The Cascading Haar Wavelet algorithm for computing the Walsh-Hadamard Transform

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The Walsh-Hadamard Transform (WHT)

- The WHT is a staple of the signal processing world.
- It is widely used...
  - ...in coding in wireless communications
  - ...in image processing
  - ...as a proxy for the Fast Fourier Transform (FFT).
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- Given $x \in \mathbb{R}^n$ where $n = 2^m$, its WHT (with coefficients in dyadic/Paley order) is equal to $H_m x$, where $H_m$ is a Walsh-Hadamard matrix defined by the recursion

$$
H_0 := 1; \quad H_{r+1} = \frac{1}{\sqrt{2}} \begin{bmatrix}
H_r \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\end{bmatrix}
$$

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\]

- Divide-and-conquer algorithms (e.g. Cooley-Tukey) exist which require exactly $n \log_2 n$ operations.
The Walsh-Hadamard Transform (WHT)

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- Given $x \in \mathbb{R}^m$ where $n = 2^m$, its Haar wavelet transform is equal to $\Psi_m x$ where $\Psi_m$ is defined by the recursion
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  \Psi_0 := 1; \quad \Psi_{r+1} = \frac{1}{\sqrt{2}} \left[ \Psi_r \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right] I_r \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ for } r \geq 0.
  \]
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  for $r \geq 0$.
- It requires $2n - 2$ operations.
The Haar wavelet transform

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Known connections between WHT and Haar

- Computing the WHT of $x$ is equivalent to computing mini-WHTs of the coefficients in each scale of the Haar wavelet transform of $x$ (Fino 1972, Falkowski/Rahardja 1996).

$$H_m \Psi^T_m = \begin{bmatrix} 1 & H_0 & H_1 & \ldots & H_{m-1} \end{bmatrix}$$
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- This gives an alternative algorithm for computing the WHT: via a detour into the Haar wavelet domain, also requiring $n \log_2 n$ operations.
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- The result also describes the asymptotic mutual coherence of the Walsh-Hadamard and Haar bases.
A new decomposition formula

Theorem (Thompson 2017):

\[ H_m = \left\{ \prod_{r=1}^{m-1} I_{r-1} \otimes \begin{bmatrix} I_{m-r} & 0 \\ 0 & \Psi_{m-r} \end{bmatrix} \right\} \Psi_m, \text{ for } m \geq 1. \]
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- The WHT can be computed by first computing the Haar wavelet transform, and then employing a divide-and-conquer approach also consisting of Haar wavelet transforms.
A new decomposition formula

Expanding the product:

\[ H_m = \left\{ \begin{bmatrix} I_1 & \Psi_1 & \ldots & \ldots & \Psi_1 \\ \Psi_1 & I_1 & \Psi_1 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \Psi_1 & \ldots & \ldots & I_1 & \Psi_1 \\ I_{m-2} & \Psi_{m-2} & I_{m-2} & \Psi_{m-2} & \ldots & \ldots & \ldots & \ldots \\ \Psi_{m-2} & I_{m-2} & \Psi_{m-2} & I_{m-2} & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \Psi_{m-1} & \ldots & \ldots & \ldots & \Psi_{m-1} \\ I_{m-1} & \Psi_{m-1} & \ldots & \ldots & \Psi_{m-1} \end{bmatrix} \right\} \Psi_m, \]
The Cascading Haar Wavelet (CHW) algorithm

\[ \Psi_m \rightarrow \Psi_{m-1} \rightarrow I_{m-1} \rightarrow \Psi_{m-2} \rightarrow I_{m-2} \rightarrow \Psi_{m-3} \rightarrow I_{m-3} \]

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Complexity

**Theorem (Thompson 2017):** The CHW algorithm can be implemented in $n \log_2 n$ operations.
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**Proof:**

- The CHW algorithm requires a single Haar wavelet transform of size \( 2^m \), and \( 2^{m-1-r} \) Haar wavelet transforms of size \( 2^r \), for \( r = 1, 2, \ldots, m - 1 \).
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- The CHW algorithm requires a single Haar wavelet transform of size \(2^m\),
  and \(2^{m-1-r}\) Haar wavelet transforms of size \(2^r\), for \(r = 1, 2, \ldots, m - 1\).
- The total number of operations is therefore

\[
2(2^m - 1) + \sum_{r=1}^{m-1} \left(2^{m-1-r} \cdot 2(2^r - 1)\right)
\]

\[
= 2^{m+1} - 2 + \sum_{r=1}^{m-1} 2^m - \sum_{r=1}^{m-1} 2^{m-r}
\]

\[
= 2^{m+1} - 2 + 2^m(m - 1) - 2(2^{m-1} - 1),
\]

which simplifies to \(m \cdot 2^m = n \log_2 n\).
Proposal for a parallel implementation

- Collapse by removing the identity transformations
  → a cascade of Haar wavelet transforms...
• There is a natural parallelization in which each of \( m - 1 \) nodes is devoted to the task of performing Haar wavelet transforms of a certain size.
• Here illustrated for \( m = 4 \).
Features of the proposed parallelization

- **Fixed tasks**: Each node only needs to be programmed once to do a single fixed task.
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- **Asynchronous**: Synchronization occurs automatically, when each node has received all of its inputs. It could therefore be implemented by a circuit which is not governed by a global clock.
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- A novel algorithm is proposed for computing the WHT which involves a cascade of Haar wavelet transforms.
- Its serial complexity is identical to the classical Cooley-Tukey algorithm.
- There is a natural way to parallelize the algorithm which has a number of potentially beneficial features.
References

- Falkowski, B. and Rahardja, S. *Walsh-like functions and their relations*. IEEE Proceedings on Vision, Image and Signal Processing (1996).

- Fino, B. *Relations Between Haar and Walsh/Hadamard Transforms*. Proceedings of the IEEE (1972).

- Haar, A. *Zur Theorie der orthogonalen Funktionensysteme*. Mathematische Annalen (1910).

- Thompson, A. *The Cascading Haar Wavelet algorithm for computing the Walsh-Hadamard Transform*. Signal Processing Letters (2017).