Consistency of Equations in the Second-order Gauge-invariant Cosmological Perturbation Theory

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Along the general framework of the gauge-invariant perturbation theory developed in the papers [K. Nakamura, Prog. Theor. Phys. 110 (2003), 723; ibid, 113 (2005), 481.], we re-derive the second-order Einstein equations on four-dimensional homogeneous isotropic background universe in gauge-invariant manner without ignoring any mode of perturbations. We consider the perturbations both in the universe dominated by the single perfect fluid and in that dominated by the single scalar field. We also confirmed the consistency of all equations of the second-order Einstein equation and the equations of motion for matter fields which are derived in the paper [K. Nakamura, arXiv:0804.3840 [gr-qc]]. This confirmation implies that the all derived equations of the second order are self-consistent and these equations are correct in this sense.

§1. Introduction

The general relativistic second-order cosmological perturbation theory is one of topical subjects in the recent cosmology. By the recent observation, the first order approximation of the fluctuations of our universe from a homogeneous isotropic one was revealed, the cosmological parameters are accurately measured, we have obtained the standard cosmological model, and so-called “the precision cosmology” has begun. The observational results also suggest that the fluctuations of our universe are adiabatic and Gaussian at least in the first order approximation. We are now on the stage to discuss the deviation from this first order approximation from the observational and the theoretical side through the non-Gaussianity, the non-adiabaticity, and so on. To carry out this, some analyses beyond linear order are required. The second-order cosmological perturbation theory is one of such perturbation theories beyond linear order.

Although the second-order perturbation theory in general relativity is old topics, a general framework of the gauge-invariant formulation of the second-order general relativistic perturbation are recently proposed by the present author. We refer these works as KN2003 and KN2005 in this paper. Further, this general framework was also applied to cosmological perturbations. We demonstrated the derivation of the second-order perturbation of the Einstein equation in gauge-invariant manner without any gauge fixing. We also showed that the above general framework is also applicable when we discuss the second-order perturbations of the equation of motion for matter fields. We also refer these works as KN2007 and KN2008. This gauge-invariant formulation of second-order cosmological perturbations is a natural extension of the first-order gauge-invariant cosmological perturbation theory.

In this paper, we re-derive all components of the second-order perturbations of
the Einstein equations without ignoring any modes of perturbations. In the first-order cosmological perturbation theory, the perturbations are classified into three types which are called the scalar-, the vector-, and the tensor-mode, respectively. In KN2007, we ignore the vector- and tensor-modes of the first order when we derive the second-order perturbations of the Einstein equation. These modes are taken into account in this paper and we show the precise mode-coupling of the three types of the first-order perturbations in the case of the universe filled with a perfect fluid and with a scalar field, respectively. In particular, in the case of a perfect fluid, we see that the any types of mode-coupling appear in the second-order perturbations of the Einstein equations, in principle.

We also confirm the consistency of all equations of the second-order Einstein equation and the equations of motion for matter fields which are derived in KN2008. Further, due to the fact that the Einstein equations are the first class constrained system, we have initial value constraints in the Einstein equations. Moreover, since the Einstein equations include the equation of motion for matter fields, the second-order perturbations of the equations of motion for matter fields are not independent equations of the second-order perturbation of the Einstein equations. Through these facts, we can check whether the derived equations of the second order are consistent or not. In this paper, we do check this consistency. Namely, we show that the second-order perturbations of the equations of motion for the matter field are consistent with the second-order perturbations of the Einstein equations through the background and the first-order perturbations of the Einstein equations. This confirmation implies that the all derived equations of the second order are self-consistent and these equations are correct in this sense.

The explicit derivations of the second-order perturbations of the Einstein equations without ignoring any mode are not only for the cosmological perturbations but also for the post-Minkowski expansion for a binary system. At least in the cosmological perturbations with the single matter field, it is well-known that the vector-mode of the first-order perturbation is a just decaying mode. Further, the tensor-mode whose wavelength is shorter than the Hubble horizon size also decays due to the expansion of the universe, though the tensor-mode whose wavelength is longer than the horizon size is frozen. Therefore, in many situations in cosmology, we may neglect these modes, safely. However, in this paper, we dare to include these modes in our considerations in spite that the expressions of the second-order perturbations of the Einstein equations become very complicated. The explicit expressions of the second-order perturbations of the Einstein equations are reduced to those for Minkowski background spacetime if we neglect the expansion of universe. Thus, formulae to derive the explicit Einstein equations of the second order will be also useful to reconsider the post-Minkowski expansion for a binary system in gauge-invariant manner. Of course, we have to reconsider the regularization procedure to treat the self-gravity of the point particles within a gauge-invariant manner to complete discussions of the post-Minkowski expansion for a binary system.

The organization of this paper is as follows. In §2 we briefly review the definitions of the gauge-invariant variables for the second-order perturbation which was defined by KN2007. We also summarize the components of the gauge-invariant
part of the first- and the second-order perturbations of the Einstein tensor in the cosmological perturbations in this section. We did not ignore any modes in these formulae. Further, we briefly review the background Einstein equations and the equations of motion for the matter field in §3 and its first-order perturbations in §4. The ingredients of these sections are used in §5. In §5, we show the explicit expression of the second-order perturbation of the Einstein equations without ignoring any modes of perturbations and check the consistency with the second-order perturbations of the equations of motion for matter field. The final section, §6, is devoted to the summary and discussions.

We employ the notation of our series of papers KN2003, KN2005, KN2007, and KN2008 and use the abstract index notation. We also employ the natural unit in which the light velocity is denoted by $c = 1$ and denote the Newton’s gravitational constant by $G$.

§2. Perturbations of Einstein tensor in terms of gauge-invariant variables

2.1. Gauge invariant variables

In any perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime $\mathcal{M} = \mathcal{M}_\lambda$ and the other is the background spacetime $\mathcal{M}_0$. Since these two spacetime manifolds are distinct from each other, we have to introduce a point-identification map $\mathcal{X}_\lambda : \mathcal{M}_0 \rightarrow \mathcal{M}_\lambda$. This point-identification map $\mathcal{X}$ is called a gauge choice in perturbation theories. Through the pull-back $\mathcal{X}^*\lambda$ of the gauge choice $\mathcal{X}_\lambda$, any physical variable $\hat{Q}_\lambda$ on the physical manifold $\mathcal{M}_\lambda$ is pulled back to a representation $\mathcal{X}^*_\lambda \hat{Q}_\lambda$ on the background spacetime $\mathcal{M}_0$. It is important to note that the gauge choice $\mathcal{X}_\lambda$ is not unique by virtue of general covariance in general relativity. When we have two different gauge choices $\mathcal{Y}_\lambda$ and $\mathcal{X}_\lambda$, we can consider the gauge transformation rule from a gauge choice $\mathcal{X}_\lambda$ to another one $\mathcal{Y}_\lambda$ through the diffeomorphism $\Phi_\lambda := (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda$. The pull-back $\Phi_\lambda^*\mathcal{X}$ of the diffeomorphism $\Phi_\lambda$ does change the representation $\mathcal{X}^*_\lambda \hat{Q}_\lambda$ of the physical variable $\hat{Q}_\lambda$ to another representation as $\mathcal{Y}_\lambda^*\hat{Q}_\lambda = \Phi_\lambda^*\mathcal{X}^*_\lambda \hat{Q}_\lambda$.

The pull-back $\mathcal{X}^*_\lambda \hat{Q}_\lambda$ is expanded as

$$\mathcal{X}^*_\lambda \hat{Q}_\lambda = Q_0 + \lambda^{(1)} Q + \frac{1}{2} \lambda^{(2)} Q + O(\lambda^3). \quad (2.1)$$

The first- and the second-order perturbations $^{(1)} Q$ and $^{(2)} Q$ are defined by this equation (2.1). For example, we expand the pulled-back $\mathcal{X}^*_\lambda \hat{g}_{ab}$ of the metric $\hat{g}_{ab}$ on the physical spacetime $\mathcal{M}_\lambda$ by the gauge choice $\mathcal{X}_\lambda$:

$$\mathcal{X}^*_\lambda \hat{g}_{ab} = g_{ab} + \lambda^2 h_{ab} + \frac{\lambda^2}{2} M_{ab} + O(\lambda^3), \quad (2.2)$$

where $g_{ab}$ is the metric on the background spacetime $\mathcal{M}_0$. In the case of the cosmological perturbations, we consider the homogeneous isotropic spacetime whose metric
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is given by

\[ g_{ab} = a^2 \left( -(d\eta)_a (d\eta)_b + \gamma_{ij} (dx^i)_a (dx^j)_b \right), \tag{2.3} \]

where \( \gamma_{ab} := \gamma_{ij} (dx^i)_a (dx^j)_b \) is the metric on the maximally symmetric three space and the indices \( i, j, k, \ldots \) for the spatial components run from 1 to 3. Henceforth, we do not explicitly express the index of the gauge choice \( X_\lambda \) in expressions if there is no possibility of confusion.

From the generic form of the Taylor expansion of the pull-back \( \Phi^* := Y^* \circ (X^*)^{-1} \), we can easily derive the gauge transformation rule of each order:

\[
\begin{align*}
(1) \quad &Y^Q - (1) X^Q = \xi^{(1)}_Q 0, \\
(2) \quad &Y^Q - (2) X^Q = 2 \xi^{(1)} (1) X^Q + \left\{ \xi^{(2)} + \xi^{2(1)} \right\} Q_0, \tag{2.4, 2.5}
\end{align*}
\]

where \( \xi^a_1 \) and \( \xi^a_2 \) are generators of the diffeomorphism \( \Phi^* \). Inspecting these gauge transformation rules \((2.4)\) and \((2.5)\), we consider the notion of the order by order gauge invariance. We call the \( p \)th-order perturbation \( (p) X^Q \) is gauge invariant iff

\[ \gamma^{(p) Q} = \gamma^X Q \tag{2.6} \]

for any gauge choice \( X_\lambda \) and \( Y_\lambda \). Employing this idea of order by order gauge invariance for each-order perturbations, we proposed a procedure to construct gauge invariant variables of higher-order perturbations in KN2003.

In the cosmological perturbation case, we can show that the first-order metric perturbation \( h_{ab} \) is decomposed as

\[ h_{ab} =: H_{ab} + \mathcal{L}_X g_{ab}, \tag{2.7} \]

where \( H_{ab} \) and \( X^a \) are the gauge-invariant and gauge-variant parts of the linear-order metric perturbations,\(^5\) i.e., under the gauge transformation \((2.4)\), these are transformed as

\[ \gamma H_{ab} - \chi H_{ab} = 0, \quad \gamma X^a - \chi X^a = \xi^a_1. \tag{2.8} \]

As shown in KN2007, the decomposition \((2.7)\) is accomplished if we assume the existence of the Green functions \( \Delta^{-1} := (D^i D_i)^{-1}, (\Delta + 2K)^{-1}, \) and \( (\Delta + 3K)^{-1} \), where \( D_i \) is the covariant derivative associated with the metric \( \gamma_{ij} \) on the maximally symmetric three space and \( K \) is the curvature constant of this maximally symmetric three space. Further, we may choose the components of the gauge-invariant part \( H_{ab} \) of the first-order metric perturbation as

\[
H_{ab} = a^2 \left\{ -2 (1) \Psi (d\eta)_a (d\eta)_b + 2 (1) \nu_i (dx^i)_a (dx^j)_b \right. \\
\left. + \left( -2 (1) \Psi \gamma_{ij} + (1) \chi_{ij} \right) (dx^i)_a (dx^j)_b \right\}, \tag{2.9}
\]

where \( \nu^i_1 \) and \( \chi_{ij} \) satisfy the properties

\[ D^i (1) \nu^i_1 = \gamma^{ij} D_i (1) \nu^j_1 = 0, \quad (1) \nu^i_1 = 0, \quad D^i (1) \chi_{ij} = 0. \tag{2.10} \]
where $\gamma^{kj}$ is the inverse of the metric $\gamma_{ij}$.

If the decomposition is true, we can easily show that the second-order metric perturbation $l_{ab}$ is also decomposed as

$$l_{ab} = \mathcal{L}_{ab} + 2\mathcal{L}_X h_{ab} + (\mathcal{L}_Y - \mathcal{L}_X^2)g_{ab}, \quad \text{(2.11)}$$

where $\mathcal{L}_{ab}$ and $Y^a$ are the gauge-invariant and gauge-variant parts of the second-order metric perturbations, i.e.,

$$\gamma\mathcal{L}_{ab} - \chi\mathcal{L}_{ab} = 0, \quad Y^a - \chi Y^a = \xi^{(2)}_a + [\xi^{(1)}, X]^a. \quad \text{(2.12)}$$

Further, in the cosmological perturbation case, we may also choose the components of the gauge-invariant part $\mathcal{L}_{ab}$ of the second-order metric perturbation as

$$\mathcal{L}_{ab} = a^2 \left\{ -2 (2) \Phi (d\eta)_a (d\eta)_b + 2 (2) \nu_i (d\eta)_a (dx^i)_b \right. \left. + \left\{ -2 (2) \Psi \gamma \gamma + (2) \chi \right\} (dx^i)_a (dx^j)_b \right\}, \quad \text{(2.13)}$$

where $(2)\nu_i$ and $(2)\chi_{ij}$ satisfy the properties

$$D^i (2)\nu_i = \gamma_{ij} D^i (2)\nu_j = 0, \quad (2)\chi^i = 0, \quad D^i (2)\chi_{ij} = 0. \quad \text{(2.14)}$$

### 2.2. Components of the Perturbative Einstein tensor

As shown in KN2003, through the above first- and the second-order gauge-variant parts, $X^a$ and $Y^a$, of the metric perturbations, we can define the gauge-invariant variables for an arbitrary field $Q$ other than the metric. The definitions in KN2003 imply that the first- and the second-order perturbation $(1)Q$ and $(2)Q$ are always decomposed into gauge-invariant part and gauge-variant part as

$$(1)Q =: (1)Q + \mathcal{L}_X Q_0, \quad \text{(2.15)}$$

$$(2)Q =: (2)Q + 2\mathcal{L}_X (1)Q + \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} Q_0, \quad \text{(2.16)}$$

respectively. Here, $(1)Q$ and $(2)Q$ are gauge-invariant parts of the first- and the second-order perturbations of $(1)Q$ and $(2)Q$ in the sense of the above “order by order gauge invariance”, respectively.

To evaluate the perturbations of the Einstein equation, we expand the Einstein tensor $\bar{G}_a^b$ on the physical spacetime $\mathcal{M}$ so that

$$X^* \bar{G}_a^b = G_a^b + \lambda^{(1)} \bar{G}_a^b + \frac{1}{2} \lambda^2 (2)\bar{G}_a^b + O(\lambda^3). \quad \text{(2.17)}$$

As shown in KN2005, the first- and second-order perturbations of the Einstein tensor are given in the same form as (2.15) and (2.16):

$$(1)\bar{G}_a^b = (1)G_a^b [\mathcal{H}] + \mathcal{L}_X G_a^b, \quad \text{(2.18)}$$

$$(2)\bar{G}_a^b = (1)G_a^b [\mathcal{L}] + (2)G_a^b [\mathcal{H}, \mathcal{H}] + 2\mathcal{L}_X (1)G_a^b + \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} G_a^b \quad \text{(2.19)}$$
where

\[(1) G_a^b [A] := -2\nabla_a H_d^{bd} [A] - A^{ac} R_{ac} \]
\[+ \frac{1}{2} \delta_a^b \left( 2\nabla_c H_d^{cd} [A] + R_{cd} A^{cd} \right), \quad (2.20)\]
\[(2) G_a^b [A, A] := \Sigma_a^b [A] - \frac{1}{2} \delta_a^b \Sigma_c^c [A], \quad (2.21)\]
\[\Sigma_a^b [A] := 2R_{ad} A_c^b A^{dc} + 4H_{a \nu}^{dc} [A] H_d^{\nu e} [A] \]
\[+ 4A_e^d \nabla_{[a} H_d^{be} [A] + 4A_c^b \nabla_{[a} H_d^{cd} [A] \]
\[\quad (2.22)\]
and

\[H_{abc} [A] := \nabla_{(a} A_{b)c} - \frac{1}{2} \nabla_c A_{ab}, \quad H_a^{bc} [A] := g^{bc} H_{aced} [A], \quad H_a^{bc} [A] := g^{bc} H_{acd} [A], \quad (2.23)\]
for arbitrary tensor \(A_{ab}\) of the second rank. The terms \((1) G_a^b [\ast]\) in Eqs. (2.18) and (2.19) are the gauge-invariant parts of the perturbative Einstein tensors, which consists of the linear combinations of the gauge-invariant variables for the metric perturbations of the first- \((\mathcal{H}_{ab})\) or the second-order \((\mathcal{L}_{ab})\). The term \((2) G_a^b [\mathcal{H}, \mathcal{H}]\) in the second-order perturbation (2.19) of the Einstein tensor consists of the quadratic terms of the gauge-invariant part of the first-order metric perturbation.

2.2.1. Components of \((1) G_a^b\)

As shown in KN2007, the components of \((1) G_a^b = (1) G_a^b [\mathcal{H}]\) in Eq. (2.18) are given in terms of the gauge-invariant variables defined in Eq. (2.9) as follows:

\[(1) G_i^j = \frac{1}{a^2} \left\{ -6\mathcal{H} \partial_i [\Psi] + 2\Delta + 6K \right\} (1) \Psi - 6H^2 (1) \Phi, \quad (2.24)\]
\[(1) G_i^i = -\frac{1}{a^2} \left\{ 2\partial_i D_i (1) \Psi + 2\mathcal{H} D_i (1) \Phi - \frac{1}{2} (\Delta + 2K) \nu_i \right\}, \quad (2.25)\]
\[(1) G_i^j = \frac{1}{a^2} \left\{ 2\partial_i D^j (1) \Psi + 2\mathcal{H} D^j (1) \Phi + \frac{1}{2} (\Delta + 2K + 4\mathcal{H}^2 - 4\partial_i \mathcal{H}) \nu^j \right\}, \quad (2.26)\]
\[(1) G_i^j = \frac{1}{a^2} \left\{ -D_i D^j (1) \Phi + D_i D^j (1) \Psi + \left\{ (\Delta + 2\partial_i^2 + 4\mathcal{H} \partial_i - 2K) \Psi \right\} \right\} \gamma_i^j \]
\[- \frac{1}{2} \partial_i \left( D_i \nu^j + D^j \nu_i \right) - \mathcal{H} \right\} \gamma_i^j \]
\[+ \frac{1}{2} \left( \partial_i^2 + 2\mathcal{H} \partial_i + 2K - \Delta \right) \chi_i^j, \quad (2.27)\]

where \(\partial_i a / a\) and \(\gamma_i^j := \gamma_{ik} \gamma^{kj}\) is the three-dimensional Kronecker's delta.
The components of \((1) G_a^b [\mathcal{L}]\) in Eq. (2.19) in terms of the gauge-invariant variables
defined in Eq. (2.13) are given by the replacement of the variables
\[
(1) \Phi \rightarrow (2) \Phi, \quad (1) \nu_i \rightarrow (2) \nu_i, \quad (1) \Psi \rightarrow (2) \Psi, \quad (1) \chi_{ij} \rightarrow (2) \chi_{ij},
\]
in the equations (2.24)–(2.27).

2.2.2. Components of \((2) G^a_b\)

From Eq. (2.21) and the components of the gauge-invariant parts (2.9) of the first-order metric perturbation, and the components of tensors \(H^b_{\ ab}[\mathcal{H}]\) and \(H^b_{\ ac}[\mathcal{H}]\), which are summarized in Appendix A in the paper KN2007,\(^7\) we can derive the components of \((2) G^b_a = \langle 2 \rangle G^b_{\ a}[^2\mathcal{H}, \mathcal{H}]\) in a straightforward manner. These components are summarized as follows:

\[
(2) G^\eta_\eta = \frac{2}{a^2} \left[-3D_k (1) \psi D^k (1) \psi - 8 (1) \psi \Delta (1) \psi - 3 \left( \partial_\eta (1) \psi \right)^2 - 12K (1) \psi - 12\mathcal{H}^2 (1) \phi^2 \right.
\]

\[
-12\mathcal{H} \left( (1) \phi - (1) \psi \right) \partial_\eta (1) \psi
\]

\[
-2D^k \left\{ \partial_\eta (1) \psi + \mathcal{H} \left( (1) \phi + (1) \psi \right) \right\} (1) \nu^k
\]

\[
\frac{1}{2}D_k (1) \nu_i D^k (1) \nu^i + \frac{1}{2} \nu_k \left( \Delta + 2K + 6\mathcal{H}^2 \right) \nu^k
\]

\[
+ D_i D_k (1) \psi \chi^{ik}
\]

\[
- \frac{1}{2}D^k (1) \psi \chi^{ik} (\partial_\eta + 4\mathcal{H}) \psi (1) \chi_{ik}
\]

\[
+ \frac{1}{8} \partial_\eta \psi^{kl} (\partial_\eta + 8\mathcal{H}) \chi^{kl} + \frac{1}{2}D_k \psi^{lm} \chi^{lm} - \frac{1}{8}D_k \psi^{lm} D^k \chi^{lm} - \frac{1}{2} \chi^{lm} (\Delta - K) \psi (1) \chi_{lm}
\]

\[
\left(2\right) G^\eta_\iota = \frac{2}{a^2} \left[ 4\mathcal{H} \left( (1) \psi - (1) \phi \right) D^i (1) \phi + 2\partial_\eta (1) \psi D^i (1) \psi + 2 (1) \psi \partial_i \psi (1) \psi + 2 (1) \psi \partial_\iota \psi (1) \psi + 2 (1) \psi \Delta (1) \phi + 4\mathcal{H} (1) \psi \right.
\]

\[
+ \left( 2\partial_\iota (1) \psi - 2 (1) \psi \Delta + (1) \phi + 4\mathcal{H} (1) \psi \right) \psi^i - D_j (1) \phi D^i (1) \phi \psi^j
\]

\[
+ \left( 2\mathcal{H} D^i (1) \psi \psi^{ij} - \partial_\iota D^i (1) \psi \phi \psi^{ij} \right) \psi^j
\]

\[
- \frac{1}{2}D_j (1) \phi \partial_\iota \psi (1) \partial_\iota \psi^{ij} - \left( \partial_\iota D_j (1) \psi + 2\mathcal{H} D_j (1) \phi \right) \psi^{ij}.
\]
\[ + \frac{1}{2} \nu_j \left( \partial_{\eta}^2 + 2\mathcal{H} \partial_{\eta} + 4\partial_{\eta} \mathcal{H} - 4\mathcal{H}^2 - 2K \right) \chi^{ji} + D_k \nu_j \chi^{kj} \\
+ D^j D^k \nu^i \chi_{klj} + \frac{1}{2} \Delta \nu_j \chi^{ij} \\
+ \frac{1}{4} \partial_{\eta} \chi_{jk} D^i \chi^{kj} + \frac{1}{2} \partial_{\eta} \partial_{\eta} D^i \chi^{kj} \right], \quad (2.30) \]

\[ \left(2 \mathcal{G}^\eta_{\ i} \right) = \frac{2}{a^2} \left[ 8\mathcal{H} \phi D_i \phi + 2D_i \left( \phi - 2 \psi \right) \partial_{\eta} \left( \psi \phi \right) + 4 \left( 1 - \psi \right) \partial_{\eta} D_i \psi \right. \\
\left. - D^j \phi D_i \nu_j + D^j \psi D_j \nu_i + \left( \psi - \phi \right) \left( \Delta + 2K \right) \nu_i \right. \\
\left. + \nu^j \left( D_i D_j + \gamma_{ij} \Delta \right) \psi \right. \\
\left. - 2\mathcal{H} \nu^j D_i \nu_j \right. \\
\left. + \frac{1}{2} D_j \left( \phi + \psi \right) \partial_{\eta} \chi^{ij} - \partial_{\eta} D^j \psi \chi_{ij} \right. \\
\left. + D_k D_i \nu_j \chi^{kj} + D^k \nu^j \left( D_j \chi_{ik} - \frac{1}{2} \nu_j \left( \Delta - 2K \right) \chi^{ji} \right) \right. \\
\left. - \frac{1}{4} \partial_{\eta} \chi_{kj} D_i \chi_{kj} + \chi_{kj} \partial_{\eta} D_j \chi_{ij} \right], \quad (2.31) \]

\[ \left(2 \mathcal{G}^j_{\ i} \right) = \frac{2}{a^2} \left[ D_i \phi D^j \left( \phi - \psi \right) - D_i \psi D^j \left( \phi - 3 \psi \right) + 4 \psi D_i D^j \psi \right. \\
\left. + 2 \left( 1 - \phi \right) D_i D^j \phi \right. \\
\left. + \left\{ -D_k \phi D^k \phi - 2D_k \psi D^k \psi - 2 \left( \phi - \psi \right) \Delta \phi \right. \\
\left. - 4 \psi \left( \Delta + K \right) \psi + \partial_{\eta} \psi \partial_{\eta} \psi \partial_{\eta} \phi \partial_{\eta} \psi \right. \\
\left. - 8\mathcal{H} \partial_{\eta} \phi \partial_{\eta} \phi - 8\mathcal{H} \left( \phi - \psi \right) \partial_{\eta} \phi \right. \\
\left. - 4 \left( 2\partial_{\eta} \mathcal{H} \right) \left( \frac{1}{\phi} \right)^2 \right\} \gamma_{ij} \right. \\
\left. + \frac{1}{2} \partial_{\eta} \left( \psi + \phi \right) \left( D_i \nu^j + D^j \nu_i \right) - D^j \psi \partial_{\eta} \nu_i + \partial_{\eta} D_i \psi \nu^j \right. \\
\left. + \left( \phi - \psi \right) \left( \partial_{\eta} + 2\mathcal{H} \right) \left( D_i \nu^j + D^j \nu_i \right) - D_i \psi \left( \partial_{\eta} + 2\mathcal{H} \right) \nu^j \right] \]
\[-\frac{1}{\nu_i} (\partial_\eta + 2\mathcal{H}) D^j \psi + 2\mathcal{H} D_i \Phi \nu^j - 2D_k \left( \partial_\eta \psi + \mathcal{H} \Phi \right) \nu^k \gamma_i^j \]
\[+ \frac{1}{2} \nu^k D_k \left( D^j \nu_i + D_i \nu^j \right) - \frac{1}{\nu_k} D_i D^j \nu^k - \frac{1}{2} D_i \nu_k D^j \nu^k \]
\[-\frac{1}{2} D^k \nu_i D_k \nu^j - \frac{1}{2} \nu^j \left( \Delta + 2K \right) \nu_i \]
\[+ \left\{ \nu_k \left( 2\mathcal{H} \partial_\eta + \Delta + 2\partial_\eta \mathcal{H} + \mathcal{H}^2 \right) \nu^k \right\} \]
\[+ \frac{1}{2} D_k \nu_l \left( D^k \nu^l + D^k \nu^j \right) \gamma_i^j \]
\[+ \left( \nu^l \nu^j \right) \left( \frac{1}{2} \partial_\eta^2 \mathcal{H} \partial_\eta \chi_i^j - \frac{1}{2} \partial_\eta \chi_i^j \partial_\eta \left( \frac{1}{2} \Phi - \Psi \right) \right) \]
\[+ \chi_i^j \left( \partial_\eta^2 + 2\mathcal{H} \partial_\eta \right) \frac{1}{2} D_k \nu_l \left( D^k \nu^l + \nu^l \right) \left( D_i \chi^j + D^j \chi_{ik} \right) \]
\[-\frac{1}{2} D^k \nu^j \left( \Phi + 3 \ Psi \right) D_k \chi_i^j - 2 \Psi \left( \Delta - 2K \right) \chi_i^j - \Delta \left( \Phi - \Psi \right) \chi_i^j \]
\[+ D_k D_i \Phi \chi^{jk} + D^m D^j \Psi \chi_{im} - D_l D_k \Phi \chi_{ik}^j \gamma_i^j \]
\[+ \frac{1}{2} \left( \partial_\eta + 2\mathcal{H} \right) \left( \nu_k D_i \chi^{kj} + \nu^k D^j \chi_{ki} - \nu^k D_k \chi_i^j \right) \]
\[+ \frac{1}{2} D^k \nu^j \partial_\eta \chi_{ik} + \frac{1}{2} D_k \nu_i \partial_\eta \chi^{jk} + \chi^{jk} \left( \partial_\eta + 2\mathcal{H} \right) D_i \nu_k \]
\[-\frac{1}{2} \nu^j \partial_\eta D_k \chi_i^j - \left\{ \chi_{ik} \partial_\eta D^k \nu^j + \frac{1}{2} D^k \nu^j \left( \partial_\eta + 4\mathcal{H} \right) \chi_{ik} \right\} \gamma_i^j \]
\[-\frac{1}{2} \partial_\eta \chi_{ik} \partial_\eta \chi^{jk} + D_k D_i \chi^{kj} + D^k \chi^{jl} \]
\[+ \frac{1}{2} \chi_{lm} D_i D^j \chi^{ml} - \frac{1}{2} \chi_{lm} D^j D_i \chi^{mj} - \frac{1}{2} \chi^l \chi_{lm} D_i D^j \chi_{mi} \]
\[+ \frac{1}{2} \chi_{lm} D_m D_i \chi_{nj} - \frac{1}{2} \chi^{lk} \left( \partial_\eta + 2\mathcal{H} \partial_\eta - \Delta + 2K \right) \chi_{ik} \]
\[+ \frac{1}{2} \left\{ \chi_{kl} \partial_\eta \partial_\eta \chi^{lk} - \frac{1}{2} D_k \chi_{lm} D^k \chi^{ml} + D_k \chi_{lm} D^l \chi^{km} \chi^{lj} \gamma_i^j \right\}. \tag{2.32} \]
In these components, the terms of different types of the mode-coupling are written in the different lines to show the different types of mode-coupling, manifestly. For example, in the expression of the component \((2)^{\eta \eta}\) in Eq. (2.29), the first two lines show the mode-coupling of the scalar-scalar type, the third line shows the scalar-vector mode-coupling, the fourth line shows the vector-vector type, the fifth line shows the scalar-tensor type, the sixth line shows vector-tensor type, and the seventh and eighth lines show the mode-coupling of the tensor-tensor type.

We have checked the perturbations of the Bianchi identity for the components \((2.24) - (2.27)\) and \((2.29) - (2.32)\) of the gauge-invariant parts \((1) G^a_b[H]\), \((1) G^a_b[L]\), and \((2) G^a_b[H, H]\). As shown in KN2005, \(^6\) the first- and the second-order perturbations of the Bianchi identity \(\bar{\nabla}^a \bar{G}^b_a = 0\) give the identities

\[
\nabla_a (1)^{\eta} G^a_b [H] = -H^a_c [H] G^c_b + H^c_a [H] G^a_b,
\]

\[
\nabla_a (2)^{\eta} G^a_b [H, H] = -2H^a_c [H] (1)^{\eta} G^c_b [H] + 2H^c_a [H] (1)^{\eta} G^a_c [H]
-2H_bde[H] \mathcal{H}^{de} G^a_c + 2H_cde[H] \mathcal{H}^{ad} G^a_d.
\]

Although we can check these identities without the explicit components of the gauge-invariant part of the metric perturbations as shown in KN2005, we can also check these identities \((2.33)\) and \((2.34)\) in the case of cosmological perturbation through the explicit components of the gauge-invariant parts \(H_{ab}\) and \(L_{ab}\) in Eqs. \((2.9)\) and \((2.13)\) of the metric perturbations. The actual calculations to confirm the identities \((2.33)\) and \((2.34)\) are straightforward. These checks of these Bianchi identities guarantee that the expressions \((2.24) - (2.27)\) of the components of \((1) G^a_b\) and the expressions \((2.29) - (2.32)\) of the components of \((2) G^a_b\) are self-consistent. In this sense, we may say that the formulae \((2.24) - (2.27)\) and \((2.29) - (2.32)\) are correct.

\section*{3. Consistency of the background equations}

In this section, we briefly review the Einstein equations and the equations for the matter field on a four-dimensional homogeneous isotropic universe whose metric is given by \((2.3)\). Further, we show the consistency between the equations of motion for matter field and the Einstein equations. These equations are used throughout this paper. As the matter contents, we consider a perfect fluid and a scalar field, respectively.

\subsection*{3.1. Perfect fluid case}

The energy momentum tensor for a perfect fluid on the background spacetime, whose metric is given by \((2.3)\), is given by Eq. \((A.14)\) (or Eq. \((A.15)\)). The Einstein equations \(G^a_b = 8\pi G T^a_b\) for this background spacetime filled with a perfect fluid are given by

\[
E^{(p)}_{(1)} := \mathcal{H}^2 + K - \frac{8\pi G}{3} \mathcal{H} a^2 \epsilon = 0,
\]

\[
E^{(p)}_{(2)} := 2\partial_\eta \mathcal{H} + \mathcal{H}^2 + K + 8\pi G a^2 p = 0.
\]
To consider the consistency of the perturbative equations, the equation
\[
\frac{1}{2} \left( 3 (p) E^{(0)}_{(1)} - (p) E^{(0)}_{(2)} \right) = H^2 + K - \partial_\eta H - 4\pi G a^2 (\epsilon + p) = 0 \tag{3.3}
\]
is also useful.

The divergence of the energy momentum tensor \(\text{(A.14)}\) gives the two equations, which are well-known as the energy continuity equation and the Euler equation. The Euler equation is trivial due to the fact that the pressure \(p\) is homogeneous (i.e., \(p = p(\eta)\)) and the integral curves of the fluid four-velocity \(u^a = g^{ab} u_b\) with the component \(\text{(A.8)}\) are geodesics on the background spacetime with the metric \((2.3)\). On the other hand, the energy continuity equation is given by
\[
a C_0^{(p)} := \partial_\eta \epsilon + 3H (\epsilon + p) = 0. \tag{3.4}
\]

Through Eqs. \((3.1)–(3.3)\), we easily verify that
\[
8\pi G a^3 C_0^{(p)} = -3\partial_\eta (p) E^{(1)} - 3H \left( (p) E^{(0)}_{(1)} - (p) E^{(0)}_{(2)} \right). \tag{3.5}
\]

We may say that the energy continuity equation for the background spacetime is consistent with the Einstein equations \((3.1)\) and \((3.2)\). We also note that the relation \((3.5)\) has nothing to do with the equation of state of the perfect fluid. This is a well-known fact and is just due to the Bianchi identity of the background spacetime. However, in §§4 and 5, we will show this kind consistency between the Einstein equations and the equations of motion for matter field in the case of the first- and the second-order perturbations. We use this kind of the relations to check the consistency of the set of perturbative equations.

3.2. Scalar field case

In this paper, we also consider the universe filled with a single scalar field. The energy momentum tensor for a scalar field on the homogeneous isotropic universe is given by Eq. \(\text{(A.31)}\) with the homogeneous condition \((\text{A.29)}\). Through this energy momentum tensor for a scalar field, the Einstein equations \(G_{ab} = 8\pi G T_{ab}\) for this background spacetime filled with a scalar field are given by
\[
^{(s)} E^{(0)}_{(1)} := \mathcal{H}^2 + K - \frac{8\pi G}{3} \left( \frac{1}{2} (\partial_\eta \varphi)^2 + a^2 V(\varphi) \right) = 0, \tag{3.6}
\]
\[
^{(s)} E^{(0)}_{(2)} := 2\partial_\eta \mathcal{H} + \mathcal{H}^2 + K + 8\pi G \left( \frac{1}{2} (\partial_\eta \varphi)^2 - a^2 V(\varphi) \right) = 0. \tag{3.7}
\]

We also note that these equations \((3.6)\) and \((3.7)\) lead
\[
\frac{1}{2} \left( 3 (s) E^{(0)}_{(1)} - (s) E^{(0)}_{(2)} \right) = \mathcal{H}^2 + K - \partial_\eta \mathcal{H} - 4\pi G (\partial_\eta \varphi)^2 = 0, \tag{3.8}
\]
\[ \frac{1}{2} \left( 3 (s) E_{(1)}^{(0)} + (s) E_{(2)}^{(0)} \right) = 2\mathcal{H}^2 + 2K + \partial_\eta \mathcal{H} - 8\pi Ga^2 V(\varphi) = 0. \quad (3.9) \]

The equations (3.8) and (3.9) are also useful when we check the consistency of equations for the first- and the second-order perturbations, respectively.

The divergence of the energy momentum tensor (A.30) gives the Klein-Gordon equation on the background spacetime. Further, the Klein-Gordon equation is also consistent with the background Einstein equations (3.6) and (3.7). This can be easily seen from the relation

\[ \frac{8\pi G}{3} \partial_\eta \varphi \left( a^2 C_K^{(0)} \right) = -\frac{8\pi G}{3} \partial_\eta \varphi \left( \partial^2 \varphi + 2\mathcal{H} \partial_\eta \varphi + a^2 \partial V / \partial \varphi \right) = \partial_\eta (s) E_{(1)}^{(0)} + \mathcal{H} \left( (s) E_{(1)}^{(0)} - (s) E_{(2)}^{(0)} \right). \quad (3.10) \]

Thus, the Klein-Gordon equation

\[ (s) C_K = 0 \quad (3.12) \]

for the background spacetime is consistent with the Einstein equations (3.6) and (3.7). Of course, this is a well-known fact and is just due to the Bianchi identity of the background spacetime as in the case of a perfect fluid. However, as we noted in the case of the perfect fluid, this kind of relations are useful to check whether the derived system of equations are consistent or not.

**§4. Consistency of the first-order equations**

Here, we consider the consistency of the first-order perturbations of the Einstein equations and equations of matter fields. The essence of this consistency check is similar to those for the background equations. However, these consistency checks for the first-order perturbations are similar to those for the second-order perturbations. Therefore, the consistency check for the set of the equations of the first order is instructive when we consider the set of equations for the second-order perturbations.

As shown in KN2007,\(^7\) we can derive the components of the first-order perturbation of the Einstein equation

\[ G^a_b [\mathcal{H}] = 8\pi G (1) \mathcal{T}_a^b \quad (4.1) \]

and the components of the equations of motion for matter field, which are derived in KN2008,\(^8\) As in the case of the background equations, we consider the cases for a perfect fluid (4.1.1) and a scalar field (4.1.2).

### 4.1. Perfect fluid case

Through the components of the linearized Einstein tensor Eqs. (4.2.1) and (4.2.7) and the components of the first-order perturbation of the energy momentum tensor
for a perfect fluid [Eqs. (A.19)–(A.22)], the components of the first-order Einstein equation (4.1) are summarized as

\( (p)E(1) \) := \(-3H\partial_\eta + \Delta + 3K\) \( (p) = -3H^2 \phi - 4\pi Ga^2 \mathcal{E} = 0 \), \hspace{1cm} (4.2)

\( (p)E(2) \) := \( (\partial_\eta^2 + 2H\partial_\eta - K - \frac{1}{3} \Delta) \phi + (H\partial_\eta + 2\partial_\eta H + H^2 + \frac{1}{3} \Delta) \phi \)

\(-4\pi Ga^2 (p) = 0 \), \hspace{1cm} (4.3)

\( (p)E(3) \) := \( (p) = 0 \), \hspace{1cm} (4.4)

\( (p)E(4) \) := \( (\partial_\eta D_i \psi + \mathcal{H} D_i \phi + 4\pi G (\epsilon + p) a^2 v = 0 \), \hspace{1cm} (4.5)

\( (p)E(5) \) := \( (\Delta + 2K) \) \( (p) = -16\pi G (\epsilon + p) a^2 V_i = 0 \), \hspace{1cm} (4.6)

\( (p)E(6) \) := \( (\partial_\eta \left( a^2 (p) \right) = 0 \), \hspace{1cm} (4.7)

\( (p)E(7) \) := \( (\partial_\eta^2 + 2H\partial_\eta + 2K - \Delta) \) \( (\chi_{ij}) = 0 \). \hspace{1cm} (4.8)

4.1.1. Continuity equations and Euler equations

As shown in KN2008, the first-order perturbation of the energy continuity equation in terms of the gauge-invariant variables is given by

\[ a^{(1)} \mathcal{C}_0^{(p)} = \partial_\eta \left( \mathcal{E} + \mathcal{H} (\mathcal{E} + \mathcal{P}) + (\epsilon + p) \left( (\Delta \psi - 3\partial_\eta \phi) = 0 \right. \] \hspace{1cm} (4.9)

Further, in terms of the gauge-invariant variables, the first-order perturbation of the Euler equation is given by

\[ a^{(1)} \mathcal{C}_i^{(p)} = (\epsilon + p) \left\{ (\partial_\eta + \mathcal{H}) (D_i \psi + V_i) + D_i \phi \right\} + D_i \mathcal{P} + \partial_\eta p \left( D_i \psi + V_i \right) = 0 \] \hspace{1cm} (4.10)

and this equation is decomposed into the scalar- and the vector-parts:

\[ a^{(1)} \mathcal{C}_i^{(pS)} := D_i \mathcal{D}^{-1} D_j^{(1)} \mathcal{C}_j^{(p)} \]

\[ = (\epsilon + p) \left\{ (\partial_\eta + \mathcal{H}) D_i \psi + D_i \phi \right\} + D_i \mathcal{P} + \partial_\eta p D_i \psi = 0 \] \hspace{1cm} (4.11)

\[ a^{(1)} \mathcal{C}_i^{(pV)} := (\epsilon + p) (\partial_\eta + \mathcal{H}) V_i + \partial_\eta p V_i \]

\[ = 0 \] \hspace{1cm} (4.12)
Now, we check the consistency of Eqs. (4.9), (4.11), and (4.12) for a perfect fluid and the Einstein equations (4.2)–(4.8). First, we consider the perturbation of the continuity equation (4.9). Substituting (3.3), (4.2), (4.3), and (4.5) into (4.9), we can easily verify the relation

\[
4\pi G a^3 (1)^{(p)} C_0 = -(\partial_\eta - 2\mathcal{H})(p)^{(1)} E_1 + 3\mathcal{H} (p)^{(1)} E_2 - \frac{1}{3} D_i (p)^{(1)} E_4 + 3 (p)^{(0)} E_1 - (p)^{(0)} E_2 \partial_\eta (1)^{\Psi}. \tag{4.13}
\]

Therefore, the first-order perturbation of the energy continuity equation is consistent with the background Einstein equations and the first-order perturbation of the Einstein equation.

Next, we consider the first-order perturbation (4.10) of the Euler equation. Through Eqs. (4.3)–(4.5), (3.4), and (3.3), we can easily derive the equation

\[
4\pi G a^2 (1)^{(p)} C_i = \partial_\eta (p)^{(1)} E_{(4)i} + 2\mathcal{H} (p)^{(1)} E_{(4)i} - D_i (p)^{(1)} E_2 - \frac{1}{3} D_i \left(\Delta + 3K\right)(p)^{(1)} E_3 - 4\pi G a^3 (p)^{(0)} C_0 D_i (p)^{(0)} E_{(5)i} - \frac{1}{2} \left(3 (p)^{(0)} E_1 - (p)^{(0)} E_2\right) D_i (1)^{\Phi}. \tag{4.14}
\]

Further, through Eqs. (4.6), (4.7), and (3.4), we can easily see that

\[
4\pi G a^2 (1)^{(p)} C_i = \frac{1}{4a^2} \left\{ -\partial_\eta \left(a^2 (p)^{(1)} E_{(5)i}\right) + (\Delta + 2K)(p)^{(1)} E_{(6)i}\right\}
- 4\pi G a^3 C_0^i (p)^{(1)} V_i. \tag{4.15}
\]

Since, the background energy continuity equation (3.4) is consistent with the background Einstein equation, Eqs. (4.14) and (4.15) show that the first-order perturbations (4.10) of the Euler equation are consistent with the set of the background and the first-order Einstein equations.

4.2. Scalar field case

Through Eqs. (2.24)–(2.27) and Eqs. (A.37)–(A.40), the linear-order Einstein equations are given as follows: For the scalar-mode, we have four equations:

\[
^{(s)}E_{(1)} := \partial_\eta^2 (1)^{\phi} + 2 \left(\mathcal{H} - \partial_\eta^2 \phi\right) \partial_\eta (1)^{\phi} - \Delta (1)^{\phi} + 2 \left(\partial_\eta \mathcal{H} - \mathcal{H} \partial_\eta \phi + 2K\right) (1)^{\phi} = 0, \tag{4.16}
\]

\[
^{(s)}E_{(2)} := \partial_\eta (1)^{\phi} + \mathcal{H} (1)^{\phi} - 4\pi G \partial_\eta \phi \phi_1 = 0, \tag{4.17}
\]

\[
^{(s)}E_{(3)} := \left(-\partial_\eta^2 - 6\mathcal{H} \partial_\eta + \Delta - 2\partial_\eta \mathcal{H} - 4\mathcal{H}^2 + 4K\right) (1)^{\phi} - 8\pi G a^3 \partial \phi \phi_1 = 0, \tag{4.18}
\]
For the vector-mode, we obtain
\[ (s)E^{(5)i} = \nu^i = 0. \]

For the tensor-mode, we obtain
\[ (s)E^{(6)ij} := \left( \partial^2_\eta + 2H\partial_\eta + 2K - \Delta \right) \chi_{ij} = 0. \]

These equations are also summarized in KN2007.\(^7\)

In Eqs. (4.16)–(4.18) for the scalar-modes, Eq. (4.16) determines the evolution of the scalar potential \( \Phi \) with appropriate boundary conditions. Equation (4.17) determines the behavior of the first-order perturbation \( \varphi_1 \) of the scalar field through the scalar potential \( \Phi \). Therefore, Eq. (4.18) is not necessary to solve this system. However, we can easily see that Eq. (4.18) is consistent with the set of two equations (4.16) and (4.17). Actually, from Eqs. (4.16), (4.17), and (3.10), Eq. (4.18) is given by

\[ (s)E^{(3)} = - (s)E^{(1)} - 2 \left( \frac{\partial^2 \varphi}{\partial^2_\eta} + 2H \right) (s)E^{(2)} + 8\pi G \varphi_1 a^2 C_K. \]

This shows that the Einstein equation (4.18) is not independent of the set of the equations Eqs. (4.16), (4.17) and the background Klein-Gordon equation (3.12). Since the background Klein-Gordon equation is derived from the set of the background Einstein equations as in Eq. (3.11), the equation (4.18) is satisfied through the background Einstein equations (3.6) and (3.7), the first-order perturbations (4.16) and (4.17) of the Einstein equations. Therefore to solve this system of the scalar-mode, we first solve Eq. (4.16) with an appropriate initial condition. These initial conditions gives the first-order perturbation of the scalar field \( \varphi_1 \) through the momentum constraint (4.17). Further, after these initial states, Eq. (4.17) gives the first-order perturbation \( \varphi_1 \) of the scalar field in terms of the scalar potential \( \Phi \) at any time. Thus, the free initial values for the linear-order perturbation of scalar-modes are only \( \Phi \) and \( \partial_\eta \Phi \) on the initial surface. There is no other degree of freedom for initial value in the scalar-mode.

4.2.1. Klein-Gordon equation

Here, we consider the first-order perturbation of the Klein-Gordon equation which given in KN2008:\(^8\)

\[ a^2 C^{(1)}_K = - \partial^2_\eta \varphi_1 - 2H\partial_\eta \varphi_1 + \Delta \varphi_1 + \partial_\eta \Phi \partial_\eta \varphi + 3\partial_\eta \Psi \partial_\eta \varphi \]
\[ + 4H \Phi \partial_\eta \varphi + 2 \Phi \partial^2_\eta \varphi - a^2 \varphi_1 \frac{\partial^2 V}{\partial \varphi^2} (\varphi). \]
\[
\begin{align*}
&= 3\partial_\eta \left(\varphi_1 \right) E_4 \partial_\eta \varphi - 2a^2 \Phi C_K \\
&- \partial_\eta^2 \varphi_1 - 2\mathcal{H}\partial_\eta \varphi_1 + \Delta \varphi_1 + 4\partial_\eta \Phi \partial_\eta \varphi \\
&- 2a^2 \Phi \partial_\eta^2 \varphi - a^2 \varphi_1 \partial_\eta^2 \varphi \\
&= 0, 
\end{align*}
\] (4.24)

where we have used the component (4.19) of the linearized Einstein equation and the background Klein-Gordon equations Eqs. (3.10).

Now, we check the consistency of the first-order perturbation (4.24) of the Klein-Gordon equation with the first-order perturbations (4.16)–(4.18) of the Einstein equations. We can easily derive the relation as follows:

\[
\begin{align*}
8\pi G (\partial_\eta \varphi) a^2 C_K &= -2 \left[ \partial_\eta + H \right] E_4 \\
+ 2 \left[ \partial_\eta^2 - 2 \left( \frac{\partial^2 \varphi}{\partial_\eta \varphi} - \mathcal{H} \right) \partial_\eta - \Delta - 2 \left( \frac{\partial^3 \varphi}{\partial_\eta^3 \varphi} - 2\partial_\eta H + 4\mathcal{H}^2 \right) \\
+ 2 \frac{\partial^2 \varphi}{\partial_\eta \varphi} \left( \frac{\partial^2 \varphi}{\partial_\eta \varphi} - \mathcal{H} \right) \right] E_2 \\
+ \left[ \left( \phi - 3\mathcal{H} \frac{\varphi_1}{\partial_\eta \varphi} \right) \right] \partial_\eta + 2 \left( 2\partial_\eta \phi - \mathcal{H} \phi \right) \\
+ 3 \frac{\varphi_1}{\partial_\eta \varphi} \left( 2\mathcal{H}^2 - \partial_\eta \mathcal{H} + \mathcal{H} \frac{\partial^2 \varphi}{\partial_\eta \varphi} \right) E_2 \\
+ 3 \left[ \frac{\varphi_1}{\partial_\eta \varphi} \left( \partial_\eta^2 - \left( \frac{\partial^2 \varphi}{\partial_\eta \varphi} + 4\mathcal{H} \right) \partial_\eta - 2\partial_\eta \mathcal{H} + 4\mathcal{H}^2 + 2\mathcal{H} \frac{\partial^2 \varphi}{\partial_\eta \varphi} \right) \\
+ 2 \phi \left( \partial_\eta - 2\mathcal{H} \right) \right] E_1 \right). 
\end{align*}
\] (4.26)

Through the background Einstein equations (3.6), (3.7), and the first-order perturbations (4.16) and (4.17) of the Einstein equation, we have seen that

\[
C_{(K)}^{(1)} = 0. 
\] (4.27)

Hence, the first-order perturbation of the Klein-Gordon equation is not independent equation of the background and the first-order perturbation of the Einstein equation. Therefore, from the point of view of the Cauchy problem, any information which are obtained from the first-order perturbation of the Klein-Gordon equation should be also obtained from the set of the background Einstein equations and the first-order perturbations of the Einstein equation, in principle.
§5. Consistency of the second-order equations

Now, we consider the consistency of the second-order perturbations of the Einstein equations

\[ (1) G_a^b [\mathcal{L}] + (2) G_a^b [\mathcal{H}, \mathcal{H}] = 8\pi G (2) T_a^b, \]

and equations of motion for matter fields derived in KN2008. The essence of this consistency check is same as those for the background equations and for the equations of the first-order perturbations as mentioned in the last section. However, these consistency checks for the second-order perturbations are necessary to guarantee that the derived equations are correct, since the second-order Einstein equations have complicated forms due to the quadratic terms of the linear-order perturbations which arise from the non-linear effects of the Einstein equations.

5.1. Perfect fluid case

First, we consider the second-order Einstein equation \((5\cdot1)\) in the case of a perfect fluid. The components of \((1) G_a^b [\mathcal{L}]\) are given by Eqs. \((2\cdot24)\)–\((2\cdot27)\) and the replacement \((2\cdot28)\) and the components of the second term in the right hand side of Eq. \((5\cdot1)\) are given by Eqs. \((2\cdot29)\)–\((2\cdot32)\). Further, the components of the right hand side of Eq. \((5\cdot1)\) are given by Eqs. \((A\cdot23)\)–\((A\cdot26)\). So, we can write down the components of the second-order perturbations of the Einstein equation. The resulting equations are completely same as those obtained in KN2007 except for the definitions of \(\Gamma_0, \Gamma_i,\) and \(\Gamma_{ij}\). Therefore, the same calculations as those in KN2007 lead

\[
(2) E_{(1)} := (-3H\partial_\eta + \Delta + 3K) \psi - 3H^2 \phi - 4\pi G a^2 \mathcal{E} - \Gamma_0 = 0, \quad (5\cdot2)
\]

\[
(2) E_{(2)} := \left( \partial_\eta^2 + 2H\partial_\eta - K - \frac{1}{3}\Delta \right) \psi + \left( \mathcal{H}\partial_\eta + 2\partial_\eta \mathcal{H} + \mathcal{H}^2 + \frac{1}{3}\Delta \right) \phi - 4\pi G a^2 \mathcal{P} - \frac{1}{6} \Gamma_k^k = 0, \quad (5\cdot3)
\]

\[
(2) E_{(3)} := \psi - \phi - \frac{3}{2} (\Delta + 3K)^{-1} \left( \Delta^{-1} D^i D^j \Gamma_{ij} - \frac{1}{3} \Gamma_k^k \right) = 0, \quad (5\cdot4)
\]

\[
(2) E_{(4)i} := \partial_\eta D_i \psi + \mathcal{H} D_i \phi - \frac{1}{2} D_i \Delta^{-1} D^k \Gamma_k + 4\pi G a^2 (\epsilon + p) D_i \nu^i = 0, \quad (5\cdot5)
\]

\[
(2) E_{(5)i} := (\Delta + 2K) \nu_i + 2 \left( \Gamma_i - D_i \Delta^{-1} D^k \Gamma_k \right) - 16\pi G a^2 (\epsilon + p) \nu_i = 0, \quad (5\cdot6)
\]
(\gamma_0^{(2)}) = \partial_\eta \left( a^2 \nu_i \right) - 2a^2 (\Delta + 2K)^{-1} \left\{ D_i \Delta^{-1} D^k D^l \Gamma_{kl} - D^k \Gamma_{ik} \right\} \\
= 0.

(\gamma_0^{(2)}) = (\partial_\eta^2 + 2\mathcal{H}\partial_\eta + 2K - \Delta) \chi_{ij} - 2\Gamma_{ij} + \frac{2}{3} \gamma_{ij} \Gamma_k^k \\
+ 3 \left( \Delta \chi_{ij} \right) (\Delta + 3K)^{-1} \left( \Delta^{-1} D^k D^l \Gamma_{kl} - \frac{1}{3} \Gamma_k^k \right) \\
- 4 \left( D_i (\Delta + 2K)^{-1} D_j \Delta^{-1} D^k D^l \Gamma_{ik} - D_i (\Delta + 2K)^{-1} D^k \Gamma_{jk} \right) \\
= 0.

Here, the definitions of \Gamma_0, \Gamma_i, and \Gamma_{ij} are as follows:

\Gamma_0 := 8\pi Ga^2 (\epsilon + p) D_i \nu^i D^i \nu^i - 3D_k \Phi D^k \Phi - 8 \Phi \Delta \Phi - 3 (\partial_\eta \Phi)^2 \\
- 12K \Phi^2 - 12\mathcal{H}^2 \Phi^2 \\
- 4 \left( \partial_\eta D_i \Phi + \mathcal{H} D_i \Phi \right) \nu^i - 2\mathcal{H} D_k \Phi \nu^k \\
+ 8\pi Ga^2 (\epsilon + p) \nu_i \nu^i + \frac{1}{2} D_k \nu^i D^i \nu^k + 3\mathcal{H}^2 \nu^k \nu_k \\
+ D_i D_k \Phi \chi^{ik} \\
- 2\mathcal{H} D^k \nu^i \chi_{kl} - \frac{1}{2} D^k \nu^i \partial_\eta \chi_{ik} \\
+ \frac{1}{2} \partial_\eta \chi_{ik} \partial_\eta \chi^{ik} + \mathcal{H} \chi_{kl} \partial_\eta \chi^{ik} - \frac{1}{8} D_k \chi_{lm} \chi^{kl} \\
+ \frac{1}{2} D_k \chi_{lm} \chi^{kl} - \frac{1}{2} \chi_{lm} (\Delta - K) \chi_{im},

(5.9)

\Gamma_i := -16\pi Ga^2 \left( \mathcal{E} + p \right) D_i \nu^i + 12 \mathcal{H} \Phi D_i \Phi \\
- 4 \Phi \partial_\eta D_i \Phi - 4\partial_\eta \Phi D_i \Phi \\
- 16\pi Ga^2 \left( \mathcal{E} + p \right) \nu_i - 2D^j \Phi D_i \nu_j + 2D_i D^j \Phi \nu_j + 2\Delta \Phi \nu_i \\
+ \Phi \Delta \nu_i + 2K \Phi \nu_i \\
- 4\mathcal{H} \nu_i D_i \nu_j \\
+ 2D^j \Phi \partial_\eta \chi_{ij} - 2\partial_\eta D^j \Psi \chi_{ij}.
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$$+2D_k D_i \nu^{(1)}_{m} \chi^{km} + 2D_k^{[i} \nu^{j]} D_j \chi^{k} + 2K \nu \chi^{i} - \nu^j \Delta \chi^{ji}$$

$$- \frac{1}{2} \delta_{\eta} \chi^{ik} D_i \chi_{kj} + 2 \chi^{j} \eta D_{ij} \chi^{jk}$$

$$(5\cdot10)$$

$$\Gamma_{ij} := 16\pi G \alpha^2 (\epsilon + p) D_i \nu^j D_j \nu^i - 4D_i \Phi D_j \Phi - 8 \Phi D_i D_j \Phi$$

$$+ 2 \left\{ 3D_k \Phi D^k \Phi + 4 \Phi \Delta \Phi + \left( \partial_\eta \Phi \right)^2 + 8 \mathcal{H} \Phi \partial_\eta \Phi \right\} \gamma_{ij}$$

$$+ 32\pi G \alpha^2 (\epsilon + p) D_i \nu_j - 4\partial_\eta \Phi D_i \nu_j + 4\partial_\eta D_i \Phi \nu_j$$

$$+ 4 \left( \partial_\eta D_k \Phi \nu^k + \mathcal{H} D_k \Phi \nu^k \right) \gamma_{ij}$$

$$+ 16\pi G \alpha^2 (\epsilon + p) \mathcal{V}_i \nu_j - 2 \nu^k D_k D_i \nu_j + 2 \nu_k D_i D_j \nu^k$$

$$+ D_i \nu^k D_j \nu^i + D_k \nu^j D_i \nu^k$$

$$- \left\{ D_k \nu^i D^k \nu^i + D_k \nu^i D^k \nu^j + 2 \nu^i \left( \Delta + 2\partial_\eta \mathcal{H} - 3 \mathcal{H}^2 \right) \nu^j \right\} \gamma_{ij}$$

$$- 4\mathcal{H} \partial_\eta \Phi \chi_{ij} - 2\partial_\eta \Phi \chi_{ij} + 4D^k \Phi \Phi D_i \chi_{jk} + 4D^k \Phi \Phi D_k \chi_{ij} - 8K \Phi \chi_{ij}$$

$$+ 4 \Phi \Delta \chi_{ij} - 4D^k D_i \Phi \chi_{jk} + 2\Delta \Phi \chi_{ij} + 2D_k \Phi \chi^{lk} \gamma_{ij}$$

$$- 2D^k \nu_{(i} \partial_\eta \chi_{jk)} - 2 \nu^i \partial_\eta D_i \chi_{jk} + 2 \nu^k \partial_\eta D_k \chi_{ij} + 2 \nu^j \partial_\eta \chi_{ik} \gamma_{ij}$$

$$+ \partial_\eta \chi_{ik} \partial_\eta \chi_{jk} + 2D_l \chi_{ik} \chi^{l} - \frac{1}{2} D_j \chi_{ik} D_i \chi^{j}$$

$$- \left( \chi^{lm} D_i D_j \chi_{lm} + 2 \chi^{lm} D_i \chi_{jm} D_l \chi_{ij} - \chi^{lm} D_m D_l \chi_{ij} \right)$$

$$- \frac{1}{4} \left( 3\partial_\eta \chi_{ik} \partial_\eta \chi^{kl} - 3D_k \chi^{lm} D^k \chi^{ml} \right)$$

$$+ 2D_k \left( \chi^{lm} D^l \chi^{mk} - 4K \chi^{lm} \chi_{im} \chi^{lk} \right) \gamma_{ij},$$

$$(5\cdot11)$$

and we denote $\Gamma_{i}^{j} = \gamma^{jk} \Gamma_{ij}$. The definitions (5.9)–(5.11) of the source terms $\Gamma_0$, $\Gamma_i$, and $\Gamma_{ij}$ in the second-order perturbations of the Einstein equations represent the precise mode-coupling: For example, the first two lines in Eq. (5.9) represent the scalar-scalar mode-coupling; the third line in Eq. (5.9) represents the scalar-
vector mode-coupling; the fourth line in Eq. (5.9) represents the vector-vector mode-coupling; the fifth line in Eq. (5.9) shows the scalar-tensor mode-coupling; the sixth line in Eq. (5.9) corresponds to the tensor-tensor mode coupling. Thus, in the second-order perturbations of the Einstein equations, any types of mode-coupling appear due to the non-linear effects of the Einstein equations. Actually, the definitions (5.9)–(5.11) of the source terms \( \Gamma_0, \Gamma_i, \) and \( \Gamma_{ij} \) do shows these any types of mode-coupling occur in the second-order perturbations of the Einstein equations.

5.1.1. Consistency with the continuity equation

Now, we consider the second-order perturbation of the energy continuity equation which is the second-order perturbation of \( \bar{u}^a \bar{\nabla}^b \bar{T}^{b}_{a} = 0. \) As shown in KN2008,\(^8\) the second-order perturbation of the energy continuity equation is given by

\[
a^{(2)} v_0^{(p)} := \partial_\eta \left( \frac{2}{3} \mathcal{H} \left( \mathcal{E}^{(2)} + (\mathcal{P}) \right) + (\epsilon + p) \left( \Delta^{(2)} v - 3\partial_\eta \Psi \right) \right) - \Xi_0 = 0. \tag{5.12}
\]

Here, \( \Xi_0 \) consists of the quadratic terms of the linear order perturbations defined by

\[
\Xi_0 := 2 \left[ \begin{array}{c}
6 \Psi \partial_\eta \Psi - 2 \Psi \Delta v - \Phi \Delta v - \nu^k D_k \Psi \\
+ \left( \frac{1}{2} \right) \left( D_k \Psi - D_k \Phi - \partial_\eta D_k \frac{1}{2} \partial_\eta D_k \Psi \right) \\
+ \chi^{ik} \left( D_i D_k \frac{1}{2} v + D_i \frac{1}{2} \nu_k - D_i \nu^k + \frac{1}{2} \partial_\eta \chi^{ik} \right) \right] (\epsilon + p) \\
- 2 \left( D_i \frac{1}{2} v + \nu^i - \nu^i \right) D_i \mathcal{E} - 2 \left( \Delta \frac{1}{2} v - 3\partial_\eta \Psi \right) \left( \mathcal{E} + (\mathcal{P}) \right). \tag{5.13}
\]

To confirm the consistency of the background and the perturbations of the Einstein equation and the energy continuity equation (5.12), we first substitute the second-order Einstein equations (5.2)–(5.5), and the background Einstein equation (3.3), into Eq. (5.12) as in the case of the first-order perturbations in \( \S 4.1.1 \). For simplicity, we first impose Eq. (4.4) on all equations. Then, we obtain

\[
4\pi G a^3 C_0^{(p)} = -\partial_\eta \left( E^{(2)} + (\mathcal{P}) \right) \left( E^{(2)} + (\mathcal{P}) \right) - 3\mathcal{H} \left( E^{(2)} + (\mathcal{P}) \right) + D^i (p) E_{(i)}^{(2)} + \frac{3}{2} \left( 3 (p) E_{(1)}^{(0)} - (p) E_{(2)}^{(0)} \right) \partial_\eta \Psi \\
- \partial_\eta \Gamma_0 - \mathcal{H} \Gamma_0 - \frac{1}{2} \mathcal{H} \Gamma_k^k + \frac{1}{2} D^k \Gamma_k - 4\pi G a^2 \Xi_0. \tag{5.14}
\]

Imposing the second-order Einstein equations (5.2), (5.3), (5.5), and the background Einstein equation (3.3), we see that the second-order perturbation (5.12) of the
energy continuity equation is consistent with the second-order and the background Einstein equations if the equation
\[
4\pi G a^2 \Xi_0 + (\partial_\eta + \mathcal{H}) \Gamma_0 + \frac{1}{2} \mathcal{H} \Gamma_k^k - \frac{1}{2} D^k \Gamma_k = 0 \quad (5.15)
\]
is satisfied under the background, the first-order Einstein equations. Actually, through the background Einstein equation (3.3) (or equivalently Eqs. (3.1) and (3.2)) and the first-order perturbations of the Einstein equations (4.2)–(4.8), we can easily see that
\[
4\pi G a^2 \Xi_0 + \partial_\eta \Gamma_0 + \mathcal{H} \Gamma_0 + \frac{1}{2} \mathcal{H} \Gamma_k^k - \frac{1}{2} D^k \Gamma_k
\]
\[
= - \left[ 6 \phi (\partial_\eta \phi + D^i \phi D_i) + 2 \frac{\phi}{v} D^i \phi - 3 \phi D^i \phi \right] \left( 3 (p) E_1^{(0)} - (p) E_2^{(0)} \right)
\]
\[
+ 2 \left( D^i \phi \partial_\eta + 2 \mathcal{H} D^i \phi - D^i \phi - 3 \phi D^i \phi \right) \left( E_1^{(0)} - (p) E_2^{(0)} \right)
\]
\[
- 6 \partial_\eta \phi (p) E_1^{(0)} - 2 \left( 3 \partial_\eta \phi + D_i \phi D_i \phi \right) \left( p E_1^{(0)} - (p) E_2^{(0)} \right)
\]
\[
+ D_i \phi \left( \nu^i - \mathcal{V}^i \phi \right) \left( 3 (p) E_1^{(0)} - (p) E_2^{(0)} \right) - 2 \nu^i D_i \left( E_1^{(0)} - (p) E_2^{(0)} \right)
\]
\[
+ 2 \mathcal{V}^i \left( \partial_\eta + 2 \mathcal{H} \right) (p) E_4^{(0)} - \frac{1}{2} \left[ D^i \phi \left( \partial_\eta + 2 \mathcal{H} \right) - D^i \phi \right] \left( p E_5^{(0)} \right)
\]
\[
+ \frac{1}{2} \mathcal{V}^i \left( \partial_\eta + 2 \mathcal{H} \right) (p) E_4^{(0)} + \frac{1}{2} \left[ \phi^i \left( \Delta + 2 K \right) - 4 \mathcal{H} D^i \phi \right] \left( p E_6^{(0)} \right)
\]
\[
- \frac{1}{2} \mathcal{V}^i \left( \partial_\eta + 2 \mathcal{H} \right) (p) E_4^{(0)} + \frac{1}{2} \left[ \phi^i \left( \Delta + 2 K \right) + 2 D^i \nu^k \right] D_i + 12 \mathcal{H} \nu^k \phi \right] \left( p E_6^{(0)} \right)
\]
\[
+ 2 \phi^i \nu^k D_i \left( p E_4^{(0)} \right)
\]
\[
+ \left( 3 (p) E_1^{(0)} - (p) E_2^{(0)} \right) D_i (p) E_5^{(0)} - \frac{1}{2} \phi^i \nu^k D_i \left( p E_5^{(0)} \right)
\]
\[
- \frac{1}{2} \left( \partial_\eta + 4 \mathcal{H} \right) \phi (p) E_6^{(0)} - \frac{1}{2} \phi^i \nu^k E_7^{(0)}
\]
\[
- \frac{1}{2} \phi^i \partial_\eta \phi \left( 3 (p) E_1^{(0)} - (p) E_2^{(0)} \right) + \frac{1}{4} \left( \partial_\eta + 4 \mathcal{H} \right) \phi (p) E_8^{(0)}.
\]
This shows that Eq. (5.15) is satisfied under the Einstein equations of the background and of the first order. Thus, this implies that the second-order perturbation (5.12) of the energy continuity equation together with the source term (5.13) is consistent with the Einstein equations of the background, the first- and the second-order, i.e., Eqs. (3.3), (4.2)–(4.8), (5.2), (5.6), and (5.5).
5.1.2. Consistency with the Euler equation

Next, we consider the second-order perturbations of the Euler equations. For simplicity, we first impose Eq. (4.4) on all equations, again. As shown in KN2008, the second-order perturbation of the Euler equation for a single perfect fluid is given in terms of gauge-invariant form as

\[
(2)C_i^{(p)} = (\epsilon + p) \left\{ (\partial_\eta + \mathcal{H}) \left( D_i (2) \nu + (2) V_i \right) + D_i (2) \Phi \right\} \\
+ D_i (2) \mathcal{P} + \partial_\eta p \left( D_i (2) \nu + (2) V_i \right) - \Xi_i^{(p)} = 0, \tag{5.17}
\]

where \( \Xi_i^{(p)} \) is defined by

\[
\Xi_i^{(p)} := -2 (1) \Phi \left\{ (1) \mathcal{P} - (\epsilon + p) (1) \Phi \right\} \\
-2 (\epsilon + p) \left( \nu^j (1) - D^j (1) \nu - (1) V^j \right) \left\{ D_i (1) \nu_j - D_j (1) \nu^i \right\} \\
-2 \left( \mathcal{E} + (1) \mathcal{P} \right) \left\{ D_i (1) \Phi + \partial_\eta \left( D_i (1) \nu + (1) V_i \right) + \mathcal{H} \left( D_i (1) \nu + (1) V_i \right) \right\} \\
-2 \left( D_i (1) \nu + (1) V_i \right) \partial_\eta (1) \mathcal{P}. \tag{5.18}
\]

This \( \Xi_i^{(p)} \) is the collection of the quadratic terms of the linear-order perturbations in the second-order perturbation of the Euler equation. As in the case of the first-order perturbations of the Euler equation, the equation (5.17) is decomposed into the scalar- and the vector-parts as

\[
(2)C_i^{(pS)} := D_i \Delta^{-1} D^j (2)C_j^{(p)} \\
= (\epsilon + p) \left\{ (\partial_\eta + \mathcal{H}) D_i (2) \nu + D_i (2) \Phi \right\} + D_i (2) \mathcal{P} + \partial_\eta p D_i (2) \nu \\
- D_i \Delta^{-1} D^j \Xi_j^{(p)} \\
= 0, \tag{5.19}
\]

\[
(2)C_i^{(pV)} := (2)C_i^{(p)} - D_i \Delta^{-1} D^j (2)C_j^{(p)} \\
= (\epsilon + p) \left( \partial_\eta + \mathcal{H} \right) (2) \nu_i + \partial_\eta p (2) V_i \\
- \Xi_i^{(p)} + D_i \Delta^{-1} D^j \Xi_j^{(p)} \\
= 0. \tag{5.20}
\]

First, we consider the scalar-part (5.19) of the Euler equation. As in the case of the first-order perturbation of the Euler equation, through the background Einstein
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equation (5.3), the background energy continuity equation (3.4), and the Einstein equations of the second order (5.3)–(5.5), we can obtain

\[ 4\pi G a^2 C^{(p_S)}_i = -4\pi G a^3 D_i (^{(2)}v) C^{(p)}_0 - \frac{1}{2} D_i (^{(2)}\Phi) \left( 3 (^{(0)}E_1) - (^{(0)}E_2) \right) - D_i (^{(2)}E_2) - \frac{1}{3} D_i (\Delta + 3K) (^{(p)}E_3) + \frac{1}{2} (^{(0)}E_4) \]

\[ - \frac{1}{2} D_i \Delta^{-1} D^j J^j, \tag{5.21} \]

where we defined

\[ J^j := 8\pi G a^2 \Xi^{(p)}_j - (\partial_\eta + 2\mathcal{H}) \Gamma^j + D^j \Gamma^j. \tag{5.22} \]

On the other hand, through the background energy continuity equation (3.4) and the Einstein equation of the second order (5.6) and (5.7), the vector-part (5.20) of the Euler equation (5.17) is given by

\[ 8\pi G a^2 C^{(p_V)}_i = -8\pi G a^3 C^{(p)}_0 V_i - \frac{1}{2} (\partial_\eta + 2\mathcal{H}) (^{(p)}E_5) + \frac{1}{2} a^2 (\Delta + 2K) (^{(p)}E_6) \]

\[ - J^i + D_i \Delta^{-1} D^j J^j = 0. \tag{5.23} \]

Eqs. (5.21) and (5.23) show that the second-order perturbations of the Euler equations is consistent with the background Einstein equations and the second-order perturbations of the Einstein equations if the equation

\[ J_i = 0 \tag{5.24} \]

is satisfied under the Einstein equations of the background and the first-order perturbations. Actually, we can easily confirm Eq. (5.24) as follows. Through Eqs. (5.3), (5.2)–(10), we can see the relation

\[ J_i = 2 \left[ 3\partial_\eta (^{(1)}\Phi) D_i (^{(1)}v) - (^{(1)}\Phi) D_i (^{(1)}\Phi) \right] \left( 3 (^{(0)}E_1) - (^{(0)}E_2) \right) \]

\[ - 4 \left[ D_i (^{(1)}v) (\partial_\eta + \mathcal{H}) - D_i (^{(1)}\Phi) \right] (^{(1)}E_1) \]

\[ + 4 \left[ (^{(1)}\Phi) D_i + D_i (^{(1)}\Phi) - 3\mathcal{H}D_i (^{(1)}v) \right] (^{(1)}E_2) \]

\[ + 4 \left[ D_i (^{(1)}v) D^j + 3\gamma^j_i (^{(1)}\Phi) \right] (^{(1)}E_{4j}) \]

\[ + 6\partial_\eta \psi V_i \left( 3 (^{(0)}E_1) - (^{(0)}E_2) \right) - 4 V_i \left[ (\partial_\eta + \mathcal{H}) (^{(1)}E_1) - 12\mathcal{H} (^{(1)}E_2) \right] \]
\[ +4 \left[ \gamma_i^k \nu^j D_j + \nu_i^j D^k + D_i \nu^k \right] (p) E_{(4)k}^{(1)} - 3 \partial_\eta \psi (p) E_{(5)i}^{(1)} \]
\[ - \frac{2}{a^2} \left[ D_i D^j \phi + \gamma_i^j \Delta \phi - D^j \phi D_i + \frac{1}{2} \gamma_i^j \phi (\Delta + 2K) \right] (p) E_{(6)j}^{(1)} \]
\[ + 2 \nu^j D_i \nu_j \left( 3 (p) E_{(1)}^{(0)} - (p) E_{(2)}^{(0)} \right) - \left[ D_i \nu^k - \gamma_i^k \nu^j D_j \right] (p) E_{(5)k}^{(1)} \]
\[ + \frac{4}{a^2} \mathcal{H} \left[ D_i \nu^j + \nu^j D_i \right] (p) E_{(6)j}^{(1)} \]
\[ - 2 D^j \phi (p) E_{(7)ij}^{(1)} \]
\[ - \frac{2}{a^2} \left[ \gamma_i^j \chi_{km}^k D_m D_j - D^j \chi_{km}^k D_j - \frac{1}{2} (\Delta - K) \chi_i^k \right] (p) E_{(6)k}^{(1)} \]
\[ + \frac{1}{2} \left[ D_i \chi_{kj}^{(p)} E_{(7)jk}^{(1)} + 4 \chi_{kj}^{(p)} D_i^{(1)} E_{(7)jk}^{(1)} \right] . \]

This represents that Eq. (5.24) is satisfied due to the background Einstein equations and the first-order perturbations of the Einstein equations, and implies that the second-order perturbation of the Euler equation is consistent with the set of the background, the first-order, and the second-order Einstein equations. From general point of view, this is just a well-known result, i.e., the Einstein equation includes the equations of motion for matter field due to the Bianchi identity. However, the above verification of the identity (5.24) implies that our derived second-order perturbations of the Einstein equation and the Euler equation are consistent. In this sense, we may say that the derived second-order Einstein equations, in particular, the derived formulae for the source terms \( \Gamma_0, \Gamma_i, \Gamma_{ij}, \Xi_0, \) and \( \Xi_i \) are correct.

5.2. Scalar field case

Next, we consider the second-order Einstein equation (5.1) in the case of a single scalar field. Through the components of \( (1)G^a_b [\mathcal{L}] \) given by Eqs. (2.24)–(2.27) with the replacement (2.28), the components of \( (2)G^a_b \) given by Eqs. (2.29)–(2.32), and the components of the second-order perturbation of the energy momentum tensor \( (2)T^a_b \) given by Eqs. (A.41)–(A.44), we can obtain the all components of the second-order perturbation of the Einstein equation (5.1) in the case of the universe filled with a single scalar field. For simplicity, we used the first-order Einstein equations (4.19) and (4.20). For the scalar-mode of the second order, we obtain the equations as follows:

\[ (2)E_{(1)}^{(s)} := \left\{ \partial_\eta^2 + 2 \left( \mathcal{H} - \frac{\partial^2 \phi}{\partial_\eta \phi} \right) \partial_\eta - \Delta - 4K + 2 \left( \partial_\eta \mathcal{H} - \frac{\partial^2 \phi}{\partial_\eta \phi} \mathcal{H} \right) \right\} (2) \phi \]
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\[ +\Gamma_0 + \frac{1}{2} \Gamma_k^k - \Delta^{-1} D^j D^i \Gamma_{ij} \]
\[- \frac{3}{2} \left[ - \partial^2_\eta + \left( 2 \frac{\partial^2 \varphi}{\partial \eta^2 \varphi} - \mathcal{H} \right) \partial_\eta \right] (\Delta + 3K)^{-1} \left( \Delta^{-1} D^j D^i \Gamma_{ij} - \frac{1}{3} \Gamma_k^k \right) \]
\[- \left( \partial_\eta - \frac{\partial^2 \varphi}{\partial \eta^2 \varphi} \right) \Delta^{-1} D^k \Gamma_k \]
\[= 0; \quad (5.26) \]

\[ (s)_E^{(2)} := 2 \partial_\eta (2)^\Psi + 2\mathcal{H} (2)^\phi - 8\pi G \partial_\eta \varphi \varphi_2 - \Delta^{-1} D^k \Gamma_k \]
\[= 0; \quad (5.27) \]

\[ (s)_E^{(3)} := \left( - \partial^2_\eta - 5\mathcal{H} \partial_\eta + \frac{4}{3} \Delta + 4K \right) (2)^\Psi - \left( \mathcal{H} \partial_\eta + 2\partial_\eta \mathcal{H} + 4\mathcal{H}^2 + \frac{1}{3} \Delta \right) (2)^\phi \]
\[-8\pi G a^2 \varphi_2 \frac{\partial V}{\partial \varphi} - \Gamma_0 + \frac{1}{6} \Gamma_k^k , \]
\[= 0; \quad (5.28) \]

\[ (s)_E^{(4)} := (2) \Psi - (2) \phi - \frac{3}{2} (\Delta + 3K)^{-1} \left( \Delta^{-1} D^j D^i \Gamma_{ij} - \frac{1}{3} \Gamma_k^k \right) \]
\[= 0. \quad (5.29) \]

For the vector-mode of the second order, we obtain a constraint equation and an evolution equation as follows:

\[ (s)_E^{(5)} := (2) \nu_i - 2 (\Delta + 2K)^{-1} \left\{ D_i \Delta^{-1} D^k \Gamma_k - \Gamma_i \right\} \]
\[= 0; \quad (5.30) \]

\[ (s)_E^{(6)} := \partial_\eta \left\{ a^2 (2) \nu_i \right\} - 2a^2 (\Delta + 2K)^{-1} \left\{ D_i \Delta^{-1} D^k D^l \Gamma_{kl} - D^k \Gamma_{ki} \right\} \]
\[= 0. \quad (5.31) \]

For the tensor-mode of the second order, we obtain the single evolution equation as follows:

\[ (s)_E^{(7)} := (2) \right( \partial^2_\eta + 2\mathcal{H} \partial_\eta + 2K - \Delta \right) \chi_{ij} - 2\Gamma_{ij} + \frac{2}{3} \Gamma_k^k \gamma_{ij} \]
\[+ 3 \left( D_i D_j - \frac{1}{3} \gamma_{ij} \Delta \right) (\Delta + 3K)^{-1} \left( \Delta^{-1} D^k D^l \Gamma_{kl} - \frac{1}{3} \Gamma_k^k \right) \]
\[- 4 \left\{ D_i (\Delta + 2K)^{-1} D_j \Delta^{-1} D^k D^l \Gamma_{kl} - D_i (\Delta + 2K)^{-1} D^k \Gamma_{j,k} \right\} \]
\[= 0. \quad (5.32) \]

Here, \( \Gamma_0 \), \( \Gamma_i \), and \( \Gamma_{ij} \) in these expressions are defined by

\[ \Gamma_0 := 4\pi G \left( (\partial_\eta \varphi_1)^2 + D_i \varphi_1 D^i \varphi_1 + a^2 (\varphi_1)^2 \frac{\partial^2 V}{\partial \varphi^2} \right) \]
First, we consider the consistency between Eqs. (5.30) and (5.31). Eq. (5.30) comes from the momentum constraints in the Einstein equations, which is an initial value

\[-4\partial_\eta\mathcal{H}\left(\frac{\partial}{\partial \phi}\right)^2 - 2 \left(\frac{\partial}{\partial \phi}\right)^2 - 3D_k \Phi D^k \Phi - 10\Phi \Delta \Phi + D_1 D_k \Phi \chi^{lk} + \frac{1}{8} \partial_\eta \chi_{ik} \partial_\eta \chi^{kl} + \mathcal{H} \chi_{kl} \partial_\eta \chi_{il} - \frac{3}{8} D_k \chi_{lm} D^k \chi^{ml} + \frac{1}{4} D_k \chi_{lm} D^l \chi^{mk}\]

\[-3 \left(\partial_\eta \Phi \right)^2 - 16K \left(\frac{\partial}{\partial \phi}\right)^2 - 8\mathcal{H}^2 \left(\frac{\partial}{\partial \phi}\right)^2\]

\[+ \frac{1}{2} \chi^{lm} \Delta \chi^{lm} + \frac{1}{2} K \chi^{lm} \chi^{lm}, \quad (5.33)\]

\[\Gamma_i := 16\pi G \partial_\eta \varphi_1 D_i \varphi_1 - 4\partial_\eta \Phi D_i \Phi + 8\mathcal{H} \Phi D_i \Phi - 8\Phi \partial_\eta D_i \Phi + 2D^j \Phi \partial_\eta \chi_{ji} - 2\partial_\eta D^j \Phi \chi_{ij} - \frac{1}{2} \partial_\eta \chi_{jk} D_i \chi^{kj} - \chi_{lk} \partial_\eta D_i \chi^{lk} + \chi^{lk} \partial_\eta D_k \chi_{il}, \quad (5.34)\]

\[\Gamma_{ij} := 16\pi GD_i \varphi_1 D^j \varphi_1 + 8\pi \mathcal{G} \left\{ \left(\partial_\eta \varphi_1\right)^2 - D_i \varphi_1 D^j \varphi_1 - a^2 \left(\varphi_1\right)^2 \frac{\partial^2 V}{\partial \varphi^2} \right\} \gamma_{ij}\]

\[-4D_i \Phi D_j \Phi - 8\Phi D_i D_j \Phi + \left(6D_k \Phi D^k \Phi + 4 \left(\frac{\partial}{\partial \phi}\right)^2 + 2 \left(\partial_\eta \Phi \right)^2 + 8\partial_\eta \mathcal{H} \left(\frac{\partial}{\partial \phi}\right)^2\right) + 16\mathcal{H}^2 \left(\frac{\partial}{\partial \phi}\right)^2 + 16\mathcal{H} \Phi \partial_\eta \Phi - 4 \Phi \partial_\eta \varphi_1 \gamma_{ij}\]

\[-4\partial_\eta \varphi_1 \Phi \chi_{ij} - 2\partial_\eta \Phi \chi_{ij} - 4D^k \Phi D_i \chi_{jk} + 4D_k \Phi D_k \chi_{ij} - 8K \Phi \chi_{ij}\]

\[+ 4 \Phi \Delta \chi_{ij} - 4D^k D_i \Phi \chi_{jk} + 2\mathcal{H} \Phi \chi_{ij} + 2D_i D_k \Phi \chi^{lk} \gamma_{ij}\]

\[+ \partial_\eta \chi_{ik} \partial_\eta \chi_{lj} - D^k \chi_{ij} + D^k \chi_{ij} + D^k \chi_{ij} - \frac{1}{2} D_i \chi^{lk} D_j \chi_{lk}\]

\[-\chi_{lm} D_i D_j \chi^{ml} + 2 \chi_{lm} D_i \chi_{jm} - \chi_{lm} D_m D_i \chi_{ij}\]

\[-\frac{1}{4} \left(3\partial_\eta \chi_{ik} \partial_\eta \chi_{jl} - 3D_k \chi_{ij} + D^k \chi_{ij} + D^k \chi_{ij} - \frac{1}{2} D_i \chi^{lk} D_j \chi_{lk}\right)\]

\[-4K \chi_{lm} \chi_{lm} \gamma_{ij}. \quad (5.35)\]

Now, we consider the consistency check in the set of equations (5.30) to (5.35).

First, we consider the consistency between Eqs. (5.30) and (5.31). Eq. (5.30) comes from the momentum constraints in the Einstein equations, which is an initial value
consistency, and should be consistent with the evolution equations in the Einstein equations from general point of view. In this sense, Eqs. (5.30) and (5.31) should be consistent with each other. Now, we explicitly check this. Through these equations, we obtain

\[
(2)_{(s)}E_{(6)i} - \partial_\eta \left( a^2 (2)_{(s)}E_{(5)i} \right) = 2a^2 (\Delta + 2K)^{-1} \left( D_i \Delta^{-1} D^k - \gamma_i^k \right) \left( \partial_\eta \Gamma_k + 2\mathcal{H} \Gamma_k - D^j \Gamma_{kl} \right) = 0. \tag{5.36}
\]

Therefore, the vector-part (5.30) of the momentum constraint is consistent with the evolution equation (5.31) if the equation

\[
\partial_\eta \Gamma_k + 2\mathcal{H} \Gamma_k - D^j \Gamma_{lk} = 0 \tag{5.37}
\]

is satisfied. Actually, through Eqs. (4.16), (4.17), (4.21), and (4.23), the left hand side of Eq. (5.37) is given by

\[
\partial_\eta \Gamma_k + 2\mathcal{H} \Gamma_k - D^j \Gamma_{lk} = -16\pi Ga^2 D_{k}\varphi_1 \mathcal{C}_{(K)} - 4\phi (1) D_k (s)E_{(1)} - 16 \left( \partial_\eta + \mathcal{H} + \frac{\partial_\eta^2 \varphi}{2\partial_\eta \varphi} \right) (1) D_k (s)E_{(2)} + 2D^j (1) \varphi (s)E_{(6)jk} - \frac{1}{2} \left( D_k \chi^j + 2\chi^{jl} D_k \right) (1)E_{(6)lj} + \chi^{jl} D_j (s)E_{(6)kl}. \tag{5.38}
\]

Since the first-order perturbation (4.24) of the Klein-Gordon equation is consistent with the Einstein equation as shown in Eq. (4.26), Eq. (5.38) shows that the initial value constraint (5.30) for the vector-mode of the second-order perturbation is consistent with the evolution equation (5.31) by virtue of the first-order perturbations of the Einstein equations. This is a trivial result from general point of view, because the Einstein equation is the first class constrained system. However, this trivial result implies that we have derived the source terms \( \Gamma_i \) and \( \Gamma_{ij} \) of the second-order Einstein equations consistently.

Next, we consider the equation (5.28). Through Eqs. (5.26) and (5.29), Eq. (5.28) is given by

\[
(2)_{(s)}E_{(3)} = \left( -\partial_\eta^2 - 5\mathcal{H} \partial_\eta + \frac{4}{3} \Delta + 4K \right) (s)E_{(4)} - (s)E_{(1)} - 2 \left( 2\mathcal{H} + \frac{\partial_\eta^2 \varphi}{\partial_\eta \varphi} \right) \left( \partial_\eta + \mathcal{H} \right) (2) \varphi \mathcal{F} - 8\pi Ga^2 \varphi_2 \partial V \partial \varphi - 3 \left( 2\mathcal{H} + \frac{\partial_\eta^2 \varphi}{\partial_\eta \varphi} \right) \partial_\eta (\Delta + 3K)^{-1} \left( \Delta^{-1} D^j D^k \Gamma_{ij} - \frac{1}{3} \Gamma_k \right) + \Delta^{-1} D^j D^k \Gamma_{ij} - \left( \partial_\eta - \frac{\partial_\eta^2 \varphi}{\partial_\eta \varphi} \right) \Delta^{-1} D^k \Gamma_k. \tag{5.39}
\]
On the other hand, from Eqs. (5.27) and (5.29), we obtain
\[
(2)_s E^{(2)} - 2\partial_\eta (2)_s E^{(4)} := 2\partial_\eta (2)_s \Phi + 2\mathcal{H} (2)_s \Phi - 8\pi G\partial_\eta \varphi \varphi_2 \\
+ 3\partial_\eta (\Delta + 3K)^{-1} \left( \Delta^{-1} D^j D^i \Gamma_{ij} - \frac{1}{3} \Gamma_k^k \right) - \Delta^{-1} D^k \Gamma_k = 0.
\] (5.40)

Through Eq. (5.40) and the background Klein-Gordon equation (3.10), the equation (5.39) is given by
\[
(2)_s E^{(3)} = \left( -\partial_\eta^2 - 5\mathcal{H}\partial_\eta + \frac{4}{3}\Delta + 4K \right) (2)_s E^{(4)} - (2)_s E^{(1)} \\
- \left( 2\mathcal{H} + \frac{\partial^2 \varphi}{\partial_\eta \varphi} \right) (2)_s E^{(2)} - 2\partial_\eta (2)_s E^{(4)} - 8\pi G\alpha^2 C_K \varphi_2 \\
- \Delta^{-1} \left( \partial_\eta D_k^k \Gamma_k + 2\mathcal{H} D_k^k \Gamma_k - D^j D^i \Gamma_{ij} \right).
\] (5.41)

This equation (5.41) shows that Eq. (5.28) is consistent with the set of the background, the first-order, and the other second-order Einstein equations if the equation (\partial_\eta + 2\mathcal{H}) D_k^k \Gamma_k - D^j D^i \Gamma_{ij} = 0 (5.42)
is satisfied under the background and the first-order Einstein equations. Actually, we have already seen Eq. (5.37) is satisfied under the background and the first-order Einstein equation. Taking the divergence of Eq. (5.37), we can easily confirm Eq. (5.42). Thus, the component (5.28) of the Einstein equation is not independent of the set of equations (5.26), (5.27), (5.29), and the first-order perturbations of the Einstein equation, i.e., Eqs. (4.16), (4.17), (4.19). As seen above, the component (4.18) of the first-order Einstein equation is derived from the set of the equations (4.16), (4.17), (4.19) and the background Einstein equations. This implies that the potential of the scalar field affects to the evolution of the system only through the background Einstein equations at least in the first order perturbations. We have also seen that the situation is also same even in the second-order perturbations, i.e., the potential of the scalar field affects to the evolution of the system only through the background Einstein equation even in the second-order perturbations.

Thus, we have seen that the derive Einstein equations of the second order (5.26)--(5.35) are consistent with each other through the equation (5.37). This fact implies that the derived source term \(\Gamma_i\) and \(\Gamma_{ij}\) of the second-order perturbations of the Einstein equations, which are defined by Eqs. (5.34) and (5.35), are correct source terms of the second-order Einstein equations. On the other hand, for \(\Gamma_0\), we have to consider the consistency between the perturbative Einstein equations and the perturbative Klein-Gordon equation as seen below.

5.2.1. Consistency with the Klein-Gordon equation

Here, we consider the consistency of the second-order perturbation of the Klein-Gordon equation and the Einstein equations. As shown in KN2008, the second-
order perturbation of the Klein-Gordon equation is given by
\[ a^2 C^{(2)}_{(K)} = -\partial^2 \varphi_2 - 2\mathcal{H}\partial_\eta \varphi_2 + \Delta \varphi_2 + \partial^2_\eta \varphi_2 + 3\partial_\eta \varphi_2 + 3\partial_\eta \Psi \partial_\eta \varphi \\
-2a^2 \frac{\partial V}{\partial \varphi_2} - a^2 \varphi_2 \frac{\partial^2 V}{\partial \varphi_2^2} + \Xi_{(K)}, \]
(5.43)
where the term \( \Xi_{(K)} \) is reduced to
\[ \Xi_{(K)} := 8\partial_\eta \varphi_1 + 8 \partial \varphi_1 + 8 \partial \varphi_1 \partial_\eta \varphi \\
-2\chi^{ij} D_j D_i \varphi_1 + \chi^{ij} \partial_\eta \chi^{ij} \partial_\eta \varphi \\
-4a^2 \frac{\partial^2 V}{\partial \varphi_2} - a^2 \varphi_1 \frac{\partial^3 V}{\partial \varphi_2^3} + a^2 \varphi_1 \frac{\partial^3 V}{\partial \varphi_2^3}. \]
(5.44)
Here, we have imposed the Einstein equations (4.19) and (4.20) of the first order, the background Klein-Gordon equation (3.10), and its first-order perturbation (4.24).

As in the case of Eq. (4.26) for the first-order perturbation of the Klein-Gordon equation, we check the consistency of the second-order perturbation (5.43) of the Klein-Gordon equation with the second-order perturbations (5.26)–(5.35) of the Einstein equation. Since the vector-mode \( \nu_i \) and the tensor-mode \( \chi^{ij} \) of the second-order do not appear in the expressions (5.43) nor (5.44) of the second-order perturbation of the Klein-Gordon equation, we may concentrate on the Einstein equations for scalar-mode of the second order, i.e., Eqs. (5.26)–(5.29), with the definitions (5.33)–(5.35) of the source terms. Further, as shown above, the equation (5.28) is not independent equation from the set of equations consists of the second-order perturbations of the Einstein equation (5.26), (5.27), (5.29), the first-order perturbations of the Einstein equation (4.16), (4.17), (4.19), and the background Einstein equations (3.6) and (3.7). Moreover, as shown in §§3.2 and 4.2.1 the background Klein-Gordon equation is also derived from the background Einstein equation, and the first-order perturbation of the Klein-Gordon equation is also derived from the background and the first-order perturbations of the Einstein equations. For these reason, the second-order perturbation of the Klein-Gordon equation should be also derived from the set of the equations which consists of the second-order perturbations of the Einstein equations (5.26), (5.27), (5.29), the first-order perturbations of the Einstein equation (4.16), (4.17), (4.19), and the background Einstein equations (3.6) and (3.7). Actually, as in the case of Eq. (4.26), we can easily derive the relation

\[ -8\pi Ga^2 \partial_\eta \varphi_2 \frac{\partial^2 \varphi}{\partial \eta \varphi} C_{(K)} = 3 \left[ \frac{\varphi_2^2}{\partial_\eta \varphi_2} + \left( \frac{\partial^2 \varphi}{\partial_\eta \varphi} + 4\mathcal{H} \right) \partial_\eta + 2\partial_\eta \mathcal{H} - 4\mathcal{H}^2 - 2\mathcal{H} \frac{\partial^2 \varphi}{\partial_\eta \varphi} \right] \]

(4.16)

\[ -2 \frac{\partial \varphi}{\partial \eta_2} - 2\mathcal{H} \left[ \right] \]
where we have used Eqs. (3.6), (3.8), (5.27), (5.29), and (5.26). Equation (5.45)
vanishes due to Eq. (5.37). Therefore, we may say that the second-order
perturbation of the Klein-Gordon equation is consistent
with the background and first-order Einstein equations. Actually, we
can derive the relation

\[ 2 (\partial_\eta + \mathcal{H}) \Gamma_0 - D^k \Gamma_k + \mathcal{H} \Gamma_k^k + 8\pi G (\partial_\eta \varphi)^3 \Xi(K) \]

\[ + \Delta^{-1} \left[ (\partial_\eta - 2\mathcal{H}) D^k \left\{ \partial_\eta \Gamma_k + 2\mathcal{H} \Gamma_k - D^l \Gamma_{lk} \right\} \right], \]

(5.45)

where we have used Eqs. (3.6), (3.8), (5.27), (5.29), and (5.26). Equation (5.45)
shows that the second-order perturbation of the Klein-Gordon equation is consistent
with the background, the second-order Einstein equations if the last two lines in
Eq. (5.45) vanish. Further, since \( \Gamma_k \) and \( \Gamma_{ij} \) satisfy Eq. (5.37), the last line in
Eq. (5.45) vanishes due to Eq. (5.37). Therefore, we may say that the second-order
perturbation of the Klein-Gordon equation is consistent with the background and
the second-order Einstein equations if the equation

\[ 2 (\partial_\eta + \mathcal{H}) \Gamma_0 - D^k \Gamma_k + \mathcal{H} \Gamma_k^k + 8\pi G (\partial_\eta \varphi)^3 \Xi(K) = 0 \]

(5.46)
is satisfied under the background and first-order Einstein equations. Actually, we
can derive the relation

\[ 2 (\partial_\eta + \mathcal{H}) \Gamma_0 - D^k \Gamma_k + \mathcal{H} \Gamma_k^k + 8\pi G (\partial_\eta \varphi)^3 \Xi(K) \]

\[ = 4 \Phi \left[ \Phi \partial_\eta + 2 (\partial_\eta + 2\mathcal{H}) \Phi \right] \left( 3 \left( s \right) E_{(1)} - \left( s \right) E_{(2)} \right) \]

\[ - 16\pi G a^2 \left[ - 2\partial_\eta \varphi \left( \partial_\eta \varphi + \partial_\eta \varphi_1 \right) \right] \]

\[ - 8 \left( \partial_\eta \right) \left\{ \partial_\eta^2 - \left( \partial_\eta \varphi \right)^2 + 4\mathcal{H} \right\} \partial_\eta + \Delta + \frac{\partial^2 \varphi}{\partial_\eta \varphi} \left( \frac{\partial^2 \varphi}{\partial \eta \varphi} - 4\mathcal{H} \right) - \frac{\partial^3 \varphi}{\partial_\eta \varphi} \]

\[ + 4\partial_\eta \left( \partial_\eta \right) \left( \partial_\eta - \partial_\eta \varphi \right) \left( s \right) E_{(2)} \]
\[
+4 \left[ (1) \Phi \partial_\eta + 4 (\partial_\eta + \mathcal{H}) (1) \Phi \right] (1) E^{(1)}_\eta \\
+4 (1) \chi^{ij} D_iD_j (s) E^{(2)}_\eta \\
- (1) \chi^l \partial_\eta (1) \chi^{ij} \left( 3 (s) E^{(1)}_\eta - (s) E^{(2)}_\eta \right) + \frac{1}{2} (\partial_\eta + 4\mathcal{H}) (1)^l \chi^{lm} (s) E^{(6)lm} \\
= 0,
\]

where we have used the background Einstein equation (3.9), the first-order perturbation (4.24) of the Klein-Gordon equation, the scalar-part of the first-order perturbation of the momentum constraint (4.17), the evolution equation (4.16) of the scalar-mode in the first-order perturbation of the Einstein equation, and the evolution equation (4.21) of the tensor-mode in the first-order perturbation of the Einstein equation.

As shown in §4.2.1, the first-order perturbation of the Klein-Gordon equation is derived from the background and the first-order perturbations of the Einstein equation. In the case of the second-order perturbation of the Klein-Gordon equation (5.43), we have derived the relation (5.45) from the background Einstein equations (3.6) and (3.7), the scalar-part of the second-order perturbation of the Einstein equation (5.26), (5.27), and (5.29). These equations include the source terms \( \Gamma_0, \Gamma_i, \Gamma_{ij}, \) and \( \Xi(K) \) due to the mode-coupling of the linear-order perturbations. The equation (5.45) gives the relation (5.46) between the source terms \( \Gamma_0, \Gamma_i, \Gamma_{ij}, \Xi(K) \) and we have also confirmed that the equation (5.46) is satisfied due to the background, the first-order perturbation of the Einstein equations, and the Klein-Gordon equation. Thus, the second-order perturbation of the Klein-Gordon equation is not independent equation of the Einstein equations if we impose on each order perturbations of the Einstein equation at any conformal time \( \eta \). This also implies that the derived formulae of the source terms \( \Gamma_0, \Gamma_i, \Gamma_{ij}, \) and \( \Xi(K) \) are consistent with each other. In this sense, we may say that the formulae (5.33)–(5.35) and (5.44) for these source terms are correct.

§6. Summary and Discussion

In summary, we derived the all components of the second-order perturbation of the Einstein equation without ignoring any modes of perturbation in the case of a perfect fluid and a scalar field. The derivation is based on the general framework of the second-order gauge-invariant perturbation theory developed in the paper KN2003\(^5\) and KN2005\(^6\). In this formulation, any gauge fixing is not necessary and we can obtain any equation in the gauge-invariant form which is equivalent to the complete gauge fixing. In other words, our formulation gives complete gauge fixed equations without any gauge fixing. Therefore, equations which are obtained in gauge-invariant manner cannot be reduced without physical restrictions any more. In this sense, the equations shown here are irreducible. This is one of the advantages of the gauge-invariant perturbation theory.
The resulting Einstein equations of the second order shows that any types of mode-coupling appears as the quadratic terms of the linear-order perturbations due to the non-linear effect of the Einstein equations, in principle. Perturbations in cosmological situations are classified into three types: scalar-, vector-, tensor-types. In the second-order perturbations, we also have these three types of perturbations as in the case of the first-order perturbations. Further, in the equations for the second-order perturbations, there are many quadratic terms of linear-order perturbations due to the nonlinear effects of the system. Due to these nonlinear effects, the above three types of perturbations couple with each other.

Actually, the source terms \( \Gamma_0, \Gamma_i, \Gamma_{ij}, \Xi_0, \) and \( \Xi_i \) defined by Eqs. (5.9)–(5.11), (5.13), and (5.18) in the perfect fluid case include all types of mode-coupling, i.e., the scalar-scalar; the scalar-vector; the scalar-tensor; the vector-vector; the vector-tensor; the tensor-tensor types. Since we concentrate only on the case of a single perfect fluid, there is no anisotropic stress in the energy momentum tensor and we have Eq. (4.4). This equation is imposed in the definitions (5.9)–(5.11). For this reason, the resulting Einstein equations are simpler than those in the case of the fluid with anisotropic stress and the source terms in Eqs. (5.9)–(5.11) are not generic form in this sense. Even in this simple case, Eqs. (5.9)–(5.11), (5.13), and (5.18) include all types of mode-coupling. Thus, we should keep in mind that all types of mode-coupling may occur in some situations. However, we may neglect vector- and tensor-modes of the linear order in many realistic situations because these modes rapidly decay due to the expansion of the universe. If we take these behaviors of each mode into account, the source terms Eqs. (5.9)–(5.11), (5.13), and (5.18) become simpler.

In the case of the single scalar field, the vector-mode of the linear order vanishes due to the first-order perturbation of the momentum constraint. Further, we also have Eq. (4.19). Due to these two facts, the source terms \( \Gamma_0, \Gamma_i, \Gamma_{ij}, \) and \( \Xi_i(K) \), which are defined Eqs. (5.33)–(5.35) and (5.44), are simpler than those in the case of a perfect fluid. As a result, the source terms (5.33)–(5.35) shows the mode-coupling of the scalar-scalar; the scalar-tensor; and the tensor-tensor types. Since the tensor-mode of the linear order is also generated due to quantum fluctuations during the inflationary phase, the mode-coupling of the scalar-tensor and the tensor-tensor types may appear in the inflation. If these mode-coupling occur during the inflationary phase, these effects will depend on the scalar-tensor ratio \( r \). If so, there is a possibility that the accurate observations of the second-order effects in the fluctuations of the scalar-type in our universe also restrict the scalar-tensor ratio \( r \) or give some consistency relations between the other observations such as the measurements of the B-mode of the polarization of CMB. This is a new effect which gives some information of the scalar-tensor ratio \( r \).

Further, we have also checked the consistency between the second-order perturbations of the equation of motion of matter field and the Einstein equations. In the case of a perfect fluid, we considered the consistency between the second-order perturbations of the energy continuity equation, the Euler equation, and the Einstein equations. As a result, we obtain the consistency relation between the source terms in these equations \( \Gamma_0, \Gamma_i, \Gamma_{ij}, \Xi_0, \) and \( \Xi_i \) which are given by Eqs. (5.15) and (5.24).
with Eq. (5.22). We also showed that these consistency relations between the source terms are satisfied through the background and the first-order perturbation of the Einstein equations. This implies that the set of all equations are self-consistent and the derived source terms $\Gamma_0$, $\Gamma_i$, $\Gamma_{ij}$, $\Xi_0$, and $\Xi_i$ are correct. We also note that these results are independent of the equation of state of the perfect fluid.

In the case of a scalar field, we checked the consistency between the second-order perturbations of the Klein-Gordon equation and the Einstein equations. As in the case of a perfect fluid, we have also obtained the consistency relation between the source terms $\Gamma_0$, $\Gamma_i$, $\Gamma_{ij}$, and $\Xi_{(K)}$ which are given by Eqs. (5.37) and (5.46). We note that the relation (5.37) comes from the initial value constraint in the Einstein equations of the second order by itself, while the relation (5.46) comes from the second-order perturbation of the Klein-Gordon equation. We also showed that these relations between the source terms are satisfied through the background and the first-order perturbation of the Einstein equations. This implies that the set of all equations are self-consistent and the derived source terms $\Gamma_0$, $\Gamma_i$, $\Gamma_{ij}$, and $\Xi_{(K)}$ are correct. We also note that these relations are independent of the details of the potential of the scalar field.

Thus, we have derived the self-consistent set of equations of the second-order perturbation of the Einstein equations and the evolution equation of matter fields in the cases of a perfect fluid and a scalar field, respectively. Therefore, in the case of the single matter field, we may say that we have been ready to clarify the physical behaviors of the second-order cosmological perturbations. The physical behavior of the second-order perturbations in the universe filled with a single matter field will be instructive to clarify the physical behaviors of the second-order cosmological perturbations in more realistic situations. We leave these issues as future works.

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**Appendix A**

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**Perturbations of energy momentum tensors**

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Since we consider the perturbations of the Einstein equations, we summarize the components of the perturbations of the energy momentum tensor for a perfect fluid and a scalar field. Though the ingredients of this section are already given in KN2007 and KN2008, we show again these components to summarize the explicit definitions of the perturbative variables which are necessary in the main text of this paper.
A.1. Perfect fluid

Here, we consider the perturbative expressions of the energy momentum tensor for a perfect fluid. The total energy momentum tensor of the fluid is characterized by the energy density $\bar{\epsilon}$, the pressure $\bar{p}$, and the four-velocity $\bar{u}^a$, and it is given by

$$(p)\bar{T}^b_a = (\bar{\epsilon} + \bar{p})\bar{u}^b \bar{u}_a + \bar{p}\delta^b_a. \quad (A.1)$$

Since $\bar{\epsilon}$, $\bar{p}$, and $\bar{u}_a$ are variables on the physical spacetime $M$, these variables are pulled back to the background spacetime through an appropriate gauge choice $X_\lambda$ to evaluate these variables on the background spacetime. We expand the fluid components $\bar{\epsilon}$, $\bar{p}$, and $\bar{u}_a$ in Eq. (A.1) as follows:

$$\bar{\epsilon} := \epsilon + \lambda \frac{\epsilon}{1} + \frac{1}{2} \lambda^2 \frac{\epsilon}{2} + O(\lambda^3); \quad (A.2)$$

$$\bar{p} := p + \lambda \frac{p}{1} + \frac{1}{2} \lambda^2 \frac{p}{2} + O(\lambda^3); \quad (A.3)$$

$$\bar{u}_a := u_a + \lambda \frac{u_a}{1} + \frac{1}{2} \lambda^2 \frac{u_a}{2} + O(\lambda^3). \quad (A.4)$$

Following to Eqs. (2.15) and (2.16), we define the gauge-invariant variable for the perturbations of the fluid components $\bar{\epsilon}$, $\bar{p}$, and $\bar{u}_a$:

$$\mathcal{E} := \epsilon - \mathcal{L}_X \epsilon; \quad \mathcal{P} := p - \mathcal{L}_X p; \quad \mathcal{U}_a := (u_a) - \mathcal{L}_X u_a; \quad (A.5)$$

$$\mathcal{E} := \epsilon - 2\mathcal{L}_X \epsilon - \{\mathcal{L}_Y - \mathcal{L}_X^2\} \epsilon; \quad \mathcal{P} := p - 2\mathcal{L}_X p - \{\mathcal{L}_Y - \mathcal{L}_X^2\} p; \quad (A.6)$$

$$\mathcal{U}_a := (u_a) - 2\mathcal{L}_X u_a - \{\mathcal{L}_Y - \mathcal{L}_X^2\} u_a. \quad (A.7)$$

where the vector fields $X_a$ and $Y_a$ are the gauge-variant parts of the first- and second-order metric perturbations, respectively, and these vector fields are defined in Eqs. (2.7) and (2.11).

Components of the background value and the first- and the second-order perturbations of the fluid four-velocity are summarized as

$$u_a = -a(d\eta)_a, \quad (A.8)$$

$$\mathcal{U}_a = -a \left( \Phi \frac{(d\eta)_a}{(dx^i)_a} + \left(D_i \frac{(d\eta)_a}{(dx^i)_a} + \frac{(d\eta)_a}{(dx^i)_a}\right) \right), \quad (A.9)$$

$$\mathcal{U}_a = \mathcal{U}_a \left( \frac{(d\eta)_a}{(dx^i)_a} + \frac{(d\eta)_a}{(dx^i)_a}\right) \left(D_i \frac{(d\eta)_a}{(dx^i)_a} + \frac{(d\eta)_a}{(dx^i)_a}\right), \quad (A.10)$$

where

$$D_i \frac{(d\eta)_a}{(dx^i)_a} = D_i \frac{(d\eta)_a}{(dx^i)_a} = 0, \quad (A.11)$$

$$\mathcal{U}_a = a \left( \Phi \right)^2 \frac{(d\eta)_a}{(dx^i)_a} - \Phi - \left(D_i \frac{(d\eta)_a}{(dx^i)_a} + \frac{(d\eta)_a}{(dx^i)_a}\right) \right) \left(D_i \frac{(d\eta)_a}{(dx^i)_a} + \frac{(d\eta)_a}{(dx^i)_a}\right), \quad (A.12)$$
Here, we have used the normalization conditions of the four-velocity $\bar{g}^{ab} \bar{u}_a \bar{u}_b = -1$, and its perturbations.

The perturbative expansion of the energy momentum tensor (A.1) is given by

$$ (p)^{b}_{a} T = (p)^{b}_{a} T + \lambda \left( (p)^{b}_{a} T + \frac{1}{2} \lambda^2 (p)^{b}_{a} T + O(\lambda^3) \right). \quad (A.13) $$

The background energy momentum tensor for a perfect fluid is given by

$$ T^{b}_{a} = \epsilon \bar{u}^{a} \bar{u}^{b} + p (\delta^{b}_{a} + \bar{u}^{a} \bar{u}^{b}) $$

$$ = -\epsilon (d\eta)_a \left( \frac{\partial}{\partial \eta} \right)^b + p \gamma^b_a, \quad (A.15) $$

where we have used

$$ \delta^b_a = (d\eta)_a \left( \frac{\partial}{\partial \eta} \right)^b + \gamma^b_a, \quad (A.16) $$

and $\gamma_{ab} := \gamma_{ij}(dx^i)_a(dx^j)_b$, $\gamma^b_a := \gamma^j_i(dx^i)_a(\partial/\partial x^j)^b$.

The first- and the second-order perturbations $\langle p \rangle T^{b}_{a}$ and $\langle 2 \rangle T^{b}_{a}$ of the energy momentum tensor are also decomposed into the form as Eqs. (2.15) and (2.16), respectively, i.e.,

$$ \langle 1 \rangle (p)^{b}_{a} T = \langle 1 \rangle (p)^{b}_{a} T + \mathcal{L}_X \langle 1 \rangle (p)^{b}_{a} T, \quad (A.17) $$

$$ \langle 2 \rangle (p)^{b}_{a} T = \langle 2 \rangle (p)^{b}_{a} T + 2\mathcal{L}_X \langle 1 \rangle (p)^{b}_{a} T + \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} \langle 1 \rangle (p)^{b}_{a} T. \quad (A.18) $$

Here, the components of the gauge-invariant parts $\langle 1 \rangle (p)^{b}_{a} T$ of the first order are given by

$$ \langle 1 \rangle (p)^{\eta}_{a} T = - \mathcal{E}, \quad (A.19) $$

$$ \langle 1 \rangle (p)^{i}_{a} T = - (\epsilon + p) \left( D^i_{\nu} + V^i_{\nu} + \nu^i \right), \quad (A.20) $$

$$ \langle 1 \rangle (p)^{i}_{a} T = (\epsilon + p) \left( D^i_{\nu} + V^i_{\nu} \right), \quad (A.21) $$

$$ \langle 1 \rangle (p)^{i}_{a} T = \mathcal{P} \delta^i_a, \quad (A.22) $$

and the components of the gauge-invariant part $\langle 2 \rangle (p)^{b}_{a} T$ of the second order are given by

$$ \langle 2 \rangle (p)^{\eta}_{a} T = - 2 (\epsilon + p) \left( D^i_{\nu} + V^i_{\nu} \right) \left( D^i_{\nu} + V^i_{\nu} \right), \quad (A.23) $$

and

$$ \langle 2 \rangle (p)^{i}_{a} T = - 2 (\epsilon + p) \left( D^i_{\nu} + V^i_{\nu} \right) \left( D^i_{\nu} + V^i_{\nu} \right), \quad (A.24) $$

and

$$ \langle 2 \rangle (p)^{i}_{a} T = - 2 (\epsilon + p) \left( D^i_{\nu} + V^i_{\nu} \right) \left( D^i_{\nu} + V^i_{\nu} \right), \quad (A.25) $$

and

$$ \langle 2 \rangle (p)^{i}_{a} T = - 2 (\epsilon + p) \left( D^i_{\nu} + V^i_{\nu} \right) \left( D^i_{\nu} + V^i_{\nu} \right), \quad (A.26) $$
\[ (p)T^{i}_{i} = 2 \left( (1)E + (1)P \right) \left( D_{i} (1)\dot{v} + \dot{V}_{i} \right) \]
\[ + (\epsilon + p) \left( D_{i} (2)\dot{v} + \dot{V}_{i} - 2 (1)\phi D_{i} (1)\dot{v} - 2 (1)\dot{\phi} V_{i} \right), \tag{A.24} \]
\[ (p)T^{i}_{\eta} = -2 \left( (1)E + (1)P \right) \left( D_{i} (1)\dot{v} + \dot{V}_{i} - \nu^{i} \right) \]
\[ + (\epsilon + p) \left\{ -D_{i} (2)\dot{v} - V^{i} + \nu^{i} - 2 (1)\phi \left( D_{i} (1)\dot{v} + \dot{V}_{i} \right) \right. \]
\[ \left. + 2 \left( -2 (1)\Psi^{ij} + \chi^{ij} \right) \left( D_{j} (1)\dot{v} + \dot{V}_{j} - \nu_{j} \right) \right\}, \tag{A.25} \]
\[ (p)T^{j}_{i} = 2 (\epsilon + p) \left( D_{i} (1)\dot{v} + \dot{V}_{i} \right) \left( D_{j} (1)\dot{v} + \dot{V}_{j} - \nu^{j} \right) + P \delta^{j}_{i}. \tag{A.26} \]

**A.2. Scalar fluid**

Next, we summarize the gauge-invariant variables for the perturbations of a scalar field. The energy momentum tensor of the single scalar field \( \bar{\phi} \) is given by
\[ \bar{T}_{b}^{a} = g^{bc} \bar{\nabla}_{a} \bar{\phi} \bar{\nabla}_{c} \bar{\phi} - \frac{1}{2} \delta^{b}_{a} \left( g^{cd} \bar{\nabla}_{c} \bar{\phi} \bar{\nabla}_{d} \bar{\phi} + 2 V(\bar{\phi}) \right), \tag{A.27} \]
where \( V(\phi) \) is the potential of the scalar field \( \phi \). Since we shall consider a homogeneous and isotropic universe with small perturbations, the scalar field must also be approximately homogeneous. In this case, the scalar field \( \bar{\phi} \) can be expanded as
\[ \bar{\phi} = \phi + \lambda \bar{\phi}_{1} + \frac{1}{2} \lambda^{2} \bar{\phi}_{2} + O(\lambda^{3}), \tag{A.28} \]
where \( \phi \) is the homogeneous function on the homogeneous isotropic universe, i.e.,
\[ \phi = \phi(\eta). \tag{A.29} \]
The background energy momentum tensor for the scalar field on the homogeneous isotropic universe is given by
\[ T_{a}^{b} = \nabla_{a} \phi \nabla^{b} \phi - \frac{1}{2} \delta_{a}^{b} \left( \nabla_{c} \phi \nabla^{c} \phi + 2 V(\phi) \right) \]
\[ = - \left( \frac{1}{2a^{2}}(\partial_{\eta} \phi)^{2} + V(\phi) \right) \left( d\eta \right)_{a} \left( \frac{\partial \lambda}{\partial \eta} \right)^{b} + \left( \frac{1}{2a^{2}}(\partial_{\eta} \phi)^{2} - V(\phi) \right) \gamma_{a}^{b}. \tag{A.31} \]
The energy momentum tensor \( (A.27) \) can also be decomposed into the background, the first-order perturbation, and the second-order perturbation:
\[ \bar{T}_{a}^{b} = T_{a}^{b} + \lambda^{(1)} \left( T_{a}^{b} \right) + \frac{1}{2} \lambda^{2(2)} \left( T_{a}^{b} \right) + O(\lambda^{3}), \tag{A.32} \]
where \((1) (T^b_a)\) is linear in matter and metric perturbations \(\hat{\varphi}_1\) and \(h_{ab}\), and \((2) (T^b_a)\) includes the second-order metric and matter perturbations \(l_{ab}\) and \(\hat{\varphi}_2\) and quadratic terms of the first-order perturbations \(\hat{\varphi}_1\) and \(h_{ab}\).

As in the case of the perfect fluid, each order perturbations of the scalar field \(\varphi\) is decomposed into the gauge-invariant part and gauge-variant part:

\[
\hat{\varphi}_1 =: \varphi_1 + \mathcal{L}_X \varphi, \tag{A.33}
\]
\[
\hat{\varphi}_2 =: \varphi_2 + 2\mathcal{L}_X \hat{\varphi}_1 + \left(\mathcal{L}_Y - \mathcal{L}_X^2\right) \varphi, \tag{A.34}
\]

where \(\varphi_1\) and \(\varphi_2\) are the first-order and the second-order gauge-invariant perturbation of the scalar field.

The perturbed energy momentum tensor of each order also decomposed into the gauge-invariant and gauge-variant parts as \((2.15)\) and \((2.16)\). Through Eqs. \((2.7)\), \((2.11)\), \((A.33)\), and \((A.34)\), we can decompose the perturbations \((1) (T^b_a)\) and \((2) (T^b_a)\) of the energy momentum tensor as

\[
(1) (T^b_a) =: (1) T^b_a + \mathcal{L}_X T^b_a, \tag{A.35}
\]
\[
(2) (T^b_a) =: (2) T^b_a + 2\mathcal{L}_X (1) T^b_a + \left(\mathcal{L}_Y - \mathcal{L}_X^2\right) T^b_a. \tag{A.36}
\]

Further, through the components \((2.9)\) of the gauge-invariant part of the first-order metric perturbation, the definition \((A.33)\) of the first-order perturbation of the scalar field, and the homogeneous background condition \((A.29)\) for the background field, the components of the first-order perturbation of the energy-momentum tensor of the scalar field are given by

\[
(1) T^{\eta \eta} = -\frac{1}{a^2} \left\{ \partial_\eta \varphi \partial_\eta \varphi_1 - \frac{(1)}{\Phi} (\partial_\eta \varphi)^2 + a^2 \partial V \partial_\varphi \varphi_1 \right\}, \tag{A.37}
\]
\[
(1) T^{\eta i} = \frac{1}{a^2} \partial_\eta \varphi \left( D^i \varphi_1 + \nu^i \partial_\eta \varphi \right), \tag{A.38}
\]
\[
(1) T^{\eta i} = -\frac{1}{a^2} D_i \varphi_1 \partial_\eta \varphi, \tag{A.39}
\]
\[
(1) T^{j i} = \frac{1}{a^2} \gamma^{j i} \left\{ \partial_\eta \varphi \partial_\eta \varphi_1 - \frac{(1)}{\Phi} (\partial_\eta \varphi)^2 - a^2 \partial V \partial_\varphi \varphi_1 \right\}. \tag{A.40}
\]

Finally, we summarize the components of the gauge-invariant part of the second-order perturbation of the energy momentum tensor for a scalar field. Through the components \((2.9)\) and \((2.13)\) of the gauge-invariant parts of the first- and the second-order metric perturbations, the definitions \((A.33)\) and \((A.34)\) of the gauge-invariant variables for the first- and the second-order perturbations of the scalar field, and the homogeneous background condition \((A.29)\), the components of the second-order perturbation of the energy-momentum tensor for a single scalar field are given by

\[
(2) T^{\eta \eta} = -\frac{1}{a^2} \left\{ \partial_\eta \varphi \partial_\eta \varphi_2 - (\partial_\eta \varphi)^2 \left(2 \frac{2}{\Phi} + a^2 \varphi_2 \partial V \partial_\varphi \right) - 4 \partial_\eta \varphi \frac{(1)}{\Phi} \partial_\eta \varphi_1 + 4(\partial_\eta \varphi)^2 \right\}. \tag{A.41}
\]
\[-(\partial_\eta \varphi)^2 \nu^i (1) \nu_i + (\partial_\eta \varphi_1)^2 + D_i \varphi_1 D^i \varphi_1 + a^2 (\varphi_1)^2 \frac{\partial^2 V}{\partial \varphi^2}\], \hspace{1cm} (A.41)

\[(2) T^{\eta}_{i} = -\frac{1}{a^2} \left\{ \partial_\eta \varphi \left( D_i \varphi_2 - 4 D_i \varphi_1 (1) \Phi \right) + 2 D_i \varphi_1 \partial_\eta \varphi_1 \right\}, \hspace{1cm} (A.42)

\[(2) T^{\eta}_{i} = \frac{1}{a^2} \left[ \partial_\eta \varphi D^i \varphi_2 + 2 \partial_\eta \varphi_1 D^i \varphi_1 + 2 \partial_\eta \varphi \left( 2 \nu^i \partial_\eta \varphi_1 + 2 \Phi \nu^i \partial_\eta \varphi_1 - \chi^i \nu^i \right) \right] + (\partial_\eta \varphi)^2 \left( \nu^i - 4 (1) \Phi \nu^i + 4 (1) \Phi \nu^i - 2 \chi^i \nu^i \right), \hspace{1cm} (A.43)

\[(2) T^{j}_{i} = \frac{2}{a^2} \left[ D_i \varphi_1 D^j \varphi_1 + D_i \varphi_1 \nu^1 \partial_\eta \varphi \right.
\left. + \frac{1}{2} \gamma^1_j \left\{ + \partial_\eta \varphi \left[ \partial_\eta \varphi_2 - 4 (1) \Phi \partial_\eta \varphi_1 - 2 \nu_1 \partial_\eta \varphi_1 \right] \right. \right.
\left. \left. + (\nabla_\eta \varphi)^2 \left( 4 \Phi^2 \nu_1 - (1) \Phi \nu_1 - (2) \Phi \right) + (\partial_\eta \varphi_1)^2 - D_i \varphi_1 D^i \varphi_1 \right. \right.
\left. \left. - a^2 \varphi_2 \frac{\partial V}{\partial \varphi} - a^2 (\varphi_1)^2 \frac{\partial^2 V}{\partial \varphi^2} \right\} \right]. \hspace{1cm} (A.44)

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