Quantum Hidden Subgroup Algorithms:
A Mathematical Perspective

Samuel J. Lomonaco, Jr. and Louis H. Kauffman

Abstract. The ultimate objective of this paper is to create a stepping stone to the development of new quantum algorithms. The strategy chosen is to begin by focusing on the class of abelian quantum hidden subgroup algorithms, i.e., the class of abelian algorithms of the Shor/Simon genre. Our strategy is to make this class of algorithms as mathematically transparent as possible. By the phrase “mathematically transparent” we mean to expose, to bring to the surface, and to make explicit the concealed mathematical structures that are inherently and fundamentally a part of such algorithms. In so doing, we create symbolic abelian quantum hidden subgroup algorithms that are analogous to those symbolic algorithms found within such software packages as Axiom, MacAley, Maple, Mathematica, and Magma.

As a spin-off of this effort, we create three different generalizations of Shor’s quantum factoring algorithm to free abelian groups of finite rank. We refer to these algorithms as wandering (or vintage $Z_Q$) Shor algorithms. They are essentially quantum algorithms on free abelian groups $A$ of finite rank $n$ which, with each iteration, first select a random cyclic direct summand $Z$ of the group $A$ and then apply one iteration of the standard Shor algorithm to produce a random character of the “approximating” finite group $\tilde{A} = Z_Q$, called the group probe. These characters are then in turn used to find either the order $P$ of a maximal cyclic subgroup $Z_P$ of the hidden quotient group $H_\phi$, or the entire hidden quotient group $H_\phi$. An integral part of these wandering quantum algorithms is the selection of a very special random transversal $\iota_\mu : \tilde{A} \to A$, which we refer to as a Shor transversal. The algorithmic time complexity of the first of these wandering Shor algorithms is found to be $O\left(n^2 (\log Q)^3 (\log \log Q)^{n+1}\right)$.

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Part 1.  Preamble

1. Introduction
2. An example of Shor’s quantum factoring algorithm
3. Definition of the hidden subgroup problem (HSP) and hidden subgroup algorithms (HSAs)

Part 2.  Algebraic Preliminaries

4. The Character Group
5. Fourier analysis on a finite abelian group
6. Implementation issues: Group algebras as Hilbert spaces

Part 3.  QRand\(\varphi()\): The Progenitor of All QHSAs

7. Implementing \(Prob_{\varphi}(\chi)\) with quantum subroutine QRAND\(\varphi()\)

Part 4.  Vintage Simon Algorithms

8. Properties of the probability distribution \(Prob_{\varphi}(\chi)\) when \(\varphi\) has a hidden subgroup
9. A Markov process \(M_{\varphi}\) induced by \(Prob_{\varphi}\)
10. Vintage Simon quantum hidden subgroup algorithms (QHSAs)

Part 5.  Vintage Shor Algorithms

11. Vintage Shor quantum hidden subgroup algorithms (QHSAs)
12. Direct summand structure
13. Vintage Shor QHSAs with group probe \(\tilde{A} = \mathbb{Z}_Q\)
14. Finding Shor transversals for vintage \(\mathbb{Z}_Q\) Shor algorithms
15. Maximal Shor transversals
16. Identifying characters of cyclic groups with points on the unit circle \(S^1\) in the complex plane \(\mathbb{C}\).
17. Group norms
18. Vintage \(\mathbb{Z}_Q\) Shor QHSAs (Cont.)
19. When are characters of \(\tilde{A} = \mathbb{Z}_Q\) close to some character of a maximal cyclic subgroup \(\mathbb{Z}_P\) of \(H_{\varphi}\)?
20. Summary of Vintage \(\mathbb{Z}_Q\) Shor QHSAs
21. A cursory analysis of complexity
22. Two alternative vintage \(\mathbb{Z}_Q\) Shor algorithms

Part 6.  Epilogue

23. Conclusion
24. Acknowledgement
25. Appendix A. Continued fractions
26. Appendix B. Probability Distributions on Integers
References

Part 1. Preamble

1. Introduction

The ultimate objective of this paper is to create a stepping stone to the development of new quantum algorithms. The strategy chosen is to begin by focusing on the class of abelian quantum hidden subgroup algorithms (QHSAs), i.e., the class of abelian algorithms of the Shor/Simon genre. Our strategy is to make this class of algorithms as mathematically transparent as possible. By the phrase “mathematically transparent,” we mean to expose, to bring to the surface, and to make explicit the concealed mathematical structures that are inherently and fundamentally a part of such algorithms. In so doing, we create a class of symbolic abelian QHSAs that are analogous to those symbolic algorithms found within such software packages as Axiom, Cayley, Magma, Maple, and Mathematica.

During this mathematical analysis, the differences between the Simon and Shor quantum algorithms become dramatically apparent. This is in spite of the fact that these two share a common ancestor, namely, the quantum random group character generator \( \text{QRand} \), described herein. While the Simon algorithm is a QHSA on finite abelian groups which produces random characters of the hidden quotient group, the Shor algorithm is a QHSA on free abelian finite rank groups which produces random characters of a group which “approximate” the hidden quotient group. It is misleading, and a frequent cause of much confusion in the open literature, to call them both essentially the same QHSA.

Surprisingly, these two very different algorithms touch an amazing array of different mathematical disciplines, from the obvious to the not-so-obvious, requiring the integration of many diverse fields of mathematics. Shor’s quantum factoring algorithm, for example, depends heavily on the interplay of two metrics on the unit circle \( S^1 \), namely the arclength metric \( \text{Arc}_{2\pi} \) and the chordal metric \( \text{Chord}_{2\pi} \). This observation greatly simplifies the analysis of the Shor factoring algorithm, while at the same time revealing more of the structure concealed within the algorithm.

As a spin-off of this effort, we create three different generalizations of Shor’s quantum factoring algorithm to free abelian groups of finite rank, found in sections 20 and 22. We refer to these algorithms as wandering (or vintage \( \mathbb{Z}_Q \)) Shor algorithms. They are essentially QHSAs on free abelian finite rank \( n \) groups \( A \) which, with each iteration, first select a random cyclic direct summand \( \mathbb{Z} \) of the group \( A \) and then apply one iteration of the standard Shor algorithm to produce a random character of the “approximating” finite group \( A \), called a group probe. These algorithms find either the order \( P \) of a maximal cyclic subgroup \( \mathbb{Z}_P \) of the hidden quotient group \( H_\varphi \), or the entire hidden quotient group \( H_\varphi \). An integral part of these wandering algorithms is the selection of a very special random
transversal $\eta : \tilde{A} \rightarrow A$, which we refer to as a Shor transversal. The algorithmic time complexity of the first of these wandering (or vintage $\mathbb{Z}_Q$) algorithms is found in theorem 11 of section 21 to be $O \left( n^2 \left( \log Q \right)^3 \left( \log \log Q \right)^{n+1} \right)$, where $n$ denotes the fixed finite rank of the free abelian group $A$. Theorem 11 is based on the assumptions also found in section 21. This asymptotic bound is by no means the tightest possible.

Throughout this paper, it is assumed that the reader is familiar with the class of quantum hidden subgroup algorithms. For an introductions to this subject, please refer, for example, to any one of the references [8], [25], [26], [29], [33], [36], [43], [44]. This paper focuses, in particular, on the abelian hidden subgroup problem (HSP), with eye toward future work by the authors on the non-abelian HSP. There is a great deal of literature on the abelian HSP, for example, [5], [14], [25], [26], [27], [29], [35], [36], [43], [44], [45]. For literature on the non-abelian hidden subgroup problem, see for example, [16], [24], [14], [26], [33], [36], [38], [39], [41], [48].

2. An example of Shor’s quantum factoring algorithm

As an example of what we would like to make mathematically transparent, consider the following instance of Peter Shor’s quantum factoring algorithm. A great part of this paper is devoted to exposing and bringing to the surface the many concealed mathematical structures that are inherently and fundamentally part of this example.

Perhaps you see them? Perhaps you find them to be self evident? If you do, then you need read no more of this paper, although you are most certainly welcome to read on. If, on the other hand, the following example leaves you with a restless, uneasy feeling of not fully understanding what is really going on (i.e., of not fully understanding what concealed mathematical structures are lurking underneath these calculations), then you are invited to read the remainder of this paper.

Peter Shor’s quantum factoring algorithm reduces the task of factoring a positive integer $N$ to first finding a random integer $a$ relatively prime to $N$, and then next to determining the period $P$ of the following function

$$
\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \mod N
$$

$$
x \mapsto a^x \mod N
$$

where $\mathbb{Z}$ denotes the additive group of integers, and where $\mathbb{Z} \mod N$ denotes the integers mod $N$ under multiplication

$\footnote{A random integer $a$ with gcd $(a,N) = 1$ is found by selecting a random integer, and then applying the Euclidean algorithm to determine whether or not it is relatively prime to $N$. If not, then the gcd is a non-trivial factor of $N$, and there is no need to proceed further. However, this possibility is highly unlikely if $N$ is large.}$

Since $\mathbb{Z}$ is an infinite group, Shor chooses to work instead with the finite additive cyclic group $\mathbb{Z}_Q$ of order $Q = 2^m$, where $N^2 \leq Q < 2N^2$, and with the
“approximating” map
\[
\mathbb{Z}_Q \xrightarrow{\bar{\varphi}} \mathbb{Z} \mod N \\
x \mapsto a^x \mod N, \quad 0 \leq x < Q
\]
Shor begins by constructing a quantum system with two quantum registers
\[\text{LEFT\_REGISTER} \otimes \text{RIGHT\_REGISTER},\]
the left intended to hold the arguments \(x\) of \(\bar{\varphi}\), the right to hold the corresponding values of \(\bar{\varphi}\). This quantum system has been constructed with a unitary transformation
\[U_{\bar{\varphi}} : |x\rangle |1\rangle \mapsto |x\rangle |\bar{\varphi}(x)\rangle\]
implementing the “approximating” map \(\bar{\varphi}\).

As an example, let us use Shor’s algorithm to factor the enormous integer \(N = 21\), assuming that \(a = 2\) has been randomly chosen. Thus, \(Q = 2^9 = 512\).

Unknown to Peter Shor, the period is \(P = 6\), and hence, \(Q = 6 \cdot 85 + 2\).

Shor proceeds by executing the following steps:

**STEP 0** Initialize
\[|\psi_0\rangle = |0\rangle |1\rangle\]

**STEP 1** Apply the Fourier transform
\[F : |u\rangle \mapsto \frac{1}{\sqrt{512}} \sum_{x=0}^{511} \omega^{ux} |x\rangle\]
to the left register, where \(\omega = \exp(2\pi i/512)\) is a primitive 512-th root of unity, to obtain
\[|\psi_1\rangle = \frac{1}{\sqrt{512}} \sum_{x=0}^{511} |x\rangle |1\rangle\]

**STEP 2** Apply the unitary transformation
\[U_{\bar{\varphi}} : |x\rangle |1\rangle \mapsto |x\rangle |2^x \mod 21\rangle\]
to obtain
\[|\psi_2\rangle = \frac{1}{\sqrt{512}} \sum_{x=0}^{511} |x\rangle |2^x \mod 21\rangle\]
STEP 3  Once again apply the Fourier transform

$$F : |x\rangle \mapsto \frac{1}{\sqrt{512}} \sum_{y=0}^{511} \omega^{xy} |y\rangle$$

to the left register to obtain

$$|\psi_3\rangle = \frac{1}{512} \sum_{x=0}^{511} \sum_{y=0}^{511} \omega^{xy} |2^x \text{ mod } 21\rangle = \frac{1}{512} \sum_{y=0}^{511} (\sum_{x=0}^{511} \omega^{xy} |2^x \text{ mod } 21\rangle)$$

$$= \frac{1}{512} \sum_{y=0}^{511} |y\rangle |\Upsilon (y)\rangle$$

where

$$|\Upsilon (y)\rangle = \sum_{x=0}^{511} \omega^{xy} |2^x \text{ mod } 21\rangle$$

STEP 4  Measure the left register. Then with Probability

$$\text{Prob}_{\tilde{\varphi}} (y) = \frac{\langle \Upsilon (y) | \Upsilon (y) \rangle}{(512)^2}$$

the state will “collapse” to $|y\rangle$ with the value measured being the integer $y$, where $0 \leq y < Q$.

Let us digress for a moment to find a more usable expression for the probability distribution $\text{Prob}_{\tilde{\varphi}} (y)$.

$$|\Upsilon(y)\rangle = \sum_{x=0}^{511} \omega^{xy} |2^x \text{ mod } 21\rangle$$

$$= \sum_{x_1=0}^{85-1} \sum_{x_0=0}^{6-1} \omega^{(6x_1+x_0)y} |2^{6x_1+x_0} \text{ mod } 21\rangle + \sum_{x_0=0}^{2} \omega^{(6 \cdot 85+x_0)y} |2^{6 \cdot 85+x_0} \text{ mod } 21\rangle$$

But the order of $a = 2$ modulo 21 is $P = 6$, i.e., $P = 6$ is the smallest positive integer such that $2^6 = 1 \text{ mod } 21$. Hence, the above expression becomes

$$|\Upsilon(y)\rangle = \left( \sum_{x_1=0}^{84} \omega^{6x_1y} \right) \sum_{x_0=0}^{5} \omega^{x_0y} |2^{x_0} \text{ mod } 21\rangle + \omega^{6 \cdot 85 y} \sum_{x_0=0}^{1} \omega^{x_0y} |2^{x_0 \text{ mod } 21\rangle}$$

$$= \left( \sum_{x_1=0}^{85} \omega^{6x_1y} \right) \sum_{x_0=0}^{1} \omega^{x_0y} |2^{x_0} \text{ mod } 21\rangle + \left( \sum_{x_1=0}^{84} \omega^{6x_1y} \right) \sum_{x_0=2}^{5} \omega^{x_0y} |2^{x_0} \text{ mod } 21\rangle$$
Since the kets \( \{ |2^{x_0} \text{mod } 21\rangle | 0 \leq x_0 < 6 \} \) are all distinct, we have

\[
\langle \Upsilon (y) | \Upsilon (y) \rangle = 2 \left| \sum_{x_1=0}^{85} \omega^{6x_1y} \right|^2 + 4 \left| \sum_{x_1=0}^{84} \omega^{6x_1y} \right|^2.
\]

After a little algebraic manipulation, we finally have the following expression for \( \text{Prob}_{\tilde{\phi}} (y) \):

\[
\text{Prob}_{\tilde{\phi}} (y) = \frac{\langle \Upsilon (y) | \Upsilon (y) \rangle}{(512)^2} = \begin{cases} 
\frac{\sin^2 \left( \frac{\pi y}{128} \right) + 2 \sin^2 \left( \frac{131072}{256} \right)}{16384} & \text{if } y \neq 0 \text{ or } 256 \\
10923 & \text{if } y = 0 \text{ or } 256
\end{cases}
\]

A plot of \( \text{Prob}_{\tilde{\phi}} (y) \) is shown in Figure 1.

Figure 1. A plot of \( \text{Prob}_{\tilde{\phi}} (y) \).

The peaks in the above plot of \( \text{Prob}_{\tilde{\phi}} (y) \) occur at the integers

\[y = 0, 85, 171, 256, 341, 427.\]

The probability that at least one of these six integers will occur is quite high. It is actually 0.78+. Indeed, the probability distribution has been intentionally engineered to make the probability of these particular integers as high as possible. And there is a good reason for doing so.

The above six integers are those for which the corresponding rational \( y/Q \) is “closest” to a rational of the form \( d/P \). By “closest” we mean that

\[
\frac{|y - d|}{Q} < \frac{1}{2Q} < \frac{1}{2P^2}.
\]
In particular, 

\[
\begin{array}{cccccccc}
0 & 85 & 171 & 256 & 341 & 427 \\
\frac{512}{512} & \frac{512}{512} & \frac{512}{512} & \frac{512}{512} & \frac{512}{512} & \frac{512}{512}
\end{array}
\]

are rationals respectively “closest” to the rationals

\[
\frac{0}{6}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}.
\]

So by theorem 12 of Appendix A, the six rational numbers 0/6, 1/6, ..., 5/6 are convergents of the continued fraction expansions of 0/512, 85/512, ..., 427/512, respectively. Hence, each of the six rationals 0/6, 1/6, ..., 5/6 can be found with the recursion given in Appendix A.

But ... , we are not searching for rationals of the form \( d/P \). Instead, we seek only the denominator \( P = 6 \).

Unfortunately, the denominator \( P = 6 \) can only be gotten from the continued fraction recursion when the numerator and denominator of \( d/P \) are relatively prime. Given that the algorithm has selected one of the random integers 0, 85, ..., 427, the probability that the corresponding rational \( d/P \) has relatively prime numerator and denominator is \( \phi(6)/6 = 1/3 \), where \( \phi(\cdot) \) denotes the Euler totient function. So the probability of finding \( P = 6 \) is actually not 0.78\(^+\), but is instead 0.23\(^-\).

From Peter Shor’s perspective, the expression for the probability distribution is not known, since the period \( P \) is not known. All that Peter sees is a random integer \( y \) produced by the probability distribution \( \text{Prob}_\varphi \). However, he does know an approximate lower bound for the probability that the random \( y \) produced by \( \text{Prob}_\varphi \) is a “closest” one, namely the approximate lower bound \( 4/\pi^2 = 0.41^- \). Also, because\(^2\)

\[
\liminf \frac{\phi(N) \ln \ln N}{N} = e^{-\gamma},
\]

where \( \gamma = 0.5772 \cdots \) denotes Euler’s constant, he knows that

\[
\frac{\phi(P)}{P} = \Omega \left( \frac{1}{\lg \lg N} \right).
\]

Hence, if he repeats the algorithm \( O(\lg \lg N) \) times\(^3\), he will obtain one of the desired integers \( y \) with probability bounded below by approximately \( 4/\pi^2 \).

However, once he has in his possession a candidate \( P' \) for the actual period \( P = 6 \), the only way he can be sure he has the correct period \( P \) is to test \( P' \) by computing \( 2^{P'} \bmod 21 \). If the result is 1, he is certain he has found the correct period \( P \). This last part of the computation is done by the repeated squaring algorithm\(^4\).

\(^2\)Please refer to reference [21, Theorem 328, Section 18.4].

\(^3\)For even tighter asymptotic bounds, please refer to [9] and [37].

\(^4\)By the repeated squaring algorithm, we mean the algorithm which computes \( a^{P'} \bmod N \) via the expression

\[
a^{P'} = \prod_j \left( a^{2^j} \right)^{P'_j},
\]

where \( P' = \sum_j P'_j 2^j \) is the radix 2 expansion of \( P' \).
3. Definition of the hidden subgroup problem (HSP) and hidden subgroup algorithms (HSAs)

We now proceed by defining what is meant by a hidden subgroup problem (HSP) and a corresponding hidden subgroup algorithm. For other perspectives on HSPs, please refer to [29], [27], [35].

**Definition 1.** A map \( \varphi : A \rightarrow S \) from a group \( A \) into a set \( S \) is said to have hidden subgroup structure if there exists a subgroup \( K_\varphi \) of \( A \), called a hidden subgroup, and an injection \( \iota_\varphi : A/K_\varphi \rightarrow S \), called a hidden injection, such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & S \\
\downarrow{\nu} & & \uparrow{\iota_\varphi} \\
A/K_\varphi & &
\end{array}
\]

is commutative, where \( A/K_\varphi \) denotes the collection of right cosets of \( K_\varphi \) in \( A \), and where \( \nu : A \rightarrow A/K_\varphi \) is the natural map of \( A \) onto \( A/K_\varphi \). We refer to the group \( A \) as the ambient group and to the set \( S \) as the target set. If \( K_\varphi \) is a normal subgroup of \( A \), then \( H_\varphi = A/K_\varphi \) is a group, called the hidden quotient group, and \( \nu : A \rightarrow A/K_\varphi \) is an epimorphism, called the hidden epimorphism.

The hidden subgroup problem can be expressed as follows:

**Problem 1 (Hidden Subgroup Problem (HSP)).** Given a map with hidden subgroup structure

\[ \varphi : A \rightarrow S, \]

determine a hidden subgroup \( K_\varphi \) of \( A \). An algorithm solving this problem is called a hidden subgroup algorithm (HSA).

The corresponding quantum form of this HSP is stated as follows:

**Problem 2 (Hidden Subgroup Problem: Quantum Version).** Let

\[ \varphi : A \rightarrow S \]

be a map with hidden subgroup structure. Construct a quantum implementation of the map \( \varphi \) as follows:

Let \( \mathcal{H}_A \) and \( \mathcal{H}_S \) be Hilbert spaces defined respectively by the orthonormal bases

\[ \{ |a\rangle \mid a \in A \} \quad \text{and} \quad \{ |s\rangle \mid s \in S \}, \]

and let \( s_0 = \varphi(0) \), where \( 0 \) denotes the identity of the ambient group \( A \). Finally, let \( U_\varphi \) be the unitary transformation

\[ U_\varphi : \mathcal{H}_A \otimes \mathcal{H}_S \rightarrow \mathcal{H}_A \otimes \mathcal{H}_S \]

\[ |a\rangle |s_0\rangle \mapsto |a\rangle |\varphi(a)\rangle. \]
Determine the hidden subgroup $K_\varphi$ with bounded probability of error by making as few queries as possible of the blackbox $U_\varphi$. A quantum algorithm solving this problem is called a quantum hidden subgroup algorithm (QHSA).

In this paper, we focus on the abelian hidden subgroup problem (AHSP), i.e., the HSP with the ambient group $A$ assumed to be a finitely generated abelian group, and where the image of the hidden morphism $\varphi$ is a finite subset of $S$. (We will also on occasion assume that the entire set $S$ is finite.)

In this paper we focus on the following two classes of abelian hidden subgroup problems:

- **Vintage Simon AHSP.** The ambient group $A$ is finite and abelian.
- **Vintage Shor AHSP.** The ambient group $A$ is free abelian of finite rank.

**Notation Convention** For notational simplicity, throughout this paper we will use additive notation for both the ambient group $A$ and the hidden subgroup $K_\varphi$, and multiplicative notation for the hidden quotient group $H_\varphi = A/K_\varphi$.\(^6\)

### Part 2. Algebraic Preliminaries

#### 4. The Character Group

Let $G$ be an abelian group. Then the character group (or, dual group) $\hat{G}$ of $G$ is defined as the group of all morphisms of $G$ into the group $S^1$, i.e.,

$$\hat{G} = \text{Hom}(G, S^1)$$

where $S^1$ denotes the group of orientation preserving symmetries of the standard circle, and where multiplication on $\hat{G}$ is defined as:

$$(f_1 f_2)(g) = f_1(g) f_2(g) \quad \text{for all } f_1, f_2 \in \hat{G}$$

The elements of $\hat{G}$ are called **characters**.\(^7\)

**Remark 1.** The group $S^1$ can be identified with

1) The multiplicative group $U(1) = \{ e^{2\pi i x} \mid x \in \mathbb{R} \}$, i.e., with multiplication defined by $e^{2\pi i \alpha} \cdot e^{2\pi i \beta} = e^{2\pi i (\alpha + \beta)}$

2) The additive group $2\pi \mathbb{R}/2\pi \mathbb{Z}$, i.e., the reals modulo $2\pi$ under addition, i.e., with addition defined as

$$2\pi \alpha + 2\pi \beta \mod 2\pi = 2\pi (\alpha + \beta \mod 1)$$

---

\(^5\)For the general abelian HSP, please refer to [8] and [29].

\(^6\)This follows the notational convention found in [43].

\(^7\)More generally, for non-abelian groups, a character is defined as the trace of a representation of the group.
Remark 2. Please note that the 1-sphere $S^1$ can be thought of as a $\mathbb{Z}$-module under the action

$$(n, 2\pi \alpha) \mapsto 2\pi (n \alpha \mod 1)$$

Theorem 1. Every finite abelian group $G$ is isomorphic to the direct product of cyclic groups, i.e.,

$$G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_\ell},$$

where $\mathbb{Z}_{m_j}$ denotes the cyclic group of order $m_j$.

Theorem 2. Let $G$ be a finite abelian group. If $G = G_1 \times G_2$, then $\hat{G} = \hat{G}_1 \times \hat{G}_2$.

Theorem 3. $\hat{\mathbb{Z}}_m \cong \mathbb{Z}_m$

Corollary 1. If $G$ is a finite abelian group, then $G \cong \hat{G}$.

Remark 3. The isomorphism $G \cong \hat{G}$ can be expressed more explicitly as follows:

Let $G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_\ell}$, and let $g_1, g_2, \ldots, g_\ell$ denote generators of $\mathbb{Z}_{m_1}, \mathbb{Z}_{m_2}, \ldots, \mathbb{Z}_{m_\ell}$, respectively. Moreover, let $\omega_1, \omega_2, \ldots, \omega_\ell$ be $m_1$-th, $m_2$-th, ..., $m_\ell$-th primitive roots of unity, respectively. Then the character $\tilde{\chi}_j$ of $\mathbb{Z}_{m_j}$ defined by

$$\tilde{\chi}_j(g_j) = \omega_j$$

generates $\hat{\mathbb{Z}}_{m_j}$ as a cyclic group, i.e., the powers $(\tilde{\chi}_j)^k$ generate $\hat{\mathbb{Z}}_{m_j}$. Moreover, the characters $\chi_j$ of $G$ defined by

$$\chi_j = \left( \prod_{i=0}^{j-1} \bar{\chi}_i \right) \tilde{\chi}_j \left( \prod_{i=j+1}^{\ell} \bar{\chi}_i \right)$$

generate $\hat{G}$. It follows that an isomorphism $G \cong \hat{G}$ is given by

$$g_j \leftrightarrow \chi_j$$

Notation Convention In general, we will not need to represent the isomorphism $G \cong \hat{G}$ as explicitly as stated above. We will use the following convention. Let $\{g_1, g_2, \ldots, g_\ell\}$ and $\{\chi_1, \chi_2, \ldots, \chi_\ell\}$ denote respectively the set of elements of $G$ and $\hat{G}$ indexed in such a way that

$$g_j \leftrightarrow \chi_j$$

is the chosen isomorphism of $G$ and $\hat{G}$. We will at times use the notation

$$\begin{cases} g \leftrightarrow \chi_g \\ \chi \leftrightarrow g_{\chi} \end{cases}$$
5. Fourier analysis on a finite abelian group

As in the previous section, let $G$ be a finite abelian group and let $\hat{G}$ denote its character group. Let $g$ and $\chi$ denote respectively elements of the groups $G$ and $\hat{G}$.

Let $\mathbb{C}G$ and $\mathbb{C}\hat{G}$ denote the corresponding group algebras of $G$ and $\hat{G}$ over the complex numbers $\mathbb{C}$. Hence, $\mathbb{C}G$ consists of all maps $f : G \to \mathbb{C}$. Addition $\cdot+$, multiplication $\cdot$, and scalar multiplication are defined as:

\[
\begin{align*}
(f_1 + f_2)(g) &= f_1(g) + f_2(g) \quad \forall g \in G \\
(f_1 \cdot f_2)(g) &= \sum_{h \in G} f_1(h) f_2(h^{-1} g) \quad \forall g \in G \quad \text{(Convolution)} \\
(\lambda f)(g) &= \lambda f(g) \quad \forall \lambda \in \mathbb{C} \text{ and } \forall g \in G
\end{align*}
\]

Caveat. Please note that the symbol $g$ has at least three different meanings:

♦ INTERPRETATION 1. The symbol $g$ denotes an element of the group $G$

♦ INTERPRETATION 2. The symbol $g$ denotes a pointwise map $g : G \to \mathbb{C}$ defined by

\[
g(g') = \begin{cases} 
1 & \text{if } g = g' \\
0 & \text{otherwise}
\end{cases}
\]

Thus,

$f = \sum_{g \in G} f(g) g$ denotes $g \mapsto f(g)$

Hence, $g \in \mathbb{C}G$. Since $G$ is isomorphic as a group to the set of pointwise maps $\{g : G \to \mathbb{C} \mid g \in G\}$ under convolution, we can and do identify the group elements of $G$ with the pointwise maps $g \in \mathbb{C}G$. Thus, interpretations 1 and 2 lead to no ambiguity at the algebraic level.

♦ INTERPRETATION 3. The symbol $g$ denotes a character of $\hat{G}$ defined by

$g(\chi) = \chi(g)$

Thus, with this interpretation, $g \in \hat{G} \subset \mathbb{C}\hat{G}$. This third interpretation can, in some instances, lead to some unnecessary confusion. When this intended interpretation is possibly not clear from context, we will resort to the notation $g^*$. 

\footnote{If $G$ is infinite, then ring multiplication $\cdot$ is not always well defined. So $\mathbb{C}G$ is not a ring, but a $\mathbb{Z}$-module, with a group of operators. One way of making $\mathbb{C}G$ into a ring, is to restrict the maps on $G$, e.g., to maps with compact support, to maps with $L^2$ norm, etc.}
for interpretation 3 of the symbol $g$. Thus, for example,

$$f^\bullet = \sum_{g \in G} f(g)g^\bullet$$

denotes the map $\chi \mapsto \sum_{g \in G} f(g)\chi(g)$

In like manner, the symbol $\chi$ has at least three different meanings:

- **Interpretation 1.** The symbol $\chi$ denotes an element of the group $\hat{G}$

- **Interpretation 2.** The symbol $\chi$ denotes a pointwise map

$$\chi : \hat{G} \to \mathbb{C}$$

defined by

$$\chi(\chi') = \begin{cases} 1 & \text{if } \chi = \chi' \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\hat{f} = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi$$

denotes $\chi \mapsto \hat{f}(\chi)$

Hence, $\chi \in \mathbb{C}\hat{G}$. Since $\hat{G}$ is isomorphic as a group to the set of pointwise maps $\{\chi : \hat{G} \to \mathbb{C} \mid \chi \in \hat{G}\}$ under convolution, we can and do identify the group elements of $\hat{G}$ with the pointwise maps $\chi \in \mathbb{C}\hat{G}$. Thus, interpretations 1 and 2 lead to no ambiguity at the algebraic level.

- **Interpretation 3.** The symbol $\chi$ denotes a character map of $G$ onto $\mathbb{C}$ defined by

$$g \mapsto \chi(g)$$

Thus, with this interpretation, $\chi \in \mathbb{C}G$. This third interpretation can, in some instances, also lead to some unnecessary confusion. When this intended interpretation is possibly not clear from context, we will resort to the notation

$$\chi^\bullet$$

for interpretation 3 of the symbol $\chi$. Thus, for example,

$$\hat{f}^\bullet = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi^\bullet$$

denotes the map $g \mapsto \sum_{\chi \in \hat{G}} \hat{f}(\chi)(g)$

We define complex inner products on the group algebras $\mathbb{C}G$ and $\mathbb{C}\hat{G}$ as follows:

$$\begin{align*}
(f_1, f_2) &= \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)} \quad \forall f_1, f_2 \in \mathbb{C}G \\
(\hat{f}_1, \hat{f}_2) &= \frac{1}{|\hat{G}|} \sum_{\chi \in \hat{G}} \hat{f}_1(\chi)\overline{\hat{f}_2(\chi)} \quad \forall \hat{f}_1, \hat{f}_2 \in \mathbb{C}\hat{G}
\end{align*}$$

where $\overline{f_2(g)}$ and $\overline{\hat{f}_2(\chi)}$ denote respectively the complex conjugates of $f_2(g)$ and $\hat{f}_2(\chi)$.
The corresponding norms are defined as
\[
\begin{align*}
\|f\| &= \sqrt{\langle f, f \rangle} \quad \forall f \in \mathbb{C}G \\
\|\hat{f}\| &= \sqrt{\langle \hat{f}, \hat{f} \rangle} \quad \forall \hat{f} \in \mathbb{C}\hat{G}
\end{align*}
\]

As an immediate consequence of the above definitions, we have:
\[
(g_1, g_2) = \begin{cases} 
1 & \text{if } g_1 = g_2 \\
0 & \text{otherwise}
\end{cases}
\quad \text{and} \quad
(\chi_1, \chi_2) = \begin{cases} 
1 & \text{if } \chi_1 = \chi_2 \\
0 & \text{otherwise}
\end{cases}
\]

It also follows from the standard character identities that
\[
(g^*, g_2) = \begin{cases} 
1 & \text{if } g^* = g_2 \\
0 & \text{otherwise}
\end{cases}
\quad \text{and} \quad
(\chi^*, \chi_2) = \begin{cases} 
1 & \text{if } \chi^* = \chi_2 \\
0 & \text{otherwise}
\end{cases}
\]

We are now in a position to define the Fourier transform on a finite abelian group \( G \).

**Definition 2.** The **Fourier transform** \( \mathcal{F} \) for a finite abelian group \( G \) is defined as
\[
\mathbb{C}G \xrightarrow{\mathcal{F}} \mathbb{C}\hat{G} \quad f \mapsto \hat{f} = \frac{1}{\sqrt{|G|}} \sum_{g \in G} f(g) \overline{\chi(g)} = \frac{1}{\sqrt{|\hat{G}|}} \sum_{\chi \in \hat{G}} \left( \sum_{g \in G} f(g) \chi(g) \right) \chi
\]

Hence,
\[
\hat{f}(\chi) = \sqrt{|G|} \langle f, \chi^* \rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} f(g) \overline{\chi(g)}
\]

**Proposition 1.**
\[
f = \frac{1}{\sqrt{|G|}} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi^*
\]

**Proof.**
\[
\frac{1}{\sqrt{|G|}} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g_0) = \frac{1}{\sqrt{|G|}} \sum_{\chi \in \hat{G}} \frac{1}{\sqrt{|G|}} \sum_{g \in G} f(g) \overline{\chi(g)} \chi(g_0) = \frac{1}{|G|} \sum_{g \in G} f(g) \frac{1}{\sqrt{|\hat{G}|}} \sum_{\chi \in \hat{G}} \overline{\chi(g)} \chi(g_0) = f(g_0)
\]

We define the inverse Fourier transform as follows:
**Definition 3.** The inverse Fourier transform $\mathcal{F}^{-1}$ is defined as

$$
\mathbb{C}^\hat{G} \xrightarrow{\mathcal{F}^{-1}} \mathbb{C}^G \quad \hat{f} \quad \longrightarrow \quad f = \frac{1}{\sqrt{|G|}} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi^* \quad \text{Hence,}
$$

$$
f(g) = \sqrt{|G|} \left( \hat{f}, \chi \right) = \frac{1}{\sqrt{|G|}} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g)
$$

**Theorem 4 (Plancherel identity).**

$$
\|f\| = \|\hat{f}\|
$$

**Proof.**

$$
\|f\|^2 = (f, f) = \frac{1}{|G|} \sum_{g \in G} |f(g)|^2 = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g) \left( \frac{1}{\sqrt{|G|}} \sum_{\chi' \in \hat{G}} \hat{f}(\chi') \chi'(g) \right)
$$

$$
= \frac{1}{|G|^2} \sum_{g \in G} \sum_{\chi \in \hat{G}} \sum_{\chi' \in \hat{G}} \hat{f}(\chi) \chi(g) \chi'(g)
$$

$$
= \frac{1}{|G|^2} \sum_{\chi \in \hat{G}} \sum_{\chi' \in \hat{G}} \hat{f}(\chi) \hat{f}(\chi') \left( \sum_{g \in G} \chi(g) \chi'(g) \right)
$$

$$
= \frac{1}{|G|} \sum_{\chi \in \hat{G}} \left| \hat{f}(\chi) \right|^2 = \|\hat{f}\|^2
$$

6. Implementation issues: Group algebras as Hilbert spaces

For implementation purposes, we will need to view group algebras also as Hilbert spaces.\(^9\)

In particular, $\mathbb{C}G$ and $\mathbb{C}^\hat{G}$ can be respectively viewed as the Hilbert spaces $\mathcal{H}_G$ and $\mathcal{H}_{\hat{G}}$ defined by the respective orthonormal bases

$$
\{|g\} \mid g \in G\} \text{ and } \{|\chi\} \mid \chi \in \hat{G}\}.
$$

\(^9\)Category theorists will recognize this as a forgetful functor.
In this context, the **Fourier transform** $\mathcal{F}$ becomes

$$|f\rangle = \sum_{g \in G} f(g) |g\rangle \quad \mapsto \quad |\hat{f}\rangle = \frac{1}{\sqrt{|G|}} \sum_{\chi \in \hat{G}} \left( \sum_{g \in G} \hat{f}(\chi) \chi(g) \right) |\chi\rangle$$

and the **inverse Fourier transform** $\mathcal{F}^{-1}$ becomes

$$|\hat{f}\rangle \quad \mapsto \quad |f\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \left( \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g) \right) |g\rangle$$

One important and useful identification is to use the Hilbert space isomorphism

$$\mathcal{H}_G \leftrightarrow \mathcal{H}_{\hat{G}}$$

$$|g\rangle \leftrightarrow |\chi g\rangle$$

$$|g\chi\rangle \leftrightarrow |\chi\rangle$$

to identify the two Hilbert spaces $\mathcal{H}_G$ and $\mathcal{H}_{\hat{G}}$. As a result, the Fourier transform $\mathcal{F}$ and it’s inverse $\mathcal{F}^{-1}$ can both be viewed as transforms taking the Hilbert space $\mathcal{H}_G$ to itself, i.e.,

$$\mathcal{H}_G \xrightarrow{\mathcal{F}} \mathcal{H}_G$$

$$\mathcal{H}_G \xleftarrow{\mathcal{F}^{-1}} \mathcal{H}_G$$

**Remark 4.** This last identification is crucial for the implementation of hidden subgroup algorithms.

**Part 3. QRand$_{\phi}()$: The Progenitor of All QHSAs**

7. Implementing $\text{Prob}_{\phi}(\chi)$ with quantum subroutine QRand$_{\phi}()$

Let

$$\phi : A \rightarrow S$$

be a map from a finite abelian group $A$ into a finite set $S$.

We use additive notation for the group $A$; and let $s_0 = \phi(0)$ denote the image of the identity 0 of $A$ under the map $\phi$.

Let $\mathcal{H}_A$, $\mathcal{H}_\hat{A}$, and $\mathcal{H}_S$ denote the Hilbert spaces respectively defined by the orthonormal bases

$$\left\{ |a\rangle \mid a \in A \right\}, \left\{ |\chi\rangle \mid \chi \in \hat{A} \right\}, \text{ and } \left\{ |s\rangle \mid s \in S \right\}.$$
We assume that we are given a quantum system which implements the unitary transformation $U_\varphi$ defined by

$$
\mathcal{H}_A \otimes \mathcal{H}_S \xrightarrow{U_\varphi} \mathcal{H}_A \otimes \mathcal{H}_S
$$

$$
|a\rangle |s_0\rangle \mapsto |a\rangle |\varphi(a)\rangle
$$

We will use the above implementation to construct a quantum subroutine $\text{QRAND}_\varphi()$ which produces a probability distribution $\text{Prob}_\varphi : \hat{A} \rightarrow [0, 1]$ on the character group $\hat{A}$ of the group $A$.

Before doing so, we will, as explained in the previous section, make use of various identifications, such as respectively identifying the Fourier and inverse Fourier transforms $\mathcal{F}_A$ and $\mathcal{F}_A^{-1}$ on the group $A$

$$
C_A = \mathcal{H}_A \xrightarrow{\mathcal{F}_A} \mathcal{H}_{\hat{A}} = C\hat{A}
$$

with

$$
C_A = \mathcal{H}_A \xleftarrow{\mathcal{F}_A^{-1}} \mathcal{H}_A = CA
$$

\begin{center}
\textbf{Quantum Subroutine QRAND}_\varphi()\end{center}

**Step 0.** Initialization

$$
|\psi_0\rangle = |0\rangle |s_0\rangle
$$

**Step 1.** Application of the inverse Fourier transform $\mathcal{F}_A^{-1}$ of $A$

$$
|\psi_1\rangle = (\mathcal{F}_A^{-1} \otimes 1_S) |\psi_0\rangle = \frac{1}{\sqrt{|A|}} \sum_{a \in A} |a\rangle |s_0\rangle
$$

where $|A|$ denotes the cardinality of the group $A$.

**Step 2.** Application of the unitary transformation $U_\varphi$

$$
|\psi_2\rangle = U_\varphi |\psi_1\rangle = \frac{1}{\sqrt{|A|}} \sum_{a \in A} |a\rangle |\varphi(a)\rangle
$$
Step 3. Application of the Fourier transform $\mathcal{F}_A$ of $A$

$$|\psi_3⟩ = (\mathcal{F}_A^{-1} \otimes 1_S)|ψ_2⟩ = \frac{1}{|A|} \sum_{a ∈ A} \sum_{χ ∈ \hat{A}} \chi(a) |χ⟩ |φ(a)⟩$$

$$= \sum_{χ ∈ \hat{A}} \frac{||φ(χ^*)||}{|A|} |χ⟩ \frac{||φ(χ^*)||}{||φ(χ^*)||}$$

where

$$|φ(χ^*)⟩ = \sum_{a ∈ A} \chi(a) |φ(a)⟩$$

Remark 5. This notation is meant to be suggestive, since under the identification $\mathcal{H}_A = \mathbb{C}A$ we have

$$|φ(χ^*)⟩ = \sum_{a ∈ A} \chi(a) |φ(a)⟩ = φ \left( \sum_{a ∈ A} \chi(a) a \right) = φ(χ^*)$$

Step 4. Measurement of the left quantum register. Thus, with probability

$$\text{Prob}_φ(χ) = \frac{||φ(χ^*)||^2}{|A|^2}$$

the character $χ$ is the resulting measured value, and the quantum system “collapses” to the state

$$|ψ_4⟩ = |χ⟩ \frac{|φ(χ^*)⟩}{||φ(χ^*)||}$$

Step 5. Output the character $χ$, and stop.

Remark 6. The quantum subroutine $\text{QRand}_φ()$ can also be viewed as a subroutine with the state $|χ⟩|φ(χ^*)⟩$ as a side effect.

As a result of the above description of $\text{QRand}_φ()$, we have the following theorem:

Theorem 5. Let

$$φ : A → S$$
be a map from a finite abelian group $A$ into a finite set $S$. Then the quantum sub-
routine $\text{QRAND}_\varphi()$ is an implementation of the probability distribution $\text{Prob}_\varphi(\chi)$ on the group $\hat{A}$ of characters of $A$ given by

$$\text{Prob}_\varphi(\chi) = \frac{\|\varphi(\chi^*)\|^2}{|A|^2},$$

for all $\chi \in \hat{A}$, where $\chi^*$ denotes

$$\chi^* = \sum_{a \in A} \chi(a) a \in \mathbb{C}A$$

**Remark 7.** Please note that the above theorem is true whether or not the map $\varphi : A \rightarrow S$ has a hidden subgroup.

We will, on occasion, refer to the probability distribution

$$\text{Prob}_\varphi : \hat{A} \rightarrow [0, 1]$$

on the character group $\hat{A}$ as the **stochastic source** $S_\varphi(\chi)$ which produces a symbol $\chi \in \hat{A}$ with probability $\text{Prob}_\varphi(\chi)$. (See [32].) Thus, $\text{QRAND}_\varphi(\chi)$ is an algorithmic implementation of the stochastic source $S_\varphi(\chi)$.

**Part 4. Vintage Simon Algorithms**

We now begin the development of the class of vintage Simon QHSAs. These are QHSAs for which the ambient group $A$ is finite abelian.

**8. Properties of the probability distribution $\text{Prob}_\varphi(\chi)$ when $\varphi$ has a hidden subgroup**

Let

$$\varphi : A \rightarrow S$$

be a map from a finite abelian group $A$ to a set $S$. We now assume that $\varphi$ has a hidden subgroup $K_\varphi$, and hence, a hidden quotient group $H_\varphi = A/K_\varphi$.

Let

$$\nu : A \rightarrow H_\varphi = A/K_\varphi$$

denote the corresponding natural epimorphism respectively. Then since $\text{Hom}_{\mathbb{Z}}(-, 2\pi\mathbb{R}/2\pi\mathbb{Z})$ is a left exact contravariant functor, the map

$$\tilde{\nu} : \hat{H}_\varphi \rightarrow \hat{A}$$

$$\eta \mapsto \eta \circ \nu$$

is a monomorphism\textsuperscript{10}.

Since $\tilde{\nu}$ is a monomorphism, each character $\eta$ of the hidden quotient group $H_\varphi$ can be identified with a character $\chi$ of $A$ for which $\chi(k) = 1$ for every element of

\textsuperscript{10}See [7].
K_\varphi$. In other words, $\widehat{H_\varphi}$ can be identified with all characters of $A$ which are trivial on $K_\varphi$.

**Theorem 6.** Let

$$\varphi : A \rightarrow S$$

be a map from a finite abelian group $A$ into a finite set $S$. If there exists a hidden subgroup $K_\varphi$ of $\varphi$, and hence a hidden quotient group $H_\varphi = A/K_\varphi$ of $\varphi$, then the probability distribution $\text{Prob}_\varphi (\chi)$ on $\hat{\mathbb{A}}$ implemented by the quantum subroutine $\text{QRand}_\varphi ()$ is given by

$$\text{Prob}_\varphi (\chi) = \begin{cases} \frac{1}{|H_\varphi|} & \text{if } \chi \in \widehat{H_\varphi} \\ 0 & \text{otherwise} \end{cases}$$

In other words, in this particular case, $\text{Prob}_\varphi (\chi)$ is nothing more than the uniform probability distribution on the character group $\widehat{H_\varphi}$ of the hidden quotient group $H_\varphi$.

**Proof.** Since $\varphi$ has a hidden subgroup $K_\varphi$, there exists a hidden injection $\iota_\varphi : H_\varphi \rightarrow S$ from the hidden quotient group $H_\varphi = A/K_\varphi$ to the set $S$ such that the diagram

$$A \xrightarrow{\varphi} S$$

$$\nu \downarrow \iota_\varphi$$

$H_\varphi$

is commutative, where $\nu : A \rightarrow H_\varphi$ denotes the hidden natural epimorphism of $A$ onto the quotient group $H_\varphi = A/K_\varphi$.

Next let

$$\iota_\nu : H_\varphi \rightarrow A$$

be a transversal map of the subgroup $K_\varphi$ in $A$, i.e., a map such that

$$\nu \circ \iota_\nu = \text{id}_{H_\varphi}.$$ 

In other words, $\iota_\nu$ sends each element $h$ of $H_\varphi$ to a unique element of the coset $\varphi^{-1}(h)$.

Recalling that

$$\varphi (\chi^\ast) = \sum_{a \in A} \chi(a) \varphi(a),$$

we have

$$\varphi (\chi^\ast) = \sum_{a \in A} \chi(a) \iota_\varphi \nu a = \sum_{h \in H_\varphi} \left( \sum_{k \in K_\varphi} \chi(\iota_\nu h + k) \right) \iota_\varphi h$$

$$= \sum_{h \in H_\varphi} \chi(\iota_\nu h) \left( \sum_{k \in K_\varphi} \chi(k) \right) \iota_\varphi h = \left( \sum_{k \in K_\varphi} \chi(k) \right) \left( \sum_{h \in H_\varphi} \chi(\iota_\nu h) \iota_\varphi h \right)$$
Thus,

$$\|\phi(\chi^\bullet)\|^2 = \left| \sum_{k \in K_\phi} \chi(k) \right|^2 \left| \sum_{h \in H_\phi} \chi(\iota_h h) \iota_h h \right|^2$$

$$= \left| \sum_{k \in K_\phi} \chi(k) \right|^2 \sum_{h \in H_\phi} |\chi(\iota_h h)|^2 = \sum_{k \in K_\phi} |\chi(k)|^2 |H_\phi|$$

But by a standard character identity\(^\text{11}\), we have

$$\sum_{k \in K_\phi} \chi(k) = \begin{cases} |K_\phi| = |A| / |H_\phi| & \text{if } \chi \in \hat{H}_\phi \\ 0 & \text{otherwise} \end{cases}$$

Hence, it follows that

$$\text{Prob}_\phi(\chi) = \frac{\|\phi(\chi^\bullet)\|^2}{|A|^2} = \begin{cases} \frac{1}{|H_\phi|} & \text{if } \chi \in \hat{H}_\phi \\ 0 & \text{otherwise} \end{cases}$$

\(\square\)

9. A Markov process \(M_\phi\) induced by \(\text{Prob}_\phi\)

Before we can discuss the class of vintage Simon quantum hidden subgroup algorithms, we need to develop the mathematical machinery to deal with the following question:

**Question.** Let \(\varphi : A \rightarrow S\) be a map from a finite abelian group \(A\) to a finite set \(S\). Assume that the map \(\varphi\) has a hidden group \(K_\varphi\), and hence a hidden quotient group \(H_\varphi\). From theorem 6 of the previous section, we know that the probability distribution

$$\text{Prob}_\varphi : \hat{A} \rightarrow [0,1]$$

is effectively the uniform probability distribution on the character group \(\hat{H}_\varphi\) of the hidden quotient group \(H_\varphi\). How many times do we need to query the probability distribution \(\text{Prob}_\varphi\) to obtain enough characters of \(H_\varphi\) to generate the entire character group \(\hat{H}_\varphi\)?

We begin with a definition:

**Definition 4.** Let

$$\text{Prob}_G : G \rightarrow [0,1]$$

be a probability distribution on a finite abelian group \(G\), and let \(G_+\) denote the subgroup of \(G\) generated by all elements \(g\) of \(G\) such that \(\text{Prob}_G(g) > 0\). The Markov

\(^{11}\)See [17].
**process** $\mathcal{M}_G$ associated with a probability distribution $\text{Prob}_G$ is the Markov process with the subgroups $G_\alpha$ of $G_+$ as states, and with transition probabilities given by

$$\text{Prob}(G_\alpha \rightsquigarrow G_\beta) = \text{Prob}_G \{g \in G_+ \mid G_\beta \text{ is generated by } g \text{ and the elements of } G_\alpha\},$$

where $G_\alpha \rightsquigarrow G_\beta$ denotes the transition from state $G_\alpha$ to state $G_\beta$. The initial state of the Markov process $\mathcal{M}_G$ is the trivial subgroup $G_0$. The subgroup $G_+$ is called the absorbing subgroup of $G$. The transition matrix $T$ of the Markov process is the matrix indexed on the states according to some chosen fixed linear ordering with $(G_\alpha, G_\beta)$-th entry $T_{\alpha\beta}$ given by $\text{Prob}(G_\alpha \rightsquigarrow G_\beta)$.

The following two propositions are immediate consequences of the above definition:

**Proposition 2.** Let

$$\text{Prob}_G : G \longrightarrow [0, 1]$$

be a probability distribution on a finite abelian group $G$. Then the Markov process $\mathcal{M}_G$ is an absorbing Markov process with sole absorbing state $G_+$, a state which once entered can never be left. The remaining states are transient states, i.e., states once left can never again be entered. Hence,

$$\lim_{n \to \infty} \text{Prob}_G \left(G_0 \rightsquigarrow n G_\alpha\right) = \begin{cases} 1 & \text{if } G_\alpha = G_+ \\ 0 & \text{if } G_\alpha \neq G_+ \end{cases}$$

In other words, if the Markov process $\mathcal{M}_G$ starts in state $G_0$, it will eventually end up permanently in the absorbing state $G_+$.

**Proposition 3.** Let $T$ be the transition matrix of the Markov process associated with the probability distribution

$$\text{Prob}_G : G \longrightarrow [0, 1]$$

Then the probability $\text{Prob} \left(G_\alpha \rightsquigarrow n G_\beta\right)$ that the Markov process $\mathcal{M}_G$ starting in state $G_\alpha$ is in state $G_\beta$ after $n$ transitions is equal to the $(G_\alpha, G_\beta)$-th entry of the matrix $T^n$, i.e.,

$$\text{Prob} \left(G_\alpha \rightsquigarrow n G_\beta\right) = (T^n)_{\alpha\beta}$$

Under certain circumstances, we can work with a much simpler Markov process.

**Proposition 4.** Let $G$ be a finite abelian group with probability distribution

$$\text{Prob}_G : G \longrightarrow [0, 1]$$

such that $\text{Prob}_G$ is the uniform probability distribution on the absorbing group $G_+$. Partition the states of the associated Markov process $\mathcal{M}_G$ into the collection of sets

$$\{G_j \mid j \text{ divides } |G_+|\},$$

where $G_j$ is the set of all states $G_\alpha$ of $\mathcal{M}_G$ of group order $j$. 

If

\[ \text{Prob} (G_i \rightarrow G_j) = \sum_{G_j \in G_i} \text{Prob} (G_i \rightarrow G_j) \]

has the same value for all \( G_i \in G_i \), then the states of \( M_G \) can be combined (lumped) to form a Markov process \( M_G^{\text{lumped}} \) with states \( \{ G_j \mid j \text{ divides } |G_+| \} \), and with transition probabilities given by

\[ \text{Prob}^{\text{lumped}} (G_i \rightarrow G_j) = \text{Prob} (G_i \rightarrow G_j) \]

where \( G_i \) is an arbitrarily chosen element of \( G_i \), and with initial state \( G_1 = \{ G_0 \} \).

Moreover, the resulting \( M_G^{\text{lumped}} \) is also an absorbing Markov process with sole absorbing state \( G|G_+| = \{ G_+ \} \), with all other states transient, and such that

\[ \text{Prob} \left( G_0 \rightarrow k G_+ \right) = \text{Prob} \left( G_1 \rightarrow k G|G_+| \right) \]

As a consequence of the above proposition and theorem 6, we have:

**Corollary 2.** Let \( \varphi : A \rightarrow S \) be a map from a finite abelian group \( A \) to a finite set \( S \), which has a hidden subgroup \( K_\varphi \), and hence a hidden quotient group \( H_\varphi \). Moreover, let the ambient group \( A \) be the direct sum of cyclic groups of the same prime order \( p \), i.e., let

\[ A = \bigoplus_{i=1}^n \mathbb{Z}_p \]

Then the combined (lumped) process \( M_A^{\text{lumped}} \) is a Markov process such that

\[ \text{Prob} \left( \hat{A}_0 \rightarrow k \hat{H}_\varphi \right) = \text{Prob} \left( \hat{G}_1 \rightarrow k \hat{G}|\hat{H}_\varphi| \right) \]

Moreover, if the states of \( M_A^{\text{lumped}} \) are linearly ordered as

\( G_i < G_j \) if and only if \( i \) divides \( j \),

then the transition matrix \( T \) of \( M_A^{\text{lumped}} \) is given by

\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 - \frac{1}{p} & \frac{1}{p} & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 - \frac{1}{p^2} & \frac{1}{p^2} & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & - \frac{1}{p^3} & \frac{1}{p^3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{p^{n+1}} & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 - \frac{1}{p^n} & \frac{1}{p^n}
\end{pmatrix}
\]

Hence,

\[ \text{Prob} \left( \hat{A}_0 \rightarrow k \hat{H}_\varphi \right) = (T^k)_{n+1} \]

from which it easily follows that

\[ \text{Prob} \left( \hat{A}_0 \rightarrow k \hat{H}_\varphi \right) > 1 - \frac{1}{p - 1} \left( \frac{1}{p} \right)^{k-n} S \geq 1 - \frac{1}{(p-1)p^2} \]

for \( k \geq n + 2 \).
10. Vintage Simon quantum hidden subgroup algorithms (QHSAs)

We are now prepared to extend Simon’s quantum algorithm to an entire class of QHSAs on finite abelian groups.

Let

$$\varphi : A \rightarrow S$$

be a map from a finite abelian group $$A$$ to a finite set $$S$$ for which there exists a hidden subgroup $$K_\varphi$$, and hence, a hidden quotient group $$H_\varphi = A/K_\varphi$$.

Following our usual convention, we use additive notation for the ambient group $$A$$ and multiplicative notation for the hidden quotient group $$H_\varphi$$.

As mentioned in section 2 of this paper, it follows from the standard theory of abelian groups (i.e., Theorem 1) that the ambient group $$A$$ can be decomposed into the finite direct sum of cyclic groups $$\mathbb{Z}_{m_0}, \mathbb{Z}_{m_1}, \ldots, \mathbb{Z}_{m_{\ell-1}}$$, i.e.,

$$A = \mathbb{Z}_{m_0} \oplus \mathbb{Z}_{m_1} \oplus \ldots \oplus \mathbb{Z}_{m_{\ell-1}}$$.

We denote respective generators of the above cyclic groups by

$$a_0, a_1, \ldots, a_{\ell-1}$$.

Consequently, each character $$\chi$$ of the ambient group $$A$$ can be uniquely expressed as

$$\chi : \sum_{j=0}^{\ell-1} \alpha_j a_j \longmapsto \exp \left( 2\pi i \sum_{j=0}^{\ell-1} \alpha_j \frac{y_j}{m_j} \right),$$

where $$0 \leq y_j < m_j$$ for $$j = 0, 1, \ldots, \ell - 1$$. Thus, we have a one-to-one correspondence between the characters $$\chi$$ of $$A$$ and $$\ell$$-tuples of rationals (modulo 1) of the form

$$\left( \frac{y_0}{m_0}, \frac{y_1}{m_1}, \ldots, \frac{y_{\ell-1}}{m_{\ell-1}} \right),$$

where

$$0 \leq y_j < m_j, \ j = 0, 1, \ldots, \ell - 1.$$ 

As a result, we can and do use the following notation to refer uniquely to each and every character $$\chi$$ of $$A$$

$$\chi = \chi(\frac{y_0}{m_0}, \frac{y_1}{m_1}, \ldots, \frac{y_{\ell-1}}{m_{\ell-1}}).$$

**Definition 5.** Let

$$A = \mathbb{Z}_{m_0} \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_{\ell-1}}$$

be a direct sum decomposition of a finite abelian group $$A$$ into finite cyclic groups. Let

$$a_0, a_1, \ldots, a_{n-1}$$

denote respective generators of the cyclic groups in this direct sum decomposition.
Then an integer matrix
\[ G = [\alpha_{ij}]_{k \times n} \mod (m_0, m_1, \ldots, m_{n-1}) \]
is said to be a **generator matrix** of a subgroup \( K \) of \( A \) provided
\[ \left\{ \sum_{j=0}^{n-1} \alpha_{ij} a_j \mid 0 \leq i < k \right\} \]
is a complete set of generators of the subgroup \( K \).

A matrix of rationals \( \mod 1 \)
\[ H = \begin{bmatrix} y_{ij} \\ m_j \end{bmatrix}_{\ell \times n} \mod 1 \]
is said to be a **dual generator matrix** of a subgroup \( K \) of \( A \) provided
\[ \left\{ \chi_{\frac{y_i}{m_0}, \frac{y_i}{m_1}, \ldots, \frac{y_i}{m_{n-1}}} \mid 0 \leq i < \ell \right\} \]
is a complete set of generators of the character group \( \hat{H} \) of the quotient group \( H = A/K \).

Let \( M_\varphi \) be the Markov process associated with the probability distribution
\[ \text{Prob}_\varphi : \hat{A} \rightarrow [0, 1] \]
on the character group \( \hat{A} \) of the ambient group \( A \).

Let \( 0 \leq \epsilon \ll 1 \) be a chosen threshold.

Then a vintage Simon algorithm is given below:

**Vintage Simon(\( \varphi, \epsilon \))**

**Step 1.** Select a positive integer \( \ell \) such that
\[ \text{Prob}_\varphi \left( \hat{A}_0 \xrightarrow{\ell} \hat{H}_\varphi \right) < 1 - \epsilon \]

**Step 2.** Initialize running dual generator matrix
\[ \mathcal{H} = [ \ ] \]

**Step 3.** Query the probability distribution \( \text{Prob}_\varphi \) \( \ell \) times to obtain \( \ell \) characters (not necessarily distinct) of the hidden quotient group \( H_\varphi \), while incrementing the running dual generator matrix \( \mathcal{H} \).
Loop $i$ From 0 To $\ell - 1$ Do

$$\chi\left(\frac{y_0}{m_0}, \frac{y_1}{m_1}, \ldots, \frac{y_{n-1}}{m_{n-1}}\right) = \text{QRAND}\varphi()$$

\[ \mathcal{H}_i = \begin{bmatrix} \frac{y_0}{m_0} & \frac{y_1}{m_1} & \ldots & \frac{y_{n-1}}{m_{n-1}} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{y_{n-1}}{m_{n-1}} & & \frac{y_0}{m_0} & \ldots \end{bmatrix} \]

Loop Lower Boundary;

**Step 4.** Compute the generator matrix $\mathcal{G}$ from the dual generator matrix $\mathcal{H}$ by using Gaussian elimination to solve the system of equations

$$\sum_{j=0}^{n-1} \frac{y_{ij}}{m_j} x_j \equiv 0 \mod 1 \quad 0 \leq i < N_0$$

for unknown $x_j \mod m_j$.

**Step 5.** Output $\mathcal{G}$ and Stop.

Part 5. Vintage Shor Algorithms

11. Vintage Shor quantum hidden subgroup algorithms (QHSAs)

Let $\varphi : A \rightarrow S$ be a map with hidden subgroup structure. We now consider QHSPs for which the ambient group $A$ is free abelian of finite rank $n$.

Since the ambient group $A$ is infinite, at least two difficulties naturally arise. One is that the associated complex vector space $\mathcal{H}_A$ is now infinite dimensional, thereby causing some implementation problems. The other is that the Fourier transform of a periodic function on $A$ does not exist as a function, but as a generalized function!

Following Shor’s lead, we side-step these annoying obstacles by choosing not to work with the ambient group $A$ and the map $\varphi$ at all. Instead, we work with a group $\tilde{A}$ and a map $\tilde{\varphi} : \tilde{A} \rightarrow S$ which are “approximations” of $A$ and $\varphi : A \rightarrow S$, respectively.

The group $\tilde{A}$ and the approximating map $\tilde{\varphi}$ are constructed as follows:

---

As a clarifying note, let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be a period $P$ function on $\mathbb{Z}$. Then $f$ on $\mathbb{Z}$ is neither of compact support, nor of bounded $L^2$ or $L^1$ norm. So the Fourier transform of $f$ does not exist as a function, but as a generalized function, i.e., as a distribution. However, the function $f$ does induce a function $\tilde{f} : \mathbb{Z}_P \rightarrow \mathbb{C}$ which does have a Fourier transform on $\mathbb{Z}_P$ which exists as a function. The problem is that we do not know the period of $\varphi$, and as a consequence, cannot do Fourier analysis on the corresponding unknown finite cyclic group.
Choose an epimorphism
\[ \mu : A \rightarrow \tilde{A} \]
of the ambient group \( A \) onto a chosen finite group \( \tilde{A} \), called a **group probe**. Next, select a **transversal**
\[ \iota_{\mu} : \tilde{A} \rightarrow A \]
of \( \mu \), i.e., a map such that
\[ \mu \circ \iota_{\mu} = \text{id}_{\tilde{A}} \]
where \( \text{id}_{\tilde{A}} \) denotes the identity map on the group probe \( \tilde{A} \). [Consequently, \( \iota_{\mu} \) is an injection, and in most cases not a morphism at all.]

Having chosen \( \mu \) and \( \iota_{\mu} \), the **approximating map** \( \tilde{\varphi} \) is defined as
\[ \tilde{\varphi} = \varphi \circ \iota_{\mu} : \tilde{A} \rightarrow S \]

Although the map \( \tilde{\varphi} \) is not usually a morphism, the quantum subroutine \( \text{QRand}_{\tilde{\varphi}}() \) is still a well defined quantum procedure which produces a well defined probability distribution \( \text{Prob}_{\tilde{\varphi}}(\chi) \) on the character group \( \hat{\tilde{A}} \) of the group probe \( \tilde{A} \). As we shall see, if the the map \( \tilde{\varphi} \) is a “reasonably good approximation” to the original map \( \varphi \), then \( \text{QRand}_{\tilde{\varphi}}() \) will with high probability produce characters \( \chi \) of the probe group \( \tilde{A} \) which are “sufficiently close” to corresponding characters \( \eta \) of the hidden quotient group \( H_{\varphi} \).

Following this basic strategy, we will now use the quantum subroutine \( \text{QRand}_{\tilde{\varphi}}() \) to build three classes of vintage Shor QHSAs, where the probe group \( \tilde{A} \) is a finite cyclic group \( \mathbb{Z}_Q \) of order \( Q \). In this way, we will create three classes of quantum algorithms which form natural extensions of Shor’s original quantum factoring algorithm.

### 12. Direct summand structure

We digress momentarily to discuss the direct sum structure of the ambient group \( A \) when it is free abelian of finite rank \( n \).

Since the ambient group \( A \) is free abelian of finite rank \( n \), the hidden subgroup \( K_{\varphi} \) is also free abelian of finite rank. Moreover, there exist compatible direct sum decompositions of \( A \) and \( K_{\varphi} \) into free cyclic groups
\[
\begin{align*}
K_{\varphi} & = P_1\mathbb{Z} \oplus \cdots \oplus P_n\mathbb{Z} \\
A & = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} 
\end{align*}
\]
where \( P_1, ..., P_n \) are non-negative integers, and where the inclusion morphism
\[ K_{\varphi} = P_1\mathbb{Z} \oplus \cdots \oplus P_n\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} = A \]
is the direct sum of the inclusion morphisms

\[ P_j \mathbb{Z} \hookrightarrow \mathbb{Z} \]

It should be mentioned that, since the group \( K_\varphi \) is hidden, the above direct sum decompositions are also hidden. Moreover, the selection of a direct sum decomposition of the ambient group \( A \) is operationally equivalent to a selection of a basis of \( A \). This leads to the following definition:

**Definition 6.** A basis \( \{a_1, a_2, \ldots, a_n\} \) of the ambient group \( A \) corresponding to the above hidden direct sum decomposition of \( A \) is called a **hidden basis** of \( A \).

**Question.** How is the hidden basis \( \{a_1, a_2, \ldots, a_n\} \) of \( A \) related to any “visible” basis \( \{a'_1, a'_2, \ldots, a'_n\} \) of \( A \) that we might choose to work with?

The group of automorphisms of the free abelian group \( A \) of rank \( n \) is isomorphic to the group

\[ SL_\pm(n, \mathbb{Z}) \]

of \( n \times n \) invertible integer matrices. This is the same as the group of \( n \times n \) integer matrices of determinant \( \pm 1 \).

**Proposition 5.** Let \( \{a_1, a_2, \ldots, a_n\} \) be a hidden basis of \( A \), and let \( \{a'_1, a'_2, \ldots, a'_n\} \) be any other basis of \( A \). Then there exists a unique element \( M \in SL_\pm(n, \mathbb{Z}) \) which carries the basis \( \{a'_1, a'_2, \ldots, a'_n\} \) into the hidden basis \( \{a_1, a_2, \ldots, a_n\} \).

Since the image of \( \varphi \) is finite, we know that \( P_j > 0 \), for all \( j \). Thus, the direct sum decomposition of the inclusion morphism becomes

\[
\begin{array}{ccc}
\text{K}_\varphi & \text{A} \\
(P_1 \mathbb{Z} \oplus \cdots \oplus P_n \mathbb{Z}) \oplus (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) & \hookrightarrow & (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \\
\text{ n} & \text{ n} & \\
\end{array}
\]

As a consequence, the hidden quotient group \( H_\varphi \) is the corresponding direct sum of finite cyclic groups

\[ H_\varphi = (\mathbb{Z}p_1 \oplus \cdots \oplus \mathbb{Z}p_\pi) \oplus (0 \oplus \cdots \oplus 0), \]

and the hidden epimorphism

\[ \nu : (P_1 \mathbb{Z} \oplus \cdots \oplus P_n \mathbb{Z}) \oplus (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \rightarrow \mathbb{Z}p_1 \oplus \cdots \oplus \mathbb{Z}p_\pi \]
is the direct sum of the epimorphisms
\[
\begin{align*}
\mathbb{Z} &\to \mathbb{Z}_{p_j} \\
\mathbb{Z} &\to 0
\end{align*}
\]

As a consequence of the above, we have:

**Definition 7.** Let \( \{a_1, a_2, \ldots, a_n\} \) be a hidden basis of \( A \). Then a corresponding **induced hidden basis** of the hidden quotient group \( H_\varphi \) is defined as
\[
\{b_1 = \nu (a_1), b_2 = \nu (a_2), \ldots, b_n = \nu (a_n)\},
\]
where \( \nu : A \to H_\varphi \) denotes the hidden epimorphism.\(^{13}\)

The above direct sum decompositions are summarized in the following diagram:

\[
\begin{array}{c}
\bigoplus_{j=1}^{n} \mathbb{Z}_{P_j} \\
\bigoplus_{j=1}^{n} \mathbb{Z}
\end{array}
\oplus
\begin{array}{c}
\bigoplus_{j=1}^{n} \mathbb{Z}
\end{array}
\to
\begin{array}{c}
\bigoplus_{j=1}^{n} \mathbb{Z}
\end{array}
\oplus
\begin{array}{c}
\bigoplus_{j=1}^{n} \mathbb{Z}_{P_j}
\end{array}
\to
\begin{array}{c}
S
\end{array}
\]

**Definition 8.** Let \( H \) be a finite abelian group. Then a **maximal cyclic subgroup** of \( H \) is a cyclic subgroup of \( H \) of highest possible order.

**Proposition 6.** Let \( b_1, b_2, \ldots, b_n \) be the above defined induced hidden basis of the hidden quotient group \( H_\varphi = \mathbb{Z}_{P_1} \oplus \mathbb{Z}_{P_2} \oplus \cdots \oplus \mathbb{Z}_{P_n} \). Then a maximal cyclic subgroup of \( H_\varphi \) is generated by
\[
b_1 \oplus b_2 \oplus \cdots \oplus b_n,
\]
and is isomorphic to the finite cyclic group \( \mathbb{Z}_P \) of order
\[
P = \text{lcm} (P_1, P_2, \ldots, P_n).
\]

\(^{13}\)Please note that the hidden basis \( \{a_1, a_2, \ldots, a_n\} \) of \( A \) is free in the abelian category. However, the induced basis \( \{b_1, b_2, \ldots, b_n\} \) of \( H_\varphi \) is not because \( H_\varphi \) is a torsion group. \( \{b_1, b_2, \ldots, b_n\} \) is a basis in the sense that it is a set of generators of \( H_\varphi \) such that
\[
b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n} = 1
\]
implies that
\[
b_j^{k_j} = 1
\]
for every \( j \). (For more information, please refer to [20].)
13. Vintage Shor QHSAs with group probe $\tilde{A} = \mathbb{Z}_Q$.

Choose a positive integer $Q$ and an epimorphism
$$\mu : A \rightarrow \mathbb{Z}_Q$$
of the free abelian group $A$ onto the finite cyclic group $\tilde{A} = \mathbb{Z}_Q$ of order $Q$.

Next we wish to select a transversal $\iota_\mu$ of the epimorphism $\mu$.

However, at this juncture we must take care. For, not every choice of the transversal $\iota_\mu$ will produce an efficient vintage Shor algorithm. In fact, most choices probably will produce highly inefficient algorithms\(^{14}\). We emphasize that the efficiency of the class of algorithms we are about to define depends heavily on the choice of the transversal $\iota_\mu$.

Following Shor’s lead once again, we select a very special transversal $\iota_\mu$.

**Definition 9.** Let $\mu : A \rightarrow \mathbb{Z}_Q$ be an epimorphism from a free abelian group $A$ of finite rank $n$ onto a finite cyclic group $\mathbb{Z}_Q$ of order $Q$, and let $\tilde{a}$ be a chosen generator of the cyclic group $\mathbb{Z}_Q$.

A transversal $\iota_\mu : \mathbb{Z}_Q \rightarrow A$ is said to be a Shor transversal provided

1) $\iota_\mu (k\tilde{a}) \mapsto k\iota_\mu (\tilde{a})$, for all $0 \leq k < Q$, and

2) There exists a basis $\{a_1', a_2', \ldots, a_n'\}$ of $A$ such that, when $\iota_\mu (\tilde{a})$ is expressed in this basis, i.e., when
$$\iota_\mu (\tilde{a}) = \sum_{j=1}^{n} \lambda_j' a_j' ,$$
it follows that
$$\gcd (\lambda_1', \lambda_2', \ldots, \lambda_n') = 1$$

**Proposition 7.** Let $\lambda_1', \lambda_2', \ldots, \lambda_n'$ be $n$ integers, and let $M$ be a non-singular $n \times n$ integral matrix, i.e., an element of $SL_\pm(n, \mathbb{Z})$. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are $n$ integers defined by
$$(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\lambda_1', \lambda_2', \ldots, \lambda_n') M ,$$
then
$$\gcd (\lambda_1, \lambda_2, \ldots, \lambda_n) = \gcd (\lambda_1', \lambda_2', \ldots, \lambda_n')$$

As a corollary, we have

\(^{14}\)For example, consider $A = \mathbb{Z}$, $P = 6$, $Q = 64$, and the transversal defined by $\iota_\mu : 6n+k \mapsto 6n+k+64 \lfloor k/2 \rfloor$ for $0 \leq n \leq 10$, where $\left\{ \begin{array}{ll} 0 \leq k < 6 & \text{if } 0 \leq n < 10 \\ 0 \leq k < 4 & \text{if } n = 10 \end{array} \right.$ One reason this is a poor choice of transversal is that the image of $\iota_\mu$ does not contain a representative of every coset of the hidden subgroup $\mathbb{Z}_P$ of the ambient group $A$. 
Proposition 8. If condition 2) is true with respect to one basis, then it is true with respect to every basis.

An another immediate consequence of the definition of a Shor traversal, we have the following lemma:

Lemma 1. If a Shor transversal

\[ \iota_\mu : \mathbb{Z}_Q \rightarrow A, \]

is used to construct the the approximating map

\[ \tilde{\varphi} = \varphi \circ \iota_\mu : \mathbb{Z}_Q \rightarrow S, \]

then the approximating map \( \tilde{\varphi} \) has the following property

\[ \tilde{\varphi}(k\tilde{a}) = [\tilde{\varphi}(\tilde{a})]^k, \]

for all \( 0 \leq k < Q \), where we have used the hidden injection \( \iota_\varphi : H_\varphi \rightarrow S \) to identify the elements \( \tilde{\varphi}(k\tilde{a}) \) of the set \( S \) with corresponding elements of the hidden quotient group \( H_\varphi \).

14. Finding Shor transversals for vintage \( \mathbb{Z}_Q \) Shor algorithms

Surprisingly enough, it is algorithmically simpler to find a Shor transversal \( \iota_\mu : \mathbb{Z}_Q \rightarrow A \) first, and then, as an after thought, to construct a corresponding epimorphism \( \mu : A \rightarrow \mathbb{Z}_Q \).

Definition 10. Let \( A \) be an ambient group, and let \( \mathbb{Z}_Q \) be a finite cyclic group of order \( Q \) with a selected generator \( \tilde{a} \). Then an injection

\[ \iota : \mathbb{Z}_Q \rightarrow A \]

is called a Shor injection provided

1) \( \iota(k\tilde{a}) = k\iota(\tilde{a}) \), for all \( 0 \leq k < Q \), and

2) There exists a basis \( \{ a'_1, a'_2, \ldots, a'_n \} \) of the ambient group \( A \) such that

\[ \gcd(\lambda'_1, \lambda'_2, \ldots, \lambda'_n) = 1, \]

where

\[ \iota(\tilde{a}) = \sum_{j=1}^{n} \lambda'_j a'_j. \]

Proposition 9. If condition 2) is true with respect to one basis, it is true with respect to all.

Next, we need to construct an epimorphism \( \mu : A \rightarrow \mathbb{Z}_Q \) for which \( \iota : \mathbb{Z}_Q \rightarrow A \) is a Shor transversal.
Proposition 10. Let $A$ be an ambient group, and let $\mathbb{Z}_Q$ be a finite cyclic group of order $Q$ with a selected generator $\tilde{a}$. Given a Shor injection
\[ \iota : \mathbb{Z}_Q \to A, \]
there exists an epimorphism
\[ \mu : A \to \mathbb{Z}_Q \]
such that $\iota$ is a Shor transversal for $\mu$, i.e., such that
\[ \mu \circ \iota = \text{id}_{\mathbb{Z}_Q}, \]
where $\text{id}_{\mathbb{Z}_Q}$ denotes the identity morphism on $\mathbb{Z}_Q$.

Proof. Select an arbitrary basis $\{a'_1, a'_2, \ldots, a'_n\}$ of $A$. Then
\[ \iota(\tilde{a}) = \sum_{j=1}^{n} \lambda'_j a'_j, \]
where
\[ \gcd (\lambda'_1, \lambda'_2, \ldots, \lambda'_n) = 1. \]
Hence, from the extended Euclidean algorithm, we can find integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ for which
\[ \sum_{j=1}^{n} \alpha_j \lambda'_j = 1. \]

Define
\[ \mu : \{a'_1, a'_2, \ldots, a'_n\} \to \mathbb{Z}_Q \]
by
\[ \mu (a'_j) = \alpha_j \tilde{a}, \quad j = 1, 2, \ldots, n. \]

Since $a'_1, a'_2, \ldots, a'_n$ is a free abelian basis of the ambient group $A$, it uniquely extends to a morphism
\[ \mu : A \to \mathbb{Z}_Q. \]
It immediately follows that $\mu$ is an epimorphism because
\[ \mu \left( \sum_{j=1}^{n} \lambda'_j a'_j \right) = \sum_{j=1}^{n} \alpha_j \lambda'_j \tilde{a} = \tilde{a}. \]

Thus the task of finding an epimorphism $\mu : A \to \mathbb{Z}_Q$ and a corresponding Shor transversal reduces to the task of finding $n$ integers $\lambda'_1, \lambda'_2, \ldots, \lambda'_n$ such that
\[ \gcd (\lambda'_1, \lambda'_2, \ldots, \lambda'_n) = 1. \]
This leads of to the following probabilistic subroutine which finds a random Shor traversal:

\begin{verbatim}
RANDOM_SHOR_TRANSVERSAL( \{a'_1, a'_2, \ldots, a'_n\}, Q, \tilde{a}, n )
\end{verbatim}
# INPUT: A basis \( \{ a'_1, a'_2, \ldots, a'_n \} \) of \( A \), a positive integer \( Q \),
# a selected generator \( \tilde{a} \) of \( A \), and the rank \( n \) of \( A \)
# OUTPUT: Shor transversal \( \iota_{\mu} : Z_Q \to A \)
# SIDE EFFECT: Epimorphism \( \mu : A \to Z_Q \)
# SIDE EFFECT: Random integers \( \lambda'_1, \lambda'_2, \ldots, \lambda'_n \)

GLOBAL: \( \mu : A \to Z_Q \)
GLOBAL: \( \lambda'_1, \lambda'_2, \ldots, \lambda'_n \)

\begin{itemize}
  \item \textbf{Step 0} If \( n = 1 \) Then (Set \( \lambda'_1 = 1 \) And Goto \textbf{Step 4})
  \item \textbf{Step 1} Select with replacement \( n \) random \( \lambda'_1, \lambda'_2, \ldots, \lambda'_n \) from \( \{1, 2, \ldots, Q\} \).
  \item \textbf{Step 2} Use the extended Euclidean algorithm to determine
    \[ d = gcd (\lambda'_1, \lambda'_2, \ldots, \lambda'_n) \]
    and integers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that \( \sum_{j=1}^{n} \alpha_j \lambda'_j = d \)
  \item \textbf{Step 3} If \( d \neq 1 \) Then Goto \textbf{Step 1} Else Goto \textbf{Step 4}
  \item \textbf{Step 4} Construct Shor transversal \( \iota_{\mu} : Z_Q \to A \) as \( \iota_{\mu} (k\tilde{a}) = k \sum_{j=1}^{n} \lambda'_j a'_j \), for \( 0 \leq k < Q \)
  \item \textbf{Step 5} Construct epimorphism \( \mu : A \to Z_Q \) as
    \[ \mu (a'_j) = \alpha_j \tilde{a} \] for all \( j = 1, 2, \ldots, n \)
  \item \textbf{Step 6} OUTPUT transversal \( \iota_{\mu} : Z_Q \to A \) and Stop
\end{itemize}

**Theorem 7.** For \( n > 1 \), the average case complexity of the \textsc{Random\_Shor\_transversal} subroutine is
\[
O \left( n \left( \lg Q \right)^3 \right).
\]

**Proof.** The computationally dominant part of this subroutine is the main loop Steps 1 through 3.

Each iteration of the main loop executes the extended Euclidean algorithm \( n \) times to find the gcd \( d \). Since the computational complexity of the extended Euclidean algorithm\(^{15}\) is \( O \left( \left( \lg Q \right)^3 \right) \), it follows that the computational cost of one iteration of steps 1 through 3 is
\[
O \left( n \left( \lg Q \right)^3 \right).
\]

But by Corollary 7 of Appendix B,
\[
\text{Prob}_Q (gcd (\lambda'_1, \lambda'_2, \ldots, \lambda'_n) = 1) = \Omega (1) .
\]

\(^{15}\)See [11, Chap. 31].
Thus, the average number of iterations before a successful exit to Step 4 is $O(1)$.

Hence, the average case complexity of steps 1 through 4 is $O(n \lg Q^3)$.

\[ \square \]

**Remark 8.** Our objective in this paper is to find reasonable asymptotic bounds, not the tightest possible bounds. For example, the above bound is by no means the tightest possible. For a tighter bound for the Euclidean algorithm is $O((\lg Q)^2)$ which can be found in [11]. Thus, the bound found in the above theorem can be tightened to at least $O(n \lg Q^2)$.

15. Maximal Shor transversals

Unfortunately, the definition of a Shor transversal is in some instances not strong enough to extend Shor’s quantum factoring algorithm to ambient groups which are free abelian groups of finite rank. From necessity, we are forced to make the following definition.

**Definition 11.** Let $a_1, a_2, \ldots, a_n$ be a hidden basis of the ambient group $A$, let $\tilde{a}$ be a chosen generator the cyclic group probe $\mathbb{Z}_Q$, and let $H_\varphi = \mathbb{Z}_{P_1} \oplus \mathbb{Z}_{P_2} \oplus \cdots \oplus \mathbb{Z}_{P_n}$ be the corresponding hidden direct sum decomposition. A **maximal Shor transversal** is a Shor transversal $\iota_\mu : \mathbb{Z}_Q \to A$ such that

$$\gcd(\lambda_j, P_j) = 1, \quad \text{for } 0 \leq j < n,$$

where the integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are defined by

$$\iota_\mu(\tilde{a}) = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n$$

**Remark 9.** Thus, for maximal Shor traversals, $\iota_\mu(\tilde{a})$ maps via the hidden epimorphism $\nu : A \to A/K_\varphi$ to a maximum order element of the hidden quotient group $H_\varphi$.

One of the difficulties of the above definition is that it does not appear to be possible to determine whether or not a Shor transversal is maximal without first knowing the hidden direct sum decomposition of the hidden quotient group $H_\varphi$. We address this important issue in the following corollary, which is an immediate consequence of corollary 8 (found in Appendix B):
Corollary 3. Let
\[ P_1, P_2, \ldots, P_n \]
be \( n \) fixed positive integers, and let \( Q \) be an integer such that
\[ Q \geq \text{lcm}(P_1, P_2, \ldots, P_n), \]
where \( n > 1 \).

If Conjecture 1 (found in Appendix B) is true, then the probability that the subroutine `Random_Shor_transversal` produces a maximal Shor transversal is
\[ \Omega\left(\frac{1}{\prod_{j=1}^{n} \lg \lg P_j}\right) = \Omega\left(\frac{1}{(\lg \lg Q)^n}\right). \]

16. Identifying characters of cyclic groups with points on the unit circle \( S^1 \) in the complex plane \( \mathbb{C} \).

We will now begin to develop an answer to the following question:

**Question. 1.** Are the characters of the group probe \( \mathbb{Z}_Q \) produced by the quantum subroutine `QRand_\phi()` “close enough” to the characters of a maximal cyclic subgroup of the hidden quotient group \( H_\phi \)?

If `QRand_\phi()` produces a character \( \chi \) of \( \mathbb{Z}_Q \) which is “close enough” to some character \( \eta \) of a maximal cyclic subgroup \( \mathbb{Z}_P \) of the hidden quotient group \( H_\phi \), then the character \( \chi \) can be used to find the corresponding closest character \( \eta \) of \( \mathbb{Z}_P \). Each time such a character \( \eta \) is found, something more is known about the hidden quotient group \( H_\phi \) and the hidden subgroup \( K_\phi \). In this way, we have the conceptual genesis of a class of vintage \( \mathbb{Z}_Q \) Shor algorithms.

But before we can answer the above question, we need to answer a more fundamental question, namely:

**Question. 2.** What do we mean by “close enough”? I.e., what do we mean by saying that a character \( \chi \) of \( \hat{\mathbb{A}} = \mathbb{Z}_Q \) is “close enough” to some character \( \eta \) of \( \mathbb{Z}_P \)?

To answer this last question, we need to introduce two additional concepts:

1) The concept of a common domain for the characters \( \chi \) of \( \mathbb{Z}_Q \) and the characters \( \eta \) of \( \mathbb{Z}_P \).
2) The concept of a group norm which is to be used to define when two characters are “close.”

In this section, we address item 1). In the next, item 2).

We begin by noting that the character group \( \hat{\mathbb{Z}} \) of the infinite cyclic group \( \mathbb{Z} \) is simply the group \( S^1 \), i.e.,
\[ \hat{\mathbb{Z}} = S^1 = \{ \chi_{\theta} : n \mapsto e^{2\pi i \theta n} \mid 0 \leq \theta < 1 \} \]
In other words, the characters of \( \mathbb{Z} \) can be identified with the points on the unit radius circle in the complex plane \( \mathbb{C} \).
Moreover, given an arbitrary epimorphism
\[ \tau : \mathbb{Z} \rightarrow \mathbb{Z}_m \]
of the infinite cyclic group \( \mathbb{Z} \) onto a finite cyclic group \( \mathbb{Z}_m \), the left exact contravariant functor\(^{16}\)

\[ \text{Hom}_\mathbb{Z}(-, 2\pi \mathbb{R}/2\pi \mathbb{Z}) \]
transforms \( \tau \) into the monomorphism
\[ \hat{\tau} : \hat{\mathbb{Z}}_m \rightarrow \hat{\mathbb{Z}} \]
\[ \eta \mapsto \eta \circ \tau \]
In this way the characters of \( \mathbb{Z}_m \) can be identified with the points of \( \hat{\mathbb{Z}} = S^1 \).

Thus, to find a common domain \( S^1 \) for the characters of the group probe \( \mathbb{Z}_Q \) and the maximal cyclic group \( \mathbb{Z}_P \), all that need be done is to find epimorphisms \( \hat{\mu} : \mathbb{Z} \rightarrow \mathbb{Z}_Q \) and \( \hat{\tau} : \mathbb{Z} \rightarrow \mathbb{Z}_P \). This is accomplished as follows:

Let \( a \) be a generator of the infinite cyclic group \( \mathbb{Z} \), and let \( a_1, a_2, \ldots, a_n \) be a hidden basis of the ambient group \( A \). Then the epimorphisms \( \hat{\mu} \) and \( \hat{\tau} \) are defined as
\[ \hat{\mu} : \mathbb{Z} \rightarrow \mathbb{Z}_Q \]
\[ ka \mapsto k\tilde{a} \]
and
\[ \hat{\tau} : \mathbb{Z} \rightarrow \mathbb{Z}_P \]
\[ ka \mapsto \nu [k (a_1 + a_2 + \ldots + a_n)] \]
where \( \tilde{a} \) is the selected generator of the group probe \( \mathbb{Z}_Q \), and where \( \nu : A \rightarrow H_\varphi \) is the hidden epimorphism.

Thus, as a partial answer to Question 2 of Section 16, a character \( \chi \) of the probe group \( \hat{A} = \mathbb{Z}_Q \) is “close” to some character \( \eta \) of the maximal cyclic subgroup \( \mathbb{Z}_P \), if the corresponding points \( \hat{\mu}(\chi) \) and \( \hat{\tau}(\eta) \) on the circle \( S^1 \) are “close.”

But precisely what do we mean by two points of \( S^1 \) being “close” to one another? To answer this question, we need to observe that Shor’s algorithm uses, in addition to the group structure of \( S^1 \), also the metric structure of \( S^1 \).

17. Group norms

We proceed to define a metric structure on the circle group \( S^1 \). To do so, we need to define what is meant by a group norm.

**Definition 12.** A **(group theoretic) norm** on a group \( G \) is a map

\[ ||-|| : G \rightarrow \mathbb{R} \]
such that

1) \( ||x|| \geq 0 \), for all \( x \), and \( ||x|| = 0 \) if and only if \( x \) is the group identity (which is 1 if we think of \( G \) as a multiplicative group, or 0 if we think of \( G \) as an additive group).

\(^{16}\)For the definition of a left exact contravariant functor, please refer to, for example, \([7]\).
2) \[ ||x \cdot y|| \leq ||x|| + ||y|| \text{ or } ||x + y|| \leq ||x|| + ||y||, \]
depending respectively on whether we think of $G$ as a multiplicative or as an additive group.

**Caveat.** The group norms defined in this section are different from the group algebra norms defined in Section 5.

**Remark 10.** Such a norm induces a metric

\[
G \times G \rightarrow \mathbb{R}
\]

\[
(x, y) \mapsto ||x \cdot y^{-1}|| \text{ or } ||x - y||
\]
depending on whether multiplicative or additive notation is used.

As mentioned in Section 4, we think of the 1-sphere $S^1$ interchangeably as the multiplicative group

\[
S^1 = \{ e^{2\pi i \alpha} \mid 0 \leq \alpha < 1 \} \subset \mathbb{C}
\]

with multiplication defined as

\[
e^{2\pi i \alpha} \cdot e^{2\pi i \beta} = e^{2\pi i (\alpha + \beta)}
\]
or as the additive group of reals $\mathbb{R}$ modulo $2\pi$, i.e., as

\[
S^1 = 2\pi \mathbb{R}/2\pi \mathbb{Z} = \{ 2\pi \alpha \mid 0 \leq \alpha < 1 \}
\]

with addition defined as

\[
2\pi \alpha + 2\pi \beta = (2\pi \alpha + 2\pi \beta) \text{ mod } 2\pi = 2\pi (\alpha + \beta \text{ mod } 1)
\]

It should be clear from context which of the two representation of the group $S^1$ is being used.

---

**Figure 2.** Two metrics on the unit circle $S^1$, $\text{ARC}_{2\pi}$ and $\text{CHORD}_{2\pi}$. 
There are two different norms on the 1-sphere $S^1$ that we will be of use to us. The first is the **arclength norm**, written $\text{Arc}_{2\pi}$, defined by

$$\text{Arc}_{2\pi}(\alpha) = 2\pi \min \{ |\alpha| - |\lfloor |\alpha| \rfloor|, |\lceil |\alpha| \rceil| - |\alpha| \} ,$$

which is simply the length of the shortest arc in the 1-sphere $S^1$ connecting the point $e^{2\pi i \alpha}$ to the point 1.

The second norm is the **chordal length norm**, written $\text{Chord}_{2\pi}$, defined by

$$\text{Chord}_{2\pi}(\alpha) = 2 |\sin (\pi \alpha)| ,$$

which is simply the length of the chord in the complex plane connecting the point $e^{2\pi i \alpha}$ to the point 1.

Shor's algorithm depends heavily on the interrelationship of these two norms. We summarize these interrelationships in the following proposition:

**Proposition 11.** The the norms $\text{Arc}_{2\pi}$ and $\text{Chord}_{2\pi}$ satisfy the following conditions:

1) $\text{Chord}_{2\pi}(\alpha) = 2 \sin \left( \frac{1}{2} \text{Arc}_{2\pi}(\alpha) \right)$

2) $\frac{2}{\pi} \text{Arc}_{2\pi}(\alpha) \leq \text{Chord}_{2\pi}(\alpha) \leq \text{Arc}_{2\pi}(\alpha)$

We need the following property of the arclength norm $\text{Arc}_{2\pi}$:

**Proposition 12.** Let $n$ be a nonzero integer. If $\text{Arc}_{2\pi}(\alpha) \leq \frac{\pi}{|n|}$, then $\text{Arc}_{2\pi}(n \alpha) = |n| \text{Arc}_{2\pi}(\alpha)$

18. Vintage $\mathbb{Z}_Q$ Shor QHSAs (Cont.)

Our next step is to look more closely at the probability distribution

$$\text{Prob}_{\tilde{\gamma}} : \tilde{A} \rightarrow [0, 1] .$$

We seek first to use this probability distribution to determine the maximal cyclic subgroup $\mathbb{Z}_P$ of the hidden quotient group $H_\phi$. However, as indicated by the following lemma, there are a number of obstacles to finding the subgroup $\mathbb{Z}_P$. 

Lemma 2. Let \( a_1, \ldots, a_n \) denote a hidden basis of the ambient group \( A = \bigoplus_{j=1}^{n} \mathbb{Z} \), and let \( P_1, P_2, \ldots, P_n \) denote the respective orders of the corresponding cyclic direct summands of the hidden quotient group \( H_{\varphi} = \bigoplus_{j=1}^{n} \mathbb{Z}_{P_j} \).

Let \( \tilde{a} \) denote a chosen generator of the group probe \( \tilde{A} = \mathbb{Z}_{Q} \), and let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) denote the unknown integers such that

\[
\iota_\mu(\tilde{a}) = \sum_{j=1}^{n} \lambda_j a_j \in A.
\]

Finally, use the hidden injection \( \iota_\varphi : H_{\varphi} \rightarrow S \) to identify the elements of the hidden quotient group \( H_{\varphi} \) with the corresponding elements of the set \( S \).

If the approximating map \( \tilde{\varphi} \) is constructed from a Shor transversal, then the order of \( \tilde{\varphi}(\tilde{a}) \in H_{\varphi} \) is \( P \), i.e.,

\[
\text{order} (\tilde{\varphi}(\tilde{a})) = P,
\]

where \( P = \text{lcm}(P_1, P_2, \ldots, P_n) \), and where \( P_j = P_j/\gcd(\lambda_j, P_j) \) for \( j = 1, 2, \ldots, n \).

Hence,

\[
\left\{ \tilde{\varphi}(k\tilde{a}) = \tilde{\varphi}(\tilde{a})^k \mid 0 \leq k < P \right\}
\]

are all distinct elements of \( S \).

Moreover, if the approximating map \( \tilde{\varphi} \) is constructed from a maximal Shor transversal, \( P = \text{lcm}(P_1, P_2, \ldots, P_n) \).

Proof.

\[
\tilde{\varphi}(\tilde{a}) = \varphi \circ \iota_\mu(\tilde{a}) = \varphi \left( \sum_j \lambda_j a_j \right) = \prod_j \varphi(a_j)^{\lambda_j} = \prod_j b_j^{\lambda_j},
\]

where we have used the hidden injection \( \iota_\varphi : H_{\varphi} \rightarrow S \) to identify the hidden basis element \( b_j \) of \( H_{\varphi} \) with the element \( \varphi(a_j) \) of the set \( S \).

Since the order of each \( b_j \) is \( P_j \), it follows from elementary group theory that the order of \( \prod_j b_j^{\lambda_j} \) must be \( P \). \( \square \)

Lemma 3. Let \( a_1, \ldots, a_n \) be a hidden basis of the ambient group \( A = \bigoplus_{j=1}^{n} \mathbb{Z} \), and let \( P_1, P_2, \ldots, P_n \) denote the respective orders of the corresponding cyclic direct summands of the hidden quotient group \( H_{\varphi} = \bigoplus_{j=1}^{n} \mathbb{Z}_{P_j} \).

Let \( \tilde{a} \) denote a chosen generator of the group probe \( \tilde{A} = \mathbb{Z}_{Q} \), let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) denote the unknown integers such that

\[
\iota_\mu(\tilde{a}) = \sum_{j=1}^{n} \lambda_j a_j \in A,
\]

and let \( \chi_{\tilde{\varphi}} \) be a character of \( \mathbb{Z}_{Q} \).

Finally, identify the elements of the hidden quotient group \( H_{\varphi} \) with the corresponding elements of the set \( S \) via the hidden injection \( \iota_\varphi : H_{\varphi} \rightarrow S \).

If the approximating map \( \tilde{\varphi} \) is constructed from a Shor transversal, then
When \( Py \neq 0 \mod Q \), we have

\[
\tilde{\phi} \left( \chi \frac{y}{Q} \right) = \pm e^{i\pi \frac{Py}{Q}} \sum_{k_0=0}^{r-1} \chi \frac{y}{Q} (k_0\bar{a}) \tilde{\phi} (k_0\bar{a})
\]

where \( P = \text{lcm} (P_1, P_2, \ldots, P_n) \), \( P_j = P_j / \gcd (\lambda_j, P_j) \) for \( j = 1, 2, \ldots, n \), and where

\[
Q = qP + r, \text { with } 0 \leq r < P.
\]

And when \( Py = 0 \mod Q \), we have

\[
\tilde{\phi} \left( \chi \frac{y}{Q} \right) = (q + 1) \sum_{k_0=0}^{r-1} \chi \frac{y}{Q} (k_0\bar{a}) \tilde{\phi} (k_0\bar{a}) + q \sum_{k_0=r}^{P-1} \chi \frac{y}{Q} (k_0\bar{a}) \tilde{\phi} (k_0\bar{a})
\]

Moreover, if the approximating map \( \tilde{\phi} \) is constructed from a maximal Shor transversal, then \( P = P = \text{lcm} (P_1, P_2, \ldots, P_n) \).

**Proof.** We begin by identifying the elements of the hidden quotient group \( H_\phi \) with the corresponding elements of the set \( S \) via injection \( \iota_\phi : H_\phi \rightarrow S \).

We first consider the case when \( Py \neq 0 \mod Q \).

Then

\[
\tilde{\phi} \left( \chi \frac{y}{Q} \right) = \tilde{\phi} \left( \sum_{k=0}^{Q-1} \chi \frac{y}{Q} (k\bar{a}) k\bar{a} \right) = \sum_{k=0}^{Q-1} \chi \frac{y}{Q} (k\bar{a}) \tilde{\phi} (k\bar{a})
\]

\[
= \sum_{k=0}^{qP-1} \chi \frac{y}{Q} (k\bar{a}) \tilde{\phi} (\bar{a})^k + \sum_{k=qP}^{Q-1} \chi \frac{y}{Q} (k\bar{a}) \tilde{\phi} (\bar{a})^k
\]

\[
= \sum_{k_1=0}^{q-1} \sum_{k_0=0}^{P-1} \chi \frac{y}{Q} \left( (k_1P + k_0) \bar{a} \right) \tilde{\phi} (\bar{a})^{k_1P + k_0} + \sum_{n_0=0}^{r-1} \chi \frac{y}{Q} (k_0) \tilde{\phi} (\bar{a})^{k_0}
\]

From Lemma 2 we have

\[
\tilde{\phi} (\bar{a})^{k_1P + k_0} = \tilde{\phi} (\bar{a})^{k_0}.
\]
So,

\[ \tilde{\varphi}(\chi \cdot \bar{a}) = \left( \sum_{k_1=0}^{q-1} \chi \left[ (k_1\bar{P}) \bar{a} \right] \right) \sum_{k_0=0}^{r-1} \chi \left( k_0 \bar{a} \right) \tilde{\varphi}(\bar{a})^{k_0} \]

\[ + \chi \left( q\bar{P} \bar{a} \right) \sum_{k_0=0}^{r-1} \chi \left( k_0 \bar{a} \right) \tilde{\varphi}(\bar{a})^{k_0} \]

\[ = \left( \sum_{k_1=0}^{q} \chi \left[ (k_1\bar{P}) \bar{a} \right] \right) \sum_{k_0=0}^{r-1} \chi \left( k_0 \bar{a} \right) \tilde{\varphi}(\bar{a})^{k_0} \]

\[ + \left( \sum_{k_1=0}^{q-1} \chi \left[ (k_1\bar{P}) \bar{a} \right] \right) \sum_{k_0=r}^{r-1} \chi \left( k_0 \bar{a} \right) \tilde{\varphi}(\bar{a})^{k_0} \]

\[ = \left( \frac{e^{2\pi i \frac{Py}{Q}(q+1)} - 1}{e^{2\pi i \frac{Py}{Q}} - 1} \right) \sum_{k_0=0}^{r-1} \chi \left( k_0 \bar{a} \right) \tilde{\varphi}(k_0\bar{a}) \]

\[ + \left( \frac{e^{2\pi i \frac{Py}{Q}q} - 1}{e^{2\pi i \frac{Py}{Q}} - 1} \right) \sum_{k_0=r}^{r-1} \chi \left( k_0 \bar{a} \right) \tilde{\varphi}(k_0\bar{a}) \]

\[ = e^{\pi \frac{Py}{Q}q} \left( \frac{e^{\pi \frac{Py}{Q}q} - e^{-\pi \frac{Py}{Q}q}}{e^{\pi \frac{Py}{Q}q} - e^{-\pi \frac{Py}{Q}q}} \right) \sum_{k_0=0}^{r-1} \chi \left( k_0 \bar{a} \right) \tilde{\varphi}(k_0\bar{a}) \]

\[ + e^{\pi \frac{Py}{Q}q} \left( \frac{e^{-\pi \frac{Py}{Q}q} - e^{\pi \frac{Py}{Q}q}}{e^{\pi \frac{Py}{Q}q} - e^{-\pi \frac{Py}{Q}q}} \right) \sum_{k_0=r}^{r-1} \chi \left( k_0 \bar{a} \right) \tilde{\varphi}(k_0\bar{a}) \]

\[ = e^{i\frac{\pi}{2} \frac{Py}{Q}q} \left( \frac{\sin \left( \pi \frac{Py}{Q}(q+1) \right)}{\sin \left( \pi \frac{Py}{Q} \right)} \right) \sum_{k_0=0}^{r-1} \chi \left( k_0 \bar{a} \right) \tilde{\varphi}(k_0\bar{a}) \]

\[ + e^{i\frac{\pi}{2} \frac{Py}{Q}q} \left( \frac{\sin \left( \pi \frac{Py}{Q}q \right)}{\sin \left( \pi \frac{Py}{Q} \right)} \right) \sum_{k_0=r}^{r-1} \chi \left( k_0 \bar{a} \right) \tilde{\varphi}(k_0\bar{a}) \]

For the exceptional case when \( Py = 0 \mod Q \), we need only observe that

\[ \sum_{k_1=0}^{q} \chi \left[ (k_1\bar{P}) \bar{a} \right] = q + 1 \] and \( \sum_{k_1=0}^{q-1} \chi \left( k_1\bar{P} \right) \bar{a} = q \).

\( \square \)

As an immediate consequence of above lemmas 2 and 3, we have:
Corollary 4. If the approximating map $\tilde{\varphi}$ is constructed from a Shor transversal, then

$$\| \tilde{\varphi} \left( \chi_{\frac{y}{Q}} \right) \|^2 = \begin{cases} \frac{r \text{Chord}^2_{\pi} \left[ \frac{T}{Q} y (q+1) \right] + (P-r) \text{Chord}^2_{\pi} \left[ \frac{T}{Q} y q \right]}{\text{Chord}^2_{\pi} \left[ \frac{T}{Q} y \right]} & \text{if } Ty \neq 0 \mod Q \\ r (q+1)^2 + (P-r) q^2 & \text{if } Ty = 0 \mod Q \end{cases}$$

Moreover, if the approximating map $\tilde{\varphi}$ is constructed from a maximal Shor transversal, then $P = P = \text{lcm} (P_1, P_2, \ldots, P_n)$.

As a consequence of the inequalities found in Proposition 11, we have:

Corollary 5. If the approximating map $\tilde{\varphi}$ is constructed from a Shor transversal, then when $Ty \neq 0 \mod Q$ we have

$$\| \tilde{\varphi} \left( \chi_{\frac{y}{Q}} \right) \|^2 \geq \frac{4}{\pi^2} \left( r \text{Arc}^2_{2\pi} \left[ \frac{T}{Q} y (q+1) \right] + (P-r) \text{Arc}^2_{2\pi} \left[ \frac{T}{Q} y q \right] \right)$$

Moreover, if the approximating map $\tilde{\varphi}$ is constructed from a maximal Shor transversal, then $P = P = \text{lcm} (P_1, P_2, \ldots, P_n)$.

Figure 3. The characters of $\mathbb{Z}_P$ and $\mathbb{Z}_Q$ as points on the circle $S^1$ of radius 1, with $P = 3$ and $Q = 8$. The characters $\chi_1, \chi_{3/8}, \chi_{5/8}$ of $\mathbb{Z}_Q$ are close respectively to characters $\chi_1, \chi_{1/3}, \chi_{2/3}$ of $\mathbb{Z}_P$. They are the characters of $\text{Arc}_{2\pi}$ distance less than $\frac{\pi}{Q} (1 - \frac{P}{Q})$ from some character of $\mathbb{Z}_P$. Also, $\chi_{1/3}$ and $\chi_{2/3}$ are the primitive...
characters of $\mathbb{Z}_P$. Unfortunately, since $Q \not\equiv P^2$, the characters $\chi_{3/8}$ and $\chi_{5/8}$ of $\mathbb{Z}_Q$ are not sufficiently close respectively to the primitive characters $\chi_1$ and $\chi_2$ of $\mathbb{Z}_P$. Hence, the continued fraction algorithm cannot be used to find $P$.

19. When are characters of $\tilde{A} = \mathbb{Z}_Q$ close to some character of a maximal cyclic subgroup $\mathbb{Z}_P$ of $H_\varphi$?

**Definition 13.** Let $\mathbb{Z}_{P_1} \oplus \mathbb{Z}_{P_2} \oplus \cdots \oplus \mathbb{Z}_{P_n}$ be the hidden direct sum decomposition of the hidden quotient group $H_\varphi$, and let $P = \text{lcm}(P_1, P_2, \ldots, P_n)$. A character $\chi_{\frac{y}{Q}}$ of the group probe $\mathbb{Z}_Q$ is said to be close to a character of the maximal cyclic subgroup $\mathbb{Z}_P$ of the hidden quotient group $H_\varphi$ provided either of the following equivalent conditions are satisfied:

**Closeness Condition 1**

There exists an integer $d$ such that

$$\text{Arc}_{2\pi} \left( \frac{y}{Q} - \frac{d}{P} \right) \leq \frac{\pi}{Q} \left( 1 - \frac{P}{Q} \right),$$

or equivalently,

**Closeness Condition 1’**

$$\text{Arc}_{2\pi} \left( \frac{Py}{Q} \right) \leq \frac{\pi P}{Q} \left( 1 - \frac{P}{Q} \right)$$

If in addition, $Q \geq P^2$, then the character $\chi_{\frac{y}{Q}}$ of $\mathbb{Z}_Q$ is said to be sufficiently close to a character of the maximal cyclic subgroup $\mathbb{Z}_P$.

It immediately follows from the theory of continued fractions [21, 33] that

**Proposition 13.** If a character $\chi_{\frac{y}{Q}}$ of $\mathbb{Z}_Q$ is sufficiently close to a character $\chi_{\frac{d}{P}}$ of $\mathbb{Z}_P$, then $\frac{d}{P}$ is a convergent of the continued fraction expansion of $\frac{y}{Q}$.

However, to determine the sought integer $P$ from the rational $\frac{d}{P}$, the numerator and denominator of $\frac{d}{P}$ must be relatively prime, i.e.,

$$\gcd (d, P) = 1.$$

This leads to the following definition:

**Definition 14.** A character $\chi_{\frac{d}{P}}$ of $\mathbb{Z}_P$ is said to be primitive provided that it is a generator of the dual group $\hat{\mathbb{Z}}_P$.

**Proposition 14.** A character $\chi_{\frac{d}{P}}$ of $\mathbb{Z}_P$ is a primitive character if and only $\gcd (d, P) = 1$. Moreover, the number of primitive characters of $\mathbb{Z}_P$ is $\phi(P)$, where $\phi(P)$ denotes Euler’s totient function, i.e., the number of positive integers less than $P$ which are relatively prime to $P$. 

Theorem 8. Assume that $Q \geq P^2$, and that the approximating map $\tilde{\varphi}$ is constructed from a maximal Shor transversal. Then the probability that $\text{QRAND}_{\tilde{\varphi}}()$ produces a character of the group probe $\mathbb{Z}_Q$ which is sufficiently close to a primitive character of the maximal cyclic subgroup $\mathbb{Z}_P$ of the hidden quotient group $H_\varphi$ satisfies the following bound

$$\text{Prob}_{\tilde{\varphi}} \left( \chi \text{ sufficiently close to some primitive character of } \mathbb{Z}_P \right) \geq \frac{4}{\pi^2} \frac{\phi(P)}{P} \left( 1 - \frac{P}{Q} \right)^2$$

Proof. Let $\chi_{\tilde{\varphi}}$ be a particular character of the group probe $\mathbb{Z}_Q$ which is sufficiently close to some character of the maximal cyclic subgroup $\mathbb{Z}_P$. We now compute the probability that $\text{QRAND}_{\tilde{\varphi}}()$ will produce this particular character.

First consider the exceptional case when $Py = 0 \mod Q$. Using the expression for $\| \tilde{\varphi} (\chi_{\tilde{\varphi}}) \|^2$ given in Corollary 5, we have

$$\| \tilde{\varphi} (\chi_{\tilde{\varphi}}) \|^2 = r (q + 1)^2 + (P - r) q^2 \geq Pq^2 = P \left( \frac{Q - r}{P} \right)^2 \geq \frac{1}{P} (Q - P)^2.$$ 

So

$$\text{Prob}_{\tilde{\varphi}} (\chi_{\tilde{\varphi}}) = \frac{\| \tilde{\varphi} (\chi_{\tilde{\varphi}}) \|^2}{Q^2} \geq \frac{1}{P} \frac{P (Q - P)^2}{Q^2} = \frac{1}{P} \left( 1 - \frac{P}{Q} \right)^2 \geq \frac{4}{\pi^2} \frac{1}{P} \left( 1 - \frac{P}{Q} \right)^2.$$ 

Next consider the non-exceptional case when $Py \neq 0 \mod Q$.

In this case, Proposition 12 can be applied to both terms in the numerator of the expression given in Corollary 5. Hence,

$$\| \tilde{\varphi} (\chi_{\tilde{\varphi}}) \|^2 \geq \frac{4}{\pi^2} \frac{r \text{ARC}_{2\pi} \left( \frac{P_0}{Q} y (q + 1) \right) + (P - r) \text{ARC}_{2\pi} \left( \frac{P_0}{Q} y q \right)}{\text{ARC}_{2\pi} \left( \frac{P_0}{Q} y \right)}$$

$$\geq \frac{4}{\pi^2} \frac{r (q + 1)^2 \text{ARC}_{2\pi} \left( \frac{P_0}{Q} y \right) + (P - r) q^2 \text{ARC}_{2\pi} \left( \frac{P_0}{Q} y q \right)}{\text{ARC}_{2\pi} \left( \frac{P_0}{Q} y \right)}$$

$$\geq \frac{4}{\pi^2} \frac{r (q + 1)^2 + \frac{4}{\pi^2} (P - r) q^2 \geq \frac{4}{\pi^2} r q^2 + \frac{4}{\pi^2} (P - r) q^2}{Pq^2 = \frac{4}{\pi^2} \frac{1}{P} (Q - r)^2 \geq \frac{4}{\pi^2} \frac{1}{P} (Q - P)^2}$$

Thus,

$$\text{Prob}_{\tilde{\varphi}} (\chi_{\tilde{\varphi}}) = \frac{\| \tilde{\varphi} (\chi_{\tilde{\varphi}}) \|^2}{Q^2} \geq \frac{4}{\pi^2} \frac{1}{P} \left( 1 - \frac{P}{Q} \right)^2.$$
So, in either case we have

\[
\text{Prob}_{\sim} (\chi \text{ sufficiently close to some primitive character of } \mathbb{Z}_P) \geq \frac{4}{\pi^2} \frac{1}{P} \left(1 - \frac{P}{Q}\right)^2.
\]

We now note that there is one-to-one correspondence between the characters of \( \mathbb{Z}_P \) and the sufficiently close characters of \( \mathbb{Z}_Q \). Hence, there are exactly \( \phi(P) \) characters of the group probe \( \mathbb{Z}_Q \) which are sufficiently close some primitive character of the maximal cyclic group \( \mathbb{Z}_P \). The theorem follows.

The following theorem can be found in [21, Theorem 328, Section 18.4]:

**Theorem 9.**

\[
\lim \inf \frac{\phi(N)}{N/\ln \ln N} = e^{-\gamma},
\]

where \( \gamma \) denotes Euler’s constant \( \gamma = 0.57721566490153286061\ldots \), and where \( e^{-\gamma} = 0.5614594836\ldots \).

As a corollary, we have:

**Corollary 6.** \( \text{Prob}_{\sim} (\chi \text{ sufficiently close to some primitive character of } \mathbb{Z}_P) \) is bounded below by

\[
\frac{4}{\pi^2 \ln 2} \frac{e^{-\gamma} - e(P)}{\lg \lg Q} \cdot \left(1 - \frac{P}{Q}\right)^2,
\]

where \( e(P) \) is a monotone decreasing sequence converging to zero. In terms of asymptotic notation,

\[
\text{Prob}_{\sim} (\chi \text{ sufficiently close to some primitive character of } \mathbb{Z}_P) = \Omega \left(\frac{1}{\lg \lg Q}\right).
\]

For a proof of the above, please refer to [33, 43].

**20. Summary of Vintage \( \mathbb{Z}_Q \) Shor QHSA**

Let \( \varphi : A \rightarrow S \) be a map with hidden subgroup structure with ambient group \( A \) free abelian of finite rank \( n \), and with image of \( \varphi \) finite. Then as a culmination of the mathematical developments in sections 11 through 19, we have the following **vintage \( \mathbb{Z}_Q \) Shor QHSA** for finding the order \( P = \text{lcm}(P_1, P_2, \ldots, P_n) \) of the maximum cyclic subgroup \( \mathbb{Z}_P \) of the hidden quotient group \( H_\varphi = \bigoplus_{j=1}^n \mathbb{Z}_{P_j} \). A flowchart of this algorithm is given in Figure 4.
P = \text{lcm} (P_1, P_2, \ldots, P_n) \text{ if hidden quotient group is } H_{\varphi} = \bigoplus_{j=1}^{n} \mathbb{Z} P_j

\begin{itemize}
  \item \textbf{Step 1} Select a basis } a'_1, a'_2, \ldots, a'_n \text{ of } A \text{ and a generator } \tilde{a} \text{ of } \mathbb{Z} Q

  \item \textbf{Step 2} \((\iota : \mathbb{Z} Q \rightarrow A) = \text{RANDOM\_SHOR\_TRANSVERSL}(\{a'_1, a'_2, \ldots, a'_n\}, Q, \tilde{a}, n)\)

  \item \textbf{Step 3} Construct } \tilde{\varphi} = \varphi \circ \iota : \mathbb{Z} Q \rightarrow S

  \item \textbf{Step 4} \(\chi_{\tilde{\varphi}} = \text{QRAND}_{\tilde{\varphi}}()\)

  \item \textbf{Step 5} \((d'', P'') = (0, 1) \text{  } \# 0\text{-th Cont. Frac. Convergent of } \frac{y}{Q}\)

  \text{INNER LOOP }

  \( (\text{Save}_d, \text{Save}_P) = (d', P') \)

  \( (d', P') = \text{NEXT\_CONT\_FRAC\_CONVERGENT}(\frac{y}{Q}, (d', P'), (d'', P'')) \)

  If } \varphi \left( P'_a_{j} \right) = \varphi (0) \text{ for all } j = 1, 2, \ldots, n \text{ Then Goto } \textbf{Step 6}

  \text{IF } \frac{d'}{P'} = \frac{y}{Q} \text{ THEN Goto } \textbf{Step 2}

  \text{INNER LOOP Boundary Goto Step 2}

  \item \textbf{Step 6} Output } P' \text{ and Stop}
21. A cursory analysis of complexity

We now make a cursory analysis of the algorithmic complexity of the vintage $\mathbb{Z}_Q$ Shor algorithm. By the word “cursory” we mean that our objective is to find an asymptotic bound which is by no means the tightest possible.

Our analysis is based on the following three assumptions:

- **Assumption 1.** Conjecture 1 (found in Appendix B) is true.

- **Assumption 2.** $U_{\bar{\varphi}}$ is of complexity $O\left(n^2 (\log Q)^3\right)$. 
- **Assumption 3.** The integer $Q$ is chosen so that $Q = 2^L \geq P^2$, where $P = \text{lcm}(P_1, P_2, \ldots, P_n)$.

The following theorem is an immediate consequence of **Assumption 2**.

**Theorem 10.** Let

\[ \tilde{\varphi} : \mathbb{Z}_Q \to S \]

be a map from the cyclic group $\mathbb{Z}_Q$ to a set $S$, where $Q = 2^L$.

If $U_{\tilde{\varphi}}$ is of algorithmic complexity

\[ O\left(n^2 (\lg Q)^3\right), \]

then the algorithmic complexity of $\text{QRAND}_{\tilde{\varphi}}()$ is the same, i.e.,

\[ O\left(n^2 (\lg Q)^3\right) \]

**Proof.** Steps 1 and 3 are each of the same algorithmic complexity as the quantum Fourier transform\(^{17}\), i.e., of complexity $O\left((\lg Q)^2\right)$. (See [36, Chapter 5].) Thus the dominant step in $\text{QRAND}_{\tilde{\varphi}}()$ is Step 2, which is by assumption of complexity $O\left(n^2 (\lg Q)^3\right)$.

The complexities of each step of the vintage $Z_Q$ Shor algorithm are given below. An accompanying abbreviated flow chart of this algorithm is shown in Figure 5.

**Step 1** Step 1 is of algorithmic complexity is $O(n)$.

**Step 2** By theorem 7 of section 14, Step 2 is of average case complexity $O\left(n^2 (\lg Q)^3\right)$. By corollary 8 of Appendix B, the probability that this step will be successful, i.e., will produce a maximal Shor transversal, is $\Omega\left(\left(\frac{1}{\lg Q}\right)^n\right)$.

**Step 3** Step 3 is of algorithmic complexity $O(n)$.

**Step 4** By theorem 10 given above, Step 4 is of algorithmic complexity $O\left(n^2 (\lg Q)^3\right)$. By corollary 6 of section 19, the probability (given that Step 2 is successful) that this step will be successful, i.e., will produce a character sufficiently close to a primitive character of the maximal cyclic group $\mathbb{Z}_P$ is $\Omega\left(\frac{1}{\lg Q}\right)$.

**Step 5** This step is of algorithmic complexity $O\left(n (\lg Q)^3\right)$. (See, for example, [30].)

---

\(^{17}\)If instead the Hadamard-Walsh transform is used in Step 1, then the complexity of Step 1 is $O(\lg Q)$. 

Step 6  For Step 5 to branch to this step, both Steps 2 and 4 must be successful. Thus the probability of branching to step 6 is $\text{Prob}_{\text{Success}}(\text{Step 2}) \cdot \text{Prob}_{\text{Success}}(\text{Step 2}) = \Omega \left( \left( \frac{1}{\lg \lg Q} \right)^{n+1} \right)$.

Since the Steps 2 through 5 loop will on average be executed $O \left( \left( \lg \lg Q \right)^{n+1} \right)$ times, the average algorithmic complexity of the Vintage $Z_Q$ Shor algorithm is $O \left( n^2 \left( \log Q \right)^3 \left( \lg \lg Q \right)^{n+1} \right)$. (This is, of course, not the tightest possible asymptotic bound.) We formalize this analysis as a theorem:

**Theorem 11.** Assuming the three assumptions given in section 21, the average algorithmic complexity of the Vintage $Z_Q$ Shor algorithm for finding the maximal...
cyclic subgroup $\mathbb{Z}_P$ of the hidden quotient group $H_\varphi = \bigoplus_{j=1}^n \mathbb{Z}_{P_j}$ is

$$O\left(n^2 (\lg Q)^3 (\lg \lg Q)^{n+1}\right)$$

22. Two alternative vintage $\mathbb{Z}_Q$ Shor algorithms

As two alternatives to the algorithm described in the last two sections, we give below two other vintage $\mathbb{Z}_Q$ Shor algorithms. Unlike the above described algorithm, these two alternative algorithms do not depend on finding a maximal Shor transversal. The first finds the order of the maximal cyclic subgroup $\mathbb{Z}_P$ of the hidden quotient group $H_\varphi$. The second finds the entire hidden subgroup $K_\varphi$. Flowcharts for these two quantum algorithms are given in Figures 6 and 7.

An optimal choice for the parameter $K$ of the following algorithm is not known at this time.

\begin{verbatim}
ALTERNATIVE1_VINTAGE_SHOR($\varphi, Q, n, K$)

# INPUT: $\varphi : A \rightarrow S, Q, \text{rank } n$ of $A$, and number of
# inner loop iterations $K$
# OUTPUT: $P = \text{lcm}(P_1, P_2, \ldots, P_n)$ if hidden quotient group
# is $H_\varphi = \bigoplus_{j=1}^n \mathbb{Z}_{P_j}$

Step 1
   SET $P = 1$

Step 2
   SELECT a basis $a'_1, a'_2, \ldots, a'_n$ of $A$ and a generator $\tilde{a}$ of $\mathbb{Z}_Q$

Step 3
   OUTER LOOP

Step 4
   INNER LOOP for $K$ iterations

Step 5
   $(\iota: Z_Q \rightarrow A) = \text{RAND_SHOR_TRANSVR}((a'_1, a'_2, \ldots, a'_n), Q, \tilde{a}, n)$

Step 6
   CONSTRUCT $\tilde{\varphi} = \varphi \circ \iota_\mu : Z_Q \rightarrow S$

Step 7
   $\chi_{\tilde{\varphi}} = \text{QRAND}_{\tilde{\varphi}}()$

Step 8
   $(d', P'') = (0, 1)$ # 0-th Cont. Frac. Converg. of $\frac{d}{P}$
   $(d', P') = \left(1, \left\lceil \frac{Q}{Q} \right\rceil \right)$ # 1-th Cont. Frac. Converg. of $\frac{d}{P}$
   INNERMOST LOOP
   $(\text{Save}_d', \text{Save}_P') = (d', P')$
   $(d'', P'') = \text{NXT_CONT_FRAC_CONV}(\frac{d'}{P'}, (d', P'), (d'', P''))$
   IF $\varphi(P' \iota_\mu(\tilde{a})) = \varphi(0)$ THEN GOTO Step 9
   IF $\frac{d''}{P''} = \frac{d}{P}$ THEN GOTO Step 4
   INNERMOST LOOP BOUNDARY
\end{verbatim}
Figure 6. The First Alternate Vintage $\mathbb{Z}_Q$ Shor QHSA. This is a Wandering Shor algorithm.
The following wandering Shor algorithm actually finds the entire hidden quotient group \( H_\varphi \), and hence the hidden subgroup \( K_\varphi \):

\[
\text{ALTERNATIVE2\_VINTAGE\_SHOR}(\varphi, Q, n)
\]

# INPUT: \( \varphi : A \rightarrow S, Q, \text{rank} \ n \) of \( A \)
# OUTPUT: A matrix \( \mathfrak{G} \) with row span equal to the hidden subgroup \( K_\varphi = \bigoplus_{j=1}^{n} P_j \mathbb{Z} \)

**Step 1** Set \( \mathfrak{G} = [ \ ] \) and NonZeroRows = 0

**Step 2** Select a basis \( a'_1, a'_2, \ldots, a'_n \) of \( A \) and a generator \( \tilde{a} \) of \( \mathbb{Z}_Q \)

**Step 3** Outer Loop Until NonZeroRows = \( n \)

**Step 4** \((\iota_\mu : Z_Q \rightarrow A) = \text{RANSHOR\_TRANSVRSL}(\{a'_1, a'_2, \ldots, a'_n\}, Q, \tilde{a}, n)\)

**Step 5** Construct \( \tilde{\varphi} = \varphi \circ \iota_\mu : Z_Q \rightarrow S \)

**Step 6** \( \chi_{\tilde{\varphi}} = \text{QRAND}_{\tilde{\varphi}}() \)

**Step 7** \((d'', P'') = (0, 1) \quad \# \) 0-th Cont. Frac. Converg. of \( \frac{\tilde{a}}{Q} \)

\((d', P') = \left(1, \left\lfloor \frac{\tilde{a}}{Q} \right\rfloor \right) \quad \# \) 1-th Cont. Frac. Converg. of \( \frac{\tilde{a}}{Q} \)

**Inner Loop**

\((\text{Save}_d', \text{Save}_P') = (d', P')\)

\((d', P') = \text{NEXT\_CONT\_Frac\_Converg}(\frac{\tilde{a}}{Q}, (d', P'), (d'', P''))\)

\((d'', P'') = (\text{Save}_d', \text{Save}_P')\)

If \( \varphi(P'\iota_\mu(\tilde{a})) = \varphi(0) \) Then Goto **Step 8**

If \( \frac{d'}{P'} = \frac{d}{P} \) Then Goto **Step 11**

**Inner Loop Boundary: Continue**

**Step 8** \( \mathfrak{G} = \left[ \begin{array}{cccc} \mathfrak{G} & \vDash & \vDash & \vDash & \vDash & \vDash & \vDash & \vDash \\ P'\lambda_1' & P'\lambda_2' & \ldots & P'\lambda_n' \end{array} \right] \)

**Step 9** \( \mathfrak{G} = \text{PUT\_IN\_ECHelon\_CANONICAL\_FORM}(\mathfrak{G}) \)

**Step 10** NonZeroRows = \text{NUMBER\_OF\_NON\_ZERO\_ROWS}(\mathfrak{G})

**Step 11** Outer Loop Lower Boundary: Continue

**Step 12** Output matrix \( \mathfrak{G} \) and Stop
Figure 7. The Second Alternate Vintage $\mathbb{Z}_Q$ Shor QHSA. This is a Wandering Shor algorithm.
Part 6. Epilogue

23. Conclusion

Each of the three vintage $\mathbb{Z}_Q$ Shor QHSAs created in this paper is a natural generalization of Shor's original quantum factoring algorithm to free abelian groups $A$ of finite rank $n$. The first two of the three find a maximal cyclic subgroup $\mathbb{Z}_P$ of the hidden quotient group $H_\varphi$. The last of the three does more. It finds the entire hidden quotient group $H_\varphi$.

We also note that these QHSAs can be viewed from yet another perspective as wandering Shor algorithms on free abelian groups. By this we mean quantum algorithms which, with each iteration, first select a random cyclic direct summand $\mathbb{Z}$ of the ambient group $A$ and then apply one iteration of the standard Shor algorithm on $\mathbb{Z}$ to produce a random character of the “approximating” group $\tilde{A} = \mathbb{Z}_Q$.

From this perspective, under the assumptions given in section 21, the algorithmic complexity of the first of these wandering QHSAs is found to be

$$O \left( n^2 \left( \log Q \right)^3 \left( \log \log Q \right)^{n+1} \right).$$

Obviously, much remains to be accomplished.

It should be possible to extend the vintage $\mathbb{Z}_Q$ Shor algorithms to quantum algorithms with more general group probes of the form

$$\tilde{A} = \bigoplus_{j=1}^{m} \mathbb{Z}_Q,$$

for $m > 1$. This would be a full generalization of Shor’s quantum factoring algorithm to the abelian category.

It is hoped that this paper will provide a useful stepping stone to the construction of QHSAs on non-abelian groups.

24. Acknowledgement

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25. Appendix A. Continued fractions

We give a brief summary of those aspects of the theory of continued fractions that are relevant to this paper. (For a more in-depth explanation of the theory of continued fractions, please refer, for example, to [21] and [31].)

Every positive rational number $\xi$ can be written as an expression in the form

$$\xi = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots + \frac{1}{aN}}}}$$

where $a_0$ is a non-negative integer, and where $a_1, \ldots, a_N$ are positive integers. Such an expression is called a (finite, simple) continued fraction, and is uniquely determined by $\xi$ provided we impose the condition $a_N > 1$. For typographical simplicity, we denote the above continued fraction by

$$[a_0, a_1, \ldots, a_N] .$$

The continued fraction expansion of $\xi$ can be computed with the following recurrence relation, which always terminates if $\xi$ is rational:

$$\begin{align*}
    a_0 &= \lfloor \xi \rfloor, \\
    \xi_0 &= \xi - a_0, \\
    a_{n+1} &= \lfloor 1/\xi_n \rfloor, \\
    \xi_{n+1} &= \frac{1}{\xi_n} - a_{n+1}
\end{align*}$$

The $n$-th convergent ($0 \leq n \leq N$) of the above continued fraction is defined as the rational number $\xi_n$ given by

$$\xi_n = [a_0, a_1, \ldots, a_n] .$$

Each convergent $\xi_n$ can be written in the form, $\xi_n = \frac{p_n}{q_n}$, where $p_n$ and $q_n$ are relatively prime integers ( $\gcd(p_n, q_n) = 1$). The integers $p_n$ and $q_n$ are determined by the recurrence relation

$$\begin{align*}
    p_0 &= a_0, \\
    q_0 &= 1, \\
    p_1 &= a_1 a_0 + 1, \\
    p_n &= a_n p_{n-1} + p_{n-2}, \\
    q_1 &= a_1, \\
    q_n &= a_n q_{n-1} + q_{n-2}
\end{align*}$$

The subroutine

```
Next_Cont_Frac_Convergent
```

found in the vintage $Z_Q$ Shor algorithm given in section 20 is an embodiment of the above recursion.

This recursion is used because of the following theorem which can be found in [21, Theorem 184, Secton 10.15]:
Theorem 12. Let $\xi$ be a real number, and let $d$ and $P$ be integers with $P > 0$. If
\[ |\xi - \frac{d}{P}| \leq \frac{1}{2P^2}, \]
then the rational number $d/P$ is a convergent of the continued fraction expansion of $\xi$.

26. Appendix B. Probability Distributions on Integers

Let
\[ \text{Prob}_Q : \{1, 2, \ldots, Q\} \to [0, 1] \]
denote the uniform probability distribution on the finite set of integers $\{1, 2, \ldots, Q\}$. Thus, the probability that a random integer $\lambda$ from $\{1, 2, \ldots, Q\}$ is divisible by a given prime $p$ is
\[ \text{Prob}_Q (p | \lambda) = \frac{|Q/p|}{Q} \leq \frac{1}{p}, \]
where ‘$[-]$’ denotes the the floor function.

The limit $\text{Prob}_\infty$, should it exist, of the probability distribution $\text{Prob}_Q$ as $Q$ approaches infinity, i.e.,
\[ \text{Prob}_\infty = \lim_{Q \to \infty} \text{Prob}_Q, \]
will turn out to be a useful tool. Since $\text{Prob}_\infty$ is not a probability distribution, we will call it a pseudo-probability distribution on the integers $\mathbb{Z}$. It immediately follows that
\[ \text{Prob}_\infty (p | \lambda) = \frac{1}{p}. \]
In this sense, we say that the pseudo-probability of a random integer $\lambda \in \mathbb{Z}$ being divisible by a given prime $p$ is $1/p$.

Theorem 13. Let $n$ be an integer greater than 1. Let $\lambda'_1, \lambda'_2, \ldots, \lambda'_n$, be $n$ integers selected randomly and independently with replacement from the set $\{1, 2, \ldots, Q\}$ according to the uniform probability distribution. Then the probability that
\[ \gcd (\lambda'_1, \lambda'_2, \ldots, \lambda'_n) = 1 \]
is
\[ \text{Prob}_Q \left( \gcd (\lambda'_1, \lambda'_2, \ldots, \lambda'_n) = 1 \right) = \sum_{k=1}^{Q} \mu (k) \left( \frac{|Q/k|}{Q} \right)^n, \]
where ‘$[-]$’ and ‘$\mu (-)$’ respectively denote the floor and Möbius functions. Moreover,
\[ \text{Prob}_\infty \left( \gcd (\lambda'_1, \lambda'_2, \ldots, \lambda'_n) = 1 \right) = \zeta (n)^{-1}, \]
where $\zeta (n)$ denotes the Riemann zeta function $\zeta (n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$. 

Proof. Let \( \text{Primes}_Q \) denote the set of primes less than or equal to \( Q \). For each prime \( p \) and integer \( 0 < j \leq n \), let \( A_{pj} \) denote the set
\[
A_{pj} = \left\{ \lambda \in \{1, \ldots, Q\}^n : p \mid \lambda_j \right\}.
\]
Since
\[
\bigcap_{p \in \text{Primes}_Q} \bigcup_{j=1}^n \overline{A}_{pj} = \{ \lambda \in \{1, \ldots, Q\}^n : \forall p \exists j \ p \mid \lambda_j \},
\]
we have
\[
\Pr_Q(\gcd(\lambda_1, \lambda_2, \ldots, \lambda_n) = 1) = \Pr_Q\left(\bigcap_{p \in \text{Primes}_Q} \bigcup_{j=1}^n \overline{A}_{pj}\right),
\]
where \( \overline{A}_{pj} \) denotes the complement of \( A_{pj} \).

We proceed to compute \( \Pr_Q\left(\bigcap_{p \in \text{Primes}_Q} \bigcup_{j=1}^n \overline{A}_{pj}\right) \) by first noting that:
\[
\Pr_Q\left(\bigcap_{p \in \text{Primes}_Q} \bigcup_{j=1}^n \overline{A}_{pj}\right) = 1 - \Pr_Q\left(\bigcup_{p \in \text{Primes}_Q} \bigcap_{j=1}^n A_{pj}\right).
\]
So by the inclusion/exclusion principle, we have
\[
\Pr_Q\left(\bigcup_{p \in \text{Primes}_Q} \bigcap_{j=1}^n A_{pj}\right) = \sum_{S \subseteq \text{Primes}_Q} (-1)^{|S|} \Pr_Q\left(\bigcap_{p \in S} \bigcap_{j=1}^n A_{pj}\right).
\]

Since the \( \lambda_j \)'s are independent random variables, we have
\[
\Pr_Q\left(\bigcap_{j=1}^n \bigcap_{p \in S} A_{pj}\right) = \prod_{j=1}^n \Pr_Q\left(\bigcap_{p \in S} A_{pj}\right)
\]
Moreover, it follows from a straightforward counting argument that
\[
\Pr_Q\left(\bigcap_{p \in S} A_{pj}\right) = \left[ Q/\prod_{p \in S} p \right]/Q,
\]
from which we obtain
\[
\Pr_Q\left(\bigcap_{j=1}^n \bigcap_{p \in S} A_{pj}\right) = \prod_{j=1}^n \left[ Q/\prod_{p \in S} p \right]/Q = \left( \left[ Q/\prod_{p \in S} p \right]/Q \right)^n.
\]
Thus,

\[ \text{Prob}_Q \left( \bigcap_{p \in \text{Primes}_Q} \bigcup_{j=1}^n \mathcal{A}_{p_j} \right) = \sum_{S \subseteq \text{Primes}_Q} (-1)^{|S|} \left( \left[ Q/ \prod_{p \in S} p \right] / Q \right)^n \]

This last expression expands to

\[ 1 - \sum_{p \leq Q} \left( \frac{|Q/p|}{Q} \right)^n + \sum_{p < p' \leq Q} \left( \frac{|Q/pq|}{Q} \right)^n - \sum_{p < p' < p'' \leq Q} \left( \frac{|Q/pp'p''|}{Q} \right)^n + \ldots, \]

which can be rewritten as

\[ \sum_{k=1}^Q \mu(k) \left( \frac{|Q/k|}{Q} \right)^n \]

since \( \mu(k) = 0 \) for all integers \( k \) that are not squarefree.

The last part of this theorem follows immediately from the fact that

\[ \lim_{Q \to \infty} \sum_{k=1}^Q \mu(k) \left( \frac{|Q/k|}{Q} \right)^n = \sum_{k=1}^\infty \mu(k) \frac{1}{k^n} = \zeta(n)^{-1}. \]

(See [40], [42], or [21].) \( \square \)

**Corollary 7.** Let \( n \) be an integer greater than 1, and let \( \lambda_1', \lambda_2', \ldots, \lambda_n' \) be \( n \) integers randomly and independently selected with replacement from the set \( \{1, 2, \ldots, Q\} \) according to the uniform probability distribution. Let \( M \) be a fixed element of the group \( SL_\pm(n, \mathbb{Z}) \) of invertible \( n \times n \) integer matrices. Finally, let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be \( n \) integers given by

\[ (\lambda_1', \lambda_2', \ldots, \lambda_n')^\text{transpose} = M (\lambda_1', \lambda_2', \ldots, \lambda_n')^\text{transpose}. \]

Then the probability that

\[ \gcd(\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \]

is

\[ \text{Prob}_Q \left( \gcd(\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \right) = \sum_{k=1}^Q \mu(k) \left( \frac{|Q/k|}{Q} \right)^n, \]

where ‘\( \lfloor \cdot \rfloor \)’ and ‘\( \mu(\cdot) \)’ respectively denote the floor and Möbius functions.

Moreover,

\[ \text{Prob}_\infty \left( \gcd(\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \right) = \zeta(n)^{-1}, \]

where \( \zeta(n) \) denotes the Riemann zeta function \( \zeta(n) = \sum_{k=1}^\infty \frac{1}{k^n} \). Hence,

\[ \text{Prob}_Q \left( \gcd(\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \right) = \Omega \left( \frac{1}{\zeta(n)^{-1}} \right) = \Omega(1) \]

**Proof.** This corollary immediately follows from the fact that the gcd is invariant under the action of \( SL_\pm(n, \mathbb{Z}) \). \( \square \)
Remark 11. We conjecture that a stronger result holds, namely that the function $\zeta(n)^{-1}$ is actually a lower bound for $\text{Prob}_Q \left( \gcd (\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \right)$ for $Q \geq n$.

We need to make the following conjecture to estimate the algorithmic complexity of Vintage $\mathbb{Z}_Q$ algorithms, also called wandering Shor algorithms.

**Conjecture 1.** Let $n$ be an integer greater than 1, let $P_1, P_2, \ldots, P_n$ be $n$ fixed positive integers, and let $\lambda'_1, \lambda'_2, \ldots, \lambda'_n$ be $n$ integers randomly and independently selected with replacement from the set $\{1, 2, \ldots, Q\}$ according to the uniform probability distribution. Let $M$ be a fixed element of the group $SL_{\pm}(n, \mathbb{Z})$ of invertible $n \times n$ integral matrices, and let

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) = M (\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$$

Then the conditional pseudo-probability

$$\text{Prob}_\infty \left( \gcd (\lambda_j, P_j) = 1 \ \forall j \mid \gcd (\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \right)$$

is given by

$$\frac{\prod_{j=1}^{n} \phi(P_j)}{\prod_{p \text{ prime}} (1 - p^{-n})} \geq \frac{\prod_{j=1}^{n} \phi(P_j)}{P_j},$$

where $\zeta(\cdot)$ and $\phi(\cdot)$ denote respectively the Riemann zeta and the Euler totient functions.

**Plausibility Argument.** (This is not a proof.)

We treat $\text{Prob}_\infty$ as if it were a probability distribution on the integers $\mathbb{Z}^n = \{(\lambda_1, \lambda_2, \ldots, \lambda_n)\}$. We assume that $M$ maps this distribution on itself, and that $\text{Prob}_\infty (p \mid \lambda_j)$ and $\text{Prob}_\infty (q \mid \lambda_j)$ are stochastically independent when $p$ and $q$ are distinct primes.

For fixed $j$, the probability $\text{Prob}_\infty (p \mid \lambda_j)$ that a given prime divisor $p$ of $P_j$ does not divide $\lambda'_j$ is

$$1 - \frac{1}{p}.$$ 

Hence, the probability that $P_j$ and $\lambda_j$ are relatively prime is

$$\text{Prob}_\infty \left( \gcd (P_j, \lambda_j) = 1 \right) = \prod_{p \mid P_j} \left(1 - \frac{1}{p}\right).$$

This can be reexpressed in terms of the Euler totient function as

$$\text{Prob}_\infty \left( \gcd (P_j, \lambda_j) = 1 \right) = \frac{\phi(P_j)}{P_j}.$$
Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are independent random variables, we have

$$\text{Prob}_\infty \left( \gcd (P_j, \lambda_j) = 1 \forall j \right) = \prod_{j=1}^{n} \frac{\varphi(P_j)}{P_j}.$$ 

On the other hand, the probability that a given prime $p$ does not divide all the integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ is

$$1 - \frac{1}{p^n}.$$ 

Thus,

$$\text{Prob}_\infty \left( p \nmid \gcd (\lambda_1, \lambda_2, \ldots, \lambda_n) \forall p \text{ s.t. } p \nmid \operatorname{lcm}(P_1, P_2, \ldots, P_n) \right)$$

is given by the expression

$$\prod_{p \mid \operatorname{lcm}(P_1, P_2, \ldots, P_n)} (1 - p^{-n}) = \frac{\zeta(n)^{-1}}{\prod_{p \mid \operatorname{lcm}(P_1, P_2, \ldots, P_n)} (1 - p^{-n})},$$

where we have used the fact [21] that

$$\zeta(n)^{-1} = \prod_{p \text{ Prime}} \left(1 - \frac{1}{p^n}\right).$$

We next note that the events $\forall j \ \gcd (P_j, \lambda_j) = 1$ and $\forall p \ nmid \operatorname{lcm}(P_1, P_2, \ldots, P_n) \rightarrow p \ nmid \gcd (P_1, P_2, \ldots, P_n)$ are stochastically independent since they respectively refer to the disjoint sets of primes $\{p : p \mid \operatorname{lcm}(P_1, P_2, \ldots, P_n)\}$ and $\{p : p \mid \operatorname{lcm}(P_1, P_2, \ldots, P_n)\}$. Hence, the probability of the joint event

$$\text{Prob}_\infty \left( \gcd (P_j, \lambda_j) = 1 \ \forall j \ \text{AND} \ \gcd (\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \right)$$

is given by the expression

$$\zeta(n)^{-1} \prod_{j=1}^{n} \frac{\varphi(P_j)}{P_j} \prod_{p \mid \operatorname{lcm}(P_1, P_2, \ldots, P_n)} (1 - p^{-n}).$$

Using exactly the same argument as that used to find an expression for

$$\text{Prob}_\infty \left( p \nmid \gcd (\lambda_1, \lambda_2, \ldots, \lambda_n) \forall p \text{ s.t. } p \nmid \operatorname{lcm}(P_1, P_2, \ldots, P_n) \right),$$

we have

$$\text{Prob}_\infty \left( \gcd (\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \right) = \zeta(n)^{-1}.$$ 

Hence the conditional probability

$$\text{Prob}_\infty \left( \gcd (P_j, \lambda_j) = 1 \ \forall j \mid \gcd (\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \right)$$
is given by the expression
\[
\prod_{j=1}^{n} \phi(P_j) P_j = \prod_{p \mid \text{lcm}(P_1, P_2, \ldots, P_n)} (1 - p^{-n})
\]

Finally, since
\[
\prod_{p \mid \text{lcm}(P_1, P_2, \ldots, P_n)} (1 - p^{-n}) \leq 1 ,
\]
it follows that the conditional probability
\[
\Pr_{\infty} \left( \gcd (P_j, \lambda_j) = 1 \forall j \mid \gcd (\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \right)
\]
is bounded below by the expression
\[
\prod_{j=1}^{n} \frac{\phi(P_j)}{P_j} .
\]

The following is an immediate corollary of the above conjecture.

**Corollary 8.** Let \( n \) be an integer greater than 1, and let \( P_1, P_2, \ldots, P_n \) be \( n \) fixed positive integers. Let \( \lambda_1', \lambda_2', \ldots, \lambda_n' \) be \( n \) integers randomly and independently selected with replacement from the set all integers \( \mathbb{Z} \) according to the uniform probability distribution. Let \( M \) be a fixed element of the group \( SL_{\pm}(n, \mathbb{Z}) \) of invertible \( n \times n \) integer matrices. Finally, let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be \( n \) integers given by
\[
(\lambda_1, \lambda_2, \ldots, \lambda_n)^{\text{transpose}} = M (\lambda_1', \lambda_2', \ldots, \lambda_n')^{\text{transpose}} .
\]

Then, assuming conjecture 1, we have
\[
\Pr_{\infty} \left( \gcd (\lambda_j, P_j) = 1 \forall j \mid \gcd (\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \right) = \Omega \left( \prod_{j=1}^{n} \frac{1}{\lg \lg P_j} \right) ,
\]
where \( \Omega (\cdot) \) denotes the asymptotic lower bound 'big-omega.' Thus, if \( Q \) is greater than each \( P_j \), we have
\[
\Pr_{Q} \left( \gcd (\lambda_j, P_j) = 1 \forall j \mid \gcd (\lambda_1, \lambda_2, \ldots, \lambda_n) = 1 \right) = \Omega \left( \left( \frac{1}{\lg \lg Q} \right)^n \right) .
\]

**Proof.** Since
\[
\lim_{n \to \infty} \frac{\phi(n) \ln \ln n}{n} = e^{-\gamma} ,
\]
where \( \gamma \) denotes Euler's constant, we have that
\[
\frac{\phi(P_j)}{P_j} = \Omega \left( \frac{1}{\lg \lg P_j} \right) .
\]

\(^{18}\text{See [21, Theorem 328, Section 18.4].}\)
Thus, an asymptotic lower bound for the above conditional probability is given by the expression

\[ \Omega \left( \prod_{j=1}^{n} \frac{1}{\log \log P_j} \right) . \]

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(Samuel J. Lomonaco, Jr.) Department of Computer Science and Electrical Engineering, University of Maryland Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250
  E-mail address, Samuel J. Lomonaco, Jr.: Lomonaco@UMBC.EDU
  URL: WebPage: http://www.csee.umbc.edu/~lomonaco

(Louis H. Kauffman) Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607-7045
  E-mail address, Louis H. Kauffman: kauffman@uic.edu
  URL: WebPage: http://math.uic.edu/~kauffman