Conformally-modified gravity and vacuum energy

Christian Henke

University of Technology at Clausthal, Department of Mathematics,
Erzstrasse 1, D-38678 Clausthal-Zellerfeld, Germany

Abstract

The paper deals with a modified theory of gravity and the cosmological consequences. Instead of concerning the field equations directly, we modify a conformally-related and equivalent equation, such that a spontaneous symmetry breaking at Planck scale occurs in the trace equation. As the consequence the cosmological constant problem is solved.

1. Introduction

Since the 1990s, the cosmological constant $\Lambda$ in Einstein’s field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad \kappa = \frac{8\pi G}{c^4},$$

has been used as a simple explanation for an expansion of the universe (see \[1, 2, 3, 4\] for further details and \[5\] for notational conventions). Due to the unknown form of the underlying energy, this anti-gravitational mechanism is called dark energy. From the source point of view the cosmological constant can be written as an energy-momentum tensor

$$T_{\mu\nu} = \left(\rho_\Lambda + \frac{p_\Lambda}{c^2}\right) u_\mu u_\nu + p_\Lambda g_{\mu\nu},$$

where $p_\Lambda = -c^4\Lambda/8\pi G$ and $\rho_\Lambda = -p_\Lambda/c^2$. Hence, in the absence of matter the cosmological constant could be interpreted as the energy density of the vacuum. In contrast to the notion of an empty space, the quantum field theory defines the vacuum as the state of lowest energy density. A comparison of both concepts by cosmological observations of $\Lambda$ and theoretical calculations of the quantum energy density uncovers a large discrepancy

$$\frac{\rho_\Lambda}{\rho_{\text{vac}}} \approx 10^{-122},$$

which is “probably the worst theoretical prediction in the history of physics” \[6\].

The purpose of this paper is to modify Einstein’s field equation with the help of ideas from particle physics and demonstrate that the large discrepancy of the energy densities disappears. The concept of minimising the energy in General Relativity has been used
for canonical energy-momentum tensors which are generated from additional scalar fields (see [3, p. 457] and [2]). Unfortunately, this approach results in a devastating fine-tuning problem.

In contrast to other scalar field theories with an additional scalar field, we use a conformally-related metric of an equivalent formulation of Einstein’s field equation. It turns out that the trace is a mass-less Klein-Gordon equation, where the cosmological constant is the coefficient of a quartic interaction term. In order to prepare the ground for the conformal modification of the equation under consideration, the paper starts in section 2 with the presentation of the classical problem in the new framework. In section 3 we introduce the conformally-modified field equation and verify that our modification falls within Einstein’s field theory. In the next section, we develop an analogy to particle physics: Balancing the mass coefficient and the cosmological constant such that the symmetry is spontaneously broken at Planck scale gives automatically an energy density of the same magnitude as the quantum energy density. Finally, we conclude with a validation from cosmological observations.

2. Conformally-related field equation

Let $M$ be a 4-dimensional metric equipped with a metric $\bar{g}_{\mu\nu}$ of signature $(-,+,+,+)$. If $u$ is a strictly positive $C^\infty(M)$--function, then the metric $g_{\mu\nu} = u^{-2} \bar{g}_{\mu\nu}$ is said to be conformally-related to $\bar{g}_{\mu\nu}$. In this section we start from the field equation for the Robertson-Walker metric $\bar{g}_{\mu\nu}$ and consider the transformed equations for the conformally-related metric $g_{\mu\nu}$. Instead of using the metric $\bar{g} = -dt^2 + a(t)^2 \left( \frac{1}{1-kr^2} dr^2 + r^2 d\Omega^2 \right)$, where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and $k$ is a curvature parameter, we carry out the transformation $r \to r \left( 1 + \frac{1}{4} kr^2 \right)^{-1}$ and consider the equivalent metric

$$\bar{g} = -dt^2 + a(t)^2 b(r)^2 \left( dr^2 + r^2 d\Omega^2 \right),$$

$$b(r) = \left( 1 + \frac{1}{4} kr^2 \right)^{-1}.$$ (1)

Here, $k$ has units of length$^{-2}$ and $a(t)$ and $b(r)$ are unitless. Introducing the notation $\nabla_\mu$ for the covariant derivative, $\Delta = g^{\mu\nu} \nabla_\mu \nabla_\nu$ and $|\nabla u|^2 = g^{\mu\nu} \nabla_\mu u \nabla_\nu u$, we note down the relation for the Ricci tensor (see [5, p. 446] and [7, p. 223]):

$$u^2 \bar{R}_{\mu\nu} = u^2 R_{\mu\nu} - 2u \nabla_\mu \nabla_\nu u + 4 \nabla_\mu u \nabla_\nu u - \left( u \Delta u + |\nabla u|^2 \right) g_{\mu\nu}.$$ (2)

Choosing the usual energy-momentum tensor of a spatially-homogeneous perfect fluid

$$\bar{T}_{\mu\nu} = \left( \rho + \frac{p}{c^2} \right) u_\mu u_\nu + pg_{\mu\nu}, \quad \nabla^\mu \bar{T}_{\mu\nu} = 0,$$

Einstein’s field equation can be written as

$$\bar{R}_{\mu\nu} - \Lambda \bar{g}_{\mu\nu} = \kappa \left( \bar{T}_{\mu\nu} - \frac{\bar{T}}{2} \bar{g}_{\mu\nu} \right).$$ (3)
Up to now the above considerations are valid for every \( u \in C^\infty(M) \). Fixing this function, equation (3) is equivalent to the specialised formulation:

**Conformally-related field equation 1.** Find a metric

\[
g = -u^{-2}dt^2 + u^{-2}a(t)b(r)^2 \left(dr^2 + r^2d\Omega^2\right),
\]

such that

\[
L_{\mu\nu} = F_{\mu\nu},
\]

is satisfied for \( u = a(t)b(r) > 0 \), where

\[
L_{\mu\nu} = u^2R_{\mu\nu} - 2u\nabla_\mu \nabla_\nu u + 4\nabla_\mu u \nabla_\nu u
\]

\[- \left(u\Delta u + |\nabla u|^2 + \Lambda u^4\right)g_{\mu\nu},
\]

\[
F_{\mu\nu} = \kappa u^2 \left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right).
\]

Using the 4-velocity vector \( \bar{u}^\mu = (c, 0, 0, 0) \), we find for the right-hand side the identity

\[
\bar{T}_{\mu\nu} - \frac{T}{2}g_{\mu\nu} = (\rho c^2 + p) \delta^0_\mu \delta^0_\nu + \frac{\rho c^2 - p}{2}u^2g_{\mu\nu}.
\]

Next we demonstrate how to derive Friedmann’s equation from equation (4). In order to do that, we introduce the notion of decomposable coordinates (c.f. [7, p. 223]).

**Definition 1.** A space-time \((M, g)\) is called \((2, 2)\)-decomposable, if and only if around any point of \(M\) local coordinates \(x^\mu = x^a, x^i\) can be found, such that

\[
g = g_{ab}(x^0, x^1) dx^a dx^b + g_{ij}(x^2, x^3) dx^i dx^j,
\]

where \(a, b, c, \ldots\) and \(i, j, k, \ldots\) denote restricted indices 0, 1 and 2, 3, respectively. Here, we define \('R = g^{cd}g^{bc}R_{abcd}\). ''\(R = g^{cd}g^{bk}R_{kcdn}\). 'u\(\Delta = g^{cd}\nabla_a \nabla_b\) and 'u\(|\nabla u|^2 = g^{cd}\nabla_a u \nabla_b u\).

Moreover, the following identities are valid

\[
R = 'R(x^0, x^1) + ''R(x^2, x^3), \quad 2R_{ab} = 'R_{gab}, \quad 2R_{ij} = ''R_{gij}.
\]

Hence, the metric from the above conformally-related field equation is \((2, 2)\)-decomposable. Specialising \((\mu, \nu) = (a, b)\) and \((\mu, \nu) = (i, j)\), it holds that

\[
L_{ab} = -2u\nabla_a \nabla_b u + 4\nabla_a u \nabla_b u
\]

\[- \left(u'\Delta u + |\nabla u|^2 - \frac{R}{2}u^2 + \Lambda u^4\right)g_{ab},
\]

\[
F_{ab} = \kappa u^2 \left((\rho c^2 + p) \delta^{0}_a \delta^0_b + \frac{\rho c^2 - p}{2}u^2g_{ab}\right),
\]

\[
L_{ij} = - \left(u'\Delta u + |\nabla u|^2 - \frac{R}{2}u^2 + \Lambda u^4\right)g_{ij},
\]

\[
F_{ij} = \kappa \frac{\rho c^2 - p}{2}u^4 g_{ij}.
\]
Lemma 1. Let \( u \) satisfy the conformally-related field equation \( 1 \) and let
\[
- (\nabla^j u)^2 g^{rr} - (\nabla^r u)^2 g^{tt} \neq 0
\]
be satisfied. Then equation (4) is equivalent to
\[
L_a^a - F_a^a = 0, \quad L_i^i - F_i^i = 0, \quad \nabla^b u (L_{ab} - F_{ab}) = 0, \quad a = t, r.
\]

Proof. Obviously, every solution \( u \) of equation (4) solves (9)-(11). Conversely, let (9)-(11) be satisfied. Using the setting \( S_{ab} = L_{ab} - F_{ab} \) and the symmetry in the lower two indices of the Christoffel symbols, we obtain \( S_{ab} = S_{ba} \). Thus, because of equation (11) we conclude
\[
- (\nabla^j u)^2 S_{tt} + (\nabla^r u)^2 S_{rr} = 0.
\]
Moreover, thanks to the diagonal metric, we write equation (9) as
\[
g^{rr} S_{tt} + g^{rr} S_{rr} = 0.
\]
From (8) it follows that \( S_{tt} = S_{rr} = 0 \). It remains to show \( S_{tr} = S_{rt} = 0 \) which now follows directly from equation (11). Finally, multiplying equation (10) by \( g_{ij}/2 \), we get \( L_{ij} = F_{ij} \).

Analysing the different arguments in the condition of Lemma 1 we see that
\[
- (\nabla^j u)^2 g^{rr} - (\nabla^r u)^2 g^{tt} = a^2 r^4 b^4 \left( (rb' + b)^2 r^{-2} b^{-2} - \dot{a}^2 \right) \neq 0.
\]
is satisfied. Next, for the 2 dimensional traces we have \( L_a^a - F_a^a = 0 \) and \( L_i^i - F_i^i = 0 \). That is
\[
-4 a' \Delta u + 2 |\nabla u|^2 + 'R \Delta u^2 - 2 \Delta u^4 + 2 \kappa u^3 p = 0, \quad -2 a' \Delta u - 2 |\nabla u|^2 + ''R \Delta u^2 - 2 \Delta u^4 - \kappa (\rho c^2 - p) u^4 = 0,
\]
where \( 'R = -2 \) and \( ''R = 2 \). From the definitions we get
\[
' \Delta u = -r^2 a \left( r b^2 \ddot{a} + r ab^3 \dot{a} + \frac{rb'^2}{b} - rb'' - b' \right),
\]
\[
|\nabla u|^2 = a^2 r^2 \left( r^2 \dot{a}^2 b^4 - r^2 b'^2 - 2 r b b' - b'^2 \right).
\]
Now using the expressions \( -12 / 6 r^4 a^2 b^4 + 13 / 3 r^4 a^2 b^4 \) and \( 12 / r^4 a^2 b^4 - 13 / 2 r^4 a^2 b^4 \), we get Friedmann’s equations
\[
\ddot{a}^2 - \frac{1}{3} \Lambda a^2 - \frac{\kappa}{3} a^2 p a^2 = \frac{b'^2}{3b^4} + \frac{b''}{3b^4} + \frac{5b'}{3rb^4} = -k, \quad 3 a\ddot{a} - \Lambda a^2 + \frac{\kappa}{2} (\rho c^2 + 3 p) a^2 = -\frac{2 b'^2}{b^4} + \frac{b''}{b^4} - \frac{b'}{rb^3} = 0.
\]
In order to show that Friedmann’s equation is equivalent to the conformally-related field equation \(1\) we check the identities
\[
\nabla_t u \left( \left( L_a^a - F_a^a \right) - \left( L_l^l - F_l^l \right) / 2 \right) = \nabla_b u \left( L_{lk} - F_{lk} \right),
\]
\[
\nabla_r u \left( L_i^i - F_i^i \right) / 2 = \nabla_b u \left( L_{rb} - F_{rb} \right).
\]
(16)

To do so, we write (16) as
\[
-u \nabla_t \left( |\nabla u|^2 \right) + \left( 2 u' \Delta u - \left( R - ''R \right) u^2 / 2 \right) \nabla_t u = 0,
\]
\[
-u \nabla_r \left( |\nabla u|^2 \right) + \left( 4 |\nabla u|^2 + \left( R - ''R \right) u^2 / 2 \right) \nabla_r u = 0,
\]
(17)

and finally as
\[
r^5 a^2 b^5 \left( -2 b'^2 / b^4 - b'' / r b^3 \right) = 0,
\]
\[
r^4 a^3 b^4 (rb' + b) \left( -2 b'^2 / b^4 - b'' / r b^3 \right) = 0,
\]
which is fulfilled by (15).

In the view of particle physics, a key characteristic of the equivalent formulation of Einstein’s field equation becomes clear when looking at the trace equation of \(4\). Adding the equations (12) and (13) and using equation (7) we get
\[
-6 u' \Delta u + Ru^2 - 4 \Lambda u^4 = \kappa \left( \rho c^2 - 3p \right) u^4,
\]
(18)

which is the Klein-Gordon equation for \(u \neq 0\). The \(\Lambda\)-term represents the quartic interaction term of the related Lagrangian. Concerning the results for \(''R\) and \(''R\), we observe that the Klein-Gordon equation is mass-less.

3. Conformally-modified field equation

In this section we present the conformally-modified gravity equation. The central idea is to introduce an additional mass term in formulation \(1\), such that
- the trace equation is the Klein-Gordon equation with a non-zero mass term;
- spontaneous symmetry breaking occurs at Planck scale.

Namely, we propose the

**Conformally-related field equation 2.** Find a metric
\[
g_{\epsilon} = -u_\epsilon^{-2} dt^2 + u_\epsilon^{-2} C(t, r)^2 \left( dr^2 + r^2 d\Omega \right),
\]
(19)
such that
\[
L_{\mu\nu} = F_{\mu\nu},
\]
(20)
is satisfied for \(u_\epsilon = C(t, r) \epsilon > 0\), where
\[
L_{\mu\nu} = u_\epsilon^2 R_{\mu\nu} - 2 u_\epsilon \nabla_\mu \nabla_\nu u_\epsilon + 4 \nabla_\mu u_\epsilon \nabla_\nu u_\epsilon - \left( u_\epsilon \Delta u_\epsilon + |\nabla u_\epsilon|^2 + cu_\epsilon^2 + \Lambda u_\epsilon^4 \right) g_{\mu\nu},
\]
(21)
\[
F_{\mu\nu} = \kappa u_\epsilon^2 \left( \nabla_\mu - \nabla_\nu \frac{1}{2} g_{\mu\nu} \right).
\]
(22)
Since the new mass parameter $\epsilon \in \mathbb{R}$ introduces an inhomogeneous term in (20), we proposed a $(2,2)$-decomposable, inhomogeneous metric $g_\epsilon$.

Returning to the Einstein framework, we get the equivalent formulation
\[
\bar{R}_{\mu\nu} - \Lambda \bar{g}_{\mu\nu} - \epsilon C^{-2} r^{-2} \bar{g}_{\mu\nu} = \kappa \left( T_{\mu\nu} - \frac{T}{2} \bar{g}_{\mu\nu} \right).
\]  
(23)

As usual, the $\epsilon$-term could be included in the energy-momentum tensor of the right-hand side. This succeeds with the substitution $\rho \to \tilde{\rho} = \rho + \rho^\epsilon$, $p \to \tilde{p} = p + p^\epsilon$, where
\[
\rho^\epsilon = \epsilon c^2 2 \kappa r^2 C^2, \quad p^\epsilon = -\frac{\epsilon}{r^2 C^2}.
\]

Therefore, the ratio $p^\epsilon / c^2 \rho^\epsilon = -1$ is independent of $\epsilon$.

In order to match the experiments which verify Einstein’s field equation, the parameters have to be sufficiently small
\[0 < |\epsilon| \ll 1, \quad 0 < \Lambda \ll 1.,\]

The trace part of equation (20) reads as follows
\[-6u_\epsilon' \Delta u_\epsilon + (R_\epsilon - 4\epsilon) u_\epsilon^2 - 4\Lambda u_\epsilon^4 = \kappa (p c^2 - 3p) u_\epsilon^4.\]
(24)

Now, specialising again $(\mu, \nu) = (a, b)$ and $(\mu, \nu) = (i, j)$, it holds that
\[
L_{ab} = -2u_\epsilon \nabla_a \nabla_b u_\epsilon + 4\nabla_a u_\epsilon \nabla_b u_\epsilon - 'L g_{ab},
\]
(25)
\[
F_{ab} = \kappa u_\epsilon^2 \left( (p c^2 + p) \delta_a^0 \delta_b^0 + \frac{\rho c^2 - p}{2} u_\epsilon^2 g_{ab} \right),
\]
(26)
\[
L_{ij} = -'L g_{ij},
\]
(27)
\[
F_{ij} = \kappa \frac{\rho c^2 - p}{2} u_\epsilon^4 g_{ij},
\]
(28)

where
\[
L = u_\epsilon' \Delta u_\epsilon + |'\nabla u_\epsilon|^2 - \frac{R_\epsilon}{2} u_\epsilon^2 + \epsilon u_\epsilon^2 + \Lambda u_\epsilon^4;
\]
\[
'L = u_\epsilon' \Delta u_\epsilon + |'\nabla u_\epsilon|^2 - \frac{R_\epsilon}{2} u_\epsilon^2 + \epsilon u_\epsilon^2 + \Lambda u_\epsilon^4;
\]

and $'R_\epsilon = 2$. Using the definition of the two dimensional scalar curvature we get
\[
R_\epsilon = -\frac{2(2r^2 C'' - r^2 CC'' + r CC' + C^2)}{C^2}.
\]

Further, the identity $F_1^1 = F_2^2$ immediately implies
\[
(R - ''R) \frac{u_\epsilon^2}{2} - 2u_\epsilon \nabla_r \nabla^r u_\epsilon + 4\nabla_r u_\epsilon \nabla^r u_\epsilon = 0.
\]
(29)

Therefore, it holds that
\[
(2r^2 C^2 - r^2 CC'' + r CC') r^2 = 0,
\]
i.e. $R_\epsilon = -2$. 6
Lemma 2. Let $u$ satisfy the conformally-related field equation and let

$$-(\nabla^t u)^2 g^{rr} - (\nabla^r u)^2 g^{tt} \neq 0$$

be satisfied. Then equation (20) is equivalent to

$$L_a^a - F_a^a = 0,$$
$$L_i^i - F_i^i = 0,$$
$$-u \nabla_a \left( \| \nabla u \|^2 \right) + 4 \| \nabla u \|^2 \nabla_a u_a = 0, \quad a = t, r.$$ 

Proof. The proof goes along the lines of Lemma 1. Further, we get again equation (17) and the application of equation (29) finishes the proof.

Moreover, a straightforward calculation gives the following result

Corollary 1. Let the assumptions of Lemma 3 be satisfied. Then equation (20) is equivalent to

$$L_a^a - F_a^a = 0,$$
$$L_i^i - F_i^i = 0,$$
$$\left( \frac{\dot{C}}{C} \right)' = 0.$$ 

Now, the two dimensional traces read as follows

$$L_a^a - F_a^a = -4 u'_\epsilon \Delta u_\epsilon + 2 \| \nabla u_\epsilon \|^2 + R_\epsilon u_\epsilon^2 - 2 u_\epsilon^2 - 2 \Lambda u_\epsilon^4 + 2 \kappa u_\epsilon^4 p = 0, \quad (31)$$
$$L_i^i - F_i^i = -2 u'_\epsilon \Delta u_\epsilon - 2 \| \nabla u_\epsilon \|^2 + R_\epsilon u_\epsilon^2 - 2 u_\epsilon^2 - 2 \Lambda u_\epsilon^4 - \kappa (\rho c^2 - p) \ u_\epsilon^4 = 0. \quad (32)$$

4. Spontaneous symmetry breaking

In the following we restrict our attention to the vacuum, where we necessarily assume that $\rho, p = 0$. We start by using the result $R_\epsilon = R_\epsilon + \rho R_\epsilon = 0$ and write the trace equation (24) divided by $u_\epsilon/6$ as

$$-\Delta u_\epsilon - \frac{dV(u_\epsilon)}{du_\epsilon} = 0, \quad V(u_\epsilon) = \frac{1}{3} u_\epsilon^2 + \frac{1}{6} \Lambda u_\epsilon^4.$$

Further the related Lagrangian density

$$\mathcal{L} = \sqrt{-g} \left( \frac{1}{2} \| \nabla u_\epsilon \|^2 - V(u_\epsilon) \right)$$

can be decompose

$$\mathcal{L} = \mathcal{L}_\Delta + \mathcal{L}_M.$$
such that the $\epsilon$- and $\Lambda$-terms represent a Lagrangian density of matter

$$\mathcal{L}_M = -\sqrt{-g} V(u_\epsilon).$$

In quantum physics the vacuum denotes the ground state (the state of the lowest possible energy) of the system. In order to realise the lowest possible energy of the Lagrangian density, we restrict ourselves to $u_\epsilon \neq 0$ and claim

$$\left. \frac{d}{du_\epsilon} V(u_\epsilon) \right|_{u_\epsilon = v} = v \left( \frac{2}{3} \epsilon + \frac{2}{3} \Lambda v^2 \right) = 0,$$

$$\left. \frac{d^2}{du_\epsilon^2} V(u_\epsilon) \right|_{u_\epsilon = v} = \frac{2}{3} \epsilon + 2 \Lambda v^2 > 0,$$

which implies

$$v^2 = -\frac{\epsilon}{\Lambda} \geq 0,$$

and $\Lambda > 0$. Combining the last relations, we get $\epsilon < 0$. As a consequence, it follows that

$$\min_{u_\epsilon} V(u_\epsilon) = V(v) = -\frac{\epsilon^2}{6\Lambda} = -\frac{\Lambda}{6} v^4 < 0.$$

Notice that a negative $V(v)$ and a non-zero $v$ follows only for $\epsilon \neq 0$ (cf. Section 2). Because $u_\epsilon$ has the unit of length, $v > 0$ is chosen (spontaneous symmetry breaking).

Now, we get from equation (31) and (32)

$$R_\epsilon - 2 \epsilon - 2 \Lambda v^2 + 2 \kappa p_{\text{vac}} v^2 = 0,$$

$$2 - 2 \epsilon - 2 \Lambda v^2 - \kappa \left( \rho_{\text{vac}} c^2 - p_{\text{vac}} \right) v^2 = 0,$$

and furthermore it follows by an application of equation (33) in the $\Lambda$-term that

$$p_{\text{vac}} = \frac{1}{\kappa v^2}, \quad \rho_{\text{vac}} = \frac{3}{\kappa v^2 c^2}. $$

In order to determine $\epsilon$ and $\Lambda$, we need two conditions. First, to be within a meaningful theory, $v$ should be resolved by the Planck length $l_p$. According to Shannon’s sampling theorem we have $v > 2 l_p$. Now we can take the lower limit with

$$v \approx 2 l_p. $$

Therefore, it holds that

$$p_{\text{vac}} \approx \frac{1}{4\kappa l_p^2}, \quad \rho_{\text{vac}} \approx \frac{3}{4\kappa l_p^2 c^2}. $$

For a comparison with the zero-point energy, we refer to [1] and [4] and consider a cut-off wave-number of $k_{\text{max}} = 1/l_p$

$$\rho_{\text{zpe}} = \frac{E}{V c^2} = \frac{\hbar k_{\text{max}}^4}{16\pi^2 c} = \frac{\epsilon^2}{16\pi^2 G l_p^2} = \frac{1}{2\pi \kappa l_p^2 c^2}. $$

Hence, there is no cosmological constant problem:

$$\frac{\rho_{\text{vac}}}{\rho_{\text{zpe}}} \approx \frac{3\pi}{2}. $$
Finally, we can give an interpretation of $\rho_{\Lambda}$, which was considered by many authors as the vacuum density. In contrast to the above dealing with equation (34) and (35), we use equation (37) for the first substitution of $v$. Using the usual setting $\rho_{\Lambda} = \Lambda/kc^2$ and $\rho_c = (2 - 2\epsilon)/8kc^2$, it follows

$$\rho_{\text{vac}} = 3(\rho_c - \rho_{\Lambda}).$$

The last equation can be interpreted in two ways. On the one hand, it is an automatically satisfied fine-tuning between $\rho_{\text{vac}}$ and $\rho_{\Lambda}$, and on the other hand, it states that the vacuum is realised by the dark energy and the perturbation effect of the conformally-modified gravity.

5. Observational validation

In this section we show that the inclusion of arguments from particle physics can still describe the accelerating expansion of the universe. In order to do that, we will derive the generalised Friedmann’s equation and providing the ground for a comparison with cosmological observations. Using the expressions $\frac{\rho_{c}}{3a^2} - \frac{\rho}{3a^2}$ and $\frac{p_c}{3a^2}$ and apply equation (30), we get

$$\ddot{C} - \frac{\epsilon}{3r^2c^2} = \frac{\Lambda}{3} = -\frac{k}{6}(\rho c^2 + 3p),$$

(38)

$$\frac{\dot{C}^2}{C^2} + \frac{K}{C^2} - \frac{\epsilon}{3r^2c^2} = \frac{\Lambda}{3} = \frac{k}{3}c^2,$$

(39)

where $K = -\frac{C''}{C^2} - 2\frac{C'}{rC} \rightarrow k$ for $\epsilon \rightarrow 0$ in the case of the Robertson-Walker metric $C(t, r) = a(t)b(r), b(r) = (1 + kr^2/4)^{-1}$. Thus, recalling the spontaneous symmetry breaking requirement $\epsilon < 0$ reveals the gravitational effect of the $\epsilon$-term in equation (38).

In order to get a consistent theory, we put the $\epsilon$-term from (23) on the right-hand side and require that

$$\nabla^\mu \tilde{T}_{\mu\nu} = 0, \quad \tilde{T}_{\mu\nu} = \left(\tilde{\rho} + \frac{\tilde{p}}{c^2}\right)u_\mu u_\nu + \tilde{p}g_{\mu\nu}.$$ 

It follows that

$$\dot{\tilde{\rho}} + \rho' + 3\frac{\dot{C}}{C}\left(\tilde{\rho} + \frac{\tilde{p}}{c^2}\right) = 0, \quad \dot{p}' + p'' = 0,$$

which is solved in the case of a matter dominated universe ($p \ll \rho c^2$) by $p = \epsilon/kr^2c^2$ and $\rho(t, r) = F(r)/C(t, r)^3 - \epsilon/kr^2$ where $F(r)$ is an arbitrary function. This allows one to write the equations (38) and (39) as

$$C\ddot{C} - \frac{\lambda}{3}C^2 = -\frac{k\rho c^2}{6C} F,$$

(40)

$$C^2 + K = \frac{\lambda}{3}C^2 = \frac{k\rho c^2}{3C} F,$$

(41)

and using the Robertson-Walker metric $C(t, r) = a(t)b(r)$ also in the conformally-modified case. Then the evaluation of the second equation at $t = t_0$ becomes

$$1 = \Omega_m + \Omega_\Lambda + \Omega_K,$$

(42)
where
\[ \Omega_m = \frac{\kappa c^4 F}{3H_0^2 C_0^3}, \]
\[ \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2}, \]
\[ \Omega_K = -\frac{Kc^2}{C_0^2 H_0^2}, \]
\[ C_0 = C(t_0, r_0), \]
\[ \frac{H_0}{c} = \frac{\dot{C}(t_0, r_0)}{C_0}. \]

denote the density parameters. Consequently, the modeling of the universe with equation (42) is consistent with usual Lambda-Dark Matter Models. Now, from (33) and (37) we can derive a realistic value of \( \epsilon \approx -10^{-123} \).

6. Concluding remarks

In this paper, it has been shown that Einstein’s field equation satisfies a mass-less Klein-Gordon equation where the cosmological constant represents the quartic interaction term. In order to complete the analogy to particle physics we have modified the field equation such that the Klein-Gordon equation with a spontaneous symmetry breaking at Planck scale is fulfilled. It was demonstrated that the scale discrepancy between the cosmological vacuum energy density and the quantum zero-point energy could be avoided without a contradiction to the measured accelerating expansion of the universe.

References

[1] S. Carroll and W. Press, Annu. Rev. Astron. Astrophys. 30, 499 (1992).
[2] S. Carroll, Living Rev. Relativity 4 (2001).
[3] V. Sahni and A. Starobinsky, Int. J. Mod. Phys. D9, 373 (2000).
[4] S. Weinberg, Rev. Mod. Phys. 61(1) (1989).
[5] R. Wald, General Relativity (The University of Chicago Press, 1984).
[6] M. Hobson, G. Efstathiou, and A. Lasenby, General Relativity (Cambridge University Press, 2006).
[7] B. Fiedler and R. Schimming, Astron. Nachr. 304, 221 (1983).