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Quasiperiodic mechanical metamaterials with extreme isotropic stiffness

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ABSTRACT

Architected metamaterials are dominantly periodic, but on the other hand, natural materials usually exhibit aperiodicity or even disordered randomness. In this study, we systematically design a novel family of mechanical metamaterials that are simply composed of n+1 sets of planar plates layered in a transversely quasiperiodic manner. Through rigorous theoretical and numerical analysis, we demonstrate that these quasiperiodic metamaterials attain the maximum isotropic elastic stiffness in the low density limit and can preserve over 96% optimal stiffness at moderate densities up to 50%. Moreover, we identify a dual family of quasiperiodic truss metamaterials by orientating bar members in the normal directions of aforementioned plate sets. These truss structures that are connected by construction are also stiffness optimal in the sense that they achieve the stiffness limit for truss microstructures. Both the material families possess superior directional yield strength compared to other existing periodic stiffness-optimal metamaterials. The quasiperiodic geometries offer possibilities in discovery of emerging 3D metamaterials in various areas of electromagnetics, mechanics, acoustics and others.

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1. Introduction

Metamaterials have attracted enormous interest for their exceptional capabilities that go beyond natural materials [1]. Rational design methods are crucial for realizing metamaterials of desired or extreme properties since the properties are strongly determined by the geometries of their microstructures. For simplicity, most design methods build metamaterials with regularly periodic blocks and the design process is implemented within a single unit cell [2,3]. However, periodicity implicitly imposes strong limitations on geometries of microstructures, and might hinder achievement of real extreme properties [4]. On the other hand, quasiperiodic structures have been observed in many natural materials [5,6]. They are well ordered but have no translational symmetry. Their geometrical freedom provides a larger design space compared to periodic cases. Despite that quasicrystals have been found to exhibit unusual physical behavior [7,8], systematic design of quasiperiodic metamaterials with extreme mechanical properties is rarely studied.

Stiffness is a fundamental material property that is commonly used to evaluate load-bearing capability of elastic materials. In case of isotropic metamaterials, their stiffness can be characterized by the effective Young’s (E∗), bulk (K∗) and/or shear (G∗) modulus. Two extreme cases are the pentamode [9,10] and dilatational metamaterials [11,12], which have comparatively large bulk (G∗ ≪ K∗) and shear (G∗ ≫ K∗) moduli, respectively, and other metamaterials with moderate G∗/K∗ values can achieve designed functionality, such as strain cloaking [13] and programmable negative Poisson’s ratios [3,14]. Notably, combining two phases of infinitely rigid and compliant constituents, it is theoretically possible to recover all allowable Cauchy elasticity tensors [10].

Looking for stiffness-optimal microstructures is of particular importance for design of multiscale structures [15,16]. Extensive efforts have been devoted to study periodic truss lattice structures (TLS) [17–20]. However, theory predicts that only closed-walled microstructures comprising of plates are able to attain the Hashin–Shtrikman (H–S) upper bound on elastic stiffness [21]. The extreme stiffness of the optimal plate microstructures can be up to three times that of their counterpart optimal truss microstructures when subjected to the same loading cases [4]. This is due to the fact that an individual plate offers planar multi-axial stiffness while bars sustain only axial forces. The coated spheres [22] and the Vigdergauz-type [23,24] microstructures can achieve the bulk modulus bound but not simultaneously the shear modulus bound. Structures composed of at most rank-6 laminates (for 3D structures) at multiple length scales are able to obtain the optimal stiffness for any loading cases [25,26], and the optimal isotropic rank-6 structure attains the Young’s modulus bound [27,28]. Physically realizable single-scale microstructures are able to recover extreme stiffness in the low density limit [29].
and have inferior stiffness up to a few percent at moderate densities [30]. Recently, a class of combined plate lattice structures (PLS) was proposed by combining two or three anisotropic elementary forms, including simple cubics (SC), body-centered cubics (BCC) and face-centered cubics (FCC) [31,32]. These microstructures achieve optimal isotropic stiffness, and the trinary combined PLSs further have almost plastic isotropy [32]. A class of optimal combined TLSs have independently been identified to attain the isotropic stiffness limit for truss microstructures [33–35].

This paper identifies a family of quasiperiodic mechanical metamaterials with extreme isotropic stiffness. They are built by synthesizing \( n + 1 \) sets of continuous plates in a transversely quasiperiodic manner, and importantly, their geometries satisfy \( n \)-fold symmetry. Theoretical analysis provides the geometric parameters resulting in elastic isotropy and attaining the theoretical H–S bound on elastic stiffness. Moreover, a dual family of quasiperiodic truss metamaterials is correspondingly identified by orientating bar members in the normal directions of the plate sets. Interestingly, they also obtain the stiffness limit for truss microstructures. Remarkably, we emphasize that the stiffness optimality of our microstructures arises from their \( n \)-fold symmetric geometries, a mechanism which is essentially different from the existing combined optimal PLSs that take directional stiffness advantage of each individual anisotropic basic form to achieve optimal isotropy [31,32]. When they are subjected to uniaxial loadings, both the plate and truss quasiperiodic metamaterials possess higher directional yield strength than other combined stiffness-optimal TLSs and PLSs [32,33].

2. Methodologies

2.1. Geometry

Our plate microstructures are simply built by synthesizing \( n + 1 \) periodic sets of planar plates. In each set, an infinite number of equidistant parallel plates are distributed to fill space (Fig. 1(c)). For illustration, we denote the single plate set by horizontal plate (HP) and the other \( n \) plate sets by oblique plates (OP) (Fig. 1(a)). All the OPs have the same plate thickness of \( t^{\text{OP}} \) and the distance between two parallel plates in the same set is \( d^{\text{OP}} \), and those quantities for the HP are \( t^{\text{HP}} \) and \( d^{\text{HP}} \). The relative thicknesses are obtained by \( q^{\text{OP}} = t^{\text{OP}}/d^{\text{OP}} \) and \( q^{\text{HP}} = t^{\text{HP}}/d^{\text{HP}} \).

In the microstructures, the OPs are oriented with the same relative angle of \( \alpha \) to the normal axis of the HP, and moreover, they are evenly rotated by \( \beta = 2\pi/n \) around the normal axis of the HP. That is, provided \( \mathbf{n}^{\text{HP}} = [0, 0, 1]^T \), the normal direction of the \( i \)th OP is \( \mathbf{n}^{\text{OP}} = [\sin \alpha \cos \beta i, \sin \alpha \sin \beta i, \cos \alpha]^T \). In this form, intersection of the OPs with any horizontal cut-off plane presents regular polygons that typically have \( n \)-fold symmetry, in which the symmetry order is equal to the number of OP sets (Fig. 1(b)). From a macroscopic view, a series of scaled polygons are observed but never repeat (Fig. 1(d)). Furthermore, in the HP normal direction, the microstructures can be represented by a repeated stack of a single layer (Fig. 1(e)), whose height meets \( H^{\text{plate}} = d^{\text{HP}} = d^{\text{OP}}/\cos \alpha \). To this end, these microstructures are recognized to be transversely quasiperiodic, i.e. quasiperiodic in the HP plane and periodic along the HP normal axis.

We also consider a dual family of truss microstructures that aligns \( n + 1 \) sets of continuous bars in the normal directions of the plate sets (Fig. 2(a)). As discussed in [36], connectivity of the bars from various sets is crucial to ensure physical truss microstructures without hanging bars. Here, the oblique bars (OBS) in each OP plane are arrayed in a rectangular manner to fill space (Fig. 2(b) top), whose periodicity along two planar directions satisfy \( l^{\text{OB}}_2 = \frac{\sin \beta}{\cos \alpha (1 - \cos \beta)} l^{\text{OP}}_2 \). This tilling pattern ensures that any bar from one OB set intersects with infinitely many bars from the two neighboring OB sets, and it also intersects with at least one bar from the other OB sets. On the other hand, the vertical bars (VB) are distributed in a parallelogram in the HP plane (Fig. 2(b) bottom), with the periodicity of \( l^{\text{VB}} = l^{\text{OP}}_2/\sin \beta \). Any one VB bar connects with infinitely many bars in at least two OB sets. Hence,
the proposed truss microstructures are connected by construction (see Supplementary Information S4 for proof of connectivity). The family of truss microstructures is again transversely quasiperiodic and can be represented by a repeated single layer along VB with the height of $H_{\text{truss}} = t^\text{HB}/\sin \alpha$ (Fig. 2(d)).

In the special case of $n = 6$, the plate and truss microstructures become fully periodic and preserve both 6-fold symmetry and translational symmetry. Their geometries are represented by two parallelepiped unit cells, as seen in Fig. 3(a) and 3(b). Each unit cell is formed by mirror/rotational transformations of a regular-triangular prism. In the PLS prism, each OP goes through one top/bottom edge and the opposite bottom/top vertex, and in the TLS prism, equilateral triangular frames form the side faces. With the prescribed unit cells, the thickness and area ratios of the 6-fold PLS and TLS are $t^\text{PLS}/t^\text{HB} = \sqrt{5}/3$ and $d^\text{TLS}/d^\text{HB} = 2\sqrt{5}/3$, respectively.

2.2. Theoretical model

The relative density of a microstructure is defined by $\rho^* = \int_{\Omega_1} \rho \, d\Omega / \int_{\Omega_2} \rho \, d\Omega$, with $\Omega_1$ and $\Omega_2$ indicating the regions occupied by solid constituent and the full microstructures, respectively. In the following analysis, we assume an isotropic solid constituent with Young’s modulus and Poisson’s ratio of $E_1$ and $\mu_1 = 1/3$, respectively. In the low density limit case $\rho^* \to 0$, the plates are purely stretched in a state of plane stress. Therefore, the effective stiffness of these plate microstructures can be obtained by directly summing up the stiffness of each individual plate set by [21,29]

$$\mathbf{D}^* = q^{\text{opt}}\mathbf{D}^\text{opt} + q^{\text{def}}\mathbf{D}^\text{def} = q^{\text{opt}}\mathbf{D}^\text{opt} + q^{\text{def}} \sum_{i=1}^{n} T_i^\text{ref} \mathbf{D}_i$$

(1)

where $T_i$ is the standard transformation matrix related to $\mathbf{n}^\text{opt}_i$ and $\mathbf{n}^\text{def}_i$ is the reference stiffness matrix for a single plate set. The relative density of the plate microstructure is analogously obtained by $\rho^* = q^{\text{opt}} + q^{\text{def}}$. The same superposition rule can also be applied to truss microstructures, by using the relative bar areas and the reference stiffness matrix for a single bar set instead.

The effective yield strength of the plate microstructures is quantified by the maximal von-Mises stress among all the plates, noted as $\sigma_{\text{max}}^\text{vm} = \max_{1\leq n\leq 1} \{\sigma_{\text{max}}^\text{vm}\}$, with $\sigma_{\text{max}}^\text{vm}$ for the von-Mises stress of the nth plate set. Smaller $\sigma_{\text{max}}^\text{vm}$ implies that the stresses in the plate members are further away from the yield stress, and therefore the microstructures have higher yield strength. Particularly, we consider the yield strength when the microstructures are subjected to uniaxial stress state. The uniaxial stress is denoted by $\sigma^* = \sigma_0 p$, where $\sigma_0$ is the magnitude and $p$ is the stress direction, stated by $p = [\cos \theta_1 \cos \theta_2, \cos \theta_1 \sin \theta_2, \sin \theta_1]^T$, with $\theta_1$ and $\theta_2$ two Euler angles. Accounting for geometric symmetry of the considered isotropic microstructures, the two angles are evaluated in interval of $0 \leq \theta_1, \theta_2 \leq \pi/2$. The

![Fig. 2. Dual family of quasiperiodic truss microstructures, with red and other colors indicating VB and OB bars, respectively. (a) Bars oriented in the normal directions of two plates; (b) distributions of (top) OBs and (bottom) VBs, with the shaded regions for periodic shapes in the plane; (c) 5-fold symmetric geometry of assemblage of OBs from various sets; and (d) the entire microstructure formed by a repeated single layer (highlighted) along VB. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)](https://doi.org/10.1016/j.eml.2019.100596)

![Fig. 3. Stiffness optimality of (a) 6-fold PLS and (b) 6-fold TLS, with the yellow highlighted parts indicating the parallelepiped unit cells. (c) Simulated $E^*/\rho^*E_i$ curves of the two 6-fold lattice structures in $\rho^* \leq 50\%$. (d) Normalized strain energy density distributions subjected to the vertical strain at $\rho^* = 20\%$, where the strain energy density is normalized by the energy density of the solid constituent.](https://doi.org/10.1016/j.eml.2019.100596)
von-Mises stress in the eth plate set is analytically obtained by

$$\sigma_{vm}^e = \frac{\sigma_0}{\rho^*} \sqrt{\frac{175}{16}} \sin^4 \theta_{se} - 5 \sin^2 \theta_{se} + 1$$  \hspace{1cm} (2)$$

with $\theta_{se}$ the relative angle between the stress and the normal direction of the eth plate set.

Analogously, the yield strength of the truss microstructures is quantified by the maximal axial stress among the bar sets $\sigma_{max}^{\alpha} = \max_{1 \leq n \leq 5} (\sigma_{vm}^{\alpha})$. The axial stress of the eth bar set is

$$\sigma_{ax}^e = \frac{3\sigma_0}{2\rho^*} (5 \cos^2 \theta_{se} - 1)$$  \hspace{1cm} (3)$$

where $\theta_{se}$ becomes the relative angle between the stress and the eth bar set. Mathematical details for derivation of Eqs. (2) and (3) are given in Supplementary Information S2.

### 2.3. Computational analysis

As the above superposition law is effective for the cases approaching low density limit, a computational homogenization method is implemented to estimate the stiffness of various periodic PLSs and TLSs for relative densities up to 50%. The finite element method imposes periodic displacement boundary conditions on the opposite faces of unit cells, and applies six unit macroscopic strains, including three normal and three shear strains, to the cell bodies. The simulations are done by running the Cell Periodicity node in COMSOL 5.4, where the microstructures are discretized by using body-fitted 2nd-order tetrahedron elements, with 300k~400k elements for the PLSs and 150k~200k elements for the TLSs. Moreover, the plates are discretized by using at least two layers of elements in their thickness directions to allow for plate bending deformations. We use a universal isotropy index [37] to measure the property deviation from the perfect isotropy for the considered microstructures under various volume fractions.

### 3. Results and discussion

#### 3.1. Stiffness optimality

In principle, the n-fold symmetric geometry ensures that the microstructures have transversely isotropic properties in cases of $n \geq 5$ [38]. Incorporating the plate normal directions into Eq. (1), it is seen that the properties of the microstructures are a function of $q^{\text{HP}}$, $q^{\text{OP}}$ and $\alpha$. The conditions for the microstructures to achieve full isotropy are analytically obtained by (see Supplementary Information S1)

$$\frac{q^{\text{HP}}}{q^{\text{OP}}} = \frac{n}{5}, \quad \sin^2 \alpha = \frac{4}{5}$$  \hspace{1cm} (4)$$

It is worth noting that these conditions are unique for the plate microstructures with any number of $n \geq 5$. It means that both assembly of OPs themselves and the cases of $n \leq 4$ cannot generate isotropy. Also note that these conditions are independent of the properties of the solid constituent. Among various n, the 5-fold ($n = 5$) microstructure recovers the optimal rank-6 laminate, which is icosahedral quasiperiodic.

The obtained Young’s modulus for any number of n is uniquely quantified by $E^*/\rho^*E_s = 1/2$. This value exactly attains the maximum Young’s modulus in theory for isotropic microstructures, which also implies that both the maximum bulk and shear moduli are simultaneously obtained [21]. In other words, we have identified a new family of quasiperiodic mechanical metamaterials that attain the theoretically maximum isotropic stiffness (in the low density limit)!

The isotropic conditions for the truss microstructures are the same as those in Eq. (4), by replacing the relative thicknesses of the plates by the relative areas of the bars and $\alpha$ becomes the relative angle between VB and OBS. The obtained Young’s modulus for any number of n is quantified by $E^*/\rho^*E_s = 1/6$. Excitingly, this value attains the maximum isotropic stiffness limit for truss microstructures [21], and hence the new quasiperiodic truss microstructures are also stiffness optimal (in the low density limit)!

Stiffness optimality of the two 6-fold lattice structures at finite densities of up to 50% are presented in Fig. 3(c). The intersection points of the two simulated curves with $\rho^* = 0$ are obtained by extrapolation from the sampling points, yielding $E^*/\rho^*E_s \approx 0.5$ and 0.17 for the 6-fold PLS and TLS, respectively. These values agree with the analytical optimal results. In the low density range of $\rho^* \leq 20\%$, both lattice structures remain stretching-dominated, in which $E^*/\rho^*E_s$ vary linearly with respect to $\rho^*$. As the density further increases, intersections from various plates or struts introduce energy concentrations in the crossing regions, thus leading to bigger stiffness deviations. Here, the 6-fold PLS is capable of obtaining 97.5% and 94.9% stiffness of the theoretical bound at $\rho^* = 20\%$ and 50%, respectively. Comparing the results with those from two combined lattice structures (SC + BCC and SC + FCC), it is found that they have similar stiffness moduli (see Fig.S.1). The obtained near-optimal stiffness can only attain the bounds for finite volume fractions by introducing multiscale features to either the entire microstructures [25,39] or to local member parts [23]. Alternatively, a common engineering approach is to smooth the sharp intersection edges for achieving more uniform energy distributions. With a fillet radius of one fifth of the OP thickness, the stiffness of the 6-fold PLS are improved to 98.2% and 96.1% of the bound at $\rho^* = 20\%$ and 50%, respectively.

Stiffness superiority of the plate microstructures can be verified by examining the distribution uniformity of the local strain energy densities (denoted by $\Sigma$). When the PLS is subjected to the vertical strain, most energy is stored in the OPs, where around 66.9% of the solid material has $\Sigma \geq 0.5$ and it is efficiently used to store 82.7% of the total strain energy (Fig. 3(d)). In this loading case, the HPs store negligible energy, since they are perpendicular to the loading direction. For comparison, in the TLS only 13.0% of the solid material is efficiently used to store energy, which is mainly delivered by the vertical struts (Fig. 3(e)). The mathematical average of $\int_{\Omega_s} \Sigma d\Omega / \int_{\Omega_t} d\Omega$ in the PLS and TLS are about 0.33 and 0.16, respectively. The energy ratio of $\approx 2$ reveals the stiffness superiority of the PLS in the considered density. The same conclusion can be made for the other test loading cases where the strain energy is distributed in the PLS in a more uniform manner than in the TLS (see Table S.1 and Fig.S.2). Note that although the PLSs always have higher stiffness than the TLSs, both of them have similar effective Poisson’s ratios of $\mu^* \approx 0.25$.

#### 3.2. Superior yield strength

Despite their similar stiffness optimality, the n-fold microstructures have quite different effective yield strength from the two combined lattice structures, which is mainly due to their distinct geometries. The polar charts of the yield strength for various microstructures are shown in Fig. 4. From Eq. (2), it is seen that $\sigma_{vm}^e$ is only related to the relative angle of $\theta_{se}$. When the stress is in the plane of any plate set, i.e. $\theta_{se} = \pi/2$, the plates in the set are fully stressed to first yield failure, generating the maximum $\sigma_{vm}^e$ by

$$\Sigma = \left(\sigma_{vm}^{\text{max}}\right) = \left(\sigma_{vm}\right)_{\theta_{se}=\pi/2} = \frac{\sigma_0}{\rho^*} \sqrt{\frac{\pi}{4}}$$  \hspace{1cm} (5)$$

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This worst case can be observed in any plate microstructure, and thus all the results are normalized by $\Sigma$.

Conversely, the highest yield strength is observed when the stress direction is farthest away from any plate direction,

$$\frac{\sigma_{\text{max}}^{\text{yield}}}{\Sigma} \approx 0.76, \frac{\sigma_{\text{max}}^{\text{yield} \Sigma \text{SB}}}{\Sigma} \approx 0.87, \frac{\sigma_{\text{max}}^{\text{yield} \Sigma \text{SF}}}{\Sigma} \approx 0.95 \quad (6)$$

In the best cases, the yield strength of the 6-fold plate microstructures is 14.2% and 24.7% higher than the SC + BCC and SC + FCC PLSs, respectively. It is worth noting that using more OPs in the plate microstructures increases the strength isotropy, but full plastic isotropy can never be achieved with these microstructures. This is because that the best cases always arise when a vertical stress is applied, thus retaining the same highest yield strength value for any $n$.

For the truss microstructures, the worst cases arise when the stress direction aligns with any one of the bar set (see Eq. (3)), yielding

$$\Sigma = (\sigma_{\text{max}}^{\text{ax}})_{n=6} = \frac{6\sigma_0}{\rho^*} \quad (7)$$

With various number of OBs, the n-fold truss microstructures have different $\sigma_{\text{max}}^{\text{ax}}/\Sigma$ in the best cases. There, the 6-fold TLS behaves the highest yield strength by

$$\frac{\sigma_{\text{max}}^{\text{ax}}}{\Sigma} \approx 0.50, \frac{\sigma_{\text{max}}^{\text{ax} \Sigma \text{SB}}}{\Sigma} \approx 0.57, \frac{\sigma_{\text{max}}^{\text{ax} \Sigma \text{SF}}}{\Sigma} \approx 0.59 \quad (8)$$

The 6-fold TLS has 13.4% and 17.8% higher yield strength than the SC + BCC and SC + FCC TLSs, respectively.

Analogously, the same procedure can be carried out for the cases when subjected to uniaxial strains. The obtained results generate the same polar charts as those in Fig. 4. To this end, the conclusion is made that the n-fold plate and truss microstructures have the highest directional yield strength among all the known stiffness-optimal microstructures.

4. Conclusions

In summary, we have identified a new family of quasiperiodic extreme mechanical metamaterials. They attain optimal isotropic stiffness owing to their $n$-fold symmetric geometries. In addition, we analytically demonstrate that these $n$-fold microstructures have superior directional yield strength. Note, however, that in low densities, both plate and truss microstructures having thin-sized members could suffer from buckling failure before yield. Precise evaluation of buckling failure for 3D quasi-periodic structures with complex geometries is an arduous task that we are currently pursuing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.eml.2019.100596.

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