Almost flat angles in surface superconductivity

Michele Correggi\textsuperscript{1,}\textsuperscript{*} and Emanuela L Giacomelli\textsuperscript{2}

\textsuperscript{1} Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
\textsuperscript{2} Department of Mathematics, Ludwig-Maximilians Universitaet Muenchen, Theresienstr. 39, 80333 Muenchen, Germany

E-mail: michele.correggi@gmail.com

Received 16 November 2020, revised 2 July 2021
Accepted for publication 8 September 2021
Published 28 September 2021

Abstract

Type-II superconductivity is known to persist close to the sample surface in presence of a strong magnetic field. As a consequence, the ground state energy in the Ginzburg–Landau theory is approximated by an effective one-dimensional model. As shown by Correggi and Giacomelli (2021 Calc. Var. Partial Differential Equations in press), the presence of corners on the surface affects the energy of the sample with a non-trivial contribution. In (Correggi and Giacomelli 2021 Calc. Var. Partial Differential Equations in press), the two-dimensional model problem providing the corner energy is implicitly identified and, although no explicit dependence of the energy on the corner opening angle is derived, a conjecture about its form is proposed. We study here such a conjecture and confirm it, at least to leading order, for corners with almost flat opening angle.

Keywords: Ginzburg–Landau theory, superconductivity, domains with corners
Mathematics Subject Classification numbers: 35Q56.

1. Introduction

The phenomenon of superconductivity was first discovered in 1911 by Kamerlingh Onnes and it is nowadays well understood: the electrical resistance of various materials, such as mercury, drops down dramatically below a critical temperature $T_c$, whose value depends on the material. Its microscopic explanation relies on BCS theory [BCS], which describes superconductivity as a quantum critical phenomenon in which the conducting electrons arrange in weakly bound pairs (Cooper pairs) and, once they are created, the pairs exhibit a collective behavior, which

\textsuperscript{*} Author to whom any correspondence should be addressed.
Recommended by Dr Claude Le Bris.
is similar to the one appearing in Bose–Einstein condensate and which is responsible of the sudden drop of resistivity.

When a superconductor is immersed in a magnetic field, however, an even richer physics may emerge: for instance, if the intensity of the external field is small enough, the material behaves like a perfect superconductor. Hence, no effect of the field is observed in the sample and the field is expelled from the material, giving rise to the famous *Meissner effect*. However, if the field is strong enough, the material loses all its superconducting properties. According to how this breakdown occurs, one can distinguish between two different kinds of superconductors: for type-I superconductors, there is only a first order phase transition, while type-II superconductors exhibit a more complex behavior and the normal and superconducting phase may coexist (mixed state).

Close enough to the critical temperature, the response of the superconductor to the external magnetic field is very well described through a macroscopic model proposed in 1950 by Ginzburg and Landau [GL]—the Ginzburg–Landau (GL) theory—, where all the information is encoded in a one-particle wave function known as *order parameter* together with the magnetic potential generating the observed magnetic field. Such a theory is clearly much simpler than the microscopic description provided by BCS theory, but nevertheless its predictions are extremely accurate: the only external parameters are indeed the intensity and direction of the applied field and a length (*London penetration depth*), which is characteristic of each material and appears in the theory through a parameter $\kappa > 0$, whose threshold value $\kappa = 1$ conventionally separates type-I from type-II superconductors.

### 1.1. Ginzburg–Landau theory

The GL free energy of a type-II superconducting sample made of an infinite wire of constant cross section $\Omega$ is given by

$$
G^{GL}_\kappa[\psi, A] = \int_{\Omega} dr \left\{ |(\nabla + ih_{\text{ex}} A)\psi|^2 - \kappa^2 |\psi|^2 + \frac{1}{2} \kappa^2 |\psi|^4 \right\} + h_{\text{ex}}^2 \int_{\mathbb{R}^2} dr |\text{curl} A - 1|^2, 
$$

where we assumed that the applied magnetic field is of uniform intensity $h_{\text{ex}}$ along the superconducting wire and that it is perpendicular to $\Omega$; $h_{\text{ex}} A : \mathbb{R}^2 \to \mathbb{R}^2$ is the vector potential generating the induced magnetic field $h_{\text{ex}} \text{curl} A = h_{\text{ex}}(\partial_1 A_2 - \partial_2 A_1)$; $\psi : \Omega \to \mathbb{C}$ is the order parameter, i.e., the center of mass wave function of the Cooper pairs. The modulus $|\psi|$ is a measure of the relative density of the superconducting Cooper pairs, i.e., $0 \leq |\psi| \leq 1$ and, wherever $|\psi| = 0$, there are no Cooper pairs (loss of superconductivity), whereas, if $|\psi| = 1$ somewhere, then all the electrons are superconducting. The phase of $\psi$ encodes the information about the stationary current flowing in the superconductor, i.e.,

$$
\mathbf{j}[\psi] := \frac{i}{2}(\psi^* \nabla \psi - \psi \nabla^* \psi) = \text{Im}(\psi^* \nabla \psi).
$$

We are interested in studying the equilibrium state of the sample, which is obtained by minimizing (1.1) w.r.t. $\psi$ and $A$. Moreover, we focus on (extreme) type-II superconductors: indeed, we study the limit $\kappa \to \infty$, also known as London limit.

The GL energy functional introduced above was extensively studied for constant or regular magnetic fields in smooth domains (see, e.g., [CDR, CR2, CR3, CR4, FK1] for the 2D case and [AG, FK2, FKP, FMP, P] for the 3D one) and in domains with corners at the boundary [Cor, CG1, CG2, Gia, HK]. More recently, 2D GL models with piecewise-constant magnetic fields were also considered in [Ass1, Ass2, AK, AKP-S].
1.2. Sample with corners

Before proceeding further, we specify the assumption we make on \( \Omega \): we consider a bounded and simply connected domain \( \Omega \) with piecewise smooth boundary, such that the unit inward normal \( \nu \) to the boundary is well defined everywhere but in a finite number of points. We call these points the corners of \( \Omega \). We now state our assumption in a more precise way (see also [Gri] for a detailed discussion of domains with non-smooth boundaries). For the sake of concreteness we assume here that the boundary is \( C^\infty \) far from the corners, but less regularity is of course sufficient.

**Assumption 1 (Piecewise smooth boundary).** Let \( \Omega \) be a bounded and simply connected open set of \( \mathbb{R}^2 \). We assume that \( \partial \Omega \) is a smooth curvilinear polygon, i.e., for every \( r \in \partial \Omega \) there exists a neighborhood \( U \) of \( r \) and a map \( \Phi : U \to \mathbb{R}^2 \), such that

- \( \Phi \) is injective;
- \( \Phi \) together with \( \Phi^{-1} \) (defined from \( \Phi(U) \)) are \( C^\infty \);
- The region \( \Omega \cap U \) coincides with either \( \{ r \in \Omega \cap U | (\Phi(r)_1 < 0 \} \) or \( \{ r \in \Omega \cap U | (\Phi(r)_2 < 0 \} \) or \( \{ r \in \Omega \cap U | (\Phi(r)_1 < 0, (\Phi(r)_2 < 0 \} \), where \( (\Phi)_j \) stands for the \( j \)th component of \( \Phi \).

**Assumption 2 (Boundary with corners).** We assume that the set \( \Sigma \) of corners of \( \partial \Omega \) is non-empty but finite and we denote by \( \beta_j \) the angle of the \( j \)th corner (measured towards the interior) for \( j = 1, \ldots, N \).

1.3. Critical fields

We now investigate the response of the superconductor to the external magnetic field. It is well known that the material exhibits a different behavior depending on the intensity of the field (see, e.g., [SJdG]). Let us first describe what occurs for samples with smooth cross sections \( \Omega \): one can identify three different critical values of the applied field marking phase transitions of the material. More precisely, the first critical value \( H_{c1} \) is such that the minimizing order parameter has at least one vortex as soon as \( h_{ex} > H_{c1} \), i.e., superconductivity is lost at isolated points; the second critical field \( H_{c2} \) is related to the transition from bulk to boundary behavior, meaning that if \( h_{ex} > H_{c2} \), superconductivity survives only near the boundary of the sample; the last critical field \( H_{c3} \) is such that above it the material behaves as a normal conductor.

Asymptotically, as \( \kappa \to +\infty \),

\[
H_{c1} \sim C_\Omega \log \kappa, \quad H_{c2} \sim \kappa^2, \quad H_{c3} \sim \frac{1}{\Theta_0} \kappa^2 \tag{1.3}
\]

where \( C_\Omega > 0 \) depends only on the domain \( \Omega \) and \( \Theta_0 \simeq 0.59 \) is a universal constant. For more details about the critical fields for smooth domains, we refer respectively to [SS, section 2] and [FH, sections 10.6 and 13].

Let us now underline the expected differences in presence of corners. The first one is a possible change of the asymptotic value of the third critical field [B-NF]. More precisely, one observes the transition to the normal state only for fields whose intensity is larger than

\[
H_{c3} = \frac{1}{\mu(\beta)} \kappa^2, \tag{1.4}
\]

where \( \mu(\beta) \) stands for the ground state energy of a Schrödinger operator with uniform magnetic field in the infinite sector \( W_\beta \) with opening angle \( \beta \), i.e.,
\[ \mu(\beta) := \inf \text{spec}_{L^2(W^\beta)} \left( -\left( \nabla + \frac{1}{2} r^\perp \right)^2 \right). \]

Note that the universal constant \( \Theta_0 \) related to the third critical field for smooth domains (see (1.3)) is nothing but the ground state energy of the same Schrödinger operator on the half-plane (i.e., for \( \beta = \pi \)).

The shift of the third critical field occurs then whenever there is a corner with angle \( \beta \) such that \( \mu(\beta) < \Theta_0 \). This is proven for \( \beta \leq \frac{\pi}{2} + \epsilon \) [Bon, Jad, ELP-O] but, based on numerical experiments [AB-N], conjectured to be true [Bon, ELP-O] for any acute angle \( 0 < \beta < \pi \). Moreover, as the applied field gets closer to (1.4) from below, the order parameter concentrates around the corner with smallest opening angle and decays exponentially on a scale \( \kappa^{-1} \) far from it [B-NF, Theorem 1.6]. Hence, before disappearing, superconductivity survives only close to the corner(s). In [CG1, CG2], we proved that, if

\[ \kappa^2 < h_{\text{ex}} < \frac{1}{\Theta_0^2}, \]

superconductivity is however uniformly distributed (in \( L^2 \) sense) along the boundary and exponentially small in the bulk, so recovering the same behavior as in smooth domains. This strongly suggests the emergence of a new critical field

\[ H_{\text{corner}} \sim \frac{1}{\Theta_0^2} \kappa^2 \]

marking the transition from surface to corner superconductivity.

2. Main results

As anticipated, we aim at studying a model sample in the surface superconductivity regime, which is identified by an intensity of the external field

\[ h_{\text{ex}} := bn^2, \quad 1 < b < \Theta_0^{-1}, \]

since, in this parameter window, superconductivity survives only near the boundary of the sample. More precisely, one can prove that, if (2.1) holds, the minimizing order parameter exponentially decays in the distance from the boundary \( \partial \Omega \). We stress that the presence of corners does not affect this behavior [FH, section 15.3.1].

2.1. Surface superconductivity in domains with corners

Before discussing further the properties of any minimizing configuration, we perform a change of units, which proves to be very convenient in the surface regime given by (2.1) above. We introduce a new parameter

\[ \varepsilon := b^{-\frac{1}{2}} \kappa^{-1} \ll 1, \]

so that the GL functional becomes

\[ \mathcal{E}_\varepsilon^{GL}[\psi, A; \Omega] := \int_\Omega dr \left\{ \left( \nabla + \frac{A}{\varepsilon^2} \right)^2 \psi^2 - \frac{1}{2b\varepsilon^2} (2|\psi|^2 - |\psi|^4) \right\} + \frac{1}{\varepsilon^4} \int_{\mathbb{R}^2} d\mathbf{r} |\text{curl} A - 1|^2. \]
We then set \( E^{GL}_\varepsilon := \min_{(\psi,A) \in \mathcal{D}_\varepsilon} E^{GL}_\varepsilon[\psi,A;\Omega] \), where the minimization domain is given by
\[
\mathcal{D}_\varepsilon := \{(\psi,A) \in H^1(\Omega) \times H^{1,4}(\mathbb{R}^2;\mathbb{R}^2) \mid \text{curl} A - 1 \in L^2(\mathbb{R}^2)\},
\]
and we denote by \((\psi^{GL},A^{GL})\) any corresponding minimizing configuration.

The precise statement of the order parameter decay in this setting takes the form of the so-called Agmon estimates:

**Theorem 2.1 (Agmon estimates [FH, section 15.3.1]).** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded and simply connected domain, if \( b > 1 \). Then, any critical point \((\psi,A)\) of the GL functional satisfies
\[
\int_{\Omega} \mathcal{F}_{\varepsilon}(\psi,A) \leq C \int_{(\text{dist}(r,\partial \Omega) \leq \varepsilon)} dr |\psi|^2
\]
for some \( c(b) > 0 \).

Thanks to the above result, one can restrict the analysis to a boundary layer of thickness of order \( \varepsilon \), since the energy contribution of the rest of the sample is small: for instance, if we restrict to the region
\[
\mathcal{A}_\varepsilon := \{ r \in \Omega \mid \text{dist}(r,\partial \Omega) \leq c_0 \varepsilon |\log \varepsilon| \},
\]
for a (arbitrarily large) constant \( c_0 > 0 \), the above theorem 2.1 guarantees that the energy in \( \Omega \setminus \mathcal{A}_\varepsilon \) is smaller than a (arbitrarily large) power of \( \varepsilon \), which we denote by saying that it is \( O(\varepsilon^\infty) \) there. The restriction to \( \mathcal{A}_\varepsilon \) is quite relevant, since it allows to use suitable tubular coordinates: let \( \gamma : [0,|\partial \Omega|) \rightarrow \partial \Omega \) be a parametrization of the boundary, then we denote by \((s,t) \in [0,|\partial \Omega|) \times [0,c_0 \varepsilon |\log \varepsilon|] \) the coordinates satisfying
\[
r(s,t) := \gamma'(s) + \nu(s), \quad \forall r \in \mathcal{A}_\varepsilon,
\]
where \( \nu \) stands for the inward normal to the boundary and \( t := \text{dist}(r,\partial \Omega) \). The relation (2.7) introduce a local diffeomorphism in the smooth portion of \( \mathcal{A}_\varepsilon \). We also denote by \((s,t)\) the \( \varepsilon \)-rescaled counterparts of \((s,t)\), i.e., \( s := s/\varepsilon, t := t \varepsilon \). The curvature of the boundary is denoted by \( \mathcal{R}(s) \) and we set \( k(s) := \mathcal{R}(\varepsilon s) \) for short, so that, e.g., \( dr = \varepsilon^2 ds/(1 - \varepsilon k(s))t \).

We now focus on the energy asymptotics in the surface superconductivity regime. The most accurate result about is proven in [CG2] (see also the review [Cor]) and reads as follows.

**Theorem 2.2 (GL energy in piecewise smooth domains [CG2, theorem 2.1]).** Let \( \Omega \subset \mathbb{R}^2 \) be any bounded simply connected domain satisfying assumptions 1 and 2. Then, for any fixed \( 1 < b < \Theta_0^{-1} \), as \( \varepsilon \rightarrow 0 \), it holds,
\[
E^{GL}_\varepsilon = \frac{|\partial \Omega|E^{1,0}_\varepsilon}{\varepsilon} - E^{\text{corr}}_{\text{b}} \int_{\partial \Omega} \mathcal{R}(s) \, ds + \sum_{j=1}^{N} E^{\text{corner},\beta_j} + o(1).
\]

The first three terms on the rhs (2.8) can be interpreted as follows:

- The leading order term of order \( \varepsilon^{-1} \) is proportional to the length of the boundary and its coefficient is given by a one-dimensional model problem (see (2.9) below), whose expression is independent of the boundary curvature;

---

\(^3\) We denote by \( C \) a positive finite constant, whose value may change from line to line.
The curvature corrections to the energy are contained in the second term, which is of order 1 and whose coefficient is also curvature-independent (see (2.11));

- The contribution of corners is also of order 1 and it is given by the third term in the energy expansion in terms of an effective model on a wedge-like region (see (2.12)).

Both the quantities \( E_{\text{e}1D} \) and \( E_{\text{cor}} \) are determined through an effective one-dimensional model describing the variation of the modulus of the order parameter along the normal to the boundary. More precisely, we set

\[
E_{\text{e}1D} := \inf_{\alpha \in \mathbb{R}} \inf_{f \in \mathcal{D}_{1D}} \mathcal{E}_{\alpha}^{1D}[f; \mathbb{R}^+].
\]

(2.9)

\[
\mathcal{E}_{\alpha}^{1D}[f; \mathbb{R}^+] := \int_0^{+\infty} \mathrm{d} t \left\{ |\partial_t f|^2 + (t + \alpha)^2 f^2 - \frac{1}{2b} (2f^2 - f^4) \right\},
\]

(2.10)

with \( \mathcal{D}_{1D} := \{ f \in H^1(\mathbb{R}) \mid f \in L^2(\mathbb{R}) \} \), and

\[
E_{\text{cor}} := \int_0^{+\infty} \mathrm{d} t \left\{ |\partial_t f|^2 + f^2 \left( -\alpha_s (t + \alpha_s) - \frac{1}{b} + \frac{1}{2b} f^2 \right) \right\}
\]

\[= \frac{1}{3} f_s^2(0)\alpha_s - E_{\text{e}1D},
\]

(2.11)

where \( \alpha_s, f_s \) is a minimizing pair realizing \( E_{\text{e}1D} \), whose existence is discussed in [CG2, section A.1]. Here, we skip most of the details about the one-dimensional models and refer to appendix A or [CR2, CR3, CG1, CG2, Gia] instead (see in particular [CR3, section A]).

2.2. Corner effective energy

The corners’ energy contribution is given in terms of the effective energy

\[
E_{\text{corner}, \beta} := \lim_{\ell \to +\infty} \lim_{L \to +\infty} \left( E_{\text{corner}, \beta}(L, \ell) \right),
\]

(2.12)

with

\[
E_{\text{corner}, \beta}(L, \ell) := -2LE_0^{1D}(\ell) + \inf_{\psi \in \mathcal{D}_{1D}(\Gamma_{\beta}(L, \ell))} \mathcal{E}_{1D}^{\psi}[\psi; \Gamma_{\beta}(L, \ell)]
\]

(2.13)

where \( \mathcal{D}_{1D} := \frac{1}{L} \mathbb{R}^+ \), the latter functional is defined as in (2.3) with \( \varepsilon = 1 \) and \( A = F \). Moreover, \( E_0^{1D}(\ell) \) is the ground state energy of the finite-interval version of (2.9), i.e.,

\[
E_0^{1D}(\ell) := \inf_{\alpha \in \mathbb{R}} \inf_{f \in H^1([0, \ell])} \mathcal{E}_{\alpha}^{1D}[f; [0, \ell]].
\]

(2.14)

The domain \( \Gamma_{\beta}(L, \ell) \) is on the other hand a sort of wedge (depicted in figure 1), whose longitudinal and tangential length equals \( L \) and \( \ell \), respectively, and whose opening angle is \( \beta \in (0, 2\pi) \). Note that in order for the construction to be possible, we have to add the condition \( \ell \leq \tan(\beta/2)L \). Such a domain is indeed meant as a blow up of the corner region, where the longitudinal boundaries of \( \Gamma_{\beta}(L, \ell) \) are the \( \varepsilon \)-rescaled (and straighten) portions of \( \partial D \) close to the corner, which is represented by the vertex in \( \Gamma_{\beta}(L, \ell) \). The other boundaries of \( \Gamma_{\beta}(L, \ell) \) are in fact fake boundaries separating the corner region from the rest of the boundary layer. The well-posedness and the existence of the limit in (2.12) is not at all a trivial question, which is still open in full generality. For our purposes, it is enough however to know that the limit \( L, \ell \to +\infty \) in (2.12) exists and it is finite when taken over two monotone sequences \( \{ \ell_n \}_{n \in \mathbb{N}} \).
The region $\Gamma_\beta(L, \ell)$, where $\beta$ is the opening angle $\hat{AVB}$, $L = |AV| = |VB|$ and $\ell = |AC| = |EB|$. 

$$
\{L_n\}_{n\in\mathbb{N}} \text{ with } L_n, \ell_n \to +\infty \text{ and such that } 1 \ll L_n \ll C_n \ell_n^a, \text{ for some } a > 0 \text{ (see [CG2, proposition 2.2]). There is indeed no loss of generality in such a choice because in a more realistic setting the size of } \ell \text{ and } L \text{ is actually dictated by the characteristic length scale of the problem (London depth) and both quantities are of the same order (see the proof of corollary 2.1).}
$$

Finally, the minimization domain for the corner problem is

$$
\mathcal{D}_*(\Gamma_\beta(L, \ell)) := \{ \psi \in H^1(\Gamma_\beta(L, \ell)), |\psi|_{\partial\Gamma_\bd \cup \partial\Gamma_\in} = \psi_\star \},
$$

where we denoted by $\partial\Gamma_\bd \cup \partial\Gamma_\in$ the inner and tangential boundaries of $\Gamma_\beta$, i.e., concretely, $\partial\Gamma_\bd = \overline{AC} \cup \overline{EB}$ and $\partial\Gamma_\in = \overline{CD} \cup \overline{DE}$, and

$$
\psi_\star(s, t) := f_0(t) \exp \left( -i\alpha_0 s - \frac{1}{2}ist \right).
$$

where $(\alpha_0, f_0)$ is a minimizing pair realizing $E_0^{\text{ID}}(\ell)$ and $(s, t)$ are boundary coordinates of $\Gamma_\beta(L, \ell)$, such that $t \in [0, \ell]$ is the normal distance to the outer boundary and $s \in [-L, L]$ the tangential coordinate (with the vertex in $s = 0$).

The heuristic motivation behind the definition (2.13) of the effective energy is that, after blow-up around the singularity, in order to extract the precise contribution due to the presence of the corner, one has first to subtract the surface energy produced by the smooth part of the boundary, which equals exactly the first term on the rhs of (2.13) (see, e.g., [CG1, theorem 1.1]). Note also that, in the minimization in (2.13), the magnetic potential is fixed and the energy is minimized only over the order parameter. This is related to the very well known fact that, in the superconductivity layer surviving at the boundary, to a very good approximation, the magnetic field equals the applied one, so that, with a clever choice of the gauge, $A$ can be replaced with $F$. We omit further details for the sake of brevity (see [CG2]).

An important consequence of the energy estimate of theorem 2.2 is that superconductivity is robust along the boundary $\partial\Omega$ with the possible exception of the corner points, i.e., $\psi^{\text{GL}}$ is non-vanishing there. More precisely, under the same assumptions of theorem 2.2, we proved [CG2, proposition 2.1] that

$$
|||\psi^{\text{GL}}(r)| - f_\star(0)|||_{L^\infty(\partial\Omega^{\text{smooth}})} = o(1),
$$

where $\partial\Omega^{\text{smooth}} := \{ r \in \partial\Omega | \text{dist}(r, \Sigma) \geq c|\log \varepsilon| \}.$
2.3. Main results

As explained in details in [CG2, section 2.2], the explicit dependence of \( E_{\text{corner},b} \) on the angle \( \beta \) is not accessible and the definition of the effective energy is rather implicit. However, based on some geometric considerations, we formulated in [CG2] the following conjecture.

**Conjecture 1 (GL corner energy).** For any \( 1 < b < \Theta_0^{-1} \) and \( \beta \in (0, 2\pi) \),

\[
E_{\text{corner},\beta} = - (\pi - \beta) E_{\text{corr}}.
\]  

(2.18)

The goal of this paper is to address (2.18) and show that, if the corner angle \( \beta \) is close to \( \pi \), (2.18) is in fact correct, at least to leading order. This is the content of the next theorem.

**Theorem 2.3 (GL corner energy for almost flat angles).** Let \( 0 \leq \delta \ll 1 \) and \( 1 < b < \Theta_0^{-1} \). Then, as \( \ell \to +\infty \) and \( L \to +\infty \),

\[
E_{\text{corner},\pi \pm \delta}(L, \ell) = \pm \delta E_{\text{corr}} + \mathcal{O}(\delta^{4/3} |\log \delta|) + \mathcal{O}(L^2 \ell^{-\infty}).
\]

(2.19)

The above result is an asymptotic confirmation of conjecture 1 for almost flat angles. Unfortunately, we are not able to show that the remainder is identically zero, as it should be if one expects the conjecture to be true. There is however an important piece of information to extract from (2.19): the fact that the correction is non-zero for both acute and obtuse angles marks the difference with the linear behavior close to \( H_{-3} \). Indeed, in that case, it is expected and numerically confirmed that the effective energy is a monotone function of the angle but stays constant for angles larger or equal to \( \pi \). Here, on the opposite, we observe a symmetric behavior for acute and obtuse angles, at least for small variations around \( \pi \): since one expects that \( E_{\text{corr}} > 0 \) for \( 1 < b < \Theta_0^{-1} \), this would imply that corners with obtuse angles increase the energy, while acute angles lower it.

The above result applies to the model problem (2.12), but it has a direct consequence on the more general setting of a sample with a curvilinear polygonal boundary, provided the corners’ angles are almost flat.

**Corollary 2.1 (GL energy for domains with almost flat angles).** Let \( 0 \leq \delta \ll 1 \) and let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with a finite number of corners along \( \partial \Omega \) with almost flat angles \( \beta_j, j \in \Sigma \), i.e., such that \( |\beta_j - \pi| \leq \delta \ll 1 \). Then, for any fixed \( 1 < b < \Theta_0^{-1} \), as \( \varepsilon \to 0 \),

\[
E_{\varepsilon}^{GL} = \frac{\lvert \partial \Omega \rvert F_{10}^{10}}{\varepsilon} - 2\pi E_{\text{corr}} + \mathcal{O}(\delta^{4/3} |\log \delta|) + o_{\varepsilon}(1).
\]

(2.20)

3. Mathematical setting and preliminaries

Before facing the proofs of our main results, we provide more details about the mathematical setting we are going to consider. We study a wedge like domain \( \Gamma_{x,\pm \delta}(L, \ell) \) described in figure 2. We recall that we denote by \( \partial \Gamma_{\text{out}} \) the outer portion of the boundary \( \partial \Omega \), while the inner part \( \overline{CDE} \) and the tangential components \( \overline{CV} \cup \overline{EB} \) are denoted by \( \partial \Gamma_{\text{in}} \) and \( \partial \Gamma_{\text{bd}} \), respectively. More precisely, using polar coordinates with origin at the vertex \( V \) and axis along one of the outer sides of \( \Gamma_{x,\pm \delta}(L, \ell) \), we get:

\[
\partial \Gamma_{\text{out}} := \{ \mathbf{r} \in \mathbb{R}^2 \mid \mathbf{r} = (\varrho, 0), 0 \leq \varrho \leq L \} \cup \{ \mathbf{r} \in \mathbb{R}^2 \mid \mathbf{r} = (\varrho, \pi \pm \delta), 0 \leq \varrho \leq L \},
\]

(3.1)

\[
\Gamma \equiv \Gamma_{x,\pm \delta}(L, \ell) := \{ \mathbf{r} \in \mathbb{R}^2 \mid \text{dist}(\mathbf{r}, \partial \Gamma_{\text{out}} \leq \ell) \},
\]

(3.2)
The decay stated in theorem 2.1 holds true in the region $\Gamma$ as well, with the obvious adaptations. For later purposes however we formulate a more specific result applying to a subregion of longitudinal length of order 1.

Lemma 3.1. Let $S^\pm \subset \Gamma^\pm$ be domains of the form

$$S^\pm := \{(s_\pm, t_\pm) \in [-L, L] \times [0, \ell] \mid \bar{s}(t) \leq s_\pm \leq \bar{s}(t) + C\}$$

for some smooth increasing function $\bar{s}(t)$. Then, for any $b > 1$, there exists a constant $c(b) > 0$, such that

$$\int_{S^\pm} dr \ e^{c(b) \text{dist}(r, \partial\Gamma_{out})} \left\{ |\psi_{\Gamma_T}|^2 + |(\nabla + i\mathbf{F}) \psi_{\Gamma_T}|^2 \right\} = O(1).$$

Proof. The proof is a simple adaptation of the argument used to prove [CG2, lemma B.4]. □

An alternative form of the Agmon estimate is provided by next lemma, whose proof is given in [CG2, lemma B.5].

Lemma 3.2. For any $b > 1$ there exists a finite constant $C$, such that

$$|\psi_{\Gamma_T}(r)| \leq C e^{-\frac{1}{2}c(b) \text{dist}(r, \partial\Gamma_{out})},$$

where $c(b)$ is the constant appearing in (3.6).

3.1. Systems of coordinates

Besides the aforementioned polar coordinates, we are going to use other sets of tubular-like coordinates we introduce here. This however calls for a split of the corner region into two subregions, in which we are allowed to use the tangential length along the boundary and the distance from the outer boundary as global coordinates. For concreteness, we consider only the case of opening angle $\pi - \delta$ and provide a graphic representation of the domains $\Gamma^\pm$ in next figure 3 (the adaptation to the case of angle $\pi + \delta$ is trivial). More precisely, we set

$$\partial \Gamma \equiv \partial \Gamma_{\pi \pm \delta}(L, \ell),$$

$$\partial \Gamma_{in} := \{r \in \partial \Gamma \mid \text{dist}(r, \partial \Gamma_{out}) = \ell\}, \quad \partial \Gamma_{bd} := \partial \Gamma \setminus (\partial \Gamma_{out} \cup \partial \Gamma_{in}).$$

The points A, B, C, D, E are not labeled.
The two subregions \( \Gamma^{\pm} \).  

\[
\partial \Gamma_+^{\text{out}} := \{ r \in \Gamma \mid \vartheta = 0, 0 \leq \varrho \leq L \}, \quad \partial \Gamma_-^{\text{out}} := \{ r \in \Gamma \mid \vartheta = \pi - \delta, 0 \leq \varrho \leq L \}, 
\]

\[
(3.8)
\]

\[
\Gamma^{\pm} := \{ r \in \mathbb{R}^2 \mid \text{dist}(r, \partial \Gamma^{\pm}_{\text{out}}) \leq \ell, 0 \leq \mp (\vartheta - \vartheta_{\text{bis}}) \leq \vartheta_{\text{bis}} \},
\]

where \( \vartheta_{\text{bis}} := (\pi - \delta)/2 \) for short. As before, we also set \( \partial \Gamma_+^{\text{in}} := \partial \Gamma^{+} \cap \partial \Gamma^{\text{in}} \) and \( \partial \Gamma_-^{\text{in}} := \partial \Gamma^{+} \cap \partial \Gamma^{\text{in}} \).

Let then \((x, y)\) be the Cartesian coordinates centered at the vertex \(V\), as in figure 3. The tubular coordinates in the regions \( \Gamma^{\pm} \) respectively read

\[
\begin{align*}
\begin{cases}
\quad s_+ := x, \\
\quad t_+ := \text{dist} (r, \partial \Gamma_+^{\text{out}}) = y, \quad \forall r \in \Gamma^+,
\end{cases}
\end{align*}
\]

\[
(3.10)
\]

\[
\begin{align*}
\begin{cases}
\quad s_- = x \cos \delta - y \sin \delta, \\
\quad t_- := \text{dist} (r, \partial \Gamma_-^{\text{out}}) = x \sin \delta + y \cos \delta, \quad \forall r \in \Gamma^-.
\end{cases}
\end{align*}
\]

\[
(3.11)
\]

Hence, in both regions \(s_\pm\) measures the tangential length along the outer boundary, while \(t_\pm\) is the normal distance to it. Inside each region \( \Gamma^{\pm} \), the coordinates \((s_\pm, t_\pm)\) identify a smooth diffeomorphism mapping \( \Gamma^{\pm} \) to the sets \( \{ 0 \leq t_+ \leq \ell, \tan(\delta/2) t_+ \leq s_+ \leq L \} \) and \( \{ 0 \leq t_- \leq \ell, -L \leq s_- \leq -t_- \tan(\delta/2) \} \), which with a little abuse of notation we still denote by \( \Gamma^+ \) and \( \Gamma^- \), respectively. Note that the vertex of the corner is identified by the tubular coordinates \(s_\pm = 0, t_\pm = 0\).

**Remark 3.1 (Continuity of the normal coordinate).** The coordinates defined above provide two patches to cover the corner region and are obviously discontinuous along the bisector. However, the discontinuity appears only in the tangential components \(s_\pm\). The values of \(t_\pm\) indeed coincide on \( \partial \Gamma^+ \cap \partial \Gamma^- \), i.e., on the line \( y = \tan \left( \frac{\pi - \delta}{2} \right) x \), where

\[
\begin{align*}
\begin{align*}
\quad t_+ &= \tan \left( \frac{\pi - \delta}{2} \right) x = \frac{x}{\tan(\delta/2)}, & t_- &= \left[ \sin \delta + \frac{\cos \delta}{\tan(\delta/2)} \right] x = \tan \left( \frac{\pi - \delta}{2} \right) x.
\end{align*}
\end{align*}
\]

7642
We anticipate that the jump discontinuity of the tangential coordinates $s_{\pm}$ is going to play a key role in the proof of our result and, in order to take it properly into account, we identify a transition region around the bisectrix (see figure 4), obtained by restricting the polar angle to the symmetric interval $[\vartheta_-, \vartheta_+]$, with
\[
\vartheta_-= \frac{1}{2}(\pi - \delta - \gamma), \quad \vartheta_+= \frac{1}{2}(\pi - \delta + \gamma)
\]
with $\gamma > 0$ a (small) parameter to be chosen later. We denote by $\Gamma_{\gamma}^\pm$ the corresponding subsets of $\Gamma^\pm$, i.e., explicitly, $\Gamma_{\gamma}^\pm := \Gamma^\pm \cap \{ |\vartheta - \vartheta_{\text{bis}}| \leq \gamma/2 \}$. The complementary sets are denoted by $\Gamma_{\gamma}^\pm := \Gamma^\pm \setminus \Gamma_{\gamma}^\pm$, respectively.

For later convenience, we observe that the relation between tubular and polar coordinates in $\Gamma^\pm$ is given by
\[
\begin{align*}
{s_+} &= \varrho \cos \vartheta, \\
{t_+} &= \varrho \sin \vartheta, \\
{s_-} &= \varrho \cos(\vartheta + \delta), \\
{t_-} &= \varrho \sin(\vartheta + \delta),
\end{align*}
\]
respectively.

### 3.2. Sketch of the proof

Our main result is obtained by combining suitable upper and lower bounds to the energy $E_{\text{corner}, \pm, \delta}(L, \ell)$ working in different ways in all the subregions we introduced above. The heuristics behind the splitting of $\Gamma_{\beta}(L, \ell)$ just introduced is indeed that the leading contribution to the energy is given by the regions $\Gamma_{\gamma}^\pm$, i.e., $2LE_0^{1D}(\ell)$, while the regions $\Gamma_{\gamma}^\pm$ are instead responsible for the correction associated with the jump of the tangential coordinate, i.e., $-\delta E_{\text{corr}}$. As discussed in detail in [CG2, section 3.1] the dominant term in the energy of a finite strip in the surface superconductivity regime is given by $E_0^{1D}(\ell)$ times the length of the outer boundary ($2L$ in our case), where $\ell$ is the width of the strip.

The presence of a non-trivial correction to such an energy is entirely due to the presence of the corner. Let us describe the underlying heuristics. Because of the singularity in the tangential coordinates $s_{\pm}$, it is indeed impossible to glue together the optimizers of the effective models in the two components $\Gamma_{\gamma}^\pm$ of the corner region. More precisely, to leading order, the boundary behavior of the order parameter is captured by a wave function of the factorized form (up to a
global phase factor
\[ f_0(t) e^{-i \alpha_0 s - \frac{1}{2} t}, \]
where \( s \) and \( t \) stand for the tangential coordinate and the normal distance to the outer boundary, respectively, and \((f_0, \alpha_0) \in H^1(\mathbb{R}^+) \times \mathbb{R}\) is a minimizing pair for the variational problem (2.14) (see also appendix A). Here, the term \(-\frac{1}{2} t\) is a gauge phase needed to map the magnetic potential \( F \) to the vector \((-t, 0)\) in tubular coordinates.

Now, the modulus of the above ansatz is continuous through the bisectrix (see remark 3.1), but, on the opposite, the phase has a jump discontinuity, inherited from the behavior of the tangential coordinates \( s_{\pm} \). In order to glue together the optimal profiles in \( \Gamma_{\pm} \) in the upper bound and then recover the leading order term of the energy, a non-trivial variation of the phase is needed, i.e., the phase of the minimizer must be a genuine two-dimensional function, unlike the model ansatz where the dependence on \((s, t)\) is factorized. Concretely, we accommodate the transition by introducing a two-dimensional phase in \( \Gamma_{\pm} \) gluing together the phases \(-\alpha_0 s_{\pm} - \frac{1}{2} s_{\pm} t_{\pm}\) in \( \Gamma_{\pm} \): with a little abuse of notation, we define \( \Phi: \Gamma \to \mathbb{R} \) as
\[
\Phi := \begin{cases} 
\Phi_{\pm}(s_{\pm}, t_{\pm}), & \text{in } \Gamma_{\pm}^c, \\
\Xi(\vartheta, \varrho), & \text{in } \Gamma_{\gamma}^\pm,
\end{cases}
\] (3.14)
where the two phases \( \Phi_{\pm} \) are defined as follows
\[ \Phi_{\pm}(s_{\pm}, t_{\pm}) := -\alpha_0 s_{\pm} - \frac{1}{2} s_{\pm} t_{\pm}, \text{ in } \Gamma_{\pm}^c, \] (3.15)
and the function gluing them together is given in polar coordinates by
\[
\Xi(\vartheta, \varrho) := \left[ \alpha_0 \varrho \sin \left( \frac{\delta + \gamma}{2} \right) + \frac{\varrho^2}{4} \sin(\delta + \gamma) \right] \left( \frac{2 \vartheta - \vartheta_- - \vartheta_+}{\gamma} \right). \] (3.16)
Note that the function \( \Xi \) has been defined in such a way that
\[
\Xi|_{\vartheta=\vartheta_-} = \Phi_+|_{\vartheta=\vartheta_-}, \quad \Xi|_{\vartheta=\vartheta_+} = \Phi_-|_{\vartheta=\vartheta_+},
\] (3.17)
which ensures that the phase is continuous (in particular \( H^1(\Gamma) \)). However, by estimating the difference \( \Xi|_{\vartheta=\vartheta_-} - \Xi|_{\vartheta=\vartheta_+} \), one immediately sees that the phase undergoes a jump of order \( O(\delta + \gamma) \) in the region \( \Gamma_{\gamma}^+ \cup \Gamma_{\gamma}^- \), which is going to be very relevant in the proof. Note however that the choice of \( \Xi \) is rather arbitrary and any other function with the same properties would work.

On the other hand, the lower bound part of the proof does not involve the phase \( \Xi \), but an apparent discontinuity along the bisectrix emerges nevertheless: the key tools in the argument are a splitting technique to extract the energy to recover, by defining a pair of unknown functions \( u_{\pm} \) via
\[
\psi_{\Gamma} := f_0(t) e^{-i \alpha_0 s - \frac{1}{2} s t} u_{\pm}, \text{ in } \Gamma_{\pm}^b,
\]
and the consequent reduction of the problem to the minimization of a weighted functional \( \tilde{E}_0 \) of \( u_{\pm} \). Note that, according to the upper bound heuristics, we expect \( u_{\pm} \) to be suitably close to 1, which would make the reduced energy \( \tilde{E}_0 \) vanish identically. The key step in the lower bound is in fact the proof of the positivity of such an energy: this is done via a suitable integration by parts of the only non-positive term of \( \tilde{E}_0 \) and exploiting the pointwise positivity of a one-dimensional
cost function (see also appendix A). There are however non-vanishing boundary terms along the bisectrix generated in the integration by parts, which provide an energy contribution necessary to recover the correction $-E_{\text{corr}}\delta$ and which are entirely due to the discontinuity of the phase $-i\alpha_0 s_- - \frac{1}{2}s_\pm t_\pm$ (and, in particular, to the jump of the coordinate $s_\pm$) on the bisectrix.

4. Proofs

We denote for short by $G$ the energy functional in (2.13), i.e.,
\[
G[\psi; \Omega] := \mathcal{E}[\psi, F; \Omega],
\]
and by $E_\Gamma$ and $\psi_\Gamma$ its ground state energy and any corresponding minimizer for $\Omega = \Gamma$, respectively (recall that $\Gamma := \Gamma_\beta(L, \ell)$):
\[
E_\Gamma := \inf_{\psi \in \mathcal{D}(\Gamma)} G[\psi; \Gamma] = G[\psi_\Gamma; \Gamma],
\]
where $\mathcal{D}(\Gamma) := \mathcal{D}_\beta(\Gamma_\beta(L, \ell))$ is defined in (2.15).

As anticipated, we combine suitable upper and lower bounds to the energy to obtain the main result. For the sake of simplicity, we are going to spell in full detail only the proof for opening angle $\pi - \delta$, while the adaptation to the angle $\pi + \delta$ is discussed in section 4.3.

4.1. Energy upper bound

We first prove an upper bound for the energy $E_\Gamma$.

**Proposition 4.1 (Energy upper bound).** For any fixed $1 < b < \Theta_0^{-1}$, as $\ell, L \to +\infty$ and $\delta \to 0$,
\[
E_\Gamma \leq 2LE_1D_0(\ell) - \delta E_{\text{corr}} + O\left(\delta^{4/3}\right) + O(\ell^{-\infty}).
\]

**Proof.** To prove the upper bound, we evaluate the energy of the trial state $\psi_\text{trial} \in \mathcal{D}(\Gamma)$ given by (again with a little abuse of notation)
\[
\psi_\text{trial} := \begin{cases} 
  f_0(t_\pm)e^{i\Phi_\pm(s_\pm, t_\pm)}, & \text{in } \Gamma_\pm, \\
  f_0(t_\pm)e^{i\Phi_\pm(\rho, \phi)}, & \text{in } \Gamma_\gamma.
\end{cases}
\]

Note that the trial state above satisfies the boundary conditions in $\mathcal{D}(\Gamma)$ and, moreover, $\psi_\text{trial}$ may be extended outside $\Gamma$ expanding the domain of tubular coordinates.

We start by working in the region $\Gamma_\pm^+$: denoting by $R^+$ the rectangle obtained by completing $\Gamma_\pm^+$, i.e.,
\[
R^+ := \{(s_+, t_+) \in [0, L] \times [0, \ell]\},
\]
we obviously get
\[
\mathcal{G} \left[ f_0 e^{i\Phi_+}; \Gamma_\pm^+ \right] = \mathcal{G} \left[ f_0 e^{i\Phi_+}; R^+ \right] - \mathcal{G} \left[ f_0 e^{i\Phi_+}; R^+ \setminus \Gamma_\pm^+ \right].
\]

Using that the boundary coordinates $(s_+, t_+)$ coincide with the Cartesian ones in $R$ (recall (3.10)) and the identity
\[
F(s_+, t_+) + \nabla_{(s_+, t_+)} \Phi_+(s_+, t_+) = \left( -\frac{1}{2} t_+ - \alpha_0 - \frac{1}{2} t_+ \right) \hat{e}_s + \left( \frac{1}{2} s_+ - \frac{1}{2} s_+ \right) \hat{e}_t \\
= -(t_+ + \alpha_0) \hat{e}_s,
\]

where we have denoted by \( \hat{e}_s \) and \( \hat{e}_t \) the unit vector in the tangential and normal directions, respectively, an easy calculation shows that (see, e.g., [CG2, section 3.1])

\[
\mathcal{G} \left[ f_0 e^{i\Phi^+} ; R^+ \right] = LE_{1D}^0(\ell),
\]

where we recall that \( E_{1D}^0(\ell) \) is the ground state energy of the 1D effective model in the interval \([0, \ell]\) introduced in (2.14). On the other hand, we compute in \( R \setminus \Gamma_c^- \)

\[
\mathcal{G} \left[ f_0 e^{i\Phi^+} ; R \setminus \Gamma_c^+ \right] = \int_0^\ell dt \int_0^t \tan \frac{s}{t} ds \left\{ |f'_0|^2 + (t + \alpha_0)^2 f_0^2 - \frac{1}{2b} (2f_0^2 - f_0^4) \right\} \\
= \frac{\delta + \gamma}{2} \int_0^\ell dt \left\{ |f'_0|^2 + (t + \alpha_0)^2 f_0^2 - \frac{1}{2b} (2f_0^2 - f_0^4) \right\} + O(\gamma^3),
\]

where we chose \( \gamma \) in such a way that

\[
\delta = O(\gamma).
\]

Note also that the integral on the rhs can be easily seen to be \( O(1) \) by exploiting the decay properties of \( f_0 \) (see (A.10)) and the uniform boundedness of \( \alpha_0 \) (see appendix A).

Let us now consider the region \( \Gamma_c^- \): using the inverse transformation of (3.11) together with the Cartesian coordinate representation of the unit vectors \( \hat{e}_s = (\cos \delta, -\sin \delta) \), \( \hat{e}_t = (\sin \delta, \cos \delta) \), we compute

\[
F(s_-, t_-) \cdot \hat{e}_s = \frac{1}{2} (s_- \sin \delta - t_- \cos \delta, s_- \cos \delta + t_- \sin \delta) \cdot (\cos \delta, -\sin \delta) \\
= -\frac{1}{2} t_-, \tag{4.10}
\]

\[
F(s_-, t_-) \cdot \hat{e}_t = \frac{1}{2} t_-, \tag{4.11}
\]

so that

\[
|F + iF| (f_0 e^{i\Phi_-})^2 = \left| f_0^2(t_-) \frac{\partial_s \Phi_-}{\partial_s \Phi_-} - \frac{1}{2} t_- \right|^2 + \left| f_0(t_-) + i \left( \frac{1}{2} s_- + \partial_s \Phi_- \right) f_0(t_-) \right|^2 \\
= |f'_0(t_-)|^2 + (t_- + \alpha_0)^2 f_0^2(t_-), \tag{4.12}
\]

Hence, we can proceed as for \( \Gamma_c^+ \) (see (4.5), (4.7) and (4.8)), to get

\[
\mathcal{G} \left[ f_0 e^{i\Phi^-} ; \Gamma_c^- \right] = LE_{1D}^0(\ell) - \frac{\delta + \gamma}{2} \\
\times \int_0^\ell dt \left\{ |f'_0|^2 + (t + \alpha_0)^2 f_0^2 - \frac{1}{2b} (2f_0^2 - f_0^4) \right\} + O(\gamma^3),
\]

7646
and combining this with the analogous result for $\Gamma_+^\gamma$, we conclude that

$$
G [\psi_{\text{trial}}; \Gamma_+^\gamma \cup \Gamma_+^\gamma] = 2LE_0^{\text{ID}}(\ell) - (\delta + \gamma) \int_0^\ell dt \left\{ |f_0'|^2 + (t + \alpha_0)^2 f_0^2 - \frac{1}{2b}(2f_0^2 - f_0^4) \right\}
+ O(\gamma^3).
$$

(4.13)

It remains to compute the energy contribution in the two regions $\Gamma^\pm_\gamma$ close to the bisectrix. There it is more convenient to work in polar coordinates. Since

$$
F \cdot \hat{e}_\varrho := \frac{1}{2} \varrho \left( -\sin \vartheta, \cos \vartheta \right) \cdot \left( \cos \vartheta, \sin \vartheta \right) = \frac{1}{2} \varrho,
$$

$$
F \cdot \hat{e}_\vartheta := \frac{1}{2} \varrho \left( -\sin \vartheta, \cos \vartheta \right) \cdot \left( -\sin \vartheta, \cos \vartheta \right) = \frac{1}{2} \varrho \sin \vartheta,
$$

(4.14)

we obtain

$$
G [\psi_{\text{trial}}; \Gamma^+_\gamma] = \int_{\Gamma^+_\gamma} d\varrho d\vartheta \varrho \left\{ |\partial_\varrho \psi_{\text{trial}}|^2 + 1 \left| \partial_\vartheta \psi_{\text{trial}} + \frac{i}{2} \varrho \psi_{\text{trial}} \right|^2 - \frac{1}{2b}(2f_0^2 - f_0^4) \right\},
$$

where we have omitted for short the dependence of $f_0$ and $\psi_{\text{trial}}$ on $\varrho$ and $\vartheta$. Furthermore, we have

$$
|\partial_\varrho \psi_{\text{trial}}|^2 = |\partial_\varrho (f_0 e^{i\varpi})|^2 = \left| \partial_\varrho f_0(\varrho \sin \vartheta) \right|^2 + \frac{\varrho^2}{2} \left| \partial_\vartheta \varpi(\varrho, \vartheta) \right|^2,
$$

and

$$
\partial_\vartheta \varpi = \left[ \alpha_0 \sin \left( \frac{\delta + \gamma}{2} \right) + \frac{\varrho}{2} \sin(\delta + \gamma) \right] \left( \frac{2\vartheta - \vartheta_< - \vartheta_>}{\gamma} \right) = O(\gamma) + O(\varrho \gamma),
$$

(4.15)

under the assumption $\delta = O(\gamma)$, so that (again by the decay (A.10) of $f_0$)

$$
\int_{\Gamma^+_\gamma} d\varrho d\vartheta \varrho |\partial_\varrho \psi_{\text{trial}}|^2 = \int_{\Gamma^+_\gamma} d\varrho d\vartheta \varrho \sin^2 \vartheta |f_0(\varrho \sin \vartheta)|^2 + O(\gamma^2).
$$

(4.16)

We now estimate the angular component of the kinetic energy in $\Gamma^+_\gamma$:

$$
\frac{1}{\varrho} \left| \partial_\varrho \psi_{\text{trial}} + \frac{i}{2} \varrho \psi_{\text{trial}} \right|^2 = \cos^2 \vartheta |f_0'(\varrho \sin \vartheta)|^2 + f_0'(\varrho \sin \vartheta) \left| \frac{1}{\varrho} \partial_\varrho \varpi + \frac{1}{2} \varrho \right|^2.
$$

Since

$$
\frac{1}{\varrho} \partial_\varrho \varpi = \frac{2}{\gamma} \left[ \alpha_0 \sin \left( \frac{\delta + \gamma}{2} \right) + \frac{\varrho}{4} \sin(\delta + \gamma) \right]
$$

$$
= \frac{\varrho}{2} + \alpha_0 + \frac{\delta}{\gamma} \left( \frac{\varrho}{2} + \alpha_0 \right) + O(\gamma^2) + O(\varrho \gamma^2),
$$

(4.17)

we find
\[
\int_{\Gamma_{\gamma}^+} d\rho \, d\vartheta \, \frac{1}{\rho} \left| \partial_{\psi} \psi_{\text{trial}} + \frac{i}{2} \psi_{\text{trial}} \right|^2 = \int_{\Gamma_{\gamma}^+} d\rho \, d\vartheta \, \left\{ \rho \cos^2 \vartheta \left| f'_0(\rho \cos \vartheta) \right|^2 + \rho f''_0(\rho \cos \vartheta) \left[ \rho \alpha_0 + \frac{\delta}{\gamma} \left( \frac{1}{2} \rho + \alpha_0 \right) \right]^2 \right\} + \mathcal{O}(\gamma^2).
\]

(4.18)

We can now apply the estimates proven in lemma A.1, to get

\[
\mathcal{G} \left[ \psi_{\text{trial}}; \Gamma_{\gamma}^+ \right] = \int_{\Gamma_{\gamma}^+} d\rho \, d\vartheta \, \varrho \left\{ |f'_0(\varrho)|^2 + f''_0(\varrho) \left[ \varrho + \alpha_0 + \frac{\delta}{\gamma} \left( \frac{1}{2} \varrho + \alpha_0 \right) \right]^2 \right\} - \frac{1}{2b} \left( 2f''_0(\varrho) - f''_0(\varrho) \right) + \mathcal{O}(\gamma^2),
\]

(4.19)

where we have also exploited the fact that \( \vartheta_+ - \vartheta_- = \gamma \). Now, we can approximate the integration domain as follows: let \( F(\varrho) \) be a function enjoying the same decay properties as \( f_0 \) or \( f'_0 \), then

\[
\int_{\Gamma_{\gamma}^+} d\rho \, d\vartheta \, \varrho \, F(\varrho) = \int_{\vartheta_-}^{\vartheta_+} d\vartheta \int_{0}^{\vartheta} d\rho \, \varrho \, F(\varrho) = \int_{\vartheta_-}^{\vartheta_+} \frac{\gamma}{2} \int_{0}^{\vartheta} d\rho \, \varrho \, F(\varrho) = \int_{\vartheta_-}^{\vartheta_+} \frac{\gamma}{2} \int_{0}^{\vartheta} d\rho \, \varrho \, F(\varrho) + \mathcal{O}(\gamma \epsilon^{-\infty}).
\]

(4.20)

where we have denoted by \( (\varrho(\vartheta), \vartheta) \), \( \vartheta \in [\vartheta_-, \vartheta_+] \), the polar coordinates of points belonging to the inner boundary of the region \( \Gamma_{\gamma}^+ \cup \Gamma_{\gamma}^- \) (note that \( \varrho(\vartheta) \geq \ell \) for any \( \vartheta \)). Hence,

\[
\mathcal{G} \left[ \psi_{\text{trial}}; \Gamma_{\gamma}^+ \right] = \frac{\gamma}{2} \int_{\vartheta_-}^{\vartheta_+} \int_{0}^{\vartheta} \varrho \, F(\varrho) \left\{ |f'_0(\varrho)|^2 + f''_0(\varrho) \left[ \varrho + \alpha_0 \right]^2 + \frac{\delta}{\gamma} \left( \varrho + \alpha_0 \right) \left( \varrho + 2\alpha_0 \right) \right\} - \frac{1}{2b} \left( 2f''_0(\varrho) - f''_0(\varrho) \right) + \mathcal{O}(\gamma^2) + \mathcal{O}(\delta^2 \gamma^{-1}) + \mathcal{O}(\gamma \epsilon^{-\infty}).
\]

(4.21)

The estimate of the energy contribution from \( \Gamma_{\gamma}^- \) is perfectly analogous and we obtain

\[
\mathcal{G} \left[ \psi_{\text{trial}}; \Gamma_{\gamma}^+ \cup \Gamma_{\gamma}^- \right] = \gamma \int_{\vartheta_-}^{\vartheta_+} \int_{0}^{\vartheta} \varrho \, F(\varrho) \left\{ |f'_0(\varrho)|^2 + \left( \varrho + \alpha_0 \right)^2 f''_0(\varrho) - \frac{1}{2b} \left( 2f''_0(\varrho) - f''_0(\varrho) \right) \right\} + \mathcal{O}(\gamma^2) + \mathcal{O}(\delta^2 \gamma^{-1}) + \mathcal{O}(\gamma \epsilon^{-\infty}).
\]

(4.22)

Combining this with (4.13), we finally get
\[ G[\psi_{\text{trial}}; \Gamma] = 2LE_{1D}(\ell) - \delta \int_0^\ell dt \left\{ |f_0'|^2 + f_0^2 [(t + \alpha_0)^2 - \alpha_0 (t + \alpha_0)] - \frac{1}{2b} (2f_0^2 - f_0^4) \right\} + \mathcal{O}(\gamma^2) + \mathcal{O}(\delta^2 \gamma^{-1}) + \mathcal{O}(\gamma \ell^{-\infty}), \] (4.23)

and, optimizing over \( \gamma \), i.e., choosing \( \gamma = \delta^{2/3} \), we conclude that
\[ E_\Gamma \leq 2LE_{1D}(\ell) - \delta E_{\text{corr}} + \mathcal{O}(\delta^{4/3}) + \mathcal{O}(\delta^2 \gamma^{-1/3} \ell^{-\infty}). \] (4.24)

4.2. Energy lower bound

We now prove the lower bound corresponding to the upper bound proved in proposition 4.1.

**Proposition 4.2 (Energy lower bound).** For any fixed \( 1 < b < \Theta_0^{-1} \), as \( \ell, L \to +\infty \) and \( \delta \to 0 \),
\[ E_\Gamma \geq 2LE_{1D}(\ell) - \delta E_{\text{corr}} + \mathcal{O}(\delta^{4/3} \log \delta) + \mathcal{O}(L^2 \ell^{-\infty}). \] (4.25)

The starting point of the lower bound proof is a suitable energy splitting, which is somewhat customary in the study of the GL and related functionals.

**Proposition 4.3 (Energy splitting).** For any fixed \( 1 < b < \Theta_0^{-1} \), let the function \( u := u_+ \mathds{1}_{\Gamma^+} + u_- \mathds{1}_{\Gamma^-} \in C(\Gamma) \) be defined via
\[ \psi_{\Gamma} := u_+ f_0 e^{\phi_+}, \quad \text{in } \Gamma^+. \] (4.26)

Then, as \( \ell, L \to +\infty \) and \( \delta \to 0 \),
\[ E_\Gamma \geq 2LE_{1D}(\ell) - \delta E_{\text{corr}} - \delta \int_0^\ell dt \left( t + 2\alpha_0 \right) f_0^2 + \mathcal{E}_0[u] + \mathcal{O}(\delta^3), \] (4.27)

where we have used the compact notation \( \mathcal{E}_0[u] = \mathcal{E}_0[u^+; \Gamma^+] + \mathcal{E}_0[u^-; \Gamma^-] \) and
\[ \mathcal{E}_0[u; D] := \int_D ds dt f_0^2 \left\{ (\partial \mu)^2 + (\partial u)^2 - 2(t + \alpha_0) j_0[u] + \frac{1}{2b} f_0^2 (1 - |u|^2)^2 \right\}, \] (4.28)

with \( j_0[u] := \mathbf{e} \cdot \mathbf{j}[u] \) and the current \( j[u] \) defined as in (1.2).

**Proof.** Let us consider first \( \Gamma^+ \); by (4.26), dropping the label + on \( u \) for short,
\[ G[\psi_{\Gamma}; \Gamma^+] = \int_{\Gamma^+} ds \, dr \left\{ |(\nabla + i\mathbf{F}) (f_0 u e^{\phi_+})|^2 - \frac{1}{2b} (2f_0^2 |u|^2 - f_0^4 |u|^4) \right\}. \]
Using (4.6), we get
\[\left|(\nabla + i\mathbf{F}) (f_0 u e^{\alpha_0})\right|^2 = \left|(\nabla - i(t + \alpha_0) \hat{\mathbf{e}}_s) (f_0 u)\right|^2\]
\[= |f'_0|^2 u^2 + f_0^2 |\partial_s u|^2 + f_0^2 |\partial_\alpha u|^2 + (t + \alpha_0)^2 f_0^2 |u|^2 - 2(t + \alpha_0) f_0^2 j_s [u] + f_0 \partial_t f_0 \partial_s |u|^2.\]

Acting as in [CR2], we perform an integration by parts of the term \(f_0 (\partial_t f_0) \partial_s |u|^2\):
\[\int_{\Gamma^+} ds \, dt \, f_0 (t) \nabla f_0 \cdot \nabla |u|^2 = \int_{\partial \Gamma^+} d\sigma \, f_0 |u|^2 \hat{n}_+ \cdot \nabla f_0 \]
\[- \int_{\Gamma^+} ds \, dt \{ |f'_0|^2 |u|^2 - f_0 f_0'' |u|^2 \}, \quad (4.29)\]
where \(\hat{n}_+\) is the outward normal unit vector along \(\partial \Gamma^+\). The boundary of \(\Gamma^+\) is composed of four segments but
\[|\hat{n}_+ \cdot \nabla f_0| = |f'_0| = 0, \quad \text{on } \partial \Gamma^+ \cap \partial \Gamma_{in} \text{ and on } \partial \Gamma^+ \cap \partial \Gamma_{out}, \quad (4.30)\]
while \(|\hat{n}_+ \cdot \nabla f_0| = |\hat{\mathbf{e}}_s \cdot \nabla f_0| = 0 \quad \text{on } \partial \Gamma^+ \cap \partial \Gamma_{bd}\). Therefore, the boundary term above reduces to the one computed over the remaining portion of \(\partial \Gamma^+\), i.e., the bisectrix of the region \(\Gamma\), that we denote by \(\partial \Gamma_{bis}\). Hence, we obtain
\[\mathcal{G} [\psi; \Gamma^+] = \int_{\Gamma^+} ds \, dt \, f_0^2 \left\{ - f_0'' |u|^2 + |\nabla u|^2 + (t + \alpha_0)^2 |u|^2 \right\}
- 2(t + \alpha_0) j_s [u] + \frac{1}{2b} (2 - f_0^2) \right\} + \int_{\partial \Gamma_{bis}} d\sigma \, f_0 |u|^2 \hat{n}_+ \cdot \nabla f_0 \]
which, via the variational equations for \(f_0\), leads to
\[\mathcal{G} [\psi; \Gamma^+] = - \frac{1}{2b} \int_{\Gamma^+} ds \, dt \, f_0^2 (t) + \mathcal{E}_0 [u; \Gamma^+] + \int_{\partial \Gamma_{bis}} d\sigma \, f_0 |u|^2 \hat{n}_+ \cdot \nabla f_0. \quad (4.31)\]

Reproducing the computation in \(\Gamma^-\) and using (4.10) and (4.11) there, we end up with a similar expression, but, since \(\hat{n}_- = -\hat{n}_+\), the two boundary terms cancel out, since \(|u_+| = |u_-|\) on \(\partial \Gamma_{bis}\). The final result is then
\[\mathcal{G} [\psi; \Gamma] = - \frac{1}{2b} \int_{\Gamma^+ \cup \Gamma^-} ds \, dt \, f_0^2 (t) + \mathcal{E}_0 [u], \quad (4.32)\]
and it just remains to compute the first term on the rhs. Actually, such a computation has already been done in the upper bound proof, therefore we only sketch it:
where we have computed

\[
- \frac{1}{2b} \int_{\Gamma^+} \frac{ds}{dr} f_0'(t) f_0(t) = - \frac{1}{2b} \int_{\Gamma^+} \frac{ds}{dr} f_0'(t) f_0(t) + \frac{1}{2b} \int_{\Gamma^+} \frac{ds}{dr} f_0'(t)
\]

\[
= 2L_0^{\text{BD}}(t) - \int_{\Gamma^+} \frac{ds}{dr} \left\{ - f_0'(t) + \frac{1}{2b} (2 - f_0^2) f_0 \right\} f_0
\]

\[
= 2L_0^{\text{BD}} - \delta E_{\text{corr}} - \delta \int_0^t dt \left( t + \alpha_0 \right) (t + 2\alpha_0) f_0^4
\]

\[
+ O(\delta^3),
\]

(4.33)

and estimated (see (4.8))

\[
\int_{R^{+ \setminus 1 \setminus \Gamma^\pm}} \frac{ds}{dr} \left\{ f_0'^2 + \frac{1}{2b} (2 - f_0^2) f_0 \right\}
\]

\[
= \delta \int_0^t dt \left\{ f_0'^2 + \frac{1}{2b} (2 - f_0^2) f_0 \right\} + O(\delta^3).
\]

After the splitting, we have almost isolated in (4.27) the energy to extract from the lower bound. Indeed, the reduced energy $\mathcal{E}_0[u]$ turns out to be positive on a rectangle, as a consequence of the pointwise positivity of the cost function $K_{0,\ell}$ (see (A.13)). There is however an additional unexpected term of order $\delta$ (the third one on the rhs of (4.27)), which must cancel out in order to recover the correct energy. This is precisely the point where the phase discontinuity described in section 3 comes into play: the region $\Gamma$ is indeed obtained by gluing together two almost rectangular regions $\Gamma^\pm$ and $\mathcal{E}_0[u]$ is positive on it up to some boundary terms on the bisectrix, which exactly cancel the unwanted term in (4.27).

**Proposition 4.4 (Lower bound on the reduced energy).** Under the same assumptions of proposition 4.3,

\[
\mathcal{E}_0[u] \geq \delta \int_0^t dt \left( t + \alpha_0 \right) (t + 2\alpha_0) f_0^4 + \frac{1}{4b} \int_\Gamma \sqrt{f_0^4 (1 - |u|^2)^2}
\]

\[
+ O(\delta^{4/3} \log \delta) + O(L^2 \ell^{-\infty}).
\]

(4.34)
Proof. The idea of the proof goes back to the works on Bose–Einstein condensates (see, e.g., [CD, CPRY1, CPRY2, CR1, CRY, Rou]) and consists in an integration by parts of the current term in the energy, which is the only non-positive term there. This allows to express e.g., \[ CD, CPRY1, CPRY2, CR1, CRY, Rou \] and consists in an integration by parts of the \[ \text{vorticity measure} \ \text{curl} \ |u| \], which in turn is pointwise bounded by the gradient square \[ \|\nabla u\|^2 \]. In conclusion, one gets a lower bound to the energy given by the integral of a suitable function times \[ \|\nabla u\|^2 \]. Such a function turns out to be the \textit{cost function} \( K_{0,t} \) discussed in appendix A (see (A.13)) and its pointwise positivity is enough to complete the argument. As anticipated, however, the shape of the domain \( \Gamma = \Gamma^+ \cup \Gamma^- \) is responsible for the emergence of some boundary terms along the bisectrix, which are actually crucial to recover the first term on the rhs of (4.34).

Recalling the definition (A.14) of \( F_0 \), we write,

\[
-2 \int_{t^+} \int_{\Gamma^+} ds \ dt (t + \alpha_0) f_0^2(j_{u^+}) = - \int_{t^+} \int_{\Gamma^+} ds \ dt \ \nabla \cdot F_0 \cdot j_{u^+} = \int_{t^+} \int_{\Gamma^+} ds \ dt \ \nabla \cdot j_{u^+} \]

where \( \tau \) are the unit tangential vectors to \( \partial \Gamma^\pm \). We first consider the first term on the rhs above: since \( \nabla \cdot j_{u} = -\partial_\tau \text{Im} (u^* \partial_t u) + \partial_s \text{Im} (u^* \partial_s u) = 2 \text{Im} (\partial_s u^* \partial_t u) \) and \( F_0 \leq 0 \),

\[
\int_{t^+} \int_{\Gamma^+} ds \ dt F_0 \nabla \cdot j_{u} \geq \int_{t^+} \int_{\Gamma^+} ds \ dt F_0(t) |\nabla u|^2,
\]

yielding

\[
E_0[u] \geq \int_{t^+} \int_{\Gamma^+} ds \ dt K_{0,t}(t) |\nabla u|^2 + \frac{1}{2b} \int_{t^+} \int_{\Gamma^+} ds \ dt f_0^2(1 - |u|^2)^2 - \int_{\partial \Gamma^\pm} ds \ F_0 \ \tau \cdot j_{u^+} \\
\geq \frac{1}{2b} \int_{t^+} \int_{\Gamma^+} ds \ dt f_0^2(1 - |u|^2)^2 - \int_{\partial \Gamma^\pm} ds \ F_0 \ \tau \cdot j_{u^+} + O(L^2 \ell^{-\infty}),
\]

where we have used the pointwise positivity of \( K_{0,t} \) in \( \mathcal{I}_f \) (see (A.15)) and the exponential decay of \( \nabla \psi_T, \psi_T, f_0 \) and \( f_0^2 \) provided by (3.6), (3.7), (A.10) and (A.12) to estimate the rest: by (A.17),

\[
\int_{t^+} \int_{\Gamma^+} ds \ dt K_{0,t}(t) |\nabla u|^2 \geq \int_{t^+} \int_{\Gamma^+} ds \ dt f_0^2(t) |\nabla u|^2 \geq -Ct \int_{\ell}^L \int_{\ell} ds \ dt f_0^2(t) |\nabla u|^2 \\
\geq -Ct \int_{\ell}^L \int_{\ell} dt \left\{ L + t + f_0^2 \ell^2 \right\} |\psi_T|^2 + |\nabla \psi_T|^2 \right\} \\
= O(L^2 \ell^{-\infty}).
\]

Concerning the boundary term in (4.37), the vanishing of \( F_0 \) at \( t = 0 \) and \( t = \ell \) together with the vanishing of the \( t \)-current of \( u \) at the boundary \( \partial \Gamma_{bd} \) thanks to the boundary conditions satisfied by \( \psi_T \), imply that the only boundary terms surviving are the ones along \( \partial \Gamma_{bd} \). Exploiting the definitions of \( u^+ \) in (4.26), which yield
\[ \hat{\tau}_+ \cdot [u_+]_{\partial \Omega_{\text{bis}}} = f_0 - 2 \left[ j_0 |\psi_T| - |\psi_T|^2 \left( \alpha_0 \cos \vartheta_{\text{bis}} + \frac{1}{2} \vartheta \sin 2 \vartheta_{\text{bis}} \right) \right], \quad (4.38) \]

\[ \hat{\tau}_- \cdot [u_-]_{\partial \Omega_{\text{bis}}} = -f_0 - 2 \left[ j_0 |\psi_T| - |\psi_T|^2 \left( \alpha_0 \cos \vartheta_{\text{bis}} + \delta + \frac{1}{2} \vartheta \sin 2 \vartheta_{\text{bis}} + 2 \delta \right) \right], \quad (4.39) \]

and the trivial identities \( \cos \vartheta_{\text{bis}} = \sin \delta/2, \cos(\vartheta_{\text{bis}} + \delta) = -\sin \delta /2 \), we obtain

\[ \hat{\tau}_+ \cdot [u_+]_{\partial \Omega_{\text{bis}}} + \hat{\tau}_- \cdot [u_-]_{\partial \Omega_{\text{bis}}} = - \frac{2}{f_0} \left[ \psi_T \right]^2 \left( \alpha_0 \sin \delta /2 + \frac{1}{2} \vartheta \sin \delta \right) = - \frac{1}{2} \delta \left( 1 + \mathcal{O}(\delta^2) \right) \left( \alpha_0 + \vartheta \right) \left( |u_+|^2 + |u_-|^2 \right)_{\partial \Omega_{\text{bis}}}. \quad (4.40) \]

Hence, the boundary terms in (4.35) become (since \( \cos \delta /2 = 1 + \mathcal{O}(\delta^2) \))

\[ - \int_{\partial \Omega^+} d\sigma F_0 \hat{\tau}_+ \cdot [u_+] - \int_{\partial \Omega^-} d\sigma F_0 \hat{\tau}_- \cdot [u_-] = - \frac{1}{2} \delta \left( 1 + \mathcal{O}(\delta^2) \right) \int_0^\delta d\vartheta F_0(\vartheta \cos \delta /2) \left( \alpha_0 + \vartheta \cos \delta /2 \right) \left( |u_+|^2 + |u_-|^2 \right)_{\partial \Omega_{\text{bis}}}. \quad (4.41) \]

By (A.17) and the decay of \( \psi_T \) in (3.7), we can get rid of the prefactor \( 1 + \mathcal{O}(\delta^2) \) up to an error of order \( \delta^3 \). Furthermore, the smoothness of \( \Phi_0 \), the bound (A.17), the decay of \( f_0 \) in (A.10) and a direct integration by parts yield

\[ - \int_0^\delta d\vartheta F_0(\vartheta \cos \delta /2) \left( \alpha_0 + \vartheta \cos \delta /2 \right) = - \int_0^\delta d\vartheta F_0(\vartheta) \left( \alpha_0 + \vartheta \right) + \mathcal{O}(\delta^2) = \int_0^\delta d\vartheta \left( \alpha_0 \vartheta + \frac{1}{2} \vartheta^2 \right) + \mathcal{O}(\delta^2) = \int_0^\delta dt \left( t + \alpha_0 \right) (t + 2 \alpha_0) f_0^2 \left( t + \alpha_0 \right) + \mathcal{O}(\delta^2), \quad (4.42) \]

and plugging this into (4.37), we find

\[ \mathcal{E}_0[u] \geq \delta \int_0^\delta dt \left( t + \alpha_0 \right) (t + 2 \alpha_0) f_0^2 \left( 1 - |u|^2 \right)^2 + \frac{1}{2} \int_0^\delta \left| f_0^2 \left( 1 - |u|^2 \right)^2 \right|_{\partial \Gamma_{\text{bis}}} + \mathcal{O}(\delta^3) + \mathcal{O}(L^2 \ell^{-\infty}). \quad (4.43) \]

Therefore, it just remains to bound from below the third term on the rhs of (4.43). Let us consider only the contribution of \( u_+ \), since the other is perfectly analogous: let \( \chi \) be a smooth function such that \( \chi = 1 \) on \( \partial \Omega_{\text{bis}} \) and which vanishes on a line parallel to \( \partial \Omega_{\text{bis}} \) and passing through \((\gamma, 0)\) for some \( \gamma > 0 \) to be chosen later. Concretely, we may take a function \( \chi(s) \)
equal to 1 for $s = t \sin \delta/2$ and vanishing at $s = t \sin \delta/2 + \gamma$. Moreover, we may assume that $|\chi'(s)| \leq C_\gamma^{-1}$ and

$$
\delta \int_0^\ell dt \, F_0(t) (\alpha_0 + t) \left(1 - |u_+|^2\right) \chi|_{\partial T_{X0}}
= \delta \int_0^\ell dt \int_{t \sin \delta/2}^{\gamma + t \sin \delta/2} ds \, \chi'(s) F_0(t) (\alpha_0 + t) \left(1 - |u_+|^2\right)
- \delta \int_0^\ell dt \int_{t \sin \delta/2}^{\gamma + t \sin \delta/2} ds \, \chi(s) F_0(t) (\alpha_0 + t) \partial_t |u_+|^2.
$$

(4.44)

We now estimate the two terms on the rhs separately: acting as in the proof of lemma A.2 and using once more (A.17), we estimate, for any $a, d > 0$,

$$
\int_0^\ell dt \int_{t \sin \delta/2}^{\gamma + t \sin \delta/2} ds \, \chi(s) F_0(t) (\alpha_0 + t) \left(1 - |u_+|^2\right)
\geq\int_0^\ell dt \int_{t \sin \delta/2}^{\gamma + t \sin \delta/2} ds \, \chi'(s) F_0(t) (\alpha_0 + t) \left(1 - |u_+|^2\right) + O(\ell^{-\infty})
\geq -\frac{C}{\gamma} \int_0^\ell dt \int_{t \sin \delta/2}^{\gamma + t \sin \delta/2} ds \, (1 + t) f_0^2 (1 - |u_+|^2) + O(\ell^{-\infty})
\geq -\frac{C}{\gamma} \left[ \frac{1}{d} \int_{\Gamma^+} ds \, d t f_0^2 (1 - |u_+|^2)^2 + a d^3 \gamma + e^{-\frac{1}{2}a^2}\right],
$$

which leads to

$$
\delta \int_0^\ell dt \int_{t \sin \delta/2}^{\gamma + t \sin \delta/2} ds \, \chi'(s) F_0(t) (\alpha_0 + t) \left(1 - |u_+|^2\right)
\geq -\frac{1}{4b} \int_{\Gamma^+} ds \, d t f_0^2 (1 - |u_+|^2)^2
+ O(\delta^2 \gamma^{-1} |\log \delta|^3),
$$

(4.45)

after taking $a = \frac{d \log \delta}{\gamma}$ and $d = |\log \delta|$. For the other term, we directly apply (3.6) to get

$$
-\delta \int_0^\ell dt \int_{t \sin \delta/2}^{\gamma + t \sin \delta/2} ds \, \chi(s) F_0(t) (\alpha_0 + t) \partial_t |u_+|^2
\geq -C \delta \int_0^\ell dt \int_{t \sin \delta/2}^{\gamma + t \sin \delta/2} ds \, (1 + t) |\psi T| |\partial_\nu \psi T| + O(\ell^{-\infty})
= O(\delta \sqrt{\gamma}) + O(\ell^{-\infty}),
$$

(4.46)

via Cauchy–Schwarz inequality. Plugging (4.45) and (4.46) into (4.44), we finally get

$$
\delta \int_0^\ell dt \, F_0(t) (\alpha_0 + t) \left(1 - |u_+|^2\right) \chi|_{\partial T_{X0}} + \frac{1}{4b} \int_{\Gamma^+} ds \, d t f_0^2 (1 - |u_+|^2)^2
= O(\delta^2 \gamma^{-1} |\log \delta|^3) + O(\delta \sqrt{\gamma}) + O(\ell^{-\infty}) = O(\delta^{4/3} |\log \delta|) + O(\ell^{-\infty}),
$$

(4.47)
after an optimization over $\gamma$, i.e., taking $\gamma = \delta^{2/3} |\log \delta|^2$. Hence, we can use the second term on the rhs of (4.43) to compensate for the negative contribution on the lhs above, to obtain the result. □

4.3. Completion of the proofs

We are now in position to complete the proofs of our mains results.

**Proof of theorem 2.3.** For corner angle $\pi - \delta$, the result is obtained via a straightforward combination of propositions 4.1, 4.2 and (2.13). Let us then comment on the adaptations needed for opening angle $\pi + \delta$. The strategy of the proof is indeed the same. In particular, in the upper bound, one has to split the region $\Gamma$ into the subregions $\Gamma_\pm$, $\Gamma_{\alpha}^\pm$ as before, with $\gamma \ll \delta$, and use the same trial state as in (4.4). The outcome is the same estimate as in (4.3) with an opposite sign in front of $\delta$, which is due to the fact that in this case $R^\pm \subset \Gamma^\pm$. The lower bound, on the other hand, is proven in exactly the same way and, in particular, the phase singularity along $\partial \Gamma_{\text{bs}}$ appears as well. □

It just remains to address corollary 2.1:

**Proof of corollary 2.1.** The result is a direct consequence of Gauss–Bonnet theorem for piecewise smooth domains combined with the energy asymptotics proved in theorems 2.2 and 2.3. In fact, by Gauss–Bonnet, one gets

$$\int_{\partial \Omega_{\text{smooth}}} ds \, r(s) + \sum_{j \in \Sigma} (\pi - \beta_j) = 2\pi. \quad (4.48)$$

If we use the asymptotics (2.19) for $E_{\text{corner,}\beta_j}$, assuming that $|\beta_j - \pi| \ll 1$, $\forall j \in \Sigma$, together with the energy asymptotics (2.8), choosing $L, \ell = O(|\log \varepsilon|)$ as in [CG2], we get (2.20). □

**Acknowledgments**

The authors are thankful to S Fournais and N Rougerie for useful comments and remarks about this work. The support of the National Group of Mathematical Physics (GNFM) of INdAM through Progetto Giovani 2016 ‘Superfluidity and Superconductivity’ and Progetto Giovani 2018 ‘Two-dimensional Phases’ is also acknowledged.

**Appendix A. One-dimensional effective models**

In this appendix we collect some well known results about the effective one-dimensional problems, which are known to play a role in surface superconductivity. We refer to [CR2, CR3, CR4, CDR, CG2] for more details (see, in particular, [CG2, appendix A] for a review).

**A.1. Effective model on the half-line**

The effective model useful to describe the behavior of the order parameter in the surface superconductivity regime is given in first approximation by the following energy functional:

$$\mathcal{E}_{\alpha,0}[f] := \int_0^{+\infty} dt \left\{ |\partial_t f|^2 + (t + \alpha)^2 f^2 - \frac{1}{2b}(2f^2 - f^4) \right\}, \quad (A.1)$$
where \( t \) is the rescaled distance from the outer boundary and \( \alpha \in \mathbb{R} \).

For any fixed \( \alpha \in \mathbb{R} \), the functional (A.1) has a unique minimizer in the domain \( \mathcal{D}^{1 \alpha} = \{ f \in H^1(\mathbb{R}^+; \mathbb{R}) \mid t f(t) \in L^2(\mathbb{R}^+) \} \), which is strictly positive and monotonically decreasing for \( t \) large enough. We set \( E_{\alpha}^{1 \mathcal{D}} = \inf_{\alpha \in \mathbb{R}} E_{\alpha \mathcal{D}}^{1 \mathcal{D}} = \inf_{\alpha \in \mathbb{R}} \inf_{f \in \mathcal{D}^{1 \alpha}} \mathcal{E}_{\alpha \mathcal{D}}^{1 \mathcal{D}}(f) \) and denote by \((\alpha^\ast, f^\ast) \in \mathbb{R} \times \mathcal{D}^{1 \mathcal{D}}\) any minimizing pair for (A.1). We recall some key properties of the minimization (A.1):

- Variational equation for \( f^\ast \):
  \[
  -f''^\ast + (t + \alpha^\ast)^2 f^\ast = \frac{1}{b} (1 - f^2) f^\ast, \quad \text{with } f'(0) = 0; \tag{A.2}
  \]

- Optimality of the phase \( \alpha^\ast \):
  \[
  \int_0^{+\infty} dt \left( t + \alpha^\ast \right) f^2(t) = 0; \tag{A.3}
  \]

- The groundstate energy can be expressed as
  \[
  E_{\alpha^\ast}^{1 \mathcal{D}} = -\frac{1}{2b} \int_0^{+\infty} dt f^2(t) < 0. \tag{A.4}
  \]

- \( f^\ast \) decays exponentially in the distance from the boundary [CR2, proposition 3.3]:
  \[
  f^\ast \leq C \exp\left\{ -\frac{1}{2} (t + \alpha^\ast)^2 \right\}. \tag{A.5}
  \]

### A.2. Effective model on a finite interval

We now discuss a variant of the 1D effective model above by minimizing the energy on a finite interval \( I_\ell := [0, \ell], \ell \gg 1 \), rather than in the whole of \( \mathbb{R}^+ \), i.e., we set

\[
\mathcal{E}_{\alpha I_\ell}^{1 \mathcal{D}}[f] := \int_0^\ell dt \left\{ \left| \partial_t f \right|^2 + (t + \alpha)^2 f^2 - \frac{1}{2b} (2 f^2 - f^4) \right\},
\]

\[
E_0^{1 \mathcal{D}} := \inf_{\alpha \in \mathbb{R}} \inf_{f \in H^1(I_\ell)} \mathcal{E}_{\alpha I_\ell}^{1 \mathcal{D}}[f]. \tag{A.6}
\]

Exactly as before, one can prove that there is a unique minimizing pair \((\alpha_0, f_0) \in \mathbb{R} \times H^1(I_\ell)\) of \( \mathcal{E}_{\alpha I_\ell}^{1 \mathcal{D}} \). Moreover, \( f_0 \) satisfies the same variational equation of \( f^\ast \) in (A.2) in the interval \([0, \ell]\) with \( \alpha_0 \) in place of \( \alpha^\ast \), i.e.,

\[
-f''_0 + (t + \alpha_0)^2 f_0 = \frac{1}{b} (1 - f^2_0) f_0. \tag{A.7}
\]

In addition, \( f_0 \) satisfies Neumann boundary conditions

\[
f'_0(0) = f'_0(\ell) = 0. \tag{A.8}
\]

Furthermore, all the properties (A.3)–(A.5) hold true for \( f_0 \) in the interval \([0, \ell]\). In particular, \( f_0 \) is monotonically decreasing for \( t \geq t_0 \), where \( t_0 \) is the unique maximum point of \( f_0 \) and it satisfies

\[
0 < t_0 \leq |\alpha_0| + \frac{1}{\sqrt{b}}. \tag{A.9}
\]
and
\[
f_0(t) \leq C \exp \left\{ -\frac{1}{2}(t + \alpha_0)^2 \right\}.
\] (A.10)

A similar estimate holds true for the derivative of \( f_0 \) (see [CG2, lemma A.1]): for any \( 1 < b < \Theta_0^{-1} \) and \( \ell \in [1, \bar{\ell}] \), there exists a finite constant \( C > 0 \), such that
\[
|f'_0(t)| \leq C \exp \left\{ \frac{1}{2} (t + \alpha_0)^2 \right\}, \quad \forall \ t \in [0, \ell],
\] (A.11)
\[
|f'_0(t)| \leq C T \ f_0(t), \quad \forall \ t \in [0, \bar{\ell}].
\] (A.12)

### A.3. Cost function

A very important object constructed over \( f_0 \) is the cost function
\[
K_0, \ell := (1 - d_\ell) f_0^2(t) + F_0(t), \quad d_\ell = \mathcal{O}(\ell^{-4}),
\] (A.13)
where (by optimality of \( \alpha_0 \))
\[
F_0(t) := 2 \int_{t}^{\ell} d\eta \ (\eta + \alpha_0) f_0^2(\eta) = -2 \int_{t}^{\ell} d\eta \ (\eta + \alpha_0) f_0^2(\eta).
\] (A.14)

The most important property of \( K_0, \ell \) is its pointwise positivity in the relevant region, i.e., where \( f_0 \) is not exponentially small in \( \ell \): for any \( 1 < b < \Theta_0^{-1} \),
\[
K_0, \ell(t) \geq 0, \quad \text{for any } t \in I_\ell,
\] (A.15)
where \( I_\ell \) is defined as
\[
I_\ell := \{ t \in (0, \ell) \mid f_0(t) \geq \ell^3 f_0(t) \} = [0, \bar{\ell}].
\]
Finally, we underline that (A.2) and (A.5), imply that \( \bar{\ell} = \ell + \mathcal{O}(1) \) and, as a consequence,
\[
f_0(t) = \mathcal{O}(\ell^{-\infty}), \quad \text{for } t \in [0, \ell \setminus I_\ell.
\] (A.16)

Note that, as a by-product of the positivity of \( K_0, \ell \), we also get
\[
|F_0(t)| \leq \begin{cases} f_0^2(t), & \text{for } t \in I_\ell, \\ Cf_0^2(t), & \text{for } t \in I_\ell^c. \end{cases}
\] (A.17)

since, inside \( I_\ell \), \( |F_0(t)| \leq f_0^2(t) \), while, for \( t \geq \bar{\ell} \), we can use the monotonicity of \( f_0 \) to bound
\[
|F_0(t)| = \int_{t}^{\ell} d\eta \ 2(\eta + \alpha_0)f_0^2(\eta) \leq Cf_0^2(t).
\]
A.4. Useful estimates close to $\partial \Omega_{\text{bis}}$

We prove here some useful bounds in the regions $\Gamma_{\pm}^\gamma$. The key observation is that, in those regions, $\vartheta = \vartheta_{\text{bis}} + O(\gamma) = \frac{\pi}{2} + O(\delta + \gamma)$, so that

$$\sin \vartheta = 1 + O(\gamma^2), \quad \cos \vartheta = O(\gamma),$$

under the assumption $\delta = O(\gamma)$ (recall (4.9)) and this leads to some other useful approximations.

Lemma A.1. Let $\delta = O(\gamma)$. Then, for any $\vartheta \in [\vartheta_-, \vartheta_+]$, there exist two finite constant $c, C > 0$, such that

$$|f_0(\varrho \sin \vartheta) - f_0(\varrho)| \leq C \gamma e^{-c \varrho^2}, \quad (A.18)$$

$$|f'_0(\varrho \sin \vartheta) - f'_0(\varrho)| \leq C \gamma^2 e^{-c \varrho^2}. \quad (A.19)$$

Proof. For the proof of (A.18) we simply observe that a straightforward application of Taylor formula yields

$$f_0(\varrho \sin \vartheta) = f_0(\varrho) + f'_0(\varrho \sin \tilde{\vartheta}) \left( \vartheta - \frac{\pi}{2} \right),$$

for some $\tilde{\vartheta} \in (\vartheta_-, \vartheta_+)$. We now use (A.11) to bound

$$\left| f'_0(\varrho \sin \tilde{\vartheta}) \right| \leq C e^{-\frac{1}{4} \varrho^2 \sin^2 \tilde{\vartheta}} \leq C e^{-c \varrho^2},$$

for some $c > 0$, which implies the result.

To prove (A.19), we use again the Taylor formula applied to $f'_0$ and exploit the variational equation (A.7) to control $f''_0$, which yields the desired estimate. We omit the computation for the sake of brevity. $\square$

Lemma A.2. Let $\delta = O(\gamma), \beta \geq 0$ and $a > 1$. Then, for any $\vartheta \in [\vartheta_-, \vartheta_+]$ and any function $u \in C(\Gamma)$, we get

$$\int_{\Gamma_{\pm}^\gamma} d\vartheta d\varrho \varrho^\beta f_0^2(\varrho \sin \vartheta)|u|^2 - \gamma \int_0^\ell d\varrho \varrho^\beta f_0^2(\varrho) \geq \frac{1}{a} \int_{\Gamma_{\pm}^\gamma} d\vartheta d\varrho \varrho f_0^3(\varrho \sin \vartheta)(1 - |u|^2)^2$$

$$+ O(\gamma^2) + O \left( \gamma a |\log a|^{\beta} \right). \quad (A.20)$$

Proof. We write

$$\int_{\Gamma_{\pm}^\gamma} d\vartheta d\varrho \varrho^\beta f_0^2(\varrho \sin \vartheta)|u|^2 = \int_{\Gamma_{\pm}^\gamma} d\vartheta d\varrho \varrho^\beta f_0^2(\varrho \sin \vartheta)$$

$$+ \int_{\Gamma_{\pm}^\gamma} d\vartheta d\varrho \varrho^\beta f_0^2(\varrho \sin \vartheta) (|u|^2 - 1) \quad (A.21)$$

and exploit the approximation of $f_0(\varrho \sin \vartheta)$ proved in lemma A.1, specifically (A.18), to obtain

$$\int_{\Gamma_{\pm}^\gamma} d\vartheta d\varrho \varrho^\beta f_0^2(\varrho \sin \vartheta) = \gamma \int_0^\ell d\varrho \varrho^\beta f_0^2(\varrho) + O(\gamma^2), \quad (A.22)$$
where we have replaced $\ell / \sin \vartheta$ and $\ell / \sin(\vartheta + \delta)$ with $\ell$ in the integration up to a small error $O(\ell^{-\infty})$.

We now estimate the second term in (A.21). Let us consider only $\Gamma^+_{\vartheta}$ since the estimate in $\Gamma^-_{\vartheta}$ is identical: for any $d \in (0, \ell)$

\[
\int_{\Gamma^+_{\vartheta}} d\vartheta d\varphi \varphi^3 f_0^3(\varphi \sin \vartheta)(|u|^2 - 1) = \int_{\vartheta_c}^{\vartheta_b} \int_{d}^{d_{\varphi}} d\vartheta d\varphi \varphi^3 f_0^3(\varphi \sin \vartheta)(|u|^2 - 1)
+ \int_{\vartheta_c}^{\vartheta_b} \int_{d}^{d_{\varphi}} d\vartheta d\varphi \varphi^3 f_0^3(\varphi \sin \vartheta)(|u|^2 - 1).
\]

(A.23)

We estimate the first term on the rhs of (A.23) as

\[
\int_{\vartheta_c}^{\vartheta_b} \int_{d}^{d_{\varphi}} d\vartheta d\varphi \varphi^3 f_0^3(\varphi \sin \vartheta)(|u|^2 - 1)
\geq -\frac{\alpha^2}{2} \int_{\vartheta_c}^{\vartheta_b} \int_{d}^{d_{\varphi}} d\vartheta d\varphi \varphi^2(1 - |u|^2)^2
\geq -\frac{1}{a} \int_{\Gamma^+_{\vartheta}} d\vartheta d\varphi \varphi^3 f_0^3(\varphi \sin \vartheta)(1 - |u|^2)^2 - Ca\gamma d^{2/3}.
\]

On the other hand, the second term on the rhs of (A.23) decays exponentially in $d$: by the exponential decay of $f_0$ in (A.10) and the fact that $\sin \vartheta = 1 + O(\gamma^2) > 0$, we deduce that

\[
\int_{\vartheta_c}^{\vartheta_b} \int_{d}^{d_{\varphi}} d\vartheta d\varphi \varphi^3 f_0^3(\varphi \sin \vartheta)(|u|^2 - 1)
\geq -\int_{\vartheta_c}^{\vartheta_b} \int_{d}^{d_{\varphi}} d\vartheta d\varphi \varphi^3 f_0^3(\varphi \sin \vartheta)
\geq -C \gamma e^{-\frac{d}{2}}.
\]

(A.24)

which completes the proof after an optimization over $d$.

\[
\square
\]

ORCID iDs

Michele Correggi
https://orcid.org/0000-0001-6522-5241

Emanuela L Giacomelli
https://orcid.org/0000-0002-8147-3250

References

[A-B-N] Alouges F and Bonnaillie-Noël V 2006 Numerical computations of fundamental eigenstates for the Schrödinger operator under constant magnetic field Numer. Methods Part. Differ. Equ. 22 1090–105

[Ass2] Assaad W 2020 Magnetic steps on the threshold of the normal state J. Math. Phys. 61 101508

[Ass1] Assaad W 2020 The breakdown of superconductivity in the presence of magnetic steps Commun. Contemp. Math. 23 2050005

[AG] Assaad W and Giacomelli E L 2021 3D-Schrödinger operators under magnetic steps (arXiv:2108.04580)
[AK] Assaad W and Kachmar A 2016 The influence of magnetic steps on bulk superconductivity
Discrete Continuous Dyn. Syst. A. 36 6623–43

[AKP-S] Assaad W, Kachmar A and Persson-Sundqvist M 2019 The distribution of superconductivity
near a magnetic barrier Commun. Math. Phys. 366 269–332

[BCS] Bardeen J, Cooper L N and Schrieffer J R 1957 Theory of superconductivity Phys. Rev. 108
1175–204

[Bon] Bonnaillie V 2005 On the fundamental state energy for a Schrödinger operator with magnetic
field in domains with corners Asymptotic Anal. 41 215–58

[B-NF] Bonnaillie-Noël V and Fournais S 2007 Superconductivity in domains with corners Rev. Math.
Phys. 19 607–37

[Cor] Correggi M 2020 Surface effects in superconductors with corners Bull. Unione Mat. Ital. 14
51–67

[CDR] Correggi M, Devanarayanan B and Rougerie N 2017 Universal and shape dependent features
of surface superconductivity Eur. Phys. J. B 90 231

[CD] Correggi M and Dimonte D 2016 On the third critical speed for rotating Bose–Einstein
condensates J. Math. Phys. 57 071901

[CG1] Correggi M and Giacomelli E L 2017 Surface superconductivity in presence of corners Rev.
Math. Phys. 29 1750005

[CG2] Correggi M and Giacomelli E L 2021 Effects of corners in surface superconductivity Calc. Var.
Partial Differential Equations in press

[CPRY1] Correggi M, Pinsker F, Rougerie N and Yngvason J 2011 Critical rotational speeds in the
Gross–Pitaevskii theory on a disc with Dirichlet boundary conditions J. Stat. Phys. 143
261–305

[CPRY2] Correggi M, Pinsker F, Rougerie N and Yngvason J 2012 Critical rotational speeds for
superfluids in homogeneous traps J. Math. Phys. 53 095203

[CR1] Correggi M and Rougerie N 2013 Inhomogeneous vortex patterns in rotating Bose–Einstein
condensates Commun. Math. Phys. 321 817–60

[CR3] Correggi M and Rougerie N 2015 Boundary behavior of the Ginzburg–Landau order parameter
in the surface superconductivity regime Arch. Ration. Mech. Anal. 219 553–606

[CR4] Correggi M and Rougerie N 2016 Effects of boundary curvature on surface superconductivity
Lett. Math. Phys. 106 445–67

[CR2] Correggi M and Rougerie N 2014 On the Ginzburg–Landau functional in the surface super-
conductivity regime Commun. Math. Phys. 332 1297–343
Correggi M and Rougerie N 2015 Commun. Math. Phys. 338 1451–2 (erratum)

[CRY] Correggi M, Rougerie N and Yngvason J 2011 The transition to a giant vortex phase in a fast
rotating Bose–Einstein condensate Commun. Math. Phys. 303 451–508

[ELP-O] Exner P, Lotoreichik V and Pérez-Obiol A 2018 On the bound states of magnetic Laplacians
on wedges Rep. Math. Phys. 82 161–85

[FDM] Fomin V M, Devereese J T and Moshchalkov V V 1998 Surface superconductivity in a wedge
Europhys. Lett. 42 553–8
Fomin V M, Devereese J T and Moshchalkov V V 1999 Europhys. Lett. 46 118–9 (erratum)

[FH] Fournais S and Helffer B 2010 Spectral Methods in Surface Superconductivity Progress in Nonlinear Differential Equations and Their Applications vol 77 (Basel: Birkhäuser)

[FK1] Fournais S and Kachmar A 2011 Nucleation of bulk superconductivity close to critical magnetic
field Adv. Math. 226 1213–58

[FK2] Fournais S and Kachmar A 2013 The ground state energy of the three dimensional
Ginzburg–Landau functional: I. Bulk regime Commun. PDE 38 339–83

[FKP] Fournais S, Kachmar A and Persson M 2013 The ground state energy of the three dimensional
Ginzburg–Landau functional: II. Surface regime J. Math. Pure Appl. 99 343–74

[FMP] Fournais S, Miqueu J-P and Pan X-B 2019 Concentration behavior and lattice structure of 3D
surface superconductivity in the half space Math. Phys. Anal. Geom. 22 12

[Gia] Giacomelli E L 2018 Surface superconductivity in presence of corners PhD Thesis Sapienza
University of Rome

[GL] Ginzburg V L and Landau L D 1950 On the theory of superconductivity Zh. Eksp. Teor. Fiz. 20
1064–82

[Gri] Grisvard P 2011 Elliptic Problems in Nonsmooth Domains Classics in Applied Mathematics
vol 69 (Philadelphia, PA: SIAM)
[HK] Helffer B and Kachmar A 2018 The density of superconductivity in domains with corners Lett. Math. Phys. 108 2169–87

[Jad] Jadallah H T 2001 The onset of superconductivity in domains with corner J. Math. Phys. 42 4101

[P] Popoff N 2013 The Schrödinger operator on an infinite wedge with a tangent magnetic field J. Math. Phys. 54 041507

[Rou] Rougerie N 2011 The giant vortex state for a Bose–Einstein condensate in a rotating anharmonic trap: extreme rotation regimes J. Math. Pure Appl. 95 296–347

[SJdG] Saint-James D and de Gennes P G 1963 Onset of superconductivity in decreasing fields Phys. Lett. 7 306–8

[SS] Sandier E and Serfaty S 2007 Vortices in the Magnetic Ginzburg–Landau Model Progress in Nonlinear Differential Equations and Their Applications vol 70 (Basel: Birkhäuser) erratum available at http://ann.jussieu.fr-serfaty/publis.html