Abstract

We construct embeddings of boundary algebras $\mathcal{B}$ into ZF algebras $\mathcal{A}$. Since it is known that these algebras are the relevant ones for the study of quantum integrable systems (with boundaries for $\mathcal{B}$ and without for $\mathcal{A}$), this connection allows to make the link between different approaches of the systems with boundaries. The construction uses the well-bred vertex operators built recently, and is classified by reflection matrices. It relies only on the existence of an $R$-matrix obeying a unitarity condition, and as such can be applied to any infinite dimensional quantum group.
1 Introduction

The problem of boundaries in integrable systems in the QISM framework was initiated by Cherednik [1]. Basically, one can distinguish two approaches: the point of view of Sklyanin [2], which relies on reflection matrices, and leads to the study boundary states (such as in [3] or [4] for instance); or the more algebraic approach of Mintchev et al [5, 6], where all the information is encoded in a boundary algebra \( B \), and which allows to compute off-shell correlation functions and to study the integrals of motion [7].

On one hand, one starts with the bulk system and implement the boundary condition through a reflection matrix, while on the other hand, one has from the very beginning a boundary algebra \( B \), which contains a reflection operator, and only after the specification of one of the several \( B \)-Fock space one gets a reflection matrix.

In the present letter, we will take the second way of tackling the problem, and try to make more clear the connection with the first approach. For that purpose, we construct embeddings of \( B \) into a Zamolodchikov-Faddeev (ZF) algebra, which is known to be the relevant algebra for the study of systems without boundary [8]. These embeddings uses the well-bred vertex operators built in [9], and they are classified by the reflection matrices of the first approach, whence the link between the two points of view.

Since one of the key point in our approach relies on the existence of so-called well-bred vertex operators, we present a generalization of this construction to the case of reflection operators.

The only assumption made for such constructions is the existence of an evaluated \( R \)-matrix (with spectral parameter) which obey the unitarity condition. It can thus be used for most of the integrable systems encountered in the literature.

The paper is organized as follows. In the section 2, we introduce the different notions we will need. From these definitions, we construct, in section 3, a boundary algebra \( B^B_R \) from the deformed oscillator algebra \( \mathcal{A}_R \). Then, we consider, in section 4, the hierarchy associated to \( B^B_R \). Section 5 deals with one example: the nonlinear Schrödinger equation with boundary. Finally, we conclude in section 7.

2 Definitions and notations

The starting point is an evaluated \( R \)-matrix, of size \( N^2 \times N^2 \), with spectral parameter, and which obeys the Yang-Baxter equation and the unitarity condition:

\[
R_{12}(k_1, k_2)R_{13}(k_1, k_3)R_{23}(k_2, k_3) = R_{23}(k_2, k_3)R_{13}(k_1, k_3)R_{12}(k_1, k_2) \quad (2.1)
\]

\[
R_{12}(k_1, k_2)R_{21}(k_2, k_1) = \mathbb{I} \otimes \mathbb{I} \quad (2.2)
\]
In the following, we will use the notation
\[ R_{12} = R_{12}(k_1, k_2) \quad (2.3) \]

**Definition 2.1 (ZF algebra \( A_R \))**

To the above \( R \)-matrix, one can associate a ZF algebra \( A_R \), with generators \( a_i(k) \) and \( a_i^\dagger(k) \) \((i = 1, \ldots, N)\) and exchange relations:

\[
\begin{align*}
  a_1 a_2 &= R_{21} a_2 a_1 \\
  a_1^\dagger a_2^\dagger &= a_2^\dagger a_1^\dagger R_{21} \\
  a_1 a_2^\dagger &= a_2^\dagger R_{12} a_1 + \delta_{12} \\
\end{align*}
\]

(2.4) \quad (2.5) \quad (2.6)

We use the notations on auxiliary spaces

\[
\begin{align*}
  a_1 &= \sum_{i=1}^N a_i(k_1) e_i \otimes \mathbb{I} \\
  a_2 &= \sum_{i=1}^N a_i(k_2) \mathbb{I} \otimes e_i \\
  a_1^\dagger &= \sum_{i=1}^N a_i^\dagger(k_1) e_i^\dagger \otimes \mathbb{I} \\
  a_2^\dagger &= \sum_{i=1}^N a_i^\dagger(k_2) \mathbb{I} \otimes e_i^\dagger \\
\end{align*}
\]

(2.7) \quad (2.8)

\[
\delta_{12} = \delta(k_1 - k_2) \sum_{i=1}^N e_i \otimes e_i^\dagger, \quad e_i^\dagger = (0, \ldots, 0, 1, 0, \ldots, 0), \quad e_i^\dagger \cdot e_j = \delta_{ij} \quad (2.9)
\]

where \( \cdot \) stands for the scalar product of vectors.

**Definition 2.2 (Boundary algebra \( B_R \))**

To the same \( R \)-matrix, one can associate another algebra, the boundary algebra \( B_R \), with generators \( \tilde{a}_i(k) \), \( \tilde{a}_i^\dagger(k) \) and \( b_{ij}(k) \) \((i, j = 1, \ldots, N)\) and exchange relations:

\[
\begin{align*}
  \tilde{a}_1 \tilde{a}_2 &= R_{21} \tilde{a}_2 \tilde{a}_1 \\
  \tilde{a}_1^\dagger \tilde{a}_2^\dagger &= \tilde{a}_2^\dagger \tilde{a}_1^\dagger R_{21} \\
  \tilde{a}_1 \tilde{a}_2^\dagger &= \tilde{a}_2^\dagger R_{12} \tilde{a}_1 + \frac{1}{2} \delta_{12} \delta + \frac{1}{2} b_{12} \\
  \tilde{a}_1 b_2 &= R_{21} b_2 R'_{12} \tilde{a}_1 \\
  b_1 \tilde{a}_2^\dagger &= \tilde{a}_2^\dagger R_{21} b_1 R'_{21} \\
  R_{12} b_1 R'_{21} b_2 &= b_2 R'_{12} b_1 R_{21} \\
  b(k)b(-k) &= \mathbb{I} \\
\end{align*}
\]

(2.10) \quad (2.11) \quad (2.12) \quad (2.13) \quad (2.14)

We have completed the notations \((2.7, 2.8, 2.9)\) by:

\[
\begin{align*}
  R'_{12} &= R_{12}(k_1, -k_2) \quad ; \quad R'_{21} = R_{21}(k_2, -k_1) \\
  R'_{12}^{-1} &= R_{12}(-k_1, k_2) \quad ; \quad R'_{21}^{-1} = R_{21}(-k_2, k_1) \\
  \hat{R}_{12} &= R_{12}(-k_1, -k_2) \quad ; \quad \hat{R}_{21} = R_{21}(-k_2, -k_1) \\
\end{align*}
\]
\[ b_{12} = \delta(k_1 + k_2) \sum_{i,j=1}^{N} b_{ij}(k_1) e_i \otimes e_j^\dagger \]

\[ b_1(k) = \sum_{i,j=1}^{N} b_{ij}(k) E_{ij} \otimes I_N ; \quad b_2(k) = \sum_{i,j=1}^{N} b_{ij}(k) I_N \otimes E_{ij}. \]

Let us stress that

\[ \bar{R}_{12}^{-1} = \left( R_{12}' \right)^{-1} = \left( R_{12}(k_1, -k_2) \right)^{-1} = R_{21}(-k_2, k_1) \neq R_{21}' \]

while \( \bar{R}_{12}^{-1} = \bar{R}_{21} \).

The \( B_R \) algebras have been introduced in [6], where they were shown to play a fundamental role in the study of integrable systems with boundaries. They allow for instance the determination of off-shell correlation functions.

Note that there is an automorphism on \( B_R \) given by [6]:

\[
\rho \left\{ \begin{array}{c}
B_R & \rightarrow & B_R \\
a(k) & \rightarrow & b(k) a(-k) \\
a^\dagger(k) & \rightarrow & a^\dagger(-k) b(-k) \\
b(k) & \rightarrow & b(k)
\end{array} \right. \tag{2.16}
\]

**Definition 2.3 (Reflection algebra \( S_R \))**

The reflection algebra \( S_R \) is the subalgebra of the boundary algebra, with generators \( b_{ij}(k) \) (\( i, j = 1, \ldots, N \)). It has exchange relations:

\[
R_{12} b_1 R_{21}' b_2 = b_2 R_{12}' b_1 R_{21} \tag{2.17}
\]

\[
b(k) b(-k) = \mathbb{I} \tag{2.18}
\]

\( S_R \) algebras enter into the class of \( ABCD \)-algebras introduced in [10]. They correspond, in the boundary algebra approach, to the symmetries of the underlying model (with boundary).

**Definition 2.4 (Well-bred vertex operator)**

It has been shown in [9], that there exist in \( A_R \) a unique so-called well-bred vertex operator \( T(k) = T^{ij}(k) E_{ij} \) such that

\[
T(k_\infty) = \mathbb{I} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a_{n-1}^\dagger T_{n1\ldots n}^{(n)} a_{1\ldots n} \tag{2.19}
\]

\[
T_1 a_2 = R_{21} a_2 T_1 \tag{2.20}
\]

\[
T_1 a_2^\dagger = a_2^\dagger R_{12} T_1 \tag{2.21}
\]

\[
R_{12} T_1 T_2 = T_2 T_1 R_{12} \tag{2.22}
\]
with
\begin{align}
a_{n \ldots 1}^\dagger &= a_{\alpha_n}^\dagger (k_n) \ldots a_{\alpha_1}^\dagger (k_1) \\
a_{1 \ldots n} &= a_{\alpha_1} (k_1) \ldots a_{\alpha_n} (k_n) \\
T^{(n)}_{\infty \ldots n} &= T^{(n)}_{\infty \alpha_1 \ldots \alpha_n} (k_\infty, k_1, \ldots, k_n) \in \left( \mathbb{C}^\otimes N^2 \right)^{\otimes (n+1)} (k_\infty, k_1, \ldots, k_n)
\end{align}

In (2.20), there is an implicit summation on the indices \( \alpha_1, \ldots, \alpha_n = 1, \ldots, N \) and an integration over the spectral parameters \( \int dk_1 \cdots dk_n \). The matrices \( T^{(n)}_{\infty 1 \ldots n} \) are built using only the evaluated R-matrix. For their exact expression, we refer to [9].

3 Construction of \( \mathcal{B}_R \) from \( \mathcal{A}_R \)

Theorem 3.1 Let \( \mathcal{A}_R \) be a ZF algebra, and \( T \) its corresponding well-bred vertex operator. Let \( B(k) \) be a \( N \times N \) matrix such that
\begin{align}
R_{12} B_1 R'_{21} B_2 &= B_2 R'_{12} B_1 \check{R}_{21} \\
B(k)B(-k) &= \mathbb{I}_N
\end{align}

Then, the following generators obey a boundary algebra \( \mathcal{B}_R \):
\begin{align}
\bar{a}(k) &= \frac{1}{2} (a(k) + b(k)a(-k)) \\
\bar{a}^\dagger (k) &= \frac{1}{2} (a^\dagger (k) + a^\dagger (-k)b(-k)) \\
b(k) &= T(k)B(k)T(-k)^{-1}
\end{align}

\( B(k) \) is called the reflection matrix.

Proof: We show the exchange relations by a direct calculation. As far as the exchange properties for \( b \) are concerned, the proof follows the lines given in [11], i.e. a repetitive use of the relation (2.22) in its various presentation, and also of (3.1). We get
\begin{align}
R_{12} b_1 R'_{21} b_2 &= b_2 R'_{12} b_1 \check{R}_{21}
\end{align}

The unitarity condition \( b(k)b(-k) = \mathbb{I} \) is obvious from the form of \( b \).

We now compute the action of \( b(k) \) on \( a(k) \) and \( a^\dagger (k) \). For compactness, we write
\begin{align}
a' &= a(-k) ; \ a'' = a^\dagger (-k) ; \ T' = T(-k) \quad \text{and} \quad b' = b(-k) = b(k)^{-1}
\end{align}

\begin{align}
a_1 b_2 &= a_1 T_2 B_2 T_2'^{-1} = R_{21} T_2 a_1 B_2 T_2'^{-1} \\
&= R_{21} T_2 B_2 a_1 T_2'^{-1} = R_{21} T_2 B_2 T_2'^{-1} R_{12} a_1
\end{align}
Thus, we obtain
\[ a_1(k_1)b_2(k_2) = R_{21}(k_2, k_1)b_2(k_2)R_{12}(k_1, -k_2)a_1(k_1) \quad \text{i.e.} \quad a_1b_2 = R_{21}b_2R_{12}a_1 \quad (3.8) \]
In the same way, we have
\[
b_1a_2^\dagger = T_1B_1T_1'^{-1}a_2^\dagger = T_1B_1a_2^\dagger T_1'^{-1}R_{21}'
\]
\[
= T_1a_2^\dagger B_1T_1'^{-1}R_{21}' = a_2^\dagger R_{12}T_1B_1T_1'^{-1}R_{21}'
\]
that is
\[
b_1(k_1)a_2^\dagger(k_2) = a_2^\dagger(k_2)R_{12}(k_1, k_2)b_1(k_1)R_{21}(k_2, -k_1) \quad \text{i.e.} \quad b_1a_2^\dagger = a_2^\dagger R_{12}b_1R_{21}' \quad (3.9)
\]
Thanks to the properties (3.6,3.8), one computes
\[
4\tilde{a}_1\tilde{a}_2 = a_1b_2 + a_1b_2a_2' + b_1a_1'a_2 + b_1a_1'b_2a_2' = R_{21}a_2a_1 + R_{21}b_2R'_{12}a_1a_2' + b_1R_{21}'a_2a_1' + (b_1R_{21}'b_2\bar{R}_{12})(a_1'a_2')
\]
\[
= R_{21}a_2a_1 + R_{21}b_2a_2'a_1' + R_{21}a_2b_1a_1' + R_{21}b_2R'_{12}b_1R_{21}'a_2'a_1
\]
\[
= R_{21}\left(a_2a_1 + b_2a_2'a_1' + a_2b_1a_1' + b_2a_2'b_1a_1'\right)
\]
\[
= 4R_{21}\tilde{a}_2\tilde{a}_1
\]
The same calculation can be done for $\tilde{a}_1\tilde{a}_2$ using properties (3.6,3.9).

We will use the unusual notation
\[
\delta'_{12} = \delta(k_1 + k_2) \sum_{i=1}^{N} e_i \otimes e_i^\dagger \quad (3.10)
\]
Generally, the prime denotes a derivative when associated to the $\delta$ distribution, but fortunately the derivative of $\delta$ never occurs in the present article, so that there will be no confusion.

Looking now at $\tilde{a}_1\tilde{a}_2$, one gets:
\[
4\tilde{a}_1\tilde{a}_2 = a_1a_2^\dagger + a_1a_2^\dagger b_2' + b_1a_1' a_2^\dagger + b_1a_1'a_2'^\dagger b_2
\]
\[
= a_2^\dagger R_{12}a_1 + \delta_{12} + (a_2^\dagger R'_{12}a_1 + \delta_{12})b_2' + b_1(a_2^\dagger a_1' + \delta_{12}) + b_1(a_2^\dagger \bar{R}_{12}a_1' + \delta_{12})b_2'
\]
\[
= a_2^\dagger R_{12}a_1 + 2\delta_{12} + 2b_2 + a_2^\dagger b_2 R_{12}a_1 + a_2^\dagger R_{12}b_1a_1' + a_2 R'_{12}b_1 R_{21}'b_2 R'_{21}^{-1}a_1
\]
\[
= 2a_2^\dagger R_{12}a_1 + 2\delta_{12} + 2b_2 + a_2^\dagger R_{12}b_1a_1' + a_2^\dagger b_2 R_{12}b_1a_1
\]
\[
= 4\tilde{a}_2 R_{12}\tilde{a}_1 + 2\delta_{12} + 2b_2
\]
Finally, from (3.6), (3.8) and (3.9), one computes the last equations, for instance:
\[
2b_1\tilde{a}_2 = b_1(a_2^\dagger + a_2^\dagger b_2') = a_2^\dagger R_{12}b_1R'_{12} + a_2^\dagger R_{12}b_1R_{21}'b_2'
\]
\[
= a_2^\dagger R_{12}b_1R'_{12} + a_2^\dagger b_2' R_{12}b_1 R_{21}' = \tilde{a}_2 R_{12}b_1 R'_{12}
\]

\[\blacksquare\]
Lemma 3.2 In $\mathcal{B}_R^\rho$, the automorphism $\rho$ given in (2.16) is the identity.

Proof: Obvious, using $b(k)b(-k) = 1$.

Remark: Strictly speaking, $\mathcal{B}_R^\rho$ corresponds to the coset (noted $\mathcal{B}_R^\rho$) of the abstract boundary algebra $\mathcal{B}_R$ (given by definition 2.2) by the relation $\rho + iI = 0$, so that the construction of $\mathcal{B}_R^\rho$ in theorem 3.1 defines inclusions of $\mathcal{B}_R^\rho$ into $\mathcal{A}_R$ (see also theorem 5.2).

Property 3.3 The Fock space $\mathcal{F}_R$ of $\mathcal{A}_R$ provides a Fock space representation for $\mathcal{B}_R^\rho$, defined by

$$\tilde{a}(k)\Omega = 0 \quad \text{and} \quad b(k)\Omega = B(k)\Omega$$

(3.11)

Proof: From the definition of the Fock space for $\mathcal{A}_R$, one has $a(k)\Omega = 0$, and thus $\tilde{a}(k)\Omega = 0$. The well-bred vertex operator $T(p)$ satisfies $T(p)\Omega = \Omega$, which implies $b(k)\Omega = B(k)\Omega$.

Remark: In [6, 7], the boundary algebra $\mathcal{B}_R$ has several Fock spaces, depending on the value of $b(k)$ on $\Omega$. In the present article, the algebra $\mathcal{B}_R^\rho$ has only one Fock space, but $B$ is within the construction of $\mathcal{B}_R^\rho$, and there are as much $\mathcal{B}_R^\rho$-algebra constructions in the present approach, as there are Fock spaces in the approach of [1, 2].

4 Vertex operator construction

As already mentioned, in [7], it has been shown that a well-bred vertex operator $T$ can be constructed as a series in $a$'s:

$$T(k_\infty) = I + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a_{\alpha_{n+1}}^\dagger T^{(n)}_{\alpha_1, \ldots, \alpha_n} a_{\alpha_1, \ldots, \alpha_n}$$

(4.1)

with

$$a_{\alpha_{n+1}}^\dagger = a_{\alpha_n}^\dagger(k_n) \cdots a_{\alpha_1}^\dagger(k_1)$$

(4.2)

$$a_{\alpha_1, \ldots, \alpha_n} = a_{\alpha_1}(k_1) \cdots a_{\alpha_n}(k_n)$$

(4.3)

$$T^{(n)}_{\alpha_1, \ldots, \alpha_n}(k_\infty, k_1, \ldots, k_n) \in \left(\mathbb{C}^N^\otimes \right)^\otimes(n+1)(k_\infty, k_1, \ldots, k_n)$$

(4.4)

and an implicit summation on the indices $\alpha_1, \ldots, \alpha_n = 1, \ldots, N$ and an integration over the spectral parameters $\int dk_1 \cdots dk_n$.

The same expansion can be done for $T^{-1}$:

$$T(k_\infty)^{-1} = I + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a_{\alpha_{n+1}}^\dagger T^{(n)}_{\alpha_1, \ldots, \alpha_n} a_{\alpha_1, \ldots, \alpha_n}$$

(4.5)
For the exact expression of $T_{\infty 1...n}^{(n)}$ and $T_{\infty 1...n}^{(n)\dagger}$, we refer to [9]. These matrices are constructed using only the evaluated $R$-matrix.

One can do the same construction for the $b$ operator:

**Property 4.1 (Reflection operators as vertex operators)**

In terms of the $A_R$ generators, the reflection operators read

$$b_0 = \mathbb{I} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a_{n...1}^{\dagger} \beta_{01...n}^{(n)} a_{1...n}$$

$$\beta_{01...n}^{(n)} = T_{01...n}^{(n)} B_0 + B_0 T_{01...n}^{(n)\dagger} + (n-1) \sum_{p=1}^{n-1} \left( \frac{n-2}{p-1} \right) T_{bp+1...n}^{(n-p)} B_0 T_{01...p}^{(p)\dagger} R_p'$$

where the prime $'$ indicates that one has to consider $-k_0$ instead of $k_0$ (as in definition 2.2) and

$$R_p' = R_{p+1}^{r-1} \cdots R_{n0}^{r-1} = \prod_{s=p+1}^{n} R_0 s(-k_0, k_s)$$

Proof: We start with $b = TBT'^{-1}$ and use the expansion of $T$. Then, from $a_1 T_2'^{-1} = T_2'^{-1} R_{12}^{-1} a_1$, one gets

$$b_0 = B_0 T_0'^{-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a_{n...1}^{\dagger} T_{01...n}^{(n)} B_0 T_0'^{-1} R_{10}^{r-1} \cdots R_{n0}^{r-1} a_{1...n}$$

Finally, using the expansion (4.5) for $T_0'^{-1}$, and relabeling the auxiliary spaces, one obtains the result.

Let us stress that the expansion is done in terms of the $A_R$ generators $a$ and $a^{\dagger}$, not in terms of $\tilde{a}$ and $\tilde{a}^{\dagger}$, generators of $B_R$. It is possible that such an expansion would lead to a more simple expression for $\beta^{(n)}$.

## 5 Hierarchy for $B_R^B$

**Property 5.1 (Hierarchy for $B_R^B$)**

Let

$$H^{(n)} = \int_{-\infty}^{\infty} dk \, k^n \tilde{a}^{\dagger}(k) \tilde{a}(k)$$

Then:

(i) $H^{(2n+1)}$ vanish identically in $B_R^B$.
(ii) $[H^{(2n)}, \tilde{a}^{\dagger}(k)] = k^{2n} \tilde{a}^{\dagger}(k)$ and $[H^{(2n)}, \tilde{a}(k)] = -k^{2n} \tilde{a}(k)$
(iii) $\{H^{(2n)}\}_{n \in \mathbb{Z}_+}$ form a commuting flow for $B_R^B$, called its hierarchy.
Proof: We first show (i):

$$H^{(2n+1)} = \int_{-\infty}^{\infty} dk k^{2n+1} \tilde{a}^\dagger(k) \tilde{a}(k) = \int_{-\infty}^{\infty} dk (-k)^{2n+1} \tilde{a}^\dagger(-k) \tilde{a}(-k)$$

$$= - \int_{-\infty}^{\infty} dk k^{2n+1} \tilde{a}^\dagger(k) b(k) b(-k) \tilde{a}(k) = -H^{(2n+1)}$$

Now using the exchange relations of $\mathcal{A}_R$, one gets

$$H^{(2n)} \tilde{a}^\dagger(k) = \int_{-\infty}^{\infty} dp p^{2n} \tilde{a}^\dagger_1(p) \tilde{a}_1(p) \tilde{a}^\dagger_2(k)$$

$$= \int_{-\infty}^{\infty} dp p^{2n} \tilde{a}^\dagger_1(p) \left( \tilde{a}^\dagger_2(k) R_{12}(p, k) \tilde{a}_1(p) + \frac{1}{2} \delta(p-k) + \frac{1}{2} b_{12}(p) \delta(p+k) \right)$$

$$= \frac{1}{2} \left( k^{2n} \tilde{a}^\dagger(k) + (-k)^{2n} \tilde{a}^\dagger(-k) b(-k) \right) + \int_{-\infty}^{\infty} dp p^{2n} \tilde{a}^\dagger_2(k) \tilde{a}^\dagger_1(p) R_{21}(k, p) R_{12}(p, k) \tilde{a}_1(p)$$

$$= k^{2n} \tilde{a}^\dagger(k) + \tilde{a}^\dagger(k) H^{(2n)}$$

where in the last step we have used the automorphism $\rho = id$. The same computation leads to $[H^{(2n)} \tilde{a}^\dagger(k)] = -k^{2n} \tilde{a}(k)$.

Note that starting with $H^{(2n+1)}$ and performing the above calculation leads to e.g. $[H^{(2n+1)}, \tilde{a}(k)] = 0$, which is compatible with (i).

Finally, using (ii), one computes

$$[H^{(2n)}, H^{(2m)}] = \int_{-\infty}^{\infty} dp p^{2n} \left( \tilde{a}^\dagger(p) [\tilde{a}(p), H^{(2m)}] + \tilde{a}(p) [\tilde{a}^\dagger(p), H^{(2m)}] \right)$$

$$= \int_{-\infty}^{\infty} dp p^{2n} \left( p^{2m-2} \tilde{a}^\dagger(p) \tilde{a}(p) - p^{2m-2} \tilde{a}(p) \tilde{a}^\dagger(p) \right)$$

$$= 0$$

Remark: On the Fock space $\mathcal{F}_R$, and considering the states $|\tilde{k} \rangle = \tilde{a}^\dagger(k) \Omega$, one has

$$H^{(2n)} \Omega = 0 \Rightarrow H^{(2n)} |\tilde{k} \rangle = H^{(2n)} [H^{(2n)}, \tilde{a}^\dagger(k)] \Omega = k^{2n} |\tilde{k} \rangle$$

which shows that $H^{(2n)} \neq 0$.

Remark 2: The hierarchy defines integrable systems with boundary defined by $B(k)$. In the framework we have adopted, the definition of the boundary is given by the data of the reflection matrix $B(k)$, as it is presented in [1], but the boundary algebra is naturally recovered here, contrarily to [1], where it is lacking.
for the calculation of off-shell correlation functions. On the other hand, in \[6, 5, 7\], the boundary algebra is the basic data (whence the possibility of computation of correlation functions), but the data of the boundary condition (i.e. the reflection matrix) is given with the choice of a Fock space \(F^B\). Thus, the present framework can be viewed as a bridge between the approaches \[4\] and \[6, 7\].

This remark is confirmed in the following theorem:

**Theorem 5.2** Let \(B\) be a reflection matrix of \(A_R\), and \(b = TBT^{-1}\) the corresponding reflection operator. Let \(\rho_B\) be defined by

\[
\rho_B(a) = ba' \quad \text{and} \quad \rho_B(a^\dagger) = a'^\dagger b'
\]  

(5.3)

Then:

(i) \(\rho_B\) is an automorphism of \(A_R\)

(ii) \(B^R\) is the coset of \(A_R\) by the ideal \(\text{Ker}(\rho_B - \text{id})\)

**Proof:** We prove (i) by direct calculation, using the results (3.8,3.9) and the exchange relations (2.10,2.14). Let \(\alpha = \rho_B(a)\) and \(\alpha^\dagger = \rho_B(a^\dagger)\):

\[
\alpha_1\alpha_2 = b_1a'_1 b_2a'_2 = b_1R'_{12}b_2R_{12}a'_1 a'_2 = R_{21}b_2R'_{12}b_1R_{21}a'_2 a'_1 = R_{21}a_1 a_2
\]

The same calculation can be done with \(\alpha_1^\dagger \alpha_2^\dagger\). For the last relation, we have:

\[
\alpha_1^\dagger \alpha_2^\dagger = b_1a'_1 a'^\dagger b'^\dagger = b_1(a'^\dagger b'_2R_{12}a'_1 a'_2 + \delta_{12})b'^\dagger
\]

\[
= a'^\dagger R_{12}b_1R_{21}b_2R_{12}b'_2R_{21}^{-1}a'_1 + \delta_{12} = a'^\dagger b'_2R_{12}b_1R_{21}R_{12}^{-1}a'_1 + \delta_{12}
\]

\[
= a'^\dagger R_{12}a_1 + \delta_{12}
\]

The proof for (ii) follows from the observation that \(\tilde{a} = \frac{1}{2}(a + \rho_B(a))\) and \(\tilde{a}^\dagger = \frac{1}{2}(a^\dagger + \rho_B(a^\dagger))\). This means

\[
a \equiv \tilde{a} \mod[\text{Ker}(\rho_B - \text{id})] \quad \text{and} \quad a^\dagger \equiv \tilde{a}^\dagger \mod[\text{Ker}(\rho_B - \text{id})] \quad (5.4)
\]

so that making the coset by \(\text{Ker}(\rho_B - \text{id})\) we recover \(B^R\).

We present now a property which was already proved in \[7\] for the case of additive \(R\)-matrix \(R_{12}(k_1 - k_2)\), but the reasoning is valid in full generality:

**Property 5.3 (Integrals of motions of the hierarchy)**

The reflection algebra \(S^B_R\) generates integrals of motion for the \(B^R\)-hierarchy.

**Proof:** By direct computation using the exchange relations (2.11), one shows that

\[
[H^{(2n)}, b(k)] = 0.
\]

Still following the lines given in \[7\], one gets:
Property 5.4 (Spontaneous symmetry breaking)

In the Fock space representation, there is a spontaneous symmetry breaking of the symmetry algebra through

\[ b(k)\Omega = B(k)\Omega \]  

(5.5)

Proof: From (5.5), one knows all the operators \( b_{ij}(k) \) which have non-vanishing value on \( \Omega \). Since the \( S^B_R \)-algebra constitutes the symmetry algebra of our problem, we are exactly faced with a mechanism of spontaneous symmetry breaking for our reflection algebra.

6 Example: the nonlinear Schrödinger equation with boundary

It has already been shown \([5, 7]\) that all the informations on the hierarchy associated to the nonlinear Schrödinger equation in 1+1 dimensions with boundary (BNLS) can be reconstructed starting from a boundary algebra \( B_R \), where \( R \) is the \( R \)-matrix of the Yangian \( Y(N) \) based on \( gl(N) \)

\[ R(k) = \frac{1}{k + ig} (k \mathbb{I}_N \otimes \mathbb{I}_N + ig P_{12}) \quad , \quad P_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji} \]  

(6.1)

This \( R \)-matrix obey an additive Yang-Baxter equation

\[ R_{12}(k_1 - k_2)R_{13}(k_1 - k_3)R_{23}(k_2 - k_3) = R_{23}(k_2 - k_3)R_{13}(k_1 - k_3)R_{12}(k_1 - k_2) \]  

(6.2)

and one shows, using \( P^2 = \mathbb{I} \), that \( R_{12}(k)R_{21}(-k) = \mathbb{I} \). Thus, the properties stated above apply.

In fact, it is well-known that the canonical field \( \Phi \) obeying the (quantum) NLS:

\[ \left( i\partial_t + \partial_x^2 \right) \Phi(x, t) = 2g : \Phi(x, t)\Phi(x, t)\Phi(x, t) : \quad \text{with} \quad \Phi(x, t) = \begin{pmatrix} \varphi_1(x, t) \\ \vdots \\ \varphi_n(x, t) \end{pmatrix} \]

has an Hamiltonian which is exactly \( H^{(2)} \), and that the reflection algebra \( S_R \) is a symmetry of the hierarchy.

To make the contact with the present point of view, one has to specify the Fock space \( \mathcal{F}^B_R \), which amount to fix a boundary matrix \( B \). Then, one can construct the boundary algebra \( B^B_R \), which will have the same Fock space. The data of \( B \) also completely determine the boundary condition for the physical field \( \Phi \) (see \([4]\)), as it should in the approach of \([4]\). The phenomenon of spontaneous symmetry breaking in BNLS was also studied in \([7]\).
7 Conclusion

We have shown that one can embed the boundary algebra $B_R$ into the ZF algebra $A_R$, and that there are as much embedding as there are reflection matrices. Such embeddings allow to link the approach of Mintchev et al [5, 6], who introduced the boundary algebras, to the original work of Cherednik [1] and lately Sklyanin [2], who studied the problem of factorized S-matrices in models with boundaries. In particular, the results presented here allow to reconstruct the boundary algebra from the ZF algebra and the data of a reflection matrix.

The construction relies only on an $R$-matrix with spectral parameter which satisfies the Yang-Baxter equation and a unitarity condition. As a consequence, it is applicable to most of the infinite dimensional quantum groups, and in particular to Yangians, and to centerless affine or elliptic quantum groups.

Taking as an example the $R$-matrix of $Y(N)$, the Yangian based on $gl(N)$, we recover by this construction the nonlinear Schrödinger equation with boundary and its symmetry. It is thus very natural to believe that the other integrable systems known in the literature can be treated with the present approach.

As an extension to our approach, the generic problem of (elliptic) quantum algebras with central extension should also be treated.

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