Superimposed XOR: Approaching Capacity Bounds of the Two-Way Relay Channels

Jianquan Liu, Meixia Tao, Youyun Xu, and Xiaodong Wang

Abstract

In two-way relay channels, bitwise XOR and symbol-level superposition coding are two popular network-coding based relaying schemes. However, neither of them can approach the capacity bound when the channels in the broadcast phase are asymmetric. In this paper, we present a new physical layer network coding (PLNC) scheme, called superimposed XOR. The new scheme advances the existing schemes by specifically taking into account the channel asymmetry as well as information asymmetry in the broadcast phase. We obtain its achievable rate regions over Gaussian channels when integrated with two known time control protocols in two-way relaying. We also demonstrate their average maximum sum-rates and service delay performances over fading channels. Numerical results show that the proposed superimposed XOR achieves a larger rate region than both XOR and superposition and performs much better over fading channels. We further deduce the boundary of its achievable rate region of the broadcast phase in an explicit and analytical expression. Based on these results, we then show that the gap to the capacity bound approaches zero at high signal-to-noise ratio.

Index Terms

Two-way relaying, capacity bound, physical layer network coding, bitwise XOR, superposition coding.

This work was partly presented in IEEE GLOBECOM 2009 [1].

Jianquan Liu, Meixia Tao and Youyun Xu are with the Department of Electronic Engineering, Shanghai Jiao Tong University, Shanghai, 200240, P. R. China (e-mails: {jianquanliu, mxtao, xuyouyun}@sjtu.edu.cn).

Xiaodong Wang is with the Department of Electrical Engineering, Columbia University, New York, NY 10027, USA (e-mail: wangx@ee.columbia.edu).
I. INTRODUCTION

Cooperative communications enables different users in a wireless network to share their antennas and cooperate in signal transmission at the physical layer. This opens up the possibilities of exploiting distributed spatial diversity and hence effectively enhancing the system performance. Cooperative communications has thus attracted significant amount of interests from both academia and industry with applications in ad-hoc as well as infrastructure-based network. A basic building block of a cooperative network is the relay channel, first proposed by van der Meulen [2] and then extensively studied from information theoretic perspectives by Cover and Gamal [3]. The classic relay channel consists of three nodes, wherein a source node communicates with a destination node with the help of a relay node. Thus far, a number of relay schemes have been proposed. Among them, three popular strategies are known as amplify-and-forward (AF), decode-and-forward (DF) and compress-and-forward (CF) respectively. However, due to the half-duplex constraint arising from practical considerations, these traditional relay schemes suffer from loss in spectral efficiency.

Two-way relaying, where two source nodes exchange information with the help of a relay node, has recently gained a lot of research interests [4]–[10]. It is shown able to overcome the half-duplex constraint and significantly improve the system spectral efficiency in relay-based cooperative networks. Upon receiving the bidirectional information flows, the relay node combines them together and then broadcasts to the two desired destinations. The operation at the relay resembles network coding [11], a technique originally developed for wireline networks. It is thus often referred to as physical layer network coding (PLNC) [8] or analog network coding (ANC) [9].

Researchers have attempted to find the best achievable rate region of two-way relay channels [12]–[14]. Oechtering, et al. obtained the capacity region of the broadcast phase in terms of the maximal probability of error [12]. The achievable rate region of a two-way relay channel considering both the multiple-access phase (i.e. the two source nodes transmit simultaneously to the relay node) and the broadcast phase was studied by Xie using random binning [13]. Kim, et al. [14], [15] further broadened the frontier of the achievable rate region by allowing time share between different transmission phases. In particular, the capacity region of a two-way relay channel with two-step time protocol is now well-known to be the intersection of the optimal
the information theoretic techniques including random binning and jointly typical set decoding are often adopted at the relay and destinations in the broadcast phase [12]–[14].

Meanwhile, a number of practical PLNC schemes for two-way relay channels have also been proposed and analyzed, such as bit-level XOR [16], [17], symbol-level superposition coding [10], [18] and AF [8], [10]. In particular, authors in [10] obtained the rate pair expressions for superposition based relaying. Authors in [16] analyzed the maximum achievable sum-rate for XOR based relaying with optimal time control. It is worth mentioning that the asymmetry in both packet size and channel gain of the two transmitting nodes are considered in [16]. Oechtering also studied the optimal time control for superposition scheme [18]. Despite all these attempts, there is still a large gap between the rate regions achieved by the practical PLNC schemes and the capacity bound in asymmetric relay channels, as shown by a recent comparative study in [19].

In this paper, we study practical and capacity approaching PLNC schemes over the two-way relay channels. In this regard, we propose a novel PLNC scheme, named as superimposed XOR, tailored for the broadcast phase of two-way relaying with asymmetric channels. Combining it with two known transmission protocols: 4-step with direct link [14] and 2-step with no direct link, we analyze its achievable rate regions over Gaussian channels. We also demonstrate their average maximum sum-rates and service delay performances over fading channels. Numerical results show that the performance of the proposed superimposed XOR outperforms the traditional XOR and superposition in terms of achievable rate region. It also closely approaches the optimal capacity bounds of the two-way relay channels in the high signal-to-noise ratio (SNR) regime. To further illustrate the capacity approaching behavior of the proposed superimposed XOR, we obtain the analytical expressions of the boundary of its achievable broadcast rate region. Based on these results, we then explicitly prove that its gap with the capacity of the broadcast phase approaches zero when the SNR is much larger than one.

The rest of the paper is organized as follows. In Section II, we present the system model of two-way relaying. In Section III, we describe the proposed superimposed XOR scheme. Section IV characterizes the rate regions over Gaussian channels. The capacity approaching performance

\footnote{The capacity region of two-way relaying with partial decoding still remains open.}
is analyzed in Section V. Finally, we conclude the paper in section VI.

II. SYSTEM MODEL

We consider a two-way relay channel which consists of two source nodes and one relay node. The source nodes, denoted as 0 and 2, wish to exchange information with the help of the relay node, denoted as 1. We assume that all the nodes operate in the half-duplex mode. The channel on each communication link is assumed to be corrupted with Rayleigh fading and additive white Gaussian noise. The SNR of the link from node $i$ to node $j$ is denoted as $\gamma_{ij}$, for $i, j \in \{0, 1, 2\}$, and it counts both channel gain and transmit power. Note that $\gamma_{ij}$ may not be equal to $\gamma_{ji}$ as the channels considered here may not be reciprocal. The channel capacity in bit/s/Hz of the link from node $i$ to node $j$ is denoted as $C_{ij}$, and determined by the SNR on the link as

$$C_{ij} \triangleq C(\gamma_{ij}) = \log_2(1 + \gamma_{ij}). \quad (1)$$

Throughout this paper, we assume that the relay needs to fully decode the information of the two source nodes. The sum-rate capacity of the multiple access channel (MAC) when nodes 0 and 2 are transmitting simultaneously to node 1 is denoted as

$$C_m \triangleq C(\gamma_{01} + \gamma_{21}) = \log_2(1 + \gamma_{01} + \gamma_{21}). \quad (2)$$

A. Time control protocols

Two-way relaying involves not only PLNC at the relay node but also time control for node transmission. In this subsection, we review some existing time control protocols. Similar to [16], we name the protocols based on the number of time steps to finish one round of information exchange between the source nodes. We focus on the 4-step and 2-step protocols in this paper.

1) 4-step protocol: In the first step, node 0 transmits for $\lambda_1$ time duration and node 1 and 2 listen. In the second step, node 2 transmits for $\lambda_2$ time duration and node 1 and 0 listen. In the third step, nodes 0 and 2 transmit simultaneously for $\lambda_3$ time duration while node 1 listens. In the fourth step, node 1 transmits for $\lambda_4$ time duration and node 0 and 2 listen. Without loss of generality, the total time duration is normalized to one, i.e., $\sum_{i=1}^4 \lambda_i = 1$.

2) 2-step protocol: In 2-step protocol, nodes 0 and 2 first transmit simultaneously for $\lambda_1$ time duration while node 1 listens. Next, node 1 transmits for $\lambda_2$ time duration and nodes 0 and 2 listen. This protocol can be regarded as a special case of the 4-step protocol by letting $\lambda_1 = \lambda_2 = 0$. No direct link is exploited here.
III. SUPERIMPOSED XOR

Upon decoding the two bit sequences $b_0$ and $b_2$ from the source nodes 0 and 2 to be exchanged, the relay node will perform physical layer network coding on $b_0$ and $b_2$ and then broadcast to the two destinations. Before introducing the proposed PLNC scheme, we briefly discuss the bit-level XOR and symbol-level superposition coding which the proposed scheme is based upon.

A. Bit-level XOR

The relay node performs bitwise XOR on the two bit sequences as $b_0 \oplus b_2$. In the case where the lengths of the two sequences are not equal, there are two methods to perform the XOR, as shown in Fig. 1(a). The first one is to pad the shorter sequence with zero bits so as to make it having the same length as the longer sequence and then perform XOR. In the second method, the longer sequence will be partitioned into two sub-sequences, with one having the same length as the short sequence. XOR is then performed on the shorter sequence and the sub-sequence with equal length. The resulting bit sequence is broadcasted to both receivers. The other sub-sequence will be transmitted alone, at a possibly higher rate, to its desired receiver. In practice, which method to use depends on the relationship of the channel gains in the broadcast phase of two-way relaying [16]. In general, if the channel quality of the receiver of the longer sequence is worse than the channel quality of the receiver of the shorter sequence, zero-padding is applied Otherwise, partitioning the longer packet is preferred.

B. Symbol-level superposition

The relay encodes the two bit sequences $b_0$ and $b_2$ separately into baseband signal sequences $x_0$ and $x_2$ with the same length, and then superimposes them together as $\sqrt{\theta}x_0 + \sqrt{1-\theta}x_2$, as shown in Fig. 1(b). Here $\theta$ is a power allocation coefficient [10]. The signal $\sqrt{\theta}x_0 + \sqrt{1-\theta}x_2$ is then broadcasted directly without further encoding. Unlike bit-level XOR, there is no need to consider the issue of asymmetry in bit length. Another essential difference between superposition and XOR is that, the information combining is carried out in the symbol level after channel coding and modulation for the former, while it is in the information bit level before channel coding for the latter.
C. Proposed superimposed XOR

As discussed in [19], for symmetrical broadcast channels ($\gamma_{10} = \gamma_{12}$), bit-level XOR is capacity-achieving whereas superposition coding is suboptimal. But for asymmetrical channels ($\gamma_{10} \neq \gamma_{12}$), bit-level XOR becomes inferior to superposition at certain rate regions. The proposed superimposed XOR is specifically designed for asymmetrical broadcast channels and it utilizes the advantages of both XOR and superposition schemes. The details are as follows.

If the lengths of two bit sequences $b_0$ and $b_2$ are equal, then the conventional bitwise XOR is performed. Otherwise, there are two methods to process the two sequences, as shown in Fig. 1(c). The first method is the same as the one in bit-level XOR scheme, that is, padding the shorter sequence, say $b_0$, with zero bits, so as to make them equal and then performing bit-level XOR. The resulting bit sequence, after channel coding and modulation, is then broadcasted to both receivers. In the second method, the relay node first partitions the longer sequence, say $b_2$, into two sub-sequences as $b_2 = [b'_2, b''_2]$, where the sub-sequence $b'_2$ has the same length as $b_0$. It then encodes the XORed sub-sequence $b_0 \oplus b'_2$ and the sub-sequence $b''_2$, separately. We denote the resulting coded symbol sequences as $x_0$ and $x_2$. Finally, the relay superimposes them together as $\sqrt{\theta}x_0 + \sqrt{1-\theta}x_2$ which is broadcasted directly to the two destinations. Here, $\theta$ is also a power allocation coefficient, $x_0$ is to be received by both destinations, and $x_2$ is to be received by one destination only. Which of the above two methods to use depends on the relationship between $\gamma_{10}$ and $\gamma_{12}$ in the broadcast channels. Suppose that the longer sequence is $b_2$ which is to be sent to node 0. Then, if $\gamma_{10} < \gamma_{12}$, we apply method one and else apply method two. We shall discuss this in more detail in the proof of the rate regions in the next section.

Note that Liu, et al. proposed a joint network coding and superposition coding (JNSC) scheme for information exchange among more than two users in a wireless relay network [20]. Therein, two XOR-ed packets generated by information from three nodes are superimposed. Our proposed superimposed XOR scheme differs from the JNSC [20] in that our scheme performs superposition on only one XOR-ed packet and the sub-packet obtained by partitioning the longer bit sequence. A similar PLNC scheme as the second method of our proposed superimposed XOR was proposed by Chen, et al. for multi-hop wireless networks in [21]. The difference however is that our scheme adaptively selects the aforementioned two methods illustrated in Fig. 1(c) according to the channel conditions in the broadcast phase while the scheme in [21] is fixed at method two.
regardless the channel conditions. Such static approach could be far from capacity-achieving as it does not exploit the channel dynamics.

In the rest of the paper, for ease of presentation, if the considered three PLNC schemes (XOR, superposition, and superimposed XOR) are combined with the 4-step time control protocol, we denote them as 4S-XOR, 4S-SUP and 4S-SuX, respectively. Likewise, when combined with 2-step protocol, they are named as 2S-XOR, 2S-SUP and 2S-SuX.

IV. ANALYSIS OF RATE REGIONS

Let $R_0$ and $R_2$ denote the data rates of the information flows $0 \to 2$ and $2 \to 0$, respectively, in the considered two-way communications. In this section, we derive the rate region $(R_0, R_2)$ of the aforementioned relay strategies. Without loss of generality, we assume $\gamma_{10} \geq \gamma_{12}$ and thus $C_{10} \geq C_{12}$ in Subsections IV-A and B.

A. Achievable rate region for superimposed XOR

Theorem 1 (4S-SuX): The rate region for 4S-SuX is the closure of the set of all rate pairs $(R_0, R_2)$ satisfying

$$(R_0, R_2) : \left\{ R_0 \leq \min \left((\lambda_1 + \lambda_3)C_{01}, \lambda_1 C_{02} + \lambda_4 C_{12}(\theta)\right), \right.$$  
$$ R_2 \leq \min \left((\lambda_2 + \lambda_3)C_{21}, \lambda_2 C_{20} + \lambda_4 C_{10}\right), \right.$$  
$$ R_2 - R_0 \leq -\lambda_1 C_{02} + \lambda_2 C_{20} + \lambda_4 C_{10}(1 - \theta), \right.$$  
$$ R_0 + R_2 \leq \lambda_1 C_{01} + \lambda_2 C_{21} + \lambda_3 C_m, \right.$$  
$$ \sum_{i=1}^{4} \lambda_i = 1, \theta \in [0, 1] \right\}.$$  

Here $C_{ij}(\theta) \triangleq C(\gamma_{ij}\theta) = \log_2(1 + \gamma_{ij}\theta)$.

Proof: For ease of comprehension, Fig. 2 is presented to assist the proof. Let $D_0$ denote the information packet to be transmitted from node 0 to 2 and its packet length in bits be denoted as $|D_0|$. Assume the message in the packet is further split into two parts, denoted as $D_0^{(1)}$ and $D_0^{(3)}$, which are transmitted in the first and third step, respectively, as illustrated in Fig. 2 (a) and (c). Likewise, we let $D_2$ denote the information packet to be transmitted from node 2 to 0, and let it be split into $D_2^{(2)}$ and $D_2^{(3)}$ for transmission in the second and third steps, as depicted in Fig. 2 (b) and (c). During the first three steps of packet transmission (i.e. the multiple-access phase), it is obvious that $|D_0^{(1)}| \leq \lambda_1 C_{01}$, $|D_2^{(2)}| \leq \lambda_2 C_{21}$, $|D_0^{(3)}| \leq \lambda_3 C_{01}$, $|D_2^{(3)}| \leq \lambda_3 C_{21}$, and
$|D_{0}^{(3)}| + |D_{2}^{(3)}| \leq \lambda_3 C_m$. Note that in the first step, due to the presence of direct link between node 0 and node 2, the desired destination node 2 is able to exact $|D_{02}| \leq \lambda_1 C_{02}$ amount of information. Thus, the total amount of information bits to be transmitted through the relay link in the fourth step to node 2 is $|D_0| - |D_{02}|$ and we denote the corresponding packet as $D'_0$.

Similarly, the total amount of information to be relayed from node 2 to node 0 in the fourth step is $|D_2| - |D_{20}|$, with $|D_{20}| \leq \lambda_2 C_{20}$, and the corresponding packet can be denoted as $D'_2$. Then, during the fourth step transmission (i.e. the broadcast phase), by comparing the packet sizes $|D'_0|$ and $|D'_2|$, two cases need to be considered.

**Case 1:** $|D'_2| \geq |D'_0|$. As shown in Fig. 2 (d), the relay node 1 partitions the packet $D'_2$ into $D''_2^{(1)}$ and $D''_2^{(2)}$ so that $|D''_2^{(1)}| = |D'_0|$ and $|D''_2^{(2)}| = |D'_2| - |D'_0|$. Packet $D''_2^{(1)}$ contains the first $|D'_0|$ bits from $D'_2$ and packet $D''_2^{(2)}$ contains the rest of the bits from $D'_2$. Now, node 1 creates $D'_1 = D''_2^{(1)} \oplus D'_0$. Then, the information bits $D'_1$ and $D''_2^{(2)}$ are encoded separately into two codewords $x'_0$ and $x'_2$ with the same length, which are then superimposed together in the complex field. Unlike XOR-based scheme, there is no extra time used to transmit $D''_2^{(2)}$ to node 0. Let $\theta$ present the power ratio allocated to the signal $x'_0$ to be transmitted to nodes 0 and 2 and $1 - \theta$ be the power ratio on the signal $x'_2$ to node 0, where $0 \leq \theta \leq 1$. Since $D'_2$ is known at node 2 and so is $x'_2$, to successfully decode the packet $D'_1$ at node 2, the transmission rate of $x'_0$ in the fourth step with a fraction $\lambda_4$ of time cannot exceed $C_{12}(\theta) = C(\theta \gamma_{12})$. Thus, we have $|D_0| - |D_{02}| = |D'_0| = |D'_1| \leq \lambda_4 C_{12}(\theta)$. Since node 0 does not know the packets $D''_2^{(1)}$ and $D''_2^{(2)}$ and only node 0 needs to decode $x'_2$, the link $1 \rightarrow 0$ can be regarded as a virtual multiple-access channel (MAC) and the channel capacity is bound by $|D''_2^{(1)}| = |D'_1| \leq \lambda_4 C_{12}(\theta)$, $|D'_2| - |D'_0| = |D'_2| - |D''_2^{(1)}| = |D''_2^{(2)}| \leq \lambda_4 C_{10}(1 - \theta)$ and $|D'_2| = |D''_2^{(1)}| + |D''_2^{(2)}| \leq \lambda_4 C(\theta \gamma_{10} + (1 - \theta) \gamma_{10}) = \lambda_4 C_{10}$.

After receiving $x_1 = \sqrt{\theta} x'_0 + \sqrt{1 - \theta} x'_2$, node 2 extracts the symbol $x'_0$ as $x'_0 = x_1 - \sqrt{1 - \theta} x'_2$, where $x'_2$ is encoded by $D''_2^{(2)}$ which already has been known. By decoding $x'_0$, we get $D'_1$. Then, node 2 extracts the packet $D'_0$ as $D'_0 = D'_1 \oplus D''_2^{(1)}$. Similarly, after receiving $x_1$, node 0 extracts $x'_0$ and $x'_2$ by fully decoding. Then, $D'_1$ and $D''_2^{(2)}$ can be obtained easily. Next, node 0 extracts the packet $D''_2^{(1)}$ as $D''_2^{(1)} = D'_1 \oplus D'_0$. Note that, for each destination, say node 2, to recover the desired packet $D_0$ from $D_{02}$ and $D'_0$, a coding method for Gaussian parallel channel should be employed [22].

With the constraints obtained from the above discussion and using the definition that $R_0 =$
$|D_0|$ and $R_2 = |D_2|$, we obtain the set of linear inequalities about $R_0$ and $R_2$ after simple manipulation:

\[
R_0 \leq \min \left( (\lambda_1 + \lambda_3)C_{01}, \lambda_1 C_{02} + \lambda_4 C_{12}(\theta) \right),
\]

\[
R_2 \leq \min \left( (\lambda_2 + \lambda_3)C_{21}, \lambda_2 C_{20} + \lambda_4 C_{10} \right),
\]

\[
R_2 - R_0 \leq -\lambda_1 C_{02} + \lambda_2 C_{20} + \lambda_4 C_{10}(1 - \theta),
\]

\[
R_0 + R_2 \leq \lambda_1 C_{01} + \lambda_2 C_{21} + \lambda_3 C_m,
\]

\[
R_0 - R_2 \leq -\lambda_1 C_{02} - \lambda_2 C_{20}.
\]

In addition, due to the total time constraint, we have $\sum_{i=1}^{4} \lambda_i = 1$. Lastly, the power ratio $\theta$ can take any value that satisfies $0 \leq \theta \leq 1$.

**Case 2**: $|D'_2| \leq |D'_0|$. As in Fig. 2(e), the packet $D'_2$ is padded with zeros to obtain the packet $D''_2$ such that $|D''_2| = |D'_0|$. Since node 0 and 2 know the size of $D'_2$, they also know how many zeros are used for padding. Node 1 creates the packet $D'_1 = D''_2 \oplus D'_0$. In Step 4 the packet $D'_1$ is broadcasted at a rate at which both node 0 and node 2 can successfully decode. Thus, we have $R_0 - R_{02} = |D_0| - |D_{02}| = |D'_0| = |D'_1| \leq \lambda_4 C_{12}$. Node 2 then extracts $D'_0$ as $D'_0 = D''_2 \oplus D'_1$, which is the desired packet sent from node 0. Similarly, node 0 can obtain $D''_2$ from $D'_1$. The packet $D'_2$ is then obtained by removing the padding zeros from $D''_2$.

Thus, in this case, we obtain the following linear inequalities that the rate pair $(R_0, R_2)$ has to satisfy:

\[
R_0 \leq \min \left( (\lambda_1 + \lambda_3)C_{01}, \lambda_1 C_{02} + \lambda_4 C_{12} \right),
\]

\[
R_2 \leq (\lambda_2 + \lambda_3)C_{21},
\]

\[
R_0 + R_2 \leq \lambda_1 C_{01} + \lambda_2 C_{21} + \lambda_3 C_m,
\]

\[
R_2 - R_0 \leq -\lambda_1 C_{02} + \lambda_2 C_{20}.
\]

Finally, combining the set of results for case 2 with the results for case 1, we obtain the rate region of 4S-SuX as given in the theorem. 

**Remark**: By setting $\lambda_1 = \lambda_2 = 0$ in Theorem 1, we obtain the rate region for 2S-SuX.
Theorem 2 (2S-SuX): The rate region for 2S-SuX is the closure of the set of all rate pairs $(R_0, R_2)$ satisfying

$$(R_0, R_2) : \left\{ R_0 \leq \min (\lambda_1 C_{01}, \lambda_2 C_{12}(\theta)), \right.$$

$$R_2 \leq \min(\lambda_1 C_{21}, \lambda_2 C_{10}),$$

$$R_2 - R_0 \leq \lambda_2 C_{10}(1 - \theta),$$

$$R_0 + R_2 \leq \lambda_1 C_m,$$

$$\sum_{i=1}^{2} \lambda_i = 1, \theta \in [0, 1].$$

B. Achievable rate region for XOR

Theorem 3 (4S-XOR): The rate region for 4S-XOR is the closure of the set of all rate pair $(R_0, R_2)$ satisfying

$$(R_0, R_2) : \left\{ R_0 \leq \min ((\lambda_1 + \lambda_3) C_{01}, \lambda_1 C_{02} + \lambda_4 C_{12}), \right.$$

$$R_0 + R_2 \leq \lambda_1 C_{01} + \lambda_2 C_{21} + \lambda_3 C_m, R_2 \leq (\lambda_2 + \lambda_3) C_{21},$$

$$R_2 - R_0 \leq -\lambda_1 C_{02} + \lambda_2 C_{20} + \lambda_5 C_{10}, \sum_{i=1}^{5} \lambda_i = 1.$$

Proof: The proof of this theorem differs from the proof of Theorem 1 mainly in Case 1 of the broadcast phase. The coding and decoding method of Case 1 are similar to those discussed in [16], [17] and are omitted.

Remark: A point to note is that five time-sharing parameters are needed when the 4-step time protocol is combined with XOR. This is because the last step (broadcast phase) in the XOR scheme can be further divided into two sub-steps, if necessary, according to the discussion in Section III-A.

Remark: By setting $\lambda_1 = \lambda_2 = 0$ in Theorem 3, we obtain the rate region for 2S-XOR.

Theorem 4 (2S-XOR): The rate region for 2S-XOR is the closure of the set of all rate pair $(R_0, R_2)$ satisfying

$$(R_0, R_2) : \left\{ R_0 \leq \min (\lambda_1 C_{01}, \lambda_2 C_{12}), R_2 \leq \lambda_1 C_{21}, \right.$$

$$R_0 + R_2 \leq \lambda_1 C_m, R_2 - R_0 \leq \lambda_3 C_{10}, \sum_{i=1}^{3} \lambda_i = 1.$$

Note that the authors in [16] also studied the rate pair of 2S-XOR, but only the maximum sum-rate is considered and no rate region is discussed. In addition, the work in [16] is only suitable for reciprocal channels with $C_{01} = C_{10}$ and $C_{21} = C_{12}$.
C. Achievable rate region for superposition coding

The achievable rate region of 2S-SUP is well studied in [10], [18], [23]. Though the achievable rate region of 4S-SUP is not studied yet in the literature, its derivation is trivial.

**Theorem 5 (4S-SUP):** The rate region for 4S-SUP is the closure of the set of all rate pair \((R_0, R_2)\) satisfying

\[
\begin{align*}
R_0 &\leq \min \left( (\lambda_1 + \lambda_3)C_{01}, \lambda_1 C_{02} + \lambda_3 C_{12} (\theta) \right), \\
R_2 &\leq \min \left( (\lambda_2 + \lambda_3)C_{21}, \lambda_2 C_{20} + \lambda_4 C_{10} (1 - \theta) \right), \\
R_0 + R_2 &\leq \lambda_1 C_{01} + \lambda_2 C_{21} + \lambda_3 C_{m}, \quad \sum_{i=1}^{4} \lambda_i = 1, \theta \in [0, 1].
\end{align*}
\]

**Theorem 6 (2S-SUP):** The rate region for 2S-SUP is the closure of the set of all rate pair \((R_0, R_2)\) satisfying

\[
\begin{align*}
R_0 &\leq \min \left( \lambda_1 C_{01}, \lambda_2 C_{12} (\theta) \right), \\
R_0 + R_2 &\leq \lambda_1 C_{m}, \\
R_2 &\leq \min \left( \lambda_1 C_{21}, \lambda_2 C_{10} (1 - \theta) \right), \quad \sum_{i=1}^{2} \lambda_i = 1, \theta \in [0, 1].
\end{align*}
\]

D. Cases with direct transmission

For 4-step strategies, it is assumed by default that the direct link is always worse than the relay link. That is, \(C_{02} < C_{01}\) and \(C_{20} < C_{21}\). In this subsection, we consider the special cases where the direct link is better.

If \(C_{02} < C_{01}\) and \(C_{20} \geq C_{21}\), the signal from node 2 will be transmitted directly to node 0 without the help of relay. This corresponds to \(\lambda_3 = 0\) and the relay transmitting to node 2 only during \(\lambda_4\) in the 4-step protocol. Thus, using Theorem 1, we obtain the sets of rate pairs \((R_0, R_2)\) satisfying

\[
(R_0, R_2) : \left\{ R_0 \leq \min \left( \lambda_1 C_{01}, \lambda_2 C_{12} (\theta) \right), R_0 + R_2 \leq \lambda_1 C_{m}, R_2 \leq \lambda_3 C_{20}, \sum_{i=1}^{3} \lambda_i = 1 \right\}.
\]

If \(C_{02} \geq C_{01}\) and \(C_{20} < C_{21}\), the signal from node 0 will be transmitted directly to node 2 without the help of relay. The sets of rate pairs \((R_0, R_2)\) satisfy

\[
(R_0, R_2) : \left\{ R_2 \leq \min \left( \lambda_2 C_{21}, \lambda_2 C_{20} + \lambda_3 C_{10} (1 - \theta) \right), R_0 \leq \lambda_1 C_{01}, \sum_{i=1}^{3} \lambda_i = 1 \right\}.
\]

If \(C_{02} \geq C_{01}\) and \(C_{20} \geq C_{21}\), direct communication between node 0 and node 2 in both ways is preferred and no relay is needed. Hence, the rate pairs are given by

\[
(R_0, R_2) : \left\{ R_0 \leq \lambda_1 C_{02}, R_2 \leq \lambda_2 C_{20}, \sum_{i=1}^{2} \lambda_i = 1 \right\}.
\]
E. Numerical results

In this subsection, we present a numerical study of the proposed superimposed XOR in terms of three performance metrics: rate regions, system average sum-rates and service delay performances.

Suppose that the channel gain on each link is modeled by the distance path loss model, given by \( \alpha_{ij} = c \cdot d_{ij}^{-n} \), where \( c \) is an attenuation constant, \( n \) is the path loss exponent and fixed at 3, and \( d_{ij} \) denotes the distance between nodes \( i \) and \( j \). For simplicity, each node uses the same transmission power \( P \), though our analytical results are suitable for the case with unequal transmit power. The noise power is assumed to one. We consider the network layout shown in Fig. 3 where the distance between nodes 0 and 2 is normalized to 1 and the location of the relay is determined using the projections \( x \) and \( y \). The source nodes 0 and 2 are located at coordinates \((-0.5, 0)\) and \((0.5, 0)\), respectively. The distances from the relay to the source nodes can be computed as \( d_{01} = \sqrt{(x + 0.5)^2 + y^2} \), and \( d_{12} = \sqrt{(x - 0.5)^2 + y^2} \).

For fading channels, the same network layout and channel model, except that small-scale fading is included. We assume that the fading on each link follows Rayleigh distribution and are independent and reciprocal for different links.

1) Rate regions in Gaussian channels: Figs. 4-6 illustrate the rate regions of different two-way relay strategies. For comparison, the achievable rate region of the hybrid broadcast (HBC) protocol (4-step protocol) [14] and the capacity of the multiple-access broadcast (MABC) protocol (2-step protocol) derived in [12]-[14] are also shown and denoted by markers only in the figures. From Fig. 4 where the two relay channels are symmetrical, we see that the SuX and XOR schemes are equivalent and capacity achieving, whereas the SUP scheme is much inferior. It also shows that 4-step schemes can achieve higher one-way rate than 2-step schemes. This is expected because 4-step schemes exploit the direct link.

From Figs. 5-6 where the two relay channels are asymmetrical, it is observed that XOR is far from capacity-achieving and that SUP schemes becomes better than XOR schemes if \( R_2 > R_0 \). On the other hand, the proposed SuX schemes closely approach the capacity (or the best achievable rate region) in the high SNR regime (Fig. 6), while there is only a minor gap in the low SNR regime (Fig. 5).
2) Maximum sum-rates (MSRs) in fading channels: The problem of maximum sum-rate is formulated as

$$\max_{(R_0(t), R_2(t)) \in \mathcal{R}(t)} R_0(t) + R_2(t)$$

where $R_k(t), k \in \{0, 2\}$, denotes the service rate of node $k$ at the $t$-th time slot, and $\mathcal{R}(t)$ stands for the rate region with respect to the channel realization at the $t$-th time slot.

Figs. 7-8 show the averaged MSRs of different two-way relay strategies, when the relay node moves alone the line between the two source nodes. It is observed that no matter where the relay is, the proposed SuX scheme always achieves the largest sum-rates among all the considered PLNC schemes and approach the corresponding optimal bounds very well, especially in the high SNR regime. Moreover, we can see that all strategies except 2S-SUP achieve their average maximum sum-rate when the relay node lies in the middle. As the relay node is approaching one source node (i.e. $x$ approaches 0.5 or -0.5), the performances of the considered PLNC schemes (XOR, SUP and SuX) converge to the optimal bound.

3) Service delay in fading channels: Here, we use the same queuing mode as in [23], in which service rate allocation is done by a cross-layer approach by taking into account both queue length and channel state. Note that, Oechtering, et al. in [23] only focus on 2S-SUP strategies and consider the queue backlog versus bit arrival rate. Chen, et al. considered delay power tradeoff in [24], and assumed all the links have the same rate and only the relay has buffer. We study the bit delay versus packet arrival rate. Suppose that the packet arrival at two source nodes follow Poisson distribution with mean $\rho$, the length of each packet is fixed as $L$ bits. Let $Q(t-1) = [Q_0(t-1), Q_2(t-1)]$ represent the remaining bits in the queues after the $(t-1)$-th time slot. Then, $Q(t) = [Q_0(t), Q_2(t)] = [Q_0(t-1) - R_0(t) + A_0(t)L, Q_2(t-1) - R_2(t) + A_2(t)L]$, where $R_k(t), A_k(t)$ ($k \in \{0, 2\}$) denote the service rates and packet arrival rates of node $k$ in $t$-th time slot. The rate allocation problem is formulated as

$$\max_{(R_0(t), R_2(t)) \in \mathcal{R}(t) \atop R_0(t) \leq Q_0(t-1), R_2(t) \leq Q_2(t-1)} Q_0(t-1)R_0(t) + Q_2(t-1)R_2(t)$$

Fig. 9 shows the system delay based on the above weighted sum-rate maximization with $L = 10$. It can be clearly seen that the proposed SuX scheme always outperforms the other two PLNC schemes and approach the corresponding best achievable bound, no matter which time control protocol is applied.
V. Analysis of Performance Gap

The numerical results in the previous section demonstrate that the rate region achieved by proposed superimposed XOR closely approach the capacity bound. In this section, we shall analytically quantify the performance gap and show that it indeed approaches zero when SNR is large.

Note that the proposed scheme only concerns the information processing at the relay and destination during the broadcast phase, no matter which time protocol is adopted. Hence, it suffices to analyze the gap on the broadcast rate region. In what follows we first deduce the boundary of the achievable broadcast rate region in an analytical expression. Then we characterize an upper bound on the capacity gap.

Let $R_{10}$ and $R_{12}$ denote the data rates of the information flows $1 \rightarrow 0$ and $1 \rightarrow 2$, respectively, in the broadcast phase. From the proof of Theorem 1, it is seen clearly that, for any given power allocation parameter $\theta$, the rate pair $(R_{10}, R_{12})$ must satisfy the following linear inequalities:

$$
R_{10} \leq C_{10},
$$
$$
R_{12} \leq C_{12}(\theta),
$$
$$
R_{10} - R_{12} \leq C_{10}(1 - \theta),
$$

for $R_{10} \geq R_{12}$, and satisfy

$$
R_{12} \leq C_{12}
$$

for $R_{10} \leq R_{12}$. Graphing the feasible set and considering that $R_{10} \geq 0$ and $R_{12} \geq 0$, we obtain the rate region as sketched in Fig. 10. Here, we have

$$
\theta' = \left[1 + \frac{1}{\gamma_{10}} - \frac{1}{\gamma_{12}}\right]^+, \tag{5}
$$

with $[x]^+ = x$ if $x \geq 0$ and $[x]^+ = 0$ otherwise. This power control value satisfies $C_{10} = C_{12}(\theta') + C_{10}(1 - \theta')$.

When $0 \leq \theta \leq \theta'$ or equivalently $C_{10} \leq C_{12}(\theta) + C_{10}(1 - \theta)$, the rate region is plotted in Fig. 10(a), with $A = (0, C_{10}(1 - \theta))$, $B = (C_{10} - C_{10}(1 - \theta), C_{10})$, $C = (C_{12}(\theta), C_{10})$ and $D = (C_{12}, C_{12})$. On the other hand, when $\theta' \leq \theta \leq 1$, or equivalently, $C_{10} \geq C_{12}(\theta) + C_{10}(1 - \theta)$, the rate region is in Fig. 10(b), where $A = (0, C_{10}(1 - \theta))$, $B = (C_{12}(\theta), C_{12}(\theta) + C_{10}(1 - \theta))$ and $D = (C_{12}, C_{12})$. 
Then, by varying $\theta$ from 0 to 1, the overall broadcast rate region is obtained as the union of the above feasible sets, which is given in Fig. 10(c). The coordinates of the boundary in the segment $\tilde{BC}$ can be characterized as:

\[
B : (C_{12}(\theta'), C_{10})
\]

\[
C : (C_{12}, C_{12})
\]

\[
\tilde{BC} : (C_{12}(\theta), C_{12}(\theta) + C_{10}(1 - \theta)), \forall \theta \in [\theta', 1].
\]

In Fig. 10(c), the capacity bound of two-way relaying in the broadcast phase is also shown for comparison. It is given by the rectangle characterized by $R_{10} \leq C_{10}$ and $R_{12} \leq C_{12}$ [12].

We now define the gap between the rate region of the superimposed XOR and the capacity bound as the area of the shadowed region as depicted in Fig. 10(c) ($\tilde{BC}, CE$ and $EB$), denoted as $\Delta$. Thus,

\[
\Delta = \int_{\theta'}^{1} \left\{ C_{10} - \left[ C_{12}(\theta) + C_{10}(1 - \theta) \right] \right\} dC_{12}(\theta)
\]

\[
= \int_{\theta'}^{1} \frac{\gamma_{12}}{(1 + \theta \gamma_{12}) \ln 2} \frac{1 + \gamma_{10}}{1 + \gamma_{12}(1 + (1 - \theta) \gamma_{10})} d\theta
\]

However, computing the exact value of $\Delta$ is involved. It can be verified easily that the gap is upper-bounded by the area of the triangular formed by $BC$, $CE$ and $EB$. Namely,

\[
\Delta \leq \frac{1}{2} (C_{10} - C_{12}) \left[ C_{12} - C_{12}(\theta') \right]
\]

\[
= \frac{1}{2} \log_2 \frac{1 + \gamma_{10}}{1 + \gamma_{12}} \frac{\gamma_{10}(1 + \gamma_{12})}{\gamma_{12}(1 + \gamma_{10})}
\]

(6)

(7)

It is seen that when both $\gamma_{10}$ and $\gamma_{12}$ are much larger than one ($\gamma_{10} \geq \gamma_{12} \gg 1$), then the term $\log_2(\gamma_{10}(1 + \gamma_{12})/(\gamma_{12}(1 + \gamma_{10})))$ approaches zero. Therefore, the upper bound approaches zero. Finally, we conclude that the proposed scheme is capacity approaching for larger SNR.

In the following we show some numerical examples for further illustration. Fig. 11 demonstrates the rate regions of superimposed XOR for different $\theta$ as well as the overall broadcast rate region by letting $\theta$ take all possible values in $[0, 1]$. In this example, we have $\gamma_{10} = 13.17$ dB and $\gamma_{12} = 5.55$ dB. The corresponding $\theta'$ is 0.77.

Fig. 12 shows the broadcast rate regions of superimposed XOR compared to the capacity bound and the rate regions obtained by conventional XOR and superposition at different SNR.
It can be seen that the proposed superimposed XOR outperforms the other two schemes, and the gap to the capacity bound vanishes as SNR increases.

VI. CONCLUSION

In this research, we proposed superimposed XOR, a novel PLNC scheme for two-way relay communications. It takes into consideration the asymmetry in both channel gain and bidirectional information length during the broadcast phase. In specific, when the receiver channel quality of the longer packet is worse than that of the shorter packet, it reduces to the conventional XOR. Otherwise, it combines the extra information bits from the longer packet with the XORed bits using superposition coding. We characterized its achievable rate region over the Gaussian channel when applied together with the 4-step and 2-step transmission protocols. We also demonstrate its average maximum sum-rate and service delay performance over fading channels. Numerical results showed that proposed PLNC scheme achieves a larger rate region than XOR and superposition when the broadcast channels are asymmetric and performs much better over fading channels. Moreover, at the high SNR region, it approaches the capacity bound. We also explicitly proved this capacity approaching behavior by deriving the analytical expressions of the boundary of the broadcast rate region.

REFERENCES

[1] J. Liu, M. Tao, Y. Xu, and X. Wang, “Superimposed xor: A new physical layer network coding scheme for two-way relay channels,” in Proc. IEEE Global Telecomm. Conf. (GLOBECOM), Nov. 2009.
[2] E. C. V. D. Meulen, “Three-terminal communication channels,” in Advances in Applied Probability, vol. 3. Applied Probability Trust, 1971, pp. 120–154.
[3] T. M. Cover and A. E. Gamal, “Capacity theorems for the relay channel,” IEEE Trans. Inform. Theory, vol. 25, no. 5, pp. 572–584, Sept. 1979.
[4] Y. Wu, P. A. Chou, and S.-Y. Kung, “Information exchange in wireless networks with network coding and physical-layer broadcast,” in Proc. 39th Conf. Inform. Sciences Systems (CISS), Mar. 2005.
[5] P. Larsson, N. Johansson, and K.-E. Sunell, “Coded bi-directional relaying,” in Proc. 5th Scandinavian Workshop Ad Hoc Networks (ADHOC), May 2005.
[6] P. Popovski and H. Yomo, “Bi-directional amplification of throughput in a wireless multi-hop network,” in Proc. IEEE Veh. Tech. Conf. (VTC), May 2006, pp. 588–593.
[7] C. Hausl and J. Hagenauer, “Iterative network and channel decoding for the two-way relay channel,” in Proc. IEEE Int. Conf. Comm. (ICC), Jun. 2006, pp. 1568–1573.
[8] S. Zhang, S. C. Liew, and P. P. Lam, “Physical-layer network coding,” in Proc. ACM 15th Annual Int. Conf. Mobile Computing Networking (MobiCom), Sept. 2006, pp. 358–365.
[9] S. Katti, H. Rahul, W. Hu, D. Katabi, M. M. edard, and J. Crowcroft, “XORs in the air: Practical wireless network coding,” in Proc. ACM SIGCOMM, Sept. 2006, pp. 243–254.

[10] B. Rankov and A. Wittneben, “Spectral efficient protocols for half-duplex fading relay channels,” IEEE Jour. Selec. Areas. Comm., vol. 25, no. 2, pp. 379–389, Feb. 2007.

[11] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, “Network information flow,” IEEE Trans. Inform. Theory, vol. 46, no. 4, pp. 1204–1216, July 2000.

[12] T. J. Oechtering, C. Schnurr, I. Bjelakovic, and H. Boche, “Broadcast capacity region of two-phase bidirectional relaying,” IEEE Trans. Inform. Theory, vol. 54, no. 1, pp. 454–458, Jan. 2008.

[13] L. L. Xie, “Network coding and random binning for multi-user channels,” in Proc. 10th Canadian Workshop Inf. Theory, June 2007, pp. 85–88.

[14] S. J. Kim, P. Mitran, and V. Tarokh, “Performance bounds for bi-directional coded cooperation protocols,” IEEE Trans. Inform. Theory, vol. 54, no. 11, pp. 5235–5241, Nov. 2008.

[15] S. J. Kim, N. Devroye, P. Mitran, and V. Tarokh, “Comparison of bi-directional relaying protocols,” in Proc. IEEE Sarnoff Symposium, April 2008, pp. 1–5.

[16] P. Popovski and H. Yomo, “Physical network coding in two-way wireless relay channels,” in Proc. IEEE Int. Conf. Comm. (ICC), June 2007, pp. 707–712.

[17] C. H. Liu and F. Xue, “Network coding for two-way relaying: rate region, sum rate and opportunistic scheduling,” in Proc. IEEE Int. Conf. Comm. (ICC), May 2008, pp. 1044–1049.

[18] T. J. Oechtering and H. Boche, “Optimal time-division for bi-directional relaying using superposition encoding,” IEEE Comm. Letters, vol. 12, no. 4, pp. 265–267, April 2008.

[19] J. Liu, M. Tao, and Y. Xu, “Rate regions of a two-way gaussian relay channel,” in Proc. Int. Conf. Comm. Networking China (ChinaCom), Aug. 2009.

[20] C. H. Liu and A. Arapostathis, “Joint network coding and superposition coding for multi-user information exchange in wireless relaying networks,” in Proc. IEEE Global Telecomm. Conf. (GLOBECOM), Nov. 2008, pp. 1–6.

[21] W. Chen, K. B. Letaief, and Z. Cao, “A cross layer method for interference cancellation and network coding in wireless networks,” in Proc. IEEE Int. Conf. Comm. (ICC), May 2006, pp. 3693–3698.

[22] A. H.-Madsen, “Capacity bounds for cooperative diversity,” IEEE Trans. Inform. Theory, vol. 52, no. 4, pp. 1522–1544, April 2006.

[23] T. J. Oechtering and H. Boche, “Stability region of an optimized bi-directional regenerative half-duplex relaying protocol,” IEEE Trans. Comm., vol. 56, no. 9, pp. 1519–1529, Sep. 2008.

[24] W. Chen, K. B. Letaief, and Z. Cao, “Opportunistic network coding for wireless networks,” in Proc. IEEE Int. Conf. Comm. (ICC), 2007, pp. 4634–4639.
Fig. 1: Three kinds of PLNC schemes for broadcast phase: (a) XOR, (b) Superposition, (c) Superimposed XOR.
1st step: 

\[ D_0^{(1)} \rightarrow D_1 \rightarrow D_2 \]

2nd step: 

\[ \rightarrow D_2^{(2)} \rightarrow D_1^{(2)} \rightarrow \]

3rd step: 

\[ D_0^{(3)} \rightarrow D_1^{(3)} \rightarrow D_2^{(3)} \]

4th step: 

\[ \text{Case 1: } |D_2'| = |D_2| - |D_{20}| \geq |D_0'| = |D_0| - |D_{02}| \]

\[ D_0' \rightarrow D_1' \rightarrow x_0' \rightarrow \sqrt{\theta} \]

\[ D_2' \rightarrow D_2^{(1)} \rightarrow D_2^{(2)} \rightarrow x_2' \rightarrow 1 - \theta \]

\[ \text{Case 2: } |D_2'| = |D_2| - |D_{20}| \leq |D_0'| = |D_0| - |D_{02}| \]

\[ \text{Fig. 2: Process flow of the 4s-Sux.} \]
Fig. 3: Layout of the two-way relaying network

Fig. 4: Rate region at \((x, y) = (0, 0)\) with \(P = 0\)dB.
Fig. 5: Rate region at \((x, y) = (-0.2, 0.3)\) with \(P = 0\) dB.

Fig. 6: Rate region at \((x, y) = (-0.2, 0.3)\) with \(P = 10\) dB.
Fig. 7: Averaged maximum sum-rates versus relay location when $-0.5 \leq x \leq 0.5, y = 0$ with $P = 0$ dB.

Fig. 8: Averaged maximum sum-rates versus relay location when $-0.5 \leq x \leq 0.5, y = 0$ with $P = 10$ dB.
Fig. 9: System service delay versus packet arrival rate: \((x, y) = (0, 0), P = 10\text{dB}\).

Fig. 10: Rate region of superimposed XOR: shadowed area in (a) or (b) is the rate region at a fixed \(\theta\); solid lines and dashed lines in (c) are the boundary of superimposed XOR and capacity bound, respectively.
Fig. 11: Rate regions of superimposed XOR at fixed $\theta$ and the boundary for $\theta \in [0, 1]$ in broadcast phase:

$$(x, y) = (-0.2, 0.3), \quad P = 2\text{dB}.$$
Fig. 12: Comparisons of rate regions between the superimposed XOR and capacity bound in broadcast phase:

\[(x, y) = (-0.2, 0.3), \quad P = \{0, 5, 10\} \text{ dB.}\]
Superimposed XOR: Approaching Capacity Bounds of the Two-Way Relay Channels

Jianquan Liu, Meixia Tao, Youyun Xu, and Xiaodong Wang

Abstract

In two-way relay channels, bitwise XOR and symbol-level superposition coding are two popular network-coding based relaying schemes. However, neither of them can approach the capacity bound when the channels in the broadcast phase are asymmetric. In this paper, we present a new physical layer network coding (PLNC) scheme, called superimposed XOR. The new scheme advances the existing schemes by specifically taking into account the channel asymmetry as well as information asymmetry in the broadcast phase. We obtain its achievable rate regions over Gaussian channels when integrated with two known time control protocols in two-way relaying. We also demonstrate their average maximum sum-rates and service delay performances over fading channels. Numerical results show that the proposed superimposed XOR achieves a larger rate region than both XOR and superposition and performs much better over fading channels. We further deduce the boundary of its achievable rate region of the broadcast phase in an explicit and analytical expression. Based on these results, we then show that the gap to the capacity bound approaches zero at high signal-to-noise ratio.

Index Terms

Two-way relaying, capacity bound, physical layer network coding, bitwise XOR, superposition coding.

This work was partly presented in IEEE GLOBECOM 2009 [?].

Jianquan Liu, Meixia Tao and Youyun Xu are with the Department of Electronic Engineering, Shanghai Jiao Tong University, Shanghai, 200240, P. R. China (e-mails: {jianquanliu, mxtao, xuyouyun}@sjtu.edu.cn).

Xiaodong Wang is with the Department of Electrical Engineering, Columbia University, New York, NY 10027, USA (e-mail: wangx@ee.columbia.edu).
I. INTRODUCTION

Cooperative communications enables different users in a wireless network to share their antennas and cooperate in signal transmission at the physical layer. This opens up the possibilities of exploiting distributed spatial diversity and hence effectively enhancing the system performance. Cooperative communications has thus attracted significant amount of interests from both academia and industry with applications in ad-hoc as well as infrastructure-based network. A basic building block of a cooperative network is the relay channel, first proposed by van der Meulen [17] and then extensively studied from information theoretic perspectives by Cover and Gamal [16]. The classic relay channel consists of three nodes, wherein a source node communicates with a destination node with the help of a relay node. Thus far, a number of relay schemes have been proposed. Among them, three popular strategies are known as amplify-and-forward (AF), decode-and-forward (DF) and compress-and-forward (CF) respectively. However, due to the half-duplex constraint arising from practical considerations, these traditional relay schemes suffer from loss in spectral efficiency.

Two-way relaying, where two source nodes exchange information with the help of a relay node, has recently gained a lot of research interests [29], [30], [31], [32], [33], [34]. It is shown able to overcome the half-duplex constraint and significantly improve the system spectral efficiency in relay-based cooperative networks. Upon receiving the bidirectional information flows, the relay node combines them together and then broadcasts to the two desired destinations. The operation at the relay resembles network coding [25], a technique originally developed for wireline networks. It is thus often referred to as physical layer network coding (PLNC) [28] or analog network coding (ANC) [26].

Researchers have attempted to find the best achievable rate region of two-way relay channels [29], [30], [31]. Oechtering, et al. obtained the capacity region of the broadcast phase in terms of the maximal probability of error [29]. The achievable rate region of a two-way relay channel considering both the multiple-access phase (i.e. the two source nodes transmit simultaneously to the relay node) and the broadcast phase was studied by Xie using random binning [33]. Kim, et al. [31], [32] further broadened the frontier of the achievable rate region by allowing time share between different transmission phases. In particular, the capacity region of a two-way relay channel with two-step time protocol is now well-known to be the intersection of the optimal
time-weighted capacity regions of the multiple-access phase and the broadcast phase. Note that
the above capacity analysis all assumed full decoding at the relay. Moreover, the information
theoretic techniques including random binning and jointly typical set decoding are often adopted
at the relay and destinations in the broadcast phase [?], [?], [?].

Meanwhile, a number of practical PLNC schemes for two-way relay channels have also been
proposed and analyzed, such as bit-level XOR [?], [?], symbol-level superposition coding [?], [?]
and AF [?], [?]. In particular, authors in [?] obtained the rate pair expressions for superposition
based relaying. Authors in [?] analyzed the maximum achievable sum-rate for XOR based
relaying with optimal time control. It is worth mentioning that the asymmetry in both packet size
and channel gain of the two transmitting nodes are considered in [?]. Oechtering also studied
the optimal time control for superposition scheme [?]. Despite all these attempts, there is still
a large gap between the rate regions achieved by the practical PLNC schemes and the capacity
bound in asymmetric relay channels, as shown by a recent comparative study in [?].

In this paper, we study practical and capacity approaching PLNC schemes over the two-way
relay channels. In this regard, we propose a novel PLNC scheme, named as superimposed XOR,
tailored for the broadcast phase of two-way relaying with asymmetric channels. Combining it
with two known transmission protocols: 4-step with direct link [?] and 2-step with no direct
link, we analyze its achievable rate regions over Gaussian channels. We also demonstrate their
average maximum sum-rates and service delay performances over fading channels. Numerical
results show that the performance of the proposed superimposed XOR outperforms the traditional
XOR and superposition in terms of achievable rate region. It also closely approaches the optimal
capacity bounds of the two-way relay channels in the high signal-to-noise ratio (SNR) regime.
To further illustrate the capacity approaching behavior of the proposed superimposed XOR, we
obtain the analytical expressions of the boundary of its achievable broadcast rate region. Based
on these results, we then explicitly prove that its gap with the capacity of the broadcast phase
approaches zero when the SNR is much larger than one.

The rest of the paper is organized as follows. In Section II, we present the system model of
two-way relaying. In Section III, we describe the proposed superimposed XOR scheme. Section
IV characterizes the rate regions over Gaussian channels. The capacity approaching performance

\footnote{The capacity region of two-way relaying with partial decoding still remains open.}
is analyzed in Section V. Finally, we conclude the paper in section VI.

II. SYSTEM MODEL

We consider a two-way relay channel which consists of two source nodes and one relay node. The source nodes, denoted as 0 and 2, wish to exchange information with the help of the relay node, denoted as 1. We assume that all the nodes operate in the half-duplex mode. The channel on each communication link is assumed to be corrupted with Rayleigh fading and additive white Gaussian noise. The SNR of the link from node $i$ to node $j$ is denoted as $\gamma_{ij}$, for $i, j \in \{0, 1, 2\}$, and it counts both channel gain and transmit power. Note that $\gamma_{ij}$ may not be equal to $\gamma_{ji}$ as the channels considered here may not be reciprocal. The channel capacity in bit/s/Hz of the link from node $i$ to node $j$ is denoted as $C_{ij}$, and determined by the SNR on the link as

$$C_{ij} \triangleq C(\gamma_{ij}) = \log_2(1 + \gamma_{ij}).$$

(1)

Throughout this paper, we assume that the relay needs to fully decode the information of the two source nodes. The sum-rate capacity of the multiple access channel (MAC) when nodes 0 and 2 are transmitting simultaneously to node 1 is denoted as

$$C_m \triangleq C(\gamma_{01} + \gamma_{21}) = \log_2(1 + \gamma_{01} + \gamma_{21}).$$

(2)

A. Time control protocols

Two-way relaying involves not only PLNC at the relay node but also time control for node transmission. In this subsection, we review some existing time control protocols. Similar to [?], we name the protocols based on the number of time steps to finish one round of information exchange between the source nodes. We focus on the 4-step and 2-step protocols in this paper.

1) 4-step protocol: In the first step, node 0 transmits for $\lambda_1$ time duration and node 1 and 2 listen. In the second step, node 2 transmits for $\lambda_2$ time duration and node 1 and 0 listen. In the third step, nodes 0 and 2 transmit simultaneously for $\lambda_3$ time duration while node 1 listens. In the fourth step, node 1 transmits for $\lambda_4$ time duration and node 0 and 2 listen. Without loss of generality, the total time duration is normalized to one, i.e., $\sum_{i=1}^{4} \lambda_i = 1$.

2) 2-step protocol: In 2-step protocol, nodes 0 and 2 first transmit simultaneously for $\lambda_1$ time duration while node 1 listens. Next, node 1 transmits for $\lambda_2$ time duration and nodes 0 and 2 listen. This protocol can be regarded as a special case of the 4-step protocol by letting $\lambda_1 = \lambda_2 = 0$. No direct link is exploited here.
III. SUPERIMPOSED XOR

Upon decoding the two bit sequences $b_0$ and $b_2$ from the source nodes 0 and 2 to be exchanged, the relay node will perform physical layer network coding on $b_0$ and $b_2$ and then broadcast to the two destinations. Before introducing the proposed PLNC scheme, we briefly discuss the bit-level XOR and symbol-level superposition coding which the proposed scheme is based upon.

A. Bit-level XOR

The relay node performs bitwise XOR on the two bit sequences as $b_0 \oplus b_2$. In the case where the lengths of the two sequences are not equal, there are two methods to perform the XOR, as shown in Fig. 1(a). The first one is to pad the shorter sequence with zero bits so as to make it having the same length as the longer sequence and then perform XOR. In the second method, the longer sequence will be partitioned into two sub-sequences, with one having the same length as the short sequence. XOR is then performed on the shorter sequence and the sub-sequence with equal length. The resulting bit sequence is broadcasted to both receivers. The other sub-sequence will be transmitted alone, at a possibly higher rate, to its desired receiver. In practice, which method to use depends on the relationship of the channel gains in the broadcast phase of two-way relaying [?]. In general, if the channel quality of the receiver of the longer sequence is worse than the channel quality of the receiver of the shorter sequence, zero-padding is applied. Otherwise, partitioning the longer packet is preferred.

B. Symbol-level superposition

The relay encodes the two bit sequences $b_0$ and $b_2$ separately into baseband signal sequences $x_0$ and $x_2$ with the same length, and then superimposes them together as $\sqrt{\theta}x_0 + \sqrt{1-\theta}x_2$, as shown in Fig. 1(b). Here $\theta$ is a power allocation coefficient [?]. The signal $\sqrt{\theta}x_0 + \sqrt{1-\theta}x_2$ is then broadcasted directly without further encoding. Unlike bit-level XOR, there is no need to consider the issue of asymmetry in bit length. Another essential difference between superposition and XOR is that, the information combining is carried out in the symbol level after channel coding and modulation for the former, while it is in the information bit level before channel coding for the latter.
C. Proposed superimposed XOR

As discussed in [?], for symmetrical broadcast channels \((\gamma_{10} = \gamma_{12})\), bit-level XOR is capacity-achieving whereas superposition coding is suboptimal. But for asymmetrical channels \((\gamma_{10} \neq \gamma_{12})\), bit-level XOR becomes inferior to superposition at certain rate regions. The proposed superimposed XOR is specifically designed for asymmetrical broadcast channels and it utilizes the advantages of both XOR and superposition schemes. The details are as follows.

If the lengths of two bit sequences \(b_0\) and \(b_2\) are equal, then the conventional bitwise XOR is performed. Otherwise, there are two methods to process the two sequences, as shown in Fig. (c). The first method is the same as the one in bit-level XOR scheme, that is, padding the shorter sequence, say \(b_0\), with zero bits, so as to make them equal and then performing bit-level XOR. The resulting bit sequence, after channel coding and modulation, is then broadcasted to both receivers. In the second method, the relay node first partitions the longer sequence, say \(b_2\), into two sub-sequences as \(b_2 = [b_2', b_2'']\), where the sub-sequence \(b_2'\) has the same length as \(b_0\). It then encodes the XORed sub-sequence \(b_0 \oplus b_2'\) and the sub-sequence \(b_2''\), separately. We denote the resulting coded symbol sequences as \(x_0\) and \(x_2\). Finally, the relay superimposes them together as \(\sqrt{\theta}x_0 + \sqrt{1-\theta}x_2\) which is broadcasted directly to the two destinations. Here, \(\theta\) is also a power allocation coefficient, \(x_0\) is to be received by both destinations, and \(x_2\) is to be received by one destination only. Which of the above two methods to use depends on the relationship between \(\gamma_{10}\) and \(\gamma_{12}\) in the broadcast channels. Suppose that the longer sequence is \(b_2\) which is to be sent to node 0. Then, if \(\gamma_{10} < \gamma_{12}\), we apply method one and else apply method two. We shall discuss this in more detail in the proof of the rate regions in the next section.

Note that Liu, et al. proposed a joint network coding and superposition coding (JNSC) scheme for information exchange among more than two users in a wireless relay network [?]. Therein, two XOR-ed packets generated by information from three nodes are superimposed. Our proposed superimposed XOR scheme differs from the JNSC [?] in that our scheme performs superposition on only one XOR-ed packet and the sub-packet obtained by partitioning the longer bit sequence. A similar PLNC scheme as the second method of our proposed superimposed XOR was proposed by Chen, et al. for multi-hop wireless networks in [?]. The difference however is that our scheme adaptively selects the aforementioned two methods illustrated in Fig. (c) according to the channel conditions in the broadcast phase while the scheme in [?] is fixed at method two.
regardless the channel conditions. Such static approach could be far from capacity-achieving as it does not exploit the channel dynamics.

In the rest of the paper, for ease of presentation, if the considered three PLNC schemes (XOR, superposition, and superimposed XOR) are combined with the 4-step time control protocol, we denote them as 4S-XOR, 4S-SUP and 4S-SuX, respectively. Likewise, when combined with 2-step protocol, they are named as 2S-XOR, 2S-SUP and 2S-SuX.

IV. ANALYSIS OF RATE REGIONS

Let $R_0$ and $R_2$ denote the data rates of the information flows $0 \rightarrow 2$ and $2 \rightarrow 0$, respectively, in the considered two-way communications. In this section, we derive the rate region $(R_0, R_2)$ of the aforementioned relay strategies. Without loss of generality, we assume $\gamma_{10} \geq \gamma_{12}$ and thus $C_{10} \geq C_{12}$ in Subsections IV-A and B.

A. Achievable rate region for superimposed XOR

**Theorem 1 (4S-SuX):** The rate region for 4S-SuX is the closure of the set of all rate pairs $(R_0, R_2)$ satisfying

$$(R_0, R_2): \left\{ R_0 \leq \min \left( (\lambda_1 + \lambda_3)C_{01}, \lambda_1 C_{02} + \lambda_4 C_{12}(\theta) \right), 
R_2 \leq \min \left( (\lambda_2 + \lambda_3)C_{21}, \lambda_2 C_{20} + \lambda_4 C_{10} \right), 
R_2 - R_0 \leq -\lambda_1 C_{02} + \lambda_2 C_{20} + \lambda_4 C_{10}(1 - \theta), 
R_0 + R_2 \leq \lambda_1 C_{01} + \lambda_2 C_{21} + \lambda_3 C_m, 
\sum_{i=1}^{4} \lambda_i = 1, \theta \in [0, 1] \right\}.$$

Here $C_{ij}(\theta) \triangleq C(\gamma_{ij}\theta) = \log_2(1 + \gamma_{ij}\theta)$.

**Proof:** For ease of comprehension, Fig. 2 is presented to assist the proof. Let $D_0$ denote the information packet to be transmitted from node 0 to 2 and its packet length in bits be denoted as $|D_0|$. Assume the message in the packet is further split into two parts, denoted as $D_0^{(1)}$ and $D_0^{(3)}$, which are transmitted in the first and third step, respectively, as illustrated in Fig. 2 (a) and (c). Likewise, we let $D_2$ denote the information packet to be transmitted from node 2 to 0, and let it be split into $D_2^{(2)}$ and $D_2^{(3)}$ for transmission in the second and third steps, as depicted in Fig. 2 (b) and (c). During the first three steps of packet transmission (i.e. the multiple-access phase), it is obvious that $|D_0^{(1)}| \leq \lambda_1 C_{01}$, $|D_0^{(2)}| \leq \lambda_2 C_{21}$, $|D_0^{(3)}| \leq \lambda_3 C_{01}$, $|D_2^{(2)}| \leq \lambda_3 C_{21}$, and...
$|D_0^{(3)}| + |D_2^{(3)}| \leq \lambda_3 C_m$. Note that in the first step, due to the presence of direct link between node 0 and node 2, the desired destination node 2 is able to exact $|D_{02}| \leq \lambda_1 C_{02}$ amount of information. Thus, the total amount of information bits to be transmitted through the relay link in the fourth step to node 2 is $|D_0| - |D_{02}|$ and we denote the corresponding packet as $D'_0$. Similarly, the total amount of information to be relayed from node 2 to node 0 in the fourth step is $|D_2| - |D_{20}|$, with $|D_{20}| \leq \lambda_2 C_{20}$, and the corresponding packet can be denoted as $D'_2$.

Then, during the fourth step transmission (i.e. the broadcast phase), by comparing the packet sizes $|D'_0|$ and $|D'_2|$, two cases need to be considered.

**Case 1:** $|D'_2| \geq |D'_0|$. As shown in Fig. 2 (d), the relay node 1 partitions the packet $D'_2$ into $D_2^{(1)}$ and $D_2^{(2)}$ so that $|D_2^{(1)}| = |D'_0|$ and $|D_2^{(2)}| = |D'_2| - |D'_0|$. Packet $D_2^{(1)}$ contains the first $|D'_0|$ bits from $D_2'$ and packet $D_2^{(2)}$ contains the rest of the bits from $D_2'$. Now, node 1 creates $D'_1 = D_2^{(1)} \oplus D'_0$. Then, the information bits $D'_1$ and $D_2^{(2)}$ are encoded separately into two codewords $x'_0$ and $x'_2$ with the same length, which are then superimposed together in the complex field. Unlike XOR-based scheme, there is no extra time used to transmit $D_2^{(2)}$ to node 0. Let $\theta$ present the power ratio allocated to the signal $x'_0$ to be transmitted to nodes 0 and 2 and $1 - \theta$ be the power ratio on the signal $x'_2$ to node 0, where $0 \leq \theta \leq 1$. Since $D'_2$ is known at node 2 and so is $x'_2$, to successfully decode the packet $D'_1$ at node 2, the transmission rate of $x'_0$ in the fourth step with a fraction $\lambda_4$ of time cannot exceed $C_{12}(\theta) = C(\theta \gamma_{12})$. Thus, we have $|D_0| - |D_{02}| = |D'_0| = |D'_1| \leq \lambda_4 C_{12}(\theta)$. Since node 0 does not know the packets $D_2^{(1)}$ and $D_2^{(2)}$ and only node 0 needs to decode $x'_2$, the link $1 \rightarrow 0$ can be regarded as a virtual multiple-access channel (MAC) and the channel capacity is bound by $|D_2^{(1)}| = |D'_1| \leq \lambda_4 C_{12}(\theta)$, $|D_2'| - |D'_0| = |D'_2| - |D_2^{(1)}| = |D_2^{(2)}| \leq \lambda_4 C_{10}(1 - \theta)$ and $|D_2'| = |D_2^{(1)}| + |D_2^{(2)}| \leq \lambda_4 C(\theta \gamma_{12} + (1 - \theta) \gamma_{10}) = \lambda_4 C_{10}$.

After receiving $x_1 = \sqrt{\theta}x'_0 + \sqrt{1 - \theta}x'_2$, node 2 extracts the symbol $x'_1$ as $x'_0 = x_1 - \sqrt{1 - \theta}x'_2$, where $x'_2$ is encoded by $D_2^{(2)}$ which already has been known. By decoding $x'_0$, we get $D'_1$. Then, node 2 extracts the packet $D'_0$ as $D'_0 = D'_1 \oplus D_2^{(1)}$. Similarly, after receiving $x_1$, node 0 extracts $x'_2$ and $x'_0$ by fully decoding. Then, $D'_1$ and $D_2^{(2)}$ can be obtained easily. Next, node 0 extracts the packet $D_2^{(1)}$ as $D_2^{(1)} = D'_1 \oplus D'_0$. Note that, for each destination, say node 2, to recover the desired packet $D_0$ from $D_{02}$ and $D'_0$, a coding method for Gaussian parallel channel should be employed [?].

With the constraints obtained from the above discussion and using the definition that $R_0 =
$|D_0|$ and $R_2 = |D_2|$, we obtain the set of linear inequalities about $R_0$ and $R_2$ after simple manipulation:

$$R_0 \leq \min \left( (\lambda_1 + \lambda_3)C_{01}, \lambda_1 C_{02} + \lambda_4 C_{12}(\theta) \right),$$

$$R_2 \leq \min \left( (\lambda_2 + \lambda_3)C_{21}, \lambda_2 C_{20} + \lambda_4 C_{10} \right),$$

$$R_2 - R_0 \leq -\lambda_1 C_{02} + \lambda_2 C_{20} + \lambda_4 (1 - \theta),$$

$$R_0 + R_2 \leq \lambda_1 C_{01} + \lambda_2 C_{21} + \lambda_3 C_m,$$

$$R_0 - R_2 \leq \lambda_1 C_{02} - \lambda_2 C_{20}.$$

In addition, due to the total time constraint, we have $\sum_{i=1}^{4} \lambda_i = 1$. Lastly, the power ratio $\theta$ can take any value that satisfies $0 \leq \theta \leq 1$.

**Case 2:** $|D'_2| \leq |D'_0|$. As in Fig. 2(e), the packet $D'_2$ is padded with zeros to obtain the packet $D''_2$ such that $|D''_2| = |D'_0|$. Since node 0 and 2 know the size of $D'_2$, they also know how many zeros are used for padding. Node 1 creates the packet $D'_1 = D''_2 \oplus D'_0$. In Step 4 the packet $D'_1$ is broadcasted at a rate at which both node 0 and node 2 can successfully decode. Thus, we have $R_0 - R_{02} = |D_0| - |D_{02}| = |D'_0| = |D'_1| \leq \lambda_4 C_{12}$. Node 2 then extracts $D'_0$ as $D'_0 = D''_2 \oplus D'_1$, which is the desired packet sent from node 0. Similarly, node 0 can obtain $D''_2$ from $D'_1$. The packet $D'_2$ is then obtained by removing the padding zeros from $D''_2$.

Thus, in this case, we obtain the following linear inequalities that the rate pair $(R_0, R_2)$ has to satisfy:

$$R_0 \leq \min \left( (\lambda_1 + \lambda_3)C_{01}, \lambda_1 C_{02} + \lambda_4 C_{12} \right),$$

$$R_2 \leq (\lambda_2 + \lambda_3)C_{21},$$

$$R_0 + R_2 \leq \lambda_1 C_{01} + \lambda_2 C_{21} + \lambda_3 C_m,$$

$$R_2 - R_0 \leq -\lambda_1 C_{02} + \lambda_2 C_{20}.$$

Finally, combining the set of results for case 2 with the results for case 1, we obtain the rate region of 4S-SuX as given in the theorem.  

**Remark:** By setting $\lambda_1 = \lambda_2 = 0$ in Theorem 1, we obtain the rate region for 2S-SuX.
**Theorem 2 (2S-SuX):** The rate region for 2S-SuX is the closure of the set of all rate pairs $(R_0, R_2)$ satisfying

$$(R_0, R_2) : \left \{ R_0 \leq \min (\lambda_1 C_{01}, \lambda_2 C_{12}(\theta)) , \right.
\left. R_2 \leq \min(\lambda_1 C_{21}, \lambda_2 C_{10}) , \right.
\left. R_2 - R_0 \leq \lambda_2 C_{10}(1 - \theta) , \right.
\left. R_0 + R_2 \leq \lambda_1 C_m , \right.
\left. \sum^2_{i=1} \lambda_i = 1 , \theta \in [0, 1] \right \} .$$

**B. Achievable rate region for XOR**

**Theorem 3 (4S-XOR):** The rate region for 4S-XOR is the closure of the set of all rate pair $(R_0, R_2)$ satisfying

$$(R_0, R_2) : \left \{ R_0 \leq \min ((\lambda_1 + \lambda_3)C_{01}, \lambda_1 C_{02} + \lambda_4 C_{12}) , \right.
\left. R_0 + R_2 \leq \lambda_1 C_{01} + \lambda_2 C_{21} + \lambda_3 C_m , \right.
\left. R_2 \leq (\lambda_2 + \lambda_3)C_{21} , \right.
\left. R_2 - R_0 \leq -\lambda_1 C_{02} + \lambda_2 C_{20} + \lambda_5 C_{10} , \right.
\left. \sum^5_{i=1} \lambda_i = 1 \right \} .$$

**Proof:** The proof of this theorem differs from the proof of Theorem 1 mainly in Case 1 of the broadcast phase. The coding and decoding method of Case 1 are similar to those discussed in [20], [21] and are omitted.

**Remark:** A point to note is that five time-sharing parameters are needed when the 4-step time protocol is combined with XOR. This is because the last step (broadcast phase) in the XOR scheme can be further divided into two sub-steps, if necessary, according to the discussion in Section III-A.

**Remark:** By setting $\lambda_1 = \lambda_2 = 0$ in Theorem 3, we obtain the rate region for 2S-XOR.

**Theorem 4 (2S-XOR):** The rate region for 2S-XOR is the closure of the set of all rate pair $(R_0, R_2)$ satisfying

$$(R_0, R_2) : \left \{ R_0 \leq \min (\lambda_1 C_{01}, \lambda_2 C_{12}) , R_2 \leq \lambda_1 C_{21} , \right.
\left. R_0 + R_2 \leq \lambda_1 C_m , \right.
\left. R_2 - R_0 \leq \lambda_3 C_{10} , \right.
\left. \sum^3_{i=1} \lambda_i = 1 \right \} .$$

Note that the authors in [22] also studied the rate pair of 2S-XOR, but only the maximum sum-rate is considered and no rate region is discussed. In addition, the work in [22] is only suitable for reciprocal channels with $C_{01} = C_{10}$ and $C_{21} = C_{12}$. 
C. Achievable rate region for superposition coding

The achievable rate region of 2S-SUP is well studied in [?], [?], [?]. Though the achieve rate region of 4S-SUP is not studied yet in the literature, its derivation is trivial.

**Theorem 5 (4S-SUP):** The rate region for 4S-SUP is the closure of the set of all rate pair \((R_0, R_2)\) satisfying

\[
\begin{align*}
R_0 &\leq \min \left( (\lambda_1 + \lambda_3)C_{01}, \lambda_1 C_{02} + \lambda_4 C_{12}(\theta) \right), \\
R_2 &\leq \min \left( (\lambda_2 + \lambda_3)C_{21}, \lambda_2 C_{20} + \lambda_4 C_{10}(1 - \theta) \right), \\
R_0 + R_2 &\leq \lambda_1 C_{01} + \lambda_2 C_{21} + \lambda_3 C_m, \\
\sum_{i=1}^{4} \lambda_i & = 1, \theta \in [0, 1]
\end{align*}
\]

**Theorem 6 (2S-SUP):** The rate region for 2S-SUP is the closure of the set of all rate pair \((R_0, R_2)\) satisfying

\[
\begin{align*}
R_0 &\leq \min \left( \lambda_1 C_{01}, \lambda_2 C_{12}(\theta) \right), \\
R_0 + R_2 &\leq \lambda_1 C_m, \\
R_2 &\leq \lambda_2 C_{12}(1 - \theta), \\
\sum_{i=1}^{3} \lambda_i & = 1, \theta \in [0, 1]
\end{align*}
\]

D. Cases with direct transmission

For 4-step strategies, it is assumed by default that the direct link is always worse than the relay link. That is, \(C_{02} < C_{01}\) and \(C_{20} < C_{21}\). In this subsection, we consider the special cases where the direct link is better.

If \(C_{02} < C_{01}\) and \(C_{20} \geq C_{21}\), the signal from node 2 will be transmitted directly to node 0 without the help of relay. This corresponds to \(\lambda_3 = 0\) and the relay transmitting to node 2 only during \(\lambda_4\) in the 4-step protocol. Thus, using Theorem 1, we obtain the sets of rate pairs \((R_0, R_2)\) satisfying

\[
\begin{align*}
R_0 &\leq \min \left( \lambda_1 C_{01}, \lambda_2 C_{12}(\theta) \right), \\
R_0 + R_2 &\leq \lambda_1 C_m, \\
R_2 &\leq \min \left( \lambda_2 C_{21}, \lambda_2 C_{20} + \lambda_3 C_{10}(1 - \theta) \right), \\
\sum_{i=1}^{3} \lambda_i & = 1, \theta \in [0, 1]
\end{align*}
\]

If \(C_{02} \geq C_{01}\) and \(C_{20} < C_{21}\), the signal from node 0 will be transmitted directly to node 2 without the help of relay. The sets of rate pairs \((R_0, R_2)\) satisfy

\[
\begin{align*}
R_0 &\leq \min \left( \lambda_1 C_{01}, \lambda_2 C_{02} + \lambda_3 C_{12} \right), \\
R_2 &\leq \lambda_2 C_{20}, \sum_{i=1}^{3} \lambda_i = 1
\end{align*}
\]

If \(C_{02} \geq C_{01}\) and \(C_{20} \geq C_{21}\), direct communication between node 0 and node 2 in both ways is preferred and no relay is needed. Hence, the rate pairs are given by

\[
\begin{align*}
R_0 &\leq \lambda_1 C_{02}, \\
R_2 &\leq \lambda_2 C_{20}, \sum_{i=1}^{2} \lambda_i = 1
\end{align*}
\]
E. Numerical results

In this subsection, we present a numerical study of the proposed superimposed XOR in terms of three performance metrics: rate regions, system average sum-rates and service delay performances.

Suppose that the channel gain on each link is modeled by the distance path loss model, given by \( \alpha_{ij} = c \cdot d_{ij}^{-n} \), where \( c \) is an attenuation constant, \( n \) is the path loss exponent and fixed at 3, and \( d_{ij} \) denotes the distance between nodes \( i \) and \( j \). For simplicity, each node uses the same transmission power \( P \), though our analytical results are suitable for the case with unequal transmit power. The noise power is assumed to one. We consider the network layout shown in Fig. 3 where the distance between nodes 0 and 2 is normalized to 1 and the location of the relay is determined using the projections \( x \) and \( y \). The source nodes 0 and 2 are located at coordinates \((-0.5, 0)\) and \((0.5, 0)\), respectively. The distances from the relay to the source nodes can be computed as \( d_{01} = \sqrt{(x + 0.5)^2 + y^2} \), and \( d_{12} = \sqrt{(x - 0.5)^2 + y^2} \).

For fading channels, the same network layout and channel model, except that small-scale fading is included. We assume that the fading on each link follows Rayleigh distribution and are independent and reciprocal for different links.

1) Rate regions in Gaussian channels: Figs. 4-6 illustrate the rate regions of different two-way relay strategies. For comparison, the achievable rate region of the hybrid broadcast (HBC) protocol (4-step protocol) \([?]\) and the capacity of the multiple-access broadcast (MABC) protocol (2-step protocol) derived in \([?], [?], [?]\) are also shown and denoted by markers only in the figures. From Fig. 4 where the two relay channels are symmetrical, we see that the SuX and XOR schemes are equivalent and capacity achieving, whereas the SUP scheme is much inferior. It also shows that 4-step schemes can achieve higher one-way rate than 2-step schemes. This is expected because 4-step schemes exploit the direct link.

From Figs. 5-6, where the two relay channels are asymmetrical, it is observed that XOR is far from capacity-achieving and that SUP schemes becomes better than XOR schemes if \( R_2 > R_0 \). On the other hand, the proposed SuX schemes closely approach the capacity (or the best achievable rate region) in the high SNR regime (Fig. 6), while there is only a minor gap in the low SNR regime (Fig. 5).
2) Maximum sum-rates (MSRs) in fading channels: The problem of maximum sum-rate is formulated as

\[
\max_{(R_0(t), R_2(t)) \in \mathcal{R}(t)} R_0(t) + R_2(t)
\]  

(3)

where \(R_k(t), k \in \{0, 2\}\), denotes the service rate of node \(k\) at the \(t\)-th time slot, and \(\mathcal{R}(t)\) stands for the rate region with respect to the channel realization at the \(t\)-th time slot.

Figs. [7][8] show the averaged MSRs of different two-way relay strategies, when the relay node moves along the line between the two source nodes. It is observed that no matter where the relay is, the proposed SuX scheme always achieves the largest sum-rates among all the considered PLNC schemes and approach the corresponding optimal bounds very well, especially in the high SNR regime. Moreover, we can see that all strategies except 2S-SUP achieve their average maximum sum-rate when the relay node lies in the middle. As the relay node is approaching one source node (i.e. \(x\) approaches 0.5 or -0.5), the performances of the considered PLNC schemes (XOR, SUP and SuX) converge to the optimal bound.

3) Service delay in fading channels: Here, we use the same queuing mode as in [?], in which service rate allocation is done by a cross-layer approach by taking into account both queue length and channel state. Note that, Oechtering, et al. in [?] only focus on 2S-SUP strategies and consider the queue backlog versus bit arrival rate. Chen, et al. considered delay power tradeoff in [?], and assumed all the links have the same rate and only the relay has buffer. We study the bit delay versus packet arrival rate. Suppose that the packet arrival at two source nodes follow Poisson distribution with mean \(\rho\), the length of each packet is fixed as \(L\) bits. Let \(Q(t-1) = [Q_0(t-1), Q_2(t-1)]\) represent the remaining bits in the queues after the \((t-1)\)-th time slot. Then, \(Q(t) = [Q_0(t), Q_2(t)] = [Q_0(t-1) - R_0(t) + A_0(t)L, Q_2(t-1) - R_2(t) + A_2(t)L]\), where \(R_k(t), A_k(t) (k \in \{0, 2\})\) denote the service rates and packet arrival rates of node \(k\) in \(t\)-th time slot. The rate allocation problem is formulated as

\[
\max_{(R_0(t), R_2(t)) \in \mathcal{R}(t)} Q_0(t-1)R_0(t) + Q_2(t-1)R_2(t)
\]  

(4)

Fig. [2] shows the system delay based on the above weighted sum-rate maximization with \(L = 10\). It can be clearly seen that the proposed SuX scheme always outperforms the other two PLNC schemes and approach the corresponding best achievable bound, no matter which time control protocol is applied.
V. ANALYSIS OF PERFORMANCE GAP

The numerical results in the previous section demonstrate that the rate region achieved by proposed superimposed XOR closely approach the capacity bound. In this section, we shall analytically quantify the performance gap and show that it indeed approaches zero when SNR is large.

Note that the proposed scheme only concerns the information processing at the relay and destination during the broadcast phase, no matter which time protocol is adopted. Hence, it suffices to analyze the gap on the broadcast rate region. In what follows we first deduce the boundary of the achievable broadcast rate region in an analytical expression. Then we characterize an upper bound on the capacity gap.

Let $R_{10}$ and $R_{12}$ denote the data rates of the information flows $1 \to 0$ and $1 \to 2$, respectively, in the broadcast phase. From the proof of Theorem 1, it is seen clearly that, for any given power allocation parameter $\theta$, the rate pair $(R_{10}, R_{12})$ must satisfy the following linear inequalities:

\[
R_{10} \leq C_{10},
\]
\[
R_{12} \leq C_{12}(\theta),
\]
\[
R_{10} - R_{12} \leq C_{10}(1 - \theta).
\]

for $R_{10} \geq R_{12}$, and satisfy

\[
R_{12} \leq C_{12}
\]

for $R_{10} \leq R_{12}$. Graphing the feasible set and considering that $R_{10} \geq 0$ and $R_{12} \geq 0$, we obtain the rate region as sketched in Fig. 10. Here, we have

\[
\theta' = \left[ 1 + \frac{1}{\gamma_{10}} - \frac{1}{\gamma_{12}} \right]^+, \tag{5}
\]

with $[x]^+ = x$ if $x \geq 0$ and $[x]^+ = 0$ otherwise. This power control value satisfies $C_{10} = C_{12}(\theta') + C_{10}(1 - \theta').$

When $0 \leq \theta \leq \theta'$ or equivalently $C_{10} \leq C_{12}(\theta) + C_{10}(1 - \theta)$, the rate region is plotted in Fig. 10(a), with $A = (0, C_{10}(1 - \theta))$, $B = (C_{10} - C_{10}(1 - \theta), C_{10})$, $C = (C_{12}(\theta), C_{10})$ and $D = (C_{12}, C_{12})$. On the other hand, when $\theta' \leq \theta \leq 1$, or equivalently, $C_{10} \geq C_{12}(\theta) + C_{10}(1 - \theta)$, the rate region is in Fig. 10(b), where $A = (0, C_{10}(1 - \theta))$, $B = (C_{12}(\theta), C_{12}(\theta) + C_{10}(1 - \theta))$ and $D = (C_{12}, C_{12})$. 
Then, by varying $\theta$ from 0 to 1, the overall broadcast rate region is obtained as the union of the above feasible sets, which is given in Fig. 11. The coordinates of the boundary in the segment $\widetilde{BC}$ can be characterized as:

\[
\begin{align*}
B : & \quad (C_{12}(\theta'), C_{10}) \\
C : & \quad (C_{12}, C_{12}) \\
\widetilde{BC} : & \quad (C_{12}(\theta), C_{12}(\theta) + C_{10}(1 - \theta)), \forall \theta \in [\theta', 1].
\end{align*}
\]

In Fig. 11, the capacity bound of two-way relaying in the broadcast phase is also shown for comparison. It is given by the rectangle characterized by $R_{10} \leq C_{10}$ and $R_{12} \leq C_{12}$.[?]

We now define the gap between the rate region of the superimposed XOR and the capacity bound as the area of the shadowed region as depicted in Fig. 11($\widetilde{BC}$, $CE$ and $EB$), denoted as $\Delta$. Thus,

\[
\Delta = \int_{\theta'}^{1} \left\{ C_{10} - \left[ C_{12}(\theta) + C_{10}(1 - \theta) \right] \right\} dC_{12}(\theta)
\]

\[
= \int_{\theta'}^{1} \frac{\gamma_{12}}{(1 + \theta \gamma_{12}) \ln 2} \log_{2} \frac{1 + \gamma_{10}}{(1 + \theta \gamma_{12})(1 + (1 - \theta) \gamma_{10})} d\theta
\]

However, computing the exact value of $\Delta$ is involved. It can be verified easily that the gap is upper-bounded by the area of the triangular formed by $\overline{BC}$, $CE$ and $EB$. Namely,

\[
\Delta \leq \frac{1}{2} (C_{10} - C_{12}) \left[ C_{12} - C_{12}(\theta') \right]
\]

\[
= \frac{1}{2} \log_{2} \frac{1 + \gamma_{10}}{1 + \gamma_{12}} \log_{2} \frac{\gamma_{10}(1 + \gamma_{12})}{\gamma_{12}(1 + \gamma_{10})}
\]

(6)

(7)

It is seen that when both $\gamma_{10}$ and $\gamma_{12}$ are much larger than one ($\gamma_{10} \geq \gamma_{12} \gg 1$), then the term $\log_{2}(\gamma_{10}(1 + \gamma_{12})/(\gamma_{12}(1 + \gamma_{10})))$ approaches zero. Therefore, the upper bound approaches zero. Finally, we conclude that the proposed scheme is capacity approaching for larger SNR.

In the following we show some numerical examples for further illustration. Fig. 12 demonstrates the rate regions of superimposed XOR for different $\theta$ as well as the overall broadcast rate region by letting $\theta$ take all possible values in $[0, 1]$. In this example, we have $\gamma_{10} = 13.17$ dB and $\gamma_{12} = 5.55$ dB. The corresponding $\theta'$ is 0.77.

Fig. 13 shows the broadcast rate regions of superimposed XOR compared to the capacity bound and the rate regions obtained by conventional XOR and superposition at different SNR.
It can be seen that the proposed superimposed XOR outperforms the other two schemes, and the gap to the capacity bound vanishes as SNR increases.

VI. CONCLUSION

In this research, we proposed superimposed XOR, a novel PLNC scheme for two-way relay communications. It takes into consideration the asymmetry in both channel gain and bidirectional information length during the broadcast phase. In specific, when the receiver channel quality of the longer packet is worse than that of the shorter packet, it reduces to the conventional XOR. Otherwise, it combines the extra information bits from the longer packet with the XORed bits using superposition coding. We characterized its achievable rate region over the Gaussian channel when applied together with the 4-step and 2-step transmission protocols. We also demonstrate its average maximum sum-rate and service delay performance over fading channels. Numerical results showed that proposed PLNC scheme achieves a larger rate region than XOR and superposition when the broadcast channels are asymmetric and performs much better over fading channels. Moreover, at the high SNR region, it approaches the capacity bound. We also explicitly proved this capacity approaching behavior by deriving the analytical expressions of the boundary of the broadcast rate region.
Fig. 1: Three kinds of PLNC schemes for broadcast phase: (a) XOR, (b) Superposition, (c) Superimposed XOR.
1st step:

2nd step:

3rd step:

4th step:

Case 1: \[|D'_2| = |D_2| - |D_{20}| \geq |D'_0| = |D_0| - |D_{02}|\]

\[D'_0 \rightarrow D'_1 \rightarrow x'_0 \sqrt{\theta} \rightarrow x'_1\]

\[D'_2^{(1)} \rightarrow D'_2 \rightarrow x'_2 \sqrt{1-\theta} \rightarrow x'_1\]

Case 2: \[|D'_2| = |D_2| - |D_{20}| \leq |D'_0| = |D_0| - |D_{02}|\]

\[D'_0 \rightarrow D'_1 \rightarrow x'_1\]

\[D'_2 \rightarrow D'_2^{(2)} \rightarrow x'_2\]

Fig. 2: Process flow of the 4s-Sux.
Fig. 3: Layout of the two-way relaying network

Fig. 4: Rate region at \((x, y) = (0, 0)\) with \(P = 0\text{dB}\).
Fig. 5: Rate region at \((x, y) = (-0.2, 0.3)\) with \(P = 0\)dB.

Fig. 6: Rate region at \((x, y) = (-0.2, 0.3)\) with \(P = 10\)dB.
Fig. 7: Averaged maximum sum-rates versus relay location when $-0.5 \leq x \leq 0.5, y = 0$ with $P = 0$dB.

Fig. 8: Averaged maximum sum-rates versus relay location when $-0.5 \leq x \leq 0.5, y = 0$ with $P = 10$dB.
Fig. 9: System service delay versus packet arrival rate: \((x, y) = (0, 0), P = 10\, \text{dB}\).

Fig. 10: Rate region of superimposed XOR (shadowed area) at a fixed \(\theta\).
Fig. 11: Rate region of superimposed XOR (solid lines) and capacity bound (dashed lines) in the broadcast phase.
Fig. 12: Rate regions of superimposed XOR at fixed $\theta$ and the boundary for $\theta \in [0, 1]$ in broadcast phase:

$$(x, y) = (-0.2, 0.3), P = 2\text{dB}.$$
Fig. 13: Comparisons of rate regions between the superimposed XOR and capacity bound in broadcast phase:

\((x, y) = (-0.2, 0.3), P = \{0, 5, 10\} \text{ dB}\).