Large deviations of glassy effective potentials

Silvio Franz\textsuperscript{1,2} and Jacopo Rocchi\textsuperscript{1,*}

\textsuperscript{1} LPTMS, Università Paris-Sud 11, UMR 8626 CNRS, Bât. 100, 91405 Orsay Cedex, France
\textsuperscript{2} Dipartimento di Fisica Università, La Sapienza, Piazzale Aldo Moro 5, I-00185 Roma, Italy

E-mail: jacoporocchi@yahoo.it

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Abstract
The theory of glassy fluctuations can be formulated in terms of disordered effective potentials. While the properties of the average potentials are well understood, the study of the fluctuations has been so far quite limited. Close to the MCT transition, fluctuations induced by the dynamical heterogeneities in supercooled liquids can be described by a cubic field theory in presence of a random field term. In this paper, we set up the general problem of the large deviations going beyond the assumption of the vicinity to $T_{\text{MCT}}$ and analyze it in the paradigmatic case of spherical ($p$-spin) glass models. This tool can be applied to study the probability of the observation of dynamic trajectories with memory of the initial condition in regimes where, typically, the correlation $C(t, 0)$ decays to zero at long times, at finite $T$ and at $T = 0$.

Keywords: large deviations, random first order transition, glassy systems, effective potentials

(Some figures may appear in colour only in the online journal)

1. Introduction

The last two decades of research have underlined the deep role of space-time fluctuations, a.k.a. dynamical heterogeneities, in the relaxational dynamics of glassy systems [1]. Unfortunately, despite many efforts [2–7], an accomplished theory of such fluctuations is still lacking. In [8], it was proposed that relevant information could be gained renouncing to describe the temporal dimension. Under a local equilibrium hypothesis, it was proposed to use an effective field theory where the role of the order parameter is played by the space dependent overlap of the actual system with a random thermalized configuration. The construction provided a field theoretical extension of the effective potential often used to describe the glass transition in terms...
of a Landau-like theory. In that paper, it was developed a theory of critical fluctuations close
to the putative mode coupling (MC) transition temperature $T_{\text{MC}}$, where MC theory predicts a
growing dynamical length, see [9] for a review. Under the strong hypothesis that any activated
processes could be neglected, a disordered effective field theory emerged, identical to that of
the spinodal point of a random field Ising model. The appearance of disorder in the description
provided a foundation to the idea of ‘self-induced disorder’, a concept often advocated
to rationalize the similarity between structural glasses and spin glasses [10–18]. Similarly, an
effective disorder was later found in the perturbative description of phase diagram of coupled
glassy systems and systems where a fraction of the particles are frozen in random positions
[19–21]. It is clear that to study the system away from -true or approximate- glassy critical
points the analysis of fluctuations has to be extended to a non-perturbative level. In two recent
papers, a brave attempt to describe the self-induced disorder at the non-perturbative level was
undertaken. Unfortunately, the analysis is based on several approximations whose range of
validity is difficult to assess [22, 23].

We study the problem of non-perturbative fluctuations in the class of long-range spin glass
models such as the spherical $p$-spin models in the pure and the mixed versions. These mod-
els are at the basis of the random first order transition (ROFT), see [24] for a review. The
principal tool of our analysis is the effective potential function, defined as the large deviation
function of the overlap probability density function (PDF) from a random reference configura-
tion chosen with the Boltzmann–Gibbs probability. In the thermodynamic limit, this function
is self-averaging with respect to the choice of the reference configuration. Its behaviour reflects
the properties of the Gibbs measure, with a single minimum in the ergodic phase and two min-
ima in glassy phases. For large but finite volumes, however, the effective potential fluctuates
from reference to reference and its large fluctuations are themselves described by a large devi-
ation principle. Ideally we would like to study the fluctuation of a function. While theoretically
conceivable, this is unfortunately a technically formidable task. As a first step, we study the
fluctuations of neighboring pairs of points of the function. This allows in particular to study the
probability that the glassy effective potential has a local minimum in regions where on average
it is either increasing or decreasing. In particular this gives the probability of the existence of
the secondary minimum in the high temperature ergodic region. In terms of the dynamics, in the
infinite size limit, the long time limit of the correlation with the initial condition $\lim_{t\to\infty} C(t,0)$
decays to zero at large temperatures. On the other hand, in finite systems, as well as in numer-
ical simulations, this may not be always true. The tools developed in the present paper allow
to study the probability to observe a dynamics that does not loose memory of the initial con-
figuration and to describe the features of such atypical dynamics. After a short introduction of
the glassy effective potential in section 2, we review the theory of fluctuations developed in
[8] and we describe how to extend it far from the MCT transition in section 3. We then discuss
the results at finite and at zero temperature in section 4 and show that they can be interpreted
in the light of the state following [17, 25, 26], an introduction of which is provided in the text.
Finally we discuss the implication that this study has to the recent analysis performed in [27].
The more technical aspects of the computation are provided in the appendix.

2. Effective potential: a short introduction

A good starting point to study fluctuations in glasses is provided by the glassy effective poten-
tial and its field theoretical generalizations, which have been described several times in the
literature. Given a system specified by all its coordinates $X$ and a Hamiltonian $H(X)$, and some
notion of local similarity -or overlap- between configurations $q_r(X,Y)$, one defines [28]
\[ V_N(\{q(x)\}|X) = \frac{T}{N}(\log Z_N(\{q(x)\}|X) - \log Z_N), \]  
(1)

\[ Z_N(\{q(x)\}|X) = \sum_{Y} e^{-\beta H[Y]} \delta_{q(x) = q(Y)}, \]  
(2)

where \( Z_N \) is the unconstrained partition function \( Z_N = \sum_{Y} e^{-\beta H[Y]} / \beta \) the inverse temperature, \( x \) a spatial coordinate. The condition on the overlap is imposed on every \( x \). For reasonable choices of the configuration \( X \), this is an effective field theoretical action for the overlap field \( q(x) \). The configuration \( X \) is chosen randomly from an equilibrium measure at the same or at a different temperature \( T' \) from the one appearing in (1), \( V(\{q(x)\}|X) \) is then a random object. The \( X \)-average of \( V \) has been well studied in mean-field models and numerical simulations [29–32]. A few papers have studied fluctuations [33, 34]. In atomistic glass forming liquids, large fluctuations of the overlap emerge as temperature decreases, consistent with the existence of the random critical point that is predicted by effective field theories. From its form one can get important insights of glassy behaviour; for example, point-to-set correlation functions of a set \( B \) [35, 36], can be obtained fixing \( q(x) = 1 \) outside the set \( B \), averaging over \( X \) and studying the overlap distribution inside \( B \) [37]. As we said, besides the mean, the fluctuations provide crucial information on the nature of glassy phases.

A systematic approach starting from mean-field, and including the space dimension in a controlled expansion seems desirable. As a preliminary step we study the finite volume fluctuations of the glassy effective potential in mean field in the large deviation regime. To keep the level of difficulty to the minimum, we use the spherical \( p \)-spin model [11, 38, 39], where metastable states can be studied with the TAP method [40, 41] and replicas give rise to closed equations. The fully connected spherical (mixed) \( p \)-spin model, is defined by the Hamiltonian

\[ H[\sigma] = \sum_{p} a_p \mathcal{J}_p, \]  
(3)

where the \( \mathcal{J}_p \) are independent Gaussian random variable with mean \( \mathcal{J}_p = 0 \) and variance \( \mathcal{J}_p^2 = (2p/(2N^{p-1}) \) and \( a_p > 0 \) coefficient such that given two spin configurations \( \sigma \) and \( \tau \), denoting \( q_{\sigma,\tau} = \sum_{i} \sigma_i \tau_i \) their overlap,

\[ H[\sigma][H[\tau]] = N f(q_{\sigma,\tau}) = \frac{1}{2} \sum_{p} a_p q_{\sigma,\tau}^p. \]  
(4)

Here and in the rest of the paper, the average over quenched disorder is indicated by an overline. The \( N \) spins \( \sigma_i \) are continuous variables constrained to be on the sphere \( \sum_{i=1}^{N} \sigma_i^2 = N \). The pure \( p \)-spin model corresponds to a single non vanishing coefficient \( a_p \), while in mixed models, more than one coefficient is non-zero. It can be studied within the framework of replicas and it may be described in terms of the so called one step replica symmetry breaking (RSB) transition [42]. Moreover, its Langevin dynamics was shown to display the interesting behaviour known as aging and weak ergodicity breaking [43, 44].

As described many times [28, 45–48], see [49–51] for reviews on the topic, while at large temperature the system is in the paramagnetic phase, an ergodicity breaking transition occurs at the dynamical temperature \( T_d \). The paramagnetic state disappears and the Boltzmann measure is replaced by an exponential number of equilibrium states, whose number \( e^{N\Sigma} \) is exponential in the system size and \( \Sigma \) is the complexity, or configurational entropy and zero overlap among each other. As mentioned above, this behavior can be interpreted in terms of the broader perspective of the RFOT [24]. The dynamics of this model is described by a set of equations.
formally identical to those obtained by MCT (mode coupling theory) for liquids, reproducing
the two steps relaxation behaviour of the correlation function and the dynamical slowing down
at \( T_{MCT} \) [39].

Since this model lacks any notion of space, the definition (1) reduces to the global effective potential

\[
V_n(p|\tau) = -\frac{T}{N} \log Z[p|\tau] - \log Z_n,
\]

\[
Z[p|\tau] = \int D\sigma e^{-\beta H(x)} \delta(q_{\sigma,\tau} - p),
\]

where \( D\sigma \) denotes the uniform measure on the sphere. In the rest of the paper, unless stated otherwise, we drop \( N \) from \( V_n(p|\tau) \) to ease the notation. We observe that the (\( \tau \) dependent) potential can be written as \( V(p|\tau) = F(p|\tau) - F \), where \( F(p|\tau) \) is the (\( \tau \) dependent) constrained free energy. We concentrate in this paper to the case where the configuration \( \tau \) is chosen from the equilibrium distribution at temperature \( T' \) larger than the Kauzmann transition temperature \( T_K \), where fluctuations with respect to the couplings are very small and not only the free-energy, but the partition function \( Z_n \) has the self-averaging property and can be computed in the annealed approximation. This property allows to introduce replicas only to deal with the logarithm of the partition function, avoiding the introduction of another set of replicas to compute the inverse of \( Z_n \). Using the relation \( \log x = \lim_{n \to 0} \partial_n x^n \), the constrained free energy reads

\[
F(p) = \bar{E}_\tau F(p|\tau) = -\frac{1}{\beta N} \lim_{n \to 0} \partial_n (Z[p|\tau])^n,
\]

where we get the limit from the analytic continuation from integer values of \( n \). Replica formalism used in the present work is outlined in the appendix, see appendix A.1. Here and in the rest of the paper, the average over the first replica is indicated by \( \bar{E}_\tau \). The typical potential \( \bar{E}_\tau V(p|\tau) = V(p) = F(p) - F \) depends on the temperatures. For \( T' = T \), at high temperature, \( T > T_d \) the potential has a single minimum at \( p = 0 \). Below \( T_d \), one sees a characteristic two minima structure with a secondary minimum at a high value of \( p = q_{EA} \), where \( q_{EA} \) is the Edwards–Anderson parameter, that signals ergodicity breaking in an exponential number of states. Similarly to the Landau phenomenological theory of the second order phase transitions, the effective potential allows to describe the glassy transition. The difference between minima, i.e. the configurational entropy times the temperature, tends to zero at the temperature of the static phase transition \( T_K \) and remains zero below.

As illustrated in figure 1, the interval \( p \in [0, 1] \) is divided into three regions [25, 52]. This separation is due to the presence of a \( p \neq 0 \) and refers to the constrained measure, the unconstrained one being at \( p = 0 \). Their physical interpretation is the following: while at small \( p \) the second replica feels a small constraint due to the first one and it can explore an exponential number of states in the dynamical phase, this is not true anymore when \( p \) is increased and \( p > p_K \). For \( p < p_K \) the complexity is larger than zero while for \( p > p_K \) the system can explore only a sub-exponential number of states. Nevertheless, when the constraint is too strong, i.e. when \( p \) is large, the second replica is forced to stay in the same state as the first one. While the first regime is called dynamical RSB, the last is replica symmetric (RS) From now on we denote by \( p_{RS}(T) \) the point at the frontier between these two regions. The stability of the RS solution can be studied from the analysis of the modes associated to a small perturbation around it. As long as the replicon mode is positive, the RS solution is stable. At \( p_{RS} \), the replicon mode is zero, \( \beta^2 f''(q)(1 - q)^2 = 1 \), with \( q \) being the overlap between replicas in the
Figure 1. $V(p)$ for the 3−spin at $\beta = 1.66$, larger than $\beta_d$, computed with the 1RSB ansatz (blue line) and the complexity $\Sigma(p)$ (orange dashed) as a function of $p$. The RS potential is also plotted for comparison (red dotted). Vertical lines divide three regions: from left to right dynamic 1RSB, static 1RSB, RS.

RS scheme discussed in appendix A.1. For large enough temperatures, $\beta < \beta_{RS}$, this equation has no real solutions, the replica symmetric instability disappears and the potential is always replica symmetric. On the other hand, when $\beta > \beta_{RS}$ the replica symmetric ansatz does not estimate the potential correctly at intermediate values of $p$ and for $\beta > \beta_d$ spurious stationary points appear, see figure 1. For $\beta < \beta_d$, besides the 3 regions discussed above and observable in figure 1, there is a preliminary replica symmetric region, for small values of $p$, as discussed more in details in the appendix, see appendix A.2.

The profile of the configurational entropy and of the potential in the dynamical phase is given in figure 1. It is possible to interpret the potential in terms of the PDF of the overlap. In fact, the secondary minimum corresponds to the unlikely event that the second replica is extracted from the same state of the first one. In this case, the overlap between these two replicas would be that of the state where the first replica is. Since the first replica is sampled at equilibrium, the overlap of this state is $q_{EA}$. Moreover, since the typical overlap between different equilibrium states is zero, one finds that the global minimum continues to be at $p = 0$.

In the thermodynamic limit, $V$ is self-averaging, both with respect to the extraction of $\tau$ and the quenched couplings $J$ of the model. However, for finite $N$, fluctuations are present. It was shown in [8] that thanks to the self-averaging of $Z_w$, the fluctuations with respect to the $J$’s are much weaker than those with respect to $\tau$ (this is one of the reasons why $p$-spin models are good models of structural glasses). In the rest of the paper, unless stated otherwise, we then concentrate in the study of the large deviations with respect to the reference configuration $\tau$.

3. Fluctuations

In this section we review the method of the effective potential to describe fluctuations in glasses. Moreover, unless stated otherwise, $T' = T$. Leaving aside the difficulties arising with the mixed model in the $T' \neq T$ case [17, 25–27] that we will discuss afterwards, the potential can be interpreted dynamically. The existence of a unique minimum is associated with ergodic behavior. If we consider relaxation dynamics with the reference replica as initial condition, the system
will evolve till it will have zero correlations with the initial configuration. Conversely, if there are two minima, the system is not ergodic and it will remain confined in the vicinity of the initial state, at the value of the overlap of the secondary minimum. The configuration space is split into ergodic components and the difference between the two minima just measures the configurational entropy (multiplied by temperature) of the ergodic components. For large $N$ the secondary minimum appears at a well defined temperature $T_d$, signaling a sharp breaking of ergodicity. In this paper we are concerned with the finite $N$ fluctuations and we ask what is the probability of initial conditions capable to confine the system for an exponentially large time, even in the paramagnetic phase. Namely, we want to compute the probability that the potential has a minimum for $T > T_d$. This is a large deviation regime where the probability is exponentially small in $N$ and we are interested in the rate function $I(T)$. We will use two strategies. The first one, valid close to $T_d$, both from below and from above, is based on the perturbative theory of glassy fluctuations developed in [8]. The second uses full fledged large deviation analysis and is valid for all $T > T_d$.

3.1. Small fluctuations

In this section we consider the temperature $T$ to be close to $T_d$. The central quantity we need is the covariance of the potential function for two different values of the overlap, with respect to the extraction of the first replica

$$W^{(2)}_{\text{het}} = \mathbb{E}_\tau \left( V(p_1|\tau) V(p_2|\tau) \right) - \mathbb{E}_\tau \left( V(p_1|\tau) \right) \mathbb{E}_\tau \left( V(p_2|\tau) \right).$$

Being small fluctuations Gaussian, this quantity specifies completely their statistics. We will not consider other sources of fluctuations. In fact, disorder or sample to sample fluctuations

$$W^{(2)}_{\text{dis}} = \mathbb{E}_\tau \left( V(p_1|\tau) \right) \mathbb{E}_\tau \left( V(p_2|\tau) \right) - \mathbb{E}_\tau \left( V(p_1|\tau) \right) \mathbb{E}_\tau \left( V(p_2|\tau) \right)$$

can be shown to be subdominant with respect to fluctuations in $\tau$ [8]. From now on, we drop the subscript ‘het’ and we denote equation (8) by $W^{(2)}_{p_1, p_2}$. Connected correlations are computed respect to the measure $\mathbb{E}_\tau$, e.g. $\mathbb{E}_\tau[AB] = \mathbb{E}_\tau AB - \mathbb{E}_\tau A \mathbb{E}_\tau B$, describing fluctuations with respect to the first replica. Eventually we will consider quantities averaged over the disorder, $\mathbb{E}_\tau[AB] = \mathbb{E}_\tau AB - \mathbb{E}_\tau A \mathbb{E}_\tau B$. We also denote by $\langle A \rangle = \mathbb{E}_\tau(A)$ and $\langle AB \rangle = \mathbb{E}_\tau(AB) - \mathbb{E}_\tau(A) \mathbb{E}_\tau(B)$. Defining $\delta V(p_1|\tau) = V(p_1|\tau) - V(p_1)$, the covariance $W^{(2)}_{\text{het}}$ at $p_1$ and $p_2$:

$$W^{(2)}_{p_1, p_2} = \mathbb{E}_\tau \left( \delta V(p_1|\tau) \delta V(p_2|\tau) \right)$$

can be computed within mean-field theory using replicas, as explained in the appendix, see appendix B.1. When $p_1$ and $p_2$ are close enough to $p_d$, location of the inflexion point that appears at $T_d$, one finds

$$NW^{(2)}_{\phi_1, \phi_2} = A + B(\phi_1 + \phi_2) + C\phi_1\phi_2,$$

where $\phi_1 = p_1 - p_d$ and $\phi_2 = p_2 - p_d$. For $T$ close enough to $T_d$, the potential has the linear behavior $V(p = p_d + \phi) = V(p_d) + \mu (T - T_d) \phi + O(\phi^2)$, with $\mu > 0$. The condition $\mu > 0$ insures that the secondary minimum disappears for $T > T_d$. On the other hand, equation (10) suggests that the $\tau$-dependent potential can be rewritten as the typical value $NV(p = p_d + \phi)$ plus a small correction given by

$$\sqrt{N} \delta V(p = p_d + \phi|\tau) = \eta \phi + \alpha,$$
where $\eta$ and $\alpha$ are correlated Gaussian random fields and, comparing with equation (10), $\langle \eta^2 \rangle = C$. While $\alpha$ represents a random correction to the value of the free energy, $\eta$ is a random temperature term. This result leads to the observation that the secondary minimum can exist even for $T > T_d$ if $\sqrt{N} \mu (T - T_d) + \eta = 0$. In other words, the random fluctuation $\eta$ allows the reappearance of the secondary minimum, even when typically it does not exists. The probability of this event can be written in terms of the rate function $I(T)$ computed in the small fluctuations regime,

$$e^{-NG_{\eta}(T)} = \int_{-\infty}^{\sqrt{N} \mu (T - T_d)} d\eta P(\eta)$$

where, denoting by $\sigma^2$ the variance of $\eta$,

$$I_{SF}(T) = \frac{1}{2} \frac{\mu^2 (T - T_d)^2}{\sigma^2}.$$ 

This result can be extended also for $T < T_d$, where typically the secondary minimum exists. In this case, random fluctuations play the opposite role, creating a saddle. The condition for this to happen is still $\sqrt{N} \mu (T - T_d) + \eta = 0$, where now $\mu < 0$.

This computation cannot be extended for a finite $|T - T_d|$ because there are no guarantees that the disorder induced by the initial condition for the dynamics can be described in terms of a Gaussian random term in a cubic field theory well above the dynamical temperature. In order to tackle this problem, we develop a novel technique, independent of this theory and based on a first principles computation. This technique is general and it may be applied in a general context, when one is interested in the computation of the rate function of the probability of the existence of stationary points in functionals depending on some source of randomness.

### 3.2. Large fluctuations

In this section we extend the results discussed previously by looking at the probability of the existence of a secondary minimum in the effective potential for an arbitrary $T$. This task requires controlling the values of the potential in multiple points. Our strategy is based on taking two points $p_1$ and $p_2$, distant $\delta p$, and considering the difference $V_2 - V_1 = V(p_2 | \tau) - V(p_1 | \tau)$. In the absence of stationary points, $V$ is locally linear and $V_2 - V_1$ of order $O(\delta p)$. This is the typical situation above $T_d$ and $p_1 \neq 0$. Conversely, close to a stationary point $V$ is quadratic. In order to describe the rare appearance of a stationary point in this regime we then look for the probability that $V_2 - V_1 = V(p_2 | \tau) - V(p_1 | \tau) \sim O(\delta p^2)$, which has a large deviation form.

We first introduce our notation and then we discuss the computation of the large deviation functional. Even if fluctuations induced by the quenched disorder are sub-dominant with respect to sample-to-sample fluctuations, to be as clear as possible we maintain the subscript $J$ in quantities that are formally dependent of the quenched disorder. We consider the probability $P(V_2 - V_1, p_1, p_2)$ that, given a configuration $\tau$, the difference between $V(p_2 | \tau)$ and $V(p_1 | \tau)$ is $V_2 - V_1$,

$$P(V_2 - V_1, p_1, p_2) = e^{-NG_{J}(V_2 - V_1, p_1, p_2)},$$

where $G_J$ is the rate function of this probability at given quenched disorder. By definition,

$$P(V_2 - V_1, p_1, p_2) = \mathbb{E}_\tau \delta \left[ V_2 - V_1 - (V(p_2 | \tau) - V(p_1 | \tau)) \right],$$

where $\mathbb{E}_\tau$ denotes the average over different realizations of the disorder $\tau$.

In order to compute the large deviations, we may use the technique of Section 3.2, which is based on the calculation of the rate function $I_{SF}(T)$ when the dynamics is quenched. In this case, the rate function $I_{SF}(T)$ is given by

$$I_{SF}(T) = \frac{1}{2} \frac{\mu^2 (T - T_d)^2}{\sigma^2},$$

where $\mu$ is a local field associated to the stationary point $V$. The rate function $I_{SF}(T)$ can be computed by solving the saddle point equation

$$\delta I_{SF}(T) = 0,$$
and using the exponential representation of the delta function and neglecting inessential factors we can write it in the form

\[ P(V_2 - V_1, p_1, p_2) = \int dm \ e^{-N\text{tr}V_2 - V_1} Z_j(m, p_1, p_2), \]  

(16)

where the last term can be written in terms of the generating function of the connected correlation functions \( \Gamma_j(m, p_1, p_2) \),

\[ Z_j(m, p_1, p_2) = \mathbb{E}_\tau e^{\text{tr}V_2 - V_1} = e^{N\Gamma_j(m, p_1, p_2)}. \]

(17)

Taking the average over \( J \), we finally obtain the average rate function

\[ G(V_2 - V_1, p_1, p_2) = \left( (V_2 - V_1)m^* - \Gamma(m^*, p_1, p_2) \right), \]

(18)

where \( \Gamma(m^*, p_1, p_2) = \Gamma_j(m^*, p_1, p_2) \) and the parameter \( m^* \) is the solution of the equation

\[ V_2 - V_1 = \frac{\partial \Gamma(m^*, p_1, p_2)}{\partial m}. \]

(19)

We notice in equation (18) that the rate function \( G(V_2-V_1, p_1, p_2) \) is the Legendre–Fenchel transform of \( \Gamma(m^*, p_1, p_2) \). \( G \) can be related to the large deviation function of the existence of the secondary minimum of the potential \( I \), as illustrated below. For the reasons discussed above, we set \( p_2 \) close to \( p_1, p_1 = p - \delta p/2, p_2 = p + \delta p/2 \). The absolute value of \( m^* \) grows as we require \( V_2 - V_1 \) to be far from its equilibrium value, being exactly 0 when this difference is chosen to be \( \langle V(p_2|\tau) - V(p_1|\tau) \rangle \). This could lead to the conclusion that it is sufficient to look at a low order truncation of the series of \( \Gamma \) in powers of \( m \), where successive terms appears to be of higher and higher order in \( \delta p \):

\[ \Gamma_j(m, p_1, p_2) = \sum_{k=1}^{\infty} \frac{m^k}{N_k!} \frac{\partial^k}{\partial m^k} \log Z_j(0, p_1, p_2) = \sum_{k=1}^{\infty} \frac{(mN)^k}{N_k!} \mathbb{E}_\tau \left( \langle V(p_2|\tau) - V(p_1|\tau) \rangle^k \right)_c, \]

(20)

where the subscript \( c \) indicates the connected component of the correlation function. Unfortunately, since we have to take a saddle point in \( m \), which therefore depends on \( \delta p \), the nominal order \( \delta p^k \) of the \( k \)th term in the expansion, does not coincide with the effective order. To understand this point let us first truncate the series to the second order in \( \delta p \),

\[ \Gamma_2 \left( m, p - \frac{\delta p}{2}, p + \frac{\delta p}{2} \right) = m \langle V'(p|\tau)\delta p \rangle + \frac{Nm^2}{2} \langle \langle V(p|\tau)\delta p \rangle^2 \rangle_c, \]

where \( \langle \rangle \) denotes the average over quenched disorder and the initial configuration. Optimizing equation (18) over \( m \) leads to

\[ m^* = \frac{(V_2 - V_1) - \langle V'(p_1|\tau) \rangle \delta p}{N \langle \langle V'(p|\tau) \rangle^2 \rangle_c \delta p^2}. \]

(21)

In the case of our interest, \( V_2 - V_1 = O(\delta p^3) \), leading to \( m^* = O(\delta p^{-1}) \), and the expansion cannot be truncated as all the terms are of the same order. If we blindly ignore this issue, the...
expression for $G$ to the second order is
\[ G_2 \left( V_2 - V_1, p - \frac{\delta p}{2}, p + \frac{\delta p}{2} \right) = \frac{1}{2N} \left( V_2 - V_1 - \langle V'(p) \rangle \right)^2. \] (22)

If on the other hand we set $V_2 - V_1$ exactly equal to zero, we get an expression similar to equation (13). Let us denote by $G_2(p)$ the zero order term of the Taylor expansion of $G_2(0, p - \delta p/2, p + \delta p/2)$ in $\delta p$,
\[ G_2(p) = \lim_{\delta p \to 0} G_2 \left( 0, p - \frac{\delta p}{2}, p + \frac{\delta p}{2} \right). \] (23)

The associated large deviation function $I(T)$ can be obtained minimizing $G_2(p)$ over $p$,
\[ G_2(p) = \frac{\langle (V'(p))^2 \rangle_{\tau}^2}{2 \langle (V'(p))^2 \rangle_{\tau}^2} \] (24)
\[ I_{LF,2}(T) = \min_p G_2(p). \] (25)

Having learned that for $V_2 - V_1 = O(\delta p^2)m$ is $O(1/\delta p)$, we need to perform the large deviation computation without resorting to a power expansion in $m$. Our main focus is the computation of $I(m, p_1, p_2)$. In fact, using equation (17), the large deviation function $G(V_2 - V_1, p_1, p_2)$ can be computed from
\[ e^{N\Gamma(m, p_1, p_2)} = \left( \langle Z[p_1] \rangle^{m/2} \langle Z[p_2] \rangle^{-m/2} \right). \] (26)

Setting $p_1 = p - \delta p/2, p_2 = p + \delta p/2$ and $m = u/\delta p$, we define
\[ \Gamma(u, p) = \lim_{\delta p \to 0} \Gamma \left[ \frac{u}{\delta p}, p - \frac{\delta p}{2}, p + \frac{\delta p}{2} \right] \] (27)
and we perform the computation of this object using replicas. The technical details are provided in the appendix, see appendix B.2. A fundamental difference with the perturbative expansion presented above is that now $u$ has to be optimised over with the other saddle point variables, whereas in the previous case $m$ is determined after the moments of the derivatives of the potential have already been computed, see equation (21). We perform the computation both in the RS and in the 1RSB scheme and using equation (18) we may define $G(p) = -\Gamma(u^*, p)$. The large deviation function $I(T)$ is obtained by taking the minimum over $p$ of $G(p)$, as in equation (25).
\[ I_{LF}(T) = \min_p G(p) \] (28)

A comparison between $G(p)$ and $G_2(p)$ can be found in the appendix, see appendix B.2. In figure 2 we plot the potential and the large deviation function for the 3-spin in the dynamical phase, where the potential has a local minimum. The first minimum of $G(p)$ on the right of the figure corresponds to the local minimum of the potential and, since $\beta > \beta_d$, $G(p) = 0$ in this point. In fact, the probability of having $V'(p) = 0$ on this point is 1. For each stationary point of the potential, $G(p)$ has a zero: $G(p)$ is zero also on the spurious stationary points of the RS potential (the first two from the left).

We study its behavior in figures 3–6 for the 3-spin and the 3 + 4-spin, at different values of $\beta < \beta_d$ for $\beta' = \beta$. We denote by $p^*(T) = \arg \min G(p)$. When $\beta' = \beta$ and $\beta < \beta_d$, the
Figure 2. Potential $V(p)$ and the large deviation function $G(p)$ computed at the RS level for the 3-spin at $\beta = 1.66$, larger than $\beta_d$. The potential has been rescaled in order to fit in the figure. The $y$-axis refers to the values of $G(p)$. The first two minima of $G(p)$ from the left are spurious, as they correspond to the first two stationary points of the potential due to the RS ansatz.

Figure 3. $G(p)$ for the 3-spin at $\beta = \beta' = 1.58$, smaller than $\beta_d$, computed with the 1RSB ansatz (dots) and the RS ansatz (line). The vertical line indicates $p_{\text{RS}}$.

minimum over $p$ of $G(p)$ appears at $p_{\text{RS}}$, the frontier between the static 1RSB region and the RS region, as can be observed in figures 3–6. The 1RSB computation differs from the RS one only in the static 1RSB regime. This is not a surprise since the RS ansatz is correct when $p > p_{\text{RS}}$. For any $\beta < \beta_d$, it is worthwhile noticing that $p^*$ does not correspond to the point where the first derivative of the potential takes the smallest value, as observed in figure 7. In this case, the value of $G$ in $p^*$ is not zero, differently from figure 2, because at $\beta < \beta_d$ the secondary minimum does not exist in the typical potential $V(p)$. Moreover, as long as $\beta > \beta_{\text{RS}}$, the replica symmetric $G(p)$ does not estimate correctly $p^*$.

The results presented above suggest that the most likely event for the appearance of a stationary point in the potential $V(p)$ for $T > T_d$ is that the first replica is picked in one of the marginal states still surviving at $T$, i.e. in one of the out-of-equilibrium states for which the replicon mode is zero, $\beta^2 p'(q)(1 - q)^2 = 1$. In the following, we denote by $q_{\text{RS}}$ the largest solution to this equation, which exists as long as $\beta > \beta_{\text{RS}}$. For smaller $\beta$, the RS ansatz is
always stable and the most likely event for the appearance of a stationary point in the potential $V(p)$ is that the first replica is chosen in one of the states for which the replicon mode is minimum.

Finally, in figure 8, we plot the large deviation functions in the small and large deviation case. We compare also the second order approximation $I_{LF2}$ of the large deviation function computed on $p_{RS}$. There are no reasons to compute $I_{LF2}$ on $p_{RS}$ but this computation gives the same results obtained from the large deviation analysis as long as $\beta > \beta_{RS}$. This property is lost for $\beta < \beta_{RS}$, where $p_{RS}$ is replaced by the value of $p$ associated to the $q$ for which the replicon mode is minimum.

4. Connection with the state following

Before discussing the results presented above, let us define the tilted potential. Ideally one would like to compute the value of the potential in $p$ conditioned on the value of the difference of the potential in two different neighbouring points, $p_1$ and $p_2$. This computations turns out to
Figure 6. $G(p)$ for the 3 + 4-spin model at $\beta = \beta' = 1.15$, smaller than $\beta_c$, computed with the 1RSB ansatz (dots) and the RS ansatz (line). The vertical lines indicate, from left to right, $p_K$ and $p_{RS}$.

Figure 7. $V(p)$ and $V'(p)$ computed at 1RSB level for the 3-spin at $\beta = 1.58$, in the paramagnetic phase. The three vertical lines separate, from the left, a first RS region followed by the dynamic 1RSB, the static 1RSB and the last RS region.

be rather ambitious and here we limit the discussion to the case $p = p_1$.

$$\tilde{V}(p_1|V_2 - V_1) = \int dV_1 V_1 P(V_1|V_2 - V_1) = \int dV_1 V_1 \frac{P(V_1, V_2 - V_1)}{P(V_2 - V_1)}, \quad (29)$$

where $P(V_1|V_2 - V_1)$ denotes the probability that the random potential $V(p_1|\tau)$ is equal to $V_1$ given that $V(p_2|\tau) - V(p_1|\tau) = V_2 - V_1$, and $P(V_2 - V_1)$ is the quantity defined in equation (14) where, to simplify the notation, we dropped $p_1$ and $p_2$ from the arguments. $\tilde{V}(p_1|V_2 - V_1)$ is the value of the potential in $p_1$ conditioned on the value of the difference $V(p_2|\tau) - V(p_1|\tau)$. Without this condition, $\tilde{V}(p_1) = V(p_1)$, the typical potential. Details on the computation of the tilted potential are provided in the appendix, see appendix C. We will use equation (29) to interpret the results after having discussed their connection with the state following.
Figure 8. Comparison between $I_{SF}$ and $I_{LF}$ for different $\beta < \beta_d$. Dots refers to the second order approximation $I_{LF, 2}$ of the large deviation function computed on $p_{RS}$. The second derivatives of the large deviation functions approaching $\beta_d$, $\lim_{T \to T_d} I''(T)$, are 0.25 in the 3-spin and 0.543 in the $3+4$-spin.

Table 1. Collection of relevant temperatures in the pure and in the mixed model.

|       | 3 spin | 3 + 4 spin |
|-------|--------|------------|
| $\beta_K$ | 1.706  | 1.312      |
| $\beta_d$  | 1.633  | 1.242      |
| $\beta_{RS}$ | 1.500  | 1.132      |

From now on we will set $V_2 - V_1 = 0$ and denote by $\tilde{V}(p)$ the value of the tilted potential for $V_2 - V_1 = 0$, i.e. $\tilde{V}(p) = \tilde{V}(p|V_2 - V_1 = 0)$. We also report in table 1 the several values of the relevant temperatures in the pure and in the mixed model.

4.1. Finite temperature

Using the notation introduced previously, we denote by $T'$ the temperature of the first replica and by $T$ the temperature of the second replica of the potential. When $T' > T_d$, no local minimum exists in the typical potential. If $T' \leq T_d$, increasing $T$ the secondary minimum disappears at a certain point and, as long as it exists, it describes situations where the second replica is in a TAP state of equilibrium at temperature $T'$, followed at $T$ [25, 26, 28, 53]. Following states at different temperatures can be given a dynamical meaning considering a system thermalized at $T'$ and whose Langevin dynamics is done at temperature $T$: in the long time limit the system relaxes inside one of the TAP states that was at equilibrium at $T'$, whose properties have
Figure 9. \( \beta' \) as a function of \( \beta \). \( \beta' \) reaches \( \beta_K \) on the left side of the plots. In the 3-spin, when \( \beta = 1.5 \), \( \beta' = \beta_d \), i.e. there is not need to increase \( \beta \) from \( \beta_d \) to find the secondary minimum as it still exists. This is true \( \forall \beta : 1.5 < \beta < \beta_d \). In the mixed model, as soon as \( \beta < \beta_d \), \( \beta' \) must be increased in order to find the secondary minimum. This is the reason why in the right border of the x-axis is \( \beta = 1.5 \) in (a) and \( \beta = \beta_d \) in (b).

changed since temperature has been shifted to \( T \) [28, 49, 54]. The minimum of the potential represents the correlation between the state where the first replica is extracted from and the same state at a different temperature \( T \neq T' \). When \( T > T' \), depending on the model, different things happen. For the pure model, if \( T' = T_d \), increasing \( T \) the secondary stationary point continues to exist (as a saddle) until \( T_{RS} = 1/\beta_{RS} \) (\( \beta_{RS} = 1.5 \) in the 3-spin model). For larger temperatures, it disappears and in order to make it re-appear, the first temperature \( T' \) must be decreased. For \( T > T_{RS} \) no \( p_{RS} \) exists, i.e. no RS instability appears. At a given \( T \), we denote by \( T'_{s} \) the largest value of \( T' \) for which the secondary minimum exists. For consistency, we set \( T'_{s} = T_d \) for \( T < T_{RS} \). In the mixed model, if \( T' = T_d \), as soon as \( T > T_d \), the secondary minimum disappears. In order to make it re-appear, the first temperature \( T' \) must be decreased from \( T_d \). This behavior is described in figure 9. In both the pure and the mixed spin models, when \( T' = T = T_d \), the second stationary point at \( p = q_{EA} \) describes marginal states, i.e. states with an overlap value \( q \) for which the replica mode is zero. The description provided above can be rephrased by saying that using the potential \( V(p) \) with two temperatures \( T' \) and \( T \), marginal states can be followed up in temperature in the pure model, but not in the mixed one. Let us denote by \( p_{min}(T', T) \) the point where the potential has the secondary stationary point when the first temperature is \( T' \) and the second \( T \), by \( V_{min}(T', T) \) the value of the potential in the local minimum and by \( q_{min}(T', T) \) the self-overlap of the state described by the secondary stationary point. In the pure model, it turns out that

\[
p_{min}(T'_{s}, T) = p^{*}(T),
\]

i.e. \( p^{*}(T) \) corresponds to the secondary stationary point of the potential when \( T' = T_{s} \) and the second temperature is \( T \). Moreover, for \( \beta > \beta_{RS} \),

\[
p_{RS}(T) = p_{min}(T_d, T) = p^{*}(T),
\]

i.e. these states are marginal. For \( \beta < \beta_{RS} \), no \( p_{RS} \) exists, the marginal mode is always positive but equation (30) is still valid. In this case, as mentioned above, \( p^{*}(T) \) corresponds to the minimum of the marginal mode.

Equation (30) suggests that for \( \beta < \beta_{d} \), lowering the first temperature of the potential to the first value where the stationary point appears, \( T'_{s} = 1/\beta'_{s} \), it is possible to describe the point where a stationary point is more likely to appear in the random potential \( V(p|\tau) \) at temperature
\[ T = \frac{1}{\beta} > T_\text{d}. \] Thus we compare the value of the tilted potential \( \tilde{V}(p) \) in \( p^* \) at temperature \( T \), \( \tilde{V}^*(T) \), with \( V_{\text{min}}(\tau, T) \). In the pure model, \( V_{\text{min}}(\tau, T) = V'(T_\tau, T) \), where the rhs of the equation is the value of the potential with first temperature equal to \( T_\tau \) and second temperature equal to \( T \) in \( p^* \). It turns out that

\[ V_{\text{min}}(\tau, T) = \tilde{V}^*(T). \quad (32) \]

This behavior is analyzed in figures 10(a) and 11(a) in the range 1.264 < \( \beta < \beta_4 \) for the 3-spin. At \( \beta = 1.264 \) the secondary minimum exists only if the first replica is taken at \( \beta^{*}_1 = \beta_K \). Decreasing further \( \beta \), we should increase the value of \( \beta^{*}_1 \) of the first replica and our approximation would not be valid anymore. The interpretation of these results is that when \( T > T_\text{d} \) and no local minimum is present in \( V(p) \), the most likely way to make it appear in an atypical realization \( V(p|\tau) \) is that the first replica \( \tau \) is taken inside one of the equilibrium states at \( T_\tau \), still existing at \( T \) as an out-of-equilibrium state. This state can be studied with the potential, lowering the first temperature from \( T \) to \( T_\tau \) in order to make the secondary minimum re-appear. In other words, the rare appearance of the secondary minimum in the random potential \( V(p|\tau) \) at temperature \( T \) is due to the sampling of \( \tau \), at temperature \( T' = T \), among the configurations that are typical at \( T_\tau \).

While for the pure model it was possible to interpret our results in terms of the potential with two temperatures, for the mixed model, this is not possible. This model, contrary to the pure model, presents the general property of level crossing and the associated temperature chaos. In other words, following states in temperature does not preserve their order in terms of their free energy [17]. As described previously, when \( T' = T_\text{d} \) the secondary stationary point disappears as soon as \( T > T_\text{d} \). We can again lower \( T' \) from \( T_\text{d} \) in order to make it appear. At this point, we may again look at the value of \( p \) where the potential has the secondary stationary point \( p_{\text{min}}(T_\tau, T) \) and we may compare it with \( p^*(T) \). In this case, there is no match between the two. While \( G(p) \) is minimum in \( p_{\text{RS}} \), as long as it exists for \( \beta > \beta_{\text{RS}} (\beta_{\text{RS}} = 1.132 \) in the 3 + 4-spin model), \( p_{\text{min}}(T_\tau, T) \) is found to be larger: increasing \( T \) above \( T_\text{d} \) in the potential, the secondary minimum disappears before that the states it describes become marginal. Nevertheless, the large deviation analysis still predicts that, when \( \beta_{\text{RS}} < \beta < \beta_4 \), in order to observe a secondary stationary point in \( V(p|\tau) \), the first replica must be chosen in one of the marginal states still existing at \( T = 1/\beta \), and when \( \beta < \beta_{\text{RS}} \) in one of the states with overlap \( q \) such that \( 1 - \beta^2 p''(q) (1 - q)^2 \) is minimum. Looking at the tilted potential, we observe that \( \tilde{V}^*(T) \) coincides with \( V'(T_\tau, T) \) as in the pure model, but at difference with the pure model, \( \tilde{V}^*(T_\tau, T) > V_{\text{min}}(T_\tau, T) \), since the local minimum is at a value of \( p \neq p^* \), as commented previously. This behavior is analyzed in figures 10(b)–11(b) in the range 0.9175 < \( \beta < \beta_4 \) for the 3 + 4-spin. At \( \beta = 0.9175 \) the secondary minimum exists only if the first replica is taken at \( \beta^{*}_1 = \beta_K \). Decreasing further \( \beta \), we should increase the value of \( \beta^{*}_1 \) of the first replica and our approximation would not be valid anymore.

In order to look for the existence of marginal states both in the pure and in the mixed model, we compute the complexity of states \( \Sigma(q) \) as a function of their overlap at temperature \( T \), using replicas [48, 55]. We show the results in figures 12 and 13. We denote by \( q_{\text{max}}(T) \) the value of \( q \) where \( \Sigma \) is maximum. This computation is described more in details in the appendix, see appendix A.3. The observation of marginal states above \( T_\text{d} \) in both the pure and in the mixed model is possible thanks to a non-orthodox prescription in the use of the clone method, used to evaluate the complexity, as will be discussed more in details in [56]. In both models, picking the first replica in a marginal state is the key to observe the re-appearance of the secondary stationary minimum for \( \beta < \beta_4 \). The main difference between the pure and the mixed model
is that while, in the former, marginal states can be obtained by following states within the framework of the potential, in the latter they cannot.

4.2. Zero temperature

State following for the mixed model when $T < T'$ is a well known open problem. In terms of the potential, when $T' = T_d$, the secondary stationary point disappears not only increasing the second temperature $T$ but also decreasing it [17, 25]. One can study the potential with $T = 0$, $V_0(p)$, at the RS level and observe that it develops the secondary minimum only for a temperature $T' = T_{SF} < T_d$. For comparison, in the pure model, this happens exactly at $T' = T_d$. The same picture holds at a finite temperature $T$: it is possible to define a $T_{rsb}(T)$ below which states multfucrate and the local minimum of the RS potential disappears [25]. The impossibility to follow states by cooling when $T' = T_d$ remains valid also taking into account the 1RSB potential [26]. On the other hand, in the 1RSB case, decreasing $T'$, the local minimum reappears at a temperature larger than $T_{SF}$, even if still smaller than $T_d$. Inspired by these anomalies, the long-time limit of the out of equilibrium dynamics of mean-field spherical mixed models has been investigated in [27]. Interestingly, considering a gradient descent ($T = 0$) dynamics starting
from a configuration at equilibrium at $T'$, a new description of the dynamical phase transition emerged. Surprisingly, for some temperatures $T' > T_d$, it was shown that the dynamics converges below the energy of the threshold states $e_{th}$. This behaviour is radically different from that of the pure model, where starting from $T' > T_d$ the zero-temperature dynamics always converges to the same value $e_{th}$ and memory of the initial condition is lost [43]. In the mixed model, this happens only for $T > T_{\text{onset}} > T_d$. Moreover, it was found that it is only for $T < T_{SF} < T_d$ that the dynamics becomes fast and the long time dynamics converges to the states described by the local minimum of the RS potential with the second temperature equal to zero. The existence of a new phase, the *hic sunt leones* phase for $T_{SF} < T < T_{\text{onset}}$, is predicted. In this phase, the relation between dynamics and static computation is missing. Differently thermalized configurations lie in basins of attraction of different marginal states. The dynamics does not lose memory of the initial condition and presents the features of aging in metabasins [25] but the analysis performed in [27] seems to exclude this scenario.

Based on these results, we performed the large deviation analysis at $T = 0$. Details of the computation are provided in the appendix, see appendix D. This analysis allows to study the rare events of the re-appearance of the local minimum when typically it does not exists, thus in
Figure 12. $\Sigma(q)$ for the 3-spin at two temperatures, above the dynamical one. The horizontal line indicates the maximum value attained by the complexity. As long as $\beta > 1.5$, in the pure model, the maximum of the complexity does not change. $q_{RS}(T), q_{\text{min}}(T', T)$ and $q_{\text{max}}(T)$ coincide and are indicated by the vertical line. For $q < q_{RS}$ the solutions are unstable in the replica space and thus unphysical.

5. Concluding remarks

In this paper we provide a first generalization of the theory of fluctuations in glasses based on the analogy with the spinodal point of the RFIM, in which the effect of the self-induced disorder in the beta regime is described by random term in a cubic field theory. This theory provides a quasi-equilibrium description and it is valid in the vicinity of the critical point, the mode coupling transition. Moreover, it relies on the approximation that fluctuations are Gaussian. In our approach, we keep the first ingredient, which allows to study fluctuations through the use of constrained equilibrium measures and their associated replica action. On the hic sunt leones phase as well. As at finite temperature, while in the pure model $p^*$ coincides with $p_{RS}$, this is not true in the mixed model. In the mixed model, $V_0$ has the local minimum in the RS region as long as $T' > T_{SF}$ and in the 1RSB region for larger $T$. When the 1RSB potential loses its minimum at $T' = 0.8041 < T_d$, the large deviation function $G_0(p)$ has still a local minimum, with $G(p^*) > 0$, as can be observed in figure 14. This implies that, in the hic sunt leones phase, results obtained from numerical simulation of the dynamics at $T = 0$ in finite systems may be influenced by the rare dynamics where memory of the initial condition is not lost. This new phase is not completely understood yet and a comparison with the results provided in [27] in finite systems is still in progress [57].
the other hand, we go beyond the assumption of the vicinity to the critical point presenting a first principle computation of the large deviation function. More precisely we look for the probability of the existence of the secondary minimum of the potential in a finite region of the paramagnetic phase above the dynamical temperature. We consider the case of spherical spin models and we show that in the vicinity of the critical point, we recover the result implied by the cubic field theory, as observed in figure 8 for both the pure and the mixed models. The large deviation analysis allows to go beyond the vicinity to $T_d$ and to show that the re-appearance of the secondary stationary point in a regime where typically it does not exist can be explained in terms of the sampling of the first replica of the potential in an out of equilibrium state. As long as marginal states exists, picking the first replica in these states dominates the probability to observe the re-appearance of the local minimum. When they disappear, this probability is dominated by states with an overlap $q$ such that the replicon mode is minimum. Moreover, we showed that this behaviour can be interpreted with the state following analysis, at least in the pure model.

In terms of the dynamics when $T$ is large and the dynamics is performed at $T = T'$, while typically the system lose memory of its initial condition, our results suggests that on finite $N$ systems it is possible to observe as a rare event that the dynamical correlation does not decay to zero, but to $p^*$. This is particularly interesting at $T = 0$ where we show that understanding

![Figure 13](image-url)
Figure 14. $G(p')$ computed on the RS (blue dashed) and on the 1RSB (orange dotted) potential with $T = 0$ for the $3 + 4$-model. The right most point on the $x$-axis is $\beta_{SF}$. The RS potential loses its local minimum for $T > T_{SF}$. The 1RSB one lose it at a larger $T = 0.8041 < T_d$. $\beta_d$ is indicated by the vertical line. As long as $V_0$ has a local minimum, $G_0$ has its local minimum on the same point. $p^*$ computed on the RS expression of $G_0$ does not coincide with the $p^*$ computed with the 1RSB expression of $G_0$.

the new phase predicted in [27] in numerical simulations may be tricky. In fact, dynamics with memory of the initial condition may be explained in term of atypical events. As with finite temperatures, looking at figure 14, these events are not so rare: if $N = 100$, $e^{-100G(p')}$ $\sim O(1)$ for a wide range of temperatures around $T_d$.

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Appendix A. Replicas

A.1. Replica action

Here we introduce the formalism used in the computation of the potential as well as of the small and the large deviation functions. Given an integer number $n$, we introduce the fixed overlap replica action

$$e^{\frac{N}{2}S[Q,n]} = \int \prod_{a=0}^{n} Dq^a e^{-\beta \sum_{a=0}^{n} H(q^a) \prod_{a,b} \delta(Q_{ab} - q(s^a,s^b))}$$

(33)

where we defined $\tau = s^0$, $q(s,s') = N^{-1} \sum_i s_is'_i$ and $Q_{ab}$ is a square symmetric matrix of size $(1+n) \times (1+n)$. All the moments of the probability of $V(p|\tau)$ can be written in terms of this action. In particular if we take all the overlaps $Q_{0a} = p$, $(a = 1, \ldots, n)$, we obtain

$$e^{\frac{N}{2}S[1]|p]} = \langle \left(Z[p|\tau]\right)^n \rangle = \int DQ_{ab} e^{\frac{N}{2}S[Q,n]} \prod_{a=1}^{n} \delta(p - Q_{0a}),$$

(34)
where $D Q_{ab}$ denotes the integration over the parameters of the overlap matrix. This quantity is used to compute the potential.

The integration over the non-constrained overlaps can be performed by considering the saddle point. Depending on the temperature and on the value of the overlap is used to compute the potential. Setting straightforward. The diagonal blocks are labelled by $l$ indexed by $l$, the replica ansatz of the constrained system. Both in the RS and in the 1RSB ansatz, the out-of-diagonal block indexed by $l$, $Q_{ab}$, $Q_{ab}$ and $Q_{ab}$, where $Q_{ab}$ is the diagonal block with all the elements equal to $1$. For every $k$, the diagonal elements of $Q$ are equal to 1. For $k = 1$, $Q_{ab} = p_1$ for $a = 1, \ldots, n_1$. For $k = 2$, $Q_{ab} = p_1$ for $a = 1, \ldots, n_1$ and $Q_{ab} = p_2$ for $a = n_1 + 1, \ldots, n_1 + n_2$. Generalization to a generic $k$ is straightforward. The diagonal block matrices are labelled by $Q_{ab}$, $Q_{ab}$ and so on. They reflect the replica ansatz of the constrained system. Both in the RS and in the 1RSB ansatz, the out-of-diagonal block indexed by $l$, $Q_{ab}$ is a rectangular $n_1 \times n_2$ matrix with all the elements equal to $q_{lm}$. For the sake of simplicity, here we describe results obtained with the RS ansatz. Generalization to 1RSB and details on the computation of the determinant are provided later.

The action $S^{(k)}$ can be expanded in a power series of $n_1, n_2, \ldots, n_k$ with the zero order term being zero. We indicate derivatives with respect to replicas, computed at a number of replicas equal to one, with a superscript notation. Recalling that

$$F(p) = - \frac{1}{\beta N} \lim_{N \to 0} \frac{\partial}{\partial q} E_i[Z[p|\tau]]^2,$$

for $k = 1$, neglecting higher order terms, $S^{(1)}[p] = n S^{(1)}[p]$ and using equation (34) one has

$$V(p) = F(p) - F = - \frac{1}{N\beta} \lim_{N \to 0} N S^{(1)}[p] - F. \tag{38}$$

Setting $p_1 = p$ and $q_1 = q$ the expression of the potential reads

$$V(p) = \frac{1}{2\beta} \left( \frac{q - p^2}{q - 1} - 2\beta' f(p) + \beta^2 f(q) - \log(1 - q) \right). \tag{39}$$

The saddle point value of $q$ at a given $p$ is given by

$$\frac{q - p^2 - (q - 1)^2 \beta^2 f'(q)}{(q - 1)^2} = 0. \tag{40}$$
Now we study the term \( k = 2 \). \( S^{(2)}[p_1, p_2] \) can be expanded as
\[
S^{(2)}[p_1, p_2] = n_1 S^{(2),(1,0)}[p_1] + n_2 S^{(2),(0,1)}[p_2] + n_1 n_2 S^{(2),(1,1)}[p_1, p_2]
\]
(41)

neglecting third order terms, and it has the following properties
\[
\begin{align*}
\partial_{q_{12}} S^{(2),(1,0)}[p_1] &= \partial_{q_{12}} S^{(2),(0,1)}[p_2] = 0, \\
\partial_{q_1} S^{(2),(0,1)}[p_1] &= \partial_{q_1} S^{(2),(1,1)}[p_2] = 0,
\end{align*}
\]
(42) \hspace{0.5cm} (43)
i.e.: \( q_{12} \) appears only in second order terms and the first order terms relative to replica 1(2) does not depend on saddle point parameters of replica 2(1). Moreover, the first order terms \( S^{(2),(1,0)}[p] \) and \( S^{(2),(0,1)}[p] \) are equal to \( S^{(1),(1)}[p] \). \( S^{(2)} \) can be expanded in a similar way. From the definition of the fluctuations,
\[
W_{p_1, p_2}^{(2)} = \mathbb{E}_r \left( V(p_1 | \tau) V(p_2 | \tau) \right) - \mathbb{E}_r \left( V(p_1 | \tau) \right) \mathbb{E}_r \left( V(p_2 | \tau) \right)
\]
(44)
we have
\[
W_{p_1, p_2}^{(2)} = \frac{1}{\beta^2 N^2} \lim_{n_1 \rightarrow 0} \lim_{n_2 \rightarrow 0} \left( \frac{\partial^2}{\partial n_1 \partial n_2} (Z^{(n)}[p_1 | \tau] Z^{(n)}[p_2 | \tau]) + \frac{\partial}{\partial n_1} (Z^{(n)}[p_1 | \tau]) \frac{\partial}{\partial n_2} (Z^{(n)}[p_2 | \tau]) \right)
\]
\hspace{0.5cm} (45) \hspace{0.5cm} (46)

Thanks to equations (41) and (35), the first term can be written as
\[
\begin{align*}
&\lim_{n_1 \rightarrow 0} \lim_{n_2 \rightarrow 0} \frac{\partial^2}{\partial n_1 \partial n_2} (Z^{(n)}[p_1 | \tau] Z^{(n)}[p_2 | \tau]) \\
&= \lim_{n_1 \rightarrow 0} \lim_{n_2 \rightarrow 0} \left[ \frac{N}{2} \frac{\partial^2 S^{(2)}[p_1, p_2]}{\partial n_1 \partial n_2} + \left( \frac{N}{2} \right)^2 \frac{\partial S^{(2)}[p_1, p_2]}{\partial n_1} \frac{\partial S^{(2)}[p_1, p_2]}{\partial n_2} \right] \\
&= \lim_{n_1 \rightarrow 0} \lim_{n_2 \rightarrow 0} \left[ \frac{N}{2} S^{(2),(1,1)}[p_1, p_2] + \left( \frac{N}{2} \right)^2 S^{(2),(1,0)}[p_1] S^{(2),(0,1)}[p_2] \right]
\end{align*}
\]
(47)
and we observe that the last term is equal to the disconnected part. Thus fluctuations are computed from
\[
W_{p_1, p_2}^{(2)} = \frac{1}{2 \beta^2 N} \lim_{n_1 \rightarrow 0} \lim_{n_2 \rightarrow 0} \frac{\partial^2}{\partial n_1 \partial n_2} S^{(2)}[p_1, p_2]
\]
(48)

and their expression reads
\[
W_{p_1, p_2}^{(2)} = \frac{1}{2 \beta^2 N} \left( \frac{q_{12} - p_1 p_2}{(q_1 - 1)(q_2 - 1)} + 2 \beta^2 f(q_{12}) \right)
\]
(49)

We use \( z \) to denote the set of all the order parameters over which we have to optimize \( S^{(2)} \). For \( k = 2 \), this set contains \( q_1, q_2 \) and the mixed term \( q_{12} \). The optimization over \( z \) leads to
\[
\frac{q_1 - p_1^2 - (q_1 - 1)\beta^2 f(q_1)}{(q_1 - 1)^2} = 0,
\]
(50)
in equation (40). In the 1RSB ansatz, on the other hand, it reads
structure. We call these components
the RS ansatz the diagonal block matrix
on the RS/1RSB ansatz used to evaluate both the potential and the large deviation function. In
this section we give more details on the computation of the determinant in equation (36) and
A.2. Replica symmetry breaking
In this section we give more details on the computation of the determinant in equation (36) and
on the RS/1RSB ansatz used to evaluate both the potential and the large deviation function. In
the RS ansatz the diagonal block matrix \( Q_{ab}^k \), for each \( k \), is parametrized by
\[
Q_{ab}^k = q_k + (1 - q_k)\delta_{ab},
\]
and \( q_k \) has to be optimized over. For each \( k \), the saddle point value of \( q \) depends on \( p \) as stated in equation (40). In the 1RSB ansatz, on the other hand, it reads
\[
Q_{ab}^k = q_k^0 + (q_k^1 - q_k^0)\epsilon_{ab} + (1 - q_k^1)\delta_{ab},
\]
where the matrix \( \epsilon_{ab} \) has ones on the diagonal block of size \( x_k \) and zeros elsewhere, with \( n/x_k \in \mathbb{N} \). To ease the notation we use \( y_k \) to denote the saddle point parameters relative to replica \( k \). For each \( k \), in the RS ansatz \( y \) is just the overlap \( q \). In the 1RSB ansatz it is the collection \( \{q^0, q^1, x\} \). On the other hand, the elements of the out-of-diagonal blocks are taken to be the same. The saddle point values of \( q^0, q^1 \) and \( x \) at a given \( p \) are given by
\[
\frac{x\left[p^2 - q^0 + (1 + q^1(x - 1) - q^0 x)^2\beta^2 f'(q^0)\right]}{1 + q^1(x - 1) - q^0 x^2} = 0,
\]
\[
\frac{\left[(p^2 - q^0)(q^1 - 1) + (q^0 - q^1)^2 x\right]}{(q^1 - 1)[1 + q^1(x - 1) - q^0 x^2]} + \beta^2 f'(q^1) \right] (x - 1) = 0,
\]
\[
q^0 - p^2(q^0 - q^1) + \frac{q^1 - q^0}{[1 + q^1(x - 1) - q^0 x^2]} + \frac{q^1 - q^0}{[1 + q^1(x - 1) - q^0 x^2]} + \beta^2 [f(q^1) - f(q^0)]
\]
\[
- \frac{1}{x^2} \log \left[1 + \frac{(q^1 - q^0)x}{1 - q^1}\right] = 0.
\]
When to use the RS or the 1RSB scheme will be discussed later.
In order to obtain the expression of \( S^{(k)}[p_1, p_2, \ldots, p_k] \) we need to compute the determinant of the \( Q \) matrix, thus its spectrum, see equation (36). Let us first discuss the simple case of \( k = 1 \). In this case, the spectrum of \( Q \) is found by looking for eigenvectors that have a \( 1 + n \) structure. We call these components \( u, v_a, a = 1, \ldots, n \), obtaining the following eigenvalue equations
\[
\begin{align*}
\sum_b Q_{ab} v_b &= \lambda v_a, \\
pu + \sum_b Q_{ab} v_b &= \lambda v_a.
\end{align*}
\]
\( (58) \)
Let us set \( u = 0 \), \( \sum_{a} v_{a} = 0 \), leading to \( n - 1 \) eigenvectors of \( Q_{ab} \). Given the 1RSB structure of \( Q_{ab} \), the equation \( \sum_{b} Q_{ab} v_{b} = \lambda v_{a} \) is equal to
\[
q^{0} \sum_{b} v_{b} + (q^{1} - q^{0}) \sum_{b} \epsilon_{ab} v_{b} + (1 - q^{1}) v_{a} = \lambda v_{a}.
\]
(59)

We look for two different kinds of solutions:

(a) We may require that \( \sum_{b} \epsilon_{ab} v_{b} = 0 \) \( \forall a \), i.e. the partial sum \( \sum_{b} (\text{block}_{a}) v_{b} = 0 \), where block\(_{a}\) is the block to which replica \( a \) belongs to. We obtain eigenvectors with eigenvalue \( \lambda = 1 - q^{1} \). Since we have \( n/x \) blocks and each one has size \( x \), the multiplicity of this eigenvalue is \( n/x(x - 1) \);

(b) We may consider the case when in each of \( n/x \) blocks we have a distinct value of \( v_{b} \) and their sum is zero. This condition leads to eigenvectors with eigenvalue equal to \( \lambda = (q^{1} - q^{0}) x + (1 - q^{1}) \), whose multiplicity is equal to \( n/x - 1 \).

Counting the degeneracies of the eigenvalues obtained up to now, we see that we have got \( n - 1 \) eigenvectors of \( Q_{ab} \). Thus, we see that we still miss one eigenvector of \( Q_{ab} \). Clearly the constant vector \( \mathbf{v} = (v_{a=1,...,n}) \) is the last eigenvector of \( Q_{ab} \), with eigenvalue equal to \( \sum_{a} Q_{ab} \), ruled out by the condition \( \sum_{a} v_{a} = 0 \). Nevertheless this vectors is not an eigenvector of the matrix \( Q \) and we observe that in order to find the 2 missing ones we need to solve the reduced eigenvalue problem
\[
\begin{cases}
u + \rho n v = \lambda u \\
p u + \sum_{b} Q_{ab} v = \lambda v
\end{cases}
\]
(60)

\( F \) can be easily computed from the condition \( V(0) = 0 \) and realizing that in this case the overlap matrix \( Q \) reduces to a single number, 1. We find \( F = -\beta f(1)/2 = -\beta/4 \). While \( p \) is a control parameter of the problem, \( q^{1}, q^{0} \) and \( x \) are parameters we need to optimize over. This task cannot be accomplished in a straightforward way [25]. In fact, for \( \beta \) in the dynamical phase, physical intuition suggests the existence of three regions:

- One at very small values of \( p \), where the second replica is weakly constrained to the first one, having the possibility to explore an exponential number of states and leading thus to a dynamical 1RSB phase, where the \( x = 1 \) and \( q^{1} \neq q^{0} \);
- One at intermediate values of \( p \), where the second replica can explore \( O(N) \) metastable states, leading to a static 1RSB phase with \( x < 1 \) and \( q^{1} \neq q^{0} \);
- One at very large values of \( p \) where the second replica is forced to be very close to the first one, exploring only the state where the first one is and leading thus to an RS phase where \( x = 1 \) and \( q^{1} = q^{0} \).

As discussed in the main text, see the section 2, the dynamical 1RSB phase is characterised by a non negative complexity. This phase is defined by \( p_{0} < p < p_{K} \). As discussed below, for temperatures smaller than the dynamical one, \( p_{0} = 0 \). This is the reason why the description of the different regimes provided above starts from the dynamical 1RSB phase. As discussed below, this will not be the case for larger temperatures. The vanishing of the complexity at \( p_{K} \) identifies the transition between the dynamical 1RSB region and the static one, while the instability of the replicon at \( p_{RS} \) identifies the transition between the static 1RSB region and the RS one. The RS instability is detected by looking at the largest solution of the replicon equation \( \beta^{2}(1 - q^{2})f''(q) = 1 \). This solution survives as long as \( \beta > \beta_{RS} \); \( \beta_{RS} \) is equal to 1.5
in the 3-spin and equal to 1.13234 in the 3 + 4-spin. This point coincides also with the point where \( p_0 \) and \( p_K \) cross, see figures 15 and 16. Thus, for \( \beta < \beta_{\text{RS}} \) the potential is always RS.

When \( \beta < \beta_d \), already in the limiting case \( p = 0 \), the second replica cannot explore an exponential number of states. This implies the existence of a preliminary RS phase at low values of \( p < p_0 \). It turns out that the RS ansatz is locally stable even for \( p > p_0 \) and thus, in order to detect \( p_0 \) we study the stability of the 1RSB solution in the dynamical 1RSB region. Thus, in general we have four regions. While in the first and in the last regions we have to fix \( x = 1, \) \( q_1^2 = q_0^2 \) and solve the RS saddle point equations, in the other ones we need to solve the 1RSB saddle point equations. Anyway, in the dynamical 1RSB region, we need to fix \( x = 1 \). In figure 17 we show the value of the saddle point values in the paramagnetic phase.

When dealing with \( k > 1 \) the same reasoning can be repeated, looking respectively for eigenvectors with a 1 + \( n_1 + n_2 \) and 1 + \( n_1 + n_2 + n_3 \) structure for \( k = 2 \) and \( k = 3 \) and setting \( u = 0, \sum a_i^2 = 0, \sum a_i^2 = 0, \sum a_i^2 = 0 \). This lead to \( n_1 + n_2 - 2 \) and \( n_1 + n_2 + n_3 - 3 \) eigenvalues, respectively, with the same multiplicity discussed above. In the first case we find, besides those mentioned above, an eigenvalue equal to 1 – \( q_2 \) with multiplicity \( n_2/x_2(x_2 - 1) \) and another one equal to \( (q_1^2 - q_2^2)x_2 + (1 - q_1^2) \) with multiplicity \( n_2/x_2 - 1 \). These are \( n_2 - 1 \) eigenvalues of the matrix \( Q_{ab}^2 \). Similarly, in the second case, we also have an eigenvalue equal
Figure 17. Saddle point values for the 3-spin for $q^1$ (blue), $q^0$ (orange) and $x$ (green) for different values of $p$ at $\beta = 1.6$, smaller than $\beta_d$. We observe the four phases discussed in the text: from left to right the preliminary RS phase, the dynamic 1RSB phase, the static 1RSB phase and the final RS phase. The three vertical lines from left to right are $p_0$, $p_K$ and $p_{RS}$.

A.3. Complexity

We describe the analysis of the number of equilibrium states accessible to a system constrained to be at a given distance from a reference configuration, using the formalism developed in [25,
It is convenient to start from the situation when \( p = 0 \), i.e. when the second replica of the potential, is not constrained by the first one. For \( \beta > \beta_d \), the system can explore an exponential number of TAP states, but increasing \( p \) this number decreases. The complexity \( \Sigma(p) \) is the logarithm of this number and can be computed introducing the free energy of \( m \) coupled replicas, constrained to be at a fixed overlap \( p \) from a reference configuration

\[
Z_m^c = \sum_\alpha e^{-\beta mNf^*_\alpha} = e^{-N\beta\Phi_c(m, T)},
\]

(63)

In fact,

\[
Z_m^c = \int df_c \sum_\alpha \delta(f_c - f^{\alpha}_c) e^{-\beta mNf^*_c} = e^{N\left(\Sigma(p, f^{\alpha}_c(m, T), m) - \beta m\xi_c\right)},
\]

(64)

where \( f^{\alpha}_c \) denotes the saddle point value of the constrained free energy and thus we see that \( T/m \) plays the role of an effective temperature in the relation between \( m^{-1}\Phi_c(m, T) \) and the complexity:

\[
m^{-1}\Phi_c(m, T) = f^{\alpha}_c - \frac{T}{m} \Sigma(p, f^{\alpha}_c(m, T), m).
\]

(65)

As the entropy can be computed from the free energy as \( S = -dF/dT \), the complexity reads

\[
\Sigma(p) \equiv \Sigma(p, f^{\alpha}_c(m, T), m)
\]

\[
= -\frac{\partial (m^{-1}\Phi_c(m, T))}{\partial (T/m)} = m^2 \frac{\partial (\beta m^{-1}\Phi_c(m, T))}{\partial m}
\]

(66)

and so all we need to do is compute \( \Phi_c(m, T) \). By definition,

\[
-\beta N\Phi_c(m, T) = \mathbb{E}_r \log \left[ \int D\sigma e^{-\beta H(\sigma)\delta \left( Np - \sum_i \sigma_i \tau_i \right)} \right]^m
\]

(67)

and thus, introducing \( n \) replicas to deal with the logarithm and using that \( \lim_{n \to 0} \log x = \lim_{m \to 0} \partial_m \mathbb{E}_r^m \) we obtain

\[
-\beta N\Phi_c(m, T) = \lim_{n \to 0} \mathbb{E}_r \left[ \int D\sigma e^{-\beta H(\sigma)\delta \left( Np - \sum_i \sigma_i \tau_i \right)} \right]^{mn}
\]

(68)

Using equation (36) with \( k = 1 \), we may write the rhs of this equation as

\[
-\beta N\Phi_c(m, T) = \lim_{n \to 0} \partial_n \mathbb{E}_r \left[ \beta^2 \sum_{a, b} \log \det Q_{a,b} \right]_{n \to m}
\]

(69)

where \( Q_{a,b} \) has size \( 1 + nm \). In fact, when \( k = 1 \), the matrix \( Q \) in equation (36) has size \( 1 + n \) and in the 1RSB ansatz, the \( n \times n \) sub-matrix has a block structure with block-size equal to \( x \). Here the role of \( x \) is played by \( m \):
\[-\beta N \Phi_c(m, T) = \lim_{n \to 0} \frac{\partial}{\partial n} \frac{1}{2} S^{(1)}[p] \bigg|_{m \to \infty} \bigg|_{x = \infty} \]
\[
= \frac{N}{2} m S^{(1)}[p] \bigg|_{x = \infty} \bigg|_{x = \infty}.
\]

Thus we get
\[
m^{-1} \Phi_c(m, T) = -\frac{T}{2} S^{(1)}[p] \bigg|_{x = \infty} \bigg|_{x = \infty}.
\]

Free energy $F$ apart, the expression for $\Phi_c(m, T)$ is equivalent to that of the potential replicated $m$ times where $x \to m$, see equation (38). Since $\Phi_c(m, T)$ is the free energy of $m$ copies of the system constrained to be at distance $p$ from a reference configuration, this could be guessed from its definition. Finally, using equation (66), the constrained complexity reads
\[
\Sigma(p) = -\frac{1}{2} \frac{\partial S^{(1)}[p]}{\partial m} \bigg|_{x = \infty},
\]
\[
\text{i.e.}
\]
\[
2 \Sigma(p) = \frac{q^0 - q^1}{m[1 + q^1(m - 1) - q^0m]} + \frac{(p^2 - q^0)[(p^0 - q^1)^2]}{[q^1 - 1 + m(q^0 - q^1)]^2}
+ \beta^2 f(q^0) - \beta^2 f(q^1) + \frac{1}{m^2} \log \left(1 + \frac{(q^0 - q^1)m}{q^1 - 1}\right).
\]

For $\beta > \beta_d$, the complexity is larger than zero for small values of $p$, i.e. when the system is not very constrained. For all the values of $p < p_{RS} \Sigma(p)$ as well as $V(p)$ has to be computed on the solutions $q^1$, $q^0$ of the 1RSB saddle point equations where $x = 1$ ($m = 1$ in $\Sigma(p)$). For $p > p_{RS}$, in order to get $\Sigma(p) = 0$, $x$ cannot be taken equal to 1 anymore and the system enters in a static 1RSB phase. As explained previously, this region lasts until when the 1RSB saddle point equations give $q^1 \neq q^0$ and corresponds to intermediate values of $p$ after which a RS region appears, because the constraint is so strong that the second replica can only explore the state of the first one. At $p = 0$ this expression gives $\Sigma(0) = TV(q_{EA})$, the unconstrained complexity. This equality comes from the observation that in the dynamical phase one has $F = F_{RS} = f - \Sigma(0)T$, where $f = F(q_{EA})$ is the TAP free energy of the system in one equilibrium TAP state, whose number is $e^{\Sigma(0)}$, and $F_{RS}$ denotes the free energy computed at the RS level. On the other hand, by definition $F(q_{EA}) = V(q_{EA}) + F$. Thus $\Sigma(0) = TV(q_{EA})$. Finally we notice that the expression of the saddle point equation (57) obtained optimizing over $x$ is equal to the expression of the complexity, see equation (74). This means that when in the static 1RSB regime we impose equation (74), we are forcing the complexity to be zero, as explained above.

When $p = 0$, $q^0$ is zero as can be observed in figure 17. The complexity is thus a function of $q^1$ and $m = x$ only. Having set $p = 0$, there are no more constraints and thus we remove the subscript $c$ from $\Phi_c$. The optimization of $\Phi$ over $q^1$ produces
\[
\frac{q^1}{(q^1 - 1)(1 + (m - 1)q^1)} + \beta^2 f'(q^1) = 0
\]

and it is possible to check that this equation corresponds to equation (56) in the limit $q^0 = 0$. Solving this equation for $m$ and using this solution in equation (74), setting $p = q_0 = 0$, we
obtain
\[ 2\Sigma(q) = [\chi(q)q^2 f'(q)]^{-1} \left[ -f(q)(q - \chi(q))^2 + q\beta^2 f'(q)^2 \right. \]
\[ \times \left. \left( (q - 1)^2 q \left( \log q - 1 \right) + (q - 1)^2 \chi(q) \right) \right] \]
(76)

where we set \( q^1 = q \) and \( \chi(q) = \beta^2(q - 1)^2 f'(q) \). This expression gives the complexity of states as a function of their overlap \( q \) at any given \( T \).

**Appendix B. Fluctuations**

**B.1. Small fluctuations**

In this section we give more details on the computation of the small fluctuations. The central quantity we need is the covariance of the potential function for two different values of the overlap, with respect to the extraction of the first replica, equation (44).

As mentioned in the main text and recalled here, for \( T \) close enough to \( T_d \), the potential has the linear behavior
\[ V(p = p_d + \phi) = V(p_d) + \mu(T - T_d)\phi + O(\phi^3) \] with \( \mu > 0 \). The condition \( \mu > 0 \) insures that the secondary minimum disappears for \( T > T_d \). On the other hand, the \( \tau \)-dependent potential can be rewritten as the typical value \( N V(p = p_d + \phi) \) plus a small correction given by
\[ \sqrt{N} \delta V(p = p_d + \phi|\tau) = \eta \phi + \alpha, \]
(77)

see equation (11), where \( \eta \) and \( \alpha \) are correlated Gaussian random fields. The probability of the re-appearance of a local minimum when typically it does not exists can be written in terms of the rate function \( I(T) \) computed in the small fluctuations regime,
\[ e^{-N I_{SF}(T)} = \int_{-\infty}^{-\sqrt{N} \mu(T - T_d)} d\eta P(\eta) \]
(78)

where, denoting by \( \sigma^2 \) the variance of \( \eta \),
\[ I_{SF}(T) = \frac{1}{2} \frac{\mu^2 (T - T_d)^2}{\sigma^2}. \]
(79)

Given their definition, the parameters \( \mu \) and \( \sigma \) can be computed from
\[ \mu(T - T_d) = \left. \frac{dV(p)}{dp} \right|_{p=p_d} \]
(80)

\[ \sigma^2 = \lim_{n_1 \to 0, n_2 \to 0} \frac{T_d^2}{2} \left. \frac{d^2 S_2[\phi_1, \phi_2]}{d\phi_1 d\phi_2} \right|_{\phi_1 = 0, \phi_2 = 0}. \]
(81)

This last equation can be discussed in relation with equation (24), which shed further light on the derivation of equation (81). In fact, by definition, we have
\[ \left\langle \left( V'(p|\tau) \right)^2 \right\rangle_c = \frac{1}{\beta^2 N^2} \lim_{n_1 \to 0, n_2 \to 0} \left. \frac{d}{dp_1} \frac{d}{dp_2} \log Z[p_1|\tau] \log Z[p_2|\tau] \right|_{\phi_1 = 0, \phi_2 = 0}. \]
(82)
and, using replicas and repeating the manipulations of equation (47), we obtain

\[
\left\langle \left( V'(p|\tau) \right)^2 \right\rangle_c = \frac{1}{2\beta^2 N} \lim_{p_1 \to p} \lim_{p_2 \to p} \frac{d}{dp_1} \frac{d}{dp_2} S^{(2),(1)}[p_1, p_2].
\]

(83)

At this point, by looking at equation (81), we observe the similarity between \( \left\langle \left( V'(p|\tau) \right)^2 \right\rangle_c \) and \( \sigma \). In fact, the two rate functions in equations (13) and (24) differ only for the point \( p \) in which we compute fluctuations. In the first case, this point is the \( p \) where the potential has the secondary minimum at \( T_d \) (\( \forall T < T_d \)). In the second case, this point is chosen by looking at the minimum of \( G_2(p) \) over \( p \) (\( \forall T > T_d \)) (figures B.1 and B.2).

In both the equations (81) and (83), the derivatives with respect to \( p_1 \) and \( p_2 \) require some care. In fact, they are total derivatives and saddle point parameters \( z \) are sensitive to shifts in \( p_1 \) and \( p_2 \). We will consider the case when \( p_1 \) and \( p_2 \) are very close. It useful to notice that in the limit \( p_1 \to p, p_2 \to p \), we expect the saddle point values of \( S^{(2)}[p_1, p_2] \) to be close to those of \( S^{(1)}[p] \). Their small perturbations can be computed taking the derivative of saddle point equations with respect to \( p_k \), \( k = 1, 2 \),

\[
\partial_{p_k} \nabla_z S^{(2)}[p_1, p_2] + H_z S^{(2)}[p_1, p_2] \partial_{p_k} z_* = 0,
\]

(84)

where \( \nabla_z \) denotes the gradient with respect to \( z = \{q_1, q_2, q_{12}\} \), \( H_z \) denotes the Hessian and \( * \) denotes the saddle point values at \( p_1 = p_2 = p \). We obtain

\[
\partial_{p_k} z_* = -[H_z S^{(2)}]^{-1} \partial_{p_k} \nabla_z S^{(2)}|_*.
\]

(85)

The explicit form of these derivatives is provided below. When \( p_k \) is shifted, in the RS anstaz \( z \) changes in the following way

\[
\frac{\partial q_k}{\partial p_k} |_{p_1=p_2=p} = -\frac{2 \sqrt{q - (q - 1)^2 \beta^2 f'(q)}}{2(q - 1) \beta^2 f'(q) + (q - 1)^2 \beta^2 f''(q) - 1}.
\]

(86)
Figure B.2. $G(p)$ and $G_2(p)$ at $\beta = \beta' = 1.51$ at the RS level for the 3-spin. The two lines cross at $p_{RS}$.

\[
\frac{\partial q_k}{\partial p_{k'}}|_{p_1=p_2=p} = 0, \quad k' \neq k; \quad (87)
\]
\[
\frac{\partial q_{12}}{\partial p_{k}}|_{p_1=p_2=p} = -\frac{\sqrt{q - (q - 1)^2}}{2(q - 1)\beta^2 f'(q) + (q - 1)^2\beta^2 f''(q) - 1}. \quad (88)
\]

We observe that order parameters relative to replica 1 (resp. $p_1$) are insensitive to small changes in $p_2$ (resp. $p_2$). Given the expansion in equation (41), changes induced in $q_k$ by $p_k$ are determined by first order terms and thus by $S^{(1)}$. We also notice that the shifts in the mixed term due to a change in $p_1$ and $p_2$ are the same. This information is very important in the large deviation computation, when we set $p_1 = p - \delta p / 2$ and $p_2 = p + \delta p / 2$ and thus we can set the shift in the mixed term to be $O(\delta p^2)$. Once obtained the shifts in the order parameters due to shifts in $p_1$ and $p_2$ around $p$, we may define the two following auxiliary functions and compute the variance in equation (83) as

\[
D_1[p_1, p_2] = \partial_{p_2} S^{(2),(1,1)}[p_1, p_2] + \nabla_z S^{(2),(1,1)}[p_1, p_2] \partial_{p_1} z
\]
\[
D_2[p_1, p_2] = \partial_{p_1} D_1[p_1, p_2] + \nabla_z D_1[p_1, p_2] \partial_{p_2} z
\]
\[
\left\langle \left( V'(|\tau|) \right)^2 \right\rangle_c = \frac{1}{2^2N_{\gamma_p}^{-1}p_{\gamma_p}^{-p}} \lim_{p_{\gamma_p}^{-p} \to p} D_2[p_1, p_2]. \quad (89)
\]

Having computed the fluctuations of the derivative of the potential, using equation (24), the final expression of $G_2(p)$ reads

\[
G_2(p) = -\frac{1}{2}(1 - q)^2[(q - 1)\beta^2[2f'(q) + (q - 1)f''(q)] - 1]^2 \\
\times \left[p + (q - 1)\beta^2 f'(p) \right]^2 \left[4(2p^2 - q)(q - 1)^2\beta^2 f'(q) \right]^2
\]
where \( q \) is given by equation (40) at any value of \( p \) and \( T > T_d \). As discussed above, this quantity gives both equations (13) and (24) at the RS level, by taking the appropriate \( p \), at any \( T > T_d \).

B.2. Large fluctuations

In this section we give more details on the computation of the large fluctuations. More precisely, we focus on the computation of equation (27). For commodity, we remind that our aim is to evaluate

\[
e^{N \Gamma(m, p_1, p_2)} = \langle \left( Z[p_1 | \tau] \right)^{m/\beta} \left( Z[p_2 | \tau] \right)^{-m/\beta} \rangle
\]

because knowing \( \Gamma(m, p_1, p_2) \) and using equation (18) leads to the large deviation function \( G(V_2 - V_1, p_1, p_2) \). We notice that taking \( m_1 \to m/\beta \) and \( m_2 \to -m/\beta \) in the definition of \( S^{(2)}[p_1, p_2] \) given in equation (35), leads to

\[
\Gamma(m, p_1, p_2) = \frac{S^{(2)}[p_1, p_2]}{2} \bigg|_{n_1 = n_2 = \frac{m}{\beta}}.
\]

For simplicity, we denote by \( S^{(2)}(u, p) \) the zero order term of the Taylor expansion of \( S^{(2)}[p_1 = p - \delta p/2, p_2 = p + \delta p/2] \) in \( \delta p \), computed at \( m = u/\delta p \),

\[
S^{(2)}(u, p) = \lim_{\delta p \to 0} S^{(2)} \left[ p - \frac{\delta p}{2}, p + \frac{\delta p}{2} \right] \bigg|_{n_1 = n_2 = \frac{m}{\beta}}.
\]

Its expression within a RS ansatz reads

\[
S^{(2)}(u, p) = u^2 \frac{f(q_1) + f(q_2) - 2 f(q_{12})}{\delta p^2} + \beta u \frac{f(q_2) - f(q_1)}{\delta p} + u \frac{\log(1 - q_1) - \log(1 - q_2)}{\beta \delta p} - 2u \beta f'(p) - \log(1 - q_1) - \log(1 - q_2) + \log \left( \frac{u^2 q_{12}^2 - 2p^2 q_{12} + p^2 q_1 + p^2 q_2 - q_1 q_2}{\beta^2 \delta p^2} \right),
\]

where \( q_1, q_2 \) and \( q_{12} \) parametrize the overlap matrix in equation (35). We observe the presence of apparent divergences in this expansion. They come from the rescaling of the parameter \( m \), that is necessary as long as we force \( V_2 - V_1 \sim O(\delta p^2) \) for \( T > T_d \), as observed in equation (21). However, these divergences are unphysical. When \( p_1 \) and \( p_2 \) are close enough to \( p \), saddle point values of the two-points function \( S^{(2)}[p_1, p_2] \) can be written as perturbations of those of \( S^{(1)}[p] \). The divergences disappear if we take into account the dependence of the...
parameters $q_1$, $q_2$ and $q_{12}$ on $\delta p$ at the saddle point
\[
\begin{align*}
q_1 &= q - (\delta p/2) \delta q \\
q_2 &= q + (\delta p/2) \delta q \\
q_{12} &= q + \delta p^2 \delta q_{12}
\end{align*}
\] (95)
where we indicate with $q$ the saddle point value of the RS potential for a value of the mutual overlap $p$. These two quantities are related by the saddle point equation
\[
p = \sqrt{q - \beta^2 f'(q)(1 - q)^2},
\] (96)
see equation (40). The perturbation in $q_{12}$ is $O(\delta p^2)$ because the two $O(\delta p)$ contributions induced by $\delta p_1 = -\delta p/2$ and $\delta p_2 = \delta p/2$ cancel out, as already observed in equations (86)–(88). The expression of $S^{(2)}$ when evaluated on equation (95) reads
\[
S^{(2)}(u, p) = \frac{u \delta q}{\beta(1 - q)} + u(\beta \delta q - 2u \delta q_{12})f'(q) - 2 \log \beta - 2 \log(1 - q) \\
+ \log \left[ q(u^2 - 2pu\beta - 2\beta^2 + 2u^2 \delta q_{12}) + pu(2\beta - u\delta q) + (q\beta)^2 + \frac{1}{4}(2\beta - u\delta q)^2 \right] \\
+ p^2(u(-2u\delta q_{12} + \beta \delta q)) - 2u\beta f'(p) + \frac{1}{4}u^2 \delta q^2 f''(q).
\] (97)
Finally we need to optimize over the parameters $\delta q$ and $\delta q_{12}$ and $u$. Thus we solve the system of equations
\[
\begin{align*}
\frac{\partial q}{\partial S^{(2)}(u, p)} &= 0 \\
\frac{\partial q_{12}}{\partial S^{(2)}(u, p)} &= 0, \\
\frac{\partial u}{\partial S^{(2)}(u, p)} &= 0
\end{align*}
\] (98)
The solution of this system of equations leads to an expression that depends only on $q$ and $p$. Using equation (92), we define $\Gamma(u^*, p) = S^{(2)}(u^*, p)/2$, where $*$ denotes the solution of equation (98),
\[
\Gamma(u^*, p) = \frac{1}{2} [(q - p^2)[f'(q) + q f''(q)]]^{-1} \left\{ (q - p^2)f'(q) \times \left( 1 + \log \frac{(q - p^2)}{\beta^2 f'(q)(1 - q)^2} + (q - 1)^2 \beta^2 f''(q) \right) - \beta f'(p) (\beta f'(q - 1) f'(p) + 2p) + \beta(q - 1)f''(q) \times \left( q - q \log \frac{(q - p^2)}{\beta^2 f'(q)(1 - q)^2} + 2\beta p(q - 1)f'(p) \right) \right\} \\
\times (q - 1)\beta f'(q) - q f'(q) + (p^2 - q)(q - 1)^2 \beta^2 f'(p)^2 f''(q).
\] (99)
and using equation (18) we also define $G(p) = -\Gamma(u^*, p)$. The large deviation function $I(T)$ is obtained by taking the minimum over $p$ of $G(p)$, as in equation (25),
\[
I_{LF}(T) = \min_p G(p)
\] (100)
The broken replica 1RSB computation is similar. As with the RS computation, the rescaling of $m$ leads to apparent divergent terms:

$$S^{(2)}(u, p) = u^2 \frac{f(q_1^4) + f(q_1^2) - 2 f(q_{12})}{\delta p^2} + \beta u \frac{u f(q_1^2) - f(q_1^1)}{\delta p}$$

$$+ \beta u \frac{x_2 f(q_1^2) - x_1 f(q_1^1)}{\delta p} - \beta u \frac{x_2 f(q_1^1) - x_1 f(q_1^1)}{\delta p}$$

$$+ u \log(1 - q_1^1) / x_2 - \log(1 - q_1^1) / x_1 - 2u \beta f(p)$$

$$+ u \log(1 + q_1^1(x_1 - 1) - q_1^0 x_1) - x_1 \beta p$$

$$- u \log(1 + q_1^1(x_2 - 1) - q_1^0 x_2) - x_2 \beta p$$

$$+ u \log(1 - q_1^1) - \log(1 - q_1^1) - \log(1 + q_1^1(x_2 - 1) - q_1^0 x_2)$$

$$+ \log \left( u + q_{12}^0 - 2p^2 q_{12} + p^2 q_1^0 + p^2 q_2^0 - q_{12}^0 q_{12}^0 \right).$$

(101)

In this case, for each $k = 1, 2$, $q_k^0$, $q_k^1$, and $x_k$ need to be shifted when $p_1$ and $p_2$ are shifted from $p$. These order parameters parametrize the 1RSB overlap matrix in equation (35). The apparent divergences are unphysical and in order to re-absorb them we set:

$$q_1^1 = q_1^1 - (\delta p/2) \delta q_1$$

$$q_2^1 = q_2^1 + (\delta p/2) \delta q_1$$

$$q_1^0 = q_1^0 - (\delta p/2) \delta q_0$$

$$q_2^0 = q_2^0 + (\delta p/2) \delta q_0$$

$$x_1 = x_1 - (\delta p/2) \delta x$$

$$x_2 = x_2 + (\delta p/2) \delta x$$

$$q_{12} = q_{12} + (\delta p^2) \delta q_{12}$$

(102)

where we indicate with $q_1^1$, $q_1^0$ and $x$ the saddle point values of the 1RSB potential for a value of the mutual overlap $p$. These quantities are determined by equations (55)–(57). When evaluating equation (101) on equation (102), the expression for $S^{(2)}$ has to be optimized over $\delta q_1^1$, $\delta q_1^0$, $\delta x$ and $\delta q_{12}$. However, these variables are not independent. This can be observed by setting

$$t = \frac{\beta(x - 1) \delta q_1 + (q_1 - q_0) \delta x}{2u}.$$  

(103)

With this replacement, $S^{(2)}$ depends only on $t$ and $\delta q_0$.

$$S^{(2)}(u, p) = \frac{u \delta q_0}{\beta(1 - q_1^1 + x \beta q_1^0 - q_1^0)} + u(x \beta q_0 - 2ut f(q_1^0) - 2 \log \beta$$

$$- 2 \log(1 + q_1^1(x - 1) - q_1^0 x) + \log(q_0 \left( u^2 - 2px \beta + 2x(1 + q_1^1 x) \beta^2 + 2u^2 t \right)$$

$$+ 1) \delta q_{12}.\]
The solution to this system of equations leads to an expression that depends only on $q^i$.

Using equation (35) we easily obtain

\[
\Delta^2 u = -2au + \frac{1}{4}a^2 \delta q^i \delta q^i
\]

and it can be optimized over these variables together with the optimization over $u$ as done in the RS computation:

\[
\begin{aligned}
\frac{\partial \log Z^{(2)}}{u} &= 0 \\
\frac{\partial S^{(2)}}{u} &= 0 \\
\frac{\partial S^{(2)}}{p} &= 0
\end{aligned}
\]

The solution to this system of equations leads to an expression that depends only on $q^i, q^j, x$ and $p$. One then proceeds as in the RS computation, setting $G(p) = -\Gamma(u^*, p)$ and computing the large deviation function $f(T)$ optimizing over $p$.

**Appendix C. Tilted potential**

In this section we derive the expression for the tilted potential discussed in section 4. Our aim here is to compute the value of the potential in a point, when we condition the potential to have a given derivative on the same point. We start with a few definitions:

\[
P(V_1, V_2 - V_1) = \langle \delta(V_1 - V(p_1|\tau))\delta(V_2 - V_1 - V(p_2|\tau)) \rangle
\]

\[
\int dV_1 V_1 P(V_1, V_2 - V_1) = \langle V(p_1|\tau)\delta(V_2 - V_1 - V(p_1|\tau) + V(p_2|\tau)) \rangle
\]

\[
= \lim_{\varepsilon \to 0} \varepsilon^{-1}\delta \langle \varepsilon \delta m \rangle Z(m^*, z, p_1, p_2)
\]

where $m^*$ denotes the saddle point value of $m$ and where we defined

\[
Z(m^*, z, p_1, p_2) = \langle e^{N \delta m (V_2 - V_1)} \rangle
\]

generalizing equation (17). Taking the logarithm of this function

\[
N\Gamma(m, z, p_1, p_2) = \log Z(m, z, p_1, p_2)
\]

we may write the integral as

\[
\int dV_1 V_1 P(V_1, V_2 - V_1) = \lim_{\varepsilon \to 0} \varepsilon^{-1} e^{N \delta m (V_2 - V_1) + N\Gamma(m^*, z, p_1, p_2)}
\]

\[
\times \partial_\varepsilon \Gamma(m^*, z, p_1, p_2).
\]

Using equation (35) we easily obtain

\[
\Gamma(m, z, p_1, p_2) = \frac{1}{2} S^{(2)}[p_1, p_2]_{\varepsilon \to 0} - zF
\]

and by taking $p_1 = p - \delta p/2$ and $p_2 = p + \delta p/2$, $m^* = u/\delta p$, denoting by

\[
\Gamma(u, z, p) = \lim_{\delta p \to 0} \Gamma \left( u/\delta p, z, p - \frac{\delta p}{2}, p + \frac{\delta p}{2} \right)
\]
\[ S^{(2)}(u, z, p) = \lim_{\delta p \to 0} S^{(2)} \left[ p - \frac{\delta p}{2}, p + \frac{\delta p}{2} \right] \]  
\[ S^{(2)}(u, z, p) = \lim_{\delta p \to 0} S^{(2)} \left[ p - \frac{\delta p}{2}, p + \frac{\delta p}{2} \right] \]  
\[ \frac{u_1}{\delta p} = \frac{u_2}{\delta p} \]  
we finally obtain the expression for the tilted potential

\[
\tilde{V}(p|V_2 - V_1) = \lim_{z \to 0} \partial_z \left[ \frac{1}{2} S^{(2)}(u, z, p) - zF \right]
\]

where the prefactor in equation (109) cancels out with the denominator \( P(V_2 - V_1) \) in equation (29), see also equation (16). \( \tilde{V}(p|V_2 - V_1) \) formally depends on the difference \( V_2 - V_1 = V(p + \delta p/2|\tau) - V(p - \delta p/2|\tau) \).

### Appendix D. Zero temperature

As discussed in the main text, the large deviation analysis can be done at \( T = 0 \) in order to study the rare events of the re-appearance of the local minimum when typically it does not exist, thus in the *hic sunt leones* phase as well. The \( T = 0 \) limit \( G_0(p) \) of the large deviation function can be done by taking \( q_1 = 1 - \chi T \) and \( x = yT \) in equation (100) and by taking the \( T = 0 \) limit. The 1RSB expression is valid for \( p < p_{RS} \), where RS breaks down. The RS marginality condition

\[
1 = \beta f''(1) (1 - q)^2
\]

at \( T = 0 \) is

\[
1 = \chi^2 f''(1). \tag{114}
\]

On the other hand, at \( T = 0 \) the RS saddle point equation relating \( q \) and \( p \), equation (96), becomes

\[
p = \sqrt{1 - \chi^2 f'(1)} \tag{115}
\]

and thus solving for \( p \) one finds \( p_{RS} = \sqrt{1 - f'(1)/f''(1)} \), i.e. \( p_{RS} = 0.707 107 \) in the 3-spin and \( p_{RS} = 0.781 736 \) in the 3 + 4-spin. For larger \( p \), the RS expression must be used, obtained by taking \( q_1 = 1 - \chi T \) in equation (99) and by taking the limit \( T \to 0 \),

\[
2G_0(p) = \left\{ \beta^2 \chi^2 f'(p)^2 \left( f'(1) - (p^2 - 1) f''(1) \right) \right. \\
- f'(1) \left[ \chi^2 (f''(1) + f'(1)) \log \left( \frac{(1 - p^2) \chi}{f'(1)} \right) \right. \\
+ \left( \chi^2 f'(1) - 1 \right) \left( \chi^2 f''(1) + 1 \right) \\
- 3 \chi^2 \log(\chi) \left( f''(1) + f'(1) \right) \\
- \left( 2 \beta p \chi f'(1)f'(p) \left( \chi^2 f''(1) + 1 \right) \right) \right. \\
\left. \times \left[ \chi^2 f'(1) \left( f''(1) + f'(1) \right) \right]^{-1} \right\} \tag{116}
\]

### ORCID iDs

Silvio Franz [https://orcid.org/0000-0001-8300-8443](https://orcid.org/0000-0001-8300-8443)

Jacopo Rocchi [https://orcid.org/0000-0002-4535-9426](https://orcid.org/0000-0002-4535-9426)
References

[1] Berthier L, Biroli G, Bouchaud J-P, Cipelletti L and van Saarloos W 2011 Dynamical Heterogeneities in Glasses, Colloids, and Granular Media vol 150 (Oxford: Oxford University Press)
[2] Franz S and Parisi G 2000 On non-linear susceptibility in supercooled liquids J. Phys.: Condens. Matter. 12 6335
[3] Donati C, Franz S, Sharon C G and Giorgio P 2002 Theory of non-linear susceptibility and correlation length in glasses and liquids J. Non-Cryst. Solids 307 215–24
[4] Bouchaud J-P and Biroli G 2004 On the Adam–Gibbs–Kirkpatrick–Thirumalai–Wolynes scenario for the viscosity increase in glasses Chem. Phys. 121 7347–54
[5] Bouchaud G and Bouchaud J-P 2004 Diverging length scale and upper critical dimension in the mode-coupling theory of the glass transition Europhys. Lett. 67 21
[6] Bouchaud J-P and Biroli G 2005 Nonlinear susceptibility in glassy systems: a probe for cooperative dynamical length scales Phys. Rev. B 72 064204
[7] Berthier L, Biroli G, Bouchaud J-P, Walter K, Miyazaki K and David R R 2007 Spontaneous and induced dynamic correlations in glass formers II. Model calculations and comparison to numerical simulations Chem. Phys. 126 184504
[8] Franz S, Parisi G, Ricci-Tersenghi F and Rizzo T 2011 Field theory of fluctuations in glasses Eur. Phys. J. E 34 102
[9] David R R and Charbonneau P 2005 Mode-coupling theory J. Stat. Mech. Theor. Exp. 2005 P05013
[10] Binder K and Peter Young A 1986 Rev. Mod. Phys. 58 801
[11] Kirkpatrick T R and Thirumalai D 1987 Phys. Rev. B 36 5388
[12] Mézard M, Parisi G and Virasoro M 1987 Spin Glass Theory and beyond: An Introduction to the Replica Method and its Applications vol 9 (Singapore: World Scientific)
[13] Kirkpatrick T R and Thirumalai D 1988 Comparison between dynamical theories and metastable states in regular and glassy mean-field spin models with underlying first-order-like phase transitions Phys. Rev. A 37 4439
[14] Kirkpatrick T R, Thirumalai D and Wolynes P G 1989 Phys. Rev. A 40 1045
[15] Fischer K H and Hertz J 1993 Spin Glasses vol 1 (Cambridge: Cambridge University Press)
[16] Parisi G 1994 Slow dynamics in glasses Il Nuovo Cimento D 16 939–47
[17] Franz S and Hertz J 1995 Glassy transition and aging in a model without disorder Phys. Rev. Lett. 74 2114
[18] Bouchaud J-P, Cugliandolo L F, Kurchan J and Mezard M 1998 Spin Glasses and Random Fields pp 161–223
[19] Franz S, Parisi G and Ricci-Tersenghi F 2013 Glassy critical points and the random field ising model J. Stat. Mech. Theor. Exp. 2013 L02001
[20] Franz S and Parisi G 2013 Universality classes of critical points in constrained glasses J. Stat. Mech. Theor. Exp. 2013 P11012
[21] Biroli G, Cammarota C, Tarjus G and Tarzia M 2014 Random-field-like criticality in glass-forming liquids Phys. Rev. Lett. 112 175701
[22] Biroli G, Cammarota C, Tarjus G and Tarzia M 2018 Random-field ising-like effective theory of the glass transition I. Mean-field models Phys. Rev. B 98 174205
[23] Biroli G, Cammarota C, Tarjus G and Tarzia M 2018 Random field ising-like effective theory of the glass transition II. Finite-dimensional models Phys. Rev. B 98 174206
[24] Biroli G and Bouchaud J-P 2012 The random first-order transition theory of glasses: a critical assessment Structural Glasses and Supercold Liquids: Theory, Experiment, and Applications ed PG Wolynes, V Lubchenko (Hoboken, NJ: Wiley) pp 31–114
[25] Barrat A, Franz S and Parisi G 1997 Temperature evolution and bifurcations of metastable states in mean-field spin glasses, with connections with structural glasses J. Phys. A: Math. Theor. 30 5593
[26] Sun YiF, Crisanti A, Krzakala F, Leuzzi L and Zdeborová L 2012 Following states in temperature in the spherical s + p-spin glass model J. Stat. Mech. Theor. Exp. 2012 P07002
[27] Folena G, Franz S and Ricci-Tersenghi F 2019 Memories from the ergodic phase: the awkward dynamics of spherical mixed p-spin models (arXiv:1903.01421)
[28] Franz S and Parisi G 1995 Recipes for metastable states in spin glasses J. Phys. I 5 1401–15
[29] Coluzzi B and Parisi G 1998 On the approach to the equilibrium and the equilibrium properties of a glass-forming model J. Phys. A: Math. Gen. 31 4349
[30] Parisi G 2009 On the replica scenario for the glass transition (arXiv:0911.2265)
[31] Cammarota C, Cavagna A, Giardina I, Gradenigo G, Grigera T S, Parisi G and Verrocchio P 2010 Phase-separation perspective on dynamic heterogeneities in glass-forming liquids Phys. Rev. Lett. 105 055703
[32] Parisi G and Seoane B 2014 Liquid-glass transition in equilibrium Phys. Rev. E 89 022309
[33] Berthier L 2013 Overlap fluctuations in glass-forming liquids Phys. Rev. E 88 022313
[34] Berthier L and Robert I J 2015 Evidence for a disordered critical point in a glass-forming liquid Phys. Rev. Lett. 114 205701
[35] Berthier L and Walter K 2012 Static point-to-set correlations in glass-forming liquids Phys. Rev. E 85 011102
[36] Biroli G, Bouchaud J-P, Cavagna A, Grigera T and Verrocchio P 2008 Thermodynamic signature of growing amorphous order in glass-forming liquids Nat. Phys. 4 771
[37] Franz S and Montanari A 2007 Analytic determination of dynamical and mosaic length scales in a kac glass model J. Phys. A: Math. Theor. 40 F251
[38] Crisanti A and Sommers H-J 1992 Z. Phys. B Condens. Matter 87 341–54
[39] Crisanti A, Horner H and Sommers H-J 1993 Z. Phys. B Condens. Matter 92 257–71
[40] Thouless D J, Anderson P W and Palmer R G 1977 Phil. Mag. 35 593–601
[41] Rieger H 1992 The number of solutions of the Thouless–Anderson–Palmer equations for p-spin-interaction spin glasses Phys. Rev. B 46 14655
[42] Mézard M, Parisi G and Virasoro M-A 1990 Spin Glass Theory and Beyond (Singapore: World Scientific)
[43] Cugliandolo L F and Kurchan J 1993 Analytical solution of the off-equilibrium dynamics of a long-range spin-glass model Phys. Rev. Lett. 71 173
[44] Bouchaud J-P 1992 Weak ergodicity breaking and aging in disordered systems J. Phys. I 2 1705–13
[45] Kirkpatrick T R and Thirumalai D 1987 Phys. Rev. Lett. 58 2091
[46] Kurchan J, Parisi G and Virasoro M-A 1993 J. Phys. I 3 1819–38
[47] Crisanti A and Sommers H-J 1995 J. Phys. I 5 805–13
[48] Monasson R 1995 Phys. Rev. Lett. 75 2847
[49] Barrat A 1997 The p-spin spherical spin glass model (arXiv:cond-mat/9701031)
[50] Castellani T and Cavagna A 2005 J. Stat. Mech. Theor. Exp. 2005 P05012
[51] Zamponi F 2010 arXiv:1008.4844
[52] Capone B, Castellani T, Giardina I and Ricci-Tersenghi F 2006 Off-equilibrium confined dynamics in a glassy system with level-crossing states Phys. Rev. B 74 144301
[53] Krzakala F and Zdeborová L 2010 Following gibbs states adiabatically, the energy landscape of mean-field glassy systems Europhys. Lett. 90 66002
[54] Barrat A, Burioni R and Mézard M 1996 J. Phys. A: Math. Gen. 29 L81
[55] Mézard M 1999 How to compute the thermodynamics of a glass using a cloned liquid Phys. Stat. Mech. Appl. 265 352–69
[56] Folena G, Franz S, Rocchi J, Ricci-Tersenghi F and Urbani P in preparation
[57] Folena G, Franz S, Rocchi J and Ricci-Tersenghi F in preparation