GENERALIZED TRANSLATION INVARIANT VALUATIONS
AND THE POLYTOPE ALGEBRA

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ABSTRACT. We study the space of generalized translation invariant valuations on a finite-dimensional vector space and construct a partial convolution which extends the convolution of smooth translation invariant valuations. Our main theorem is that McMullen’s polytope algebra is a subalgebra of the (partial) convolution algebra of generalized translation invariant valuations. More precisely, we show that the polytope algebra embeds injectively into the space of generalized translation invariant valuations and that for polytopes in general position, the convolution is defined and corresponds to the product in the polytope algebra.

1. Introduction

Let \( V \) be an \( n \)-dimensional vector space, \( V^* \) the dual vector space, \( \mathcal{K}(V) \) the set of non-empty compact convex subsets in \( V \), endowed with the topology induced by the Hausdorff metric for an arbitrary Euclidean structure on \( V \), and \( \mathcal{P}(V) \) the set of polytopes in \( V \). A valuation is a map \( \mu : \mathcal{K}(V) \to \mathbb{C} \) such that

\[
\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)
\]

whenever \( K, L, K \cup L \in \mathcal{K}(V) \). Continuity of valuations will be with respect to the Hausdorff topology.

Examples of valuations are measures, the intrinsic volumes (in particular the Euler characteristic \( \chi \)) and mixed volumes.

Let \( \text{Val}(V) \) denote the (Banach-)space of continuous, translation invariant valuations. It was the object of intensive research during the last few years, compare \( [3, 9, 11, 12, 14, 15, 16, 17, 19, 21] \) and the references therein.

Valuations with values in semi-groups other than \( \mathbb{C} \) have also attracted a lot of interest. We only mention the recent papers \( [1, 2, 18, 23, 25, 30, 31, 32, 33, 34] \) to give a flavor on this active research area.

The group \( \text{GL}(V) \) acts in the natural way on \( \text{Val}(V) \). The dense subspace of \( \text{GL}(V) \)-smooth vectors in \( \text{Val}(V) \) is denoted by \( \text{Val}^{\infty}(V) \). It carries a Fréchet topology which is finer than the induced topology.

In \( [10] \), a convolution product on \( \text{Val}^{\infty}(V) \otimes \text{Dens}(V^*) \) was constructed. Here and in the following, \( \text{Dens}(W) \) denotes the 1-dimensional space of densities on a linear space \( W \). Note that \( \text{Dens}(V) \otimes \text{Dens}(V^*) \cong \mathbb{C} \): if \( \text{vol} \) is any choice of Lebesgue measure on \( V \), and \( \text{vol}^* \) the corresponding dual measure on \( V^* \), then \( \text{vol} \otimes \text{vol}^* \in \text{Dens}(V) \otimes \text{Dens}(V^*) \) is independent of the choice of \( \text{vol} \). If \( \phi_i(K) = \text{vol}(K + A_i) \otimes \text{vol}^* \) with smooth compact strictly convex bodies \( A_1, A_2 \), then \( \phi_1 \ast \phi_2(K) = \text{vol}(K + A_1 + A_2) \otimes \text{vol}^* \).

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vol(\mathcal{K} + A_1 + A_2) \otimes \text{vol}^\ast$. By Alesker’s proof \cite{Alesker} of McMullen’s conjecture, linear combinations of such valuations are dense in the space of all smooth valuations. The convolution extends by bilinearity and continuity to $\text{Val}^\infty(V) \otimes \text{Dens}(V^\ast)$.

By \cite{Minkowski}, the convolution product is closely related to additive kinematic formulas. It was recently used in the study of unitary kinematic formulas \cite{Unitary}, local unitary additive kinematic formulas \cite{Local, Local2} and kinematic formulas for tensor valuations \cite{Tensor}.

In this paper, we will extend the convolution to a (partially defined) convolution on the space of generalized translation invariant valuations.

**Definition 1.1.** Elements of the space

$\text{Val}^{-\infty}(V) := \text{Val}^\infty(V)^\ast \otimes \text{Dens}(V)$

are called generalized translation invariant valuations.

By the Alesker-Poincaré duality \cite{Alesker}, $\text{Val}^\infty(V)$ embeds in $\text{Val}^{-\infty}(V)$ as a dense subspace. More generally, it follows from \cite{Alesker} Proposition 8.1.2 that $\text{Val}(V)$ embeds in $\text{Val}^{-\infty}(V)$, hence we have the inclusions

$\text{Val}^\infty(V) \subset \text{Val}(V) \subset \text{Val}^{-\infty}(V)$.

Generalized translation invariant valuations were introduced and studied in the recent preprint \cite{Generalized}. Note that another notion of generalized valuation was introduced by Alesker in \cite{Alesker, Alesker2, Alesker3, Alesker4}. In the next section, we will construct a natural isomorphism between the space of translation invariant generalized valuations in Alesker’s sense and the space of generalized translation invariant valuations in the sense of the above definition.

Given a polytope $P$ in $V$, there is an element $M(P) \in \text{Val}^{-\infty}(V) \otimes \text{Dens}(V^\ast) \cong \text{Val}^\infty(V)^\ast$ defined by

$\langle M(P), \phi \rangle = \phi(P)$.

Let $\Pi(V)$ be McMullen’s polytope algebra \cite{McMullen}. As a vector space, $\Pi(V)$ is generated by all symbols $[P]$, where $P$ is a polytope in $V$, modulo the relations $[P] \equiv [P + v], v \in V$, and $[P \cup Q] + [P \cap Q] = [P] + [Q]$ whenever $P, Q, P \cup Q$ are polytopes in $V$. The product is defined by $[P] \cdot [Q] := |P + Q|$.

The map $M : \mathcal{P}(V) \to \text{Val}^{-\infty}(V) \otimes \text{Dens}(V^\ast)$ extends to a linear map

$M : \Pi(V) \to \text{Val}^{-\infty}(V) \otimes \text{Dens}(V^\ast)$.

Our first main theorem shows that McMullen’s polytope algebra is a subset of $\text{Val}^{-\infty}(V) \otimes \text{Dens}(V^\ast)$.

**Theorem 1.** The map $M : \Pi(V) \to \text{Val}^{-\infty}(V) \otimes \text{Dens}(V^\ast)$ is injective. Equivalently, the elements of $\text{Val}^\infty(V)$ separate the elements of $\Pi(V)$.

In Section \ref{Transversality} we will introduce a notion of transversality of generalized translation invariant valuations. Our second main theorem is the following.

**Theorem 2.** There exists a partial convolution product $\ast$ on $\text{Val}^{-\infty}(V) \otimes \text{Dens}(V^\ast)$ with the following properties:

i) If $\phi_1, \phi_2 \in \text{Val}^{-\infty}(V) \otimes \text{Dens}(V^\ast)$ are transversal, then $\phi_1 \ast \phi_2 \in \text{Val}^{-\infty}(V) \otimes \text{Dens}(V^\ast)$ is defined.

ii) If $\phi_1 \ast \phi_2$ is defined and $g \in \text{GL}(V)$, then $(g \ast \phi_1) \ast (g \ast \phi_2)$ is defined and equals $g \ast (\phi_1 \ast \phi_2)$. 
Whenever the convolution is defined, it is bilinear, commutative, associative and of degree $-n$.

The restriction to the subspace $\text{Val}\,^\infty(V) \otimes \text{Dens}(V^*)$ is the convolution product from \cite{10}.

If $x, y$ are elements in $\Pi(V)$ in general position, then $M(x), M(y)$ are transversal in $\text{Val}\,^{-\infty}(V) \otimes \text{Dens}(V^*)$ and

$$M(x \cdot y) = M(x) * M(y).$$

Stated otherwise, the maps in the diagram

$$\text{Val}\,^\infty(V) \otimes \text{Dens}(V^*) \xrightarrow{\text{Val}\,^{-\infty}(V) \otimes \text{Dens}(V^*)} \Pi(V)$$

have dense images and are compatible with the (partial) algebra structures.

Remarks:

i) In Prop. 4.7 we will show that, under some technical conditions in terms of wave fronts, the convolution on generalized translation invariant valuations from Theorem 2 is the unique jointly sequentially continuous extension of the convolution product on smooth translation invariant valuations.

ii) In \cite{10}, it was shown that the space $V^{-\infty}(X)$ of generalized valuations on a smooth manifold $X$ admits a partial product structure extending the Alesker product of smooth valuations on $X$. If $X$ is real-analytic, then the space $\mathcal{F}(X)$ of $\mathbb{C}$-valued constructible functions on $X$ embeds densely into $V^{-\infty}(X)$. It was conjectured that whenever two constructible functions meet transversally, then the product in the sense of generalized valuations exists and equals the generalized valuation corresponding to the product of the two functions. The relevant diagram in this case is

$$\text{Val}\,^\infty(X) \xrightarrow{\text{Val}\,^{-\infty}(X) \otimes \text{Dens}(V^*)} \mathcal{F}(X),$$

where both maps are injections with dense images and are (conjecturally) compatible with the partial product structure. Theorem 5 in \cite{10} gives strong support for this conjecture.

iii) The Alesker-Fourier transform from \cite{9} extends to an isomorphism $F : \text{Val}\,^{-\infty}(V^*) \rightarrow \text{Val}\,^{-\infty}(V) \otimes \text{Dens}(V^*)$, compare \cite{12}. Another natural partially defined convolution on $\text{Val}\,^{-\infty}(V) \otimes \text{Dens}(V^*)$ would be

$$\psi_1 * \psi_2 := F^{-1} \psi_1 \cdot F^{-1} \psi_2,$$

where the dot is the partially defined product on $\text{Val}\,^{-\infty}(V^*)$ from \cite{10}. It seems natural to expect that this convolution coincides with the one from Theorem 2, but we do not have a proof of this fact.

Plan of the paper. In the next section, we introduce and study the space of generalized translation invariant valuations and explore its relation to generalized valuations from Alesker’s theory. In Section 3 we show that McMullen’s polytope algebra embeds into the space of generalized translation invariant valuations. A partial convolution structure on this space is constructed in Section 4. Section 5 is devoted to the proof of the fact that the embedding of the polytope algebra is compatible with the two convolution (product) structures. Finally, in the appendix we construct a certain current on the sphere which is related to the volumes of spherical joins. This current plays a major role in Section 6. Its construction is based on geometric measure theory and is independent from the rest of the paper.
2. Preliminaries

In this section $X$ will be an oriented $n$-dimensional smooth manifold and $S^*X$ the cosphere bundle over $X$. It consists of all pairs $(x, [\xi])$ where $x \in X$, $\xi \in T_x^*X$, $\xi \neq 0$ and where the equivalence relation is defined by $[\xi] = [\tau]$ if and only if $\xi = \lambda \tau$ for some $\lambda > 0$.

The projection onto $X$ is denoted by $\pi : S^*X \to X$, $(x, \xi) \mapsto x$. The antipodal map $s : S^*X \to S^*X$ is defined by $(x, [\xi]) \mapsto (x, [-\xi])$.

The push-forward map (also called fiber integration) $\pi_* : \Omega^k(S^*X) \to \Omega^{k-(n-1)}(X)$ satisfies

$$\int_{S^*X} \pi^* \gamma \wedge \omega = \int_X \gamma \wedge \pi_* \omega, \quad \gamma \in \Omega^{2n-k-1}(X).$$

If $V$ is a vector space, then $S^*V \cong V \times \mathbb{P}_+(V^*)$, where $\mathbb{P}_+(V^*) := V^* \setminus \{0\}/\mathbb{R}_+^*$ is the sphere in $V^*$. Moreover, if $V$ is an Euclidean vector space of dimension $n$, we identify $S^*V$ and $SV = V \times S^{n-1}$ and write $\pi = \pi_1 : SV \to V$, $\pi_2 : SV \to S^{n-1}$ for the two projections.

2.1. Currents. Let us recall some terminology from geometric measure theory. We refer to [20, 27] for more information.

The space of $k$-forms on $X$ is denoted by $\Omega^k(X)$, the space of compactly supported $k$-forms is denoted by $\Omega^k_c(X)$. Elements of the dual space $D_k(X) := \Omega^k_c(X)^*$ are called $k$-currents. A 0-current is also called distribution.

The boundary of a $k$-current $T \in D_k(X)$ is defined by $\langle \partial T, \phi \rangle = \langle T, d\phi \rangle$, $\phi \in \Omega^{k-1}_c(X)$. If $\partial T = 0$, $T$ is called a cycle. If $T \in D_k(X)$ and $\omega \in \Omega^l(X)$, $l \leq k$, then the current $T \wedge \omega \in D_{k-l}(X)$ is defined by $\langle T \wedge \omega, \phi \rangle := \langle T, \omega \wedge \phi \rangle$.

If $f : X \to Y$ is a smooth map between smooth manifolds $X,Y$ and $T \in D_k(X)$ such that $f|_{\text{spt}\, T}$ is proper, then the push-forward $f_*T \in D_k(Y)$ is defined by $\langle f_*T, \phi \rangle := \langle T, \zeta f^* \phi \rangle$, where $\zeta \in C^\infty_c(X)$ is equal to 1 in a neighborhood of $\text{spt}\, T \cap \text{spt}\, f^* \omega$. It is easily checked that

$$\partial([X]|_l, \omega) = (-1)^{\deg \omega+1} [[X]|_l, (d\omega) \frac{|\omega|}{\omega}, \ldots (d\omega) \frac{|\omega|}{\omega}], \quad (1)$$

and

$$\pi_* ([S^*X]|_l, \omega) = (-1)^{(n+1)(\deg \omega+1)} [[X]|_l, \pi_* \omega.] \quad (2)$$

Every oriented submanifold $Y \subset X$ of dimension $k$ induces a $k$-current $[Y]$ such that $\langle [Y], \phi \rangle = \int_Y \phi$. By Stokes' theorem, $\partial[Y] = [\partial Y]$. A smooth current is a current of the form $[X]|_l, \omega \in D_{n-k}(X)$ with $\omega \in \Omega^k(X)$.

If $X$ and $Y$ are smooth manifolds, $T \in D_k(X)$, $S \in D_l(Y)$, then there is a unique current $T \times S \in D_{k+l}(X \times Y)$ such that $\langle T \times S, \pi^*_1 \omega \wedge \pi^*_2 \phi \rangle = \langle T, \omega \rangle \cdot \langle S, \phi \rangle$, for all $\omega \in \Omega^k(X)$, $\phi \in \Omega^l(Y)$. Here $\pi_1, \pi_2$ are the projections from $X \times Y$ to $X$ and $Y$ respectively.

If $T = [[X]|_l, \omega, S = [[Y]|_l, \phi$, then

$$T \times S = (-1)^{(\dim X - \deg \omega) \deg \phi} [[X \times Y]|_l, (\omega \wedge \phi). \quad (3)$$

The boundary of the product is given by

$$\partial(T \times S) = \partial T \times S + (-1)^k T \times \partial S, \quad (4)$$

compare [20, 4.1.8].
If $X$ is a Riemannian manifold, the mass of a current $T \in D^k(X)$ is

$$M(T) := \sup\{ \langle T, \phi \rangle : \phi \in \Omega^k_c(X), \|\phi(x)\|_* \leq 1, \forall x \in X\},$$

where $\| \cdot \|_*$ denotes the comass norm.

Currents of finite mass having a boundary of finite mass are called normal currents.

The flat norm of $T$ is defined by

$$F(T) := \sup\{ \langle T, \phi \rangle : \phi \in \Omega^k_c(X), \|\phi(x)\|_* \leq 1, \|d\phi(x)\|_* \leq 1, \forall x \in X\}.$$

If $X$ is compact, then the $F$-closure of the space of normal $k$-currents is the space of real flat chains.

2.2. Wave fronts. We refer to [22] and [24] for the general theory of wave fronts and its applications. For the reader’s convenience and later reference, we will recall some basic definitions and some fundamental properties of wave fronts, following [24].

First let $X$ be a linear space of dimension $n$, $T$ a distribution on $X$.

The cone $\Sigma(T) \subset X^*$ is defined as the closure of the complement of the set of all $\eta \in X^*$ such that for all $\xi$ in a conic neighborhood of $\eta$ we have

$$\|\hat{T}(\xi)\| \leq C_N(1 + \|\xi\|)^{-N}, \quad N \in \mathbb{N}$$

(with constants $C_N$ only dependent on $N$ and the chosen neighborhood). Here $\hat{T}$ denotes the Fourier transform of $T$, and the norm is taken with respect to an arbitrary scalar product on $X^*$.

Next, for an affine space $X$ and a point $x \in X$, the set $\Sigma_x(T) \subset T^*_x X$ is defined by

$$\Sigma_x(T) := \bigcap_{\phi} \Sigma(\phi T),$$

where $\phi$ ranges over all compactly supported smooth functions on $X$ with $\phi(x) \neq 0$. Note that one uses the canonical identification $T^*_x X = X^*$.

The wave front set of $T$ is by definition

$$WF(T) := \{(x, [\xi]) \in S^* X : \xi \in \Sigma_x(T)\}.$$

The set $\text{singsupp}(T) := \pi(WF(T)) \subset X$ is called singular support.

Let $(x_1, \ldots, x_n)$ be coordinates on $X$. Given a current $T \in D^k(X)$, we may write

$$T = \sum_{I=\{i_1, \ldots, i_k\}} T_I \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}$$

with distributions $T_I$. Then the wave front of $T$ is defined as

$$WF(T) := \bigcup_I WF(T_I).$$

A current $T$ is smooth, i.e. given by integration against a smooth differential form, if and only if $WF(T) = \emptyset$.

**Definition 2.1.** Let $\Gamma \subset S^* X$ be a closed set. Then we set

$$D^k_{\Gamma}(X) := \{ T \in D^k(X) : WF(T) \subset \Gamma \}.$$ 

A sequence $T_j \in D^k_{\Gamma}(X)$ converges to $T \in D^k_{\Gamma}(X)$ if (writing $T_j, T$ as above)

$T_j, I \to T_I$ weakly in the sense of distributions and for each compactly supported
function $\phi \in C_c^\infty(X)$ and each closed cone $A$ in $X^*$ such that $\Gamma \cap (\text{spt } \phi \times A) = \emptyset$ we have
\[
\sup_{\xi \in A} \| \phi T_j(\xi) - \phi T_{j,1}(\xi) \| \to 0, \quad j \to \infty
\]
for all $N \in \mathbb{N}$.

**Proposition 2.2** ([24] Thm. 8.2.3]). Let $T \in \mathcal{D}_{k,T}(X)$. Then there exists a sequence of compactly supported smooth $k$-forms $\omega_i \in \Omega^{n-k}(X)$ such that $[[X]]_\ast \omega_i \to T$ in $\mathcal{D}_{k,T}(X)$. In other words, smooth forms are dense in $\mathcal{D}_{k,T}(X)$.

**Proposition 2.3** ([24] Thm. 8.2.10]). Let $T_1 \in \mathcal{D}_{k_1}(X), T_2 \in \mathcal{D}_{k_2}(X)$ such that the following transversality condition is satisfied:
\[
\text{WF}(T_1) \cap s\text{WF}(T_2) = \emptyset.
\]
Then the intersection current $T_1 \cap T_2 \in \mathcal{D}_{k_1+k_2-n}(X)$ is well-defined. More precisely, if $[[X]]_\ast \omega_1^j \to T_j$ in $\mathcal{D}_{k_j,\text{WF}(T_j)}(X)$ with $\omega_1^j \in \Omega^{n-k_j}(X)$, $j = 1, 2$, then $[[X]]_\ast (\omega_1^1 \wedge \omega_2^2) \to T_1 \cap T_2$ in $\mathcal{D}_{k_1+k_2-n,\Gamma}(X)$, where
\[
\Gamma := \text{WF}(T_1) \cup \text{WF}(T_2) \cup \{ (x, [\xi_1 + \xi_2]) : (x, [\xi_1]) \in \text{WF}(T_1), (x, [\xi_2]) \in \text{WF}(T_2) \}.
\]
The boundary of the intersection is given by
\[
\partial(T_1 \cap T_2) = (-1)^{n-k_2} \partial T_1 \cap T_2 + T_1 \cap \partial T_2. \tag{5}
\]

The wave front of a distribution is defined locally and behaves well under coordinate changes. Using local coordinates, one can define the wave front set $\text{WF}(T) \subset S^*X$ for a distribution $T$ on a smooth manifold $X$. Definition 2.1 and Propositions 2.2 and 2.3 remain valid in this greater generality.

We will need a special case of these constructions.

**Proposition 2.4.** Let $Y \subset X$ be a compact oriented $k$-dimensional submanifold. Then
\[
\text{WF}([[Y]]) = N_X(Y) = \{(x, [\xi]) \in S^*X|_Y : \xi|_{T_xY} = 0 \}.
\]

An example for Proposition 2.3 is when $T_i = [[Y_i]]$ with oriented submanifolds $Y_1, Y_2 \subset X$ intersecting transversally (in the usual sense). Then Proposition 2.3 implies that the transversality condition in Proposition 2.3 is satisfied, and $T_1 \cap T_2 = [[Y_1 \cap Y_2]]$.

It is easily checked using (3), that for currents $A_1, A_2$ on an $n$-dimensional manifold $X$ and currents $B_1, B_2$ on an $m$-dimensional manifold $Y$, we have
\[
(A_1 \times B_1) \cap (A_2 \times B_2) = (-1)^{(n - \deg A_1)(m - \deg B_2)}(A_1 \cap A_2) \times (B_1 \cap B_2) \tag{6}
\]
whenever both sides are well-defined.

Given a differential operator, we have $\text{WF}(PT) \subset \text{WF}(T)$ with equality in case $P$ is elliptic [24] (8.1.11 and Corollary 8.3.2). In particular, it follows that for a current on a manifold $X$, we have
\[
\text{WF}(\partial T) \subset \text{WF}(T). \tag{7}
\]
If $T$ is a current on the sphere $S^{n-1}$ and $\Delta$ the Laplace-Beltrami operator, then
\[
\text{WF}(\Delta T) = \text{WF}(T). \tag{8}
\]
2.3. **Valuations.** Let us now briefly recall some notions from Alesker’s theory of valuations on manifolds, referring to [5, 6, 7, 8, 13] for more details.

Let $X$ be a smooth manifold of dimension $n$, which for simplicity we suppose to be oriented. Let $\mathcal{P}(X)$ be the space of all compact differentiable polyhedra on $X$. Given $P \in \mathcal{P}(X)$, the conormal cycle $N(P)$ is a Legendrian cycle in the cosphere bundle $S^*X$ (i.e. $\partial N(P) = 0$ and $N(P) \cap \alpha = 0$ where $\alpha$ is the contact form on $S^*X$). A map of the form

$$ P \mapsto \int_{N(P)} \omega + \int_P \gamma, \quad P \in \mathcal{P}(X), \omega \in \Omega^{n-1}(S^*X), \gamma \in \Omega^n(X) $$

is called a smooth valuation on $X$. The space of smooth valuations on $X$ is denoted by $V^\infty(X)$. It carries a natural Fréchet space topology. The subspace of compactly supported smooth valuations is denoted by $V^\infty_c(X)$. The valuation defined by the above equation will be denoted by $\nu(\omega, \gamma)$.

We remark that, without using an orientation, we can still define $\nu(\omega, \phi)$, where $\omega \in \Omega^{n-1}(S^*X) \otimes \text{or}(X), \phi \in \Omega^n(X) \otimes \text{or}(X)$. Here $\text{or}(X)$ is the orientation bundle over $X$.

Elements of the space

$$ V^{\infty}(X) := (V^\infty_c(X))^* $$

are called generalized valuations [8]. Each compact differentiable polyhedron $P$ defines a generalized valuation $\Gamma(P)$ by

$$ \langle \Gamma(P), \phi \rangle := \phi(P), \quad \phi \in V^\infty_c(X). $$

A smooth valuation can be considered as a generalized valuation by Alesker-Poincaré duality [8, Thm. 6.1.1.].

We thus have injections

$$ V^\infty(X) \hookrightarrow V^{\infty}(X) \hookrightarrow \mathcal{P}(X). $$

By the results in [15] and [10], a generalized valuation $\phi \in V^{\infty}(X)$ is uniquely described by a pair of currents $E(\phi) = (T(\phi), C(\phi)) \in D_{n-1}(S^*X) \times D_n(X)$ such that

$$ \partial T = 0, \pi_* T = \partial C, T \text{ is Legendrian, i.e. } T \cap \alpha = 0. \quad (9) $$

Note that, in contrast to different uses of the word Legendrian in the literature, $T$ is not assumed to be rectifiable.

Given $(T, C)$ satisfying these conditions, we denote by $E^{-1}(T, C)$ the corresponding generalized valuation.

If $\mu$ is a compactly supported smooth valuation on $X$, then we may represent $\mu = \nu(\omega, \gamma)$ with compactly supported forms $\omega, \gamma$. If $E(\phi) = (T, C)$, then

$$ \langle \phi, \mu \rangle = T(\omega) + C(\gamma). $$

In particular, the generalized valuation $\Gamma(P)$ corresponding to $P \in \mathcal{P}(X)$ satisfies

$$ E(\Gamma(P)) = (N(P), [[P]]). \quad (10) $$

If $\phi = \nu(\omega, \gamma)$ is smooth, then

$$ T(\phi) = [[S^*X]] \iota s^*(D\omega + \pi^*\gamma), \quad (11) $$

$$ C(\phi) = [[X]] \iota \pi_* \omega , \quad (12) $$

where $D$ is the Rumin operator and $s$ is the involution on $S^*X$ given by $[\xi] \mapsto [-\xi]$ [10, 28].
Let us specialize to the case where \( X = V \) is a finite-dimensional vector space. We denote by \( \mathcal{V}^\infty(V)^{tr} \) and \( \mathcal{V}^{-\infty}(V)^{tr} \) the spaces of translation invariant elements. Alesker has shown in [6] that

\[
\mathcal{V}^\infty(V)^{tr} \cong \text{Val}^\infty(V).
\]

A similar statement for generalized valuations is shown in the next proposition.

**Proposition 2.5.** The transpose of the map

\[
F : \mathcal{V}_c^\infty(V) \to \text{Val}^\infty(V) \otimes \text{Dens}(V^*)
\]

\[
\mu \mapsto \int_V \mu(\bullet + x) d\text{vol}(x) \otimes \text{vol}^*
\]

induces an isomorphism

\[
F^* : \text{Val}^{-\infty}(V) \xrightarrow{\cong} \mathcal{V}^{-\infty}(V)^{tr}.
\]

The diagram

\[
\begin{array}{ccc}
\text{Val}^{-\infty}(V) & \xrightarrow{F^*} & \mathcal{V}^{-\infty}(V)^{tr} \\
\uparrow & & \downarrow \cong \\
\text{Val}^\infty(V) & \xrightarrow{\cong} & \mathcal{V}^\infty(V)^{tr}
\end{array}
\]

commutes and the vertical maps have dense images.

**Proof.** First we show that the diagram is commutative, i.e. that the restriction of \( F^* \) to \( \text{Val}^\infty(V) \) is the identity.

Let \( \phi \in \text{Val}^\infty(V) \) be a smooth valuation. We will show that for any \( \mu \in \mathcal{V}_c^\infty(V) \) one has

\[
\langle F^* \phi, \mu \rangle_{\mathcal{V}^\infty(V)} = \langle \phi, \mu \rangle_{\text{Val}^\infty(V)},
\]

or equivalently that

\[
\langle \phi, F \mu \rangle_{\text{Val}^\infty(V)} = \langle \phi, \mu \rangle_{\text{Val}^\infty(V)}.
\]

Fix a Euclidean structure on \( V \), which induces canonical identifications \( \text{Dens}(V) \cong \mathbb{C} \) and \( S^*V \cong V \times S^{n-1} \). We may assume by linearity that \( \phi \) is \( k \)-homogeneous. Represent \( \mu = \nu(\omega, \gamma) \) with some compactly supported forms \( \omega \in \Omega^{n-1}_c(S^*V) \), \( \gamma \in \Omega^n_0(V) \).

We may write

\[
F \mu = \nu \left( \int_V x^* \omega d\text{vol}(x), \int_V x^* \gamma d\text{vol}(x) \right).
\]

If \( k < n \), then \( \phi = \nu(\beta, 0) \) for some form \( \beta \in \Omega^{n-1}(S^*V)^{tr} \) by the irreducibility theorem [3]. By the product formula from [10] we have

\[
\langle \phi, \mu \rangle_{\text{Val}^\infty(V)} = \int_{S^*V} \omega \wedge \pi^* \beta + \int_V \gamma \wedge \pi^* \beta \tag{13}
\]

and

\[
\langle \phi, F \mu \rangle_{\text{Val}^\infty(V)} = \pi_* \left( \int_V x^* \omega d\text{vol}(x) \wedge \pi^* \beta \right) + \int_{S^{n-1}} \beta \cdot \int_V x^* \gamma d\text{vol}(x). \tag{14}
\]

The second summand in (13) is

\[
\int_V \gamma \wedge \pi^* \beta = \int_{S^{n-1}} \beta \cdot \int_V \gamma.
\]
which coincides with the second summand of \([14]\).

Denoting \(\psi := s^* D \beta \in \Omega^n(S^* V)^{tr}\) and \(\tau := \omega \wedge \psi \in \Omega_{\nu}^{n-1}(S^* V)\), it remains to verify that

\[
\pi_* \left( \int_{V} x^* \omega \, d \mathrm{vol}(x) \wedge \psi \right) = \int_{S^* V} \omega \wedge \psi,
\]

which is equivalent to

\[
\pi_* \left( \int_{V} x^* \tau \, d \mathrm{vol}(x) \right) = \left( \int_{S^* V} \tau \right) \mathrm{vol}.
\]

Write \(\tau = f(y, \theta) d \mathrm{vol}(y) d \theta\) for \(y \in V, \ theta \in S^{n-1}\) and \(d \theta\) the volume form on \(S^{n-1}\). Then

\[
\begin{align*}
\pi_* \left( \int_{V} x^* \tau \, d \mathrm{vol}(x) \right) (y) &= \left( \int_{S^{n-1}} \left( \int_{V} f(y + x, \theta) \, d \mathrm{vol}(x) \right) d \theta \right) \mathrm{vol}(y) \\
&= \left( \int_{S^{n-1}} \left( \int_{V} f(x, \theta) \, d \mathrm{vol}(x) \right) d \theta \right) \mathrm{vol}(y) \\
&= \left( \int_{S^* V} \tau \right) \mathrm{vol}(y),
\end{align*}
\]

as required.

Now assume \(k = n\), so \(\phi = \nu(0, \lambda \, \mathrm{vol})\) is a Lebesgue measure on \(V\). Then

\[
(\phi, F \mu)_{\nu^\infty(V)} = \lambda \pi_* \left( \int_{V} x^* \omega \, d \mathrm{vol}(x) \right)
\]

and

\[
(\phi, \mu)_{\nu^\infty(V)} = \lambda \int_{V} \pi_* \omega \, \mathrm{vol}.
\]

Since the right hand sides coincide, the commutativity of the diagram follows.

We proceed to show surjectivity of \(F\). Let \(\phi\) be a smooth translation invariant valuation on \(V\). We fix translation invariant differential forms \(\omega \in \Omega^{n-1}(S^* V), \gamma \in \Omega^n(V)\) with \(\phi = \nu(\omega, \gamma)\).

Let \(\mathrm{vol}\) be a density on \(V\) and let \(\beta\) be a compactly supported smooth function such that \(\int_{V} \beta(x) \, d \mathrm{vol}(x) = 1\).

The valuation \(\mu := \nu(\pi^*(\beta) \wedge \omega, \beta \gamma)\) is smooth, compactly supported and satisfies \(F(\mu) = \phi \otimes \mathrm{vol}^*\). This shows that \(F\) is onto. Thus \(F\) induces an isomorphism

\[
\tilde{F} : \nu^\infty(V) / \ker F \xrightarrow{\cong} \nu^\infty(V) \otimes \mathrm{Dens}(V^*).
\]

The transpose of \(\tilde{F}\) is an isomorphism

\[
\tilde{F}^* : \nu^{-\infty}(V) \otimes \mathrm{Dens}(V^*) \to \nu^{-\infty}(V).
\]

The proof will be finished once we can show \((\ker F)^\perp = \nu^{-\infty}(V)^{tr}\).

If \(\mu \in \nu^\infty(V)\), then \((t_v)_* \mu - \mu |_{\ker F}\) for every \(v \in V\), where \(t_v\) is the translation by \(v\). It follows that \((\ker F)^\perp \subset \nu^{-\infty}(V)^{tr}\).

Let \(\phi \in \nu^{-\infty}(V)^{tr}\). Fix a compactly supported approximate identity \(f_\varepsilon\) in \(\mathrm{GL}(n)\) and set \(\phi_\varepsilon := \phi \ast f_\varepsilon\). Then \(T(\phi_\varepsilon) = T(\phi) \ast f_\varepsilon\) and \(C(\phi_\varepsilon) = C(\phi) \ast f_\varepsilon\) are smooth currents. By \([10]\) Lemma 8.1], \(\phi_\varepsilon \in \nu^\infty(V)\) and \(\phi_\varepsilon \to \phi\). This shows that \(\nu^{-\infty}(V) \cong \nu^{-\infty}(V)^{tr}\) is dense in \(\nu^{-\infty}(V)^{tr}\).

Clearly \(\text{Im} (F^* : \nu^\infty(V) \to \nu^{-\infty}(V)^{tr}) \subset (\ker F)^\perp\). Since the image is dense in \(\nu^{-\infty}(V)^{tr}\), it follows that \(\nu^{-\infty}(V)^{tr} \subset (\ker F)^\perp\). \(\square\)
Definition 2.6. For \( \phi \in \text{Val}^{-\infty}(V) \) we set \( \text{WF}(\phi) := \text{WF}(T(\phi)) \subset S^*(S^*V) \). Given \( \Gamma \subset S^*(S^*V) \) a closed set, we define
\[
\text{Val}^{-\infty}_{\Gamma}(V) := \{ \phi \in \text{Val}^{-\infty}(V) : \text{WF}(\phi) \subset \Gamma \}.
\]

Lemma 2.7. The subspace \( \text{Val}^\infty(V) \subset \text{Val}^{-\infty}_{\Gamma}(V) \) is dense.

Proof. This follows by [10, Lemma 8.2], noting that in the proof of that lemma, a translation invariant generalized valuation is approximated by translation invariant smooth valuations. \( \square \)

3. Embedding McMullen’s polytope algebra

Let \( V \) denote an \( n \)-dimensional real vector space, and \( \Pi(V) \) the McMullen polytope algebra on \( V \). It is defined as the abelian group generated by polytopes, with the relations of the inclusion-exclusion principle and translation invariance. The product is defined on generators by \([P] \cdot [Q] := [P + Q]\). It is almost a graded algebra over \( \mathbb{R} \). We refer to [26] for a detailed study of its properties.

For \( \lambda \in \mathbb{R} \), the dilatation \( \Delta(\lambda) \) is defined by \( \Delta(\lambda)[P] = [\lambda P] \). The \( k \)-th weight space is defined by
\[
\Xi_k := \{ x \in \Pi : \Delta(\lambda)x = \lambda^k x \text{ for some rational } \lambda > 0, \lambda \neq 1 \}.
\]

McMullen has shown ([26], Lemma 20) that \( \Pi(V) = \bigoplus_{k=0}^{n} \Xi_k \).

The aim of this section is to prove Theorem [11] namely that \( \Pi(V) \) embeds into \( \text{Val}^{-\infty}(V) \otimes \text{Dens}(V^*) \).

Recall that the map \( M : \Pi(V) \to \text{Val}^{-\infty}(V) \otimes \text{Dens}(V^*) \cong \text{Val}^\infty(V)^* \) is defined by
\[
\langle M([P]), \phi \rangle = \phi(P), \quad P \in \mathcal{P}(V).
\]

We will denote by \( M_k : \Pi(V) \to \text{Val}_k(V) \otimes \text{Dens}(V^*) \) the \( k \)-homogeneous component of the image of \( M \), and similarly \( [\phi]_k \) is the \( k \)-homogeneous component of a valuation \( \phi \).

Lemma 3.1. \( M([P]) = \text{vol}(\bullet - P) \otimes \text{vol}^* \).

In particular \( \text{Im}(M) \subset \text{Val}(V) \otimes \text{Dens}(V^*) \).

Proof. We claim that \( \langle \text{PD}(\psi), \text{vol}(\bullet - K) \otimes \text{vol}^* \rangle = \psi(K) \) for \( \psi \in \text{Val}^\infty(V) \) and \( K \in \mathcal{K}(V) \), where \( \text{PD} : \text{Val}^\infty(V) \to \text{Val}^{-\infty}(V) \) denotes the natural embedding given by the Poincaré duality of Alesker [4].

Indeed, let first \( K \) be smooth with positive curvature. Then, by ([14], (15)),
\[
\psi \cdot \text{vol}(\bullet - K) = \int_V \psi((y + K) \cap \bullet) d\text{vol}(y)
\]
and hence
\[
\langle \text{PD}(\psi), \text{vol}(\bullet - K) \otimes \text{vol}^* \rangle = \psi(K).
\]

By approximation, this holds for non-smooth \( K \) as well, showing the claim. Since by definition \( \langle M[P], \psi \rangle = \psi(P) \), the statement follows. \( \square \)
Lemma 3.2. Let $P$ be a polytope and $\Gamma(P) \in V^{-\infty}(V)$ the associated generalized valuation, i.e. $(\Gamma(P), \phi) = \phi(P)$ for $\phi \in V_c^{-\infty}(V)$. Then

$$F^* M([P]) = \int_V \Gamma(P + x) d\text{vol}(x) \otimes \text{vol}^*,$$

with $F$ as in Proposition 2.5.

Proof. Let $\phi \in V_c^{-\infty}(V)$. Then

$$\langle F^* M([P]), \phi \rangle = \langle M([P]), F\phi \rangle = F(\phi)(P) = \int_V \phi(P + x) d\text{vol}(x) \otimes \text{vol}^* = \int_V \langle \Gamma(P + x), \phi \rangle d\text{vol}(x) \otimes \text{vol}^* = \left\langle \int_V \Gamma(P + x) d\text{vol}(x) \otimes \text{vol}^*, \phi \right\rangle.$$

\[\square\]

For the following, we fix a Euclidean structure and an orientation on $V$ and denote by vol the corresponding Lebesgue measure on $V$.

Denote by $C_j$ the collection of $j$-dimensional oriented submanifolds $N \subset S^{n-1}$, obtained by intersecting $S^{n-1}$ with a $(j + 1)$-dimensional polytopal cone $\hat{N}$ in $V$.

Given $v \in \Lambda^k V$, define the current $[v] \in D_k(V)$ by

$$\langle [v], \omega \rangle := \int_V \omega|_x(v) d\text{vol}(x), \omega \in \Omega^k_c(V).$$

Let $\Lambda^k V$ denote the cone of simple $k$-vectors in $V$. Given a pair $(v, N) \in \Lambda^k V \times C_{n-k-1}$, we define the current $A_{v,N} = [v] \times [[N]] \in D_{n-1}(V \times S^{n-1})$, where $[[N]]$ is the current of integration over $N$.

Observe that changing the sign of $v$ and the orientation of $N$ simultaneously leaves the current $A_{v,N}$ invariant.

Let $Y_k \subset D_{n-1}(V \times S^{n-1})$ be the $\mathbb{C}$-span of currents of the form $A_{v,N}, v \in \Lambda^k V, N \in C_{n-k-1}$ such that $\text{Span}(v) \oplus \text{Span}(\hat{N}) = V$ as oriented spaces.

Given a polytope $P \subset V$, we let $F_k$ be the set of $k$-faces of $P$. Each face is assumed to possess some fixed orientation. If $F$ is a face of $P$ of dimension strictly less than $n$, we let $n(F, P)$ be the normal cone of $F$, and $\hat{n}(F, P) := n(F, P) \cap S^{n-1}$ is oriented so that the linear space parallel to $F$, followed by $\text{Span}(\hat{n}(F, P))$, is positively oriented. Moreover, let $v_F$ be the unique $k$-vector in the linear space parallel to $F$ such that $|v_F| = \text{vol}_k F$, the sign determined by the orientation of $F$.

Lemma 3.3. Let $P$ be a polytope. Then

$$E(M_0([P])) = (0, \text{vol}(P)[[V]]),$$

$$E(M_{n-k}([P])) = \left( \sum_{F \in F_k(P)} A_{v_F, \hat{n}(F,P)}, 0 \right), \quad 0 \leq k \leq n - 1.$$
Proof. It follows from (10) and Lemma 3.2 that
\[ T(M([P])) = \int_V T(\Gamma(P + x)) d\text{vol}(x) = \sum_{F \in \mathcal{F}_{n-1}(P)} \Lambda_{vF, \bar{n}(F, P)} \]
\[ C(M([P])) = \int_V C(\Gamma(P + x)) d\text{vol}(x) = \text{vol}(P)(|V|). \]
From this the statement follows. \(\square\)

Let \(T_k : \text{Im}(M_{n-k}) \to Y_k, \quad 0 \leq k \leq n - 1\) be the restriction of \(T\) to \(\text{Im}(M_{n-k})\). For a linear subspace \(L \subseteq V\), we will write \(S(L) = S_{n-1} \cap L\).

Let \(F(V)\) denote the space of \(\mathbb{Z}\)-valued constructible functions on \(V\), i.e. functions of the form \(\sum_{i=1}^N n_i 1_{P_i}\) with \(n_i \in \mathbb{Z}, P_i \in \mathcal{P}(V)\) (compare [8]). Let \(F_{a.e.}(V)\) be the set of congruence classes of constructible functions where \(f \sim g\) if \(f - g = 0\) almost everywhere.

Lemma 3.4. Denote by \(Z\) the abelian group generated by all formal integral combination of compact convex polytopes in \(V\). Let \(W \subseteq Z\) denote the subgroup generated by lower-dimensional polytopes and elements of the form \([P \cup Q] + [P \cap Q] - [P] - [Q]\) where \(P \cup Q\) is convex. Then the map
\[ Z/W \to F_{a.e.}(V) \]
\[ \sum_i n_i [P_i] \mapsto \sum_i n_i 1_{P_i}, \quad n_i \in \mathbb{Z} \]
is injective.

Proof. It is easily checked that the map is well-defined. To prove injectivity, it is enough to prove that \(f := \sum_i 1_{P_i} \sim \sum_j 1_{Q_j}\) implies \(\sum_i [P_i] \equiv \sum_j [Q_j]\). Decompose the connected components of
\[ (\bigcup_i P_i \cup \bigcup_j Q_j) \setminus (\bigcup_i \partial P_i \cup \bigcup_j \partial Q_j) \]
into simplices \(\{\Delta\}\), disjoint except at their boundary. Then, by the inclusion-exclusion principle,
\[ \sum_i [P_i] \equiv \sum_i \sum_{\Delta \subseteq P_i} [\Delta] \equiv \sum_k (-1)^{k+1} \sum_{\Delta \subseteq \bigcup_{i < \ldots < i_k} P_{i_k}} [\Delta] \]

By examining the superlevel set \(\{f \geq k\}\) we see that
\[ \bigcup_{i_1 < \ldots < i_k} (P_{i_1} \cap \ldots \cap P_{i_k}) \sim \bigcup_{j_1 < \ldots < j_k} (Q_{j_1} \cap \ldots \cap Q_{j_k}), \]
where \(A \sim B\) means the sets \(A, B\) coincide up to a set of measure zero.

Therefore,
\[ \sum_i [P_i] \equiv \sum_j [Q_j] \mod W, \]
as claimed. \(\square\)

Remark 3.5. The same claim and proof apply if we replace \(Z\) with the free abelian group of polytopal cones with vertex in the origin.
Let us recall some notions from [26]. Let $L$ be a subspace of $V$. The cone group $\hat{\Sigma}(L)$ is the abelian group with generators $[[C]]$, where $C$ ranges over all convex polyhedral cones in $L$, and with the relations

denote the nearest-point projection.

Thus on the generators of $Y$, $\hat{\Sigma}$

The first thing to note is that if $A_{v_1,N_1} = A_{v_2,N_2}$, then either $v_1 = v_2$, $N_1 = N_2$ or $v_1 = -v_2$ and $N_1 = N_2$, where $N_j$ is $N_j$ with reversed orientation.

Thus on the generators of $Y_k$, $\hat{\Sigma}(A_{v,N}) := [v] \otimes [N]$ is well-defined.

Now assume that $\sum_{j} c_j A_{v_j,N_j} = 0$. We shall show that $\sum_{j} c_j |v_j| \otimes [N_j] = 0$.

Note that for all $\rho \in \Omega^k_e(V)$ and $\omega \in \Omega^{n-k-1}(S^{n-1})$, one has

More generally, suppose that we have

for some coefficients $\lambda_j \in \mathbb{C}$ and for all $\omega \in \Omega^{n-k-1}(S^{n-1})$.

Let $L \subset V$ be a linear subspace of dimension $n - k$. Let $U_\epsilon \subset S^{n-1}$ denote the $\epsilon$-neighborhood of $L \cap S^{n-1}$, where $\epsilon$ is sufficiently small. Let $p_\epsilon : U_\epsilon \rightarrow L \cap S^{n-1}$ denote the nearest-point projection.

Fix some $\beta \in C^\infty(0,1]$ such that $\beta(x) = 1$ for $0 \leq x \leq \frac{1}{4}$ and $\beta(x) = 0$ for $\frac{2}{3} \leq x \leq 1$. Let $\beta_\epsilon \in C^\infty(U_\epsilon)$ be given by $\beta_\epsilon(x) = \beta(\text{dist}(x,p(x))/\epsilon)$.

Given a form $\sigma \in \Omega^{n-k-1}(L \cap S^{n-1})$, we set $\omega := \beta_\epsilon \sigma \in \Omega^{n-k-1}(S^{n-1})$.

If $N_j \subset L \cap S^{n-1}$, then

while if $N_j$ does not lie in $L \cap S^{n-1}$ then $\int_{N_j} \omega \rightarrow 0$ as $\epsilon \rightarrow 0$.

Thus, letting $\epsilon \rightarrow 0$, we obtain

The full cone group is given by

where the sum extends over all linear subspaces of $L$.

**Lemma 3.6** (Lemma 39 from [26]). The map

such that $\Phi_k : \Pi(V) \rightarrow \mathbb{C} \otimes _\mathbb{Z} \hat{\Sigma}$

restricts to an injection on $\Xi_k$.

**Proposition 3.7.** For all $0 \leq k \leq n-1$, there is a linear map $\Phi_k : Y_k \rightarrow \mathbb{C} \otimes _\mathbb{Z} \hat{\Sigma}$ such that $\Phi_k(A_{v,N}) = [v] \otimes [N]$.

**Proof.** The first thing to note is that if $A_{v_1,N_1} = A_{v_2,N_2}$, then either $v_1 = v_2$, $N_1 = N_2$ or $v_1 = -v_2$ and $N_1 = N_2$, where $N_j$ is $N_j$ with reversed orientation.

Thus on the generators of $Y_k$, $\Phi_k(A_{v,N}) := [v] \otimes [N]$ is well-defined.

Now assume that $\sum_{j} c_j A_{v_j,N_j} = 0$. We shall show that $\sum_{j} c_j |v_j| \otimes [N_j] = 0$.

Note that for all $\rho \in \Omega^k_e(V)$ and $\omega \in \Omega^{n-k-1}(S^{n-1})$, one has

More generally, suppose that we have

for some coefficients $\lambda_j \in \mathbb{C}$ and for all $\omega \in \Omega^{n-k-1}(S^{n-1})$.

Let $L \subset V$ be a linear subspace of dimension $n - k$. Let $U_\epsilon \subset S^{n-1}$ denote the $\epsilon$-neighborhood of $L \cap S^{n-1}$, where $\epsilon$ is sufficiently small. Let $p_\epsilon : U_\epsilon \rightarrow L \cap S^{n-1}$ denote the nearest-point projection.

Fix some $\beta \in C^\infty(0,1]$ such that $\beta(x) = 1$ for $0 \leq x \leq \frac{1}{4}$ and $\beta(x) = 0$ for $\frac{2}{3} \leq x \leq 1$. Let $\beta_\epsilon \in C^\infty(U_\epsilon)$ be given by $\beta_\epsilon(x) = \beta(\text{dist}(x,p(x))/\epsilon)$.

Given a form $\sigma \in \Omega^{n-k-1}(L \cap S^{n-1})$, we set $\omega := \beta_\epsilon \sigma \in \Omega^{n-k-1}(S^{n-1})$.

If $N_j \subset L \cap S^{n-1}$, then

while if $N_j$ does not lie in $L \cap S^{n-1}$ then $\int_{N_j} \omega \rightarrow 0$ as $\epsilon \rightarrow 0$.

Thus, letting $\epsilon \rightarrow 0$, we obtain

The full cone group is given by

where the sum extends over all linear subspaces of $L$. 

...
where the sum is over all $N_j$ contained in $L$.

Now fix an orientation on $L$ and assume without loss of generality that all $N_j \subset L$ have the induced orientation. Going back to the original equation we may write

$$\sum_{N_j \subset L} c_j \int_{N_j} \sigma = 0.$$ 

Let $v_0 \in \Lambda^k L^\perp \subset \Lambda^k V$ be the unique vector with $|v_0| = 1$ and $v_0^\perp = L$ (the last equality understood with orientation). Then $v_j = |v_j| v_0$ for all $j$ with $N_j \subset L$.

Therefore, the above equation implies that

$$\sum_{N_j \subset L} c_j |v_j| \int_{N_j} \sigma = 0$$

for all $\sigma \in \Omega^{n-k-1}(L \cap S^{n-1})$ and this implies that

$$\sum_{N_j \subset L} c_j |v_j| 1_{N_j} = 0$$

almost everywhere.

By Lemma 3.4 we have in $\mathbb{C} \otimes \hat{\Sigma}(L)$

$$\sum_{N_j \subset L} c_j |v_j| \otimes [\hat{N}_j] = 0.$$ 

Since $L$ was arbitrary, we deduce that in $\mathbb{C} \otimes \hat{\Sigma}$ we have

$$\sum_j c_j |v_j| \otimes [\hat{N}_j] = 0,$$

as required. \[\Box\]

**Proof of Theorem 4**. By Lemma 3.6 the map

$$\sigma_k : \Xi_k \to \mathbb{C} \otimes \hat{\Sigma}$$

is injective. For $0 \leq k \leq n-1$ we have

$$\sigma_k = \Phi_k \circ T_k \circ M_{n-k},$$

while $\sigma_n$ is the volume functional on $\Pi(V)$ restricted to $\Xi_n$, so identifying the space of Lebesgue measures on $V$ with $\mathbb{C}$ allows us to write $\sigma_n = C \circ M_0$. We conclude that $(M_{n-k})|_{\Xi_k}$ is injective for each $k$, and hence $M$ is injective. \[\Box\]

4. **PARTIAL CONVOLUTION PRODUCT**

Let us first describe the convolution in the smooth case, see [16], but using a more intrinsic approach.

Let $V$ be an $n$-dimensional vector space. Let $\text{Dens}(V)$ denote the 1-dimensional $\text{GL}(V)$-module of densities on $V$. The orientation bundle $\text{or}(V)$ is the real 1-dimensional linear space consisting of all functions $\rho : \Lambda^k \to \mathbb{R}$ such that $\rho(\lambda \omega) = \text{sign}(\lambda) \rho(\omega)$ for all $\lambda \in \mathbb{R}$ and $\omega \in \Lambda^k V$. Note that there is a canonical isomorphism $\text{or}(V) \cong \text{or}(V^*)$.

Let $\mathbb{P}_+(V)$ be the space of oriented 1-dimensional subspaces of $V$. Then $S^*V := V \times \mathbb{P}_+(V^*)$ has a natural contact structure.

We have a natural non-degenerate pairing

$$\Lambda^k V^* \otimes \Lambda^{n-k} V^* \xrightarrow{\Delta} \Lambda^n V^* \cong \text{Dens}(V) \otimes \text{or}(V),$$
which induces an isomorphism
\[ * : \Lambda^k V^* \otimes \text{or}(V) \otimes \text{Dens}(V^*) \xrightarrow{\sim} (\Lambda^{n-k} V^*)^* \cong \Lambda^{n-k} V. \]

Let
\[ *_1 : \Omega(S^* V)^{tr} \otimes \text{or}(V) \otimes \text{Dens}(V^*) \xrightarrow{\sim} \Omega(V^* \times \mathbb{P}_+(V^*))^{tr} \]
be defined by
\[ *_1(\pi_1^* \gamma_1 \wedge \pi_2^* \gamma_2) := (-1)^{\frac{n-2}{2}} \pi_1^* (\gamma_1) \wedge \pi_2^* \gamma_2 \]
for \( \gamma_1 \in \Omega(V)^{tr} \otimes \text{or}(V) \otimes \text{Dens}(V^*), \gamma_2 \in \Omega(\mathbb{P}_+(V^*)). \)

Let \( \phi_j = \nu(\omega_j, \gamma_j) \in \text{Val}^\infty(V) \otimes \text{Dens}(V^*), j = 1, 2, \) where \( \omega_j \in \Omega^{n-1}(S^* V)^{tr} \otimes \text{or}(V) \otimes \text{Dens}(V^*), \gamma_j \in \Omega^n(V)^{tr} \otimes \text{or}(V) \otimes \text{Dens}(V^*). \) Then the convolution product \( \phi_1 * \phi_2 \) is defined as \( \nu(\omega, \gamma) \in \text{Val}^\infty(V) \otimes \text{Dens}(V^*), \) where \( \omega \) and \( \gamma \) satisfy
\[ D\omega + \pi^* \gamma = *_1^{-1} (*_1(D\omega_1 + \pi^* \gamma_1) \wedge *_1(D\omega_2 + \pi^* \gamma_2)) \]
\[ \pi_\omega = \pi_\omega \circ *_1^{-1} (*_1 \kappa_1 \wedge *_1(D\omega_2)) + (*_1 \kappa_1 \pi_\omega_2 + (*_2 \pi_\omega_1). \]

Here \( \kappa_1 \in \Omega^{n-1}(S^* V)^{tr} \) is any form such that \( d\kappa_1 = D\omega_1. \)

The convolution extends to a partially defined convolution on the space \( \text{Val}^{-\infty}(V) \otimes \text{Dens}(V^*) \) as follows.

The space \( D_*(S^* V) \) admits a bigrading
\[ D_*(S^* V) = \bigoplus_{k=0}^{n} \bigoplus_{l=0}^{n-1} D_{k,l}(S^* V), \]
and for \( T \in D_*(S^* V), \) we denote by \( [T]_{k,l} \) the component of bidegree \( (k, l). \)

We consider now the GL(V)-module of translation-invariant currents \( D(S^* V)^{tr} = D(V)^{tr} \otimes D(\mathbb{P}_+(V^*)). \) One has the natural identification \( D_k(V)^{tr} = \Omega^{n-k}(V)^{tr} \otimes \text{or}(V), \) and a non-degenerate pairing \( D_k(V)^{tr} \otimes \Omega^k(V)^{tr} \to \text{Dens}(V). \)

We define \( T \in D_{k,l}(S^* V)^{tr} \otimes \text{or}(V) \otimes \text{Dens}(V^*) \) the element \( *_1 T \in D_{n-k-1,l}(V^* \times \mathbb{P}_+(V^*))^{tr} \) by
\[ (*_1 T, \delta) := (-1)^{nk+nl+n+l+2} [T]_{k,l}^{-1} \delta \in \text{Dens}(V^*) \]
for all \( \delta \in \Omega^{n-k-1,l}(V^* \times \mathbb{P}_+(V^*))^{tr}. \) With this definition of \( *_1 \) the diagram
\[ \Omega^{k,l}(S^* V)^{tr} \otimes \text{or}(V) \otimes \text{Dens}(V^*) \xrightarrow{*_1} \Omega^{n-k,l}(V^* \times \mathbb{P}_+(V^*))^{tr} \]
\[ D_{n-k-1,l}(S^* V)^{tr} \otimes \text{or}(V) \otimes \text{Dens}(V^*) \xrightarrow{*_1} D_{k,n-l-1}(V^* \times \mathbb{P}_+(V^*))^{tr} \]
commutes. Equivalently,
\[ *_1 ([[S^* V]]_{k,l} \gamma) = [[S^* V]]_{k,n} \gamma \]
for all \( \gamma \in \Omega(S^* V)^{tr} \otimes \text{or}(V) \otimes \text{Dens}(V^*). \) Clearly we have \( \text{WF}(*_1 T) = \text{WF}(T) \) for \( T \in D(S^* V)^{tr} \)

**Remark 4.1.** Without the choice of an orientation and a Euclidean scalar product, we may intrinsically define \( A_{v,N} \in D_{k,n-1-k}(S^* V)^{tr} \otimes \text{or}(V) \otimes \text{Dens}(V^*) \) as follows.

Let \( v \in \Lambda_v^k V \) and let \( N \subset \mathbb{P}_+(V^*) \) be an \( (n-1-k) \)-dimensional, geodesically convex polytope contained in \( v^* \cap \mathbb{P}_+(V^*). \) Then define
\[ (A_{v,N}, \pi_1^* \gamma \wedge \pi_2^* \delta \otimes \sigma \otimes \epsilon) = \int_V \langle x^* \omega, v \rangle d\sigma(x) \int_{N^*} \delta \]
for $\gamma \in \Omega^k(V), \delta \in \Omega^{n-k-1}(\mathbb{P}_+(V^*)), \epsilon \in \text{or}(V)$ and $\sigma \in \text{Dens}(V)$. Here $N_\epsilon$ equals $N$ with a choice of orientation such that the orientation of the pair $(v, N_\epsilon)$ is induced by $\epsilon$.

Given a Euclidean trivialization, this reduces to the previous definition of $A_{v,N}$.

**Lemma 4.2.** Given a Legendrian cycle $T \in \mathcal{D}_{n-k,k-1}(S^*V)^{tr}$ with $1 \leq k \leq n-1$, there exists $\tilde{T} \in \mathcal{D}_{n-k,k}(S^*V)^{tr}$ with $T = \partial \tilde{T}$ and $\text{WF}(\tilde{T}) = \text{WF}(T)$.

**Proof.** Let us use a Euclidean scalar product and an orientation on $V$. Then we may identify $\text{Dens}(V) \cong \mathbb{C}, \text{or}(V) \cong \mathbb{C}, S^*V \cong SV = V \times S^{n-1}$.

Let $\phi \in \text{Val}_k^\infty(V)$ be the valuation represented by $(T, 0)$. Choose a sequence $\phi_j \in \text{Val}_k^\infty(V)$ such that $\phi_j \to \phi$. Let $\omega_j \in \Omega^{k-n-k-1}(SV)^{tr} = \Omega^{n-k-1}(S^{n-1}) \otimes \Lambda^kV^*$ be a form representing $\phi$ and such that $D\omega_j = d\omega_j$.

Note that $\mathcal{D}_{n-k,k-1}(SV)^{tr} = \mathcal{D}_{k-1}(S^{n-1}) \otimes \Lambda^kV$. Then $d\omega_j \in \Omega^{n-k}(S^{n-1}) \otimes \Lambda^kV^* \subset \mathcal{D}_{k-1}(S^{n-1}) \otimes \Lambda^{n,k-V}$ converges weakly to $T$.

Let $G : \Omega^*(S^{n-1}) \to \Omega^*(S^{n-1})$ denote the Green operator on $S^{n-1}$ and $\delta : \Omega^*(S^{n-1}) \to \Omega^{*(1)}(S^{n-1})$ the codifferential. We define $\beta_j := G(d\omega_j) \in \Omega^{n-k}(S^{n-1}) \otimes \Lambda^{n,k-V}$, i.e. $\Delta \beta_j = d\omega_j$. Then $\Delta d\beta_j = d\Delta \beta_j = 0$, hence $d\beta_j$ is harmonic, which implies that $\delta d\beta_j = 0$.

We define

$$\tilde{T} := (-1)^{n+k} \lim_{j \to \infty} [S^{n-1}]d\beta_j \in \mathcal{D}_k(S^{n-1}) \otimes \Lambda^{n-k}V \subset \mathcal{D}_{n-k,k}(SV)^{tr}.$$ 

Then

$$\partial \tilde{T} = \lim_{j \to \infty} [S^{n-1}]d\beta_j = \lim_{j \to \infty} [S^{n-1}]\Delta \beta_j = \lim_{j \to \infty} [S^{n-1}]d\omega_j = T.$$

From $\tilde{T} = \delta \circ G(T)$ and (8), we infer that $\text{WF}(\tilde{T}) \subset \text{WF}(T)$. Conversely, from (7) we deduce that $\text{WF}(T) = \text{WF}(\partial \tilde{T}) \subset \text{WF}(\tilde{T})$. □

Recall that $s : S^*V \to S^*V$ is the antipodal map.

**Definition 4.3.** Let $\phi_j \in \text{Val}^{\infty}(V) \otimes \text{Dens}(V^*), E(\phi_j) =: (T_j, C_j), j = 1, 2$. We call $\phi_1, \phi_2$ transversal if

$$\text{WF}(T_1) \cap s(\text{WF}(T_2)) = \emptyset.$$ 

**Proposition 4.4.** Let $\phi_j \in \text{Val}^{\infty}(V) \otimes \text{Dens}(V^*), j = 1, 2$ be transversal and $(T_j, C_j) := E(\phi_j), j = 1, 2$. Decompose $T_j = t_j + T'_j$, where $t_j = \alpha_j \pi^*([V]) \cdot \text{vol}_n$ \otimes $\text{vol}^n_\alpha \in \mathcal{D}_0(n-1)(S^*V)^{tr} \otimes \Lambda^nV^*$ is the corresponding $(0,n-1)$-component, and let $T_1 \in \mathcal{D}_0(S^*V)^{tr} \otimes \Lambda^nV^*$ be a current such that $\partial \tilde{T}_1 = T'_1$ and $\text{WF}(\tilde{T}_1) = \text{WF}(T_1)$, guaranteed to exist by Lemma 4.2. Then the currents

$$T := *_{-1}^{-1}(*_1 T_1 \cap *_1 T_2)$$

$$C := \pi_* \left( *_{1}^{-1} \left( *_{1} T_1 \cap *_{1} T_2' \right) \right) + \alpha_1 C_2 + \alpha_2 C_1$$

are independent of the choice of $\tilde{T}_1$ and satisfy the conditions (9).

**Proof.** Note first that $\partial$ commutes (up to sign) with $*_1$ on translation invariant currents. By (5), $T$ equals (up to sign) the boundary of the $n$-current $S := *_{1}^{-1} \left( *_{1} T_1 \cap *_{1} T_2 \right)$. In particular, $T$ is a cycle. Moreover, $\pi_* T = \pm \partial \pi_* S = 0$, since $\pi_* S$ is a translation invariant $n$-current, hence a multiple of the integration
current on $V$ which has no boundary. For the same reason, $\partial C = 0$, hence the condition $\pi_*T = \partial C$ is trivially satisfied.

Note that whenever $Q \in \mathcal{D}_n(S^*V)$ is a boundary, then $\pi_*Q = 0$. Indeed, let $Q = \partial R$ and $\rho \in \Omega^n_\circ(V)$. Then

$$\langle \pi_*Q, \rho \rangle = \langle Q, \pi^\ast \rho \rangle = \langle \partial R, \pi^\ast \rho \rangle = \langle R, \pi^\ast d\rho \rangle = 0.$$ (15)

By Lemma 4.2 there exists a translation invariant $n$-current $\tilde{T}_2$ with $\partial \tilde{T}_2 = T_2'$. Suppose that $\partial \tilde{T}_1 = \partial \tilde{T}_1 = T_1'$ for two $n$-currents $\tilde{T}_1$ and $\tilde{T}_1$. Then the $n$-current $Q := \ast^{-1}_1 \left( \ast_1 (\tilde{T}_1 - \tilde{T}_1) \cap \ast_1 T_2' \right)$ is (up to a sign) the boundary of the $(n+1)$-current $R := \ast^{-1}_1 \left( \ast_1 (\tilde{T}_1 - \tilde{T}_1) \cap \ast_1 \tilde{T}_2 \right)$. By (15), it follows that $\pi_*Q = 0$, which shows that $C$ is independent of the choice of $\tilde{T}_1$.

Let us finally show that $T$ is Legendrian. Fix sequences $(\phi_j^i)$ of smooth and translation invariant valuations converging to $\phi_1, j = 1, 2$. Let $\phi_1^i$ be represented by the forms $(\omega_1^i, \gamma_1^i)$. Then $E(\phi_1^i) = (T_1^i, C_1^i)$ is given by the formulas (11), (12) and hence $\phi_1^i \ast \phi_2^i$ is represented by the current $T^i = [[S^*V]] \ast s^\ast \kappa^i$, with

$$s^\ast \kappa^i := \ast^{-1}_1 \ast_1 \ast_1^\ast (S^\ast \omega_1^i + \pi^\ast \gamma_1^i) \ast_1 \ast_1^\ast (S^\ast \omega_2^i + \pi^\ast \gamma_2^i).$$ (16)

It is easily checked that $\kappa^i$ is a horizontal, closed $n$-form (compare also [16], Eq. (37)). It follows that $T^i$ is Legendrian.

Note that $[[S^*V]] \ast s^\ast (D\omega_j^i + \pi^\ast \gamma_j)$ converges to $T_j$. By the definition of the intersection current, $[[S^*V]] \ast s^\ast \kappa^i$ converges to $T$ and hence $T$ is Legendrian. □

**Definition 4.5.** In the same situation, the convolution product $\phi_1 \ast \phi_2 \in \text{Val}^{-\infty}(V) \otimes \text{Dens}(V^\ast)$ is defined as $\phi_1 \ast \phi_2 := E^{-1}(T, C)$.

**Proposition 4.6.** If $\phi_1, \phi_2 \in \text{Val}^{\infty}(V) \otimes \text{Dens}(V^\ast) \subset \text{Val}^{-\infty}(V) \otimes \text{Dens}(V^\ast)$, then the convolution of Definition 4.5 coincides with the convolution from [16].

**Proof.** For $\phi_j \in \text{Val}^{\infty}(V)$ given by the pairs $(\omega_j, \gamma_j) \in \Omega^{n-1}(S^*V)^{fr} \otimes \text{Dens}(V)$, the corresponding currents are $E(\phi_j) = (\ast(\omega_j), \ast(\gamma_j), [[S^\ast V]] \ast s^\ast (D\omega_j^i + \pi^\ast \gamma_j), [[V]] \ast \pi^\ast \omega_j)$.

We consider two cases:

If $\omega_1 = 0$ and $\gamma_1 = c \cdot \text{vol}$, then $\phi_1 = c \cdot \text{vol}$, and $\phi_1 \ast \phi_2 = c \phi_2$ by the original definition of convolution.

By the new definition, $E(\phi_1) = (\ast(\omega_1), \pi^\ast \gamma_1, 0)$, and

$$\ast \pi^\ast \gamma_1 = c \in \Omega^0(V \times S^{n-1}).$$

Write

$$\phi_1 \ast \phi_2 = E^{-1}(\ast(\omega_1), \pi^\ast \gamma_1, 0) \ast E^{-1}(0, C_2) + E^{-1}(\ast(\omega_1), \pi^\ast \gamma_1, 0) \ast E^{-1}(T_2, 0).$$

If $E^{-1}(0, C_2) = \lambda X$, by definition the first summand equals $c \lambda X = c E^{-1}(0, C_2)$. The second summand is $c \cdot E^{-1}(T_2, 0)$, and so $\phi_1 \ast \phi_2 = c E^{-1}(T_2, C_2) = c \phi_2$, as required.

In the remaining case, we may assume $\gamma_1 = \gamma_2 = 0$, so $E(\phi_j) = (\ast(\omega_j), [[S^\ast V]] \ast s^\ast (D\omega_j), [[V]] \ast \pi^\ast \omega_j)$. Moreover, we may assume that $d\omega_1, d\omega_2$ are vertical.

The original definition of convolution gives

$$E(\phi_1 \ast \phi_2) = (\ast(\omega_1 \land \ast_1 D\omega_2)) \quad \text{with}$$

$$D\omega = \ast^{-1}_1 (\ast_1 D\omega_1 \land \ast_1 D\omega_2).$$
Since $E^{-1}(\{\nu\mid s^*D\nu_j, 0\}) = 0$, it remains to verify that by the new definition,

$$E^{-1}(\{\nu\mid s^*D\nu_k, 0\}) = E^{-1}(\{\nu\mid s^*D\nu_k, 0\})$$

By homogeneity, we may assume that $\deg\omega_1 = (k, n - 1 - k)$ and $\deg\omega_2 = (l, n - 1)$. If $k + l < n$ then by dimensional considerations $\omega = 0$. If $k + l > n$ then $\pi_s \omega = 0$, and the new definition of convolution gives

$$E(\phi_1 * \phi_2) = \{\nu\mid s^*D\nu_k, 0\} = (\{\nu\mid s^*D\nu_k, 0\})$$

as required.

Finally, if $k + l = n$, then $1 \leq k \leq n - 1$, $D\omega = 0$, and by the new definition $T(\phi_1 * \phi_2) = 0$. Since $T_1 = \{\nu\mid s^*D\nu_k, 0\}$ and $\omega_1 \in \Omega^{-1}(S^*V)$, using (1) one can take $\tilde{T_1} = (-1)^n([-S^*V]\nu\omega_1)$. Using (2), the fact that the operations $s^*$, $*$ and $V$ commute, while $\pi_s \omega = 0$, we obtain

$$C(\phi_1 * \phi_2) = (-1)^n\pi_s \{\nu\mid s^*D\nu_k, 0\}$$

completing the verification. \qed

**Proposition 4.7.** Let $\Gamma_1, \Gamma_2 \subset S^*(S^*V)$ be closed sets with $\Gamma_1 \cap s\Gamma_2 = \emptyset$ and set $\Gamma := \Gamma_1 \cup \Gamma_2 \cup \{(x, [\xi], [\eta_1 + \eta_2]) : (x, [\xi]) \in S^*V, (x, [\xi], [\eta_1]) \in \Gamma_1, (x, [\xi], [\eta_2]) \in \Gamma_2\}$. Then the convolution is a (jointly sequentially) continuous map

$$Val_{\Gamma_1}^{-\infty}(V) \times Val_{\Gamma_2}^{-\infty}(V) \to Val_{\Gamma}^{-\infty}(V) \times Val_{\Gamma}^{-\infty}(V).$$

**Proof.** In the notations of Proposition 4.4, we have $WF((*)_1) = WF((*)_1) \subset WF(T_1) \subset \Gamma_1$ and $WF((*)_2) \subset WF(T_2) \subset \Gamma_2$. Since the intersection of currents is a jointly sequentially continuous map $D_{\Gamma_1}^{-\infty}(S^*V) \times D_{\Gamma_2}^{-\infty}(S^*V) \to D_{\Gamma}^{-\infty}(S^*V)$, the statement follows. \qed

**Corollary 4.8.** Whenever it is defined, the convolution is commutative and associative.

**Proof.** Let $\phi_j \in Val_{\Gamma_j}^{-\infty}(V) \otimes Val_{\Gamma_2}^{-\infty}(V)$, $j = 1, 2$. By Lemma 2.7 there exist sequences $\phi_j^* \in Val^{-\infty}(V) \otimes Val_{\Gamma_2}^{-\infty}(V)$, $j = 1, 2$ converging to $\phi_j$ in $Val_{\Gamma_j}^{-\infty}(V) \otimes Val_{\Gamma}^{-\infty}(V)$. By Proposition 4.7 $\phi_1 \ast \phi_2$ converges to $\phi_1 \ast \phi_2$ in $Val_{\Gamma_j}^{-\infty}(V)$, while $\phi_2 \ast \phi_1$ converges to $\phi_1 \ast \phi_2$. Since the convolution on smooth valuations is commutative, it follows that $\phi_1 \ast \phi_2 = \phi_2 \ast \phi_1$. For associativity, let $\Gamma_1, \Gamma_2, \Gamma_3 \subset S^*(S^*V)$ be closed sets such that if $(x, [\xi]) \in S^*V, (x, [\xi], [\eta]) \in \Gamma_i, i = 1, 2, 3$, then

$$\eta_1 + \eta_2 \neq 0, \eta_1 + \eta_3 \neq 0, \eta_2 + \eta_3 \neq 0, \eta_1 + \eta_2 + \eta_3 \neq 0.$$ 

One easily checks that both maps $Val_{\Gamma_j}^{-\infty}(V) \times Val_{\Gamma_j}^{-\infty}(V) \times Val_{\Gamma_j}^{-\infty}(V) \to Val_{\Gamma_j}^{-\infty}(V)$ are well-defined. An approximation argument as above shows that they agree. \qed
5. Compatibility of the algebra structures

From now on, we fix an orientation and a Euclidean scalar product on $V$ and identify $\text{Dens}(V) \cong \mathbb{C}$, or$(V) \cong \mathbb{C}$, $S^*V \cong SV = V \times S^{n-1}$. It then holds that

$$\langle *_1 T, \omega \rangle = (-1)^{nk+nl+k}(T, *_1 \omega)$$

for $T \in \mathcal{D}_{k,l}(S^*V)^{tr}$, $\omega \in \Omega^{k,n-1,l}(S^*V)^{tr}$.

**Proposition 5.1.** Let $v_i \in \Lambda^k V$ and $T_i \in \mathcal{D}_i(S^{n-1}), i = 1, 2$ be currents on the sphere such that $T_1 \cap T_2 \in \mathcal{D}_{i,1+l_2-l_1}(S^{n-1})$ is defined. Then

$$*_1 ([v_1] \times T_1) \cap *_1 ([v_2] \times T_2) = (-1)^{k_1(l_2-l_1)}*_1 ([v_1 \wedge v_2] \times (T_1 \cap T_2)).$$

In particular, for $A_{v_1, N_1} \in Y_k$, with $N_1$ and $N_2$ being transversal, one has

$$*_1 A_{v_1, N_1} \cap *_1 A_{v_2, N_2} = *_1 A_{v_1 \wedge v_2, N_1 \cap N_2}.$$

**Proof.** Let $v \in \Lambda^k V, T \in \mathcal{D}_i(S^{n-1})$. We compute for $\gamma_1 \in \Omega^{n-k}_c(V), \gamma_2 \in \Omega^l(V)$

$$(*_1 ([v] \times T), \pi_1^* \gamma_1 \wedge \pi_2^* \gamma_2) = (-1)^{nk+nl+k}([v] \times T, *_1(\pi_1^* \gamma_1 \wedge \pi_2^* \gamma_2))$$

$$= (-1)^{nk+nl+k+l}([v], *_1 \gamma_1) \cdot (T, \gamma_2)$$

$$= (-1)^{nk+nl+k+(l)}([v], \gamma_1) \cdot (T, \gamma_2)$$

$$= (-1)^{l+nl}([v] \times T, \pi_1^* \gamma_1 \wedge \pi_2^* \gamma_2),$$

i.e.

$$*_1 ([v] \times T) = (-1)^{l}([v] \times T).$$

Let now $v_i \in \Lambda^k V, T_i \in \mathcal{D}_i(S^{n-1}), i = 1, 2$. We have $[*v_1] \cap [*v_2] = [*(v_1 \wedge v_2)]$. Using (13) it follows that

$$(*_1 v_1 \times T_1) \cap (*_1 v_2 \times T_2) = (-1)^{k_1(n-1-l_2)}[*(v_1 \wedge v_2)] \times (T_1 \cap T_2).$$

The statement now follows from (17). \qed

A geodesically convex polytope on $S^{n-1}$ is the intersection of a proper convex closed polyhedral cone in $V$ with $S^{n-1}$. We let $\mathcal{P}(S^{n-1})$ denote the set of oriented geodesically convex polytopes on $S^{n-1}$. For $I \in \mathcal{P}(S^{n-1})$ of dimension $k$, denote by $E_I = \text{Span}_V I \cap S^{n-1}$ the $k$-dimensional equator that it spans. For $I, J \subset S^{n-1}$ geodesically convex polytopes, we let $\text{conv}(I, J) \subset S^{n-1}$ denote the union of all shortest geodesic intervals having an endpoint in $I$ and an endpoint in $J$. If $\dim I + \dim J = n-2$, both $I, J$ are oriented and $E_I \cap E_J = \emptyset$, one has a natural orientation on $\text{conv}(I, J)$, by comparing the orientation of $\text{Span}_V (I) \oplus \text{Span}_V (J) = V$ with the orientation of $V$. The geodesically convex polytope $-I$ is oriented in such a way that the antipodal map $I \mapsto -I$ is orientation preserving. Note that $\partial(-I) = -\partial I$ whenever $\dim I > 0$, while when $\dim I = 0$ we have $\partial(-I) = \partial I$. Here and in the following, $\partial$ denotes the extended singular boundary operator, i.e. for a positively oriented point $I$ we have $\partial I = 1$. If $\varepsilon = \pm 1$, we write $\text{conv}(I, \varepsilon) = I^\varepsilon$, where $I^{-1}$ denotes orientation reversal.

We denote $A_{n-1}(I, J) := \text{vol}_{n-1}(\text{conv}(I, J))$ provided that $(-I) \cap J = \emptyset$. Note that whenever the orientation of $\text{conv}(I, J)$ is not well-defined, it is a set of volume zero. Note also that $A_{n-1}$ is a partially-defined bi-valuation in $I, J$, so we may extend $A_{n-1}$ as a partially-defined bilinear functional on chains of polytopes. The following proposition shows that in fact $A_{n-1}$ can be uniquely extended as a bilinear functional on all chains.
Proposition 5.2. Given $I \in \mathcal{P}(S^{n-1})$ of dimension $k$, where $0 \leq k \leq n-2$, there exists a unique $L^1$-integrable form $\omega_I \in \Omega^{n-k-2}(S^{n-1} \setminus I)$, such that for any $J \in \mathcal{P}(S^{n-1})$ of dimension $(n-k+2)$ with $J \cap I = \emptyset$ one has
\[
\int_J \omega_I = A_{n-1}(-I,J).
\] (18)

The current $T_I \in \mathcal{D}_{k+1}(S^{n-1})$ with
\[
\langle T_I, \phi \rangle := \int_{S^{n-1}} \omega_I \wedge \phi, \quad \phi \in \Omega^{k+1}(S^{n-1})
\]
has the following properties:

i) $T_I$ is additive (i.e. $T_I = T_{I_1} + T_{I_2}$ whenever $I = I_1 \cup I_2$ with geodesically convex polytopes $I_1, I_2$ such that $I_1 \cap I_2$ is a common face of $I_1$ and $I_2$);

ii) the singular support of $T_I$ equals $I$;

iii) If $k > 0$, then
\[
\partial T_I = (-1)^{n+k+1} \text{vol}(S^{n-1})[[I]] + (-1)^{n+1} T_{\partial I},
\] where $[[I]] \in \mathcal{D}_k(S^{n-1})$ is the $k$-current of integration over $I$;

iv) If $k = 0$,
\[
\partial T_I = -\text{vol}(S^{n-1})[[I]] + (-1)^n T_{\partial I},
\] where we adopt the convention $\omega_{\pm 1} := \mp \text{vol}_{n-1} \in \Omega^{n-1}(S^{n-1}), T_{\pm 1} = \mp \text{vol}_{n-1} \in \mathcal{D}_0(S^{n-1})$. Then (18) holds also for $I = \pm 1$.

For further use, we give the following current-theoretic interpretation of (18). If $\phi \in \Omega^{k+1}_c(S^{n-1} \setminus I)$, then $T_I \cap ([[S^{n-1}]] \cdot \phi) = [[[S^{n-1}]] \cdot (\omega_I \wedge \phi)]$ and hence
\[
\langle T_I \cap ([[S^{n-1}]] \cdot \phi), 1 \rangle = \int_{S^{n-1}} \omega_I \wedge \phi = (-1)^{(n-k-2)(k+1)} \langle [[[S^{n-1}]] \cdot \phi, \omega_I] \rangle.
\]

Let $\phi_i \in \Omega^{k+1}_c(S^{n-1} \setminus I)$ be a sequence with $[[S^{n-1}]] \cdot \phi_i \to [[J]]$ in $\mathcal{D}_{n-k-2,\text{WF}([[J]])}(S^{n-1})$
Since $\text{WF}(T_I) \cap s \text{WF}([[J]]) = \emptyset$ by the second item, $(T_I \cap [[[S^{n-1}]] \cdot \phi_i], 1) \to (T_I \cap [[J]], 1)$, while $([[S^{n-1}]] \cdot \phi_i, \omega_I) \to \int_J \omega_I = A_{n-1}(-I,J)$. Therefore
\[
\langle T_I \cap [[J]], 1 \rangle = (-1)^{(n+k+1)}A_{n-1}(-I,J).
\] (20)

Proposition 5.3. Let $P$ be a polytope and let $(T,C) := E(M([P]))$ the associated currents. Decompose $T = t + T'$ as in Proposition 5.2 where $t$ is the $(0,n-1)$-component of $T$ and $T' = \sum_{k=1}^{n-1} \sum_{F \in \mathcal{F}(I(n-1))} A_{kF,\tilde{n}(F,P)}$. Let
\[
\tilde{T} := \frac{1}{\text{vol}(S^{n-1})} \sum_{k=1}^{n-1} (-1)^{nk+k+1} \sum_{F \in \mathcal{F}(I(n-1))} [\text{WF}] \times T_{\tilde{n}(F,P)} \in \mathcal{D}_n(SV).
\]

Then
\[
\partial \tilde{T} = T'.
\]

Proof. The proof of these two propositions is postponed to the appendix. \hfill \square

Definition 5.4. Two elements $x, y \in \Pi(V)$ are in general position, if any two normal cones to a pair of faces of $x$ and $y$ are transversal.

Proposition 5.5. Given two elements $x, y \in \Pi(V)$ in general position, the convolution $M(x) * M(y)$ is well-defined, and $M(x \cdot y) = M(x) * M(y)$. 


Proof. By linearity, it suffices to consider \( x = [P] \) and \( y = [Q] \) for some polytopes \( P, Q \).

The wavefront of the current \( A_{v,N} \in Y_k \) is the conormal bundle to \( N \) in \( S^{n-1} \).
Thus \( \text{WF}(A_{v_1,N_1}) \cap \text{WF}(A_{v_2,N_2}) = \emptyset \) if and only if for all \( x \in N_1 \cap N_2 \) we have

\[
T_x N_1 + T_x N_2 = T_x S^{n-1},
\]

that is, if and only if \( N_1 \) and \( N_2 \) are transversal.

Recall from Lemma 3.3 that for \( 0 \leq k \leq n-1 \)

\[
T(M_{n-k}[P]) = \sum_{F \in \mathcal{F}_k(P)} A_{v_F, \hat{n}(F,P)},
\]

Thus, given \([P],[Q] \in \Pi(V)\) in general position, the normal cones in \( S^{n-1} \) have disjoint wavefronts.

Let \( F \) be a face of \( P \) and \( G \) a face of \( Q \). If \( \dim F + \dim G \geq n \) then \( \hat{n}(F,P) \cap \hat{n}(G,Q) = \emptyset \) by transversality.

Thus, by Proposition 5.1

\[
*_1 T(M[|P|] * M(|Q|)) = \sum_{\dim F + \dim G < n} *_1 A_{v_F, \hat{n}(F,P)} \cap *_1 A_{v_G, \hat{n}(G,Q)}
\]

and so

\[
T(M[P] * M[Q]) = \sum_{H \in \mathcal{F}(P+Q)} A_{v_H, \hat{n}(H,P+Q)} = T(M[P + Q]).
\]

It remains to verify that \( C(M[P] * M[Q]) = C(M[P + Q]) \). We set

\[
T_1 = t_1 + T'_1 := T(M[P]), T_2 = t_2 + T'_2 := T(M[Q])
\]

as in Proposition 4.3. Then \( t_1 = \sum_{F \in \mathcal{F}(P)} A_{F,n(F,P)} = \pi^*([V] \wedge \text{vol}_n) \), hence \( \alpha_1 = 1 \). Similarly \( \alpha_2 = 1 \).

Let \( F \) be a \( k \)-dimensional face of \( P \) and \( G \) an \( (n-k) \)-dimensional face of \( Q \), where \( 1 \leq k \leq n-1 \). By Proposition 5.1

\[
*_1([v_F] \times T_{\hat{n}(F,P)}) \cap *_1(A_{G,\hat{n}(G,Q)}) = *_1([v_F \wedge v_G] \times (T_{\hat{n}(F,P)} \cap [[\hat{n}(G,Q)])].
\]
Lemma A.1. Let $\partial^n F$ be an oriented geodesically convex polytope of dimension $k$. Suppose it does not intersect $\text{Span}_{\mathbb{R}^n} (I) \cap S^{n-1}$ for all $I \in \partial^n F$, for all $F \in \mathcal{F}_k (P)$. Let $\omega \in \Omega^k_+ (\mathbb{R}^n)$. Then

$$\sum_{F \in \mathcal{F}_k (P)} \langle [v_F], \omega \rangle A_{n-1} (-\partial^n F, J) = 0.$$  

(21)

Remark: Note that if $-\partial^n F = \sum_j I_j$ is the decomposition of $\partial^n F$ into geodesically convex polytopes, then by definition

$$A_{n-1} (-\partial^n F, J) = \sum A_{n-1} (I_j, J).$$
is the sum of the oriented volumes. Note also that in formula (21) the orientation of $F$ in each summand appears twice, and so the summands are well-defined.

Proof. For $k = n - 1$, this is the well-known statement that

$$\sum_{F \in \mathcal{F}_{n-1}(P)} [v_F] = 0,$$

where the orientation is given by fixing the outer normals to $P$.

Now for $k < n - 1$, fix an arbitrary orientation for all faces of dimension $k$ and $(k+1)$. For a pair of faces $F \in \mathcal{F}_k(P)$, $G \in \mathcal{F}_{k+1}(P)$, s.t. $F \subset \partial G$, define $\text{sign}(F,G) = \pm 1$ according to the orientation. Then

$$A_{n-1}(-n_F, J) = \sum_{G \supset F} A_{n-1}(-n_G, J) \text{sign}(F,G),$$

where the sum is over all $G \in \mathcal{F}_{k+1}(P)$ containing $F$ in their boundary, and therefore

$$\sum_{F \in \mathcal{F}_k(P)} \langle [v_F], \omega \rangle A_{n-1}(-n_F, J) = \sum_{G \in \mathcal{F}_{k+1}(P)} A_{n-1}(-n_G, J) \sum_{F \subset G} \text{sign}(F,G)([v_F], \omega),$$

but the internal sum is obviously zero by the case $k = n - 1$. □

Lemma A.2. Let $I, J \subset S^{n-1}$ be oriented, geodesically convex polytopes with $\dim I = k$, $\dim J = n - 1 - k$, such that $J \cap E_I = \emptyset$. Then

$$A_{n-1}(\partial I, J) = (-1)^k A_{n-1}(I, \partial J).$$

Proof. Let us consider $I, J$ as singular cycles, such that the singular boundary operator on an oriented point equals its sign. Denote $\partial I = \sum_i I_i$, $\partial J = \sum_j J_j$, where $I_i, J_j$ are geodesically convex. Choose any point $x \in S^{n-1}$ outside $I \cup J$, and let $H = S^{n-1} \setminus \{x\}$. Choose a form $\beta \in \Omega^{n-2}(H)$ such that $d\beta = \text{vol}_{n-1}$. Then

$$\partial \text{conv}(I_i, J) = \text{conv}(\partial I_i, J) + (-1)^k \sum_j \text{conv}(I_i, J_j),$$

and since $\partial^2 = 0$, we can write

$$\partial \sum_i \text{conv}(I_i, J) = (-1)^k \sum_{i,j} \text{conv}(I_i, J_j).$$

Similarly,

$$\partial \sum_j \text{conv}(I, J_j) = \sum_{i,j} \text{conv}(I_i, J_j).$$

Therefore

$$A_{n-1}(\partial I, J) = \left\langle \text{vol}_{n-1}, \sum_i \text{conv}(I_i, J) \right\rangle$$

$$= \left\langle \beta, \sum_i \text{conv}(I_i, J) \right\rangle$$

$$= (-1)^k \left\langle \beta, \sum_i \sum_j \text{conv}(I_i, J_j) \right\rangle$$

$$= (-1)^k A_{n-1}(I, \partial J),$$

concluding the proof. □
Proof of Proposition 5.2 Step 1. Define for \( v \in S^{n-1} \) the hemisphere \( H_v := \{ p \in S^{n-1} : \langle p, v \rangle > 0 \} \).

Let \( W \subset (S^{n-1})^n \) be the set of \( n \)-tuples \( (p_1, \ldots, p_n) \) belonging to some \( H_v \). Define \( F : W \to \mathbb{R} \) by \( F(p_1, \ldots, p_n) = \text{vol}_{n-1}(\Delta(p_1, \ldots, p_n)) \), the oriented volume of the geodesic simplex \( \Delta(p_1, \ldots, p_n) \) with vertices \( p_1, \ldots, p_n \). \( F \) is well defined and smooth, since all \( p_i \) lie in one hemisphere.

For two non-antipodal points \( q, p \in S^{n-1} \), we define
\[
\omega_{q,p} \in \Lambda^k T_q S^{n-1} \otimes \Lambda^{n-k-2} T_q S^{n-1}
\]
by setting, for \( u_1, \ldots, u_k, v_{n-k-2} \in T_p S^{n-1} \)
\[
\omega_{q,p}(u_1, \ldots, u_k, v_1, \ldots, v_{n-k-2}) := \frac{d^{n-2}}{ds^k} \bigg|_{s,t=0} F(q, \gamma_1(s), \ldots, \gamma_k(s), p, \delta_1(t), \ldots, \delta_{n-k-2}(t)),
\]
where \( \gamma_i \) resp. \( \delta_j \) are any smooth curves through \( q \) resp. \( p \) such that \( \gamma_i'(0) = u_i \), \( \delta_j'(0) = v_j \). It is immediate that the definition is independent of the choice of such curves, and that \( \omega_{q,p} \) defines a unique element \( \omega \in \Omega^{k,n-k-2}(S^{n-1} \times S^{n-1} \setminus \overline{\Sigma}) \), where \( \Sigma = \{(q,-q) : q \in S^{n-1}\} \subset S^{n-1} \times S^{n-1} \) denotes the skew-diagonal. Note that
\[
F(q, \gamma_1(\epsilon), \ldots, \gamma_k(\epsilon), p, \delta_1(\epsilon), \ldots, \delta_{n-k-2}(\epsilon)) = \frac{1}{k!(n-k-2)!} \omega_{q,p}(u_1, \ldots, u_k, v_1, \ldots, v_{n-k-2}) \epsilon^{n-2} + o(\epsilon^{n-2}).
\]

Given an oriented geodesic \( k \)-dimensional polytope \( I \subset S^{n-1} \), define \( \omega_I \in \Omega^{n-k-2}(S^{n-1} \setminus I) \) by
\[
\omega_I|_p = \int_I \omega_{q,p} dq.
\]

Let us verify that \( \int_J \omega_I = A_{n-1}(-I, J) \) for an \((n-k-2)\)-dimensional geodesic polytope \( J \) such that \( J \cap I = \emptyset \). Since both sides are additive in both \( I, J \), we may assume that \( I, J \) are geodesic simplices. We may further assume that there are vector fields \( U_1, \ldots, U_k \) on \(-I\) that are orthonormal and tangent to \(-I\), and \( V_1, \ldots, V_{n-k-2} \) on \( J \) orthonormal and tangent to \( J \). These vector fields define flow curves on \(-I, J\).

For \( \epsilon > 0 \) one can use those curves to define a grid on \(-I\), resp. \( J \) denoted \( \{-q_i\} \) resp. \( \{p_j\} \), defining parallelograms \( -Q_i \) resp. \( P_j \) of volumes \( \epsilon^k + o(\epsilon^k) \) resp. \( \epsilon^{n-k-2} + o(\epsilon^{n-k-2}) \). Note that the volume of the convex hull of two \( \epsilon \)-simplices is equal, up to \( o(\epsilon^{n-2}) \), to \( \frac{1}{\epsilon^2(n-k-2)!} \) times the volume of the convex hull of the corresponding parallelograms. Thus the total volume is given by
\[
A(-I, J) = \sum_i A(-Q_i, P_j)
= \sum_{i,j} (\omega_{-q_i,p_j}(U_1, \ldots, U_k, V_1, \ldots, V_{n-k-2})) \epsilon^{n-2} + o(\epsilon^{n-2})
= \int_{(-I) \times J} \omega + o(1).
\]
Taking \( \epsilon \to 0 \), this proves the claim.
Step 2. Let $J \subset S^{n-1} \setminus I$ be a geodesic polytope of dimension $(n-k-1)$. Then

$$
\int_I \omega_I = \int_{\partial J} \omega_I = A_{n-1}(-I, \partial J) = (-1)^k A_{n-1}(-\partial I, J) \quad \text{by Lemma A.2}
$$

It follows that on $S^{n-1} \setminus I$ we have

$$
d\omega_I = (-1)^k \omega_{\partial I}.
$$

Step 3. Let us verify that $\omega_I$ is an integrable section of $\Omega^{n-k-2}(S^{n-1} \setminus I)$, and therefore admits a unique extension to all of $S^{n-1}$ as a current of finite mass.

Introduce spherical coordinates

$$
\Phi_n : [0, 2\pi] \times [0, \pi]^{n-2} \to S^{n-1}
$$

$$(\theta_0, \theta_1, \ldots, \theta_{n-2}) \mapsto \Phi_n(\theta_0, \theta_1, \ldots, \theta_{n-2}),$$

which are inductively defined by

$$
\Phi_2(\theta_0) := (\cos \theta_0, \sin \theta_0),
$$

$$
\Phi_n(\theta_0, \theta_1, \ldots, \theta_{n-2}) := (\sin \theta_{n-2} \Phi_{n-1}(\theta_0, \theta_1, \ldots, \theta_{n-3}), \cos \theta_{n-2}).
$$

Note that $\theta_{n-2}$ is defined on the whole sphere $S^{n-1}$ and smooth outside $\{\theta_{n-2} = 0, \pi \} = S^{n-1}$, while for $i > 0$, $\theta_{n-2-i}$ is undefined in $\{\theta_{n-1-i} = 0, \pi \} \cup \{\theta_{n-1-i} \text{ undefined} \} = S^{n-1}$, and constitutes a coordinate outside $\{\theta_{n-2-i} = 0, \pi \}$.

The volume form of $S^{n-1}$ is given by

$$
\text{vol}_{n-1} = \prod_{i=0}^{n-3} \sin^{n-2-i} \theta_{n-2-i} \prod_{i=0}^{n-2} d\theta_{n-2-i}.
$$

Define for $0 \leq i \leq n-2$ the vector fields

$$
X_{n-2-i} = \frac{1}{\prod_{j=0}^{n-1-i} \sin^{n-2-j} \theta_{n-2-j} \partial \theta_{n-2-i}} \partial.
$$

The vector field $X_{n-2-i}$ is well defined outside the set $\{\theta_{n-2-i} = 0, \pi \}$. Whenever two such vector fields are defined, they are pairwise orthonormal. Now $\omega_I$ is integrable if

$$
\int_{S^{n-1}} |\omega_I(X_{i_1}, \ldots, X_{i_{n-k-2}})| \text{vol}_{n-1} < \infty
$$

for all $i_1, \ldots, i_{n-k-2}$. Let $j_1, \ldots, j_{k+1}$ be the indices not appearing in $\{i_1, \ldots, i_{n-k-2}\}$. We consider the common level sets $C = C(\theta_{j_1}, \ldots, \theta_{j_{k+1}})$, with volume element $\sigma_C$, so that

$$
\text{vol}_{n-1} = \left( \prod_{l=1}^{k+1} \sin^{j_l} \theta_{j_l} \prod_{l=1}^{k+1} d\theta_{j_l} \right) \wedge \sigma_C.
$$
Then
\[
\int_{S^{n-1}} |\omega_I(X_{i_1}, \ldots, X_{i_{n-k-2}})| \, \text{vol}_{n-1} = \int_{\theta_{j_1} \cdots \theta_{j_{k+1}}} \prod_{l=1}^{k+1} \sin^{n-1} \theta_{j_l} \int_{C(\theta_{j_1} \cdots \theta_{j_{k+1}})} |\omega_I(X_{i_1}, \ldots, X_{i_{n-k-2}})| \, \sigma_C.
\]

While \(C(\theta_{j_1}, \ldots, \theta_{j_{k+1}})\) is not a geodesic polytope in \(S^{n-1}\), it nevertheless holds by the definition of \(\omega_I\) that the internal integral is bounded by the total area of the sphere. Thus the entire integral is finite. We can therefore define the current \(T_I \in D_{k+1}(S^{n-1})\) by
\[
\langle T_I, \phi \rangle := \int_{S^{n-1} \setminus I} \omega_I \wedge \phi, \quad \phi \in \Omega^{n-k-2}(S^{n-1}).
\]

**Step 4.**
We prove (19) by induction on \(k\). For the induction base \(k = 0\), recall that \(T_I = -[[S^{n-1}]]_\text{vol}_{n-1}, \omega_I = -\text{vol}_{n-1}\).

The zero-dimensional geodesic polytope \(I\) is just a point, which we may suppose to be positively oriented. Then \(-I\) is the positively oriented antipodal point. Let \(S^{n-1}_\varepsilon\) be the sphere \(S^{n-1}\) minus the geodesic ball of radius \(\varepsilon\) centered at \(I\). For \(g \in C^\infty(S^{n-1})\), we compute
\[
\langle \partial T_I, g \rangle = \langle T_I, dg \rangle = \int_{S^{n-1} \setminus I} \omega_I \wedge dg = \lim_{\varepsilon \to 0} \int_{S^{n-1}_\varepsilon} \omega_I \wedge dg = \lim_{\varepsilon \to 0} \left[ (-1)^{n+1} \int_{S^{n-1}_\varepsilon} d\omega_I \wedge g + (-1)^n \int_{\partial S^{n-1}_\varepsilon} \omega_I \wedge g \right] = (-1)^{n+1} \int_{S^{n-1}} g \, \text{vol}_{n-1} + (-1)^n \lim_{\varepsilon \to 0} \int_{\partial S^{n-1}_\varepsilon} \omega_I \wedge g.
\]

The boundary of \(S^{n-1}_\varepsilon\) is an \((n-2)\)-dimensional geodesic sphere around \(I\). Since \(\int_{\partial S^{n-1}_\varepsilon} \omega_I = A_{n-1}(-I, \partial S^{n-1}_\varepsilon) = (-1)^{n-1} \text{vol} S^{n-1}_\varepsilon\) (note that Lemma (A.2) does not apply here, as \(S_\varepsilon\) is not geodesically convex), the second integral tends to \((-1)^{n-1} g(I)\) times the volume of \(S^{n-1}_\varepsilon\).

It follows that
\[
\partial T_I = -\text{vol}(S^{n-1})[[I]] + (-1)^n T_I,
\]
as claimed.

**Step 5.** Suppose now that \(k > 0\) and that (19) holds for all polytopes of dimension strictly smaller than \(k\).

Define the current \(U_I := \partial T_I + (-1)^n T_{\partial I} \in D_k(S^{n-1})\). By step 2 and equation (11), \(U_I\) is supported on \(I\). We have to show that \(U_I = (-1)^{nk+1} \text{vol}(S^{n-1})[[I]]\).
Choose a family of closed neighborhoods $I_\epsilon$ with smooth boundary such that $I_\epsilon$ converges to $I$ as $\epsilon \to 0$. Define currents $T_{I_\epsilon}, V_{I_\epsilon}, U_{I_\epsilon}$ on $S^{n-1}$ by

\[(T_{I_\epsilon}, \phi) := \int_{S^{n-1}\setminus I_\epsilon} \omega_I \wedge \phi, \quad \phi \in \Omega^{k+1}(S^{n-1}),\]

\[(V_{I_\epsilon}, \phi) := \int_{S^{n-1}\setminus I_\epsilon} \omega_{\partial I} \wedge \phi, \quad \phi \in \Omega^k(S^{n-1}),\]

\[(U_{I_\epsilon}, \phi) := (-1)^{n+k} \int_{\partial(S^{n-1}\setminus I_\epsilon)} \omega_I \wedge \phi, \quad \phi \in \Omega^k(S^{n-1}).\]

By Step 3, $M(T_{I_{\epsilon}-T_I}) \to 0$, $M(V_{I_{\epsilon}-T_{\partial I}}) \to 0$ as $\epsilon \to 0$. Since $U_{I_{\epsilon}}$ is given by integration of a smooth form on a compact smooth manifold, it is a normal current (i.e. its mass and the mass of its boundary are finite).

By Stokes’ theorem and Step 2, we have $U_{I_\epsilon} = \partial T_{I_\epsilon} + (-1)^n V_{I_\epsilon}$. Therefore

\[F(U_{I_{\epsilon}-U_I}) = F(\partial(T_{I_{\epsilon}-T_I}) + (-1)^n(V_{I_{\epsilon}-T_{\partial I}})) \leq M(T_{I_{\epsilon}-T_I}) + M(V_{I_{\epsilon}-T_{\partial I}}) \to 0.\]

It follows that $U_I$ is a real flat $k$-chain supported on the $k$-dimensional spherical polytope $I$. By induction, we have $\partial T_{\partial I} = (-1)^{nk+n+1} \operatorname{vol}(S^{n-1})[\partial T_I]$ and hence $\partial U_I = (-1)^n \partial T_{\partial I} = (-1)^{nk+1} \operatorname{vol}(S^{n-1})[\partial T_I]$. The constancy theorem \cite[4.1.31], \cite[Proposition 4.9]{} implies that $U_I = (-1)^{nk+1} \operatorname{vol}(S^{n-1})[I]$, as claimed.

\[\Box\]

**Proof of Proposition 5.3.** By \cite{1} and Proposition 5.2 we have

\[\partial T = \frac{1}{\operatorname{vol}(S^{n-1})} \sum_{k=1}^{n-1} (-1)^{nk+k+1} \sum_{F \in \mathcal{F}_k(P)} \partial([v_F] \times T_{\bar{c}_{F,P}})\]

\[= \frac{1}{\operatorname{vol}(S^{n-1})} \sum_{k=1}^{n-1} (-1)^{nk+1} \sum_{F \in \mathcal{F}_k(P)} [v_F] \times \partial T_{\bar{c}_{F,P}}\]

\[= \sum_{k=1}^{n-1} \sum_{F \in \mathcal{F}_k(P)} A_{v_F, \bar{c}_{F,P}} + \frac{1}{\operatorname{vol}(S^{n-1})} \sum_{k=1}^{n-1} \sum_{F \in \mathcal{F}_k(P)} (-1)^{nk+n} [v_F] \times T_{\partial n_F}.\]

Let $1 \leq k \leq n-1$ be fixed and let $J \subset S^{n-1}$ be a $(k-1)$-dimensional geodesic polytope not intersecting any $\partial n_F$ for $F \in \mathcal{F}_k(P)$. Lemma A.1 implies that

\[\sum_{F \in \mathcal{F}_k(P)} \langle [v_F], \omega \rangle T_{\partial n_F} \cap [J] = 0\]

for all $\omega \in \Omega^k_1(R^n)$. It follows that $\sum_{F \in \mathcal{F}_k(P)} [v_F] \times T_{\partial n_F} = 0$, and the statement follows. $\Box$

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