The Casimir effect with quantized charged scalar matter in background magnetic field

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Abstract

We study the influence of a background uniform magnetic field and boundary conditions on the vacuum of a quantized charged massive scalar matter field confined between two parallel plates; the magnetic field is directed orthogonally to the plates. The admissible set of boundary conditions at the plates is determined by the requirement that the operator of one-particle energy squared be self-adjoint and positive definite. We show that, in the case of a weak magnetic field and a small separation of the plates, the Casimir force is either attractive or repulsive, depending on the choice of a boundary condition. In the case of a strong magnetic field and a large separation of the plates, the Casimir force is repulsive, being independent of the choice of a boundary condition, as well as of the distance between the plates.

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1 Introduction

The Casimir effect \cite{1, 2} is a macroscopic effect of quantum field theory, which is caused by the polarization of the vacuum owing to the presence of material boundaries of a quantization volume, see review in Ref.\cite{3}. The effect has been confirmed experimentally with a sufficient precision, see, e.g. Refs.\cite{4, 5}, as well as other publications cited in Ref.\cite{3}, and this opens a way for various applications in modern nanotechnology.

Generically, the interest was focused on the Casimir effect with quantized electromagnetic field, whereas the Casimir effect with other (nonelectromagnetic) quantized fields was mostly regarded as an academic exercise that could hardly be validated in laboratory. However, the nonelectromagnetic fields can be charged, and this opens a new prospect allowing one to consider the Casimir effect as that caused by the polarization of the vacuum of quantized charged matter owing to the presence of both the material boundaries and a background electromagnetic field inside the quantization volume.

In this respect it should be recalled that the effect of the background uniform electromagnetic field alone on the vacuum of quantized charged matter was studied long ago, see Refs.\cite{6, 7, 8, 9, 10} and Refs.\cite{11, 12} for reviews. The case of a background field filling the whole (infinite) space is hard to be regarded as realistic, whereas the case of a background field confined to the bounded quantization volume for charged matter looks much more reasonable and even can be regarded as realizable in laboratory. Moreover, there is no way to detect the energy density which is induced in the vacuum in the first case, whereas in the second case it results in the pressure from the vacuum on the boundary, and the latter is in principle detectable. One may suggest intuitively that the pressure, at least in certain circumstances, is positive, i.e. directed from the inside to the outside of the quantization volume. A natural question is, whether the pressure depends on a boundary condition imposed on the quantized charged matter field at the boundary?

Thus, an issue of a choice of boundary conditions acquires a primary importance, requiring a thorough examination. To deal with this issue comprehensively, one has to care for the self-adjointness of a differential operator corresponding to the physical observable of a quantum system. A quest for the self-adjointness is stipulated by the mere fact that a multiple action is well-defined for the self-adjoint operator only, allowing for the construction of functions of the operator, such as evolution, zeta-function and heat kernel operators, etc. A relevant basic operator in the present context is that of
one-particle energy (or energy squared in the case of quantized relativistic bosonic fields). The requirement of its self-adjointness renders the most general set of boundary conditions, which may be further restricted by some additional physical constraints.

In the present paper we consider the Casimir effect with a quantized charged massive scalar matter field in the background of an external uniform magnetic field; both the quantized and external fields are confined between two parallel plates, and the external field is orthogonal to the plates. It should be noted that exactly this problem has been studied more than a decade ago in Refs.\[13, 14, 15\], however, results obtained there are incomplete, and, therefore, somewhat misleading. In particular, according to them, there is no room for the validation of the aforementioned intuitive suggestion: the pressure in all circumstances is negative, i.e. the plates are attracted. On the contrary, we show that, even in the case of a weak magnetic field and a small distance between the plates, the pressure is either negative or positive, depending on the choice of a boundary condition. A much more essential distinction from Refs.\[13, 14, 15\] is that, in the case of a strong magnetic field and a large distance between the plates, the pressure is positive, being independent of the choice of a boundary condition and even of the distance between the plates.

In the next section we consider in general the problem of the self-adjointness for the operator of one-particle energy squared, which is the same as that for the covariant Laplace operator. In Section 3 we discuss the vacuum energy density which is induced by an external uniform magnetic field and compare the appropriate expressions in the cases of the unbounded quantization volume and the quantization volume bounded by two parallel plates. A choice of boundary conditions for the quantized scalar field is considered in Section 4. The expressions for the Casimir energy and force are obtained in Section 5. The conclusions are drawn and discussed in Section 6. We relegate some details of performing the infinite summation over discrete eigenvalues to the Appendix.
2 Self-adjointness of the Laplace operator

Defining a scalar product as
\[(\tilde{\chi}, \chi) = \int_D d^3r \tilde{\chi}^* \chi,\]
we get, using integration by parts,
\[(\tilde{\chi}, \nabla^2 \chi) = (\nabla^2 \tilde{\chi}, \chi) + \int_{\partial D} \, d\sigma [\tilde{\chi}^* (\nabla \chi) - (\nabla \tilde{\chi})^* \chi], \quad (1)\]
where \(\partial D\) is a two-dimensional surface bounding the three-dimensional spatial region \(D\), \(\nabla\) is the covariant derivative involving both affine and bundle connections. The covariant Laplace operator, \(\nabla^2\), is Hermitian (or symmetric in mathematical parlance),
\[(\tilde{\chi}, \nabla^2 \chi) = (\nabla^2 \tilde{\chi}, \chi), \quad (2)\]
if
\[\int_{\partial D} \, d\sigma [\tilde{\chi}^* (\nabla \chi) - (\nabla \tilde{\chi})^* \chi] = 0. \quad (3)\]
The latter condition can be satisfied in various ways by imposing different boundary conditions for \(\chi\) and \(\tilde{\chi}\). However, among the whole variety, there may exist a possibility that a boundary condition for \(\tilde{\chi}\) is the same as that for \(\chi\); then operator \(\nabla^2\) is self-adjoint. The spectral theorem is valid for self-adjoint operators only, and this allows one to construct appropriate unitary operator exponentials playing the key role in defining the dynamical evolution of quantum systems, see, e.g., Ref.\[16\]. In the case of a disconnected noncompact boundary consisting of two components, \(\partial D^{(+)}\) and \(\partial D^{(-)}\), condition (3) takes the form,
\[\int_{\partial D^{(+)}} \, d\sigma [\tilde{\chi}^* (\nabla \chi) - (\nabla \tilde{\chi})^* \chi] - \int_{\partial D^{(-)}} \, d\sigma [\tilde{\chi}^* (\nabla \chi) - (\nabla \tilde{\chi})^* \chi] = 0, \quad (4)\]
where normals to surfaces \(\partial D^{(+)\)} and \(\partial D^{(-)}\) are chosen to point in the same direction, i.e. outwards for \(\partial D^{(+)\)} and inwards for \(\partial D^{(-)}\). One can introduce coordinates \(r = (x, y, z)\) in such a way, that \(y\) and \(z\) are tangential to the
boundary, while \( x \) is normal to it, then the position of \( \partial D(\pm) \) is identified with, say, \( x = \pm a \). In this way we obtain

\[
(\tilde{\chi}, \nabla^2 \chi) - (\nabla^2 \tilde{\chi}, \chi) = \\
\int \, dy \, dz \left( \tilde{\chi}^*|a \, \nabla_x \chi|_a - \nabla_x \tilde{\chi}^*|a \, \chi - \tilde{\chi}^*|_a \nabla_x \chi|_-a + \nabla_x \tilde{\chi}^*|_a \chi - \tilde{\chi}^*|_-a \right) = \\
= \frac{i}{2a} \int \, dy \, dz (\tilde{\chi}^*|_a \chi - \tilde{\chi}^*|_a \chi + \tilde{\chi}^*|_-a \chi - \tilde{\chi}^*|_-a),
\]

where

\[
\chi_\pm = a \nabla_x \chi \pm i \chi, \quad \tilde{\chi}_\pm = a \nabla_x \tilde{\chi} \pm i \tilde{\chi}.
\]

The integrand in (5) vanishes when the following condition is satisfied:

\[
\begin{align*}
(\chi|_-a, \chi|_a) &= U(\chi|_-a, \chi|_a), \\
(\tilde{\chi}^*|_a \chi - \tilde{\chi}^*|_-a \chi + \tilde{\chi}^*|_-a \chi - \tilde{\chi}^*|_a, \chi) &= U(\tilde{\chi}^*|_a \chi - \tilde{\chi}^*|_-a \chi + \tilde{\chi}^*|_-a \chi - \tilde{\chi}^*|_a)
\end{align*}
\]

where

\[
U = e^{-i\alpha} \left( \begin{array}{cc} u & v \\ -v^* & u^* \end{array} \right), \quad 0 < \alpha < \pi, \quad |u|^2 + |v|^2 = 1.
\]

Thus, the explicit form of the boundary condition ensuring the self-adjointness of the Laplace operator is

\[
\{[1 - e^{-i\alpha}(u^* \pm v)]a \nabla_x + i[1 + e^{-i\alpha}(u^* \pm v)]\} \chi|_a = \\
= \{[1 - e^{-i\alpha}(u \mp v^*)]a \nabla_x \pm i[1 + e^{-i\alpha}(u \mp v^*)]\} \chi|_-a
\]

(8)

(\text{the same condition is for } \tilde{\chi}).

Similar considerations in a somewhat simplified form apply also to the operator of the covariant momentum component in the normal to the boundary direction, \(-i\nabla_x\):

\[
(\tilde{\chi}, -i\nabla_x \chi) = (-i\nabla_x \tilde{\chi}, \chi) - i \int_{\partial D} \, d\sigma^x \tilde{\chi}^* \chi.
\]

The explicit form of the boundary condition ensuring the self-adjointness of \(-i\nabla_x\) is

\[
\chi|_a = \bar{u} \chi|_-a, \quad |\bar{u}|^2 = 1
\]

(10)
(the same condition is for $\tilde{\chi}$).

In the one-dimensional case (when dimensions along the $y$ and $z$ axes are ignored), the deficiency index is $\{2, 2\}$ in the case of $\nabla^2$, see Ref. [17], and $\{1, 1\}$ in the case of $-i\nabla_x$, see, e.g., Ref. [16]. The four real parameters from (7) and the one real parameter from (10) are the self-adjoint extension parameters. When the account is taken for dimensions along the boundary, these parameters become arbitrary functions of $y$ and $z$. However, in such a general case, conditions (8) and (10) cannot be regarded as the ones determining the spectrum of the momentum in the $x$-direction. Therefore, we assume that the self-adjoint extension parameters are independent of $y$ and $z$.

Moreover, there are further restrictions which are due to physical reasons. For instance, one may choose a one-parameter family of boundary conditions, ensuring the self-adjointness of the momentum operator in the $x$-direction, see (10), then the Laplace operator is surely self-adjoint. However, this choice is too restrictive, with a lack of physical motivation, and we shall follow another way. A solution to the stationary Klein-Fock-Gordon equation, $\psi(r)$, will be chosen in such a manner that its phase is independent of the coordinate which is normal to the boundary. Then $U^*U = I$ which results in

$$v^* = -v,$$

and the number of the self-adjoint extension parameters is diminished to 3. The density of the conserved current in the $x$-direction,

$$j^x(r) = -i[\psi^*(\nabla_x \psi) - (\nabla_x \psi)^* \psi],$$

in this case vanishes at the boundary:

$$j^x|_a = j^x|_{-a} = 0,$$

and, thus, the matter is confined within the boundaries.

A much more stringent restriction is condition $U^2 = I$ which results in

$$(\psi, -\nabla^2 \psi) = (-i\nabla \psi, -i\nabla \psi),$$

meaning that the spectrum of the Laplace operator is non-positive-definite. Condition (14) ensures that the values of the one-particle energy squared exceed the value of the mass squared.
3 Induced vacuum energy density in the magnetic field background

The operator of a charged massive scalar field which is quantized in a static background is presented in the form

\[ \hat{\Psi}(t, r) = \sum_\lambda \frac{1}{\sqrt{2\omega_\lambda}} [e^{-i\omega_\lambda t}\psi_\lambda(r)\hat{a}_\lambda + e^{i\omega_\lambda t}\psi_\lambda^*(r)\hat{b}_\lambda^\dagger], \quad (15) \]

where \( \hat{a}_\lambda^\dagger \) and \( \hat{a}_\lambda \) (\( \hat{b}_\lambda^\dagger \) and \( \hat{b}_\lambda \)) are the scalar particle (antiparticle) creation and destruction operators, satisfying commutation relations

\[ [\hat{a}_\lambda, \hat{a}_\lambda^\dagger]_- = [\hat{b}_\lambda, \hat{b}_\lambda^\dagger]_- = \langle \lambda | \lambda' \rangle; \]

\( \lambda \) is the set of parameters (quantum numbers) specifying the state; \( \omega_\lambda > 0 \) is the energy of the state; symbol \( \sum_\lambda \) denotes summation over discrete and integration (with a certain measure) over continuous values of \( \lambda \); wave functions \( \psi_\lambda(r) \) form a complete set of solutions to the stationary Klein-Fock-Gordon equation

\[ [-\nabla^2 + m^2 + \xi R(r)]\psi_\lambda(r) = \omega_\lambda^2 \psi_\lambda(r), \quad (16) \]

where \( R(r) \) is the scalar curvature of space-time; \( m \) is the particle mass. The temporal component of the energy-momentum tensor is given by expression

\[ \hat{T}^{00} = [\partial_0 \hat{\Psi}^\dagger, \partial_0 \hat{\Psi}]_+ - \left[ \frac{1}{4}(\partial_0^2 - \nabla^2) + \xi(\nabla^2 + R^{00}) \right] [\hat{\Psi}^\dagger, \hat{\Psi}]_+, \quad (17) \]

where \( R^{00}(r) \) is the temporal component of the Ricci tensor. Thence, the formal expression for the vacuum energy density is

\[ \varepsilon = \langle \text{vac}|\hat{T}^{00}|\text{vac} \rangle = \sum_\lambda \omega_\lambda \psi_\lambda^*(r)\psi_\lambda(r) + \]

\[ + \left[ \left( \frac{1}{4} - \xi \right) \nabla^2 - \xi R^{00}(r) \right] \sum_\lambda \omega_\lambda^{-1} \psi_\lambda^*(r)\psi_\lambda(r). \quad (18) \]

It should be noted that, in general, the energy-momentum tensor and its vacuum expectation value remain dependent on the coupling \( (\xi, \text{see (16)}) \) of
the scalar field to the scalar curvature of space-time even in the case of flat space-time, i.e. when the Ricci tensor and the Ricci scalar vanish. This is an evidence for some arbitrariness in the definition of the energy-momentum tensor for the scalar field in flat space-time. One can add term $\xi \nabla_\rho \theta^\mu\nu$, where $\theta^\mu\nu = -\theta^\nu\mu$, to the canonically-defined energy-momentum tensor, $T^\mu\nu_{\text{can}}$. The whole construction at $\xi = 1/6$ is known as the improved energy-momentum tensor which is adequate for the implementation of conformal invariance in the $m = 0$ case \[18, 19\]. Physical observables are certainly independent of this arbitrariness, i.e. of $\xi$.

Let us consider the quantization of the charged massive scalar field in the background of a uniform magnetic field ($B$) in flat space-time ($R = 0, R_{00} = 0$), then the covariant derivative is defined as

$$\nabla \Psi = (\partial - ieA)\Psi, \quad \nabla \Psi^\dagger = (\partial + ieA)\Psi^\dagger, \quad B = \partial \times A,$$

(19)
e is the particle charge. Directing the magnetic field along the $x$-axis, $B = (B, 0, 0)$, we choose the gauge with $A^x = A^y = 0, A^z = yB$. Then the solution to the Klein-Fock-Gordon equation takes form

$$\psi_{k_nq}(r) = X_k(x)Y_n(q)Z_q(z), \quad -\infty < k < \infty, \quad -\infty < q < \infty, \quad n = 0, 1, 2, ...,$$

(20)

where

$$X_k(x) = (2\pi)^{-1/2}e^{ikx}, \quad Z_q(z) = (2\pi)^{-1/2}e^{iqz},$$

(21)

and

$$Y_n(q) = \sqrt{\frac{|eB|^{1/2}}{2^n n! \pi^{1/2}}} \exp \left[ -\frac{|eB|}{2} \left( y + \frac{q}{eB} \right) \right] H_n \left[ \sqrt{|eB|} \left( y + \frac{q}{eB} \right) \right],$$

(22)

$H_n(w)$ is the Hermite polynomial. Wave functions $\psi_{k_nq}(r)$ satisfy the conditions of orthonormality

$$\int d^3 r \psi_{k_nq}^*(r)\psi_{k'n'q'}(r) = \delta(k - k')\delta_{nn'}\delta(q - q'),$$

(23)

and completeness

$$\int \frac{dk}{-\infty} \int \frac{dq}{-\infty} \sum_{n=0}^{\infty} \psi_{k_nq}^*(r)\psi_{k_nq}(r') = \delta^3(r - r').$$

(24)
The one-particle energy spectrum (Landau levels) is the following
\[
\omega_{kn} = \sqrt{|eB|(2n + 1) + k^2 + m^2}.
\] (25)

With the use of relations
\[
\int_{-\infty}^{\infty} dq Y_{nq}^2(y) = |eB|, \quad \int_{-\infty}^{\infty} dq \partial_y^2 Y_{nq}^2(y) = 0,
\] (26)
the formal expression for the vacuum energy density in the present case is readily obtained:
\[
\varepsilon^\infty = \frac{|eB|}{(2\pi)^2} \int_{-\infty}^{\infty} dk \sum_{n=0}^{\infty} \omega_{kn},
\] (27)
where the superscript indicates that the external magnetic field fills the whole (infinite) space; note that dependence on \(\xi\) has disappeared (just owing to the second relation in (26)). The integral and the sum in (27) are divergent and require regularization and renormalization. This problem has been solved long ago by Weisskopf [9] (see also Ref. [10]), and we just list here his result
\[
\varepsilon_{\text{ren}}^\infty = -\frac{1}{(4\pi)^2} \int_{0}^{\infty} \frac{d\tau}{\tau} e^{-\tau} \left[ \frac{eBm^2}{\tau \sinh (\frac{eB\tau}{m^2})} - \frac{m^4}{\tau^2} + \frac{1}{6} e^2 B^2 \right];
\] (28)
note that the renormalization procedure includes subtraction at \(B = 0\) and renormalization of the charge.

Let us turn now to the quantization of the charged massive scalar field in the background of a uniform magnetic field in spatial region \(D\) bounded by two parallel surfaces \(\partial D^+\) and \(\partial D^-\); the position of \(\partial D^{(\pm)}\) is identified with \(x = \pm a\), and the magnetic field is orthogonal to the boundary. Then the solution to the Klein-Fock-Gordon equation takes form
\[
\psi_{lnq}(r) = X_l(x)Y_{nq}(y)Z_q(z), l = 0, \pm 1, \pm 2, \ldots, -\infty < q < \infty, n = 0, 1, 2, \ldots,
\] (29)
where \(Y_{nq}(y)\) and \(Z_q(z)\) are the same as in the previous case, while \(X_l(x)\) is the real solution to equation
\[
(\partial_x^2 + k_l^2)X_l(x) = 0,
\] (30)
and the discrete spectrum of $k_l$ is determined by the boundary condition for $X_l(x)$, see (8) with (11):

$$\{[1 - e^{-\imath(u^* \pm v)}]a \partial_x + i[1 + e^{-\imath}(u^* \pm v)]\} X_l|_a =$$

$$= \{ \mp[1 - e^{-\imath}(u \pm v)]a \partial_x \mp i[1 + e^{-\imath}(u \pm v)]\} X_l|_{-a} \quad (v^* = -v). \quad (31)$$

Wave functions $\psi_{lnq}(\mathbf{r})$ satisfy the conditions of orthonormality

$$\int_D d^3 \mathbf{r} \psi_{lnq}^*(\mathbf{r}) \psi_{l'n'q'}(\mathbf{r}) = \delta_{ll'} \delta_{nn'} \delta(q - q'), \quad (32)$$

and completeness

$$\int dq \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \psi_{lnq}^*(\mathbf{r}) \psi_{lnq}(\mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}'). \quad (33)$$

The formal expression for the vacuum energy density appears to be $\xi$-dependent, cf. (27),

$$\varepsilon = \frac{|eB|}{2\pi} \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \left[ \omega_{ln} + \left( \frac{1}{4} - \xi \right) \omega_{ln}^{-1} \partial_x^2 \right] X_l^2(x), \quad (34)$$

where

$$\omega_{ln} = \sqrt{|eB|(2n + 1) + k_l^2 + m^2}. \quad (35)$$

4 Choice of boundary conditions for the Casimir effect

The Casimir energy is defined as the induced vacuum energy per unit area of the boundary surface:

$$\frac{E}{S} = \frac{\int_a^q d\mathbf{x} \varepsilon}{-a}. \quad (36)$$

In view of the normalization condition, we obtain the formal expression for the Casimir energy

$$\frac{E}{S} = \frac{|eB|}{2\pi} \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \left[ \omega_{ln} + \left( \frac{1}{4} - \xi \right) \omega_{ln}^{-1} I_l \right], \quad (37)$$
where

\[ I_l = \int_{-a}^{a} dx \, \partial_x^2 X_l^2(x). \]  \hspace{1cm} (38)

By rewriting the integrand in (38) as \( 2X_l(\partial_x^2 X_l) + 2(\partial_x X_l)^2 \), one immediately recognizes that condition (14) ensures the vanishing of \( I_l \). Thus, this condition guarantees that the Casimir energy in flat space is independent of the \( \xi \)-parameter, as it should be expected for a physically meaningful quantity.

Without loss of generality, one may take \( X_l(x) \) in the form

\[ X_l(x) = \frac{1}{\sqrt{a}} \sin(k_l x + \delta_l), \]  \hspace{1cm} (39)

with phase \( \delta_l \), as well as momentum \( k_l \), being determined from boundary condition (31). Substituting (39) into (38), one gets

\[ I_l = k_l a \cos(2\delta_l) \sin(2k_l a). \]  \hspace{1cm} (40)

The condition of the vanishing of \( I_l \) (40) restricts the number and the range of self-adjoint extension parameters.

Let us consider the following cases:

I. \( v = 0 \), then

\[ U = \exp(-i\alpha I + i\tilde{\alpha} \sigma_3), \quad 0 < \alpha < \pi, \quad 0 < \tilde{\alpha} < 2\pi, \]  \hspace{1cm} (41)

where parametrization \( \text{Re} u = \cos \tilde{\alpha}, \text{Im} u = \sin \tilde{\alpha} \) is used;

II. \( \text{Re} u = 0 \) and \( \alpha = \frac{\pi}{2} \), then

\[ U = \sigma_1 \cos \beta + \sigma_3 \sin \beta, \quad 0 < \beta < 2\pi, \]  \hspace{1cm} (42)

where parametrization \( \text{Im} u = \sin \beta, \text{Im} v = \cos \beta \) is used;

III. \( \text{Im} u = 0 \), then

\[ U = \exp(-i\alpha I + i\tilde{\beta} \sigma_1), \quad 0 < \alpha < \pi, \quad -\pi < \tilde{\beta} < \pi, \]  \hspace{1cm} (43)

where parametrization \( \text{Re} u = \cos \tilde{\beta}, \text{Im} v = \sin \tilde{\beta} \) is used; here \( \sigma_j \) are the Pauli matrices, \( I \) is the unity matrix, the endpoints of the ranges are identified.

Boundary condition (31) takes the form

\[ k_l a \cos(\pm k_l a + \delta_l) \sin \left[ \frac{1}{2} (\alpha \pm \tilde{\alpha}) \right] \pm \sin(\pm k_l a + \delta_l) \cos \left[ \frac{1}{2} (\alpha \pm \tilde{\alpha}) \right] = 0 \]  \hspace{1cm} (44)
in case I, 
\[
\begin{align*}
\cos(k_l a + \delta_l) \cos \left( \frac{1}{2} \beta - \frac{\pi}{4} \right) + \cos(-k_l a + \delta_l) \sin \left( \frac{1}{2} \beta - \frac{\pi}{4} \right) &= 0 \\
\sin(k_l a + \delta_l) \sin \left( \frac{1}{2} \beta - \frac{\pi}{4} \right) + \sin(-k_l a + \delta_l) \cos \left( \frac{1}{2} \beta - \frac{\pi}{4} \right) &= 0
\end{align*}
\]
(45)

in case II,
\[
\begin{align*}
\begin{cases}
  k_l a \cos(k_l a) \sin \left[ \frac{1}{2} (\alpha - \tilde{\beta}) \right] + \sin(k_l a) \cos \left[ \frac{1}{2} (\alpha - \tilde{\beta}) \right] = 0 \\
  k_l a \sin(k_l a) \sin \left[ \frac{1}{2} (\alpha + \tilde{\beta}) \right] - \cos(k_l a) \cos \left[ \frac{1}{2} (\alpha + \tilde{\beta}) \right] = 0
\end{cases}
\end{align*}
\]
(46)

in case III.

In case I, condition $\alpha = \tilde{\alpha} = 0$ corresponds to the Dirichlet boundary condition ($X_l|_{-a} = X_l|_a = 0$) which yields the spectrum
\[
\delta_l = 0, \quad k_l = \frac{l \pi}{a}, \quad l = 1, 2, \ldots,
\]
(47)
\[
\delta_l = \frac{\pi}{2}, \quad k_l = \left( l + \frac{1}{2} \right) \frac{\pi}{a}, \quad l = 0, 1, 2, \ldots.
\]
(48)

In case I, condition $\alpha = \tilde{\alpha} - \pi = 0$ corresponds to the Neumann boundary condition ($\partial_x X_l|_{-a} = \partial_x X_l|_a = 0$) which yields the spectrum
\[
\delta_l = (l - 1) \frac{\pi}{2}, \quad k_l = \frac{l \pi}{a}, \quad l = 1, 2, \ldots.
\]
(49)

In case I, condition $\alpha = \tilde{\alpha} = \frac{\pi}{2}$ corresponds to the mixed Dirichlet-Neumann boundary condition ($X_l|_{-a} = \partial_x X_l|_a = 0$) which yields the spectrum
\[
\delta_l = \left( l + \frac{1}{2} \right) \frac{\pi}{2}, \quad k_l = \left( l + \frac{1}{2} \right) \frac{\pi}{2a}, \quad l = 0, 1, 2, \ldots.
\]
(50)

The mixed Neumann-Dirichlet boundary condition ($\partial_x X_l|_{-a} = X_l|_a = 0$; $\alpha = \tilde{\alpha} - \pi = \frac{\pi}{2}$) yields the same spectrum as in (50) with the opposite phase.

In case II the spectrum is determined for the whole range of $\beta$:
\[
\delta_l = \frac{\pi}{4}, \quad k_l = \frac{l \pi}{a} + \frac{\beta}{2a}, \quad l = 0, \pm 1, \pm 2, \ldots.
\]
(51)

The boundary condition in case III, see (46), is independent of phase $\delta_l$; condition $\alpha = \tilde{\beta} = \frac{\pi}{2}$ corresponds to the periodicity boundary condition ($X_l|_{-a} = X_l|_a$) yielding the spectrum
\[
k_l = \frac{l \pi}{a}, \quad l = 0, \pm 1, \pm 2, \ldots,
\]
(52)
while condition $\alpha = -\tilde{\beta} = \frac{\pi}{2}$ corresponds to the antiperiodicity boundary condition $(X_l|_{-a} = -X_l|_{a})$ yielding the spectrum

$$k_l = \left( l + \frac{1}{2} \right) \frac{\pi}{a}, \quad l = 0, \pm 1, \pm 2, \ldots . \quad (53)$$

Results (52) and (53) are also obtained in case II at $\beta = 0$ and $\beta = \pi$, respectively, without any restriction on phase $\delta_l$.

The boundary conditions yielding the spectra of $k_l$ in (47)-(53) ensure the vanishing of $I_l$ (40). This is a complete list of conditions giving $U^2 = I$.

It should be noted that the self-adjointness of the momentum in the $x$-direction, see (9) and (10), implies the vanishing of quantity

$$\tilde{I}_l = \int_{-a}^{a} dx \partial_x X_l^2(x) = \frac{1}{a} \sin(2\delta_l) \sin(2k_l a). \quad (54)$$

Thus, the operator $-i\partial_x$ is not self-adjoint under the mixed (Dirichlet-Neumann or Neumann-Dirichlet) boundary condition, when (50) holds, and under the one-parameter family of boundary conditions, when (51) holds, unless $\beta = 0, \pi$.

5 Casimir energy and force

Employing the boundary conditions specified in the previous section, we obtain the following expression for the Casimir energy

$$\frac{E}{S} = \frac{|eB|}{2\pi} \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \omega_{ln}. \quad (55)$$

The expression is ill-defined, as was already noted, since infinite sums in (55) are divergent. To tame the divergence, a factor containing the regularization parameter is inserted in (55).

Let us perform calculations for case II of the previous section, when the boundary condition is given by (45) and the spectrum of $k_l$ is given by (51). The summation over $l$ is made with the use of the following version of the Abel-Plana formula which is derived in Appendix (see also Ref.[20]):
\[\sum_{l=-\infty}^{\infty} f \left[ \left( \pi l + \frac{\beta}{2} \right)^2 \right] = \frac{1}{\pi} \int_{-\infty}^{\infty} d\mu f(\mu^2) - \frac{i}{\pi} \int_{0}^{\infty} d\nu \left\{ f[(\nu - i\nu)^2] - f[(i\nu)^2] \right\} \frac{\cos \beta - e^{-2\nu}}{\cosh 2\nu - \cos \beta}, \quad (56)\]

where \( f(w^2) \) as a function of complex variable \( w \) is decreasing sufficiently fast at large distances from the origin of the complex \( w \)-plane. The regularization in the last integral on the right-hand side of (56) can be removed; then

\[\frac{i}{\pi} \int_{0}^{\infty} d\nu \left\{ f[(\nu - i\nu)^2] - f[(i\nu)^2] \right\} = |eB| \sum_{n=0}^{\infty} \sqrt{\left( \frac{\nu}{a} \right)^2 - |eB|(2n+1) + m^2} \quad (57)\]

with the range of \( \nu \) restricted to \( \nu > a\sqrt{|eB|(2n+1) + m^2} \). Introducing variable \( k = \mu/a \) in the first integral on the right-hand side of (56), one immediately recognizes that this integral is the same as quantity \( \varepsilon_{\text{ren}}^{\infty} \) (27) multiplied by \( 2a \). Hence, the problem of regularization and removal of the divergency in expression (55) is reduced to that in the case of no boundaries, when the magnetic field fills the whole space. This is owing to the Abel-Plana formula (56) which effectively reduces the contribution of the boundaries to the term (last integral on the right-hand side) that is free of the divergency. Thus we obtain the following expression for the renormalized Casimir energy:

\[\frac{E_{\text{ren}}}{S} = 2a\varepsilon_{\text{ren}}^{\infty} - \frac{|eB|}{\pi^2 a} \sum_{n=0}^{\infty} \int_{aM_n}^{\infty} d\nu \sqrt{\nu^2 - a^2 M_n^2} \frac{\cos \beta - e^{-2\nu}}{\cosh 2\nu - \cos \beta}, \quad (58)\]

where \( \varepsilon_{\text{ren}}^{\infty} \) is given by (28) and

\[M_n = \sqrt{|eB|(2n+1) + m^2}. \quad (59)\]

In particular, in the case of the periodicity boundary condition when the spectrum is given by (52), we get \( \beta = 0 \) and

\[\frac{E_{\text{ren}}}{S} = 2a\varepsilon_{\text{ren}}^{\infty} - \frac{2|eB|}{\pi^2 a} \sum_{n=0}^{\infty} \int_{aM_n}^{\infty} d\nu \frac{\sqrt{\nu^2 - a^2 M_n^2}}{e^{2\nu} - 1}, \quad (60)\]
while, in the case of the antiperiodicity boundary condition when the spectrum is given by (53), we get $\beta = \pi$ and

$$E_{\text{ren}} = 2a\varepsilon_{\text{ren}}^\infty + \frac{2|eB|}{\pi^2a} \sum_{n=0}^{\infty} \int \frac{d\nu}{\sqrt{\nu^2 - a^2M_n^2}} \frac{\sqrt{\nu^2 - a^2M_n^2}}{e^{2\nu} + 1}. \quad (61)$$

The case of the mixed (either Dirichlet-Neumann or Neumann-Dirichlet) boundary condition when the spectrum is given by (50) is obtained from (61) by an appropriate rescaling of the integration variable

$$E_{\text{ren}} = 2a\varepsilon_{\text{ren}}^\infty + \frac{2|eB|}{\pi^2a} \sum_{n=0}^{\infty} \int \frac{d\nu}{\sqrt{\nu^2 - a^2M_n^2}} \frac{\sqrt{\nu^2 - a^2M_n^2}}{e^{4\nu} + 1}. \quad (62)$$

The case of the Dirichlet boundary condition which is given by (44) at $\alpha = \tilde{\alpha} = 0$ deserves a special attention. The modes in this case are divided into two series of opposite parity. For the modes of even parity, see (48), using

$$\sum_{l=0}^{\infty} f \left[ \left( \pi l + \frac{\pi}{2} \right)^2 \right] = \frac{1}{2} \sum_{l=-\infty}^{\infty} f \left[ \left( \pi l + \frac{\pi}{2} \right)^2 \right], \quad (63)$$

we obtain

$$E_{S}|_{\text{even}} = a\varepsilon_{\text{ren}}^\infty + \frac{|eB|}{\pi^2a} \sum_{n=0}^{\infty} \int \frac{d\nu}{\sqrt{\nu^2 - a^2M_n^2}} \frac{\sqrt{\nu^2 - a^2M_n^2}}{e^{2\nu} + 1}. \quad (64)$$

For the modes of odd parity, see (47), using

$$\sum_{l=1}^{\infty} f \left[ (\pi l)^2 \right] = \frac{1}{2} \sum_{l=-\infty}^{\infty} f \left[ (\pi l)^2 \right] - \frac{1}{2} f(0), \quad (64)$$

we obtain

$$E_{S}|_{\text{odd}} = a\varepsilon_{\text{ren}}^\infty - \frac{|eB|}{\pi^2a} \sum_{n=0}^{\infty} \int \frac{d\nu}{\sqrt{\nu^2 - a^2M_n^2}} \frac{\sqrt{\nu^2 - a^2M_n^2}}{e^{2\nu} - 1} - \frac{|eB|}{4\pi} \sum_{n=0}^{\infty} M_n. \quad (65)$$

Summing the contributions of both parities, we get

$$E_{S} = 2a\varepsilon_{\text{ren}}^\infty - \frac{2|eB|}{\pi^2a} \sum_{n=0}^{\infty} \int \frac{d\nu}{\sqrt{\nu^2 - a^2M_n^2}} \frac{\sqrt{\nu^2 - a^2M_n^2}}{e^{4\nu} - 1} - \frac{|eB|}{4\pi} \sum_{n=0}^{\infty} M_n. \quad (66)$$
Hence, by removing the divergency in the same manner as in the case of no boundaries, we arrive at the expression containing infinities,

\[
\frac{\tilde{E}_{\text{ren}}}{S} = 2a\varepsilon_{\text{ren}}^{\infty} - \frac{2|eB|}{\pi^2a} \sum_{n=0}^{\infty} \int_{aM_n}^{\infty} d\nu \frac{\nu^2 - a^2 M_n^2}{e^{4\nu} - 1} - \frac{|eB|}{4\pi} \sum_{n=0}^{\infty} M_n; \quad (65)
\]
the last sum in (65) is divergent.

The same situation is encountered in the case of the Neumann boundary condition, see (44) at \(\alpha = \tilde{\alpha} - \pi = 0\), when the spectrum is given by (49). Using (64), we arrive finally at the same expression as (65). Moreover, if a zero mode of even parity, \(X_0 = \frac{1}{\sqrt{2a}}\), is added, then the divergent sum enters with the opposite sign:

\[
\frac{\tilde{E}_{\text{ren}}}{S} = 2a\varepsilon_{\text{ren}}^{\infty} - \frac{2|eB|}{\pi^2a} \sum_{n=0}^{\infty} \int_{aM_n}^{\infty} d\nu \frac{\nu^2 - a^2 M_n^2}{e^{4\nu} - 1} + \frac{|eB|}{4\pi} \sum_{n=0}^{\infty} M_n. \quad (66)
\]

However, a physically measurable characteristics of the Casimir effect is the Casimir force which is defined as the force per unit area of the boundary, or pressure:

\[
F = -\frac{1}{2} \frac{\partial}{\partial a} \frac{E_{\text{ren}}}{S}. \quad (67)
\]

The divergent pieces of the Casimir energy in (65) and (66) do not contribute to the force, since they are independent of \(a\). Hence, we obtain the following expression for the Casimir force in the case of either Dirichlet or Neumann boundary condition:

\[
F = -\varepsilon_{\text{ren}}^{\infty} - \frac{|eB|}{\pi^2a^2} \sum_{n=0}^{\infty} \int_{aM_n}^{\infty} d\nu \frac{\nu^2}{\sqrt{\nu^2 - a^2 M_n^2} e^{4\nu} - 1}. \quad (68)
\]

In the case of the mixed boundary condition the Casimir force is

\[
F = -\varepsilon_{\text{ren}}^{\infty} + \frac{|eB|}{\pi^2a^2} \sum_{n=0}^{\infty} \int_{aM_n}^{\infty} d\nu \frac{\nu^2}{\sqrt{\nu^2 - a^2 M_n^2} e^{4\nu} + 1}. \quad (69)
\]

while in the cases of the periodicity and antiperiodicity boundary conditions it takes forms

\[
F = -\varepsilon_{\text{ren}}^{\infty} - \frac{|eB|}{\pi^2a^2} \sum_{n=0}^{\infty} \int_{aM_n}^{\infty} d\nu \frac{\nu^2}{\sqrt{\nu^2 - a^2 M_n^2} e^{2\nu} - 1} \quad (70)
\]
and

\[ F = -\varepsilon_\text{ren}^\infty + \frac{|eB|}{\pi^2 a^2} \sum_{n=0}^\infty \int_a^{aM_n} d\nu \frac{\nu^2}{\sqrt{\nu^2 - a^2 M_n^2}} \frac{1}{e^{2\nu} + 1}, \]  

(71)

respectively.

It should be noted that the integrals in (70) and (71) can be taken after expanding the last factors as \( \sum_{j=1}^{\infty} (\pm 1)^{j-1} e^{-2j\nu} \). In this way, we obtain the following expressions for the Casimir energy

\[ \frac{E_\text{ren}}{S} = 2a \varepsilon_\text{ren}^\infty \mp \frac{|eB|}{\pi} \sum_{n=0}^\infty \sum_{j=1}^{\infty} (\pm 1)^{j-1} \frac{1}{j} K_1(2j a M_n) \]  

(72)

and the Casimir force

\[ F = -\varepsilon_\text{ren}^\infty \mp \frac{|eB|}{\pi^2} \sum_{n=0}^\infty M_n^2 \sum_{j=1}^{\infty} (\pm 1)^{j-1} \left[ K_0(2j a M_n) + \frac{1}{2j a M_n} K_1(2j a M_n) \right], \]  

(73)

where the upper (lower) sign corresponds to the periodicity (antiperiodicity) boundary condition, \( K_\rho(t) \) is the Macdonald function of order \( \rho \). Similarly, we obtain an alternative to (68) and (69) representation for the Casimir force in the cases of the Dirichlet or Neumann boundary condition (upper sign) and the mixed boundary condition (lower sign):

\[ F = -\varepsilon_\text{ren}^\infty \mp \frac{|eB|}{\pi^2} \sum_{n=0}^\infty M_n^2 \sum_{j=1}^{\infty} (\pm 1)^{j-1} \left[ K_0(4j a M_n) + \frac{1}{4j a M_n} K_1(4j a M_n) \right]. \]  

(74)

Finally, we present the Casimir force in the case of the one-parameter family of boundary conditions given by (45):

\[ F = -\varepsilon_\text{ren}^\infty - \frac{|eB|}{2\pi^2 a^2} \sum_{n=0}^\infty \int_a^{aM_n} d\nu \frac{\nu^2}{\sqrt{\nu^2 - a^2 M_n^2}} \frac{\cos \beta - e^{-2\nu}}{\cosh 2\nu - \cos \beta}. \]  

(75)

6 Conclusion and discussion

In the present paper, we have considered the influence of a background uniform magnetic field and boundary conditions on the vacuum of a quantized
charged massive scalar matter field confined between two parallel plates separated by distance 2a. The magnetic field is assumed to be directed orthogonally to the plates, then the covariant Laplace operator is self-adjoint under a set of boundary conditions depending on four arbitrary functions of two coordinates which are tangential to the plates. Ignoring this functional dependence and imposing a condition that the flow of quantized matter outside the bounding plates is absent, see (13), we arrive at the set of boundary conditions depending on three arbitrary parameters, see (7) with (11). A further restriction, see (14), is due to a physical requirement that the operator of one-particle energy squared be positive definite, which makes the Casimir effect to be independent of ξ – the coupling of the scalar field to the scalar curvature of space-time. This reduces the $U(2)$-matrix defining the boundary condition, see (6), to the form given by (42), rendering finally the set of boundary conditions depending on one parameter, β, in the range $0 < \beta < 2\pi$ with the endpoints identified; under these circumstances the Casimir force is shown to take the form of (75). In particular, the Casimir force in the cases of the periodicity and the antiperiodicity boundary conditions is obtained at $\beta = 0$ and $\beta = \pi$, respectively, see (70) and (71), or alternatively (73); while the Casimir force in the cases of the Dirichlet (or Neumann) and the mixed Dirichlet-Neumann (or Neumann-Dirichlet) boundary conditions is obtained from the two preceding ones by change $a \rightarrow 2a$, see (68) and (69), or alternatively (74).

In the limit of a weak magnetic field, $|B| \ll m^2|e|^{-1}$, one has (see Ref.[9])

$$\varepsilon_{\text{ren}}^\infty = -\frac{7}{8} \frac{1}{720\pi^2} \frac{e^4B^4}{m^4}. \quad (76)$$

Thus, at $|B| \rightarrow 0$ the first term on the right-hand side of (75) vanishes, and, substituting the sum in the remaining part there by integral $\int_0^\infty dn$, we get

$$F|_{B=0} = -\frac{1}{2\pi^2a^4} \int_{am}^\infty d\nu \nu^2 \sqrt{\nu^2 - a^2m^2} \frac{\cos \beta - e^{-2\nu}}{\cosh 2\nu - \cos \beta}, \quad (77)$$

which in the limits of large and small distances between the plates takes the forms:
\[ F|_{B=0} = -\frac{1}{4a^{5/2}} \left( \frac{m}{\pi} \right)^{3/2} \left\{ \cos \beta e^{-2am} \left[ 1 + O \left( \frac{1}{am} \right) \right] + \frac{\cos 2\beta}{2\sqrt{2}} e^{-4am} \left[ 1 + O \left( \frac{1}{am} \right) \right] + O(e^{-6am}) \right\}, \quad am \gg 1 \] (78)

and

\[ F|_{B=0} = -\frac{1}{2\pi^2 a^4} \int_0^\infty d\nu \nu^3 \frac{\cos \beta - e^{-2\nu}}{\cosh 2\nu - \cos \beta} = -\frac{\pi^2}{8a^4} \left[ \frac{1}{30} - \frac{\beta^2}{(2\pi)^2} \left( 1 - \frac{\beta}{2\pi} \right)^2 \right], \quad am \ll 1. \] (79)

The standard results for the Casimir force in the case of the massless charged scalar field are obtained from (79) at \( \beta = 0 \) (periodicity boundary condition), see, e.g., Ref.\[3\],

\[ F|_{B=0} = -\frac{\pi^2}{240} \frac{1}{a^4} \] (80)

and at \( \beta = \pi \) (antiperiodicity boundary condition),

\[ F|_{B=0} = \frac{7}{8} \frac{\pi^2}{240} \frac{1}{a^4}; \] (81)

the results for \( F|_{B=0} \) under the Dirichlet (or Neumann) and the mixed Dirichlet-Neumann (or Neumann-Dirichlet) conditions are obtained from (80) and (81), respectively, by changing \( a \to 2a \).

In the limit of a strong magnetic field, \(|B| \gg m^2 e^{-1}\), one has (see, e.g., Ref.\[12\])

\[ \epsilon_{\text{ren}}^\infty = -\frac{e^2 B^2}{96\pi^2} \ln \frac{2|eB|}{m^2}, \] (82)

while the remaining piece of the force is

\[ (F + \epsilon_{\text{ren}}^\infty)|_{m=0} = -\frac{|eB|^2}{2\pi^2 a^2} \sum_{n=0}^\infty \int_0^\infty d\nu \frac{\nu^2}{\sqrt{\nu^2 - \left| eB(2n+1) \right|^2}} \times \] \[ \times \frac{\cos \beta - e^{-2\nu}}{\cosh 2\nu - \cos \beta}. \] (83)
The latter expression in the limits of large and small distances between the plates takes the forms:

\[
(F + \varepsilon_{\text{ren}}) \big|_{m=0} = -\frac{|eB|^{7/4}}{2\pi^{3/2}a^{1/2}} \sum_{n=0}^{\infty} (2n + 1)^{3/4} \left\{ \cos \beta e^{-2a\sqrt{|eB|(2n+1)}} \left[ 1 + O\left(\frac{1}{a\sqrt{|eB|(2n+1)}}\right)\right] + \frac{\cos 2\beta}{\sqrt{2}} e^{-4a\sqrt{|eB|(2n+1)}} \left[ 1 + O\left(\frac{1}{a\sqrt{|eB|(2n+1)}}\right)\right] + O\left(\frac{1}{a\sqrt{|eB|(2n+1)}}\right) \right\}, \quad a\sqrt{|eB|} \gg 1 \tag{84}
\]

and

\[
(F + \varepsilon_{\text{ren}}) \big|_{m=0} = -\frac{\pi^2}{8a^4} \left[ \frac{1}{30} - \frac{\beta^2}{(2\pi)^2} \left( 1 - \frac{\beta}{2\pi} \right)^2 \right], \quad a\sqrt{|eB|} \ll 1. \tag{85}
\]

We can conclude that the Weisskopf term, \(\varepsilon_{\text{ren}}\) (28), is dominating at a relatively large separation of the plates, \(2a \gg 2m^{-1}\), at a nonweak magnetic field. In this case the Casimir force, \(F \approx -\varepsilon_{\text{ren}}\), is repulsive (the pressure from the vacuum is positive), being independent of the choice of boundary conditions at the plates, as well as of the distance between the plates. In the opposite case of a relatively small separation of the plates, \(2a \ll 2m^{-1}\), at a sufficiently weak magnetic field, \(|B| \ll m^2|e|^{-1}\), the Weisskopf term is negligible, and the Casimir force, being power dependent on the distance between the plates as \((2a)^{-4}\), see (79), is either attractive or repulsive, depending on the choice of boundary conditions. A numerical analysis of a regime which is intermediate between the two above will be considered elsewhere.

Let us compare our results with those of our predecessors, see Refs. [13, 14, 15]. It should be noted in the first place that these authors have disregarded the dependence on the choice of boundary conditions, restricting themselves to the choice of the Dirichlet one. Secondly, they present expressions for the Casimir energy only, and the latter lacks immediate physical meaning. In particular, the author of Ref. [15] uses the Abel-Plana formula to obtain the Casimir energy, but simply drops without any explanation (see transition from (18) to (19) in Ref. [15]) a divergency given by the last sum in (65); fortunately, this divergency has no effect on the Casimir force, as we have discussed in Section 5. An approach of the authors of Refs. [13, 14] is different, and by using some analytical methods of regularization they obtain a finite
piece of the Casimir energy, which is actually given by before the last sum in (65). A common fault of Refs. [13, 14, 15] is that the Weisskopf-term contribution to the Casimir energy, \(2a\varepsilon_{\text{ren}}\), is completely ignored.

Finally, let us discuss the contribution of the quantized electromagnetic matter field to the Casimir effect. The Casimir force which is due to it is

\[
F = -\frac{\pi^2}{240} \frac{1}{(2a)^4},
\]

(86)

Usually, the Casimir effect is validated experimentally for the separation of parallel plates to be of order of \(10^{-8} - 10^{-5}\) m, see, e.g., Ref. [3]. Even if the mass of the quantized charged matter field is estimated to be as small as the electron mass (the Compton wavelength be of order of \(10^{-12}\) m), then the contribution of the latter to the Casimir effect is damped as \(e^{-10^4} - e^{-10^7}\), see (78), and with stronger exponents, see (84). The contribution of (86) becomes negligible at larger separations, where the contribution of the quantized charged matter field dominates due to its independence of the separation distance, see (76) or (82) (and (28) in general). Although supercritical values of the magnetic field, \(|B| > m^2|e|^{-1}\), are hardly feasible in laboratory (but may be attainable in some astrophysical objects such as magnetars), the contribution of (76) may dominate for large enough but still subcritical values of the magnetic field, \(|B| \ll m^2|e|^{-1}\), which can be attained in future in laboratory.

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21
Appendix. Abel-Plana summation formula

We start by presenting the infinite sum over $l$ as an integral over a contour in complex $w$-plane:

$$\sum_{l=-\infty}^{\infty} f \left[ \left( \frac{\pi l + \beta}{2} \right)^2 \right] = \frac{1}{\pi} \int_{C_{\pm}} dw f(w^2)G(w), \quad (A.1)$$

where contour $C_{\pm}$ consists of two parallel infinite lines going closely on the lower and upper sides of the real axis, see Fig.1, and $G(w)$ possesses simple poles on the real axis at $w = \pi l + \frac{\beta}{2}$:

$$G(w) = -\frac{1}{e^{2w+\beta} - 1}. \quad (A.2)$$

By deforming the parts of contour $C_{\pm}$ on the $w$-plane into contours $C_{\cap}$ and $C_{\cup}$ enclosing the lower and upper imaginary semiaxes, see Fig.1, we get

$$\sum_{l=-\infty}^{\infty} f \left[ \left( \frac{\pi l + \beta}{2} \right)^2 \right] = \frac{1}{\pi} \int_{C_{\cap}} dw f(w^2)G(w) + \frac{1}{\pi} \int_{C_{\cup}} dw f(w^2)G(w), \quad (A.3)$$

where it is implied that all singularities of $f$ as a function of $w$ lie on the imaginary axis at some distances from the origin. In view of obvious relation

$$\lim_{\epsilon \to 0^+} (\epsilon \pm i\nu)^2 = \lim_{\epsilon \to 0^+} (-\epsilon \mp i\nu)^2 = (\pm i\nu)^2$$

for real positive $\nu$ and $\epsilon$, we obtain

$$\sum_{l=-\infty}^{\infty} f \left[ \left( \frac{\pi l + \beta}{2} \right)^2 \right] = \frac{i}{\pi} \int_0^\infty d\nu \{ f[(-i\nu)^2] - f([i\nu]^2) \} \{ G(-i\nu) - G(i\nu) \}. \quad (A.4)$$

Let us note relation

$$G(-i\nu) + G(i\nu) = 1 - g(\nu^2), \quad (A.5)$$

where

$$G(\mp i\nu) = -\frac{1}{e^{\pm 2\nu - \beta} - 1}. \quad (A.6)$$
Figure 1: Contours $C_=$, $C_\cap$ and $C_\cup$ on the complex $w$-plane; the positions of poles of $G(w)$ are indicated by crosses.

and

$$g(\nu^2) = \frac{i \sin \beta}{\cosh 2\nu - \cos \beta}; \quad (A.7)$$

note also that both $G(-i\nu)$ and $g(\nu^2)$ are exponentially decreasing at large $\nu$. With the use of (A.5) we get

$$\sum_{l=-\infty}^{\infty} f \left[ \left( \pi l + \frac{\beta}{2} \right)^2 \right] = \frac{i}{\pi} \int_{0}^{\infty} d\nu \{ f[(-i\nu)^2] - f[(i\nu)^2]\} [2G(-i\nu) + g(\nu^2)] -$$

$$-\frac{i}{\pi} \int_{0}^{\infty} d\nu f[(-i\nu)^2] + \frac{i}{\pi} \int_{0}^{\infty} d\nu f[(i\nu)^2]. \quad (A.8)$$

By rotating the paths of integration in the last and before the last integrals in (A.8) by $90^\circ$ in the clockwise and anticlockwise directions, respectively, we finally get
\[
\sum_{l=-\infty}^{\infty} f \left( \left( \pi l + \frac{\beta}{2} \right)^2 \right) = \frac{i}{\pi} \int_{0}^{\infty} d\nu \left\{ f\left[\left(-i\nu\right)^2\right] - f\left[(i\nu)^2\right]\right\} \left[ 2G(-i\nu) + g(\nu^2) \right] + \\
+ \frac{2}{\pi} \int_{0}^{\infty} d\mu f(\mu^2). \tag{A.9}
\]

In view of relation

\[
2G(-i\nu) + g(\nu^2) = -\frac{\cos \beta - e^{-2\nu}}{\cosh 2\nu - \cos \beta}, \tag{A.10}
\]

and the evenness of the integrand in the integral over \( \mu \), we obtain (56).

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