Invasion Percolation with Temperature and the Nature of SOC in Real Systems

Andrea Gabrielli1,2, Guido Caldarelli2 and Luciano Pietronero2

1 Laboratoire de Physique de la Matière Condensée, Ecole Polytechnique, 91128-Palaiseau Cedex, France
2 INFM - Unità di Roma 1 "La Sapienza", P.le A. Moro 2, 00185 - Roma, Italy

We show that the introduction of thermal noise in Invasion Percolation (IP) brings the system outside the critical point. This result suggests a possible definition of SOC systems as ordinary critical systems where the critical point correspond to set to 0 one of the parameters. We recover both IP and EDEN model, for $T \to 0$, and $T \to \infty$ respectively. For small $T$ we find a dynamical second order transition with correlation length diverging when $T \to 0$.

62.20.M, 05.40.+j, 02.50.-r

The spontaneous development of complex and fractal structures has been studied on the basis of several models which manifest the property of Self-Organized Criticality (SOC) [1]. This concept is very intriguing and its very meaning has been highly debated. The combination of different properties as for example stochastic and quenched disorder, usually brings out of the criticality. Henceforth, the distinction with ordinary critical phenomena (instead of self-organized) seems to become elusive. In order to clarify these basic questions we consider here one of the classical models of self-organization, the Invasion Percolation (IP) model [2]. IP describes the displacement of a fluid in a disordered net of random throats due to another immiscible fluid pushed with a vanishing pressure rate.

In this letter we study this model when a temperature-like noise $T$ is present. The reason for this generalization is twofold. On one side, one is interested in the robustness of SOC with respect to external solicitations [3-5]. In particular we find that for $T \neq 0$ a finite correlation length appears and the process goes out of criticality. This result (together with the observation that in other SOC model the scaling properties are limited by the “slow driving” [6]) suggests a possible definition for SOC phenomena in real systems. A system or a dynamical process is SOC if the critical value of the driving parameter is 0, instead of another real number. The reason why such a value makes such a large difference is because the driving parameter of these process is always a ratio (grain of sand added with respect to the total number of sites for the sandpiles, sites whose “value” is changed for IP, DLA [6], Bak and Sneppen [7] etc.) and any value smaller than a certain threshold can be considered equal to 0. For this reason the zero value tends to occupy a much larger region of the phase space than the other real numbers.

On the other side IP model is the most famous and simple example of evolution in quenched disorder. IP dynamics of evolution is deterministic and extremal in the sense that at each time step the fluid invades the environment by selecting the minimum throat available. This limiting case is particularly instructive since the extremal dynamics is suitable to be studied analytically. By using the tool of the Run Time Statistics (RTS) [8], we can approach also the more real stochastic case, where fluctuations affect the dynamics of invasion.

It is useful at this point to describe in some detail the IP model that we are studying [2]. (a) in a lattice of size $L$ a random number $x_i$ (extracted from the uniform distribution $p_0(x) = 1$ for $x \in [0,1]$) is assigned to each bond $i$; (b) at time $t = 0$ the dynamics starts from a finite connected set of bonds $C_0$ (in the asymptotic regime the system does not depend upon $C_0$). $C_t$ is the connected set of bonds grown until time $t$. At each time step the interface $\partial C_t$ of the growing cluster $C_t$ is the set of non-grown bonds in contact with the cluster. Only bonds belonging to $\partial C_t$ grow at the time step $t$; (c) At time $t$ the bond $i \in \partial C_t$ with the lowest $x_i$ grows; then $C_{t+1} = C_t \cup \{i\}$ and the interface is updated. The dynamics stops when the cluster percolates the lattice.

This simple growth model develops spontaneously geometrical and dynamical critical features. In particular: (1) the asymptotic cluster is a fractal (i.e. it has an infinite correlation length) with fractal dimension $D_f \simeq 1.89$ in a 2d lattice, which is the same fractal dimension of the infinite cluster of percolation at the critical point; (2) the normalized histogram $\phi_t(x)$, of the interface variables, has the following asymptotic shape:

$$\phi_t(x) = \frac{1}{1 - p_c} \theta(x - p_c), \quad (1)$$

while the initial shape is obviously $\phi_0(x) = 1$. $p_c$ coincides with the critical threshold of the percolation in the lattice; (3) the asymptotic dynamics evolves for critical avalanches. Any bond $i$ growing at time $t$ is the initiator of an own avalanche. An avalanche is a temporal consecutive sequence of causally and geometrically connected growth events starting with the growth of the initiator (for a detailed definition of avalanche see e.g. [6]). Note that $x$ of the initiator, due to the shape of the asymptotic histogram and of the acceptance profile must be $x \leq p_c$ if $t$ is very large. The size distribution $D(s; x)$ (where $x$ is the random number of the initiator) of the avalanche has the following behavior:

$$D(s; x) = s^{-\tau} f(s^\sigma |x - p_c|), \quad (2)$$

where $f(x) = constant$ for $x \ll 1$ and decay exponentially for $x \gg 1$ (i.e. for $s > s_0 = |x - p_c|^{-1/\sigma}$), with $\tau = 1.57 \pm 0.03$ and $\sigma = 1 - \tau + 2/D_f = 0.49 \pm 0.03$. It may be observed that if $x = p_c$ the size distribution...
is a power law and the characteristic size diverges. As a consequence, this kind of avalanches are called critical avalanches.

We now generalize the IP model by introducing the presence of thermal noise. The numerical study of an analogous application to the Bak-Sneppen model [4] can be found in [5] whilst the case of sandpiles has been considered in [6]. The first effect of a finite temperature $T$ is that the deterministic dynamics becomes stochastic, in such a way that the larger the temperature the larger the stochasticity. The definitions of $C_t$ and $\partial C_t$ in this model are the same as IP, but the growth rule is different: each bond $i \in \partial C_t$ has the following growth probability depending on the realization of the quenched disorder:

$$\eta_{i,t}(\{x\} \partial C_t) = \frac{e^{-\beta x_i}}{\sum_{j \in \partial C_t} e^{-\beta x_j}}$$

(3)

where $\beta = 1/T$ and $\{x\} \partial C_t$ the realization of the quenched disorder on the interface $\partial C_t$. The larger is $T$ the more $\eta_{i,t}$ is independent on $x_i$. Hereafter, we shall indicate with $||C_t||$ the number of bonds belonging to $C_t$, and with $||\partial C_t||$ the number of bonds belonging to $\partial C_t$.

It is important to study the two different limits $T \to \infty$ and $T \to 0$. In the first limit we have:

$$\lim_{T \to \infty} \eta_{i,t} = \frac{1}{||\partial C_t||}$$

(4)

where $||\partial C_t||$ is the total number of bonds belonging to the growth interface at time $t$. Eq.(4) means that all the bonds on the interface have the same probability to grow. This model is well known and usually called Eden model [8]: this dynamical growth generates a compact cluster (fractal dimension equal to the space dimension) with a rough surface (interface). In the second limit we have:

$$\lim_{T \to 0} \eta_{i,t} = \prod_{j \in \partial C_t - \{i\}} \theta(x_j - x_i)$$

(5)

where $\partial C_t - \{i\}$ means the interface $\partial C_t$ minus the bond $i$. Eq.(5) provides nothing else the deterministic extremal growth rule of IP: $\eta_{i,t} = 1$ if $x_i$ is the extremal (minimum) value and zero otherwise. In this paper we address mainly the study of the behavior for small values of $T$, i.e. the transition of the model towards IP. In particular we will study the case of a 2d square bond lattice. We started by studying some Monte-Carlo simulations of this model. The presence of the temperature introduces a characteristic length $\xi(T)$ the effect of which is quite clear in Fig.1 where percolating clusters for different values of $\beta = 1/T$ are shown. The differences between the clusters can be explained by characterizing qualitatively the dynamical evolution of the growth.

For any value of $T$, a characteristic time $t^*(T)$ exists such that, for $t < t^*(T)$ the dynamics of the model is the IP dynamics, i.e. even if the dynamical rule given by Eq.3 is not deterministic, the effect of stochasticity is still negligible and the effective dynamics is almost extremal. On the other hand for $t > t^*(T)$ the effect of the stochastic noise begins to be more and more important and the deviation from IP and then from fractality, becomes larger.

If we suppose that $T \ll 1$, and then $t^*(T) \gg 1$, it is clear that $t^*(T)$ represents the correlation time of the system. Since one bond is removed for each time-step, $t^*(T)$ represents also the number of bonds $s_0(T)$ in a correlated region of the cluster when $t \gg t^*(T)$. This is in agreement with the idea that at $T > 0$ IP is the repulsive fixed point of the dynamics under a spatio-temporal coarse-graining transformation, and the Eden model is the trivial attractive fixed point characterized by $T \to \infty$. These features can be checked by looking at the dynamical evolution of the histogram $\phi_t(x)$. Obviously $\phi_0(x) = 1$; for $t < t^*(T)$ as previously noted, the evolution is the same of IP, that is $\phi_t(x)$ evolves in the step-function given by Eq.(1). At $t = t^*(T)$, $\phi_t(x)$ is a smoothed step function (the size of the smoothed interval around $p_c$ increases with $T$). For $t > t^*(T)$, because of stochasticity, the growth of bonds with $x$ well larger $p_c$ are permitted and the histogram $\phi_t(x)$ shifts towards high values of $x$. We have measured through simulations $t^*(T)$ by measuring the time step when $\phi_t(x)$ start to shifts and we obtain the scaling law $t^*(T) \equiv s_0(T) \sim T^{-\gamma}$ with $\gamma = 1.9 \pm 0.2$. In the following we find the same behavior theoretically and we link it to the correlation length of the structure.

In order to study analytically the model, we formulate the generalization to stochastic growth dynamics of the Run Time Statistics (RTS) [9] that we call Generalized Run Time Statistics (GRTS). The usual RTS is a probabilistic technique based on the concept of conditional probability, introduced to study IP-like dynamics, i.e. deterministic extremal dynamics with quenched disorder. With the GRTS approach one can solve the following problem: suppose to fix the time-ordered path $C_t$ followed by the dynamics, and to ignore the realization of the disorder: then one can compute the joint probability density function $P_t(\{x\} \partial C_t)$ of all the variables $x_i$ of the bonds $i$ belonging to the interface $\partial C_t$, conditioned to the history $C_t$. Furthermore one can compute the conditioned probability of any possible next growth step. This joint Probability Density Function (PDF) $P_t(\{x\} \partial C_t)$ plays a central role, since from it one can compute the probability (conditioned to the whole past history, i.e. to all the previous steps of the path) of any possible next growth step. After that, one updates consequently the joint probability density obtaining $P_{t+1}(\{x\} \partial C_{t+1})$. Here we expose an approximated version of GRTS. The approximation consists in assuming that at any time-step the PDF can be written as the product of single bond density functions $p_k(x_k)$

$$P_t(\{x\} \partial C_t) = \prod_{k \in \partial C_t} p_k(x_k)$$

This means that one is assuming that all the information about the history can be contained in the set of effective single bond density functions. Usually this is not the
case, in fact, even if we start the dynamics with independent variables (as in this case), the information about the dynamical history generates correlations among the interface variables \[12\]. However, it can be seen that this approximation works very well even for IP where the effect of this correlation is the maximum, due to the extremal nature of the dynamics \[13\].

Starting from the PDF’s we want to compute the conditional probability \( \mu_{i,t} \) that a certain bond \( i \in \partial C_t \) grows at time \( t \). Let us suppose to have the “effective” probabilities density functions \( p_{k,t}(x_k) \) for each \( k \in \partial C_t \). The functions \( p_{k,t}(x_k) \) are determined by the whole past history up to time \( t \) (in particular, for \( t = 0 \) each \( p_{k,t}(x) = p_0(x) = 1 \) because there is no information yet on the dynamics). Starting from functions \( p_{k,t}(x_k) \) we write the conditioned probability \( \mu_{i,t} \) as:

\[
\mu_{i,t} = \int_0^1 \int_0^1 \sum_{k \in \partial C_t} \left[ dx_k \right] p_{k,t}(x_k) \frac{e^{-\beta x_i}}{\sum_{k \in \partial C_t} e^{-\beta x_k}} \tag{6}
\]

Eq.\[8\] provides the Growth Probability Distribution (GPD) conditioned to the past growth history up to time \( t \). At this point we may iterate the procedure by updating the PDF’s, that is by updating the “effective” probability density functions conditioned to the last growth-event. In order to do that, we have to distinguish three cases: (a) the last bond grown \( i \), (b) the other bonds \( j \) belonging to \( \partial C_t \), and finally (c) the bonds just entered in the new interface \( \partial C_{t+1} \) because of the growth of \( i \):

(a) In this case, \( i \) does not belong to \( \partial C_{t+1} \); we introduce the new symbol \( m_{i,t+1} \) analogous to \( p_{k,t} \):

\[
m_{i,t+1}(x) = \frac{1}{\mu_{i,t}} \int_0^1 \int_0^1 \sum_{k \in \partial C_t} \left[ dx_k \right] p_{k,t}(x_k) \cdot \frac{e^{-\beta x_i}}{\sum_{k \in \partial C_t} e^{-\beta x_k}} \tag{7}
\]

(b) In this case we have:

\[
p_{j,t+1}(x) = \frac{1}{\mu_{i,t}} \int_0^1 \int_0^1 \sum_{k \in \partial C_t} \left[ dx_k \right] p_{k,t}(x_k) \cdot \frac{e^{-\beta x_i}}{\sum_{k \in \partial C_t} e^{-\beta x_k}} \tag{8}
\]

(c) Finally, we have \( p_{j,t+1}(x) = p_0(x) = 1 \); let us call \( n_{i,t} \) the number of these bonds. Note that the following relations hold: \( ||C_t|| = t \) and \( ||\partial C_{t+1}|| = ||\partial C_t|| + n_{i,t} - 1 \). Here we shall call \( \Omega_t \) and \( n_t \) the average values over different histories respectively of \( ||C_t|| \) and \( n_{i,t} \).

Using Eqs.\[8\],\[9\],\[10\] and the rule that bonds just entered the interface have simply \( p_0(x) = 1 \) as “effective” density function, one can describe from a conditional probability point of view any possible dynamical history, knowing only \( p_0(x) \) and the dynamical rule given by Eq.\[8\]. In \[8\] these approach in the \( T = 0 \) limit has been used to study IP to evaluate both \( D_f \) and \( \tau \).

Now we use this generalized approach to study the transition towards IP (stochastic-extremal transition). We introduce, then, the histogram \( h_t(x) \). \( h_t(x) \) is the distribution of \( x \)'s on the interface at time \( t \)

\[
h_t(x) \, dx = \# \text{ of interface bonds with throat } \in [x, x + dx]
\]

If we fix a history up to time \( t \), we can write:

\[
h_t(x) = \sum_{i \in \partial C_t} p_{i,t}(x)
\]

where the functions \( p_{i,t}(x) \) must be evaluated through the “algorithm” provided by Eqs.\[6\]-\[8\] for the given history. Note that \( \int_0^1 dx h_t(x) = ||\partial C_t|| \). Since the disorder is quenched (i.e. time-independent), the dynamical equation for \( h_t(x) \) is

\[
h_{t+1}(x) = h_t(x) - m_{i,t+1}(x) + n_{i,t} p_0(x) \tag{9}
\]

It is convenient to study the normalized histogram \( \phi_t(x) \) defined as \( \phi_t(x) = h_t(x)/||\partial C_t|| \) (i.e. \( \int_0^1 dx \phi_t(x) = 1 \)). Since \( \phi_t(x) \) is an almost self-averaging quantity for small \( T \), we can take the average of Eq.\[8\],\[9\] over all the possible histories in order to evaluate it. After some algebra and some approximations, it is possible to derive the following equations:

\[
\Omega_{t+1} + \phi_{t+1}(x) = \Omega_t \phi_t(x) - \Omega_t \phi_t(x) \frac{1}{1 + \Omega_t e^{\beta(x - \Omega_t)/n_t}} \tag{10}
\]

where \( \Omega_{t+1} = \Omega_t + n_t - 1 \). To obtain Eq.\[10\] we have assumed that \( \Omega_t \gg 1 \) and \( e^\beta \gg \Omega_t \). Clearly the dynamical evolution of the histogram is strictly related to that of \( n_t \); in IP for \( t \gg 1 \) we have \( n_t \approx 1/p_c \). For \( t \gg T \) the evolution of \( \phi_t(x) \) will be very slow (i.e. \( |\phi_{t+1}(x) - \phi_t(x)|/\phi_t(x) \ll 1 \)), because of the increasing effect of stochasticity. The main effect is geometrical and it is related to the fact that, as the system pass from fractal to homogeneous, \( n_t \rightarrow 1 \) (it can be shown that \( n_t - 1 \) represents the asymptotic value of the ratio between the number of bonds belonging to the interface and the bonds belonging to the cluster). Here we have:

\[
\phi_t(x) \approx \frac{n_t}{n_t - 1 + \frac{1}{n_t e^{\beta(x - \Omega_t)/n_t}}} \tag{11}
\]

\( \phi_t(x) \) is a smoothed step function centered at \( x = 1/n_t \) and large \( \Delta x \sim T \). For \( t = t^*(T) \gg 1 \) we have \( n_t \approx 1/p_c \), as the dynamics is IP-like \[10\],\[12\]. Then

\[
\phi_{t^*}(x) \approx \frac{1}{1 - p_c + \frac{p_c}{n_t e^{\beta(x - p_c)/n_t}}} \tag{12}
\]

This function differs from eq.\[11\] only in a region of extension \( \Delta x \sim T \) just around \( x = p_c \). The agreement between this function and the numerical data is very good for a wide range of \( T \) (Fig.2). From Eq.\[12\] and from the exponent of IP, we obtain the behavior of \( s_0(T) \)
and $\xi(T)$ for small values of $T$. In IP an avalanche with initiator different from $p_c$ of a quantity $\Delta x$ has a typical size $s_0(\Delta x) \sim \Delta x^{-1/\gamma}$. Here we have a natural value $\Delta x \sim T$ even for the maximal sequence of correlated growth events. Hence $s_0(T) \sim T^{-1/\gamma} = T^{-\gamma}$ with $\gamma = 2.0 \pm 0.1$ in agreement with the simulations. Finally because of the fractality of IP, we have $s_0(T) \sim \xi(T)^\nu$, hence $\xi(T) \sim T^{-\nu}$ with $\nu = \gamma / D = 1.10 \pm 0.05$.

In conclusion, we presented here a generalization of the IP model where stochasticity, by means of a temperature-like parameter $T$ is introduced. The model produces structures that are fractal and self-organized only by tuning this parameter to 0, otherwise a finite correlation length exists. This behaviour (similar to that observed for the Bak and Sneppen model \cite{7} by M. Vergeles \cite{3}), supports the hypothesis that SOC models are closely related to ordinary critical systems, where parameters have to be tuned to their critical value. The fundamental difference, in our opinion, is in the feasibility of this tuning. For SOC models, one has typically to consider limits to 0 instead of some other real number, the larger probability to achieve this 0 value with respect to any other value is linked to the nature of the driving parameter that it is usually a density for such systems.

[1] P.Bak, C.Tang and K.Weisenfeld, Phys. Rev. Lett. 59, 381 (1987). M.Paczuski, P.Bak and S.Maslov, Phys. Rev. E 53, 414 (1996).
[2] D.Wilkinson, J.P.Willemsen, J. Phys. A London 26, 3365 (1993).
[3] M.Vergeles, Phys. Rev. Lett. 75, 1969 (1995).
[4] G.Caldarelli, M.Vendruscolo, A.Maritan, Europhys. Lett. 35, 481 (1996). G.Caldarelli Physica A 252, 295 (1998).
[5] R.Dickman, A.Vespignani and S.Zapperi, Phys. Rev. E, 57, 5095 (1998).
[6] T.A.Witten, L.M.Sander, Phys. Rev. Lett. 47, 1400 (1981).
[7] P. Bak and K. Sneppen, Phys. Rev. Lett. 71, 4083 (1993).
[8] M.Eden, 4th Berkeley Symp. on Mathematical Stat. and Prob. pag.223 (1961):
[9] M.Marsili, J. Stat. Phys. 77, 733 (1994): A.Gabrielli, M.Marsili, R.Caferro and L.Pietronero, J. Stat. Phys. 84, 889 (1996).
[10] A.Gabrielli, R.Caferro, M.Marsili and L.Pietronero, Europhys. Lett. 38, 491 (1997).
[11] A.Gabrielli, R.Caferro, G.Caldarelli, Europhys. Lett. 45, 13 (1999).
[12] “Memory effects in stochastic growth dynamics with quenched disorder: the Generalized Run Time Statistics”, A.Gabrielli, in preparation.
[13] R.Caferro, A.Gabrielli and M.Marsili, Phys. Rev. E, 55, 7745 (1997).
[14] R.Caferro, G.Caldarelli and A.Gabrielli, Phys. Rev. E, 56, R1291 (1997).
[15] J.T. Chayes, L. Chayes and C. M. Newman, Comm. Math. Phys. 101, 383 (1985).