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| **Author(s)** | Lovejoy, Jeremy; Osburn, Robert |
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QUADRATIC FORMS AND FOUR PARTITION FUNCTIONS MODULO 3

JEREMY LOVEJOY AND ROBERT OSBURN

Abstract. Recently, Andrews, Hirschhorn and Sellers have proven congruences modulo 3 for four types of partitions using elementary series manipulations. In this paper, we generalize their congruences using arithmetic properties of certain quadratic forms.

1. Introduction

A partition of a non-negative integer $n$ is a non-increasing sequence whose sum is $n$. An overpartition of $n$ is a partition of $n$ where we may overline the first occurrence of a part. Let $p(n)$ denote the number of overpartitions of $n$, $p_o(n)$ the number of overpartitions of $n$ into odd parts, $ped(n)$ the number of partitions of $n$ without repeated even parts and $pod(n)$ the number of partitions of $n$ without repeated odd parts. The generating functions for these partitions are

$$\sum_{n \geq 0} p(n)q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}},$$

$$\sum_{n \geq 0} p_o(n)q^n = \frac{(-q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

$$\sum_{n \geq 0} ped(n)q^n = \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}},$$

and

$$\sum_{n \geq 0} pod(n)q^n = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}}.$$

where as usual

$$(a; q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

The infinite products in (1.1)–(1.4) are essentially the four different ways one can specialize the product $(-aq; q)_{\infty}/(bq; q)_{\infty}$ to obtain a modular form whose level is relatively prime to 3.

A series of four recent papers examined congruence properties for these partition functions modulo 3 [1, 5, 6, 7]. Among the main theorems in these papers are the following congruences (see Theorem 1.3 in [6], Corollary 3.3 and Theorem 3.5 in [1], Theorem 1.1 in [5] and Theorem 3.2 in [7], respectively). For all $n \geq 0$ and $\alpha \geq 0$ we have

$$p_o(3^{2\alpha}(An + B)) \equiv 0 \pmod{3},$$

where $An + B = 9n + 6$ or $27n + 9$.

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\[ \text{ped}\left(3^{2\alpha+3}n + \frac{17 \cdot 3^{2\alpha+2} - 1}{8}\right) \equiv \text{ped}\left(3^{2\alpha+2}n + \frac{19 \cdot 3^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod{3}, \quad (1.6) \]

\[ \overline{p}(3^{2\alpha}(27n + 18)) \equiv 0 \pmod{3} \quad (1.7) \]

and

\[ \text{pod}\left(3^{2\alpha+3} + \frac{23 \cdot 3^{2\alpha+2} + 1}{8}\right) \equiv 0 \pmod{3}. \quad (1.8) \]

We note that congruences modulo 3 for \(p(n), \overline{p}_o(n)\) and \(\text{ped}(n)\) are typically valid modulo 6 or 12. The powers of 2 enter trivially (or nearly so), however, so we do not mention them here.

The congruences in (1.5)–(1.8) are proven in [1, 5, 6, 7] using elementary series manipulations.

If we allow ourselves some elementary number theory, we find that much more is true.

With our first result we exhibit formulas for \(\overline{p}_o(3n)\) and \(\text{ped}(3n + 1)\) modulo 3 for all \(n \geq 0\). These formulas depend on the factorization of \(n\), which we write as

\[ n = 2^{a_3}b \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j}, \quad (1.9) \]

where \(p_i \equiv 1, 5, 7\) or 11 (mod 24) and \(q_j \equiv 13, 17, 19\) or 23 (mod 24). Further, let \(t\) denote the number of prime factors of \(n\) (counting multiplicity) that are congruent to 5 or 11 (mod 24). Let \(R(n, Q)\) denote the number of representations of \(n\) by the quadratic form \(Q\).

**Theorem 1.1.** For all \(n \geq 0\) we have

\[ \overline{p}_o(3n) \equiv f(n) R(n, x^2 + 6y^2) \pmod{3} \]

and

\[ \text{ped}(3n + 1) \equiv (-1)^{n+1} R(8n + 3, 2x^2 + 3y^2) \pmod{3}, \]

where \(f(n)\) is defined by

\[ f(n) = \begin{cases} -1, & n \equiv 1, 6, 9, 10 \pmod{12}, \\ 1, & \text{otherwise}. \end{cases} \]

Moreover, we have

\[ \overline{p}_o(3n) \equiv f(n)(1 + (-1)^{a+b+t}) \prod_{i=1}^{r}(1 + v_i) \prod_{j=1}^{s}\left(\frac{1 + (-1)^{w_j}}{2}\right) \pmod{3} \quad (1.10) \]

and

\[ (-1)^n \text{ped}(3n + 1) \equiv \overline{p}_o(48n + 18) \pmod{3}. \quad (1.11) \]

There are many ways to deduce congruences from Theorem 1.1. For example, calculating the possible residues of \(x^2 + 6y^2\) modulo 9 we see that

\[ R(3n + 2, x^2 + 6y^2) = R(9n + 3, x^2 + 6y^2) = 0, \]

and then (1.10) implies that \(\overline{p}_o(27n) \equiv \overline{p}_o(3n) \pmod{3}\). This gives (1.5). The congruences in (1.6) follow from those in (1.5) after replacing \(48n + 18\) by \(3^{2\alpha}(48(3n+2) + 18)\) and \(3^{2\alpha}(48(9n + 6) + 18)\) in (1.11). We record two more corollaries, which also follow readily from Theorem 1.1.
Corollary 1.2. For all \( n \geq 0 \) and \( \alpha \geq 0 \) we have
\[
\overline{p}_o(2^{2\alpha}(An + B)) \equiv 0 \pmod{3},
\]
where \( An + B = 24n + 9 \) or \( 24n + 15 \).

Corollary 1.3. If \( \ell \equiv 1, 5, 7 \) or \( 11 \pmod{24} \) is prime, then for all \( n \) with \( \ell \nmid n \) we have
\[
\overline{p}_o(3\ell^2 n) \equiv 0 \pmod{3}.
\] (1.12)

For the functions \( \overline{p}(3n) \) and \( \text{pod}(3n + 2) \) we have relations not to binary quadratic forms but to \( r_5(n) \), the number of representations of \( n \) as the sum of five squares. Our second result is the following.

Theorem 1.4. For all \( n \geq 0 \) we have
\[
\overline{p}(3n) \equiv (-1)^n r_5(n) \pmod{3}
\]
and
\[
\text{pod}(3n + 2) \equiv (-1)^n r_5(8n + 5) \pmod{3}.
\] Moreover, for all odd primes \( \ell \) and \( n \geq 0 \), we have
\[
\overline{p}(3\ell^2 n) \equiv \left( \ell - \ell \left( \frac{n}{\ell} \right) + 1 \right) \overline{p}(3n) - \ell \overline{p}(3n/\ell^2) \pmod{3} \quad (1.13)
\] and
\[
(-1)^{n+1} \text{pod}(3n + 2) \equiv \overline{p}(24n + 15) \pmod{3}, \quad (1.14)
\]
where \( \left( \frac{\bullet}{\ell} \right) \) denotes the Legendre symbol.

Here we have taken \( \overline{p}(3n/\ell^2) \) to be 0 unless \( \ell^2 \mid 3n \). Again there are many ways to deduce congruences. For example, (1.7) follows readily upon combining (1.13) in the case \( \ell = 3 \) with the fact that
\[
r_5(9n + 6) \equiv 0 \pmod{3},
\]
which is a consequence of the fact that \( R(9n + 6, x^2 + y^2 + 3z^2) = 0 \). One can check that (1.8) follows similarly. For another example, we may apply (1.13) with \( n \) replaced by \( n\ell \) for \( \ell \equiv 2 \pmod{3} \) to obtain

Corollary 1.5. If \( \ell \equiv 2 \pmod{3} \) is prime and \( \ell \nmid n \), then
\[
\overline{p}(3\ell^3 n) \equiv 0 \pmod{3}.
\]

2. Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.1. On page 364 of [6] we find the identity
\[
\sum_{n \geq 0} \overline{p}_o(3n)q^n = \frac{D(q^3)D(q^6)}{D(q)^2},
\]
where
\[
D(q) := \sum_{n \in \mathbb{Z}} (-1)^n q^n.
\]
Reducing modulo 3, this implies that
\[
\sum_{n \geq 0} p_5(3n)q^n \equiv \sum_{x,y \in \mathbb{Z}} (-1)^{x+y}q^{x^2+6y^2} \pmod{3}
\]
\[
\equiv \sum_{n \geq 0} f(n)R(n,x^2+6y^2)q^n \pmod{3}.
\]
Now it is known (see Corollary 4.2 of [3], for example) that if \(n\) has the factorization in (1.9), then
\[
R(n,x^2+6y^2) = (1 + (-1)^{a+b+t}) \prod_{i=1}^{r}(1 + v_i) \prod_{j=1}^{s} \left( \frac{1 + (-1)^{w_j}}{2} \right).
\]
(2.1)
This gives (1.10). Next, from [1] we find the identity
\[
\sum_{n \geq 0} ped(3n+1)q^n = \frac{D(q^3)\psi(-q^3)}{D(q^2)},
\]
where
\[
\psi(q) := \sum_{n \geq 0} q^{n(n+1)/2}.
\]
Reducing modulo 3 and replacing \(q\) by \(-q\) yields
\[
\sum_{n \geq 0} (-1)^{n+1} ped(3n+1)q^{3n+3} \equiv \sum_{n \geq 0} R(3n+3,2x^2+3y^2)q^{3n+3} \pmod{3}.
\]
It is known (see Corollary 4.3 of [3], for example) that if \(n\) has the factorization given in (1.9), then
\[
R(n,2x^2+3y^2) = (1 - (-1)^{a+b+t}) \prod_{i=1}^{r}(1 + v_i) \prod_{j=1}^{s} \left( \frac{1 + (-1)^{w_j}}{2} \right).
\]
Comparing with (2.1) finishes the proof of (1.11).

**Proof of Theorem 1.4.** On page 3 of [5] we find the identity
\[
\sum_{n \geq 0} \tilde{p}(3n)q^n \equiv \frac{D(q^3)^2}{D(q)} \pmod{3}.
\]
Reducing modulo 3 and replacing \(q\) by \(-q\) yields
\[
\sum_{n \geq 0} (-1)^n \tilde{p}(3n)q^n = \sum_{n \geq 0} r_5(n)q^n \pmod{3}.
\]
It is known (see Lemma 1 in [4], for example) that for any odd prime \(\ell\) we have
\[
r_5(\ell^2n) = \left( \ell^3 - \ell \left( \frac{n}{\ell} \right) + 1 \right) r_5(n) - \ell^3 r_5(n/\ell^2).
\]
Here \( r_5(n/\ell^2) = 0 \) unless \( \ell^2 \mid n \). Replacing \( r_5(n) \) by \( (-1)^n p(3n) \) throughout gives \((1.13)\). Now equation (1) of [7] reads

\[
\sum_{n \geq 0} (-1)^n pod(3n + 2)q^n = \frac{\psi(q^3)^3}{\psi(q)^4}.
\]

Reducing modulo 3 we have

\[
\sum_{n \geq 0} (-1)^n pod(3n + 2)q^n \equiv \psi(q)^5 \pmod{3}
\]

\[
\equiv \sum_{n \geq 0} r_5(8n + 5)q^n \pmod{3}
\]

\[
\equiv -\sum_{n \geq 0} p(24n + 15)q^n \pmod{3},
\]

where the second congruence follows from Theorem 1.1 in [2]. This implies \((1.14)\) and thus the proof of Theorem 1.4 is complete.

\[\square\]

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CNRS, LIAFA, Université Denis Diderot - Paris 7, Case 7014, 75205 Paris Cedex 13, FRANCE

School of Mathematical Sciences, University College Dublin, Belﬁeld, Dublin 4, Ireland

E-mail address: lovejoy@liafa.jussieu.fr
E-mail address: robert.osburn@ucd.ie