Rational solutions of the fourth and fifth Painlevé hierarchies

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We will consider four hierarchies of higher order analogues of the fourth (P4) and fifth (P5) Painleve equations. The necessary and sufficient conditions for having rational solutions will be presented. Also the algorithm for obtaining such solutions will be described.

The hierarchies obtained in [1] generalize the property of having an extended affine Weyl group of Bäcklund transformations. The hierarchy of (P4) is represented by the system of ordinary differential equations

\[
\begin{align*}
f'_0 &= f_0(f_1 - f_2 + f_3 - f_4 + \ldots + f_{n-2} - f_{n-1}) + \alpha_0 \\
f'_1 &= f_1(f_2 - f_3 + f_4 - f_5 + \ldots + f_{n-1} - f_0) + \alpha_1 \\
&\vdots \\
f'_{n-1} &= f_{n-1}(f_0 - f_1 + f_2 - f_3 + \ldots + f_{n-3} - f_{n-2}) + \alpha_{n-1},
\end{align*}
\]

where \(\alpha_i\) are such complex parameters, that \(\sum_{i=0}^{n-1} \alpha_i = h \neq 0\), \(f_i\) are unknown functions and \(n\) is an odd number defining different members of the hierarchy. Further description of (1) and its properties connected to this paper could be found in [5], [8] and [9].

Another hierarchy of analogues of the (P4) was considered by Shabat [1], Veselov [2] and Adler [3]. It was obtained as the result of periodic closing of the dressing chain for linear Schrödinger equation.

\[
\begin{align*}
g'_0 + g'_1 &= g_1^2 - g_0^2 + \alpha_0 \\
g'_1 + g'_2 &= g_2^2 - g_1^2 + \alpha_1 \\
&\vdots \\
g'_{2n+1} &= g_{2n+1} - g_0^2 + \alpha_{n-1},
\end{align*}
\]

where \(\alpha_i\) are also such complex parameters that \(\sum_{i=0}^{n-1} \alpha_i = h \neq 0\) and \(n\) is odd. With the scaling change of variables in (1) as also in (2) one can vary the value of \(h\). Considering this we will further use \(h = 1\) without the loss of generality.

As analogues of (P5) we will again consider the hierarchies of Noumi and Yamada from [4] with \((2n+1)\alpha_i = h \neq 0\) and arbitrary natural \(n\) defining the number of a member in the hierarchy. After change of variables it takes form

\[
\begin{align*}
zf'_i = z^2 f_i \Phi_i - A_{Mod(i,2)} f_0 + \alpha_i C_{Mod(i,2)}, & i = 0, 2n + 1 \\
\sum_{r=0}^{n} f_{2r} = C_0, & \sum_{s=0}^{n} f_{2s+1} = C_1
\end{align*}
\]

where

\[
\Phi_i^{(n)} = \sum_{1 \leq r \leq s \leq n} f_{i+2r-1} f_{i+2s} - \sum_{1 \leq r \leq s \leq n} f_{i+2r} f_{i+2s+1},
\]

\[
A_0^{(n)} = \sum_{i=0}^{n} \alpha_{2r}, \quad A_1^{(n)} = \sum_{s=0}^{n} \alpha_{2s+1}.
\]

Some properties of (3) connected to this paper are presented in [1], [10] and [11]. We will explain the correspondence of (3) to the systems of ordinary differential equations obtained by Shabat [1], Veselov [2] and Adler [3] with \((2n+1)\alpha_i = h \neq 0\).

\[
\begin{align*}
g'_0 + g'_1 &= g_1^2 - g_0^2 + \alpha_0 \\
g'_1 + g'_2 &= g_2^2 - g_1^2 + \alpha_1 \\
&\vdots \\
g'_{2n+1} &= g_{2n+1} - g_0^2 + \alpha_{2n+1},
\end{align*}
\]

Without the loss of generality we will again assume for (3) and (5) that \(h = 1\).

When considering these hierarchies with real-valued parameters \(\alpha_i\), there exist a Bäcklund transformation, which leads to the system with \(0 \leq \alpha_i \leq 1\) e.g. [8]. In this case the following sufficient conditions on \(\alpha_i\) for existing of rational solutions are known: for the system (1) with odd \(n\)

\[
\begin{align*}
(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_n) & \quad (f_0, f_1, f_2, f_3, f_4, \ldots, f_n) \\
(1, 0, 0, 0, 0, \ldots, 0) & \quad (t, 0, 0, 0, 0, \ldots, 0), \\
(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \ldots, 0) & \quad \left(\frac{t}{3}, \frac{t}{3}, \frac{t}{3}, 0, 0, \ldots, 0\right), \\
(\ldots) & \quad (\ldots, \ldots, \ldots, \ldots, \ldots, \ldots) \\
(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) & \quad \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right).
\end{align*}
\]
For the system (3) with $C_0 \neq 0$ and $C_1 \neq 0$ we also have

\[
\begin{pmatrix}
    (a_0, & a_1, & a_2, & a_3, & a_4, & \ldots & a_{2n+1}) \\
    (a_0, & 1-a_0, & 0, & 0, & 0, & \ldots & 0), & a_0 \in [0, 1] \\
    (a_0, & \frac{1}{2}(1-2a_0), & a_0, & \frac{1}{2}(1-2a_0), & 0, & \ldots & 0), & a_0 \in [0, \frac{1}{2}] \\
    (\ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots) \\
    (a_0, & \frac{1}{n+1}-a_0, & a_0, & \frac{1}{n+1}-a_0, & a_0, & \ldots & \frac{1}{n+1}-a_0), & a_0 \in [0, \frac{1}{n+1}] \\
\end{pmatrix}
\]

And if in the rows of the tables (6) and (7) we admit all possible permutations of all couples of zeroes this will describe the complete set of the parameters $a_i$, with which the hierarchies (11) and (3) admit rational solutions. Considering the relations between the both P4-hierarchies (1), (2) and also between the both P5-hierarchies (3), (5) the tables could be used for obtaining the conditions of having rational solutions for the systems (2) and (5).

Further we well consider two more hierarchies of the (P4). We well show that both of them admit the solution in the form of $1/x$. These hierarchies are the one that was introduced by Kudryashov in [6] as the compatibility condition

\[ 4PA_x + 2AP_x + \lambda Ay_x + 2\lambda y A_x - yy_x A + A_{xxx} + \lambda^2 + 2y_x A_x - Ay_{xx} - \lambda^2 A_x - y^2 A_x - \lambda y = 0 \]  

for the system of linear partial differential equations

\[
\begin{cases}
    \Psi_{xx} = (\lambda - y(x))\Psi_x - P(x)\Psi \\
    \lambda\Psi_L = A(x, \lambda)\Psi_x + B(x, \lambda)\Psi,
\end{cases}
\]

where $A(x, \lambda)$ is a polynomial of $\lambda$. The degree of $A(x, \lambda)$ as a polynomial of $\lambda$ corresponds to the member’s number in the hierarchy. For example, the second member $(n = 2)$ has the form:

\begin{align*}
(y'' - 2xy' - y^3 - \beta)y^2(y^{(4)} - \frac{1}{4}y^2(y^{(3)})^2 + 2y^2 + 8y^3y' + 4xyy' - y'y'' + \beta y'y^{(3)} - \frac{\lambda}{2}y(y')^2 + 3xy^2 + 3\beta y - \frac{3}{2}y^2 + \frac{3}{2}(y')^2)(y'' + \beta y^4 - 2y^2y' - 12y^2y^3 - 2\beta^2y + 10xy^3 - 3\beta(y')^2 + 10y^7 - 4xy(y')^2 - 4\beta xy^2)y'' + 2(\beta - 4\beta^3)y^4y' + (4\beta xy + 8xy^4 + \frac{3}{2}\beta^2 + 12\beta y^3)(y')^2 - \frac{10}{2}y^{10} - 8xy^8 - 2\beta^2y - 6x^2y^6 - 2\beta xy^5 + \frac{2}{2}y^2 - 2 + 9\delta - \frac{1}{2}x^3) + \beta xy^2 + \frac{1}{2}\beta^3 y = 0,
\end{align*}

In this equation $\beta$ and $\delta$ are complex parameters.

And the last considered here analogues of the (P4) were obtained by Gordoa, Joshi and Pickering in [7] from nonsisospectral scattering problem

\[
\begin{cases}
    L_n.x - 2K_n - (u + \frac{2\alpha_{n+1}}{g_n})L_n = g_n - 2\alpha_n \\
    L_nK_n + uL_n^2 + K_n - L_nK_n(u + \frac{2\alpha_{n+1}}{g_n})L_nK_n = \left(\frac{1}{2}g_n - \alpha_n\right)^2 - \frac{1}{4}\beta_n^2,
\end{cases}
\]

where

\[
\begin{pmatrix}
    K_n \\
    L_n
\end{pmatrix} = B^{-1} \begin{pmatrix}
    R^{n-1}U_x + g_n \begin{pmatrix}
    1 \\
    0
\end{pmatrix} + \sum_{i=0}^{n-2} \left(\frac{-2g_{n+i}}{g_n}\right)^{n-i-1} R^iU_x \end{pmatrix} + 2\left(\frac{-2g_{n+1}}{g_n}\right)^{n-i-1} \begin{pmatrix}
    1 \\
    0
\end{pmatrix}
\]

\[
U = \begin{pmatrix}
    u \\
    v
\end{pmatrix} = \begin{pmatrix}
    u(x) \\
    v(x)
\end{pmatrix}, \quad B = \begin{pmatrix}
    0 & \partial_x \\
    \partial_x & 0
\end{pmatrix}, \quad R = \frac{1}{2} \begin{pmatrix}
    \partial_x u \partial^{-1} x & 2 \\
    2v + \partial_x u \partial^{-1} x & u + \partial_x
\end{pmatrix}.
\]

In this problem $\alpha_n, \beta_n$ and $g_i, i = 0, \ldots, n+1$ are complex parameters among which $g_{n+1} = 0$ and $g_n \neq 0$. The number $n$ denotes the member’s place in the resulting hierarchy.

The last result shows that all the considered P4-hierarchies have the common property: they assume the solution $1/x$ as also the original P4 does.

References

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