Modified Exponential Weighted Moving Average (EWMA) Control Chart on Autocorrelation Data

Erna Tri Herdiani, Geysa Fandrilla, Nurtiti Sunusi
Study Program Statistics, Faculty Mathematics and Natural Sciences, Hasanuddin University
E-mail: herdiani.erna@gmail.com

Abstract. In general, observations of the statistical process control are assumed to be mutually independence. However, this assumption is often violated in practice. Consequently, statistical process controls were developed for interrelated processes, including Shewhart, Cumulative Sum (CUSUM), and exponentially weighted moving average (EWMA) control charts in the data that were autocorrelation. One researcher stated that this chart is not suitable if the same control limits are used in the case of independent variables. For this reason, it is necessary to apply the time series model in building the control chart. A classical control chart for independent variables is usually applied to residual processes. This procedure is permitted provided that residuals are independent. In 1978, Shewhart modification for the autoregressive process was introduced by using the distance between the sample mean and the target value compared to the standard deviation of the autocorrelation process. In this paper we will examine the mean of EWMA for autocorrelation process derived from Montgomery and Patel. Performance to be investigated was investigated by examining Average Run Length (ARL) based on the Markov Chain Method.

1. Introduction
In general, observations of the statistical process control are assumed to be mutually independence. However, this assumption is often violated in practically. Consequently, statistical process control observations developed for autocorrelation processes include Shewhart control charts, Cumulative Sum (CUSUM), and exponentially weighted moving average (EWMA)[1, 2]. State that this chart is not suitable if the same control limits are used in the case of independent variables [3]. For this reason, it is necessary to apply the time series model in building the control chart.

A classical control chart for independent variables is usually applied to residual processes. This procedure is permitted provided that residuals are independent. In 1978, [4] and Stamboulis introduce shewhart modifications to the autoregressive process. The distance between the sample mean and the target value is compared to the standard deviation of the autocorrelation process.

In the year of 1991, [5] has discussed the autocorrelation process for the Cumulative Sum (CUSUM) control chart, [6]) has examined the mean of EWMA for autocorrelation process,and Patel (2011) perfectly by searching for variance, control chart and ARL from EWMA with autocorrelation process. The patel formulation hereinafter referred to as patel modification, whose contents combine Shewart and EWMA control charts that are useful for detecting small variables of industrial processes that have data with a single-order autocorrelation process. Statistics proposed by adding values in EWMA statistics. The result shows that Patel modification is more sensitive than EWMA in autocorrelation process, where the formulation of EWMA used is in [2]. In this paper, we will propose
another modification of EWMA from Patel modification, which is then referred to as the Proposed modification. The result will be compared to modification of Patel and Montgomery.

2. EWMA Control Chart

EWMA control chart was first introduced by Roberts in 1959. EWMA control chart is optimal for the process with the mean in the period t associated with the mean in the period \((t-1)\) \[2\]. EWMA is also one of the Shewart control charts that are widely applied especially in case of time series \[2\]. EWMA control chart is a graphic alternative to the Shewart controller in detecting small shifts in the mean process. Specifically EWMA is used on individual observations. It is assumed observation of the process on the variable \(X\) distributed normal with mean \(\mu\) and variance \(\sigma^2\). Defined EWMA controls chart accordingly \[2\] to be:

\[
Z_t = \lambda X_t + (1 - \lambda)Z_{t-1}
\]

With \(Z_t\) = average from historical data, \(\lambda\) = weighted parameters that are worth between 0 and 1, \(X_t\) = value of observation to \(t\), \(t = 1, 2, \ldots\), \(Z_{t-1}\) = target. Upper control limit (UCL) and lower control limit (LCL) EWMA control chart as follows.

\[
UCL = \mu_0 + L\sigma \sqrt{\frac{\lambda}{2 - \lambda}} (1 - (1 - \lambda)^{2t})
\]

\[
CL = \mu_0
\]

\[
LCL = \mu_0 - L\sigma \sqrt{\frac{\lambda}{2 - \lambda}} (1 - (1 - \lambda)^{2t})
\]

With: \(\mu_0\) = average, \(\sigma\) = standard deviation, \(L\) = width of the control limit, \(\lambda\) = weighting parameters, \(t = 1, 2, \ldots\)

3. Modification EWMA Patel Control Chart

Modification EWMA control chart developed by Patel in the year 2011. The properties of the control chart are easy to apply and effective for detecting shifts of all sizes as per specifications. Patel combines Shewart and EWMA control charts that are useful for detecting small changes from industrial processes that have data with a single-order autocorrelation process. Statistics proposed by adding values in EWMA statistics. EWMA modification control chart according to Patel (2011) is:

\[
Z_t = \lambda X_t + (1 - \lambda)Z_{t-1} + (X_t - X_{t-1})
\]

With \(Z_t\) = average from historical data, \(\lambda\) = weighted parameters that are worth between 0 and 1, \(X_t\) = value of observation to \(t\), \(t = 1, 2, \ldots\), \(X_{t-1}\) = value of previous observations, \(Z_{t-1}\) = target. Upper control limit (UCL) and lower control limit (LCL) of modification EWMA Patel is:

\[
UCL = \mu_0 + L\sigma \sqrt{\frac{\lambda}{2 - \lambda} + \frac{2\lambda(1 - \lambda)}{2 - \lambda}}
\]

\[
CL = \mu_0
\]

\[
LCL = \mu_0 - L\sigma \sqrt{\frac{\lambda}{2 - \lambda} + \frac{2\lambda(1 - \lambda)}{2 - \lambda}}
\]
With $\mu_0 =$ average, $\sigma =$ standard deviation, $L =$ width of control limit, $\lambda =$ weighted parameter.

4. Modification EWMA Proposed Control Chart

Its Known autocorrelation process order 1 or AR (1) is as follows

$$\bar{x}_t = \phi x_{t-1} + \epsilon_t$$

Modified EWMA patel is an ewma control chart with error following AR Process (1) which has value $\phi = 1$, the result can be seen in equation (2). Meanwhile, the proposed ewma modification is to add the process AR (1) to the data in the EWMA process. The advantages of EWMA modification proposed, the value of $\phi$ involved wider is $-1 \leq \phi \leq 1$, while modification EWMA patel only in the state $\phi = 1$. The results of the proposed EWMA modification are as follows:

$$Z_t = \lambda \bar{x}_t + (1 - \lambda)Z_{t-1}$$
$$= \lambda(\phi x_{t-1} + \epsilon_t) + (1 - \lambda)Z_{t-1}$$
$$= \lambda\phi x_{t-1} + \lambda\epsilon_t + \lambda(1 - \lambda)\phi x_{t-1} + \lambda(1 - \lambda)\epsilon_{t-2}$$
$$+ \lambda(1 - \lambda)^2\phi x_{t-2} + \lambda(1 - \lambda)^2\epsilon_{t-2} + (1 - \lambda)^3Z_{t-3}$$

If set again for $Z_{t-k}, k = 4, 5, 6, ..., t$, so that $Z_t$ can be expressed by

$$Z_t = \lambda\phi x_t + \lambda\phi(1 - \lambda)x_{t-1} + \lambda\phi(1 - \lambda)^2x_{t-2} + ...$$
$$+ \lambda\phi(1 - \lambda)^{t-2}x_{t(t-2)} + \lambda\phi(1 - \lambda)^{t-1}x_{t(t-1)} + (1 - \lambda)^tZ_{t-t}$$
$$= \lambda\phi x_t + \lambda\phi(1 - \lambda)x_{t-1} + \lambda\phi(1 - \lambda)^2x_{t-2} + ...$$
$$+ \lambda\phi(1 - \lambda)^{t-2}x_2 + (1 - \lambda)^{t-1}x_1 + (1 - \lambda)^tZ_0$$

Simply to $t = 1, 2, 3, ..., Z_t$ can be expressed by

$$Z_t = \lambda\phi \sum_{k=0}^{t-1}(1 - \lambda)^k x_{t-k} + \lambda(1 - \lambda)^k \epsilon_{t-k} + (1 - \lambda)^tZ_0$$

In this chapter, we will determine the mean and variance of $Z_t$. Each subgroup, the average subgroup value is actually expected to be a value $Z_0$ or mathematically means for each subgroup $i, i = 1, 2, ..., (t - 2), (t - 1), t$ for $t = 1, 2, 3, ...$ then $E[x_i] = \mu_0$. Because, $E[x_1] = E[x_2] = ... = E[x_{t-1}] = E[x_t] = \mu_0$ and $Z_0$ is value of expectation. So, $E[Z_0] = \mu_0$ part $\sum_{k=0}^{t-1}(1 - \lambda)^k$ to be geometric series with initial value $a = 1$, comparison of the second value with initial value to be $r = (1 - \lambda)$, and number of values $n = t$. So obtained,

$$\sum_{k=0}^{t-1}(1 - \lambda)^k = \frac{a(1-r^n)}{1-r} = \frac{\Phi[1-(1-\lambda)^t]}{1-(1-\lambda)} = \frac{[1-(1-\lambda)^t]}{\lambda}$$

Therefore,

$$E(Z_t) = E \left( \lambda \sum_{k=0}^{t-1}(1 - \lambda)^k x_{t-k} + (1 - \lambda)^tZ_0 \right) = \mu_0$$
So the mean value of EWMA to autocorrelation process is $\mu_0$. After getting the centralization measure $Z_t$, next will be searched for variability value $Z_t$, namely variance. If for each $x_t$, $t = 1, 2, 3, ..., $ independence and has a standard deviation $\sigma_x$. Then variance of $Z_t$ is

$$
\sigma^2_{Z_t} = \text{Var}(Z_t)
$$

$$
= \text{Var}\left( \lambda \phi \sum_{k=0}^{t-1} (1 - \lambda)^k x_{t-k} \right) + \text{Var}((1 - \lambda)^t Z_0)
$$

$$
+ 2\text{cov}(\lambda \phi \sum_{k=0}^{t-1} (1 - \lambda)^k x_{t-k}, (1 - \lambda)^t Z_0)
$$

Because, $\text{Var}(c) = c$ for c is constant then $\text{Var}((1 - \lambda)^t Z_0) = 0$ and for each $\bar{x}_t$ independence, then

$$
\text{cov}\left( \lambda \phi \sum_{k=0}^{t-1} (1 - \lambda)^k x_{t-k}, (1 - \lambda)^t Z_0 \right) = 0
$$

Furthermore,

$$
\sigma^2_{\hat{x}_t} = \text{Var}\left( \lambda \phi \sum_{k=0}^{t-1} (1 - \lambda)^k x_{t-k} \right)
$$

$$
= \lambda^2 \phi \sum_{k=0}^{t-1} (1 - \lambda)^{2k} \sigma^2_{\hat{x}_t}
$$

Part $\sum_{k=0}^{t-1} (1 - \lambda)^{2k}$ is a geometric series with initial value $a = 1$, comparison of second value with initial value is $r = (1 - \lambda)^2$, and number of value is $n = t$. So obtained

$$
\sum_{k=0}^{t-1} (1 - \lambda)^{2k} = \frac{a(1-r^n)}{1-r} = \frac{1-(1-\lambda)^{2t}}{2(1-\lambda)^2}
$$

So that,

$$
\sigma^2_{\hat{x}_t} = \lambda^2 \phi \sum_{k=0}^{t-1} (1 - \lambda)^{2k} \sigma^2_{\hat{x}_t} = \lambda^2 \phi \sigma^2_{\hat{x}_t} \frac{1-(1-\lambda)^{2t}}{2(1-\lambda)^2}
$$

Let $\sigma^2_{\hat{x}_t} = \frac{\sigma^2}{n}$ then $\sigma^2_{Z_t} = \frac{\lambda \phi \sigma^2}{n} \left( \frac{1-(1-\lambda)^{2t}}{2(1-\lambda)^2} \right)$.

So the value of the variance of EWMA is as follows:

$$
\sigma^2_{Z_t} = \frac{\lambda \phi \sigma^2}{n} \left( \frac{1-(1-\lambda)^{2t}}{2(1-\lambda)^2} \right)
$$

**Lower Control Limit (LCL)** and **Upper Control Limit (UCL)** to Modification EWMA control chart with autocorrelation data is
\[ UCL = \mu_0 - 3\sigma \sqrt{\frac{\lambda \phi}{n} \left( \frac{1 - (1 - \lambda)^{2t}}{2 - \lambda} \right)} \]

\[ CL = \mu_0 \]

\[ LCL = \mu_0 + 3\sigma \sqrt{\frac{\lambda \phi}{n} \left( \frac{1 - (1 - \lambda)^{2t}}{2 - \lambda} \right)} \]

Modification EWMA control chart on autocorrelation data, hereinafter referred to as Modification EWMA Proposed Control Chart. In the next section, we will determine the value of the average run length (ARL) to see which control charts are more sensitive in detecting the mean changes in the data.

5. Calculation of ARL Value

Calculation of ARL value will use the markov chain method as contained in the paper that has been written by [7] and [8]. The sequence is as follows:

1. It is known that the mean value of the data is value \( a = 0 \), the standard deviation value of the data is \( b = 1 \), \( n \) denotes the number of observations with \( n = 1 \), \( p \) addresses the selected sub interval of 5, and \( \lambda \) represents the weight by taking \( \lambda = 0.02 \). Based on the information obtained \( \delta \) value as follows:

\[ \delta = \frac{UCL - LCL}{2p} = \frac{5.4414 - 4.9266}{2(5)} = \frac{0.5148}{10} = 0.0515 \]

2. The matrix transition will be determined by determining the matrix \( Q \). The matrix elements \( Q \) for rows and columns are marked according to the five sub-intervals selected respectively -2, -1, 0, 1 and 2. Thus the matrix \( Q \) will contain elements as in below:

\[
Q = \begin{pmatrix}
Q_{-2,-2} & Q_{-2,-1} & Q_{-2,0} & 0 & 0 \\
0 & Q_{-1,-1} & Q_{-1,0} & Q_{-1,1} & 0 \\
0 & 0 & Q_{0,0} & Q_{0,1} & Q_{0,2} \\
0 & 0 & 0 & Q_{1,1} & Q_{1,2} \\
0 & 0 & 0 & 0 & Q_{2,2}
\end{pmatrix}
\]

\[
Q = \begin{pmatrix}
0.008 & 0.989 & 0.005 & 0 & 0 \\
0 & 0.006 & 0.990 & 0.008 & 0 \\
0 & 0 & 0.005 & 0.989 & 0.010 \\
0 & 0 & 0 & 0.003 & 0.989 \\
0 & 0 & 0 & 0 & 0.002
\end{pmatrix}
\]

While the elements of \( Q_{-2,-2} \) obtained by means

\[
Q_{-2,-2} = F_N \left( \frac{H_{-2} + \delta - (1 - \lambda)H_{-2}}{\lambda} - a \right) \sqrt{\frac{n}{b}}
\]
\[ F_{N}\left(\frac{H_{-2} - \delta - (1 - \lambda)H_{-2}}{\lambda} - a\right) \frac{\sqrt{n}}{b} \]

\[ Q_{-2,-2} = F_{N}\left(\frac{4.9781 + 0.0515 - (1 - 0.02)4.9781}{0.02} - 0\right) \frac{\sqrt{1}}{1} - F_{N}\left(\frac{4.9781 - 0.0515 - (1 - 0.02)4.9781}{0.02} - 0\right) \frac{\sqrt{1}}{1} \]

\[ Q_{-2,-2} = F_{N}(7.5531) - F_{N}(2.4031) \]

\[ Q_{-2,-2} = 1 - 0.992 = 0.008 \]

3. From the matrix results Q, ARL values can be obtained as follows

\[ ARL = (Q^t + 1)^t \times ((I - Q)^{-1} + 1) \]

Next ARL from each statistic can be seen in Table 4.

**Table 1. ARL Value of EWMA Control Chart**

| \(\lambda\) value | Montgomery | Modification Patel | Modification Proposed |
|-------------------|------------|--------------------|-----------------------|
| 0.05              | 6.4268     | 2.8410e+03         | 1.4519                |
| 0.10              | 3.8301     | 59.7436            | 1.3433                |
| 0.20              | 2.1333     | 46.7014            | 1.2129                |
| 0.50              | 0.2981     | 7.2852             | 0.7586                |
| 0.70              | 0.0781     | 3.4342             | 0.4505                |

Table 1 describe ARL values obtained for modification EWMA proposed control charts is the smallest of the ARL values obtained for the EWMA Montgomery control chart and Modification EWMA Patel control chart, when the \(\lambda\) value is less than 0.5. While for \(\lambda\) is greater or equal with 0.5 then EWMA Montgomery is better used, because its ARL value is the smallest of modification EWMA Patel and modification EWMA proposed.

6. **Conclusion**

Modification EWMA Proposed Control Chart formed by considering the mean and standard deviation values on autocorrelation and standardized normal data. Mean data is \(\mu_0\) and deviation standard is \(\sqrt{n}\left(\frac{1-(1-\lambda)^{2t}}{2-\lambda}\right)\). Based on this, we obtained a limit of control for the autocorrelation data as follows.

\[ UCL = \mu_0 + 3\sigma_{\sqrt{n}} \left(\frac{\lambda\phi}{2-\lambda}\right) \]

\[ CL = \mu_0 \]
$$LCL = \mu_0 - 3\sigma \sqrt{\frac{\lambda \phi \left( \frac{1 - (1 - \lambda)^{2t}}{2 - \lambda} \right)}{n}}$$

ARL value comparison for $\lambda = 0.05$, $\lambda = 0.1$ and $\lambda = 0.2$ then the proposed Modification EWMA is more sensitive than Montgomery or Patel method. While $\lambda = 0.5$ and $\lambda = 0.7$ then Montgomery method is more sensitive than Patel and the proposed.

References

[1] Box G E P, Jenkins G M and MacGregor J F 1974 Some recent advances in forecasting and control *II J Roy Statist Soc Ser C* 23 158-179
[2] Montgomery D C 2009 *Introduction to Statistical Quality Control Sixth edition* (New York: Wiley)
[3] Alwan L C and Roberts H V 1988 Time series modeling for statistical process control *J. Business and Economic Statist* 6 87-95
[4] Vasilopoulos A V and Stamboulis A P 1978 Modification of control chart limits in the presence of data correlation *J. Quality Technology* 10 20-30
[5] Harris T J and Ross W H 1991 Statistical process control procedures for correlated observations *Canadian J. Chemical Engineering* 69 48-57
[6] Lu C W 1999 Control Chart for Monitoring the Mean and Variance of Autocorrelated Processes *Journal of Quality Technology* 259-274
[7] Yang S F, Tsai W C, Huang T M, Yang C C and Cheng S 2011 Monitoring process mean with a new EWMA control chart *Produção* vol 21 no 2 217-222
[8] Chananet C, Sukparungsee S, and Areepong Y 2014 The ARL of EWMA Chart for Monitoring ZINB Model Using Markov Chain Approach *International Journal of Applied Physics and Mathematic* 4