On the overall polarisation properties of Poincaré beams

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Abstract
We analyse the polarisation properties of Poincaré beams. We consider different configurations, such as Laguerre–Poincaré (LP), Bessel–Poincaré (BP), and Lambert–Poincaré (LaP) beams. The former considers the well-known cylindrical vector beams and full-Poincaré beams produced by a collinear superposition of two Laguerre–Gauss beams with orthogonal polarisations. For this configuration, we describe the Stokes statistics and overall invariant parameters. Similarly, BP beams are produced by the collinear superposition of Bessel beams with orthogonal polarisations. We describe their properties under propagation and show that they behave as a free-space polarisation attractor transforming elliptical polarisations to linear polarisations. We also propose a novel type of full Poincaré pattern, one which is generated by a Lambert projection of the Poincaré sphere on the transverse plane, and hence we call them LaP. This configuration, contrary to the LP, provides a finite region containing all polarisation states uniformly distributed on the Poincaré sphere.

Keywords: Laguerre–Gauss, Bessel–Gauss, polarisation, Stokes statistics, Poincaré sphere, full Poincaré beams

1. Introduction
Poincaré beams refer to light beams with space-variant polarisation distribution [1, 2]. They can be visualised as the mapping of points on the Poincaré sphere onto points in the transverse plane. Poincaré beams containing all states of polarisation are known as full-Poincaré (FP) beams [3–6]. Their polarisation properties have attracted attention in beam shaping and singular optics, where they can be used to generate complex topologies such as Möbius strips [7]. Other studies analyse the optical forces arising from the curl of the spin angular momentum [8]. They can be applied in laser cutting, where the flat-top intensity is used to realise a clean cut [9]. Potential applications have been shown for quantum communications [10], polarimetry measurements [11, 12], and polarisation speckles [13].

FP beams were originally realised by Beckley et al [3]. They are formed by a coherent and collinear superposition of two Laguerre–Gauss (LG) beams with either linear or circular orthogonal polarisations. For the circular basis, two distinct polarisation singularities can be generated, that is, lemon and star singularities. Higher-order FP beams were studied by Galvez et al [4], where superpositions of higher-order LG beams were used to produce the mapping of multiple Poincaré spheres on the transverse plane. Other types of FP beam include the Bessel–Poincaré (BP) studied by Shvedov et al [5] and Holmes et al [6], which is formed by a superposition of Bessel beams. Similarly, there are Mathieu–Poincaré beams, which are formed by using nondiffracting Mathieu beams [14].

In this work, we describe interesting polarisation properties regarding Poincaré beams. We consider the polarisation structures derived from the superposition of two LG modes with orthogonal polarisations, such as cylindrical vector [15]...
and FP beams. We call them Laguerre–Poincaré (LP) to distinguish them from the other structures of interest, which are BP and Lambert–Poincaré (LP) beams. The latter is realised through a Lambert projection of the Poincaré sphere on the transverse plane. Section 2 describes the Stokes statistics that we use to describe the overall polarisation properties of the Poincaré beams. Inspired by the work of Martínez-Herrero et al. [16], we found an invariant parameter that relates the local degree of polarisation (LDoP) with the averaged Stokes parameters. We use this formalism to study the properties of LP beams in section 3. We describe the Stokes variances according to the azimuthal and radial indices and show that higher-order LP beams are predominantly circularly polarised. Furthermore, we study their propagation and describe the evolution of the transverse polarisation distribution as a rotation of the Poincaré sphere. Similarly, section 4 studies the propagation of BP beams and show that they behave as a polarisation attractor [17], which transforms elliptical polarisation states to linear polarisations. Finally, section 5 explains the realisation of the LaP polarisation pattern, which provides a uniform distribution of polarisation states on the Poincaré sphere [18].

2. Global parameters and invariants for the characterisation of space-variant polarised beams

In this section, we briefly describe the Stokes statistics in order to establish basic formulas and notation. The Stokes parameters, $S_j$ with $j \equiv 0, 1, 2, 3$, are determined by six power measurements. Each power measurement is realised after the beam passes through an ideal polariser. The Stokes parameters are defined as [19]

$$ S = (P, Q, U, V) = (P_H + P_V, P_H - P_V, P_{135} - P_{45}, P_R - P_L), $$

where $P$ is equal to the total power of the beam. $Q$ is the power transmitted after a horizontal polariser ($P_H$) minus the power transmitted through a vertical polariser ($P_V$). Similarly, $U$ is the power transmitted after a $45^\circ$ polariser ($P_{45}$) minus the power after a $135^\circ$ polariser ($P_{135}$). Finally, $V$ measures the difference in power between the right-handed circular component ($P_R$) and the left-handed circular component ($P_L$). We label these global Stokes parameters as $P$, $Q$, $U$, $V$ to distinguish them from the space-dependent Stokes parameters defined later.

For space-variant polarised beams, e.g. Poincaré beams, each power measurement is equal to the spatial integration of their respective intensities. For instance

$$ P = P_H + P_V = \int \left( I_H(x, y) + I_V(x, y) \right) dx dy = \int I(x, y) dx dy, $$

where $I(x, y)$ is the total intensity of the beam, $I_H(x, y)$ and $I_V(x, y)$ are the horizontal and vertical intensity components, respectively. Correspondingly, $I_{45}$, $I_{135}$, $I_R$ and $I_L$ are the intensity components for $P_{45}$, $P_{135}$, $P_R$ and $P_L$. The intensities are measured with a CCD camera whereas the power or flux of energy is measured using a photodetector. To keep the notation simpler, we will disregard the space dependence $(x, y)$ unless is necessary for clarity.

We define the normalised and space-variant Stokes parameters

$$ s_1(x, y) = (I_H - I_V) / I, $$

$$ s_2(x, y) = (I_{45} - I_{135}) / I, $$

$$ s_3(x, y) = (I_R - I_L) / I. $$

Inspired by the work of Martínez-Herrero et al. [16], we consider in our analyses the weighted average of the normalised Stokes parameters, which are given by

$$ \langle s_j \rangle = \frac{\int \int s_j(x, y) I(x, y) dx dy}{P}, $$

with $j \equiv 1, 2, 3$. Notice that $(x, y)$ in this definition is equivalent to a density function. In addition, the variance $\langle \sigma_j^2 \rangle$ is defined as

$$ \langle \sigma_j^2 \rangle = \frac{\int \int (s_j(x, y) - \langle s_j \rangle)^2 I(x, y) dx dy}{P}. $$

We notice that Martínez-Herrero et al. used these definitions to characterise space-variant polarised beams in terms of one or two Stokes parameters [16, 20]. However, in this work, we show that considering the three Stokes parameters provide information regarding the distribution of polarisation states over the Poincaré sphere.

In what follows, we will repeatedly use the definition

$$ \langle A \rangle = \frac{\int \int A(x, y) I(x, y) dx dy}{P}, $$

with $A \equiv s_j$, $s_j^2$, $\sigma_j^2$, etc. By expanding equation (7), we can write $\langle \sigma_j^2 \rangle$ in the alternative form

$$ \langle \sigma_j^2 \rangle = \langle s_j^2 \rangle - \langle s_j \rangle^2. $$

We now consider the contribution from the three Stokes parameters. Let us define

$$ \Omega = \sum_{j=1}^{3} \langle \sigma_j^2 \rangle = \sum_{j=1}^{3} (\langle s_j^2 \rangle - \langle s_j \rangle^2), $$

as the total variance of polarisation states. We observe that the first term on the right-hand side

$$ \sum_{j=1}^{3} \langle s_j^2 \rangle = \frac{\int \int (s_1^2 + s_2^2 + s_3^2) I(x, y) dx dy}{P}, $$

is equal to the expected value of the squared of the LDoP, defined as (see [16])

$$ \text{LDoP}(x, y) = \sqrt{s_1^2(x, y) + s_2^2(x, y) + s_3^2(x, y)}. $$

Thus, we have that

$$ \sum_{j=1}^{3} \langle s_j^2 \rangle = \langle \text{LDoP}^2 \rangle. $$

Then, the addition of the squared of the average values, turns out to be equal to the squared of the global degree of
polari...zation (DoP), i.e.

\[ \sum_{j=1}^{3} (s_j)^2 = \frac{Q^2 + U^2 + V^2}{P^2} = \text{DoP}^2. \]  

Therefore, equation (10) can be written as

\[ \Omega = (L\text{DoP}^2) - \text{DoP}^2. \]  

By knowing that the global DoP is invariant under propagation through non-polarising optical elements [19], it is convenient to write the previous equation as

\[ \text{DoP}^2 = (L\text{DoP}^2) - \Omega. \]  

Equation (16) is the first important result of this work. It helps to characterise space-variant polarised beams. The DoP describes the overall state of polarisation whereas the variances describe the spread of the polarisation states on the principal Stokes axes. The DoP has an intuitive interpretation. If we graph the polarisation states on the Poincaré sphere, the DoP resembles the concept of center of mass. Therefore, if the polarisation states span the Poincaré sphere or one of its great circles, it results that DoP = 0, as is the case for FP and cylindrical vector beams. The variances give information about the predominant state of polarisation according to the intensity of the beam. For example, values of \( (\sigma_1^2) \approx (\sigma_2^2) \approx (\sigma_3^2) \) means that the beam is circularly polarised in regions where the intensity is more significant.

In the examples that follow, we consider temporally and spatially coherent light beams (e.g. monochromatic laser beams). It means that the light is locally polarised and hence \( L\text{DoP} = 1 \). Therefore \( \text{DoP}^2 + \Omega = 1 \), which is written as

\[ (\sigma_1^2) + (\sigma_2^2) + (\sigma_3^2) = 1 - \text{DoP}^2. \]  

Figure 1 shows examples of the application of the previous equation. We can see that for homogeneously polarised beams DoP = 1 and hence \( (\sigma_1^2, \sigma_2^2, \sigma_3^2) = 0 \). For FP and cylindrical vector beams DoP = 0. Cylindrical vector beams span the equator of the Poincaré sphere, meaning that \( (\sigma_1^2) = 0 \) and \( (\sigma_2^2) = (\sigma_3^2) = 1/2 \) (see figure 1(b)). Figure 1(c) shows a LaP beam which uniformly covers the Poincaré sphere and hence \( (\sigma_1^2) = (\sigma_2^2) = (\sigma_3^2) = 1/3 \).

3. Polarisation properties of LP beams

Under the paraxial approximation, an LG beam is written in a cylindrical coordinate system \( r = (r, \phi, z) \) as [21]

\[
L_n^m(r, \phi, z) = \sqrt{\frac{2(2n!)}{\pi(n + |m|)!}} \frac{1}{r^{|m|}} \frac{\sqrt{2r}}{w_c} \exp \left( -\frac{r^2}{2w_c^2} \right) \times L_n^{|m|} \left( \frac{2r^2}{w_c^2} \right) \exp \left( i \frac{k r^2}{2 R_z} \right) \exp (-i \Phi) \exp (i m \phi),
\]

where \( n \) and \( m \) are the radial and azimuthal quantum numbers, respectively, and they also determine the order \( n \) and degree \( m \) of the generalised Laguerre polynomial \( L_n^{|m|} \). Moreover,

\[
z_R = \frac{1}{2} k w_0^2,
\]

\[
w_c = w_0 \sqrt{1 + \left( z/z_R \right)^2},
\]

\[
R_z = \frac{z^2 + z_R^2}{z},
\]

\[
\Phi = (2n + |\ell| + 1) \arctan(z/z_R),
\]

where \( z_R \) is the Rayleigh range, \( k = 2 \pi / \lambda \) is the wave number, \( w_c \) is the beam waist (so \( w_0 \) is the beam waist at \( z = 0 \)), \( R_z \) is the radius of curvature, and \( \Phi \) is the Gouy phase. Furthermore, equation (18) is normalised such that \( \int \int |LG|^2 \, dx \, dy = 1 \).

LP beams are produced with a coherent and collinear superposition of LG beams with orthogonal polarisations. We consider a circular polarisation basis and use the unit vectors \( \hat{e}_R = (\hat{x} - i \hat{y}) / \sqrt{2} \) and \( \hat{e}_L = (\hat{x} + i \hat{y}) / \sqrt{2} \), which correspond to the right-handed and left-handed circular polarisations, respectively. Thus, an LP beam with a circular polarisation basis is given by [4]

\[
\text{LP} = \frac{1}{\sqrt{2}} (L_n^m \hat{e}_R + L_n^m \hat{e}_L),
\]

where we consider that both components can have different quantum numbers [22]. Another polarisation basis can be used to generate LP beams, whose difference is observed in the transverse distribution of the polarisation states. The factor of \( 1 / \sqrt{2} \) ensures that \( \int \int |\text{LP}|^2 \, dx \, dy = 1 \). Notice that equation (23) does not span all polarisation states for all combinations of \( m_R, m_L, n_R, n_L \). In fact, cylindrical vector beams are obtained with \( m_R = -m_L \) and \( n_R = n_L \) [15]. Higher-order cylindrical vector beams correspond to \( |m_R| > 1 \) (or \( |m_L| > 1 \)). FP beams are obtained with \( |m_R| = |m_L| \) and is independent of the relation between the radial indices. First-order FP modes correspond to the cases with \( n_R = n_L = 0 \) and \( |m_R| - |m_L| = \pm 1 \) [3]. Finally, higher-order FP modes correspond to the cases where \( n_R + n_L > 0 \) [9].
Figure 2 shows the properties for different LP beams with both polarisation components carrying a zero radial quantum number (i.e. \( n_R = n_L = 0 \)). The first, second and third columns show the transverse polarisation distribution, the polarisation states visualised on the Poincaré sphere and the Stokes variances (see equation (7)), respectively. The calculations were done at the plane \( z = 0 \) with \( w_0 = 4 \) mm and \( \lambda = 633 \) nm. For the transverse polarisation distributions, the gray colormap represents the intensity, whereas the red (blue) ellipses represent polarisation states with right (left) handedness. The graph of the polarisation states on the Poincaré sphere is realised by mapping each pixel of the transverse plane on a unit sphere using the normalised Stokes parameters (see equations (3)–(5)).

Figure 2(a) shows a first-order LP beam with a star singularity \((m_R = 0, \ m_L = 1)\). From the transverse pattern, notice that the north hemisphere, the one containing right-handed polarisation states, is located where the intensity of the beam is significant. Specifically, the north hemisphere has about 90 percent of the total power (it was calculated at the radial position \( r = w_0/\sqrt{2} \) that satisfies \( s_3 = 0 \), which is the location of the equator). Therefore, the left-handed hemisphere covers a region with practically null intensity. This causes \( \langle \sigma_3^2 \rangle \) to be less significant than \( \langle \sigma_1^2 \rangle \) and \( \langle \sigma_2^2 \rangle \), as shown in the third column.

Figures 2(b) and (c) show the results for higher-order LP beams with \((m_R, \ m_L) = (0, 2)\) and \((m_R, \ m_L) = (0, 3)\) (see [23]). Since the LG beam radius increases with the azimuthal quantum number due to the factor \( (\sqrt{2} r/\omega_c)^m \) in equation (18), the overlap between the two LG components decreases for high values of \( m_R \). Therefore, the circular polarisation states are dominant for higher-order LP beams. Notice the accumulation of polarisation states on the poles of the Poincaré sphere. Similarly, the value of \( \langle \sigma_3^2 \rangle \) becomes more significant than \( \langle \sigma_1^2 \rangle \) and \( \langle \sigma_2^2 \rangle \).

Due to the circular polarisation basis, the results discussed in the previous paragraphs satisfy \( \sum \langle \sigma_i^2 \rangle = \langle \sigma_3^2 \rangle \). Furthermore, notice that in all cases \( \sum \langle \sigma_i^2 \rangle = 1 \) is satisfied. Of course, we can infer similar conclusions according to the polarisation basis that we use. For example, if we use a linear polarisation basis
\[ \hat{x}, \hat{y} \text{ it is expected that } \langle \sigma_2^x \rangle = \langle \sigma_2^y \rangle, \text{ and the polarisation states on the Poincaré sphere will accumulate on } s_1 = \pm 1 \text{ for higher-order modes (and therefore } \langle \sigma_1^z \rangle > \langle \sigma_2^z \rangle). \]

To extend our previous analysis, figure 3 shows more combinations of \( m_R, m_L \) including negative values. The cases with \( m_R = -m_L \), where \( \langle \sigma_1^z \rangle = \langle \sigma_2^z \rangle = 1/2 \) correspond to cylindrical vectors beams. The results confirm that \( \langle \sigma_1^z \rangle = \langle \sigma_2^z \rangle \) and the predominant variance is \( \langle \sigma_3^z \rangle \) for values of \( |m_L| \gg |m_R| \) (or vice versa \( |m_R| \gg |m_L| \)).

Next, we study the propagation effects on the Stokes statistics. It has been shown that under propagation the first-order LP beams experience a rotation of their polarisation states equivalent to a rotation of the Poincaré sphere. This rotation depends on the polarisation basis and is due to the Gouy phase (see equation (22)). For circular polarisation basis, the rotation is with respect to the s3 axis (which are the poles of the Poincaré sphere). Since \( \langle \sigma_1^z \rangle = \langle \sigma_2^z \rangle \) we can expect that this rotation does not change the values of the variances, which is only true for LP beams that cover the Poincaré sphere or its equator. Thus, the rotation does not change the polarisation state distribution on the sphere. Nevertheless, there are cases where the polarisation states cover a great circle of the Poincaré sphere other than the equator. For these cases, we can infer that the rotation will change the values of \( \langle \sigma_1^z \rangle \) and \( \langle \sigma_2^z \rangle \). To show this effect, figure 4 shows the Stokes variances for an LP beam with azimuthal quantum numbers \( m_R = m_L = 0 \) and radial numbers \( n_R = 3, n_L = 2 \). The plot shows the variances as a function of \( z \) up to one Rayleigh distance. We confirm that \( \langle \sigma_3^z \rangle \) remains constant on propagation, but \( \langle \sigma_1^z \rangle \) and \( \langle \sigma_2^z \rangle \) oscillate with a spatial period equal to 2\( z_R \). As shown in figure 4(b), the polarisation states at \( z = 0 \) cover the great circle that passes through \( s_1 = \pm 1 \) and \( s_2 = \pm 1 \). During propagation, the states rotate with respect to the s3 axis and at \( z = 2z_R \) the states cover the great circle that passes through \( s_3 = \pm 1 \).

We can summarise the observations for LP beams as follows: (i) cases with \( n_R = n_L = 0 \) and \( |m_R| = |m_L| \) correspond to LP beams that span all polarisation states. For a circular polarisation basis \( \langle \sigma_1^z \rangle = \langle \sigma_2^z \rangle \) and \( \langle \sigma_3^z \rangle \) is predominant if \( |m_R| - |m_L| \gg 1 \). (ii) Higher-order cylindrical vector beams are given with \( n_R = n_L \) and \( m_R = -m_L \). They satisfy \( \langle \sigma_1^z \rangle = \langle \sigma_2^z \rangle = 1/2 \) and \( \langle \sigma_3^z \rangle = 0 \). (iii) For \( m_R = m_L \) and \( n_R = n_L \) the polarisation states cover a great circle different from the equator. Under propagation, the variances for the cases (i) and (ii) remain constant, but for (iii) the variances \( \langle \sigma_1^z \rangle, \langle \sigma_2^z \rangle \) oscillate while \( \langle \sigma_3^z \rangle \) remains constant. We remark that in all cases DoP = 0.

4. Polarisation properties of BP beams

Here, we describe the polarisation properties of BP beams. The spatial modes that we use are paraxial Bessel–Gauss (BG) beams instead of nondiffracting Bessel beams. This is with the purpose of describing a beam profile closely related to an experimental realisation. BG are described by the expression [24]

\[
BG_m = \exp \left( -i \frac{k_r z}{2k \mu} \right) \exp(\im \phi)
\]

where

\[
GB(r) = \exp \left( \frac{i k z}{\mu} \right) \frac{\exp \left( -\frac{r^2}{\mu^2} \right)}{\mu}
\]

is a Gaussian beam that apodises the Bessel function \( J_m \). The index \( m \) is the azimuthal quantum number that describes the amount of orbital angular momentum. Furthermore,

\[
\mu = 1 + iz_0/\kappa_R
\]

(26)

\[
\gamma = k_l w_0 / 2
\]

(27)

where \( z_R = k w_0^2 / 2 \) is the Rayleigh distance for the GB apodization. However, the distance where the beam behaves as a nondiffracting beam is given by

\[
z_{\text{max}} = \kappa_R / \gamma.
\]

(28)

The transverse wavevector \( k_t \) is related to \( z_{\text{max}} \) as \( k_t = k w_0 / z_{\text{max}} \).

A first-order BP beam in a circular polarisation basis reads as

\[
BP = \frac{1}{\sqrt{2}} (BG_0 \hat{r} + BG_1 \hat{L}).
\]

(29)

This configuration spans the Poincaré sphere and therefore its DoP = 0. The variances at \( z = 0 \) are equal to \( \langle \sigma_1^z \rangle = \langle \sigma_2^z \rangle = 0.24 \) and \( \langle \sigma_3^z \rangle = 0.52 \). An interesting difference between BP and LP beams appears on propagation. We notice that for LP beams, the transverse polarisation distribution on propagation changes according to a rigid rotation of the Poincaré sphere (see section 3). It is known that the far-field transverse amplitude of a BG beam corresponds to a ring of intensity. In other words, the angular spectrum of a Bessel beam is a ring in \( k \) space [25]. Since the ring radius is independent of the azimuthal quantum number [26], the far-field intensity distribution of both components of equation (29) are the same. An equal amplitude superposition means that the resulting polarisation pattern contains linear polarisation.
states. Therefore, during the propagation of a BP beam, the elliptical polarisation states evolve to linear polarisations.

Figure 5 shows the variances as a function of the propagation distance \( z \) for a first-order BP beam. In the calculations we use \( \lambda = 633 \text{ nm} \), \( w_0 = 2 \text{ mm} \), and \( k_s = k \sin(0.1^\circ) \) with \( k = 2\pi/\lambda \). The results confirm our previous comments. Initially, the polarisation states cover the full Poincaré sphere. During propagation, however, the polarisation states are attracted to the equator. This behaviour is compared to a polarisation attractor in optical fibres, where the states are attracted to a particular point or line on the sphere [27]. Notice that at all planes the DoP = 0 and \( \langle \sigma_1^2 \rangle = \langle \sigma_2^2 \rangle \) due to the circular polarisation basis. However, contrary to the LP beams, the value of \( \langle \sigma_3^2 \rangle \) is not constant during propagation. It turns out that, since the polarisation states are attracted to the equator, the value of \( \langle \sigma_3^2 \rangle = 0 \) in the far field and \( \langle \sigma_1^2 \rangle = \langle \sigma_2^2 \rangle = 1/2 \). Therefore, the final distribution is a particular case of a vector beam. In the example of figure 5(b), the far-field polarisation distribution is similar to a fractional vector beam of index 1/2 [28].

5. Polarisation properties of LaP patterns

Here, we show a novel type of FP pattern that covers the Poincaré sphere with a uniform distribution of polarisation states on the sphere. The LP beams have been described as a stereographic projection of the Poincaré sphere on the transverse plane. Nonetheless, this projection maps one of the hemispheres to a region of infinite extension, and thus they are not physically realisable. Instead, we propose a Lambert projection to map the polarisation states on a finite region. Of course, if the field is confined in a finite region, it will diffract during propagation. However, there are several methods to create this pattern in the image plane using spatial light modulators [29–31].

We define a first-order LaP pattern using circular polarisation components as

\[
\text{LaP} = \frac{r/2}{1 - (r/2)^2} \hat{e}_r + \sqrt{1 - (r/2)^2} \hat{e}_L,
\]

where \( 0 \leq \phi \leq 2\pi \) is the azimuthal coordinate and \( r \) is a dimensionless radial coordinate confined in the range \( 0 \leq r \leq 2 \). Certainly, higher-order LaP beams can be created by simply adding a larger azimuthal phase variation.

Using the definition for the Stokes parameters in terms of circular polarisation components

\[
s_0 = |E_R|^2 + |E_L|^2,
\]
\[
s_1 = 2 \text{Re} \left( E_R^* E_L \right)/s_0,
\]
\[
s_2 = -2 \text{Im} \left( E_R^* E_L \right)/s_0,
\]
\[
s_3 = \left( |E_R|^2 - |E_L|^2 \right)/s_0,
\]

we can show that \( s_0 = 1 \) and

\[
\text{LaP} = \frac{r}{1 - (r/2)^2} \hat{e}_r + \sqrt{1 - (r/2)^2} \hat{e}_L,
\]

Figure 4. (a) Stokes variances during the propagation of a LP beam with \((n_R, n_L) = (3, 2)\) and \(m_R = m_L = 0\). Red, green, blue curves correspond to \( \langle \sigma_1^2 \rangle, \langle \sigma_2^2 \rangle, \langle \sigma_3^2 \rangle \), respectively. (b) Transverse polarisation distribution and its representation on the Poincaré sphere for the initial \((z = 0)\) and final plane \((z = z_R)\).
Using the previous results, it is straightforward to show that $P = 4\pi$, DoP $= 0$ and $\langle \sigma_1^2 \rangle = \langle \sigma_2^2 \rangle = \langle \sigma_3^2 \rangle = 1/3$.

An important property of the LaP pattern is that the polarisation states cover the Poincaré sphere with a uniform distribution [18, 32]. Figure 6(a) shows the transverse polarisation pattern of a first-order LaP containing a lemon singularity (see equation (30)). Figure 6(b) shows the polarisation states on the Poincaré sphere. To show that the polarisation states are uniformly distributed on the sphere, figure 6(c) shows the histogram of polarisation states for the Stokes parameters. The histogram for each Stokes component uses 10 equally spaced bins between $-1$ and 1. In our calculations, we use a grid of $64 \times 64$ points. The results confirm that the polarisation states are uniformly distributed on the Poincaré sphere.

6. Discussion and conclusions

We have studied the polarisation properties of Poincaré beams according to the Stokes statistics developed in section 2. Through the use the Stokes variances, we have characterised LP, BP and LaP patterns. When using a circular polarisation basis, higher-order LaP beams showed specific properties according to the radial and azimuthal quantum numbers of its components. On the one hand, equal azimuthal quantum numbers produce a polarisation pattern that covers one great circle of the Poincaré sphere and its Stokes variances change under propagation. On the other hand, dissimilar azimuthal quantum numbers produce full Poincaré patterns whose Stokes statistics are propagation invariant. We also showed that BP beams behave as free-space polarisation attractors, converting elliptical polarisation states to linear polarisation states. In terms of its Stokes statistics, we showed that the variances reach stable values at the far-field, where the polarisation pattern is similar to a cylindrical vector beam of fractional order. Therefore, we have shown a method to generate fractional vector beams by propagating a superposition of integer-order Bessel beams. Furthermore, we have introduced a LaP polarisation pattern that contains all possible states of polarisation in a finite transverse region. It is true that there are other polarisation structures that contain all states of polarisation in a finite region, for example, an LP beam with $n_R = n_L = 1$ and $(m_R, m_L) = (1, 0)$ [4] or a BP beam with $(m_R, m_L) = (1, 0)$ [6]. However, these cases have repeated polarisation states, whereas the LaP field contains a single realisation of the Poincaré sphere with the interesting property of covering the surface of the Poincaré sphere with a uniform density. This is demonstrated with the histogram of the Stokes parameters which shows a flat distribution. The experimental realisation and further properties of LaP patterns are the subjects of future work. Finally, we emphasize that the aforementioned Stokes variances can be obtained with current procedures to measure the Stokes parameters. Therefore, they can be used as figures of merit to characterise current and novel space-variant polarised beams [33, 34].

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Figure 6. First-order Lambert–Poincaré pattern (see equation (30)). (a) Polarisation distribution, (b) transverse polarisation states represented on the Poincaré sphere, and (c) histogram showing the distribution of the Stokes parameters.
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