Twisted elliptic genera of $\mathcal{N} = 2$ SCFTs in two dimensions

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Abstract

The elliptic genera of two-dimensional $\mathcal{N} = 2$ superconformal field theories can be twisted by the action of the integral Heisenberg group if their $U(1)$ charges are fractional. The basic properties of the resulting twisted elliptic genera and the associated twisted Witten indices are investigated with due attention to their behaviors in orbifoldization. Our findings are illustrated by and applied to several concrete examples. We obtain a better understanding of the previously observed duality phenomenon for certain Landau–Ginzburg models. We revisit and prove an old conjecture of Witten which states that every $ADE$ Landau–Ginzburg model and the corresponding minimal model share the same elliptic genus. Mathematically, we establish $ADE$ generalizations of the quintuple product identity.

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1. Introduction

Besides being basic tools for constructing superstring vacua, $\mathcal{N} = 2$ superconformal field theories (SCFTs) in two dimensions are interesting subjects in their own right and have long been studied by many authors. Among other things, their elliptic genera have drawn attention since they are often amenable to exact computations and help us to explore possible relations among different models.

In this paper, we discuss certain, less well-known, aspects of such genera. We consider the twisted versions of elliptic genera with the twisting caused by the action of the integral Heisenberg group. Actually, this sort of elliptic genera previously appeared in the process of orbifoldization [1], but we will provide a systematic study of them here. The twisted elliptic genera we consider are of interest only when the $U(1)$ charges of the theory are fractional as in Landau–Ginzburg (LG) models or, say, Kazama–Suzuki models. If instead the $U(1)$ charges are integral as is the case with any $\mathcal{N} = 2$ sigma model with a compact Calabi–Yau (CY) target space, the twisted elliptic genera trivially reduce to the ordinary one.
We begin by briefly reviewing in section 2 the fundamental notion of the spectral flow for the $\mathcal{N} = 2$ superconformal algebra (SCA) [2]. In section 3, we recast the functional properties of elliptic genera (as proposed in [1]) into suitable forms so that the role of the Jacobi group [3] is apparent. We introduce the twisted elliptic genera by applying the elements of the integral Heisenberg group to the elliptic genera in section 4. For the twisted elliptic genera thus defined we can also introduce the twisted Witten indices in exactly the same way as the Witten index is associated with the ordinary elliptic genus. We explain how the twisted Witten indices are related to the $\chi_g$-genus. Then, we investigate in section 5 the behaviors of the twisted elliptic genera and the twisted Witten indices under orbifoldization.

When $\hat{c} < 1$ (with $\hat{c}$ being the one third of the central charge), the information from the twisted Witten indices turns out to be especially useful, since the elliptic genera of any two theories with the same $\hat{c} < 1$ satisfying the same functional properties ought to be identical if their twisted Witten indices coincide. This is an easy consequence of the lemma of Atkin and Swinnerton-Dyer [4] as will be explained in section 6.

In the remaining sections, our general findings are illustrated by and applied to several concrete examples. We treat general LG models and their orbifolds in section 7 and explain in section 8 how the insight gained in this work helps us to understand the somewhat mysterious phenomenon observed in [5] concerning the dualities (or mirror symmetries) of certain LG models. In section 9 we discuss the $\mathcal{N} = 2$ minimal models and prove an old conjecture of Witten [6] which states that every ADE LG model and the corresponding minimal model share the same elliptic genus. We also explain how this result can be interpreted mathematically as ADE generalizations of the quintuple product identity (QPI). We point out that the identities of the A-series are essentially Bailey’s generalizations of the QPI [7].

**Convention.** By a $\mathcal{N} = 2$ SCFT we always mean a $\mathcal{N} = (2, 2)$ SCFT.

**Notations.** We write $e[x]$ for exp$(2\pi \sqrt{-1} x)$. For a positive integer $n$ we set $\xi_n = e[1/n]$ and write $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ for the integers modulo $n$. We denote by $\delta_{n,b}$ the Kronecker delta symbol modulo $n$. Namely, $\delta_{n,b}$ is equal to 1 if $a \equiv b \pmod{n}$ and vanishes otherwise.

2. **The spectral flow of the $\mathcal{N} = 2$ SCA**

The $\mathcal{N} = 2$ SCA $\mathfrak{g}_\varphi$ with the mode parameter $\varphi \in \mathbb{R}$ is a super Lie algebra over $\mathbb{C}$ linearly spanned by the bosonic center $\hat{C}$ whose eigenvalue is commonly written as $\hat{c}$, the bosonic Virasoro generators $\{L_n\}_{n \in \mathbb{Z}}$, the bosonic $U(1)$ current generators $\{J_n\}_{n \in \mathbb{Z}}$, and the fermionic supercurrent generators $\{G^\pm_k\}_{k \in \mathbb{Z} + \varphi}$. The non-vanishing super commutation relations are given by

$$
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{m^3 - m}{4} \delta_{m+n,0} \hat{C}, \\
[L_m, J_n] &= -nJ_{m+n}, \\
[J_m, J_n] &= m \delta_{m+n,0} \hat{C}, \\
[L_m, G^\pm_k] &= \left(\frac{m}{2} - k\right) G^\pm_{m+k}, \\
[J_m, G^\pm_k] &= \pm G^\pm_{m+k}, \\
[G^+_k, G^-_l] &= 2L_{k+l} + (k-l)J_{k+l} + \left(\frac{k^2 - 1}{4}\right) \delta_{k+l,0} \hat{C}.
\end{align*}
$$

(2.1)

The algebra $\mathfrak{g}_\varphi$ is called Ramond if $\varphi \in \mathbb{Z}$ and Neveu–Schwarz if $\varphi \in \mathbb{Z} + 1/2$. The so-called twisted $\mathcal{N} = 2$ SCA is of no concern to us in this paper.
Actually, we have an isomorphism $\mathfrak{A}_\psi \cong \mathfrak{A}_{\psi'}$ as super Lie algebras for any $\psi, \psi' \in \mathbb{R}$. If we set $\eta = \psi - \psi'$, this isomorphism is given by the spectral flow $\sigma_{\eta} : \mathfrak{A}_\psi \to \mathfrak{A}_{\psi'}$ which one defines as

$$\sigma_{\eta}(\hat{C}) = \hat{C},$$

(2.2)

$$\sigma_{\eta}(L_n) = L_n + \eta J_n + \frac{\eta^2}{2} \delta_{n,0} \hat{C},$$

(2.3)

$$\sigma_{\eta}(J_n) = J_n + \eta \delta_{n,0} \hat{C},$$

(2.4)

$$\sigma_{\eta}(G^\pm_{-\infty}) = G^\pm_{-\infty}$$

(2.5)

together with $\mathbb{C}$-linearity [2, 8]. The inverse of $\sigma_{\eta}$ is given by $\sigma_{-\eta}$. If $\rho_{\psi'}$ is a (highest weight) representation of $\mathfrak{A}_{\psi'}$, we also obtain a representation $\rho_{\psi}$ of $\mathfrak{A}_{\psi}$ via $\sigma_{\eta}$. An important point to keep in mind is that although we have $\mathfrak{A}_{\psi} \cong \mathfrak{A}_{\psi'}$, the two representations $\rho_{\psi}$ and $\rho_{\psi'}$ are in general not equivalent. This is so since the modes of $G^\pm_{-\infty}$ shift as in (2.5) modifying the highest weight conditions.

For instance, we will meet the following situation: consider the Ramond $N = 2$ SCA $\mathfrak{A}_0$ and fix its representation $\rho_0$ on some space $\mathcal{H}$. The spectral flow $\sigma_r : \mathfrak{A}_r \to \mathfrak{A}_0$ with $r \in \mathbb{Z}$ induces a representation $\rho_r$ of another copy of the Ramond $N = 2$ SCA $\mathfrak{A}_r$ on $\mathcal{H}$. We can consider the subspaces $\mathcal{V}_0 \subset \mathcal{H}$ and $\mathcal{V}_r \subset \mathcal{H}$ of the Ramond ground states for $\rho_0$ and $\rho_r$ which are killed by the respective $G^\pm_{-\infty}$. However, $\mathcal{V}_0$ and $\mathcal{V}_r$ in general have different dimensions since $\mathcal{V}_r$ corresponds to the states annihilated by $G^\pm_{-\infty}$ in the representation $\rho_0$. So, in particular, $\rho_0$ and $\rho_r$ will have different Witten indices and they are different representations in general. Obviously, a similar statement applies for $\rho_r$ and $\rho_{r'}$ with integers $r \neq r'$.

Before ending this brief section, we recall the most prominent application of the spectral flow. The spectral flow $\sigma_{1/2} : \mathfrak{A}_{1/2} \to \mathfrak{A}_0$ determines a representation $\rho_{1/2}$ of the Neveu–Schwarz algebra $\mathfrak{A}_{1/2}$. As amply discussed in [8], $\mathcal{V}_0$ is bijectively mapped to the chiral ring $\mathcal{R}$ annihilated by $G^\pm_{-1/2}$ for $\rho_{1/2}$.

### 3. Elliptic genera of $N = 2$ SCFTs

We begin with the setup of a general $N = 2$ SCFT to be studied in this work.

Let $\mathfrak{A}_0$ and $\mathfrak{A}_r$ be respectively the left and right Ramond $N = 2$ SCAs of our $N = 2$ SCFT which is assumed to have a non-negative rational left=right central charge $\hat{c}$. (Below we follow the convention such that, given some object for the left-mover, the corresponding one for the right-mover is written with a tilde.) Let $\mathfrak{A}_{0,0}$ be the representation of $(\mathfrak{A}_0, \mathfrak{A}_0)$ underlying the theory. Let also $\mathfrak{A}_{r,r'}$ ($r, r' \in \mathbb{Z}$) be the representation of $(\mathfrak{A}_r, \mathfrak{A}_{r'})$ induced from $\mathfrak{A}_{0,0}$ by the spectral flow $(\sigma_r, \sigma_{r'})$. We assume the existence of a positive integer $h$ such that

- The central charge $\hat{c}$ can be expressed for some integers $D$ and $\delta$ as
  $$\hat{c} = D - \frac{2\delta}{h}.$$  
  (3.1)

- The left and right $U(1)$ charges in $\mathfrak{A}_{r,r'}$ belong to $-\hat{c}/2 + (1/h)\mathbb{Z}$.

- There exist periodic isomorphisms $\mathfrak{A}_{r+p,r'+p'} \cong \mathfrak{A}_{r,r'}$ for any $p, p' \in h\mathbb{Z}$.

We choose $h$ to be the smallest possible in the following.

Let $\mathbb{H}$ be the upper half-plane $\{ \tau \in \mathbb{C} \mid \text{Im } \tau > 0 \}$. According to Witten, the elliptic genus is defined by

$$Z(\tau, z) = \text{Tr}_{\mathfrak{A}_{0,0}} (-1)^F (-1)^\hat{c} q^{\frac{1}{2}} \bar{q}^{\frac{1}{2}} \tilde{z}_0 \tilde{z}_1 \tilde{y}, \quad (\tau, z) \in \mathbb{H} \times \mathbb{C}$$  

(3.2)
where \((-1)^F\) and \((-1)^F\) are the usual left and right fermion parity operators and we have set \(q = e^r\) and \(y = e^z\). Due to supersymmetric cancellations between bosonic and fermionic states above the ground level for the right-mover we may and will assume that the elliptic genus \(Z(\tau, z)\) is a holomorphic function on \(\mathbb{H} \times \mathbb{C}\) having a Fourier expansion of the form

\[
Z(\tau, z) = \sum_{n \geq 0} 3_n(z)q^n, \quad 3_n(z) \in \mathbb{Z}[y^{1/h}, y^{-1/h}]. \tag{3.3}
\]

In [1], we proposed that \(Z(\tau, z)\) should in addition satisfy

\[
Z\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = e\left[\frac{c}{2} (\tau^2 + 2r)\right] Z(\tau, z), \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \text{SL}_2(\mathbb{Z}), \tag{3.4}
\]

\[
Z(\tau, z + \tau + s) = (-1)^{\delta(y+s)} e\left[-\frac{c}{2} (r^2 \tau + 2rz)\right] Z(\tau, z), \quad (r, s) \in (h\mathbb{Z})^2. \tag{3.5}
\]

These properties have been checked for many examples and will be postulated in this work. One can almost derive (3.5) from our assumption except the subtlety for the factor \((-1)^{\delta}\) which should be understood from the behavior of \((-1)^F\) under the left spectral flow \(\sigma_\tau\). In fact, if the eigenvalues of \(J_0 - \tilde{J}_0\) are always integers so that we can identify \((-1)^F\) with \(e\left[\frac{1}{2}(J_0 - \tilde{J}_0)\right]\), we see easily that \((-1)^{\delta}\) arises from \(\sigma_\tau\). This choice of \((-1)^F\) has been discussed, for instance, in [8].

The left \(U(1)\) charge spectrum of the chiral ring (or the \((c, c)\)-ring to be precise [8]) is captured by the \(\chi_\tau\)-genus\(^1\). In view of the spectral flow \(\sigma_{\pm 1/2}\), we should have

\[
3_0(z) = y^{-\tau/2} \chi_\tau, \quad \chi_\tau \in \mathbb{Z}[y^{1/h}]. \tag{3.6}
\]

Since \(Z(\tau, -z) = Z(\tau, z)\) from (3.4) we have the duality \(\chi_{-1} = y^{-z} \chi_\tau\). As usual, the Witten index \(\mathcal{X}\) is defined by

\[
\mathcal{X} = Z(\tau, 0) = \chi_{\tau}|_{y=1}. \tag{3.7}
\]

To develop our formalism below, it is expedient to rephrase the properties of (3.4) and (3.5) in the language of the Jacobi group. The standard reference for this purpose is the monograph by Eichler and Zagier [3] (though we need a small extension of the materials treated there). We first introduce the symplectic product on \(\mathbb{Z}^2\) by

\[
v \wedge v' = vJ(v')^T = \det \begin{pmatrix} v & v' \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.8}
\]

where \(v, v' \in \mathbb{Z}^2\) are interpreted as row vectors. We have \(v \wedge v' = -v' \wedge v\) and \(vM \wedge v'M = v \wedge v'\) for any \(M \in \text{SL}_2(\mathbb{Z}) \cong \text{Sp}_2(\mathbb{Z})\). The integral Heisenberg group \(H(\mathbb{Z})\) is the set \(\mathbb{Z}^3 = \{(v, \kappa) \mid v \in \mathbb{Z}^2, \kappa \in \mathbb{Z}\}\) equipped with the group multiplication

\[
(v, \kappa) * (v', \kappa') = (v + v', \kappa + \kappa' + v \wedge v'). \tag{3.9}
\]

The modular (or symplectic) group \(\text{SL}_2(\mathbb{Z})\) acts on \(H(\mathbb{Z})\) by \((v, \kappa)M = (vM, \kappa)\). Then the Jacobi group \(G' = \text{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z})\) is the group with the multiplication law

\[
[M, X] * [M', X'] = [MM', XM' * X] \tag{3.10}
\]

where \(M, M' \in \text{SL}_2(\mathbb{Z})\) and \(X, X' \in H(\mathbb{Z})\). For a positive integer \(n\), we also need to introduce \(H(n\mathbb{Z}) = (n\mathbb{Z})^3 = \{(v, \kappa) \mid v \in (n\mathbb{Z})^2, \kappa \in n\mathbb{Z}\}\) which is a subgroup of \(H(\mathbb{Z})\). Accordingly, we set \(G'_n = \text{SL}_2(\mathbb{Z}) \ltimes H(n\mathbb{Z})\).

\(^1\) For a \(N = 2\) sigma model with a compact Calabi–Yau target space \(V\), we have \(h = 1\) and \(\chi_e = \sum (-1)^{y+h/2} (V)^y\). Thus our convention for the \(\chi_\tau\)-genus differs from Hirzebruch’s original by the sign of \(y\).
Let \( \epsilon : H(\mathbb{Z}) \to \{1, -1\} \) be defined by
\[
\epsilon : ((r, s), \kappa) \mapsto (-1)^{r+s+rs+\kappa}. \tag{3.11}
\]
As one may easily confirm, this is a group homomorphism satisfying \( \epsilon(\lambda M) = \epsilon(X) \) for any \( M \in SL_2(\mathbb{Z}) \). For \( \ell \in \{0, 1\}, m \in \mathbb{Q}, \) and \( \phi(\tau, z) \) a holomorphic function on \( \mathbb{H} \times \mathbb{C} \), we set
\[
\phi_{\ell,m}(\tau, z) = \epsilon(X) \phi[r^2\tau + rz + rs + \kappa]\phi(\tau, z + r\tau + s) \tag{3.12}
\]
where \( X = ((r, s), \kappa) \in H(\mathbb{Z}) \). This gives a Schrödinger type representation of \( H(\mathbb{Z}) \). Namely, we have
\[
\phi_{\ell,m}(\tau, z) = \phi_{\ell,m}(X' \cdot X), \quad X, X' \in H(\mathbb{Z}). \tag{3.13}
\]
In the following it is useful to remember
\[
\phi_{\ell,m}(X' \cdot X) = \phi_{\ell,m}(X) \cdot \phi_{\ell,m}(X'), \quad X, X' \in H(\mathbb{Z}). \tag{3.14}
\]
and as well as
\[
\phi_{\ell,m}(v, \kappa) = \epsilon[(\ell/2 + m)\kappa] \phi_{\ell,m}(v, 0), \tag{3.15}
\]
\[
\phi_{\ell,m}(v, \kappa) = \epsilon[(\ell/2 + m)(v + v')] \phi_{\ell,m}(v + v', \kappa + \kappa'). \tag{3.16}
\]
If we introduce
\[
\phi_{k,m}(\tau, z) = (c\tau + d)^{-k} e^{-\frac{mz^2}{c\tau + d}} \phi[\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}] \tag{3.17}
\]
for \( k \in \mathbb{Z}, m \in \mathbb{Q} \), and \( M = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z}) \), we have
\[
\phi_{k,m}(M) = \phi_{k,m}(MM'), \quad M, M' \in SL_2(\mathbb{Z}) \tag{3.18}
\]
which yields a representation of \( SL_2(\mathbb{Z}) \).
Since a straightforward calculation shows that
\[
\phi_{k,m}(M) = \phi_{k,m}(M) \phi_{k,m}(M), \tag{3.19}
\]
we obtain a representation of \( G' \) by setting
\[
\phi_{k,m}(M, X) = \phi_{k,m}(M) \phi_{k,m}(X), \tag{3.20}
\]
Now we can rephrase the properties (3.4) and (3.5) as
\[
Z_{\frac{\ell}{2}, \frac{m}{2}} M = Z, \quad M \in SL_2(\mathbb{Z}), \tag{3.21}
\]
\[
Z_{\frac{\ell}{2}, \frac{m}{2}} X = Z, \quad X \in H(h\mathbb{Z}) \tag{3.22}
\]
or more concisely,
\[
Z_{\frac{\ell}{2}, \frac{m}{2}} [M, X] = Z, \quad [M, X] \in G'_d \tag{3.23}
\]
where \( \tilde{D} \in \{0, 1\} \) is determined by \( \tilde{D} \equiv \tilde{D} \pmod{2} \).
Let \( G_m \) be the multiplicative group of non-zero complex numbers and let \( G_m^\times \) be the group ring of \( G_m \) over \( \mathbb{C} \). Given a \( \xi \in G_m \), the corresponding element in \( G_m^\times \) is denoted by \( (\xi) \). For a monic polynomial \( p(t) = \prod_{i=1}^{d}(t - \xi) \) with an indeterminate \( t \) and \( \xi \in G_m \), define \( \text{Div}(p(t)) \) \( G_m^\times \) by \( \langle \xi_1 \rangle + \cdots + \langle \xi_k \rangle \). Set \( \Lambda_d = \text{Div}(t^d - 1) = \sum_{i \in \mathbb{Z}_d} \langle \zeta_d^i \rangle \) for a positive integer \( d \). We will sometimes identify \( \Lambda_1 = \langle 1 \rangle \) with 1.
Given a $\mathcal{N} = 2$ SCFT we define $\Upsilon \in \mathbb{C}G_{\Delta}$ as follows\(^2\). Suppose that $y^{1/h} x_r \in \mathbb{Z}[y^{1/h}, y^{-1/h}]$ is written explicitly as $y^{1/h} x_r = \sum n_0 y^{1/h}$. Then we put $\Upsilon = \sum n_0 |q_{\mu}^{(1)}|$. This can be uniquely expanded as\(^3\)

$$\Upsilon = \sum_{d|h} u_d \Lambda_d$$  \hspace{1cm} (3.24)

for some $u_d \in \mathbb{Z}$. Notice that, for any $s \in \mathbb{Z}$, we have

$$y^{1/h} x_r |_{z=ns} = \sum_{d|s,d|h} u_d d.$$  \hspace{1cm} (3.25)

In particular, the case $s = 0$ leads to

$$\mathcal{X} = \sum_{d|h} u_d d.$$  \hspace{1cm} (3.26)

Note also that we have $\Upsilon = u_1 \Lambda_1 = \mathcal{X} \Lambda_1$ if $h = 1$.

4. Twisted elliptic genera

We define the twisted elliptic genus $Z_v(\tau, z)$ for any $v \in \mathbb{Z}^2$ by the action of $(v, 0) \in H(\mathbb{Z})$ on $Z(\tau, z)$, namely,

$$Z_v := Z_{\hat{\partial}, (v, 0)}. \hspace{1cm} (4.1)$$

(In view of (3.15) twisting by a more general element $(v, \kappa) \in H(\mathbb{Z})$ does not lead to an essentially new possibility.)

It is straightforward to show that

$$Z_{v|[0,\frac{1}{2}] M} = Z_{v M}, \hspace{1cm} M \in \text{SL}_2(\mathbb{Z}). \hspace{1cm} (4.2)$$

$$Z_{v, \delta} X = Z_v, \hspace{1cm} X \in H(h \mathbb{Z}). \hspace{1cm} (4.3)$$

In fact, (4.2) readily follows from (3.19) and (3.21) while (4.3) follows from (3.14) and (3.22). We should also note that

$$Z_{v, \delta} X = \zeta_h^{-(k^2 + \ell^2)(v \cdot v')} Z_{v + v'}, \hspace{1cm} X = (v', \kappa') \in H(\mathbb{Z})$$

which can be seen from (3.15) and (3.16). Combined with (4.3), this implies the periodicity

$$Z_v = Z_{v + v'}, \hspace{1cm} v' \in (h \mathbb{Z})^2. \hspace{1cm} (4.4)$$

Throughout this work we assume the following important property:

**Assumption 1.** The twisted Witten index $\Upsilon_v$ defined by $\Upsilon_v := Z_v(\tau, 0)$ for any $v \in \mathbb{Z}^2$ is independent of $\tau$.

In order to explain why we believe this assumption to be a reasonable one, we first write down the explicit expression of $Z_v$ as

$$Z_{(r,0)}(\tau, z) = (-1)^D(r+s+r) \left[ \frac{1}{2} (r^2 \tau + 2rz + rs) \right] Z(\tau, z + rt + s). \hspace{1cm} (4.6)$$

Then we recognize, after taking into account the properties of the spectral flow reviewed in section 2, that $Z_{(r,0)}(\tau, z)$ is essentially the ordinary elliptic genus with the trace in (3.2) being

\(^2\) The motivation behind this definition can be found in section 7 where we will see that $\Upsilon$ of a LG model reduces to the quantity known in the context of the Milnor monodromy of a singularity.

\(^3\) To see this, let $l_d$ be defined by $l_d = u_d / d$ if $d | h$ and $l_d = 0$ if otherwise. Then, we should have $n_t = \sum_{d|h} l_d$, which can be inverted as $l_d = \sum_{d|h} \mu(d) n_t / d$ with the aid of the Möbius function $\mu$. 

over $\varphi_{0,0}$ instead of $\varphi_{0,0}$. Therefore $Z_{(\tau,0)}(\tau,0)$ must be a constant for any $r \in \mathbb{Z}$. Suppose that $(r,s) \in \mathbb{Z}^2$ is not equal to $(0,0)$. Then there exist integers $m$ and $n$ such that $mr+ns = \gcd(r,s)$. If we put

$$M = \begin{pmatrix} r/\gcd(r,s) & s/\gcd(r,s) \\ -n & m \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

then (4.2) means

$$Z_{(\gcd(r,s),0)|0,d/2}M = Z_{(r,s)}.$$

Since the LHS of this is constant when $z = 0$, we see that $Z_{(r,s)}(\tau,0)$ must be constant as well. In any case, assumption 1 can be directly confirmed for all the examples we meet below.

Once the assumption is accepted, we find rather stringent constraints on $X_v$. For instance, if $Z_v(\tau, z) = y^{-r/2}(\chi_y)_v + O(q)$, then we should have $X_v = (\chi_y)_v|_{z=1}$. More striking is the fact that $X_v$ is essentially determined by $\Upsilon$ (hence by $\chi_y$). Observe first that

$$X_v = X_{v,M}, \quad M \in \text{SL}_2(\mathbb{Z}),$$

(4.9)

$$X_v = X_{v+v'}, \quad v' \in (h\mathbb{Z})^2.$$  

(4.10)

These follow from (4.2) and (4.5). We see that $X_{(r,s)} = X_{(\gcd(r,s),0)}$, for any $(r,s) \in \mathbb{Z}^2$. Indeed, if $(r,s) = (0,0)$ this is trivially true while if $(r,s) \neq (0,0)$ this follows from (4.8) by setting $z = 0$. That $X_{(r,s)} = X_{(\gcd(r,s),0)}$ for any $(r,s) \in \mathbb{Z}^2$ implies $X_{(r,s)} = X_{(r,\tau)}$ for any $(r,s) \in \mathbb{Z}^2$.

Consequently, $X_v$ actually has a larger symmetry than (4.9)$^4$.

$$X_v = X_{v,M}, \quad M \in \text{GL}_2(\mathbb{Z}).$$

(4.11)

Since $X_{(\gcd(r,s),0)} = X_{(0,\gcd(r,s))}$, we also have $X_{(r,s)} = X_{(0,\gcd(r,s))}$. Furthermore, since we have $X_{(0,\gcd(r,s))} = X_{(h,\gcd(h,r,s))}$ by (4.10), we may employ the same logic to see $X_{(r,s)} = X_{(0,\gcd(h,r,s))}$.

Notice then that

$$X_{(0,\gcd(h,r,s))} = (-1)^{\frac{d}{2}\gcd(h,r,s)} y^{-r/2} X_{\frac{d}{2}\gcd(h,r,s)} = y^{\frac{d}{2}} X_{\frac{d}{2}\gcd(h,r,s)}.$$ 

(4.12)

By applying (3.25) to this, we therefore obtain

$$X_{(r,s)} = \sum_{d|\gcd(h,r,s)} u_d d^r.$$  

(4.13)

In other words, we have shown, quite in parallel to (3.24), that

$$X_v = \sum_{d|h} u_d (\Delta_d)_v$$

(4.14)

where we have introduced

$$(\Delta_d)_v = \begin{cases} d & \text{if } d \mid r \text{ and } d \mid s \text{ for } v = (r,s), \\ 0 & \text{otherwise}. \end{cases}$$

(4.15)

Note that this in particular means $X_v \in \mathbb{Z}$. Since we can easily see that $(\Delta_d)_v = (\Delta_d)_{v,M}$ ($M \in \text{SL}_2(\mathbb{Z})$), $(\Delta_d)_{(r,s)} = (\Delta_d)_{(r,\tau)}$ and $(\Delta_d)_v = (\Delta_d)_{v+v'}$ ($v' \in (h\mathbb{Z})^2$) for $d \mid h$, it is obvious that (4.14) indeed satisfies (4.11) and (4.10).

It should be noted that the expansion of the form (4.14) is unique$^5$. Therefore, once either (3.24) or (4.14) is known, the other has to follow. The expansion (4.14) turns out to be useful when we investigate how $X_v$ (hence $\Upsilon$) behaves under the orbifoldization in the next section.

$^4$ Recall that generators of $\text{GL}_2(\mathbb{Z})$ can be taken as those of $\text{SL}_2(\mathbb{Z})$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$^5$ This can be seen as follows. Suppose that we have (4.14). Then we have $X_{(0,0)} = \sum_{d|r,h} u_dd = \sum_{d|h} u_dd$ where we have extended the definition of $u_d$ by setting $u_d = 0$ if $d \nmid h$. Then we have $u_d = \frac{1}{h} \sum_{\mu|d} \mu(r) X_{(r,\mu)}$ by the M"{o}bius inversion formula.
5. Orbifolds

The orbifold elliptic genus is defined by averaging the twisted elliptic genera \[ Z_{\text{orb}}(\tau, z) = \frac{1}{h} \sum_{v \in (\mathbb{Z}^2)^{\perp}} Z_v(\tau, z). \] (5.1)

Just like the ordinary elliptic genus, it obeys

\[ Z_{\text{orb}}|_{D/\mathbb{Z}^2} = Z_{\text{orb}}(M) = Z_{\text{orb}}, \quad M \in \text{SL}_2(\mathbb{Z}), \] (5.2)

or

\[ Z_{\text{orb}}|_{D/\mathbb{Z}^2} X = Z_{\text{orb}}, \quad X \in H(h_{\text{orb}}/\mathbb{Z}) \] (5.3)

where \( h_{\text{orb}} = h/\gcd(\delta, h) \). We see that (5.2) follows from (4.2). To prove (5.3) we need to use (4.4).

By definition \( h_{\text{orb}} \) is some divisor of \( h \), but several extreme cases are noteworthy. If \( \delta = 0 \) (or \( \delta = D \)) we have \( h_{\text{orb}} = 1 \). In this case, our orbifold procedure corresponds to Gepner’s method for achieving charge integrality. Sometimes, the orbifold procedure can be regarded as the mirror transformation. Namely \( Z_{\text{orb}} \) may be interpreted as the elliptic genus of the mirror theory within a sign. For this to make sense, it is necessary to have \( h_{\text{orb}} = h \) since the mirror transformation must be involutive. For instance, the relation \( h_{\text{orb}} = h \) is obviously satisfied when \( \delta = \pm 1 \). Later we will discuss these extreme cases in more detail taking examples from LG models.

One can also consider the twisted version \( Z_{\text{orb}}^\delta \) which apparently obeys the same functional equations as \( Z_v \). For example, one has to replace \( h \) by \( h_{\text{orb}} \). By using (4.4) we see that

\[ Z_{\text{orb}}^\delta(\tau, z) = \frac{1}{h} \sum_{v' \in (\mathbb{Z}^2)^{\perp}} \zeta_{\delta}^{v \cdot v'} Z_{v'}(\tau, z). \] (5.5)

From this it is obvious that \( Z_{\text{orb}}^\delta \) also satisfies assumption 1. Thus, for \( X_{\text{orb}}^\delta := Z_{\text{orb}}^\delta(\tau, 0) \), we have

\[ X_{\text{orb}}^\delta = \frac{1}{h} \sum_{v' \in (\mathbb{Z}^2)^{\perp}} \zeta_{\delta}^{v \cdot v'} X_{v'} \] (5.6)

Here we claim that

\[ X_{\text{orb}}^\delta = \sum_{d|h} u_{h/d, \gcd(\delta, d)} \Delta_{h/d, \gcd(\delta, d)} \] (5.7)

which implies in particular \( X_{\text{orb}}^\delta \in \mathbb{Z} \) and

\[ X_{\text{orb}} := X_{\text{orb}}^0 = \sum_{d|h} u_{h/d, d}. \] (5.8)

To see this, we expand \( X_{v+v'} \) in (5.6) by using \( X_{v+v'} = \sum_{d|h} u_{h/d} \Delta_{h/d} \Delta_{h/d+v'} \). Then it suffices to show that

\[ \frac{1}{h} \sum_{v' \in (\mathbb{Z}^2)^{\perp}} \zeta_{\delta}^{v \cdot v'} \Delta_{h/d+v'} = \gcd(\delta, d) \Delta_{h/d \gcd(\delta, d)}. \] (5.9)

6 Of course, depending on the symmetry of the model, we may consider more general types of orbifold theories and in fact such orbifolds are needed, say, when we consider mirror symmetry à la Greene–Plesser. However, we restrict ourselves to the most fundamental one in this paper.
Since \((\Delta_{h/d})_{v+v'}\) is equal to \(h/d\) if \(v + v' = (h/d)v''\) for \(v'' \in (\mathbb{Z}_d)^2\) or 0 otherwise, we have

\[
\frac{1}{h} \sum_{v' \in (\mathbb{Z}_d)^2} \hat{\chi}_h((v, v') \delta) (\Delta_{h/d})_{v+v'} = \frac{1}{d} \sum_{v' \in (\mathbb{Z}_d)^2} \xi_{h'}((v, v') \delta).
\]

Then (5.9) readily follows from (4.15).

Since \(h^{\text{orb}}/(d/\gcd(\delta, d)) = \operatorname{lcm}(\delta, h)/\operatorname{lcm}(\delta, d)\), we see that \(d/\gcd(\delta, d)\) is a divisor of \(h^{\text{orb}}\) if \(d \mid h\). So (5.7) can be recast in the form

\[
\mathcal{X}^{\text{orb}}_v = \sum_{d \mid h^{\text{orb}}} u_d^{\text{orb}} (\Delta_d)_v
\]

with an appropriate choice of integers \(u_d^{\text{orb}}\). For instance, if \(\delta = 0\) for which \(h^{\text{orb}} = 1\) we have \(u_1^{\text{orb}} = \mathcal{X}^{\text{orb}} = \sum_{d \mid h} u_{h/d} d\) while if \(\delta = \pm 1\) for which \(h^{\text{orb}} = h\), we have \(u_d^{\text{orb}} = u_{h/d}\).

Previously, we defined \(\gamma\) from the data of \(\chi\) and studied its relation to \(\mathcal{X}_v\). We may similarly associate \(\gamma^{\text{orb}}\) with \(\chi^{\text{orb}}\). Then \(\gamma^{\text{orb}}\) has to be similarly related to \(\mathcal{X}_{v'}^{\text{orb}}\). So we should have

\[
\gamma^{\text{orb}} = \sum_{d \mid h^{\text{orb}}} \Lambda_d^{\text{orb}}
\]

or

\[
\gamma^{\text{orb}} = \sum_{d \mid h} u_{h/d} \gcd(\delta, d) \Lambda_{d/\gcd(\delta, d)}.
\]

6. Some criteria for the equality of two elliptic genera

Suppose that there are two \(\mathcal{N} = 2\) SCFTs that are suspected to be equivalent. Denote respectively their elliptic genera, twisted Witten indices and \(\chi\)-genera by \(Z(i)(\tau, z), \mathcal{X}^{(i)}_v\) and \(\chi^{(i)}_v\) \((i = 1, 2)\). Suppose furthermore that \(Z(1)(\tau, z)\) and \(Z(2)(\tau, z)\) are known explicitly and are confirmed to satisfy (3.4) and (3.5) with the same \(\hat{c}\) and \(h\). Then, it is natural to ask if \(Z(1)(\tau, z) = Z(2)(\tau, z)\) exactly.

With regard to this issue, we recall a frequently-used approach which is best suited when \(\hat{c}h^2\) is relatively small. (Note that \(\hat{c}h^2\) is a non-negative integer due to (3.1).) If we put \(\phi^{(i)}(\tau, z') = Z(i)(\tau, \hat{h}z')\) then \(\phi^{(i)}(\tau, z')\) is a weak Jacobi form of weight zero and index \(\hat{c}h^2/2\) (possibly with multipliers). The structure theorem over \(\mathbb{C}\) on the ring of weak Jacobi forms with even weights and integral indices was proved in [3, theorem 9.3]. It is not difficult to extend this theorem for the case with indices taking their values to \(\mathbb{Z}/2\). To apply this sort of structure theorem to the present problem we only have to check whether \(\phi^{(1)}(\tau, z')\) and \(\phi^{(2)}(\tau, z')\) have the same \(q\)-expansions up to a certain order. However, the disadvantage of this approach is that as \(\hat{c}h^2\) becomes large, the necessary order for the \(q\)-expansions also becomes large and we may not be able to carry out the test in practice. Notice also that the approach is quite general but rather weak since it does not utilize the properties of the elliptic genera imposed by assumption 1.

Here we explain another approach which also suffers from its own limitations but is especially powerful when combined with assumption 1 and applied to the cases with \(\hat{c} < 1\). It employs the fundamental lemma of Atkin and Swinnerton–Dyer [4]:

**Lemma 1.** Fix a complex number \(q\) with \(0 < |q| < 1\) and set \(A = \{x \in \mathbb{C} \mid |q| < |x| \leq 1\}\).

Suppose that a meromorphic function \(F(x)\) on \(\mathbb{C}^+\) satisfies \(F(qx) = Kx^{-n}F(x)\) where \(n\) is an integer and \(K\) is some constant as a function of \(x\). Then either \(F(x)\) has exactly \(n\) more zeros than poles in \(A\) or \(F(x)\) vanishes identically.
Proposition 1 (cf [3], theorem 1.2). The elliptic genus $Z(\tau, z)$ of a $\mathcal{N} = 2$ SCFT either has exactly $\hat{c}h^2$ zeros in $\mathcal{P} := \{(s + t\tau) | (s, t) \in [0, h)^2\}$ as a function of $z$ or vanishes identically.

Proof. Set $F(x) = Z(\tau, z)$ with $y = x^0$. Obviously, we have

$$F(qx) = (-1)^{\hat{c}h^2/2} x^{\hat{c}h^2} F(x).$$

(6.1)

Since $F(x)$ is holomorphic in $\mathcal{A}$ (by our assumption on $Z(\tau, z)$), the lemma implies that it has exactly $\hat{c}h^2$ zeros in $\mathcal{A}$ or is identically zero.

Proposition 2. Given the elliptic genera $Z^{(i)}(\tau, z)$ and $Z^{(i')} (\tau, z)$ of two $\mathcal{N} = 2$ SCFTs satisfying (3.4) and (3.5) with the same $\hat{c}$ and $h$, we have the following criteria for their equality.

(i) If $Z^{(1)}(\tau, z) = Z^{(2)}(\tau, z)$ at more than $\hat{c}h^2$ values of $z$ in $\mathcal{P}$, then $Z^{(1)}(\tau, z) = Z^{(2)}(\tau, z)$ identically.

(ii) If $\hat{c} < 1$ and $X^{(1)}_v = X^{(2)}_v$ for all $v \in (\mathbb{Z}_h)^2$, then $Z^{(1)}(\tau, z) = Z^{(2)}(\tau, z)$ identically.

(iii) If $\hat{c} < 1$ and $\chi^{(1)}_i = \chi^{(2)}_i$, then $Z^{(1)}(\tau, z) = Z^{(2)}(\tau, z)$ identically.

Proof. Put $F(x) = F_{Z^{(1)}}(x) - F_{Z^{(2)}}(x)$ where $F_{Z^{(i)}}(x) (i = 1, 2)$ is as in the proof of the previous proposition. Since $F(x)$ is holomorphic in $\mathcal{A}$, the statement (i) follows from the lemma. Suppose that $\hat{c} < 1$. In view of (4.6), $X^{(1)}_v = X^{(2)}_v$ implies $Z^{(1)}(\tau, z) = Z^{(2)}(\tau, z)$ at $h^2$ integral points $\{ r\tau + s \mid r, s = 0, 1, \ldots, h - 1 \} \subset \mathcal{P}$. Hence we obtain (ii) from (i). The relation we studied between $\chi_\tau$ and $\chi_i$ implies that (iii) is a consequence of (ii) (so long as we can confirm our assumption 1).

7. LG models and their orbifolds

We next turn to LG models and their orbifolds to illustrate our general formalism. We start by fixing some notation. We introduce the normalized Jacobi theta function $\vartheta$ on $\mathbb{H} \times \mathbb{C}$ by

$$\vartheta (\tau, u) = (x^{-1/2} - x^{1/2}) \prod_{n=1}^\infty \frac{1 - q^n x(1 - q^n x^{-1})}{(1 - q^n)^2}$$

(7.1)

where $x = e^u$. This has no poles but simple zeros at $r\tau + s \ (r, s \in \mathbb{Z})$ as a function of $u$ and obeys

$$\vartheta |_{-1, \frac{1}{2}} M = \vartheta, \quad M \in \text{SL}_2(\mathbb{Z}),$$

(7.2)

$$\vartheta |_{1, \frac{1}{2}} X = \vartheta, \quad X \in \text{H}(\mathbb{Z}),$$

(7.3)

Let $A = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials. Consider a weighted homogeneous polynomial $f \in A$ satisfying $f(0, \ldots, 0) = 0$ and

$$f(t^{a_1} x_1, \ldots, t^{a_n} x_n) = tf(x_1, \ldots, x_n), \quad t \in \mathbb{C}^*$$

(7.4)

where the weight $a_i$ are positive rational numbers. We express $a_i$ as an irreducible fraction $b_i/a_i$ so that $a_i, b_i > 0$ and $\gcd(a_i, b_i) = 1$. Set $N = \text{lcm}(a_1, \ldots, a_n)$. Let $I_f \subset A$ be the ideal generated by the partial derivatives $\partial f, \ldots, \partial_n f$. We assume that $(0, \ldots, 0) \in \mathbb{C}^n$ is the unique common zero of $\partial f, \ldots, \partial_n f$.

The elliptic genus of the $\mathcal{N} = 2$ LG model with superpotential $f$ is given by

$$Z(\tau, z) = \prod_{i=1}^n \frac{\vartheta (\tau, (1 - a_i)z)}{\vartheta (\tau, a_i z)}.$$  

(7.5)
This is a straightforward generalization of Witten’s result [6] for the simplest case $f(x_1) = x_1^a$ with $\omega_1 = 1/h$. Beyond this expression is the so-called $hc\beta\gamma$ realization of $N = 2$ SCA [9, 10, 6, 1]. It is easy to see that (7.5) obeys the transformation laws (3.4) and (3.5) with $\hat{c} = \sum_{i=1}^{n}(1 - 2\omega_i)$ and an appropriate choice of $h$. The actual value of $h$ depends on the detail of $f$ but we should have $h \mid N$. We may then take $D$ and $\delta$ so that $D \equiv n \pmod{2}$ and $\delta/h = \sum \omega_i \in \mathbb{Z}$. One can also readily confirm that (7.5) has only removable singularities and is in fact holomorphic. It has $\sum (1 - \omega_j)^2 h^2 - \sum \omega_i^2 h^2 = \hat{c} h^2$ zeros in $\mathcal{P}$ as expected.

The chiral ring $\mathcal{R}$ is nothing but the Jacobi ring $J_f := \Lambda/I_f$ which has the grading $J_f = \oplus \nu(J_f)\nu$ induced from (7.4). The $\chi_\nu$-genus is simply the Poincaré polynomial of $J_f$:

$$\chi_\nu = \sum_v \dim(J_f)_v y^\nu = \prod_{i=1}^{n} \frac{1 - y^{1-\omega_i}}{1 - y^{\nu}}.$$  

(7.6)

The Witten index is equal to the Milnor number: $\mathcal{X} = \prod_{i=1}^{n} \left( \frac{1}{\omega_i} - 1 \right)$. On the other hand, the space of Ramond ground states $V_0$ should rather be identified with $\Omega^1_f/\text{d}f \wedge \Omega^{n-1}_f$ where $\Omega^1_f$ is the complex of Kähler differentials. The $U(1)$ charge assignments for $x_i$ and $\text{d}x_i$ are respectively $\omega_i$ and $1/\omega_i - 1/2$. Thus $\text{d}x_1 \wedge \cdots \wedge \text{d}x_n$ has the $U(1)$ charge $\sum (1/\omega_i - 1/2) = -\hat{c}/2$. This explains the necessary shift of the $U(1)$ charge under the spectral flow.

Before turning to the explicit formula of $\mathcal{Y}$, let us observe that, in the present case, we have $\mathcal{Y} = \text{Div}(\Phi(t))$ where $\Phi(t)$ is the characteristic polynomial of the Milnor monodromy [11]. To see this, set $l(v) = v + \sum \omega_i - 1$ for any $v \in \mathbb{Q}$. Let $\{\phi_i\}$ be a basis of $J_f$ as a $\mathbb{C}$-vector space such that $\phi_i \in (J_f)_i$. Since the $(n-1)$th cohomology group of the Milnor fiber is spanned by the Gelfand–Leary forms $[\phi_i, \text{d}x_1 \wedge \cdots \wedge \text{d}x_n/\text{d}f]$, the (cohomological) Milnor monodromy has the spectrum $\{e[l(v_i)]\} [12, 13]$. Consequently, we have

$$\text{Div}(\Phi(t)) = \sum_v \dim(J_f)_v \langle e[l(v)] \rangle.$$  

(7.7)

Then this is easily seen to coincide with our definition of $\mathcal{Y}$:

$$\mathcal{Y} = \langle e[\delta/h] \rangle \sum_v \dim(J_f)_v \langle e[v] \rangle.$$  

(7.8)

In practice, we can calculate $\mathcal{Y}$ from $\chi_\nu$ following the trick of Orlik and Solomon [14]. Clearly, we should have

$$\mathcal{Y} = \langle e[\delta/h] \rangle \lim_{y \to 1} \prod_{i=1}^{n} \frac{1 - \langle e[-\omega_i]\rangle y^{1-\omega_i}}{1 - \langle e[\omega_i]\rangle y^{\omega_i}} = \lim_{y \to 1} \prod_{i=1}^{n} \frac{\langle e[\omega_i]\rangle - y^{1-\omega_i}}{1 - \langle e[\omega_i]\rangle y^{\omega_i}}.$$  

(7.9)

The evaluation of the limit is rather subtle, but if we notice that

$$\frac{\langle e[\omega_i]\rangle - y^{1-\omega_i}}{1 - \langle e[\omega_i]\rangle y^{\omega_i}} = y^{-\omega_i} \left( \frac{1 - y}{1 - \langle e[\omega_i]\rangle y^{\omega_i}} - 1 \right)$$

$$= y^{-\omega_i} \left( \frac{1 - y}{1 - y^{\omega_i}} \sum_{k=0}^{a_i-1} \langle e[k\omega_i]\rangle y^{k\omega_i} - 1 \right)$$

one finds that

$$\mathcal{Y} = \prod_{i=1}^{n} \left( \frac{1}{b_i} \lambda_{\omega_i} - 1 \right).$$  

(7.11)

This is precisely the formula of Milnor and Orlik [15] for $\text{Div}(\Phi(t))$ in accordance with our anticipation. By repeatedly employing a useful formula [15]

$$\Lambda_a \Lambda_b = \gcd(a, b) \Lambda_{\text{lcm}(a, b)}$$  

(7.12)
one obtains the expansion (3.24) with
\[ u_1 = (-1)^{n_d} \quad \text{and} \quad u_d = \frac{1}{d} \sum_{\text{lcm}(\alpha_{i_1}, \ldots, \alpha_{i_d}) = \alpha_1} (-1)^{n_d - \ell} \] for \( d > 1 \). (7.13)

Note that \( u_d \) given by (7.13) has to vanish for a divisor \( d \) of \( N \) greater than \( h \).

**Remark 1.** The work of A’Campo [16] enables us to interpret the expansion (3.24) in terms of the data associated with the resolution of the singularity. On the other hand, from the perspective of \( N \geq 2 \) SCFTs, (3.24) is related to the Coulomb gas decomposition in the case of ADE minimal models (cf (9.12)). We pointed out this parallelism before [5] but a satisfactory understanding is still lacking.

Since one can check assumption 1, our previous argument implies that \( X_v \) must have the expansion (4.14) with the \( u_d \) given by (7.13). This can also be seen in a direct computation. Indeed, by substituting (7.5) into (4.6) it is straightforward to show
\[ X_v = (-1)^{n_d} \prod_{\ell \in S_v} \left( 1 - \frac{1}{\omega_{i, j}} \right) \] (7.14)
where \( S_v \) is defined by \( S_{v, (d)} = \{ i \in \{1, \ldots, n \} \mid r_{\alpha_i} \in \mathbb{Z} \text{ and } s_{\alpha_i} \in \mathbb{Z} \} \). Then we can easily expand this expression to obtain the expected result. It should be noted that we have the (quasi) periodicity:
\[ X_v [f + x^2_{r+1}] = \epsilon((v, 0)) X_v [f], \quad X_v [f + x^2_{r+1} + x^2_{r+2}] = X_v [f]. \] (7.15)

As we have already seen, we are bound to have (5.7) and (5.13) for the orbifold theory. Nevertheless, here we prove (5.13) directly for the sake of completeness. By explicitly calculating the expansion \( Z_v (\tau, z) = y^{-\gamma/2} (X_v)_{v} + O(q) \) one finds [17] that
\[ (X_v)_{v} = \prod_{i \in S_v} -y^{-\frac{1}{2}(1-2\omega_i) - \langle \omega_{\alpha} \rangle} \prod_{i \in S_v} e^{\langle \omega_{\alpha} \rangle} y^{-\langle \omega_{\alpha} \rangle} \] (7.16)
where we have set \( S_v = \{ i \mid \omega_{\alpha} r \in \mathbb{Z} \} \) and \( \langle \omega_{\alpha} \rangle = \omega - \lfloor \omega \rfloor - 1/2 \) with \( \lfloor \omega \rfloor = \max\{n \in \mathbb{Z} \mid n \leq \omega \} \). Therefore,
\[ X^{\text{orb}}_{v} = \frac{1}{h} \sum_{v \in (\mathbb{Z}_2)^2} (X_v)_{v}. \] (7.17)

This reproduces Vafa’s expression for \( X^{\text{orb}}_{v} \) [18] originally found by a judicious physical argument but without recourse to the elliptic genus. As before, one can calculate \( \Upsilon^{\text{orb}} \) from this expression of \( X^{\text{orb}}_{v} \) as
\[ \Upsilon^{\text{orb}} = \frac{1}{h} \sum_{v \in (\mathbb{Z}_2)^2} \Upsilon_v \] (7.18)
where
\[ \Upsilon_{v, (x)} = (s^{\langle \omega \rangle}) \lim_{y \to 1} y^{-1} \prod_{i \in S_v} -y^{-\frac{1}{2}(1-2\omega_i) - \langle \omega_{\alpha} \rangle} \prod_{i \in S_v} e^{\langle \omega_{\alpha} \rangle} \frac{1 - e^{\langle \omega_{\alpha} \rangle} y^{-\langle \omega_{\alpha} \rangle}}{1 - e^{\langle \omega_{\alpha} \rangle} e^{\langle \omega_{\alpha} \rangle} y^{-\langle \omega_{\alpha} \rangle}} \] (7.19)
To evaluate the limit we use as before
\[ \frac{e^{\langle \omega_{\alpha} \rangle} e^{\langle \omega_{\alpha} \rangle} y^{-\langle \omega_{\alpha} \rangle}}{1 - e^{\langle \omega_{\alpha} \rangle} e^{\langle \omega_{\alpha} \rangle} y^{-\langle \omega_{\alpha} \rangle}} = \left( y^{-\langle \omega \rangle} \frac{1 - y}{1 - e^{\langle \omega_{\alpha} \rangle} e^{\langle \omega_{\alpha} \rangle} y^{-\langle \omega_{\alpha} \rangle}} - 1 \right) = \left( y^{-\langle \omega \rangle} \frac{1 - y^{a_{\alpha} - 1}}{1 - y} \sum_{k=0}^{\infty} e^{\langle \omega_{\alpha} \rangle} e^{\langle \omega_{\alpha} \rangle} y^{a_{\alpha} k} - 1 \right). \] (7.20)
Then it turns out that

$$\Upsilon_{\tau,z} = (-1)^n [\frac{1}{\chi]\prod_{i=5}^{\infty} \left(1 - \frac{1}{b_j^{a_j}} \sum_{k=0}^{b_j} e^{[a_j,sk]} e^{[a_j,k]} \right). \right.$$  \hfill (7.21)

By plugging this into (7.18) and expanding the resulting expression, we obtain

$$\Upsilon_{\text{orb}} = \frac{(-1)^n}{h} \sum_{(\tau,z) \in \mathbb{Z}^2} \left[ \left(\frac{1}{\chi}\right) \sum_{\ell, i_1, \ldots, i_l \in \mathbb{Z}} \left(\frac{(-1)^\ell}{b_{i_1} \cdots b_{i_l}} \right) \times \sum_{k_1=0}^{a_{i_1}-1} \cdots \sum_{k_l=0}^{a_{i_l}-1} e^{\left(\sum_{j=1}^l \omega_{i_j} k_j \right) s} \right) \left(\sum_{j=1}^l \omega_{i_j} k_j \right). \right.$$  \hfill (7.22)

We then perform the sum over $s$ to find

$$\Upsilon_{\text{orb}} = \sum_{d|h} u_d \sum_{r \in \mathbb{Z}^A \ (h/d)r} \left(\frac{1}{\chi}\right) \sum_{\ell, i_1, \ldots, i_l \in \mathbb{Z}} \left(\frac{(-1)^\ell}{b_{i_1} \cdots b_{i_l}} \right) \times \sum_{k_1=0}^{a_{i_1}-1} \cdots \sum_{k_l=0}^{a_{i_l}-1} e^{\left(\sum_{j=1}^l \omega_{i_j} k_j \right) s} \right) \left(\sum_{j=1}^l \omega_{i_j} k_j \right). \right.$$  \hfill (7.23)

where the $u_d$ are as given by (7.13). We can confirm without difficulty that

$$\sum_{r \in \mathbb{Z}^A \ (h/d)r} \left(\frac{1}{\chi}\right) \sum_{\ell, i_1, \ldots, i_l \in \mathbb{Z}} \left(\frac{(-1)^\ell}{b_{i_1} \cdots b_{i_l}} \right) \times \sum_{k_1=0}^{a_{i_1}-1} \cdots \sum_{k_l=0}^{a_{i_l}-1} e^{\left(\sum_{j=1}^l \omega_{i_j} k_j \right) s} \right) \left(\sum_{j=1}^l \omega_{i_j} k_j \right) = \gcd(\delta, d) \Lambda_{d/gcd(\delta, d)} \quad \text{if } d | h. \right.$$  \hfill (7.24)

So we have completed the proof of (5.13).

8. Some LG examples with $\delta = 0$, $\pm 1$ and their dualities

The orbifold theories of some $\mathcal{N} = 2$ SCFTs have the chance to be equivalent or at least closely related to some other known $\mathcal{N} = 2$ SCFTs. In order to appreciate what we have learnt so far, we give below some examples of LG models with $\delta = 0, \pm 1$ for which $(-1)^D \mathbb{Z}^{\text{orb}}(\tau, z)$ is (expected to be) equal to the elliptic genus of another familiar $\mathcal{N} = 2$ SCFT.

First, consider a LG model with $\delta = 0$, $\pm 1$ for which $(-1)^D \mathbb{Z}^{\text{orb}}(\tau, z)$ is the elliptic genus of another familiar $\mathcal{N} = 2$ SCFT.

Next we discuss several (well-known) LG models with $\delta = \pm 1$ and explain how the duality or mirror phenomena observed long before in [5] can be understood from the vantage viewpoint of the present work. The LG models we consider are in three variables and are distinguished by their types $T$. We express their weights $\omega_i$ as $d_i/h$ for $i = 1, 2, 3$.

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7 The sign here is purely conventional and one may prefer to include this in the definition of $\mathbb{Z}^{\text{orb}}$. 

We say that $T$ is simple or of ADE type if it is given by the following data:

| $T$ | $(h, d_1, d_2, d_3)$ | $f$ |
|-----|-----------------|-----|
| $A_{n-1}$ | $(h, 1, \frac{h}{2}, \frac{h}{2})$ | $x_1^h + x_2^h + x_3^h$ |
| $D_{2+1}$ | $(h, 2, \frac{h}{2} - 1, \frac{h}{2})$ | $x_1^h + x_1x_2^h + x_3^h$ |
| $E_6$ | $(12, 3, 4, 6)$ | $x_1^4 + x_2^6 + x_3^2$ |
| $E_7$ | $(18, 4, 6, 9)$ | $x_1^3x_2 + x_3^3 + x_4^2$ |
| $E_8$ | $(30, 6, 10, 15)$ | $x_1^5 + x_2^3 + x_3^2$ |

where we assume that $h \geq 2$ for $A_{n-1}$ and $h \geq 6, h \in 2\mathbb{Z}$ for $D_{2+1}$ and $h$ is the Coxeter number of the Lie algebra corresponding to $T$. We have $\hat{c} = 1 - 2/h$ so that we may take $D = 1$ and $\delta = 1$. We also see that $X = \delta$ where $\delta$ is the rank of the Lie algebra and that $\chi_\delta = \sum_{i=1}^{\delta} t^{m_i}$ with $y = t^{1/h}$ where the $m_i$ are the Coxeter exponents:

| $T$ | $m_i$ |
|-----|-------|
| $A_{n-1}$ | $1, 2, \ldots, h - 1$ |
| $D_{2+1}$ | $1, 3, \ldots, h - 3, h - 1, \frac{h}{2}$ |
| $E_6$ | $1, 4, 5, 7, 8, 11$ |
| $E_7$ | $1, 5, 7, 9, 11, 13, 17$ |
| $E_8$ | $1, 7, 11, 13, 17, 19, 23, 29$ |

With the help of (7.11) it is easy to see that

| $T$ | $\Gamma$ |
|-----|-------|
| $A_{n-1}$ | $\Lambda_h - \Lambda_1$ |
| $D_{2+1}$ | $\Lambda_h - \Lambda_{\frac{h}{2}} + \Lambda_2 - \Lambda_1$ |
| $E_6$ | $\Lambda_{12} - \Lambda_6 - \Lambda_4 + \Lambda_3 + \Lambda_2 - \Lambda_1$ |
| $E_7$ | $\Lambda_{18} - \Lambda_9 - \Lambda_6 + \Lambda_3 + \Lambda_2 - \Lambda_1$ |
| $E_8$ | $\Lambda_{30} - \Lambda_{15} - \Lambda_{10} - \Lambda_6 - \Lambda_4 + \Lambda_3 + \Lambda_2 - \Lambda_1$ |

We say that $T$ is exceptional if it is in the list of Arnold’s 14 exceptional unimodal singularities [19]:

| $T$ | $(h, d_1, d_2, d_3)$ | $f$ |
|-----|-----------------|-----|
| $U_{13}$ | $(12, 4, 4, 3)$ | $x_1^2 + x_2^2 + x_3^2$ |
| $S_{12}$ | $(13, 4, 3, 3)$ | $x_1^2x_3 + x_2^2x_3 + x_3x_4^3$ |
| $Q_{12}$ | $(15, 6, 5, 3)$ | $x_1^3x_3 + x_2^3 + x_4^4$ |
| $S_{11}$ | $(16, 5, 4, 6)$ | $x_1^3x_3 + x_2^4x_3 + x_4^2$ |
| $W_{11}$ | $(16, 4, 3, 8)$ | $x_1^2 + x_1x_4^2 + x_3^2$ |
| $Q_{11}$ | $(18, 7, 6, 4)$ | $x_1^3x_3 + x_2^3 + x_3x_4^3$ |
| $Z_{13}$ | $(18, 5, 3, 9)$ | $x_1^2x_2 + x_2^3 + x_4^3$ |
| $W_{12}$ | $(20, 5, 4, 10)$ | $x_1^4 + x_2^5 + x_3^3$ |
| $Z_{12}$ | $(22, 6, 4, 11)$ | $x_1^2x_2 + x_4^2 + x_3^2$ |
| $Q_{10}$ | $(24, 9, 8, 6)$ | $x_1^3x_3 + x_2^3 + x_4^4$ |
| $E_{14}$ | $(24, 8, 3, 12)$ | $x_1^4 + x_2^4 + x_3^3$ |
| $Z_{11}$ | $(30, 8, 6, 15)$ | $x_1^2x_2 + x_2^3 + x_4^3$ |
| $E_{13}$ | $(30, 10, 4, 15)$ | $x_1^2 + x_1x_4^2 + x_3^3$ |
| $E_{12}$ | $(42, 14, 6, 21)$ | $x_1^4 + x_2^2 + x_3^2$ |
For these singularities, we have \( \delta = 1 + 2/h \) so that we can take \( D = 1 \) and \( \delta = -1 \). The \( \chi \)-genera with \( t = y^{1/h} \) are as follows:

\[
\begin{array}{c|c}
T & \chi_T \\
\hline
U_{12} & 1 + t^2 + 2t^6 + t^8 + 2t^6 + t^8 + 2t^{10} + t^{11} + t^{14} \\
S_{12} & 1 + t^2 + t^6 + t^8 + t^{10} + t^{11} + t^{12} + t^{14} + t^{17} \\
Q_{12} & 1 + t^2 + t^6 + t^8 + t^{10} + t^{11} + t^{12} + t^{14} + t^{17} \\
S_{11} & 1 + t^2 + t^6 + t^8 + t^{10} + t^{11} + t^{12} + t^{14} + t^{18} \\
W_{13} & 1 + t^2 + t^6 + t^8 + t^{10} + t^{11} + t^{12} + t^{14} + t^{15} + t^{18} \\
Q_{11} & 1 + t^2 + t^6 + t^8 + t^{10} + t^{11} + t^{12} + t^{14} + t^{15} + t^{18} \\
Z_{13} & 1 + t^2 + t^6 + t^8 + t^{10} + t^{11} + t^{12} + t^{14} + t^{15} + t^{17} + t^{20} \\
W_{12} & 1 + t^2 + t^6 + t^8 + t^{10} + t^{11} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20} + t^{24} \\
Z_{12} & 1 + t^2 + t^6 + t^8 + t^{10} + 2t^{12} + t^{14} + t^{16} + t^{18} + t^{20} + t^{24} \\
Q_{10} & 1 + t^6 + t^8 + t^9 + t^{12} + t^{14} + t^{16} + t^{18} + t^{20} + t^{26} \\
E_{14} & 1 + t^2 + t^6 + t^8 + t^{10} + t^{11} + t^{12} + t^{14} + t^{15} + t^{17} + t^{18} + t^{20} + t^{23} + t^{26} \\
Z_{11} & 1 + t^2 + t^6 + t^8 + t^{12} + t^{14} + t^{16} + t^{18} + t^{20} + t^{24} + t^{26} + t^{32} \\
E_{13} & 1 + t^2 + t^6 + t^8 + t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20} + t^{22} + t^{24} + t^{28} + t^{32} \\
E_{12} & 1 + t^6 + t^{10} + t^{14} + t^{18} + t^{20} + t^{24} + t^{26} + t^{30} + t^{32} + t^{36} + t^{38} + t^{44} \\
\end{array}
\]

The computation of \( \Upsilon \) is again straightforward [5]:

\[
\begin{array}{c|c}
T & \Upsilon \\
\hline
U_{12} & \Lambda_{12} + \Lambda_4 - \Lambda_3 - \Lambda_1 \\
S_{12} & \Lambda_{13} - \Lambda_1 \\
Q_{12} & \Lambda_{15} - \Lambda_5 + \Lambda_3 - \Lambda_1 \\
S_{11} & \Lambda_{16} - \Lambda_8 + \Lambda_4 - \Lambda_1 \\
W_{13} & \Lambda_6 - \Lambda_4 + \Lambda_2 - \Lambda_1 \\
Q_{11} & \Lambda_{18} - \Lambda_9 + \Lambda_3 - \Lambda_1 \\
Z_{13} & \Lambda_{18} - \Lambda_6 + \Lambda_2 - \Lambda_1 \\
W_{12} & \Lambda_{20} - \Lambda_{10} + \Lambda_5 + \Lambda_4 + \Lambda_2 - \Lambda_1 \\
Z_{12} & \Lambda_{22} - \Lambda_{11} + \Lambda_3 - \Lambda_1 \\
Q_{10} & \Lambda_{24} - \Lambda_{12} - \Lambda_8 + \Lambda_4 + \Lambda_3 - \Lambda_1 \\
E_{14} & \Lambda_{24} - \Lambda_8 - \Lambda_6 + \Lambda_3 + \Lambda_2 - \Lambda_1 \\
Z_{11} & \Lambda_{30} - \Lambda_{15} - \Lambda_{10} + \Lambda_5 + \Lambda_2 - \Lambda_1 \\
E_{13} & \Lambda_{30} - \Lambda_{15} - \Lambda_6 + \Lambda_3 + \Lambda_2 - \Lambda_1 \\
E_{12} & \Lambda_{42} - \Lambda_{21} - \Lambda_{14} + \Lambda_7 - \Lambda_6 + \Lambda_3 + \Lambda_2 - \Lambda_1 \\
\end{array}
\]

To make manifest the dependence on the type under consideration, we write \( \Upsilon \) for type \( T \) as \( \Upsilon[T] \) etc in the following. If \( T \) is simple, let its dual \( T^* \) be \( T \) itself. If \( T \) is exceptional, \( T^* \) is defined to be the dual of \( T \) in the sense of Arnold’s strange duality [19]. For instance, we have \( S_{11} = W_{13} \) and \( E_{12} = E_{12} \). Note that \( T \) and \( T^* \) share the same \( h \). We observed in [5] that if \( T \) is either simple or exceptional with \( \Upsilon[T] = \sum_{d|h} m_d \Lambda_d \), then we have

\[ \Upsilon[T^*] = -\sum_{d|h} m_d \Lambda_d. \]

Moreover, it was anticipated there that this observation should somehow be related to orbifoldization. From the results in the preceding sections it is now easy to understand why this phenomenon must occur. Indeed, we already saw in [17] that

\[ \chi^{\text{orb}}_T[T] = -\chi_T[T^*]. \]

It is then obvious that

\[ \Upsilon^{\text{orb}}[T] = -\Upsilon[T^*]. \]
On the other hand, we have
\[ Y^\text{orb}[T] = \sum_{d|h} u_{h/d} \Delta_d \] (8.11)
since \( \delta = \pm 1 \) implies \( h^\text{orb} = h \) and \( u_d^\text{orb} = u_{h/d} \). So (8.8) must hold by consistency. Since \( \mathcal{X}_v^\text{orb}[T] = -\mathcal{X}_v[T^*] \), we see in a similar fashion that
\[ \mathcal{X}_v[T^*] = -\sum_{d|h} u_{h/d}(\Delta_d) v. \] (8.12)

Of course, it is tempting to suppose that
\[ Z^\text{orb}[T](\tau, z) = -Z[T^*](\tau, z) \] (8.13)
from which (8.9) follows. If \( T \) is simple, in which case we have \( \hat{c} < 1 \), it is obvious that this holds by our reasoning in section 6.

9. The \( \mathcal{N} = 2 \) minimal models and Witten’s conjecture

We begin this section by reviewing the elliptic genera of the \( \mathcal{N} = 2 \) minimal models following the description in [1]. For a positive integer \( n \) and \( m \in \mathbb{Z}_{2n} \), the theta function \( \theta_{m,n}(\tau, u) \) is defined by
\[ \theta_{m,n}(\tau, u) = \sum_{j \in \mathbb{Z} + \frac{z}{2}} \exp[n(j^2 \tau + ju)]. \] (9.1)

Fix a non-negative integer \( k \) and set \( h = k + 2 \). The integrable representations at level \( k \) of the untwisted affine Lie algebra \( \hat{A}_1 \) are labeled by \( \ell \in \{1, 2, \ldots, h - 1\} \) so that \( (\ell - 1)/2 \) is the spin of the underlying representation of \( \hat{A}_1 \). The associated Weyl–Kac characters are given by
\[ \chi_\ell(\tau, u) = \frac{\theta_{k,h}(\tau, u) - \theta_{-k,h}(\tau, u)}{\theta_{1,2}(\tau, u) - \theta_{-1,2}(\tau, u)}. \] (9.2)

Then the string functions \( c_\ell^m(\tau) \) are defined through [20]
\[ \chi_\ell(\tau, u) = \sum_{m \in \mathbb{Z}_{2h}} c_\ell^m(\tau) \theta_{m,k}(\tau, u). \] (9.3)

Note that \( c_\ell^m(\tau) \) vanishes unless \( \ell - 1 \equiv m \pmod{2} \).

The branching relation introduced by Gepner [21] for the \( \mathcal{N} = 2 \) minimal models has to be slightly extended [1] so that the dependence on the variable \( z \) which measures the \( U(1) \) charge is manifest:
\[ \chi_\ell(\tau, u) \theta_{n,2}(\tau, u - z) = \sum_{m \in \mathbb{Z}_{2h}} \chi_{\ell,a}^m(\tau, z) \theta_{m,h}\left(\tau, u - \frac{2z}{h}\right) \] (9.4)
where \( a \in \mathbb{Z}_4 \). This can be solved by the multiplication formula of theta functions as
\[ \chi_{\ell,a}^m(\tau, z) = \sum_{j \in \mathbb{Z}} c_\ell^{m-a-4j}(\tau) q^{2j}(z^{2} - z^{2j}) y^{2} - z^{2j}. \] (9.5)

We then define
\[ \Pi_\ell^m(\tau, z) = \chi_{\ell,1}^m(\tau, z) - \chi_{\ell,-1}^m(\tau, z). \] (9.6)
Note that \( \Pi_\ell^m(\tau, z) \) vanishes unless \( \ell \equiv m(\text{mod } 2) \). Actually, \( \Pi_\ell^m(\tau, z) \) coincides with the (twisted) character in the representation theory of the Ramond \( \mathcal{N} = 2 \) SCA [22–24]. Though this is expected from our definition, we will provide an explicit proof shortly. It is easy to check
\[ \Pi_\ell^m(\tau, z) = -\Pi_{-m}^\ell(\tau, -z) = -\frac{h}{h-m}(\tau, z) \] as well as
\[ \Pi_{m}^m(\tau, 0) = \delta_{m,\ell} = \delta_{m,-\ell}. \] (9.7)
As is well-known, the modular invariants of the $\hat{A}_1$ Wess–Zumino–Witten models at level $k$ fall into the ADE pattern [25–28]:

$$\sum_{\ell,c} \mathcal{N}_{\ell,c} \chi_\ell \chi_c$$

where we assume as before that $h > 2$ for $A_{h-1}$ and $h \geq 6$, $h \in 2\mathbb{Z}$ for $D_{\frac{h}{2}+1}$. Note the symmetry $\mathcal{N}_{\ell,\epsilon} = \mathcal{N}_{\ell,\epsilon'}$. After picking up a modular invariant from this list the elliptic genus is given by

$$Z(\tau, z) = \sum_{\ell, c} \mathcal{N}_{\ell,c} \frac{\eta^h}{\eta^{\ell}}(\tau, z).$$

That this expression satisfies the desired functional properties (3.4) and (3.5) with $\hat{c} = 1 - 2/h$ was confirmed in [1].

It is straightforward to find the expressions for the twisted elliptic genera:

$$Z_{(r,s)}(\tau, z) = \sum_{\ell, c} \mathcal{N}_{\ell,c} \frac{\eta^h}{\eta^{\ell}}(\tau, z), \quad (r, s) \in (\mathbb{Z}_h)^2.$$  

From this it is easy to show that $Z_{(r,s)}(\tau, z) = -Z(\tau, z)$ as already claimed in [1].

Now we are in a position to prove Witten’s conjecture [6] which amounts to say that the elliptic genera of the $\mathcal{N} = 2$ minimal model and the LG model both labeled by the same ADE type must coincide. We should first confirm that the elliptic genera of both theories are holomorphic with respect to $z$. For the LG model this has already been seen. As for the minimal model this can be confirmed for instance from proposition 4 below. That assumption 1 is satisfied by both theories is also apparent. Since the two theories have the same central charge $\hat{c} = 1 - 2/h$ and their elliptic genera satisfy the same functional equations, we can apply proposition 2. Since $\hat{c} < 1$, it suffices to show that both theories have either the same twisted Witten indices or the same $X_\ell$-genus. That both have the same $X_\ell$-genus is easy to show and in fact well-known: $X_\ell = \sum_{c} \mathcal{N}_{\ell,c} (\ell-1)/h = \sum_{c} X_{(\ell-1)/h}^{(\ell-1)/h}$. So we are done.

It is also not too difficult to show directly that both have the same twisted Witten indices. To explain this, it is convenient to extend the definition of $\mathcal{N}_{\ell,c}$ for all $\ell, \ell' \in \mathbb{Z}$ by

$$\mathcal{N}_{\ell,c} = \epsilon' \mathcal{N}_{\ell,c} \quad (\epsilon, \epsilon' = \pm 1), \quad \mathcal{N}_{\ell+2p,c+2p'} = \mathcal{N}_{\ell,c} \quad (p, p' \in h\mathbb{Z}).$$

We then recall\(^8\) that

$$\mathcal{N}_{\ell,c} = \sum_{d|h} u_d(\Omega_d) \chi_\ell \chi_c$$

[8] Historically speaking, this is the way the $\mathcal{N}_{\ell,c}$ were first found in [25] and the $(\Omega_d)_{\ell,c}$ are associated with the modular invariants for theta functions.
where
\[
(\Omega_d)_{\ell, \ell'} = \begin{cases} 
1 & \text{if } \frac{h}{d} \left| \frac{\ell + \ell'}{2} \right. 	ext{ and } d \left| \frac{\ell - \ell'}{2} \right. \\
0 & \text{otherwise}
\end{cases}
\] (9.13)
and the \( u_d \) are precisely the ones given in (8.4). Note that this expression is consistent with (9.11) since \( u_d = -u_{h/d} \) and \((\Omega_d)_{\ell, -\ell'} = (\Omega_{h/d})_{\ell, \ell'}\). It follows from (9.10) that
\[
X_{(r, \ell)} = \frac{1}{2} \sum_{(\ell, \ell') \in (Z_2)^2} \mathcal{N}_{\ell, \ell'} \xi_h^{(\ell + \ell') / 2} \delta_{(\ell, \ell') \rightarrow (h, h)}
= \frac{1}{2} \sum_{(\ell, \ell') \in (Z_2)^2} \mathcal{N}_{\ell, \ell'} \xi_h^{(\ell + \ell') / 2} \delta_{\ell, \ell' \rightarrow h, h}.
\] (9.14)
Therefore, all we have to show is
\[
\frac{1}{2} \sum_{(\ell, \ell') \in (Z_2)^2} \mathcal{N}_{\ell, \ell'} \xi_h^{(\ell + \ell') / 2} \delta_{\ell, \ell' \rightarrow h, h} = (\Delta_d)_{(r, \ell)},
\] (9.15)
but this can be readily checked.

In the remainder of this section we reformulate Witten’s conjecture which we have just proved as mathematical identities generalizing the QPI. For this purpose, it is convenient to borrow several notations from \( q \)-analysis. If \( q \) and \( x \) are complex numbers with \(|q| < 1\), set
\[
(x; q)_\infty = \prod_{n=0}^{\infty} (1 - xq^n), \quad (q; q)_\infty = (q; q)_\infty
\] (9.16)
and if in addition \( x \neq 0 \), put
\[
\theta(x, q) = \frac{(x; q)_\infty (q/x; q)_\infty}{(q; q)_\infty}.
\] (9.17)
By Jacobi’s triple product identity, we have
\[
(x; q)_\infty (q/x; q)_\infty (q; q)_\infty = (q)^{3/4} \theta(x, q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(n-1)/2} x^n
\] (9.18)
where \( \binom{n}{2} = \frac{n(n-1)}{2} \). The following identities easily follow from the definitions:
\[
\theta(q/x, q) = \theta(x, q), \quad (9.19)
\]
\[
\theta(x^{-1}, q) = -x^{-1} \theta(x, q), \quad (9.20)
\]
\[
\theta(q^i x, q) = (-1)^i q^{i(i-1)/2} x^{-i} \theta(x, q) \quad (i \in \mathbb{Z}). \quad (9.21)
\]
Moreover, for a positive integer \( h \), we have
\[
\theta(x, q) = \frac{(q^h)^{2h} h^{-1}}{(q^2)^{2h}} \prod_{x=0}^{h-1} \theta(q^ix, q^h), \quad (9.22)
\]
By comparing the behaviors of both sides as \( x \to 1 \), we see that
\[
1 = \frac{(q^h)^{2h} h^{-1}}{(q^2)^{2h}} \prod_{x=1}^{h-1} \theta(q^ix, q^h).
\] (9.23)
It thus follows that
\[
\frac{\theta(x, q)}{\theta(x, q^h)} = \prod_{i=1}^{h-1} \frac{\theta(q^ix, q^h)}{\theta(q^i, q^h)}.
\] (9.24)
We also quote the following results of Kronecker [29]:
Proposition 3. For $0 < |q| < |x| < 1$ and $y$ neither $0$ nor an integral power of $q$,\[
\sum_{r \in \mathbb{Z}} \frac{x^r}{1 - q^r y} = \frac{\theta(xy, q)}{\theta(x, q) \theta(y, q)}.
\] (9.25)

For $|q| < |x| < 1$ and $|q| < |y| < 1$,\[
\left( \sum_{r \geq 0} - \sum_{r < 0} \right) q^r x^r y^r = \frac{\theta(xy, q)}{\theta(x, q) \theta(y, q)}.
\] (9.26)

While our definition of $\mathbb{I}_m^\ell$ has the advantage of an easy access to its modular properties, its description in terms of string functions is rather awkward. Fortunately, there is an alternative expression:

Proposition 4. We have\[
\mathbb{I}_m^\ell(t, z) = q^{\frac{c_\ell}{4\pi^2} \frac{m^2}{2} - \frac{i}{4} \frac{\ell}{2}} \frac{\theta(y, q) \theta(q, q^2)}{\theta(q y^{-1}, q^2) \theta(y, q^2)} = 1
\] (9.27)

where $\ell \equiv m \pmod{2}$ should be understood.

Proof. Consider first the case $h = 2$. If $(\ell, m) = (1, 1)$, then the RHS of (9.27) reads\[
\frac{\theta(y, q) \theta(q, q^2)}{\theta(q y^{-1}, q^2) \theta(y, q^2)} = 1
\] (9.28)

where we used (9.24) in the last equality. On the other hand $\mathbb{I}_1^1(t, z) = 1$, so the assertion holds. The situation for $(\ell, m) = (1, -1)$ is similar.

Suppose next that $h > 2$. The string function $c_\ell^m$ is well-known to be expressed as [20]\[
c_\ell^m(t) = \frac{q^{d_\ell,m}}{(q)^\infty} \left( \sum_{r \geq 0} - \sum_{r < 0} \right) (-1)^{s+r} q^{(\ell+1)z} + h z^s - \frac{\ell}{2} - r s
\] (9.29)

where $d_\ell,m = \frac{c_\ell}{4\pi^2} \frac{m^2}{2} - \frac{1}{8}$. Note that some terms in the sum are spurious due to the identity $\sum_{n \in \mathbb{Z}} (-1)^n q^{(\ell+1)z} = 0$ $(a \in \mathbb{Z})$ which follows from (9.18) and $\theta(q^a, q) = 0$. We thus obtain from the definition of $\mathbb{I}_m^\ell(t, z)$ that\[
(q^{\frac{c_\ell}{4\pi^2} \frac{m^2}{2} - \frac{i}{4} \frac{\ell}{2}})^{-1} \mathbb{I}_m^\ell(t, z) = \frac{1}{(q)^\infty} \sum_{r \geq 0} - \sum_{r < 0} (-1)^{s+r} q^{(\ell+1)z} + h z^s - \frac{\ell}{2} - r s
\] (9.30)

On the other hand, we see from (9.18) and (9.26) that\[
\theta(y, q) \frac{\theta(q, q^2)}{\theta(q y^{-1}, q^2) \theta(q^2, q^2)}
\]
\[= \frac{1}{(q)^\infty} \sum_{r \geq 0} - \sum_{r < 0} q^{h r s} (q^{\frac{c_\ell}{4\pi^2} \frac{m^2}{2} - \frac{i}{4} \frac{\ell}{2}})^s (q^{\frac{c_\ell}{4\pi^2} \frac{m^2}{2} - \frac{i}{4} \frac{\ell}{2}})^{r s}
\]
\[= \frac{1}{(q)^\infty} \sum_{r \geq 0} - \sum_{r < 0} (-1)^{s+r} q^{(\ell+1)z} + h z^s - \frac{\ell}{2} - r s
\] (9.31)

By replacing $i$ by $n + r - s$ this is seen to coincide with (9.30). \hfill \square
Remark 2. As promised, the RHS of (9.27) is in fact the product expression found in [23] for the (twisted) character of the Ramond $\mathcal{N}=2$ SCA.

With this provision, we can restate our result as

**Theorem 1.** Suppose that $\mathcal{T}$ is of ADE type with $(h, d_1, d_2, d_3)$ and $\mathcal{N}_{i,c}$ given respectively by (8.2) and (9.8). Then,

\[
\prod_{i=1}^{3} \frac{\theta(x^{b-d_i}, q)}{\theta(x^{b}, q)} = \sum_{i, \ell=1}^{h-1} \mathcal{N}_{i,c} J_{\ell}^c(x, q)
\]

(9.32)

where

\[
\frac{J_{\ell}^c(x, q)}{\theta(x, q)} = q^{\frac{\ell^2 - \ell}{2}} x^{\ell-1} \frac{\theta(x^{b}, q)\theta(q^{1b}, q^{b})}{\theta(q^{\ell x^{-b}}, q^{b})\theta(q^{\ell x^{b}}, q^{b})}.
\]

(9.33)

Remark 3. Note that the LHS of (9.32) is a redundant expression and can actually be simplified as $\prod_{i=1}^{3} \frac{\theta(x^{b-d_i}, q)}{\theta(x^{b}, q)}$ where $n'=1$ if $\mathcal{T}=A_{h-1}$ and $n'=2$ if otherwise.

This theorem can be interpreted as ADE generalizations of the QPI for the following reason. (See [30] for a comprehensive survey on the history, many available proofs and some generalizations of the QPI.) When $\mathcal{T}=A_{h-1}$, the explicit form of (9.32) is

\[
\frac{\theta(x^{b-1}, q)}{\theta(x, q)} = \sum_{i=1}^{h-1} x^{i-1} \frac{\theta(x^{b}, q)\theta(q^{1b}, q^{b})}{\theta(q^{i x^{-b}}, q^{b})\theta(q^{i x^{b}}, q^{b})}.
\]

(9.34)

As before, it is easy to confirm the case $h=2$. The first non-trivial case $h=3$ turns out to be just the QPI (expressed in one among many other possible ways [30]). By using the $h$ term relation of the Weierstraß $\sigma$-function, Bailey [7] was able to generalize the QPI as

\[
\frac{\theta(x^{b-1}, q)}{\theta(x, q)} = \prod_{i=1}^{h} \frac{\theta(q^{i x^{b}}, q^{b})}{\theta(q^{i x^{-b}}, q^{b})} \sum_{i=1}^{h-1} x^{i-1} \frac{\theta(q^{i x^{-b}}, q^{b})}{\theta(q^{i x^{b}}, q^{b})}.
\]

(9.35)

With the aid of (9.24), it is easy to see the equivalence between (9.34) and (9.35).

It should be mentioned that we can prove (9.34) by using (9.25) as well. Indeed, if one notices $\theta(q^{b}, q^{b})=0$, the RHS of (9.34) can be rewritten as

\[
\frac{\theta(x^{b}, q)}{\theta(x, q)} \sum_{i=1}^{h} x^{i-1} \frac{\theta(q^{i}, q^{b})}{\theta(q^{i x^{-b}}, q^{b})\theta(x^{b}, q^{b})} = \theta(x^{b}, q) \sum_{i=1}^{h} x^{i-1} \sum_{r \in \mathbb{Z}} \frac{(q^{i} x^{-b})^{r}}{1-q^{i} x^{b}}
\]

\[
= \theta(x^{b}, q) \sum_{r \in \mathbb{Z}} \frac{(q^{b}/x)^{r}}{1-q^{b} x}.
\]

(9.36)

On the other hand, we have

\[
\frac{\theta(x^{b-1}, q)}{\theta(x, q)} = \theta(x^{b}, q) \frac{\theta(q^{x^{b-1}}, q)}{\theta(q^{x^{b}}, q)} = \theta(x^{b}, q) \sum_{r \in \mathbb{Z}} \frac{(q^{b}/x)^{r}}{1-q^{b} x}.
\]

(9.37)

In view of the fact that the QPI admits a variety of derivations [30] it might be interesting to pursue alternative proofs of the theorem.

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