Full Counting Statistics of Multiple
Andreev Reflections in incoherent diffusive
superconducting junctions

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Abstract  We present a theory for the full distribution of current fluctuations in incoherent diffusive superconducting junctions, subjected to a voltage bias. This theory of full counting statistics of incoherent multiple Andreev reflections is valid for arbitrary applied voltage. We present a detailed discussion of the properties of the first four cumulants as well as the low and high voltage regimes of the full counting statistics. The work is an extension of the results of Pilgram and the author, Phys. Rev. Lett. 94, 086806 (2005).
1 Introduction

The mechanism for charge transport in voltage biased superconducting junctions is Multiple Andreev Reflections (MAR) \[1,2\]. Coherent MAR-theory \[3,4,5\] has provided a quantum mechanical picture for current transport in mesoscopic superconducting junctions. The predictions of the theory were found to be in impressive agreement with experimental results \[6,7,8\]. This development, with the emphasis on the properties of atomic point contact junctions, was recently reviewed \[9\]. Additional insight in coherent MAR transport, as e.g. the effective charge transferred, has been obtained by investigations of the current noise \[10\], both theoretical \[11,12,13,14\] and experimental \[15,16,17\]. Recently the theoretical interest has turned to the full distribution of current fluctuations \[18\], the Full Counting Statistics (FCS) of coherent MAR-transport \[19,20,21\].

In several types of superconducting junctions, e.g. long diffusive superconducting-normal-superconducting (SNS) junctions, quasiparticles lose phase coherence when traversing the normal part of the junction and both the ac and dc Josephson currents are suppressed. This is the regime of incoherent MAR-transport. The properties of the current in the incoherent MAR-regime have been thoroughly studied, see e.g. \[22\] and references therein. In addition, in several works the current noise has been investigated, both theoretically \[23,24,25\] and experimentally \[26,27,28,29\]. Recently experimental data on the third current cumulant in incoherent diffusive SNS junctions was presented by Reulet et al \[30\]. Moreover, Pilgram and the author \[31\] presented
a general theory for FCS of incoherent MAR. The theoretical result for a
diffusive SNS junction showed qualitative similarities with the experimental
data [30]. The results in Ref. [31] were mainly derived with a stochastic path
tegral technique [32, 33], however the extension of the FCS for diffusive
SNS to arbitrary voltage bias was based on a more intuitive picture of inco-
herent MAR-transport. Appealing to this intuitive picture, in this work we
first present a detailed derivation and then a thorough analysis of the FCS
of incoherent MAR in diffusive SNS. Moreover, a formal derivation based
on the stochastic path integral approach is presented in the appendix.

2 Model and theory

We consider a junction with a diffusive normal conductor connected to
two superconducting terminals. The superconductors have an energy gap $\Delta$
and a voltage $V$ is applied between the superconductors. The phase break-
ing length in the normal conductor is taken to be much shorter than the
length of the conductor and the transport is consequently incoherent [34,
35]. We make the simplifying assumptions that the junction is perfectly
voltage biased and that the scattering is completely elastic. The normal-
superconducting(NS) interfaces are assumed to be transparent. Due to the
diffusive nature of the transport, this gives that the effective Andreev re-
flexion is perfect inside the superconducting gap (at quasiparticle energies
$|E| < \Delta$) and negligible outside the gap [36,37]. Throughout the paper we
take the temperature to be well below the gap, $kT \ll \Delta$, and consider only zero frequency transport properties.

In the paper we will appeal to a qualitative scattering picture of MAR, a more formal derivation based on the approach of Ref. is presented in the appendix. Within the scattering approach, the transport of quasiparticles in the junction can be described as MAR-transport in energy space: a quasiparticle below the gap injected from the superconductor into the normal conductor gain energy by successive Andreev reflections before escaping out into the superconductors at energies above the gap. This is illustrated to the left in fig. [1] for an applied voltage $\Delta < eV < 2\Delta$. Due

![Fig. 1](image)

**Fig. 1** Left: MAR-transport in energy space: An electron-like quasiparticle is injected from the left superconducting terminal at an energy below the gap, $E < -\Delta$. The quasiparticle propagates via MAR to an energy above the gap, $E > \Delta$, at which it escapes out into one of the superconducting terminals. Right: Energy windows for the transfer of an effective charge $2e$ and $3e$.

to the perfect effective Andreev reflection at the normal-superconducting interfaces, there are two possible processes by which a quasiparticle, injected
from the left, can propagate from filled states below the gap to empty states above the gap.

* for injection energies \(-\Delta - eV < E < -\Delta - 2eV\) a quasiparticle traverses the gap three times (gaining an energy \(3eV\)) as electron \(\rightarrow\) hole \(\rightarrow\) electron, effectively transferring a charge \(3e\) from left to right.

* for injection energies \(\Delta - 2eV < E < -\Delta\) a quasiparticle traverses the gap twice (gaining an energy \(2eV\)) as electron \(\rightarrow\) hole, effectively transferring a charge \(2e\) from left to right.

The two transport processes are shown to the right in Fig. [1]. Importantly, only these processes contribute to the net charge transport. We emphasize that in the scattering processes the net charge transferred across the junction is given by the electron charge multiplied by the net number of quasiparticle traversals or equivalently, the gained energy divided by the voltage \(V\). This holds independently on the injection energy and the type of quasiparticle injected [19]. We also note that for quasiparticles injected from the right there are, due to the symmetry of the junction, two equivalent transport processes. Below, we for simplicity consider only the left injected quasiparticles and multiply all the derived transport properties with a factor of two.
2.1 Generalization to arbitrary voltage

The scenario presented above can be generalized to arbitrary applied voltage. We have that for a given voltage

\[
\frac{2\Delta}{n+1} < eV < \frac{2\Delta}{n}, \quad n = 0, 1, 2, ... \tag{1}
\]

there are two relevant transport processes for left injected quasiparticles.

- quasiparticles injected at energies

\[-\Delta - eV < E < \Delta - (n+1)eV \tag{2}\]

traverse the junction \(n+2\) times, transferring a charge \((n+2)e\) from left to right (gaining an energy \((n+2)eV\)).

- quasiparticles injected at energies

\[\Delta - (n+1)eV < E < -\Delta \tag{3}\]

traverse the junction \(n+1\) times, transferring a charge \((n+1)e\) from left to right (gaining an energy \((n+1)eV\)). As emphasized above, no other processes transferring charge between occupied and unoccupied states are possible. Consequently, only these two processes \((2)\) and \((3)\) contribute to the net charge transport.

2.2 Effective diffusive transport

The MAR-transport in real space is diffusive. Since the transport is incoherent, neither the energy of the quasiparticle inside the normal conductor nor the electron or hole character of the quasiparticle is of importance for the
transport properties. As a consequence, the process where a quasiparticle traverses the junction a net $n$ number of times, transferring a net charge $ne$, gives the same (zero frequency) transport statistics as a process where a quasiparticle with an effective charge $q_n = ne$ is transported through $n$ normal diffusive conductors in series. The $n$ conductors in series have a conductance $G_n = G/n$, with $G$ the conductance of the individual conductors [40]. We point out that this effective model, illustrated in fig. 2 can be seen as a straightforward extension and combination of the noise models of refs. [24][25].

\[ 2\Delta - eV \]

\[ 3e \]

**Fig. 2** Left: MAR-process transporting a charge $3e$ from left to right. Right: Equivalent normal circuit, conductance $G_3 = G/3$, with quasiparticles with effective charge $3e$.

The “effective voltage” across the equivalent normal conductor is just the energy window available for the process (divided by $e$). Moreover, each of the two transport processes possible for a given voltage are independent. Consequently, the the total transport statistics is obtained by just summing
up the contributions from the two processes. Importantly, since the FCS, or the transport statistics of diffusive conductors is known \[42\], the above observation allows us to directly derive the FCS of the diffusive superconducting junction.

3 Cumulants of the current

Considering only positive voltages, we start with the average current. The current is the sum of the currents of two transport processes. For a voltage \(2\Delta/(n+1) < eV < 2\Delta/n\) we have

\[
I(V) = V_{n+1}^{(1)} G_{n+1} \frac{q_{n+1}}{e} + V_{n+2}^{(2)} G_{n+2} \frac{q_{n+2}}{e} \\
= \left[ (n+1)V - \frac{2\Delta}{e} \right] \frac{G}{n+1} (n+1) + \left[ \frac{2\Delta}{e} - nV \right] \frac{G}{n+2} (n+2) \\
= GV
\]

which is just Ohms law. Here, for clarity, we introduced the effective voltages for the two processes \(V_{n+1}^{(1)} = V(n+1) - 2\Delta/e\) and \(V_{n+2}^{(2)} = 2\Delta/e - nV\). As is clear from Eq. (4), although the individual processes depend in a nontrivial way on the applied voltage via \(n\), the total current is just linear in voltage.

For the noise, the transport processes are associated with a certain noise power \(P\). For a diffusive conductor the noise power for \(n\) conductors in series is \(P_n = 2eG_n/3\), where the \(1/3\) suppression factor comes from the diffusive nature of the transport \[43,44\]. In addition, the noise is proportional to the effective charge squared. We can thus write the total noise

\[
S(V) = V_{n+1}^{(1)} P_{n+1} \left( \frac{q_{n+1}}{e} \right)^2 + V_{n+2}^{(2)} P_{n+2} \left( \frac{q_{n+2}}{e} \right)^2
\]
\[
\begin{align*}
&= \frac{2e}{3} \left[ (n+1)V - \frac{2\Delta}{e} \right] G_{n+1}(n+1)^2 + \left[ \frac{2\Delta}{e} - nV \right] G_{n+2}(n+2)^2 \\
&= \frac{2e}{3} G \left[ V([n+1]^2 - n[n+2]) + \frac{2\Delta}{e} ([n+2] - [n+1]) \right] \\
&= \frac{4\Delta}{3} G \left( 1 + \frac{eV}{2\Delta} \right) 
\end{align*}
\]

which is the known result of Refs. \[24, 25\]. It is interesting to note that again the total noise shows no \(n\)-dependence, it is a linear function of voltage, with a zero voltage offset. However, going to the third cumulant \(C_3\), the result does depend on \(n\), i.e. it displays a subharmonic gap structure in the form of kinks at voltages \(2\Delta/em\) with \(m = 1, 2, ...\). The expression for the third cumulant is given, along the same lines as for current and noise, by

\[
C_3(V) = \frac{e^2}{15} G \left[ eV(1 - n - n^2) + \Delta(6 + 4n) \right] 
\]

In the same way we also get the fourth cumulant

\[
C_4(V) = \frac{e^3}{105} G \left[ eV(-1 + 4n + 6n^2 + 2n^3) - 2\Delta(7 + 9n + 3n^2) \right] 
\]

The first four cumulants are plotted in fig. 3. We see that, just as for a normal diffusive conductor, the first three cumulants are manifestly positive but the fourth cumulant is negative. In addition, while the first two cumulants are finite at low voltages, the third and fourth cumulants diverge for \(V \to 0\).

Interestingly, in the experimental data \[30\] the third cumulant shows a clear increase with decreasing voltage, down to a cut-off voltage. In this context we note that e.g. inelastic scattering, not accounted for in our model, will remove the low voltage divergences, similar to the situation for the noise \[12, 24, 25\].
The subharmonic gap structure in the voltage dependence of the third and fourth cumulant is hardly visible in fig. 3. However, turning to the differential cumulants, the derivatives of the cumulants with respect to the voltage, the subharmonic gap structure becomes apparent. The differential cumulants are shown in fig. 4. We note that the third and the fourth cumulants display a clear step structure. From Eqs. (6) and (7) we have

\[
\frac{dC_3(V)}{dV} = \frac{e^3}{15} G(1 - n - n^2)
\]  

(8)

and

\[
\frac{dC_4(V)}{dV} = \frac{e^4}{105} G(-1 + 4n + 6n^2 + 2n^3)
\]  

(9)

We point out that in contrast to the normalized cumulants in coherent single-mode junctions in the tunneling regime [13,14,19,20,21], the height of the
**Fig. 4** The first four differential cumulants $dC_p(V)/dV$ as a function of voltage.

The third and fourth cumulants show a clear step structure.

steps of the differential cumulants can not directly be related to an effective charge.

### 4 Full counting statistics

With the knowledge of the FCS of a normal diffusive conductor, the scheme above can readily be extended to all higher cumulants. The cumulant generating function, from which the cumulants follow by successive derivatives with respect to the counting field $\chi$, is then given by the sum of the generating functions for the two MAR-processes

$$F(\chi) = \frac{e}{e^2} G \left( \frac{(n + 1)eV - 2\Delta}{n + 1} \text{asinh}^2 \sqrt{e^{\chi(n+1)} - 1} \right) + \frac{2\Delta - neV}{n + 2} \text{asinh}^2 \sqrt{e^{\chi(n+2)} - 1} \right). \quad (10)$$
where $\tau$ is the measurement time. The current is $I = (e/\tau) dF/d\chi|_{\chi=0}$, the noise $S = 2(e^2/\tau) d^2 F/d\chi^2|_{\chi=0}$, $C_3 = (e^3/\tau) d^3 F/d\chi^3|_{\chi=0}$ etc. We note that the distribution of transferred charge $Q$ is given by a Fourier transform

$$P(Q) = \int d\chi e^{F(i\chi)-iQ\chi}$$

(11)

Here we however do not further investigate the probability distribution, we instead turn to the low and high voltage limits of the cumulant generating function.

5 Voltage limits

5.1 Low voltage, $eV \ll \Delta$

In the limit of low voltage, $eV \ll \Delta$, we have $n \approx 2\Delta/eV$ and the cumulant generating function in Eq. (10) is given by

$$F(\chi) = \tau \frac{V^2 G}{2\Delta} \text{asinh}^2 \sqrt{e\chi^2/eV} - 1$$

(12)

This is just the FCS for a normal diffusive conductor with an effective charge $e(2\Delta/eV)$ transferred. Importantly, as was found in Ref. [31], this result (with different $G$) holds for an arbitrary incoherent superconducting junction. The origin of this universal behavior is that charge transport in incoherent superconducting junctions at low voltage bias can be described as diffusion in energy space. This energy diffusion was already noted for diffusive junctions, in the context of noise, in Refs. [24][25].

We note that the different cumulants $C_p$ behave as

$$C_p \propto V^{2-p}$$

(13)
This is different from the coherent (short) diffusive SNS-junctions investigated in Refs. [45, 19], where \( C_p \propto V^{3/2-p} \).

### 5.2 High voltage, \( eV > 2\Delta \)

In the limit of high voltage, \( eV > 2\Delta \), giving \( n = 0 \) in Eq. (11), the cumulant generating function in Eq. (10) is given by

\[
F(\chi) = G \left( [eV - 2\Delta] \sinh^2 \sqrt{e\chi - 1} + \Delta \sinh^2 \sqrt{e^2\chi - 1} \right) .
\]

The physical interpretation of this result follows directly from the model: the first term is due to single particle transport (\( \chi \) in the exponent) with an onset at \( eV = 2\Delta \) while the second term is two-particle transport (\( 2\chi \) in the exponent). The “excess” fluctuations, the difference between the cumulant generating functions for the diffusive junction with the terminals in the superconducting and normal state respectively, is thus given by

\[
F_{\text{exc}}(\chi) = F(\chi) - F_N(\chi) = G\Delta \left( \sinh^2 \sqrt{e^2\chi - 1} - 2\sinh^2 \sqrt{e\chi - 1} \right) .
\]

We note that this result is independent on voltage. It gives the excess cumulants \( I_{\text{exc}} = 0 \), \( S_{\text{exc}} = 4\Delta G/3 \), \( C_{3\text{exc}} = 2e\Delta G/5 \), etc.

### 6 Conclusions

In conclusion, we have investigated the full counting statistics of incoherent multiple Andreev reflections in diffusive SNS-junctions. An expression for the cumulant generating function has been derived. A careful analysis of the
first four cumulants and the low and high voltage limits of the cumulant generating function has been presented. The results are valid under the assumptions of perfect voltage bias and the absence of inelastic scattering. To allow for a quantitative comparison with experiments for the full range of applied voltages, a theoretical model where these assumptions can be relaxed would be of great interest.

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8 Appendix

The cumulant generating function in Eq. (10) can be conveniently derived in a more formal way using the stochastic path integral approach [32,33] applied to incoherent MAR [31]. We consider for simplicity a wire geometry, length $L$, of the diffusive conductor (for a more general diffusive geometry, see Ref. [33]). The starting point is to discretize the diffusive wire into $K \gg 1$ nodes, contacted via quantum point contacts supporting $M \gg 1$ transport modes with transparency $T$. This discretization is shown in Fig. 5. The transparency $T$ and the number of transport modes $M$ are chosen to give a conductance of the $K$ nodes in series $[MT/K]2e^2/h$ equal to the conductance $G$ of the wire.
Fig. 5  Left: discretization of the diffusive wire into $K$ nodes. The contact between the nodes has transparency $T$ and the contact between the nodes and the superconducting electrodes has transparency $\Gamma$. Right: the node structure along a part of the MAR-ladder. The distribution functions $f_{j,k}$ and counting fields $\lambda_{j,k}$ of the nodes labeled with $j,k$ are shown.

The resulting node structure in energy space, along the MAR-ladder, is also shown in Fig. 5. Each node is characterized by a quasiparticle distribution function $f_{j,k}$ and a quasiparticle counting field $\lambda_{j,k}$, where $1 \leq k \leq K$ is the number of the node and $j$ denotes the rung of the MAR-ladder. The $j$:th rung goes between NS-interfaces at energies $E_{j-1}$ and $E_j$ where $E_j = E + jeV$. We take $j = 0$ for energies just below the gap $-\Delta < E < -\Delta - 2eV$, on the left side. The quasiparticles are thus electrons for rungs with $j$ odd and holes for $j$ even. Note also that $k$ is counted along the ladder, upwards in energy.

Following the scheme for FCS of incoherent MAR in Ref. [31], we write the total generating function as a sum (energy integral) over the generating functions for different MAR-ladders, which contribute independently to the
total generating function. This gives

\[ S[\chi, \{f\}, \{\lambda\}] = \int_{-\Delta - 2eV}^{-\Delta} dE \sum_{j=-\infty}^{\infty} S_j^{NS} + \sum_{k=1}^{K-1} S_j^{N S} \] (16)

where the generating function for a given MAR-ladder is a sum of the generating functions \( S_j^{N S} \) and \( S_j^{NS} \) along the ladder. Here

\[ S_j^{N S} = \frac{M \tau}{\hbar} \ln \left[ f_{j,k} f_{j,k+1} + f_{j,k} (1 - f_{j,k+1}) \right] \left[ Te^{\lambda_{j,k+1} - \lambda_{j,k}} + R \right] \\
+ f_{j,k+1} (1 - f_{j,k}) \left[ Te^{\lambda_{j,k} - \lambda_{j,k+1}} + R \right] \\
+ (1 - f_{j,k}) (1 - f_{j,k+1}) \] (17)

is the generating function \([46]\) for the quantum point contact between nodes in the wire, with \( R = 1 - T \). The generating function \( S_j^{NS} \) describes the connection between nodes at the NS-interfaces \([47, 48]\). At energies inside the gap, \(|E_j| < \Delta\), we have

\[ S_j^{NS} = \frac{M \tau}{\hbar} \ln \left[ f_{j,K} f_{j+1,1} + f_{j,K} (1 - f_{j+1,1}) \right] \left[ R_A e^{2\chi_j + \lambda_{j+1,1} - \lambda_{j,K}} + R_N \right] \\
+ f_{j+1,1} (1 - f_{j,K}) \left[ R_A e^{-2\chi_j + \lambda_{j,K} - \lambda_{j+1,1}} + R_N \right] \\
+ (1 - f_{j,K}) (1 - f_{j+1,1}) \] (18)

with \( R_A = R_A(E, \Gamma) \) the Andreev reflection probability and \( R_N = R_N(E, \Gamma) = 1 - R_A \) the normal reflection probability, where \( \Gamma \) is the (normal) transparency of NS-interface. The counting field \( \chi_j \) is taken equal to \( \chi \) for \( j \) odd and zero for \( j \) even, i.e. the transferred charge is counted on the right side of the junction. It is important to note that \( \chi \) counts electrical charge while \( \lambda_{j,k} \) counts quasiparticles.
Outside the gap, at $|E_j| > \Delta$, the Andreev reflection can effectively be neglected (as discussed above) and we can for simplicity consider a generating function for a perfect normal interface, i.e. a NS-interface with the superconductor taken in the normal state. This gives

$$S^{NS}_j = \frac{M\tau}{h}\ln\left[f_{j,K}f_j + f_{j,K}(1-f_j)\left(e^{\chi_j-\lambda_j,K} + 1\right)\right. + f_j(1-f_{j,K})\left[e^{-\chi_j+\lambda_j,K} + 1\right] + (1-f_{j,K})(1-f_j)]$$

(19)

above the gap, $E_j > \Delta$ and similarity below the gap. Here $f_j = f(E_j)$, with $f(E)$ the quasiparticle distribution function of the superconducting electrodes. For $kT \ll \Delta$ considered here, we have $f(E) = 1$ below the gap and $f(E) = 0$ above the gap.

To evaluate the generating functions for the ladders we first perform a transformation

$$\lambda_{j,k} \rightarrow \lambda_{j,k} - j\chi, \quad j = 2, 4, 6...$$
$$\lambda_{j,k} \rightarrow \lambda_{j,k} - (j-1)\chi, \quad j = 1, 3, 5....$$

(20)

[Under the stated conditions, only $\lambda_{j,k}$ with $j \geq 1$ are of interest]. This transformation removes the counting field $\chi$ from the generating functions of the NS-interfaces at energies inside the gap, in Eq. (15). In addition, the generating function for the NS-interface above the gap, Eq. (19) is modified as $\chi_j \rightarrow j\chi$. This means that the counting field $\chi$ is transferred away from the interior of the ladder to the upper boundary.
Second, we note that the structure of the generating functions for the quantum point contact between the nodes in the wire, Eq. (17), and for the NS-interface at energies inside the gap, Eq. (18), are formally identical when $\chi$ is transferred away (however with $R_A$ instead of $T$). The NS-interface thus effectively connects the end-nodes $j, K$ and $j + 1, 1$ of the two rungs $j$ and $j + 1$.

In addition, we emphasize that the boundary conditions (in the superconductors) for the distribution functions as well as the counting fields [after the transformation in Eq. (20)] are independent on the electron or hole character of the quasiparticles, they only depend on the energy $E_j$. Consequently, the fact that distribution functions and counting fields describe electrons for $j$ odd and holes for $j$ even becomes irrelevant for the charge transfer and one thus only need to consider the position $j, k$ of the node in the MAR-ladder.

Taken together, the MAR ladder can thus be represented as a series of $nK$ nodes, with $n$ equal to the number of rungs on the ladder, or equivalently the net number of absorbed energy quanta $eV$ when traversing the junction. Note that since the resistance of the NS-interface is much smaller than the resistance of the diffusive wire, the details of the connection at the NS-interface ($R_A$ instead of $T$) can be neglected. The only difference from a discretized normal diffusive wire of length $nL$ is then the boundary condition for the counting field at the end of the wire, being $n\chi$ instead of just $\chi$ for a normal wire.
Having performed this mapping to a discretized normal wire with renormalized boundary conditions we can continue with the standard procedure \cite{33} to derive the total generating function in Eq. (16). We first denote the nodes with a single index $k$ running from 1 to $nK$ (dropping the rung index). In the limit $K \gg 1$ the difference between distribution functions as well as counting fields of two neighboring nodes $k$ and $k+1$ is of the order $1/K$ and we can expand $f_{k+1} = f_k + \Delta f_k$ and $\lambda_{k+1} = \lambda_k + \Delta \lambda_k$. Inserting this expansion into the generating function in Eq. (17) we have to leading order in $\Delta f_k$ and $\Delta \lambda_k$

$$S_N^k = \frac{M \tau}{\hbar} T [\Delta \lambda_k \Delta f_k - f_k (1 - f_k) \Delta \lambda_k^2]$$  \hspace{1cm} (21)

Taking the continuum limit and summing over all generating functions for different $k$ we get the generating function for a given energy $E$ as \cite{33}

$$S[\chi, f, \lambda] = \frac{M \tau}{\hbar} T \int_0^1 dx \left[ \frac{d\lambda}{dx} \frac{df}{dx} - f (1 - f) \left( \frac{d\lambda}{dx} \right)^2 \right]$$  \hspace{1cm} (22)

where we have changed from summation over nodes to an integral over the dimensionless position $0 < x < 1$ along the wire, i.e $f = f(x)$ and $\lambda = \lambda(x)$.

Second, this generating function is varied over all possible configurations of $f$ and $\lambda$. This results in a stochastic path integral formulation for the averaged generating function $F(\chi)$

$$\exp(F[i\chi]) = \int \mathcal{D}f \mathcal{D}\lambda \exp(S[i\chi, f, i\lambda])$$  \hspace{1cm} (23)

Under the semiclassical conditions considered here, this path integral can be solved in the saddle point approximation. The saddle point equations
are given by the functional derivatives
\[
\frac{\delta S[\chi, f, \lambda]}{\delta \lambda} = 2 \frac{d}{dx} \left[ \frac{d\lambda}{dx} f (1 - f) \right] - \frac{d^2 f}{dx^2} = 0
\]
\[
\frac{\delta S[\chi, f, \lambda]}{\delta f} = (1 - 2f) \left( \frac{d\lambda}{dx} \right)^2 + \frac{d^2 \lambda}{dx^2} = 0
\] (24)

Solving these equations for \(\lambda\) and \(f\) in terms of \(\chi\), inserting the solutions back into the generating function and performing the integration over \(x\) then yields (see Ref. [33] for details)
\[
F(\chi) = \frac{\tau G}{e^2 n} \text{asinh}^2 \sqrt{e^{\chi n} - 1}
\] (25)

describing the FCS of a diffusive wire with conductance \(G_n = G/n\) and a renormalized charge \(e \rightarrow ne\). Finally, performing the integral over energy in Eq. (16), taking into account energy window for the two MAR-ladders, gives the generating function in Eq. (10). This concludes the formal derivation.

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