Analytical stabilization of modulated optical similaritons in a tapered graded-index waveguide

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Abstract. We note that the self-similar waves recently reported, in a tapered graded-index waveguide, severely suffer from extreme increase in the amplitude, after traversing a finite distance. In this work, we theoretically described that by adding a suitable homogeneous background in the expression for widths of the self-similar solutions which is achievable through the modulation of tapering profile, we could avert this situation. We further find that the self-similar waves propagate in the nonlinear waveguide, quite a distance stably, albeit they undergo self-compression.

1. Introduction

In recent years, significant efforts have been posed in the study of design and properties of the tapered optical waveguides both theoretically as well as experimentally. Tapering is one of the best mechanisms for improving the coupling efficiency between the fibers and waveguides [1]. It produces high nonlinear effects even for low power intensity due to small mode area. One such effect is the enhancement of the two-photon absorption in the nonlinear tapered optical fibers in rubidium vapors [2]. Propagation of soliton and parabolic pulses through the tapered silica core optical fiber has been investigated numerically [3, 4]. It has significant effect on the wave dynamics and intensity of wave profile. Hence, a lot of work is being done on the propagation of the self-similar waves, waves which maintain their shape but the amplitude and width change with the modulating system parameters [5, 6], in the tapered graded index waveguides [7, 8, 9]. Tapering modulates the refractive index of the optical waveguide. For the tapered graded index optical waveguide, the refractive index will be of the form

\[ n(x, z) = n_0 + n_1 F(z) x^2 + n_2 |u|^2, \]

where the first two terms correspond to the linear part of refractive index and the last term is Kerr-type nonlinearity. The function \( F(z) \) describe the geometry of tapered waveguide along the waveguide axis. The quadratic variation of the refractive index in transverse direction is a good approximation to the true refractive index distribution, and is often known as lens-like medium. Such waveguides have applications in image-transmitting processes because they possess a very wide bandwidth. The shape of a taper i.e. \( F(z) \) can be modeled appropriately depending upon the practical requirements.

In recent work, authors have studied the wave propagation through tapered graded index waveguides and reported the existence of self-similar waves [7, 8, 10]. The width of these waves obeys the second order differential equation identical to a wave equation governing the modes of an inhomogeneous
planar waveguide with the refractive index profile given by the function \( F(z) \). They have considered the lowest-order mode of sech\(^2\)-profile waveguides to study the propagation of self-similar waves. A close inspection reveals that for long distance propagation, these waves undergo catastrophic collapse. In this paper, we show that by modulating the tapering and width functions one can suppress the collapse of these self-similar waves. This has been achieved by adding a suitable homogeneous background in the expression of width function which can be experimentally achieved through the modulation of tapering profile. This analysis has been done for self-similar 1- and 2- soliton solutions, and first- and second-order rogue waves. Solitons [11] and rogue waves [12, 13] are the solutions of standard nonlinear Schrödinger equation which appear under different initial conditions. Modifying the width and tapering functions help us to propagate the self-similar solitons quite a long distance stably and also to control the exponential growth of background in self-similar rogue waves. Due to this modification in width function, the high amplitude feature of rogue waves no longer remains prominent.

2. Model equation and self-similar solutions

The propagation of beam through tapered graded-index nonlinear waveguide amplifier is governed by generalized nonlinear Schrödinger equation (GNLSE)

\[
\frac{i}{\partial z} \frac{\partial u}{\partial z} + \frac{1}{2k_0} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} k_0 n_1 F(z) u^2 - \frac{i}{2} \alpha(z) u + k_0 n_2 |u|^2 u = 0,
\]

where \( u(x, z) \) is the complex envelope of the electrical field, \( g \) and \( \alpha \) account for linear gain and loss, respectively, \( k_0 = 2\pi n_0 / \lambda \), \( \lambda \) being the wavelength of the optical source generating the beam, \( n_1 \) is the linear defocusing parameter \((n_1 > 0)\), and \( n_2 \) represents Kerr-type nonlinearity. The dimensionless profile function \( F(z) \) can be negative or positive, depending on whether the graded-index medium acts as a focusing or defocusing linear lens. Introducing the normalized variables \( X = x/w_0 \), \( Z = z/L_D \), \( G = [g(z) - \alpha(z)] L_D \), and \( U = (k_0 n_2 |L_D|)^{1/2} u \), where \( L_D = k_0 w_0^2 \) is the diffraction length associated with the characteristic transverse scale \( w_0 = (k_0 n_1)^{-1/4} \), equation (2) can be rewritten in a dimensionless form [7].

\[
\frac{i}{\partial Z} \frac{\partial U}{\partial Z} + \frac{1}{2} \frac{\partial^2 U}{\partial X^2} + F(Z) \frac{X^2}{2} U - \frac{i}{2} G(Z) U + |U|^2 U = 0.
\]

The similariton solutions for equation (3) can be obtained by transforming it into standard homogeneous NLSE using the gauge and similarity transformation [7, 14]

\[
U(X, Z) = \frac{1}{W(Z)} \Psi[\chi(X, Z), \zeta(Z)] e^{i\Phi(X, Z)},
\]

where \( W(Z) \) and \( X_c \) are the dimensionless width and position of the self-similar wave center. Substituting equation (4) into equation (3), one obtains the NLSE

\[
\frac{i}{\partial \zeta} \frac{\partial \Psi}{\partial \zeta} + \frac{1}{2} \frac{\partial^2 \Psi}{\partial X^2} + |\Psi|^2 \Psi = 0.
\]

Here the similarity variable, guiding-center position, effective propagation distance and phase are given respectively as [14], \( \chi(X, Z) = \frac{x-X_c}{W(Z)} \), \( X_c(Z) = W(Z) \left( \frac{X_0}{W_0} + C_{02} \int_0^Z ds \frac{dW}{W^2} \right) \), \( \zeta(Z) = \zeta_0 + \int_0^Z \frac{dW}{W^2} \), \( \Phi(X, Z) = \frac{x^2}{2W} + c_{02} X - c_{01} \int_0^Z \frac{dW}{W^2} \), where \( C_{02}, X_0 \) and \( W_0 \) are constant of integration. Further, the tapering function, gain and similariton width \( W(Z) \) are related as

\[
d^2 W/dZ^2 - F(Z) W = 0,
\]
and

\[ G(Z) = -d[\ln W(Z)]/dZ. \]  

As stated earlier, transformation given by equation (4) reduces equation (3) to the standard NLSE, so for all the localized solutions of NLSE, 1-, 2- soliton [11] and first- and second-order rational solutions [13], the corresponding optical self-similar solutions of equation (3) can be obtained by means of the reverse transformation variables and functions.

The general expression of the intensity for self-similar 1- and 2-soliton solutions \((I_{1S} \text{ and } I_{2S}, \text{ respectively})\) is given by

\[ I_{1S}(X, Z) = \frac{\eta^2}{W^2} \text{sech}^2[\eta(\chi - v\zeta)], \]  

where \(\eta \text{ and } v\) are the soliton amplitude and velocity.

\[ I_{2S}(X, Z) = \frac{16}{W^2} \frac{|N(\chi, \zeta)|^2}{D(\chi, \zeta)^2}, \]  

where \(N(\chi, \zeta) = \eta_1 \cosh \xi_2 \exp(-i\chi_1)[(k_2 - k_1)^2 + 2i\eta_2(k_2 - k_1) \tanh \xi_2 + \eta_1^2 - \eta_2^2] + \eta_2 \cosh \xi_1 \exp(-i\xi_2)\[(k_2 - k_1)^2 - 2i\eta_1(k_2 - k_1) \tanh \xi_1 + \eta_1^2 + \eta_2^2]\), \(D(\chi, \zeta) = \cosh(\xi_1 + \xi_2)\[(k_2 - k_1)^2 + (\eta_2 - \eta_1)^2] + \cosh(\xi_1 - \xi_2)\[(k_2 - k_1)^2 + (\eta_2 + \eta_1)^2\] - 4\eta_1\eta_2 \cos(\chi_2 - \chi_1).\) The arguments and the phases are given as \(\xi_i(\chi, \zeta) = 2\eta_i \chi + 4\eta_i \kappa_i \zeta\) and \(\chi_i(\chi, \zeta) = 2\kappa_i \chi + 2(\kappa_i^2 - \eta_i^2) \zeta\), respectively, where, \(\eta_i \text{ and } \kappa_i\) defines the amplitude and the velocity of the \(i\)th soliton \((i = 1, 2)\) respectively.

The intensity expression for first- and second-order rational solutions \((I_{R1} \text{ and } I_{R2}, \text{ respectively})\) is given by [8]

\[ I_{R1}(X, Z) = \frac{1}{W^2} \left[ 1 + 8 \frac{1 + 4\chi^2 - 4\zeta^2}{(1 + 4\chi^2 + 4\zeta^2)^2} \right], \]  

\[ I_{R2}(X, Z) = \frac{1}{W^2} \left[ \left( \frac{D - G}{D} \right)^2 + \left( \frac{H}{D} \right)^2 \right], \]  

where \(D = \frac{1}{4} \chi^2 + \zeta^2)^3 + \frac{1}{4} \chi^2 - 3\zeta^2)^2 + \frac{3}{41} \left( 12\chi^2 + 44\zeta^2 + 1 \right), \quad G = \chi^2 + \zeta^2 + \frac{3}{4} \left( \chi^2 + 5\zeta^2 + \frac{3}{4} \right) - \frac{3}{4}, \quad H = \zeta^2 - 3\chi^2 + 2(\chi^2 + \zeta^2)^2 - \frac{13}{7}.\)

From equation (6), the width of similaritons obeys the second order differential equation identical to a wave equation governing the modes of an inhomogeneous planar waveguide with the refractive index profile given by the function \(F(z)\). In earlier works, authors have considered the sech-type [7, 8, 14] and sech tanh-type [10] width profiles. The former in this case vanishes at infinity and the latter has a node at \(Z = 0\) which following equation (8) results in blow up of intensity. Here we show that adding a homogeneous background in the width profile, which in turn modifies the tapering profile of the waveguide, can help to avert this problem. We discuss these two cases elaborately below.

**Case (a):** By adding a constant background \((\alpha)\) into the width function, the modified width and tapering functions satisfying equation (6) can be written as

\[ W(Z) = \alpha + \text{sech}(\beta Z) \]  

and

\[ F(Z) = \frac{\beta^2 \text{sech}(\beta Z) \left( 1 - 2 \text{sech}^2(\beta Z) \right)}{\alpha + \text{sech}(\beta Z)}. \]  

The gain can be found from equation (7) as

\[ G(Z) = \frac{\beta \text{sech}(\beta Z) \tanh(\beta Z)}{\alpha + \text{sech}(\beta Z)}. \]
For $\alpha = 0$ and $\beta = 1$, these functional forms reduce to the one considered in Refs. [7, 8, 14]. The variation of the magnitude of the width, tapering and gain functions along the propagation direction, for different choices of $\alpha$, is shown in figure 1.

![Figure 1](image)

**Figure 1.** Profiles of width, tapering and gain given by equations (12) to (14) for different choices of $\alpha$. Curve A (solid) for $\alpha = 0$; Curve B (dashed) for $\alpha = 1.01$; Curve C (dotted) for $\alpha = 2$. All the curves are plotted for $\beta = 1$.

It can be seen from the plots of width function that for $\alpha = 0$ width profile approaches zero for finite $Z$. But for non-zero values of $\alpha$, width profiles approach to some finite value which will help us to control the collapse of self-similar waves. The addition of homogeneous background into the width profile also has a significant effect on the plots of tapering and gain functions. For non-zero values of $\alpha$, both tapering and gain profiles approaches to zero unlike for the case $\alpha = 0$. It means that we need both tapering and gain medium only in small range of $Z$. It is also clear that tapering is initially negative and crosses zero at some value of $Z$, implying that the linear inhomogeneity of the waveguide is changing from focusing to defocusing type, but the magnitude of tapering is small for non-zero values of $\alpha$ which is desirable as it is easy to fabricate them, for practical applications.

![Figure 2](image)

**Figure 2.** Comparison of intensity profiles of self-similar 1-soliton solution (equation (8)) for width function, $W(Z) = \alpha + \text{sech}(\beta Z)$ for (a) $\alpha = 1.01$ (Present work), (b) $\alpha = 0$ (Collapse). The parameters used in the plots are $\beta = 1$, $\eta = 1$, $\zeta = 0$, $\nu = 0.3$ and $\zeta = 0$.

In the following, we discuss the effect of modified width function on the intensity of self-similar waves for tapering given by equation (13). The intensity profile of self-similar 1- and 2-soliton solutions is shown in figures 2 and 3, respectively. Figures 2(a) (for $\alpha = 1.01$) and 2(b) ($\alpha = 0$), depicts the evolution of bright similariton, such as figure 2(a) depicts the self-compression of similariton for low values of $z$ and it has constant intensity profile due to effect of modified tapering, where as figure 2(b) depicts the continuous self-compression of bright similariton and undergo catastrophic collapse. The similar comparison has also been made for evolution of self-similar 2-soliton solution, shown in figure 3. Hence, for both 1- and 2-soliton solutions, comparing the profiles for $\alpha = 1.01$ (figures 2(a) and 3(a)) with the profiles for $\alpha = 0$ (figures 2(b) and 3(b)), respectively, we find that the addition of homogeneous background in the width profile prevents the collapse of self-similar waves and allows
them to propagate stably to a large distance. After simplifying the intensity expression it is easy to convince that \( \alpha \in \mathbb{R} - [-1, 1] \), so as to have well behaved solutions. Choice of \( \alpha \) poses inverse effect on intensity, hence its value should be adjusted as per physical requirement.

In figures 4 and 5, we have shown the intensity profiles of self-similar first- and second-order rational solutions, respectively. These solutions are localized in both directions and have amplitude significantly larger as compared to the background. In general, these rational solutions are called rogue waves, but due to modification in width function, the high amplitude feature of rogue waves no longer remains prominent here, and we called them only rational solutions. In earlier work [8], it is found that these solutions are well behaved with \( \alpha = 0 \) for low range of \( z \) but for large propagation distance it shows exponential rise in the background. Here also the modified width function has significant effect on the intensity profiles of rational solutions as it prevents this exponential rise of background, as shown in figures 4(a) and 5(a), for \( \alpha = 1.01 \), unlike for the case with \( \alpha = 0 \) (figures 4(b) and 5(b)). Thus it makes the features of these waves more prominent.

**Case (b):** Corresponding to the choice of \( W(Z) = \text{sech}(\beta Z) \tan h(\beta Z) \) [10], adding a homogeneous background to the width profile, the new width function can be written as

\[
W(Z) = \alpha + \text{sech}(\beta Z)\tan h(\beta Z)
\]

and hence following equation (6), we obtain the following expression for tapering profile:

\[
F(Z) = \frac{\beta^2 \text{sech}(\beta Z)\tan h(\beta Z)(1 - 6\text{sech}^2(\beta Z))}{\alpha + \text{sech}(\beta Z)\tan h(\beta Z)}.
\]

The gain function can be found from equation (7) as

\[
G(Z) = \frac{\beta \text{sech}(\beta Z)(1 - 2\text{sech}^2(\beta Z))}{\alpha + \text{sech}(\beta Z)\tan h(\beta Z)}.
\]

For \( \alpha = 0 \) and \( \beta = 1 \), these functional forms are considered in Ref. [10]. The variation of magnitude of the width, tapering and gain functions along the propagation direction, for different choices of \( \alpha \), is shown in figure 6. Here also addition of background in width profile has same effect on all the three functions as in the previous case. For non-zero values of \( \alpha \), width profiles approach to some finite value which is useful for the evolution of self-similar waves at large distances. The tapering and gain profiles approaches to zero at large propagation distances. It means waveguide should be tapered and doped with some gain medium only for finite range of \( z \). Here, for non-zero values of \( \alpha \), tapering is always negative which implies that the linear inhomogeneity of the waveguide is only focusing. From figure 6 it is clear that non-zero values of \( \alpha \) can be experimentally achieved by implementing the tapering and gain only in a small region of waveguide.
Figure 4. Comparison of intensity profiles of self-similar first-order rational solution (equation (10)) for width function, $W(Z) = \alpha + \text{sech}(\beta Z)$ for (a) $\alpha = 1.01$ (Present work), (b) $\alpha = 0$ (Collapse). The parameters used in the plots are $\beta = 1, C_{02} = 0.3, X_0 = 0$ and $\zeta_0 = 0$.

Figure 5. Comparison of intensity profiles of self-similar second-order rational solution (equation (11)) for width function, $W(Z) = \alpha + \text{sech}(\beta Z)$ for (a) $\alpha = 1.01$ (Present work), (b) $\alpha = 0$ (Collapse). The parameters used in the plots are $\beta = 1, C_{02} = 0.3, X_0 = 0$ and $\zeta_0 = 0$.

Figure 6. Profiles of width, tapering and gain given by equations (15) to (17) for different choices of $\alpha$. Curve A (solid) for $\alpha = 0$; Curve B (dashed) for $\alpha = 1.01$; Curve C (dotted) for $\alpha = 2$. All the curves are plotted for $\beta = 1$.

As in the previous case, the intensity of self-similar waves can be found for tapering given by equation (16). In recent work [10], for $\alpha = 0$ and $\beta = 1$, a close inspection reveals that the intensity of bright similariton blow up at $z = 0$ and experiences a catastrophic collapse for large propagation distance. Due to the appearance of width function in the intensity expression of self-similar waves explicitly and also implicitly through $\chi$ and $\zeta$, $\alpha$ plays an important role in the dominant appearance of wave over the background. By doing an exhaustive study of its available range, we find that wave appears more prominently for large values of $z$. Here we are specifically plotting the intensity profiles of self-similar
1- and 2-soliton, and first- and second-order rational solutions (figures 7 and 8, respectively), for $\alpha = 2$. Here too we observe that addition of a suitable homogeneous background in the width profile prevents the blow up of intensity and allows the wave to propagate to a large distance stably.

Figure 7. Intensity profiles of self-similar (a) 1-soliton and (b) 2-soliton solution for width function, $W(Z) = \alpha + \text{sech}(\beta Z)\tanh(\beta Z)$, for $\alpha = 2$, $\beta = 1$. The other parameters used are same as earlier.

Figure 8. Intensity profiles of self-similar (a) first-order and (b) second-order rational solution for width function, $W(Z) = \alpha + \text{sech}(\beta Z)\tanh(\beta Z)$, for $\alpha = 2$, $\beta = 1$. The other parameters used are same as earlier.

3. Conclusion
We have noticed that the addition of a suitable homogeneous background in the expression for widths of the self-similar waves in a graded-index waveguide, through the modulation of tapering profile, enhances the stabilization of intensity profiles of the similaritons. Furthermore, we find that these self-similar waves propagate in the nonlinear waveguide, quite a distance stably, albeit they undergo self-compression. We hope that these similaritons can be launched in long-haul telecommunication networks, involving tapered graded-index waveguides.

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