Use of Genetic Algorithms to solve Inverse Problems in Relativistic Hydrodynamics

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Abstract. We present the use of Genetic Algorithms (GAs) as a strategy to solve inverse problems associated with models of relativistic hydrodynamics. The signal we consider to emulate an observation is the density of a relativistic gas, measured at a point where a shock is traveling. This shock is generated numerically out of a Riemann problem with mildly relativistic conditions. The inverse problem we propose is the prediction of the initial conditions of density, velocity and pressure of the Riemann problem that gave origin to that signal. For this we use the density, velocity and pressure of the gas at both sides of the discontinuity, as the six genes of an organism, initially with random values within a tolerance. We then prepare an initial population of $N$ of these organisms and evolve them using methods based on GAs. In the end, the organism with the best fitness of each generation is compared to the signal and the process ends when the set of initial conditions of the organisms of a later generation fit the Signal within a tolerance.

1. Introduction
In high energy astrophysics, given the experiments cannot be repeated, the explanation and modeling of an event requires the solution of inverse problems, that is, one is interested in tracking the model and the initial conditions of an event, that fit the data of observations within certain error bars.

Simple type of observational data are time-series. These are collection of time dependent data, for instance the light curve of an astrophysical event in a certain frequency band. The models to explain -for instance- gamma ray bursts are of great variety [1], including various ingredients of matter, radiation, interaction between matter an radiation, models in one, two and three dimensions, etc. Each model attempts to solve an inverse problem, with certain parameters that are always faced with data.

The complexity of solving an inverse problem is strongly correlated to the simplicity of the model. For instance, when a set of experimental data has to be fit with a given function with $d$ parameters, the problem reduces to minimizing the error function defined in a parameter space of dimension $d$. However, more complicated data modeling depends on the solution of Ordinary Differential Equations or Partial Differential Equations (PDEs). For instance, in [2], a model of long gamma ray bursts is given as a solution of the Radiation Relativistic Hydrodynamics equations for a fine tuned set of initial conditions and a highly relativistic shock. This means that instead of trying to fit data with a given function, the data are assumed to be the result of the solution of a complicated system of non-linear PDEs that depends on some initial conditions.
A case illustrating how complicated an inverse problem can be, is the inverse problem of paramount current interest after the Gravitational Wave detection by the LIGO [3]. This problem is the reconstruction of the parameters of a binary black hole system out of the observed Gravitational Wave signal [4, 5].

Given the complexity of some inverse problems, more sophisticated and efficient methods are required that can be used with the current hardware technology. This is the case of GAs. The GA method is an abstraction of biological evolution for moving from one population of organisms (chromosomes, in our case chains of numbers) to a new generation of organisms by using a kind of natural selection together with the genetics inspired operations of crossover and mutation. Each chromosome is made of genes (in our case the pressure, density or velocity of a state of a relativistic gas at initial time). The method has many applications, and for the purposes of the present paper we apply it for an optimization process, that is, the rules of evolution and crossover will be defined in terms of how optimal the genes of a DNA organism are at minimizing an error function, namely the difference between the observed and computed signal.

The problem we choose to test GAs as inverse problem solvers is the solution of a relativistic hydrodynamics shock-tube that we describe below.

2. Numerical methods
2.1. Numerical solution of Relativistic Euler equations
The Riemann problem is an initial value problem for a gas with discontinuous initial data, whose evolution is ruled by Euler’s equations. The set of Relativistic Euler’s equations determine the evolution of the density of gas, its velocity field and either its pressure or total energy. The model we use is that of a perfect fluid, described by the stress energy tensor

\[ T^{\mu\nu} = \rho_0 h u^\mu u^\nu + p \eta^{\mu\nu}, \]

where \( \rho_0 \) is the rest mass density of a fluid element, \( u^\mu \) its four velocity, \( p \) the pressure, \( h = 1 + \varepsilon + p/\rho_0 \) is the specific enthalpy and \( \eta^{\mu\nu} \) are the components of the metric describing Minkowski space-time.

The set of relativistic Euler equations is obtained from the local conservation of the rest mass and the local conservation of the stress energy tensor of the fluid, which are respectively

\[ (\rho_0 u^\mu)_{,\mu} = 0, \]
\[ (T^{\mu\nu})_{,\nu} = 0, \]

where \( u^\mu = W(1, v^x, 0, 0) \) and \( W = \frac{1}{\sqrt{1 - v^x v^x}} \) is the Lorentz factor and \( v^x \) is the Eulerian velocity of the fluid elements. It is possible to arrange these equations as a flux balance set of equations as [6, 7]

\[ \partial_t \mathbf{u} + \partial_x \mathbf{F}(\mathbf{u}) = 0, \]

where conservative variables are defined by \( \mathbf{u} = (D,S^x,\tau)^T \) and the resulting fluxes are \( \mathbf{F} = (Dv,Sv + p,S) \), where we assume that specifically \( v = v^x \) and \( S = S^x \), since we are only considering one spatial dimension. The conservative variables are defined in terms of the primitive ones \((\rho_0, v, p)\) as follows

\[ D = \rho_0 W; \]
\[ S = \rho_0 h W^2 v; \]
\[ \tau = \rho_0 h W - p. \]
In terms of these variables the flux balance equations are explicitly \[6, 7\]

\[
\begin{align*}
\partial_t D + \partial_x (D v) &= 0, \\
\partial_t S + \partial_x (S v + p) &= 0, \\
\partial_t \tau + \partial_x S &= 0.
\end{align*}
\]  

(4) \hspace{1cm} (5) \hspace{1cm} (6)

We assume a $\Gamma$-law Equation of State, $p = \rho_0 \varepsilon (\Gamma - 1)$, where $\Gamma$ is the adiabatic index $\Gamma = c_p/c_v$.

The initial data of the Riemann problem is defined by

\[
\mathbf{u} = \begin{cases} 
\mathbf{u}_L, & x < x_0 \\
\mathbf{u}_R, & x > x_0,
\end{cases}
\]  

(7)

where $\mathbf{u}_L$ and $\mathbf{u}_R$ represent the values of the gas properties on a chamber at the left and at the right from an interface between the two states at $x = x_0$ that exists only at initial time.

The domain used to prepare the initial conditions and track the evolution is $x \in [0, 1]$, $t \geq 0$ and the interface is initially located at $x_0 = 0.5$. We prepare the initial conditions on the left and right through the primitive variables $(\rho_L, v_L, p_L)$ and $(\rho_R, v_R, p_R)$, where for simplicity we have omitted the subscript zero that identifies the rest mass density.

2.2. Generation of the signal

What we call an observation is actually a numerical solution of the above system, that will be unknown to the GA. The parameters we choose are those of a standard tests for a mildly relativistic shock ($\rho_L = 10, v_L = 0, p_L = 13.33$) and ($\rho_R = 1, v_R = 0, p_R = 0$), using the value $\Gamma = 5/3$ for the adiabatic index. The shock wave moves toward the right of the domain as seen in the snapshots of Fig. 1. We choose the observed signal $\text{Signal}(t)$ to be the value of the density measured with a detector at $x_d = 0.75$ as a function of time. This a is pretty close example of how a scalar quantity is measured during a simulation at a given point, for instance in [2] this is how the loss of radiated energy is estimated during the propagation of a strong shock with radiation.

![Figure 1](image.png)

**Figure 1.** On the left we show the density, first it is discontinuous and once the system starts evolving a shock moves to the right with a mild velocity of 75% the speed of light. On the right we show the value of the density at $x_d = 0.75$ measured in time.

We calculate this numerical solution using a High Resolution Shock Capturing method, with a method of lines that uses the Harten-Lax-van Leer-Einfeldt flux formula, minmod reconstructor and third order Runge-Kutta time integrator (see for instance [8, 9, 10] for some applications).
2.3. Genetic algorithm

The inverse problem we want to solve is the problem of the cause. That is, we consider that the system generating the signal is ruled by the relativistic Euler equations above and that the initial conditions are those of a Riemann problem. However the cause is the set of initial conditions that produces $\text{Signal}(t)$. Physically, the observer only knows that the signal was generated by a Riemann problem of the relativistic 1D Euler equations and wonders about the initial conditions that generated the signal.

We define the DNA of an organism as a vector whose entries are the initial conditions. We say the organism $i$ is $(\rho_L, v_L, p_L, \rho_R, v_R, p_R)$, where each entry plays the role of a gene.

Then we define a population of organisms $i = 1, ..., N$, that is, we choose $N$ sets of initial conditions that define the DNA of each organism and run a simulation for each of them. We define the fitness of organism $i$ as $F_i = 1/L_1(E_i)$, where $E_i = (\rho(x_d = 0.75) - \text{Signal}(t))_i$ is the error of the observed density at $x = 0.75$ with respect to $\text{Signal}(t)$ and $L_1$ indicates the norm one.

For this paper we select the best fit $N/2$ organisms that we keep for the next generation. We complete such new generation with $N/2$ organisms resulting from the crossover of these well fitted $N/2$ organisms. The crossover between organisms is as follows, we take the first three values of the DNA of the organism $i$ and join them to the last three values of the organism $j$ to produce the child $(\rho_L, v_L, p_L, 0, 0, 0)_i + (0, 0, 0, \rho_R, v_R, p_R)_j$. To this new generation we apply a random mutation that may spontaneously give a stronger DNA. Then we run a new set of $N$ simulations and diagnose the fitness, then crossover again and repeat during $N_G$ generations.

2.4. Preparation of the populations

The entries of the organisms can in principle be arbitrary and generated randomly. In practice one knows at least the order of magnitude of the parameters to be found or classified in an inverse problem, in our case here, these are the initial conditions that gave rise to the Time Series. Since we generated the Time Series with initial conditions we know, we can systematize the efficacy of the GA by allowing the values of the initial entries of each organism to have random values within a range around the correct initial condition. We present two cases, one in which we allow the initial entries to have random values within a band of $\pm 20$ and $\pm 30\%$ from the correct initial conditions and a special case where half of the genes are selected randomly within $\pm 50\%$ and the other half within $\pm 10\%$. In all the three cases we set the population to $N = 10$.

3. Results

In Fig. 2 we present the results for the three cases described above. We show the signal detected for the best fitted organism of the first generation, and the best organism of a given latest generation depending on the case, 300, 100 and 300 generations. This illustrates how the initial data of the best initial conditions provides a significantly different signal compared to $\text{Signal}(t)$ and how the best fit organism of a later generation fits better the data of $\text{Signal}(t)$. We also show how the error of the best fit organism decreases with the number of generations and how it reaches a minimum, after which neither the mutation nor the evolution can improve the fitting. We say that at this stage the DNA of the best fitted organism of the last generation has genes pretty close to the initial conditions that generated the observed $\text{Signal}(t)$.

4. Final comments

We have presented a first exploration of the use of GAs in the solution of inverse problems involving PDEs, already similar to those related to realistic long gamma ray burst sources. The computing time seems to limit the type of problems that can be tackled using this approach. For instance, one of the cases using 300 generations with $N = 10$ organisms each,
Figure 2. Results for the three different scenarios discussed in this paper. On the left there is the value of the density measured at $x_d = 0.75$ for the organism with the best fitness of the first and the latest generation, and compare it with $\text{Signal}(t)$. On the right we show the difference between the best fit organism of each generation using the $L_1$ norm of the error.

requires the execution of 10 simulations per generation, that is, 3,000 simulations of a shock-tube evolution. The problem uses only one spatial dimension, and with our methods for solving the Relativistic Euler system, each simulation takes about 2 seconds in average, including the GA analysis. This means that the whole process takes about 6,000 seconds, less 2 hours.

In a more realistic scenario, considering more than 6 genes, with a less educated restriction on the initial values of the genes (in our cases within 20%, 30% and a combination of 50% and 10%), it might take a considerably longer number of generations and possibly a bigger population would be needed. Another potential and important limitation is also that increasing the number of dimensions, say from one to two, will increase the number of calculations in orders of magnitude,
and therefore the computing time of each solution of the PDE system. These are cases in which either more involved computing methods, like the use of parallel code or programming with GPUs are needed, or more sophisticated methods including pattern recognition and machine learning need to be applied as we have done for Gravitational Waves [5].

At the moment, we have shown here that for problems involving one spatial dimension our GA method is usable in the solution of inverse problems associated realistic astrophysical scenarios.

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