The $l_q/l_p$ Hankel norms of discrete-time positive systems across switching

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ABSTRACT
In this study, we focus on the $l_q/l_p$ Hankel norms of linear time-invariant (LTI) discrete-time positive systems across a single switching. The $l_q/l_p$ Hankel norms are defined as the induced norms from vector-valued $l_p$ past inputs to vector-valued $l_q$ future outputs across a system switching and are applicable even when the systems are (positive) switched. A closed-form characterization of the $l_2/l_2$ Hankel norm in this switching setting for general LTI systems can readily be derived as the natural extension of the standard $l_2/l_2$ Hankel norm. Thanks to the strong positivity property, we show that we can successfully characterize the $l_q/l_p$ Hankel norms for the positive system switching case even in some combinations of $p, q$ being $1, 2, \infty$. In particular, some of them are given in the form of linear programming (LP) and semidefinite programming (SDP). These LP- and SDP-based characterizations are particularly useful for the analysis of the $l_q/l_p$ Hankel norms where the systems of interest are affected by parametric uncertainties.

1. Introduction
This study is concerned with the analysis of the $l_q/l_p$ Hankel norms of discrete-time positive systems across a single switching. Early studies on positive systems focused on controllability and reachability analysis [1,2], positive realization [3,4], and positive stabilization [5]. Then, the analysis and synthesis of positive systems using convex optimization have attracted great attention, and some fruitful results have been obtained to this date. Those results include the induced norm analysis using linear programming [6,7], Kalman–Yakubovich–Popov lemma with diagonal Lyapunov variables [8,9], positive system synthesis using geometric programming [10], and positive system analysis using copositive programming [11]. Surveys on recent studies of positive systems can be found at [12,13].

Analysis and synthesis of switched positive systems form a rather new and active research area in the theory of positive systems [14–17]. In [18], relatively new results on the induced norms of positive systems shown in [6,7,19,20] are also successfully extended to the switched case. On the other hand, the author of [21,22] considered the case where a general (i.e. nonpositive) continuous-time LTI system switches to another one at the time instant zero, and introduced the $l_2/l_2$ Hankel norm as the induced norm from vector-valued $l_2$ past inputs to vector-valued $l_2$ future outputs. Here, the past input is injected to the system before switching, driving the initial state of the system after switching to some nonzero values along with the state transition at the time instant zero, and the future output corresponds to the initial response of the system after switching. By using this norm, we can quantitatively evaluate the magnitude of the “bumpy response” caused by switching.

The goal of this study is to derive the explicit characterizations of the $l_q/l_p$ Hankel norms in the discrete-time positive system switching setting. These norms are defined in exactly the same manner as [21,22], even though we evaluate the past inputs with $l_p$ norm and the future outputs with $l_q$ norm where $p, q$ being $1, 2$ or $\infty$. Namely, we deal with the $l_q/l_p$ Hankel norms across a single switching, and these can be regarded (interpreted) as natural extensions of the standard (nonswitched) $l_q/l_p$ Hankel norms. We focus only on single switching cases since in multiple switching cases such interpretation as Hankel norms is not possible. Similarly to [21,22], a closed-form characterization of the $l_2/l_2$ Hankel norm in the general setting can readily be derived as the natural extension of the standard $l_2/l_2$ Hankel norm of LTI systems. We then move on to the main issues and show that, thanks to the strong positivity property, we can successfully characterize the $l_q/l_p$ Hankel norms for the positive system switching even in some combinations of $p, q$ being $1, 2, \infty$. In particular, some of them are given in the form of linear programming (LP) and semidefinite programming (SDP). These LP- and SDP-based characterizations are particularly useful for the analysis of the $l_q/l_p$ Hankel norms where the systems of interest are affected by parametric uncertainties. We finally note that the
continuous-time system counterpart of this study has been discussed in [23].

We use the following notation. The set of $n \times m$ real matrices is denoted by $\mathbb{R}^{n \times m}$, and the set of $n \times m$ entrywise nonnegative (strictly positive) matrices is denoted by $\mathbb{R}_{+}^{n \times m}$ ($\mathbb{R}_{++}^{n \times m}$). For a matrix $A$, we also write $A \geq 0$ ($A > 0$) to denote that $A$ is entrywise nonnegative (strictly positive). We denote by $I^n \in \mathbb{R}^n$ the all-ones vector. The set of $n \times n$ real symmetric matrices is denoted by $\mathcal{S}^n$. For a matrix $A \in \mathcal{S}^n$, we write $A > 0$ ($A < 0$) to denote that $A$ is positive (negative) definite. For a matrix $A \in \mathcal{S}^n$, we also denote by $\lambda_{\text{max}}(A)$ and $d_{\text{max}}(A)$ the maximum eigenvalue and the maximum diagonal entry of $A$, respectively. The maximum singular value of $A \in \mathbb{R}^{n \times m}$ is denoted by $\sigma(A)$. Finally, for $A \in \mathbb{R}^{n \times \nu}$, we denote by $\rho(A)$ the spectral radius of $A$ and we further define $\text{He}(A) = A + A^T$.

2. Definition of the $l_q/l_p$ Hankel norms across switching

Suppose two stable LTI systems $\Sigma_p$ and $\Sigma_f$ are given, which are the models of the system before and after switching at the time $k = 0$, respectively (see Figure 1). We assume that the state space realizations of $\Sigma_p$ and $\Sigma_f$ are given, respectively by

$$
\Sigma_p : \quad x_p(k+1) = A_px_p(k) + B_pw(k) \quad (k < 0),
$$

$$
\Sigma_f : \quad \begin{cases} x_f(k+1) = A_fx_f(k), \\ z(k) = C_fx_f(k) \quad (k \geq 0). \end{cases}
$$

Here, $A_p \in \mathbb{R}_+^{n_p \times n_p}$, $B_p \in \mathbb{R}_+^{n_p \times n_w}$, $A_f \in \mathbb{R}_+^{n_f \times n_f}$, $C_f \in \mathbb{R}_+^{2 \times n_f}$ with $\rho(A_p) < 1$ and $\rho(A_f) < 1$. We consider the case where the system $\Sigma_p$ switches to the system $\Sigma_f$ at $k = 0$ along with the state transition described by

$$
x_f(0) = Sx_p(0).
$$

Here, $S \in \mathbb{R}_+^{n_f \times n_p}$ is a given matrix.

For the input signal $w$ and the output signal $z$, we define

$$
\|w\|_{1-} := \sum_{k=-\infty}^{-1} |w(k)|_1,
$$

$$
\|z\|_{1+} := \sum_{k=0}^\infty |z(k)|_1.
$$

Figure 1. Switching from $\Sigma_p$ to $\Sigma_f$ along with state transition.

For $p, q = 1, 2, \infty$ we also define

$$
l_{p-} := \{w : \|w\|_{p-} < \infty\}, \quad l_{q+} := \{z : \|z\|_{q+} < \infty\},
$$

$$
l_{p+} := \{w : w \in l_{p-}, w(k) \geq 0 \ (\forall k \leq -1)\},
$$

$$
l_{q+} := \{z : z \in l_{q+}, z(k) \geq 0 \ (\forall k \geq 0)\}.
$$

Then, the $l_q/l_p$ Hankel norm across switching from $\Sigma_p$ to $\Sigma_f$ with the state transition matrix $S \in \mathbb{R}_+^{n_f \times n_p}$ is defined by

$$
y_{q/p} := \sup_{w \in l_{p-}, \|w\|_{p-} = 1} \|z\| q+ \quad \text{s.t. (1), (2), (3). (4)}
$$

Note that $x_p(-\infty) = 0$ is tacitly assumed. In the following, we partition $B_p \in \mathbb{R}_+^{n_p \times n_w}$ and $C_f \in \mathbb{R}_+^{n_f \times n_f}$ as follows:

$$
B_p = [B_{p,1} \cdots B_{p,n_w}] \quad (B_{p,j} \in \mathbb{R}_+^{n_p \times 1}, j = 1, \ldots, n_w),
$$

$$
C_f = [C_{f,1} \cdots C_{f,n_f}] \quad (C_{f,i} \in \mathbb{R}_+^{1 \times n_f}, i = 1, \ldots, n_z).
$$

3. The standard $l_q/l_p$ Hankel norms of positive systems

3.1. Basics of discrete-Time positive systems

Let us consider the LTI system $G$ described by

$$
G : \quad x(k+1) = Ax(k) + Bw(k), \quad z(k) = Cx(k)
$$

where $A \in \mathbb{R}_+^{n \times n}$, $B \in \mathbb{R}_+^{n \times n_w}$, $C \in \mathbb{R}_+^{n_z \times n}$. The impulse response $g$ of the system $G$ is given by

$$
g(k) = \begin{cases} 0 & (k \leq 0), \\ CA^{k-1}B & (k > 0). \end{cases}
$$

The definition of the positivity of $G$ and its characterization are given as follows.

Definition 3.1 ([24]): The LTI system $G$ given by (5) is called internally positive if its state $x(k)$ and output $z(k)$
are nonnegative for \( k \geq 0 \) for any nonnegative input \( w(k) \) for \( k \geq 0 \) and nonnegative initial state \( x(0) \).

**Proposition 3.2 ([24]):** The system \( G \) given by (5) is internally positive iff

\[
A \in \mathbb{R}^{n \times n}_+, \quad B \in \mathbb{R}^{n \times nw}, \quad C \in \mathbb{R}^{n_s \times n}\tag{7}
\]

**Definition 3.3 ([24]):** The LTI system \( G \) given by (5) is called externally positive if its output \( z(k) \) is nonnegative for \( k \geq 0 \) for any nonnegative input \( w(k) \) for \( k \geq 0 \) and the zero initial state \( x(0) = 0 \).

**Proposition 3.4 ([24]):** The system \( G \) given by (5) is externally positive iff its impulse response (6) is nonnegative, i.e.

\[
g(k) \geq 0 \quad (\forall k > 0). \tag{8}
\]

In the following, we often identify \( \Sigma_p \) and \( \Sigma_f \) with \( \hat{\Sigma}_p \) and \( \hat{\Sigma}_f \), respectively, where

\[
\hat{\Sigma}_p(z) := \begin{bmatrix} A_p & B_p \\ I_{np} & 0 \end{bmatrix}, \quad \hat{\Sigma}_f(z) := \begin{bmatrix} A_f & I_{n_s} \\ C_f & 0 \end{bmatrix}.
\]

Then, we say that \( \Sigma_p \) and \( \Sigma_f \) are both internally positive iff \( \hat{\Sigma}_p \) and \( \hat{\Sigma}_f \) are both internally positive, i.e.

\[
A_p \in \mathbb{R}^{np \times np}_+, \quad B_p \in \mathbb{R}^{np \times nw}, \quad A_f \in \mathbb{R}^{n_s \times n_f}_+, \quad C_f \in \mathbb{R}^{n_s \times n_f}.
\]

Similarly, we say that \( \Sigma_p \) and \( \Sigma_f \) are both externally positive iff \( \hat{\Sigma}_p \) and \( \hat{\Sigma}_f \) are both externally positive, i.e.

\[
A_p^{k-1}B_p \geq 0, \quad C_fA_f^{-1} \geq 0 \quad (\forall k > 0). \tag{9}
\]

**3.2. Existing results on the standard \( l_q/l_p \) Hankel norms positive systems**

In the case where

\[
A_p = A_f = A, \quad B_p = B, \quad C_f = C, \quad S = I_n, \tag{11}
\]

we can see that the \( l_q/l_p \) Hankel norm \( \gamma_{q/p} \) defined by (4) reduces to the standard \( l_q/l_p \) Hankel norm of the system \( G \) which is denoted by \( \|G\|_{q/p} \). In the continuous-time system case, the \( l_q/l_p \) Hankel norms of general (i.e. nonpositive) LTI systems are studied by [25,26], and some of those results are made explicit for positive systems in [27]. Then, very recently, the counterpart results of [27] for discrete-time positive systems are derived in [28]. Some of them are given in the next proposition, where \( X \in \mathbb{S}^n \) and \( P \in \mathbb{S}^n \) stand for the controllability and observability Gramians of the system \( G \) given by (5), respectively. These are the unique solutions of the Lyapunov equations

\[
- X + AXA^T + BB^T = 0, \quad - P + A^TPA + C^TC = 0. \tag{12}
\]

**Proposition 3.5 ([28]):** For the stable and externally positive system \( G \) given by (5) and (8), we have

\[
\|G\|_{1/1} = | - A^T_n C(A - I)^{-1}B|_\infty. \tag{13}
\]

\[
\|G\|_{2/1} = \sqrt{\max_{k \geq 0} | g_k(0) |}. \tag{14}
\]

\[
\|G\|_{\infty/1} = \max_{k \geq 0} | g_k(0) |. \tag{15}
\]

\[
\|G\|_{1/2} = \sqrt{\lambda_{\max}(P^X)}. \tag{16}
\]

\[
\|G\|_{\infty/2} = \sqrt{\lambda_{\max}(CXC^T)}. \tag{17}
\]

\[
\|G\|_{1/\infty} = \sqrt{1_n^T C(A - I)^{-1}B1_{nw}}. \tag{19}
\]

\[
\|G\|_{2/\infty} = \sqrt{1_n^T B^T (A - I)^{-T}P(A - I)^{-1}B1_{nw}}. \tag{20}
\]

\[
\|G\|_{\infty/\infty} = | - C(A - I)^{-1}B1_{nw}|_\infty. \tag{21}
\]

The characterizations (14), (15), (17), and (18) are valid even for general LTI systems and these are direct counterparts of the continuous-time results in [25]. The rest characterizations are direct counterparts of [27]. We finally note that explicit closed-form characterization of \( \|G\|_{\infty/1} \) is hardly available even for internally positive systems.

**4. The \( l_q/l_p \) Hankel norms across switching: positive system results**

We say that the system switching described by (1), (2), and (3) is positive if the following conditions hold.

(i) Both systems \( \Sigma_p \) and \( \Sigma_f \) are stable and externally positive, i.e. (10) holds.

(ii) The matrix \( S \) in (3) is nonnegative, i.e.

\[
S \in \mathbb{R}^{n \times np}_+. \tag{22}
\]

In the following, we denote by \( X_p \in \mathbb{S}^{np} \) and \( P_t \in \mathbb{S}^{n_f} \) the controllability gramian of \( \Sigma_p \) and the observability Gramian of \( \Sigma_f \), respectively. These are the unique solutions of the Lyapunov equations

\[
- X_p + A_pX_pA_p^T + B_pB_p^T = 0, \quad - P_t + A_f^TP_tA_f + C_f^TC_f = 0.
\]

**4.1. Characterization of \( \gamma_{2/2} \) for general system switching**

We first note that \( \gamma_{2/2} \) is given by

\[
\gamma_{2/2} = \sigma_{\max}(P_t^{1/2}SP_p^{1/2}). \tag{23}
\]

This is the counterpart of the continuous-time system result by [22] and valid for general (i.e. nonpositive) switching. To provide a concise proof of (23), let us
denote by $\Sigma$ the linear operator from $w \in l_p^-$ to $z \in l_q^+$ in the system switching defined by (1), (2), (3). Namely, 
$$\Sigma : l_p^- \ni w \mapsto z \in l_q^+,$$

$$(\Sigma w)(k) = C_I A_f^k S \sum_{l=1}^{\infty} A_p^{l-1} B_p w(-l) = z(k) (k \geq 0).$$

(24)

Then the result (23) readily follows from (24) as 

$$\gamma_{2/2} = \sup_{w \in l_2^-, \|w\|_{l_2^-} = 1} \|z\|_{l_2^+}$$

$$= \sup_{w \in l_2^-, \|w\|_{l_2^-} = 1} \left\| P_{1/2}^{1/2} \sum_{l=1}^{\infty} A_p^{l-1} B_p w(-l) \right\|_{l_2}$$

$$= \sqrt{\lambda_{\max}(P_{1/2}^{1/2} S X_p S^T P_{1/2}^{1/2})}$$

$$= \sigma_{\max}(P_{1/2}^{1/2} S X_p^{1/2}).$$

The worst-case input $w^* \in l_{2^-}$ that attains (23) can be given explicitly by 

$$w^*(k) = B_p^T (A_p^{-1})^{k-1} v^* (k \leq -1)$$

where $v^* \in \mathbb{R}^p$ is the eigenvector corresponding to the eigenvalue $\lambda_{\max}(S X_p S^T P_{1/2}^{1/2})$ with $\|v^*\|_{l_2^+} = 1$. It is very clear that (23) reduces to (17) in the standard Hankel norm setting (11).

In the case where $(p, q) \neq (2, 2)$, on the other hand, explicit closed-form characterizations of $\gamma_{q/p}$ are hardly available for general switching. The difficulty in comparison with the standard $l_q/l_p$ Hankel norms comes from the time-varying nature of the underlying system in the switched case. However, in the case of positive system switching defined by (10) and (22), we can still derive explicit characterizations for some combinations of $(p, q)$. The key feature is that the linear operator $\Sigma$ given by (24) is positive under (10) and (22) in the sense that $w \in l_2^-$ leads to $z \in l_q^+$. This drastically facilitates the treatment of $\gamma_{q/p}$ particularly when $p = \infty$ and $q = 1$. The next lemma plays a key role in analyzing the $l_q/l_p$ Hankel norm $\gamma_{q/p}$ for the positive system switching. The proof of this lemma is given in the appendix section.

**Lemma 4.1:** For the positive system switching from $\Sigma_p$ to $\Sigma_f$ described by (1), (2), (3), (10), and (22), suppose there exists an input $w \in l_p^-$ with $\|w\|_{l_p^-} = 1$ such that the corresponding output $z \in l_q^+$ satisfies $\|z\|_{l_q^+} = \gamma$ for a given $\gamma > 0$. Then, there exists an input $\hat{w} \in l_p^-$ with $\|\hat{w}\|_{l_p^-} = 1$ such that the corresponding output $\hat{z} \in l_q^+$ satisfies $\|\hat{z}\|_{l_q^+} \geq \gamma$.

We also note that the next lemma holds. The proof of this lemma is given in the appendix section.

**Lemma 4.2:** For the positive system switching from $\Sigma_p$ to $\Sigma_f$ described by (1), (2), (3), (10), and (22), suppose there exists an input $w \in l_p^-$ with $\|w\|_{l_p^-} = 1$ such that the corresponding output $z \in l_q^+$ satisfies $\|z\|_{l_q^+} = \gamma$ for a given $\gamma > 0$. Then, there exists an input $\hat{w} \in l_p^-$ with $\|\hat{w}\|_{l_p^-} = 1$ such that the corresponding output $\hat{z} \in l_q^+$ satisfies $\|\hat{z}\|_{l_q^+} \geq \gamma$.

**4.2. Characterizations of $\gamma_{q/\infty}$ for positive system switching**

In the case where we consider the $l_q/l_p$ Hankel norms with $p = \infty$, we can readily see from Lemmas 4.1–4.3 that the next strong result holds.

**Lemma 4.3:** For the positive system switching from $\Sigma_p$ to $\Sigma_f$ described by (1), (2), (3), (10), and (22), suppose inputs $w_1, w_2 \in l_p^{-}$ yield outputs $z_1, z_2 \in l_q^+$, respectively. Then, if $w_1(k) \geq w_2(k)$ $(\forall k < 0)$, we have $\|z_1\|_{l_q^+} \geq \|z_2\|_{l_q^+}$.

**Lemma 4.4:** For the positive system switching from $\Sigma_p$ to $\Sigma_f$ described by (1), (2), (3), (10), and (22), the $l_q/l_p$ Hankel norms with $q$ being 1, 2, $\infty$ are attained by the input $w^* \in l_q^{-}$ given by $w^*(k) = 1_{n_w}$ $(\forall k < 0)$. This input leads to the initial state before switching

$$x_p(0) = -(A_p - I)^{-1} B_p 1_{n_w} \in \mathbb{R}_+^{n_q}$$

and the initial state after switching

$$x_f(0) = -S(A_p - I)^{-1} B_p 1_{n_w} \in \mathbb{R}_+^{n_q}.$$

From this lemma, we can readily obtain the next theorems.

**Theorem 4.5:** For the positive system switching from $\Sigma_p$ to $\Sigma_f$ described by (1), (2), (3), (10), and (22), we have

$$\gamma_{1/\infty} = 1_{n_f}^{T} C_f (A_f - I)^{-1} S (A_p - I)^{-1} B_p 1_{n_w}. \quad (28)$$

Moreover, the following conditions are equivalent for a given $\gamma > 0$.

(i) $\gamma_{1/\infty} < \gamma$.

(ii) There exists $F \in \mathbb{R}^{(n_p+n_r) \times (n_q+n_r+1)}$ such that

$$-2\gamma' \begin{bmatrix} 0 & \begin{bmatrix} 1_{n_q}^{T} C_f 0 \\ 0 & A_f - I \end{bmatrix} \\ B_p 1_{n_w} & 0 \end{bmatrix} + \text{He}
\begin{bmatrix} \begin{bmatrix} 1_{n_q}^{T} C_f 0 \\ 0 & A_p - I \end{bmatrix} \\ S \end{bmatrix} F < 0. \quad (29)$$

Furthermore, the next condition is also equivalent to (i) and (ii) if $\Sigma_p$ and $\Sigma_f$ are both internally positive, i.e. (9) holds.
(iii) There exist $f_p \in \mathbb{R}^{n_p}$, $f_0 \in \mathbb{R}^{n_0}$, and $f_I \in \mathbb{R}^{n_I}$ such that
\[
(A_p - I)f_p + B_p I_n_w < 0, \quad S f_p < f_0,
\]
\[
(A_I - I)f_I + f_0 < 0, \quad 1^T_{n_z} C_I f_I < \gamma. \quad (30)
\]

**Theorem 4.6:** For the positive system switching from $\Sigma_p$ to $\Sigma_f$ described by (1), (2), (3), (10), and (22), we have
\[
\gamma_{2/\infty} = \sqrt{1^T_{n_w} B_p^T (A_p - I) - T^T S P (A_p - I)^{-1} B_p 1_{n_w}}.
\]
Moreover, the following conditions are equivalent for a given $\gamma > 0$.

(i) $\gamma_{2/\infty} < \gamma$.

(ii) There exist $Q_I \in \mathbb{S}^{n_I}_{++}$, $F_I \in \mathbb{R}^{(n_p + n_I) \times (n_p + n_I + 1)}$ and $F_2 \in \mathbb{R}^{n_I \times 2n_I}$ such that
\[
\begin{align*}
-\gamma^2 & \quad 0 \\
0 & \quad 0_{n_p} \\
0 & \quad 0_{n_I} \\
& \quad Q_I + \text{He} \left( \begin{bmatrix} 1^T_{n_w} B_p^T (A_p - I) & 0 \\ (A_p - I)^T S & I_{n_I} \end{bmatrix} F_I \right) \begin{bmatrix} 0 \\ S \end{bmatrix} \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \begin{bmatrix} 0 \\ Q_I \end{bmatrix} \right) \begin{bmatrix} -Q_I + C_I^T C_I \\ 0 \\ 0 \\ Q_I \end{bmatrix} \begin{bmatrix} 0 \\ F_2 \end{bmatrix} < 0, \\
& \quad F_2 < 0. \quad (32)
\end{align*}
\]

**Theorem 4.7:** For the positive system switching from $\Sigma_p$ to $\Sigma_f$ described by (1), (2), (3), (10), and (22), we have
\[
\gamma_{\infty/\infty} = \max_{k_0 \geq 0} \left| -C_f A_f^k (A_p - I)^{-1} B_p 1_{n_w} \right|_{\infty}.
\]

The results (28), (31), and (33) readily follow by considering the initial response corresponding to the initial state (27). The proof for the equivalence of (i), (ii), and (iii) in Theorem 4.5 and (i) and (ii) in Theorem 4.6 are given in the appendix section. Important remarks on Theorems 4.5, 4.6, and 4.7 are as follows.

**Remark 4.1:** (i) It is clear that (28) and (31) reduce to (19) and (20), respectively, in the case of (11). On the other hand, (33) looks much more complicated than (21), and we see that (21) can be obtained from (33) by assuming (11) and the maximum in (33) is attained at $k_f = 0$. In the time-invariant case (11), it is allowed to consider the “shift” of the input signal $w$ due to the time-invariant nature of the system and this intuitively explains the reason why the maximum is attained at $k_f = 0$. In fact, if we assume (11) in (33), we have
\[
\begin{align*}
-CA^k (A - I)^{-1} B 1_{n_w} = -CA^k (A - I)^{-1} B 1_{n_w} \\
-CA^{k+1} (A - I)^{-1} B 1_{n_w}
\end{align*}
\]
and hence the maximum is actually attained at $k_f = 0$. However, in the switched case, the intrinsic time-varying nature of the system does not allow us to conclude in such a way and we have to take the maximum over $k_f \geq 0$ as in (33).

(ii) The SDP- and LP-based characterizations (29) and (30) are useful in analysing the $l_1/l_\infty$ Hankel norm where the positive systems $\Sigma_p$ and $\Sigma_f$ as well as the matrix $S$ are affected by parametric uncertainties. Similarly for (32). See Section 5 for concrete examples.

### 4.3. Characterizations of $\gamma_{1/\rho}$ for positive system switching

When considering the $l_\rho/l_\infty$ Hankel norms for the positive system switching, we can confine ourselves to nonnegative input signals from Lemma 4.2. This leads to $x_p(0) \in \mathbb{R}^{n_p}_{++}$ and $x_I(0) = S x_p(0) \in \mathbb{R}^{n_I}_{++}$, and hence we have $z(k) = CA^k x_p(0) \in \mathbb{R}^{n_z}_{++}$ ($V_k \geq 0$). It follows that $\|z\|_{1+} = -I^T_{n_z} C_f (A_f - I)^{-1} S x_p(0)$. Namely, we can characterize $\gamma_{1/\rho}$ as follows:
\[
\gamma_{1/\rho} = \sup_{w \in l_\rho_{++}, \|w\|_{l_\rho} = 1} \left| -I^T_{n_z} C_f (A_f - I)^{-1} S \right| \times \sum_{l=1}^{\infty} A_{l-1}^{-1} B_p w(-I).
\]
From this expression, we can see that $\gamma_{1/\rho}$ is identical to the $l_{\infty/\rho}$ Hankel norm $\|G\|_{\infty/\rho}$ of the single-output, stable and externally positive LTI system $G$ given by
\[
\tilde{G}(z) := \begin{bmatrix} A_p & B_p \\ -I^T_{n_z} C_f (A_f - I)^{-1} S & 0 \end{bmatrix}.
\]
From this key observation and Proposition 3.5, we can obtain the next theorems.

**Theorem 4.8:** For the positive system switching from $\Sigma_p$ to $\Sigma_f$ described by (1), (2), (3), (10), and (22), we have
\[
\gamma_{1/1} = \max_{k_\rho \geq 0} \left| -I^T_{n_z} C_f (A_f - I)^{-1} S \right|_{l_\rho} |A_{l-1}^{-1} B_p|_{\infty}. \quad (34)
\]

**Theorem 4.9:** For the positive system switching from $\Sigma_p$ to $\Sigma_f$ described by (1), (2), (3), (10), and (22), we have
\[
\gamma_{1/2} = \sqrt{I^T_{n_z} C_f (A_f - I)^{-1} S A_{l-1}^{-1} S f_p S^T (A_f - I)^{-1} C_I^T 1_{n_z}}. \quad (35)
\]
Moreover, the following conditions (i) and (ii) are equivalent for a given $\gamma > 0$. 
There exist $Y_p \in \mathbb{S}^{n_p+}_+$, $F_1 \in \mathbb{R}^{(n_1+n_p) \times (n_1+n_p+1)}$ and $F_2 \in \mathbb{R}^{n_p \times 2n_p}$ such that

$$
\begin{bmatrix}
-\gamma^2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & Y_p
\end{bmatrix}
+ \text{He}
\begin{bmatrix}
I_{n_1}^T C_f & 0 \\
A_f - I & S \\
0 & I_{n_p}
\end{bmatrix}
F_1 < 0,
$$

and

$$
\begin{bmatrix}
-Y_p + B_p B_p^T & 0 \\
0 & 0
\end{bmatrix}
+ \text{He}
\begin{bmatrix}
A_p & 0 \\
0 & -I_{n_p}
\end{bmatrix}
F_2 < 0.
$$

(36)

We can verify (34) from $\gamma_{1/1} = \|\hat{G}\|_{\infty/1}$ and (15). Similarly, we can verify (35) from $\gamma_{1/2} = \|\hat{G}\|_{\infty/2}$ and (18). The equivalence of (i) and (ii) in Theorem 4.6 follows by exactly the same argument used in the proof of the equivalence of (i) and (ii) in Theorem 4.9. It should be noted that the expression of $\gamma_{1/\infty}$ given by (28) can also be obtained from the fact that $\gamma_{1/\infty} = \|\hat{G}\|_{\infty/\infty}$ and (21). Important remarks on Theorems 4.8 and 4.9 are as follows.

**Remark 4.2:**

(i) We can see that (35) reduces to (16) in the case of (11). On the other hand, we see that (34) can be reduced to (13) by assuming (11) and the maximum in (34) is attained at $k_p = 0$. In fact, in the case where (11) holds in (34), we see that

$$
(-1)_{x_{n_1}}^TC(A-I)^{-1}A^{kp}B
- (-1)_{x_{n_1}}^TC(A-I)^{-1}A^{kp+1}B
= -1_{x_{n_1}}^TC(A-I)^{-1}(A-I)A^{kp}B
= 1_{x_{n_1}}^TC A^{kp} B
\geq 0
$$

and hence the maximum is actually attained at $k_p = 0$.

(ii) The worst-case input $w^* \in l_{-\infty}$ that attains (35) can be given explicitly by

$$
w^*(k) = -\frac{B_p^T A_p^{-k} S^T (A_f - I)^{-T} C_f^T 1_{n_x}}{\sqrt{1_{x_{n_1}}^T C_f (A_f - I)^{-1} S^T S (A_f - I)^{-T} C_f^T 1_{n_x}}} (k < 0).
$$

(iii) Obviously, the duality holds between $\gamma_{2/\infty}$ given by (31) and $\gamma_{1/2}$ given by (35). Namely, we see that the $l_2/\infty_1$ Hankel norm for the positive system switching from $\Sigma_p$ to $\Sigma_f$ via $S \in \mathbb{R}^{n_p \times n_p}$ is equivalent to the $l_1/l_2$ Hankel norm on the positive system switching from $\Sigma_f$ to $\Sigma_p$ via $S^T \in \mathbb{R}^{n_p \times n_p}$ where

$$
\tilde{\Sigma}_f: \tilde{\xi}_p(k+1) = A_f^T \tilde{\xi}_p(k) + C_f^T \tilde{w}(k) (k \leq 0)
$$

and

$$
\tilde{\Sigma}_p: \left\{ \begin{array}{ll}
\tilde{\xi}_f(k+1) = A_f^T \tilde{\xi}_f(k), \\
\tilde{z}(k) = B_f^T \tilde{\xi}_f(k) (k \geq 0)
\end{array} \right.
$$

(38)

### 4.4. The Hankel norms $\gamma_{\infty/1}$, $\gamma_{\infty/2}$ and $\gamma_{2/1}$ for general switching

In this subsection, we provide explicit characterizations of $\gamma_{\infty/1}$, $\gamma_{\infty/2}$, and $\gamma_{2/1}$. The results in this subsection can be derived without relying on the positivity and hence they are valid even for general (i.e. nonpositive) switching cases.

#### 4.4.1. Characterization of $\gamma_{\infty/1}$ and $\gamma_{\infty/2}$

For the characterization of $\gamma_{\infty/p}$ ($p = 1, 2$), for each $k_f \geq 0$, let us define

$$
v_{\infty/p}(k_f) := \sup_{w \in l_{-p}, \|w\|_{l_{-p}} = 1} \|z(k_f)\|_{\infty}
$$

s.t. (1), (2), (3).

(39)

Then, we have

$$
\gamma_{\infty/p} = \max_{k_f \geq 0} v_{\infty/p}(k_f).
$$

(40)

On the other hand, in view of the fact that $z(k)$ ($k \geq 0$) can be written explicitly as (24), let us define for each $k_f \geq 0$ the LTI positive system $	ilde{F}_{k_f}$ by

$$
\tilde{F}_{k_f}(z) := \begin{bmatrix} A_p & B_p \\ C_f A_f^k S & 0 \end{bmatrix} (k_f \geq 0).
$$

(41)

Then, we see from (39), (24), (41), and (15) that

$$
v_{\infty/1}(k_f) = \|\tilde{F}_{k_f}\|_{\infty/1} = \max_{k_p \geq 0, j} \|C_{i_f} A_{i_f}^k S P_{i_f}^j B_{j_f} \|_{1,2},
$$

(42)

Similarly, we see from (39), (24), (41), and (18) that

$$
v_{\infty/2}(k_f) = \|\tilde{F}_{k_f}\|_{\infty/2} = \sqrt{d_{\max}(C_{i_f} A_{i_f}^k S P_{i_f} S^T (A_f^k)^{j_f} C_f^T)}.
$$

(43)

It follows from (40), (42) and (43) that the next results hold.

**Theorem 4.10:** For the system switching from $\Sigma_p$ to $\Sigma_f$ described by (1), (2), and (3), we have

$$
\gamma_{\infty/1} = \max_{k_f \geq 0} \max_{k_p \geq 0} \max_{i_f,j_f} \|C_{i_f} A_{i_f}^k S P_{i_f}^j B_{j_f} \|_{1,2},
$$

(44)

$$
\gamma_{\infty/2} = \max_{k_f \geq 0} \sqrt{d_{\max}(C_{i_f} A_{i_f}^k S P_{i_f} S^T (A_f^k)^{j_f} C_f^T)}.
$$

(45)
Remark 4.3: (i) We note that the expression of \( \gamma_{\infty}/\infty \) given by (33) can also be obtained from the fact that

\[
\gamma_{\infty}/\infty = \max_{k_1 \geq 0} \nu_{\infty}/\infty(k_1) = \max_{k_1 \geq 0} \|P_{k_1}\|_{\infty}/\infty
\]

and (21).

(ii) We can see that (44) reduces to (15) in the case of (11). On the other hand, we see that (45) can be reduced to (18) by assuming (11) and the maximum in (45) is attained at \( k_f = 0 \). In fact, in the case where (11) holds in (45), we see that

\[
\begin{align*}
CA_k X(A_k)^T C^T - CA_{k+1} X(A_{k+1})^T C^T &= CA_k X - AXA^T(A_k)^T C^T \\
&= CA_k BB^T(A_k)^T C^T \\
&\geq 0
\end{align*}
\]

(46)

and hence the maximum is actually attained at \( k_f = 0 \).

4.4.2. Characterization of \( \gamma_{2/1} \)

We next consider the characterization of \( \gamma_{2/1} \). To this end, we define

\[
\begin{align*}
\|z\|_{(\infty,2)+} &:= \sup_{0 \leq k < \infty} |z(k)|_2, \\
l_{(\infty,2)+} &:= \left\{ z : \|z\|_{(\infty,2)+} < \infty \right\}.
\end{align*}
\]

Then we can obtain the next lemma that is the discrete-time system counterpart of the result in [29] dealing with continuous-time systems.

Lemma 4.11: Let us consider the stable LTI system \( G \) given by (5) with \( x(0) = 0 \) and define its induced norm \( \|G\|_{(\infty,2)/1} \) from \( w \in l_{1+} \) to \( z \in l_{(\infty,2)+} \) by

\[
\|G\|_{(\infty,2)/1}^{ind} := \sup_{w \in l_{1+}, \|w\|_{1+} = 1} \|z\|_{(\infty,2)+}.
\]

Then, we have

\[
\|G\|_{(\infty,2)/1}^{ind} = \max_{k_p \geq 0} \sqrt{\inf_{l_{1+} = 1} d_{\max}(B^T(A_k)^T \gamma_k C^T CA_k B)}.
\]

We now go back to the analysis of Hankel norm \( \gamma_{2/1} \). If we define \( \tilde{C}_f := P_{1/2}^T \in R^{n_l \times n_p} \), we can see from (24) that

\[
\begin{align*}
\gamma_{2/1} &= \sup_{w \in l_{1+}, \|w\|_{1+} = 1} \left| \tilde{C}_f \sum_{l=1}^{\infty} A_{l}^{-1} B_{w}(-l) \right|_2 \\
&= \sup_{w \in l_{1+}, \|w\|_{1+} = 1} \left| \tilde{C}_f \sum_{l=1}^{\infty} A_{l}^{-1} B_{w}(l) \right|_2 \\
&= \sup_{w \in l_{1+}, \|w\|_{1+} = 1} \left| \tilde{C}_f \sum_{l=0}^{\infty} A_{l}^T B_{w}(l) \right|_2
\end{align*}
\]

and (47)

Remark 4.4: (i) We can see that (47) reduces to (14) by assuming (11) and the maximum in (47) is attained at \( k_p = 0 \). We can verify this similarly to (46).

(ii) Obviously, the duality holds between \( \gamma_{\infty/2} \) given by (45) and \( \gamma_{2/1} \) given by (47).

5. Numerical examples

Let us consider the case where the systems \( \Sigma_p \), \( \Sigma_f \), and the matrix \( S \) in (1), (2), and (3), respectively, are affected by polytopic-type uncertainty of the form

\[
\begin{bmatrix}
C_{\ell} & 0 & 0 \\
A_{\ell} & S & 0 \\
0 & A_p & B_p
\end{bmatrix}
\in \sum_{l=1}^{N} \alpha_l \begin{bmatrix}
C_{l}^{[l]} & 0 & 0 \\
A_{l}^{[l]} & S^{[l]} & 0 \\
0 & A_p^{[l]} & B_p^{[l]}
\end{bmatrix} : \alpha \in \alpha
\]

Here, we assume that the given matrices \( A_{l}^{[l]} \), \( B_{p}^{[l]} \), \( A_{p}^{[l]} \), \( C_{l}^{[l]} \), and \( S^{[l]} \) (\( l = 1, \ldots, N \) ) that define the vertices of the polytope satisfy \( A_{l}^{[l]} \in R_{+}^{n_l \times n_p} \), \( B_{p}^{[l]} \in R_{+}^{n_p \times n_w} \), \( A_{p}^{[l]} \in R_{+}^{n_l \times n_t} \), \( C_{l}^{[l]} \in R_{+}^{n_t \times n_t} \), and \( S^{[l]} \in R_{+}^{n_t \times n_t} \). In the following, we denote by \( \Sigma_{p,\alpha} \), \( \Sigma_{f,\alpha} \), and \( S_{\alpha} \) the positive systems and the nonnegative matrix corresponding to the parameter \( \alpha \in \alpha \). We assume that both \( \Sigma_{p,\alpha} \) and \( \Sigma_{f,\alpha} \) are stable for any \( \alpha \in \alpha \). Under these assumptions, we denote by \( \gamma_{q/p}(\alpha) \) the \( l_q/l_p \) Hankel norm for the positive system switching from \( \Sigma_{p,\alpha} \) to \( \Sigma_{f,\alpha} \) via \( S_{\alpha} \).

The problem we consider in this section is to compute the worst case \( l_q/l_p \) Hankel norm \( \gamma_{q/p}^{*} \) defined by \( \gamma_{q/p}^{*} := \max_{\alpha \in \alpha} \gamma_{q/p}(\alpha) \). Even though exact and efficient computation of \( \gamma_{q/p}^{*} \) is hard, we can compute its upper bound efficiently by using the SDP and LP characterizations provided in the preceding section. For instance, from (29), we see that we can obtain an upper bound of \( \gamma_{q/p}^{*} \) by solving the following SDP:

\[
\begin{align*}
\gamma_{1/\infty}^{*} := \inf_{\gamma \in \gamma_{p}^{*}} \gamma & \text{ subject to }
\end{align*}
\]
As a concrete example, let us consider the case where $N = 2$ and
\[
\begin{pmatrix}
-2\gamma & 0 & 1_n^T P_0 P_1 & T \\
0 & 0 & 0 & 0 \\
B_0^T & 1_n & 0 & 0 \\
\end{pmatrix} + \text{He}
\begin{pmatrix}
1_n^T C_0 & 0 & 0 & 0 \\
A_0 & -I & S_0 & I \\
0 & 0 & A_0 & -I \\
\end{pmatrix} F < 0
\]
(\(l = 1, \ldots, N\)).

\section{Conclusion}

In this paper, we analysed the $l_q/l_p$ Hankel norms of positive systems across a single switching. We derived explicit representations of the $l_q/l_p$ Hankel norms for $p$, $q$ being 1, 2, $\infty$, where those new results for $(q, p) = \{(\infty, 1), (\infty, 2), (2, 1)\}$ are valid even for general (nonpositive) switching cases. In particular, for $(q, p) = \{(1, \infty), (2, \infty), (1, 2)\}$, we provided LP- and SDP-based characterizations. By numerical examples, we illustrated the usefulness of the SDP-based characterizations for the analysis of the $l_q/l_p$ Hankel norms where the systems of interest are affected by parametric uncertainties.

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\section*{Notes on contributor}

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Here in the fourth equality, we used the fact that \((\Sigma w_\pm)(k) \geq 0 \quad (\forall k \geq 0)\) holds for \(w_\pm \in l^\infty_{\mathbb{R}^+}\). This completes the proof.

\section*{Appendix 2. Proof of Lemma 4.3}

\textbf{Proof of Lemma 4.3:} For \(w_1, w_2 \in l^\infty_{\mathbb{R}^+}\) with \(w_1(k) \geq w_2(k) \geq 0 \quad (\forall k < 0)\), we have \(z_1(k) \geq z_2(k) \geq 0 \quad (\forall k \geq 0)\) and hence \(\|z_1\|_{l_q^+} \geq \|z_2\|_{l_q^+}\) holds.

\section*{Appendix 3. Proof of Theorem 4.5}

In the following, for a matrix \(E \in \mathbb{R}^{n \times n}\) with \(n \geq m\) and \(\text{rank}(E) = r\), we denote by \(E^+\) a full-rank matrix such that \(E^+ \in \mathbb{R}^{(m-r) \times n}\) and \(E^+ E = 0\).

\textbf{Proof of Lemma Theorem 4.5:} In the following, we prove the equivalence of (i)-(iii).

\[(i) \iff (ii)\] In view of the fact that
\[
\begin{bmatrix}
I_{n^2} & C_f & 0 \\
A_f - I & S & 0 \\
0 & A_p - I
\end{bmatrix}^+ \]
\[
= \begin{bmatrix} 1 - 1_{n^2} C_f (A_f - I)^{-1} 1_{n^2} C_f (A_f - I)^{-1} S (A_p - I)^{-1} \end{bmatrix},
\]
the equivalence (i)\iff(ii) follows from (28) and \(S\)-variable LMI results shown in [30].

\[(i) \implies (ii)\] We first prove (i)\implies(ii). To this end, suppose (i) holds, i.e. \(1_{n^2} C_f (A_f - I)^{-1} S (A_p - I)^{-1} B_p 1_{n_p} < \gamma^t\). Then, there exist sufficiently small \(\varepsilon > 0\) \((i = 1, 2, 3)\) and \(f_p \in R^m_{\mathbb{R}^+}, f_0 \in R^m_{\mathbb{R}^+}\), and \(f_1 \in R^m_{\mathbb{R}^+}\) such that
\[
(A_p - I) f_p = -(A_f - I)^{-1} B_p 1_{n_p} - \varepsilon_1 1_{n_p}, \quad f_0 = S f_p + \varepsilon_2 1_{n_p}, \\
(A_f - I) f_t = -(A_f - I)^{-1} B_f 1_{n_p},
\]
\[
1_{n^2} C_f g_{f_t} < \gamma.
\]
Here we used the fact that \((A_p - I)(A_f - I)^{-1} \leq 0\) and \((A_f - I)(A_f - I)^{-1} \leq 0\). It is clear that \(f_p \leq f_p, f_0 \leq f_0,\) and \(f_1 \leq f_1\) satisfy (30) and hence (ii) hold. We next prove that (ii)\implies(i). To this end, we note from (30) that
\[
1_{n^2} C_f g_{f_t} < \gamma, \quad f_1 > -(A_f - I)^{-1} f_0, \quad S f_p < f_0, \quad f_p > -(A_p - I)^{-1} B_p 1_{n_p}.
\]
It follows that the proof can be completed by
\[
\gamma > 1_{n^2} C_f g_{f_t} \geq -1_{n^2} C_f (A_f - I)^{-1} f_0 \geq -1_{n^2} C_f (A_f - I)^{-1} S f_p \geq 1_{n^2} C_f (A_f - I)^{-1} S (A_p - I)^{-1} B_p 1_{n_p}.
\]

\section*{Appendix 4. Proof of Theorem 4.6}

\textbf{Proof of Theorem 4.6:} In the following, we prove the equivalence of (i) and (ii).
(i)⇔(ii) In view of the fact that
\[
\begin{bmatrix}
1_{n_w}^T B_p^T & 0 \\
(A_p - I)^T & S^T
\end{bmatrix}
\begin{bmatrix}
w_n & 0 \\
0 & I_{n_t}
\end{bmatrix}
= \begin{bmatrix}
1 & -1_{n_w}^T B_p^T (A_p - I)^{-T} \\
0 & 1_{n_t}
\end{bmatrix}
= \begin{bmatrix}
A_f^T & 0 \\
-I_{n_t}
\end{bmatrix}
\begin{bmatrix}
I_{n_t} & A_f^T
\end{bmatrix},
\]
we see from $S$-variable LMI results shown in [30] that (ii) holds if and only if there exists $Q_f \in S_{++}^{n_f}$ such that
\[-\gamma^2 + 1_{n_w}^T B_p^T (A_p - I)^{-T} S^T Q_f S (A_p - I)^{-1} B_p 1_{n_w} < 0,
- Q_f + A_f^T Q_f A_f + C_f^T C_f < 0.
\]
Then, it is an elementary fact that the above condition holds if and only if (i) with (31) holds. □