Normal mode analysis for scalar fields in BTZ black hole background

Sayan K Chakrabarti, Pulak Ranjan Giri and Kumar S. Gupta

Theory Division, Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Calcutta 700064, India
(Dated: July 15, 2008)

We analyze the possibility of inequivalent boundary conditions for a scalar field propagating in the BTZ black hole space-time. We find that for certain ranges of the black hole parameters, the Klein-Gordon operator admits a one-parameter family of self-adjoint extensions. For this range, the BTZ space-time is not quantum mechanically complete. We suggest a physically motivated method for determining the spectra of the Klein-Gordon operator.

PACS numbers: 04.70.Dy, 03.65.Ge

INTRODUCTION

Probing a black hole space-time with scalar fields reveals important information about thermodynamical properties of the system, including Hawking radiation and black hole entropy [1, 2, 3, 4, 5, 6, 7, 8, 9]. In this approach, the black hole geometry provides an external background on which the scalar field dynamics is analyzed. There are however several problems with such an analysis. The behaviour of the scalar field at the event horizon may not be well behaved. The free energy of the scalar field may show divergence due to infinite number of modes contributing to it near the horizon [1]. Finally, for certain geometries, the space-time may not be globally hyperbolic, thereby leading to difficulties in the predicting the field propagation [11]. A possible way to address these issues is by imposing suitable boundary conditions on the scalar fields. In particular, the brick wall condition that the scalar field vanishes at a certain distance away from the horizon leads to a well defined prescription for the corresponding free energy and entropy [3]. Such a boundary condition is by no means unique and the physical results may depend on the particular choice made. It is therefore a relevant exercise to classify all the possible boundary conditions that lead to a well posed scalar field propagation and with the view to ultimately find the effect of these different conditions on various physical quantities of interest.

In this paper we shall study the possible choices of boundary conditions and their consequences for the scalar field propagation in the BTZ black hole background, which exists in 2+1 dimensions with a negative cosmological constant [12]. This system exhibits many of the subtleties as mentioned above and yet, the scalar field equation is exactly solvable in this geometry and the normal mode analysis can be performed [4, 13, 14]. This allows for the possibility to analyze the effect of different boundary conditions analytically, which is difficult for most other geometries. We shall use the method of deficiency indices due to von Neumann [15] to classify the various boundary conditions that can be imposed on the scalar field. We find that for certain range of the system parameters, there exists a one parameter family of self-adjoint extensions of the corresponding Klein-Gordon equation. It often happens that the determination of the domain using this technique leads to inequivalent quantizations of the system [16, 17, 18, 19, 20]. Our analysis will show that there are certain difficulties is adopting the usual technique for the case of BTZ.

It has been argued that a space-time that is classically incomplete may or may not be so in a quantum mechanical sense [21]. The particular prescription in [21] proposes to define quantum mechanical completeness in terms of the existence of a unique time evolution of a scalar field probe. This idea is related to the issue of the quantum mechanical completeness of a differential operator [15]. Our analysis clearly indicates that for certain range of the system parameters, the time evolution of a scalar field in the BTZ background is not unique and the corresponding space-time is not quantum mechanically complete. In static geometries in which there is a lack of global hyperbolicity, there exists a technique using self-adjoint extensions [11] which leads to a well posed problem. This technique, which has also been used to analyze quantum singularities [21], is very similar to the one we use here. However, we would like to emphasize that BTZ space-time with a non-zero angular momentum is not static and hence the technique of [11] cannot be used here to address the issue of global non-hyperbolicity, which has to be addressed by the choice of suitable boundary conditions at spatial infinity [4, 10].

This paper is organized as follows. In Section 2 we briefly review the scalar field propagation problem in the BTZ space-time. In Section 3 we use von Neumann’s method of self-adjoint extension to find the deficiency indices of the radial part of the Klein-Gordon operator. In Section 4 we discuss the implications of the self-adjoint extensions and it physical relevance. Section 5 concludes the paper with some discussions and an outlook.
SCALAR FIELD IN BTZ BLACK HOLE SPACE-TIME

We begin our discussion with the BTZ black hole, which is (2 + 1)-dimensional space-time obtained from Einstein Equation with negative cosmological constant $\Lambda = -\frac{1}{l^2}$. The space-time is defined by the metric

$$ds^2 = \left(-M + \frac{J^2}{4r^2} + \frac{J}{2r^2 \ell^2} \right) dt^2 + \frac{dr^2}{\left(-M + \frac{J^2}{4r^2} + \frac{J}{2r^2 \ell^2} \right)} + r^2 \left(-\frac{J}{2r^2} dt + d\phi \right)^2$$

(1)

where $M$ and $J$ denote the black hole mass and angular momentum respectively. For $0 < |J| < M \ell$, there are two horizons, the outer and inner horizons, corresponding respectively to $r = r_+$ and $r = r_-$, where

$$r^2_{\pm} = \frac{M \ell^2}{2} \left\{ 1 \pm \left[ 1 - \left( \frac{J}{M \ell} \right)^2 \right]^{1/2} \right\}.$$  

(2)

The dynamics of a scalar field in this black hole space-time is given by the Klein-Gordon equation

$$\left(\Box - \mu l^{-2}\right)\Psi(x) = 0.$$  

(3)

This Klein-Gordon equation can be solved exactly. To see this, we first use the separation of variables

$$\Psi(x) = e^{-iE_t}e^{im\theta} U_E(r),$$  

where $m \in \mathbb{Z}$. Next we use the ansatz

$$U_E(r) = \left(\frac{r^2}{l^2} - \frac{r^2_+}{l^2} \right)^\alpha \left(\frac{r^2}{l^2} - \frac{r^2_-}{l^2} \right)^\beta f(r),$$  

where

$$\alpha = \frac{i l^2}{2(r^2_+ - r^2_-)} \left( r_+ \cdot E - \frac{r_-}{l} \cdot n \right),$$  

$$\beta = \frac{i l^2}{2(r^2_+ - r^2_-)} \left( r_- \cdot E - \frac{r_+}{l} \cdot n \right).$$  

(6)

Finally, introducing the variable $z = \frac{r^2-r^2_-}{r^2_+ - r^2_-}$, the radial part of the scalar field equation becomes

$$H f(z) = 0,$$  

where

$$H = z(z - 1) \frac{d^2}{dz^2} + \{c - (a + b + 1)z\} \frac{d}{dz} - ab$$  

(8)

and

$$a = \alpha + \beta + \frac{1}{2} (1 + \sqrt{1 + \mu}),$$  

$$b = \alpha + \beta + \frac{1}{2} (1 - \sqrt{1 + \mu}),$$  

$$c = 2\beta + 1.$$  

(9)

From (7) and (8) we see that the function $f(z)$ satisfies the hypergeometric equation [22]. We shall restrict our attention to the analysis of the Klein-Gordon equation in the region $[r_+, \infty)$.

The Klein-Gordon equation for the BTZ has been analyzed in detail [4, 13, 14], which requires specification of the boundary conditions. The BTZ black hole can be obtained by a discrete quotienting of the universal covering space of $AdS^3$, which has a timelike spatial infinity. This feature leads to the lack of global hyperbolicity for the BTZ. Following [10], this issue was handled in [4] by requiring that the solution of the Klein-Gordon equation vanishes sufficiently rapidly at spatial infinity. We shall also adopt the same boundary condition at spatial infinity. The solution of (7) that vanishes at the spatial infinity can be analytically continued to the outer horizon $r_+$, where it
delves. In order to regulate this divergence, it is customary to introduce a brick wall parameter $\epsilon$ and require that the wave-function vanishes at $x = z - 1 = \epsilon$, where $z = 1$ is the outer horizon and $x$ is the near-horizon coordinate $r$. This set of boundary conditions lead to a well posed problem with a finite solution as a function of the brick wall cutoff. The corresponding thermodynamic quantities can then be evaluated, which clearly depends on the specific choice of the boundary conditions [4].

As stated earlier, our main purpose here is to investigate if other boundary conditions exist which may also lead to sensible physics, which is what we discuss in the next section.

**INEQUIVALENT BOUNDARY CONDITIONS**

In this Section we shall try to find all possible boundary conditions that lead to a sensible physical description for the scalar field propagation in the BTZ background. As mentioned earlier, the BTZ space-time is not globally hyperbolic. In order to circumvent the difficulties associated with the non global hyperbolicity, following [4], we shall choose the solution of the wave equation (7) which vanishes at infinity. The boundary condition at the brick wall is however not fixed by this criterion. Instead of requiring the the solution vanishes at the brick wall, we shall demand that the solution is square integrable near the outer horizon $r_+$ and that the corresponding Klein-Gordon operator is self-adjoint. We shall assume standard periodic boundary conditions on the angular variables and investigate the possibility of any new boundary conditions for the radial part of the Klein-Gordon operator, denoted by $A$. We do this following von Neumann’s method of self-adjoint extensions [13], whose basic features are recalled below.

Let $T$ be an unbounded symmetric differential operator acting on a Hilbert space $\mathcal{H}$ and let $D(T)$ be the domain of $T$. There exists a criterion due to von Neumann for determining if $T$ is self-adjoint in $D(T)$. For this purpose let us define the deficiency subspaces $K_\pm = \text{Ker}(i \mp T^*)$ and the deficiency indices $n_\pm(T) = \dim[K_\pm]$. Then $T$ falls in one of the following categories. i) $T$ is (essentially) self-adjoint iff $(n_+, n_-) = (0, 0)$. ii) $T$ has self-adjoint extensions iff $n_+ = n_-$. There is a one-to-one correspondence between self-adjoint extensions of $T$ and unitary maps from $K_+$ into $K_-$. iii) If $n_+ \neq n_-$, then $T$ has no self-adjoint extensions.

In order to find the deficiency index $n_+$, we shall replace $E$ with $i$ everywhere in $A$ and find the square integrable solutions of the corresponding equation. We are interested in the physics in the region outside the outer horizon $r_+$. $A$ is an unbounded differential operator defined in $[r_+, \infty)$. It is a symmetric operator in the domain $D(A) \equiv \{ \phi(x = \epsilon) = \phi'(x = \epsilon) = 0, \phi, \phi' \text{ absolutely continuous}, \phi \in L^2(\sqrt{gdr}) \}$. We would next like to determine if $A$ is self-adjoint in $D(A)$, for which we proceed to find the deficiency indices of $A$.

Consider first the operator $A_+$, in which the eigenvalues have been replaced with $+i$. Denote the resulting operator with $A_+$. The equation which determines the deficiency index $n_+$ can then be written as

$$A_+ \phi_+ = 0,$$

where $\phi_+$ is a function of $r$ (or equivalently of $z$ or $x$). We emphasize that the rhs of (10) is zero as the eigenvalues, which have been replaced with $+i$, already appear in $A_+$. The solution of the corresponding radial equation which vanishes at infinity is given by

$$\phi_+ = \left( \frac{r^2 - r_+^2}{l^2} \right)^{\alpha_+} \left( \frac{r^2 - r_+^2}{l^2} \right)^{\beta_+} z^{-\alpha_+} F(a_+, a_+ - c_+, a_+ - b_+ + 1, \frac{1}{z}),$$

where

$$\alpha_+ = \frac{il^2}{2(r_+^2 - r_+^2)} \left( r_+ i - \frac{r_+}{l} \right),$$

$$\beta_+ = \frac{il^2}{2(r_+^2 - r_+^2)} \left( r_+ i - \frac{r_+}{l} \right).$$

and $a_+, b_+$ and $c_+$ are obtained by substituting $\alpha_+$ and $\beta_+$ in eqn. (10).

The solution $\phi_+$ can be analytically continued near the outer horizon where it takes the form

$$\phi_+ \sim (z - 1)^{\alpha_+} z^{\beta_+} e^{-i\xi} F(a_+, b_+, 2\alpha_+ + 1, 1 - z) + A_+ e^{i\xi} (z - 1)^{-\alpha_+} z^{-\beta_+} F(1 - b_+, 1 - a_+, 1 - 2\alpha_+, 1 - z),$$

where

$$A_+ e^{2i\xi} = \frac{\Gamma(1 - b_+)\Gamma(c_+ - b_+)}{\Gamma(a_+ - c_+ + 1)} \frac{\Gamma(a_+ - c_+ + 1)\Gamma(c_+ - a_+ - b_+)}{\Gamma(a_+ - c_+ + 1)\Gamma(c_+ - a_+ - b_+)}.$$
In terms of the near-horizon coordinate \( x = z - 1 \), the behaviour of \( \phi_+ \) near the outer horizon defined by \( z = 1 \) can be written as

\[
\phi_+ \sim e^{-i\xi} x^{\alpha_+} - A_+ e^{i\xi} x^{-\alpha_+}.
\]

(15)

In the near-horizon region, we therefore get that

\[
|\phi_+|^2 = x^{2\Re(\alpha_+)} + A_+^2 x^{-2\Re(\alpha_+)} + A_+^2,
\]

(16)

where \( \Re(\alpha_+) = -\frac{r_+^2}{r_+^2 - r_H^2} \) denotes the real part of \( \alpha_+ \) in (12), which is a negative number. Note that in the near horizon region, the measure \( \sqrt{-g} dr \sim r dr \sim dx \) Thus, the solution \( \phi_+ \) is square integrable near the horizon if \( 0 > \Re(\alpha_+) > -\frac{1}{2} \). Under this condition, we see that the deficiency index \( n_+ = 1 \). A similar analysis shows that \( n_- = 1 \) as well. This in the region \( 0 > \Re(\alpha_+) > -\frac{1}{2} \), the operator \( \mathcal{A} \) admits a one parameter family of self-adjoint extensions, the latter being parametrized by a phase denoted by \( e^{i\gamma} \), \( \gamma \in R \) (mod 2\( \pi \)).

In the definition of the deficiency subspaces, instead of \( \pm i \), in general one can consider an arbitrary complex number \( \lambda \) and its complex conjugate, although it is usually sufficient to consider \( \pm i \) alone. If we define the deficiency indices in terms of \( \lambda \) and its complex conjugate, we find that the solutions spanning the deficiency subspaces are square-integrable near the horizon only with a band of values of the imaginary part of \( \lambda \). This is an unusual situation for the BTZ space-time for which we do not have any definite interpretation.

**IMPLICATIONS OF BOUNDARY CONDITIONS**

Application of the method of von Neumann has led us to the conclusion that for certain range of the system parameters, the operator \( \mathcal{A} \) is not self-adjoint but admits a one parameter family of self-adjoint extensions. In addition, using von Neumann’s analysis [13], it is possible to construct the domain in which the operator \( \mathcal{A} \) would be self-adjoint when \( 0 > \Re(\alpha_+) > -\frac{1}{2} \). Since we have \( n_+ = n_- = 1 \), the inequivalent domains are characterized by a 1 \( \otimes \) 1 unitary matrix, which is just a phase denoted by \( e^{i\gamma} \). The domain \( D_+(\mathcal{A}) \) in which the operator \( \mathcal{A} \) is self-adjoint is then given by \( D_+(\mathcal{A}) = D(\mathcal{A}) \oplus C(\phi_+ + e^{i\gamma} \phi_-) \), where \( C \) is an arbitrary complex constant. For each value of \( \gamma \in [0, 2\pi] \), we have a domain \( D_+(\mathcal{A}) \), which provides an inequivalent set of boundary conditions compatible with the self-adjointness of the Klein-Gordon operator.

In the usual problems of this type, the physical solution can be made to belong to the domain of self-adjointness which leads to the inequivalent quantizations of the system [16, 17, 18, 19, 20]. For the case of the BTZ however this prescription for the spectra does not seem to work. It can be shown that near the outer horizon a typical example of the domain \( D_+(\mathcal{A}) \) has both oscillatory and non-oscillatory parts whereas the physical solution is purely oscillatory [4]. For this reason we are not able to obtain the spectra corresponding to the \( D_+(\mathcal{A}) \) directly. We shall make some further comments about this at the end of this section.

The issue of quantum mechanical completeness of the BTZ space-time can be addressed using the analysis presented here. We use the basic idea of [15, 21] where the quantum mechanical completeness of a given geometry is related to the existence of a unique well defined quantum time evolution of a scalar field probe in that background. This idea can be formulated in terms of the self-adjoint extension of \( \mathcal{A} \). We have seen above that outside the parameter range \( 0 > \Re(\alpha_+) > -\frac{1}{2} \), the deficiency indices \( n_+ = n_- = 0 \). This means that outside this range of parameters \( \mathcal{A} \) is essentially self-adjoint in \( D(\mathcal{A}) \), which means that it has a unique self-adjoint extension. Hence, outside the range \( 0 > \Re(\alpha_+) > -\frac{1}{2} \), a scalar field in the BTZ space-time has a unique time evolution and the corresponding geometry is quantum mechanically complete. On the other hand, when \( 0 > \Re(\alpha_+) > -\frac{1}{2} \), \( n_+ = n_- = 1 \) and the Klein-Gordon operator admits a one parameter family of self-adjoint extensions. The key issue is that within the context of the scalar field dynamics in the BTZ background, there is no preferred choice among the possible self-adjoint extensions. The corresponding dynamics therefore suffers from this one parameter ambiguity, which cannot be resolved within the context of this problem alone. This implies that the BTZ space-time in not quantum mechanically complete when \( 0 > \Re(\alpha_+) > -\frac{1}{2} \). In drawing this conclusion we have assumed that the lack of global hyperbolicity for the BTZ can be addressed by assuming a suitable boundary condition at infinity [4, 10].

Finally, we shall attempt to give a proposal for finding alternate boundary conditions at the outer horizon, from which a spectra different from that of [4] can be obtained. Our prescription is not directly related to the domain of self-adjointness found above, but is motivated by physical considerations [24].

Let us first note that around the outer horizon \( r_+ \), given by \( z = 1 \), the two linearly independent expressions for \( U_E(z) \) are given by \((z-1)^\alpha z^\beta F(1+b-c, 1+a-c, a+b+1-c, 1-z)\) and \((z-1)^{-\alpha} z^{-\beta} F(1-a, 1-b, c-a-b+1, 1-z)\).
As $z \to 1$, the hypergeometric functions tend to 1. Therefore a general expression for $U_E(z)$ near $z = 1$ can be given by

$$U_E(z) \sim (z - 1)^\alpha z^\beta + D(z - 1)^{-\alpha} z^{-\beta}. \quad (17)$$

On the other hand, the solution $g$ which vanishes at infinity can be analytically continued near the outer horizon and has the form

$$g(z) \sim (z - 1)^\alpha z^\beta + e^{-2i\pi \theta_0} (z - 1)^{-\alpha} z^{-\beta}, \quad (18)$$

where

$$e^{-2i\pi \theta_0} = -\frac{\Gamma(1 - b)\Gamma(c - b)\Gamma(a + b - c)}{\Gamma(a)\Gamma(a - c + 1)\Gamma(c - a - b)}. \quad (19)$$

These two expressions physically denote the same quantity, namely the behaviour of the scalar field near the outer horizon, and they would be compatible if $D$ is chosen as a pure phase $-e^{-2i\pi \delta}$. Furthermore, in order to regulate the wildly oscillatory behaviour of the functions near the horizon, we introduce a brick wall cutoff at $x = z - 1 = \epsilon$. By comparing the two expressions near the horizon we get

$$\sin(\alpha \ln(\epsilon) + \pi \theta_0) = \sin(\alpha \ln(\epsilon) + \pi \gamma). \quad (20)$$

A particular choice of the boundary condition could be that the solutions vanishes at $x = \epsilon$, which will recover the results of [4]. However, instead of imposing a definite boundary condition at a given point, we would also calculate the spectrum by demanding that

$$\delta = \theta_0 + 2\pi k, \quad (21)$$

where $k$ is an integer. In the above equation, $\theta_0$ depends on the system parameters as well as on the frequency, while $\delta \in [0, 2\pi]$ is an arbitrary constant. For fixed values of the system parameters, any choice of $\gamma$ would lead to a different spectrum, which can be obtained numerically.

CONCLUSIONS

In this paper we have considered the possibility of inequivalent boundary conditions for the radial part of the Klein-Gordon equation of a scalar field in the BTZ background. Although the radial equation in this case is exactly solvable, the analysis of self-adjoint extension appears to be somewhat unusual. First, we have seen that the solutions of the deficiency indices depend on the range of the imaginary part of the complex quantity $\lambda$ that goes in the calculation of the deficiency subspaces. This was certainly an unexpected feature of this analysis. Second and even more surprising, the domain that we obtain using von Neumann’s analysis turns out to be such that the physical solution does not appear to belong to it. Thus, even though we found that the Klein-Gordon operator admits a one parameter family of self-adjoint extensions, we could not obtain the spectrum from the usual technique of von Neumann.

Due to this reason, we have proposed a way of obtaining the inequivalent quantization of the spectrum motivated by a physical approach towards the problem [24]. In this approach, the spectrum depends on an undetermined parameter and is independent of any short distance cutoff. This is however somewhat misleading as the equation determining the spectrum presupposes the existence of a brick wall type cutoff. Thus, while in the usual analysis of [4] the dependence of the spectrum on the brick wall parameter is explicit, in our general case, it is only implicit.

There is a proposal to classify the quantum completeness of space-time using the self-adjoint extensions of the corresponding scalar field equations [21]. By that criterion, our analysis indicates that the space-time associated with a rotating BTZ black hole is quantum mechanically incomplete.

Finally, it appears likely that a renormalization technique used to study certain singular potentials [23] can be used for this geometry to obtain a beta function for the gravitational coupling. Such an idea has already been used in the literature [3]. In the usual spectrum of [4], if we demand that the Newton’s constant depends on the cutoff and demand that the dependence is such that any particular energy level is independent of the cutoff as the latter goes to zero, we would obtain a beta function for the Newton’s constant. In this case, it would be however difficult to obtain an analytical expression due to the nature of the spectrum.

* Electronic address: sayan.chakrabarti@saha.ac.in
[1] G. 't Hooft, Nucl. Phys. B256, 727 (1985).
[2] L. Bombelli, R. K. Koul, J.-H Lee, R. D. Sorkin Phys. Rev. D34, 373 (1986).
[3] L. Susskind and J. Uglum, Phys. Rev. D50, 2700 (1994).
[4] I. Ichinose and Y. Satoh, Nucl. Phys. B447, 340 (1995).
[5] G. 't Hooft, Int. J. Mod. Phys. A11, 4623 (1996).
[6] G. 't Hooft, hep-th/0003004.
[7] D. Birmingham, Kumar S. Gupta and S. Sen, Phys. Lett. B505, 191 (2001).
[8] Kumar S. Gupta and S. Sen, Phys. Lett. B526, 121 (2002).
[9] Sayan K. Chakrabarti, Kumar S. Gupta and S. Sen, arXiv:0708.1667 [hep-th]
[10] S. J. Avis, C. J. Isham and D. Storey, Phys. Rev. D 18, 3565 (1978).
[11] R. M. Wald, Jour. Math. Phys. 21, 2802 (1980).
[12] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992).
[13] M. Kenmoku, M. Kuwata and K. Shigemoto, Class. Quant. Grav. 25, 145016 (2008).
[14] M. Kuwata, M. Kenmoku, K. Shigemoto, Prog. Theor. Phys. 119, 939 (2008).
[15] M. Reed and B. Simon, Fourier Analysis, Self-Adjointness (New York :Academic, 1975).
[16] R. Jackiw, M. A. B. Beg Memorial Volume, edited by A. Ali and P. Hoodbhoy (World Scientific, Singapore, 1991).
[17] B. Basu-Mallick, P. K. Ghosh, and K. S. Gupta, Nucl. Phys. B659, 437 (2003).
[18] B. Basu-Mallick, P. K. Ghosh, and K. S. Gupta, Phys. Lett. A311, 87 (2003).
[19] P. R. Giri, Phys. Rev. A76, 012114 (2007).
[20] P. R. Giri, Eur. Phys. J C56, 147 (2008).
[21] T. Horowitz and D. Marolf, Phys. Rev. D52, 5670 (1995).
[22] A. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (New York, Dover, 1972).
[23] K. Gupta and S.G. Rajeev, Phys. Rev. D48, 5940 (1993).
[24] M. Alford, J. March-Russell and F. Wilczek, Nucl. Phys. B328, 140 (1989).