Exact Solutions of a One-dimensional Quantum Spin Chain with $SO(5)$-Symmetry

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A new exactly solvable one-dimensional spin-3/2 Heisenberg model with $SO(5)$-invariance is proposed. The eigenvalues and Bethe ansatz equations of the model are obtained by using the nested algebraic Bethe ansatz approach. Several exotic elementary excitations in the antiferromagnetic region such as neutral spinon with zero spin, heavy spinon with spin-3/2 and dressed spinon with spin-1/2 are found.

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I. INTRODUCTION

The one-dimensional(1D) quantum spin chains play a very important role in the strongly correlated systems and low-dimensional quantum magnetism and show many interesting behaviors. For example, the ground state of the spin-1/2 Heisenberg chain with antiferromagnetic couplings is expected to be a spin liquid rather than Néel ordered state due to the strong quantum fluctuations. The elementary excitations of such a system are usually described by the spinons which carry spin-1/2 rather than spin waves. In another hand, Haldane conjectured that the spin systems with half-integer spins have gapless excitation spectra, while those with integer spins have gapped spectra[1]. The spin-1/2 Heisenberg model was exactly solved by Bethe[2]. By using Bethe’s hypothesis, Yang and Yang solved the $XXZ$ Heisenberg model successfully[3]. Subsequently, Takhtajan and Faddeev developed the algebraic Bethe ansatz method and several spin chain models have been exactly solved[3,5,7,8].

Recently, much attention has been focused on the high spin systems because not only many peculiar quantum orders and exotic collective excitations can appear in this kind of systems, but also some materials such as CsVBr$_3$, CsVCl$_3$, CsVI$_3$ and AgCrP$_2$S$_6$[17] in nature can be modeled by the spin-3/2 chain quite well. With the developments of experimental technique of laser cooling and magnetic traps, atoms with high nuclear spin such as $^{87}$Rb and $^{23}$Na with spin-1, $^{132}$Cs, $^{9}$Be, $^{135}$Ba, $^{137}$Ba, and $^{53}$Cr with spin-3/2 can be trapped. Using the Feshbach resonance techniques, one can tune the scattering lengths among the atoms, which make it possible to simulate the traditional solid systems with various interactions. These progress provide us a ideal platform to study the physics in high spin systems and many interesting quantum phenomena are found[18,19,20].

It is well-known that the spin-s chain with the $SU(2s+1)$-symmetry can be solved exactly, for the Hamiltonian can be mapped onto the summation of permutation operators. Takhtajan and Babudjian found that besides the $SU(3)$ integrable point, the spin-1 system can still be solved at a special $SU(2)$-invariant point[4,6]. They also showed that the elementary excitation of the system is gapless. This motivate us to seek other integrable points in the high spin systems. In this paper, we show that besides the $SU(4)$ integrable point, there is another exactly solvable model of the spin-3/2 chain with $SO(5)$ symmetry. By using the nested algebraic Bethe ansatz method, we obtain the exact solutions of the model. Based on the exact solutions, several exotic excitations such as the heavy spinon with fractional spin 3/2, the neutral spinon with spin zero and the dressed spinon with spin 1/2 are found, which are quite different from those of the $SU(4)$ integrable spin chain.

The paper is organized as follows. We introduce the model and its symmetry in Sec. II. The nested algebraic Bethe ansatz approach for the model is shown in Sec. III. The thermodynamic properties of the system are analyzed in Sec. IV. The ferromagnetic and antiferromagnetic ground states are discussed in Sec. V. The elementary excitations are given in Sec. VI. Sec. VII is a brief summary.

II. THE MODEL

As mentioned above, two integrable models of 1D spin-3/2 Heisenberg chains, i.e., the $SU(2)$-invariant one and the $SU(4)$-invariant one, have been found and solved. In this paper, we introduce another integrable spin-3/2 chain model with $SO(5)$-symmetry. Our model Hamiltonian reads

$$H = J \sum_{i=1}^{N} \left[ \frac{25}{8} \vec{S}_i \cdot \vec{S}_{i+1} - \frac{7}{3} \left( \vec{S}_i \cdot \vec{S}_{i+1} \right)^2 - \frac{2}{3} \left( \vec{S}_i \cdot \vec{S}_{i+1} \right)^3 \right],$$

where $J$ is a coupling constant; $\vec{S}_i$ is the spin-3/2 operator at site $i$, $i = 1, 2, \cdots, N$ and $N$ is the length of the system. Here, we adopt the periodic boundary condition, i.e., $\vec{S}_{N+1} = \vec{S}_1$. 

III. THE MODEL
To show the \( SO(5) \) symmetry of our model, we introduce the following Dirac matrices

\[
\Gamma^1 = -\frac{1}{\sqrt{3}}(S_xS_y + S_yS_x), \quad \Gamma^2 = \frac{1}{\sqrt{3}}(S_zS_x + S_xS_z), \quad \Gamma^3 = \frac{1}{\sqrt{3}}(S_zS_y + S_yS_z), \\
\Gamma^4 = S_z^2 - 5/4, \quad \Gamma^5 = \frac{1}{\sqrt{3}}(S_x^2 - S_y^2).
\]

(2)

The 10 generators of the \( SO(5) \) Lie algebra can be expressed by the Dirac matrices as \( \Gamma^{a,b} = -\frac{i}{2}[\Gamma^a, \Gamma^b] \). The explicit form of these generators are [21]

\[
\Gamma^{1,(2,3,4)} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \Gamma^{1,5} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \\
\Gamma^{3,4;4;2;3} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \Gamma^{2,(3,4,5)} = \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\vec{\sigma} & 0 \end{pmatrix}.
\]

(3)

After some algebra, we find that the Hamiltonian (1) commutes with the generators (3), \( [H, \Gamma^{a,b}] = 0 \). Thus the model has the \( SO(5) \) symmetry. Different from the \( SU(4) \) integrable spin chain, there only exist three conserved quantities and the number of spins with individual components is no longer conserved. After some detailed analysis, we find that the following quantities are conserved

\[
J_1 = N_{3/2} + N_{1/2} + N_{-1/2} + N_{-3/2}, \\
J_2 = N_{3/2} - N_{3/2}, \\
J_3 = N_{1/2} - N_{-1/2}.
\]

(4)

The \( R \)-matrix of this model reads

\[
R_{ab}(\lambda) = -\frac{2\lambda + 3i}{2\lambda - 3i}P^0_{ab} + P^1_{ab} - \frac{2\lambda + i}{2\lambda - i}P^2_{ab} + P^3_{ab}
\]

where \( \lambda \) is the spectral parameter; \( P^a_{ab} \) is the projection operator in the total spin-\( s \) channel and acts on the two coupled spin space \( V_a \otimes V_b \). Just as the \( SU(4) \) one, the non-trivial scattering processes only exist in the total spin-0 and 2 channels, but here the scattering strengths are different in these two channels.

This \( R \)-matrix satisfies the Yang-Baxter equation [8, 22]

\[
R_{ab}(\lambda)R_{bc}(\lambda + \mu)R_{ab}(\mu) = R_{bc}(\mu)R_{ab}(\lambda + \mu)R_{bc}(\lambda).
\]

(6)

In the framework of quantum inverse scattering method (QISM), the Lax operators of the system are

\[
L_n(\lambda) = R_{0n}(\lambda)P_{0n} = \frac{2\lambda + 3i}{2\lambda - 3i}P^0_{ab} + P^1_{ab} + \frac{2\lambda + i}{2\lambda - i}P^2_{ab} + P^3_{ab},
\]

where \( V_0 \) is an auxiliary space and \( V_n \) is the quantum space. \( P_{0n} \) is the permutation operator, which can be expressed by the projection operators as \( P_{0n} = -P^0_{ab} + P^1_{ab} - P^2_{ab} + P^3_{ab} \). The monodromy matrix \( T \) is constructed by the Lax operators as

\[
T(\lambda) = L_N(\lambda)L_{N-1}(\lambda)\cdots L_1(\lambda).
\]

(8)

The monodromy matrix \( T \) satisfies the Yang-Baxter relation

\[
R(\lambda - \mu)T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda)R(\lambda - \mu),
\]

(9)

Taking trace in the auxiliary space of \( T \), we obtain the transfer matrix

\[
t(\lambda) = \text{tr}_0T(\lambda).
\]

(10)

From the Yang-Baxter relation, one can prove that the transfer matrices with different spectral parameters commute with each other,

\[
[t(\lambda), t(\mu)] = 0.
\]

(11)

These transfer matrices are the infinite conserved quantities of this system. Thus the system is integrable.

Taking the derivative of the logarithm of the transfer matrix, we arrive at the Hamiltonian (1) [23, 24]

\[
H = \frac{9J}{4} \frac{\partial \ln T(\lambda)}{\partial \lambda} \bigg|_{\lambda=0} - \frac{99}{8} JN = -3J \left( P_{i,i+1}^0 + 3P_{i,i+1}^2 \right) - \frac{99}{8} JN
\]

(12)

Here, we have used the fact that the Lax operator [23] degenerates into the permutation operator when the spectral parameter is zero, \( L_{ab}(\lambda)|_{\lambda=0} = P_{ab} \), and put a constant \( J \) into the Hamiltonian. The eigenvalue problem of the Hamiltonian is therefore turned into the diagonalization of transfer matrices. Suppose the eigenvalues of the Hamiltonian and the transfer matrix are \( E \) and \( \Lambda \), then the eigenvalue \( E \) can be determined by the eigenvalue \( \lambda \) as

\[
E = -\frac{9J}{4} \frac{\partial \ln \Lambda(\lambda)}{\partial \lambda} = -\frac{99}{8} JN.
\]

(13)
III. NESTED ALGEBRAIC BETHE ANSATZ

In this section we show that the system (1) can be solved exactly by using the nested algebraic Bethe ansatz method similar to that used for the $sp(2|1)$ supersymmetric one.\cite{22}

The local vacuum state of the $i$th site is chosen as $|0\rangle_i = \{|3/2\rangle_i = (1, 0, 0, 0)^i\}$, where $t$ means the transport. The Lax operator on the site $i$ can be written into following matrix form in the auxiliary space

$$L_i(\lambda) = \begin{pmatrix}
\lambda & i & i & i F(\lambda) \\
i A(\lambda) & i B_1(\lambda) & i B_2(\lambda) & i F(\lambda) \\
i C(\lambda) & i D_1(\lambda) & i D_2(\lambda) & i B^{*1}(\lambda) \\
i C(\lambda) & i D_1(\lambda) & i D_2(\lambda) & i B^{*2}(\lambda)
\end{pmatrix}, \quad (14)
$$

where $i A, i D_1, i B_j, i B^*_j, i C, i C^*_j, i G, i V$ and $i F$ are operators in the quantum space $V_i$. The Lax operator $L_i$ acting on the local vacuum state gives

$$L_i(\lambda) |0\rangle_i = \begin{pmatrix}
1 & i B_1(\lambda) |0\rangle_i & i B_2(\lambda) |0\rangle_i & i F(\lambda) |0\rangle_i \\
0 & f(\lambda) |0\rangle_i & 0 & i B^{*1}(\lambda) |0\rangle_i \\
0 & 0 & f(\lambda) |0\rangle_i & i B^{*2}(\lambda) |0\rangle_i \\
0 & 0 & 0 & p(\lambda) |0\rangle_i
\end{pmatrix}, \quad (15)
$$

where $f(\lambda) = 2\lambda/(2\lambda + i)$. The operators $i B_1(\lambda)$ and $i B_2(\lambda)$ are

$$i B_1(\lambda) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & m(\lambda) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad i B_2(\lambda) = \begin{pmatrix}
m(\lambda) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad (16)
$$

where $m(\lambda) = i/(2\lambda + i)$ and $s(\lambda) = -2i/(2\lambda + 3i)/(2\lambda + i)$. Thus the actions of $i B_1$ and $i B_2$ get a flip of one and two spin quanta respectively. Obviously, the whole state space can be obtained by these two operators. This is quite different from the $SU(4)$ one in which $|−3/2\rangle$ can not be reached with the flips made by $i B_1$ and $i B_2$. So that the construction of the eigenstates might only need these two operators in this model. This makes that only one nesting is needed in the algebraic Bethe ansatz of the model (1). In fact $i F$ is also needed in the construction of the eigenstates but plays an assistant role and doesn’t affect the number of nestings.

The global monodromy matrix of the system \cite{11} is the production of local vacuum states $|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_N$. The monodromy matrix $T$ acting on the global vacuum state gives

$$T_N(\lambda) |0\rangle = L_N(\lambda)L_{N-1}(\lambda) \cdots L_1(\lambda) |0\rangle = \begin{pmatrix}
1 & \sum_i g_i^1(\lambda) \lambda B_1(\lambda) & \sum_i g_i^2(\lambda) \lambda B_2(\lambda) & \sum_i g_i^3(\lambda) \lambda B^{*1}(\lambda) \\
0 & f^N(\lambda), & 0, & \sum_i g_i^4(\lambda) \lambda B^{*2}(\lambda) \\
0, & 0, & f^N(\lambda), & \sum_i g_i^4(\lambda) \lambda B^{*2}(\lambda) \\
0, & 0, & 0, & p^N(\lambda)
\end{pmatrix} |0\rangle, \quad (17)
$$

where $g_i^j$ are some coefficients, and $*$ represents the nonzero element. The matrix form of the monodromy matrix $T$ in the auxiliary space is

$$T(\lambda) = \begin{pmatrix}
A(\lambda) & B_1(\lambda) & B_2(\lambda) & F(\lambda) \\
C(\lambda) & D_1(\lambda) & D_2(\lambda) & B^{*1}(\lambda) \\
C(\lambda) & D_1(\lambda) & D_2(\lambda) & B^{*2}(\lambda) \\
G(\lambda) & C^*_1(\lambda) & C^*_2(\lambda) & V(\lambda)
\end{pmatrix}, \quad (18)
$$

where $A, B_j, B^{*j}, G, C^j, C^*_j, D_j, F$ and $V$ are operators in the Hilbert space $V_1 \otimes V_2 \otimes \cdots \otimes V_N$. Acting these elements on the vacuum state, we obtain

$$C(\lambda) |0\rangle = 0, \quad C^{*j}(\lambda) |0\rangle = 0, \quad G(\lambda) |0\rangle = 0, \quad D_2(\lambda) |0\rangle = D_2(\lambda) |0\rangle = 0, \quad (19)
$$

$$A(\lambda) |0\rangle = |0\rangle, \quad D_1(\lambda) |0\rangle = D_2(\lambda) |0\rangle = f^N(\lambda) |0\rangle, \quad V(\lambda) |0\rangle = p^N(\lambda) |0\rangle. \quad (20)
$$

The operators $C_j, C^*_j, G, D_1$ and $D_2$ acting on the vacuum state $|0\rangle$ gives zero and the operators $A, D_1, D_2$ and $V$ acting on the vacuum state $|0\rangle$ give the eigenvalues. The elements $B_j, F$ and $B^{*j}$ acting on the vacuum state $|0\rangle$ will generate other spin-flipped states, and thus can be regarded as the generating operators of the multi-particle eigenstates of the transfer matrix.

Now we turn to the eigenvalue problem of transfer matrix $T$. From the definition \cite{13}, the transfer matrix can be expressed in the form of

$$T(\lambda) = A(\lambda) + D_1(\lambda) + D_2(\lambda) + V(\lambda). \quad (21)$$
The eigenvalues of $T$ can be determined by the eigenvalues of operators $B, D_1, D_2$ and $V$. These operators are shown to be diagonalized in vacuum state $|0\rangle$ in Eq. (20). Obviously, $|\psi_0\rangle = |0\rangle$ is an eigenstate of $t(\lambda)$, and the corresponding eigenvalue is

$$A_0(\lambda) = \left[1 + 2f^N(\lambda) + p^N(\lambda)\right].$$

(22)

Here $|\psi_0\rangle$ is called zero-particle state. To construct the other eigenstates of the transfer matrix, the generating operators $B_j, B^{*j}$ and $F$ should be used. $B_j$ and $F$ are enough to generate these states as shown below. Assume that the eigenstates of the transfer matrix have the form $|\psi\rangle = \psi(B_j, F)|0\rangle$. To go further, we need need the commutation relations of $A(\lambda)B_a(\mu), D_0^b(\lambda)B_c(\mu), V(\lambda)B_a(\mu), B(\lambda)F(\mu), A_0^a(\lambda)F(\mu)$ and $V(\lambda)F(\mu)$. From the Yang-Baxter relation [7] we obtain [25]

$$A(\lambda)B_a(\mu) = \frac{m}{f}B_a(\lambda)A(\mu) + \frac{1}{f}B_a(\mu)A(\lambda),$$

(23)

$$D_0^b(\lambda)B_c(\mu) = \frac{1}{f}B_c(\mu)D_0^b(\lambda) + \frac{m}{fB_0(\lambda)D_0^b(\mu) + \frac{s}{p}B^{*a}(\lambda)A(\mu) - \frac{m}{p}F(\lambda)C^a(\mu) - \frac{1}{p}F(\mu)C^a(\lambda)\right] \xi_{bc}.$$}

(24)

Where $p(\lambda) = 4\lambda(\lambda + i)/(2\lambda + 3i)(2\lambda + i), \xi$ is a vector of $2 \otimes 2$-dimension, $\xi = (0, 1, -1, 0), r$ is the matrix

$$r(\lambda) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b(\lambda) & a(\lambda) & 0 \\
0 & a(\lambda) & b(\lambda) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

(25)

where $a(\lambda) = \lambda(\lambda + i), b(\lambda) = i(\lambda + i)$. Here the parameter $(\lambda - \mu)$ is omitted for short and the parameter $(\mu - \lambda)$ is also omitted by adding a tilde above the corresponding function. The commutation relation of $V(\lambda)B_a(\mu)$ is

$$V(\lambda)B_a(\mu) = \frac{f}{p}B_a(\mu)V(\lambda) + \frac{m}{p}F(\mu)C^a(\mu) - \frac{n}{p}F(\lambda)C^a(\mu) - \frac{s}{p}\xi_{bc}A^{*a}(\lambda)B_a(\mu),$$

(26)

where $n(\lambda) = i(\lambda + 3i)/(2\lambda + 3i)(2\lambda + i)$.

$$A(\lambda)F(\mu) = \frac{1}{p}F(\mu)A(\lambda) - \frac{\hat{m}}{f}F(\lambda)A(\mu) + \xi_{ab},$$

(27)

$$D_0^a(\mu)F(\mu) = (1 - \frac{m^2}{f^2})F(\mu)D_0^a(\lambda) + \frac{m^2}{f^2}F(\lambda)D_0^a(\mu) + \frac{s}{p}[B^{*a}(\lambda)B_0(\mu) - B_0(\lambda)B^{*a}(\mu)].$$

(28)

The commutation relations of $D_0^aF$ are contained in them.

$$V(\lambda)F(\mu) = \frac{1}{p}F(\mu)V(\lambda) - \frac{n}{p}F(\lambda)V(\mu) - \frac{s}{p}\xi_{ab}B^{*a}(\lambda)B^{*b}(\mu),$$

(29)

For $C_a$ and $C^a_a$ appear in the left hand of (24), the commutation relation of them with the operators $B_a$ and $F$ are also needed in the discussion. For only one and two-particle cases are discussed in detail here, the commutation rules used in this article are

$$C^a_a(\lambda)B_b(\mu) = \frac{n}{p}B_b(\mu)C^a_a(\lambda) + B_0(\mu)C^a_a(\lambda)$$

$$+ \frac{s}{p}[A(\mu)V(\lambda) - F(\mu)G(\lambda)\xi_{ab} - \frac{n}{p}B_a(\lambda)C^a(\mu) - \frac{s}{p}\xi_{cd}D_0^a(\lambda)D_0^a(\mu)],$$

(30)

$$C^a_a(\lambda)B_b(\mu) = B_0(\mu)C^a_a(\lambda) + \frac{m}{f}[A(\mu)D_0^a(\lambda) - A(\lambda)D_0^a(\mu)].$$

(31)

$$B_0(\lambda)B_b(\mu) = B_0(\mu)B_0(\lambda)r(\lambda - \mu) + \frac{s}{p}[F(\lambda)A(\mu) - F(\mu)A(\lambda)]\xi_{ab},$$

(32)

$$B_0(\lambda)B^{*b}(\mu) = B^{*b}(\mu)B_0(\lambda) + \frac{m}{f}[F(\mu)D_0^b(\lambda) - F(\lambda)D_0^b(\mu)]\xi_{ab}.$$
Additionally, in the discussion of the three-particle wave functions, the following commutation relations are needed,

\[ B_\alpha(\lambda)F(\mu) = \frac{1}{f}F(\mu)B_\alpha(\lambda) - \frac{\tilde{m}}{f}F(\lambda)B_\alpha(\mu). \]  

(34)

From the commutation rules listed above, the eigenstates and eigenvalues of the transfer matrices can be discussed. The one particle state can be defined by \( B_\lambda \) operator acting on the vacuum state as usual,

\[ |\psi_1(\lambda_1)| = B_\alpha(\lambda_1)W^\alpha|0\rangle, \]

(35)

where \( W^\alpha \) the coefficients of \( B_\alpha(\lambda_1) \). The transfer matrix acting on the one particle state arrives

\[ t(\lambda)|\psi_1(\lambda_1)\rangle = [A(\lambda) + D_{11}(\lambda) + D_{22}(\lambda) + V(\lambda)]B_\alpha(\lambda_1)W^\alpha|0\rangle \]

\[ = \left\{ \frac{1}{f(\lambda_1 - \lambda)} + \frac{f(\lambda - \lambda_1)}{p(\lambda - \lambda_1)}\rho^N(\lambda) + \frac{f^N(\lambda)}{f(\lambda - \lambda_1)}[1 + s(\lambda - \lambda_1)] \right\} |\psi_1(\lambda_1)\rangle \]

\[ - \left\{ \frac{m(\lambda - \lambda_1)}{f(\lambda_1 - \lambda)} + \frac{m(\lambda - \lambda_1)}{f(\lambda_1 - \lambda)}\xi_\alpha \right\} B_\alpha(\lambda)W^\alpha|0\rangle \]

\[ - \frac{s(\lambda - \lambda_1)}{p(\lambda_1 - \lambda)}[f_N(\lambda_1) - 1]\xi_\alpha B^{\alpha b}(\lambda)W^b|0\rangle, \]

(36)

If the one particle state is a eigenstate of the transfer matrix, terms including \( B_\alpha(\lambda)W^\alpha|0\rangle \) and \( B_\beta(\lambda)\xi_\alpha W^\alpha|0\rangle \) should be canceled with each other. The condition that the unwanted terms cancel with each other gives

\[ f^L(\lambda_1) - 1 = 0. \]

(37)

The Eq. (37) is the Bethe ansatz equation. If the parameter \( \lambda_1 \) satisfies the Bethe ansatz equation (37), the one particle state (35) is an eigenstate of the system. The eigenvalue \( A_1(\lambda, \lambda_1) \) of the transfer matrix at one-particle state is

\[ A_1(\lambda, \lambda_1) = \frac{1}{f(\lambda_1 - \lambda)} + \frac{f(\lambda - \lambda_1)}{p(\lambda - \lambda_1)}\rho^N(\lambda) + \frac{f^N(\lambda)}{f(\lambda - \lambda_1)}[1 + s(\lambda - \lambda_1)]. \]

(38)

To generate the two particle states, operator \( F \) is needed additionally. The two-particle eigenstate is assumed to be

\[ |\psi_2(\lambda_1, \lambda_2)\rangle = B_\alpha(\lambda_1)B_{a_2}(\lambda_2)W^{a_2a_1}|0\rangle + h(\lambda_1, \lambda_2)F(\lambda_1)\xi_{a_2a_1}W^{a_2a_1}|0\rangle, \]

(39)

where \( h(\lambda_1, \lambda_2) \) is a undetermined function. Acting the transfer matrix \( t_N(\lambda) \) on the assumed state (39), we obtain

\[ t_N(\lambda)|\psi_2(\lambda_1, \lambda_2)\rangle = [A(\lambda) + D_{1}(\lambda) + D_{2}(\lambda) + V(\lambda)]|\psi_2(\lambda_1, \lambda_2)\rangle \]

\[ = |\psi_2^0\rangle + |\psi_2^1\rangle + |\psi_2^2\rangle + |\psi_2^3\rangle + |\psi_2^4\rangle + |\psi_2^5\rangle + |\psi_2^6\rangle + |\psi_2^7\rangle. \]

(40)

Here, \( |\psi_2^0\rangle \) denotes the eigenstate which including the operators \( B_\alpha(\lambda_1)B_{a_2}(\lambda_2) \) and \( F(\lambda_1) \). \( |\psi_2^1\rangle, |\psi_2^2\rangle, |\psi_2^3\rangle \), \( |\psi_2^4\rangle, |\psi_2^5\rangle, |\psi_2^6\rangle \) and \( |\psi_2^7\rangle \) denote the unwanted terms including \( B_\alpha^a(\lambda)B_{a_2}(\lambda_1), B_{m}(\lambda)B_{m}(\lambda_1), B_{a_3}(\lambda)B_{a_2}(\lambda_2), B_{m}(\lambda)B_{a_2}(\lambda_2), B_{m}(\lambda)B_{a_2}(\lambda_2), B_{m}(\lambda)B_{a_2}(\lambda_2), B_{m}(\lambda)B_{m}(\lambda_1) \) and \( F(\lambda) \) respectively. The unwanted terms should be canceled with each other, which gives the form of the function \( h \) and the Bethe ansatz equations. The unwanted terms \( |\psi_2^1\rangle \) and \( |\psi_2^2\rangle \) can be explicitly expressed as

\[ |\psi_2^1\rangle = \frac{s(\lambda - \lambda_1)}{p(\lambda - \lambda_1)}\left[ h(\lambda_1, \lambda_2) - \frac{s(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda_2)}\xi_{b_1b_2}W^{b_1b_2}B_{a_1a_2}(\lambda)B^{a_2}(\lambda_2)|0\rangle, \]

(41)

\[ |\psi_2^2\rangle = \frac{m(\lambda - \lambda_1)}{f(\lambda_1 - \lambda)}\left[ h(\lambda_1, \lambda_2) - \frac{s(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda_2)}\xi_{a_1a_2}W^{a_1a_2}B_{m}(\lambda)B^{*m}(\lambda_2)|0\rangle. \]

(42)

If the function \( h \) satisfies

\[ h(\lambda_1, \lambda_2) = \frac{s(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda_2)}. \]

(43)
The unwanted term $|\psi_2^2\rangle$ and $|\psi_2^4\rangle$ are canceled with each other. This constraint determines the values of $\lambda$ introduced in the two-particle eigenstate (39). The unwanted terms $|\psi_2^3\rangle$ and $|\psi_2^4\rangle$ are

$$|\psi_2^3\rangle = \frac{m(\lambda_1 - \lambda)}{f(\lambda_1 - \lambda)f(\lambda_2 - \lambda)}W^{a_1a_2}$$
$$- \frac{m(\lambda - \lambda_1)f^N(\lambda_1)}{f(\lambda - \lambda_1)f(\lambda_2 - \lambda)}r(\lambda_1 - \lambda_2)_b^{a_2a_1}W^{b_1b_2}B_{a_1}(\lambda)B_{a_2}(\lambda_2)|0\rangle,$$

$$|\psi_2^4\rangle = \left[ - \frac{s(\lambda - \lambda_1)f^N(\lambda_1)}{f(\lambda - \lambda_1)f(\lambda_2 - \lambda)}\xi_{a_1}r(\lambda_1 - \lambda_2)_b^{a_2a_1}W^{b_1b_2} \right] B_m^*(\lambda)B_{a_2}(\lambda_2)|0\rangle.$$

After some tedious calculations, we find that if the parameter $\lambda_1$ and $\lambda_2$ satisfy the following Bethe ansatz equations

$$\frac{1}{f^N(\lambda_1)}\frac{f(\lambda_1 - \lambda_2)}{f(\lambda_2 - \lambda)}W^{a_1a_2} = r(\lambda_1 - \lambda_2)_b^{a_2a_1}W^{b_1b_2},$$

the terms $|\psi_2^3\rangle$ and $|\psi_2^4\rangle$ are also canceled with each other. The unwanted term $|\psi_2^5\rangle$ is

$$|\psi_2^5\rangle = \left\{ h(\lambda_1, \lambda_2)\frac{s(\lambda_1 - \lambda)}{p(\lambda_1 - \lambda)}\xi_{b_1}W^{b_1b_2} + \frac{m(\lambda_1 - \lambda)m(\lambda_2 - \lambda)}{f(\lambda_1 - \lambda)f(\lambda_2 - \lambda)}W^{a_1a_2} \right.$$

$$- \frac{m(\lambda_2 - \lambda)}{f(\lambda_2 - \lambda)f(\lambda_1 - \lambda)}r(\lambda_1 - \lambda)_b^{a_1a_2}W^{b_1b_2}$$

$$- \frac{m(\lambda - \lambda_2)}{f(\lambda - \lambda_1)f(\lambda - \lambda_2)}r(\lambda_1 - \lambda)_b^{a_1a_2}r(\lambda - \lambda_1)_b^{a_1a_2}f^N(\lambda_2)W^{c_1c_2}$$

$$+ \frac{m(\lambda - \lambda_1)m(\lambda_2 - \lambda)}{f(\lambda - \lambda_1)f(\lambda_2 - \lambda)}f^N(\lambda_2)W^{a_2a_1} \right\} B_{a_2}(\lambda_1)B_{a_2}(\lambda),$$

We find that if the parameter $\lambda_1$ and $\lambda_2$ satisfy Eq. (46) and

$$\frac{1}{f^N(\lambda_2)}\frac{f(\lambda_2 - \lambda_1)}{f(\lambda_1 - \lambda_2)}W^{a_1a_2} = r(\lambda_2 - \lambda_1)_b^{a_2a_1}W^{b_1b_2}.$$ 

The unwanted term $|\psi_2^5\rangle$ is zero. The Eqs. (46) and (48) are the two-particle Bethe ansatz equations, which can be written into a uniform form

$$\frac{1}{f^N(\lambda_1)}\frac{f(\lambda_1 - \lambda_2)}{f(\lambda_2 - \lambda_1)}W^{a_1a_2} = r(\lambda_1 - \lambda_2)_b^{a_2a_1}W^{b_1b_2}, \quad (i, j = 1, 2, \ i \neq j).$$

The function $W^{a_1a_2}$ can be determined by the nested algebraic Bethe ansatz. Together with Eq. (48), the
following unwanted terms $|\psi_2^0\rangle$ and $|\psi_2^7\rangle$ can be canceled,

\[
|\psi_2^0\rangle = \left\{ \begin{array}{l}
\frac{s(\lambda - \lambda_1)}{p(\lambda - \lambda_1)} \frac{m(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda_2)} - \frac{f(\lambda - \lambda_1) s(\lambda - \lambda_2)}{p(\lambda - \lambda_1) p(\lambda - \lambda_2)} \end{array} \right\} f^N(\lambda_2) \xi_{m_2a} W^{ca_2} \\
- \frac{s(\lambda - \lambda_1)}{p(\lambda - \lambda_1)} \frac{m(\lambda_1 - \lambda_2)}{p(\lambda_1 - \lambda_2)} \xi_{ma_1} \xi_{m_2a} + \frac{1}{f(\lambda - \lambda_1) p(\lambda - \lambda_2)} \xi_{b_2a_2} r(\lambda - \lambda_1) c_{b_2} \\
\times W^{a_1a_2} B_{\lambda_1} B_{\lambda_2} |0\rangle + h(\lambda_1, \lambda_2) \frac{m(\lambda - \lambda_1)}{f(\lambda - \lambda_1)} \xi_{a_2a_1} W^{a_1a_2} B_{\lambda_1} B_{\lambda_2} |0\rangle ,
\]

\[
|\psi_2^7\rangle = \left\{ \begin{array}{l}
\frac{n(\lambda - \lambda_1)}{p(\lambda - \lambda_1)} \left[ h(\lambda_1, \lambda_2) - \frac{s(\lambda - \lambda_2)}{p(\lambda_1 - \lambda_2)} \right] p^N(\lambda_1) \xi_{a_1a_2} W^{a_1a_2} |0\rangle \\
+ \frac{1}{f(\lambda - \lambda_1) f(\lambda - \lambda_2)} \xi_{a_1a_2} W^{a_1a_2} \end{array} \right\} \times F(\lambda) |0\rangle + \frac{m(\lambda - \lambda_1) s(\lambda - \lambda_2)}{f(\lambda - \lambda_1) p(\lambda_1 - \lambda_2)} \left[ h(\lambda_1, \lambda_2) \frac{n(\lambda - \lambda_1)}{p(\lambda_1 - \lambda_2)} \right] W^{a_1a_2} F(\lambda) |0\rangle \\
+ \frac{1}{f(\lambda - \lambda_1) p(\lambda_1 - \lambda_2)} \xi_{a_1a_2} F(\lambda) |0\rangle + \frac{1}{f(\lambda - \lambda_1) p(\lambda - \lambda_2)} \frac{m(\lambda - \lambda_2)}{f(\lambda - \lambda_1)} \xi_{a_2a_1} W^{a_1a_2} F(\lambda) |0\rangle .
\]

With the above results, the explicit form of the two-particle eigenstate of the system is

\[
|\Psi_2^0\rangle = \left\{ \begin{array}{l}
\left\{ \begin{array}{l}
\frac{1}{f(\lambda_1 - \lambda_1) f(\lambda_2 - \lambda_2)} + \frac{f(\lambda - \lambda_1) f(\lambda - \lambda_2)}{p(\lambda_1 - \lambda) p(\lambda_2 - \lambda)} p^N(\lambda) \\
+ \frac{f^N(\lambda)}{f(\lambda - \lambda_1) f(\lambda - \lambda_2)} \xi_{a_1a_2} W^{a_1a_2} |0\rangle \\
+ \left\{ \begin{array}{l}
h(\lambda_1, \lambda_2) \frac{1}{p(\lambda_1 - \lambda_1) p(\lambda_2 - \lambda_2)} + \frac{p^N(\lambda)}{f(\lambda_1 - \lambda) f(\lambda_2 - \lambda)} \left[ \frac{1}{2} f^N(\lambda) \right] \xi_{a_1a_2} \\
+ \frac{1}{f(\lambda_1 - \lambda_1) f(\lambda_2 - \lambda_2)} \xi_{a_1a_2} W^{a_1a_2} \end{array} \right\} \times F(\lambda) |0\rangle \\
+ \frac{1}{f(\lambda_1 - \lambda_1) p(\lambda_1 - \lambda_2)} \xi_{a_1a_2} W^{a_1a_2} \end{array} \right\} |0\rangle, \quad (52)
\]

Considering Eqs. (43) and (48), the eigen-equation of the transfer matrix becomes

\[
t(\lambda) |\psi_2(\lambda_1, \lambda_2)\rangle = \prod_{i=1}^{2} \frac{1}{f(\lambda_i - \lambda)} + p^N(\lambda) \prod_{i=1}^{2} \frac{f(\lambda - \lambda_i)}{p(\lambda - \lambda_i)} + \prod_{i=1}^{2} \frac{f^N(\lambda)}{f(\lambda - \lambda_i)} \times r(\lambda - \lambda)^{a_1 b_1} d_1 r(\lambda - \lambda)^{a_2 b_2} W^{d_1 d_2} |B_{\lambda_1} B_{\lambda_2} + h(\lambda_1, \lambda_2) F(\lambda_1) \xi_{a_1a_2}| |0\rangle . \quad (53)
\]

The eigen-value of two-particle state is

\[
A_2(\lambda, \{\lambda_1, \lambda_2\}) = \prod_{i=1}^{2} \frac{1}{f(\lambda_i - \lambda)} + p^N(\lambda) \prod_{i=1}^{2} \frac{f(\lambda - \lambda_i)}{p(\lambda - \lambda_i)} \\
+ f^N(\lambda) \prod_{i=1}^{2} \frac{1}{f(\lambda - \lambda_i)} A^{(2)}_2(\lambda, \{\lambda_1, \lambda_2\}), \quad (54)
\]

where $A^{(1)}_2$ is the eigen-value of $t^{(1)}(\lambda, \{\lambda_1, \lambda_2\})$. 
From the above discussions, we see that the construction of eigenstates of the $SO(5)$-invariant quantum spin chain is quite different from that of the $SU(4)$-invariant one. The spin-flipped operators are very complicated. The symmetry analysis of the states is helpful to construct the eigenstates \cite{25}. Now, we construct the many-particle eigenstates. We define $\tilde{\psi}_n$ as the creation operators of the $n$-particle state for convenience

$$|\psi_n(\{\lambda_i\})\rangle = \tilde{\psi}_n \tilde{W} |0\rangle,$$

where $\tilde{W} = (W^1, W^2, \ldots, W^n)^T$ are some vectors. Obviously, $\tilde{\psi}_0 = 1$ for $|\psi_0\rangle = |0\rangle$. From Eqs. \eqref{eq:55} and \eqref{eq:59}, we know the creation operators of one and two-particle eigenstates are

$$\tilde{\psi}_1 = \tilde{B}(\lambda_1), \quad \tilde{\psi}_2(\lambda_1, \lambda_2) = \tilde{B}(\lambda_1) \otimes \tilde{B}(\lambda_2) + h(\lambda_1, \lambda_2) F(\lambda_1) \tilde{\xi}.$$  

From the commutation relations \eqref{eq:62}, the relation of $\tilde{\psi}(\lambda_1, \lambda_2)$ and $\tilde{\psi}(\lambda_2, \lambda_1)$ is

$$\tilde{\psi}_2(\lambda_2, \lambda_1) = \tilde{\psi}_2(\lambda_1, \lambda_2) r(\lambda_2 - \lambda_1), \quad \tilde{\psi}_3(\lambda_1, \lambda_2) = \tilde{\psi}_3(\lambda_2, \lambda_1) r(\lambda_1 - \lambda_2).$$  

After a more detailed analysis, we find that the three-particle states should have the same symmetry. Assume the three-particle eigenstates as

$$\tilde{\psi}_3(\lambda_1, \lambda_2, \lambda_3) = \tilde{B}(\lambda_1) \otimes \tilde{\psi}_2(\lambda_2, \lambda_3) - h(\lambda_1, \lambda_2) F(\lambda_1) \tilde{\xi} \otimes \tilde{\psi}_1(\lambda_3) \frac{1}{f(\lambda_3 - \lambda_2)}$$

$$- h(\lambda_1, \lambda_3) F(\lambda_1) \tilde{\xi} \otimes \tilde{\psi}_1(\lambda_2) \frac{1}{f(\lambda_2 - \lambda_3)} r_{23}(\lambda_2 - \lambda_3),$$

which satisfies

$$|\psi_3(\lambda_2, \lambda_3)\rangle = |\psi_3(\lambda_1, \lambda_2, \lambda_3)\rangle r_{12}(\lambda_1 - \lambda_2),$$

$$|\psi_3(\lambda_1, \lambda_3)\rangle = |\psi_3(\lambda_1, \lambda_2, \lambda_3)\rangle r_{23}(\lambda_2 - \lambda_3).$$  

Borrowed the ideas in \cite{25}, the generator of multi-particle state is assumed as

$$\tilde{\psi}_{M_1}(\lambda_1, \lambda_2, \cdots, \lambda_{M_1}) = \tilde{B}(\lambda_1) \otimes \tilde{\psi}_{M_1-1}(\lambda_2, \lambda_3, \cdots, \lambda_{M_1})$$

$$- F(\lambda_1) \tilde{\xi} \otimes \sum_{j=2}^{M_1} h(\lambda_1 - \lambda_j) \prod_{k=2, k \neq j}^{M_1} \frac{1}{f(\lambda_k - \lambda_j)}$$

$$\times \tilde{\psi}_{M_1-2}(\lambda_2, \cdots, \lambda_{j-1}, \cdots, \lambda_{j+1}, \lambda_{M_1}) \prod_{k=2}^{j-1} r_{k,k+1}(\lambda_k - \lambda_j),$$

which satisfies

$$\tilde{\psi}_{M_1}(\lambda_1, \lambda_2, \cdots, \lambda_{M_1}) = \tilde{\psi}_{M_1}(\lambda_1, \cdots, \lambda_{j+1}, \lambda_j, \cdots, \lambda_{M_1}) r_{j,j+1}(\lambda_j - \lambda_{j+1}).$$

By using the same process as that for the two-particle case, we obtain the following Bethe ansatz equations

$$\frac{1}{f^N(\lambda_j)} W^a_1 \cdots a_{M_1} = \prod_{i \neq j}^{M_1} \frac{f(\lambda_i - \lambda_j)}{f(\lambda_i - \lambda_j - \lambda_1)}$$

$$\times A_{M_1-1}^{(1)} (\lambda_j, \{\lambda_{j+1}, \cdots, \lambda_{M_1}, \lambda_1, \cdots, \lambda_{j-1}\}) W^{d_1 \cdots d_{M_1}},$$

$$A_{M_1}(\lambda, \{\lambda_i\}) = \prod_{k=1}^{M_1} \frac{1}{f(\lambda_k - \lambda)} A_{M_1-1}(\lambda, \{\lambda_i\}),$$

where $A_{M_1}^{(1)} (\lambda, \{\lambda_i\})$ is the eigenvalue of a series production of $r$ matrices

$$r(\lambda - \lambda_1)^{a_1 b_1} r(\lambda - \lambda_2)^{a_2 b_2} \cdots r(\lambda - \lambda_n)^{a_n b_n} W^{d_1 \cdots d_n} = A_{M_1}^{(1)} (\lambda, \{\lambda_i\}) W^{a_1 \cdots a_n}.$$  

Now, we diagonalize the eigen-equation \eqref{eq:64}. One can easily check that the nested $R$ matrix satisfies the Yang-Baxter equation \cite{55}. In fact, it is the 6-vertex $R$-matrix in space $V^{(1)} \otimes V^{(1)}$ with $V^{(1)}$ a two dimensional space. The corresponding Lax operator is

$$L^{(1)}(\lambda) = R_{a\alpha b\beta}(\lambda).$$
The nested monodromy matrix is defined as
\[ T_{M_1}^{(1)}(\lambda, \{\lambda_j\}) = L_{0,M_1}^{(1)}(\lambda - \lambda_1) L_{0,M_1-1}^{(1)}(\lambda - \lambda_2) \cdots L_{0,1}^{(1)}(\lambda - \lambda_{M_1}), \]  
which satisfies the Yang-Baxter relation \[ 10]. The nested transfer matrix is
\[ t_{M_1}^{(1)}(\lambda) = \text{tr} T_{M_1}^{(1)}(\lambda). \]
The transfer matrices with different spectral parameters commute with each other
\[ [t_{M_1}^{(1)}(\lambda), t_{M_1}^{(1)}(\mu)] = 0. \]

Using the standard Bethe ansatz method for the six-vertex model, we obtain the eigenvalues of nested transfer matrix \[ t_{M_1}^{(1)} \] as
\[ A_{M_2}^{(1)}(\lambda, \{\mu_j\}) = \prod_{i=1}^{M_2} a(\mu_i - \lambda) + \prod_{i=1}^{M_1} a(\lambda - \lambda_i) \prod_{i=1}^{M_2} a(\lambda - \mu_i), \]
where the parameter \[ \mu_i \] should satisfy the Bethe ansatz equation
\[ \prod_{i=1}^{m} a(\mu_j - \lambda_i) a(\mu_j - \lambda_j) = \prod_{i=1}^{n} a(\mu_j - \lambda_i). \]

Substituting Eq. \[ 69 \] into \[ 62 \] and \[ 63 \], we obtain the eigenvalues of \[ T_N(\lambda) \] as
\[ A_{M_1,M_2}(\lambda, \{\lambda_i\}, \{\mu_j\}) = \prod_{i=1}^{M_1} f(\lambda_i - \lambda) + \prod_{i=1}^{M_2} f(\lambda - \lambda_i) \prod_{i=1}^{M_1} a(\lambda - \lambda_i) \prod_{j=1}^{M_2} a(\lambda - \mu_j) + \prod_{j=1}^{M_2} a(\mu_j - \lambda). \]
The Bethe ansatz equations are Eq. \[ 70 \] and
\[ \frac{1}{f(\lambda_i)} = \prod_{j \neq i} \frac{f(\lambda_i - \lambda_j)}{f(\lambda_j - \lambda_i)} \prod_{k=1}^{M_1} \frac{1}{a(\mu_k - \lambda_i)}. \]

Putting \[ \lambda_i \to \lambda_i - i/4, \mu_j \to \mu_j + 3i/4 \], the Bethe ansatz equations \[ 70 \] and \[ 72 \] can be written as
\[ \left[ \begin{array}{c} \lambda_j + i/4 \\ \lambda_j - i/4 \end{array} \right] N \prod_{i=1}^{M_2} \lambda_j - \lambda_i + i/2 = - \prod_{i=1}^{M_1} \lambda_j - \lambda_i + i/2, \]
\[ \left[ \begin{array}{c} \mu_j - \lambda_i + i/2 \\ \mu_j - \lambda_i - i/2 \end{array} \right] = - \prod_{i=1}^{M_2} \mu_j - \mu_i + i/2. \]

From the relation \[ 71 \], we obtain the eigenvalue \[ E \] of the Hamiltonian \[ H \] as
\[ E = -\frac{9J}{4} \left( \left. \frac{\partial \ln T(\lambda)}{\partial \lambda} \right|_{\lambda=0} + \frac{11}{2} N \right) = -\frac{9J\pi}{2} \sum_{\lambda=1}^{M_1} a_{1/4}(\lambda) - \frac{99}{8} J N, \]
where \[ a_{1/4}(x) = t/[\pi(x^2 + t^2)] \] and \[ \lambda_j \] should satisfy the Bethe ansatz equations \[ 73 \]. The momentum \[ P \] is
\[ P = \sum_{j=1}^{M_1} k_j \mod 2\pi, \]
where \[ k_j \] are parameterized by \[ e^{ik_j} = (\lambda_j - i/4)/(\lambda_j + i/4). \]

The basis of Hilbert space \[ V_i \] are \([3/2]_i, [1/2]_i, -[1/2]_i, \] and \([-3/2]_i \). Thus the vacuum state is a ferromagnetic state \([0] = [3/2]_1 \otimes [3/2]_2 \otimes \cdots \otimes [3/2]_N \). The \[ M_1 + M_2 \] is the total number of flipped spins from this ferromagnetic state, so the total spin along the z-component and magnetization are
\[ S^z = 3N/2 - (M_1 + M_2), \]
\[ m = 3/2 - (M_1 + M_2)/N = 3/2 - n_\lambda - n_\mu. \]

From Eq. \[ 74 \], if \[ J < 0 \], the \[ \lambda \] will lead to addition to the energy. Therefore, the ground state of this case is the ferromagnetic state \([0] \). While if \[ J > 0 \], the \[ \lambda \] will contribute a negative value to the energy, so the ground state might be a anti-ferromagnetic state. In order to obtain the ground state configuration of the system, we first consider the above. The Bethe ansatz and magnetic states are consistent with the spin variables, so the ground state is the ferromagnetic state.
IV. THERMODYNAMICS

Now, we solve the Bethe ansatz equations \(73\). In the thermodynamic limit, both the total particle number and the system size tend to infinity while the ratio \(N/L\) keeps a non-zero constant. The Bethe ansatz equations can have the complex solutions, i.e., strings. Checking of the Bethe ansatz equations \(73\) in detail, we find that the string hypothesis of the Bethe ansatz equations \(73\) is

\[
\lambda^\beta_{\beta,j} = \lambda^\beta_0 + \frac{i}{4}(n + 1 - 2j), \quad j = 1, \ldots, n,
\]

\[
\mu^m_{\nu,j} = \mu^m_0 + \frac{i}{2}(n + 1 - 2j), \quad j = 1, \ldots, m,
\]

where \(\lambda^\beta_0\) and \(\mu^m_0\) are the real parts of the \(n\)-string of \(\lambda\) and the \(m\)-string of \(\mu\), respectively. From Eq. \(74\), we see that the contributions of \(n\)-string of \(\lambda\) to the energy is

\[
e_n(\lambda^\beta) = -\frac{9}{2}J\pi a_{n/4}(\lambda^\beta).
\]

The energy \(E\) and the momentum \(P\) are

\[
E = \sum_{n,z} e_n(\lambda^\beta) - \frac{99}{8}JN = -\frac{9}{2}J\pi \sum_{n,z} a_{n/4}(\lambda^\beta) - \frac{99}{8}JN,
\]

\[
P = \sum_{n,z} \pi - \theta_{n/4}(\lambda^\beta) \mod 2\pi,
\]

where \((x - it)/(x + it) = -e^{i\theta_i(x)}\) and \(\theta_i(x) = 2\arctan(x/t)\).

Substituting the string solutions into the Bethe ansatz equations \(73\), we have

\[
 e^{-iN\varphi}(\lambda^\beta) \prod_{m,y} e^{-iB_{m,n}^4(\lambda^\beta - \mu^m)} = (-)^{\delta_{\beta n}} \prod_{m,y} e^{-iA_{m,n}^4(\lambda^\beta - \lambda^m)},
\]

\[
\prod_{m,y} e^{-iB_{m,n}^4(\mu^m - \lambda^\beta)} = (-)^{\delta_{\beta n}} \prod_{m,y} e^{-iA_{m,n}^4(\mu^m - \mu^m)},
\]

where

\[
B_{m,n}^4(x) = \varphi_{\lambda^\beta}((2m+n-1) + \varphi_{\lambda^\beta}((2m+n-3) + \varphi_{\lambda^\beta}((2m+n-1)),
\]

\[
A_{m,n}^4(x) = \varphi_{\lambda^\beta}(m+n) + 2\varphi_{\lambda^\beta}(m+n-2) + \cdots + 2\varphi_{\lambda^\beta}(m-n)) + \varphi_{\lambda^\beta}((m-n)),
\]

and \(\delta\) gives the phase shifts \(\pi\) or \(0\) according to the numbers of production terms. Taking the logarithm of \(83\), we have

\[
2\pi I^\lambda_{n,z} = N\varphi_{\lambda^\beta}(\lambda^\beta) + \sum_{m,y} B_{m,n}^4(\lambda^\beta - \mu^m) - \sum_{m} A_{m,n}^4(\lambda^\beta - \lambda^m),
\]

\[
2\pi I^\mu_{m,y} = \sum_{n,z} B_{m,n}^4(\mu^m - \lambda^\beta) - \sum_{n,z} A_{m,n}^4(\mu^m - \mu^m),
\]

where \(I^\lambda_{n,z}\) and \(I^\mu_{m,y}\) are the integers or half-integers which determine the eigenstates. A set of \(\{I^\lambda_{n,z}, I^\mu_{m,y}\}\) satisfying \(86\) gives a highest weight eigenstate of Hamiltonian \(H\). The momentum \(75\) can be written in terms of \(I^\lambda_{n,z}\) and \(I^\mu_{m,y}\)

\[
P = 2\pi \frac{1}{N} \sum_{n,z} (I^\lambda_{n,z} + I^\lambda_{n,z}) \mod 2\pi.
\]

In the thermodynamic limit, the summations become integrations. Denoting \(\eta_n\) and \(\sigma_m\) as the densities of \(\lambda\) \(n\)-strings and \(\mu\) \(m\)-strings in the thermodynamic limit, and \(\eta^\beta_n\) and \(\sigma^\beta_m\) as the corresponding densities of holes, the Bethe ansatz equations read

\[
I^\lambda_N = \theta_{n/4}(\lambda) + \sum_{m} B_{m,n}^4 + \sigma^m(\lambda) - \sum_{m} A_{m,n}^4 + \eta^m(\lambda),
\]

\[
I^\mu_N = \sum_{n,z} B_{m,n}^4 + \eta^m(\mu) - \sum_{n,z} A_{m,n}^4 + \sigma^m(\mu),
\]

\[N\]
where the densities of $\lambda^u_\nu$ and $\mu^u_\nu$ are denoted as $\eta^u(\lambda_z)$ and $\eta^u(\mu_z)$, respectively, $I$s are the quantum numbers, and the operator $*$ is defined by

$$\int a_t(x - y) f(y) dy = |t| * f(x).$$  \hfill (89)

Taking the differentials of Eq. (88), we obtain the integral form of the Bethe ansatz equations

$$\eta^u_n(k) = a_{n/4}(k) + \sum_{m=1} \hat{B}^4_{m,n} * \sigma^m(k) - \sum_{m=1} \hat{A}^4_{m,n} * \eta^m(k),$$

$$\sigma^u_n(k) = \sum_{m=1} \hat{B}^4_{n,m} * \eta^m(k) - \sum_{m=1} \hat{A}^2_{m,n} * \sigma^m(k).$$  \hfill (90)

where $\hat{A}^t_{m,n}$ and $\hat{B}^t_{m,n}$ are the integral operators, $\hat{A}^t_{m,n} = a_{(m+n)/t} + 2a_{(m+n-2)/t} + \cdots + 2a_{(m-n-2)/t} + a_{(m-n)/t}$; $\hat{B}^t_{m,n} = a_{(2m-n-1)/t} + a_{(2m-n-3)/t} + \cdots + a_{(2m-n+1)/t}$. In the derivation, we have used the relation $\frac{\partial}{\partial x} \theta_j(x) = 2\pi a_t(x)$.

At temperature $T$, the Gibbs free energy of the system (1) with an external magnetic field $h$ reads

$$F = E - h \left( \frac{3}{2} N - M_1 - M_2 \right) - TS,$$  \hfill (91)

where

$$M_1 = N \sum_n \int \eta_n(\lambda) d\lambda, \quad M_2 = N \sum_m \int \sigma_m(\mu) d\mu$$  \hfill (92)

$$E = -\frac{9}{2} \pi JN \sum_n \int d\lambda a_{n/4}(\lambda) \eta^u(\lambda) - \frac{99}{8} N,$$  \hfill (93)

$$S = N \sum_n \int d\lambda [\eta_n + \eta^h_n \ln(\eta_n + \eta^h_n) - \eta_n \ln \eta_n - \eta^h_n \ln \eta^h_n]$$

$$+ N \sum_m \int d\mu [\sigma_m + \sigma^h_m \ln(\sigma_m + \sigma^h_m) - \sigma_m \ln \sigma_m - \sigma^h_m \ln \sigma^h_m].$$  \hfill (94)

Minimizing the Gibbs free energy at the thermal equilibrium, we obtain the following thermodynamic Bethe ansatz equations

$$\ln \hat{\eta}_1 = -\frac{9}{2} \pi J \frac{G_4(\lambda)}{T} + G_4 * \ln[(1 + \hat{\eta}_2)(1 + \hat{\rho}^{-1})^{-1}],$$

$$\ln \hat{\sigma}_1 = G_2 * \ln \frac{1 + \hat{\sigma}_2}{(1 + \hat{\eta}_1)(1 + \hat{\rho}^{-1})} - \frac{G_2}{G_4} * \ln(1 + \hat{\eta}_2^{-1}),$$

$$\ln \hat{\eta}_n |_{even} = G_4 * \ln \frac{(1 + \hat{\eta}_{n-1})(1 + \hat{\rho})}{1 + \hat{\rho}^{-1/2}},$$

$$\ln \hat{\eta}_n |_{odd} = G_4 * \ln [(1 + \hat{\eta}_{n-1})(1 + \hat{\eta}_{n+1})],$$

$$\ln \hat{\sigma}_m = G_2 * \ln \frac{(1 + \hat{\sigma}_{m-1})(1 + \hat{\rho})}{(1 + \hat{\eta}_{2m-1})(1 + \hat{\eta}_{2m+1})} - \frac{G_2}{G_4} * \ln(1 + \hat{\eta}_{2m}^{-1}),$$

$$\lim_{n \to \infty} \frac{\ln \hat{\eta}_n}{n} = \frac{h}{T}, \quad \lim_{m \to \infty} \frac{\ln \hat{\sigma}_m}{m} = \frac{h}{T},$$  \hfill (95)

where $\hat{\eta}_n = \eta^h_n/\eta_n$, $\hat{\sigma}_m = \sigma^h_m/\sigma_m$, $G_n = a_{1/n}/(a_0 + a_{2/n})$ and $a_0 = \delta(x)$. In the derivation, we have used the
The ground state string distribution of the system can be obtained by taking the limit of
\[ \lim_{T \to \infty} \text{energy}. \]
In the thermodynamic limit, we obtain the following thermodynamic Bethe ansatz equations for the dressed
\[ \zeta_n(\lambda) = T \ln \tilde{\eta}_n(\lambda), \quad \varsigma_m(\mu) = T \ln \tilde{\sigma}_m(\mu). \]
From Eqs. (100) and (101), we find that the dressed energies should satisfy
\[ \zeta_n(\lambda) = -\frac{9}{2} \pi J a_{n/4}(\lambda) + h n - \sum_{m=1}^{\infty} \hat{B}_{m,n}^4 \ln \left[ 1 + e^{-\zeta_m(\lambda)/T} \right] \]
\[ + \sum_{m=1}^{\infty} \hat{A}_{m,n}^4 \ln \left[ 1 + e^{-\varsigma_m(\lambda)/T} \right], \quad (98) \]
\[ \varsigma_n(k) = h n - \sum_{m=1}^{\infty} \hat{B}_{m,n}^4 \ln \left[ 1 + e^{-\zeta_m(\lambda)/T} \right] + \sum_{m=1}^{\infty} \hat{A}_{m,n}^4 \ln \left[ 1 + e^{-\varsigma_m(\lambda)/T} \right]. \quad (99) \]
In the thermodynamic limit, we obtain the following thermodynamic Bethe ansatz equations for the dressed energy
\[ \zeta_1 = -\frac{9}{2} \pi J G_4(\lambda) + T G_4 \ln \left[ 1 + e^{\xi_\lambda(\lambda)/T} \right], \]
\[ \varsigma_1 = T G_1 \ln \left[ 1 + e^{\xi_\lambda(\lambda)/T} \right] - \ln \left[ 1 + e^{-\zeta_\lambda(\lambda)/T} \right] - \ln \left[ 1 + e^{-\varsigma_\lambda(\lambda)/T} \right], \]
\[ \zeta_{n \in \text{even}} = T G_4 \ln \left[ 1 + e^{\zeta_{n-1}/T} \right] + \ln \left[ 1 + e^{\zeta_{n+1}/T} \right] - \ln \left[ 1 + e^{-\zeta_{n-2}/T} \right], \quad (100) \]
\[ \zeta_{n \in \text{odd}} = T G_4 \ln \left[ 1 + e^{\zeta_{n-1}/T} \right] + \ln \left[ 1 + e^{\zeta_{n+1}/T} \right] - \ln \left[ 1 + e^{-\zeta_{2n+1}/T} \right], \]
\[ - \ln \left[ 1 + e^{-\zeta_{n-1}/T} \right] - T G_2 \frac{G_4}{G_2} \ln \left[ 1 + e^{-\zeta_{n-2}/T} \right], \]
\[ \lim_{n \to \infty} \zeta_n = h, \quad \lim_{m \to \infty} \varsigma_m = h. \]
The dressed energies can be expressed into two parts \( \epsilon^+(k) \) and \( \epsilon^-(k) \),
\[ \epsilon^+(k) = \begin{cases} \epsilon(k) & \text{if } \epsilon(k) > 0, \\ 0 & \text{if } \epsilon(k) < 0 \end{cases}, \quad \epsilon^-(k) = \begin{cases} \epsilon(k) & \text{if } \epsilon(k) < 0, \\ 0 & \text{if } \epsilon(k) > 0. \end{cases} \quad (101) \]
The ground state string distribution of the system can be obtained by taking the limit of \( T \to 0 \) and \( h \to 0 \). When \( T \to 0 \) and \( h \to 0 \), we have \( \lim_{T \to 0} e^{\xi_\lambda(\lambda)/T} = 1 \) and \( \lim_{h \to 0} e^{\xi_\lambda(\lambda)/T} = 1 \).
\[ e^{-\epsilon(\lambda)/T} e^{-\epsilon(\lambda_1)/T} \]. When \( T \to 0 \), both \( e^{-\epsilon(\lambda)/T} \) and \( e^{-\epsilon(\lambda_1)/T} \) tend to one. Thus \( \ln[1+e^{\epsilon(\lambda)/T}] = \ln[1+e^{\epsilon(\lambda_1)/T}] \) and \( \ln[1+e^{-\epsilon(\lambda)/T}] = \ln[1+e^{-\epsilon(\lambda_1)/T}] \).

If \( J < 0 \), the ground state of the system is the ferromagnetic state \(|0\rangle = \otimes_{j=1}^{N} |3/2\rangle \). It is easy to understand from the eigen energy \([9]\) for a \( n \)-string \( \lambda \) give a positive contribution \( e_n(\lambda) \) to the eigen energy. In the ground state, the total spin along \( z \)-direction and the magnetization are \( S^z = 3N/2 \) and \( m = 3/2 \), respectively. It is a highest weight representation of the Yang-algebra \([9]\). The ground state energy and momentum are \( E = -99JN/8 \) and \( P = 0 \), respectively.

FIG. 1: The ground state energy of the system. There is a quantum phase transition at the critical point \( J = 0 \). The system is in the ferromagnetic phase if \( J < 0 \) and is in the antiferromagnetic phase if \( J > 0 \).

If \( J > 0 \), we find that some \( \lambda s \) are real while the others form 2-strings, and the \( \mu s \) are real. Such a ground state configuration is quite different from that of the \( SU(4) \) Sutherland model where there is no string or spin bound state in the ground state. In the present \( SO(5) \) case, part of the spectral parameters form 2-strings which heavily affect the spin excitations as we shall show below. To make the ground state energy lowest, all these strings are filled up and no holes left. This can be understood from the entropy \( S \) of the system. The completely filled string configurations contribute zero entropy, and make the system in a most stable state.

From Eqs. \([50]\), the densities \( n_0^m \) and \( \sigma_0^m \) satisfy the following integral equations

\[ \tilde{\rho}_0(\lambda) = \tilde{g}(\lambda) + \tilde{K} * \tilde{\rho}_0(\lambda), \]  

where \( \tilde{\rho}_0(\lambda) = [\eta_01(\lambda), \eta_02(\lambda), \sigma_01(\lambda)]^t \), \( \tilde{g}(\lambda) = [a_{1/4}(\lambda), a_{2/4}(\lambda), 0]^t \) and

\[ \tilde{K} = \begin{pmatrix} -a_{3/4} & -a_{1/4} & +a_{1/2} \\ -a_{1/2} + a_3 & -a_{3/4} & -a_{1/4} \\ a_{3/4} & a_{1/2} & -a_{1/2} \\ \end{pmatrix}. \]

Taking the inverse of Eq. \( (102) \), we obtain

\[ \tilde{\rho}_0(\lambda) = \tilde{F} * \tilde{g}(\lambda), \]  

where \( \tilde{F} = 1/(1 - \tilde{K}) \). The solution of \( (103) \) is

\[ \eta_01(\lambda) = \cosh(\pi \lambda), \]

\[ \eta_02(\lambda) = \frac{1}{18\pi} \cosh(2\pi \lambda) \left[ 4\sqrt{3} \sinh \left( \frac{4\pi \lambda}{3} \right) - 12\pi \lambda \right], \]

\[ \sigma_01(\lambda) = \cosh(\pi \lambda/3)/3. \]

Because all the density functions are even, the ground state string configuration \( \{I_0^1(\lambda^{(1)})\}, \{I_0^2(\lambda^{(2)})\}, \{I_0^3(\mu^{(1)})\} \) is symmetric around the origin. Taking the integration, we obtain the densities of strings \( n_{01}^0 = M_{1}^{1}/N \), \( n_{02}^0 = M_{2}^{1}/N \) and \( n_{01}^\sigma = M_{1}^{2}/N \) as

\[ n_{01}^0 = \frac{1}{2}, \quad n_{02}^0 = \frac{1}{4}, \quad n_{01}^\sigma = \frac{1}{2}. \]

After some derivations, we find \( M_{1}^{1} = N \) and \( M_{2}^{1} = N/2 \), which mean that the total spin of the ground state is \( S = 3N/2 \) and \( M = 3/2 \). The ground state is an eigenstate.
and the momentum are
\begin{align}
E^A_0 &= -\frac{9}{2} \pi J N F(0) - \frac{99}{8} J N, \quad F(\lambda) = a_{1/4} \ast \eta_{01}(\lambda) + a_{1/2} \ast \eta_{02}(\lambda), \\
P^A_0 &= 2\pi \frac{1}{N} \sum_{n,z} (I_{\lambda_n^2} + I_{\lambda_n^2}) \mod 2\pi = 0.
\end{align}

From Eq. (98), the ground state dressed energies satisfy the following equations
\begin{equation}
\vec{\epsilon}(\lambda) = \begin{bmatrix} \zeta_1(\lambda), \zeta_2(\lambda), \varsigma_1(\lambda) \end{bmatrix}^T = \frac{9}{2} \pi J F^* \ast \vec{g}(\lambda) = -\frac{9}{2} \pi J \vec{\rho}_0(\lambda),
\end{equation}
where \(\vec{\epsilon}(\lambda) = [\zeta_1(\lambda), \zeta_2(\lambda), \varsigma_1(\lambda)]^T\). Recalling the definition of the dressed energy \(\epsilon = T \ln(\rho^h/\rho)\), we see that the dressed energies are negative in the limit of \(T \to 0\), which means that the corresponding strings are completely filled.

**VI. ELEMENTARY EXCITATIONS**

Based on the ground state configuration, the elementary excitations of the system can be studied exactly. For the ferromagnetic case \((J < 0)\), elementary excitations are spin waves with dispersion relation \(\Delta E = -18(J/n) \cos^2(n\Delta P/2)\). The excitations in anti-ferromagnetic sector \((J > 0)\) are somehow complicated. In the language of Bethe ansatz, these can be described by the changes of the string distributions. These excitations are very different from those of the \(SU(4)\) model, for the ground state configuration contains 2-strings. In fact, the spin excitations can be described by adding some holes or high strings into the ground state configuration.

The holes and extra strings lead to redistributions of \(\lambda s\) and \(\mu s\). Formally, the extra strings contribute nothing to the energy because the contribution of such strings is exactly canceled by the rearrangement of the ground state distribution though they do contribute to the spin quanta carried by the excitations.
The excited states can be determined by the integral Bethe ansatz equations with holes and high strings in the \( \lambda_1, \lambda_2 \) and \( \mu_2 \) sectors

\[
\bar{\rho}(\lambda) = \bar{F} \ast [\bar{g}(\lambda) - \bar{\rho}_h(\lambda)],
\]

(110)

where \( \bar{\rho}(\lambda) = [\eta_1(\lambda), \eta_2(\lambda), \sigma_1(\lambda)]^T \) and \( \bar{\rho}_h(\lambda) = [\eta_h^1(\lambda), \eta_h^2(\lambda), \sigma_h^1(\lambda)]^T \). In the thermodynamic limit the density of holes are

\[
\eta^h_a(\nu) = \sum_{i=1}^{m_a} \frac{1}{N} \delta(\nu - \nu^h_a),
\]

(111)

where \( m_{1,2,3} \) represent the numbers of holes in real \( \lambda \) sea, in 2-string \( \lambda \) sea and in real \( \mu \) sea, \( \nu^1_h, \nu^2_h \) and \( \nu^3_h \) are the positions of the corresponding charges and holes. The excitations lead to the redistributions of densities \( \bar{\rho}_0(\lambda), \)

\[
\bar{\rho}(\lambda) + \bar{\rho}_0(k) = \bar{K} \ast [\bar{\rho}(\lambda) - \bar{\rho}_0(k)].
\]

(112)

Denoting the charges of the densities as \( \Delta \rho(k) = \bar{\rho}(\lambda) - \bar{\rho}_0(\lambda) \), which satisfies

\[
\Delta \bar{\rho}(\lambda) = -\bar{F} \ast \bar{\rho}_h(\lambda).
\]

(113)

Form Eq. (112), the excited energy is

\[
\Delta E = \Delta E^A = -N \sum_{a} \epsilon_a \ast \rho^h_a(\lambda)|_{\lambda=0} = -\sum_{a=1}^{3} \sum_{i=1}^{m_a} \epsilon_a(\nu^h_a).
\]

(114)

Thus the excited energies are the summation of dressed energies carried by the holes with a inverse sign. The excited momentum is

\[
\Delta P = P - P_0^A = \sum_{a=1}^{3} \sum_{i=1}^{m_a} \int_0^{\nu^h_a} \rho_{0a}(\lambda) d\lambda \mod 2\pi.
\]

(115)

The spin quanta carried by the spin excitation is

\[
S = \frac{3}{2} m_2 + 2 m_3 + \sum_{l=3} \sum_{t=2} (2 - l)m_{\lambda(l)} + \sum_{l=2} \sum_{t=2} (1 - t)m_{\mu(l)},
\]

(116)

where \( m_{\lambda(l)} \) and \( m_{\mu(l)} \) are the numbers of \( \lambda \) \( l \)-strings and the \( \mu \) \( t \)-strings formed in the excitations, respectively.

Because the energies, momenta and spins of the holes are additive, the thermodynamic behaviors of the system are mainly determined by the dispersion relations of the individual holes, which are shown in Fig. 3. From the Fig.3 we find that the single \( \lambda \) 2-string hole carries the lowest energy with spin 3/2, which is named as heavy spinon here. These heavy spinons dominate the low temperature thermodynamics of the system. Surprisingly, the holes in the real \( \lambda \) sector carry zero spin, corresponding to a new kind of neutral spin excitations. The spin quanta carried by each \( \mu \) hole are 2. Different from the \( SU(4) \) model, the \( \lambda \) 2-string heavy spinons cover one quarter of the Brillouin zone.

**FIG. 4:** The single-hole excitations of the system. Here \( J = 2/9\pi \), \( \Delta E \) and \( P \) are the energy and the momentum carried by a single hole, respectively. The dotted dashed line is the single-hole excitation of real \( \lambda \). The solid line is that of \( \lambda \) 2-string and the dashed line is that of real \( \mu \).
TABLE I: Some possible configurations of the excitations. Here, $\Delta M_a = \sum_\lambda \Delta M_{a\lambda}$ and $m_a^\lambda$ are basic excitations. $m_1^\lambda$ and $m_2^\lambda$ are the high string excitations. Some possible elementary excitations $m_1 = m_1^0 + m_\lambda$, $m_2 = m_2^0 + m_\lambda$, $m_3 = m_3^0 + 2m_\lambda$, $m_4 = m_4^0 + m_\lambda^2 + 2m_\mu$ and $m_5 = m_5^0 + m_\lambda + m_\mu$ are shown. Here $a \times b$ means the number of $b$-strings added is $a$.

| $m$ | $m_{1}$ | $m_{2}$ | $m_{3}$ | $m_{4,\lambda}$ | $m_{5,\mu}$ | $\Delta M_{1}^\lambda$ | $\Delta M_{2}^\lambda$ | $\Delta M_{2}^\mu$ | $\Delta M_{2}$ | $\Delta S$ |
|-----|---------|---------|---------|----------------|-------------|----------------|----------------|----------------|-----------|--------|
| $m_1$ | 0 0 0 0 0 | -2 | 1 | 0 | 0 | 0 | |
| $m_2$ | 1 0 1 0 0 | -1 | 0 | -1 | -1 | 2 | |
| $m_3^0$ | 1 2 0 0 0 | 0 | -1 | -1 | -2 | -1 | 3 | |
| $m_4^0$ | 0 2 1 0 0 | 1 | -2 | -2 | -2 | -2 | 5 | |
| $m_5^0$ | 0 4 0 0 0 | 2 | -3 | -2 | -4 | -2 | 6 | |
| $m_6^0$ | 0 0 2 0 0 | 0 | -1 | -2 | -2 | -2 | 4 | |
| $m_7^0$ | 0 0 1 1 × t | 0 | -1 | 0 | t-1 | 0 | -t | |
| $m_8^0$ | 0 0 0 0 1 × t | 0 | 0 | -1 | t-1 | 0 | -t | |
| $m_9^0$ | 1 0 1 0 1 × 2 | -1 | 0 | -2 | -1 | 0 | 1 | |
| $m_{10}^0$ | 1 2 0 1 × 4 | 0 | -2 | -1 | 0 | -1 | 1 | |
| $m_{11}^0$ | 1 2 0 2 × 3 | 0 | -3 | -1 | 0 | -1 | 1 | |
| $m_{12}^0$ | 0 4 0 1 × 6 | 2 | -4 | -4 | 0 | 0 | 0 | |
| $m_{13}^0$ | 0 4 0 1 × 6 | 2 | -4 | -3 | 0 | 0 | 0 | |

The numbers of holes and strings added are not independent but satisfy some constraints determined by the Bethe ansatz equations (118),

\begin{align}
\Delta M_1^1 &= -m_1 + \frac{1}{2} m_2, \\
\Delta M_1^2 &= \frac{1}{2} m_1 - \frac{3}{4} m_2 - \frac{1}{2} m_3 - \frac{1}{2} \sum_{i \geq 3} m_{\lambda(i)}, \\
\Delta M_2^1 &= \frac{1}{2} m_2 - m_3 - \frac{1}{2} \sum_{i \geq 2} m_{\mu(i)},
\end{align}

(117)

where $m_1, m_2, m_3$ are the numbers of holes in the real $\lambda$, 2-string $\lambda$ and real $\mu$-sea, $m_{\lambda(i)}$ and $m_{\mu(i)}$ are the number of $t$-string in the rapidity $\lambda$ and $t$-string in the rapidity $\mu$, respectively. Thus $\Delta M^1_{1,2}$ are some integers which indicate the number changes of $\lambda, \mu$ l-strings. For convenience, we define $\Delta M_a = \sum_i \Delta M_{a\lambda}$ and denote these excitations as $m = [(m_1, m_2, m_3), (m_4^0, m_4^\lambda), \cdots, (m_{10}^0, m_{10}^\lambda)]$. As we mentioned above, the excited momenta, excited energies and the spins are additive, $\Delta p_{m+n} = \Delta p_m + \Delta p_n, \Delta E_{m+n} = \Delta E_m + \Delta E_n$ and $S_{m+n}^z = S_m^z + S_n^z$. The number changes of $\Delta M^b_a$ are also additive

\begin{align}
\Delta M_{a\mu+m}^b &= \Delta M_{a\mu}^b + \Delta M_{a\mu}^b, \\
\Delta M_{a\mu+m'}^b &= \Delta M_{a\mu} + \Delta M_{a\mu}^b.
\end{align}

Some possible hole configurations are listed in Table. We denote $[(a, b, c), (0, 0, \cdots), (0, 0, \cdots)] = [(a, b, c)]$ for short. We find that the excitations $m_0^0 = [(2, 0, 0)], m_1^0 = [(1, 0, 1)], m_2^0 = [(1, 2, 0)], m_3^0 = [(0, 0, 2)], m_4^0 = [(0, 2, 1)]$ and $m_5^0 = [(0, 4, 0)]$ are the basic excitations. The additional single high strings $m_\lambda^1$ are not independent. They can form the excitations with $m_\mu^0$, such as $m_1, m_2, m_3, m_4$ and $m_5$.

The low-lying excitations $m_\mu^0$ are shown in Fig. The simplest spin excitation is a real $\lambda$ hole-pair $m^0_\lambda$ that is two-neutron spinon excitation, corresponding to the two domain walls of a single excited domain (Fig. (a)). The $\lambda$ 2-string hole pair can not exist independently. They must be associated with a neutral spinon, i. e. $m_0^0$ (Fig. (c)). Accompanied by a real $\mu$ hole, the $\lambda$ 2-string hole pair is also a possible excitation $m_\mu^0$ (Fig. (c)). Further, if we add a $\lambda$ 4-string into a 2-string hole pair and a real $\lambda$ hole in the $\lambda$-sea, the total spin of this excitation is 1 (Fig. (b)). In this case each of the $\lambda$ 2-string holes carries a spin 1/2. Such a dressed hole is quite similar to the ordinary spinon, and it is named as dressed spinon here. Four $\lambda$ 2-string holes $m_\mu^0$ may exist independently (Fig. (d)). If we put further one $\lambda$ 6-string and one $\mu$ 3-strings into this four hole configuration (m5 in Table. (1)), we obtain the SO(5) spin singlet excitation. The simplest excitation in the $\mu$ sector is a pair of real $\mu$ holes $m^0_\mu$ (Fig. (d)). This excitation is quite similar to a real $\lambda$ hole pair but each of the real $\mu$ hole carries a spin 2. Joint pair of a real $\lambda$ hole and a real $\mu$ hole $m^0_{\lambda\mu}$ may also happen as shown in Table. (1). Other kinds of spin excitations such as $m_1, m_3$ and $m_4$ can be constructed similarly.
FIG. 5: The low-lying excitations of the system. Here $J = 2\pi/9$, $\Delta E$ and $\Delta P$ are the energy and the momentum carried by the excitation, respectively. (a) $m_0^1 = [(200)]$; (b) $m_0^2 = [(101)]$; (c) $m_0^3 = [(120)]$; (d) $m_0^4 = [(002)]$; (e) $m_0^4 = [(021)]$; (f) $m_0^5 = [(040)]$.

VII. CONCLUSION

In conclusion, we propose an integrable spin-3/2 chain model with $SO(5)$ symmetry. By using the nested quantum inverse scattering method, we obtain the exact solutions of the system. Different from the $SU(4)$ integrable spin chain, there only exist three conserved quantities. Based on the exact solutions, the ground state and thermodynamic properties of the system are discussed. Several new kinds of spin excitations such as the neutral spin excitations, heavy spinons and dressed spinons are found.

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