Webs of integrable theories

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Abstract

We present an intuitive diagrammatic representation of a new class of integrable $\sigma$-models. It is shown that to any given diagram corresponds an integrable theory that couples a certain number of each of the following four fundamental integrable models, the PCM, the YB model, both based on a group $G$, the isotropic $\sigma$-model on the symmetric space $G/H$ and the YB model on the symmetric space $G/H$. To each vertex of a diagram we assign the matrix of one of the aforementioned fundamental integrable theories. Any two vertices may be connected with a number of "propagators" having momenta $k_i$, with each of the propagators being associated with an asymmetrically gauged WZW model at an arbitrary integer level $k_i$. Gauge invariance of the full action is translated to momentum conservation at the vertices. We also show how to immediately read from the diagrams the corresponding $\sigma$-model actions. The most generic of these models depends on at least $n^2 + 1$ parameters, where $n$ is the total number of vertices/fundamental integrable models. Finally, we discuss the case where the momentum conservation at the vertices is relaxed and the case where the deformation matrix is not diagonal in the space of integrable models.
1 Introduction

Integrability plays a pivotal role in obtaining exact results in quantum field theory (QFT). One of the most studied examples in which integrability was greatly exploited is that of $\mathcal{N} = 4$ SYM, the maximally supersymmetric gauge theory in four spacetime dimensions. Employing a variety of integrability-based techniques ranging from the asymptotic Bethe ansatz [2] and the thermodynamic Bethe ansatz [3] to the Y-system [4], the planar anomalous dimensions of gauge invariant operators was determined essentially for all values of the 't Hooft coupling $\lambda = g^2_{YM} N$. Further developments on integrability and the AdS/CFT correspondence can be found in [5] and references therein.

Integrable non-linear $\sigma$-models play an instrumental role in the context of gauge/ gravity dualities. This happens because, thanks to the duality, the strongly coupled dynamics of gauge theory can be translated to the weakly coupled dynamics of an integrable two-dimensional non-linear $\sigma$-model. The prototypical example of such an integrable $\sigma$-model is the principal chiral model (PCM) based on a semi-simple group $G$, with or without a Wess-Zumino (WZ) term. In [6–8] it was shown that the PCM
based on a semi-simple group $G$ admits an integrable deformation depending on an additional continuous parameter. These integrable models are called Yang-Baxter (YB) models and for the case of symmetric and semi-symmetric spaces they were studied in [9–11]. There are also two parameter integrable deformations of the PCM. These are the YB $\sigma$-model with a WZWN term [12] and the bi-YB model [8]. Furthermore, integrable deformations of the PCM with three or more parameters were studied in [13]. It is a remarkable fact that all these models can be put under the unifying description of the so-called $E$-models [15, 16].

Recently, the systematic construction of a large class of integrable two-dimensional field theories based on group, symmetric and semi-symmetric spaces and having an explicit Lagrangian formulation was deployed in a series of papers [17–24]. These models may contain several couplings, for small values of which they take the form of one or more WZW models [25] perturbed by current bilinears. Following their construction, the quantum properties of these theories were studied in great detail in [27, 29, 26, 30, 31]. In this context many observables of these theories, including their $\beta$-functions [35, 36, 30, 32, 37, 38], anomalous dimensions of currents and primary operators [27, 29, 39, 40] and three-point correlators of currents and/or primary fields [27, 40] were computed as exact functions of the deformation parameters. Subsequently, the Zamolodchikov’s C-function [41] of these models were calculated also as exact functions of the deformation parameters [42, 43].

To get these exact results for the aforementioned observables a variety of complementary methods were employed. One way [28, 27, 29] to obtain exact expressions for the anomalous dimensions of currents and primary operators, as well as for the three-point correlators involving currents and primaries was to combine low order perturbation theory around the conformal point with certain non-perturbative symmetries [33, 20, 21, 29] which these theories generically exhibit in the space of couplings. Another method developed was based on the geometry in the space of couplings [39]. This method makes no use of perturbation theory and allows, in principle, the calculation of the anomalous dimensions of composite operators made from an arbitrary number of currents. The essence of the method relies on the ability to construct the all-

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\footnote{These results although exact in the deformation parameters provide only the leading contribution in the $1/k$-expansion. More recently, the subleading terms in $1/k$-expansion were obtained for the $\beta$-functions in [44, 46] and for the C-function and the anomalous dimensions of the operators perturbing the CFT in the cases of group and coset spaces in [44].}
loop effective action of these models [39]. Even more recently, yet another method for calculating exact results in this class of models was initiated in [40]. The method consists of expanding the known all-loop effective actions of the theories around the unit group element and keeping only a few leading terms in the expansion. The advantage of this method is that one ends up performing perturbative calculations around a free field theory and not around the conformal point, which is a much easier task. In addition, all deformation effects are captured by the couplings of the interaction vertices. Subsequently, the applicability of this method to the case of deformed coset CFTs was demonstrated in [45].

Let us mention that the main virtue of the models constructed in [19–21] for deformations based on current algebras and in [47] for deformations of coset CFTs, compared to the prototype single λ-deformed model of [17] (for the group $SU(2)$ the λ-deformed model was found earlier in [48]) is that the RG flows of the former have a rich structure consisting of several fixed points, with different CFTs sitting at different fixed points. It remains an open problem to fully classify these CFTs according to their symmetry groups. In [49], this goal was achieved for a generalisation of the cyclic λ-deformed models of [26] in which arbitrary different levels for the WZW models were allowed.

In a parallel development, an interesting relation between λ-deformations and η-deformations for group and coset spaces was uncovered in [50, 51], [52, 15, 14, 53]. In particular, the λ-deformed models are related to the η-deformed models via Poisson-Lie T-duality\footnote{Poisson-Lie T-duality has been introduced for group spaces in [53] and extended to coset spaces in [56].} and appropriate analytic continuations. Finally, D-branes regarded as boundary configurations preserving integrability were introduced in the context of λ-deformations in [54].

The plan of the paper is as follows. In section 2, we will construct the σ-model actions of a general class of integrable models that couple an arbitrary number of the following four fundamental theories, that is $n_1$ different copies of the PCM, $n_2$ different copies of the YB model, both based on a group $G$, $n_3$ different copies of the the isotropic σ-model on the symmetric coset space $G/H$ and $n_4$ different copies of the YB model on the symmetric space $G/H$. The coupling is achieved by gauging the left global symmetry of the aforementioned fundamental integrable models and connect-
ing them with asymmetrically gauged WZW models. The latter depend on both the
gauge fields of the fundamental integrable theories which they connect. In this way,
webs of integrable theories are obtained. We show that a diagrammatic representation
of these webs is possible. The virtue of this diagrammatic representation is that one
can, at the back of the envelope, draw any diagram and directly write down from it the
corresponding integrable theory. The most generic of these models depends on at least
\( n^2 + 1 + n_2 + n_4 \) parameters, where \( n \) is the total number of vertices/fundamental in-
tegrable models. In section 3, we will prove that the theories constructed in section
2 are indeed classically integrable by finding the corresponding Lax pairs. In section
4, we will consider two more general situations. In the first one we focus on the case
in which the deformation matrix is not diagonal in the space of the fundamental theo-
ries, in distinction with the models of section 2. In the second we examine the case in
which, although the deformation matrix is diagonal in the space of the fundamental
theories, momentum conservation at the vertices is relaxed. In both cases we were
able to prove integrability only when all the deformation matrices are proportional to
the identity in the group space, that is when only when all the theories we couple are
all of the PCM-type. Finally, in section 5 we will present our conclusions.

2 Coupling integrable theories

In this section we will construct the effective actions of our models and establish their
diagrammatic representation. In section 3, we will derive the corresponding equa-
tions of motion and prove that the theories presented in this section are classically
integrable.

2.1 Constructing the models and their diagrammatic representation

Our starting point is to consider the sum of \( n \) integrable models based on group ele-
ments \( \tilde{g}_i, i = 1, 2, \ldots, n \) each of which has a left global symmetry \( \dot{g}_i \rightarrow \Lambda_i^{-1}\tilde{g}_i \), which
will be eventually gauged. Thus we start from the action

\[
S_{E_i}(\tilde{g}_i) = -\frac{1}{\pi i} \int d^2\sigma \left( \dot{\tilde{g}}_i^{-1} \partial_+ \tilde{g}_i \right)_a E_{ij}^{ab} \left( \dot{\tilde{g}}_j^{-1} \partial_- \tilde{g}_j \right)_b, \quad E_{ij}^{ab} = \delta_{ij} E_i^{ab} \quad (2.1)
\]
where the indices \( i, j = 1, 2, \ldots, n \) enumerate the different integrable models while the indices \( a, b = 1, 2, \ldots, \dim(G) \) denote group indices. Furthermore, although in most of this paper we will assume that the matrix \( E_{ij} \sim \delta_{ij} \) in all algebraic manipulations we will keep its most general non-diagonal in the space of models form, in anticipation of the analysis in section 4.1. The integrable models appearing in (2.1) will be the basic building blocks of our construction and will be called the fundamental integrable models. In the sum (2.1) there can be \( n_1 \) different copies of the PCM, \( n_2 \) different copies of the YB model both based on the same semi-simple group \( G \), \( n_3 \) different copies of the isotropic \( \sigma \)-model on the symmetric coset space \( G/H \) and \( n_4 \) different copies of the YB model on the symmetric space \( G/H \), with \( n_1 + n_2 + n_3 + n_4 = n \).

The corresponding \( E_i \) matrices acquire the following forms, \( E_{i}^{ab} = E_i \delta^{ab} \) for the PCM, \( E_i = \frac{1}{t_i} (1 - \eta_i R_i)^{-1} \) for the YB, \( E_{i}^{ab} = \text{diag}(E_i^{g/h} \delta^{ab}, 0_h) \) for the isotropic symmetric space \( G/H \) and \( E_i = \text{diag}(\frac{1}{t_i} (1 - \eta_i R_i)^{-1}|_{g/h}, 0_h) \) for the YB on the symmetric space \( G/H \). In the last case \( R_i \) is an antisymmetric matrix of dimension \( \dim(G) - \dim(H) \) which one can think of as being the projection to the coset \( G/H \) of an \( R \)-matrix obeying the modified Yang-Baxter equation \([52]\)\(^3\). In addition, \( R_i \) should be such that it obeys the condition (3.13).

The question we would like to answer in this section is the following. Is it possible to couple the aforementioned fundamental integrable models appearing in (2.1) in such a way that the resulting \( \sigma \)-model is also integrable? The answer to this question is affirmative. The first step to achieve this coupling is to gauge the left global symmetry of (2.1) mentioned above. As a result the action (2.1) becomes

\[
S_{E_i}(\tilde{g}_i, A^{(i)}_{\pm}) = -\frac{1}{\pi} \int d^2 \sigma \left( \tilde{g}_i^{-1} \tilde{D}_+ \tilde{g}_i \right)_a E_{ij}^{ab} \left( \tilde{g}_j^{-1} \tilde{D}_- \tilde{g}_j \right)_b, \quad E_{ij}^{ab} = \delta_{ij} E_i^{ab} \tag{2.2}
\]

where the covariant derivatives are defined as \( \tilde{D}_\pm \tilde{g}_i = (\partial_\pm - A^{(i)}_{\pm}) \tilde{g}_i \). Then the coupling of the fundamental building blocks is realised by connecting them with asymmetrically gauged WZW models at arbitrary integer levels. To be more precise con-

\(^3\)For the YB theories the group indices \( a, b \) have been suppressed in the corresponding expressions for \( E_i \).
Consider the asymmetrically gauged WZW model \cite{57}

\begin{equation}
S_{k_{ij}}^{(l_{ij})}(S_{ij}^{(l_{ij})}, A^{(i)}, A^{(j)}) = S_{k_{ij}}^{(l_{ij})}(g_{ij}^{(l_{ij})}) + \frac{k_{ij}^{(l_{ij})}}{\pi} \int d^2 \sigma \text{Tr}(A^{(i)}J_{+ij} - A^{(j)}J_{-ij}) + A^{(i)}g_{ij}^{(l_{ij})}A^{(j)}(g_{ij}^{(l_{ij})})^{-1} - \frac{1}{2} A^{(i)}A^{(i)} - \frac{1}{2} A^{(j)}A^{(j)},
\end{equation}

(2.3)

where we have defined the currents

\begin{equation}
J_{+ij}^{(l_{ij})} = J_{+}(g_{ij}^{(l_{ij})}) = \partial_+ g_{ij}^{(l_{ij})}(g_{ij}^{(l_{ij})})^{-1}, \quad J_{-ij}^{(l_{ij})} = J_{-}(g_{ij}^{(l_{ij})}) = (g_{ij}^{(l_{ij})})^{-1} \partial_- g_{ij}^{(l_{ij})},
\end{equation}

(2.4)

and where \(S_{k_{ij}}^{(l_{ij})}(g_{ij}^{(l_{ij})})\) is the WZW model at level \(k_{ij}^{(l_{ij})}\). The notation in (2.3) and (2.4) should be self-explanatory. The asymmetrically gauged WZW functional depends on the group element \(g_{ij}^{(l_{ij})}\) and connects the fundamental integrable model at site \(i\) to that at site \(j\) since it depends also on \(A^{(i)}\) and \(A^{(j)}\). The corresponding WZW level is denoted by \(k_{ij}^{(l_{ij})}\). The superscript \(l_{ij}\) counts how many different gauged WZW models connecting site \(i\) to site \(j\) one has. In the case where there is just one such model the superscript \(l_{ij}\) is superfluous and can be omitted (see, for example, figure 3). Furthermore, due to the asymmetry of the gauging one can assign a direction to the WZW model, and as a consequence to the flow of the level \(k_{ij}^{(l_{ij})}\), which we choose to be from the site \(i\) to the site \(j\). Notice that \(k_{ij}^{(l_{ij})}\) is generically different from \(k_{ji}^{(l_{ij})}\) due to the asymmetry mentioned above, the former connects sites \(i\) and \(j\) having direction from \(i\) to \(j\) while the latter connects the same sites but with opposite direction. An important comment is in order. In the case where \(i \equiv j\) (2.3) becomes the usual vectorially gauged WZW model at level \(k_{ii}^{(l_{ij})}\).

The group elements of the asymmetrically gauged WZW models have the following transformations \(g_{ij}^{(l_{ij})} \rightarrow \Lambda_i^{-1}g_{ij}^{(l_{ij})}\Lambda_j\). Needless to say that as it stands the action (2.3) is not gauge invariant. Its variation under the infinitesimal form of the gauge transformations

\begin{equation}
\delta g_{ij}^{(l_{ij})} = g_{ij}^{(l_{ij})}u_j - u_i g_{ij}^{(l_{ij})}, \quad \delta A^{(i)} = -\partial_+ u_i + [A^{(i)}_+, u_i], \quad \delta A^{(j)} = -\partial_- u_j + [A^{(j)}_-, u_j],
\end{equation}

(2.5)

\footnote{Regarding the WZW action and the Polyakov-Wiegmann identity we follow the conventions of \cite{18,19}.}
is given by
\[
\delta S_{k_{ij}}(g_{ij}^{(l_{ij})}, A^{(i)}_+, A^{(i)}_-) = \frac{k_{ij}^{(l_{ij})}}{2\pi} \int d^2 \sigma \text{Tr}\left[(A^{(i)}_+ \partial_- u_i - A^{(i)}_+ \partial_+ u_i - (A^{(j)}_- \partial_- u_j - A^{(j)}_- \partial_+ u_j)\right].
\] (2.6)

Notice that in the special case where \(i \equiv j\), \(\delta S_{k_{ii}}(g_{ii}^{(l_{ii})}, A^{(i)}_+, A^{(i)}_-) = 0\).

Consider now the complete action
\[
S_t = S_E(g_i, A^{(i)}_+) + \sum_{i,j} \sum_{l_{ij}} S_{k_{ij}}(g_{ij}^{(l_{ij})}, A^{(i)}_+, A^{(i)}_-). \quad (2.7)
\]

The variation of this action under the transformations (2.5) is given by
\[
\delta S_t = \sum_{i,j} \sum_{l_{ij}} k_{ij}^{(l_{ij})} \int d^2 \sigma \text{Tr}\left[(A^{(i)}_+ \partial_- u_i - A^{(i)}_+ \partial_+ u_i - (A^{(j)}_- \partial_- u_j - A^{(j)}_- \partial_+ u_j)\right]. \quad (2.8)
\]

By exchanging \(i \leftrightarrow j\) in the second parenthesis of (2.8) and by gathering identical terms we deduce that
\[
\delta S_t = 0 \iff \sum_{j \neq i} k_{ij}^{(l_{ij})} = \sum_{j \neq i} k_{ji}^{(l_{ji})}, \forall i. \quad (2.9)
\]

In the spirit of the discussion below (2.4) this relation can be interpreted as momentum conservation at each site \(i\). We have, thus, seen that gauge invariance of the action is equivalent to the requirement that the momentum that flows towards any site should be equal to the momentum that flows away from it. At this point it would be useful to define the following quantities
\[
\tilde{k}_i = \sum_{j \neq i} k_{ij}^{(l_{ij})}, \quad \hat{k}_i = \sum_{j \neq i} k_{ji}^{(l_{ji})}, \forall i. \quad (2.10)
\]

One may now fix the gauge by choosing \(\tilde{g}_i = 1\) for \(i = 1, 2, \ldots, n\). Alternatively, one could have chosen to set to the unit element one or more of the group elements \(g_{ij}^{(l_{ij})}\) appearing in the WZW models leaving, as a result, some of the \(\tilde{g}_i\) intact, that is leaving them as dynamical degrees of freedom. Notice that this is possible due to the fact that most of the WZW models are asymmetrically gauged. We have not checked explicitly but most probably, and up to global issues, this second choice should be related to the first one by a coordinate transformation, as it happens in the case of non-abelian
T-duality. After the gauge fixing one ends up with the following action

\[
S_{gf} = -\frac{1}{\pi} \int d^2\sigma \left[ A_+^{(i)} (\lambda^{-1})_{ij}^{ab} A_-^{(j)} + \sum_{ij} \left( S_{k_{ij}}^{(i)} S_{ij}^{(l_{ij})} + \frac{k_{ij}^{(l_{ij})}}{\pi} \int d^2\sigma \text{Tr} \left( A_-^{(i)} f_+^{(l_{ij})} - A_+^{(j)} f_-^{(l_{ij})} + A_-^{(i)} g_+^{(l_{ij})} A_+^{(j)} \frac{S_{ij}^{(l_{ij})}}{\pi} \right)^{-1} \right) \right],
\]

(2.11)

where

\[
(\lambda^{-1})_{ij}^{ab} = \frac{1}{2} (\tilde{k}_i + \hat{k}_i) \delta_{ij} \delta^{ab} + E_{ij}^{ab} = \delta_{ij} (\lambda^{-1})_{ij}^{ab}, \quad \text{since} \quad E_{ij}^{ab} = \delta_{ij} E_i^{ab}.
\]

(2.12)

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Figure 1: Diagram of an integrable web coupling any two of the fundamental theories through four asymmetrically and four anomaly free gauged WZW models. The two fundamental theories are sitting at the two vertices. There are four different kinds of vertices corresponding to the four fundamental theories, the PCM, the YB model, the isotropic $\sigma$-model on the symmetric space $G/H$ and the YB model on the symmetric space $G/H$. The "propagators" (blue lines) connecting the vertices are associated with WZW models at arbitrary levels subject to the condition that "momentum" at each vertex is conserved. The diagram corresponds to an integrable theory with action of the form (2.11) and (2.16).

We are now in position to present a diagrammatic representation of the action (2.11). Namely,

- With every action functional of the form (2.11) we associate a certain diagram (see, for example, figure 1, figure 2 and figure 3 for the coupling of two, three and four fundamental integrable theories, respectively).

- To each vertex of a diagram we assign the matrix $\lambda_i^{-1}$ of one of the fundamental
integrable theories. These vertices represent the first line of (2.11). The number of vertices \( n = n_1 + n_2 + n_3 + n_4 \) is equal to the number of the fundamental integrable theories we intend to couple, that is \( n_1 \) different copies of the PCM, \( n_2 \) different copies of the YB model \(^5\), both based on a group \( G \), \( n_3 \) different copies of the symmetric coset space \( G/H \) and \( n_4 \) different copies of the YB model on the symmetric space \( G/H \).

- To each "propagator" connecting two vertices \( i \) and \( j \) we assign one of the asymmetrically gauged WZW models in the second line of (2.11). The momentum of the propagator corresponds to the level \( k_{ij}^{(l_{ij})} \) of the WZW model. The flow of the momentum is defined to be from \( i \) to \( j \). \(^6\) There can be more than one propagators with the same direction connecting the vertex \( i \) to the vertex \( j \). The superscript \( l_{ij} \) counts how many different propagators connecting \( i \) and \( j \) and having direction from \( i \) towards \( j \) the diagram has.

- One may also have tadpole-like propagators connecting the vertex \( i \) to itself (see figures 1 and 2). In this case the corresponding WZW model is gauged in the usual anomaly free way.

- Finally, as mentioned above, gauge invariance of the action, before fixing the gauge of course, is equivalent to momentum conservation at each vertex. This fact imposes \( n - 1 \) constraints on the momenta circulating in the diagram, namely that \( \tilde{k}_i = \hat{k}_i \), \( \forall i \). Notice that if one imposes momentum conservation to \( n - 1 \) vertices then momentum conservation of the remaining vertex is automatically satisfied.

Let us now comment on figures 1, 2 and 3. The diagram of figure 1 has two vertices, i.e. \( n = 2 \), and as a result represents the coupling of any two of the fundamental theories. This coupling is achieved through eight gauged WZW models four of which are asymmetrically gauged. Two of the latter have direction from the vertex/model 1 to the vertex/model 2 while the other two from the vertex/model 2 to the vertex/model 1. The four tadpole-like parts of the diagram correspond to four anomaly free vectorially gauged WZW models coupling the two vertices to themselves. Figure 2 depicts the coupling of three fundamental theories \( n = 3 \) through seven asymmetrically and three vectorially gauged WZW models. Finally, figure 3 is an example of an integrable web that couples four, i.e. \( n = 4 \) of the fundamental integrable theories where each

\(^5\)Each of the YB models can have different parameters and different \( R \) matrices obeying the modified YB equation.

\(^6\)Notice that the flow of momentum to the opposite direction from vertex \( j \) to vertex \( i \) is related to the WZW with group element \( g_{ji}^{(l_{ij})} \) at level \( k_{ji}^{(l_{ji})} \).
Figure 2: An example of an integrable web coupling any three of the fundamental theories through seven asymmetrically and three vectorially gauged WZW models. The three fundamental theories are sitting at the three vertices. The "propagators" (blue lines) connecting the vertices are associated with WZW models at arbitrary levels subject to the condition that "momentum" at each vertex is conserved. The diagram corresponds to an integrable theory.

of the vertices is connected to all others. This diagram has a total of sixteen propagators/WZW models.

In order to obtain the $\sigma$-model action one should integrate out the gauge fields $A^{(i)}_{\pm}$ from (2.11). To this end we evaluate

$$\frac{\delta S_{gf}}{\delta A^{(i)}_{+}} = 0 \implies A^{(i)}_{+} = -\left( \frac{1}{\lambda^{-1} - D^{T}} \right)_{ij} J^{(j)}_{-}, \quad J^{(j)}_{-} = \sum_{n,l_{nj}} k^{(l_{nj})}_{nj} J_{-}(g_{nj}^{(l_{nj})}), \quad (2.13)$$

and

$$\frac{\delta S_{gf}}{\delta A^{(i)}_{-}} = 0 \implies A^{(i)}_{-} = \left( \frac{1}{\lambda^{-1} - D} \right)_{ij} J^{(j)}_{+}, \quad J^{(j)}_{+} = \sum_{n,l_{jn}} k^{(l_{jn})}_{jn} J_{+}(g_{jn}^{(l_{jn})}), \quad (2.14)$$
where we have also defined the matrices

\[ D_{ij} = \sum_{l_{ij}} k^{(l_{ij})}_{ij} D(l_{ij}), \quad D^T_{ij} = \sum_{l_{ji}} k^{(l_{ji})}_{ji} D^T(l_{ji}), \quad D^{ab}(g) = \text{tr}(t^a g t^b g^{-1}). \]

Thus, every entry of the matrix \( D_{ij} \) is the sum of the \( D^{ab} \) matrices of the group elements which connect the corresponding vertices/models weighted appropriately by their WZW levels. At this point we should stress that the transpositions in (2.13), (2.14) and (2.15) apply only to the suppressed group indices \( a, b = 1, \ldots, \text{dim}(G) \). Therefore, the entries of the matrices \( \lambda^{-T} - D \) and \( \lambda^{-1} - D^T \) are matrices themselves with their indices taking values in the group \( G \). Consequently, their inversion is to be understood as an inversion in the space of the fundamental integrable models keeping in mind that their entries are non-commutative objects. One can now substitute the gauge fields (2.13) and (2.14) in (2.11) to get the \( \sigma \)-model

\[
S_{\sigma-\text{mod.}} = -\frac{1}{\pi} \int d^2 \sigma \sum_{ij} \left( \frac{1}{\lambda^{-1} - D^T} \right)_{ij} J^{(j)} + \sum_{i,j} \sum_{l_{ij}} S(l_{ij})(g^{(l_{ij})}_{ij}). \tag{2.16}
\]

As an example the matrix \( \lambda^{-1} - D^T \) corresponding to the diagram of figure 2 takes the form

\[
\lambda^{-1} - D^T = \begin{pmatrix}
\lambda_1^{-1} - k^{(1)}_{11} D^T_{11} & -k^{(1)}_{21} D^T_{21} & -k^{(2)}_{21} D^T_{21} & -k^{(1)}_{31} D^T_{31} \\
-k^{(1)}_{12} D^T_{12} & \lambda_2^{-1} - k^{(1)}_{22} D^T_{22} & -k^{(2)}_{22} D^T_{22} & -k^{(1)}_{32} D^T_{32} \\
-k^{(1)}_{13} D^T_{13} & -k^{(1)}_{23} D^T_{23} & \lambda_3^{-1} - k^{(1)}_{33} D^T_{33} & \\
\end{pmatrix}. \tag{2.17}
\]

An important comment is in order. Note that although (2.16) is similar in form to equation (2.13) of [21], the models of the present work certainly do not belong to the subclass of the integrable sector of the models presented in [21] and most probably they do not belong at all to the general class of the models constructed in [21]. A first hint comes from inspecting of the matrix \( D \) which in our case may generically have non-zero entries everywhere in contradistinction to [21] where \( D \) has entries only along the diagonal. The reason behind this difference is that in the models of [21], as well as in those of [60] the number of the WZW terms is equal to that of the integrable theories one couples while in our case the number of the WZW models is strictly greater or equal to that of the integrable theories we couple.

In order to be able to compare with the theories of [21], notice that the the first term
in (2.16) can be rewritten as an $N \times N$ matrix coupling the $N$ currents $j^{(l)}_{i=1,...,N}$ with the $N$ currents $j^{(l)}_{i=1,...,N}$, where $N$ is the total number of the WZW models/propagators. In doing so one can verify that the aforementioned $N \times N$ matrix has zero determinant, it is not invertible and as a result it can never be written as the inverse of an $N \times N$ matrix of the form $\Lambda^{-T} - D$ for some regular $\Lambda^{-T}$ and a diagonal $D$ matrix as it is required by the models constructed in [21] and in section 4.1 of [60]. To demonstrate this fact with an example consider the integrable theory of figure 1. In this case the first term in (2.16) can be rewritten as

$$
\begin{pmatrix}
J^{(1)}_{+11}, J^{(2)}_{+11}, J^{(1)}_{+12}, J^{(2)}_{+12}, J^{(1)}_{+22}, J^{(2)}_{+22}, J^{(1)}_{+21}, J^{(2)}_{+21}
\end{pmatrix} \cdot M_{8 \times 8},
$$

(2.18)

where the $8 \times 8$ matrix $M_{8 \times 8}$ is

$$
M_{8 \times 8} =
\begin{pmatrix}
\Sigma_{11}(k^{(1)}_{11})^2 & \Sigma_{11}(k^{(1)}_{11})k^{(2)}_{11} & \Sigma_{11}k^{(1)}_{11}k^{(1)}_{21} & \Sigma_{11}k^{(1)}_{11}k^{(2)}_{21} & \Sigma_{11}k^{(1)}_{11}k^{(1)}_{22} & \Sigma_{11}k^{(1)}_{11}k^{(2)}_{22} & \Sigma_{12}k^{(1)}_{11}k^{(1)}_{12} & \Sigma_{12}k^{(1)}_{11}k^{(2)}_{12} \\
\Sigma_{11}k^{(1)}_{11}k^{(2)}_{11} & \Sigma_{11}(k^{(2)}_{11})^2 & \Sigma_{11}k^{(2)}_{11}k^{(1)}_{21} & \Sigma_{11}k^{(2)}_{11}k^{(2)}_{21} & \Sigma_{11}k^{(2)}_{11}k^{(1)}_{22} & \Sigma_{11}k^{(2)}_{11}k^{(2)}_{22} & \Sigma_{12}k^{(2)}_{11}k^{(1)}_{12} & \Sigma_{12}k^{(2)}_{11}k^{(2)}_{12} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix},
$$

(2.19)

where $\Sigma_{ij} = \left(\frac{1}{\lambda_{1} - \lambda_{2}} \right)_{ij}$, $i, j = 1, 2$ and we have explicitly written only the first two lines. It is evident from (2.19) that the second row of the matrix is equal to the first multiplied by $k^{(2)}_{11}/k^{(1)}_{11}$. This implies that the determinant of $M_{8 \times 8}$ is zero and thus this matrix is not invertible, as discussed above. This result is a consequence of the fact that only certain linear combinations of the currents $J^{(l)}(i, j)$ enter the first term of the action (2.16).
2.2 Reading the $\sigma$-model from its diagram

We are now in position to reverse the argument. Given any diagram one can immediately write down the corresponding integrable $\sigma$-model action. The steps are as follows.

- Draw a diagram with any number of vertices and to each vertex assign one of the four fundamental integrable theories (see, for example, figure 3).
- Connect the vertices with any number of propagators you wish in such a way that momentum at each vertex is conserved.
- For each propagator write a WZW model at the level dictated by the momentum of the propagator (2nd term in (2.20)).
- For each vertex write the incoming and outgoing currents $J^{(i)}_-$ and $J^{(i)}_+$, respectively.
- Finally, couple these currents through the matrix $(\lambda^{-1} - D^T)^{-1}$, where $D$ is defined in (2.15), to get the special case of (2.16) that corresponds to the diagram at hand.

For the convenience of the reader we copy from the previous section the final form of the $\sigma$-model action

$$S_{\sigma-\text{mod.}} = -\frac{1}{\pi} \int d^2\sigma \sum_{ij} (\lambda^{-1})_{ij} J^{(i)}_+ J^{(j)}_- + \sum_{ij} \sum_{l_{ij}} S_{k_{ij}} (g^{(l_{ij})}_{ij}). \quad (2.20)$$

We should, of course, mention that the inverse of the matrix $\lambda^{-1} - D^T$ has to be evaluated in a case by case basis. Finally, let us mention that for small values of the entries of the matrix $(\lambda^{-1})_{ij}$ the action becomes

$$S_{\sigma-\text{mod.}} = -\frac{1}{\pi} \int d^2\sigma \sum_{ij} (\lambda^{-1})_{ij} J^{(i)}_- J^{(j)}_+ + \sum_{ij} \sum_{l_{ij}} S_{k_{ij}} (g^{(l_{ij})}_{ij}) + O(\lambda^2). \quad (2.21)$$

Notice that in our conventions $-iJ^{(i)}_-$ and $-iJ^{(i)}_+$ generate two Kac-Moody currents at levels $\hat{k}_i$ and $\tilde{k}_j$ respectively. As discussed above these levels are equal due to momentum/level conservation. Finally, the same arguments with those below (2.17) apply to the linearised action (2.21). When the first term in (2.21) is written in a form involving an $N \times N$ matrix, where $N$ is the number of the WZW models, the corresponding $(\Lambda^{-1})_{ij}$ where now $i, j = 1, \ldots, N$ is a non-invertible matrix. This means that there is no matrix $\Lambda^{-1}$ with regular entries such that our model can be straightforwardly
Figure 3: An example of an integrable web representing the coupling of four of the fundamental theories through twelve asymmetrically and four vectorially gauged WZW models. The four fundamental theories are sitting at the four vertices. The "propagators" (blue lines) connecting the vertices are associated with WZW models at arbitrary levels subject to the condition that "momentum" at each vertex is conserved. The intersections of the red with the blue "propagatots" do not represent vertices. The diagram corresponds to an integrable theory.

obtained from the models of [21]. The same holds for the integrable models in section 4.1 of [60] which have the same structure as those in [21] but with a more general $\Lambda$ matrix. Last, but not least, we would like to stress that in the construction of the present paper some of the fundamental theories that serve as building blocks of the final integrable theory are the isotropic $\sigma$-models on the symmetric space $G/H$ or the YB models on the symmetric space $G/H$. This was not the case neither in the construction of [60] nor in that of [21].

\footnote{It might be possible that the models of the present work could be obtained as special decoupling limits of those in [60] but only in the case where all the fundamental theories we couple are of the PCM-type.}
3 Proof of integrability

In this section, we prove that the theories constructed in the previous section are integrable in the case where the coupling $\lambda_{ij}^{-1} = \delta_{ij} \lambda_i^{-1}$. However in most of the manipulations and in anticipation of the results of the following sections we will treat $\lambda_{ij}^{-1}$ as being a general matrix.

We start with the equations of motion for $A^{(i)}_\pm$. These can be easily brought to the form

$$\sum_{i,j} k_{ij}^{(l)} (g^{(l)}_{ij})^{-1} D_+ g^{(l)}_{ij} = - (\lambda_{jn} - \hat{k}_n \delta_{jn}) A^{(n)}_- \quad (3.1)$$

and

$$\sum_{j,i} k^{(l)}_{ij} D_+ g^{(l)}_{ij} (g^{(l)}_{ij})^{-1} = (\lambda_{ni} - \hat{k}_n \delta_{in}) A^{(n)}_+ \quad (3.2)$$

where $\hat{k}_n$ and $\tilde{k}_n$ are defined in (2.10) and the transposition in (3.2) refers only to the suppressed group indices. Notice also that although $\tilde{k}_n = \hat{k}_n$ due to momentum conservation we have not imposed this condition in (3.1) and (3.2) yet. Finally, the covariant derivatives on the WZW group elements read $D_+ g^{(l)}_{ij} = \partial_+ g^{(l)}_{ij} - A^{(i)}_- g^{(l)}_{ij} + g^{(l)}_{ij} A^{(j)}_+$. 

In addition, we will need the equations of motion for the group elements of the WZW models. These turn out to be

$$\frac{\delta S_{gf}}{\delta g^{(l)}_{ij}} = 0 \implies D_- \left( D_+ g^{(l)}_{ij} (g^{(l)}_{ij})^{-1} \right) = F^{A^{(i)}_+}_+ \iff D_+ \left( (g^{(l)}_{ij})^{-1} D_- g^{(l)}_{ij} \right) = F^{A^{(i)}_-}_+ \quad (3.3)$$

where the field strength are defined as usual, $F^{A^{(i)}_+}_+ = \partial_+ A^{(i)}_- - \partial_- A^{(i)}_+ - [A^{(i)}_+, A^{(i)}_-]$ and where the left covariant derivative in the second and third equation of (3.3) are acting to its arguments according to their transformation properties, namely $D_- \cdot = \partial_- \cdot - [A^{(i)}_+, \cdot]$ and $D_+ \cdot = \partial_+ \cdot - [A^{(j)}_+, \cdot]$ respectively. Multiplying the second and third equation in (3.3) by $k^{(l)}_{ij}$ and summing over $j, l, i$ and $i, l, i$ respectively we arrive at

$$\sum_{j, l, i} k^{(l)}_{ij} D_+ \left( D_+ g^{(l)}_{ij} (g^{(l)}_{ij})^{-1} \right) = \tilde{k}_i F^{A^{(i)}_+}_+ \quad (3.4)$$

$$\sum_{i, l, i} k^{(l)}_{ij} D_+ \left( (g^{(l)}_{ij})^{-1} D_- g^{(l)}_{ij} \right) = \hat{k}_i F^{A^{(i)}_-}_+ \quad (3.4)$$
Substituting now (3.1) and (3.2) in (3.4) we get after some algebra the equations of motion of the system expressed solely in terms of the gauge fields. These read
\[ \tilde{k}_i \partial_+ A_+^{(i)} - \lambda^{-T}_{ni} \partial_- A_+^{(n)} = [\lambda^{-T}_{ni} A_+^{(n)}, A_-^{(i)}], \]
\[ \lambda^{-1}_{in} \partial_+ A_-^{(n)} - \hat{k}_i \partial_- A_-^{(i)} = [A_-^{(i)}, \lambda^{-1}_{in} A_-^{(n)}]. \]

(3.5)

In the case where \( \lambda^{-1}_{ij} = \delta_{ij} \lambda_i^{-1} \) and the momentum at each vertex is conserved, i.e. \( \tilde{k}_i = \hat{k}_i \), which is precisely the case we consider in this section, the equations in (3.5) decouple in the space of models and become
\[ \partial_+ A_-^{(i)} - \lambda_i^{-T} \partial_- A_-^{(i)} = [\lambda_i^{-T} A_-^{(i)}, A_-^{(i)}], \]
\[ \lambda_i^{-1} \partial_+ A_-^{(i)} - \partial_- A_-^{(i)} = [A_-^{(i)}, \lambda_i^{-1} A_-^{(n)}]. \]

(3.6)

where \( \lambda_i^{-1} = 1 + \frac{E_i}{k_i} = \frac{\lambda_i}{k_i} \). In the last relation we have used momentum conservation to define \( k_i = \tilde{k}_i = \hat{k}_i \). Thus we see that the equations of motion of our theory reduce to \( n = n_1 + n_2 + n_3 + n_4 \) decoupled sets of equations. Each of these sets correspond to the equations of motion of a single \( \lambda \)-deformed model with \( n_1 \) of the \( \lambda_i^{-1} \)'s being the isotropic matrices of PCMs, \( n_2 \) of the \( \lambda_i^{-1} \)'s being the matrices of YB models, \( n_3 \) of the \( \lambda_i^{-1} \)'s being the isotropic matrices of a symmetric coset space \( G/H \) and \( n_4 \) of the \( \lambda_i^{-1} \)'s being the matrices of the YB model on the symmetric space \( G/H \).

Notice that despite the decoupling of the equations of motion when these are expressed in terms of the gauge fields, the \( \sigma \)-model action assumes a non-trivial form in which the group elements and the deformation matrices \( \lambda_i^{-1} \) are coupled in a very complicated way the details of which depend on the topology of the corresponding diagram. More precisely, to the same set of \( n = n_1 + n_2 + n_3 + n_4 \) fundamental integrable models and WZW models at certain levels correspond many different coupled \( \sigma \)-models since there are many different ways/diagrams to connect the vertices (fundamental models) with the propagators (asymmetrically gauged WZW models). Furthermore, the Hamiltonian density of our models can not be written solely in terms of the gauge fields \( A_+^{(i)} \) and the couplings \( \lambda_i^{-1} \) as it was possible in the case of doubly and cyclic \( \lambda \)-deformed models which were shown to be canonically equivalent to the sum of two or more single \( \lambda \)-deformed models [18, 26]. This essential difference can be traced to the fact that in our models the group degrees of freedom \( g_i^{(li)} \) are generically strictly greater than the number of the fundamental theories we couple and thus
greater than the number of the gauge fields of the theory (see for example (2.7) or (2.11)).

The number of independent parameters that the models of this section possess is at least \( n^2 + 1 + n_2 + n_4 \). This number comes out as follows. Since each of the vertices can be connected to all other vertices including itself the most generic diagram has at least \( n^2 \) parameters which are the levels \( k_{ij}^{(l)} \). Momentum conservation imposes \( n - 1 \) constraints on the levels (see last bullet point below (2.12)). Lastly, one has \( n + n_2 + n_4 \) continuous parameters in the definitions of the \( \hat{\lambda}_i^{-1} \) matrices. Putting everything together we get that our models depend on at least \( n^2 + 1 + n_2 + n_4 \) parameters.

Equations (3.6) imply the existence of \( n \) independent Lax pairs satisfying

\[
\partial_+ L_{-}^{(i)} - \partial_- L_{+}^{(i)} - [L_{+}^{(i)}, L_{-}^{(i)}] = 0. \tag{3.7}
\]

These are given by

\[
L_{\pm}^{(i)} = \frac{2}{1 + \lambda_i} \frac{z_i}{1 \mp z_i} A_{\pm}^{(i)}, \quad 1 \leq i \leq n_1 \tag{3.8}
\]

in the case where \( \lambda_i \) is the coupling obtained from the isotropic matrix of a PCM, whereas in the case where \( \lambda_i = 1 + \frac{1}{\eta_i} (1 - \eta_i R_i)^{-1} \) and the coupling is obtained from a YB model the Lax pair is given by \([52]\]

\[
L_{\pm}^{(i)} = \left( (\alpha_1^{(i)} + \alpha_2^{(i)} \frac{z_i}{z_i \mp 1}) \mp \eta_i R_i \right) (\mp \eta_i R_i)^{-1} A_{\pm}^{(i)}, \quad n_1 < i \leq n_1 + n_2
\]

\[
\alpha_1^{(i)} = \alpha_i - \sqrt{\alpha^2 - c^2 \eta_i^2}, \quad \alpha_2^{(i)} = 2 \sqrt{\alpha_i^2 - c^2 \eta_i^2}, \quad \alpha_i = \frac{1 + c^2 \eta_i^2 \rho_i}{1 + \rho_i}, \quad \rho_i = \frac{k_i t_i}{1 + k_i t_i}. \tag{3.9}
\]

In (3.9) \( c^2 = 0, \pm 1 \) and the skew symmetric matrix \( R_i \) satisfies the modified Yang-Baxter equation \([R_i A, R_i B] - R_i ([R_i A, B] + [A, R_i B]) = -c^2 [A, B], \forall A, B \in \mathcal{L}(G)\). The third possibility is when the fundamental integrable model sitting at a vertex is the isotropic \( \sigma \)-model on a symmetric space of the coset form \( G/H \). In this case the
The equation of motion (3.6) become
\[
\partial_\pm A^{(i)g/h}_\pm = -[A^{(i)g/h}_\pm , A^{(i)h}_\pm ] , \quad \partial_\pm A^{(i)h}_\pm - \partial_- A^{(i)h}_+ - [A^{(i)h}_+, A^{(i)h}_-] = \frac{1}{\hat{\lambda}_i} [A^{(i)g/h}_+, A^{(i)h}_-] ,
\]
\[
A^{(i)}_\pm = A^{(i)h}_\pm + z^{\pm 1}_i A^{(i)g/h}_\pm , \quad n_1 + n_2 < i \leq n_1 + n_2 + n_3 \tag{3.10}
\]
These equations imply the existence of a Lax connection of the following form [22]
\[
\mathcal{L}^{(i)}_\pm = A^{(i)h}_\pm + z^{\pm 1}_i A^{(i)g/h}_\pm , \quad n_1 + n_2 < i \leq n_1 + n_2 + n_3 \tag{3.11}
\]
where \(z_i\) is, as usual, the spectral parameter.

The fourth and last possibility is when the fundamental integrable model sitting at a vertex is a YB model based on the symmetric space of the coset form \(G/H\). In this case the equation of motion (3.6) imply the existence of a Lax connection of the form [52]
\[
\mathcal{L}^{(i)}_\pm = A^{(i)h}_\pm + z^{\pm 1}_i (\frac{1}{\sqrt{p_i}} + \eta_i \rho_i^{\pm 1} \mathcal{R}_i) (\mathbf{1} \pm \eta_i \mathcal{R}_i)^{-1} A^{(i)g/h}_\pm , \quad n_1 + n_2 + n_3 < i \leq n \tag{3.12}
\]
given that the projection of the \(\mathcal{R}_i\)-bracket in the subalgebra \(h\) vanishes, namely that
\[
([\mathcal{R}_i X, Y] + [X, \mathcal{R}_i Y])|_h = 0 , \quad X, Y \in g/h . \tag{3.13}
\]

We close this section with an important comment. The infinite tower of conserved charges obtained from any of the above \(n = n_1 + n_2 + n_3 + n_4\) Lax pairs are in involution with those obtained from all the remaining \(n - 1\) Lax pairs. To see this one can define the following dressed currents that obey commuting Kac-Moody algebras [58, 18, 26]
\[
\mathcal{J}_{-ji}^{(lji)} = - (S_{ji}^{(lji)})^{-1} D_- S_{ji}^{(lji)} + A_-^{(i)} - A_+^{(i)} \tag{3.14}
\]
\[
\mathcal{J}_{+ij}^{(lji)} = D_+ S_{ij}^{(lji)} (S_{ij}^{(lji)})^{-1} + A_+^{(i)} - A_-^{(i)} .
\]

Multiplying the first equation by \(k_{ji}^{(lji)}\) and summing over \(j\) and \(l_{ji}\) and the second equation by \(k_{ij}^{(lji)}\) and summing again over \(j\) and \(l_{ij}\) one obtains, after substituting in
the constraints (3.1) and (3.2), the following relations

\[
\sum_{j,l} k_{ji}^{(lji)} \mathcal{J}_{-ji}^{(lji)} = (\lambda_i^{-1} A_-(i) - \tilde{k}_i A_+^{(i)})
\]

\[
\sum_{j,l} k_{lj}^{(lji)} \mathcal{J}_{+ji}^{(lji)} = (\lambda_i^{-T} A_+^{(i)} - \tilde{k}_i A_-^{(i)}).
\]

(3.15)

This set of equations can be now solved for the gauge fields \( A_{\pm}^{(i)} \) in terms of \( \sum_{j,l} k_{ji}^{(lji)} \mathcal{J}_{-ji}^{(lji)} \) and \( \sum_{j,l} k_{lj}^{(lji)} \mathcal{J}_{+ji}^{(lji)} \). Given that the Poisson brackets \( \{ \mathcal{J}_{+ji}^{(lji)}, \mathcal{J}_{-ji}^{(lji)} \}_{PB} = 0 = \{ \mathcal{J}_{-ji}^{(lji)}, \mathcal{J}_{-ki}^{(lki)} \}_{PB} \) when \( i \neq \hat{i} \) and that \( \{ \mathcal{J}_{+\ldots}, \mathcal{J}_{-\ldots}^{(l)} \}_{PB} = 0 \) we deduce that \( \{ A_+^{(i)}, A_-^{(i)} \}_{PB} = 0 \) for \( i \neq \hat{i} \).

As a result of the last equation we have that \( \{ L_+^{(i)}, L_-^{(i)} \}_{PB} = 0 \) for \( i \neq \hat{i} \) which in turn implies that the conserved charges obtained from different Lax connections are in involution.

## 4 Coupling isotropic integrable theories

In this section, we consider the special case where all the vertices are of the PCM-type, that is when the coupling matrices \( (\lambda^{-1})_{ij}^{ab} \) are diagonal and isotropic in the group space, namely \( (\lambda^{-1})_{ij}^{ab} = (\lambda^{-1})_{ij} \delta_{ab} \). In section 4.1, we will consider the integrable case where the deformation matrix is non-diagonal in the space of theories with momentum conservation imposed at each vertex. In the next section 4.2, we will focus on the integrable case where the deformation matrix is diagonal in both the group space and the space of models, that is when \( (\lambda^{-1})_{ij}^{ab} = \lambda_i^{-1} \delta_{ij} \delta_{ab} \), but momentum conservation at the vertices is not imposed.

### 4.1 Non-diagonal in the space of models deformation matrix

In this case after we make the following redefinitions

\[
\tilde{A}_-^{(i)} = \sqrt{k_i} A_-^{(i)}, \quad \tilde{A}_+^{(i)} = \sqrt{k_i} A_+^{(i)}, \quad \lambda_{ij}^{-1} = \frac{1}{\sqrt{k_i k_j}} \lambda_i^{-1}
\]

(4.1)
the general equations of motion \eqref{eq:gen_eqs} become
\[
\partial_+ \tilde{A}^{(i)}_+ - \tilde{\lambda}^{-1}_{ij} \partial_- \tilde{A}^{(j)}_- = \frac{1}{\sqrt{\tilde{k}_i}} [\tilde{A}^{(j)}_-, \tilde{A}^{(i)}_+] ,
\]
\[
\tilde{\lambda}^{-1}_{ij} \partial_+ \tilde{A}^{(j)}_- - \partial_- \tilde{A}^{(i)}_+ = \frac{1}{\sqrt{\tilde{k}_i}} [\tilde{A}^{(i)}_+, \tilde{\lambda}^{-1}_{ij} \tilde{A}^{(j)}_-] .
\]

If we now impose momentum conservation $k_i = \tilde{k}_i = \hat{k}_i$ we see that in the case of isotropic $\lambda$ the equations of motion of our model take precisely the form of the equations of motion of the most general $\lambda$-deformed model constructed in [21] (see eq. (2.9) of this work). This by no means that these two classes of theories are trivially identical since, as mentioned above, in our construction the number of WZW models is strictly greater or equal to the number of gauge fields while in the construction of [21] these two number are precisely equal. Thus, the degrees of freedom of our models are generically greater than those of the models in [21] for the same number of gauge fields.

Given the form of the equations of motion \eqref{eq:gen_eqs} we immediately deduce that when the matrix $\tilde{\lambda}^{-1}$ has the form\footnote{Notice that the $\lambda^{-1}_{ij}$ in \eqref{eq:lambda_inv} that follow are not the same with the $\lambda^{-1}_{ij}$ of \eqref{eq:lambda_inverse}. We have used the same letter so that we do not have proliferation of symbols.}
\[
\tilde{\lambda}^{-1}_{ij} = \begin{pmatrix}
\lambda^{-1}_{11} & 0 & \cdots & 0 \\
\lambda^{-1}_{21} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda^{-1}_{(n-1)1} & 0 & \cdots & 0 \\
0 & \lambda^{-1}_{n2} & \cdots & \lambda^{-1}_{nn}
\end{pmatrix}
\]

the theory is integrable, exactly as it happened in \cite{21}, with the Lax pairs given by
\[
\mathcal{L}^{(1)}_+ = \sum_{i=1}^{n-1} c^{(i)}_+(z) \tilde{A}^{(i)}_+ , \quad \mathcal{L}^{(1)}_- = z \tilde{A}^{(1)}_- ,
\]

where
\[
c^{(i)}_+ = \frac{\lambda^{-1}_{11}(\lambda_{i1}^{-1} - \mu_{i1})}{(\lambda_{i1}^{-1} - z \sqrt{k_i})} \frac{z}{d + d_1} , \quad i = 1, 2, \ldots, n-1 , \quad d_1 = \sum_{j=1}^{n-1} \frac{\lambda^{-2}_{j1}(\lambda_{j1}^{-1} - \mu_{j1})}{\lambda_{j1}^{-1} - z \sqrt{k_j}} , \quad \mu_{i1} = \sqrt{\frac{k_i}{k_1}}
\]

\[
(4.5)
\]
and

\[ \mathcal{L}_-^{(2)} = \sum_{i=2}^{n} c_{-}^{(i)} \tilde{A}_-^{(i)}, \quad \mathcal{L}_+^{(2)} = z \tilde{A}_+^{(n)}, \]  

(4.6)

where

\[ c_{-}^{(i)} = \frac{\lambda_{ni}^{-1}(\lambda_{ni}^{-1} - \mu_{ni})}{(\lambda_{ni}^{-1} - z \sqrt{k_i})} \frac{z}{d + d_1}, \quad i = 2, \ldots, n, \quad \hat{d}_1 = \sum_{j=2}^{n} \frac{\lambda_{nj}^{-2}(\lambda_{nj}^{-1} - \mu_{nj})}{\lambda_{nj}^{-1} - z \sqrt{k_j}}, \quad \mu_{ni} = \sqrt{\frac{k_i}{k_n}}. \]

(4.7)

Finally, in passing let us note that strong integrability of the models presented in [21] with a deformation matrix of the form (4.3) has been proven in [59].

### 4.2 Relaxing momentum conservation

In this section we consider the case where one drops the requirement of momentum conservation at the vertices of the diagrams, that is we no longer impose the conditions \( \tilde{k}_i = \hat{k}_i, \forall i \). In this case and in order to end up with an integrable theory one must demand that the fundamental theories one couples are all of the PCM-type and that the coupling matrix is diagonal in the space of theories, namely that \( (\tilde{\lambda}^{-1})_{ij} = \lambda_i^{-1} \delta_{ij} \delta^{ab} \).

To proceed we make in (4.2) the following redefinitions \( \tilde{A}_+^{(i)} = \sqrt{\hat{k}_i} A_+^{(i)} \) and \( \tilde{A}_-^{(i)} = \sqrt{k_i} A_-^{(i)} \) to get

\[ \partial_+ A_+^{(i)} - (\lambda_0^{(i)})^{-1} \lambda_i^{-1} \partial_- A_+^{(i)} = (\lambda_0^{(i)})^{-1}[\lambda_i^{-1} A_+^{(i)}, A_-^{(i)}], \]

\[ \lambda_0^{(i)} \lambda_i^{-1} \partial_+ A_-^{(i)} - \partial_- A_+^{(i)} = \lambda_0^{(i)} [A_+^{(i)}, A_-^{(i)}], \quad \lambda_0^{(i)} = \sqrt{\frac{k_i}{k_i}}. \]

(4.8)

Notice that this is precisely \( n \) copies of the equations of motion (3.6) of the model presented in [19]. The above equations of motion imply the existence of a Lax connection of the form [19]

\[ \mathcal{L}_-^{(i)} = \frac{2z_i}{z_i + 1} \tilde{A}_-^{(i)}, \quad \tilde{A}_+^{(i)} = \frac{1 - (\lambda_0^{(i)})^{-1} \lambda_i^{-1}}{1 - \lambda_i^2} A_+^{(i)}, \quad \tilde{A}_-^{(i)} = \frac{1 - \lambda_0^{(i)} \lambda_i^{-1}}{1 - \lambda_i^2} A_-^{(i)}, \quad z_i \in \mathbb{C}. \]

(4.9)

As in section 3, one can straightforwardly show by imitating the discussion below (3.14) that the conserved charges obtained from the different Lax connections of (4.9)
are in involution.

A final comment is in order. Note that the cyclic $\lambda$-deformed models of \cite{26} and \cite{49} belong to the class of the models of this subsection. In particular, the diagrams representing the aforementioned models are canonical polygons where each vertex is connected only to the adjacent ones.

5 Conclusions

In this work, we constructed the $\sigma$-model actions of a general class of integrable models. These models couple an arbitrary number of the following fundamental integrable theories, namely $n_1$ different copies of the PCM, $n_2$ different copies of the YB model, both based on a group $G$, $n_3$ different copies of the isotropic $\sigma$-model on the symmetric coset space $G/H$ and $n_4$ different copies of the YB model on the symmetric space $G/H$. The coupling is achieved by gauging the left global symmetry of the aforementioned fundamental integrable theories and connecting them with asymmetrically gauged WZW models. The action of the latter depends on both the gauge fields of the fundamental integrable theories which they connect. In this way, webs of integrable theories are obtained. We show that a diagrammatic representation of these webs is possible. To each vertex of a diagram we assigned the matrix of one of the aforementioned fundamental integrable theories. Any two vertices may be connected with a number of "propagators" having momenta $k_i$ with each of the propagators being associated to an asymmetrically gauged WZW model at an arbitrary level $k_i$. Gauge invariance of the full action is translated to momentum conservation at the vertices. The virtue of this diagrammatic representation is that one can, at the back of the envelope, draw any diagram and directly write down from it the corresponding integrable theory. The most generic of these models depends on at least $n_1^2 + 1 + n_2 + n_4$ parameters, where $n$ is the total number of vertices/fundamental integrable models. Next, we proved that the theories constructed are indeed classically integrable by finding the corresponding Lax pairs.

Subsequently, we considered two more general settings. In the first one, we focused on the case in which the deformation matrix is not diagonal in the space of the fundamental theories, in distinction to the theories of the previous sections. In the
second we examined the case in which, although the deformation matrix is diagonal in the space of the fundamental theories, momentum conservation at the vertices is relaxed. In both cases we were able to prove integrability only when all the deformation matrices are proportional to the identity in the group space, that is when only when all the theories we are coupling are all of the PCM-type.

There is a couple of interesting questions remaining to be addressed. The first one concerns the quantum properties of the models presented. Although the \( \beta \)-functions of the couplings and the anomalous dimensions of the single currents can be straightforwardly deduced from the works \cite{27, 29} using perturbation theory around the conformal point \(^{10}\) the calculation of the exact anomalous dimensions of composite operators made from currents belonging to different WZW models, as well as those of the primary operators is certainly much more demanding since the result will be a non-trivial function of the couplings \( \lambda_{ij}^{-1} \). The same holds for the three-point correlators involving currents and/or primary fields. Notice that for these calculations the methods developed in \cite{39} and \cite{40} are more appropriate compared to the ones used in \cite{27, 29} since for the theories of the present work we do not have the non-perturbative symmetries in the space of couplings which we had in the \( \lambda \)-deformed models with one or more parameters \cite{33, 20, 21, 29}. A second question concerns the Poisson-Lee T-dual theories of our models. Given the relation between the \( \lambda \)- and \( \eta \)-deformations via Poisson-Lie T-duality and appropriate analytic continuations it is natural to wonder if there are new integrable \( \sigma \)-models of the \( \eta \)-type to be constructed and which their relation will be to the ones constructed based on the interpretation of integrable field theories as realisations of the affine Gaudin models \cite{61–63}. In that respect it would be important to study the details of the algebraic and Hamiltonian structure of our theories.

\(^{10}\) The \( \beta \)-functions and the single currents anomalous dimensions of the models of section 2 are the same with those of the corresponding single \( \lambda \)-deformed models while the \( \beta \)-functions and the single currents anomalous dimensions of section 4.1 can be straightforwardly obtained from the analogous expressions in \cite{21}. Finally, the \( \beta \)-functions and the single current anomalous dimensions of the models in section 4.2 are identical to the corresponding expressions of the two-level asymmetric construction of \cite{29, 21} after identifying \( \lambda_0 \) of the latter paper with \( \lambda_0^{(i)} \) of \cite{18}. The reason behind these identifications is that perturbation theory around the conformal points of the different theories is organised in precisely the same way since it is of the current-current or parafermion-parafermion type.
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