The local invariant for scale structures on mapping spaces

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Abstract
A scale Hilbert space is a natural generalization of a Hilbert space which considers not only a single Hilbert space but a nested sequence of subspaces. Scale structures were introduced by H. Hofer, K. Wysocki, and E. Zehnder as a new concept of smooth structures in infinite dimensions. In this paper, we prove that scale structures on mapping spaces are completely determined by the dimension of domain manifolds. We also give a complete description of the local invariant introduced by U. Frauenfelder for these spaces. Product mapping spaces and relative mapping spaces are also studied. Our approach is based on the spectral resolution of Laplace type operators together with the eigenvalue growth estimate.

Keywords Scale structures · Frauenfelder’s local invariant · Mapping spaces

Mathematics Subject Classification 46T05 · 58B15

1 Introduction

While dimension is a complete invariant for finite-dimensional vector spaces, there is no invariant for infinite-dimensional separable Hilbert spaces, such as $L^2(\mathbb{R}^n)$, since all of them...
are isometric to the $\ell^2$-space. However if we consider a scale structure on $L^2(\mathbb{R}^n)$, i.e. a nested sequence of Hilbert spaces

$$L^2(\mathbb{R}^n) \supset W^{1,2}(\mathbb{R}^n) \supset W^{2,2}(\mathbb{R}^n) \supset \cdots \supset \bigcap_{k \in \mathbb{N}} W^{k,2}(\mathbb{R}^n),$$

there is an invariant introduced by Frauenfelder [6], which encodes information on how each space of the sequence is embedded in other spaces. In this paper, we focus on mapping spaces which admit natural scale Hilbert structures. We give a full description of the invariant due to Frauenfelder for mapping spaces. As a consequence, we show that scale structures on mapping spaces are completely determined by the dimension of domain manifolds.

**Main Theorem** Two mapping spaces $\operatorname{Map}(N_1, M_1)$ and $\operatorname{Map}(N_2, M_2)$ are locally scale isomorphic if and only if $\dim N_1 = \dim N_2$.

The precise statement of this Main Theorem will appear as Theorem A below. Scale structures of product mapping spaces are also studied in Theorem B using the $\ast$-operation on scale Hilbert spaces. Moreover, these results hold also for relative mapping spaces with suitable boundary conditions, see Corollary B. In fact, scale structures on mapping spaces are relevant to the order of elliptic self-adjoint operators as discussed in the appendix.

**Definition 1.1**

1 A scale smooth structure on a Hilbert space $H$ is a sequence of pairs

$$\mathcal{H} = \left\{ (H_k, \langle \cdot, \cdot \rangle_k) \right\}_{k \in \mathbb{N}_0}$$

where $(H_k, \langle \cdot, \cdot \rangle_k), k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ are real Hilbert spaces, and they form a nested sequence

$$H = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_\infty := \bigcap_{k=0}^{\infty} H_k$$

with the following two properties.

(i) For each $k \in \mathbb{N}_0$, the inclusion

$$(H_{k+1}, \langle \cdot, \cdot \rangle_{k+1}) \hookrightarrow (H_k, \langle \cdot, \cdot \rangle_k)$$

is a compact operator.

(ii) The subspace $H_\infty$ is dense in $(H_k, \langle \cdot, \cdot \rangle_k)$ for every $k \in \mathbb{N}_0$.

Scale structures were introduced by H. Hofer, K. Wysocki, and E. Zehnder to give a new concept of a smooth structure in infinite dimensions, see [13,14]. We point out that there is a unique scale structure on a finite-dimensional Hilbert space $H$, namely $H_k = H$ for all $k \in \mathbb{N}_0$ due to property (ii) in Definition 1.1. We simply call $\mathcal{H}$ a scale Hilbert space. Given $\mathcal{H}$, we denote by $\mathcal{H}^j$ for $j \in \mathbb{N}_0$ the scale smooth structure on $H_j$ given by

$$\{ (H_{j+k}, \langle \cdot, \cdot \rangle_{j+k}) \}_{k \in \mathbb{N}_0}.$$  

Note that $\mathcal{H}^0 = \mathcal{H}$. The scale product of two scale Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, $\mathcal{H} \oplus_{sc} \mathcal{H}'$ is defined by

$$((H \oplus_{sc} H')_k, \langle \cdot, \cdot \rangle_k) = (H_k \oplus H'_k, \langle \cdot, \cdot \rangle_{H_k} \oplus \langle \cdot, \cdot \rangle_{H'_k}).$$

1 Our and Frauenfelder’s definition of scale Hilbert spaces is slightly different from Hofer–Wysocki–Zender’s. The latter definition only requires that the zeroth level $H_0$ is a Hilbert space.

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A scale Hilbert space \( \mathcal{Y} := \{ Y_k, (\cdot, \cdot)_k \}_{k \in \mathbb{N}_0} \) is said to be a scale subspace of \( \mathcal{H} \) if \( Y_k \) is a subspace of \( H_k \) for all \( k \in \mathbb{N}_0 \). Moreover if \( \mathcal{Y} \) is a closed scale subspace of \( \mathcal{H} \), the orthogonal complement of \( \mathcal{Y} \) is defined by \( \mathcal{Y}^\perp := \{ Y_k^\perp(\cdot, \cdot)_k \} \) where \( Y_k^\perp(\cdot, \cdot)_k \) stands for the orthogonal complement of \( Y_k \) with respect to \( (\cdot, \cdot)_k \). The definition of scale Hilbert manifolds is the obvious modification from the definition of standard manifolds, or see [13]. The product operation for scale Hilbert manifolds is also defined in a similar vein and denoted by \( \times_{sc} \).

**Definition 1.2** Let \( \mathcal{H} \) and \( \mathcal{H}' \) be two scale Hilbert spaces. A scale operator \( T : \mathcal{H} \rightarrow \mathcal{H}' \) is a bounded linear operator \( T : H_0 \rightarrow H'_0 \) such that it maps \( H_k \) to \( H'_k \) for each \( k \in \mathbb{N} \) and the restriction

\[
T|_{H_k} : H_k \rightarrow H'_k, \quad k \in \mathbb{N}_0
\]

is also a bounded linear operator. A scale operator \( T : \mathcal{H} \rightarrow \mathcal{H}' \) is said to be a scale isomorphism if it is invertible, i.e. there exists a scale operator \( T^{-1} : \mathcal{H}' \rightarrow \mathcal{H} \) such that

\[
T^{-1} \circ T = \text{Id}_{\mathcal{H}}, \quad T \circ T^{-1} = \text{Id}_{\mathcal{H}'}
\]

where \( \text{Id}_{\mathcal{H}} \) and \( \text{Id}_{\mathcal{H}'} \) are scale operators which induce the identity operator on each level. If there is a scale isomorphism between \( \mathcal{H} \) and \( \mathcal{H}' \), then we say that they are scale isomorphic and denote by \( \mathcal{H} \overset{sc}{=} \mathcal{H}' \).

We recall the notion of fractal structures on scale Hilbert spaces studied in [7]. We define a Hilbert space \( \ell_f^2 \) for a monotone and unbounded function \( f : \mathbb{N} \rightarrow (0, \infty) \) by

\[
\ell_f^2 := \left\{ x = (x_1, x_2, \cdots) \left| x_\mu \in \mathbb{R}, \mu \in \mathbb{N}, \sum_{\mu=1}^{\infty} f(\mu)x_\mu^2 < \infty \right. \right\}
\]

with the inner product

\[
\langle x, y \rangle_f := \sum_{\mu=1}^{\infty} f(\mu)x_\mu y_\mu, \quad x, y \in \ell_f^2.
\]

We denote by \( \tilde{\mathcal{F}} \) the set of functions \( f : \mathbb{N} \rightarrow (0, \infty) \) which are monotone and unbounded. We define an equivalence relation on this space: Two functions \( f_1, f_2 \in \tilde{\mathcal{F}} \) are called equivalent (written \( f_1 \sim f_2 \)) if there exists a constant \( c > 0 \) such that

\[
\frac{1}{c} f_1(\mu) \leq f_2(\mu) \leq cf_1(\mu), \quad \text{for all } \mu \in \mathbb{N}.
\]

The quotient set of \( \tilde{\mathcal{F}} \) by \( \sim \) is denoted by

\[
\mathcal{F} := \tilde{\mathcal{F}} / \sim = \{ [f] \mid f \in \tilde{\mathcal{F}} \}.
\]

**Definition 1.3** A scale Hilbert space \( \mathcal{H} \) is fractal if there exists \( f \in \tilde{\mathcal{F}} \) such that \( \mathcal{H} \) is scale isomorphic to the scale Hilbert space \( \ell_{f}^2 \) given by

\[
\ell_{f}^2 := \{ (\ell_{f_k}^2, (\cdot, \cdot)_{f_k}) \}_{k \in \mathbb{N}_0},
\]

where \( f^k \) refers to the \( k \)th power of \( f \).
One can easily check that $\ell^{2, f_1}$ and $\ell^{2, f_2}$ are scale isomorphic if $f_1 \sim f_2$. In other words, an equivalence class $[f] \in \mathcal{F}$ determines the structure of fractal scale Hilbert spaces.

In order to define Frauenfelder’s invariant for scale Hilbert spaces, we consider a scale Hilbert pair which consists of a pair of Hilbert spaces

$$\mathcal{H}_2 = \{(H_0, \langle \cdot, \cdot \rangle_0), (H_1, \langle \cdot, \cdot \rangle_1)\}$$

such that there exists a compact dense inclusion $H_1 \hookrightarrow H_0$. Let

$$\mathcal{S}_2 := \{\mathcal{H}_2 | \dim H_0 = \infty\}/\sim_2$$

where $\sim_2$ stands for the equivalence relation given by scale isomorphisms. It is proved in [6] that there exists a bijection

$$\Phi : \mathcal{F} \longrightarrow \mathcal{S}_2$$

$$[f] \longmapsto [\ell^2, \ell^2_j].$$

In particular, for a scale Hilbert space $\mathcal{H}$, every Hilbert space $(H_k, \langle \cdot, \cdot \rangle_k), k \in \mathbb{N}_0$ is separable. Since every separable Hilbert space is isometric to the $\ell^2$-space, there is no invariant for separable Hilbert spaces. However, scale Hilbert spaces do have an invariant as Frauenfelder introduced: Let $\mathcal{S}$ be the set of infinite-dimensional scale Hilbert spaces modulo scale isomorphisms and let $\Lambda$ be the upper triangle of $\mathbb{N}_0 \times \mathbb{N}_0$, i.e.

$$\mathcal{S} := \{\mathcal{H} | \dim H_0 = \infty\}/\sim_2, \quad \Lambda := \{(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0 | i < j\}.$$ 

The following map can be regarded as an invariant for scale Hilbert spaces:

$$\mathcal{K} : \mathcal{S} \longrightarrow \text{Map}(\Lambda, \mathcal{F})$$

defined for $\mathcal{H} \in \mathcal{S}, (i, j) \in \Lambda$ by

$$\mathcal{K}([\mathcal{H}]) (i, j) := \Phi^{-1}([H_i, H_j]).$$

This also gives a local invariant for scale Hilbert manifolds and we use the same symbol $\mathcal{K}$ for that. The (local) invariant $\mathcal{K}$ can be computed in fractal scale Hilbert spaces (or manifolds). If $\mathcal{H}$ is scale isomorphic to $\ell^2, f$ for some $f \in \mathcal{F}$, then the invariant for $\mathcal{H}$ is of the form $\mathcal{K}([\mathcal{H}]) (i, j) = [f^{j-i}]$.

**Definition 1.4** A scale operator $T$ between two scale Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ is said to be Fredholm if the following conditions hold.

1. ker $T$ is finite-dimensional scale subspace of $\mathcal{H}$.
2. im $T$ is a closed scale subspace of $\mathcal{H}'$.
3. coker $T := \text{im} T^\perp$ is a finite-dimensional scale subspace of $\mathcal{H}'$.

The index of a scale-Fredholm operator $T : \mathcal{H} \rightarrow \mathcal{H}'$ is

$$\text{ind} T := \dim \text{ker} T - \dim \text{coker} T.$$ 

As mentioned above, finite-dimensional spaces ker $T$ and im$T^\perp$ admit unique scale structures, namely $(\text{ker} T)_k = \text{ker} T$ and $(\text{im} T^\perp)_k = \text{im} T^\perp$ for all $k \in \mathbb{N}_0$.

**Remark 1.5** Since $\dim \text{ker} T < \infty$, there exist closed subspaces $Y_k \subset H_k, k \in \mathbb{N}_0$ such that $H_k = \text{ker} T \oplus Y_k$. It was proved that $Y_k$ can be chosen so that $\mathcal{Y} = \{Y_k, \langle \cdot, \cdot \rangle_k\}$ is indeed a scale subspace of $\mathcal{H}$ and $\mathcal{H} = \text{ker} T \oplus_{sc} \mathcal{Y}$, see [13]. It also holds that $\mathcal{H}' = \text{im} T \oplus_{sc} \text{im} T^\perp$. Moreover by the open mapping theorem, $T|_{\mathcal{Y}} : \mathcal{Y} \rightarrow \text{im} T$ is a scale isomorphism.
From the definition of scale-Fredholm, we have the following regularity property: If \( T : \mathcal{H} \to \mathcal{H}' \) is a scale-Fredholm operator and there are \( e \in H_0 \) and \( f \in H_j \) for some \( j \in \mathbb{N}_0 \) such that \( Te = f \), then \( e \in H_j \). See [13] for the proof.

**Definition 1.6** A scale operator \( T : \mathcal{H}^1 \to \mathcal{H}^0 \) is called a scale Hessian operator if it is a scale-Fredholm operator of index zero and symmetric, i.e. \( \langle T \xi, \xi \rangle_0 = \langle \xi, T \xi \rangle_0 \) for any \( \xi, \zeta \in H_1 \).

Frauenfelder gave the following evidence in support of the claim that fractal structure is the right structure for a general setup of Floer theory.

**Theorem A** Let \( E \) be a vector bundle over a closed Riemannian manifold \((N, g)\). The scale Hilbert space \( \mathcal{X}(N, E) \) is scale isomorphic to \( \ell^2, f \) for \( f (\mu) = \mu^{2/\dim N} \), \( \mu \in \mathbb{N} \). In particular, the invariant \( \mathcal{R} \) is given by

\[
\mathcal{R} ([\mathcal{X}(N, E)])(i, j) = [\mu^{2(j-i)/\dim N}].
\]

This theorem will be proved in Sect. 3. Corollary A below follows immediately from Theorem A. It is worth mentioning that this result shows that components of mapping spaces are locally scale isomorphic.

**Corollary A** In consequence of Theorem A, the local invariant \( \mathcal{R} \) for \( \text{Map}(N, M) \) is

\[
\mathcal{R} ([\text{Map}(N, M)])(i, j) = [\mu^{2(j-i)/\dim N}].
\]

Moreover, let \((N_1, g_1)\) and \((N_2, g_2)\) be closed Riemannian manifolds and \( M_1 \) and \( M_2 \) be any manifolds. Then \( \dim N_1 = \dim N_2 \) if and only if \( \text{Map}(N_1, M_1) \) is locally scale isomorphic to \( \text{Map}(N_2, M_2) \).

**Theorem B** Let \( E_1 \) and \( E_2 \) be vector bundles over closed Riemannian manifolds \( N_1 \) and \( N_2 \) respectively and let \( \dim N_1 \leq \dim N_2 \). A product of scale Hilbert spaces \( \mathcal{X}(N_1, E_1) \oplus_{sc} \mathcal{X}(N_2, E_2) \) is scale isomorphic to \( \mathcal{X}(N_2, E_2) \). Accordingly, \( \text{Map}(N_1, M_1) \times_{sc} \text{Map}(N_2, M_2) \) is locally scale isomorphic to \( \text{Map}(N_2, M_2) \) for arbitrary manifolds \( M_1, M_2 \).
Even if \( N \) has nonempty boundary, we can draw the same conclusion as above by imposing the mixed boundary condition which generalizes both Dirichlet and Neumann boundary conditions, see (4.1) and (4.2).

**Corollary B** If a compact manifold \( N \) has nonempty boundary, Theorem A, Corollary A, and Theorem B are true under the mixed boundary condition.

In the appendix, we discuss relations between scale structures (and hence the local invariant) of mapping spaces and the order of elliptic self-adjoint operators on elliptic complexes.

## 2 Preliminaries

### 2.1 Spectral resolution

Let \((N, g)\) be an \( n \)-dimensional closed Riemannian manifold and let \( E \) be a vector bundle over \( N \) equipped with a bundle metric \( \langle \cdot , \cdot \rangle_E \). We denote the spaces of smooth sections of \( E \) resp. \( T^*N \otimes E \) by \( \Gamma(N, E) \) resp. \( \Gamma(N, T^*N \otimes E) \). We denote by \( \Gamma^2(N, E) \) the completion of \( \Gamma(N, E) \) with respect to the the \( L^2 \)-product given by
\[
\langle \phi, \psi \rangle_{L^2(N, E)} = \int_N \langle \phi, \psi \rangle_E \, d\text{vol}_N, \quad \phi, \psi \in \Gamma(N, E).
\]
We also need the \( L^2 \)-product on \( \Gamma(T^*N \otimes E) \)
\[
\langle \phi, \psi \rangle_{L^2(N, T^*N \otimes E)} = \int_N \langle \phi, \psi \rangle_{T^*N \otimes E} \, d\text{vol}_N, \quad \phi, \psi \in \Gamma(N, T^*N \otimes E).
\]
where \( \langle \cdot , \cdot \rangle_{T^*N \otimes E} \) is the bundle metric on \( T^*N \otimes E \) induced by \( g \) and \( \langle \cdot , \cdot \rangle_E \). If there is no confusion, we shall write \( L^2 \) instead of \( L^2(N, E) \) and \( L^2(N, T^*N \otimes E) \). We consider a Riemannian connection \( \nabla : \Gamma(N, E) \rightarrow \Gamma(N, T^*N \otimes E) \) and take the formal \( L^2 \)-adjoint operator of \( \nabla \),
\[
\nabla^* : \Gamma(N, T^*N \otimes E) \rightarrow \Gamma(N, E), \quad \langle \nabla \phi, \psi \rangle_{L^2} = \langle \phi, \nabla^* \psi \rangle_{L^2}.
\]
Then the **Bochner Laplacian** is defined by
\[
\Delta := \nabla^* \nabla : \Gamma(N, E) \rightarrow \Gamma(N, E).
\]
This can be equivalently defined by \( \Delta \phi = -\text{trace} \nabla^2 \phi \) where \( \nabla^2 \phi \) is the second covariant derivative of \( \phi \in \Gamma(N, E) \) induced by the connection \( \nabla \) on \( E \) together with the Levi-Civita connection on \( T^*N \).

A real number \( \lambda \in \mathbb{R} \) is called an **eigenvalue** if there is some nonzero \( \phi \in \Gamma(N, E) \) satisfying \( \Delta \phi = \lambda \phi \). Such a \( \phi \in \Gamma(N, E) \) is called an **eigensection** associated to \( \lambda \). The set of all eigenvalues of \( \Delta \) is called the **spectrum** and denoted by
\[
\text{Spec}(N) = \text{Spec}(N, g) = \{ \lambda_\mu \}_{\mu \in \mathbb{N}} = \{ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\mu \leq \cdots \}.
\]
We say that \( \{ \phi_\mu, \lambda_\mu \}_{\mu \in \mathbb{N}} \) is a **discrete spectral resolution** of \( \Delta \) if the set \( \{ \phi_\mu \}_{\mu \in \mathbb{N}} \) is a complete orthonormal basis for \( \Gamma^2(N, E) \) where \( \phi_\mu \in \Gamma(N, E) \) so that \( \Delta \phi_\mu = \lambda_\mu \phi_\mu \).

**Theorem 2.1** Let \( \Delta \) be the Bochner Laplacian on \( E \). Then the following hold:

(i) There exists a discrete spectral resolution of \( \Delta \), \( \{ \phi_\mu, \lambda_\mu \}_{\mu \in \mathbb{N}} \).
(ii) There are only finitely many non-positive eigenvalues and \(\lambda_\mu \sim C \mu^{2/n}\) for some constant \(C > 0\) as \(\mu \to \infty\).

**Proof** The assertions hold for general elliptic self-adjoint operators of order two (e.g. self-adjoint Laplace type operators), see Theorem 5.1. We refer to the Gilkey’s book [8, Chapter 1] or [10] for the proof. \(\square\)

**Remark 2.2** The simplest one among Laplace type operators is the Laplace–Beltrami operator \(\Delta_0 : C^\infty(N) \to C^\infty(N)\) on smooth function spaces defined by

\[\Delta_0 u = -\text{div}_g \nabla_g u = -\text{trace}_g \text{Hess} u\]

where \(\text{div}_g\), \(\nabla_g\), and \(\text{Hess}\) stands for the divergence, the gradient, and the Hessian respectively. In this case, the first assertion of the above theorem is proved by examining the Rayleigh quotient and the second assertion is nothing but Weyl’s asymptotic formula, see [1] or [2].

The contractible component of a mapping space \(\text{Map}(N, M)\) is modeled on a scale Hilbert space \(\text{Map}(N, C^\infty(M))\) where \(m = \dim M\), and hence one can work with the Laplace–Beltrami operator (instead of the Bochner Laplacian) for the contractible component.

### 2.2 Equivalence of Sobolev spaces

A section of a vector bundle \(E \to N\) is said to be of class \(W^{k,p}\) if all its local coordinate representations are in \(W^{k,p}\). This definition is independent of the choice of coordinate charts even if \(kp \leq n\). But in order to make a definition of maps of class \(W^{k,p}\) between manifolds which does not depend on the choice of coordinate charts, we need the following well-known proposition which holds only for \(kp > n\), see [18, Appendix B]. For such a reason, we only deal with \(W^{k,p}\)-maps between manifolds for \(kp > n\).

**Proposition 2.3** Let \(\Omega \subset \mathbb{R}^n\) be a bounded open domain with \(C^k\)-boundary. If \(kp > n\) and \(\varphi \in C^\infty(\mathbb{R})\), then we have the following smooth map between Banach spaces.

\[\bar{\varphi}_{k,p} : W^{k,p}(\Omega) \to W^{k,p}(\Omega), \quad \bar{\varphi}_{k,p}(u) = \varphi \circ u.\]

The following theorem is about the Bochner–Weitzenböch formula.

**Theorem 2.4** Let \(\Delta_0\) be the Laplace–Beltrami operator on \(N\). Then locally \(\Delta = \Delta_0 + R\) where \(R\) is an endomorphism of \(E\) involving only the curvature tensor.

**Proof** The proof can be found in [8, Chapter 4]. \(\square\)

Next, we recall the Calderon–Zygmund inequality for the Laplace–Beltrami operator.

**Theorem 2.5** Let \(1 < p < \infty\), \(k \geq 0\) be integers, and let \(\Delta_0\) be the Laplace–Beltrami operator on an open domain \(\Omega \subset \mathbb{R}^n\). There exists a constant \(c > 0\) such that for every \(u \in C^\infty_c(\Omega)\), there holds

\[\|u\|_{W^{k+2,p}(\Omega)} \leq c\left(\|\Delta_0u\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)}\right).\]

Accordingly, if \(\Delta_0\) is the Laplace–Beltrami operator on a closed manifold \(N\), then there exists a constant \(c > 0\) such that for every \(u \in C^\infty(N)\), there holds

\[\|u\|_{W^{k+2,p}(N)} \leq c\left(\|\Delta_0u\|_{W^{k,p}(N)} + \|u\|_{L^p(N)}\right).\]

Here the constant \(c > 0\) depends only on \(k, p, \text{ and } \Omega \) (or \(N\)).
\textbf{Proof} The proof can be found in [15, Chapter 8] or [18, Appendix B]. \hfill \Box

\textbf{Definition 2.6} The \( \Delta^{k,p} \)-norm on \( \Gamma(N, E) \) is defined by

\[
||u||_{\Delta^{k,p}(N,E)} := ||u||_{L^p} + ||\nabla u||_{L^p} + ||\Delta u||_{L^p} + \cdots + ||\nabla^{2(k-2)} u||_{L^p}.
\]

Here \( ||\cdot||_{L^p} \) is either \( ||\cdot||_{L^p(N,E)} \) or \( ||\cdot||_{L^p(N,T^*N \otimes E)} \). In particular, the \( \Delta^{k,2} \)-norm is induced from the \( \Delta^{k,2} \)-product given by

\[
\langle u, v \rangle_{\Delta^{k,2}(N,E)} = \langle u, v \rangle_{L^2} + \langle \nabla u, \nabla v \rangle_{L^2} + \langle \Delta f, \Delta h \rangle_{L^2} + \langle \nabla \Delta u, \nabla \Delta v \rangle_{L^2} + \cdots + \langle \nabla^{2(k-2)} u, \nabla^{2(k-2)} v \rangle_{L^2}.
\]

\textbf{Corollary 2.7} On a vector bundle \( E \) over a closed Riemannian manifold \( N \), the \( W^{k,p} \)-norm and the \( \Delta^{k,p} \)-norm are equivalent for \( 1 < p < \infty \). In particular, the Sobolev spaces defined via the completion of these norms on the space of smooth sections coincide:

\[
\Gamma^{k,p}(N, E) := \overline{\Gamma(N, E)}^||\cdot||_{W^{k,p}} = \overline{\Gamma(N, E)}^||\cdot||_{\Delta^{k,p}}.
\]

\textbf{Proof} It is easy to see that there exists a constant \( c > 0 \) such that

\[
||\phi||_{\Delta^{k,p}(N,E)} \leq c ||\phi||_{W^{k,p}(N,E)}, \quad \phi \in \Gamma(N, E),
\]

since locally \( \nabla = d + A \) where \( d \) is the trivial connection and \( A \) is a matrix of 1-forms whose entries are Christoffel symbols. Next, we prove an inequality in opposite direction. An immediate consequence of the Bochner–Weitzenböch formula (Theorem 2.4) and the Calderon–Zygmund inequality (Theorem 2.5) is that there exists \( c_0 > 0 \) satisfying

\[
||\phi||_{W^{k,p}(N,E)} \leq c_0 (||\Delta \phi||_{W^{k-2,p}(N,E)} + ||\phi||_{W^{k-2,p}(N,E)}).
\]

Therefere there exist constants \( c_0, c_1, \ldots, C > 0 \) satisfying

\[
||\phi||_{W^{k,p}} \leq c_0 (||\Delta \phi||_{W^{k-2,p}} + ||\phi||_{W^{k-2,p}}) \\
\leq c_0 (c_1 (||\Delta \phi||_{W^{k-4,p}} + ||\phi||_{W^{k-4,p}}) + c_2 (||\Delta \phi||_{W^{k-4,p}} + ||\phi||_{W^{k-4,p}})) \\
\vdots \\
\leq C ||\phi||_{\Delta^{k,p}}.
\]

\hfill \Box

\textbf{Remark 2.8} There is an alternative way to prove the preceding corollary. It can be proved that the \( \Delta^{k,p} \)-norm and the norm \( ||\cdot||_{\nabla,k,p} \) defined by

\[
||\phi||_{\nabla,k,p} := ||\phi||_{L^2(N,E)} + ||\nabla \phi||_{L^2(N,T^*N \otimes E)} + \cdots + ||\nabla^k \phi||_{L^2(N,T^*N \otimes^k \otimes E)}, \quad \phi \in \Gamma(N, E)
\]

are equivalent by examining the commutator

\[
\nabla^* \nabla - \nabla \nabla^* : \Gamma(N, T^*N \otimes^j \otimes E) \to \Gamma(N, T^*N \otimes^{j+1} \otimes E), \quad j \in \mathbb{N}.
\]

An advantage of this approach is that the preceding corollary can be proved even for noncompact complete manifolds whose curvature tensors and their covariant derivatives are bounded. See Theorem 1.3 in [3] (or section 2 in [20]) but a wrong identity was used in the proof of [3]; later on, it was repaired by [20].
3 Fractal scale structures on mapping spaces

The objective of this section is to explore the geography of fractal scale structures on a scale Hilbert space $X(N, E)$ which consists of

$$(\Gamma^{k_0, 2}(N, E), \langle \cdot, \cdot \rangle_{W^{k_0, 2}}) \supset (\Gamma^{k_0 + 1, 2}(N, E), \langle \cdot, \cdot \rangle_{W^{k_0 + 1, 2}}) \supset \cdots \supset \Gamma(N, E)$$

where $k_0$ is the smallest natural number satisfying $2k_0 > n = \dim N$.

**Theorem 3.1** The scale Hilbert space $X(N, E)$ is fractal. More precisely, it is scale isomorphic to $\ell^{2, f}$ for $f(\mu) = \lambda_\mu$, $\mu \in \mathbb{N}$ where $\{\lambda_\mu\}_{\mu \in \mathbb{N}}$ is the spectrum of the Bochner Laplacian on $E$.

**Proof** According to Theorem 2.1, eigensections $\{\phi_\mu\}_{\mu \in \mathbb{N}}$ of the Bochner Laplacian $\Delta$ form an $L^2$-orthonormal basis for $\Gamma^2(N, E)$. Let $\lambda_\mu \in \mathbb{R}$ be an eigenvalue associated to $\phi_\mu$, $\mu \in \mathbb{N}$. We can take the $A^k$-product instead of the $W^k$-product due to Corollary 2.7. It is easy to see that $\{\phi_\mu\}_{\mu \in \mathbb{N}}$ form a $A^k$-orthogonal basis for $\Gamma^k(N, E)$, $k \in \mathbb{N}$ as well: For $i, j \in \mathbb{N}$, we compute

$$\langle \phi_i, \phi_j \rangle_{A^2} = \langle \phi_i \rangle_{L^2} + (\nabla \phi_i, \nabla \phi_j)_{L^2} + (\Delta \phi_i, \Delta \phi_j)_{L^2} + \cdots + (\nabla^{2(k/2-1)} \phi_i, \nabla^{2(k/2-1)} \phi_j)_{L^2}$$

$$= \langle \phi_i, \phi_j \rangle_{L^2} + (\Delta \phi_i, \phi_j)_{L^2} + \lambda_i \lambda_j \langle \phi_i, \phi_j \rangle_{L^2}$$

$$+ \cdots + (\lambda_i \lambda_j)^{(k/2)} (\nabla^{2(k/2-1)} \phi_i, \phi_j)_{L^2}$$

$$= (1 + \lambda_i + \lambda_j^2 + \cdots + \lambda_i^k) \delta_{ij}.$$

Let $f(\mu) = \lambda_\mu$, $\mu \in \mathbb{N}$ and consider the following map between two scale Hilbert spaces.

$$\Phi : X(E, N) \longrightarrow \ell^{2, f}$$

$$\psi \longmapsto \left( \cdots, \frac{1}{\sqrt{\sum_{j=0}^{k_0} \lambda_\mu^j}} \langle \psi, \phi_\mu \rangle_{A^{k_0, 2}}, \cdots \right)$$

Then the map $\Phi$ is a scale isomorphism since for $\lambda_\mu \geq 1$,

$$\lambda_\mu^{k_0} \leq 1 + \lambda_\mu + \lambda_\mu^2 + \cdots + \lambda_\mu^{k_0} \leq (1 + k_0) \lambda_\mu^{k_0}.$$  

**Remark 3.2** In the case of Map($S^1$, $\mathbb{R}$) whose levels are

$$(L^2(S^1, \mathbb{R}), \langle \cdot, \cdot \rangle_{L^2}) \supset (W^{1, 2}(S^1, \mathbb{R}), \langle \cdot, \cdot \rangle_{W^{1, 2}}) \supset \cdots \supset C^\infty(S^1, \mathbb{R}),$$

the usual Fourier basis forms a $W^k$-orthogonal basis for $W^k(S^1, \mathbb{R})$, $k \in \mathbb{N}$ as well as an $L^2$-orthonormal basis for $L^2(S^1, \mathbb{R})$. The previous theorem and Theorem 2.1 yield that Map($S^1$, $\mathbb{R}$) is scale isomorphic to $\ell^{2, f}$ for $f(\mu) = \mu^2$, $\mu \in \mathbb{N}$. This also can be shown by a straightforward computation with Fourier basis.

We have not introduced the notion of differential of functions or maps in the scale-world since this is not our main concern. Instead, we just recall the following useful criterion for scale-smoothness.
Theorem 3.3 [13] Let \( \mathcal{H} \) and \( \mathcal{H}' \) be scale Hilbert spaces and let \( \mathcal{V} \) be an open subset of \( \mathcal{H} \). Assume that a map \( T : \mathcal{V} \to \mathcal{H}' \) is scale continuous and \( T|_{V_{m+k}} : V_{m+k} \to H'_m \) is of class \( C^{k+1} \) for all \( m, k \geq 0 \). Then \( T \) is scale smooth.

Proposition 3.4 The mapping space \( \text{Map}(N, M) \) is a scale Hilbert manifold with local charts on \( \mathcal{X}(N, u^*TM) \) for \( u \in C^\infty(N, M) \).

Proof The proof immediately follows from Eliasson’s work [4] together with the previous theorem. We first pick a smooth map \( u \in C^\infty(N, M) \); since \( C^\infty(N, M) \) is dense in \( W^{k,2}(N, M) \) for all \( k \in \mathbb{N} \), it suffices to find open covering charts near smooth maps. We denote by \( \tilde{u} \) the bundle map \( u^*TM \to TM \) induced by \( u : N \to M \). Let \( D_*TM \) be the \( \epsilon \)-disk subbundle of \( TM \). There exists a small \( \epsilon > 0 \) such that the exponential map \( \exp_{D_*TM} \) is a diffeomorphism onto an open neighborhood of \( p \in M \). Then we have the following parametrization:

\[
\begin{align*}
\exp^k_u : & \Gamma^{k,k}_{u^*T\mathcal{H}}(N, u^*D_*TM) \to W^{k,k}_{u^*D_*TM}(N, M) \\
& \phi(\cdot) \longmapsto \exp(\tilde{u}(\phi(\cdot)))
\end{align*}
\]

It turns out that \( W^{k,k}_{u^*D_*TM}(N, M) \) is a Hilbert manifold and its differentiable structure is given by \( \{\mathcal{V}_k, (\exp^k_u)^{-1}\}_{k \in \mathbb{N}} \) where \( \mathcal{V}_k := \exp^k_u(\Gamma^{k,k}_{u^*T\mathcal{H}}(N, u^*D_*TM)) \), see [4,17]. In order to prove that \( \text{Map}(N, M) \) is a scale Hilbert manifold, we take a close look at the following map:

\[\text{Exp}_{uu'} : \mathcal{V} \to \mathcal{X}(N, u'^*TM)\]

where \( \mathcal{V} \) is an open subset in \( \mathcal{X}(N, u'^*TM) \) given by \( \mathcal{V}_k := (\exp^k_u)^{-1}(\mathcal{V}_k \cap \mathcal{V}_k') \) and

\[\text{Exp}_{uu'}|_{\mathcal{V}_k} := (\exp^k_{u'})^{-1} \circ \exp^k_u : (\exp^k_u)^{-1}(\mathcal{V}_k \cap \mathcal{V}_k') \to (\exp^k_{u'})^{-1}(\mathcal{V}_k \cap \mathcal{V}_k').\]

Due to [4], we know that each \( \text{Exp}_{uu'}|_{\mathcal{V}_k} \) for all \( k \in \mathbb{N} \) is of class \( C^\infty \) and \( \text{Exp}_{uu'} \) is obviously scale continuous. Thus, applying Theorem 3.3, we prove that \( \text{Exp}_{uu'} \) is scale smooth for all \( u, u' \in C^\infty(N, M) \) and hence the proposition. \( \square \)

Proof of Theorem A According to Theorem 3.1, \( \mathcal{X}(N, E) \) is scale isomorphic to \( \ell^{2,f} \) for \( f(\mu) = \lambda_\mu \), \( \mu \in \mathbb{N} \). As \( \mu \to \infty \), \( \lambda_\mu \) converges asymptotically to \( \mu^{2/n} \) due to Theorem 2.1. Therefore \( \mathcal{X}(N, E) \) is scale isomorphic to \( \ell^{2,f} \) for \( f(\mu) = \mu^{2/n} \), \( \mu \in \mathbb{N} \). \( \square \)

Proof of Corollary A The proof follows from Theorem A and Proposition 3.4. \( \square \)

In what follows, we shall define an operation on \( \mathcal{F} \) to study fractal scale structures of product mapping spaces. See the introduction for the definitions of sets \( \mathcal{F} \) and \( \mathcal{F}^\prime \). The \(*\)-operation between \( f, h \in \mathcal{F}^\prime \) is defined by

\[
\begin{align*}
f \ast h(1) &= \min\{f(\mu), h(\mu) \mid \mu \in \mathbb{N}\} \\
f \ast h(2) &= \min\{\{f(\mu), h(\mu) \mid \mu \in \mathbb{N}\} \setminus \{f \ast h(1)\}\} \\
&\vdots \\
f \ast h(i) &= \min\{\{f(\mu), h(\mu) \mid \mu \in \mathbb{N}\} \setminus \{f \ast h(1), \ldots, f \ast h(i-1)\}\} \quad \vdots
\end{align*}
\]

(3.1)
Lemma 3.5 Let \( f, f' \in \mathcal{F} \) satisfy \( f \sim f' \). Then there exists a constant \( c > 0 \) such that for every \( h \in \mathcal{F} \),
\[
\frac{1}{c} (f' \ast h)(\mu) \leq f \ast h(\mu) \leq c (f' \ast h)(\mu)
\]
for all \( \mu \in \mathbb{N} \).

Proof Since \( f \sim f' \), there exists \( c > 0 \) such that
\[
\frac{1}{c} f'(\mu) \leq f(\mu) \leq c f'(\mu), \quad \mu \in \mathbb{N}.
\]
We claim that the assertion holds for this \( c > 0 \). Assume on the contrary that \( f \ast h(\eta) > c (f' \ast h)(\eta) \) for some \( \eta \in \mathbb{N} \). The only nontrivial case is as follows: \( r, s \in \mathbb{N} \),
\[
\{ f \ast h(1), \ldots, f \ast h(\eta) \} = \{ f(1), \ldots, f(r), h(1), \ldots h(\eta - r) \},
\]
\[
\{ f' \ast h(1), \ldots, f' \ast h(\eta) \} = \{ f'(1), \ldots, f'(s), h(1), \ldots h(\eta - s) \}.
\]
If \( f \ast h(\eta) = f(r) \), by assumption, \( f(r) > c (f' \ast h)(\eta) \geq c f'(s) \) and thus \( r > s \). This implies that \( \eta - r < \eta - s \) and \( h(\eta - r) \leq h(\eta - s) \). But then \( f(r) \leq h(\eta - s) \) and this leads to a contradiction
\[
f \ast h(\eta) = f(r) \leq h(\eta - s) \leq f' \ast h(\eta).
\]
Suppose that \( f \ast h(\eta) = h(\eta - r) \), then by assumption, \( h(\eta - r) > c (f' \ast h)(\eta) \geq ch(\eta - s) \). Thus \( r < s \) and \( f(s) \geq h(\eta - r) \). But then we get
\[
f \ast h(\eta) = h(\eta - r) \leq f(s) \leq c f'(s) \leq c (f' \ast h)(\eta)
\]
which contradicts our assumption. We have proved that \( f \ast h(\mu) \leq c (f' \ast h)(\mu) \) for all \( \mu \in \mathbb{N} \). In a similar way, one can prove \( f \ast h(\mu) > \frac{1}{c} (f' \ast h)(\mu) \) for all \( \mu \in \mathbb{N} \) and this completes the proof. \( \square \)

This lemma yields that the \( \ast \)-operation descends to \( \mathcal{F} \):
\[
[f] \ast [h] := [f \ast h].
\]

This operation is commutative and associative. We endow a partial order on \( \mathcal{F} \) as follows: \([f_1] \leq [f_2] \) if there is \( c > 0 \) such that \( f_1(\mu) \leq cf_2(\mu) \) for all \( \mu \in \mathbb{N} \). Then \( \ast \)-operation preserves this partial order, i.e. \([f_1] \ast [h] \leq [f_2] \ast [h] \) if \([f_1] \leq [f_2] \). Moreover it holds that \([f] \ast [h] \leq [h] \) for any \([f] \in \mathcal{F} \). If we allow \( \mathcal{F} \) to include an element \( e(\mu) = \infty \) for all \( \mu \in \mathbb{N} \), then \( (\mathcal{F}, \ast) \) becomes a partially ordered commutative monoid with the identity element \( e \).

Proposition 3.6 An element \([f] \in \mathcal{F} \) which can be represented by a polynomial is an idempotent element with respect to the \( \ast \)-operation.

Proof There is no loss of generality in assuming that \( f(\mu) = \mu^k \). We note that
\[
f \ast f(\mu) = f\left( \left\lfloor \frac{\mu - 1}{2} \right\rfloor + 1 \right)
\]
and thus we have
\[
\left( \frac{\mu}{2} \right)^k \leq f \ast f(\mu) \leq \left( \frac{\mu + 1}{2} \right)^k.
\]
Since \( \mu \in \mathbb{N} \), \((4\mu)^k \geq (\mu + 1)^k \), and hence
\[
\left( \frac{1}{2} \right)^k f(\mu) = \left( \frac{1}{2} \right)^k \mu^k \leq f \ast f(\mu) \leq 2^k \mu^k = 2^k f(\mu).
\]
This implies that $[f] * [f] = [f]$ in $\mathcal{F}$ and thus the proposition is proved.

\[\Box\]

**Lemma 3.7** A product of fractal scale Hilbert spaces is fractal again.

**Proof** It suffices to show that the product of $\ell^2,f_1$ and $\ell^2,f_2$, $f_1, f_2 \in \mathcal{F}$, is scale isomorphic to $\ell^2,h$ for some $h \in \mathcal{F}$. An element in $(\ell^2,f_1 \oplus_{sc} \ell^2,f_2)_k = \ell^2_{f_1} \oplus \ell^2_{f_2}$ is a sequence $(x, y) : \mathbb{N} \to \mathbb{R} \times \mathbb{R}$ such that

$$\sum_{\mu=1}^{\infty} f_1^j(\mu)x_\mu^2 + f_2^j(\mu)y_\mu^2 < \infty, \quad x = (x_\mu)_{\mu \in \mathbb{N}}, \quad y = (y_\mu)_{\mu \in \mathbb{N}}.$$ 

Then the following map is a scale isomorphism by definition of $*$-operation.

$$\ell^2,f_1 \oplus_{sc} \ell^2,f_2 \longrightarrow \ell^2,f_1 * f_2$$

$$(x, y) \longmapsto z$$

where $z_\mu := x_j$ resp. $z_j$ if $f_1 * f_2(\mu) = f_1(j)$ resp. $f_2(j)$ for $j \in \mathbb{N}$.

\[\Box\]

**Proof of Theorem B** Due to Theorem 3.1, there exist a scale isomorphism

$$\mathcal{X}(N_1, E_1) \oplus_{sc} \mathcal{X}(N_2, E_2) \cong \ell^2,f_1 \oplus_{sc} \ell^2,f_2.$$ 

for $f_i(\mu) = \lambda_i^\mu$ where $\{\lambda_i^\mu\}_{\mu \in \mathbb{N}}$ is the spectrum of the Bochner Laplacians on $\Gamma(N_i, E_i)$ for $i \in \{1, 2\}$. Then Lemma 3.7 yields that

$$\mathcal{X}(N_1, E_1) \oplus_{sc} \mathcal{X}(N_2, E_2) \cong \ell^2,f_1 \oplus_{sc} \ell^2,f_2 \cong \ell^2,f_1 * f_2.$$ 

In addition, due to Theorem 2.1, $f_1(\mu) \sim \mu^{2/n_1}$ and $f_2(\mu) \sim \mu^{2/n_2}$ as $\mu \to \infty$ where $n_1 = \text{dim } N_1$ and $n_2 = \text{dim } N_2$. Let us assume that $n_1 \leq n_2$, i.e. $[f_1] \geq [f_2]$. Since $[f_2] * [f_2] = [f_2]$ by Proposition 3.6, we have

$$[f_2] = [f_2] * [f_2] \leq [f_1] * [f_2] \leq [f_2].$$

This shows that $\ell^2,f_1 * f_2$ and $\ell^2,f_2$ are scale isomorphic and hence the theorem is proved:

$$\mathcal{X}(N_1, E_1) \oplus_{sc} \mathcal{X}(N_2, E_2) \cong \ell^2,f_1 \oplus_{sc} \ell^2,f_2 \cong \ell^2,f_1 * f_2 \cong \ell^2,f_2 \cong \mathcal{X}(N_2, E_2).$$

\[\Box\]

## 4 Relative mapping spaces

This section is devoted to study relative mapping spaces. Let $N$ be an $n$-dimensional compact manifold with nonempty boundary $\partial N$. In the presence of boundary, most part of spectral theory continues to work with nice boundary conditions. Here we consider the mixed boundary condition which generalizes both Dirichlet and Neumann boundary conditions. Let $L_p$'s be submanifolds in a manifold $M$ parametrized by $p \in \partial N$ and set $L := \bigsqcup_{p \in \partial N} L_p$. We are interested in relative mapping spaces of the following form:

$$\text{Map}((N, \partial N), (M, L)) = \{(W_d^{k + k_0, 2}, \langle \cdot, \cdot \rangle_{W_d^{k + k_0, 2}})\}_{k \in \mathbb{N}_0}$$

where $k_0$ is the smallest natural number satisfying $2k_0 > n$ as before and $W_d^{k,2}$'s are Hilbert manifolds given by

$$W_d^{k, 2} := \{u \in W^{k,2}(N, M) \mid u(p) \in L_p, \partial \nu u(p) \in N_{u(p)}L_p, \ p \in \partial N\}.$$  

\[\Box\ Springer\]
Here $N_{u(p)}L_p$ is the normal bundle of $L_p \subset M$ at $u(p)$ and $v$ stands for the outward pointing unit normal vector field of $N$ at $\partial N$. This scale Hilbert manifold is modeled on the following scale Hilbert space:

$$\mathcal{X}(N, \partial N), (u^*TM, u^*TL) = \left\{ \left( \Gamma^{k+k_0,2}_\partial, \langle \cdot, \cdot \rangle_{W^{k+k_0,2}} \right) \right\}_{k \in \mathbb{N}_0}$$

where Hilbert spaces $\Gamma^{k,2}_\partial$'s are given by

$$\Gamma^{k,2}_\partial := \left\{ \phi \in \Gamma^{k,2}(N, u^*TM) \mid \phi(p) \in T_{u(p)}L_p, \partial_v \phi(p) \in N_{u(p)}L_p, \ p \in \partial N \right\}.$$ This kind of boundary condition is said to be the mixed boundary condition; $\phi$ satisfies the Dirichlet boundary condition on $N_{u(p)}L_p$ and the Neumann boundary condition on $T_{u(p)}L_p$, i.e.

$$\phi(p)|_{N_{u(p)}L_p} = 0 \quad \& \quad \partial_v \phi(p)|_{T_{u(p)}L_p} = 0, \quad p \in \partial N. \quad (4.2)$$

Of course $L_p$'s can be a single submanifold, i.e. $L_p = L_q$ for all $p, q \in \partial N$. But in Floer theory, boundary points often map to different Lagrangian submanifolds, see [5] for Lagrangian Floer homology and see the end of the first section in [12] for Hyperkähler Floer homology with the Lagrangian boundary condition.

When we prove Theorem A, Theorem 2.1 and Theorem 2.5 played crucial roles. Corresponding theorems go through for relative mapping spaces under the mixed boundary condition.

**Theorem 4.1** Let $\Delta$ be the Bochner Laplacian on $\bigcap_{k \in \mathbb{N}} \Gamma^{k,2}_\partial$. Then the followings hold:

(i) $\Delta$ is self-adjoint with respect to $L^2$-metric.

(ii) There exists a discrete spectral resolution of $\Delta$ for $\Gamma^{0,2}_\partial, \{ \phi_\mu, \lambda_\mu \}_{\mu \in \mathbb{N}}$.

(iii) There are only finitely many non-positive eigenvalues and $\lambda_\mu \sim C\mu^{2/n}$ for some constant $C > 0$ as $\mu \to \infty$.

**Proof** See [9, Chapter 1] with [11] or [10, Theorem 2.8.4].

**Theorem 4.2** Let $(N, \partial N)$ be a compact manifold with nonempty boundary. There exists a constant $c > 0$ such that for $u \in C^\infty(N)$ with either $u|_{\partial N} \equiv 0$ or $\partial_v u|_{\partial N} \equiv 0$,

$$||u||_{W^{k+2,2}(N)} \leq c(||\Delta_0 u||_{W^{k,2}(N)} + ||u||_{W^{k,2}(N)}).$$

**Proof** The proof can be found in [16] and [21].

**Proof of Corollary B** Making use of the above two theorems together with the Bochner–Weitzenböck formula (Theorem 2.4), the corollary is proved by following through the arguments of Sect. 3.

**Lagrangian Floer homology** In this subsection, we justify that the boundary condition described above is reasonable for Lagrangian Floer theory. Here we only consider the simplest case and refer to [5,19] for a more general set-up. Let $I$ be an interval $[0, 1]$ and let $\omega_{\mathbb{C}^n}$ be the standard symplectic structure on $(\mathbb{C}^n, i)$ with the compatible metric $g(\cdot, \cdot) = \omega_{\mathbb{C}^n}(\cdot, i \cdot)$. The two-form $\omega_{\mathbb{C}^n}$ is exact, i.e. $\omega_{\mathbb{C}^n} = d\lambda$ for some 1-form $\lambda$ on $\mathbb{C}^n$, and $L = \mathbb{R}^n \times \{0\}$ is a Lagrangian submanifold. We consider the space of $W^{k,2}$-paths, $k \in \mathbb{N}$, satisfying the Lagrangian boundary condition.

$$\Omega^k(L : \mathbb{C}^n) := \left\{ \gamma \in W^{k,2}(I, \mathbb{C}^n) \mid \gamma(p), i\partial_I \gamma(p) \in L = T_{\gamma(p)}L, \ p \in \{0, 1\} \right\}.$$

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This space carries the following action functional.

\[ A : \Omega^1(L : \mathbb{C}^n) \rightarrow \mathbb{R}, \quad A(\gamma) := \int_L \gamma^* \lambda. \]

Next we consider the $W^{k,2}$-tangent bundle, $k \in \mathbb{N}_0$, along $\gamma \in \Omega^1(L : \mathbb{C}^n)$:

\[ \Omega^k_\gamma = \Omega^k_\gamma(L : \mathbb{C}^n) := \{ \xi \in W^{k,2}(I, \gamma^* T\mathbb{C}^n) \mid \xi(p), i\partial_t \xi(p) \in L = T\gamma(p)L, \ p \in \{0, 1\} \}. \]

This boundary condition yields that $\xi$ satisfies the Dirichlet boundary condition on the second $\mathbb{R}^n$ factor of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{C}^n$ and the Neumann boundary condition on the first $\mathbb{R}^n$ ($= L$). A direct computation shows that the Hessian of $A$ at $\gamma$ is given by

\[ \mathcal{H}_\gamma : \Omega^{k+1}_\gamma \rightarrow \Omega^k_\gamma, \quad \xi \mapsto i\partial_t \xi. \]

The boundary condition we imposed is necessary for the Hessian $\mathcal{H}_\gamma$ to be well-defined:

\[ \mathcal{H}_\gamma[\xi](p) = i\partial_t \xi(p) \in L. \]

5 Appendix: Some remarks on the local invariant

As we have observed in the introduction, the invariant $\mathcal{R}$ has a simple formula for fractal scale Hilbert spaces, namely

\[ \mathcal{R}([\ell^{2, f}])(i, j) = [f^{j-i}]. \]

Thus the growth types of fractal functions $f : \mathbb{N} \rightarrow (0, \infty)$ determine the (local) invariant for fractal scale Hilbert spaces (or manifolds). In Theorem A, we gave a complete description of the local invariant $\mathcal{R}$ for mapping spaces $\text{Map}(N, M)$:

\[ \mathcal{R}([\text{Map}(N, M)])(i, j) = [\mu^{2(j-i)/\dim N}], \quad \mu \in \mathbb{N}. \]

In this appendix, we construct mapping spaces which are fractal scale Hilbert spaces and whose fractal functions have different growth types from $\text{Map}(N, M)$ we have considered. Thus we provide concrete examples of fractal scale Hilbert spaces with a variety of the invariant formulas.

For a scale Hilbert space $\mathcal{H}$ which is scale isomorphic to $\ell^{2, f}$, we set

\[ \mathcal{H}[j] := \{(H_{jk}, \langle \cdot, \cdot \rangle_{jk})\}_{k \in \mathbb{N}_0}, \quad j \in \mathbb{N}. \]

Then $\mathcal{H}[j]$ is a scale Hilbert subspace of $\mathcal{H}$ that is scale isomorphic to $\ell^{2, f^j}$. From this simple observation, we can easily construct mapping spaces with various polynomial growth types; for example,

\[ \text{Map}(N, \mathbb{R})[j] = \{(W^{j,2}_{jk}(N, \mathbb{R}), \langle \cdot, \cdot \rangle_{W^{j,2}_{jk}})\}_{k \in \mathbb{N}_0} \cong \ell^{2, f}, \quad f(\mu) = \mu^{2j/\dim N}. \]

In this example, the growth type of fractal functions of mapping spaces is determined by the growth type of eigenvalues of the Laplace–Beltrami operator; since an elliptic operator of Laplace type is of order two, $\text{Map}(N, \mathbb{R})$ (or $\text{Map}(N, M)$ in general) has the growth type $f(\mu) = \mu^{2j/\dim N}$. Thus using the following theorem, we can construct mapping spaces whose fractal functions are of arbitrary polynomial growth types.
Theorem 5.1 Let $P$ be an elliptic self-adjoint operator on $\Gamma(N, E)$ of order $d > 0$. Then the following holds:

(i) There exists a discrete spectral resolution of $P$ for $\Gamma^2(N, E)$, $\{\phi_\mu, \lambda_\mu\}_{\mu \in \mathbb{N}}$.

(ii) There are only finitely many non-positive eigenvalues and $\lambda_\mu \sim C\mu^{d/n}$ for some constant $C > 0$ as $\mu \to \infty$.

Proof The proof can be found in [8, Chapter 1]

Using this theorem, we deduce that scale structures (and hence the invariant) on the spaces of sections of the following form are determined by the order of elliptic self-adjoint operators $P$ and the dimension of domain manifolds $N$:

$$X_P(N, E) := \left\{ \left( \Gamma^k_P(N, E), \langle \cdot, \cdot \rangle_{P, k, 2} \right) \right\}_{k \in \mathbb{N}_0}$$

where each level and metric are given by

$$\Gamma^k_P(N, E) := \{ \phi \in \Gamma^2(N, E) \mid P^j\phi \in \Gamma^2(N, E), 1 \leq j \leq k \},$$

$$\langle \phi, \psi \rangle_{P, k, 2} := \sum_{j=0}^{k} \langle P^j\phi, P^j\psi \rangle_{L^2}.$$

Following through the argument of the previous sections, we can prove that

$$X_P(N, E) \cong \ell^2, f, \quad f(\mu) = \mu^{\text{ord} P / \dim N},$$

and thus the invariant is

$$\mathcal{R}(\{X_P(N, E)\})(i, j) = [\mu^{\text{ord} P(j-i) / \dim N}].$$

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