Specifying nonlocality of a pure bipartite state and analytical relations between measures for bipartite nonlocality and entanglement

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Abstract

For a multipartite quantum state, the maximal violation of all Bell inequalities constitutes a measure of its nonlocality [Loubenets, J. Math. Phys. 53, 022201 (2012)]. In the present article, for the maximal violation of Bell inequalities by a pure bipartite state, possibly infinite-dimensional, we derive a new upper bound expressed in terms of the Schmidt coefficients of this state. This new upper bound allows us also to specify general analytical relations between the maximal violation of Bell inequalities by a bipartite quantum state, pure or mixed, and such entanglement measures for this state as "negativity" and "concurrence". To our knowledge, no any general analytical relations between measures for bipartite nonlocality and entanglement have been reported in the literature though, for a general bipartite state, specifically such relations are important for the entanglement certification and quantification scenarios. As an example, we apply our new results to finding upper bounds on nonlocality of bipartite coherent states intensively discussed last years in the literature in view of their experimental implementations.

1 Introduction

Ever since the seminal paper of Bell [11] quantum violation of Bell inequalities was analyzed, analytically and numerically, in many papers and is now used in many quantum information processing tasks. It is well known that quantum violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [2] cannot exceed $\sqrt{2}$ for any bipartite quantum state, possibly infinite-dimensional. It was also recently proved [7, 8] that the maximal quantum violation of the original Bell inequality [1] is equal to $\frac{3}{2}$.

¹For the original Bell inequality see also [9].
More generally, quantum violation of any (unconditional) correlation bipartite Bell inequality cannot exceed the real Grothendieck’s constant $K_G^{(R)} \in [1.676, 1.783]$ but this is not already the case for quantum violation of a bipartite Bell inequality on joint probabilities and, more generally, quantum violation of a Bell inequality of an arbitrary form, a general bipartite Bell inequality [10], and last years finding upper bounds on violation by a bipartite quantum state of any general Bell inequality was intensively discussed within different mathematical approaches [11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

At present, the following general analytical results on quantum violation of bipartite Bell inequalities are known in the literature:

(i) for an arbitrary bipartite state, pure or mixed, on a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, $\dim \mathcal{H}_n = d_n$, violation of any general Bell inequality with $S_n$ settings at $n = 1, 2$ sites cannot exceed

$$2 \min\{d_1, d_2, S_1, S_2\} - 1$$

– in case of generalized quantum measurements (see Eq. 64 in [13] and [21, 22])

$$\min\{d^S, 3\}, \quad \text{for } S = 2,$$

$$\min\{d^S, 2 \min\{d, S\} - 1\}, \quad \text{for } S \geq 3,$$

– in case of projective quantum measurements and $S$ settings per site;

(ii) for the two-qudit Greenberger-Horne-Zeilinger (GHZ) state $\frac{1}{\sqrt{d}} \sum_{j=1}^{d} |j\rangle \otimes |j\rangle$, violation of any general Bell inequality with an arbitrary number of measurement settings at each of sites admits (Theorem 0.3 in [17]) cannot exceed the bound $Cd/\sqrt{\ln d}$ where $C$ is an unknown constant independent on a dimension $d$.

From (1) it follows that, for every bipartite quantum state, possibly infinite-dimensional, violation of any general Bell inequality with fixed numbers $S_1, S_2 \geq 1$ of settings at each of two sites is upper bounded by the value

$$2 \min\{S_1, S_2\} - 1,$$

whereas, for an arbitrary finite-dimensional bipartite state, pure or mixed, violation of any general Bell inequality is bounded from above by

$$2 \min\{d_1, d_2\} - 1.$$

In the present article, based on the local quasi hidden variable (LqHV) formalism developed in [13, 14, 15], for violation of any general Bell inequality by a pure bipartite state we find a new upper bound which is expressed via the Schmidt coefficients of this pure state and is tighter than the upper bound (1) valid for any state, pure or mixed.

Based on this new general result, we further specify for a general bipartite quantum state the analytical relations between its maximal violation of Bell inequalities on one side and “negativity” and “concurrence” of this state from the other side. To our knowledge, no any general analytical relations between measures for bipartite non-locality and entanglement have been reported in the literature, though, for a general

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2This follows from the definition of the Grothendieck’s constant $K_G^{(R)}$ and Theorem 2.1 in [4].
bipartite state, such relations are specifically important for finding the minimal amount of entanglement via the collected experimental data on Bell violation – the goals of the semi-device-independent scenario and the device-independent scenario for the entanglement certification and quantification [23, 24, 25, 26].

As an example, we apply our new results to finding the upper bounds on nonlocality of bipartite entangled coherent states intensively discussed last years in the literature in view of their experimental implementations, see [27] and references therein.

The article is organized as follows.

In Section 2, we recall the notion of a general Bell inequality and specify (i) the state parameter characterizing nonlocality a quantum state and (ii) the analytical upper bound (Theorem 1) on the maximal violation of Bell inequalities by a bipartite quantum state derived in [13].

In Section 3, based on Theorem 1, we derive for the maximal violation of Bell inequalities by an arbitrary pure state new upper bounds (Theorem 2, Corollary 1) expressed in terms of the Schmidt coefficients of this state.

In Section 4, we specify the general analytical relations (Proposition 1 and Theorem 3) between measures for nonlocality and entanglement of a general bipartite state, pure or mixed.

In Section 5, we apply our new results to finding upper bounds (Proposition 2) on the maximal violation of Bell inequalities by bipartite coherent states.

In Section 6, we summarize the main results of the present article.

2 Preliminaries: general Bell inequalities and nonlocality of a quantum state

Consider a general bipartite correlation scenario where each of two participants performs $S_n \geq 1$, $n = 1, 2$, different measurements, indexed by numbers $s_n = 1, ..., S_n$ and with outcomes $\lambda_n \in \Lambda$. We refer to this correlation scenario as $S_1 \times S_2$-setting and denote by $P_{(s_1,s_2)}(\cdot)$ the probability distribution of outcomes $(\lambda_1, \lambda_2) \in \Lambda := \Lambda_1 \times \Lambda_2$ under the joint measurement specified by a tuple $(s_1, s_2)$ of settings where each $n$-th participant performs a measurement $s_n$ at the $n$-th site. The complete probabilistic description of such an $S_1 \times S_2$-setting correlation scenario is given by the family

$$P_{S,\Lambda} := \{P_{(s_1,s_2)} \mid s_n = 1, ..., S_n, \quad n = 1, 2\}, \quad (5)$$

of joint probability distributions. A correlation scenario admits a local hidden variable (LHV) model if each of its joint probability distributions $P_{(s_1,s_2)} \in P_{S,\Lambda}$ admits the representation

$$P_{(s_1,s_2)}(d\lambda_1 \times d\lambda_2) = \int_{\Omega} P_{1,s_1}(d\lambda_1|\omega) \cdot P_{2,s_2}(d\lambda_2|\omega) \nu(d\omega) \quad (6)$$

On the probabilistic description of a general correlation scenario, see [23].

For the definition of this notion under a general correlation scenario, see Definition 4 in [28].
in terms of a unique probability distribution $\nu(d\omega)$ of some variables $\omega \in \Omega$ and conditional probability distributions $P_{n,s_n}(\cdot|\omega)$, referred to as “local” in the sense that each $P_{n,s_n}(\cdot|\omega)$ at $n$-th site depends only on a measurement $s_n = 1, ..., S_n$ at an $n$-th site.

Under an $S_1 \times S_2$-setting correlation scenario described by a family of joint probability distributions \[\mathcal{F},\] consider a linear combination \[\Phi_{S,A} := \sum_{s_1,s_2} \langle \phi(s_1,s_2)(\lambda_1,\lambda_2) \rangle_{P_{s_1,s_2}} \] of the mathematical expectations

\[\langle \phi(s_1,s_2)(\lambda_1,\lambda_2) \rangle_{P_{s_1,s_2}} := \int \phi(s_1,s_2)(\lambda_1,\lambda_2) P_{s_1,s_2}(d\lambda_1 \times d\lambda_2) \]

of an arbitrary form, specified by a collection

\[\Phi_{S,A} = \{\phi(s_1,s_2) : \Lambda \rightarrow \mathbb{R} | s_n = 1, ..., S_n; \ n = 1, 2\} \]

of bounded real-valued functions $\phi(s_1,s_2)$ on $\Lambda = \Lambda_1 \times \Lambda_2$. Depending on a choice of a bounded function $\phi(s_1,s_2)$ and types of outcome sets $\Lambda_n$, $n = 1, 2$, expression \[\Phi_{S,A}\] can constitute either the probability of some observed event or if $\Lambda_n \subset \mathbb{R}$, $n = 1, 2$, the mathematical expectation (mean) of the product of observed outcomes (called in quantum information as a correlation function) or have a more complicated form.

If an $S_1 \times S_2$-setting correlation scenario \[\mathcal{F}\] admits the LHV modelling in the sense of representation \[\mathcal{A}\], then every linear combination \[\Phi_{S,A}\] of its mathematical expectations \[\Phi_{S,A}\] satisfies the “tight” LHV constraints \[\Phi_{S,A} \leq B_{\Phi,S,A}(\mathcal{P}_{S,A}) \]

\[\sup_{\mathcal{P}_{S,A} \in \mathcal{F}} \Phi_{S,A}(\mathcal{P}_{S,A}) \leq B_{\Phi,S,A}(\mathcal{P}_{S,A}) \]

\[B_{\Phi,S,A}(\mathcal{P}_{S,A}) = \sup_{\mathcal{P}_{S,A} \in \mathcal{F}} \Phi_{S,A}(\mathcal{P}_{S,A}) = \sup_{\Lambda_n(s_n) \in \Lambda_n, \forall s_n, n=1,2} \sum_{s_1,s_2} \phi(s_1,s_2)(\lambda_1^{(s_1)},\lambda_2^{(s_2)}), \]

\[\inf_{\mathcal{P}_{S,A} \in \mathcal{F}} B_{\Phi,S,A}(\mathcal{P}_{S,A}) = \sup_{\mathcal{P}_{S,A} \in \mathcal{F}} \Phi_{S,A}(\mathcal{P}_{S,A}) \]

\[\inf_{\Lambda_n(s_n) \in \Lambda_n, \forall s_n, n=1,2} \sum_{s_1,s_2} \phi(s_1,s_2)(\lambda_1^{(s_1)},\lambda_2^{(s_2)}). \]

\[\text{Here, the word a “tight” LHV constraint means that, in a LHV frame, the bounds established by this constraint cannot be improved. On the difference between the terms an “extreme” LHV constraint and a “tight” LHV constraint see section 2.1 of \[10\].} \]
Here, $\mathcal{G}^{lhv}_{S,A}$ denotes the set of all families of joint probability distributions describing $S_1 \times S_2$-setting correlation scenarios with outcomes in $\Lambda = \Lambda_1 \times \Lambda_2$ admitting the LHV modeling.

Depending on a form of functional (7), which is specified by a family $\Phi_{S,A}$ of bounded functions (10), some of the LHV constraints in (10) can hold for a wider (than LHV) class of correlation scenarios, some may be simply trivial, i.e. fulfilled under all correlation scenarios.

**Definition 1 [10, 13]** Each of the tight linear LHV constraints in (10) that can be violated under a non-LHV correlation scenario is referred to as a general Bell inequality. Bell inequalities on correlation functions (like the CHSH inequality) and Bell inequalities on joint probabilities constitute particular classes of general Bell inequalities.

If, under a bipartite correlation scenario, all joint measurements $(s_1, s_2)$ are performed on a quantum state $\rho$ on a Hilbert space $H_1 \otimes H_2$, then each joint probability distribution $P_{(s_1, s_2)}$ in (13) takes the form

$$P_{(s_1, s_2)}(d\lambda_1 \times d\lambda_2) = \text{tr}[\rho\{M_1^{(s_1)}(d\lambda_1) \otimes M_2^{(s_2)}(d\lambda_2)]],$$

where $M_n^{(s_n)}(\cdot)$, $M_n^{(s_n)}(\Lambda_n) = I_{H_n}$, is a normalized positive operator-valued (POV) measure, describing $s_n$-th quantum measurement at $n$-th site. For this correlation scenario, we denote the family (13) of joint probability distributions by

$$P_{(\rho, m_{S,A})}^{(\rho, m_{S,A})} := \{\text{tr}[\rho\{M_1^{(s_1)}(d\lambda_1) \otimes M_2^{(s_2)}(d\lambda_2)]], s_n = 1, \ldots, S_n, \ n \in 1, 2\},$$

where

$$m_{S,A} := \{M_n^{(s_n)} | s_n = 1, \ldots, S_n, \ n \in 1, 2\}$$

is the collection of all local POV measures at two sites, describing this quantum correlation scenario.

For a quantum $S_1 \times S_2$-setting correlation scenario [13] performed on a state $\rho$ on $H_1 \otimes H_2$, possibly infinite-dimensional, every linear combination (7) of its mathematical expectations (8) satisfies the \textit{“tight”} constraints [13]:

$$B_{\Phi_{S,A}}^{inf} - \frac{\gamma_{S_1 \times S_2}(\rho, \Lambda)}{2}(B_{\Phi_{S,A}}^{sup} - B_{\Phi_{S,A}}^{inf}) \leq B_{\Phi_{S,A}}(P_{S,A}^{\rho, m_{S,A}}) \leq B_{\Phi_{S,A}}^{sup} + \frac{\gamma_{S_1 \times S_2}(\rho, \Lambda)}{2}(B_{\Phi_{S,A}}^{sup} - B_{\Phi_{S,A}}^{inf}),$$

where

$$1 \leq \gamma_{S_1 \times S_2}(\rho, \Lambda) := \sup_{m_{S,A}, \Phi_{S,A}, \mathcal{G}^{lhv}_{S,A} \neq 0} \frac{B_{\Phi_{S,A}}(P_{S,A}^{\rho, m_{S,A}})}{B_{\Phi_{S,A}}^{lhv}}$$

See Eq. (48) and Lemma 3 in [13].
is the maximal violation by a state $\rho$ of all $S_1 \times S_2$-setting general Bell inequalities with outcomes $(\lambda_1, \lambda_2) \in \Lambda$.

Denote by $\Upsilon$ in Eq. (17)

$$1 \leq \Upsilon_{S_1 \times S_2}^{(\rho)} := \sup_{\Lambda} \Upsilon_{S_1 \times S_2}^{(\rho, \Lambda)}$$

the maximal violation by a state $\rho$ of all $S_1 \times S_2$-setting general Bell inequalities for any type of outcomes, discrete or continuous, at each of two sites.

If, for a state $\rho$, the maximal violation $\Upsilon_{S_1 \times S_2}^{(\rho)}$ is bounded from above by a value independent on numbers $S_1, S_2 \geq 1$, then the parameter

$$1 \leq \Upsilon := \sup_{S_1, S_2} \Upsilon_{S_1 \times S_2}^{(\rho, \Lambda)}$$

constitutes the maximal violation by a state $\rho$ of all general Bell inequalities with any numbers of settings at each site and constitutes a measure for nonlocality of a state $\rho$ under measurements with all possible numbers of setting at each of sites.

**Definition 2** A multipartite quantum state is called nonlocal if it violates a Bell inequality.

Definition 2 and Eq. (17) imply [13] [15].

**Criterion 1** A bipartite state $\rho$ is local if and only if parameter $\Upsilon_{S_1 \times S_2}^{(\rho)} = 1$ for all $S_1, S_2 \geq 1$ and is nonlocal if and only if $\Upsilon_{S_1 \times S_2}^{(\rho)} > 1$ for some numbers $S_1, S_2$ of measurement settings at each of sites.

Therefore, parameter $\Upsilon_{S_1 \times S_2}^{(\rho)}$ constitutes a measure for nonlocality of a bipartite quantum state $\rho$ under correlation scenarios with fixed numbers $S_1, S_2$ of settings at each of two sites while parameter $\Upsilon$ is a measure for nonlocality of a state $\rho$ under correlation scenarios with any numbers of settings at each of two sites.

According to the upper bound (1), for a finite-dimensional bipartite state, pure or mixed, on $\mathcal{H}_1 \otimes \mathcal{H}_2$, dim $\mathcal{H}_n = d_n$,

$$\Upsilon \leq 2 \min\{d_1, d_2\} - 1.$$  

(19)

Let $T_{S_1 \times S_2}^{(\rho)}$ be a self-adjoint trace class dilation of a state $\rho$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ to the Hilbert space $\mathcal{H}_1^{\otimes S_1} \otimes \mathcal{H}_2^{\otimes S_2}$. By its definition

$$\text{tr} \left[ T_{S_1 \times S_2}^{(\rho)} \left\{ I_{\mathcal{H}_1^{\otimes k_1}} \otimes X_1 \otimes I_{\mathcal{H}_1^{\otimes (S_1 - 1 - k_1)}} \otimes I_{\mathcal{H}_2^{\otimes k_2}} \otimes X_2 \otimes I_{\mathcal{H}_2^{\otimes (S_2 - 1 - k_2)}} \right\} \right]$$

(20)

$$\text{tr} \left[ \rho \{ X_1 \otimes X_2 \} \right], \quad k_n = 0, \ldots, (S_n - 1), \quad n = 1, 2,$$
for all bounded operators $X_n$ on $\mathcal{H}_n$, $n = 1, 2$. Clearly, $T^{(\rho)}_{1 \times 1} = \rho$, $\text{tr}[T^{(\rho)}_{S_1 \times S_2}] = 1$ and $\|T^{(\rho)}_{S_1 \times S_2}\|_1 \geq 1$, where $\|\cdot\|_1$ means the trace norm.

In [13, 29], we call a self-adjoint trace class operator $T^{(\rho)}_{S_1 \times S_2}$ as an $S_1 \times S_2$-setting source operator for a state $\rho$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. As proved in [13], for every bipartite state $\rho$ and arbitrary integers $S_1, S_2 \geq 1$, a source operator $T^{(\rho)}_{S_1 \times S_2}$ exists.

**Remark 1** For a separable quantum state, there always exists a positive source operator. However, for an arbitrary bipartite quantum state, a source operator $T^{(\rho)}_{S_1 \times S_2}$ on $\mathcal{H}_1^{\otimes S_1} \otimes \mathcal{H}_2^{\otimes S_2}$ does not need to be either positive or, more generally, tensor positive. The latter general notion introduced in [13] means that

$$\text{tr}\left[ T^{(\rho)}_{S_1 \times S_2} \{ A_1 \otimes \cdots \otimes A_{S_1} \otimes B_1 \otimes \cdots \otimes B_{S_2} \} \right] \geq 0$$

for all positive bounded operators $A_k, B_m$ on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Theorem 3 in [13] and, more precisely, the second line of Eq. (53) in this theorem, imply the following analytical upper bound on $\Upsilon^{(\rho)}_{S_1 \times S_2}$.

**Theorem 1** For an arbitrary bipartite quantum state $\rho$, possibly infinite dimensional, and any integers $S_n \geq 1$, $n = 1, 2$, the maximal violation $\Upsilon^{(\rho)}_{S_1 \times S_2}$ by state $\rho$ of all $S_1 \times S_2$-setting general Bell inequalities satisfies the relations

$$1 \leq \Upsilon^{(\rho)}_{S_1 \times S_2} \leq \min \left\{ \inf_{T^{(\rho)}_{1 \times 1}} \left\| T^{(\rho)}_{S_1 \times 1} \right\|_1, \inf_{T^{(\rho)}_{1 \times S_2}} \left\| T^{(\rho)}_{1 \times S_2} \right\|_1 \right\}$$

$$\leq \inf_{T^{(\rho)}_{S_1 \times S_2}} \left\| T^{(\rho)}_{S_1 \times S_2} \right\|_1,$$  

where $\|\cdot\|_1$ is the trace norm and $T^{(\rho)}_{S_1 \times 1}$, $T^{(\rho)}_{1 \times S_2}$, $T^{(\rho)}_{S_1 \times S_2}$ are source operators of state $\rho$ on Hilbert spaces $\mathcal{H}_1^{\otimes S_1} \otimes \mathcal{H}_2$, $\mathcal{H}_1 \otimes \mathcal{H}_2^{\otimes S_2}$ and $\mathcal{H}_1^{\otimes S_1} \otimes \mathcal{H}_2^{\otimes S_2}$, respectively.

In the following Section, based on Theorem 1, we find for a pure bipartite state $|\psi\rangle\langle\psi|$ the new upper bounds on its maximal Bell violation $\Upsilon^{(|\psi\rangle\langle\psi|)}_{S_1 \times S_2}$. These new upper bounds are expressed in terms of the Schmidt coefficients of a pure state and are tighter than the upper bound (1) valid for any bipartite state, pure or mixed.

### 3 New upper bounds for a pure state

Recall that, for any pure bipartite state $|\psi\rangle\langle\psi|$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, $\dim \mathcal{H}_n = d_n$, the non-zero eigenvalues $0 < \lambda_k(\psi) \leq 1$ of its reduced states on $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide and have the

This follows from Proposition 1 in [13] for a general $N$-partite case.
same multiplicity while vector $|\psi\rangle \in H_1 \otimes H_2$ admits the Schmidt decomposition

$$|\psi\rangle = \sum_{1 \leq k \leq r_{sch}(\psi)} \sqrt{\lambda_k(\psi)} |e_k^{(1)}\rangle \otimes |e_k^{(2)}\rangle, \quad \sum_{1 \leq k \leq r_{sch}(\psi)} \lambda_k(\psi) = 1,$$

where each eigenvalue of the reduced states is taken in this sum according to its multiplicity and $|e_k^{(n)}\rangle \in H_n$, $n = 1, 2$, are the normalized eigenvectors of the reduced states of a pure state $|\psi\rangle\langle\psi|$. Parameters $\sqrt{\lambda_k(\psi)}$ and $1 \leq r_{sch}(\psi) \leq d := \min\{d_1, d_2\}$ are called the Schmidt coefficients and the Schmidt rank of $|\psi\rangle$, respectively. For a separable pure bipartite state, its Schmidt rank equals to 1.

It is easy to check that the self-adjoint trace class operators

$$T_{1 \times S_2}^{(\psi)} := \sum_{k, k_1} \sqrt{\lambda_k(\psi)} \lambda_{k_1}(\psi) |e_k^{(1)}\rangle \langle e_{k_1}^{(1)}| \otimes W_{kk_1}^{(2, S_2)},$$

$$T_{S_1 \times 1}^{(\psi)} := \sum_{k, k_1} \sqrt{\lambda_k(\psi)} \lambda_{k_1}(\psi) W_{kk_1}^{(1, S_1)} \otimes |e_k^{(2)}\rangle \langle e_{k_1}^{(2)}|,$$

on $H_1 \otimes H_2^{\otimes S_2}$ and $H_1^{\otimes S_1} \otimes H_2$, respectively, where

$$W_{kk_1}^{(n, S_n)} := \left( |e_k^{(n)}\rangle \langle e_k^{(n)}| \right)^{\otimes S_n},$$

$$W_{k \neq k_1}^{(n, S_n)} := \frac{\left( |e_k^{(n)} + e_{k_1}^{(n)}\rangle \langle e_k^{(n)} + e_{k_1}^{(n)}| \right)^{\otimes S_n} - \left( |e_k^{(n)} - e_{k_1}^{(n)}\rangle \langle e_k^{(n)} - e_{k_1}^{(n)}| \right)^{\otimes S_n}}{2^{S_n + 1}} - \frac{i \left( |e_k^{(n)} + i e_{k_1}^{(n)}\rangle \langle e_k^{(n)} + i e_{k_1}^{(n)}| \right)^{\otimes S_n} - \left( |e_k^{(n)} - i e_{k_1}^{(n)}\rangle \langle e_k^{(n)} - i e_{k_1}^{(n)}| \right)^{\otimes S_n}}{2^{S_n + 1}},$$

$n = 1, 2,$

constitute the $1 \times S_2$-setting and $S_1 \times 1$-setting source operators of a pure bipartite state $|\psi\rangle\langle\psi|$. For source operators $[24]$, the trace norms admit the bound

$$\|T_{1 \times S_2}^{(\psi)}\|_1, \|T_{S_1 \times 1}^{(\psi)}\|_1 \leq 2 \left( \sum_k \sqrt{\lambda_k(\psi)} \right)^2 - 1,$$

which does not depend on numbers $S_1, S_2$ of measurement settings at each of two sites. Note that

$$2 \left( \sum_k \sqrt{\lambda_k(\psi)} \right)^2 - 1 \leq 2 r_{sch}(\psi) - 1.$$

In view of relations $[22], [20]$ and $[27]$ and bound $[11]$, we derive the following new result.
Theorem 2  For an arbitrary pure bipartite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, the maximal violation $\Upsilon_{S_1 \times S_2}^{(|\psi\rangle\langle\psi|)}$ of $S_1 \times S_2$-setting general Bell inequalities for any number and type of outcomes at each of sites admits the bound

\[
\Upsilon_{S_1 \times S_2}^{(|\psi\rangle\langle\psi|)} \leq 2 \min \left\{ \sum_{1 \leq k \leq r_{sch}^{(\psi)}} \sqrt{\lambda_k^{(\psi)}}, S_1, S_2 \right\} - 1 \tag{28}
\]

\[
\leq 2 \min \left\{ r_{sch}^{(\psi)}, S_1, S_2 \right\} - 1, \tag{29}
\]

where $\sqrt{\lambda_k^{(\psi)}}$ are the Schmidt coefficients and $r_{sch}^{(\psi)}$ is the Schmidt rank of a pure bipartite state $|\psi\rangle$.

In view of the relation

\[
2 \min \left\{ r_{sch}^{(\psi)}, S_1, S_2 \right\} - 1 \leq 2 \min \left\{ d_1, d_2, S_1, S_2 \right\} - 1, \tag{30}
\]

the upper bounds (28), (29) for a pure state are tighter than the upper bound (1) valid for any state, pure or mixed.

Theorem 2 and relation (18) imply

Corollary 1  For an arbitrary pure bipartite state $|\psi\rangle\langle\psi|$, possibly infinite dimensional, the maximal violation $\Upsilon_{|\psi\rangle\langle\psi|}$ by this state of general Bell inequalities for any type of outcomes and any number of settings at each of sites admits the bounds

\[
1 \leq \Upsilon_{|\psi\rangle\langle\psi|} \leq 2 \left( \sum_{1 \leq k \leq r_{sch}^{(\psi)}} \sqrt{\lambda_k^{(\psi)}} \right)^2 - 1 \tag{31}
\]

\[
\leq 2r_{sch}^{(\psi)} - 1, \tag{32}
\]

4  Analytical relations between nonlocality and entanglement

Recall shortly the notions of "negativity" and "concurrence" – well-known entanglement measures for a bipartite state.

Negativity $\mathcal{N}_\rho$ of a bipartite state $\rho$ on a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, $d_n = \dim \mathcal{H}_n$, is given by the relation

\[
2\mathcal{N}_\rho = \|\rho_{T_n}\|_1 - 1, \tag{33}
\]

where $\rho_{T_n}$, $n = 1,2$, is a partial transpose of $\rho$ with respect to the subsystem described by a Hilbert space $\mathcal{H}_n$ and $\|\rho_{T_1}\|_1 = \|\rho_{T_2}\|_1$. For a pure bipartite state $|\psi\rangle\langle\psi|$ with the
Schmidt decomposition (23), we have
\[ \| (|\psi\rangle\langle\psi|)_n \|_1 = \left( \sum_{1 \leq k \leq r_{\text{sch}}^{(\psi)}} \sqrt{\lambda_k(\psi)} \right)^2 \] (34)
and, hence,
\[ 2N_{|\psi\rangle\langle\psi|} = \left( \sum_{1 \leq k \leq r_{\text{sch}}^{(\psi)}} \sqrt{\lambda_k(\psi)} \right)^2 - 1. \] (35)

Concurrence \( C_{|\psi\rangle\langle\psi|} \) of a pure state \( |\psi\rangle\langle\psi| \) with the Schmidt decomposition (23) is defined by
\[ C_{|\psi\rangle\langle\psi|} = \sqrt{2 \left( 1 - \sum_{1 \leq k \leq r_{\text{sch}}^{(\psi)}} \lambda_k^2(\psi) \right)} = \sqrt{2 \sum_{k \neq m} \lambda_k(\psi) \lambda_m(\psi)}. \] (36)

If the concurrence of a pure state is normalized to the unity for maximally entangled quantum states, then it takes the form
\[ C_{|\psi\rangle\langle\psi|}^{(\text{normal})} = \sqrt{\frac{d}{d-1} \left( 1 - \sum_{1 \leq k \leq r_{\text{sch}}^{(\psi)}} \lambda_k^2(\psi) \right)} = \sqrt{\frac{d}{d-1} \sum_{k \neq m} \lambda_k(\psi) \lambda_m(\psi)}, \] (37)
where \( d := \min\{d_1, d_2\} \).

**Lemma 1** For an arbitrary pure bipartite state \( |\psi\rangle\langle\psi| \) with the Schmidt rank \( r_{\text{sch}}^{(\psi)} \),
\[ C_{|\psi\rangle\langle\psi|} \geq \frac{8}{r_{\text{sch}}^{(\psi)}(r_{\text{sch}}^{(\psi)} - 1)} N_{|\psi\rangle\langle\psi|} \] (38)
\[ \geq \frac{8}{d(d-1)} N_{|\psi\rangle\langle\psi|}, \]
where \( d := \min\{d_1, d_2\} \).

**Proof.** From relation (8) in [30] it follows
\[ 2r_{\text{sch}}^{(\psi)}(r_{\text{sch}}^{(\psi)} - 1) \sum_{k \neq m} \lambda_k(\psi) \lambda_m(\psi) \geq 2 \left( \sum_{k} \sqrt{\lambda_k(\psi)} \right)^2 - 1 \] (39)
\footnotetext{5}{See, for example, in [30].}
\footnotetext{9}{For this notion, see [8] and references therein.}
\footnotetext{10}{See in section 7 of [31].}
This, Eq. (36) and relation \( r_{s_{\text{ch}}}^{(\psi)} \leq d \) imply

\[
C_{|\psi\rangle\langle\psi|} \geq \frac{2}{r_{s_{\text{ch}}}^{(\psi)}(r_{s_{\text{ch}}}^{(\psi)} - 1)} \left\{ \left( \sum_{k} \sqrt{\lambda_{k}(\psi)} \right)^{2} - 1 \right\} \tag{40}
\]


\[
= \frac{8}{r_{s_{\text{ch}}}^{(\psi)}(r_{s_{\text{ch}}}^{(\psi)} - 1)} N_{|\psi\rangle\langle\psi|} \geq \sqrt{\frac{8}{d(d - 1)}} N_{|\psi\rangle\langle\psi|}.
\]

This proves the statement. \( \blacksquare \)

From Theorem 2 and Eqs. (35)–(38) it follows.

**Proposition 1** For an arbitrary pure state \(|\psi\rangle\langle\psi|\) on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), \( d := \dim \mathcal{H}_n \), \( d = \min \{d_1, d_2\} < \infty \), negativity (35) and concurrence (36) satisfy the relations

\[
N_{|\psi\rangle\langle\psi|} \geq \frac{\Upsilon_{|\psi\rangle\langle\psi|} - 1}{4}, \tag{41}
\]

and

\[
C_{|\psi\rangle\langle\psi|} \geq \frac{\Upsilon_{|\psi\rangle\langle\psi|} - 1}{\sqrt{2r_{s_{\text{ch}}}^{(\psi)}(r_{s_{\text{ch}}}^{(\psi)} - 1)}} \geq \frac{\Upsilon_{|\psi\rangle\langle\psi|} - 1}{\sqrt{2d(d - 1)}} \tag{42}
\]

where \( \Upsilon_{|\psi\rangle\langle\psi|} \) is the maximal violation by this state of all Bell inequalities defined by (18).

For a general state \( \rho \), pure or mixed, concurrence \( C_{\rho} \) and negativity \( N_{\rho} \) are defined by relations\( ^{11} \)

\[
C_{\rho} := \inf_{\{\alpha_i, \psi_i\}} \sum \alpha_i C_{|\psi_i\rangle\langle\psi_i|}, \tag{43}
\]

\[
N_{\rho} := \inf_{\{\alpha_i, \psi_i\}} \sum \alpha_i N_{|\psi_i\rangle\langle\psi_i|}, \tag{44}
\]

where \( \rho = \sum \alpha_i |\psi_i\rangle\langle\psi_i| \), \( \sum \alpha_i = 1 \), \( \alpha_i > 0 \), is a possible convex decomposition of a state \( \rho \) via pure states.

**Theorem 3** For an arbitrary finite-dimensional state \( \rho \), pure or mixed, on a Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), \( d := \dim \mathcal{H}_n \), \( d = \min \{d_1, d_2\} < \infty \), concurrence (43) and negativity (44) satisfy the relations

\[
C_{\rho} \geq \frac{\Upsilon_{\rho} - 1}{\sqrt{2d(d - 1)}}, \tag{45}
\]

\[
N_{\rho} \geq \frac{\Upsilon_{\rho} - 1}{4}, \tag{46}
\]

where \( \Upsilon_{\rho} \), defined by (18), is the maximal violation by this state of all Bell inequalities.

\( ^{11} \)See [30, 32] and references therein.
Proof. From definition (18) of $\Upsilon_\rho$ and linearity in $\rho$ of the functional (7) in case of quantum probability distributions (12), we have

$$\Upsilon_\rho \leq \sum \alpha_i \Upsilon_{|\psi_i\rangle\langle\psi_i|}$$

for each possible convex decomposition $\rho = \sum \alpha_i |\psi_i\rangle\langle\psi_i|$, $\sum \alpha_i = 1$, $\alpha_i > 0$. This implies

$$\Upsilon_\rho \leq \inf_{\{\alpha_i, \psi_i\}} \sum \alpha_i \Upsilon_{|\psi_i\rangle\langle\psi_i|}. \quad (48)$$

Taking into account in (48) relations (42) and (43), we derive

$$\Upsilon_\rho \leq \inf_{\{\alpha_i, \psi_i\}} \sum \alpha_i \Upsilon_{|\psi_i\rangle\langle\psi_i|} \leq 1 + \frac{\sqrt{2(d-1)}}{2d} \inf_{\{\alpha_i, \psi_i\}} \sum \alpha_i C_{|\psi_i\rangle\langle\psi_i|} \cdot \quad (49)$$

Similarly, by using relations (48), (41) and (44), we have

$$\Upsilon_\rho \leq \inf_{\{\alpha_i, \psi_i\}} \sum \alpha_i \Upsilon_{|\psi_i\rangle\langle\psi_i|} \leq 1 + \frac{\sqrt{2(d-1)}}{2d} \inf_{\{\alpha_i, \psi_i\}} \sum \alpha_i N_{|\psi\rangle\langle\psi|} \quad (50)$$

This proves the statement. ■

Since $C_\rho \leq \sqrt{\frac{2(d-1)}{d}}$, relation (46) immediately implies the upper bound (1), derived otherwise in [13].

Remark 2 The analytical bounds (45) and (46) for a general bipartite state are specifically important for finding the minimal amount of entanglement via the collected experimental data on Bell violation – the goals of the semi-device-independent scenario and the device-independent scenario for the entanglement certification and quantification [23, 24, 25, 26].

5 Example

In this section, we apply the new results of Proposition 1 for specifying upper bounds on the maximal violation of Bell inequalities by infinite-dimensional bipartite coherent states of the Bell states like forms

$$|\Phi_1(\alpha)\rangle = \frac{|\alpha\rangle \otimes |\alpha\rangle + |\alpha\rangle \otimes |\alpha\rangle}{\sqrt{2(1 + e^{-4\alpha^2})}}, \quad (51)$$

$$|\Phi_2(\alpha)\rangle = \frac{|\alpha\rangle \otimes |\alpha\rangle + |\alpha\rangle \otimes |\alpha\rangle}{\sqrt{2(1 + e^{-4\alpha^2})}},$$

where

$$|\pm \alpha\rangle = e^{-\frac{\alpha^2}{2}} \sum_{m=0}^{\infty} \frac{(\pm \alpha)^m}{\sqrt{m!}} |m\rangle \quad (52)$$

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are the normalized binary coherent states with parameter $\alpha > 0$ and $\{|m\}, \ m = 0, 1, \ldots\}$ are the Fock vectors. For $\alpha \to 0$, each of bipartite coherent states $|\Phi_j\rangle$ tends to the product state $|0\rangle \otimes |0\rangle$.

For states $|\Phi_j\rangle$, the nonzero eigenvalues of their reduced states are nondegenerate and are equal to (see Appendix)

$$\lambda_{\pm}(\Phi_1(\alpha)) = \lambda_{\pm}(\Phi_2(\alpha)) = \frac{(1 \pm e^{-2\alpha^2})^2}{2(1 + e^{-4\alpha^2})},$$

(53)

for all $\alpha > 0$ and the Schmidt ranks $r_{\text{sch}}^{(\Phi_j)} = 2, j = 1, 2$.

From Eq. (53) it follows

$$\sum_k \sqrt{\lambda_k(\Phi_j)} = \sqrt{\lambda_{+}(\Phi_j(\alpha)) + \lambda_{-}(\Phi_j(\alpha))} = \sqrt{\frac{2}{1 + e^{-4\alpha^2}}},$$

(54)

$$\sum_k \lambda_k^2(\Phi_j) = \frac{1}{2} + \frac{2e^{-2\alpha^2}}{(1 + e^{-4\alpha^2})^2},$$

(55)

From equalities (54), (55) and Eqs. (53), (56) it follows:

$$2N_{|\Phi_j\rangle\langle\Phi_j|} = \left(\sum_k \sqrt{\lambda_k(\Phi_j)}\right)^2 - 1 = \frac{1 - e^{-4\alpha^2}}{1 + e^{-4\alpha^2}},$$

(56)

$$C_{|\Phi_j\rangle\langle\Phi_j|} = \sqrt{2 \left(1 - \sum_k \lambda_k^2(\Phi_j)\right)} = \frac{1 - e^{-4\alpha^2}}{1 + e^{-4\alpha^2}} = 2N_{|\Phi_j\rangle\langle\Phi_j|},$$

(57)

The latter equality is consistent with Lemma 1 if $d \to \infty$ and $r_{\text{sch}}^{(\psi)} = 2$. Note that, for each of states $|\Phi_j\rangle$, concurrence $C_{|\Phi_j\rangle\langle\Phi_j|}$ is an increasing function of a parameter $\alpha$ tending to 1 for $\alpha \to \infty$.

Relations (56), (57) and Proposition 1 imply.

**Proposition 2** For each of infinite-dimensional bipartite coherent states $|\Phi_j(\alpha)\rangle, j = 1, 2$, the maximal violation of Bell inequalities satisfies the relation

$$1 \leq \Upsilon_{|\Phi_j\rangle\langle\Phi_j|} \leq \frac{3 - e^{-4\alpha^2}}{1 + e^{-4\alpha^2}}, \quad j = 1, 2,$$

(58)

for all $\alpha > 0$. 

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6 Conclusion

In the present article, based on the local quasi hidden variable (LqHV) formalism developed in [13, 14, 15], we find a new upper bound (28) (Theorem 2) on the maximal violation \( \Upsilon_{S_1 \times S_2}^{(|\psi\rangle\langle\psi|)} \) of all \( S_1 \times S_2 \)-setting Bell inequalities by a pure bipartite state \(|\psi\rangle\langle\psi|\), possibly infinite-dimensional. This upper bound is expressed in terms of the Schmidt coefficients of this state and numbers \( S_1, S_2 \) of measurement settings at each of two sites and implies (Corollary 1) that the maximal violation \( \Upsilon_{|\psi\rangle\langle\psi|} \) of all Bell inequalities by a pure bipartite state \(|\psi\rangle\langle\psi|\) cannot exceed the value 
\[
2 \left( \sum_k \sqrt{\lambda_k(\psi)} \right)^2 - 1.
\]

Based on the new results of Theorem 2, we further find for "negativity" and "concurrence" of a general bipartite state, pure or mixed, the new lower bounds (Proposition 1 and Theorem 3) expressed via the maximal violation by this state of all Bell inequalities. To our knowledge, no any general analytical relations between measures for bipartite nonlocality and entanglement have been reported in the literature, though, for a general bipartite state, such relations as the analytical bounds (45) and (46) are specifically important for finding the minimal amount of entanglement via the collected experimental data on Bell violation – the goals of the semi-device-independent scenario and the device-independent scenario for the entanglement certification and quantification [23, 24, 25, 26].

As an example, we specify (Proposition 2) the upper bound on the maximal violation of general Bell inequalities by bipartite entangled coherent states (51) intensively discussed last years in the literature in view of their experimental implementations [27].

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Appendix

The vectors
\[
|u_1\rangle := |\alpha\rangle, \quad |w_2\rangle := \frac{|-\alpha\rangle - |u_1|\alpha|u_1\rangle}{\sqrt{1 - |\langle u_1|-\alpha\rangle|^2}},
\]
(A1)
where vector $|u_2\rangle$ is due to the Gram-Schmidt orthonormalization process between vectors $|\alpha\rangle$ and $|-\alpha\rangle$, constitute the orthonormal basis of the linear span of vectors $|\alpha\rangle$ and $|-\alpha\rangle$. For $\alpha > 0$,

$$
|u_2\rangle = \frac{|-\alpha\rangle - e^{-2\alpha^2}|\alpha\rangle}{\sqrt{1 - e^{-4\alpha^2}}}
$$

and bipartite coherent states (51) admit the following decompositions:

$$
|\Phi_1(\alpha)\rangle = \frac{(1 + e^{-4\alpha^2})|u_1\rangle \otimes |u_1\rangle + e^{-2\alpha^2}\sqrt{1 - e^{-4\alpha^2}}|u_1\rangle \otimes |u_2\rangle}{\sqrt{2(1 + e^{-4\alpha^2})}}
$$

$$
+ \frac{e^{-2\alpha^2}\sqrt{1 - e^{-4\alpha^2}}|u_2\rangle \otimes |u_1\rangle + (1 - e^{-4\alpha^2})|u_2\rangle \otimes |u_2\rangle}{\sqrt{2(1 + e^{-4\alpha^2})}},
$$

$$
|\Phi_2(\alpha)\rangle = \frac{2e^{-2\alpha^2}|u_1\rangle \otimes |u_1\rangle + \sqrt{1 - e^{-4\alpha^2}}(|u_1\rangle \otimes |u_2\rangle + |u_2\rangle \otimes |u_1\rangle)}{\sqrt{2(1 + e^{-4\alpha^2})}}.
$$

The nonzero eigenvalues of the reduced states of $|\Phi_j(\alpha)\rangle\langle\Phi_j(\alpha)|$, $j = 1, 2$ can be easily calculated and are nongenerate and given by

$$
\lambda_{\pm}(\Phi_j(\alpha)) = \frac{(1 \pm e^{-2\alpha^2})^2}{2(1 + e^{-4\alpha^2})}, \quad j = 1, 2.
$$

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