Renormalization scheme and gauge (in)dependence of the generalized Crewther relation: what are the real grounds of the $\beta$-factorization property?

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Abstract: The problem of scheme and gauge dependence of the factorization property of the renormalization group $\beta$-function in the $SU(N_c)$ QCD generalized Crewther relation (GCR), which connects the flavor non-singlet contributions to the Adler and Bjorken polarized sum rule functions, is investigated at the $O(a_4^4s)$ level of perturbation theory. It is known that in the gauge-invariant renormalization $\overline{\text{MS}}$-scheme this property holds in the QCD GCR at least at this order. To study whether this factorization property is true in all gauge-invariant schemes, we consider the $\overline{\text{MS}}$-like schemes in QCD and the QED-limit of the GCR in the $\overline{\text{MS}}$-scheme and in two other gauge-independent subtraction schemes, namely in the momentum MOM and the on-shell OS schemes. In these schemes we confirm the existence of the $\beta$-function factorization in the QCD and QED variants of the GCR. The problem of the possible $\beta$-factorization in the gauge-dependent renormalization schemes in QCD is studied. To investigate this problem we consider the gauge non-invariant mMOM and MOMgggg-schemes. We demonstrate that in the mMOM scheme at the $O(a_3^3s)$ level the $\beta$-factorization is valid for three values of the gauge parameter $\xi$ only, namely for $\xi = -3$, $-1$ and $\xi = 0$. In the $O(a_4^4)$ order of PT it remains valid only for case of the Landau gauge $\xi = 0$. The consideration of these two gauge-dependent schemes for the QCD GCR allows us to conclude that the factorization of RG $\beta$-function will always be implemented in any MOM-like renormalization schemes with linear covariant gauge at $\xi = 0$ and $\xi = -3$ at the $O(a_3^3)$ approximation. It is demonstrated that if factorization property for the MS-like schemes is true in all orders of PT, as theoretically indicated in the several works on the subject, then the factorization will also occur in the arbitrary MOM-like scheme in the Landau gauge in all orders of perturbation theory as well.

Keywords: Perturbative QCD, Renormalization Group

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1 Introduction

The triangle Green function, composed from axial-vector-vector (AVV) fermion currents, is one of the most interesting quantities in the modern theory of electromagnetic and strong interactions. From experimental point of view these studies in different kinematic regimes
allow to obtain the important information on say $\pi^0 \to \gamma \gamma$ decay amplitude and measurable formfactors of light mesons. From theoretical point of view the detailed investigation of the fundamental consequences, obtained from consideration of the AVV Green function, leads to the discovery of axial anomaly and property of its non-renormalizability. As was shown in Ref.\[1\] the result of application of the conformal symmetry (CS) transformations to the triangle diagram with one flavor non-singlet (NS) axial and two vector fermion currents without the internal gauge particles coincides with result, obtained at the lowest order of perturbation theory for triangle diagram, which is proportional to $\pi^0 \to \gamma \gamma$ decay amplitude (and the number of the quark colors $N_c$ in particular). In the work \[2\] within the massless quark-parton model it was proved that in the Born approximation the application of the operator product expansion (OPE) approach to the AVV triangle diagram in this CS limit leads to the following identity:

$$D_{\text{Born}}^{\text{NS}} C_{\text{Bjp,Born}}^{\text{NS}} = 1 .$$

(1.1)

Here $D_{\text{Born}}^{\text{NS}}$ is the Born approximation of the flavor NS contribution to the normalized Euclidean characteristic of the $e^+ e^- \to \gamma \to \text{hadrons}$ process, namely the Adler function $D(a_s)$, which has the following general massless renormalized PT expression:

$$D(a_s) = N_c \left( \sum_f Q_f^2 D^{\text{NS}}(a_s) + \left( \sum_f Q_f \right)^2 D^{\text{SI}}(a_s) \right) ,$$

(1.2)

where the number of colors $N_c$ is identical to the dimension of the quark representation of the Lie algebra of the $SU(N_c)$ color group, $Q_f$ is the electric charge of the active quark with flavor $f$, $a_s = \alpha_s / \pi$ and $\alpha_s$ is the QCD coupling constant, $D^{\text{SI}}(a_s)$ is the singlet contribution to the Adler function, that begins to manifest itself from the $\mathcal{O}(a_s^3)$ level \[3\].

The whole Adler function is related to the measurable in the Minkowski region characteristic of the electron-positron annihilation to hadrons process $R$-ratio:

$$R(s) = \frac{\sigma(e^+ e^- \to \gamma \to \text{hadrons})}{\sigma_{\text{Born}}(e^+ e^- \to \gamma \to \mu^+ \mu^-)} ,$$

(1.3)

where $\sigma_{\text{Born}}(e^+ e^- \to \mu^+ \mu^-) = 4\pi\alpha_{EM}^2/(3s)$ is the Born massless normalization factor with fixed expression of the QED coupling constant $\alpha_{EM} \approx 1/137$. The Adler function and the $R$-ratio are related through the following Källen–Lehmann dispersion representation

$$D(Q^2) = \int_0^\infty ds \frac{R(s)}{(s + Q^2)^2} ,$$

(1.4)

and begin to differ in the Minkowski region from three-loop level due to the effects of the analytical continuation, studied in Refs.\[4–6\]:

$$R(s) = D(s) - \frac{\pi^2}{3} d_1 \beta_0^2 a_s^3(s) - \pi^2 (d_2 \beta_0^2 + \frac{5}{6} d_1 \beta_1 \beta_0) a_s^4(s) + \mathcal{O}(a_s^5) .$$

(1.5)

The second term $C_{\text{Bjp,Born}}^{\text{NS}}$ in the l.h.s. of Eq.(1.1) is the Born approximation for the NS contribution to the normalized characteristic of the pure Euclidean process of deep inelastic
scattering (DIS) of polarized leptons on nucleons, namely to the Bjorken polarized sum rule, which is defined as

$$\int_0^1 \left( g^{lp}_1(x, Q^2) - g^{ln}_1(x, Q^2) \right) dx = \frac{1}{6} g_A \left| g_V \right| C_{Bjp}(Q^2).$$  \hspace{1cm} (1.6)

Here $g^{lp}_1(x, Q^2)$ and $g^{ln}_1(x, Q^2)$ are the structure functions of these deep-inelastic scattering processes, which characterize the spin distribution of quarks and gluons inside nucleons, $g_A$ and $g_V$ are the axial and vector neutron $\beta$-decay constants with $g_A/g_V = -1.2723 \pm 0.0023$ correspondingly \cite{7}. The application of CVC hypothesis leads to the result $g_V=1$. Its application is sometimes silently assumed in the definition of the r.h.s. of Eq.(1.6). The coefficient function of the polarized Bjorken sum rule $C_{Bjp}$ can be represented in the following form:

$$C_{Bjp}(a_s) = C_{Bjp}^{NS}(a_s) + N_c \sum_f Q_f C_{Bjp}^{SI}(a_s),$$  \hspace{1cm} (1.7)

where $C_{Bjp}^{SI}$ is the singlet contribution, which appears first at the $O(a_s^4)$ level of PT \cite{8} with the coefficient, analytically evaluated in Ref\cite{9}. Note, that since in this work our main aim is to study theoretical relations between coefficients of the PT series for the $D^{NS}(a_s)$ and $C_{Bjp}^{NS}(a_s)$ functions, which follows from the CS limit and its violation by the procedure of renormalizations, we neglect in Eq.(1.2) and (1.6) the massive and massless $O(1/Q^2 k)$ contributions with $k \geq 1$. Indeed, these effects are the extra sources of the violation of the CS in QCD.

At the second stage of theoretical studies it is interesting to learn what will happen with the Crewther relation (1.1) in QED-limit in the case when the AVV triangle amplitude will contain internal propagators of photons without internal fermion loops. In the work of \cite{10} it was correctly assumed that these insertions do not renormalize the AVV triangle graph. At two-loop level this assumption was confirmed in Ref\cite{11} by the demonstration of the cancellations of the $O(\alpha_s)$ corrections to the AVV triangle Green function in the most general kinematics. In the case of perturbative quenched QED (pqQED) the application of the theoretically motivated surmise of Ref\cite{10} leads to the conclusion that the Crewther relation (1.1) should be rewritten at the two-loop level as

$$D^{NS}(a)C_{Bjp}^{NS}(a) = 1,$$  \hspace{1cm} (1.8)

where $a = \alpha/\pi$ is a fixed scale-independent QED coupling constant. Since at this time the two-loop and three-loop pqQED expressions for the photon vacuum polarization function were analytically calculated in Refs\cite{12}, \cite{13}, the application of Eq.(1.8) allowed the authors of Ref\cite{10} to predict the analytical expression for the $O(a^2)$ corrections to the $C_{Bjp}^{NS}(a)$-function in pqQED. It was observed in Ref\cite{14} that these results are in agreement with pqQED limit of the $SU(N_c)$ leading-order (LO) and next-to-leading order (NLO) QCD results, obtained previously in Refs\cite{15} and \cite{16} correspondingly.

In all orders of pqQED the validity of the relation (1.8) was proved in Ref\cite{17}, where it was clarified that a fixed scale-independent coupling constant $a$ should be considered
as the unrenormalized (bare) QED coupling in the CS limit, which is realized when the charge renormalization constant of QED is fixed to unity in all orders of perturbation theory, namely $Z_3 = 1$. In the work [17] it was also explained how to define this pqQED limit, which is equivalent to the language of the investigated some time ago phenomenologically unrealized but theoretically important finite QED program [20], used in the process of the analytical four-loop computations of Ref.[21], which confirmed the analytical expression of the scheme-independent four-loop contribution to the renormalization group QED $\beta$-function in the MS-like and MOM-schemes, previously evaluated in Ref.[22] by direct diagram-by-diagram calculations.

However, if one will now consider massless limit of the based on $SU(N_c)$ color group renormalizable QCD or of the realized in nature real QED (which presumes the account of the coupling constant renormalization by the ultraviolet divergent expression of the photon renormalization constant $Z_3$), the Crewther relation, written down in the form of either (1.1) or (1.8), should be modified. The theoretical reason for this modification is related to the appearance inside AVV triangle graph of the internal divergent subgraphs, which should be renormalized. It is known that the procedure of renormalizations violates the CS and leads to the appearance of the conformal anomaly [2], [23], which as demonstrated almost at the same time in Refs.[24–27] is fixed by the perturbative expression of the RG $\beta$-function, coinciding with anomalous dimension of the trace of energy momentum tensor. In the case of $SU(N_c)$ QCD it is usually evaluated in the related to dimensional regularization [28, 29] variant of the MS-like schemes [30], namely in the $\overline{\text{MS}}$-scheme [31]. This RG $\beta$-function is responsible for the evolution of the corresponding coupling constant $a_s(\mu^2) = \alpha_s(\mu^2)/\pi$ and can be defined as:

$$\beta(a_s) = \mu^2 \frac{d a_s}{d \mu^2} = - \sum_{i \geq 0} \beta_i a_s^{i+2}.$$  (1.9)

During rather long period it was unclear how the violation of the CS will manifest itself in high orders of the PT for the QCD generalization of the Crewther (GCR) relation in the $\overline{\text{MS}}$-scheme. As was discovered in Ref.[33] in this gauge-independent scheme the Crewther relation receives additional conformal symmetry breaking contribution $\Delta_{\text{csb}}(a_s)$, first appearing at the $O(a_s^2)$ level:

$$D^{NS}(a_s)C_{Bjp}^{NS}(a_s) = 1 + \Delta_{\text{csb}}(a_s),$$  (1.10)

Moreover, in Ref.[33] it was shown that at least at the $O(a_s^3)$ level this extra term can be written down in the following factorized form:

$$\Delta_{\text{csb}} = \left( \frac{\beta(a_s)}{a_s} \right) K(a_s),$$  (1.11)

here $\beta(a_s)$ is the two-loop expression for the defined in Eq.(1.9) $\beta$-function of $SU(N_c)$-theory, analytically calculated in the $\overline{\text{MS}}$-scheme at the one-loop level in Refs.[34, 35] and at

\begin{itemize}
  \item Note, that the consideration of the consequences of Eq.(1.8), performed in Ref.[14], turned out to be rather important for the first confirmation of the validity of the complicated analytical calculation of the $O(a_s^2)$ contribution to $D^{NS}(a_s)$ function in QCD [18], or to be more precise, of the term, containing unexpected previously transcendental $\zeta_3$-contribution to its QED limit, firstly discovered in Ref.[19].
  \item Its more formal definition is given in Ref.[32].
\end{itemize}
the two-loop level in Refs.[36–38]. The function \( K(a_s) \) can be represented as the polynomial in powers of \( a_s \):
\[
K(a_s) = \sum_{i\geq 1} K_i a_s^i
\]
(1.12)
with the depending on the \( SU(N_c) \) group structures coefficients \( K_i \), computed in Ref.[33] for \( i = 1, 2 \). Its first term \( K_1 \) depends only on the defined in the fundamental representation quadratic Casimir operator \( C_F \), while \( K_2 \) is expressed through \( C_F^2, C_FC_A \) and \( C_FT_Fn_f \) color structures, where \( C_A \) is the Casimir operator in the adjoint representation of the Lie algebra, \( T_F \) is the Dynkin index and \( n_f \) is the number of active quark flavors. These coefficients were obtained from Eqs.(1.10-1.12) due to the knowledge of the \( SU(N_c) \) structure of the \( \mathcal{O}(a_s^2) \) approximations for \( D^{NS}(a_s) \) and \( C^{NS}_{Bjp}(a_s) \) functions in the \( \overline{\text{MS}} \)-scheme and the two-loop analytical expression for the QCD \( \beta \)-function\(^3\).

The corresponding expression for \( D^{NS}(a_s) \) was known thanks to the analytical calculations of the leading-order (LO) \( \mathcal{O}(a_s) \)-correction [40, 41], the next-to-leading order (NLO) \( \mathcal{O}(a_s^2) \)-correction [42, 43] (which agrees with the numerical result for this contribution, independently obtained in Ref.[44]) and next-to-next-to-leading order (NNLO) \( \mathcal{O}(a_s^3) \)-correction, evaluated in Ref.[3] and confirmed in Ref.[45] and later on in Ref.[46] using a bit different technique. The analytical \( \mathcal{O}(a_s^2) \) approximation for the \( C^{NS}_{Bjp}(a_s) \) function in the \( \overline{\text{MS}} \)-scheme, which was used in Ref.[33], included the information on the obtained in Ref.[15] LO corrections and the NLO and NNLO-terms, evaluated in Ref.[16] and Ref.[47] respectively.

Seventeen years later the validity of the \( \beta \)-factorization property in \( \Delta_{cab} \) term of Eq.(1.10), discovered in Ref.[33], was detected at the \( \mathcal{O}(a_s^3) \) level in Ref.[48] after getting analytical four-loop \( SU(N_c) \) expressions for the \( D^{NS}(a_s) \)-function (confirmed recently by the independent calculation of Ref.[49]) and for the \( C^{NS}_{Bjp}(a_s) \)-function, and taking into account the explicit expression for the three-loop \( SU(N_c) \) RG \( \beta \)-function, computed in Ref.[50] and confirmed in Ref.[51] in the \( \overline{\text{MS}} \)-scheme. The presented in Ref.[48] \( \mathcal{O}(a_s^3) \) expression for the conformal symmetry breaking term contains the third coefficient \( K_3 \) of \( \Delta_{cab} \) term, which is composed of six color structures, namely \( C_F^3, C_F^2C_A, C_FC_A^2, C_F^2T_Fn_f, C_FC_AT_Fn_f \) and \( C_FT_F^2n_f^2 \).

Theoretical arguments in favor of the validity of the \( \beta \)-factorization property in the \( \overline{\text{MS}} \)-scheme were given in Ref.[52] in all orders of PT. At more solid theoretical level this statement was analyzed in the work [53] and in the review [54]. However, the detailed understanding and proof of the origin of the existence of this fundamental feature of the GCR is still absent.

In this work we study this problem at the four-loop level using the method of theoretical experiment and try to shed some light on the origin of this feature analyzing scheme-dependence of the analytical expression of the GCR in QCD and QED.

\(^3\) The appearance of the term proportional to \( \beta_0 K_1 \) in Eq.(1.11) is in agreement with the performed later on in Ref.[39] explicit analytical calculations of the three-loop contributions to the AVV correlator, which contain the internal sub-diagrams responsible for the running of the coupling constant and the emergence of the \( \beta_0 \)-coefficient.
We start our analysis from the consideration of the PT theoretical QCD expressions for the NS contributions \(D^{NS}\) and \(C^{NS}_{Bjp}\) to the Adler and Bjorken polarized sum rule functions, which are defined as:

\[
D^{NS}(a_s) = 1 + \sum_{k\geq 1} d_k a_s^k , \quad (1.13a) \\
C^{NS}_{Bjp}(a_s) = 1 + \sum_{k\geq 1} c_k a_s^k . \quad (1.13b)
\]

It is already known that the property of the factorization of the \(SU(N_c)\) RG \(\beta\)-function in the GCR is scheme-dependent even within the framework of a gauge-invariant renormalization schemes. Indeed, as was shown in Ref.[55], the transformation to the considered in Ref.[56] \'{t} Hooft scheme with its RG \(\beta\)-function, which contains two nonzero scheme-independent PT coefficients only (the rest are assumed to be zero by finite renormalizations of charge), spoils the property of pure \(\beta\)-factorization in the GCR. In view of this the natural question arises whether there do exist theoretical requirements to the choice of the ultraviolet subtraction schemes, which provide the realization of the fundamental property of the \(\beta\)-factorization in the GCR, associated with the manifestation of the conformal symmetry breaking.

In Sec.2 we study this question in two renormalized gauge theories, namely in \(SU(N_c)\) QCD and in QED, based on the Abelian \(U(1)\) gauge group. In this section we arrive to the conclusion that the sufficient condition for factorization of the RG \(\beta\)-function in the GCR at the fourth order of PT at least is the choice of the gauge-invariant renormalization scheme for evaluating PT expressions for the \(D^{NS}\), \(C^{NS}_{Bjp}\)-functions and of the corresponding RG \(\beta\)-function as well. We clarify the features of the considered analytical PT expressions for the \(D^{NS}\) and \(C^{NS}_{Bjp}\)-functions in various gauge-independent schemes, including the consideration of the \(\overline{\text{MS}}\)-results, discussed previously in Refs.[33], [57–62] and in Refs.[63, 64], [48], based on the effective charges approach, developed in Refs.[65, 66]. Remind that the class of gauge-invariant schemes includes not only the original MS-scheme and widely used in QCD \(\overline{\text{MS}}\)-scheme, but also formulated for simplifying multi-loop calculations G-scheme [67] and applied at high-order computations in the chiral perturbation theory \(\overline{\text{MS}} + 1\)-scheme [68]. All these schemes are the elements of the considered in Ref.[69] infinite set of \(\delta\)-Renormalization gauge-independent MS-like schemes.

In the QED studies to be presented in this work we use three gauge-invariant schemes only, namely the \(\overline{\text{MS}}\)-scheme, the momentum subtraction (MOM) scheme and the physical-motivated on-shell (OS) scheme. Also we consider the scheme dependence of the coefficients \(d_k\) and \(c_k\), determined in Eqs.(1.13a) and (1.13b), within MS-like schemes up to four-loop level. After this we examine the scheme dependence of the coefficients \(K_i\) of the conformal symmetry breaking term in Eq.(1.12) in this class of the gauge-invariant schemes.

In Sec.3 we first investigate the question of the possibility of factorization of the RG \(\beta\)-function in the generalized Crewther relation in the gauge non-invariant renormalization schemes. For this purpose we turn to consideration of the popular at present the gauge-dependent subtraction scheme, called the miniMOM (mMOM) scheme, introduced in Ref.[70]. We calculate the \(\mathcal{O}(a_s^4)\) expression for the NS flavor contribution to the Adler
and Bjorken polarized sum rule functions and we find the improved convergence of these
PT series for the cases of \( n_f = 3, 4, 5 \) in comparison with the similar results, obtained in
the \( \overline{\text{MS}} \)-scheme, especially for the Bjorken coefficient function.

In Sec.4 we present an exact recipe for determination of those values of the gauge
parameter, for which the \( \beta \)-factorization is performed in the GCR at the \( \mathcal{O}(a_s^2) \) and \( \mathcal{O}(a_s^4) \)
levels. By explicit calculation we demonstrate that in the mMOM-scheme the gauges \( \xi = 0, -1, -3 \) are distinguished from an infinite set of values of gauge parameter \( \xi \) at the
\( \mathcal{O}(a_s^2) \) level of PT, because exactly for these values the factorization of \( \beta \)-function in the
GCR is fulfilled. At the \( \mathcal{O}(a_s^4) \) level the factorization property holds in the mMOM-scheme
in the Landau gauge only. If we fix \( \xi = -1 \) or \( \xi = -3 \), we observe at this level the
partial factorization only (see Appendix B). Then we consider another gauge-dependent
renormalization scheme, namely the MOMgggg-scheme, and discover definite similarities
between these completely different MOM-type schemes. We find that at \( \xi = 0 \) and \( \xi = -3 \)
the \( \beta \)-factorization property holds in the \( \mathcal{O}(a_s^2) \) approximation in the MOMgggg-scheme as
well. Based on these at the first glance surprising results (indeed, it is not immediately
clear why the values \( \xi = 0 \) and \( \xi = -3 \) are highlighted), we conclude that in the \( \mathcal{O}(a_s^2) \)
approximation the factorization of the \( \beta \)-function is always possible in an arbitrary
MOM-like renormalization scheme with linear covariant gauges \( \xi = 0 \) and \( \xi = -3 \). Moreover, we
prove that if the \( \beta \)-factorization is valid in MS-like schemes in all orders of PT, then in the
MOM-like schemes with the Landau gauge this factorization property persists in all orders
as well.

2 Scheme-dependence of the GCR in the gauge-invariant schemes

2.1 Scheme-dependence of the PT series in the \( SU(N_c) \) QCD

Consider first the case of \( SU(N_c) \) theory. In this extended version of QCD the most effective
method of performing high-order PT calculations is based on the application of dimensional
regularization the class of the MS-like gauge-invariant ultraviolet renormalization schemes.
Let us find out how the coefficients of the physical quantities discussed above in Eqs.(1.13a)
and (1.13b) change under the transition from one to another scale \( \mu \to \hat{\mu} \) in this class of
schemes. It is known that the transformation from the \( \overline{\text{MS}} \)-scheme to another representative
of the class of MS-like schemes (or following terminology of Ref.[69] \( R_\beta \)-schemes) can be
accomplished by the proper change of the renormalization scale. The transition from the \( \overline{\text{MS}} \)
to MS requires following shift \( \hat{\mu}^2 = \mu^2 \exp(\log 4\pi - \gamma_E) \), where \( \gamma_E \approx 0.577 \) is the constant
of Euler-Mascheroni. This immediately implies that the PT coefficients for the Adler
and Bjorken polarized sum rule functions in the MS-scheme will contain additional terms,
proportional to the absent in the \( \overline{\text{MS}} \)-scheme transcendental \( (\log 4\pi - \gamma_E) \)-factor. Indeed, the
solution of the RG equation (1.9) at the \( \mathcal{O}(a_s^4) \)-level

\[
as_s(\hat{\mu}^2) = a_s(\mu^2) \left( 1 - \beta_0 L a_s(\mu^2) + (\beta_0^2 L^2 - \beta_1 L) a_s^2(\mu^2) + (-\beta_0^3 L^3 + \frac{5}{2} \beta_0 \beta_1 L^2 - \beta_2 L) a_s^3(\mu^2) \right) \tag{2.1}
\]
depends on the RG logarithm term \( L = \log \hat{\mu}^2 / \mu^2 \). Using the property of the RG-invariance
of the \( D^{NS}(a_s) \)-function and taking into account Eq.(2.1), we obtain the following relations
reflecting the transformation law of the coefficients $d_k$ at $\mu \to \bar{\mu}$ in the MS-like schemes:

\begin{align}
\tilde{d}_1 &= d_1 \\
\tilde{d}_2 &= d_2 + \beta_0 d_1 L \\
\tilde{d}_3 &= d_3 + (\beta_1 d_1 + 2 \beta_0 d_2) L + \beta_0^2 d_1 L^2 \\
\tilde{d}_4 &= d_4 + (\beta_2 d_1 + 2 \beta_1 d_2 + 3 \beta_0 d_3) L + \left(\frac{5}{2} \beta_0 \beta_1 d_1 + 3 \beta_0^2 d_2\right) L^2 + \beta_0^3 d_1 L^3 .
\end{align}

(2.2a-2.2d)

It is obvious that coefficients $c_k$ in Eq.(1.13b) for $C_{BJP}^{NS}(a_s)$ function will obey the same transformation laws after the replacement of $d_k$ by $c_k$.

Let us now study the scheme-dependence of the GCR, defined by Eqs.(1.10-1.11) in the case of application of the gauge-invariant MS-like schemes. Substituting the $O(a_s^4)$ PT approximations for $D^{NS}(a_s)$ and $C^{NS}(a_s)$-functions, given in Eqs.(1.13a-1.13b), into the GCR, presented in Eqs.(1.10-1.11), we get the following relations:

\begin{align}
d_1 + c_1 &= 0 , \\
d_2 + c_2 + d_1 c_1 &= - \beta_0 K_1 , \\
d_3 + c_3 + d_1 c_2 + c_1 d_2 &= - \beta_1 K_1 - \beta_0 K_2 , \\
d_4 + c_4 + d_1 c_3 + c_1 d_3 + d_2 c_2 &= - \beta_2 K_1 - \beta_1 K_2 - \beta_0 K_3 ,
\end{align}

(2.3a-2.3d)

where $\beta_i$ are the coefficients of the $SU(N_c)$ RG $\beta$-function, defined in Eq.(1.9). The nullification of Eq.(2.3a) is the consequence of the conformal symmetry and it reflects the feature of the absence of the $O(a_s)$-corrections to the CSB term $\Delta_{csb}$. The expressions (2.3b-2.3d) are valid when the property of factorization of the $\beta$ function in the GCR is true\(^4\).

Now everything is ready to study the scheme-dependence of the coefficients $K_i$, included in the CSB term, within the class of MS-like schemes. Using equations (2.2a-2.2d), the relations (2.3a-2.3d) and taking into account that these formulas remain valid for any scale\(^5\), we obtain:

\begin{align}
\bar{K}_1 &= K_1 , \\
\bar{K}_2 &= K_2 + 2 \beta_0 K_1 L , \\
\bar{K}_3 &= K_3 + 3 (\beta_1 K_1 + \beta_0 K_2) L + 3 \beta_0^2 K_1 L^2 .
\end{align}

(2.4a-2.4c)

Note, that the coefficient $K_1$ is invariant in the MS-like schemes and the remaining coefficients $K_2$, $K_3$ are not MS-like schemes invariants. The expressions (2.4a-2.4c) coincide with the results, obtained recently in Ref.[71]. It should be emphasized that the scale transformations do not violate the property of the factorization of the $SU(N_c)$ $\beta$-function in the PT expression for the CSB term $\Delta_{csb}(a_s)$ of the GCR (1.10) within the gauge-independent MS-like schemes, but only modify expressions for coefficients $K_i$ according to formulas (2.4a-2.4c). This gives us the idea that the gauge-invariance of the subtraction schemes

\(^4\)Note that due to the absence of overall minus in the definition of the QED RG $\beta$-function in Eq.(1.9) the QED analogues of these formulas contain pluses instead minuses in r.h.s. of these expressions.

\(^5\)Remind that transformations $\mu \to \bar{\mu}$ do not affect the expressions for the coefficients of the RG $\beta$-function of Eq.(1.9) in MS-like schemes.
is the sufficient condition for the $\beta$-factorization in the CSB massless perturbative term of the GCR. To study the status of this statement we extend the considerations, reported in Ref.[72], and analyze the GCR in QED, where the number of widely applicable gauge-independent schemes is larger than in QCD. In the process of these considerations we use three concrete renormalization schemes, namely for $\overline{\text{MS}}$, momentum subtraction (MOM) and on-shell (OS) gauge-invariant schemes. 

Let us now move aside slightly and consider the \{$\beta$\}-expansion approach [73] for representing PT coefficients of the RG invariant quantity, namely for $d_k$-coefficients of the PT $D^6$\{$a_s$\}-series, proposed and used in Ref.[73], which is the high-order PT QCD generalization of the suggested $\overline{\text{MS}}$-version of the BLM scale-fixing approach in Ref.[74]. At $Q^2 = \mu^2$ the \{$\beta$\}-expanded representations for $d_k$ have the following form [73]:

$$d_1 = d_1[0]\, , \quad d_2 = d_2[0] + \beta_0 d_2[1]\, , \quad d_3 = d_3[0] + \beta_0 d_3[1] + \beta_0^2 d_3[2] + \beta_1 d_3[0, 1]\, ,$$
$$d_4 = d_4[0] + \beta_0 d_4[1] + \beta_0^2 d_4[2] + \beta_0^3 d_4[3] + \beta_1 d_4[0, 1] + \beta_1 \beta_0 d_4[1, 1] + \beta_2 d_4[0, 0, 1]\, .$$

In particular, the \{$\beta$\}-expansion formalism was used for the construction of the QCD multiloop generalization of the $\overline{\text{MS}}$-version of the BLM approach, called the Principle of Maximal Conformality (PMC) [76]. The idea of applying the term PMC belongs to authors of Ref.[76], who correctly realized that within the class of MS-like (or $R_\delta$) schemes the $\beta_k$-independent contributions in \{$\beta$\}-expanded coefficients $d_k$, namely $d_k[0]$ in Eqs.(2.5a-2.5b), are scheme-independent and obey the same properties as the coefficients of Green functions within studied previously finite QED program [20]. As was explained in Ref.[17] this property is related to the possibility of defining the CS limit in this Abelian model.

The presented above RG-based expressions (2.2a-2.2d) clarify the scheme-independent property of the terms $d_k[0]$ within the class of MS-like schemes. Actually, taking into account the \{$\beta$\}-expanded formalism (2.5a-2.5b) and expanding relations (2.2a-2.2d) at an appropriate scale in accordance with this formalism, we obtain that the $\beta_k$-dependent terms of Eqs.(2.5a-2.5b) depend on the choice of the concrete MS-like prescription of fixing $\mu$:

$$\tilde{d}_1[0] = d_1[0]\, , \quad \tilde{d}_2[0] = d_2[0]\, , \quad \tilde{d}_2[1] = d_2[1] + d_1[0]L\, ,$$
$$\tilde{d}_3[0] = d_3[0]\, , \quad \tilde{d}_3[1] = d_3[1] + 2d_2[0]L\, ,$$
$$\tilde{d}_3[0, 1] = d_3[0, 1] + d_1[0]L\, , \quad \tilde{d}_3[2] = d_3[2] + 2d_2[1]L + d_1[0]L^2\, ,$$
$$\tilde{d}_4[0] = d_4[0]\, , \quad \tilde{d}_4[1] = d_4[1] + 3d_3[0]L\, ,$$
$$\tilde{d}_4[0, 1] = d_4[0, 1] + 2d_2[0]L\, , \quad \tilde{d}_4[2] = d_4[2] + 3d_3[1]L + 3d_2[0]L^2\, ,$$
$$\tilde{d}_4[0, 0, 1] = d_4[0, 0, 1] + d_1[0]L\, ,$$
$$\tilde{d}_4[1, 1] = d_4[1, 1] + (2d_2[1] + 3d_3[0, 1])L + \frac{5}{2} d_1[0]L^2\, ,$$
$$\tilde{d}_4[3] = d_4[3] + 3d_3[2]L + 3d_2[1]L^2 + d_1[0]L^3.$$ 

The similar relations holds for the \{$\beta$\}-expanded coefficients of PT series of any RG-invariant physical quantity, including \{$\beta$\}-expanded coefficients $c_k$ of the NS Bjorken po-

\footnote{For the discussion on the scheme-dependence of the original BLM procedure within QCD see e.g. [75].}
labeled sum rule function $C_{Bjp}^{NS}(a_s)$, studied from different points of view in the works of [57],[59],[71],[77].

In accordance with the proposals of Ref.[73],[76],[69] all $\beta_k$-dependent contributions in Eqs.(2.5a-2.5b) should be absorbed into the set of scales of the initially defined in the MS-scheme QCD coupling constant. In view of the scheme-dependence of these $\beta_k$-dependent terms, the scales of the resulting QCD-expansion parameter become scheme-dependent as well and do not respect the CS approximations. Therefore, the term Principle of Maximal Conformality, used in the number of the related works on this topic [58],[69],[71],[76–78], should be used with care.

Note that in Ref.[59] the NNLO PMC-type procedure to both $D_{NS}(a_s)$ and $C_{Bjp}^{NS}(a_s)$-functions was applied in the QCD-type model, based on the $SU(N_c)$ color gauge theory, supplemented with multiplet of SUSY gluino. The analytical results for the $\{\beta\}$-expanded expressions of $d_2$ and $d_3$ coefficients in this model were first obtained in Ref.[73] and confirmed later on in Ref.[78]. In Ref.[60] it was demonstrated how to define the expressions for $\{\beta\}$-expanded coefficients of the Adler and Bjorken functions at the $O(a_4^2)$ level in $SU(N_c)$ QCD without any gluino. The careful QCD $O(a_4^2)$ reconsideration of the results of Ref.[59] and Refs.[69],[71],[76–78] with taking into account that the QCD anomalous dimension of the photon vacuum polarization function has its own well-defined $\beta$-expansion structure (for the detailed clarification of this point see Ref.[61]) is on the agenda and will be considered elsewhere.

We will not discuss anymore these phenomenologically oriented problems, but will return to one of the theoretical aims of this work, namely to the study of the scheme (in)dependence of the CSB PT contribution to the GCR in the factorized by RG $\beta$-function form in the different schemes, including the ones commonly used in QED. In order to analyze this problem we should get the $O(a^4)$ analytical PT approximations for the NS Adler function and Bjorken polarized sum rule function in QED in the gauge-invariant renormalization schemes we are interested in.

### 2.2 The NS Adler function in the $\overline{\text{MS}}$, MOM and OS schemes in QED

In order to obtain the $O(a^4)$ expression for the non-singlet QED Adler function in the $\overline{\text{MS}}$ scheme we use the results of Ref.[48]. We consider QED with $N$ types of charged leptons ($N = 1$, 2, 3 for $e$, $\mu$, $\tau$-leptons correspondingly). Fixing $C_F = 1$, $C_A = 0$, $T_F = 1$, $d_F^{abcd} = 1$, $d_A^{abcd} = 0$, $d_R = 1$, $n_f = N$, we obtain the following four-loop analytical approximation for the $D_{NS}^{a}(a)$-function in the $\overline{\text{MS}}$-scheme, which is related through the Källen–Lehmann dispersion representation with the massless PT expression for the total cross-section of the process $e^+e^- \rightarrow \gamma \rightarrow l^+l^-$ (here $l$ denotes one of the charged leptons $e, \mu$ or $\tau$-lepton):

$$D_{\overline{\text{MS}}}^{NS}(a) = 1 + \sum_{k=1}^{4} d_{k}^{\overline{\text{MS}}} (a^{\overline{\text{MS}}})^k,$$  

(2.7a)
\[ d_1^{\text{MS}} = \frac{3}{4}, \quad d_2^{\text{MS}} = -\frac{3}{32} + \left( -\frac{11}{8} + \zeta_3 \right) N, \]  
\[ d_3^{\text{MS}} = -\frac{69}{128} + \left( -\frac{29}{64} + \frac{19}{4} \zeta_3 - 5\zeta_5 \right) N + \left( \frac{151}{54} - \frac{19}{9} \zeta_3 \right) N^2, \]  
\[ d_4^{\text{MS}} = \frac{4157}{2048} + \frac{3}{8} \zeta_3 + \left( \frac{689}{384} + \frac{67}{32} \zeta_3 - \frac{115}{4} \zeta_5 + \frac{105}{4} \zeta_7 \right) N \]  
\[ + \left( \frac{5713}{1728} - \frac{581}{24} \zeta_3 + \frac{125}{6} \zeta_3 + 3\zeta_5^2 \right) N^2 + \left( -\frac{6131}{972} + \frac{203}{54} \zeta_3 + \frac{5}{3} \zeta_5 \right) N^3. \]

The energy behavior of the QED coupling constant \( a^{\text{MS}} = \alpha^{\text{MS}}/\pi \) in Eq.(2.7a) is determined by the four-loop expression of the corresponding \( \beta \)-function, which reads

\[ \beta^{\text{MS}}(a^{\text{MS}}) = \frac{1}{3} N (a^{\text{MS}})^2 + \frac{1}{4} N (a^{\text{MS}})^3 + \left( -\frac{1}{32} N - \frac{11}{144} N^2 \right) (a^{\text{MS}})^4 \]
\[ + \left( -\frac{23}{128} N + \left[ \frac{95}{864} - \frac{13}{36} \zeta_3 \right] N^2 - \frac{77}{3888} N^3 \right) (a^{\text{MS}})^5. \]

Note, that the first two scheme-independent coefficients, included in Eq.(2.8), were obtained in Ref.[79] from the two-loop approximation of the photon vacuum polarization function \( \Pi(Q^2/\mu^2, a) \), evaluated previously at the same level in Ref.[12]. The \( N \)-dependence of the three-loop coefficient in the \( \overline{\text{MS}} \)-scheme was obtained in Ref.[80]. At \( N = 1 \) it agrees with the results of calculations, independently performed in Refs.[67] and [81]. The four-loop term was obtained in Ref.[22].

To transform the results of Eqs.(2.7b-2.7d) to the MOM and OS-schemes we use the following RG-based relations

\[ \beta^{\text{MOM}}(a^{\text{MOM}}) = \beta^{\text{MS}}(a^{\text{MS}}) \frac{\partial a^{\text{MOM}}}{\partial a^{\text{MS}}}, \quad \beta^{\text{OS}}(a^{\text{OS}}) = \beta^{\text{MS}}(a^{\text{MS}}) \frac{\partial a^{\text{OS}}}{\partial a^{\text{MS}}}, \]

and take into account the properties of the RG-invariance and scheme-independence of the QED invariant charge \( a/(1 + \Pi(Q^2/\mu^2, a)) \) or to be more precise of its expression, related to the \( \overline{\text{MS}}, \text{MOM} \) and OS schemes. Note also that in the MOM scheme, determined by subtractions of the UV divergences of the photon vacuum polarization function at the non-zero Euclidean momentum, the QED \( \beta \)-function coincides with the Gell-Mann–Low function, which governs the energy behavior of the QED invariant charge.

The four-loop expressions for the QED \( \beta \)-functions in the MOM and OS schemes are known and have the following form:

\[ \beta^{\text{MOM}}(a^{\text{MOM}}) = \frac{1}{3} N (a^{\text{MOM}})^2 + \frac{1}{4} N (a^{\text{MOM}})^3 + \left( -\frac{1}{32} N - \left[ \frac{23}{72} - \frac{1}{3} \zeta_3 \right] N^2 \right) (a^{\text{MOM}})^4 \]
\[ + \left( -\frac{23}{128} N + \left[ \frac{13}{32} + \frac{2}{3} \zeta_3 - \frac{5}{3} \zeta_5 \right] N^2 + \left[ \frac{1}{2} - \frac{1}{3} \zeta_3 \right] N^3 \right) (a^{\text{MOM}})^5, \]
\[ \beta^{\text{OS}}(a^{\text{OS}}) = \frac{1}{3} N (a^{\text{OS}})^2 + \frac{1}{4} N (a^{\text{OS}})^3 + \left( -\frac{1}{32} N - \frac{7}{18} N^2 \right) (a^{\text{OS}})^4 + \left( -\frac{23}{128} N \right) \]
\[ + \left[ \frac{1}{48} - \frac{5}{6} \zeta_2 + \frac{4}{3} \zeta_2 \log 2 - \frac{35}{96} \zeta_3 \right] N^2 + \left[ \frac{901}{1296} - \frac{4}{9} \zeta_2 - \frac{7}{9} \zeta_3 \right] N^3 \]  
\[ (a^{\text{OS}})^5. \]
The proportional to $N$ scheme-independent contribution to the three-loop corrections of Eqs.(2.11) and (2.12) was calculated long time ago in Ref.[13], while the full three-loop corrections to Eqs.(2.11) and (2.12) were evaluated in Refs.[80] and [83] correspondingly. The analytical expressions for the four-loop contributions to (2.11) and (2.12) were calculated in the works of [22] and [84].

Fixing $Q^2 = \mu^2_{\text{MS}} = \mu^2_{\text{MOM}} = m^2_{\text{OS}}$ (that reflects theoretical freedom in fixing scale parameters of the $\text{MS}$ and $\text{MOM}$-schemes in QED) and taking into account the scheme-independence of the constant term of the single lepton-loop contribution to the photon vacuum polarization function included in the QED invariant charge [84–86], and applying the results of Eqs.(2.8-2.12), one can obtain the following transformation relations:

$$a_{\text{MS}} = a_{\text{MOM}} + \left( \frac{35}{48} - \frac{\zeta_3}{2} \right) N (a_{\text{MOM}})^3 + N \left( - \frac{4}{9} - \frac{37}{24} \zeta_3 + \frac{5}{2} \zeta_5 \right) (a_{\text{MOM}})^4 \quad (2.13)$$

$$a_{\text{MS}} = a_{\text{OS}} + \frac{15}{16} N (a_{\text{OS}})^3 + \left( \frac{77}{576} + \frac{5}{4} \zeta_2 - 2 \zeta_2 \log 2 + \frac{\zeta_3}{192} \right) N \quad (2.14)$$

Using expressions (2.7a-2.7d), the obtained expansions (2.13-2.14) and taking into consideration that the flavor NS Adler function is the renormalization group invariant quantity, we get the values of the coefficients $d_k$ of the QED PT series for Eq.(1.13a) in the MOM-scheme:

$$D_{\text{MOM}}^{\text{NS}}(a_{\text{MOM}}) = 1 + \sum_{k=1}^{4} d^\text{MOM}_k (a_{\text{MOM}})^k \quad (2.15a)$$

$$d^\text{MOM}_1 = \frac{3}{4}, \quad d^\text{MOM}_2 = - \frac{3}{32} + \left( - \frac{11}{8} + \zeta_3 \right) N, \quad (2.15b)$$

$$d^\text{MOM}_3 = - \frac{69}{128} + \left( \frac{3}{32} + 4 \zeta_3 - 5 \zeta_5 \right) N + \left( \frac{151}{54} - \frac{19}{9} \zeta_3 \right) N^2, \quad (2.15c)$$

$$d^\text{MOM}_4 = \frac{4157}{2048} + \frac{3}{8} \zeta_3 + \left( \frac{339}{256} + \frac{9}{8} \zeta_3 - \frac{215}{8} \zeta_5 - \frac{105}{4} \zeta_7 \right) N$$

$$+ \left( \zeta_3 + \frac{157}{384} - \frac{125}{6} \zeta_3 + \zeta_5^2 \right) N^2 + \left( - \frac{6131}{972} + \frac{203}{54} \zeta_3 + \frac{5}{3} \zeta_5 \right) N^3, \quad (2.15d)$$

and in the physically motivated OS-scheme:

$$D_{\text{OS}}^{\text{NS}}(a_{\text{OS}}) = 1 + \sum_{k=1}^{4} d^\text{OS}_k (a_{\text{OS}})^k \quad (2.16a)$$

$$d^\text{OS}_1 = \frac{3}{4}, \quad d^\text{OS}_2 = - \frac{3}{32} + \left( - \frac{11}{8} + \zeta_3 \right) N, \quad (2.16b)$$

$^7$For $N=1$ the obtained in Ref.[80] three-loop analytical result coincides with the one, previously computed in Ref.[82].
\[ d_3^{\text{OS}} = -\frac{69}{128} + \left( \frac{1}{4} + \frac{19}{4} \zeta_3 - 5 \zeta_5 \right) N + \left( \frac{151}{54} - \frac{19}{9} \zeta_3 \right) N^2, \]

\[ d_4^{\text{OS}} = \frac{4157}{2048} + \frac{3}{8} \zeta_8 + \left( \frac{55}{32} + \frac{15}{16} \zeta_2 + \frac{537}{256} \zeta_3 - \frac{115}{4} \zeta_5 + \frac{105}{4} \zeta_7 - \frac{3}{2} \zeta_2 \log 2 \right) N \]

\[ + \left( -\frac{11}{144} + \frac{\zeta_2}{2} - \frac{17089}{768} \zeta_3 + \frac{125}{6} \zeta_5 + 3 \zeta_3^2 \right) N^2 + \left( -\frac{6131}{972} + \frac{203}{54} \zeta_3 + \frac{5}{3} \zeta_3^2 \right) N^3, \]

where \( a^{\text{MOM}} \) and \( a^{\text{OS}} \) are the QED running coupling constants of the MOM and OS schemes, which, at the studied by us level are the solutions of the corresponding RG equation, with the MOM and OS four-loop expressions of the QED \( \beta \)-functions, presented in Eqs.(2.11-2.12).

It is worth emphasizing that the analytical expressions for the coefficients of the \( O(a) \) and \( O(a^2) \) terms for the QED expressions of the \( D^{NS} \)-function in the \( \overline{\text{MS}} \), MOM and OS-schemes, are the same. This circumstance is a consequence of the presented in Eqs.(2.13-2.14)) relations between QED running coupling constants of \( \overline{\text{MS}} \), MOM and OS-schemes, which do not contain \( O(a) \)-corrections in the case, when we choose \( \mu^2_{\overline{\text{MS}}} = \mu^2_{\text{MOM}} = m^2_{\text{OS}} \).

Let us comment the observations, which follow from the comparison of the analytical expressions for the \( D^{NS} \) coefficients in Eqs.(2.7b-2.7d), (2.15b-2.15d) and (2.16b-2.16d) in three gauge-invariant schemes mentioned above.

1. The scheme-independence of the proportional to \( N^0 \)-contributions to the the QED PT expression for the NS Adler function follows from the CS, which is valid in the case of consideration of the pqQED approximation of the PT series for the RG invariant quantities (for detailed explanation see Ref.[17]).

2. As explained above, the \( O(a^2) \) QED PT contributions to \( D^{NS} \) are scheme-independent.

3. The scheme-independence of the leading on \( N \) corrections to the coefficients of \( d_k \) are the consequence of the scheme-independence of the leading renormalon contributions to the QED \( D^{NS} \)-function.

4. Note also the scheme-independence of the proportional to \( \zeta_3, \zeta_5 \) and \( \zeta_7 \) high transcendence contributions to the proportional to \( N \) terms in the PT coefficients \( d_2, d_3 \) and \( d_4 \). This interesting feature is not yet understood.

### 2.3 The NS Bjorken function in the \( \overline{\text{MS}} \), MOM and OS schemes in QED

Consider now the \( O(a^4) \) approximations to the QED analytical expressions for the non-singlet coefficient function of the Bjorken polarized sum rule in three schemes we are interested in. Using the inverse \( O(a^4) \) \( SU(N_c) \) QCD \( \overline{\text{MS}} \)-scheme results given in Ref.[48], one can find the following analytical expression for the QED corrections to the Bjorken polarized sum rule function in the \( \overline{\text{MS}} \)-scheme:

\[ C_{Bjp, \overline{\text{MS}}}^{\text{NS}}(a^{\overline{\text{MS}}}) = 1 + \sum_{k=1}^{4} c_k^{\overline{\text{MS}}}(a^{\overline{\text{MS}}})^k, \]

\[ c_1^{\overline{\text{MS}}} = -\frac{3}{4} \ , \quad c_2^{\overline{\text{MS}}} = \frac{21}{32} + \frac{N}{2}, \]

\[ c_3^{\overline{\text{MS}}} = -\frac{11}{144} + \frac{\zeta_2}{2} - \frac{17089}{768} \zeta_3 + \frac{125}{6} \zeta_5 + 3 \zeta_3^2 \]

\[ + \left( -\frac{6131}{972} + \frac{203}{54} \zeta_3 + \frac{5}{3} \zeta_3^2 \right) N^2 + \left( -\frac{6131}{972} + \frac{203}{54} \zeta_3 + \frac{5}{3} \zeta_3^2 \right) N^3, \]
\[ c_3^{\text{MS}} = -\frac{3}{128} + \left( -\frac{133}{576} - \frac{5}{12} \zeta_3 \right) N - \frac{115}{216} N^2, \quad (2.17c) \]
\[ c_4^{\text{MS}} = \frac{4823}{2048} - \frac{3}{8} \zeta_3 + \left( \frac{2711}{2304} + \frac{547}{96} \zeta_3 - \frac{205}{24} \zeta_5 \right) N \]
\[ + \left( -\frac{265}{576} + \frac{29}{24} \zeta_3 \right) N^2 + \frac{605}{972} N^3. \quad (2.17d) \]

Applying now the relations (2.13-2.14) between the QED running coupling constants of the \( \overline{\text{MS}}, \text{MOM} \) and \( \text{OS} \) schemes, we get the following analytical \( \mathcal{O}(a^4) \) PT QED approximations for \( C_{\text{Bjp}}^{\text{NS}} \)-function in the MOM and OS-schemes:

\[
C_{\text{Bjp, MOM}}^{\text{NS}}(a^{\text{MOM}}) = 1 + \sum_{k=1}^{4} c_k^{\text{MOM}}(a^{\text{MOM}})^k, \quad (2.18a)
\]
\[
c_1^{\text{MOM}} = -\frac{3}{4}, \quad c_2^{\text{MOM}} = \frac{21}{32} + \frac{N}{2}, \quad (2.18b)
\]
\[
c_3^{\text{MOM}} = -\frac{3}{128} + \left( -\frac{7}{9} + \frac{\zeta_3}{3} \right) N - \frac{115}{216} N^2, \quad (2.18c)
\]
\[
c_4^{\text{MOM}} = \frac{4823}{2048} - \frac{3}{8} \zeta_3 + \left( \frac{1421}{576} + \frac{133}{24} \zeta_3 - \frac{125}{12} \zeta_5 \right) N \]
\[ + \left( \frac{2951}{3456} - \frac{\zeta_3}{6} \right) N^2 + \frac{605}{972} N^3. \quad (2.18d) \]

\[
C_{\text{Bjp, OS}}^{\text{NS}}(a^{\text{OS}}) = 1 + \sum_{k=1}^{4} c_k^{\text{OS}}(a^{\text{OS}})^k, \quad (2.19a)
\]
\[
c_1^{\text{OS}} = -\frac{3}{4}, \quad c_2^{\text{OS}} = \frac{21}{32} + \frac{N}{2}, \quad (2.19b)
\]
\[
c_3^{\text{OS}} = -\frac{3}{128} + \left( -\frac{269}{288} - \frac{5}{12} \zeta_3 \right) N - \frac{115}{216} N^2, \quad (2.19c)
\]
\[
c_4^{\text{OS}} = \frac{4823}{2048} - \frac{3}{8} \zeta_3 + \left( \frac{5315}{2304} - \frac{15}{16} \zeta_2 + \frac{4373}{768} \zeta_3 - \frac{205}{24} \zeta_5 + \frac{3}{2} \zeta_2 \log 2 \right) N \]
\[ + \left( \frac{2215}{1728} - \frac{\zeta_2}{2} + \frac{865}{768} \zeta_3 \right) N^2 + \frac{605}{972} N^3. \quad (2.19d) \]

Note, that three from four noticed in Sec.2.2 properties of the analytical structure of the PT series for the \( D^{\text{NS}}(a) \)-function are also valid in the case of the \( C_{\text{Bjp}}^{\text{NS}}(a) \)-function, considered in the gauge-invariant \( \overline{\text{MS}}, \text{MOM} \) and \( \text{OS} \) schemes. The fourth observed feature is violated. Indeed, the proportional to \( N \) contributions to the \( c_2, c_3 \) and \( c_4 \) coefficients of Eq.(1.13b) do not contain high-transcendental functions \( \zeta_3, \zeta_5 \) and \( \zeta_7 \).

2.4 The QED generalized Crewther relation in the \( \overline{\text{MS}}, \text{MOM} \) and \( \text{OS} \) schemes

Having now at hand the analytical \( \mathcal{O}(a^4) \) QED results of Sec.2.2 for the \( D^{\text{NS}}(a) \)-function in the \( \overline{\text{MS}}, \text{MOM} \) and \( \text{OS} \) schemes and the presented in Sec.2.3 similar PT expressions for the

\[ * \]These high-transcendental terms are contained in the proportional to \( C_A \) non-abelian contributions to the corresponding coefficients of the PT \( SU(N_c) \) expressions for the \( C_{\text{Bjp}}^{\text{NS}}(a) \)-function in the \( \overline{\text{MS}} \)-scheme (see Ref.[48]), which is not studied in this section.

\[ \text{– 14 –} \]
$C_{Bij}^{NS}(a)$-function we are able to verify whether the the conformal symmetry breaking term $\Delta_{cab}$ of the GCR obeys the property of the $\beta$-factorization in the class of the MS-like schemes only, or it is also fulfilled (at least in QED) in other gauge-invariant schemes, namely MOM and OS schemes. To understand this we assume, that in QED the transformations from $\overline{\text{MS}}$-scheme to MOM and OS schemes will not spoil the structure of the GCR given in Eqs.(1.10-1.12), which is valid at the $O(a_s^3)$ level in the MS-like schemes for sure and leads to the Eqs.(2.3a-2.3d), obtained in Sec.2.1. It is clear that Eq.(2.3a), which is one of the CS relations, encoded in the studied in Ref.[10] pqQED variant of the original Crewther relation, is valid for the scheme-independent coefficients $c_1$ and $d_1$ in MOM and OS-schemes.

Note, that the used in this work and defined in Eqs.(2.8), (2.11), (2.12) PT expressions for the QED $\beta$-functions in the $\overline{\text{MS}}$, MOM and OS-schemes differ from introduced in Eq.(1.9) determination of the QCD RG $\beta$-function by the overall sign. Keeping this in mind, using the given in Sec.2.2 and 2.3 analytical QED expressions for the coefficients $d_k$ and $c_k$ (with $1 \leq k \leq 4$) in the $\overline{\text{MS}}$, MOM and OS-schemes and substituting the corresponding expressions for two scheme-independent coefficients $\beta_0$ and $\beta_1$ of the RG QED $\beta$-function and scheme-dependent coefficient $\beta_2$ in the system of equations (2.3b-2.3d), we obtain the following analytical expressions for the $K_i$-terms:

$$K_1^{\overline{\text{MS}}} = K_1^{\text{MOM}} = K_1^{\text{OS}} = -\frac{21}{8} + 3\zeta_3,$$  \hspace{1cm} (2.20)

$$K_2^{\overline{\text{MS}}} = K_2^{\text{MOM}} = K_2^{\text{OS}} = \frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5 + \left(\frac{163}{24} - \frac{19}{3}\zeta_3\right)N,$$  \hspace{1cm} (2.21)

$$K_3^{\overline{\text{MS}}} = \frac{2471}{768} + \frac{61}{8}\zeta_3 - \frac{715}{8}\zeta_5 + \frac{315}{4}\zeta_7 + \left(-\frac{7729}{1152} - \frac{917}{16}\zeta_3 + \frac{125}{2}\zeta_5 + 9\zeta_3^2\right)N,$$  \hspace{1cm} (2.22)

$$+ \left(-\frac{307}{18} + \frac{203}{18}\zeta_3 + 5\zeta_5\right)N^2,$$

$$K_3^{\text{MOM}} = \frac{2471}{768} + \frac{61}{8}\zeta_3 - \frac{715}{8}\zeta_5 + \frac{315}{4}\zeta_7 + \left(-\frac{1793}{144} - \frac{343}{8}\zeta_3 + \frac{125}{2}\zeta_5\right)N,$$  \hspace{1cm} (2.23)

$$+ \left(-\frac{307}{18} + \frac{203}{18}\zeta_3 + 5\zeta_5\right)N^2,$$

$$K_3^{\text{OS}} = \frac{2471}{768} + \frac{61}{8}\zeta_3 - \frac{715}{8}\zeta_5 + \frac{315}{4}\zeta_7 + \left(-\frac{8117}{576} - \frac{391}{8}\zeta_3 + \frac{125}{2}\zeta_5 + 9\zeta_3^2\right)N,$$  \hspace{1cm} (2.24)

$$+ \left(-\frac{307}{18} + \frac{203}{18}\zeta_3 + 5\zeta_5\right)N^2.$$

These results are convincing us that the factorization property of the $\beta$-function in the conformal symmetry breaking term $\Delta_{cab}$ to the GCR in QED is valid at least at the fourth order of massless PT at least in three widely used in QED gauge-invariant schemes, namely in the $\overline{\text{MS}}$ (and therefore MS-like), MOM and OS schemes. Moreover, we discover that in QED the first two coefficients $K_1$ and $K_2$ in expansion of $\Delta_{cab}$ term are scheme-independent and find that the $K_3$-contributions, computed in the $\overline{\text{MS}}$, MOM and OS-schemes, differ from each other in the linear in the number of leptons $N$ term only:

$$K_3^{\overline{\text{MS}}} = K_3^{\text{MOM}} + \left(\frac{735}{128} - \frac{231}{16}\zeta_3 + 9\zeta_3^2\right)N = K_3^{\text{OS}} + \left(\frac{945}{128} - \frac{135}{16}\zeta_3\right)N.$$  \hspace{1cm} (2.25)
Completing this Section we arrive to the conclusion that really the factorization of the \( \beta \)-function in the GCR takes place in the gauge-invariant renormalization schemes, such as MS-like, MOM, OS-schemes in QED and MS-like schemes in QCD. However, another important problem arises, namely whether gauge invariance of the renormalization subtraction schemes is a necessary condition for factorization of the \( \beta \)-function in the GCR. This problem is studied below for the case of QCD with \( SU(N_c) \) color gauge group.

3 May the \( \beta \)-factorization property manifest itself in the generalized Crewther relation in the gauge-dependent schemes in QCD?

It is known that in QCD the MS-like subtractions schemes are distinguished by their gauge-independence. However, in theoretical and phenomenological QCD applications the gauge-dependent MOM-like schemes are used as well [87–92]. Among the number of MOM subtractions schemes one of the most applicable at present is the miniMOM (mMOM) scheme, which was originally formulated in Ref.[70]. It is widely used in different QCD-oriented studies [93–100],[49]. In this Section we will use this mMOM-scheme in the \( SU(N_c) \) theory to find out the constraints imposed by its gauge-dependence on the conditions of existence of the fundamental property of the \( \beta \)-function factorization in the GCR.

3.1 The reminding notes on the miniMOM scheme

The basic requirement, which is lying beyond the definition of mMOM-like schemes, is that the renormalization constant of the gluon-ghost-antighost vertex is fixed by the its equality to the renormalization constant of the same vertex, computed in the MS-scheme:

\[
Z_{cg}^{mMOM}(\alpha_s^{MOM}, \epsilon) = Z_{cg}^{MS}(\alpha_s^{MS}, \epsilon)
\]

(3.1)

where \(2\epsilon = 4 - D\). This is the most important requirement which allocates this scheme among all the variants of MOM-like schemes in QCD and greatly simplifies concrete calculations. Let us fix the following notations for the QCD renormalization constants:

\[
A_{B, \mu}^a = \sqrt{Z_A A_{\mu}^a}, \quad c_B = \sqrt{Z_c c^a}, \quad g_B = \mu^\epsilon Z_g g, \quad \xi_B = Z_\xi^{-1} Z_A \xi
\]

(3.2)

where \(A_{B, \mu}^a, c^a\) are the renormalized gluon and ghost fields correspondingly, \(g\) is the coupling constant, \(\xi\) is the gauge parameter, which is included in the QCD Lagrangian in the form of additive term \((\partial_{\mu} A_{B, \mu}^a)^2/(2\xi)\), \(\mu\) is a scale parameter of the dimensional regularization. As usually the index “\(B\)” denotes the bare unrenormalized quantities. One should emphasize that we work within the theory with linear covariant gauge and this fact means that we imply \(Z_\xi = 1\) in all orders of PT. The defined in Eq.(3.2) renormalization constants are related to the renormalization constant of the gluon-ghost-antighost vertex as \(Z_{cg} = Z_g Z_A^{1/2} Z_c\).

In general the MOM-like schemes are determined by the requirement that at \(Q^2 = -q^2 = \mu^2\) the residues for the gluon and ghost propagators are equal to unity. The renormalized two-point Green functions for the gluon and ghost fields can be written down in
the following form:

\[ G^{\mu\nu}_{ab}(q) = -\frac{\delta_{ab}}{q^2} \left[ \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) \frac{1}{1 + \Pi_A(q^2)} - \frac{\xi q^\mu q^\nu}{q^2} \right] , \]  

(3.3)

\[ \Delta^{ab}(q) = -\delta_{ab} \frac{1}{q^2} \frac{1}{1 + \Pi_c(q^2)} , \]  

(3.4)

where \( \Pi_A(q^2) \) and \( \Pi_c(q^2) \) are the gluon and ghosts self-energy functions.

At the subtraction point \( q^2 = -\mu^2 \) in the mMOM scheme the requirements for the residues of these propagators imply the fulfillment of the following conditions:

\[ \Pi_{\text{mMOM}}^A(q^2 = -\mu^2) = 0 , \quad \Pi_{\text{mMOM}}^c(q^2 = -\mu^2) = 0 . \]  

(3.5)

The relation between unrenormalized and renormalized two-point Green functions can be presented in the following form:

\[ 1 + \Pi_{\text{mMOM}}^A(a_s, \xi) = Z_{\text{mMOM}}^A(a_s, \xi, \epsilon) \left( 1 + \Pi_B^A(a_s^B, \xi^B(\epsilon)) \right) , \]  

(3.6)

\[ 1 + \Pi_{\text{mMOM}}^c(a_s, \xi) = Z_{\text{mMOM}}^c(a_s, \xi, \epsilon) \left( 1 + \Pi_B^c(a_s^B, \xi^B(\epsilon)) \right) , \]  

(3.7)

where the QCD coupling constant \( a_s = a_s(\mu^2) \) and the gauge parameter \( \xi = \xi(\mu^2) \) are renormalized in the mMOM-scheme.

The similar relations hold in the \( \overline{\text{MS}} \)-scheme but with a replacement of the mMOM quantities \( a_s(\mu^2), \xi(\mu^2) \) by their analogies, defined in the \( \overline{\text{MS}} \)-scheme.

Combining the given above definition of the renormalization constant of gluon-ghost-antighost vertex with (3.1) and (3.2), one gets the following relation between coupling constants of the mMOM and \( \overline{\text{MS}} \) schemes:

\[ a_s^{\text{mMOM}}(\mu^2) = \frac{Z_{\text{mMOM}}^A}{Z_{\text{\overline{MS}}}^A} \left( \frac{Z_{\text{mMOM}}^c}{Z_{\text{\overline{MS}}}^c} \right)^2 a_s^{\text{\overline{MS}}}(\mu^2) , \]  

(3.8)

where the gauge-dependence on \( \xi_{\text{\overline{MS}}} \) enters in the ratios of \( Z_A \) and \( Z_c \) in two considered schemes. One should emphasize that this relation requires knowledge of the renormalization constants of the gluon and ghost fields only, but not of any vertex structures. Using Eqs.(3.6), (3.7) and (3.8), one can obtain the following relations [70],[93],[100]:

\[ a_s^{\text{mMOM}}(\mu^2) = \frac{a_s^{\text{\overline{MS}}}(\mu^2)}{\left( 1 + \Pi_{\text{\overline{MS}}}^A(a_s^{\text{\overline{MS}}}(\mu^2), \xi_{\text{\overline{MS}}}(\mu^2)) \right)^2} , \]  

(3.9)

\[ \xi^{\text{mMOM}}(\mu^2) = \left( 1 + \Pi_{\text{\overline{MS}}}^A(a_s^{\text{\overline{MS}}}(\mu^2), \xi_{\text{\overline{MS}}}(\mu^2)) \right) \xi_{\text{\overline{MS}}}(\mu^2) . \]  

(3.10)

The three-loop results for self-energies \( \Pi_{\text{\overline{MS}}}^A \) and \( \Pi_{\text{\overline{MS}}}^c \) were calculated in Ref.[101] with explicit dependence on the gauge parameter \( \xi_{\text{\overline{MS}}} \) taken into account. The analogous four-loop results were obtained in the recent work of Ref.[100]. Using results of these computations

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in the \( \mathcal{O}(a_s^3) \) approximation and the expansions (3.9-3.10), we obtain the following relation for \( \alpha_s^{\text{MS}} \) coupling constant, expressed through \( \alpha_s^{\text{mMOM}} \) and \( \xi = \xi^{\text{mMOM}} \) (see Appendix A):

\[
a_s^{\text{MS}} = a_s^{\text{mMOM}} + b_1^{\text{mMOM}}(a_s^{\text{mMOM}})^2 + b_2^{\text{mMOM}}(a_s^{\text{mMOM}})^3 + b_3^{\text{mMOM}}(a_s^{\text{mMOM}})^4, \tag{3.11a}
\]

\[
b_1^{\text{mMOM}} = \left[ -\frac{169}{144} - \frac{1}{8} - \frac{1}{16} \frac{\xi}{\epsilon_2} \right] C_A + \frac{5}{9} T F n_f, \tag{3.11b}
\]

\[
b_2^{\text{mMOM}} = \left[ -\frac{18941}{20736} + 39 \frac{\zeta_3}{128} + \left( \frac{889}{2304} - \frac{11}{64} \frac{\zeta_3}{3} \right) \xi + \left( \frac{203}{2304} + \frac{3}{128} \frac{\zeta_3}{3} \right) \xi^2 - \frac{3}{256} \xi^3 \right] C_A^2 \right.
\]

\[
\left. + \left[ -\frac{107}{648} - \frac{\zeta_3}{2} - \frac{5}{36} - \frac{5}{72} \xi^2 \right] C_A T F n_f + \left[ \frac{55}{48} - \frac{\zeta_3}{3} \right] C_F T F n_f + \frac{25}{81} T F n_f^2, \tag{3.11c}
\]

\[
b_3^{\text{mMOM}} = \left[ -\frac{1935757}{2985984} + 7495 \frac{\zeta_3}{18432} + \frac{7805}{12288} \xi + \left( \frac{4877}{36864} - \frac{611}{1536} \frac{\zeta_3}{3} + \frac{295}{1024} \frac{\zeta_3}{5} \right) \xi \right.
\]

\[
\left. + \left[ \frac{17315}{110592} - \frac{47}{768} \frac{\zeta_3}{3} + \frac{175}{6144} \xi \right] \xi^2 \right] C_A^3 \right.
\]

\[
\left. + \left[ \frac{235}{36864} - \frac{5}{6144} \frac{\zeta_3}{3} - \frac{35}{12288} \xi \right] \xi^3 \right] C_A^2 T F n_f \right.
\]

\[
\left. + \left[ \frac{5}{2304} \xi^4 \right] C_A^2 T F n_f + \left[ \frac{1235}{15552} + \frac{13}{36} \frac{\zeta_3}{3} + \left( \frac{19}{216} - \frac{1}{9} \frac{\zeta_3}{3} \right) \xi - \frac{25}{432} \xi^2 \right] C_A T F n_f^2 \right.
\]

\[
\left. + \left[ \frac{1249}{2592} - \frac{11}{18} \frac{\zeta_3}{3} \right] C_F T F n_f^2 + \frac{25}{729} T F n_f^3 \right]. \tag{3.11d}
\]

In the necessary order of accuracy, required for our goals, the corresponding PT relation between gauge parameters, defined in the \( \text{MS} \) and mMOM-schemes reads:

\[
\xi^{\text{MS}} = \xi^{\text{mMOM}} \left( 1 + \eta_1^{\text{mMOM}} a_s^{\text{mMOM}} + \eta_2^{\text{mMOM}} (a_s^{\text{mMOM}})^2 \right), \tag{3.12a}
\]

\[
\eta_1^{\text{mMOM}} = \left[ \frac{97}{144} + \frac{1}{8} \frac{\xi}{\epsilon_2} \right] C_A - \frac{5}{9} T F n_f, \tag{3.12b}
\]

\[
\eta_2^{\text{mMOM}} = \left[ \frac{5591}{4608} - \frac{3}{16} \frac{\zeta_3}{3} + \left( \frac{121}{1536} + \frac{1}{8} \frac{\zeta_3}{3} \right) \xi + \frac{7}{256} \xi^2 + \frac{7}{256} \xi^3 \right] C_A^2 + \left[ \frac{5}{576} - \frac{\zeta_3}{2} \right] C_A T F n_f + \left[ -\frac{55}{48} + \frac{\zeta_3}{3} \right] C_F T F n_f. \tag{3.12c}
\]

One should note that expressions similar to (3.11a) and (3.12a) were presented in the works [70], [93] and [100] in terms of \( \xi^{\text{MS}} \) for the \( SU(N_c) \) color group. In our further analysis we will use the results, totally related to the mMOM-scheme, transforming everywhere the dependence on the gauge parameter \( \xi^{\text{MS}} \) to its analog in the MOM-scheme.

The mMOM-scheme \( \beta \)-function can be computed with the explicit dependence on \( \xi^{\text{mMOM}} \) using the following relation:

\[
\beta^{\text{mMOM}}(a_s^{\text{mMOM}}, \xi^{\text{mMOM}}) = \beta^{\text{MS}}(a_s^{\text{MS}}) \frac{\partial a_s^{\text{mMOM}}}{\partial a_s^{\text{MS}}} + \xi^{\text{MS}} \xi^{\text{MS}}(a_s^{\text{MS}}, \xi^{\text{MS}}) \frac{\partial a_s^{\text{mMOM}}}{\partial \xi^{\text{MS}}}, \tag{3.13}
\]
where $\gamma_\xi = d \log \xi / d \log \mu^2$ is the anomalous dimension of gauge, which due to the Slavnov–Taylor identities coincides with the expression for the gluon anomalous dimension taken with the opposite sign. The three-loop $\overline{\text{MS}}$-results for this anomalous dimension are known for an arbitrary $SU(N_c)$ color group in Ref.[51]. The corresponding expression for the RG $\beta$-function with the arbitrary gauge in mMOM-scheme, that can be obtained from Eq.(3.13), was presented in Ref.[70] in the case of $SU(N_c)$ color gauge group with explicit dependence on number of colors $N_c$ and number of active flavors $n_f$, and in Refs.[93],[100] for $SU(N_c)$ group when the corresponding Casimir operators were kept non-expanded on number of colors. As in all previously used MOM-like QCD schemes (see e.g. [89, 90]) the mMOM-scheme $\beta$-function starts to depend on the gauge from the two-loop level:

$$
\begin{align*}
\beta_0^\text{mMOM} &= \frac{11}{12} C_A - \frac{1}{3} T_F n_f, \\
\beta_1^\text{mMOM} &= \left[ \frac{17}{24} - \frac{13}{192} \xi - \frac{5}{96} \xi^2 + \frac{1}{64} \xi^3 \right] C_A^2 - \frac{1}{4} C_F T_F n_f, \\
\beta_2^\text{mMOM} &= \left[ \frac{9655}{4608} - \frac{143}{512} \xi^3 + \left( - \frac{1097}{6144} + \frac{33}{512} \xi \right) \xi + \left( - \frac{725}{6144} + \frac{13}{512} \xi \right) \xi^2 \right] C_A^2 + \frac{1}{32} C_F^2 T_F n_f + \left[ \frac{23}{96} + \frac{3}{6} \xi^3 \right] C_A T_F n_f^2,
\end{align*}
$$

\text{(3.14a)}
\text{(3.14b)}
\text{(3.14c)}

The two-loop coefficient $\beta_1^\text{mMOM}$ contains cubic term $\xi^3$ and coincides with its $\overline{\text{MS}}$ analogue at $\xi^\text{mMOM} = 0$, $-1$ only, while the three-loop coefficient $\beta_2^\text{mMOM}$, evaluated at $\xi = 0$ and $\xi = -1$, differs from its gauge-independent $\overline{\text{MS}}$ analogue, presented in Ref.[50], [51]. We remind here that at fixed renormalization point $q^2 = -\mu^2$ the values of the gauge parameter and the coupling constant are fixed and start to run when the energy scales move away from their initial normalization values $\mu^2$.

In order to study what will happen with the property of the factorization of $\beta$-function in the GCR in the gauge-dependent mMOM-scheme at the $\mathcal{O}(a_s^4)$ level we need to analyze the product of the analytical general $SU(N_c)$ expressions for the non-singlet contributions to the Adler function and Bjorken polarized sum rule in this gauge-dependent scheme. These expressions will be obtained below.

### 3.2 The NS Adler function in the mMOM-scheme in the $\mathcal{O}(a_s^4)$ approximation

Using the $\mathcal{O}(a_s^4)$ result for the $D^{NS}(a_s)$-function in the $\overline{\text{MS}}$-scheme, presented in Ref.[48], and the results of Eqs.(3.11a-3.12c), we obtain the following analytical expression for this Euclidean characteristic of the $e^+e^- \rightarrow \gamma \rightarrow \text{hadrons}$ process in the mMOM-scheme:
\(D_{NS}^{\text{mMOM}} = 1 + \sum_{k=1}^{4} d_{m\text{MOM}}^{\text{mMOM}} (a_{n}^{\text{mMOM}})^{k},\)  

(3.15a)

\[d_{1}^{\text{mMOM}} = \frac{3}{4} C_{F},\]  

(3.15b)

\[d_{2}^{\text{mMOM}} = -\frac{3}{32} C_{F}^{2} + \left[\frac{569}{192} - \frac{11}{4} \xi_{3} - \frac{3}{32} \xi - \frac{3}{64} \xi^{2}\right] C_{F} C_{A} + \left[-\frac{23}{24} + \xi_{3}\right] C_{F} T_{Fn_{f}}\]  

(3.15c)

\[d_{3}^{\text{mMOM}} = -\frac{69}{128} C_{F}^{3} + \left[-\frac{1355}{768} - \frac{143}{16} \xi_{3} + \frac{55}{4} \xi_{5} + \frac{3}{128} \xi + \frac{3}{256} \xi^{2}\right] C_{F}^{2} C_{A}\]  

(3.15d)

\[d_{4}^{\text{mMOM}} = \frac{\eta d_{F}^{\text{bPed}} d_{A}^{\text{bPed}}}{N_{c}} \left[\frac{3}{16} - \frac{\xi_{3}}{4} - \frac{5}{4} \xi_{5}\right] + n_{f} \frac{d_{F}^{\text{bPed}} \eta d_{F}^{\text{bPed}}}{N_{c}} \left[-\frac{13}{16} - \xi_{3} + \frac{3}{2} \xi_{5}\right]\]  

(3.15e)
Note that in $SU(N_c)$ theory we have:

\[
C_F = \frac{N_c^2 - 1}{2N_c}, \quad C_A = N_c, \quad T_F = \frac{1}{2},
\]

\[
\frac{d_F^{abcd}d_A^{abcd}}{N_c} = \frac{(N_c^2 - 1)(N_c^2 + 6)}{48}, \quad \frac{d_F^{abcd}d_F^{abcd}}{N_c} = \frac{(N_c^2 - 1)(N_c^4 - 6N_c^2 + 18)}{96N_c^3},
\]

while in the $SU_c(3)$ case $C_F = 4/3$, $C_A = 3$, $d_F^{abcd}d_A^{abcd}/N_c = 5/2$, $d_F^{abcd}d_F^{abcd}/N_c = 5/36$.

In the particular case of the Landau gauge $\xi = 0$ with $n_f=3$ the numerical expressions for the coefficients (3.15b-3.15e) are consistent with the ones, obtained in Ref.[70]. The detailed study of the numerical behavior of these PT series for other $n_f$ and other values of gauges will be presented below. The scheme-dependence of the PT series for the $R$-ratio, defined in Eq.(1.3) in Minkowski region and related to Euclidean Adler function, was analyzed in the mMOM-scheme at $\xi = 0$ in Refs.[94] and [96].

3.3 The NS Bjorken function in the mMOM-scheme in the $O(a_s^4)$ approximation

Let us now turn to the calculation of the $C_{NS Bjp}^{N_S}(a_s)$-function in the $O(a_s^4)$-approximation in the mMOM-scheme in the $SU(N_c)$ version of QCD. Its analytical expression can be obtained from the summarized in Ref.[48] $\overline{\text{MS}}$-scheme results and the presented in Sec.3.1 transformations of the QCD coupling constant $a_s^{\overline{\text{MS}}}$ from the $\overline{\text{MS}}$-scheme to the gauge-dependent mMOM-scheme with the gauge parameter $\xi$, defined in this renormalization scheme.

The corresponding expression for the PT coefficient function of the polarized Bjorken sum rule, calculated in the mMOM-scheme, reads

\[
C_{Bjp, \text{MOM}}^{NS} = 1 + \sum_{k=1}^{4} c_k^{\text{MOM}}(a_s^{\text{MOM}})^k, \quad (3.16a)
\]

Its coefficients $c_k$ have the following form

\[
c_1^{\text{MOM}} = -\frac{3}{4} C_F, \quad (3.16b)
\]

\[
c_2^{\text{MOM}} = \frac{21}{32} C_F^2 + \left(-\frac{107}{192} + \frac{3}{32} \xi + \frac{3}{64} \xi^2 \right) C_F C_A + \frac{1}{12} C_F T_{FNF_f}, \quad (3.16c)
\]

\[
c_3^{\text{MOM}} = -\frac{3}{128} C_F^3 + \left[\frac{1415}{2304} - \frac{11}{12} \zeta_3 - \frac{21}{128} \xi - \frac{21}{256} \xi^2 \right] C_F^2 C_A
\]

\[
+ \left[\frac{13}{36} + \frac{\zeta_3}{3} \right] C_F^2 T_{FNF_f} + \left[\frac{13}{9} + \frac{3}{8} \zeta_3 - \frac{5}{6} \zeta_5 - \frac{1}{48} \xi - \frac{1}{96} \xi^2 \right] C_F C_A T_{FNF_f}
\]

\[
+ \left[\frac{-20585}{9216} - \frac{117}{512} \zeta_3 + \frac{55}{24} \zeta_5 + \left(\frac{215}{3072} + \frac{33}{256} \zeta_3 \right) \xi + \left(\frac{349}{3072} - \frac{9}{512} \zeta_3 \right) \xi^2
\]

\[
+ \frac{9}{1024} \zeta_3 C_F C_A - \frac{5}{24} C_F T_{FNF_f}^2 \right], \quad (3.16d)
\]
\[C^{\text{mMOM}}_4 = \frac{d_{F A}^{abcd} d_{A (a_{	ext{MS}})}^{abcd}}{N_c} \left[ -\frac{3}{16} + \frac{\zeta_3}{4} + \frac{5}{4} \zeta_5 \right] + n_f d_{F A}^{abcd} d_{A (a_{\text{MS}})}^{abcd} \left[ \frac{13}{16} + \zeta_3 - \frac{5}{2} \zeta_5 \right] \]  
(3.16e)

\[
+ \left[ -\frac{4823}{2048} - \frac{3}{8} \zeta_3 \right] C_F^A + \left[ -\frac{13307}{18432} - \frac{971}{96} \zeta_3 + \frac{1045}{48} \zeta_5 + \frac{9}{2048} \frac{9}{\xi^2} \right] C_F^C A
\]

\[
+ \left[ \frac{2543485}{221184} + \frac{90169}{144} \zeta_3 - \frac{1375}{144} \zeta_5 - \frac{385}{16} \zeta_7 + \left( -\frac{1339}{12288} + \frac{121}{1024} \xi \right) \right] C^2_F C_A
\]

\[
+ \left( \frac{1117}{6144} + \frac{415}{2048} \zeta_3 \right) \xi^2 - \frac{21}{4096} \xi^3 + \frac{21}{8192} \xi^4 \right] C^2_F C_A^2 + \left[ \frac{-3927799}{442368} + \frac{49763}{73728} \zeta_3 \right]
\]

\[
+ \left[ \frac{345555}{147456} \zeta_3 + \frac{385}{64} \zeta_5 + \left( \frac{107569}{147456} + \frac{1623}{2048} \zeta_3 - \frac{4405}{4096} \zeta_5 \right) \xi \right]
\]

\[
+ \left[ \frac{28305}{49152} - \frac{25}{512} \zeta_3 - \frac{3695}{8192} \zeta_5 \right] \xi^2 + \left[ \frac{151}{3072} - \frac{59}{2048} \zeta_3 - \frac{5}{4096} \zeta_5 \right] \xi^3 \right]
\]

\[
+ \left( \frac{41}{49152} + \frac{5}{8192} \zeta_3 + \frac{35}{16384} \zeta_5 \right) \xi^4 \right] C_F C_A^3 + \left[ \frac{317}{144} + \frac{109}{4} \zeta_3 - \frac{95}{12} \zeta_5 \right] C_F T_{\xi n_f}^2
\]

\[
+ \left[ \frac{6229}{864} - \frac{1739}{288} \zeta_3 + \frac{205}{72} \zeta_5 + \frac{35}{4} \zeta_7 + \left( \frac{13}{96} - \frac{\zeta_3}{8} \right) \xi + \left( \frac{13}{192} - \frac{\zeta_3}{16} \right) \xi^2 \right] C_F^2 C_A T_{\xi n_f}^2
\]

\[
+ \left[ \frac{12265}{1728} - \frac{1237}{512} \zeta_3 + \frac{15}{16} \zeta_5 + \frac{35}{16} \zeta_7 + \frac{11}{12} \zeta_3 \right] \xi^2 + \left( \frac{8257}{18432} - \frac{49}{96} \zeta_3 + \frac{5}{16} \zeta_5 \right) \xi \right]
\]

\[
+ \left( \frac{869}{3072} - \frac{33}{512} \zeta_3 + \frac{5}{32} \zeta_5 \right) \xi^2 + \frac{1}{1536} \xi^3 + \frac{1}{3072} \xi^4 \right] C_F C_A T_{\xi n_f}^3
\]

\[
+ \left[ \frac{1283}{864} + \frac{85}{36} \zeta_3 - \frac{35}{36} \zeta_5 - \frac{\zeta_3}{6} \right] \xi + \left( \frac{11}{192} + \frac{\zeta_3}{12} \right) \xi + \frac{5}{128} \xi^2 \right] C_F C_A T_{\xi n_f}^3 n_f^2
\]

\[
+ \left[ \frac{1891}{3456} - \frac{\zeta_3}{36} \right] C_F^2 T_{\xi n_f}^2 n_f^2 + \frac{5}{72} C_F T_{\xi n_f}^3 n_f^3 \right]
\]

We will use the analytical results, presented in Sec.3.1, 3.2 and 3.3, for two purposes. The first problem we are interested in is whether the unexplained yet from the first principles of gauge quantum field theory MS-scheme structure of the GCR, discovered in Ref.[33], will be essentially modified in the mMOM-scheme. The second problem is more landed and is related to the comparison of the asymptotic behavior of the \(O(a_s^4)\)-approximations of the PT series for \(D^{NS}(a_s)\) and \(C^{NS}_{BIP}(a_s)\)-functions in the MS-scheme and the mMOM-scheme for different numbers of quark flavors \(n_f\) and in several fixed gauges.

### 3.4 About the gauge-dependence of the generalized Crewther relation in the mMOM-scheme

Consider now the question whether (or when) the RG gauge-dependent \(\beta^{mMOM}\)-function is factorized in the GCR. This problem will be analyzed using the analytical results of Eqs.(3.14a–3.16e). Taking first into account the relation (2.3b) we conclude that in the \(O(a_s^4)\) approximation the conformal symmetry breaking term \(\Delta_{mab}\) can be factorized at any value of the gauge parameter \(\xi\) thanks to the the fulfillment of Eq.(2.3a), which follows from the property of scheme-independence of this relation. This implies that the coefficient
$K_1$ does not depend on $\xi$ and coincides with its $\overline{\text{MS}}$ expression, namely

$$K_1^{\text{mMOM}} = \left( -\frac{21}{8} + 3\zeta_3 \right) C_F . \quad (3.17)$$

Using now Eq.(2.3c) we find that the $O(a_s^3)$ mMOM-scheme coefficient in $\Delta_{csb}$ cannot be represented in the form (1.11) for any $\xi$. Indeed, equation (2.3c) imposes certain restrictions on the factorization conditions in mMOM-scheme. Taking into account the concrete analytical results for the coefficients $c_k^{\text{mMOM}}$ from Sec.3.2 and for the coefficients $c_k^{\text{mMOM}}$ from Sec.3.3 with $1 \leq k \leq 3$ we find that the two-loop $\beta^{\text{mMOM}}$-function is factorized in the GCR only for certain values of gauge parameter, which are fixed by the solution of the following equation:

$$\xi^3 + 4\xi^2 + 3\xi = 0 , \quad (3.18)$$

namely for three specific values $\xi = 0, -1, -3$. The extra more detailed theoretical clarification of these foundations will be presented below in the separate Section.

It should be stressed that the Landau gauge $\xi = 0$ is often used in multiloop calculations (see e.g. [70],[93, 94],[96],[100]). Its most vivid feature is the validity of the property of the non-renormalization of the gluon-ghost-antighost vertex [102]. That is why the renormalization constant $Z_{cg}$ in the mMOM-scheme is chosen the same as in the $\overline{\text{MS}}$-scheme. Note also that in the Landau gauge the longitudinal part of the renormalized gluon propagator vanishes and therefore its PT approximation has a transverse structure, namely $q_\mu G_{ab}^{\mu\nu}(q) = 0$.

Other two gauges $\xi = -1$ and $\xi = -3$ are less studied. However, some attractive features of using in the PT QCD expansions of measurable physical quantities the class of MOM-schemes with anti-Feynman gauge $\xi = -1$ (more precise of the class of gauges with $|\xi| \leq 1$) was noticed in the work of [88], where it was shown that for these values of gauge the one-loop QCD corrections to MOM-scheme effective charges, defined as the combination of Green functions, are rather small at $n_f = 4$. The anti-Yennie gauge $\xi = -3$ was first used in QCD by Stefanis [103, 104] to clarify the special features of renormalizations of gauge-invariant definition of QCD quark correlator, formulated with the help of Wilson line. This gauge was independently applied later on by Mikhailov in Refs.[105, 106], where it was demonstrated that when this gauge is chosen the one-loop correction to the renormalization constant of the gluon field is proportional to the first scheme-independent coefficient of the QCD $\beta$-function. In what follows we call this anti-Yennie gauge $\xi = -3$ as the Stefanis–Mikhailov gauge.

Taking now into account Eq.(2.3c) and the analytical mMOM results of (3.14a-3.16e), specified to the cases of the Landau, anti-Feynman and Stefanis–Mikhailov gauges, we obtain the explicit analytical expressions of the $K_2^{\text{mMOM}}$ coefficient, included in Eq.(1.12), for these three gauges, which do not violate the property of the factorization of the two-loop mMOM $\beta$-function in the CSB term of Eq.(1.11) in the $O(a_s^3)$ approximation of the GCR. The concrete results and the analysis of their structure will be presented in Section 4.

The dependence on $\xi$ of the $O(a_s^4)$ coefficient of the GCR is more complicated. We found that for the case of the Landau gauge the fundamental property of factorization of
the three-loop mMOM-scheme $\beta$-function of $SU(N_c)$ theory in the conformal symmetry breaking term $\Delta_{\text{csb}}$ of the mMOM-variant of the GCR takes place. The corresponding analytical expression of the $K_3^{\text{mMOM}}$-term at $\xi = 0$ will be also presented in Sec.4.

In the case of $\xi = -1$ and $\xi = -3$ only partial factorization of three-loop mMOM $\beta$-function in the analytical expression for the CSB term of the GCR is observed. Indeed, the analytical expression of the $K_3$-term contains six color structures, first revealed in the case of application of the gauge-independent $\overline{\text{MS}}$-scheme in Ref.[48]. In the mMOM-scheme when the Landau gauge is chosen, $K_3$ also contains these six color structures and we conclude that in this case total factorization of the three-loop QCD $\beta$-function also persists in the GCR. However, in the case of anti-Feynman and Stefanis–Mikhailov gauges only five from these six color structures may be found from Eq.(2.3d) and the one is not determined, namely the coefficient, proportional to $C_F C_A T_{FN_f}$-contribution (see Appendix B) Therefore for these two gauges the three-loop approximation of the mMOM-scheme $\beta$-function obeys the property of partial factorization only.

Thus from the study of the structure of the GCR in the mMOM-scheme in the Landau gauge we come to conclusion that the property of gauge invariance of the renormalization schemes is the sufficient but not a necessary property for the factorization of the $SU(N_c)$ $\beta$-function in the $\mathcal{O}(a_s^3)$-expression of the generalized Crewther relation in QCD, presented in the concrete renormalization schemes, which are fixed by kinematics conditions of the subtractions of renormalizations of the Green functions, related to the QCD vertexes.

3.5 Asymptotic behavior of the NS Adler and Bjorken functions at the fourth-loop level: the mMOM vs $\overline{\text{MS}}$-scheme

Let us study the asymptotic behavior of the $\mathcal{O}(a_s^4)$-approximation for two basic functions $D_{\text{NS}}^{\mathcal{O}}(a_s)$ and $C_{\text{NS Bj}}^{\mathcal{O}}(a_s)$, which enter the GCR and both were extracted from the concrete experimental data (see [107, 108]). From phenomenological point of view one may realize that these studies are of interest for the values $n_f = 3, 4, 5$, while the results, summarized above may attract definite interest to the behavior of the corresponding PT series, obtained within mMOM scheme with three specific values $\xi = 0, -1, -3$ of the gauge parameter.

These results are compared with the $\overline{\text{MS}}$-results in the $\mathcal{O}(a_s^4)$-approximation and are presented in the Table 1. Note, that the mMOM-scheme numerical results for $D_{\text{NS}}^{\mathcal{O}}(a_s)$ agree with the ones, obtained in Ref.[70] for the case of the Landau gauge. It was also recently checked that the given in the Table 1 $\mathcal{O}(a_s^4)$ mMOM expressions for $C_{\text{Bj}}^{\mathcal{O}}$ at $\xi = 0$, presented in Ref.[109], are consistent with the results of Ref.[110], based on the QCD calculation within the variant of Analytical Perturbation Theory (APT), developed previously in Ref.[111]. The existing at present point of view on the structure of the asymptotic PT QCD series for the physical quantities, defined in the Euclidean region, indicates that in the $\overline{\text{MS}}$-scheme the coefficients of these PT expansions obey the pattern of infrared renormalon (IRR) generated sign-constant factorial growth up to the level, when they are starting to compete with sign-alternating factorial contributions, generated in the related PT QCD series by the corresponding ultraviolet renormalon (UVR) effects (for

\footnotesize{\textsuperscript{9}We are grateful to G. Cvetič for the confirmation of this agreement.}}
| $\xi$ | $n_f$ | The flavor NS Adler function $D^{NS}$ in mMOM and $\overline{\text{MS}}$ schemes | The flavor NS Bjorken function $C^{NS}_{BjP}$ in mMOM and $\overline{\text{MS}}$ schemes |
|-------|-------|--------------------------------|--------------------------------------------------|
| 0     | 3     | $1 + a_s - 1.048a_s^2 - 4.8241a_s^3 + 3.12575a_s^4$ | $1 - a_s - 0.896a_s^2 + 1.4262a_s^3 - 22.96225a_s^4$ |
| 1     | 4     | $1 + a_s - 0.885a_s^2 - 5.8133a_s^3 + 11.71854a_s^4$ | $1 - a_s - 0.840a_s^2 + 3.0375a_s^3 - 12.34185a_s^4$ |
| 2     | 5     | $1 + a_s - 0.723a_s^2 - 6.6039a_s^3 + 18.25793a_s^4$ | $1 - a_s - 0.785a_s^2 + 4.5099a_s^3 - 3.61660a_s^4$ |
| -1    | 3     | $1 + a_s - 0.860a_s^2 - 4.3579a_s^3 + 6.38863a_s^4$ | $1 - a_s - 1.083a_s^2 + 0.2312a_s^3 - 31.54404a_s^4$ |
| -2    | 4     | $1 + a_s - 0.698a_s^2 - 5.2862a_s^3 + 13.28190a_s^4$ | $1 - a_s - 1.028a_s^2 + 1.8633a_s^3 - 18.49192a_s^4$ |
| -3    | 5     | $1 + a_s - 0.535a_s^2 - 6.0159a_s^3 + 18.39217a_s^4$ | $1 - a_s - 0.972a_s^2 + 3.3566a_s^3 - 7.57175a_s^4$ |
| -4    | 3     | $1 + a_s - 1.610a_s^2 - 0.1797a_s^3 + 15.90258a_s^4$ | $1 - a_s - 0.333a_s^2 - 1.0317a_s^3 - 44.09174a_s^4$ |
| -5    | 4     | $1 + a_s - 1.448a_s^2 - 1.3517a_s^3 + 23.10284a_s^4$ | $1 - a_s - 0.278a_s^2 + 0.5170a_s^3 - 31.54819a_s^4$ |
| -6    | 5     | $1 + a_s - 1.285a_s^2 - 2.3251a_s^3 + 28.39054a_s^4$ | $1 - a_s - 0.222a_s^2 + 1.9269a_s^3 - 21.14145a_s^4$ |
| -7    | 3     | $1 + a_s + 1.640a_s^2 + 6.3710a_s^3 + 49.07570a_s^4$ | $1 - a_s - 3.583a_s^2 - 20.2153a_s^3 - 175.74950a_s^4$ |
| -8    | 4     | $1 + a_s + 1.525a_s^2 + 2.7586a_s^3 + 27.38880a_s^4$ | $1 - a_s - 3.250a_s^2 - 13.8503a_s^3 - 102.40204a_s^4$ |
| -9    | 5     | $1 + a_s + 1.409a_s^2 - 0.6814a_s^3 + 9.21018a_s^4$ | $1 - a_s - 2.917a_s^2 - 7.8402a_s^3 - 41.95977a_s^4$ |

TABLE 1. The comparison of the $O(a_s^3)$ PT expansions for the $D^{NS}$ and $C^{NS}_{BjP}$ functions, evaluated in QCD with $n_f = 3, 4, 5$ active flavors in mMOM-scheme with $\xi = 0, -1, -3$ and $\overline{\text{MS}}$-scheme ($\bar{a}_s$ corresponds to the calculation in the $\overline{\text{MS}}$-scheme).

The detailed discussions see reviews [112, 113]). The concrete $\overline{\text{MS}}$-scheme calculations of the Borel image for the $D^{NS}(a_s)$-function [114] and the one for the $C^{NS}_{BjP}(a_s)$-function [33] demonstrate the interplay of the IRR and UVR effects are manifesting itself in the case of $D^{NS}(a_s)$ at the level above $O(a_s^4)$ contribution and for the $C^{NS}_{BjP}(a_s)$ even at more high level of the related PT expansions [112].

The comparison of numerical $\overline{\text{MS}}$-results, presented in Table 1, give extra argument in flavor of this renormalon-motivated guess. Indeed, in the case of the Bjorken polarized sum rule the the sign-constant pattern and the related asymptotic growth of the coefficients of $O(a_s^4)$-approximation for all considered values of $n_f = 3, 4, 5$ are more pronounced, than in the case of $D^{NS}(a_s)$-function. Moreover, for $n_f = 5$ in the latter case these features are even violated at the $O(a_s^4)$ approximation. The inconsistency of signs of PT series in the mMOM-scheme is observed for both physical quantities (apart of the case of mMOM-PT series for $C^{NS}_{BjP}(a_s)$ with $n_f = 3$ and $\xi = -3$). However, without any estimates of the numerical values of the unknown at present $O(a_s^5)$ corrections we can not make definite conclusion whether in the mMOM-scheme the considered asymptotic PT QCD series have sign-alternating or sign-nonregular structure. Next, on the contrary to the $\overline{\text{MS}}$ PT approximations for $D^{NS}(a_s)$-function of their mMOM-scheme analogs are growing when $n_f$ is increasing from $n_f = 3$ to $n_f = 5$. The situation for the Bjorken polarized sum rule function is somewhat
different: here with growth of $n_f$ the values of the $O(a_s^4)$ coefficients are decreasing modulo in both schemes. One more interesting feature of the mMOM-scheme PT series for $C_{Bjp}^{NS}$ catches the eyes: the absolute values of the $a_s^2$, $a_s^3$ and $a_s^4$ coefficients are much smaller than the ones, obtained in $\overline{MS}$-scheme. This difference may be essential in the process of study of the scheme-dependence uncertainties of the possible new more careful analysis of the experimental data for the Bjorken polarized sum rule (see Ref.[110]), which should include virtual heavy-quark massive effects, calculated at the leading order of PT in Ref.[115] and next-to-leading order in Ref.[116] and in the most detailed work of Ref.[117].

4 The generalized Crewther relation in the MOM-like schemes in QCD

4.1 The $O(a_s^4)$ $\beta$-factorization of the generalized Crewther relation in the mMOM-scheme

Let us return to analysis of the analytical structure of the $O(a_s^4)$ approximation of the GCR in $SU(N_c)$ QCD in the case of application of the gauge-dependent mMOM-scheme, described in Sec.3.4. We remind that in the class of gauge-invariant MS-like schemes at the $O(a_s^4)$ level of PT QCD the GCR relation is defined as

$$D_{NS}(a_s)C_{Bjp}^{NS}(a_s) = 1 + \left(\frac{\beta(a_s)}{a_s}\right)K(a_s).$$

As already mentioned above in Sec.3.4, the direct computations, performed by us in the mMOM-scheme with using Eqs.(2.3c-2.3d), lead to the conclusion that in the GCR the factorization of the RG $\beta$-function in the conformal symmetry breaking term is possible only for the certain specific values of the gauge parameter. To study this property in more detail we write down the following equality, which allows us to find out what values of the gauge parameter respect the $\beta$-function factorization property at definite orders of PT:

$$\frac{\beta_{MS}(a_s)C_{Bjp}^{MS}(a_s)}{\beta_{mMOM}(a_s)C_{Bjp}^{mMOM}(a_s)} = 1 + \left(\frac{\beta(a_s)}{a_s}\right)K(a_s).$$ (4.1)

This equation permits us to obtain the relations between coefficients $K_i$, determined in the mMOM-scheme, and their analogs, evaluated in the $\overline{MS}$-scheme. Naturally, the equality (4.1) is valid only under the assumption of the $\beta$-factorization of the CSB term $\Delta_{csb}$ in Eq.(1.11) and is not satisfied for any values of $\xi$. Indeed, the Eq.(4.1) allow us to find all these values of $\xi$, for which the factorization is possible.

Now we find out the criteria for $\beta$-factorization in gauge-dependent mMOM scheme. Using the expansion $a_s^{MS}$ through $a_s^{mMOM}$ in the arbitrary gauge (3.11a), Eq.(3.14a) and the relation (4.1) we reproduce the result, presented above in Eq.(3.17) for any value of $\xi$:

$$K_1^{mMOM} = K_1^{MS}.\quad (4.2)$$

For the coefficient $K_2^{mMOM}$ the relation with $K_2^{MS}$ can be found from Eqs.(3.11a), (3.14a), (3.14b), (4.1) and looks like as:

$$K_2^{mMOM} = K_2^{MS} + \left(\frac{\beta^{MS}_{21} - \beta^{mMOM}_{21}}{\beta_0} + 2b_1^{mMOM}\right)K_1^{MS}.\quad (4.3)$$
In this equation the coefficient $K_2^{\text{mMOM}}$ does not contain terms proportional to $1/\beta_0$, if the difference $\beta_{1\text{MS}}^2 - \beta_1^{\text{mMOM}}$ is proportional to the leading coefficient $\beta_0$ of the RG $\beta$-function, namely $\beta_{1\text{MS}}^2 - \beta_1^{\text{mMOM}} = \theta \beta_0 C_A$, where $\theta$ is some real number. Using the explicit expressions for the one and two-loop coefficients of the RG $\beta$-function in the $\overline{\text{MS}}$ and mMOM schemes, we get the following equations

$$
\beta_{1\text{MS}}^2 - \beta_1^{\text{mMOM}} = \left( \frac{13}{192} \xi + \frac{5}{96} \xi^2 - \frac{1}{64} \xi^3 \right) C_A^2 - \frac{1}{24} (\xi + \xi^2) C_A T_{Ff} = \frac{11}{12} \theta C_A^2 - \frac{1}{3} \theta C_A T_{Ff}.
$$

Hence, we obtain the following system of equations:

$$
\begin{align*}
\frac{11}{12} \theta &= \frac{13}{192} \xi + \frac{5}{96} \xi^2 - \frac{1}{64} \xi^3, \\
\frac{1}{3} \theta &= \frac{1}{24} \xi + \frac{1}{24} \xi^2,
\end{align*}
$$

This system leads to the presented above single equation (3.18) and has the following solutions $(\xi, \theta) = (0, 0), (-1, 0), (-3, 3/4)$. Thus, we demonstrate that the $\beta$-factorization property for $\mathcal{O}(a_s^4)$ approximation of the CSB contribution $\Delta_{\text{csb}}$ to the GCR in the mMOM-scheme is possible only for three values of the gauge parameter $\xi$, namely for $\xi = 0, -1, -3$.\footnote{In the case of $\xi = 0, -1$ this conclusion also follows from the fact that for these two values of the gauge parameter the two-loop coefficient $\beta_1^{\text{mMOM}}$ coincides with $\beta_1^{\text{MS}}$ identically and therefore the difference $\beta_{1\text{MS}}^2 - \beta_1^{\text{mMOM}}$ in Eq.(4.3) is nullified.}

The corresponding $SU(N_c)$ analytical expressions of the $K_2^{\text{mMOM}}$ coefficients, defined in Eq.(1.12) in the mMOM-scheme, have the following form:

$$
\begin{align*}
K_{2, \xi=0}^{\text{mMOM}} &= \left[ \frac{397}{96} + \frac{17}{2} \zeta_3 - 15 \xi \right] C_F^2 + \left[ - \frac{2591}{192} + \frac{91}{8} \xi \right] C_F C_A + \left[ \frac{31}{8} - 3 \xi \right] C_F T_{Ff} \quad (4.4) \\
K_{2, \xi=-1}^{\text{mMOM}} &= \left[ \frac{397}{96} + \frac{17}{2} \zeta_3 - 15 \xi \right] C_F^2 + \left[ - \frac{1327}{96} + \frac{47}{4} \xi \right] C_F C_A + \left[ \frac{31}{8} - 3 \xi \right] C_F T_{Ff} \quad (4.5) \\
K_{2, \xi=-3}^{\text{mMOM}} &= \left[ \frac{397}{96} + \frac{17}{2} \zeta_3 - 15 \xi \right] C_F^2 + \left[ - \frac{695}{48} + \frac{25}{2} \xi \right] C_F C_A + \left[ \frac{31}{8} - 3 \xi \right] C_F T_{Ff} \quad (4.6)
\end{align*}
$$

The coincidence of the analytical terms, proportional to $C_F^2$-factor, in Eqs.(4.4-4.6) in the QED-limit with the results, obtained in three gauge-invariant renormalizations schemes in QED in Eq.(2.21) is quite understandable. Other typical features of Eqs.(4.4-4.6) are the gauge-dependence of the $C_F C_A$-term to the $K_2^{\text{mMOM}}$-expression and the gauge-independence of the $C_F T_{Ff}$-structure. However, as one can see these $C_F T_{Ff}$-contributions differ from its QED analogs, included in Eq.(2.21). The reason for this difference becomes clear upon careful consideration of the Eq.(4.3). Indeed, due to the scheme-independence of the first two coefficients of the RG $\beta$-function in the QED-limit the contribution, proportional to $\beta_{1\text{MS}}^2 - \beta_1^X$, is nullified (here under the $X$ we mean either MOM or OS renormalization schemes, defined in QED). Unlike the QCD case the second term, proportional to $b n_{f}^{\text{mMOM}}$ (and as a consequence to $n_f$), can always be chosen to be zero in QED due to the corresponding normalizations. That is why these $C_F T_{Ff}$-contributions are different.
Comparing it with Eq.(4.3) we notice that this relation contains three new additional fractions \( \beta_2^{\MS} - \beta_2^{\mMOM} \)/\( \beta_0 \), \( (3 \beta_1^{\MS} - 2 \beta_1^{\mMOM}) b_1^{\mMOM} / \beta_0 \) and \( (\beta_1^{\mMOM} (\beta_1^{\mMOM} - \beta_1^{\MS})) / \beta_0^3 \), which did not enter into Eq.(4.3). Thus, it is clear that the question of factorization of the three-loop mMOM \( \beta \)-function in the \( O(a_s^4) \)-approximation of the GCR reduces to investigating the divisibility of these fractions. Indeed, if all these fractions are contractible in sum or separately, then we can confidently state the existence of the \( \beta \)-factorization property in the GCR at \( O(a_s^4) \) level. We study this problem for \( \xi = 0, -1, -3 \) only, since at these values of gauge parameter the \( \beta \)-factorization property holds at \( O(a_s^4) \) level at least. The extra appearing fraction \( (\beta_1^{\mMOM} (\beta_1^{\mMOM} - \beta_1^{\MS})) / \beta_0^3 \) is equal to zero in case of \( \xi = 0 \) and \( \xi = -1 \), and differs from zero for Stefanis–Mikhailov gauge \( \xi = -3 \). Moreover, at \( \xi = -3 \) this fraction is irreducible. The remaining two fractions can not be divided individually at all considered values of \( \xi \). However, in Landau gauge the sum of these two fractions are divided by \( \beta_0 \). Nothing like this is observed for \( \xi = -1 \) and \( \xi = -3 \). Therefore we find that the \( \beta \)-factorization property holds at \( O(a_s^4) \) level in the mMOM-scheme in Landau gauge \( \xi = 0 \) only and it is not performed for the anti-Feynman gauge \( \xi = -1 \) and Stefanis–Mikhailov gauge \( \xi = -3 \).

Summarizing the foregoing, we get the following analytical expression for \( K_3 \)-term, computed in the Landau gauge:

\[
K_3^{\mMOM} = K_3^{\MS} + \left( \frac{\beta_1^{\MS} - \beta_1^{\mMOM}}{\beta_0} + 3 b_1^{\mMOM} \right) K_2^{\MS} + \left( 2 b_2^{\mMOM} + (b_1^{\mMOM})^2 \right) K_1^{\MS} + \left( \frac{\beta_2^{\MS} - \beta_2^{\mMOM}}{\beta_0} + (3 \beta_1^{\MS} - 2 \beta_1^{\mMOM}) b_1^{\mMOM} / \beta_0 \right) + \frac{\beta_1^{\mMOM} (\beta_1^{\mMOM} - \beta_1^{\MS})}{\beta_0^3} \right) K_1^{\MS} \tag{4.7}
\]

Comparing it with Eq.(4.3) we notice that this relation contains three new additional fractions \( \beta_2^{\MS} - \beta_2^{\mMOM} \)/\( \beta_0 \), \( (3 \beta_1^{\MS} - 2 \beta_1^{\mMOM}) b_1^{\mMOM} / \beta_0 \) and \( (\beta_1^{\mMOM} (\beta_1^{\mMOM} - \beta_1^{\MS})) / \beta_0^3 \), which did not enter into Eq.(4.3). Thus, it is clear that the question of factorization of the three-loop mMOM \( \beta \)-function in the \( O(a_s^4) \)-approximation of the GCR reduces to investigating the divisibility of these fractions. Indeed, if all these fractions are contractible in sum or separately, then we can confidently state the existence of the \( \beta \)-factorization property in the GCR at \( O(a_s^4) \) level. We study this problem for \( \xi = 0, -1, -3 \) only, since at these values of gauge parameter the \( \beta \)-factorization property holds at \( O(a_s^4) \) level at least. The extra appearing fraction \( (\beta_1^{\mMOM} (\beta_1^{\mMOM} - \beta_1^{\MS})) / \beta_0^3 \) is equal to zero in case of \( \xi = 0 \) and \( \xi = -1 \), and differs from zero for Stefanis–Mikhailov gauge \( \xi = -3 \). Moreover, at \( \xi = -3 \) this fraction is irreducible. The remaining two fractions can not be divided individually at all considered values of \( \xi \). However, in Landau gauge the sum of these two fractions are divided by \( \beta_0 \). Nothing like this is observed for \( \xi = -1 \) and \( \xi = -3 \). Therefore we find that the \( \beta \)-factorization property holds at \( O(a_s^4) \) level in the mMOM-scheme in Landau gauge \( \xi = 0 \) only and it is not performed for the anti-Feynman gauge \( \xi = -1 \) and Stefanis–Mikhailov gauge \( \xi = -3 \).

Summarizing the foregoing, we get the following analytical expression for \( K_3 \)-term, computed in the Landau gauge:

\[
K_3^{\mMOM} = \left( \frac{2471}{768} + \frac{61}{8} \xi_3 - \frac{715}{8} \xi_5 + \frac{315}{4} \xi_7 \right) C_F^3 + \left( \frac{132421}{4608} + \frac{451}{8} \xi_3 - \frac{3685}{48} \xi_5 - \frac{105}{8} \xi_7 \right) C_F^2 C_A + \left( \frac{1840145}{18432} + \frac{152329}{3072} \xi_3 + \frac{2975}{48} \xi_5 - \frac{2113}{128} \xi_3^2 \right) C_F C_A^2 + \left( \frac{71251}{1152} - \frac{539}{24} \xi_3 - \frac{125}{3} \xi_5 + \frac{5}{12} \xi_3^2 \right) C_F C_A T_F n_f + \left( \frac{1273}{144} - \frac{599}{24} \xi_3 + \frac{75}{2} \xi_5 \right) C_F^2 T_F n_f + \left( -\frac{49}{6} + \frac{7}{2} \xi_3 + 5 \xi_5 \right) C_F T_F^2 n_f^2 . \tag{4.8}
\]

For the gauges \( \xi = -1 \) and \( \xi = -3 \) the mMOM \( O(a_s^4) \) contributions to the GCR obey the property of the partial \( \beta \)-factorization only. It is violated by the extra non-factorized contributions, proportional to the \( SU(N_c) \) monomials \( C_F C_A T_F n_f \) in the corresponding expressions for the coefficients \( K_3^{\mMOM} \), \( K_3, \xi=-1 \) and \( K_3, \xi=-3 \) (the more detailed clarification and derivation of these statements are given in the Appendix B below).

One should emphasize that relations (4.2), (4.3), (4.7) are valid not only for the mMOM-scheme, but and for any MOM-like renormalization schemes (AS) in QCD. To achieve this
goal it is only necessary to replace all quantities, calculated in mMOM scheme, by the corresponding quantities in any other MOM-like scheme \((β_{1\text{MOM}}^{\text{mMOM}}, b_{1\text{MOM}}^{\text{mMOM}}) \rightarrow (β_{1\text{AS}}^{\text{MOM}}, b_{1\text{AS}}^{\text{MOM}})\).

Note, for the QED case all coefficients of \(β\)-function in any renormalization schemes are proportional to the number of charged leptons \(N\). Therefore, the differences \(β_1^{\text{MS}} - β_1^{\text{AS}}, β_2^{\text{MS}} - β_2^{\text{AS}}, 3β_3^{\text{MS}} - 2β_3^{\text{AS}}\) are always divided by \(β_0\)-factor, and the expression \(β_1^{\text{AS}}(β_1^{\text{AS}} - β_1^{\text{MS}})\) is always divided by \(β_0^2\). This means that observed in Sec.2.4 \(β\)-factorization property of the QED version of the GCR will be valid in all renormalization schemes in QED.

Moreover, in the case of QED, the relations (4.3), (4.7) can be rewritten in more compact form. Actually, taking into account the scheme-independence of the first two coefficients of the RG \(β\)-function of QED, the discussed above possibility of fixing the values of scale parameters, which are leading to nullification the first coefficient \(b_1\) in the expression which relates the QED coupling constants in the \(\overline{\text{MS}}\)-scheme with arbitrary scheme \(X\) (the QED analog of the QCD relation (3.11a)) and the special feature, that in QED the second and third terms \(b_i\) in these expressions can be represented through values of corresponding coefficients of \(β\)-function, namely

\[
K_{2,\text{QED}}^X = \frac{\beta_2^{\text{MS}} - \beta_2^X}{β_0}, \quad K_{3,\text{QED}}^X = \frac{β_3^{\text{MS}} - \beta_3^X}{2β_0},
\]

we get that in QED the coefficients \(K_{2,\text{QED}}^X\) and \(K_{3,\text{QED}}^X\), calculated in the \(X\) renormalization scheme, have the following form:

\[
K_{2,\text{QED}}^X = K_{2,\text{QED}}^{\text{MS}}, \quad K_{3,\text{QED}}^X = K_{3,\text{QED}}^{\text{MS}} + 3K_1 \frac{β_2^{\text{MS}} - β_2^X}{β_0}.
\]

The expressions (4.10) are in full agreement with the results, obtained in Sec.2.4 (see also the unpublished work, presented in the talk of Ref.[72]).

4.2 The \(β\)-function factorization of the \(SU_c(3)\) QCD GCR in the \(O(a_s^3)\) approximation in the MOMggggs scheme

It is important to find out whether there are other MOM-schemes in QCD, which respect the property of the RG \(β\)-function factorization in the GCR for the concrete choice of the gauge parameter. Let us consider the \(O(a_s^3)\) approximation for the GCR in the MOMggggs scheme, defined by renormalization of the quartic gluon vertex through subtractions of ultraviolet divergences in the symmetric subtraction point, investigated and used in the concrete calculations in Refs.[94],[118].

Using Eq.(4.3) and taking into account the explicit form of the RG \(β\)-function in this MOMggggs-scheme, calculated in Ref.[118] at the two-loop approximation in terms of powers of number of colors \(N_c\), we are convinced that the \(O(a_s^3)\) level \(β\)-function factorization property is also valid in this scheme for the Landau and Stefanis–Mikhailov gauges and is violated for anti-Feynman gauge. Therefore we conclude that the property of the factorization of the QCD \(β\)-function in the \(O(a_s^3)\) approximation for the fixed anti-Feynman gauge is the peculiarity of the mMOM-scheme.

In general the two-loop coefficient for the MOMggggs-scheme QCD \(β\)-function has more complicated analytical expression than its mMOM-scheme analog. Indeed, the two-loop
coefficient, calculated in the \textit{MOMgggg}-scheme, depends on the fourth power of gauge \(\xi^4\) and contains additional transcendental logarithmic and Clausen-function terms \cite{118}. The complicated structure remains even in the case of QCD with \(SU_c(3)\) group when the Stefanis–Mikhailov gauge is chosen. Indeed, fixing in the results of Ref.\cite{118} \(N_c = 3\) and \(\xi = -3\), we get the following analytical expression:

\[
\beta_{1,\xi=-3, N_c=3}^{\text{MOMgggg}} = \frac{4173}{80} + \frac{38907}{800} \log\left(\frac{4}{3}\right) - \frac{99}{16} \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) - \frac{373923}{25600} \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)
\]

\[
+ \left[ -\frac{107}{30} - \frac{1179}{400} \log\left(\frac{4}{3}\right) + \frac{3}{8} \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) + \frac{11331}{12800} \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \right] n_f,
\]

where the contributions \(\Phi_1(3/4, 3/4), \Phi_1(9/16, 9/16)\) are expressed through the Clausen function \(\text{Cl}_2(\Theta)\) \cite{118} as\(^{11}\):

\[
\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) = \sqrt{2} \left[ 2\text{Cl}_2\left(2\arccos\left(\frac{1}{\sqrt{3}}\right)\right) + \text{Cl}_2\left(2\arccos\left(\frac{1}{3}\right)\right) \right],
\]

\[
\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) = \frac{4}{\sqrt{5}} \left[ 2\text{Cl}_2\left(2\arccos\left(\frac{2}{3}\right)\right) + \text{Cl}_2\left(2\arccos\left(\frac{1}{9}\right)\right) \right],
\]

\[
\text{Cl}_2(\Theta) = -\int_0^\Theta dx \log \left| 2 \sin \frac{x}{2} \right|.
\]

The numerical values of these extra terms are \(\Phi_1(3/4, 3/4) \approx 2.832045\) and \(\Phi_1(9/16, 9/16) \approx 3.403614\).

Initially there was no indication that at \(\xi = -3\) the factorization of the \(\beta\)-function in the GCR will be valid in this scheme at \(O(a^3s)\) level. However, considering the \textit{MOMgggg} analog of the r.h.s. of Eq.(4.3) in the \(SU_c(3)\) case, we obtain the following relation:

\[
\frac{\beta_{1,\xi=-3, N_c=3}^{\text{MS}} - \beta_{1,\xi=-3}^{\text{MOMgggg}}}{\beta_0} = -\frac{333}{20} - \frac{3537}{200} \log\left(\frac{4}{3}\right) + \frac{9}{4} \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) + \frac{33993}{6400} \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right).
\]

Thus, like in the mMOM-scheme, in the \textit{MOMgggg}-scheme in the Stefanis–Mikhailov gauge the difference \(\left(\beta_{1,\xi=-3}^{\text{MS}} - \beta_{1,\xi=-3}^{\text{MOMgggg}}\right)\) is divided by the \(\beta_0\)-factor without any residue. As was already discussed above, this feature is essential for the \(\beta\)-function factorization in the GCR at the \(O(a^3s)\) level. Using now the two-loop results of Ref.\cite{118} for relating the coupling constants of the \textit{MOMgggg} and MS-schemes, taking into account equations from the Appendix A, getting the analytical expression for one-loop coefficient in the \textit{MOMgggg}-analog of Eq.(3.11a), namely \(b_{1,\xi=-3}^{\text{MOMgggg}}\)-term, substituting it and Eq.(4.15) into the \textit{MOMgggg}-version of relation (4.3), we find the following analytical expression of the coefficient \(K_2\),

\(^{11}\)One should note that the analytical three-loop QCD results for \(R\)-ratio in the \textit{MOMgggg}-scheme, computed in Ref.\cite{94}, were expressed through more familiar functions such as the derivative of the logarithm of the Euler \(\Gamma\)-function and the imaginary part of polylogarithm function with argument \(\exp(iz)/\sqrt{3}\), but not through \(\Phi_1(x,y)\)-functions.
defined at $\xi = -3$:

$$K_{2, \xi=-3, N_c=3}^{\text{MOMggg}} = \frac{9337}{270} + \frac{13769}{400} \log \left(\frac{4}{3}\right) - \frac{35}{32} \Phi_1 \left(\frac{3}{4}, \frac{3}{4}\right) - \frac{2191}{12800} \Phi_1 \left(\frac{9}{16}, \frac{9}{16}\right)$$

$$+ \zeta_3 \left(\frac{2108}{45} - \frac{1967}{50} \log \left(\frac{4}{3}\right) + \frac{5}{4} \Phi_1 \left(\frac{3}{4}, \frac{3}{4}\right) + \frac{313}{1600} \Phi_1 \left(\frac{9}{16}, \frac{9}{16}\right)\right) - \frac{80}{3} \zeta_5$$

$$+ \left[\frac{65}{36} - \frac{49}{24} \log \left(\frac{4}{3}\right) - \frac{49}{96} \Phi_1 \left(\frac{3}{4}, \frac{3}{4}\right) + \frac{7}{96} \Phi_1 \left(\frac{9}{16}, \frac{9}{16}\right)\right] n_f.$$  

In the case of the Landau gauge $\xi = 0$ all transcendental functions included into the two-loop coefficient of the MOMggg RG $\beta$-function are nullified and we arrive to the following result

$$\beta_{1, \xi=0, N_c=3}^{\text{MOMggg}} = \frac{51}{8} - \frac{19}{24} n_f,$$

which as expected is equal to the $\overline{\text{MS}}$-expression for the $\beta_1$-coefficient. Following the described above considerations, we also obtain the following expression for $K_2$-coefficient in the Landau gauge:

$$K_{2, \xi=0, N_c=3}^{\text{MOMggg}} = \frac{280073}{8640} + \frac{3017}{100} \log \left(\frac{4}{3}\right) - \frac{595}{256} \Phi_1 \left(\frac{3}{4}, \frac{3}{4}\right) - \frac{50533}{51200} \Phi_1 \left(\frac{9}{16}, \frac{9}{16}\right)$$

$$+ \zeta_3 \left(\frac{15973}{360} - \frac{862}{25} \log \left(\frac{4}{3}\right) + \frac{85}{32} \Phi_1 \left(\frac{3}{4}, \frac{3}{4}\right) + \frac{7219}{6400} \Phi_1 \left(\frac{9}{16}, \frac{9}{16}\right)\right) - \frac{80}{3} \zeta_5$$

$$+ \left[\frac{65}{36} - \frac{49}{24} \log \left(\frac{4}{3}\right) - \frac{49}{96} \Phi_1 \left(\frac{3}{4}, \frac{3}{4}\right) + \frac{7}{96} \Phi_1 \left(\frac{9}{16}, \frac{9}{16}\right)\right] n_f.$$  

Thus the consideration of completely different gauge-dependent renormalization schemes, namely the mMOM and MOMggg schemes, allows us to discover that at $\xi = 0$ and $\xi = -3$ the factorization of the RG $\beta$-function holds in the GCR at $O(a_s^2)$ level. Therefore, it is important to understand why this happens not only for the Landau gauge, where the two-loop coefficient of the QCD $\beta$-function coincides with the one of the $\overline{\text{MS}}$-scheme, but also for the Stefanis–Mikhailov gauge as well. Next, in Sec.4.1 we have shown that in the Landau gauge only the $\beta$-function factorization property is valid in the mMOM-scheme at the $O(a_s^2)$ level. These foundations give us the hint that for the gauges $\xi = 0$ and $\xi = -3$ the fundamental property of factorization will be true at the $O(a_s^2)$ level in all MOM-like schemes, while for the Landau gauge this property will be fulfilled in all MOM-like schemes in the $O(a_s^4)$ approximation at least. Moreover, we may make the guess that in the higher orders of PT the property of $\beta$-function factorization will take place in MOM-like schemes in the case of Landau gauge as well. In the next section we will study these assumptions in more detail.
4.3 Special features of the Landau and Stefanis-Mikhailov gauges in the QCD MOM-like schemes

The summarized above assumptions can be proved using the following relation, which is similar to the one of Eq.(3.13):

$$\beta^{AS}(a_s^{AS}, \xi^{AS}) = \beta^{\text{MS}}(a_s^{\text{MS}}) \frac{\partial a_s^{AS}}{\partial a_s^{\text{MS}}} + \xi \frac{\gamma_0}{\xi} \frac{\partial a_s^{AS}}{\partial \xi} \bigg|_{\text{MS} \to \text{AS}}$$

where $\beta^{\text{MS}}$ denotes any MOM-like renormalization scheme with linear covariant gauge. Considering Eq.(4.19) and the arbitrary formal PT series, containing the expansion of the coupling constant, defined in the $\text{MS}$-scheme, through the coupling constant of another $AS$-scheme, namely $a_s^{\text{MS}} = a_s^{AS} + \sum_{k=1} \beta_k (a_s^{AS})^{k+1}$, and taking into account the formulas of Appendix A, we get:

$$\beta_1^{AS} = \beta_1^{\text{MS}} - \xi \gamma_0^{\text{MS}} \left( \frac{\partial b_1(\xi)}{\partial \xi} \right),$$

where $\xi = \xi^{AS}$ and $\gamma_0^{\text{MS}} = (-13/24 + \xi^{\text{MS}}/8)C_A + T_F n_f / 3$ is the one-loop coefficient of the anomalous dimension of gauge. Due to its linear covariance it coincides with the corresponding coefficient of the anomalous dimension but with the opposite sign. As follows from Sec.4.1, to study the validity of the property of the $\beta$-function factorization in the GCR at the $O(a_s^3)$-level we should consider the issue whether the expression for $(\beta_1^{\text{MS}} - \beta_1^{AS})$ is divided by $\beta_0$-factor. It is clear from Eq.(4.20) that at $\xi = 0$ $\beta^{AS} = \beta_1^{\text{MS}}$ and therefore the factorization of $\beta$-function in the GCR is always possible in the Landau gauge at this level in the $AS$ scheme. Next, since at $\xi = -3$ we have $\gamma_0 = -11C_A / 12 + T_F n_f / 3 = -\beta_0$, it is obviously from Eq.(4.20) that in the Stefanis–Mikhailov gauge the expression $(\beta_1^{\text{MS}} - \beta_1^{AS})$ will always be divided by $\beta_0$-factor. Therefore at $\xi = -3$ the $\beta$-factorization property is always valid at the $O(a_s^3)$ level in the GCR in the $AS$ scheme with linear covariant gauge. Note, that these statements do not contradict that the $\beta$-function factorization may also hold at other values of $\xi$. Indeed, as was already demonstrated above this feature holds in the mMOM-scheme at $\xi = -1$ as well. Now we can make more strict statement that other values of $\xi$, which respect the $\beta$-function factorization property at the $O(a_s^3)$ level, depend on the concrete renormalization scheme and are determined by the special behavior of the $\partial b_1/\partial \xi$-term.

Taking into account the PT relation between gauge parameters, defined in the $\text{MS}$ and arbitrary $AS$ MOM-like scheme, namely $\xi^{\text{MS}} = \xi^{AS} + \xi^{AS} \sum_{k=1} \eta_k (a_s^{AS})^k$, and using the transformation equations from Appendix A, we obtain the following relation between three-loop coefficients of the $AS$ and $\text{MS}$-schemes:

$$\beta_2^{AS} = \beta_2^{\text{MS}} + \beta_1^{\text{MS}} b_1(\xi) + \beta_0 \left( b_2^0(\xi) - b_2(\xi) \right) - \xi \gamma_0^{\text{MS}}(\xi) \frac{\partial b_2(\xi)}{\partial \xi}$$

$$+ \xi \frac{\partial b_1(\xi)}{\partial \xi} \left( \beta_0 \eta_1(\xi) + \gamma_0^{\text{MS}}(\xi) b_1(\xi) - \gamma_1^{\text{MS}}(\xi) \right) \frac{\partial \eta_1(\xi)}{\partial \xi} - \xi \eta_1(\xi) \frac{\partial \gamma_0^{\text{MS}}(\xi)}{\partial \xi}.$$\(^{12}\)

The fact that at $\xi = -3$ the one-loop expression for renormalization constant of gluon field is proportional to $\beta_0$ was noted in Refs.[105, 106].
Here $\xi = \xi^{AS}$, $\gamma_1^{\overline{MS}}(\xi)$ is the two-loop coefficient of the anomalous dimension of gauge [101]. The application of the relation (4.21) allows to confirm the results of calculations of the three-loop coefficient of $\beta$-function in the mMOM-scheme [70],[93]. Since we have already found out that the factorization of $\beta$-function does not occur at the $\mathcal{O}(a_s^4)$ level in the mMOM-scheme at $\xi = -3$, at this PT level the Landau gauge will be only applied. At $\xi = 0$ in an arbitrary AS-scheme we obtain the following simplified version of Eq.(4.21):

$$\beta_2^{AS,\xi=0} = \beta_2^{\overline{MS}} + \beta_1^{\overline{MS}}b_1(0) + \beta_0\left(b_1^2(0) - b_2(0)\right).$$

(4.22)

In the limit $\xi = 0$ the analog of the relation (4.7) has the following form:

$$K_3^{AS} = K_3^{\overline{MS}} + 3b_1(0)K_2^{\overline{MS}} + \left(2b_2(0) + b_1^2(0) + \frac{\beta_2^{\overline{MS}} - \beta_2^{AS}}{\beta_0} + \frac{\beta_1^{\overline{MS}}b_1(0)}{\beta_0}\right)K_1^{\overline{MS}}.$$  

(4.23)

Substituting the expression for $\beta_2^{AS,\xi=0}$ from Eq.(4.22) into Eq.(4.23), we find that

$$K_3^{AS} = K_3^{\overline{MS}} + 3b_1(0)K_2^{\overline{MS}} + 3b_2(0)K_1^{\overline{MS}}.$$  

(4.24)

Thus, the relation (4.24) proves the validity of the factorization of the RG $\beta$-function in the Landau gauge in the GCR in arbitrary AS MOM-type scheme at the $\mathcal{O}(a_s^4)$ level.

This fact leads us to the natural assumption of the realization of the property of the $\beta$-function factorization in the GCR, evaluated in the AS MOM-like scheme with linear covariant gauge in any order of perturbation theory in the Landau gauge. Indeed, using the relation (4.19), we obtain the following expressions for the four- and five-loop coefficients of $\beta$-function in AS-scheme when the Landau gauge is chosen:

$$\beta_3^{AS,\xi=0} = \beta_3^{\overline{MS}} + 2\beta_2^{\overline{MS}}b_1(0) + \beta_1^{\overline{MS}}b_1^2(0) + 2\beta_0\left(2b_1(0)b_2(0) - b_1^3(0) - b_3(0)\right),$$  

(4.25)

$$\beta_4^{AS,\xi=0} = \beta_4^{\overline{MS}} + \beta_2^{\overline{MS}}\left(b_2(0) + 2b_1^2(0)\right) + \beta_1^{\overline{MS}}\left(3b_1(0)b_2(0) - b_1^3(0) - b_3(0)\right) + 3\beta_3^{\overline{MS}}b_1(0) + \beta_0\left(4b_1^3(0) - 11b_2(0)b_1^2(0) + 6b_3(0)b_1(0) + 4b_4^2(0) - 3b_4(0)\right).$$  

(4.26)

Keeping in mind the presented in Refs.[52–54] arguments in favor of the validity of the factorization of the RG $\beta$-function for the GCR in gauge-invariant $\overline{MS}$-scheme in all orders of PT, using Eqs.(4.2),(4.3),(4.22),(4.24-4.26) and taking into account the relation (4.1), we obtain the $\mathcal{O}(a_s^4)$ and $\mathcal{O}(a_s^5)$-coefficients of the polynomial $K(a_s)$ of Eq.(1.12), which are defined in the AS-scheme at $\xi = 0$:

$$K_4^{AS} = K_4^{\overline{MS}} + 4b_1(0)K_3^{\overline{MS}} + \left(4b_2(0) + 2b_1^2(0)\right)K_2^{\overline{MS}} + 4b_3(0)K_1^{\overline{MS}},$$  

(4.27)

$$K_5^{AS} = K_5^{\overline{MS}} + 5b_1(0)K_4^{\overline{MS}} + 5\left(b_2(0) + b_1^3(0)\right)K_3^{\overline{MS}} + 5\left(b_3(0) + b_1(0)b_2(0)\right)K_2^{\overline{MS}} + 5b_4(0)K_1^{\overline{MS}}.$$  

(4.28)
Unlike the relations (4.3), (4.7) the obtained expressions (4.27-4.28) do not contain the terms, proportional to the powers of $1/\beta_0$-factor. Therefore we conclude that in the MOM-type schemes in the Landau gauge the factorization of $\beta$-function holds in the GCR at the $\mathcal{O}(a_s^5)$ and $\mathcal{O}(a_s^6)$ level. There are no obstacles to obtain similar expressions for any value of order of PT. Thus we conclude that since the $\beta_{\text{MS}}$-function factorization property of GCR is most probably true in all orders of PT, this property also takes place in Landau gauge in all MOM-type schemes in any order of PT. The explanation of this new feature from the first principles of perturbative QCD is the important opened problem.

5 Conclusion

In this work we find out that the property of factorization of the RG $\beta$-function in the generalized Crewther relation really holds true not only in gauge-invariant renormalization schemes in QCD and QED, but in the gauge-dependent subtraction schemes in QCD as well. Considering two gauge non-invariant QCD schemes, namely mMOM- and MOMgggg-schemes, we discover that the $\beta$-factorization property of the GCR is valid at the $\mathcal{O}(a_s^3)$ level in all MOM-like renormalization schemes with linear covariant gauge at the Landau $\xi = 0$ and the Stefanis–Mikhailov $\xi = -3$ gauges. We also find out that in the mMOM-scheme in this order of PT approximation the property of the $\beta$-function factorization is also valid in one more extra gauge, namely anti-Feynman gauge with $\xi = -1$. However, in the $\mathcal{O}(a_s^4)$ approximation the mMOM $SU(N_c)$ expressions for the GCR with $\xi = -3$ and $\xi = -1$ satisfy the property of the partial $\beta$-function factorization only, whereas in Landau gauge this problem do not manifest itself. Moreover, we conclude that if the factorization of the RG $\beta$-function in the GCR in the $\text{MS}$-scheme persists in all orders of PT, then it is also true in all MOM-like scheme with linear covariant Landau gauge. Therefore the gauge-invariance of the renormalization schemes is sufficient but not necessary condition for the manifestation of the $\beta$-function factorization in the conformal symmetry breaking term of the QCD GCR. We obtain the explicit $SU(N_c)$ analytical $\mathcal{O}(a_s^4)$ approximations for Adler and Bjorken functions in the mMOM-scheme with the arbitrary covariant gauge. We also show that in QED the $\beta$-factorization property remains valid in all orders of PT in any ultraviolet subtraction scheme. We expect, that the similar QCD and QED properties will be true for the proposed in Ref.[57] variant of the GCR, with its conformal symmetry breaking term, expressed through the two-fold series in powers of the conformal anomaly and the coupling constant of these fundamental gauge theories.

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A The transformation relations

In this part we present transformation relations, which allow us to obtain the PT coefficients in one particular renormalization scheme, if coefficients in another scheme are known. Let us explain this in details, considering the following example. Suppose that we know the explicit form of the coefficients $\upsilon_i(\xi)$ and $\omega_i(\xi)$, depending on gauge parameter $\xi$ defined in some bar-renormalization scheme $\bar{A}S$, in the following expressions:

\begin{align}
a_s = \pi_s \left( 1 + \upsilon_1(\xi)\pi_s + \upsilon_2(\xi)\pi_s^2 + \upsilon_3(\xi)\pi_s^3 + \mathcal{O}(\pi_s^4) \right), \\
\xi = \bar{\xi} \left( 1 + \omega_1(\xi)\pi_s + \omega_2(\xi)\pi_s^2 + \mathcal{O}(\pi_s^3) \right).
\end{align}

(A.1) (A.2)

Our aim is to find analogies $b_i(\xi)$, $\eta_i(\xi)$ of coefficients $\upsilon_i(\xi)$ and $\omega_i(\bar{\xi})$ correspondingly, determined in another scheme $A\bar{S}$ at the same order of perturbation theory:

\begin{align}
a_s = a_s \left( 1 + b_1(\xi)a_s + b_2(\xi)a_s^2 + b_3(\xi)a_s^3 + \mathcal{O}(a_s^4) \right), \\
\xi = \xi \left( 1 + \eta_1(\xi)a_s + \eta_2(\xi)a_s^2 + \mathcal{O}(a_s^3) \right).
\end{align}

(A.3) (A.4)

Using the Taylor expansion it is straightforward to obtain the following set of transformation equations:

\begin{align}
b_1(\xi) &= -\upsilon_1(\xi), \\
b_2(\xi) &= -\upsilon_2(\xi) + 2\upsilon_3^2(\xi) + \xi \omega_1(\xi) \frac{d\upsilon_1(\xi)}{d\xi}, \\
\eta_2(\xi) &= -\omega_2(\xi) + \omega_3^2(\xi) + \omega_1(\xi)\nu_1(\xi) + \xi \omega_1(\xi) \frac{d\omega_1(\xi)}{d\xi}, \\
b_3(\xi) &= -\upsilon_3(\xi) + 5\nu_2(\xi)\nu_1(\xi) - 5\nu_3^2(\xi) + \xi \omega_1(\xi) \frac{d\nu_2(\xi)}{d\xi} - \frac{1}{2} \xi^2 \omega_1^2(\xi) \frac{d^2\nu_1(\xi)}{d\xi^2} \\
&+ \xi \frac{d\nu_1(\xi)}{d\xi} \left( \omega_2(\xi) - \omega_3^2(\xi) - 5\omega_1(\xi)\nu_1(\xi) - \xi \omega_1(\xi) \frac{d\omega_1(\xi)}{d\xi} \right).
\end{align}

(A.5) (A.6) (A.7) (A.8)

The presented relations (A.5–A.8) solve the stated problem of finding the required coefficients $b_i(\xi)$ and $\eta_i(\xi)$ in the $A\bar{S}$-scheme (note, that in the context of our studies, presented in this work, under the $A\bar{S}$ scheme we mean either the $m\text{MOM}$ or $\text{MOM}_{gggg}$-schemes, and under the $\bar{A}S$ we imply the $\bar{m}\text{MS}$-scheme).
B About partial factorization in the mMOM-scheme at the $\mathcal{O}(a_s^4)$ level

B.1 The case of the Stefanis–Mikhailov gauge $\xi = -3$

Using the relations (2.3d), (3.14b), (3.14c), (3.15a), (3.16a), (3.17), (4.6) we obtain the following mMOM-expression, defined at the $\mathcal{O}(a_s^4)$ level in the Stefanis–Mikhailov gauge:

$$
d_4 + c_4 + d_1c_3 + c_1d_3 + d_2c_2 + \beta_1 K_2 + \beta_2 K_1 \right|_{\xi = -3}^{\text{mMOM}} = \quad (B.1)
$$

$$
= \left[ - \frac{27181}{9216} - \frac{671}{96} \zeta_3 + \frac{7865}{96} \zeta_5 - \frac{1155}{16} \zeta_7 \right] C_F^2 C_A + \left[ \frac{2471}{2304} + \frac{61}{24} \zeta_3 - \frac{715}{24} \zeta_5 + \frac{105}{4} \zeta_7 \right] C_F^2 T_F n_f
$$

$$
+ \left[ \frac{747967}{27648} - \frac{20405}{284} \zeta_3 + \frac{10505}{144} \zeta_5 + \frac{385}{32} \zeta_7 \right] C_F^2 C_A^2 + \left[ - \frac{1273}{432} - \frac{599}{72} \zeta_3 + \frac{25}{2} \zeta_5 \right] C_F^2 T_F^2 n_f
$$

$$
+ \left[ \frac{10031531}{9216} - \frac{376445}{576} \zeta_3 + \frac{32725}{384} \zeta_5 + \frac{4499}{384} \zeta_7 \right] C_F C_A^3 + \left[ - \frac{49}{18} + \frac{7}{6} \zeta_3 + \frac{5}{3} \zeta_5 \right] C_F T_F^3 n_f
$$

$$
+ \left[ \frac{124009}{6912} + \frac{6077}{144} \zeta_3 - \frac{4385}{72} \zeta_5 - \frac{35}{8} \zeta_7 \right] C_F C_A T_F n_f
$$

$$
+ \left[ - \frac{2499749}{27648} + \frac{27673}{768} \zeta_3 + \frac{2825}{48} \zeta_5 - \frac{629}{96} \zeta_7 \right] C_F C_A^2 T_F n_f
$$

$$
+ \left[ \frac{12245}{432} - \frac{1567}{144} \zeta_3 - \frac{665}{36} \zeta_5 + \frac{5}{6} \zeta_7 \right] C_F C_A T_F^2 n_f
$$

According to Eq.(2.3d) in the case of existence of the $\beta$-function factorization property at the $\mathcal{O}(a_s^4)$ level of the GCR the expression (B.1) must be equal to the $-\beta_0 K_3$ term at $\xi = -3$. In this case the coefficient $K_3$ must contain the same group structures as in the Landau gauge (4.8), namely

$$
K_3 = \theta_1 C_F^3 + \theta_2 C_F^2 C_A + \theta_3 C_F C_A^2 + \theta_4 C_F^2 T_F n_f + \theta_5 C_F C_A T_F n_f + \theta_6 C_F T_F^2 n_f^2 . \quad (B.2)
$$

where $\theta_1 - \theta_6$ are unknown analytical coefficients. Taking into account expression (3.14a) one can find the following expression:

$$
-\beta_0 K_3 = - \frac{11}{12} \theta_1 C_F^3 C_A + \frac{11}{12} \theta_2 C_F^2 C_A^2 + \frac{11}{12} \theta_3 C_F C_A^3 + \frac{1}{3} \theta_4 C_F^2 T_F n_f
$$

$$
+ \left( \frac{1}{3} \theta_2 - \frac{11}{12} \theta_4 \right) C_F^2 C_A T_F n_f + \left( \frac{1}{3} \theta_3 - \frac{11}{12} \theta_5 \right) C_F C_A^2 T_F n_f
$$

$$
+ \left( \frac{1}{3} \theta_5 - \frac{11}{12} \theta_6 \right) C_F C_A T_F^2 n_f^2 + \frac{1}{3} \theta_4 C_F^2 T_F^2 n_f^2 + \frac{1}{3} \theta_6 C_F T_F^3 n_f^3
$$

Equating the expression (B.1) with $-\beta_0 K_3$ term we obtain for Stefanis–Mikhailov gauge:

$$
\theta_1^{\xi=-3} = \frac{2471}{768} + \frac{61}{8} \zeta_3 - \frac{715}{8} \zeta_5 + \frac{315}{4} \zeta_7 ,
$$

$$
\theta_2^{\xi=-3} = \frac{67997}{2304} + \frac{1855}{32} \zeta_3 - \frac{955}{12} \zeta_5 - \frac{105}{8} \zeta_7 ,
$$

$$
\theta_3^{\xi=-3} = - \frac{10031531}{101376} + \frac{376445}{8448} \zeta_3 + \frac{2975}{48} \zeta_5 - \frac{409}{32} \zeta_7 ,
$$

$$
\theta_4^{\xi=-3} = \frac{1273}{144} - \frac{599}{24} \zeta_3 + \frac{75}{2} \zeta_5 ,
$$

$$
\theta_6^{\xi=-3} = - \frac{49}{6} + \frac{7}{2} \zeta_3 + 5 \zeta_5 .
$$
The system of equations containing the $\theta_5$ contribution at the group weights $C_F C_A^2 T_F n_f$ and $C_F C_A^2 T_F^2 n_f^2$ in $-\beta_0 K_3$ term is incompatible. Therefore the coefficient $\theta_5$ can not be found indicating the absence of the $\beta$ factorization for $\xi = -3$ at $O(a_s^4)$ level. However, one should emphasize that there holds a partial factorization for five of the six possible coefficients $\theta_i$ in group structures in (B.2) except the $\theta_5 C_F C_A T_F n_f$ term.

B.2 The case of the anti-Feynman gauge $\xi = -1$

Similarly one can obtain:

$$d_4 + c_4 + d_1 c_3 + c_1 d_3 + d_2 c_2 + \beta_1 K_2 + \beta_2 K_1 \bigg|_{\xi = -1}^{\text{mMOM}} = \left[ \begin{array}{l}
\left( -\frac{27181}{9216} - \frac{671}{96} \zeta_3 + \frac{7865}{96} \zeta_5 - \frac{1155}{16} \zeta_7 \right) C_F^2 C_A + \left( \frac{2471}{2304} + \frac{61}{24} \zeta_3 - \frac{715}{24} \zeta_5 + \frac{105}{4} \zeta_7 \right) C_F^2 T_F n_f

+ \left( -\frac{747967}{27648} - \frac{20405}{384} \zeta_3 + \frac{10505}{144} \zeta_5 + \frac{385}{32} \zeta_7 \right) C_F^2 C_A^2 + \left( -\frac{1273}{432} - \frac{599}{72} \zeta_3 + \frac{25}{2} \zeta_5 \right) C_F^2 T_F^2 n_f^2

+ \left( \frac{1287481}{13824} - \frac{26255}{576} \zeta_5 + \frac{32725}{576} \zeta_5 + \frac{335}{24} \zeta_3 \right) C_F C_A^3 + \left( -\frac{49}{18} + \frac{7}{6} \zeta_3 + \frac{5}{3} \zeta_5 \right) C_F T_F^3 n_f^3

+ \left( \frac{124009}{6912} + \frac{6077}{144} \zeta_3 - \frac{4385}{72} \zeta_5 - \frac{35}{8} \zeta_7 \right) C_F^2 C_A T_F n_f

+ \left( -\frac{2523437}{27648} + \frac{9671}{256} \zeta_3 + \frac{2825}{48} \zeta_5 - \frac{713}{96} \zeta_3 \right) C_F C_A^2 T_F n_f

+ \left( -\frac{12245}{432} - \frac{1567}{144} \zeta_3 - \frac{665}{36} \zeta_5 + \frac{5}{6} \zeta_5 \right) C_F C_A T_F^2 n_f^2
\end{array} \right].$$

It is interesting to note that in this expression among the nine possible terms in $SU(N_c)$ group structures the seven coincide with the results for Stefanis–Mikhailov gauge, namely contributions which is proportional to $C_F^2 C_A$, $C_F^2 C_A^2$, $C_F^2 T_F n_f$, $C_F^2 C_A T_F n_f$, $C_F C_A^2 T_F n_f$, $C_F C_A T_F^2 n_f^2$, $C_F C_A T_F^3 n_f^3$ in anti-Feynman gauge are identically equal to the results for $\xi = -3$ presented in (B.1). Separately it should be explained that of these seven contributions, the three are gauge-dependent, namely those that are proportional to $C_F^2 C_A$, $C_F^2 C_A^2 T_F n_f$, $C_F C_A^2 T_F n_f^2$ group weights. Nevertheless it is quite surprising that these gauge-dependent terms coincide at $\xi = -3$ and $\xi = -1$.

For $\xi = -1$ we obtain the following values of the $\theta_i$ coefficients:

$$\theta_1^{\xi = -1} = \frac{1}{3} \zeta_5, \quad \theta_2^{\xi = -1} = \frac{1}{3} \zeta_5, \quad \theta_3^{\xi = -1} = \frac{1}{3} \zeta_5, \quad \theta_4^{\xi = -1} = \frac{1}{3} \zeta_5, \quad \theta_5^{\xi = -1} = \frac{1}{3} \zeta_5.$$

The $\beta$ factorization property is again violated by the term $\theta_5 C_F C_A T_F n_f$ in expansion (B.2). Therefore in this case we observe the partial factorization of the $O(a_s^4)$ mMOM-scheme approximation of the GCR as well.

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