Schubert calculus and Intersection theory of Flag manifolds

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Abstract

Hilbert’s 15th problem called for a rigorous foundation of Schubert’s calculus, in which a long standing and challenging part is Schubert’s problem of characteristics. In the course of securing the foundation of algebraic geometry, Van der Waerden and André Weil attributed the problem to the determination of the intersection theory of flag manifolds.

This article surveys the background, content, and resolution of the problem of characteristics. Our main results are a unified formula for the characteristics, and a system description for the intersection rings of flag manifolds. We illustrate the effectiveness of the formula and the algorithm via explicit examples.

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1 Introduction

Hilbert’s 15th problem [43] is an inspiring and far-reaching one. It promotes the enumerative geometry of the 19th century growing into the algebraic geometry founded by Van der Waerden and André Weil, and makes Schubert calculus integrated deeply into many branches of mathematics. However, despite great many achievements in the 20th century (e.g. [37, 47, 64]), the part of the problem of finding an effective rule performing the calculus has been stagnant for a long time, notably, the Schubert’s problem of characteristics [70, §8], or the Weil’s problem [71, p.331] on the intersection theory of flag manifolds $G/P$, where $G$ is a compact connected Lie group and $P$ a parabolic subgroup [4].

In the series of works [18, 19, 21, 27, 29], we have addressed both of the problem of characteristics and Weil’s problem, implying that the 15th problem has been solved satisfactorily [30, Remark 6.3]. The purpose of this article is to give an overview of the background, content and the resolution of Schubert’s characteristics. In §2 we glimpse the development from Apollonius’s work ”Tangencies” to Lefschetz’s homology theory, which reflects the evolution of the basic ideas from enumerative geometry to intersection theory. Section §3 summarises the pioneer contributions of Van der Waerden, Ehresmann, Weil,
Bernstein-Gel’fand-Gel’fand to Schubert calculus, which led to a great clarification of Schubert’s characteristics. Our main results are introduced in §4, where we present a formula expressing the characteristics of a flag manifold $G/P$ as a polynomial in the Cartan numbers of the Lie group $G$ (Theorem 4.5), develop a system description for the intersection rings of flag manifolds (Theorem 4.8), and illustrate the effectiveness of our programs in Examples 4.6, 4.9 and 4.12. In particular, since our approach uses the Cartan matrices of Lie groups as the main input, the solutions can be implemented successfully by computer programs, so that the intersection theory of flag manifolds becomes easily accessible by a broad readers.

2 An introduction to intersection theory

In the 2nd century B.C. Apollonius of Perga obtained the following enumerative result in the paper “Tangencies”.

**Apollonius’s Theorem.** The number of circles tangent to three general circles in plane is $8$. □

The original proof of Apollonius was lost, but a record of the theorem by Pappus dated in the 4th century survived. During the Renaissance different proofs were founded respectively by Francois Viete, Adriaan van Roomen, Joseph Diaz Gergonne and Isaac Newton [12, p.159]. For a pictorial illustration of the theorem see the cover-page story of the book “3264 and all that” [33].

Descartes’s discovery of the Euclidean coordinates makes it possible for geometers (e.g. Maclaurin, Euler, Bezout) to exploit polynomial system to characterize geometric figures that satisfy a system of incidence conditions. Consequently, many enumerative problems admit the following algebraic formulation.

**Problem 2.1.** Given a system of $n$ polynomials in $n$ variables with complex coefficients

$$
\begin{align*}
&f_1(x_1, \ldots, x_n) = 0 \\
&\vdots \\
&f_n(x_1, \ldots, x_n) = 0
\end{align*}
$$

(2.1)

find the number of solutions to the system. □

Problem 2.1 is a fundamental one in algebra. In the case $n = 1$ Gauss proved in the 1820’s that the number of zero’s of a polynomial in a single variable is the degree of that polynomial, well-known as the fundamental theorem of algebra.

Letting $g_i$ be the homogenization of the polynomial $f_i$ in (2.1), we get in the $n$-dimensional complex projective space $\mathbb{C}P^n$ a hypersurface $N_i := g_i^{-1}(0)$. In general, the zero locus of a homogeneous system on a complex projective space is called a projective variety. Naturally, the study of problem 2.1 leads to the fundamental problem of intersection theory.
Problem 2.2. Given $k$ subvarieties $N_1,\ldots,N_k$ in a smooth projective manifold $M$ that satisfy the dimension constraint $\dim N_1 + \cdots + \dim N_k = (k-1) \dim M$, find the number $|N_1 \cap \cdots \cap N_k|$ of the common intersection points when the subvarieties $N_i$’s are in general position.

In the course of studying Problem 2.2 S. Lefschetz developed the homology theory for the cellular complexes \cite{50}, 1926. In the perspective of this theory let $\alpha_i \in H^{\dim M - \dim N_i}(M)$ be the Poincaré dual of the cycle class represented by the subvariety $N_i \subset M$.

Problem 2.3. Given $k$ projective subvarieties $N_1,\ldots,N_k$ in a smooth projective manifold $M$ that satisfy the dimension constraint $\dim N_1 + \cdots + \dim N_k = (k-1) \dim M$, compute the Kronecker pairing
\[
\langle \alpha_1 \cup \cdots \cup \alpha_k, [M] \rangle = ?
\]
where $\cup$ means the cup product on the cohomology ring $H^*(M)$, and where $[M]$ denotes the orientation class of $M$.

Through problems 2.1 to 2.3 we have briefly reviewed three seemingly different approaches to the problems of enumerative geometry. Given that the effective computability is the primary task of enumerative geometry, a natural question is: which approach is the mostly calculable one? The development of the intersection theory shall tell us the answer.

3 Schubert’s problem of characteristics

Hermann Schubert (1848-1911) received his Ph.D. from the University of Halle, Germany in 1870. His doctoral thesis “The theory of characteristics” \cite{58} is about enumerative geometry. Prior to this he had shown that there are 16 spheres tangent to 4 general spheres in space, a direct extension of the Apollonius theorem.

In 1879 Schubert published the celebrated book “Calculus of Enumerative Geometry” \cite{59} that represents the summit of intersection theory in the late 19th century \cite{33}, p.2]. While developing M. Chasles’s work on conics \cite{15} he demonstrated amazing applications of intersection theory to enumerative geometry, such as
i) The number of conics tangent to 8 general quadrics in space is 4,407,296;

ii) The number of quadrics tangent to 9 general quadrics in space is 666,841,088.

iii) The number of twisted cubic curves tangent to 12 general quadrics in space is 5,819,539,783,680.

Nevertheless, in addition to the extensive use of the controversial “principle of conservation of numbers” \cite{45, 69}, Schubert’s exposition was so sketching that gave “no definition of intersection multiplicity, no way to find it nor to calculate it” \cite{68}. At the outset of the 20th century Hilbert made finding rigorous foundations for Schubert calculus one of his celebrated problems, where he praised also the advantage of the calculus to foresee the final degree of a polynomial system before carrying out the actual process of elimination \cite{43}.

In order to gain insight into the central part of Schubert’s approach to those spectacle enumerative numbers, we resort to the table of the characteristics for the variety of complete conics in space from his book \cite{59, p.95}:

| $\mu^3 \nu^5$ | $\mu^2 \nu^6$ | $\mu \nu^7$ | $\nu^8$ |
|----------------|----------------|--------------|---------|
| $= 1$          | $= 8$          | $= 34$       | $= 92$  |
| $\mu^3 \nu^4 \rho = 2$ | $\mu^2 \nu^5 \rho = 14$ | $\mu \nu^6 \rho = 52$ | $\nu^7 \rho = 116$ |
| $\mu^3 \nu^3 \rho^2 = 2$ | $\mu^2 \nu^4 \rho^2 = 24$ | $\mu \nu^5 \rho^2 = 76$ | $\nu^6 \rho^2 = 128$ |
| $\mu \nu \rho^3 = 2$ | $\mu^2 \nu \rho^3 = 16$ | $\mu \nu \rho^4 = 48$ | $\nu^3 \rho^4 = 104$ |
| $\mu \rho^5 = 1$ | $\mu^2 \rho^6 = 4$ | $\mu \rho^7 = 12$ | $\nu \rho^8 = 16$ |
| $= 1$          | $= 8$          | $= 6$        | $= 8$   |
| $\mu^3 \rho = 1$ | $\mu^2 \rho = 4$ | $\mu \rho = 6$ | $\rho = 4$ |

Table 1. The characteristics of the space of complete conics on $CP^3$.

where the symbols $\mu$, $\nu$ and $\rho$ stand for the subvarieties of conics passing through a given point, intersecting a given line, and tangent to a given plane, respectively. The table consists of the equalities evaluating the monomials $\mu^m \nu^n \rho^8 - m - n$ by integers, which were called characteristics by Schubert, and the Schubert’s symbolic equations by earlier researchers. Schubert emphasized that the problem of characteristics is the fundamental one of enumerative geometry \cite{46, 58, 59, 61}. However, to state the problem in its natural simplicity and generality, one has to wait until 1950’s for the celebrated “basis theorem of Schubert calculus”. Let us recall the relevant works on the subject.

The study of the characteristics began with the Italian school headed by Segre, Enriques and Severi. Two representing papers of the school are “The principle of conservation of numbers” and “The foundation of enumerative geometry and the theory of characteristics” due to Severi \cite{62, 63}. Regarding these works Van de Wareden \cite{69} commented that they “erected an admirable structure, but its logical foundation was shaky. The notions were not well-defined, and the proofs were insufficient”.

In the pioneer work “Topological foundation of enumerative geometry” \cite{70} 1930 Van der Waerden interpreted the Schubert’s characteristics in the perspective of the homology theory developed by Lefschetz \cite{50} (e.g.Problem 2.3). He
had the following crucial observations that enlightened the course of the later studies on the 15th problem:

1) Each Schubert’s symbolic equation is a relation on the homology of a projective manifold;

2) The solvability of Schubert’s characteristic problem relies on a finite basis of the homology of the relevant manifold;

3) The determination of the intersection products in homology is the goal of all enumerative methods.

C. Ehresmann [34, 1934] went two important steps further: he discovered that

4) The parameter spaces of the geometric figures of Schubert are in principle certain types of flag manifolds $G/P$ (see Remark 3.6);

5) For the Grassmannian $G_{n,k}$ of $k$-planes on the $n$-space $\mathbb{C}^n$ the Schubert symbols form exactly a basis of the homology $H_\ast(G_{n,k})$.

In what follows we denote by $W(P,G)$ the set of left cosets of the Weyl group $W(G)$ of $G$ by the Weyl group $W(P)$ of $P$, and let $l : W(P,G) \to \mathbb{Z}$ be the associated length function [4]. With the in-depth research on the structures of Lie groups (e.g. [7]) the vague term “Schubert symbols” in the early literature was gradually replaced by such rigorous defined geometric objects as “Schubert cells” or “Schubert varieties”. In particular, extending Ehresmann’s work [34] on the Grassmannian $G_{n,k}$, the following result was announced by Chevalley [16] for the complete flag manifold $G/T$ (where $T \subset G$ is a maximal torus), and extended to all flag manifolds $G/P$ by Bernstein-Gel’fand-Gel’fand [4, Proposition 5.1].

**Theorem 3.1.** Every flag manifold $G/P$ has a canonical decomposition into the Schubert cells $\mathcal{S}_w$, parameterized by the elements $w$ of $W(P,G)$,

\[(3.1) \quad G/P = \bigcup_{w \in W(P,G)} S_w, \quad \dim S_w = 2l(w),\]

where the closure $X_w$ of each cell $S_w$ is a subvariety of $G/P$, called the Schubert variety on $G/P$ associated to $w \in W(P,G)$. □

Since only even dimensional cells are involved in the partition (3.1), the set $\{[X_w], w \in W(P,G)\}$ of fundamental classes forms an additive basis of the homology $H_\ast(G/P)$. The co-cycle classes $s_w \in H^\ast(G/P)$ Kronecker dual to the basis (i.e. $\langle s_w, [X_u] \rangle = \delta_{w,u}, w, u \in W(P,G)$) gives rise to the Schubert class associated to $w \in W(P,G)$. Theorem 3.1 implies the following result, which is expected by Van der Waerden [70, §8], and is well-known as the “basis theorem of Schubert calculus”.

**Theorem 3.2.** ([4, Proposition 5.2]) The set $\{s_w, w \in W(P,G)\}$ of Schubert classes forms a basis of the cohomology $H^\ast(G/P)$. □
An immediate consequence of the basis theorem is that any product $s_{u_1} \cdots s_{u_k}$ in the Schubert classes can be uniquely expressed as a linear combination of the basis elements

\begin{equation}
(3.2) \quad s_{u_1} \cdots s_{u_k} = \sum_{w \in W(P,G), l(w) = l(u_1) + \cdots + l(u_k)} c^w_{u_1, \ldots, u_k} \cdot s_w, \quad c^w_{u_1, \ldots, u_k} \in \mathbb{Z}
\end{equation}

by which Schubert’s problem of characteristics \cite{34, 60, 61, 70} has the following concise expression.\footnote{By Coolidge \cite[p.184]{13} “The fundamental problem which occupies Schubert is to express the product of two of these symbols in terms of others linearly. He succeeds in part.”}

**Problem 3.3.** Given any monomial $s_{u_1} \cdots s_{u_k}$ in the Schubert classes, determine the characteristics numbers $c^w_{u_1, \ldots, u_k}$ in the linear expansion (3.2). \hfill \Box

In the momentous treatise “Foundations of Algebraic Geometry \cite{71}” A. Weil completed the definition of intersection multiplicities for the first time, and summarized the task of Schubert calculus in the context of the modern intersection theory \footnote{The classical Schubert calculus amounts to the determination of the intersection-rings on Grassmann varieties and on the so-called flag manifolds of projective geometry \cite[p.331]{71}, where we note that for a flag manifold $G/P$ the Chow ring $A^*(G/P)$ is canonically isomorphic to the cohomology $H^*(G/P)$}.

**Problem 3.4.** Determine the intersection rings of flag manifolds $G/P$. \hfill \Box

Weil commented his problem as “the modern form taken by the topic formerly known as enumerative geometry” \cite[p.331]{71}. We show that

**Theorem 3.5.** For the flag manifolds $G/P$ the Weil’s problem is equivalent to the Schubert’s one.

**Proof.** A ring is an abelian group $R$ that is furnished with a multiplication $R \times R \to R$. By the basis theorem the cohomology $H^*(G/P)$ has a canonical basis consisting of Schubert classes. Therefore, the multiplication on $H^*(G/P)$ is uniquely determined by the product among the basis elements, which is handled by the characteristics $c^w_{u_1, \ldots, u_k}$. \hfill \Box

**Remark 3.6.** For the case $k = 2$ the characteristics $c^w_{u_1, \ldots, u_k}$ admit various interpretations. They are called the Schubert’s structure constants of the flag manifold $G/P$ in topology; and the Littlewood-Richardson coefficients in representation theory \cite{51}.

In certain cases the parameter spaces of the geometric figures concerned by Schubert \cite[Chapt.IV]{59} fail to be flag manifolds, but can be constructed by performing finite steps of blow-ups on flag manifolds, see examples in Fulton \cite[Section 10.4]{37}, Eisenbud-Harris\cite[Chap.13]{33}, or in \cite{24} for the constructions of the parameter spaces of the complete conics and quadrics on $\mathbb{C}P^3$. As results the relevant characteristics can be computed from those of flag manifolds via strict transformations (e.g \cite[Examples 5.11; 5.12]{24}). \hfill \Box
4 Intersection theory of flag manifolds

To secure the foundation of a “calculus” it suffices to decide the objects to be calculated, and to determine accordingly the rules of the calculation (e.g. [2], Chap.2], [3]). As for the Schubert’s calculus we have seen from §3 that the objects to be calculated have been clarified to be the Schubert symbols, or Schubert varieties. In this section we determine the rule of the calculus by a unified formula computing the characteristics, and apply the formula to complete the intersection theory of flag manifolds.

4.1 Observation and expectation

The major difficulties that one encounters with the problem of characteristics are fairly transparent:

i) The simply-connected simple Lie groups $G$ consist of the three infinite families of classical Lie groups $Spin(n), Sp(n), SU(n)$, as well as the five exceptional ones $G_2, F_4, E_6, E_7, E_8$;

ii) for a simple Lie group $G$ with rank $n$ there are precisely $2^n - 1$ parabolic subgroups $P$ on $G$.

That is, there exist plenty of flag manifolds $G/P$ whose geometries and topologies vary considerably with respect to different choices of $G$ and $P$. In addition

iii) the number of Schubert classes of $G/P$ agrees with the Euler characteristic $\chi(G/P)$, which is normally very large, not to mention the number of the relevant characteristics. For instance, for an exceptional Lie group $G$ with a maximal torus $T$ the Euler characteristic $\chi(G/T)$ of the flag manifold $G/T$ is given in the table below

| $G$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|-----|-------|-------|-------|-------|-------|
| $\chi(G/T)$ | $12$ | $1152$ | $2^9 \cdot 3^3 \cdot 5$ | $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ | $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ |

Summarizing, studies case by case can never reach a complete solution to the problem.

On the other hand, according to E. Cartan’s beautiful work on compact Lie groups, associate to each simple Lie group $G$ there is a Cartan matrix $C$, which acts the role of “the cosmological constants” to classify all flag manifolds $G/P$, see discussions in the coming section. As examples, for the five exceptional Lie groups those matrices are

$G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, $F_4 : \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$, $E_6 : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$
This raises the following question: can one express the characteristic numbers, as well as the intersection ring of a flag manifold \(G/P\), merely in terms of the Cartan matrix of the Lie group \(G\)? In this section we fulfill this expectation.

### 4.2 Numerical construction of a Weyl group

Therefore, let \(C = (c_{i,j})_{n \times n}\) be the Cartan matrix of some compact simple Lie group \(G\), and let \(\mathbb{R}^n\) be the \(n\)-dimensional real vector space with basis \(\{\omega_1, \ldots, \omega_n\}\). Define in terms of \(C\) the endomorphisms \(\sigma_i \in \text{End}(\mathbb{R}^n), 1 \leq i \leq n\), by the formula

\[
\sigma_i(\omega_k) = \begin{cases} 
\omega_k & \text{if } i \neq k; \\
\omega_k - (c_{k,1}\omega_1 + c_{k,2}\omega_2 + \cdots + c_{k,n}\omega_n) & \text{if } i \neq k.
\end{cases}
\]

By general properties of Cartan matrices we have \(\sigma_i^2 = id\), implying that \(\sigma_i \in \text{Aut}(\mathbb{R}^n)\). It can be further shown that

**Lemma 4.1.** The subgroup of \(\text{Aut}(\mathbb{R}^n)\) generated by the \(\sigma_i\)'s is isomorphic to the Weyl group \(W(G)\) of \(G\). \(\square\)

For each subset \(K \subset \{1, \ldots, n\}\) there is a parabolic subgroup \(P = P_K\), unique up to the conjugations on \(G\), whose Weyl group \(W(P)\) is generated by those generators \(\sigma_j \in W(G)\) with \(j \notin K\). Resorting to the length function \(l\) on \(W(G)\) we can furthermore embed the set \(W(P; G)\) as the subset of the group \(W(G)\) [4]

\[
W(P; G) = \{w \in W(G) \mid l(w_1) \geq l(w), w_1 \in wW(P)\},
\]

and put \(W^m(P; G) := \{w \in W(P; G) \mid l(w) = m\}\). By Lemma 4.1 each element \(w \in W^m(P; G)\) admits a factorization of the form

\[
w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_m}\ 	ext{with } 1 \leq i_1, \ldots, i_m \leq n,
\]

hence can be denoted by \(w = \sigma_I\), where \(I = (i_1, \ldots, i_m)\). Such expressions of \(w\) may not be unique, but the ambiguity can be dispelled by employing the following notion. Furnish the set \(D(w) := \{I = (i_1, \ldots, i_m) \mid w = \sigma_I\}\) with the lexicographical order \(\preceq\) on the multi-indices \(I\)'s. We call a decomposition \(w = \sigma_I\) minimized if \(I \in D(w)\) is the minimal one. Clearly we have (e.g. [9])
**Lemma 4.2.** Every $w \in W(P; G)$ has a unique minimized decomposition. □

It follows that the set $W^m(P; G)$ is also ordered by the lexicographical order ≤ on the multi-index $I$'s, hence can be uniquely presented as

\[(4.1) \quad W^m(P; G) = \{w_{m,i} \mid 1 \leq i \leq \beta(m)\}, \quad \beta(m) := |W^m(P; G)|,\]

where $w_{m,i}$ denotes the $i^{th}$ element in $W^m(P; G)$. In [24] the package “Decomposition” in MATHEMATICA is compiled, whose function is stated below.

**Algorithm I. Decomposition.**

**Input:** The Cartan matrix $C = (a_{ij})_{n \times n}$ of $G$, and a subset $K \subset \{1, \ldots, n\}$ to specify a parabolic subgroup $P$.

**Output:** The set $W(P; G)$ being presented by the minimized decompositions of its elements, together with the index system (4.1) imposed by the order ≤. □

**Example 4.3.** Let $G = SU(n)$ be the special unitary group, and let $k \in \{1, \ldots, n-1\}$. The flag manifold $G/P_{(k)}$ is the Grassmannian manifold $G_{n,k}$. Applying the Decomposition to the case $G_{9,4}$ we obtain the following table of minimized decompositions, as well as the order imposed by (4.1), for the elements $w \in W(P_4; SU(9))$ with $l(w) \leq 8$.

| $w_{i,j}$ | decomposition | $w_{i,j}$ | decomposition | $w_{i,j}$ | decomposition |
|----------|---------------|----------|---------------|----------|---------------|
| $w_{1,1}$ | [4]           | $w_{2,1}$ | [3, 4]        | $w_{2,2}$ | [5, 4]        |
| $w_{3,1}$ | [2, 3, 4]     | $w_{3,2}$ | [3, 5, 4]     | $w_{3,3}$ | [6, 5, 4]     |
| $w_{4,1}$ | [1, 2, 3, 4]  | $w_{4,2}$ | [2, 3, 5, 4]  | $w_{4,3}$ | [3, 6, 5, 4]  |
| $w_{4,4}$ | [4, 3, 5, 4]  | $w_{4,5}$ | [7, 6, 5, 4]  | $w_{5,1}$ | [1, 2, 3, 5, 4]|
| $w_{5,2}$ | [2, 3, 6, 5, 4]| $w_{5,3}$ | [2, 4, 3, 5, 4]| $w_{5,4}$ | [3, 7, 6, 5, 4]|
| $w_{5,5}$ | [4, 3, 6, 5, 4]| $w_{5,6}$ | [8, 7, 6, 5, 4]| $w_{6,1}$ | [1, 2, 3, 6, 5, 4]|
| $w_{6,2}$ | [1, 2, 4, 3, 5, 4]| $w_{6,3}$ | [2, 3, 7, 6, 5, 4]| $w_{6,4}$ | [2, 4, 3, 6, 5, 4]|
| $w_{6,5}$ | [3, 2, 4, 3, 5, 4]| $w_{6,6}$ | [3, 8, 7, 6, 5, 4]| $w_{6,7}$ | [4, 3, 7, 6, 5, 4]|
| $w_{6,8}$ | [5, 4, 3, 6, 5, 4]| $w_{7,1}$ | [1, 2, 3, 7, 6, 5, 4]| $w_{7,2}$ | [1, 2, 4, 3, 6, 5, 4]|
| $w_{7,3}$ | [1, 3, 2, 4, 3, 5, 4]| $w_{7,4}$ | [2, 3, 8, 7, 6, 5, 4]| $w_{7,5}$ | [2, 4, 3, 7, 6, 5, 4]|
| $w_{7,6}$ | [2, 5, 4, 3, 6, 5, 4]| $w_{7,7}$ | [3, 2, 4, 3, 6, 5, 4]| $w_{7,8}$ | [4, 3, 8, 7, 6, 5, 4]|
| $w_{7,9}$ | [5, 4, 3, 7, 6, 5, 4]| $w_{8,1}$ | [1, 2, 3, 8, 7, 6, 5, 4]| $w_{8,2}$ | [1, 2, 4, 3, 7, 6, 5, 4]|
| $w_{8,3}$ | [1, 2, 5, 4, 3, 6, 5, 4]| $w_{8,4}$ | [1, 3, 2, 4, 3, 6, 5, 4]| $w_{8,5}$ | [2, 1, 3, 2, 4, 3, 5, 4]|
| $w_{8,6}$ | [2, 4, 3, 8, 7, 6, 5, 4]| $w_{8,7}$ | [2, 5, 4, 3, 7, 6, 5, 4]| $w_{8,8}$ | [3, 2, 4, 3, 7, 6, 5, 4]|
| $w_{8,9}$ | [3, 2, 5, 4, 3, 6, 5, 4]| $w_{8,10}$ | [5, 4, 3, 8, 7, 6, 5, 4]| $w_{8,11}$ | [6, 5, 4, 3, 7, 6, 5, 4]|

For more examples of the results produced by Decomposition we refer to [28] Sections 1.1–7.1.

Geometrically, for any $w \in W(P; G)$ the Schubert variety $X_w$ can be explicitly constructed in terms of its minimized decomposition [4] [19]. □
4.3 A unified formula for Schubert’s characteristics

Given an element \( w \in W(P_K; G) \) with minimized decomposition

\[
w = \sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_m}, \quad 1 \leq i_1, i_2, \ldots, i_m \leq n,
\]

the structure matrix of \( w \) is the strictly upper triangular matrix \( A_w = (a_{s,t})_{m \times m} \) defined by the Cartan matrix \( C = (c_{i,j})_{n \times n} \) of \( G \) as

\[
a_{s,t} = 0 \text{ if } s \geq t, \quad -c_{i_s,i_t} \text{ if } s < t.
\]

As examples recall that the Cartan matrix of the exceptional Lie group \( G_2 \) is

\[
C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.
\]

By Lemma 4.1 the Weyl group \( W(G_2) \) has two generators \( \sigma_1, \sigma_2 \). Consider the following elements with length 4:

\[
u = \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 \quad \text{and} \quad v = \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1.
\]

From the Cartan matrix \( C \) one reads

\[
A_u = \begin{pmatrix} 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_v = \begin{pmatrix} 0 & 3 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Let \( \mathbb{Z}[x_1, \ldots, x_m] \) be the ring of polynomials in \( x_1, \ldots, x_m \) that is graded by \( \deg x_i = 1 \), and let \( \mathbb{Z}[x_1, \ldots, x_m]^{(m)} \) be its subgroup spanned by the monomials with degree \( m \). Given a \( m \times m \) strictly upper triangular integer matrix \( A = (a_{i,j}) \) the triangular operator \( T_A \) associated to \( A \) is the linear map

\[
T_A : \mathbb{Z}[x_1, \ldots, x_m]^{(m)} \to \mathbb{Z}
\]

defined recursively by the following elimination rules:

i) If \( m = 1 \) (i.e. \( A = (0) \)) then \( T_A(x_1) = 1 \);

ii) If \( h \in \mathbb{Z}[x_1, \ldots, x_{m-1}]^{(m)} \) then \( T_A(h) = 0 \);

iii) If \( h \in \mathbb{Z}[x_1, \ldots, x_{m-1}]^{(m-r)} \) with \( r \geq 1 \) then

\[
T_A(h \cdot x_m^r) = T_{A_1}(h \cdot (a_{1,m}x_1 + \cdots + a_{m-1,m}x_{m-1})^{r-1}),
\]

where \( A_1 \) is the \((m - 1) \times (m - 1)\) (strictly upper triangular matrix) obtained from \( A \) by deleting both of the \( m^{th} \) column and row. Since every polynomial \( h \in \mathbb{Z}[x_1, \ldots, x_{m-1}]^{(m)} \) admits the unique expansion

\[
h = \sum_{0 \leq r \leq m} h_r \cdot x_m^r \quad \text{with} \quad h_r \in \mathbb{Z}[x_1, \ldots, x_{m-1}]^{(m-r)},
\]
the operator $T_A$ is well-defined by the rules i), ii) and iii). It follows that

**Lemma 4.4.** For any polynomial $h \in \mathbb{Z}[x_1, \ldots, x_m]^{(m)}$, the number $T_A(h)$ is a polynomial in the entries of the matrix $A$ with degree $m$. □

Extending the main results of [18, 19, 21] we have shown in [30, Theorem 2.4] the following formula that expresses the Schubert’s characteristics of a flag manifold $G/P$ as polynomials in the Cartan numbers of the group $G$.

**Theorem 4.5.** Let $w \in W(P;G)$ be an element with minimized decomposition $\sigma_{i_1} \circ \cdots \circ \sigma_{i_m}$ and structure matrix $A_w$. For any monomial $s_{u_1} \cdots s_{u_k}$ in the Schubert classes with total degree $m$ one has

$$
(4.2) \ c^w_{u_1,\ldots,u_k} = T_{A_w} \left( \prod_{i=1,\ldots,k} \left( \sum_{|I|=l(u_i), I \subseteq \{1,\ldots,m\}} x_I \right) \right),
$$

where for a multi-index $I = \{j_1, \ldots, j_t\}$ we have set $|I| := t$ and

$$
\sigma_I := \sigma_{i_{j_1}} \circ \cdots \circ \sigma_{i_{j_t}} \in W(G), \ x_I := x_{i_{j_1}} \cdots x_{i_{j_t}} \in \mathbb{Z}[x_1, \ldots, x_m]. \quad □
$$

Since the matrix $A_w$ is constructed from the Cartan matrix of the group $G$ in term of the minimized decomposition of $w$, while the operator $T_{A_w}$ is evaluated easily by the elimination rules i)-iii) stated above, the formula (4.2) indicates an effective algorithm to evaluate the numbers $c^w_{u_1,\ldots,u_k}$. Combining these ideas the package “Characteristics” in MATHEMATICA has been compiled (e.g. [24]) whose function is described as follows.

**Algorithm II: Characteristics.**

**Input:** The Cartan matrix $C = (a_{ij})_{n \times n}$ of $G$, and a subset $K \subset \{1, \ldots, n\}$ to specify a parabolic subgroup $P$.

**Output:** The characteristics $c^w_{u_1,\ldots,u_k}$ of $G/P$. □

**Example 4.6.** The characteristics of the Schubert monomials at the top degree. Let $G/P$ be a flag manifold with dim$_C G/P = m$. According to the basis theorem there exists a unique element $w_0 \in W(P;G)$ so that $l(w_0) = m$, and that the Schubert class $s_{w_0}$ generates the top degree cohomology $H^{2m}(G/P) = \mathbb{Z}$. It follows that, for any monomial $s_{u_1} \cdots s_{u_k}$ in the Schubert classes with total degree $m$, the characteristic number $c^w_{u_1,\ldots,u_k}$ is given by the equality (see in Problem 2.3):

$$
\langle s_{u_1} \cdots s_{u_k}, [G/P] \rangle = c^w_{u_1,\ldots,u_k},
$$

which will be abbreviated by $s_{u_1} \cdots s_{u_k} = c^w_{u_1,\ldots,u_k}$. In addition, for an element $w \in W(P;G)$ with minimized decomposition $w = \sigma_I$ we can use the notion $s_I$ to denote the Schubert class $s_w$. 

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The cohomology of the Grassmannians $G_{n,k}$ are the most classical and archetypal examples of intersection theory \[60, 61, 33\] p.4. Traditionally, the characteristics $e_{w_{n,k}}$ are given by the combinatorial Littlewood-Richardson rule \[51\], rather than a closed formula. In contrast our formula (4.2) is practical for numerical computation. In term of the convention above we set

$$c_r := s_{k-r+1,k-r+2,\ldots,k} \in H^{2r}(G_{n,k}), r = 1, \ldots, k.$$ Then $c_r$ is also the $r^{th}$ Chern class of the canonical $k$-dimensional complex vector bundles on $G_{n,k}$ \[55\]. Applying Characteristics to the case $G_{9,4}$ we obtain the following table of characteristics for the monomials in the Chern classes at the top degree $\dim \mathbb{C} G_{9,4} = 20$.

| $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
|-------|-------|-------|-------|-------|
| 1     | $c_2$ | 1     | $c_2 c_3$ | 9     |
| $c_2 c_4$ | 1     | $c_2 c_4$ | 3     |
| 45    | 26    | 16    | $c_2 c_4$ |
| 126   | $c_2 c_4$ | 1     | $c_2 c_4$ | 4     |
| $c_1 c_3 c_4$ | 3     | $c_1 c_3 c_4$ | 29    |
| $c_1 c_3 c_4$ | 10    | $c_1 c_3 c_4$ | 76    |
| $c_1 c_3 c_4$ | 2     | $c_1 c_3 c_4$ | 1     |
| $c_1 c_3 c_4$ | 26    | $c_1 c_3 c_4$ | 89    |
| $c_1 c_3 c_4$ | 17    | $c_1 c_3 c_4$ | 47    |
| $c_1 c_3 c_4$ | 1     | $c_1 c_3 c_4$ | 12    |
| $c_1 c_3 c_4$ | 98    | $c_1 c_3 c_4$ | 59    |
| $c_1 c_3 c_4$ | 26    | $c_1 c_3 c_4$ | 141   |
| $c_1 c_3 c_4$ | 175   | $c_1 c_3 c_4$ | 45    |
| $c_1 c_3 c_4$ | 436   | $c_1 c_3 c_4$ | 4     |
| $c_1 c_3 c_4$ | 59    | $c_1 c_3 c_4$ | 29    |
| $c_1 c_3 c_4$ | 2962  | $c_1 c_3 c_4$ | 9     |
| $c_1 c_3 c_4$ | 164   | $c_1 c_3 c_4$ | 1744  |
| $c_1 c_3 c_4$ | 10302 | $c_1 c_3 c_4$ | 49    |
| $c_1 c_3 c_4$ | 496   | $c_1 c_3 c_4$ | 14    |
| $c_1 c_3 c_4$ | 300   | $c_1 c_3 c_4$ | 3437  |
| $c_1 c_3 c_4$ | 20887 | $c_1 c_3 c_4$ | 2025  |
| $c_1 c_3 c_4$ | 11853 | $c_1 c_3 c_4$ | 252   |
| $c_1 c_3 c_4$ | 4102  | $c_1 c_3 c_4$ | 392    |
| $c_1 c_3 c_4$ | 23662 | $c_1 c_3 c_4$ | 462   |
| $c_1 c_3 c_4$ | 28417 | $c_1 c_3 c_4$ | 4332  |
| $c_1 c_3 c_4$ | 28417 | $c_1 c_3 c_4$ | 4332  |

Table 2. The characteristics of the Grassmannian $G_{9,4}$

The Characteristics works equally well for other types of flag manifolds. For example consider the flag manifold $E_6/P(2)$, where $P(2) = S^1 \cdot SU(6)$. Following Bourbaki’s numbering of simple roots \[9\] let $y_2, y_3, y_4, y_6$ be respectively the Schubert classes $s_I$ with

$$I = \{2\}, \{5, 4, 2\}, \{6, 5, 4, 2\}, \{1, 3, 6, 5, 4, 2\},$$

Then the cohomology $H^*(E_6/P(2))$ is generated by $y_1, y_3, y_4, y_6$ by \[27\] Theorem 3. Applying Characteristics we obtain the following table of characteristics for all the monomials in the Schubert generators $y_I$'s at the top degree $\dim \mathbb{C} E_6/P(2) = 21$. 

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Let \( H = \text{the homogeneous elements of positive degrees} \), and let \( D \).

Proof. 

Geometrically, the operator \( T_A \) in (4.2) handles the integrations along the Schubert cell \( X_w \).

The formula (4.2) has been extended in [20] to compute the products of the basis elements of the Grothendieck \( K \)-theory of flag manifolds, and to evaluate the Steenrod operations on Schubert classes in [23].

**Remark 4.7.** Geometrically, the operator \( T_A \) in (4.2) handles the integrations along the Schubert cell \( X_w \) [4][19].

The formula (4.2) has been extended in [20] to compute the products of the basis elements of the Grothendieck \( K \)-theory of flag manifolds, and to evaluate the Steenrod operations on Schubert classes in [23].

**4.4 The intersection rings of flag manifolds**

As in Example 4.6 let \( c_i \in H^{2i}(G_{n,k}) \) be the \( i \)-th Chern class. Borel [4] has shown that

\[
H^*(G_{n,k}) = \mathbb{Z}[c_1, \ldots, c_k]/\langle c_{n-k+1}^{-1}, \ldots, c_n^{-1} \rangle,
\]

where \( c_j^{-1} \) denotes the component of the formal inverse of \( 1+c_1+\cdots+c_k \) in degree \( j \), and where \( \langle \cdot, \cdot \rangle \) denotes the ideal generated by the enclosed polynomials.

Comparing formula (4.3) with the contents in Table 2 reveals the following phenomena: the characteristic numbers are essential for enumerative geometry, but fail to be a concise way to characterize the structure of the ring \( H^*(G_{n,k}) \).

It is the Weil’s problem that motivates us the following extension of Borel’s formula (4.3) to all flag manifolds.

**Theorem 4.8.** For each flag manifold \( G/P \) there exist Schubert classes \( y_1, \cdots, y_n \) such that

\[
H^*(G/P) = \mathbb{Z}[y_1, \cdots, y_n]/\langle f_1, \cdots, f_m \rangle,
\]

where \( f_i \in \mathbb{Z}[y_1, \cdots, y_n], 1 \leq i \leq m \), and where the numbers \( n \) and \( m \) are minimum subject to the presentation.

**Proof.** Let \( H^+(G/P) \) be the subring of the cohomology \( H^*(G/P) \) spanned by the homogeneous elements of positive degrees, and let \( D(H^*(G/P)) \) be the

| \( y_1y_6 \) | 3 |  |  | 21 | 21 | 150 |
|----------|---|---|---|----|----|-----|
| \( y_1^2 \) | 1158 | 2 |  | 42 | 66 | 2328 |
| \( y_1^2y_6 \) | 9 | 2 |  | 18 | 6 | 132 |
| \( y_1^3 \) | 56 | 84 |  | 624 | 4677 |
| \( y_1^3y_6 \) | 168 | 84 |  | 264 | 168 |
| \( y_1^4 \) | 1248 | 9390 |  | 813 | 528 |
| \( y_1^4y_6 \) | 2496 | 18837 |  | 1638 | 7917 |
| \( y_1^5 \) | 37752 | 15912 |  | 75582 | 151164 |

Table 3. The characteristics of the flag manifold \( E_6/S^1 \cdot SU(6) \).
ideal of the decomposable elements of the ring $H^+(G/P)$. Since the cohomology $H^+(G/P)$ is torsion free and has a basis consisting of Schubert classes, there exist Schubert classes $y_1, \cdots, y_n$ on $G/P$ that correspond to a basis of the quotient group $H^+(G/P)/D(H^*(G/P))$. It follows that the inclusion $\{y_1, \cdots, y_n\} \subset H^*(G/P)$ induces a ring epimorphism

$$\pi : \mathbb{Z}[y_1, \cdots, y_n] \to H^*(G/P).$$

By the Hilbert’s basis theorem there exist finitely many polynomials $f_1, \cdots, f_m \in \mathbb{Z}[y_1, \cdots, y_n]$ such that $\ker \pi = \langle f_1, \cdots, f_m \rangle$. We can of course assume that the number $m$ is minimum with respect to the formula (4.4).

As the cardinality of a basis of the quotient group $H^+(G/P)/D(H^*(G/P))$ the number $n$ is an invariant of $G/P$. In addition, if one changes the generators $y_1, \cdots, y_n$ to $y'_1, \cdots, y'_n$, then each old generator $y_i$ can be expressed as a polynomial $g_i$ in the new ones $y'_1, \cdots, y'_n$. The invariance of the number $m$ is shown by the presentation

$$H^*(G/P) = \mathbb{Z}[y'_1, \cdots, y'_n]/\langle f'_1, \cdots, f'_m \rangle,$$

where $f'_j$ is obtained from $f_j$ by substituting $g_i$ for $y_i, 1 \leq j \leq m$. □

The proof of Theorem 4.8 singles out two crucial steps in resolving Weil’s problem:

i) Find a minimal set of Schubert classes $\{y_1, \cdots, y_n\}$ on $G/P$ that generates the quotient group $H^+(G/P)/D(H^*(G/P))$;

ii) Seek a Hilbert basis $\{f_1, \cdots, f_m\}$ of the ideal $\ker \pi$.

Since both of the tasks can be implemented by the Characteristics (e.g. [30] Section 4.4), we obtain therefore the package “Chow-ring” in MATHEMATICA [27, 30] whose function is stated below.

**Algorithm III: Chow-ring.**

**Input:** The Cartan matrix $C = (a_{ij})_{n \times n}$ of $G$, and a subset $K \subset \{1, \ldots, n\}$ to specify a parabolic subgroup $P$.

**Output:** A presentation (4.4) of the cohomology $H^*(G/P)$. □

**Example 4.9.** If $G$ is a simple Lie group of rank $n$ and if $K = \{1, \cdots, n\}$, then the parabolic subgroup $P_K$ of $G$ is a maximal torus $T$ on $G$, and the flag manifold $G/T$ is called the complete flag manifold of the group $G$. As applications of the Chow-ring the cohomologies $H^*(G/T)$ for the exceptional Lie groups have been determined in terms of a minimal system of generators and relations in the Schubert classes (e.g. [29]). We present below the results for $G = F_4, E_6, E_7$. 

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Lemma 2.4); the $c_\lambda$ is also the set of fundamental dominant weights of the relevant group $\G / T$ whose geometric implication will be made transparent in Example 4.12 below,

\begin{align*}
H^* (F_4 / T) &= \mathbb{Z}[\omega_1, \cdots, \omega_4, y_3, y_4] / \langle \rho_2, \rho_4, \rho_3, \rho_4, r_6, r_8, r_{12} \rangle, \text{ where} \\
\rho_2 &= c_2 - 4\omega_1^2; \\
\rho_3 &= 3y_4 + 2\omega_1 y_3 - c_4; \\
r_3 &= 2y_3 - \omega_1^3; \\
r_4 &= y_3^2 + 2c_6 - 3\omega_1^3 y_4; \\
r_6 &= 3y_4^2 - \omega_1^2 c_6; \\
r_8 &= y_4^2 - c_6^2. \\
r_{12} &= y_4^2 - c_6^2.
\end{align*}

\begin{align*}
H^* (E_6 / T) &= \mathbb{Z}[\omega_1, \cdots, \omega_6, y_3, y_4] / \langle \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, r_8, r_9, r_{12} \rangle, \text{ where} \\
\rho_2 &= 4\omega_2^2 - c_2; \\
\rho_3 &= 2y_3 + 2\omega_2^3 - c_3; \\
\rho_4 &= 3y_4 + \omega_2^3 - c_4; \\
\rho_5 &= 2\omega_2 y_3 - \omega_2 c_4 + c_5; \\
r_6 &= y_3^2 - \omega_2 c_6 + 2c_6; \\
r_8 &= 3y_4^2 - 2c_5 y_3 - \omega_2^2 c_6 + \omega_2^3 c_5; \\
r_9 &= 2y_3 c_6 - \omega_2^3 c_6; \\
r_{12} &= y_4^2 - c_6^2.
\end{align*}

\begin{align*}
H^* (E_7 / T) &= \mathbb{Z}[\omega_1, \cdots, \omega_7, y_3, y_4, y_5, y_9] / \langle \rho_1, r_j \rangle, \text{ where} \\
\rho_2 &= 4\omega_2^2 - c_2; \\
\rho_3 &= 2y_3 + 2\omega_2^3 - c_3; \\
\rho_4 &= 3y_4 + \omega_2^3 - c_4; \\
\rho_5 &= 2y_5 - 2\omega_2 y_3 + \omega_2 c_4 - c_5; \\
r_6 &= y_3^2 - \omega_2 c_6 + 2c_6; \\
r_8 &= 3y_4^2 + 2y_3 y_5 - 2y_3 y_5 + 2\omega_2 c_7 - \omega_2^2 c_6 + \omega_2^3 c_5; \\
r_9 &= 2y_9 + 2y_3 y_5 - 2y_3 c_6 - \omega_2^2 c_7 + \omega_2^3 c_6; \\
r_{10} &= y_5^2 - 2y_3 c_7 + \omega_2^2 c_7; \\
r_{12} &= y_4^2 - 4y_3 c_7 - c_6^2 - 2y_3 y_9 - 2y_3 y_4 y_5 + 2\omega_2 y_5 c_6 + 3\omega_2 y_4 c_7 + c_5 c_7; \\
r_{14} &= c_7^2 - 2y_5 y_9 + 2y_3 y_4 c_7 - \omega_2^3 y_4 c_7; \\
r_{18} &= y_5^2 + 2y_5 c_6 c_7 - y_4 c_7^2 - 2y_4 y_5 y_9 + 2y_3 y_5^2 - 5\omega_2 y_5^2 c_7.
\end{align*}

where the set $\{\omega_i, 1 \leq i \leq \text{rank}(G)\}$ is the Schubert’s basis of $H^2 (G / T)$, which is also the set of fundamental dominant weights of the relevant group $G$ (e.g. [22, Lemma 2.4]); the $c_\lambda$’s are certain polynomials in $\omega_1, \cdots, \omega_n$ defined in [30, (5.17)] whose geometric implication will be made transparent in Example 4.12 below, and where in terms of Bourbaki’s numbering of simple roots [9], the $y_i$’s are the Schubert classes $s_i$ on $G / T$ specified in the table below:
For more examples of the applications of Chow-ring to computing with partial flag manifolds $G/P$ we refer to [27, Theorems 1-7]. □

4.5 Schubert polynomials

In the classical enumerative geometry, the Chern class $c_i$ of the Grassmannian $G_{n,k}$ arises firstly as the Poincaré dual of the variety of the $k$-planes meeting a general $(n-k-i)$-plane on the $n$-space $\mathbb{C}^n$, well-known as the $i^{th}$ special Schubert class of $G_{n,k}$. The celebrated Giambelli formula [40, 1902], expressing an arbitrary Schubert class $s_w$ on $G_{n,k}$ as a determinant (i.e. a polynomial) in the special ones, can be praised as the beginning of the idea of Schubert polynomials.

In general, suppose that $G/P$ is a flag manifold for which a solution (4.4) to Weil’s problem has been available. Then, every Schubert class $s_w$ can be expressed as a polynomial in the generators $y_1,\ldots,y_n$. Inspired by the Giambelli formula, we may call the generators $y_1,\ldots,y_n$ the special Schubert classes of $G/P$, and ask the question of expressing an arbitrary Schubert class $s_w$ on $G/P$ as a polynomial $G_w$ in the special ones, where $\deg G_w = 2l(w)$. In particular, starting from Borel’s presentation of the cohomology ring of the flag manifold $U(n)/T$ [6]

$$H^*(U(n)/T) = \mathbb{Z}[x_1,\ldots,x_n]/\langle e_1,\ldots,e_n \rangle,$$

where $e_1,\ldots,e_n$ are the elementary symmetric polynomials in $x_1,\ldots,x_n$, Lascoux and Schützenberger [52] proposed a solution to the problem by the following rules:

i) Take the generator $G_0 = x_1^{n-1}x_2^{n-2}\cdots x_{n-1} \in H^{n(n-1)}(U(n)/T) = \mathbb{Z}$ as the top degree Schubert polynomial.

ii) Define $G_w := D_w G_0$, where $w_0$ denotes the longest element of the Weyl group $W(U(n))$, and where $D_u$ is the divided difference operator associated to $u \in W(U(n))$ [4].

Strictly speaking, Borel’s generators $x_1,\ldots,x_{n-1}$ are the simple roots of the group $SU(n)$, rather than the fundamental dominant weights (i.e. the special Schubert classes of $U(n)/T$). However, this deficiency is not serious, because the linear transformation between these two sets of generators of the ring $H^*(U(n)/T)$ is given by the Cartan matrix of $U(n)$ [22, Lemma 2.3]. Extending the work of Lascoux-Schützenberger, the Schubert polynomials of the complete flag manifolds $G/T$ have been defined by Billey-Haiman [5] for $G = Spin(n), Sp(n)$, and independently by Fomin-Kirillov [35] for $G = Spin(2n+1)$.
In addition, as early as in 1974 Marlin \cite{54} has determined the ring $H^*(G/T)$ for $G = \text{Spin}(n)$ in the context of Schubert calculus.

In the traditional approach to Schubert polynomials, there are two necessary prerequisites (e.g. \cite{5}):

a) Identify the ring $H^*(G/P)$ explicitly in terms of a system of special Schubert classes $y_1, \ldots, y_n$ on $G/P$;

b) Specify a polynomial $G_0$ in $y_1, \ldots, y_n$ representing the top degree Schubert class of $G/P$.

Nevertheless, granted with Characteristics, we have alternatively a linear algorithm to achieve Schubert polynomials, without resorting to a top degree Schubert polynomial $G_0$, and to the complicated operators $D_w$.

**Algorithm IV: Schubert polynomials**

**Input:** A set \{\(y_1, \ldots, y_n\)\} of special Schubert classes on $G/P$ and an integer $m > 0$.

**Output:** a Schubert polynomial $G_w$ for $w \in W^m(H; G)$.  

We clarify the details of Algorithm IV. Let $\mathbb{Z}[y_1, \ldots, y_n]^{(m)}$ be the group of the polynomials of degree $m$ in the special Schubert classes $y_1, \ldots, y_n$, and examine the map

$$
\pi_m : \mathbb{Z}[y_1, \ldots, y_n]^{(m)} \to H^{2m}(G/P)
$$

induced by the inclusion $y_1, \ldots, y_n \in H^*(G/T)$. Let $B(m) := \{y^{\alpha_1}, \ldots, y^{\alpha_{b(m)}}\}$ be the monomial basis of the group $\mathbb{Z}[y_1, \ldots, y_n]^{(m)}$, and recall by (4.1) that the Schubert basis of the group $H^{2m}(G/P)$ is $\{s_{m,1}, \ldots, s_{m,\beta(m)}\}$. Since each $y^\alpha \in B(m)$ is a monomial in the special Schubert classes, Algorithm II is applicable to expand it linearly in $\{s_{m,1}, \ldots, s_{m,\beta(m)}\}$ to get a $b(m) \times \beta(m)$ matrix $M(\pi_m)$ that satisfies the linear system

$$
\begin{pmatrix}
y^{\alpha_1} \\
\vdots \\
y^{\alpha_{b(m)}}
\end{pmatrix} = M(\pi_m)
\begin{pmatrix}
s_{m,1} \\
\vdots \\
s_{m,\beta(m)}
\end{pmatrix}.
$$

Moreover, since the map $\pi_m$ surjects, the matrix $M(\pi_m)$ has a $\beta(m) \times \beta(m)$ minor equal to \(\pm1\). Thus, the standard integral row and column operation diagonalizing $M(\pi_m)$ \cite[p.162-164]{57} provides us with two invertible matrices $P = P_{b(m) \times b(m)}$ and $Q = Q_{\beta(m) \times \beta(m)}$ satisfying the relation

$$
(4.8) \quad PM(\pi_m)Q = \begin{pmatrix} I_{\beta(m)} \\ C \end{pmatrix}_{b(m) \times \beta(m)},
$$

where $I_{\beta(m)}$ denotes the identity matrix of rank $\beta(m)$. Summarizing, Algorithm IV can be realized by following procedure.
Step 1. Compute $M(\pi_m)$ using the Characteristics;

Step 2. Diagonalize $M(\pi_m)$ to get the matrices $P$ and $Q$ in (4.8);

Step 3. Set
\[
\begin{pmatrix}
G_{m,1} \\
\vdots \\
G_{m,\beta(m)}
\end{pmatrix} := Q \cdot [P] \begin{pmatrix}
y^{n_1} \\
\vdots \\
y^{n_{\beta(m)}}
\end{pmatrix},
\]

where $[P]$ is the $\beta(m) \times \beta(m)$ matrix formed by the first $\beta(m)$ rows of $P$. It is clear that

**Theorem 4.11.** The polynomial $G_{m,k}(y_1, \ldots, y_n)$ is a Schubert polynomial of the Schubert class $s_{m,k}$, $1 \leq k \leq \beta(m)$.

**Example 4.12.** Algorithm IV is suitable for those flag manifolds $G/P$ whose top degree Schubert polynomials are absent (or in question).

For the exceptional Lie group $G = E_n$ with $n = 6, 7, 8$ the parabolic subgroup $P_{(2)}$ has a canonical $n$-dimensional complex representation, that gives rise to the canonical complex $n$-bundle $\xi_n$ on the flag manifold $E_n/P_{(2)}$. According to Borel-Hirzebruch [8, Section 10] the Chern classes $c_i(\xi_n)$ can be expressed as a (rather lengthy) polynomial in positive roots of the group $E_n$. However, with respect to the special Schubert classes of $E_n/P_{(2)}$ specified in the table

| $y_i$ | $E_n/P_{(2)}$, $n = 6, 7, 8$ |
|-------|-------------------------------|
| $y_1$ | $s_{(2)}$, $n = 6, 7, 8$     |
| $y_3$ | $s_{(5,4,2)}$, $n = 6, 7, 8$ |
| $y_4$ | $s_{(6,5,4,2)}$, $n = 6, 7, 8$ |
| $y_5$ | $s_{(7,6,5,4,2)}$, $n = 7, 8$ |
| $y_6$ | $s_{(1,3,6,5,4,2)}$, $n = 6, 7, 8$ |
| $y_7$ | $s_{(1,3,7,6,5,4,2)}$, $n = 7, 8$ |
| $y_8$ | $s_{(1,3,8,7,6,5,4,2)}$, $n = 8$ |

we get from Algorithm IV the following concise expressions of the Chern classes $c_i(\xi_n)$ as polynomials in the special Schubert classes:

| $c_i$ | $E_6/P_{(2)}$  | $E_7/P_{(2)}$  | $E_8/P_{(2)}$  |
|-------|----------------|----------------|----------------|
| $c_1$ | $3y_1$         | $3y_1$         | $3y_1$         |
| $c_2$ | $4y_1^2$       | $4y_1^2$       | $4y_1^2$       |
| $c_3$ | $2y_1 + 2y_1^2$ | $2y_1 + 2y_1^2$ | $2y_1 + 2y_1^2$ |
| $c_4$ | $3y_4 + y_4^2$ | $3y_4 + y_4^2$ | $3y_4 + y_4^2$ |
| $c_5$ | $3y_1y_4 - 2y_1^3y_3 + y_4^2$ | $2y_5 + 3y_1y_4 - 2y_1^3y_3 + y_4^2$ | $2y_5 + 3y_1y_4 - 2y_1^3y_3 + y_4^2$ |
| $c_6$ | $y_6$          | $y_6 + 2y_1y_5$ | $y_6 + 2y_1y_5$ |
| $c_7$ | $0$            | $y_7$          | $y_7 + 4y_1y_6 + 2y_1y_5 + 4y_1^2y_5 - 6y_1y_4 + 4y_1^2y_3 - 2y_1^3$ |
| $c_8$ | $0$            | $0$            | $y_8$          |

By the way, the Schubert polynomials of the flag manifold $E_6/P_{(2)}$ in degrees $m = 8, 9$ are
Remark 4.13. Presumably, Schubert polynomials may be useful to compute the characteristics numbers. However, in our approach the Schubert’s characteristics is a preliminary step toward Schubert polynomials (e.g. Algorithm IV).

Currently, the theory of Schubert polynomials is a powerful tool for discovering combinatorial structures of the Littlewood-Richardson coefficients [17, 10, 11], and is essential to the geometric topic of degeneracy loci of maps between vector bundles [39, 66]. There have also been extensive studies of Schubert polynomials in quantum cohomology [36], and in the $K$-theory of flag manifolds. For recent progresses of this branch of contemporary Schubert calculus, we refer to the articles Kirillov-Narus [48], and Smirnov-Tutubalina [65].

4.6 Applications to the topology of homogeneous spaces

For a compact Lie group $G$ with a closed subgroup $H$ the quotient space $G/H$ is called a homogeneous space of $G$. In contrast to the flag manifolds the cohomology of a homogeneous space may be nontrivial in odd degrees, and may contain torsion elements.

A classical problem of topology is to express the cohomology of a Lie group $G$, or a homogeneous space $G/H$, by a minimal system of explicit generators and relations. The traditional approaches due to H.Cartan, A.Borel, P.Baum and H.Toda utilize various spectral sequence techniques [3, 6, 49, 67, 72], and the calculation encounters the same difficulties when applied to a Lie group $G$ whose integral cohomology has torsion elements, in particular when $G$ is one of the exceptional Lie groups.

Schubert calculus makes the cohomology theory of homogeneous spaces appearing in a new light. For examples, inputting the formulae (4.5) – (4.7) into the second page of the Serre spectral sequence of the fibration $G \to G/T$

$$E_2^{2*}(G) = H^*(G/T) \otimes H^*(T),$$

the integral cohomology $H^*(G)$, as well as the Hopf algebra structure on the mod $p$ cohomology $H^*(G; \mathbb{Z}_p)$, has been determined by computing with the Schubert classes on $G/T$ [26, 31]. For more examples of the extension of Schubert calculus to computing with the homogeneous spaces, see in [27, Sect.5].
5 Concluding remarks

Throughout the ages a common hope of geometers is to find calculable mechanisms among the geometric entities they are caring of (e.g. algebraic varieties, cellular complexes, vector bundles, or the cobordism classes of smooth manifolds). The emergence of Schubert’s calculus, or the birth of the intersection theory, had catered to this demand. Today, Schubert’s calculus has widely integrated into many branches of mathematics, and has profoundly affected the trajectories of the development of those fields, such as the theory of characteristic classes [55], the string theory [44], and algebraic combinatorics [38]. All of these vigorously witnessed Hilbert’s broad vision and foresight, and at the same time, put forward the essential request to explore effective rules performing the computation.

Subject to the plan this article recalled the earlier studies on Schubert calculus, presented a resolution of the characteristics, and illustrated a passage from the Cartan matrices of Lie groups to the intersection theory of flag manifolds, in which the characteristics play a central role. For the historic significance and rigorous treatments of the enumerative examples of Schubert mentioned in Section 3, we refer to the survey articles S. Kleiman [45, 47], or the relevant sections of the books Fulton [37] and Eisenbud-Harris [33] on intersection theory. For other computer systems that may be used to perform certain computations in the intersection rings of flag manifolds, see for example Nikolenko-Semenov [56] (the package ChowMaple06), Grayson et al. [41, 42] (the package Schubert2 in Macaulay2), and Decker et al. [32] (the library Schubert in SINGULAR).

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References

[1] Atiyah M., Hirzebruch F., Vector bundles and homogeneous spaces, 1961 Proc. Sympos. Pure Math., Vol. III pp. 7–38.
[2] Baker H. F., Principles of geometry, Vol. 6, Cambridge, 1936.
[3] Baum P., On the cohomology of homogeneous spaces, Topology 7(1968), 15–38.
[4] Bernstein I. N., Gel’fand I. M., Gel’fand S. I. Schubert cells and cohomology of the spaces $G/P$, Russian Math. Surveys, 28: 3 (1973), 1–26.
[5] S. Billey and M. Haiman, Schubert polynomials for the classical groups, J. Amer. Math. Soc. 8 (1995), no. 2, 443–482.
[6] Borel A., Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de Lie compacts, Ann. Math. 57(1953), 115–207.
[7] Borel A., Kählerian coset spaces of semisimple Lie groups, Proc. Nat. Acad. Sci. U.S.A. 40(1954), 1147-1151.

[8] Borel A., Hirzebruch F., Characteristic classes and homogeneous space I, Amer. J. Math., vol 80 (1958), 458–538.

[9] Bourbaki N., Elements de mathematique, Chapitre I-Chapitre VIII, Groupes et algebres de Lie, Hermann, Paris, 1960-1975.

[10] Buch, A. S., Mutations of puzzles and equivariant cohomology of two-step flag varieties. Ann. of Math. (2)182(2015), no. 1, 173–220.

[11] Buch A.S., Kresch A., Purbhoo K., Tamvakis H., The puzzle conjecture for the cohomology of two-step flag manifolds, J. Algebraic Combin. 44 (2016), no. 4, 973–1007.

[12] Boyer C.B., A History of Mathematics, Wiley, New York, 1968.

[13] Coolidge J.L., A history of geometrical methods, Oxford Univ. press, 1940.

[14] Chaput P.E., Perrin N., Towards a Littlewood-Richardson rule for Kac-Moody homogeneous spaces, J. Lie Theory 22(2012), no. 1, 17–80.

[15] Chasles M., Construction des coniques qui satisfont à cinque conditions, C. R. Acad. Sci. Paris, 58(1864), 297–308.

[16] Chevalley C., Sur les Decompositions Cellulaires des Espaces G/B, Proc. Symp. in Pure Math. 56 (part 1) (1994), 1–26.

[17] Coskun I., A Littlewood-Richardson rule for two-step flag varieties. Invent. Math. 176 (2009), no. 2, 325–395.

[18] Duan H., The degree of a Schubert variety, Adv. in Math., 180(2003), 112–133.

[19] Duan H., Multiplicative rule of Schubert classes, Invent. Math. 159(2005), no. 2, 407–436.

[20] Duan H., Multiplicative rule in the Grothendieck cohomology of a flag variety, J. Reine Angew. Math. 600 (2006), 157–176.

[21] Duan H., Zhao X., Erratum: Multiplicative rule of Schubert classes, Invent. Math.; 177(2009), no.3, 683–684.

[22] Duan H., On the Borel transgression in the fibration $G \to G/T$, Homology, Homotopy and Applications, Vol.20(1)(2018), 79–86.

[23] Duan H., Li B., Topology of Blow-ups and Enumerative Geometry, arXiv:0906.4152.

[24] Duan H., Zhao X., Algorithm for multiplying Schubert classes, Internat. J. Algebra Comput. 16(2006), no.6, 1197–1210, arXiv:math/0309158.
[25] Duan H., Zhao X., A unified formula for Steenrod operations in flag manifolds. Compos. Math. 143(1), (2007), 257–270.

[26] Duan H., Zhao X., Schubert calculus and cohomology of Lie groups, Part I. the 1-connected Lie groups, arXiv:0711.2541.

[27] Duan H., Zhao X., The Chow rings of generalized Grassmannians, Found. Math. Comput. 10(2010), no.3, 245–274.

[28] Duan H., Zhao X., Appendix to The Chow rings of generalized Grassmannians, arXiv: math.AG/0510085.

[29] Duan H., Zhao X., Schubert presentation of the cohomology ring of flag manifolds $G/T$, LMS J. Comput. Math. 18(2015), no.1, 489–506.

[30] Duan H., Zhao X., On Schubert’s Problem of Characteristics, in Schubert calculus and its applications in Combinatorics and representation theory, Springer proceedings in Mathematics and Statistics, Vol. 332(2020), 43–71.

[31] Duan H., Zhao X., Schubert calculus and the Hopf algebra structures of exceptional Lie groups, Forum. Math. Vol.26, no.1(2014), 113–140.

[32] Decker W., Greuel G.-M., Pfister G., and Schönemann H., “SINGULAR 4-0-2-A computer algebra system for polynomial computations”, 2015. Available at http://singular.uni-kl.de.

[33] Eisenbud D., Harris J., 3264 and all that: A Second Course in Algebraic Geometry, Cambridge University Press, 2016.

[34] Ehresmann C., Sur la topologie de certains espaces homogenes, Ann. of Math. (2) 35 (1934), 396–443.

[35] Fomin S., Kirillov A. N., Combinatorial Bn-analogues of Schubert polynomials. Trans. Amer. Math. Soc. 348 (1996), no. 9, 3591–3620.

[36] Fomin S.; Gelfand S., Postnikov A., Quantum Schubert polynomials, J. Amer. Math. Soc., Vol.10 no. 3(1997), 565–596.

[37] Fulton W., Intersection theory, Springer-Verlag, 1998.

[38] Fulton W., Young tableaux, Cambridge Univ. Press, 1997.

[39] Fulton W., Pragacz P., Schubert varieties and degeneracy loci, Lecture Notes in Math. 1689, Springer-Verlag, 1998.

[40] Giambelli G., Risoluzione del problema degli spazi secanti, Mem.R.Acc. Torino (2), 52(1902), 171–211.

[41] Grayson D., Seceleanu A., Stillman M., Computations in intersection rings of flag bundles, 2012. Available at http://arxiv.org/abs/1205.4190.
[42] Grayson D., Stillman A., "Macaulay2: a software system for research in algebraic geometry", 2015. Available at http://math.uiuc.edu/Macaulay2.

[43] Hilbert D., Mathematische Probleme, Bull. AMS. 8(1902), 437–479.

[44] Katz, S., Enumerative Geometry and String Theory, Student Mathematical Library, 32, American Mathematical Society, 2006.

[45] Kleiman S., Problem 15: Rigorous foundation of Schubert’s enumerative calculus, Proc. Symp. Pure Math., 28, Amer. Math. Soc. (1976), 445–482.

[46] Kleiman S., Book review on “Intersection Theory by W. Fulton”, Bull. AMS, 12(1985), no.1, 137–143.

[47] Kleiman S., Intersection theory and enumerative geometry: A decade in review, Proc. Symp. Pure Math., 46:2, Amer. Math. Soc. (1987), 321–370.

[48] Kirillov A. N., Naruse H., Construction of double Grothendieck polynomials of classical types using IdCoxeter algebras, Tokyo J. Math. Volume 39, Number 3 (2017), 695–728.

[49] Husemoller D., Moore J., Stasheff J., Differential homological algebra and homogeneous spaces, J. Pure Appl. Algebra 5 (1974), 113–185.

[50] Lefschetz S., Intersections and transformations of complexes and manifolds, Trans. AMS, Vol.28, no.1 (1926), 1–49.

[51] Littlewood D. E., Richardson A.R., Group characters and algebra, Philos. Trans. Royal Soc. London., 233 (1934), 99–141.

[52] Lascoux L., Schützenberger M.P., Polynomes de Schubert. C. R. Acad. Sci. Paris Ser. I Math., 294(13):447–450, 1982.

[53] Manin Ju I., On Hilbert fifteenth problem, Hilbert’s problems (Russian), Izdat. “Nauka”, Moscow (1969), 175–181.

[54] Marlin R., Anneaux de Chow des groupes algébriques $SU(n), Sp(n), SO(n)$, $Spin(n), G_2, F_4$, C. R. Acad. Sci. Paris, A 279(1974), 119–122.

[55] Milnor J., Stasheff J., Characteristic classes, Ann. of Math. Studies 76, Princeton Univ. Press, 1975.

[56] Nikolenko S. I., Semenov N.S., Chow ring structure made simple, arXiv: math.AG/0606335.

[57] Scherk J., Algebra. A computational introduction, Studies in Advanced Mathematics, Chapman-Hall/CRC, Boca Raton, FL, 2000.

[58] Schubert H., Zur Theorie der Charakteristike, Celles Journ. 71(1870), 366–386.
[59] Schubert H., Kalkül der abzählenden Geometrie, Berlin, Heidelberg, New York: Springer-Verlag (1979).

[60] Schubert H., Anzahlbestimmungen für lineare Räume beliebiger Dimension, Acta Math., 8 (1886), 97–118.

[61] Schubert H., a Lösung des Characteristiken-Problems für lineare Räume beliebiger Dimension, Mitteilungen der Mathematische Gesellschaft in Hamburg 1 (1886), 134–155.

[62] Severi F., Il Principio della Conservazione del Numero, Rendiconti Circolo Mat. Palermo 33, 313 (1912).

[63] Severi F., Sui fondamenti della geometria numerativa e sulla teoria delle caratteristiche, Atti Ist. Veneto, 75 (1916), 1121–1162.

[64] Sottile, F., Schubert calculus, Springer Encyclopedia of Mathematics, 2001.

[65] Smirnov E., Tutubalina A., Pipe dreams for Schubert polynomials of the classical groups, Math: arXiv 2009.14120.

[66] Tamvakis H., Giambelli and degeneracy locus formulas for classical $G/P$ spaces. Mosc. Math. J. 16 (2016), no. 1, 125–177.

[67] Toda T., On the cohomology ring of some homogeneous spaces, J. Math. Kyoto Univ. 15(1975), 185–199.

[68] Yvonne, D. S., Interview with Bartel Leendert van der Aaerden, Notice of the AMS, Vol.44, No.3 (1997), 313–321.

[69] van der Waerden B L., The foundation of algebraic geometry from Severi to André Weil, Archive for History of Exact Sciences, 1971.

[70] van der Waerden B L., Topologische Begründung des Kalküls der abzählenden Geometrie, Math. Ann. 102 (1930), no. 1, 337–362.

[71] Weil A., Foundations of algebraic geometry, American Mathematical Society, Providence, R. I. 1962.

[72] Wolf J., The cohomology of homogeneous spaces, Amer. J. Math. 99 (1977), no. 2, 312–340.

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