ON SOME INEQUALITIES OF CHEBYSHEV TYPE

ANDRIY L. SHIDLICH, STANISLAV O. CHAIChENKO

Abstract. We obtain some new inequalities of Chebyshev Type.

1. Introduction.

Let \( f, g : [a, b] \to \mathbb{R} \) be integrable functions, both increasing or both decreasing. Further, let \( p : [a, b] \to \mathbb{R}_0^+ \) be an integrable function. Then (see, for example, [1, Chap. IX])
\[
\int_a^b p(x)f(x)g(x)dx \geq \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \left( \int_a^b p(x)dx \right)^{-1}.
\]

If one of the functions \( f \) or \( g \) is nonincreasing and the other nondecreasing the reversed inequality is true, i.e.,
\[
\int_a^b p(x)f(x)g(x)dx \leq \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \left( \int_a^b p(x)dx \right)^{-1}.
\]

Inequalities (1) and (2) are known as Chebyshev’s inequalities. These inequalities were obtained by P.L. Chebyshev [2, 3] and they attracted great interest of the researchers. So, a lot of analogues and generalizations of inequalities (1) and (2) is known. In particular, these results can be found in Chapter IX of the book [1] by D.S. Mitrinović, J.E. Pečarić and A. M. Fink which trace completely the historical and chronological developments of Chebyshev’s and related inequalities (see also [4, 5]). Also we would like to recommend the article of H.P. Heinig and L. Maligranda [6], where one can found a lot of important results on Chebyshev’s inequalities for strongly increasing functions, positive convex and concave functions as well as on Chebyshev’s inequalities in Banach function spaces and symmetric spaces.

In [7], these investigations were developed in the following direction: the author found necessary and sufficient conditions on the function \( g : [a, b] \to \mathbb{R}_0^+ \) and \( p : [a, b] \to \mathbb{R}^+ \) such that for any monotone function \( f : [a, b] \to \mathbb{R}_0^+ \) the relations
\[
\int_a^b p(x)f(x)g(x)dx \geq \left( \int_a^b p^r(x)f^r(x)dx \right)^{1/r} \left( \int_a^b p(x)f(x)dx \right)^{1/r} \left( \int_a^b p^r(x)dx \right)^{-1/r}
\]
and
\[
\int_a^b p(x)f(x)g(x)dx \leq \left( \int_a^b p^r(x)f^r(x)dx \right)^{1/r} \left( \int_a^b p(x)f(x)dx \right)^{1/r} \left( \int_a^b p^r(x)dx \right)^{-1/r}
\]
hold with \( r \) being an arbitrary positive number.

In this paper we continue the study of the inequalities of the type (1)–(4), namely, we obtain the following assertions:
THEOREM 1. Assume that $g: [a, b] \to \mathbb{R}_0^+$ and $p: [a, b] \to \mathbb{R}^+$ are integrable functions such that the product $p \cdot g$ is also integrable on $[a, b]$ function. Let $f: [a, b] \to \mathbb{R}_0^+$ be a nonincreasing function. Then for any convex function $M: [0, \infty) \to \mathbb{R}$ such that $M(0) = 0$, the following inequality is true:

$$
\int_a^b p(x)g(x)M(f(x))dx \leq \sup_{s \in [a, b]} \left\{ M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}\right) \int_a^s p(x)g(x)dx \right\},
$$

(5)

and for any concave function $M: [0, \infty) \to \mathbb{R}$ such that $M(0) = 0$, the following inequality is true:

$$
\int_a^b p(x)g(x)M(f(x))dx \geq \inf_{s \in [a, b]} \left\{ M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}\right) \int_a^s p(x)g(x)dx \right\}.
$$

(6)

Furthermore, if the function $f(x) \equiv c, c \geq 0$, then relations (1) and (2) are equalities.

Putting $M(t) = t^{1/r}, r > 0$, from Theorem 1 we obtain the following corollaries:

COROLLARY 1. Let $r \in (0, 1]$, and let $g: [a, b] \to \mathbb{R}_0^+$ and $p: [a, b] \to \mathbb{R}^+$ be integrable functions such that for all $s \in (a, b)$,

$$
\frac{\int_a^s p(x)g(x)dx}{\left(\int_a^s p(x)dx\right)^{1/r}} \leq \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}.
$$

(7)

Then for any nonincreasing function $f: [a, b] \to \mathbb{R}_0^+$,

$$
\int_a^b p(x)g(x)f(x)dx \leq \left(\int_a^b p(x)f(x)dx\right)^{1/r} \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}.
$$

(8)

COROLLARY 2. Let $r \geq 1$, and let $g: [a, b] \to \mathbb{R}_0^+$ and $p: [a, b] \to \mathbb{R}^+$ be integrable functions such that for all $s \in (a, b)$,

$$
\frac{\int_a^s p(x)g(x)dx}{\left(\int_a^s p(x)dx\right)^{1/r}} \geq \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}.
$$

(9)

Then for any nonincreasing function $f: [a, b] \to \mathbb{R}_0^+$,

$$
\int_a^b p(x)g(x)f(x)dx \geq \left(\int_a^b p(x)f(x)dx\right)^{1/r} \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}.
$$

(10)

If in corollaries 1 and 2, we put $r = 1$, then we see that relations (8) and (10) are the Chebyshev’s classical inequalities (11) and (12). Furthermore, it should be noted that conditions on the functions $p$ and $g$ of the form (7) and (9) for validity of inequalities (11) and (12) were considered in the papers [7] and [8].

In the case, where the function $M(f(x))$ is nonincreasing and the function $g$ is nondecreasing (or nonincreasing), we can apply the Chebyshev’s classical inequalities to the integral $\int_a^b p(x)g(x)M(f(x))dx$ on the left-hand side of relations (11) (or (13)). Respectively, we obtain

$$
\int_a^b p(x)g(x)M(f(x))dx \leq \frac{\int_a^b p(x)M(f(x))dx}{\int_a^b p(x)dx} \int_a^b p(x)g(x)dx
$$

(11)
Furthermore, if exact upper (or lower) bound on the right-hand side of (5) (or (6)) is realized for (13) and (14) of the integral

Here, it should be noted that by virtue of Jensen’s inequality (see, for example, [1, Chap. I]), estimations then from relations (5) and (6) we get

and for any concave function

\[M \geq a \geq \ldots \geq a_m\] and \(p_k > 0\). Then for any convex function \(M : [0, \infty) \to \mathbb{R}\) such that \(M(0) = 0\), the following inequality is true:

\[
\sum_{k=1}^{m} p_k b_k M(a_k) \leq \max_{s \in [1, m]} \left\{ M \left( \frac{\sum_{k=1}^{m} p_k a_k}{\sum_{k=1}^{m} p_k} \right) \sum_{k=1}^{s} p_k b_k \right\},
\]  
(15)

and for any concave function \(M : [0, \infty) \to \mathbb{R}\) such that \(M(0) = 0\), the following inequality is true:

\[
\sum_{k=1}^{m} p_k b_k M(a_k) \geq \min_{s \in [1, m]} \left\{ M \left( \frac{\sum_{k=1}^{m} p_k a_k}{\sum_{k=1}^{m} p_k} \right) \sum_{k=1}^{s} p_k b_k \right\}.
\]  
(16)

Furthermore, if the sequence \(a_k \equiv c\), \(c \geq 0\), then relations (15) and (16) are equalities.

Proof. Consider the case, where the function \(M\) is convex (in the case, where the function \(M\) is concave, the proof is similar). Let us prove by the induction on \(m\) the proposition that for any convex function \(M : [0, \infty) \to \mathbb{R}\) such that \(M(0) = 0\), inequality (15) holds.

The case \(m = 1\) is obvious.

Also consider the case \(m = 2\).

Put

\[c = p_1 a_1 + p_2 a_2, \quad x_0 = p_1 a_1, \quad \alpha_k = p_k b_k, \quad \beta_k = p_k^{-1}, \quad k = 1, 2,\]  
(17)

and consider on the interval \([0, c]\) the function

\[h(x) = \alpha_1 M(\beta_1 x) + \alpha_2 M(\beta_2 (c - x)).\]  
(18)
Due to convexity of the function $M(t)$, the function $h(x)$ is also convex on $[0, c]$. Hence, this function attains its maximum value on any interval $[x_1, x_2] \subseteq [0, c]$ at one of its endpoints. Thus

$$h(x) \leq \max\{h(x_1), h(x_2)\} \quad \forall x \in [x_1, x_2].$$

(19)

Setting $x_1 := \beta_2 c(\beta_1 + \beta_2)^{-1}$ and $x_2 := c$, we see that the number $x_0$ (by virtue of monotonicity of the sequence $a$) belongs to the interval $[x_1, x_2]$.

Therefore, in view of relations (17)–(19) and of the equality $M(0) = 0$, we get

$$\sum_{k=1}^{2} p_k b_k M(a_k) = h(x_0) \leq \max\{h(x_1), h(x_2)\} =$$

$$= \max\left\{M\left(\frac{p_1 a_2 + p_2 a_2}{p_1 + p_2}\right)(p_1 b_2 + p_2 b_2), M\left(\frac{p_1 a_2 + p_2 a_2}{p_1}\right)p_1 b_1\right\}.$$ 

Hence, for $m = 2$, inequality (15) holds.

Now, assume that for $m = n - 1 \geq 1$, the proposition is true.

Let us show that for $m = n$, it is also true. Let us use notations (17) and consider on the interval $[0, c]$ the function $h(x)$ of the form as in (18). Setting $x_1 := \beta_2 c(\beta_1 + \beta_2)^{-1}$ and $x_2 := c - a_3 / \beta_2$, we see that the number $x_0$ (by virtue of monotonicity of the sequence $a$) belongs to the interval $[x_1, x_2]$. Thus in view of relations (17)–(19),

$$\sum_{k=1}^{n} p_k b_k M(a_k) = h(x_0) + \sum_{k=3}^{n} p_k b_k M(a_k) \leq \max\{h(x_1), h(x_2)\} + \sum_{k=3}^{n} p_k b_k M(a_k).$$

(20)

Further, in the case, where $h(x_1) \geq h(x_2)$, we set

$$p_k' = \begin{cases} p_1 + p_2, & k = 1, \\ p_{k+1}, & k = 2, m - 1; \end{cases}$$

$$b_k' = \begin{cases} (p_1 b_1 + p_2 b_2)/(p_1 + p_2), & k = 1, \\ b_{k+1}, & k = 2, m - 1; \end{cases}$$

(21)

$$a_k' = \begin{cases} (p_1 a_1 + p_2 a_2)/(p_1 + p_2), & k = 1, \\ a_{k+1}, & k = 2, m - 1. \end{cases}$$

(22)

Then by virtue of (20), we conclude that the following relation is true:

$$\sum_{k=1}^{m} p_k b_k M(a_k) \leq \sum_{k=1}^{m-1} p_k' b_k' M(a_k').$$

(23)

In the case, where $h(x_1) < h(x_2)$, relation (23) holds for the sequences $a'$, $b'$ and $p'$ of the form:

$$p_k' = \begin{cases} p_1, & k = 1, \\ p_2 + p_3, & k = 2, \\ p_{k+1}, & k = 3, m - 1; \end{cases}$$

$$b_k' = \begin{cases} b_1, & k = 1, \\ (p_2 b_2 + p_3 b_3)(p_2 + p_3)^{-1}, & k = 2, \\ b_{k+1}, & k = 3, m - 1; \end{cases}$$

$$a_k' = \begin{cases} (p_1 a_1 + p_2 a_2 - p_2 a_3)/p_1, & k = 1, \\ a_{k+1}, & k = 2, m - 1. \end{cases}$$

(24)

(25)
The sum on the right-hand side of (23) contains \( n - 1 \) items. Furthermore, in both cases, for the sequences \( a', b' \) and \( p' \), the induction assumption is satisfied. Thus, taking into account (21)–(25), we obtain the necessary estimate (15):

\[
\sum_{k=1}^{n} p_k b_k M(a_k) \leq \sum_{k=1}^{n-1} p'_k b'_k M(a'_k) \leq \sup_{s \in [1,n-1]} \left\{ M \left( \frac{\sum_{k=1}^{n-1} p'_k a'_k}{\sum_{k=1}^{s} p'_k} \right) \sum_{k=1}^{s} p'_k b'_k \right\} \leq \sup_{s \in [1,n]} \left\{ M \left( \frac{\sum_{k=1}^{n} p_k a_k}{\sum_{k=1}^{s} p_k} \right) \sum_{k=1}^{s} p_k b_k \right\}.
\]

Lemma is proved.

3. Proof of Theorem 1.

Consider the case, where the function \( M \) is convex (in the case, where the function \( M \) is concave, the proof is similar). First, let us verify that inequality (5) holds for any function \( f \) such that for a certain \( m \in \mathbb{N} \),

\[ f(x) = a_k, \quad x \in [l_{k-1}, l_k), \quad k = 1, 2, \ldots, m, \]

where \( a_1 > a_2 > \ldots > a_m \geq 0 \) and \( a = l_0 < l_1 < \ldots < l_m = b \).

For any \( k = 1, 2, \ldots, m \), we put

\[ p_k = \int_{l_{k-1}}^{l_k} p(x) dx, \quad b_k = \int_{l_{k-1}}^{l_k} p(x) g(x) dx \left( \int_{l_{k-1}}^{l_k} p(x) dx \right)^{-1}. \]

Then by virtue of Lemma 1, we get (5):

\[
\int_{a}^{b} p(x) g(x) M(f(x)) dx = \sum_{k=1}^{m} \int_{l_{k-1}}^{l_k} p(x) g(x) M(f(x)) dx = \sum_{k=1}^{m} p_k b_k M(a_k) \leq \sup_{s \in [1,m] \cap \mathbb{N}} \left\{ M \left( \frac{\sum_{k=1}^{m} p_k a_k}{\sum_{k=1}^{s} p_k} \right) \sum_{k=1}^{s} p_k b_k \right\} = \sup_{s \in [1,m] \cap \mathbb{N}} \left\{ M \left( \frac{\int_{a}^{b} p(x) f(x) dx}{\int_{a}^{l_{s}} p(x) dx} \right) \int_{a}^{l_{s}} p(x) g(x) dx \right\} \leq \sup_{s \in [a,b]} \left\{ M \left( \frac{\int_{a}^{b} p(x) f(x) dx}{\int_{a}^{l_{s}} p(x) dx} \right) \int_{a}^{l_{s}} p(x) g(x) dx \right\}.
\]

Let us prove the validity of inequality (5) in general case. First, note that if the functions \( M \) and \( f \) satisfy the conditions of Theorem 1, then there exists the number \( n_0 = n_0(M, f) \in \mathbb{N} \) such that for any \( n > n_0 \) and for all \( x \in [a; b] \), the inequality \( |M(f(x))| < n \) holds.

For any \( n > n_0 \), consider the system of points \( l_0^{(n)} < l_1^{(n)} < \ldots < l_m^{(n)} = b \), defined in the following way: we put \( l_0^{(n)} := a \) and for any \( k \in [1; m] \cap \mathbb{N} \) the value \( l_k^{(n)} \) is a greatest positive number such that

\[ l_k^{(n)} > l_{k-1}^{(n)} \quad \text{and for all} \quad x \in [l_{k-1}^{(n)}; l_k^{(n)}), \quad \text{the following relation} \quad \text{is true:} \]

\[ |M(f(l_{k-1}^{(n)})) - M(f(x))| \leq \frac{1}{n}. \]
By virtue of the conditions on the function $M$ and $f$, this system of points always exist and $m \leq 2n^2$.

Further, consider the functions $f_n = f_n(x)$ such that

$$f_n(x) \equiv \lim_{t \to t_k^{(n)}} f(t), \quad \text{for all } x \in [l_{k-1}^{(n)}, l_k^{(n)}], \ k = 1, 2, \ldots, m. \quad (26)$$

We see that the inequality $|M(f(x)) - M(f_n(x))| \leq \frac{1}{n}$ holds for all $n > n_0$ and $x \in [a, b]$. Due to integrability on $[a, b]$ of the product $p(x)g(x)$, the values

$$\int_a^b p(x)g(x)[M(f(x)) - M(f_n(x))] \, dx$$

converge to zero as $n \to \infty$. Furthermore, for any $n > n_0$, the function $f_n(x)$ is nonincreasing and it takes finitely many values on $[a, b]$. Hence, this function satisfies the conditions of the proposition proved above.

Thus, in view of (26) and continuity of the function $M$, we conclude that for any $\varepsilon > 0$ and for all sufficiently great $n$ ($n > n_1(\varepsilon)$),

$$\int_a^b p(x)g(x)M(f(x)) \, dx = \int_a^b p(x)g(x)M(f_n(x)) \, dx + \int_a^b p(x)g(x)(M(f(x)) - M(f_n(x))) \, dx \leq \sup_{s \in [a, b]} \left\{ M\left( \frac{\int_a^s p(x)f_n(x) \, dx}{\int_a^s p(x) \, dx} \right) \right\} + \frac{\varepsilon}{2} \leq \sup_{s \in [a, b]} \left\{ M\left( \frac{\int_a^s p(x)f(x) \, dx}{\int_a^s p(x) \, dx} \right) \right\} + \varepsilon.$$

Hence, relation (5) is true.

Analyzing the proof of Theorem 1, we see that the similar statement is also true in the case, where $b = \infty$.

**THEOREM 1’.** Assume that $g : [a, b] \to \mathbb{R}_0^+$ and $p : [a, b] \to \mathbb{R}_0^+$ (where $b \in (a, \infty]$) are integrable functions such that the product $p \cdot g$ is also integrable on $[a, b]$ function. Let also $f : [a, b] \to \mathbb{R}_0^+$ be a nonincreasing function. Then for any convex (or concave) function $M : [0, \infty) \to \mathbb{R}$ such that $M(0) = 0$, inequality (5) (or inequality (3)) is true.

Analogically, one can obtain the statement, similar to Lemma 1, in the case, where $n = \infty$.

**LEMMA 1’.** Let $a = \{a_k\}_{k=1}^\infty$, $b = \{b_k\}_{k=1}^\infty$ and $p = \{p_k\}_{k=1}^\infty$ be nonnegative number sequences such that $a_1 \geq a_2 \geq \ldots$, $p_k > 0$ and the series $\sum_{k=1}^\infty p_kb_k$ is convergent. Then for any convex function $M : [0, \infty) \to \mathbb{R}$ such that $M(0) = 0$, the following inequality is true:

$$\sum_{k=1}^\infty p_kb_kM(a_k) \leq \sup_{s \in [1, \infty)} \left\{ M\left( \frac{\sum_{k=1}^s p_kb_k}{\sum_{k=1}^s p_k} \right) \right\}, \quad (15)$$

and for any concave function $M : [0, \infty) \to \mathbb{R}$ such that $M(0) = 0$, the following inequality is true:

$$\sum_{k=1}^\infty p_kb_kM(a_k) \geq \inf_{s \in [1, \infty)} \left\{ M\left( \frac{\sum_{k=1}^s p_kb_k}{\sum_{k=1}^s p_k} \right) \right\}. \quad (16)$$
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