Lehmer’s totient problem over $\mathbb{F}_q[x]$

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Abstract: In this paper, we consider the function field analogue of the Lehmer’s totient problem. Let $p(x) \in \mathbb{F}_q[x]$ and $\varphi(q, p(x))$ be the Euler’s totient function of $p(x)$ over $\mathbb{F}_q[x]$, where $\mathbb{F}_q$ is a finite field with $q$ elements. We prove that $\varphi(q, p(x)) | (q^{\deg(p(x))}-1)$ if and only if (i) $p(x)$ is irreducible; or (ii) $q=3$, $p(x)$ is the product of any 2 non-associate irreducibles of degree 1; or (iii) $q=2$, $p(x)$ is the product of all irreducibles of degree 1, all irreducibles of degree 1 and 2, and the product of any 3 irreducibles one each of degree 1, 2 and 3.

Keywords: Euler’s totient function, Lehmer’s totient problem, cyclotomic polynomial.

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1. Introduction

Throughout this paper, let $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the field of rational numbers, the ring of rational integers and the set of nonnegative integers, respectively. Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. As usual, let $\text{ord}_p$ denote the normalized $p$-adic valuation of $\mathbb{Q}_p$.

Lehmer’s totient problem Let $\varphi$ be the Euler’s totient function. In [6], Lehmer discussed the equation

$$k\varphi(n) = n - 1,$$

where $k$ is an integer. In his pioneering paper [6], Lehmer showed that if $n$ is a solution of (1), then $n$ is a prime or the product of seven or more distinct primes. One is tempted to believe that an integer $n$ is a prime if and only if $\varphi(n)$ divides $n - 1$. This problem has not been solved to this day. But some progress has been made in this direction. In the literature, some authors call these composite numbers $n$ satisfying equation (1) the Lehmer numbers. Lehmer’s totient problem is to determine the set of Lehmer numbers. To the best of our knowledge, the current

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best result is due to Richard G. E. Pinch (see [9]), that the number of prime factors of a Lehmer number \( n \) must be at least 15 and there is no Lehmer number less than \( 10^{30} \). For further results on this topic we refer the reader to ([1], [2], [5], [7], [10]).

J. Schettler [11] generalizes the divisibility condition \( \varphi(n)|(n-1) \), constructs reasonable notion of Lehmer numbers and Carmichael numbers in a PID and gets some interesting results. Let \( R \) be a PID with the property: \( R/(r) \) is finite whenever \( 0 \neq r \in R \). Denote the sets of units, primes and (non-zero) zero divisors, in \( R \), by \( U(R) \), \( P(R) \) and \( Z(R) \), respectively; additionally, define

\[
L_R := \{ r \in R \setminus \{0\} \cup U(R) \cup P(R) : |U(R/(r))| \mid |Z(R/(r))| \}. \tag{2}
\]

Note that when \( R = \mathbb{Z} \), \( L_{\mathbb{Z}} \) is the set of Lehmer numbers. An element of \( L_R \) is also called a Lehmer number of \( R \). Let \( \mathbb{F}_q \) is a finite field with \( q \) elements. Then \( \mathbb{F}_q[x] \) is a PID. Schettler obtains some properties of elements of \( L_{\mathbb{F}_q[x]} \) as follows.

**Proposition 1.1.** ([11], Theorems 5.1, 5.2, 5.3) (1) Suppose \( f(x) \in L_{\mathbb{F}_q[x]} \), \( p(x) \in P(\mathbb{F}_q[x]) \) and \( p(x)|f(x) \). Then \( \deg(p(x))|\deg(f(x)) \).

(2) Suppose \( f(x) \in L_{\mathbb{F}_q[x]} \). Then \( f(x) \) has at least \( \lceil \log_2(q+1) \rceil \) distinct prime factors.

(3) There exists a PID \( R \) such that \( L_R \neq \emptyset \). (E.g., \( f(x) = x(x+1) \in L_{\mathbb{Z}/2\mathbb{Z}} \)).

Our work is inspired by above proposition, in this paper, our goal is to determine the set \( L_{\mathbb{F}_q[x]} \).

**Euler’s totient function over** \( \mathbb{F}_q[x] \). Let \( f(x) \in \mathbb{F}_q[x] \) with \( m = \deg(f(x)) \geq 1 \). Put

\[
\Phi(f(x)) = \{ g(x) \in \mathbb{F}_q[x] \mid \deg(g(x)) \leq m-1, (f(x), g(x)) = 1 \}.
\]

The Euler’s totient function \( \varphi(q, f(x)) \) of \( f(x) \) is defined as follows:

\[
\varphi(q, f(x)) = q^{\deg(f(x))} - 1.
\]

If \( f(x) \in \mathbb{F}_q[x] \) is irreducible, then \( \varphi(q, f(x)) = q^{\deg(f(x))} - 1 \). It is easy to see that the functions \( \varphi(q, f(x)) \) and \( \varphi(n) \) have the following similar properties:

**Proposition 1.2.** Let \( f(x) = p_1(x)^{r_1} \cdots p_k(x)^{r_k} \in \mathbb{F}_q[x] \) of degree \( n \geq 1 \), where \( p_1(x), \ldots, p_k(x) \in P(\mathbb{F}_q[x]) \) are non-associate, \( \deg(p_i(x)) = n_i \) and \( r_i \geq 1, 1 \leq i \leq k \). Then we have

(1) \( \varphi(q, f(x)) = q^n \prod_{i=1}^{k} (1 - \frac{1}{q^{r_i}}) \);

(2) If \( g(x) \in \mathbb{F}_q[x] \) and \( (f(x), g(x)) = 1 \), then \( g(x)^{\varphi(q, f(x))} \equiv 1 \pmod{f(x)} \);

(3) If \( \varphi(q, f(x))(q^n - 1) \), then \( r_i = 1 \), for all \( 1 \leq i \leq k \).
Hence it is natural to consider the Lehmer’s totient problem over \( \mathbb{F}_q[x] \):

Determine \( f(x) \in \mathbb{F}_q[x] \) such that \( \varphi(q, f(x))|(q^{\deg(f(x))} - 1) \).

Set

\[
\mathcal{L}_{\mathbb{F}_q} = \{ f(x) \in \mathbb{F}_q[x] \setminus \{0\} \mid \deg(f(x)) \geq 1, \varphi(q, f(x))|(q^{\deg(f(x))} - 1) \}.
\]

By the definition (2), it is easy to see that

\[
\mathcal{L}_{\mathbb{F}_q[x]} = \{ f(x) \in \mathbb{F}_q[x] \setminus \{0\} \mid f(x) \text{ is reducible, } \varphi(q, f(x))|(q^{\deg(f(x))} - 1) \}.
\]

Hence \( \mathcal{L}_{\mathbb{F}_q} = P(\mathbb{F}_q[x]) \cup \mathcal{L}_{\mathbb{F}_q[x]} \).

For \( q = 2, 3 \), Lv Hengfei \[8\] gave some polynomials \( f(x) \in \mathcal{L}_{\mathbb{F}_q[x]} \) as follows:

1. \( q = 2 \), \( f(x) = x(x+1)(x^2+x+1) \), then \( \varphi(2, f(x)) = 3 \), hence \( \varphi(2, f(x))|(2^4 - 1) \).
2. \( q = 3 \), \( f(x) = x(x+1) \), then \( \varphi(3, f(x)) = 4 \), hence \( \varphi(3, f(x))|(3^2 - 1) \).

In this paper, we give the necessary and sufficient conditions for \( f(x) \in \mathcal{L}_{\mathbb{F}_q[x]} \) as follows.

**Main Theorem**

1. Assume \( q \geq 4 \). Then \( \mathcal{L}_{\mathbb{F}_q[x]} = \emptyset \).
2. Assume \( q = 3 \). Then \( \mathcal{L}_{\mathbb{F}_3[x]} \) consists of the products of any 2 non-associate irreducibles of degree 1, i.e.,

   \[
   \mathcal{L}_{\mathbb{F}_3[x]} = \{ ax(x+1), ax(x-1), a(x+1)(x-1), a = 1, 2 \}.
   \]

3. Assume \( q = 2 \). Then \( \mathcal{L}_{\mathbb{F}_2[x]} \) consists of the products of all irreducibles of degree 1, the products of all irreducibles of degree 1 and 2, and the products of any 3 irreducibles one each of degree 1, 2, and 3, i.e.,

   \[
   \mathcal{L}_{\mathbb{F}_2[x]} = \{ x(x+1), x(x+1)(x^2+x+1), x(x^2+x+1)(x^3+x+1), \\
   (x+1)(x^2+x+1)(x^3+x+1), x(x^2+x+1)(x^3+x^2+1), \\
   (x+1)(x^2+x+1)(x^3+x^2+1) \in \mathbb{F}_2[x] \}.
   \]

The proof is essentially to give the necessary and sufficient conditions for \( \varphi(q, f(x))|(q^{\deg(f(x))} - 1) \) which will be divided into two cases \( q \geq 3 \) and \( q = 2 \).

2. **Properties of cyclotomic polynomials**

Let \( n \in \mathbb{N}^* \) and \( \zeta_n \) be a primitive \( n \)-th root of unity. The polynomial

\[
\Phi_n(x) = \prod_{(i, n)=1} (x - \zeta_n^i)
\]

3
is called the $n$-th cyclotomic polynomial. It is well-known that $\Phi_n(x)$ is an irreducible polynomial of degree $\varphi(n)$ in $\mathbb{Z}[x]$ and

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

(3)

Note that the polynomial factorization in (3) is complete. But it does not follow that the factorization

$$a^n - 1 = \prod_{d|n} \Phi_d(a), \quad a \in \mathbb{Z},$$

(4)

is complete, since the integer $\Phi_d(a)$ may not be prime.

**Definition 2.1.** Suppose $a > b > 0$ are coprime integers. A prime divisor $p$ of $a^n - b^n$, $n \geq 2$, is called primitive if $p \nmid a^k - b^k$; for any $k < n$. Otherwise, it is called algebraic.

It is well-known that the following Bang-Zsigmondy’s Theorem provides the existence of a primitive prime factor.

**Bang-Zsigmondy’s Theorem** ([14]) Suppose $a > b > 0$ are coprime integers. Then for any natural number $n > 1$ there is a primitive prime divisor $p$ of $a^n - b^n$ with the following exceptions:

$$a = 2, \quad b = 1, \text{ and } n = 6; \text{ or}$$

$$a + b \text{ is a power of two, and } n = 2.$$

It is clear that for any $n$, and $d|n$, that any prime $p$ dividing $\phi_d(a)$ will be an algebraic divisor of $[14]$, since $p$ must divide $a^d - 1$ as $\phi_d(a)$ does. On the other hand, any primitive factor of $a^n - 1$ will have to divide $\Phi_n(a)$. It is not true, however, that every prime factor of $\Phi_n(a)$ is primitive.

**Lemma 2.2.** ([3]) Let $p$ be a prime and $m \in \mathbb{N}^*$ with $(p, m) = 1$. Suppose $v \in \mathbb{N}^*$ and $a \in \mathbb{Z}$. Then $p|\Phi_{mp^v}(a)$ if and only if $p|\Phi_m(a)$. Furthermore, if $p|\Phi_m(a)$, then $\text{ord}_p(\Phi_{mp^v}(a)) = 1$.

**Lemma 2.3.** Let $p$ be a prime and $n \in \mathbb{N}^*$. Suppose $n = p^v m$ with $v = \text{ord}_p(n)$. Then $p|\Phi_n(a)$ for some $a \in \mathbb{Z}$ if and only if $m|(p - 1)$.

**Proof.** It is obvious from Lemma 2.2 and ([13], Lemmas 2.9, 2.10).
Corollary 2.4. Let $p$ be a prime and $a \in \mathbb{Z}$, $v \in \mathbb{N}$. Then $p|\Phi_{p^v}(a)$ if and only if $p|(a - 1)$.

Corollary 2.5. Let $m > n$ be positive integers. For any $a \in \mathbb{Z}$, we obtain that $(\Phi_n(a), \Phi_m(a)) = 1$ or $(\Phi_n(a), \Phi_m(a))$ is a prime. Furthermore, if $(\Phi_n(a), \Phi_m(a)) = p$ is a prime, then $m = p^v n$ for some $v \geq 1$.

Lemma 2.6. Let $a, m \in \mathbb{N}^*$ and $a \geq 2$. Then $|\Phi_m(a)| = 1$ if and only if $m = 1$, $a = 2$.

Proof. By the formula $\Phi_m(a) = \prod_{(j, m) = 1} (a - \zeta_m^j)$, we know that $|a - \zeta_m^j| > 1$ for all $a \geq 2$ and $m \geq 2$, hence $|\Phi_m(a)| > 1$. On the other hand, $\Phi_1(x) = x - 1$. Therefore $\Phi_m(a) = 1$ if and only if $m = 1$, $a = 2$. □

To end this section, we recall an estimate for $\Phi_n(a)$.

Lemma 2.7. ([12], Theorem 5) For any integers $n \geq 2$ and $a \geq 2$, we have
\[
\frac{1}{2} a^{\varphi(n)} \leq \Phi_n(a) \leq 2 \cdot a^{\varphi(n)}.
\]

3. Main Results

Proposition 3.1. Let $a, n \in \mathbb{N}^*$ and $a \geq 3$, $n \geq 2$. Assume $s \geq 2$ and $e_1, e_2, \ldots, e_s \in \mathbb{N}^*$ with $\sum_{i=1}^s e_i = n$. Then $\prod_{i=1}^s (a^{e_i} - 1)|(a^n - 1)$ if and only if $n = s = 2$, $e_1 = e_2 = 1$ and $a = 3$.

Proof. The sufficiency is trivial. It is sufficient to show the necessity. Suppose $\prod_{i=1}^s (a^{e_i} - 1)|(a^n - 1)$. First, we have
\[
\frac{x^n - 1}{\prod_{i=1}^s (x^{e_i} - 1)} = \prod_{d \in T} \Phi_d(x) \prod_{d' \in T'} \Phi_{d'}(x) = \prod_{d \in T} \Phi_d(x) = \frac{P(x)}{Q(x)},
\]
where $T = \{d > 1 \mid d|n, \ d \nmid e_i, \ 1 \leq i \leq s\}$, $P(x) = \prod_{d \in T} \Phi_d(x)$ and $Q(x) = \prod_{d' \in T'} \Phi_{d'}(x)$ for some index set $T'$, and $T'' = \{d' \in T' \mid d' \geq 2\}$.

We have
(i) \((P(x), Q(x)) = 1\) and \(\deg(P(x)) = \deg(Q(x))\);

(ii) For any \(d' \in T'\), we have

\[d'|e_i \text{ for some } 1 \leq i \leq s, \text{ and } (\Phi_{d'}(x), \Phi_d(x)) = 1 \text{ for all } d \in T;\]

(iii) For any \(d \in T\) and \(d' \in T'\), we have \(d \nmid d'\).

(iv) For any \(d \in T\) and \(d_1', d_2' \in T'\) such that

\[(\Phi_d(a), \Phi_{d_1'}(a)) \neq 1 \text{ and } (\Phi_d(a), \Phi_{d_2'}(a)) \neq 1.\]

Then \((\Phi_d(a), \Phi_{d_1'}(a)) = (\Phi_d(a), \Phi_{d_2'}(a)) = p\) for some prime \(p\) and \(d = p^{v_1}d_1' = p^{v_2}d_2'\) for some \(v_1, v_2 \in \mathbb{N}^*\). Furthermore, \(\text{ord}_p(\Phi_d(a)) = 1\).

The statements (i), (ii) and (iii) are obvious. We only prove (iv). In fact, by Corollary 2.5 there exist primes \(p_1\) and \(p_2\) such that \((\Phi_d(a), \Phi_{d_1'}(a)) = p_1\) and \((\Phi_d(a), \Phi_{d_2'}(a)) = p_2\). If \(p_1 \neq p_2\), then by (iii) and Corollary 2.5, we have \(d = p_1^{r_1}p_2^{r_2}d''\) for some \(r_1, r_2, d'' \in \mathbb{N}^*\) with \(p_1, p_2d'' = (p_2, p_1d'') = 1\). By Lemma 2.3 we have \(p_2^{r_2}d''|p_1 - 1\) and \(p_1^{r_1}d''|p_2 - 1\). This is a contradiction. Hence we obtain \((\Phi_d(a), \Phi_{d_1'}(a)) = (\Phi_d(a), \Phi_{d_2'}(a)) = p\) for some prime \(p\). From (iii) and Corollary 2.5, we have \(d = p^{v_1}d_1' = p^{v_2}d_2'\) for some \(v_1, v_2 \in \mathbb{N}^*\). By Lemma 2.2 we have \(\text{ord}_p(\Phi_d(a)) = 1\). Thus we complete the proof of (iv).

By assumption, we have

\[
\frac{a^n - 1}{\prod_{i=1}^{s} (a^{e_i} - 1)} = \frac{\prod_{d \in T} \Phi_d(a)}{(a - 1)^{s-1}} \cdot \frac{\prod_{d' \in T'} \Phi_{d'}(a)}{\Phi_d(a)} = \frac{P(a)}{Q(a)} \in \mathbb{N}^*.
\]

Let \(p\) be a prime such that \(p^r || (a - 1)\) for some \(r \in \mathbb{N}^*\). Then \(p^{r(s-1)}|P(a)\). By Lemma 2.2 and Corollary 2.5 there exist positive integers \(1 \leq j_1 < j_2 < \cdots < j_{r(s-1)}\) such that

\(p^{j_1}, p^{j_2}, \ldots, p^{j_{r(s-1)}} \in T, \text{ and } \text{ord}_p(\Phi_{p^{j_k}}(a)) = 1, 1 \leq k \leq r(s-1)\).

Now we define a map \(f : T'' \to T\) as follows. By Lemma 2.6, for any \(d' \in T''\), we have \(|\Phi_{d'}(a)| \neq 1\). Choose a prime factor of \(\Phi_{d'}(a)\), say \(p'|\Phi_{d'}(a)\), there exists \(d = p^v d' \in T\) for some \(v \geq 1\). Define \(f(d') = d\). By Lemma 2.2 we have \(\text{ord}_{p'}(\Phi_d(a)) = 1\). By (iv), the map \(f\) is injective and \(f(d') \neq p^k, 1 \leq k \leq r(s-1)\). It is clear that

\[
\sum_{k=1}^{r(s-1)} \deg(\Phi_{p^{j_k}}(x)) \geq s - 1, \text{ and } \deg(\Phi_{f(d')}(x)) \geq \deg(\Phi_{d'}(x)), \text{ } d' \in T''.
\]
Hence the equality \( \deg(P(x)) = \deg(Q(x)) \) implies that

\[
\sum_{k=1}^{r(s-1)} \deg(\Phi_{p^k}(x)) = s - 1 \quad \text{and} \quad a - 1 = p^r.
\]

It is easy to verify that \( \sum_{k=1}^{r(s-1)} \deg(\Phi_{p^k}(x)) = s - 1 \) if and only if \( a = 3, p = 2, s = 2, r = 1 \) and \( e_1 = e_2 = 1 \). This completes the proof. \( \square \)

Lemma 3.2. Let \( n \in \mathbb{N}^* \) and \( n \geq 2 \). Assume \( s \geq 2 \) and \( e_1, e_2, \ldots, e_s \in \mathbb{N}^* \) with \( \sum_{i=1}^{s} e_i = n \). If \( \prod_{i=1}^{s} (2^{e_i} - 1) \mid (2^n - 1) \), then \( e_i \mid n \) for all \( 1 \leq i \leq s \), and \( (e_1, \ldots, e_s) = 1 \).

Proof. The assumption \( \prod_{i=1}^{s} (2^{e_i} - 1) \mid (2^n - 1) \) implies that

\[
\frac{2^n - 1}{\prod_{i=1}^{s} (2^{e_i} - 1)} = \frac{\prod_{d \in T} \Phi_d(2)}{\prod_{d' \in T''} \Phi_{d'}(2)} = \frac{P(2)}{Q(2)} \in \mathbb{N}^*,
\]

where the sets \( T \) and \( T'' \) are defined by the formula (5). Suppose that there exists \( e_{i_0} \) for some \( 1 \leq i_0 \leq s \) such that \( e_{i_0} \nmid n \). Hence there is a prime \( p \) and \( r \in \mathbb{N}^* \) such that \( p^r \mid e_{i_0} \) and \( p^r \nmid n \). Thus \( p^r \in T'' \). By Lemma 2.6, we have \( |\Phi_{p^r}(2)| \neq 1 \). Let \( q \) be a prime such that \( q \mid \Phi_{p^r}(2) \). Then there exists \( d \in T \) such that \( q \mid \Phi_d(2) \). From (iii) of the proof of Proposition 3.1 and Corollary 2.5, we have \( d = q^r p^r \) for some \( v \in \mathbb{N}^* \). Therefore \( q^r p^r \mid n \). This contradicts the fact \( p^r \nmid n \). Hence we have \( e_i \mid n \) for all \( 1 \leq i \leq s \).

Assume \( (e_1, \ldots, e_s) = d > 1 \). Put \( a = 2^d, e_i = e'_i d, 1 \leq i \leq s, n = n'd \). Then \( a \geq 4 \) and \( n' = \sum_{i=1}^{s} e'_i \). By Proposition 3.1, we have \( \prod_{i=1}^{s} (a^{e'_i} - 1) \nmid (a^{n'} - 1) \), hence \( \prod_{i=1}^{s} (2^{e_i} - 1) \nmid (2^n - 1) \). This contradicts the assumption \( \prod_{i=1}^{s} (2^{e_i} - 1) \mid (2^n - 1) \). Therefore we have \( (e_1, \ldots, e_s) = 1 \).

\( \square \)

Lemma 3.3. Let \( n \in \mathbb{N}^* \) and \( h(n) = \frac{\sigma(n)}{n} \), where \( \sigma(n) = \sum_{d \mid n} d \). Then we have \( h(n) < 1.28n^{\frac{1}{4}} \), for all \( n \in \mathbb{N}^* \).
Proof. Let $p \geq 5$ be a prime and $a \in \mathbb{N}^*$. It is easy to see that $\frac{h(p^a)}{p^a} < 1$. For $p = 2, 3$, we get

$$h(2^a) \begin{cases} < 1.262, & \text{if } a = 1, \\ < 1.238, & \text{if } a = 2, \\ < 1.115, & \text{if } a = 3, \\ < 1, & \text{if } a \geq 4. \end{cases}$$

and

$$h(3^a) \begin{cases} < 1.014, & \text{if } a = 1, \\ < 1, & \text{if } a \geq 2. \end{cases}$$

Hence we have $h(n) < 1.262 \times 1.014n^{\frac{1}{4}} < 1.28n^{\frac{1}{4}}$, for all $n \in \mathbb{N}^*$.

Lemma 3.4. Let $n \in \mathbb{N}^*$. Set

$$c(n) = \begin{cases} 0.59, & \text{if } \text{ord}_2(n) = 1, \\ 0.70, & \text{if } \text{ord}_2(n) = 2, \\ 0.84, & \text{if } \text{ord}_2(n) = 3, \\ 1, & \text{if } \text{ord}_2(n) \geq 4, \text{ or } \text{ord}_2(n) = 0. \end{cases}$$

Then $\varphi(n) > c(n)n^{\frac{3}{4}}$, for any integer $n \geq 2$.

Proof. If $p$ is an odd prime, then $\varphi(p^a) > p^{\frac{3a}{2}}$ for any $a \in \mathbb{N}^*$. On the other hand, we have

$$\varphi(2^a) \begin{cases} > 0.59, & \text{if } \text{ord}_2(n) = 1, \\ > 0.70, & \text{if } \text{ord}_2(n) = 2, \\ > 0.84, & \text{if } \text{ord}_2(n) = 3, \\ > 1, & \text{if } \text{ord}_2(n) \geq 4. \end{cases}$$

Hence $\varphi(n) > c(n)n^{\frac{3}{4}}$, for any integer $n \geq 2$.

Proposition 3.5. Let $n \geq s \geq 2$, $e_1 \leq e_2 \leq \cdots \leq e_s$ be positive integers such that $\sum_{i=1}^{s} e_i = n$. For each $d | n$, $d < n$, let $u_d = \#\{e_i \mid e_i = d, 1 \leq i \leq s\}$. Assume that $u_1 \leq 2$ and $u_d \leq \frac{2^{d-1}}{d}$ for any $d \geq 2$. Then $\prod_{i=1}^{s} (2^{e_i} - 1)|(2^n - 1)$ if and only if (1) $n = 2, s = 2, e_1 = e_2 = 1$; or (2) $n = 4, s = 3, e_1 = e_2 = 1, e_3 = 2$; or (3) $n = 6, s = 3, e_1 = 1, e_2 = 2, e_3 = 3$. 

Proof. The sufficiency is trivial. It is sufficient to show the necessity. Set

\[ R = \frac{2^n - 1}{\prod_{i=1}^{s} (2^{e_i} - 1)} \in \mathbb{N}^*. \]

(1) Assume \( 2 \leq n \leq 6 \). It is easy to show the necessity by Lemma 3.2.

(2) Assume \( n \geq 7 \). The primitive part \( M \) of \( 2^n - 1 \) can not be reduced with the denominator, so \( R \geq M \). By Lemma 2.7, we have

\[ R \geq M \geq \Phi_n(2) \geq \frac{2^{\varphi(n)}}{2n}. \]

On the other hand, we have

\[ R = \frac{2^n - 1}{2^n} \prod_{i=1}^{s} (1 - 2^{-e_i})^{-1} < \prod_{i=1}^{s} (1 - 2^{-e_i})^{-1}. \]

By assumption, \( u_1 \leq 2, u_2 \leq 1 \), hence

\[
\log R < 2 \log 2 + \delta(n) \log \frac{4}{3} - \sum_{e_i \geq 3} \log (1 - 2^{-e_i})
< \log 4 + \delta(n) \log \frac{4}{3} + \sum_{e_i \geq 3} \frac{1}{2^{e_i} - 1}
< \log 4 + \delta(n) \log \frac{4}{3} + \sum_{d|n, 3 \leq d < n} \frac{\mu(d)}{2^{d-1}}
\leq \log 4 + \delta(n) \log \frac{4}{3} + \sum_{d|n, 3 \leq d < n} \frac{1}{d}
= \log 4 + \delta(n) \log \frac{4}{3} - 1 - \frac{\delta(n)}{2} - \frac{1}{n} + h(n),
\]

where \( \delta(n) = \begin{cases} 
1, & \text{if } n \equiv 0 \pmod{2}, \\
0, & \text{if } n \equiv 1 \pmod{2}.
\end{cases} \)

By Lemmas 3.4, 3.5, we have

\[
\log R > \varphi(n) \log 2 - \log 2n > c(n) \log 2 \cdot n^{\frac{3}{4}} - \log 2n,
\]

\[
\log R < \log 4 + \delta(n) \log \frac{4}{3} - 1 - \frac{\delta(n)}{2} - \frac{1}{n} + 1.28n^{\frac{3}{4}}.
\]

It is easy to calculate that the inequality

\[
\log 4 + \delta(n) \log \frac{4}{3} - 1 - \frac{\delta(n)}{2} - \frac{1}{n} + 1.28n^{\frac{3}{4}} > c(n) \log 2 \cdot n^{\frac{3}{4}} - \log 2n
\]

holds for \( n \geq 7 \) if and only if \( n \in \{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 30, 34, 38, 42, 46, 50, 54 \} \).

Hence the inequality

\[
\log 4 + \delta(n) \log \frac{4}{3} - 1 - \frac{\delta(n)}{2} - \frac{1}{n} + h(n) > \varphi(n) \log 2 - \log 2n
\]
holds for $n \geq 7$ if and only if $n \in D = \{8, 9, 10, 12, 14, 18, 20, 24, 30\}$. By Lemma 3.2, we can straightly calculate that there is no $n \in D$ meeting the assumptions. This completes the proof.

We are now in the position to prove the main theorem.

**Proof of Main Theorem** The sufficiency is trivial. We need only show the necessity. Assume that $p(x) \in \mathbb{F}_q[x]$ is reducible and of degree $n \geq 1$. Let

$$p(x) = p_1(x)^{r_1} \cdots p_k(x)^{r_k}$$

be the standard decomposition, where $p_i(x)$ is irreducible and of degree $n_i \geq 1$, $r_i \geq 1$, $1 \leq i \leq k$. By (3) of Proposition 1.2, we have $r_1 = r_2 = \cdots = r_k = 1$. Hence

$$p(x) = p_1(x) \cdots p_k(x) \quad \text{and} \quad n = \sum_{i=1}^k n_i.$$

If $q \geq 3$, then, by Proposition 3.1, we have $q = 3$, $k = 2$, $n_1 = n_2 = 1$, hence $p(x)$ is the product of any 2 non-associate irreducibles of degree 1; i.e.,

$$L_{\mathbb{F}_3[x]} = \{ax(x+1), ax(x-1), a(x+1)(x-1) \in \mathbb{F}_3[x], a = 1, 2\}.$$

If $q = 2$, then the $n'_i$s satisfy the assumptions of Proposition 3.5 hence we have (i) $n = 2, k = 2, n_1 = n_2 = 1$; or (ii) $n = 4, k = 3, n_1 = n_2 = 1, n_3 = 2$; or (iii) $n = 6, k = 3, n_1 = 1, n_2 = 2, n_3 = 3$. On the other hand, the irreducibles of degree 1 are $x$ and $x+1$; $x^2 + x + 1$ is the unique irreducible of degree 2; the irreducibles of degree 3 are $x^3 + x + 1$ and $x^3 + x^2 + 1$. Hence

$$L_{\mathbb{F}_2[x]} = \{x(x+1), x(x+1)(x^2 + x + 1), x(x^2 + x + 1)(x^3 + x + 1),$$

$$\quad (x+1)(x^2 + x + 1)(x^3 + x + 1), x(x^2 + x + 1)(x^3 + x^2 + 1),$$

$$\quad (x+1)(x^2 + x + 1)(x^3 + x^2 + 1) \in \mathbb{F}_2[x]\}.$$

This completes the proof.

\[\square\]

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