Hopf-algebra description of noncommutative-spacetime symmetries

Alessandra AGOSTINI

Dipartimento di Fisica, Università di Napoli “Federico II” and INFN, Sez. Napoli, Monte S. Angelo, Via Cintia, 80126 Napoli, Italy

Abstract

I give a brief summary of the results reported in [1], in collaboration with G. Amelino-Camelia and F. D’Andrea. I focus on the analysis of the symmetries of $\kappa$-Minkowski noncommutative space-time, described in terms of a Weyl map. The commutative-spacetime notion of Lie-algebra symmetries must be replaced by the one of Hopf-algebra symmetries. However, in the Hopf-algebra sense, it is possible to construct an action in $\kappa$-Minkowski which is invariant under a 10-generators Poincaré-like symmetry algebra.

1 Introduction

In recent research much attention has been devoted to the implications of noncommutativity for the classical Poincaré symmetries of Minkowski spacetime $\mathcal{M}$.

For the simplest NCSTs (noncommutative spacetimes), the canonical one ($[x_\mu, x_\nu] = i\theta_{\mu\nu}$), a full understanding has been matured, and in particular it has been established that the Lorentz-sector symmetries are broken [2]. But already at the next level of complexity, the one of Lie-algebra type ($[x_\mu, x_\nu] = i\zeta^{\sigma}_{\mu\nu}x_\sigma$), our present understanding of the fate of Poincaré symmetries is still unsatisfactory. Some progress on this problem was reported in Ref. [1], by Amelino-Camelia, D’Andrea and myself, focusing on the illustrative example of the “$\kappa$-Minkowski Lie-algebra noncommutative spacetime” $\mathcal{M}_\kappa$ [3, 4]:

$$[x_j, x_0] = \frac{i}{\kappa}x_j, \quad [x_j, x_k] = 0 \quad j, k = 1, 2, 3$$ (1)

In some mathematical studies [4, 5] it emerges that the symmetries of $\mathcal{M}_\kappa$ can be described by any one of a large number of $\kappa$-Poincaré Hopf algebras, but this degeneracy (based on “duality” axioms) remains obscure from a physics perspective. This issue has recently taken central stage also in research on the physical proposal [6] of relativistic theories with two invariants, where $\mathcal{M}_\kappa$ is being considered as a possible

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spacetime underlying these theories, and the possibility of observable consequences is being explored [7].

We proposed in Ref. [1] a new approach in which symmetries are introduced directly at the level of the action. We illustrated this idea in the simple case of a free scalar theory in $\mathcal{M}_\kappa$, and I intend to give a brief summary here of those results.

## 2 Hopf-Algebra description of symmetries

As in the familiar context of CSTs (commutative spacetimes) one can describe an external symmetry as a transformation of the coordinates that leaves invariant the action of the theory.

Let us consider the symmetry analysis for a commutative free scalar theory

$$S(\phi) = \int d^4x \phi(\Box - M^2)\phi \quad (\Box = \partial_\mu \partial^\mu)$$

(2)

The most general infinitesimal transformation generated by $T$ we can consider is:

$$x'_\mu = (1 - i\epsilon T)x \quad \phi'(x) = \phi(x) + (x_\mu - x'_\mu)\partial^\mu \phi(x) = (1 + i\epsilon T)\phi \quad (\epsilon)$$

(3)

Actually the action is invariant under $T$-generated transformations if and only if the variation of the action is zero. At the leading order in $\epsilon$:

$$S(\phi') - S(\phi) = i\epsilon \int d^4x \left( T\{\phi(\Box - M^2)\phi\} + \phi[\Box, T]\phi \right) = 0.$$

(4)

In Minkowski spacetime the symmetries of this action are fully described in terms of the classical Poincaré Lie algebra $\mathcal{P}$, and the operator $\Box = -P_\mu P^\mu$.

However an algebra can be promoted to Hopf algebra introducing some coalgebraic structures\footnote{A Hopf-algebra is an algebra endowed with a coproduct ($\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$), a counit ($\epsilon : \mathcal{A} \to C$) and an antipode ($S : \mathcal{A} \to \mathcal{A}$), with some compatibility conditions. A trivial Hopf-algebra is characterized by trivial structure of counit, coproduct and antipode over the generators $T$: $\Delta(T) = T \otimes 1 + 1 \otimes T$ $\epsilon(T) = 0$ $S(T) = -T$}. From this perspective $\mathcal{P}$ is equivalent to a “trivial Hopf algebra”.

Then for theories in CST the symmetries can always be described in terms of a trivial Hopf algebra. This property is connected with the commutativity of functions. In fact, from $f \cdot g = g \cdot f$, it follows that $\Delta$ is cocommutative (trivial). In general in a NCST $\Delta$ is not cocommutative, and the Lie-algebra description cannot be maintained, since it would not provide a sufficient set of rules to handle consistently the laws of symmetry transformation of products of (NC) functions.
3 Symmetries in $\kappa$-Minkowski: free scalar theory

One can introduce the noncommutative fields $\Phi \in \mathcal{M}_\kappa$ through the Weyl map. It is well known that the Weyl map is not unique and in order to explore the possible dependence of the symmetry analysis on the Weyl map it is useful to consider two explicit choices, respectively the "time-to-the right" $\Omega_R$ [8] and the "time-symmetrized" $\Omega_S$[9] map, defined in the following way:

$$\Phi_{R/S} = \Omega_{R,S}(\phi) = \int \tilde{\phi}(p) \Omega_{R,S}(e^{ipx}) d^4p$$

$$\Omega_R(e^{ipx}) = e^{ip\vec{x}} e^{-ip_0x_0} \quad \Omega_S(e^{ipx}) = e^{-ip_0 \frac{\kappa}{4}} e^{ip_0 \frac{\kappa}{4}} = \Omega_R(e^{i\vec{p}\cdot\vec{x}} e^{-ip_0 x_0})$$

where $\tilde{\phi}(p)$ is the inverse Fourier transform of $\phi(x)$. Concerning the rule of integration one can adopt the "right-integral" $\int d^4x \; \Omega_R(\phi) = \int d^4x \; \phi(x)$.

At this point one can already formulate an educated guess for the action [1]

$$S(\Phi) = \int d^4x \; \Phi(\Box_\kappa - M^2) \Phi \quad \Phi \in \mathcal{M}_\kappa$$

where $\Box_\kappa$ is a (differential) operator which should reproduce $\Box$ in the limit $\kappa \to \infty$.

By straightforward generalization of the results (4), it is natural to describe a set of transformations $T$ as symmetries if (and only if) they close a Hopf-algebra structure and

$$\int d^4x \; \left( T \cdot \{ \Phi(\Box_\kappa - M^2) \Phi \} + \Phi[\Box_\kappa, T] \Phi \right) = 0$$

The search of a maximally-symmetric action can be structured in two steps. In the first step one looks for a Hopf algebra whose generators $T$ satisfy

$$\int d^4x \; T \{ \Phi \Box_\kappa \Phi \} = 0$$

for each differential operator $\Box_\kappa$. In the second step one looks for an operator $\Box_\kappa$ that is invariant ($[\Box_\kappa, T] = 0$) under the action of this algebra.

In introducing the concepts of translations and rotations we chose [1] to follow as closely as possible the analogy with the well-established commutative case in which:

$$P_\mu(e^{ikx}) = k_\mu e^{ikx}, \quad M_j(e^{ikx}) = -i\epsilon_{jkl} x_k \partial_l e^{ikx}$$

It appears natural to define translations and rotations in $\mathcal{M}_\kappa$ by straightforward "quantization" of their classical actions (9) through the Weyl map, but the non-uniqueness of the Weyl map does not allow to implement uniquely these definitions:

$$P^{R/S}_\mu \Omega_{R/S}(e^{ikx}) = k_\mu \Omega_{R/S}(e^{ikx}), \quad M^{R/S}_j \Omega_{R/S}(e^{ikx}) = \Omega_{R/S}(e^{ikx}(-i\epsilon_{jkl} x_k \partial_l e^{ikx}))$$

The alternative definition $\int \Omega_S(\phi) = \int d^4x \phi(x)$ turns out to be equivalent [1].
Although introduced differently (respectively in terms of the action on right-ordered functions and on symmetrically-ordered functions) \( M^R_\kappa \) and \( M^S_\kappa \) are actually identical.

In fact applying \( M^{R/S}_j \) to the same element of \( \mathcal{M}_\kappa \) (for ex., \((e^{i\vec{x}_0 \cdot \vec{e}_{-i\vec{x}_0}})\)) one finds \( M^R_j(e^{i\vec{x}_0 \cdot \vec{e}_{-i\vec{x}_0}}) = M^S_j(e^{i\vec{x}_0 \cdot \vec{e}_{-i\vec{x}_0}}) \). This applies also to \( P^{R/S}_{0j} \). We therefore remove the indices \( R/S \) for these operators. However the ambiguity we are facing in defining spatial translations is certainly more serious. In fact, the two candidates as translation candidates as \( R/S \) (the indices \( R \)) the indices \( R/S \) finds a non-trivial coalgebra sector:

In these algebras the rotations turn out to be completely classical.

One can easily verify \([1]\) that both the 7-generators of operators \((P^R_\mu, M_j)\) and \((P^S_\mu, M_j)\) satisfy the condition \((8)\) and do give rise to genuine Hopf algebras of translation-rotation symmetries. In these algebras the rotations turn out to be completely classical (undeformed) both in algebra and in co-algebra sector, whereas for translations one finds a non-trivial coalgebra sector:

\[
\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \quad \Delta P^R_j = P^R_j \otimes 1 + e^{-\frac{\rho_0}{\kappa}} \otimes P^R_j \quad \Delta P^S_j = P^S_j \otimes e^{\frac{\rho_0}{\kappa}} + e^{-\frac{\rho_0}{\kappa}} \otimes P^S_j
\]

Still, the action of rotations on energy-momentum is undeformed \([M^R/S_j, P^R/S_\mu] = i\delta_{\mu j} \varepsilon_{jkl} P^S_{lR/S} \).

In including also boosts to obtain 10-generator symmetry algebras one finds that the action on functions in \( \mathcal{M}_\kappa \) cannot be obtained by “quantization” of the classical action \( N^{R/S}_j \) \( \Omega_{R/S}(f) = \Omega_{R/S}(i[x_0 \partial_j - x_j \partial_0|[f]) \). In fact \( N^{R/S}_j \) do not close a consistent Hopf algebra structure: the coproduct \( \Delta(N^{R/S}_j) \) is not an element of the algebraic tensor product of the algebra generated by \((P^R_\mu, M_j, N^R_j)\). Therefore the “classical” choice \( N^{R/S}_j \) cannot be combined with \((P^R/S_j, M^R_j/S)\). However, a 10-generator symmetry-algebra extension does exist, but it requires nonclassical boosts.

We considered \([1]\) the most general form of deformed boost generators \( N_j \) that transform as vectors under rotations

\[
N_j \Omega(\phi) = \Omega[ix_0 A(-i\partial_x) \partial_j + \kappa x_j B(-i\partial_x) - \frac{2i}{\kappa} C(-i\partial_x) \partial_i \partial_j - i\varepsilon_{jkl} x_k D(-i\partial_x) \partial_l|\phi]\]

where \( A, B, C, D \) are unknown functions of \( P^R_\mu \) (in the classical limit \( A = i, D = 0; \) moreover, as \( \kappa \rightarrow \infty \) one obtains the classical limit if \( C/\kappa \rightarrow 0 \) and \( B \rightarrow \kappa^{-1} P_0 \)).

Imposing consistency of the 10-generator Hopf-algebra structure and imposing that the classical Lorentz-subalgebra relations are preserved one obtains some constraints on \( A, B, C, D \). The solution is

\[
N^{R/S}_j \Omega_{R/S}(f) = \Omega_{R/S}([ix_0 \partial_j + \frac{\kappa}{2} x_j(1 - e^{2i\theta_0/\kappa} - \frac{\nabla^2}{\kappa^2}) - \frac{x_i}{\kappa} \partial_j] \partial_j] f) \quad (11)
\]

\[
N^{R/S}_j \Omega_{R/S}(f) = \Omega_{R/S}([ix_0 \partial_j - x_j(\kappa \sinh(i\theta_0/\kappa)) + \frac{\nabla^2}{2\kappa} + \frac{x_i}{\kappa} \partial_j] e^{i\theta_0/\kappa} f) \quad (12)
\]

As in the case of the rotations one can easily verify that \( N^{R/S}_j \), \( N^{S}_j \) are equivalent, and it is therefore appropriate to remove the label \( R/S \). It is easy to verify that the
Hopf algebras \((P^R_\mu, M_j, N_j)\) and \((P^S_\mu, M_j, N_j)\) both satisfy all the requirements for a candidate symmetry-algebra for theories in \(\mathcal{M}_\kappa\). In summary we have two candidate Hopf algebras of 10-generator Poincaré-like symmetries: \((P^R_\mu, M_j, N_j)\) is the well-known Majid-Ruegg bicrossproduct \(\kappa\)-Poincaré basis[3], while \((P^S_\mu, M_j, N_j)\) is a new type of bicrossproduct basis which had not previously emerged in the literature [1].

The final step is to look for a differential operator \(\Box_\kappa\) suitable for a maximally-symmetric action. It is easy to verify that the proposal \(\Box_\kappa = (2\kappa \sinh \frac{P_0^R}{2\kappa})^2 - e^{\frac{P_0^R}{2\kappa}} P_0^R\) satisfies \([\Box_\kappa, T] = 0\) for every \(T\) both in \((P^R_\mu, M_j, N_j)\) and \((P^S_\mu, M_j, N_j)\). Therefore, it makes the action (6) invariant both under \((P^R_\mu, M_j, N_j)\) transformations and under \((P^S_\mu, M_j, N_j)\) transformations.

In this analysis the ambiguity associated with the choice of a Weyl map led to consideration of two Hopf algebras, \((P^R_\mu, M_j, N_j)\) and \((P^S_\mu, M_j, N_j)\), which originate from two different choices of ordering in \(\kappa\)-Minkowski (in the sense codified in the Weyl maps \(\Omega_R\) and \(\Omega_S\)). Of course, one could consider other types of ordering conventions. This would lead to other candidates \(P^*_\mu\) as translation generators and, correspondingly, other candidate 10-generator Hopf algebras of Poincaré-like symmetries for \(\mathcal{M}_\kappa\) of the type \((P^*_\mu, M_j, N_j)\).

4 More on the description of translations

The results obtained above were based on the natural symmetry requirement (7), that however deserves a few more comments. Let us consider an infinitesimal translation generated by \(T = -i\epsilon^\mu \partial_\mu\) (with an expansion parameter \(\alpha \in R\)):

\[
x \rightarrow x' = x - \alpha \epsilon, \quad \Phi(x) \rightarrow \Phi'(x) = \Phi(x) + i\alpha T\Phi(x) + O(\alpha^2)
\]

Following the analogy with corresponding analyses in CSTs there are actually two possible starting points for a description of \(T\) as a symmetry of the action:

\[I) \quad \delta IS(\Phi) = i \int d^4x T \cdot \{\Phi(\Box - M^2)\Phi\} = 0 \quad II) \quad \delta II S(\Phi) = S(\Phi') - S(\Phi) = 0\]

In the context of theories in CSTs the conditions I) II) are easily shown to be equivalent. But in a NCST this is not necessarily the case. By a straightforward calculation one can see that assuming commutative translation parameters \(\epsilon, \delta S_I \neq \delta S_{II}\). If one wants to preserve the double description I) II) of symmetry under translation transformations it is necessary [1] to introduce noncommutative transformation parameters. In fact, it is easy to verify that assuming \([\epsilon_j, x_0] = i\kappa^{-1} \epsilon_j, [\epsilon_j, x_k] = 0\), one finds that the conditions I) II) are equivalent. It appears plausible that other choices of noncommutative transformation parameters would preserve the double description of symmetry. But it is interesting that this choice of noncommutativity of the transformation parameters allows to describe them as differential forms\(^4\), \(\epsilon_\mu = dx_\mu\). This connection with

\(^4\)Note that this is one of the two differential calculi introduced in Ref. [10].
differential forms leads to the following description of translations

\[ x_{\mu} \rightarrow x'_{\mu} = x_{\mu} + dx_{\mu} \quad \Phi(x) \rightarrow \Phi'(x) = \Phi(x) + i dx_{\mu} P^\mu \Phi \]

where the \( dx_{\mu} \) describe the proper concept of differential forms for \( \mathcal{M}_\kappa \) and the \( P^\mu \) act as in (10). This is rather satisfactory from a conceptual perspective, since even in CST an infinitesimal translation is most properly described as “addition” of a differential form. The differentials satisfy the relations \( [dx_{\mu}, x_{\nu}] = i \delta_{\mu j} \delta_{\nu 0} \kappa^{-1} dx_j \) as required for our translations to preserve the commutators of \( \mathcal{M}_\kappa \). An infinitesimal translation \( \Phi' \equiv \Phi + d\Phi \) associates to each element of \( \mathcal{M}_\kappa \) an element of the algebra \( \mathcal{M}_\kappa \oplus \Gamma \) defined over a vector space that is direct sum of \( \mathcal{M}_\kappa \) and the bimodule \( \Gamma \), over \( \mathcal{M}_\kappa \), with product rule \( (\Phi + d\Phi)(\Psi + d\Psi) = \Phi\Psi + \Phi.d\Psi + d\Phi.\Psi = \Phi\Psi + d(\Phi\Psi) \). This algebra is isomorphic to \( \mathcal{M}_\kappa \) through the map \( 1 + d \). Then an infinitesimal translation transforms an element of \( \mathcal{M}_\kappa \) in an element of a “second copy” of \( \mathcal{M}_\kappa \). It is a transformation internal to the same abstract algebra. This abstract algebra is our “space of functions of the spacetime coordinates”.

5 Closing remarks

We introduced [1] a concept of NCST symmetry, which follows very closely the one adopted in CSTs, and is naturally analyzed in terms of a Weyl map. We did find 10-generators symmetries of a free scalar theory in \( \mathcal{M}_\kappa \). These symmetries can be formulated in terms of Hopf-algebra versions of the classical Poincaré symmetries.

The form of the commutation relations of \( \mathcal{M}_\kappa \) clearly suggest that classical rotations can be implemented as a symmetry, and this finds confirmation also at the level of the analysis of the action. Instead the \( \kappa \)-Minkowski commutation relations are clearly not invariant under classical translations. Still, we have shown that one can construct theories in \( \mathcal{M}_\kappa \) that enjoy a deformed (Hopf-algebra) translational symmetry. For boosts something analogous to what happens for translations occurs: classical boosts are not a symmetry of \( \mathcal{M}_\kappa \), but, as we showed, there is a deformed version of boosts that are symmetries.

Our analysis allowed us to clarify the nature of the ambiguity in the description of the symmetries of theories in these NCSTs, but it appears that we are left with a choice between different realizations of the concept of translations. It remains to be seen whether this ambiguity can be removed at some deeper level of analysis. A natural context in which to explore this issue might be provided by attempting to construct gauge theories in \( \mathcal{M}_\kappa \) following the approach here advocated.

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