Criticality in non-linear one-dimensional maps: RG universal map and non-extensive entropy

A. Robledo
Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, México 01000 D.F., Mexico.

Abstract

We consider the period-doubling and intermittency transitions in iterated nonlinear one-dimensional maps to corroborate unambiguously the validity of Tsallis’ non-extensive statistics at these critical points. We study the map $x_{n+1} = x_n + u |x_n|^z$, $z > 1$, as it describes generically the neighborhood of all of these transitions. The exact renormalization group (RG) fixed-point map and perturbation static expressions match the corresponding expressions for the dynamics of iterates. The time evolution is universal in the RG sense and the non-extensive entropy $S_Q$ associated to the fixed-point map is maximum with respect to that of the other maps in its basin of attraction. The degree of non-extensivity - the index $Q$ in $S_Q$ - and the degree of non-linearity $z$ are equivalent and the generalized Lyapunov exponent $\lambda_q$, $q = 2 - Q^{-1}$, is the leading map expansion coefficient $u$. The corresponding deterministic diffusion problem is similarly interpreted. We discuss our results.

1 Introduction

The proposal, in 1988, by Tsallis [1] of a non-extensive generalization of statistical mechanics unleashed an unprecedented discussion [2] on the foundations of this most basic branch of physics. This is an exceptional event, since its inception, with Boltzmann and Gibbs (BG), in the long and heretofore unchallenged history of statistical mechanics. Many recent studies [2] have
offered experimental and numerical evidences indicating limiting conditions for the domain of validity of the canonical BG theory. But, so far there have been no firm developments to tilt the balance towards universal acceptance of these new ideas, nor a definite understanding of the deep-lying physical reasons (believed in some cases to be a breakdown in the chain of increasing randomness from non-ergodicity to completely developed chaoticity) for the failure of the BG statistics and the competence of the non-extensive generalization. However, here we present exact analytical results for a class of critical points in non-linear maps from which a rigorous corroboration and understanding of the non-extensive theory can be drawn [3] - [7].

The renowned period-doubling and intermittency routes to chaos are now classical examples of how the use as starting points of simple non-linear discrete maps have often led to important developments in the theory of non-linear dynamical systems [8]. The discovery of the universal properties associated to these mechanisms, comparable with those of conventional critical phenomena in statistical physics, triggered, over two decades ago, an explosion of activity in the field and now both routes are well understood and verified experimentally. Here we might add yet other universal aspects to their properties (explained in more detail below) the non-extensivity marked by the power-law growth or decay of the sensitivity to initial conditions, and the entropy extrema associated to each of the fixed points that define the RG universality classes.

The well-known period-doubling and intermittency transitions are based on the pitchfork and the tangent bifurcations. In particular, the one-dimensional logistic map and its generalization to non-linearity of order $z > 1$, $f_\mu(x) = 1 - \mu |x|^z$, $-1 \leq x \leq 1$, exhibit these two different types of bifurcations an infinite number of times as the control parameter $\mu$ reaches critical values. At these critical points universal scaling laws and indexes hold, independently of the details of the map, and are therefore shared by all maps of the same order $z$. As with other statistical mechanical systems an explanation of universality is provided via the renormalization group (RG) method, which has been successfully applied to this type of iterated map. The RG doubling transformation, consisting of functional composition and rescaling, was first devised to study the cascade of period-doubling transitions and its accumulation point, and was later applied to the intermittency transition. As pointed out [8] the latter case is one of the rare examples where the RG equations can be solved exactly. Here we indicate that this analytic solution holds
generally not only for tangent bifurcations (as known time ago [8]) but also for pitchfork bifurcations (as noticed recently [4]). We also point out that the fixed-point map can be determined as an extremum of the non-extensive entropy, thus providing evidence for a possible connection between extremal properties of entropy expressions and those for the fixed points obtained from the usual RG recursive method [9].

As required, at each of the map critical points the Lyapunov \( \lambda_1 \) exponent vanishes, and the sensitivity to initial conditions \( \xi_t \) for large iteration time \( t \) ceases to obey exponential behavior, exhibiting instead power-law behavior [10]. As the means to describe the dynamics at such critical points, and based on the non-extensive entropy of Tsallis \( S_Q \) (see Eq. (16) below), the \( q \)-exponential expression

\[
\xi_t = \exp_q(\lambda_q t) \equiv [1 - (q - 1)\lambda_q t]^{1/(q-1)}
\]

containing a \( q \)-generalized Lyapunov exponent \( \lambda_q \) was proposed [11], together with generalizations for the rate of entropy production (referred here as the Kolmogorov-Sinai (KS) entropy [8]) \( K_Q \) (see Eq. (15) below) and for the Pesin identity \( \lambda_q = K_Q \) [11]. The standard expressions are recovered when \( q, Q \to 1 \). The use of the two indexes \( q \) and \( Q \) in our application will become clear below. Furthermore, recent studies [11]-[14], focusing mainly at the onset of chaos (the accumulation point of the period-doubling bifurcations) of the \( z \)-logistic map have revealed a series of connections between the Tsallis entropic index \( q \) and the non-linearity \( z \), the fractal dimension \( d_f \) of the chaotic attractor, and the end points of its multifractal singularity spectrum \( f(\alpha) \). Here we identify \( q \) and \( \lambda_q \) in terms of the tangent and pitchfork critical map parameters.

The properties of dynamical systems at critical points can also be probed by considering the deterministic diffusive processes that take place in nonlinear maps with discrete translational symmetry. These maps consist of a sequence of cells containing each elements of a basic nonlinear map. One such construction [15] makes use of the tangent bifurcation, and, interestingly, anomalous diffusional behavior has been observed close to the intermittency transition [15], [16]. Continuous-time random-walk theory concepts have been applied [15], [16] and, characteristically, the associated waiting-time distribution function \( \psi(t) \) has been shown to have non-exponential power-law decay. The validity of the non-extensive approach to criticality has been
recently checked by examination of the anomalous diffusion at the intermittency transition [16], [17]. The key results in these studies center on the analytical form predicted for $\psi(t)$ which bears a close relationship to the power-law Eq. (1). Here we point out that the analytical forms for this and related distributions actually correspond to those obtained by the repeated application of the RG transformation. As mentioned, if the fixed point solution can also be determined via an entropy optimization method the expressions for these distributions are indicative of the same entropy extremum.

## 2 Outline of results

The main points in the following analysis are:

1) The connection between the RG fixed-point map expression for the iterate $x_t$ associated to trajectories $x_t$ with $x_0 \to 0$ at $t = 0$ and the expression in Eq. (1) for the sensitivity of initial conditions $\xi_t$. This leads to simple identifications of the non-extensive parameters as $q = 2 - z^{-1}$ and $\lambda_q \sim a$. We corroborate the earlier known results $q = 3/2$ and $q = 5/3$ for the tangent and pitchfork bifurcations for the logistic map $z = 2$, respectively [11].

2) The connection between the RG fixed-point map recursion relation for the iterate $x_t$ and the expression obtained from $S_Q$ for the temporal evolution of the number of cells $W_t$ occupied by an ensemble of initial conditions spread over a finite-length interval $[0, x_0]$. (See the definition of $S_Q$ in Eq. (16), and the explanatory comments at the end of this section about the relationship $q = 2 - Q^{-1}$ between the indexes $q$ and $Q$). This leads to the identifications $Q = z$ and $K_q \sim a$, confirming $q = 2 - z^{-1}$ and giving support to the validity of the generalized Pesin identity $\lambda_q = K_Q$ at the critical transitions. See Ref. [7] for a recent proof of the validity of this generalized identity at the edge of chaos of the logistic map $z = 2$.

3) The general property of the non-extensive entropy function which decreases monotonically as the RG transformation flows away from the fixed point where it attains its maximum value.

4) The maximum entropy property of the distribution $\Psi(\tau)$, $\Psi(\tau) = \int_0^\tau dt \psi(t)$, of cell residence time intervals $[0, \tau]$ in the critical diffusive processes.

For clarity of presentation it is convenient to advance the following prop-
erty of the $q$-exponential function. For this function the ordinary derivative $d \exp(x)/dx = \exp(x)$ and inverse $[\exp(x)]^{-1} = \exp(-x)$ properties become, respectively, $d \exp_q(x)/dx = [\exp_q(x)]^q$ and $[\exp_q(x)]^{-1} = \exp_{2-q}(-x)$, and these combine into the identity

$$\exp_q(x) \equiv \left[\exp\left(\frac{x}{Q}\right)\right]^Q,$$

(2)

where $q = 2 - Q^{-1}$. Thus, in a given problem where the $q$-exponential function applies there appear related values of the $q$-index, that become the same (and the above identity becomes trivial) as $q \to 1$.

3 Critical dynamics from fixed-point map

So, we recall [8], [19] the solution of Hu and Rudnick to the Feigenbaum RG recursion relation obtained for the case of the tangent bifurcation. For the transition to periodicity of order $n$ consider the $n$-th composition $f^{(n)}$ of the original map $f$ in the neighborhood of one of the $n$ points tangent to the line with unit slope and shift the origin of coordinates to that point. Then, one obtains

$$f^{(n)}(x) = x + u |x|^z + o(|x|^z),$$

(3)

where $u > 0$ is the leading expansion coefficient (the right-hand-side in Eq. (3) may depend on $n$ only through the omitted terms). The RG fixed-point map $x' = f^*(x)$ was found to be

$$x' = x \exp_2(u x^{z-1}) = x [1 - (z - 1) u x^{z-1}]^{-1/(z-1)},$$

(4)

as it satisfies $f^*(f^*(x)) = \alpha^{-1} f^*(\alpha x)$ with $\alpha = 2^{1/(z-1)}$ and has a power-series expansion in $x$ that coincides with Eq. (3) in the two lowest-order terms. (We have used $x^{z-1} \equiv |x|^{z-1} \text{sgn}(x)$). Eq. (4) can be rewritten as $x' = x^{-[z-1]} - (z - 1) u$, a translation of the power law $y = x^{-[z-1]}$, the property that led Hu and Rudnick in their derivation. The effect of a perturbation of the form $x^{-p}$ transforms this into

$$x'^{-[z-1]} + \varepsilon' x'^{-p} = x^{-[z-1]} + \varepsilon x^{-p} - (z - 1) u,$$

(5)

or, to lowest order in $\varepsilon' = \varepsilon$,

$$x' = x \exp_2(u x^{z-1}) - \frac{\varepsilon x^{z-p}}{z-1} [\exp_2(u x^{z-1})]^z.$$
\[ \frac{z}{z - 1} \left[ \exp_z (ux^{z-1}) \right]^{z-p}. \]  

The recurrence relation in Eq. (5) remains invariant under iteration and rescaling by \( \alpha \) when \( \epsilon' = \epsilon \) is multiplied by the factor \( \gamma = 2^{(p-z+1)/(z-1)} \). As it is known [18], [19], the leading relevant perturbation about this critical point is that which corresponds to a shift from tangency, for this case the lowest order term in \( x \) in Eq. (6) must be of order one, and so we fix \( p = z \) for the perturbation exponent (in the unnormalized eigenfunction).

Fittingly, it can be readily concluded that the RG recursion relation in the neighborhood of the pitchfork bifurcations \( df^{(2k-1)}(x)/dx \) \( x=0 \) conforms with the above scheme and that its solution is also applicable to these transitions provided the sign of \( u \) is changed for \( x > 0 \). It is interesting to point out that at the time the RG method was first applied to the tangent bifurcation it was thought that the simple closed form obtained for the fixed point map, the \( q \)-exponential in Eq. (4), would most likely prove not to be fruitful for the study of period doubling [8], [19]. Not only is this solution applicable to each period doubling transition with the minor modification of the indicated sign change, but also, as it has been recently demonstrated [6], the time evolution at their accumulation point, the edge of chaos, is expressible in terms of similar \( q \)-exponentials. Below we comment on the caveats that appear in relating \( z \) above to the \( z \) in the initial map e.g. \( f_\mu(x) = 1 - \mu |x|^3 \).

When the number of iterations (after an initial \( x_0 \)) is large and the continuous time \( t \) approximation is taken (\( x' = x_t, x_0 = x \) and \( u = at \) in Eq. (4)) the fixed-point map can be written as

\[ x_t = x_0 [1 - (z - 1)ax_0^{z-1}t]^{1/(z-1)}, \]  

with properties \( dx_t/dt = ax_t^z \) and \( dx_t/dx_0 = (x_t/x_0)^z \). Thus, the fixed-point and perturbation relations can be used to describe the time evolution of intervals containing iterates, an initial interval \([0, x_0 < x_+]\) of length \( \Delta x_0 \) evolves into \([0, x_t < x_+]\) with length \( \Delta x_t \) where \( x_t \) is the image of \( x_0 \) at time \( t \). Accordingly, Eq. (5) with \( p = z \) becomes

\[ \Delta x_t^{-(z-1)} + \epsilon_t \Delta x_t^{-z} = \Delta x_0^{-(z-1)} + \epsilon_0 \Delta x_0^{-z} - (z - 1)at. \]  

We should like to emphasize that the expression for time evolution Eq. (7) has the same universal form as that for the fixed-point map Eq. (4). In the
neighborhood of the points of tangency $f^{(n)}(0)$ the iterates follow monotonic paths with time shape set by the map shape itself, i.e. long time dynamics follow the static solution of Hu and Rudnick. It should also be kept in mind that Eq. (8) applies only to small perturbations as this corresponds to the perturbation recursion relation obtained to linear order around the RG fixed point. We also comment that our use here of the continuous time approximation (valid for large iteration time $t$) in deriving Eq. (4) is only for expediency reasons and has no effect in the generality of our results as this equation is equally obtained from the scaling property

$$x_m = f^{*(m)}(x) = m^{-1/z-1} f^*(m^{1/z-1}x), \quad m = 1, 2, ..., (9)$$

that in turn is obtained from $f^*(f^*(x)) = \alpha^{-1} f^*(\alpha x)$ with $\alpha = 2^{1/z-1} [5]$.

4 Sensitivity to initial conditions

We have reached a point at which a straightforward proof of the validity of Tsallis’ Eq. (1) can be obtained. This is done by direct evaluation of the sensitivity to initial conditions from its definition $\xi_t \equiv \lim_{\Delta x_0 \to 0} (\Delta x_t/\Delta x_0)$. From Eq. (7) and $dx_t/dx_0 = (x_t/x_0)^z$ one obtains

$$\xi_t(x_0) = [1 - (z - 1)ax_0^{z-1}t]^{-z/(z-1)}, \quad (10)$$

and comparison with Eq. (1) yields $q = 2 - z^{-1}$ and $\lambda_q(x_0) = zax_0^{z-1}[16]$.

We can also confirm the values of $q$ obtained numerically by Tsallis et al [11] for both the tangent and pitchfork bifurcations when $z = 2$. The composition $f^{(n)}_{\mu}$ for the 3-logistic map $f_{\mu}(x) = 1 - \mu |x|^3$, $3 > 1$, at the tangent bifurcations is $f^{(n)}_{\mu}(x) = x + ux^2 + o(x^2)$, $u > 0$, and from $z = 2$ one has $q = 3/2$. For the pitchfork bifurcations one has instead $f^{(n)}_{\mu}(x) = x + ux^3 + o(x^3)$, because $d^2 f^{(2k)}_{\mu}/dx^2 = 0$ at these transitions and $u < 0$ is now the coefficient associated to $d^3 f^{(2k)}_{\mu}/dx^3 < 0$. In this case we have $z = 3$ in $q = 2 - z^{-1}$ and one obtains $q = 5/3$. Notably, these specific results for the index $q$ are valid for all $3 > 1$ and therefore define the existence of only two universality classes for unimodal maps, one for the tangent and the other one for the pitchfork bifurcations [5].

Further, if one uses the fact [20] that the invariant distribution of $f^{(n)}$ in Eq. (3) is of the form $\rho(x) \sim x^{-(z-1)}$, the average $\overline{\lambda_q}$ of $\lambda_q(x_0)$ over $x_0$
yields the constant $\bar{\lambda}_q \sim a$. In relation to this we observe that this average is compatible with
\[
\bar{\lambda}_q = \int dx \rho(x) \ln_q \left| \frac{df^*(x)}{dx} \right|,
\] (11)
the $q$-extension of the customary [8] expression for $\lambda_1$ as the average of $\ln|df^*(x)/dx|$ over $\rho(x)$. By taking the $q$-logarithm (where $\ln_q y \equiv (y^{1-q} - 1)/(1 - q)$ is the inverse of $\exp_q(y)$) of both sides of
\[
\xi_t(x_0) = [1 - (q - 1)\lambda_q(x_0)t]^{-1/(q-1)},
\] (12)
and by recalling the definition of $\xi_t(x_0)$ one obtains
\[
\lambda_q(x_0) = t^{-1} \ln_q \left| \frac{df^*_{\xi_t}(x_0)}{dx_0} \right|,
\] (13)
where $\ln_q |df^*_{\xi_t}(x_0)/dx_0| \sim t$ as $\lambda_q(x_0)$ does not depend on $t$. (Notice that the limit $t \to \infty$ of Eq. (13) is the $q$-extension of the regular definition of $\lambda_1(x_0)$ [8]). Next, use of the fixed-point relation $f^*_{\xi_t}(x_0) = t^{1/z} f^*(t^{1/z} - 1, x_0)$ and $t = (x/x_0)^{z-1}$ into (13) leads to $\lambda_q(x_0) = (x/x_0)^{-(z-1)} \ln_q |df^*(x)/dx|$ and Eq. (11).

5 Rate of entropy change

Even though our basic results have been derived, it is useful to extend our analysis and reveal more details about the dynamics at these critical points. Our following arguments focus on $x_t$ rather than $dx_t/dx_0$ and imply $Q = z$, thus confirming $q = 2 - z^{-1}$. They also indicate that the rate of entropy change is $K_Q \sim a$, suggesting the validity of the $q$-generalization of the Pesin $\bar{\lambda}_q = K_Q$ at these critical states. Therefore, let’s consider a cell partition ($N \to \infty$ cells of length $l \to 0$) in the ‘entire’ phase space $[0, x_+]$. Consider also the time evolution of occupied cells $W_t$ by an ensemble of iterates with an extended initial occupation $W_0$ that spans $[0, x_0 < x_+]$. We write the cell distribution of iterates $p_i$ as
\[
p_i = \frac{1 + \delta_i}{W}, \quad \sum_i^W \delta_i = 0,
\] (14)
as this is a convenient form when considering normalized perturbations to the uniform distribution. It is a general expression when no further conditions are imposed on the $\delta_i$ but below we require small $\delta_i$. When Eq. (14) is introduced into the KS $q$-entropy difference

$$K_Q t = S_Q(t) - S_Q(0), \; t \text{ large},$$

where

$$S_Q = \frac{1 - \sum_i W_i^Q}{Q - 1}$$

is the Tsallis entropy, we obtain

$$W_t^{-Q} \sum_i W_i (1 + \delta_{i,t})^Q = W_0^{-Q} \sum_i (1 + \delta_{i,0})^Q - (Q - 1)K_Q t.$$  

To lowest order in $\delta_i$ this reduces to

$$W_t^{-(Q-1)} + \frac{1}{2} Q(Q - 1) \delta_t^2 W_t^{-Q} =$$

$$W_0^{-(Q-1)} + \frac{1}{2} Q(Q - 1) \delta_0^2 W_0^{-Q} - (Q - 1)K_Q t,$$

with $\delta^2 \equiv \sum_i W_i \delta_i^2$.

Remarkably, Eqs. (18) and (8) can be seen to be equivalent if $Q = z$ and if $W_{t,0} \sim \Delta x_{t,0}$, $K_Q \sim a$ and $\frac{1}{2} Q(Q - 1) \delta_{t,0}^2 \sim \varepsilon_{t,0}$. The entropic index $Q$ is simply given by the non-linearity $z$ while $K_Q$ is the expansion coefficient $a$ (and through the proviso that $\lambda_q = K_Q$ this would imply $\lambda_q \sim a$). Our comparison between Eqs. (18) and (8) is consistent in that each is the lowest-order expression valid for small perturbations, the first to the uniform cell distribution of iterates and the second to the fixed-point map solution. We can draw the following conclusion from the association of Eqs. (18) and (8): The entropy $S_Q^*$ for the fixed-point map is maximum, as $S_Q^* - S_Q = Q\delta^2/2W_0^q > 0$, and the distribution becomes uniform, $p_i = W^{-1}$. The RG flow is away from the fixed-point since $\gamma > 1$ and the entropy decreases as the RG transformation is applied. The proposed equivalence between Eqs. (18) and (8) is justified by making the following connection with Eq. (1). For a uniform distribution $W_{t,0} \sim \Delta x_{t,0}$, as the ratio of the number of cells $W_t$ occupied at time $t$ to the distance $\Delta x_t$ is the length $l$, and in the limit
$N \to \infty$ and $l \to 0$, so that $\Delta x_0 \to 0$, with say $\Delta x_0/l = b$, one obtains $W_t = b \xi_t$. This explicit link between the occupation numbers $W_t$ and $\xi_t$ is in line with the all-important connection between the loss of information embodied by the KS entropy and the sensitivity to initial conditions in non-critical regimes [8], and with the parallel connection suggested by Tsallis et al [11], [12] for the analogous quantities at critical points.

It is worth the mention that the use of $q$-exponential (and the inverse $q$-logarithm) functions in the derivation of the properties of a given system will in general introduce pairs of conjugate indexes such as $q = 2 - Q^{-1}$ above. As pointed out, taking derivatives and inverse operations lead to expressions involving one or the other of these simply related values of the entropic index (that become trivially the same $q = Q = 1$ for the usual exponential and logarithmic functions). Consequently, while some theoretical features are expressed through $q$ some others appear expressed via $Q$, but in all cases their occurrence and meaning is apparent.

6 Deterministic diffusion

We comment now on the deterministic diffusion process that takes place for the variable $x_t$ in the map $x_{t+1} = f(x_t)$, where $f(m+x) = m+f(x)$, $m$ integer and $f(-x) = -f(x)$. Geisel and Thomae [15] found that when the unit-cell map $f(0 \leq x \leq 1/2)$ is chosen to have the (intermittency transition) form $x_{t+1} = x_t + ax_t^z$ within $[0, x_+ \leq 1/2]$ anomalous diffusion, of the dispersive type, is generated when $z > 2$ since then the mean-square displacement $< \Delta x^2(t) > \sim t^{1/(z-1)}$. This behavior follows [15] after the distribution of cell residence times $\psi(t)$ is found to be

$$\psi(t) = ax_t^{z-1}[1 + (z - 1)ax_t^{z-1}t]^{-z/(z-1)}.$$  \hspace{1cm} (19)

Grigolini and co-workers [16] recognized the relationship between the expressions for $\psi(t)$ and $\xi_t$. They also analyzed, and verified, the validity of the non-extensive scheme involving the $q$-generalized KS entropy its equality with the $q$-generalized Lyapunov exponent [16], [17]. Here we can recap these results as follows: The properties of the diffusion process driven by the intermittency transition are universal since they stem from the RG fixed-point map expression. Namely, $\psi(t)$ is, besides normalization, the derivative of the
inverse fixed-point map \( dx_0/dx_t \) with \( x_t = x_+ \) and is therefore inversely proportional to \( \xi_t = dx_t/dx_0 \). The distribution \( \Psi(\tau) = \int_0^\tau dt \psi(t) \sim < \Delta x^2(\tau) > \) is in turn proportional to the inverse of the fixed-point map \( x_0(x_\tau) \). Also,

\[
\Psi(\tau) = A[1 + (z - 1)ax_+^{z-1}\tau]^{-1/(z-1)},
\]

has the form of the distribution obtained by optimization of the non-extensive entropy with index \( Q = z \) for fixed \( ax_+^{z-1}\tau \). The related entropy index \( q = 2 - Q^{-1} = 2 - z^{-1} \) applies to the residence time distribution \( \psi(t) \).

7 Summary and discussion

Thus, for the unimodal maps analyzed we have indicated that the exact static solution of the RG equations for the tangent and pitchfork bifurcations also describe the dynamics of iterates. Further, these fixed-point expressions are the same \( q \)-exponential expressions describing the temporal evolution of ensembles of iterates prescribed by the non-extensive formalism for non-linear maps at criticality. Therefore the RG fixed-point map for the universality classes determined by \( z \) can be constructed via Tsallis entropic arguments. The equivalence rests on the derivation of Eq. (1) exclusively from RG procedures. The study of the pitchfork and period-doubling transitions has been expanded via detailed derivation of their \( q \)-generalized Lyapunov exponents \( \lambda_q \) and interpretation of the different types of sensitivity \( \xi_t \) [5]. Likewise, the properties of the intricate trajectories at the edge of chaos of the logistic map \( z = 2 \) have been analytically obtained leading too to the determination of \( \lambda_q \) and interpretation of the dynamics at the strange attractor [6]. Further, the validity of the \( q \)-generalized Pesin identity at this important state has been recently rigorously proved [7]. More generally, we addressed here the properties of the fixed-point map with the purpose of finding additional evidence [9] for an existing relationship between the variational properties of entropy expressions and the RG approach when applied to systems with scale invariance properties. We found that the non-extensive entropy function decreases monotonically as the RG transformation flows away from the fixed point where it attains its maximum value.

The ergodic hypothesis lies at the foundation of statistical mechanics implying that trajectories in phase space cover uniformly the entire pertinent regions. But is this hypothesis always correct? Already many years ago the
answer to this question has been probed by the study of simple dynamical systems with only a few degrees of freedom [8]. Besides their uncomplicated description these systems display extremely convoluted motion in phase space that is neither regular nor simply ergodic, and the mechanism by which ergodicity emerges in these and more complex deterministic systems has been effectively explored by studying the sensitivity to initial conditions and the associated Lyapunov exponents [8]. The distinction between periodic and chaotic motion is signaled, respectively, by the long-time exponential approach or departure of trajectories with close initial positions. Here we have analyzed the borderline critical states in one-dimensional non-linear maps at which the exponential sensitivity law stops working and find that the universal dynamical behavior at these states precisely follow the predictions of the generalized non-extensive theory.

The currently developed non-extensive generalization [1], [2] of BG statistical mechanics entails a rare examination of the domain of validity of this most fundamental base of physics. The suggested physical circumstances for which BG statistics fails to be applicable are thought to be associated to states that lack the full degree of chaotic irregular dynamics that probes phase space thoroughly and that is necessary for true equilibrium. Such anomalous states are signalled by the vanishing of the largest Lyapunov exponent and exhibit non-ergodicity or unusual power-law mixing. One particular case in point is that of critical states in nonlinear unimodal maps. Here impeded or incomplete mixing in phase space (the interval $-1 \leq x \leq 1$) arises from the special ‘tangency’ shape of the map at the pitchfork and tangent transitions that produces the monotonic trajectories given by Eq. (7). This has the effect of confining or expelling trajectories causing anomalous phase-space sampling, in contrast to the thorough coverage in generic states with $\lambda_1 > 0$.

By construction the dynamics at the intermittency transitions, describe a purely nonextensive regime, since the map studied here, Eq. (3), does not consider access of trajectories to an adjacent or neighboring chaotic region, as in the setting of Refs. [16], [17] or as in trajectories in conservative maps with weakly developed chaotic regions [21]. Hence there is no reappearance of trajectories from chaotic regions that would cause the relaxation from the nonextensive regime with vanishing ordinary Lyapunov exponent to an extensive regime with a positive one at some crossover iteration time $\tau$.

The links we have exhibited through exact analytical results between the various properties of one-dimensional non-linear maps at critical conditions
provide a clear-cut corroboration, with a universal attribute, in the RG sense, of the validity of non-extensive statistics at such states.

Acknowledgments. I would like to thank C. Tsallis and F. Baldovin for useful discussions and comments. I gratefully acknowledge the hospitality of the Centro Brasileiro de Pesquisas Fisicas where this work was carried out and the financial support given by the CNPq processo 300894/01-5 (Brazil). This work was also partially supported by CONACyT grant P-40530-F (Mexico).

References

[1] C. Tsallis, J. Stat. Phys. 52, 479 (1988).

[2] For a recent review see, C. Tsallis, in Nonextensive Statistical Mechanics and Its Applications, eds. S. Abe and Y. Okamoto, Lecture Notes in Physics 560, 3 (Springer, Berlin, 2001). See http://tsallis.cat.cbpf.br/biblio.htm for full bibliography.

[3] V. Latora, A. Rapisarda and A. Robledo, Letters to the Editor, Science 300, 250 (2003).

[4] A. Robledo, Physica A 314, 437 (2002).

[5] F. Baldovin and A. Robledo, Europhys. Lett. 60, 518 (2002).

[6] F. Baldovin and A. Robledo, Phys. Rev. E 66, 045104-1(R) (2002).

[7] F. Baldovin and A. Robledo, cond-mat/0304410.

[8] See, for example, H.G. Schuster, Deterministic Chaos. An Introduction, 2nd Revised Edition (VCH Publishers, Weinheim, 1988).

[9] A. Robledo, Phys. Rev. Lett. 83, 2289 (1999).

[10] P. Grassberger and M. Scheunert, J. Stat. Phys. 26 (1981) 697; T. Schneider, A. Politi and D. Wurtz, Z. Phys. B 66, 469 (1987); G. Anania and A. Politi, Europhys. Lett. 7 (1988) 119; H. Hata, T. Horita and H. Mori, Progr. Theor. Phys. 82, 897 (1989).
[11] C. Tsallis, A.R. Plastino and W.-M. Zheng, Chaos, Solitons and Fractals 8, 885 (1997).

[12] U.M.S. Costa, M.L. Lyra, A.R. Plastino and C. Tsallis, Phys. Rev. E 56, 245 (1997).

[13] M.L. Lyra and C. Tsallis, Phys. Rev. Lett. 80, 53 (1998).

[14] F.A.B.F. de Moura, U. Tirnakli and M.L. Lyra, Phys. Rev. E 62, 6361 (2000).

[15] T. Geisel and S. Thomae, Phys. Rev. Lett. 52, 1936 (1984).

[16] M. Buiatti, P. Grigolini and A. Montagnini, Phys. Rev. Lett. 82, 3383 (1999).

[17] M. Ignaccolo, P. Grigolini and A. Rosa, Phys. Rev. E 64, 026210 (2001).

[18] J.E. Hirsch, M. Nauenberg and D.J. Scalapino, Phys. Lett. 87A, 391 (1982).

[19] B. Hu and J. Rudnick, Phys. Rev. Lett. 48, 1645 (1982).

[20] P. Gaspard and X.J. Wang, Proc. Natl. Acad. Sci. USA, 85, 4591 (1988).

[21] F. Baldovin, E. Brigatti and C. Tsallis, cond-mat/ 0302559.