Adler-Kostant-Symes systems as Lagrangian gauge theories

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Abstract

It is well known that the integrable Hamiltonian systems defined by the Adler-Kostant-Symes construction correspond via Hamiltonian reduction to systems on cotangent bundles of Lie groups. Generalizing previous results on Toda systems, here a Lagrangian version of the reduction procedure is exhibited for those cases for which the underlying Lie algebra admits an invariant scalar product. This is achieved by constructing a Lagrangian with gauge symmetry in such a way that, by means of the Dirac algorithm, this Lagrangian reproduces the Adler-Kostant-Symes system whose Hamiltonian is the quadratic form associated with the scalar product on the Lie algebra.

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1 Introduction

The Adler-Kostant-Symes (AKS) construction associates Hamiltonian systems that are in many cases integrable with certain Lie algebraic data [1, 2, 3]. As found by Reyman and Semenov-Tian-Shansky [4], these systems may be viewed as symmetry reductions of corresponding Hamiltonian systems on cotangent bundles of Lie groups generated by Hamiltonians invariant under left and right translations. An advantage of such a viewpoint is that it leads to a natural regularization of some AKS systems whose Hamiltonian vector field is incomplete [4, 5, 6, 7].

The aim of this paper is to provide a Lagrangian description for an important subclass of the AKS systems. Our construction requires the underlying Lie algebra to be self-dual and a further technical condition must hold. These conditions are satisfied, for example, in the case of the open Toda lattices and their generalizations that are among the most studied integrable systems. The conformal Toda field theories were treated in a similar Lagrangian manner in [8], which actually served as the starting point for the present work. Our Lagrangian may be used in the future to perform a path integral quantization of the AKS systems, and it may permit interesting generalizations in the field theoretical case in analogy with the Toda systems.

Let $G$ be a connected real Lie group whose Lie algebra $G$ is equipped with a nondegenerate, symmetric, $G$-invariant bilinear form $\langle \cdot, \cdot \rangle$. Identify $G^*$ with $G$ by means of the `scalar product' $\langle \cdot, \cdot \rangle$. Suppose that $A, B \subset G$ are Lie subalgebras in such a way that as a vector space $G = A + B$. (1.1)

This induces the decomposition $G = A^\perp + B^\perp$, (1.2)

which gives rise to the further identifications $A^* \cong B^\perp$ and $B^* \cong A^\perp$. We denote by $\pi_A$, $\pi_B$ and by $\pi_{A^\perp}$, $\pi_{B^\perp}$ the projection operators on $G$ associated with these decompositions. Let $A, B \subset G$ be the connected Lie subgroups corresponding to $A, B$ and fix elements $\mu \in A^*$ and $\nu \in B^*$. The phase space of the AKS system of our interest, designated as $M_{\mu, \nu}$, consists of those elements $X \in G$ that have the following form:

$$X = X_A^* + X_B^*$$

with $X_A^* \in \mathcal{O}_A^- (\mu)$, $X_B^* \in \mathcal{O}_B^+ (\nu)$, (1.3)

where

$$\mathcal{O}_A^- (\mu) = \{ \pi_B (g\mu g^{-1}) | \forall g \in A \}, \quad \mathcal{O}_B^+ (\nu) = \{ \pi_{A^\perp} (g\nu g^{-1}) | \forall g \in B \}$$

are the coadjoint orbits of $A$ and $B$ through $\mu$ and $\nu$, respectively. The plus/minus superscripts indicate that these orbits are equipped with opposite Lie-Poisson brackets. In the AKS construction the Poisson brackets, denoted here by $\{ \cdot, \cdot \}_*$, are postulated to be

$$\{ \langle X_A^*, \xi \rangle, \langle X_A^*, \xi' \rangle \}_* = -\langle X_A^*, [\xi, \xi'] \rangle, \quad \forall \xi, \xi' \in A,$$

$$\{ \langle X_B^*, \eta \rangle, \langle X_B^*, \eta' \rangle \}_* = \langle X_B^*, [\eta, \eta'] \rangle, \quad \forall \eta, \eta' \in B,$$

$$\{ \langle X_A^*, \xi \rangle, \langle X_B^*, \eta \rangle \}_* = 0.$$ (1.5)

For the structure of such ‘self-dual’ Lie algebras, see e.g. [5].
The main point is that the $G$-invariant functions on $G^*$ yield a commuting family with respect to $\{ , \}_{\ast}$ and generate Hamiltonian systems on $\mathcal{M}_{\mu,\nu}$ that are often integrable in the Liouville sense [1, 2, 3, 4, 5, 6]. In our case a distinguished $G$-invariant Hamiltonian is furnished by

$$H(\mathcal{X}) := \frac{1}{2} \langle \mathcal{X}, \mathcal{X} \rangle.$$  (1.6)

The evolution equation associated with the Hamiltonian system $(\mathcal{M}_{\mu,\nu}, \{ , \}_{\ast}, H)$ reads as

$$\dot{\mathcal{X}} = \{ \mathcal{X}, H \}_{\ast} = -[\pi_A(\mathcal{X}), \mathcal{X}] = [\pi_B(\mathcal{X}), \mathcal{X}].$$  (1.7)

In this paper we present a Lagrangian model of the system given by $(\mathcal{M}_{\mu,\nu}, \{ , \}_{\ast}, H)$. The equivalence of the Lagrangian and Hamiltonian descriptions is established at the level of the equations of motion in section 2. Then the Poisson bracket aspect is dealt with by applying the Dirac algorithm [10, 11] to the Lagrangian in section 3. Examples are contained in section 4. In addition to the above-mentioned data, our construction relies on the existence of an open submanifold $\tilde{G} \subset G$ which is diffeomorphic to $A \times B$ by the map $A \times B \ni (g_A, g_B) \mapsto g_A g_B$. A typical example, related to Toda type systems, for which this condition is satisfied is $A = G_{>0}$ and $B = G_{\leq 0}$ for some integral gradation $G = \oplus_{n \in \mathbb{Z}} G_n$ of a semisimple Lie algebra $G$.

A remark is in order here concerning our notations. Throughout the paper, we pretend that $G$ is a matrix group to simplify notations. This is not a real restriction in any sense since one can rewrite all equations in a more general notation. For instance, $\pi_B(\mu g^{-1})$ in (1.4) would then be replaced by $(\text{Ad}_g^*(\mu))$ to denote the coadjoint action of $g \in A$ on $\mu \in A^*$ and so on.

## 2 The AKS system as a gauge theory

Motivated by the work on Toda theories [8], we propose to consider the following Lagrangian:

$$L(g, \dot{g}, \alpha, \beta) := \frac{1}{2} \langle \dot{g} g^{-1}, \dot{g} g^{-1} \rangle + \langle \alpha, \dot{g} g^{-1} - \mu \rangle + \langle \beta, g^{-1} \dot{g} - \nu \rangle$$

$$+ \langle \alpha, g \beta g^{-1} \rangle + \frac{1}{2} \langle \alpha, \alpha \rangle + \frac{1}{2} \langle \beta, \beta \rangle.$$  (2.1)

Here $g \in G$, $\dot{g} \in T_g G$ and $\alpha \in A$, $\beta \in B$. The first term is the Lagrangian of a free particle moving on the group manifold $G$. The variables $\alpha$ and $\beta$ act essentially as Lagrange multipliers that impose the constraints that appear in the Hamiltonian reduction treatment [4, 5, 6] of the AKS system. The terms in the second line are chosen so as to equip the Lagrangian with the gauge symmetry that we describe next.

The little groups of the constants $\mu \in A^*$ and $\nu \in B^*$ are given by

$$A_{\mu} = \{ a \in A | \pi_B(\mu a^{-1}) = \mu \}, \quad B_{\nu} = \{ b \in B | \pi_A(\nu b^{-1}) = \nu \}.$$  (2.2)

We associate a gauge transformation with any curve $a(t) \in A_{\mu}$, $b(t) \in B_{\nu}$ by letting any curve $(g(t), \alpha(t), \beta(t))$ in the configuration space of our Lagrangian system transform as

$$g(t) \mapsto a(t) g(t) b^{-1}(t)$$
\[\alpha(t) \mapsto a(t)\alpha(t)a^{-1}(t) - \dot{a}(t)a^{-1}(t)\]
\[\beta(t) \mapsto b(t)\beta(t)b^{-1}(t) + \dot{b}(t)b^{-1}(t).\]
\[(2.3)\]

One can check that \(L\) changes by a total time derivative under these transformations for any \((a(t), b(t)) \in A_\mu \times B_\nu\), and \(A_\mu \times B_\nu \subset A \times B\) is the maximal subgroup with this property.

For the further analysis it is convenient to introduce the quantities \(J_r\) and \(J_l\) by
\[J_r := g^{-1}_g + g\beta g^{-1} + \alpha, \quad J_l := g^{-1}_r g = g^{-1}_r \dot{g} + g^{-1}_r \alpha g + \beta.\]
\[(2.4)\]

Under (2.3) their gauge transformation properties are
\[J_r(t) \mapsto a(t)J_r(t)a^{-1}(t), \quad J_l(t) \mapsto b(t)J_l(t)b^{-1}(t).\]
\[(2.5)\]

The Euler-Lagrange equations of \(L\) obtained by varying \(\alpha\) and \(\beta\), respectively, are
\[\pi_B \perp (J_r) = \mu \quad \text{and} \quad \pi_A \perp (J_l) = \nu.\]
\[(2.6)\]

The equations that result by varying \(g\) are encoded by either of the following two relations:
\[\dot{J}_r = [J_r, \alpha] \quad \text{and} \quad \dot{J}_l = [\beta, J_l],\]
\[(2.7)\]

which are actually equivalent among each other. It can be verified that the gauge transformations (2.3) map any solution of (2.6), (2.7) into another solution.

We remark that the derivation of (2.7) is very easy in the case for which \(G = gl_n\) and \(\langle X, Y \rangle = \text{tr}(XY)\), since in this case one can parametrize \(g \in GL_n\) by its matrix elements. In general one derives the Euler-Lagrange equations by using some arbitrary local coordinates on \(G\), and then rewrites those equations in the coordinate independent form (2.7).

By assumption, there exists an open submanifold \(\tilde{G} \subset G\) diffeomorphic to \(A \times B\) by the factorization map
\[A \times B \ni (g_A, g_B) \mapsto g_A g_B \in \tilde{G}.\]
\[(2.8)\]

From now on we restrict \(g\) to belong to \(\tilde{G}\). By using the decomposition \(g = g_A g_B\), the first line of the transformation rule (2.3) becomes
\[g_A(t) \mapsto a(t)g_A(t), \quad g_B(t) \mapsto b(t)g_B^{-1}(t).\]
\[(2.9)\]

If \(g \in \tilde{G}\), we can define the gauge invariant quantity
\[X := g_A^{-1}J_r g_A = g_B J_l g_B^{-1}.\]
\[(2.10)\]

We next show that \(X\) satisfies the evolution equation (1.7) of the AKS system.

First, notice that by using the Euler-Lagrange equations (2.3) \(X\) can be written as
\[X = \pi_B \perp (g_A^{-1} \mu g_A) + \pi_A \perp (g_B \nu g_B^{-1}).\]
\[(2.11)\]

\footnote{It may happen that \(\tilde{G} = G\), examples are mentioned in section 4.}
This follows from \( \pi_{B^+}(X) = \pi_{B^+}(g_A^{-1}J^rg_A) \) by inserting that \( J^r = \mu + \pi_{A^+}(J^r) \) where the second term does not contribute since \( g_A^{-1}A^+g_A \subset A^+ \); \( \pi_{A^+}(X) \) is determined similarly. Upon comparison with (1.3), we see that \( X(t) \) belongs to the AKS phase space \( \mathcal{M}_{\mu,\nu} \). Second, let us show that (2.7) implies
\[
\dot{X} = -[\pi_A(X),X].
\] (2.12)
For this note from (2.10) that
\[
\pi_A(X) = g_A^{-1}\dot{g}_A + g_A^{-1}\alpha g_A, \quad \pi_B(X) = \dot{g}_B g_B^{-1} + g_B \beta g_B^{-1}.
\] (2.13)
By using the first equation in (2.7) we obtain
\[
\dot{X} = \frac{d}{dt}(g_A^{-1}J^rg_A) = g_A^{-1}J^rg_A - [g_A^{-1}\dot{g}_A, X] = -[g_A^{-1}\alpha g_A, X] - [g_A^{-1}\dot{g}_A, X],
\] (2.14)
which gives (2.12) on account of (2.13). A similar calculation using \( X = g_B J^r g_B^{-1} \) and the second relation in (2.7) yields \( \dot{X} = [\pi_B(X),X] \), which is plainly equivalent to (2.12).

In conclusion, we have shown that if \( g(t) \in \mathcal{G} \) and \( (g(t),\alpha(t),\beta(t)) \) satisfies the Euler-Lagrange equations of \( L \) in (2.1), then the gauge invariant function \( X(t) \) belongs to \( \mathcal{M}_{\mu,\nu} \) and satisfies the same evolution equation as defined by the AKS system \( (\mathcal{M}_{\mu,\nu}, \{ \, \}^*, H) \). Next we explain that the Lagrangian \( L \) encodes the Hamiltonian structure of the system as well.

3 Dirac analysis of the Lagrangian

The Lagrangian \( L \) (2.1) is singular since it does not depend on the velocities of \( \alpha \) and \( \beta \). Thus one has to apply the Dirac algorithm [10, 11] to associate a Hamiltonian system with \( L \). In this manner we below recover the AKS system.

The phase space corresponding to the configuration space \( G \times A \times B \) of our Lagrangian system is the cotangent bundle \( \mathcal{M} := T^*G \times T^*A \times T^*B \). By identifying \( T^*G \) with \( G \times G^* \) with the aid of right translations on \( G \) and using as earlier that \( G^* \cong \mathcal{G} \), we have
\[
\mathcal{M} = G \times \mathcal{G} \times A \times A^* \times B \times B^* = \{(g, J^r, \alpha, \pi_\alpha, \beta, \pi_\beta)\}.
\] (3.1)
Let \( \{\theta_a\} \) denote a basis of \( \mathcal{G} \) with dual basis \( \{\theta^a\} \). \( \{\theta_a\} \) can be chosen as \( \{\theta_a\} = \{\xi_m\} \cup \{\eta_n\} \), where \( \{\xi_m\} \) and \( \{\eta_n\} \) are bases of \( A \) and \( B \), respectively. Then \( \{\theta^a\} = \{\xi^m\} \cup \{\eta^n\} \), where \( \{\xi^m\} \) is a basis of \( A^* \cong B^\perp \) and \( \{\eta^n\} \) is a basis of \( B^* \cong A^\perp \). Now the fundamental Poisson brackets on \( \mathcal{M} \) are given by
\[
\{g, \langle J^r, \theta_a \rangle \} = \theta_ag \\
\{\langle J^r, \theta_a \rangle, \langle J^r, \theta_b \rangle \} = \langle J^r, [\theta_a, \theta_b] \rangle \\
\{\langle \alpha, \xi^m \rangle, \langle \pi_\alpha, \xi^n \rangle \} = \delta^m_n \\
\{\langle \beta, \eta^s \rangle, \langle \pi_\beta, \eta_t \rangle \} = \delta^s_t.
\] (3.2)
The other Poisson brackets between \( g, \mathcal{J}^r, \alpha, \pi_\alpha, \beta \) and \( \pi_\beta \) vanish. We introduce
\[
\mathcal{J}^l := g^{-1} \mathcal{J}^r g,
\]
and note that it has the Poisson brackets
\[
\{ g, \langle \mathcal{J}^l, \theta_a \rangle \} = g \theta_a \\
\{ \langle \mathcal{J}^l, \theta_a \rangle, \langle \mathcal{J}^l, \theta_b \rangle \} = -\langle \mathcal{J}^l, [\theta_a, \theta_b] \rangle \\
\{ \langle \mathcal{J}^r, \theta_a \rangle, \langle \mathcal{J}^l, \theta_b \rangle \} = 0.
\]
(3.4)

If \( q^i \) denotes local coordinates on some \( U \subset G \) and \( q^i, p_j \) are the corresponding canonical coordinates on \( T^*U \subset T^*G \), then on \( T^*U \) we have
\[
\mathcal{J}^r(q, p) = \mathcal{E}^{-1}(q)^i_a p_i \theta^a \quad \text{and} \quad \mathcal{J}^l(q, p) = \mathcal{F}^{-1}(q)^i_a p_i \theta^a,
\]
where \( \mathcal{E}^{-1} \) and \( \mathcal{F}^{-1} \) are the inverse matrices to \( \mathcal{E} \) and \( \mathcal{F} \) defined by
\[
\frac{\partial g(q)}{\partial q^i} g^{-1}(q) = \mathcal{E}(q)^a_i \theta_a \quad \text{and} \quad g^{-1}(q) \frac{\partial g(q)}{\partial q^i} = \mathcal{F}(q)^a_i \theta_a.
\]
(3.6)

The local Poisson brackets \( \{ q^i, p_j \} = \delta^i_j \) on \( T^*U \) are equivalent to the Poisson brackets of \( g, \mathcal{J}^r \) and \( \mathcal{J}^l \) in (3.2), (3.4).

Later we shall restrict ourselves to the open submanifold \( \tilde{\mathcal{M}} = T^*\tilde{G} \times T^*A \times T^*B \subset \mathcal{M} \), where the factorization \( g = g_A g_B \) is valid (3.3). We use also the decompositions
\[
\mathcal{J}^r = \mathcal{J}^r_A + \mathcal{J}^r_B, \quad \mathcal{J}^l = \mathcal{J}^l_A + \mathcal{J}^l_B,
\]
(3.7)
where \( \mathcal{J}^r_A = \pi_{B^*}(\mathcal{J}^r), \mathcal{J}^l_B = \pi_{A^*}(\mathcal{J}^l) \) and similarly for \( \mathcal{J}^l \). On \( \tilde{\mathcal{M}} \) we thus obtain,
\[
\{ g_A, \langle \mathcal{J}^r_A, \xi_m \rangle \} = \xi_m g_A \\
\{ g_B, \langle \mathcal{J}^r_A, \xi_m \rangle \} = 0 \\
\{ g_B, \langle \mathcal{J}^l_B, \eta_r \rangle \} = g_B \eta_r \\
\{ g_A, \langle \mathcal{J}^l_B, \eta_r \rangle \} = 0.
\]
(3.8)

Now we apply the Dirac algorithm to the Lagrangian \( L \) in (2.1). This will lead to a Hamiltonian system on \( \mathcal{M} \) with constraints. In fact, in the first step we obtain the primary Hamiltonian
\[
H_P = \frac{1}{2} \langle \mathcal{J}^r, \mathcal{J}^r \rangle + \langle \alpha, \mu - \mathcal{J}^r_A \rangle + \langle \beta, \nu - \mathcal{J}^l_B \rangle + \langle v_\alpha, \pi_\alpha \rangle + \langle v_\beta, \pi_\beta \rangle
\]
(3.9)

together with the primary constraints
\[
\pi_\alpha = 0 \quad \text{and} \quad \pi_\beta = 0.
\]
(3.10)

In addition to being a function on \( \mathcal{M} \), the Hamiltonian \( H_P \) contains \( v_\alpha \in A \) and \( v_\beta \in B \), which are to be regarded as arbitrary parameters. We note that \( H_P \) is derived from the relation
\[
H_P = p_i q^i + \langle v_\alpha, \pi_\alpha \rangle + \langle v_\beta, \pi_\beta \rangle - L \quad \text{with} \quad p_i = \frac{\partial L}{\partial \dot{q}^i},
\]
(3.11)
if we restrict to some coordinate neighbourhood $U \subset G$. Incidentally, by substituting the explicit formula
\[ p_i = \langle \frac{\partial g(q)}{\partial q^i} g^{-1}(q), \alpha + g(q) \beta g^{-1}(q) + \dot{g}(q) g^{-1}(q) \rangle \] (3.12)
into the definition (3.3), $J^r$ and $J^i$ get converted into $J^r$ and $J^i$ as defined in (2.4). The primary constraints express the fact that $L$ (2.1) does not depend on the velocities of $\alpha$ and $\beta$.

According to Dirac [10, 11], we next have to apply a consistency analysis to the system $(\mathcal{M}, \{ \cdot, \cdot \}, H_P)$ to obtain a constrained manifold $\mathcal{M}_c \subset \mathcal{M}$ which is preserved by the Hamiltonian vector field generated by $H_P$. By computing the Poisson brackets $\{ \pi, H_P \}$ and $\{ \pi, H_P \}$ and noting that these must vanish upon restriction to $\mathcal{M}_c$, we get the secondary constraints:
\[ J_r^a - \mu = 0 \quad \text{and} \quad J_l^b - \nu = 0. \] (3.13)
The derivatives of these constraints also must vanish along the restriction of the Hamiltonian vector field of $H_P$ to $\mathcal{M}_c$. It is not difficult to see that this requirement leads to the conditions that $\alpha \in A_\mu$ and $\beta \in B_\nu$, where $A_\mu$ and $B_\nu$ are the Lie algebras of the little groups $A_\mu$ and $B_\nu$ defined in (2.2), respectively. This means that we must impose the further secondary constraints
\[ \langle \pi, \xi \rangle = 0 \quad \forall \xi \in A_\mu \cap B_\nu \quad \text{and} \quad \langle \pi, \eta \rangle = 0 \quad \forall \eta \in B_\mu \cap A_\nu. \] (3.14)
(For any subspace $W \subset G$, $W^\perp \subset G$ consists of those $\zeta \in G$ for which $\langle \zeta, w \rangle = 0$ holds $\forall w \in W$.) It is clear that these constraints will be preserved by the flow generated by the Hamiltonian vector field of $H_P$, if we choose the so far arbitrary parameters $v_\alpha$ and $v_\beta$ so as to satisfy
\[ v_\alpha \in A_\mu \quad \text{and} \quad v_\beta \in B_\nu. \] (3.15)
The consistency analysis stops at this point. To summarize, we have arrived at the submanifold $\mathcal{M}_c \subset \mathcal{M}$ defined by imposing the constraints given by (3.10), (3.13) and (3.14). The restriction of the Hamiltonian vector field of $H_P$ to $\mathcal{M}_c$ is tangent to $\mathcal{M}_c$ due to these constraints together with the restriction (3.15).

To continue the Dirac procedure, we have to select the first class constraints and then find the gauge invariant quantities. Recall that a constraint $\phi = 0$ is first class if the Hamiltonian vector field $V_\phi$, given by $V_\phi[f] = \{ f, \phi \}$ for any $f \in C^\infty(\mathcal{M})$, is tangent to $\mathcal{M}_c$. A function $F$ on $\mathcal{M}_c$ is gauge invariant if its derivative is zero with respect to $V_\phi|\mathcal{M}_c$ for all first class constraints $\phi$. In our case it is not difficult to see that the first class constraints are
\[ \langle \pi, \xi \rangle = 0, \quad \langle \pi, \eta \rangle = 0, \quad \forall \xi \in A_\mu, \eta \in B_\nu, \] (3.16)
and
\[ \langle J^a_r - \mu, \xi \rangle = 0, \quad \langle J^b_l - \nu, \eta \rangle = 0, \quad \forall \xi \in A_\mu, \eta \in B_\nu. \] (3.17)
The momentum constraints (3.16) correspond to the gauge transformations
\[ (g, J^r, \alpha, \pi_\alpha, \beta, \pi_\beta) \mapsto (g, J^r, \alpha + \xi, \pi_\alpha, \beta + \eta, \pi_\beta) \quad \text{with some} \quad \xi \in A_\mu, \eta \in B_\nu, \] (3.18)
while the gauge transformations generated by the constraints in (3.17) operate as
\[(g, J^r, \alpha, \pi_\alpha, \beta, \pi_\beta) \mapsto (agb^{-1}, a J^r a^{-1}, \alpha, \pi_\alpha, \beta, \pi_\beta) \quad \text{with some} \quad a \in A_\mu, \, b \in B_\nu. \quad (3.19)\]
As a consequence,
\[\mathcal{J}^l \mapsto b \mathcal{J}^l b^{-1}. \quad (3.20)\]

The translations in (3.18) define an action of the abelian group \(A_\mu \times B_\nu\) on \(\mathcal{M}\), where the group structure is given by the obvious addition, and (3.19) yields an action of the group \(A_\mu \times B_\nu\) on \(\mathcal{M}\). Of course, these gauge transformations map \(\mathcal{M} \subset \mathcal{M}\) to itself. On \(\mathcal{M}\) under the gauge transformations (3.19). It follows that the function \(\tilde{\mathcal{X}}: \check{\mathcal{M}} \to \mathcal{G}\) given by
\[\tilde{\mathcal{X}} := g_A^{-1} J^r g_A = g_B J^l g_B^{-1}\]
is gauge invariant. The formula of \(\tilde{\mathcal{X}}\) can be rewritten as
\[\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_A + \tilde{\mathcal{X}}_B, \quad \tilde{\mathcal{X}}_A = \pi_A^{-1} (g_A^{-1} J^r_A g_A), \quad \tilde{\mathcal{X}}_B = \pi_A^{-1} (g_B J^l_B g_B^{-1}). \quad (3.24)\]
Defining the function \(\mathcal{X}: \check{\mathcal{M}}_c \to \mathcal{G}\) by \(\mathcal{X} := \tilde{\mathcal{X}}|_{\check{\mathcal{M}}_c}\), we obtain
\[\mathcal{X} = \pi_A^{-1} (g_A^{-1} \mu g_A) + \pi_A^{-1} (g_B \nu g_B^{-1}). \quad (3.25)\]

The components of \(\mathcal{X}\) form a complete set among the gauge invariant functions on \(\check{\mathcal{M}}_c\). In fact, \(\mathcal{X}\) parametrizes the space of the gauge orbits in \(\check{\mathcal{M}}_c\), since this space can be naturally identified with the double coset space
\[A_\mu \backslash \mathcal{G}/B_\nu \cong A_\mu \backslash A \times B/B_\nu \cong \mathcal{O}_A(\mu) + \mathcal{O}_B(\nu) \subset \mathcal{G}, \quad (3.26)\]
where \(\mathcal{O}_A(\mu)\) and \(\mathcal{O}_B(\nu)\) appear (1.4). We obtain this identification by using that \(\alpha\) and \(\beta\) can be set to zero by the gauge transformations in (3.19), and that on \(\check{\mathcal{M}}_c\) \(J^r\) is uniquely determined by \(g\) as \(J^r = g_A \mathcal{X} g_A^{-1}\).

The Dirac brackets of the components of \(\mathcal{X}\), which encode a Poisson structure \(\{\ , \\}^*\) on the above space of orbits, can be found by restricting the Poisson brackets of \(\mathcal{X}\) to \(\check{\mathcal{M}}_c\):
\[\{\langle \mathcal{X}, \theta \rangle, \langle \mathcal{X}, \theta' \rangle\}^* = \{\langle \tilde{\mathcal{X}}, \theta \rangle, \langle \tilde{\mathcal{X}}, \theta' \rangle\}|_{\check{\mathcal{M}}_c} \quad \forall \theta, \theta' \in \mathcal{G}. \quad (3.27)\]
This relation follows from the standard formula of the Dirac bracket since \(\mathcal{X} = \tilde{\mathcal{X}}|_{\check{\mathcal{M}}_c}\) and \(\tilde{\mathcal{X}}\) have zero Poisson brackets on \(\check{\mathcal{M}}\) with all (not only the first class) constraints that define \(\mathcal{M}_c \subset \check{\mathcal{M}}\). To calculate the right hand side of (3.27), notice that
\[\langle \tilde{\mathcal{X}}, \xi \rangle = \langle g_A^{-1} J^r_A g_A, \xi \rangle \quad \forall \xi \in A, \quad \langle \tilde{\mathcal{X}}, \eta \rangle = \langle g_B J^l_B g_B^{-1}, \eta \rangle \quad \forall \eta \in B. \quad (3.28)\]
By using this, (3.2), (3.4) and (3.8) easily lead to the relations
\[
\{\langle \bar{\mathcal{X}}, \xi \rangle, \langle \bar{\mathcal{X}}, \xi' \rangle \} = -\langle \bar{\mathcal{X}}, [\xi, \xi'] \rangle, \quad \forall \xi, \xi' \in \mathcal{A},
\]
\[
\{\langle \bar{\mathcal{X}}, \eta \rangle, \langle \bar{\mathcal{X}}, \eta' \rangle \} = \langle \bar{\mathcal{X}}, [\eta, \eta'] \rangle, \quad \forall \eta, \eta' \in \mathcal{B},
\]
\[
\{\langle \bar{\mathcal{X}}, \xi \rangle, \langle \bar{\mathcal{X}}, \eta \rangle \} = 0.
\]
(3.29)

Thus (3.27) implies that the Dirac brackets \{ , \}∗ of the components of \mathcal{X} are identical to the Poisson brackets \{ , \}∗ (1.5) that appear in the definition of the AKS system. To identify also the respective Hamiltonians, we note that
\[
\{\bar{\mathcal{X}}, H_P\} = \{\bar{\mathcal{X}}, \frac{1}{2}\langle \bar{\mathcal{X}}, \bar{\mathcal{X}} \rangle\} \quad \text{on} \quad \tilde{\mathcal{M}}.
\]
(3.30)

Indeed, the last four terms in \( H_P \) (3.9) have zero Poisson brackets with \( \bar{\mathcal{X}} \) and \( \langle J^r, J^r \rangle = \langle \bar{\mathcal{X}}, \bar{\mathcal{X}} \rangle \). We conclude from (3.30) that the Hamiltonian
\[
H(\mathcal{X}) = \frac{1}{2}\langle \mathcal{X}, \mathcal{X} \rangle
\]
(3.31)
generates the time evolution of the gauge invariant functions on \( \tilde{\mathcal{M}}_c \) through the Dirac bracket.

In general, the outcome of the Dirac algorithm can be viewed as an effective Hamiltonian system on a reduced phase space. The above considerations show that (with the restriction to \( \tilde{G} \subset G \)) the effective Hamiltonian system that belongs to the Lagrangian \( L \) in (2.1) is the AKS system described in the introduction.

We remark that if \( \tilde{G} \subset G \) is a proper submanifold but the restriction to \( \tilde{G} \) is not imposed, or the unique factorization appearing in (2.3) is not valid globally on \( A \times B \), then the application of the Dirac algorithm to the Lagrangian (2.1) leads to the same Hamiltonian system that results also by the corresponding Hamiltonian reduction of \( T^*G \) considered in [4, 5, 6, 7].

4 Conclusion

The construction described in this paper yields an interpretation of certain AKS systems as Lagrangian gauge theories. This interpretation is available if the Hamiltonian is the quadratic form of a scalar product on a self-dual Lie algebra and the factorization in (2.8) exists.

There are many examples (see [3]) to which our construction is applicable. The most familiar case is that of \( G = sl(n, \mathbb{R}) \) with \( \mathcal{A} \) and \( \mathcal{B} \) being the strictly upper-triangular subalgebra and the lower-triangular Borel subalgebra, respectively. In this case \( \tilde{G} \) consists of the Gauss-decomposable elements of \( SL(n, \mathbb{R}) \). These data can be generalized by replacing \( sl(n, \mathbb{R}) \) with the normal real form of a simple Lie algebra, and by using any integral gradation to define a triangular decomposition of \( G \). Another well known example is furnished by taking \( \mathcal{A} = so(n, \mathbb{R}) \subset sl(n, \mathbb{R}) = G \) and \( \mathcal{B} \) the Borel subalgebra as before. This example generalizes to any simple Lie algebra, too, and in the so-obtained cases \( \tilde{G} = G \) due to the global nature of the
Iwasawa decomposition. The open Toda lattices and their various generalizations appear among the AKS systems associated with the aforementioned Lie algebraic data. Further examples can be found, for instance, by using the theory of Drinfeld doubles.

Our definition of the Lagrangian (2.1) was motivated by the ‘point particle version’ of the gauged WZNW model [8] that provides a Lagrangian realization of the Hamiltonian reduction of the WZNW model to a conformal Toda field theory. Since the Lagrangian (2.1) is not restricted to Toda systems, it could be interesting to search for new gauged WZNW models that would yield field theoretical generalizations of the AKS systems treated in this paper.

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References

[1] M. Adler, Invent. Math. 50, 219-248 (1979).
[2] B. Kostant, Adv. Math. 34, 195-338 (1979).
[3] W.W. Symes, Invent. Math. 59, 13-51 (1980).
[4] A.G. Reyman and M.A. Semenov-Tian-Shansky, Invent. Math. 51, 81-100 (1979).
[5] A.G. Reyman, J. Sov. Math. 19, 1507-1545 (1982).
[6] A.G. Reyman and M.A. Semenov-Tian-Shansky, in: Encyclopedia of Mathematical Sciences, Vol. 16, V.I. Arnold and S.P. Novikov (editors), Springer, 1994.
[7] L. Fehér and I. Tsutsui, J. Geom. Phys. 21, 97-135 (1997).
[8] J. Balog, L. Fehér, L. O’Raifeartaigh, P. Forgács and A. Wipf, Ann. Phys. (N.Y.) 203, 76-136 (1990).
[9] J.M. Figueroa-O’Farrill and S. Stanciu, J. Math. Phys. 37, 4121-4134 (1996).
[10] P.A.M. Dirac, Lectures on Quantum Mechanics, Yeshia University Press, New York, 1964.
[11] K. Sundermeyer, Constrained Dynamics, Lecture Notes in Physics 169, Springer, 1982.