NON-FLOQUET INVARIANT TORI IN REVERSIBLE SYSTEMS

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Abstract. In this paper we obtain a theorem about the persistence of non-floquet invariant tori of analytic reversible systems by an improved KAM iteration. This theorem can be applied to solve the persistence problem of completely hyperbolic-type degenerate invariant tori for a class of reversible system.

1. Introduction and main results. Consider the existence of $n$-dimensional invariant tori of the following dynamical system:

$$
\begin{align*}
\dot{x} &= \omega(y) + f^1(x, y, u, v), \\
\dot{y} &= f^2(x, y, u, v), \\
\dot{u} &= A(y)v + f^3(x, y, u, v), \\
\dot{v} &= B(y)u + f^4(x, y, u, v),
\end{align*}
$$

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p (n \leq m)$, $\omega(y) = (\omega_1(y), \cdots, \omega_n(y)) \in \mathbb{R}^n$ is called frequency vector, and $y \in \mathcal{M} \subset \mathbb{R}^m$, where $\mathcal{M}$ is a bounded open domain. $A$ and $B$ are $p \times p$ matrices, $f^1$, $f^2$, $f^3$ and $f^4$ are small perturbations.

If $f^j = 0$ $(j = 1, 2, 3, 4)$, then the system (1.1) becomes

$$
\begin{align*}
\dot{x} &= \omega(y), \\
\dot{y} &= 0, \\
\dot{u} &= A(y)v, \\
\dot{v} &= B(y)u,
\end{align*}
$$

which has an invariant subspace $\{u = 0, v = 0\}$. Moreover, this space is foliated by a family of invariant tori $\mathbb{T}^n \times \{y_0\} \times \{0\} \times \{0\}$ with the frequency $\omega(y_0)$ for all $y_0 \in \mathcal{M}$. If the tangential frequencies and the normal frequencies are independent of angle variables $x$, then the invariant tori $\mathbb{T}^n \times \{y_0\} \times \{0\} \times \{0\}$ are said to be floquet invariant tori. Otherwise, they are called non-floquet invariant tori. The persistence of these floquet invariant tori under small perturbations of (1.2) is an important problem in the perturbation theory and is studied by many authors.

When (1.1) is a Hamiltonian system (in this case, $m = n$), the persistence of lower dimensional tori has been investigated extensively. For instance, if all eigenvalues of

$$
\Omega = \begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
$$

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are not purely imaginary, Moser [16], Graff [8], and Zehnder [40, 41] prove that, for any \( \omega = (\omega_1, \cdots, \omega_n) \) satisfying the Diophantine condition
\[
|\langle k, \omega \rangle| \geq \frac{\alpha}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\},
\]
where \( \alpha > 0 \) and \( \tau > n - 1 \) are some constants, there is an \( \omega^* \) close to \( \omega \) such that (1.1) at \( \omega^* \) has an invariant \( n \)-torus with prescribed frequencies \( \omega \) if the perturbations \( f_j = 0 \) \( (j = 1, 2, 3, 4) \) are sufficiently small. If all the eigenvalues of \( \Omega \) are purely imaginary, Eliasson [7], Kuksin [11], Pöschel [13], Xu [37] and You [38], et al. get the existence of elliptic lower dimensional tori for Hamiltonian system of the form (1.1) and obtained many kinds of KAM theorems.

In this paper, we suppose that (1.1) are reversible with respect to the involution \( G : (x, y, u, v) \rightarrow (x, y, -u, v) \), that is,
\[
DG \cdot F = -F \circ G,
\]
where \( F = (\omega + f_1, f_2, Av + f_3, Bu + f_4)^T \). By (1.4), it follows that the system (1.1) is reversible if
\[
f_j(-x, y, -u, v) = (-1)^{j+1} f_j(x, y, u, v), \quad j = 1, 2, 3, 4.
\]
A mapping \( \Phi : (x, y, u, v) \rightarrow (x, y, +, u, +, v) \) is called compatible with the involution \( G \) if \( \Phi \) and \( G \) commute. The compatible transformations transform reversible systems into systems reversible with respect to the same involution.

During the last 50 years, many authors study the persistence of invariant tori for reversible systems of form (1.1) and obtained many kinds of KAM theorems (see [1, 2, 3, 4, 5, 6, 10, 15, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 35] and the references therein).

In the special case of \( p = 0 \) and \( m \geq n \), that is, where there is no normal frequency, Arnold [1] and Sevryuk [21] proved the existence of invariant tori under the non-degeneracy condition that the unperturbed frequency map \( y \rightarrow \omega(y) \) is submersive in \( \mathcal{M} \), i.e.,
\[
\text{Rank} \frac{\partial \omega}{\partial y} = n, \quad \text{for all } y \in \mathcal{M}.
\]
In the case of \( p > 0 \), the invariant \( n \)-tori of reversible system (1.2) are called lower dimensional. Many authors studied the persistence of lower dimensional invariant tori under the non-degeneracy condition (1.5) and obtained many kinds of KAM theorems [2, 3, 4, 5, 22, 23, 24, 35].

The above mentioned results about the persistence of invariant tori in reversible systems deal only with the Diophantine condition:
\[
|\langle k, \omega \rangle| \geq \frac{\alpha}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\},
\]
where \( \alpha > 0 \) and \( \tau > n - 1 \) are some constants. The natural question to ask is whether the Diophantine (1.6) can be further weakened and in what case the frequencies of the persisting invariant tori for reversible systems can persist. Recently, Zhang, Xu and Wang obtained a result for the above problem under Brjuno-Rüssmann’s non-resonant condition [42].

The persistence of invariant tori for reversible systems of the form (1.1) requires the condition that the matrix \( A(y) \) is non-singular. When the matrix \( A \) is non-singular, we can use the linear term \( Av \) to control the shift of lower-order terms from small perturbation in KAM steps and so we can completely control the shift
of equilibrium point. When $A(y)$ is singular, the invariant tori are said to be degenerate.

There are already some results on degenerate lower-dimensional invariant tori for Hamiltonian systems \([9, 12, 39]\). Li and Yi \([12]\) studied the persistence of lower dimensional tori of general type in Hamiltonian systems of general normal forms and proved a persistence result, under a Melnikov type of non-resonance condition, which particularly allows multiple and degenerate normal frequencies of the unperturbed lower dimensional tori. Han, Li and Yi \([9]\) considered the persistence problem of lower-dimensional, possibly degenerate, invariant tori for Hamiltonian systems of the following form:

$$H(x, y, z) = \langle \omega, y \rangle + \frac{1}{2} \langle z, M(\omega)z \rangle + \epsilon P(x, y, z, \omega, \epsilon),$$

where $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2m}$, $\epsilon$ is a small parameter, and $M(\omega)$ can be singular. The result \([9]\) replaced the condition $\det M \neq 0$ by the following restrictive condition on the perturbation:

$$M(\omega)z_\epsilon(\omega) + \epsilon \partial_z[P](\cdot, 0, z_\epsilon(\omega), \omega, 0) = 0,$$

with a real analytic family $z_\epsilon$. Here $[P]$ is the mean value of the function $P(x, 0, z_\epsilon(\omega), \omega, 0)$ over $\mathbb{T}^n(\text{see (1.12) below}).$

Recent years, the existence of degenerate lower dimensional invariant tori for reversible systems has been studied by many authors. Liu \([15]\) considered the reversible systems of the following form:

$$\begin{aligned}
\dot{x} &= \omega + P^1(x, y, u, v, \omega), \\
\dot{y} &= D(\omega)u + P^2(x, y, u, v, \omega), \\
\dot{u} &= C(\omega)y + A(\omega)v + P^3(x, y, u, v, \omega), \\
\dot{v} &= B(\omega)u + P^4(x, y, u, v, \omega),
\end{aligned}$$

(1.7)

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^q$, $\omega$ is an independent parameter varying over a positive measure set $M \subset \mathbb{R}^n$. Liu \([15]\) replaced the condition $\det A \neq 0$ by the condition $\text{Rank}(A, C) = p$ in the reversible system (1.7). A natural question is what happens when $\text{Rank}(C, A) < p$? Wang, Xu and Zhang \([30, 31]\) obtained some results about the persistence of degenerate lower-dimensional tori in reversible systems of the following form:

$$\begin{aligned}
\dot{x} &= \omega + Q(x)y + P^1(x, y, u, v), \\
\dot{y} &= P^2(x, y, u, v), \\
\dot{u} &= y^{m+1} + v^{n+1} + P^3(x, y, u, v), \\
\dot{v} &= Bu + P^4(x, y, u, v),
\end{aligned}$$

(1.8)

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$ $(m \geq n+1)$, $y = (y_1, y_2, \cdots, y_m) \in \mathbb{R}^m$, $n_1$ and $n_2$ is a positive integer, $P^1$, $P^2$, $P^3$ and $P^4$ are small perturbations. The system (1.8) is actually partially degenerate case. If $B = 0$, the equilibrium point of the unperturbed system (1.8) is completely degenerate case.

In this paper, we are mainly interested in the persistence of non-floquet invariant tori. There are already some results on non-floquet invariant tori for Hamiltonian systems\([8, 41]\). Zehnder \([41]\) considered the Hamiltonian function of the following form:

$$H = e + \langle \omega, y \rangle + \frac{1}{2} \langle Q(x)y, y \rangle + \langle \Omega(x)z_+, z_- \rangle + P(x, y, z_+, z_-),$$
where \((x, z_+) \in \mathbb{T}^n \times \mathbb{R}^m\) and \((y, z_-) \in \mathbb{R}^n \times \mathbb{R}^m\) are canonically conjugate variables, 
\(P\) is a small perturbation, \(Q\) is an \(n \times n\)-matrix valued function on \(\mathbb{T}^n\), \(\bar{\Omega}\) is an \(m \times m\)-matrix-valued function satisfying

\[
\text{Det}[Q] \neq 0, \quad (1.9)
\]
\[
\text{Re}(\bar{\Omega}(x, \xi, \xi)) \geq \sigma|\xi|^2, \quad (1.10)
\]

for all \(\xi \in \mathbb{C}^m\) and some \(\sigma > 0\). The author proved that all the hyperbolic lower dimensional tori with Diophantine frequency of Hamiltonian systems survive small perturbations. For a similar result also see [8].

Motivated by [8, 41], we want to obtain some information on the persistence of non-floquet invariant tori in reversible system. To be more precise, we consider the reversible system of the following form:

\[
\begin{align*}
\dot{x} &= \omega + Q(x)y + C(x)z + f^1(x, y, z), \\
\dot{y} &= f^2(x, y, z), \\
\dot{z} &= \left(\Omega + \bar{\Omega}(x)\right)z + g(x, y, z),
\end{align*}
\]

\( (x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^p (n \leq m)\). \(A, B, C, Q, \bar{\Omega}\) and \(f^1, f^2, f^3\) and \(f^4\) are small perturbations. The corresponding involution \(G\) is \((x, y, u, v) \mapsto (x, y, -u, v)\). Assume that \(Q(x), C(x)\) and \(\bar{\Omega}(x)\) are real analytic in \(x\) on a complex domain

\[
D(s) = \left\{ x \in \mathbb{C}^n / 2\pi\mathbb{Z}^n \mid \text{Im}x \leq s \right\}.
\]

Moreover, the perturbation terms \(f^1, f^2\) and \(g\) are all real analytic with respect to \((x, y, u, v)\) on a complex domain \(D(s, r)\), where

\[
D(s, r) = \left\{ (x, y, u, v) \mid \text{Im}x \leq s, \text{Im}y \leq r, |u| \leq r, |v| \leq r \right\}
\]

\[
\subset \mathbb{C}^n / 2\pi\mathbb{Z}^n \times \mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}^p.
\]

Obviously, the frequencies of the invariant tori for the unperturbed system admit the different dimensions of angle variables, then the invariant tori for the unperturbed system are said to be non-floquet invariant tori. The purpose of this paper is to study the persistence of the non-floquet invariant torus with given frequencies \(\omega\).

Before formulating our theorem, we first give some notations and assumptions.

Let \(f(x_1, \cdots, x_n)\) be a continuous function with period \(2\pi\) in every \(x_i, i = 1, 2, \cdots, n\), denote the average of \(f\) by

\[
[f] = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(x)dx_1 \cdots dx_n.
\]

(1.12)

If \(f(x)\) is analytic in \(x\) on \(D(s)\), then \(f\) can be expanded as

\[
f(x) = \sum_{k \in \mathbb{Z}^n} f_k e^{i \langle k, x \rangle}.
\]

Denote the norm of \(f\) on \(D_s\) by \(\|f\|_s = \sum_{k \in \mathbb{Z}^n} |f_k| e^{s|k|} \). Let

\[
Q(x) = \begin{pmatrix} q_{ij}(x) \end{pmatrix}_{1 \leq i \leq n, 1 \leq j \leq m}
\]
be an \( n \times m \) matrix with all \( q_{ij}(x) \) are analytic in \( x \) on \( D(s) \). Define a norm of \( Q \) by
\[
\|Q\|_s = \max \left\{ \sum_{j=1}^m |q_{1j}|_s, \sum_{j=1}^m |q_{2j}|_s, \ldots, \sum_{j=1}^m |q_{nj}|_s \right\}.
\]
If \( f(x, y, u, v) \) is analytic in \( (x, y, u, v) \) on \( D(s, r) \), then \( f \) can be expanded as
\[
f(x, y, u, v) = \sum_{k \in \mathbb{Z}^n, l \in \mathbb{Z}_+^n, i, j \in \mathbb{Z}_+^n} f_{klij} y^i u^j e^{\sqrt{-1}(k,x)}.
\]
Define a norm of \( f \) by
\[
\|f\|_{s,r} = \sum_{k \in \mathbb{Z}^n} |Mf_k|_{r} e^{n|k|},
\]
where \( Mf_k = \sum_{i,j} |f_{klij}| y^i u^j \) and \( |Mf_k|_r \) denotes the sup-norm of \( Mf_k \) over the domains \( D(s, r) \) with respect to \( y, u, v \). Let \( f = (f^1, f^2, g) \) with \( g = (f^3, f^4) \). Define
\[
\|f\|_{s,r} = \|f^1\|_{s,r} + \|f^2\|_{s,r} + \|g\|_{s,r}.
\]
Let \( B_s \) denote the space of all real analytic functions \( h(x) \) defined in the complex domain \( D(s) \), that is
\[
B_s = \left\{ h(x) \mid h(x) = \sum_{k \in \mathbb{Z}^n} h_k e^{\sqrt{-1}(k,x)}, \|h\|_s < \infty \right\}.
\]
Then it is easy to see that \( B_s \) is a Banach space. We assume that \( \Omega \) is hyperbolic on \( M \). Define a linear operator \( L \) on \( B_s \) by
\[
L(h(x)) = \sum_{k \in \mathbb{Z}^n} \left( \sqrt{-1}(k, \omega) - \Omega \right)^{-1} h_k e^{\sqrt{-1}(k,x)}
\]
with \( h(x) \in B_s \). Denote the norm of the linear operator \( L \) on \( B_s \) by
\[
\|L\| = \sup_{0 \neq \|h\|_s \leq 1} \left\| L(h(x)) \right\|_s.
\]
Remark 1. If
\[
\Omega = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}
\]
where \( A = \text{diag}(\lambda_1, \ldots, \lambda_p) \) with \( |\text{Re}\lambda_i| \geq \sigma \) for \( i = 1, 2, \ldots, p \), it is easy to see that \( \Omega \) is hyperbolic. Moreover, we have \( \|L\| \leq \frac{1}{\sigma} \).

To state the main result, we need the following assumptions:

**Assumption (1).** Non-degeneracy conditions. Let \( [Q] \) be the average of \( Q(x) \). We suppose that the rank of the matrix \( [Q] \) is \( n \).

**Assumption (2).** Hyperbolic conditions. There is a \( \delta \geq 0 \) such that
\[
\|L\| \cdot \|\hat{\Omega}\|_s \leq \delta < 1.
\] (1.13)

**Remark 2.** The condition (1.13) is used to guarantee that \( \Omega + \hat{\Omega}(x) \) is hyperbolic on \( D(s) \).

**Assumption (3).** Non-resonance conditions. The frequency vector \( \omega \) satisfies the Diophantine condition
\[
|\langle k, \omega \rangle| \geq \frac{\alpha}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\},
\] (1.14)
where \( \alpha > 0 \) and \( \tau > n - 1 \) are some constants.
Theorem 1.1. Consider the reversible system (1.11). Suppose that the above assumptions (1)-(3) hold. Then, there exists a sufficiently small \( \gamma > 0 \), which is independent of \( \alpha \) and usually depends on \( \delta, \tau, \sigma, |Q|, T_0 = \max\{\|Q\|, \|C\|\} \), \( n, p \), such that for any \( 0 < \alpha < 1 \), if
\[
\|f\|_{s,r} \leq \epsilon r = \alpha^3 \gamma s^{3r+1} r,
\]
there is a compatible transformation
\[
\Phi_s(\cdot, \cdot, \cdot, \cdot) : D(s/2, r/2) \to D(s, r)
\]
which transforms the reversible system (1.11) into
\[
\dot{x} = \omega + f^1_s, \quad \dot{y} = f^2_s, \quad \dot{u} = Av + f^3_s, \quad \dot{v} = Bu + f^4_s,
\]
where \( f^2 \) satisfies \( f^2(x, 0, 0, 0) = 0, \ (j = 1, 2, 3, 4) \). Hence, \( \Phi_s(\mathbb{T}^n, 0, 0, 0) \) is an invariant torus of reversible system (1.11) with frequencies \( \omega \).

Remark 3. An example to which the above theorem can be applied is \( A = B = \text{diag}(8, -8, 8, -8, \cdots, 8, -8) \). Assume that \( \text{Rank}[Q] = n \) and \( \omega \) satisfies the non-resonant conditions (1.14). If \( \|\Omega\|_s = 7 \), then it follows that
\[
\|\mathcal{L}\| \cdot \|\Omega\|_s \leq \frac{7}{8} < 1.
\]
By Theorem 1.1, the non-floquet invariant torus with given frequencies \( \omega \) survives small perturbations. In contrast to the conditions of Zehnder and Graff [8, 41], our condition (1.13) does not require \( \Omega(x) \) to satisfy the positivity condition (1.10).

Moreover, if \( m > n \), we have analytic \((m - n)\)-parameter families of invariant \( n \)-tori with frequency vector \( \omega \), since we only need \( n \) components of \( y \) to control the shift of frequency and the other \( n - m \) components can introduce some parameters.

Remark 4. The key features of the system (1.11) are that the tangential frequencies and the normal frequencies of the invariant tori for the unperturbed system admit angle variables, that is, the presence of the terms \( Q(x)y \) and \( \tilde{\Omega}(x)z \) and the dependence of \( Q \) and \( \tilde{\Omega} \) on \( x \). If \( \Omega = 0 \) and \( \tilde{Q}(x) = Q(x) - |Q| = 0 \), that is \( Q = \text{const} \), consider the rescaling
\[
u \rightarrow \varepsilon u, v \rightarrow \varepsilon v, f^i \rightarrow \varepsilon f^i, i = 1, 2, 3, 4,
\]
for small \( \varepsilon > 0 \). Then the system (1.11) becomes
\[
\begin{aligned}
\dot{x} &= \omega + Qy + \varepsilon C_1(x)u + \varepsilon C_2(x)v + \varepsilon f^1(x, y, \varepsilon u, \varepsilon v), \\
\dot{y} &= \varepsilon f^2(x, y, \varepsilon u, \varepsilon v), \\
\dot{u} &= Av + \varepsilon f^3(x, y, \varepsilon u, \varepsilon v), \\
\dot{v} &= Bu + \varepsilon f^4(x, y, \varepsilon u, \varepsilon v),
\end{aligned}
\]
where the terms \( \varepsilon C_1(x), \varepsilon C_2(x)v \) can be treated as perturbations. So Theorem 1.1 can be regarded as a particular case of the previous results [2, 3, 15, 22, 23, 24, 29]. However, if \( \tilde{Q}(x) \neq 0 \) and \( \tilde{\Omega}(x) \neq 0 \), the previous results cannot tell whether the reversible system (1.11) has a torus with prescribed frequencies \( \omega \). Our result shows that the reversible system (1.11) has a torus with the frequencies \( \omega \).

As an application of Theorem 1.1, we give a result about the persistence problem of lower-dimensional, possibly degenerate (i.e. the matrix \( A \) may be singular),
invariant tori for reversible system of the form:

\[
\begin{cases}
\dot{x} = \omega + Q(x)y + \epsilon P^1(x, y, u, v, \epsilon) \\
\dot{y} = \epsilon P^2(x, y, u, v, \epsilon) \\
\dot{u} = Av + \epsilon P^3(x, y, u, v, \epsilon), \\
\dot{v} = Bu + \epsilon P^4(x, y, u, v, \epsilon),
\end{cases}
\]

where \((x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^q(n \leq m), \epsilon \in (0, 1)\) is a small parameter. \(\epsilon P^1, \epsilon P^2, \epsilon P^3\) and \(\epsilon P^4\) are small perturbations. The corresponding involution \(G\) is \((x, y, u, v) \rightarrow (-x, y, -u, v)\). Assume that \(Q(x)\) is real analytic in \(x\) on a complex domain \(D(s)\). Moreover, the perturbation terms \(\epsilon P^1, \epsilon P^2, \epsilon P^3\) and \(\epsilon P^4\) are all real analytic with respect to \((x, y, u, v, \epsilon)\) on a complex domain \(D(s, r) \times (0, 1)\).

**Theorem 1.2.** Consider the reversible system (1.15) with \(\text{Rank}[Q] = n\). Suppose that

(i) There exists a real function \(v_\epsilon : (0, 1) \rightarrow \{v : |v| < r\}\) such that

\[
Av_\epsilon + \epsilon [P^3(\cdot, 0_{m+p}, v_\epsilon, \epsilon)] = O(\epsilon^2).
\]

If \(\text{Det}(B) = 0\), suppose that

\[
\left[\frac{\partial P^2(\cdot, 0_{m+p}, v_\epsilon, \epsilon)}{\partial u}\right] = O(\epsilon).
\]

(ii) Let \(\lambda_i, i = 1, 2, \ldots, 2p\), be eigenvalues of

\[
\Omega = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.
\]

Suppose that frequency vector \(\omega\) and \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2p})\) satisfying the following non-resonance conditions:

\[
|\sqrt{-1}(k, \omega) - (l, \lambda)| \geq \frac{\alpha}{|k|^r},
\]

where \(k \in \mathbb{Z}^n \setminus \{0\}, l \in \mathbb{Z}^2p\) and \(|l| \leq 2\).

(iii) Let \(\lambda^*_i, i = 1, 2, \ldots, 2p\), be eigenvalues of

\[
\Omega_\epsilon = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} + \epsilon \left[\frac{\partial (P^3, P^4)}{\partial (u, v)}\right]_{y=0, u=0, v=v_\epsilon}
\]

and there is a \(\sigma > 0\) such that

\[
|\text{Re}\lambda^*_i| \geq \epsilon \sigma, \text{ for } i = 1, 2, \ldots, 2p.
\]

Then there exists sufficiently small positive constant \(\epsilon_0\) such that for each \(\epsilon \in (0, \epsilon_0)\), the system (1.15) has a torus with frequencies \(\omega\).

**Remark 5.** It is easy to see that the condition (1.16) is automatically satisfied if \(A\) is nonsingular. In the case that \(A\) becomes singular, invariant \(n\)-tori can be destroyed if condition (1.16) fails. For example, consider reversible system of the form:

\[
\begin{align*}
\dot{x} &= \omega + Q(x)y, \quad \dot{y} = 0, \quad \dot{u} = \epsilon v^2, \quad \dot{v} = 0,
\end{align*}
\]

where \((x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}\). Then for all \(\epsilon > 0\), the reversible system (1.19) has no invariant torus. Moreover, if \(B\) is nonsingular, we can use the linear term \(Bu\) to remove the term
by a compatible transformation. Therefore, the condition (1.17) is not necessary if $\text{Det}(B) \neq 0$.

**Remark 6.** Set
$$\mathcal{B}_s^0 = \left\{ h(x) \mid h(x) \in \mathcal{B}_s, \ |h| = 0 \right\}.$$ Consider the reversible system (1.15) with $\epsilon P^2 \in \mathcal{B}_s^0$, $\Omega = 0$ and
$$\epsilon P^3 = (\epsilon^2, \epsilon^2, \cdots, \epsilon^2)^T + cv + \epsilon \tilde{P}^3, \ \epsilon P^4 = \epsilon u + \epsilon \tilde{P}^4,$$ where $\tilde{P}^3 \in \mathcal{B}_s^0$ and $\tilde{P}^4 \in \mathcal{B}_s^0$. It is easy to see that $v_\epsilon = 0$ and
$$\Omega_\epsilon = \epsilon \left( \begin{array}{cc} 0 & I_p \\ I_p & 0 \end{array} \right)$$ is hyperbolic. Here $I_p$ denotes the $p \times p$ identity matrix. By Theorem 1.2, there exists an $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$, the system (1.15) has a degenerate torus with frequencies $\omega$.

**Remark 7.** Consider the reversible system (1.15) with $\epsilon P^2 \in \mathcal{B}_s^0$, $\Omega = 0$ and
$$\epsilon P^3 = r(\epsilon, \epsilon, \cdots, \epsilon)^T + 2cv + \epsilon \tilde{f}^3, \ \epsilon P^4 = \epsilon u + \epsilon \tilde{f}^4,$$ where $\tilde{f}^3 \in \mathcal{B}_s^0$ and $\tilde{f}^4 \in \mathcal{B}_s^0$. Let $v_\epsilon = -\frac{r}{2}(1, 1, \cdots, 1)^T$ and
$$\Omega_\epsilon = \epsilon \left( \begin{array}{cc} 0 & 2I_p \\ I_p & 0 \end{array} \right).$$ By Theorem 1.2, there exists an $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$, the system (1.15) has a degenerate torus with frequencies $\omega$.

The rest of the paper is organized as follows. In Section 2 the detailed proof of Theorem 1.1 is described, which consists of KAM step, setting the parameters and iteration, and convergence of iteration. In Section 3, we prove that Theorem 1.2 can be reduced to Theorem 1.1.

2. **Proof of Theorem 1.1.** In this section we use an improved KAM iteration to prove Theorem 1.1. In the proof of this theorem, we can remove the shifts of frequencies $\omega$ by a small translation of coordinates of $y$ in KAM steps. The existence of such translation of coordinates can be guaranteed by the condition that $\text{Rank}[Q] = n$. The condition (1.13) is used to ensure that $\Omega + \tilde{\Omega}(x)$ is hyperbolic on $D(s)$.

2.1. **KAM-step.** In this section, we outline the formal process of one cycle of the KAM iteration. To simplify notations, in what follows, the quantities without subscripts refer to those at the $j$-th step, while the quantities with subscripts “+” denote the corresponding ones at the $(j+1)$-th step. We will use the same notation $c$ to indicate different constants, which are independent of the iteration process.

Suppose at the $j$-th step, we have arrived at the following reversible system:
$$\begin{align*}
\dot{x} &= \omega + Q(x)y + C(x)z + f^1(x, y, z), \\
\dot{y} &= f^2(x, y, z), \\
\dot{z} &= \left( \Omega + \tilde{\Omega}(x) \right)z + g(x, y, z),
\end{align*}$$ (2.1)
where
$$z = \left( \begin{array}{c} u \\ v \end{array} \right), \quad \Omega = \left( \begin{array}{cc} 0 & A \\ B & 0 \end{array} \right), \quad g = \left( \begin{array}{c} \tilde{f}^3 \\ \tilde{f}^4 \end{array} \right),$$
Lemma 2.1. Let us consider the above reversible system (2.1) with $||f||_{s,r} \leq c r = \alpha^3 \rho^{3r+1} E r$. We assume that $||L|| \cdot ||\Omega||_s \leq \frac{4+1}{2} < 1$ and

$$
||Q - M|| \leq c E_0, \quad ||Q||_s \leq T_0 + 1, \quad ||C||_s \leq T_0 + 1, \quad (2.2)
$$

where $M \in \mathbb{R}^{n \times m}$ is a matrix with rank($M$) = $n$ and $E_0$ is a parameter that can be chosen sufficiently small. Set

$$
\eta = E_+^2, \quad s_+ = s - 4 \rho, \quad \rho_+ = \frac{\rho}{2}, \quad r_+ = \eta r, \quad E_+ = c E_+^2, \quad \epsilon_+ = \alpha^3 \rho^{3r+1} E_+.
$$

Then, for any $\omega$ satisfying the Diophantine condition (1.14), there exists a compatible transformation $\Phi(\cdot, \cdot, \cdot) : D(s_+, r_+) \rightarrow D(s, r)$ which changes the reversible system (2.1) into

$$
\begin{cases}
\dot{x} = \omega + Q_+(x)y + C_+(x)z + f_+(x, y, z), \\
\dot{y} = f_+(x, y, z), \\
\dot{z} = \left(\Omega + \bar{\Omega}_+(x)\right)z + g_+(x, y, z),
\end{cases}
$$

with $||f_+||_{s_+, r_+} \leq \epsilon_+$. Moreover, we have

$$
\|\Xi(\Phi - id)\|_{s_+, r_+} \leq c E, \quad \|\Xi(D\Phi - I_{m+n+2p})\|_{s_+, r_+} \leq c E, \quad (2.4)
$$

$$
\max\{||Q_+ - Q||_{s_+}, ||C_+ - C||_{s_+}, ||\bar{\Omega}_+ - \bar{\Omega}||_{s_+}\} \leq c E, \quad (2.5)
$$

where $\Xi = \text{diag}(I_n, \frac{1}{\rho} I_m, \frac{1}{\rho} I_p, \frac{1}{\rho} I_p)$.

The above lemma is actually one KAM step. We divide the proof of Lemma 2.1 into several parts.

A. Constructing compatible transformation. In the following, we will construct a compatible transformation $\Phi$ which changes the reversible system (2.1) into (2.3). Let $\Phi$ be defined by

$$
\begin{cases}
x = x_+ + h(x_+), \\
y = y_+ + a_1(x_+) + b_1(x_+)y_+ + d(x_+)z_+, \\
z = z_+ + a_2(x_+) + b_2(x_+)y_+,
\end{cases}
$$

Let $a(x) = (a_1(x), a_2(x))$, $b(x) = (b_1(x), b_2(x))$, $S_1 = \text{diag}(I_m, -I_p, I_p)$ and $S_2 = \text{diag}(-I_p, I_p)$. It is easy to see that $\Phi$ is compatible with the involution $G$ if and only if

$$
h(-x) = -h(x), \quad S_1 a(-x) = a(x), \quad S_1 b(-x) = b(x), \quad S_2 d(-x) = d(x). \quad (2.7)
$$

Let $h(x) \in B_s$, define $\partial_\Delta h$ by

$$
\partial_\Delta h = \sum_{k \in \mathbb{Z}^n} \sqrt{-1}(k, \Delta) h_k e^{y \cdot \tau(k, x)}. \quad (2.8)
$$
Let $\Omega(x) = \Omega + \hat{\Omega}(x)$ and set

\[
Q_+ = \left( I_n + \partial_x h(x) \right)^{-1} \left( Q(x + h)(1 + b_1(x)) + C(x + h)b_2(x) \right) + f^1_+(x, 0),
\]

\[
C_+ = \left( I_n + \partial_x h(x) \right)^{-1} \left( C(x + h) + Q(x + h)d(x) + f^2_+(x, 0) \right),
\]

\[
\Omega_+ = \Omega(x + h) + \partial_{C_+} a_2(x) + g(x, 0),
\]

where $f^1_+(x, 0) = \frac{\partial f^1}{\partial y} |_{(y,z)=0}$, $f^2_+(x, 0) = \frac{\partial f^2}{\partial z} |_{(y,z)=0}$, and $g(x, 0) = \frac{\partial g}{\partial z} |_{(y,z)=0}$. Under the transformation $\Phi$ the system (2.1) is changed into

\[
\begin{align*}
\dot{x} &= \omega + \left( I_n + \partial_x h(x) \right)^{-1} \left( -\partial_\omega h(x) + f^1(x, 0) + Q(x)a_1(x) + \\
&\quad + C(x)a_2(x) \right) + Q_+ x + C_+ x + f^1_+, \\
\dot{y} &= \left( I_m + b_1(x) \right)^{-1} \left( -\partial_\omega a_1(x) + f^2(x, 0) + \left( -\partial_\omega b_1(x) + f^2_+(x, 0) + \\
&\quad + \partial_x a_1(x)Q(x) \right) + C_+ x \right) + f^2_+, \\
\dot{z} &= -\partial_\omega a_2(x) + \Omega_+ a_2(x) + g(x, 0) + \left( -\partial_\omega b_2(x) + \Omega(x)b_2(x) \right) + \\
&\quad + g_+(x, 0) + \partial_{Q_+} a_2(x) \right) y + \Omega_+ z + g_+,
\end{align*}
\]

where

\[
f^1_+ = \left( I_n + \partial_x h(x) \right)^{-1} \left( f^1 \circ \Phi - f^1(x, 0) - f^1_+(x, 0) \right) w + \\
\left( Q(x + h) - Q(x) \right) a_1(x) + \left( C(x + h) - C(x) \right) a_2(x),
\]

\[
f^2_+ = \left( I_m + b_1(x) \right)^{-1} \left( f^2 \circ \Phi - f^2(x, 0) - f^2_+(x, 0) \right) w - \\
\partial_{Q_+} a_2(x) + \left( C(x + h) - C(x) \right) a_2(x),
\]

\[
g_+ = \left( \Omega(x + h) - \Omega(x) \right) a_2(x) + \partial_{Q_+} a_2(x) + \\
+ \partial_{Q_+} a_2(x) + \left( C(x + h) - C(x) \right) a_2(x),
\]

with $w = (y, z)$, $\hat{Q}(x) = Q_+(x) - Q(x)$, $\hat{C}(x) = C_+(x) - C(x)$ and $\hat{\Omega}(x) = \Omega_+(x) - \Omega(x)$. Note that we have used $(x, y, z)$ instead of the new variables $(x_+, y_+, z_+)$ in the transformed equations for simplicity.

We hope to find $h(x)$, $a_1(x)$, $a_2(x)$, $b_1(x)$, $b_2(x)$ and $d(x)$ such that

\[
\partial_\omega h(x) = f^1(x, 0) + Q(x)a_1(x) + C(x)a_2(x),
\]

\[
\partial_\omega a_1(x) = f^2(x, 0),
\]

\[
\partial_\omega b_1(x) = f^2_+(x, 0) + \partial_x a_1(x)Q(x),
\]

\[
\partial_\omega a_2(x) = \Omega(x) a_2(x) = g(x, 0),
\]

\[
\partial_\omega b_2(x) - \Omega(x) b_2(x) = g_+(x, 0) + \partial_{Q_+} a_2(x),
\]

\[
\partial_\omega d(x) + d(x) \Omega(x) = f^2(x, 0) + \partial_a a_1(x) C(x).
\]

Then the system (2.12) becomes (2.3).
B. Solving linear homological equations. In the following, we solve the linear homological equations (2.16)-(2.21).

We first solve the equation (2.17). Note that $\|f^2\|_{s,r} \leq \epsilon r$ and $f^2$ is odd function with respect to $x$. Therefore $|f^2| = 0$. Let

$$f^2(x,0) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} f_k^2 e^{\sqrt{-1} \langle k, \omega \rangle}, \quad a_1(x) = \sum_{k \in \mathbb{Z}^n} a_k^1 e^{\sqrt{-1} \langle k, x \rangle}.$$  

Then we have

$$a_k^1 = \frac{f_k^2}{\sqrt{-1} \langle k, \omega \rangle}, \quad k \in \mathbb{Z}^n \setminus \{0\}.$$  

It follows that

$$\|a_1 - a_0^1\|_{s-r} = \| \sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_k^1 e^{\sqrt{-1} \langle k, x \rangle} \|_{s-r} \leq \frac{c \epsilon r}{\alpha \rho^s},$$  

with $a_0^1$ being determined later.

Since the system (2.1) is reversible, we have

$$Q(-x) = Q(x), \quad f^2(-x,0) = -f^2(x,0), \quad \partial_y f^2(-x,0) = -\partial_y f^2(x,0).$$  

By (2.17) it follows that $a_1(-x) = a_1(x)$, which yields that

$$[-\partial_x a_1(\cdot) Q(\cdot) + \partial_y f^2(\cdot,0,0)] = 0.$$  

Similar to solve the equation (2.17), the equation (2.18) has a unique solution $b_1(x) \in B^0_{s-2\rho}$ with

$$\|b_1\|_{s-2\rho} \leq \frac{c}{\alpha \rho^s} \left( \frac{c \epsilon r}{\alpha \rho^s} + \epsilon \right).$$  

Next we solve the equation (2.19). Noting that $\Omega(x) = \Omega + \tilde{\Omega}(x)$, we set $A(x) = \tilde{\Omega}(x) a_2(x)$. Let

$$a_2(x) = \sum_{k \in \mathbb{Z}^n} a_k^2 e^{\sqrt{-1} \langle k, x \rangle}, \quad A(x) = \sum_{k \in \mathbb{Z}^n} A_k e^{\sqrt{-1} \langle k, x \rangle}, \quad g(x,0) = \sum_{k \in \mathbb{Z}^n} g_k e^{\sqrt{-1} \langle k, x \rangle}.$$  

By (2.19) it follows that

$$\sum_{k \in \mathbb{Z}^n} \left( \sqrt{-1} \langle k, \omega \rangle - \Omega \right)^{-1} (g_k + A_k) e^{\sqrt{-1} \langle k, x \rangle} = a_2(x).$$  

Define a mapping $T$ on Banach space $B_s$ by

$$Ta = Lg + L\tilde{\Omega}(x)a.$$  

It is easy to see that $T : B_s \rightarrow B_s$. Then the problem is reduced to finding the fixed point of $T$ on $B_s$. Obviously,

$$\|T a_1 - Ta_2\|_s = \| L\tilde{\Omega}(x)(a_1 - a_2)\|_s \leq \| L\| \cdot \|\tilde{\Omega}(x)\|_s \cdot \|a_1 - a_2\|_s \leq \frac{1 + \delta}{2} \|a_1 - a_2\|_s.$$  

In view of $\frac{1 + \delta}{2} < 1$, by the contraction mapping theorem, there exists a unique fixed point $a$ of the mapping $T$ on $B_s$. Moreover, we have

$$\|a_2\|_s = \|Lg + \tilde{\Omega}(x)a_2\|_s \leq \|Lg\|_s + \frac{1 + \delta}{2} \beta_0 \|a_2\|_s \leq \|L\| \cdot \|g\|_s + \frac{1 + \delta}{2} \|a_2\|_s.$$  

Then

$$\|a_2\|_s \leq \frac{2 \|L\|}{1-\delta} \|g\|_s \leq c \epsilon r.$$
Similarly, we can solve the equations (2.20) and (2.21). Moreover we have the following estimates:

\[ \|b_2\|_s \leq c\varepsilon, \quad \|d\|_s \leq c\varepsilon. \]

Now we consider the equation (2.16). First of all, we choose suitable \( a_\varepsilon \) such that the homological equation (2.16) is also solvable. Let \( Q(x) = \sum_{k \in \mathbb{Z}^n} Q^k e^{2\pi i k(x)} \) and we have

\[ [f^1(\cdot, 0) + Q(\cdot) a_1(\cdot) + C(\cdot)a_2(\cdot)] = Q^0 a_0^1 + \sum_{|k| \neq 0} Q^k a_{-k}^1 + [C(\cdot)a_2(\cdot) + f^1(\cdot, 0)]. \]

Note that \( |Q| - M| = |Q^0 - M| \leq cE_0 \) with rank\( (M) = n \). We write \( M \) and \( Q^0 \) in the block form:

\[ M = (M_1, M_2), \quad Q^0 = (\bar{Q}, \bar{Q}), \]

where \( M_1, M_2, \bar{Q}, \bar{Q} \) are \( n \times n, n \times (m - n), n \times n \) and \( n \times (m - n) \) matrices, respectively. Without loss of generality, we assume that \( M_1 \) is a non-singular matrix. It is easy to see that \( |Q - M_1| \leq cE_0 \). So it follows that \( \bar{Q} \) is also a non-singular matrix if \( E_0 \) is sufficiently small. Note that we already obtained the \( a_\varepsilon^1 (|k| \neq 0) \) and \( a_\varepsilon^2 (x) \) from (2.17) and (2.19). Let

\[ a_0 = -\left( \bar{Q}^{-1} \sum_{|k| \neq 0} Q^k a_{-k}^1 + [C(\cdot)a_2(\cdot) + f^1(\cdot, 0)], 0, \cdots, 0 \right)^T. \]

Then we have \( [f^1(\cdot, 0) + Q(\cdot) a_1(\cdot) + C(\cdot)a_2(\cdot)] = 0 \) and \( |a_0^1| \leq \frac{c\varepsilon}{\alpha^2 \rho^2}. \) Note that \( \|f^1(x, 0) + Q(x) a_1(x) + C(x) a_2(x)\|_{s-\rho} \leq \frac{c\varepsilon}{\alpha^2 \rho^2} \). Then the homological equation (2.16) has a solution \( h(x) \in B_0^{s-2\rho} \) with

\[ \|h_1\|_{s-2\rho} \leq \frac{c\varepsilon}{\alpha^2 \rho^2}. \]

Now we already obtained \( h(x), a_1(x), a_2(x), b_1(x), b_2(x) \) and \( d(x) \) from the homological equations (2.16)-(2.21). Moreover, we have

\[ \|h\|_{s-2\rho} \leq \frac{c\varepsilon}{\alpha^2 \rho^2}, \quad \|a_1\|_{s-\rho} \leq \frac{c\varepsilon}{\alpha \rho^2}, \quad \|a_2\|_s \leq c\varepsilon, \tag{2.22} \]

\[ \|b_1\|_{s-2\rho} \leq \left( \frac{c\varepsilon}{\alpha \rho^2} + \epsilon \right), \quad \|b_2\|_s \leq c\varepsilon, \quad \|d\|_s \leq \frac{c\varepsilon}{\alpha^2 \rho^2+1}. \tag{2.23} \]

Therefore, the transformation \( \Phi \) is well defined by (2.6). The symmetry of (2.7) can be proved in the same way as in [15, 27, 30]. Hence, \( \Phi \) is a compatible transformation.

Set \( s_+ = s - 4\rho, r_+ = \eta \rho \). In view of \( \frac{c\varepsilon}{\alpha^2 \rho^2} \leq cE_0 \rho < \rho \) and \( \frac{c\varepsilon}{\alpha \rho^2} \leq E_0 \frac{\varepsilon}{\rho} \eta \rho < \frac{\varepsilon}{\alpha}, \) it is easy to see that

\[ \Phi : (x, w) \in D(s_+, r_+) \rightarrow (x, w) \in D(s - 3\rho, 2\eta \rho) \subset D(s, r). \]

Moreover, we have

\[ \|\Xi(\Phi - id)\|_{s_+, r_+} \leq cE, \quad \|\Xi(D\Phi - I_{m+n+2p})\Xi^{-1}\|_{s_+, r_+} \leq cE, \]

where \( \Xi = \text{diag}(I_n, \frac{1}{\rho} I_m, \frac{1}{\rho} I_p, \frac{1}{\rho} I_p). \)

\[ C. \text{Estimates of perturbation terms.} \]

By the expressions of \( Q_+, \Omega_+ \) and \( C_+ \), we have

\[ \max \left( \|\tilde{Q}\|_{s_+}, \|\tilde{C}\|_{s_+} \right) \leq \frac{c\varepsilon}{\alpha^2 \rho^2 + 1} < cE, \tag{2.24} \]

\[ \|\tilde{\Omega}(x)\|_{s_+} \leq \frac{c\varepsilon}{\alpha^2 \rho^2 + 1} + c\varepsilon \leq cE. \tag{2.25} \]
Moreover, it is easy to see that \( \|Q_+\|_{s_+} \) and \( \|C_+\|_{s_+} \) and \( \Omega_+ \) are bounded.

Now we first give an estimate of the new perturbation term \( f_+^1 \). By (2.2), (2.13), (2.22) and (2.23), we have
\[
\|f_+^1\|_{s_+, r_+} \leq \frac{c \epsilon_r}{\alpha^2 \rho^{3r+1}} + \|f^1 \circ \Phi(x, w) - f^1(x, 0) - (\partial_w f^1(x, 0), w)\|_{s_+, r_+}. \tag{2.26}
\]
Next we give an estimate of \( \|f^1 \circ \Phi(x, w) - f^1(x, 0) - (\partial_w f^1(x, 0), w)\|_{s_+, r_+} \). Obviously, we have
\[
\|f^1 \circ \Phi(x, w) - f^1(x, 0) - (\partial_w f^1(x, 0), w)\|_{s_+, r_+} = \mathcal{F}^1 + \mathcal{F}^2, \tag{2.27}
\]
where
\[
\mathcal{F}^1 = f^1 \circ \Phi - f^1(x, w),
\]
\[
\mathcal{F}^2 = f^1(x, w) - f^1(x, 0) - (\partial_w f^1(x, 0), w).
\]
By the Cauchy estimates and noting that \( \epsilon = \alpha^3 \rho^{3r+1} E \), it follows that
\[
\|\mathcal{F}^1\|_{s_+, r_+} \leq c \left( \frac{\epsilon_r}{\rho} \cdot \frac{\epsilon_r}{\alpha^2 \rho^{3r}} + \frac{\epsilon_r}{\alpha \rho^r} \right) \leq c E \epsilon r \text{ and } \|\mathcal{F}^2\|_{s_+, r_+} \leq \eta^2 \epsilon r.
\]
In view of \( \eta = E^{\frac{1}{2}} \), we have
\[
\|f^1 \circ \Phi - f^1(x, 0) - (\partial_w f^1(x, 0), w)\|_{s_+, r_+} \leq c \eta^2 \epsilon r.
\]
By (2.26) we have
\[
\|f_+^1\|_{s_+, r_+} \leq c \eta^2 \epsilon r.
\]
Now we estimate the new perturbation term \( f_+^2 \) and \( g_+ \). Similar to the above estimates of \( f_+^1 \), it is easy to see that \( \|(I_m + b_1)^{-1}\|_{s_+, r_+} \leq 2 \) if \( E \) is sufficiently small. Combining (2.2), (2.14), (2.15), (2.22), (2.23), (2.24) and (2.25), it follows that
\[
\|f_+^2\|_{s_+, r_+} \leq 2 \left\| f^2 \circ \Phi(x, w) - f^2(x, 0) - f_+^2(x, 0) w \right\|_{s_+, r_+} + c \eta^2 \epsilon r + c E \left( \|f_+^1\|_{s_+, r_+} + \|g_+\|_{s_+, r_+} \right)
\]
\[
\|g_+\|_{s_+, r_+} \leq \left\| g \circ \Phi(x, w) - g(x, 0) - g_w(x, 0) w \right\|_{s_+, r_+} + c \eta^2 \epsilon r + c E \left( \|f_+^1\|_{s_+, r_+} + \|f_+^2\|_{s_+, r_+} \right).
\]
In the same way as (2.27), we have
\[
\|f_+^2\|_{s_+, r_+} \leq c \eta^2 \epsilon r + c E \|g_+\|_{s_+, r_+}
\]
\[
\|g_+\|_{s_+, r_+} \leq c \eta^2 \epsilon r + c E \|f_+^2\|_{s_+, r_+}.
\]
Noting that \( E \) is sufficiently small, it follows easily that
\[
\|f_+^2\|_{s_+, r_+} \leq c \eta^2 \epsilon r, \quad \|g_+\|_{s_+, r_+} \leq c \eta^2 \epsilon r.
\]
Therefore, we have
\[
\|f_+^1\|_{s_+, r_+} = \sum_{i=1}^4 \|f_+^i\|_{s_+, r_+} \leq c \eta \epsilon r_+ = \alpha^3 \rho^{3r+1} c E^2 r_+ = \alpha^3 \rho^{3r+1} E r_+ = \epsilon r_+,
\]
where \( E_+ = c \epsilon^2 \). Thus, Lemma 2.1 is proved. \( \Box \)
2.2. Convergence of iteration. Now we choose some suitable parameters so that the above iteration can go on infinitely. At the initial step, we set \( Q_0(x) = Q(x) \), \( M = [Q_0], C_0(x) = C(x), \tilde{\Omega}_0(x) = \Omega(x), f_0^1 = f^1, f_0^2 = f^2, g_0 = (f^3, f^4)^T, s_0 = s, E_0 = 16^{3r+1} \gamma \) and \( \epsilon_0 = \alpha^3 \rho_0^{3r+1} E_0 \). Let
\[
 s_j = s_0 \left( \frac{1}{2} + \left( \frac{1}{2} \right)^{j+1} \right), \quad \rho_j = \frac{s_j - s_{j+1}}{4}, \quad \eta_j = E_j^{\frac{j}{2}}, \quad r_{j+1} = \eta_j r_j, \quad E_{j+1} = c E_j^{\frac{3}{2}}, \quad \epsilon_{j+1} = \alpha^3 \rho_j^{3r+1} E_{j+1}.
\]

Then, it is easy to see that \( s_j, r_j, \rho_j, E_j, \epsilon_j \) are all well defined for \( j \geq 0 \). In the following we are going to check all assumptions in the iteration lemma 2.1 to ensure that KAM steps are valid for all \( j \geq 0 \).

By the definition of \( E_j \), we have \( E_j \leq (c^3 E_0)^{\frac{2}{3}} \). By (2.5), if \( c^3 E_0 < \frac{1}{2} \), it follows that
\[
 \| Q_j - Q_0 \|_{s_j} \leq \sum_{\nu=0}^{j-1} \| Q_{\nu+1} - Q_\nu \|_{s_{\nu+1}} \leq c \sum_{\nu=0}^{j-1} E_\nu \leq 2c \sum_{\nu=0}^{\infty} E_\nu \leq 2c E_0
\]
for \( \forall j \geq 1 \). Similarly, we obtain that
\[
 \| C_j - C_0 \|_{s_j} \leq 2c E_0, \quad \| \tilde{\Omega}_j - \tilde{\Omega}_0 \|_{s_j} \leq 2c E_0 \quad \text{for} \quad \forall j \geq 0.
\]

Noting that \( \| \mathcal{L} \| \cdot \| \tilde{\Omega}_0 \|_{s_0} \leq \delta < 1 \) and \( M = [Q_0] \), then, it is easy to see that if \( 2c E_0 \leq \frac{1}{2} \), it is sufficiently small, the assumptions that
\[
 \| \mathcal{L} \| \cdot \| \tilde{\Omega}_j \|_s \leq \frac{\delta + 1}{2} < 1, \quad \| [Q_j] - M \| \leq c E_0, \quad \| Q_j \|_s \leq T_0 + 1, \quad \| C_j \|_s \leq T_0 + 1
\]
holds for all \( j \geq 0 \).

By the KAM-step, we have a compatible transformation
\[
 \Phi_j(\cdot, \cdot, \cdot, \cdot) : D(s_{j+1}, r_{j+1}) \rightarrow D(s_j, r_j).
\]

Let \( \Phi_j = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{j-1} \) with \( j \geq 1 \) and \( \Phi_0 = id \). With the aid of (2.4), we can prove that the transformation \( \Phi_j \) is convergent to \( \Phi_* \) on \( D(s/2, r/2) \). The proof is the same as in the case of Hamiltonian systems (in fact simpler), so we omit the details and refer the reader to [19, 20]. By (2.5) and in view of \( E_j \rightarrow 0 \) as \( j \rightarrow \infty \), we have \( Q_j \rightarrow Q_\infty, C_j \rightarrow C_\infty \) and \( \tilde{\Omega}_j \rightarrow \tilde{\Omega}_\infty \) as \( j \rightarrow \infty \). Then it is easy to see that \( \Phi_* \) transforms the reversible system (2.1) into the following form:
\[
 \begin{align*}
 \dot{x} &= \omega + Q_\infty(x)y + C_\infty(x)z + f^1_\infty(x, y, z), \\
 \dot{y} &= f^2_\infty(x, y, z), \\
 \dot{z} &= \tilde{\Omega}_\infty(x)z + g_\infty(x, y, z).
\end{align*}
\]

Set \( f^3_* = Q_\infty(x)y + C_\infty(x)z + f^1_\infty(x, y, z), f^2_* = f^2_\infty \) and
\[
 \begin{pmatrix}
 f^3_* \\
 f^4_*
 \end{pmatrix}
 = \tilde{\Omega}_\infty(x) 
 \begin{pmatrix}
 u \\
 v
 \end{pmatrix} + \begin{pmatrix}
 f^3_\infty \\
 f^4_\infty
 \end{pmatrix}.
\]

Noting that \( \epsilon_j \rightarrow 0 \) as \( j \rightarrow \infty \), it is easy to see that \( f^j_\infty(x, 0, 0, 0) = 0, (j = 1, 2, 3, 4) \). This completes the proof of Theorem 1.1.
3. **Proof of Theorem 1.2.** By compatible transformations, we prove that the reversible system (1.15) can be reduced to a suitable normal form which Theorem 1.1 can be applied.

Define a compatible transformation \( \Phi_1 \) by

\[
x = x_+, \ y = y_+, \ u = u_+, \ v = v_+ + v_\epsilon.
\]

the system 1.1 is transformed into

\[
\begin{align*}
\dot{x} &= \omega + Q(x)y + f^1(x, y, u, v, \epsilon) \\
\dot{y} &= f^2(x, y, u, v, \epsilon) \\
\dot{u} &= Av + f^3(x, y, u, v, \epsilon), \\
\dot{v} &= Bu + f^4(x, y, u, v, \epsilon),
\end{align*}
\]

(3.1)

where \( f^j = \epsilon P^j(x, y, u, v + v_\epsilon, \epsilon) (j = 1, 2, 4) \) and \( f^3 = \epsilon P^3(x, y, u + v_\epsilon, \epsilon) + Av_\epsilon \).

Let

\[
Q(x) = \left( Q(x), 0_{n \times 2p} \right), \ w = \begin{pmatrix} y \\ u \end{pmatrix}, \ A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix}, \ G = \begin{pmatrix} f^2 \\ f^3 \\ f^4 \end{pmatrix}.
\]

Then system (3.1) is written as

\[
\dot{x} = \omega + Q(x)w + f^1(x, w, \epsilon), \ \dot{w} = Aw + G(x, w, \epsilon), \quad (3.2)
\]

where \( f^1 = O(\epsilon) \) and \( G = O(\epsilon) \). Let \( \Phi_2 : (x_+, z_+) \rightarrow (x, z) \) be defined by

\[
\begin{align*}
x &= x_+ + h(x_+), \\
y &= y_+ + a_1(x_+) + b_{11}(x_+)y_+ + b_{12}(x_+)u_+ + b_{13}(x_+)v_+, \\
u &= u_+ + a_2(x_+) + b_{21}(x_+)y_+ + b_{22}(x_+)u_+ + b_{23}(x_+)v_+, \\
v &= v_+ + a_3(x_+) + b_{31}(x_+)y_+ + b_{32}(x_+)u_+ + b_{33}(x_+)v_+.
\end{align*}
\]

(3.3)

Denote \( w_+ = (y_+, u_+, v_+)^T \). \( \Phi_2 \) is written in a more compact form:

\[
x = x_+ + h(x_+), \ w = w_+ + a(x_+) + b(x_+)w_+,
\]

(3.4)

where

\[
a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad b = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.
\]

Let \( S = \text{diag}(I_m, -I_p, I_p) \). It is easy to see that \( \Phi_2 \) is compatible with the involution \( G \) if and only if

\[
h(-x) = -h(x), \quad Sa(-x) = a(x), \quad Sb(-x)S = b(x). \quad (3.5)
\]

Under the transformation \( \Phi_2 \) the system (3.2) is changed into

\[
\begin{align*}
\dot{x} &= \omega + \left( I_n + \partial_x h(x) \right)^{-1} \left( -\partial_x h(x) + f^1(x, 0) + Q(x)a(x) \\
&\quad + (Q(x + h_1) + b(x)) + f^1_+(x, 0) \right) + f^1_+(x, w), \\
\dot{w} &= A_+ w + \left( I_{m+2p} + b(x) \right)^{-1} \left( \dot{u} = A_+ w + \left( I_{m+2p} + b(x) \right)^{-1} \left( -\partial_x a(x) + Aa(x) + G(x, 0) \right) - [G(-x, 0)] \right)
\end{align*}
\]

(3.6)

+ \left( \dot{u} = A_+ w + \left( I_{m+2p} + b(x) \right)^{-1} \left( -\partial_x a(x) + Aa(x) + G_w(x, 0) - \partial Q a(x) - A \right) \right) w + \mathcal{G}_+(x, w),
\]
where \( \mathcal{A}_+ = \mathcal{A} + \hat{\mathcal{A}} \) with \( \hat{\mathcal{A}} \) being decided later, \( f^1_w(x, 0) = \frac{\partial f^1_w}{\partial w}|_{w=0} \), \( \mathcal{G}_w(x, 0) = \frac{\partial \mathcal{G}_w}{\partial w}|_{w=0} \) and

\[
f^1_+(x, w) = \left( I_n + \partial_x h(x) \right)^{-1} \left( f^1 \circ \Phi(x, w) - f^1(x, 0) - f^1_w(x, 0)w \right) + (Q(x + h) - Q(x))a(x), \tag{3.7}
\]

\[
\mathcal{G}_+(x, w) = \left( I_{m+2p} + b(x) \right)^{-1} \left( \mathcal{G} \circ \Phi(x, w) - \mathcal{G}(x, 0) - \mathcal{G}_w(x, 0)w \right) + [\mathcal{G}(\cdot, 0)] - \partial_{Q_w + f^1_+} a(x) - \partial_{Q_w + f^1_+} b(x)w + b(x)\hat{A}w, \tag{3.8}
\]

where \( \hat{Q}(x) = Q_+(x) - Q(x) \). Note that we have used \( (x, w) \) instead of the new variables \( (x_+, w_+) \) in the transformed equations for simplicity. Let

\[
Q_+(x) = \left( I_n + \partial_x h(x) \right)^{-1} \left( Q(x + h)(1 + b(x)) + f^1_w(x, 0) \right).
\]

We hope to find \( h(x), a(x) \) and \( b(x) \) such that

\[
\begin{align*}
\partial_w h(x) &= f^1(x, 0) + Q(x)a(x), \tag{3.9} \\
\partial_w a(x) - Aa(x) &= \mathcal{G}(x, 0) - [\mathcal{G}(\cdot, 0)], \tag{3.10} \\
\partial_w b(x) - Ab(x) &= b(x)A = \mathcal{G}_w(x, 0) - \partial_Q a(x) - \hat{A}. \tag{3.11}
\end{align*}
\]

Then the system \((3.6)\) becomes

\[
\dot{x} = \omega + Q_+(x)w + f^1_+(x, w, \epsilon), \quad \dot{w} = \mathcal{A}_+w + \mathcal{G}_+(x, w, \epsilon). \tag{3.12}
\]

Since the system \((2.1)\) is reversible we have

\[
[\mathcal{G}(\cdot, 0)] = \begin{pmatrix} 0_{m \times 1} \\ K_x \\ 0_{p \times 1} \end{pmatrix}
\]

with \( K_x = A\epsilon + \epsilon[P^3(\cdot, 0_m \times p, \nu_x, \epsilon)]_{p \times 1} = O(\epsilon^2) \). Then we have \( [\mathcal{G}(\cdot, 0)] = O(\epsilon^2) \). Thus, it follows that \( f^1_+ = O(\epsilon^2) \) and \( \mathcal{G}_+ = O(\epsilon^2) \) are much smaller perturbations than before.

By \((1.18)\) and in the same way as in [15, 27, 35], it follows that the equation \((3.10)\) has a solution \( a(x) \in \mathcal{B}_{s-p}^0 \) with \( a(x) = O(\epsilon) \). Next we choose suitable \( a \) such that \( [f^1(\cdot, 0) + Q(\cdot)a(\cdot)] = 0 \). Therefore, the homological equation \((3.9)\) has a solution \( h(x) \in \mathcal{B}_{s-2p}^0 \) with \( h(x) = O(\epsilon) \).

For \(|k| = 0\) solving the equation \((3.11)\) is more complicated. Noting the system \((2.1)\) is reversible and \( Q = (Q, 0_{m \times 2p}) \), we have

\[
[\mathcal{G}_w(\cdot, 0)] = \epsilon \begin{pmatrix} 0_{M \times N_{m \times p}} & 0_{N_{m \times p} \times p} & 0_{p \times 0} \\ 0_{M \times 0} & 0_{0 \times N_{m \times p}} & 0_{p \times 0} \\ 0_{M \times 0} & 0_{0 \times N_{m \times p}} & 0_{p \times 0} \end{pmatrix}
\]

and

\[
[\partial_Q(\cdot)a(\cdot)] = \epsilon \begin{pmatrix} 0_{M \times 0} & 0_{0 \times N_{m \times p}} & 0_{p \times 0} \\ 0_{M \times 0} & 0_{0 \times N_{m \times p}} & 0_{p \times 0} \end{pmatrix}.
\]

Set

\[
\hat{A} = \epsilon \begin{pmatrix} 0_{M \times M_{p \times m}} & 0_{M_{p \times m} \times p} & 0_{p \times 0} \\ 0_{M \times 0} & 0_{0 \times N_{m \times p}} & 0_{p \times 0} \\ 0_{M \times 0} & 0_{0 \times N_{m \times p}} & 0_{p \times 0} \end{pmatrix}
\]
If \( \det B = 0 \), by (1.17) it follows that
\[
\epsilon N_{m \times p} = \epsilon \left[ \frac{\partial P^2(\cdot, 0_{m+p}, v, \epsilon)}{\partial z} \right] = O(\epsilon^2).
\]
We put \( \epsilon N_{m \times p} \) into the small perturbation term \( f_k^2 \). If \( \det B \neq 0 \), let
\[
[b] = \epsilon \begin{pmatrix} 0 & 0 & B^{-1}N \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Then we have
\[
-A[b] + [b]A = \epsilon \begin{pmatrix} 0 & N & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
If \( |k| \neq 0 \), in the same way as in [15, 27, 35], it follows that the equation (3.11) has a solution \( b(x) \in B_{s-2p}^0 \) with \( b(x) = O(\epsilon) \). In the same way as in [15, 27, 30], we can prove that \( \Phi_2 \) is a compatible transformation. Then the reversible system (3.12) is written as
\[
\begin{align*}
\dot{x} &= \omega + \left( Q(x)y + O(\epsilon)w \right) + f_+^1(x, y, u, v, \epsilon), \\
\dot{y} &= f_+^2(x, y, u, v, \epsilon), \\
\dot{u} &= \epsilon(M - \tilde{M})y + (A + \epsilon E)v + f_+^3(x, y, u, v, \epsilon), \\
\dot{v} &= (B + \epsilon F)u + f_+^4(x, y, u, v, \epsilon),
\end{align*}
\]
(3.13)
where \( f_+^i = O(\epsilon^2)(i = 1, 2, 3, 4) \). Define a compatible transformation \( \Phi_3 \) by
\[
x = x_+, \quad y = y_+, \quad u = u_+, \quad v = v_+ - \epsilon(A + \epsilon E)^{-1}(M - \tilde{M})y_+.
\]
Under the transformation \( \Phi_3 \), the system (3.13) is changed to new system
\[
\begin{align*}
\dot{x} &= \omega + \left( Q(x)y + O(\epsilon)w \right) + f_{new}^1(x, y, u, v, \epsilon), \\
\dot{y} &= f_{new}^2(x, y, u, v, \epsilon), \\
\dot{u} &= (A + \epsilon E)v + f_{new}^3(x, y, u, v, \epsilon), \\
\dot{v} &= (B + \epsilon F)u + f_{new}^4(x, y, u, v, \epsilon),
\end{align*}
\]
where \( f_{new}^i = O(\epsilon^2)(i = 1, 2, 3, 4) \). Moreover, we have
\[
\Omega_\epsilon = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} + \epsilon \left[ \frac{\partial (P^3, P^4)}{\partial (u, v)} \right]_{y=0, u=0, v=v_\epsilon} = \begin{pmatrix} 0 & A + \epsilon E_{p \times p} \\ B + \epsilon F_{p \times p} & 0 \end{pmatrix}.
\]
Define a linear operator \( \mathcal{L}_{\epsilon} \) on \( B_s \) by
\[
\mathcal{L}_{\epsilon} \left( h(x) \right) = \sum_{k \in \mathbb{Z}^n} \left( \sqrt{-1} \langle k, \omega \rangle - \Omega_\epsilon \right)^{-1} h_k e^{\sqrt{-1} \langle k, x \rangle}
\]
with \( h(x) \in B_s \). By the assumption (iii) in Theorem 1.2, we have
\[
\| \mathcal{L}_{\epsilon} \| \leq \frac{c}{\sigma \epsilon}.
\]
Obviously, the shifts of normal matrix \( \tilde{\Omega}(x) \) satisfies \( \| \tilde{\Omega}(x) \|_s = O(\epsilon^2) \). Thus, the assumption
\[
\| \mathcal{L}_{\epsilon} \| \cdot \| \tilde{\Omega} \|_s \leq \epsilon c < 1
\]
holds in KAM steps. In the same way as Theorem 1.1, we can prove Theorem 1.2.
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