Complex Field Formulation of the Quantum Estimation Theory

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Abstract. We present a complex field formulation of the quantum Fisher estimation theory that works natively with complex statistics on the dependence of complex parameters. This states new complex versions of the main quantities and results of the estimation theory depending on complex parameters, such as Fisher information matrices and Cramér-Rao bounds. This can be useful in contexts where the quantum states are described through complex parameters, as coherent states or squeezed states. We show an application of our theory in quantum communication with coherent states.

Keywords: estimation theory, statistic, Fisher information matrix, Cramér-Rao bound, standard quantum limit.
1. Introduction

The accurate determination of quantities plays a key role in the development of physical theories and its applications. For instance, high-precision measurement of the gyromagnetic factor of the electron and muon and its comparison with theoretical predictions is an important low-energy test of the relativistic quantum field theory of electrodynamics. In this context, estimation theory has become nowadays an ubiquitous tool. This theory, which deals with estimating the values of parameters based on experimentally acquired data that exhibits randomness, leads to the celebrated Cramér-Rao bound for the best achievable precision. A classic example of this bound is the standard or shot-noise limit for the interferometric determination of a phase. In the last two decades, advances in quantum information theory and foundations of quantum mechanics have led to consider quantum states and measurements as resources that can be used to further improve the accuracy of estimation processes. This has motivated the formulation of a quantum estimation theory [1, 2] that finds application when a relevant quantity is encoded in a quantum state and must be estimated using data obtained through quantum measurements. This theory plays a relevant role in quantum metrology [3–5], quantum sensing [7], quantum natural gradient [8–10], and quantum tomography [11–18], and it has already been experimentally implemented in optical interferometry [19–21], trapped ions [22], and condensed matter [23, 24], among others. The quantum estimation theory allows to exceed the standard limit and reach the Heisenberg limit, which represents a quadratic increase in precision with respect to the standard limit.

The estimation theory and its quantum extension are formulated in the field of real numbers. Despite the latter being successful [25–31], this adaptation seems unnatural since quantum mechanics is naturally formulated in the field of complex numbers. For instance, coherent and squeezed states are described by complex coefficients that are functions of complex variables and quantum measurements are described by the complex-valued Choi matrices. Recently, the advantages of estimating complex quantities using their native representation has been studied in the case of radio interferometric gain calibration [32] and optimization on complex variables has been applied to signal analysis [33] and neural networks [34–35]. Furthermore, complex numbers have been shown to play a fundamental role in the formulation of quantum mechanics [36–37], that is, a formulation of quantum mechanics on real numbers leads in certain scenarios to different predictions than quantum mechanics formulated on complex numbers. Alternative formulations of estimation theory have been proposed [38, 39], where the Fisher information matrix and the Cramér-Rao bound are adapted to consider complex statistics of complex parameters. This is implemented using W. Wirtinger’s complex calculus [40] and a particular complex map [38]. Wirtinger’s calculus is developed on the joint basis of the complex variables and their conjugates, allowing to define a complex differentiation theory in a similar way to their real counterpart. This approach has been employed in the estimation of pure quantum
states via optimization on the complex numbers \([17, 41]\). The complex map connects
the main results of the theory of quantum estimation formulated in real numbers with
that formulated on complex numbers, such as the Fisher information matrix and the
Cramér-Rao bound.

Motivated by the above, in this article we develop a complex field formulation of
the quantum estimation theory based on Wirtinger’s calculus and an extension of the
complex map, working with complex statistics which depend on complex parameters.
This formulation gives a natural framework to study estimation problems where the
quantum states are native complex functions of complex parameters. We define complex
field versions of the quantum Fisher information matrices, both symmetric and right,
and the Cramér-Rao bounds for the covariance matrix and the weighted mean square
error. We also specialize all the above results to pure states. We apply our theory to
an example in quantum communications, that is, the optimal coding and decoding of a
complex parameter encoded in a coherent state of the electromagnetic field \([42]\).

The paper is organized as follows: In section II we introduce some essential
preliminary results such as the Wirtinger Calculus, the estimation theory for complex
statistics dependent on complex parameters, and the quantum estimation theory.
In section III, we develop the complex field formulation of the quantum estimation
theory. In section IV, we apply of our formulation to the problem of optimal quantum
communications using coherent state. In section V, we conclude and summarize.

2. Preliminary results

In this section we briefly review the main results of estimation theory. First, we introduce
the estimation theory based on real parameters, which studies complex statistics through
a real transformation of themselves and their parameters. We present recent results on
the estimation problem of complex statistics for complex parameters \([40]\). In the case of
estimating parameters from quantum states, the quantum properties can be exploited
to formulate a quantum version of the estimation theory. We also summarize the basic
elements of the quantum estimation theory.

2.1. Wirtinger derivatives

The concept of derivation of a function of complex variables given by complex calculus
\([43, 44]\) is limited to holomorphic functions, that is, functions that are infinitely
differentiable in a neighborhood of the point. This is a strong restriction that leaves
many interesting complex functions without derivative, for instance \(f(z) = |z|^2\), which
plays a key role in quantum mechanics. There is, however, a generalization of the
derivation of complex functions through the Wirtinger operators or Wirtinger derivatives
\([40]\), which are defined by

\[
\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{z^*} = \frac{1}{2}(\partial_x + i\partial_y),
\]
where \( z = x + iy \in \mathbb{C} \), and \( z^* \) denote its complex conjugate. These operators can be applied to any function \( f(z) = f(z, z^*) = f(x, y) \) whose real and imaginary parts have partial real derivatives. An important property of these operators lies in its behavior, which emulates the partial derivative of a real-valued function of a real variable [40].

In the case of a multivariable complex function \( F : \mathbb{C}^n \rightarrow \mathbb{C}^m \), its Jacobian matrix is defined by

\[
\mathcal{D}_F(z_0) := \mathcal{D}_z \hat{F}(z_0) = \begin{bmatrix} \mathcal{D}_z F(z_0) & \mathcal{D}_z F(z_0)^* \\ \mathcal{D}_z^* F(z_0) & \mathcal{D}_z^* F(z_0)^* \end{bmatrix} \in \mathbb{C}^{2m \times 2n},
\]

for each \( z_0 \in \mathbb{C}^n \), where \( \hat{F} = [F, F^*]^\top \), \( \hat{z} = [z, z^*]^\top \), and

\[
(\mathcal{D}_z F(z_0))_{ij} = \partial_{z_i} F^j(z_0), \quad (\mathcal{D}_z^* F(z_0))_{ij} = \partial_{z^*_j} F^i(z_0),
\]

for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \). Let us note that \( \mathcal{D}_F \) involves the derivatives of \( F \) and also of \( F^* \), because \( (\mathcal{D}_z F(z_0))^* = \mathcal{D}_F F^*(z_0) \) and \( (\mathcal{D}_z F(z_0))^* = \mathcal{D}_z F^*(z_0) \).

2.2. Estimation theory for complex parameters

Estimation theory and its applications have been intensively studied [45–52], the Cramér-Rao bound been one of its most important results. This theory deals with estimating the value of a parameter \( \theta = \alpha + i\beta \in \mathbb{C}^k \) given empirical data obtained from a random variable \( \rho(\theta) \).

This problem is usually stated in the field of the real numbers. Considering the real representation of \( \theta \) given by

\[
\hat{\theta} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{R}^{2k},
\]

the probability density functions of \( \rho(\theta) \) and \( \rho(\hat{\theta}) \) are equivalent, and the Fisher information matrix (FIM) of \( \rho \) for the real parameter \( \hat{\theta} \) is defined by

\[
I_{\hat{\theta}} = \mathbb{E} \left( \nabla_{\hat{\theta}} \ln f(\rho(\hat{\theta})) \nabla_{\hat{\theta}} \ln f(\rho(\hat{\theta}))^\top \right),
\]

where \( f \) is the likelihood function. To obtain information about \( \rho \), we consider a complex statistic \( \bar{t} \), that is, a random variable that describes a characteristic of \( \rho(\theta) \). Its real representation \( \bar{t} \) has a covariance matrix given by

\[
\text{Cov}(\bar{t}) = \mathbb{E} \left( (\bar{t} - \mathbb{E}(\bar{t}))(\bar{t} - \mathbb{E}(\bar{t}))^\top \right).
\]

The Cramér-Rao inequality (CRI) for \( \bar{t} \) states that

\[
\text{Cov}(\bar{t}) \geq D_g I_{\hat{\theta}}^{-1} D_g^\top,
\]

where \( \hat{g} \) is the real representation of \( g(\theta) := \mathbb{E}(\bar{t}) \) and \( D_g \) is the Jacobian matrix of \( \hat{g} \). The term at the right side of (7) is the Cramér-Rao bound (CRB) and represents the smallest uncertainty with which the statistic \( \bar{t} \) can be measured.

The above analysis for complex statistics can be replicated working in its native complex field [38]. Consider the continuous map \( \langle \cdot \rangle_{\mathbb{C}} : \mathbb{C}^{2d \times 2k} \rightarrow \mathbb{C}^{2d \times 2k} \), defined for all \( G \in \mathbb{C}^{2d \times 2k} \) by

\[
\langle G \rangle_{\mathbb{C}} = 2M_{2d}^1 GM_{2k},
\]

where \( M_{2d}^1 \) is a matrix in \( \mathbb{C}^{2d \times 2d} \) and \( G \) is a matrix in \( \mathbb{C}^{2d \times 2d} \).
where \( M_{2l} \) is a complex matrix of size \( 2l \times 2l \) given by

\[
M_{2l} := \frac{1}{2} \begin{bmatrix} I_l & I_l \\ -iI_l & iI_l \end{bmatrix},
\]

for any \( l \in \mathbb{N} \), and \( I_l \) is the \( l \times l \) identity matrix. Note that \( M_{2l} \) is invertible, its inverse is given by \( M_{2l}^{-1} = 2M_{2l}^\dagger \). The action of the map onto a block matrix

\[
G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}
\]

(10)

where \( G_{jk} \in \mathbb{C}^{d \times k} \) leads to

\[
\langle G \rangle_C = \begin{bmatrix} G_{11} + iG_{21} - i(G_{12} + iG_{22}) & G_{11} + iG_{21} + i(G_{12} + iG_{22}) \\ G_{11} - iG_{21} - i(G_{12} - iG_{22}) & G_{11} - iG_{21} + i(G_{12} - iG_{22}) \end{bmatrix}.
\]

(11)

Other properties of the map \( \langle \cdot \rangle_C \), which will be very useful later, are presented in the following proposition:

**Proposition 2.1** Let \( G, H \in \mathbb{C}^{2d \times 2k} \), \( \sigma_{2k} \) a rotation matrix given by

\[
\sigma_{2k} = \begin{bmatrix} 0 & \mathbb{I}_{2k} \\ \mathbb{I}_{2k} & 0 \end{bmatrix},
\]

and \( f \) a function defined on the spectrum of a matrix, that is,

\[
f(G) = U^{-1} f(\Lambda) U,
\]

for each symmetric matrix \( G \); \( \forall G \in \mathbb{C}^{2d \times 2k} \), where \( G = U^{-1} \Lambda U \), with \( \Lambda \) the diagonal matrix with the eigenvalues of \( G \). Then

(i) \( \langle G + \lambda H \rangle_C = \langle G \rangle_C + \lambda \langle H \rangle_C \), \( \forall \lambda \in \mathbb{C} \);

(ii) \( G = G^\dagger \Longleftrightarrow \langle G \rangle_C = \langle G \rangle_C^\dagger \).

(iii) \( G \) invertible \( \Longleftrightarrow \langle G \rangle_C \) invertible. Moreover \( \langle G \rangle_C^{-1} = \frac{1}{4} \langle G^{-1} \rangle_C \);

(iv) \( G \succeq 0 \Longleftrightarrow \langle G \rangle_C \succeq 0 \), for each symmetric matrix \( G \);

(v) \( \langle G^\dagger \rangle_C = \langle G \rangle_C^\dagger \).

(vi) \( \langle G_1 G_2 \rangle_C = \frac{1}{2} \langle G_1 \rangle_C \langle G_2 \rangle_C \).

(vii) \( \text{Tr}(G) = \frac{1}{2} \text{Tr}(\langle G \rangle_C) \).

(viii) \( \langle f(G) \rangle_C = 2f \left( \frac{1}{2} \langle G \rangle_C \right) \).

(ix) \( \langle G^* \rangle_C = \sigma_{2d} \langle G \rangle_C \sigma_{2k} \).

Consider the conjugated extension of the statistic \( t \), given by

\[
\hat{t} = \begin{bmatrix} t \\ t^* \end{bmatrix}.
\]

(13)

Notice that \( \bar{t} = M_{2d} \hat{t} \). Besides, the FIM for the complex parameter \( \hat{\theta} = [\theta, \theta^*]^\top \) is

\[
\mathcal{I}_\theta = \begin{bmatrix} I_\theta & P_\theta \\ P_\theta^* & I_\theta^* \end{bmatrix},
\]

(14)
where
\[ I_\theta = \mathbb{E} \left( \nabla_{\theta^*} \ln f(z|\theta) \nabla_{\theta^*} \ln f(z|\theta)^\dagger \right) \quad \text{and} \quad P_\theta = \mathbb{E} \left( \left( \nabla_{\theta^*} \ln f(z|\theta) \nabla_{\theta^*} \ln f(z|\theta)^\dagger \right) \right), \quad (15) \]
are the FIM of \( \theta \) and the pseudo-FIM (pFIM) of \( \theta \), respectively, and \( \nabla_{\theta^*} = [\partial_{\theta_1^*}, \partial_{\theta_2^*}, \ldots, \partial_{\theta_n^*}]^\top \). Then, the complex map relates the real transformation and the conjugate extension of \( t \) through the expressions
\[ \text{Cov} \left( \hat{t} \right) := \mathbb{E} \left( \left( \hat{t} - \mathbb{E} (\hat{t}) \right) \left( \hat{t} - \mathbb{E} (\hat{t}) \right)^\dagger \right) = \langle \text{Cov}(\bar{t}) \rangle_C, \quad (16) \]
\[ D_{\hat{g}} := \begin{bmatrix} D_{\theta^*}g & D_{\theta^*}g^* \\ (D_{\theta^*}g)^* & (D_{\theta^*}g)^* \end{bmatrix} = \frac{1}{2} \langle D_{\bar{g}} \rangle_C, \quad (17) \]
\[ I_{\hat{\theta}} = \frac{1}{4} \langle I_{\bar{\theta}} \rangle_C, \quad (18) \]
where \( D_{\hat{g}} \) is the Wirtinger Jacobian matrix of the conjugate extension of \( g \), whose components are given by
\[ (D_{\theta^*}g)_{ij} = \partial_{\theta_j}g_i, \quad (D_{\theta^*}g)^*_{ij} = \partial_{\theta_j^*}g_i, \quad i = 1, \ldots, d, \ j = 1, \ldots, k. \quad (19) \]
Applying the map \( \langle \cdot \rangle_C \) to (7) and using (16), (17) and (18) we obtain the CRI to the complex statistic \( \hat{t} \), working directly on the complex field, that is,
\[ \text{Cov} \left( \hat{t} \right) \geq D_{\hat{g}} I_{\hat{\theta}}^{-1} D_{\hat{g}}^\dagger. \quad (20) \]

2.3. Quantum estimation theory for real parameters

Quantum estimation theory uses main objects from quantum mechanics to find statistics with better uncertainty [2,25,53]. Typically, the quantity to be estimated is encoded in a quantum state and a sample statistics is generated through a measurement. According to Born’s rule, the measurement outcomes depend not only on the state but also on the chosen measurement. The fundamental object of the quantum estimation theory is the quantum version of the FIM. However, unlike the classical case, the quantum FIM involves quantum states instead of probability density functions. Due to this, two different types of logarithmic derivatives appear, the symmetric and the right. Each one defines different FIM and CRI and, depending on the problem studied, any of the derivatives can be useful.

In first place, the Symmetric Quantum FIM (SQFIM) of the quantum state \( \rho \) for the real parameter \( \bar{\theta} \) is given by
\[ \left[ J_{\bar{\theta}}^S \right]_{ij} = \frac{1}{2} \text{Tr} \left( \rho (L_{\theta_i^*}^S L_{\theta_j^*}^S + L_{\theta_j^*}^S L_{\theta_i^*}^S) \right), \quad i, j = 1, \ldots, 2k, \quad (21) \]
where \( \bar{\theta} \) is the real representation of \( \theta \), and \( L_{\theta_i^*}^S \) is a hermitian matrix called symmetric logarithmic derivative, which is implicitly defined by the equation [2]
\[ \partial_{\theta_i^*} \rho = \frac{1}{2} (\rho L_{\theta_i^*}^S + L_{\theta_i^*}^S \rho), \quad i = 1, \ldots, 2k. \quad (22) \]
The Right Quantum FIM (RQFIM) of \( \rho \) for the real parameter \( \bar{\theta} \) is defined by [27,28]
\[ \left[ J_{\bar{\theta}}^R \right]_{ij} = \text{Tr} \left( \rho L_{\theta_i^r}^R L_{\theta_j^r}^{R\dagger} \right), \quad i, j = 1, \ldots, 2k, \quad (23) \]
where $L_{R}^{R}$ is the complex matrix called right logarithmic derivative. This is defined by the equation

$$\partial_{\bar{\theta}}\rho = \rho L_{R}^{R} \quad i = 1, \ldots, 2k. \tag{24}$$

Both quantum FIMs provide an extension of the CRI (7) to the case of quantum systems, called quantum Cramér-Rao inequality (QCRI), which establishes

$$I_{\bar{\theta}} \leq J_{X}^{\bar{\theta}}, \tag{25}$$

for any $X \in \{S, R\}$, where $I_{\bar{\theta}}$ is the FIM of $\bar{\rho}$ for the real parameters $\bar{\theta}$ given by (5).

The QCRI (25) is very useful to study systems with few parameters. However, for systems with many parameters, it is more appropriate to work with a different figure of merit rather than the covariance matrix. An important figure of merit is the weighted mean squared error for an unbiased estimator $\bar{t}$, which for the real variable $\bar{t}$ is defined by

$$w(\bar{t}) = \text{Tr} (W_{\bar{\theta}} \text{Cov}(\bar{t})), \tag{26}$$

where $W_{\bar{\theta}} \in \mathbb{R}^{2k \times 2k}$ is a weighting matrix. It has been shown [27, 31] that the following two quantities are both lower bounds to $w$,

$$w^{S} = \text{Tr} (W_{\bar{\theta}} (J_{\bar{\theta}}^{S})^{-1}), \tag{27}$$

$$w^{R} = \text{Tr} (W_{\bar{\theta}} \text{Re} [(J_{\bar{\theta}}^{R})^{-1}]) + \text{Tr} \text{Abs} (W_{\bar{\theta}} \text{Im} [(J_{\bar{\theta}}^{R})^{-1}]), \tag{28}$$

where $\text{Abs}(G)$ is defined by

$$\text{Abs}(G) = O^{-1} \begin{bmatrix} |g_1| \\ \vdots \\ |g_n| \end{bmatrix} O, \tag{29}$$

where $\{g_k\}_{k=1}^{n}$ are the eigenvalues of $G \in \mathbb{C}^{n \times n}$ diagonalizable and $O$ is the matrix that diagonalizes $G$.

An important case is the estimation problem of parameters encoded on pure states. It is well-known that the SQFIM for a pure state $|\psi\rangle$, which depends on the parameters $\theta \in \mathbb{C}^{k}$, is given by

$$[J_{\bar{\theta}}^{S}]_{ij} = 4\text{Re} \langle \partial_{\bar{\theta}} \psi | (\mathbb{I} - |\psi\rangle\langle\psi|) | \partial_{\bar{\theta}} \psi \rangle, \tag{30}$$

where $\bar{\theta}$ is the real representation of $\theta$ and $|\partial_{\bar{\theta}} \psi\rangle := \partial_{\bar{\theta}} |\psi\rangle$. Notice that the SQFIM is proportional to the Hessian of the Fubini-Study metric [8, 54]

$$J_{\bar{\theta}}^{S} = -2 \left[ \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} \langle \psi(\bar{\theta})|\psi(\bar{\theta})\rangle \right)^{\top} \right]_{\bar{\theta} = \bar{\theta}}. \tag{31}$$

Furthermore, a necessary and sufficient condition to attain the symmetric quantum CRB of $\bar{t}$ for pure states is given by [30, 55]

$$\langle \psi | [L_{\bar{\theta}}, L_{\bar{\theta}}] | \psi \rangle = 0, \quad i, j = 1, \ldots, k, \tag{32}$$

with $[A, B] = AB - BA$ the commutator between $A$ and $B$. When this condition is fulfilled, the symmetric logarithmic derivatives $\{L_{\bar{\theta}}\}$ are an optimal statistic, or equivalently, an optimal measurement.
It is not possible to define a RQFIM for a low-rank state, since (24) has no solution in this case \[27\]. However, for the QCRB only the inverse of the FIM is needed, which can be obtained approaching the pure state \(|\psi\rangle\) by a mixed state \(\rho(\epsilon)\) such that \(|\psi\rangle\langle\psi| = \lim_{\epsilon \to 0} \rho(\epsilon)\). It has been proved \[25\] that the inverse of the RQFIM for a pure state is given by

\[
\lim_{\epsilon \to 0} \left( J_R^\theta(\epsilon) \right)^{-1} = \left( J_S^\theta \right)^{-1} + i \left( J_S^\theta \right)^{-1} K_\theta \left( J_S^\theta \right)^{-1}
\]

where

\[
[K_\theta]_{ij} = 4\text{Im} \left[ \langle \partial_{\bar{\theta}_j} \psi | \partial_{\bar{\theta}_i} \psi \rangle - \langle \partial_{\theta_i} \psi | \psi \rangle \langle \psi | \partial_{\theta_j} \psi \rangle \right],
\]

and \(J_S^\theta\) is the SQFIM for \(|\psi\rangle\).

3. Quantum estimation theory for complex parameters.

The space state of quantum mechanics is typically formulated over the field of the complex numbers. A quantum state provides all information required to predict the real-valued probabilities associated to the outcomes of any conceivable experiment, which is described through a generalized measurement. In general, any characteristics or information from a quantum state will be represented by a real number, which is a function of the complex coefficients that define a particular state. It seems thus convenient and consistent to formulate a quantum estimation theory that works directly with complex statistics depending on complex parameters. Here, we present an extension of the quantum estimation theory, briefly summarized in the section \[23\] working on the field of complex numbers by means of Wirtinger’s calculus.

We define the SQFIM of the state \(\rho\) for the complex parameter \(\hat{\theta}\) as

\[
J_S^\theta = \begin{bmatrix} J_S^\theta & Q_S^\theta \\ (Q_S^\theta)^* & (J_S^\theta)^* \end{bmatrix},
\]

where the block matrices are the SQFIM and the pseudo-SQFIM of the state \(\rho\) for the real parameter \(\theta\), which are given by

\[
[J_S^\theta]_{ij} = \frac{1}{2} \text{Tr} \left( \rho(L_{\alpha_i} L_{\bar{\alpha}_j}^S + L_{\alpha_j}^S L_{\bar{\alpha}_i}^S) \right); \quad [Q_S^\theta]_{ij} = \frac{1}{2} \text{Tr} \left( \rho(L_{\alpha_i}^S L_{\bar{\alpha}_j}^S + L_{\alpha_j}^S L_{\bar{\alpha}_i}^S) \right),
\]

for each \(i, j = 1, \ldots, k\), with \(\{L_{\alpha_i}^S, L_{\bar{\alpha}_i}^S\}\) the complex symmetric logarithm derivatives implicitly defined by

\[
\partial_{\alpha_i} \rho = \frac{1}{2} (\rho L_{\alpha_i}^S + L_{\alpha_i}^S \rho); \quad \partial_{\bar{\alpha}_i} \rho = \frac{1}{2} (\rho L_{\alpha_i}^S + L_{\bar{\alpha}_i}^S \rho).
\]

Let us note that, unlike the case of the real parameters, the complex symmetric logarithm derivatives are not hermitian. Instead, these satisfy the property \(L_{\bar{\alpha}_i}^S = L_{\alpha_i}^{S \dagger}\). Using \(11\) and \(22\), we can show that there exists a relation between the symmetric logarithmic derivatives of the complex parameter \(\hat{\theta}\) and its real counterparts, that is,

\[
L_{\theta_i}^S = \frac{1}{2} (L_{\alpha_i}^S - i L_{\beta_i}^S), \quad L_{\bar{\theta}_i}^S = \frac{1}{2} (L_{\alpha_i}^S + i L_{\beta_i}^S), \quad i = 1, \ldots, 2k.
\]

This result allows us to establish the following theorem:
Theorem 3.1 Let $\tilde{\theta}$ be the real representation of $\theta$ and $\tilde{\theta}$ its conjugate extension, $J^{S}_\theta$ the SQFIM of state $\rho$ for the parameter $\theta$, and $\hat{J}^{S}_\theta$ the SQFIM of state $\rho$ for the parameter $\tilde{\theta}$. Then
\[
J^{S}_\theta = \frac{1}{4} \langle J^{S}_\theta \rangle \mathcal{C}.
\] (39)

To proof this, we notice that $J^{S}_\theta$ can be expressed as a block matrix, that is,
\[
J^{S}_\theta = \begin{bmatrix}
J^{S}_{\alpha\alpha} & J^{S}_{\beta\alpha} \\
J^{S}_{\alpha\beta} & J^{S}_{\beta\beta}
\end{bmatrix},
\]
where $[J^{S}_{\alpha\beta}]_{ij} = \text{Tr} (\rho [L^{S}_{\alpha_i} L^{S}_{\beta_j} + L^{S}_{\beta_j} L^{S}_{\alpha_i}])/2$ and $a, b \in \{\alpha, \beta\}$. Then, using the map $\langle \cdot \rangle \mathcal{C}$ we obtain
\[
\langle J^{S}_\theta \rangle \mathcal{C} = \begin{bmatrix}
J^{S}_{\alpha\alpha} + iJ^{S}_{\beta\alpha} - i(J^{S}_{\alpha\beta} + iJ^{S}_{\beta\beta}) & J^{S}_{\alpha\beta} + iJ^{S}_{\beta\alpha} + i(J^{S}_{\alpha\beta} + iJ^{S}_{\beta\beta}) \\
J^{S}_{\alpha\beta} - iJ^{S}_{\beta\alpha} - i(J^{S}_{\alpha\beta} - iJ^{S}_{\beta\beta}) & J^{S}_{\alpha\alpha} - iJ^{S}_{\beta\alpha} + i(J^{S}_{\alpha\beta} - iJ^{S}_{\beta\beta})
\end{bmatrix},
\] (40)

where from (38), the first block is given by
\[
[J^{S}_{\alpha\alpha} + iJ^{S}_{\beta\alpha} - i(J^{S}_{\alpha\beta} + iJ^{S}_{\beta\beta})]_{ij} = \frac{1}{2} \text{Tr} \left( \rho \left( (L^{S}_{\alpha_i} + iL^{S}_{\beta_j})(L^{S}_{\alpha_j} - iL^{S}_{\beta_j}) \\
+ (L^{S}_{\alpha_j} - iL^{S}_{\beta_j})(L^{S}_{\alpha_i} + iL^{S}_{\beta_j}) \right) \right),
\]
\[
= \frac{1}{2} \text{Tr} \left( \rho (L^{S}_{\alpha_i} L^{S}_{\alpha_j} + L^{S}_{\beta_j} L^{S}_{\beta_i}) \right).
\]

Similarly with the remaining blocks, we obtain
\[
\langle J^{S}_\theta \rangle \mathcal{C} = 4 \begin{bmatrix}
\frac{1}{2} \text{Tr} \left( \rho (L^{S}_{\alpha_i} L^{S}_{\alpha_j} + L^{S}_{\beta_j} L^{S}_{\beta_i}) \right) & \frac{1}{2} \text{Tr} \left( \rho (L^{S}_{\alpha_i} L^{S}_{\alpha_j} + L^{S}_{\beta_j} L^{S}_{\beta_i})^* \right) \\
\frac{1}{2} \text{Tr} \left( \rho (L^{S}_{\alpha_j} L^{S}_{\beta_j} + L^{S}_{\beta_i} L^{S}_{\beta_i}) \right) & \frac{1}{2} \text{Tr} \left( \rho (L^{S}_{\alpha_j} L^{S}_{\beta_j} + L^{S}_{\beta_i} L^{S}_{\beta_i})^* \right)
\end{bmatrix},
\]
and conclude the proof from (35) and (36).

The above analysis can be applied to the quantum estimation theory based on the right logarithmic derivative. Motivated by (15) and (23), we define the RQFIM of state $\rho$ for the complex parameters $\tilde{\theta}$ as
\[
J^{R}_\theta = \begin{bmatrix}
J^{R}_{\theta} & Q^{R}_{\theta} \\
Q^{R}_{\theta} & J^{R}_{\theta^*}
\end{bmatrix},
\] (41)

where the RQFIM and the pseudo-RQFIM of state $\rho$ for the the parameter $\theta$ are given by
\[
[J^{R}_\theta]_{ij} = \text{Tr} \left( \rho \left[ L^{R}_{\theta_i} L^{R\dagger}_{\theta_j} \right] \right), \quad [Q^{R}_\theta]_{ij} = \text{Tr} \left( \rho \left[ L^{R\dagger}_{\theta_i} L^{R}_{\theta_j} \right] \right),
\] (42)
for each $i, j = 1, \ldots, k$, with $\{L^{R}_{\theta_i}, L^{R\dagger}_{\theta_i}\}$ the complex right logarithm derivatives implicitly define by
\[
\partial_{\theta_i} \rho = \rho L^{R}_{\theta_i}, \quad \partial_{\theta_i^*} \rho = \rho L^{R\dagger}_{\theta_i}, \quad i = 1, \ldots, k.
\] (43)

Let us note that analogously at the case of the real parameters, $L^{R}_{\theta_i}$ are not hermitian. As in the symmetric scenario, using (1) and (24) we can shown that the relation between
the right logarithmic derivatives of the complex parameter \( \hat{\theta} \) and its counterparts of real parameters is

\[
\mathcal{L}_{\hat{\theta}}^R = \frac{1}{2}(L_{\alpha_i}^R - iL_{\beta_i}^R), \quad \mathcal{L}_{\hat{\theta}}^S = \frac{1}{2}(L_{\alpha_i}^R + iL_{\beta_i}^S), \quad i = 1, \ldots, k. \tag{44}
\]

This result allows us to establish the following theorem.

**Theorem 3.2** Let \( \hat{\theta} \) be the real representation of \( \theta \) and \( \hat{\theta} \) its conjugate extension, \( \mathcal{J}_{\hat{\theta}}^R \) the RQFIM of \( \rho \) for the parameter \( \theta \) and \( \mathcal{J}_{\hat{\theta}}^X \) the SQFIM of \( \theta \) for the parameter \( \hat{\theta} \). Then

\[
\mathcal{J}_{\hat{\theta}}^R = \frac{1}{4} \langle J_{\hat{\theta}}^R \rangle_C. \tag{45}
\]

To proof this, we consider the block matrix

\[
\mathcal{J}_{\hat{\theta}}^R = \begin{bmatrix} J_{\alpha\alpha}^R & J_{\alpha\beta}^R \\ J_{\beta\alpha}^R & J_{\beta\beta}^R \end{bmatrix},
\]

where \([J_{ab}^R]_{ij} = \text{Tr} \left( \rho L_{ij}^R L_{ab}^R \right)\) and \(a, b \in \{\alpha, \beta\}\). Then applying the map \( \langle \cdot \rangle_C \), we have

\[
\langle J_{\hat{\theta}}^R \rangle_C = \begin{bmatrix} J_{\alpha\alpha}^R + \text{i}J_{\beta\alpha}^R - \text{i}(J_{\alpha\beta}^R + \text{i}J_{\beta\beta}^R) & J_{\alpha\alpha}^R + \text{i}J_{\beta\alpha}^R + \text{i}(J_{\alpha\beta}^R + \text{i}J_{\beta\beta}^R) \\ J_{\beta\alpha}^R - \text{i}J_{\alpha\beta}^R - \text{i}(J_{\alpha\beta}^R - \text{i}J_{\beta\beta}^R) & J_{\beta\beta}^R - \text{i}J_{\alpha\beta}^R + \text{i}(J_{\alpha\beta}^R - \text{i}J_{\beta\beta}^R) \end{bmatrix}, \tag{46}
\]

working on the first block of (46) and recalling (44), we have that its components are given by

\[
[J_{\alpha\alpha}^R + \text{i}J_{\beta\alpha}^R - \text{i}(J_{\alpha\beta}^R + \text{i}J_{\beta\beta}^R)]_{ij} = \text{Tr} \left( \rho [(L_{\alpha\alpha}^R - \text{i}L_{\beta\alpha}^R)(L_{\alpha\alpha}^R - \text{i}L_{\beta\beta}^R)] \right) = 4 \text{Tr} \left( \rho \left( L_{\alpha\alpha}^R L_{\alpha\alpha}^R \right) \right).
\]

Analogously with the other blocks, we conclude that

\[
\langle J_{\hat{\theta}}^R \rangle_C = 4 \begin{bmatrix} \text{Tr} \left( \rho \left( L_{\hat{\theta} \alpha}^R L_{\hat{\theta} \alpha}^R \right) \right) & \text{Tr} \left( \rho \left( L_{\hat{\theta} \alpha}^R L_{\hat{\theta} \beta}^R \right) \right) \\ \text{Tr} \left( \rho \left( L_{\hat{\theta} \beta}^R L_{\hat{\theta} \alpha}^R \right) \right) & \text{Tr} \left( \rho \left( L_{\hat{\theta} \beta}^R L_{\hat{\theta} \beta}^R \right) \right) \end{bmatrix},
\]

and finish the proof from (41) and (42).

Since we have defined the SQFIM and the RQFIM for complex parameters, we can state the quantum Cramér-Rao inequality (QCRI) for complex parameters as follows:

**Theorem 3.3** Let \( \hat{\theta} \) be the conjugate extension of state \( \theta \), \( \mathcal{I}_\theta \) the classical FIM of \( \rho \) for the parameter \( \hat{\theta} \) and \( \mathcal{J}_\theta^X \) the SQFIM or RQFIM of state \( \rho \) for the parameter \( \hat{\theta} \). Then

\[
\mathcal{I}_\theta \leq \mathcal{J}_\theta^X, \quad X \in \{S, R\}. \tag{47}
\]

To obtain (47), we apply the map \( \langle \cdot \rangle_C \) to (25) and use the definitions (15), (21) and (23). This result and (7) lead us to the following inequalities:

**Corollary 3.1** Let \( \hat{t} \) be the real transformation of \( t \) and \( \hat{t} \) its conjugate extension, \( \mathcal{I}_\theta \) the classical FIM of state \( \rho \) for the parameter \( \hat{\theta} \), and \( \mathcal{J}_\theta^X \) the SQFIM of state \( \rho \) for the parameter \( \hat{\theta} \). Then

\[
\text{Cov} \left( \hat{t} \right) \geq \mathcal{D}_g \left( \mathcal{I}_\theta \right)^{-1} \mathcal{D}_g^\dagger \geq \mathcal{D}_g \left( \mathcal{J}_\theta^X \right)^{-1} \mathcal{D}_g^\dagger. \tag{48}
\]
Moreover, the quantum Cramér-Rao bounds are equivalent, that is
\[
\text{Cov}(i) \geq D_g \left(J^X_\theta\right)^{-1} D_g^T \iff \text{Cov}(i) \geq D_g \left(J^X_\theta\right)^{-1} D_g^\dagger,
\]
(49)
and both are simultaneously attained, that is, if \(\bar{i}\) attains the real quantum Cramér-Rao bound, then \(\hat{i}\) attains the complex quantum Cramér-Rao bound, and vice versa.

In general, it is not known whether the right limit or the symmetric limit is better for a given problem, so one has to test both on a case-by-case basis.

The previous corollary provides us with a compact bound for the covariance of \(\hat{i}\), which includes a bound for \(i\). However, to obtain an explicit bound for the covariance of \(t\), which is our studied statistics, we need to work with the block form of \(J^X_\theta\), with \(X\) in \(\{S, R\}\).

**Corollary 3.2** Let \(\hat{\theta}\) be the conjugate extension of \(\theta\), and \(J^S_\theta\) the SQFIM of state \(\rho\) for the parameter \(\hat{\theta}\) given by (35) and (36), and considering
\[
(J^S_\theta)^{-1} = \left[ \begin{array}{cc} J^S_\theta & Q^S_\theta \\ (Q^S_\theta)^* & (J^S_\theta)^* \end{array} \right]^{-1} = \left[ \begin{array}{cc} (E^S_\theta)^{-1} & -F^S_\theta (E^S_\theta)^{-1} \\ -(F^S_\theta)^* (E^S_\theta)^{-1} & (E^S_\theta)^{-1} \end{array} \right],
\]
(50)
where \(E^S_\theta = J^S_\theta - Q^S_\theta (J^S_\theta)^{-1} (Q^S_\theta)^*\) and \(F^S_\theta = (J^S_\theta)^{-1} Q^S_\theta \). Then
\[
\text{Cov}(t) \geq D_\theta g \left( E^S_\theta \right)^{-1} (D_\theta g)^\dagger - D_\theta^* g (F^S_\theta)^* \left( E^S_\theta \right)^{-1} \left( F^S_\theta \right)^* (D_\theta g)^\dagger \
- D_\theta^* g F^S_\theta \left( E^S_\theta \right)^{-1} (D_\theta^* g)^\dagger + D_\theta^* g \left( E^S_\theta \right)^{-1} (D_\theta^* g)^\dagger.
\]
(51)
An analogous result can be obtained for the right derivative as follows:

**Corollary 3.3** Let \(\hat{\theta}\) be the conjugate extension of \(\theta\), and \(J^R_\theta\) the RQFIM of state \(\rho\) for the parameter \(\hat{\theta}\) given by (41) and (42), and considering
\[
(J^R_\theta)^{-1} = \left[ \begin{array}{cc} J^R_\theta & Q^R_\theta \\ Q^R_* & J^R_* \end{array} \right]^{-1} = \left[ \begin{array}{cc} \left( E^R_\theta \right)^{-1} & -F^R_\theta \left( E^R_\theta \right)^{-1} \\ -F^R_* \left( E^R_* \right)^{-1} & \left( E^R_* \right)^{-1} \end{array} \right],
\]
(52)
where \(E^R_\theta = J^R_\theta - Q^R_\theta (J^R_\theta)^{-1} \) and \(F^R_\theta = Q^R_\theta (J^R_\theta)^{-1} \). Then
\[
\text{Cov}(t) \geq D_\theta g \left( E^R_\theta \right)^{-1} (D_\theta g)^\dagger - D_\theta^* g F^R_* \left( E^R_* \right)^{-1} (D_\theta g)^\dagger \
- D_\theta^* g F^R_\theta \left( E^R_* \right)^{-1} (D_\theta^* g)^\dagger + D_\theta^* g \left( E^R_* \right)^{-1} (D_\theta^* g)^\dagger.
\]
(53)
Unlike the symmetric case, the matrices \(E^S_\theta\) and \(E^R_*\) are independent. Besides, in both cases, when the pseudo-FIM \(Q^X_\theta\) vanishes, the matrices \((J^X_\theta)^{-1}\) become block diagonal, and in addition, if \(t\) is an unbiased estimator, then the inequality is simplified to
\[
\text{Cov}(t) \geq \left( J^X_\theta \right)^{-1}, \quad X \in \{S, R\}. \tag{54}
\]

In order to get a lower bound for the weighted mean square error \(w\) in the complex parameter case, we define
\[
w(i) = \text{Tr} \left( W_\theta \text{Cov}(i) \right),
\]
(55)
where \(W_\theta\) is the complex weighting matrix for the complex parameter \(\hat{\theta}\) defined as
\[
W_\theta = \begin{bmatrix} W_\theta & X_\theta \\ (X_\theta)^* & (W_\theta)^* \end{bmatrix},
\]
(56)
with \(W_\theta\) and \(X_\theta\) complex weighting matrices for the complex parameter \(\theta\).
Theorem 3.4 Let $t$ be the real transformation of $t$, and $\hat{t}$ its conjugate extension. If the real and complex weighting matrices fulfill

$$W_\theta = \frac{1}{4} \langle W_\theta \rangle_C,$$  \hfill (57)

then

$$w(\hat{t}) = w(t).$$  \hfill (58)

To proof this, we use (16) and the property $\text{Tr}(AB) = \text{Tr}(BA)$ for $A$ and $B$ complex matrices. From this result, we have that (27) and (28) are also lower bounds for $w(\hat{t})$.

Theorem 3.5 Let $w^S$ and $w^R$ be the bounds to $w$ defined in (27) and (28), $\hat{\theta}$ the conjugate extension of $\theta$, $W_\hat{\theta}$ the complex weighting matrix satisfying (17), $J_\hat{\theta}^S$ and $J_\hat{\theta}^R$ the SQFIM and RQFIM of state $\rho$ for the parameter $\hat{\theta}$. Then

$$w^S = \text{Tr} \left(W_\hat{\theta} \left(J_\hat{\theta}^S \right)^{-1} \right);$$

$$w^R = \frac{1}{2} \text{Tr} \left(W_\hat{\theta} \left[(J_\hat{\theta}^R)^{-1} + \sigma_{2k}(J_\hat{\theta}^R)^{-*}\sigma_{2k}\right] \right)$$

$$+ \frac{1}{2} \text{Tr} \text{Abs} \left(W_\hat{\theta} \left[(J_\hat{\theta}^R)^{-1} - \sigma_{2k}(J_\hat{\theta}^R)^{-*}\sigma_{2k}\right] \right).$$

To proof (59), we use (57) and apply the trace property (12) of the map $\langle \cdot \rangle_C$. On the other hand, to proof (60), we note that using properties (12a), (12g) and (12h), and (45) and (57) on the first term of (28), we obtain

$$\text{Tr} \left(W_\hat{\theta} \left[(J_\hat{\theta}^R)^{-1} + (J_\hat{\theta}^R)^{-*}\right] \right) = \text{Tr} \left(\langle W_\hat{\theta} \rangle_C \left(\left(J_\hat{\theta}^R\right)^{-1} + \langle (J_\hat{\theta}^R)^* \rangle^{-1}_C \right) \right)$$

$$= \text{Tr} \left(W_\hat{\theta} \left[(J_\hat{\theta}^R)^{-1}, + \sigma_{2k}(J_\hat{\theta}^R)^{-*}\sigma_{2k}\right] \right).$$

Besides, using properties (12a), (12g), (12h), and (12f), and (45) and (57) on the second term of (28), we have that

$$\text{Tr} \text{Abs} \left(W_\hat{\theta} \left[(J_\hat{\theta}^R)^{-1} - (J_\hat{\theta}^R)^{-*}\right] \right) = \frac{1}{2} \text{Tr} \text{Abs} \left(\langle W_\hat{\theta} \rangle_C \left(\left(J_\hat{\theta}^R\right)^{-1} - \langle (J_\hat{\theta}^R)^* \rangle^{-1}_C \right) \right)$$

$$= \text{Tr} \text{Abs} \left(\langle W_\hat{\theta} \rangle_C \left[\left(J_\hat{\theta}^R\right)^{-1} - \langle (J_\hat{\theta}^R)^* \rangle^{-1}_C \right) \right)$$

$$= \text{Tr} \text{Abs} \left(W_\hat{\theta} \left[(J_\hat{\theta}^R)^{-1} - \sigma_{2k}(J_\hat{\theta}^R)^{-*}\sigma_{2k}\right] \right).$$

Ending the proof by replacing (61) and (62) in (28).

These bounds can explicitly be written considering the block forms of $W_\hat{\theta}$ and $J_\hat{\theta}^X$.

Corollary 3.4 Let $w^S$ and $w^R$ be the lower bounds to $w$ defined in (27) and (28). If we consider $J_\hat{\theta}^S$ and $J_\hat{\theta}^R$ in its block form (35) and (44), then

$$w^S = 2 \text{Tr} \left(\text{Re} \left(W_\theta (E_\theta^S)^{-1} + X_\theta (F_\theta^S)^* (E_\theta^S)^{-*}\right) \right);$$

$$w^R = \text{Tr} \left(\text{Re} \left(W_\theta (E_\theta^R)^{-1} + X_\theta F_\theta^R (E_\theta^R)^{-1} \right) + X_\theta F_\theta^R (E_\theta^R)^{-1} \right)$$

$$+ \text{Tr} \left(\text{Abs} \text{Im} \left(W_\theta (E_\theta^R)^{-1} + X_\theta (E_\theta^R)^{-1} \right) + \text{Abs} \text{Im} \left(X_\theta F_\theta^S (E_\theta^S)^{-1} + X_\theta F_\theta^R (E_\theta^R)^{-1} \right) \right).$$


Furthermore, when the pseudo-FIM $Q^X_\theta$ is null, for $X \in \{S,R\}$, the bounds for the weighted mean square error become

\begin{align}
    w^S &= 2 \text{Tr} \left( \text{Re} \left( W_\theta (J^S_\theta)^{-1} \right) \right), \\
    w^R &= \text{Tr} \left( \text{Re} \left( W_\theta (J^R_\theta)^{-1} + \tilde{W}_\theta (J^R_{\tilde{\theta}})^{-1} \right) \right) \\
        &\quad + \text{Tr} \left( \text{Abs} \text{Im} \left( W_\theta (J^R_\theta)^{-1} \right) + \text{Abs} \text{Im} \left( \tilde{W}_\theta (J^R_{\tilde{\theta}})^{-1} \right) \right).
\end{align}

Finally, we specialize our previous results to the case of pure states. We denote the Wirtinger derivative of pure states as $\left| \partial_z \psi \right\rangle = \partial_z \left| \psi \right\rangle$ for $z \in \mathbb{C}$. Let us note that $\left| \partial_z \psi \right\rangle \dagger = \langle \partial_z, \psi \rangle$.

**Theorem 3.6** Let $\hat{\theta}$ be the conjugate extension of $\theta$, and $J^S_\theta$ the SQFIM of state $|\psi\rangle$ for the parameter $\hat{\theta}$. If we consider $J^S_\theta$ in its block form (65), then its components are given by

\begin{align}
    (J^S_\theta)_{jk} &= 2 \left( \langle \partial_{\theta_j} \psi | (| - | \psi \rangle \langle \psi |) | \partial_{\theta_k} \psi \rangle + \langle \partial_{\theta_k} \psi | (| - | \psi \rangle \langle \psi |) | \partial_{\theta_j} \psi \rangle \right). \tag{67}
\end{align}

\begin{align}
    (Q^S_\theta)_{jk} &= 2 \left( \langle \partial_{\theta_j^*} \psi | (| - | \psi \rangle \langle \psi |) | \partial_{\theta_k^*} \psi \rangle + \langle \partial_{\theta_k^*} \psi | (| - | \psi \rangle \langle \psi |) | \partial_{\theta_j^*} \psi \rangle \right). \tag{68}
\end{align}

To obtain (67), we employ (30) in (10), and use (35) and (39). Similarly, we obtain (68).

**Corollary 3.5** Let $\hat{\theta}$ be the conjugate extension of $\theta$, and $J^S_\theta$ the SQFIM of state $|\psi\rangle$ for the parameter $\hat{\theta}$. Then

\begin{align}
    J^S_\theta &= -2 \left[ \frac{\partial}{\partial \theta} \left( \frac{\partial \langle \psi(\hat{\theta}) | \psi(\hat{\theta}) \rangle^2}{\partial \theta} \right) \right] |_{\hat{\theta}=\hat{\theta}}. \tag{69}
\end{align}

**Corollary 3.6** When the state $|\psi\rangle$ is only a function of $\theta$, and not of $\theta^*$, the pseudo-FIM $Q^S_\theta$ is identically zero.

**Theorem 3.7** Let $|\psi\rangle$ be a pure state which depends on complex parameters $\theta$. Let $\hat{t}$ and $\hat{\theta}$ the conjugate extensions of $t$ and $\theta$, respectively, and $\{L_{\hat{\theta}_j}\}$ the complex symmetric logarithmic derivatives of $|\psi\rangle$. A necessary and sufficient condition to attain the symmetric quantum Cramér-Rao bound, that is, $\text{Cov}(\hat{t}) = (J^S_\theta)^{-1}$, is

\begin{align}
    \langle \psi | [L_{\hat{\theta}_j}, L_{\hat{\theta}_k}] | \psi \rangle &= 0. \tag{70}
\end{align}

To show this we use (52) and (38).

In order to obtain the right QFIM of complex parameter for pure states we compute $(J^R_\theta)^{-1}$ as in (33). Then, we define

\begin{align}
    \mathcal{K}_\theta &= \begin{bmatrix} K_\theta & R_\theta \\ R_\theta & K_\theta \end{bmatrix}, \tag{71}
\end{align}

where

\begin{align}
    K_\theta &= -2i \left[ \langle \partial_{\theta^*_j} | (| - | \psi \rangle \langle \psi |) | \partial_{\theta^*_k} \rangle - \langle \partial_{\theta^*_k} | (| - | \psi \rangle \langle \psi |) | \partial_{\theta^*_j} \rangle \right]. \tag{72}
\end{align}
Complex parameter estimation plays a critical role in many quantum protocols. Consequently, our proposal could have a wide range of applications. Next, we present an application of the complex field formulation of the quantum estimation theory to quantum communication. In particular, we consider the transmission of a complex number $z$ encoded in a coherent state of the electromagnetic field, that is,

$$ |\alpha\rangle = D(\alpha)|0\rangle, $$

(77)

where $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is the displacement operator. A reasonable communication protocol must satisfy two conditions. First, the communication must be secure, so we will encode $z$ in the parameter $\alpha$ of the coherent state as

$$ \alpha = \epsilon z + \eta z^*, $$

(78)

where $(\epsilon, \eta)$ are complex parameters that play the role of key. This is previously agreed by the sender and the receiver by quantum key distribution [56]. Second, decoding must be accurate, that is, the measurements that the receptor performs on the coherent state to acquire $z$ have the least possible uncertainty. Therefore, an optimal coding and estimation strategy is given by the one that reaches the quantum Cramér-Rao bound.
Since this problem is originally formulated on the complex field numbers, we can use our proposed theory to solve it.

From (70) we have that the symmetric quantum Cramér-Rao bound is attained if
\[ \langle \alpha | [ L_z, L_{z^*}] | \alpha \rangle = 0, \] (79)
where \( L_z \) are the complex symmetric logarithmic derivatives of \( | \alpha \rangle \), which are implicitly given by (37), that is,
\[ \partial_z (| \alpha \rangle \langle \alpha |) = \frac{1}{2} (| \alpha \rangle \langle \alpha | L_z + L_z^* | \alpha \rangle \langle \alpha |). \] (80)

To compute \( L_z \) we must calculate the Wirtinger derivatives of \( | \alpha \rangle \). Using (77), we obtain
\[ | \partial_z | \alpha \rangle \rangle = D(\alpha) (\epsilon a^\dagger - \frac{1}{2} [\alpha \eta - \alpha^* \epsilon]) | 0 \rangle, \] (81)
\[ | \partial_{z^*} | \alpha \rangle \rangle = D(\alpha) (\eta a^\dagger - \frac{1}{2} [\alpha \epsilon - \alpha^* \eta]) | 0 \rangle. \] (82)

Thereby, the complex symmetric logarithmic derivatives are
\[ L_z = 2D(\alpha)(\eta^* a + \epsilon a^\dagger)D(\alpha)^\dagger, \quad L_{z^*} = L_{z^*}. \] (83)

Replacing the above relations on (79), we obtain the following optimality condition
\[ | \eta |^2 = | \epsilon |^2. \] (84)

In order to attain the CRB, we must calculate the SQFIM for \( \hat{z} \). Employing (35), (67) and (68), we obtain
\[ J_S^{\hat{z}} = 2 \left[ \frac{| \epsilon |^2 + | \eta |^2}{2 \epsilon^* \eta} \begin{bmatrix} 2 \epsilon^* \eta & -2 \epsilon \eta \cr -2 \epsilon \eta^* & | \epsilon |^2 + | \eta |^2 \end{bmatrix} \right]. \] (85)

Therefore, the symmetric QCRI reads
\[ \text{Cov}(\hat{z}) \geq (J_S^{\hat{z}})^{-1} = \frac{1}{2(| \epsilon |^2 - | \eta |^2)^2} \begin{bmatrix} | \epsilon |^2 + | \eta |^2 & -2 \epsilon^* \eta \\
-2 \epsilon \eta^* & | \epsilon |^2 + | \eta |^2 \end{bmatrix}. \] (86)

Let us note that \( J_S^{\hat{z}} \) is invertible, and then the symmetric CRB exists if its determinant does not vanish, that is,
\[ \det(J_S^{\hat{z}}) = 4(| \epsilon |^2 - | \eta |^2)^2 \neq 0. \] (87)

We arrive thus to a contradiction: in the case where we achieve optimality, the FIM cannot be inverted. Therefore, we conclude that there is no encoding/decoding scheme that achieves the CRB. This result is not a surprise, since obtaining \( z \) is equivalent to estimate its real and imaginary parts through quadrature operators, which can not be measured simultaneously. The protocol can be improved using a two-mode coherent state \( | \alpha_1 \rangle | \alpha_2 \rangle \), where the symmetric QCRI can be saturated \[42\]. In this case, we consider the encoding \( \alpha_1 = \epsilon_1 z + \eta_1 z^* \) and \( \alpha_2 = \epsilon_2 z + \eta_2 z^* \) with key \( (\epsilon_1, \eta_1, \epsilon_2, \eta_2) \). Defining the two-mode displacement operator as \( D(\alpha_1, \alpha_2) = D(\alpha_1) \otimes D(\alpha_2) \), we have that the SLDs are
\[ L_z = 2D(\alpha_1, \alpha_2)(\eta_1^* a_1 + \epsilon_1 a_1^\dagger + \eta_2^* a_2 + \epsilon_2 a_2^\dagger)D(\alpha_1, \alpha_2)^\dagger, \quad L_{z^*} = L_{z^*}. \] (88)
and that the optimally condition is
\[|\eta_1|^2 + |\eta_2|^2 = |\epsilon_1|^2 + |\epsilon_2|^2.\] (89)

The SQFIM becomes in this case
\[J^S_z = 2 \begin{bmatrix} |\epsilon_1|^2 + |\eta_1|^2 + |\epsilon_2|^2 + |\eta_2|^2 & 2\epsilon_1^*\eta_1 + 2\epsilon_2^*\eta_2 \\ 2\epsilon_1\eta_1^* + 2\epsilon_2\eta_2^* & |\epsilon_1|^2 + |\eta_1|^2 + |\epsilon_2|^2 + |\eta_2|^2 \end{bmatrix},\] (90)

and its determinant is given by
\[\det(J^S_z) = 4(|\epsilon_1|^2 + |\eta_1|^2 + |\epsilon_2|^2 + |\eta_2|^2)^2 - 16|\epsilon_1^*\eta_1 + \epsilon_2^*\eta_2|^2.\] (91)

Thus, in this case it is possible to find keys that simultaneously satisfy the optimality condition and define a FIM whose determinant does not vanish. For instance, setting \((\epsilon_1, \eta_1, \epsilon_2, \eta_2) = (1, 0, 0, 1)\) the CRB reads
\[\text{Cov}(\hat{z}) \geq (J^S_z)^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\] (92)

This bound can be attained performing a homodyne measurement in the quadrature operators
\[Q_1 = \frac{1}{2}(a_1 + a_1^\dagger), \quad Q_2 = -\frac{i}{2}(a_2 - a_2^\dagger).\] (93)

This is not the only solution for the encoding of a secret encryption key on a coherent two-mode state. An alternative is to study the problem by means of the complex field formulation of the right QCRI. Using (71), (72) and (73), and the derivatives (81) and (82), we have obtain
\[K_z = -2i \begin{bmatrix} |\epsilon|^2 - |\eta|^2 & 0 \\ 0 & -|\epsilon|^2 + |\eta|^2 \end{bmatrix}.\] (94)

Using (75), the right quantum CRB is given by
\[\text{Cov}(\hat{z}) \geq (J^R_z)^{-1} = \frac{1}{(|\epsilon|^2 - |\eta|^2)^2} \begin{bmatrix} |\epsilon|^2 & -\epsilon^*\eta \\ -\epsilon\eta^* & |\eta|^2 \end{bmatrix}.\] (95)

We can see that the symmetric and right CRIs deliver different bounds for the covariance matrix, so they give us different information about the problem. Particularly, the matrix \((J^R_z)^{-1}\) has rank one. Thereby, only partial information about \(z\) can optimally be estimated. For instance, we could optimally measure the real part of \(z\), while its imaginary part remains indeterminate.

Knowing a priori the fundamental bound of the problem is not a question with a clear answer since we work with matrix inequalities. This difficulty can partially be solved using the weighted mean square error (55). Considering a diagonal block weighting matrix \(W_z = \text{diag}(W_z, W_z^*)\), the SQFIM establishes the lower bound (59), that is,
\[w^S = \frac{|\epsilon|^2 + |\eta|^2}{2(|\epsilon|^2 - |\eta|^2)^2}(W_z + W_z^*),\] (96)
while the RQFIM defines the lower bound \( \tilde{w} \) given by

\[
\tilde{w}^R = \frac{|\epsilon|^2 + |\eta|^2}{2(|\epsilon|^2 - |\eta|^2)^2}(W_z + W^*_z) + \frac{|W_z|}{||\epsilon|^2 - |\eta|^2||}. \tag{97}
\]

We can see that symmetric and right bounds are related by \( w^R = w^S + |W_z|/||\epsilon|^2 - |\eta|^2|| \). Therefore, for a diagonal weighting error, the estimation theory based on the right derivative give us the fundamental bound for the problem.

5. Conclusion and Outlook

In this article, we have formulated the quantum estimation theory of complex statistics on dependence of complex parameters, which is natively developed on the field of complex numbers. This formulation is based on manipulating the complex parameters through their conjugate extension instead of their representation in its real and imaginary parts, and using the Wirtinger calculus instead of the real calculus. This formulation state new versions of the main quantities of quantum estimation theory, such as logarithmic derivatives, both symmetric and right, Fisher information matrices and Cramér-Rao bounds. We define lower bounds for the weighted square error, which is widely used in multi-parametric estimation. Furthermore, working the Cramér-Rao bounds by block matrices, we state lower bounds for the studied statistic and not only for its conjugate extension or its real representation. All of the above results are particularized for the case of a pure state. An application in quantum communication was studied, which consists of the optimal coding and decoding of a complex parameter in a coherent state.

This theory is equivalent to its real counterpart since the main results, such as Fisher information matrices and Cramér-Rao inequalities, can be connected through the map \( \langle \_ \_ \_ \rangle_C \). However, our theory is self-contained, that is, we can study a quantum estimation problem just using it, independently of its real counterpart. This can be exploited in problems where the parameter are natively complex, as is the case of coherent states, squeeze states or pure qudit states. Our theory provides new interpretations of the estimated complex parameters, specially when they are incompatible. Thereby, it defines a new approach to the standard techniques in quantum estimation theory, such as the optimization of the Cramér-Rao bound or the search for quasi-optimal strategies. This can be seen in our example on quantum communication, where the symmetric Cramér-Rao bound cannot be attained. However, we show that using the mean square error as a figure of merit, the right Cramér-Rao bound defines the optimal strategy. This result could be improved by searching a pseudo-optimal strategy based on the symmetric Cramér-Rao bound or optimizing the right Cramér-Rao bound in order to reduce the uncertainly.
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