A PDE for the Multi-Time Joint Probability of the Airy Process

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Abstract

This paper gives a PDE for multi-time joint probability of the Airy process, which generalizes Adler and van Moerbeke’s result on the 2-time case. As an intermediate step, the PDE for the multi-time joint probability of the Dyson Brownian motion is also given.

1 Introduction

The Airy process can be defined purely stochastically as the limit of the Dyson Brownian motion, as we are going to do later. However, it also appears in various statistical physical models, such as the polynuclear growth process [12], [7] and the Domino tiling model [8]. Since the Airy process is stationary with continuous sample path [12], we can pick any time $t$ and consider the probability of all particles being in $(-\infty, u)$, denoted by $P(u)$, and find that the probability is given by the GUE Tracy-Widom distribution [13]

$$P(u) = e^{-\int_{-\infty}^{u} (s-u)q^2(s)ds},$$  (1)

with $q(s)$ the solution of the Painlevé II equation

$$q''(s) = sq(s) + 2q^2(s), \quad q(s) \simeq \begin{cases} \frac{e^{-(2/3)s^{3/2}}}{\sqrt{2\sqrt{s}+1}} & \text{for } s \to \infty, \\ \sqrt{-s/2} & \text{for } s \to -\infty. \end{cases}$$  (2)

In their study of the joint probability for several times of the Airy process, Prähofer and Spohn [12] posed the problem to find a PDE for the joint probability. Adler and van Moerbeke [2] solved the problem for the 2-time case, and assuming a plausible conjecture of the boundary condition, got the asymptotic expansion of the probability function $P(t, u, v)$, which is the probability that all particles are in $(-\infty, u)$ initially and in $(-\infty, v)$ after a large time $t$. Their solution was obtained by a previous result of them on the spectrum of coupled random matrices [11]. They regarded the joint distribution for the Dyson Brownian motion of 2-time as a $\tau$ function of the two-Toda lattice, and construct a PDE with variables in times and boundary points of the Dyson Brownian motion as a consequence of identities for $\tau$ functions and Virasoro identities specific to the situation. Then they got the PDE for the Airy process by taking the limit.

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This paper generalizes their result to the multi-time case, and the technical heart is the same identity for $\tau$ functions, although in the generalized case we need more elaborate work to fit differential operators in times and boundary points of the Dyson Brownian motion into the structure of two-Toda $\tau$ functions.

After the description of the problem, we state the PDEs for both the Dyson Brownian motion with finite particles and its limit, the Airy process with infinitely many particles, and an example for the 3-time ($m = 2$) case for the Airy process. Section 2 derives the result for the Dyson process and section 3 derives the result for the Airy process by taking limit.

1.1 Description of the model

The free Brownian motion process is determined by the transition probability distribution

$$P(t, \tilde{X}, X) = \frac{1}{\sqrt{(2\pi t)/\beta}} e^{-\frac{(X-\tilde{X})^2}{2\beta t}},$$

where $\tilde{X}$ and $X$ are initial and terminal coordinates of the particle, and $\beta$ is the diffusion constant. The probability distribution $P(t, \tilde{X}, X)$, as a function of $t$ and $X$, satisfied the diffusion equation

$$\frac{\partial P}{\partial t} = \frac{1}{2\beta} \frac{\partial^2}{\partial X^2} P.$$

If we add a harmonic potential $\rho X^2/2$ to the process, then the probability distribution $P(t, \tilde{X}, X)$ satisfies (see e.g. [5])

$$\frac{\partial P}{\partial t} = \left(\frac{1}{2\beta} \frac{\partial^2}{\partial X^2} - \frac{\partial}{\partial X} (-\rho X)\right) P,$$

and the process is determined by ($c = e^{-\rho t}$)

$$P(t, \tilde{X}, X) = \frac{1}{\sqrt{\pi (1-c^2)/\rho \beta}} e^{-\frac{(X-\tilde{X})^2}{\rho (1-c^2)/\beta}}.$$

While the free Brownian motion process is dispersive, the Brownian motion process in the harmonic potential well has a stationary distribution

$$P(X) = \frac{e^{-\rho \beta X^2}}{\sqrt{\pi/\rho \beta}}.$$

Now we can define the Ornstein-Uhlenbeck process [11] of an $n \times n$ Hermitian matrix $B$, in which all the $n^2$ real variables—$n$ for real diagonal entries, $n(n-1)/2$ for the real parts of off diagonal entries, and the other $n(n-1)/2$ for the imaginary parts of them—are in independent Brownian motion in harmonic potential wells. The $\rho$ for them are uniformly 1, and $\beta$ are 1 for the $n$ diagonal variables and 2 for the $n(n-1)$ off diagonal variables. Therefore for $i, j$ in $\{1, \ldots, n\}$, ($c = e^{-t}$)

$$
\begin{align*}
&P_{ii}(t, \tilde{B}_{ii}, B_{ii}) = \frac{1}{\sqrt{\pi(1-c^2)}} e^{-\frac{(B_{ii} - \tilde{B}_{ii})^2}{(1-c^2)^2}}, \\
&P_{ij}(t, \Re B_{ij}, \Re B_{ij}) = \frac{1}{\sqrt{\pi(1-c^2)/2}} e^{-\frac{(\Re B_{ij} - \Re B_{ij})^2}{(1-c^2)^2}/2}, \\
&P_{ij}(t, \Im B_{ij}, \Im B_{ij}) = \frac{1}{\sqrt{\pi(1-c^2)/2}} e^{-\frac{(\Im B_{ij} - \Im B_{ij})^2}{(1-c^2)^2}/2},
\end{align*}
$$

(8)
and we can write the joint transition probability distribution as

\[ P(t, \bar{B}, B) = \prod_{i=1}^{n} P_{ii} \prod_{1 \leq i < j \leq n} (P_{ij} \neq P_{ji}) = \frac{C^{-1}}{(1-c^2)^{n^2/2}} e^{-\frac{\text{Tr}((\bar{B}-cB)^2)}{1-c^2}}. \]  

(9)

We consider the multi-time transition function with the initial state \( B_0 \) at \( t_0 = 0 \), the terminal state \( B_m \) and a series of intermediate states \( B_1, \ldots, B_{m-1} \), and the time between state \( B_0 \) and \( B_i \) being \( t_i \), if we denote

\[ s_i = \begin{cases} 
0 & i = 0, \\
t_1 & i = 1, \\
t_i - t_{i-1} & i = 2, \ldots, m,
\end{cases} \]

(10)

and

\[ c_i = e^{-s_i}, \]

(11)

then

\[ P(t_1, \ldots, t_m, B_0, \ldots, B_m) = C^{-1} \prod_{i=1}^{m} e^{-\frac{\text{Tr}((B_i - c_i B_{i-1})^2)}{1-c_i^2}}. \]

(12)

The Ornstein-Uhlenbeck process has a stationary distribution

\[ P(B) = C^{-1} e^{-\text{Tr}B^2}. \]

(13)

Since the Ornstein-Uhlenbeck process is invariant under the unitary transformation, we define the process of the eigenvalues as the Dyson Brownian motion process \( \text{[4]} \), whose multi-time transition probability distribution is \((0 = t_0 < t_1 < \ldots < t_m)\)

\[ P(t_1, \ldots, t_m, \lambda^{(0)}, \ldots, \lambda^{(m)}) = \frac{\text{The transition probability of the } n \times n \text{ Hermitian matrix with eigenvalues initially } \lambda^{(0)} = (\lambda^{(0)}_1, \ldots, \lambda^{(0)}_n) \text{ and } \lambda^{(1)} \text{ after time } t_1, \lambda^{(2)} \text{ after time } t_2 \ldots \text{ and } \lambda^{(m)} \text{ after the total time } t_m. \]

If we change the coordinates of the \( \mathbb{R}^{n^2} \) space of \( n \times n \) Hermitian matrices in to the eigenvalue-angle coordinates \( \lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_{n(n-1)} \), with the Jacobian identity (see e.g. \( \text{[10]} \))

\[ \prod_{i=1}^{n} dx_{ii} \prod_{1 \leq i < j \leq n} (d\Re(x_{ij}) d\Im(x_{ij})) = V(\lambda)^2 \prod_{i=1}^{n} d\lambda_i \prod_{i=1}^{n(n-1)} d\theta_i, \]

(15)

where \( V(\lambda) = \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j) \) is the Vandermonde, we find the explicit formula for \( P(t_1, \ldots, t_m, \lambda^{(0)}, \ldots, \lambda^{(m)}) \):

\[ P(t_1, \ldots, t_m, \lambda^{(0)}, \ldots, \lambda^{(m)}) = \frac{1}{C} \int \ldots \int \prod_{i=1}^{m} e^{-\frac{\text{Tr}(\theta(\lambda^{(i)} - \lambda^{(i-1)})) - c_i B_{i-1}^2)}{1-c_i^2}} \prod_{i=1}^{m} V(\lambda^{(i)})^2 \prod_{i=1}^{m} d\theta^{(i)}, \]

(16)

where \( \theta^{(0)} \) does not appear in the integral since the transition probability is independent of \( \theta^{(0)} \) for the unitary invariant property.

By the Harish-Chandra-Itzykson-Zuber (HCIZ) formula \( \text{[3]} \)

\[ \int_{U(n)} e^{\text{Tr}(XYU^{-1})} dU = C \frac{\det(e^{x_i, y_j})}{V(x)V(y)}, \]

(17)

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where \( X = \text{diag}(x_1, \ldots, x_n) \) and \( Y = \text{diag}(y_1, \ldots, y_n) \) are diagonal matrices, we can evaluate the multi-time transition probability density as

\[
P(t_1, \ldots, t_m, \lambda^{(0)}, \ldots, \lambda^{(m)}) = \frac{1}{C} V(\lambda^{(0)})^{-1} V(\lambda^{(m)}) \prod_{l=1}^{m} \det \left( e^{\frac{2\epsilon}{1-c_l^2} \lambda^{(l)}_{i-1} \lambda^{(l)}_i} \right) e^{-\frac{1}{1-c_l^2} \sum_{i=1}^{n} \lambda^{(l)}_i \sum_{j=1}^{m} \lambda^{(l)}_j}.
\]

If we take the initial state with eigenvalue \( \lambda^{(0)} \) from the stationary distribution \( \mathbb{P}^{(0)} \), which is

\[
\mathbb{P}^{(0)} = \frac{1}{C} V(\lambda^{(0)})^2 e^{-\sum_{i=1}^{n} \lambda^{(0)}_i^2},
\]

we get the multi-time correlation function in the stationary Dyson process

\[
P(t_1, \ldots, t_m, \lambda^{(0)}, \ldots, \lambda^{(m)}) = \mathbb{P}^{(0)} P(t_1, \ldots, t_m, \lambda^{(0)}, \ldots, \lambda^{(m)})
\]

\[
e^{\frac{-1}{1-c_l^2} \sum_{i=1}^{n} \lambda^{(0)}_i \sum_{j=1}^{m} \lambda^{(0)}_j} e^{-\frac{1}{1-c_l^2} \sum_{i=1}^{n} \lambda^{(0)}_i \sum_{j=1}^{m} \lambda^{(0)}_j}.
\]

If we want to find the probability of all \( \lambda^{(l)}_i \)'s being in \( U^{(l)} = (a^{(l)}_1, a^{(l)}_2, \ldots, a^{(l)}_{2r_l-1}, a^{(l)}_{2r_l}) \), with \( -\infty < a^{(l)}_1 < a^{(l)}_2 < \cdots < a^{(l)}_{2r_l} \leq \infty \), for \( l = 0, 1, \ldots, m \) and \( i = 1, \ldots, n \), which is

\[
\mathbb{P}^{\text{Dyson}}(t_1, \ldots, t_m; a^{(0)}_1, \ldots, a^{(m)}_{2r_m}) = \int \cdots \int_{U^{(0)} \times \cdots \times U^{(m)}_n} \mathbb{P}(t_1, \ldots, t_m, \lambda^{(0)}, \ldots, \lambda^{(m)}) \prod_{l=1}^{m} \prod_{k=1}^{n} d\lambda^{(l)}_k,
\]

we can simplify it by the symmetry of \( \lambda^{(l)}_1, \ldots, \lambda^{(l)}_n \), for all \( l = 0, 1, \ldots, m \) and get

\[
\mathbb{P}^{\text{Dyson}}(t_l; a^{(l)}_i) = \frac{1}{C} \int \cdots \int_{U^{(0)} \times \cdots \times U^{(m)}_n} V(\lambda^{(0)}) V(\lambda^{(m)}) e^{-\frac{1}{1-c_l^2} \sum_{i=1}^{n} \lambda^{(0)}_i \sum_{j=1}^{m} \lambda^{(0)}_j} e^{-\frac{1}{1-c_l^2} \sum_{i=1}^{n} \lambda^{(0)}_i \sum_{j=1}^{m} \lambda^{(0)}_j}.
\]

We are going to give a PDE satisfied by \( \log \mathbb{P}^{\text{Dyson}} \) with variables \( t_l \) and \( a^{(l)}_i \).

The Airy process can be defined as the limit of the Dyson process on the edge \( \mathbb{E} \). As \( n \to \infty \), we can prove that the right-most particle in the Dyson process is almost surely around \( \sqrt{2n} \) with the fluctuation scale \( n^{1/6} \) \( \mathbb{E} \). If we take the rescaling

\[
\tilde{t}_l = n^{1/3} t_l,
\]

\[
\tilde{\lambda}^{(l)}_k = \sqrt{2n^{1/6}} (\lambda^{(l)}_k - \sqrt{2n})
\]

\[
\tilde{a}^{(l)}_i = \sqrt{2n^{1/6}} (a^{(l)}_i - \sqrt{2n}),
\]

then for fixed \( \tilde{t}_l \) and \( \tilde{a}^{(l)}_i \), \( \mathbb{P}^{\text{Dyson}} \) converges to a function defined by the Fredholm determinant of a matrix integral operator \( \mathbb{E} \). If we take the rescaling

\[
\lim_{n \to \infty} \mathbb{P}^{\text{Dyson}}(t_l; a^{(l)}_i) = \mathbb{P}^{\text{Airy}}(\tilde{t}_1, \ldots, \tilde{t}_m; \tilde{a}^{(0)}_1, \ldots, \tilde{a}^{(m)}_{2r_m}) = \det \left( I - (\tilde{\lambda}^{(l)}_k K^{(l)}_{ij} \tilde{\lambda}^{(l)}_j)_{1 \leq i, j \leq m} \right),
\]
where $\chi^c_l$ is an indicator function defined as

$$
\chi^c_l(t) = \begin{cases} 
0 & t \in \bigcup_{i=1}^{l} (a_{2i-1}, a_{2i}), \\
1 & \text{otherwise.}
\end{cases}
$$

(27)

and ($A_i$ is the Airy function)

$$
K_{ij}^A(x, y) = \begin{cases} 
\int_0^\infty \text{Ai}(x + z)\text{Ai}(y + z)dz & \text{if } i = j, \\
\int_0^\infty e^{-z(t_i - t_j)}\text{Ai}(x + z)\text{Ai}(y + z)dz & \text{if } i > j, \\
-\int_{-\infty}^0 e^{z(t_j - t_i)}\text{Ai}(x + z)\text{Ai}(y + z)dz & \text{if } i < j.
\end{cases}
$$

(28)

Then we can define the Airy process, which contains infinitely many particles by the probability function (26). Furthermore, we are going to give a PDE satisfied by $\tau$ with variables $\bar{t}_i$ and $\bar{a}_i^{(l)}$.

**Remark 1.** To make the definition (26) meaningful, we need $\bar{a}_i^{(l)}$ to be $-\infty$ for all $l$. Otherwise the left hand side of (26) is 0 and the right hand side is not well defined.

### 1.2 Statement of main results

With notations defined in subsection (1.1), we define differential operators ($l = 0, 1, \ldots, m$)

$$
D^{l,1} = \sum_{i=1}^{2r_l} \frac{\partial}{\partial a_i^{(l)}}, \quad D^{l,2} = \sum_{i=1}^{2r_l} a_i^{(l)} \frac{\partial}{\partial a_i^{(l)}},
$$

(29)

if all $a_i^{(l)}$ are finite; otherwise we drop the $a_i^{(l)}$ (resp. $a_{2r_l}$) part if $a_i^{(l)} = -\infty$ (resp. $a_{2r_l} = \infty$). And then denote

$$
A_1 = \sum_{l=0}^{m} e^{-t_l} D^{l,1},
$$

(30)

$$
B_1 = \sum_{l=0}^{m} e^{t_l - t_m} D^{l,1},
$$

(31)

$$
A_2 = \sum_{l=0}^{m} e^{-2t_l} D^{l,2} + \sum_{l=1}^{m} (1 - e^{-2t_l}) \frac{\partial}{\partial t_l} - e^{-2t_m},
$$

(32)

$$
B_2 = \sum_{l=0}^{m} e^{2(t_l - t_m)} D^{l,2} + \sum_{l=1}^{m} (e^{2(t_l - t_m)} - e^{-2t_m}) \frac{\partial}{\partial t_l} - e^{-2t_m}.
$$

(33)

We now state

**Theorem 1** (Dyson Brownian motion). Given $t_1, \ldots, t_m$, the logarithm of the joint distribution for the stationary Dyson Brownian motion $P_n^{\text{Dyson}}$ defined in (21) (abbreviated as $\log P_n$) satisfies a third order non-linear PDE in times and boundary points of $U^{(l)}$

$$
\begin{align*}
A_1 \frac{B_2 A_1 \log P_n}{B_1 A_1 \log P_n + 2ne^{-t_m}} &= B_1 \frac{A_2 B_1 \log P_n}{A_1 B_1 \log P_n + 2ne^{-t_m}},
\end{align*}
$$

(34)
Similarly with the notations
\[ D = \sum_{l=0}^{m} D_{l,1}, \]  
\[ D_{1L} = \sum_{l=0}^{m} (t_m - t_l)D_{l,1}, \]  
\[ D_{2R} = \sum_{l=0}^{m} t_l D_{l,1}, \]  
\[ D_1 = = D_{1L} - D_{1R} = \sum_{l=0}^{m} (t_m - 2t_l)D_{l,1}, \]  
\[ D_2 = \sum_{l=0}^{m} (t_m - t_l)^2 + t_l^2 \]  
\[ D_3 = \sum_{l=0}^{m} (t_m - t_l)^3 - t_l^3 \]  
\[ E = \sum_{l=0}^{m} D_{2,1}, \]  
\[ E_1 = \sum_{l=0}^{m} (t_m - 2t_l)D_{l,1}, \]  
\[ T_1 = 2 \sum_{l=1}^{m} t_l \frac{\partial}{\partial t_l}, \]  
\[ T_2 = 2 \sum_{l=1}^{m} t_l (t_m - t_l) \frac{\partial}{\partial t_l}, \]

we state the result for the Airy process

**Theorem 2** (Airy process). Given \( t_1, \ldots , t_m \), the logarithm of the joint distribution for the Airy process \( \mathbb{P}^{\text{Airy}} \) defined in (27) (abbreviated as \( \log \mathbb{P} \)) satisfies a third order non-linear PDE in times and boundary points of \( U^{(1)} \)

\[ D^2[E_1 + D_3 + T_2] \log \mathbb{P} - DD_1[E + D_2 + T_1] \log \mathbb{P} - 2D_{1L}D_{1R}D_1 \log \mathbb{P} = \{D^2 \log \mathbb{P}, DD_1 \log \mathbb{P}\}_D. \]

In the case of \( m = 1 \), our result agrees with that in [2]. Especially, if \( U^{(1)} = (-\infty, u), U^{(1)} = (\infty, v) \) and denote \( t_1 = t \), then the result for \( \log \mathbb{P}^{\text{Airy}}(t, u, v) \) is

**Corollary 1** ([2]). The logarithm of the 2-time joint probability for the Airy process \( \mathbb{P}^{\text{Airy}}(t, u, v) \) (abbreviated as \( \log \mathbb{P} \)) satisfies a third order non-linear PDE in variables \( u, v \) and \( t \)

\[ \left( v - u \right) \left[ \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right] \frac{\partial^2}{\partial u \partial v} + t \left[ \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right] \frac{\partial}{\partial t} + t^2 \left[ \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right] \frac{\partial^2}{\partial u \partial v} \right] \log \mathbb{P} = \frac{1}{2} \left[ \left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) \log \mathbb{P} \right] \frac{\partial}{\partial u} + \frac{\partial}{\partial v} + \frac{\partial}{\partial t}. \]

\(^1\) in terms of the Wronskian, defined in subsection 1.3
In the \( m = 2 \) case, if \( U^{(0)} = (\infty, u), U^{(1)} = (\infty, v), U^{(2)} = (\infty, w), t_1 = t \) and \( t_2 = s \) the result for \( \log P^{\text{Airy}}(t, s, u, v, w) \) is

**Corollary 2.** The logarithm of the 3-time joint probability for the Airy process \( P^{\text{Airy}}(t, s, u, v, w) \) (abbreviated as \( \log P \)) satisfies a third order non-linear PDE in variables \( u, v, w, t \) and \( t \)

\[
\left[ (u - v) \frac{\partial^2}{\partial u \partial v} + (s - t)(v - w) \frac{\partial^2}{\partial u \partial w} + \left[ -s \frac{\partial}{\partial u} + (2t - s) \frac{\partial}{\partial v} + s \frac{\partial}{\partial w} \right] \left( \frac{t}{\partial t} + s \frac{\partial}{\partial s} \right) \right. \\
+ t(s - t)D \frac{\partial}{\partial t} \bigg] D \log P + \left[ -t^3 \frac{\partial^3}{\partial u^2 \partial v} - s^3 \frac{\partial^3}{\partial u^2 \partial w} + t^3 \frac{\partial^3}{\partial u \partial v^2} \\
+ (2t - s)(2s - t)(s + t) \frac{\partial^3}{\partial u \partial v \partial w} + s^3 \frac{\partial^3}{\partial u \partial w^2} - (s - t)^3 \frac{\partial^3}{\partial v^2 \partial w} + (s - t)^3 \frac{\partial^3}{\partial v \partial w^2} \bigg] \log P = \\
\frac{1}{2} \left\{ -s \frac{\partial}{\partial u} + (2t - s) \frac{\partial}{\partial v} + s \frac{\partial}{\partial w} \bigg) D \log P, D^2 \log P \right\} . \tag{47}
\]

### 1.3 Notational convenience

Throughout this paper, parentheses \((\ldots)\) always include numbers and functions; brackets \([\ldots]\) always include operators; braces \(\{\ldots\}\) are always for Wronskians: \( \{f, g\}_D = gDf - fDg \), where \( D \) is a differential operator.

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### 2 The joint probability in the Dyson Brownian motion

To get the PDE, we need to consider a generalized integral such that indices \( i \) and \( j \) can be any positive integers

\[
\tau_n(t, i, c_i, c_i, a_i) = \frac{1}{C} \int \ldots \int_{U^{(0)} \times \ldots \times U^{(m)}} V(\lambda^{(0)})V(\lambda^{(m)}) \\
\prod_{l=0}^m e^{\sum_{i=1}^{\infty} t^{(i)} l^{(i)} c^{(i)} l^{(i)}} \prod_{l=1}^m e^{\sum_{i,j=1}^{\infty} c^{(i)} c^{(j)} l^{(i-j)} l^{(i-j)}} \prod_{l=0}^m \prod_{k=1}^n d\lambda_k^{(l)} , \tag{48}
\]

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with $C$ a normalization constant such that $\mathbb{P}_n^{\text{Dyson}} = \tau_n|_{\mathcal{L}}$, where the locus $\mathcal{L}$ is defined as $(l = 1, 2, \ldots, m - 1,
k = 1, 2, \ldots, m, c_k = e^{-s_k})$

$$\mathcal{L} = \begin{cases} 
  t_2^{(0)} = -\frac{1}{1 - c_1^2}, \\
t_2^{(l)} = -\left(\frac{1}{1 - c_l^2} + \frac{c_{l+1}^2}{1 - c_{l+1}^2}\right), \\
t_2^{(m)} = -\frac{1}{1 - c_m^2}, \\
c_{1,1}^{(k)} = \frac{2c_k}{1 - c_k^2}, \\
\end{cases} \quad (49)$$

**Remark 2.** All $t_i^{(l)}$ and $c_{i,j}^{(k)}$ are variables of $\mathbb{P}_n^{\text{Dyson}}$ in latter part of the paper, though most of them assume the value 0. Therefore it is legitimate to consider $\frac{\partial}{\partial t_i^{(l)}}\mathbb{P}_n^{\text{Dyson}}$ etc.

**Remark 3.** Since we allow $s_1^{(l)}$ to be $-\infty$ and $a_2^{(l)}$ to be $+\infty$, the integral in (48) may be divergent for general values of $t_i^{(l)}$ and $c_{i,j}^{(k)}$. However, if we assume $t_i^{(l)} = 0$ for $i > 2$, $c_{i,j}^{(k)} = 0$ for $\max(i, j) > 1$, and values of $t_2^{(l)}$, $t_1^{(l)}$, and $c_{1,1}^{(k)}$ are near to the locus $\mathcal{L}$, then the integral is convergent, and all algebraic operations in latter part of the paper can be taken in this restricted setting, so they are legitimate.

Now we consider actions of $D^{l,1}$ on $\tau_n$. Since $D^{l,1}$ acts on the integral domains of $\lambda_1^{(l)}, \ldots, \lambda_n^{(l)}$, by the formula

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right) \int_a^b f(x)dx = f(b) - f(a) = \int_a^b f'(x)dx, \quad (50)$$
we get

\[ D^{0,1} \tau_n = \frac{1}{C} \sum_{i=1}^{2n} \left[ \frac{\partial}{\partial a_i^{(0)}} \right] \prod_{l=0}^{m} e^{\sum_{i=1}^{n} \lambda_i^{(l)} \sum_{j=1}^{n} \lambda_j^{(l)}} \int \cdots \int_{U^{(0)} \times U^{(m)}} V(\lambda^{(0)}) V(\lambda^{(m)}) \]
Now we define an $m+1 \times m+1$ matrix
\[
J = \begin{pmatrix}
2t_2^{(0)} & c_{1,1}^{(1)} & c_{1,1}^{(2)} & \cdots & c_{1,1}^{(m)} \\
2t_2^{(1)} & c_{1,1}^{(1)} & & & \\
2t_2^{(2)} & c_{1,1}^{(2)} & & & \\
\vdots & & & & \vdots \\
c_{1,1}^{(m)} & & & & 2t_2^{(m)}
\end{pmatrix},
\]
whose rows and columns are indexed from 0 to $m$. On the locus
\[
J|_{L} = \begin{pmatrix}
-\frac{2}{1-c_1} & \frac{2c_1}{1-c_1} & \frac{2c_2}{1-c_2} & \cdots & \frac{2c_m}{1-c_m} \\
\frac{2c_1}{1-c_1} & -\frac{2}{1-c_1} & \frac{2c_2}{1-c_2} & \cdots & \frac{2c_m}{1-c_m} \\
\frac{2c_2}{1-c_2} & \frac{2c_1}{1-c_1} & -\frac{2}{1-c_1} & \cdots & \frac{2c_m}{1-c_m} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\frac{2c_m}{1-c_m} & \cdots & \frac{2c_2}{1-c_2} & \frac{2c_1}{1-c_1} & -\frac{2}{1-c_1}
\end{pmatrix},
\]
We can find the entries of the first and the last row of $J$ on the locus explicitly,
\[
J_{0,t}|_{L} = -\frac{1}{2} \prod_{i=1}^{l} c_{i} = -\frac{1}{2} e^{-t_{i}},
\]
\[
J_{m,t}|_{L} = -\frac{1}{2} \prod_{i=1}^{m-i} c_{m-i+1} = -\frac{1}{2} e^{t_{i}-t_{m}},
\]
and especially
\[
J_{0,m}|_{L} = J_{m,0}|_{L} = -\frac{1}{2} \prod_{i=1}^{m} c_{i} = -\frac{1}{2} e^{-t_{m}}.
\]
Then let
\[
\begin{pmatrix}
E^{0,1} \\
E^{1,1} \\
\vdots \\
E^{m,1}
\end{pmatrix} = J \begin{pmatrix}
D^{0,1} \\
D^{1,1} \\
\vdots \\
D^{m,1}
\end{pmatrix},
\]
we have

**Lemma 1.**
\[
E^{0,1} E^{m,1} \log P_{n} = E^{m,1} E^{0,1} \log P_{n} = \frac{\partial^2 \log P_{n}}{\partial t_1^{(0)} \partial t_1^{(m)}} - \frac{n}{2} e^{-t_{m}}.
\]

**Proof.** First, since $E^{0,1}$ and $E^{m,1}$ are linear combinations of $D^{l,1}$’s, they are differential operators of order 1, and we have
\[
E^{0,1} E^{m,1} \log P_{n} = -\frac{E^{0,1} P_{n}}{P_{n}^2} + \frac{E^{0,1} E^{m,1} P_{n}}{P_{n}^2}.
\]
By (54) – (56), (59) and (60), we get
\[ E^{0,1}P_n = \left( \sum_{i=0}^{m} J_{0,i} D^{i,1} \right) P_n = \frac{\partial P_n}{\partial t_1^{(0)}}. \] (65)
and
\[ E^{m,1}P_n = \left( \sum_{i=0}^{m} J_{m,i} D^{i,1} \right) P_n = \frac{\partial P_n}{\partial t_1^{(m)}}. \] (66)
Therefore
\[ E^{0,1} E^{m,1} \log P_n = -\frac{\partial P_n}{\partial t_1^{(0)}} \frac{\partial P_n}{\partial t_1^{(m)}} + E^{0,1} \frac{\partial P_n}{\partial t_1^{(m)}} P_n. \] (67)
Here we need to be careful about the term \( E^{0,1} \frac{\partial P_n}{\partial t_1^{(m)}} P_n \). By (51) – (53), the action of \( E^{m,1} \) on \( \tau_n \) is equivalent to that of a differential operator which does not contain \( a_i^{(l)} \) explicitly. On the locus \( \mathcal{L} \), all terms of the differential operator except for \( \frac{\partial}{\partial t_1^{(m)}} \) vanish, so we can ignore them and replace \( E^{m,1} \) by \( \frac{\partial}{\partial t_1^{(m)}} \) between \( E^{0,1} \) and \( P_n \).
Since \( E^{0,1} \) and \( \frac{\partial}{\partial t_1^{(m)}} \) commute,
\[ E^{0,1} \frac{\partial}{\partial t_1^{(m)}} P_n = \frac{\partial}{\partial t_1^{(m)}} E^{0,1} P_n, \] (68)
by (61) – (63), (59) and (62), we have the identity for the action of \( E^{0,1} \) on \( \tau_n \)
\[ E^{0,1} \tau_n = J_{0,0} \left[ n t_1^{(0)} + \sum_{i=2}^{m} \tau_i^{(0)} \frac{\partial}{\partial t_1^{(0)}} + \sum_{i=1}^{m} c_{1,i}^{(1)} \frac{\partial}{\partial t_1^{(1)}} + \sum_{i=2}^{m} \sum_{j=1}^{m} i c_{i,j}^{(1)} \frac{\partial}{\partial c_{i,j}^{(1)}} \right] \tau_n \]
\[ + \sum_{i=1}^{m-1} J_{0,i} \left[ n t_1^{(i)} + \sum_{i=2}^{m} \tau_i^{(i)} \frac{\partial}{\partial t_1^{(i)}} + \sum_{i=1}^{m} c_{1,i}^{(i)} \frac{\partial}{\partial t_1^{(i-1)}} + \sum_{j=2}^{m} \sum_{i=1}^{m} j c_{i,j}^{(i)} \frac{\partial}{\partial c_{i,j}^{(i-1)}} \right] \tau_n \]
\[ + \sum_{i=1}^{m-1} c_{1,i}^{(i+1)} \frac{\partial}{\partial t_1^{(i+1)}} + \sum_{i=2}^{m} \sum_{j=1}^{m} i c_{i,j}^{(i+1)} \frac{\partial}{\partial c_{i,j}^{(i+1)}} \right] \tau_n \]
\[ + J_{0,m} e^{-tm} \left[ n t_1^{(m)} + \sum_{i=2}^{m} \tau_i^{(m)} \frac{\partial}{\partial t_1^{(m)}} + \sum_{i=1}^{m} c_{1,i}^{(m)} \frac{\partial}{\partial t_1^{(m-1)}} + \sum_{j=2}^{m} \sum_{i=1}^{m} j c_{i,j}^{(m)} \frac{\partial}{\partial c_{i,j}^{(m-1)}} \right] \tau_n \]
\[ = \left( \frac{\partial}{\partial t_1^{(m)}} + n J_{0,m} t_1^{(m)} + \ldots \right) \tau_n, \]
with coefficients of all terms except for \( \frac{\partial}{\partial t_1^{(m)}} + n J_{0,m} t_1^{(m)} \) of the right-hand side operator vanishing on the locus \( \mathcal{L} \) and not containing \( t_1^{(m)} \) explicitly. So on the locus
\[ \frac{\partial}{\partial t_1^{(m)}} E^{0,1} P_n = \frac{\partial}{\partial t_1^{(m)}} \left( \frac{\partial P_n}{\partial t_1^{(0)}} + n J_{0,m} t_1^{(m)} + \ldots \right) P_n = \frac{\partial^2 P_n}{\partial t_1^{(0)} \partial t_1^{(m)}} + n J_{0,m} \mathcal{L} P_n, \] (70)
and
\[ E^{0,1} E^{m,1} \log P_n = -\frac{\partial P_n}{\partial t_1^{(0)}} \frac{\partial P_n}{\partial t_1^{(m)}} \frac{\partial^2 P_n}{\partial t_1^{(0)} \partial t_1^{(m)}} + n J_{0,m} \mathcal{L} P_n = \frac{\partial^2 \log P_n}{\partial t_1^{(0)} \partial t_1^{(m)}} - \frac{n}{2} e^{-tm}. \] (71)
Similarly to \((51) - (53)\), with the help of the formula
\[
\left( a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} \right) \int_a^b f(x)dx = bf(b) - af(a) = \int_a^b (xf(x))'dx,
\]
we get \(\left[ \sum_{k=1}^n \frac{2}{\partial \lambda_k^{(i)}} \lambda_k^{(0)} \right]\) is regarded as an operator

\[
D^{0,2} \tau_n = \frac{1}{C} \sum_{i=1}^{2^{2n}} \left[ \frac{\partial}{\partial t_i^{(0)}} \right] \int \cdots \int_{U^{(m)} \times \cdots \times U^{(m)}} V(\lambda^{(0)})V(\lambda^{(m)}) \prod_{i=1}^{2^{2n}} \lambda_k^{(i)} \prod_{i=0}^{2n-1} \tau_n
\]

\[
= \frac{1}{C} \int \cdots \int_{U^{(m)} \times \cdots \times U^{(m)}} \left[ \sum_{i=1}^{\infty} i t_i^{(0)} \lambda_k^{(i)} \right] \int \cdots \int_{U^{(m)} \times \cdots \times U^{(m)}} V(\lambda^{(0)})V(\lambda^{(m)}) \prod_{i=1}^{2^{2n}} \lambda_k^{(i)} \prod_{i=0}^{2n-1} \tau_n
\]

\[
D^{n,2} \tau_n = \left[ \sum_{i=1}^{\infty} \frac{\partial}{\partial t_i^{(m)}} \lambda_k^{(m)} \right] + \sum_{i=1}^{\infty} j c_{i,j}^{(m)} \frac{\partial}{\partial c_{i,j}^{(m)}} \tau_n
\]

and for \(l = 1, \ldots, m - 1\)

\[
D^{l,2} \tau_n = \left[ \sum_{i=1}^{\infty} \frac{\partial}{\partial t_i^{(l)}} + \sum_{i=1}^{\infty} j c_{i,j}^{(l)} \frac{\partial}{\partial c_{i,j}^{(l)}} + \sum_{i=1}^{\infty} j c_{i,j}^{(l+1)} \frac{\partial}{\partial c_{i,j}^{(l+1)}} \right] \tau_n.
\]
On the locus $L$ we get $(l = 1, \ldots, m - 1)$

\[
D^{0.2} \mathbb{P}_n = \left[ -\frac{2}{1 - c^2} \frac{\partial}{\partial t^{(0)}} + \frac{2c_1}{1 - c^2} \frac{\partial}{\partial c_{1,1}^{(l)}} + \frac{n(n + 1)}{2} \right] \mathbb{P}_n,
\]

\[
D^{1.2} \mathbb{P}_n = \left[ -\left( \frac{2}{1 - c^2} + \frac{c_{l+1}^2}{1 - c_{l+1}^2} \right) \frac{\partial}{\partial t^{(l)}} + \frac{2c_l}{1 - c^2} \frac{\partial}{\partial c_{1,1}^{(l)}} + \frac{2c_{l+1}}{1 - c_{l+1}^2} \frac{\partial}{\partial c_{1,1}^{(l+1)}} + n \right] \mathbb{P}_n,
\]

\[
D^{m.2} \mathbb{P}_n = \left[ -\frac{2}{1 - c_m^2} \frac{\partial}{\partial t^{(m)}} + \frac{2c_m}{1 - c_m^2} \frac{\partial}{\partial c_{1,1}^{(m)}} + \frac{n(n + 1)}{2} \right] \mathbb{P}_n.
\]

If we define $(l = 1, 2, \ldots, m - 1)$

\[
E^{0.2} = D^{0.2} - c_{1,1}^{(1)} \frac{\partial}{\partial c_{1,1}^{(1)}},
\]

\[
E^{l.2} = D^{l.2} - c_{1,1}^{(l)} \frac{\partial}{\partial c_{1,1}^{(l)}} - c_{1,1}^{(l+1)} \frac{\partial}{\partial c_{1,1}^{(l+1)}},
\]

\[
E^{m.2} = D^{m.2} - c_{1,1}^{(m)} \frac{\partial}{\partial c_{1,1}^{(m)}},
\]

we have

**Lemma 2.** For $k, l = 0, 1, \ldots, m$,

\[
E^{k.2} E^{l.1} \log \mathbb{P}_n = 2t_2^{(k)} |c| \frac{\partial^{2p}}{\partial t_2^{(k)} \partial t_1^{(l)}} + \delta_k^l E^{l.1} \log \mathbb{P}_n.
\]

**Proof.** With arguments similar to those for (65) and (67), we get

\[
E^{k.2} E^{l.1} \log \mathbb{P}_n = \frac{E^{k.2} \mathbb{P}_n E^{l.1} \mathbb{P}_n}{\mathbb{P}_n^2} + \frac{E^{k.2} E^{l.1} \mathbb{P}_n}{\mathbb{P}_n} = -\frac{\left( 2t_2^{(k)} |c| \frac{\partial^{2p}}{\partial t_2^{(k)} \partial t_1^{(l)}} + C \mathbb{P}_n \right) \frac{\partial \mathbb{P}_n}{\partial t_1^{(l)}}}{\mathbb{P}_n^2} + \frac{E^{k.2} \frac{\partial}{\partial t_1^{(l)}} \mathbb{P}_n}{\mathbb{P}_n}.
\]

Here

\[
C = \begin{cases} -\frac{n(n+1)}{2} & k = 1 \text{ or } m, \\ -n & \text{otherwise}. \end{cases}
\]

Similar to (68) and (69), we have

\[
E^{k.2} \frac{\partial}{\partial t_1^{(l)}} \mathbb{P}_n = \frac{\partial}{\partial t_1^{(l)}} E^{k.2} \mathbb{P}_n
\]

and

\[
E^{k.2} \tau_n = \left( 2t_2^{(k)} \frac{\partial}{\partial t_2^{(k)}} + t_1^{(k)} \frac{\partial}{\partial t_1^{(k)}} + C + \ldots \right) \tau_n,
\]

with coefficients of all terms other than $2t_2^{(k)} \frac{\partial}{\partial t_2^{(k)}}$, $t_1^{(k)} \frac{\partial}{\partial t_1^{(k)}}$, or $C$ of the right-hand side operator vanishing.
on \( \mathcal{L} \) and not containing \( t_1^{(l)} \) explicitly. Therefore with an argument similar to that for (70)

\[
E^{k,2} E^{l,1} \log \mathbb{P}_n = - \left( \frac{2t_2^{(k)} \left( \frac{\partial^2 \varphi}{\partial t_2^{(k)} \partial t_2^{(l)}} \right) \frac{\partial \varphi}{\partial t_2^{(l)}} + \frac{\partial E_{n}}{\partial t_2^{(l)}}}{\mathbb{P}_n^2} \right) \mathbb{P}_n + \frac{\partial E_{n}}{\partial t_2^{(l)}} \left( 2t_2^{(k)} \frac{\partial}{\partial t_2^{(k)}} + t_1^{(k)} \frac{\partial}{\partial t_1^{(k)}} + C \right) \mathbb{P}_n
\]

\[
= 2t_2^{(k)} \left( \frac{\partial^2 \log \mathbb{P}_n}{\partial t_2^{(k)} \partial t_2^{(l)}} + \frac{\delta_k}{\mathbb{P}_n} \right) + 2 \frac{\partial^2 \log \mathbb{P}_n}{\partial t_2^{(k)} \partial t_2^{(l)}} + \delta_k E^{l,1} \log \mathbb{P}_n,
\]

since

\[
E^{l,1} \log \mathbb{P}_n = \frac{E^{l,1 \mathbb{P}_n}}{\mathbb{P}_n} = \frac{\partial \varphi}{\partial t_2^{(l)}}.
\]  

Since on the locus \( \mathcal{L} \), \( c_{1,1}^{(k)} \) and \( t_2^{(l)} \) are functions of \( e^{-s_1}, \ldots, e^{-s_m} \) defined in (79), by the chain rule we get as operators on \( \mathbb{P}_n \) (\( l = 1, 2, \ldots, m \))

\[
\frac{\partial}{\partial s_l} = \frac{2c_l^2}{1 - c_l^2} \frac{\partial}{\partial t_2^{(l)}} + \frac{2c_l^2}{1 - c_l^2} \frac{\partial}{\partial t_2^{(l)}} - \frac{2c_l(1 + c_l^2)}{(1 - c_l^2)^2} \frac{\partial}{\partial t_2^{(l)}}
\]

(89)

and

\[
c_{1,1} \frac{\partial}{\partial c_{1,1}^{(l)}} = \frac{2c_l^2}{1 - c_l^2} \frac{\partial}{\partial t_2^{(l)}} + \frac{2c_l^2}{1 - c_l^2} \frac{\partial}{\partial t_2^{(l)}} - \frac{1 - c_l^2}{1 + c_l^2} \frac{\partial}{\partial t_2^{(l)}}
\]

(90)

Therefore by (79) - (81) we get on \( \mathcal{L} \) that (\( l = 1, 2, \ldots, m - 1 \))

\[
E^{0,2} = D^{0,2} - \frac{2c_1^2}{1 - c_1^2} \frac{\partial}{\partial t_2^{(0)}} - \frac{2c_1^2}{1 - c_1^2} \frac{\partial}{\partial t_2^{(0)}} + \frac{1 - c_1^2}{1 + c_1^2} \frac{\partial}{\partial s_1},
\]

(91)

\[
E^{l,2} = D^{l,2} - \frac{2c_l^2}{1 - c_l^2} \frac{\partial}{\partial t_2^{(l)}} - \frac{2c_l^2}{1 - c_l^2} \frac{\partial}{\partial t_2^{(l)}} + \frac{1 - c_l^2}{1 + c_l^2} \frac{\partial}{\partial s_l},
\]

(92)

\[
E^{m,2} = D^{m,2} - \frac{2c_m^2}{1 - c_m^2} \frac{\partial}{\partial t_2^{(m-1)}} - \frac{2c_m^2}{1 - c_m^2} \frac{\partial}{\partial t_2^{(m-1)}} + \frac{1 - c_m^2}{1 + c_m^2} \frac{\partial}{\partial s_m}.
\]

(93)

Now we denote (\( l = 1, 2, \ldots, m - 1 \))

\[
E^{0,2} = D^{0,2} + \frac{1 - c_1^2}{1 + c_1^2} \frac{\partial}{\partial s_1},
\]

(94)

\[
E^{l,2} = D^{l,2} + \frac{1 - c_l^2}{1 + c_l^2} \frac{\partial}{\partial s_l} + \frac{1 - c_{l+1}^2}{1 + c_{l+1}^2} \frac{\partial}{\partial s_{l+1}},
\]

(95)

\[
E^{m,2} = D^{m,2} + \frac{1 - c_m^2}{1 + c_m^2} \frac{\partial}{\partial s_m},
\]

(96)

and we have
Lemma 3. For \( l = 1, \ldots, m - 1 \),

\[
F^{0,2}E^{m,1}\log P_n = \left[ -\frac{2}{1-c_1}\frac{\partial^2}{\partial t_2^{(0)} \partial t_1^{(m)}} + \frac{2c_1^2}{1-c_1}\frac{\partial^2}{\partial t_2^{(1)} \partial t_1^{(m)}} \right] \log P_n,
\]

(97)

\[
F^{l,2}E^{m,1}\log P_n = \left[ \frac{2c_l^2}{1-c_l}\frac{\partial^2}{\partial t_2^{(l-1)} \partial t_1^{(m)}} - \left( \frac{2}{1-c_l^2} + \frac{2c_{l+1}^2}{1-c_{l+1}} \right) \frac{\partial^2}{\partial t_2^{(l)} \partial t_1^{(m)}} \right] \log P_n,
\]

(98)

\[
F^{m,2}E^{m,1}\log P_n = \left[ \frac{2c_m^2}{1-c_m}\frac{\partial^2}{\partial t_2^{(m-1)} \partial t_1^{(m)}} - \frac{2}{1-c_m^2}\frac{\partial^2}{\partial t_2^{(m)} \partial t_1^{(m)}} \right] \log P_n + E^{m,1}\log P_n.
\]

(99)

Proof. We only prove (97). By (91), on \( \mathcal{L} \) we have

\[
F^{0,2} = E^{0,2} + \frac{2c_1^2}{1-c_1^2}\frac{\partial}{\partial t_2^{(0)}} + \frac{2c_1^2}{1-c_1^2}\frac{\partial}{\partial t_2^{(1)}},
\]

(100)

and similar to (97) and (100), we have

\[
\frac{\partial}{\partial t_2^{(k)}} E^{1,1}\log P_n = \frac{\partial}{\partial t_2^{(0)}} E^{1,1}\log P_n + \frac{\partial}{\partial t_2^{(1)}} E^{1,1}\log P_n = \frac{\partial^2}{\partial t_2^{(k)}} \log P_n.
\]

(101)

so that with the result of (88),

\[
F^{0,2}E^{m,1}\log P_n = E^{0,2}E^{m,1}\log P_n + \frac{2c_1^2}{1-c_1^2}\frac{\partial}{\partial t_2^{(0)}} E^{m,1}\log P_n + \frac{2c_1^2}{1-c_1^2}\frac{\partial}{\partial t_2^{(1)}} E^{m,1}\log P_n
\]

\[
= 2t_2^{(0)} \frac{\partial^2}{\partial t_2^{(0)} \partial t_1^{(m)}} \log P_n + \frac{2c_1^2}{1-c_1^2} \frac{\partial^2}{\partial t_2^{(0)} \partial t_1^{(m)}} \log P_n + \frac{2c_1^2}{1-c_1^2} \frac{\partial^2}{\partial t_2^{(1)} \partial t_1^{(m)}} \log P_n
\]

\[
= \left[ -\frac{2}{1-c_1^2} \frac{\partial^2}{\partial t_2^{(0)} \partial t_1^{(m)}} + \frac{2c_1^2}{1-c_1^2} \frac{\partial^2}{\partial t_2^{(1)} \partial t_1^{(m)}} \right] \log P_n.
\]

(102)

Finally we define

\[
\begin{pmatrix}
G^{0,2} \\
G^{1,2} \\
\vdots \\
G^{m,2}
\end{pmatrix} = K
\begin{pmatrix}
F^{0,2} \\
F^{1,2} \\
\vdots \\
F^{m,2}
\end{pmatrix},
\]

(103)

where

\[
K = \begin{pmatrix}
-\frac{2}{1-c_1^2} & \frac{2c_1^2}{1-c_1^2} & \frac{2c_1^2}{1-c_1^2} & \cdots \\
\frac{2c_1^2}{1-c_1^2} & -\frac{2}{1-c_1^2} & \frac{2c_1^2}{1-c_1^2} & \cdots \\
\frac{2c_1^2}{1-c_1^2} & \frac{2c_1^2}{1-c_1^2} & -\frac{2}{1-c_1^2} & \cdots \\
\cdots & \cdots & \cdots & \ddots \\
\frac{2c_m^2}{1-c_m^2} & \cdots & \cdots & \frac{2c_m^2}{1-c_m^2}
\end{pmatrix}^{-1}
\]

(104)
We can get $K^{-1}$ by substituting each $c_i$ in $J^{-1}|_{\mathcal{L}}$ by $c_i^2$, so we have

$$K_{0,l} = -\frac{1}{2} \prod_{i=1}^{l} c_i^2, \quad K_{m,l} = -\frac{1}{2} \prod_{i=1}^{m-l} c_{m-i+1}^2.$$ \hfill (105)

and get by (107) – (109).

**Lemma 4.**

$$G^{0,2}E^{m,1} \log P_n = \frac{\partial^2}{\partial t_2^{(0)} \partial t_1^{(m)}} \log P_n + K_{0,m}E^{m,1} \log P_n.$$ \hfill (106)

Symmetrically, we can get by the same method

**Lemma 5.**

$$G^{m,2}E^{1,1} \log P_n = \frac{\partial^2}{\partial t_2^{(m)} \partial t_1^{(0)}} \log P_n + K_{m,0}E^{0,1} \log P_n.$$ \hfill (107)

By the result of [1],

$$\frac{\partial}{\partial t_1^{(0)}} \log \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\partial^2}{\partial t_2^{(0)} \partial t_1^{(m)}} \log \tau_n,$$ \hfill (108)

$$\frac{\partial}{\partial t_1^{(m)}} \log \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\partial^2}{\partial t_2^{(m)} \partial t_1^{(0)}} \log \tau_n,$$ \hfill (109)

we get the differential equation on the locus

$$E^{0,1} \log \frac{\tau_{n+1}}{\tau_{n-1}}|_{\mathcal{L}} = \frac{G^{0,2}E^{m,1} \log \tau_n|_{\mathcal{L}} - K_{0,m}E^{m,1} \log \tau_n|_{\mathcal{L}}}{E^{0,1}E^{m,1} \log \tau_n|_{\mathcal{L}} - nJ_{m,0}},$$ \hfill (110)

$$E^{m,1} \log \frac{\tau_{n+1}}{\tau_{n-1}}|_{\mathcal{L}} = \frac{G^{m,2}E^{1,1} \log \tau_n|_{\mathcal{L}} - K_{m,0}E^{1,1} \log \tau_n|_{\mathcal{L}}}{E^{m,1}E^{1,1} \log \tau_n|_{\mathcal{L}} - nJ_{0,m}},$$ \hfill (111)

where the result

$$\frac{\partial}{\partial t_1^{(0)}} \log \frac{\tau_{n+1}}{\tau_{n-1}}|_{\mathcal{L}} = E^{0,1} \log \frac{\tau_{n+1}}{\tau_{n-1}}|_{\mathcal{L}}$$ \hfill (112)

can be proved like lemma [1]. By the identity

$$E^{0,1}E^{m,1} \log \frac{\tau_{n+1}}{\tau_{n-1}}|_{\mathcal{L}} = E^{m,1}E^{0,1} \log \frac{\tau_{n+1}}{\tau_{n-1}}|_{\mathcal{L}},$$ \hfill (113)

we get the final result

$$E^{0,1}G^{m,2}E^{0,1} \log \tilde{\tau}_n - K_{m,0}E^{0,1} \log \tilde{\tau}_n = E^{m,1}G^{0,2}E^{m,1} \log \tilde{\tau}_n - K_{0,m}E^{m,1} \log \tilde{\tau}_n.$$ \hfill (114)

Now we denote

$$A_1 = -2E^{0,1},$$ \hfill (115)

$$B_1 = -2E^{m,1},$$ \hfill (116)

$$A_2 = -2(G^{0,2} - K_{0,m}),$$ \hfill (117)

$$B_2 = -2(G^{m,2} - K_{m,0}),$$ \hfill (118)

and we get the equation [34].
Remark 4. In the 2-time case, i.e., \( m = 1 \),

\[
\begin{align*}
A_1 &= D^{0,1} + c_1 D^{1,1}, \quad \text{(119)} \\
B_1 &= c_1 D^{0,1} + D^{1,1}, \quad \text{(120)}
\end{align*}
\]

\[
\begin{align*}
A_2 &= F^{0,1} + c_1^2 F^{1,1} = D^{0,1} + c_1^2 D^{1,1} + (1 - c_1^2) \frac{\partial}{\partial t_1} - c_1^2, \quad \text{(121)} \\
B_2 &= c_1^2 F^{0,1} + F^{1,1} = c_1^2 D^{0,1} + D^{1,1} + (1 - c_1^2) \frac{\partial}{\partial t_1} - c_1^2. \quad \text{(122)}
\end{align*}
\]

Our PDE (34) agrees with that in [2].

3 The joint probability in the Airy process

In this section we adapt notations defined in (23)–(25), and by remark 1, \( a^{(l)}_1 = \bar{a}^{(l)}_1 = -\infty \). We denote \( (l = 0, 1, \ldots, m) \)

\[
\begin{align*}
\bar{D}^{l,1} &= \sum_{k=1}^{2r_l} \frac{\partial}{\partial \bar{a}^{(l)}_k}, \quad \bar{D}^{l,2} = \sum_{k=1}^{2r_l} \frac{\partial}{\partial \bar{a}^{(l)}_k}, \quad \text{(123)}
\end{align*}
\]

if all \( a^{(l)}_{r_l} < +\infty \), otherwise drop the \( a^{(l)}_{r_l} \) part. we can write the differential operators defined for the Dyson process as

\[
\begin{align*}
A_1 &= \sqrt{2\bar{n}} \sum_{l=0}^{m} e^{-\bar{t}_l/\bar{n}} \bar{D}^{l,1}, \quad \text{(124)} \\
B_1 &= \sqrt{2\bar{n}} \sum_{l=0}^{m} e^{(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,1}, \quad \text{(125)}
\end{align*}
\]

\[
\begin{align*}
A_2 &= \sum_{l=0}^{m} e^{-2\bar{t}_l/\bar{n}} \bar{D}^{l,2} + 2\bar{n}^2 \sum_{l=0}^{m} e^{-2\bar{t}_l/\bar{n}} \bar{D}^{l,1} + \bar{n} \sum_{l=1}^{m} (1 - e^{-2\bar{t}_l/\bar{n}}) \frac{\partial}{\partial \bar{t}_l} - e^{-2\bar{t}_m/\bar{n}}, \quad \text{(126)} \\
B_2 &= \sum_{l=0}^{m} e^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,2} + 2\bar{n}^2 \sum_{l=0}^{m} e^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,1} + \bar{n} \sum_{l=1}^{m} (e^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} - e^{-2\bar{t}_m/\bar{n}}) \frac{\partial}{\partial \bar{t}_l} - e^{-2\bar{t}_m/\bar{n}}. \quad \text{(127)}
\end{align*}
\]

It is not difficult to see that (34) implies

\[
[ A_1 B_2 A_1 - B_1 A_2 B_1 ] \log \mathbb{P}_n \cdot ( A_1 B_1 \log \mathbb{P}_n + 2ne^{-t_m} ) = \\
B_2 A_1 \log \mathbb{P}_n \cdot A_1 B_1 A_1 \log \mathbb{P}_n - A_2 B_1 \log \mathbb{P}_n \cdot B_1 A_1 B_1 \log \mathbb{P}_n. \quad \text{(128)}
\]
Take substitutions (124)–(127) into (128), we get

\[
\left( \sum_{l=0}^{m} e^{-\tilde{t}_l/\bar{n}} \tilde{D}_l^{1.1} \right) \left( \sum_{l=0}^{m} e^{2(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.2} + 2\bar{n}^2 \sum_{l=0}^{m} e^{2(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1} \right) \\
+ \bar{n} \sum_{l=1}^{m} \left( e^{(\tilde{t}_l-\bar{t}_m)/\bar{n}} - e^{-2\bar{t}_m/\bar{n}} \right) \frac{\partial}{\partial t_l} - e^{-2\bar{t}_m/\bar{n}} \left( \sum_{l=0}^{m} e^{-\tilde{t}_l/\bar{n}} \tilde{D}_l^{1.1} \right) \log \mathbb{P}_n \\
- \left( \sum_{l=0}^{m} e^{(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1} \right) \left( \sum_{l=0}^{m} e^{2(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.2} + 2\bar{n}^2 \sum_{l=0}^{m} e^{-2\bar{t}_m/\bar{n}} \tilde{D}_l^{1.1} \right) \\
+ \bar{n} \sum_{l=1}^{m} \left( 1 - e^{-2\tilde{t}_l/\bar{n}} \right) \frac{\partial}{\partial t_l} - e^{-2\bar{t}_m/\bar{n}} \left( \sum_{l=0}^{m} e^{(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1} \right) \log \mathbb{P}_n \right) \\
\times \left( \sum_{l=0}^{m} e^{-\tilde{t}_l/\bar{n}} \tilde{D}_l^{1.1} \right) \left( \sum_{l=0}^{m} e^{(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1} \right) \log \mathbb{P}_n + \bar{n}^2 e^{-\bar{t}_m/\bar{n}} \right)
\]

which becomes

\[
\left( \sum_{l=0}^{m} e^{2(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.2} + 2\bar{n}^2 \sum_{l=0}^{m} e^{2(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1} + \bar{n} \sum_{l=1}^{m} \left( e^{2(\tilde{t}_l-\bar{t}_m)/\bar{n}} - e^{-2\bar{t}_m/\bar{n}} \right) \frac{\partial}{\partial t_l} - e^{-2\bar{t}_m/\bar{n}} \right) \\
\left( \sum_{l=0}^{m} e^{(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1} \right) \log \mathbb{P}_n \times \left( \sum_{l=0}^{m} e^{-\tilde{t}_l/\bar{n}} \tilde{D}_l^{1.1} \right) \left( \sum_{l=0}^{m} e^{(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1} \right) \left( \sum_{l=0}^{m} e^{-\tilde{t}_l/\bar{n}} \tilde{D}_l^{1.1} \right) \log \mathbb{P}_n \\
- \left( \sum_{l=0}^{m} e^{-2\tilde{t}_l/\bar{n}} \tilde{D}_l^{1.2} + 2\bar{n}^2 \sum_{l=0}^{m} e^{-2\tilde{t}_l/\bar{n}} \tilde{D}_l^{1.1} + \bar{n} \sum_{l=1}^{m} \left( 1 - e^{-2\tilde{t}_l/\bar{n}} \right) \frac{\partial}{\partial t_l} - e^{-2\bar{t}_m/\bar{n}} \right) \\
\sum_{l=0}^{m} e^{(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1} \right) \log \mathbb{P}_n \times \left( \sum_{l=0}^{m} e^{(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1} \right) \left( \sum_{l=0}^{m} e^{-\tilde{t}_l/\bar{n}} \tilde{D}_l^{1.1} \right) \sum_{l=0}^{m} e^{(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1} \right) \log \mathbb{P}_n.
\]

Since we have commutator formulas

\[
\left[ \bar{n} \left( e^{2(\tilde{t}_l-\bar{t}_m)/\bar{n}} - e^{-2\bar{t}_m/\bar{n}} \right) \frac{\partial}{\partial t_l}, \sum_{l=0}^{m} e^{-\tilde{t}_l/\bar{n}} \tilde{D}_l^{1.1} \right] = \sum_{l=0}^{m} e^{(\tilde{t}_l-2\bar{t}_m)/\bar{n}} - e^{-(\tilde{t}_l+2\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1},
\]

\[
\left[ \bar{n} \sum_{l=1}^{m} \left( 1 - e^{-2\tilde{t}_l/\bar{n}} \right) \frac{\partial}{\partial t_l}, \sum_{l=0}^{m} e^{(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1} \right] = \sum_{l=0}^{m} e^{(\tilde{t}_l-3\bar{t}_m)/\bar{n}} - e^{-(\tilde{t}_l+\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1},
\]

\[
\left[ \sum_{l=0}^{m} e^{-\tilde{t}_l/\bar{n}} \tilde{D}_l^{1.1}, \sum_{l=0}^{m} e^{2(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.2} \right] = \sum_{l=0}^{m} e^{(\tilde{t}_l-2\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1},
\]

\[
\left[ \sum_{l=0}^{m} e^{(\tilde{t}_l-\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1}, \sum_{l=0}^{m} e^{2\tilde{t}_l/\bar{n}} \tilde{D}_l^{1.2} \right] = \sum_{l=0}^{m} e^{-(\tilde{t}_l+\bar{t}_m)/\bar{n}} \tilde{D}_l^{1.1},
\]

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we can write (129) as

\[
\left( \sum_{l=0}^{m} e^{-\tilde{t}_l / \bar{n}} \bar{D}^{l,1} \right)^2 \left[ \sum_{l=0}^{m} e^{2(\tilde{t}_l - \tilde{t}_m) / \bar{n}} \bar{D}^{l,2} + 2\bar{n}^2 \sum_{l=0}^{m} e^{2(\tilde{t}_l - \tilde{t}_m) / \bar{n}} \bar{D}^{l,1} + \bar{n} \sum_{l=1}^{m} \left( e^{2(\tilde{t}_l - \tilde{t}_m) / \bar{n}} - e^{-2\tilde{t}_l / \bar{n}} \right) \frac{\partial}{\partial t_l} \right] \log \mathbb{P}_n
\]

\[
\left[ \sum_{l=0}^{m} e^{\tilde{t}_l / \bar{n}} \bar{D}^{l,1} \right]^2 \left[ \sum_{l=0}^{m} e^{-2\tilde{t}_l / \bar{n}} \bar{D}^{l,2} + 2\bar{n}^2 \sum_{l=0}^{m} e^{-2\tilde{t}_l / \bar{n}} \bar{D}^{l,1} + \bar{n} \sum_{l=1}^{m} \left( 1 - e^{-2\tilde{t}_l / \bar{n}} \right) \frac{\partial}{\partial t_l} \right] \log \mathbb{P}_n
\]

\[
\times \left( \sum_{l=0}^{m} e^{-\tilde{t}_l / \bar{n}} \bar{D}^{l,1} \right) \left[ \sum_{l=0}^{m} e^{(\tilde{t}_l - \tilde{t}_m) / \bar{n}} \bar{D}^{l,1} \right] \log \mathbb{P}_n + \bar{n}^2 e^{-\tilde{t}_m / \bar{n}}
\]

\[
= \left( \sum_{l=0}^{m} e^{-\tilde{t}_l / \bar{n}} \bar{D}^{l,1} \right)^2 \left[ \sum_{l=0}^{m} e^{2(\tilde{t}_l - \tilde{t}_m) / \bar{n}} \bar{D}^{l,2} + 2\bar{n}^2 \sum_{l=0}^{m} e^{2(\tilde{t}_l - \tilde{t}_m) / \bar{n}} \bar{D}^{l,1} + \bar{n} \sum_{l=1}^{m} \left( e^{2(\tilde{t}_l - \tilde{t}_m) / \bar{n}} - e^{-2\tilde{t}_l / \bar{n}} \right) \frac{\partial}{\partial t_l} \right] \log \mathbb{P}_n
\]

\[
- \left[ \sum_{l=0}^{m} e^{(\tilde{t}_l - 2\tilde{t}_m) / \bar{n}} \bar{D}^{l,1} \right] \log \mathbb{P}_n \times \left[ \sum_{l=0}^{m} e^{-\tilde{t}_l / \bar{n}} \bar{D}^{l,1} \right]^2 \left[ \sum_{l=0}^{m} e^{(\tilde{t}_l - \tilde{t}_m) / \bar{n}} \bar{D}^{l,1} \right] \log \mathbb{P}_n
\]

\[
- \left[ \sum_{l=0}^{m} e^{(\tilde{t}_l - \tilde{t}_m) / \bar{n}} \bar{D}^{l,1} \right] \log \mathbb{P}_n \times \left[ \sum_{l=0}^{m} e^{-2\tilde{t}_l / \bar{n}} \bar{D}^{l,2} + 2\bar{n}^2 \sum_{l=0}^{m} e^{-2\tilde{t}_l / \bar{n}} \bar{D}^{l,1} + \bar{n} \sum_{l=1}^{m} \left( 1 - e^{-2\tilde{t}_l / \bar{n}} \right) \frac{\partial}{\partial t_l} \right] \log \mathbb{P}_n
\]

\[
- \left[ \sum_{l=0}^{m} e^{(\tilde{t}_l + \tilde{t}_m) / \bar{n}} \bar{D}^{l,1} \right] \log \mathbb{P}_n \times \left[ \sum_{l=0}^{m} e^{-\tilde{t}_l / \bar{n}} \bar{D}^{l,1} \right] \left[ \sum_{l=0}^{m} e^{(\tilde{t}_l - \tilde{t}_m) / \bar{n}} \bar{D}^{l,1} \right]^2 \log \mathbb{P}_n.
\]

(134)

Since all terms of the PDE involves \( \bar{n} \), we can expand the PDE with respect to \( \bar{n} \), with formulas (* can be 1 or 2)

\[
\sum_{l=0}^{m} e^{-\tilde{t}_l / \bar{n}} \bar{D}^{l,1} = \sum_{l=0}^{m} \bar{D}^{l,1} - \frac{1}{\bar{n}} \sum_{l=0}^{m} \tilde{t}_l \bar{D}^{l,1} + \frac{1}{2\bar{n}^2} \sum_{l=0}^{m} \tilde{t}_l^2 \bar{D}^{l,1} - \frac{1}{6\bar{n}^3} \sum_{l=0}^{m} \tilde{t}_l^3 \bar{D}^{l,1} + O \left( \frac{1}{\bar{n}^4} \right),
\]

(135)

\[
\sum_{l=0}^{m} e^{-2\tilde{t}_l / \bar{n}} \bar{D}^{l,1} = \sum_{l=0}^{m} \bar{D}^{l,1} - \frac{2}{\bar{n}^2} \sum_{l=0}^{m} \tilde{t}_l \bar{D}^{l,1} + \frac{2}{3\bar{n}^3} \sum_{l=0}^{m} \tilde{t}_l^2 \bar{D}^{l,1} + O \left( \frac{1}{\bar{n}^4} \right),
\]

(136)

\[
\sum_{l=0}^{m} e^{(\tilde{t}_l - \tilde{t}_m) / \bar{n}} \bar{D}^{l,1} = \sum_{l=0}^{m} \bar{D}^{l,1} + \frac{1}{\bar{n}} \sum_{l=0}^{m} (\tilde{t}_l - \tilde{t}_m) \bar{D}^{l,1} + \frac{1}{2\bar{n}^2} \sum_{l=0}^{m} (\tilde{t}_l - \tilde{t}_m)^2 \bar{D}^{l,1}
\]

\[
+ \frac{1}{6\bar{n}^3} \sum_{l=0}^{m} (\tilde{t}_l - \tilde{t}_m)^3 \bar{D}^{l,1} + O \left( \frac{1}{\bar{n}^4} \right),
\]

(137)

\[
\sum_{l=0}^{m} e^{2(\tilde{t}_l - \tilde{t}_m) / \bar{n}} \bar{D}^{l,1} = \sum_{l=0}^{m} \bar{D}^{l,1} + \frac{2}{\bar{n}} \sum_{l=0}^{m} (\tilde{t}_l - \tilde{t}_m) \bar{D}^{l,1} + \frac{2}{3\bar{n}^2} \sum_{l=0}^{m} (\tilde{t}_l - \tilde{t}_m)^2 \bar{D}^{l,1}
\]

\[
+ \frac{4}{3\bar{n}^3} \sum_{l=0}^{m} (\tilde{t}_l - \tilde{t}_m)^3 \bar{D}^{l,1} + O \left( \frac{1}{\bar{n}^4} \right),
\]

(138)

\[
\bar{n} \sum_{l=1}^{m} \left( e^{2(\tilde{t}_l - \tilde{t}_m) / \bar{n}} - e^{-2\tilde{t}_l / \bar{n}} \right) \frac{\partial}{\partial t_l} = 2 \sum_{l=1}^{m} \tilde{t}_l \frac{\partial}{\partial t_l} + \frac{2}{\bar{n}} \sum_{l=1}^{m} (\tilde{t}_l - 2\tilde{t}_m) \frac{\partial}{\partial t_l} + O \left( \frac{1}{\bar{n}^2} \right),
\]

(139)

\[
\bar{n} \sum_{l=1}^{m} (1 - e^{-2\tilde{t}_l / \bar{n}}) \frac{\partial}{\partial t_l} = 2 \sum_{l=1}^{m} \tilde{t}_l \frac{\partial}{\partial t_l} - \frac{2}{\bar{n}} \sum_{l=1}^{m} \tilde{t}_l^2 \frac{\partial}{\partial t_l} + O \left( \frac{1}{\bar{n}^2} \right).
\]

(140)
Although the left hand side of (129) contains $O(\bar{n}^4)$ terms, after careful calculation we find all $O(\bar{n}^4)$, $O(\bar{n}^3)$ and $O(\bar{n}^2)$ terms disappear, and the equation becomes

$$
\left[ \sum_{l=0}^{m} D_l^{1,1} \right]^2 \left[ \sum_{l=0}^{m} (\bar{t}_m - 2\bar{t}_l) D_l^{1,2} + \sum_{l=0}^{m} ((\bar{t}_m - \bar{t}_l)^3 - \bar{t}_l^3) D_l^{1,1} + 2 \sum_{l=1}^{m} \bar{t}_l (\bar{t}_m - \bar{t}_l) \frac{\partial}{\partial \bar{t}_l} \right] \log \mathbb{P}_n
\hspace{1cm} + \left[ \sum_{l=0}^{m} D_l^{1,1} \right] \left[ \sum_{l=0}^{m} (2\bar{t}_l - \bar{t}_m) D_l^{1,1} \right] \left[ \sum_{l=0}^{m} (\bar{t}_m - \bar{t}_l \bar{t}_l) D_l^{1,1} \right] - \left[ \sum_{l=0}^{m} (\bar{t}_m - \bar{t}_l) D_l^{1,1} \right] \left[ \sum_{l=0}^{m} \bar{t}_l^2 D_l^{1,1} \right] \log \mathbb{P}_n
\hspace{1cm} + 2 \left[ \sum_{l=0}^{m} \bar{t}_l D_l^{1,1} \right] \left[ \sum_{l=0}^{m} (\bar{t}_m - \bar{t}_l) D_l^{1,1} \right] \left[ \sum_{l=0}^{m} (2\bar{t}_l - \bar{t}_m) D_l^{1,1} \right] \log \mathbb{P}_n
\hspace{1cm} = \left\{ \left[ \sum_{l=0}^{m} (2\bar{t}_l - \bar{t}_m) D_l^{1,1} \right] \left[ \sum_{l=0}^{m} D_l^{1,1} \right] \log \mathbb{P}_n, \left[ \sum_{l=0}^{m} D_l^{1,1} \right] \left[ \sum_{l=0}^{m} D_l^{1,1} \right] \log \mathbb{P}_n \right\} \sum_{l=0}^{m} D_l^{1,1} + O\left(\frac{1}{\bar{n}}\right).
$$

The term $O\left(\frac{1}{\bar{n}}\right)$ in (141) is a quadratic function in term of $\log \mathbb{P}_n$ and its derivatives with coefficients $O\left(\frac{1}{\bar{n}}\right)$. By the definition of $\mathbb{P}_{\text{Airy}}$ in (26) and the convergence result in [2], we take the limit $n \to \infty$, and get the PDE (14) after the changing of notations, i.e., cleaning all “bars” for variables and operators.

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