Conserved Currents, Consistency Relations and Operator Product Expansions in the Conformally Invariant $O(N)$ Vector Model

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Abstract

We discuss conserved currents and operator product expansions (OPE’s) in the context of a $O(N)$ invariant conformal field theory. Using OPE’s we find explicit expressions for the first few terms in suitable short-distance limits for various four-point functions involving the fundamental $N$-component scalar field $\phi^\alpha(x), \alpha = 1, 2, .., N$. We propose an alternative evaluation of these four-point functions based on graphical expansions. Requiring consistency of the algebraic and graphical treatments of the four-point functions we obtain the values of the dynamical parameters in either a free theory of $N$ massless fields or a non-trivial conformally invariant $O(N)$ vector model in $2 < d < 4$, up to next-to-leading order in a $1/N$ expansion. Our approach suggests an interesting duality property of the critical $O(N)$ invariant theory. Also, solving our consistency relations we obtain the next-to-leading order in $1/N$ correction for $C_T$ which corresponds to the normalisation of the energy momentum tensor two-point function.

\[1\]Partially supported by PPARC.
Introduction

There exists an enormous literature (see [1, 2, 3] and references therein) devoted to the study of conformal field theories (CFT’s). Such discussions have been primarily motivated by the observation that scale invariance at the fixed points of the renormalisation group also implies conformal invariance for most quantum field theories [4]. An interesting possibility is that CFT’s may be formulated in terms of algebras of local fields closed under operator product expansions which, if the OPE coefficients are known, enable the evaluation of all correlation functions in the theory without using a Lagrangian [5]. Such a formulation has been successfully developed for a large class of interesting CFT’s in two dimensions [6, 2, 3] giving, among other things, important information concerning the properties of two-dimensional statistical mechanical systems. Although not many explicitly constructed non-trivial CFT’s are known in \( d > 2 \), statistical mechanical systems at criticality enjoying at least scale and in many cases conformal invariance exist in the real world in more than two dimensions. Non-perturbative field theoretic methods for studying these systems are still rudimentary, however the investigation of CFT’s provides a viable framework for understanding their properties in \( d > 2 \).

In [7] the subject of CFT’s in \( d > 2 \) was reconsidered but only free field theories were discussed as explicit examples in the context of that work. In the present work we explore a more complicated CFT in \( d > 2 \), the \( O(N) \) invariant model of a \( N \)-component scalar field \( \phi^\alpha(x), \alpha = 1, 2, .., N \). In this model the most singular contributions in the OPE of \( \phi^\alpha(x) \) with itself are given by a conserved vector current, the energy momentum tensor and a scalar field \( O(x) \) of low dimension \( \eta_0 < d \). As we also will see, this model naturally suggests an approximation scheme based on a \( 1/N \) expansion. Our main purpose in the context of the present work is to provide evidence that various four-point functions involving the fundamental field \( \phi^\alpha(x) \) can be consistently treated, at least in certain short-distance limits, on the basis of conformally invariant OPE’s and also with the aid of graphical expansions built using internal lines corresponding to the two-point functions of the field \( \phi^\alpha(x) \) and of another scalar field \( \tilde{O}(x) \) having dimension \( \tilde{\eta}_0 \) with \( 0 < \tilde{\eta}_0 < d \). The latter field can be identified either with the scalar field \( O(x) \) appearing in the OPE of \( \phi^\alpha(x) \) with itself or with its shadow field \( O_s(x) \) of dimension \( d - \eta_0 \).

The paper is organised as follows. In section [1] we derive Ward identities associated with a conserved vector current and the energy momentum tensor for Euclidean CFT’s in \( d > 2 \). In section [2] we focus on the discussion of the \( O(N) \) invariant field theory at its critical point assuming conformal invariance. We show that Ward identities relate the
couplings in the three-point functions of $\phi^\alpha(x)$ with a conserved vector current or the energy momentum tensor to the dimension $\eta$ and the normalisation constant $C_\phi$ of the two-point function of $\phi^\alpha(x)$. Next, making a simple ansatz for the OPE of $\phi^\alpha(x)$ with itself we obtain explicit expressions for the first few terms in a suitable short-distance limit of the four-point function $\langle \phi \phi \phi \phi \rangle$. In section 3 we show consistency of our results for the four-point function with explicit calculations in the context of the trivial CFT of $N$ massless free fields in any dimension $d$. Motivated by the form of the OPE ansatz we assume in section 4 that the four-point function $\langle \phi \phi \phi \phi \rangle$ in a non-trivial $O(N)$ invariant CFT in $2 < d < 4$ can alternatively be evaluated using a skeleton graph expansion with internal lines corresponding to the two-point functions of $\phi^\alpha(x)$ and of a scalar field $\tilde{O}(x)$ having dimension $\tilde{\eta}_o$ with $0 < \tilde{\eta}_o < d$. The first few graphs in such an expansion represent the interaction of $\phi^\alpha(x)$ with $\tilde{O}(x)$ via a unique vertex having coupling constant $g_*$. Requiring agreement of the algebraic and graphical treatments of the four-point function we obtain consistency relations for the dynamical parameters of the theory which we solve using a $1/N$ expansion. Our next-to-leading order in $1/N$ results agree with previous calculations \cite{10,19} in the context of the $O(N)$ sigma model in $2 < d < 4$. When $d \to 4$, our result for the next-to-leading order in $1/N$ correction for the normalisation of the energy momentum tensor two-point function $C_T$, agrees with the $\epsilon$-expansion results \cite{8,9}. Furthermore, the graphical expansion is shown to possess a shadow symmetry which, assuming that it holds to all orders, suggests an interesting duality property of the critical $O(N)$ vector model. In section 5 we present the graphical and algebraic treatments of the four-point function $\langle \phi \phi O O \rangle$ involving the dimension $\eta_o < d$ field $O(x)$ which appears in the OPE of $\phi^\alpha(x)$ with itself. The consistency relations in this case provide a non-trivial check for the associativity of the OPE in $d > 2$. Our detailed calculations are in many respects similar to those of Lang and Rühl \cite{10,11} and in large part we agree on their results, however our perspective is rather different. Discussions concerning possible implications of our results and concluding remarks are presented in section 6. Four Appendices contain technical details.

1 Conserved Currents and Ward Identities in CFT

We consider Euclidean CFT in dimensions $d > 2$ which is invariant under the global action of a compact semisimple finite-dimensional Lie group $G$. The field algebra of such a theory consists of quasiprimary fields $O^\alpha_i(x)$, with spacetime index $i$ and internal index $\alpha$, together with their derivatives \cite{5}. Quasiprimary fields of dimension $\eta$ transform under a finite-dimensional representation of the Euclidean conformal group, (which is isomorphic to the
pseudo-orthogonal group $O(d + 1, 1)$, as

$$O^\alpha_i(x) \rightarrow O'^\alpha_i(x') = [\Omega(x)]^\alpha_j D^j_i(R(x)) O^\alpha_j(x), \tag{1.1}$$

where $\Omega(x)$ is the Jacobian of the conformal transformation $x_\mu \rightarrow x'_\mu(x)$ and $D^j_i(R)$ with $R_{\mu\nu}(x) = \Omega(x) \partial x'_\mu/\partial x_\nu$, $R_{\mu\nu} \in O(d)$, belongs to a representation of the rotation group associated with that transformation. We only consider fields transforming under irreducible representations of $O(d)$ here. Correlation functions of the fields $O^\alpha_i(x)$ are defined to be statistical averages with Boltzmann weight $e^{-S[\phi]}$, formally

$$\langle O^\alpha_i(x) \cdots \rangle_S = \int (D\phi) \, O^\alpha_i(x) \cdots e^{-S[\phi]} . \tag{1.2}$$

The action (Hamiltonian) $S[\phi]$ is a functional of some fundamental fields $\phi^\alpha_i(x)$ and their derivatives.

Conserved currents play an important role in CFT [12] and in the present work we focus on the discussion of the energy momentum tensor $T_{\mu\nu}(x)$ and the vector current $J_\kappa^{\mu}(x)$, where $\kappa$ denotes indices in the adjoint representation of $G - T_{\mu\nu}(x)$ is a singlet under the action of $G$. The conservation equations read $\partial_\mu T_{\mu\nu}(x) = \partial_\mu J_\kappa^{\mu}(x) = 0$. The energy momentum tensor is here defined as the response of the action $S[\phi]$ to arbitrary infinitesimal spacetime transformations $x_\mu \rightarrow x_\mu + a_\mu(x)$ inducing suitable infinitesimal deformations $\delta_a \phi(x)$ of the fields which reduce to the form (1.1) for $a_\mu(x)$ conformal transformations, namely

$$S \rightarrow S + \delta_a S , \quad \delta_a S = \int d^d x (\partial_\mu a_\nu(x)) T_{\mu\nu}(x). \tag{1.3}$$

This definition of the energy momentum tensor is arbitrary up to $T_{\mu\nu}(x) \rightarrow T_{\mu\nu}(x) + \partial_\lambda X_{\lambda\mu\nu}(x)$ with $X_{\lambda\mu\nu}(x) = -X_{\mu\lambda\nu}(x)$. We assume that $T_{\mu\nu}(x)$ can be made symmetric and traceless for a suitable choice of $X_{\lambda\mu\nu}(x)$.

The conserved vector current $J_\kappa^{\mu}(x)$ is defined as the response of the action $S[\phi]$ to arbitrary infinitesimal group transformations parametrised by a local field $\epsilon^\kappa(x)$, namely

$$S \rightarrow S + \delta_\epsilon S , \quad \delta_\epsilon S = \int d^d x (\partial_\mu \epsilon^\kappa(x)) J_\kappa^{\mu}(x). \tag{1.4}$$

Then, the Ward identities follow from the requirement that correlation functions in the theory should satisfy

$$\langle O^\alpha_i(x) \cdots \rangle_S = \langle O^\alpha_i(x) + \delta_a O^\alpha_i(x) \cdots \rangle_{S + \delta_a S}, \tag{1.5}$$
and similarly for $\delta_x O^a_i(x), \delta_x S$. By virtue of (1.2) and (1.3), (1.4) we obtain from (1.5) for $n$-point functions

$$\sum_{k=1}^{n} \langle O^a_{i_1}(x_1) \cdots \delta_x O^a_{i_k}(x_k) \cdots \rangle_S = \int d^d x \langle \partial_x a_{\nu}(x) \rangle \langle T_{\mu\nu}(x) O^a_{i_1}(x_1) \cdots \rangle_S, \quad (1.6)$$

$$\sum_{k=1}^{n} \langle O^a_{i_1}(x_1) \cdots \delta_x O^a_{i_k}(x_k) \cdots \rangle_S = \int d^d x \langle \partial_x \epsilon^x(x) \rangle \langle J^a_{\mu}(x) O^a_{i_1}(x_1) \cdots \rangle_S. \quad (1.7)$$

Given the particular representation of $G$ under which the fields in the field algebra transform, the deformations $\delta_x O^a_i(x)$ are easily obtained. However, when $a_\mu(x)$ corresponds to a conformal transformation e.g. $\partial_\mu a_\nu(x) + \partial_\nu a_\mu(x) \propto \delta_{\mu\nu}$, the r.h.s. of (1.6) vanishes due to the tracelessness of $T_{\mu\nu}(x)$. For example, in the case of spacetime dilatations with $a_\mu(x) \propto x_\mu$, then $\delta_x$ becomes the configuration space analog of the Callan-Symanzik operator with vanishing beta functions acting on the fields in a massless theory at a fixed point of the renormalisation group and we obtain

$$\sum_{k=1}^{n} \left( x_\mu^k \frac{\partial}{\partial x_\mu^k} + \eta_k \right) \langle O^a_{i_1}(x_1) \cdots O^a_{i_k}(x_k) \cdots \rangle_S \equiv 0. \quad (1.8)$$

More generally (1.6) and (1.7) determine Ward identities for $\partial_\mu \langle T_{\mu\nu}(x) \cdots \rangle$ and $\partial_\mu \langle J^a_{\mu}(x) \cdots \rangle$ or equivalently control the leading terms in the OPE’s of $T_{\mu\nu}(x)$ and $J^a_{\mu}(x)$ with the quasiprimary fields. Note that for $x \neq x_i, i = 1, \ldots, n$, (1.3) requires $\partial_\mu \langle T_{\mu\nu}(x) \cdots \rangle = 0$. If the transformations $a_\mu(x)$ ($\epsilon^x(x)$) satisfy the conformal invariance property $\partial_\mu a_\nu(x) + \partial_\nu a_\mu(x) \propto \delta_{\mu\nu}$ (are $x$-independent group transformations) in the region $|x - x_i| \leq r$ for $|x_i - x_j| > r, i \neq j,$ $r > 0$, then we obtain

$$\langle \delta_x O^a_{i_1}(x_1) \cdots \rangle_S = - \int dS_\mu a_\nu(x) \langle T_{\mu\nu}(x) O^a_{i_1}(x_1) \cdots \rangle_S, \quad (1.9)$$

$$\langle \delta_x O^a_{i_1}(x_1) \cdots \rangle_S = - \epsilon^x \int dS_\mu \langle J^a_{\mu}(x) O^a_{i_1}(x_1) \cdots \rangle_S, \quad (1.10)$$

where $dS_\mu = d\Omega r_\mu r^{d-2}$ denotes integration on the surface of the ball $|x - x_1| = r$ and the angular integration is normalised as $\int d\Omega = S_d$ with $S_d = 2\pi^{d/2}/\Gamma(d/2)$ the surface of the unit $d$-dimensional sphere. We find equations (1.9) and (1.10) to be a useful form of the Ward identities expressing conformal and group symmetry of $n$-point correlation functions and we will use them for explicit calculations in what follows. Note that the r.h.s. of (1.9) and (1.10) does not depend on $r$ as $r \to 0$ because the difference of two surface integrals
on the balls with radii \( r \) and \( r' \), when \( r > r' \), is a vanishing volume integral. We may also remark that with our presentation of the Ward identities

\[ \langle T_{\mu\nu}(x) \cdots \rangle = 0 \text{ for all } x. \]

2 Conserved Currents and Operator Product Expansions in the Conformally Invariant \( O(N) \) Vector Model

2.1 General Remarks

We consider the \( O(N) \) invariant CFT having as fundamental field the \( N \)-component vector, \( O(d) \) scalar field \( \phi^\alpha(x) \), \( \alpha = 1, 2, \ldots, N \). In CFT the two- and three-point correlation functions are fixed up to some arbitrary constants \([7]\). The two-point function of \( \phi^\alpha(x) \) in this model is diagonal in the \( O(N) \) indices and can be written as

\[
\langle \phi^\alpha(x_1)\phi^\beta(x_2) \rangle = \Phi^{\alpha\beta}(x_{12}) = C_{\phi} \frac{1}{(x_{12})^2} \delta^{\alpha\beta}, \quad x_{12} = x_1 - x_2, \tag{2.1}
\]

with \( C_{\phi} \) a normalisation constant and \( \eta \) the dimension of \( \phi^\alpha(x) \). The three-point function of the fundamental field in the conformally invariant \( O(N) \) vector model is zero. Conformal invariance alone does not fix the form of \( n \)-point functions with \( n \geq 4 \). One can show that conformal \( n \)-point functions will in general depend on \( n(n - 3)/2 \) variables \([13, 3]\). The four-point function of \( \phi^\alpha(x) \) in the \( O(N) \) vector model can in general be written as

\[
\langle \phi^\alpha(x_1)\phi^\beta(x_2)\phi^\gamma(x_3)\phi^\delta(x_4) \rangle \equiv \Phi^{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) = \Phi(x_1, x_2, x_3, x_4) \delta^{\alpha\beta} \delta^{\gamma\delta} \]

\[
+ \Phi(x_1, x_3, x_2, x_4) \delta^{\alpha\gamma} \delta^{\beta\delta} + \Phi(x_1, x_4, x_3, x_2) \delta^{\alpha\delta} \delta^{\beta\gamma}, \tag{2.2}
\]

where \( \Phi(x_1, x_2, x_3, x_4) \) has the following crossing symmetry properties

\[
\Phi(x_1, x_2, x_3, x_4) = \Phi(x_2, x_1, x_3, x_4) = \Phi(x_3, x_4, x_1, x_2). \tag{2.3}
\]

By virtue of conformal invariance and (2.3) we may then write

\[
\Phi(x_1, x_2, x_3, x_4) = H(\eta, x) F(u, v) \tag{2.4}
\]

\footnote{In \([\text{III}]\), alternative Ward identities were derived leading to contributions to \( \langle T_{\mu\nu}(x)O(x_1)\cdots \rangle \) proportional to \( \delta^d(x - x_1)(O(x_1)\cdots) \) for suitable fields \( O(x) \). Our approach is equivalent to the one of \([\text{III}]\) up to a redefinition of \( \langle T_{\mu\nu}(x)O(x_1)\cdots \rangle \) using sums of terms proportional to \( \delta^d(x - x_1)(O(x_1)\cdots) \).}
with
\[ H(\eta, x) = \frac{1}{(x_{12}^2 x_{34}^2 x_{13}^2 x_{24}^2 x_{14}^2 x_{23}^2)^{1/\eta}}. \] (2.5)

where \( F(u, v) = F(v, u) \) is an arbitrary function of the two independent invariant ratios
\[ u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad \text{and} \quad v = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}. \] (2.6)

We may also cast (2.2) into a form convenient for what follows
\[ \Phi^{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) = H(\eta, x)F_S(u, v)\delta^{\alpha\beta}\delta^{\gamma\delta} \\
+ H(\eta, x)F_V(u, v)\frac{1}{2}(\delta^{\alpha\gamma}\delta^{\beta\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma}) \\
+ H(\eta, x)F_T(u, v)\frac{1}{2}(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma} - \frac{2}{N}\delta^{\alpha\beta}\delta^{\gamma\delta}), \] (2.7)

where
\[ F_S(u, v) = F(u, v) + \frac{1}{N}\left(F\left(\frac{1}{u}, \frac{v}{u}\right) + F\left(\frac{u}{v}, \frac{1}{v}\right)\right), \]
\[ F_V(u, v) = F\left(\frac{1}{u}, \frac{v}{u}\right) - F\left(\frac{u}{v}, \frac{1}{v}\right), \]
\[ F_T(u, v) = F\left(\frac{1}{u}, \frac{v}{u}\right) + F\left(\frac{u}{v}, \frac{1}{v}\right). \] (2.8)

Correlation functions involving the group symmetry conserved current \( J_\mu^\alpha(x) \) and the energy momentum tensor \( T_{\mu\nu}(x) \) are of special interest in CFT. In the conformally invariant \( O(N) \) vector model the conserved current of the \( O(N) \) symmetry has \( O(d) \) spin-1, dimension \( d - 1 \) and transforms irreducibly under the action of the adjoint representation of \( O(N) \), hence it may be written as \( J_\mu^{\alpha\beta}(x) = -J_\mu^{\beta\alpha}(x) \). The energy momentum tensor \( T_{\mu\nu}(x) \) has \( O(d) \) spin-2 and dimension \( d \). Therefore, the two-point functions of \( T_{\mu\nu}(x) \) and \( J_\mu^{\alpha\beta}(x) \) can be written as
\[ \langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2) \rangle = C_T \frac{I_{\mu\nu,\rho\sigma}(x_{12})}{x_{12}^{2d}}, \] (2.9)
\[ \langle J_\mu^{\alpha\beta}(x_1)J_\nu^{\gamma\delta}(x_2) \rangle = C_J \frac{I_{\mu\nu}(x_{12})}{x_{12}^{2(d-1)}} (\delta^{\alpha\gamma}\delta^{\beta\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma}). \] (2.10)
The functions $I_{\mu,\rho\sigma}(x)$ and $I_{\mu
u}(x)$ are determined from the conservation, symmetry and tracelessness properties of the currents and have been explicitly given in [7]

\[ I_{\mu,\rho\sigma}(x) = \frac{1}{2} \left( I_{\mu\rho}(x) I_{\nu\sigma}(x) + I_{\mu\sigma}(x) I_{\nu\rho}(x) \right) - \frac{1}{d} \delta_{\mu\nu} \delta_{\rho\sigma}, \] (2.11)

\[ I_{\mu\nu}(x) = \delta_{\mu\nu} - 2 \frac{x_{\mu} x_{\nu}}{x^2}. \] (2.12)

The quantities $C_T$ and $C_J$, which may be considered as the normalisations of the conserved currents $T_{\mu\nu}(x)$ and $J_{\mu\beta}^\alpha(x)$, are fixed by the defining relations (1.3) and (1.4) and they should be given in terms of the dynamical variables of the theory i.e. the dimensions of the fields and the couplings [1]. In the following we provide evidence in favour of this assertion.

It will also be useful for what follows to consider three-point functions of the fundamental field $\phi^\alpha(x)$ involving one $T_{\mu\nu}(x)$, one $J_{\mu\beta}^\alpha(x)$ or one scalar field $O(x)$ of dimension $\eta_0$. Conformal invariance fixes the form of these three-point functions up to a constant [7]

\[ \langle \phi^\alpha(x_1) \phi^\beta(x_2) O(x_3) \rangle = \frac{g_{\phi\phi O}}{(x_{12}^2)^{\eta - \frac{1}{2} \eta_0 (x_{13}^2 x_{23}^2)^{\frac{1}{2} \eta_0}}} \delta^{\alpha\beta}, \] (2.13)

\[ \langle \phi^\alpha(x_1) \phi^\beta(x_2) T_{\mu\nu}(x_3) \rangle = \frac{-g_{\phi\phi T}}{(x_{12}^2)^{\eta - \mu + 1} (x_{13}^2 x_{23}^2)^{\mu - 1}} \left[ (X_{12})_{\mu} (X_{12})_{\nu} - \frac{1}{d} \delta_{\mu\nu} (X_{12})^2 \right] \delta^{\alpha\beta}, \] (2.14)

\[ \langle \phi^\alpha(x_1) \phi^\beta(x_2) J_{\mu}^{\gamma\delta}(x_3) \rangle = \frac{g_{\phi\phi J}}{(x_{12}^2)^{\eta - \mu + 1} (x_{13}^2 x_{23}^2)^{\mu - 1}} (X_{12})_{\mu} \left( \delta^{\alpha\gamma} \delta^{\beta\delta} - \delta^{\alpha\delta} \delta^{\beta\gamma} \right), \] (2.15)

where

\[ (X_{12})_{\mu} = \frac{(x_{13})_{\mu}}{x_{13}^2} - \frac{(x_{23})_{\mu}}{x_{23}^2}. \] (2.16)

The couplings $g_{\phi\phi O}$, $g_{\phi\phi T}$ and $g_{\phi\phi J}$ above depend on the dynamics of the particular CFT model. However, in the case of (2.14) and (2.15) the Ward identities (1.9) and (1.10) relate $g_{\phi\phi T}$ and $g_{\phi\phi J}$ to the dimension $\eta$ and the normalisation constant $C_\phi$ of the two-point function of $\phi^\alpha(x)$. To see this we use the following infinitesimal field deformations

\[ a_\mu(x) = \lambda x_\mu \Rightarrow \delta_\alpha \phi^\alpha(x) = \lambda (\eta + x_\mu \partial_\mu) \phi^\alpha(x), \] (2.17)

\[ \delta_\epsilon \phi^\alpha(x) = \epsilon^{\alpha\beta} \phi^\beta(x), \] (2.18)

where $\lambda$ and $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ are arbitrary infinitesimal parameters. By virtue of the antisymmetry of $\epsilon^{\alpha\beta}$ we may now take in (1.10) $e^\epsilon J_{\mu}^\alpha(x) \rightarrow \frac{1}{2} \epsilon^{\alpha\beta} J_{\mu}^{\beta\alpha}(x)$ and then from (1.9) and (1.10)
we obtain
\[
\langle \delta \phi(x_1) \phi(x_2) \rangle = -\lambda \int d\Omega r^{d-2} r_\nu r_\rho \langle T_{\nu\rho}(x) \phi(x_1) \phi(x_2) \rangle,
\]  
\[
\langle \delta \phi(x_1) \phi(x_2) \rangle = -\frac{1}{2} \gamma_{\delta} \int d\Omega r^{d-2} r_\nu \langle J_{\nu}^\delta(x) \phi(x_1) \phi(x_2) \rangle,
\]  
where \( r = |x - x_1| \). Performing the integrals in (2.19) and (2.20) in the limit \( r \to 0 \) we obtain
\[
g_{\phi T} = \frac{1}{S_d} \frac{d\eta}{d-1} C_\phi,
\]
\[
g_{\phi J} = \frac{1}{S_d} C_\phi.
\]
It is clear that in the place of \( \phi(x) \) in (2.19) and (2.20) we could have had any \( O(d) \) scalar field. We conclude that the couplings (such as \( g_{\phi T} \) and \( g_{\phi J} \) above) of all \( O(d) \) scalar fields with the conserved currents \( T_{\mu\nu}(x) \) and \( J_\mu^\kappa(x) \) in any CFT can be found solely from the knowledge of the complete two-point functions of those scalar fields.

2.2 Operator Product Expansions and the Four-Point Function in the Conformally Invariant \( O(N) \) Vector Model

Within non-Lagrangian approaches to quantum field theory and in particular to CFT the operator product expansion is a basic starting point. In the context of the present work we begin our discussion with the assumption of the following OPE
\[
\phi^\alpha(x_1) \phi^\beta(x_2) = C_{\phi} \frac{1}{(x_{12})^\eta} \delta^{\alpha\beta} + \sum_{O_n} C_{\phi O_n}(x_{12}, \partial_2) O_n^\beta(x_2),
\]
where we have suppressed \( O(d) \) indices and the index \( n \) here denotes all different quasi-primary fields which may appear in the OPE above. For the purposes of the present work we follow [13] and assume that the fields \( O_n^{\alpha\beta}(x) \) are realised by symmetric \( O(d) \) tensors \( O^{\alpha\beta}_{\mu_1\mu_2...\mu_k}(x) \) of rank \( k \) and dimension \( \eta_k \). Moreover, these fields can be \( O(N) \) singlets, symmetric traceless \( O(N) \) tensors or antisymmetric \( O(N) \) tensors. Inserting (2.23) into (2.2) we obtain formally
\[
\Phi^{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) = C_{\phi}^2 \frac{1}{(x_{12} x_{34})^{\eta}} \delta^{\alpha\beta} \delta^{\gamma\delta}
\]  
\[
+ \sum_{O_n} C_{\phi O_n}(x_{12}, \partial_2) C_{\phi O_n}(x_{34}, \partial_4) \langle O_n^{\alpha\beta}(x_2) O_n^{\gamma\delta}(x_4) \rangle,
\]  
\[
\Phi^{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) = C_{\phi}^2 \frac{1}{(x_{12} x_{34})^{\eta}} \delta^{\alpha\beta} \delta^{\gamma\delta}
\]  
\[
+ \sum_{O_n} C_{\phi O_n}(x_{12}, \partial_2) C_{\phi O_n}(x_{34}, \partial_4) \langle O_n^{\alpha\beta}(x_2) O_n^{\gamma\delta}(x_4) \rangle,
\]  
\[
\Phi^{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) = C_{\phi}^2 \frac{1}{(x_{12} x_{34})^{\eta}} \delta^{\alpha\beta} \delta^{\gamma\delta}
\]  
\[
+ \sum_{O_n} C_{\phi O_n}(x_{12}, \partial_2) C_{\phi O_n}(x_{34}, \partial_4) \langle O_n^{\alpha\beta}(x_2) O_n^{\gamma\delta}(x_4) \rangle,
\]  
\[
\Phi^{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) = C_{\phi}^2 \frac{1}{(x_{12} x_{34})^{\eta}} \delta^{\alpha\beta} \delta^{\gamma\delta}
\]  
\[
+ \sum_{O_n} C_{\phi O_n}(x_{12}, \partial_2) C_{\phi O_n}(x_{34}, \partial_4) \langle O_n^{\alpha\beta}(x_2) O_n^{\gamma\delta}(x_4) \rangle,
\]  
\[
\Phi^{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) = C_{\phi}^2 \frac{1}{(x_{12} x_{34})^{\eta}} \delta^{\alpha\beta} \delta^{\gamma\delta}
\]  
\[
+ \sum_{O_n} C_{\phi O_n}(x_{12}, \partial_2) C_{\phi O_n}(x_{34}, \partial_4) \langle O_n^{\alpha\beta}(x_2) O_n^{\gamma\delta}(x_4) \rangle,
\]  
\[9\]
where we have used the selection rule for quasiprimary fields \[13, 7\]

\[
\langle O_{\mu_1\mu_2\ldots\mu_k}(x_1) O_{\nu_1\nu_2\ldots\nu_l}(x_2) \rangle \neq 0 \text{ only if } k = l \text{ and } \eta_k = \eta_l,
\]

(2.25)

and we have also assumed that if \( k = l \) and \( \eta_k = \eta_l \) the two-point functions (2.25) are diagonal in the \( O(N) \) representation. On the r.h.s. of (2.24), the first term represents the contribution from the unit field in the field algebra and the sum involves the two-point functions of all fields in the OPE (2.23). Clearly, the OPE (2.23) and the resulting expression (2.24) determine the form of the functions \( F_S(u,v) \), \( F_V(u,v) \) and \( F_T(u,v) \) in (2.7). Next, we make the following ansatz for the OPE of the fundamental field \( \phi^\alpha(x) \) with itself

\[
\phi^\alpha(x_1)\phi^\beta(x_2) = C_\phi \frac{1}{x_{12}^{2\eta}} \delta^{\alpha\beta} + C_\phi\phi O(x_{12}, \partial_2) O(x_2) \delta^{\alpha\beta} + \frac{g_{\phi\phi J}}{C_J} \frac{(x_{12})_\mu}{(x_{12}^2)^{\eta-\mu+1}} J^\alpha_{\mu}(x_2) + \cdots
\]

\[
- \frac{g_{\phi\phi T}}{C_T} \frac{(x_{12})_\mu(x_{12})_\nu}{(x_{12}^2)^{\eta-\mu+1}} T_{\mu\nu}(x_2) \delta^{\alpha\beta} + \cdots
\]

(2.26)

Namely, the most singular terms as \( x_{12}^2 \to 0 \) in the OPE (2.23) are assumed to be, apart from the contribution of the unit field, the coefficients of the conserved vector current \( J^\alpha_{\mu}(x) \), of the energy momentum tensor \( T_{\mu\nu}(x) \) and also of some scalar field \( O(x) \) of dimension \( \eta_o \) with \( 0 < \eta_o < d \) whose two-point function in normalised as

\[
\langle O(x_1) O(x_2) \rangle = \Phi_O(x_{12}) = C_O \frac{1}{x_{12}^{2\eta_o}}.
\]

(2.27)

Other possible fields neglected in (2.26) include all symmetric traceless rank-2 \( O(N) \) tensors. The dots in the second (third) row of (2.26) stand for derivatives of \( J^\alpha_{\mu}(x) \) \( (T_{\mu\nu}(x)) \) having for coefficients less singular functions of \( x_{12}^2 \) as \( x_{12}^2 \to 0 \) and also for other \( O(N) \) singlets and antisymmetric tensors. The leading OPE coefficients of \( J^\alpha_{\mu}(x) \) and \( T_{\mu\nu}(x) \) are determined from the conservation, symmetry and tracelessness properties of these currents. The normalisation coefficients \( g_{\phi\phi T}/C_T \) and \( g_{\phi\phi J}/C_J \) are determined by requiring consistency of (2.26) with (2.14) and (2.13).

The OPE coefficient \( C_\phi\phi O(x_{12}, \partial_2) \) is determined \[13\] by requiring consistency of (2.26) with the three-point function (2.13). If we let

\[
C_O C_\phi\phi O(x_{12}, \partial_2) = g_{\phi\phi O} \frac{1}{(x_{12}^2)^{\eta-\frac{1}{2}\eta_o}} C_{\phi\phi O}(x_{12}, \partial_2),
\]

(2.28)
then we require
\[ C^{\eta_0}(x_{12}, \partial_2) \frac{1}{x_{23}^{2\eta_0}} = \frac{1}{(x_{13}^2 x_{23}^2)^{\frac{1}{2}\eta_0}}. \]  

(2.29)

To determine \( C^{\eta_0}(x_{12}, \partial_2) \) we use \[15\]
\[
\frac{1}{(ab)^\rho} = \frac{1}{B(\rho, \rho)} \int_0^1 dt [t(1-t)]^{\rho-1} \frac{1}{[a t + (1-t)b]^{2\rho}},
\]

(2.30)

and we find from (2.29)
\[
C^{\eta_0}(x_{12}, \partial_2) \frac{1}{x_{23}^{2\eta_0}} = \frac{1}{B(\frac{1}{2}\eta_0, \frac{1}{2}\eta_0)} \int_0^1 dt [t(1-t)]^{\frac{1}{2}\eta_0-1} \frac{1}{[t(x_{23} + x_{12})^2 + (1-t)x_{23}^2]^{\eta_0}}
\]

\[
= \frac{1}{B(\frac{1}{2}\eta_0, \frac{1}{2}\eta_0)} \int_0^1 dt [t(1-t)]^{\frac{1}{2}\eta_0-1} \sum_{m=0}^\infty \frac{(\eta_0)_m}{m!} \frac{[-\frac{1}{4}x_{12}^2 t(1-t)]^m}{(x_{23} + t x_{12})^{2\eta_0+m}}.
\]

(2.31)

where \((a)_m = \Gamma(a+m)/\Gamma(a)\) is the Pochhammer symbol. Then, using
\[
(\partial^2)^m \frac{1}{x^{2\alpha}} = 4^m (a)_m (a + 1 - \mu)_m \frac{1}{(x^2)^{a+m}},
\]

(2.32)

we obtain from (2.31)
\[
C^{\eta_0}(x_{12}, \partial_2) \frac{1}{x_{23}^{2\eta_0}} = \frac{1}{B(\frac{1}{2}\eta_0, \frac{1}{2}\eta_0)} \int_0^1 dt [t(1-t)]^{\frac{1}{2}\eta_0-1} \sum_{m=0}^\infty \frac{1}{m!} \frac{1}{(\eta_0 + 1 - \mu)_m}
\]

\[
\times \left[ -\frac{1}{4} x_{12}^2 t(1-t) \right]^m \partial_2^{2m} \frac{1}{(x_{23} + t x_{12})^{2\eta_0}}.
\]

(2.33)

Comparing (2.29) with (2.33) we finally obtain
\[
C^{\eta_0}(x_{12}, \partial_2) = \frac{1}{B(\frac{1}{2}\eta_0, \frac{1}{2}\eta_0)} \int_0^1 dt [t(1-t)]^{\frac{1}{2}\eta_0-1} \sum_{m=0}^\infty \frac{1}{m!} \frac{1}{(\eta_0 + 1 - \mu)_m}
\]

\[
\times \left[ -\frac{1}{4} x_{12}^2 t(1-t) \right]^m \partial_2^{2m} e^{tx_{12}\partial_2},
\]

\[
= 1 + \frac{1}{2} (x_{12})_\mu \partial_2, + \frac{n_0 + 2}{8(n_0 + 1)} (x_{12})_\mu (x_{12})_\nu \partial_2, \partial_2, \partial_2, - \frac{n_0}{16(n_0 + 1)(n_0 + 1 - \mu)} x_{12}^2 \partial_2^2 + O(x_{12}^3, \partial_2^3).
\]

(2.34)
Only the first few most singular terms as $x_{12}^2 \to 0$ of the differential operator $C_{\eta}(x_{12}, \partial_2)$ are essential for our purposes and are explicitly displayed on the r.h.s. of (2.34). The contribution to the four-point function (2.24) from such a scalar field in the OPE can be found using

$$C_{\eta}(x_{12}, \partial_2) C_{\eta}(x_{34}, \partial_4) \frac{1}{x_{24}^2} = \frac{1}{(x_{12}^2 x_{24}^2)^{2\eta}} \mathcal{H}_\eta(u, v),$$

(2.35)

where

$$\mathcal{H}_\eta(u, v) = (\frac{v}{u})^{\frac{1}{2}\eta} \sum_{n=0}^{\infty} \frac{v^n}{n!} (\frac{1}{2}\eta)_n^4 2F_1(\frac{1}{2}\eta_n + n, \frac{1}{2}\eta_n + n; \eta_0 + 2n; 1 - \frac{v}{u}).$$

(2.36)

which is proved in Appendix A. Recalling the following property of the hypergeometric function

$$2F_1(a, b; c; z) = (1 - z)^{-a} 2F_1(a, c - b; c; \frac{z}{z - 1}),$$

(2.37)

we easily verify that (2.35) is symmetric in $(x_1, x_2) \leftrightarrow (x_3, x_4)$ or $u \leftrightarrow v$ as it should be. We emphasise once more that under our basis assumption, $O(x)$ is the only scalar field with dimension $< d$ contributing to the OPE (2.26).

The leading contribution of the energy momentum tensor $T_{\mu\nu}(x)$ to the four-point function in the limit $x_{12}^2, x_{34}^2 \to 0$ is also seen from (2.9) and (2.26) to be

$$\frac{g^2}{C_T} \frac{1}{(x_{12}^2 x_{24}^2)^{\eta - \mu + 1}} \frac{1}{(x_{24}^2)^{2\mu}} (x_{12})_{\mu}(x_{12})_{\nu}(x_{34})_{\rho}(x_{34})_{\sigma} I_{\mu\nu,\rho\sigma}(x_{24}) \delta^{\alpha\beta} \delta^{\gamma\delta}

\sim \frac{g^2}{C_T} \eta(x, y)^{1/2} H(\eta, x) (uv)^{-1/2} \eta + 1/2 \left(\frac{x_{12}}{2} \cdot \frac{x_{34}}{2} \cdot \frac{x_{24}}{2}ight) \left[\frac{(x_{12} \cdot x_{34}) - 2(x_{12} \cdot x_{24})(x_{34} \cdot x_{24})}{x_{24}^4} \right]^2 - \frac{1}{d} \frac{x_{12}^2 x_{34}^2}{x_{24}^4} \left(2(x_{12} \cdot x_{34}) - 2(x_{12} \cdot x_{24})(x_{34} \cdot x_{24}) + \cdots\right) \delta^{\alpha\beta} \delta^{\gamma\delta}.$$

(2.38)

Similarly, the leading contribution of the conserved current $J^\mu(x)$ to the four-point function in the limit $x_{12}^2, x_{34}^2 \to 0$ can be found from (2.10) and (2.26) to be

$$\frac{g^2}{C_J} \frac{1}{(x_{12}^2 x_{24}^2)^{\eta - \mu + 1}} \frac{1}{(x_{24}^2)^{2\mu - 1}} (x_{12})_{\mu}(x_{24})_{\nu} I_{\mu\nu}(x_{24}) \left(\delta^{\alpha\gamma} \delta^{\beta\delta} - \delta^{\alpha\delta} \delta^{\beta\gamma}\right)

\sim \frac{g^2}{C_J} \eta(x, y)^{1/2} H(\eta, x) (uv)^{-1/2} \eta + 1/2 \left(\frac{x_{12}}{2} \cdot \frac{x_{34}}{2} \cdot \frac{x_{24}}{2}ight) \left[\frac{(x_{12} \cdot x_{34}) - 2(x_{12} \cdot x_{24})(x_{34} \cdot x_{24})}{x_{24}^4} \right]^2 - \frac{1}{d} \frac{x_{12}^2 x_{34}^2}{x_{24}^4} \left(2(x_{12} \cdot x_{34}) - 2(x_{12} \cdot x_{24})(x_{34} \cdot x_{24}) + \cdots\right) \left(\delta^{\alpha\gamma} \delta^{\beta\delta} - \delta^{\alpha\delta} \delta^{\beta\gamma}\right).$$

(2.39)
The dots in (2.38) and (2.39) stand for less singular terms as \( x_{12}^2 \to 0 \) and \( x_{34}^2 \to 0 \) independently. Equations (2.35), (2.38) and (2.39) suggest that it is convenient for what follows to consider \( H^{-1}(\eta, x) \Phi^{\alpha \beta \gamma \delta}(x_1, x_2, x_3, x_4) \) as a function of the two new independent variables

\[
Y = 1 - \frac{v}{u} = 2 \frac{1}{x_{24}^3} \left[ (x_{12} \cdot x_{34}) - 2 \frac{(x_{12} \cdot x_{24})(x_{34} \cdot x_{24})}{x_{24}^2} \right] + \cdots, \tag{2.40}
\]

\[
W = (uv)^{1/2} = \frac{x_{12}^2 x_{34}^2}{x_{24}^3} + \cdots, \tag{2.41}
\]

where the dots stand for terms which tend to zero faster as \( x_{12}^2 \to 0 \) and \( x_{34}^2 \to 0 \). Note that as \( x_{12}, x_{34} \to 0 \), then \( Y, W \to 0 \). Also note that \( Y^2 \) and \( W \) are of the same order in the above limit. Comparing (2.7) with (2.35), (2.38), (2.39) and expanding \( H_{\eta o}(u, v) \) for \( u, v \to 0 \) with \( v/u \to 1 \), we obtain

\[
F_S(u, v) \equiv \mathcal{F}_S(Y, W) = C_o^2 W^{-\frac{2}{3}}
\]

\[
+ g_o^2 \frac{\Phi_O}{C_O} W^{\frac{1}{2} \eta_o - \frac{1}{2} \eta} \left( 1 - \frac{\eta_o^2}{32(\eta_o + 1)} Y^2 + \frac{\eta_o^3}{16(\eta_o + 1)(\eta_o + 1 - \mu)} W \right) + \cdots, \tag{2.42}
\]

\[
F_V(u, v) \equiv \mathcal{F}_V(Y, W) = \frac{1}{2} g_o^2 \frac{\Phi_J}{C_J} W^{\frac{1}{2} \eta} \left( \frac{1}{4} Y^2 - \frac{1}{d} W \right) + \cdots, \tag{2.43}
\]

where the dots stand for less singular terms in the limit \( Y, W \to 0 \). The discussion of \( F_T(u, v) \) is outside the scope of the present work. The field dimensions and the couplings appearing in (2.42) and (2.43) are the dynamical parameters of the theory and can only be determined within the context of explicit CFT models.

---

3The crossing symmetry properties (2.3) imply among other things that \( F_S(u, v), (F_V(u, v)) \), is symmetric (antisymmetric) in \( u \leftrightarrow v \). These functions can be expanded as infinite power series in \((1 - v/u)\). Truncating such infinite series to a finite order makes the crossing symmetry properties of the resulting expressions less transparent. Alternatively, we might consider the expansion in the variable \( \tilde{Y} = \left[ (u/v)^{1/2} - (v/u)^{1/2} \right] = Y + O(Y^3) \).
\[ \Phi_f(x_1, x_2, x_3, x_4) = C_\phi^2 \frac{1}{(x_{12}^2 x_{34}^2)_{12}} H(\eta, x) F_f(u, v) = G_0 \]

Figure 1: The Graphical Expansion for \( \Phi_f(x_1, x_2, x_3, x_4) \)

3 The Four-Point Function in the Free Field Theory

Consider the trivial theory of \( N \) massless free scalar fields \( \phi^\alpha(x) \) with the normalisation of the fundamental two-point function being as in (2.1). In this case an explicit expression for the four-point function (2.2) can be found using Wick’s theorem with elementary contraction (2.1). It is then easy to see that

\[ F_f(u, v) = C_\phi^2 (uv)^{-\frac{4}{3}\eta}, \]  

where the subscript \( f \) stands for “free field theory”. We may also represent graphically the above result as shown in Fig. 1 where the solid lines stand for the two-point function (2.1) (without the \( O(N) \) indices). It is then easy to obtain

\[
F_{S,f}(u, v) = F_f(u, v) + \frac{1}{N} \left( F_f\left(\frac{1}{u}, \frac{v}{u}\right) + F_f\left(\frac{1}{v}, \frac{u}{v}\right) \right) \\
\equiv F_{S,f}(Y, W) = C_\phi^2 W^{-\frac{2}{3}\eta} + \frac{1}{N} C_\phi^2 W^{\frac{2}{3}\eta} \left( 2 + \frac{1}{4} \eta^2 Y^2 + \cdots \right),
\]

and

\[
F_{V,f}(u, v) = F_f\left(\frac{1}{u}, \frac{v}{u}\right) - F_f\left(\frac{1}{v}, \frac{u}{v}\right) \\
\equiv F_{V,f}(Y, W) = \eta C_\phi^2 W^{\frac{2}{3}\eta} Y + \cdots.
\]

Equations (3.2) and (3.3) have to be compared with (2.42) and (2.43) respectively. Consistency of (3.2) and (2.42) requires

\[
\mu - 1 - \frac{2}{3}\eta = \frac{1}{3}\eta \quad \Rightarrow \quad \eta = \mu - 1, \\
\frac{1}{2}\eta_o - \frac{2}{3}\eta = \frac{1}{3}\eta \quad \Rightarrow \quad \eta_o = 2\eta,
\]

\[
g_{\phi\phi O}^2 = \frac{2}{N} C_O C_\phi^2.
\]
Comparing the coefficient of $Y^2$ in (3.2) and (2.42) we obtain by virtue of (3.4)-(3.6) using also (2.21) for $g_{\phi\phi T}$

$$\frac{\eta^2}{N} C^2_\phi = - \frac{g^2_{\phi\phi O}}{C_O} \frac{\eta_0^2}{8(\eta_0 + 1)} + \frac{d^2 \eta^2}{(d-1)^2 S_d^2 C_T} C^2_\phi, \quad \Rightarrow \quad C_T = N \frac{d}{(d-1) S_d^2}. \quad (3.7)$$

Using the results above we also see that

$$\frac{g^2_{\phi\phi O}}{C_O} \frac{\eta_0^3}{16(\eta_0 + 1)(\eta_0 + 1 - \mu)} - \frac{g^2_{\phi\phi T}}{C_T} \frac{1}{d} = 0. \quad (3.8)$$

as required for consistency of the coefficients of $W$ in (3.2) and (2.42).

Comparing (3.3) and (2.43) we obtain by virtue of the results above and also (2.22) for $g_{\phi\phi J}$

$$C_J = \frac{2}{(d-2) S_d^2}. \quad (3.9)$$

The values for the dimensions $\eta$ and $\eta_0$, for the coupling $g_{\phi\phi O}$ and for the normalisations of the conserved currents $C_T$ and $C_J$ obtained above, are in agreement with the results given e.g. in [7] for the theory on $N$ massless free scalar fields in any dimension $d$.

4 The Four-Point Function in a Non-Trivial CFT in $2 < d < 4$

4.1 The Graphical Expansion

Although the theory considered in the previous section is trivial, it suggests a possible treatment for a non-trivial conformally invariant $O(N)$ vector model. In particular, we showed that consistency of the (trivial) graphical representation in Fig. 1 with the OPE (2.20) and the resulting expressions (2.42) and (2.43) for the four-point function, determines the values of the dimensions $\eta$, $\eta_0$ and the other dynamical parameters to be the corresponding ones for a free theory of $N$ massless scalar fields.

We propose that a graphical expansion for a non-trivial $\Phi(x_1, x_2, x_3, x_4)$ in (2.2) may be obtained by introducing a conformally invariant vertex in the theory. This vertex is assumed to describe the interaction of $\phi^\alpha(x)$ with an arbitrary scalar field $\bar{O}(x)$, which is a $O(N)$
singlet and has dimension $\tilde{\eta}_o$ with $0 < \tilde{\eta}_o < d$, whose two-point function is normalised as

$$
\langle \tilde{O}(x_1)\tilde{O}(x_2) \rangle \equiv \Phi_{\tilde{O}}(x_{12}) = C_{\tilde{O}} \frac{1}{x_{12}^{2\tilde{\eta}_o}}, \quad (4.1)
$$

and it is represented diagrammatically as a dashed line. Then, the full three-point function of $\tilde{O}(x)$ with $\phi^\alpha(x)$ is analogously to (2.13)

$$
\langle \phi^\alpha(x_1)\phi^\beta(x_2)\tilde{O}(x_3) \rangle = g^* \frac{1}{x_{12}^{\eta - \frac{1}{2}\tilde{\eta}_o}} \frac{1}{x_{23}^{\eta - \frac{1}{2}\tilde{\eta}_o}} \delta^{\alpha\beta}. \quad (4.2)
$$

Diagrammatically this can be represented as shown in Fig. 2. The coupling constant $g^*$ has to be determined from the dynamics of the theory. Next, we assume that the amplitudes for $n$-point functions of $\phi^\alpha(x)$ with $n \geq 4$ in our non-trivial CFT are constructed in terms of skeleton graphs, with no self-energy or vertex insertions, with internal lines corresponding to the two-point functions of $\phi^\alpha(x)$ and $\tilde{O}(x)$. Symmetry factors are determined as in the usual Feynman perturbation expansion. Graphical expansions in field theory are usually connected with a Lagrangian formalism. In the present work however, we consider a formulation for a non-trivial CFT based on a graphical expansion without explicit reference to an underlying Lagrangian.

For use in the graphical expansion it is necessary to require amputation of the three-point function (4.2) for any leg linked to an internal line of a graph. For example, the inverse kernel for the two-point function (4.1) is

$$
\Phi_{\tilde{O}}^{-1}(x_{12}) = \frac{1}{C_{\tilde{O}}} \rho(\tilde{\eta}_o) \frac{1}{x_{12}^{d - \tilde{\eta}_o}}, \quad (4.3)
$$

where

$$
\rho(\tilde{\eta}_o) = \frac{1}{\pi^d} \frac{\Gamma(d - \tilde{\eta}_o)\Gamma(\tilde{\eta}_o)}{\Gamma(\tilde{\eta}_o - \mu)\Gamma(\mu - \tilde{\eta}_o)}, \quad (4.4)
$$

and it is represented diagrammatically as a dashed line. Then, the full three-point function of $\tilde{O}(x)$ with $\phi^\alpha(x)$ is analogously to (2.13)

$$
\langle \phi^\alpha(x_1)\phi^\beta(x_2)\tilde{O}(x_3) \rangle = g^* \frac{1}{x_{12}^{\eta - \frac{1}{2}\tilde{\eta}_o}} \frac{1}{x_{23}^{\eta - \frac{1}{2}\tilde{\eta}_o}} \delta^{\alpha\beta}. \quad (4.2)
$$

Diagrammatically this can be represented as shown in Fig. 2. The coupling constant $g^*$ has to be determined from the dynamics of the theory. Next, we assume that the amplitudes for $n$-point functions of $\phi^\alpha(x)$ with $n \geq 4$ in our non-trivial CFT are constructed in terms of skeleton graphs, with no self-energy or vertex insertions, with internal lines corresponding to the two-point functions of $\phi^\alpha(x)$ and $\tilde{O}(x)$. Symmetry factors are determined as in the usual Feynman perturbation expansion. Graphical expansions in field theory are usually connected with a Lagrangian formalism. In the present work however, we consider a formulation for a non-trivial CFT based on a graphical expansion without explicit reference to an underlying Lagrangian.

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$$
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$$

where

$$
\rho(\tilde{\eta}_o) = \frac{1}{\pi^d} \frac{\Gamma(d - \tilde{\eta}_o)\Gamma(\tilde{\eta}_o)}{\Gamma(\tilde{\eta}_o - \mu)\Gamma(\mu - \tilde{\eta}_o)}, \quad (4.4)
$$

and it is represented diagrammatically as a dashed line. Then, the full three-point function of $\tilde{O}(x)$ with $\phi^\alpha(x)$ is analogously to (2.13)

$$
\langle \phi^\alpha(x_1)\phi^\beta(x_2)\tilde{O}(x_3) \rangle = g^* \frac{1}{x_{12}^{\eta - \frac{1}{2}\tilde{\eta}_o}} \frac{1}{x_{23}^{\eta - \frac{1}{2}\tilde{\eta}_o}} \delta^{\alpha\beta}. \quad (4.2)
$$

Diagrammatically this can be represented as shown in Fig. 2. The coupling constant $g^*$ has to be determined from the dynamics of the theory. Next, we assume that the amplitudes for $n$-point functions of $\phi^\alpha(x)$ with $n \geq 4$ in our non-trivial CFT are constructed in terms of skeleton graphs, with no self-energy or vertex insertions, with internal lines corresponding to the two-point functions of $\phi^\alpha(x)$ and $\tilde{O}(x)$. Symmetry factors are determined as in the usual Feynman perturbation expansion. Graphical expansions in field theory are usually connected with a Lagrangian formalism. In the present work however, we consider a formulation for a non-trivial CFT based on a graphical expansion without explicit reference to an underlying Lagrangian.
and it is represented diagrammatically by a dotted line. We may then amputate the three-point function \((4.2)\) on the \(\tilde{O}\) leg obtaining the following vertex

\[
V_{\phi\phi\tilde{O}}^{\alpha\beta}(x_1, x_2, x_3) = \int d^d x \langle \phi^\alpha(x_1) \phi^\beta(x_2) \tilde{O}(x) \rangle \Phi^{-1}(x - x_3)
\]

\[
= \frac{g^*}{C_\tilde{O}} \rho(\tilde{\eta}_o) U(\frac{1}{2} \tilde{\eta}_o, \frac{1}{2} \tilde{\eta}_o, d - \tilde{\eta}_o) \frac{1}{(x_{12}^2)^{\eta - \mu + \frac{1}{2} \tilde{\eta}_o} (x_{13}^2 x_{23}^2)^{\mu - \frac{1}{2} \tilde{\eta}_o}} \delta^{\alpha\beta}. \tag{4.5}
\]

This amputation can be diagrammatically represented as shown in Fig. 3. In obtaining (4.5) we have used the D’EPP formula [16]

\[
\int d^d x \frac{1}{(x_1 - x)^{2a_1} (x_2 - x)^{2a_2} (x_3 - x)^{2a_3}} = U(a_1, a_2, a_3) \frac{1}{(x_{12}^a)^{a-a_3} (x_{13}^a)^{a-a_2} (x_{23}^a)^{a-a_1}}, \tag{4.6}
\]

which is valid for \(a_1 + a_2 + a_3 = d\), with

\[
U(a_1, a_2, a_3) = \pi^\mu \frac{\Gamma(\mu - a_1) \Gamma(\mu - a_2) \Gamma(\mu - a_3)}{\Gamma(a_1) \Gamma(a_2) \Gamma(a_3)}. \tag{4.7}
\]

We denote by \(\Phi(\tilde{\eta}_o)(x_1, x_2, x_3, x_4)\) the amplitude of interest in our graphical treatment of the four-point function \((2.2)\). The first few graphs in the skeleton expansion for this amplitude in increasing order according to the number of vertices are displayed in Fig. 4 where the dots stand for graphs with more than four vertices. The solid lines stand for the two-point function \((2.1)\) and the dashed lines stand for the two-point function \((4.1)\). The vertices (dark blobs) formed by two solid and one dashed lines stand for the three-point function \((4.2)\) with amputation on all legs. Integration over the coordinates of all internal vertices is understood. The superscript \((\tilde{\eta}_o)\) denotes skeleton graphs built using dashed lines corresponding to the two-point function of the scalar field \(\tilde{O}(x)\) of dimension \(\tilde{\eta}_o\).
Note that the amplitude \( \Phi^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4) \) as given by the skeleton expansion of Fig. 4 satisfies the crossing symmetry properties (2.3). The graphs displayed in Fig. 4 have the interesting property

\[
\Phi^{(d - \tilde{\eta}_o)}(x_1, x_2, x_3, x_4) = \mathcal{G}_0(x_1, x_2, x_3, x_4) + C(\tilde{\eta}_o)\mathcal{G}_1^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4) + [C(\tilde{\eta}_o)]^2 \left( \mathcal{G}_2^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4) + \mathcal{G}_2^{(\tilde{\eta}_o)}(x_1, x_2, x_4, x_3) \right) + \cdots (4.8)
\]

The superscript \((d - \tilde{\eta}_o)\) on the l.h.s. of (4.8) denotes the same graphical expansion as previously but using for the internal dashed lines the two-point function (4.1) with dimension \(d - \tilde{\eta}_o\) and the same normalisation \(C_0\), while the dark blobs now correspond to the interaction of \(\phi^\alpha(x)\) with a scalar field \(\tilde{O}_s(x)\) of dimension \(d - \tilde{\eta}_o\) via a unique vertex having the same coupling constant \(g_*\). The coefficient \(C(\tilde{\eta}_o)\) is given by

\[
C(d - \tilde{\eta}_o) = C^{-1}(\tilde{\eta}_o) = \frac{\rho(\tilde{\eta}_o)U(\frac{1}{2}\tilde{\eta}_o, \frac{1}{2}\tilde{\eta}_o, d - \tilde{\eta}_o)}{\rho(d - \tilde{\eta}_o)U(\mu - \frac{1}{2}\tilde{\eta}_o, \mu - \frac{1}{2}\tilde{\eta}_o, \tilde{\eta}_o)} = \frac{\Gamma(\tilde{\eta}_o)\Gamma(\tilde{\eta}_o - \mu)\Gamma^4(\mu - \frac{1}{2}\tilde{\eta}_o)}{\Gamma(2\mu - \tilde{\eta}_o)\Gamma(\mu - \tilde{\eta}_o)\Gamma^4(\frac{1}{2}\tilde{\eta}_o)}. \tag{4.9}
\]

The shadow symmetry properties of the skeleton graphs \(\mathcal{G}_1^{(\tilde{\eta}_o)}\) and \(\mathcal{G}_2^{(\tilde{\eta}_o)}\) required for obtaining (4.8) are proved in Appendices B and C. An obvious conjecture is that each graph in the full skeleton expansion \(\Phi^{(d - \tilde{\eta}_o)}(x_1, x_2, x_3, x_4)\) is proportional to the corresponding graph in the skeleton expansion for \(\Phi^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4)\) with proportionality constant \([C(\tilde{\eta}_o)]^n\) where \(2n\) is the number of vertices formed by two solid and one dashed lines in the graph. In the following we will see that assuming this symmetry property of the skeleton graph expansion holds to all orders we arrive at an interesting duality of the \(O(N)\) theory.

\[\Phi^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4) = \mathcal{G}_0 + \mathcal{G}_1^{(\tilde{\eta}_o)} + \mathcal{G}_2^{(\tilde{\eta}_o)} + \cdots (x_3 \leftrightarrow x_4) + \cdots\]

\[\text{Figure 4: The Skeleton Graph Expansion for } \Phi^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4)\]
4.2 The Consistency Relations

The crucial consistency requirement regarding the present work is that amplitudes constructed according to graphical expansions such as the one in Fig. 4 correspond to a CFT having operator content in agreement with the OPE ansatz (2.26) and are therefore compatible with amplitudes obtained by straightforward application of this ansatz on \( n \)-point functions. Without further input at this point we have no intrinsic means in estimating the magnitude of \( g_* \) and hence we cannot hope to obtain a weak coupling expansion. However, on account of the \( O(N) \) symmetry we subsequently see that the assumption \( g_*^2 = O(1/N) \) leads naturally to a well defined perturbation expansion in \( 1/N \) for the theory. Therefore, from now on we consider the theory for large \( N \).

An important point concerning the consistency relations is that one has to conduct the calculation of the amplitudes corresponding to skeleton graphs like the ones in Fig. 4, in such a way that the resulting expressions can be compared with the OPE results (2.42) and (2.43). In the context of this subsection this means that one must ensure that these amplitudes are explicitly expressed as functions of \( Y \) and \( W \) which allow for the transparent and unambiguous evaluation of the limit as \( Y \to 0, W \to 0 \) independently. Our subsequent calculations will clarify further this point.

Then, by virtue of the results in the Appendices we can write

\[
\mathcal{F}_{S,1}^{(\tilde{\eta}_o)}(Y, W) = C_\phi^2 W^{\frac{2}{3} \eta} + \frac{1}{N} C_\phi^2 W^{\frac{1}{3} \eta} \left[ (1 - Y)^{-\frac{1}{2} \eta} + (1 - Y)^{\frac{1}{2} \eta} \right] + \frac{g_*^2}{C_\theta} W^{-\frac{2}{3} \eta} \left[ \mathcal{H}_{\tilde{\eta}_o}(u, v) + C(d - \tilde{\eta}_o) \mathcal{H}_{d - \tilde{\eta}_o}(u, v) \right] + \frac{1}{N} g_*^2 \mathcal{F}_{S,1}^{(\tilde{\eta}_o)}(Y, W) + \left( \frac{g_*^2}{C_\theta} \right)^2 \mathcal{F}_{S,2}^{(\tilde{\eta}_o)}(Y, W) + O\left( \frac{1}{N^2} \right), \tag{4.10}
\]

where

\[
\frac{g_*^2}{C_\theta} \mathcal{F}_{S,1}^{(\tilde{\eta}_o)}(Y, W) = H^{-1}(\eta, x) \left[ \mathcal{G}_1^{(\tilde{\eta}_o)}(x_1, x_3, x_2, x_4) + \mathcal{G}_1^{(\tilde{\eta}_o)}(x_1, x_4, x_3, x_2) \right], \tag{4.11}
\]

\[
\left( \frac{g_*^2}{C_\theta} \right)^2 \mathcal{F}_{S,2}^{(\tilde{\eta}_o)}(Y, W) = H^{-1}(\eta, x) \left[ \mathcal{G}_2^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4) + \mathcal{G}_2^{(\tilde{\eta}_o)}(x_1, x_2, x_4, x_3) \right]. \tag{4.12}
\]

We also find

\[
\mathcal{F}_{V,1}^{(\tilde{\eta}_o)}(Y, W) = C_\phi^2 W^{\frac{1}{2} \eta} \left[ (1 - Y)^{-\frac{1}{2} \eta} - (1 - Y)^{\frac{1}{2} \eta} \right] + \frac{g_*^2}{C_\theta} \mathcal{F}_{S,1}^{(\tilde{\eta}_o)}(Y, W) + O\left( \frac{1}{N^2} \right), \tag{4.13}
\]
where
\[
\frac{g_s^2}{C_\tilde{O}} \mathcal{F}_{V,1}^{(\tilde{\eta}_o)}(Y, W) = H^{-1}(\eta, x) \left[ \mathcal{G}_{1}^{(\tilde{\eta}_o)}(x_1, x_3, x_2, x_4) - \mathcal{G}_{1}^{(\tilde{\eta}_o)}(x_1, x_4, x_3, x_2) \right].
\] (4.14)

To $O(1/N)$ we just need to consider the first two lines on the r.h.s. of (4.10). From the results for free fields in section 3 we see that the first line on the r.h.s. of (4.10) is compatible with an OPE for a field $\phi^a(x)$ of dimension $\eta = \mu - 1$ with itself, including the energy momentum tensor and a scalar field of dimension $2\eta = d - 2$. Next, we note that for $2 < d < 4$ and $0 < \tilde{\eta}_o < d$ it is easy to show that $C(d - \tilde{\eta}_o) < 0$. Therefore, to leading order in $1/N$ if we take $\tilde{\eta}_o = 2$ we must impose
\[
\frac{g_s^2}{C_\tilde{O}} C(d - \tilde{\eta}_o) + \frac{2}{N} C_\phi^2 = 0,
\] (4.15)
to ensure that only one scalar field appears in the OPE (2.20) in accordance with our basic requirement. By virtue of the required agreement of $\mathcal{F}_{S}^{(\tilde{\eta}_o)}(Y, W)$ in (4.10) with the operator product expansion result $\mathcal{F}_{S}(Y, W)$ in (2.42), we may therefore identify the field $O(x)$ in the (2.20) with the field $\tilde{O}(x)$ which is associated with the graphical expansion, so we set $\eta_o = \tilde{\eta}_o$ and $g_{\phi\phi O}^2 = g_s^2$. From (4.15) this implies that to leading order in $1/N$
\[
g_{\phi\phi O}^2 = \frac{2}{N} C(2) C_\phi^2 C_\tilde{O}
= \frac{2}{N} \frac{\Gamma(2\mu - 2)}{\Gamma(3 - \mu) \Gamma(3) (\mu - 1)} C_\phi^2 C_\tilde{O}.
\] (4.16)

For simplicity we henceforth use the fact that in CFT one can arbitrarily adjust the normalisations of the two-point functions, (except of those of the conserved currents), by rescaling the relevant fields, and we set $C_\phi = C_O = C_{\tilde{O}} = 1$ (in the skeleton expansion these are not modified). The discussion of (1.13) to $O(1/N)$ and (4.10) to $O(1/N^2)$ is achieved assuming a $1/N$ expansion for the dynamical parameters of the theory as follows

\[
\eta = \mu - 1 + \frac{1}{N} \eta_1, \tag{4.17}
\]
\[
\tilde{\eta}_o = \eta_o = 2 + \frac{1}{N} \eta_{o,1}, \tag{4.18}
\]
\[
g_s^2 = g_{\phi\phi O}^2 = \frac{2}{N} \frac{\Gamma(2\mu - 2)}{\Gamma(3 - \mu) \Gamma^3(\mu - 1)} \left( 1 + \frac{1}{N} g_{*,1} \right), \tag{4.19}
\]
\[ \begin{align*}
C_J &= \frac{2}{(d-2)S_d^2} \left( 1 + \frac{1}{N} C_{J,1} \right), \quad (4.20) \\
C_T &= N \frac{d}{(d-1)S_d^2} \left( 1 + \frac{1}{N} C_{T,1} \right). \quad (4.21)
\end{align*} \]

It is simpler to first consider \( \mathcal{F}_{V}^{(\tilde{\eta}_o)}(Y,W) \) in (4.13). Using the results in Appendix A we can expand \( g_s^2 \mathcal{F}_{V}^{(\tilde{\eta}_o)}(Y,W) \) in powers of \( Y \) and then, by virtue of the required agreement of \( \mathcal{F}_{V}^{(\tilde{\eta}_o)}(Y,W) \) in (4.13) with \( \mathcal{F}_V(Y,W) \) in (2.43), we obtain to \( O(1/N) \) using (2.22)

\[ \begin{align*}
&\frac{1}{2} \frac{1}{S_d^2 C_J} W^{\mu-1-\frac{2}{d} \eta} Y = \eta W^\frac{1}{d} Y \\
+ & g_s^2 \frac{\Gamma^2(\mu - \frac{1}{2} \tilde{\eta}_o) \Gamma(\tilde{\eta}_o)}{\Gamma^2(\frac{1}{2} \tilde{\eta}_o) \Gamma(\mu - \tilde{\eta}_o) \Gamma(\mu)} W^\frac{1}{d} Y \left[ \left( \eta - \frac{\tilde{\eta}_o(d - \tilde{\eta}_o)}{d} \right) (b_{00} - \ln W) \\
+ & 1 - 4 \frac{\tilde{\eta}_o(d - \tilde{\eta}_o)}{d^2} \right], \quad (4.22)
\end{align*} \]

where

\[ b_{00} = 2 \left( \psi(1) + \psi(\mu) - \psi\left(\frac{1}{2} \tilde{\eta}_o\right) - \psi\left(\mu - \frac{1}{2} \tilde{\eta}_o\right) \right). \quad (4.23) \]

Note that (4.22) is manifestly symmetric in \( \tilde{\eta}_o \leftrightarrow d - \tilde{\eta}_o \). In the first line of (4.22) we expand \( \eta \) and \( C_J \) as in (4.17) and (4.20) respectively. From the results for free fields in section 3 the \( O(1) \) terms are in agreement in both sides of (4.22). To \( O(1/N) \) we have also to consider the last two lines of (4.22) using (4.16) and setting \( \eta = \mu - 1, \tilde{\eta}_o = 2 \) when \( b_{00} = 2/(\mu - 1) \).

Consistency of the \( O(1/N) \) terms in both sides of (4.22) requires

\[ \begin{align*}
\eta_1 &= \frac{2\Gamma(2\mu - 2)}{\Gamma(1 - \mu) \Gamma(\mu) \Gamma(\mu + 1) \Gamma(\mu - 2)}, \quad (4.24) \\
C_{J,1} &= -\frac{2(2\mu - 1)}{\mu(\mu - 1)} \eta_1. \quad (4.25)
\end{align*} \]

Next, agreement of \( \mathcal{F}_{S}^{(\tilde{\eta}_o)}(Y,W) \) in (4.10) with \( \mathcal{F}_S(Y,W) \) in (2.42) to \( O(1/N^2) \) requires, by virtue of (2.21)

\[ \begin{align*}
&\left( \frac{d\eta}{d-1} \right)^2 \frac{1}{S_d^2 C_T} W^{\mu-1-\frac{2}{d} \eta} \left( \frac{1}{4} Y^2 - \frac{1}{d} W \right) = \frac{1}{N} W^{\frac{1}{d} \eta} \left( \frac{1}{4} + \frac{1}{d} \eta^4 Y^2 \right) \\
+ & g_s^2 \phi_0 \psi C(d - \eta_0) W^{\frac{1}{d} (d - \eta)} \left( 1 - \frac{(d - \eta_0)^2}{32(d - \eta_0 + 1)} Y^2 + \frac{(d - \eta_0)^3}{16(d - \eta_0 + 1)(\mu - \eta_0 + 1)} W \right).
\end{align*} \]
\[ + \frac{1}{N} g^2_{\phi\phi} F_{S,1}^{(\eta_0)}(Y, W) + g^4_{\phi\phi} F_{S,2}^{(\eta_0)}(Y, W). \] (4.26)

In obtaining (4.26) we have used the result (2.35) for \( C^{\eta_0}(x_{12}, \partial_1) C^{\eta_0}(x_{34}, \partial_4)(1/x_{24}^{-2\eta_0}) \) in \( F_{S}(Y, W) \) to cancel the term \( g^2 W^{-3/2} \mathcal{H}_{\tilde{\eta}_0}(u, v) \) in \( F_{S}^{(\tilde{\eta}_0)}(Y, W) \) when \( \eta_0 = \tilde{\eta}_0 \) and \( g^2_{\phi\phi} = g^2_\ast \).

We have also expanded \( \mathcal{H}_{d-\tilde{\eta}_0}(u, v) \) as in (2.42). From the results in Appendices B and C using (4.16) and setting \( \eta = \mu - 1 \) and \( \tilde{\eta}_0 = 2 \) we also obtain after some algebra

\[
\frac{1}{N} g^2_{\phi\phi} F_{S,1}^{(\eta_0)}(Y, W) + g^4_{\phi\phi} F_{S,2}^{(\eta_0)}(Y, W)
\]

\[ = \frac{1}{N^2} \frac{2\mu(4\mu - 5)}{\mu - 2} \eta_1 W^{3/2(\mu - 1)}
\]

\[ \times \left[ (A_1 - \ln W) + \frac{(\mu - 1)^2}{\mu(4\mu - 5)} (B_1 - \ln W) W - \frac{(\mu - 1)^3}{8(4\mu - 5)} (C_1 - \ln W) Y^2 \right] + \cdots \] (4.27)

where the dots stand for terms \( O(Y^3, W^2) \) and

\[
A_1 = \frac{8\mu - 11}{(\mu - 1)(4\mu - 5)} + \frac{2(2\mu - 3)}{4\mu - 5} \mathcal{C}(\mu),
\]

\[
B_1 = \frac{2\mu^2 + \mu - 2}{\mu(\mu - 1)(\mu + 1)} + \frac{\mu^2 - 2}{(\mu - 1)(\mu + 1)} \mathcal{C}(\mu),
\]

\[
C_1 = \frac{\mu^2 - 3\mu + 4}{\mu(\mu - 1)(\mu + 1)} + \frac{2(\mu - 1)}{\mu + 1} \mathcal{C}(\mu),
\]

with

\[
\mathcal{C}(\mu) = \psi(3 - \mu) + \psi(2\mu - 1) - \psi(1) - \psi(\mu), \quad \psi(x) = \Gamma'(x)/\Gamma(x).
\] (4.31)

In the first two lines of (4.26) we expand \( \eta, \eta_0, g^2_{\phi\phi} \) and \( C_T \) as in (4.17)-(4.19) and (4.21) respectively. The leading order terms are in agreement in both sides of (4.26) as expected from the results for free fields in section 3. Agreement of the \( O(1/N^2) \) terms in both sides of (4.26) requires (4.24) again and also

\[
\eta_{0,1} = 4 \frac{(2\mu - 1)(\mu - 1)}{\mu - 2} \eta_1,
\]

\[
g_{\ast,1} = -2 \left( \frac{2\mu^2 - 3\mu + 2}{\mu - 2} \mathcal{C}(\mu) + \frac{8\mu^3 - 24\mu^2 + 21\mu - 2}{2(\mu - 1)(\mu - 2)} \right) \eta_1,
\]

\[
C_{T,1} = - \left( \frac{2}{\mu + 1} \mathcal{C}(\mu) + \frac{\mu^2 + 3\mu - 2}{\mu(\mu - 1)(\mu + 1)} \right) \eta_1.
\] (4.34)

\(^5\)These terms are actually \( O(1/N) \) here.
The discussion of the consistency relations presented above, between the amplitudes constructed according to the graphical expansion in Fig. 4 and the ones obtained from straightforward application of OPE (2.26) on the four-point function, is not unique. To the order in \(1/N\) considered in the present work and by virtue of the shadow symmetry properties (B.3) and (C.7) proved in Appendices B and C respectively, we could have chosen instead \(\tilde{\eta}_o \equiv \tilde{\eta}_o' = d - \eta_o\) and \(g^2 = g^2_{\phi O} C(\tilde{\eta}_o)\) in the discussion of (4.10). This would have been equivalent to identifying the field \(O(x)\) in the OPE (2.26) with the shadow field\(^6\) of \(\tilde{O}(x)\) in the graphical expansion or vice versa. Such a choice would lead to the same results (4.24), (4.25) and (4.32) for the dynamical parameters \(\eta, C_J\) and \(C_T\) respectively of the theory. In this case we also obtain

\[
\tilde{\eta}_o' = d - 2 - \frac{1}{N} \frac{4(2\mu - 1)(\mu - 1)}{\mu - 2} \eta_1, 
\]

\[
\lambda^2_sthat = - \frac{2}{N} \left[ 1 + \frac{1}{N} \left( \frac{2\mu(2\mu - 3)}{\mu - 2} C(\mu) + \frac{\mu(8\mu - 11)}{(\mu - 1)(\mu - 2)} \right) \eta_1 \right].
\]

Note that, at least for \(N\) large enough, \(\lambda_s\) is purely imaginary when \(2 < d < 4\) and therefore the underlying field theory is non-unitary. However, it is perhaps interesting to remark that such a non-unitary theory may be related, at least to leading order in \(1/N\), to the free theory of section 3 through the correspondence \(\tilde{O}(x) \rightarrow \lambda_s \phi^2(x)/2\) where

\[
\phi^2(x) = :\phi^\alpha(x)\phi^\alpha(x):.
\]

Therefore, on account of our conjecture at the end of section \(4.1\), we arrive at a possible surprising duality property of the \(O(N)\) invariant theory. Namely, the field theory underlying the graphical expansion in Fig. 4, which corresponds to the interaction of the field \(\phi^\alpha(x)\) with a scalar field \(\tilde{O}(x)\) of dimension \(0 < \tilde{\eta}_o < d\) via a unique vertex having coupling constant \(g_s\), may be equivalent to a field theory underlying an identical graphical expansion but corresponding to the interaction of \(\phi^\alpha(x)\) with a scalar field \(\tilde{O}_s(x)\) of dimension \(\tilde{\eta}'_o = d - \tilde{\eta}_o\) via a unique vertex having an imaginary coupling coupling constant \(\lambda_s = |C(\tilde{\eta}'_o)|^{\frac{1}{2}} g_s\). The equivalence holds presumably in \(2 < d < 4\). From (4.35), (4.36) and (4.37) one may view the theory dual to the \(O(N)\) vector model as a possible non-unitary \(1/N\) deformation of the free field theory of section 3.

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\(^6\)This is an \(O(d)\) scalar field associated with \(O(x)\) and having dimension \(d - \tilde{\eta}_o\). It is crucial to avoid having both fields and their shadows in the OPE. For the notion of shadow fields in CFT see [17] and references therein.
5 Four-Point Function Involving the Field $O(x)$

In the previous section we argued for the possibility of constructing the amplitude for the four-point function $\langle \phi \phi \phi \phi \rangle$ in terms of skeleton graphs with internal lines corresponding to the full two-point functions of the fields $\phi^\alpha(x)$ and $\tilde{O}(x)$, the latter being identified either with the field $O(x)$ appearing in the OPE (2.26) or with its shadow field. However, up to the order considered only graphs involving the vertex formed by two $\phi$'s and one $\tilde{O}$ were necessary. In higher orders, graphs involving the vertex formed by three $\tilde{O}(x)$ fields will also appear in any consistent graphical treatment of $n$-point functions for $\phi^\alpha(x)$ with $n \geq 4$, based on a skeleton expansion with no self-energy or vertex insertions. This vertex introduces a new coupling constant $g_{\tilde{O}}$ which has to be determined in our approach by requiring consistency of the algebraic and the graphical treatments of $n$-point functions involving $O(x)$ or its shadow field. To this end, consider the following four-point function

$$\langle \phi^\alpha(x_1)\phi^\beta(x_2)O(x_3)O(x_4) \rangle = Y(x_1, x_2, x_3, x_4)\delta^{\alpha\beta}, \quad (5.1)$$

which is symmetric in $x_1 \leftrightarrow x_2$ and $x_3 \leftrightarrow x_4$. By virtue of conformal invariance we can write

$$Y(x_1, x_2, x_3, x_4) = \frac{1}{x_{12}^{2\eta_2} x_{34}^{2\eta_4}} Y(u, v), \quad (5.2)$$

where $Y(u, v) = Y(v, u)$ is an arbitrary function of the usual invariant ratios $u, v$ given in (2.6).

There are basically two independent algebraic treatments of (5.1) based on OPE’s. One is to consider the OPE $^7$ for $\phi^\alpha(x_1)\phi^\beta(x_2)$ in (2.26) together with

$$O(x_1)O(x_2) = \frac{1}{x_{12}^{2\eta_2}} + g_O \frac{1}{(x_{12}^{2\eta_2})^{\frac{1}{2}\eta_0}} C^{\eta_0}(x_{12}, \partial_2)O(x_2) + \cdots, \quad (5.3)$$

where the dots stand for other possible quasiprimary fields. Following the procedure of subsection 2.2 it is easily seen that the ansatz (5.3) is consistent with the conformally invariant three point function

$$\langle O(x_1)O(x_2)O(x_3) \rangle = g_O \frac{1}{(x_{12} x_{13} x_{23})^{\frac{1}{2}\eta_0}}, \quad (5.4)$$

---

$^7$ In this section we set to one the normalisations of the two-point functions of all fields unless explicitly stated otherwise.
for $C^{\eta_0}(x_1, \partial_2)$ as in (2.34). Substituting (2.26) and (5.3) into (5.1) we obtain, using the results in Appendix A,

$$Y(u, v) = 1 + g_{\phi O} g_{O \eta_0} H_{\eta_0}(u, v) + \cdots.$$  \hspace{1cm} (5.5)

Clearly, (5.5) is appropriate in considering the limit of $Y(u, v)$ as $u, v \to 0$ and the dots stand for less singular terms.

The other way of algebraically evaluating the four-point function (5.1) is to consider the OPE ansatz for $\phi^\alpha(x_1)O(x_3)$ which has a leading term

$$\phi^\alpha(x_1)O(x_3) = g_{\phi O} \frac{1}{(x_{13}^2)^{\frac{\eta_0}{2}}} C^{\eta_0}(x_{31}, \partial_1) \phi^\alpha(x_1) + \cdots,$$  \hspace{1cm} (5.6)

where, for compatibility with (2.13), $C^{a,b}(y, \partial)$ generalising (2.27) is defined by the requirement

$$C^{a,-b}(x_{21}, \partial_1) \frac{1}{x_{21}^{a_0}} = C^{a,b}(x_{12}, \partial_2) \frac{1}{x_{12}^{a_0}} = \frac{1}{x_{23}^{a_0} x_{23}^{a_0}}, \quad a_\pm = \frac{1}{2}(a \pm b),$$  \hspace{1cm} (5.7)

and, following a similar argument to the one in subsection 2.2, explicitly given as

$$C^{a,b}(y, \partial) = \frac{1}{B(a_+, a_-)} \int_0^1 dt t^{a_+-1}(1-t)^{a_--1}$$

$$\times \sum_{m=0}^\infty \frac{1}{m! (a + 1 - \mu)_m} \left[-\frac{1}{4} y^2 t(1-t)\right]^m \partial^2m e^t y \partial.$$  \hspace{1cm} (5.8)

The OPE for two scalar fields $A(x_1), B(x_2)$ with arbitrary dimensions $\eta_A, \eta_B$ then has the form

$$A(x_1)B(x_2) = \cdots + g_{ABC} \frac{1}{(x_{12}^2)^{\frac{1}{2}(\eta_A+\eta_B-\eta_C)}} C^{\eta_C, \eta_A-\eta_B}(x_{12}, \partial_2)C(x_1) + \cdots.$$  \hspace{1cm} (5.9)

The corresponding generalisation of (2.35) is then

$$C^{a,b}(x_{12}, \partial_2) C^{a,b}(x_{34}, \partial_4) \frac{1}{x_{24}^{a_0}} = \frac{1}{x_{24}^{a_0} x_{24}^{a_0}} E_{a,b}(u, v),$$  \hspace{1cm} (5.10)

where

$$E_{a,b}(u, v) = E_{a,-b}(u, v)$$

$$= \left(\frac{v}{u}\right)^{a_+} \sum_{n=0}^\infty \frac{\left(a_+\right)_n^2 \left(a_-\right)_n^2}{n! (a + 1 - \mu)_n} 2F_1(a_+ + n, a_+ + n; a + 2n; 1 - \frac{v}{u}).$$  \hspace{1cm} (5.11)
Note that for $a = \eta_0$ and $b = 0$, (5.11) coincides with (2.34). By virtue of (5.6) and (5.11) we then find

$$Y(u,v) = g^2_{\phi\phi O} u^{2\eta_0} \mathcal{E}_{\eta_0} \left( \frac{1}{u}, \frac{v}{u} \right) + \cdots. \quad (5.12)$$

Clearly, the result (5.12) is appropriate in considering the limit of $Y(u,v)$ as $1/u, v/u \to 0$ or equivalently $v/u \to 0$ and $v \to 1$ and the dots stand for less singular terms. We assume that no other fields having dimensions $< d$ contribute in this limit to the OPE of $\phi^\alpha(x_1)O(x_3)$.

In the spirit of the present work we will require consistency of (5.5) and (5.12) with an evaluation of (5.1) based on a graphical expansion. It is crucial that both consistency relations obtained should give the same results for the couplings and the field dimensions of the theory. As previously, we assume that a graphical expansion for the four-point function (5.1) can be constructed using graphs with internal lines corresponding to the full propagators of $\phi^\alpha(x)$ and $O(x)$, the latter being the field appearing in the OPE (2.26), which are glued together using the vertex discussed in subsection 4.1 and also the new vertex obtained from the full leg amputation of the three-point function (5.4). The first few graphs in increasing order according to the total number of vertices are shown in Fig. 5 where the new vertex in represented by a crossed circle. Without further input the magnitude of the new coupling $g_\tilde{O} \equiv g_O$ is undetermined. Nevertheless, we assume that $g^2_{\tilde{O}} \sim O(1/N)$ so that the graphs displayed in Fig. 5 give the full four-point function (5.1) up to $O(1/N)$. This assumption will be justified in what follows by the consistency of our approach.

The evaluation of the amplitudes corresponding to the graphs in Fig. 5 is straightforward if one uses the results of Appendix A. However, as it should be clear by now, one must conduct the calculations of these amplitudes having in mind the corresponding short-distance limit of (5.1) with which the resulting expressions have to be compared. All our calculations are based on formula (B.11) with the basic amplitudes corresponding to the graphs given in Appendix B. Thus to $O(1/N)$

$$Y(u,v) = 1 + g_{\phi\phi O} g_O K_1(u,v)$$
\[ + g_{O\phi O}^2 \left( \mathcal{K}_2(u, v) + \mathcal{K}_2(v, u) \right) + \cdots, \]  

(5.13)

where

\[
\mathcal{K}_1(u, v) = u^{\frac{3}{2} \eta_o} \mathcal{H}_{\eta_o}(u, v) + C(d - \eta_o) u^{\mu - \frac{1}{2} \eta_o} \mathcal{H}_{d-\eta_o}(u, v)
\]

(5.14)

\[
\mathcal{K}_2(u, v) = \frac{\Gamma(\eta) \Gamma(\eta_o - \eta) \Gamma^2(\mu - \frac{1}{2} \eta_o)}{\Gamma(\mu - \eta) \Gamma(\mu + \eta - \eta_o) \Gamma^2(\frac{1}{2} \eta_o)} \times \left( u^{\eta} \mathcal{E}_{\mu + \eta - \eta_o, \mu - \eta}(u, v) + C(\mu + \eta - \eta_o, 2\mu - \eta_o) u^{\eta_o} \mathcal{E}_{\mu - \eta + \eta_o, \mu - \eta}(u, v) \right),
\]

(5.15)

\[
C(\eta, \eta_o) = \frac{\Gamma(\eta) \Gamma(\eta - \mu) \Gamma^2(\mu - \eta + \frac{1}{2} \eta_o) \Gamma^2(\mu - \frac{1}{2} \eta_o) \Gamma(2\mu - \eta) \Gamma(\eta - \frac{1}{2} \eta_o) \Gamma^2(\frac{1}{2} \eta_o)}{\Gamma(\mu - \eta) \Gamma(2\mu - \eta) \Gamma^2(\eta - \frac{1}{2} \eta_o) \Gamma^2(\frac{1}{2} \eta_o)},
\]

(5.16)

with the same \( \mathcal{E}_{a,b}(u, v) \) as in (5.11). \( \mathcal{K}_2(u, v) \) corresponds to the third graph in Fig. 5 and \( \mathcal{K}_2(v, u) \) to the fourth (not drawn) graph.

In the limit \( u, v \to 0 \) with \( v/u \to 1 \), which can be found using (5.14), (5.15), it is necessary to cancel the shadow singularities coming from the \( \mathcal{H}_{d-\eta_o}(u, v) \) term in (5.14). This cancellation is achieved by the first term in (5.15) if one requires \( \mu - \frac{1}{2} \eta_o = \eta \) which is satisfied for \( \eta = \mu - 1, \eta_o = 2 \), and then using the expression (4.13) for \( C(d - \eta_o) \) as well as (5.16) above we easily find

\[ g_O = 2 \left( 2\mu - 3 \right) g_{O\phi O}, \]

(5.17)

where \( g_{O\phi O}^2 \) is given by (4.16). Therefore, our assumption \( g_O^2 \sim O(1/N) \) is justified. Only the terms \( \propto u^{\eta} \) in \( \mathcal{K}_2(u, v) \) in (5.15) were used in obtaining (5.17). The terms \( \propto u^{\eta_o} \) in \( \mathcal{K}_2(u, v) \) correspond to contributions in the four-point function from a scalar field appearing in the OPE’s (2.26) and (5.3), having dimension 4 to leading order in \( 1/N \).

Using (5.17) then we find from (5.13)

\[ Y(u, v) = 1 + g_{O\phi O} g_O \mathcal{H}_2(u, v) + \frac{1}{N} \frac{4\mu(\mu - 1)}{(2\mu - 1)} \left( \frac{1}{4} \left( 1 - \frac{v}{u} \right)^2 - \frac{1}{2\mu} \left( uv \right)^{\frac{1}{2}} \right) + \cdots. \]

(5.18)

The last term in (5.18) corresponds exactly to the contribution of the energy momentum tensor in the OPE’s (2.26) and (5.4) with \( C_T \) given to leading order by (3.7) or (4.21) again, which is another consistency check for our approach.

To discuss \( Y(u, v) \) in the limit as \( x_{13}^2, x_{24}^2 \to 0 \) or \( v/u \to 0 \) and \( v \to 1 \), it is necessary to find an alternative expansion for \( \mathcal{K}_1(u, v) \) and \( \mathcal{K}_2(u, v) \). This is achieved in much the same way as the one used in Appendix 3 to obtain different expansions for the one-particle exchange.
amplitudes $g_1^{(\eta_o)}$. Of course, the resulting expressions should be equal to the corresponding ones given in (5.14) and (5.15), but they are appropriate in considering the limit as $v/u \to 0$ and $v \to 1$. We obtain

$$
\mathcal{K}_1(u, v) = g_{\phi\phi O} g_{O} v^{\frac{1}{2}\eta_o} \sum_{n,m=0}^{\infty} \frac{(\frac{v}{u})^n (1-v)^m}{n!m!} a_{nm}[-\ln \frac{v}{u} + b_{nm}],
$$

(5.19)

$$
\mathcal{K}_2(u, v) = g_{\phi\phi O}^2 \left[ u^{\frac{1}{2}\eta_o} E_{\eta,\eta} \left( \frac{1}{u} \right) + C(\eta, \eta_o) u^{\eta+\frac{1}{2}\eta_o-\mu} E_{d-\eta,\eta-\eta_o} \left( \frac{1}{u} \right) \right],
$$

(5.20)

$$
\mathcal{K}_2(v, u) = g_{\phi\phi O}^2 v^n \sum_{n,m=0}^{\infty} \frac{(\frac{v}{u})^n (1-v)^m}{n!m!} C_{nm}[-\ln \frac{v}{u} + D_{nm}],
$$

(5.21)

with the same $C(\eta, \eta_o)$ as in (5.16). The coefficients $a_{nm}$ and $b_{nm}$ are given by (B.14) and (B.15) while $C_{nm}$ and $D_{nm}$ are given by (D.3) and (D.4) respectively.

Discussion of this limit requires more care. The important point is that $C(\eta, \eta_o)$ in (5.20) as given by (5.16) is $O(N)$ since it contains the factor $\Gamma(\eta - \mu)$ in the numerator. To leading order

$$
C(\eta, \eta_o) \to -\frac{N}{\eta_1} \frac{(\mu - 2)^2}{\mu(\mu - 1)} = -\frac{1}{g_{\phi\phi O}^2}.
$$

(5.22)

Therefore, $g_{\phi\phi O}^2 C(\eta, \eta_o) = -1 + O(1/N)$ and this will generate terms which are $O(1)$. These are essential to ensure the cancellation of terms corresponding to the shadow field of $\phi^\alpha(x)$ which has dimension $d - \eta$, and hence obtain the correct structure for the OPE (5.6). Writing in (5.15)

$$
u^{\eta+\frac{1}{2}\eta_o-\mu} = 1 + \frac{1}{N}(\eta_1 + \frac{1}{2}\eta_o)\ln u + \cdots,
$$

(5.23)

then in this limit $\mathcal{K}_2(u, v) \to -1 + O(1/N)$ which ensures the cancellation of the first term on the r.h.s. of (5.13). In order to cancel the $O(1/N) \ln u$ terms present in (5.19)-(5.21) it is necessary that

$$
g_{\phi\phi O}^2 (\mu - 2)^2 \eta_1 + \frac{1}{2}\eta_o = 0.
$$

(5.24)

It is easily seen that, by virtue of (4.16), (4.24), (4.32) and the results in Appendix D, (5.24) requires (5.17) again. Agreement of the two approaches of evaluating the four-point function

---

8We only consider the $n = m = 0$ contributions in the series in (5.19)-(5.21) which suffice for our purposes. Discussion of the $n, m \neq 1$ contributions requires the introduction of more quasiprimary fields in the OPE (5.6).
(5.1) is essentially a check on the associativity of the OPE. It is perhaps interesting to note that $g_O$ in (5.17) vanishes for $d = 3$, at least to leading order in $1/N$, which implies the possible existence of a discrete symmetry $O(x) \leftrightarrow -O(x)$ in this dimension in the sector generated by $O(x)$.

For completeness we briefly discuss the free field theory of section 3. In this case we take

$$O(x) = \frac{1}{\sqrt{2N}} :\phi^2(x):,$$  \hspace{1cm} (5.25)

since we set to one the normalisation of the two-point function of $O(x)$. Therefore, from Wick’s theorem we easily find

$$Y(u,v)_f = 1 + \frac{2}{N}(u^\eta + v^\eta).$$  \hspace{1cm} (5.26)

In the limit $u, v \to 0$ with $u/v \to 1$ we find

$$Y(u,v)_f = 1 + \frac{2}{N}v^\eta(2 + \eta(1 - \frac{v}{u}) + \cdots).$$  \hspace{1cm} (5.27)

Requiring agreement of (5.27) with (5.5) we obtain

$$\eta_o = 2\eta, \quad g_O = 2g_{\phi\phi O} = 2\sqrt{\frac{2}{N}},$$  \hspace{1cm} (5.28)

which is consistent with free field theory and $g_{\phi\phi O}^2$ in (3.6).

To discuss the limit $v/u \to 0, \quad v \to 1$ of (5.26) one needs to introduce an additional quasiprimary field in the free field theory version of the OPE (5.6), namely

$$\phi^\alpha(x_1)O(x_3) = g_{\phi\phi O} \frac{1}{x_{13}}\phi^\alpha(x_1) + g_{\phi OF} F^\alpha(x_1) + \cdots.$$  \hspace{1cm} (5.29)

The new $O(N)$ vector $O(d)$ scalar field $F^\alpha(x)$ having dimension $3\eta$ may be taken to be

$$F^\alpha(x) = C :\phi^\alpha(x)\phi^2(x):.$$  \hspace{1cm} (5.30)

Normalising to one its two-point function requires $2(N+2)C^2 = 1$ which in turn implies that $g_{\phi OF} = \sqrt{(N+2)/N}$. Then, by virtue of (5.29), the relevant terms of $Y(u,v)_f$ in the limit as $v/u \to 0, \quad v \to 1$ are

$$Y(u,v)_f = 1 + \frac{2}{N} + \frac{2}{N}u^\eta + \cdots,$$  \hspace{1cm} (5.31)
which agrees with the corresponding limit of (5.26).

Regarding the possible duality property of the $O(N)$ vector model we remark that one could have built the non-trivial graphs in the skeleton expansion in Fig. 2 for $Y(x_1, x_2, x_3, x_4)$ using dashed lines corresponding to the shadow field of $O(x)$. Then, the dark blobs would correspond to the coupling $\lambda_\ast$ and the crossed circle to a coupling $\lambda_O$ while $\eta_o \equiv \eta_o' = d - 2$ to leading order in $1/N$. In this case one can show that $\lambda_O = 2\lambda_\ast$ at least to leading order in $1/N$. But this is exactly the relationship (5.28) between the couplings $g_{\phi\phi O}$ and $g_O$ in the free field theory of section 3. This is in accord with our conjecture that the theory dual to the $O(N)$ vector model is somehow related to the theory of $N$ massless free scalars.

Clearly, one can discuss many more consistency checks regarding the non-trivial $O(N)$ invariant CFT in $2 < d < 4$. For example, one may try to reproduce our next-to-leading order in $1/N$ result (4.34) for $C_T$ by introducing the energy momentum tensor into the OPE (5.3). Our formalism can also be applied to discussing other four-point functions such as $\langle OOOO \rangle$ or those involving conserved currents. We leave these investigations for future developments of our approach.

6 Discussion and Concluding Remarks

The results of the present work may have possible applications to other conformally invariant field theories in $d > 2$. Starting from the OPE ansatz (2.26) in a $O(N)$ invariant CFT we were able to find explicit expressions for the four-point function of the fundamental field $\phi^\alpha(x)$ in a suitably chosen short-distance limit. However, these expressions contain undetermined dynamical parameters of the theory. Motivated by the form of the ansatz (2.26), we assumed that the four-point function of $\phi^\alpha(x)$ can alternatively be calculated using a skeleton graph expansion with internal lines corresponding to the two-point functions of the fields $\phi^\alpha(x)$ and $\tilde{O}(x)$ where the latter is a scalar field of dimension $\tilde{\eta}_o$ with $0 < \tilde{\eta}_o < d$. To low orders only the interaction of $\phi^\alpha(x)$ with $\tilde{O}(x)$ via a unique vertex having coupling constant $g_\ast$ is essential and our assumption was shown to be valid when $\tilde{O}(x)$ is identified with the field $O(x)$ in the OPE (2.26) and $g_\ast^2 = g_{\phi\phi O}^2$. Since a consistent skeleton graphical expansion for $n$-point functions with $n \geq 4$ requires the introduction of the triple $O(x)$ vertex, we have also discussed a four-point function involving the scalar field $O(x)$ and we showed that our approach of requiring consistency of algebraically and graphically based evaluations of correlation functions is consistent in this case as well. This last case provided a non-trivial check for the associativity property of OPE’s in $d > 2$. Our treatment of the four-point
functions may be considered as an application of the well known bootstrap program of CFT [18, 19, 20] and as such we expect that it is also applicable [22] in the case of other known examples of non-trivial CFT’s in $2 < d < 4$, like the Four-Fermi or the Gross-Neveu model in the $1/N$ expansion [20, 21] at its critical point.

Requiring consistency of the graphical expansions with the algebraic treatments of four-point functions resulted in a set of consistency relations which determine the dynamical parameters of the theory at least within the context of a $1/N$ expansion. More specifically, our results (4.24) and (4.32) for $\eta$ and $\eta_0$ agree with the well known results (e.g.see [10] and references therein) for the anomalous dimensions of the fundamental and auxiliary fields respectively in the $O(N)$ sigma model at its critical point which exists in the context of a $1/N$ expansion for $2 < d < 4$. We conclude that the $O(N)$ sigma model and the non-trivial theory considered in section 4 coincide since they seem to have the same field algebra. Therefore, we may identify the auxiliary field $\sigma(x)$ which enforces the constraint $\phi^2(x) = 1$ in the $O(N)$ sigma model, with the field $O(x)$ which appears in the OPE (2.26), e.g. both these fields have dimension close to 2 for large $N$. Note that for large $N$ the field algebras of the trivial and the non-trivial $O(N)$ invariant CFT’s are quite different since in the former the low dimension $< d$ scalar field which appears in the OPE (2.26) has dimension close to $d - 2$. Consequently, these two theories correspond to different universality classes in $2 < d < 4$ although strong evidence indicates that they coincide for $d = 4$. The results (4.33) for the coupling $g_{\phi\phi\phi}$ and (4.23) for $C_J$ agree with previously obtained results in [11].

The duality property of the graphical expansion mentioned at the end of section 4.2 is another novel result of the present work. It seems that the non-unitary field theory which underlies the graphical expansion corresponding to the interaction of $\phi^2(x)$ with $\tilde{O}_s(x)$ via a unique vertex having coupling constant $\lambda_*$ as given in (4.36), is related to the free field theory of section 3.

Apart from these general considerations it is of interest to discuss our next-to-leading order in $1/N$ results for the important quantities $C_J$ and $C_T$. The normalisation of the energy momentum tensor two-point function $C_T$ has been considered [4, 23] as one possible generalisation of the two-dimensional Virasoro central charge in higher-dimensional CFT’s. Similarly, (e.g.see [24]) one may view $C_J$ as one possible generalisation in higher dimensions of the Kac-Moody algebra level of a two-dimensional conformal WZW model. These generalisations refer to the property of both the central charge and the algebra level to be the fixed point values of important quantities which are monotonously decreasing along the renormalisation group flow from UV to IR fixed points of unitary two-dimensional quantum
field theories [24]. Such a property of the corresponding $d > 2$ quantities $C_J$ and $C_T$ is no longer true in general in higher dimensions. However, it is well known [25] that each order in the $1/N$ expansion includes contributions from all orders in the usual weak coupling perturbation expansion. Hence, our next-to-leading order in $1/N$ results (4.34) and (4.25) give non-perturbative information for the renormalisation group flow from the UV fixed point (free theory of section 3) to the IR one (non-trivial theory of section 4), when looked at from the point of view of weak coupling expansions.

In Fig. 6 we plot $C_{T,1}$ for $2 < d < 4$ and by virtue of (4.21) we see that for this range of dimensions $C_T$ at the UV fixed point (gaussian theory) is always greater than $C_T$ at the IR fixed point (non-trivial theory) at least to next-to-leading order in a $1/N$ expansion. A similar conclusion can be drawn for $C_J$ based on the plot of $C_{J,1}$ for $2 < d < 4$ in Fig. 7.

It is also interesting to consider the case when $d \to 4$ and to compare our results for $C_T$ and $C_J$ with known results in the context of $\epsilon$-expansion when $\epsilon = 4 - d > 0$. In four dimensions, if the theory is defined for a background metric $g_{\mu\nu}$ and has a gauge field $A_\mu^a$ coupled to the conserved vector current, even for a conformal theory there is a trace anomaly.
Figure 7: $C_{J,1}$ for $2 < d < 4$.

\[ g^{\mu \nu} \langle T_{\mu \nu} \rangle = -\beta_a F - \kappa \frac{1}{4} F_{\mu \nu}^\alpha F^{\alpha, \mu \nu} + \cdots, \]  

(6.1)

where $F$ is the square of the Weyl tensor and terms which are irrelevant here are neglected. The quantities $\beta_a$ and $\kappa$ can be perturbatively calculated and for a $O(N)$ invariant renormalisable field theory with $\frac{1}{24} g (\phi^2)^2$ interaction, a three loop calculation yields

\[ \beta_a = -\frac{1}{16\pi^2} \frac{N}{120} \left(1 - \frac{5}{108} (N + 2) u^2\right), \]  

(6.2)

\[ \kappa = \frac{1}{3} \frac{1}{16\pi^2} R \left(1 - \frac{1}{12} (N + 2) u^2\right), \]  

(6.3)

where $u = g/16\pi^2$ with $g$ the renormalised coupling and $\text{tr}(t^\alpha t^\beta) = -\delta^{\alpha \beta} R$. For the adjoint representation of $O(N)$ we have $(t^\alpha t^\beta)^\gamma = -(\delta^{\alpha \gamma} \delta^{\beta \delta} - \delta^{\alpha \delta} \delta^{\beta \gamma})$ and $R = 2$. The results of our previous work \[7\] show that for a conformal theory when $d = 4$

\[ C_T = -\frac{640}{\pi^2} \beta_a , \quad C_J = \frac{6}{\pi^2} \kappa. \]  

(6.4)
In general we suppose that we may write $C_T(\epsilon, u_\ast)$, $C_J(\epsilon, u_\ast)$ where $u_\ast$ is the critical coupling. The free or Gaussian field theory results (3.7), (3.9) correspond to $C_{T,f} = C_T(\epsilon, 0)$ and $C_{J,f} = C_J(\epsilon, 0)$ while (6.4) gives $C_T(0, u)$ and $C_J(0, u)$. Using then (6.2) and (6.3) with $u_\ast = 3\epsilon/(N + 8) + O(\epsilon^2)$ [25] gives the leading corrections in the $\epsilon$-expansion

$$C_T = C_{T,f} \left(1 - \frac{5}{12} \frac{N + 2}{(N + 8)^2} \epsilon^2 + O(\epsilon^3)\right),$$  \hfill (6.5)

$$C_J = C_{J,f} \left(1 - \frac{3}{4} \frac{N + 2}{(N + 8)^2} \epsilon^2 + O(\epsilon^3)\right).$$  \hfill (6.6)

As $d \to 4$ we see from (4.24) that $\eta_1 \sim \epsilon^2/4$ and then we can easily show that our results (4.34) and (4.25) agree correspondingly with (6.5) and (6.6), something which is a remarkable independent check for their validity at least up to the order considered here.

Finally, it was shown in [27] that $C_T$ parametrises universal finite size effects of statistical systems at their critical points in two dimensions which provides a natural method for its measurement both numerically and experimentally. For $d > 2$, although Cardy [27] has pointed out that $C_T$ may be in principle measurable, the finite scaling of the free energy is parametrised [27, 28] by a universal number $\tilde{c}$ whose relation with $C_T$ is not clear. Sachdev [28] has calculated $\tilde{c}$ for the $O(N)$ vector model to leading order in $1/N$ for $d = 3$ and found it to be a rational number however different from the leading order in $1/N$ value for $C_T$ [27]. Using our results (4.34) and (4.25) we obtain for $d = 3$

$$C_T|_{d=3} = N \frac{3}{2S_3^2} \left(1 - \frac{1}{N} \frac{40}{9\pi^2}\right),$$  \hfill (6.7)

$$C_J|_{d=3} = \frac{2}{S_3^2} \left(1 - \frac{1}{N} \frac{32}{9\pi^2}\right).$$  \hfill (6.8)

Noting from (1.3) and (2.34) that with our normalisation $C_T$ has to be multiplied by $S_d^2/2$ to agree with the corresponding quantity in [28], we can answer by virtue of (6.7) part of the question addressed in that reference: $C_T$ does not seem to be a rational number for finite $N$ in three dimensions.

---

9The result (1.3) for $N = 1$ was found in [8].

10It is interesting to point out that the leading order in $1/N$ value for $C_T$ coincides with the Gaussian theory value $C_{T,f}$ in any dimension whereas as shown in [28] the leading order in $1/N$ and the Gaussian values for $\tilde{c}$ differ in $2 < d < 4$. 
Acknowledgements

I would like to thank my supervisor Hugh Osborn for much guidance and helpful suggestions during the preparation of this manuscript and also for providing me with a copy of [31].

Appendix

A Scalar Field Contribution to the Operator Product Expansion

We describe here details of the calculation leading to (2.35). Using (2.30) we readily find

\[ C^{\eta_0}(x_{12}, \partial_2)C^{\eta_0}(x_{34}, \partial_4) \frac{1}{x_{24}^{\eta_0}} \]

\[ = \frac{1}{\left[ B\left( \frac{1}{2}\eta_0, \frac{1}{2}\eta_0 \right) \right]^2} \]

\[ \times \int_0^1 dt ds [t(1-t)s(1-s)]^{\frac{1}{2}\eta_0 - 1} \left( \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(\eta_0 + 1 - \mu)_k} \left[ -\frac{1}{4} x_{12}^2 t(1-t) \partial_2^2 \right]^k \right) \]

\[ \times \left( \sum_{l=0}^{\infty} \frac{1}{l!} \frac{1}{(\eta_0 + 1 - \mu)_l} \left[ -\frac{1}{4} x_{34}^2 t(1-t) \partial_4^2 \right]^l \right) \]

\[ \frac{1}{[tx_1 + (1-t)x_2 - sx_3 - (1-s)x_4]^{2\eta_0}} \]  

(A.1)

Next we write

\[ [tx_1 + (1-t)x_2 - sx_3 - (1-s)x_4]^2 = \Lambda^2 - A^2 - B^2, \quad (A.2) \]

where

\[ \Lambda^2 = tsx_{12}^2 + t(1-s)x_{14}^2 + s(1-t)x_{23}^2 + (1-t)(1-s)x_{24}^2, \quad (A.3) \]

\[ A^2 = x_{12}^2 t(1-t), \quad (A.4) \]

\[ B^2 = x_{34}^2 s(1-s). \quad (A.5) \]

Using (2.32) and the notation above we may cast (A.1) into the following form

\[ \frac{1}{\left[ B\left( \frac{1}{2}\eta_0, \frac{1}{2}\eta_0 \right) \right]^2} \int_0^1 dt ds [t(1-t)s(1-s)]^{\frac{1}{2}\eta_0 - 1} \left[ \frac{1}{\Lambda^2 - A^2 - B^2} \right]^{\eta_0} \]

\[ \times \sum_{k,l=0}^{\infty} \frac{1}{k!!l!!} \frac{(\eta_0)_k + (\eta_0 + 1 - \mu)_k}{(\eta_0 + 1 - \mu)_k} \left[ \frac{-A^2}{\Lambda^2 - A^2 - B^2} \right]^k \left[ \frac{-B^2}{\Lambda^2 - A^2 - B^2} \right]^l \]
\[
\begin{align*}
&= \frac{1}{[B(\frac{1}{2}\eta_0, \frac{1}{2}\eta_0)]^2} \int_0^1 dt ds[t(1-t)s(1-s)]^{\frac{1}{2}\eta_0-1} \left[ \frac{1}{A^2 - A^2 - B^2} \right]^\eta_0 \\
&\times F_4 \left( \eta_0, \eta_0 + 1 - \mu, \eta_0 + 1 - \mu, \eta_0 + 1 - \mu; \frac{-A^2}{A^2 - A^2 - B^2}, \frac{-B^2}{A^2 - A^2 - B^2} \right), \tag{A.6}
\end{align*}
\]

where

\[
F_4(a, b; c, d; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m(b)_n}{m!n!} x^m y^n,
\tag{A.7}
\]
is one of the Appell functions (hypergeometric functions of two variables). Then, by virtue of the following property \[29\]

\[
F_4(a, b; b, b; x, y) = (1 - x - y)^{-a} \binom{a}{2} \binom{1}{b} \binom{b + 1}{b} \frac{4xy}{(1 - x - y)^2}, \tag{A.8}
\]

(A.6) becomes

\[
\frac{1}{[B(\frac{1}{2}\eta_0, \frac{1}{2}\eta_0)]^2} \times \int_0^1 dt ds[t(1-t)s(1-s)]^{\frac{1}{2}\eta_0-1} \frac{1}{A^2} F_1 \left( \frac{1}{2}\eta_0, \frac{1}{2}(\eta_0 + 1); \eta_0 + 1 - \mu; \frac{4A^2B^2}{A^4} \right). \tag{A.9}
\]

Next, we use the following representation for the hypergeometric function

\[
_2F_1(a, b; c; z) = \frac{1}{2\pi i} \Gamma(c) \Gamma(a) \Gamma(b) \int_{-\infty}^{\infty} dx \Gamma(-x) \frac{\Gamma(a + x)\Gamma(b + x)}{\Gamma(c + x)} (-z)^x, \tag{A.10}
\]

obtaining

\[
\frac{1}{B^2(\frac{1}{2}\eta_0, \frac{1}{2}\eta_0)} \times \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \Gamma(-x) \left( \frac{\Gamma(\frac{1}{2}\eta_0 + x)\Gamma(\frac{1}{2}(\eta_0 + 1) + x)\Gamma(\eta_0 + 1 - \mu)}{\Gamma(\eta_0 + 1 - \mu + x)\Gamma(\frac{1}{2}\eta_0)\Gamma(\frac{1}{2}\eta_0 + \frac{1}{2})} \right)^x \times \int_0^1 dt ds \frac{[t(1-t)s(1-s)]^{\frac{1}{2}\eta_0+x-1}}{tx^2_{13} + t(1-s)x^2_{14} + s(1-t)x^2_{23} + (1-s)(1-t)x^2_{24}}. \tag{A.11}
\]

We may then successively integrate over \(t\) and \(s\) using

\[
\int_0^1 dt \ t^{a-1}(1-t)^{b-1}[tx + (1-t)y]^{-a-b} = x^{-a}y^{-b}B(a, b), \tag{A.12}
\]

\[
\int_0^1 ds \ s^{a-1}(1-s)^{b-1}(1-sx)^{-\rho}(1-sy)^{-\sigma} = B(a, b) F_1(a, \rho, \sigma; a + b; x, y), \tag{A.13}
\]

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and also
\[ F_1(a, b, c; b + c; x, y) = (1 - y)^{-a} F_1(a, b; b + c; \frac{x - y}{1 - y}), \] (A.14)
\[ 4^n \left( \frac{1}{2} \eta_o \right)_n \left( \frac{1}{2} \eta_o + \frac{1}{2} \right)_n = (\eta_o)_{2n}, \] (A.15)
to finally obtain
\[ C^{\eta_o}(x_{12}, \partial_2) C^{\eta_o}(x_{34}, \partial_4) \frac{1}{x_{24}^{2\eta_o}} = \frac{1}{(x_{12} x_{24})^{\frac{1}{2} \eta_o}} \mathcal{H}_{\eta_o}(u, v), \] (A.16)
where
\[ \mathcal{H}_{\eta_o}(u, v) = \left( \frac{v}{u} \right)^{\frac{1}{2} \eta_o} \sum_{n=0}^{\infty} \frac{v^n}{n!} \left( \frac{1}{2} \eta_o \right)_n \left( \eta_o + 1 - \mu \right)_n 2 F_1 \left( \frac{1}{2} \eta_o + n, \frac{1}{2} \eta_o + n; \eta_o + 2n; 1 - \frac{v}{u} \right), \] (A.17)
which is the result used in the text.

**B The One-Particle Exchange Graphs** \( G_1^{(\tilde{\eta}_o)} \)

After simple manipulations as described in the text we obtain the amplitude
\[ G_1^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4) = \int d^d x d^d y \langle \phi(x_1) \phi(x_2) \tilde{O}(x) \rangle \Phi_{\tilde{O}}^{-1}(x - y) \langle \phi(x_3) \phi(x_4) \tilde{O}(y) \rangle \]
\[ = \frac{g_s^2}{C_{\tilde{O}}^{(\tilde{\eta}_o)}} \rho(\tilde{\eta}_o) U(\frac{1}{2} \tilde{\eta}_o, \frac{1}{2} \tilde{\eta}_o, d - \tilde{\eta}_o) \frac{1}{(x_{12})^{\eta - \frac{1}{2} \tilde{\eta}_o} (x_{34})^{\eta + \frac{1}{2} \tilde{\eta}_o}} \]
\[ \times S_4(x_1, \frac{1}{2} \tilde{\eta}_o; x_2, \frac{1}{2} \tilde{\eta}_o; x_3, \mu - \frac{1}{2} \tilde{\eta}_o; x_4, \mu - \frac{1}{2} \tilde{\eta}_o), \] (B.1)
where we have used once the D’EPP formula (4.6) and have defined
\[ S_4(x_1, a_1; x_2, a_2; x_3, a_3; x_4, a_4) \]
\[ = \int d^d x \frac{1}{(x_1 - x)^{2a_1}(x_2 - x)^{2a_2}(x_3 - x)^{2a_3}(x_4 - x)^{2a_4}}, \] (B.2)
which is conformally invariant if \( a_1 + a_2 + a_3 + a_4 = d \). In (B.1), from (1.3) and (1.7)
\[ \pi^\mu \rho(\tilde{\eta}_o) U(\frac{1}{2} \tilde{\eta}_o, \frac{1}{2} \tilde{\eta}_o, d - \tilde{\eta}_o) = \frac{\Gamma(\tilde{\eta}_o) \Gamma(\mu - \frac{1}{2} \tilde{\eta}_o)}{\Gamma(\mu - \tilde{\eta}_o) \Gamma(\frac{1}{2} \tilde{\eta}_o)} \] (B.3)
Formula (B.1) clearly demonstrates the “shadow symmetry” property for the graphs \( G_1^{(\tilde{\eta}_o)} \) since we easily see that
\[
G_1^{(d - \tilde{\eta}_o)}(x_1, x_2, x_3, x_4) = C(\tilde{\eta}_o) G_1^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4),
\]
where the quantity \( C(\tilde{\eta}_o) \) was defined in (4.9). The integral (B.4) can be performed in various ways when \( a_1 + a_2 + a_3 + a_4 = d \). Here we apply Symanzik’s [30] method for its evaluation. Using standard procedures we obtain
\[
S_4(x_1, a_1; x_2, a_2; x_3, a_3; x_4, a_4)
= \frac{\pi^\mu}{\prod_{i=1}^4 \Gamma(\alpha_i)} \int_0^\infty d\lambda_1 \ldots d\lambda_4 \prod_{i=1}^4 \left[ \lambda_i^{\alpha_i - 1} \right] \left[ S_\lambda \right]^{-\mu} \exp \left[ -\frac{1}{S_\lambda} \sum_{j>i=1}^4 \left( \lambda_i \lambda_j x_{ij} \right) \right],
\]
where
\[
S_\lambda = \sum_{i=1}^4 \lambda_i.
\]

The conformal invariance of (B.5) manifests itself in the fact that on the r.h.s one may replace \( S_\lambda \) with \( \sum_{i=1}^4 c_i^2 \lambda_i \) and obtain the same result for arbitrary positive \( c_i^2, i = 1, \ldots, 4 \). Following Symanzik we then choose \( c_1^2 = 1 \) and \( c_i^2 = 0 \), for \( i = 2, 3, 4 \), or \( S_\lambda = \lambda_1 \). Next, using the representation
\[
\exp[-z] = \frac{1}{2}\pi i \int_{c-i\infty}^{c+i\infty} ds \Gamma(-s) z^s, \quad (c < 0, \arg z < \frac{1}{2}\pi),
\]
for the two exponentials involving \( x_{24}^2 \) and \( x_{34}^2 \), we can perform the \( \lambda \)-integrations obtaining
\[
S_4(x_1, a_1; x_2, a_2; x_3, a_3; x_4, a_4)
= \frac{\pi^\mu}{\prod_{i=1}^4 \Gamma(\alpha_i)} \frac{1}{(x_{23}^2)^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \mu} (x_{14}^2)^{\mu - a_4} (x_{34}^2)^{\alpha_3 + a_4 - \mu} (x_{24}^2)^{a_2 + a_4 - \mu}}
\times \left( \frac{1}{2\pi i} \right)^2 \int_{c-i\infty}^{c+i\infty} dtds \left( \Gamma(-s) \Gamma(-t) \Gamma(a_3 + a_4 - \mu - t) \right)
\times \Gamma(a_2 + a_4 - \mu - s) \Gamma(\mu - a_4 + s + t) \Gamma(a_1 + s + t) \left( \frac{v}{u} \right)^s v^t,
\]
where
\[
v = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}, \quad \frac{v}{u} = \frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{23}^2}.
\]
The s-integration can be performed using

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma(a + s) \Gamma(b + s) \Gamma(c - a - b - s) \Gamma(-s)(1 - z)^s
\]

\[= \Gamma(c - a) \Gamma(b - a) \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} {}_2F_1(a, b; c; z), \tag{B.10}\]

and the remaining t-integration results in the sum of two infinite series of poles of the Gamma functions \(\Gamma(-t)\) and \(\Gamma(a_3 + a_4 - \mu - t)\) situated at the points \(t = n\) and \(t = a_3 + a_4 - \mu + n\) respectively with \(n = 0, 1, 2, \ldots\) The general result, valid for \(a_1 + a_2 + a_3 + a_4 = d\), reads

\[
S_4(x_1, a_1; x_2, a_2; x_3, a_3; x_4, a_4) = \frac{\pi^\mu}{\prod_{i=1}^{4} \Gamma(a_i)} \frac{1}{(x_{23}^2)_{a_3-\mu}(x_{14}^2)_{a_1} (x_{34}^2)_{a_3+a_4-\mu}(x_{24}^2)_{a_2+a_4-\mu}} \times 
\left( \Gamma(\mu - a_1 - a_2) \sum_{n=0}^{\infty} \frac{v^n}{n!} \frac{\Gamma(1 - \mu + a_1 + a_2) \Gamma(\mu - a_3 + n) \Gamma(a_2 + n)}{\Gamma(1 - \mu + a_1 + a_2 + n)} \times \gamma(\mu - a_4 + n) \Gamma(a_1 + n) \Gamma(a_1 + a_2 + 2n) \right) \times \Gamma(\mu - a_2 + n) \times \Gamma(\mu - a_3 + n) \Gamma(\mu - a_3 + a_4) \Gamma(a_4 + n) \Gamma(\mu - a_1 + n) \Gamma(a_3 + n)
\]

\[\times \Gamma(\mu - a_2 + n) {}_2F_1(a_3 + n, \mu - a_2 + n; a_3 + a_4 + 2n; 1 - \frac{v}{u}) \right) \]. \tag{B.11}\]

Note that we have conducted the calculation leading to (B.11) so that one can unambiguously consider the limits as \(uv \to 0\), \(1 - v/u \to 0\) independently. The amplitude for the graph \(G_1(\tilde{\eta}_0)(x_1, x_2, x_3, x_4)\) is obtained using (B.11) when \(a_1 = a_2 = \frac{1}{2}\tilde{\eta}_0\) and \(a_3 = a_4 = \mu - \frac{1}{2}\tilde{\eta}_0\). The result is

\[
G_1(\tilde{\eta}_0)(x_1, x_2, x_3, x_4) = \frac{g_5^2}{C_{\tilde{\eta}_0}} \frac{1}{(x_{12}^2 x_{23}^2)^{\frac{1}{4}}} \left[ u^{\frac{1}{2} \tilde{\eta}_0} \mathcal{H}_{\tilde{\eta}_0}(u, v) + C(d - \tilde{\eta}_0) u^{\mu - \frac{1}{2} \tilde{\eta}_0} \mathcal{H}_{d - \tilde{\eta}_0}(u, v) \right], \tag{B.12}\]

with \(\mathcal{H}(u, v)\) defined in (2.30).

The amplitude for the graph \(G_1(\tilde{\eta}_0)(x_1, x_4, x_3, x_2)\) is obtained using (B.11) when \(a_1 + a_2 - \mu = \Delta = \mu - a_3 - a_4\) in the limit \(\Delta \to 0\). In this case, the Gamma functions \(\Gamma(\mu - a_1 - a_2)\) and \(\Gamma(\mu - a_3 - a_4)\) have poles \(\pm \frac{1}{2\Delta}\) which cancel between the two terms in (B.11). The finite result reads

\[
G_1(\tilde{\eta}_0)(x_1, x_4, x_3, x_2) = \frac{g_5^2}{C_{\tilde{\eta}_0}} \frac{1}{(x_{12}^2 x_{23}^2)^{\eta}} \sum_{n,m=0}^{\infty} \frac{v^n}{n! m!} (1 - \frac{v}{u})^m a_{nm} [-\ln v + b_{nm}], \tag{B.13}\]

39
\[ G_2^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4) = \left( \frac{c_s^2}{c_G c_o} \right)^2 \rho^2(\eta) \rho^2(\tilde{\eta}_o) U^2(\frac{1}{2} \tilde{\eta}_o, \frac{1}{2} \tilde{\eta}_o, d - \tilde{\eta}_o) \times \]

Figure 8: The amplitude for \( G_2^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4) \).

where the overall coefficient is given in (B.3) and

\[ a_{nm} = \pi \rho(\tilde{\eta}_o) U(\frac{1}{2} \tilde{\eta}_o, \frac{1}{2} \tilde{\eta}_o, d - \tilde{\eta}_o) \frac{(\frac{1}{2} \tilde{\eta}_o)_n (\mu - \frac{1}{2} \tilde{\eta}_o)_n (\frac{1}{2} \tilde{\eta}_o)_n + (\mu - \frac{1}{2} \tilde{\eta}_o)_n + m}{\Gamma(1 + n) \Gamma(\mu + 2 n + m)}, \quad (B.14) \]

\[ b_{nm} = 2 \psi(1 + n) + 2 \psi(\mu + m + 2 n) - \psi(\frac{1}{2} \tilde{\eta}_o + n) - \psi(\mu - \frac{1}{2} \tilde{\eta}_o + n) - \psi(\frac{1}{2} \tilde{\eta}_o + n + m) - \psi(\mu - \frac{1}{2} \tilde{\eta}_o + n + m). \quad (B.15) \]

The amplitude for the graph \( G_1^{(\tilde{\eta}_o)}(x_1, x_3, x_2, x_4) \) is simply obtained from (B.13) by the interchange \( x_3 \leftrightarrow x_4 \) or \( u \leftrightarrow v \). The results in this Appendix are in accord with similar results given in [10].

C The Box-Graphs \( G_2^{(\tilde{\eta}_o)} \)

Using the diagrammatic representations in Fig. 3 for the amputation of the three-point function \( \langle \phi \phi \tilde{O} \rangle \) in the \( \tilde{O} \) leg we obtain the amplitude shown in Fig. 8 with

\[ a_1 = \eta - \frac{1}{2} \tilde{\eta}_o, \quad (C.1) \]
\[ a_2 = \frac{1}{2} \tilde{\eta}_o, \quad (C.2) \]
\[ a_3 = \mu - \frac{1}{2} \tilde{\eta}_o, \quad (C.3) \]
\[ a_4 = \eta + \frac{1}{2} \tilde{\eta}_o - \mu, \quad (C.4) \]
\[ a_5 = d - \eta. \quad (C.5) \]

The lines in Fig. 8 represent powers of the separation with exponent twice the attached numbers given in (C.1)-(C.5). The internal vertices are simple integration vertices. It is then clear that the shadow symmetry property

\[ G_2^{(d-\tilde{\eta}_o)}(x_1, x_2, x_3, x_4) = [C(\tilde{\eta}_o)]^2 G_2^{(\tilde{\eta}_o)}(x_1, x_2, x_3, x_4), \quad (C.6) \]
can be easily proved in general on account of the symmetry property

\[ G_2(\tilde{\eta}_o)(x_1, x_2, x_3, x_4) = G_2(\tilde{\eta}_o)(x_3, x_4, x_1, x_2). \] (C.7)

The graph in Fig. 8 has not been evaluated for general values of the exponents \( a_1, \ldots a_5 \). However, in certain cases it reduces to a simpler box-graph for which an analytic expression in closed form has been found using techniques similar to the ones applied for the evaluation of the one-particle exchange graphs in Appendix B. Such is the case when

\[ \tilde{\eta}_o = d - 2\eta \implies a_4 = 0. \] (C.8)

Then, using only D'EPP formula (4.6) we find the equation shown in Fig. 9. Moreover, it can be seen either from the shadow symmetry property (C.6) or by direct calculation that the graph in Fig. 8 reduces to a simple box-graph with unique vertices when

\[ \tilde{\eta}_o = 2\eta \implies a_1 = 0 \] (C.9)

The box-graph in Fig. 9 has been evaluated by Lang [31]. In the context of the present work and to the order in \( 1/N \) we are interested, we just need the value of this box-graph when \( \tilde{\eta}_o = 2 \). Putting it all together and using the general result of [31] we find

\[
G_2(\tilde{\eta}_o)(x_1, x_2, x_3, x_4) = \left( \frac{g_2^2}{C_\phi C_{\tilde{O}}} \right)^2 A(d) \frac{1}{(x_{12}^2 x_{34}^2)^{\mu-1}}
\]

11An integration vertex made out of lines which are powers of the separation is called unique if the sum of the exponents of these lines is \( 2d \). One can easily prove that scalar amplitudes which correspond to graphs made out of power lines and unique vertices behave like conformal scalars (i.e.see (1.1)) under conformal transformations of the external coordinates.
\[ \times v^{\mu-1} \sum_{n,m=0}^{\infty} \frac{v^n(1 - \frac{\mu}{v})^m}{n!m!} [-c_{nm}\ln v + d_{nm}], \quad (C.10) \]

with

\[
c_{nm} = \pi^{4\mu} \frac{n!\Gamma(3 - \mu)\Gamma^2(\mu - 1 + n + m)}{(\mu - 2)^2\Gamma(\mu - 2)\Gamma^3(\mu - 1)\Gamma(\mu + m + 2n)}
\times \sum_{s=0}^{n} \frac{1}{s!} \frac{\Gamma(\mu - 2 + s)\Gamma(\mu - 1 + n + m + s)}{\Gamma(2\mu - 3 + n + m + s)}, \quad (C.11)\]

\[
d_{nm} = -\pi^{4\mu} \frac{\Gamma(\mu - 2)\Gamma^2(\mu - 1 + n + m)}{\Gamma^6(\mu - 1)}
\times \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{1}{r!s!(n-r-s)!} \frac{\Gamma(1 + n - r)\Gamma(\mu - 2 + r)\Gamma(\mu - 1 + n + m + s)}{\Gamma(1 - r)\Gamma(\mu + m + 2n - r)}
\times \frac{\Gamma(\mu - 2 + r + s)\Gamma(1 + n - r - s)\Gamma(3 - \mu - r)}{\Gamma(2\mu - 3 + n + m + r + s)}
\times \left[ \psi(\mu - 1 + n + m + s) + \psi(1 + n - r) + \psi(1 - r) 
+ \psi(1 + n - r - s) + 2\psi(\mu - 1 + n + m) - \psi(3 - \mu - r) - 2\psi(1 + n) 
- \psi(2\mu - 3 + n + m + r + s) - 2\psi(\mu + m + 2n - r) \right], \quad (C.12)\]

where

\[
A(d) = \left[ \rho(\mu - 1) \rho(2) U(1, 1, 2\mu - 2) U(\mu + 1, 1, \mu - 2) \right]^2 
\quad = \frac{1}{\pi^{4\mu}} \left[ (\mu - 2)\Gamma(\mu - 1) \right]^4. \quad (C.13)\]

The amplitude for the graph \( G_2^{(\eta_0)}(x_1, x_2, x_3, x_4) \) is obtained from \( G_2^{(\eta_0)}(x_1, x_2, x_3, x_4) \) by the interchange \( x_3 \leftrightarrow x_4 \leftrightarrow u \leftrightarrow v \). Note that in evaluating \( d_{10} \) the \( r = 1 \) contribution does not vanish despite the presence of \( \Gamma(1 - r) \) in the denominator of \( (C.12) \), by virtue of

\[
\frac{1}{\Gamma(1 - r)} \psi(1 - r) = \frac{1 - r}{\Gamma(2 - r)} \psi(2 - r) - \frac{1}{\Gamma(2 - r)}, \quad (C.14)\]

and we obtain

\[
d_{10} = \pi^{4\mu} \frac{\Gamma(3 - \mu)\Gamma(\mu - 2)(\mu - 1)^2(\mu^2 - 2)}{\Gamma(\mu)\Gamma(2\mu - 2)2\mu(\mu + 1)(\mu - 2)^3} \left( \frac{2\mu^3 + 3\mu^2 - 5\mu - 4}{\mu(\mu + 1)(\mu^2 - 2)} + C(\mu) \right), \quad (C.15)\]

where \( C(\mu) \) is defined in \( (4.31) \).
The Graphs in the Four-Point Function $\langle \phi \phi O O \rangle$

For the graphs shown in Fig. 5, after simple manipulation as described in the text, we obtain

$$K_1(x_1, x_2, x_3, x_4) = g\phi\phi O \rho(\eta_0) U(\tfrac{1}{2} \eta_0, \tfrac{1}{2} \eta_0, d - \eta_0) \frac{1}{(x_{12}^2)^{\mu - \frac{1}{2} \eta_0} (x_{34}^2)^{\mu - \frac{1}{2} \eta_0}}$$
$$\times S_4(x_1, \frac{1}{2} \eta_0; x_2, \frac{1}{2} \eta_0; x_3, \mu - \frac{1}{2} \eta_0; x_4, \mu - \frac{1}{2} \eta_0).$$  \hspace{1cm} (D.1)

The limit $x_{12}^2, x_{34}^2 \to 0$ leads to (5.14). The limit $x_{13}^2, x_{24}^2 \to 0$ corresponds to the case explained just before (B.13) when poles appear in the evaluation of $S_4$ which however cancel giving as finite result (5.19).

Similarly

$$K_2(x_1, x_2, x_3, x_4) = g^2\phi\phi O \rho(\eta) U(\eta - \frac{1}{2} \eta_0, \frac{1}{2} \eta_0, d - \eta) \frac{1}{(x_{13}^2)^{\frac{1}{2} \eta_0} (x_{24}^2)^{\eta - \frac{1}{2} \eta_0}}$$
$$\times S_4(x_1, \eta - \frac{1}{2} \eta_0; x_2, \mu - \frac{1}{2} \eta_0; x_3, \frac{1}{2} \eta_0; x_4, \mu - \eta + \frac{1}{2} \eta_0).$$  \hspace{1cm} (D.2)

Three different limits are relevant here. When $x_{12}^2, x_{34}^2 \to 0$ we obtain (5.15). When $x_{13}^2, x_{24}^2 \to 0$ we obtain (5.20) while the case when $x_{14}^2, x_{23}^2 \to 0$ corresponds to a cancellation of poles in the integral as before and leads to the finite result (5.21) with

$$C_{nm} = \pi^m \rho(\eta) U(\eta - \frac{1}{2} \eta_0, \frac{1}{2} \eta_0, d - \eta) \frac{(\mu - \eta + \frac{1}{2} \eta_0)_n (\frac{1}{2} \eta_0)_n (\eta - \frac{1}{2} \eta_0)_n (\mu - \frac{1}{2} \eta_0)_n}{\Gamma(1 + n) \Gamma(\mu + 2n + m)},$$

$$D_{nm} = 2\psi(1 + n) + 2\psi(\mu + 2n + m) - \psi(\mu - \eta + \frac{1}{2} + n) - \psi(\frac{1}{2} \eta_0 + n)$$
$$- \psi(\eta - \frac{1}{2} \eta_0 + n + m) - \psi(\mu - \frac{1}{2} \eta_0 + n + m).$$  \hspace{1cm} (D.3)

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