ASYMPTOTIC FOR A SECOND ORDER EVOLUTION EQUATION WITH VANISHING DAMPING TERM AND TIKHONOV REGULARIZATION

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Abstract. Let $\mathcal{H}$ be a real Hilbert space. We investigate the long time behavior of the trajectories $x(.)$ of the vanishing damped nonlinear dynamical system with Tikhonov regularizing term

$$\text{(GAVD}_{\gamma, \varepsilon}\text{)}$$

$$x''(t) + \gamma(t)x'(t) + \nabla \Phi(x(t)) + \varepsilon(t)\nabla U(x(t)) = 0,$$

where $\Phi, U : \mathcal{H} \to \mathbb{R}$ are two convex continuously differentiable functions, $\varepsilon(.)$ is a decreasing function satisfying $\lim_{t \to +\infty} \varepsilon(t) = 0$, and $\gamma(.)$ is a nonnegative function which behaves, for $t$ large enough, like $\frac{K}{t^\theta}$ where $K > 0$ and $0 \leq \theta \leq 1$. The main contribution of this paper is the following control result: If $\int_0^{+\infty} \frac{\varepsilon(t)}{\gamma(t)} dt = +\infty$, $U$ is strongly convex and its unique minimizer $x^*$ is also a minimizer of $\Phi$ then every trajectory $x(.)$ of (GAVD$_{\gamma, \varepsilon}$) converges strongly to $x^*$ and the rate of convergence of its energy function

$$W(t) = \frac{1}{2} \|x'(t)\|^2 + \Phi(x(t)) - \min \Phi$$

is of order to $o(1/t^{1+\theta})$. Moreover, we prove a new result concerning the weak convergence of the trajectories of (GAVD$_{\gamma, \varepsilon}$) to a common minimizer of $\Phi$ and $U$ (if one exists) under a simple condition on the speed of decay of the Tikhonov factor $\varepsilon(t)$ to 0 with respect to $\gamma(t)$.

1. Introduction and statement of the main results

Let $\mathcal{H}$ be a real Hilbert space endowed with the inner product $\langle ., . \rangle$ and the associated norm $\| . \|$. Let $\Phi, U : \mathcal{H} \to \mathbb{R}$ be two convex continuously differentiable functions and $\gamma, \varepsilon$ be two real positive functions defined on a fixed time interval $[t_0, +\infty)$ for some $t_0 > 0$. Motivated by the recent work [3] of Attouch, Chbani, and Riahi on the asymptotic behavior of the trajectories of the asymptotic vanishing damping dynamical system with regularizing Tikhonov term

$$\text{(AVD}_{\alpha, \varepsilon}\text{)}$$

$$x''(t) + \frac{\alpha}{t} x'(t) + \nabla \Phi(x(t)) + \varepsilon(t)x(t) = 0,$$

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we investigate in this paper the long time behavior, as $t \to +\infty$, of the trajectories of the following generalized version of the (AVD$_{\alpha,\varepsilon}$) dynamical system

$$
(\text{GAVD}_\gamma, \varepsilon) \quad x''(t) + \gamma(t)x'(t) + \nabla \Phi(x(t)) + \varepsilon(t)\nabla U(x(t)) = 0.
$$

For the importance and the applications of these two dynamical systems and many other related dynamical systems in Mechanics and Optimization, we refer the reader to [2], [3], [6], [15] and references therein.

Throughout this paper, we assume the following general hypothesis:

$\mathbf{H}_1$: The functions $\Phi, U : \mathcal{H} \to \mathbb{R}$ are convex, differentiable, and bounded from below. We set $\Phi^* = \inf_{x \in \mathcal{H}} \Phi(x)$ and $U^* = \inf_{x \in \mathcal{H}} U(x)$.

$\mathbf{H}_2$: The set $S_\Phi := \text{argmin}_x \Phi = \{z \in \mathcal{H} : \Phi(z) = \Phi^*\}$ is not empty.

$\mathbf{H}_3$: The gradient functions $\nabla \Phi$ and $\nabla U$ of $\Phi$ and $U$ are Lipschitz on bounded subsets of $\mathcal{H}$.

$\mathbf{H}_4$: The function $\gamma : [t_0, +\infty) \to (0, +\infty)$ is absolutely continuous and satisfies the following property: there exist $t_1 \geq t_0$ and two real constants $K_1, K_2 > 0$ such that

$$
\gamma(t) \geq \frac{K_1}{t} \quad \text{and} \quad \gamma'(t) \leq \frac{K_2}{t^2}
$$

for almost every $t \geq t_1$.

$\mathbf{H}_5$: The function $\varepsilon : [t_0, +\infty) \to (0, +\infty)$ is absolutely continuous, nonincreasing, and satisfies

$$
\lim_{t \to +\infty} \varepsilon(t) = 0.
$$

Proceeding as in the proof of [Theorem 3.1, [3]] and using the classical Cauchy-Lipschitz and the energy function

$$
W(t) = \frac{1}{2} \|x'(t)\|^2 + \Phi(x(t)) - \Phi^* + \varepsilon(t)(U(x(t)) - U^*),
$$

one can easily prove that for every initial data $(x_0, v_0) \in \mathcal{H} \times \mathcal{H}$, the dynamical system (GAVD$_{\gamma, \varepsilon}$) has a unique solution $x(.) \in C^2([t_0, +\infty); \mathcal{H})$ which satisfies $x(t_0) = x_0$ and $x'(t_0) = v_0$. Therefore, we assume in what follows that $x(.)$ is a classical global solution of (GAVD$_{\gamma, \varepsilon}$) and we focus our attention on the study of the long time behavior of $x(t)$ as $t$ goes to infinity. Before setting the contributions of this work in this direction, let us first recall some well known results on the asymptotic behavior of solutions of a variant dynamical systems related to (GAVD$_{\gamma, \varepsilon}$). In the pioneer work [1], Avarez considered the case where $\gamma(.)$ is constant and $\varepsilon = 0$. He established that the trajectory $x(t)$ converges weakly to some element $\bar{x}$ of $S_\Phi$. In this case, the rate of convergence of $\Phi(x(t))$ to $\Phi^*$ is of order $o(1/t)$ (see [2]). To overcome the drawback of the weak convergence to a non identified minimizer of $\Phi$, Attouch and Cazerniki [5] proved that, up to adding a Tikhonov regularizing term $\varepsilon(t)x(t)$ with $\int_{t_0}^{+\infty} \varepsilon(t)dt = +\infty$, any trajectory $x(t)$ of the system

$$
x''(t) + \gamma(t)x'(t) + \nabla \Phi(x(t)) + \varepsilon(t)x(t) = 0
$$

(1.2)
converges strongly to the element $x^*$ of minimum norm of the set $S_\Phi$. Using a different approach, Jendoubi and May [10] proved that this strong convergence result remains true even a perturbation integrable term $g(t)$ is added to the equation (1.2). In other direction, in order to improve the rate of convergence of $\Phi(x(t))$ to $\Phi^*$, Su, Boyd, and Candes [15] introduced the dynamical system which is the continuous version of the Nestrov’s accelerated minimization method [13]

$$
(1.3) \
 x''(t) + \frac{\alpha}{t} x'(t) + \nabla \Phi(x(t)) = 0.
$$

They proved that if $\alpha \geq 3$ then

$$
\Phi(x(t)) - \Phi^* = O(1/t^2).
$$

This result was later improved in [3] and [12]. In fact it was proved that if $\alpha > 3$ then $x(t)$ of converges weakly to some element $\bar{x}$ of $S_\Phi$ and that

$$
\Phi(x(t)) - \Phi^* = o(1/t^2).
$$

Recently, in order to benefit at the same time of the quick minimization property $\Phi(x(t)) - \Phi^* = o(1/t^2)$ due to the presence of the vanishing damping term $\gamma(t) = \frac{\alpha}{t}$ in (1.3) and the strong convergence of the trajectories of (1.2) to a particular minimizer of $\Phi$ which is a consequence of the regularizing Tikhonov term $\varepsilon(t)x(t)$, Attouch, Chbani, and Riahi [4] have considered the dynamical system (AVD$_{\alpha,\varepsilon}$) and they have established some properties of the asymptotic behavior of its trajectories which we can summarize in the following theorem.

**Theorem 1.1** (Attouch, Chbani and Riahi). Let $x \in C^2(t_0, +\infty; \mathcal{H})$ be a solution of (AVD$_{\alpha,\varepsilon}$).

**A:** If $\alpha > 1$ and $\int_{t_0}^{+\infty} \frac{\varepsilon(t)}{t} dt < +\infty$, then $\int_{t_0}^{+\infty} \frac{\|x'(t)\|^2}{t} dt < +\infty$, $\lim_{t \to +\infty} x'(t) = 0$ and $\lim_{t \to +\infty} \Phi(x(t)) = \Phi^*$.

**B:** If $\alpha > 3$ and $\int_{t_0}^{+\infty} t\varepsilon(t) dt < +\infty$, then $x(t)$ converges weakly to some element of $S_\Phi$.

Furthermore, the associated energy function $W(t) = \frac{1}{2} \|x'(t)\|^2 + \Phi(x(t)) - \Phi^*$ satisfies $W(t) = o(1/t^2)$ and $\int_{t_0}^{+\infty} tW(t) dt < +\infty$.

**C:** If the function $\varepsilon$ satisfies moreover one of the following hypothesis

- **H$_{5a}$**: $\lim_{t \to +\infty} t^2 \varepsilon(t) = +\infty$ if $\alpha = 3$
- **H$_{5b}$**: $t^2 \varepsilon(t) \geq c > \frac{4}{9} \alpha (\alpha - 3)$ if $\alpha > 3$
- **H$_{5c}$**: $\int_{t_0}^{+\infty} \frac{\varepsilon(t)}{t} dt = +\infty$
\[ \lim_{t \to +\infty} \|x(t) - x^*\| = 0 \text{ where } x^* \text{ is the element of minimal norm of the set } S_\Phi. \]

In this paper, we improve and extend these results to the general dynamical system \((GAVD_{\gamma, \varepsilon})\). Moreover, we discover some new asymptotic properties of the trajectories of \((GAVD_{\gamma, \varepsilon})\).

Our first result is a general minimization property of \((GAVD_{\gamma, \varepsilon})\) which is a slight improvement of the assertion (A) in the previous theorem.

**Theorem 1.2** (A general minimization property of \((GAVD_{\gamma, \varepsilon})\)). Let \(x(.)\) be a classical solution of \((GAVD_{\gamma, \varepsilon})\). Then
\[ \int_{t_0}^{+\infty} \gamma(t) \|x'(t)\|^2 dt < +\infty, \]
and the energy function \(W(t)\), defined by (1.1), converges to 0 as \(t \to +\infty\). In particular \(\lim_{t \to +\infty} x'(t) = 0\) and \(\lim_{t \to +\infty} \Phi(x(t)) = \Phi^*\).

Our second main result concerns the weak convergence property of the trajectories of \((GAVD_{\gamma, \varepsilon})\). The first part of this result is similar to the assertion B in Theorem 1.1. Our proof, which is different from the arguments given by Attouch, Chbani, and Riahi [Theorem 3.1, [4]], provide an other confirmation of the fact, noticed recently in many works as [4, 2, 12] and [15], that the value \(\alpha = 3\) in the the system (1.3) is critical and somehow mysterious. The second part of the theorem is a simple result on the weak convergence to a common minimizer of the two convex functions \(\Phi\) and \(U\) which, at our knowledge, is not known even in the case where the damping term \(\gamma\) is constant. A comparable result was proved by Cabot (see [Proposition 2.5, [7]]) for the first order system \(x'(t) + \nabla \Phi(x(t)) + \varepsilon(t) \nabla U(x(t)) = 0\).

**Theorem 1.3** (Weak convergence properties of \((GAVD_{\gamma, \varepsilon})\)). Assume that there exist \(t_1 \geq t_0, \, 0 \leq \theta \leq 1, \alpha > 0\) with \(\alpha > 3\) if \(\theta = 1\) such that
\[ (1.4) \quad \gamma(t) \geq \frac{\alpha}{t^{\theta}} \text{ for every } t \geq t_1 \text{ and } \int_{t_0}^{+\infty} \left[ (t^\theta \gamma(t))' \right]^+ dt < +\infty \]
where \([t^\theta \gamma(t)]' = \max\{0, (t^\theta \gamma(t))'\}\). Let \(x(.)\) be a classical solution of \((GAVD_{\gamma, \varepsilon})\). Then the two following properties hold:

- **P1:** If \(\int_{t_0}^{+\infty} t^\theta \varepsilon(t) dt < +\infty\) then \(x(t)\) converges weakly to some element of \(S_\Phi\).
- **P2:** If \(S_\Phi \cap S_U \neq \emptyset\) and \(\liminf_{t \to +\infty} t^{1+\theta} \varepsilon(t) > 0\) then \(x(t)\) converges weakly to some element of \(S_\Phi \cap S_U\).

Moreover, in both case, the energy function \(W\) satisfies
\[ (1.5) \quad W(t) = o(1/t^{1+\theta}) \text{ and } \int_{t_0}^{+\infty} t^\theta W(t) dt < +\infty. \]

Our last mean result deals with the strong convergence of the trajectories of \((GAVD_{\gamma, \varepsilon})\) to a minimizer of the function \(U\) on the set of minimizers of \(\Phi\).
Theorem 1.4 (Strong convergence properties of (GAVD_{\gamma,\varepsilon})). Assume that \( U \) is strongly convex and \( \gamma(t) = \frac{\alpha}{t^\theta} \) with \( \alpha > 0 \) if \( 0 \leq \theta < 1 \) and \( \alpha > 3 \) if \( \theta = 1 \). Suppose in addition that \( \int_0^{+\infty} t^\theta \varepsilon(t) dt = +\infty \). Let \( x(\cdot) \) be a classical solution of (GAVD_{\gamma,\varepsilon}). Then the two following assertions hold:

\( Q_1 \): If \( x'(t) = o(1/t^\theta) \) and \( \int_0^{+\infty} t^\theta \| x'(t) \|^2 dt < +\infty \) then \( x(t) \) converges strongly to the unique minimizer \( p^*_\Phi \) of \( U \) on \( S_\Phi \).

\( Q_2 \): If the unique minimizer \( x^* \) of \( H \) belongs to \( S_\Phi \) then \( x(t) \) converges strongly to \( x^* \) and the energy function \( W \) satisfies the asymptotic properties (1.5).

Remark 1.5. In the case \( \gamma(t) = \gamma \) is constant (which correspond the case \( \theta = 0 \)), combining Theorem 1.2 and the assertion \( Q_1 \) of the Theorem 1.4 yields a generalization of the strong convergence result of Attouch and Cazernicki [Theorem 2.3, [9]].

2. A General Minimization Property of (GAVD_{\gamma,\varepsilon})

This section is devoted to the proof of Theorem 1.2 which is inspired from the arguments of [Theorem 3.1, [9]] and [Theorem 2.1, [11]]. Notice that the assumption \( (H_2) \) can be excluded.

Proof. Differentiating the energy function \( W \) and using the equation (GAVD_{\gamma,\varepsilon}), we obtain

\[
W'(t) = -\gamma(t) \| x'(t) \|^2 + \varepsilon(t)(U(x(t)) - U^*)
\leq -\gamma(t) \| x'(t) \|^2.
\]

Hence \( W(t) \) is decreasing and approaching to some nonnegative real number \( W_\infty \) as \( t \to +\infty \). Moreover, we have

\[
\int_0^{+\infty} \gamma(t) \| x'(t) \|^2 dt < \infty.
\]

To conclude, we just have to show that \( W_\infty \leq 0 \). Let \( v \) be an arbitrarily element of \( \mathcal{H} \). Consider the function

\[
h_v(t) \equiv \frac{1}{2} \| x(t) - v \|^2.
\]

Using the equation (GAVD_{\gamma,\varepsilon}) and the convexity of \( \Phi \) and \( U \), one can easily check that

\[
h_v''(t) + \gamma(t)h_v'(t) = \| x'(t) \|^2 + \langle \nabla \Phi(x(t)), v - x(t) \rangle + \varepsilon(t)\langle \nabla U(x(t)), v - x(t) \rangle
\leq \| x'(t) \|^2 + \Phi(v) - \Phi(x(t)) + \varepsilon(t)(U(v) - U(x(t)))
\]

\[
= \frac{3}{2} \| x'(t) \|^2 - W(t) + \Phi(v) - \Phi^* + \varepsilon(t)(U(v) - U^*).
\]

Recalling that \( W(t) \geq W_\infty \), we get

\[
A_\infty \leq -h_v''(t) - \gamma(t)h_v'(t) + \frac{3}{2} \| x'(t) \|^2 + \varepsilon(t)(U(v) - U^*)
\]

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where \( A_\infty = W_\infty + \Phi^* - \Phi(v) \).

Integrating the last inequality over \([t_0, t]\) and using the fact that \( \gamma h_v \geq 0 \) and the assumption \( \gamma(t) \leq \frac{K_2}{t^2} \), we find

\[
(2.4) \quad (t - t_0)A_\infty \leq h'_v(t_0) - \gamma(t_0)h_v(t_0) + h'_v(t) + \frac{3}{2} \int_{t_0}^{t} \|x'(s)\|^2 \, ds + \int_{t_0}^{t} f_v(s) \, ds,
\]

where \( f_v(s) = \varepsilon(s)(U(v) - U^*) + \frac{K_2}{s} h_v(s) \).

From (2.2), we deduce that

\[
\int_{t_0}^{+\infty} \frac{\|x'(s)\|^2}{s} \, ds < \infty \quad \text{which implies (see [Lemma 3.2, [9]) that}
\]

\[
(2.5) \quad \int_{t_0}^{t} \|x'(s)\|^2 \, ds = o(t).
\]

Using now the Cauchy-Schwartz inequality, we infer

\[
\|x(t)\| \leq \|x(t_0)\| + \sqrt{t - t_0} \left( \int_{t_0}^{t} \|x'(s)\|^2 \, ds \right)^{\frac{1}{2}}
\]

\[
= o(t).
\]

Therefore \( \lim_{t \to +\infty} f_v(t) = 0 \) and as a consequence

\[
(2.7) \quad \int_{t_0}^{t} f_v(s) \, ds = o(t).
\]

Recalling now that since \( W \) is bounded, \( x' \) is also bounded. Thus, we get by (2.6)

\[
(2.8) \quad h'_v(t) = 2 \langle x'(t), x(t) - v \rangle = o(t).
\]

Finally, dividing the inequality (2.4) by \( t \), using the estimates (2.5), (2.7), (2.8) and letting \( t \to +\infty \), we obtain \( A_\infty \leq 0 \), which implies that \( W_\infty \leq \Phi(v) - \Phi^* \). Since this holds for every \( v \in \mathcal{H} \), the required result \( W_\infty \leq 0 \) follows.

**Remark 2.1.** Let us notice that, if \( S_\phi \) is empty, any solution \( x(.) \) of the (GAVD\( _{\gamma,\varepsilon} \)) system is unbounded. Indeed, if else, there exists a sequence \( (t_n) \) tending to \( +\infty \) so that \( (x(t_n))_n \) converges weakly to an element \( \bar{x} \in \mathcal{H} \). From the lower semi-continuity property it follows that

\[
\Phi(\bar{x}) \leq \liminf_{n \to +\infty} \Phi(x(t_n)),
\]

which means that \( \Phi(\bar{x}) \leq \Phi^* \). This contradicts that \( S_\phi = \emptyset \).

3. **Weak convergence properties of (GAVD\( _{\gamma,\varepsilon} \))**

In this section, we give a proof of Theorem 1.3 which relies on the classical Opial’s lemma and the following important lemma which will be also useful in the study of the strong convergence properties of the trajectories of (GAVD\( _{\gamma,\varepsilon} \)) in the next section.
Lemma 3.1. Assume that the function $\gamma(.)$ satisfies the assumption (I.4) in Theorem I.3. Let $x(.)$ be a classical solution of (GAVD$\gamma,\varepsilon$) and let $v \in S_{\Phi}$ such that $[t^\theta r_v(t)]^+$ belongs to $L^1(t_0, +\infty; \mathbb{R})$ where $r_v(t) \equiv \varepsilon(t)(U(v) - U(x(t)))$. Then the distance function $h_v(t) \equiv \frac{1}{2}||x(t) - v||^2$ converges as $t \to +\infty$ and the energy function $W$ satisfies the asymptotic property (I.3).

Proof. First, we notice that up to take $t_1$ large enough we can assume that

$$\gamma(t) \geq \frac{K}{t} \text{ for every } t \geq t_1$$

with $K > 3$ and $K = \alpha$ if $\theta = 1$.

Let $\lambda(t) = t^{1+\theta}$. Using (2.1) and the above inequality, we find

$$(\lambda W)' \leq \lambda' W - \lambda \gamma \|x\|^2$$

$$\leq \lambda' W - \frac{K}{1+\theta} \lambda \|x\|^2$$

$$\leq \lambda' W - \frac{K}{2} \lambda \|x\|^2.$$  

(3.1)

Therefore,

$$\frac{3}{2} \lambda \|x\|^2 \leq \frac{3}{K} \lambda' W - \frac{3}{K} (\lambda W)'.$$

Multiplying (2.3) by $\lambda'(t)$ (we recall that, since $v \in S_{\Phi}$, $\Phi(v) = \Phi^*$) and using the above inequality, we obtain

$$(1 - \frac{3}{K}) \lambda' W + \frac{3}{K} (\lambda W)' \leq -\lambda'' h_v - \lambda' \gamma h_v + \lambda [r_v]^+.$$
Using now the fact that
\[ |h'(t)| \leq \|x'(t)\| \|x(t) - v\| \leq 2\sqrt{W(t)} \sqrt{h_v(t)}, \]
it follows, from the estimate (3.4) and the elementary inequality
\[ bx - ax^2 \leq \frac{b^2}{4a} \quad \forall a > 0, \quad (x, b) \in \mathbb{R}^2, \]
that for every \( t \geq t_1 \)
\[ -\lambda'(t) h_v'(t) + (\lambda'' - \lambda' \gamma)(t) h_v(t) \leq \frac{(\lambda'(t))^2 W(t)}{A(\theta)} - \mu(\theta) h_v(t) \]
(3.5)
\[ = B(\theta, t) \lambda(t) W(t) - \mu(\theta) h_v(t), \]
where
\[ B(\theta, t) = \frac{(\theta + 1)^2 t^{\theta-1}}{A(\theta)}. \]

Inserting (3.5) in the inequality (3.2), we obtain
(3.6)
\[ (1 - \frac{3}{K}) \int_{t_1}^t \lambda'(s) W(s) ds + \left( \frac{3}{K} - B(\theta, t) \right) \lambda(t) W(t) + \mu(\theta) h_v(t) \leq C_0 + \int_{t_1}^t g_{\theta}(s) h_v(s) ds \]
Let us notice that if \( 0 \leq \theta < 1 \) then \( \lim_{t \to +\infty} B(\theta, t) = 0 \) and in the case where \( \theta = 1 \), since \( \alpha > 3 \), one can choose \( 0 < \mu(1) < \frac{2}{3}(\alpha - 3) \) to get
\[ \frac{3}{K} - B(1, t) = \frac{3}{\alpha} - \frac{4}{A(1)} > 0. \]

Hence, up to take \( t_1 \) large enough we assume that, for every \( 0 \leq \theta \leq 1 \), there exists a constant \( \nu(\theta) > 0 \) such that
\[ \frac{3}{K} - B(\theta, t) \geq \nu(\theta), \quad \text{for all } t \geq t_1. \]

Recalling that the function \( g_{\theta} \) is integrable over \([t_1, +\infty)\) and applying the Gronwall lemma to the inequality (3.6), we deduce that the function \( h_v \) is bounded and as a consequence we get
\[ \sup_{t \geq t_1} \lambda(t) W(t) < +\infty \]
and
(3.7)
\[ \int_{t_1}^{+\infty} \lambda'(s) W(s) ds < +\infty. \]
Now, using the fact that the energy function $W$ is decreasing, we deduce from (3.7) that $t^{1+\theta}W(t) \to 0$ as $t \to +\infty$ in fact for every $t \geq t_1$ we have

$$(1 + \theta) \left( \frac{t}{2} \right)^{1+\theta} W(t) \leq \int_{t}^{t_1} \lambda'(s)W(s)ds.$$  

To conclude, it remains to prove that $\lim_{t \to +\infty} h_v(t)$ exists. From (2.3), the function $h_v$ satisfies the differential inequality

$$h''_v(t) + \gamma(t)h'_v(t) \leq \zeta(t)$$

where $\zeta(t) = \frac{3}{2} \|x'(t)\|^2 + [r_v(t)]^+$. The assumption on the function $r_v$ and the estimate (3.7) imply that $t^\theta \zeta(t) \in L^1(a, +\infty; \mathbb{R}^+)$, then the existence of $\lim_{t \to +\infty} h_v(t)$ follows from the following lemma.

**Lemma 3.2.** Let $a > 0$ and $w : [a, +\infty) \to \mathbb{R}^+$ be a continuous function satisfying

$$w(t) \geq \frac{\alpha}{t^\theta} \forall t \geq a$$

where $\alpha$ and $\theta$ are nonnegative constants with $0 \leq \theta \leq 1$ and $\alpha > 1$ if $\theta = 1$. Let $\varphi \in C^2(a, +\infty; \mathbb{R}^+)$ satisfy the differential inequality

$$(3.8) \quad \varphi''(t) + w(t)\varphi'(t) \leq \psi(t)$$

with $t^\theta \psi(t) \in L^1(a, +\infty; \mathbb{R}^+)$. Then $\lim_{t \to +\infty} \varphi(t)$ exists.

**Proof.** From (3.8), we have for every $t \geq a$

$$(3.9) \quad \varphi'(t) \leq e^{-\Gamma(t,a)} \varphi'(a) + \int_{a}^{t} e^{-\Gamma(t,s)}\psi(s)ds;$$

where

$$\Gamma(t, s) = \int_{s}^{t} w(\tau)d\tau.$$  

Similarly to as in the proof of (Lemma 3.14,[8]), one can easily check that

$$\int_{s}^{+\infty} e^{-\Gamma(t,s)}dt \leq M \quad s^\theta \forall s \geq a,$$

where $M > 0$ is an absolute constant. We deduce from (3.9) and Fubini’s Theorem that the positive part $[\varphi']^+$ of $\varphi'$ belongs to $L^1(a, +\infty; \mathbb{R}^+)$ which implies that $\lim_{t \to +\infty} \varphi(t)$ exists.

Before starting the proof of Theorem 1.3 let us recall the classical Opial’s lemma.

**Lemma 3.3** (Opial’s lemma). Let $x : [t_0, +\infty) \to \mathcal{H}$. Assume that there exists a nonempty subset $S$ of $\mathcal{H}$ such that:

i) if $t_n \to +\infty$ and $x(t_n) \rightharpoonup x$ weakly in $\mathcal{H}$, then $x \in S$,

ii) for every $z \in S$, $\lim_{t \to +\infty} \|x(t) - z\|$ exists.
Then there exists \( z_\infty \in S \) such that \( x(t) \rightharpoonup z_\infty \) weakly in \( \mathcal{H} \) as \( t \to +\infty \).

For a simple proof of Opial’s lemma, we refer the reader to [14].

**Proof of Theorem 1.3 Step 1:** Proof of the property \((P_1)\). Since \( r_v(t) \leq \varepsilon(t)(U(v) - U^*) \), then, according to Lemma 3.1 \( \lim_{t \to +\infty} h_v(t) \) exists for every \( v \in S_\Phi \) and the energy function \( W \) satisfies (1.5). Let \( t_n \to +\infty \) such that \( x(t_n) \) converges weakly in \( \mathcal{H} \) to some \( \bar{x} \). Since \( \Phi(x(t)) \to \Phi^* \) as \( t \to +\infty \), the weak lower semi-continuity of \( \Phi \) implies that \( \Phi(\bar{x}) \leq \Phi^* \) which means that \( \bar{x} \in S_\Phi \). By Opial’s lemma, we deduce that \( x(t) \) converges weakly in \( \mathcal{H} \) as \( t \to +\infty \) to some element of \( S_\Phi \).

**Step2:** Proof of the property \((P_2)\). Let \( v \in S = S_\Phi \cap S_U \). Since \( r_v \) is nonpositive, then Lemma 3.1 implies that \( \lim_{t \to +\infty} h_v(t) \) exists and \( W \) satisfies (1.3). Thus, in view of the assumption \( \lim_{t \to +\infty} \inf \varepsilon^{t+1}(t) > 0 \), we have \( U(x(t)) \to U^* \) as \( t \to +\infty \). Therefore the lower semi-continuity of \( \Phi \) and \( U \) gives, as in the above step, that every sequential weak cluster point of \( x(t) \), as \( t \to +\infty \), belongs to the subset \( S \). This completes the proof of the property \((P_2)\) due to Opial’s lemma.

\[ \square \]

4. Strong Convergence Properties of \((\text{GAVD}_{\gamma,\varepsilon})\)

This section is devoted to the proof of Theorem 1.4. Before proving separately the two properties \((Q_1)\) and \((Q_2)\), let us first recall some general facts about strongly convex functions and the Tikhonov approximation method [16]. The function \( U \) is strongly convex then there exists a positive real \( m \) such that \( U(x) - \frac{m}{2} \|x\|^2 \) is convex (we say that \( U \) is \( m \)-strongly convex). Moreover, for every nonempty, convex and closed subset \( C \) of \( \mathcal{H} \), the function \( U \) has a unique minimizer \( x^*_C \) on \( C \). Let \( x^* \) be the minimizer on \( \mathcal{H} \) and \( p^* \) its minimizer on \( S_\Phi \). For every \( t \geq t_0 \), we consider the function \( \Phi_t \) defined on \( \mathcal{H} \) by

\[ \Phi_t(x) = \Phi(x) + \varepsilon(t)U(x). \]

Clearly, \( \Phi_t \) is \( \varepsilon(t)m \)-strongly convex. Therefore, \( \Phi_t \) satisfies the convex inequality

\[ (4.1) \quad \Phi_t(z) \geq \Phi_t(y) + \langle \nabla \Phi_t(y), z - y \rangle + \frac{m}{2} \varepsilon(t) \|z - y\|^2, \]

and has a unique global minimizer which we denote by \( x_{\varepsilon(t)} \). Adopting the Tikhonov method, we can prove that \( x_{\varepsilon(t)} \) converges strongly to \( p^* \) as \( t \to +\infty \). Indeed, since

\[ (4.2) \quad \Phi_t(x_{\varepsilon(t)}) \leq \Phi_t(p^*) \]

and

\[ \Phi(p^*) \leq \Phi(x_{\varepsilon(t)}), \]

then

\[ (4.3) \quad U(x_{\varepsilon(t)}) \leq U(p^*). \]

Furthermore, seeing that \( U \) is coercive, the last inequality implies that \( (x_{\varepsilon(t)})_{t \geq t_0} \) is bounded. So, let \( \bar{x} \in \mathcal{H} \) be a weak limit of a sequence \( (x_{\varepsilon(t_n)}) \) where \( t_n \to +\infty \). Using the weak lower semi-continuity of the two convex functions \( \Phi \) and \( U \) and letting
\( t = t_n \to +\infty \) in the inequalities (4.2) and (4.3), we deduce that \( \Phi(\tilde{x}) \leq \Phi(p^*) \) and \( U(\tilde{x}) \leq U(p^*) \) which is, from the definition of \( p^* \), is equivalent to \( \tilde{x} = p^* \). Consequently, we infer that \( x_{\varepsilon(t)} \) converges weakly to \( p^* \) as \( t_n \to +\infty \). Now, for the reason that \( U \) is \( m \)-strongly convex, we have

\[
U(x_{\varepsilon(t)}) \geq U(p^*) + \langle \nabla U(p^*), x_{\varepsilon(t)} - p^* \rangle + \frac{m}{2} \| x_{\varepsilon(t)} - p^* \|^2.
\]

Hence, by (4.3), we deduce that

\[
\lim_{t \to +\infty} \| x_{\varepsilon(t)} - p^* \| = 0
\]

which completes the proof of the claim.

Proof of Theorem 1.4. Let us first prove the property (Q_1). We consider the function

\[
h(t) = h_{p^*}(t) = \frac{1}{2} \| x(t) - p^* \|^2.
\]

Using the equation (GAVD_{\gamma, \varepsilon}) and the convex inequality (4.1) we obtain

\[
h''(t) + \gamma(t)h'(t) = \| x'(t) \|^2 + \langle \nabla \Phi_t(x(t)), p^* - x(t) \rangle
\]

\[
\leq \| x'(t) \|^2 + \Phi_t(p^*) - \Phi_t(x(t)) - m \varepsilon(t)h(t)
\]

\[
\leq \| x'(t) \|^2 + \Phi_t(p^*) - \Phi_t(x_{\varepsilon(t)}) - m \varepsilon(t)h(t)
\]

\[
\leq \| x'(t) \|^2 + \varepsilon(t)(U(p^*) - U(x_{\varepsilon(t)})) - m \varepsilon(t)h(t).
\]

(4.4)

In the last inequality we have used the fact that \( p^* \) is also a minimizer of \( \Phi \). Set

\[
\sigma(t) \equiv U(x_{\varepsilon(t)}) - U(p^*) + m h(t).
\]

The inequality (4.4) becomes

\[
(4.5) \quad h''(t) + \gamma(t)h'(t) + \varepsilon(t)\sigma(t) \leq \| x'(t) \|^2.
\]

Let us prove that \( \lim_{t \to +\infty} h(t) = 0 \). We argue by contradiction. As consequence of

\[
\lim_{t \to +\infty} \| x_{\varepsilon(t)} - p^* \| = 0,
\]

there exists \( t_2 \geq t_0 \) large enough and \( \mu > 0 \) such that \( \sigma(t) \geq \mu \) for every \( t \geq t_2 \). Therefore the differential inequality (4.5) implies that, for every \( t \geq t_2 \), we have

\[
h(t) + \mu \int_{t_2}^t \int_{t_2}^\tau e^{-\Gamma(t,s)} \varepsilon(s)dsd\tau \leq h(t_2) + \int_{t_2}^t e^{-\Gamma(t,t_2)} d\tau h'(t_2) + \int_{t_2}^t \int_s^\tau e^{-\Gamma(t,s)} \| x'(s) \|^2 dsd\tau,
\]

where

\[
\Gamma(t, s) = \int_s^t \gamma(\tau)d\tau.
\]

Applying Fubini’s theorem, we then infer that

\[
(4.6) \quad \mu \int_{t_2}^{+\infty} \varepsilon(s) \int_s^{+\infty} e^{-\Gamma(t,s)} d\tau ds \leq h(t_2) + |h'(t_2)| \int_{t_2}^{+\infty} e^{-\Gamma(t,t_2)} d\tau + \int_{t_2}^{+\infty} \| x'(s) \|^2 \int_s^{+\infty} e^{-\Gamma(t,s)} d\tau ds.
\]

A simple computation ensures the existence of two real constants \( B_\theta > A_\theta > 0 \) so that

\[
A_\theta s^\theta \leq \int_s^{+\infty} e^{-\Gamma(t,s)} d\tau \leq B_\theta s^\theta.
\]
Hence, combining the inequality (4.6) and the assumption \( \int_{t_0}^{+\infty} s^\theta \| x'(s) \|^2 \, ds < +\infty \), we get \( \int_{t_0}^{+\infty} s^\theta \varepsilon(s) \, ds < +\infty \), a contradiction. Consequently

\[
(4.7) \quad \liminf_{t \to +\infty} h(t) = 0.
\]

Now let us suppose that

\[
(4.8) \quad \limsup_{t \to +\infty} h(t) > 0.
\]

The continuity of the function \( h \) combined with (4.7) and (4.8) ensure the existence of two real numbers \( \lambda < \delta \) and two positive real sequences \( (s_n)_n \) and \( (t_n)_n \) such that for every \( n \in \mathbb{N} \) we have

\[
\max \{ t_*, n \} < s_n < t_n,
\]

\[
h(t_n) = \delta,
\]

\[
h(s_n) = \lambda,
\]

\[
h(s) \in [\lambda, \delta] \text{ on } [s_n, t_n],
\]

where \( t_* > t_2 \) is a fixed positive number such that \( U(x_{\varepsilon(t)}) - U(p^*) \geq -m\lambda \) for all \( t \geq t_* \) (for more details see [Theorem 5.1 [10]]). We deduce from (4.5) that for every \( n \in \mathbb{N} \) and for all \( t \in [s_n, t_n] \)

\[
h''(t) + \frac{\alpha}{t^\theta} h'(t) \leq \| x'(t) \|^2.
\]

Multiplying the last differential inequality by \( t^\theta \) and integrating over \([s_n, t_n] \), we obtain

\[
t_n^\theta h'(t_n) - s_n^\theta h'(s_n) + \theta s_n^{\theta-1} \lambda \theta t_n^{\theta-1} \lambda + \alpha (\delta - \lambda) + \theta (\theta - 1) \int_{s_n}^{t_n} t^{\theta-2} h(t) \, dt \leq \int_{s_n}^{t_n} t^\theta \| x'(t) \|^2.
\]

Using now the facts

\[
|h'(t_n)| \leq \| x'(t_n) \| \sqrt{2h(t_n)} = \| x'(t_n) \| \sqrt{2\delta},
\]

\[
|h'(s_n)| \leq \| x'(s_n) \| \sqrt{2\lambda},
\]

\[
\int_{s_n}^{t_n} t^{\theta-2} h(t) \, dt \leq \delta s_n^{\theta-1} \frac{1}{1 - \theta} \text{ if } 0 \leq \theta < 1,
\]

and letting \( n \) goes to \(+\infty\) in the the inequality (4.9), we get

\[
(\alpha - 1)(\delta - \lambda) \leq 0 \text{ if } \theta = 1,
\]

\[
\alpha (\delta - \lambda) \leq 0 \text{ if } 0 \leq \theta < 1.
\]

This contradicts the assumption \( \delta > \lambda \). We therefore conclude that \( \lim_{t \to +\infty} h(t) = 0 \), which completes the proof of the property \((Q_1)\). \( \square \)
5. Numerical Experiments

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