CHARACTERIZATIONS OF SMOOTH PROJECTIVE 
HOROSPHERICAL VARIETIES OF PICARD NUMBER ONE

JAEHYUN HONG AND SHIN-YOUNG KIM

Abstract. Let \( X \) be a smooth projective horospherical variety of Picard number one. We show that a uniruled projective manifold of Picard number one is biholomorphic to \( X \) if its variety of minimal rational tangents at a general point is projectively equivalent to that of \( X \). To get a local flatness of the geometric structure arising from the variety of minimal rational tangents, we apply the methods of \( W \)-normal complete step prolongations. We compute the associated Lie algebra cohomology space of degree two and show the vanishing of holomorphic sections of the vector bundle having this cohomology space as a fiber.

1. Introduction

Let \( M \) be a projective uniruled algebraic variety over \( \mathbb{C} \). Given a covering family \( \mathcal{K} \) of minimal rational curves, by collecting the tangent directions of rational curves in \( \mathcal{K} \) passing through each point in \( M \), we define a subbundle \( \mathcal{C}(M) \) of the projectivization \( \mathbb{P}(TM) \) of the tangent bundle, called the variety of minimal rational tangents associated to \( \mathcal{K} \). For a precise definition, see Definition 2.1.

The theory of varieties of minimal rational tangents, introduced by Hwang and Mok, has played an important role in the complex geometry of a uniruled projective manifold of Picard number one, that is, a Fano manifold of Picard number one. A general philosophy in this theory is that one should be able to recover the complex geometry of a uniruled projective manifold of Picard number one, such as the deformation rigidity and the stability of the tangent bundles, from the projective geometry of its varieties of minimal rational tangents. There are many results manifesting this philosophy, one of which is recognizing rational homogeneous varieties of Picard number one by their varieties of minimal rational tangents.

Theorem 1.1 (Main Theorem of [19], Main Theorem of [5]). Let \( X \) be a rational homogeneous variety associated to a long root and let \( \mathcal{C}_o(X) \subset \mathbb{P}T_oX \) be the variety of minimal rational tangents at a base point \( o \in X \). Let \( M \) be a Fano manifold of Picard number one and \( \mathcal{C}_x(M) \subset \mathbb{P}T_xM \) be the variety of minimal rational tangents at a general point \( x \in X \) associated to a minimal dominant rational component. Suppose that \( \mathcal{C}_x(M) \subset \mathbb{P}T_xM \) are projectively equivalent to \( \mathcal{C}_o(X) \subset \mathbb{P}T_oX \) for general \( x \in M \). Then \( M \) is biholomorphic to \( X \).

Here, for a rational homogeneous variety \( X \) associated to a long root, there is a canonical choice of a covering family \( \mathcal{K} \) of minimal rational curves and \( \mathcal{C}(X) \) is the variety of minimal rational tangents associated with \( \mathcal{K} \).

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Two ingredients of the proof of Theorem 1.1 are the following:

- (Local equivalence problem) existence of a biholomorphism \( \phi \) from a connected open subset \( U \) of \( M \) onto an open subset of \( X \) whose differential \( d\phi \) maps \( C_x(M) \) onto \( C_{\phi(x)}(X) \) for any \( x \in U \);
- (Extension problem) extension of a biholomorphism between connected open subsets of two Fano manifolds of Picard number one which preserves varieties of minimal rational tangents.

The second problem has an answer we can apply to any Fano manifold of Picard number one under mild geometric conditions (Theorem 2.4), while the first problem should be treated, case by case, depending on \( X \). We will focus on this local equivalence problem in this paper.

Given a quasi-homogeneous variety \( X \) of Picard number one which is uniruled and smooth, we may ask the same question, whether we can characterize \( X \), or, the open dense orbit \( X_0 \) of the automorphism group of \( X \), in terms of the variety of minimal rational tangents. To make this a reasonable question, it is natural to assume that there is a minimal rational curve in \( X \) contained in \( X_0 \) completely. This happens, for example, when the boundary \( X \setminus X_0 \) has codimension \( \geq 2 \) in \( X \). Among such quasi-homogeneous varieties, we will consider smooth projective horospherical varieties of Picard number one.

For a reductive algebraic group \( L \), a normal \( L \)-variety \( X \) is said to be horospherical if it has an open \( L \)-orbit \( L/H \) whose isotropy group \( H \) contains the unipotent part of a Borel subgroup of \( L \). Then the normalizer \( P \) of \( H \) in \( L \) is a parabolic subgroup of \( L \) and the open orbit \( L/H \) is isomorphic to a \((\mathbb{C}^*)^r\)-bundle over the rational homogeneous variety \( L/P \) for some \( r \in \mathbb{Z}_{\geq 0} \).

Pasquier classified smooth projective horospherical varieties of Picard number one and obtained the following result.

**Theorem 1.2** (Theorem 0.1 of [22]). Let \( L \) be a reductive group. Let \( X \) be a smooth nonhomogeneous projective horospherical \( L \)-variety with Picard number one. Then \( X \) is uniquely determined by its two closed \( L \)-orbits \( Y \) and \( Z \), isomorphic to \( L/P_Y \) and \( L/P_Z \), respectively; and \((L, P_Y, P_Z)\) in one of the triples listed below.

1. \((B_m, \alpha_{m-1}, \alpha_m)\) for \( m \geq 3 \);
2. \((B_3, \alpha_1, \alpha_3)\);
3. \((C_m, \alpha_{i+1}, \alpha_i)\) for \( m \geq 2, i \in \{1, \ldots, m-1\} \);
4. \((F_4, \alpha_2, \alpha_3)\);
5. \((G_2, \alpha_2, \alpha_1)\).

Here, we take the convention that \( \alpha_3, \alpha_4 \) are short roots when \( L \) is \( F_4 \) and \( \alpha_1 \) is a short root when \( L \) is \( G_2 \).

The main result in this paper is to characterize smooth horospherical varieties by using the variety of minimal rational tangents as Theorem 1.1. As before, there is a canonical choice of a covering family of minimal rational curves on them (Proposition 1.6).

**Theorem 1.3.** Let \( X \) be a smooth projective horospherical variety of Picard number one. Let \( C_o(X) \) denote the variety of minimal rational tangents at a base point \( o \) of \( X \). Let \( M \) be a Fano manifold of Picard number one and \( C_x(M) \subset \mathbb{P}T_xM \) be the variety of minimal rational tangents at a general point \( x \in X \) associated to a minimal dominant rational component. Suppose that \( C_x(M) \subset \mathbb{P}T_xM \) are projectively equivalent to \( C_o(X) \subset \mathbb{P}T_oX \) for general \( x \in M \). Then \( M \) is biholomorphic to \( X \).
For the smooth horospherical varieties $(C_m, \alpha_{i+1}, \alpha_i)$ and $(G_2, \alpha_2, \alpha_1)$, Hwang and Li solve the characterization problem, proving that these horospherical varieties are characterized by their varieties of minimal rational tangents ([8], [9]). In this paper we prove the same characterization for other smooth horospherical varieties in the list of Theorem 1.2.

As we said at the beginning, the main issue is how to obtain the local equivalence of geometric structures modeled on $C(X) \subset \mathbb{P}(TX)$.

One effective way to solve the local equivalence problem of geometric structures is by constructing Cartan connections. In the cases dealt with Theorem 1.1 the existence of a Cartan connection solving local equivalence problem of geometric structures modeled on the subbundle $C(X) \subset \mathbb{P}(TX)$ was proved by Tanaka (Theorem 3.12). Later on, Morimoto extended the theory of Cartan connections to geometric structures satisfying the condition (C) (Theorem 3.13). Hwang and Li developed a way to solve the local equivalence problem from the vanishing of certain types of sections (Theorem 3.14). In this paper we use the prolongation methods (Theorem 3.15 and Proposition 3.16) together with the computation of the Lie algebra cohomology.

The computation of Lie algebra cohomology $H^2(m, \Gamma)$ is of independent interest, where $m$ is the negative part of a graded Lie algebra $g$ and $\Gamma$ is a representation of $g$. When $g$ is not semisimple, we cannot apply the theory of Kostant directly to compute it ([17]). We suggest a way to compute the Lie algebra cohomology $H^2(m, \Gamma)$ by reducing it to the computation of the Lie algebra cohomology associated with the restriction of the action to the semisimple part of $g$. It is worth comparing our procedure with the Lyndon-Hochschild-Serre spectral sequence, which is a tool to compute the Lie algebra cohomology when $g$ is not semisimple ([1], [3]).

The horospherical variety of type $(B_3, \alpha_1, \alpha_3)$ is a smooth hyperplane section of the Spinor variety $S_5$ of dimension 10 and its variety of minimal rational tangents is a hyperplane section of the Grassmannian $Gr(2, 5)$. Any two smooth hyperplane sections of $S_5$ are projectively equivalent ([2]). Theorem 1.3 implies that the smooth hyperplane section of $S_5$ can be characterized by the variety of minimal rational tangents. However, a smooth codimension two linear section of $S_5$ does not have this property. Indeed, there are two non-isomorphic smooth codimension two linear sections of $S_5$, both of which are quasi-homogeneous (Proposition 4.8 of [1]) and have the same variety of minimal rational tangents being codim 2 linear section of $Gr(2, 5)$.

This paper is organized as follows. We review the theory of varieties of minimal rational tangents and collect results on the second fundamental forms and the third fundamental forms of varieties of minimal rational tangents in Section 2. Section 3 devotes to the theory of geometric structures, correspondence between $G_0$-structures and $S$-structures, and we review the prolongation methods which gives the local equivalence under the vanishing of sections of vector bundles associated to Lie algebra cohomology of degree 2. We describe the varieties of minimal rational tangents of smooth horospherical varieties of Picard number one in Section 4.

In the remaining sections, we prove Theorem 1.3 in the case when $X$ is $(B_m, \alpha_{m-1}, \alpha_m)$ or $(F_4, \alpha_2, \alpha_3)$ or $(B_3, \alpha_1, \alpha_3)$. For notational purposes, we break the case into two parts: The first two cases and the last case. Restricting ourselves to horospherical varieties $(B_m, \alpha_{m-1}, \alpha_m)$ and $(F_4, \alpha_2, \alpha_3)$, we confirm that the requirements to apply theories in
Section 2 and Section 3 hold (Section 5), we compute the Lie algebra cohomology $H^2(m, g)$ (Section 6), and we show that any section of the associated vector bundle with fiber $H^2(m, g)$ vanishes, which completes the proof of main Theorem (Section 7). In the final section, we repeat the same process for $X = (B_3, \alpha_1, \alpha_3)$. The computations are relatively shorter since the structure of the Lie algebra of $\text{Aut}(X)$ and the projective geometry of the variety of minimal rational tangents are less complicated.

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2. **Varieties of minimal rational tangents**

2.1. **Definitions and properties.** We recall definitions and properties of varieties of minimal rational tangents. For details, see Section 1 of [11] and Section 1 of [19].

**Definition 2.1.** Let $M$ be a uniruled projective manifold and let $K$ be a minimal rational component, i.e., an irreducible component of the space of rational curves on $M$ such that the rational curves in $K$ sweep out a Zariski open subset of $M$ and such that, with respect to a fixed ample line bundle on $M$, the degree of the rational curves in $K$ is minimal among all such irreducible components. Any rational curve $C$ in $K$ can be represented by a parameterized rational curve $f : \mathbb{P}^1 \to M$ so that $C = f(\mathbb{P}^1)$.

Denote by $\rho : U \to K$ and $\mu : U \to M$ the universal family associated to $K$. We call a rational curve $C$ standard if $f^*TM$ is decomposed as $f^*TM = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ where $f : \mathbb{P}^1 \to M$ is a parameterized rational curve representing $C$. Then there is a smallest closed subvariety $E \subset M$ such that, for every $x \in M \setminus E$, generic rational curve $C$ in $K$ passing through $x$ is standard. We call $E$ the bad locus of $K$.

Define the tangent map $\tau : U \dashrightarrow \mathbb{P}(TM)$ by $\tau([f]) = [df(T_0\mathbb{P}^1)]$ for any $f : \mathbb{P}^1 \to M$ with $df(0) \neq 0$. For $x \in M$ the image $\mathcal{C}_x$ of the rational map $\tau_x : U_x := \mu^{-1}(x) \dashrightarrow \mathbb{P}(TM)$ is called the variety of minimal rational tangents of $M$ at $x$. The proper image $\mathcal{C} \subset \mathbb{P}TM$ of $\tau : U \dashrightarrow \mathbb{P}TM$ is called the (compactified) total space of varieties of minimal rational tangents.

We will use the following results on varieties on minimal rational tangents $\mathcal{C}_x$ and the distribution defined by the linear span of its affine cone $\hat{\mathcal{C}}_x$ for a general point $x \in M$. By a distribution on a complex manifold $M$ we mean a subbundle of the tangent bundle $TM$ of $M$.

**Proposition 2.2** (Proposition II.3.7 of [14]). Let $M$ be a uniruled projective manifold and $K$ be a minimal rational component on $M$. Let $Z \subset M$ be a subvariety of codimension $\geq 2$. Then a general $C \in K$ does not intersect $Z$.

**Proposition 2.3** (Proposition 13 of [10], Proposition 1.2.2 of [11]). Let $M$ be a uniruled projective manifold of Picard number one and $K$ be a minimal rational component on $M$. Assume that the variety $\mathcal{C}_x$ of minimal rational tangents at a general point $x \in M$ is linearly degenerate, that is, the affine cone $\hat{\mathcal{C}}_x$ does not span the whole tangent space.
Theorem 2.4 (Theorem 1.2 of [12]). Let $M_1$ and $M_2$ be uniruled projective manifolds of Picard number one and $K_1$ and let $K_2$ be minimal rational components on $M_1$ and $M_2$, respectively. Let $E_1$ and $E_2$ be the bad loci of $K_1$ and $K_2$ and let $C_1$ and $C_2$ be the varieties of minimal rational tangents of $(M_1,K_1)$ and $(M_2,K_2)$. Assume that the general fiber $C_{1,x}$ is positive dimensional and the Gauss map on $C_{1,x}$ is generically finite.

Let $U_1 \subset M_1 \setminus E_1$ and $U_2 \subset M_2 \setminus E_2$ be connected open subset and $f : U_1 \to U_2$ be a biholomorphism such that $[df](C_1|_{U_1}) = C_2|_{U_2}$. Then there is a unique biholomorphic map $F : M_1 \to M_2$ such that $F|_{U_1} = f$.

For the definition of the Gauss map of a projective variety, see Definition 2.5. For example, if $C_x$ is smooth and not linear, then the Gauss map on $C_x \subset \mathbb{P}(T_x M)$ is generically finite. See [12] for more details.

2.2. Fundamental forms. Fundamental forms are basic invariants of projective varieties. We investigate behaviors of relative second and third fundamental forms of the varieties of minimal rational tangents along the liftings of standard minimal rational curves.

Definition 2.5. Let $U$ be a vector space and $Z \subset \mathbb{P}(U)$ be a subvariety of dimension $n$. For a point $z$ in the smooth locus $Z^0$ of $Z$, denote by $\tilde{T}_z Z$ the affine tangent space of $Z$ at $z$. Then the intrinsic tangent space at $z$ is $T_z Z = \tilde{z}^* \otimes (\tilde{T}_z Z / \tilde{z})$ and the normal space is $N_z = T_z(\mathbb{P}(U))/T_z Z = \tilde{z}^* \otimes U/\tilde{T}_z Z$, where $\tilde{z}$ is the one-dimensional subspace of $U$ corresponding to the point $z$. The Gauss map $\gamma : Z^0 \to \text{Gr}(n + 1, U)$ is defined by sending $z \in Z^0$ to the affine tangent space $\tilde{T}_z Z$ of $Z$ at $z$.

The differential $d_z \gamma : \tilde{T}_z Z \to (\tilde{T}_z Z)^* \otimes (U/\tilde{T}_z Z)$ is symmetric in the sense that $d_z \gamma(v)(w) = d_z \gamma(w)(v)$ for any $v, w \in \tilde{T}_z Z$ and thus defines a linear map $II_z : S^2 T_z Z \to N_z$, called the second fundamental form of $Z$ at $z$. The image of $II_z$ is called the second normal space $N_z^{(2)}$ of $Z$ at $z$. The second osculating affine tangent space $T_z^{(2)} Z$ is defined by the subpace of $U$ whose quotient space by $\tilde{T}_z Z$ is $\tilde{z} \otimes \text{Im} II_z \subset U/\tilde{T}_z Z$.

Let $Z^{00}$ denote the points in $Z$ where the rank of the second fundamental form does not drop. The second Gauss map $\gamma^{(2)} : Z^{00} \to \text{Gr}(n^{(2)} + 1, U)$ is defined by sending $z \in Z^{00}$ to the second osculating affine tangent space $\tilde{T}_z^{(2)} Z$ of $Z$ at $z$. The differential $d_\gamma^{(2)}$ defines a linear map $III_z : S^3 T_z Z \to \tilde{z}^* \otimes U/\tilde{T}_z^{(2)} Z = N_z/\tilde{N}_z^{(2)}$, called the third fundamental form of $Z$ at $z$. For any $k \geq 4$ the $k$-th fundamental form is defined in a similar way.

Definition 2.6. Let $\mathcal{U}$ be a vector bundle on a manifold $M$ and $\pi : \mathbb{P} \mathcal{U} \to M$ be the projection map from its projectivization. Let $Z \subset \mathbb{P}(\mathcal{U})$ be a subvariety and let $\varpi : Z \to M$ be the restriction of $\pi$ to $Z$.

Let $Z^0 \subset Z$ be an open subset such that $\varpi|_{Z^0} : Z^0 \to M^0 := \varpi(Z^0)$ is a submersion and for each $t \in M^0$, $Z_t := \varpi^{-1}(t)$ is immersed in $\mathbb{P} \mathcal{U}_t := \pi^{-1}(t)$ at any point in $Z^0_0 := \varpi^{-1}(t) \cap Z^0$. Define a vector bundle $T^\varpi$ on $Z^0$ by $T^\varpi := \cup_{z \in Z^0} T_z Z_{\varpi(z)}$, which is called the relative tangent bundle of $Z \subset \mathbb{P}(\mathcal{U})$. In a similar way we can define the relative affine tangent bundle $\tilde{T}^\varpi$, relative normal bundles $\mathcal{N}$ and relative second fundamental form $II^\varpi$. Then $II^\varpi$ is a section of $\text{Hom}(\text{Sym}^2 T^\varpi, \mathcal{N})$. 
Assume that the rank of $II^\varpi$ is constant. Then the image of $II^\varpi$ defines a subbundle $\mathcal{N}^{(2)}$ of $\mathcal{N}$, called the second normal bundle. Similarly, we can define the relative third fundamental form $III^\varpi$ as a section of $\text{Hom}(\text{Sym}^3 T^\varpi, \mathcal{N}/\mathcal{N}^{(2)})$.

Let $M$ be a uniruled projective manifold and $C$ be the variety of minimal rational tangents associated to a choice of a minimal rational component $\mathcal{K}$ on $M$. For a rational curve $C$ in $\mathcal{K}$ represented by $f : \mathbb{P}^1 \to M$, we will denote by $f^\sharp$ the map $\mathbb{P}^1 \to \mathbb{P}(TM)$ mapping $z \in \mathbb{P}^1$ to $df(T_z \mathbb{P}^1)$ and by $C^\sharp$ the image $f^\sharp(\mathbb{P}^1)$ of $f^\sharp$.

**Proposition 2.7** (Proposition 2.2 of [19]). Let $M$ be a uniruled projective manifold and let $\mathcal{C}$ be the variety of minimal rational tangents associated to a choice of a minimal rational component $\mathcal{K}$ on $M$. Denote by $\varpi : \mathcal{C} \to M$ the restriction of the projection map $\mathbb{P}(TM) \to M$. Let $C = [f]$ be a standard rational curve on $X$ so that $f^*TM = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some $p$ and $q$. Then

1. the pull-back $(f^\sharp)^*\mathcal{T}^\varpi$ of the relative affine tangent bundle $\mathcal{T}^\varpi$ of $\mathcal{C} \subset \mathbb{P}(TM)$ is the positive part $\varpi := \mathcal{O}(2) \oplus \mathcal{O}(1)^p$ of $f^*TM$;
2. the relative 2nd fundamental form $II^\varpi$ of $\mathcal{C}[C]$ is constant and the pull-back $(f^\sharp)^*\mathcal{T}(2)^\varpi$ of the relative second osculating affine bundle $\mathcal{T}(2)^\varpi$ of $\mathcal{C} \subset \mathbb{P}(TM)$ is a subbundle $P^{(2)} = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^r$ of $f^*TM$, where $r$ is the dimension of the image of $II^\varpi$.

Proposition 2.7 implies that the second fundamental forms of the varieties of minimal rational tangents are constant along the lifting of standard minimal rational curves. However, the third fundamental forms can vary. We will show the consistency of the third fundamental forms under some assumptions (Proposition 2.9).

Let $D$ be the distribution on $M$ defined by the linear span of the affine cone $\mathcal{C}_x$. Then outside a subvariety $\text{Sing}(D)$ of codimension $\geq 2$, $D$ is a subbundle of $TM$ and we may think of relative fundamental forms of $\mathcal{C} \subset \mathbb{P}(D)$. The kernel of the Frobenius bracket $[, ] : D \land D \to TM/D$ is related to the projective geometry of the variety of minimal rational tangents as follows.

**Proposition 2.8** (Proof of Proposition 1.2.1 and Proposition 1.3.1 and Proposition 1.3.2 of [11]). Let $M$ be a uniruled projective manifold and $\mathcal{K}$ be a minimal rational component on $M$. Assume that at a general point $x \in M$ the variety of minimal rational tangents $\mathcal{C}_x$ is linearly degenerate. Denote by $D_x \subset T_x M$ the linear span of the affine cone $\mathcal{C}_x$ of $\mathcal{C}_x$ and by $[ , ] : \land^2 D_x \to T_x M/D_x$ the Frobenius Lie bracket. Then for a generic $\alpha \in \mathcal{C}_x$ and $\xi, \eta \in T_x \mathcal{C}_x$,

$$\alpha \land \xi, \alpha \land II(\xi, \eta), \xi \land \eta$$

are contained in the kernel of $[, ] : \land^2 D_x \to T_x M/D_x$.

Proposition 2.8 implies that the second osculation affine space $\mathcal{T}(2)^\alpha_x \mathcal{C}_x$ is contained in the kernel of $[\alpha, ] : D_x \to T_x M/D_x$, and thus the rank of $[\alpha, ] : D_x \to T_x M/D_x$ is less than or equal to the codimension of $\mathcal{T}(2)^\alpha_x \mathcal{C}_x$ in $D_x$.

**Proposition 2.9** (cf. Proposition 3.1 of [19], Proposition 5.1 of [5]). Let $M$ be a uniruled projective manifold of Picard number one and let $\mathcal{C}$ be the variety of minimal rational tangents associated to a choice of a minimal rational component $\mathcal{K}$ on $M$. Let $D$ be the distribution on $M$ defined by the linear span of the affine cone $\mathcal{C}_x$. By Proposition 2.2 we may take a standard minimal rational curve $C = f(\mathbb{P}^1)$ with $C \cap \text{Sing}(D) = \emptyset$. Let $s$
be the codimension of the second osculating affine tangent space of $C_x$ in $\mathbb{P}(D_x)$. Assume that

1. the third fundamental form of $C_x \subset \mathbb{P}(D_x)$ is surjective for generic $x \in C$;
2. the rank of $[\mathcal{O}(2)_{x,1}] : D_x \to T_x M/D_x$ is $s$ for generic $x \in C$.

Then the relative 3rd fundamental form $III^\omega$ is constant along $C^\omega$ and we have $f^* D = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^r \oplus \mathcal{O}(-1)^s$ and $f^* TM/D = \mathcal{O}(1)^p \oplus \mathcal{O}^r$.

**Proof.** If $s = 0$, then $D$ is integrable and thus $D = TM$ (Proposition 2.3). Assume that $s > 0$. Since $x$ is contained in $D_x$, $P^{(2)}$ is a subbundle of $f^* D$. Furthermore, $f^* D/P^{(2)}$ is a subbundle of $f^* TM/P^{(2)} \simeq \mathcal{O}^{s-r}$ and thus is $\mathcal{O}(b_1) \oplus \ldots \mathcal{O}(b_s)$, where $0 \geq b_1 \geq \ldots \geq b_s$. It follows that the short exact sequence $0 \to P^{(2)} \to f^* D \to f^* D/P^{(2)} \to 0$ is split and

$$f^* D = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^r \oplus \mathcal{O}(b_1) \oplus \ldots \mathcal{O}(b_s),$$

where $0 \geq b_1 \geq \ldots \geq b_s$.

If all $b_i$ are zero, then the Frobenius Lie bracket $[,] : D \times D \to TM/D$ is zero and thus $D$ is integrable, a contradiction. Therefore, $b_s < 0$.

The relative 3rd fundamental form $III^\omega$ of $\mathcal{C}|_{C^\omega}$ is a section of

$$\text{Hom}(\text{Sym}^3 T^{\omega}, \text{Hom}(f^* f^* D/P^{(2)}))|_{C^\omega} \subset \text{Hom}(\text{Sym}^3 T^{\omega}, N/N^{(2)})|_{C^\omega},$$

which is isomorphic to $\text{Hom}(\text{Sym}^3 \mathcal{O}(-1)^p, \mathcal{O}(b_1-2) \oplus \ldots \mathcal{O}(b_s-2))$. By the surjectivity of the third fundamental forms $III$, we have $b_s \geq -1$. Thus $b_s = -1$ and $f^* D = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^r \oplus \mathcal{O}(-1)^s$ for some $r' \geq r$ and $s' \leq s$.

The Chern number of $f^*(TM/D)$ is $s'$. Since every factor of $f^*(TM/D)$ has nonnegative degree, the rank of the positive part $f^*(TM/D)^+_+$ of $f^*(TM/D)$ is $\leq s'$.

On the other hand, under the Frobenius bracket $[,] : D \times D \to TM/D$, the image $[\mathcal{O}(2), f^* D]$ is contained in $f^*(TM/D)^+_+$ and has dimension $\leq s'_+ \leq s$. By the condition that the rank of $[\mathcal{O}(2), 1] : D_x \to T_x M/D_x$ is $s$, we have $s \leq s'_+$ and thus we have $s = s'_+$. Consequently, $r = r'_+$ and $f^* D = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^r \oplus \mathcal{O}(-1)^s$ and $f^*(TM/D) = \mathcal{O}(1)^p \oplus \mathcal{O}^{s-r-2s}$. Therefore, $III^\omega$ is a section of a trivial vector bundle and thus is constant. \hfill \Box

### 3. Geometric structures

3.1. $G_0$-structures and S-structures. Let $m = \bigoplus_{p<0} g_p$ a fundamental graded Lie algebra, that is, a graded Lie algebra with $[g_p, g_{-1}] = g_{p-1}$ for any $p < 0$.

**Definition 3.1.** Let $D$ be a distribution on a manifold $M$. Define $D^p$ for $p < 0$ inductively by the following property:

$$D^p = [D^{p+1}, D^{-1}] + D^{p+1},$$

where $D^r$ is the sheaf of local sections of the vector bundle $D^r$. Then $\text{Sym}_x(D) := \sum_{p<0} D^p(x)/D^{p+1}(x)$ is endowed with a structure of graded Lie algebra, called the symbol algebra of $D$.

A distribution $D$ on a manifold $M$ is called of type $m$ if for each $x \in M$ the symbol algebra $\text{Sym}_x(D)$ is isomorphic to $m$ as a graded Lie algebra. In this case, the pair $(M, D)$ is called a filtered manifold of type $m$.

For each $x \in M$, let $R_x$ be the set of all isomorphisms $r : m \to \text{Sym}_x(D)$ of graded Lie algebras. Then $R := \bigcup_{x \in M} R_x$ is a principal $G_0(m)$-bundle on $M$, where $G_0(m)$ is the automorphism group of the graded Lie algebra $m$. We call $R$ the frame bundle of $(M, D)$. 
Definition 3.2. Let \((M, D)\) be a filtered manifold of type \(\mathfrak{m}\). Given a closed subgroup \(G_0 \subset G_0(\mathfrak{m})\), a \(G_0\)-structure on \((M, D)\) is a \(G_0\)-subbundle of the frame bundle \(\mathcal{R}\) of \((M, D)\). Two \(G_0\)-structures \(\mathcal{P}_1\) on \((M_1, D_1)\) and \(\mathcal{P}_2\) on \((M_2, D_2)\) are equivalent if there is a biholomorphism \(\varphi: M_1 \to M_2\) such that \(d\varphi: TM_1 \to TM_2\) induces an isomorphism from \(\mathcal{P}_1\) onto \(\mathcal{P}_2\). The local equivalence of two \(G_0\)-structures is defined similarly for open sets \(U_1 \subset M_1\) and \(U_2 \subset M_2\).

Definition 3.3. Let \((M, D)\) be a filtered manifold of type \(\mathfrak{m}\). Let \(S\) be a nondegenerate subvariety of \(\mathbb{P}\mathfrak{g}_{-1}\). A fiber subbundle \(S \subset \mathbb{P}D\) is called an \(S\)-structure on \((M, D)\) if for each \(x \in M\), the fiber \(S_x \subset \mathbb{P}D_x\) is isomorphic to \(S \subset \mathbb{P}\mathfrak{g}_{-1}\) under a graded Lie algebra isomorphism \(\mathfrak{m} \to \text{Symb}_x(D)\).

Two \(S\)-structures \(S_1\) on \((M_1, D_1)\) and \(S_2\) on \((M_2, D_2)\) are said to be equivalent if there exists a biholomorphism \(\phi: M_1 \to M_2\) such that \(d\phi: \mathbb{P}TM_1 \to \mathbb{P}TM_2\) sends \(S_1 \subset \mathbb{P}TM_1\) to \(S_2 \subset \mathbb{P}TM_2\). The local equivalence of two \(S\)-structures is defined similarly for open subsets \(U_1 \subset M_1\) and \(U_2 \subset M_2\).

An \(S\)-structure can be interpreted as a \(G_0\)-structure, and the local equivalence of \(S\)-structures can be checked by using the local equivalence of the corresponding \(G_0\)-structures under some conditions.

Definition 3.4. Let \(U\) be a vector space and and let \(S\) be a nondegenerate subvariety of \(\mathbb{P}\mathfrak{U}\). Consider the graded free Lie algebra \(F(\mathfrak{U})\) generated by \(U\). Denote by \(I(\mathfrak{S})\) the ideal of \(F(\mathfrak{U})\) generated by the relation \([v, w] = 0\) for \(v, w \in U\) such that \(v \in \hat{S}\) and \(w \in T_v\hat{S}\). We call the quotient graded Lie algebra \(\mathfrak{m}(\mathfrak{S}, \mathbb{P}\mathfrak{U}) := F(\mathfrak{U})/I(\mathfrak{S})\) the graded Lie algebra determined by \(\mathfrak{S} \subset \mathbb{P}\mathfrak{U}\).

Definition 3.5. Let \(\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p\) be a fundamental graded Lie algebra and let \(S\) be a nondegenerate subvariety of \(\mathbb{P}\mathfrak{g}_{-1}\). If the graded Lie algebra determined by \(S \subset \mathbb{P}\mathfrak{g}_{-1}\) is isomorphic to \(\mathfrak{m}\), we say that \(\mathfrak{m}\) is determined by \(S \subset \mathbb{P}\mathfrak{g}_{-1}\).

Example 3.6. Let \(\mathfrak{g} = \bigoplus_{-\mu \leq i \leq \mu} \mathfrak{g}_i\) be a simple graded Lie algebra. Let \(S \subset \mathbb{P}\mathfrak{g}_{-1}\) be the projectivization of the cone of highest weight vectors of the irreducible \(\mathfrak{g}_0\)-module \(\mathfrak{g}_{-1}\). Then \(\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p\) is determined by \(S \subset \mathbb{P}\mathfrak{g}_{-1}\) (Proposition 7 of [13]).

Let \(G(\hat{S})\) denote the linear automorphism group of \(\hat{S} \subset \mathfrak{g}_{-1}\), i.e., the subgroup of \(\text{GL}(\mathfrak{g}_{-1})\) consisting of linear automorphism of \(\hat{S} \subset \mathfrak{g}_{-1}\). Then \(G(\hat{S})\) acts on \(\mathfrak{m}(S, \mathbb{P}\mathfrak{g}_{-1})\) preserving the graded Lie algebra structure. This \(G(\hat{S})\)-action defines a homomorphism \(G(\hat{S}) \to G_0(\mathfrak{m})\) induced by the isomorphism between \(\mathfrak{m}(S, \mathbb{P}\mathfrak{g}_{-1})\) and \(\mathfrak{m}\). The induced map \(G(\hat{S}) \to G_0(\mathfrak{m})\) is injective.

Proposition 3.7. Let \(\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p\) be a fundamental graded Lie algebra determined by a nondegenerate subvariety \(S\) of \(\mathbb{P}\mathfrak{g}_{-1}\). Let \(G(\hat{S})\) be the linear automorphism group of \(\hat{S} \subset \mathfrak{g}_{-1}\). Consider the induced map \(G(\hat{S}) \to G_0(\mathfrak{m})\) and let \(G_0 \subset G_0(\mathfrak{m})\) denote its image. Then, there is a one-to-one correspondence between \(G_0\)-structures and \(S\)-structures on filtered manifolds of type \(\mathfrak{m}\). Furthermore, two \(G_0\)-structures \(\mathcal{P}_1\) and \(\mathcal{P}_2\) are equivalent if and only if the corresponding \(S\)-structures \(S_1\) and \(S_2\) are equivalent.

Proof. Let \((M, D)\) be a filtered manifold of type \(\mathfrak{m}\). Given a \(G_0\)-structure \(\mathcal{P}\) on \((M, D)\), define \(S_x\) by \([r](S)\) for any \(r \in \mathcal{P}_x\), where \([r]\) is the isomorphism \(\mathbb{P}\mathfrak{g}_{-1} \to \mathbb{P}D_x\) induced by
the isomorphism \( r : \mathfrak{m} \to \text{Symb}_\phi(D) \). Then \( S_x \) is well defined and the union \( S = \cup_{x \in M} S_x \) defines an \( S \)-structure on \((M, D)\).

Conversely, let \( S \subset \mathbb{P}D \) be an \( S \)-structure on \((M, D)\). Define \( \mathcal{R}_x \) by \( \{ r \in \mathcal{R}_x : \varphi(S) = S_x \} \). Then \( \mathcal{R}_x \) is well defined and the union \( \mathcal{R} = \cup_{x \in M} \mathcal{R}_x \) defines a \( G_0 \)-subbundle of the frame bundle \( \mathcal{R} \) of \((M, D)\).

### 3.2. Lie algebra cohomologies.

**Notation 3.8.** Let \( \mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p \) be a nilpotent Lie algebra and \( \Gamma \) be a representation space of \( \mathfrak{m} \). Define a complex

\[
\begin{align*}
0 & \to \Gamma \to \text{Hom}(\mathfrak{m}, \Gamma) \to \text{Hom}(\wedge^2 \mathfrak{m}, \Gamma) \to \ldots 
\end{align*}
\]

by

\[
\partial \phi(X_1, \ldots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} X_i \phi(X_1, \ldots, \hat{X}_i, \ldots, X_{q+1}) + \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \phi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{q+1})
\]

for \( \phi \in \text{Hom}(\wedge^q \mathfrak{m}, \Gamma) \) and \( X_1, \ldots, X_{q+1} \in \mathfrak{m} \). The cohomology space

\[
H^q(\mathfrak{m}, \Gamma) := \frac{\text{Ker} (\partial : \text{Hom}(\wedge^q \mathfrak{m}, \Gamma) \to \text{Hom}(\wedge^{q+1} \mathfrak{m}, \Gamma))}{\text{Im} (\partial : \text{Hom}(\wedge^{q-1} \mathfrak{m}, \Gamma) \to \text{Hom}(\wedge^q \mathfrak{m}, \Gamma))}
\]

is called the **Lie algebra cohomology space associated to the representation** \( \Gamma \) of \( \mathfrak{m} \).

Assume that \( \Gamma \) has a gradation such that \( \mathfrak{g}_p, \Gamma \leq \Gamma_{p+\ell} \) for \( p < 0 \). Then \( \text{Hom}(\wedge^q \mathfrak{m}, \Gamma) \) has an induced grading

\[
\text{Hom}(\wedge^q \mathfrak{m}, \Gamma)_{\Gamma} = \bigoplus_{\ell} \text{Hom}(\wedge^q_{\Gamma} \mathfrak{m}, \Gamma_{\ell+\Gamma}),
\]

where

\[
\wedge^q_{\Gamma} \mathfrak{m} = \sum_{j_1 + \ldots + j_q = j} \mathfrak{g}_{j_1} \wedge \ldots \wedge \mathfrak{g}_{j_q}.
\]

For each \( \ell \) the complex (1) restricts to the complex

\[
0 \to \Gamma_{\ell} \xrightarrow{\partial} \text{Hom}(\mathfrak{m}, \Gamma)_{\ell} \xrightarrow{\partial} \text{Hom}(\wedge^2 \mathfrak{m}, \Gamma)_{\ell} \xrightarrow{\partial} \ldots
\]

so that \( H^q(\mathfrak{m}, \Gamma) \) has a gradation

\[
H^q(\mathfrak{m}, \Gamma) = \bigoplus_{\ell} H^q(\mathfrak{m}, \Gamma)_{\ell}
\]

### 3.3. Prolongation methods.

We review the theory of Cartan connections and prolongation methods in [24], [20], [8], [23], and [6].

Let \( \mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p \) be a fundamental graded Lie algebra and \( G_0 \) be a connected subgroup of \( G_0(\mathfrak{m}) \) with Lie algebra \( \mathfrak{g}_0 \). Then there is a unique maximal transitive graded Lie algebra \( \mathfrak{g} = \bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}_{\ell} \) extending \( \mathfrak{m} \oplus \mathfrak{g}_0 \), called the **prolongation of** \((\mathfrak{m}, \mathfrak{g}_0)\). For \( \ell \geq 1 \), \( \mathfrak{g}_{\ell} \) is given by

\[
\{ \alpha \in \bigoplus_{p < 0} \text{Hom}(\mathfrak{g}_p, \mathfrak{g}_{p+\ell}^*) : \alpha([u, v]) = [\alpha(u), v] + [u, \alpha(v)] \text{ and } \alpha(u) \in \mathfrak{g}_{\ell-1} \text{ for all } u, v \in \mathfrak{g}_{-1} \},
\]

and the Lie bracket \([ , ] : \mathfrak{g}_{\ell} \times \mathfrak{g}_k \to \mathfrak{g}_{\ell+k}\) is given by:
Definition 3.10. Let \( (P, \theta) \) be a Cartan connection of type \( G/G^0 \) which contains \( G_0 \) with Lie algebras \( \mathfrak{g} \) and \( \mathfrak{g}^0 := \bigoplus_{\ell \geq 0} \mathfrak{g}_\ell \).

Definition 3.11. Assume that \( \mathfrak{g} \) is finite dimensional. Then there is a Lie group \( G \) and its subgroup \( G\theta \) isomorphic to \( (P, \theta) \).

Definition 3.12. Let \( \mathfrak{g} = \bigoplus_{\ell = -\mu} g_\ell \) be a simple graded Lie algebra and let \( \mathfrak{m} \) be the negative part \( \bigoplus_{\ell < 0} \mathfrak{g}_\ell \). Let \( G_0 \subset G_0(\mathfrak{m}) \) be the subgroup with Lie algebra \( \mathfrak{g}_0 \). Define a Hermitian metric \( (\, , \, ) \) on \( \mathfrak{g} \) induced by the Killing form of \( \mathfrak{g} \). Denote by \( \partial^* \) the adjoint of \( \partial \) with respect to \( (\, , \, ) \). Then

\[
\text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g}) = \partial \text{Hom}(\mathfrak{m}, \mathfrak{g}) \oplus \text{Ker} \partial^*
\]

A Cartan connection \( (P, \theta) \) of type \( G/G^0 \) is said to be normal if its curvature \( K \) satisfies that its component \( K_{\ell+1} \) of degree \( \ell + 1 \) has values in \( (\text{Ker} \partial^*)_{\ell+1} \) for any \( \ell \geq 0 \).

Theorem 3.12 (Theorem 2.7 and Theorem 2.9 of [24]). Let \( \mathfrak{m} \) and \( G_0 \) be as in Definition 3.17. Assume that \( \mathfrak{g} \) is the prolongation of \( (\mathfrak{m}, \mathfrak{g}_0) \). Then for any \( G_0 \) structure \( \mathcal{P} \) on a filtered manifold \( (M, D) \) of type \( \mathfrak{m} \), there is a normal Cartan connection \( (P, \theta) \) of type \( G/G^0 \). Furthermore, given two \( G_0 \)-structures \( \mathcal{P}_1 \) on \( (M_1, D_1) \) and \( \mathcal{P}_2 \) on \( (M_2, D_2) \), \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are locally equivalent if and only if the corresponding normal Cartan connections \( (P_1, \theta_1) \) and \( (P_2, \theta_2) \) are locally isomorphic.

Given a \( G_0 \)-structure \( \mathcal{P} \) on a filtered manifold \( (M, D) \) of type \( \mathfrak{m} \), define a vector bundle \( \mathcal{H}^2_k \) on \( M \) by \( \mathcal{H}^2_k := \mathcal{P} \times_{\mathfrak{g}_0} \mathcal{H}^2(\mathfrak{m}, \mathfrak{g})_k \) for \( k \geq 1 \). If \( H^0(M, \mathcal{H}^2_k) \) is zero for all \( k \geq 1 \), then the corresponding Cartan connection \( (P, \theta) \) is flat, and \( \mathcal{P} \) is locally equivalent to the standard \( G_0 \)-structure on \( G/G^0 \).

Theorem 3.12 is extended to the case when \( (\mathfrak{m}, G_0) \) satisfies the condition (C) or \( \mathcal{P} \) satisfies a pseudo-concavity type condition.
Theorem 3.13 (Theorem 3.10.1 of [20]). Assume that \((m, G_0)\) satisfies the condition (C), that is, there is a subspace \(W = \oplus_{\ell \geq 0} W_{\ell+1}\) of \(\text{Hom}(\wedge^2 m, g)\) with
\[
\text{Hom}(\wedge^2 m, g)_{\ell+1} = W_{\ell+1} \oplus \partial \text{Hom}(m, g)_{\ell+1}
\]
for any \(\ell \geq 0,\)
which is stable under the action of \(G^0.\) Then for any \(G_0\)-structure \(\mathcal{P}\) on a filtered manifold \((M, D)\) of type \(m,\) there is a Cartan connection \((P, \theta)\) of type \(G/G^0,\) whose curvature has value in \(W.\) Furthermore, given two \(G_0\)-structures \(\mathcal{P}_1\) on \((M_1, D_1)\) and \(\mathcal{P}_2\) on \((M_2, D_2),\) \(\mathcal{P}_1\) and \(\mathcal{P}_2\) are locally equivalent if and only if the corresponding Cartan connections \((P_1, \theta_1)\) and \((P_2, \theta_2)\) are locally isomorphic.

For \(m\) and \(G_0\) as in Definition 3.11, \(\text{Ker} \partial^\ast\) is stable under the action of \(G^0\) (Lemma 1.12 of [24]), and thus thus \((m, G_0)\) satisfies the condition (C).

Theorem 3.14 (Theorem 2.17 of [8] and Theorem 2.6 of [9]). Let \(g = \bigoplus_{\ell = -\nu}^\mu g_\ell\) be the prolongation of \((m, g_0).\) Given a \(G_0\)-structure \(\mathcal{P}\) on a filtered manifold \((M, D)\) of type \(m,\)
define define a vector bundle \(A^2_k\) on \(M\) by \(A^2_\ell := \mathcal{P} \times_{G_0} (\text{Hom}(\wedge^2 m, g)_{\ell}/\partial \text{Hom}(m, g)_{\ell})\) for \(k \geq 1.\) If \(H^0(M, A_k) = 0\) for \(1 \leq k \leq \mu + \nu,\) then there is a Cartan connection \((P, \theta).\) If, furthermore, \(H^0(M, A_k) = 0\) for \(k \geq \mu + \nu + 1,\) then the corresponding Cartan connection \((P, \theta)\) is flat.

Theorem 3.12 Theorem 3.13 and Theorem 3.14 enables us to transform the local equivalence problem of geometric structures to the local isomorphism problem of Cartan connections, and the latter is more systematic than the former. To deal with more general cases, we weaken the requirement that \(P\) should be a principal bundle on \(M\) as follows.

A geometric structure of order 0 of type \((m, G_0)\) is a \(G_0\)-structure \(\mathcal{P}\) on a filtered manifold \((M, D)\) of type \(m.\) For \(\ell \geq 1,\) we call a sequence of principal bundles
\[
P^{(i)} : \mathcal{P}^{(i)} \longrightarrow \mathcal{P}^{(i-1)} \longrightarrow \cdots \longrightarrow \mathcal{P}^{(0)} \longrightarrow M
\]
a geometric structure of order \(\ell\) of type \((m, G_0, \ldots, G_\ell)\) if for \(0 \leq i \leq \ell - 1,\)
- \(P^{(i)} : \mathcal{P}^{(i)} \longrightarrow \mathcal{P}^{(i-1)} \longrightarrow \cdots \longrightarrow M\) is a geometric structure of type \((m, G_0, \ldots, G_i);\)
- \(\mathcal{P}^{(i+1)} \longrightarrow \mathcal{P}^{(i)}\) is a principal \(G_{i+1}\)-subbundle of the universal frame bundle \(\mathcal{J}^{(i+1)} \mathcal{P}^{(i)} \longrightarrow \mathcal{J}^{(i)} \mathcal{P}^{(i)}\) of \(\mathcal{P}^{(i)}\) of order \(i + 1.\)

For the definition of the universal frame bundle \(\mathcal{J}^{(i+1)} \mathcal{P}^{(i)}\) of order \(i + 1\) of a geometric structure \(\mathcal{P}^{(i)},\) see Definition 2.1 of [8]. The property we use is that a map between two geometric structures \(\mathcal{P}^{(i)}, Q^{(i)}\) of order \(i\) induces a map between their universal frame bundles \(\mathcal{J}^{(i+1)} \mathcal{P}^{(i)}, \mathcal{J}^{(i+1)} Q^{(i)}\) of order \(i + 1.\)

The equivalence of two geometric structures \(\mathcal{P}^{(i)}\) and \(\mathcal{Q}^{(i)}\) is defined inductively as follows. Two geometric structures \(\mathcal{P}^{(i)} : \mathcal{P}^{(i)} \longrightarrow \mathcal{P}^{(i-1)} \longrightarrow \cdots \longrightarrow M\) and \(\mathcal{Q}^{(i)} : \mathcal{Q}^{(i)} \longrightarrow \mathcal{Q}^{(i-1)} \longrightarrow \cdots \longrightarrow M\) are equivalent if their truncations \(\mathcal{P}^{(i-1)}\) and \(\mathcal{Q}^{(i-1)}\) are equivalent and the lifting \(\mathcal{J}^{(i)} \mathcal{P}^{(i-1)} \longrightarrow \mathcal{J}^{(i)} \mathcal{Q}^{(i-1)}\) of their equivalence maps \(\mathcal{J}^{(i)}\) onto \(\mathcal{J}^{(i)}\)

Fix a set of subspaces \(W = \{W_{\ell_1}, W_{\ell_2}\}_{\ell \geq 0}\) such that
\[
\text{Hom}(m, g)_{\ell} = W_{\ell} \oplus \partial g_{\ell}
\]
\[
\text{Hom}(\wedge^2 m, g)_{\ell+1} = W_{\ell+1} \oplus \partial \text{Hom}(m, g)_{\ell+1}.
\]
Note that we don’t require that the complement \(\oplus_{\ell \geq 0} W_{\ell+1}\) of \(\partial \text{Hom}(m, g)_{\ell+1}\) should be stable under the action of \(G^0.\)
Theorem 3.15 (Theorem 8.3 of [23], Theorem 3.1 of [6]). Let $\mathcal{P}$ be a $G_0$-structure on a filtered manifold $(M, D)$ of type $\mathfrak{m}$. Then for each $\ell \geq 1$, there is a geometric structure

$$
\mathcal{I}_W^{(\ell)} \mathcal{P} \xrightarrow{G_1} \mathcal{I}_W^{(\ell-1)} \mathcal{P} \rightarrow \cdots \rightarrow \mathcal{I}_W^{(1)} \mathcal{P} \xrightarrow{G_0} \mathcal{P} \xrightarrow{G_0} M
$$

of type $(g_-, G_0, \ldots, G_t)$. Furthermore, two $G_0$-structures $\mathcal{P}$ and $\mathcal{Q}$ are equivalent if and only if the corresponding geometric structures $\mathcal{I}_W^{(\ell)} \mathcal{P}$ and $\mathcal{I}_W^{(\ell)} \mathcal{Q}$ are equivalent.

We call the limit $\mathcal{I}_W \mathcal{P} = \lim_\ell \mathcal{I}_W^{(\ell)} \mathcal{P}$ the $W$-normal complete step prolongation of $\mathcal{P}$.

As in the case of $G_0$-structures modeled on a rational homogeneous variety $G/G^0$, we get a local equivalence of geometric structures by the vanishing of sections of vector bundles $\mathcal{H}_k = \mathcal{P} \times_{G_0} H^2(\mathfrak{m}, \mathfrak{g})_k$.

Proposition 3.16 (Theorem 7.4 of [6]). Let $\mathcal{P}$ be a $G_0$-structure on a filtered manifold $(M, D)$ of type $\mathfrak{m}$. If $H^0(M, \mathcal{H}_k^\alpha)$ is zero for all $k \geq 1$, then the $W$-normal complete step prolongation $\mathcal{I}_W \mathcal{P}$ of $\mathcal{P}$ is a Cartan connection of type $G/G^0$ which is flat, and $\mathcal{P}$ is locally equivalent to the standard $G_0$-structure on $G/G^0$.

We will use Proposition 3.16 to prove Theorem 1.3 (see Section 7 and Section 8.3).

4. Smooth horospherical varieties of Picard number one

4.1. Classifications. Let $\mathfrak{l}$ be a semisimple Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ and let $\Phi$ be the set of roots of $\mathfrak{l}$ relative to $\mathfrak{h}$. The root space decomposition of $\mathfrak{l}$ is given by

$$
\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{l}_\alpha,
$$

where $\mathfrak{l}_\alpha$ is the root space of $\alpha \in \Phi$. For any root $\alpha$, let $U_\alpha$ be the root group of $\alpha$.

Definition 4.1. Let $\{\alpha_1, \ldots, \alpha_m\}$ be a set of simple roots of $\mathfrak{l}$. We define the characteristic element associated with $\alpha_i$ as an element $E_{\alpha_i}$ in $\mathfrak{h}$ such that $\alpha_j(E_{\alpha_i}) = \delta_{ij}$ for $i, j = 1, \ldots, m$. Define a gradation $\mathfrak{l} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{l}_p$ on $\mathfrak{l}$ by $\mathfrak{l}_p = \{v \in \mathfrak{l} : [E_{\alpha_i}, v] = pv\}$ for $p \in \mathbb{Z}$, which is called the gradation associated with $\alpha_i$. In general, given a representation $\mathcal{V}$ of $\mathfrak{l}$ we define the gradation associated with $\alpha_i$ in a similar way.

Notation 4.2. Given a set $\{\alpha_1, \ldots, \alpha_m\}$ of simple roots of $\mathfrak{l}$, let $\{\varpi_1, \ldots, \varpi_m\}$ be the set of fundamental weights. For each $i = 1, \ldots, m$, let $P_{\alpha_i}$ denote the maximal parabolic subgroup associated to $\alpha_i$ and let $V_{\varpi_i}$ denote the irreducible representation with highest weight $\varpi_i$ of semisimple Lie group $L$ corresponding to $\mathfrak{l}$.

For a reductive algebraic group $L$, a normal $L$-variety is said to be horospherical if it has an open $L$-orbit $L/H$ whose isotropy group $H$ contains the unipotent part of a Borel subgroup of $L$.

Theorem 4.3 (Theorem 0.1 and Theorem 1.11 of [22]). Let $L$ be a reductive group. Let $X$ be a smooth nonhomogeneous projective horospherical $L$-variety with Picard number one. Then $X$ is uniquely determined by its two closed $L$-orbits $Y$ and $Z$, isomorphic to $L/P_Y$ and $L/P_Z$, respectively; and $(L, \alpha, \beta)$ in one of the triples of the following list, where $P_Y = P^\alpha$ and $P_Z = P^\beta$ for simple roots $\alpha$ and $\beta$.

1. $(B_m, \alpha_{m-1}, \alpha_m)$ for $m \geq 3$;
(2) \((B_3, \alpha_1, \alpha_3)\);
(3) \((C_m, \alpha_i+1, \alpha_i)\) for \(m \geq 2, \, i \in \{1, \ldots, m-1\}\);
(4) \((F_4, \alpha_2, \alpha_3)\);
(5) \((G_2, \alpha_2, \alpha_1)\).

Moreover, the automorphism group \(\text{Aut}(X)\) of \(X\) is \((SO(2m+1) \times \mathbb{C}^*) \ltimes V_{\varpi_m}, \, (SO(7) \times \mathbb{C}^*) \ltimes V_{\varpi_3}, \, ((Sp(2m) \times \mathbb{C}^*)/\{\pm 1\}) \ltimes V_{\varpi_4}, \, (F_4 \times \mathbb{C}^*) \ltimes V_{\varpi_4}\) and \((G_2 \times \mathbb{C}^*) \ltimes V_{\varpi_1}\), respectively.

Finally, \(\text{Aut}(X)\) has two orbits in \(X\) and \(Z\), the complement of \(Z\) in \(X\).

In Theorem 4.3, \(Y = L/P_Y\) is contained in the open orbit \(X^0\) of \(\text{Aut}(X)\) in \(X\). In particular, the base point \(o\) of \(Y = L/P_Y\) is contained in \(X^0\). We will take \(o\) as the base point of the quasi- homogeneous variety \(X\). Let \(g = (1 + \mathbb{C}) \supset U\) be the Lie algebra of \(\text{Aut}(X)\), where \(U\) is the Lie algebra of \(L\). The characteristic element associated with \(\alpha\) as in Definition 4.1 gives a gradation on \(I\) and a gradation on \(U\). Then we shift the gradation on \(U\) to identify the part \(U_{-1} \oplus \bigoplus_{p \geq 0} U_p\) with the tangent space of \(X\) at \(o \in X\).

**Proposition 4.4** (Section 2, Proposition 48 and Proposition 49 of [16]). Let \(X\) be a smooth nonhomogeneous projective horospherical variety \((L, \alpha, \beta)\) of Picard number one. Let \(G = \text{Aut}(X)\) and let \(g = (1 + \mathbb{C}) \supset U\) be the corresponding Lie algebra. Then,

1. there is a grading on \(I\) and \(U\),

\[
I = \bigoplus_{k=-\mu}^{\mu} I_k \quad \text{and} \quad U = \bigoplus_{k=-1}^{\nu} U_k,
\]

such that, with the grading being defined by

\[
g_0 := (I_0 \oplus \mathbb{C}) \supset U_0
\]

\[
g_p := I_p \oplus U_p \quad \text{for} \quad p \neq 0,
\]

the negative part \(m = \bigoplus_{p<0} g_p\) of \(g\) is identified with the tangent space of \(X\) at the base point \(o\) of \(X\). By convention we set \(I_k = 0\) for \(k < -\mu\) or \(k > \mu\), and \(U_k = 0\) for \(k < -1\) or \(k > \nu\).

2. \(H^1(m, g)_p = 0\) for \(p > 0\).

3. \(g = \bigoplus_{\ell \in \mathbb{Z}} g_\ell\) is the prolongation of \((m, g_0)\).

As \(I_0\)-representations, \(I_k\) and \(U_k\) are irreducible, and as \(g_0\)-representations, \(g_k\) is irreducible (the proof of Lemma 27 of [16]). Due to (2) and (3), we could consider the prolongation methods of section 3.3 for a smooth nonhomogeneous projective horospherical variety of Picard number one.

4.2. Varieties of minimal rational tangents. In this section we will describe the varieties of minimal rational tangents of horospherical varieties in the list of Theorem 4.3. We will use the same notions as in Proposition 4.4. As we mentioned in the previous subsection, we take the base point \(o\) of \(Y = L/P_Y\) as the base point of \(X\). For the root \(\alpha\), we define \(C_{\alpha} := \overline{U_{-\alpha}} \subset Y\). Then, \(C_{\alpha}\) is a minimal rational curve in \(Y\) and thus in \(X\).

For an arbitrary reductive group \(L\) and for a finitely many irreducible \(L\)-representation spaces \(V_i\) \((i = 1, \ldots, r)\), let \(\mathcal{H}_L(\bigoplus_{i=1}^r V_i)\) denote the closure of the sum of highest weight vectors \(v_i\) of \(V_i\) in \(\mathbb{P}(\bigoplus_{i=1}^r V_i)\). For example, \(\mathcal{H}_L(V)\) for an irreducible representation space \(V\) is the highest weight orbit, and \((L, \alpha_i, \alpha_j)\) is \(\mathcal{H}_L(V_{\varpi_i} \oplus V_{\varpi_j})\), where \(V_{\varpi_i}\) (respectively, \(V_{\varpi_j}\)) is the irreducible representation of \(L\) of highest weight \(\varpi_i\) (respectively, \(\varpi_j\)).
Let $L_0$ be the subgroup of $G = \text{Aut}(X)$ with Lie algebra $l_0$. We need the descriptions of the representation $U_1 \oplus l_1$ of $L_0$ and $\mathcal{H}_{L_0}(U_1 \oplus l_1)$ for $(B_m, \alpha_{m-1}, \alpha_m)$, $(B_3, \alpha_1, \alpha_3)$, $(C_m, \alpha_m, \alpha_{m-1})$, $(F_4, \alpha_2, \alpha_3)$ and $(G_2, \alpha_2, \alpha_1)$ and also the representation $U_1 \oplus l_1 \oplus l_2$ of $L_0$ and $\mathcal{H}_{L_0}(U_1 \oplus l_1 \oplus l_2)$ for $(C_m, \alpha_{i+1}, \alpha_i)$, $1 < i + 1 < m$. The following are parts of Lemma 3.5.1 and Proposition 3.5.2 of [13].

**Lemma 4.5.**

1. $(B_m, \alpha_{m-1}, \alpha_m)$, $m > 2$ where $U = V_{\sigma_m}$; Denote $L_0 = A_1 \times A_{m-2}$. Let $V$ be the standard representation of $A_1$ and $W^*$ be the standard representation of $A_{m-2}$. Then

\[
\begin{align*}
L_1 & = \text{Sym}^2 V \otimes W \\
U_1 & = V.
\end{align*}
\]

The closure $\mathcal{H}_{L_0}(U_1 \oplus l_1)$ of the $L_0$-orbit at $v + v^2 \otimes w$ is

\[
\mathcal{H}_{L_0}(U_1 \oplus l_1) = \mathbb{P}\{cv + v^2 \otimes w : c \in \mathbb{C}, v \in V, w \in W\}
\]

\[
\simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}(V)}(-1) \oplus \mathcal{O}_{\mathbb{P}(V)}(-2)^{m-1}),
\]

where $\dim V = 2$ and $\dim W = m - 1$.

2. $(B_3, \alpha_1, \alpha_3)$ where $U = V_{\sigma_3}$; Denote $L_0 = B_2$. Let $V$ be the spin representation of $B_2$. Let $W$ be the standard representation of $B_2$. Then

\[
\begin{align*}
L_1 & = W \\
U_1 & = V.
\end{align*}
\]

The closure $\mathcal{H}_{L_0}(U_1 \oplus l_1)$ of the $L_0$-orbit at $v + w$ is the horospherical variety of type $(C_2, \alpha_2, \alpha_1)$, the odd symplectiv Grassmannian $\text{Gr}_w(2, \mathbb{C}^5)$ of isotropic 2-subspaces in $\mathbb{C}^5$.

3. $(C_m, \alpha_{i+1}, \alpha_i)$, $1 < i + 1 < m$ where $U = V_{\sigma_1}$; Denote $L_0 = A_i \times C_{m-i-1}$. Let $V^*$ be the standard representation of $A_i$, let $Q^*$ be the standard representation of $C_{m-i-1}$ and $W := \mathbb{C} \oplus Q$. Then

\[
\begin{align*}
L_2 & = \text{Sym}^2 V \\
U_1 \oplus l_1 & = V \otimes W.
\end{align*}
\]

The closure $\mathcal{H}_{L_0}(U_1 \oplus l_1 \oplus l_2)$ of the $L_0$-orbit at $v \otimes w + v^2$ is

\[
\mathcal{H}_{L_0}(U_1 \oplus l_1 \oplus l_2) = \mathbb{P}\{cv \otimes w + v^2 : c \in \mathbb{C}, v \in V, w \in W\}
\]

\[
\simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}(V)}(-1)^{2m-2i-1} \oplus \mathcal{O}_{\mathbb{P}(V)}(-2)),
\]

where $\dim V = i + 1$ and $\dim W = 2m - 2i - 1$.

3. $(C_m, \alpha_m, \alpha_{m-1})$ where $U = V_{\sigma_1}$; Denote $L_0 = A_{m-1}$. Let $V^*$ be the standard representation of $A_{m-1}$. Then

\[
\begin{align*}
L_1 & = \text{Sym}^2 V \\
U_1 & = V.
\end{align*}
\]

The closure $\mathcal{H}_{L_0}(U_1 \oplus l_1)$ of the $L_0$-orbit at $v + v^2$ is

\[
\mathcal{H}_{L_0}(U_1 \oplus l_1) = \mathbb{P}\{cv + v^2 : c \in \mathbb{C}, v \in V\}
\]

\[
\simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}(V)}(-1) \oplus \mathcal{O}_{\mathbb{P}(V)}(-2)),
\]

where $\dim V = m$. 


(4) \((F_{4},\alpha_{2},\alpha_{3})\) where \(U = V_{w_{4}}\); Denote \(L_{0} = A_{1} \times A_{2}\). Let \(V\) be the standard representation of \(A_{1}\) and \(V^{*} = V\) and let \(W^{*}\) be the standard representation of \(A_{2}\). Then

\[
\begin{align*}
L_{-1} &= \text{Sym}^{2} V \otimes W \\
U_{-1} &= V.
\end{align*}
\]

The closure \(\mathcal{H}_{L_{0}}(U_{-1} \oplus L_{-1})\) of the \(L_{0}\)-orbit at \(v + v^{2} \otimes w\) is

\[
\begin{align*}
\mathcal{H}_{L_{0}}(U_{-1} \oplus L_{-1}) &= \mathbb{P}\{cv + v^{2} \otimes w : c \in \mathbb{C}, v \in V, w \in W\} \\
&\simeq \mathbb{P}(O_{\mathbb{P}(V)}(-1) \oplus O_{\mathbb{P}(V)}(-2)^{2}),
\end{align*}
\]

where \(\dim V = 3\) and \(\dim W = 2\).

(5) \((G_{2},\alpha_{2},\alpha_{1})\) where \(U = V_{w_{1}}\); Denote \(L_{0} = A_{1}\). Let \(V\) be the standard representation of \(A_{1}\) such that \(V^{*} = V\). Then

\[
\begin{align*}
L_{-1} &= \text{Sym}^{3} V \\
U_{-1} &= V.
\end{align*}
\]

The closure \(\mathcal{H}_{L_{0}}(U_{-1} \oplus L_{-1})\) of the \(L_{0}\)-orbit at \(v + v^{3}\) is

\[
\begin{align*}
\mathcal{H}_{L_{0}}(U_{-1} \oplus L_{-1}) &= \mathbb{P}\{cv + v^{3} : c \in \mathbb{C}, v \in V\} \\
&\simeq \mathbb{P}(O_{\mathbb{P}(V)}(-1) \oplus O_{\mathbb{P}(V)}(-3)),
\end{align*}
\]

where \(\dim V = 2\).

**Proposition 4.6** (Proposition 3.5.2 of [15]). Let \(X\) be a smooth nonhomogeneous projective horospherical variety \((L,\alpha,\beta)\) of Picard number one. Let \(\mathcal{C}_{o}(X) \subset \mathbb{P}(T_{o}X)\) denote the variety of minimal rational tangents of \(X\) at the base point \(o\). Then

\[
\mathcal{C}_{o}(X) = \begin{cases} 
\mathcal{H}_{L_{0}}(U_{-1} \oplus L_{-1} \oplus L_{-2}) & \text{if } X \text{ is } (C_{m},\alpha_{i+1},\alpha_{i}) \text{ for } 1 \leq i < m \\
\mathcal{H}_{L_{0}}(U_{-1} \oplus L_{-1}) & \text{otherwise.}
\end{cases}
\]

**Proof.** We note that the variety \(\mathcal{C}_{o}(Y)\) of minimal rational tangents of \(Y\) at \(o\) is contained in the variety \(\mathcal{C}_{o}(X)\) of minimal rational tangents of \(X\) at \(o\).

Assume that \(X\) is not \((C_{m},\alpha_{i+1},\alpha_{i}), m > 2, i = 1, \ldots, m - 2\). Since \(Y = L/P_{Y}\) is associated with a long simple root \(\alpha\), by Proposition 1 of Hwang-Mok ([13]), the variety \(\mathcal{C}_{o}(Y)\) of minimal rational tangents of \(Y\) is \(\mathcal{H}_{L_{0}}(L_{-1})\). Furthermore, \(L_{0} \triangleright U_{0}\) acts invariantly on \(\mathcal{C}_{o}(X)\). From \([U_{0},L_{-1}] \subset U_{-1}\), it follows that the highest weight vector of \(U_{-1}\) is contained in \(\mathcal{C}_{o}(X)\). Therefore, \(\mathcal{H}_{L_{0}}(U_{-1} \oplus L_{-1})\) is contained in \(\mathcal{C}_{o}(X)\) so that \(\dim \mathcal{H}_{L_{0}}(U_{-1} \oplus L_{-1})\) is less than or equal to \(\dim \mathcal{C}_{o}(X)\). However, \(\dim \mathcal{C}_{o}(X)\) cannot exceed \(\dim \mathcal{H}^{0}(\mathcal{C}_{o}, N_{\mathcal{C}_{o}|X}|(-1))\) which is equal to \(K_{X}^{-1} \cdot C_{\alpha} - 2\). Now by comparing the dimension of \(\mathcal{H}_{L_{0}}(U_{-1} \oplus L_{-1})\) with \(K_{X}^{-1} \cdot C_{\alpha} - 2\), we get the desired results.

| types | \(K_{X}^{-1} \cdot C_{\alpha} - 2\) |
|-------|-------------------------------|
| \((B_{m},\alpha_{m-1},\alpha_{m})\) | \(m\) |
| \((B_{3},\alpha_{1},\alpha_{3})\) | \(5\) |
| \((C_{m},\alpha_{m};\alpha_{m-1})\) | \(m\) |
| \((C_{m},\alpha_{i+1},\alpha_{i})_{1 < i+1 < m}\) | \(2m - i - 1\) |
| \((F_{4},\alpha_{2},\alpha_{3})\) | \(4\) |
| \((G_{2},\alpha_{2},\alpha_{1})\) | \(2\) |
Here, we use the description of $\mathcal{H}_{L_0}(U_{-1} \oplus L_{-1})$ in Lemma 4.5.

Assume that $X$ is $(C_m, \alpha_{i+1}, \alpha_i)$, $1 < i + 1 < m$. Then we have $C_o(Y) = \mathcal{H}_{L_0}(L_{-1} \oplus L_{-2})$. By a similar argument, after replacing $\mathcal{H}_{L_0}(L_{-1})$ by $\mathcal{H}_{L_0}(L_{-1} \oplus L_{-2})$, we get the desired result. \hfill $\Box$

5. Projective geometry of varieties of minimal rational tangents

In this section, let $X$ be either $(B_m, \alpha_{m-1}, \alpha_m)$, where $m \geq 3$, or $(F_4, \alpha_2, \alpha_3)$. Let $g = \bigoplus_{p \in \mathbb{Z}} g_p$ be the Lie algebra of Aut$(X)$ with the gradation given as in Proposition 1.4 and $m := \bigoplus_{p < 0} g_p$ be its negative part. As in Lemma 4.5, let $V$ and $W$ be the vector spaces with $(\dim V, \dim W) = (2, m - 1)$ for $(B_m, \alpha_{m-1}, \alpha_m)$ and $(\dim V, \dim W) = (3, 2)$ for $(F_4, \alpha_2, \alpha_3)$, and set

$$U := V \oplus (\text{Sym}^2(V) \otimes W).$$

Then $U$ can be identified with $g_{-1}$. Let

$$S = \mathbb{P}\{v + v^2 \otimes w : v \in V, w \in W\} \subset \mathbb{P}U$$

be the variety of minimal rational tangents of $X$ at the base point as in Proposition 4.6.

5.1. Projective geometries of VMRTs. We will show that the variety $S \subset \mathbb{P}U$ of minimal rational tangents of $X$ at the base point satisfies the conditions in Proposition 2.9.

Lemma 5.1. Let $s$ be the codimension of $T^{(2)}\hat{S}$ in $U = g_{-1}$.

(1) The third fundamental form of $S \subset \mathbb{P}(U)$ is surjective;

(2) The dimension of $[g_{-\alpha}, g_{-1}]$ is $s$, where $\alpha$ is the simple root which gives the gradation on $g$.

Proof. (1) The tangent space $T_\beta\hat{S}$ at $\beta = v + v^2 \otimes w \in \hat{S}$ is given by

$$T_\beta\hat{S} = \{v' + 2v \circ v' \otimes w + v^2 \otimes w' : v' \in V, w' \in W\}.$$

The second fundamental form $II_\beta : \text{Sym}^2 T_\beta\hat{S} \rightarrow U/T_\beta\hat{S}$ is

$$II_\beta(v' + 2v \circ v' \otimes w, v'' + 2v \circ v'' \otimes w) = 2v' \circ v'' \otimes w$$

$$II_\beta(v', 2v \circ v' \otimes w, v^2 \otimes w') = 2v \circ v' \otimes w'$$

$$II_\beta(v^2 \otimes w', v^2 \otimes w') = 0$$

where $v', v'' \in V$ and $w', w'' \in W$. The third fundamental form $III_\beta : \text{Sym}^3 T_\beta\hat{S} \rightarrow U/T_\beta^{(2)}\hat{S}$ is zero except

$$III_\beta(v' + 2v \circ v' \otimes w, v'' + 2v \circ v'' \otimes w, v^2 \otimes w') = 2v' \circ v'' \otimes w' \mod T_\beta^{(2)}\hat{S},$$

where $v', v'' \in V$ and $w' \in W$. Thus the third osculating space is the whole space $U$.

(2) The dimension $r$ of the image of $II_\beta$ is $m$ in the first case, and is 7 in the second case. $(\dim U, \dim \hat{S}) = (\dim D_x, p + 1)$ is $(2 + 3(m - 1) = 3m - 1, m + 1)$ in the first case, and is $(15, 5)$ in the second case. Thus $s = \dim g_{-1} - (1 + p + r)$ is $m - 2$ in the first case, and is 3 in the second case.

On the other hand, the minimum of $\dim[\xi, D_x]$ occurs when $\xi \in g_{-\alpha}$ where $\alpha$ is the simple root which gives the gradation on $g$. Furthermore, the dimension of $[\xi, D_x] = [\xi, L_{-1}]$
is equal to the number of roots $\gamma$ with $\mathfrak{g}_{-\gamma} \subset \mathfrak{l}_{-1}$ such that $\alpha + \gamma$ is again a root. One can compute directly that this number is $m - 2$ in the first case, and is $3$ in the second case. \hfill \square

From the exact sequence of $G_0$-modules where $G_0$ is the subgroup of $G = \text{Aut}(X)$ with Lie algebra $\mathfrak{g}_0$

\[ 0 \to V \to U \to \text{Sym}^2 V \otimes W \to 0, \]

we get the following exact sequences.

**Lemma 5.2.** Let $\beta = v + v^2 \otimes w$ be an element of $\hat{\mathcal{S}}$, where $v \neq 0 \in V$ and $w \neq 0 \in W$. Denote by $V_0$ the subspace of $V$ generated by $v$ and by $W_0$ the subspace of $W$ generated by $w$. Then we have the following exact sequences.

\[
\begin{align*}
0 \to & \quad \mathbb{C}\beta \quad \to \text{Sym}^2 V_0 \otimes W_0 \to 0 \\
0 \to & \quad V_0 \to \quad T_\beta \hat{\mathcal{S}} / \mathbb{C}\beta \quad \to (V_0 \odot (V/V_0) \otimes W_0) \oplus (\text{Sym}^2 V_0 \otimes (W/W_0)) \to 0 \\
0 \to & \quad V/V_0 \to \quad T_\beta^{(2)} \hat{\mathcal{S}} / T_\beta \hat{\mathcal{S}} \quad \to (\text{Sym}^2 (V/V_0) \otimes W_0) \oplus (V_0 \odot (V/V_0)) \otimes (W/W_0) \to 0 \\
0 \to & \quad U / T_\beta^{(2)} \hat{\mathcal{S}} \quad \to \text{Sym}^2 (V/V_0) \otimes (W/W_0) \to 0
\end{align*}
\]

**Proof.** From the proof of Lemma 5.1 we get the following exact sequences.

\[
\begin{align*}
0 \to & \quad \mathbb{C}\beta \quad \to \text{Sym}^2 V_0 \otimes W_0 \to 0 \\
0 \to & \quad V_0 \to \quad T_\beta \hat{\mathcal{S}} \quad \to V_0 \odot V \otimes W_0 + \text{Sym}^2 V_0 \otimes W \to 0 \\
0 \to & \quad V \to \quad T_\beta^{(2)} \hat{\mathcal{S}} \quad \to \text{Sym}^2 V \otimes W_0 + V_0 \odot V \otimes W \to 0.
\end{align*}
\]

Here, we remark that $V_0 \odot V \otimes W_0 + \text{Sym}^2 V_0 \otimes W$ and $\text{Sym}^2 V \otimes W_0 + V_0 \odot V \otimes W$ are not direct sums:

\[
\begin{align*}
(V_0 \odot V \otimes W_0) \cap (\text{Sym}^2 V_0 \otimes W) &= \text{Sym}^2 V_0 \otimes W_0 \\
(\text{Sym}^2 V \otimes W_0) \cap (V_0 \odot V \otimes W) &= V_0 \odot V \otimes W_0.
\end{align*}
\]

By taking the quotients we get the desired exact sequences. \hfill \square

**Remark 5.3.** Let $p = \dim \mathcal{S}$, $q = \dim \mathcal{Q} / T_\beta \hat{\mathcal{S}}$, $r = \dim T_\beta^{(2)} \hat{\mathcal{S}} / T_\beta \hat{\mathcal{S}}$, $s = \dim U / T_\beta^{(2)} \hat{\mathcal{S}}$ as above, and $t = q - r - 2s$, where $\mathcal{Q} := \mathfrak{m}$ as a vector space. Then we have the following table.

| $(B_m, (\alpha_{m-1}, \alpha_m))$ | $p$ | $q$ | $r$ | $s$ | $t$ |
|----------------------------------|-----|-----|-----|-----|-----|
| $(F_4, \alpha_2, \alpha_3)$     | 4   | 10  | 7   | 3   | 5   |

5.2. **Fundamental graded Lie algebras determined by VMRTs.** We will show that $\mathcal{S}$ and $\mathfrak{m}$ satisfies the conditions in Proposition 3.7 so that there is a one-to-one correspondence between $G_0$-structures and $\mathcal{S}$-structures on filtered manifolds of type $\mathfrak{m}$.

**Proposition 5.4.**

1. The graded Lie algebra $\mathfrak{m}$ is fundamental and determined by $\mathcal{S} \subset \mathbb{P}\mathfrak{g}_{-1}$.
2. Let $\mathfrak{g}(\hat{\mathcal{S}})$ be the Lie algebra of the linear automorphism group $G(\hat{\mathcal{S}})$. Then the induced homomorphism $\mathfrak{g}(\hat{\mathcal{S}}) \to \mathfrak{g}_0(\mathfrak{m})$ is injective and its image is $\mathfrak{g}_0$. 
In the remaining part of this subsection we will prove Proposition 5.3. Recall that $U$ is given by
\[ U = V \oplus (\text{Sym}^2(V) \otimes W) \]
where $V$ and $W$ are vector spaces with $(\dim V, \dim W) = (2, m - 1)$ for $(B_m, \alpha_{m-1}, \alpha_m)$ and $(\dim V, \dim W) = (3,2)$ for $(F_4, \alpha_2, \alpha_3)$ and that $S$ is given by
\[ S = \mathbb{P}\{v + v^2 \otimes w : v \in V, w \in W\} \cong \mathbb{P}(O(-1) \oplus O(-2)^k) \subset \mathbb{P}U, \]
where $k = \dim W$. Let $G := SL(V) \times SL(W)$ be a group acting on $S$. Then, $S$ is a smooth horospherical $G$-variety of rank one. Moreover, it has two closed $G$-orbits, $Z := \mathbb{P}(V)$ corresponding to $\mathbb{P}(O(-1))$ and
\[ Y := \mathbb{P}\{\lambda^2 \otimes \mu \in \text{Sym}^2 V \otimes W | \lambda \in V, \mu \in W\} \]
corresponding to a choice of $\mathbb{P}(O(-2)^k)$, has one open $G$-orbit the complement of $Y \cup Z$ in $S$.

Denote that $Y$ is the variety of minimal rational tangents of $Y = L/P_Y$ that is a rational homogeneous associated with a long simple root of type $(B_m, \alpha_{m-1})$ (respectively, of type $(F_4, \alpha_2)$).

**Proposition 5.5** (Proposition 1 and Proposition 7 of [13], Proposition 4.1 of [5]). Let $Y$ be a rational homogeneous space of type $(I, \alpha)$ where $\alpha$ is a long simple root. Let $n = \bigoplus_{p \leq 0} l_p$ be the negative part of the graded Lie algebra $l = \bigoplus_{p \in \mathbb{Z}} l_p$ with gradation associated to $\alpha$. Let $D_o \subset T_o Y$ be the linear span of the homogeneous cone of the variety $C_o$ of minimal rational tangents at a base point $o \in Y$. Then $n$ is the fundamental graded Lie algebra determined by $C_o \subset \mathbb{P}D_o$.

Since $\wedge^2(\text{Sym}^2 V \otimes W) = \wedge^2(\text{Sym}^2 V) \otimes \text{Sym}^2 W \oplus \text{Sym}^2(\text{Sym}^2 V) \otimes \wedge^2 W$, if we decompose $\text{Sym}^2(\text{Sym}^2 V) = \text{Sym}^4 V \oplus (\text{Sym}^4 V)\perp$, then the Lie bracket $[,] : \wedge^2 l_1 \to l_2$ is given by the projection map
\[ \nu : \wedge^2(\text{Sym}^2 V \otimes W) \to (\text{Sym}^4 V)\perp \otimes \wedge^2 W. \]
If $X$ is of type $(B_m, \alpha_{m-1}, \alpha_m)$, then $(\text{Sym}^4 V)\perp \otimes \wedge^2 W = \mathbb{C} \otimes \wedge^2 W = \wedge^2 W$. If $X$ is of type $(F_4, \alpha_2, \alpha_3)$, then $(\text{Sym}^4 V)\perp \otimes \wedge^2 W = \text{Sym}^2(\wedge^2 V) \otimes \mathbb{C} = \text{Sym}^2(\wedge^2 V)$. Thus $\wedge^2(\text{Sym}^2 V \otimes W)$ is the direct sum of $\text{Ker} \nu$ and an irreducible representation of $SL(V) \times SL(W)$.

Note that $g_{-1} = U$ and $g_{-2} = l_2$. The Lie bracket $[,] : \wedge^2 g_{-1} \to g_{-2}$ defines
\[ \omega : \wedge^2(V \oplus \text{Sym}^2 V \otimes W) \rightarrow l_2, \]
where $\omega|_{\wedge^2(\text{Sym}^2 V \otimes W)} = \nu$ and $\omega(V, V) = \omega(V, \text{Sym}^2 V \otimes W) = 0$. Then, $\wedge^2 U = \wedge^2(V \oplus \text{Sym}^2 V \otimes W)$ is decomposed as
\[ \wedge^2(V \oplus \text{Sym}^2 V \otimes W) = \wedge^2 V \oplus (V \wedge (\text{Sym}^2 V \otimes W)) \oplus \wedge^2(\text{Sym}^2 V \otimes W) \]
and we have $\text{Ker} \omega = \wedge^2 V \oplus (V \wedge (\text{Sym}^2 V \otimes W)) \oplus \text{Ker} \nu$.

**Lemma 5.6.** The kernel $\text{Ker} \omega$ is spanned by $\wedge^2 P \subset \wedge^2 U$ where $P$ is 2-dimensional subspace of $U$ tangent to $\mathbb{S}$.

**Proof.** Let $\Xi$ be the subspace of $\wedge^2(V \oplus \text{Sym}^2 V \otimes W)$ spanned by
\[ \{\wedge^2 P \subset \wedge^2(V \oplus \text{Sym}^2 V \otimes W)|P \text{ is tangent to } \mathbb{S}, \dim P = 2\}. \]
Since $\hat{S} \subset V \oplus \text{Sym}^2 V \otimes W$ is a $SL(V) \times SL(W)$-invariant subvariety, $\Xi$ is also $SL(V) \times SL(W)$-invariant subspace of $\wedge^2(V \oplus \text{Sym}^2 V \otimes W)$. Furthermore, $\wedge^2(V \oplus \text{Sym}^2 V \otimes W)$ is the direct sum of $\text{Ker}\,\omega$ and an irreducible representation of $SL(V) \times SL(W)$. We will show that $\text{Ker}\,\omega$ is contained in $\Xi$. If so, since $\Xi$ is an $SL(V) \times SL(W)$-invariant subspace and is proper, we get the desired equality.

As the subspace $V$ is contained in $\hat{S}$, the component $\wedge^2 V$ is contained in $\Xi$. By Proposition 5.5, the kernel of $\nu$ is spanned by $\wedge^2 P$, where $P$ is tangent to $\hat{Y}$, and thus is contained in $\Xi$. We claim that $V \wedge (\text{Sym}^2 V \otimes W)$ is contained in $\Xi$, which completes the proof.

Let $\beta_t = v_t + v_t^2 \otimes w_t$ be a curve in $\hat{S}$, where $v_t \in V$ and $w_t \in W$. By definition of $\Xi$,

$$\beta_0 \land \beta'_0 = (v_0 + v_0^2 \otimes w_0) \land (v'_0 + v'_0 \otimes v_0 \otimes w_0 + v_0 \otimes v'_0 \otimes w_0 + v_0^2 \otimes w'_0)$$

is contained in $\Xi$. Since $\alpha_t := v_t^2 \otimes w_t$ is a curve in $\hat{Y} \subset \hat{S}$,

$$\alpha_0 \land \alpha'_0 = (v_0^2 \otimes w_0) \land (v'_0 \otimes v_0 \otimes w_0 + v_0 \otimes v'_0 \otimes w_0 + v_0^2 \otimes w'_0)$$

is contained in $\Xi$. Hence, $\beta_0 \land \beta'_0 - \alpha_0 \land \alpha'_0 - v_0 \land v'_0$, which is equal to

$$v_0 \land (v'_0 \otimes v_0 \otimes w_0 + v_0 \otimes v'_0 \otimes w_0 + v_0^2 \otimes w'_0) + (v_0^2 \otimes w_0) \land v'_0$$

is contained in $\Xi$. Note that

$$v_0 \land (v'_0 \otimes v_0 + v_0 \otimes v'_0)$$

$$= v_0 \otimes v'_0 \otimes v_0 + v_0 \otimes v_0 \otimes v'_0 - v'_0 \otimes v_0 \otimes v_0 - v_0 \otimes v'_0 \otimes v_0$$

$$= v_0^2 \land v'_0$$

and

$$v_0 \land v_0^2 = v_0 \otimes (v_0 \otimes v_0) - (v_0 \otimes v_0) \otimes v_0 = 0$$

Therefore, $(v'_0 \land v_0^2) \otimes w_0$ is contained in $\Xi$. Here, we consider $V \wedge (\text{Sym}^2 V \otimes W)$ as $(V \wedge \text{Sym}^2 V) \otimes W$. This is true for arbitrary $v_0, v'_0 \in V$ and $w_0 \in W$, it follows that $V \wedge (\text{Sym}^2 V \otimes W)$ is contained in $\Xi$. \hfill $\Box$

**Lemma 5.7.** \(\text{Aut}^0(S) = ((SL(V) \times SL(W))/\text{Z}(G)) \times C^* \triangleright V^* \otimes W^*\) where $\text{Z}(G)$ is the center of $G = SL(V) \times SL(W)$.

**Proof.** Since $S$ is a smooth horospherical $G$-variety of rank one with two closed $G$-orbits $Z$ and $Y$, we will apply the same arguments as in the proof of Lemma 1.1. of [22].

| type               | dim $S$ | dim $Y$ | dim $Z$ | $N_{Y|S}$ | $N_{Z|S}$ |
|--------------------|---------|---------|---------|-----------|-----------|
| \((B_m, \alpha_{m-1}, \alpha_m)\) | $m$     | $m-1$   | $1$     | $O(1)$    | $O(-1)^{m-1}$ |
| \((F_1, \alpha_2, \alpha_3)\)     | $4$     | $3$     | $1$     | $O(1)$    | $O(-1)^2$  |

Thus $H^0(Y, N_{Y|S}) = V^* \otimes W^*$ and $H^0(Z, N_{Z|S}) = 0$.

By the same arguments as in the proof of Lemma 1.1. of [22], the closed $G$-orbit $Z$ is stable under the action of $\text{Aut}^0(S)$ and we have

$$\text{Aut}^0(S) = \left(\frac{G}{\text{Z}(G) \times C^*}\right) \triangleright H^0(Y, N_{Y|S})$$

$$= \left((SL(V) \times SL(W))/\text{Z}(G)) \times C^* \triangleright V^* \otimes W^*\right.$$
Proof of Proposition 5.4 (1). Proposition 5.5 says that $L = \oplus_{k<0} I_k$ is the fundamental graded Lie algebra determined by $Y \subset \mathbb{P}(L_1)$. Since $m = U_{-1} \oplus L$ and $[U_{-1}, m] = 0$, by Proposition 5.5 and Lemma 5.6, the graded Lie algebra $m$ is fundamental and determined by $S \subset \mathbb{P} g_{-1}$. \hfill \Box

Proof of Proposition 5.4 (2). We recall that $g_0 = (l_0 \oplus \mathbb{C}) \triangleright U_0 \subset g_0(m)$ from Proposition 4.4. The center of $l_0$ is of dimension one, the semisimple part of $l_0$ is $sl(V) + sl(W)$ and the vector space $U_0$ is $V^* \otimes W^*$. We compare $g_0$ with the Lie algebra of the neutral component $\text{Aut}^0(S)$ of the automorphism group of the cone $\hat{S} \subset g_1$ over $S$. Then the rest of proof follows from Lemma 5.7 that is, $\text{Aut}^0(\hat{S})$ is equal to the linear automorphism group $G(\hat{S})$ and the induced map $g(\hat{S}) \to g_0(m)$ is injective whose image in $g_0(m)$ agrees with $g_0 \subset g_0(m)$. \hfill \Box

5.3. Parallel transports of VMRTs. We will show that $S$ is not changed under the deformation keeping the second fundamental form and the third fundamental form constant (Proposition 5.10). We adapt arguments in the proof of Proposition 8.9 of [7], which proves the same statement as in Proposition 5.10 for the case when $X$ is $(G_2, \alpha_2, \alpha_3)$.

Assume that $X$ is $(F_4, \alpha_2, \alpha_3)$. Then $U$ and $S$ are given by $U = V \oplus (\text{Sym}^2(V) \otimes W)$ and

$$S = \mathbb{P}\{cv + v^2 \otimes w : c \in \mathbb{C}, v \in V, w \in W\},$$

where $V$ is a vector space of dimension 3 and $W$ is a vector space of dimension 2.

Consider $S$ as a projective bundle $\mathbb{P}(O(-1) \oplus O(-2)^2)$ over $\mathbb{P}(V)$. Let $\psi : S \to \mathbb{P}(V)$ denote this $\mathbb{P}^2$-fibration and let $\xi$ denote the dual tautological line bundle on $\psi : S \to \mathbb{P}(V)$. Then $H^0(S, \xi) = H^0(\mathbb{P}(V), \psi_\ast \xi) = V^* \oplus (\text{Sym}^2 V^* \otimes W^*)$. Hence $\xi$ induces the embedding $S \subset \mathbb{P}(U)$.

The $O(-1)$-factor defines a subvariety $Z = \mathbb{P}(O(-1)) \subset S$, which is isomorphic to $\mathbb{P}^2$ and is a section of $\psi$. A choice of an $O(-2)$-factor gives a section $B$ of $\psi$ whose linear span $A$ is isomorphic to $\mathbb{P}^5$ and disjoint from $Z$. We call such a section a complementary section.

Fix an $O(-2)$-factor and denote by $B_0$ the corresponding complementary section. Then the complement $S - Z$ of $Z$ is biholomorphic to the total space of the vector bundle $O(1)^2$ on $\mathbb{P}(V)$ whose zero section corresponds to $B_0$. A complementary section corresponds to a section of $O(1)^2$. Thus, for a given triple $(s_1^1, s_1^2, s_1^3)$ in $S - Z$ such that $\psi(s_1^1), \psi(s_1^2), \psi(s_1^3)$ are distinct, there is a unique complementary section $B_1$ with $s_1^1, s_1^2, s_1^3 \in B_1$.

Proposition 5.8. When $X$ is $(F_4, \alpha_2, \alpha_3)$, $S \subset \mathbb{P}^{14}$ is obtained from the following four data:

(i) a plane $Z \subset \mathbb{P}^{14}$,
(ii) two disjoint linear spaces $A_1 \cong \mathbb{P}^5$ and $A_2 \cong \mathbb{P}^5$ such that the union $Z \cup A_1 \cup A_2$ span $\mathbb{P}^{14}$,
(iii) two subvarieties $B_1 \subset A_1$ and $B_2 \subset A_2$, each of which is isomorphic to the Veronese surface $\nu(\mathbb{P}^2) \subset \mathbb{P}^5$,
(iv) two birational maps $\xi_1 : Z \to B_1$ and $\xi_2 : Z \to B_2$.

Proof. The locus of planes generated by $z \in Z$ and $\xi_1(z), \xi_2(z)$ is a subvariety of $\mathbb{P}^{14}$ projectively equivalent to $S \subset \mathbb{P}^{14}$. \hfill \Box
Similarly we have the following.

**Proposition 5.9.** When $X$ is $(B_m, \alpha_{m-1}, \alpha_m)$, $S \subset \mathbb{P}^{3m-2}$ is obtained from the following four data:

(i) a line $Z \subset \mathbb{P}^{3m-2}$,

(ii) $m-1$ linear spaces $A_i \cong \mathbb{P}^3$, where $1 \leq i \leq m-1$, such that the union $Z \cup \bigcup_{i=1}^{m-1} A_i$ spans $\mathbb{P}^{3m-2}$,

(iii) $m-1$ subvarieties $B_i \subset A_i$, where $1 \leq i \leq m-1$, each of which is isomorphic to the conic $\nu(\mathbb{P}^1) \subset \mathbb{P}^2$,

(iv) $m-1$ birational maps $\epsilon_i : Z \to B_i$, where $1 \leq i \leq m-1$.

**Proposition 5.10.** Let $S \subset \mathbb{P}U$ be the variety of minimal rational tangents of $(B_m, \alpha_{m-1}, \alpha_m)$, $m \geq 3$ or $(F_4, \alpha_2, \alpha_3)$ at the base point.

Let $\pi : \mathbb{P}U \to \mathbb{P}^1$ be the projectivization of a holomorphic vector bundle $U$ over $\mathbb{P}^1$ and let $C \subset \mathbb{P}U$ be an irreducible subvariety. Denote by $\varpi$ the restriction of $\pi$ to $C$. Assume that

1. $C_t := \varpi^{-1}(t) \subset \mathbb{P}U_t := \pi^{-1}(t)$ is projectively equivalent to $S \subset \mathbb{P}U$ for all $t \in \mathbb{P}^1 - \{t_1, \ldots, t_k\}$;

2. for a general section $\sigma \subset C$ of $\varpi$, the relative second fundamental forms and the relative third fundamental forms of $C$ along $\sigma$ are constants.

Then for any $t \in \mathbb{P}^1$, $C_t \subset \mathbb{P}(U_t)$ is projectively equivalent to $S \subset \mathbb{P}(U)$.

**Proof.** Assume that $C_t \subset \mathbb{P}U_t$ is projectively equivalent to $S \subset \mathbb{P}U$ for all $t$ in the unit disc $\Delta \subset \mathbb{P}^1$ except for $t = 0$. Assume further that for a general section $\sigma \subset C$ of $\varpi$, the relative second fundamental forms and the relative third fundamental forms of $C$ along $\sigma$ are constants. Then there is an open submanifold $C^0 \subset C$ such that $C^0_t \subset C_t$ corresponds to an open subset of $S - Z \subset S$ for $t \neq 0 \in \Delta$. We claim that $C_0 \subset \mathbb{P}U_0$ is also projectively equivalent to $S \subset \mathbb{P}U$. It suffice to show that we get the four data in Proposition 5.9 at $t = 0$ from $C_0 = \varpi^{-1}(0) \subset \mathbb{P}U_0$.

Assume that $X = (F_4, \alpha_2, \alpha_3)$. Let $\beta_t \in C_t$ be a section of $\varpi$ corresponding to a general point $\beta \in S$ for $t \neq 0$ and let $\beta_0$ be the limit. According to the computation of the second fundamental form in the proof of Lemma 5.8 (1), we see that $\text{Base locus}(II_{\beta_0})$ gives us a foliation whose leaves are isomorphic to $\mathbb{P}^2$-fiber of the projective bundle $S$ over $\mathbb{P}^2$. By the assumption (2), there is also a foliation on the central fiber given by $\text{Base locus}(II_{\beta_0})$, which is exactly $\mathbb{P}^2$ in $C_0 = \varpi^{-1}(0) \subset \mathbb{P}U_0$. We call these subvarieties $\mathbb{P}^2$ of $C_t$ for $t \in \Delta$, corresponding to the $\mathbb{P}^2$-fiber of $S$, the planes of the rulings.

(i) Let $\psi_t : C_t \to \mathbb{P}^2$ be the corresponding fibration for $t \neq 0$ of the projective bundle $S = \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)) \to \mathbb{P}^2$. Then $Z \subset S$ gives family of distinguished planes $Z_t \simeq \mathbb{P}(\mathcal{O}(-1)) \subset C_t \subset \mathbb{P}U_t$ for $t \neq 0$, which is a section of $\psi_t$. Then the limit $Z_0$ is also a $\mathbb{P}^2$. Hence, there are $\mathbb{P}^2$ subbundle $Z \subset C$ of $\pi : \mathbb{P}U \to \mathbb{P}^1$.

(ii) Pick 3 distinct points $s^1_1, s^1_2, s^1_3 \in C_0 - Z_0$ that lie in 3 distinct planes of the rulings and choose local sections $\sigma^1_1, \sigma^1_2, \sigma^1_3$ of $\varpi : C^0 \to \Delta$ such that $\sigma^1_i(0) = s^1_i$ for $i = 1, 2, 3$. Then there exists a unique complementary section $B_{1,t}$ with $\sigma^1_1(t), \sigma^1_2(t), \sigma^1_3(t) \in B_{1,t}$ for any $t \neq 0$. 
Let $A_{1,t}$ be the linear span of $B_{1,t}$. Then, $B_{1,t} \subset A_{1,t}$ is isomorphic to the Veronese surface $\nu(\mathbb{P}^2) \subset \mathbb{P}^5$. The limit $A_{1,0}$ of $A_{1,t} \simeq \mathbb{P}^5 \subset \mathbb{P}U_t$ is also a projective space $\mathbb{P}^5 \subset \mathbb{P}U_0$, which contains $s_1, s_2, s_3$.

Choose 3 distinct points $s_1^i, s_2^i, s_3^i$ in the complement $\mathcal{C}_0 - (Z_0 \cup A_{1,0})$ such that each line span$(s_1^i, s_2^i)$ for $i = 1, 2, 3$ is in the same plane of the rulings. Choose local sections $\sigma_2^i$ of $\nu$ such that $\sigma_2^i(t) \in \mathcal{C}_t - (Z_t \cup A_{1,t})$ and $\sigma_2^i(0) = s_1^i$ for $i = 1, 2, 3$. Then there exist a unique complementary section $B_{2,t}$ such that $\nu_{A}^1(t), \nu_{A}^2(t), \nu_{A}^3(t) \in B_{2,t}$.

Let $A_{2,t}$ be the linear span of $B_{2,t}$. The two complementary sections $B_{1,t} \subset A_{1,t}$ and $B_{2,t} \subset A_{2,t}$ are isomorphic to the Veronese surface $\nu(\mathbb{P}^2) \subset \mathbb{P}^5$ for $t \neq 0$.

The limit $A_{j,0}$ of $A_{j,t} \simeq \mathbb{P}^5 \subset \mathbb{P}U_t$ is also projective space $\mathbb{P}^5 \subset \mathbb{P}U_0$, $j = 1, 2$ such that $s_1^1, s_2^1, s_3^1 \in A_{1,0}$ and $s_1^2, s_2^2, s_3^2 \in A_{2,0}$. Then the union $Z_0 \cup A_{1,0} \cup A_{2,0}$ spans $\mathbb{P}U_0$. For, otherwise, there is hyperplane such that $Z_0 \cup A_{1,0} \cup A_{2,0} \subset \mathbb{P}^{13} \subset \mathbb{P}U_0$ which contradicts to the constancy of second and third fundamental form.

(iii) Consider the limit $B_{j,0}$ of $B_{j,t}$, which is a surface in $A_{j,0} = \mathbb{P}^5$ of degree less then 4.

The planes of rulings on $\mathcal{C}_0$ intersecting $A_{j,0}$, give a analytic surface $B'_{j,0} \subset B_{j,0}$. Let $A'_{j,0}$ be the linear span of $B'_{j,0}$. If $B'_{j,0}$ lies in a hyperplane of $A_{j,0}$, then $\mathbb{P}(Z_0 \oplus A_{1,0} \oplus A'_{j,0})$ is contained in a hyperplane of $\mathbb{P}U_0$ which contradiction to the fundamental forms. Hence, $B'_{j,0}$ must be non-degenerate in $A_{j,0}$ and this means $B_{j,0}$ is irreducible non-degenerate surface.

Since the Veronese surface is minimal degree surface, a Veronese surface for $j = 1, 2$.

(iv) With a choice of $Z_t, B_{j,t}$ and $A_{1,t}$ for $j = 1, 2$ as above in (i) - (ii), we have birational morphisms $\varepsilon_{j,t}: Z_t \to B_{j,t}$ for $t \neq 0$ such that the planes of rulings on $\mathcal{C}_t$ are the plane spanned by $z_t \in Z_t, \varepsilon_{1,t}(z_t) \in B_{1,t}$ and $\varepsilon_{2,t}(z_t) \in B_{2,t}$. Let $\mathcal{P}$ be the blow up of the bundle $\pi: \mathbb{P}U \to \Delta$ along the submanifold $Z$ and let $\mathcal{E}$ be the exceptional divisor, which is biholomorphic to $Z \times_{\Delta} \mathbb{P}(\hat{A}_1 \oplus \hat{A}_2)$. Let $E_t$ be the exceptional divisor of the blow up of $\mathbb{P}U_t$, which is $Z_t \times \mathbb{P}(A_{1,t} \oplus A_{2,t})$. For $t \neq 0$, the proper transform of the plane joining $z \in Z_t, \varepsilon_{1,t}(z) \in B_{1,t}$ and $\varepsilon_{2,t}(z) \in B_{2,t}$ intersects $E_t$ at a point $(z, \varepsilon_t(z)) \in E_t$, where $\varepsilon_t(z) := \varepsilon_{1,t}(z) \oplus \varepsilon_{2,t}(z)$. The surface $\Gamma_t := \{ (z, \varepsilon_t(z)) | z \in Z_t \}$ corresponds to the graph of $\varepsilon_t$. Let $\Gamma_t \cap E_t$ be the limit of $\Gamma_t$. Then there exists an irreducible component $\Gamma'_0$ of $\Gamma_0$ which is the intersection of $E_t$ with the proper transform of the planes in the planes of ruling at $t = 0$. By definition, $\Gamma'_0$ dominate $B_{j,0}$ for $j = 1, 2$. $\Gamma'_0$ dominates $Z_0$, too. For, otherwise, the intersection with the planes of ruling of $Z_0$ is codimension $\geq 1$, which contradicts to the constancy of the second fundamental forms and the third fundamental forms.

On $\mathcal{E}$, let $\mathcal{L}^Z$ be the line bundle which is the pull back of the relative $\mathcal{O}(1)$ bundle on $Z$ and let $\mathcal{L}^{A_j}$ be the line bundle which is the pull back of the relative $\mathcal{O}(1)$ bundle on $A_j$. Then $\mathcal{L}^Z \otimes \mathcal{L}^{A_1} \otimes \mathcal{L}^{A_2}$ be a relative ample line bundle on $\mathcal{E}$ which has degree 9 with respect to $\Gamma_t$ for $t \neq 0$. Since $\Gamma'_0$ dominate $Z_0$ and $B_{j,0}$ for $j = 1, 2$, we have $\Gamma'_0 \cdot \mathcal{L}^Z \geq 1$ and $\Gamma'_0 \cdot \mathcal{L}^{A_j} \geq 4$. Hence, $\Gamma'_0 = \Gamma_0$ and $\Gamma_0$ determines birational map $\varepsilon_{j,0}: Z_0 \to B_{j,0}$ for $j = 1, 2$.

In case when $X = (B_m, \alpha_{m-1}, \alpha_m)$, $m > 2$,

$$S = \mathbb{P}\{ cv + v^2 \otimes w : c \in \mathbb{C}, v \in V, w \in W \} \simeq \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)^{m-1})$$

where $\dim V = 2$ and $\dim W = m - 1$. Since $\nu_2(\mathbb{P}^1) \subset \mathbb{P} \text{Sym}^2 V \simeq \mathbb{P}^2$ is rational normal curve with minimal degree, we conclude the result in a similar method. □
6. $H^2$-COHOMOLOGIES

We keep the same assumptions and notations as in Section 5. We write $\mathbb{C}$ as $\mathfrak{z}$ to emphasize that $\mathbb{C}$ commutes with $1$.

6.1. Reductions. A main ingredient is that the computation of $H^2(m, g)$ can be reduced to that of $H^2(L, g)$ and $H^1(L, g)$, which can be computed by applying Kostant’s theory.

**Proposition 6.1.** Let $k \geq 1$. Let $\phi \in \text{Hom}(\wedge^2 m, g)_k$ be such that $\partial \phi = 0$. Then there is $\eta \in \text{Hom}(m, g)_k$ and $\zeta \in \text{Hom}(\wedge^2 g_{-1}, U)_k$ such that

$$\phi = \partial \eta + \zeta$$

and $\zeta|_{\wedge^2 L} \in H^2(L, U)_k$ and $\zeta(Y^{-1}, \cdot)|_L \in H^1(L, U)_{k-1} + H^1(L, L)_{k-1}$ for any $Y^{-1} \in U_{-1}$ and $\zeta|_{\wedge^2 U_{-1}} \in H^2(U_{-1}, U)_1$. Furthermore, the choice of $\zeta$ in the expression $\phi = \partial \eta + \zeta$ is unique.

We postpone the proof of Proposition 6.1 until Section 6.3 and proceed with the computation of the cohomology $H^2(m, g)$.

**Lemma 6.2** (Computation of $H^1(L, g)$). Let $k \geq 1$.

(i) $H^1(L, \mathfrak{z})_{k-1}$ vanishes except for $k = 1$ of type $(B_3, \alpha_2, \alpha_3)$ and $H^1(L, \mathfrak{z})_0 \subset \text{Hom}(L_{-1}, L_{-1})$.

(ii) $H^1(L, \mathfrak{z})_{k-1}$ vanishes except for $k = 2$ and $H^1(L, \mathfrak{z})_1 \subset \text{Hom}(L_{-1}, \mathfrak{z})$.

(iii) $H^1(L, U)_{k-1}$ vanishes except for $k = 1$ and $H^1(L, U)_0 \subset \text{Hom}(L_{-1}, U_{-1})$.

**Lemma 6.3** (Computation of $H^2(L, g)$). Let $k \geq 1$.

(i) $H^2(L, \mathfrak{z})_k$ vanishes.

(ii) $H^2(L, \mathfrak{z})_k$ vanishes except for $k = 2$, and $H^2(L, \mathfrak{z})_2 \subset \wedge^2 l_{-1} \otimes \mathfrak{z} \subset (\wedge^2 l_{-1} \otimes U_{-1})^* \otimes U_{-1}$

(iii) (1) If $(m, g_0)$ is of type $(B_m, \alpha_{m-1}, \alpha_{m})$, then $H^2(L, U)_k$ vanishes except for $k = 1, 2$, and we have

$$H^2(L, U)_1 \subset \wedge^2 l_{-1} \otimes U_{-1}$$

$$H^2(L, U)_2 \subset \wedge^2 l_{-1} \otimes U_0 \subset (\wedge^2 l_{-1} \otimes l_{-1})^* \otimes U_{-1}$$

(2) If $(m, g_0)$ is of type $(F_4, \alpha_2, \alpha_3)$, then $H^2(L, U)_k$ vanishes except for $k = 1$, and we have

$$H^2(L, U)_1 \subset \wedge^2 l_{-1} \otimes U_{-1}$$

**Proof.** Use Kostant’s theory ([17]).

We remark that, in the proof of Proposition 6.1, we use the property that both $H^1(L, \mathfrak{z})_{k-1}$ and $H^2(L, \mathfrak{z})_k$ vanish for any $k \geq 1$ (Lemma 6.2 (i) and Lemma 6.3 (i)).

**Proposition 6.4** (Computation of $H^2(m, g)$). Let $k \geq 1$. 

(1) If \((m, g_0)\) is of type \((B_3, \alpha_2, \alpha_3)\), then \(H^2(m, g)_k\) vanishes except for \(k = 1, 2\), and

\[
H^2(m, g)_1 \subset \wedge^2 g^*_{-1} \otimes g_{-1}
\]

\[
H^2(m, g)_2 \subset \wedge^2 g^*_{-1} \otimes U_0 \subset (\wedge^2 g_{-1} \otimes g_{-1})^* \otimes U_{-1}.
\]

(2) If \((m, g_0)\) is of type \((B_m, \alpha_{m-1}, \alpha_m)\), where \(m > 3\), then \(H^2(m, g)_k\) vanishes except for \(k = 1, 2\), and

\[
H^2(m, g)_1 \subset \wedge^2 g^*_{-1} \otimes U_{-1}
\]

\[
H^2(m, g)_2 \subset \wedge^2 g^*_{-1} \otimes U_0 \subset (\wedge^2 g_{-1} \otimes g_{-1})^* \otimes U_{-1}.
\]

(3) If \((m, g_0)\) is of type \((F_4, \alpha_2, \alpha_3)\), then \(H^2(m, g)_k\) vanishes except for \(k = 1\), and

\[
H^2(m, g)_1 \subset \wedge^2 g^*_{-1} \otimes U_{-1}
\]

Proof. By Proposition \([6.1]\) and Lemma \([6.2]\) and Lemma \([6.3]\) \(H^2(m, g)_k\) vanishes for any \(k \geq 3\). By the uniqueness of \(\zeta\) in the expression \(\phi = \partial \eta + \zeta\) in Proposition \([6.1]\) the nonvanishing \(H^2(m, g)_k\) can be thought of as a subspace of the spaces in the right hand side. In the first case, we can regards \(\zeta(Y^{-1}, \cdot) \in \text{Hom}(L_{-1}, L_{-1})\) because \(\text{dim} L_{-2} = 1\) implies that \(\zeta(Y^{-1}, \cdot) \in \text{Hom}(L_{-2}, L_{-2})\) is just a constant multiple which comes from a boundary.

6.2. Technical Lemmata. We denote the restriction of \(\partial\) to \(L_{-}\) by \(\partial_0\), so that we have a subcomplex

\[
0 \to \partial_0 g \to \text{Hom}(L_{-}, g) \to \text{Hom}(\wedge^2 L_{-}, g) \to \ldots
\]

Recall that in the proof of the vanishing of \(H^1(m, g)_k\) for positive \(k\) (Proposition 48 of \([16]\)) a main difficulty is to show that the nonvanishing cohomology \(H^1(L_{-}, g)_1\) does not contribute to the cohomology \(H^1(m, g)_1\), and a crucial Lemma is the following.

Lemma 6.5 (Lemma 27 (3) of \([16]\)). For \(A \in U_0\), if the image of \(\partial_0 A : L_{-1} \to U_{-1}\) has dimension \(\leq 1\), then we have \(\partial_0 A = 0\).

Under the assumption that \(X\) is either of type \((B_m, \alpha_{m-1}, \alpha_m)\) or of type \((F_4, \alpha_2, \alpha_3)\), Lemma \([6.5]\) can be improved.

Lemma 6.6. For \(A \in U_0\), if the image of \(\partial_0 A : L_{-1} \to U_{-1}\) has dimension \(\leq 2\), then we have \(\partial_0 A = 0\).

Similarly, we have the following Lemma.

Lemma 6.7. For \(A \in \text{Hom}(L_{-}, U)_1\), if the image of \(\partial_0 A : \wedge^2 L_{-1} \to U_{-1}\) has dimension \(\leq 1\), then we have \(\partial_0 A = 0\).

Proof. Let \(\{x_{-\alpha}\}\) be a basis of \(L_{-1}\) consisting of root vectors. We may assume that \([x_{-\alpha}, x_{-\beta}] = c_{\alpha\beta} x_{-\alpha-\beta}, \quad c_{\alpha\beta} \in \mathbb{C}\), form a basis of \(L_{-2}\). Let \(\{u_{\mu}\}\) (\(\{u_{\lambda}\}\), respectively) be a basis of \(U_{-1}\) (\(U_0\), respectively) consisting of weight vectors. We may assume that \([x_{-\alpha}, u_{\lambda}] = u_{-\alpha + \lambda}\) if \(-\alpha + \lambda\) is a weight.
For $A \in \text{Hom}(\mathcal{L}, U)_1$, we have

$$A(x_{-\alpha}) = \sum_{\lambda} A_{\lambda,\alpha} u_\lambda$$
$$A(x_{-\alpha-\beta}) = \sum_{\mu} A_{\mu,\alpha+\beta} u_\mu,$$

for some matrices $A_{\lambda,\alpha}$ and $A_{\mu,\alpha+\beta}$. Then

$$(\partial_0 A)(x_{-\alpha}, x_{-\beta}) = [x_{-\alpha}, A(x_{-\beta})] - [x_{-\beta}, A(x_{-\alpha})] - A([x_{-\alpha}, x_{-\beta}])$$

$$= [x_{-\alpha}, \sum_{\lambda} A_{\lambda,\alpha} u_\lambda] - [x_{-\beta}, \sum_{\lambda} A_{\lambda,\alpha} u_\lambda] - A(x_{-\alpha+\beta}c_{\alpha\beta})$$

$$= \sum_{\lambda} A_{\lambda,\beta} u_{-\alpha-\lambda} - \sum_{\lambda} A_{\lambda,\alpha} u_{-\beta-\lambda} - \sum_{\mu} A_{\mu,\alpha+\beta} c_{\alpha\beta} u_\mu$$

$$= \sum_{\mu} \left( A_{\mu+\alpha,\beta} - A_{\mu+\beta,\alpha} + A_{\mu,\alpha+\beta} c_{\alpha\beta} \right) u_\mu.$$ 

The condition that the image of $\partial_0 A$ has dimension $\leq 1$ implies that for any choice of a pair $(x_{-\alpha}, x_{-\beta})$, $(\partial_0 A)(x_{-\alpha}, x_{-\beta})$ is parallel to each other.

1. If $(\mathfrak{m}, \mathfrak{g}_0)$ is of type $(B_m, \alpha_{m-1}, \alpha_m)$, the action $I_1 \times U_{-1} \to U_0$ is given by

$$(\text{Sym}^2 V^* \otimes W^*) \times V \to V^* \otimes W^*$$

$$(v_i^2 \otimes w^*, v_j) \mapsto \delta_{ij} v_i \otimes w^*$$

where $\{v_i : i = 1, 2\}$ is a basis of $V$ and $w$ is an element of $W$. Furthermore, for $i = 1, 2$, the Lie bracket of any two elements of $\{v_i^2 \otimes w : w \in W\}$ is zero. Write $v_1, v_2$ as $u_\mu, u_\nu$. Then $(\partial_0 A)(x_{-\alpha}, x_{-\beta})$ is given by

$$\left( A_{\mu+\alpha,\beta} - A_{\mu+\beta,\alpha} + A_{\mu,\alpha+\beta} c_{\alpha\beta} \right) u_\mu + \left( A_{\nu+\alpha,\beta} - A_{\nu+\beta,\alpha} + A_{\nu,\alpha+\beta} c_{\alpha\beta} \right) u_\nu.$$ 

If we take $x_{-\alpha}, x_{-\beta}$ in $\{v_1^2 \otimes w : w \in W\}$, then the coefficient of $u_\mu$ is zero. If we take $x_{-\alpha}, x_{-\beta}$ in $\{v_2^2 \otimes w : w \in W\}$, then the coefficient of $u_\nu$ is also zero. Since $(\partial_0 A)(x_{-\alpha}, x_{-\beta})$ is parallel to each other for any choice of a pair $(x_{-\alpha}, x_{-\beta})$, both coefficients are zero for any $\alpha, \beta$.

2. If $(\mathfrak{m}, \mathfrak{g}_0)$ is of type $(F_4, \alpha_2, \alpha_3)$, the action $I_1 \times U_{-1} \to U_0$ is given by

$$(\text{Sym}^2 V^* \otimes W^*) \times V \to V^* \otimes W^*$$

$$(v_i^2 \otimes w^*, v_k) \mapsto (\delta_{ik} v_i^* + \delta_{jk} v_i^*) \otimes w^*.$$ 

In particular, $(v_i^2 \otimes w^*, v_j)$ maps to $\delta_{ij} v_i^* \otimes w^*$, where $\{v_i : i = 1, 2, 3\}$ is a basis of $V$ and $w$ is an element of $W$. Furthermore, for $i = 1, 2, 3$, the Lie bracket of any two elements in $\{v_i^2 \otimes w : w \in W\}$ is zero. Write $v_1, v_2, v_3$ as $u_\mu, u_\nu, u_\xi$. Then $(\partial_0 A)(x_{-\alpha}, x_{-\beta})$ is given by

$$\left( A_{\mu+\alpha,\beta} - A_{\mu+\beta,\alpha} + A_{\mu,\alpha+\beta} c_{\alpha\beta} \right) u_\mu + \left( A_{\nu+\alpha,\beta} - A_{\nu+\beta,\alpha} + A_{\nu,\alpha+\beta} c_{\alpha\beta} \right) u_\nu$$

$$+ \left( A_{\xi+\alpha,\beta} - A_{\xi+\beta,\alpha} + A_{\xi,\alpha+\beta} c_{\alpha\beta} \right) u_\xi$$

By the same arguments as in (1), we get that all coefficients are zero.
6.3. A proof of Proposition 6.1. We list up properties of \( g = U + \mathfrak{j} + \mathfrak{l} \).

(P\(_0\)) \([\mathfrak{j}, \mathfrak{j}] = 0\) and \([\mathfrak{l}, U_-] = 0\).

(P\(_1\)) For \( X \in \mathfrak{l}_{-\mu+1} + U_{\geq 0} \), if \([\mathfrak{l}, X] = 0\), then \( X = 0 \).

Proof of Proposition 6.1. Let \( \phi \in \text{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g})_k \) be such that \( \partial \phi = 0 \), where \( k \geq 1 \). We will show that there exist \( \eta \in \text{Hom}(\mathfrak{m}, \mathfrak{g})_k \), \( \zeta = \zeta_U + \zeta_U \) with \( \zeta_U \in H^2(U_-, U)_k \) and \( \zeta_U \in H^2(U_-, U)_1 \), and \( \xi = \xi_U + \xi_U \) with \( \xi_U(X^{-1}, \cdot) \in H^1(I, \mathfrak{l})_{k-1} \) and \( \xi(X^{-1}, \cdot) \in H^1(I, \mathfrak{l})_{k-1} \) for \( X^{-1} \in U_- \) satisfying that

\[
\phi = \partial \eta + \zeta + \xi.
\]

In other words, for any \( X^{-1}, Y^{-1} \in U_- \) and \( X^0, Y^0 \in \mathfrak{l} \), we have

\[
\phi(X^{-1} + X^0, Y^{-1} + Y^0) = [X^{-1} + X^0, \eta(Y^{-1} + Y^0)] - [Y^{-1} + Y^0, \eta(X^{-1} + X^0)] - \eta([X^0, Y^0]) + \zeta(X^0, Y^0) + \xi(X^0, Y^{-1}) + \xi(X^{-1}, Y^0) + \xi(X^{-1}, Y^{-1}),
\]

or equivalently,

\[
\phi_{\mathfrak{l}+3}(X^{-1} + X^0, Y^{-1} + Y^0) = [X^0, \eta_{\mathfrak{l}+3}(Y^{-1} + Y^0)] - [Y^0, \eta_{\mathfrak{l}+3}(X^{-1} + X^0)] - \eta_{\mathfrak{l}+3}([X^0, Y^0]) + \zeta_{\mathfrak{l}}(X^0, Y^{-1}) + \zeta_{\mathfrak{l}}(X^{-1}, Y^0)
\]

\[
\phi_U(X^{-1} + X^0, Y^{-1} + Y^0) = [X^{-1}, \eta_U(Y^{-1} + Y^0)] - [Y^{-1}, \eta_U(X^{-1} + X^0)] + [X^0, \eta_U(Y^{-1} + Y^0)] - [Y^0, \eta_U(X^{-1} + X^0)] - \eta_U([X^0, Y^0]) + \zeta_U(X^0, Y^0) + \zeta_U(X^0, Y^{-1}) + \zeta_U(X^{-1}, Y^0) + \zeta_U(X^{-1}, Y^{-1}).
\]

To do this, we decompose the identity

\[
\partial \phi(X^{-1} + X^0, Y^{-1} + Y^0, Z^{-1} + Z^0) = 0
\]

into the sum of two identities:

(I) \([X^0, \phi_{\mathfrak{l}+3}(Y^{-1} + Y^0, Z^{-1} + Z^0)] - [Y^0, \phi_{\mathfrak{l}+3}(X^{-1} + X^0, Z^{-1} + Z^0)] + [Z^0, \phi_{\mathfrak{l}+3}(X^{-1} + X^0, Y^{-1} + Y^0)] - \phi_{\mathfrak{l}+3}([X^0, Y^0], Z^{-1} + Z^0) + \phi_{\mathfrak{l}+3}([X^0, Z^0], Y^{-1} + Y^0] - \phi_{\mathfrak{l}+3}([Y^0, Z^0], X^{-1} + X^0) = 0\).

(II) \([X^0, \phi_U(Y^{-1} + Y^0, Z^{-1} + Z^0)] - [Y^0, \phi_U(X^{-1} + X^0, Z^{-1} + Z^0)] + [Z^0, \phi_U(X^{-1} + X^0, Y^{-1} + Y^0)] + [X^{-1}, \phi_{\mathfrak{l}+3}(Y^{-1} + Y^0, Z^{-1} + Z^0)] - [Y^{-1}, \phi_{\mathfrak{l}+3}(X^{-1} + X^0, Z^{-1} + Z^0)] + [Z^{-1}, \phi_{\mathfrak{l}+3}(X^{-1} + X^0, Y^{-1} + Y^0)] - \phi_U([X^0, Y^0], Z^{-1} + Z^0] + \phi_U([X^0, Z^0], Y^{-1} + Y^0) - \phi_U([Y^0, Z^0], X^{-1} + X^0) = 0\).

R1. There exist \( \eta_{\mathfrak{l}+3} + \eta_U \in \text{Hom}(\mathfrak{l}, \mathfrak{g})_k \) and \( \zeta_{\mathfrak{l}+3} + \zeta_U \in H^2(\mathfrak{l}, \mathfrak{g})_k \) such that

\[
\phi_{\mathfrak{l}+3}(X^0, Y^0) = [X^0, \eta_{\mathfrak{l}+3}(Y^0)] - [Y^0, \eta_{\mathfrak{l}+3}(X^0)] - \eta_{\mathfrak{l}+3}([X^0, Y^0]) + \zeta_{\mathfrak{l}+3}(X^0, Y^0)
\]

\[
\phi_U(X^0, Y^0) = [X^0, \eta_U(Y^0)] - [Y^0, \eta_U(X^0)] - \eta_U([X^0, Y^0]) + \zeta_U(X^0, Y^0)
\]

Indeed, putting \( X^{-1} = Y^{-1} = Z^{-1} = 0 \) into (I) and (II), we get \( \partial_0(\phi_{\mathfrak{l}}) = 0 \).

Note that \( \zeta_{\mathfrak{l}+3} = \zeta_{\mathfrak{j}} \) if \( k = 2 \), and \( \zeta_{\mathfrak{l}+3} = 0 \), otherwise, by Lemma 6.3 (i) and (ii).
R2. We have \( \phi_r(Y^{-1}, Z^{-1}) = 0 \) for \( r \geq -\mu + 1 \) and \( r \neq 0 \). Indeed, putting \( Y^0 = Z^0 = 0 \) to (I) for \( k \neq 2 \), we get \([X^0, \phi_r(Y^{-1}, Z^{-1})] = 0\) for \( r \neq 0 \). The desired vanishing follows from \((P_1)\).

R3. We may extend \( \eta_{k+3} \) from \( \mathbb{L} \) to \( \mathfrak{m} = \mathbb{L} + U_+ \) so that

\[
\phi_1(Y^{-1}, \cdot) = [\eta_{k+3}(Y^{-1}), \cdot] - \zeta_{k+3}^{-1}(\cdot) \text{ in } \text{Hom}(\mathbb{L}, I + \mathfrak{z})_{k-1}
\]

for some \( \zeta_{k+3}^{-1} \in H^1(\mathbb{L}, I + \mathfrak{z})_{k-1} \). Indeed, putting \( X^{-1} = Y^0 = Z^{-1} = 0 \) to (I) we get

\[
[X^0, \phi_{k+3}(Y^{-1}, Z^0)] - [Z^0, \phi_{k+3}(Y^{-1}, X^0)] - \phi_{k+3}(Y^{-1}, [X^0, Z^0]) = 0,
\]

or equivalently,

\[
\partial_0(\phi_{k+3}(Y^{-1}, \cdot)) = 0.
\]

Thus there exist \( \eta_{k+3}^{-1} \in (I + \mathfrak{z})_{k-1} \) and \( \zeta_{k+3}^{-1}(\cdot) \in H^1(\mathbb{L}, I + \mathfrak{z})_{k-1} \) such that

\[
\phi_{k+3}(Y^{-1}, \cdot) = [\eta_{k+3}^{-1}, \cdot] - \zeta_{k+3}^{-1}(\cdot).
\]

Note that \( \zeta_{k+3}^{-1} = \zeta_{l+1}^{-1} \) if \( k = 1 \) (for \( B_3 \) type), \( \zeta_{k+3}^{-1} = \zeta_{l+1}^{-1} \) if \( k = 2 \) and \( \zeta_{k+3}^{-1} = 0 \) otherwise, by Lemma \([6,2]\) (i) and (ii). Define \( \eta_{k+3}(Y^{-1}) \) by \( \eta_{k+3}(Y^{-1}) := \eta_{k+3}^{-1} \).

R4. We have \( \phi_{l+3}(Y^{-1}, Z^{-1}) = 0 \). Indeed, putting \( Y^0 = Z^0 = 0 \) to (I) for \( k = 2 \), we get \( [X^0, \phi_{l+3}(Y^{-1}, Z^{-1})] = 0 \) and putting \( X^0 = Y^0 = Z^0 = 0 \) to (II) for \( k = 2 \) we get

\[
[X^{-1}, \phi_{l+3}(Y^{-1}, Z^{-1})] - [Y^{-1}, \phi_{l+3}(X^{-1}, Z^{-1})] + [Z^{-1}, \phi_{l+3}(X^{-1}, Y^{-1})] = 0.
\]

Take independent vectors \( X^{-1}, Y^{-1}, Z^{-1} \) in \( U_+ \) and arbitrary \( X^0 \in \mathbb{L} \) to get the desired result.

R5. We may extend \( \zeta_{U} \) from \( \mathbb{L} \) to \( \mathfrak{m} = \mathbb{L} + U_+ \) such that \( \zeta_{U} |_{\mathbb{L}^{\perp U_+}} \in H^2(U_+, U_+) \), satisfying

\[
\phi_{U}(Y^{-1}, Z^{-1}) = [Y^{-1}, \eta_{l+3}(Z^{-1})] - [Z^{-1}, \eta_{l+3}(Y^{-1})] + \zeta_{U}(Y^{-1}, Z^{-1})
\]

and

\[
\phi_{U}(X^0, Z^{-1}) = 0.
\]

Indeed, putting \( X^{-1} = Y^0 = Z^0 = 0 \) to (II) we get

\[
[X^0, \phi_{U}(Y^{-1}, Z^{-1})] - [Y^{-1}, \phi_{l+3}(X^0, Z^{-1})] + [Z^{-1}, \phi_{l+3}(X^0, Y^{-1})] = 0.
\]

Thus

\[
[X^0, \phi_{U}(Y^{-1}, Z^{-1})]
\]

\((R3)\)

\[
[Y^{-1}, [X^0, \eta_{l+3}(Z^{-1})]] + \zeta_{l+3}^{-1}(X^0) - [Z^{-1}, [X^0, \eta_{l+3}(Y^{-1})]] + \zeta_{l+3}^{-1}(X^0)
\]

\((P_0)\)

\[
[X^0, [Y^{-1}, \eta_{l+3}(Z^{-1})]] - [X^0, [Z^{-1}, \eta_{l+3}(Y^{-1})]] + [Y^{-1}, \zeta_{l+3}^{-1}(X^0)] - [Z^{-1}, \zeta_{l+3}^{-1}(X^0)].
\]

Hence

\[
[X^0, \phi_{U}(Y^{-1}, Z^{-1})] - [X^0, [Y^{-1}, \eta_{l+3}(Z^{-1})]] + [X^0, [Z^{-1}, \eta_{l+3}(Y^{-1})]]
\]

\[
= [Y^{-1}, \zeta_{l+3}^{-1}(X^0)] - [Z^{-1}, \zeta_{l+3}^{-1}(X^0)].
\]

Therefore, we have

\[
\partial_0(\phi_{U}(Y^{-1}, Z^{-1}) - [Y^{-1}, \eta_{l+3}(Z^{-1})] + [Z^{-1}, \eta_{l+3}(Y^{-1})]) = [Y^{-1}, \zeta_{l+3}^{-1}(\cdot)] - [Z^{-1}, \zeta_{l+3}^{-1}(\cdot)].
\]
We claim that both sides vanish.

By Lemma 6.2 (i) and (ii), \( \xi_{t+3}^{Y^{-1}} \) is \( \xi_{t}^{Y^{-1}} \) or zero if \( k = 1 \), \( \xi_{t+3}^{Y^{-1}} \) is \( \xi_{t}^{Y^{-1}} \) if \( k = 2 \), and is zero otherwise. In the last case,

\[
\partial_{0} (\phi_{U}(Y^{-1}, Z^{-1}) - [Y^{-1}, \eta_{t+3}(Z^{-1})] + [Z^{-1}, \eta_{t+3}(Y^{-1})]) = 0.
\]

In the second case,

\[
A^{Y^{-1}, Z^{-1}} := \phi_{U}(Y^{-1}, Z^{-1}) - [Y^{-1}, \eta_{t+3}(Z^{-1})] + [Z^{-1}, \eta_{t+3}(Y^{-1})]
\]

is an element of \( U_{0} \) and we have

\[
\partial_{0} A^{Y^{-1}, Z^{-1}} = [Y^{-1}, \xi_{t}^{Z^{-1}}(\cdot)] - [Z^{-1}, \xi_{t}^{Y^{-1}}(\cdot)].
\]

Both sides vanish by Lemma 6.6 because \([Y^{-1}, \xi - Z^{-1}, \xi] \) has dimension two. In the first case, it is clear that both sides vanish because \([U, g_{-}] = 0 \).

Therefore, we have

\[
\phi_{U}(Y^{-1}, Z^{-1}) = [Y^{-1}, \eta_{t+3}(Z^{-1})] - [Z^{-1}, \eta_{t+3}(Y^{-1})] + \zeta_{U}(Y^{-1}, Z^{-1}) \text{ by (P)}
\]

and

\[
\phi_{3}(X^{0}, Z^{-1}) = \xi_{t}^{Z^{-1}}(X^{0}) = 0.
\]

Note that \( \zeta_{U}(Y^{-1}, Z^{-1}) \in U_{-} \).

**R6.** We may extend \( \eta_{U} \) from \( I_{-} \) to \( m = I_{-} + U_{-} \) so that

\[
\phi_{U}(Y^{-1}, \cdot) - [Y^{-1}, \eta_{t+3}(\cdot)] = -[\cdot, \eta_{U}(Y^{-1})] \text{ in Hom}(I_{-}, U)_{k-1},
\]

and

\[
\zeta_{3}(X^{0}, Y^{0}) = 0.
\]

Indeed, putting \( X^{-1} = Y^{0} = Z^{-1} = 0 \) we get

\[
[X^{0}, \phi_{U}(Y^{-1}, Z^{0}) + [Z^{0}, \phi_{U}(X^{0}, Y^{-1})] - [Y^{-1}, \phi_{t+3}(X^{0}), Z^{0}]]
\]

\[
\phi_{U}([X^{0}, Z^{0}], Y^{-1}) = 0.
\]

Thus

\[
[X^{0}, \phi_{U}(Y^{-1}, Z^{0})] + [Z^{0}, \phi_{U}(X^{0}, Y^{-1})] + \phi_{U}([X^{0}, Z^{0}], Y^{-1})
\]

\[
= [Y^{-1}, \phi_{t+3}(X^{0}, Z^{0})]
\]

\[
\text{R1.} [Y^{-1}, [X^{0}, \eta_{t+3}(Z^{0})] - [Z^{0}, \eta_{t+3}(X^{0})] - \eta_{t+3}([X^{0}, Z^{0}]) + \zeta_{t+3}(X^{0}, Z^{0})]
\]

\[
= [X^{0}, [Y^{-1}, \eta_{t+3}(Y^{0})]] - [Y^{-1}, [Z^{0}, \eta_{t+3}(Y^{0})]] - [Y^{-1}, \eta_{t+3}([X^{0}, Z^{0}])] + [Y^{-1}, \zeta_{t+3}(X^{0}, Z^{0})]
\]

Therefore, we have

\[
\partial_{0} (\phi_{U}(Y^{-1}, \cdot) - [Y^{-1}, \eta_{t+3}(\cdot)])(\cdot, \cdot) = [Y^{-1}, \zeta_{t+3}(\cdot, \cdot)].
\]

We claim that both sides vanish.

By Lemma 6.3 (i) and (ii), \( \zeta_{t+3} \) is \( \zeta_{3} \) if \( k = 2 \), and is zero, otherwise. In the second case,

\[
\partial_{0} (\phi_{U}(Y^{-1}, \cdot) - [Y^{-1}, \eta_{t+3}(\cdot)])(\cdot, \cdot) = 0.
\]

In the first case,

\[
A^{Y^{-1}} := \phi_{U}(Y^{-1}, \cdot) - [Y^{-1}, \eta_{t+3}(\cdot)]
\]
is an element of \( \text{Hom}(L_-, U)_1 \), and we have
\[
\partial_0 A_{Y^{-1}}^{Y^{-1}} = [Y^{-1}, \zeta(\cdot, \cdot)].
\]
Both sides vanish by Lemma 6.7 because \([Y^{-1}, \zeta] \) has dimension one.

By Lemma 6.2 there exist \( \eta_{U}^{Y^{-1}} \in U_{k-1} \) and \( \xi_{U}^{Y^{-1}} \in H^1(L_-, U)_{k-1} \) such that
\[
\phi_U(Y^{-1}, \cdot) - [Y^{-1}, \eta_{U}^{Y^{-1}}] = -[\cdot, \eta_{U}^{Y^{-1}}] + \xi_{U}^{Y^{-1}}(\cdot) \quad \text{in} \quad \text{Hom}(L_-, U)_{k-1},
\]
and
\[
\zeta(\cdot, \cdot) = 0.
\]
Define \( \eta_U(Y^{-1}) \) by \( \eta_U(Y^{-1}) := \eta_{U}^{Y^{-1}} \) and define \( \xi_U \in \text{Hom}(L_1 \cap U_{-1}, U_{-1}) \) by \( \xi_U(Y^{-1}, X^0) = -\xi_U(X^0, Y^{-1}) := \xi_{U}^{Y^{-1}}(X^0) \).

Consequently, there exist \( \eta = \eta_{h_1} + \eta_U \in \text{Hom}(m, g)_k, \zeta = \zeta_U \in H^2(L_-, U)_{k+H^2(U_-, U)}_1 \) and \( \zeta_U + \zeta \in (L_1 \cap U_{-1})^* \otimes (U_{-1} + L_1) \) satisfying
\[
\begin{align*}
\phi_{l+t}(X^0, Y^0) &\equiv R_{1, R_6} [X^0, \eta_{l+t}(Y^0)] - [Y^0, \eta_{l+t}(X^0)] - \eta_{l+t}([X^0, Y^0]) \\
\phi_{l+t}(X^0, Y^{-1}) &\equiv R_{3, R_5} [X^0, \eta(Y^{-1})] + \xi_l(X^0, Y^{-1}) \\
\phi_{l-\mu+t}(X^1, Y^{-1}) &\equiv R_{2, R_4} 0 \\
\phi_U(X^0, Y^0) &\equiv R_1 [X^0, \eta_U(Y^0)] - [Y^0, \eta_U(X^0)] - \eta_U([X^0, Y^0]) + \zeta_U(X^0, Y^0) \\
\phi_U(X^0, Y^{-1}) &\equiv R_6 -[Y^{-1}, \eta_{h}(X^0)] + [X^0, \eta_{U}(Y^{-1})] + \xi_U(X^0, Y^{-1}) \\
\phi_U(X^1, Y^{-1}) &\equiv R_5 [X^1, \eta_{h}(Y^{-1})] - [Y^{-1}, \eta_{h}(X^1)] + \zeta_U(Y^{-1}, Z^{-1}).
\end{align*}
\]
It remains to show that the choice of \( \zeta \) in the expression \( \phi = \partial \eta + \zeta \) is unique. Indeed, if \( \partial \eta + \zeta = \partial \eta' + \zeta' \), then \( \partial(\eta - \eta') = \zeta' - \zeta \). It suffices to show that if \( \partial \eta = \zeta \), then \( \zeta \) is zero. From
\[
(\partial \eta)(X^{-1} + X^0, Y^{-1} + Y^0) = \zeta(X^{-1} + X^0, Y^{-1} + Y^0)
\]
it follows that
\[
\begin{align*}
\zeta(X^0, Y^0) &= [X^0, \eta(Y^0)] - [Y^0, \eta(X^0)] - \eta([X^0, Y^0]) \\
\zeta(X^0, Y^{-1}) &= [X^0, \eta(Y^{-1})] - [Y^{-1}, \eta(X^0)].
\end{align*}
\]
The first identity implies that \( \zeta = \partial \eta |_{L_-} \). Since \( \zeta |_{L_2 \cap L_-} \in H^2(L_-, U)_k \), \( \zeta |_{L_2 \cap L_-} \) vanishes and so does \( \partial \eta |_{L_-} \). Since \( H^1(L_-, g)_k = 0 \) for any \( k \geq 1 \), there is \( \chi \) such that \( \eta |_{L_-} = \partial_0 \chi \). Then \( \eta(X^0) = [\chi, X^0] \) for \( X^0 \in L_- \).

The second identity becomes
\[
\begin{align*}
\zeta(X^0, Y^{-1}) &= [X^0, \eta(Y^{-1})] - [Y^{-1}, [\chi, X^0]] \\
&= [X^0, \eta(Y^{-1}) + [Y^{-1}, \chi]] \quad \text{because} \quad [L_-, U_-] = 0.
\end{align*}
\]
Since \( \zeta(\cdot, Y^{-1}) |_{L_-} \in H^1(L_-, U)_{k-1} \), we have \( \zeta(\cdot, Y^{-1}) |_{L_-} = 0 \).

This completes the proof of Proposition 6.1. \( \square \)
7. Local equivalence of geometric structures

We keep the same assumptions and notations as in Section 5.

**Proposition 7.1.** Let $M$ be a Fano manifold of Picard number one and $\mathcal{C}_x(M) \subset \mathbb{P}T_xM$ be the variety of minimal rational tangents at a general point $x \in X$ associated with a minimal dominant rational component $\mathcal{K}$. Suppose that $\mathcal{C}_x(M) \subset \mathbb{P}T_xM$ is projectively equivalent to $S \subset \mathbb{P}m$ for general $x \in M$. Denote by $D \subset TM$ the distribution on $M$ obtained by the linear span of the affine cone of $\mathcal{C}_x(M)$ in $T_xM$. Then there is a Zariski open subset $M^0 \subset M$ such that a general member of $\mathcal{K}$ lies on $M^0$, satisfying the following properties:

1. $D|_{M^0}$ is of type $m$;
2. $C|_{M^0} \subset \mathbb{P}D|_{M^0}$ defines an $S$-structure on $(M^0, D|_{M^0})$ and there corresponds to a $G_0$-structure on $(M^0, D|_{M^0})$, where $G_0 = G(S)$.

For the definition of $S$-structures and $G_0$-structures, see Section 3.1.

**Proof.** The distribution $D$ is holomorphic outside its singular set $\text{Sing}(D) \subset M$ of codimension $\geq 2$. Let $M' \subset M - \text{Sing}(D)$ be a Zariski open subset such that for all $x \in M'$ satisfying $\mathcal{C}_x \subset \mathbb{P}D_x$ is projectively equivalent to $S \subset \mathbb{P}g_{-1}$.

Let $C$ be a standard minimal rational curve represented by $f : \mathbb{P}^1 \to M$ with $C \cap M' \neq \emptyset$ and $C \not\subset \text{bad}(\mathcal{K})$ and $C \cap \text{Sing}(D) = \emptyset$. Then for a generic point $y \in C$, $\mathcal{C}_y \subset \mathbb{P}D_y$ is projectively equivalent to $S \subset \mathbb{P}g_{-1}$. By Proposition 2.7 and Proposition 2.9 and Lemma 5.1, the relative second fundamental form and the relative third fundamental form of $\mathcal{C}(M)$ along the lifting $C^\ast$ of $C$ is constant, and we have

$$f^*D = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^r \oplus \mathcal{O}(-1)^s$$

$$f^*(TM/D) = \mathcal{O}(1)^s \oplus \mathcal{O}^t.$$  

For the values $p, q, r, s, t$, see Remark 5.2. Furthermore, the pull-back $(f^\ast)\nabla^\omega$ of the relative affine tangent bundle $\nabla^\omega$ of $C \subset \mathbb{P}(TM)$ is the positive part $P := \mathcal{O}(2) \oplus \mathcal{O}(1)^p$ of $f^*D$, and the pull-back $(f^\ast)\nabla^{(2),\omega}$ of the relative second osculating affine bundle $\nabla^{(2),\omega}$ of $C \subset \mathbb{P}(TM)$ is the subbundle $P^{(2)} := \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^r$ of $f^*D$.

By Proposition 5.10 for any $y \in C$, $\mathcal{C}_y \subset \mathbb{P}D_y$ is projectively equivalent to $S \subset \mathbb{P}g_{-1}$. Let $D^{-1} := D$ and $D^{-2} := D + [D, D]$. For generic $x \in M$, the Frobenius bracket $[,]$ is given by

$$[,]_1 : \wedge^2 D_x^{-1} \to D_x^{-2}/D_x \subset T_xM/D_x$$

$$[,]_2 : D_x^{-1} \wedge D_x^{-2} \to (T_xM/D_x)/(D_x^{-2}/D_x) = T_xM/D_x^{-2}.$$  

In particular, if $X$ is $(B_m, \alpha_{m-1}, \alpha_m)$, then $D^{-2} = TM$ and simply the bracket is

$$[,] : \wedge^2 D_x \to T_xM/D_x.$$  

For generic $v \in \hat{\mathcal{C}}_x$ and $w \in T_v(\hat{\mathcal{C}}_x)$, we have $[v, w] = 0$ by Proposition 2.8. Hence, by Proposition 5.4 (1) and its proof, the symbol algebra $\text{Sym}_y(D)$ is isomorphic to $m$ at general $x$. It remains to show that $\text{Sym}_y(D)$ is isomorphic to $m$ for any $y \in C$. For this, we need to know the decompositions of vector bundles related to the Frobenius bracket. From now on, we use notations $D|_C, TM/D|_C$, etc., instead of $f^*D, f^*(TM/D)$, for simplicity.

Recall that
(1) when $X$ is $(B_m, \alpha_{m-1}, \alpha_m)$,
   (a) $\dim \mathfrak{g}_{-1} = 3m - 1$, $\dim \mathfrak{S} = m + 1$, $\dim \mathfrak{g}_{-2} = (m - 1)(m - 2)/2$,
   (b) $D|_C = \mathcal{O}(2) \oplus \mathcal{O}(1)^m \oplus \mathcal{O}^m \oplus \mathcal{O}(-1)^{m-2}, TM/D|_C = \mathcal{O}(1)^{m-2} \oplus \mathcal{O}^t$, where $t = (m - 2)(m - 3)/2$;
(2) when $X$ is $(F_4, \alpha_2, \alpha_3)$,
   (a) $\dim \mathfrak{g}_{-1} = 15$, $\dim \mathfrak{S} = 5$, $\dim \mathfrak{g}_{-2} = 6$, $\dim \mathfrak{g}_{-3} = 2$
   (b) $D|_C = \mathcal{O}(2) \oplus \mathcal{O}(1)^4 \oplus \mathcal{O}^7 \oplus \mathcal{O}(-1)^3$, $TM/D|_C = \mathcal{O}(1)^3 \oplus \mathcal{O}^5$.

We need the following Lemma to complete the proof of Proposition 7.1.

**Lemma 7.2.** Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on $M'$ associated with $V$ and $W$. Let $C$ be a general member of $\mathcal{K}$ passing through $x \in M'$.

1. When $X$ is $(B_m, \alpha_{m-1}, \alpha_m)$, we have
   $$\mathcal{V}|_C = \mathcal{O}(1) \oplus \mathcal{O} \quad \text{and} \quad \mathcal{W}|_C = \mathcal{O} \oplus \mathcal{O}(-1)^{m-2}.$$  
2. When $X$ is $(F_4, \alpha_2, \alpha_3)$, we have
   $$\mathcal{V}|_C = \mathcal{O}(1) \oplus \mathcal{O}^2 \quad \text{and} \quad \mathcal{W}|_C = \mathcal{O} \oplus \mathcal{O}(-1).$$

Set
$$\mathcal{E} := (\text{Sym}^4 \mathcal{V})^\perp \otimes \mathcal{W} = \text{Sym}^2(\mathcal{V} \otimes \mathcal{W}).$$

By Lemma 7.2 we have
$$\mathcal{E}|_C = \begin{cases} \mathcal{O}(2) \otimes (\mathcal{O}(-1)^{m-2} \oplus \mathcal{O}(-2)^t) \quad \text{when} \quad X = (B_m, \alpha_{m-1}, \alpha_m), \\
\text{Sym}^2(\mathcal{O}(1)^2 \oplus \mathcal{O}) \otimes \mathcal{O}(-1) \quad \text{when} \quad X = (F_4, \alpha_2, \alpha_3). \end{cases}$$

The Frobenius bracket $[,] : \wedge^2 D|_C \to TM/D|_C$ induces a map
$$\mathcal{E}|_C \to TM/D|_C,$$
which is an isomorphism onto its image for generic $y \in C$ by Proposition 5.4 (1).

When $X$ is $(B_m, \alpha_{m-1}, \alpha_m)$, since $\mathcal{E}|_C$ and $TM/D|_C$ have the same splitting type, $\mathcal{E}|_C \to TM/D|_C$ is an isomorphism at any point of $C$.

When $X$ is $(F_4, \alpha_2, \alpha_3)$, $\mathcal{E}|_C \to TM/D|_C$ is an isomorphism onto $D^{-2}/D|_C$ at any point of $C$. Now we have
$$0 \to D^{-2}/D \to TM/D \to TM/D^{-2} \to 0.$$  
From $D^{-2}/D|_C = \mathcal{O}(1)^3 \oplus \mathcal{O} \oplus \mathcal{O}(-1)$ and $TM/D|_C = \mathcal{O}(1)^3 \oplus \mathcal{O}^5$, it follows that $TM/D^{-2}|_C = \mathcal{O}(1) \oplus \mathcal{O}$.

Set
$$\mathcal{E}^3 := \text{Sym}^2(\mathcal{V} \otimes \mathcal{W})$$

Then $\mathcal{E}^3|_C = \mathcal{O}(1) \oplus \mathcal{O}$. The Frobenius bracket $[,] : D \wedge (D^{-2}/D) \to TM/D^{-2}$ induces a map
$$\mathcal{E}^3|_C \to TM/D^{-2}|_C$$
which is an isomorphism for generic $y \in C$ by Proposition 5.4 (1). Hence it is an isomorphism at any point of $C$ because $TM/D^{-2}|_C = \mathcal{O}(1) \oplus \mathcal{O}$ has same splitting type as $\mathcal{E}^3|_C$.

Consequently, $\text{Symb}_y(D)$ is isomorphic to $\mathfrak{m}$ for any $y \in C$. 

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In sum, there exists a Zariski open subset $M' \subset M^0 \subset M$ such that a general member of $K$ lies on $M^0$ and the varieties of minimal rational tangents $C_{|M^0} \subset \mathbb{P}D_{|M^0}$ defines an $S$-structure on $(M^0, D_{|M^0})$ of type $m$. Proposition 3.7 together with Proposition 5.4 implies that there exist $G_0$-structure on $(M^0, D_{|M^0})$ corresponding to the $S$-structure, where $G_0 = G(\hat{S})$. This completes the proof of Proposition 7.1. □

Proof of Lemma 7.2. We adapt an argument in the proof of Proposition 7.3 of [9].

Recall that we have the following exact sequence:

$$0 \to V|C \to D|C = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}(-1)^s \to \text{Sym}^2 V \otimes W|C \to 0.$$  

Let $v \in V$ and $w \in W$ be vectors with $[T_xC] = [v + v^2 \otimes w]$. Then they define vector subbundles $V_0 \subset V|C$ and $W_0 \subset W|C$. Set $V_0^{(1)} := V|C/V_0$ and $W_0^{(2)} := W|C/W_0$. By Lemma 5.2, we get the following exact sequences.

$$0 \to \mathcal{O}(2) \to Q^{(0)} \to 0$$

$$0 \to V_0 \to \mathcal{O}(1)^p \to Q^{(1)} \to 0$$

$$0 \to V_0^{(1)} \to \mathcal{O}^r \to Q^{(2)} \to 0$$

$$0 \to \mathcal{O}(-1)^s \to Q^{(3)} \to 0$$

with

$$Q^{(0)} = \text{Sym}^2 V_0 \otimes W_0$$

$$\deg Q^{(1)} = \deg \left(V_0 \circ V_0^{(1)} \otimes W_0\right) \oplus \left(\text{Sym}^2 V_0 \otimes W_0^{(1)}\right)$$

$$\deg Q^{(2)} = \deg \left(\text{Sym}^2 V_0^{(1)} \otimes W_0\right) \oplus \left(V_0 \circ V_0^{(1)} \otimes W_0^{(1)}\right)$$

$$Q^{(3)} = \text{Sym}^2 V_0^{(1)} \otimes W_0^{(1)}.$$  

Furthermore,

any direct summand of $V_0^{(1)}$ has degree $\leq 0$, and

any direct summand of $\text{Sym}^2 V_0^{(1)} \otimes W_0^{(1)}$ has degree $-1$.

The first statement follows from the third exact sequence, and the second statement follows from the last one.

When $X$ is $(B_m, \alpha_{m-1}, \alpha_m)$, write

$$V_0 = \mathcal{O}(a_1), V_0^{(1)} = \mathcal{O}(a_2), W_0 = \mathcal{O}(b_1), \text{ and } W_0^{(1)} = \mathcal{O}(b_2) \oplus \cdots \oplus \mathcal{O}(b_{m-1}).$$

Then we have

$$2 = 2a_1 + b_1$$

$$m = a_1 + (a_1 + a_2 + b_1) + \sum_{i=2}^{m-1} (2a_1 + b_i) = (2m - 2)a_1 + a_2 + \sum_{i=1}^{m-1} b_i$$

$$0 = a_2 + (2a_2 + b_1) + \sum_{i=2}^{m-1} (a_1 + a_2 + b_i) = (m - 2)a_1 + (m + 1)a_2 + \sum_{i=1}^{m-1} b_i$$

$$-m + 2 = \sum_{i=2}^{m-1} (2a_2 + b_i) = 2(m - 2)a_2 + \sum_{i=2}^{m-1} b_i.$$
Therefore, \(a_1 = 1\) and \(a_2 = 0\) and \(b_1 = 0\) and \(\sum_{i=2}^{m-1} b_i = -m + 2\). Since the degree of a direct summand of \(\text{Sym}^2 \mathcal{V}_0^{(1)} \otimes \mathcal{W}_0^{(1)}\) is \(-1\), we have \(b_i = -1\) for all \(2 \leq i \leq m - 1\).

Since the exact sequences

\[
0 \to \mathcal{V}_0 \to \mathcal{V}_C \to \mathcal{V}_0^{(1)} \to 0 \\
0 \to \mathcal{W}_0 \to \mathcal{W}_C \to \mathcal{W}_0^{(1)} \to 0
\]

split, we have the desired decompositions of \(\mathcal{V}|_C\) and \(\mathcal{W}|_C\).

When \(X = (F_4, \alpha_2, \alpha_3)\), write

\[
\mathcal{V}_0 = \mathcal{O}(a_1), \mathcal{V}_0^{(1)} = \mathcal{O}(a_2) \oplus \mathcal{O}(a_3), \mathcal{W}_0 = \mathcal{O}(b_1), \text{ and } \mathcal{W}_0^{(1)} = \mathcal{O}(b_2).
\]

Then we have

\[
\begin{align*}
2 &= 2a_1 + b_1 \\
4 &= a_1 + (a_1 + a_2 + b_1) + (a_1 + a_3 + b_1) + (2a_1 + b_2) \\
&= 5a_1 + (a_2 + a_3) + 2b_1 + b_2 \\
0 &= a_2 + a_3 + (3(a_2 + a_3) + 3b_1) + (a_1 + a_2 + b_2) + (a_1 + a_3 + b_2) \\
&= 2a_1 + 5(a_2 + a_3) + 3b_1 + 2b_2 \\
-3 &= 3(a_2 + a_3) + 3b_2.
\end{align*}
\]

Therefore, \(a_1 = 1\) and \(b_1 = 0\) and \(a_2 + a_3 = 0\) and \(b_2 = -1\). Since the degree of a direct summand of \(\mathcal{V}_0^{(1)}\) is \(\leq 0\), we get \(a_2 = a_3 = 0\). By the same reason as in the case of \((B_m, \alpha_{m-1}, \alpha_m)\), we have the desired decompositions of \(\mathcal{V}|_C\) and \(\mathcal{W}|_C\).

This completes the proof of Lemma 7.2. \(\square\)

By Proposition 7.1, there is a Zariski open subset \(M^0 \subset M\) such that a general member of \(\mathcal{K}\) lies on \(M^0\), satisfying the following properties:

1. \(D|_{M^0}\) is of type \(m\);
2. \(\mathcal{C}|_{M^0} \subset \mathbb{P}D|_{M^0}\) defines an \(S\)-structure on \((M^0, D|_{M^0})\) and there corresponds to a \(G_0\)-structure \(\mathcal{P}\) on \((M^0, D|_{M^0})\), where \(G_0 = G(\hat{S})\).

Define a vector bundle \(H^2_k\) on \(M^0\) by \(H^2_k := \mathcal{P} \times_{G_0} H^2(m, g)_k\).

**Lemma 7.3.** \(H^0(M^0, H^2_k)\) is zero for all \(k \geq 1\).

**Proof.** By Proposition 6.4, it suffices to show that the followings;

1. \(H^0(M^0, \wedge^2 D^* \otimes \mathcal{V}) = 0\) when \((m, g_0)\) is of type \((B_m, \alpha_{m-1}, \alpha_m)\) for \(m > 3\) or of type \((F_4, \alpha_2, \alpha_3)\);
2. \(H^0(M^0, \wedge^2 D^* \otimes D) = 0\) when \((m, g_0)\) is of type \((B_3, \alpha_2, \alpha_3)\);
3. \(H^0(M^0, \wedge^2 D^* \otimes D)^* \otimes \mathcal{V}) = 0\) when \((m, g_0)\) is of type \((B_m, \alpha_{m-1}, \alpha_m)\) for \(m \geq 3\).

We will adapt an argument similar to the proof of Proposition 6.2 of [5]. Assume that \((m, g_0)\) is either of type \((B_m, \alpha_{m-1}, \alpha_m)\) for \(m > 3\) or of type \((F_4, \alpha_2, \alpha_3)\). Let \(\varphi: \wedge^2 D \to \mathcal{V}\) be a nontrivial vector bundle map. For \(x \in M^0\) and \(\beta \in D_x\) with \([\beta] \in C_x\), take \(C\) to be a member of \(\mathcal{K}\) passing through \(x\) with \([T_x C] = [\beta]\). Then \(T_C \wedge D|_C\) is decomposed as a sum of \(\mathcal{O}(a)\)'s with \(a \geq 1\). Thus \(\varphi|_C\) maps \(T_C \wedge D|_C\) into the \(\mathcal{O}(1)\)-factor of \(\mathcal{V}|_C\), the intersection of \(\mathcal{V}_x\) with \(T_{\beta} \hat{\mathcal{C}}_x\). Applying this argument to a general \([\beta] \in C_x\), we see that the image of \(\varphi_x\) is contained in the intersection

\[
\bigcap_{[\beta] \in C_x} \left( T_{\beta} \hat{\mathcal{C}}_x \cap \mathcal{V}_x \right),
\]
which is $G_0$-invariant and degenerate in $V_x$, contradicting to the fact that $V$ is an irreducible $G_0$-bundle. Therefore, $\varphi_x$ is zero.

When $(m, g_0)$ is of type $(B_3, \alpha_2, \alpha_3)$, Let $\varphi : \wedge^2 D \rightarrow V$ and $\psi : \wedge^2 D \rightarrow \text{Sym}^2 V \otimes W$ be nontrivial vector bundle maps. For $x \in M^0$ and $\beta \in D_x$ with $[\beta] \in C_x$, take $C$ to be a member of $K$ passing through $x$ with $[T_x C] = [\beta]$. Then $TC \wedge D|_C$ is decomposed as a sum of $O(a)$’s with $a \geq 1$. Thus $\varphi|_C$ maps $TC \wedge D|_C$ into the $O(1)$-factor of $V|_C$ and $\psi|_C$ maps $TC \wedge D|_C$ into the positive-factors of $\text{Sym}^2 V \otimes W|_C$, the intersection of $V_x$ with $T_\beta \hat{C}_x$ and the intersection of $\text{Sym}^2 V_x \otimes W_x$ with $T_\beta \hat{C}_x$. Applying this argument to $\text{span}(\beta)$ $\in C_x$, we see that the image of $\varphi_x$ and $\psi_x$ are contained in the intersections

$$\cap_{[\beta] \in C_x} (T_\beta \hat{C}_x \cap V_x) \text{ and } \cap_{[\beta] \in C_x} \{ T_\beta \hat{C}_x \cap (\text{Sym}^2 V_x \otimes W_x) \}$$

which are $G_0$-invariant and degenerate in $V_x$ and $\text{Sym}^2 V_x \otimes W_x$ respectively, contradicting to the fact that $V$ and $\text{Sym}^2 V \otimes W$ are irreducible $G_0$-bundles. Therefore, $\varphi_x$ and $\psi_x$ are zero. Hence, $H^0(M^0, \wedge^2 D^* \otimes D) = 0$.

Assume that $(m, g_0)$ is of type $(B_m, \alpha_{m-1}, \alpha_m)$ for $m \geq 3$. Let $\psi : \wedge^2 D \otimes D \rightarrow V$ be a vector bundle map. Then, by the same argument as above, for $x \in M^0$ and $[\beta] \in C_x$ with $[\beta] = [T_x C]$, $\psi_x$ maps $\beta \wedge T_\beta \hat{C}_x \otimes D_x$ into the intersection $V_x \cap T_\beta \hat{C}_x$. Thus the restriction of $\psi_x$ to $(\text{span}(\beta \wedge T_\beta \hat{C}_x : \beta \in C_x)) \otimes D_x$ is zero. By Proposition 2.8, $\text{span}(\beta \wedge T_\beta \hat{C}_x : \beta \in C_x)$ is the kernel of the Frobenius bracket $[\, ,] : \wedge^2 D \rightarrow TM/D$ at each point $x \in M^0$, and thus $\psi$ induces a bundle map

$$\mathcal{E} \otimes D \rightarrow V.$$

On the other hand, for $x \in M^0$ and $[\beta] \in C_x$ with $[\beta] = [T_x C]$, we have $\mathcal{E}|_C = O(1)^{m-2} \oplus O'$. Therefore, by the induced map $\mathcal{E} \otimes D \rightarrow V$, the image from $\mathcal{E}|_C \otimes TC$ to $V|_C = O(1) \oplus O$ is zero. Since $[\beta] \in C_x$ span $D_x$, the map $\psi_x$ is zero.

This completes the proof of Lemma 7.3. \qed

Proof of Theorem 1.3 in the case when $X$ is $(B_m, \alpha_{m-1}, \alpha_m)$ for $m \geq 3$ or $(F_4, \alpha_2, \alpha_3)$. By Proposition 3.10 and Lemma 7.3, $G_0$-structure $\mathcal{T}$ on $(M^0, D|_{M^0})$ is locally equivalent to the standard one. By Proposition 3.7 together with Proposition 5.4, the $S$-structure on $M^0$ defined by $C(M)|_{M^0}$ is locally equivalent to the standard one. By Theorem 2.4, a local map preserving the varieties of minimal rational tangents can be extended to a global biholomorphism. Hence, $M$ is biholomorphic to $X$. This completes the proof of Theorem 1.3 in the case when $X$ is $(B_m, \alpha_{m-1}, \alpha_m)$ for $m \geq 3$ or $(F_4, \alpha_2, \alpha_3)$. \qed

8. $(B_3, \alpha_1, \alpha_3)$ CASE

In this section, we assume that $X$ is the horospherical variety $(B_3, \alpha_1, \alpha_3)$. We prove Theorem 1.3 in this case by the method we use for the horospherical varieties $(B_m, \alpha_{m-1}, \alpha_m)$ for $m \geq 3$ or $(F_4, \alpha_2, \alpha_3)$ in Section 5, Section 6 and Section 7.

Since the structure of the Lie algebra of $\text{Aut}(X)$ and the projective geometry of the variety of minimal rational tangents are less complicated, the computations are relatively shorter.

Let $V$ be the spin representation of $L_0 = B_2$ and let $W$ be the standard representation of $B_2$. By the isomorphism $B_2 \cong C_2$, we may consider $V$ as the standard representation.
of $C_2$ with a nondegenerate skew symmetric bilinear form $\omega$ and $W$ as the subspace $\Lambda^2 V$ of $\Lambda^2 V$ generated by $v \wedge u$ with $\omega(v, u) = 0$.

Set

$$U := V \oplus W = V \oplus \Lambda^2 V$$

Let $S \subset \mathbb{P}U$ be the variety of minimal rational tangents of $X$ at the base point (Proposition 4.6):

$$S = \mathbb{P}\{v + v \wedge u : v, u \in V, \omega(v, u) = 0\} \subset \mathbb{P}U.$$ 

Let $\mathfrak{g} = (I + \mathbb{C}) \triangleright U$ be the Lie algebra of $\text{Aut}(X)$ graded as in Proposition 4.4 and $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be its negative part.

8.1. Projective geometry of varieties of minimal rational tangents. We list up properties of $S$ as a projective subvariety of $\mathbb{P}U$ as in Section 4.

**Lemma 8.1** (Lemma 1.17 of [22] or Theorem 1.1 of [15]). The variety $S$ is the odd symplectic Grassmannian $\text{Gr}_\omega(2, 5)$ and the automorphism group $\text{Aut}^0(\widehat{S})$ of the cone $\widehat{S}$ is $\text{PSp}(5) = (\text{Sp}(4) \times \mathbb{C}^*)/\{\pm 1\} \ltimes \mathbb{C}^4$.

The tangent space $T_{\beta}\widehat{S}$ at $\beta = v + v \wedge u \in \widehat{S}$ is given by

$$T_{\beta}\widehat{S} = \{v' + v' \wedge u + v \wedge u' \in U : v', u' \in V\}.$$

The second fundamental form $II_{\beta} : \text{Sym}^2 T_{\beta}\widehat{S} \rightarrow U/T_{\beta}\widehat{S}$ is

$$II_{\beta}(v' + v' \wedge u, v'' + v'' \wedge u) = 0$$

$$II_{\beta}(v' + v' \wedge u, v \wedge u') = v' \wedge u'$$

$$II_{\beta}(v \wedge u', v \wedge u'') = 0,$$

where $v', v'', u', u'' \in V$. Thus the second osculating space $T_{\beta}^{(2)}\widehat{S}$ is $W + T_{\beta}\widehat{S} = U$. Therefore, the third fundamental form $III_{\beta} : \text{Sym}^3 T_{\beta}\widehat{S} \rightarrow U/T_{\beta}^{(2)}\widehat{S}$ is zero.

Let $G_0$ be the subgroup of $G = \text{Aut}(X)$ with Lie algebra $\mathfrak{g}_0$. From the exact sequence of $G_0$-modules

$$0 \rightarrow V \rightarrow U \rightarrow W \rightarrow 0$$

we get the following exact sequences.

**Lemma 8.2.** Let $\beta = v_1 + v_1 \wedge v_2$ be an element of $\widehat{S}$, where $v_1, v_2 \in V$ such that $v_1 \wedge v_2 \neq 0$. For $i = 1, \ldots, 4$, denote by $V_i$ the subspace of $V$ generated by $v_i$ and $\omega(v_i, v_j) = \delta_{i+2,j}$ for $i = 1, 2$ and $1 \leq j \leq 4$. Then we have the following exact sequences.

$$0 \rightarrow \mathbb{C} \beta \rightarrow V_1 \wedge V_2 \rightarrow 0$$

$$0 \rightarrow V_1 \oplus V_2 \rightarrow T_{\beta}\widehat{S} / \mathbb{C} \beta \rightarrow \{(V/V_1) \wedge V_2 + V_1 \wedge (V/V_2)\}_\omega \rightarrow 0$$

$$0 \rightarrow V/(V_1 \oplus V_2) \rightarrow U/T_{\beta}\widehat{S} \rightarrow V_3 \wedge V_4 \rightarrow 0$$

**Proof.** We get the following exact sequences.

$$0 \rightarrow \mathbb{C} \beta \rightarrow V_1 \wedge V_2 \rightarrow 0$$

$$0 \rightarrow V_1 \oplus V_2 \rightarrow T_{\beta}\widehat{S} \rightarrow \{V \wedge V_2 + V_1 \wedge V\}_\omega \rightarrow 0$$

Denote that $(V \wedge V_2) \cap (V_1 \wedge V) = V_1 \wedge V_2$. By taking the quotients we get the desired exact sequences. \qed
Lemma 8.3.

(1) \( \dim \mathfrak{g}_{-1} = 9 \) and \( \dim \mathfrak{g}_k = 0 \) for \( k > 1 \)

(2) \( \mathfrak{g}_0 = \text{aut}^0(\hat{S}) \)

Proof. Since \( \mathfrak{m} = \mathfrak{g}_{-1} \) is isomorphic to \( \mathfrak{U} \) as vector space, (1) follows. Since \( \text{Aut}^0(\hat{S}) \) is equal to the linear automorphism group \( G(\hat{S}) \) and the induced map \( \mathfrak{g}(\hat{S}) \to \mathfrak{g}_0(\mathfrak{m}) \) is injective whose image in \( \mathfrak{g}_0(\mathfrak{m}) \) agrees with \( \mathfrak{g}_0 \subset \mathfrak{g}_0(\mathfrak{m}) \). \( \square \)

Proposition 8.4. Let \( S \subset \mathbb{P}U \) be the variety of minimal rational tangents of \( (B_3, \alpha_1, \alpha_3) \) at the base point. Let \( \pi : \mathbb{P}U \to \mathbb{P}^1 \) be the projectivization of a holomorphic vector bundle \( U \) over \( \mathbb{P}^1 \) and let \( C \subset \mathbb{P}U \) be an irreducible subvariety. Denote by \( \varpi \) the restriction of \( \pi \) to \( C \). Assume that

(1) \( C_t := \varpi^{-1}(t) \subset \mathbb{P}U_t := \pi^{-1}(t) \) is projectively equivalent to \( S \subset \mathbb{P}U \) for all \( t \in \mathbb{P}^1 - \{ t_1, \ldots, t_k \} \);

(2) for a general section \( \sigma \subset C \) of \( \varpi \), the relative second fundamental forms of \( C \) along \( \sigma \) are constants.

Then for any \( t \in \mathbb{P}^1 \), \( C_t \subset \mathbb{P}(U_t) \) is projectively equivalent to \( S \subset \mathbb{P}(U) \).

Proof. The Picard number of the odd Lagrangian Grassmannian \( \mathbb{G}_n(n, 2n+1) \) is one and \( S \subset \mathbb{P}(U) \) is the minimal embedding by the line bundle \( \mathcal{O}(1) \). The deformation rigidity of the odd Lagrangian Grassmannian \( \mathbb{G}_n(n, 2n+1) \) is also known in Theorem 1.2 of [21] and hence, for any \( t \in \mathbb{P}^1 \), \( C_t \) is biholomorphic to \( S \). The constancy of the second fundamental form implies that \( C_t \subset T^{(2)}_{\varpi}C_t = U_t \) is non-degenerate. Hence, \( C_t \subset \mathbb{P}(U_t) \) is projectively equivalent to the minimal embedding \( S \subset \mathbb{P}(U) \) for all \( t \). \( \square \)

8.2. \( H^2 \)-cohomology. Now we compute Lie algebra cohomologies as in Section 6.

Lemma 8.5. For \( X = (B_3, \alpha_1, \alpha_3) \), we have the followings:

(i) \( H^1(L, I)_{-1} \) vanishes except for \( k = 1 \) and

\[
H^1(L, I)_{-1} \subset \text{Hom}(L_{-1}, L_1).
\]

(ii) \( H^2(L, U)_{-1} \) vanishes except for \( k = 1 \), and we have

\[
H^2(L, U)_{-1} \subset \wedge^2 L_{-1} \otimes U_{-1}.
\]

(iii) except (i) and (ii), Lemma 6.2 and Lemma 6.3 are satisfies.

Proof. Apply the Kostant theory to get the desired result. \( \square \)

Lemma 8.6. For \( A \in \text{Hom}(L, U)_1 \), if the image of \( \partial_0 A : \wedge^2 L_{-1} \to U_{-1} \) has dimension \( \leq 1 \), then we have \( \partial_0 A = 0 \).

Proof. We recall the notions: let \( \{ x_\alpha \} \) be a basis of \( L_{-1} \) consisting of root vectors and let \( \{ u_\lambda \} \) \( \{ u_\lambda \} \) respectively be a basis of \( U_{-1} \) \( (U_0 \), respectively) consisting of weight vectors. We may assume that \( [x_\alpha, u_\lambda] = u_{\alpha + \lambda} \) if \( -\alpha + \lambda \) is a weight.

For \( A \in \text{Hom}(L, U)_1 \), we have

\[
A(x_\alpha) = \sum \lambda A_{\lambda, \alpha} u_\lambda.
\]

If \( (\mathfrak{m}, \mathfrak{g}_0) \) is of type \( (B_3, \alpha_1, \alpha_3) \), the action \( L_{-1} \times U_1 \to U_0 \) (equivalently, \( L_1 \times U_{-1} \to U_0 \)) is given by
From Lemma 8.2, we have the exact sequences

\[ \text{Proof.} \]

Lemma 8.8.

of the relative second osculating affine bundle \( \hat{\mathcal{V}} \) for \( \mathcal{P} \) and \( \mathcal{C} \) of \( u \) lifting \( \mathcal{C} \)

Proposition 8.7. \( H^2(\mathfrak{m}, \mathfrak{g})_k \) vanishes except for \( k = 1 \), and

\[ H^2(\mathfrak{m}, \mathfrak{g})_1 \subset \wedge^2 \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}. \]

Proof. By Lemma 8.5 and Lemma 8.6 the same argument as in the proof of Proposition 6.1 apply to the case when \( X \) is \( (B_3, \alpha_1, \alpha_3) \) to get the desired result.

8.3. Local equivalence of geometric structures. We complete the proof of Theorem 1.3 as in Section 7.

Let \( M' \subset M \) be a Zariski open subset such that for all \( x \in M' \) satisfying \( \mathcal{C}_x \subset \mathbb{P}TM_x \) is projectively equivalent to \( \mathcal{S} \subset \mathbb{P}\mathfrak{g}_{-1} \).

Let \( C \) be a standard minimal rational curve represented by \( f : \mathbb{P}^1 \to M \) with \( C \cap M' \neq \emptyset \) and \( C \not\subset \bad(\mathcal{K}) \). Then for a generic point \( y \in C \), \( \mathcal{C}_y \subset \mathbb{P}D_y \) is projectively equivalent to \( \mathcal{S} \subset \mathbb{P}\mathfrak{g}_{-1} \). By Proposition 2.7, the relative second fundamental form of \( \mathcal{C}(M) \) along the lifting \( C' \) of \( C \) is constant, and we have

\[ f^*TM = \mathcal{O}(2) \oplus \mathcal{O}(1)^5 \oplus \mathcal{O}^3. \]

Furthermore, the pull-back \( (f^*)^*\hat{T}_\omega \) of the relative affine tangent bundle \( \hat{T}_\omega \) of \( C \subset \mathbb{P}(TM) \) is the positive part \( P := \mathcal{O}(2) \oplus \mathcal{O}(1)^5 \) of \( f^*TM \), and the pull-back \( (f^*)^*\hat{T}^{(2)}_{\omega} \) of the relative second osculating affine bundle \( \hat{T}^{(2)}_{\omega} \) of \( C \subset \mathbb{P}(TM) \) is the subbundle \( P^{(2)} := \mathcal{O}(2) \oplus \mathcal{O}(1)^5 \oplus \mathcal{O}^3 \) of \( f^*TM \).

By Proposition 8.4 for any \( y \in C \), \( \mathcal{C}_y \subset \mathbb{P}TM_y \) is projectively equivalent to \( \mathcal{S} \subset \mathbb{P}\mathfrak{g}_{-1} \).

Lemma 8.8. Let \( \mathcal{V} \) and \( \mathcal{W} \) be vector bundles on \( M' \) associated with \( \mathcal{V} \) and \( \mathcal{W} \). Let \( C \) be a general member of \( \mathcal{K} \) passing through \( x \in M' \). When \( X \) is \( (B_3, \alpha_1, \alpha_3) \), we have

\[ \mathcal{V}|_C = \mathcal{O}(1)^2 \oplus \mathcal{O}^2 \text{ and } \mathcal{W}|_C = \mathcal{O}(2) \oplus \mathcal{O}(1)^3 \oplus \mathcal{O}. \]

Proof. From Lemma 8.2, we have the exact sequences

\[
\begin{array}{cccc}
0 & \to & \mathcal{O}(2) & \to \mathcal{Q}^{(0)} \to 0 \\
0 & \to & \mathcal{F}_0 & \to \mathcal{O}(1)^5 \to \mathcal{Q}^{(1)} \to 0 \\
0 & \to & \mathcal{V}/\mathcal{F}_0 & \to \mathcal{O}^3 \to \mathcal{Q}^{(2)} \to 0 \\
\end{array}
\]
with
\[
Q^{(0)} = \wedge^2 F_0 \\
\deg Q^{(1)} = \deg (F_0 \wedge V/F_0) \\
\deg Q^{(2)} = \deg \wedge^2 (V/F_0).
\]
Write \( F_0 = \mathcal{O}(a_1) + \mathcal{O}(a_2) \), \( V/F_0 = \mathcal{O}(a_3) + \mathcal{O}(a_4), \) \( Q^{(0)} = \mathcal{O}(b_1), \) \( Q^{(1)} = \mathcal{O}(b_2) + \mathcal{O}(b_3) + \mathcal{O}(b_4) \) and \( Q^{(2)} = \mathcal{O}(b_5). \) Then \( 2 = b_1 = a_1 + a_2, \) \( 5 = (a_1 + a_2) + b_2 + b_3 + b_4 \) and \( 0 = (a_3 + a_4) + b_5 \) with \( b_5 = a_3 + a_4. \) Thus \( b_5 = 0. \)

From the second exact sequence, \( a_i \leq 1 \) for \( i = 1, 2 \) and \( b_j \geq 0 \) for \( j = 2, 3, 4. \) Thus the second exact sequence splits and \( a_1 = a_2 = b_2 = b_3 = b_4 = 1. \) From the third exact sequence we get \( a_3 = a_4 = 0. \)

In sum, there exists a Zariski open subset \( M' \subset M^0 \subset M \) such that a general member of \( K \) lies on \( M^0 \) and the varieties of minimal rational tangents \( C|_{M^0} \subset \mathbb{P}TM|_{M^0} \) defines an \( S \)-structure on \( M^0. \) Proposition 3.7 and Lemma 8.9 implies that there exist \( G_0 \)-structure on \( (M^0, D|_{M^0}) \) corresponding to the \( S \)-structure, where \( G_0 = G(S). \)

Define a vector bundle \( \mathcal{H}^2_k \) on \( M^0 \) by \( \mathcal{H}^2_k := \mathcal{P} \times G_0 H^2(\mathfrak{m}, \mathfrak{g})_k. \)

**Lemma 8.9.** \( H^0(M^0, \mathcal{H}^2_k) \) is zero for all \( k \geq 1. \)

**Proof.** By proposition 8.7 it suffices to show the vanishing \( H^0(M^0, \wedge^2 D^* \otimes D) = 0. \) Let \( \varphi : \wedge^2 D \to V \) and \( \psi : \wedge^2 D \to W \) be nontrivial vector bundle maps. For \( x \in M^0 \) and \( \beta \in D_x \) with \( [\beta] \in C_x, \) take \( C \) to be a member of \( K \) passing through \( x \) with \( [T_x C] = [\beta]. \) Then \( TC \wedge D|_C \) is decomposed as a sum of \( \mathcal{O}(a) \)'s with \( a \geq 1. \) Thus \( \varphi|_C \) maps \( TC \wedge D|_C \) into the \( \mathcal{O}(1) \)-factor of \( V|_C \) and \( \psi|_C \) maps \( TC \wedge D|_C \) into the positive-factors of \( W|_C, \) the intersection of \( V_x \) with \( T_x \hat{C}_x \) and the intersection of \( W_x \) with \( T_x \hat{C}_x. \) Applying this argument to a general \( [\beta] \in C_x, \) we see that the image of \( \varphi_x \) and \( \psi_x \) are contained in the intersections
\[
\cap_{[\beta] \in C_x} (T_x \hat{C}_x \cap V_x) \text{ and } \cap_{[\beta] \in C_x} (T_x \hat{C}_x \cap W_x)
\]
which are \( G_0 \)-invariant and degenerate in \( V_x \) and \( W_x \) respectively, contradicting to the fact that \( V \) and \( W \) are irreducible \( G_0 \)-bundles. Therefore, \( \varphi_x \) and \( \psi_x \) are zero. Hence, \( H^0(M^0, \wedge^2 D^* \otimes D) = 0. \)

**Proof of Theorem 1.3 in the case when \( X \) is \( (B_3, \alpha_1, \alpha_3). \)** By Proposition 3.16 and Lemma 8.9, \( G_0 \)-structure \( \mathcal{P} \) on \( M^0 \) is locally equivalent to the standard one. By Proposition 3.7, the \( S \)-structure on \( M^0 \) defined by \( C(M)|_{M^0} \) is locally equivalent to the standard one. By Theorem 2.4, a local map preserving the varieties of minimal rational tangents can be extended to a global biholomorphism. Hence, \( M \) is biholomorphic to \( X. \) This completes the proof of Theorem 1.3 in the case when \( X \) is \( (B_3, \alpha_1, \alpha_3). \)
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Center for Complex Geometry, Institute for Basic Science (IBS), 55 Expo-ro, Yuseong-gu, Daejeon, Korea 34126

Email address: jhhong00@ibs.re.kr

Center for Geometry and Physics, Institute for Basic Science (IBS), 77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk, Korea 37673

Email address: shinyoungkim@ibs.re.kr