APPLICATION OF MOUNTAIN PASS THEOREM TO SUPERLINEAR EQUATIONS WITH FRACTIONAL LAPLACIAN CONTROLLED BY DISTRIBUTED PARAMETERS AND BOUNDARY DATA

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ABSTRACT. In the paper we consider a boundary value problem involving a differential equation with the fractional Laplacian \((-\Delta)^{\alpha/2}\) for \(\alpha \in (1,2)\) and some superlinear and subcritical nonlinearity \(G_z\) provided with a nonhomogeneous Dirichlet exterior boundary condition. Some sufficient conditions under which the set of weak solutions to the boundary value problem is nonempty and depends continuously in the Painlevé-Kuratowski sense on distributed parameters and exterior boundary data are stated. The proofs of the existence results rely on the Mountain Pass Theorem.

1. Introduction. The problems with the fractional Laplacian attracted in recent years a lot of attention as they naturally arise in various areas of applications. The fractional Laplacian naturally appears in probabilistic framework as well as in mathematical finance as infinitesimal generators of stable Lévy processes \([1, 6, 32]\). One can find the problems involving the fractional Laplacian in mechanics and in elastostatics, to mention only, a Signorini obstacle problem originating from linear elasticity \([5, 13]\). Then concerning fluid mechanics and hydrodynamics the nonlocal fractional Laplacian appears, for instance, in the quasi-geostrophic fractional Navier-Stokes equation \([15]\) and in the hydrodynamic model of the flow in some porous media \([7, 33]\).

In the paper we consider problems modelled by the differential equation with the fractional Laplacian \((-\Delta)^{\alpha/2}\) and some nonlinearity \(G_z\) of the form

\[(-\Delta)^{\alpha/2} z(x) = G_z(x, z(x), u(x)) \text{ in } \Omega\]  \hspace{1cm} (1)

with the nonhomogeneous Dirichlet exterior condition

\[z(x) = v(x) \text{ in } \mathbb{R}^n \setminus \Omega\]  \hspace{1cm} (2)

where \(\alpha \in (1,2)\) is fixed, \(\Omega \subset \mathbb{R}^n\) for \(n \geq 2\) is a bounded domain with a Lipschitz boundary, \(G_z\) is the partial derivative of the function \(G\) with respect to \(z\) variable which is a suitable Carathéodory function, \(u : \Omega \to \mathbb{R}^m\), \(m \geq 1\), is a distributed parameter, \(v : \mathbb{R}^n \to \mathbb{R}\) is boundary data and the fractional Laplace operator \((-\Delta)^{\alpha/2}\) is defined like, for example, in \([31]\) as

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\[-\Delta^{\alpha/2}z(x) = c(n,\alpha) \int_{\mathbb{R}^n} \frac{2z(x) - z(x + y) - z(x - y)}{|y|^{n+\alpha}} dy\] (3)

where \(c(n,\alpha) = \Gamma((n + \alpha)/2) / (\Gamma(-\alpha/2) \pi^{n/2} 2^{1-\alpha})\) is a positive normalizing constant, cf. [6, 11, 33] and [17] for other normalizations. By using variational methods in an appropriate abstract framework developed by Servadei and Valdinoci [29], first of all, we prove the existence results to (1) - (2) for a certain class of boundary data and distributed parameters. Without going into details we examine the existence of all, we prove the existence results to (1) in an appropriate abstract framework developed by Servadei and Valdinoci [29], first of all, we prove the existence results to (1) - (2) for a certain class of boundary data and distributed parameters. Without going into details we examine the existence of the weak solution \(z\) of (1) - (2) such that \(z - v \in X_0\) where \(v \in V \subset X \cap L^2(\mathbb{R}^n)\) and \(u \in U \subset L^\infty\). Next, we address the stability issue for problem (1) - (2). By stability here we mean the continuous dependence of solutions \(z\) on distributed parameters \(u\) and boundary data \(v\). It is possible to prove that under some suitable assumptions, for an arbitrary pair \((u,v)\) there exists a weak solution \(z_{u,v}\) to problem (1) - (2) which is stable with respect to the distributed parameters \(u\) and the boundary data \(v\). In general, a weak solution is not unique and therefore by stability here we understand upper semicontinuous dependence of sets of weak solutions \(S_{u,v}\) to problem (1) - (2) on distributed parameters \(u\) and boundary data \(v\). In other words, we prove that \(z_{u,v} \rightarrow z_{u_0,v_0}\) in \(X \cap L^2(\mathbb{R}^n)\), if solutions are unique, which means in general case that \(0 \neq \limsup S_{u,v} \subset S_{u_0,v_0}\) in \(X \cap L^2(\mathbb{R}^n)\), provided that \(u\) tends to \(u_0\) in \(L^\infty\) and \(v\) tends to \(v_0\) in \(L^\infty(\mathbb{R}^n)\). The main stability result for problem (1) - (2) is a direct consequence of Theorem 5.2 presented in Section 5.

It should be noted that the weak formulation of system (1) with zero homogeneous exterior boundary condition, i.e. \(v = 0\), corresponds to the Euler-Lagrange equation for the following integral functional

\[
F(z) = \frac{c(n,\alpha)}{2} \int_{\mathbb{R}^{2n}} \frac{|z(x) - z(y)|^2}{|x - y|^{n+\alpha}} dx dy - \int_{\Omega} G(x, z(x), u(x)) dx
\] (4)

where \(z \in X_0\), cf. [28]. The above functional is referred to as the functional of action or the functional of energy. On the function \(G\) we impose, besides some technical, growth and regularity assumptions, the following superlinearity assumption

\[
a < pg(x, z, u) \leq zG_z(x, z, u)
\] (5)

which is satisfied for some \(a > 0\), \(p > 2\) and \(|z|\) sufficiently large. This condition guarantees that problem (1) - (2) can be referred to as a superlinear exterior boundary value problem and as illustrated in Remark 1 the nonlinear functional (4), in general, can be unbounded from above and below. Thus we cannot adopt the approach to the existence and stability issue of Dirichlet problem involving the fractional Laplacian presented for example in [9] where the coercive functional bounded from below was studied, while in [18] only the linear case was treated.

In general, in the theory of boundary value problems and its applications we consider, first of all, the problem of the existence of a solution and next questions of stability, uniqueness, smoothness, asymptotics etc.. The problem of existence of solutions to equation (1) with the homogenous Dirichlet boundary condition corresponding to critical point of mountain pass type was considered for example in the recent papers [28, 29]. For more references on the existence results for problems involving nonlocal fractional Laplacian equation with subcritical nonlinearities, see, for example [27] as well as [4] for problems with critical nonlinearities. Moreover, the asymptotically linear case was investigated in [19] whereas in [23] one can find a bifurcation result in the fractional setting. We also refer the interested reader
to [3, 7, 11, 12, 14, 18, 21, 24, 25, 30] for other results related to the fractional Laplacian. In the paper we apply to the functional defined in (4) the renowned Mountain Pass Theorem presented, for example, in [22] which enables us to obtain the existence result for problem related to (1) – (2) similarly as in [28, 29].

As far as the continuous dependence results of solutions on parameters and boundary data for equation (1) are concerned, up to our best knowledge, the subject in fractional setting seems to have received almost no attention in the literature. Some continuous dependence results for homogenous Dirichlet boundary problem involving the fractional Laplacian one can find in [9] where coercive case is examined by the direct method of calculus of variations. Differentiable continuous dependence on parameters, or in other words robustness result are presented in [10] where the application of theorem on global diffeomorphism leads to the stability result for the problem involving one-dimensional fractional Laplacian with zero boundary condition. In the present paper we obtain the existence and the continuous dependence results for the exterior boundary value problem involving the equation with the fractional Laplacian by adopting the approach presented in [8] where superlinear elliptic boundary value problem with the nonhomogeneous Dirichlet boundary condition was examined.

The structure of the paper reads as follows. Section 2 contains some useful information on functional spaces introduced in [29] by Servadei and Valdinoci with an appropriate extension. The variational formulation of the problem and some standard assumptions are presented in Section 3, whereas in Section 4, our attention is focused on proving some auxiliary lemmas which are of a paramount importance to the rest of the paper. Some sufficient condition for the existence and continuous dependence of solutions to the exterior boundary value problem involving the equation with the fractional Laplacian on distributed parameters and boundary data can be found in Section 5.

2. Functional setup. In this section we introduce the notation and give some preliminary results which will be useful in the sequel. We now recall, following [4, 17, 26], the definition of the classical fractional Sobolev space. Let $D$ be an open, possibly unbounded, domain in $\mathbb{R}^n$ for $n > \alpha$ with suitably smooth boundary, for example Lipschitz (in our case $D = \Omega$ or $D = \mathbb{R}^n$). For $\alpha \in (1, 2)$, by $H^{\alpha/2}(D)$, we denote the following space

$$H^{\alpha/2}(D) = \left\{ z \in L^2(D) : \frac{z(x) - z(y)}{|x - y|^{n+\alpha}/2} \in L^2(D \times D) \right\}. \quad (6)$$

The space $H^{\alpha/2}(D)$ is a Hilbert space placed between $L^2(D)$ and $H^1(D)$ endowed with the norm

$$\|z\|_{H^{\alpha/2}(D)} = \|z\|_{L^2(D)} + \left( \int_{D \times D} \frac{|z(x) - z(y)|^2}{|x - y|^{n+\alpha}} \, dx \, dy \right)^{1/2}. \quad (7)$$

$H^{\alpha/2}_0(D)$ can be defined as completion of $C_0^\infty(D)$ with respect to the norm in $H^{\alpha/2}(D)$ or $H^{\alpha/2}(\mathbb{R}^n)$ and one can extend the functions from $H^{\alpha/2}_0(D)$ with 0 to $\mathbb{R}^n$ as presented in [18]. It should be mentioned for completeness that for domains with non-Lipschitz boundary or for $\alpha \in (0, 1]$ various definitions of the space of the fractional order might lead to non-equivalent formulations, see, for example [7, 18].
Due to the nonlocal character of the fractional Laplacian, we will consider spaces $X^{\alpha/2}$, $X_0^{\alpha/2}$ introduced in [29] and denoted therein by $X$, $X_0$, respectively. However, in the paper we shall work with the specific kernel of the form $K(x) = |x|^{-(n+\alpha)}$. Let $\Omega$ be a bounded domain with a Lipschitz boundary and denote by $Q$ the set as

$$Q = \mathbb{R}^n \setminus ((\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega)).$$

We define

$$X^{\alpha/2} = \left\{ z : \mathbb{R}^n \to \mathbb{R} : z|_{\Omega} \in L^2(\Omega) \text{ and } \frac{z(x) - z(y)}{|x - y|^{(n+\alpha)/2}} \in L^2(Q) \right\}$$

with the norm

$$||z||_{X^{\alpha/2}} = ||z||_{L^2(\Omega)} + \left[ \frac{\int_{Q} \left| \frac{z(x) - z(y)}{|x-y|^{n+\alpha}} \right|^2 \, dx \, dy}{|x-y|^{(n+\alpha)/2}} \right]^{1/2}. \quad (8)$$

For the proof that $|||z||_{X^{\alpha/2}}$ is a norm on $X^{\alpha/2}$, see, for instance, [29]. Obviously, $Q \supset \Omega \times \Omega$ and it implies that $X^{\alpha/2}$ and $H^{\alpha/2}(\Omega)$ are not equivalent as the norms $(7)$ and $(8)$ are not the same. We also consider the linear subspace of $X^{\alpha/2}$

$$X_0^{\alpha/2} = \left\{ z \in X^{\alpha/2} : z = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \right\}$$

with the norm

$$||z||_{X_0^{\alpha/2}} = \left( \int_{Q} \left| \frac{z(x) - z(y)}{|x-y|^{n+\alpha}} \right|^2 \, dx \, dy \right)^{1/2}. \quad (9)$$

We remark that $X^{\alpha/2}$, $X_0^{\alpha/2}$ are nonempty, since, by [30, Lemma 11], $C^2_0(\Omega) \subseteq X_0^{\alpha/2}$. Moreover, the space $X_0^{\alpha/2}$ is a Hilbert space, for the proof of this, see, [18, Lemma 2.3] or [29, Lemma 7] and the inner product has the form

$$(z_1, z_2)_{X_0^{\alpha/2}} = \left( \int_{Q} \frac{(z_1(x) - z_1(y))(z_2(x) - z_2(y))}{|x-y|^{n+\alpha}} \, dx \, dy \right)^{1/2}.$$ 

Furthermore, we have that $X^{\alpha/2} \subset H^{\alpha/2}(\Omega)$, $H^{\alpha/2}(\mathbb{R}^n) \subset X^{\alpha/2}$ and $X_0^{\alpha/2} \subset H^{\alpha/2}(\mathbb{R}^n) \cap H_0^{\alpha/2}(\Omega)$ (cf. [18, 29]).

In order to consider nonhomogenous exterior boundary data we assume that these values are prescribed by a function $v : \mathbb{R}^n \to \mathbb{R}$. For the functional treatment of this problem we need a modification of the space $X^{\alpha/2}$ that turns this normed space into a Hilbert space with the appropriate inner product. For that reason we define the following space

$$Y^{\alpha/2} = X^{\alpha/2} \cap L^2(\mathbb{R}^n)$$

with the norm

$$||z||_{Y^{\alpha/2}} = ||z||_{L^2(\mathbb{R}^n)} + \left( \int_{Q} \left| \frac{z(x) - z(y)}{|x-y|^{n+\alpha}} \right|^2 \, dx \, dy \right)^{1/2}. \quad (10)$$

By analogy with the proof of Lemma 2.3 in [18] or the proof of Lemma 7 in [29] it can be seen that this space is a separable Hilbert space with the inner product

$$(z_1, z_2)_{Y^{\alpha/2}} = (z_1, z_2)_{L^2(\mathbb{R}^n)} + (z_1, z_2)_{X_0^{\alpha/2}}. \quad (11)$$
Immediately, from the definition we have the following inclusions
\[ H^{\alpha/2}(\mathbb{R}^n) \subset Y^{\alpha/2} \subset X^{\alpha/2}. \]

It is worth reminding the reader that for a bounded domain \( \Omega \subset \mathbb{R}^n \) with a Lipschitz boundary, the space \( X_0^{\alpha/2} \) is compactly embedded into \( L^s(\Omega) \) for \( s \in [1,2^*_n) \) where \( 2^*_n = 2n/(n-\alpha) \) and the inequality holds
\[ \|z\|_{L^s(\Omega)} \leq C \|z\|_{X_0^{\alpha/2}} \tag{12} \]
for \( n > \alpha \) and any \( z \in X_0^{\alpha/2} \), cf. Lemma 8 in [29] or Corollary 7.2 in [17].

For further details on the fractional Sobolev spaces we refer the reader to [17] and the references therein, while for other details on \( X^{\alpha/2} \) and \( X_0^{\alpha/2} \) we refer to [30], where these functional spaces were introduced and various properties of these spaces were proved.

3. Variational formulation of the problem and standing assumptions. In the paper we shall consider a problem involving a weak formulation of the following equation with the fractional Laplacian of the form
\[
\begin{cases}
( -\Delta )^{\alpha/2} z(x) = G_z(x,z(x),u(x)) \quad & \text{in } \Omega \subset \mathbb{R}^n \\
z(x) = v(x) \quad & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\tag{13}
\]
where the exterior boundary condition will be ascertained by claiming that \( z - v \in X_0^{\alpha/2} \). \( G_z : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \) and \( \Omega \subset \mathbb{R}^n \), \( n > \alpha \) is a bounded domain with a Lipschitz boundary. Let \( v_0 \) be a fixed element from the space \( Y^{\alpha/2} \). By \( \mathcal{V} \) we denote the following set
\[ \mathcal{V} = \left\{ v \in Y^{\alpha/2} : \|v-v_0\|_{Y^{\alpha/2}} \leq l_1 \right\} \]
for \( l_1 > 0 \) and \( \mathcal{U} \) denotes the set of distributed parameters \( u \) of the form
\[ \mathcal{U} = \{ u \in L^\infty : u(x) \in U \subset \mathbb{R}^m \text{ for a.e. } x \in \Omega \text{ and } \|u\|_{L^\infty} \leq l_2 \} \]
for \( l_2 > 0 \) and some fixed subset \( U \) of \( \mathbb{R}^m \) with \( m \geq 1 \) where \( (C2)-(C5) \) hold.

Besides, one additionally requires that the mapping \( v \) is chosen such that \( v \in \left( X_0^{\alpha/2} \right)^\perp \), while we have the following orthogonal decomposition
\[ Y^{\alpha/2} = X_0^{\alpha/2} \oplus \left( X_0^{\alpha/2} \right)^\perp. \tag{14} \]

As it was announced we look for a weak solution of (13) such that \( z - v \in X_0^{\alpha/2} \).

Let \( w = z - v \), then the problem in (13) can be rewritten as
\[
\begin{cases}
( -\Delta )^{\alpha/2} w(x) + ( -\Delta )^{\alpha/2} v(x) = G_w(x,(w+v)(x),u(x)) \quad & \text{in } \Omega \subset \mathbb{R}^n \\
w(x) = 0 \quad & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\tag{15}
\]

Next, we say that \( w \in X_0^{\alpha/2} \) is a weak solution or an energy solution to (15) if
\[
\int_Q \frac{(w(x) - w(y)) (v(x) - v(y))}{|x-y|^{n+\alpha}} \, dx \, dy + \int_Q \frac{(v(x) - v(y)) (w(x) - w(y))}{|x-y|^{n+\alpha}} \, dx \, dy
\]
\[= \frac{1}{c(n,\alpha)} \int_{\Omega} G_w(x,(w+v)(x),u(x)) \varphi(x) \, dx \tag{16} \]
holds for every \( \varphi \in X^{\alpha/2}_0 \). Then the functional of action defined on \( X^{\alpha/2}_0 \) reads as

\[
F_{u,v}(w) = c(n, \alpha) \left( \int_Q \frac{|w(x) - w(y)|^2}{2|x - y|^{n+\alpha}} \, dx \, dy + \int_Q \frac{(v(x) - v(y)) (w(x) - w(y))}{|x - y|^{n+\alpha}} \, dx \, dy \right)
- \int_{\Omega} G(x, (w + v)(x), u(x)) \, dx
\]

and is related to \( F \) defined in (4) by

\[
F_{u,v}(w) = F(w + v) - c(n, \alpha) \int_Q \frac{|v(x) - v(y)|^2}{2|x - y|^{n+\alpha}} \, dx \, dy.
\]

On the function \( G \) we shall impose the following conditions:

(C1) \( G, G_z \) are Carathéodory functions, i.e. they are measurable with respect to \( x \) for any \( (z, u) \in \mathbb{R} \times \mathbb{R}^m \) and continuous with respect to \( (z, u) \in \mathbb{R} \times \mathbb{R}^m \) for a.e. \( x \in \Omega \);

(C2) for any bounded subset \( U_0 \subset U \), there exists \( c > 0 \) such that

\[ |G(x, z, u)| \leq c(1 + |z|^n), \quad |G_z(x, z, u)| \leq c \left( 1 + |z|^{s-1} \right), \]

for \( z \in \mathbb{R} \), \( u \in U_0 \) and a.e. \( x \in \Omega \), where \( s \in (2, 2^*_n) \) with \( 2^*_n = 2n/(n - \alpha) \) for \( n > \alpha \) and \( \alpha \in (1, 2) \);

(C3) there exist \( p > 2, a > 0 \) and \( R > 0 \) such that

\[ a < pG(x, z, u) \leq zG_z(x, z, u) \]

for a.e. \( x \in \Omega \), any \( u \in U \) and \( |z| \geq R \);

(C4) there exist \( \zeta > 0 \) and \( 0 < b < \frac{c(n, \alpha)}{2} \) such that

\[ \left| G(x, z, u) + \frac{c(n, \alpha)}{2} z^2 \right| \leq \frac{b}{2} |z - v_0(x)|^2 \]

for \( |z| \leq \zeta, u \in U \) and a.e. \( x \in \Omega \) and ess sup |\( v_0 | < \zeta \);

(C5) for any \( u_0 \in U \) and \( \varepsilon > 0 \), there exists a constant \( c > 0 \) such that

\[ |G(x, z, u_1) - G(x, z, u_2)| \leq c \left( 1 + |z|^2 \right) |u_1 - u_2| \]
\[ |G_z(x, z, u_1) - G_z(x, z, u_2)| \leq c(1 + |z|) |u_1 - u_2| \]

for a.e. \( x \in \Omega \), any \( z \in \mathbb{R} \) and \( u_1, u_2 \in U \) such that \( |u_1 - u_0| < \varepsilon \) and \( |u_2 - u_0| < \varepsilon \).

In short, conditions (C1)–(C4), as we shall demonstrate, guarantee the existence of weak solution to problem (13) corresponding to the critical points of mountain pass type of the associated functional of action. If, additionally, condition (C5) is satisfied, it is feasible to prove that these solutions depend continuously (or in general case upper semicontinuously) on \( u \) and \( v \) in appropriate topologies.
4. Verification of Mountain Pass Theorem assumptions. In this section we focus our attention on proving some auxiliary results which are of a key importance to the rest of the paper. First of all, we recall some definitions. Let $I : E \rightarrow \mathbb{R}$ be a functional of $C^1$-class defined on a real Banach space $E$. A point $w \in E$ is a critical point of the functional $I$ if $I'(w) = 0$. Moreover, a value $c = I(w)$ is referred to as a critical value of the functional $I$ related to a critical point $w$.

In what follows we will need some compactness properties of the functional $I$ guaranteeing for example by the Palais-Smale condition. Now we recall what this means. A sequence $\{w_k\} \subset E$ is referred to as a Palais-Smale sequence for a functional $I$ if for some $C > 0$, any $k \in \mathbb{N}$, $|I(w_k)| \leq C$ and $I'(w_k) \rightarrow 0$ as $k \rightarrow \infty$. We say that $I$ satisfies the Palais-Smale condition if any Palais-Smale sequence possesses a convergent subsequence in $E$. For more details on the Palais-Smale condition we refer the reader to Chapter 4.2 in book [22] by Mawhin and Willem.

Moreover, we shall use the following version of the Mountain Pass Theorem.

**Theorem 4.1.** If
(a) there exist $\omega_0, \omega_1 \in E$ and a bounded neighborhood $B$ of $\omega_0$, such that $\omega_1 \in E \setminus B$,
(b) $\inf_{\omega \in \partial B} I(\omega) > \max \{I(\omega_0), I(\omega_1)\}$,
(c) $c = \inf_{g \in M} \max_{t \in [0,1]} I(g(t)), M = \{g \in C([0,1], E) : g(0) = \omega_0, g(1) = \omega_1\}$,
(d) $I$ satisfies the Palais-Smale condition,
then $c$ is a critical value of $I$ and $c > \max \{I(\omega_0), I(\omega_1)\}$.

Throughout this section we shall use the following notation and definitions. First,
$$M_r = \{g \in C([0,1], E_r) : g(0) = \omega_0, g(1) = \omega_1\}$$
where $\omega_0, \omega_1 \in E_r$ and $E_r = \{w \in X_0^{0/2} : \|w\|_{X_0^{0/2}} < r\}$ and $r > 0$. Next, for $k \in \mathbb{N}_0$, let
$$I_k : X_0^{0/2} \rightarrow \mathbb{R}$$
denote an arbitrary sequence of $C^1$-class functionals and $c_k(r)$ be the value defined by setting
$$c_k(r) = \inf_{g \in M_r} \max_{t \in [0,1]} I_k(g(t)).$$
Moreover, for $k \in \mathbb{N}_0$, let $W_k(r)$ denote the set of all critical points in $E_r$ corresponding to the value $c_k(r)$, i.e.
$$W_k(r) = \{w \in E_r : I_k(w) = c_k(r) \text{ and } I_k'(w) = 0\}.$$  \hfill (19)

In what follows, we shall establish the properties of the upper limit of sets $W_k(r)$ in order to state stability results for the problem under consideration. Let us recall that by the Painlevé-Kuratowski upper limit of sets or, in short, the upper limit of sets $S_k$, denoted by $\limsup S_k$, we understand the set of all cluster points with respect to the strong topology of $E$ of a sequence $\{s_k\}$ such that $s_k \in S_k$ for $k \in \mathbb{N}$. In particular, $\limsup W_k(r)$ is the upper limit of the sets $W_k(r)$, $k \in \mathbb{N}$, hence the set of all cluster points with respect to the strong topology of $X_0^{0/2}$ of a sequence $\{w_k\}$ such that $w_k \in W_k(r)$ for $k \in \mathbb{N}$. For more details on the Painlevé-Kuratowski upper limits of sets we refer the reader to the book [2] by Aubin and Frankowska.

Now we prove, under some assumptions imposed on the sequences $\{I_k\}, \{I_k'\}$, that the upper limit in $X_0^{0/2}$ of sets $W_k(r)$ is nonempty and is a subset of $W_0(r)$.

**Lemma 4.2.** Assume that
(a) for any $k \in \mathbb{N}_0$, the functional $I_k$, is of $C^1$-class
(b) the functional $I_0$ satisfies the Palais-Smale condition,
(c) the sequences \( \{ I_k \} \), \( \{ I'_k \} \) tend uniformly on the ball \( B_r \) to \( I_0 \), \( I'_0 \), respectively, (d) for any sufficiently large \( k \in \mathbb{N}_0 \), the sets \( W_k(r) \) are nonempty.

Then any sequence \( \{ w_k \} \) such that \( w_k \in W_k(r) \), \( k \in \mathbb{N} \) is relatively compact in \( X_0^\alpha/2 \) and \( \text{Lim sup} W_k(r) \subset W_0(r) \).

**Proof.** In the proof we shall follow the lines of the proof of Lemma 3.1 from [8]. First of all, one can prove that \( \text{Lim sup} W_k(r) \) is not empty. To do this, let \( \{ w_k \} \) be an arbitrary sequence such that \( w_k \in W_k(r) \) for \( k \in \mathbb{N}_0 \). Such a sequence exists by (d). Moreover, by (c), \( 0 = \lim k \to \infty \mathcal{I}_0'(w_k) \) as \( \mathcal{I}'_k(w_k) = 0 \) for \( k \in \mathbb{N}_0 \). Furthermore, \( \| w_k \|_{X_0^\alpha/2} < r \) hence the sequence \( \mathcal{I}_0(w_k) \) is bounded. Since \( \mathcal{I}_0 \) satisfies the Palais-Smale condition, as assumed in (b), the sequence \( \{ w_k \} \) is relatively compact in \( X_0^\alpha/2 \), that is, \( \text{Lim sup} W_k(r) \) is not empty. Next, again, by (c), as in Lemma 3.1 from [8] taking into account (18) and (19) for \( k \in \mathbb{N}_0 = \{0 \} \cup \mathbb{N} \), we get

\[
\lim_{k \to \infty} c_k(r) = c_0(r).
\]

Moreover, for any sequence \( \{ w_k \} \) such that \( w_k \in W_k(r) \) for \( k \in \mathbb{N} \), we have \( \mathcal{I}_0(w_k) - \mathcal{I}_k(w_k) \to 0 \) as \( k \to \infty \). From the convergence in (20), we conclude that \( \lim k \to \infty \mathcal{I}_0(w_k) = c_0(r) \). Since the set \( \text{Lim sup} W_k(r) \) is not empty, choose \( \tilde{w} \) from this set, so that \( \tilde{w} \) is a cluster point of some sequence \( \{ w_k \} \) such that \( w_k \in W_k(r) \) for \( k \in \mathbb{N} \). Therefore, passing to a subsequence, if necessary, we may assume that \( w_k \to \tilde{w} \) as \( k \to \infty \). Suppose that \( \tilde{w} \not\in W_0(r) \), i.e. \( \mathcal{I}_0(\tilde{w}) \neq c_0(r) \) or \( \mathcal{I}'_0(\tilde{w}) \neq 0 \). Let us observe that the second condition is false. Indeed, assumption (c) and the first part of our proof allow us to write

\[
\mathcal{I}'_0(\tilde{w}) = \lim_{k \to \infty} (\mathcal{I}'_0(w_k) - \mathcal{I}'_k(w_k)) = 0.
\]

By putting \( \delta = \mathcal{I}_0(\tilde{w}) - \mathcal{I}_0(w_0) \), where \( w_0 \in W_0(r) \) and \( \delta \neq 0 \), we arrive at

\[
c_k(r) - c_0(r) = [\mathcal{I}_k(w_k) - \mathcal{I}_0(w_k)] + [\mathcal{I}_0(w_k) - \mathcal{I}_0(\tilde{w})] + \delta.
\]

From (20) and by (a) and (c), we have that \( c_k(r) - c_0(r) \to 0 \), \( \mathcal{I}_k(w_k) - \mathcal{I}_0(w_k) \to 0 \) and \( \mathcal{I}_0(w_k) - \mathcal{I}_0(\tilde{w}) \to 0 \) as \( k \to \infty \). This contradicts the fact that \( \delta \neq 0 \). Thus \( \tilde{w} \in W_0(r) \) and consequently \( \text{Lim sup} W_k(r) \subset W_0(r) \), which concludes the proof. \( \square \)

What we need at this point of our consideration is to examine a specific form of the functional \( \mathcal{I}_k \) derived from the functional of action given by (17).

Let \( \{ v_k \} \) be a sequence of boundary data and \( \{ u_k \} \) a sequence of parameters such that \( \{ v_k \} \in \mathcal{V} \), \( \{ u_k \} \in \mathcal{U} \), for \( k \in \mathbb{N}_0 \). Furthermore, let \( \{ F_k \} \) stand for the sequence of functionals of the form

\[
F_k(w) = F_{u_k,v_k}(w) + \int_{\Omega} G(x, v_k(x), u_k(x)) \, dx
\]

for which we define the value

\[
c_k = \inf_{g \in M} \max_{t \in [0,1]} F_k(g(t)),
\]

where

\[
M = \left\{ g \in C \left( [0,1], X_0^{\alpha/2} \right) : g(0) = \omega_0, \, g(1) = \omega_1 \right\}
\]

and \( \omega_0, \omega_1 \in X_0^{\alpha/2} \).
Here and throughout the paper, for \( k \in \mathbb{N}_0 \), let \( W_k \) denote the set of critical points corresponding to the value \( c_k \), that is, the set of the form
\[
W_k = \left\{ w \in X_0^{\alpha/2} : \mathcal{F}_k(w) = c_k \text{ and } \mathcal{F}_k'(w) = 0 \right\}.
\] (23)
In Section 5, we shall prove that for each \( k \in \mathbb{N} \), the set \( W_k \) is not empty and the sequence of sets \( \{W_k\} \) possesses nonempty upper limit in \( X_0^{\alpha/2} \) such that \( \limsup W_k \subset W_0 \). In the proof of that results we need the following lemma in which the boundedness of the sequence \( \{W_k\} \) is claimed.

**Lemma 4.3.** If the function \( G \) satisfies conditions (C1) – (C3), then for any boundary data \( v_k \in V \) and for any parameter \( u_k \in U \) there exists a ball \( B_\rho = \left\{ w \in X_0^{\alpha/2} : \|w\|_{X_0^{\alpha/2}} < \rho \right\} \) with \( \rho > 0 \) such that \( W_k \subset B_\rho \).

**Proof.** First of all, let us observe that the set of values \( \{c_k : v_k \in V, u_k \in U\} \) is bounded from above. Indeed, for any \( k \in \mathbb{N} \) and \( g(t) = (1 - t)\omega_0 + t\omega_1 \) on \([0,1]\), conditions (C2), (C3) enable us to infer that
\[
c_k = \inf_{g \in M} \max_{t \in [0,1]} \mathcal{F}_k(g(t)) \leq \max_{t \in [0,1]} \mathcal{F}_k((1 - t)\omega_0 + t\omega_1)
\leq \max_{t \in [0,1]} \left( 2c(n, \alpha)(1 - t)^2 \|\omega_0\|_{X_0^{\alpha/2}}^2 + 2c(n, \alpha) t^2 \|\omega_1\|_{X_0^{\alpha/2}}^2 \right.
\left. - \frac{2}{p} \left| \Omega_+ \right| + c(1 + R^*) \left| \Omega_- \right| \right) + \frac{c(n, \alpha)}{2} \|v_k\|_{X_0^{\alpha/2}}^2 + c(1 + R^*) \|\omega_0\|_{X_0^{\alpha/2}}^2 + c(1 + R^*) \|\omega_1\|_{X_0^{\alpha/2}}^2
\leq 2c(n, \alpha) \max \left\{ \|\omega_0\|_{X_0^{\alpha/2}}^2, \|\omega_1\|_{X_0^{\alpha/2}}^2 \right\} + D \leq \bar{c},
\] and therefore
\[
c_k \leq \bar{c}
\] where \( D, c, d, \bar{c} \) are some constants, and the sets \( \Omega_+^-, \Omega_-^+ \) are defined as
\[
\Omega_+^+ = \left\{ x \in \Omega : |(g(t) + v_k)(x)| \geq R \right\}, \Omega_-^- = \left\{ x \in \Omega : |(g(t) + v_k)(x)| < R \right\}.
\]
Then, for any \( v_k \in V, u_k \in U \) and \( w \in W_k \) we have by (C2) and (C3)
\[
p\bar{c} \geq p c_k = p \mathcal{F}_k(w) - \langle \mathcal{F}_k'(w), w + v_k \rangle
\geq c(n, \alpha) \left( \frac{p - 2}{2} \|w\|_{X_0^{\alpha/2}}^2 - (p - 2) \|v_k\|_{X_0^{\alpha/2}}^2 \right) - pc \left| \Omega_+ \right| - pc \|v_k\|_{L^p(\Omega)}^p + D \|
\geq c(n, \alpha) \frac{p - 2}{2} \|w\|_{X_0^{\alpha/2}}^2 - D_1 \|w\|_{X_0^{\alpha/2}} - D_2.
\] where \( D_1, D_2 \) are some positive constants and
\[
\Omega_+^+ = \left\{ x \in \Omega : |(w + v_k)(x)| \geq R \right\}, \Omega_-^- = \left\{ x \in \Omega : |(w + v_k)(x)| < R \right\}.
\]
and \( \langle \cdot, \cdot \rangle \) is a dual pair in \( X_0^{\alpha/2} \). Thus
\[
p\bar{c} \geq c(n, \alpha) \frac{p - 2}{2} \|w\|_{X_0^{\alpha/2}}^2 + D_1 \|w\|_{X_0^{\alpha/2}} + D_2.
\] (24)
By condition (C3), \( p - 2 > 0 \), and consequently there exists \( \rho > 0 \) such that \( w \in B_\rho \). Hence, \( W_k \subset B_\rho \) for any \( v_k \in V \) and \( u_k \in U \), which completes the proof. \( \square \)

After necessary shift resulting in (15) and \( \mathcal{F}_k \) defined by (21), we can assume, without loss of generality, that from this point \( \omega_0 = 0 \). It is possible to demonstrate that there exist a bounded neighborhood \( B \) of \( \omega_0 \) in \( X_0^{\alpha/2} \) and some point \( \omega_1 \notin B \) such that the assumptions of Theorem 4.1 are fulfilled for \( \mathcal{F}_k \).
Lemma 4.4. Suppose that conditions (C1)–(C4) are satisfied, the sequence \{v_k\} \subset \mathcal{V} tends to \(v_0\) in \(Y^{\alpha/2}\) and the sequence \{u_k\} \subset \mathcal{U} tends to \(u_0\) in \(L^\infty\). Then for any sufficiently large \(k \in \mathbb{N}\), \(v_k \in \mathcal{V}\) and \(u_k \in \mathcal{U}\), there exist a ball \(B_\eta \subset X_0^{\alpha/2}\) and an element \(\omega_1 \notin \overline{B_\eta}\) such that \(\inf_{w \in \partial B_\eta} \mathcal{F}_k (w) > 0\) and \(\mathcal{F}_k (\omega_1) < 0\), where \(B_\eta = \left\{ w \in X_0^{\alpha/2} : \|w\|_{Y^{\alpha/2}} < \eta \right\}\) for \(\eta > 0\).

Proof. From (C3), in a similar fashion as in the proof of Lemma 4 in [28], there exists a constant \(a_0 > 0\) such that
\[
G (x, z, u) \geq a_0 |z|^p,
\]
for any \(|z| \geq R, u \in U\), a.e. \(x \in \Omega\) and \(R > 0\) as in (C3) with \(p > 2\). On combining the above inequality with (C2) we get that there exists a constant \(a_1 > 0\) such that
\[
G (x, z, u) \geq a_0 |z|^p - a_1
\]
for \(z \in \mathbb{R}, u \in U\) and a.e. \(x \in \Omega\), with \(p \in (2, 2^*_\alpha)\). By (C2), (C4) similarly to the proof of Lemma 3 in [29] there exist \(b \in \left(0, \frac{c(n, \alpha)}{2}\right)\) and \(a_2 > 0\) such that
\[
\left| G (x, z, u) + \frac{c(n, \alpha)}{2} z^2 \right| \leq b \|z - v_0 (x)\|^2 + a_2 \|z - v_0 (x)\|^s
\]
for any \(z \in \mathbb{R}, u \in U\), a.e. \(x \in \Omega\) and \(s \in (2, 2^*_\alpha)\).

For fixed \(k \in \mathbb{N}\), the orthogonality condition \((w, v_k)_{Y^{\alpha/2}} = 0\) and inequality in (26) lead to the following estimate
\[
\mathcal{F}_k (w) = \frac{c(n, \alpha)}{2} \|w\|^2_{Y^{\alpha/2}} + \frac{c(n, \alpha)}{2} (w + v_k)^2 - G (x, v_k, u_k) - \frac{c(n, \alpha)}{2} v_k^2
\]
\[
\geq \left( \frac{c(n, \alpha)}{2} - b \right) \|w\|^2_{Y^{\alpha/2}} - C_1 \|w\|^s_{Y^{\alpha/2}} - C_2 \|v_k - v_0\|^2_{Y^{\alpha/2}} - C_3 \|v_k - v_0\|^s_{Y^{\alpha/2}}
\]
where constants \(C_1, C_2, C_3\) are positive. Since, by (C4), \(b < \frac{c(n, \alpha)}{2}\) and \(v_k \to v_0\) in \(Y^{\alpha/2}\) while \(s > 2\), it follows that there exists a constant \(\eta > 0\) such that \(\inf_{w \in \partial B_\eta} \mathcal{F}_k (w) \geq \varepsilon > 0\) for any \(k\) sufficiently large.

To finish the proof it is enough to show that for any \(v_k \in \mathcal{V}\) and \(u_k \in \mathcal{U}\) there exists \(\omega_1 \notin \overline{B_\eta}\) such that \(\mathcal{F}_k (\omega_1) < 0\). For a fixed nonzero \(w \in X_0^{\alpha/2}\) and \(l > 0\), from (C2) and (C3) hence (25) we have the following estimates
\[
\mathcal{F}_k (lw) \leq \frac{c(n, \alpha)}{2} \|w\|^2_{Y^{\alpha/2}} + c(n, \alpha) l C_4 \|w\|_{Y^{\alpha/2}} - \int \Omega (a_0 |lw + v_k|^p - a_1) \ dx
\]
\[
+ \int \Omega G (x, v_k, u_k) \ dx
\]
\[
\leq \frac{c(n, \alpha)}{2} \|w\|^2_{Y^{\alpha/2}} + c(n, \alpha) l C_4 \|w\|_{Y^{\alpha/2}} - \frac{a_0 l^p}{p} \|w + \frac{\eta}{l}\|^p_{L^p} + C_5
\]
where \(C_4, C_5, a_0 > 0\). Since \(p \in (2, 2^*_\alpha)\) and \(a_0 > 0\) from (25) we get \(\lim_{l \to \infty} \mathcal{F}_k (lw) = -\infty\). Therefore, there exists \(l_0 > 0\) such that for \(\omega_1 = l_0 w\) we have \(\|\omega_1\|_{Y^{\alpha/2}} \geq \eta\) and \(\mathcal{F}_k (\omega_1) < 0\) for any \(v_k \in \mathcal{V}\) and \(u_k \in \mathcal{U}\), which completes the proof.

Lemmas 4.3 and 4.4 give that the geometry of Mountain Pass Theorem is fulfilled by \(\mathcal{F}_k\). Therefore, in order to apply such Mountain Pass Theorem, we are left with checking the validity of the Palais-Smale condition. This will be accomplished in the forthcoming lemma.
Lemma 4.5. Suppose that conditions (C1) – (C4) are satisfied. Then for any $k \in \mathbb{N}_0$, $F_k$ satisfies the Palais-Smale condition.

Proof. Let $k$ be fixed and $\{w_i\}$ be a Palais-Smale sequence so that $\{F_k(w_i)\}$ is bounded and $F'_k(w_i) \to 0$ as $i \to \infty$. Thus, there exist constants $C_1, C_2 > 0$ such that $|F_k(w_i)| \leq C_1$ and $\|F'_k(w_i)\| \leq C_2$ for all $i \in \mathbb{N}$. In the same manner as in the proof of Lemma 4.3, we obtain the following estimates

$$pC_1 + C_2 \|v_k\|_{Y^{\alpha/2}} + C_3 \|w_i\|_{X_0^{\alpha/2}} \geq pC_1 + C_2 \|w_i + v_k\|_{Y^{\alpha/2}}$$

$$\geq pF_k(w_i) - \langle F'_k(w_i), w_i + v_k \rangle \geq c(n, \alpha) \frac{p^2}{2} \|w_i\|_{X_0^{\alpha/2}}^2 - D_1 \|w_i\|_{X_0^{\alpha/2}} - D_2,$$

where constants $D_1, D_2 > 0$ are from the proof of Lemma 4.3 and $C_3 > 0$. Hence

$$\|w_i\|_{X_0^{\alpha/2}}^2 \leq \frac{2}{c(n, \alpha)(p-2)} \left( C_4 \|w_i\|_{X_0^{\alpha/2}} + pC_1 + C_2 \|v_k\|_{Y^{\alpha/2}} + D_2 \right)$$

for $i \in \mathbb{N}$, where $C_4 > 0$. Therefore, the sequence $\{w_i\}$ is bounded in $X_0^{\alpha/2}$ and as such it contains a subsequence, still denoted by $\{w_i\}$, such that $w_i$ tends to $w_0$ weakly in $X_0^{\alpha/2}$, since the space $X_0^{\alpha/2}$ is reflexive. From the fact that the space $X_0^{\alpha/2}$ is compactly embedded into the space $L^s(\Omega)$ with $s \in [1, \frac{\alpha}{\alpha-n})$, we may assume after passing to a subsequence, still labelled $\{w_i\}$, that $w_i \to w_0$ in $L^s(\Omega)$. Consequently,

$$\langle F'_k(w_i) - F'_k(w_0), w_i - w_0 \rangle \to 0.$$

The equality

$$\langle F'_k(w_i) - F'_k(w_0), w_i - w_0 \rangle = c(n, \alpha) \|w_i - w_0\|_{X_0^{\alpha/2}}^2$$

$$\int_\Omega (G_w(x, w_0 + v_k, u_k) - G_w(x, w_i + v_k, u_k)) (w_i - w_0) \, dx$$

and the growth condition (C2) lead by Hölder inequality to

$$\left| \int_\Omega (G_w(x, w_0 + v_k, u_k) - G_w(x, w_i + v_k, u_k)) (w_i - w_0) \, dx \right|$$

$$\leq \|w_i - w_0\|_{L^s} \left( \int_\Omega |G_w(x, w_i + v_k, u_k) - G_w(x, w_0 + v_k, u_k)|^{\frac{s}{s-1}} \, dx \right)^{\frac{s-1}{s}}.$$

Let us notice that the right hand side of the above inequality tends as $i \to \infty$ to 0, by the growth condition (C2) and since $w_i \to w_0$ in $L^s(\Omega)$. As a result, $w_i \to w_0$ in $X_0^{\alpha/2}$. In that way we get that for any $k \in \mathbb{N}_0$, $F_k$ satisfies the Palais-Smale condition guaranteeing the required compactness property.

5. The main continuity result. In this section we state and prove some sufficient conditions under which critical points of mountain pass type of the functional $F_k$ defined in (21) exist and depend continuously on parameters. First of all, we state some sufficient conditions guaranteeing a uniform convergence on any ball in the space $X_0^{\alpha/2}$ of a sequence of functionals together with a sequence of their derivatives. This is in fact verification of the assumption (b) of Lemma 4.2.

Lemma 5.1. If conditions (C1), (C2), (C5) are satisfied, the sequence $\{v_k\} \subset \mathcal{V}$ tends to $v_0$ in $Y^{\alpha/2}$ while the sequence $\{u_k\} \subset \mathcal{U}$ tends to $u_0$ in $L^\infty$, then $\{F_k\}$, $\{F'_k\}$ tend uniformly on any ball from $X_0^{\alpha/2}$ to $F_0$ and $F'_0$, respectively.
Theorem 5.2. Suppose that the function

Proof. For any \( B_\rho \subset X^0_{\alpha/2} \) and \( w \in B_\rho \), we have

\[
|F_k(w) - F_0(w)| \leq c(n, \alpha) \rho \|v_k - v_0\|_{Y^\alpha/2} \\
+ \int_\Omega |G(x, v_k, u_0) - G(x, v_0, u_0)| \, dx + \int_\Omega |G(x, w + v_k, u_0) - G(x, w + v_0, u_0)| \, dx \\
+ \|u_k - u_0\|_{L^\infty}(D_1 + D_2 \|v_k\|_{Y^\alpha/2}^2) < \varepsilon
\]

for any fixed \( \varepsilon > 0 \) and sufficiently large \( k \). Indeed, let us recall that \( v_k \) tends to \( v_0 \) in \( Y^\alpha/2 \), hence up to subsequence, it converges in \( L^s \) with \( s \in (2, 2^*_\alpha) \). By conditions (C1), (C2), (C5), the Krasnosielskii Theorem on continuity of Niemyckii’s operator, cf. [20], implies that the right hand side of the above inequality tends to 0 for any \( w \in B_\rho \). It means that the sequence \( \{F_k\} \) tends uniformly to \( F_0 \) on a ball \( B_\rho \).

A similar reasoning holds for the case of a uniform convergence of the sequence \( \{F'_k\} \) to \( F'_0 \) on a ball from \( X^0_{\alpha/2} \). Let us take any ball \( B_\rho \subset X^0_{\alpha/2} \). For any \( w \in B_\rho \) and \( h \in X^0_{\alpha/2} \) such that \( h \in B_1 \) simple calculations lead to

\[
|(F'_k(w) - F'_0(w), h)| \\
\leq c(n, \alpha) \|v_k - v_0\|_{Y^\alpha/2} + \left( \int_\Omega |G_w(x, w + v_k, u_0) - G_w(x, w + v_0, u_0)|^{\alpha/2} \, dx \right)^{1/\alpha} \\
+ \|u_k - u_0\|_{L^\infty}(D_3 + D_4 \|v_k\|_{Y^\alpha/2}) < \varepsilon
\]

for sufficiently large \( k \), and the claim of the lemma follows.

Now we employ Lemmas 4.2, 4.3, 4.4, 4.5, 5.1 and Theorem 4.1 to prove:

Theorem 5.2. Suppose that the function \( G \) satisfies conditions (C1) – (C5) and moreover the sequence \( \{v_k\} \subset V \) tends to \( v_0 \) in \( Y^\alpha/2 \) and the sequence \( \{u_k\} \subset U \) tends to \( u_0 \) in \( L^\infty \). Then

(i) for any \( k \) large enough, the set of critical points \( W_k \) of the functional \( F_k \) is nonempty and does not contain zero and in fact any point with zero value,

(ii) any sequence \( \{w_k\} \) such that \( w_k \in W_k, k \in \mathbb{N} \), is relatively compact in \( X^0_{\alpha/2} \) and \( \limsup W_k \subset W_0 \).

Proof. First, we shall prove the first part of the assertion of Theorem 5.2, that is for any \( k \) sufficiently large, the set of critical points \( W_k \) of the functional \( F_k \) is nonempty and does not contain zero and any point with zero value. Obviously, the functional \( F_k, k \in \mathbb{N}_0 \) is of \( C^1 \)–class on \( X^0_{\alpha/2} \). Moreover, from Lemma 4.4 it follows that there exist the ball \( B_\rho \) and the point \( \omega_1 \in X^0_{\alpha/2} \), independent of the choice of \( v_k, u_k \), such that \( \omega_1 \notin \overline{B_\rho} \) and \( \inf_{w \in \partial B_\rho} F_k(w) > 0 = \max \{F_k(0), F_k(\omega_1)\} \), \( k \in \mathbb{N}_0 \) and furthermore conditions (C2), (C3) guarantee that the functional \( F_k \) satisfies the Palais-Smale condition for \( k \in \mathbb{N}_0 \) as stated in Lemma 4.5. At this point we apply the Mountain Pass Theorem 4.1, with \( \omega_0 = 0 \) and \( c = c_k \), to deduce that for any \( v_k \) and \( u_k \), the set of critical points for which a critical value of the functional \( F_k \) denoted by \( c_k \) is attained, is not empty, i.e.

\[
W_k = \left\{ w \in X^0_{\alpha/2} : F_k(w) = c_k \text{ and } F'_k(w) = 0 \right\} \neq \emptyset.
\]

Moreover, \( c_k = \inf_{g \in M} \max_{t \in [0,1]} F_k(g(t)) > 0 = \max \{F_k(0), F_k(\omega_1)\} = 0 \), so \( w = 0 \) does not belong to the set \( W_k \) for all \( k \in \mathbb{N}_0 \) and actually any point with zero value.

What is left is to demonstrate that \( \limsup W_k \neq \emptyset \) in \( X^0_{\alpha/2} \) and \( \limsup W_k \subset W_0 \).
This part of the assertion of Theorem 5.2 follows directly from Lemma 4.2. Indeed, by invoking Lemma 4.3, we obtain that there exists a ball $B_\rho \subset X_0^{\alpha/2}$ with $\rho > 0$ such that $W_k \subset B_\rho$ for all $k \in \mathbb{N}_0$, and subsequently there exists a ball $B_\rho \subset X_0^{\alpha/2}$ with $r \geq \rho$ such that $w_1 \in B_r$, i.e. $W_k (r) = W_k$, where $W_k (r)$ is given by (19) with $I_k = F_k$, $k \in \mathbb{N}_0$. $F_k$ satisfies the Palais-Smale condition and each functional $F_k$ is of $C^1$–class, and corresponding set $W_k (r) = W_k$ is nonempty for $k \in \mathbb{N}_0$. Finally, Lemma 5.1 implies that the sequences $\{F_k\}$, $\{F'_k\}$ tend uniformly on any ball from $X_0^{\alpha/2}$ to $F_0$ and $F'_0$, respectively, which completes the proof.

Let us notice that, for $k \in \mathbb{N}_0$, the critical value of the functional $F_{u_k,v_k}$ defined in (17) denoted by $c_k$ satisfies the following relation

$$c_k = c_k - \int_{\Omega} G (x, v_k (x), u_k (x)) \, dx$$

where $c_k$ is defined in (22) as the critical value of $F_k$. The set of critical points of the functional $F_{u_k,v_k}$ for which the critical value $c_k$ is attained has the form

$$W_{u_k,v_k} = \left\{ w \in X_0^{\alpha/2} : F_{u_k,v_k} (w) = c_k \text{ and } F'_{u_k,v_k} (w) = 0 \right\}.$$ 

Immediately from Theorem 5.2 we get the following corollaries characterizing, first of all, the set of critical points of mountain pass type of the functional of action without shift and then the set of corresponding weak solutions of the problem involving the equation with the fractional Laplacian with exterior homogeneous Dirichlet boundary data.

**Corollary 1.** If all assumptions of Theorem 5.2 are satisfied, then for any $k$ sufficiently large the set $W_{u_k,v_k}$ is nonempty and does not contain zero and any point with zero value, $\limsup W_{u_k,v_k} \neq \emptyset$ in $X_0^{\alpha/2}$ and $\limsup W_{u_k,v_k} \subset W_{u_0,v_0}$.

In other words, the set valued mapping $L^\infty \times Y^{\alpha/2} \ni (u, v) \mapsto W_{u,v} \subset X_0^{\alpha/2}$ is upper semicontinuous with respect to the norm topologies of $L^\infty$, $Y^{\alpha/2}$ and $X_0^{\alpha/2}$.

Let us denote by $S^w_{u_k,v_k}$ the set of the weak solutions to problem (13) defined in (16) corresponding to the critical value $c_k$.

**Corollary 2.** If all assumptions of Theorem 5.2 are satisfied, then for any $k$ sufficiently large the set $S^w_{u_k,v_k}$ is nonempty, $\limsup S^w_{u_k,v_k} \neq \emptyset$ in $X_0^{\alpha/2}$ and $\limsup S^w_{u_k,v_k} \subset S^w_{u_0,v_0}$.

Furthermore, it is easy to observe that $S^z_{u_k,v_k} = S^w_{u_k,v_k} + v_k$, $k \in \mathbb{N}_0$ is a set of weak solutions to problem (1) with nonhomogenous exterior boundary data corresponding to the critical value $c_k$.

**Corollary 3.** If all assumptions of Theorem 5.2 are satisfied, then for any $k$ the set $S^w_{u_k,v_k}$ is nonempty and does not contain $v_k$, $\limsup S^z_{u_k,v_k} \neq \emptyset$ in $Y^{\alpha/2}$ and $\limsup S^z_{u_k,v_k} \subset S^z_{u_0,v_0}$.

**Example 5.3.** Let $\Omega = (0,1)^3$. Consider the equation

$$
\begin{cases}
(-\Delta)^{3/4} z (x) = \frac{7}{2} z^{5/2} (x) - \gamma u (x) z (x) - \frac{5}{2} u (x) z^{3/2} (x) \sin^2 |x| \text{ in } \Omega \subset \mathbb{R}^3 \\
z (x) = v (x) \text{ in } \mathbb{R}^3 \setminus \Omega,
\end{cases}
$$

(27)
where \( u \in \mathcal{U} = \{ u \in L^\infty (\Omega, \mathbb{R}) : u (x) \in U \subset (\hat{c} - b) / \gamma, (\hat{c} + b) / \gamma \text{ a.e.} \} \) with \( \gamma > 0 \), \( \hat{c} = \frac{15}{2^{11/2} \pi^{9/2}} \) and \( b \) is like in (C4) while \( v \in \mathcal{V} = \{ v \in Y^{3/4} : v (x) \in [0, 1], \| v \| \leq 1 \} \). Let us notice that the functional of action for the equation with homogenous exterior boundary condition related to \( F_{u,v} \) defined in (17), and to \( \mathcal{F}_k \) defined in (21) has the following form

\[
F (z) = \frac{c}{2} \int_Q \frac{|z(x) - z(y)|^2}{|x - y|^{3/2}} dx dy - \int_\Omega \left[ z^2 (x) - \frac{\gamma}{2} u (x) z^2 (x) - u (x) z^2 (x) \sin^2 |x| \right] dx
\]

where \( z \in X^{3/4}_0 \). It is easy to check that all assumption \((C1) - (C5)\) are satisfied by

\[
G(x, z, u) = z^{7/2} - \frac{\gamma}{2} u z^2 - u z^5/2 \sin^2 |x|.
\]

By Corollary 3, for any \( u \in \mathcal{U} \) and \( v \in \mathcal{V} \), there exists a weak solution \( z_{u,v} \in Y^{3/4} \) to problem (27) and the set of weak solutions continuously, or upper semicontinuously, depends on boundary data \( v \to 0 \) and distributed parameter \( u \).

**Remark 1.** Let us observe that for \( F \) defined in (28) putting, \( \tilde{z}_k (x) = k \rho_1 (x) \) where \( \rho_1 \) is the positive first eigenfunction of the fractional Laplace operator defined in (3) with \( \alpha = 1/2 \) we get for some \( \delta > 0 \)

\[
F (\tilde{z}_k) \leq \frac{\hat{c}}{2} k^2 \| \rho_1 \|^2_{X^{3/2}} - k^{7/2} \| \rho_1 \|^2_{L^{7/2}} - \frac{\gamma}{2} \| u \|_{L^2} k^2 \| \rho_1 \|^2_{L^4} + k^{5/2} \| u \|_{L^\infty} \| \rho_1 \|^2_{L^{5/2}}
\]

\[
\leq -k^{7/2} \left( \| \rho_1 \|^2_{L^{7/2}} + k^{-\delta} A \right) \to -\infty
\]

as \( k \to \infty \). Moreover, for a sequence \( \pi_k (x) = \rho_k (x) \) such that \((-\Delta)^{3/4} \rho_k = \lambda_k \rho_k\), \( \| \rho_k \|_{L^2} = 1 \) and \( \| \rho_k \|_{L^\infty} \leq C \) for some \( C > 0 \) we obtain

\[
F (\pi_k) \geq \frac{\hat{c}}{2} \lambda_k \| \rho_k \|^2_{L^2} - C = \frac{\hat{c}}{2} \lambda_k - C \to \infty
\]

as \( k \to \infty \) and \( \lambda_k \sim k^{1/2} \) (cf. \[16\]) while \( C \) depends on \( \| u \|_{L^\infty} \) and \( \| \rho_k \|_{L^\infty} \) in a bounded way. Consequently, the functional \( F \) is unbounded both from above and below. For that reason we cannot use methods applied, for example, in \[10\].

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