On Solutions to the Wave Equation on Non-globally Hyperbolic Manifold

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Abstract

We consider the Cauchy problem for the wave equation on a non-globally hyperbolic manifold of the special form (Minkowski plane with a handle) containing closed timelike curves (time machines). We prove that the classical solution of the Cauchy problem exists and is unique if and only if the initial data satisfy to some set of additional conditions.
1 Introduction

There is currently a quite developed theory of Cauchy problem for hyperbolic equations on globally hyperbolic manifolds [1]–[4]. *Globally hyperbolic manifold* is a spacetime oriented with respect to time (i.e., a pair \((M, g)\), where \(M\) is a smooth manifold and \(g\) is the Lorentz metric) if \(M\) is diffeomorphic to \(\mathbb{R}^1 \times \Sigma\), where \(\Sigma\) is a Cauchy surface. This definition is equivalent to Lerays definition of global hyperbolicity [3, 5].

Hyperbolic equations on non-globally hyperbolic spacetimes have been studied considerably less, although numerous examples of such spacetimes are described by well-known solutions of gravitation field equations; such are the solutions of Gödel, Kerr, Gott and many others [5]–[18]. These manifolds contain closed timelike curves (time machine) and are non-globally hyperbolic.

Elementary examples of non-globally hyperbolic spacetimes are \(S^1_t \times \mathbb{R}^3_x\), with the Minkowski metric in which time argument passes a circle, and also anti-de Sitter space. There are several papers where the hyperbolic equations on non-globally hyperbolic manifolds were discussed [19]–[21].

Our purpose here is to study the wave equation on a manifold containing closed time-like curves (CTC). We consider the Minkowski plane with two slits whose edges are glued in a specific manner (plane with a handle). In paper [22] the Cauchy problem for the wave equation on the Minkowski plane with handle was considered and it was proved that there exists a solution, which is generally discontinuous on the characteristics emerging from the conical points.

In this work we establish the necessary and sufficient conditions on the initial data for existence and uniqueness of the classical (i.e. smooth) solution to the Cauchy problem in the half-plane \(t \geq 0\) with exception of slits.

Our motivation is related with studying the possibility of creating “wormholes” and non-globally hyperbolic regions (mini time machines) in collisions of the high-energy particles [23], also see [21].

Formation of CTC is related with the violation of the null energy condition [25].

Problems of boundary control for wave equation are considered in [26]. Nonstandard boundary conditions for dynamic equations are discussed in [27, 28].

We use in this work the following method to obtain the classical solution: we divide upper half-plane into 7 regions \(D_1, ..., D_7\), write out the general
solutions of wave equation in each of these regions and then we try to satisfy the gluing conditions and initial data, solving a certain system of linear equations. The specific conditions on the initial data (12)–(17) appear in this case. Thus we obtain the classical solution to the wave equation on the plane with handle (theorem 3.1). We also give another method to solving problem using theory of distributions (theorem 5.3). The results obtained by these two methods are equivalent.

In this work we consider boundary value problems for the wave equation on the Minkowski plane with the handle. It would be interesting to establish relationship with the known theory of the boundary value problems for Laplace operator on the Riemann surfaces, see for example, [29, 30].

2 Setting the problem

We consider two vertical intervals \( \gamma_1 \) and \( \gamma_2 \) with length \( \ell > 0 \) in a half-plane \( \mathbb{R}_2^+ = \{(x, t) \in \mathbb{R}^2| t > 0 \} \):

\[
\gamma_1 = \{(x, t) \in \mathbb{R}_2^+| x = a_1, \ b_1 < t < b_1 + \ell \}, \quad (1)
\]

\[
\gamma_2 = \{(x, t) \in \mathbb{R}_2^+| x = a_2, \ b_2 < t < b_2 + \ell \} \quad (2)
\]

We suppose that

\[
a_2 > a_1, \ b_2 > b_1 + \ell + a_2 - a_1. \quad (3)
\]

We assume that the edges of the intervals are glued as it is shown in Fig.1. The resulting manifold has two conic points – the ends of the intervals.

Every continuous field on this manifold will satisfy certain gluing conditions on the slits \( \gamma_1 \) and \( \gamma_2 \). Conversely, if the field is continuous in domain \( \Omega = \mathbb{R}_2^+ \setminus \bar{\gamma}_1 \cup \bar{\gamma}_2 \) and satisfies those gluing conditions then it is continuous on the manifold.

Consider the wave equation on that manifold for the function \( u = u(x, t) \)

\[
u_{tt} - u_{xx} = 0, \ (x,t) \in \Omega \quad (4)
\]

with initial conditions

\[
u(x,0) = \varphi(x), \quad (5)
\]

\[
u_t(x,0) = \psi(x), \quad (6)
\]
Figure 1: Minkowski plane with two slits glued in a specific way: the “inner” edges of the slits are glued one with another, while the “outer” edges of the slits are glued with each other. The identifications of the “outer” and “inner” edge points are shown by arrows. There is also drawn a light cone emerging out of the point $S_1$ with coordinates $(a_1, b_1)$. We assume that the vector $I$ generating identifications is time-like. The point $S_2$ has coordinates $(a_2, b_2)$.

where $\varphi \in C^2(\mathbb{R})$, $\psi \in C^1(\mathbb{R})$. Let us set the following gluing conditions:

\begin{align}
  u(a_1 - 0, t) &= u(a_2 + 0, t + b_2 - b_1), \quad (7) \\
  u(a_1 + 0, t) &= u(a_2 - 0, t + b_2 - b_1), \quad (8) \\
  u_x(a_1 - 0, t) &= u_x(a_2 + 0, t + b_2 - b_1), \quad (9) \\
  u_x(a_1 + 0, t) &= u_x(a_2 - 0, t + b_2 - b_1), \quad (10)
\end{align}

where $b_1 < t < b_1 + \ell$ and the indicated limits exist. We will show below that no extra conditions needed.

Let us define the classical solution:

**Definition 2.1.** A function $u \in C^2(\Omega) \cap C^1(\Omega \cup \{t = 0\})$ is called the classical solution to the problem (4)–(10) if it satisfies conditions (4)–(10), provided the indicated left- and right-hand side limits exist.

Using characteristic half-lines emerging out of the ends of the intervals $\tilde{\gamma}_1, \tilde{\gamma}_2$, 

\[ \text{Figure 1: Minkowski plane with two slits glued in a specific way: the “inner” edges of the slits are glued one with another, while the “outer” edges of the slits are glued with each other. The identifications of the “outer” and “inner” edge points are shown by arrows. There is also drawn a light cone emerging out of the point $S_1$ with coordinates $(a_1, b_1)$. We assume that the vector $I$ generating identifications is time-like. The point $S_2$ has coordinates $(a_2, b_2)$.} \]
we divide the upper half-plane $\mathbb{R}^2_+$ into 7 simply connected domains $D_1, ..., D_7$ (see Fig.2):

$D_1 : x > a_i, a_i - b_i - l < x - t < a_i - b_i,$
$D_{i+2} : x < a_i, a_i + b_i < x + t < a_i + b_i + l, \ i = 1, 2,$
$D_5 : \ 0 < t < |x - a_1| + b_1,$
$D_6 : \ |x - a_1| + b_1 + l < t < |x - a_2| + b_2,$
$D_7 : \ t > |x - a_2| + b_2 + l.$

It will be shown that if the initial data meet certain conditions, then the classical solution to the problem (11) exists, is unique, and is of the form

$$u(x,t) = f(x-t+A_i) + g(x+t+B_i) + C_i \quad (11)$$

in each domain $D_i, \ i = 1, ..., 7,$ where $A_i, B_i, C_i$ are constants and $f, g$ are functions defined by d’Alembert’s formulae.

3 Theorem of existence of classical solution

Hereafter we will use notations:

$$a = a_2 - a_1, \quad b = b_2 - b_1,$$

and

$$c_i = a_i - b_i, \quad d_i = a_i + b_i.$$
where \( i \) can be 1, 2 or empty (in particular, \( c = a_2 - a_1 - b_2 + b_1 \)).

We will prove that the existence of a classical solution is equivalent to fulfilling the conditions
\[
\varphi(c_2 - \ell) - \varphi(c_1 - \ell) + \int_{c_2 - \ell}^{c_1 - \ell} \psi(s) \, ds = \varphi(d_1) - \varphi(d_2) - \int_{d_1}^{d_2} \psi(s) \, ds, \tag{12}
\]
\[
\varphi(d_2 + \ell) - \varphi(d_1 + \ell) + \int_{d_1 + \ell}^{d_2 + \ell} \psi(s) \, ds = \varphi(c_1) - \varphi(c_2) - \int_{c_2}^{c_1} \psi(s) \, ds. \tag{13}
\]

and conditions of smoothness on characteristics
\[
\begin{align*}
\varphi^{(i)}(c_1) - \psi^{(i-1)}(c_1) &= \varphi^{(i)}(c_2) - \psi^{(i-1)}(c_2) \tag{14} \\
\varphi^{(i)}(d_1) + \psi^{(i-1)}(d_1) &= \varphi^{(i)}(d_2) + \psi^{(i-1)}(d_2) \tag{15} \\
\varphi^{(i)}(c_1 - \ell) - \psi^{(i-1)}(c_1 - \ell) &= \varphi^{(i)}(c_2 - \ell) - \psi^{(i-1)}(c_2 - \ell) \tag{16} \\
\varphi^{(i)}(d_1 + \ell) + \psi^{(i-1)}(d_1 + \ell) &= \varphi^{(i)}(d_2 + \ell) + \psi^{(i-1)}(d_2 + \ell), \quad i = 1, 2, \tag{17}
\end{align*}
\]

Namely, the following theorem holds.

**Theorem 3.1.** The classical solution to the problem (4)–(10) exists if and only if the conditions (12)–(17) for \( \varphi, \psi \) hold. Given this, the classical solution is unique and is given by the formula
\[
u(x, t) = u_i(x, t) \quad \text{if} \quad (x, t) \in D_i, \quad i = 1, \ldots, 7, \tag{18}\]

where
\[
\begin{align*}
u_1(x, t) &= f(\eta + c) + g(\xi) + f(c_1) - f(c_2) \tag{19} \\
u_2(x, t) &= f(\eta - c) + g(\xi + d) + g(d_1) - g(d_2) \tag{20} \\
u_3(x, t) &= f(\eta) + g(\xi + d) + g(d_1) - g(d_2) \tag{21} \\
u_4(x, t) &= f(\eta) + g(\xi - d) + f(c_1) - f(c_2) \tag{22} \\
u_5(x, t) &= f(\eta) + g(\xi) \tag{23} \\
u_6(x, t) &= f(\eta) + g(\xi) + g(d_1) - g(d_2) + f(c_1) - f(c_2) \tag{24} \\
u_7(x, t) &= f(\eta) + g(\xi); \tag{25}
\end{align*}

here \( \xi = x + t, \ \eta = x - t, \)
\[
f(x) = \frac{1}{2} \left[ \varphi(x) - \int_{x_0}^{x} \psi(s) \, ds \right] \tag{26}
\]
and
\[ g(x) = \frac{1}{2} \left[ \varphi(x) + \int_{x_0}^x \psi(s) \, ds \right]. \] (27)

Proof. An arbitrary solution to equation \((\text{I})\) in domain \(D_i, i = 1, \ldots, 7\) is given by
\[ u(x, t) = f_i(x - t) + g_i(x + t), \quad \text{when } (x, t) \in D_i, \ i = 1, \ldots, 7, \]
where \(f_i(x - t), \ g_i(x + t) \in C^2(D_i), \ i = 1, \ldots, 7\). We will show that conditions \((\text{II})-\text{(I)}\) impose quite strong restrictions to \(f_i\) and \(g_i\).

Functions \(f_5\) and \(g_5\) are calculated directly from \(\varphi, \psi\) via the d’Alembert formulae:
\[ f_5(x) = f(x), \quad g_5(x) = g(x). \] (28)

From now on we will evaluate \(f_i, g_i\) through \(f, g\) in a manner to make the solution \(u\) twice continuously differentiable on \(\Omega\), including eight characteristics \(\Gamma_{ij} = \overline{D_i} \cap \overline{D_j} \cap \Omega\); here \(i, j\) take up such values from 1, \ldots, 7 that \(\Gamma_{ij}\) is open half-line. Let us write continuity conditions on \(\Gamma_{51}\), i.e.
\[ u_1 = u_5, \quad (x, t) \in \Gamma_{51}. \] (29)

Analytically half-line \(\Gamma_{51}\) is given by \(\{ x - a_1 = t - b_1 > 0 \}\). Thus we can write (29) as
\[ f_1(a_1 - b_1) + g_1(2t + a_1 - b_1) = f_5(a_1 - b_1) + g_5(2t + a_1 - b_1), \quad t > b_1. \]

Using our notations, we evaluate \(g_1\):
\[ g_1(2t + c_1) = g_5(2t + c_1) + G_{51}, \quad t > b_1, \]
where
\[ G_{51} = f_5(c_1) - f_1(c_1). \]

Therefore, we have defined function \(g_1(\xi)\) when \(\xi > a_1 + b_1\); thus it is also defined when \((x, t) \in D_1\); in addition, \(g_1(\xi)\) equals \(g(\xi)\) up to constant.

Similarly, using continuity conditions on \(\Gamma_{16}, \Gamma_{62}, \Gamma_{27}\) we get functions \(g_6, g_2, g_7\) defined when \(\xi\) is greater than \(d_1 + \ell, \ d_2, \ d_2 + \ell\) respectively and equal \(g(\xi)\) up to constant.

In a similar way, it is easy to show that functions \(f_3, \ f_6, \ f_4, \ f_7\) of \(\eta\) are defined when \(\eta > c_1, \ c_1 - \ell, \ c_2, \ c_2 - \ell\) and are equal to \(f(\eta)\) up to constants.
**Gluing conditions.** Now we apply the gluing conditions for functions (7):

\[ u_1(a_1, t) = u_4(a_2, t + b), \]

i. e.

\[ f_1(a_1 - t) + g_1(a_1 + t) = f_4(a_2 - (t + b)) + g_4(a_2 + t + b). \]  

(30)

And gluing conditions for derivatives are

\[ f_1'(a_1 - t) + g_1'(a_1 + t) = f_4'(a_2 - (t + b)) + g_4'(a_2 + t + b). \]  

(31)

Let us differentiate (30) on \( t \) and add it to (31). We obtain

\[ g_4'(a_1 + t) = g_4'(a_2 + t + b). \]

Thus,

\[ g_4(\xi) = g_1(\xi - a - b) + \text{const}. \]

Let us note that as this equation holds for

\[ a_2 + b_2 < \xi < a_2 + b_2 + \ell, \]

it defines \( g_4(x + t) \) for \((x, t) \in D_4\).

From (30) we obtain

\[ f_1(a_1 - t) = f_4(a_2 - t - b) + \text{const}. \]

Therefore, the function

\[ f_1(\eta) = f_4(\eta + a - b) + \text{const}, \]

is defined for all \( \eta = x - t \) when \((x, t) \in D_1\).

Finally, we have

\[ g_4(\xi) = g_1(\xi - d) + \text{const} \]

and

\[ f_1(\eta) = f_4(\eta + c) + \text{const}. \]

Similarly, using the gluing conditions for \( u_2 \) and \( u_3 \), we have

\[ g_3(\xi) = g_2(\xi + d) + \text{const} \]

and

\[ f_2(\eta) = f_3(\eta - c) + \text{const}. \]
Evaluating constants. We have obtained solution in the form

\begin{align*}
  u_1(x,t) &= f(\eta + c) + g(\xi) + U_1, \quad (32) \\
  u_2(x,t) &= f(\eta - c) + g(\xi) + U_2, \quad (33) \\
  u_3(x,t) &= f(\eta) + g(\xi + d) + U_3, \quad (34) \\
  u_4(x,t) &= f(\eta) + g(\xi - d) + U_4, \quad (35) \\
  u_5(x,t) &= f(\eta) + g(\xi), \quad (36) \\
  u_6(x,t) &= f(\eta) + g(\xi) + U_6, \quad (37) \\
  u_7(x,t) &= f(\eta) + g(\xi) + U_7. \quad (38)
\end{align*}

Now we have to find the constants \( U_i \).

It follows from (30) that

\[ U_1 = U_4; \]

similarly,

\[ U_2 = U_3. \]

Now we will find \( U_1 \) and \( U_2 \), by employing the continuity conditions for solution on the half-lines \( \Gamma_{51} \) and \( \Gamma_{53} \) respectively. We have \( \eta = c_1 \) on \( \Gamma_{51} \), so we can write

\[ u_1 = u_5 \]

as

\[ f(c_1 + c) + g(\xi) + U_1 = f(c_1) + g(\xi). \]

Recalling \( c_1 + c = c_2 \), we have

\[ U_1 = f(c_1) - f(c_2). \]

In a similar manner we get

\[ U_2 = g(d_1) - g(d_2). \]

Now we consider the half-lines \( \Gamma_{16} \) and \( \Gamma_{36} \). Continuity condition on \( \Gamma_{16} \) is written as

\[ f(c_1 - \ell + c) + g(\xi) + U_1 = f(c_1 - \ell) + g(\xi) + U_6, \]

wherefrom

\[ U_6 = f(c_2 - \ell) - f(c_1 - \ell) + f(c_1) - f(c_2). \]
Similarly, the continuity on $\Gamma_{36}$ is written as
\[ f(\eta) + g(d_1 + \ell + d) + U_2 = f(\eta) + g(d_1 + \ell) + U_6, \]
wherefrom, bearing in mind $d_1 + d = d_2$, we get
\[ U_6 = g(d_2 + \ell) - g(d_1 + \ell) + g(d_1) - g(d_2). \]

We have obtained the condition for the functions $f, g$:
\[ f(c_2 - \ell) - f(c_1 - \ell) + f(c_1) - f(c_2) = g(d_2 + \ell) - g(d_1 + \ell) + g(d_1) - g(d_2). \] (39)

As we will notice, we need two conditions for the continuous solution; the obtained condition will necessarily follow from those two.

So, let us consider the half-lines $\Gamma_{62}$ and $\Gamma_{64}$. We have $\eta = c_2$ on $\Gamma_{62}$. Let us insert it into
\[ u_6 = u_2. \]
We get
\[ f(c_2) + U_6 = f(c_2 - c) + U_2. \]
Inserting the found constants, we get
\[ f(c_2) + f(c_2 - \ell) - f(c_1 - \ell) + f(c_1) - f(c_2) = f(c_1) + g(d_1) - g(d_2). \]
Thus we have found the first condition:
\[ f(c_2 - \ell) - f(c_1 - \ell) = g(d_1) - g(d_2). \] (40)

If we express $f, g$ through $\varphi, \psi$, we will have exactly (12).

Consider $\Gamma_{64}$. We have $\xi = d_2$ on it; computing similarly, we obtain the second condition
\[ f(c_1) - f(c_2) = g(d_2 + \ell) - g(d_1 + \ell). \] (41)

Easy to see that if we add (40) to (41) we will obtain precisely the condition (39).

We are left to find the last constant $U_7$. We consider conditions on $\Gamma_{27}$: let us insert $\eta = c_2 - \ell$ into
\[ u_2|_{\Gamma_{27}}(\eta) = u_7|_{\Gamma_{27}}(\eta). \]
We obtain
\[ f(c_1 - \ell) + U_2 = f(c_2 - \ell) + U_7. \]

Recalling (40), we get
\[ U_7 = 0. \]

One can easily check that the continuity condition on \( \Gamma_{47} \) also yields zero \( U_7 \).

Hence, inserting obtained \( U_i \) into (32)–(38), we get the solution \( u \) given by (19)–(25).

**Differentiability conditions.** We will find the conditions for differentiability of the solutions on the half-lines \( \Gamma_{ij} \). The partial derivatives along half-lines \( \Gamma_{ij} \) exist, as it follows directly from the formulae (19)–(25). Let us write the conditions for continuity of partial derivatives of solution along normals to corresponding half-lines.

\[
\begin{align*}
    f^{(i)}(c_1) &= f^{(i)}(c_2) \quad (42) \\
    g^{(i)}(d_1) &= g^{(i)}(d_2) \quad (43) \\
    f^{(i)}(c_1 - \ell) &= f^{(i)}(c_2 - \ell) \quad (44) \\
    g^{(i)}(d_1 + \ell) &= g^{(i)}(d_2 + \ell), \quad i = 1, 2. \quad (45)
\end{align*}
\]

These conditions are equivalent to (14)–(17).

Theorem 3.1 is proved. ■

**3.1 Example**

We will discuss an example when all conditions of the theorem are satisfied, and thus, the classical solution exists. We will look for the solution of the right-mode form:
\[ u = f(x - t). \]

From \( g \equiv 0 \) it follows that we should pick such initial conditions:
\[ \psi = -\varphi'. \]

Then \( f = \varphi \). We choose as \( \varphi \) bump function with support in \([c_1 - \ell, c_1]\):
\[
\varphi(x) = \begin{cases} 
\exp \left( -\frac{\ell^2}{2(x-c_1+\ell/2)^2} \right), & x \in (c_1 - \ell, c_1), \\
0, & x \notin (c_1 - \ell, c_1)
\end{cases}
\]

Conditions (12)–(17) are fulfilled. The solution is right-travelling wave, coming into the lower slit and leaving out of the upper one.
4 Discontinuity jumps at slits

In the next section we will study the problem (4)–(10) by means of theory of distributions. We will generalize the method of analysis of the Cauchy problem from [4] to our case of plane with slits. Our method can be of interest in the analysis of generalized solutions of the problem concerned. Here we shall confine ourselves to study some properties of classical solutions of problem (4)–(10) in the “strengthened” setting.

We will use the following notations for the “one-sided” limits and discontinuity jumps of functions:

\[(x,t) \to (A-0,B) \iff (x,t) \to (A,B) \mid x < A\]
\[(x,t) \to (A+0,B) \iff (x,t) \to (A,B) \mid x > A\]

\[F(x,t)]_{x=A} \equiv [F]_{x=A}(t) = \lim_{(x,\tau) \to (A+0,t)} F(x,\tau) - \lim_{(x,\tau) \to (A-0,t)} F(x,\tau).
\]

(46)

For convenience we shall introduce the following class \(\mathcal{K}\) of functions:

**Definition 4.1** A function \(u(x,t)\) belongs to the class \(\mathcal{K}\) if \(u(x,t) \in C^2(\Omega) \cap C^1(\Omega \cup \{t = 0\})\) and there exist the following limits:

\[\lim_{(x,\tau) \to (a_i \pm 0, b_i + t)} Du(x,\tau),\]

where \(i = 1, 2\), \(Du = \{u, u_x, u_t\}, 0 \leq t \leq \ell\).

**Definition 4.2** (“strengthened” setting of problem (4)–(10)) The solution \(u(x,t)\) of the problem (4)–(6) is called strengthened classical solution of the problem (4)–(10) if \(u(x,t) \in \mathcal{K}\) and the following conditions are satisfied:

\[\lim_{(x,\tau) \to (a_1-0,b_1+t)} u(x,\tau) = \lim_{(x,\tau) \to (a_2+0,b_2+t)} u(x,\tau),\]  \hspace{1cm} (47)
\[\lim_{(x,\tau) \to (a_1+0,b_1+t)} u(x,\tau) = \lim_{(x,\tau) \to (a_2-0,b_2+t)} u(x,\tau),\]  \hspace{1cm} (48)
\[\lim_{(x,\tau) \to (a_1-0,b_1+t)} u_x(x,\tau) = \lim_{(x,\tau) \to (a_2+0,b_2+t)} u_x(x,\tau),\]  \hspace{1cm} (49)
\[\lim_{(x,\tau) \to (a_1+0,b_1+t)} u_x(x,\tau) = \lim_{(x,\tau) \to (a_2-0,b_2+t)} u_x(x,\tau),\]  \hspace{1cm} (50)

where \(t \in [0, \ell]\).
It is not difficult to see that the conditions (7)–(10) are weaker than the conditions (17)–(50).

Let us formulate the main properties of functions from class $\mathcal{K}$ which comply with the conditions (47)–(50):

**Theorem 4.1** Let $u(x,t) \in \mathcal{K}$ satisfies the conditions (47)–(50). Let $\nu(t)$ and $\omega(t)$ denote the discontinuity jumps of function $u(x,t)$ and its derivative $u_x(x,t)$ at the upper slit $\gamma_2$ respectively:

$$
\nu(t) = [u]_{x=a_2} (b_2 + t), \quad \omega(t) = [u_x]_{x=a_2} (b_2 + t).
$$

Then one has:

1. $u(x,t) \in L^1_{1,loc}(\mathbb{R}^2_{t \geq 0})$.

2. $\omega(t) \in C(\mathbb{R})$, $\nu(t) \in C^1(\mathbb{R})$, and for the discontinuity jumps at the lower slit $\gamma_1$ we have

$$
[u]_{x=a_1} (b_1 + t) = -\nu(t),
[u_x]_{x=a_1} (b_1 + t) = -\omega(t),
$$

moreover for $t \notin [0, \ell]$ we have $\nu(t) = \omega(t) = 0$.

3. Time derivatives satisfy the following gluing conditions:

$$
\lim_{(x,\tau) \to (a_1 - 0, b_1 + t)} u_\tau(x, \tau) = \lim_{(x,\tau) \to (a_2 + 0, b_2 + t)} u_\tau(x, \tau),
\lim_{(x,\tau) \to (a_1 + 0, b_1 + t)} u_\tau(x, \tau) = \lim_{(x,\tau) \to (a_2 - 0, b_2 + t)} u_\tau(x, \tau),
$$

(51)

where $t \in [0, \ell]$.

Note that the conditions (51), in contrast to the conditions (47)–(50), are imposed on time derivatives instead of space derivatives. Hence, in the “strengthened” setting of the problem there is no need in additional gluing conditions for the solution $u(x,t)$ at the slits $\gamma_1$ and $\gamma_2$.

## 5 Classical and generalized solutions

In this section we will derive the equation which will be satisfied by every strengthened classical solution of the problem (1)–(10) in sense of distributions $\mathcal{D}'(\mathbb{R}^2)$. We will use the following notation for the d’Alembert operator: $\square \equiv \partial_t^2 - \partial_x^2$. Also let $\mathcal{D}'(\mathbb{R}^2)$ denote the set of distributions from $\mathcal{D}'(\mathbb{R}^2)$ which equal to 0 for $t < 0$. 


Theorem 5.1 Let \( u(x,t) \) be a strengthened classical solution of the problem (4) – (10). Then the function

\[
\tilde{u}(x,t) = \begin{cases} u(x,t), & t \geq 0, \\ 0, & t < 0. \end{cases}
\]

satisfies the following equation in the sense of \( D'(\mathbb{R}^2) \):

\[
\Box \tilde{u}(x,t) = F(x,t),
\]

where

\[
F(x,t) = \varphi(x) \cdot \delta'(t) + \psi(x) \cdot \delta(t) - [u]_{x=a_1} \cdot \delta'(x-a_1) - [u_x]_{x=a_1} \cdot \delta(x-a_1)
\]

\[
- [u]_{x=a_2} \cdot \delta'(x-a_2) - [u_x]_{x=a_2} \cdot \delta(x-a_2)
\]

The proof is similar to the derivation of the generalized Cauchy problem setting given in [4]. It relies on the fact that \( \tilde{u}(x,t) \in L_{1,loc}(\mathbb{R}^2_{t \geq 0}) \), which follows from Theorem 4.1.

Recall the following formula [4]:

\[
\Delta f = \{\Delta f\} + \left[ \frac{\partial f}{\partial n} \right]_S \delta_S + \frac{\partial}{\partial n} ([f]_S \delta_S),
\]

in sense of \( D'(\mathbb{R}^n) \). Here \( \Delta \) denotes the Laplace operator in \( \mathbb{R}^n \), function \( f \in C^2(\bar{G}) \cap C^2(\bar{G}_1) \), domain \( G \) in \( \mathbb{R}^n \) has partially smooth boundary \( S \), \( G_1 = \mathbb{R}^n \setminus \bar{G} \), \{\Delta f\} denotes the action of the classical Laplace operator and \([f]_S \) denotes the discontinuity jump of \( f \) at the surface \( S \). We have obtained the analog of this formula for the d’Alembert operator on the plane with the slits.

Next, by virtue of theorem 4.1

\[
[u]_{x=a_1}(b_1 + t) = -[u]_{x=a_2}(b_2 + t) = -\nu(t) \in C^1(\mathbb{R}); \quad \nu(t) = 0, \ t \notin [0, \ell]
\]

\[
[u_x]_{x=a_1}(b_1 + t) = -[u_x]_{x=a_2}(b_2 + t) = -\omega(t) \in C(\mathbb{R}); \quad \omega(t) = 0, \ t \notin [0, \ell].
\]

Hence, in the “strengthened” setting problem (4) – (10) is equivalent to the following problem:

Find functions \( \omega(t) \in C(\mathbb{R}) \) and \( \nu(t) \in C^1(\mathbb{R}) \), equal to 0 for \( t \notin [0, \ell] \), such that the generalized solution in \( D'(\mathbb{R}^2) \) of the equation

\[
\Box u(x,t) = F(x,t),
\]

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\[ F(x,t) = \varphi(x) \cdot \delta'(t) + \psi(x) \cdot \delta(t) + \nu(t - b_1) \cdot \delta'(x - a_1) + \omega(t - b_1) \cdot \delta(x - a_1) - \\
\nu(t - b_2) \cdot \delta'(x - a_2) - \omega(t - b_2) \cdot \delta(x - a_2), \quad (57) \]

belongs to class \( K \) and satisfies conditions (17) and (50).

Note that the conditions (49) and (48) will be satisfied automatically by virtue of conditions (55).

So, the problem of existence and uniqueness of solution \( u(x,t) \) has converted to the problem of existence and uniqueness of the discontinuity jumps \( \omega(t) \) and \( \nu(t) \) satisfying specific conditions. To obtain these conditions we will first find the general solution of equation (56).

5.1 Solution of equation (56)

As is known [4], the solution of the generalized Cauchy problem for equation (56) exists, is unique and is given by a convolution of the fundamental solution \( \mathcal{E}_1 \) with the right hand side \( F \) defined in (57):

\[ u(x,t) = \mathcal{E}_1 * F(x,t). \quad (58) \]

Here

\[ \mathcal{E}_1(x,t) = \frac{1}{2} \theta(t - |x|) \]

is the fundamental solution of operator \( \Box \), where \( \theta(t) \) denotes Heaviside step function; \( \theta(t) = 1 \) for \( t > 0 \) and \( \theta(t) = 0 \) for \( t < 0 \).

Let us write out an explicit formula for the convolution (58). For this purpose we use the following formulae:

\[ \mathcal{E}_1 * \varphi(x) \delta'(t) = \frac{1}{2} \left[ \varphi(x + t) + \varphi(x - t) \right], \quad (59) \]

\[ \mathcal{E}_1 * \psi(x) \delta(t) = \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds, \]

\[ \mathcal{E}_1 * \omega(t) \delta(x) = \frac{1}{2} \theta(t - |x|) \int_{0}^{t-|x|} \omega(\tau) d\tau, \]

\[ \mathcal{E}_1 * \nu(t) \delta'(x) = \frac{\partial}{\partial x} [\mathcal{E}_1 * \nu(t) \delta(x)] = -\theta(t - |x|) \frac{\text{sign} x}{2} \nu(t - |x|). \]

Therefore denoting

\[ U(x,t) = \frac{1}{2} \theta(t - |x|) \int_{0}^{t-|x|} \omega(\tau) d\tau - \theta(t - |x|) \frac{\text{sign} x}{2} \nu(t - |x|), \quad (60) \]
we obtain the solution of equation (56) in the following form:

$$u(x,t) = u^D(x,t) + U(x - a_1, t - b_1) - U(x - a_2, t - b_2).$$  

(61)

Here $u^D$ denotes the solution of classical Cauchy problem for wave equation defined by d’Alembert’s formula:

$$u^D(x,t) = \frac{1}{2} [\varphi(x+t) + \varphi(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds = f(x-t) + g(x+t).$$  

(62)

### 5.2 Gluing conditions

Let us now define the functions $\nu(t)$ and $\omega(t)$ using the gluing conditions. Conditions at the slits (47) (50) take the form

$$u^D(a_1, b_1 + t) + U(-0, t) - U(a_1 - a_2 - 0, b_1 - b_2 + t) = (63)$$

$$= u^D(a_2, b_2 + t) + U(a_2 - a_1, b_2 - b_1 + t) - U(+0, t),$$

$$u^D_x(a_1, b_1 + t) + U_x(-0, t) = u^D_x(a_2, b_2 + t) - U_x(+0, t),$$  

(64)

where $0 < t < \ell$.

Note that from (60) follows that

$$U(a_1 - a_2 - 0, b_1 - b_2 + t) = 0,$$

because $-b_2 + b_1 + a_2 - a_1 + \ell < 0$ (see (3)) and that $U(a_2 - a_1, b_2 - b_1 + t) = \text{const}$ for $0 < t < \ell$, precisely:

$$U(a_2 - a_1, b_2 - b_1 + t) = \frac{1}{2} \int_0^\ell \omega(\tau) d\tau.$$

We also have (“one-sided” limits are meant is sense of (46)):

$$U(\pm 0, t) = \frac{1}{2} \int_0^\ell \omega(\tau) d\tau \mp \frac{1}{2} \nu(t),$$  

(65)

$$U_x(\pm 0, t) = \mp \frac{1}{2} \omega(t) + \frac{1}{2} \nu'(t).$$

Therefore gluing conditions (63) and (64) take the form

$$\int_0^\ell \omega(\tau) d\tau = \frac{1}{2} \int_0^\ell \omega(\tau) d\tau + D_1(t),$$  

(66)
\[ \nu'(t) = D_2(t), \]  

where

\[ D_1(t) = u^D(a_2, b_2 + t) - u^D(a_1, b_1 + t), \]
\[ D_2(t) = u^D_x(a_2, b_2 + t) - u^D_x(a_1, b_1 + t). \]

Problems (66), (67) have unique solutions respectively

\[ \omega(t) = D_1'(t), \]  
\[ \nu(t) = \int_0^t D_2(\tau) d\tau. \]

These solutions are sufficiently smooth (recall that \( \omega(t) \in C^1(\mathbb{R}) \), \( \nu(t) \in C^2(\mathbb{R}) \) and also \( \omega(t) = \nu(t) = 0 \) for \( t \notin [0, \ell] \)) when and only when the following conditions are satisfied:

\[ D_1'(0) = D_1'(\ell) = 0, \quad D_1(0) + D_1(\ell) = 0, \quad D_1''(0) = D_1''(\ell) = 0, \]  
\[ \int_0^\ell D_2(\tau) d\tau = 0, \quad D_2(0) = D_2(\ell) = 0, \quad D_2'(0) = D_2'(\ell) = 0. \]

These conditions are derived by direct substitution \( t = 0, \ell \) into (66), (67).

Therefore we have obtained the following result:

**Theorem 5.2** There exists a unique strengthened classical solution of the problem (4)–(10) if and only if the conditions (70), (71) to the initial data are satisfied. This solution is given by

\[ u(x, t) = u^D(x, t) + U(x - a_1, t - b_1) - U(x - a_2, t - b_2), \]

where

\[ u^D(x, t) = \frac{1}{2}[\varphi(x + t) + \varphi(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds, \]
\[ U(x, t) = \frac{1}{2} \theta(t - |x|) \int_0^{t-|x|} \omega(\tau) d\tau - \theta(t - |x|) \frac{\text{sign} x}{2} \nu(t - |x|), \]
\[ \omega(t) = \theta(t) \theta(\ell - t) \cdot (u^D_t(a_2, b_2 + t) - u^D_t(a_1, b_1 + t)), \]
\[ \nu(t) = \theta(t) \theta(\ell - t) \cdot \int_0^t (u^D_{x}(a_2, b_2 + \tau) - u^D_{x}(a_1, b_1 + \tau)) d\tau. \]
One can show that

- the conditions (70) are equivalent to the conditions (12)–(17);
- the strengthened classical solution \( u(x, t) \) given by (72) is identical to the classical solution given by (18).

Note that if we drop the gluing conditions (49) and (50) for the derivative \( u_x \) then the solution of the problem concerned will not be unique. Indeed, in this case we can substitute arbitrary \( \omega(t) \) (such that \( \omega(t) \in C^1(\mathbb{R}) \) and also \( \omega(t) = 0 \) for \( t \notin [0, \ell] \)) into (72).

Let us present an example (belonging to T. Ishiwatari) of non-trivial solution \( u^D \) and parameters \( a_i, b_i \), for which conditions (70) are satisfied:

**Example 1** Let \( a_1 - b_1 = a_2 - b_2 + 2\pi k, \ell = 1 \) and \( a_1 + b_1 = a_2 + b_2 + 2\pi l \), where \( k, l \in \mathbb{Z} \) are such that (3) is satisfied. Then for the initial conditions \( \varphi(x) = \sin(x) + \cos(x), \psi(x) = -\cos(x) - \sin(x) \) there exists a unique strengthened classical solution of problem (4)–(10) and it is given by

\[
    u(x, t) = u^D(x, t) = \sin(x - t) + \cos(x + t).
\]

Indeed, conditions (70) are satisfied because \( D_1(t) \equiv D_2(t) \equiv 0 \). In addition it follows from (68) that \( \omega(t) \equiv \nu(t) \equiv 0 \). In other words the solution is continuous and differentiable at the slits (discontinuity steps are equal to zero).

Let us present another example when there exists a nontrivial strengthened classical solution of problem (4)–(10) with nonzero discontinuity steps at the slits.

**Example 2** Let \( \ell = 1 \) and \( h(t) \in C^\infty(\mathbb{R}) \) be such that \( h(t) = 0 \) for \( t \notin [0, 1] \). For the initial conditions \( \varphi(x) = h(x + \alpha), \psi(x) = -h'(x + \alpha) \), where \( \alpha = b_2 - a_2 + 1 \), there exists a unique strengthened classical solution of problem (4)–(10) and it is given by

\[
    u(x, t) = u^D(x, t) + v_1(x, t) + v_2(x, t),
\]

where

\[
    u^D(x, t) = h(x - t + b_2 - a_2 + 1),
    v_1(x, t) = \theta(-b_1 + a_1 + t - x)h(1 + b_1 - a_1 - t + x)\text{sign}(x - a_1),
    v_2(x, t) = -\theta(-b_2 + a_2 + t - x)h(1 + b_2 - a_2 - t + x)\text{sign}(x - a_2).
\]
The solution is right-travelling wave, coming into the upper slit and leaving out of the lower one.

To conclude we would like to note that one may interpret the obtained conditions for initial data as saying that the classical solution exists for “almost all” initial data from the functional space of initial data. It would be interesting to study generalized solutions of Cauchy problem on the Minkowski plane with the slits and also to study the wave equation on more general non-globally hyperbolic manifolds.

**Acknowledgements**

This work was started at the seminar of Scientific Education Center in the Steklov Mathematical Institute. It is partially supported by Grants of RFBR 08-01-00727-a, NSh-3224.2008.1, AVCP 3341, DFG Project 436 RUS 113/951. The work of O.V.G. is partially supported by “Leading Scientific Schools” program NSh-691.2008.1.

**References**

[1] J. Hadamard, *Lectures on Cauchy’s Problem in Linear Partial Differential Equations*, Dover Publications, 2003.

[2] I.G. Petrowsky, Uber das Cauchysche Problem für Systeme von partiellen Differentialgleichungen, Mat. Sb., Vol. 2(1937), 815-870.

[3] J. Leray,, *Hyperbolic differential equations*, Institute for Advanced Study, Princeton, 1953.

[4] V.S. Vladimirov, Equations of Mathematical Physics, Marcel Dekker, New York, 1973.

[5] S.W. Hawking and G.F.R. Ellis, Large scale structure of space-time, Cambridge University Press, 1975.

[6] M. Visser, Lorentzian Wormholes, Springer-Verlag, 1995.

[7] J. R. Gott, Time Travel in Einsteins Universe, Houghton Mifflin, New York, 2001.

[8] S. Deser, R. Jackiw, *Time Travel?*, Comments Nucl.Part.Phys. 20:337-354, 1992; [arXiv:hep-th/9206094](http://arxiv.org/abs/hep-th/9206094).
[9] I.Ya. Aref’eva, High-energy scattering in the brane world and black hole production, Part.Nucl.31 (2000) 169; hep-th/9910269.

[10] M. Cvetic, G.W. Gibbons, H. Lu, C.N. Pope, Rotating Black Holes in Gauged Supergravities; Thermodynamics, Supersymmetric Limits, Topological Solitons and Time Machines, arXiv:hep-th/0504080.

[11] R. J. Gleiser, M. Gurses, A. Karasu, O. Sarioglu, Closed timelike curves and geodesics of Godel-type metrics, Class.Quant.Grav. 23 (2006) 2653-2664; arXiv:gr-qc/0512037.

[12] B.S. Kay, Quantum field theory in curved spacetime, Encyclopedia of Mathematical Physics, J.-P. Francoise, G. Naber and T.S. Tsou, eds., Academic (Elsevier) Amsterdam, New York and London 2006, Vol. 4, pages 202-214; gr-qc/0601008.

[13] A. Ori, Formation of closed timelike curves in a composite vacuum/dust asymptotically-flat spacetime, Phys.Rev.D76:044002,2007; arXiv:gr-qc/0701024.

[14] V. M. Rosa, P. S. Letelier, Stability of Closed Timelike Curves in Goedel Universe, Gen.Rel.Grav. 39:1419-1435, 2007; arXiv:gr-qc/0703100.

[15] S. Slobodov, Unwrapping Closed Timelike Curves, arXiv:0808.0956.

[16] A. DeBenedictis, R. Garattini, F. S. N. Lobo, Phantom stars and topology change, Phys.Rev.D78:104003, 2008; arXiv:0808.0839.

[17] Li-Fang Li, Jian-Yang Zhu, Averaged null energy condition in Loop Quantum Cosmology, arXiv:0812.3532.

[18] G. Gibbons, H. Kodama, Repulsons in the Myers-Perry Family, arXiv:0901.1203.

[19] J. Friedman, M.S. Morris, I.D. Novikov, F. Echeverria, G. Klinkhammer, K.S. Thorne and U. Yurtsever, Cauchy problem in spacetimes with closed timelike curves Phys. Rev. D42 1915 (1990).

[20] D. Deutsch, Quantum mechanics near closed timelike lines. Phys. Rev. D44 3197 (1991).

[21] H.D. Politzer, Path integrals, density matrices, and information flow with closed timelike curves, Phys. Rev. D49 3981 (1994).
[22] I.Ya. Aref’eva, T. Ishiwatari, I.V. Volovich, *Cauchy problem on non-globally hyperbolic space-times*, Theor. Math. Phys., 2008, 157:3, 334-344, arXiv:0903.0567

[23] I.Ya. Aref’eva, I.V. Volovich, *Time Machine at the LHC*, Int.J.Geom.Meth.Mod.Phys. 05:641-651, 2008; arXiv:0710.2696.

[24] A. Mironov, A. Morozov, T.N. Tomaras, *If LHC is a Mini-Time-Machines Factory, Can We Notice?*, arXiv:0710.3395.

[25] I.Ya. Aref’eva, I.V. Volovich, *On the null energy condition and cosmology*, Theor. Math. Phys., 155:1 (2008), 312; arXiv:hep-th/0612098.

[26] V.A. Il’in, E.I. Moiseev, *Optimization of boundary controls of string vibrations*, Russ. Math. Surveys, 60:6(366) (2005), 89114.

[27] V.V. Kozlov, I.V. Volovich, *Finite Action Klein-Gordon Solutions on Lorentzian Manifolds*, Int.J.Geom.Meth.Mod.Phys. 3 (2006) 1349-1358; arXiv:gr-qc/0603111.

[28] H.B. Nielsen, M. Ninomiya, *Future Dependent Initial Conditions from Imaginary Part in Lagrangian*, arXiv:hep-ph/0612032.

[29] B.A. Dubrovin, S.P. Novikov, A.T. Fomenko, *Modern Geometry. Methods and Applications*, Springer-Verlag, 1990.

[30] O. Forster, Lectures on Riemann Surfaces, Springer-Verlag, 1995.