CURRENT ALGEBRA AND EXOTIC
STATISTICS IN 6 DIMENSIONS

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ABSTRACT By studying ordinary chiral fermions in background gauge fields we show that in the case of gauge group $SU(3)$ and space-time dimension $5 + 1$ localized solitons obey $q -$ commutation relations with $q$ not equal to $\pm 1$ but a third root of unity.

1. INTRODUCTION

A topological explanation of the spin-statistics relation in quantum field theory is an old idea which was proposed by Finkelstein and Rubinstein in 1968, [FR]. Later, in 1984 Witten studied a concrete model (Wess-Zumino model) in $3 + 1$ space-time dimensions and showed that there are solitonic field configurations which behave like fermions under rotations. The essential property of the model is that there is a nonlocal (in $3 + 1$ dimensions) term which can be written as an integral of a 5-dimensional topological density, [W]. One can also show that an interchange of a two nonoverlapping solitons of odd degree produces a phase factor $-1$ of the corresponding states in Fock space. This is a consequence of the properties of the way the gauge group acts in the Fock space, [M1].

In this paper we shall study a more exotic statistics in higher space-time dimensions. In $3 + 1$ dimensions classical solitons can be quantized only as fermions or bosons. In $1 + 1$ it is known that there are models with fractional statistics. We shall show that also in $5 + 1$ dimensions there is an option to quantize solitons
such that they do not obey neither canonical commutation nor anticommutation relations but relations of the type

\[ q(f_1)q(f_2) = e^{2\pi i/3}q(f_2)q(f_1). \]

This relation is obtained starting from a quantization of ordinary chiral fermions in external gauge field with the gauge group \( SU(3) \) and looking at the states which are produced from the vacuum by the action of topologically nontrivial gauge transformations. As in Finkelstein’s and Rubinstein’s original idea, everything goes back to a computation of certain homotopy groups. In Witten’s case the solitons were classified by a winding number corresponding to an element in \( \pi_3(SU(2)) \) and the fermionic property under rotations is explained using \( \pi_4(SU(2)) = \mathbb{Z}_2 \). In our case we just increase dimensions; \( SU(3) \) solitons in 5 dimensions are labelled by \( \pi_5(SU(3)) = \mathbb{Z} \) and the exotic statistics corresponds to \( \pi_6(SU(3)) = \mathbb{Z}_6 \).

2. ACTION OF THE GAUGE GROUP ON FERMIONIC VACUA

In this section we shall explain some results in [CMM1, CMM2] which are necessary for the discussion in Section 3.

Let \( M \) be the compactified physical space which is assumed to be a manifold without boundary, of odd dimension \( d = 2n + 1 \) and with a fixed metric, spin structure, and orientation. In order to describe the minimal coupling of fermions to external Yang-Mills potentials we define the fermionic one-particle Hilbert space as \( H = L^2(S) \otimes V \), where the first factor in the tensor product is the space of square-integrable sections of the spin bundle (with fiber dimension \( 2^n \)) and \( V \) is a complex finite dimensional inner product space. A compact gauge group \( G \) acts in \( V \). A vector potential is a smooth 1-form \( A \) on \( M \) with values in the Lie algebra \( g \) of \( G \). We denote by \( A \) the affine space of all vector potentials. Each \( A \in A \) defines a Weyl-Dirac hamiltonian \( D_A \) acting on smooth sections in \( H \). Note that we are here dealing with massless (left handed, say) fermions.

The Hilbert space \( H \) splits to \( H = H_+ \oplus H_- \), where \( H_+ \) corresponds to positive energies for a fixed (background) hamiltonian \( D_0 = D_{A_0} \) and \( H_- \) is its orthogonal complement. This polarization defines a representation of the canonical anticommutation relations with a Dirac vacuum \( |0> \). The technical difficulty is that for
different vector potentials $A$ the representations of the CAR algebra are inequivalent. Geometrically, the Fock spaces for different potentials form a smooth vector bundle $\mathcal{F}$ where the fibers $\mathcal{F}_A$ carry inequivalent CAR representations. However, the bundle can be trivialized as $\mathcal{F} \simeq \mathcal{A} \times \mathcal{F}_0$ because the base space $\mathcal{A}$ is affine. Except for the case $d = 1$ there is no natural trivialization.

In general, the gauge action of $\mathcal{G} = Map(M, G)$ in $\mathcal{A}$ cannot be lifted to the total space of the bundle $\mathcal{F}$ such that $g^{-1} \hat{D}_A g = \hat{D}_{Ag}$ for all $A \in \mathcal{A}$ and $g \in \mathcal{G}$. Here $\hat{D}_A$ is the second quantized Hamiltonian and $A^g = g^{-1} A g + g^{-1} d g$. Instead, there is a nontrivial extension $\hat{\mathcal{G}}$ which acts in $\mathcal{F}$. The fiber of the extension is $Map(A, S^1)$. If $d = 1$ then one can reduce the fiber to the subgroup $S^1$ of constant functions on $\mathcal{A}$, defining a central extension of the loop group $LG$ for $M = S^1$.

On the Lie algebra level, the extension of the group of gauge transformations is given by a 2-cocycle (Schwinger term),

$$[q(X), q(Y)] = q([X, Y]) + c(X, Y; A),$$

where $q(X)$ denotes the second quantization of the infinitesimal gauge transformation $X$ and the cocycle $c$ is a linear function of $X, Y$, antisymmetric in the arguments $X, Y$, and satisfies

$$c(X, [Y, Z]; A) + \mathcal{L}_X c(Y, Z; A) + \text{cycl. permutations of } X, Y, Z = 0.$$

Here $\mathcal{L}_X f$ denotes the infinitesimal gauge variation of a function $f = f(A)$. If $d = 1$ then

$$c(X, Y; A) = \frac{1}{2\pi} \int \text{tr } X dY$$

does not depend on $A$. The trace is computed in the vector space $V$. For $d = 3$ one has, [M2, F-Sh],

$$c(X, Y; A) = \frac{i}{24\pi^2} \int \text{tr } A[dX, dY].$$

Given a vector $\psi \in \mathcal{F}_A$ we have a connection in the complex line bundle $E$ over the gauge orbit $A^g$; the fiber at $A^g$ is spanned by the vector $\hat{g}^{-1} \psi$, where $\hat{g}$ is any element in the fiber in $\hat{\mathcal{G}}$ over $g$. The curvature of the line bundle at $A$ is given by the 2-cocycle $c$.

We shall study the case when the local anomaly vanishes, that is, the Schwinger term $c(X, Y; A) \equiv 0$. There still might be a global anomaly which manifests itself
as a nontrivial holonomy along gauge orbits in vacuum subbundles of the Fock bundle. (In the case of Witten’s anomaly this was analyzed in [NA].) As discussed in [CMM2], this leads to an extension of $G$ by a finite center $Z$. The center is equal to $\pi_{d+1}(G)$. Whether it acts faithfully or not in Fock spaces has to be determined separately for each choice of $G$, dimension $d$, and the representation of $G$ in $V$. We shall concentrate to the physically interesting case $G = SU(3)$. If $d = 3$ we have $\pi_4(G) = 0$ and there is no global anomaly. The first nontrivial case is $d = 5$ and $\pi_6(G) = \mathbb{Z}_6$. It is known that for this choice the extension $\hat{G}$ acts faithfully in the Fock bundle, [CMM2]. In other words, a parallel transport of a vector in the Fock space around a closed loop in a gauge orbit in $A$ produces a phase $e^{k \cdot 2\pi i/6}$ corresponding to the element $k \in \mathbb{Z}_6$ defined by the loop in $\text{Map}(M,G).

3. GENERALIZED STATISTICS

We now pose the following problem. Consider a pair of localized soliton configurations $f_1, f_2 \in G = \text{Map}(M,G)$. According to Section 2 we can construct quantum operators $\hat{f}_1$ and $\hat{f}_2$ acting on second quantized fermions, or more precisely, we have operators acting on sections of the bundle of fermionic Fock spaces parametrized by external vector potentials. The operators $\hat{f}_i$ are uniquely determined up to a phase, which in the case when $\pi_1(G) = \mathbb{Z}_n$ and $H^2(G) = 0$ can be taken as a $n$th root of unity. Since the group of quantum gauge transformations is a $\mathbb{Z}_n$ extension of $G$ we have

$$\hat{f}_1 \cdot \hat{f}_2 = e^{i\beta_{12}} \hat{f}_1 \hat{f}_2 \text{ with } n\beta/2\pi \in \mathbb{Z}.$$ 

If we assume that $f_1, f_2$ have nonoverlapping support then $f_1f_2 = f_2f_1$ and it follows that

$$\hat{f}_1 \cdot \hat{f}_2 = e^{i\alpha} \hat{f}_2 \cdot \hat{f}_1, \text{ with } \alpha = \beta_{12} - \beta_{21}.$$ 

In [M1] it was proven that in the case of space-time dimension equal to $3 + 1$ and $G = SU(2)$ the phase factor is equal to $-1$ if the winding numbers of the solitons are odd; the phase is equal to $+1$ if the winding number of at least one of the solitons is even. Thus odd solitons behave like fermions and even solitons like bosons.

We shall next generalize the above result to higher dimensional space-times and other gauge groups. The main result is that in higher dimensions one can construct
anyonic statistics, the phase $e^{i\alpha}$ is some root of unity. This is contrary to the common wisdom according to which anyonic statistics occurs only in $1 + 1$ and $2 + 1$ dimensional field theory models and in the framework of Wightman axioms all particles are either fermions or bosons in higher dimensions. Here we are not considering a quantization of point particles but extended smooth solitonic objects, so there is no contradiction with Wightman axioms.

We shall start by studying the case $d + 1 = 5 + 1$ and $G = SU(3)$. The physical space $M$ is assumed to be compactified as the sphere $S^5$. Since $\pi_5(SU(3)) = \mathbb{Z}$ there are solitons in $\text{Map}(M,G)$ and these are classified, up to homotopy, by integers.

Let $g(t)$, with $0 \leq t \leq 1$, be a path in $SU(3)$ connecting the indentity to a nontrivial element of the center $Z_3 \subset SU(3)$. For $f_0 \in \text{Map}(M,G)$ we can define a function $f(t, x)$ on $S^1 \times S^5$ by

$$f(t, x) = g(t)f_0(x)g(t)^{-1}.$$ 

Note that $f(t, x)$ is periodic in $t$ because $g(1)$ commutes with everything. We want to show that this map represents an element of order 3 in $\pi_6(G)$. For a discussion of the homotopy groups of classical Lie groups needed in this paper see e.g. [DP].

First we observe that $f^3 = g(t)^3f_0(x)^3g(t)^{-3}$ and $g(t)^3$ is a closed loop in $G$ because of $g(1)^3 = 1$. But $SU(3)$ is simply connected and therefore $g(t)^3$ contracts to a constant loop and so $f^3$ is homotopic to the identity element in the group $\pi_6(G)$. It remains to show that $f$ itself is not homotopic to a constant loop in $\text{Map}(M,G)$. This is most conveniently done by computing the integral

$$I(h) = \left(\frac{1}{2\pi}\right)^4 \frac{1}{840} \int_B \text{tr}(dhh^{-1})^7,$$

where $D$ is the unit disk and $h : D \times S^5 \to SU(4)$ is a map such that at the boundary $S^1 \times S^5$ $h(t, r = 1, x) = f(t, x)$. The extension $h$ exists because $\pi_6(SU(4)) = 0$. The integral $I(h)$ is a homotopy invariant on a closed 7-manifold. On the boundary $S^1 \times S^5$ the integral defines a homotopy invariant for $f$ modulo integers. This is because 1) the form $\text{tr}(dfj^{-1})^7$ vanishes identically on $SU(3)$, 2) gluing together a pair of $h$’s with the same boundary values produces a map for which the value of the integral is an integer.

The integral $I(h)$ can be evaluated by observing first that we can replace $g(t)f_0(x)g(t)^{-1}$ by $g_1(t)f_0(x)g_1(t)^{-1}$ where $g_1(t)$ is a closed loop in the bigger group $SU(4)$. This follows from the fact that the path $g(t)$ is homotopic (with end points fixed) to the path
\[ \tilde{g}(t) = \exp(itX), \text{ where } X = \text{diag}(2\pi/3, 2\pi/3, -4\pi/3). \]

Here we have chosen (from the two nontrivial elements in the center) \( z = \text{diag}(e^{2\pi i/3}, e^{2\pi i/3}, e^{-4\pi i/3}) \) as the end point at \( t = 1 \). For this path it is easily checked that \( \tilde{g}(t)f_0\tilde{g}(t)^{-1} = g_1(t)f_0g_1(t)^{-1} \) with \( g_1(t) = \exp(itY) \),

\[
Y = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -2\pi & 0 \\
0 & 0 & 2\pi
\end{pmatrix}.
\]

We can then choose a contraction \( g_1(t, r) \) (with \( 0 \leq r \leq 1 \)) of the loop \( g_1(t) \) to a constant loop. This produces a homotopy \( f(t, r, x) \) connecting \( f(t, x) \) to the constant loop \( f_0(x) \).

One can check by a straight-forward computation that for the above choices

\[ \tr (dhh^{-1})^7 = 14d \tr (df_0f_0^{-1})^5(dg_1g_1^{-1}). \]

Thus we may apply Stokes’ theorem to obtain

\[ \int \tr (dhh^{-1})^7 = 14 \int_{S^5} \tr (df_0f_0^{-1})^5iY. \]

The value of the integral is invariant under the left action \( f_0 \mapsto af_0 \) for any \( a \in SU(3) \). In particular, we may choose \( a \) such that the matrix \( \int (df_0f_0^{-1})^5 \) is of the form \( a \cdot 1 + \lambda \), where \( \lambda \) is orthogonal with respect to \( Y \), \( \tr \lambda Y = 0 \), and \( a \in \mathbb{C} \). Thus

\[ \left( \frac{1}{2\pi} \right)^4 \frac{1}{840} \int \tr (dhh^{-1})^7 = (-2\pi i) \cdot \frac{1}{3} \cdot 14 \cdot \left( \frac{1}{2\pi} \right)^4 \frac{1}{840} \int_{S^5} \tr (df_0f_0^{-1})^5 = -n(f_0)/3, \]

where

\[ n(f_0) = \left( \frac{-i}{2\pi} \right)^3 \frac{1}{60} \int_{S^5} \tr (df_0f_0^{-1})^5 \]

is the winding number of the soliton \( f_0 \) in five dimensions. The factor \( 1/3 \) in the second expression is due to the fact that the trace of the product \( Y \cdot 1_{3 \times 3} \) is \(-2\pi \) times one third of the trace of the unit matrix in \( SU(3) \).

The loop \( t \mapsto f(t, x) = g(t)f_0(x)g(t)^{-1} \) can be viewed as a rotation on the group manifold \( SU(3) \). The above analysis shows that the action of \( \hat{G} \) in the fermionic Fock spaces, in the case of \( f(t, \cdot) \), induces a parallel transport around a closed loop resulting in a phase shift by \( e^{-2\pi in(f_0)/3} \). When \( n(f_0) \) is not divisible by 3 this is an element of order 3 in \( S^1 \).

We shall show next that the interchange of two 5-dimensional solitons in \( SU(3) \) of odd winding numbers leads to the same phase shift in the Fock space.
We assume that \( f_1, f_2 \) are functions on \( S^5 \) with values in \( SU(3) \) such that the set of points \( x \in S^5 \) such that both \( f_1(x), f_2(x) \neq 1 \) is empty. Let us also assume that \( n(f_1) = n(f_2) = 1 \). Let \( f_0 \) be a fixed soliton with \( n(f) = 1 \). We can choose paths \( g_k(t, x) \) such that \( g_k(0, x) = f_0 \) and \( g_k(1, x) = f_k(x) \). We want to show that the parallel transport along \( g_1(t)g_2(t) \) (at the end point we obtain the quantum state \( q(f_1)q(f_2)\psi \) corresponding to an initial state \( \psi \)) is \( e^{2\pi i/3} \) times the parallel transport along \( g_2(t)g_1(t) \) in the Fock bundle (this defines the quantum state \( q(f_2)q(f_1)\psi \)).

Because the parallel transport depends only on the homotopy class of the path, with end points fixed, we need to show that the loop \( h(t) = g_1(t)g_2(t)g_1(t)^{-1}g_2(t)^{-1} \) represents of nontrivial class in \( \pi_6(SU(3)) \) of degree 3.

Moving around the initial point \( f_0 \) of the paths \( g_k \) does not change the homotopy class of the loop and therefore we may take \( f_0 = f_1 \), for the sake of simplicity. Moreover, we may assume that \( f_0 \) is a soliton concentrated in a small neighborhood of the unit element in \( SU(3) \). As a consequence, we can take \( g_1(t, x) = f_1(x) \) for all \( t \) and the homotopy class of \( h(t) \) is equal to the class of \( f_1g_2(t)f_1^{-1}g_2(t)^{-1} \).

Next we note that we can freely move the end point of \( g_2(t) \) at \( t = 1 \) so long it does not overlap with the soliton \( f_1(x) = f_0(x) \) near the unit element. Thus we may choose \( g_2(t, x) = a(t)f_1(x) \) where \( a(t) \) is path in \( SU(3) \) connecting the unit element \( a(0) \) to the element \( z \) of degree 3 in the center. Thus after these deformations,

\[
h(t, x) = f_1(x)a(t)f_1(x)^{-1}a(t)^{-1}.
\]

It follows that the path \( h(t, \cdot) \) is, except for the first constant factor \( f_1 \), a rotation of an (anti)soliton \( f_1(x)^{-1} \) in the group manifold of \( SU(3) \) as discussed above. Because the curvature in the group extension is zero, the parallel transport depends only on the endpoints and the homotopy class of the path and consequently the parallel transport is given by the same phase shift \( e^{-2\pi i n(f_1)^{-1}/3} = e^{2\pi i/3} \) as in the case of a rotation.

The homotopy group \( \pi_6(SU(3)) \) has a generator which is of order 6, so the element of order 3 which we have constructed is the square of the generator. However, the square root of our homotopy class is not described by a simple explicate formula.

One can perform a similar analysis for other groups and dimensions. For example, \( \pi_{2n-1}(SU(n)) = \mathbb{Z} \) and there are \( SU(n) \) solitons in \( 2n - 1 \) dimensions; \( \pi_{2n}(SU(n)) = \mathbb{Z}_n! \) and from this one can reduce that there are soliton configura-
tions with a factor \( q \neq 1 \) in the commutation relations \( \hat{f}_1 \cdot \hat{f}_2 + q\hat{f}_2 \cdot \hat{f}_1 = 0 \) such that \( q^k = 1 \) when \( k \) divides \( n! \). Actually, this class of homotopy groups has been discussed in the context of a space-time formulation of global gauge anomalies, [EN].

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