Abstract

We give a general formula for the $C$–transfinite diameter $\delta_C(K)$ of a compact set $K \subset \mathbb{C}^2$ which is a product of univariate compacta where $C \subset (\mathbb{R}^+)^2$ is a convex body. Along the way we prove a Rumely type formula relating $\delta_C(K)$ and the $C$–Robin function $\rho_{V_{C,K}}$ of the $C$–extremal plurisubharmonic function $V_{C,K}$ for $C \subset (\mathbb{R}^+)^2$ a triangle $T_{a,b}$ with vertices $(0,0), (b,0), (0,a)$. Finally, we show how the definition of $\delta_C(K)$ can be extended to include many nonconvex bodies $C \subset \mathbb{R}^d$ for $d$–circled sets $K \subset \mathbb{C}^d$, and we prove an integral formula for $\delta_C(K)$ which we use to compute a formula for the $C$–transfinite diameter of the Euclidean unit ball $B \subset \mathbb{C}^2$.

1 Introduction

In the recently developed pluripotential theory associated to a convex body $C \subset (\mathbb{R}^+)^d$ (cf., [1]), notions of $C$–extremal plurisubharmonic (psh) function $V_{C,K}$ and $C$–transfinite diameter $\delta_C(K)$ of a compact set $K \subset \mathbb{C}^d$ generalize the corresponding notions in the standard setting. Their definitions are recalled in the next section, and we include a brief discussion of Ma’u’s recent work [12] on $C$–transfinite diameter. We also recall the notion of $C$–Robin function $\rho_{V_{C,K}}$ associated to $V_{C,K}$ as defined in [9] for $C \subset (\mathbb{R}^+)^2$ a triangle $T_{a,b}$ with vertices $(0,0), (b,0), (0,a)$. The $C$–Robin function describes the precise asymptotic behavior of $V_{C,K}$; i.e., the behavior of $V_{C,K}(z)$ for $|z|$ large.

In classical pluripotential theory, which corresponds to the special case where $C$ is the standard unit simplex $\Sigma \subset (\mathbb{R}^+)^d$, it is very difficult to find explicit formulas for extremal psh functions $V_K$ (and hence their Robin functions) or to find precise values of transfinite diameters $\delta_d(K)$ for $K \subset \mathbb{C}^d$. In 1962, Schiffer and Siciak [13] proved that if $K = E_1 \times \cdots \times E_d \subset \mathbb{C}^d$ is a product of planar compact sets $E_j$, then $\delta_d(K) = \prod_{j=1}^d D(E_j)$ where $D(E_j)$ is the univariate transfinite diameter of $E_j$. Their proof used an intertwining of univariate Leja sequences for the sets $E_j$. Then in 1999, Bloom and Calvi [4] proved a more general result: if $K = E \times F$ where $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$, then

$$\delta_{n+m}(K) = (\delta_m(E)^m, \delta_n(F)^n)^{\frac{1}{m+n}}. \quad (1.1)$$

Their proof used orthogonal polynomials associated to certain measures, called Bernstein-Markov measures, on $K$. In 2005, Calvi and Phung Van Manh [6] recovered the Bloom-Calvi result (1.1) by generalizing the Schiffer-Siciak method in introducing “block” Leja sequences for the component sets.
In [10], Rumely gave a remarkable formula relating transfinite diameter and Robin function in this classical setting. Using this formula, Blocki, Edigarian and Siciak [3] gave a very short proof of the general product formula (1.1). In section 3, based on results in [1] and [9], we prove a Rumely type formula relating \( \delta_C(K) \) and \( \rho_{V,C,K} \) for \( C = T_{a,b} \subset (\mathbb{R}^+)^2 \) and we use this in section 4 to prove a formula for \( \delta_C(K) \) when \( K = E \times F \) is a product of univariate compacta. We modify the Bloom-Calvi proof using orthogonal polynomials in section 5 to give a product formula for the \( C \)-transfinite diameter when \( C \) is a general convex body in \( (\mathbb{R}^+)^2 \). In particular, for such \( C \) which are symmetric with respect to the line \( y = x \), we obtain the striking result that the \( C \)-transfinite diameter of \( K = E \times F \) is the same for these \( C \). Finally, in section 6, we show how the \( C \)-transfinite diameter \( \delta_C(K) \) can be extended to include many nonconvex bodies \( C \subset \mathbb{R}^d \) for \( d \)-circled sets \( K \subset \mathbb{C}^d \), and we exhibit an integral formula for \( \delta_C(K) \). We use this to directly compute a formula for \( \delta_C p(B) \) for the Euclidean unit ball \( B \subset \mathbb{C}^2 \) for a natural one-parameter family of symmetric \( C = C_p \) (section 6) which explicitly yields different values for different \( p \).

2 \( C \)-transfinite diameter and \( C \)-Robin function

Let \( C \) be a convex body in \( (\mathbb{R}^+)^d \). We assume throughout that

\[
\epsilon \Sigma \subset C \subset \delta \Sigma \text{ for some } \delta > \epsilon > 0
\]

where

\[
\Sigma := \{ (x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1, \sum_{j=1}^d x_i \leq 1 \}.
\]

We set

\[
\text{Poly}(nC) = \left\{ p(z) = \sum_{J \in nC \cap \mathbb{N}^d} c_J z^J = \sum_{J \in nC \cap \mathbb{N}^d} c_J z_1^{j_1} \cdots z_d^{j_d}, \ c_J \in \mathbb{C} \right\}, \quad n = 1, 2, \ldots
\]

and for a nonconstant polynomial \( p \) we define

\[
\deg_C(p) = \min\{ n \in \mathbb{N} : p \in \text{Poly}(nC) \}.
\]

Next, we define the logarithmic indicator function

\[
H_C(z) := \sup_{J \in C} \log |z^J| := \sup_{(j_1, \ldots, j_d) \in C} \log \left( |z_1|^{j_1} \cdots |z_d|^{j_d} \right)
\]

in order to define

\[
L_C = L_C(\mathbb{C}^d) := \{ u \in PSH(\mathbb{C}^d) : u(z) - H_C(z) = O(1), \ |z| \to \infty \},
\]

and

\[
L_C^+ = L_C^+(\mathbb{C}^d) = \{ u \in L_C(\mathbb{C}^d) : u(z) \geq H_C(z) + C_u \}
\]

where \( PSH(\mathbb{C}^d) \) denotes the class of plurisubharmonic functions on \( \mathbb{C}^d \). In particular,

if \( p \in \text{Poly}(nC) \) then \( u(z) := \frac{1}{\deg_C(p)} \log |p(z)| \in L_C \).
These classes are generalizations of the classical Lelong classes $L := L_{\Sigma}$, $L^+ := L^+_{\Sigma}$ when $C = \Sigma$. The $C$-extremal function of a compact set $K \subset \mathbb{C}^d$ is defined as the uppersemicontinuous (usc) regularization $V_{C,K}^*(z) := \limsup_{\zeta \to z} V_{C,K}(\zeta)$ of
\[
V_{C,K}(z) := \sup\{u(z) : u \in L_C, u \leq 0 \text{ on } K\}.
\]
If $C = \Sigma$, we simply write $V_K := V_{\Sigma,K}$. As in this classical setting, $V_{C,K}^* \equiv +\infty$ if and only if $K$ is pluripolar; and when this is not the case, the complex Monge-Ampère measure $(dd^c V_{C,K}^*)^d$ is supported in $K$. We call $K$ regular if $V_K = V_K^*$; i.e., $V_K$ is continuous. This is equivalent to $V_{C,K}$ being continuous for any $C$. Our definition of $dd^c$ is such that $(dd^c \log^+ \max[|z_1|, \ldots, |z_d|])^d$ is a probability measure.

We recall the definition of $C$–transfinite diameter $\delta_C(K)$ of a compact set $K \subset \mathbb{C}^d$. Letting $d_n$ be the dimension of $\text{Poly}(nC)$, we have
\[
\text{Poly}(nC) = \text{span}\{e_1, \ldots, e_{d_n}\}
\]
where $\{e_j(z) := z^{\alpha(j)} = z_1^{\alpha_1(j)} \cdots z_d^{\alpha_d(j)}\}_{j=1, \ldots, d_n}$ are the standard basis monomials in $\text{Poly}(nC)$ in any order. For points $\zeta_1, \ldots, \zeta_{d_n} \in \mathbb{C}^d$, let
\[
\text{VD}_{\Sigma}(\zeta_1, \ldots, \zeta_{d_n}) := \det[e_i(\zeta_j)]_{i,j=1, \ldots, d_n}
= \det \begin{bmatrix}
e_1(\zeta_1) & e_1(\zeta_2) & \ldots & e_1(\zeta_{d_n}) \\
\vdots & \vdots & \ddots & \vdots \\
e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \ldots & e_{d_n}(\zeta_{d_n})
\end{bmatrix}
\]
and for a compact subset $K \subset \mathbb{C}^d$ let
\[
V_n = V_n(K) := \max_{\zeta_1, \ldots, \zeta_{d_n} \in K} |\text{VD}_{\Sigma}(\zeta_1, \ldots, \zeta_{d_n})|.
\]
Then
\[
\delta_C(K) := \limsup_{n \to \infty} V_n^{1/l_n}
\]
(2.3)
is the $C$–transfinite diameter of $K$ where $l_n := \sum_{j=1}^{d_n} \deg(e_j)$.

The existence of the limit is not obvious. In this generality it was proved in [1]. In the classical ($C = \Sigma$) case, Zaharjuta [14] verified the existence of the limit by introducing directional Chebyshev constants $\tau(K,\theta)$ and proving
\[
\delta_{\Sigma}(K) = \exp\left(\frac{1}{|\sigma|} \int_{\sigma^0} \log \tau(K,\theta) d|\sigma|(\theta)\right)
\]
where $\sigma := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1, \sum_{i=1}^d x_i = 1\}$ is the extreme “face” of $\Sigma$; $\sigma^0 := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 < x_i < 1, \sum_{i=1}^d x_i = 1\}$; and $|\sigma|$ is the $(d-1)$–dimensional measure of $\sigma$. We will utilize results from [12] where a Zaharjuta-type proof of the existence of the limit in the general $C$–setting is given. There it is shown that
\[
\delta_C(K) = \left[\exp\left(\frac{1}{\text{vol}(C)} \int_{\sigma^0} \log \tau_C(K,\theta) dm(\theta)\right)\right]^{1/A_C}
\]
(2.4)
where the directional Chebyshev constants $\tau_C(K,\theta)$ and the integration in the formula are over the interior $C^o$ of the entire $d$–dimensional convex body $C$ and $A_C$ is a positive constant depending only on $C$ and $d$ (defined in (2.9)).
Apriori, in the definition of $\tau_{C}(K, \theta)$ the standard grlex (graded lexicographic) ordering $\preceq$ on $\mathbb{N}^d$ (i.e., on the monomials in $\mathbb{C}^d$) was used. This was required to obtain the submultiplicativity of the “monic” polynomial classes

$$M_k(\alpha) := \{ p \in \text{Poly}(kC) : p(z) = z^\alpha + \sum_{\beta \in kC \cap \mathbb{N}^d, \beta \prec \alpha} c_\beta z^\beta \} \quad (2.5)$$

for $\alpha \in kC \cap \mathbb{N}^d$; i.e., $M_{k_1}(\alpha_1) \cdot M_{k_2}(\alpha_2) \subset M_{k_1+k_2}(\alpha_1+\alpha_2)$. Defining Chebyshev constants

$$T_k(K, \alpha) := \inf \{ \|p\|_K : p \in M_k(\alpha) \}^{1/k}, \quad (2.6)$$

for $\theta \in C^\circ$, this submultiplicativity allows one to verify existence of the limit

$$\tau_{C}(K, \theta) := \lim_{k \to \infty, \alpha/k \to \theta} T_k(K, \alpha) \quad (2.7)$$

as well as convexity of the function $\theta \to \ln \tau_{C}(K, \theta)$ on $C^\circ$.

In the proof that $\lim_{n \to \infty} V_n^{1/l_n}$ exists in [12], it is shown that

$$\lim_{n \to \infty} V_n^{1/nd_n} = \lim_{n \to \infty} \left( \prod_{j=1}^{d_n} T_n(K, \alpha(j))^n \right)^{1/nd_n}. \quad (2.8)$$

The asymptotic relation between $nd_n$ and $l_n$ is that

$$\lim_{n \to \infty} l_n/nd_n = A_C := \frac{1}{\text{vol}(C)} \cdot \int_C (x_1 + \cdots + x_d)dx_1 \cdots dx_d =: M_C/\text{vol}(C). \quad (2.9)$$

The following propositions will be useful in the sequel.

**Proposition 2.1.** For $t > 0$,

$$\delta_{tC}(K) = \delta_C(K).$$

**Proof.** We first observe that if $t \in \mathbb{N}$, since the limit in (2.3) exists,

$$\delta_C(K) = \lim_{n \to \infty} V_n^{1/l_n} = \lim_{n \to \infty} V_n^{1/lt_n} = \delta_{tC}(K).$$

Similarly, if $t \in \mathbb{Q}$ we have $\delta_{tC}(K) = \delta_C(K)$. To verify the result for $t \in \mathbb{R}$, we proceed as follows. If $t_1 < t < t_2$, from the definitions of $M_k(\alpha)$, $T_k(K, \alpha)$ and $\tau_C(K, \theta)$, we have the following:

1. for $\theta \in t_1C^\circ$, $\tau_{t_1C}(K, \theta) \geq \tau_{tC}(K, \theta)$; and
2. for $\theta \in tC^\circ$, $\tau_{tC}(K, \theta) \geq \tau_{tC_2}(K, \theta)$.

Taking a sequence $\{t_{1,j}\} \subset \mathbb{Q}$ with $t_{1,j} \uparrow t$ and a sequence $\{t_{2,j}\} \subset \mathbb{Q}$ with $t_{2,j} \downarrow t$, using the above inequalities together with (2.4) and (2.9),

$$\lim_{j \to \infty} \delta_{t_{1,j}C}(K) = \lim_{j \to \infty} \delta_{t_{2,j}C}(K) = \delta_{tC}(K).$$

We can use the Hausdorff metric on the family of our convex bodies $C$ satisfying (2.1) considered as compact sets in $\mathbb{R}^d$. Using similar ideas from the previous proof, we verify the next result.
Proposition 2.2. Given $K \subset \mathbb{C}^d$, the mapping $C \to \delta_C(K)$ is continuous.

Proof. Taking a sequence $\{C_j\}$ of convex bodies satisfying (2.1) converging to $C$ in the Hausdorff metric, we can find $\epsilon_j \to 0$ with

$$(1 - \epsilon_j)C \subset C_j \subset (1 + \epsilon_j)C, \; j = 1, 2,...$$

As in the proof of Proposition 2.1 we have

1. for $\theta \in (1 - \epsilon_j)C^o$, $\tau_{(1-\epsilon_j)}C(K, \theta) \geq \tau_{C_j}(K, \theta)$; and
2. for $\theta \in C_j^o$, $\tau_{C_j}(K, \theta) \geq \tau_{(1+\epsilon_j)}C(K, \theta)$.

Since $\epsilon_j \to 0$ implies $\text{vol}(C_j) \to \text{vol}(C)$ and $M_{C_j} \to M_C$, using the above inequalities together with (2.4) and (2.9), we find $\lim_{j \to \infty} \delta_{C_j}(K) = \delta_C(K)$. 

For most of the subsequent sections, we work in $\mathbb{C}^2$. First, recall the definition of the Robin function $\rho_u$ associated to $u \in L(\mathbb{C}^2)$:

$$\rho_u(z) := \limsup_{|\lambda| \to \infty} [u(\lambda z) - \log |\lambda|].$$

For $z = (z_1, z_2) \neq (0,0)$ we define

$$\rho_u(z) := \limsup_{|\lambda| \to \infty} [u(\lambda z) - \log |\lambda z|] = \rho_u(z) - \log |z|$$

so that $\rho_u(tz) = \rho_u(z)$ for $t \in \mathbb{C} \setminus \{0\}$. Here $|z|^2 = |z_1|^2 + |z_2|^2$. We can consider $\rho_u$ as a function on $\mathbb{P}^1 = \mathbb{P}^2 \setminus \mathbb{C}^2$ where to $p = (p_1, p_2)$ with $|p| = 1$ we associate the point where the complex line $\lambda \to \lambda p$ hits $\mathbb{P}^1$.

For a special class of convex bodies, there is a generalization of the notion of Robin function. Following [9], if we let $C$ be the triangle $T_{a,b}$ with vertices $(0,0), (b,0), (0,a)$ where $a, b$ are relatively prime positive integers, we have the following:

1. $H_C(z_1, z_2) = \max[\log^+ |z_1|^b, \log^+ |z_2|^a]$ (note $H_C = 0$ on the closure of the unit polydisk $\mathcal{P}^2 := \{(z_1, z_2) : |z_1|, |z_2| < 1\}$), and, indeed, $H_C = V_{C, \mathcal{P}^2} = V_{C, T^2}$ where $T^2 := \{(z_1, z_2) : |z_1|, |z_2| = 1\}$;

2. defining $\lambda \circ (z_1, z_2) := (\lambda^a z_1, \lambda^b z_2)$, we have

$$H_C(\lambda \circ (z_1, z_2)) = H_C(z_1, z_2) + ab \log |\lambda|$$

for $(z_1, z_2) \in \mathbb{C}^2 \setminus \mathcal{P}^2$ and $|\lambda| \geq 1$.

Definition 2.3. Given $u \in L_C$, we define the $C$–Robin function of $u$:

$$\rho_u(z_1, z_2) := \limsup_{|\lambda| \to \infty} [u(\lambda \circ (z_1, z_2)) - ab \log |\lambda|]$$

for $(z_1, z_2) \in \mathbb{C}^2$. 
Applying the transformation formula Theorem 4.1 of [9] in the case where \( d = 2; \)
\( C \) is our triangle with vertices \((0,0), (b,0), (0,a)\); \( C' = ab\Sigma; \) and we consider the proper polynomial mapping
\[ F(z_1, z_2) = (z_1^a, z_2^b), \]
we obtain
\[ ab V_{F^{-1}}(K)(z_1, z_2) = V_{C,K}(z_1^a, z_2^b) \]
so that
\[ ab \rho_{V_{F^{-1}}(K)}(z_1, z_2) = \limsup_{|\lambda| \to \infty} [V_{C,K}(\lambda(z_1^a, z_2^b)) - ab \log |\lambda|] \]
\[ = \limsup_{|\lambda| \to \infty} [V_{C,K}(\lambda \circ (z_1^a, z_2^b)) - ab \log |\lambda|] \]
\[ = \rho_{V_{C,K}}(z_1^a, z_2^b) = \rho_{V_{C,K}}(F(z_1, z_2)). \]

More generally, letting \( \zeta = (\zeta_1, \zeta_2) = F(z) = F(z_1, z_2) = (z_1^a, z_2^b), \) for \( u \in L_C, \) we have
\( \tilde{u}(z) := u(F(z_1, z_2)) \in abL \) and
\[ \rho_u(\zeta) = \rho_u(F(z_1, z_2)) = ab \rho_{u/ab}(z) \]
where \( \rho_{u/ab} \) is the standard Robin function of \( \tilde{u}/ab \in L. \) Note that if \( u \in L_C^+ \) then \( \tilde{u} \in abL^+. \) We apply these results in the next section.

3 \( C - \) Rumely formula for \( C = T_{a,b} \)

In this section, we let \( C = T_{a,b}. \) We begin with some integral formulas associated to functions in \( L^+(\mathbb{C}^2). \) The integral formula Theorem 5.5 of [2] in this setting is the following.

**Theorem 3.1. (Bedford-Taylor)** Let \( u, v, w \in L^+(\mathbb{C}^2). \) Then
\[ \int_{\mathbb{C}^2} (udd^c v - vdd^c u) \wedge dd^c w = \int_{\mathbb{P}^1} (\rho_u - \rho_v)(dd^c \rho_w + \omega) \]
where \( \omega \) is the standard Kähler form on \( \mathbb{P}^1. \)

Next, following the arguments in [7], we get a symmetrized integral formula involving Robin functions \( \rho_u, \rho_v \) for \( u, v \in L^+(\mathbb{C}^2) \) and their projectivized versions \( \rho_{u/ab}, \rho_{v/ab}; \)
\[ \int_{\mathbb{P}^1} (\rho_u - \rho_v)[(dd^c \rho_u + \omega) + (dd^c \rho_v + \omega)] \]
\[ = \int_{\mathbb{C}^1} \rho_u(1,t)dd^c \rho_u(1,t) + \rho_u(0,1) - \int_{\mathbb{C}^1} \rho_v(1,t)dd^c \rho_v(1,t) + \rho_v(0,1)]. \]

From (2.10), if \( u, v, w \in L_C^+, \)
\[ ab \int_{\mathbb{C}^2} (udd^c v - vdd^c u) \wedge dd^c w = \int_{\mathbb{C}^2} (\tilde{u}dd^c \tilde{v} - \tilde{v}dd^c \tilde{u}) \wedge dd^c \tilde{w}. \]
We apply Theorem 3.1 to the right-hand-side, multiplying by factors of $ab$ since $\bar{u}, \bar{v}, \bar{w} \in abL^+$, to obtain, with the aid of (2.11), the desired integral formula (cf., (6.3) in [9]):

$$\int_{C^2} (udd^c v - vdd^c u) \wedge dd^c w = (ab)^2 \int_{\mathbb{R}^1} (\rho_{\bar{u}/ab} - \rho_{\bar{v}/ab})(dd^c \rho_{\bar{w}/ab} + \omega). \quad (3.2)$$

Next, for $u, v \in L_C$, we define the mutual energy

$$E(u, v) := \int_{C^2} (u - v)[(dd^c u)^2 + dd^c u \wedge dd^c v + (dd^c v)^2].$$

(cf., (3.1) in [1]). We connect this notion with $C$–transfinite diameter by recalling the following formula from [1].

**Theorem 3.2.** Let $K \subset \mathbb{C}^2$ be compact and nonpluripolar. Then

$$\log \delta_C(K) = -\frac{1}{c} E(V_{C}^*, H_C)$$

where $c = 3!M_C$ with $M_C := \iint_C (x + y) dxdy$.

**Remark 3.3.** This formula is actually valid in $\mathbb{C}^d$ for $d > 1$ for any convex body $C \subset (\mathbb{R}^+)^d$ satisfying (2.1) with the appropriate definitions of $E$ and $c$.

Our goal in this section is to rewrite $E(V_{C}^*, H_C)$ using the integral formulas in order to get a formula relating $\delta_C(K)$ and $\rho_{V_{C,K,ab}}$ more in the spirit of Proposition 3.1 in [7]. This will be used in the next section to prove a formula for the $C$–transfinite diameter $\delta_C(K)$ of a product set $K = E \times F$.

**Proposition 3.4.** We have

$$E(V_{C,K}^*, H_C) = (ab)^2 [\int_{C^1} \rho_{V_{C,K,ab}}(1, t) dd^c \rho_{V_{C,K,ab}}(1, t) - \rho_{V_{C,K,ab}}(0, 1)].$$

Hence from Theorem 3.2

$$-3!M_C \log \delta_C(K) = (ab)^2 [\int_{C^1} \rho_{V_{C,K,ab}}(1, t) dd^c \rho_{V_{C,K,ab}}(1, t) - \rho_{V_{C,K,ab}}(0, 1)]. \quad (3.3)$$

**Proof.** Applying the formula (3.2) with $w = u$ and $w = v$ and adding, we obtain

$$\int_{C^2} (udd^c v - vdd^c u) \wedge dd^c (u + v) = (ab)^2 \int_{\mathbb{R}^1} (\rho_{\bar{u}/ab} - \rho_{\bar{v}/ab})[(dd^c \rho_{\bar{w}/ab} + \omega) + (dd^c \rho_{\bar{w}/ab} + \omega)].$$

We claim from the definition of $E(u, v)$, it follows that

$$E(u, v) = \int_{C^2} [(udd^c v)^2 - v(dd^c u)^2] + (ab)^2 \int_{\mathbb{R}^1} (\rho_{\bar{u}/ab} - \rho_{\bar{v}/ab})[(dd^c \rho_{\bar{u}/ab} + \omega) + (dd^c \rho_{\bar{v}/ab} + \omega)]. \quad (3.4)$$

To see this, using the previous formula it suffices to show

$$E(u, v) - \int_{C^2} u(dd^c u)^2 + \int_{C^2} v(dd^c v)^2 = \int_{C^2} (udd^c v - vdd^c u) \wedge dd^c (u + v).$$

In verifying this, all integrals are over $\mathbb{C}^2$. We write

$$E(u, v) = \int_{C^2} [(dd^c u)^2 + (dd^c v \wedge dd^c (u + v)]$$
To verify (3.5), we begin by observing that since
\[ \rho \]
In particular, since
\[ \rho \]
Thus
\[ \rho \]
We finish this proof by working with the sum of the last two integrals:
\[
\int (u - v)dd^c v \cup dd^c(u + v) + \int v[(dd^c v)^2 - (dd^c u)^2].
\]
so desired.
Letting \( u = V_{C,K_1} \) and \( v = V_{C,K_2} \) in (3.4) where \( K_1, K_2 \) are regular compact sets in \( \mathbb{C}^2 \),
\[
E(V_{C,K_1}, V_{C,K_2}) = (ab)^2 \int_{\mathbb{C}} (\rho_{V_{C,K_1}/ab} - \rho_{V_{C,K_2}/ab})[(dd^c \rho_{V_{C,K_1}/ab}) + (dd^c \rho_{V_{C,K_2}/ab}) + \omega].
\]
In particular, since \( H_C = V_{C,T^2} = V_{C,T^2} \) where \( T^2 \) is the unit torus in \( \mathbb{C}^2 \),
\[
E(V_{C,K_1}, H_C) = (ab)^2 \int_{\mathbb{C}} (\rho_{V_{C,K_1}/ab} - \rho_{H_C/ab})[(dd^c \rho_{V_{C,K_1}/ab}) + (dd^c \rho_{H_C/ab}) + \omega].
\]
The result will follow from (3.1) once we verify
\[
\int_{\mathbb{C}} \rho_{H_C/ab}(1, t) dd^c \rho_{H_C/ab}(1, t) + \rho_{H_C/ab}(0, 1) = 0. \tag{3.5}
\]
To verify (3.5), we begin by observing that since
\[
H_C(z_1, z_2) = \max(\log^+ |z_1|^b, \log^+ |z_2|^a), \quad H_C(z_1, z_2) := H_C(z_1^a, z_2^b),
\]
for \( (z_1, z_2) \in \mathbb{C}^2 \setminus (P^2)^o \),
\[
\rho_{H_C}(z_1^a, z_2^b) = H_C(z_1^a, z_2^b) = ab \rho_{H_C/ab}(z_1, z_2).
\]
In particular,
\[
\rho_{H_C/ab}(0, 1) = \frac{1}{ab} H_C(0, 1) = 0 \quad \text{and} \quad \rho_{H_C/ab}(1, t) = \frac{1}{ab} H_C(1, t^b) = \frac{1}{ab} \max(0, ab \log |t|).
\]
Thus \( dd^c \rho_{H_C/ab}(1, t) \) is supported on \( |t| = 1 \) where it is (normalized) arclength measure. On this set, we have \( \rho_{H_C/ab}(1, t) = 0 \) and (3.5) follows.
4 Product formula for $C$ a triangle

We first use (3.3) to prove a formula for the $C$–transfinite diameter of a product set when $C$ is a triangle $T_{a,b}$ with vertices $(0,0), (b,0), (0,a)$. Then in the next section we give a (conceptually) simpler proof that is valid for general convex bodies.

**Theorem 4.1.** Let $K = E \times F$ where $E, F \subset C$ are compact. Then for $C = T_{a,b}$,

$$-\log \delta_C(K) = \frac{ab}{a+b} \left( \frac{-\log D(E)}{a} + \frac{-\log D(F)}{b} \right); \ i.e.,$$

$$\delta_C(K) = D(E)^{b/(a+b)}D(F)^{a/(a+b)}$$

where $D(E), D(F)$ are the univariate transfinite diameters of $E, F$.

**Proof.** We first assume $a, b$ are positive integers and use Proposition 3.4. To this end, we compute $\rho_{C,K/ab}$ for $K = E \times F$. We can assume $E, F$ are regular compact sets in $\mathbb{C}$ and we let $\rho_E = -\log D(E)$ and $\rho_F = -\log D(F)$ be the Robin constants of these sets. From Proposition 2.4 of [5],

$$V_{C,K}(z_1, z_2) = \max \{bg_E(z_1), ag_F(z_2)\}$$

where $g_E, g_F$ are the Green functions for $E, F$. Note that

$$\rho_E = \lim_{|z_1| \to \infty} [g_E(z_1) - \log |z_1|] \text{ and } \rho_F = \lim_{|z_2| \to \infty} [g_F(z_2) - \log |z_2|].$$

Thus from Definition 2.3

$$\rho_{V_{C,K}}(z_1, z_2) = \limsup_{|\lambda| \to \infty} \max \left\{ b g_E(\lambda^a z_1), a g_F(\lambda^b z_2) \right\} - ab \log |\lambda|$$

$$= \max \left\{ b(\rho_E + \log |z_1|), a(\rho_F + \log |z_2|) \right\}$$

so that

$$\rho_{V_{C,K/ab}}(z_1, z_2) = \frac{1}{ab} \rho_{V_{C,K}}(z_1^{a}, z_2^{b}) = \max \left\{ \frac{1}{a}\rho_E + \log |z_1|, \frac{1}{b}\rho_F + \log |z_2| \right\}.$$ 

Hence

$$\rho_{V_{C,K/ab}}(1, t) = \max \left( \frac{1}{a}\rho_E, \frac{1}{b}\rho_F + \log |t| \right)$$

so that $dd^c \rho_{V_{C,K/ab}}(1, t)$ is normalized arclength measure on a circle where the value of the function $\rho_{V_{C,K/ab}}(1, t) = \frac{1}{a}\rho_E$. Finally, $\rho_{V_{C,K/ab}}(0, 1) = \frac{1}{b}\rho_F$ and the result when $a, b$ are positive integers follows from Proposition 3.4 since

$$(ab)^2 \int_C \rho_{V_{C,K/ab}}(1, t)dd^c \rho_{V_{C,K/ab}}(1, t) - \rho_{V_{C,K/ab}}(0, 1) = (ab)^2 \left( \frac{1}{a}\rho_E + \frac{1}{b}\rho_F \right)$$

and a calculation shows that $M_C = (ab/6)(a+b)$ so that $3!M_C = (ab)(a+b)$. If $a, b \in \mathbb{Q}$, the result follows from Proposition 2.1 finally, the general case when $a, b \in \mathbb{R}$ follows from Proposition 2.2. \qed
5 Product formula for general \( C \)

In this section, we give an alternate proof of Theorem 4.1 which is applicable in a much more general setting. We assume that \( C \) is a convex body satisfying (2.1) which is a lower set: whenever \((j_1, j_2) \in nC \cap \mathbb{N}^2\) we have \((k_1, k_2) \in nC \cap \mathbb{N}^2\) for all \(k_l \leq j_l\), \(l = 1, 2\). For example, the triangles \( T_{a,b} \) are lower sets. This proof is modeled on that of Bloom-Calvi in [4]. As in the previous section, we take \( K = E \times F \) where \( E, F \) are compact sets in \( \mathbb{C} \). Let \( \mu_E, \mu_F \) be Bernstein-Markov measures for \( E, F \): recall \( \nu \) is a Bernstein-Markov measure for \( E \) if for any \( \epsilon > 0 \), there exists a constant \( c_\epsilon \) so that

\[
\|p_n\|_K \leq c_\epsilon (1 + \epsilon)^n \|p_n\|_{L^2(\nu)}, \quad n = 1, 2, \ldots
\]

where \( p_n \) is any polynomial of degree \( n \). If \( E, F \) are regular, one can take, e.g., \( \mu_E \) and \( \mu_F \) to be the distributional Laplacians of the Green functions \( g_E \) and \( g_F \). Let \( \mu := \mu_E \otimes \mu_F \). Let \( \{p_j(z)\}_{j=0,1,2,\ldots} \) be monic orthogonal polynomials for \( L^2(\mu_E) \) and let \( \{q_k(w)\}_{k=0,1,2,\ldots} \) be monic orthogonal polynomials for \( L^2(\mu_F) \); then \( \{p_j(z)q_k(w)\}_{j,k=0,1,2,\ldots} \) are orthogonal in \( L^2(\mu) \). Using the grex ordering \( \prec \) on \( \mathbb{N}^2 \) and the lower set property of \( C \), it is easy to see that each \( L^2(\mu) \)–orthogonal polynomial \( p_j(z)q_k(w) \) is in a class \( M_{\nu}(\alpha) \) (recall (2.5)) where \( \alpha = (j, k) \) and \( l = \deg_C(z^jw^k) \). Here and below \( j, k \) are nonnegative integers.

We want to use (2.8): the asymptotics of \( V_n \) and \( \prod_{j=1}^{d_n} T_n(K, \alpha(j))^n \) are the same; i.e., the limits of their \( nd_n \)-th roots coincide. If \( \mu \) is a Bernstein-Markov measure on \( K \), it follows readily that one can replace the sup-norm minimizers \( T_{k}(K, \alpha) \) by \( L^2(\mu) \)–norm minimizers

\[
\widetilde{T}_{k}(K, \alpha) := \inf \{\|p\|_{L^2(\mu)} : p \in M_{\nu}(\alpha)\}^{1/k}.
\]

In our setting, for \( \alpha = (j, k) \) the polynomial \( p_j(z)q_k(w) \) is the minimizer and

\[
\|p_j q_k\|_{L^2(\mu)} = \|p_j\|_{L^2(\mu_E)} \cdot \|q_k\|_{L^2(\mu_F)}.
\]

Moreover, we know from the univariate theory that

\[
\lim_{j \to \infty} \|p_j\|_{L^2(\mu_E)}^{1/j} = D(E) \quad \text{and} \quad \lim_{k \to \infty} \|q_k\|_{L^2(\mu_F)}^{1/k} = D(F).
\]

For simplicity, we write

\[
p_j := \|p_j\|_{L^2(\mu_E)} \quad \text{and} \quad q_k := \|q_k\|_{L^2(\mu_F)}.
\]

In this notation, to utilize (2.8), we consider

\[
\left( \prod_{(j,k) \in nC} p_j q_k \right)^{1/nd_n}.
\]

We suppose that \((b, 0)\) and \((0, a)\) are extreme points of \( C \) and that the outer face \( F_C \) of \( C \); i.e., the portion of the topological boundary of \( C \) outside of the coordinate axes, can be written both as a graph \( \{(x, f(x)) : 0 \leq x \leq b\} \) and as a graph \( \{(g(y), y) : 0 \leq y \leq a\} \).

**Theorem 5.1.** Let \( K = E \times F \) where \( E, F \) are compact subsets of \( \mathbb{C} \). Then

\[
\delta_C(K) = D(E)^{A/(A+B)} \cdot D(F)^{B/(A+B)} \quad \text{ (5.1)}
\]

where \( A = \int_0^b u f(u)du \) and \( B = \int_0^a u g(u)du \). Hence for any convex body \( C \) with \( A = B \) we obtain

\[
\delta_C(K) = [D(E)D(F)]^{1/2}.
\]
Proof. The outer face $F_nC$ of $nC$ can be written as
\[
\{(x, y) : y = f_n(x) := nf(x/n), \ 0 \leq x \leq nb\} = \{(x, y) : x = g_n(y) := ng(y/n), \ 0 \leq y \leq na\}.
\]
Then the product $\prod_{(j, k) \in nC} q_j q_k$; i.e., the product over the integer lattice points in $nC$, is asymptotically given by
\[
\left( p_1^{f_n(1)/n} p_2^{f_n(2)/n} \cdots p_{nb}^{f_n(nb)/n} q_1^{g_n(1)/n} q_2^{g_n(2)/n} \cdots q_{na}^{g_n(na)/n} \right)^n.
\]
For simplicity in the calculation, we concentrate on the product
\[
p_1^{f_n(1)/n} p_2^{f_n(2)/n} \cdots p_{nb}^{f_n(nb)/n}
\]
Then using the fact that $p_j \asymp D(E)^j$,
\[
p_1^{f_n(1)/n} p_2^{f_n(2)/n} \cdots p_{nb}^{f_n(nb)/n} \asymp D(E)^{f_n(1)/n+2f_n(2)/n+\cdots+nf_n(nb)/n} = D(E)^{n^2 f_n(nb)/n} \asymp D(E)^{f_n(1)+2f_n(2)+\cdots+nb f_n(nb)/n} = D(E)^{n^2 \int_0^b u f(u) du}.
\]
Similarly, since $q_k \asymp D(F)^k$, we have
\[
q_1^{g_n(1)/n} q_2^{g_n(2)/n} \cdots q_{na}^{g_n(na)/n} \asymp D(F)^{n^2 \int_0^a u g(u) du}.
\]
Hence
\[
\prod_{(j, k) \in nC} p_j q_k \asymp D(E)^{n^3 \int_0^b u f(u) du} \cdot D(F)^{n^3 \int_0^a u g(u) du}.
\]
Using (2.8) and (2.9), since
\[
nd_n A_C \asymp \frac{n^2 \text{area}(C)}{\text{area}(C)} \cdot \frac{M_C}{\text{area}(C)} = n^3 M_C,
\]
and
\[
M_C = \iint_C x \, dy \, dx + \iint_C y \, dx \, dy = \int_0^b \int_0^{f(x)} x \, dy \, dx + \int_0^a \int_0^{g(y)} y \, dx \, dy
\]
\[
= \int_0^b x f(x) \, dx + \int_0^a y g(y) \, dy = A + B,
\]
(5.1) follows. \hfill \square

Remark 5.2. Note that $A = B$ occurs whenever $a = b$ and $f = g$; i.e., the convex body is symmetric about the line $y = x$. As special cases of this, we can take
\[
C = C_p := \{(x, y) : x, y \geq 0, \ x^p + y^p \leq 1\}, \ 1 \leq p < \infty \quad (5.2)
\]
as well as
\[
C_\infty := \{(x, y) : 0 \leq x, y \leq 1\}.
\]
Remark 5.3. We can use Theorem 5.1 to verify our result in Theorem 4.1 for \( C = T_{a,b} \). Here \( y = f(x) = a(1 - x/b) \), \( 0 \leq x \leq b \) and \( x = g(y) = b(1 - y/a) \), \( 0 \leq y \leq a \). Then

\[
\int_0^b x f(x) \, dx = \frac{ab^2}{6}; \quad \int_0^a y g(y) \, dy = \frac{ba^2}{6};
\]

and

\[
M_{T_{a,b}} = \int_0^b \int_0^{a(1-x/b)} (x+y) \, dy \, dx = (ab/6)(a+b).
\]

Hence

\[
\delta_{T_{a,b}}(K) = D(E)^{b/(a+b)} D(F)^{a/(a+b)}.
\]

Moreover, the calculations in Theorem 5.1 — and the resulting formula — are valid (and much easier) in a special case where \( F_C \) cannot be written as a graph \( \{ (x,f(x)) : 0 \leq x \leq b \} \) (nor as a graph \( \{ (g(y),y) : 0 \leq y \leq a \} \)). Namely, the rectangle \( C = R_{a,b} \) with vertices \( (0,0) \), \( (b,0) \), \( (0,a) \) and \( (b,a) \). Here we take \( y = f(x) = a \), \( 0 \leq x \leq b \) and \( x = g(y) = b \), \( 0 \leq y \leq a \) and the calculations in the proof of Theorem 5.1 yield

\[
\int_0^b x f(x) \, dx = \frac{ab^2}{2}; \quad \int_0^a x y g(y) \, dy = \frac{ba^2}{2};
\]

and

\[
M_{R_{a,b}} = \int_0^b \int_0^a (x+y) \, dy \, dx = \frac{ab}{2}(a+b).
\]

Hence, for this rectangle we recover the same product formula as for \( T_{a,b} \):

\[
\delta_{R_{a,b}}(K) = D(E)^{b/(a+b)} D(F)^{a/(a+b)}.
\]

6 The case of \( d \)-circled sets

One might wonder, given Remark 5.2, whether we always have equality of \( \delta_C(K) \) for all convex bodies \( C \) that are symmetric about the line \( y = x \) (e.g., \( C_p \) for \( 1 \leq p \leq \infty \)), i.e., for any compact set \( K \), not just product sets. This is not the case as we will illustrate for

\[
\mathbb{B} := \{(z_1, z_2) : |z_1|^2 + |z_2|^2 \leq 1\},
\]

the closed Euclidean unit ball in \( \mathbb{C}^2 \). This is an example of a 2-\textit{circled} set. We say a set \( E \subset \mathbb{C}^d \) is \( d \)-\textit{circled} if

\[
(z_1, ..., z_d) \in E \text{ implies } (e^{i\beta_1} z_1, ..., e^{i\beta_d} z_d) \in E, \text{ for all real } \beta_1, ..., \beta_d.
\]

For a compact, \( d \)-circled set \( K \), it is easy to see from the Cauchy estimates that

\[
\inf\{\|p\|_K : p \in M_k(\alpha)\} = \|z^\alpha\|_K
\]

where recall

\[
M_k(\alpha) := \{p \in \text{Poly}(k\mathbb{C}) : p(z) = z^\alpha + \sum_{\beta \in k\mathbb{N}^d, \beta < \alpha} c_\beta z^\beta\}.
\]
Then for any convex body $C$ satisfying (2.1), and any $\theta = (\theta_1, \ldots, \theta_d) \in C^o$, we have
\[
\tau_C(K, \theta) := \lim_{k \to \infty, \alpha/k \to \theta} T_k(K, \alpha) = \lim_{k \to \infty, \alpha/k \to \theta} \|z^\alpha\|_K^{1/k} = \max_{z \in K} |z_1|^{\theta_1} \cdots |z_d|^{\theta_d}.
\]

Thus, for a given $d$–circled set $K$, if we can explicitly determine these values, we can use (2.4) to compute $\delta_C(K)$.

Indeed, an elementary calculation for $K = \mathbb{B} \subset \mathbb{C}^2$ shows that
\[
\tau_C(\mathbb{B}, \theta) = \left(\frac{\theta_1}{\theta_1 + \theta_2}\right)^{\theta_1/2} \left(\frac{\theta_2}{\theta_1 + \theta_2}\right)^{\theta_2/2}.
\]

It follows readily from (2.4) that $\delta_C(\mathbb{B}) = e^{-1/4}$.

We next show that the main result in [12], specifically, equation (2.4) in our Section 1, remains valid even for certain nonconvex sets $C$ and all $d$–circled sets $K$. To this end, let $C \subset (\mathbb{R}^+)^d$ be the closure of an open, connected set satisfying (2.1). As examples, one can take $C_p$ as in [52] for $0 < p < 1$. Here, the definition of
\[
\text{Poly}(nC) = \{p(z) = \sum_{J \in nC \cap \mathbb{N}^d} c_J z^J, \ c_J \in \mathbb{C}\}, \quad n = 1, 2, \ldots
\]

makes sense; and we have $\text{Poly}(nC) = \text{span}\{e_1, \ldots, e_{d_n}\}$ where $e_j(z) := z^{\alpha(j)}$ are the standard basis monomials in $\text{Poly}(nC)$ and $d_n$ is the dimension of $\text{Poly}(nC)$. Using the same notation
\[
VDM(\zeta_1, \ldots, \zeta_{d_n}) := \det[e_i(\zeta_j)]_{i,j=1,\ldots,d_n}
\]
as in the convex setting, for a compact and $d$–circled set $K \subset \mathbb{C}^d$, we have the same notions of maximal Vandermonde $V_n = V_n(K)$; $C$–transfinite diameter $\delta_C(K)$; “monic” polynomial classes $M_k(\alpha)$ and corresponding Chebyshev constants $T_k(K, \alpha)$; and directional Chebyshev constants $\tau_C(K, \theta)$ for $\theta \in C^o$ as in (2.2), (2.3), (2.5), (2.6) and (2.7).

**Proposition 6.1.** For $C \subset (\mathbb{R}^+)^d$ the closure of an open, connected set satisfying (2.1) and for any $d$–circled set $K \subset \mathbb{C}^d$, we have:

1. for $\theta \in C^o$,
\[
\tau_C(K, \theta) := \lim_{k \to \infty, \alpha/k \to \theta} T_k(K, \alpha),
\]
i.e., the limit exists; and

2. $\lim_{n \to \infty} V_n^{1/n d_n}$ exists and equals $\lim_{n \to \infty} [\prod_{j=1}^{d_n} T_n(K, \alpha(j))^n]^{1/n d_n}$;

3. $\delta_C(K) = [\exp \left(\frac{1}{\text{vol}(C)} \int_{C^o} \log \tau_C(K, \theta) d\mu(\theta)\right)]^{1/A_C}$ where $A_C$ is a positive constant defined in (2.9).

**Proof.** Because $\inf\{\|p\|_K : p \in M_k(\alpha)\} = \|z^\alpha\|_K$, all the arguments in Lemmas 4.4 and 4.5 of [12] work to show
\[
\prod_{j=1}^{d_n} T_n(K, \alpha(j))^n \leq V_n \leq d_n! \cdot \prod_{j=1}^{d_n} T_n(K, \alpha(j))^n.
\]

The only other ingredients needed to complete the rest of the proof are simply to observe that even though the polynomial classes $M_k(\alpha)$ are not submultiplicative, the monomials $z^\alpha$ themselves are; i.e., $z^\alpha z^\beta = z^{\alpha + \beta} \in M_k(\alpha + \beta)$. This is all that is needed to show 1.; then the proof in [12] gives 2. and 3. 

\[\square\]
From the general formula

\[ \tau_C(K, \theta) = \max_{z \in K} |z_1|^\theta_1 \cdots |z_d|^\theta_d \]

for any \( d \)-circled set \( K \subset \mathbb{C}^d \),

\[ \delta_C(K) = \left( \exp \left( \frac{1}{\text{vol}(C)} \int_{C^o} \log \left( \max_{z \in K} |z_1|^\theta_1 \cdots |z_d|^\theta_d \right) dm(\theta) \right) \right)^{1/AC}. \]

Using (6.1), for \( K = B \subset \mathbb{C}^2 \),

\[ \delta_C(B) = \left( \exp \left( \frac{1}{\text{area}(C)} \int_{C^o} \log \left( \left( \frac{\theta_1}{\theta_1 + \theta_2} \right)^{\theta_1/2} \left( \frac{\theta_2}{\theta_1 + \theta_2} \right)^{\theta_2/2} \right) dm(\theta) \right) \right)^{1/AC}. \] (6.2)

Note that for \( C = C_p \) this gives a formula for the \( C_p \)-transfinite diameter of the ball \( B \) in \( \mathbb{C}^2 \) valid for all \( 0 < p \leq \infty \). We return to this approach to computing \( C \)-transfinite diameter using directional Chebyshev constants in Proposition 6.5.

We can also use orthogonal polynomials as in Section 5 to compute \( \delta_C(B) \) for general \( C \) as in Proposition 6.1 this we do next.

**Proposition 6.2.** For \( C \) as in Proposition 6.1, the \( C \)-transfinite diameter of the ball \( B \) in \( \mathbb{C}^2 \) is equal to

\[ \delta_C(B) = \exp \left( \frac{1}{4} \left( I_1 + I_2 - I_3 - \frac{\log 2\pi}{2} \text{area}(C) \right) \right), \] (6.3)

where

\[ I_1 = \iiint_{C \times [0,1]} \log \Gamma(x+z) dx dy dz, \quad I_2 = \iiint_{C \times [0,1]} \log \Gamma(y+z) dx dy dz, \] (6.4)

and

\[ I_3 = \iiint_{C \times [0,1]} \log \Gamma(x+y+z) dx dy dz, \quad I_4 = M_C = \int_C (x+y) dx dy. \] (6.5)

**Proof.** Let \( \mu \) be normalized surface area on \( \partial B \). Then the monomials \( z^a w^b \), \( a, b \) nonnegative integers, are orthogonal and

\[ \|z^a w^b\|^2_{L^2(\mu)} = \frac{a! b!}{(a+b+1)!}, \]

see [11] Propositions 1.4.8 and 1.4.9. Let us estimate

\[ Q_n = \log \prod_{(a,b) \in nC} \frac{a! b!}{(1+a+b)!}. \]

We have

\[ \log \prod_{(a,b) \in nC} a! = \sum_{(a,b) \in nC} \log \Gamma(a+1). \]

Recall the multiplication formula for the Gamma function. For \( \text{Re}(z) > 0 \), we have

\[ \Gamma(nz) = (2\pi)^{(1-n)/2} n^{(2nz-1)/2} \Gamma(z) \Gamma \left( z + \frac{1}{n} \right) \Gamma \left( z + \frac{2}{n} \right) \cdots \Gamma \left( z + \frac{n-1}{n} \right). \]
Applying the formula with \( z = (a + 1)/n \), we get

\[
\log \prod_{(a,b) \in nC} a! = \sum_{(a,b) \in nC} \sum_{k=1}^{n} \log \Gamma \left( \frac{a + k}{n} \right) + \sum_{(a,b) \in nC} \frac{1 - n}{2} \log 2\pi + \sum_{(a,b) \in nC} \frac{2a - 1}{2} \log n.
\]

Recalling that \( d_n \), the number of elements of \( nC \cap \mathbb{N}^2 \), is the dimension of \( \text{Poly}(nC) \),

\[
\log \prod_{(a,b) \in nC} a! = \sum_{(a,b) \in nC} \sum_{k=1}^{n} \log \Gamma \left( \frac{a + k}{n} \right) - nd_n \log 2\pi + \frac{d_n}{2} \log 2\pi + n \log n \sum_{(a,b) \in nC} \left( \frac{a}{n} \right) - \frac{d_n}{2} \log n.
\]

Interpreting the sums over the pairs \( (a,b) \) as Riemann sums, we get

\[
\sum_{(a,b) \in nC} \sum_{k=1}^{n} \log \Gamma \left( \frac{a + k}{n} \right) = n^3 I_1 + \mathcal{O}(n^2), \quad \sum_{(a,b) \in nC} \left( \frac{a}{n} \right) = n^2 I_2 + \mathcal{O}(n)
\]

with \( I_1 \) and \( I_2 \) given in (6.4). Together with the estimate \( d_n = n^2 \text{area}(C) + \mathcal{O}(n) \), we get

\[
\log \prod_{(a,b) \in nC} a! = (n^3 \log n) I_5 + n^3 \left( I_1 - \frac{\log 2\pi}{2} \text{area}(C) \right) + \mathcal{O}(n^2 \log n)
\]

where \( I_5 = \int_C x \, dx \, dy \). Similarly,

\[
\log \prod_{(a,b) \in nC} b! = (n^3 \log n) I_6 + n^3 \left( I_1 - \frac{\log 2\pi}{2} \text{area}(C) \right) + \mathcal{O}(n^2 \log n)
\]

where \( I_6 = \int_C y \, dx \, dy \). Moreover,

\[
\log \prod_{(a,b) \in nC} (1 + a + b)! = \sum_{(a,b) \in nC} \sum_{k=2}^{n+1} \log \Gamma \left( \frac{a + b + k}{n} \right) + \sum_{(a,b) \in nC} \frac{1 - n}{2} \log 2\pi + \sum_{(a,b) \in nC} \frac{2a + 2b + 3}{2} \log n.
\]

\[
= \sum_{(a,b) \in nC} \sum_{k=2}^{n+1} \log \Gamma \left( \frac{a + b + k}{n} \right) + \left( \frac{1 - n}{2} \right) d_n \log 2\pi + \frac{3}{2} d_n \log n + n \log n \sum_{(a,b) \in nC} \left( \frac{a + b}{n} \right) - \frac{3}{2} d_n \log n.
\]

\[
= (n^3 \log n) I_5 + n^3 \left( I_1 + I_2 - I_3 - \frac{\log 2\pi}{2} \text{area}(C) \right) + \mathcal{O}(n^2 \log n)
\]

with \( I_3 \) and \( I_4 \) given in (6.5). Hence,

\[
\mathcal{Q}_n = n^3 \left( I_1 + I_2 - I_3 - \frac{\log 2\pi}{2} \text{area}(C) \right) + \mathcal{O}(n^2 \log n).
\]

Now,

\[
\log \delta_{C}(\mathbb{B}) = \lim_{n \to \infty} \frac{\text{area}(C)}{2nd_n M_C} \mathcal{Q}_n
\]
Figure 1: $\log \delta_{C_p}(B)$ as a function of $p$. When $p$ goes to $\infty$, $\log \delta_{C_p}(B)$ tends to $(1 - 4 \log 2)/6 \simeq -0.295$.

where $d_n = \dim \text{Poly}(nC) \simeq n^2 \text{area}(C)$ and

$$M_C = \iint_C (x + y)dxdy = I_4.$$ 

Hence

$$\delta_C(B) = \exp \left( \frac{1}{2I_4} \left( I_1 + I_2 - I_3 - \frac{\log 2\pi}{2} \text{area}(C) \right) \right),$$

which is (6.3).

**Remark 6.3.** We have

$$\log \delta_{C_p}(B) = \lim_{n \to \infty} \frac{\text{area}(C_p)}{2nd_nM_p} Q_{n,p}$$

where $d_n = \dim \text{Poly}(nC_p) \simeq n^2 \text{area}(C_p)$ and

$$M_{C_p} = \iint_{C_p} (x + y)dxdy = 2I_2.$$ 

Hence

$$\delta_{C_p}(B) = \exp \left( \frac{3p}{4B(1/p, 2/p)} \left( 2I_1 - I_3 - \frac{\log 2\pi}{4p} B(1/p, 1/p) \right) \right),$$

where $B(x, y)$ denotes the Beta function.

**Remark 6.4.** The integrals $I_1$, $I_2$ and $I_3$ can be simplified, eliminating the Gamma function from the integrand. We illustrate this with $I_1$. To this end, let

$$F(x) := \int_0^1 \log \Gamma(x + z)dz.$$ 

Then

$$F'(x) = \int_0^1 \frac{\Gamma'(x + z)}{\Gamma(x + z)}dz = \log \Gamma(x + 1) - \log \Gamma(x) = \log x.$$
Thus $F(x) = x(\log x - 1) + c$ and it follows from the Raabe integral of the Gamma function that $c = \int_0^1 \log \Gamma(z) \, dz = \frac{1}{2} \log 2\pi$. Hence

$$I_1 = \iint_C F(x) \, dxdy = \iint_C \left( x(\log x - 1) + \frac{1}{2} \log 2\pi \right) \, dxdy.$$ 

In a similar fashion,

$$I_2 = \iint_C F(y) \, dxdy = \iint_C \left( y(\log y - 1) + \frac{1}{2} \log 2\pi \right) \, dxdy$$

and

$$I_3 = \iint_C F(x + y) \, dxdy = \iint_C \left( (x + y)[\log (x + y) - 1] + \frac{1}{2} \log 2\pi \right) \, dxdy.$$ 

Using these relations and (2.9), we recover (6.2).

Making use of (6.2), we get the following result for the case of $C = C_p$, $0 \leq p < \infty$.

**Proposition 6.5.** We have

$$\log \delta_{C_p}(B) = \frac{3p}{2B(1/p, 2/p)} \left( \iint_{C_p} x \log x \, dxdy - \iint_{C_p} x \log(x + y) \, dxdy \right),$$

where $B(x, y)$ denotes the Beta function. In particular, for $p = 1$, $p = 2$ and $p = \infty$ we get

$$\delta_{C_1}(B) = e^{-1/4}, \quad \delta_{C_2}(B) = \sqrt{2}(\sqrt{2} - 1)^{1/\sqrt{2}}, \quad \delta_{C_\infty}(B) = 2^{-2/3}e^{1/6}.$$  

**Proof.** One has

$$\text{area}(C_p) = \iint_{C_p} \, dxdy = \frac{1}{2p} B(1/p, 1/p), \quad I_2 = \frac{1}{3p} B(1/p, 2/p),$$

and the given formula follows. The particular values for $p = 1, 2, \infty$ follow from computing the two integrals for these cases. \qed

**7 Final remarks**

As noted in [9], the results given here in sections 2 and 3 on $C$–Robin functions and $C$–transfinite diameter for triangles $C$ in $\mathbb{R}^2$ with vertices $(0, 0), (b, 0), (0, a)$ where $a, b$ are relatively prime positive integers should generalize to the case of a simplex $C$ which is the convex hull of points $\{(0, \ldots, 0), (a_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_d)\}$ in $(\mathbb{R}^+)^d$ with $a_1, \ldots, a_d$ pairwise relatively prime (using the appropriate definition of the $C$–Robin function as defined in Remark 4.5 of [9]). For a product set $K = E_1 \times \cdots \times E_d$ in $\mathbb{C}^d$ where $E_j$ are compact sets in $\mathbb{C}$, Proposition 2.4 of [5] gives that

$$V_{C,K}(z_1, \ldots, z_d) = \max[a_1 g_{E_1}(z_1), \ldots, a_d g_{E_d}(z_d)]$$

where $g_{E_j}$ is the Green function for $E_j$. Hence a generalization of Theorem 4.1 will follow. However, unlike the standard ($C = \Sigma$) case, there is no known nor natural way to express a formula for the $C$–extremal function of a product set $K$ when not all of the component sets are planar compacta; e.g., in the simplest such case, $K = E \times F \subset \mathbb{C}^3$ with $E \subset \mathbb{C}^2$ and $F \subset \mathbb{C}$. Nevertheless, it seems that the techniques adopted in sections 5 and 6 using orthogonal polynomials and/or restricting to $d$–circled sets could likely be utilized to find more general product formulas for $C$–transfinite diameters.
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