Studying wave optics in the light curves of exoplanet microlensing

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Received 2013 January 12; in original form 2012 July 28

ABSTRACT

We study the wave optics features of gravitational microlensing by a binary lens composed of a planet and a parent star. In this system, the source star near the caustic line produces a pair of images in which they can play the role of secondary sources for the observer. This optical system is similar to the Young double-slit experiment. The coherent wavefronts from a source on the lens plane can form a diffraction pattern on the observer plane. This diffraction pattern has two modes from the close- and wide-pair images. From the observational point of view, we study the possibility of detecting this effect through the Square Kilometre Array (SKA) project in the resonance and high-magnification channels of binary lensing. While the red giant sources do not seem to satisfy the spatial coherency condition, during the caustic crossing a small part of a source traversing the caustic line can produce coherent pair images. Observations of wave optics effects at longer wavelengths accompanied by optical observations of a microlensing event provide extra information on the parameter space of the planet. These observations can provide a new basis for the study of exoplanets.

Key words: gravitational lensing: microlensing – planets and satellites: detection – Galaxy: bulge – submillimetre: general.

1 INTRODUCTION

Gravitational lensing is caused by the bending of light rays due to the gravitational effect of a foreground mass. Depending on the distribution of mass on the lens plane and on the relative distances of the lens and source from the observer, multiple images or distortion in the source shape can be formed. In the case of star–star lensing inside the Milky Way, the separation between images is less than few milliarceseconds, and the images are unresolvable for ground-based telescopes. This type of gravitational lensing is termed gravitational microlensing.

Einstein (1936) derived the gravitational lensing equation, but it was decades later before the first gravitational lensing was observed in 1979. The source of this lensing was a quasar, and observations were performed at radio frequencies (Walsh, Carswell & Weymann 1979). A few years later, Paczynski proposed studying the MACHO (massive astrophysical compact halo objects) population in the Galactic halo by the method of gravitational microlensing (Paczynski 1986). His suggestion was to observe stars in the Large and Small Magellanic Clouds, counting the number of microlensing events and measuring their transit times (Einstein crossing time). Based on this observation, it is possible to measure the contribution of MACHOs to the mass of the Galactic halo. In addition to dark matter studies, another interesting astrophysical application of microlensing was suggested by Mao & Paczyński (1991), namely the use of gravitational microlensing to aid in the discovery of exoplanets.

Microlensing effects due to single or multiple lenses have been studied mainly using geometric optics. An important study of wave optics features of gravitational lensing was undertaken by Ohanian (1983), who investigated the magnification of a radio point source when a galaxy acts as a gravitational lens. He showed that wave optics smoothes singular features of the light curve at the position of the caustic lines. In another work, Jaroszyński & Paczyński (1995) studied the caustic crossing of quasar Q2237+0305 by a galaxy composed of individual stars. By studying the diffraction images of this system, they could put a limit on the size of the quasar. Wave optics observations of gravitational lensing inside the Milky Way also have astrophysical applications, for example studying the limb darkening of small sources such as white dwarfs (Zabel & Peterson 2003). Recently, Heyl (2010, 2011a,b) discussed the possibility of detecting wave optics signals in microlensing light curves with a single substellar lens.

In this work our aim is to extend the application of wave optics to the conventional method of extra-solar planet detection by gravitational microlensing. Here we assume a binary lens composed of a lensing star and a planet. The crossing of the caustic lines of this system by the source star produces high magnification in the light curve. Moreover, owing to the small separation of the images on the lens plane, the gravitational lensing system resembles a multiple-slit optical system at astronomical scales. With a coherent condition for the wavefronts on the lens plane, the result would be a diffraction pattern on the observer plane. We study the applications of this
method in both the resonance and high-magnification channels of exoplanet detection. Observations of the contrast in the fringes and the transit time of the fringes enable us to break the degeneracy between the lens parameters. We also study the possibility of observing the wave optics features of binary microlensing using the future Square Kilometre Array (SKA) project.

In Section 2, we introduce the wave optics formalism in gravitational lensing and calculate the wave optics light curve for a binary lens system. In Section 3, we carry out semi-analytic calculations of the wave optics features for microlensing near the caustic lines and study the temporal and spatial coherency conditions. We also numerically compute wave optics light curves and compare them with the results of geometric optics. In Section 4, we discuss the possibility of detecting microlensing wave optics signals by a binary lens in which one of the lenses is a planet. Our study uses observations at radio or micrometre wavelengths and planned future observations with SKA. We also discuss the possibility of degeneracy breaking between the lens parameters in the resonance and high-magnification channels of exoplanet detection. Conclusions and a summary are given in Section 5.

2 WAVE OPTICS IN GRAVITATIONAL LENSING

In geometrical optics, the locations of the images in terms of the position of the source can be obtained from the lens equation

\[ y = x - \alpha(x), \]  

(1)

where \( x \) and \( y \) are respectively the angular positions of the image and source normalized to the projected Einstein angle in each plane, and \( \alpha(x) \) is the deflection angle, which depends on the distribution of matter as

\[ \alpha(x) = \frac{1}{\pi} \int \kappa(x') \frac{x - x'}{|x - x'|^2} \, d^2 x', \]

(2)

where \( \Sigma(x) \) is the surface mass density of the lens, \( \kappa(x) = \Sigma(x)/\Sigma_{\text{cr}} \) and \( \Sigma_{\text{cr}} = (4\pi G D_1 D_\odot)/(c^2 D_e) \). Here, \( D_1, D_\odot, D_e \) are the source–observer, lens–source and lens–observer distances, respectively. Another method for deriving the lens equation is to use the Fermat principle. For the stationary points of the Fermat potential, the position of images in terms of the position of the source is obtained from

\[ \nabla_x \phi(x, y) = 0, \]

(3)

where Fermat’s potential is given by

\[ \phi(x, y) = \frac{1}{2} (y - x)^2 - \psi(x), \]

(4)

and the deflection angle is

\[ \alpha(x) = \nabla_x \psi(x). \]

(5)

We use the determinant of the Jacobian of the mapping function from the source plane to the lens plane. The magnification of a point-like source can be obtained as

\[ \mu(x_i) = \frac{1}{|\det J(x_i)|} \left| \frac{\phi_{11}(x_i) - \phi_{21}(x_i)}{\phi_{11}(x_i)} \right|, \]

(6)

where \( x_i \) is the location of the \( i \)th image. For an extended source, we should calculate the magnification over the source area. One of the methods of calculating the magnification by an extended source is Green’s theorem. In this theorem, a two-dimensional integration over the source reduces to a one-dimensional integration on the boundary of the images (Dominik 2007).

From the Huygens principle in wave optics, every point on the lens plane can be considered as a secondary source. The amplitude of the electromagnetic wave at each point of the observer plane is composed of the superposition of the infinitesimal sources on the lens plane. From the Kirchhoff integral, we can obtain the amplitude of the electromagnetic wave \( E_\nu \) if we know the boundary condition on the lens plane (Born & Wolf 2002). Multiplying the superposition of the electromagnetic wave by its complex conjugate results in the magnification of the wave on the observer plane as follows (Schneider, Ehlers & Falco 1992):

\[ \mu(y) = \frac{f^2}{4\pi^2} \left| \int e^{i\phi(x, y)} \, d^2 x \right|^2, \]

(7)

where \( f(x, y) \) corresponds to the phase of the electromagnetic waves emitted from a source, deflected from the lens plane and received by the observer. Here \( f \) is given by \( f = 2kR \), where \( k \) is the wavenumber, and \( R = 2GM/d^2 \) is the Schwarzschild radius of the lens. This formula has been obtained for a monochromatic electromagnetic wave.

In order to show the compatibility of the wave optics formalism with that of geometric optics and the transition from the wave optics to the geometric optics formalism, we expand the Fermat potential in the lens plane using a Taylor series. For a point-like source, the expansion around the image is given by

\[ \phi(x, y) = \phi^{(0)}(x) + (x - x_i) \nabla_x \phi^{(0)}(x) + \frac{1}{2} \left[ (x_1 - x_1)^2 \phi^{(0)}_{11} + (x_2 - x_2)^2 \phi^{(0)}_{22} + 2(x_1 - x_1)(x_2 - x_2) \phi^{(0)}_{12} \right] + \ldots, \]

(8)

where the superscript \((0)\) represents the Fermat potential at the position of the image and subscripts represent the derivatives with respect to two directions on the lens plane. Substitute from equation (8) into (7), the first term of equation (8) after multiplication of \( \exp[i\phi(x_i, y)] \) by its complex conjugate results in unity. The second term is zero from the Fermat principle, and finally the third term as a non-zero term results in the magnification in the geometric optics as equation (6). The third and higher orders of derivatives in the Fermat potential result in the wave optics features in the light curve.

One of the important issues in the wave optics formalism is that, in reality, the source is not completely coherent and we need to define a coherent time-scale of \( \Delta \tau = \Delta \omega^{-1} \), where \( \Delta \omega \) is the width of the spectrum. The amplitude of a non-chromatic source on the observer plane is given by

\[ V(x, y, \phi) \propto \int g(\omega) e^{i2\pi R \phi(x, y) \omega} \, d\omega, \]

where the magnification is given by

\[ \mu(y) = \frac{R^2}{\pi^2} \left| \int d^2 x \int g(\omega) e^{i2\pi R \phi(x, y) \omega} \, d\omega \right|^2. \]

(9)

We set the speed of light to \( c = 1 \). For a coherent monochromatic source \( g(\omega) = \delta(\omega - \omega_0) \) when substituting this specific spectrum in equation (9), we can recover equation (7). Assuming a non-zero temperature for a monochromatic source, the Doppler broadening can change the spectrum of a Dirac-delta spectrum to a Gaussian distribution.

An important issue regarding the observability of the wave optics effect is the coherency of light arriving at the observer from different parts of the lens plane, the so-called temporal coherency (Mandzhos 1981). In addition, we need to have coherency between different parts of an extended source, termed spatial coherency. Assuming that a source has zero angular size, in order to examine the temporal
coherence between different images we need the time delay between the light rays from the source to the observer and compare it with the coherent time of the source. The time difference between the two light rays received by the observer is given by

$$\Delta t = 2R_\star [\phi(x_{11}, y) - \phi(x_{12}, y)],$$

where the source position is fixed and $x_{11}$ and $x_{12}$ are the positions of the images. The difference between the Fermat potentials, $\Delta \phi$, for two distant images is of the order of unity. During the caustic crossing, however, two pairs of close images can be formed with $\Delta \phi$ of the order of $10^{-3}$. Quantifying the time delay for the wide and close pair of images in the caustic crossing, $\Delta t$ is given as

$$\Delta t \sim 1 \times 2R_\star \sim 10^{-5} \frac{M}{M_\odot} \text{s},$$

for wide images

$$\Delta t \sim 0.001 \times 2R_\star \sim 10^{-6} \frac{M}{M_\odot} \text{s},$$

for close images,

where in the former case images appear around the star-lens and in the latter case images appear around the planet-lens. Here we take the mass ratio of the planet to the parent star to be of the order of $10^{-3}$. We will discuss this in detail in Section (3). As the source approaches the caustic line (i.e. $\Delta t \rightarrow 0$), the time differences between the light rays become shorter. For a source with a non-zero temperature $T_s$, the coherent time in terms of the bandwidth of the spectrum is given by $\tau_c \Delta \omega \sim 1$ (Mehta 1963). The dispersion velocity of the gas is related to the surface temperature of the caustic by $\sigma \sim \sqrt{T_s}$; on the other hand, the dispersion velocity is related to the frequency dispersion as $\sigma = \Delta \omega / \omega$, and hence the coherent time relates to the temperature of the source as (Guenther 1990):

$$\tau_c \propto \frac{1}{\sqrt{T_s}^v} \Rightarrow \tau_c = \frac{2.8 \times 10^{-4}}{v_{(GHz)}} \sqrt{\frac{3000}{T_s}}.$$  

In a double-slit experiment with a point-like source for a diffraction pattern, the time difference between the two light rays received by the observer should not be longer than the coherent time. Now we can constrain $f = 2R_\star$ with a temporal coherency condition for the wide and close pair of images. Comparing equations (11) and (12) with the coherent time in equation (13) results in

$$f_{\text{wide}} \leq 10^6 \sqrt{\frac{3000}{T_s}},$$

$$f_{\text{close}} \leq 10^4 \sqrt{\frac{3000}{T_s}}.$$  

More details of the temporal coherency in wave optics gravitational lensing are given in Appendix A.

In order to calculate the magnification by a binary lens composed of a parent star and a planet, we apply the following Fermat potential for a binary lens in equation (7):

$$\phi(x, y) = \frac{q}{2} (x - y)^2 - m_\star \ln(|x - x_\star|) - m_p \ln(|x - x_p|),$$

where $m_\star$ and $m_p$ are the relative masses of the star and the planet to the overall mass of the system, respectively. The positions of the parent star and planet are given by $x_\star$ and $x_p$. We take lenses along the $x_1$-axis and put the centre of mass of this system at $x = 0$. The positions of the parent star and planet are given by

$$x_\star = \left( \frac{d - q}{1 + q} , \ 0 \right), \quad x_p = \left( - \frac{d}{1 + q} , \ 0 \right).$$

where $d$ is the projected distance between the two lenses normalized to the Einstein radius and $q = m_p / m_\star$. We can identify the track of the source on the lens plane by two parameters: the minimum impact parameter $u_0$ with respect to the centre of mass of the lenses, and its direction with respect to the $x$-axis, $\alpha$. For the case of a single lens, the integral in equation (7) has an analytical solution; for two point-mass lenses, however, we perform numerical computations to obtain the light curve of a source moving with respect to the lens plane. In the next section, we utilize the Fourier expansion of the Fermat potential up to relevant terms, and study the Fermat potential of a binary lens near the caustic lines.

For a point-like source lensed by a binary system, we use equation (7) and plot the amplification pattern as a function of time in Fig. 1. This is a typical microlensing light curve with the wave optics features compared with those of geometric optics. Here we have two oscillating modes owing to the interference between the wide and close images. As the source gets closer to the caustic line, the longer mode is magnified and after caustic crossing it becomes dimmer. In Fig. 2 we depict the two-dimensional pattern of fringes on the observer plane when the relative motion of the observer with respect to this pattern produces the light curve in Fig. 1. In the

![Figure 1](https://example.com/figure1.png)  
Figure 1. The wave optics light curve of a point-like coherent source (upper panel) with parameters $d = 0.8$, $q = 0.1$, $u_0 = 0$, $f = 2000$, and $f = 1000$. The lower panel shows the light curve for the geometric optics. The light curves depict the magnification of the source star during the caustic crossing.

![Figure 2](https://example.com/figure2.png)  
Figure 2. Two-dimensional luminosity pattern from a point-like source on the observer plane lensed by a binary system. The fringes are demonstrated near a critical line. The overall flux results from the sum of the close images (as shown in this figure) plus the flux from the incoherent wide images.
following section, we include the finite-size effect of the source star in our calculations.

2.1 Finite-size effect

The majority of microlensing observations in recent years have been carried out by two Microlensing Observations in Astrophysics (MOA) and the Optical Gravitational Lensing Experiment (OGLE) observational groups, monitoring millions of stars towards the Galactic Centre. There are other follow-up telescopes around the world that cover ongoing events for 24 h. These telescopes with high-cadence observations identify anomalies in the microlensing light curves to discover extra-solar planets. While there are all types of stars as microlensing targets towards the Galactic bulge, there is a selection bias for the red giant population compared to main sequence stars (Rahal et al. 2009; Moniez et al., in preparation). In the direction of the Galactic bulge, red clump stars contain the main part of the source stars used for microlensing events (Hamadache et al. 2006). This selection of sources for the microlensing events is due to the brightness of red giants, which enables us to observe them from a distance. Consequently, the averaged Einstein angle corresponding to this type of source star is larger than that for main sequence stars, making the transit time of these events longer. In addition to the visual band, red giants can optically pump the interstellar medium and produce Maser emissions a few astronomical units away from the source star (Messineo et al. 2005; Vlemmings, van Langevelde & Diamond 2005).

In contrast to simple point-like sources, red giants are extended objects in which the extended source effect not only decreases the strength of the magnification for geometric optics, but also decreases the enhancements of the fringes for wave optics (Schneider & Schmid-Burgk 1985). The overall magnification for an extended source is given by

$$\mu = \frac{\int S(\rho) d^2 \rho}{\pi \rho^2}$$

(Schneider et al. 1992), where $S(\rho)$ is the magnification of the multiple images at the source position of $\rho$, and $\rho$ is the angular radius of a source normalized to the Einstein angle. A detailed expression for $\mu$ is given in equations (A6) of the Appendix.

The contrast in the interference fringes in the observer plane depends on the size of the source star in such a way that increasing the source size decreases the contrast of the interference pattern in the light curve. This effect can be seen in the Young experiment when we increase the size of the pinhole as the source in the double-slit experiment. The superposition of the fringes from different parts of a source in the observer plane is an indicator for the spatial coherency of the source. A mathematical criterion for losing spatial coherency is that the constructive interference from one part of the source overlaps with the destructive interference from the other part of the source in the observer plane.

We now apply the spatial coherency condition of the Young experiment to the microlensing effect. Let us consider a source with size $L_s$ located at distance of $D_0$ from the lens plane. For light rays arriving at the lens plane within a domain of radius $h$ from the optical axis of the system, the spatial coherency condition is met if

$$L_s < \frac{D_0 \lambda}{2h}.$$  

(19)

For a single lens, the angular separation between the images is given by $\Delta \theta = \sqrt{\beta^2 + 4\theta_E^2}$, where $\beta$ is the impact parameter. For high-

Table 1. Coherent size of a source in kilometres for a single microlensing system. Here $D_0 = 8.5$ kpc and $D_1 = 4$ kpc. The observation is performed with two wavelengths of 3 cm and 1 mm for the cases of Earth- and Jupiter-mass lenses. The content of the table shows the size of coherent sources.

| $\lambda$ | 3 cm | 1 mm |
|----------|------|------|
| $L_s(M_\odot)$ | $3.4 \times 10^5$ | 114 |
| $L_s(M_\odot)$ | $2.0 \times 10^5$ | 6.7 |

magification events, $\Delta \theta \simeq 2 \theta_E$ and the separation between the images is given by $2h = D_0 \Delta \theta \simeq 2D_0 \theta_E$. Rewriting equation (19) in terms of the source size and Einstein angle, the constraint on the spatial coherency of the source is given by

$$L_s < \frac{D_0 \lambda}{2\theta_E}.$$  

(20)

Using the definition of the Einstein angle, we have

$$L_s < \frac{\lambda}{2} \sqrt{\frac{D_0 D_1}{2 \pi R_s D_0}}.$$  

(21)

where $R_s$ is the Schwarzschild radius of the lens. This calculation is done for a single lens. In the next section we recalculate the coherency condition for a binary lens, using the Fermat potential. The advantage of a binary lens is that this lensing system can produce very close images during the caustic crossing. These images may satisfy the spatial coherency condition of the source star. Before studying the spatial coherency of a source in the binary lensing, let us estimate the coherent size of the source for a single lens.

We assume a lens at the middle of the distance between the observer and source, where the probability of microlensing observation is maximum (i.e. $D_1 = D_0$). For two typical planets with masses of the Earth and Jupiter, the Schwarzschild radius is about 1 and 286 cm, respectively. For a source star in the Galactic bulge, $D_0 = 8.5$ kpc. Table 1 shows the coherent size of sources at various wavelengths. For $\lambda = 3$ cm, we can observe the wave optics features of an Earth-mass lens with a solar-type source star. We can also observe the wave optics effect for a Jupiter-mass planet and a smaller source. At micron wavelengths, the spatial coherency decreases to 10–100 km, enabling the detection of the diffraction pattern of smaller structures such as granules on the surface of a source star (Yu et al. 2011).

As already noted, in binary lensing, during the approach of the source star to a caustic line, the distance between the pair of images can be very small compared to the case of a single lens. At the same time we may have wide images located a few astronomical units away from each other. Hence, the light rays received from the lens plane to the observer are a mixture of close coherent images and wide incoherent images. In the next section we discuss the possibility of producing a diffraction pattern by a binary lens.

3 LIGHT CURVE NEAR A CAUSTIC LINE: BINARY LENSES

In this section we use numerical and semi-analytical methods to study the light curve of a microlensing event by a binary lens. During the caustic crossing, where images form at the critical lines,
we can write the lens equation. In other words, the first derivatives of the Fermat potential are zero:

$$\phi_1^{(0)} = \phi_2^{(0)} = 0. \quad (21)$$

Diagonalizing the Fermat potential with respect to the second derivatives, we can set $\phi_{12} = \phi_{21} = 0$. In order to satisfy a singular Jacobian transformation on the critical lines, from equation (6), either $\phi_{11}$ or $\phi_{22}$ should be zero. We set $\phi_{22}^{(0)} = 0$ and $\phi_{11}^{(0)} \neq 0$. Ignoring $x^2$ in the geometric term compared to the $y^2$ term, we perform a Taylor expansion of the Fermat potential around the critical line as (Schneider et al. 1992; Jaroszynski & Paczyński 1995):

$$\phi(x, y) = \phi^{(0)} + 1/2 y^2 - x y + 1/6 \phi_{11}^{(0)} x^2 + 1/2 \phi_{12}^{(0)} x y + 1/6 \phi_{22}^{(0)} x^2 + \ldots \quad (22)$$

Using the Fermat principle of $\delta \phi/\delta x_i = 0$, we obtain the positions of the images as a function of the position of the source:

$$y_1 = \phi_{11}^{(0)} x_1 + 1/2 \phi_{111}^{(0)} x_1^3 + \phi_{112}^{(0)} x_1 y_1 + \phi_{22}^{(0)} x_1 y_1^2 + \phi_{222}^{(0)} x_1 y_1^3 + \phi_{2222}^{(0)} x_1 y_1^4 + \ldots \quad (23)$$

$$y_2 = 1/2 \phi_{222}^{(0)} x_1 + \phi_{212}^{(0)} x_1 y_1 + \phi_{2222}^{(0)} x_1 y_1^2 \quad (24)$$

The singularity for the new Jacobian of transformation implies the constraint of $\phi_{111}^{(0)} x_1 + \phi_{112}^{(0)} x_1 y_1 + \phi_{22}^{(0)} x_1 y_1^2 + \phi_{222}^{(0)} x_1 y_1^3 = 0$, where we have ignored the higher-order terms of $x$.

From equations (23) and (24), we obtain the position of the images as follows:

$$x_{\text{images}} = \left( \frac{y_1}{\phi_{11}^{(0)}} \pm \sqrt{\frac{2y_2}{\phi_{11}^{(0)}}} \right) \frac{\phi_{22}^{(0)}}{\phi_{11}^{(0)}}, \quad (25)$$

where $y_1$ is chosen along the caustic line and $y_2$ is perpendicular to the caustic line. On the positive side of $y_2$ we have two images, while for the negative side there is no image. Substituting the position of the images in equation (22), we can calculate the Fermat potential for the nearby images during the caustic crossing. The difference between the Fermat potentials of the two pairs of images is given by

$$\Delta \phi = \frac{2}{3} \frac{2y_2}{\phi_{22}^{(0)}}^{3/2} \left( \frac{\phi_{22}^{(0)}}{\phi_{11}^{(0)}} \right)^{1/2}. \quad (26)$$

Here $\Delta \phi$ is a function of $y_2$ and the third derivative of the Fermat potential on the critical line. Assuming that the trajectory of the source (i.e. $\tilde{y}(t)$) has a direction given by the angle $\gamma$ with respect to the caustic line, the position of the source in the direction perpendicular to the caustic line is given by $y_2^* = (\sin \gamma)(t - t_c)/t_E$. On the other hand, from the Fermat potential for a binary system in equation (16), we can calculate $\phi_{222}^{(cm)}$ in the centre of the mass coordinate system as follows:

$$\phi_{222}^{(cm)} = m_p \left[ \frac{6x_2}{(x_1 - x_p)^2 + x_2^2} - \frac{1}{2} \frac{8x_2^3}{(x_1 - x_p)^2 + x_2^2} \right]. \quad (27)$$

Here $x_1$ and $x_2$ represent the positions of the critical lines on which images form during the caustic crossing. The position of images on the critical line depends on the location of the source, and double images can form along the critical line, either around the star (wide images) or around the planet (close images). These pairs of images split from a single image during the caustic crossing. Because our calculation has been done in the local coordinates of the critical line where images form, we perform a coordinate transformation of the Fermat potential from the centre of mass coordinate system to the local diagonalized coordinate system at the image position. First we do a boost along the $x$-axis to one of the lens positions. The second boost is along the radial direction, equal to the Einstein radius of the lens. Finally, we perform a rotation with $R_\gamma$ to diagonalize $\phi_j$. The third derivative of the Fermat potential, which is the relevant parameter in the Fermat potential in equation (26), can be obtained after the coordinate transformation as follows: $\phi_{222}^{(0)} = R_2 R_3 R_3 R_2 \phi_{222}^{(cm)}$.

The corresponding coordinate transformations to the location of images around the critical line of the planet is given by $x_1 = x_p + R_0 \sin \theta$ and $x_2 = R_0 \cos \theta$, where $R_0$ is the Einstein radius of the planet, $\theta$ is the polar angle with respect to the line connecting the two lenses. Because $|R_\gamma| \sim 1$, the magnitude of the third-order derivative of the Fermat potential does not change so much by the coordinate transformation; that is, $\phi_{222}^{(0)} \simeq \phi_{222}^{(cm)}$. Substituting boosts in equation (28), since $m_s \gg m_p$, the first term dominates as $R_0^{(0)}$ appears in the denominator, and hence

$$\phi_{222}^{(0)} \simeq -m_p \left( \frac{R_0^{(0)}}{R_E} \right)^{-3}. \quad (28)$$

Replacing the ratio of the Einstein radius of the planet to the Einstein radius of the star with the corresponding mass ratio and using a normalized planet mass with $m_p = q/(1 + q)$, equation (28) can be written as

$$\phi_{222}^{(0)} \simeq -\frac{1}{\sqrt{q}}. \quad (29)$$

Comparing the images that form around the planet and the parent star on the lens plane, the nearby images around the planet are more suitable for producing a diffraction pattern on the observer plane. Having a smaller $q$ results in a larger $\phi_{222}^{(0)}$ and consequently a smaller $\Delta \phi$. We now substitute equation (29) into equation (26) and replace the difference in the Fermat potential with the difference in the time delay between the trajectory of the two pairs of images as follows:

$$\Delta t = 2 \times R_0 \Delta \phi$$

$$= 5 \times 10^{-9} \times \left( \frac{t - t_c}{1h} \frac{\sin \gamma}{(40 \text{ days})} \right)^{3/2} \left( \frac{R_E}{0.001} \frac{M}{M_\odot} \right)^{-3/2} \left( \frac{\lambda}{0.001} \right)^{1/4} \left( \frac{M}{M_\odot} \right) \text{ s,}$$

where $(t - t_c)/1h$ is the time corresponding to the relative distance of the source from the caustic line normalized to one hour. We note that, unlike for the single lens where $\Delta t$ is of the order of the light-crossing time of the Schwarzschild radius, in the case of a binary lens the factor $q$ decreases the corresponding time-scale (Heyl 2010). On the other hand, as $t \to t_c$, $\Delta t$ approaches zero. The characteristic time difference in the Fermat potential is about $\Delta t \sim 5 \times 10^{-9}$ s, corresponding to $\Delta t \sim 150$ cm. This length scale corresponds to a frequency of $0.2 \text{ GHz}$. Here wavelengths larger than this threshold $\Delta t$ satisfy temporal coherency and can produce a diffraction pattern from the images on the observer plane. Repeating this calculation for distant images on the lens plane, where $q$ is of the order of one, we obtain a larger value for $\Delta t$, destructive for producing the wave.
Critical lines (top left panel), caustic lines (top right panel) and light curve for geometric optics (lower panel) for a binary system with parameters $q = 0.001$, $d = 1$. Here we take a source star of size $\rho = 0.002$. The large loop (in the top left panel) corresponds to the critical line of the parent star lens, and the smaller loop corresponds to the critical line of the planet. The straight dashed line (in the top right panel) indicates the path of the source in the source plane.

Figure 3. Critical lines (top left panel), caustic lines (top right panel) and light curve for the geometric optics (lower panel) for a binary system with parameters $q = 0.001$, $d = 1$. Here we take a source star of size $\rho = 0.002$. The large loop (in the top left panel) corresponds to the critical line of the parent star lens, and the smaller loop corresponds to the critical line of the planet. The straight dashed line (in the top right panel) indicates the path of the source in the source plane.

and $B$, producing close and wide images. Fig. 4 also compares the light curve in the wave optics and the geometric optics formalism for two wavelengths at points $A$ and $B$. The light curves are obtained from the Fresnel–Kirchhoff integration. For the close images, the time variation of the fringes is slower than for the wide images. This effect can be seen from the relative velocity between the separation of the pair images from equation (30) as follows:

$$\dot{\gamma} = v_t \frac{2h}{|t - t_c|} \times \frac{\sin \gamma}{\phi^{(0)}_{222}},$$

where $v_t = R_{\xi}/R_\alpha$ is the relative velocity of the source with respect to the lens. For the close-pair images, we have a larger $\phi^{(0)}_{222}$ and for the wide-pair images this term is smaller. Hence not only close images produce large modes of fringes, there time variation is also slower and provides enough time for the observer to detect this effect with a suitable cadence of observations.

### 4 OBSERVATIONAL PROSPECTS

In this section we study the possibility of follow-up observations of a binary microlensing system during the caustic crossing by the future SKA project. For a single lens, the parameters involved in the light curve are the Einstein crossing time $t_\xi$, the minimum impact parameter $u_0$, and the time for the maximum magnification $t_0$. Amongst these parameters, the only parameter that contains the physical information of the system is $t_\xi$, which is a function of the mass of the lens, the relative distance of the lens with respect to the source and observer, and the transverse velocity of the lens with respect to our line of sight. Using additional information such as the parallax effect due to the annual motion of the Earth around the Sun (Gould 1998), one can partially break the degeneracy between the lens parameters (Rahvar et al. 2003). However, the finite-size effect of the source can also provide extra information to break the degeneracy between the lens parameters (Roulet & Mollerach 1997).

Increasing the number of lenses from one to two increases the number of parameters of the lensing system. The additional
parameters are (i) the projected distance between the lenses normalized to the Einstein radius, ‘$d$’, (ii) the relative mass of the lenses, ‘$q$’, and (iii) the angle ‘$\alpha$’ defining the trajectory of the source with respect to the line joining the two lenses. For a binary system, we have a total of six parameters to fit the light curve. The probability of microlensing observation of a binary lens depends on the size of the caustic, and for the case of resonance where the distance between the lenses is of the order of the Einstein ring, we will have the maximum probability of detection.

The main problem with resonance events is that, in spite of occasional observations of planets, owing to our lack of knowledge about observational efficiency it is difficult to analyse the distribution function of the parameters of the planet. There have been some efforts to develop a fully deterministic strategy through automated searching systems for exoplanets (Dominik et al. 2010). Having such a system to cover all the known microlensing candidates will enable us to obtain a correct statistical distribution of the parameters of the planet. The other important channel for exoplanet observation is that of high-magnification events (Griest & Safizadeh 1998). Almost all the very high-magnification events can be flagged by microlensing surveys, and follow-up telescope surveys monitor them with a high sampling rate and better photometric precision. Unlike the case for low-magnification events, for these events the detection efficiency function is almost known and statistical analysis in the parameter space can be applied (Gould et al. 2010). One of the main problems with binary lenses is the $d$↔$d^{-1}$ degeneracy problem, whereby we can have almost the same light curve for close and wide binary lenses.

In the caustic classification of binary systems we have three types of topologies for the caustics and corresponding critical lines (Schneider & Weiss 1986), the so-called the ‘close’, ‘wide’ and ‘intermediate’ or resonance binaries. Fig. 5 shows these three categories of caustic lines for three values of the planet to star mass ratio. To study the wave optics features during the caustic crossing, we generate synthetic light curves for the three categories and compare the results with the geometric optics features. Our aim is to study the wave optics signals in the resonance and high-magnification channels.

In order to quantify the wave optics features in the light curve, we use the $\chi^2$ difference from the best fit of the wave optics and geometric optics. Assuming $\sigma_i$, as the error bar for each data point, $\mu_i^{(g)}$ as the magnification for geometric optics and $\mu_i^{(w)}$ as the magnification for wave optics, the difference between the $\chi^2$s is given by

$$
\Delta \chi^2 = \chi^2_{g} - \chi^2_{w} = \sum_{i=1}^{N} \frac{1}{\sigma_i^2} (\mu_i^{(w)} - \mu_i^{(g)}) \left(2 \mu_i^{(exp)} - \mu_i^{(w)} - \mu_i^{(g)}\right).
$$  \hspace{1cm} (35)

Having a threshold for $\Delta \chi^2$, we can distinguish the wave optics light curve from that for the geometric optics. An important element in equation (35) is the estimation of the photometric error, which depends on the source flux, integration time and the size of the radio telescope. Amongst the various sources, red giants and super-giants can emit electromagnetic waves at longer wavelengths. Radio-loud quasars at cosmological scales are also bright radio sources.

Detailed studies on radio sources are performed for single-lens wave optics microlensing in Heyl (2011a). In what follows we adapt that classification. For the red giants, a closer star such as Arcturus...
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(Perryman et al. 1997) emits at the wavelengths of 2 and 6 cm with 0.68 and 0.28 mJy, respectively (Drake & Linsky 1986). The spectrum of this type of star is given by

\[ f_c \simeq 24 \left( \frac{\nu}{\text{GHz}} \right)^{0.8} \left( \frac{\text{kpc}}{D} \right)^2 \text{nJy}. \]  

(36)

Another class is that of low-mass late-type stars as asymptotic giants. For example, Mira is an example of this class and its spectrum is given by (Perryman et al. 1997; Reid et al. 1997)

\[ f_c \simeq 72 \left( \frac{\nu}{\text{GHz}} \right)^2 \left( \frac{\text{kpc}}{D} \right)^2 \text{nJy}. \]  

(37)

Finally super-giants have strong radio emission. Betelgeuse is a red super-giant located at a distance of 197 pc (Newell & Hjellming 1982; Harper, Brown & Guinan 2008). The spectrum of this star normalized to the kiloparsec distance is

\[ f_c \simeq 9.3 \left( \frac{\nu}{\text{GHz}} \right)^{1.32} \left( \frac{\text{kpc}}{D} \right)^2 \text{\mu Jy}. \]  

(38)

The radii of super-giants are of the order of a few astronomical units. Assuming these stars are in the Galactic Centre at \( \sim 8 \text{kpc} \) distance from us, from equation (32) we can obtain coherent images for close images on the lens plane. As mentioned above, at the caustic crossing only a small part of the source contributes to the production of coherent close-pair images.

In equation (35), we need an estimation for the photometric error bar. For the SKA project, the noise corresponding to Nyquist sampling is 0.27 Jy (Schilizzi et al. 2007). This sampling is defined such that the integration time multiplied by the band width is equal to one (i.e. \( \Delta \nu \Delta \tau = 1 \)). Because noise decreases with the square root of time and the band width as \( 1/\sqrt{\Delta \nu \Delta \tau} \), we can write the noise in terms of these two parameters as follows:

\[ \Delta f = 0.14 \left( \frac{\Delta \nu}{\text{GHz}} \right)^{-1/2} \left( \frac{\Delta \tau}{1 \text{h}} \right)^{-1/2} \text{\mu Jy}. \]  

(39)

We use the six parameters of the binary microlensing system to simulate light curves. In addition, the source star is assumed to radiate at longer wavelengths. We compare the simulated data in the geometric optics case with the wave optics case and use the criterion of \( \Delta \chi^2 > 5 \) between the two theoretical light curves. For an ensemble of light curves, we identify caustic lines in the three categories of binaries, as shown in Fig. 5. Those light curves with caustic crossing satisfying the criterion for the wave optics are identified in this figure. Our analysis shows that the wave optics feature is sensitive to specific parts of the caustic lines of a binary lens.

As noted above, there are two main channels for exoplanet observations. In Fig. 5 we identify the area of caustic lines in each channel with the wave optics feature. Having a small \( q \), wide and close binaries can produce almost the same geometric optics light curves. According to Fig. 5, there is no strong wave optics effect in the light curve of the high-magnification events. On the other hand, for the intermediate regime (resonance) a larger area of the caustic lines is suitable for the wave optics effect. In this case, we will have a combination of close and wide images on the lens plane.

We perform a Monte Carlo simulation to generate an ensemble of light curves and study the wave optics effect in the simulated light curves. Fig. 6 shows a sample of light curves in this simulation with a cadence rate of 45 min and signal-to-noise ratio of \( S/N = 7 \). The relevant parameters of the wave optics from equation (33) are the wavelength in the Airy function, \( Y_0 \), and the maximum magnification of the light curve, \( \mu_{\text{max}} \). These parameters depend on.

**Figure 6.** Simulation of the microlensing light curve for a binary lens. The parameters of the lens are taken as for the light curve in Fig. 3. The data points are simulated according to the noise in SKA. Here we have a signal-to-noise ratio of 7 with a 45-min integration time. The observations are performed at 10 GHz. The solid line represents the best fit in the geometric optics, and the dashed line represents the best fit of the light curve including the wave optics effect.

**Figure 7.** Wave optics features are excluded in the area above the curves in the parameter space of \( f \) and \( \phi_{222} \) according to the criterion \( \Delta \chi^2 > 10 \). The sizes of the source stars normalized to the Einstein radius are noted in the key. Smaller stars are more favourable for the observation of wave optics features in microlensing.

\( f \) and \( \phi_{222} \). For an ensemble of light curves distributed uniformly in parameter space, we calculate the \( \chi^2 \) difference between the wave optics and the geometric optics from equation (35). In order to study the sensitivity of the discriminating parameter, \( \Delta \chi^2 \), in terms of \( f \) and \( \phi_{222} \), we identify the area of parameter space that satisfies \( \Delta \chi^2 > 10 \), where the source sizes are chosen as \( \rho = 0.001, 0.005 \) and 0.002; see Fig. 7. Here the area above the curves does not satisfy our criterion and is excluded. Having smaller \( f \) means a longer wavelength for the observation. On the other hand, \( \phi_{222} \) relates to the maximum magnification and size of modes from the wave optics. By measuring how the transit time-scale of the fringes changes with respect to the observer \( \Delta \tau \), we can determine \( Y_0(f, \phi_{222}) = \Delta \tau/t_W \). However, \( \mu_{\text{max}}(\phi_{222}) \) can be measured directly from the light curve.

The physical parameters involved in the wave optics light curves are the overall mass of the system \( M \), the mass ratio \( q \) and the trajectory of the source with respect to the lens. In contrast, from observations in the geometric optics case, we can find six
parameters of the light curve with a degree of degeneracy. Having extra information from the wave optics will constrain the $M_l$ and $q$ parameters, and subsequently we can identify the parameters of the planet with better accuracy.

We now want to look at the sensitivity of the wave optics signals in terms of the physical parameters of the binary system, $q$ and $d$. From the Monte Carlo simulation, we select a fraction of events that satisfy the condition $\Delta \chi^2 > 10$. According to Fig. 8, a suitable area of the parameter space for the wave optics features is in the resonance area where the separation between the lens and the planet is of the order of the Einstein radius. This result is compatible with our preliminary analysis for the sensitivity of the wave optics signal in terms of the parameter space in Fig. 5. In order to estimate the overall number of microlensing events with the wave optics signals, we should multiply the efficiency function with the real distribution of binary lenses in terms of $q$ and $d$.

Finally, we want to extract physical information from a typical wave optics light curve, assuming that we have observational data at both visual and radio wavelengths. From the data of geometric optics, by fitting to the light curve we can extract $q$, $d$, and the trajectory of the source. On the other hand, from the wave optics our relevant variables are $f$ and $\phi_{222} \sim 1/\sqrt{q}$. Measurement of these two parameters provides directly the value of $q$ and the overall mass of the lenses. Assuming a set of simulated data points of the light curve, let us extract the observable parameters from this light curve. We fit the simulated data in Fig. 9 with the theoretical wave optics light curve. Here the theoretical value of $f$ is 49475, assuming that observations are performed at $\nu = 10$ GHz, and from the likelihood function we obtain $f = 51400^{+1655}_{-1600}$, providing 6 per cent uncertainty in mass measurement.

This effect is an example of Young's double-slit experiment at astronomical scales. We derived the wave optics features of a binary lens, showing that it depends only on the third-order derivatives of the Fermat potential $\phi_{222}$ and $f = 2kR_e$.

We take red giants and super-giants as the source stars of gravitational microlensing towards the Galactic bulge, as there is a natural selection bias for observing this type of source star. We suggested using the future SKA project for the observation of wave optics signals in the light curve. In this observational program, radio observatories accompany the microlensing follow-up telescopes in the visual bands. These two observations at long and short wavelengths can provide a complimentary program for studying binary microlensing events to break the degeneracy of binary systems. We discussed the problem of the spatial coherency of sources in binary lensing and showed that only the part of the source that crosses the caustic line contributes to the formation of close-pair images. While an extended giant star may have no spatial coherency, the spatial coherency condition holds for the small part of the source crossing the caustic line.

We studied the observability of the wave optics parameters in a Monte Carlo simulation by fitting the simulated microlensing light curves with the theoretical wave optics light curve. Out of the two channels for the detection of exoplanets by microlensing, namely (i) the high-magnification channel and (ii) the resonance channel, we showed that the wave optics observation is in favour of resonance binary microlensing events. The extra information from the wave optics light curve enables us to solve for the lens parameters with better accuracy. Our analysis has shown that the use of radio telescopes for observations of planetary microlensing events will open a new window for studies of exoplanets.

5 CONCLUSION

In this work we studied the wave optics effect of gravitational microlensing by a binary lens composed of a lens star and a companion planet. The lensing effect of a planet during the caustic crossing produces close images, a suitable configuration for the images on the lens plane to generate a diffraction pattern on the observer plane.

Figure 8. Efficiency function in terms of $d$ and $q$, comparing the wave optics features with the geometric optics. The criterion for the wave optics light curve from equation (35) is $\Delta \chi^2 > 10$. The simulation is done for a source star with a finite size of $\rho = 0.005$ and observations at $\nu = 10$ GHz.

Figure 9. The synthetic light curve for radio wavelengths is generated with a cadence of 30 min and error bar of 0.15 and fitted with the wave optics theoretical light curve. The initial value of $f$ is taken as 49475. We use $\nu = 10$ GHz in the simulated light curve for the observations. The likelihood function shows the best fit with 1$\sigma$ and 2$\sigma$ confidence levels. For 1$\sigma$, we have $f = 51400^{+1655}_{-1600}$, providing 6 per cent uncertainty in mass measurement.

ACKNOWLEDGEMENTS

AM thanks Sharif University of Technology for providing high-performance computational facilities. We thank Avery Broderick, Vahid Karimipour and Mir Abbas Jalali for providing useful comments and improving the text of the paper. We also would like to thank the anonymous referee for valuable comments. This research was supported by the Perimeter Institute for Theoretical Physics.
and the John Templeton Foundation. Research at the Perimeter Institute was supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation.

REFERENCES

Arnold V. I., 1989, Mathematical Methods of Classical Mechanics, 2nd edn. Springer-Verlag, New York
Born M., Wolf E., 2002, Principles of Optics, 7th edn. Cambridge Univ. Press, Cambridge
Dominik M., 2007, MNARS, 377, 1679
Dominik M. et al., 2010, Astronomische Nachrichten, 331, 671
Drake S. A., Linsky J. L., 1986, AJ, 91, 602
Einstein A., 1936, Sci, 84, 506
Erde H., Schneider P., 1993, A&A, 268, 453
Gould A., 1998, ApJ, 506, 253
Gould A. et al., 2010, ApJ, 720, 1073
Griest K., Safizadeh N., 1998, ApJ, 500, 37
Guenther B. D., 1990, Modern Optics. Wiley, New York
Hamadache C. et al., 2006, A&A, 454, 185
Harper G. M., Brown A., Guinan E. F., 2008, AJ, 135, 1430
Heyl J. S., 2010, MNRAS, 402, L39
Heyl J. S., 2011a, MNRAS, 411, 1780
Heyl J. S., 2011b, MNRAS, 411, 1787
Jaroszynski M., Paczyński B., 1995, ApJ, 455, 443
Mandzhos A. A., 1981, Pis’ma Astron. Zh., 7, 387
Mao S., Paczyński B., 1991, ApJ, 374, L37
Mehta C., 1963, Nuovo Cimento, 28, 401
Messineo M., Hbing H. J., Menten K. M., Omont A., Sjouwerman L. O., Bertoldi F., 2005, A&A, 435, 575
Newell R. T., Hjellming R. M., 1982, ApJ, 263, L85
Ohanian H. C., 1983, ApJ, 271, 551
Paczyński B., 1986, ApJ, 304, 1
Perryman M. A. C. et al., 1997, Astron. Astrophys., 323, L49
Rahal Y. R. et al., 2009, A&A, 500, 1027
Rahvar S., Moniez M., Ansari R., Perdereau O., 2003, A&A, 412, 81
Reid M. J., Menten K. M., 1997, ApJ, 476, 327
Rolett E., Mollerach S., 1997, Phys. Rep., 279, 67
Schilicher R. M. et al., 2007, IEEE Antennas and Propagation Society International Symposium 2006, 9
Schiulzi L. M. et al., 1998, A&A, 435, 575
Schneider P., Schmid-Burgk J., 1985, A&A, 148, 369
Schneider P., Weiß A., 1986, A&A, 164, 237
Schneider P., Ehlers J., Falco E. E., 1992, Gravitational Lenses. Springer-Verlag, Berlin
Vlemmings W. H. T., van Langevelde H. J., Diamond P. J., 2005, Memorie della Societá Astronomica Italiana, 76, 462
Walsh D., Carswell R. F., Weymann R. J., 1979, Nat, 279, 381
Yu D., Xie Z., Hu Q., Yang S., Zhang J., Wang J., 2011, ApJ, 743, 58
Zabel S. A., Peterson J. B., 2003, ApJ, 594, 456

APPENDIX A: COHERENCY IN WAVE OPTICS – GRAVITATIONAL LENSING

In this Appendix we adopt the wave optics notation used in Schneider et al. (1992). Starting from the amplitude of electromagnetic waves on the lens plane, we can obtain the amplitude on the observer plane from the superposition principle as

$$V = \int e^{i\phi(x,y)} \mathrm{d}^2x.$$  \hspace{1cm} (A1)

We perform a Taylor expansion of the Fermat potential around images and diagonalize the second-order derivatives of the potential. The amplitude of the electromagnetic waves on the observer plane is

$$V = \int e^{if(\phi(0)+\frac{1}{2}\phi''(0)\xi^2+\phi''(0)\xi^2)} \mathrm{d}^2x.$$  \hspace{1cm} (A2)

Integrating from equation (A2), we find

$$V = \frac{2\pi i}{f} \frac{1}{\sqrt{\det|J|}} e^{\frac{1}{2} \det|J|}.$$  \hspace{1cm} (A3)

where $\det|J|$ is the determinant of $\phi''(0)$, which is given in equation (6). Here $n$ refers to the type of image and can be equal to 0, 1 or 2, depending on the number of focal points transverse from the source to the observer (Arnold 1989). Now if we have $N$ images from the lensing, the overall amplitude is given by

$$V = \frac{2\pi i}{f} \sum_{i=1}^{N} \frac{1}{\sqrt{\det|J_i|}} e^{\frac{1}{2} \det|J_i|}.$$  \hspace{1cm} (A4)

We note that this equation is valid while the source is out of the caustic lines (i.e. $\det|J| = 0$). We assume a spectrum for the source and replace $f$ with $2\omega R_s$; the overall amplitude can be written as

$$V = \frac{2\pi i}{f} \sum_{i=1}^{N} \frac{1}{\sqrt{\det|J|}} \int \frac{1}{f(\omega)} e^{i(2\omega R_s \phi''(0)\xi^2)} \mathrm{d}\omega.$$  \hspace{1cm} (A5)

Finally, the overall magnification is obtained from $\mu = \mathbf{V}^2$. After averaging over time, the magnification is given by

$$\mu = \sum_{i=1}^{N} \mu_i + \sum_{i \neq j} \frac{4\pi^2}{\sqrt{\det|J_i|} \det|J_j|} \int \frac{g(\omega)^2}{f(\omega)} e^{i(\omega \phi''(0)\xi^2 - \phi''(0)\xi^2)} \mathrm{d}\omega.$$  \hspace{1cm} (A6)

Here, the time integration is done over the oscillating terms of $\phi''(0)$, and for the cross terms the result of integration is a Dirac-delta function. The phase term of $f(\omega)(\phi''(0) - \phi''(0))$ in the integrand depends on the phase difference between the $i$th and $j$th sources. For the case that this phase is larger than the coherent time of the source, the result of this integral is zero and only the first term of equation (A5) is non-zero, representing the geometric optics contribution to the magnification. For the simple case of two images, $N = 2$, let us assume a Gaussian spectrum for $g(\omega)$ with a width given by $\Delta \omega$. The magnification for this simple case is obtained as follows:

$$\mu = \mu_1 + \mu_2 + 2e^{-\frac{\pi}{2} \omega^2} \frac{\mu_1 \mu_2}{\sqrt{\mu_1 \mu_2}} \cos \left(\omega_0 \Delta t - \frac{\pi}{2} (n_1 - n_2)\right).$$  \hspace{1cm} (A7)

where we replaced the width of the spectrum with the coherent time as $\Delta \omega = 1/\tau$. For $\Delta t \ll \tau$, we have an oscillating mode that reveals the wave optics feature from the superposition of the waves. In contrast, for $\Delta t \gg \tau$ the exponential suppresses the oscillating term and we will have a geometric term for the magnification.

Now let us consider an extended source, in which each incoherent point on the source contributes to the amplitude of the electromagnetic waves on the observer plane. Hence for this case we can write equation (A5) for each point of the source, assigning it by $V(s)$. The overall amplitude can be written as

$$|V|^2 = \frac{4\pi^2}{S^2} \int \int \sum_{i,j=1}^{N} \frac{\mathrm{d} s \mathrm{d}s'}{\sqrt{\det|J(s)| \det|J(s')|}} \times \int \int \frac{\left(g(\omega)g(\omega')\right)}{f(\omega)f(\omega')} e^{i(2\omega R_s \phi''(0)(s)\xi^2 - \phi''(0)(s')\xi^2)(n_1 - n_2)\tau)} \mathrm{d}\omega \mathrm{d}\omega'.$$  \hspace{1cm} (A8)
where $S$ is the area of the source and averaging is performed over time; hence $\langle g(\omega)g(\omega') \rangle = g(\omega)^2 \delta(\omega - \omega')$. Because the differential elements on the source are spatially uncorrelated, the cross terms in equation (A8) will cancel, and only light rays propagating from the individual elements of the source contribute in this summation. Mathematically, we can write

$$\langle \phi_i^{(0)}(s)\phi_j^{(0)}(s') \rangle \sim \delta(s - s').$$

Hence equation (A8) simplifies to

$$\mu = \frac{1}{S} \int ds \int \frac{g(\omega)^2}{f(\omega)} \sum_{i,j=1}^{N} e^{\left[ f(\omega) \phi_i^{(0)}(s) - f(\omega) \phi_j^{(0)}(s) - (n_i - n_j) \pi / 2 \right]} \frac{1}{\sqrt{|\det J_i(s)| \det J_j(s)|}} d\omega; \tag{A9}$$

or, in other words, the overall magnification can be written as

$$\mu = \frac{1}{S} \int \mu(s) ds. \tag{A10}$$

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