off}} = \sum_{i \neq j} |\mathbf{X}_{ij}| \), and \( \|\mathbf{X}\| \) denote the off-diagonal \( \ell_0, \ell_1 \) norms and spectral norm, respectively. The vectorization of \( \mathbf{X} \) is defined as \( \text{vec}(\mathbf{X}) = (\mathbf{X}_{11}, \ldots, \mathbf{X}_{d_11}, \ldots, \mathbf{X}_{1d_2}, \ldots, \mathbf{X}_{d_1d_2})^\top \). We use \( \mathbf{x}_{i,j} \) or \( \mathbf{x}(i,j) \) to denote \( (i,j) \)-th entry of \( \mathbf{X} \), and \( \sigma_{\min}(\cdot) \) and \( \sigma_{\max}(\cdot) \) denote the smallest and largest eigenvalues of a matrix, respectively. Given a tensor \( \mathcal{X} \in \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_M} \), its Frobenius norm is defined as \( \|\mathcal{X}\|_F = \left( \sum_{i_1, \ldots, i_M} (\mathcal{X}_{i_1 \ldots i_M}^2) \right)^{1/2} \), and its max norm is defined
as $\|\mathbf{X}\|_{\text{max}} = \max_{i_1, \ldots, i_M} |\mathbf{X}_{i_1, \ldots, i_M}|$. For two positive sequences $a_n$ and $b_n$, write $a_n \preceq b_n$ or $a_n = O(b_n)$ if there exist $c > 0$ and $N > 0$ such that $a_n < cb_n$ for all $n > N$, and $a_n = o(b_n)$ if $a_n/b_n \to 0$ as $n \to \infty$; moreover, write $a_n \asymp b_n$ if $a_n \preceq b_n$ and $b_n \preceq a_n$.

Given a third-order tensor $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$, its mode-1, 2 and 3 fibers are denoted as $\mathbf{X}_{ijk}$, $\mathbf{X}_{i,k}$ and $\mathbf{X}_{ij}$, respectively. Given a tensor $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_M}$, the mode-$m$ unfolding, denoted as $\mathbf{X}_{(m)}$, arranges the mode-$m$ fibers to be the columns of the resulting matrix. For example, the mold-1 unfolding of a third-order tensor $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ can be written as $\mathbf{X}_{(1)} = [X_{11}, \ldots, X_{d_21}, \ldots, X_{d_2d_3}] \in \mathbb{R}^{d_1 \times (d_2 d_3)}$. The vectorization of tensor $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_M}$, denoted as $\text{vec}(\mathbf{X})$, is obtained by stacking the mode-1 fibers of $\mathbf{X}$. For example, given an order-three tensor $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$, we have $\text{vec}(\mathbf{X}) = (X_{11}^\top, \ldots, X_{d_21}^\top, \ldots, X_{d_2d_3}^\top)^\top$. See Figure 1 for an example of mode-1 fibers, mold-1 unfolding and vectorization. For $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_M}$, define their inner product as $\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i_1, \ldots, i_M} \mathbf{X}_{i_1, \ldots, i_M} \mathbf{Y}_{i_1, \ldots, i_M}$. For a tensor $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_M}$ and a matrix $\mathbf{A} \in \mathbb{R}^{J \times d_m}$, the $m$-mode tensor matrix product is denoted as $\times_m$ and element-wise we have $(\mathbf{X} \times_m \mathbf{A})_{i_1, \ldots, i_m-1, \beta, i_m+1, \ldots, i_M} = \sum_{i_m=1}^{d_m} \mathbf{X}_{i_1, \ldots, i_M} \mathbf{A}_{i_m}$. It is easy to see that if $\mathbf{U} = \mathbf{X} \times_m \mathbf{A}$, then $\mathbf{U}_{(m)} = \mathbf{A} \mathbf{X}_{(m)}$. Given a tensor $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_M}$ and a list of matrices $\mathbf{A} = \{\mathbf{A}_1, \ldots, \mathbf{A}_M\}$, where $\mathbf{A}_m \in \mathbb{R}^{d_m \times d_m}$, $m \in [M]$, their product $\mathbf{X} \times \mathbf{A}$ is defined as $\mathbf{X} \times \mathbf{A} = \mathbf{X} \times_1 \mathbf{A}_1 \times_2 \cdots \times_M \mathbf{A}_M$ (Kolda and Bader, 2009).

### 2.2 Separable covariance and tensor normal distribution

We start our introduction with third-order tensors (i.e., $M = 3$). A random tensor $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ is said to have a separable covariance structure if $\text{Cov}(\text{vec}(\mathbf{X})) = \Sigma_3 \otimes \Sigma_2 \otimes \Sigma_1$, where $\otimes$ denotes the Kronecker product. In this parameterization, $\Sigma_1 \in \mathbb{R}^{d_1 \times d_1}$, $\Sigma_2 \in \mathbb{R}^{d_2 \times d_2}$, $\Sigma_3 \in \mathbb{R}^{d_3 \times d_3}$, and $\mathbf{X}$ follows the tensor normal distribution with parameter $\mathbf{X} \sim \text{TN}(\Sigma_3 \otimes \Sigma_2 \otimes \Sigma_1, n_1, n_2, n_3)$, where $n_1, n_2, n_3$ are the sizes of the three modes of $\mathbf{X}$. When $\Sigma_1, \Sigma_2, \Sigma_3$ are diagonal matrices, $\mathbf{X}$ is said to have a separable structure.
To ease presentation, we focus on third-order tensors.

### 2.3 High-dimensional heterogeneous tensor mixture model

Let $\sum \sigma_k$ be a sum of tensor normal distributions with heterogeneous covariances such that $\sum \sigma_k$ generalizes to higher-order tensors is straightforward. Assume that there are $K$ mixtures of tensor normal distributions with independent mean $\mu_k \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ and covariance $\Sigma_k = \{\Sigma_1, \Sigma_2, \Sigma_3\}$, if $\text{vec}(\mathbf{X}) \sim N(\text{vec}(\mathbf{U}), \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_3)$.

We denote this tensor normal distribution as $\mathbf{X} \sim N_T(\mathbf{U}, \Sigma)$. Let $\mathbf{Z} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ be a random tensor with independent standard normal entries. If $\mathbf{X} \sim N_T(\mathbf{U}, \Sigma)$, we have $\mathbf{X} = \mathbf{U} + \mathbf{Z} \times_1 \Sigma_1^{1/2} \times_2 \Sigma_2^{1/2} \times_3 \Sigma_3^{1/2}$.

More generally, we say $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_M}$, $M \geq 2$, follows a tensor normal distribution with mean $\mathbf{U} \in \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_M}$ and covariance $\Sigma = \{\Sigma_1, \ldots, \Sigma_M\}$, if $\text{vec}(\mathbf{X}) \sim N(\text{vec}(\mathbf{U}), \Sigma_M \otimes \cdots \otimes \Sigma_1)$. In this case, the probability density function of $\mathbf{X}$ is

$$f(\mathbf{X}|\mathbf{U}, \Sigma) = (2\pi)^{-d/2} \left( \prod_{m=1}^{M} |\Sigma_m|^{-d/(2d_m)} \right) \exp \left( -\frac{\|\text{vec}(\mathbf{X}) - \text{vec}(\mathbf{U})\|_F^2}{2\Sigma^{-1/2}} \right),$$

where $d = \prod_{m=1}^{M} d_m$ and $\Sigma^{-1/2} = \{\Sigma_1^{-1/2}, \ldots, \Sigma_M^{-1/2}\}$.

#### 2.3.1 High-dimensional heterogenous tensor mixture model

To ease presentation, we focus on third-order tensors $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ in this section. The generalization to higher-order tensors is straightforward. Assume that there are $K$ mixtures of tensor normal distributions with heterogeneous covariances such that

$$Z \sim \text{Multinomial } (\pi_1, \ldots, \pi_K),$$

$$\mathbf{X}|Z = k \sim N_T(\mathbf{U}_k^*, \Sigma_k^*),$$

where $\sum_k \pi_k = 1$, $\mathbf{U}_k^* \in \mathbb{R}^{d_1 \times d_2 \times d_3}$, and $\Sigma_k^* = \{\Sigma_{k,1}^*, \Sigma_{k,2}^*, \Sigma_{k,3}^*\}$ with $\Sigma_{k,1} \in \mathbb{R}^{d_1 \times d_1}$, $\Sigma_{k,2} \in \mathbb{R}^{d_2 \times d_2}$, $\Sigma_{k,3} \in \mathbb{R}^{d_3 \times d_3}$, $k \in [K]$. In tensor clustering problems, $\mathbf{X}$ is observable but $Z$ is not.

Suppose we have $n$ unlabeled tensor observations $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ generated independently and identically from the mixture model in (2), that is,

$$\mathbf{X}_1, \ldots, \mathbf{X}_n \overset{i.i.d.}{\sim} \pi_1 N_T(\mathbf{U}_1^*, \Sigma_1^*) + \cdots + \pi_K^* N_T(\mathbf{U}_K^*, \Sigma_K^*)$$
Figure 2: An illustration of rank-$R$ sparse CP decomposition for a third-order tensor.

Given $\mathcal{X} = \{\mathcal{X}_1, \ldots, \mathcal{X}_n\}$, our objective is to jointly perform clustering, i.e., estimate the cluster label $z = (z_1, \ldots, z_n)$, and model estimation, i.e., estimate $\pi^*_k$, $\mathcal{U}^*_k$ and $\Sigma^*_k$ for all $k$.

One unique challenge in modeling tensor data is the inherit high-dimensionality of the problem. In tensor data analysis, the number of free parameters is usually much larger than the number of observations and thus it is imperative to employ effective dimension reduction assumptions that enable estimability and interpretability.

**Low-rankness on $\mathcal{U}^*_k$.** We assume that $\mathcal{U}^*_k$, $k \in [K]$, admits a rank-$R$ CP decomposition structure (Kolda and Bader, 2009), in that,

$$
\mathcal{U}^*_k = \sum_{r=1}^{R} \omega_{k,r}^* \mathcal{B}_{k,r,1}^* \circ \mathcal{B}_{k,r,2}^* \circ \mathcal{B}_{k,r,3}^*,
$$

where $\circ$ denotes the outer product, $\omega_{k,r}^*$ is a positive scalar, $\mathcal{B}_{k,r,1}^* \in \mathbb{R}^{d_1}$, $\mathcal{B}_{k,r,2}^* \in \mathbb{R}^{d_2}$, and $\mathcal{B}_{k,r,3}^* \in \mathbb{R}^{d_3}$, $r \in [R]$. To ensure identifiability, $\mathcal{B}_{k,r,1}^*$, $\mathcal{B}_{k,r,2}^*$ and $\mathcal{B}_{k,r,3}^*$ are assumed to be unit-norm vectors, that is, $\|\mathcal{B}_{k,r,m}^*\|_2 = 1$ for all $k, r$ and $m$. The CP low-rank structure is one of the most commonly employed tensor structures (Kolda and Bader, 2009), and is widely adopted in tensor data analysis, such as imaging analysis (Zhou et al., 2013), facial image recognition (Cao et al., 2014), and recommendation system (Bi et al., 2018).

**Internal sparsity on $\mathcal{U}^*_k$.** Besides low-rankness, it is often desirable to have sparsity in tensor parameters. This can further reduce the number of free parameters and improve model interpretability, as a sparse pattern is plausible in many tensor data problems, such as advertisement placement (Zhou et al., 2020; Hao et al., 2021) and imaging data analysis (Sun and Li, 2019). Encouraging sparsity by directly adding a $\ell_1$-norm penalty on the tensor mean may be computationally infeasible, due to the large number of parameters involved in the penalty term. Alternatively, we consider achieving sparsity under the CP structure.
Specifically, based on (3), we assume that $\beta^*_k, r, m$’s are sparse. See Figure 2 for an illustration of the sparse CP decomposition structure. To differentiate from the usual element-wise sparsity, we refer to the sparsity of $\beta^*_k, r, m$’s as the internal sparsity of $U^*_k$.

**Separable $\Sigma^*_k$ with conditionally sparsity.** An attractive feature of the separable covariance structure is that the precision matrix also enjoys a separable structure, that is,

$$(\Sigma^*_{k,3} \otimes \Sigma^*_{k,2} \otimes \Sigma^*_{k,1})^{-1} = \Omega^*_{k,3} \otimes \Omega^*_{k,2} \otimes \Omega^*_{k,1},$$

where $\Omega^*_{k,m} = \Sigma^*_{k,m}^{-1}$ for all $k, m$. We assume the separable precision matrices are sparse, which reduces the number of free parameters and enhances interpretability. Specifically, the sparse entries in $\Omega^*_{k,m}$ relates to the conditional dependence between entities along the $m$-th mode in the $k$-th mixture. Estimating such a conditional independence is of interest in many applications. For example, in import-export studies, it is helpful to understand the dependencies across different countries and commodities (Leng and Tang, 2012). To ensure identifiability, we assume $\|\Omega^*_{k,m}\|_F / \sqrt{d_m} = 1$ for $m = 1, 2$, while $\|\Omega^*_{k,3}\|_F$ remains unconstrained. This identifiability condition does not alter the sparsity structures of $\Omega^*_{k,m}$’s.

**Remark 1.** The two sparsity assumptions on means $\beta^*_k, r, m$’s and precisions $\Omega^*_{k,m}$’s facilitate estimability when the tensor dimensions exceed the sample size. If desired, these two assumptions can be omitted from our proposal, in which case the ECM algorithm and its theoretical analysis both simplify by excluding regularizations. With only low-rankness in the mean and separability in the covariance, the theoretical results in Section 4 still hold by replacing the sparsity parameters with the respective dimension parameters.

### 3 High-Dimensional ECM Estimation

Denote $\Theta = (\pi_1, \ldots, \pi_K, \theta_1, \ldots, \theta_K)^T$, where $\theta_k$ collects all parameters in the $k$-th mixture with $\theta_k = (\beta^T_{k,1,1}, \ldots, \beta^T_{k,1,M}, \omega_{k,1}, \ldots, \beta^T_{k,R,1}, \ldots, \beta^T_{k,R,M}, \omega_{k,R}, \text{vec}(\Omega_{k,1})^T, \ldots, \text{vec}(\Omega_{k,M})^T)$. If the true label $z = (z_1, \ldots, z_n)$ were observed together with $X = \{X_1, \ldots, X_n\}$, the log-likelihood for the complete data $(X, z)$ is given by

$$\ell (\Theta | X, z) = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{K} 1(z_i = k) [\log (\pi_k) + \log \{f_k(X_i | \theta_k)\}],$$

(4)
where \( f_k(\cdot) \) is defined as in (1) with \( \mathbf{U}_k \) and \( \Sigma_{k,1}, \ldots, \Sigma_{k,M} \). When \( z \) is unknown, it is common to pose (4) as a missing data problem, where the latent label \( z \) is treated as missing data. To estimate \( \Theta \) in the presence of missing data, a useful approach is the expectation-maximization (EM) algorithm (Dempster et al., 1977), which encounters two major challenges when applied to our estimation problem.

**Computational Challenges.** First, in the M-step, given \( \Theta^{(t)} \) estimated from the previous EM update, one needs to maximize \( \mathbb{E}_{Z|X, \Theta^{(t)}} \{ \ell(\Theta|X, Z) \} \) with respect to \( \Theta \). This is a challenging problem as the loss function is non-convex and there is no closed-form solution. Second, the dimension of each \( \theta_k \) is in the order of \( O(R \sum_m d_m + \sum_m d_m^2) \). When dimensions \( d_1, \ldots, d_M \) are large and \( n \) is small, regularization is needed in the M-step to ensure the sparsity of \( \beta_{k,r,m}'s \) and \( \Omega_{k,m}'s \). The standard EM algorithm cannot incorporate such sparsity structures into parameter estimation. To overcome these two challenges, we propose a high-dimensional expectation conditional maximization (HECM) algorithm that breaks the M-step optimization problem into several less challenging conditional maximization problems. These conditional maximization problems enjoy closed-form formulas in the updates and permit penalty terms that involve much less parameters.

Next, we detail the HECM algorithm. Consider the \((t+1)\)-th step of the HECM iteration.

**E-step.** In the E-step, given \( \Theta^{(t)} \) estimated from the previous HECM update, we have

\[
\tau_{ik}(\Theta^{(t)}) = \mathbb{P}(z_i = k| X_i, \Theta^{(t)}) = \frac{\pi_k^{(t)} f_k(X_i|\theta_k^{(t)})}{\sum_k \pi_k^{(t)} f_k(X_i|\theta_k^{(t)})}, \quad k \in [K].
\]

Next, define

\[
Q_n(\Theta|\Theta^{(t)}) = \mathbb{E}_{Z|X, \Theta^{(t)}} \{ \ell(\Theta|X, Z) \}, \text{ which can be written as}
\]

\[
Q_n(\Theta|\Theta^{(t)}) = \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^K \tau_{ik}(\Theta^{(t)}) [\log(\pi_k) + \log\{f_k(X_i|\theta_k)\}].
\]

Correspondingly, the objective function in the maximization step can be written as

\[
Q_n(\Theta|\Theta^{(t)}) - \mathcal{P}^{(t+1)}(\Theta),
\]

where \( \mathcal{P}^{(t+1)}(\Theta) = \sum_{k,r,m} \lambda_0^{(t+1)} \|\beta_{k,r,m}\|_1 + \sum_{k,m} \lambda_{m}^{(t+1)} \|\Omega_{k,m}\|_{1,\text{off}} \) is a penalty term that encourages sparsity in \( \beta_{k,r,m} \) and \( \Omega_{k,m} \) for all \( k, r \) and \( m \), and \( \lambda_0^{(t+1)} \) and \( \lambda_{m}^{(t+1)} \)'s are tuning parameters to be discussed in Section A of the supplement. It is easy to see that the update of \( \pi_k \) can be calculated as

\[
\pi_k^{(t+1)} = \frac{1}{n} \sum_{i=1}^n \tau_{ik}(\Theta^{(t)}). \quad (7)
\]
HCM-step. The high-dimensional conditional maximization (HCM) step then proceeds by solving the following conditional optimizations. First, for $k \in [K], r \in [R], m \in [M]$, let

$$
\tilde{\beta}_{k,r,m}^{(t+1)} = \arg \max_{\beta_{k,r,m}} Q_n(\beta_{k,r,m}, \Theta_{(t+1)}^{(t+1)} - \beta_{k,r,m}, \Theta^{(t)}) - \lambda_0^{(t+1)} \|\beta_{k,r,m}\|_1,
$$

(8)

where $\Theta_{(t+1)}^{(t+1)} - \beta_{k,r,m}$ is $\Theta$ with $\beta_{k,r,m}$ removed; moreover, parameters in $\Theta_{(t+1)}^{(t+1)} - \beta_{k,r,m}$ that are updated before $\beta_{k,r,m}$ take values from the $(t+1)$-th step and parameters in $\Theta_{(t+1)}^{(t+1)} - \beta_{k,r,m}$ that are not yet updated take values from the $t$-th step. See Figure 3 for the ordering in conditional updates when $M = 3$. The update $\tilde{\beta}_{k,r,m}^{(t+1)}$ can be calculated in closed-form with the $j$-th entry

$$
\tilde{\beta}_{k,r,m}^{(t+1)}(j) = \begin{cases} 
g_{k,r,m}^{(t+1)}(j) - \frac{n_k^{(t)} \text{sign}(\beta_{k,r,m}^{(t+1)}(j))}{n_k^{(t)} C_{k,r,m}^{(t+1)}(j,j)} & \text{if } |h_{k,r,m}^{(t+1)}(j)| > \lambda_0^{(t+1)}, \\
0 & \text{otherwise}, \end{cases}
$$

(9)

where $n_k^{(t)} = \sum_{i=1}^n \tau_{ik}(\Theta^{(t)})$ and expressions for $g_{k,r,m}^{(t+1)}, C_{k,r,m}^{(t+1)}$ and $h_{k,r,m}^{(t+1)}$ are given in Proposition 2 in the supplement. The estimate $\tilde{\beta}_{k,r,m}^{(t+1)}$ is then normalized to ensure the unit-norm constraint, and we have $\beta_{k,r,m}^{(t+1)} = \tilde{\beta}_{k,r,m}^{(t+1)}/\|\tilde{\beta}_{k,r,m}^{(t+1)}\|_2$.

Next, we consider the update of $\omega_{k,r}$’s. Define operators $\prod_{m \in [M]} \beta_{k,r,m} = \beta_{k,r,1} \circ \cdots \circ \beta_{k,r,M}$ and $\prod_{m \in [M]} \Omega_{k,m} = \Omega_{k,M} \otimes \cdots \otimes \Omega_{k,1}$. Let $\Theta_{(t+1)}^{(t)} - \omega_{k,r}$ be $\Theta$ with $\omega_{k,r}$ removed and parameters updated before and after $\omega_{k,r}$ take values from the $(t+1)$-th and $t$-th steps, respectively. Maximizing $Q_n(\omega_{k,r}, \Theta_{(t+1)}^{(t)} - \omega_{k,r}, \Theta^{(t)})$ with respect to $\omega_{k,r}$ is equivalent to solving

$$
\max_{\omega_{k,r}} \sum_{i=1}^n \tau_{ik}(\Theta^{(t)}) \left\| \left( X_{k,r}^{(t+1)} - \omega_{k,r} \prod_{m \in [M]} \beta_{k,r,m}^{(t+1)} \right) \times \Omega_k^{(t)} \right\|_F^2,
$$

Figure 3: Parameter updates (left to right) for the $k$-th cluster in the HCM step.
where \( X_{i,-r}^{(t+1)} \) is given in Proposition 2. Some straightforward algebra yields
\[
\omega_{k,r}^{(t+1)} = \frac{\sum_{i=1}^{n} \tau_{ik}(\Theta^{(t)}) \text{vec} \left( X_{i,-r}^{(t+1)} \right)^\top \left( \prod_{m \in [M]} \Omega_{k,m}^{(t)} \right) \text{vec} \left( \prod_{m \in [M]} \beta_{k,r,m}^{(t+1)} \right)}{\eta_k^{(t)} \text{vec} \left( \prod_{m \in [M]} \beta_{k,r,m}^{(t+1)} \right)^\top \left( \prod_{m \in [M]} \Omega_{k,m}^{(t)} \right) \text{vec} \left( \prod_{m \in [M]} \beta_{k,r,m}^{(t+1)} \right)}. \tag{10}
\]

Finally, we consider the update of \( \Omega_{k,m} \)'s. Let \( \Theta_{-\Omega_{k,m}}^{(t+1)} \) be \( \Theta \) with \( \Omega_{k,m} \) removed and parameters updated before and after \( \Omega_{k,m} \) take values from the \((t+1)\)-th and \(t\)-th steps, respectively. We consider, for \( k \in [K], m \in [M], \)
\[
\hat{\Omega}_{k,m}^{(t+1)} = \arg\max_{\Omega_{k,m}} Q_n(\Omega_{k,m}, \Theta_{-\Omega_{k,m}}^{(t)}, \Theta^{(t)}) - \lambda_m^{(t+1)} \| \Omega_{k,m} \|_{1,\text{off}}.
\]
With some straightforward algebra, the above objective function can be written as
\[
\hat{\Omega}_{k,m}^{(t+1)} = \arg\min_{\Omega_{k,m}} \frac{\eta_k^{(t)}}{d_{m,n}} \left\{ -\log |\Omega_{k,m}| + \text{tr} \left( S_{k,m}^{(t+1)} \Omega_{k,m} \right) \right\} + \lambda_m^{(t+1)} \| \Omega_{k,m} \|_{1,\text{off}}, \tag{11}
\]
where \( S_{k,m}^{(t+1)} = \frac{d_{m,n}}{d_{k,n}} \sum_{i=1}^{n} \tau_{ik}(\Theta^{(t)}) X_{i,k,m}^{(t+1)} X_{i,k,m}^{(t+1)\top} \), \( X_{i,k,m}^{(t+1)} = \left( X_i - U_k^{(t+1)} \right)_{(m)} A_{k,m}^{(t+1)} \) and
\[
A_{k,m}^{(t+1)} = \left( \prod_{m' > m} \Omega_{k,m'}^{(t)} \right) \left( \prod_{m' < m} \Omega_{k,m'}^{(t+1)} \right)^{1/2}.
\]
The optimization problem in (11) is convex and can be carried out efficiently using the GLasso algorithm (Friedman et al., 2008). We refer to Friedman et al. (2008) for the exact updating formulas of the GLasso. To satisfy the identifiability constraint, \( \hat{\Omega}_{k,m}^{(t+1)} \) is first normalized such that \( \hat{\Omega}_{k,m}^{(t+1)} = \sqrt{d_{m,n} \hat{\Omega}_{k,m}^{(t+1)}} / \| \hat{\Omega}_{k,m}^{(t+1)} \|_F \) for all \( m \). We then find the weight \( \eta_k^{(t+1)} \) and assign it to the last mode, i.e.,
\[
\Omega_{k,m}^{(t+1)} = \begin{cases} 
\hat{\Omega}_{k,m}^{(t+1)}, & m \leq M - 1, \\
\eta_k^{(t+1)} \bar{\Omega}_{k,M}^{(t+1)}, & m = M,
\end{cases}
\tag{12}
\]
where \( \eta_k^{(t+1)} = n_k^{(t)} d \left\{ \sum_{i=1}^{n} \tau_{ik}(\Theta^{(t)}) \text{vec} \left( X_i - U_k^{(t+1)} \right)^\top \prod_{m \in [M]} \Omega_{k,m}^{(t+1)} \text{vec} \left( X_i - U_k^{(t+1)} \right) \right\}^{-1} \). The above estimation procedure is summarized in Algorithm 1. It is seen that the HECM algorithm can be carried out efficiently as there are closed-form updating formulas for each HCM step. In practice, to speed up convergence, one may repeat steps 2.2-2.3 several times before exiting the HCM step. Such a heuristic procedure may reduce the number of steps needed to reach convergence in our experiments. Given the estimated \( \hat{\tau}_{ik}(\Theta) \) from the last HECM iterate, we may estimate the class labels using
\[
\hat{z}_i = \arg\max_k \hat{\tau}_{ik}(\Theta), \quad i \in [n].
\]
Algorithm 1 The HECM Algorithm for Heterogeneous Tensor Mixture Model

**Input:** data \( \{x_i\}_{i \in [n]} \), number of clusters \( K \), rank \( R \), maximum number of iterations \( T \) and tuning parameters \( \lambda_0^{(t)} , \{ \lambda_m^{(t)} \}_{m=1}^M \), \( t \in [T] \).

**Initialization:** calculate \( \pi_k^{(0)} , \beta_{k,r,m}^{(0)} , \omega_{k,r}^{(0)} \) and \( \Omega_{k,m}^{(0)} \), for all \( k, r \) and \( m \).

**Repeat** the following steps for \( t \in [T] \),

1. **E-step:** compute \( \tau_{ik}(\Theta^{(t)}) \) using (5) for all \( i \) and \( k \),
2. **HCM-step:**
   - **2.1:** update \( \pi_k^{(t+1)} \) using (7) for all \( k \);
   - **2.2:** update \( \tilde{\omega}_{k,r}^{(t+1)} \) given \( \Theta^{(t+1)} \) using (9) and set \( \beta_{k,r,m}^{(t+1)} = \tilde{\beta}_{k,r,m}^{(t+1)} / \| \tilde{\beta}_{k,r,m}^{(t+1)} \|_2 \); update \( \omega_{k,r}^{(t+1)} \) given \( \Theta^{(t+1)} \) using (10) for all \( k, r \) and \( m \);
   - **2.3:** update \( \tilde{\Omega}_{k,m}^{(t+1)} \) given \( \Theta^{(t+1)} \) using (11) and set \( \Omega_{k,m}^{(t+1)} \) using (12) for all \( k \) and \( m \);

**Stop** if the algorithm has converged.

**Output:** Cluster label \( \hat{z} \), cluster mean \( \hat{U}_k \) and precision matrices \( \hat{\Omega}_k \), \( k \in [K] \).

**Initialization, stopping rule and tuning.** In Algorithm 1, given the tuning parameters, we need to determine the initial values \( \pi_k^{(0)} , \beta_{k,r,m}^{(0)} , \omega_{k,r}^{(0)} \) and \( \Omega_{k,m}^{(0)} \) for all \( k, r \) and \( m \). In our implementation, when \( d_1, \ldots, d_m \) are moderate, we initialize the cluster label via \( K \)-means on the vectorized tensor observations \( \{ \text{vec}(x_i) \}_{i \in [n]} \) to find \( \{ z_i^{(0)} \}_{i \in [n]} \). We set \( \pi_k^{(0)} = \frac{1}{n} \sum_{i=1}^n 1(z_i^{(0)} = k), k \in [K] \). After that, we estimate \( \omega_{k,r}^{(0)} , \beta_{k,r,m}^{(0)} \) using the standard CP decomposition (Kolda and Bader, 2009) on each \( U_k^{(0)} = \frac{1}{n \pi_k} \sum_{i=1}^n 1(z_i^{(0)} = k) x_i \). We first let \( \tilde{\Omega}_{k,m}^{(0)} = \left\{ \frac{1}{n \pi_k^2} \sum_{i=1}^n 1(z_i^{(0)} = k) (x_i - U_k^{(0)}) (x_i - U_k^{(0)})^\top \right\}^{-1} \), and the \( \tilde{\Omega}_{k,m}^{(0)} \)'s are then normalized as in (12) to give \( \Omega_{k,m}^{(0)} \) for all \( k, m \). In our experiments, this initialization leads to good numerical performances. A similar procedure was also considered in Mai et al. (2021).

When \( d_1, \ldots, d_M \) are large, we may avoid the high computational cost from performing \( K \)-means on high-dimensional vectors and alternatively consider the tensor clustering method in Sun and Li (2019), which applies \( K \)-means on the output from tensor decomposition. Due to space limitations, the stopping rule and parameter tuning are discussed in Section A.

### 3.1 Connection to existing EM-type algorithms

Comparing with the standard EM algorithm, it is seen that the HCM-step of Algorithm 1 does not rely on finding \( \arg \max_{\Theta} Q_n(\Theta|\Theta^{(t)}) \) in the maximization step, which is an
intractable non-convex optimization under our setting. Instead, we break the maximization task into a sequence of conditional optimization problems, each of which is convex and has a much reduced dimension. This strategy dramatically improves the computational feasibility, although it brings significant challenges to our theoretical analysis. Specifically, the HCM-step does not yield $\arg\Theta Q_n(\Theta|\Theta^{(t)})$ but instead gives solutions to a sequence of conditional optimization problems $\arg\vartheta Q_n(\vartheta, \bar{\Theta}_-|\Theta^{(t)})$. Thus, existing techniques (Balakrishnan et al., 2017; Hao et al., 2017) that analyze EM iterates assuming $\Theta^{(t+1)} = \arg\Theta Q_n(\Theta|\Theta^{(t)})$ are not directly applicable.

The proposed HECM algorithm is also related to high-dimensional EM algorithms that incorporate regularization in the M-step (Wang et al., 2015; Yi and Caramanis, 2015). Specifically, the high-dimensional M-step maximizes $Q_n(\Theta|\Theta^{(t)}) - P(\Theta)$, where $P(\cdot)$ is some penalty term that encourages sparsity. Similar to the standard EM algorithm, the high-dimensional M-step still involves an intractable non-convex optimization. In comparison, our HECM incorporates regularization into the sequence of conditional maximizations in (8) and (11).

Finally, the HECM algorithm is closely related to the expectation-conditional-maximization (ECM) algorithm (Meng and Rubin, 1993; Meng, 1994). Convergence of the ECM algorithm to some arbitrary fixed point has been studied (Meng, 1994); however, to our knowledge, its convergence to the unknown true parameter has not been investigated, even in the low-dimensional regime. This analysis turns out to be highly challenging due to the dual non-convexity from both the EM-type estimation and the objective function in the M-step; see Section 4.2. Moreover, besides the aforementioned challenges, our theoretical analysis of HECM is further complicated by considering sparsity structures in the estimation.

4 Theoretical Analysis

This section establishes statistical guarantees for the local convergence of the HECM estimator. We first develop theory when rank $R = 1$ and then generalize our results to the more challenging case of rank $R > 1$. All proofs are collected in the supplement.

Let $\Theta^*$ denote the true parameters located in a non-empty compact convex set. Define $\omega_{\max} = \max_{k,r} \omega_{k,r}^*, \omega_{\min} = \min_{k,r} \omega_{k,r}^*$ and $d_{\max} = \max_{m \in [M]} d_m$. Denote the sparsity param-
eters for the tensor mean and the precisions as $s_1 = \max_{k,r,m} \|\beta_{k,r,m}^*\|_0$ and $s_2 = \max_{k,m} \|\Omega_{k,m}^*\|_{0,\text{off}}$, respectively. Without loss of generality, we assume $\|\Omega_{k,m}^*\|_F / \sqrt{d_m} = 1$ for all $k$ and $m$ (see, for example, Lyu et al. (2019)). Define the normalized distance metric (noting $\|\beta_{k,r,m}^*\| = 1$)

$$D(\Theta, \Theta^*) = \max_{k,r,m} \left\{ \|\beta_{k,r,m} - \beta_{k,r,m}^*\|_2, \frac{|\omega_{k,r} - \omega_{k,r}^*|}{|\omega_{k,r}|}, \frac{\|\Omega_{k,m} - \Omega_{k,m}^*\|_F}{\|\Omega_{k,m}^*\|_F} \right\}$$

(13)

and let $B_\alpha(\Theta^*)$ denote the ball around $\Theta^*$ with $D(\Theta, \Theta^*) \leq \alpha$. Next, we introduce several regularity conditions common for both $R = 1$ and $R > 1$.

**Condition 1.** Assume $\min_k \pi_k^* > r_0$ for some constant $r_0 > 0$, $\max_k \|U_k^*\|_{\max} = O(1)$ and $\omega_{\max}/\omega_{\min} = O(1)$. Furthermore, assume there exist some positive constants $\phi_1, \phi_2$ such that $\phi_1 \leq \sigma_{\min}(\Omega_{k,m}^*) \leq \sigma_{\max}(\Omega_{k,m}^*) \leq \phi_2$, for $k \in [K]$, $m \in [M]$.

The condition $\max_k \|U_k^*\|_{\max} = O(1)$, which bounds the tensor mean element-wisely, is needed to control errors from estimating the mean, which in turn regulates errors from estimating the precision matrices; we refer to Wu and Yang (2020) for more discussions. The condition $\omega_{\max}/\omega_{\min} = O(1)$ stipulates that weights $\omega_{k,1}, \ldots, \omega_{k,R}$ are of the same order, as commonly done in the tensor clustering literature (e.g., Sun and Li, 2019). Lastly, the bounded eigenvalue condition on the precision matrices is a regularity condition that has been employed in the literature (Leng and Tang, 2012; Lyu et al., 2019).

**Condition 2.** The initial values $\beta_{k,r,m}^{(0)}, \omega_{k,r}^{(0)}, \Omega_{k,m}^{(0)}$ for all $k, r$ and $m$ satisfy

$$D(\Theta^{(0)}, \Theta^*) \leq \min \left\{ \frac{1}{2} \left( \frac{C_0 \omega_{\min}}{(R-1)\omega_{\max}} \right) \frac{1}{\pi - 1} \right\},$$

where $C_0 \in [0, 1/3]$ is a positive constant depending on $\phi_1, \phi_2$ and $\frac{\|\Omega_{k,m}^{(0)} - \Omega_{k,m}^*\|_2}{\sigma_{\min}(\Omega_{k,m}^*)} \leq 1/2$.

This condition requires the initial values to be reasonably close to the true parameters. Such an initial error condition is commonly considered in the non-convex optimization literature (Zhang and Xia, 2018; Zhang, 2019; Mai et al., 2021). When $R = 1$, the bound on $D(\Theta^{(0)}, \Theta^*)$ is reduced to $D(\Theta^{(0)}, \Theta^*) \leq \frac{1}{2}$, which is comparable to the initial condition employed in Balakrishnan et al. (2017). Assuming $D(\Theta^{(0)}, \Theta^*) \leq \frac{1}{2}$ is a mild condition as $\beta_{k,r,m}^{(0)}$’s and $\beta_{k,r,m}^*$’s are normalized to have a unit norm and $\omega_{k,r}, \Omega_{k,m}$ are both normalized in $D(\Theta, \Theta^*)$.
The next condition generalizes the signal-to-noise condition in Balakrishnan et al. (2017), which considers a vector Gaussian mixture model with a homogeneous isotropic covariance. Recall that $\tau_{ik}(\Theta) = \mathbb{P}(z_i = k | X_i, \Theta)$. With some straightforward algebra, the derivative of $\tau_{ik}(\Theta)$ with respect to $(\theta_1, \ldots, \theta_K)$ can be written as

$$\nabla_{\theta_l} \tau_{ik}(\Theta) = \begin{cases} -\tau_{ik}(\Theta) \mu_l(\Theta) J_i(\theta_l), & \text{for } l \neq k, \\ \tau_{ik}(\Theta)(1 - \tau_{ik}(\Theta)) J_i(\theta_k), & \text{for } l = k, \end{cases}$$

while the specific form of $J_i(\theta_l)$ is given in (A25). For $\Theta, \Theta' \in B_{\frac{1}{2}}(\Theta^*)$, define $g\{J_i(\theta_l), J_i(\theta'_k)\}$ as in (A26) such that it multiplies the scaled $\ell_2$ norms of $J_i(\theta_l)$ and $J_i(\theta'_k)$.

**Condition 3 (Separability Condition).** For $\Theta, \Theta' \in B_{\frac{1}{2}}(\Theta^*)$, it holds that

$$\mathbb{E} \{W_{i1} \tau_{i1}(\Theta) \tau_{i1}(\Theta')\}^2 \leq \frac{\gamma^2}{24^2K^4(R + 1)^4(M + 1)^2}, \quad l \neq k \in [K],$$

(14)

where $W_{i1} = g\{J_i(\theta_l), J_i(\theta'_k)\}$ and $\gamma > 0$ is a sufficiently small separability parameter.

Condition 3 stipulates that the clusters are sufficiently separated and the probability that a data point belongs to two different clusters cannot be both large. Under a simple two-component vector mixture model, Condition 3 can be reduced to the commonly employed signal to noise condition, which is formally stated as follows.

**Plausibility of Condition 3.** Consider a vector Gaussian mixture model with $X \sim \frac{1}{2} N(\mu^*, \sigma^2 I_d) + \frac{1}{2} N(-\mu^*, \sigma^2 I_d)$, where $\mu^* \in \mathbb{R}^d$ and $d$ is fixed. Then, as stated in Proposition 1, Condition 3 holds under the the signal-to-noise ratio condition that requires $\frac{\|\mu^*\|_2}{\sigma}$ to be sufficiently large (see, for example, Balakrishnan et al. (2017)).

**Proposition 1.** Assume $X \sim \frac{1}{2} N(\mu^*, \sigma^2 I_d) + \frac{1}{2} N(-\mu^*, \sigma^2 I_d)$ and $\mu \in \{\mu \mid \|\mu - \mu^*\|_2 \leq \frac{1}{4} \|\mu^*\|_2\}$ as in Balakrishnan et al. (2017). When $\frac{\|\mu^*\|_2}{\sigma}$ is lower bounded by a sufficiently large constant, it holds that

$$\mathbb{E} \{W_{i1} \tau_{i1}(\mu) \tau_{i2}(\mu)\}^2 \leq 2 \exp \left\{ 4 \log \left( \frac{\|\mu^*\|_2}{\sigma} \right) - 4 \frac{\|\mu^*\|_2^2}{\sigma^2} \right\},$$

where $W_{i1}, \tau_{i1}(\mu)$ and $\tau_{i2}(\mu)$ are as defined in (14).

It is seen from Proposition 1 that $\mathbb{E} \{W_{i1} \tau_{i1}(\mu) \tau_{i2}(\mu)\}^2$ can be made sufficiently small if the signal-to-noise ratio $\frac{\|\mu^*\|_2}{\sigma}$ is sufficiently large.
Algorithm 2 The HECM Algorithm with Sample Splitting

**Input:** data \( \{X_i\}_{i \in [n]} \), number of clusters \( K \), rank \( R \), maximum number of iterations \( T \) and tuning parameters \( \lambda_0^{(t)} \), \( \{\lambda_m^{(t)}\}_{m=1}^{M} \), \( t \in [T] \).

**Initialization:** calculate \( \pi_k^{(0)} \), \( \beta_k^{(0)}_r \), \( \omega_k^{(0)}_r \) and \( \Omega_k^{(0)}_r \), for all \( k, r \) and \( m \). Split the dataset into \( T \) subsets with sample size \( n_0 = n/T \), which is assumed to be integer.

**Repeat** the following steps for \( t \in [T] \),

1. **E-step:** using the \( t \)th data split, compute \( \tau_{i,k}^{(t)}(\Theta^{(t)}) \) using (5) for all \( i \) and \( k \),
2. **HCM-step:** using the \( t \)th data split,
   2.1: update \( \pi_k^{(t+1)} \) using (7) for all \( k \);
   2.2: update \( \tilde{\beta}_k^{(t+1)}_r \) given \( \Theta_k^{(t+1)}_r \) using (9) and set \( \beta_k^{(t+1)}_r = \frac{\tilde{\beta}_k^{(t+1)}_r}{\|eta_k^{(t+1)}_r\|_2} \); update \( \omega_k^{(t+1)}_r \) given \( \Theta_k^{(t+1)}_r \) using (10) for all \( k, r \) and \( m \);
   2.3: update \( \tilde{\Omega}_k^{(t+1)}_r \) given \( \Theta_k^{(t+1)}_r \) using (11) and set \( \Omega_k^{(t+1)}_r \) using (12) for all \( k \) and \( m \);

**Stop** if the algorithm has converged.

### 4.1 Theory with rank \( R = 1 \)

To simplify the technical analysis of the HECM algorithm, we focus on its sample-splitting version, which is illustrated in Algorithm 2. Similar to many other work on EM algorithms (Yi and Caramanis, 2015; Wang et al., 2015; Balakrishnan et al., 2017), sample splitting is used to facilitate the theoretical analysis in order to derive the estimation consistency. In Algorithm 2, we divide the \( n \) samples into \( T \) subsets of size \( \lfloor n/T \rfloor \) and update the parameter using a fresh subset of samples in each iteration. See more remark after Theorem 1.

**Condition 4.** Recall \( n_0 = n/T \). The sample size \( n_0 \) satisfies

\[
    n_0 \gtrsim \max \left\{ \frac{s_1 \log d}{\omega_{\min}^2}, (s_2 + d_{\max}) \log d \right\}. \tag{15}
\]

The sample complexity condition in (15) is to guarantee that the statistical error is not excessive so that each HECM update leads to a reduced error, meanwhile satisfying the initial error condition required in Condition 2. The first term \( s_1 \log d/\omega_{\min}^2 \) in the sample complexity lower bound is related to estimating the low-rank and sparse tensor means while the second term \( (s_2 + d_{\max}) \log d \) is related to estimating the sparse separable precision matrices.

We next state the main theory for the HECM iterates in Algorithm 1 when \( R = 1 \).
Theorem 1. Suppose Conditions 1-4 hold with $\gamma d_{\max} \leq C_1$ for some constant $C_1 > 0$. Let
\[
\lambda_0^{(t)} = 4\epsilon_0 + \tau_0 \frac{D(\Theta^{(t-1)}, \Theta^*)}{\sqrt{s_1}}, \quad \lambda_m^{(t)} = 4\epsilon_m + 3\tau_1 \frac{D(\Theta^{(t-1)}, \Theta^*)}{2\sqrt{s_2 + d_m}},
\]
where $\tau_0 = C_\tau \gamma$, $\tau_1 = d\tau_0$, $d = \prod_m d_m$, $\epsilon_0 = c_1 \omega_{\max} \sqrt{T \log d/n}$ and $\epsilon_m = c_2 (d/d_m) \sqrt{T \log d/n}$ for some constants $C_\tau, c_1, c_2 > 0$. The estimator $\Theta^{(t)}$ from the $t$-th iteration of Algorithm 2 satisfies with probability $1 - o(1)$,
\[
D(\Theta^{(t)}, \Theta^*) \leq \rho D(\Theta^{(0)}, \Theta^*) + \frac{C_2}{1 - \rho} \left\{ \frac{1}{\omega_{\min}} \sqrt{T s_1 \log d/n} + \max_m \sqrt{T (s_2 + d_m) \log d/nd_m} \right\},
\]
where $C_2 > 0$ is a constant, the contraction parameter $\rho$ given in (A78) satisfies $0 < \rho \leq 1/3$ and the maximum number of iterations $T \lesssim \log d$.

Several important implications are provided as follows.

*Computational error and statistical error trade-off.* The non-asymptotic error bound in (17) involves two terms, the first of which is the computational error and it decreases geometrically in the iteration number $t$, whereas the second term is the statistical error and is independent of $t$. Thus, the HECM iterates are guaranteed to converge geometrically to a neighborhood that is within statistical precision of the unknown true parameter. When the iterations $t$ reaches its maximum $T$, the computation error is dominated by the statistical error and the algorithm can be terminated.

*Statistical errors.* Considering the statistical error, apart from the term $T \lesssim \log d$, the first error term $\frac{1}{\omega_{\min}} \sqrt{s_1 \log d/n}$ is related to estimating the low-rank and sparse tensor means, which matches with the optimal rate $\sqrt{s \log d/\log n_0}$ in high-dimensional models with sparsity parameter $s$, dimension $d$ and sample size $n_0$ (Wainwright, 2019), and the second error term $\max_m \sqrt{(s_2 + d_m) \log d/nd_m}$ is related to estimating the sparse tensor precisions. The sample splitting scheme uses a fresh subset of the data at each iteration and it is a technique commonly considered in analyzing EM algorithms (Yi and Caramanis, 2015; Wang et al., 2015; Balakrishnan et al., 2017). In Theorem 1, it is seen that the iteration number $T$ does not affect the contraction in the computational error though it increases the statistical error by at most a factor of $\log d$. We expect this logarithm factor can be eliminated by directly analyzing Algorithm 1, which however incurs significant technical
complexity as it requires the statistical error bound in Lemma 3b to hold uniformly over $B_1^2(\Theta^*)$ with high probability.

4.2 Proof outline and key technical challenges

The conditional updating scheme in HECM requires a delicate treatment in order to establish the contractions of the iterations. As HECM cannot access the maximizer of $Q_n(\cdot|\cdot)$ in the M-step, existing arguments and techniques in the population and sample-based analysis of the standard EM algorithms (Yi and Caramanis, 2015; Wang et al., 2015; Balakrishnan et al., 2017) are not directly applicable. To put our discussions in context, we first give a brief review of the population and sample-based analysis of the standard EM algorithm, which utilizes properties of the sample function $Q_n(\cdot|\cdot)$ and population function $Q(\cdot|\cdot)$. Specifically, the contraction of one EM iterate is established using three key ingredients including a restricted strong concavity condition,

$$Q_n(\Theta'|\Theta) - Q_n(\Theta^*|\Theta) - \langle \nabla Q_n(\Theta^*|\Theta), \Theta' - \Theta^* \rangle \leq -\gamma_n \|\Theta' - \Theta\|^2,$$

where $\gamma_n \geq 0$, $\Theta^*$ is the true unknown parameter and $\Theta, \Theta' \in B_\alpha(\Theta^*)$ for some $\alpha > 0$, a gradient stability condition $\|\nabla Q(\Theta^*|\Theta) - \nabla Q(\Theta^*|\Theta^*)\| \leq \tau \|\Theta - \Theta^*\|_2$ where $\tau \geq 0$ and a statistical error condition quantifying the difference $\nabla Q_n(\Theta^*|\Theta) - \nabla Q(\Theta^*|\Theta)$ in an appropriate norm. To ease notation, we write $B_\alpha(\Theta^*_{-\beta_{k,m}})$ as $B_\alpha(\Theta^*)$ when there is no ambiguity.

In our analysis, the employment of conditional maximizations requires the consideration of a sequence of conditional $Q$ functions including $Q_n(\beta'_{k,m}, \Theta_{-\beta_{k,m}}|\Theta^*)$, $Q_n(\omega'_{k}, \Theta_{-\omega_{k}}|\Theta^*)$ and $Q_n(\Omega'_{k,m}, \Theta_{-\Omega_{k,m}}|\Theta^*)$ for all $k, m$. For example, regarding the update of $\beta_{k,m}$, the sample conditional $Q$ function is expressed as $Q_n(\beta'_{k,m}, \Theta_{-\beta_{k,m}}|\Theta)$, where $\beta'_{k,m}$ is the parameter to be updated and $\Theta_{-\beta_{k,m}}$ collects all other parameters being conditioned on, with some already updated and some yet to be updated. Computational and statistical properties of the conditional $Q$ functions thus need to be established uniformly over all $\Theta_{-\beta_{k,m}} \in B_\alpha(\Theta^*)$ for some $\alpha > 0$. For example, we demonstrate in Lemma 1b that the restricted strong concavity of $Q_n(\beta'_{k,m}, \Theta_{-\beta_{k,m}}|\Theta)$ with respect to $\beta_{k,m}$ holds uniformly over all $\Theta \in B_\alpha(\Theta^*)$ and in Lemma 2b that the following stability condition holds uniformly over all $\Theta \in B_\alpha(\Theta^*)$,

$$\|\nabla_{\beta_{k,m}} Q(\beta'_{k,m}, \Theta_{-\beta_{k,m}}|\Theta) - \nabla_{\beta_{k,m}} Q(\beta'_{k,m}, \Theta_{-\beta_{k,m}}|\Theta^*)\|_2 \leq \tau_0 \cdot D(\Theta, \Theta^*),$$

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where $\tau_0 > 0$, $\Theta', \Theta \in B_\alpha(\Theta^*)$ and $D(\Theta, \Theta^*)$ is defined as in (13); the statistical error shown in Lemma 3b quantifies $\|\nabla_{\beta_{k,m}} Q_n(\beta'_{k,m}, \Theta - \beta_{k,m} | \Theta) - \nabla_{\beta_{k,m}} Q(\beta'_{k,m}, \Theta - \beta_{k,m} | \Theta)\|_{P_1^*}$ uniformly over $\Theta \in B_\alpha(\Theta^*)$ under the sparsity assumptions, where $P_1^*$ is a dual norm. Finally, utilizing these computational and statistical properties of the conditional $Q$ functions, Lemma 4b establishes a critical result that ensures contraction after one HECM update. Unlike analysis of the standard EM algorithm, our one-step contraction result requires carefully balancing the maximizer of a sequencing of conditional $Q$ functions. One major challenge in this analysis is that we are not able to leverage the useful property that $\nabla_{\Theta} Q(\Theta^* | \Theta^*) = 0$. Instead, we need to precisely characterize $\nabla_{\beta_{k,m}} Q(\beta^*_{k,m}, \Theta - \beta_{k,m} | \Theta^*)$, $\nabla_{\omega_k} Q(\omega^*_{k}, \Theta - \omega_k | \Theta^*)$ and $\nabla_{\Omega_{k,m}} Q(\Omega^*_{k,m}, \Theta - \Omega_{k,m} | \Theta^*)$ for all $k, m$ and $\Theta \in B_\alpha(\Theta^*)$; see proofs in Section D6.

The above technically challenging are nontrivial and require new proof strategies and techniques. To our knowledge, this is the first theoretical analysis of the local convergence of EM-type algorithms with conditional updates in the M-step. Additionally, we note that, as we consider a high-dimensional parameter estimation, a careful calibration of regularization is also needed in the analysis.

### 4.3 Theory with rank $R > 1$

Next, we extend our theory to the case of a general rank $R > 1$. The theoretical analysis of $R > 1$ is more challenging as components from different rank are not generally not orthogonal. To quantify the correlation between decomposed components $\beta^*_{k,r,m}$’s across different ranks, we define the following incoherence parameter

$$\xi = \max_{k, r \neq r', m} \left| \langle \beta^*_{k,r',m}, \beta^*_{k,r,m} \rangle \right|. \quad (18)$$

For example, when $\xi = 0$, the components $\beta^*_{k,r,m}$’s are orthogonal (as they are unit-norm vectors). In our theoretical analysis, we impose an upper bound condition on $\xi$ that allows the decomposed components to be correlated but only to a certain degree. Similar incoherence conditions have been assumed in low-rank tensor models (Anandkumar et al., 2014; Sun and Li, 2019; Cai et al., 2020; Xia et al., 2021).

**Theorem 2.** Suppose Conditions 1-4 hold with $\gamma d_{\text{max}} \leq C_1$ and $R \xi^M \lesssim (\log d)^{-1}$, where $C_1$
is as defined in Theorem 1. Let
\[
\lambda_0^{(t)} = 4\epsilon_0 + \tau_0 \frac{D(\Theta^{(t-1)}, \Theta^*)}{s_1}, \quad \lambda_m^{(t)} = 4\epsilon_m + 3\tau_1 \frac{D(\Theta^{(t-1)}, \Theta^*)}{2s_1 + d_m},
\]
where \(\tau_0 = C'\gamma\), \(\epsilon_0 = \epsilon_1\omega_{max} \sqrt{T \log d/n}\) for some constants \(C', \epsilon_1 > 0\), and \(\tau_1\) and \(\epsilon_m\) are as defined in (16). The estimator \(\Theta^{(t)} \) from the \(t\)-th iteration of Algorithm 2 satisfies with probability \(1 - o(1)\),
\[
D(\Theta^{(t)}, \Theta^*) \leq \rho R D(\Theta^{(0)}, \Theta^*) + \frac{C'_2}{1 - \rho R} \left\{ \frac{1}{\omega_{\min}} \sqrt{T s_1 \log d/n} + \max_m \sqrt{T (s_2 + d_m) \log d/nd_m} \right\},
\]
where \(C'_2 > 0\) is a constant, \(\rho R\) given in (A89) satisfies \(\rho \leq \rho R \leq 1/2\) with \(\rho\) in (17), and the maximum number of iterations \(T \gtrsim (-\log \rho R)^{-1} \log (d_{max} n \cdot D(\Theta^{(0)}, \Theta^*))) \gtrsim \log d\).

Theorem 2 gives the non-asymptotic error bound for the HECM estimator in the general rank case. Similar to Theorem 1, when the number of iterations \(t\) reaches \(T\), the computational error will be dominated by the statistical error, leading to \(\rho R D(\Theta^{(0)}, \Theta^*) \gtrsim \frac{1}{1 - \rho R} \left\{ \frac{1}{\omega_{\min}} \sqrt{T s_1 \log d/n} + \max_m \sqrt{T (s_2 + d_m) \log d/nd_m} \right\}\). Compared to Theorem 1, it is seen that the contraction parameter \(\rho R\) is bounded below by \(\rho\), which indicates that the contraction rate can be slower in the general rank case. Correspondingly, more iterations are need to reach convergence as \((-\log \rho R)^{-1} \log (d_{max} n \cdot D(\Theta^{(0)}, \Theta^*)))\) is larger. This agrees with the expectation that, as the tensor recovery problem becomes more challenging, the algorithm has a slower convergence rate. Finally, it is worth noting that \(\lambda_m^{(t)}\) in Theorem 2 remains the same as that in the rank one case, as regularizations for the sparse precision matrices are not affected by the tensor mean estimation.

5 Numerical Experiments

In this section, we investigate the finite-sample performance of the proposed HECM algorithm and compare it with three existing solutions, including \texttt{Kmeans} which applies K-means clustering directly to the vectorized tensor samples, the dynamic tensor clustering method (referred to as \texttt{DTC}) proposed by Sun and Li (2019) and the doubly enhanced EM algorithm.
regulates the signal strength in tensor mean estimation. We set the covariance matrices 

\[ \beta_R \]

and the true (TPR) and false positive rates (FPR) in recovering the nonzero entries, i.e.,

\[ \text{TPR} = \frac{1}{K} \sum_k \sum_{i<j} \frac{1}{\sum_{i<j} 1} \frac{1(\hat{\Omega}_{k,m}(i,j) \neq 0, \hat{\Omega}_{k,m}(i,j) \neq 0)}{\sum_{i<j} 1(\hat{\Omega}_{k,m}(i,j) = 0, \hat{\Omega}_{k,m}(i,j) = 0)}, \]

\[ \text{FPR} = \frac{1}{K} \sum_k \sum_{i<j} \frac{1}{\sum_{i<j} 1} \frac{1(\Omega_{k,m}(i,j) = 0, \hat{\Omega}_{k,m}(i,j) \neq 0)}{\sum_{i<j} 1(\Omega_{k,m}(i,j) = 0, \hat{\Omega}_{k,m}(i,j) = 0)}. \]

The CE measures the probability of disagreement between the estimated and true cluster labels, and it is commonly considered for evaluating clustering accuracy (Sun and Li, 2019). The CME and COVME measure the estimation errors for the tensor means and covariance matrices, respectively, while TPR and FPR evaluate the selection accuracy in recovering nonzero entries in the precision matrices.

We consider the third-order case \((M = 3)\) and generate \(n\) tensor samples \(X_i \in \mathbb{R}^{10 \times 10 \times 10}, \) \(i \in [n],\) from the model in (2) with four equal-sized clusters. Write \(\beta_{ii}^{*3} = \beta_{ii}^* \circ \beta_{ii}^* \circ \beta_{ii}^*. \) We let rank \(R = 4\) and set \(U_1^*, U_2^*, U_3^*\) and \(U_4^*\) as

\[
U_1^* = \sum_{i=1}^{4} \beta_{ii}^{*3}, \quad U_2^* = \sum_{i=1}^{4} (-1)^{i-1} \beta_{ii}^{*3}, \quad U_3^* = -\sum_{i=1}^{4} \beta_{ii}^{*3}, \quad U_4^* = \sum_{i=1}^{4} (-1)^{i} \beta_{ii}^{*3},
\]

where \(\beta_{11}^* = (\mu, \mu, \mu, 0, \ldots, 0), \) \(\beta_{22}^* = (0, 0, \mu, \mu, \mu, 0, \ldots, 0), \) \(\beta_{33}^* = (0, \ldots, 0, \mu, \mu, 0, 0, 0)\) and \(\beta_{44}^* = (0, \ldots, 0, \mu, \mu, 0, 0, 0). \) The parameter \(\mu\) in the decomposed components controls the signal strength of these four cluster centers. That is, when \(\mu\) is large, the four clusters are more separated and hence the clustering task is less challenging. Meanwhile, \(\mu\) also regulates the signal strength in tensor mean estimation. We set the covariance matrices \(\Sigma_{k,m}, k \in [K], m \in [M],\) as

\[ \Sigma_{k,m} = \begin{pmatrix} \Sigma_0(\nu) & 0 \\ 0 & \Sigma_0(\nu) \end{pmatrix}, \quad \Sigma_0(\nu) = \nu 11^\top + (1 - \nu) I, \]
Table 1: Clustering error (CE), cluster mean error (CME), cluster covariance error (COVME), true positive rate (TPR) and false positive rate (FPR) of four methods with varying cluster mean parameter $\mu$ and cluster covariance parameter $\nu$. HECM is the proposed algorithm; Kmeans applies K-means clustering directly to the vectorized tensor samples; DTC is proposed by Sun and Li (2019); and DEEM is proposed by Mai et al. (2021).

We fix $n = 400$ and set $\mu = 0.8, 0.85$ and $\nu = 0.3, 0.6$. Table 1 reports the mean evaluation criteria with the standard errors in the parentheses, based on 50 data replications. Since Kmeans and DTC do not give estimates for the covariance matrices or precision matrices directly, we first obtain cluster membership from these algorithms and then estimate the covariance within each cluster. As Kmeans, DTC and DEEM do not consider the sparsity for the covariance matrices or precision matrices, the TPR and FPR are not reported for these three methods. Our proposed HECM method is seen to achieve the best performance among all competing methods, in terms of both estimation accuracy and clustering accuracy. We see that clustering errors (CE) from all three methods decrease as $\mu$ increases, as the cluster centers become more separated when $\mu$ is large. The clustering performance from Kmeans is

| $\mu$ | $\nu = 0.3$ | $\nu = 0.6$ |
|-------|------------|------------|
|       | CE        | CME        | COVME     | TPR       | FPR       |
|       | HECM      | Kmeans    | DTC       | DEEM      | HECM      | Kmeans    | DTC       | DEEM      | HECM      | Kmeans    | DTC       | DEEM      |
| 0.80  | $0.114(0.017)$ | $0.481(0.036)$ | $0.001(0.000)$ | $1.000(0.000)$ | $0.059(0.005)$ | $0.001(0.000)$ | $0.216(0.020)$ | $0.001(0.000)$ | $1.000(0.000)$ | $0.037(0.002)$ | $0.001(0.000)$ | $0.216(0.020)$ | $0.001(0.000)$ | $1.000(0.000)$ | $0.037(0.002)$ |
| 0.85  | $0.132(0.011)$ | $0.750(0.041)$ | $0.001(0.000)$ | $1.000(0.000)$ | $0.124(0.014)$ | $0.132(0.011)$ | $0.621(0.046)$ | $0.001(0.000)$ | $1.000(0.000)$ | $0.125(0.014)$ | $0.132(0.011)$ | $0.621(0.046)$ | $0.001(0.000)$ | $1.000(0.000)$ | $0.125(0.014)$ |
very sensitive to correlation strength $\nu$, as the standard K-means algorithm treats the data space as isotropic (Hao et al., 2017). When the covariance matrix in the mixture model is non-diagonal, the distribution within each cluster is highly non-spherical. In this case, the K-means algorithm is expected to produce an unsatisfactory clustering result. It is worth noting that the HECM enjoys a good performance even when its initialization calculated from Kmeans performs poorly. For example, when $n = 400$, $\mu = 0.8$ and $\nu = 0.6$, the CE from Kmeans is 0.365 and it is reduced to 0.132 for HECM. The DTC method does not account for correlations among variables and it assumes a different statistical model than ours (see discussions in Section 1). Therefore, its performance is not as competitive. For DEEM method, although it considers a tensor mean structure, it relies on a critical assumption that the discriminant tensors are sparse. We conjecture the unsatisfying performance of DEEM when $\nu = 0.3$ is due to the model misspecification.

Figure 4: Errors with varying sample size $n$ from the HECM method. The left panel shows errors from estimating the cluster means (CME) and the right panel shows errors from estimating the cluster covariances (COVME). The “$\times$” marks the mean error for each setting.

Finally, we note that the cluster mean error (CME) decreases as $\mu$ increases, which agrees with Theorem 2 as a larger $\mu$ implies a larger $\omega_{\text{min}}$. Furthermore, in Figure 4 we show the empirical error rates (red dotted line) of the proposed HECM with varying sample sizes when $\mu = 0.85$ and $\nu = 0.3$. Clearly, the estimation errors of HECM decrease as $n$ increases and the empirical error rates for both CME and COVME align well with the theoretical rate of $n^{-1/2}$, while all other model parameters are fixed. These again agree with our theoretical result in Theorem 2.
| Windows $T$ | HECM  | Kmeans | DTC  | DEEM |
|-------------|--------|--------|------|------|
| 1           | 24/57  | 27/57  | 26/57| NA   |
| 15          | 22/57  | 27/57  | 23/57| NA   |
| 30          | 17/57  | 27/57  | 22/57| NA   |

Table 2: Clustering errors from HECM, DTC and Kmeans in the ABIDE data.

6 Real Data Analysis

In this section, we apply our proposed method to a brain connectivity analysis using resting-state functional magnetic resonance imaging (fMRI). The data are from the Autism Brain Imaging Data Exchange (ABIDE; Di Martino et al., 2014), a study of autism spectrum disorder (ASD). The ABIDE data were obtained from multiple imaging sites. We choose to focus on the fMRI data from the University of Utah School of Medicine (USM) site, since the sample size is relatively large ($n = 57$) meanwhile not too large to apply K-means clustering to the vectorized data for comparison. The data at the USM site consist of the resting-state fMRI from 57 subjects with 22 ASD subjects and 35 normal controls. For each subject, the fMRI data are preprocessed into a three order tensor $X_i \in \mathbb{R}^{116 \times 116 \times T}$ where $T$ is the number of temporal windows. More details of data preparation are included in Section E.

We cluster the subjects using the proposed HECM algorithm and then compare the estimated clustering resulting with each subject’s diagnosis status, which is treated as the true label in this analysis. For a fair comparison, we fix the number of clusters as $K = 2$ in all methods and compare the clustering results with the true diagnosis status. We report the clustering error of our method in Table 2, along with errors from Kmeans, DTC and DEEM. It is seen that HECM outperforms the Kmeans and DTC as it gives a smaller clustering error. For DEEM, the algorithm stops after one iteration and assigns all data points to one cluster, which gives a degenerated solution. This issue persists even when we use the true label to initialize DEEM. It is also interesting to see that both HECM and DTC give smaller errors when the number of windows increases from $T = 1$ to $T = 30$. This gain in clustering accuracy when increasing the number of temporal windows suggests that the underlying brain connectivity in this study is likely time-varying rather than static.
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Supplementary Materials of “Heterogeneous Tensor Mixture Models in High Dimensions”

In the supplement, we first state some important technical lemmas, then provide the detailed proofs of the main theorems, followed by the proofs of the technical lemmas. We conclude with computational details of our main algorithm and additional results of real data analysis.

A Implementation details

Update in the M-step. Recall \( \prod_{m \in [M]}^{\otimes} \beta_{k,r,m} = \beta_{k,r,1} \circ \cdots \circ \beta_{k,r,M} \) and \( \prod_{m \in [M]}^{\otimes} \Omega_{k,m} = \Omega_{k,M} \otimes \cdots \otimes \Omega_{k,1} \). Let \( a_{k,r,m}^{(t+1)} = \text{vec}(\prod_{m' \in [M]}^{\otimes} \beta_{k,r,m'} \circ \prod_{m' > m}^{\otimes} \beta_{k,r,m}), b_{k,r,m}^{(t+1)} = \sum_{r' < r} (\beta_{k,r',m}^{(t+1)} \beta_{k,r,m}^{(t)}) \omega_{k,r'}^{(t+1)} \text{vec}(\prod_{m' \neq m}^{\otimes} \beta_{k,r,m'}) \) and

\[
n_k^{(t)} = \sum_{i=1}^n \tau_{ik}(\Theta)^{(t+1)}.\]

We use \( \beta_{k,r,m}(j) \) to denote the \( j \)-th element of \( \beta_{k,r,m} \) and \( \Omega_{k,m}(j,l) \) to denote the \( (j,l) \)-th element of \( \Omega_{k,m} \). The unconstrained (i.e., without the unit-norm constraint) update of \( \beta_{k,r,m} \) is given in the following proposition with its proof delayed to Section D11.

**Proposition 2.** Let \( \tilde{\beta}_{k,r,m}^{(t+1)} = \text{arg max}_{\beta_{k,r,m}} Q_{\eta,T}(\beta_{k,r,m}, \Theta_{-\beta_{k,r,m}}|\Theta^{(t)}) - \lambda_0^{(t+1)} \| \beta_{k,r,m} \|_1 \). We have, for each \( k, r, m \),

\[
\tilde{\beta}_{k,r,m}^{(t+1)}(j) = \begin{cases} 
\frac{g_{k,r,m}^{(t+1)}(j) - n_k^{(t+1)} \text{sign}(\tilde{\beta}_{k,r,m}^{(t+1)}(j))}{n_k^{(t+1)} C_{k,r,m}^{(t+1)}(j,j)} \Omega_{k,m}^{(t+1)}(j,j) & \text{if } |h_{k,r,m}^{(t+1)}(j)| > \lambda_0^{(t+1)}, \\
0 & \text{otherwise,}
\end{cases}
\]

where

\[
g_{k,r,m}^{(t+1)}(j) = \sum_{i=1}^n \tau_{ik}(\Theta)^{(t)} g_{k,r,m}^{(t+1)}(j) - \sum_{l=1}^{d_m} \Omega_{k,m}^{(t)}(j,l) \beta_{k,r,m}^{(t)}(l) + \beta_{k,r,m}^{(t)},
\]

\[
h_{k,r,m}^{(t+1)}(j) = \sum_{i=1}^n \tau_{ik}(\Theta)^{(t)} g_{k,r,m}^{(t+1)}(j) - C_{k,r,m}^{(t+1)}(j,j) \sum_{l \neq j} \Omega_{k,m}^{(t)}(j,l) \beta_{k,r,m}^{(t)}(l).
\]

Here \( g_{k,r,m}^{(t)} = \Omega_{k,m}^{(t)}(X_{i-r}^{(t+1)})_{m}^{\otimes} \Omega_{k,m}^{(t)}(b_{k,r,m}^{(t+1)}) C_{k,r,m}^{(t+1)} = \omega_{k,r}^{(t)} a_{k,r,m}^{(t+1)^T} \sum_{m' \neq m}^{\otimes} \Omega_{k,m'}^{(t)} b_{k,r,m'}^{(t+1)}, \)

and \( X_{i-r}^{(t+1)} = X_i - \sum_{r' < r} \omega_{k,r'}^{(t+1)} \beta_{k,r',m}^{(t+1)} - \sum_{r' > r} \omega_{k,r'}^{(t)} \prod_{m \in [M]}^{\otimes} \beta_{k,r',m}. \)
**Stopping rule.** In Algorithm 1, the maximum number of iterations $T$ needs to be specified. In our implementation, we set $T = 20$. In practice, it is recommended to run Algorithm 1 when the distance between $\Theta^{(t)}$ and $\Theta^{(t-1)}$ becomes less than a pre-specified tolerance level. The tolerance level is set to $10^{-4}$ in our experiments and we find the algorithm usually converges within 10 iterations.

**Parameter tuning.** The proposed Algorithm 1 involves a number of tuning parameters, including the rank $R$ and sparsity parameters $\lambda^{(t)}_0$, $\lambda^{(t)}_1$, ..., $\lambda^{(t)}_M$. Having different parameters $\lambda^{(t)}_0$, $\lambda^{(t)}_1$, ..., $\lambda^{(t)}_M$ in each iteration is due to theoretical considerations, as the estimation error, which determines the level of regularization, changes at each iteration. Such an iterative regularization has also been considered in Yi and Caramanis (2015); Mai et al. (2021). In practice, tuning for these parameters at each HECM iteration can significantly increase the computational cost. For practical considerations, we fix $\lambda^{(t)}_0 = \lambda_0$ and $\lambda^{(t)}_1 = \ldots = \lambda^{(t)}_M = \lambda_1$ in our experiments. We note that this simplification is commonly employed in high-dimensional EM algorithms (e.g., Mai et al., 2021) and is found to give a satisfactory performance in our experiments. To tune $R$, $\lambda_0$, and $\lambda_1$, we consider minimizing the following extended BIC selection criterion (Chen and Chen, 2008),

$$eBIC = -2 \log \left\{ \sum_{k=1}^{K} \pi_k f_{k}(X_i|\theta_k) \right\} + \left\{ \log(n) + \frac{1}{2} \log(p_{\Theta}) \right\} s_{\text{tot}}, \quad (A1)$$

where $p_{\Theta}$ is the total number of parameters in $\Theta$ and $s_{\text{tot}}$ is the total number of non-zero parameters for a given $\Theta$. To further speed up the computation, we tune parameters $R$, $\lambda_0$ and $\lambda_1$ sequentially. That is, among the set of values for $R$, $\lambda_0$, $\lambda_1$, we first tune $R$ while $\lambda_0$, $\lambda_1$ are fixed at their minimum values. Given the selected $R$, we then tune $\lambda_0$ while $\lambda_1$ is fixed at its minimum. Finally, given the selected $R$, $\lambda_0$, we tune $\lambda_1$. Such a sequential tuning procedure enjoys a good performance and is commonly employed in high dimensional problems (Danaher et al., 2014; Sun and Li, 2019; Chi et al., 2020; Zhou et al., 2020).
B Technical Lemmas

B1 Supporting lemmas

We first state a number of supporting technical lemmas. Proofs of Lemmas S1 and S8 are delayed to Sections D1 and D2, respectively.

Lemma S1. If a tensor $\mathbf{U} \in \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_M}$ admits the following decomposition

$$\mathbf{U} = \omega \cdot \beta_1 \circ \cdots \beta_M,$$

then the mode-$m$ matricization of $\mathbf{U}$ can be written as

$$\mathbf{U}_{(m)} = \omega \cdot \beta_m \circ \text{vec}(\beta_1 \circ \cdots \beta_{m-1} \circ \beta_{m+1} \cdots \beta_M)^\top.$$

Lemma S2 (Lemma 2.7.7 of Vershynin (2018)). Let $X, Y$ be two sub-Gaussian random variables. Then $Z = X \cdot Y$ is sub-exponential random variable. Moreover, there exists a constant $C$ such that

$$\|Z\|_{\psi_1} \leq C \|X\|_{\psi_2} \cdot \|Y\|_{\psi_2}.$$

Lemma S3 (Remark 5.18 of Vershynin (2010)). Let $X$ be sub-Gaussian random variable and $Y$ be sub-exponential random variables. Then $X - \mathbb{E}(X)$ is also sub-Gaussian; $Y - \mathbb{E}(Y)$ is also sub-exponential. Moreover, we have

$$\|X - \mathbb{E}(X)\|_{\psi_2} \leq 2\|X\|_{\psi_2}, \|Y - \mathbb{E}(Y)\|_{\psi_1} \leq 2\|Y\|_{\psi_1}.$$

Lemma S4 (Theorem 2.6.2 of Vershynin (2018)). Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. centered sub-Gaussian random variables with $\|X_1\|_{\psi_2} \leq K$. Then for every $t \geq 0$, we have

$$\mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^{n} X_i\right| \geq t\right) \leq e \cdot \exp\left(-\frac{C n t^2}{K^2}\right),$$

where $C$ is an absolute constant.

Lemma S5 (Corollary 2.8.3 of Vershynin (2018)). Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. centered sub-exponential random variables with $\|X_1\|_{\psi_1} \leq K$. Then for every $t \geq 0$, we have

$$\mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^{n} X_i\right| \geq t\right) \leq 2 \cdot \exp\left(-C \min\left\{\frac{t^2}{K^2}, \frac{t}{K}\right\} n\right),$$

where $C$ is an absolute constant.
Lemma S6 (Theorem 2.2.6 of Vershynin (2018)). Hoeffding’s inequality suppose $X_1, X_2, \ldots, X_n$ are independent random variable, $a_i \leq X_i \leq b_i$, then we can have
\[
\mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^{n} (X_i - \mathbb{E}X_i) \right| \geq \epsilon\right) \leq 2 \exp\left\{-\frac{2n^2 \epsilon^2}{\sum_{i=1}^{n} (b_i - a_i)^2}\right\}.
\]
Moreover, if $a_i = 0$ and $b_i = 1$, then we have
\[
\mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^{n} (X_i - \mathbb{E}X_i) \right| \geq \epsilon\right) \leq 1 - 2e^{-2n\epsilon^2}.
\]

Lemma S7 (Theorem 5.1.4 of Vershynin (2018)). Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, where $x_1, \ldots, x_n \in \mathbb{R}$ are i.i.d. with standard norm. Consider function $f : \mathbb{R}^n \to \mathbb{R}$ with Lipschitz constant $L$, that is, for any vectors $v_1, v_2 \in \mathbb{R}^n$, there exists $L > 0$ such that $|f(v_1) - f(v_2)| \leq L \|v_1 - v_2\|_2$. Then, for any $t > 0$, we have
\[
\mathbb{P}\{|f(x) - \mathbb{E}(f(x))| > t\} \leq 2 \exp\left(-\frac{t^2}{2L^2}\right).
\]

Lemma S8. For $Y \sim \mathcal{N}_{T\{(0_{d_1 \times d_2}; I_{d_1}, I_{d_2})\}}$ and any matrix $D \in \mathbb{R}^{d_2 \times d_2}$, it holds that $\mathbb{E}(YDY^\top) = \text{tr}(D)I_{d_1}$.

**B2 Key technical lemmas**

Next, we introduce several key technical lemmas used in the proof of Theorem 1-2. The proofs of Lemmas 1b, 2b, 3b, 4b, 5b, 6b, 7b and 8b are delayed to Sections D3, D4, D5, D6, D7, D8, D9 and D10, respectively.

We start with some new notation. Write $S_\alpha(\Omega^*_{k,m}) = \left\{\Omega_{k,m} | \frac{\|\Omega_{k,m} - \Omega^*_{k,m}\|_2}{\sigma_{\min}(\Omega^*_{k,m})} \leq \alpha\right\}$ and $c_\alpha = \left(\frac{C_{0}\min_{(R-1)\omega_{\max}}}{(R-1)\omega_{\max}}\right)^{\frac{1}{M-1}}$, where $C_0$ is as defined in Condition 2. Define the population Q-function $Q(\Theta'|\Theta)$ as
\[
Q(\Theta'|\Theta) = \mathbb{E}\left[\sum_{k=1}^{K} \tau_{ik}(\Theta)\{\log(\pi_k') + \log f_k(x_{i} | \theta_k')\}\right].
\]
(A2)

Let $P_1(\beta_{k,m}) = \|\beta_{k,m}\|_1$, $P_2(\Omega_{k,m}) = \|\Omega_{k,m}\|_{1,\text{off}}$ and let $P_1^*, P_2^*$ be the dual norms of $P_1$, $P_2$, respectively.

**Lemma 1b** (Restricted Strong Concavity for $Q_{n/T}$). Suppose $R = 1$ and Conditions 1 and 3 hold. Let $\Theta, \bar{\Theta} \in B_2(\Theta^\ast)$ and satisfies $\Omega_{k,m}, \bar{\Omega}_{k,m} \in S_{\frac{1}{2}}(\Omega^*_{k,m})$ for all $k,m$. For any
\( \Theta' \) and \( \Theta'' \) satisfying \( \Omega'_{k,m}, \Omega''_{k,m} \in S_1(\Omega^*_{k,m}) \) for all \( k, m \), it holds with probability at least 
\( 1 - 1/(\log(nd))^2 \) that,

\[
Q_{n/T}(\beta''_{k,m}, \Theta - \beta_{k,m}|\Theta) - Q_{n/T}(\beta'_{k,m}, \Theta - \beta_{k,m}|\Theta) - \langle \nabla_{\beta_{k,m}} Q_{n/T}(\beta'_{k,m}, \Theta - \beta_{k,m}|\Theta), \beta''_{k,m} - \beta'_{k,m} \rangle \\
\leq - \frac{\gamma_0}{2} \left\| \beta''_{k,m} - \beta'_{k,m} \right\|_2^2 ,
\]

\[
Q_{n/T}(\omega''_{k}, \Theta - \omega_k | \Theta) - Q_{n/T}(\omega'_{k}, \Theta - \omega_k | \Theta) - \langle \nabla_{\omega_{k}} Q_{n/T}(\omega'_{k}, \Theta - \omega_k | \Theta), \omega''_{k} - \omega'_{k} \rangle \\
\leq - \frac{\gamma''_0}{2} \left| \omega''_{k} - \omega'_{k} \right|^2 ,
\]

\[
Q_{n/T}(\Omega''_{k,m}, \Theta - \Omega_{k,m} | \Theta) - Q_{n/T}(\Omega'_{k,m}, \Theta - \Omega_{k,m} | \Theta) - \langle \nabla_{\Omega_{k,m}} Q_{n/T}(\Omega'_{k,m}, \Theta - \Omega_{k,m} | \Theta), \Omega''_{k,m} - \Omega'_{k,m} \rangle \\
\leq - \frac{\gamma_m}{2} \left\| \Omega''_{k,m} - \Omega'_{k,m} \right\|_F^2 ,
\]

(A3)

where \( \gamma_0 = \frac{c_0 \omega^2}{4 \omega^2} \), \( \gamma''_0 = c_0 (\phi_1/2)^M \) and \( \gamma_m = c_0 d_{\omega}^2 (6\phi_2)^{-2} \) for some constant \( c_0 > 0 \).

**Lemma 2b** (Gradient Stability for Q). Suppose \( R = 1 \) and Condition 3 holds for \( \gamma > 0 \). Let \( \Theta, \Theta \in B_2(\Theta^*) \) and satisfies \( \Omega_{k,m}, \bar{\Omega}_{k,m} \in S_2(\Omega^*_{k,m}) \) for all \( k, m \). For any \( \Theta' \in B_2(\Theta^*) \) satisfying \( \Omega'_{k,m} \in S_2(\Omega^*_{k,m}) \) for all \( k, m \), it holds that

\[
\left\| \nabla_{\beta_{k,m}} Q(\beta'_{k,m}, \Theta - \beta_{k,m}|\Theta) - \nabla_{\beta_{k,m}} Q(\beta''_{k,m}, \Theta - \beta_{k,m}|\Theta^*) \right\|_2 \leq \tau_0 \cdot D(\Theta, \Theta^*),
\]

\[
\left\| \nabla_{\omega_k} Q(\omega'_k, \Theta - \omega_k|\Theta) - \nabla_{\omega_k} Q(\omega''_k, \Theta - \omega_k|\Theta^*) \right\|_2 \leq \tau''_0 \cdot D(\Theta, \Theta^*),
\]

\[
\left\| \nabla_{\Omega_{k,m}} Q(\Omega'_{k,m}, \Theta - \Omega_{k,m}|\Theta) - \nabla_{\Omega_{k,m}} Q(\Omega''_{k,m}, \Theta - \Omega_{k,m}|\Theta^*) \right\|_F \leq \tau_1 \cdot D(\Theta, \Theta^*). \tag{A4}
\]

where \( \tau_0 = \frac{\gamma}{12\sqrt{K(R+1)(M+1)}}, \tau''_0 = \frac{\gamma_{\max}}{12\sqrt{K(R+1)(M+1)}} \) and \( \tau_1 = \frac{\gamma_d}{12\sqrt{K(R+1)(M+1)}} \).

**Lemma 3b.** Suppose \( R = 1 \) and Condition 1 and 4 hold. Let \( \Theta, \Theta \in B_2(\Theta^*) \) and satisfies \( \Omega_{k,m}, \bar{\Omega}_{k,m} \in S_2(\Omega^*_{k,m}) \) for all \( k, m \). For any \( \Theta' \in B_2(\Theta^*) \) satisfying \( \Omega'_{k,m} \in S_2(\Omega^*_{k,m}) \) for all \( k, m \), it holds with probability at least \( 1 - K(2K+1)/(\log(nd))^2 \) that

\[
\left\| \nabla_{\beta_{k,m}} Q_{n/T}(\beta'_{k,m}, \Theta - \beta_{k,m}|\Theta) - \nabla_{\beta_{k,m}} Q_{n/T}(\beta''_{k,m}, \Theta - \beta_{k,m}|\Theta) \right\|_{p^*} \leq c_1 \omega_{\max} \sqrt{T \frac{\log d}{n}}, \tag{A5}
\]

with probability at least \( 1 - K/(\log(nd))^2 \) that

\[
|\nabla_{\omega_k} Q_{n/T}(\omega'_k, \Theta - \omega_k|\Theta) - \nabla_{\omega_k} Q_{n/T}(\omega''_k, \Theta - \omega_k|\Theta)| \leq c''_1 \omega_{\max} \sqrt{T \frac{\log \log(nd)}{n}}, \tag{A6}
\]

and with probability at least \( 1 - K(8K+2)/(\log(nd))^2 \) that

\[
\left\| \nabla_{\Omega_{k,m}} Q_{n/T}(\Omega'_{k,m}, \Theta - \Omega_{k,m}|\Theta) - \nabla_{\Omega_{k,m}} Q_{n/T}(\Omega''_{k,m}, \Theta - \Omega_{k,m}|\Theta) \right\|_{p^*} \leq c_2 \frac{d}{d_m} \sqrt{T \frac{\log d}{n}}, \tag{A7}
\]

where \( c_1, c''_1, c_2 \) are positive constants.
Lemma 4b (One-step Contraction). Suppose $R = 1$ and Conditions 1-4 hold with $\gamma d_{\text{max}} \leq C_1$ for some constant $C_1 > 0$. Let $\lambda_0^{(1)} = 4\epsilon_0 + \tau_0 \frac{D(\Theta(0), \Theta^*)}{\sqrt{n}}$, $\lambda_m^{(1)} = 4\epsilon_m + 3\tau_1 \frac{D(\Theta(0), \Theta^*)}{2\sqrt{2s_2 + d_m}}$, where $\tau_0$ and $\tau_1$ are as defined in (A4) and $\epsilon_0 = c_1\omega_{\text{max}}\sqrt{\log d \cdot T/n}$ and $\epsilon_m = c_2(d/d_m)\sqrt{\log d \cdot T/n}$ for $c_1$ in (A5) and $c_2$ in (A7). The estimator of Algorithm 1 after one-step update satisfies with probability at least $1 - C_3/\{\log(nd)\}^2$,

$$D(\Theta^{(1)}, \Theta^*) \leq \rho D(\Theta^{(0)}, \Theta^*) + \frac{C_2}{1 - \rho} \left\{ \frac{1}{\omega_{\min}} \sqrt{T \frac{s_1 \log d}{n}} + \max_m \sqrt{T \frac{(s_2 + d_m) \log d}{ndm}} \right\},$$

where $C_2, C_3 > 0$ are constants and $\rho$ given in (A78) satisfies $0 < \rho \leq 1/3$.

Lemma 5b. Suppose $R > 1$ and Conditions 1 and 3 hold. Let $\Theta, \bar{\Theta} \in B_{c_a}(\Theta^*)$ and satisfies $\Omega_{k,m}, \bar{\Omega}_{k,m} \in \mathcal{S}_2(\Omega^*_{k,m})$ for all $k, m$. For any $\Theta'$ and $\Theta''$, it holds with probability at least $1 - 1/\{\log(nd)\}^2$ that

$$Q_{n/T}(\beta''_{k,r,m}; \bar{\Theta}_{k,r,m} | \Theta) - Q_{n/T}(\beta'_{k,r,m}; \bar{\Theta}_{k,r,m} | \Theta)
- \langle \nabla_{\beta_{k,r,m}} Q_{n/T}(\beta'_{k,r,m}; \bar{\Theta}_{k,r,m} | \Theta), \beta'_{k,r,m} - \beta''_{k,r,m} \rangle \leq -\frac{\gamma_0'}{2} \| \beta''_{k,r,m} - \beta'_{k,r,m} \|_2^2,$$

where $\gamma_0' = c_0(\phi_1/2)^{M-1} \{(1 - c_\alpha)^2\omega_{\min}^2 - (1 + c_\alpha)^2\omega_{\max}^2(R - 1)R(\xi + 2c_\alpha + c_\alpha^2)^{M+1}\}$.

Lemma 6b. Suppose $R > 1$ and Condition 3 holds for $\gamma > 0$. Let $\Theta, \bar{\Theta} \in B_{c_a}(\Theta^*)$ and satisfies $\Omega_{k,m}, \bar{\Omega}_{k,m} \in \mathcal{S}_2(\Omega^*_{k,m})$ for all $k, m$. For any $\Theta' \in B_{c_a}(\Theta^*)$, it holds that

$$\| \nabla_{\beta_{k,r,m}} Q(\beta'_{k,r,m}; \bar{\Theta}_{k,r,m} | \Theta) - \nabla_{\beta_{k,r,m}} Q(\beta'_{k,r,m}; \bar{\Theta}_{k,r,m} | \Theta^*) \|_2 \leq \tau'_0 \cdot D(\Theta, \Theta^*),$$

where $\tau'_0 = \{1 + (R - 1)(\xi + 2c_\alpha + c_\alpha^2)\} \tau_0$ and $\tau_0$ is as defined in (A4).

Lemma 7b. Suppose $R > 1$ and Condition 1 and Condition 4 hold. Let $\Theta, \bar{\Theta} \in B_{c_a}(\Theta^*)$ and satisfies $\Omega_{k,m}, \bar{\Omega}_{k,m} \in \mathcal{S}_2(\Omega^*_{k,m})$ for all $k, m$. For any $\Theta' \in B_{c_a}(\Theta^*)$, it holds with probability at least $1 - K(2K + 1)/\{\log(nd)\}^2$ that

$$\| \nabla_{\beta_{k,r,m}} Q_{n/T}(\beta'_{k,r,m}; \bar{\Theta}_{k,r,m} | \Theta) - \nabla_{\beta_{k,r,m}} Q(\beta'_{k,r,m}; \bar{\Theta}_{k,r,m} | \Theta) \|_{\mathcal{P}_1} \leq c'_1 \omega_{\max} \sqrt{T \frac{\log d}{n}},$$

with $c'_1$ is some positive constant.
Lemma 8b. Suppose $R > 1$, Conditions 1-4 hold with $\gamma d_{\text{max}} \leq C_1$ for some constant $C_1 > 0$ and $R \mathcal{C}_m \lesssim (\log d)^{-1}$. Let $\lambda_0^{(1)} = 4\epsilon_0' + \tau_0' D(\Theta^{(0)})$, $\lambda_m^{(1)} = 4\epsilon_m + 3\tau_1' D(\Theta^{(0)})$, where $\tau_0'$ is as defined in (A9), $\epsilon_0' = C_1\omega_{\text{max}} \sqrt{\log d \cdot T/n}$ and $\tau_1$ and $\epsilon_m$ are as defined in Lemma 4b. The estimator of Algorithm 1 after one-step update satisfies with probability at least $1 - C_3/(\log(nd))^2$,

$$D(\Theta^{(1)}, \Theta^*) \leq \rho_R D(\Theta^{(0)}, \Theta^*) + C_2' \left\{ \frac{1}{\omega_{\text{min}}} \sqrt{T s_1 \log d/n} + \max_m \sqrt{T(s_2 + d_m) \log d/nd_m} \right\},$$

where $C_2', C_3 > 0$ are constants and $\rho$ given in (A90) satisfies $\rho \leq \rho_R \leq 1/2$ with $\rho$ given in Lemma 4b.

C Proof of Main Results

C1 Proof of Proposition 1

Balakrishnan et al. (2017) considers $X \sim \frac{1}{2} \mathcal{N}(\mu^*, \sigma^2 1_d) + \frac{1}{2} \mathcal{N}(-\mu^*, \sigma^2 1_d)$ and $\mu \in \{ \mu | \| \mu - \mu^* \|_2 \leq \frac{1}{4} \| \mu^* \|_2 \}$. They assumed $\sigma^2$ is known, and thus, without loss of generality, we let $\sigma^2 = 1$ in this proof. Under this model, the parameter vector $\Theta$ reduces to $\mu$. Note that in this case of $M = 1$, we do not need to normalize $\mu$ as there is no identifiability issue.

Suppose $k = 2$ and $l = 1$. For $\mu, \mu' \in \{ \mu | \| \mu - \mu^* \|_2 \leq \frac{1}{4} \| \mu^* \|_2 \}$, we have

$$W_{ikl}^2 = \| X_i - \mu \|_2^2 \| X_i + \mu' \|_2^2.$$ 

By the definition of $\tau_k(\mu)$, we have

$$\tau_{i1}(\mu) \tau_{i2}(\mu) = \frac{1}{\{ \exp(-\eta_1(\mu) + \eta_2(\mu)) + \exp(\eta_1(\mu) - \eta_2(\mu)) \}^2}$$

with $\eta_1(\mu) = \frac{1}{4} \| X_i - \mu \|_2^2$ and $\eta_2(\mu) = \frac{1}{4} \| X_i + \mu \|_2^2$.

Define $A_1 = \{ X_i : (1 - c)\eta_1(\mu) \geq \eta_2(\mu) \}, A_2 = \{ X_i : (1 - c)\eta_2(\mu) \geq \eta_1(\mu) \}$ and $A_3 = (A_1 \cup A_2)^c$, where $A^c$ is the complement of $A$ and $c \in (0, 1)$. Then we have

$$\mathbb{E} \left\{ W_{i21} \tau_{i1}(\mu) \tau_{i2}(\mu) \right\}^2 = \mathbb{E} \left[ \{ W_{i21} \tau_{i1}(\mu) \tau_{i2}(\mu) \}^2 | A_1 \right] \mathbb{P}(A_1) + \mathbb{E} \left[ \{ W_{i21} \tau_{i1}(\mu) \tau_{i2}(\mu) \}^2 | A_2 \right] \mathbb{P}(A_2)$$

$$+ \mathbb{E} \left[ \{ W_{i21} \tau_{i1}(\mu) \tau_{i2}(\mu) \}^2 | A_3 \right] \mathbb{P}(A_3).$$
Let \( \alpha_0 = \|\mu^*\|_2^2/16 \) and \( c = 1/2 \). In what follows, we discuss the upper bounds of terms (i), (ii) and (iii) respectively.

**Part (i).** Conditioning on \( A_1 \), it is seen that \( \tau_{i1}(\mu)\tau_{i2}(\mu) \leq \exp(-2c\eta_1(\mu)) \). Moreover, by noting \( \eta_1(\mu) > \eta_2(\mu) \) under \( A_1 \) and \( \|\mu' - \mu\|_2^2 \leq 4\alpha_0 \), it holds that

\[
4\eta_2(\mu') = \|X_i + \mu'|_2^2 \leq 2 \|X_i + \mu\|_2^2 + 2 \|\mu' - \mu\|_2^2 \leq 8\eta_1(\mu) + 8\alpha_0.
\]

Correspondingly, assuming \( \|u^*\|_2 \) is sufficiently large (e.g. \( \|u^*\|_2 \geq 8/3 \)), we have

\[
\mathbb{E} \left[ \{W_{i21}\tau_{i1}(\mu)\tau_{i2}(\mu)\}^2 | A_1 \right] \mathbb{P}(A_1) \leq \frac{1}{2} \mathbb{E} \left\{ \frac{\eta_1(\mu)\eta_2(\mu')}{{\exp(2\eta_1(\mu))}} | A_1 \right\} + \frac{\alpha_0\eta_1(\mu)}{{\exp(2\eta_1(\mu))}} | A_1 \right\} \leq \frac{\|\mu\|_2^4 + 4\alpha_0 \|\mu\|_2^2}{16} \exp(- \|\mu\|_2^2/2),
\]

where the last inequality is due to \( \eta_1(\mu) \geq 0 \) \( \|u^*\|_2^2 \) as \( 4\|\mu - X_i\|_2^2 \geq 2\|\mu - X_i\|_2^2 + 2\|X_i + \mu\|_2^2 \geq 2\mu^2 \) under \( A_1 \) and \( \|\mu\|_2 \geq \|\mu^*\|_2 - \|\mu - \mu^*\|_2 \geq \frac{3}{4}\|\mu^*\|_2 \), and \( \sup_{t \geq t^*} \exp(-at) = \exp(-at^*) \), \( \sup_{t \geq t^*} t^2 \exp(-at) = (t^*)^2 \exp(-at^*) \) when \( t^* \geq 2/a \).

**Part (ii).** Using a similar argument as in Part (i), we can get

\[
\mathbb{E} \left[ \{W_{i21}\tau_{i1}(\mu)\tau_{i2}(\mu)\}^2 | A_2 \right] \mathbb{P}(A_2) \leq \frac{\|\mu\|_2^4 + 4\alpha_0 \|\mu\|_2^2}{16} \exp(- \|\mu\|_2^2/2).
\]

**Part (iii).** Define \( B_j = \{X_i | (j-1)/2 \leq \eta_1(\mu) \leq j/2\} \) for \( j = 1, 2, \ldots \). It then holds that \( A_3 = \bigcup_{j=1}^\infty A_3 \cap B_j \). Conditioning on \( A_3 \cap B_j \), it is seen that \( \eta_1(\mu) \leq j/2\) and

\[
4\eta_2(\mu') = \|X_i + \mu'|_2^2 \leq 2\|X_i + \mu\|_2^2 + 2\|\mu' + \mu\|_2^2 \leq (8j + 10)\|\mu^*\|_2^2,
\]

where the last inequality holds due to \( \|\mu' + \mu\|_2^2 \leq 2\|\mu + \mu^*\|_2^2 + 2\|2\mu^*\|_2^2 \leq 10\|\mu^*\|_2^2 \). As \( \tau_{i1}(\mu)\tau_{i2}(\mu) \leq \frac{1}{4} \), we can write

\[
\mathbb{E} \left[ \{W_{i21}\tau_{i1}(\mu)\tau_{i2}(\mu)\}^2 | A_3 \cap B_j \right] \mathbb{P}(A_3 \cap B_j) \leq \frac{j(8j + 10)}{64} \|u^*\|_2^4 \times \mathbb{P}(A_3 \cap B_j). \quad (A11)
\]

Next, we bound \( \mathbb{P}(A_3 \cap B_j) \) for a given \( j \). By the definition of \( A_1 \) and \( A_2 \), we have

\[
A_3 = \{X : \eta_2(\mu)/2 \leq \eta_1(\mu) \leq 2\eta_2(\mu)\}
\]

\[
= \{X : \eta_1(\mu)/2 \leq \eta_2(\mu) \leq 2\eta_1(\mu)\}.
\]
We also have \( \eta_1(\mu) + \eta_2(\mu) \geq \frac{1}{2} ||\mu||_2^2 \), as \( 2 ||\mu||_2^2 \leq ||\mu - X_i||_2^2 + ||X_i + \mu||_2^2 \). With \( \eta_2(\mu) \leq 2\eta_1(\mu) \), we obtain that \( ||X_i - \mu||_2^2 \geq \frac{\eta}{2} ||\mu||_2^2 \). Similarly, \( ||X_i + \mu||_2^2 \geq \frac{\eta}{2} ||\mu||_2^2 \) also holds. Correspondingly, conditioning on \( A_3 \cap B_j \), we have

\[
4\eta_1(\mu) = ||X_i - \mu||_2^2 : \quad j||\mu^*||_2^2 \geq 4\eta_1(\mu) \geq \max\{2||\mu||_2^2 /3, (j-1)||\mu^*||_2^2\};
\]

\[
4\eta_2(\mu) = ||X_i + \mu||_2^2 : \quad 2j||\mu^*||_2^2 \geq 4\eta_2(\mu) \geq \max\{2||\mu||_2^2 /3, (j-1)||\mu^*||_2^2/2\}.
\]

Letting \( Z_i \) denote the latent cluster label of \( X_i \), we can then write \( \mathbb{P}(A_3 \cap B_j) = \mathbb{P}(A_3 \cap B_j | Z_i = 1)P(Z_i = 1) + \mathbb{P}(A_3 \cap B_j | Z_i = 2)P(Z_i = 2) \).

For \( \mathbb{P}(A_3 \cap B_j | Z_i = 1) \), it can be bounded as

\[
\mathbb{P}(A_3 \cap B_j | Z_i = 1) \leq \mathbb{P}\left( \left( \sqrt{\frac{2}{3}} \right) ||\mu^*||_2 \geq \frac{\sqrt{2}}{\sqrt{3}} ||\mu||_2, \frac{\sqrt{2}}{\sqrt{3}} \geq \frac{\sqrt{2}}{\sqrt{3}} \right) | Z_i = 1 \)
\]

\[
= \frac{1}{\Gamma(\frac{3}{2})} \int_{l_j||\mu^*||_2^2}^{u_j||\mu^*||_2^2} t^{\frac{d}{2}-1} e^{-t/2} dt
\]

where \( l_j = \left( \sqrt{j - 1} - \frac{1}{4} \right) \) and \( u_j = \left( \sqrt{j + 1} + \frac{1}{4} \right) \) and the second inequality holds due to \( ||\mu - \mu^*||_2 \leq \frac{1}{2} ||\mu^*||_2 \). We claim that, for any \( j \), it holds for some \( a_j \in [l_j, u_j] \) that

\[
\int_{l_j||\mu^*||_2^2}^{u_j||\mu^*||_2^2} t^{\frac{d}{2}-1} e^{-t/2} dt \leq 4j||\mu^*||_2^2 \int_{a_j||\mu^*||_2^2}^{(a_j+1)||\mu^*||_2^2} t^{\frac{d}{2}-1} e^{-t/2} dt.
\]

This claim can be shown by considering three scenarios by noting \( \int t^{\frac{d}{2}-1} e^{-t/2} dt \) is proportional to the pdf of \( \chi^2_d \). As the mode of \( \int t^{\frac{d}{2}-1} e^{-t/2} dt \) is \( d - 2 \), the function is increasing in \( (0, d-2] \) and decreasing in \([d-2, \infty) \). We consider: (a) \( u_j||\mu^*||_2^2 \leq d-2 \), (b) \( l_j||\mu^*||_2^2 \geq d-2 \) and (c) \( l_j||\mu^*||_2^2 < d-2 < u_j||\mu^*||_2^2 \).

**Case (a).** In this case, noting \( l_j < u_j - 1 \), we have

\[
\int_{l_j||\mu^*||_2^2}^{u_j||\mu^*||_2^2} t^{\frac{d}{2}-1} e^{-t/2} dt = \int_{l_j||\mu^*||_2^2}^{(a_j-1)||\mu^*||_2^2} t^{\frac{d}{2}-1} e^{-t/2} dt + \int_{(a_j-1)||\mu^*||_2^2}^{u_j||\mu^*||_2^2} t^{\frac{d}{2}-1} e^{-t/2} dt. \quad (A12)
\]
Since \( t^{d-1}e^{-t/2} \) is an increasing function in \( [l_j\|\mu^*\|_2^2, u_j\|\mu^*\|_2^2] \), we can get that
\[
\int_{l_j\|\mu^*\|_2^2}^{(u_j-1)\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt \leq (u_j - l_j - 1)\|\mu^*\|_2^2 \cdot (u_j - 1)\|\mu^*\|_2^2 \frac{d}{2} e^{-(u_j-1)\|\mu^*\|_2^2/2};
\]
\[
\int_{(u_j-1)\|\mu^*\|_2^2}^{u_j\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt \geq \|\mu^*\|_2^2 \cdot (u_j - 1)\|\mu^*\|_2^2 \frac{d}{2} e^{-(u_j-1)\|\mu^*\|_2^2/2}.
\]
Combining the above results together, it then follows that
\[
\int_{l_j\|\mu^*\|_2^2}^{(u_j-1)\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt \leq (u_j - l_j - 1) \int_{(u_j-1)\|\mu^*\|_2^2}^{u_j\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt.
\]
Plugging this into (A12) and we have
\[
\int_{l_j\|\mu^*\|_2^2}^{u_j\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt \leq (u_j - l_j)\|\mu^*\|_2^2 \int_{(u_j-1)\|\mu^*\|_2^2}^{u_j\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt.
\]
Letting \( a_j = u_j - 1 \) and by noting \( (u_j - l_j)/j \leq 4 \), our claim can be verified under case (a). Case (b) can be verified similarly and we omit the detailed derivations here.

Case (c). We further consider under this case two scenarios, namely, (c.1) \( d+2 - l_j\|\mu^*\|_2^2 \geq \frac{1}{2}\|\mu^*\|_2^2 \) and \( u_j\|\mu^*\|_2^2 - d-2 \geq \frac{1}{2}\|\mu^*\|_2^2 \) and (c.2) \( d+2 - l_j\|\mu^*\|_2^2 \leq \frac{1}{2}\|\mu^*\|_2^2 \) or \( u_j\|\mu^*\|_2^2 - d-2 \leq \frac{1}{2}\|\mu^*\|_2^2 \). Under (c.1), following a similar argument as in Case (a), we have
\[
\int_{l_j\|\mu^*\|_2^2}^{u_j\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt = \int_{l_j\|\mu^*\|_2^2}^{d+2} t^{d-1}e^{-t/2}dt + \int_{d+2}^{u_j\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt
\]
\[
\leq 2(d + 2 - l_j\|\mu^*\|_2^2) \int_{d+2}^{d+2 + \frac{1}{2}\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt + 2(u_j\|\mu^*\|_2^2 - d-2) \int_{d+2}^{d+2 + \frac{1}{2}\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt
\]
\[
\leq 2(u_j - l_j)\|\mu^*\|_2^2 \int_{d+2}^{d+2 + \frac{1}{2}\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt. \tag{A13}
\]
Letting \( a_j = d+2 - l_j\|\mu^*\|_2^2 \) and by noting \( 2(u_j - l_j)/j \leq 4 \), our claim can be verified under case (c.1). Under (c.2), show the claim for \( d+2 - l_j\|\mu^*\|_2^2 \leq \frac{1}{2}\|\mu^*\|_2^2 \). The case of \( u_j\|\mu^*\|_2^2 - d-2 \leq \frac{1}{2}\|\mu^*\|_2^2 \) follows a similar argument. We have
\[
\int_{l_j\|\mu^*\|_2^2}^{u_j\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt = \int_{l_j\|\mu^*\|_2^2}^{d+2} t^{d-1}e^{-t/2}dt + \int_{d+2}^{u_j\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt
\]
\[
\leq \int_{l_j\|\mu^*\|_2^2}^{d+2} t^{d-1}e^{-t/2}dt + 2(u_j\|\mu^*\|_2^2 - d-2) \int_{d+2}^{d+2 + \frac{1}{2}\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt \tag{A14}
\]
\[
\leq 2(u_j - l_j)\|\mu^*\|_2^2 \int_{l_j\|\mu^*\|_2^2}^{(l_j+1)\|\mu^*\|_2^2} t^{d-1}e^{-t/2}dt.
\]
Letting $a_j = l_j$ and by noting $2(u_j - l_j)/j \leq 4$, our claim can be verified under case (c.2).

Putting together cases (a), (b) and (c), we have

$$\mathbb{P}(A_3 \cap B_j | Z_i = 1) \leq \frac{4j\|\mu^*\|_2^2}{\Gamma(d/2)2^{d/2}} \int_{a_j\|\mu^*\|_2}^{(a_j+1)\|\mu^*\|_2} t^{d/2-1}e^{-t/2}dt$$

Using a similar argument, we can also show that

$$\mathbb{P}(A_3 \cap B_j | Z_i = 2) \leq \frac{8j\|\mu^*\|_2^2}{\Gamma(d/2)2^{d/2}} \int_{a_j\|\mu^*\|_2}^{(a_j+1)\|\mu^*\|_2} t^{d/2-1}e^{-t/2}dt,$$

where $a_j' \in [(\sqrt{j-1})/2 - 1/4, (\sqrt{j}/2 + 1)/4]$. As $\mathbb{P}(Z_i = 1) = \mathbb{P}(Z_i = 2) = \frac{1}{2}$, we can conclude that

$$\mathbb{P}(A_3 \cap B_j) \leq \frac{8j\|\mu^*\|_2^2}{\Gamma(d/2)2^{d/2}} \int_{\min\{a_j, a_j'\}\|\mu^*\|_2}^{(\max\{a_j, a_j'\}+1)\|\mu^*\|_2} t^{d/2-1}e^{-t/2}dt.$$

Plugging this result into (A11), we have

$$\mathbb{E}\{W_{i1}\tau_{i1}(\mu)\tau_{i2}(\mu)\} \leq \frac{4j\|\mu^*\|_2^2}{\Gamma(d/2)2^{d/2}} \int_{\min\{a_j, a_j'\}\|\mu^*\|_2}^{(\max\{a_j, a_j'\}+1)\|\mu^*\|_2} t^{d/2-1}e^{-t/2}dt$$

Combining Steps (i), (ii) and (iii), it holds that

$$\mathbb{E}\{W_{i1}\tau_{i1}(\mu)\tau_{i2}(\mu)\} \leq \frac{\|\mu\|_2^4 + 4\alpha_0\|\mu\|_2^2}{8}\exp(-\|\mu\|_2^2/2) + c_5\mathbb{P}(\chi_{d+6}^2 > \frac{1}{16}\|\mu^*\|_2^2),$$

and we arrive at the desired result by noting $\|\mu - \mu^*\|_2 \leq \frac{1}{4}\|\mu^*\|_2$.

### C2 Proof of Theorem 1

We consider the induction method for this proof. At $t = 1$, given Condition 1-4, Lemma 4b ensures that it holds with probability at least $1 - C_3/\{\log(nd)\}^2$.

$$D(\Theta^{(1)}, \Theta^*) \leq \epsilon + \rho D(\Theta^{(0)}, \Theta^*),$$

where
where $\rho$ is as defined in Lemma 4b and
\[
\epsilon = C_2 \left\{ \frac{1}{\omega_{\min}} \sqrt{Ts_1 \log d \over n} + \max_m \sqrt{(s_2 + d_m) \log d \cdot T \over nd_m} \right\},
\]
where $C_2$ is as defined in Lemma 4b. At step $t > 1$, suppose it holds with probability at least $1 - C_3 t / \{\log(nd)\}^2$ that
\[
D(\Theta^{(t)}, \Theta^*) \leq \frac{1 - \rho^t}{1 - \rho} \epsilon + \rho^t D(\Theta^{(0)}, \Theta^*).
\]
Then using the same argument as in Step 2 of the proof for Lemma 4b, it holds that $\Theta^{(t)}$ satisfies Condition 2. Applying Lemma 4b for $D(\Theta^{(t+1)}, \Theta^*)$, it follows that
\[
D(\Theta^{(t+1)}, \Theta^*) \leq \epsilon + \rho D(\Theta^{(t)}, \Theta^*)
\]
\[
\leq \epsilon + \rho \left\{ \frac{1 - \rho^t}{1 - \rho} \epsilon + \rho^t D(\Theta^{(0)}, \Theta^*) \right\}
\]
\[
= \frac{1 - \rho^{t+1}}{1 - \rho} \epsilon + \rho^{t+1} D(\Theta^{(0)}, \Theta^*).
\]
holds with probability at least $1 - C_3 (t + 1) / \{\log(nd)\}^2$. As such, the contraction inequality also holds for step $t + 1$.

It is then seen that $D(\Theta^{(t+1)}, \Theta^*) \leq \frac{1}{1 - \rho} \epsilon + \rho^{t+1} D(\Theta^{(0)}, \Theta^*)$ for $t = 1, \ldots, T$. Since $\rho \in (0, 1/3]$, the term $\frac{1}{1 - \rho} \epsilon$ will dominate when it reaches $T = \log(\frac{\epsilon}{\rho}) / \log \rho$ steps. From $\log d \asymp \log d_{\max}$ and $\omega_{\min} \lesssim d^{M/2}$ by Condition 1, it then holds that $\log(1/\epsilon) \lesssim \log(nd_{\max})$. Therefore, $T \lesssim (-\log \rho)^{-1} \log(d_{\max} n D(\Theta^{(0)}, \Theta^*))$. For $t \leq T$, the probability for the contraction inequality to hold can be calculated as
\[
\frac{C_3 t}{\{\log(nd)\}^2} \lesssim C_3 \frac{\log(d_{\max} n D(\Theta^{(0)}, \Theta^*))}{\log(\rho) \{\log(nd)\}^2} = o(1).
\]
Putting the above results together, we arrive at that, for $t \leq T$,
\[
D(\Theta^{(t+1)}, \Theta^*) \leq \frac{1}{1 - \rho} \epsilon + \rho^{t+1} D(\Theta^{(0)}, \Theta^*),
\]
holds with probability $1 - o(1)$.

C3 Proof of Theorem 2

We consider the induction method for this proof. At $t = 1$, given Condition 1-4, Lemma 8b ensures that it holds with probability at least $1 - C_3' / \{\log(nd)\}^2$,
\[
D(\Theta^{(1)}, \Theta^*) \leq \epsilon' + \rho_R D(\Theta^{(0)}, \Theta^*),
\]
where $\rho_R$ is as defined in Lemma 8b and

$$
\epsilon' = C'_2 \left\{ \frac{1}{\omega_{\min}} \sqrt{T \frac{s_1 \log d}{n}} + \max_m \sqrt{\frac{T (s_2 + d_m) \log d}{nd_m}} \right\},
$$

where $C'_2$ is as defined in Lemma 8b. At step $t > 1$, suppose it holds with probability at least $1 - C'_3 t / \{\log(nd)\}^2$ that

$$
D(\Theta(t), \Theta^*) \leq \frac{1 - \rho_R^t \epsilon'}{1 - \rho_R} + \rho_R^t D(\Theta(0), \Theta^*).
$$

Then using the same argument as in Step 2 of the proof for Lemma 8b, it holds that $\Theta(t)$ satisfies Condition 2. Applying Lemma 8b for $D(\Theta(t+1), \Theta^*)$, it follows that

$$
D(\Theta(t+1), \Theta^*) \leq \epsilon' + \rho_R D(\Theta(t), \Theta^*) \leq \epsilon' + \rho_R \left\{ \frac{1 - \rho_R^t \epsilon'}{1 - \rho_R} + \rho_R^t D(\Theta(0), \Theta^*) \right\} \leq \frac{1 - \rho_R^{t+1} \epsilon'}{1 - \rho_R} + \rho_R^{t+1} D(\Theta(0), \Theta^*).
$$

holds with probability at least $1 - C'_3 (t+1) / \{\log(nd)\}^2$. As such, the contraction inequality also holds for step $t+1$.

It is then seen that $D(\Theta(t+1), \Theta^*) \leq \frac{1}{1 - \rho_R} \epsilon' + \rho_R^{t+1} D(\Theta(0), \Theta^*)$ for $t = 1, \ldots, T$. Since $\rho_R \in (0, 1/2]$, the term $\frac{1}{1 - \rho_R} \epsilon'$ will dominate when it reaches $T = \log(\frac{\epsilon'}{\rho_R D(\Theta(0), \Theta^*)}) / \log \rho_R$ steps. Since $\log d \asymp \log d_{\max}$ and $\omega_{\min} \asymp d^{M/2}$ by Condition 1, it then holds that $\log(1/\epsilon') \asymp \log(nd_{\max})$. Therefore, $T \asymp (-\log \rho_R)^{-1} \log(d_{\max} n D(\Theta(0), \Theta^*))$. For $t \leq T$, the probability for the contraction inequality to hold can be calculated as

$$
\frac{C' t}{\{\log(nd)\}^2} \lesssim C'_3 \frac{\log(d_{\max} n D(\Theta(0), \Theta^*))}{\log(\rho) \{\log(nd)\}^2} = o(1).
$$

Putting the above results together, we arrive at that, for $t \leq T$,

$$
D(\Theta(t+1), \Theta^*) \leq \frac{1}{1 - \rho_R} \epsilon' + \rho_R^{t+1} D(\Theta(0), \Theta^*),
$$

holds with probability $1 - o(1)$.

**D Proof of Technical Lemmas**

**D1 Proof of Lemma S1**

Without loss of generality, we assume $\omega = 1$. If $\omega \neq 1$, we can always reparametrize by setting $\tilde{U} = U / \omega$. To ease notation, we let $M = 3$ and $m = 1$. The proof holds for a general
and $M$ with straightforward extensions. Following the definition of mode-1 matricization, we have

$$
\mathbf{U}_{(1)} = \begin{bmatrix}
\mathbf{u}_{1,1,1} & \cdots & \mathbf{u}_{1,d,1} & \cdots & \mathbf{u}_{1,d,3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{u}_{d,1,1} & \cdots & \mathbf{u}_{d,1,2} & \cdots & \mathbf{u}_{d,1,3}
\end{bmatrix}
$$

Note that

$$
\mathbf{U}_{j_1,j_2,j_3} = \beta_1(j_1)\beta_2(j_2)\beta_3(j_3),
$$

where $\beta_m(j_m)$ is the $j_m$-th element of $\beta_m$. Then $\mathbf{U}_{(1)}$ can be expressed as

$$
\mathbf{U}_{(1)} = \begin{bmatrix}
\beta_1(1)\beta_2(1)\beta_3(1) & \cdots & \beta_1(1)\beta_2(2)\beta_3(1) & \cdots & \beta_1(1)\beta_2(d)\beta_3(1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_1(1)\beta_2(1)\beta_3(1) & \cdots & \beta_1(1)\beta_2(2)\beta_3(1) & \cdots & \beta_1(1)\beta_2(d)\beta_3(1)
\end{bmatrix}
= \left(\begin{array}{c}
\beta_1(1) \\
\vdots \\
\beta_1(d_1)
\end{array}\right)
\begin{bmatrix}
\beta_2(1)\beta_3(1), \cdots, \beta_2(2)\beta_3(1), \cdots, \beta_2(d)\beta_3(1)
\end{bmatrix}
= \beta_1 \text{vec} (\beta_2 \circ \beta_3)^\top,
$$

where the last equality follows the definition of vectorization of a tensor.

**D2  Proof of Lemma S8**

Write $\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_{1,1} & \cdots & \mathbf{Y}_{1,d_2} \\ \vdots & \vdots & \vdots \\ \mathbf{Y}_{d_1,1} & \cdots & \mathbf{Y}_{d_1,d_2} \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} \mathbf{D}_{1,1} & \cdots & \mathbf{D}_{1,d_2} \\ \vdots & \vdots & \vdots \\ \mathbf{D}_{d_2,1} & \cdots & \mathbf{D}_{d_2,d_2} \end{bmatrix}$. Then $\mathbf{YD} \in \mathbb{R}^{d_1 \times d_2}$ can be written as

$$
\mathbf{YD} = \begin{bmatrix}
\sum_{i=1}^{d_2} \mathbf{Y}_{i,1} \mathbf{D}_{i,1} & \cdots & \sum_{i=1}^{d_2} \mathbf{Y}_{i,1} \mathbf{D}_{i,d_2} \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{d_2} \mathbf{Y}_{d_1,1} \mathbf{D}_{i,1} & \cdots & \sum_{i=1}^{d_2} \mathbf{Y}_{d_1,1} \mathbf{D}_{i,d_2}
\end{bmatrix}.
$$

For the matrix $\mathbb{E}(\mathbf{YDY}^\top) \in \mathbb{R}^{d_1 \times d_1}$, denote each element as $\mathbb{E}(\mathbf{YDY}^\top)_{l,k}$. If $l = k$, we have

$$
\mathbb{E}(\mathbf{YDY}^\top)_{l,l} = \sum_{j=1}^{d_2} \sum_{i=1}^{d_2} \mathbb{E}(\mathbf{Y}_{l,j} \mathbf{D}_{i,j} \mathbf{Y}_{l,j}) = \sum_{i=1}^{d_2} \mathbf{D}_{i,j} \mathbb{E}(\mathbf{Y}_{l,j}^2) = \text{tr}(\mathbf{D}).
$$

If $l \neq k$, we have

$$
\mathbb{E}(\mathbf{YDY}^\top)_{l,k} = \sum_{j=1}^{d_2} \sum_{i=1}^{d_2} \mathbb{E}(\mathbf{Y}_{l,i} \mathbf{D}_{i,j} \mathbf{Y}_{k,j}) = \sum_{j=1}^{d_2} \sum_{i=1}^{d_2} \mathbf{D}_{i,j} \mathbb{E}(\mathbf{Y}_{l,i} \mathbf{Y}_{k,j}) = 0.
$$

Putting all $(l,k)$ pairs, it arrives at that

$$
\mathbb{E}(\mathbf{YDY}^\top) = \text{tr}(\mathbf{D})\mathbf{I}_{d_1}.
$$
D3 Proof of Lemma 1b

In this proof, we establish, under $R = 1$, restricted strong concavity with respect to $\beta_{k,m}$, $\omega_k$ and $\Omega_{k,m}$, respectively. First, we give the first- and second-order partial derivatives of the sample Q-function $Q_{n/T}$ with respect to $\beta_{k,m}$, $\omega_k$ and $\Omega_{k,m}$.

Recall $\text{vec}(\prod_m \beta_{k,m}) = \text{vec}(\beta_{k,1} \circ \cdots \circ \beta_{k,M})$ and $\prod_m \Omega_{k,m} = \Omega_{k,M} \otimes \cdots \otimes \Omega_{k,1}$.

First-order:

$$\nabla_{\beta_{k,m}} Q_{n/T}(\beta'_{k,m}, \bar{\Theta} - \beta_{k,m} | \Theta) = \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \Omega_{k,m} \left\{ \text{vec}(\mathcal{X}_i) - \omega_k \beta'_{k,m} \text{vec}(\prod_{m' \neq m} \bar{\beta}_{k,m'}) \right\} \left( \prod_m \bar{\Omega}_{k,m'} \right) \bar{\omega}_k \text{vec}(\prod_{m' \neq m} \bar{\beta}_{k,m'}) ;$$

$$\nabla_{\omega_k} Q_{n/T}(\omega'_k, \bar{\Theta} - \omega_k | \Theta) = \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \left\{ \text{vec}(\mathcal{X}_i) - \omega_k \text{vec}(\prod_m \bar{\beta}_{k,m}) \right\} \left( \prod_m \bar{\Omega}_{k,m} \right) \text{vec}(\prod_m \bar{\beta}_{k,m});$$

$$\nabla_{\Omega_{k,m}} Q_{n/T}(\Omega'_{k,m}, \bar{\Theta} - \Omega_{k,m} | \Theta) = \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \left\{ \frac{d}{2d_m} (\Omega'_{k,m})^{-1} - \frac{1}{2} (\mathcal{X}_i - \bar{\mathcal{U}}_k) (\prod_{m' \neq m} \bar{\Omega}_{k,m'}) (\mathcal{X}_i - \bar{\mathcal{U}}_k)^\top \right\} .$$

Second-order:

$$\nabla^2_{\beta_{k,m}} Q_{n/T}(\beta'_{k,m}, \bar{\Theta} - \beta_{k,m} | \Theta) = -\frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \bar{\omega}_k^2 \left\{ \text{vec}(\prod_{m' \neq m} \bar{\beta}_{k,m'})^\top \left( \prod_{m' \neq m} \bar{\Omega}_{k,m'} \right) \text{vec}(\prod_{m' \neq m} \bar{\beta}_{k,m'}) \right\} \bar{\Omega}_{k,m};$$

$$\nabla^2_{\omega_k} Q_{n/T}(\omega'_k, \bar{\Theta} - \omega_k | \Theta) = -\frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \text{vec}(\prod_m \bar{\beta}_{k,m})^\top \left( \prod_m \bar{\Omega}_{k,m} \right) \text{vec}(\prod_m \bar{\beta}_{k,m});$$

$$\nabla^2_{\Omega_{k,m}} Q_{n/T}(\Omega'_{k,m}, \bar{\Theta} - \Omega_{k,m} | \Theta) = -\frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \left\{ \frac{d}{2d_m} (\Omega'_{k,m})^{-1} \otimes (\Omega'_{k,m})^{-1} \right\} .$$

First, we consider restricted strong concavity with respect to $\beta_{k,m}$. According to Taylor expansion, we can expand $Q_{n/T}(\beta''_{k,m}, \bar{\Theta} - \beta_{k,m} | \Theta)$ around $\beta'_{k,m}$ to obtain

$$Q_{n/T}(\beta''_{k,m}, \bar{\Theta} - \beta_{k,m} | \Theta) = Q_{n/T}(\beta'_{k,m}, \bar{\Theta} - \beta_{k,m} | \Theta) + \left\langle \nabla_{\beta_{k,m}} Q_{n/T}(\beta'_{k,m}, \bar{\Theta} - \beta_{k,m} | \Theta), \beta''_{k,m} - \beta'_{k,m} \right\rangle + \frac{1}{2} (\beta''_{k,m} - \beta'_{k,m})^\top \nabla^2_{\beta_{k,m}} Q_{n/T}(\mathbf{z}, \bar{\Theta} - \beta_{k,m} | \Theta)(\beta''_{k,m} - \beta'_{k,m}) ,$$

(A17)
where \( z = t\beta'_{k,m} + (1-t)\beta''_{k,m} \) with \( t \in [0,1] \). It follows from (A16) that

\[
\nabla_{\beta_{k,m}}^2 Q_{n/T}(z; \Theta_0; \theta_{k,m} | \Theta) = - \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \omega_k^2 \left\{ \text{vec} \left( \prod_{m' \neq m} \beta_{k,m'} \right) \right\}^\top \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \right\} \Omega_{k,m}. 
\]

Correspondingly, (A17) can be rewritten as

\[
Q_{n/T}(\beta''_{k,m}; \Theta_0; \theta_{k,m} | \Theta) - Q_{n/T}(\beta'_{k,m}; \Theta_0; \theta_{k,m} | \Theta) - \langle \nabla_{\theta_{k,m}} Q_{n/T}(\beta'_{k,m}; \Theta_0; \theta_{k,m} | \Theta), \beta''_{k,m} - \beta'_{k,m} \rangle = - \left( \frac{T}{2n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \right) \omega_k^2 T_{k,m}(\beta''_{k,m} - \beta'_{k,m})^\top \Omega_{k,m}(\beta''_{k,m} - \beta'_{k,m}), \tag{A18}
\]

where \( T_{k,m} = \text{vec} \left( \prod_{m' \neq m} \beta_{k,m'} \right) \right\} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \right\} \text{vec} \left( \prod_{m' \neq m} \beta_{k,m'} \right) \). By Hoeffding’s inequality and noting that \( \tau_{ik}(\Theta) \in [0,1] \), we can get that

\[
\mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) - \mathbb{E}(\tau_{ik}(\Theta)) \right| \leq t \right) \geq 1 - 2e^{-2nt^2/T}. \tag{A19}
\]

Let \( p_n = 1/\{\log(nd)\}^2 \) and \( t = \sqrt{\log(2/p_n)/(2n/T)} \), it arrives at

\[
\left| \frac{1}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) - \mathbb{E}(\tau_{ik}(\Theta)) \right| \leq \sqrt{\log(2/p_n)/(2n/T)},
\]

with probability at least \( 1 - p_n \). Recall that \( \mathbb{E}(\tau_{ik}(\Theta^*)) = \pi_k^* \) and \( \pi_k^* \)’s bounded below by a constant as assumed in Condition 1. Under Condition 3 with a sufficiently small \( \gamma > 0 \), there exists some constant \( c_0 > 0 \) such that \( \min_{k \in [K]} \mathbb{E}(\tau_{ik}(\Theta)) \geq c_0 \) for \( \Theta \in B_{2/\gamma}(\Theta^*) \) (Hao et al., 2017). Next, as \( \sqrt{\log(2/p_n)/(2n/T)} = o(1) \), there exists some constant \( c_0 > 0 \) such that

\[
\mathbb{E}(\tau_{ik}(\Theta)) - \sqrt{\log(2/p_n)/(2n/T)} \geq c_0,
\]

when \( n \) is large. By the fact that \(|a - b| \geq a - b| \), we have

\[
\frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \geq \mathbb{E}(\tau_{ik}(\Theta)) - \sqrt{\log(2/p_n)/(2n/T)} \geq c_0. \tag{A20}
\]

Furthermore, by Condition 1 and \( \Omega_{k,m} \in S_{1/2}(\Omega^*_k,m) \), it holds that

\[
\sigma_{\min}(\Omega_{k,m}) \geq \sigma_{\min}(\Omega^*_k,m) - \|\Omega^*_k,m - \Omega_{k,m}\|_2 \geq \phi_1/2.
\]
Correspondingly, we have
\[
\text{vec}(\prod_{m'\neq m} \vec{\beta}_{k,m'})^\top \left( \prod_{m'\neq m} \Omega_{k,m'} \right) \text{vec}(\prod_{m'\neq m} \vec{\beta}_{k,m'}) \geq \prod_{m'\neq m} \sigma_{\min}(\Omega_{k,m'}) \left\| \vec{\beta}_{k,1} \cdots \vec{\beta}_{k,m-1} \vec{\beta}_{k,m+1} \cdots \vec{\beta}_{k,M} \right\|^2_F \geq (\phi_1/2)^{M-1},
\]
where the first inequality follows from the fact \( \sigma_{\min}(A \otimes B) \geq \sigma_{\min}(A) \sigma_{\min}(B) \) and \( \|\text{vec}(U)\|_F^2 = \|\text{vec}(U)\|^2_2 \) and the last inequality, by noting \( \vec{\beta}_{k,m} \)'s are unit-norm vectors, is due to
\[
\left\| \vec{\beta}_{k,1} \cdots \vec{\beta}_{k,m-1} \vec{\beta}_{k,m+1} \cdots \vec{\beta}_{k,M} \right\|^2_F = 1.
\]

As \( \Theta \in \mathbb{B}_{1/2}(\Theta^*) \), we have \( \bar{\omega}_k \geq \omega_k^* - |\bar{\omega}_k - \omega_k^*| \geq \omega_{\min}/2 \). Setting \( \gamma_0 = \frac{\omega_0^2}{4} \omega_{\min}(\phi_1/2)^M \), the following holds with probability at least \( 1 - 1/(\log(nd))^2 \),
\[
Q_{n/T}(\beta''_{k,m}, \Theta - \beta_{k,m} | \Theta) - Q_{n/T}(\beta'_{k,m}, \Theta - \beta_{k,m} | \Theta) - \langle \nabla_{\beta_{k,m}} Q_{n/T}(\beta', \Theta - \beta_{k,m} | \Theta), \beta''_{k,m} - \beta'_{k,m} \rangle \\
\leq -\frac{\gamma_0}{2} \left\| \beta''_{k,m} - \beta'_{k,m} \right\|^2_2.
\]

The restricted strong concavity with respect to \( \omega_k \) and \( \Omega_{k,m} \) can be shown using similar arguments. Letting \( \gamma''_0 = c_0(\phi_1/2)^M \), it holds with probability at least \( 1 - 1/(\log(nd))^2 \) that
\[
Q_{n/T}(\omega''_k, \Theta - \omega_k | \Theta) - Q_{n/T}(\omega'_k, \Theta - \omega_k | \Theta) - \langle \nabla_{\omega_k} Q_{n/T}(\omega'_k, \Theta - \omega_k | \Theta), \omega''_k - \omega'_k \rangle \\
\leq -\frac{\gamma''_0}{2} (\omega''_k - \omega'_k)^2.
\]

For \( \Omega_{k,m} \), the Taylor expansion can be expressed as
\[
Q_{n/T}(\Omega''_{k,m}, \Theta - \Omega_{k,m} | \Theta) - Q_{n/T}(\Omega'_{k,m}, \Theta - \Omega_{k,m} | \Theta) - \langle \nabla_{\Omega_{k,m}} Q_{n/T}(\Omega'_{k,m}, \Theta - \Omega_{k,m} | \Theta), \Omega''_{k,m} - \Omega'_{k,m} \rangle \\
\leq -\frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \text{vec}(\Delta)^\top \left\{ \frac{d}{2d_m}(\Omega'_{k,m} + t\Delta)^{-1} \otimes (\Omega'_{k,m} + t\Delta)^{-1} \right\} \text{vec}(\Delta),
\]
where \( \Delta = \Omega''_{k,m} - \Omega'_{k,m} \) and \( t \in [0, 1] \). Note that
\[
\sigma_{\min}\left\{ (\Omega'_{k,m} + t\Delta)^{-1} \otimes (\Omega'_{k,m} + t\Delta)^{-1} \right\} = \left[ \sigma_{\min}\left\{ (\Omega'_{k,m} + t\Delta)^{-1} \right\} \right]^2 \\
= \left\{ \sigma_{\max}(\Omega'_{k,m} + t\Delta) \right\}^{-2} \geq \left( \left\| \Omega'_{k,m} \right\|_2 + \|t\Delta\|_2 \right)^{-2} \geq (6\phi_2)^{-2};
\]
where the last inequality is due to \( \| \Omega'_{k,m} \|_2 \leq \| \Omega''_{k,m} \|_2 + \| \Omega'''_{k,m} - \Omega''_{k,m} \|_2 \leq 2\phi_2 \) and \( \| \Delta \|_2 \leq \| \Omega'_{k,m} - \Omega''_{k,m} \|_2 + \| \Omega''_{k,m} - \Omega'_{k,m} \|_2 \leq 4\phi_2 \).

Setting \( \gamma_m = c_0 \frac{n}{d_m} (6\phi_2)^{-2} \) and plugging \( (A20) \) and \( (A23) \) into \( (A22) \), it follows that
\[
Q_n/T(\Omega''_{k,m}, \Theta - \Omega_{k,m} | \Theta) - Q_n/T(\Omega'_{k,m}, \Theta - \Omega_{k,m} | \Theta) - \langle \nabla_{\Omega_{k,m}} Q_n/T(\Omega'_{k,m}, \Theta - \Omega_{k,m} | \Theta), \Omega''_{k,m} - \Omega'_{k,m} \rangle \\
\leq \frac{-\gamma_m}{2} \| \Omega''_{k,m} - \Omega'_{k,m} \|_F^2,
\]
holds with probability at least \( 1 - 1/(\log(nd))^2 \).

**D4 Proof of Lemma 2b**

We establish, under \( R = 1 \), gradient stability with respect to \( \beta_{k,m}, \omega_k \) and \( \Omega_{k,m} \), respectively.

First, we introduce the population version of the first-order gradient in \( (A15) \),
\[
\nabla_{\beta_{k,m}} Q(\beta'_{k,m}, \Theta - \beta_{k,m} | \Theta) \\
= \mathbb{E} \left[ \tau_{ik}(\Theta) \Omega_{k,m} \left( (X_i)_{(m)} - \omega_k \beta'_{k,m} \text{vec}(\prod_{m' \neq m} \beta_{k,m'})^\top \right) \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \omega_k \text{vec}(\prod_{m' \neq m} \beta_{k,m'}) \right], \\
\nabla_{\omega_k} Q(\omega'_{k}, \Theta - \omega_{k} | \Theta) \\
= \mathbb{E} \left[ \tau_{ik}(\Theta) \left\{ \text{vec}(X_i) - \omega_k \text{vec}(\prod_{m} \beta_{k,m})^\top \right\} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \text{vec}(\prod_{m} \beta_{k,m}) \right], \\
\nabla_{\Omega_{k,m}} Q(\Omega'_{k,m}, \Theta - \Omega_{k,m} | \Theta) \\
= \mathbb{E} \left[ \frac{d}{2d_m}(\Omega'_{k,m})^{-1} - \frac{1}{2} \left( (X_i - \bar{U}_k)_{(m)} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) (X_i - \bar{U}_k)^\top_{(m)} \right) \right],
\]
(A24)

where \( \text{vec}(\prod_{m} \beta_{k,m}) \) and \( \prod_{m} \Omega_{k,m} \) are defined in \( (A15) \). In what follows, we show \( (A4) \) for \( \beta_{k,m}, \omega_k \) and \( \Omega_{k,m} \), respectively.

(I) Gradient stability for \( \beta_{k,m} \).

First, we extend \( \nabla_{\beta_{k,m}} Q(\beta'_{k,m}, \Theta - \beta_{k,m} | \Theta) - \nabla_{\beta_{k,m}} Q(\beta'_{k,m}, \Theta - \beta_{k,m} | \Theta^*) \) as
\[
\nabla_{\beta_{k,m}} Q(\beta'_{k,m}, \Theta - \beta_{k,m} | \Theta) - \nabla_{\beta_{k,m}} Q(\beta'_{k,m}, \Theta - \beta_{k,m} | \Theta^*) \\
= \mathbb{E} \left[ D_r(\Theta, \Theta^*) \Omega_{k,m} \left( (X_i)_{(m)} - \omega_k \beta'_{k,m} \text{vec}(\prod_{m' \neq m} \beta_{k,m'})^\top \right) \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \omega_k \text{vec}(\prod_{m' \neq m} \beta_{k,m'}) \right],
\]
(A25)

where \( D_r(\Theta, \Theta^*) = \tau_{ik}(\Theta) - \tau_{ik}(\Theta^*) \). We can write \( \nabla_{\Theta} \tau_{ik}(\Theta) \) as
\[
\nabla_{\Theta} \tau_{ik}(\Theta) = \left( [\nabla_{\theta_1} \tau_{ik}(\Theta)]^\top, \cdots, [\nabla_{\theta_K} \tau_{ik}(\Theta)]^\top \right)^\top,
\]

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where
\[ \nabla_{\theta_k} \tau_{ik}(\Theta) = \begin{cases} -\tau_{ik}(\Theta) \tau_{il}(\Theta) J_i(\theta_l), & \text{when } l \neq k, \\ \tau_{ik}(\Theta)(1 - \tau_{ik}(\Theta)) J_i(\theta_l), & \text{when } l = k. \end{cases} \]

We write \( J_i(\theta_l) = (J_{i,1}(\theta_l), J_{i,2}(\theta_l), J_{i,3}(\theta_l)) \) and have
\[
\begin{align*}
J_{i,1}(\theta_l) &= \Omega_{l,m}(\mathbf{X}_i - \mathcal{U}_l)(m) \left( \prod_{m' \neq m}^\otimes \Omega_{l,m'} \right) \omega_l \text{vec}(\prod_{m' \neq m}^\otimes \beta_{l,m'}) \quad \text{for } m \in [M], \\
J_{i,2}(\theta_l) &= \text{vec}(\mathbf{X}_i - \mathcal{U}_l)^\top \left( \prod_{m}^\otimes \Omega_{l,m} \right) \text{vec}(\prod_{m' \neq m}^\otimes \beta_{l,m'}), \\
J_{i,3}(\theta_l) &= \text{vec} \left\{ \frac{d}{2d_m} \Omega_{l,m}^{-1} - \frac{1}{2} (\mathbf{X}_i - \mathcal{U}_l)(m) \left( \prod_{m' \neq m}^\otimes \Omega_{l,m'} \right) (\mathbf{X}_i - \mathcal{U}_l)(m)^\top \right\} \quad \text{for } m \in [M].
\end{align*}
\]

Finally, we define \( g(J_i(\theta_l), J_i(\theta_k^*)) \) as
\[
g(J_i(\theta_l), J_i(\theta_k^*)) = g_1\{J_i(\theta_l)\} \times g_2\{J_i(\theta_k^*)\},
\]
where
\[
g_1\{J_i(\theta_l)\} = \|J_{i,1}(\theta_l)\|_2 + \omega^*_\max \|J_{i,2}(\theta_l)\|_2 + \sqrt{d_m} \|J_{i,3}(\theta_l)\|_2,
\]
\[
g_2\{J_i(\theta_k^*)\} = \|J_{i,1}(\theta_k^*)\|_2 + \|J_{i,2}(\theta_k^*)\|_2/\omega^*_\max + \|J_{i,3}(\theta_k^*)\|_2/\sqrt{d_m}.
\]

Let \( \tilde{\beta}_{k,m} = \beta_{k,m}, \tilde{\omega}_k = \omega_k/\omega^*_k \) and \( \tilde{\Omega}_{k,m} = \Omega_{k,m}/\|\Omega_{k,m}\|_F \). We note that \( \nabla_{\omega^*} \tau_{ik}(\Theta) \) and \( \nabla_{\tilde{\omega}} \tau_{ik}(\Theta) \) differ by a factor of \( \omega^*_i \), and \( \nabla_{\tilde{\Omega}_m} \tau_{ik}(\Theta) \) and \( \nabla_{\Omega_m} \tau_{ik}(\Theta) \) differ by a factor of \( \sqrt{d_m} \). By Taylor expansion, we have
\[
\tau_{ik}(\Theta) - \tau_{ik}(\Theta^*) = (\nabla_{\tilde{\Theta}} \tau_{ik}(\Theta)) \tilde{\Theta} \|\tilde{\Theta} - \tilde{\Theta}^*\|_2,
\]
where \( \tilde{\Theta}^* = \tilde{\Theta}^* + \delta \Delta \) with \( \delta \in [0, 1] \) and \( \Delta = \tilde{\Theta} - \tilde{\Theta}^* \).

Plugging (A28) into (A25), we have
\[
\|\nabla_{\tilde{\beta}_{k,m}} Q_{\tilde{\beta}_{k,m}}(\tilde{\Theta} - \tilde{\beta}_{k,m} | \Theta) - \nabla_{\beta_{k,m}} Q_{\beta_{k,m}}(\beta_{k,m}^* - \beta_{k,m} | \Theta^*)\|_2^2 \leq \tau^2_0 \|\tilde{\Theta} - \tilde{\Theta}^*\|_2^2
\]
where
\[
\tau^2_0 = \mathbb{E} \left\{ \left\| \tilde{\Omega}_{k,m} (\mathbf{X}_i - \mathcal{U}_l)(m) \left( \prod_{m' \neq m}^\otimes \tilde{\Omega}_{l,m'} \right) \tilde{\omega}_k \text{vec}(\prod_{m' \neq m}^\otimes \tilde{\beta}_{l,m'}) \right\|_2 \|\nabla_{\tilde{\Theta}} \tau_{ik}(\Theta)\|_2 \right\}.
\]

By the definition of \( \nabla_{\Theta} \tau_{ik}(\Theta) \), we have
\[
\|\nabla_{\Theta} \tau_{ik}(\Theta)\|_2 \leq \sum_{l \neq k} (\tau_{ik}(\Theta) \tau_{il}(\Theta))^2 \|g\{J_i(\theta_l^*)\}\|_2^2 + (\tau_{ik}(\Theta)(1 - \tau_{ik}(\Theta)))^2 \|g\{J_i(\theta_k^*)\}\|_2^2.
\]

(A30)
Letting $W_1 = g_1\{J_i(\theta_l)\}\|J_i,1(\beta', \tilde{\theta}_l, -\beta_{k,m})\|_2$ and by nothing $W_1 < W_{ikl}$ and Condition 3, we have

$$\tau_0^2 \leq \mathbb{E} \left[ \sum_{l \neq k} W_1^2(\tau_{ik}(\theta)\tau_{il}(\theta)) \right] + \mathbb{E} \left[ W_1^2(\tau_{ik}(\theta)(1 - \tau_{ik}(\theta))) \right]^2 \leq \sum_{l \neq k} \frac{\gamma^2}{24^2 K^4(R + 1)^4(M + 1)^2} + \frac{\gamma^2(K - 1)^2}{24^2 K^4(R + 1)^4(M + 1)^2} \leq \frac{\gamma^2}{144 K^2(R + 1)^4(M + 1)^2}.$$

By the definition of $D(\Theta, \Theta^*)$, it holds that $\|\tilde{\Theta} - \tilde{\Theta}^*\|_2 \leq K(RM + R + M)D(\Theta, \Theta^*)^2$. Putting the above results together, we have

$$\|\nabla_{\beta_{k,m}} Q(\beta', \Theta_{-\beta_{k,m}} | \Theta) - \nabla_{\beta_{k,m}} Q(\beta', \Theta_{-\beta_{k,m}} | \Theta^*)\|_2 \leq \tau_0 D(\Theta, \Theta^*)$$

where $\tau_0 = \frac{\gamma}{12\sqrt{K(R+1)(M+1)}}$.

**(II) Gradient stability for $\omega_k$.**

Similar to (A29), we can write

$$\frac{1}{\omega_{max}^2} \|\nabla_{\omega_k} Q(\omega', \Theta_{-\omega_k} | \Theta) - \nabla_{\omega_k} Q(\omega', \Theta_{-\omega_k} | \Theta^*)\|_2 \leq (\tau_0'')^2 \|\Theta - \Theta^*\|_2, \quad (A31)$$

where $(\tau_0'')^2 = \frac{1}{\omega_{max}} \mathbb{E} \left[ \left\{ \text{vec}(X_i) - \omega_k \text{vec}(\prod_m \tilde{\beta}_{k,m}) \right\}^\top \left( \prod_m \tilde{\Omega}_{k,m} \right) \text{vec}(\prod_m \tilde{\beta}_{k,m}) \|\nabla_{\Theta} \tau_{ik}(\theta)\|_2 \right].$

Let $W_2 = \frac{1}{\omega_{max}} g_1\{J_i(\theta_l)\}\|J_i,2(\omega', \tilde{\theta}_l, -\omega_k)\|_2$. By $W_2 < W_{ikl}$, (A30) and (A31), it holds that

$$(\tau_0'')^2 \leq \mathbb{E} \left[ \sum_{l \neq k} W_2^2(\tau_{ik}(\theta)\tau_{il}(\theta)) \right] + \mathbb{E} \left[ W_2^2(\tau_{ik}(\theta)(1 - \tau_{ik}(\theta))) \right]^2 \leq \sum_{l \neq k} \frac{\gamma^2}{24^2 K^4(R + 1)^4(M + 1)^2} + \frac{\gamma^2(K - 1)^2}{24^2 K^4(R + 1)^4(M + 1)^2} \leq \frac{\gamma^2}{144 K^2(R + 1)^4(M + 1)^2}.$$

This implies that

$$\|\nabla_{\omega_k} Q(\omega', \Theta_{-\omega_k} | \Theta) - \nabla_{\omega_k} Q(\omega', \Theta_{-\omega_k} | \Theta^*)\|_2 \leq \tau_0'' D(\Theta, \Theta^*)$$

where $\tau_0'' = \frac{\gamma \omega_{max}}{12\sqrt{K(R+1)(M+1)}}$. 

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(III) Gradient stability for $\Omega_{k,m}$. 

Similar to (A29), we can write

$$
\frac{1}{d^2} \left\| \nabla_{\Omega_{k,m}} Q'(\Omega'_{k,m}, \Theta_{-\Omega_{k,m}} | \Theta) - \nabla_{\Omega_{k,m}} Q'(\Omega'_{k,m}, \Theta_{-\Omega_{k,m}} | \Theta^*) \right\|_F^2 \leq \tau_1^2 \| \Theta - \Theta^* \|_2, \quad (A32)
$$

where

$$
\tau_1^2 = \mathbb{E} \left\{ \left\| \frac{1}{2d_m}(\Omega'_{k,m})^{-1} - \frac{1}{2d} (\mathbf{x}_i - \mathbf{u}_k) (\prod_{m' \neq m} \Omega_{k,m'}) (\mathbf{x}_i - \mathbf{u}_k)^\top \right\|_F \| \nabla_{\Theta^*} \tau_{ik}(\Theta) \|_2 \right\}.
$$

Letting $W_3 = g_1 \{ J_i(\theta_i) \} \| J_{i,3}(\Omega'_{k,m}, \Theta_{-\Omega_{k,m}}) \|_2 / d$ and by nothing $W_3 < W_{ikl}$ and Condition 3, it holds that

$$
\tau_1^2 \leq \mathbb{E} \left[ \sum_{l \neq k} W_3^2 (\tau_{ik}(\Theta) \tau_{il}(\Theta))^2 \right] + \mathbb{E} \left[ W_3^2 \{ \tau_{ik}(\Theta)(1 - \tau_{ik}(\Theta)) \}^2 \right] \\
\leq \sum_{l \neq k} \frac{\gamma^2}{24^2 K^4(R + 1)^4(M + 1)^2} + \frac{\gamma^2(K - 1)^2}{24^2 K^4(R + 1)^4(M + 1)^2} \\
\leq \frac{\gamma^2}{144 K^2(R + 1)^4(M + 1)^2}.
$$

This implies that

$$
\left\| \nabla_{\Omega_{k,m}} Q'(\Omega'_{k,m}, \Theta_{-\Omega_{k,m}} | \Theta) - \nabla_{\Omega_{k,m}} Q'(\Omega'_{k,m}, \Theta_{-\Omega_{k,m}} | \Theta^*) \right\|_F \leq \tau_1 D(\Theta, \Theta^*)
$$

where $\tau_1 = \frac{\gamma d}{12 \sqrt{K(R + 1)(M + 1)}}$.

**D5 Proof of Lemma 3b**

We first introduce some notation. Since $\Omega^*_{k,m}$ and $\Sigma^*_{k,m}$ are symmetric matrices, we have

$$
\| \Omega^*_{k,m} \|_{\text{max}} \leq \max_{k,m} \| \Omega^*_{k,m} \|_2 \leq \phi_2, \quad \| \Sigma^*_{k,m} \|_{\text{max}} \leq \max_{k,m} \| \Sigma^*_{k,m} \|_2 \leq 1 / \phi_1. \quad (A33)
$$

To ease notation, we define

$$
h_{\Theta, \Phi}(\beta^*_{k,m}) = \nabla_{\beta_{k,m}} Q_{n/T}(\beta^*_{k,m}, \Theta - \beta_{k,m} | \Theta) - \nabla_{\beta_{k,m}} Q(\beta^*_{k,m}, \Theta - \beta_{k,m} | \Theta),
$$

$$
h_{\Theta, \Phi}(\omega^*_{k}) = \nabla_{\omega_{k}} Q_{n/T}(\omega^*_{k}, \Theta - \omega_{k} | \Theta) - \nabla_{\omega_{k}} Q(\omega^*_{k}, \Theta - \omega_{k} | \Theta),
$$

$$
h_{\Theta, \Phi}(\Omega^*_{k,m}) = \nabla_{\Omega_{k,m}} Q_{n/T}(\Omega^*_{k,m}, \Theta - \Omega_{k,m} | \Theta) - \nabla_{\Omega_{k,m}} Q(\Omega^*_{k,m}, \Theta - \Omega_{k,m} | \Theta).
$$
Recall \( P_1(\beta_{k,m}) = \|\beta_{k,m}\|_1 \) and \( P_2(\Omega_{k,m}) = \|\Omega_{k,m}\|_{1,\text{off}} \), we have that

\[
\|h_{\Theta,\Theta}(\beta'_{k,m})\|_{P_1} \leq \max_k \|h_{\Theta,\Theta}(\beta'_{k,m})\|_1, \quad \|h_{\Theta,\Theta}(\omega'_k)\| \leq \max_k \|h_{\Theta,\Theta}(\omega'_k)\|_1,
\]

\[
\|h_{\Theta,\Theta}(\Omega'_{k,m})\|_{P_2} \leq \max_k \left\|h_{\Theta,\Theta}(\Omega'_{k,m})\right\|_{\max},
\]

where \( P_1, P_2 \) be the dual norms of \( P_1, P_2 \), respectively.

**(I) Bounding** \( h_{\Theta,\Theta}(\beta'_{k,m}) \).

Recall (A15), and we have

\[
h_{\Theta,\Theta}(\beta'_{k,m}) = \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \Omega_{k,m} \left\{ (\mathbf{X}_i)_{(m)} - \hat{\omega}_k \beta'_{k,m} \right\} \left( \prod_{m' \neq m} \bar{\Omega}_{k,m'} \right) \left( \prod_{m' \neq m} \hat{\beta}_{k,m'} \right)
\]

\[
- \mathbb{E} \left[ \tau_{ik}(\Theta) \Omega_{k,m} \left\{ (\mathbf{X}_i)_{(m)} - \hat{\omega}_k \beta'_{k,m} \right\} \left( \prod_{m' \neq m} \bar{\Omega}_{k,m'} \right) \left( \prod_{m' \neq m} \hat{\beta}_{k,m'} \right) \right].
\]

By the triangle inequality, term (I) can be bounded by

\[
\|\Omega_{k,m}\|_2 \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) (\mathbf{X}_i)_{(m)} - \mathbb{E} \left\{ \tau_{ik}(\Theta) (\mathbf{X}_i)_{(m)} \right\} \right\|_{\max} \left\| \left( \prod_{m' \neq m} \bar{\Omega}_{k,m'} \right) \hat{\omega}_k \right\|_2
\]

\[
+ \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) - \mathbb{E} (\tau_{ik}(\Theta)) \right\|_\infty \left\| \left( \prod_{m' \neq m} \bar{\Omega}_{k,m'} \right) \hat{\omega}_k \beta'_{k,m} \right\|_\infty \left( \prod_{m' \neq m} \bar{\Omega}_{k,m'} \right) \hat{\omega}_k \beta'_{k,m} \right\|_\infty.
\]

Consider the set of missing data \( \{Z_i, i \in [n]\} \), we have

\[
\mathbf{X}_i | Z_i = k' \sim \mathcal{N}(\mathbf{u}_{k',*}, \Sigma_{k'}),
\]

\[
\mathbb{P}(Z_i = k') = \pi_{k'}, \quad \sum_{k'=1}^K \pi_{k'} = 1.
\]

Correspondingly, the \( j \)-th coordinate of \( \text{vec}(\mathbf{X}_i) \) can be written as

\[
\text{vec}(\mathbf{X}_i)_j = \sum_{k'=1}^K I(Z_i = k') (\text{vec}(\mathbf{u}_{k',*})_j + V_{j,k'}). \tag{A35}
\]

Here \( \text{vec}(\mathbf{u}_{k',*})_j = \mathbb{E} \{\text{vec}(\mathbf{X}_i)_j | Z_i = k'\} \) and \( V_{j,k'} \sim \mathcal{N}(0, \text{var}(\text{vec}(\mathbf{X}_i)_j | Z_i = k')) \).
Denote that
\[ M = \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta)(\vec{X}_i)_{(m)} - \mathbb{E} \left\{ \tau_{ik}(\Theta)(\vec{X}_i)_{(m)} \right\}, \]
where \( M \in \mathbb{R}^{d_m \times \frac{d_m}{n}} \) and let \( \text{vec}(M)_j \) be the \( j \)-th element of \( \text{vec}(M) \). Plugging (A35) into \( \text{vec}(M)_j \), it can be bounded as below.

\[
|\text{vec}(M)_j| = \left| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta)\text{vec}(\vec{X}_i)_j - \mathbb{E} \left\{ \tau_{ik}(\Theta)\text{vec}(\vec{X}_i)_j \right\} \right|
\leq \sum_{k'=1}^{K} \left| \frac{T}{n} \sum_{i=1}^{n/T} I(Z_i = k')\tau_{ik}(\Theta)\text{vec}(U_{k'}^*)_j - \mathbb{E} \left\{ I(Z_i = k')\tau_{ik}(\Theta)\text{vec}(U_{k'}^*)_j \right\} \right|_{M_1(j)}
+ \sum_{k'=1}^{K} \left| \frac{T}{n} \sum_{i=1}^{n/T} I(Z_i = k')\tau_{ik}(\Theta)V_{j,k'}^* - \mathbb{E} \left\{ I(Z_i = k')\tau_{ik}(\Theta)V_{j,k'}^* \right\} \right|_{M_2(j)}.
\]

We bound \( M_1(j) \) first. From the fact that \( |I(Z_i = k')\tau_{ik}(\Theta)\text{vec}(U_{k'}^*)_j| \leq ||U_{k'}^*||_{\max} \), it holds that \( I(Z_i = k')\tau_{ik}(\Theta)U_{k'}^* \) is a sub-Gaussian random variable with sub-Gaussian norm bounded above by \( ||U_{k'}^*||_{\max} \), i.e., \( ||I(Z_i = k')\tau_{ik}(\Theta)\text{vec}(U_{k'}^*)_j||_{\psi_2} \leq ||U_{k'}^*||_{\max} \), where \( || \cdot ||_{\psi_2} \) denotes the sub-Gaussian norm. By Lemma S3, we get that
\[
||I(Z_i = k')\tau_{ik}(\Theta)\text{vec}(U_{k'}^*)_j - \mathbb{E} \left\{ I(Z_i = k')\tau_{ik}(\Theta)\text{vec}(U_{k'}^*)_j \right\}||_{\psi_2} \leq 2 ||U_{k'}^*||_{\max}.
\]
Standard concentration results give that, for some positive constant \( D_1 \) and any \( t > 0 \),
\[
\mathbb{P}(\|M_1(j)\| \geq t) \leq e \cdot \exp \left( -\frac{D_1 nt^2}{4T||U_{k'}^*||_{\max}^2} \right).
\]
which implies that, with probability at least \( 1 - p_n \),
\[
|\text{vec}(M)_j| \leq \sqrt{\frac{4}{D_1} ||U_{k'}^*||_{\max} \sqrt{\frac{\log(e/p_n)T}{n}}}.
\]
We then move to bound term \( M_2(j) \). Similarly, \( \tau_{ik}(\Theta)I(Z_i = k') \) is a sub-Gaussian random variable, since that \( ||\tau_{ik}(\Theta)I(Z_i = k')||_{\psi_2} \leq 1 \). Moreover, \( V_{j,k'}^* \) is a Gaussian random variable with sub-Gaussian norm \( ||V_{j,k'}^*||_{\psi_2} \leq 1/\phi_1^{M/2} \). Then by Lemma S2, it holds that \( I(Z_i = k')\tau_{ik}(\Theta)V_{j,k'}^* \) is sub-exponential random variable. Moreover, there exists a positive constant \( D_2 \) such that
\[
||I(Z_i = k')\tau_{ik}(\Theta)V_{j,k'}^*||_{\psi_1} \leq D_2/\phi_1^{M/2}.
\]
Applying Lemma S3, we can get that
\[
\|I(Z_i = k')\tau_{ik}(\Theta)V_{j,k'} - \mathbb{E}\{I(Z_i = k')\tau_{ik}(\Theta)V_{j,k'}\}\|_{\psi_1} \leq 2D_2/\phi_1^{M/2}.
\]
Following the concentration inequality of sub-exponential random variables (Vershynin, 2018), there exists some positive constant $D_3$ such that the following inequality
\[
P(|M_2(j)| \geq t) \leq 2\exp\left(-D_3\min\left\{\frac{t^2}{4D_2^2/(\phi_1^M)}, \frac{t}{2D_2/(\phi_1^{M/2})}\right\}\frac{n}{T}\right),
\]
holds for any $t \geq 0$. For a sufficiently small $t$, the above inequality reduces to
\[
P(|M_2(j)| \geq t) \leq 2\exp\left(-D_3\frac{nt^2}{4TD_2^2/(\phi_1^M)}\right),
\]
which implies that
\[
|M_2(j)| \leq \phi_1^{-M/2}\sqrt{\frac{4D_2^2}{D_3}}\sqrt{n\log(2/p_n)T}\frac{\log(2/p_n)T}{n},
\]
with probability at least $1 - p_n$.

Plugging (A36) and (A37) into $\text{vec}(M)_j$, it arrives that
\[
\text{vec}(M)_j \leq |M_1(j)| + |M_2(j)| \leq \sqrt{\frac{4}{D_0} \sum_k (\|U^*_k\|_{\max} + \phi_1^{-M/2})} \sqrt{\frac{\log(e/p_n)T}{n} + \log(\log(2/p_n)T/n)},
\]
with probability at least $1 - 2Kp_n$, where $D_0 = \min\{D_1, D_3/D_2^2\}$. Jointly for all $j$, we have
\[
I_1 \leq \sqrt{\frac{4}{D_0} \sum_k (\|U^*_k\|_{\max} + \phi_1^{-M/2})} \sqrt{\frac{\log(e/p_n)T + \log d}{n/T}}, \quad (A38)
\]
with probability at least $1 - 2Kp_n$.

Next, we consider term $I_2$, i.e.,
\[
I_2 = \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) - \mathbb{E}(\tau_{ik}(\Theta)) \right\|_{\infty}.
\]
By noting $\tau_{ik}(\Theta) \in [0, 1]$, Hoeffding’s inequality gives,
\[
P\left(\left| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) - \mathbb{E}(\tau_{ik}(\Theta)) \right|\leq t\right) \geq 1 - 2e^{-2nt^2/T},
\]
which implies, with probability at least $1 - p_n$,
\[
\left| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) - \mathbb{E}(\tau_{ik}(\Theta)) \right| \leq \sqrt{\frac{1}{2} \log(2/p_n)T/n}. \quad (A39)
\]
By noting the bounds of $I_1$ and $I_2$ in (A38) and (A39), respectively, there exists some constant $D_4 > 0$ such that $I_2 \leq D_4 I_1$. Letting $\varphi_{K0} = \sum_k (\|\mathcal{U}_k\|_\infty + \phi_1^{-M/2})$, we have

$$I \lesssim \|\Omega_{k,m}\|_2 \| \left( \prod_{m' \neq m}^{\hat{\Omega}_{k,m'}} \right) \text{vec} \left( \prod_{m' \neq m}^{\hat{\beta}_{k,m'}} \right) \| \varphi_{K0} \sqrt{\frac{\log(e/p_n) + \log d}{n/T}}. \quad (A40)$$

holds with probability at least $1 - (2K + 1)p_n$. By (A33) and Condition 2, we have $\|\Omega_{k,m}\|_2 \leq \frac{3}{2} \phi_2$ and $\left\| \left( \prod_{m' \neq m}^{\hat{\Omega}_{k,m'}} \right) \text{vec} \left( \prod_{m' \neq m}^{\hat{\beta}_{k,m'}} \right) \right\|_2 \leq (3\phi_2/2)^M$. Since $p_n = 1/\{\log(nd)\}^2$, we have $\log(e/p_n)/\log d = o(1)$ and it holds with probability at least $1 - K(2K + 1)/\{\log(nd)\}^2$ that,

$$\max_k I \leq c_1 \omega_{\max} \sqrt{\frac{T \log d}{n}}, \quad (A41)$$

where $c_1$ is some positive constant.

**I) Bounding $h_{\Theta, \Theta}(\omega'_k)$.**

Recall (A15), and we have

$$h_{\Theta, \Theta}(\omega'_k) = \left\{ \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) - \mathbb{E}(\tau_{ik}(\Theta)) \right\} \omega'_k \text{vec} \left( \prod_{m}^{\hat{\beta}_{k,m}} \right)^\top \left( \prod_{m}^{\hat{\Omega}_{k,m}} \right) \text{vec} \left( \prod_{m}^{\hat{\beta}_{k,m}} \right).$$

By (A39) and $\text{vec} \left( \prod_{m}^{\hat{\beta}_{k,m}} \right)^\top \left( \prod_{m}^{\hat{\Omega}_{k,m}} \right) \text{vec} \left( \prod_{m}^{\hat{\beta}_{k,m}} \right) \leq (3\phi_2/2)^M$, it holds with probability at least $1 - K/\{\log(nd)\}^2$ for term (II) that

$$\max_k \Pi \leq c''_1 \omega_{\max} \sqrt{\frac{T \log(\log(nd))}{n}},$$

where $c''_1$ is some positive constant.

**I) Bounding $h_{\Theta, \Theta}(\Omega'_{k,m})$.**

Recall (A15), and the term $h_{\Theta, \Theta}(\Omega'_{k,m})$ can be written as

$$\frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \frac{d}{2d_m} (\Omega'_{k,m})^{-1} - \frac{1}{2n} \sum_{i=1}^{n} \tau_{ik}(\Theta) \left( \mathcal{X}_i - \bar{U}_k \right)_{(m)} \left( \prod_{m' \neq m}^{\hat{\Omega}_{k,m'}} \right) \left( \prod_{m' \neq m}^{\hat{\beta}_{k,m'}} \right)^\top$$

$$- \mathbb{E}(\tau_{ik}(\Theta)) \frac{d}{2d_m} (\Omega'_{k,m})^{-1} - \frac{1}{2} \mathbb{E} \left[ \tau_{ik}(\Theta) \left( \mathcal{X}_i - \bar{U}_k \right)_{(m)} \left( \prod_{m' \neq m}^{\hat{\Omega}_{k,m'}} \right) \left( \mathcal{X}_i - \bar{U}_k \right)^\top_{(m)} \right].$$
Correspondingly, writing $\tilde{X}_{i,k} = X_i - \bar{U}_k$ and term (III) can be decomposed as

\[
\text{III} \leq \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) - \mathbb{E}(\tau_{ik}(\Theta)) \right\|_{\max} \frac{d}{2d_m}(\Omega'_{k,m})^{-1} + \text{III}_1
\]

\[
\text{III}_1 = \frac{1}{2} \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \left( \tilde{X}_{i,k} \right)_{(m)} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( \tilde{X}_{i,k} \right)_{(m)\top} - \mathbb{E} \left\{ \tau_{ik}(\Theta) \left( \tilde{X}_{i,k} \right)_{(m)} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( \tilde{X}_{i,k} \right)_{(m)\top} \right\} \right\|_{\max}
\]

By (A39) and Condition 2, we have, with probability at least $1 - p_n$,

\[
\text{III}_1 \leq \sqrt{\frac{1}{2(\phi_1/2)^2} \log(2/p_n) \sqrt{(d/d_m)^2 n/T}}.
\]

For III$_2$, it can be bounded as III$_2 < $ III$_{21} + $ III$_{22} + $ III$_{23} + $ III$_{24}$, that is,

\[
\text{III}_2 = \frac{1}{2} \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \left( X_i \right)_{(m)} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( X_i \right)_{(m)\top} - \mathbb{E} \left\{ \tau_{ik}(\Theta) \left( X_i \right)_{(m)} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( X_i \right)_{(m)\top} \right\} \right\|_{\max}
\]

\[
+ \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \left( \bar{U}_k \right)_{(m)} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( \bar{U}_k \right)_{(m)\top} - \mathbb{E} \left\{ \tau_{ik}(\Theta) \left( \bar{U}_k \right)_{(m)} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( \bar{U}_k \right)_{(m)\top} \right\} \right\|_{\max}
\]

\[
+ \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \left( \bar{U}_k \right)_{(m)} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( X_i \right)_{(m)\top} - \mathbb{E} \left\{ \tau_{ik}(\Theta) \left( \bar{U}_k \right)_{(m)} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( X_i \right)_{(m)\top} \right\} \right\|_{\max}
\]

\[
+ \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \left( \bar{U}_k \right)_{(m)} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( \bar{U}_k \right)_{(m)\top} - \mathbb{E} \left\{ \tau_{ik}(\Theta) \left( \bar{U}_k \right)_{(m)} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( \bar{U}_k \right)_{(m)\top} \right\} \right\|_{\max}
\]

We introduce some notations. Let $Y_i = (X_i)_{(m)} \left\{ \prod_{m' \neq m} (\Omega_{k,m'})^{1/2} \right\}$. It is seen that $Y_i$ is a matrix of dimension $d_m \times (\frac{d}{d_m})$. If $Z_i = k'$, we have

\[
Y_i \sim \mathcal{N}_T(\bar{U}_{k'}, \tilde{\Sigma}_{k'})
\]

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with $\tilde{U}_{k'} = (U_{k'})_{(m)} \left\{ \prod_{m' \neq m} (\Omega_{k,m'})^{1/2} \right\}$ and

$$\tilde{\Sigma}_{k'} = \left\{ \Sigma_{k',m} \prod_{m' \neq m} (\Omega_{k',m'})^{1/2} \right\} \left\{ \prod_{m' \neq m} (\Sigma_{k',m'})^{1/2} \right\} \left\{ \prod_{m' \neq m} (\Omega_{k',m'})^{1/2} \right\}.$$

Let $Y_i(l,j)$ be the $(l,j)$-th element of $Y_i$. It can then be expressed as

$$Y_i(l,j) = \sum_{k'=1}^{K} I(Z_i = k') \{ \tilde{U}_{k'}(l,j) + \tilde{V}_{l,j,k'} \},$$

where $\tilde{U}_{k'}(l,j) = \mathbb{E}\{Y_i(l,j)|Z_i = k'\}$ and $\tilde{V}_{l,j,k'} \sim \mathcal{N}(0, \text{var}(Y_i(l,j)|Z_i = k'))$. Denote $Y_i(l,\cdot) \in \mathbb{R}^{d/d_m}$ as the $l$-th row of $Y_i$, and then we may write $Y_i Y_i^\top$ as

$$\begin{pmatrix}
Y_i(1,\cdot)^\top & \cdots & Y_i(1,\cdot)^\top Y_i(d_m,\cdot) \\
\vdots & \ddots & \vdots \\
Y_i(d_m,\cdot)^\top & \cdots & Y_i(d_m,\cdot)^\top Y_i(d_m,\cdot)
\end{pmatrix},$$

where

$$Y_i(l,\cdot)^\top Y_i(l',\cdot) = \sum_{j=1}^{d/d_m} \sum_{k'=1}^{K} I(Z_i = k') I(Z_i = \tilde{k}') \{ \tilde{U}_{k'}(l,j) + \tilde{V}_{l,j,k'} \} \{ \tilde{U}_{k'}(l',j) + \tilde{V}_{l',j,k'} \}$$

$$= \sum_{j=1}^{d/d_m} \sum_{k'=1}^{K} I(Z_i = k') \{ \tilde{U}_{k'}(l,j) + \tilde{V}_{l,j,k'} \} \{ \tilde{U}_{k'}(l',j) + \tilde{V}_{l',j,k'} \}$$

$$= \sum_{j=1}^{d/d_m} \sum_{k'=1}^{K} I(Z_i = k') \{ \tilde{U}_{k'}(l,j)\tilde{U}_{k'}(l',j) + \tilde{U}_{k'}(l,j)\tilde{V}(l',j) + \tilde{V}_{l,j,k'}\tilde{U}_{k'}(l',j) + \tilde{V}_{l,j,k'}\tilde{V}_{l',j,k'} \}.$$
Plugging in the expressions of $Y_i(l, l')$, $\hat{M}(l, l')$ can be expressed as

$$
\hat{M}(l, l') = \sum_{i=1}^{n/T} \sum_{k'=1}^{K} \left[ \frac{T}{n} I(Z_i = k') \tau_{ik}(\Theta) \tilde{U}_{k'}^*(l, \cdot)^T \tilde{U}_{k'}^*(l', \cdot) - \mathbb{E}\left\{ I(Z_i = k') \tau_{ik}(\Theta) \tilde{U}_{k'}^*(l, \cdot)^T \tilde{U}_{k'}^*(l', \cdot) \right\} \right]
$$

Next, we will bound terms $\tilde{M}_1(l, l')$, $\tilde{M}_2(l, l')$, $\tilde{M}_3(l, l')$ and $\tilde{M}_4(l, l')$ separately. We begin with $\tilde{M}_1(l, l')$. Since $|I(Z_i = k') \tau_{ik}(\Theta) \tilde{U}_{k'}^*(l, \cdot)^T \tilde{U}_{k'}^*(l', \cdot)| \leq \max_{l,l'} |\tilde{U}_{k'}^*(l, \cdot)^T \tilde{U}_{k'}^*(l', \cdot)|$, it is seen that $I(Z_i = k') \tau_{ik}(\Theta) \tilde{U}(l, \cdot)^T \tilde{U}(l', \cdot)$ is a sub-Gaussian random variable with

$$
\left| I(Z_i = k') \tau_{ik}(\Theta) \tilde{U}_{k'}^*(l, \cdot)^T \tilde{U}_{k'}^*(l', \cdot) - \mathbb{E}\left\{ I(Z_i = k') \tau_{ik}(\Theta) \tilde{U}_{k'}^*(l, \cdot)^T \tilde{U}_{k'}^*(l', \cdot) \right\} \right|_{\psi_2} \leq 2 \max_{l,l'} |\tilde{U}_{k'}^*(l, \cdot)^T \tilde{U}_{k'}^*(l', \cdot)|.
$$

By the concentration inequality in Lemma S4, we have for any $t > 0$,

$$
\mathbb{P}\left( |\tilde{M}_1(l, l')| \leq t \right) \geq 1 - e \cdot \exp\left( -\frac{C n t^2}{4 T \max_{l,l'} |\tilde{U}_{k'}^*(l, \cdot)^T \tilde{U}_{k'}^*(l', \cdot)|^2} \right).
$$

Since $a^T b \leq \|a\|_2 \|b\|_2$ and $\|\Omega_{k,m}\|_2 \leq 3\phi_2/2$, we have

$$
\max_{l,l'} |\tilde{U}_{k'}^*(l, \cdot)^T \tilde{U}_{k'}^*(l', \cdot)| \leq \max_{l,l'} |U_{k'}(l, \cdot)^T U_{k'}^*(l', \cdot)| (3\phi_2/2)^{M-1} \leq \max_{l} \|((U_{k'}^*)_{(m)}(l, \cdot))\|_2^2 (3\phi_2/2)^{M-1}.
$$

Therefore, for any pair $(l, l')$, it holds that

$$
|M_1(l, l')| \leq \sqrt{4/C} \max_l \|((U_{k'}^*)_{(m)}(l, \cdot))\|_2^2 (3\phi_2/2)^{M-1} \sqrt{\frac{\log(d_m) + \log(e/p_n)}{n/T}} \tag{A42}
$$
with probability at least $1 - p_n$. Note that both $I(Z_i = k') \tau_{ik}(\Theta) \tilde{U}_k^*(l, j) \tilde{V}_{l', j, k'}$ and $I(Z_i = k') \tau_{ik}(\Theta) \tilde{V}_{l', j, k'}$ are sub-exponential random variables with

$$\left\| I(Z_i = k') \tau_{ik}(\Theta) \tilde{U}_k^*(l, \cdot)^\top \tilde{V}_{k'}(l', \cdot) - \mathbb{E}\{I(Z_i = k') \tau_{ik}(\Theta) \tilde{U}_k^*(l, \cdot)^\top \tilde{V}_{k'}(l', \cdot)\} \right\|_{\psi_1} \leq 2 \max_l \| (\mathcal{U}_k^*)_{(m)}(l, \cdot) \|_2 (3 \phi_2/2)^{M-1/2} \frac{(3 \phi_2/2)^{(M-1)/2}}{\phi_1^{M/2}}$$

$$= \max_l \| (\mathcal{U}_k^*)_{(m)}(l, \cdot) \|_2 (3 \phi_2/2)^{M-1} / \phi_1^{M/2}.$$  

Similar to the argument used in (A37), there exist one positive constant $D_5$ such that

$$\max\{|M_2(l, l'), |M_3(l, l')|\} \leq \sqrt{4/D_5} \max_l \| (\mathcal{U}_k^*)_{(m)}(l, \cdot) \|_2 (3 \phi_2/2)^{M-1} / \phi_1^{M/2} \sqrt{T d_m \log(2/p_n)} / nd,$$

with probability at least $1 - p_n$. Taking the union bound for all pairs $(l, l')$ gives

$$\max\{|M_2(l, l'), |M_3(l, l')|\} \leq \sqrt{4/D_5} \max_l \| (\mathcal{U}_k^*)_{(m)}(l, \cdot) \|_2 (3 \phi_2/2)^{M-1} / \phi_1^{M/2} \sqrt{T d_m \log(2/p_n)} / nd,$$

(A43)

with probability at least $1 - p_n$. Finally, we discuss $M_4(l, l')$. By the fact that both $I(Z_i = k') \tau_{ik}(\Theta) \tilde{V}_{l', j, k'}$ and $\tilde{V}_{l', j, k'}$ are sub-Gaussian random variables, we have $I(Z_i = k') \tau_{ik}(\Theta) \tilde{V}_{l', j, k'}$ is sub-exponential with parameter $(3 \phi_2/2)^{(M-1)/\phi_1 M}$. By Lemma S5, there exists some positive constant $D_6$ such that the following inequality

$$\mathbb{P}\left(|M_4(l, l')| \geq t\right) \leq 2 \exp\left(- \frac{D_6 nt^2}{4T d_m (3 \phi_2/2)^{2(M-1)/\phi_1^2 M}}\right),$$

holds for a sufficiently small $t > 0$. When $n$ is sufficiently large, it holds for any pair $(l, l')$ that

$$|M_4(l, l')| \leq \sqrt{4/D_5} (3 \phi_2/2)^{M-1} \sqrt{T d_m (2 \log(d_m) + \log(2/p_n)) / nd},$$

(A44)

with probability at least $1 - p_n$.

Combining (A42), (A43) and (A44) together, $\Pi_{21}$ can be bounded by

$$\Pi_{21} \lesssim \sum_{k'} \left( \max_l \| (\mathcal{U}_k^*)_{(m)}(l, \cdot) \|_2^2 \sqrt{d/d_m} + 1 \right) \times \sqrt{T d_m (2 \log(d_m) + \log(2/p_n)) / nd}$$

(A45)

with probability at least $1 - 4K p_n$. 

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For $\text{III}_{22}$, it holds with probability at least $1 - 2Kp_n$ that

$$\text{III}_{22} \leq \left\| \sum_{i=1}^{n/T} \frac{\tau_{ik}(\Theta)}{n/T} (\mathbf{x}_{(i)}(m) - \mathbb{E}\left\{ \tau_{ik}(\Theta)(\mathbf{x}_{(i)}(m) \right\}) \left( \prod_{m' \neq m} \Omega_{k,m'} \right) (\bar{\mathbf{u}}_k)_{(m)}^\top \right\|_{\text{max}}$$

where $\varphi_k(\cdot)$ is as defined in (A40) and the third inequality is due to (A38) and the fact that $\|a\|_2 \leq \sqrt{d} \max_j |a(j)|$ for any $a \in \mathbb{R}^d$. Term $\text{III}_{23}$ can be bounded similarly. For $\text{III}_{24}$, we have

$$\text{III}_{24} \leq \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) - \mathbb{E}(\tau_{ik}(\Theta)) \right\|_{\text{max}} \sqrt{\frac{T d \log d}{n d_m/T}},$$

with probability at least $1 - p_n$. By Condition 2, it holds that that

$$\|\mathbf{u}_{(m)}(l, \cdot)\|_2 \leq \|\mathbf{u}_{(m)}^*(l, \cdot)\|_2 + \|\bar{\mathbf{u}}_k(l, \cdot) - (\mathbf{u}_{(m)}^*)_k(l, \cdot)\|_2 \leq \|\mathbf{u}_{(m)}^*(l, \cdot)\|_2.$$ 

Putting (A45), (A46) and (A47) together, we have

$$\text{III}_2 \leq \left\{ \sqrt{T d \log d} \right\}_{\text{max}} \frac{T d \log d}{n d_m} + \sum_{k'} \max_i \|\mathbf{u}_{(m)}^*(l, \cdot)\|_2 \left\| \frac{T \log d}{n} \right\|,$$

with at least probability $1 - (8K + 1)p_n$. Define that $\varphi_K = \max_{k,m,l} \|\mathbf{u}_{(m)}^*(l, \cdot)\|_2$. With the upper bounds of $\text{III}_1$ and $\text{III}_2$, we have

$$\max_k \|\mathbf{u}_k\|_{\text{max}} \leq \varphi_K (\varphi_K(\varphi_K) + 1) \sqrt{\frac{T d \log d}{n d_m}} + \varphi_k^2 \frac{\log(\frac{d_m}{d}) + \log(2/p_n)}{T \log d},$$

with probability at least $1 - K(8K + 2)p_n$. Since $\|\mathbf{u}_k^*\|_{\text{max}} < \infty$, it is easily seen that $\varphi_k \leq \max_k \|\mathbf{u}_k^*\|_{\text{max}} \sqrt{\frac{d}{d_m}}$. Therefore, for some constant $c_2 > 0$, it holds that

$$\max_k \|\mathbf{u}_k\|_{\text{max}} \leq c_2 \sqrt{\frac{T d \log d}{n}},$$

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with probability at least $1 - K(8K + 2)/\log(nd)^2$.

D6  Proof of Lemma 4b

In this proof, given $\Theta^{(0)}$, we in turn bound $\|\bar{\beta}_{k,m}^{(1)} - \beta_{k,m}^{*}\|_2$, $\|\omega_k^{(1)} - \omega_k^{*}\|$ and $\|\Omega_{k,m}^{(1)} - \Omega_{k,m}^{*}\|_F$ using results from Lemmas 1b-3b. To bound each term, the proof can be divided into three steps. Take $\beta_{k,m}$ as an example. In step 1, we bound $\|\bar{\beta}_{k,m}'' - \beta_{k,m}^{*}\|_2$, where $\beta_{k,m}'' = \frac{\bar{\beta}_{k,m}}{\|\bar{\beta}_{k,m}\|_2}$ and $\bar{\beta}_{k,m} = \arg\max_{\beta_{k,m}} Q(\beta_{k,m}, \Theta - \beta_{k,m}, |\Theta|^\tau_{\beta_{k,m}}) - \lambda_0\|\beta_{k,m}\|_1$. In step 2, let $\Theta = \Theta^{(0)}$ and $\Theta' - \beta_{k,m} = \Theta_{-\beta_{k,m}}^{(1)}$, we can verify that $\Theta'' = (\beta_{k,m}, \Theta_{-\beta_{k,m}}^{(1)})^\top$ still satisfies Condition 2, which guarantees the initial condition for the next parameter update. In step 3, jointly considering all parameters, we establish the contraction inequality in Lemma 4b. To ease notation, we denote $\alpha = D(\Theta^{(0)}, \Theta^*)$. In what follows, we first discuss Step 1 for $\beta_{k,m}$, $\omega_k$ and $\Omega_{k,m}$, respectively and then move to Steps 2-3.

Step 1 for $\beta_{k,m}$:

Let $\bar{\beta}_{k,m}^*$ satisfies $\nabla_{\beta_{k,m}} Q(\bar{\beta}_{k,m}^*, \Theta' - \beta_{k,m}, |\Theta|) = 0$, without the unit norm constraint. First, we show that $\bar{\beta}_{k,m}^*/\|\bar{\beta}_{k,m}^*\|_2 = \beta_{k,m}^*$. Define $\Psi = (\pi_1, \ldots, \pi_k, U_1, \ldots, U_K, \Sigma_1, \ldots, \Sigma_K)$. With a slight abuse of notation, we may write $Q(\Theta|\Theta^*)$ as $Q'(\Psi|\Psi^*)$. Since $\nabla u_k Q'(U_k|\Psi^*) = 0$, we can get that

$$2E\left[\tau_{ik}(\Theta^*) \{\vec(x_i) - \vec(u_k^*)\} \prod_m \Omega_{k,m}^*\right] = 0, \quad (A50)$$

which implies that $E\{\tau_{ik}(\Theta^*)\} = E\{\tau_{ik}(\Theta^*)\}U_k^*$, as $\Omega_{k,m}^*$’s are positive definite. Plugging this into $\nabla_{\beta_{k,m}} Q(\bar{\beta}_{k,m}^*, \Theta' - \beta_{k,m}, |\Theta|) = 0$, we have

$$\bar{\beta}_{k,m}^* E\{\tau_{ik}(\Theta^*)\} \left\{\omega_k^* \text{vec}(\prod_{m' \neq m} \beta_{m',m'}^*)^\top \left(\prod_{m' \neq m} \Omega_{k,m'}^*\right) \omega_k^* \text{vec}(\prod_{m' \neq m} \beta_{m',m'}^*)\right\}$$

$$= E\{\tau_{ik}(\Theta^*)\} \{\prod_{m' \neq m} \Omega_{k,m'}^*\} \omega_k^* \text{vec}(\prod_{m' \neq m} \beta_{m',m'}^*)$$

$$= \bar{\beta}_{k,m}^* E\{\tau_{ik}(\Theta^*)\}^\top \left\{\omega_k^* \text{vec}(\prod_{m' \neq m} \beta_{m',m'}^*) \left(\prod_{m' \neq m} \Omega_{k,m'}^*\right) \omega_k^* \text{vec}(\prod_{m' \neq m} \beta_{m',m'}^*)\right\}.$$  

It can be seen from the above equality that $\bar{\beta}_{k,m}^* = c\bar{\beta}_{k,m}$. Combined with $\|\bar{\beta}_{k,m}^*\|_2 = 1$, we can get that $\bar{\beta}_{k,m}^*/\|\bar{\beta}_{k,m}^*\|_2 = \beta_{k,m}^*$. By Lemma 1b, with probability at least $1 - 1/\log(nd)^2$,  

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it holds for any $k$ and $m$ that
\[
\frac{\gamma_0}{2} \left\| \tilde{\beta}_{k,m} - \tilde{\beta}_{k,m}^* \right\|^2_2 \leq \left< \nabla \beta_{k,m} Q_n/T (\tilde{\beta}_{k,m}^*, \Theta' - \beta_{k,m} | \Theta), \tilde{\beta}_{k,m} - \tilde{\beta}_{k,m}^* \right> \\
+ Q_n/T (\tilde{\beta}_{k,m}^*, \Theta' - \beta_{k,m} | \Theta) - Q_n/T (\tilde{\beta}_{k,m}, \Theta' - \beta_{k,m} | \Theta). \tag{A51}
\]

First, we discuss the upper bound of $(i)$. Since $\nabla_{\beta_{k,m}} Q(\tilde{\beta}_{k,m}^*, \Theta' - \beta_{k,m} | \Theta) = 0$, we have
\[
(i) = \left< \nabla \beta_{k,m} Q_n/T (\tilde{\beta}_{k,m}^*, \Theta' - \beta_{k,m} | \Theta) - \nabla \beta_{k,m} Q(\tilde{\beta}_{k,m}^*, \Theta' - \beta_{k,m} | \Theta), \tilde{\beta}_{k,m} - \tilde{\beta}_{k,m}^* \right>
\]

Statistical Error (SE)
\[
+ \left< \nabla \beta_{k,m} Q(\tilde{\beta}_{k,m}^*, \Theta' - \beta_{k,m} | \Theta) - \nabla \beta_{k,m} Q(\tilde{\beta}_{k,m}, \Theta' - \beta_{k,m} | \Theta^*), \tilde{\beta}_{k,m} - \tilde{\beta}_{k,m}^* \right>.
\]

Optimization Error (OE)

For SE, by Lemma 3b and letting $\epsilon_0 = c_1 \omega_{\text{max}} \sqrt{\log d/n}$, it holds that
\[
|\text{SE}| \leq \| \nabla \beta_{k,m} Q_n/T (\tilde{\beta}_{k,m}^*, \Theta' - \beta_{k,m} | \Theta) - \nabla \beta_{k,m} Q(\tilde{\beta}_{k,m}^*, \Theta' - \beta_{k,m} | \Theta) \| \mathcal{P}_1 (\tilde{\beta}_{k,m} - \tilde{\beta}_{k,m}^*) \\
\leq \epsilon_0 \mathcal{P}_1 (\tilde{\beta}_{k,m} - \tilde{\beta}_{k,m}^*) \tag{A52}
\]

with probability at least $1 - K(2K + 1)/\{\log(nd)\}^2$. For OE, by Lemma 2b, it holds that
\[
|\text{OE}| \leq \| \nabla \beta_{k,m} Q(\tilde{\beta}_{k,m}^*, \Theta' - \beta_{k,m} | \Theta) - \nabla \beta_{k,m} Q(\tilde{\beta}_{k,m}, \Theta' - \beta_{k,m} | \Theta^*) \|_2 \| \tilde{\beta}_{k,m} - \tilde{\beta}_{k,m}^* \|_2 \\
\leq \tau_0 D(\Theta, \Theta^*) \| \tilde{\beta}_{k,m} - \tilde{\beta}_{k,m}^* \|_2. \tag{A53}
\]

Plugging (A52) and (A53) into term $(i)$, it arrives that
\[
(i) \leq \epsilon_0 \mathcal{P}_1 (\tilde{\beta}_{k,m} - \tilde{\beta}_{k,m}^*) + \tau_0 D(\Theta, \Theta^*) \| \tilde{\beta}_{k,m} - \tilde{\beta}_{k,m}^* \|_2. \tag{A54}
\]

with probability at least $1 - (2K^2 + K + 1)/\{\log(nd)\}^2$.

Next, we consider $(ii)$. Since $\tilde{\beta}_{k,m} = \arg \max_{\beta_{k,m}} Q_n/T (\beta_{k,m}, \Theta' - \beta_{k,m} | \Theta) - \lambda_0 \mathcal{P}_1 (\beta_{k,m})$, some straightforward algebra gives
\[
Q_n/T (\tilde{\beta}_{k,m}^*, \Theta' - \beta_{k,m} | \Theta) - Q_n/T (\tilde{\beta}_{k,m}, \Theta' - \beta_{k,m} | \Theta) \leq \lambda_0 \left( \mathcal{P}_1 (\tilde{\beta}_{k,m}^*) - \mathcal{P}_1 (\tilde{\beta}_{k,m}) \right). \tag{A55}
\]

Let $\mathcal{M}_{\beta_{k,m}}$ be the support space of $\beta_{k,m}^*$ and $\mathcal{M}_{\beta_{k,m}}^\perp$ be the corresponding orthogonal space.
The right-hand side of (A55) can be bounded as
\[
P_1(\tilde{\beta}^*_{k,m} - \beta_{k,m})
= P_1(\tilde{\beta}^*_{k,m} - \beta^*_{k,m} + \beta_{k,m})
\]
\[
= P_1(\beta^*_{k,m}) - P_1\left(\left(\tilde{\beta}^*_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m} + \left(\beta_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m} + \beta^*_{k,m}\right)
\]
\[
\leq P_1\left(\left(\tilde{\beta}^*_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m}\right) - P_1\left(\left(\beta_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m}\right)
\]
where the third equality holds due to \(P_1(\beta) = P_1(\beta_{\mathcal{D}}) + P_1(\beta_{\mathcal{D}^c})\) and the last inequality follows from the fact that \(P_1(\beta_1 + \beta_2) \leq P_1(\beta_1) + P_1(\beta_2)\). Next, it holds that
\[
P_1\left(\left(\tilde{\beta}^*_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m}\right)
\leq P_1(\tilde{\beta}_{k,m}) - P(\beta^*_{k,m}) + P_1\left(\left(\tilde{\beta}^*_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m}\right)
\]
\[
\leq \frac{1}{\lambda_0^{(1)}} \left( Q_{n/T}(\hat{\beta}_{k,m}, \Theta'_{\hat{\beta}_{k,m}} | \Theta) - Q_{n/T}(\beta^*_{k,m}, \Theta'_{\beta^*_{k,m}} | \Theta) \right) + P_1\left(\left(\tilde{\beta}^*_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m}\right)
\]
\[
\leq \frac{1}{\lambda_0^{(1)}} \left\{ e_0 P_1(\tilde{\beta}_{k,m} - \beta^*_{k,m}) + \tau_0 D(\Theta, \Theta^*) \left\| \tilde{\beta}_{k,m} - \beta^*_{k,m} \right\|_2 \right\} + P_1\left(\left(\tilde{\beta}^*_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m}\right)
\]
with probability at least \(1 - (2K^2 + K + 1)/\{\log(nd)\}^2\), where the second inequality is from (A55), the third inequality is from (A18) and the last inequality is a direct result of (A54).

Given that \(\lambda_0^{(1)} = 4e_0 + \frac{\tau_0 D(\Theta, \Theta^*)}{\sqrt{s_1}}\), (A56) can be written as
\[
3P_1\left(\left(\tilde{\beta}^*_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m}\right) \leq 5P_1\left(\left(\tilde{\beta}^*_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m}\right) + 4s_1 \left\| \tilde{\beta}_{k,m} - \beta^*_{k,m} \right\|_2. \tag{A57}
\]
Then, we have that
\[
P_1(\beta_{k,m} - \beta^*_{k,m})
\leq P_1\left(\left(\tilde{\beta}^*_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m}\right) + P_1\left(\left(\beta_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m}\right)
\]
\[
\leq \frac{8}{3} P_1\left(\left(\tilde{\beta}^*_{k,m} - \beta^*_{k,m}\right)M_{\mathcal{D}_k,m}\right) + \frac{4}{3} s_1 \left\| \tilde{\beta}_{k,m} - \beta^*_{k,m} \right\|_2 \leq 4s_1 \left\| \tilde{\beta}_{k,m} - \beta^*_{k,m} \right\|_2.
\]
with probability at least \(1 - (2K^2 + K + 1)/\{\log(nd)\}^2\), where the second inequality is due to (A57) and the last inequality is due to
Correspondingly, it holds that
\[
\frac{\gamma_0}{2} \| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2^2 \\
\leq \epsilon_0 P_1(\tilde{\beta}_{k,m} - \beta^*_{k,m}) + \tau_0 D(\Theta, \Theta^*) \| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 + \lambda_0^{(1)} \left\{ P_1(\beta^*_{k,m}) - P_1(\tilde{\beta}_{k,m}) \right\} \\
\leq 4\epsilon_0 \sqrt{s_1} \| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 + \gamma_0^{(1)} \left\{ P_1((\tilde{\beta}_{k,m} - \beta^*_{k,m}), \mathcal{M}_{\tilde{\beta}_{k,m}}) - P_1((\tilde{\beta}_{k,m} - \beta^*_{k,m}), \mathcal{M}_{\beta^*_{k,m}}) \right\} \\
\leq 4\epsilon_0 \sqrt{s_1} \| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 + \tau_0 D(\Theta, \Theta^*) \| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 + \lambda_0^{(1)} P_1((\tilde{\beta}_{k,m} - \beta^*_{k,m}), \mathcal{M}_{\tilde{\beta}_{k,m}}) \\
\leq 2\lambda_0^{(1)} \sqrt{s_1} \| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 ,
\] (A60)

with probability at least \(1 - (2K^2 + K + 1)/\{\log(nd)\}^2\), where the first inequality is by (A54) and (A55). Dividing both sizes of (A60) by \(\| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2\), it follows that
\[
\| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 \leq \frac{16\sqrt{s_1}\epsilon_0}{\gamma_0} + \frac{4\tau_0 D(\Theta, \Theta^*)}{\gamma_0},
\] (A61)

with probability at least \(1 - (2K^2 + K + 1)/\{\log(nd)\}^2\). Since \(\tilde{\beta}_{k,m}/\| \tilde{\beta}_{k,m} \|_2 = \beta^*_{k,m}\), we have
\[
\| \beta_{k,m} - \beta^*_{k,m} \|_2 \leq \| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 + \| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 + \| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 \\
\leq \frac{2\lambda_0^{(1)} \sqrt{s_1}}{\| \tilde{\beta}_{k,m} \|_2} \| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 ,
\] (A62)

where the last inequality uses that
\[
\| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 \leq \| \tilde{\beta}_{k,m} \|_2 \| \beta_{k,m} - \beta^*_{k,m} \|_2 \\
= \| \tilde{\beta}_{k,m} \|_2 \| \beta_{k,m} - \beta^*_{k,m} \|_2 \\
\leq \frac{1}{\| \tilde{\beta}_{k,m} \|_2} \| \tilde{\beta}_{k,m} - \beta^*_{k,m} \|_2 .
\]

Recall \(\alpha = D(\Theta^{(0)}, \Theta^*)\). Next, we have
\[
\| \tilde{\beta}_{k,m} \|_2 = \omega_k^\prime \text{vec}( \prod_{m' \neq m} \beta^*_{k,m'} )^\top \left( \sum_{m' \neq m} \Omega_{k,m'} \right) \omega_k^\prime \text{vec}( \prod_{m' \neq m} \beta_{k,m'} ) \\
\geq \omega_k^\prime \frac{\phi_1}{3\phi_2} \left( \sum_{m' \neq m} \beta^*_{k,m'} \right)^{M-1} \text{vec}( \prod_{m' \neq m} \beta^*_{k,m'} ) \\
\geq (1 + \alpha)^{-1} \left( \sum_{m' \neq m} \beta_{k,m'} \right)^{M-1} \left( \sum_{m' \neq m} \beta^*_{k,m'} \right)^{M-1} \left( 1 - \| \beta_{k,m'} - \beta^*_{k,m'} \|_2^2 \right) \geq \left( \frac{\phi_1}{3\phi_2} \right)^{M-1} \frac{(1 - \alpha)^{M-1}}{1 + \alpha} .
\]

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where the first inequality uses that \( \frac{\phi}{2} \leq \sigma_{\min}(\Omega_{k,m}') \leq \sigma_{\min}(\Omega_{k,m}) \leq \frac{3\phi_2}{2} \) and the last inequality uses the fact that \( \text{vec}(\prod_{m' \neq m} \beta^{*}_{k,m'}) \text{vec}(\prod_{m' \neq m} \beta^{*}_{k,m'}) \geq \prod_{m' \neq m}(1 - \|\beta^{*}_{k,m'} - \beta^{*}_{k,m'}\|_2^2). \) This fact can be easily shown. For example, when \( m = M = 3 \), it holds that (general cases follow similarly)

\[
\text{vec}(\beta^{*}_{k,1} \circ \beta^{*}_{k,2}) \text{vec}(\beta^{*}_{k,1} \circ \beta^{*}_{k,2}) = \sum_{l_1,l_2} \beta^{*}_{k,1}(l_1) \beta^{*}_{k,2}(l_2) \beta^{*}_{k,1}(l_1) \beta^{*}_{k,2}(l_2)
\]

\[
= \sum_{l_1,l_2} \beta^{*}_{k,1}(l_1) \{ \beta^{*}_{k,1}(l_1) - \beta^{*}_{k,1}(l_1) + \beta^{*}_{k,1}(l_1) \} \beta^{*}_{k,2}(l_2) \beta^{*}_{k,2}(l_2)
\]

\[
= \sum_{l_2} \beta^{*}_{k,2}(l_2) \beta^{*}_{k,2}(l_2) \sum_{l_1} \{ \beta^{*}_{k,1}(l_1) \}^2 - \{ \beta^{*}_{k,1}(l_1) \} \{ \beta^{*}_{k,1}(l_1) - \beta^{*}_{k,1}(l_1) \}
\]

\[
\geq \sum_{l_2} \beta^{*}_{k,2}(l_2) \beta^{*}_{k,2}(l_2) (1 - \|\beta^{*}_{k,2} - \beta^{*}_{k,2}\|_2^2) \geq (1 - \|\beta^{*}_{k,1} - \beta^{*}_{k,1}\|_2^2) (1 - \|\beta^{*}_{k,2} - \beta^{*}_{k,2}\|_2^2).
\]

By \((A61)\) and \( \Theta \in B_\alpha(\Theta^*) \), we have \( \|\hat{\beta}_{k,m} - \tilde{\beta}_{k,m}'\|_2 \leq \frac{1}{4} \|\hat{\beta}_{k,m}\|_2 + \frac{\alpha}{3\sqrt{K(R+1)(M+1)}} \) when \( n \) is sufficiently large and \( \gamma \leq \gamma_0 \). Thus, there exists one positive constant \( C \) such that

\[
\|\hat{\beta}_{k,m}\|_2 \geq \|\hat{\beta}_{k,m}'\|_2 - \|\hat{\beta}_{k,m}' - \hat{\beta}_{k,m}\|_2 \geq 2C.
\]

Plugging this into \((A62)\), we have

\[
\|\beta^{*}_{k,m} - \beta^{*}_{k,m}\|_2 \leq \frac{16\sqrt{s}\epsilon_0}{C\gamma_0} + \frac{4\tau_0 D(\Theta, \Theta^*)}{C\gamma_0}, \tag{A63}
\]

with probability at least \( 1 - (2K^2 + K + 1)/\{\log(nd)\}^2 \).

**Step 1 for \( \omega_k \):**

It holds from Lemma 1b to obtain that

\[
\gamma_0'' \|\omega_k'' - \omega_k'\|_2^2 \leq \langle \nabla_{\omega_k} Q_{n/T}(\omega_k'', \Theta_{\omega_k}^*| \Theta), \omega_k'' - \omega_k' \rangle + Q_{n/T}(\omega_k'', \Theta_{\omega_k}^*| \Theta) - Q_{n/T}(\omega_k'', \Theta_{\omega_k}^*| \Theta), \tag{iii}
\]

\[
\text{with probability at least } 1 - 1/\{\log(nd)\}^2. \quad \text{We will bound terms } (\text{iii}) \text{ and } (\text{iv}) \text{ respectively.} \]

Since that \( \omega_k'' \) is the maximizer, we have \( Q_{n/T}(\omega_k'', \Theta_{\omega_k}^*| \Theta) \geq Q_{n/T}(\omega_k'', \Theta_{\omega_k}^*| \Theta) \), which implies that

\[
(\text{iv}) = Q_{n/T}(\omega_k'', \Theta_{\omega_k}^*| \Theta) - Q_{n/T}(\omega_k'', \Theta_{\omega_k}^*| \Theta) \leq 0. \tag{A65}
\]

Let \( \epsilon_0'' = c_1'' \omega_{\max} \sqrt{\log \log(nd)/n} \), where \( c_1'' \) is as defined in Lemma 3b. Similar to (i), we can
get that

\[
(iii) = \langle \nabla_{\omega_k} Q_{m/T}(\omega^*_k, \Theta_{-\omega_k} \mid \Theta), \omega''_k - \omega^*_k \rangle \\
+ \langle \nabla_{\omega_k} Q(\omega^*_k, \Theta_{-\omega_k} \mid \Theta), \omega''_k - \omega^*_k \rangle \\
+ \langle \nabla_{\omega_k} Q(\omega^*_k, \Theta_{-\omega_k} \mid \Theta^*), \omega''_k - \omega^*_k \rangle \\
\leq \epsilon_0'' |\omega''_k - \omega^*_k| + \tau_0'' D(\Theta, \Theta^*) |\omega''_k - \omega^*_k| + \langle \nabla_{\omega_k} Q(\omega^*_k, \Theta_{-\omega_k} \mid \Theta^*), \omega''_k - \omega^*_k \rangle
\]

with probability at least \(1 - (K + 1)/\{\log(nd)\}^2\). By (A24) and (A50), we have that

\[
|\nabla_{\omega_k} Q(\omega^*_k, \Theta_{-\omega_k} \mid \Theta)|
\leq \omega_k^* (3\phi_2/2)^{M-1} \left\{ \text{vec}(\prod_m \beta^*_{k,m}) - \text{vec}(\prod_m \beta'_{k,m}) \right\}^T \left( \prod_{m'} \Omega'_{k,m'} \right) \text{vec}(\prod_m \beta'_{k,m})
\leq \omega_k^* (3\phi_2/2)^{M-1} \sqrt{M} \sum_m \|\beta'_{k,m} - \beta^*_{k,m}\|_2;
\]

with probability at least \(1 - (K + 1)/\{\log(nd)\}^2\), where the first inequality is obtained by \(\|\Omega'_{k,m}\|_2 \leq \|\Omega^*_{k,m}\|_2 + \|\Omega'_{k,m} - \Omega^*_{k,m}\|_2 \leq 3\phi_2/2\), the second inequality follows from the fact that \(\|\text{vec}(\prod_m \beta^*_{k,m}) - \text{vec}(\prod_m \beta'_{k,m})\|_2 \leq \sqrt{M} \sum_m \|\beta'_{k,m} - \beta^*_{k,m}\|_2\). This fact is easy to verify. For example, when \(M = 3\), we have (general cases follow similarly)

\[
\|\text{vec}(\prod_m \beta^*_{k,m}) - \text{vec}(\prod_m \beta'_{k,m})\|_2^2 = \|\prod_m \beta^*_{k,m} - \prod_m \beta'_{k,m}\|_F^2
\]
\[
= \sum_{l_1,l_2,l_3} \{\beta^*_{k,1}(l_1)\beta^*_{k,2}(l_2)\beta^*_{k,3}(l_3) - \beta'_{k,1}(l_1)\beta'_{k,2}(l_2)\beta'_{k,3}(l_3)\}^2
\leq 3 \sum_{l_1,l_2,l_3} \{\beta^*_{k,1}(l_1) - \beta'_{k,1}(l_1)\}^2 \{\beta^*_{k,2}(l_2)\}^2 \{\beta^*_{k,3}(l_3)\}^2 + 3 \sum_{l_1,l_2,l_3} \{\beta^*_{k,2}(l_2) - \beta'_{k,2}(l_2)\}^2 \{\beta^*_{k,1}(l_1)\}^2 \{\beta^*_{k,3}(l_3)\}^2
+ 3 \sum_{l_1,l_2,l_3} \{\beta^*_{k,3}(l_3) - \beta'_{k,3}(l_3)\}^2 \{\beta^*_{k,1}(l_1)\}^2 \{\beta^*_{k,2}(l_2)\}^2
\]
\[
= 3 \sum_{m=1}^3 \|\beta'_{k,m} - \beta^*_{k,m}\|_2^2.
\]

Combining (A66) and (A65), we can get that, for any \(k\),

\[
\frac{|\omega''_k - \omega^*_k|}{|\omega_k^*|} \leq \frac{2\epsilon_0''}{\omega_k^*} + \frac{2\tau_0''}{\omega_k^*} D(\Theta, \Theta^*) + 2(3\phi_2/2)^{M-1} \sqrt{M} \sum_m \|\beta'_{k,m} - \beta^*_{k,m}\|_2,
\]

with probability at least \(1 - (2K^2 + 2K + 1)/\{\log(nd)\}^2\). 

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Step 1 for $\Omega_{k,m}$:

By Lemma 1b, with probability at least $1 - \{\log(nd)\}^2$, it holds for any $k$ and $m$ that

\[
\frac{\gamma_m}{2} \left\| \hat{\Omega}_{k,m} - \Omega_{k,m}^* \right\|^2 \leq \left\langle \nabla_{\Omega_{k,m}} Q_{n/T}(\Omega_{k,m}^*, \Theta'_{-\Omega_{k,m}}, \Theta), \hat{\Omega}_{k,m} - \Omega_{k,m}^* \right\rangle \\
+ \frac{Q_{n/T}(\Omega_{k,m}^*, \Theta'_{-\Omega_{k,m}}, \Theta)}{\epsilon_m - \epsilon_m^2} - Q_{n/T}(\hat{\Omega}_{k,m}, \Theta'_{-\Omega_{k,m}}, \Theta) \right\rangle .
\] (A67)

First, we consider term $(v)$. Letting $\epsilon_m = c_2(d/d_m)\sqrt{\log d/n}$, it holds that

\[
(v) = \left\langle \nabla_{\Omega_{k,m}} Q_{n/T}(\Omega_{k,m}^*, \Theta'_{-\Omega_{k,m}, \Theta}), \hat{\Omega}_{k,m} - \Omega_{k,m}^* \right\rangle \\
+ \left\langle \nabla_{\Omega_{k,m}} Q(\Omega_{k,m}^*, \Theta'_{-\Omega_{k,m}, \Theta}'), \hat{\Omega}_{k,m} - \Omega_{k,m}^* \right\rangle \\
+ \left\langle \nabla_{\Omega_{k,m}} Q(\Omega_{k,m}^*, \Theta'_{-\Omega_{k,m}, \Theta}'), \hat{\Omega}_{k,m} - \Omega_{k,m}^* \right\rangle \\
\leq \epsilon_m P_2(\hat{\Omega}_{k,m} - \Omega_{k,m}^* + \tau_1 D(\Theta, \Theta^*) \left\| \hat{\Omega}_{k,m} - \Omega_{k,m}^* \right\|_F \\
+ \left\langle \nabla_{\Omega_{k,m}} Q(\Omega_{k,m}^*, \Theta'_{-\Omega_{k,m}, \Theta}'), \hat{\Omega}_{k,m} - \Omega_{k,m}^* \right\rangle ,
\]

with probability at least $1 - (8K^2 + 2K + 1)/\{\log(nd)\}^2$, where the last inequality holds due to Lemmas 2b-3b. Since $\nabla_{\Omega_{k,m}} Q(\Omega_{k,m}^*, \Theta'_{-\Omega_{k,m}, \Theta}^*) = 0$, we can get that

\[
\nabla_{\Omega_{k,m}} Q(\Omega_{k,m}^*, \Theta'_{-\Omega_{k,m}, \Theta}^*) = \nabla_{\Omega_{k,m}} Q(\Omega_{k,m}^*, \Theta'_{-\Omega_{k,m}, \Theta}^*) - \nabla_{\Omega_{k,m}} Q(\Omega_{k,m}^*, \Theta'_{-\Omega_{k,m}, \Theta}^*) \\
= \frac{1}{2} \left[ \tau_{ik}(\Theta^*) \left\{ \frac{d}{2d_m}(\Omega_{k,m}^*)^{-1} - \frac{1}{2} (\mathbf{x}_i - \mathbf{u}_k^*)_m \left( \prod_{m' \neq m}^\otimes \Omega_{k,m'}^* \right) (\mathbf{x}_i - \mathbf{u}_k^*)^\top_m \right\} \\
- \tau_{ik}(\Theta^*) \left\{ \frac{d}{2d_m}(\Omega_{k,m}^*)^{-1} - \frac{1}{2} (\mathbf{x}_i - \mathbf{u}_k^*)_m \left( \prod_{m' \neq m}^\otimes \Omega_{k,m'}^* \right) (\mathbf{x}_i - \mathbf{u}_k^*)^\top_m \right\} \right] \\
= \frac{1}{2} \left[ \tau_{ik}(\Theta^*) (\mathbf{x}_i - \mathbf{u}_k^*)_m \left( \prod_{m' \neq m}^\otimes \Omega_{k,m'}^* \right) (\mathbf{x}_i - \mathbf{u}_k^*)^\top_m \right] \\
+ \frac{1}{2} \left[ \tau_{ik}(\Theta^*) \left( \prod_{m' \neq m}^\otimes \Omega_{k,m'}^* \right) (\mathbf{u}_k^*)^\top_m \right] \\
= \frac{1}{2} \left[ \tau_{ik}(\Theta^*) \left( \prod_{m' \neq m}^\otimes \Omega_{k,m'}^* \right) (\mathbf{u}_k^*)^\top_m \right] .
\]

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We discuss these three terms $A_1$, $A_2$ and $A_3$, respectively. First, $A_1$ can be written as

$$A_1 = \mathbb{E} \left\{ \tau_{ik}(\Theta^*) (\mathbf{x}_i - \mathbf{u}_{k}(m)) \left( \prod_{m' \neq m} \Omega_{k,m'} - \prod_{m' \neq m} \Omega^*_{k,m'} \right) (\mathbf{x}_i - \mathbf{u}_{k}(m))^\top | Z_i \right\}.$$ 

If $Z_i = k'$, then $\mathbf{x}_i \sim \mathcal{N}(\mathbf{u}_{k'}^*, \Sigma_{k'}^*)$. We may write

$$\mathbf{x}_i = \sum_{k' = 1}^{K} I(Z_i = k') (\mathbf{u}_{k'}^* + \mathbf{v}_{k'}),$$

where $\mathbf{v}_{k'} \sim \mathcal{N}(0, \Sigma_{k'}^*)$. Correspondingly, $A_1$ can be expressed as

$$A_1 = \left[ \mathbb{E} \left\{ \tau_{ik}(\Theta^*)^2 (\mathbf{v}_{k}(m)) \left( \prod_{m' \neq m} \Omega_{k,m'} - \prod_{m' \neq m} \Omega^*_{k,m'} \right) \right\} \right]_{A_{11}}$$

$$+ \sum_{l \neq k} \mathbb{E} \left\{ \tau_{ik}(\Theta^*) \tau_{il}(\Theta^*) (\mathbf{x}_i - \mathbf{u}_{k}(m)) \left( \prod_{m' \neq m} \Omega_{k,m'} - \prod_{m' \neq m} \Omega^*_{k,m'} \right) \left( \mathbf{x}_i - \mathbf{u}_{k}(m) \right)^\top \right\}_{A_{12}}.$$ 

Let $Y = (\Omega_{k,m}^*)^{1/2} (\mathbf{v}_{k}(m)) \left( \prod_{m' \neq m} (\Omega_{k,m'}^*)^{1/2} \right)$, where $\Omega_{k,m} = (\Sigma_{k,m}^*)^{-1}$. Now $A_{11}$ can be bounded by

$$\|A_{11}\|_F \leq \| \mathbb{E} \left\{ (\Sigma_{k,m}^*)^{1/2} Y D Y^\top (\Sigma_{k,m}^*)^{1/2} \right\} \|_F = \| \text{tr}(D) \Sigma_{k,m}^* \|_F \leq \sqrt{d m} \phi_1^{-1} |\text{tr}(D)|,$$

where $D = \left( \prod_{m' \neq m} (\Sigma_{k,m'}^*)^{-1/2} \right) \left( \prod_{m' \neq m} \Omega_{k,m'}^* - \prod_{m' \neq m} \Omega_{k,m'}^* \right) \left( \prod_{m' \neq m} (\Sigma_{k,m'}^*)^{-1/2} \right)$, and the second equality is by Lemma S8 and the last inequality follows from the fact that $\|B\|_F \leq \sqrt{m} \|B\|_2$ for any matrix $B \in \mathbb{R}^{m \times m}$ and $\sigma_{\text{max}}(\Sigma_{k,m}^*) \leq \phi_1^{-1}$. Also, we have that

$$\left\| \prod_{m' \neq m} \Sigma_{k,m'}^* \right\|_F \leq \sqrt{d m} \left\| \prod_{m' \neq m} \Sigma_{k,m'}^* \right\|_2 \leq \phi_1^{1-M} \sqrt{\frac{d}{d m}}.$$

By the fact that $\text{tr}(AB) = \sum_{i,j} A_{i,j} B_{j,i}$, we have

$$|\text{tr}(D)| \leq \sum_{i,j} \left( \prod_{m' \neq m} \Sigma_{k,m'}^* \right)_{i,j} \left( \prod_{m' \neq m} \Omega_{k,m'}^* - \prod_{m' \neq m} \Omega_{k,m'}^* \right)_{j,i} \leq \left\| \prod_{m' \neq m} \Sigma_{k,m'}^* \right\|_F \left\| \prod_{m' \neq m} \Omega_{k,m'}^* - \prod_{m' \neq m} \Omega_{k,m'}^* \right\|_F \leq \frac{d}{d m} \sum_{m' \neq m} \frac{\phi_1^{1-M}}{\sqrt{d m}} \left\| \Omega_{k,m'}^* - \Omega_{k,m'}^* \right\|_F.$$
where the last inequality is due to (in the example of \( m = 3 \) and \( M = 3 \) while general cases follow similarly)

\[
\| \Omega'_{k,2} \otimes \Omega'_{k,1} - \Omega^*_{k,2} \otimes \Omega^*_{k,1} \|_F = \| \Omega'_{k,2} \otimes (\Omega'_{k,1} + \Omega^*_{k,1}) - (\Omega'_{k,2} - \Omega^*_{k,2}) \otimes \Omega^*_{k,1} \|_F \\
\leq \| \Omega'_{k,2} \otimes (\Omega'_{k,1} + \Omega^*_{k,1}) \|_F + \| (\Omega'_{k,2} - \Omega^*_{k,2}) \otimes \Omega^*_{k,1} \|_F \\
= \sqrt{\frac{d}{d_3 d_1}} \| \Omega'_{k,1} - \Omega^*_{k,1} \|_F + \sqrt{\frac{d}{d_3 d_2}} \| \Omega'_{k,2} - \Omega^*_{k,2} \|_F.
\]

As now \( \| A_{11} \|_F \leq \frac{d}{\sqrt{d_m}} \sum m' \neq m \frac{\phi_1}{\sqrt{d_m}} \| \Omega'_{k,m'} - \Omega^*_{k,m'} \|_F \), it holds that \( \| A_{11} \|_F/D(\Theta, \Theta^*) = o(d) \).

Using similar arguments as in (A29) and (A68), we can also establish that \( \| A_{12} \|_F/D(\Theta, \Theta^*) = o(d) \). Correspondingly, we have \( A_1/D(\Theta, \Theta^*) = o(d) \).

Next, we consider terms \( A_2 \) and \( A_3 \). By (A50), \( A_2 \) can be written as

\[
A_2 = E \left[ \tau_{ik}(\Theta^*) (\mathbf{X}_i)_{(m)} \right] \left( \prod_{m' \neq m} \Omega^*_{k,m'} \right) (\mathbf{U}'_k - \mathbf{U}^*_k)_{(m)}^T

= E \left[ \tau_{ik}(\Theta^*) \right] (\mathbf{U}'_k)_{(m)} \left( \prod_{m' \neq m} \Omega^*_{k,m'} \right) (\mathbf{U}'_k - \mathbf{U}^*_k)_{(m)}^T.
\]

Combining \( A_2 \) and \( A_3 \), we have that

\[
A_2 + A_3 = -\frac{1}{2} E \left[ \tau_{ik}(\Theta^*) \right] (\mathbf{U}'_k - \mathbf{U}^*_k)_{(m)} \left( \prod_{m' \neq m} \Omega^*_{k,m'} \right) (\mathbf{U}'_k - \mathbf{U}^*_k)_{(m)}^T.
\]

Recall that \( \mathbf{U}'_k = \omega'_k \beta'_{k,1} \circ \cdots \circ \beta'_{k,M} \) and \( \mathbf{U}^*_k = \omega^*_k \beta^*_{k,1} \circ \cdots \circ \beta^*_{k,M} \). Letting \( \mathbf{U}''_k = \omega''_k \beta''_{k,1} \circ \cdots \circ \beta''_{k,M} \), we have

\[
\| A_2 + A_3 \|_F \leq \frac{1}{2} \left\| (\mathbf{U}'_k - \mathbf{U}''_k + \mathbf{U}''_k - \mathbf{U}''_k)(m) \right\|_F \left( \prod_{m' \neq m} \Omega^*_{k,m'} \right) (\mathbf{U}'_k - \mathbf{U}''_k)(m)^T

\leq (\omega_k^*)^2 \left\| \left( \prod_{m' \neq m} \beta^*_{k,m'} \right) \left( \prod_{m' \neq m} \Omega^*_{k,m'} \right) \left( \prod_{m' \neq m} \beta^*_{k,m'} \right)(m)^T \right\|_F

+ (\omega'_k - \omega_k^*)^2 \left\| \left( \prod_{m' \neq m} \beta'_{k,m'} \right) \left( \prod_{m' \neq m} \Omega^*_{k,m'} \right)(m)^T \right\|_F

\leq (\omega_k^*)^2 \phi_2^{M-1} \left\| \prod_{m' \neq m} \beta^*_{k,m'} - \prod_{m' \neq m} \beta''_{k,m'} \right\|_F^2 + (\omega'_k - \omega_k^*)^2 \phi_2^{M-1}

\leq (\omega_k^*)^2 \phi_2^{M-1} \left( \sum_{m'} \| \beta'_{k,m'} - \beta''_{k,m'} \|_2 \right)^2 + (\omega'_k - \omega_k^*)^2 \phi_2^{M-1},
\]

(A68)
where the first inequality uses the fact that \(2a_1^TBA_1 + 2a_2^TBA_2 - (a_1 + a_2)^TBA_1 + a_2 = (a_1 - a_2)^TB(a_1 - a_2) \geq 0\) for non-negative definite matrix \(B\), the second inequality is due to the fact that \(\|AB\|_F^2 \leq \|A\|_F^2\|B\|_F^2\) for any matrix \(A, B \in \mathbb{R}^{n \times n}\) and the last inequality uses the fact that \(\|\prod_{m'} \beta_{k,m'} - \prod_{m'} \beta_{k,m'}\|_F \leq \sum_{m'} \|\beta_{k,m'} - \beta_{k,m'}^*\|_2\). This fact can be verified as follows when \(M = 3\) while general cases follows similarly.

\[
\left\| \prod_{m'} \beta_{k,m'} - \prod_{m'} \beta_{k,m'}^* \right\|_F = \left\| \prod_{m'} \beta_{k,m'} - \beta_{k,1} \prod_{m' \neq 1} \beta_{k,m'} + \beta_{k,1} \prod_{m' \neq 1} \beta_{k,m'} - \prod_{m'} \beta_{k,m'}^* \right\|_F \\
\leq \|\beta_{k,1} - \beta_{k,1}^*\|_2 \left\| \prod_{m' \neq 1} \beta_{k,m'} \right\|_F + \|\beta_{k,1}^*\|_2 \left\| \prod_{m' \neq 1} \beta_{k,m'} - \prod_{m'} \beta_{k,m'}^* \right\|_F \\
\leq \|\beta_{k,1} - \beta_{k,1}^*\|_2 + \left\| \prod_{m' \neq 1} \beta_{k,m'} - \prod_{m'} \beta_{k,m'}^* \right\|_F \\
\leq \|\beta_{k,1} - \beta_{k,1}^*\|_2 + \cdots \leq \sum_{m'} \|\beta_{k,m'} - \beta_{k,m'}^*\|_2.
\]

Given that \(\|\beta_{k,m'} - \beta_{k,m'}^*\|_2 \leq \alpha, \frac{\omega_{k} - \omega_{k}^*}{\omega_{k}^*} \leq \alpha\) and \(s^M_1 \|U_k^*\|_{\text{max}} = o(d^{1/2})\), we have that \(\|A_2 + A_3\|_F/D(\Theta, \Theta^*) = o(d)\). Together with \(A_1\), we have that

\[
\|\nabla_{\Omega_{k,m}} Q(\Omega_{k,m}^*, \Theta'_{k,m} | \Theta^*) - \nabla_{\Omega_{k,m}} Q(\Omega_{k,m}^*, \Theta^*_{k,m} | \Theta^*)\|_F/D(\Theta, \Theta^*) = o(d).
\]

As \(\tau_1 = O(d)\), it then holds that

\[
\|\nabla_{\Omega_{k,m}} Q(\Omega_{k,m}^*, \Theta'_{k,m} | \Theta^*) - \nabla_{\Omega_{k,m}} Q(\Omega_{k,m}^*, \Theta^*_{k,m} | \Theta^*)\|_F \leq \frac{\tau_1}{2} D(\Theta, \Theta^*).
\]

Similar as term \((ii)\) in Step 1 for \(\beta_{k,m}\), term \((vi)\) can be bounded considering

\[
Q_{n/T}(\Omega_{k,m}^*, \Theta'_{k,m} | \Theta) - Q_{n/T}(\hat{\Omega}_{k,m}, \Theta'_{k,m} | \Theta) \leq \lambda_m^{(1)} \left( P_2(\Omega_{k,m}^*) - P_2(\hat{\Omega}_{k,m}) \right).
\]

(A69)

Given \(\lambda_m^{(1)} = 4\epsilon_m + \frac{3\tau_1 D(\Theta, \Theta^*)}{2\sqrt{\epsilon_2 + \epsilon_m}}\), similar arguments as (A61) give

\[
\left\| \hat{\Omega}_{k,m} - \Omega_{k,m}^* \right\|_F \leq \frac{16\sqrt{s_2 + d_m\epsilon_m}}{\gamma_m} + \frac{6\tau_1 D(\Theta, \Theta^*)}{\gamma_m}.
\]

(A70)

with probability at least \(1 - (8K^2 + 2K + 1)/\{\log(nd)\}^2\), and \(\gamma_m = c_0(2\phi_2)^{-2}d/d_m\). Since \(\Omega_{k,m}'' = \sqrt{d_m \Omega_{k,m}}/\|\Omega_{k,m}\|_2\), we get that

\[
\frac{\|\Omega_{k,m}'' - \Omega_{k,m}^*\|_F}{\|\Omega_{k,m}''\|_F} \leq \frac{\|\hat{\Omega}_{k,m} - \Omega_{k,m}^*\|_F}{\|\Omega_{k,m}^*\|_F} \\
\leq \frac{\|\hat{\Omega}_{k,m} - \Omega_{k,m}^*\|_F}{\|\Omega_{k,m}\|_2} + \frac{\|\Omega_{k,m}^* - \Omega_{k,m}''\|_F}{\|\Omega_{k,m}^*\|_F} \\
\leq \frac{2}{\|\Omega_{k,m}\|_F} \|\Omega_{k,m} - \Omega_{k,m}^*\|_F.
\]

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The last inequality uses that
\[
\left\| \Omega^*_{k,m} - \Omega^*_{k,m} \right\|_{\mathcal{F}} = \left\| \Omega^*_{k,m} \right\|_{\mathcal{F}} \left( \left\| \Omega^*_{k,m} \right\|_{\mathcal{F}} - \left\| \Omega^*_{k,m} \right\|_{\mathcal{F}} \right) \leq \frac{1}{\left\| \Omega^*_{k,m} \right\|_{\mathcal{F}}} \left\| \Omega^*_{k,m} - \Omega^*_{k,m} \right\|_{\mathcal{F}}.
\]
By (A70), we have \( \left\| \Omega^*_{k,m} - \Omega^*_{k,m} \right\|_{\mathcal{F}} \leq \left\| \Omega^*_{k,m} \right\|_{\mathcal{F}}/4 + \frac{\alpha}{2\sqrt{K(R+1)(M+1)}} \) when \( n \) is sufficiently large and \( \gamma \leq \gamma_m \). Note that \( \left\| \Omega^*_{k,m} \right\|_{\mathcal{F}} = \sqrt{\gamma_m} \), then there exists a positive constant \( C' \) such that \( \left\| \Omega^*_{k,m} \right\|_{\mathcal{F}} \geq 2C'\sqrt{\gamma_m} \). Now we can claim that
\[
\left\| \Omega^*_{k,m} - \Omega^*_{k,m} \right\|_{\mathcal{F}} \leq \frac{16\sqrt{s_1 + d_m}\epsilon_m}{C'\sqrt{\gamma_m}} + \frac{6\tau_1D(\Theta, \Theta^*)}{C'\sqrt{\gamma_m}},
\]
with probability at least \( 1 - (8K^2 + 2K + 1)/\{\log(nd)\}^2 \).

**Step 2 for \( \beta_{k,m} \):**

Let \( \Theta = \Theta^{(0)} \), \( \Theta'_{-k,m} = \Theta^{(1)}_{-k,m} \). Plugging them into Step 1, we can obtain that
\[
\left\| \beta_{k,m}^{(1)} - \beta_{k,m}^* \right\|_2 \leq \frac{16\sqrt{s_1\epsilon_0}}{C\gamma_0} + \frac{4\tau_0D(\Theta^{(0)}, \Theta^*)}{C\gamma_0},
\]
with probability at least \( 1 - (2K^2 + K + 1)/\{\log(nd)\}^2 \). By Lemmas 1b-2b, we know that
\[
\frac{4\tau_0}{C\gamma_0} = \frac{\gamma}{3Cc_0\sqrt{K(R+1)(M+1)}(1-\alpha)^2\omega^{2}\min(\phi_1/2)^M}.
\]
Letting \( \gamma \leq Cc_0\sqrt{K(R+1)(M+1)}(1-\alpha)^2\omega^{2}\min(\phi_1/2)^M \), it then follows that \( \frac{4\tau_0D(\Theta^{(0)}, \Theta^*)}{C\gamma_0} \leq \frac{2\alpha}{3} \). In addition, when \( n/T \) is sufficiently large, we get that
\[
\frac{16\sqrt{s_1\epsilon_0}}{C\gamma_0} \leq \frac{64c_1\omega_{\max}}{Cc_0(\phi_1/2)^M\omega_{\min}} \cdot \frac{1}{\omega_{\min}} \left( \frac{s_1\log d}{n/T} \right) \leq \frac{2\alpha}{3},
\]
(A73)

Thus, we have \( \left\| \beta_{k,m}^{(1)} - \beta_{k,m}^* \right\|_2 \leq \alpha \) which implies that \( \Theta'' = (\beta_{k,m}^{(1)}, \Theta^{(1)}_{-k,m}) \in \mathcal{B}_\alpha(\Theta^*) \), ensuring the initial guarantee for the next parameter update.

**Step 2 for \( \omega_m \):**

Similarly, let \( \Theta = \Theta^{(0)} \), \( \Theta'_{-k} = \Theta^{(1)}_{-k} \). Plugging them into Step 1, we can obtain that
\[
\left\| \omega_k^{(1)} - \omega_k^* \right\|_2 \leq \frac{2\epsilon_0'}{\omega_k'^{\gamma_0}} + \frac{2\sigma_0'}{\omega_k'^{\gamma_0}}D(\Theta^{(0)}, \Theta^*) + \frac{2(3\phi_2/2)^{M-1}\sqrt{M}}{\gamma_0} \sum_m \left\| \beta_{k,m}^{(1)} - \beta_{k,m}^* \right\|_2 \leq \frac{2\epsilon_0'}{\omega_k'^{\gamma_0}} + \frac{2\sigma_0'}{\omega_k'^{\gamma_0}}D(\Theta^{(0)}, \Theta^*) + \frac{2(3\phi_2/2)^{M-1}\sqrt{M}}{\gamma_0} \left\{ \frac{16\sqrt{s_1\epsilon_0}}{C\gamma_0} + \frac{4\tau_0D(\Theta^{(0)}, \Theta^*)}{C\gamma_0} \right\},
\]
(A74)
with probability at least $1 - (2K^2 + 2K + 1) / \{ \log(nd) \}^2$. Setting $\gamma$ as a sufficiently small constant and then we have

$$\left\{ \frac{2\tau''}{\omega^*_k \gamma'_0} + \frac{2(3\phi_2/2)^{M-1}M^{3/2}}{\gamma'_0 C\gamma_0} \right\} \text{D}(\Theta^{(0)}, \Theta^*) \leq \frac{1}{3} \alpha.$$  

Also, when $n/T$ is sufficiently large, we have

$$\frac{2\tau''}{\omega^*_k \gamma'_0} + \frac{2(3\phi_2/2)^{M-1}M^{3/2}}{\gamma'_0 C\gamma_0} \leq \frac{2^{M+1}c_1 \omega_{\text{max}}}{c_0 \phi_1 M \omega_{\text{min}}} \sqrt{\log(2/\{ \log(nd) \}^2)} + \frac{2^{M+9}M^{3/2}(3\phi_2)^{M-1}}{c_0^2 C\psi_1^2 M} \cdot \frac{1}{\omega_{\text{min}}} \sqrt{s_1 \log d} \leq \frac{2\alpha}{3}. \quad (A75)$$

Thus, we have $|\omega^{(1)}_k - \omega_k| \leq \alpha$ which implies that $\Theta'' = (\omega^{(1)}_k, \Theta^{(1)}_{-\omega_k}) \in B_\alpha(\Theta^*)$, ensuring the initial guarantee for the next parameter update.

**Step 2 for $\Omega_{k,m}$:**

Let $\Theta = \Theta^{(0)}, \Theta'_{-\Omega_{k,m}} = \Theta^{(1)}_{-\Omega_{k,m}}$. Plugging them into Step 1, we can obtain that

$$\left\| \Omega^{(1)}_{k,m} - \Omega^*_k \right\|_F \leq \frac{16\sqrt{s_2} + d_m \epsilon_m}{C'\sqrt{d_m \gamma_m}} + \frac{6\tau_1 \text{D}(\Theta^{(0)}, \Theta^*)}{C'\sqrt{d_m \gamma_m}}, \quad (A76)$$

with probability at least $1 - (8K^2 + 2K + 1) / \{ \log(nd) \}^2$. Since $\| \Omega^{(1)}_{k,m} - \Omega^*_k \|_2 \leq \| \Omega^{(1)}_{k,m} - \Omega^*_k \|_F$, we have

$$\left\| \Omega^{(1)}_{k,m} - \Omega^*_k \right\|_2 \leq \frac{16\sqrt{s_2} + d_m \epsilon_m}{C'\phi_1 \gamma_m} + \frac{6\tau_1 \text{D}(\Theta^{(0)}, \Theta^*)}{C'\phi_1 \gamma_m}.$$  

Let $\gamma \leq \frac{C\phi_1 \sqrt{K(\gamma + 1)(M + 1)}}{54\phi_2 d_m}$ and then we have $\frac{6\tau_1 \text{D}(\Theta^{(0)}, \Theta^*)}{C'\phi_1 \gamma_m} \leq \frac{1}{3} \alpha$. Also, when $n/T$ is sufficiently large, it holds that

$$\frac{16\sqrt{s_2} + d_m \epsilon_m}{C'\phi_1 \gamma_m} \leq \frac{16c_2 \sqrt{s_2} + d_m}{C'c_0 \phi_1 (6\phi_2)^{-2}\alpha} \sqrt{\frac{\log d}{n/T}} \leq \frac{2\alpha}{3}. \quad (A77)$$

Thus, we have $\left\| \Omega^{(1)}_{k,m} - \Omega^*_k \right\|_2 \leq \alpha$. It then follows that $\left\| \Omega^{(1)}_{k,m} - \Omega^*_k \right\|_F \leq \alpha$, as $\| \Omega^*_k \|_F = \sqrt{d_m}$ and $\sigma_{\text{min}}(\Omega^*_k)$ is bounded below by a positive constant $\phi_1$. This implies that $\Theta'' = (\Omega^{(1)}_{k,m}, \Theta^{(1)}_{-\Omega_{k,m}}) \in B_\alpha(\Theta^*)$ and satisfies Condition 2, ensuring the initial guarantee for the next parameter update.
Step 3: Given $\Theta^{(0)} \in B_\alpha(\Theta^*)$, Step 2 gives that the updated parameter $\Theta^{(1)}$ still satisfies the initial condition in Condition 2. Thus, with probability at least $1 - C_3 / \{\log(nd)\}^2$ for some constant $C_3 > 0$, it holds that

$$\text{D}(\Theta^{(1)}, \Theta^*) \leq \epsilon + \rho \text{D}(\Theta^{(0)}, \Theta^*),$$

(A78)

where

$$\epsilon = \max \left\{ \frac{16 \sqrt{s_1} \epsilon_0}{C \gamma_0}, \frac{2 (3 \phi_2 / 2)^{M-1} M^{3/2} 16 \sqrt{s_1} \epsilon_0}{C \sqrt{d_m} \gamma_m}, \frac{16 \sqrt{s_2 + d_m} \epsilon_m}{C' \sqrt{d_m} \gamma_m} \right\}$$

and

$$\rho = \max \left\{ \frac{4 \tau_0}{C \gamma_0}, \frac{2 (3 \phi_2 / 2)^{M-1} M^{3/2} 4 \tau_0}{C' \gamma_0}, \frac{6 \tau_1}{C' \sqrt{d_m} \gamma_m}, \frac{6 \tau_1}{C' \phi_1 \gamma_m} \right\}.$$ 

Following the discussions in Step 2, there exists a constant $C_1 > 0$ such that $\rho \leq \frac{1}{3}$ when $\gamma \leq C_1 / \max$. By (A73), (A75), (A77), there exists one positive constant $C_2$ such that

$$\epsilon \leq C_2 \left\{ \frac{1}{\omega_{\min}} \sqrt{T \frac{s_1 \log d}{n}} + \max_m \sqrt{\left( \frac{s_2 + d_m \log d \cdot T}{n d_m} \right)} \right\}.$$ 

(A79)

Moreover, under Condition 4, we have $\epsilon \leq \frac{2 \alpha}{3}$, which gives that $\text{D}(\Theta^{(1)}, \Theta^*) \leq \alpha$.

D7 Proof of Lemma 5b

In this proof, we show the strong concavity with respect to $\beta_{k,r,m}$ for a general rank. First, we introduce the first- and second-order derivatives of $Q_n/T(\beta_{k,r,m}, \Theta - \beta_{k,r,m} | \Theta)$ with respect to $\beta_{k,r,m}$.

First-order:

$$\nabla_{\beta_{k,r,m}} Q_n/T(\beta'_{k,r,m}, \Theta - \beta_{k,r,m} | \Theta) = -\frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \Omega_{k,m} \left\{ (X_i - \bar{U}_{k,r-m})_m - \bar{\omega}_k \beta'_{k,r,m} \text{vec} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \right\} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) a_{k,r,m},$$

(A80)

Second-order:

$$\nabla^2_{\beta_{k,r,m}} Q_n/T(\beta'_{k,r,m}, \Theta - \beta_{k,r,m} | \Theta) = -\frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \left\{ a_{k,r,m}^\top \left( \prod_{m' \neq m} \Omega_{k,m'} \right) a_{k,r,m} \right\} \Omega_{k,m},$$

(A81)
where \( \mathbf{U}_{k, r} = \sum_{r' \neq r} \bar{\omega}_{k, r'} \mathbf{b}_{k, r', 1} \circ \circ \mathbf{b}_{k, r', M} \) and \( \mathbf{a}_{k, r, m} = \sum_{r' = 1}^{R} \xi_{k, m, r'} \mathbf{w}_{k, r'} \mathbf{vec}( \prod_{m' \neq m} \bar{\mathbf{b}}_{k, r', m'} ) \) with \( \xi_{k, m, r' r} = \langle \bar{b}_{k, r', m}, \bar{b}_{k, r, m} \rangle \).

Expand \( Q_{n/\tau}(\beta'_{k, r, m'), \Theta_{-k, r, m} | \Theta) \) around \( \beta'_{k, r, m} \) using Taylor expansion, we have

\[
Q_{n/\tau}(\beta'_{k, r, m'), \Theta_{-k, r, m} | \Theta) = Q_{n/\tau}(\beta'_{k, r, m}, \Theta_{-k, r, m} | \Theta) + \langle \nabla \beta_{k, r, m} Q_{n/\tau}(\beta'_{k, r, m}, \Theta_{-k, r, m} | \Theta), \beta''_{k, r, m} - \beta'_{k, r, m} \rangle + \frac{1}{2} (\beta''_{k, r, m} - \beta'_{k, r, m})^\top \nabla^2 Q_{n/\tau}(z, \Theta_{-k, r, m} | \Theta) (\beta''_{k, r, m} - \beta'_{k, r, m})
\]

where \( z = t^2 \beta''_{k, r, m} + (1 - t) \beta''_{k, r, m} \) with \( t \in [0, 1] \). By (A81), we have

\[
\nabla^2 Q_{n/\tau}(z, \Theta_{-k, r, m} | \Theta) = -\frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \left\{ \mathbf{a}_{k, r, m}^\top \left( \prod_{m' \neq m} \Omega_{k, m'} \right) \mathbf{a}_{k, r, m} \right\} \Omega_{k, m}.
\]

By (A20), with probability as least \( 1 - p_n, \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \geq c_0 \). Noting \( \sigma_{\min}(\Omega_{k, m}) \geq \phi/2 \) from Conditions 1-2, we have

\[
\left\{ \sum_{r' = 1}^{R} \xi_{k, m, r'} \bar{\omega}_{k, r'} \mathbf{vec}( \prod_{m' \neq m} \bar{\mathbf{b}}_{k, r', m'} ) \right\} \left\{ \prod_{m' \neq m} \Omega_{k, m'} \right\} \left\{ \sum_{r'' = 1}^{R} \xi_{k, m, r''} \bar{\omega}_{k, r''} \mathbf{vec}( \prod_{m' \neq m} \bar{\mathbf{b}}_{k, r'', m'} ) \right\} \geq (\phi/2)^{M-1} \sum_{r_1 = 1}^{R} \sum_{r_2 = 1}^{R} \xi_{k, m, r_1} \xi_{k, m, r_2} \bar{\omega}_{k, r_1} \bar{\omega}_{k, r_2} \mathbf{vec}( \prod_{m' \neq m} \bar{\mathbf{b}}_{k, r_1, m'} ) \mathbf{vec}( \prod_{m' \neq m} \bar{\mathbf{b}}_{k, r_2, m'} )
\]

To ease notation, we discuss \( \mathbf{vec}( \prod_{m' \neq m} \bar{\mathbf{b}}_{k, r_1, m'} ) \mathbf{vec}( \prod_{m' \neq m} \bar{\mathbf{b}}_{k, r_2, m'} ) \) when \( M = 3 \) and \( m = 1 \) while general cases follow similarly.

When \( r_1 = r_2 \), we have \( \mathbf{vec}( \prod_{m' \neq m} \bar{\mathbf{b}}_{k, r_1, m'} ) \mathbf{vec}( \prod_{m' \neq m} \bar{\mathbf{b}}_{k, r_2, m'} ) = 1. \) Otherwise, we have

\[
\mathbf{vec}( \prod_{m' \neq m} \bar{\mathbf{b}}_{k, r_1, m'} ) \mathbf{vec}( \prod_{m' \neq m} \bar{\mathbf{b}}_{k, r_2, m'} ) = \sum_{l_3 = 1}^{d_3} \left\{ (\bar{\mathbf{b}}_{k, r_1, 2})^\top \mathbf{b}_{k, r_2, 2} \right\} \mathbf{b}_{k, r_1, 3}(l_3) \mathbf{b}_{k, r_2, 3}(l_3)
\]

\[
= \langle \bar{\mathbf{b}}_{k, r_1, 2}, \bar{\mathbf{b}}_{k, r_2, 2} \rangle \langle \mathbf{b}_{k, r_1, 3}, \mathbf{b}_{k, r_2, 3} \rangle = \prod_{m' \neq 1} \langle \mathbf{b}_{k, r_1, m'}, \mathbf{b}_{k, r_2, m'} \rangle.
\]

By Condition 2, we have

\[
\langle \mathbf{b}_{k, r_1, m'}, \mathbf{b}_{k, r_2, m'} \rangle = \langle \beta_{k, r_1, m'}^*, \beta_{k, r_2, m'}^* \rangle + \langle \beta_{k, r_1, m} - \beta_{k, r_1, m'}^*, \beta_{k, r_2, m}^* \rangle
\]

\[
+ \langle \beta_{k, r_1, m'}, \beta_{k, r_2, m} - \beta_{k, r_2, m'}^* \rangle + \langle \beta_{k, r_1, m} - \beta_{k, r_1, m'}^*, \beta_{k, r_2, m} - \beta_{k, r_2, m'}^* \rangle
\]

\[
\leq \xi + \| \beta_{k, r_1, m} - \beta_{k, r_1, m'}^* \|_2 + \| \beta_{k, r_2, m} - \beta_{k, r_2, m'}^* \|_2 + \| \beta_{k, r_1, m} - \beta_{k, r_1, m'}^* \|_2 \| \beta_{k, r_2, m} - \beta_{k, r_2, m'}^* \|_2
\]

\[
\leq \xi + 2c_{\alpha} + c_{\alpha}^2.
\]
Then it arrives at

$$\left\{ \begin{array}{l}
\sum_{r' = 1}^R \xi_{k,m,r',r'} \text{vec} \left( \prod_{m' \neq m}^{\theta} \bar{\beta}_{k,r',m'} \right) \\
\sum_{r' = 1}^R \xi_{k,m,r',r'} \text{vec} \left( \prod_{m' \neq m}^{\theta} \bar{\beta}_{k,r',m'} \right) \\
\end{array} \right\} \geq (\phi/2)^{M-1} \left\{ \begin{array}{l}
\sum_{r' = 1}^R (\xi_{k,r',r'} \omega_{k,r'}^2) - \sum_{r_1 \neq r_2} \xi_{k,r_1,m} \xi_{k,r_2,m} \omega_{k,r_1} \omega_{k,r_2} \prod_{m' \neq m} \langle \bar{\beta}_{k,r_1,m}, \bar{\beta}_{k,r_2,m} \rangle \\
\end{array} \right\}
$$

or arrives at that

$$Q_n/T(\beta''', \Theta - \beta_{k,r,m} | \Theta) - Q_n/T(\beta'', \Theta - \beta_{k,r,m} | \Theta) - \langle \nabla_{k,r,m} Q_n/T(\beta_{k,r,m}, \Theta - \beta_{k,r,m} | \Theta), \beta'' - \beta'_{k,r,m} \rangle \leq -\frac{\gamma_0}{2} \left\| \beta''_{k,r,m} - \beta'_{k,r,m} \right\|^2,
$$

with probability at least $1 - 1/\{\log(nd)\}^2$.

### D8 Proof of Lemma 6b

In this proof, we establish the gradient stability for $\beta_{k,r,m}$. Recalling (A80) and we have

$$\nabla_{k,r,m} Q(\beta_{k,r,m}, \Theta - \beta_{k,r,m} | \Theta) = \mathbb{E} \left[ \tau_{ik}(\Theta) \Omega_{k,m} \left\{ (x_i - U_{k,-r})_{(m)} - \omega_k \beta'_{k,r,m} \text{vec} \left( \prod_{m' \neq m}^{\theta} \bar{\beta}_{k,r',m'} \right) \right\} \left( \prod_{m' \neq m}^{\theta} \Omega_{k,m'} \right) a_{k,r,m} \right],$$

where $a_{k,r,m} = \sum_{r'} \omega_{k,r'} \xi_{k,m,r'} \text{vec} \left( \prod_{m' \neq m}^{\theta} \bar{\beta}_{k,r',m'} \right)$.

First, we expand $\nabla_{k,r,m} Q(\beta_{k,r,m}, \Theta - \beta_{k,r,m} | \Theta)$ as

$$\mathbb{E} \left[ D_r(\Theta, \Theta^*) \Omega_{k,m} \left\{ (x_i - U_{k,-r})_{(m)} - \omega_k \beta'_{k,r,m} \text{vec} \left( \prod_{m' \neq m}^{\theta} \bar{\beta}_{k,r',m'} \right) \right\} \left( \prod_{m' \neq m}^{\theta} \Omega_{k,m'} \right) a_{k,r,m} \right],$$

where $D_r(\Theta, \Theta^*) = \tau_{ik}(\Theta) - \tau_{ik}(\Theta^*)$. By the definition of $\tau_0$, we can obtain that

$$\left\| \mathbb{E} \left[ \Omega_{k,m} \left\{ (x_i - U_{k,-r})_{(m)} - \omega_k \beta'_{k,r,m} \text{vec} \left( \prod_{m' \neq m}^{\theta} \bar{\beta}_{k,r',m'} \right) \right\} \left( \prod_{m' \neq m}^{\theta} \Omega_{k,m'} \right) a_{k,r,m} (\nabla \delta^T \tau_{ik}(\Theta)) \right] \right\|_2 \leq \{1 + (R - 1)(\xi + 2c_\alpha + c_\alpha^2)\}^2 \tau_0,$$

where the inequality holds due to $|\xi_{k,m,r',r'}| \leq \xi + 2c_\alpha + c_\alpha^2$. Correspondingly, we have

$$\left\| \nabla_{k,r,m} Q(\beta_{k,r,m}, \Theta - \beta_{k,r,m} | \Theta) - \nabla_{k,r,m} Q(\beta_{k,r,m}, \Theta - \beta_{k,r,m} | \Theta') \right\|^2_2 \leq \{1 + (R - 1)(\xi + 2c_\alpha + c_\alpha^2)\}^2 \tau_0^2 D(\Theta, \Theta^*) \leq \tau_0^2 D(\Theta, \Theta^*),$$

where $\tau_0' = \{1 + (R - 1)(\xi + 2c_\alpha + c_\alpha^2)\} \tau_0$.  

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D9 Proof of Lemma 7b

Similar as in Lemma 3b, define

\[ h_\Theta(\beta'_{k,r,m}) = \nabla_{\beta'_{k,r,m}} Q_{n/T}(\beta'_{k,r,m}, \Theta_{-\beta_{k,r,m}}|\Theta) - \nabla_{\beta_{k,r,m}} Q(\beta_{k,r,m}, \Theta_{-\beta_{k,r,m}}|\Theta). \]

Based on the definition of dual norm \( P_1^* \), we have that

\[ \|\nabla_{\beta_{k,r,m}} Q_{n/T}(\beta'_{k,r,m}, \Theta_{-\beta_{k,r,m}}|\Theta) - \nabla_{\beta_{k,r,m}} Q(\beta'_{k,r,m}, \Theta_{-\beta_{k,r,m}}|\Theta)\|_{P_1^*} \leq \max_k \|h_\Theta(\beta'_{k,r,m})\|_\infty, \]  

(A85)

Recalling (A80) and we have

\[ h_\Theta(\beta'_{k,r,m}) = \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \Omega_{k,m} \left\{ (\tilde{X}_i - \bar{U}_{k,-r})_{(m)} - \bar{\omega}_k \beta'_{k,r,m} \text{vec}(\prod_{m' \neq m} \bar{\beta}_{k,r,m'})^T \right\} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) a_{k,r,m} \]

- \mathbb{E} \left[ \tau_{ik}(\Theta) \Omega_{k,m} \left\{ (\tilde{X}_i - \bar{U}_{k,-r})_{(m)} - \bar{\omega}_k \beta'_{k,r,m} \text{vec}(\prod_{m' \neq m} \bar{\beta}_{k,r,m'})^T \right\} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) a_{k,r,m} \right].

By the triangle inequality, \( h_\Theta(\beta'_{k,r,m}) \) can be bounded as

\[ h_\Theta(\beta'_{k,r,m}) \leq \|\Omega_{k,m}\|_{\max} \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) \left( (\tilde{X}_i - \bar{U}_{k,-r})_{(m)} \right) - \mathbb{E} \left\{ \tau_{ik}(\Theta) \left( (\tilde{X}_i)_{(m)} \right) \right\} \right\|_{\max} \left\| \left( \prod_{m' \neq m} \Omega_{k,m'} \right) a_{k,r,m} \right\|_{\infty} \]

+ \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) - \mathbb{E}(\tau_{ik}(\Theta)) \right\|_{\infty} \left\| \Omega_{k,m} \left\{ (\tilde{U}_{k,-r})_{(m)} + \bar{\omega}_k \beta'_{k,r,m} \text{vec}(\prod_{m' \neq m} \bar{\beta}_{k,r,m'})^T \right\} \left( \prod_{m' \neq m} \Omega_{k,m'} \right) a_{k,r,m} \right\|_{\infty}. \]

By (A38), we have

\[ I \leq \sqrt{4/D_0 \varphi_{K_0} \sqrt{\log(c/p_n) + \log d}} \frac{n}{n/T}, \]

with probability at least \( 1 - 2Kp_n \). Applying the result in (A39) to II, we have

\[ \left\| \frac{T}{n} \sum_{i=1}^{n/T} \tau_{ik}(\Theta) - \mathbb{E}(\tau_{ik}(\Theta)) \right\| \leq \sqrt{\frac{1}{2} \log(2/p_n)T/n}, \]

with probability at least \( 1 - p_n \).
Note that the bound for term I is \( O_P \left( \sqrt{\frac{\log(d)T}{n}} \right) \) while the bound from II is \( O_P \left( \sqrt{\frac{\log(2/pn)T}{n}} \right) \), thus

\[
h_{\Theta_1}(\beta'_{k,r,m}) \lesssim I \times \left\| \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \alpha_{k,r,m} \right\|_\infty
\]

\[
\lesssim I \times \left\| \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \sum_{r' = 1}^R \xi_{k,m,r' \r} \tilde{\omega}_{k,r'} \vec{\prod}_{m' \neq m} \beta_{k,m',r'} \right\|_\infty
\]

(A86)

with probability at least \( 1 - (2K^2 + K + 1)/\{\log(nd)\}^2 \). Give probability at least 1 - \((2K + 1)p_n\). By (A33) and \( \|\Omega_{k,m}\|_2 \leq 3\phi_2/2 \), we have

\[
\left\| \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \sum_{r' = 1}^R \xi_{k,m,r' \r} \tilde{\omega}_{k,r'} \vec{\prod}_{m' \neq m} \beta_{k,m',r'} \right\|_\infty \leq \sum_{r' = 1}^R |\xi_{k,m,r' \r} | \tilde{\omega}_{k,r'} (3\phi_2/2)^{M-1}
\]

\[
\leq (1 + c_\alpha) \omega_{\max}(3\phi_2/2)^{M-1} \left\{ 1 + (R - 1)(\xi + 2c_\alpha + c_\alpha^2) \right\}.
\]

Therefore, there exist some constant \( c'_1 > 0 \) such that

\[
\max_k I \leq c'_1 \omega_{\max} \sqrt{T \log d / n},
\]

with probability at least \( 1 - K(2K + 1)/\{\log(nd)\}^2 \).

**D10 Proof of Lemma 8b**

Similar as in Lemma 4b, this proof can be divided into three steps. We focus on \( \beta_{k,r,m} \) and \( \omega_{k,r} \) in Steps 1-2 as having a general rank does not affect the results involving \( \Omega_{k,m} \)'s. Recall \( \alpha = D(\Theta^{(0)}, \Theta^*) \).

**Step 1 for \( \beta_{k,r,m} \):**

Using Lemma 5b, we can bound the difference \( \|\beta''_{k,r,m} - \beta^*_{k,r,m}\|_2 \), where \( \beta''_{k,r,m} = \tilde{\beta}_{k,r,m}/\|\tilde{\beta}_{k,r,m}\|_2 \) and \( \tilde{\beta}_{k,r,m} = \arg \max_{\beta_{k,r,m}} Q_n/T(\beta_{k,r,m}, \Theta - \beta_{k,r,m} | \Theta) - \lambda_{\Theta}^{(1)} \|\beta_{k,r,m}\|_1 \). Let \( \epsilon_{R,0} = c'_1 \omega_{\max} \sqrt{\frac{\log d}{n/T}} \), where \( c'_1 \) is as defined in Lemma 7b. Similar as in Step 1 for \( \beta_{k,m} \) in Section D6, we can get

\[
\|\beta''_{k,r,m} - \beta^*_{k,r,m}\|_2 \leq \frac{16 \sqrt{\gamma_1} \epsilon_{R,0}}{C\gamma_{\Theta}^0} + \frac{4\tau_0 D(\Theta, \Theta^*)}{C\gamma_{\Theta}'}
\]

with probability at least \( 1 - (2K^2 + K + 1)/\{\log(nd)\}^2 \).

**Step 1 for \( \omega_{k,r} \):**

For \( \omega_{k,r} \), Lemmas 1b-3b still holds for the general rank case. By (A64), (A65) and (A66),
we have
\[
\frac{\tau_0''}{2}|\omega_{k,r}' - \omega_{k,r}^*|^2 \leq \epsilon_0' |\omega_{k,r}' - \omega_{k,r}^*| + \tau_0'' D(\Theta, \Theta^*)|\omega_{k,r}' - \omega_{k,r}^*| + \left\langle \nabla_{\omega_{k,r}} Q(\omega_{k,r}' - \omega_{k,r}, \Theta^*), \omega_{k,r}' - \omega_{k,r}^* \right\rangle
\]
with probability at least \(1 - (K + 1)/\{\log(nd)\}^2\). By (A24) and (A50), we have that
\[
|\nabla_{\omega_{k,r}} Q(\omega_{k,r}' - \omega_{k,r}, \Theta^*)| = \mathbb{E} \{ \tau_{ik}(\Theta^*) \} \left| \omega_{k,r}' - \omega_{k,r}^* \right|
\]
\[
= \mathbb{E} \{ \tau_{ik}(\Theta^*) \} \left| \omega_{k,r}' - \omega_{k,r}^* \right| \left| \text{vec}(\prod_m \beta_{k,r,m}^r') - \text{vec}(\prod_m \beta_{k,r,m}^r) \right| \left( \prod_m \Omega_{k,m}^r \right) \text{vec}(\prod_m \beta_{k,r,m}^r)
\]
\[
+ \mathbb{E} \{ \tau_{ik}(\Theta^*) \} \sum_{r' \neq r} \left| \omega_{k,r'}^* \text{vec}(\prod_m \beta_{k,r',m}^r) - \omega_{k,r'}^* \text{vec}(\prod_m \beta_{k,r',m}^r) \right| \left( \prod_m \Omega_{k,m}^r \right) \text{vec}(\prod_m \beta_{k,r,m}^r)
\]
Similar as in Step 1 for \(\omega_k\) in Section D6, \(A_1 \leq \omega_{k,r}'(3\phi_2/2)^{M - 1} M^{1/2} \sum_m \|\beta_{k,r,m}' - \beta_{k,r,m}^*\|_2\),
with probability at least \(1 - 1/\{\log(nd)\}^2\). Since \(\mathbb{E} \{ \tau_{ik}(\Theta^*) \} \leq 1\) and \(\|\Omega_{k,m}'\|_2 \leq 3\phi_2/2\), it holds that
\[
A_2 \leq (R - 1)(3\phi_2/2)^M \max_{r' \neq r} \left| \omega_{k,r'}^* \text{vec}(\prod_m \beta_{k,r',m}^r) - \omega_{k,r'}^* \text{vec}(\prod_m \beta_{k,r',m}^r) \right| \left( \prod_m \Omega_{k,m}^r \right) \text{vec}(\prod_m \beta_{k,r,m}^r)
\]
\[
\leq (R - 1)(3\phi_2/2)^M \max_{r' \neq r} \left| \omega_{k,r'}^* \text{vec}(\prod_m \beta_{k,r',m}^r) - \omega_{k,r'}^* \text{vec}(\prod_m \beta_{k,r',m}^r) \right| \left| \prod_m \beta_{k,r,m}^r \right| \left( \prod_m \Omega_{k,m}^r \right) \text{vec}(\prod_m \beta_{k,r,m}^r)
\]
\[
+ 2(R - 1)(3\phi_2/2)^M (1 + \alpha) \omega_{\text{max}} \left| \text{vec}(\prod_m \beta_{k,r,m}^r) - \text{vec}(\prod_m \beta_{k,r,m}^r) \right|_2.
\]
where the second inequality uses the fact that \(\left| \omega_{k,r}^* \text{vec}(\prod_m \beta_{k,r',m}^r) - \omega_{k,r}^* \text{vec}(\prod_m \beta_{k,r',m}^r) \right|_2 \leq 2(1 + \alpha) \omega_{\text{max}}\). Similar as in \(A_1\), we can get that
\[
A_{22} \leq 2(R - 1)(3\phi_2/2)^M (1 + \alpha) \omega_{\text{max}} \sum_m \|\beta_{k,r,m}' - \beta_{k,r,m}^*\|_2.
\]
By (A83), we have that
\[
A_{21} = \omega_{k,r}' \left| \prod_m \langle \beta_{k,r',m}^r, \beta_{k,r,m}^r \rangle - \prod_m \langle \beta_{k,r',m}^r, \beta_{k,r,m}^r \rangle \right| + \left| \omega_{k,r}' - \omega_{k,r}' \prod_m \langle \beta_{k,r',m}^r, \beta_{k,r,m}^r \rangle \right|
\]
\[
\leq \max\{\xi^M, \xi^{M-1}(1 + \alpha), (\alpha + \xi)^M(1 + \alpha)\} \omega_{\text{max}} D(\Theta^{(t-1)}, \Theta^*).
\]
This is true, because \( \prod_{m} |\langle \beta_{k,r,m}', \beta_{k,r,m}^{*} \rangle| \leq \xi^{M} \) and
\[
\left| \prod_{m} \langle \beta_{k,r,m}', \beta_{k,r,m}^{*} \rangle - \prod_{m} \langle \beta_{k,r,m}', \beta_{k,r,m}^{*} \rangle \right| \leq \max \{ \xi^{M-1}, (\alpha + \xi)^{M-1} \} \sum_{m} \| \beta_{k,m}' - \beta_{k,m}^{*} \|_{2},
\]
which can be verified as follows in the case of \( M = 3 \) while general cases follow similarly.

Since \( |\langle \beta_{k,r,m}', \beta_{k,r,m}^{*} \rangle| \leq \xi \) and \( |\langle \beta_{k,r,m}', \beta_{k,r,m}^{*} \rangle| \leq \xi + \alpha \), we have
\[
\left| \prod_{m} \langle \beta_{k,r,m}', \beta_{k,r,m}^{*} \rangle - \prod_{m} \langle \beta_{k,r,m}', \beta_{k,r,m}^{*} \rangle \right| \leq \max \{ \xi^{2}, \xi(\alpha + \xi), (\alpha + \xi)^{2} \} \sum_{m} \| \beta_{k,r,m}' - \beta_{k,r,m}^{*} \|_{2}
= \max \{ \xi^{M-1}, (\alpha + \xi)^{M-1} \} \sum_{m} \| \beta_{k,m}'(t) - \beta_{k,m}^{*} \|_{2}.

Correspondingly, term \( A_{2} \) can be bounded by
\[
A_{2} \leq \tau_{0}''D(\Theta, \Theta^{*}) + 2(R - 1)(3\phi_{2}/2)^{M-1}M^{3/2}(1 + \alpha)\omega_{\max} \sum_{m} \| \beta_{k,r,m}' - \beta_{k,r,m}^{*} \|_{2},
\]
where \( \tau_{0}'' = \omega_{\max}(R - 1)(3\phi_{2}/2)^{M} \max \{ \xi^{M}, \xi^{M-1}(1 + \alpha), (\alpha + \xi)^{M-1}(1 + \alpha) \} \). Now we can conclude that
\[
|\nabla_{\omega_{k,r}} Q(\omega_{k,r}', \Theta'_{-\omega_{k,r}} | \Theta^{*})| \leq 2R(3\phi_{2}/2)^{M-1}M^{1/2}(1 + \alpha)\omega_{\max} \sum_{m} \| \beta_{k,r,m}' - \beta_{k,r,m}^{*} \|_{2} + \tau_{0}''D(\Theta, \Theta^{*}).
\]

Combining (A66) and (A65), we can get that, for any \( k \),
\[
\left| \frac{\omega_{k,r}' - \omega_{k,r}^{*}}{\omega_{k,r}^{*}} \right| \leq \frac{2\epsilon_{0}''}{\omega_{k,r}^{*}\gamma_{0}'} + \frac{2(\epsilon_{0}'' + \tau_{0}'' D(\Theta, \Theta^{*}))}{\omega_{k,r}^{*}\gamma_{0}''} + \frac{2R(3\phi_{2}/2)^{M-1}M^{1/2}(1 + \alpha)}{\gamma_{0}''} \sum_{m} \| \beta_{k,r,m}' - \beta_{k,r,m}^{*} \|_{2}
\]
with probability at least \( 1 - (2K^{2} + 2K + 1) / \{ \log(nd) \}^{2} \).

**Step 2 for \( \beta_{k,r,m} \):**

Let \( \Theta = \Theta^{(0)} \) and \( \Theta'_{-\beta_{k,m}} = \Theta^{(1)}_{-\beta_{k,m}} \). Plugging them into Step 1, we can obtain that
\[
\| \beta_{k,m}' - \beta_{k,m}^{*} \|_{2} \leq \frac{16\sqrt{s_{1}\epsilon_{R,0}}}{C\gamma_{0}'} + \frac{4\epsilon_{0}'' D(\Theta, \Theta^{*})}{C\gamma_{0}'} + \frac{2R(3\phi_{2}/2)^{M-1}M^{1/2}(1 + \alpha)}{\gamma_{0}''} \sum_{m} \| \beta_{k,r,m}' - \beta_{k,r,m}^{*} \|_{2},
\]
with probability at least \( 1 - (2K^{2} + K + 1) / \{ \log(nd) \}^{2} \). By Lemmas 5b-6b and \( \gamma \leq C_{1}/d_{\max} \), it is easy to verify that \( \frac{4\epsilon_{0}'' D(\Theta^{(0)}, \Theta^{*})}{C\gamma_{0}'} \leq \frac{\alpha}{3} \). In addition, when \( n \) is sufficiently large, we have
\[
\frac{16\sqrt{s_{1}\epsilon_{R,0}}}{C\gamma_{0}'} \leq \frac{64\epsilon_{0}'' \omega_{\max}}{C\omega_{min}(\phi_{1}/2)^{M} \omega_{\min} + (R - 1)(\xi + 2c_{\alpha} + c_{\alpha}^{2}) \omega_{\min}} \cdot \frac{1}{\omega_{\min} \sqrt{s_{1} \log d / nT}} \leq \frac{2\alpha}{3}.
\]
Thus, we have \( \| \bm{\beta}^{(1)}_{k,r,m} - \bm{\beta}^*_{k,r,m} \|_2 \leq \alpha \) which implies that \( \Theta'' = (\bm{\beta}^{(1)}_{k,r,m}, \Theta_{-\beta_{k,r,m}}^{(1)}) \in B_\alpha(\Theta^*) \), ensuring the initial guarantee for the next parameter update.

**Step 2 for \( \omega_{k,r} \):**

Similarly, let \( \Theta = \Theta^{(0)} \) and \( \Theta_{-\omega_{k,r}} = \Theta^{(1)}_{-\omega_{k,r}} \). Plugging them into Step 1, we can obtain that

\[
\frac{|\omega^{(1)}_{k,r} - \omega^*_{k,r}|}{|\omega^*_{k,r}|} \leq \frac{2\epsilon_0''}{\omega^*_{k,r} \gamma_0''} + \frac{2(\tau_0'' + \tau_0'')}{\omega^*_{k,r} \gamma_0''} \alpha + \frac{2R(3\phi_2/2)^{M-1}M^{3/2}(1 + \alpha)}{\gamma_0''} \sum_m \| \bm{\beta}^{(1)}_{k,r,m} - \bm{\beta}^*_{k,r,m} \|_2
\]

\[
\leq \frac{2\epsilon_0''}{\omega^*_{k,r} \gamma_0''} + \frac{2(\tau_0'' + \tau_0'')}{\omega^*_{k,r} \gamma_0''} \alpha + \frac{2R(3\phi_2/2)^{M-1}M^{3/2}(1 + \alpha)}{\gamma_0''} \left\{ \frac{16\sqrt{s_1\epsilon_{R,0}}}{C'\gamma_0'} + \frac{4\gamma_0'}{C'\gamma_0'} \right\},
\]

with probability at least \( 1 - (2K^2 + 2K + 1)/\{\log(nd)\}^2 \). Noting \( \gamma \leq C_1/d_{\max} \) and letting \( C_0 = \min \{ \epsilon_0''/\phi_2, \sqrt{R-1}/9R \} \) and \( \alpha \leq \left( \frac{C_{\log_{\min}}}{(R-1)\omega_{\max}} \right)^{M-1} \), we have

\[
\left\{ \frac{2\tau_0''}{\omega^*_{k,r} \gamma_0''} + \frac{2R(3\phi_2/2)^{M-1}M^{3/2}}{\gamma_0''} \frac{4\tau_0'}{C'\gamma_0'} + \frac{2\tau_0''}{\omega^*_{k,r} \gamma_0''} \right\} D(\Theta^{(0)}, \Theta^*) \leq (\frac{1}{3} + \frac{1}{6})\alpha = \frac{1}{2}\alpha.
\]

When \( n \) is large enough, we can get that

\[
\frac{2\epsilon_0''}{\omega^*_{k,r} \gamma_0''} + \frac{2R(3\phi_2/2)^{M-1}M^{3/2}}{\gamma_0''} \frac{16\sqrt{s_1\epsilon_{R,0}}}{C'\gamma_0'} \leq \frac{1}{2}\alpha.
\]

Correspondingly, we have that \( |\omega^{(1)}_{k,r} - \omega^*_{k,r}|/|\omega^*_{k,r}| \leq \alpha \) which implies that \( \Theta'' = (\omega^{(1)}_{k,r}, \Theta^{(1)}_{-\omega_{k,r}}) \in B_\alpha(\Theta^*) \).

**Step 3:** Given \( \Theta^{(0)} \in B_\alpha(\Theta^*) \), Step 2 gives that the updated parameter \( \Theta^{(1)} \) still satisfies the initial condition in Condition 2. Thus, with probability at least \( 1 - C_3'/\{\log(nd)\}^2 \) for some constant \( C_3' > 0 \), it holds that

\[
D(\Theta^{(1)}, \Theta^*) \leq \epsilon' + \rho_R D(\Theta^{(0)}, \Theta^*),
\]

where

\[
\epsilon' = \max \left\{ \frac{16\sqrt{s_1\epsilon_{R,0}}}{C'\gamma_0'}, \frac{2\epsilon_0''}{\omega^*_{k,r} \gamma_0''} + \frac{2R(3\phi_2/2)^{M-1}M^{3/2}}{\gamma_0''} \frac{16\sqrt{s_1\epsilon_{R,0}}}{C'\gamma_0'} \right\},
\]

and

\[
\rho_R = \max \left\{ \frac{4\gamma_0'}{C'\gamma_0'}, \frac{2(\tau_0'' + \tau_0'')}{\omega^*_{k,r} \gamma_0''} + \frac{2R(3\phi_2/2)^{M-1}M^{3/2}}{\gamma_0''} \frac{4\gamma_0'}{C'\gamma_0'} \right\}.
\]
From the discussion in Step 2 of Lemma 4b-8b, when $\gamma \leq C_1/d_{\max}$, we have $\rho_R \leq \frac{1}{2}$. From the definition of $\rho_R$, it is easy to know that $\rho_R \geq \rho$. By (A87), (A89), (A77), there exists some constant $C_2' > 0$ such that

$$
\epsilon' \leq C_2' \left\{ \frac{1}{\omega_{\min}} \sqrt{TS_1 \log d \over n} + \max_m \sqrt{(s_2 + d_m) \log d} \cdot T \right\}.
$$

(A91)

D11 Proof of Proposition 2

Updating $\beta_{k,r,m}$ in the M-step leads to solving the following problem

$$
\arg \min_{\beta_{k,r,m}} \frac{1}{2n} \sum_{i=1}^{n} \tau_k(\Theta) \left\| (x_i - u_k) \times \Sigma_k^{-1/2} \right\|^2_F + \lambda_1 \| \beta_{k,r,m} \|_1.
$$

(A92)

Following Kolda and Bader (2009), we define $V_{x,m} = (x_i)_{(m)} \left( \prod_{m' \neq m} \Omega^{1/2}_{k,m'} \right)^T$ and $V_{u,m} = (u_k)_{(m)} \left( \prod_{m' \neq m} \Omega^{1/2}_{k,m'} \right)^T$. Next, we have that

$$
\left\| (x_i - u_k) \times \Sigma_k^{-1/2} \right\|^2_F
= \text{tr} \left\{ (V_{x,m} - V_{u,m}) \Omega_{k,m} (V_{x,m} - V_{u,m}) \right\}
= \text{tr} \left( \left( \prod_{m' \neq m} \Omega^{1/2}_{k,m'} \right) (x_i)_{(m)} \Omega_{k,m} (x_i)_{(m)} \right) - 2 \text{tr} \left( \left( \prod_{m' \neq m} \Omega^{1/2}_{k,m'} \right) V_{x,m} \Omega_{k,m} V_{u,m} \right) + \text{tr} \left( \left( \prod_{m' \neq m} \Omega^{1/2}_{k,m'} \right) V_{u,m} \Omega_{k,m} V_{u,m} \right).
$$

In what follows, we will take derivatives of $I_1$, $I_2$ and $I_3$ with respect to $\beta_{k,r,m}$, respectively. First, it is easy to see that the derivative of $I_1$ with respect to $\beta_{k,r,m}$ is zero. Next, the term $I_2$ can be calculated as follows

$$
\text{tr} (V_{x,m}^T \Omega_{k,m} V_{u,m})
= \sum_{r'=1}^{R} \omega_{k,r'} \text{tr} \left\{ V_{x,m}^T \Omega_{k,m} \beta_{k,r,m'}^T \left( \prod_{m' \neq m} \Omega^{1/2}_{k,m'} \right)^T \right\}
= \sum_{r'=1}^{R} \omega_{k,r'} \text{vec} \left( \prod_{m' \neq m} \beta_{k,r,m'} \right)^T \left( \prod_{m' \neq m} \Omega^{1/2}_{k,m'} \right) \left( x_i \right)_{(m)} \Omega_{k,m} \beta_{k,r,m'},
$$

where the first equality is due to Lemma S1 and the second equality is due to the fact that $\text{tr}(AB) = \text{tr}(BA)$. Correspondingly, the first derivative of $I_2$ with respect to $\beta_{k,r,m}$ is

$$
\Omega_{k,m} \left( x_i \right)_{(m)} \left( \prod_{m' \neq m} \Omega^{1/2}_{k,m'} \right) \sum_{r'=1}^{R} \xi_{k,m',r'} \omega_{k,r'} \text{vec} \left( \prod_{m' \neq m} \beta_{k,r,m'} \right)
$$

(A93)
Similarly, $I_3$ can be calculated as
\[
\text{tr}(V_{\mathbf{t},k,m}^\top \Omega_{k,m} V_{\mathbf{t},k,m}) \\
= \text{tr}\left\{ \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( \sum_{r_1=1}^R \omega_{k,r_1} \text{vec}(\prod_{m' \neq m} \beta_{k,r_1,m'}) \beta_{k,r_1,m'}^\top \right) \Omega_{k,m} \left( \sum_{r_2=1}^R \omega_{k,r_2} \text{vec}(\prod_{m' \neq m} \beta_{k,r_2,m'}) \beta_{k,r_2,m'}^\top \right) \right\}
\]
\[
= \sum_{r_1=1}^R \sum_{r_2=1}^R \omega_{k,r_1} \omega_{k,r_2} \left\{ \text{vec}(\prod_{m' \neq m} \beta_{k,r_1,m'})^\top \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \text{vec}(\prod_{m' \neq m} \beta_{k,r_2,m'}) \right\} \beta_{k,r_1,m}^\top \Omega_{k,m} \beta_{k,r_2,m}.
\]

Correspondingly, the first derivative of $I_3$ with respect to $\beta_{k,r,m}$ can be written as
\[
2 \Omega_{k,m} \left\{ \left( \sum_{r_1=1}^R \omega_{k,r_1} \beta_{k,r_1,m} \text{vec}(\prod_{m' \neq m} \beta_{k,r_1,m'}) \right) \left( \prod_{m' \neq m} \Omega_{k,m'} \right) \left( \sum_{r_2=1}^R \xi_{k,r_2,m} \omega_{k,r_2} \text{vec}(\prod_{m' \neq m} \beta_{k,r_2,m'}) \right) \right\}.
\]  

Combining (A93) and (A94) and given $\tau_{ik}(\Theta^{(t)})$ and $\Theta^{(t)}$, the subgradient of the objective function in (A92) with respect to $\beta_{k,r,m}$ is
\[
- \frac{1}{n} \sum_{i=1}^n \tau_{ik}(\Theta^{(t)}) \hat{g}_{k,r,m}^{(t+1)} + \frac{n_k}{n} C_{k,r,m}^{(t+1)} \Omega_{k,m}^{(t)} \beta_{k,r,m} + \lambda_0^{(t+1)} \text{sign}(\beta_{k,r,m}),
\]

where $\hat{g}_{k,r,m}^{(t+1)}$, $C_{k,r,m}^{(t+1)}$ are as defined in (9). Hence, given $\Theta^{(t)}$, the updating formula for $\beta_{k,r,m}$ is given as in (9).

### E Additional Real Data Analysis

The fMRI data have been preprocessed and are summarized as a $116 \times 236$ spatial-temporal matrix for each subject. In the matrix, the 116 rows correspond to 116 regions from the Anatomical Automatic Labeling (AAL) atlas (Tzourio-Mazoyer et al., 2002) and the 236 columns correspond to the fMRI measures taken at 236 time points. For each subject, the tensor object is constructed by stacking a sequence of Fisher-transformed correlation matrices of dimension $116 \times 116$ over $T$ sliding windows, each summarizing the connectivity between 116 brain regions in a given window. We vary the number of sliding windows $T$ among \{1, 15, 30\}. When $T = 1$, each subject only has one correlation matrix calculated based on the entire spatial-temporal matrix. For $T = 15$ and 30, we let the length of the window be 20, as suggested in Sun and Li (2019), to balance the number of samples in each window and the overlap between adjacent windows.
Figure 5: Brain connectivity in ASD (columns 1-2) and normal control (NC; columns 3-4) groups at three representative windows ordered by time when $T = 30$. Columns 1, 3 and Columns 2, 4 show the connectivity in the left and right hemispheres, respectively. Edge widths in the plot are proportional to edge values.

We further explore the difference of connectivity between ASD and normal control groups. Figure 5 shows the estimated brain connectivity for ASD subjects and normal controls at three representative windows (i.e., 3, 14, 27) when $T = 30$. For each brain network, we report the identified edges with absolute values greater than 0.3. It is seen that the ASD subjects and the normal controls show notable differences in their brain connectivity. The brain networks from the ASD group are less connected and exhibit less changes across different windows, which agree with the existing finding that ASD subjects are usually found less active in brain connectivity (Rudie et al., 2013). It is seen that the occipital lobe is a relatively active area for both ASD subjects and normal controls; this agrees with existing findings that the occipital is important in posture and vision perception, and it tends to be
more active during fMRI data collections (Ouchi et al., 1999). Notably, the frontal area of the normal controls is active, both within itself and in its connection to other areas, and this activity first decreases and then increases for normal controls. Such a change in activity in the frontal area is not observed for the ASD subjects, as the frontal area appears inactive in all three windows. This agrees with existing findings that the frontal area, which may underlie impaired social and communication behaviors, shows reduced connectivity in ASD subjects (Monk et al., 2009). Moreover, we observe that the connectivity for normal controls in both hemispheres first increases and then decreases, by comparing the number of edges in the three windows, while the connectivity for ASD subjects remains relatively unchanged over time. These findings suggest some interesting resting-state connectivity patterns that warrant more in-depth investigation and validation.

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