Numerical Solution of Fourth-Order Boundary Value Problems for Euler–Bernoulli Beam Equation using FDM

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Abstract. Euler–Bernoulli beam equation is widely used in engineering, especially civil and mechanical engineering to determine the deflection or strength of bending beam. In physical science and engineering, to predict the deflection for beam problem, bending moment, soil settlement and modeling of viscoelastic flows, fourth-order ordinary differential equation (ODE) is widely used. The analytical solution of most of the higher order ordinary differential equations with complicated boundary condition that occur in any engineering problems is not easy way. Therefore, numerical technique based on finite difference method (FDM) is comparatively easy and important for solving the boundary value problems (BVP). In this study four boundary conditions (Neumann condition) are considered for solving BVP. Absolute error calculation, numerical stability and convergence are discussed. Two examples are considered to illustrate the finite difference method for solving fourth order BVP. The numerical results are rapidly converged with exact results. The results shows that the FDM is appropriate and reliable for such type of problems. Thus present study will enhance the mathematical understanding of engineering students along with an application in different field.

1. Introduction
In several branches of science and engineering, fourth order boundary value problems arise specially in elastic stability beam theory. BVP is involving ordinary differential equations with satisfying particular conditions. Analytically particular class of differential equations can be solved. Those problems which consists of higher order differential equation and complicated boundary conditions can only be solved by numerical methods to obtain closed solution. In literature, Balagurusamy [5], Hildebrand [6], Jain [8], Kelesoglu [9], Levy [11], Sastry [13, 14], Scheid [15] developed the finite difference methods for solving the initial-boundary value problems. Second order and third order BVP with ODE for Dirichlet boundary condition are solved by Hossain [7], Lakshmi [10], Muhammad [12], Siddiqi [16] and Xu [17]. Adak [1-4] studied finite difference methods for solving partial differential equation along with convergence of numerical techniques.

From the previous research work, it is found that in most of the cases, the BVP of second order differential equation with Dirichlet boundary condition was investigated. Therefore, in this study, fourth
order differential equation for beam-theory (bending equation) has been considered and solved using FDM.

2. Problem identification

In the beam theory, deflection of beam is determined by Euler Bernoulli’s equation i.e. the fourth order ordinary differential equation. Euler beam equation arises from the combination of four following equations of beam theory which are kinematic, constitutive, force result and equilibrium. That means, 

\[
\text{Kinematic} \rightarrow \text{constitutive} \rightarrow \text{force resultant} \rightarrow \text{equilibrium} = \text{Beam equation}.
\]

Where,

\[
\begin{align*}
\text{Kinematics} & : \quad k = -\theta = -\frac{dw}{dx} \\
\text{Constitutive} & : \quad \sigma(x,y) = E \varepsilon(x,y) \\
\text{Resultant} & : \quad M(x) = \iint y\sigma(x,y) \, dy \, dz \\
\text{Equilibrium} & : \quad \frac{dM}{dx} = V \quad \frac{dv}{dx} = -p 
\end{align*}
\]

Combining the two equilibrium equations (4), we obtain

\[
\frac{d^2M}{dx^2} = -p
\]

Now, In equation (5), moment M is replaced by equation (3) given by

\[
\frac{d^2}{dx^2} \left( \iint y\sigma \, dy \, dz \right) = -p
\]

To eliminate \( \sigma \) from equation (6), use constitutive relation (2) and then use kinematics (1) to replace \( \varepsilon \) in the normal displacement \( w \), thus obtained,

\[
\frac{d^2}{dx^2} \left( E \iint y^2 \, dy \, dz \right) = -p
\]

Putting, moment of inertia \( I = \iint y^2 \, dy \, dz \) in equation (7), obtain

\[
E I \frac{d^4w}{dx^4} = p
\]

Here, EI represents bending moment which is constant, then equation (8) becomes

\[
EI \frac{d^4w}{dx^4} = p
\]

In elastic theory beam problem, bending equation consists of fourth order ODE with specific four boundary conditions (Neumann condition) to simulate the practical problem. Considering the linear fourth order ODE as given by

\[
y^{(iv)} + f(x)y''' + g(x)y'' + h(x)y = r(x), \quad a < x < b
\]

with boundary conditions \( y(a) = \alpha_1, \quad y'(a) = \alpha_2, \quad y(b) = \beta_1, \quad y'(b) = \beta_2. \)

3. Finite difference method for BVP

The concept of finite difference method based on replacement of derivatives by finite difference approximation in the differential equation as well as in the boundary conditions. Simplifying the finite difference approximations in differential equation and subsequent result is to provide linear system of equations which are solved by a standard procedure.

The interval [a, b] or \([x_0, x_n]\) is divided into n number of equal subintervals of width h to solve the BVP defined by (9), so that \( x_i = x_0 + ih, \quad i = 1, 2, 3, \ldots, n. \)

The corresponding value of y at these points are denoted by

\[
y(x_i) = y_i = y(x_0 + ih), \quad i = 0, 1, 2, \ldots, n.
\]

Using Taylor’s expansion, values of \( y'(x) \), \( y''(x) \) and \( y^{(iv)}(x) \) at the point \( x = x_i \) can be written as
The given differential equation is approximated as:

\[
y'i = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2)\]

\[
y''i = \frac{y_{i+2} - 2y_{i+1} + y_{i+1}}{h^2} + O(h^2)\]

\[
y''''i = \frac{y_{i-4} - 4y_{i-1} + 6y_{i} - 4y_{i+1} + y_{i+2}}{h^4} + O(h^4)
\]

Satisfying the differential equation at the point \(x = x_i\), we get

\[
y'' + f(x)y' + g(x)y + h(x)y = r(x)\]

Substituting the expressions for \(y''\) and \(y'(x)\), obtain

\[
y'' = \frac{y_{i+2} - 2y_{i+1} + y_{i+1}}{h^2} + g_i \frac{y_{i+1} - y_{i-1}}{2h} + h_i y_i = r_i, \quad i = 1, 2, \ldots n
\]

where \(y(a) = \alpha_1\), \(y'(a) = \frac{y_{i+1} - y_{i-1}}{2h} = \alpha_2\),
\(y(b) = \beta_1\), \(y'(b) = \frac{y_{i+1} - y_{i-1}}{2h} = \beta_2\).

3.1 Error Calculation

Exact solution is calculated through C.F (Complementary function) and P.I. (Particular Integral). Then

Absolute Error = \(|\text{Exact solution} - \text{Numerical Solution}|\)

Relative Error = \(\frac{\text{Exact solution} - \text{Numerical Solution}}{\text{Exact solution}}\)

Percentage Error = \(\frac{\text{Exact solution} - \text{Numerical Solution}}{\text{Exact solution}} \times 100\).

3.2 Numerical Stability

If the difference between the numerical solution and the exact solution remains bounded as the number of steps tends to infinity, numerical scheme is called stability.

3.3 Numerical Convergence

In the iterative numerical technique, if each successive iteration results are progressively closer to the true solution, it is known as convergence. A numerical method is not always guaranteed to produce converging results. Convergence is subject to satisfying certain conditions. If these conditions are not matched, it is known as divergence.

3.4 Numerical Consistency

A numerical scheme is said to be consistent if the finite difference representation converges to the differential equation. We are trying to solve as the space step tends to very small value.

We have explained the method with three types of boundary conditions. In several practical problems, derivative boundary conditions may be prescribed, and this requires a modification in the procedures which are described above. The following examples illustrate the application of finite-difference method.

4. Test problems and verification

Example 1

Consider the fourth order linear nonhomogeneous ODE with four conditions defined by

\[
y''''(x) - 16y = x, \quad 0 \leq x \leq 1, \text{ for } y(x_j), \quad x_j = 0.25, 0.5, 0.75, \text{ with boundary conditions}
\]

\[
y(0) = 0, y''(0) = 0 \text{ and } y(1) = 0, y'(1) = 0.
\]

Solution: To use finite difference method

Here interval is \(0 \leq x \leq 1\), ie, [0, 1]. Mesh size \(h = 0.25\) along length is considered.

Length is discretized into four points which are given by \(x_0 = 0, x_1 = x_0 + h = 0.25, \quad x_2 = x_0 + 2h = 0.5, x_3 = x_0 + 3h = 0.75, x_4 = x_0 + 4h = 1\).

Boundary conditions are given by \(y(0) = 0, y''(1) = 0 \text{ and } y(1) = 0, y'(1) = 0\).

To find \(y_1, y_2, y_3\).

The given differential equation is approximated as

\[
y''''i = \frac{y_{i-4} - 4y_{i-1} + 6y_{i} - 4y_{i+1} + y_{i+2}}{h^4} - 16y_i = x_i
\]
\[ y_{i-2} - 4y_{i-1} + \frac{95}{16}y_i - 4y_{i+1} + y_{i+2} = \frac{1}{256}x_i \]  \hspace{1cm} (10)

Boundary conditions (B.C.S) are given by

\[ y_0 = y(0) = 0 \]  \hspace{1cm} (11)

and

\[ y''(0) = y''_0 = \left[ \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \right]_{i=0} = 0 \]

\[ y_{-1} - 2y_0 + y_1 = 0 \]  \hspace{1cm} (12)

\[ y_{-1} = -y_1 \]  \hspace{1cm} (13)

\[ y'(1) = y'_4 = \left[ \frac{y_{i+1} - y_{i-1}}{2h} \right]_{i=4} = 0 \]

\[ y_5 = y_3 \]  \hspace{1cm} (14)

Putting \( i = 1, 2, 3 \) in equation (10), using conditions (11), (12), (13), (14), equation (10) reduce to following linear simultaneous equations

\[ \frac{79}{16}y_1 - 4y_2 + y_3 = \frac{1}{1024} \]  \hspace{1cm} (15)

\[ -4y_1 + \frac{95}{16}y_2 - 4y_3 = \frac{1}{512} \]  \hspace{1cm} (16)

\[ y_1 - 4y_2 + \frac{111}{16}y_3 = \frac{3}{1024} \]  \hspace{1cm} (17)

Solving equations (15, 16) and (17), required solution of BVP is

\[ y_1 = y(0.25) = 0.00255, \quad y_2 = y(0.5) = 0.0034, \quad y_3 = y(0.75) = 0.00202 \]

Example 2

Consider the fourth order linear non-homogeneous boundary value problem with boundary conditions defined by

\[ y'''(x) = x, \quad 0 \leq x \leq 1, \text{ for } y(x_i), \ x_i = 0.25, 0.5, 0.75, \]

\[ y(0) = 0, \quad y''(0) = 0 \quad \text{and} \quad y(1) = 0, \quad y'(1) = 0 \]

Solution:

Here interval is \( 0 \leq x \leq 1, \text{ ie, } [0, 1]. \)

Points are given by \( x_0 = 0, \ x_1 = x_0 + h = 0.25, \ x_2 = x_0 + 2h = 0.5, \ x_3 = x_0 + 3h = 0.75, \ x_4 = x_0 + 4h = 1. \)

Therefore, \( h = 0.25 \)

Boundary conditions are given by \( y(0) = 0, \ y''(1) = 0 \quad \text{and} \quad y(1) = 0, \ y'(1) = 0. \)

To find \( y_1, \ y_2, \ y_3. \)

The given differential equation is approximated as

\[ \frac{y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}}{h^4} = x_i \]  \hspace{1cm} (18)

Using B.C.S

\[ y_0 = y(0) = 0 \]  \hspace{1cm} (19)

\[ y''(0) = y''_0 = \left[ \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \right]_{i=0} = 0 \]

\[ y_{-1} - 2y_0 + y_1 = 0 \]  \hspace{1cm} (20)

\[ y_{-1} = -y_1 \]  \hspace{1cm} (21)

\[ y'(1) = y'_4 = \left[ \frac{y_{i+1} - y_{i-1}}{2h} \right]_{i=4} = 0 \]

\[ y_5 = y_3 \]  \hspace{1cm} (22)

Putting \( i = 1, 2, 3 \) in Eq. (18), using conditions (19), (20), (21), (22), the following system of linear equations are obtained.

\[ 5y_1 - 4y_2 + y_3 = \frac{1}{1024} \]  \hspace{1cm} (23)

\[ -4y_1 + 6y_2 - 4y_3 = \frac{1}{512} \]  \hspace{1cm} (24)
\[ y_1 - 4y_2 + 7y_3 = \frac{3}{1024} \]  
\[(25)\]

Solving equations (23), (24) and (25), desired numerical solutions of BVP are

\[ y_1 = y(0.25) = 0.0023, \quad y_2 = y(0.5) = 0.0031 \quad y_1 = y(0.75) = 0.00186 \]

5. Result Discussion

In this Study approximate numerical solution of fourth order ordinary boundary value problem which is arising from beam theory has been performed using FDM. Non homogeneous problems are handled and results are given in table 1 and table 2. If mesh size is reduced then error should be reduced as shown in Table 3.

| Table 1. | Comparison results for mesh size \( h = 0.25 \) in example 2. |
|----------|---------------------------------------------------------------|
| x        | Analytical solution | Numerical solution (FDM) | Error          |
| 0        | 0                  | 0                          | 0              |
| 0.25     | 0.00183            | 0.0023                     | 0.000469       |
| 0.5      | 0.002343           | 0.0031                     | 0.000756       |
| 0.75     | 0.001196           | 0.00186                    | 0.00066        |
| 1        | 0                  | 0                          | 0              |

| Table 2. | Comparison results for mesh size \( h = 0.25 \) in example 1. |
|----------|---------------------------------------------------------------|
| x        | Analytical solution | Numerical solution (FDM) | Error          |
| 0        | 0                  | 0                          | 0              |
| 0.25     | 0.00195            | 0.00255                    | 0.0006         |
| 0.5      | 0.002445           | 0.0034                     | 0.00095        |
| 0.75     | 0.001187           | 0.00202                    | 0.00083        |
| 1        | 0                  | 0                          | 0              |

| Table 3. | Comparison results for mesh size \( h = 0.125 \) in example 2. |
|----------|---------------------------------------------------------------|
| x        | Analytical solution | Numerical solution (FDM) | Error          |
| 0        | 0                  | 0                          | 0              |
| 0.125    | 0.001009           | 0.0013169                  | 0.0003079      |
| 0.25     | 0.00183            | 0.0021236                  | 0.0002936      |
| 0.375    | 0.002307           | 0.0028536                  | 0.000546       |
| 0.5      | 0.002343           | 0.002689                   | 0.000346       |
| 0.625    | 0.00193            | 0.002095                   | 0.000165       |
| 0.75     | 0.001196           | 0.001534                   | 0.000338       |
| 0.875    | 0.000400           | 0.000435                   | 0.0001227      |
| 1        | 0                  | 0                          | 0              |

6. Conclusion

In this study, it is cleared that FDM can be applied to determine the solution of fourth order linear homogeneous and nonhomogeneous boundary value problems which arise in civil and mechanical engineering from the beam bending theory. Numerical solution converges rapidly to the exact solution.
Results also show that if the mesh size is reduced finite difference method will give the better accuracy. Hence, this technique can be successfully applied in more complicated geometries in future work. In addition, for the calculation of FDM, software like Mathematica, Matlab, Maple can be used for larger domain.

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