Anomalous Quantum Hall Effect of 4D Graphene in Background Fields

L.B Drissi $^{1,3}$, H. Mhamdi $^{1,2}$, E.H Saidi $^{1,2,4}$

1.INANOTECH, Institute of Nanomaterials and Nanotechnology (MAScIR), Morocco,
2.Lab HEP-Modeling and Simulation, FSR, Univ Mohammed V-Agdal, Rabat, Morocco,
3.International Centre for Theoretical Physics, ICTP, Trieste, Italy,
4.Centre of Physics and Mathematics, CPM-CNESTEN, Rabat, Morocco,
E-mails: ldrissi@ictp.it, h-saidi@fsr.ac.ma

Abstract: Boriçi-Creutz (BC) model describing the dynamics of light quarks in lattice QCD has been shown to be intimately linked to the four dimensional extension of 2D graphene refereed below to as four dimensional graphene (4D- graphene). Borrowing ideas from the field theory description of the usual 2D graphene, we study in this paper the anomalous quantum Hall effect (AQHE) of the BC fermions in presence of a constant background field strength $F_{\mu\nu}$ with a special focus on the case $F_{\mu\nu} = B\epsilon_{\mu\nu34} + E\epsilon_{12\mu\nu}$ with $B$ and $E$ two real moduli and $\det F_{\mu\nu} = B^2 \times E^2$. First, we revisit the anomalous 2D graphene by using QFT method. Then, we consider the AQHE of BC fermions for both regular $\det F_{\mu\nu} \neq 0$ and singular $\det F_{\mu\nu} = 0$ cases. We show, amongst others, that the exact solutions of the BC fermions coupled to constant $F_{\mu\nu}$ have a 5D interpretation; and the filling factor $\nu_{BC}$ of the BC model coupled to constant $F_{\mu\nu}$ is given by $24(N+1)(2M+1)$ with $N, M$ positive integers. Others features, such as $F_{\mu\nu}^{QCD} \neq 0$ and the extension of the obtained results to the lattice fermions like Karsten-Wilzeck (KW) fermions and naive ones, are also discussed.

Keywords: Lattice QCD, Borii-Creutz fermions, Anomalous Quantum Hall Effect, filling factor, four dimensional graphene, Index theorem, Spectral flow.
Contents

1. Introduction

2. AQHE in 3D relativistic systems: case of 2D graphene
   2.1 Dirac equation in 3D
   2.2 Filling factor

3. Boriçi-Creutz fermions in background fields
   3.1 BC model and 4D graphene
   3.2 BC fermions in background fields
     3.2.1 From $SO(1,3)$ to $SO(4)$ and then to $SO(1,4)$
     3.2.2 Dirac equation coupled to $F_{\mu\nu}$

4. The Lie algebra of the gauge covariant derivatives
   4.1 uncoupled case: $F_{\mu\nu} = B\varepsilon_{\mu34} + \mathcal{E}\varepsilon_{12\mu\nu}$
   4.2 coupled case: $F_{\mu\nu} = B^{ij}\varepsilon_{\mu\nu ij} + \mathcal{E}^{[4i]}\varepsilon_{4\mu\nu}$

5. More on AQHE in BC model
   5.1 Spectrum of BC Hamiltonian $H_{BC}$
   5.2 Filling factor $\nu_{BC}$ of BC fermions in background fields
   5.3 Link with spectral flow and topological index

6. Conclusion and comments

7. Appendix A: Index of Dirac operator
   7.1 case of 2D graphene
     7.1.1 the index theorem: continuous limit
     7.1.2 2D fermions on lattice
   7.2 Index in 4D lattice QCD
     7.2.1 Fermions on 4D space
     7.2.2 4D hypercube

8. Appendix B: Index, spectral flow and filling factor
   8.1 Index theorem and spectral flow
   8.2 Filling factor and chiral anomaly

9. Acknowledgements
1. Introduction

Few years ago M. Creutz made a remarkable observation about links between the physics of 2D graphene and four dimensional lattice chromodynamics (QCD) \[1,2\]. The two Dirac valleys of 2D graphene \[3,4\] have an analogue in the 4D extension of Creutz; and are interpreted as the up and down quarks of the physics of QCD \[1, 5, 6, 7, 8, 9, 10\]. This correspondence opened a window on applying constructions of 2D graphene modelings to 4D lattice QCD formulated on the hyperdiamond lattice \[11,12\].

In this paper, we contribute to this matter by looking for the generalization of some specific properties of graphene to the case of 4D lattice QCD with a special focus on Boris-Creutz (BC) fermions \[13,14\] in particular the aspect regarding the anomalous quantum Hall effect (AQHE) \[15,16\]. Recall that QHE in higher dimensions has been first considered in \[27\] for the case of the 4-sphere and has been further developed in \[28,29,30\] and refs therein for other higher D- manifolds. These models are non relativistic systems generalizing the well known 2D Hall fluid \[32,33\] where, due to disorder, the conductivity has plateaus resulting from the non uniformity of the spatial potential. Recall also that the quarks \(u_\alpha(x)\) and \(d_\alpha(x)\) are light particles that play a central role in lattice QCD simulations; they are described by 4D Dirac spinors \(\Psi_\alpha(x)\) with integral color quantum numbers as required by the \(SU_c(3)\) non abelian gauge symmetry; but fractional electromagnetic charges respectively equal to \(Q_u = \frac{2}{3}\) and \(Q_d = -\frac{1}{3}\). Interactions between quarks are mediated by gauge fields; in particular by \(A_\mu^G = A_\mu^{em}Q_{em} + \sum I A_I^G T_I\) valued in the adjoint representation of the gauge symmetry \(G = U_{em}(1) \times SU_c(3)\) with the \(8\) hermitian \(3 \times 3\) matrices \(T_I\) standing for the basis generators of \(SU_c(3)\). In the QCD regime \[37,38,39\] where the \(SU_u(3) \times U_{em}(1)\) gauge interactions can be approximated by constant background fields \(F_{\mu\nu}^G = (F_{\mu\nu}^{SU_u(3)}, F_{\mu\nu}^{U_{em}(1)})\), one is left with a physics quite similar to the one of the AQHE of the delocalized electrons of 2D graphene. Therefore, one expects the light quark’s dynamics to show as well an anomalous quantum Hall type phenomenon\(^1\) in the presence of a constant \(F_{\mu\nu}^G\). This tensor reads generally in terms of the gauge potential as \(\partial_\mu A_\nu^G - \partial_\nu A_\mu^G + [A_\mu^G, A_\nu^G]\); but for explicit computations, we will mainly focus on the abelian part of the gauge symmetry \(G\).

Because of the uniformity condition \(\frac{\partial}{\partial x} F_{\mu\nu}^G = 0\), and also by restricting \(F_{\mu\nu}^G\) to take values in the Cartan subalgebra \(U_c^2(1) \times U_{em}(1) \subset G\), the abelian part of the gauge potential \(A_\mu^I(x)\) that obeys the Lorentz condition \(\partial_\mu A^{hI} = 0\) reads, up to irrelevant numbers, as follows \(\frac{1}{2} F_{\mu\nu}^I x^\nu\). Notice by the way that unlike 2D graphene, the underlying space time of 4D lattice QCD has four real euclidian dimensions leading to several possibilities for the allowed directions of \(F_{\mu\nu}\). This richness, which has also a physical interpretation in terms of interactions (see section 4), may be fixed by looking for configurations that permit the diagonalization of the BC- hamiltonian \(H_{BC}\) which acts on \(SO(4)\) spinorial states \(|\Psi_E\rangle\) like \(H_{BC} |\Psi_E\rangle = E |\Psi_E\rangle\) with \(H_{BC} = \frac{1}{2} \gamma^\mu \left(\partial_\mu - i \frac{2}{e} A_\mu \right)\) and where the four matrices \(\gamma^\mu\) are the usual Dirac \(4 \times 4\) matrices in the 4D euclidian space. By choosing the constant

\(^1\)By QHE, we mean a quantized conductivity of the system following from the existence of a discrete energy spectrum and a discrete filling factor due to the background field.
background fields like,

\[
\mathcal{F}_{\mu\nu}^{em} = E_{\varepsilon_{\mu\nu}34} + E_{\varepsilon_{12\mu\nu}}, \quad \det \mathcal{F}_{\mu\nu}^{em} = B^2 \times E^2
\]

(1.1)

\[
\mathcal{F}_{\mu\nu}^{QCD} = \sum_{I=1}^{2} h_{I} F_{\mu\nu}^{I} + \sum_{\text{su3 roots } \alpha} E^{-\alpha} F_{\mu\nu}^{\alpha} = 0
\]

breaking down the \( SO(4) \) symmetry of the euclidian \( \mathbb{R}^4 \) down to \( SO(2) \times SO(2) \), one ends with a diagonal form of the squared operator \( H_{BC}^2 \); as well as remarkable factorized relations that allow to perform the explicit computation of the exact hamiltonian spectrum. Notice that, viewed from the 4D space time with \( SO(4) \) symmetry, the constants \( B \) and \( E \), appearing in the above relations, are "magnetic" and "electric" components of \( F_{\mu\nu}^{U_{em}(1)} \) respectively normal to the \( x^1-x^2 \) and \( x^3-x^4 \) real planes of \( \mathbb{R}^4 \). However, from a \((1+4)\)-dimensional space time with \( SO(1,4) \) isotropy symmetry, both of \( B \) and \( E \) behave as magnetic components that couple to left \( \Psi_L \) and right \( \Psi_R \) handed fermions respectively; and so lead to QHE phenomenon in four dimensions. This 5D interpretation will be discussed with some details in section 3 and in conclusion. With the above diagonal choice of \( \mathcal{F}_{\mu\nu}^{em} \), which contains as particular cases the singular limit \( \det \mathcal{F}_{\mu\nu}^{em} = 0 \) describing chiral configurations with solely \( \Psi_L \) or \( \Psi_R \), we find, amongst others and besides the 5D interpretation of the BC model, the two following features:

1. the energy spectrum the BC fermions in the constant background fields (1.1) is discrete provided the \( \mathcal{F}_{\mu\nu}^{em} \) tensor is non degenerate; that is as far as the product \( B \times E \neq 0 \). In this case, the energy spectrum is given by

\[
E^{\pm}_{n,m}(\omega, \omega') = \pm \hbar \sqrt{n\omega^2 + m\omega'^2}, \quad n, m \geq 0
\]

with the oscillator frequencies \( \omega^2 = \frac{2Q|B|}{e}, \omega'^2 = \frac{2Q|E|}{e} \); and where \( Q = qe \) stand for the electric fractional charge of the quark \( u_\alpha(x) \) or the quark \( d_\alpha(x) \) living at each of the two valleys of the BC model. Clearly the appearance of the frequency \( \omega' \) is due to euclidian nature of the 4D lattice and to the underlying 5D interpretation where \( E \) is thought of as an external "magnetic" field that couple to right handed \( \Psi_R \). In the singular limit \( \det \mathcal{F}_{\mu\nu}^{em} = 0 \), for instance in case where \( E \rightarrow 0 \), the above energy spectrum gets modified as

\[
E^{\pm}_{n}(\omega, k_z) = \pm \hbar \omega \sqrt{n + \frac{k_z^2 + k_r^2}{\omega^2}},
\]

with \( \hbar k_z \) being the momenta of \( \Psi_R \) along the \( i \)-th direction. These energies define a family of pairs of opposite paraboloids separated by the gaps \( \Delta E_n = 2\sqrt{n\omega^2 + k_z^2 + k_r^2} \), and touch on the fundamental state \( n = 0, k_z = 0, k_r = 0 \) leading to a zero gap system; see also fig.1 for illustration.

\[\text{In 1+4 dimensions, the antisymmetric tensor } \mathcal{F}_{MN} \text{ has 10 components; 6 of them given by } \mathcal{F}_{\mu\nu} \text{ are magnetic type and the other 4 ones given by } \mathcal{F}_{5\mu} \text{ are of electric type.}\]
(2) The $BC$ fermion’s filling factor $\nu_{BC}$ of the anomalous quantum Hall effect induced by the background fields in the band energy $0 \leq E_{n,m}^{+} \leq E_{N,M}^{+}$, with $N$, $M$ positive integers, is given by,

$$\nu_{BC} = k_V \times N_c \times g_L \times g_R \times \frac{(2N + 1)(2M + 1)}{2}.$$  \hspace{1cm} (1.3)

where $k_V$ is the number of Dirac valleys, which takes the value $k_V = 2$ for minimally doubled fermions including BC and KW ones; $k_V = 16$ for naive fermions and $k_V = 4$ for staggered fermions; $N_c$ the quark’s color number which is equal to 3; and $g_L = g_R = 2$ the number of spin polarizations of the left and right handed fermions of the $SO(4)$ spinor. Notice that in the case where, in addition to $F_{\mu\nu}^{em} = B\varepsilon_{\mu\nu34} + E\varepsilon_{12\mu\nu}$, we also have $F_{\mu\nu}^{SU_c(3)} \neq 0$ along the two directions of the Cartan subalgebra of the $SU_C(3)$ gauge symmetry, BC fermions coupled to the background fields are described by 6 oscillators (2 for each color; i.e: for each $\Psi^c_L$ and $\Psi^c_R$, with $c = 1, 2, 3$) and the above filling factor generalizes as follows,

$$\nu_{BC}^{gen} = k_V \times g_L \times g_R \times \frac{1}{2} \prod_{i=1}^{6} (2N_i + 1).$$  \hspace{1cm} (1.4)

The presentation is as follows. In section 2, we review briefly the anomalous quantum Hall effect (AQHE) of graphene; this section aims also to revisit useful aspects of graphene using relativistic field theory in $(1 + 2)$D; and also to describe our approach on a simple system. In section 3, we study the minimally doublet fermions in a constant background field by focusing on BC fermions. In particular we study the BC model of 4D lattice QCD; first as a lattice field theory on the hyperdiamond; and second as a $(1 + 4)$D extension of the graphene near the Dirac points. In section 4, we study the algebra of the gauge covariant derivatives and its highest weight representations. These representations are used in section 5 to determine the spectrum of the BC-hamiltonian in presence of background fields; and determine the filling factor of the AQHE of the BC fermions. We also give the relation with the spectral flow hamiltonian considered in [34, 35, 36]. Last section is devoted to conclusion and comments. In the appendix A and appendix B, we develop further the link between our study and the index theorem.

2. AQHE in 3D relativistic systems: case of 2D graphene

In this section, we study the anomalous quantum Hall effect of a relativistic fermionic system described by the $(1 + 2)$ Dirac equation in a constant magnetic background field. This concerns a particular QED$_3$ model where the electromagnetic field strength $F_{\mu\nu}$ takes a constant value with direction normal to the 2D space surface where live the sheet of graphene. We assume that the magnitude of the external field $B$ is bounded as follows $14$ Tesla $\leq |B| \leq 20$ Tesla so that the Zeeman coupling can be ignored [40].

2.1 Dirac equation in 3D

First consider the $(1 + 2)$ space time $\mathbb{R}^{1,2}$ with local coordinates $x^\mu = (t, x, y)$, embedded in the usual $(1 + 3)$ dimensional space time $\mathbb{R}^{1,3}$ parameterized by $X^M = (x^\mu, z)$; then
focus on the Dirac equation of a fermionic particle, described by the complex field doublet
\[ \psi^a = (\phi, \chi) \]
living in \( \mathbb{R}^{1,2} \), in the presence of an external constant magnetic field \( B \) taken along the z-direction of \( \mathbb{R}^{1,3} \),

\[
\sum_{\mu=0}^{2} \sum_{b=1}^{2} i\sigma_{ab}^\mu \left(D_\mu \psi^b\right) = 0. \tag{2.1}
\]

In this system of equations, the hermitian \( 2 \times 2 \) matrix \( D = i\sigma_{ab}^\mu D_\mu \) with \( iD_\mu = i(\partial_\mu - i\gamma A_\mu) \) and \((iD_\mu)^+ = iD_\mu\), is the gauged Dirac operator in \((1 + 2)\) spacetime dimensions, \( \sigma^0 = I_2 \) the unity matrix and \( \sigma^i = (\sigma^1, \sigma^2) \) the usual \( 2 \times 2 \) Pauli matrices obeying the 2D Clifford algebra \( \sigma^i \sigma^j + \sigma^j \sigma^i = 2\delta^{ij} \) as well as the \( SU(2) \) commutation relations type \( [\sigma^1, \sigma^2] = 2i\sigma^3 \) giving the \( \pm \) chiralities of the component fields \( \phi \) and \( \chi \). Moreover, \( A_\mu \) is the gauge potential which in the present case reads as \( F_{\mu\nu} \) where the constant field strength tensor

\[
F_{\mu\nu} = \frac{e}{c} [D_\mu, D_\nu] \tag{2.2}
\]
is given by the magnetic field \( B \) along the z-direction. Here, the rank 4 tensor \( \varepsilon_{\alpha\mu\nu\beta} \) is the completely antisymmetric \( SO(1,3) \) invariant tensor; and the restricted \( \varepsilon_{0\mu\nu\beta} \equiv \varepsilon_{\mu\nu} \) is the antisymmetric \( SO(2) \) invariant \( 2 \times 2 \) matrix. Substituting \( F_{\mu\nu} = B \varepsilon_{\mu\nu} \), we see that the linear gauge potential lives in the x-y plane with components as given below,

\[
A_1 = +\frac{B}{2} y, \quad A_0 = 0, \quad A_2 = -\frac{B}{2} x, \quad \partial_\mu A^\mu = 0. \tag{2.3}
\]

From the above relations, we learn also that \([D_1, D_2] = i\gamma B\) showing amongst others that \( B \) acts as a deformation parameter on the free spectrum of the Dirac operator; and the algebra of the gauge covariant derivatives, is up to a scale factor, the well known Heisenberg algebra of the harmonic oscillator; i.e \([A, A^\dagger] = \hbar I\). Furthermore, factorizing the time dependence of the wave function by summing over all possible frequencies as

\[
\psi^a(t, x, y) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi\hbar} e^{-\frac{i}{\hbar}E t} \psi^a_E(x, y) \tag{2.4}
\]
or equivalently by using discrete spectrum notation like

\[
\psi^a(t, x, y) = \sum_{n=-\infty}^{+\infty} e^{-\frac{i}{\hbar}E_n t} \psi^a_n(x, y), \tag{2.5}
\]

with \( E_n = \hbar \omega_n \); and \( \psi^a_{+|n|}(x, y), \psi^a_{-|n|}(x, y) \) the wave functions associated with positive and negative frequencies; then using the fact that scalar potential \( A_0 = 0 \), it is convenient to rewrite the Dirac equation of \( \psi^a_n \) as follows

\[
\begin{pmatrix}
0 & i (D_1 - iD_2) \\
i (D_1 + iD_2) & 0
\end{pmatrix}
\begin{pmatrix}
\phi_n \\
\chi_n
\end{pmatrix} = \omega_n \begin{pmatrix}
\phi_n \\
\chi_n
\end{pmatrix}. \tag{2.6}
\]

The solutions of this matrix equation give the possible frequencies \( \omega_n \) and the corresponding wave function doublets \( (\phi_n, \chi_n) \), with the integer \( n \) referring to the quantized values of
energy and momentum due to the background field $B$. To that end, acting on the above 2×2 matrix equation (2.6) by the Dirac operator $i \sum \sigma^l D_l$, we can bring it into the following diagonal one

$$
\begin{pmatrix}
D_- D_+ & 0 \\
0 & D_+ D_-
\end{pmatrix}
\begin{pmatrix}
\phi_n \\
\lambda_n
\end{pmatrix}
= \omega^2_n
\begin{pmatrix}
\phi_n \\
\lambda_n
\end{pmatrix}
$$

(2.7)

where we have set $D_\pm = i (D_1 \pm i D_2)$ with the property $(D_\pm)^+ = D_\mp$. We also have

$$
D_+ = i \left( 2 \frac{\partial}{\partial u} - \frac{eB}{2c} u \right), \quad u = x + iy, \quad \frac{\partial}{\partial u} = \frac{1}{2} \frac{\partial}{\partial x} - i \frac{\partial}{\partial y},
$$

$$
D_- = i \left( 2 \frac{\partial}{\partial u} + \frac{eB}{2c} \bar{u} \right), \quad \bar{u} = x - iy, \quad \frac{\partial}{\partial \bar{u}} = \frac{1}{2} \frac{\partial}{\partial x} + i \frac{\partial}{\partial y},
$$

(2.8)

together with the angular momentum $L_z$ around the z-axis reading like

$$
L_z = \left( u \frac{\partial}{\partial u} - \bar{u} \frac{\partial}{\partial \bar{u}} \right) = \frac{1}{i} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right),
$$

(2.9)

and acting on $D_\pm$ and $D_- D_+$ as follows

$$
[L_z, D_\pm] = \pm D_\pm, \quad [L_z, D_- D_+] = 0.
$$

(2.10)

The relation $[L_z, D_- D_+] = 0$ shows that the wave functions carry, in addition to energy, also an angular momentum charge that we skip below for simplicity. Notice moreover that the $D_\pm$ operators satisfy the commutation relation $[D_-, D_+] = \frac{2e}{c} B$ which, up on using the normalization

$$
D_- = A \sqrt{\frac{2e}{c} B}, \quad D_+ = A^\dagger \sqrt{\frac{2e}{c} B},
$$

(2.11)

with positive $B$, can be put in the usual Heisenberg form

$$
[A, A^\dagger] = I, \quad [A, A] = [A^\dagger, A^\dagger] = 0,
$$

(2.12)

describing the algebra of the quantum harmonic oscillator with $A^\dagger$ the creation operator, $A$ the annihilation operator; and $\hbar I$ (with $\hbar$ set to one) the diagonal one that captures the energy quantum number $n$. The highest weight representations of this algebra are well known; they are given by semi infinite dimensional vector spaces generated by the normalized basis vectors $\{ \xi_{\pm n}(x, y) = |\xi_{\pm n} > \}_{n \geq 0}$ based on the highest weight vector $|\xi_0 >$ obeying the conditions

$$
A |\xi_0 > = 0, \quad \hbar I |\xi_0 > = \hbar |\xi_0 >,
$$

(2.13)

solved generally like $\xi_0 (u, \bar{u}) \sim u^l e^{- \frac{\Delta u}{\hbar \Delta u}}$ with $l$ the value of the angular momentum which we set below to zero. The other states of the representation are given by successive application of the creation operator $A^\dagger$ as given below,

$$
A^\dagger |\xi_n > = \sqrt{n+1} |\xi_{n+1} >, \quad n \geq 0
$$

$$
A |\xi_n > = \sqrt{n} |\xi_{n-1} >, \quad N = A^\dagger A
$$

(2.14)

with $|\xi_{-1} > = 0$ as well as

$$
N |\xi_n > = n |\xi_n >, \quad \xi_n |\xi_m > = \delta_{nm}.
$$

(2.15)
Substituting the operators $D_{\pm}$ in terms of their expression using $A$ and $A^\dagger$, we can split the matrix equation (2.7) in two eigenstate equations as follows

\begin{align*}
(A^+ A + I) \phi_n &= \frac{\omega^2 c}{2eB} \phi_n, \\
A^+ A \chi &= \frac{\omega^2 c}{2eB} \chi_n.
\end{align*}

The wave functions of these relations are naturally solved by the basis states \(\{|\xi_n >\}\) by taking \(\phi_n\) and \(\chi_n\) like

\begin{equation}
\begin{pmatrix}
\phi_n \\
\chi_n
\end{pmatrix} = \begin{pmatrix}
|\xi_{n-1} > \\
|\xi_n >
\end{pmatrix},
\end{equation}

(2.17)

describing two neighboring polarized particles. The energies follow from the relation between the squared frequencies \(\omega^2_n = \frac{E^2_n}{\hbar^2}\) and the integer \(n\). We have

\begin{equation}
\omega^2_n = \frac{2ecBn}{\hbar}, \quad E^\pm_n = \pm \sqrt{\frac{2eBc}{\hbar}|n|},
\end{equation}

(2.18)

with \(E^+_n\) and \(E^-_n\) respectively associated with the waves \(\psi^a_{+|n|}(x,y)\) and \(\psi^a_{-|n|}(x,y)\) of eq(2.3). Notice that for \(n = 0\), we have a zero mode with negative chirality

\begin{equation}
\begin{pmatrix}
0 \\
\chi_0
\end{pmatrix}
\end{equation}

(2.19)

### 2.2 Filling factor

As was discovered in 1980, the Hall conductivity \(\sigma_{xy}\) of a 2D electron gas in a strong transverse magnetic field develops plateaus at values quantized in units of \(\frac{e^2}{\hbar}\); i.e \(\sigma_{xy} = \nu \frac{e^2}{\hbar}\) with \(\nu = \frac{N_f}{N_\phi}\) standing for the filling factor \([3, 4]\). In the case of 2D graphene, the filling factor \(\nu_{\text{graph}}\) of the QHE is given by the ratio of the number \(N_F\) of particle states with respect the number of quantum fluxes \(N_\phi = \int_{S^2} F_2\) where the 2-form \(F_2\) is given by \(Bdx \wedge dy\). We have,

\begin{equation}
\nu_{\text{graph}} = \frac{N_f}{N_\phi},
\end{equation}

(2.20)

A direct way to get this factor is to compute the number \(N_F\) per unit quantum flux \(N_\phi = 1\). For that purpose we proceed as follows: First recall that the \(\psi_{+|n|}\) and \(\psi_{-|n|}\) components of the expansion of the quantum wave function \(\psi = \sum_n \psi_n \) with \(\psi_n (t,x,y) = e^{-i\omega_n t}\psi_n (x,y)\) as in eq(2.3) describe respectively particles and holes that live on the energy levels \(E^+_n\) and \(E^-_n\) of eq(2.18). Second, the total number of \(N_F\) polarized particles and \(N_F\) holes, per unit quantum flux, that fill the band energy

\begin{equation}
0 \leq E^2_n \leq E^2_N, \quad E^-_N \leq E_n \leq E^+_N,
\end{equation}

(2.21)

with \(E^\pm_N = \pm \sqrt{\frac{2eBc}{\hbar}|N|}\) is, because of symmetry, precisely \(2N_F\); and is given by the relation

\begin{equation}
2N_F = \int_{E^-_n \leq E_n \leq E^+_N} dtd^2x |\psi (t,x,y)|^2
\end{equation}

(2.22)
together with the expansion
\[
\psi(t, x, y) = \sum_{n=-N}^{+N} e^{-\omega_n t} \psi_n(x, y).
\] (2.23)

Putting back (2.23) into (2.22) and using the orthogonality property
\[
\int d^2x \, \bar{\psi}_n(x, y) \psi_m(x, y) = \delta_{n,m},
\]
the relation (2.22) reduces to
\[
2N_F = \sum_{-N}^{+N} 1 = (2N + 1).
\] (2.24)

From the above relations, we learn that the positive energy values \(E_n^+\) are associated with the wave functions indexed by positive integers \(n \geq 0\) namely \(\psi_{+n}^a(t, x, y) = e^{-\frac{\hbar}{2} E_n^+ t} \psi_{+n}^a(x, y)\); they correspond, in the language of condensed matter physics, to the conducting band where live particle states. The negative energy values \(E_n^-\) are associated with the wave functions indexed by negative integers \(n = -|n| \leq 0\), \(\psi_{-|n|}^a(t, x, y) = e^{-\frac{\hbar}{2} E_n^- t} \psi_{-|n|}^a(x, y)\); they are associated with the valence band where live the holes. In the particular case \(n = 0\), both energies \(E_n^\pm\) vanish and the conducting and valence bands touch. So the fundamental state \(\psi_0^a\) with zero energy would be filled by \(\frac{1}{2}\) particle and \(\frac{1}{2}\) hole; this property can be viewed as another statement of the chiral anomaly.

Therefore the number of polarized particles filling the band energy \(0 \leq E_n^+ \leq \sqrt{\frac{2\hbar eB}{c} |N|}\) is equal to \(N + \frac{1}{2}\). So, by taking into account the fact that electrons has two spin polarizations, the general expression of the filling factor \(\nu_{graph}\) reads as follow:
\[
\nu_{graph} = 2k_V \left( N + \frac{1}{2} \right) \equiv 4N + 2,
\] (2.25)

where \(k_V\) stands for the number of Dirac valleys which is equal to 2 in the case of graphene.

3. Boric-Creutz fermions in background fields

In this section, we extend known results on 2D graphene to BC fermions of 4D lattice QCD; but the analysis applies straightforwardly to KW and naive fermions. First, we recall the BC fermions and its two Dirac valleys as being a simple model for lattice QCD simulations. Then we study some remarkable properties of the Dirac equation in the background field (1.1). The explicit computation of the spectrum of BC fermions in background fields will be given in next sections.

3.1 BC model and 4D graphene

Boric-Creutz fermions is a simple four dimensional lattice QCD model for studying and simulating the interacting dynamics of the two light quarks up \(u_\alpha(x)\) and down \(d_\alpha(x)\); see ref. for other specific features of BC fermions; in particular the explicit derivation of the \(SU(2)\) flavor\(^3\) symmetry \((u_\alpha, d_\alpha)\). The BC model can be derived as a particular model

\^3To be precise the species in lattice fermions should be treated as point splitting fields as proposed in and used extensively in.
resulting from the following general euclidian lattice action

$$S_{BC} = \frac{1}{a} \sum_{x_i \in L} \left( \sum_{\mu=1}^{4} \Psi^+_x \gamma^\mu \Omega^\mu_{\mu} \Psi_x + v_i - \Psi^+_x \gamma^\mu \Omega^\mu_{\mu} \Psi_x - v_i \right) , \tag{3.1}$$

where $L$ is the $4D$ hyperdiamond with lattice parameter $a$; $\Psi_{x_i}$ is a $4D$ euclidian Dirac spinor at the site $x_i$; $\gamma^\mu$ the usual $4 \times 4$ Dirac matrices to be specified later; $v_i$ are five relative vectors parameterizing the first nearest neighbors; and where $\Omega^\mu_{\mu}$ is a $4 \times 5$ tensor that reads like

$$\Omega^\mu_{\mu} = \begin{pmatrix} -\frac{1}{2} & -1+2i & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 1 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}. \tag{3.2}$$

Expanding the Dirac spinors in Fourier modes, we get $S_{BC} = \int \frac{d^4k}{(2\pi)^4} \Psi^+_k D_{BC} \Psi_k$ where the $4 \times 4$ matrix $D_{BC}$ is the $BC$ operator in the reciprocal space; it reads as follows,

$$D_{BC} \sim \frac{1}{a} \sum_{\mu=1}^{4} \gamma^\mu \sin ak_{\mu} - \frac{1}{a} \sum_{\mu=1}^{4} \gamma^\mu \cos ak_{\mu} + \frac{1}{a} \sum_{\mu=1}^{4} \Gamma \cos ak_{\mu} - \frac{2}{a} \Gamma , \tag{3.3}$$

with

$$\Gamma = \frac{1}{2} \left( \gamma^1 + \gamma^2 + \gamma^3 + \gamma^4 \right). \tag{3.4}$$

From these expressions, one can check that the operator $D_{BC}$ has two zero modes given by the two following four component wave vectors $k_{\mu},$

$$K = (k_1, k_2, k_3, k_4) = (0, 0, 0, 0), \quad \text{and} \quad K' = (k_1, k_2, k_3, k_4) = \left( \frac{\pi}{2a}, \frac{\pi}{2a}, \frac{\pi}{2a}, \frac{\pi}{2a} \right). \tag{3.5}$$

The first zero mode is associated with the quark $\bar{u}_a (k)$; and the second with the quark $\bar{d}_a (k)$, these fields are just the Fourier transform of $u_a (x)$ and $d_a (x)$ respectively. Expanding eq.(3.3) near the zero modes (3.5-i) and (3.5-ii), one gets at the first order in $k_{\mu},$

$$D_{BC} \simeq \sum_{\mu=1}^{4} \gamma^\mu k_{\mu} \sim \sum_{\mu=1}^{4} \gamma^\mu \frac{\partial}{\partial x^\mu}, \tag{3.6}$$

which is precisely the Dirac operator of a free fermion. As such, near each one of the two Dirac points the $BC$ model reads in the continuous space as follows

$$S_{BC} = \int_{\mathbb{R}^4} d^4x \Psi^+ (x) \gamma^\mu \frac{\partial \Psi (x)}{\partial x^\mu} , \tag{3.7}$$

with $\Psi (x)$ standing for the quarks $u (x)$ and $d (x)$ respectively given by integrals over momentum hyper-balls $\mathbb{B}$ centered at $K$ and $K'$ as reported below; see [1] for technical details,

$$u (x) = \sum_y \int_{B_{K}} \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} u (y) , \quad d (x) = \sum_y \int_{B_{K'}} \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} d (y). \tag{3.8}$$
Notice that the way this the above relation is a naive definition of the flavors; a more
concise way to do is that done in [9]. Although not used here below, this method is based
on point splitting fields in the reciprocal space. For later use, it is interesting to notice as
well the following remarkable properties:

First, one has to think about the field \( \Psi (x) \) as the zero mode of the hamiltonian equation

\[
\langle x | H_{BC} | \Psi_E \rangle = E \Psi_E (x), \quad \langle x | H_{BC} | \Psi_0 \rangle = 0,
\]

with hamiltonian\(^4\)

\[
H_{BC} = \gamma^\mu \frac{\partial}{i\partial x^\mu}.
\]

To interpret eq(3.9) as an equation of motion, one has to modify the action \( S_{BC} \) (3.3) by
adding an extra term of the form \( \frac{E}{i} \Psi_E^\dagger (x) \Psi_E (x) \).

Second, since the parameter \( E \) can take any real value, we will also think about the Dirac
spinor \( \Psi_E (x) \) as describing modes of a spinor field living on a 5D space time as given below

\[
\Psi (x,t) = \int \frac{dE}{2\pi \hbar} e^{iEt} \Psi_E (x),
\]

so that the 4D action (3.9) looks as resulting from the following five dimensional one

\[
S_{5D} = \int_{\mathbb{R}^{1+4}} dt d^4x \left\{ \Psi^\dagger (x,t) \left( \sum_{\mu=1}^4 \gamma^\mu \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial t} \right) \Psi (x,t) \right\},
\]

where \( t \) stands for the five dimensional coordinate; and which could be interpreted as a
"time" coordinate encoding non zero energy deformations near the Dirac points. This 5D
interpretation will be developed with further details later on; it will be also linked with the
so called spectral flow hamiltonian given in sub-section 5.3 and appendix B.

3.2 BC fermions in background fields

We start by giving some useful tools on spinors in the euclidian space time \( \mathbb{R}^4 \); then we
describe the Dirac equation of the BC fermions in a generic background field \( \mathcal{F}_{\mu\nu} \).

3.2.1 From \( SO (1,3) \) to \( SO (4) \) and then to \( SO (1,4) \)

First, recall that the quarks are confined relativistic particles described by four dimensional
space time fields \( u_\alpha (x) \) and \( d_\alpha (x) \) transforming as complex 4-component Dirac spinors,

\[
\Psi_\alpha (x) = \begin{pmatrix} \Psi_1 (x) \\ \Psi_2 (x) \\ \Psi_3 (x) \\ \Psi_4 (x) \end{pmatrix}.
\]

\(^4\)Notice that \( H_{BC} \) has been defined as an invariant under the \( SO (4) \) symmetry that follows from the
continuum limit of the hyperdiamond. If one insists on defining the hamiltonian in terms the translation
generator along the 4-th direction, one has to break \( SO (4) \) down to \( SO (3) \); and ends amongst others either
with \( \det \mathcal{F}_{\mu\nu} = 0 \) or with a quantization condition on the background fields \( \mathcal{B} \) and \( \mathcal{E} \).
These particles carry several charges, in particular the sp in charge of the $SO(1,3)$ space time symmetry as well as the electric and color charges respectively given by the $U_{em}$ (1) and the $SU_c(3)$ gauge symmetries. The color charge is understood in eq(3.13); since each component $\Psi_\alpha$ should be read as $\Psi_\alpha^c$ with $c = 1, 2, 3$ the color index; i.e:

$$\Psi_\alpha = \begin{pmatrix} \Psi_1^\alpha \\ \Psi_2^\alpha \\ \Psi_3^\alpha \end{pmatrix}.$$  \hspace{1cm} (3.14)

Recall also that in 4D lattice QCD, one uses euclidian time rather than the Minkowski one, so the standard $SO(1,3)$ Lorentz symmetry gets mapped to $SO(4)$ which is locally isometric to the product

$$SU_L(2) \times SU_R(2).$$ \hspace{1cm} (3.15)

So $SO(1,3)$ scalars type $k_\mu x^\mu = k \cdot r - k_4 x^4$ get replaced by the $SO(4)$ ones namely $k \cdot r + k_4 x^4$; similarly the $SO(1,3)$ semi-norms $k_2^2 - k_4^2 = \zeta^2$ transform into positive norms $k_2^2 + k_4^2 = \zeta^2$ leading to $k_4 = \pm \sqrt{\zeta^2 - k_2^2}$ with imaginary values in the case where $\zeta = 0$. Moreover, the four component $SO(4)$ Dirac spinor $\Psi_\alpha$ (3.13) is essentially made by the direct sum of the two $SU(2)$ spinors

$$\Psi_\alpha = \psi_a \oplus \bar{\xi}_\dot{a}, \hspace{1cm} \Psi = \Psi_L \oplus \Psi_R,$$ \hspace{1cm} (3.16)

where undotted fermions refer to $SU_L(2)$ and dotted ones to $SU_R(2)$. Each one of these Weyl spinors have two components as described in the following table,

| symmetry groups | $SU_L(2) \times SU_R(2)$ | $\simeq SO(4)$ | $SU_c(3)$ |
|-----------------|--------------------------|---------------|-----------|
| representations | $\left( \frac{1}{2}, 0 \right)$ | $\left( \frac{1}{2}, 0 \right) \oplus \left( 0, \frac{1}{2} \right)$ | 3 |
| fields         | $\psi_a$ | $\bar{\xi}_\dot{a}$ | $\Psi_\alpha = \begin{pmatrix} \psi_a \\ \bar{\xi}_\dot{a} \end{pmatrix}$ | $\begin{pmatrix} \Psi_1^a \\ \Psi_2^a \\ \Psi_3^a \end{pmatrix}$ |

The fields $\psi_a(x)$ and $\bar{\xi}_\dot{a}(x)$ are just the left handed and right handed quarks $\Psi_L(x)$ and $\Psi_R(x)$; they carry opposite chiralities under $\gamma_5$ and live on the four dimensional Euclidean space time $\mathbb{R}^4$ parameterized by the local coordinates $x = (x, y, z, \tau)$ with diagonal metric $\delta^\mu\nu = diag(+ + + +)$. These fermions have integer color charges; but fractional electric ones given by $Q_u = \frac{2}{3}$ and $Q_d = -\frac{1}{3}$.

Moreover, in lattice QCD the $BC$-fermions $\tilde{\Psi}(k)$ living at the Dirac points (3.5) have zero energy as they are zero modes of the Dirac operator. Therefore non zero energy excitations induced by deformations near the two Dirac points may be interpreted in terms
of the 5-dimensional direction $t$ introduced above; so that the space with local coordinates $x = (x, y, z, \tau)$ gets promoted to a (1+4) space time with coordinates $$(x; t) \equiv (x, y, z, \tau; t).$$

As such the four component field $\Psi(x)$ gets also promoted to $\Psi(x; t) = \Psi(x, y, z, \tau; t)$ whose expansion with respect to the fifth coordinate $t$ is given by eq[3.11].

### 3.2.2 Dirac equation coupled to $F_{\mu\nu}$

The Dirac equation describing the dynamics of the four component Dirac fermion $\Psi^c(x)$ coupled to the background vector gauge potential $A_\mu(x)$ is given by the following system of four coupled equations

$$\sum_{\mu=1}^{4} \sum_{\beta=1}^{4} \sum_{d=1}^{3} i (\gamma^\mu)^{\beta}_{\alpha} \left[ \delta_{cd} \partial_\mu - \frac{ig_G}{c} (A^G_\mu)^d_{cd} \right] \Psi^d_\beta = E \Psi^c_\alpha, \quad (3.18)$$

where $(A^G_\mu)^d_{cd} = \sum I A^{I}_\mu T^I_{cd}$ with the generators of $G = SU_c(3) \times U_{em}(1)$ and $g_G$ the gauge coupling constant. Below, we will restrict to the particular case where $A^{U_{em}(1)}_\mu \neq 0$ and $A^{SU_c(3)}_\mu = 0$; the case with $F^{SU_c(3)}_{\mu\nu} \neq 0$ will be discussed in conclusion section. In the case $A^{U_{em}(1)}_\mu \neq 0$ and $A^{SU_c(3)}_\mu = 0$, the above equation reduces to,

$$i\gamma^\mu \left( \partial_\mu - i \frac{Q}{c} A_\mu \right) \Psi = E \Psi, \quad (3.19)$$

where now $A_\mu$ stands for the electromagnetic gauge potential $A^{U_{em}(1)}_\mu$. Notice also the two following things:

First, by help of the constraint equation $\partial F_{\mu\nu} = 0$ expressing the fact that $F_{\mu\nu}$ is a constant, it is not difficult to check that the electromagnetic potential vector $A_\mu$ is linear in space time coordinate positions $x^\mu$ as follows,

$$A_\mu = \frac{1}{2} F_{\mu\nu} x^\nu, \quad \partial^\nu A_\mu = \frac{\partial A_\mu}{\partial x^\nu} = \frac{1}{2} F_{\mu\nu} \delta^{\mu\nu} = \frac{1}{2} Tr (F) = 0, \quad (3.20)$$

where we have dropped out the irrelevant integration constants. Generally speaking, the antisymmetric tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ capture six real degrees of freedom, three components for the "magnetic" field $B$ and three components of the "electric" field $E$,

$$F_{\mu\nu} = \begin{pmatrix} 0 & +B_3 & -B_2 & +E_1 \\ -B_3 & 0 & +B_1 & +E_2 \\ +B_2 & -B_1 & 0 & +E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix}, \quad \det F_{\mu\nu} = (E \cdot B)^2. \quad (3.21)$$

From the 5-dimensional view, this matrix should be reads as a 4×4 submatrix of the following 5×5 antisymmetric one,

$$F_{MN} = \begin{pmatrix} F_{\mu\nu} & F_{\mu5} \\ F_{\nu5} & 0 \end{pmatrix}. \quad (3.22)$$
where now the $F_{µν}$’s are the 4 components of the electric field in 5D; and $F_{µν}$ the 6 components of the magnetic tensor. Below, we will focus on particular situations where $F_{µν}$ has two degrees of freedom as in eq (3.1). This corresponds to choose the four components of the electromagnetic gauge potential $A_µ$ as follows:

$$A_1 = \frac{B}{2} y, \quad A_2 = \frac{E}{2} x, \quad A_3 = \frac{E}{2} \tau, \quad A_4 = -\frac{E}{2} z,$$

leading to the following constant electromagnetic field strength,

$$F_{µν} = \begin{pmatrix} 0 & -B & 0 & 0 \\ +B & 0 & 0 & 0 \\ 0 & 0 & 0 & -E \\ 0 & 0 & +E & 0 \end{pmatrix},$$

with $\det F_{µν} = B^2 \times E^2$. In this relation, $B$ and $E$ are a priori arbitrary real numbers; but to have a quantum Hall effect, they have to be large enough so that interacting energy is much greater with respect to kinetic energy.

The second comment concerns the four $γ^µ$’s of (3.19); these are the 4D euclidian $4 \times 4$ Dirac matrices obeying the usual 4D Clifford algebra,

$$γ^µγ^ν + γ^νγ^µ = 2δ^µν I_4,$$

with $I_4$ the $4 \times 4$ identity matrix. Notice that the $SO (4) \simeq SU_L (2) \times SU_R (2)$ has 6 generators, 3 of them generate $SU_L (2)$ and the three others generate $SU_R (2)$. In the spinor representation, these are given by the Pauli matrices $τ^i$ and $σ^i$ with generic $2 \times 2$ matrix representation as follows,

$$q^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad q^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad q^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with $q^i$ standing for both $τ^i$ and $σ^i$. The matrices $q^1, q^2$ satisfy the 2D Clifford algebra $q^i q^j + q^j q^i = 2δ^{ij} I_2$ and the $SU (2)$ symmetry bracket $[q^1, q^2] = 2iσ^3$. Notice moreover that the euclidian $γ^µ$ matrices can be realized in terms of tensor products of $τ^i$ and $σ^i$ as follows,

$$γ^i = τ^2 \otimes σ^i, \quad γ^4 = τ^1 \otimes σ^4, \quad γ^5 = τ^3 \otimes σ^4, \quad γ^0 = τ^4 \otimes σ^4,$$

with $τ^4, σ^4 = I_2 ≡ I$ and $γ^5 = γ^1 γ^2 γ^3 γ^4$. More explicitly, we have

$$γ^k = \begin{pmatrix} 0 & -iσ^k \\ iσ^k & 0 \end{pmatrix}, \quad γ^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad γ^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad γ^0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

The anticommutator of the $γ^µ$ matrices verify the $4D$ Clifford algebra (3.25) and their commutators $γ^{[µν]}$ give precisely the 6 generators of the spinorial representation of the $SO (4)$ symmetry,

$$γ^i γ^j - γ^j γ^i = 2iε^{ijk} (τ^2 \otimes σ^k),$$

$$γ^4 γ^i - γ^i γ^4 = 2iε^{123} (τ^3 \otimes σ^i).$$

\[ \text{– 13 –} \]
where $\varepsilon^{ijk}$ is the completely antisymmetric 3D Levi-Civita tensor.

Now, we turn to determine the spectrum of BC fermions in background fields by solving eq(3.19); but before that it is interesting to study the properties of the non commutative algebra of the covariant derivatives; then turn back to compute the spectrum.

4. The Lie algebra of the gauge covariant derivatives

As we will see, the algebra of the $D_{\mu}$ covariant derivatives in constant background fields has much to do with Heisenberg algebra of quantum harmonic oscillators. Because of the tensor structure of $\mathcal{F}_{\mu\nu}$, we distinguish two basic cases: first the case of two uncoupled quantum oscillators associated with the choice $[3.24]$ and $\det \mathcal{F}_{\mu\nu} \neq 0$. Then we consider the general case $[3.21]$ with $\det \mathcal{F}_{\mu\nu} = (\vec{E}.\vec{B})^2 \neq 0$ describing two coupled quantum harmonic oscillators.

4.1 uncoupled case: $\mathcal{F}_{\mu\nu} = \mathcal{B}\varepsilon_{\mu34} + \mathcal{E}\varepsilon_{12\mu}$

Generally, the four gauge covariant derivatives $D_1, D_2, D_3, D_4$ satisfy the generic commutation relations

$$[D_{\mu}, D_{\nu}] = -i\frac{Q}{c} \mathcal{F}_{\mu\nu},$$

but because of the choice $[3.24]$ of the background fields, they reduce to,

$$[D_1, D_2] = i\frac{Q\mathcal{B}}{c}, \quad [D_3, D_4] = i\frac{Q\mathcal{E}}{c},$$

and all others vanishing. These are very special relations; first because their right hand sides are non zero constant numbers showing amongst others that the large limits of $\mathcal{B}$ and $\mathcal{E}$ induce a non commutative geometry in the euclidian space,

$$[x^\mu, x'^\nu] = \frac{4ic}{Q} \mathcal{G}^{\mu\nu}$$

with $\mathcal{G}^{\mu\nu}$ given by

$$\mathcal{G}^{\mu\nu} = \frac{1}{\mathcal{E}\mathcal{B}} \begin{pmatrix} 0 & -\mathcal{E}_3 & \mathcal{E}_2 & -\mathcal{B}_1 \\ -\mathcal{E}_3 & 0 & -\mathcal{E}_1 & -\mathcal{B}_2 \\ -\mathcal{E}_2 & -\mathcal{E}_1 & 0 & -\mathcal{B}_3 \\ \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{B}_3 & 0 \end{pmatrix}$$

Second, the four covariant derivatives organize in $2+2$ capturing a complex structure as follows,

$$D_1 - iD_2 = \frac{\partial}{\partial u} + \frac{Q\mathcal{B}}{2c}\bar{u}, \quad D_1 + iD_2 = \frac{\partial}{\partial \bar{u}} - \frac{Q\mathcal{B}}{2c} u,$$

$$D_3 - iD_4 = \frac{\partial}{\partial v} + \frac{Q\mathcal{E}}{2c}\bar{v}, \quad D_3 + iD_4 = \frac{\partial}{\partial \bar{v}} - \frac{Q\mathcal{E}}{2c} v,$$

and their adjoint conjugates. In getting eq(4.5), we have used the explicit expressions $D_1 = \partial_1 - i\frac{Q\mathcal{B}}{2c} y, D_2 = \partial_2 + i\frac{Q\mathcal{B}}{2c} x$ together with similar relations
for $D_3$, $D_4$; and we have set

$$u = x + iy, \quad \frac{\partial}{\partial u} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right),$$
$$v = z + i\tau, \quad \frac{\partial}{\partial v} = \frac{1}{2} \left( \frac{\partial}{\partial z} - i \frac{\partial}{\partial \tau} \right). \quad (4.6)$$

To study the representations of eqs (4.2), one has to specify the sign of $\frac{Q_B}{c}$ and $\frac{Q_E}{c}$. Assuming $\frac{Q_B}{c} > 0$, $\frac{Q_E}{c} > 0$ for simplicity and which means that $Q, B, E$ have the same sign; and setting

$$i (D_1 - iD_2) = A \sqrt{\frac{2Q_B}{c}}, \quad i (D_1 + iD_2) = A^\dagger \sqrt{\frac{2Q_B}{c}},$$
$$i (D_3 - iD_4) = C \sqrt{\frac{2Q_E}{c}}, \quad i (D_3 + iD_4) = C^\dagger \sqrt{\frac{2Q_E}{c}}, \quad (4.7)$$

the commutation relations (4.2) read also as

$$[A, A^\dagger] = I, \quad [C, C^\dagger] = I, \quad [A, C] = 0, \quad [A, C^\dagger] = 0; \quad (4.8)$$

These relations (4.8) show that the Dirac fermion in the background field $F_{\mu\nu}$ (3.24) describe a priori two quantum harmonic oscillators with oscillation frequencies

$$\varpi = \sqrt{\frac{2Q_B}{c}}, \quad \varpi' = \sqrt{\frac{2Q_E}{c}}. \quad (4.9)$$

The operators $A^\dagger A$ and $C^\dagger C$ giving the number of energy excitations in $\varpi$ and $\varpi'$ units respectively read in terms of the gauge covariant derivatives as follows

$$\frac{2Q_B}{c} A^\dagger A = (D_1)^2 + (D_2)^2 + i [D_1, D_2] = (D_1)^2 + (D_2)^2 - \frac{Q_B}{c}, \quad (4.10)$$

and similarly

$$\frac{2Q_E}{c} C^\dagger C = (D_3)^2 + (D_4)^2 + i [D_3, D_4] = (D_3)^2 + (D_4)^2 - \frac{Q_E}{c}. \quad (4.11)$$

Using the space time variables $(x, y, z, \tau)$ and their derivatives $\partial_{\mu}$, these operators read more explicitly like

$$(D_1)^2 + (D_2)^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \left| \frac{Q_B}{2c} \right|^2 \left( x^2 + y^2 \right) + \left| \frac{Q_B}{2c} \right| L_{xy},$$
$$= -\frac{\partial^2}{\partial u^2} + \frac{1}{2} \left| \frac{Q_B}{2c} \right|^2 u\bar{u} + \left| \frac{Q_B}{2c} \right| L_{u\bar{u}}, \quad (4.12)$$

and

$$(D_3)^2 + (D_4)^2 = -\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2} + \left| \frac{Q_E}{2c} \right|^2 \left( z^2 + \tau^2 \right) + \left| \frac{Q_E}{2c} \right| L_{z\tau},$$
$$= -\frac{\partial^2}{\partial \bar{v}^2} + \frac{1}{2} \left| \frac{Q_E}{2c} \right|^2 (v\bar{v}) + \left| \frac{Q_E}{2c} \right| L_{v\bar{v}}, \quad (4.13)$$

– 15 –
where one recognizes the harmonic potential energy induced by the electromagnetic flux and the angular momentum operator components $L_{xy}$ and $L_{z\tau}$

$$L_{xy} = -i \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right), \quad L_{z\tau} = -i \left( z \frac{\partial}{\partial \tau} - \tau \frac{\partial}{\partial z} \right), \quad (4.14)$$
describing respectively rotations around of the oscillations of the particles the $x$-$y$ and $z$-$\tau$ planes of the 4D space-time. Notice that the hermitian operators $L_{xy}$ and $L_{z\tau}$ are nothing but the charge operators of the abelian $U_L(1)$ and $U_R(1)$ sub-symmetries of the $SU_L(2)$ and $SU_R(2)$ respectively. Indeed, using the complex variables (4.6) and putting back into (4.14), we find,

$$L_{xy} = u \frac{\partial}{\partial u} - \bar{u} \frac{\partial}{\partial \bar{u}}, \quad L_{z\tau} = v \frac{\partial}{\partial v} - \bar{v} \frac{\partial}{\partial \bar{v}}, \quad (4.15)$$

acting on the monomials $u^{n_1} \bar{u}^{n_2} v^{m_1} \bar{v}^{m_2}$ as follows

$$[L_{xy}, u^{n_1} \bar{u}^{n_2} v^{m_1} \bar{v}^{m_2}] = (n_1 - n_2) u^{n_1} \bar{u}^{n_2} v^{m_1} \bar{v}^{m_2},$$

$$[L_{z\tau}, u^{n_1} \bar{u}^{n_2} v^{m_1} \bar{v}^{m_2}] = (m_1 - m_2) u^{n_1} \bar{u}^{n_2} v^{m_1} \bar{v}^{m_2}. \quad (4.16)$$

Notice that for the particular cases $n_1 = n_2$ and $m_1 = m_2$, there is no $U_L(1)$ nor $U_R(1)$ charges. So wave functions $\Psi_\alpha = \Psi_\alpha(u, v, \bar{u}, \bar{v})$ that depend only on $u\bar{u}$ and $v\bar{v}$ have no angular momenta.

Moreover, using the expression of the matrices $\gamma^\mu$, we can write this matrix operator $(H_{BC})_{\alpha\beta} = i\gamma^\mu_{\alpha\beta} D_\mu$ in terms of the creations operators $A^\dagger, C^\dagger$ and the annihilation ones $A, C$ as follows:

$$(H_{BC})_{\alpha\beta} = \frac{1}{i} \begin{pmatrix} 0 & 0 & -\varpi^\prime C^\dagger & -\varpi A \\ 0 & 0 & -\varpi A^\dagger & \varpi^\prime C \\ \varpi^\prime C & \varpi A & 0 & 0 \\ \varpi A^\dagger & -\varpi^\prime C^\dagger & 0 & 0 \end{pmatrix}, \quad (4.17)$$

with $H_{BC}^\dagger = H_{BC}$. Moreover, using the commutation properties $CA^\dagger = A^\dagger C$ and $CA = AC$ and their adjoint conjugates following from the choice (3.24), the squared BC Hamiltonian $H_{BC}^2$ reads as follows,

$$\begin{pmatrix} \varpi^2 AA^\dagger + \varpi^2 C^\dagger C & 0 & 0 & 0 \\ 0 & \varpi^2 A^\dagger A + \varpi^2 CC^\dagger & 0 & 0 \\ 0 & 0 & \varpi^2 AA^\dagger + \varpi^2 CC^\dagger & 0 \\ 0 & 0 & 0 & \varpi^2 A^\dagger A + \varpi^2 C^\dagger C \end{pmatrix}$$

which allow to determine spectrum of the Borici-Creutz spectrum; this will be done after studying the coupled case.
4.2 coupled case: \( \mathcal{F}_{\mu \nu} = \mathcal{B}^{ij} \varepsilon_{\mu \nu ij} + \mathcal{E}^{[4]} \varepsilon_{4 \mu \nu} \)

In the case where the 6 components \( \mathcal{B}^{ij} \) and \( \mathcal{E}^{[4]} \), \((i, j = 1, 2, 3)\), of the background field \( \mathcal{F}_{\mu \nu} \) are non zero as in (3.21), the previous two Heisenberg algebras (4.8) get coupled. Indeed, using the relation \([D_\mu, D_\nu] = -i \frac{Q}{c} \mathcal{F}_{\mu \nu} \) and eq (4.7), we find in addition to (4.18)

\[
\begin{align*}
[A, A^\dag] = I \quad , \\
[C, C^\dag] = I 
\end{align*}
\]

the following couplings

\[
\begin{align*}
[A, C] &= \lambda, \\
[A^\dag, C^\dag] &= -\bar{\lambda}, \\
[A, C^\dag] &= \zeta, \\
[A^\dag, C] &= -\bar{\zeta},
\end{align*}
\]

where we have set,

\[
\begin{align*}
\lambda &= \frac{1}{2\sqrt{BC}} \left[ (\mathcal{F}_{14} + \mathcal{F}_{23}) + i (\mathcal{F}_{13} - \mathcal{F}_{24}) \right], \\
\zeta &= \frac{1}{2\sqrt{BC}} \left[ (\mathcal{F}_{23} - \mathcal{F}_{14}) + i (\mathcal{F}_{13} + \mathcal{F}_{24}) \right].
\end{align*}
\]

Using the BC hamiltonian (4.17), we find that its square \( H^2_{BC} \) is given by,

\[
\begin{pmatrix}
\omega^2 A A^\dag + \omega^2 C C^\dag & -\omega \omega' \zeta \\
-\omega \omega' \bar{\zeta} & \omega^2 A^\dag A + \omega^2 C C^\dag \\
0 & 0 \\
0 & 0 \\
0 & \omega^2 A A^\dag + \omega^2 C C^\dag & -\omega \omega' \lambda \\
0 & -\omega \omega' \bar{\lambda} & \omega^2 A^\dag A + \omega^2 C^\dag C
\end{pmatrix}
\]

and is no longer diagonal. Moreover, the operator numbers \( A^\dag A \) and \( C^\dag C \) do no longer commute; a difficulty that may be overcome by treading \( \lambda \) and \( \zeta \) as small perturbation parameters.

5. More on AQHE in BC model

First we compute the spectrum of BC fermions in the constant background fields; then we determine the filling factor \( \nu_{BC} \) of the associated quantum Hall effect. After that, we study the link with spectral flow method of [34, 4].

5.1 Spectrum of BC Hamiltonian \( H_{BC} \)

Starting from the BC fermion’s equation \( H_{BC} |\Psi\rangle = E |\Psi_E\rangle \), then substituting the matrices \( \gamma^\mu \) by their expressions (3.28) and the Dirac fermion \( \Psi_\alpha \) by its two Weyl spinors \( (\psi_a, \bar{\xi}_{\bar{a}}) \), we get

\[
\begin{pmatrix}
0 & \mathcal{O} \\
\mathcal{O}^\dag & 0
\end{pmatrix}
\begin{pmatrix}
\psi_a \\
\bar{\xi}_{\bar{a}}
\end{pmatrix}
= E
\begin{pmatrix}
\psi_a \\
\bar{\xi}_{\bar{a}}
\end{pmatrix},
\]

(5.1)
where the $O$ and $O^\dagger$ operators are given by

$$O = i \begin{pmatrix} -\varpi C^\dagger & -\varpi A \\ -\varpi A^\dagger & \varpi C \end{pmatrix}, \quad O^\dagger = i \begin{pmatrix} \varpi C & \varpi A \\ \varpi A^\dagger & -\varpi C^\dagger \end{pmatrix}. \quad (5.2)$$

These operators can be also written in the following condensed form

$$O = \sigma^4 D^4 + i \sum_{k=1}^3 \sigma^k D, \quad O^\dagger = \sigma^4 D^4 - i \sum_{k=1}^3 \sigma^k D, \quad (5.3)$$

which let understand that solutions may be as well obtained by using quaternions. To deal with eqs (5.1), we will use an explicit method relying on acting on its both sides of by $H_{BC}$. We get the following set of equations

$$OO^\dagger \psi = E^2 \psi, \quad O^\dagger O \bar{\xi} = E^2 \bar{\xi}, \quad (5.4)$$

showing that the two Weyl spinors $\psi_a$ and $\bar{\xi}_a$ can be treated separately; thank to the choice (3.24). This property was also expected as there is no mass term that couples the left and right components of the Dirac fermion. To solve (5.4), we use the following expression of $OO^\dagger$ and $O^\dagger O$,

$$OO^\dagger = \begin{pmatrix} \varpi^2 C^\dagger C + \varpi^2 [A^\dagger A + 1] & 0 \\ 0 & \varpi^2 [C^\dagger C + 1] + \varpi^2 A^\dagger A \end{pmatrix}, \quad (5.5)$$

and

$$O^\dagger O = \begin{pmatrix} \varpi^2 [C^\dagger C + 1] + \varpi^2 [A^\dagger A + 1] & 0 \\ 0 & \varpi^2 C^\dagger C + \varpi^2 A^\dagger A \end{pmatrix}. \quad (5.6)$$

Notice that these matrix operators don’t commute; they obey the commutation relation

$$OO^\dagger - O^\dagger O = -\varpi^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.7)$$

Then putting the expression of $OO^\dagger$ and $O^\dagger O$ back into eq (5.4), we obtain the following system of 4 equations which allow to determine the 4 component of the Dirac spinor

$$[\varpi^2 C^\dagger C + \varpi^2 [A^\dagger A + 1]] \psi_1 = E^2 \psi_1, \quad \psi_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$[\varpi^2 [C^\dagger C + 1] + \varpi^2 A^\dagger A] \psi_2 = E^2 \psi_2, \quad \psi_2 = \begin{pmatrix} \psi_2 \\ \bar{\xi}_2 \end{pmatrix},$$

$$[\varpi^2 [C^\dagger C + 1] + \varpi^2 [A^\dagger A + 1]] \bar{\xi}_1 = E^2 \bar{\xi}_1, \quad \bar{\xi}_1 = \begin{pmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \end{pmatrix},$$

$$[\varpi^2 C^\dagger C + \varpi^2 A^\dagger A] \bar{\xi}_2 = E^2 \bar{\xi}_2. \quad \bar{\xi}_2 = \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_2 \end{pmatrix}. \quad (5.8)$$
To solve these relations, it is useful to introduce the wave function basis \( \{ \theta_{n,m} \} \) of two uncoupled quantum harmonic oscillators. These wave functions are factorized as

\[
\mathbf{A}_{n,m}(u, \bar{u}, v, \bar{v}) = \theta_n(u, \bar{u}) \times \theta'_m(v, \bar{v}), \quad n, m \geq 0, 
\]

(5.9)

with the constraint relations

\[
A \times \theta_0(u, \bar{u}) = \left( \frac{\partial}{\partial u} + \frac{Q_R}{2\epsilon} \bar{u} \right) \theta_0(u, \bar{u}) = 0 
\]

(5.10)

\[
C \times \theta'_0(v, \bar{v}) = \left( \frac{\partial}{\partial v} + \frac{Q_R}{2\epsilon} \bar{v} \right) \theta'_0(v, \bar{v}) = 0 
\]

solved as follows

\[
\theta_0(u, \bar{u}) = \mathcal{N}_0(u) e^{-\frac{Q_R}{4\epsilon} u \bar{u}}, \quad \theta'_0(v, \bar{v}) = \mathcal{N}'_0(v) e^{-\frac{Q_R}{4\epsilon} v \bar{v}} 
\]

(5.11)

where the complex functions \( \mathcal{N}_0(u) \) and \( \mathcal{N}'_0(v) \) obey the holomorphic conditions \( \frac{\partial \mathcal{N}_0(u)}{\partial u} = 0 \) and \( \frac{\partial \mathcal{N}'_0(v)}{\partial v} = 0 \); and are generally fixed by the charge of the angular momentum operators and the normalization condition of the wave functions. The others wave functions \( \theta_n(u, \bar{u}) \) and \( \theta'_m(v, \bar{v}) \) are obtained as usual by applying the creation operators.

\[
\theta_n(u, \bar{u}) \simeq \left( \frac{2\partial}{\partial u} - \frac{Q_R}{2\epsilon} u \right)^n \theta_0(u, \bar{u}), \\
\theta'_m(v, \bar{v}) \simeq \left( \frac{2\partial}{\partial v} - \frac{Q_R}{2\epsilon} v \right)^m \theta'_0(v, \bar{v}).
\]

(5.12)

Because of the algebraic properties \( [A^\dagger A, (A^\dagger)^n] = n (A^\dagger)^n \) and \( [C^\dagger C, (C^\dagger)^m] = m (C^\dagger)^m \), the \( |\mathbf{A}_{n,m}\rangle \) states obey amongst others the following properties

\[
A^\dagger A |\mathbf{A}_{n,m}\rangle = n |\mathbf{A}_{n,m}\rangle, \\
C^\dagger C |\mathbf{A}_{n,m}\rangle = m |\mathbf{A}_{n,m}\rangle, \\
A^\dagger AC^\dagger C |\mathbf{A}_{n,m}\rangle = nm |\mathbf{A}_{n,m}\rangle.
\]

(5.13)

Taking the two Weyl spinors \( \psi_a \) and \( \xi_{\bar{a}} \) like

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\xi_1 \\
\xi_2
\end{pmatrix}
= 
\begin{pmatrix}
\mathbf{A}_{n-1,m} \\
\mathbf{A}_{n,m-1} \\
\mathbf{A}_{n-1,m-1} \\
\mathbf{A}_{n,m}
\end{pmatrix},
\]

(5.14)

with remarkable chiral zero mode

\[
\begin{pmatrix}
0 \\
0 \\
0 \\
\mathbf{A}_{0,0}
\end{pmatrix}
\]

(5.15)

and putting back into \( \mathcal{O} \mathcal{O}^\dagger \psi_a = E_{n,m}^2 \psi_a \) and \( \mathcal{O}^\dagger \mathcal{O} \xi_{\bar{a}} = E_{n,m}^2 \xi_{\bar{a}} \), we obtain the following energy spectrum,

\[
E_{n,m}^{\pm} (\varphi, \varphi') = \pm \hbar \sqrt{n \varphi^2 + m \varphi'^2}.
\]

(5.16)
This relation is comparable to (2.18) of graphene namely 
\[ \varepsilon_n^{2D-graphene} = \pm \sqrt{\frac{2\mu B}{c}} |n| \]. Notice also that in the case of the background fields \( \mathcal{B} \neq 0 \); but \( \mathcal{E} = 0 \), the tensor \( F_{\mu\nu} \) is degenerate since \( \det F_{\mu\nu} \) and the algebras of the covariant derivatives \( (4.2) \) reduces to 
\[ [D_1, D_2] = i \frac{Q B}{c}, \quad D_3 = \frac{\partial}{\partial z}, \quad D_4 = \frac{\partial}{\partial \tau}, \] (5.17)
and all remaining others which are just commutation relations vanish. Therefore one has only one quantum harmonic oscillators with frequency \( \varpi \) whose energy spectrum is given by,
\[ E_n^\pm (\varpi, k) = \pm \hbar \sqrt{n\varpi^2 + k^2} \] (5.18)
with \( k^2 = k_3^2 + k_4^2 \) and \( p_3 = h k_3, p_4 = h k_4 \) are the momenta along the \( z- \) and \( \tau- \) directions. Notice that for the case \( p_3 = p_4 = 0 \), the energy levels are given by \( E_n^\pm (\varpi, k = 0) = \pm \hbar \varpi \sqrt{n} \) with \( n \) a positive integer. Notice also that for the fundamental state \( n = 0 \), the energies vanish \( E_0^+ (\varpi, k = 0) = E_0^- (\varpi, k = 0) = 0 \); and the valence and conducting bands touch; for an illustration see fig. 1.

![Figure 1: the first levels of the energy spectrum](image)

5.2 Filling factor \( \nu_{BC} \) of BC fermions in background fields

Before computing \( \nu_{BC} \), we start by recalling that the wave function \( \Psi(x, t) \) of the BC fermions in the background fields (3.24) is given by
\[ \Psi(x, t) = \sum_{n,m=-\infty}^{\infty} e^{-\frac{i}{\hbar} E_{n,m} t} \Psi_{n,m}(x) \] (5.19)
with wave modes \( \Psi_{n,m} = (\psi_{n,m}, \bar{\xi}_{n,m}) \) and energies \( E_{n,m} \) respectively given by eqs (5.14) and (5.16). Since the waves \( \Psi_{n,m} \) form a basis and using probability interpretation of \( \Psi(x, t) \) we also have,
\[ \int_{\mathbb{R}^4} d^4 x \Psi_{n,m}^*(x) \Psi_{n',m'}(x) = \delta_{n,n'} \delta_{m,m'} \] (5.20)
These relations can be also derived by help of well known orthogonality of the waves functions $\theta_n(x, y)$ and $\theta'_m(z, \tau)$ of the quantum harmonic oscillators. Now, using these relations, one can compute the filling factor $\nu_{BC}$ of the AQHE of the Borghi-Creutz fermions in the constant background fields $B$ and $E$. This factor is given as usual by $\frac{N_F}{N_\phi}$; that is the ratio of the number $N_F$ of particle states with respect the number of quantum fluxes $N_\phi$. The latter number reads as follows:

$$N_\phi = \frac{1}{2\pi} \int_{S^2} dx^\mu \wedge dy^\nu F_{\mu\nu}.$$  \hspace{1cm} (5.21)

and, by using eq (3.24), can be split as $N_B = \int_{S^2} B dx \wedge dy$ and $N_E = \int_{S^2} E dz \wedge d\tau$. For a unit flux $N_\phi = 1$, the number $N_F$ of polarized particles that fill the band energy $0 \leq E_{n,m}^2 \leq E_{N,M}^2$ is given by the relation

$$N_F = \frac{1}{2} \int_{E_{N,M}^2 \leq E_{n,m}^2 \leq E_{N,M}^+} dt d^4x |\Psi(x, t)|^2.$$  \hspace{1cm} (5.22)

Substituting $\Psi(x, t)$ by eq (5.19) and using the orthogonality properties, we get

$$N_F = \frac{(2N + 1) (2M + 1)}{2}.$$  \hspace{1cm} (5.23)

In the particular case $N = M = 0$, the energies $E_{0,0}^-$ and $E_{0,0}^+$ vanish; and the conducting and valence coincide; so the fundamental state $\Psi_{0,0}$ with zero energy is filled by $\frac{1}{2}$ particle and $\frac{1}{2}$ hole. The number of polarized particles filling the band energy $0 \leq E_{n,m}^2 \leq E_{N,M}^2$ is equal to $N_F$. Therefore, taking into account the fact that a $BC$ fermion has four spin polarizations; and each fermion has three colors, the general expression of the filling factor $\nu_{BC}$ reads as follow:

$$\nu_{BC} = \frac{12k_V}{2} \frac{(2N + 1) (2M + 1)}{2},$$  \hspace{1cm} (5.24)

where $k_V$ stands for the number of Dirac valleys which, for BC fermions, is equal to 2 as in 2D graphene.

5.3 Link with spectral flow and topological index

In this subsection, we want to comment on the relation between the hamiltonian, that we have used in this study and which has been motivated from 2D graphene, and the so called spectral flow hamiltonian considered recently in [34] in connection with the theoretical foundation for the index theorem for fermions on lattices. More analysis on this matter and other issues are further developed in the appendices A and B.

In studying the AQHE in 4-dimensions, we used the equation (3.13) describing the dynamics of a fermion near a generic Dirac point corresponding to a zero mode solving $D\Psi = 0$. This equation, that we put in the following simple form,

$$(D - E) \Psi = 0,$$  \hspace{1cm} (5.25)

depends on the spectral parameter $E$, which in relativistic theory, has a dimension of mass; it captures deformations away from the Dirac point and allows to open the gap between...
Figure 2: On left the Dirac point where conducting and valence bands touch. On right the gap between conducting and valence bands induced by the mass term.

the conducting and the valence bands as shown on fig 2. Recall that in eq(5.25), the Dirac operator in the background field $F_{\mu\nu}$ as in eq(3.24) is an antihermitian operator given by

$$D = \gamma^\mu D_\mu = \gamma^\mu (\partial_\mu - iA_\mu).$$

Notice that the corresponding hamiltonian used in this study namely $H (E) = (D - E)$ looks quite similar to the one used in the method of spectral flow of [34]. There the spectral flow hamiltonian is given by the following

$$H_{sp} (m) = \gamma_5 (D - m)$$

with

$$\gamma_5 (D - m) \Psi = \lambda (m) \Psi,$$

and the eigenvalues $\lambda (m)$ depending on the spectral parameter $m$. Clearly the hamiltonian $H (E)$ and the spectral one $H_{sp} (m)$ are close cousins and are related as follows,

$$H_{sp} (m) = \gamma_5 H (m).$$

The trick of multiplication by $\gamma_5$ has two nice and remarkable effects; it makes $H_{sp} (m)$ hermitian and allows to lift the degeneracy of energies with opposite chiralities. Indeed, by splitting the Dirac spinor $\Psi$ into its two chiral components $\Psi_\pm$ like

$$\Psi_+ = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \quad \gamma_5 \Psi_\pm = \pm \Psi_\pm$$

and restricting the study to the Dirac zero modes $D\Psi = 0$, the eigenvalue equation of the spectral flow hamiltonian reads as follows

$$H_{sp} (m) \Psi_\pm = \mp m \Psi_\pm,$$

and captures perfectly the data of fig 2 contrary to $H (E)$ which fails in this matter since we have

$$H (m) \Psi_\pm = -m \Psi_\pm.$$
It happens that the spectral flow hamiltonian $H_{sp} (m)$ is a powerful ingredient as it leads to exactly the topological index $\text{Ind} (D)$ of the Dirac operator. Following [34, 9] and the analysis developed in the appendices A and B, one can show that the spectrum of $H_{sp} (m)$ is, up to an irrelevant sign, the topological index

$$\text{Tr} \left( \gamma_5 e^{-tD} \right) = N_+ - N_-.$$  

(5.33)

In this relation, $N_+$ is the number of zero modes with positive chirality; and $N_-$ is the number of zero modes with negative chirality. The difference $N_+ - N_-$ is equal to the topological charge $Q_{\text{top}}$ of the gauge field; i.e: $N_+ - N_- = Q_{\text{top}}$. For the case of the hamiltonian $H (m)$, the corresponding quantity (restricted to the zero modes) is given by

$$\text{Tr} \left( e^{-tD} \right) \mid_{\text{zero modes}} = N_+ + N_-,$$

(5.34)

it is not a topological quantity; it gets extra contribution from non zero modes. We end by noting that in the present 4D case, we have

| quarks $q$ | $n_{q+}$ | $n_{q-}$ | degeneracy | $N_{q+}$ | $N_{q-}$ |
|---|---|---|---|---|---|
| $u (x)$ | 0 | 1 | $|Q_x| |Q_z|$ | 0 | $|Q_x| |Q_z|$ |
| $d (x)$ | 0 | 1 | $|Q_x| |Q_z|$ | 0 | $|Q_x| |Q_z|$ |

(5.35)

where the integer $|Q_x| |Q_z|$ is the degeneracy of the zero modes; see eq(7.36-7.37) of appendix A. The index theorem reads therefore $N_+ - N_- = (N_{u+} + N_{d+}) - (N_{u-} + N_{d-}) = 2 |Q_x| |Q_z|$ in agreement with results of [9]. More comments and technical details are reported in appendices A and B.

6. Conclusion and comments

In this paper, we have studied the anomalous quantum Hall effect in 4D lattice QCD with a special focus on Borici-Creutz fermions in presence of a constant background field strength $F_{\mu \nu}$. Recall that $BC$ fermions were proposed to simulate the dynamics of light quarks using four dimensional lattice QCD and where shown to be intimately linked the 4D extension of graphene on hyperdiamond. We have shown that in the neighborhood of the two Dirac valleys $[3, 3]$, the $BC$ model has a 5D field theoretical interpretation with the following features:

(1) the background fields given by antisymmetric tensor $F_{\mu \nu}$ appear as the $4 \times 4$ submatrix of a 5D field strength $F_{MN}$ having $4 + 6$ components distributed like

$$F_{MN} = \begin{pmatrix} F_{\mu \nu} & F_{\mu 5} \\ -F_{\mu 5} & 0 \end{pmatrix}.$$  

In this 5D interpretation, the four components $F_{\mu 5}$ describing a 5D electric field are equal to zero ($F_{\mu 5} = 0$); while the remaining 6 components $F_{\mu \nu}$, which describe the components
of a 5D magnetic field, have been used to study the AQHE of BC fermions. This property explains also the behavior of the background fields $B$ and $E$ as a magnetic fields respectively rotating left and right fermions $\Psi_L$ and $\Psi_R$; and leads to AQHE with cyclotron frequencies

$$\varpi = \sqrt{\frac{2Q_em}{c}} \text{ and } \varpi' = \sqrt{\frac{2Q_em}{c}}.$$ 

(2) The Hamiltonian $H_{BC}$ is precisely given by the euclidian 4D Dirac operator $\sum_{\mu=1}^{4} i\gamma^{\mu}D_{\mu}$ with $D_{\mu} = (\partial_{\mu} - i\frac{Q_em}{2c}F_{\mu\nu}x^\nu)$ capturing interactions between BC fermions and the "5D magnetic" tensor $F_{\mu\nu}$. Because of the non zero flux induced by the background field $F_{\mu\nu}$, the commuting algebra of the four flat space translations $\partial_{\mu}$ turns into the non commutative $[D_{\mu}, D_{\nu}] \sim iF_{\mu\nu}$, $[D_{\mu}, F_{\nu\rho}] = 0$. In the case where $F_{\mu\nu}$ is chosen as $B\varepsilon_{\mu\nu34} + E\varepsilon_{12\mu\nu}$, the algebra of the $D_{\mu}$'s splits into two uncoupled Heisenberg ones,

$$[D_1, D_2] = -i\frac{Q_em}{2c}I \quad , \quad [D_3, D_4] = -i\frac{Q_em}{2c}I \quad ,$$

whose spectrum as well as the associated AQHE have been explicitly studied in this paper. This splitting teaches us moreover that $[D_{\mu}, D_{\nu}] \sim iF_{\mu\nu}$ and describes in general two coupled quantum harmonic oscillators which can be studied by using perturbation theory if the following conditions hold

$$|F_{14} + F_{23}|^2 + |F_{13} - F_{24}|^2 << 2 |F_{12}F_{34}| \quad ,

|F_{14} - F_{23}|^2 + |F_{13} + F_{24}|^2 << 2 |F_{12}F_{34}| \quad .$$

We conclude this work by giving a comment on the extension of the result given in this study to the case where the background gauge potential is valued in the Cartan subalgebra of $SU_c(3) \times U_{em}(1)$,

$$U^2(1) \times U_{em}(1) \subset SU_c(3) \times U_{em}(1).$$

This subalgebra is a 3-dimensional abelian algebra generated, in addition to $h_uem(1) = Q_{em}I$, by the two diagonal hermitian $3 \times 3$ matrices $h_1$ and $h_2$ of $SU_c(3)$ which read in the Gell-Mann vector basis as follows

$$h_{em} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad h_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$ 

In presence of non zero constant background fields $F_{\mu\nu}^{(em)} \neq 0$, $F_{\mu\nu}^{(1)} \neq 0$, $F_{\mu\nu}^{(2)} \neq 0$, the gauge covariant derivatives reads as follows,

$$D_{\mu} = \partial_{\mu} - x^{\nu} \left( \frac{ig}{2c}h_1F_{\nu\mu}^{(1)} + \frac{ig}{2c}h_2F_{\nu\mu}^{(2)} + \frac{iQ_{em}F_{\mu\nu}^{(em)}}{2c} \right)$$

obeying the non commutative commutation relations $[D_{\mu}, D_{\nu}] \sim iF_{\mu\nu}$; but now with $F_{\mu\nu} = \sum_{I} h_{I}F_{\mu\nu}^{I}$ that reads in matrix notation like

$$\begin{pmatrix} F_{\mu\nu}^{(em)} + F_{\mu\nu}^{(1)} + F_{\mu\nu}^{(2)} & 0 & 0 \\ 0 & F_{\mu\nu}^{(em)} - F_{\mu\nu}^{(1)} + F_{\mu\nu}^{(2)} & 0 \\ 0 & 0 & F_{\mu\nu}^{(em)} - 2F_{\mu\nu}^{(2)} \end{pmatrix}$$
Moreover, because the $h_I$'s are diagonal matrices; each gauge covariant derivative $D_\mu$ splits into 3 components $D^{(1)}_\mu$, $D^{(2)}_\mu$, $D^{(3)}_\mu$ as given below,

\[
D^{(1)}_\mu = \partial_\mu - \frac{i}{2e} x^\nu \left( Q_{em} F^{(em)}_{\nu\mu} + g F^{(1)}_{\mu\nu} + g F^{(2)}_{\mu\nu} \right)
\]
\[
D^{(2)}_\mu = \partial_\mu - \frac{i}{2e} x^\nu \left( Q_{em} F^{(em)}_{\nu\mu} - g F^{(1)}_{\mu\nu} + g F^{(2)}_{\mu\nu} \right)
\]
\[
D^{(3)}_\mu = \partial_\mu - \frac{i}{2e} x^\nu \left( Q_{em} F^{(em)}_{\nu\mu} - 2g F^{(2)}_{\mu\nu} \right)
\]

and one is left with 6 quantum harmonic oscillators whose uncoupled realization is given by the choice

\[ F^{I}_{\mu\nu} = B^I \varepsilon_{\mu\nu34} + \varepsilon^I \varepsilon_{12\mu\nu}. \]

In this case the filling factor of the AQHE of BC fermions induced by these $F^{I}_{\mu\nu}$'s is given by eq(1.4). In the end, it would be interesting to study the analogue of the Zeeman effect of 2D graphene in 4D lattice QCD by using models like the BC fermions we have considered in this study.

We close this conclusion by two more comments. The first comment concerns the relation between our results and the index theorem. This matter has been described in a condensed form throughout the paper; in particular in subsection 5.3. It is developed with details in appendices A and B. The second comment concerns QHE in lattice QCD as a whole; the present study focused in BC fermions should be viewed as a first step which itself need further investigations; several issues like the explicit breaking of discrete symmetries in BC fermions [10] as well as spin Hall effect and the associated topological indices [50, 51] haven’t been addressed here.

7. Appendix A: Index of Dirac operator

In this section, we describe the index theorem of the Dirac operator $D$ in background gauge configuration. This index provides the relationship between physical properties and the topology of the space in which live the fermions. First, we describe the index in 2-dimensions; both in continuum and lattice QFTs. Then, we consider the main lines of the index in 4-dimensions and its relationship with the so called spectral flow hamiltonian. After that, we make an heuristic interpretation of the filling factors obtained in this study in terms of the index of the Dirac operator.

7.1 case of 2D graphene

We begin by describing the topological index theorem of the Dirac operator in 2-dimensions. Then we apply the construction to the case of 2D graphene.

7.1.1 the index theorem: continuous limit

In the case of fermions living on a 2-dimensional surface in presence of external gauge fields, the index theorem gives a relation between zero modes of the gauged Dirac operator $D = \sigma_\alpha^\mu D_\mu$, with gauge covariant derivative $D_\mu = (\partial_\mu - i A_\mu)$, and the flux $Q = \frac{1}{2\pi} \int_S F_2$.
going through $S$ ($F_2 = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$). Writing this Dirac operator as $2 \times 2$ matrix $\mathcal{D}$ as follows,

$$
\mathcal{D} = \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}, \quad D_+ = D_1 + iD_2, \quad D_- = D_1 - iD_2 \tag{7.1}
$$

and the flux $Q$ through $S$ like $Q = \frac{\mathcal{B} S}{2 \pi}$ with $\mathcal{B} = \frac{1}{2} \varepsilon^{3\mu\nu} F_{\mu\nu}$, the index of the Dirac operator reads explicitly in terms of the number $N_0^+$ ($N_0^-$) of zero modes with positive (negative) chirality and the magnetic flux as

$$
N_0^+ - N_0^- = Q. \tag{7.2}
$$

This index is defined like $\text{Ind}(\mathcal{D}) = Tr(\sigma^3 e^{-t\lambda^D})$ with $t$ some spectral parameter and leads to the above equality which can be got by computing $\text{Ind}(\mathcal{D})$ in two different ways and equate the two results. Let us describe rapidly how this is done.

(1) use the property that the Dirac operator $\mathcal{D}$ and its square $\mathcal{D}^2$ have the same number of zero modes. Since $\mathcal{D}^2$ has a diagonal form

$$
\mathcal{D}^2 = \eta^{\mu\nu} D_\mu D_\nu - \frac{i}{4} [\sigma^{\mu\nu}] F_{\mu\nu} \tag{7.3}
$$
as clearly seen on the matrix representation

$$
\mathcal{D}^2 = \begin{pmatrix} D_+D_- & 0 \\ 0 & D_-D_+ \end{pmatrix}, \quad \mathcal{D}^2 \Psi = E^2 \Psi \tag{7.4}
$$

one can do explicit calculations by using this property. Let show rapidly how this works. From above equation, we learn that the operators $D_+D_-$ and $D_-D_+$ have the same non zero eigenvalues $E \neq 0$; this means that they have the number of non zero modes $N_n^+$ and $N_n^-$

$$
N_n^+ = N_n^-, \quad n \neq 0, \tag{7.5}
$$

with $\pm$ refereing to positive and negative chiralities. Denoting by $\Psi = (\Psi_+, \Psi_-)$ the 2D spinor with $\Psi_+$ and $\Psi_-$ standing for the positive and negative chiralities; and assuming that

$$
D_+D_- \Psi_+ = \lambda \Psi_+, \quad \lambda \neq 0 \tag{7.6}
$$

then $D_- \Psi_+$ is an eigenstate of $D_-D_+$ with the same eigenvalue; but with opposite chirality. This feature follows directly by multiplying both sides of eq. (7.6) by $D_-$. However in the case where $\lambda = 0$; that is $D_+D_- \Psi_+ = 0$, the equality $N_n^+ = N_n^-$ doesn’t necessary hold for zero modes since $D_- \Psi_+$ may be itself equal to zero; so $N_0^+ - N_0^-$ may be different from zero. Moreover, using (7.4), it is not difficult to see that

$$
Tr \left( \sigma^3 e^{-tD^2} \right) = Tr \left( e^{-tD_+D_-} \right) - Tr \left( e^{-tD_-D_+} \right). \tag{7.7}
$$

Expanding the trace in terms of zero modes and non zero modes the above relation can be also put in the form

$$
\text{Ind}(\mathcal{D}) = \left( \sum_{E_+^{(0)}} e^{-tE_+^{(0)}} - \sum_{E_-^{(0)}} e^{-tE_-^{(0)}} \right) + \left( \sum_{E_+^{(\neq 0)}} e^{-tE_+^{(\neq 0)}} - \sum_{E_-^{(\neq 0)}} e^{-tE_-^{(\neq 0)}} \right),
$$
leading afterwards to
\[ \text{Tr} \left( \tau^3 e^{-tD^2} \right) = N_0^+ - N_0^- . \] (7.8)

(2) To get the right hand side of eq (7.2), we use two more features of the index; first
the remarkable independence of \( \text{Tr} \left( \tau^3 e^{-tD^2} \right) \) on the spectral parameter \( t \) as explicitly
exhibited by (7.8). Second use the so called heat expansion method
\[ \text{Ind}(D) = \frac{1}{4\pi t} \sum_{n \geq 0} t^n b_n(D) \] (7.9)
where the \( b_n(D) \)'s are expansion coefficients. Since this expression should be \( t \)-invariant;
it follows that \( \text{Ind}(D) = \frac{b_2(D)}{4\pi} \); which by using eq (7.8) leads to
\[ b_2(D) = \text{Tr} \left[ \sigma^3 \left( \frac{-i}{4} [\sigma^\mu, \sigma^\nu] F_{\mu\nu} \right) \right] = 2 \int B dS \] (7.10)
with \( [\sigma^\mu, \sigma^\nu] = 2i\epsilon^{\mu\nu\alpha} \sigma^\alpha \) and \( \frac{1}{2\pi} \int B dS = Q \) producing the total magnetic monopole charge
in discrete values inside the surface.

7.1.2 2D fermions on lattice
To get the topological index for 2D graphene, one starts from the analysis given in previous
subsection; then works out the extension fermions on honeycomb. The index on a square
lattice has been studied in [46]; and the one on honeycomb has been considered recently
in [47]. Below we give a brief description of the main lines of the two constructions.

fermions on a square lattice
In the case of a finite \( L \times L' \) square lattice where \( L = Na, \ L' = N'a \) with \( a \) the spacing
lattice parameter and individual \( S_{\text{square}} = a^2 \), the gauge configuration is chosen as follows:
\[ A_1(x, y) = -\omega y, \ A_2(x, y) = 0, \ F_{12} = B \] (7.11)
with flux
\[ Q = \frac{1}{2\pi} \int dxdyF_{12} = \frac{BS}{2\pi} . \] (7.12)
which, for later use, we write it as follows,
\[ Q = \frac{BLL'}{2\pi}, \ B = \frac{2\pi}{LL'}Q, \ S = LL' , \] (7.13)
We also have the following boundary conditions
\[ A_1|_{y=0} = A_1|_{y=L'} + i\Omega \frac{\partial}{\partial x} \Omega^{-1} \] ,
\[ \Omega(x) = e^{iBL'x} , \ \Omega(x)|_{x=0} = \Omega(x)|_{y=L} \] , (7.14)
leaving to \( BLL' = 2\pi n \) with \( n \) integer. Notice that the discontinuity in the vector potential
is captured by a gauge transformation. Notice also that the periodicity condition on the
transition function \( \Omega(x) \) requires \( \frac{B}{2\pi}S = Q \in \mathbb{Z} \); it teaches us that the field strength \( B \)
and the topological charge \( Q \) are quantized. By following [48], the general solution of the Dirac
equation of the 2-dimensional fermions on the square lattice satisfying the above boundary conditions is given by

$$
\Psi_n^\pm (x,y) = \sum_{k \in \mathbb{Z}} C_k^\pm e^{\frac{2 \pi i}{\sqrt{d}} x} e^{-\frac{1}{\xi^2} H_n (\xi)} , \quad \xi = \sqrt{|B|} \left( y + \frac{k}{\xi} L' \right)
$$

(7.15)

where $H_n (\xi)$ are Hermite polynomials of order $n$; and where the $C_k^\pm$ coefficients are constrained by the recurrent relations $C_k^\pm = C_{k-2}^\pm Q$; which result from boundary conditions and showing that only $C_0^\pm , \ldots , C_{|Q|-1}^\pm$ which can be chosen arbitrary. The eigenvalues associated with the $\Psi_n^\pm (x,y)$'s are given by

$$
\begin{align*}
(E_n^+)^2 &= 2 (n + 1) |B| - B , \\
(E_n^-)^2 &= 2 (n + 1) |B| + B .
\end{align*}
$$

(7.16)

The link between chirality and zero modes depends on the sign of the topological charge $Q$ or the sense of the external magnetic field $B$. For the case $B > 0$, the above relations read as $(E_n^+)^2 = 2nB$ and $(E_n^-)^2 = 2 (n + 1) B$. So only the wave function with positive chirality has zero modes with degree of degeneracy equal to $|Q|$. For the case $B < 0$, the zero modes have a negative chirality with $|Q|$ degeneracy.

For the case of $N$ continuum flavors described by the Dirac operator in the magnetic background field as specified above, the index theorem reads therefore as follows

$$
Ind (D_{\text{square}}) = N |Q| .
$$

(7.17)

As such, the index of the Dirac operators for minimally doubled fermions; in particular for BC fermions is $2 |Q|$. For naive fermions it is equal to $16 |Q|$.

**honeycomb fermions**

On the honeycomb, the situation is quite similar; the main difference comes form the crystallographic structure. The honeycomb is given by the superposition of two sublattices $A_{\text{gra}}$ and $B_{\text{gra}}$. An interesting way to parameterize these sublattices is in terms of the two simple roots $\alpha_1$ and $\alpha_2$ of $SU(3)$ and the weight vectors $\lambda_1$, $\lambda_2$, $\lambda_3$ of its fundamental representations $[47, 11, 48, 49]$. More precisely sites $r_n$ in $A$ and $r'_n$ in $B$ are expanded as follows

$$
\begin{align*}
A : & \quad r_n^A = n_1 \alpha_1 + n_2 \alpha_2 , \quad n = (n_1, n_2) \in \mathbb{Z}^2 \\
B : & \quad r'_n = r_n + s , \quad d = a \frac{\sqrt{3}}{2}
\end{align*}
$$

(7.18)

where $\alpha_1$, $\alpha_2$ and their sum $\alpha_3 = \alpha_1 + \alpha_2$ which is also a root but not simple; can be taken like

$$
\alpha_1 = \left( \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \right) , \quad \alpha_2 = (0, \sqrt{2}) , \quad \alpha_3 = \left( \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right) ,
$$

(7.19)

satisfying the usual root’s properties of $SU(3)$; in particular $\alpha_i^2 = 2$. Moreover, the shift vector $s$ is related to one of the weight vectors like $\lambda = s \frac{\sqrt{3}}{2}$; it can be any one of the three following

$$
s_1 = a (1,0) , \quad s_2 = a \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right) , \quad s_3 = a \left( -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right) .
$$

(7.20)

Notice that the sum $s_1 + s_2 + s_3 = 0$, its captures the traceless property of the fundamental representation of $SU(3)$; we also have the relations

$$
(s_1 - s_2) = \sqrt{\frac{3}{2}} \alpha_1 , \quad (s_2 - s_3) = \sqrt{\frac{3}{2}} \alpha_2 , \quad (s_3 - s_1) = \sqrt{\frac{3}{2}} \alpha_0 .
$$

(7.21)
with \( \alpha_0 = -\alpha_3 \). Plaquettes in the 2D honeycomb are hexagonal with area \( S_{\text{hexa}} = a^2 \sqrt{3}/2 \), so the magnetic field is quantized as follows

\[
B = \frac{4\pi}{NN' a^2 \sqrt{3} Q}.
\]  

(7.22)

In [47], the two lattice axes of the honeycomb were chosen as generated by the roots \((\alpha_0, -\alpha_1)\), i.e: \( r = x_0 \alpha_0 - x_1 \alpha_0 \); and the boundary conditions to have a finite translationally invariant 2D graphene lattice \( L_0 \times L_1 \) were taken like

\[
F (r + L_0 \alpha_0) = F (r) ,
\]

\[
F (r + L_1 \alpha_1) = F (r) .
\]  

(7.23)

Moreover, the link field configuration depending on these boundary conditions are given by

\[
U (r, s_1) = e^{-i \frac{Ba^2 \sqrt{3}}{2} x_0} ,
\]

\[
U (r, s_2) = 1 ,
\]

\[
U (r, s_3) = 1 ,
\]  

(7.24)

for all cells except those of the last row with \( x_0 = L_0 - 1 \) where it is required moreover

\[
U (x_0 = L_0 - 1, x_1, s_3) = e^{-i \frac{Ba^2 \sqrt{3}}{2} L_0 x_1} .
\]  

(7.25)

Like in the square lattice, the topological index is also given by \( \text{Ind} (D_{\text{gra}}) = 2 |Q| \); the main difference is that it is given by \( \Psi^+ \Sigma^3 \Psi \) where \( \Sigma^1, \Sigma^2, \Sigma^3 \) are the generators of the \( SU (2) \) flavor symmetry rotating the two Dirac points; for an explicit analysis see [47].

### 7.2 Index in 4D lattice QCD

In 4-dimensions, the determination of the index of the Dirac operator of fermions living on 4D spaces follows the same approach as in 2D case. Let us describe briefly the main lines of the method.

#### 7.2.1 Fermions on 4D space

To get the topological index in a \( SU (N) \) background gauge field configuration one has to compute both sides of the 4D analogue of eq(7.2) namely

\[
N_0^+ - N_0^- = 2 |Q| ,
\]  

(7.26)

where \( N_0^+ \) are the number of chiral zero modes and \( Q \) the flux. Let us compute the topological charge \( Q \) that gives the right hand side of this relation. It is generally given by:

\[
Q = \frac{1}{32\pi^2} \int_{S_4} d^4 x \varepsilon_{\mu\nu\rho\sigma} T r (F^{\mu\nu} F^{\rho\sigma}) ,
\]  

(7.27)

with field strength \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu] \) valued in the adjoint representation of \( SU (N) \); i.e \( F_{\mu\nu} = \sum_a T_a F_{\mu\nu}^a \). Making a simple choice of this gauge invariant field as follows

\[
F_{\mu\nu} = \begin{pmatrix}
0 & B_1 & 0 & 0 \\
-B_1 & 0 & 0 & 0 \\
0 & 0 & 0 & B_2 \\
0 & 0 & -B_2 & 0
\end{pmatrix} \otimes T ,
\]

(7.28)
where $T$ is one of the generators of $SU(N)$ normalized to $\text{Tr}(T^2) = 2$, one can determine, up to a gauge transformation, the corresponding gauge potentials. Notice that a gauge configuration that is appropriate to lattice computations is given by,

$$
A_1(x, y, z, \tau) = -B_1 y T, \quad A_2(x, y, z, \tau) = 0
$$

$$
A_3(x, y, z, \tau) = -B_2 z T, \quad A_4(x, y, z, \tau) = 0
$$

(7.29)

Putting (7.28) back into (7.27) by using,

$$
\tilde{F}^{\mu\nu} = \begin{pmatrix}
0 & -B_2 & 0 & 0 \\
B_2 & 0 & 0 & 0 \\
0 & 0 & 0 & -B_1 \\
0 & 0 & B_1 & 0
\end{pmatrix} \otimes T,
$$

(7.30)

which is (anti) self dual for $B_2 = \pm B_1$, one gets the explicit expression of the topological charge $Q$ in terms of the fields $B_1$ and $B_2$ and the volume of the 4-dimensional compact hyper surface. We have,

$$
Q = \frac{16B_1 B_2}{32\pi^2} \text{Vol}(S_4) \in \mathbb{Z}.
$$

(7.31)

### 7.2.2 4D hypercube

This relation can be given a more explicit form by working by considering fermions on lattices. In the case of the 4D hypercube $L_1 \times L_2 \times L_3 \times L_4$, the previous relations reads as

$$
Q = \frac{16B_1 B_2}{32\pi^2} \prod_{i=1}^4 L_i
$$

(7.32)

Moreover use the fact that the field strengths $B_1$ and $B_2$ are quantized as

$$
B_1 = \frac{2\pi N_1}{L_1 L_2}, \quad B_2 = \frac{2\pi N_2}{L_3 L_4}, \quad N_{1,2} \in \mathbb{Z},
$$

(7.33)

we end with the following topological charge

$$
Q = 2N_1 N_2,
$$

(7.34)

which is independent from the nature of the 4D lattice; and is then also valid for the 4D hyperdiamond of 4D lattice QCD. Regarding the left hand side of (7.26); it is determined by solving the Dirac equation. We find

$$
\Psi_{n,m}^{\pm}(x, y) \sum_{(k_x, k_z) \in \mathbb{Z}^2} C_{k_x, k_z}^{\pm} \exp\left( i \left( \frac{2\pi k_x}{L_1} x + \frac{2\pi k_z}{L_3} z \right) \right) e^{-\frac{i}{2}(\xi^2 + \zeta^2)} H_{n,m},
$$

(7.35)

with

$$
H_{n,m} = H_n(\xi) \times H_m(\zeta)
$$

$$
\xi = \sqrt{|B_1|} \left( y + \frac{k_x}{L_2} L_2 \right)
$$

$$
\zeta = \sqrt{|B_2|} \left( \tau + \frac{k_z}{L_4} L_4 \right)
$$

(7.36)

where $H_n(\xi)$ are Hermite polynomials; and where the coefficients $C_{k_x, k_z}^{\pm}$ are constrained by the recurrent relations

$$
C_{k_x, k_z}^{\pm} = C_{k_x - Q_x, k_z - Q_z}^{\mp},
$$

(7.37)

showing that the degeneracy of the chiral zero mode is $|Q_x Q_z|$. 


8. Appendix B: Index, spectral flow and filling factor

8.1 Index theorem and spectral flow

In this section, we describe the main lines of the spectral flow method to get the topological index which can be symbolically stated as

\[ \text{Ind}(D) = -\text{spf}(\mathcal{H}) \]  

(8.1)

where \( \text{spf}(H) \) stands for spectral flow of \( \mathcal{H} \) which, reads in terms of the 4-dimensional Dirac operator \( D = \gamma^\mu (\partial_\mu - iA_\mu) \), as follows

\[ \mathcal{H}(m) = \gamma_5 (D - m) , \quad m \in \mathbb{R} . \]  

(8.2)

This approach was first considered in [34] for staggered lattice fermions where the would-be chiral zero-modes has been identified away from the continuum limit. Then, it has extended in [8, 9] to minimally doubled fermions as well as the naive ones by using the point splitting method for implementing flavored mass terms. The spectral flow method detects exactly the index of the would-be zero modes fixing the gauge field topology; it has been explicitly checked numerically for 2D and 4D staggered, minimally doubled and naive fermions. Below, we will mainly focus on 4D; but the results can be extended to all even dimensions.

The key idea of the spectral flow method relies on defining a hermitian spectral hamiltonian depending on two basic things. (1) The Dirac equation for zero modes namely

\[
\begin{pmatrix}
0 & \tilde{D}^\dagger \\
-D & 0
\end{pmatrix}
\begin{pmatrix}
\phi \\
\chi
\end{pmatrix} = 0,
\]

(8.3)

where each block is a \( 2 \times 2 \) matrix and where the operators \( D \) and \( D^\dagger \) are as in eq(4.5,4.7) and

\[ \Psi = \begin{pmatrix}
\phi \\
\chi
\end{pmatrix} \]  

(8.4)

being the Dirac spinor in 4-dimensions with the two chiral components \( \Psi_\pm \) given below,

\[ \Psi_+ = \begin{pmatrix}
\phi \\
0
\end{pmatrix}, \quad \Psi_- = \begin{pmatrix}
0 \\
\chi
\end{pmatrix}, \quad \gamma_5 \Psi_\pm = \pm \Psi_\pm. \]  

(8.5)

(2) the spectral hamiltonian operator \( \mathcal{H}(m) = \gamma_5 (D - m) \) depending on a real spectral parameter \( m \) which can be thought of as mass. In matrix notation, we have,

\[ \mathcal{D} = \begin{pmatrix}
0 & \tilde{D}^\dagger \\
-D & 0
\end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix}
m & \tilde{D}^\dagger \\
-D^\dagger & -m
\end{pmatrix} , \]  

(8.6)

Notice also the traceless property of the spectral hamiltonian operator

\[ Tr\mathcal{H} = 0, \]  

(8.7)
which, a priori, should be independent of basis change in the spinorial representation space, turns out to play a crucial role since the sum of its eigenvalues should be equal to zero. Notice also that for the Dirac zero modes \( \Psi \), we have \( \mathcal{H}\Psi = -m\gamma^5 \Psi \) and so a zero-mode of \( D \) with \( \pm \) chirality is also an eigenmode of \( \mathcal{H} \) with eigenvalue \( \lambda (m) = \pm m \) as given here below

\[
\mathcal{H}\Psi_\pm = \mp m\Psi_\pm.
\] (8.8)

These eigenvalues cross the axis \( \lambda (m) = 0 \) with two possible slopes \( \pm 1 \) at \( m = 0 \). Moreover, from the property \( \mathcal{H}^2 = D^\dagger D + m \), it follows that that \( \lambda (m) = \pm m \) are the only eigenvalues of \( \mathcal{H} \) that cross the origin at any value of \( m \). Therefore the spectral flow of \( \mathcal{H} (m) \), defined as the net number of \( \lambda (m) \)'s that cross the origin, counted with sign \( \pm \) depending on the slope of the crossing, comes entirely from eigenvalue crossings at \( m = 0 \) and equals \( N_- - N_+ = -\text{Ind} (D) \). Numerical results showed that the topological index is indeed given by minus the spectral flow hamiltonian; for more details see [9, 34].

### 8.2 Filling factor and chiral anomaly

First recall the expression of the filling factor \( \nu \) in terms of the number \( N_f \) of fermions and the flux number \( N_\phi \),

\[
\nu = \frac{N_f}{N_\phi}.
\] (8.9)

In 2-dimensions, we found that the filling factor \( \nu_{2D} = g_{2D} \times (2N + 1) \) where \( g_{2D} \) is some degeneracy factor giving the number of species and their quantum numbers. The integer \( (2N + 1) \) is the sum of two contributions \( (N + \frac{1}{2}) \) coming from electrons and \( (N + \frac{1}{2}) \) from holes. The two half integers \( \frac{1}{2} \) are associated with the fundamental state \( N = 0 \) where the conducting (electrons) and valence (holes) bands touch; this state is a chiral zero mode as shown on eq(2.19). Notice that the background field is given by

\[
\mathcal{F}^{(2D)}_{\mu\nu} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}
\] (8.10)

with flux \( \Phi = BS \) through a surface \( S \). The same situation happens in 4-dimensions where we have found that \( \nu_{4D} = g_{4D} \times (2N_1 + 1)(2N_2 + 1) \). The main difference is that in 4D the background field involves two kinds of magnetic fields

\[
\mathcal{F}^{(4D)}_{\mu\nu} = \begin{pmatrix} 0 & B_1 & 0 & 0 \\ -B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2 \\ 0 & 0 & -B_2 & 0 \end{pmatrix}
\] (8.11)

and then two fluxes \( \Phi_1 = B_1 S \) and \( \Phi_2 = B_2 S \). The numbers \( (2N_1 + 1) \) and \( (2N_2 + 1) \) are then associated with the fluxes \( \Phi_1 \) and \( \Phi_2 \). Here also the zero modes are chiral and so contribute to the topological index.

More explicit relations can be written down by working on lattices on which the magnetic...
fields and the fluxes are quantized as follows,

| Lattice | magnetic fields | fluxes |
|---------|----------------|--------|
| 2D  | \[
\frac{1}{B} = \frac{L_1 L_2}{2\pi N_1},
\]
| \[Q_{2D} = N_1\] |
| 4D  | \[
\frac{1}{B_1} = \frac{L_1 L_2}{2\pi N_1}, \quad \frac{1}{B_2} = \frac{L_3 L_4}{2\pi N_2},
\]
| \[Q_{4D} = 2N_1 N_2\] |

(8.12)

On the other hand, using the fact that the momenta \(p_\mu = \hbar k_\mu\) of a particle of coordinate \((x_\mu)\) in background fields is given by the gauge covariant derivatives \(iD_\mu = i\partial_\mu + \frac{1}{2}F_{\mu\nu}x^\nu\); we learn that in the large magnetic field this momenta (wave vector) is dominated by the cyclotronic term \(\frac{1}{2}F_{\mu\nu}x^\nu\). This limit leads to

\[
[x^\mu, x^\nu] = 4iG^{\mu\nu},
\]

(8.13)

where \(G^{\mu\nu}\) is as in eq(4.4). This relation is a typical phase space relations; it teaches us that in presence of a strong background field the space gets discretized into fundamental cells of area \(l_B^2 = 4G^{\mu\nu}\). In 2-dimensions, we have, up to a normalization factor, the following non commutative geometry relation

\[
[x, y] = -\frac{i}{B} = -\frac{iL_1 L_2}{2\pi N},
\]

(8.14)

where the second equality, which is valid for \(N \neq 0\) in agreement with the large \(B\) limit, follows from (8.12). From this relation, we learn the area \(l_B^2\) of the fundamental cell

\[
l_B^2 = \frac{L_1 L_2}{2\pi N},
\]

(8.15)

and so one is left with \(N\) electrons (\(N\) holes) coupled to the quantum flux leading to the filling \(\nu_{2D}^* = g_{2D} \times 2N\). But this is not exactly the computed value \(\nu_{2D} = g_{2D} \times (2N + 1)\); the latter may be then viewed as the quantum version of \(\nu_{2D}^*\) and, due to the chiral anomaly, corresponds to shifting the magnetic length as

\[
l_B^2 \rightarrow \frac{L_1 L_2}{2\pi (N + \frac{1}{2})},
\]

(8.16)

which in turns corresponds to shifting the topological charge as \(Q = N \rightarrow N + \frac{1}{2}\).

In 4-dimensions, the situation is quite similar to 2D case; except that here we have two kinds of commutations relations

\[
[x, y] = -\frac{i}{B_1} = -\frac{iL_1 L_2}{2\pi N_1},
\]

\[
[z, \tau] = -\frac{i}{B_2} = -\frac{iL_3 L_4}{2\pi N_2},
\]

(8.17)

and so two kinds of magnetic lengths namely \(l_{B_1}^2 = \frac{L_1 L_2}{2\pi N_1}\) and \(l_{B_2}^2 = \frac{L_3 L_4}{2\pi N_2}\) leading afterwards to a classical relation of the filling factor \(\nu_{4D}^* = g_{4D} \times 4N_1 N_2\). However to get the right expression of the filling factor namely \(\nu_{4D} = g_{4D} \times 4(N_1 + \frac{1}{2}) (N_2 + \frac{1}{2})\), one has to shift the topological charges and the magnetic lengths as in (8.16); this behavior is also a manifestation of the 4D chiral anomaly.
9. Acknowledgements

The authors would like to thank Drs M. Bousmina, M. Daoud and A. Jellal for discussions.
L.B Drissi thanks the associateship program of ICTP, Trieste, Italy. E.H Saidi thanks URAC 09/CNRST.

References

[1] Michael Creutz, *Four dimensional graphene and chiral fermions*, JHEP0804:017,2008,
arXiv:0712.1201,

[2] A.Borici, Phys. Rev. D78 (2008) 074504, [arXiv:0712.4401],

[3] P.R Wallace, Phys Rev 71, (1947), 622,

[4] A.H.Castro-Neto et al. Rev. Mod. Phys.81, 109 (2009),

[5] L.B Drissi, E.H Saidi, M.Bousmina, Nucl Phys B, Vol 829, (2010) p.523-533,

[6] P.F Bedaque, M.I Buchoff, B.C Tiburzi, A.Walker-Loud, Phys. Rev. D78 (2008) 017502,[arXiv:0804.1145],

[7] P.F.Bedaque, M.I.Buchoff, B.C.Tiburzi, A.Walker-Loud, Phys. Lett. B662 (2008) 449,
[arXiv:0801.3361],

[8] Michael Creutz, *Minimal doubling and point splitting*, PoS Lattice2010:078,2010,
arXiv:1009.3154,

[9] Michael Creutz, Taro Kimura, Tatsuhiro Misumi, *Index Theorem and Overlap Formalism
with Naive and Minimally Doubled Fermions*, JHEP 1012:041,2010, arXiv:1011.0761,

[10] Michael Creutz, Taro Kimura, Tatsuhiro Misumi, *Aoki Phases in the Lattice Gross-Neveu
Model with Flavored Mass terms*, Phys.Rev.D83:094506,2011, arXiv:1101.4239,

[11] L.B Drissi, E.H Saidi, M. Bousmina, *4D Graphene*, Phys.Rev.D84:014504,2011,
arXiv:1106.5222,

[12] L.B Drissi, E.H Saidi, M. Bousmina, J. Math. Phys. 52, 022306 (2011)

[13] Taro Kimura, Tatsuhiro Misumi, Prog.Theor.Phys.123: 63-78, (2010), arXiv:0907.3774,

[14] T. Kimura and T. Misumi, Prog.Theor.Phys.124: 415-432, (2010), arXiv:0907.1371,

[15] Michael Creutz, Tatsuhiro Misumi, *Classification of Minimally Doubled Fermions*,
Phys.Rev.D82:074502,2010, arXiv:1007.3328,

[16] S.Capitani, J.Weber, H.Wittig, Phys.Lett.B 681, 2009, 105, arXiv:0907.2825

[17] S. Capitani, M. Creutz, J. Weber, H.Wittig, JHEP 1009:027,2010, arXiv:1006.2009

[18] L.H. Karsten, Phys. Lett. B104 (1981) 315,

[19] F. Wilczek, Phys. Rev. Lett. 59 (1987) 2397,

[20] J.B Kogut, L. Susskind, Phys Rev D11, 395, (1975)

[21] H.B Nielson, M. Ninomiya, Nucl Phys B185, (1980), 20, B195,(1982) 541

[22] K. S. Novoselov, A. K. Geim, et al., Nature 438, 197 (2005),
[23] Y. Zhang et al., Nature 438, 201 (2005),
[24] K. S. Novoselov, E. McCann, et al., Nature Physics 2, 177 (2006),
[25] Y. Zhang, et al., Phys. Rev. Lett. 96, 136806 (2006),
[26] Z. Jiang, Y. Zhang, H. L. Stormer, and P. Kim , Phys. Rev. Lett. 99, 106802 (2007),
[27] S.C. Zhang and J.P. Hu, Science 294 (2001) 823; J.P. Hu and S.C. Zhang, condmat/0110572,
[28] Dimitra Karabali, V.P. Nair, Quantum Hall Effect in Higher Dimensions, Nucl.Phys. B641 (2002) 533-546, arXiv:hep-th/0203264,
[29] Dimitra Karabali, V.P. Nair, Quantum Hall Effect in Higher Dimensions, Matrix Models and Fuzzy Geometry, J.Phys.A39:12735-12764,2006, arXiv:hep-th/0606161,
[30] M. Daoud, A. Jellal, Quantum Hall Effect on the Flag Manifold $F_2$, Int.J.Mod.Phys.A23:3129-3154,2008, arXiv:hep-th/0610157,
[31] A. Jellal, Anomalous Quantum Hall Effect on Sphere, Nucl. Phys. B804 (2008) 361, arXiv:0709.4126,
[32] R.B Laughlin, Phys Rev Lett 50, 1395 (1983)
[33] R. Prange and S.M Girvin, The Quantum Hall Effect, Spinger Verlag, Berlin, Germany 1990,
[34] D. H. Adams, Phys. Rev. Lett. 104, 141602 (2010) [arXiv:0912.2850],
[35] D. H. Adams, Phys.Lett.B699:394-397,2011, [arXiv:1008.2833],
[36] C. Hoelbling, Phys. Lett. B 696, 422 (2011) [arXiv:1009.5362],
[37] William Detmold, Brian C. Tiburzi, Andre Walker-Loud, Phys. Rev. D81: 054502, (2010), arXiv:1001.1131
[38] Dale S. Roberts, Patrick O. Bowman, Waseem Kamleh, Derek B. Leinweber, Phys.Rev.D83:094504,2011, arXiv:1011.1975
[39] William Detmold, Brian C. Tiburzi, Andre Walker-Loud, Lattice QCD in Background Fields, arXiv:0908.3626
[40] Paolo Cea, On the Quantum Hall Effect in Graphene, arXiv:1101.5703
[41] L.B Drissi, E.H Saidi, Dirac Zero Modes in Hyperdiamond Model, Phys.Rev.D84:014509,2011, arXiv:1103.1316
[42] Simon Catterall, David B. Kaplan, Mithat Unsal, Exact lattice supersymmetry, arXiv:0903.4881, To be published in Physics Reports,
[43] Simon Catterall, Eric Dzienkowski, Joel Giedt, Anosh Joseph, Robert Wells, Perturbative renormalization of lattice $N=4$ super Yang-Mills theory, arXiv:1102.1725
[44] Aziz El Rhalami, El Hassan Saidi, JHEP10(2002)039, arXiv:hep-th/0208144,
[45] Kee-Su Park, Topological effects, index theorem and supersymmetry in graphene, arXiv:1009.6033,
[46] J. Smit and J. C. Vink, Remnants of the Index Theorem on the Lattice, Nucl. Phys. B 286, 485 (1987),
[47] Dipankar Chakraborti, Simon Hands, Antonio Rago, Topological Aspects of Fermions on a Honeycomb Lattice, JHEP 0906:060,2009, arXiv:0904.1310.
[48] Lalla Btissam Drissi, El Hassan Saidi, *Graphene, its Homologues and Their Classification*, arXiv:1106.5933,

[49] Lalla Btissam Drissi, El Hassan Saidi, Mosto Bousmina, *Graphene and Cousin Systems*, Chapter of the book ”Graphene Simulation” Edited by: J.R. Gong, InTech Publishing, Rijeka, Croatia, (2011), ISBN 978-953-307-556-3, arXiv:1108.1748,

[50] C.L. Kane and E.J. Mele Phys. Rev. Lett. 95, 226801, (2005),

[51] Liang Fu, C.L. Kane, E.J. Mele, *Topological Insulators in Three Dimensions*, Phys. Rev. Lett. 98, 106803 (2007).