RANKING GRAPHS THROUGH MARKOV CHAINS AND HITTING TIMES.
APPLICATIONS TO PENNEY-TYPE GAMES

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Abstract. For a given collection \( S \) of random variables we construct the associated ranking oriented graph \( G(S) \). We show that for any assigned oriented graph \( H \) there exist a 1-dependent uniform chain \( X \) and some identically distributed hitting times, \( T \) of \( X \), such that \( G(T) = H \). This result is strictly related with the classical one about ranking tournaments (see \cite{17}), but within the framework of Markov chains and hitting times. As an application of our theory, we present the analysis of a generalization of the Penney’s game. Sufficient and necessary conditions are given in order to obtain favourable, fair and unfavourable games for the two players.

Keywords: 1-Dependent Markov Chain; Ordering; Ranking Graphs.
AMS MSC 2010: 60J10, 91A10, 91B06.

1. Introduction

The study of a hitting time \( T \) to a target set \( A \) is a classical topic widely discussed in the literature (see for instance \cite{11, 15, 16}). We will study some order properties between identically distributed hitting times. Thus, the order will be a consequence of the dependence between these hitting times.

We start with some notation. Let \( \mathbb{N} \) be the set of positive integers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \([n]\) be the set \( \{1, \ldots, n\} \), for \( n \in \mathbb{N} \). We denote an oriented graph by \( G = (V, \vec{E}) \), where \( V \) is the set of vertices and \( \vec{E} \subset V \times V \) is the set of arrows. For any \( i \in V \), \( (i, i) \notin \vec{E} \) (no loops), moreover for distinct vertices \( i, j \in V \), if \( (i, j) \in \vec{E} \) then \( (j, i) \notin \vec{E} \) (no 2-cycles).

When the cardinality of the vertices \( |V| \) is equal to \( n \), we identify \( V \) with \([n]\). Thus we will write \( G = ([n], \vec{E}) \). A tournament \( T = ([n], \vec{E}) \), with \( n \) vertices, is an oriented graph such that, for any two distinct vertices \( i, j \in [n] \), either \( (i, j) \in \vec{E} \) or \( (j, i) \in \vec{E} \).

Given an oriented graph \( G = ([n], \vec{E}) \), for two disjoint sets \( A, B \subset [n] \) we say that \( A \) dominates \( B \) in \( G \) if for any vertex \( a \in A \) and any vertex \( b \in B \), \( (a, b) \) belongs to \( \vec{E} \). When \( A \) dominates \( B \) in \( G \) we write \( A \to B \).

For \( r_1, r_2 \in \mathbb{N} \), we say that an oriented graph \( G = ([n], \vec{E}) \) is \((r_1, r_2)\)-directional if for any \( A \subset [n] \) with \( |A| = r_1 \) there exists \( B \subset [n] \setminus A \) with \( |B| = r_2 \) such that \( A \to B \) (see \cite{12} and \cite{4} for similar definitions). In Theorem 2 we prove the existence of \((r_1, r_2)\)-directional graphs for any \( r_1, r_2 \in \mathbb{N} \).

We will also consider some collections of random variables \( \mathcal{Y}_n = \{Y_1, \ldots, Y_n\} \) on a same probability space with the no-tie property, that is \( \mathbb{P}(Y_i = Y_j) = 0 \) for any pair of distinct indices \( i, j \in [n] \). For such a collection \( \mathcal{Y}_n \), we define the ranking oriented graph \( G(\mathcal{Y}_n) = ([n], \vec{E}(\mathcal{Y}_n)) \) as follows. For any \( i, j \in [n] \),

\[ (i, j) \in \vec{E}(\mathcal{Y}_n) \iff \mathbb{P}(Y_j < Y_i) > \frac{1}{2}. \]  

(1)

It is clear from \cite{11} that \( G(\mathcal{Y}_n) \) does not have loops or 2-cycles, thus it is an oriented graph.
The construction of the oriented graph $\mathcal{G}((Y_n))$ is analogous to those realized with a list of linear orders $P_1, P_2, \ldots, P_{2k-1}$ where there is an arrow $(i, j) \in \vec{E}$ if and only if $j$ lies above $i$ in at least $k$ of the orders $P_1, P_2, \ldots, P_{2k-1}$ (in this last case either $(i, j) \in \vec{E}$ or $(j, i) \in \vec{E}$, see [17]).

It was shown in [17] that given any tournament $T = ([n], \vec{E})$ there exist a finite $k = k(n)$ and some linear orders $P_1, P_2, \ldots, P_{2k(n)-1}$ such that the oriented graph, constructed through these linear orders, coincides with $T$. Many improvements were achieved in [19], [13], [3] and [4] giving bounds from above and below of $k(n)$, the number of linear orders needed to realize any assigned tournament with $n$ vertices.

In our construction we will use hitting times related to some Markov chains. Even in our more specific setting, we will obtain that for any oriented graph $\mathbb{H} = ([n], \vec{E})$ there exist a Markov chain $X = (X_t : t \in \mathbb{N}_0)$ and a family of hitting times $T_n = \{T_1, \ldots, T_n\}$ such that $\mathcal{G}(T_n) = \mathbb{H}$.

Before presenting our results, we need some further definitions.

For $m \in \mathbb{N}_0$, a process $(L_i)_{i \in \mathbb{N}_0}$ is said to be $m$-dependent if

$$(L_i)_{i=0, \ldots, n} \text{ is independent from } (L_{m+i})_{i \geq n+m+1}, \text{ for any } n \in \mathbb{N}_0.$$

See [1], [2] for a careful analysis and characterization of these processes through Markov chains. Notice that the $0$-dependent case coincides with the independent one. Among the $\ell$-dependent Markov chains we consider those having a unique invariant distribution which is furthermore uniform on the state space. In particular, the irreducibility of the Markov chain is guaranteed by the uniqueness of the invariant distribution.

If an $\ell$-dependent Markov chain has initial distribution equal to the uniform one, which is also invariant, it will be called an $\ell$-dependent uniform chain. We will denote by $\mathcal{M}_\ell$ the set of $\ell$-dependent uniform chains. We call $0$-dependent uniform chain a Markov chain $(X_t : t \in \mathbb{N}_0)$ formed by i.i.d. random variables having uniform distribution over a finite set. Notice that $\mathcal{M}_\ell \subset \mathcal{M}_{\ell+1}$ for any $\ell \in \mathbb{N}_0$. In the following we will concentrate on the class $\mathcal{M}_1$. Furthermore, we consider families of identically distributed hitting times $T_n = \{T_1, \ldots, T_n\}$ of a chain $X \in \mathcal{M}_1$, i.e. the hitting times share the same marginal distribution.

In this specific setting, for any oriented graph $\mathbb{H} = ([n], \vec{E})$, we will construct a Markov chain $X \in \mathcal{M}_1$ and a family of identically distributed hitting times $T_n = \{T_1, \ldots, T_n\}$ of $X$ such that $\mathcal{G}(T_n) = \mathbb{H}$ (see Theorem [1]). This result is natural in many contexts and goes beyond the ranking tournament problem.

In the second part of the paper we present an application to a generalized Penney-game.

The classical Penney’s game concerns the occurrence of different strings in a sequence of independent draws of letters. This kind of problem was studied and solved in [7] and [14] (see also [10]) for a version of the game with many players). In [14], among other results, the Authors give the construction for the optimal reply or optimal string to every string chosen by the first player. The game is always favourable to the second player. The cause of this behavior lies in the absence of transitivity for the stochastic precedence order. Recall that in the probabilistic literature the stochastic precedence, denoted by $X \leq_{sp} Y$, means $\mathbb{P}(X < Y) > \frac{1}{2}$. This is a non-transitive relation, i.e. one can construct random variables $X, Y, Z$ such that $X \leq_{sp} Y, Y \leq_{sp} Z$ and $Z \leq_{sp} X$ (see e.g. [20], [13], [8]).

Let $T_n = \{T_1, T_2, \ldots, T_n\}$ a collection of no-tie random variables. For $A \subset [n]$ we write

$$T_{\min}^{(A)} = \min\{T_i : i \in A\}.$$

By the no-tie property, if the subsets $A, B \subset [n]$ are disjoint then $\mathbb{P}(T_{\min}^{(A)} = T_{\min}^{(B)}) = 0$.

Let $r_1, r_2 \in \mathbb{N}$, $X \in \mathcal{M}_1$ and let $T_n$ be a collection of identically distributed hitting times of $X$ with $n \geq r_1 + r_2$. We define the stochastic zero-sum game $G_{r_1, r_2}(X, T_n)$ as follows:
Step 1. Player I chooses a set $A \subset [n]$ with $|A| = r_1$.

Step 2. Player II chooses a set $B \subset [n] \setminus A$ with $|B| = r_2$.

Step 3. Player I chooses two nonempty sets $A' \subset A$ and $B' \subset B$.

Step 4. If $T_{\min}^{(A')} < T_{\min}^{(B')}$ then Player II pays $|B'|$ dollars to Player I, otherwise Player I pays $|A'|$ dollars to Player II.

After the choice of $A'$ and $B'$, the expected payoff of the first player is given by

$$|B'| \cdot \mathbb{P}(T_{\min}^{(A')} < T_{\min}^{(B')}) - |A'| \cdot \mathbb{P}(T_{\min}^{(A')} > T_{\min}^{(B')}).$$

The idea underlying this payoff is that each player pays one dollar for betting on any hitting time in the final stage and the winner takes all the stakes.

Note that for given $X \in M_1$ and for a collection of hitting times $T_n$ the expected payoff of the first player is a nondecreasing function of $r_1$ and $r_2$, until $r_1 + r_2$ is smaller or equal than $n$. The proof of this fact is almost obvious. Indeed, when $r_1' > r_1$ or $r_2' > r_2$, the first player can mimic, for $G_{r_1',r_2'}(X, T_n)$, the strategies used in $G_{r_1,r_2}(X, T_n)$. Therefore his expected payoff is a monotone increasing function in $r_1$ and $r_2$.

It is quite easy to construct for given $r_1$, $r_2$ and $n \geq r_1 + r_2$ games of this kind that are fair or favourable to Player I. We will come back to this point in Section 5. It is more difficult to answer the following question.

**Q**: For given $r_1, r_2 \in \mathbb{N}$, is there $n$ such that, for suitable choices of $X \in M_1$ and identically distributed hitting times $T_1, \ldots, T_n$, the game will be favourable to the second player?

We will answer this question in the affirmative. Moreover, for $r_1, r_2 \in \mathbb{N}$, we will find a function $(r_1, r_2) \mapsto S(r_1, r_2)$ such that there exist games $G_{r_1,r_2}(X, T_n)$ favourable to the second player if and only if $n \geq S(r_1, r_2)$. The function $S(r_1, r_2)$ will be characterized through the use of $(r_1, r_2)$-directional graphs.

The paper is organized as follows. In Section 2, we define the patterns (a class of matrices with integers entries), a class of 1-dependent uniform chains and some identically distributed hitting times. In Section 3, for a given oriented graph $\tilde{G} = ([n], \tilde{E})$, we construct some $X \in M_1$ and some identically distributed hitting times $T_n$ of $X$ such that $\mathbb{G}(T_n) = \tilde{G}$. In Section 4, we prove the existence of $(r_1, r_2)$-directional graphs, for any $r_1, r_2 \in \mathbb{N}$ when the number of vertices of the graph is large enough. In Section 5, we give the definition of 2-determined random variables and finally we show that for $n \geq S(r_1, r_2)$ there are games $G_{r_1,r_2}(X, T_n)$ that are favourable to the second player.

2. 1-dependent uniform Markov chains, patterns and identically distributed hitting times

In order to avoid some difficulties and present the result in the most simple way we will realize some specific constructions, in any case some of the following results could be presented in a more general framework.

Let us consider a Markov chain with state space $I_{N,k}$, where $I_{N,k}$ is the collection of all the matrices of the form

$$A = (a_{i,j} \in [N] : i \in [k], j \in [2]).$$

The cardinality of the state space $I_{N,k}$ is $N^{2k}$. We consider the transition matrix $P_{N,k} = (p_{A,B} : A, B \in I_{N,k})$ defined by

$$p_{A,B} = \begin{cases} \frac{1}{N^k} & \text{if } a_{i,2} = b_{i,1} \text{ for } i = 1, \ldots, k; \\ 0 & \text{otherwise}; \end{cases}$$

(3)
where \( A = (a_{i,j} \in [N] : i \in [k], j \in [2]) \) and \( B = (b_{i,j} \in [N] : i \in [k], j \in [2]) \) are elements of \( I_{N,k} \).

It is clear that this transition matrix is irreducible and bi-stochastic, therefore the unique invariant distribution is uniform on \( I_{N,k} \). The uniform chain having transition matrix \( P_{N,k} \), with uniform initial distribution, is denoted by \( X^{(N,k)} = (X_m^{(N,k)} : m \in \mathbb{N}_0) \).

Notice that the collection of random variables \( (X_m : m \in \mathbb{N}_0) \) forms a 1-dependent family of random variables, therefore for any sequence \( m_1 < m_2 < \ldots m_\ell \) with \( m_{i+1} - m_i \geq 2 \) the random variables \( \{X_{m_1}^{(N,k)}, \ldots, X_{m_\ell}^{(N,k)}\} \) are independent. One can think to the states of this Markov chain as vectors \( (U_n \in [N]^k : n \in \mathbb{N}_0) \) sequentially postponed where the components of the vectors are independent and uniform distributes on \([N]\) (see Example 1); in formula \( X_m^{(N,k)} = (U_m, U_{m+1}) \).

We also notice that the Markov chain is reversible, in any case this last property will not play any role in our future analysis.

2.1. Patterns. For \( M \geq 2 \) and \( k \in \mathbb{N} \), a pattern \( Q = (q_{i,j} \in [M] \cup \{0\} : i \in [k], j \in [2]) \) is a \( k \times 2 \) matrix having entries in \([M] \cup \{0\}\), more precisely any element \( q_{i,j} \) of the first column of a pattern is in \([M]\), for \( j \in [k] \), and all the elements of the second column are zero with the exception of one that takes value in \([M]\). The collection of all the patterns that are matrices with \( k \) rows and entries in \([M] \cup \{0\}\) is denoted by \( P_{M,k} \). In particular, any pattern in \( P_{M,k} \) has a number of entries different from zero which is equal to \( k + 1 \).

2.2. Hitting time of a pattern. For two matrices of the form \( R = (r_{i,j} \in \mathbb{R} : i \in [k], j \in [2]) \) and \( S = (s_{i,j} \in \mathbb{R} : i \in [k], j \in [2]) \) we define the relation

\[
R \times S \iff \sum_{i=1}^{k} \sum_{j=1}^{2} r_{i,j} s_{i,j} |s_{i,j} - r_{i,j}| = 0
\]

Clearly, this relation is reflexive and symmetric but non transitive.

For \( N \geq M \geq 2, k \geq 1 \) let us consider \( X^{(N,k)} = (X_m^{(N,k)} : m \in \mathbb{N}_0) \in \mathcal{M}_1 \) and a pattern \( R \in P_{M,k} \), we define the hitting time to the target pattern \( R \) as

\[
T_R = \inf\{m \in \mathbb{N}_0 : X_m^{(N,k)} \times R\}.
\]

The hitting time \( T_R \) can be interpreted as the first time in which the pattern \( R \) occurs in the random sequence \( (X_m^{(N,k)})_{m \in \mathbb{N}_0} \). From the finiteness of state space \( I_{N,k} \) and the irreducibility of the Markov chain \( X^{(N,k)} \) follows that \( T_R \) is finite almost surely under the hypothesis \( N \geq M \).

2.3. Overlap. For two different patterns

\[
R = (r_{i,j} : i \in [k], j \in [2]), S = (s_{i,j} : i \in [k], j \in [2]) \in P_{M,k},
\]

we define the overlap \( O(R,S) \) as the vector over \( \{0,1\} \) having 2 components that is defined as follows.

For \( \ell \in [2] \), the \( \ell \)-th component of \( O(R,S) \) is

\[
O(R,S)_\ell = \begin{cases} 
1, & \text{if } \sum_{m=\ell}^{2} \sum_{h=1}^{k} s_{h,m} r_{h,m+1-\ell} |s_{h,m} - r_{h,m+1-\ell}| = 0; \\
0, & \text{otherwise}.
\end{cases}
\]

The overlap \( O(R,S) \) is in general different from \( O(S,R) \), moreover it makes sense to consider the overlap of a pattern with itself. In this case the first component of the overlap is always equal to 1.

For patterns \( R_1, \ldots, R_n \in P_{M,k} \), it is clear that

\[
\text{no-tie property of } T_{R_1}, \ldots, T_{R_n} \iff \text{for distinct } i, j \in [n], O(R_i, R_j)_1 = 0.
\]

In this case, for sake of simplicity, we say that the collection of patterns \( R_1, \ldots, R_n \) is no-tie.
To familiarize with the notions and definitions we present the following example

**Example 1.** Let

\[
R = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},
\]

be two patterns in \(P_{3,2}\). The overlap \(O(R, S) = (0, 0)\) and \(O(S, R) = (0, 1)\). The collection of patterns \(\{R, S\}\) is no-tie.

Suppose that

\[
X^{(2,3)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & \cdots \\ 2 & 2 & 1 & 2 & 1 & \cdots \\ 2 & 1 & 2 & 1 & 2 & 1 & \cdots \end{bmatrix}
\]

Then \(T_S = 2\) being

\[
S \neq \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 1 \end{bmatrix}, \quad S \neq \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad S \neq \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}.
\]

Similarly, \(T_R = 4\).

3. **Oriented graphs through 1-dependent uniform chains and identically distributed hitting times**

Let \(R\) be a pattern with the property that \(O(R, R) = (1, 0)\). Following the proof of Theorem 2.1 in [6] we compute recursively the discrete distribution of \(T_R\).

**Lemma 1.** Let \(k, M, N\) be integers such that \(N \geq M \geq 2\) and \(k \geq 1\). Let \(R \in \mathcal{P}_{M,k}\) with \(O(R, R) = (1, 0)\) and consider the 1-dependent uniform chain \(X^{(N,k)}\). Define \(w(t) := \mathbb{P}(T_R = t)\). The probabilities \((w(t) : t \in \mathbb{N}_0)\) are recursively determined by

\[
w(t) = N^{-k-1} - N^{-k-1} \sum_{s=0}^{t-2} w(s), \quad t \geq 2,
\]

with \(w(0) = w(1) = N^{-k-1}\).

**Proof.** By construction \(w(0) = w(1) = N^{-k-1}\). For \(t \geq 2\), consider the event \(\{X_t^{(N,k)} \propto R\}\). This event happen with probability \(N^{-k-1}\). Moreover, it can be holden in three disjoint ways:

(i) \(\{T_R = t\}\);
(ii) \(\{X_t^{(N,k)} \propto R\} \cap \{T_R = s\}\), for \(s < t - 1\);
(iii) \(\{X_t^{(N,k)} \propto R\} \cap \{T_R = t - 1\}\).

The probability of \(\{T_R = t\}\) is by definition \(w(t)\).

For \(s < t - 1\), as a consequence of 1-dependence, one has

\[
\mathbb{P}((X_t^{(N,k)} \propto R) \cap \{T_R = s\}) = w(s) \mathbb{P}(X_t^{(N,k)} \propto R) = N^{-k-1} w(s).
\]

By hypothesis \(O(R, R)_2 = 0\), it follows that the two events in (iii) are incompatible, thus the probability of the event in (iii) is null. Therefore, for \(t \geq 2\),

\[
N^{-k-1} = w(t) + N^{-k-1} \sum_{s=0}^{t-2} w(s),
\]

which corresponds to (\[\Box\]).
Remark 1. Let \(R_1, \ldots, R_n \in \mathcal{P}_{M,k}\) such that \(O(R_i, R_i) = (1,0)\), for each \(i \in [n]\). Let \(N \geq M\) and consider the \(1\)-dependent uniform chain \(X^{(N,k)}\). Then, by Lemma 3 all the hitting times \(T_{R_1}, \ldots, T_{R_n}\) share the same distribution determined by [3].

Given an oriented graph \(\bar{G} = ([n], \bar{E})\), we now construct a collection of associated patterns \(\{R_u \in \mathcal{P}_{n+1,n+1} : u \in [n]\}\). For \(\ell \in [n]\), pattern \(R_\ell = (r_{i,j}^{(\ell)} : i \in [n+1], j \in [2])\) is constructed in the following way:

1. \(r_{1,1}^{(\ell)} = r_{\ell+1,2}^{(\ell)} = \ell\);
2. for any \(j \in [n+1] \setminus \{\ell + 1\}\), \(r_{j,2}^{(\ell)} = 0\);
3. for any \(j \in [n]\),
   \[
   r_{j+1,1}^{(\ell)} = \begin{cases} j, & \text{if } (\ell, j) \in \bar{E}; \\ n + 1, & \text{otherwise.} \end{cases}
   \] (9)

We will say that the patterns \(R_1, \ldots, R_n\) are generated by the graph \(\bar{G} = ([n], \bar{E})\).

We now consider the properties of the overlap for patterns \(R_1, \ldots, R_n\) generated by the oriented graph \(\bar{G} = ([n], \bar{E})\).

Lemma 2. Let \(n \geq 2\) and \(\bar{G} = ([n], \bar{E})\) be an oriented graph. The overlaps of the patterns \(R_1, \ldots, R_n\) generated by \(\bar{G} = ([n], \bar{E})\) are

\[
O(R_i, R_j) = \delta_{i,j}(1,0) + (1 - \delta_{i,j})((0,1)1_{\{(i,j) \in \bar{E}\}} + (0,0)1_{\{(i,j) \notin \bar{E}\}}),
\]

for any \(i, j \in [n]\).

Proof. Case \(i = j\), then \(O(R_i, R_i)_1 = 1\). The second component \(O(R_i, R_i)_2 = 0\) since \(r_{i+1,1}^{(i)} = n + 1 \neq i = r_{i+1,2}^{(i)}\) (see (2)).

Case \(i \neq j\). \(O(R_i, R_j)_1 = 0\) since \(r_{1,1}^{(i)} = i \neq j = r_{1,1}^{(j)}\).

If \((i, j) \in \bar{E}\) then \(O(R_i, R_j)_2 = 1\). Indeed, the sum in (6) becomes

\[
\sum_{h=1}^{n+1} r_{h,2}^{(j)} \cdot r_{h,1}^{(i)} \cdot |r_{h,2}^{(j)} - r_{h,1}^{(i)}| = r_{j+1,2}^{(i)} \cdot r_{j+1,1}^{(i)} \cdot |r_{j+1,2}^{(i)} - r_{j+1,1}^{(i)}|
\]

the previous sum is null since \(r_{j+1,2}^{(i)} = r_{j+1,1}^{(i)} = j\).

If \((i, j) \notin \bar{E}\) then \(O(R_i, R_j)_2 = 0\). Indeed, the sum in (6) becomes

\[
\sum_{h=1}^{n+1} r_{h,2}^{(j)} \cdot r_{h,1}^{(i)} \cdot |r_{h,2}^{(j)} - r_{h,1}^{(i)}| = r_{j+1,2}^{(j)} \cdot r_{j+1,1}^{(i)} \cdot |r_{j+1,2}^{(j)} - r_{j+1,1}^{(i)}| \neq 0
\]

the previous sum is larger than zero since \(r_{j+1,2}^{(j)} = j\) and \(r_{j+1,1}^{(i)} = n + 1\).

\[\square\]

Let us consider no-tie collection of patterns \(\{R_1, \ldots, R_\ell\}\) belonging to \(\mathcal{P}_{M,k}\) and the corresponding hitting times \(T_\ell = \{T_{R_1}, \ldots, T_{R_\ell}\}\) of \(X^{(N,k)}\), with \(N \geq M\). We are interested to upper and lower bound

\[
p_i(T_\ell) = p_i(\{T_{R_1}, \ldots, T_{R_\ell}\}) := P \left( \bigcap_{j \in [\ell]} \{T_{R_i} \leq T_{R_j}\} \right) \text{ for } i \in [\ell].
\]

(10)

First notice that by the no-tie property one has \(\sum_{i \in [\ell]} p_i(T_\ell) = 1\). The probabilities \(p_i(T_\ell) : i \in [\ell]\) could be explicitly computed through a linear system but for our purposes it will be more useful to have good upper and lower bounds.
Lemma 3. Let \( N \geq M \geq 2 \) and \( k, \ell \geq 2 \). Let \( R_1, \ldots, R_\ell \in \mathcal{P}_{M,k} \) be a collection of no-tie patterns and let \( O(R_i, R_i)_2 = 0 \), for \( i \in [\ell] \). Let us take the uniform Markov chain \( X^{(N,k)} \) and the hitting times \( T_\ell = \{ T_{R_i} : i \in [\ell] \} \). Then \( T_\ell \) are identically distributed. Moreover, for \( i \in [\ell] \),

\[
p_i(T_\ell) = \frac{v_i}{\sum_{j \in [\ell]} v_j}
\]

where

\[
1 - \frac{1}{N^k} \sum_{j \in [\ell]} O(R_i, R_j)_2 \leq v_i \leq 1 - \frac{1}{N^k} \left( 1 - \frac{\ell - 1}{N^k} \right) \sum_{j \in [\ell]} O(R_i, R_j)_2.
\]

Proof. Lemma 1 Remark 1 and the fact that \( O(R_i, R_i)_2 = 0 \), for each \( i \), imply that \( T_\ell \) are identically distributed.

Let us define the sequence of stopping times \( (Z_h : h \in \mathbb{N}_0) \), as

\[
Z_0 = \inf\{ m \geq 0 : X_m \succ R_i \text{ for some } i \in [\ell] \}
\]

and recursively, let

\[
Z_{h+1} = \inf\{ m \geq Z_h + 2 : X_m \succ R_i \text{ for some } i \in [\ell] \}.
\]

Notice that any hitting time \( Z_h \) is finite almost surely. For any \( i \in [\ell] \), by definition

\[
p_i(T_\ell) = \mathbb{P}(T_{R_i} = Z_0).
\]

From the fact that the sequence of random variables \( (X_m)_{m \in \mathbb{N}_0} \) is 1-dependent also

\[
p_i(T_\ell) = \mathbb{P}(X_{Z_h} \succ R_i),
\]

for any \( h \in \mathbb{N} \). Indeed, more precisely one has that

\[
\mathbb{P}(X_{Z_h} \succ R_i, Z_h - Z_{h-1} = s | X_{Z_{h-1}} = j, Z_{h-1} = t), \text{ for } h, s \in \mathbb{N}, \text{ and } i \in [\ell],
\]

does not depend on \( h, t \in \mathbb{N} \) and \( j \in [n] \). Again this is a consequence of the 1-dependent structure.

We define, for any \( i \in [\ell] \), the sets of random times

\[
V_{i,t} := \{ r < t : r = Z_s \text{ for some } s, X_{Z_s} \succ R_i \}, \quad N_{i,t} := \{ s < t : X_s \succ R_i \}
\]

where \( t \in \mathbb{N} \cup \{ +\infty \} \). The cardinalities are

\[
V_{i,t} := |V_{i,t}| = \sum_{s=0}^{\infty} 1_{\{ Z_s \leq t-1 \}} 1_{\{ X_{Z_s} \succ R_i \}}, \quad N_{i,t} := |N_{i,t}| = \sum_{s=0}^{t-1} 1_{\{ X_s \succ R_i \}}
\]

By (15) and by the ergodic theorem for renewal process, one has

\[
\lim_{t \to \infty} \frac{V_{i,t}}{\sum_{j \in [\ell]} V_{j,t}} = p_i(T_\ell) \text{ a.s.}, \quad \lim_{t \to \infty} \frac{N_{i,t}}{t} = \frac{1}{N^{k+1}} \text{ a.s.}
\]

We define the quantities \( (v_i > 0 : i \in [\ell]) \) as

\[
v_i := \lim_{t \to \infty} \frac{V_{i,t} N^{k+1}}{t} \text{ a.s.},
\]

by hypothesis \( N \geq M \) one has \( v_i \leq 1 \), for each \( i \in [\ell] \).

The equalities in (18) and the previous definition give (11).

We notice that

\[
N_{i,t} = V_{i,t} + \sum_{j \in [\ell] \setminus \{ i \}} \sum_{s=0}^{\infty} 1_{\{ Z_s \leq t-2 \}} 1_{\{ X_{Z_s} \succ R_j \}} 1_{\{ X_{Z_s+1} \succ R_i \}},
\]

(20)
indeed if \( X_{Z_s + 1} \sim R_i \), for some \( s \), the time \( (Z_s + 1) \) belongs to \( \mathcal{N}_{i, \infty} \) but it is not in \( \mathcal{V}_{i, \infty} \). Let us multiply by \( N^{k+1}/t \) the previous formula and take the limit for \( t \to \infty \), then, by the ergodic theorem, by (18) and (19), one obtains

\[
1 = v_i + \lim_{t \to \infty} \frac{N^{k+1}}{t} \sum_{j \in [\ell] : j \neq i} \sum_{s=0}^{\infty} 1_{\{Z_s \leq t-2\}} 1_{\{X_{Z_s} \neq R_j\}} 1_{\{X_{Z_s + 1} \sim R_i\}}. \tag{21}
\]

We notice that

\[
\mathbb{P}(X_{Z_s + 1} \sim R_i | X_{Z_s} \neq R_j) = \frac{O(R_i, R_j)_2}{N^k}. \tag{22}
\]

By the ergodic theorem one has

\[
1 = v_i + \sum_{j \in [\ell] : j \neq i} \frac{O(R_i, R_j)_2}{N^k} v_j. \tag{23}
\]

Thus,

\[
1 \leq v_i + \sum_{j \in [\ell] : j \neq i} \frac{O(R_i, R_j)_2}{N^k} = v_i + \sum_{j \in [\ell]} \frac{O(R_i, R_j)_2}{N^k}. \tag{24}
\]

The inequality (24) corresponds to the first inequality in (12). In particular \( v_j \geq 1 - (n - 1)/N^k \), for any \( j \in [\ell] \). Thus, for any fixed \( j \in [\ell] \)

\[
\lim_{t \to \infty} \sum_{s=0}^{\infty} 1_{\{Z_s \leq t-2\}} 1_{\{X_{Z_s} \neq R_j\}} \geq 1 - (\ell - 1)/N^k. \tag{25}
\]

Now, by (20)–(25) and ergodicity one has

\[
v_i \leq 1 - \frac{1}{N^k} \left( 1 - \frac{\ell - 1}{N^k} \right) \sum_{j \in [\ell]} O(R_i, R_j)_2. \]

This end the proof. \( \square \)

We are now ready to present the following result on the construction of any oriented graph through the ranking graphs of uniform chains and identically distributed hitting times.

**Theorem 1.** For any oriented graph \( \mathcal{G} = ([n], E) \) there exists a 1-dependent uniform chain \( \mathbf{X} = (X_m : m \in \mathbb{N}_0) \) and a collection of identically distributed hitting times \( \mathcal{T}_n = \{T_1, T_2, \ldots, T_n\} \) such that \( \mathcal{G}(\mathcal{T}_n) = \mathcal{G} \).

**Proof.** Let us consider the 1-dependent uniform chain \( \mathbf{X}^{(N,n+1)} \) with \( N \geq n + 1 \). By Lemma 2 the patterns \( R_1, \ldots, R_n \) generated by \( \mathcal{G} \) have the no-tie property.

For any couple of distinct indices \( i, j \)

\[
p_{a}(\{T_{R_i}, T_{R_j}\}) = \mathbb{P}(T_{R_a} = Z_1),
\]

for \( a \in \{i, j\} \).

In the case that \((i, j)\) and \((j, i)\) does not belong to the oriented graph \( \mathcal{G} \) then, by Lemma 2

\[
O(R_i, R_j)_2 = O(R_j, R_i)_2 = O(R_i, R_i)_2 = O(R_j, R_j)_2 = 0. \text{ Thus, by (20) of Lemma 3 follows that} \quad N_{i,t} = V_{i,t} \text{ and } N_{j,t} = V_{j,t}. \text{ Hence by (18) and (19) one has}
\]

\[
p_i(\{T_{R_i}, T_{R_j}\}) = p_j(\{T_{R_i}, T_{R_j}\}) = \frac{1}{2},
\]

Hence, \((i, j)\) and \((j, i)\) does not belong to the oriented graph \( \mathcal{G}(\mathcal{T}_n) \).
We now consider the case: \((i, j)\) is in \(\hat{G}\). By Lemma 2, \(O(R_j, R_i) = (0, 0)\) and \(O(R_j, R_i) = (0, 1)\).

Thus, by (20) follows that \(N_{j,t} = V_{j,t}\) while

\[
N_{i,t} = V_{i,t} + \sum_{s=0}^{\infty} 1\{Z_s \leq t-2\} 1\{X_{Z_s} = R_j\} 1\{X_{Z_s+1} = R_i\},
\]

Now, defining \(v_i\) and \(v_j\) as in the proof of Lemma 3, \(v_j = 1\) while

\[
v_i \leq 1 - \frac{1}{Nk} \left( 1 - \frac{1}{Nk} \right) < 1.
\]

Therefore

\[
p_i(\{T_{R_i}, T_{R_j}\}) \frac{v_i}{v_i + v_j} < \frac{v_j}{v_i + v_j} = p_j(\{T_{R_i}, T_{R_j}\}).
\]

Hence, \((i, j)\) belongs to the oriented graph \(\hat{G}(T_n)\).

\[\square\]

4. Existence of \((r_1, r_2)\)-directional graphs

In [12], Erdős analyses a problem that correspond to the existence of \((r_1, 1)\)-directional graphs (see also [5]). The probabilistic method developed in [12] can be easily adapted in our case.

For any \(r_1, r_2 \in \mathbb{R}\), let us define

\[
S(r_1, r_2) := \inf \left\{ k \geq r_1 + r_2 : \text{there exists a } (r_1, r_2)\text{-directional tournament } ([k], \vec{E}) \right\}.
\]

**Theorem 2.** For any \(r_1, r_2 \in \mathbb{N}\) and any \(n \geq S(r_1, r_2)\) there exists a \((r_1, r_2)\)-directional tournament \(T = ([n], \vec{E})\). Moreover

\[
S(r_1, r_2) \leq \inf \left\{ n \geq r_1 + r_2 : \binom{n}{r_1} \left( 1 - \frac{1}{2r_1r_2} \right)^{\left\lfloor \frac{n-1}{r_2} \right\rfloor} < 1 \right\} < \infty.
\]

**Proof.** For \(r_1, r_2 \in \mathbb{N}\), we first assume that for a specific \(n_0 \in \mathbb{N}\) there exists a \((r_1, r_2)\)-directional tournament \(T_{n_0} = ([n_0], \vec{E})\). Then, we prove that for any \(n > n_0\) there is a tournament \(T_n = ([n], \vec{E}_n)\) which is \((r_1, r_2)\)-directional. The proof is by induction.

Suppose that for \(n - 1 \geq n_0\) there is a \((r_1, r_2)\)-directional tournament \(T_{n-1} = ([n-1], \vec{E}_{n-1})\), then we will construct a tournament \(T_n = ([n], \vec{E}_n)\) that is \((r_1, r_2)\)-directional.

For any distinct \(i, j \in [n-1]\) let \((i, j)\) be in \(\vec{E}_{n-1}\) if and only if \((i, j) \in E_{n-1}\). Moreover, for any \(i \in [n-1]\), we impose that \((n, i)\) belongs to \(\vec{E}_{n}\). It is clear that if \(T_{n-1}\) is a \((r_1, r_2)\)-directional tournament then also \(T_n\) is an \((r_1, r_2)\)-directional tournament. Indeed, if \(A \subset [n-1]\) with \(|A| = r_1\) then one can select \(B \subset [n-1]\) with \(|B| = r_2\) with \(A \to B\), as in \(T_{n-1}\). On the other hand if we consider an \(A \subset [n]\) such that \(n \in A\) and \(|A| = r_1\) then one can take \(B \subset [n-1]\) such that \((A \setminus \{n\}) \to B\) and \(|B| = r_2\). In any case the relation \((A \setminus \{n\}) \to B\) implies \(A \to B\) because \((n, i) \in \vec{E}_{n}\) for any \(i \in [n-1]\).

Now, we prove formula (27), by the probabilistic method (see e.g. [5]). For this purpose, we will construct a random tournament and we will show that it is \((r_1, r_2)\)-directional with positive probability. For \(n \in \mathbb{N}\), let us construct a random tournament \(T(n) = ([n], \vec{E}(n))\) in the following way.

For two distinct vertices \(u, v\), either \((u, v) \in \vec{E}(n)\) or \((v, u) \in \vec{E}(n)\); both these events occur with probability 1/2. Moreover all the events involving distinct edges are assumed independent.

For given \(r_1, r_2 \in \mathbb{N}\) let \(\vec{V} \subset [n]\) with \(|\vec{V}| = r_1\), we define the event

\[
A_{\vec{V}} := \{ \exists V' \subset [n] \setminus \vec{V} : \vec{V} \to V', \text{ with } |V'| = r_2 \}.
\]
Now, for a given \( \tilde{V} \) having cardinality \( r_1 \), let us choose a family of sets of vertices

\[
\left( V_i : |V_i| = r_2, V_i \subset [n] \setminus \tilde{V}, \ i = 1, \ldots, \left\lfloor \frac{n-r_1}{r_2} \right\rfloor \right),
\]

with \( V_i \cap V_j = \emptyset \), for \( i \neq j \).

By independence of the random directions involving different edges one has

\[
\mathbb{P}(A_{\tilde{V}}^c) \leq \mathbb{P} \left( \bigcap_{i=1}^{n-r_1/r_2} \{ \tilde{V} \to V_i \}^c \right) \leq \left( 1 - \frac{1}{2^{r_1 r_2}} \right)^{n-r_1/r_2},
\]

for any \( \tilde{V} \subset [n] \) with \( |	ilde{V}| = r_1 \).

By subadditivity of the probability measure one has

\[
\mathbb{P} \left( \bigcap_{\tilde{V} \subset [n] ; |\tilde{V}| = r_1} A_{\tilde{V}} \right) = 1 - \mathbb{P} \left( \bigcup_{\tilde{V} \subset [n] ; |\tilde{V}| = r_1} A_{\tilde{V}}^c \right) \geq 1 - \left( \frac{n}{r_1} \right) \left( 1 - \frac{1}{2^{r_1 r_2}} \right)^{n-r_1/r_2}. \tag{28}
\]

For any \( r_1, r_2 \in \mathbb{N} \),

\[
\lim_{n \to \infty} \left( \frac{n}{r_1} \right) \left( 1 - \frac{1}{2^{r_1 r_2}} \right)^{n-r_1/r_2} = 0. \tag{29}
\]

Formulas (28) and (29) imply (27).

It is clear that, for a given \( n \in \mathbb{N} \), if \( G = ([n], \tilde{E}) \) is an \((r_1, r_2)\)-directional oriented graph and \( G' = ([n], \tilde{E}') \) an oriented graph with \( \tilde{E} \subset \tilde{E}' \) then also \( G' \) is an \((r_1, r_2)\)-directional oriented graph. This easy observation and Theorem 2 imply

**Corollary 1.** For \( r_1, r_2 \in \mathbb{N} \), all the \((r_1, r_2)\)-directional oriented graphs have the number of vertices larger than or equal to \( S(r_1, r_2) \).

5. FAVOURABLE, FAIR AND UNFAVOURABLE GAMES

We start the analysis of the game presented in the introduction. In the following, we take the point of view of the second player so we declare *favourable* the game if the expected value of the payoff of the second player is positive, when both players adopt the best strategies. Analogously, we declare *fair* the game when the expected value of the payoff is zero and *unfavourable* when the expected payoff of the second player is negative.

We start with some easy comments and remarks. We first construct, for any \( n \geq r_1 + r_2 \) a fair game \( G_{r_1,r_2}(X, T_n) \). Let \( X = (X_m : m \in \mathbb{N}_0) \) be a sequence of i.i.d. random variables taking value on \([n] \), with \( X_0 \) uniformly distributed on \([n] \). Then \( X \) is a 1-dependent uniform chain. Let \( T_n = \{T_1, \ldots, T_n\} \) be the collection of identically distribute hitting times, where

\[
T_i = \inf \{ m \in \mathbb{N}_0 : X_m = i \}.
\]

It is clear that for any strategy of the players the game has an expected payoff equal to zero, therefore the game is trivially fair.

Now we construct an unfavourable game for any \( n \geq r_1 + r_2 \). Let us consider a tournament \( T = ([n], \tilde{E}) \) such that \((i,n) \in \tilde{E}\) for any \( i \in [n-1] \). If the first player takes the set \( A \) such that \( n \in A \) then for any choice \( B \) of the second player the first player can select \( A' = \{n\} \). By construction \( B \rightarrow \{n\} \), therefore the first player has guaranteed a positive expected payoff if he select any \( B' \subset B \) such that \( |B'| = 1 \) and \( A' = \{n\} \). This strategy of the first player could be suboptimal but it is enough to guarantee a negative expected payoff for the second player.
In order to find for which \( n \) there exist favorable games we need some more discussions and definitions. The following definition and Definition 2 are similar to others given in [9] but they will be used for different applications.

**Definition 1.** Let us consider two finite sets of random variables \( S_A = \{S_i : i \in A\} \) and \( S_B = \{S_i : i \in B\} \) on the same probability space, such that \( S_A \cup S_B \) has the no-tie property. We say that \( S_A \) is small with respect to \( S_B \) if

\[
\frac{1}{|A|} \sum_{i \in A} p_i(S_A \cup S_B) > \frac{1}{|B|} \sum_{i \in B} p_i(S_A \cup S_B).
\]

(30)

First we present an example showing that the collective behaviour cannot be deduced by pair relations.

**Example 2.** Let \( S_1 = \frac{49}{100} \) and let \( S_2, S_3 \) be independent r.v. with uniform law on \([0, 1]\). The collection \( \{S_1, S_2, S_3\} \) has the no-tie property. The r.v. \( S_1 \) is small with respect to \( S_i \), for \( i = 2, 3 \), because \( P(S_1 < S_2) = P(S_1 < S_3) = \frac{51}{100} \). But

\[
\frac{1}{2} \sum_{i=2}^{3} p_i(\{S_1, S_2, S_3\}) = \frac{1}{2} - \frac{1}{2} \left( \frac{51}{100} \right)^2 > \left( \frac{51}{100} \right)^2 = p_1(\{S_1, S_2, S_3\}).
\]

Therefore \( \{S_2, S_3\} \) is small with respect to \( S_1 \). This example shows that the analysis of the smallness property cannot be reduced to the study of pair relations. In fact, these can be completely reversed when we move on to consider collections of random variables.

For the previous example we think that the study of a generic game \( G_{r_1,r_2}(X, T_n) \) could in general become difficult. In order to avoid some difficulties in the construction of favorable games we define some special systems of random variables.

**Definition 2.** Let \( S_n = \{S_1, \ldots, S_n\} \) be a collection of random variables and let \( \mathbb{G}(S_n) = ([n], \vec{E}) \) be the associated ranking oriented graph. We say that \( S_n \) is 2-determined if for any two disjoint \( A, B \subset [n] \) such that \( A \to B \) then \( \mathcal{S}_B = \{S_i : i \in B\} \) is small with respect to \( \mathcal{S}_A = \{S_i : i \in A\} \).

**Theorem 3.** Let \( n \geq 2 \) and let \( \mathbb{G} = ([n], \vec{E}) \) be given. Let \( R_1, \ldots, R_n \) be patterns in \( \mathcal{P}_{n+1,n+1} \) generated by the oriented graph \( \mathbb{G} = ([n], \vec{E}) \). For \( N \geq n + 1 \), let us consider the 1-dependent uniform chain \( X^{(N,n+1)} \) and the identically distributed hitting times \( T_n = \{T_{R_1}, \ldots, T_{R_n}\} \). Then the collection of hitting times \( T_n \) is 2-determined when \( N \) is large enough.

**Proof.** We need to show that \( T_B := \{T_i : i \in B\} \) is small with respect to \( T_A := \{T_i : i \in A\} \), for any \( A, B \subset [n] \) when \( A \) dominates \( B \) in \( \mathbb{G} \).

Thus, we have to prove

\[
\frac{1}{|A|} \sum_{i \in A} p_i(T_A \cup T_B) < \frac{1}{|B|} \sum_{i \in B} p_i(T_A \cup T_B),
\]

(31)

when \( A \to B \). Inequality (31) holds true if and only if

\[
\frac{1}{|B|} \sum_{i \in B} v_i - \frac{1}{|A|} \sum_{i \in A} v_i > 0.
\]

For any \( i \in A \), by Lemma 3 one has

\[
v_i \leq 1 - \frac{1}{N^{n+1}} \left( 1 - \frac{|A| + |B| - 1}{N^{n+1}} \right) \frac{|B|}{N^{n+1}}.
\]
Hence, 
\[ \frac{1}{|A|} \sum_{i \in A} v_i \leq 1 - \frac{|B|}{N^{n+1}} + \frac{(|A| + |B| - 1)|B|}{N^{2n+2}}. \] (32)

Analogously, by the first inequality in (12) and the hypothesis that \( A \to B \), one has 
\[ \frac{1}{|B|} \sum_{i \in B} v_i \geq 1 - \frac{1}{N^{n+1}} \sum_{i \in B} \sum_{j \in B \setminus \{i\}} 1. \] (33)

Hence, 
\[ \frac{1}{|B|} \sum_{i \in B} v_i \geq 1 - \frac{|B| - 1}{2N^{n+1}}. \] (34)

Therefore one obtains
\[ \frac{1}{|B|} \sum_{i \in B} v_i - \frac{1}{|A|} \sum_{i \in A} v_i > \frac{|B|}{2N^{n+1}} - \frac{(|A| + |B| - 1)|B|}{N^{2n+2}}. \]

Thus, for \( N \) large enough, the previous quantity is larger than zero for any choice of disjoint sets \( A, B \subset [n] \). \■

We give the key result that will allow the construction of favourable games.

**Theorem 4.** Let \( r_1, r_2 \in \mathbb{N}, n \geq r_1 + r_2 \), let \( X \) be a 1-dependent uniform chain and \( \mathcal{T}_n = \{T_1, \ldots, T_n\} \) a collection of hitting times.

i. The game \( G_{r_1, r_2}(X, \mathcal{T}_n) \) is favourable \( \Rightarrow \) \( G(\mathcal{T}_n) \) is \( (r_1, r_2) \)-directional.

Moreover, let us suppose that \( \mathcal{T}_n \) is 2-determined then

ii. The game \( G_{r_1, r_2}(X, \mathcal{T}_n) \) is favourable \( \iff \) \( G(\mathcal{T}_n) \) is \( (r_1, r_2) \)-directional.

**Proof.** Item i. Suppose that the oriented graph \( G_1(\mathcal{T}_n) = ([n], \mathcal{E}(\mathcal{T}_n)) \) is not \((r_1, r_2)\)-directional. Then the first player can select a set \( A \), with cardinality \( r_1 \), such that for any \( B \) with cardinality \( r_2 \) the set \( A \) does not dominate \( B \) in \( G_1(\mathcal{T}_n) \). Therefore for any choice of \( B \), done by the second player, the first player can select \( i \in A \) and \( j \in B \) such that \((i, j) \notin \mathcal{E}(\mathcal{T}_n) \). Hence, \( \Pr(T_j < T_i) \leq \frac{1}{2} \).

Therefore, by choosing \( A' = \{i\} \) and \( B' = \{j\} \) the first player has guaranteed that the game is either fair or in his favour. Obviously, this strategy could be suboptimal but it gives an expected payoff to the first player that is not negative for any choice of the second player. Therefore the game \( G_{r_1, r_2}(X, \mathcal{T}_n) \) is not favourable.

Item ii. Suppose that the ranking oriented graph \( G(\mathcal{T}_n) \) is \((r_1, r_2)\)-directional and \( \mathcal{T}_n \) is 2-determined. For any set \( A \) having cardinality \( r_1 \), the second player can select a set \( B \), with cardinality \( r_2 \), such that \( A \to B \). Let us consider such a \( B \). From the fact that the system is 2-determined for any choice of no empty \( A' \subset A \) and \( B' \subset B \) one obtains
\[ \frac{1}{|A'|} \sum_{i \in A'} p_i(T_{A' \cup B'}) < \frac{1}{|B'|} \sum_{i \in B'} p_i(T_{A' \cup B'}). \]

By (2), the expected payoff of the second player is
\[ |A'| \sum_{i \in B'} p_i(T_{A' \cup B'}) - |B'| \sum_{i \in A'} p_i(T_{A' \cup B'}) > 0, \]
for all \( A', B' \) with \( A' \subset A \), \( B' \subset B \). We are not investigating the best strategy, in any case this suboptimal solution shows that the expected payoff of the second player is positive. \■
Now we collect all the previous results in order to construct a favourable game. For given \( r_1, r_2 \) if \( n < S(r_1, r_2) \), by Corollary 1 all the graphs cannot be \((r_1, r_2)\)-directional; then by Theorem 1 the considered game is fair or favourable to the first player.

If \( n \geq S(r_1, r_2) \), by Theorem 2 one construct an oriented graph \( \vec{G} = ([n], \vec{E}) \) that is \((r_1, r_2)\)-directional. Then the patterns \( R_1, \ldots, R_n \) in \( P_{n+1,n+1} \) generated by \( \vec{G} \) are considered. Then, by Theorem 1 we know that \( G(\{T_{R_1}, \ldots, T_{R_n}\}) = \vec{G} \). whenever the parameter \( N \) of the uniform chain \( X^{(N,n+1)} \) is larger than or equal to \( n+1 \). By Theorem 3 \( N \) is taken large enough to guarantee that the identically distributed hitting times \( T_{R_1}, \ldots, T_{R_n} \) are 2-determined.

Finally, by the item ii of Theorem 4 the graph \( G_{r_1,r_2}(X^{(N,n+1)}, \{T_{R_1}, \ldots, T_{R_n}\}) \) is favourable to the second player. We have proved that

**Corollary 2.** For \( r_1, r_2 \in \mathbb{N} \), if \( n \geq S(r_1, r_2) \) then there exist \( X \in M_1 \) and a family of identically distributed hitting times \( T_n \) of \( X \) such that the game \( G_{r_1,r_2}(X, T_n) \) is favourable.

We have introduced the ranking graph in connection with uniform chains in \( M_1 \) and identically distributed hitting times. We have shown that, given an oriented graph \( \vec{G} \), one can construct the ranking oriented graph equal to \( \vec{G} \). Our result is optimal in the sense that it is impossible to generate these graphs with uniform chains in \( M_0 \) (a sequence of independent uniform random variables) and identically distributed hitting times.

In the second part of the paper we have applied this result in order to analyze a Penney type game. In particular, we characterize the existence of favourable games in this class. The aspect that we consider most interesting is that many things can be said only from the knowledge of the discrete parameters of the game. In this way we can do a qualitative analysis that does not involve the difficulties of Nash’s equilibria.

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