Iterative Adaptive Spectroscopy of Short Signals

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We develop an iterative, adaptive frequency sensing protocol based on Ramsey interferometry of a two-level system. Our scheme allows one to estimate unknown frequencies with a high precision from short, finite signals. It avoids several issues related to processing of decaying signals and reduces the experimental overhead related to sampling. High precision is achieved by enhancing the Ramsey sequence to prepare with high fidelity both the sensing and readout state and by using an iterative procedure built to mitigate systematic errors when estimating frequencies from Fourier transforms.

Introduction — The precision of any coherent sensor, classical or quantum, is limited by the maximal measurement time-window $t_w$ over which a signal of interest can be sampled. This limitation originates from unwanted interactions of the sensor with noisy environmental degrees of freedom, which lead to decoherence and thus, to the decay of the measured signal [1].

Strategies extending the coherence time [2–7] or using states with no classical analogs [8–13], e.g., entangled or squeezed states, to improve the precision have been put forward. However, these strategies do not guarantee that the sensor operates with the highest precision attainable. This is only possible if all steps involved in the sensing protocol are optimized [14].

In this work, we demonstrate how to enhance the ubiquitous Ramsey interferometry of a two-level system [15] (see Fig. 1) to estimate an unknown frequency from a signal with a short $t_w$. There are two reasons to impose a short $t_w$: First, it is experimentally simpler to sample short signals (smaller number of datapoints required) and secondly, from a signal processing point of view, it is easier to deal with non-decaying signals to enhance the frequency estimation through the application of window functions [16] [17].

Enhancing Ramsey interferometry is done by reducing the error in preparing both the sensing and readout states [steps (II) and (IV) in Fig. 1 (a)]. To find frequency-sweeps yielding high fidelity state preparation, we use the recently proposed Magnus-based strategy for control [18, 19].

To improve frequency estimation, one needs to overcome two main shortcomings associated to discrete Fourier transforms: (FT1) Spectral leakage [20], which makes a simple, single frequency spectrum appear with multiple frequency components and can lead to a shift of the maxima of the Fourier transform, and (FT2) finite frequency resolution [21–23], which follows from having signals that are a discrete, finite set of points separated by a finite time-interval. We stress that (FT1) is not an issue when $1/t_w$ is much smaller than the frequency we seek to estimate. The Fourier uncertainty principle [24] guarantees then that the resulting signal spectrum in frequency domain is highly localized.

To mitigate systematic errors originating from (FT1) and (FT2) we develop an iterative, adaptive sensing protocol. Each interaction consists in performing Ramsey interferometry and processing the acquired signal with a window function tailored to either enhance the amplitude of the Fourier spectrum or frequency estimation. After each iteration, we use the newly found frequency estimate to update steps II, III, and IV of the Ramsey sequence [see Fig. 1 (a) and (c)].

Updating steps II and IV entails running the Magnus-based strategy for control with the new frequency estimate to update steps II, III, and IV of the Ramsey sequence [see Fig. 1 (a) and (c)].

Figure 1. Enhanced Ramsey sequence. (a) The five steps of a Ramsey sequence [15] to sense the coupling strength $\Omega_0/2$ between a two-mode (or two-level) system as depicted in (b). The normal (slow) sequence is indicated by gray arrows, while the enhanced (fast, high-fidelity) sequence follows the orange arrows. (c) Typical time-dependent frequency-sweep to generate a Ramsey sequence. (d) Comparison of the leading edge of the uncorrected (green) and modified (red and blue) frequency-sweeps allowing one to prepare the sensing state [step (II) in (a)].

Model — We consider a parametrically coupled two-
and readout state preparation protocols to be

$$D(t) = \frac{1}{2} (\Delta(t) \sigma_x + [\Omega_0 + \delta\Omega(t)] \sigma_z),$$  \hspace{1cm} (1)$$

where $\sigma_j$, $j \in \{x, y, z\}$, are Pauli matrices, $\Delta(t) = \omega_2(t) - \omega_1(t)$ is the controllable frequency difference between mode 1 and 2 [see Fig. 1 (b)], $\Omega_0/2$ is the unknown coupling strength we seek to estimate [see Fig. 1 (b)], and $\delta\Omega(t)$ is a real stochastic process describing how noise arising, e.g., from thermal fluctuations or fluctuations of the fields the system is subjected to, affects the coupling between the modes. Neglecting noise, the eigenfrequencies of Eq. (1) are given by $\lambda_{\pm}(t) = \pm \sqrt{\Delta^2(t) + \Omega_0^2}/2$ [see Fig. 1 (b)].

The dynamical matrix $D(t)$ is formally equivalent to a two-level Hamiltonian $\hat{H}(t)$ [26–28] subject to classical noise. The model we use, thus, describes both coherent classical systems [25, 29–36] and quantum systems [37–44].

Since our goal is to develop a sensing protocol whose duration $t_s$ [see Fig. 1 (c)] is short, i.e., $(\Omega_0/2\pi)t_s < 5$, we assume that $\delta\Omega(t) \approx \delta\Omega$ does not change appreciably for one realization of the protocol (frozen environment approximation). Within this framework, averaged Ramsey signals are obtained by performing statistical averaging, i.e.,

$$\langle s(t) \rangle_{\delta\Omega} = \int_{-\infty}^{\infty} d\delta\Omega \rho(\delta\Omega)s(t),$$  \hspace{1cm} (2)$$

where $\rho(\delta\Omega)$ is the probability distribution of $\delta\Omega$. Here, we assume $\rho(\delta\Omega)$ to be a Gaussian distribution with zero mean and standard deviation $\sigma_{\delta\Omega}$,

$$\rho(\delta\Omega) = \frac{1}{\sqrt{2\pi}\sigma_{\delta\Omega}} \exp\left[-\frac{1}{2} \frac{\delta\Omega^2}{\sigma_{\delta\Omega}^2}\right].$$  \hspace{1cm} (3)$$

In this work we choose $\sigma_{\delta\Omega} = \Omega_0/10$.

**Sensing and readout state preparation** — We consider a generic frequency-sweep for the Ramsey interferometer given by [see Fig. 1 (c)]

$$\Delta(t) = \begin{cases} 
\Delta_s(t) = \Delta_0 f(t), & \text{for } 0 \leq t \leq t_s, \\
0, & \text{for } t_s < t < t_t, \\
\Delta_s(t) = \Delta_0 [1 - f(t - t_t)], & \text{for } t_t \leq t \leq t_r.
\end{cases}$$  \hspace{1cm} (4)$$

where $\Delta(0) = \Delta(t_s) = \Delta_0$ is the initial (final) value of the frequency difference [see Fig. 1 (b)] and $f(t)$ is a smooth sweep function obeying $f(0) = f(t_t) = 1$ and $f(t_s) = f(t_r) = 0$. We define the measurement time-window $t_w = t_t - t_s$ and the total sensing time $t_s = 2t_s + t_w$, where we choose the duration of both the sensing and readout state preparation protocols to be $t_s$. [see Fig. 1 (c)]

By choosing $\Delta_0 \gg \Omega_0$, we can initialize the system in mode $a_1 = (0, 1)^T$ at $t = 0$ [see step (I) in Figs. 1 (a) and (c)]. This is also the sensing state $a_s$ we would like to use to probe $\Omega_0$, i.e., $a_s = a_1$. At the avoided crossing, $a_1$ can be expressed as an equal superposition of the eigenmodes of Eq. (1), which is the state maximizing the visibility of the Ramsey fringes (see Supplemental Material).

To prepare the sensing state $a_s = a_1$ at $t = t_s$ and the readout state $a_r$ at $t = t_r$ (step II and IV in Figs. 1 (a) and (c), respectively) one needs to choose $f(t)$ such that the evolution generated by Eq. (1) corresponds to the identity in the intervals $0 \leq t \leq t_s$ and $t_t \leq t \leq t_r$. This can theoretically be realized with a frequency-sweep whose leading and trailing edges duration fulfills the condition $\Omega_0 t_s \ll 1$ (quasi instantaneous sweep). However, faithful reproduction of fast sweeps in the lab environment are limited by the maximum bandwidth of wave generators. The Landau-Zener model [45–48], where one sets the sweep function to be linear in time, $f(t) = 1 - t/t_s$, is a perfect example: Faster and faster sweeps require more and more Fourier components to accurately reproduce $f(t)$ (see Supplemental Material).

The alternative is to use a frequency-sweep with an adiabatic leading edge followed by a $\pi/2$ pulse [see, e.g., Ref. [27] and extra step I’ in Fig. 1 (a)]. While such a protocol is not limited by bandwidth constraints, it is limited by adiabaticity and the fidelity of the resonant $\pi/2$ pulse. A high-fidelity adiabatic pulse must fulfill the condition $t_0 t_s \gg 1$. This renders high fidelity preparation of the sensing state in the presence of noise unsustainable [49].

To design a fast, bandwidth-limited protocol yielding a high-fidelity state preparation [orange arrows in Fig. 1 (a)], we start from the single-tone function [green line in Fig. 1 (d)]

$$f(t) = \frac{1}{2} \left[1 + \cos\left(\frac{\pi}{t_s}\right)\right],$$  \hspace{1cm} (5)$$

and choose a $t_s$ which respects the bandwidth limitation imposed by various experimental components including arbitrary waveform generators (AWG), filters, amplifiers, and other passive and active circuit components in a laboratory environment. This implies that the allowed maximal $t_s$ in Eq. (5) is in general still too slow to realize a quasi instantaneous sweep, but yet much shorter than the $t_s$ required to fulfill the adiabatic criterion.

In this intermediate regime, where the generated evolution is coherent, we use the recently proposed Magnus-based strategy for control [18, 19] to cancel on average transitions to mode $a_2$. This yields a function $f_{\text{mod}}(t)$ that defines a modified frequency-sweep that allows one to achieve high-fidelity state preparation.

For the rest of this work we consider two different modified frequency-sweeps that we label Mod1 and Mod2. The leading edges of the uncorrected (green), Mod1
Frequency estimation with trivial signal processing —

Although our modified frequency-sweeps allow us to prepare the ideal sensing and readout states, they do not allow us to correctly estimate unknown frequencies from short-time signals, i.e., $2 \leq (\Omega_0/2\pi)\delta \omega \leq 4$.

We illustrate this in Fig. 2 (c)-(e) where we compare the modulus squared of the discrete signal Fourier transform (spectral density), $|\mathcal{F}[s(t)]|^2$, for different case scenarios. We stress that the spectra were obtained assuming that we know exactly the value of $\Omega_0$ to capture only the effects of the shortcomings associated to Fourier transforms of short-time signals (FT1).

The modified frequency-sweeps (red and blue traces) lead to spectra where the global maximum (with the 0-frequency peak excluded) can be more easily identified. However, the global maximum is not located at $\Omega/\Omega_0 = 1$. Our results also show that small changes in $t_w$ can result in different spectra with maxima located at very different frequencies, which obviates a correct frequency estimation.

In the following, we show how an iterative procedure combining both an update on the estimate for $\Omega_0$ and different windowing schemes [16] to process the measured signal solves the issues outlined above and yields a high-precision estimate of the unknown frequency.

Iterative, adaptive frequency estimate procedure —

Windows, or tapers, are weighting functions designed to simplify the analysis of harmonic signals in the presence of noise and harmonic interference. In particular, the window functions apply selective weights to reduce spectral leakage associated with finite measurement windows [16, 20].

In this context, the so-called Blackman-Harris window [16, 20] is notably known to effectively reduce spectral leakage. However, as a downside, the measurement-time window must be chosen such that $(\Omega_0/2\pi)\delta \omega \geq 4$ and the amplitude of the spectral density is reduced, which can render frequency detection problematic, especially for short-time, noisy signals. Finally, and we cannot stress this enough, while windowing reduces spectral leakage, it can never completely suppress it. Thus, even with windowing, high-precision frequency estimation is still limited by artifacts linked to discrete Fourier transforms of short-time signals.

To overcome this limitation we use the iterative, adaptive sensing (IAS) protocol depicted in Fig. 3 (a). Each interaction consists in performing Ramsey interferometry [rhombi in Fig. 3 (a)] with a frequency-sweep that takes into account our current knowledge of the frequency estimate [circles in Fig. 3 (a)]. This way we can iteratively suppress systematic frequency shift errors originating from spectral leakage when $t_w$ [step III in Fig. 1 (a)] is not an integer multiple of the period. This is done by updating after each iteration the measurement-time
The window functions for signal processing are chosen to either enhance the amplitude of the Fourier spectrum or by using spectroscopic methods. The other source of systematic errors come from dealing with a signal which is constructed from a finite number of sampling points (measurements) \( n_{\text{samp}} \) (FT2). A small \( n_{\text{samp}} \) will lead to scalloping, i.e., the \([n_{\text{samp}}/2]\)-point discrete Fourier transform does not resolve the real maxima of the spectrum [16]. This is, however, easily fixable by using zero-padding, as described in signal processing textbooks, e.g., in Ref. [16]. Zero-padding consists in extending the signal with \( n_{\text{pad}} \) zeros yielding a \(((n_{\text{samp}} + n_{\text{pad}})/2)\)-point discrete Fourier transform. To further reduce the effects of scalloping we use interpolation of the padded discrete spectrum [52] (see also Supplemental Material).

Figure 3 (b) shows the relative error of the frequency estimate \( \epsilon_\Omega^{(1)} = \log_{10}(1 - \bar{\Omega}_{(1)}/\bar{\Omega}_0) \) as a function of \( n_{\text{samp}} \) for \( n = 4 \) for the first iteration \( m = 1 \) in Eqs. (8) and (9) of our iterative sensing scheme. Here \( n_{\text{pad}} \) is chosen such that \( n_{\text{samp}} + n_{\text{pad}} = 1000 \). Independently of the frequency-sweep used, doubling \( n_{\text{samp}} \) only leads to a small variation of the relative error. This allows us to identify \([n_{\text{samp}}/n] = 8 \geq 2\) as a good compromise between the error and experimental cost, i.e., the number of measurements. The results show the advantage of using the modified frequency-sweeps Mod1 (red trace) and Mod2 (blue trace) over the uncorrected one (green trace); the smaller the error in preparing both the sensing and readout state the smaller the relative error of the first frequency estimate for a signal of identical duration.

In Fig. 3 (c) we plot the relative error \( \epsilon_\Omega^{(m)} \) after each iteration of our adaptive scheme (IAS 1, light blue squares). We also included the error on our prior estimate \( \bar{\Omega}^{(0)} \) for reference (orange circle). The iterative procedure converges to a value of the frequency estimate whose error is smaller than the initial estimate. Convergence indicates that we reached the spectral resolution allowed by our measurement time-window after just a few iterations.

To show that the choice of window for the first iteration has no influence on the results, we also plotted in Fig. 3 (c) the relative error obtained by using at every step the Blackman-Harris window for signal processing (IAS 2, dark blue triangles).

Finally, we plot in Fig. 3 (d) the signal-to-noise (SNR) ratio defined as

\[
\text{SNR} = \sqrt{N} \frac{\mathcal{F}[(s^{(n)}(t)f^{(n)}(t))]}{\left[\sum_{j=1}^{N} \left(\mathcal{F}[s_j^{(n)}(t)f^{(n)}(t)] - \mathcal{F}[(s^{(n)}(t)f^{(n)}(t))]ight)\right]^2}^{1/2},
\]
which is a measure of the confidence level on the frequency estimate. More precisely, in this context, the SNR quantifies the degree of confidence we have in identifying the global maximum of the spectral density (0-frequency component excluded). Our results show that as the relative error on the frequency estimate decreases, we become more and more confident in identifying the frequency associated to the global maximum of the spectral density in spite of having a noisy signal.

**Conclusion** — We have developed an iterative, adaptive sensing protocol based on enhanced Ramsey interferometry of two-level systems. Our scheme allows one to get precise estimates of an unknown frequency by considering short, finite-time signals under realistic assumptions of experimental bandwidth limitations. Specifically, our scheme avoids shortcomings both related to dealing with decaying signals and experimental constraints related to the sampling and could be implemented, e.g., in coupled mechanical oscillators [25, 29–31], optomechanical systems [32, 33], hybrid optomechanical systems [34], coupled optical modes [35], and qubits [53, 54] under the influence of classical noise, just to name a few.

The main ingredients of our method are the use of the Magnus-based strategy for control to find frequency-sweeps that allow one to prepare with high fidelity both the sensing and readout state and an iterative procedure built to mitigate systematic errors when using Fourier transforms to extract frequency components. We stress that independently of how the sensing and readout state are prepare, our iterative, adaptive sensing protocol can always be applied to enhance frequency estimates.

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[1] Note that the measured signal (or signal) refers to the data recorded by the sensor, whereas the signal of interest represents the quantity to be detected.
[2] P. Fisk, M. Sellars, M. Lawn, C. Coles, A. Mann, and D. Blair, Very high Q microwave spectroscopy on trapped 171yb+ ions: Application as a frequency standard, IEEE Transactions on Instrumentation and Measurement 44, 113 (1995).
[3] K. Saeedi, S. Simmons, J. Z. Salvail, P. Dhuby, H. Riemann, N. V. Abrosimov, P. Becker, H.-J. Pohl, J. J. L. Morton, and M. L. W. Thewalt, Room-temperature quantum bit storage exceeding 39 minutes using ionized donors in silicon-28, Science 342, 830 (2013).
[4] M. Zhong, M. P. Hedges, R. L. Ahlefeldt, J. G. Bartholomew, S. E. Beavan, S. M. Wittig, J. J. Longdell, and M. J. Sellars, Optically addressable nuclear spins in a solid with a six-hour coherence time, Nature 517, 177 (2015).
[5] A. Barfuss, J. Köbl, L. Thiel, J. Teissier, M. Kasperczyk, and P. Maletinsky, Phase-controlled coherent dynamics of a single spin under closed-contour interaction, Nature Physics 14, 1087 (2018).
[6] J. Köbl, A. Barfuss, M. S. Kasperczyk, L. Thiel, A. A. Clerk, H. Ribeiro, and P. Maletinsky, Initialization of single spin dressed states using shortcuts to adiabaticity, Phys. Rev. Lett. 122, 090502 (2019).
[7] E. D. Herbschleb, H. Kato, Y. Maruyama, T. Danjo, T. Makino, S. Yamasaki, I. Ohki, K. Hayashi, H. Morishita, M. Fujiwara, and N. Mizuochi, Ultra-long coherence times amongst room-temperature solid-state spins, Nature Communications 10, 3766 (2019).
[8] M. Tse and et al., Quantum-enhanced advanced ligo detectors in the era of gravitational-wave astronomy, Phys. Rev. Lett. 123, 231107 (2019).
[9] M. Malhau, D. A. Palken, B. M. Brubaker, L. R. Vale, G. C. Hilton, and K. W. Lehnert, Squeezed vacuum used to accelerate the search for a weak classical signal, Phys. Rev. X 9, 021023 (2019).
[10] T. L. scientific Collaboration, A gravitational wave observatory operating beyond the quantum shot-noise limit, Nature Physics 7, 962 (2011).
[11] J. Aasi and et al., Enhanced sensitivity of the ligo gravitational wave detector by using squeezed states of light, Nature Photonics 7, 613 (2013).
[12] M. Werninghaus, D. J. Egger, F. Roy, S. Machnes, F. K. Wilhelm, and S. Filipp, Leakage reduction in fast superconducting qubit gates via optimal control, npj Quantum Information 7, 14 (2021).
[13] B. J. Lawrie, P. D. Lett, A. M. Marino, and R. C. Pooser, Quantum Sensing with Squeezed Light, ACS Photonics 6, 1307 (2019).
[14] J. Liu, M. Zhang, H. Chen, L. Wang, and H. Yuan, Optimal scheme for quantum metrology, Advanced Quantum Technologies 5, 2100080 (2022).
[15] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, Reviews of Modern Physics 89, 1 (2017).
[16] K. M. M. Prabhu, Window functions and their applications in signal processing (1st ed.) (Taylor & Francis, 2014).
[17] The decay envelope of the signal can be viewed as an “uncontrolled” window function applied to the signal oscillating component. The presence of an “uncontrolled” window function can negate all the benefits of using specific window functions tailored to enhance frequency estimation.
[18] H. Ribeiro, A. Baksic, and A. A. Clerk, Systematic magnus-based approach for suppressing leakage and nonadiabatic errors in quantum dynamics, Physical Review X 7, 1 (2017).
[19] T. Figueiredo Roque, A. A. Clerk, and H. Ribeiro, Engineering fast high-fidelity quantum operations with constrained interactions, npj Quantum Information 7, 10.1038/s41534-020-00349-z (2021).
[20] F. Harris, On the use of windows for harmonic analysis with the discrete fourier transform, Proceedings of the IEEE 66, 51 (1978).
[21] E. T. Whittaker, Xviii.—on the functions which are represented by the expansions of the interpolation-theory, Proceedings of the Royal Society of Edinburgh 35, 181–194 (1915).
[22] H. Nyquist, Certain topics in telegraph transmission theory, Transactions of the American Institute of Electrical Engineers 47, 617 (1928).
[23] C. Shannon, Communication in the presence of noise, Proceedings of the IRE 37, 10 (1949).
[24] J. A. Hogan, Fourier uncertainty principles, in Time-Frequency and Time-Scale Methods: Adaptive Decompositions, Uncertainty Principles, and Sampling
M. J. Seitner, H. Ribeiro, J. Kölbl, T. Faust, and E. M. Weig, Finite-time Stückberg interferometry with nanomechanical modes, New Journal of Physics 19, 10.1088/1367-2630/aa5a3f (2017).

L. Novotny, Strong coupling, energy splitting, and level crossings: A classical perspective, American Journal of Physics 78, 1199 (2010).

T. Faust, J. Rieger, M. J. Seitner, J. P. Kotthaus, and E. M. Weig, Coherent control of a classical nanomechanical two-level system, Nature Physics 9, 485 (2013).

M. Frimmer and L. Novotny, The classical Bloch equations, American Journal of Physics 82, 947 (2014).

T. Faust, J. Rieger, M. J. Seitner, and P. Krenn, Nonadiabatic Dynamics of Two Strongly Coupled Nanomechanical Resonator Modes, Physical Review Letters 037205, 1 (2012).

H. Okamoto, A. Gourgout, C.-Y. Chang, K. Onomitsu, I. Mahboob, E. Y. Chang, and H. Yamaguchi, Coherent phonon manipulation in coupled mechanical resonators, Nature Physics 9, 480 (2013).

F. R. Braakman, N. Rossi, G. Tüttüncüoglu, A. Fontcuberta, and M. Poggio, Coherent Two-Mode Dynamics of a Nanowire Force Sensor, Physical Review Applied 9, 54045 (2018).

J. D. Teufel, D. Li, M. S. Allman, K. Ciaak, a. J. Siros, J. D. Whittaker, and R. W. Simmonds, Circuit cavity electromechanics in the strong-coupling regime., Nature 471, 204 (2011).

A. Ranfagni, P. Vezio, M. Calamai, A. Chowdhury, F. Marino, and F. Marin, Vectorial polaritons in the quantum motion of a levitated nanosphere, Nature Physics 17, 1120 (2021).

I. Yeo, P.-L. de Assis, a. Gloppe, E. Dupont-Ferrier, P. Verlot, N. S. Malik, E. Dupuy, J. Claudon, J.-M. Gérard, a. Auffèves, G. Nogues, S. Seidelin, J.-p. Poizat, O. Anciet, and M. Richard, Strain-mediated coupling in a quantum dot-mechanical oscillator hybrid system., Nature Nanotechnology 9, 106 (2014).

S. M. Thon, M. T. Rakher, H. Kim, J. Gudat, W. T. Irvine, P. M. Petroff, and D. Bouwmeester, Strong coupling through optical positioning of a quantum dot in a photonic crystal cavity, Applied Physics Letters 94, 1 (2009).

M. Perpentiener, P. Schmidt, D. Schwienbacher, R. Gross, and H. Hübner, Frequency control and coherent excitation transfer in a nanostring-resonator network, Phys. Rev. Applied 10, 034007 (2018).

C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, Remote state preparation, Physical Review Letters 87, 77902 (2001).

J. Koch, T. M. Yu, J. Gambetta, A. A. Houck, D. I. Schuster, J. Majer, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Charge-insensitive qubit design derived from the cooper pair box, Phys. Rev. A 76, 042319 (2007).

P. Rabl, S. J. Kolkowitz, F. H. Koppens, J. G. Harris, P. Zoller, and M. D. Lukin, A quantum spin transducer based on nanoelectromechanical resonator arrays, Nature Physics 6, 602 (2010).

M. S. Barson, P. Peddibhotla, P. Ovartchaiyapong, K. Ganesan, R. L. Taylor, M. Gebert, Z. Mielenz, B. Koslowski, D. A. Simpson, L. P. McGuinness, J. McCaffey, S. Prawer, S. Onoda, T. Ohshima, A. C. Bleszynski Jayich, F. Jelezko, N. B. Manson, and M. W. Doherty, Nanomechanical Sensing Using Spins in Diamond, Nano Letters 17, 1496 (2017).

P. Kurpiers, P. Magnard, T. Walter, B. Royer, M. Pechal, J. Heinsoo, Y. Salathé, A. Akin, S. Storz, J. C. Bess, S. Gasparinetti, A. Blais, and A. Wallraff, Deterministic quantum state transfer and remote entanglement using microwave photons, Nature 558, 264 (2018).

A. Zrenner, E. Beham, S. Stufler, F. Findeis, M. Bichler, and G. Abstreiter, Coherent properties of a two-level system based on a quantum-dot photodiode, Nature 418, 612 (2002).

J. M. Boss, K. S. Cujia, J. Zopes, and C. L. Degen, Quantum sensing with arbitrary frequency resolution, Science 356, 837 (2017).

P. Foggiali, P. Cappellaro, and N. Fabbri, Optimal control for one-qubit quantum sensing, Phys. Rev. X 8, 021059 (2018).

L. D. Landau, Zur Theorie der Energieubertragung, II, Phys. Z. Sowjetunion 2, 46 (1932).

C. Zener, Non-adiabatic crossing of energy levels, Proc. R. Soc. A 137, 696 (1932).

E. C. G. Stückberg, Theorie der unelastischen Stösse zwischen Atomen, Helv. Phys. Acta 5, 239 (1932).

E. Majorana, Atomi orientati in campo magnetico variabile, Nuovo Cimento 9, 43 (1932).

P. Nalbach and M. Thorwart, Landau-zener transitions in a dissipative environment: Numerically exact results, Phys. Rev. Lett. 103, 220401 (2009).

M. J. Seitner, H. Ribeiro, J. Kölbl, T. Faust, and E. M. Weig, Finite-time stickelberg interferometry with nanomechanical modes, New Journal of Physics 19, 033011 (2017).

From the point of view of signal processing applying a rectangular window is the same as applying no window.

M. Gasior and J. L. Gonzalez, Improving ft frequency measurement resolution by parabolic and gaussian spectrum interpolation, AIP Conference Proceedings 732, 276 (2004).

W. D. Oliver, Y. Yu, J. C. Lee, K. K. Berggren, L. S. Levitov, and T. P. Orlando, Mach-zehnder interferometry in a strongly driven superconducting qubit, Science 310, 1653 (2005).

J. R. Petta, H. Lu, and A. C. Gossard, A coherent beam splitter for electronic spin states, Science 327, 669 (2010).
Supplemental Material: Iterative Adaptive Spectroscopy of Short Signals

Appendix A: Using $a_1$ as sensing state

In this section we briefly show the advantages of using the sensing state $a_s = a_1$.

The advantage is that $a_1$ maximizes the visibility of the Ramsey signal. This can be readily verified by assuming that one can prepare with unit fidelity both the sensing and readout state. In this case the Ramsey signal is given by

$$s(t) = |a_1^T \exp(-i\Omega_0 t a_1)|^2 = \cos^2[(\Omega_0/2)t],$$

(A1)

which is a function oscillating between 0 and 1, and thus with unit visibility.

To understand why this property is important in the context of frequency estimation from short signals, let us consider the situation where we prepare any arbitrary state at $t=0$, which is a function oscillating between 0 and 1, and thus with unit visibility. For any other rotation, the Ramsey signal can be viewed as the sum of two signals: A constant signal $s_1(t) = 1 + v(\alpha, \beta, \theta)$, and an oscillating signal $s_2(t) = v(\alpha, \beta, \theta) \sin^2[(\Omega_0/2)t]$ with visibility $0 \leq |v(\alpha, \beta, \theta)| < 1$. Thus, we can write the Fourier spectrum as

$$|\mathcal{F}[s_0(t)]|^2 = |\mathcal{F}[s_1(t)]|^2 + |\mathcal{F}[s_2(t)]|^2 + 2\Re[\mathcal{F}[s_1(t)]\mathcal{F}[s_1(t)^*]].$$

(A4)

Equation (A4) shows that one can interpret the spectrum of a finite $s_0(t)$ as an interferometric pattern. As a result, $|\mathcal{F}[s_0(t)]|^2$ does not necessarily have a maximum located at $\omega = \Omega_0$. This is yet another type of systematic error in the sensing result which cannot be eliminated unless one knows exactly what the values of $\alpha$, $\beta$, and $\theta$ are.

Finally, we note that in the infinite measurement-time window limit, i.e., $t_w \rightarrow \infty$, this issues vanishes since the discrete Fourier transform of $s_1(t)$ would reduce to the Kronecker delta function. Consequently, in the limit defined by $\Omega_0 t_w \gg 1$, the induced systematic error is negligible.

Appendix B: Magnus-based strategy for control

In this section, we detail how we obtained the modified protocols, Mod1 and Mod2, using the Magnus-based strategy for control [18, 19]. In particular, we show how we find the modification for the leading and trailing edge of the frequency-sweep.

The first step consists in finding a partition of the dynamical matrix $D(t) = D_0(t) + V(t)$ (see Eq. (1) of the main text), where $D_0(t)$ generates the desired dynamics and $V(t)$ describes the spurious coupling disrupting the desired dynamics. For the problem at hand, and neglecting noise, we have

$$D_0(t) = \frac{1}{2} \Delta(t) \sigma_z,$$

(B1)
\[ V(t) = \frac{1}{2} \Omega_0 \sigma_x. \]  

(B2)

The second step consists in introducing a control \( W(t) \) which cancels on average the spurious effects generated by \( V(t) \). Formally, this leads to a modified dynamical matrix \( D_{\text{mod}}(t) = D(t) + W(t) \), which generates a flow \( \Phi_{\text{mod}}(t) \). The control \( W(t) \) must be chosen such that

\[ \Phi_{\text{mod}}(t_f) = \Phi_0(t_f), \]  

(B3)

where \( \Phi_0(t) \) is the flow generated by \( D_0(t) \) [see Eq. (B1)].

Following the procedure detailed in [19], we consider \( W(t) = \Delta_{\text{corr}}(t) \sigma_z \), which is compatible with the constraints of the problem; we can only control in time the field coupling to \( \sigma_z \). Taking advantage of the mirror symmetry of the frequency-sweep around \( t = t_r/2 \), we can express \( \Delta_{\text{corr}}(t) \) as

\[ \Delta_{\text{corr}}(t) = \begin{cases} \Delta_{\text{even}}(t) + \Delta_{\text{odd}}(t) & \text{for } 0 \leq t \leq t_s, \\ 0 & \text{for } t_f < t < t_r, \\ \Delta_{\text{even}}(t) - \Delta_{\text{odd}}(t) & \text{for } t_f \leq t \leq t_r, \end{cases} \]  

(B4)

where \( \Delta_{\text{even}}(t) \) and \( \Delta_{\text{odd}}(t) \) are parametrized as finite Fourier series

\begin{align*}
\Delta_{\text{even}}(t) &= \sum_{k=1}^{k_{\text{max}}} c_k \left[ 1 - \cos \left( 2\pi k \frac{t}{t_f} \right) \right], \\
\Delta_{\text{odd}}(t) &= \sum_{l=1}^{l_{\text{max}}} d_l \sin \left( 2\pi l \frac{t}{t_f} \right).
\end{align*}  

(B5)

Here \( c_k \) and \( d_l \) are the free Fourier coefficients one must find in order to fulfill Eq. (B3). The number of free coefficients is set by choosing appropriate values for \( k_{\text{max}} \) and \( l_{\text{max}} \), e.g., one might want to constraint bandwidth. We stress that we parametrized Eq. (B4) such that the coefficients \( c_k \) and \( d_l \) are the same for the leading and trailing edge of \( \Delta_{\text{corr}}(t) \).

Equations for \( c_k \) and \( d_l \) are found by transforming \( D_{\text{mod}}(t) \) to the interaction picture defined by \( \Phi_0(t) \), i.e., \( D(t) \rightarrow D_{\text{mod},1}(t) = \Phi_0(t) D_{\text{mod}}(t) \Phi_0(t) - i \Phi_0(t) \dot{\Phi}_0(t) \). We find \( D_{\text{mod},1}(t) = V_1(t) + W_1(t) \), where

\[ V_1(t) = \frac{\Omega_0}{2} \cos \left( \int_0^t dt_1 \Delta(t_1) \right) \sigma_x - \sin \left( \int_0^t dt_1 \Delta(t_1) \right) \sigma_y \]  

(B6)

and

\[ W_1(t) = W(t). \]  

(B7)

By comparing \( W_1(t) \) with \( V_1(t) \), we notice that the control dynamical matrix \( W(t) \) is singular [19]. Thus, we follow the strategy outlined in section “Singular or ill-conditioned correction Hamiltonians” of Ref. [19] to obtain a set of nonlinear equations for the coefficients \( c_k \) and \( d_l \).

Here, we look for the coefficients \( c_k \) and \( d_l \) which fulfilled the coupled equations

\begin{align*}
\frac{1}{2} \text{Tr} \left[ \sum_{k=1}^{4} \Omega_k^{(1)}(t_f) \sigma_x \right] &= 0, \\
\frac{1}{2} \text{Tr} \left[ \sum_{k=1}^{4} \Omega_k^{(1)}(t_f) \sigma_y \right] &= 0,
\end{align*}  

(B8)

where \( \Omega_k^{(1)}(t_f) \) are the elements of the Magnus series generated by \( D_{\text{mod},1}(t) \). Since we only want to prevent (coherent) transitions on average from mode 1 to mode 2 and vice versa, we only look for the coefficients \( c_k \) and \( d_l \) that cancel the off-diagonal elements of the Magnus expansion up to fourth order.

Since the non-linear system of equations in Eq. (B8) has in general more than one solution, one can choose the solution that minimizes the norm of the vector of free parameters, i.e., the function \( g = \sum_{k,l} (c_k^2 + d_l^2) \). In particular, it is numerically more efficient to directly minimize \( g \) under the constraints defined by Eq. (B8).
Appendix C: Bandwidth limitation of the linear ramp pulse

In this section we briefly discuss the bandwidth requirements to use a linear sweep function \( f(t) \) (see Eq. (4) of the main text). Linear ramps are obtained by approximating a triangular wave with a period equal to \( T \). The linear ramp is then obtained by considering the time-interval \( t \in [0, t_s = T/4] \). The Fourier series of the triangle wave is given by

\[
 s_{\text{triangle}}(t) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{2\pi(2n-1)}{T} t \right),
\]

\[
 c_n = -\frac{8}{\pi^2} \frac{(-1)^n}{(2n-1)^2}.
\]

This indicates that one needs to use an infinite number of harmonics to faithfully reproduce a linear ramp (see also Fig. S1).

Figure S1. Plot of the Fourier amplitude \( |c_n| \) [see Eq. (C1)] as a function of the harmonic \( n \), for one tone (green) and linear ramp \( s_{\text{triangle}}(t) \) (purple).

Appendix D: State preparation errors

In this section we discuss how noise and low fidelity state preparation affects the sensing protocol.

1. Noise-induced errors

In the main text, we showed the noise averaged sensing and readout state fidelity error (see Fig. 2 (a) and (b) of the main text). Here, to give a sense on how noise hinders the preparation of both states, we present in Fig. S2 (a) and (b) \( \varepsilon_{s,j} \) and \( \varepsilon_{r,j} \) (see Eq. (6) of the main text) obtained in the absence of noise \( \delta\Omega = 0 \) in Eq. (1) of the main text).

Let us first consider the sensing state fidelity error for the uncorrected protocol [green trace in Fig. 2 (a) of the main text and Fig. S2 (a)]. In this case, the presence of noise favors the state preparation by preventing the generation of the ideal coherent evolution and thus partially suppressing coherent transitions to mode \( a_2 \). On the other hand, for the modified protocols [red and blue traces in Fig. 2 (a) of the main text and and Fig. S2 (a)] that rely on coherent evolution to average out the effects of the spurious interaction [see Eqs. (B2) and (B6)] noise reduces the state preparation fidelity. However, as the results of the main text show, the fidelity errors obtained with the modified protocols in the presence of noise are still orders of magnitude smaller than the uncorrected one. This fact can be attributed to having protocols that are designed to be shorter than the decoherence time set by the noise.

For the readout state preparation, one would expect the same observations as above. This is, however, not the case. Comparison of Fig. 2 (b) and Fig. S2 (b) reveals that noise hinders the readout state preparation to a much
Figure S2. (a) Comparison of the sensing state error fidelity for $\delta \Omega = 0$ between the uncorrected (green) Mod1 (red), and Mod2 (blue) frequency-sweeps as a function of $t_s$. (b) Same as (a) for the readout state. (c) Evolution of the probability $P_1(t)$ of measuring mode $a_1$ as a function of time for $\Omega_0 t_s / 2\pi = 0.5$. The green dots and arrows are a visual indicator to show how much the ideal coherent evolution is corrupted by coherent errors when using the uncorrected pulse in both sensing and readout state.

greater extent than it does for the sensing state preparation. This difference originates from the dependence of the state fidelity error on the initial state, which for preparing the sensing state is simply $a_1$ and for preparing the readout state is $a(t_f) = \Phi_j(t_f) a_1$. The latter is a coherent superposition state and is therefore more susceptible to noise-induced decoherence.

2. Coherent errors

To illustrate how low fidelity state preparation affects the sensing protocol, we plot in Fig. S2 (c) the probability $P_1(t_w, t)$ of measuring mode $a_1$ as a function of time for a fixed measurement-time window $t_w$. We note that the Ramsey signal $s(t_w)$ is constructed from the values of $P_1(t_w, t_r)$ when $t_w$ is varied, i.e., $s(t_w) = P_1(t_w, t_r)$. We have

$$P_{1,j}(t) = |a_1^T \Phi_j(t) a_1|^2,$$

where $j \in \{0, 1, 2\}$ labels, as in the main text, which detuning sweep is used to obtain $\Phi_j(t)$. We recall that $j = 0$ labels the uncorrected detuning-sweep, while $j = 1, 2$ labels the detuning-sweep coined Mod1 and Mod2, respectively.

The uncorrected detuning-sweep [green trace in Fig. S2 (c)] shows how coherent errors propagate and lead to the “wrong” Ramsey signal. Using Mod1 (red trace) or Mod2 (blue trace) which allow for high fidelity state preparation of both the sensing and readout state, coherent errors are reduced and a more faithful Ramsey signal can be constructed.

Appendix E: Spectral leakage and scalloping losses

To visualize the shortcomings associated with Fourier transforms of short signals, it is useful to consider a simple sinusoidal signal

$$s_1(t) = \cos(\Omega_0 t),$$

over a finite interval of time. We consider two different intervals of time defined as $t_{1,\text{max}} = 4T$ and $t_{2,\text{max}} = 4.5T$, where $T = 2\pi / \Omega_0$ [see Fig. S3 (a)-(b)].
Let us first consider a situation where one could continuously measure the signal. Since having a finite-time signal is equivalent to the pointwise multiplication of an infinite signal with a rectangular window, the Fourier transform produces a spectrum whose value at \( \omega = \omega_0 \) is the sum of all the spectral contributions of the signal weighted by the spectrum of the window centered at \( \omega_0 \) (convolution theorem). As a result, even a simple, single frequency spectrum appears with multiple frequency components [see Fig. S3 (c)].

![Figure S3](image)

**Figure S3.** Time and frequency domain representation of \( s_1(t) \) for a time interval of \( 4T \) (purple) and \( 4.5T \) (pink). (a-b) Continuous time measurement with finite time windows and their corresponding continuous Fourier transforms in (c). (d-e) Discrete sampling of a continuous signal for both time intervals. The associated discrete Fourier transforms (dots) and discrete-time Fourier transforms (solid lines) are shown in (f). (g-h) 0-padding of the sampled signal and corresponding interpolated discrete Fourier transforms in (i). Post-processing of the sampled signal with a Blackmann-Harris window and 0-padding (j-k). The resulting discrete Fourier transform is less susceptible to spectral leakage (l).

Experimentally, however, it is not always possible to measure a continuous signal, but the signal can be sampled at a certain rate [Fig. S3 (d) and (e)]. In this situation, one uses the discrete Fourier transform to extract information about frequency components. This results in a spectrum with a finite number of points [dotted points in Fig. S3 (f)], from which frequency estimation is limited due to having a finite frequency resolution set by \( \delta f = 1/t_{\text{max}} \). This is known as scalloping loss.

The continuous lines in Fig. S3 (f) show the discrete-time Fourier transform associated to the sampled signals in (d-e). The discrete Fourier transforms [dots in Fig. S3 (f)] samples the continuous spectrum generated by the discrete-time Fourier transform with a frequency interval \( \delta f \).

To minimize scalloping losses, the common practice requires one to use 0-padding [Fig. S3 (g) - (h)], which effectively reduces \( \delta f \). Additionally, 0-padding forces the signal to appear aperiodic, which mitigates spectral leakage when the time interval is not an integer multiple of the period [see Fig. S3 (i)]. However, the accuracy of the frequency estimate
is still limited by the use of a rectangular window (or no-window), which as we discussed above, also induces spectral leakage [inset in Fig. S3 (i)].

Using specific window functions, e.g., the Blackmann-Harris window [Fig. S3 (j)-(k)] [20], one further mitigates spectral leakages. This leads to a better estimation of the signal frequency as seen in Fig. S3 (l). As discussed in the main text, using interpolation of the Fourier spectrum further enhances the accuracy of the frequency estimate.

Appendix F: Iterative procedure for an idealized Ramsey signal

In this section, we show that the iterative, adaptive sensing protocol we developed in the main text is less efficient when using rectangular windows only.

To show this, we use a toy model which assumes an ideal Ramsey signal, \( s(t) = \cos^2(\Omega_0/2t) \) and consider three different IAS protocols. We consider the protocols IAS 1 and 2 discussed in the main text along with the the protocol IAS 3, which consists in using only rectangular windows. For convenience, they are defined again below:

- **IAS 1**: \( f^{(m)}(t) = \Theta(t) - \Theta(t - t_{w1}^{(1)}) \) for \( m = 1 \) and \( f^{(m)}(t) = f_{BH}(t/t_w - 1/2) \) for \( m \geq 1 \),
- **IAS 2**: \( f^{(m)}(t) = f_{BH}(t/t_{wm} - 1/2) \) \( \forall m \),
- **IAS 3**: \( f^{(m)}(t) = \Theta(t) - \Theta(t - t_{w1}^{(1)}) \) \( \forall m \).

We assume the starting estimate to be \( \hat{\Omega}(0) = 1.1\Omega_0 \) and, as in the main text, after each iteration we update the measurement time-window \( t_w \) based on the new estimate.

In Fig. S4 we show the relative error of the frequency estimate (see main text) using the previously defined IAS protocols. Using IAS 3 results in a less accurate estimation due to using rectangular windows, which lead to spectral leakage. The results generated by IAS 1 and IAS 2 lead to the same relative error for \( m > 1 \), in agreement with the results shown in the main text.

As explained in the main text, we find IAS 1 to be the protocol of choice, since it allows one to make the first frequency estimate from a spectrum whose main frequency component has a larger Fourier amplitude.

![Figure S4. Relative error \( \epsilon^{(m)}_{\Omega} \) as a function of iteration number, \( m \) for three different protocols.](image-url)