Learning Dynamic Mechanisms in Unknown Environments: A Reinforcement Learning Approach

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Abstract

Dynamic mechanism design studies how mechanism designers should allocate resources among agents in a time-varying environment. We consider the problem where the agents interact with the mechanism designer according to an unknown Markov Decision Process (MDP), where agent rewards and the mechanism designer’s state evolve according to an episodic MDP with unknown reward functions and transition kernels. We focus on the online setting with linear function approximation and attempt to recover the dynamic Vickrey-Clarke-Grove (VCG) mechanism over multiple rounds of interaction. A key contribution of our work is incorporating reward-free online Reinforcement Learning (RL) to aid exploration over a rich policy space to estimate prices in the dynamic VCG mechanism. We show that the regret of our proposed method is upper bounded by \(\tilde{O}(T^{2/3})\) and further devise a lower bound to show that our algorithm is efficient, incurring the same \(\tilde{O}(T^{2/3})\) regret as the lower bound, where \(T\) is the total number of rounds. Our work establishes the regret guarantee for online RL in solving dynamic mechanism design problems without prior knowledge of the underlying model.

1 Introduction

Mechanism design is a branch of economics that studies the allocation of goods among rational agents (Myerson, 1989). Its sub-field, dynamic mechanism design, focuses on the setting where the environment, such as agents’ preferences, may vary with time (Bergemann and Välimäki, \(^\ast\)Equal Contribution. Names are listed in alphabetical order.
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Dynamic mechanism design has attracted significant research interest from economists and computer scientists alike (Pavan et al., 2014; Parkes and Singh, 2003). Many real-world problems, such as Uber’s surge pricing, the wholesale energy market, and congestion control, have all been studied under this framework (Chen and Sheldon, 2016; Bejestani and Annaswamy, 2014; Barrera and Garcia, 2014). However, existing work usually requires prior knowledge of key parameters or functionals in the problem, such as the optimal policy or the agents’ valuations of goods (Parkes and Singh, 2003; Pavan et al., 2009). Such requirements may be unrealistic in real life.

A promising, emerging research direction is to learn dynamic mechanisms from repeated interactions with the environment. Drawing inspiration from Bergemann and Välimäki (2010) and Parkes and Singh (2003), we propose the first algorithm that can learn a dynamic mechanism from repeated interactions via reinforcement learning (RL), with zero prior knowledge of the problem.

As a first attempt, we focus on learning a dynamic generalization of the classic Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1979). More specifically, we consider the case where the interaction between a group of agents and a single seller is modeled as an episodic linear Markov Decision Process (MDP) (Jin et al., 2019; Yang and Wang, 2019; Jin et al., 2020b), where the seller takes actions to determine the allocation of a class of scarce resources among agents. Our task is to learn an ideal mechanism from repeated interactions via online RL (Jin et al., 2019; Cai et al., 2019). The mechanism we consider implements the policy that maximizes social welfare, and charges each agent according to the celebrated Clarke pivot rule (Clarke, 1971). A slight variant of the mechanism has been discussed, under known MDP dynamics, in Parkes (2007) and we describe the mechanism in full detail in Section 2.

A key challenge we resolve is estimating the VCG price with no prior knowledge of the MDP. In particular, the VCG price charged to each agent $i$ is characterized by the externality of that agent, that is, the difference between the maximum social welfare of the whole group and that when agent $i$ is absent (Karlin and Peres, 2017; Groves, 1979). In other words, it is the loss that an agent’s participation incurs on other agents’ welfare. Estimating the VCG price, in our dynamic setting, involves learning the optimal policy of the fictitious problem where agent $i$ is absent. Such a policy is never executed by the seller and thus it is challenging to assess its uncertainty from data. For instance, existing confidence-bound-based algorithms would not explore such a fictitious policy as these methods reduce uncertainty in a trial-and-error fashion, thereby precluding efficient estimation of VCG prices via a direct application of online RL algorithms (Jin et al., 2019; Cai et al., 2019; Zhou et al., 2021a).

To address this challenge, our algorithm incorporates a reward-free exploration subroutine to ensure sufficient coverage over the policy space, thereby reducing the uncertainty of all policies, ensuring that we can even reduce the uncertainty about the fictitious policies (Jin et al., 2020a; Wang et al., 2020; Qiu et al., 2021; Zhang et al., 2021; Kaufmann et al., 2021). However, such a
reward-free approach comes at a price—our proposed approach attains $\tilde{O}(T^{2/3})$ regret in terms of social welfare, agent utility, and seller utility, as opposed to the common $\tilde{O}(T^{1/2})$ regret in online RL (Jin et al., 2019). Moreover, we further derive a matching lower bound for the regrets, showing that our algorithm is minimax optimal up to multiplicative factors of problem dependent terms.

To summarize, our contributions are threefold. First, we introduced the first reinforcement learning algorithm that can recover an optimal dynamic mechanism with no prior knowledge of the problem. In particular, our algorithm is separated into two phases, namely, exploration and exploitation. In the exploration phase, we propose to learn the underlying model via reward-free exploration. Then, in the exploitation phase, the algorithm executes a data-driven policy by solving a planning problem using the collected dataset. Moreover, our algorithm is able to handle large state spaces by incorporating linear function approximation. Second, we prove that the proposed algorithm achieve sublinear regret upper bounds in term of the various regret notions such as the welfare regret and individual regret of the seller and buyers. Moreover, our algorithm is proven to asymptotically satisfy the three key mechanism design desiderata — truthfulness, individual rationality, and efficiency. Finally, we demonstrate that the $\tilde{O}(T^{2/3})$ regret has the minimax optimal dependency in $T$ by establishing a matching regret lower bound. To our best knowledge, we seem to establish the first provably efficient reinforcement learning algorithm for learning a dynamic mechanism.

Related Works. There is a wealth of literature in dynamic mechanism design. Parkes and Singh (2003); Parkes et al. (2004) are two of the earliest works that analyzes dynamic mechanism design from an MDP perspective, and the proposed mechanism is applied to a real-world problem in Friedman and Parkes (2003). Bergemann and Välimäki (2006) generalize the VCG mechanism based on the marginal contribution of each agent and derives a mechanism that is truth-telling in every period. Bapna and Weber (2005) focus on the dynamic auction setting and formulate the problem as a multi-arm bandit problem. Athey and Segal (2013) adapt the d’Aspremont-Gerard-Varet (AGV) mechanism (d’Aspremont and Gérard-Varet, 1979) to the dynamic setting and design an efficient, budget balanced, and Bayesian incentive compatible mechanism. Pavan et al. (2009) derive the first order conditions of incentive compatibility in dynamic mechanisms. Cavallo (2008) devises a dynamic allocation rule for auctions in the multi-arm bandits setting, where a single good is distributed among agents over multiple rounds. Cavallo et al. (2009) study the truthful implementation of efficient policies when agents have dynamic types. Pavan et al. (2014) extend the seminal work of Myerson (1989) and characterize perfect Bayesian equilibrium-implementable allocation rules in the dynamic regime. Cavallo (2009); Bergemann and Pavan (2015); Bergemann and Välimäki (2019) provide useful surveys of dynamic mechanism research.

Kandasamy et al. (2020) studies online learning of the VCG mechanism with stationary multi-arm bandits. Our work considers the more challenging setting, where the agents’ rewards are
state-dependent and may evolve over time within each episode. More importantly, Kandasamy et al. (2020) estimates the VCG price via uniformly exploring over all arms, which cannot be directly applied to the dynamic setting (Wang et al., 2020). Rather than uniformly bounding the uncertainty over all actions, our approach bounds the uncertainty over all implementable policies via a variant of least-squares value iteration, and enjoys provably efficiency under the function approximation setting.

There are many recent works concerning provably efficient RL for linear MDPs in the absence of generative models (Yang and Wang, 2019; Du et al., 2019; Yang and Wang, 2020; Jin et al., 2019; Cai et al., 2019). Jin et al. (2019) provides the first provably efficient RL algorithm for linear MDPs and Cai et al. (2019) designs for linear MDP a provably efficient policy optimization algorithm that incorporates exploration. Zhou et al. (2021b) provides a provably efficient algorithm for infinite-horizon, discounted linear MDPs. Ayoub et al. (2020) studies a model-based regime where the transition kernel belongs to a family of models known to the learning agent. Zhou et al. (2021a) proposes a computationally efficient nearly minimax optimal algorithm for linear MDPs whose transition kernel is a linear mixture model.

Reward-free exploration in reinforcement learning has also seen great research interests recently. Specifically, Jin et al. (2020a) introduces the problem of reward-free exploration in RL and proposes a sample-efficient algorithm for tabular MDPs. Ménard et al. (2021); Kaufmann et al. (2021) provide improved algorithms and tighter rates, also for tabular MDPs. Zhang et al. (2021) further improves the analysis and obtains nearly minimax-optimal sample complexity bounds. Wang et al. (2020); Zanette et al. (2020); Chen et al. (2021); Wagenmaker et al. (2022) study reward-free RL algorithms for linear or linear mixture MDPs and Qiu et al. (2021) for kernel and neural function approximations. Moreover, Kong et al. (2021) proposes reward-free algorithms for RL with general function approximation under the setting of bounded eluder dimension. Miryoosefi and Jin (2021) investigates the problem of reward-free RL with constraints. Wu et al. (2021) then proposes a reward-free algorithm for the multi-objective RL problem. In addition, Bai and Jin (2020); Liu et al. (2021); Qiu et al. (2021) further study the reward-free RL algorithms under the multi-agent setting.

Furthermore, we would like to emphasize that directly extending the existing results on reward-free exploration (see, e.g., Wang et al. (2020); Qiu et al. (2021)) to learning the dynamic VCG mechanism seems infeasible. The main reason is that these works focus only on estimating the optimal value functions corresponding to different reward functions. In contrast, in the context of mechanism design, we have multiple desiderata, namely the truthfulness, individual rationality, and efficiency, which mathematically translates into the various different regret notions such as the welfare regret and individual regret of the seller and the buyers. Showing that the proposed algorithm approximately satisfies these desiderata requires bounding these regret notions using the properties of the dynamic VCG mechanism as well as the results of reward-free exploration. Finally,
our work is related to Lyu et al. (2022), which focuses on learning the Markov VCG mechanism using offline RL from a set of collected trajectories. Under the offline setting, exploration is out of the scope and thus our core challenge caused the fictitious policy is absent in Lyu et al. (2022).

2 Problem Setup

Consider an episodic MDP defined by \( \mathcal{M}(\mathcal{S}, \mathcal{A}, H, \mathbb{P}, r) \), where \( \mathcal{S} \) and \( \mathcal{A} \) are state and action spaces, \( H \) the length of each episode, \( \mathbb{P} = \{ \mathbb{P}_h \}_{h=1}^H \) the transition kernel, and \( r = \{ r_{i,h} \}_{i=0, h=1}^n \) the reward functions. We use \( r_{0,h} : \mathcal{S} \times \mathcal{A} \mapsto [0, R_{\text{max}}] \) to denote the reward function of the seller at the step \( h \) and let \( r_{i,h} : \mathcal{S} \times \mathcal{A} \mapsto [0, 1] \) be the reward function of agent (buyer) \( i \) at the step \( h \) for \( i \in [n] \), where \( n \) is the number of agents.

Let \( \pi = \{ \pi_h \}_{h=1}^H \) denote the seller’s policy, where for each \( h \in [H] \), \( \pi_h : \mathcal{S} \mapsto \mathcal{A} \) maps a given state to an action. For each step \( h \in [H] \) and reward function \( r = \{ r_h \}_{h=1}^H \), we define the value function \( V^\pi_h(x; r) : \mathcal{S} \mapsto \mathbb{R} \) for all \( x \in \mathcal{S} \) as

\[
V^\pi_h(x; r) := \sum_{h'=h}^H \mathbb{E}\left[r_{h'}(s_{h'}, \pi_{h'}(s_{h'})) | s_h = x\right].
\]

We stress that while the MDP we consider contain multiple reward functions and interaction between multiple agents, our problem settings differs from the Markov game setting, as we assume that the seller is the only participant that can take action (Littman, 1994).

Dynamic Mechanism Design. We now describe how agents interact with the mechanism designer (seller) in our setting. At the beginning of each episode, the mechanism starts from the initial state \( x_1 \). At each step \( h \in [H] \), the seller observes some state \( x_h \in \mathcal{S} \), picks an action \( a_h \in \mathcal{A} \), and receives a reward \( r_{0,h}(x_h, a_h) \). Each agent (buyer) receives their own reward \( r_{i,h}(x_h, a_h) \). At the end of each episode, the seller charges each customer some price \( p_i \). For any policy \( \pi \) and prices \( \{ p_i \}_{i=1}^n \), we define agent \( i \)’s utility as

\[
u_i := \mathbb{E}_\pi \left[ \sum_{h=1}^H r_{i,h}(x_h, a_h) \right] - p_i = V^\pi_1(x_1; r_i) - p_i.
\]

That is, agent \( i \)’s utility is equal to the difference between the expected total reward and the charged price. The seller’s utility is similarly defined as

\[
u_0 := V^\pi_1(x_1; r_0) + \sum_{i=1}^n p_i.
\]
The social welfare, $W^\pi$, is defined as the sum of the agents and the seller’s utilities, given by

$$W^\pi(x_1) = \sum_{i=0}^{n} V^\pi_1(x_1; r_i) = V^\pi(x_1; \sum_{i=0}^{n} r_i),$$

which is equivalent to the expectation of the sum of all rewards as the prices cancel out. For convenience, we let $u_{it}, u_{0t}, p_{it}$ be the observed values of $u_i, u_0, p_i$ at the $t$-th round, respectively.

**Markov VCG Mechanism.** Suppose that the transition kernel is known, all agents and the seller know their own reward functions $r_{i,h}$ for all $(i, h) \in [n] \times [H]$, and the agents submit the true reward functions to the seller as bids. The VCG mechanism demands that we choose the welfare-maximizing policy $\pi^*$. Each agent $i$ is subsequently charged a price $p_{i*}$, which is the loss her presence causes to others. Hence we have the following definitions:

$$\pi_* := \arg \max_{\pi} V_{1}^\pi(x_1; R), \quad \pi_*^{-i} := \arg \max_{\pi} V_{1}^\pi(x_1; R^{-i}),$$

$$p_{i*} := V_{1}^\pi^{-i}(x_1; R^{-i}) - V_{1}^{\pi*}(x_1; R^{-i}),$$

where $R = \sum_{i=0}^{n} r_i$ is the total reward function, $R^{-i} = \sum_{j=0,j\neq i}^{n} r_j$ is the sum of reward function except agent $i$.

With the notations for optimal policies and various reward functions defined, we may now formally introduce the Markov VCG Mechanism, an extension of the classical VCG mechanism to the dynamic setting. Similar to the static counterpart, it satisfies a few ideal properties. The following lemma, obtained from concurrent work (Lyu et al., 2022), introduces the properties of the Markov VCG mechanism.

**Lemma 2.1** The Markov VCG mechanism satisfies the following desiderata in mechanism design:

1. **Truthfulness:** A mechanism is truthful if the utility $u_i$ of agent $i$ is maximized when, regardless of other agents’ bids, agent $i$ reports her rewards truthfully.
2. **Individual rationality:** A mechanism is individually rational, if it charges an agent no more than her bid, or equivalently, the utility $u_i$ of agent $i$ is non-negative, when agent $i$ is truthful.
3. **Efficiency:** A mechanism is efficient, if the mechanism maximizes the welfare when all agents are truthful.

An agent is truthful if this agent submits her reward functions and bids truthfully.

**Proof** The detailed proof for these three properties can be found in Appendix B of Lyu et al. (2022). We also include a sketch of the proof in Appendix B for completeness.
Mechanism Design with Unknown MDP. Consider the case where the agents’ value functions and the MDP’s transition kernel are unknown and the procedure is repeated for multiple rounds. We denote the sum of the utilities of agent \( i \) and the mechanism respectively over \( T \) rounds with \( U_{iT} \) and \( U_{0iT} \), that is, \( U_{iT} = \sum_{t=1}^{T} u_{it} \) and \( U_{0iT} = \sum_{t=1}^{T} u_{0t} \). An agent participates in the mechanism in one of the following ways:

1. **By bids**: Ahead of time, they may submit bids \( b_i = \{b_{ih}\}_{h=1}^{H} \), where for each \( h \in [H] \), \( b_{ih} : S \times A \mapsto [0,1] \) maps a given state-action pair to a number in \([0,1]\) (not necessarily truthful).

2. **By rewards**: Alternatively, they may report a realised reward \( \tilde{r}_{it} \) (not necessarily truthful) after the seller has taken actions.

Our goal is to design an algorithm that respects the three mechanism design desiderata over multiple rounds even when the true reward functions and transition kernels are unknown, as well as achieving sublinear regret for the agents, the seller, and in terms of welfare. The following metrics are used to quantify the algorithm’s performance:

\[
\begin{align*}
\text{Reg}_W^T &= TV_1^*(x_1; R) - \sum_{t=1}^{T} V_1^{\pi_t}(x_1; R) \\
\text{Reg}_{0T} &= Tu_{0*} - U_{0iT}, \quad \text{Reg}_{iT} = Tu_{i*} - U_{iT}, \quad \text{Reg}_s^T = \sum_{i=1}^{n} \text{Reg}_{iT}.
\end{align*}
\]

Here \( V_1^*(\cdot; r) \) is the maximum of the value function at the first step for a reward function \( r \). Additionally we let \( u_{0*} = V_1^*(x_1; r_0) + \sum_{i=1}^{n} p_{i*} \) and \( u_{i*} = V_1^*(x_1; r_i) - p_{i*} \) be the utilities of the seller and agent \( i \) respectively in the VCG mechanism, assuming the reward functions and transition kernels are known a priori. \( \text{Reg}_W^T \) is the welfare regret over \( T \) rounds, \( \text{Reg}_{0T} \) the seller regret, and \( \text{Reg}_{iT} \) the agent \( i \)'s regret, respectively. Additionally, let \( \text{Reg}_s^T \) be the sum of regrets over all agents.

In addition to attaining sublinear regret bounds, we would also like to achieve the desiderata we formulated for mechanism design. Here we state asymptotic variants of those desiderata attainable in our settings, which are adapted from the static case (Kandasamy et al., 2020).

1. **Truthfulness**: Let \( U_{iT} \) be the sum of utilities when agent \( i \) is truthful and \( \tilde{U}_{iT} \) the sum of utilities when agent \( i \) is untruthful. A mechanism is *asymptotically truthful* if \( \tilde{U}_{iT} - U_{iT} = o(T) \) even when other agents are not truthful.

2. **Individual rationality**: When agent \( i \) reports truthfully, the mechanism is *asymptotically individual rational* if \( \lim_{T \to \infty} U_{iT}/T \geq 0 \) even if other agents are untruthful.

3. **Efficiency**: A mechanism is *asymptotically efficient* if \( \text{Reg}_W^T = o(T) \) when all agents are truthful.

Note that an agent adopts untruthful reward reporting strategy means she reports her rewards with reward functions \( \{\tilde{r}_{ih}\} \) instead of the true reward functions \( \{r_{ih}\} \).
To handle the potentially large state and action spaces $S, A$, our work focuses on the linear MDP setting, which we introduce below formally.

**Linear MDP.** We assume that there exists a feature map $\phi : S \times A \mapsto \mathbb{R}^d$ where, for any $h \in [H]$, there are $d$ unknown measures $\mu_h = (\mu^1_h, \cdots, \mu^d_h)$ over $S$ and $n + 1$ unknown vectors $\{\theta_{ih}\}_{i=0}^n$ with each $\theta_{ih} \in \mathbb{R}^d$, such that for any $(x, a, x') \in S \times A \times S$, we have

$$P_h(x'|x, a) = \langle \phi(x, a), \mu_h(x') \rangle$$

$$E[r_{i,h}(x, a)] = \langle \phi(x, a), \theta_{ih} \rangle, \quad \forall i = 0, 1, \cdots, n.$$  

(2)

Following standard assumptions in the prior literature (Jin et al., 2019; Cai et al., 2019; Jin et al., 2020b), we assume $\|\phi(x, a)\| \leq 1$ for all $(x, a) \in S \times A$, max$\{\|\mu_h(S)\|, \|\theta_{ih}\|\} \leq \sqrt{d}$ for all $h \in [H], 0 \leq i \leq n$. Recall that the linear MDP assumption implies that the value functions and action-value functions are both linear in the feature space defined by $\phi$ (Jin et al., 2019).

### 3 Algorithm

In this section, we introduce our proposed algorithm for VCG mechanism learning on linear MDPs (VCG-LinMDP). The general learning framework of our algorithm is summarized in Algorithm 1, which is composed of two phases: the exploration phase and the exploitation phase. The exploration and exploitation phases are summarized in Algorithm 2, Algorithm 3, and Algorithm 4, whose detailed illustration is deferred to the appendix.

#### 3.1 Algorithmic Framework

**Markov VCG with Function Approximation.** For the Markov VCG learning, in this paper, we consider a learning framework with function approximation. In the above framework, the reward-free exploration phase aims to efficiently explore the environment with a wide coverage over the underlying policy space. The exploitation phase targets at utilizing the collected data to update the seller’s policy and estimate the prices to charge the agents. We remark that this learning framework fits any linear or nonlinear function approximators. Such a learning framework is summarized as follows:

1. Exploration for multiple rounds to collect an initial dataset. The exploration is performed via a reward-free least-square value iteration (LSVI) with function approximation (Jin et al., 2020a; Wang et al., 2020; Qiu et al., 2021).
2. Exploitation with the collected data. At each round $t$ of Exploitation phase:
   - Update the seller’s policy $\hat{\pi}^t$ via a Planning subroutine implemented as optimistic LSVI with function approximation w.r.t. reward $R$. 

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• Update $F_t^{-i}$ by the value function via a Planning subroutine implemented as optimistic or pessimistic LSVI with function approximation w.r.t. reward $R^{-i}$.
• Update $G_t^{-i}$ by the value function via a Policy Evaluation subroutine by optimistic or pessimistic evaluation with function approximation at the learned policy $\tilde{\pi}^t$ w.r.t. reward function $R^{-i}$.
• Estimate the price $p_{it} = F_t^{-i} - G_t^{-i}$ for all $i \in [n]$.
• Take actions following the learned policy $\hat{\pi}^t$ and charge each agent $i$ a price $p_{it}$ for $i \in [n]$.
• Determine whether we should update the dataset with the new trajectory.

This paper focuses on a special case, i.e., Markov VCG with linear function approximation named VCG-LinMDP, as shown in Algorithm 1. Then, the associated Exploration phase in implemented in Algorithm 2 and the Exploitation phase is implemented in Algorithms 3 and 4, where we adopt LSVI with linear function approximation. In particular, Algorithms 3 and 4 are the Planning and Policy Evaluation subroutines respectively. As shown in Algorithm 1, there are multiple hyper-parameters as input argument. Specifically, $\zeta_1$ is the hyper-parameter to control the overall learning strategy of VCG-LinMDP. The option ETC indicates the explore-then-commit strategy, where we only exploit the data generated during the exploration phase. EWC indicates explore-while-commit strategy, where we learn by the data generated not only during the exploration but also in the exploitation phase. The options OPT and PES for the hyper-parameters $\zeta_2$ and $\zeta_3$ refer to optimistic and pessimistic exploitation approaches respectively. Moreover, the hyper-parameter $R$ controls whether the input reward function is $R = \sum_{i=0}^{n} r_i$ or $R^{-i} = \sum_{j=0, j \neq i}^{n} r_j$.

**Least-Square Value Iteration.** For any function approximation class $\mathcal{F}$, at the $t$-th episode, we have $t-1$ transition tuples, $\{(x_{\tau}^t, a_{\tau}^t, x_{\tau+1}^t)\}_{\tau \in [t-1]}$. LSVI with function approximation (Jin et al., 2019; Yang et al., 2020) propose to estimate the Q-function by $\tilde{f}_h^t$ following the least-squares regression problem as below

$$
\tilde{f}_h^t = \arg \min_{f \in \mathcal{F}} \sum_{\tau = 1}^{t-1} [r_h(x_{\tau}^t, a_{\tau}^t) + V_h^t(x_{\tau}^t) - f_h(x_{\tau}^t, a_{\tau}^t)]^2 + \text{pen}(f),
$$

where pen$(f)$ is some penalty term and $r_h$ is some reward function. We further let $f_h^t = \text{truncate}\{\tilde{f}_h^t\}$ where truncate{$\cdot$} is some truncation operator to guarantee that the approximation function is in a correct scale. Then, for optimistic LSVI, we construct optimistic Q-function as

$$
Q_h^t = \text{truncate}\{f_h^t + u_h^t\},
$$

where truncate is some truncation operator to guarantee $Q_h^t$ is in a correct scale and $u_h^t$ is an
Algorithm 1 VCG-LinMDP

**Input:** $\zeta_1 \in \{ETC, EWC\}$, $\zeta_2, \zeta_3 \in \{OPT, PES\}$, $\mathcal{R} \in \{R, R^{-i}\}$, and $K$.

1: //Exploration Phase
2: Reward-free exploration for $K$ rounds via Algorithm 2 and obtain $\mathcal{D} = \{(x_h^k, a_h^k)\}_{k,h} \cup \{r_{i,h}(x_h^k, a_h^k)\}_{i,h,k}$.
3: //Exploitation Phase
4: for $t = K + 1, \ldots, T$ do
5: Update policy $\hat{\pi}_t$ by the returned policy of Algorithm 3 with input parameters $(R, \zeta_1, OPT, \mathcal{D})$.
6: Update $F_{t}^{-i}$ by the returned value function of Algorithm 3 with parameters $(R^{-i}, \zeta_1, \zeta_2, \mathcal{D})$ for all $i \in [n]$.
7: Update $G_{t}^{-i}$ by the returned value function of Algorithm 4 with parameters $(R^{-i}, \zeta_1, \zeta_3, \mathcal{D}, \hat{\pi}_t)$ for all $i \in [n]$.
8: Calculate the price $p_{it} = F_{t}^{-i} - G_{t}^{-i}$ for all $i \in [n]$.
9: Take action $a_{t}^i = \hat{\pi}_t^i(x_{t}^i)$, receive rewards $\{r_{i,h}(x_{t}^i, a_{t}^i)\}_{i,h}$, and observe $x_{h+1}^i \sim \mathbb{P}_h(\cdot|x_{h}^i, a_{h}^i)$ from $h = 1$ to $H$.
10: Charge each agent $i$ a price $p_{it}$ for all $i \in [n]$.
11: if $\zeta_1 = EWC$ then
12: $\mathcal{D} \leftarrow \mathcal{D} \cup \{(x_{t}^i, a_{t}^i)\}_{i,h} \cup \{r_{i,h}(x_{t}^i, a_{t}^i)\}_{i,h}$
13: else if $\zeta_1 = ETC$ then
14: Keep $\mathcal{D}$ unchanged.
15: end if
16: end for

associated UCB bonus term. Then, the pessimistic Q-function is constructed as

$$Q_{h}^t = \text{truncate}\{f_{h}^t - u_{h}^t\}.$$  

We update the value function by a greedy strategy as

$$V_{h}^t(\cdot) = \arg \max_{a \in A} Q_{h}^t(\cdot, a),$$

for optimistic Q-function or pessimistic Q-function respectively. For the liner function approximation, we let $f(\cdot, \cdot) = w^\top \phi(\cdot, \cdot)$ for any $f \in \mathcal{F}$ and $\text{pen}(f) = \lambda \|w\|^2$.

### 3.2 Exploration Phase

Inspired by Wang et al. (2020), we design a reward-free exploration algorithm as in Algorithm 3, incorporating the linear structure of the MDP. Specifically, to handle multiple reward functions from the seller and $n$ agents, we propose to explore the environment without using the observed rewards from it. Instead, we define an exploration-driven rewards $l_{h}^k$ as a scaled bonus term $u_{h}^k$ to encourage exploration by further taking into account the uncertainty of estimating the environment.
Algorithm 2 Exploration

Input: Failure probability $\delta > 0$, $K$, and $\lambda > 0$
1: $\beta = c_n dH \sqrt{\log(n d HK/\delta)}$ and $B = H(n + R_{max})$
2: for $k = 1, 2 \cdots, K$ do
3: $Q_{H+1}^k(\cdot, \cdot) = 0$ and $V_{H+1}^k(\cdot) = 0$.
4: for $h = H, H - 1 \cdots, 1$ do
5: $\Lambda_h^k = \sum_{\tau=1}^{k-1} \phi(x^\tau_h, a^\tau_h)\phi(x^\tau_h, a^\tau_h)^\top + \lambda I$.
6: $u_h^k(\cdot, \cdot) = \min \{ \beta[\phi(\cdot, \cdot)(\Lambda_h^k)^{-1}\phi(\cdot, \cdot)]^{1/2}, B \}$.
7: Define exploration-driven reward function $l_h^k(\cdot, \cdot) = u_h^k(\cdot, \cdot)/H$.
8: $w_h^k = (\Lambda_h^k)^{-1} \sum_{\tau=1}^{k-1} \phi(x^\tau_h, a^\tau_h)V_{h+1}^k(x^\tau_h+1)$.
9: $Q_h^k(\cdot, \cdot) = \min \{ \Pi_{[0,B]}[(w_h^k)^\top \phi(\cdot, \cdot)] + l_h^k(\cdot, \cdot) + u_h^k(\cdot, \cdot), B \}$.
10: $V_h^k(\cdot) = \max_{a \in A} Q_h^k(\cdot, a)$.
11: $\pi_h^k(\cdot) = \arg \max_{a \in A} Q_h^k(\cdot, a)$.
12: end for
13: Take action $a_h^k = \pi_h^k(x_h^k)$, receive rewards $\{r_{i,h}(x_h^k, a_h^k)\}_i$, and observe $x_{h+1}^k \sim P_h(\cdot|x_h^k, a_h^k)$ from $h = 1$ to $H$.
14: end for
15: return $D = \{ (x_h^k, a_h^k) \}_{k,h} \cup \{ r_{i,h}(x_h^k, a_h^k) \}_{i,h,k}$

The bonus term computed in Line 6 quantifies the uncertainty of estimation with a linear function approximator. Based on the exploration-driven rewards $l_h^k = u_h^k/H$ and the bonus term $u_h^k$ as well as the linear function approximation, we calculate an optimistic Q-function and perform the optimistic reward-free LSVI to generate the exploration policy. Note that in Algorithm 2 and the subsequent Algorithms 3 and 4, we define a truncation operator $\Pi_{[0,x]}[\cdot] := \max\{\min\{\cdot, x\}, 0\}$. Different from the standard LSVI introduced above, the reward-free LSVI only considers value function as the regression target, i.e., we solve a least-square regression problem in the following form

$$\arg \min_{f \in F_{lin}} \sum_{\tau=1}^{k-1} [V_{h}^k(x^\tau_h) - f_h(x^\tau_h, a^\tau_h)]^2 + pen(f),$$

where $F_{lin}$ is the linear function class. Then, we obtain the coefficient vector $w_h^k$ for linear function approximation. Moreover, for the optimistic Q-function in Line 9, we construct it by combining not only the linear approximation function and the exploration bonus $u_h^k$ but also the exploration-driven reward $l_h^k$. Meanwhile, we collect the trajectories of visited state-action pairs and the corresponding reward feedbacks of $r_i, \forall i = 0, 1, \cdots, n$, which is $D = \{(x_h^k, a_h^k)\}_{k,h} \cup \{ r_{i,h}(x_h^k, a_h^k) \}_{i,h,k}$ for the subsequent Exploitation phase in Algorithms 3 and 4.

3.3 Exploitation Phase

The exploitation phase is separated into two subroutines, namely Planning in Algorithm 3 and Policy Evaluation in Algorithm 4. These algorithms are two general subroutines whose realization
varies according to different inputs.

The basic idea of Planning subroutine in Algorithm 3 is an optimistic or pessimistic LSVI with linear function approximation to generate a greedy policy and the associated value function. In Line 6 of Algorithm 3, we compute a bonus $u_t^h$ to quantify the uncertainty in estimation. And in Lines 7 and 8, we obtain the coefficient vector $w_t^h$ for linear function approximation and the approximator $f_t^h$. Lines 10 and 12 give the optimistic and pessimistic Q-functions respectively. Different from Algorithm 3, the Policy Evaluation subroutine in Algorithm 4 only evaluates any input policy $\pi$ by computing the value function under $\pi$ with linear function approximation based on the optimistic or pessimistic construction of Q-function.

Moreover, for both subroutines, the composition of the set $P_t$ in Lines 5 and 7 is controlled by $\zeta_1$, which is defined as

$$P_t := \begin{cases} 
\{1, 2, \cdots, K\} & \text{if } \zeta_1 = \text{ETC} \\
\{1, 2, \cdots, t - 1\} & \text{if } \zeta_1 = \text{EWC}. 
\end{cases}$$

(3)

It indicates whether we should use the original exploration dataset or the updated dataset to construct the bonus term and the linear function approximator. The function $\alpha_h(\mathcal{R})$ in both algorithms controls the truncation constant, which is equal to the supremum of the corresponding reward function. Precisely, we define

$$\alpha_h(\mathcal{R}) := \begin{cases} 
(n + R_{\max})(H - h + 1) & \text{if } \mathcal{R} = R \\
(n - 1 + R_{\max})(H - h + 1) & \text{if } \mathcal{R} = R^{-i} \text{ for any } i \in [n]. 
\end{cases}$$

(4)

**Remark 3.1** Note that Algorithm 3 and Algorithm 4 are two generic subroutines for the exploitation phase, whose concrete implementation is corresponding to distinct input hyper-parameters. For brevity, we denote all the value functions and Q-functions in Algorithm 3 and Algorithm 4 calculated in step $t$ by $V_t^h(\cdot; \cdot)$ and $Q_t^h(\cdot, \cdot; \cdot)$ respectively. Specifically, in the rest of this work, for different realizations of Algorithm 3 and Algorithm 4, we let $\{\hat{V}_h^t,*,(\cdot; \mathcal{R}), \hat{Q}_h^t,*,(\cdot,\cdot; \mathcal{R})\}$ and $\{\check{V}_h^t,*,(\cdot; \mathcal{R}), \check{Q}_h^t,*,(\cdot,\cdot; \mathcal{R})\}$ correspond to $\zeta_2 = \text{OPT}$ and $\zeta_2 = \text{PES}$ respectively, which are generated by Algorithm 3; and let $\{\hat{V}_h^t,\pi,(\cdot; \mathcal{R}), \hat{Q}_h^t,\pi,(\cdot,\cdot; \mathcal{R})\}$ and $\{\check{V}_h^t,\pi,(\cdot; \mathcal{R}), \check{Q}_h^t,\pi,(\cdot,\cdot; \mathcal{R})\}$ be associated with $\zeta_3 = \text{OPT}$ and $\zeta_3 = \text{PES}$ respectively, which are generated by Algorithm 4 with arbitrary input policy $\pi$. Therefore, in Algorithm 1, we have

$$F_t^{-i} = \begin{cases} 
\hat{V}_1^t,*,(x_1; R^{-i}) & \text{if } \zeta_2 = \text{OPT} \\
\check{V}_1^t,*,(x_1; R^{-i}) & \text{if } \zeta_2 = \text{PES}, 
\end{cases}$$

$$G_t^{-i} = \begin{cases} 
\hat{V}_1^{t,\pi},(x_1; R^{-i}) & \text{if } \zeta_3 = \text{OPT} \\
\check{V}_1^{t,\pi},(x_1; R^{-i}) & \text{if } \zeta_3 = \text{PES}. 
\end{cases}$$

**Remark 3.2** We remark that VCG-LinMDP (Algorithm 1) is not a direct extension of reward-free
RL algorithms with function approximation (e.g., Jin et al. (2020a); Wang et al. (2020); Qiu et al. (2021)) which focus only on estimating the optimal value functions corresponding to different reward functions. In contrast, for learning the dynamic mechanism, we aim to achieve multiple desiderata as introduced in Section 2 with minimizing the corresponding regrets. In particular, as the dynamic VCG mechanism involve the fictitious policy defined as the optimal policy in the absence of each agent $i$, whose trajectories are never observed, we adopt reward-free exploration to learn such a policy. Furthermore, to show that the final policy output by the Exploitation phase enjoy the desired desiderata requires the particular structure of the dynamic mechanism. Besides, the Exploitation phase (Algorithm 3 and Algorithm 4) allows for optimism and pessimism in an online setting, inducing different price estimation strategies as discussed in Remark 3.1. Moreover, Algorithm 2 differs from standard reward-free RL algorithms by recording the received rewards of different agents during exploration and utilizing these collected rewards to learn the welfare-maximizing policy and the agents’ prices.

4 Main Results

In this section, we discuss our main theoretical results. We first state the results corresponding to the three desiderata in mechanism design when $\zeta_i = \text{ETC, EWC}$ respectively. Then we present the
lower bound of our problem. We begin with the results when $\zeta_1 = \text{ETC}$.

**Theorem 4.1** When $\zeta_1 = \text{ETC}$, with probability at least $1 - \delta$, the following results hold after executing Algorithm 1 for $T$ rounds:

1. Assuming all agents report truthfully, for all $\zeta_2, \zeta_3 \in \{\text{OPT}, \text{PES}\}$, the welfare regret satisfies
   \[ \text{Reg}_W^T = \tilde{O}\left((nH + d^{2/3}H^3)T^{2/3}\right). \]

2. Assuming all agents report truthfully, for all $\zeta_2, \zeta_3 \in \{\text{OPT}, \text{PES}\}$, the regret of agent $i$ satisfies
   \[ \text{Reg}_{i\text{IT}}^T = \tilde{O}\left((nH + d^{2/3}H^3)T^{2/3}\right). \]

3. Assume all agents report truthfully. When $(\zeta_2, \zeta_3) = (\text{PES}, \text{OPT})$, the regret of the seller satisfies
   \[ \text{Reg}_{0\text{IT}}^T = \tilde{O}\left((nH + nd^{2/3}H^3)T^{2/3}\right). \]

4. *Algorithm 1 is asymptotically individually rational.*

5. *Algorithm 1 is asymptotically truthful.*

In Algorithm 1, we use the hyperparameters $\zeta_2, \zeta_3$ to control the trade-off between the utility of the seller and the agents. When $\zeta_2 = \text{OPT}$ and $\zeta_3 = \text{PES}$, the price charged by the seller from the agents will be large, and thus the utility of the seller will be large while the utility of the agents will be small. On the other hand, when $\zeta_2 = \text{PES}$ and $\zeta_3 = \text{OPT}$, we can draw the opposite conclusion on the utility of the seller and the agents, i.e., the utility of the seller will be small while the utility of the agents will be large. However, since the welfare does not depend on the price, the choices of $(\zeta_2, \zeta_3)$ thus have no impact on the welfare regret. Although there are four different options for $(\zeta_2, \zeta_3)$ and we have analyzed all four cases in the appendix, we only highlight two most representative choices of $(\zeta_2, \zeta_3)$ here. As for the individual rationality and truthfulness in mechanism design, we present two asymptotic results with respect to the number of rounds $T$ here.

We further present the results for $\zeta_1 = \text{EWC}$ as follows.

**Theorem 4.2** When $\zeta_1 = \text{EWC}$, with probability at least $1 - \delta$, the following results hold after executing Algorithm 1 for $T$ rounds:

1. Hereafter, we use $\tilde{O}$ to hide the logarithmic dependence on $T, d, n, H,$ and $1/\delta$. 

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1. Assuming all agents report truthfully, for all \( \zeta_2, \zeta_3 \in \{\text{OPT}, \text{PES}\} \), the welfare regret satisfies

\[
\text{Reg}^W_T = \tilde{O}(nHT^{2/3} + d^{2/3}H^3T^{1/2})
\]

2. Assume all agents report truthfully. When \( (\zeta_2, \zeta_3) = (\text{OPT}, \text{PES}) \), the regret of agent \( i \) satisfies

\[
\text{Reg}_{iT} = \tilde{O}(nHT^{2/3} + d^{2/3}H^3(T^{1/2} + T^{2/3})).
\]

When \( (\zeta_2, \zeta_3) = (\text{PES}, \text{OPT}) \), the regret of agent \( i \) satisfies

\[
\text{Reg}_{iT} = \tilde{O}(nHT^{2/3} + d^{2/3}H^3T^{1/2}).
\]

3. Assume all agents report truthfully. When \( (\zeta_2, \zeta_3) = (\text{PES}, \text{OPT}) \), the regret of the seller satisfies

\[
\text{Reg}_0T = \tilde{O}((nH + nd^{2/3}H^3)T^{2/3}).
\]

When \( (\zeta_2, \zeta_3) = (\text{OPT}, \text{PES}) \), the regret of the seller satisfies

\[
\text{Reg}_0T = \tilde{O}(nHT^{2/3}).
\]

4. Algorithm 1 is asymptotically individually rational.

5. Algorithm 1 is asymptotically truthful.

Most of the results when \( \zeta_1 = \text{EWC} \) are the similar to those when \( \zeta_1 = \text{ETC} \). The most significant improvement is that we can get an \( \tilde{O}(nHT^{2/3} + d^{2/3}H^3T^{1/2}) \) upper bound for the welfare regret, which is strictly tighter than the \( \tilde{O}((nH + d^{2/3}H^3)T^{2/3}) \) one when \( \zeta_1 = \text{ETC} \). This improvement results from the use of all the data gathered up to time step \( t \) in \text{EWC} setting rather than only use the data collected in the exploration phase in \text{ETC} setting. The bounds for agent regrets are also improved, albeit to a lesser extent, which results from the tighter welfare regret.

At last, we provide the lower bound on the maximum of the three regret terms in (1) when all agents are truthful, which is the lower bound of our mechanism design problem. To formalize this, let \( \Theta \) and \( \text{Alg} \) be the class of problems under our problem and the class of algorithms for this setting respectively.

**Theorem 4.3** Let \( \text{Reg}^W_T, \text{Reg}^2_T, \text{Reg}_0T \) be as defined in (1). Let all agents be truthful. We have:

\[
\inf_{\text{Alg}} \sup_{\Theta} \mathbb{E} \left[ \max\left( n\text{Reg}^W_T, \text{Reg}^2_T, \text{Reg}_0T \right) \right] \geq \Omega(n^{4/3}H^{2/3}T^{2/3} + n^2d\sqrt{HT}),
\]

for \( T \geq \max\{16(n - 1)/(H - 1), 64(d - 3)^2H\} \), \( H \geq 2, d \geq 4 \) and \( n \geq 3 \).
The lower bound constructed in Theorem 4.3 matches the $\tilde{O}(T^{2/3})$ upper bounds of the three regrets established in Theorem 4.1 and Theorem 4.2, although there is still a gap between the upper and lower bounds in terms of the multiplicative factors $n, d, \text{ and } H$. We leave the derivation of matching upper and lower bounds as an open question for our future work.

Our work features several prominent contributions over existing literature in mechanism design learning and online learning of linear MDPs. As shown in Theorem 4.1 and Theorem 4.2, our work proposes the first algorithm capable of learning a dynamic mechanism with no prior knowledge. In particular, we further show that the mechanism learned by Algorithm 1 simultaneously satisfies asymptotic efficiency, asymptotic individual rationality, and asymptotic truthfulness. As we will demonstrate in the sequel, the satisfaction of the asymptotic versions of the three mechanism design desiderata are demonstrated through novel decomposition approaches. Moreover, Theorem 4.3 demonstrates that the seller and agents’ regrets are minimax optimal up to problem-dependent constants.

5 Theoretical Analysis

In this section we outline the analysis of our theorems. The formal proof is deferred to the appendix.

5.1 Proof Sketch of Theorem 4.1

Welfare Regret. We can decompose the regret into two parts: the regret incurred in the exploration phase and the regret incurred in the exploitation phase as

$$\text{Reg}^W_T = \sum_{t=1}^{K} \text{reg}^W_t + \sum_{t=K+1}^{T} \text{reg}^W_t, \tag{5}$$

where $\text{reg}^W_t := V_1^*(x_1; R) - V_1^{\pi_t}(x_1; R)$ is the instantaneous welfare regret. For the first part, we bound the instantaneous regret $\text{reg}^W_t$ at each time step with $H(n + R_{\text{max}})$, the maximum of the instantaneous regret at each round. Then, we can obtain an $\tilde{O}(nHK)$ upper bound for the first summation in Equation (5) where we set $K = T^{2/3}$ as a trade-off between the regrets for the exploration and exploitation steps. For the second summation in Equation (5), we can bound each term, namely the instantaneous welfare regret $\text{reg}^W_t$ at round $t$, by analyzing the optimistic LSVI for exploitation on the linear MDP inspired by (Wang et al., 2020). We then obtain an $\tilde{O}(d^{3/2}H^3T^{2/3})$ upper bound of the second summation in (5). Combining the two parts, we obtain an $\tilde{O}((nH + d^{3/2}H^3)T^{2/3})$ upper bound for the welfare regret $\text{Reg}^W_T$ over $T$ rounds.

Agent Regret. We can also decompose the agent regret into two parts: the regret incurred in the
exploration phase and the regret incurred in the exploitation phase as

\[
\text{Reg}_{it} = \sum_{t=1}^{K} \text{reg}_{it} + \sum_{t=K+1}^{T} \text{reg}_{it},
\]

where \( \text{reg}_{it} := u_{is} - u_{it} \) is the instantaneous regret of agent \( i \). For the first part, we can bound it in the same way as we bound the first term in Equation (5) and obtain an \( \tilde{O}(nHT^{2/3}) \) upper bound. For the second part, we first decompose the instantaneous regret of agent \( i \) at round \( t \) to several simple terms as follows

\[
\text{reg}_{it} = \left[ F_t^{-i} - V_1^*(x_1; R^{-i}) \right] + \left[ V_1^\pi_t(x_1; R^{-i}) - G_t^{-i} \right] + \left[ V_1^*(x_1; R) - V_1^\pi_t(x_1; R) \right],
\]

where the first two terms can be viewed as the errors when we estimate the price \( p_i \) and the last term is the welfare regret. When we set \((\zeta_2, \zeta_3) = (\text{PES}, \text{OPT})\), the first two terms are non-positive since \( F_t^{-i} \) and \( G_t^{-i} \) are the pessimistic and optimistic estimations of \( V_1^*(x_1; R^{-i}) \) and \( V_1^\pi_t(x_1; R^{-i}) \) respectively. Then, we can bound \( \sum_{t=K+1}^{T} \text{reg}_{it} \) by the upper bound of \( \sum_{t=K+1}^{T} \text{reg}^W \) which is in the order \( \tilde{O}(d^{3/2}H^3T^{2/3}) \). When we set \((\zeta_2, \zeta_3) = (\text{OPT}, \text{PES})\), \( F_t^{-i} \) and \( G_t^{-i} \) are the optimistic and pessimistic estimations of \( V_1^*(x_1; R^{-i}) \) and \( V_1^\pi_t(x_1; R^{-i}) \) respectively, and the summation over \( T - K \) rounds in the exploitation phase of these two terms can be bounded by \( \tilde{O}((d^{3/2}H^3)T^{2/3}) \) also. Then we can bound \( \sum_{t=K+1}^{T} \text{reg}_{it} \) by \( \tilde{O}(d^{3/2}H^3T^{2/3}) \). Combining the two parts, we obtain an \( \tilde{O}((nH + d^{3/2}H^3)T^{2/3}) \) upper bound for the agent regret under the two choices of \((\zeta_2, \zeta_3)\).

**Seller Regret.** We also decompose the seller regret into two parts: the regret incurred in the exploration phase and the regret incurred in the exploitation phase as

\[
\text{Reg}_{0t} = \sum_{t=1}^{K} \text{reg}_{0t} + \sum_{t=K+1}^{T} \text{reg}_{0t},
\]

where \( \text{reg}_{0t} := u_{0s} - u_{0t} \) is the instantaneous regret of the seller. For the first part, we similarly bound it in the way of bounding the first term in Equation (5) and obtain an \( \tilde{O}(nHT^{2/3}) \) upper bound. For the second part, we further decompose the instantaneous regret of the seller at round \( t \) as follows

\[
\text{reg}_{0t} = (n - 1) \left[ V_1^\pi_t(x_1; R) - V_1^*(x_1; R) \right] + \sum_{i=1}^{n} \left[ V_1^*(x_1; R^{-i}) - F_t^{-i} \right] + \sum_{i=1}^{n} \left[ G_t^{-i} - V_1^\pi_t(x_1; R^{-i}) \right],
\]

where the first term can be viewed as the error of estimating the cumulative reward of the seller and the last two terms can be seen as the error of estimating the prices of all the agents. Notice that the first term is non-positive since \( V_1^* \) is the maximum of the value function. When \((\zeta_2, \zeta_3) = (\text{OPT}, \text{PES})\), the last two terms are non-positive since \( F_t^{-i} \) and \( G_t^{-i} \) are the optimistic and pessimistic estimations of
we just need to consider the regrets incurred in the exploration phase and thus can bound Reg_{0T} by \(\tilde{O}(nHT^{2/3})\). When \((\kappa_2, \kappa_3) = (PES, OPT)\), \(F_t^{-i}\) and \(G_t^{-i}\) are the pessimistic and optimistic estimations of \(V_1^\pi^+(x_1; R^{-i})\) and \(V_1^\pi^-(x_1; R^{-i})\) respectively, and the summation over \(T - K\) rounds in the exploitation phase of the last two terms can be bounded by \(\tilde{O}((nd^{3/2}H^3)T^{2/3})\) also. Then we can bound Reg_{0T} by \(\tilde{O}((nH + nd^{3/2}H^3)T^{2/3})\) by adding the regrets incurred in the exploration phase.

Next, we provide the proof sketches of individually rationality and truthfulness. Note that in the following analysis, we do not assume the agents are reporting truthfully. We denote the reported reward of agent \(i\) at \(h\) step by \(\tilde{r}_{ih}\) and then \(\tilde{r}_i = \{\tilde{r}_{ih}\}_{h=1}^H\). Let \(\tilde{R}^{-i} = \sum_{j=1, j \neq i}^{n} \tilde{r}_j\) be the reported reward of \(R^{-i}\).

Individual Rationality. As shown in the appendix, the instantaneous utility \(u_{it}\) of agent \(i\) can be decomposed as

\[
u_{it} = \left[V_1^{\pi^+(x_1; R^{-i})} - F_t^{-i}\right] + \left[G_t^{-i} - V_1^{\pi^-(x_1; R^{-i})}\right],
\]

where the first term is the error when we estimate the value function \(V_1^{\pi^+(x_1; R)}\) in the pricing part of VCG mechanism, and the second term is the error when we estimate \(V_1^{\pi^-(x_1; R^{-i})}\). Note that we do not charge the agents in the exploration phase, the utilities in this phase are always non-negative. In this way, we just need to consider the utilities in the exploitation phase and we can regard the underlying MDP as the one with the reported rewards from the agents. Thus, we can then lower bound the summation of the above two terms over \(T\) episodes by \(\tilde{O}( - d^{2/3}H^3T^{2/3})\) respectively. Combining these two parts, we have \(\lim_{T \to \infty} U_{iT}/T \geq 0\) where \(U_{iT} = \sum_{t=1}^{T} u_{it}\). This implies our algorithm is asymptotically individual rational.

Truthfulness. We first declare some notations here. We let \(\nu_{i}\) be a strategy used by agent \(i\) for reporting rewards when she is untruthful and \(\nu_{i}\) be the strategy used by agent \(i\) for reporting rewards when she is truthful. Furthermore, let \(\nu_i^+\) be the reporting strategy that an agent first follows \(\nu_{i}\) for round 1, 2, \(\cdots\), \(r - 1\) and switches to \(\nu_i\) for round \(r, \cdots, T\). Let \(\tilde{u}_{it}\) and \(u_{it}^{-1}\) be the instantaneous utilities of agent \(i\) in round \(t\) using reporting strategy \(\tilde{\nu}_{i}\) or \(\nu_i^{-1}\). Let \(\tilde{U}_{IT}\) be the summation of utilities \(\tilde{u}_{it}\) using reporting strategy \(\tilde{\nu}_{i}\). We then have the following decomposition

\[
\tilde{U}_{IT} - U_{IT} = \sum_{t=1}^{T} [\tilde{u}_{it} - u_{it}^{-1}] + \sum_{t=2}^{T} [u_{it}^{-1} - u_{it}].
\]
We can further decompose the term in the first summation in Equation (9) as

\[
\tilde{u}_{it} - u_{it} = \left[ V_{1}^{\pi_{t-1}}(x_1; r_i + \tilde{R}^{-i}) - \hat{V}_{1}^{\pi_{t}}(x_1; r_i + \tilde{R}^{-i}) \right] \\
+ \left[ \tilde{V}_{1}^{\pi_{t-1}}(x_1; r_i + \tilde{R}^{-i}) - V_{1}^{\pi_{t-1}}(x_1; r_i + \tilde{R}^{-i}) \right] \\
+ \left[ G_{t}^{i} - V_{1}^{\pi_{t}}(x_1; \tilde{R}^{-i}) \right] + \left[ V_{1}^{\pi_{t-1}}(x_1; \tilde{R}^{-i}) - G_{t}^{i} \right],
\]

(10)

where the first two terms represent the error when estimating the value functions and the last two terms represent the error when estimating \( G \) functions. For the term in the second summation in Equation (9), we have the following decomposition,

\[
u_{it}^{t-1} - u_{it} = \left[ V_{1}^{\pi_{t-1}}(x_1; r_i) - V_{1}^{\pi_{t}}(x_1; r_i) \right] \\
+ \left[ V_{1}^{\pi_{t-1}}(x_1; \tilde{R}^{-i}) - F_{t}^{i} \right] + \left[ F_{t}^{i} - V_{1}^{\pi_{t}}(x_1; \tilde{R}^{-i}) \right] \\
+ \left[ G_{t}^{i} - V_{1}^{\pi_{t-1}}(x_1; \tilde{R}^{-i}) \right] + \left[ V_{1}^{\pi_{t}}(x_1; \tilde{R}^{-i}) - G_{t}^{i} \right],
\]

(11)

where the first term is the difference between the cumulative rewards of agent \( i \) when adopting different reporting strategies, and the following four terms are the error terms for estimating the price. Then we can bound all the terms in Equations (10) and (11) using the same technique in the analysis of individual rationality and the regrets. In this way, we get the conclusion that \( \tilde{U}_{iT} - U_{iT} = \tilde{O}(T^{2/3}) \), which implies our algorithm is asymptotically truthful.

### 5.2 Proof Sketch of Theorem 4.2

The proof idea of Theorem 4.2 is nearly identical to the one of Theorem 4.1. The key difference is that we use all the information gathered up to round \( t \) instead of the first \( K \) rounds. This leads to a tighter upper bound when we bound the welfare regret terms in the exploitation phase, i.e., \( \sum_{t=K+1}^{T} \text{reg}_{i}^{W} \) in Equation (5) and the summation of \( V_{1}^{\pi}(x_1; R) - V_{1}^{\pi_{t}}(x_1; \tilde{R}) \) in Equation (7). Precisely, we can bound the summation of the instantaneous welfare regret in the exploitation phase by \( \tilde{O}(nd^{2/3}H^{3}T^{1/2}) \) using elliptical potential lemma (Abbasi-Yadkori et al., 2011) and Hoeffding inequality. Combining the regret in the exploration phase, which is \( \tilde{O}(nHT^{2/3}) \), and the regret in the exploitation phase as above, we obtain a tighter \( \tilde{O}(nHT^{2/3} + d^{2/3}H^{3}T^{1/2}) \) upper bound for the welfare regret.

### 5.3 Proof Sketch of Theorem 4.3

Our lower bound construction is divided into two parts. The first part is inspired by the proof of the lower bound for VCG learning under the bandit setting (Kandasamy et al., 2020). Extending the two instances under the bandit setting in the proof of Theorem 1 in Kandasamy et al. (2020),
we construct two problems $\theta_0$ and $\theta_1$ with the underlying MDPs sharing the same feature mappings and transition kernels but have differences in reward functions, which allows us to take the horizon $H$ into consideration. In this way, we obtain the lower bound in an order of $\Omega(n^{4/3}H^{2/3}T^{2/3})$. For the second part, notice that $\max\{n\text{Reg}_T^W, \text{Reg}_T^+, \text{Reg}_T^0\} \geq n\text{Reg}_T^W$ always holds, which implies we can lower bound this term following the idea of the lower bound for the regret of learning linear MDPs with dimension $d$ (Zhou et al., 2020). Thus, inspired by the aforementioned work, we give the lower bound of $n\text{Reg}_T^W$ as $\Omega(n(n+1)d\sqrt{HT})$, where the factor $n+1$ reflects the impact of $n+1$ reward functions in our problem on the lower bound. Combining the above two parts, we obtain the lower bound for our mechanism design problem which is $\Omega(n^{4/3}H^{2/3}T^{2/3} + n^2d\sqrt{HT})$.

6 Conclusion

We consider the dynamic mechanism design problem on an unknown linear MDP. We focus on the online setting and attempt to recover the dynamic VCG mechanism over multiple rounds of interaction. Our work incorporates the reward-free online RL to aid exploration over a rich policy space to estimate prices in the dynamic VCG mechanism. We further show that our algorithm is efficient and incurs an $\tilde{O}(T^{2/3})$ regret, matching a lower bound proved in this work. Our work is the first work that directly tackles dynamic mechanism design problems without prior knowledge of reward functions, transition kernel, or optimal policy.

References

ABBASI-YADKORI, Y., PÁL, D. and SZEPESVÁRI, C. (2011). Improved algorithms for linear stochastic bandits. In NIPS, vol. 11.

ATHEY, S. and SEGAL, I. (2013). An efficient dynamic mechanism. Econometrica 81 2463–2485.

AYOUB, A., JIA, Z., SZEPESVARI, C., WANG, M. and YANG, L. (2020). Model-based reinforcement learning with value-targeted regression. In International Conference on Machine Learning. PMLR.

BAI, Y. and JIN, C. (2020). Provable self-play algorithms for competitive reinforcement learning. In International conference on machine learning. PMLR.

BAPNA, A. and WEBER, T. A. (2005). Efficient dynamic allocation with uncertain valuations. Available at SSRN 874770.

BARRERA, J. and GARCIA, A. (2014). Dynamic incentives for congestion control. IEEE Transactions on Automatic Control 60 299–310.
Bejestani, A. K. and Annaswamy, A. (2014). A dynamic mechanism for wholesale energy market: Stability and robustness. *IEEE Transactions on Smart Grid* **5** 2877–2888.

Bergemann, D. and Pavan, A. (2015). Introduction to symposium on dynamic contracts and mechanism design. *Journal of Economic Theory* **159** 679–701.

Bergemann, D. and Välimäki, J. (2006). Efficient dynamic auctions. Tech. rep., Cowles Foundation for Research in Economics, Yale University.

Bergemann, D. and Välimäki, J. (2010). The dynamic pivot mechanism. *Econometrica* **78** 771–789.

Bergemann, D. and Välimäki, J. (2019). Dynamic mechanism design: An introduction. *Journal of Economic Literature* **57** 235–74.

Cai, Q., Yang, Z., Jin, C. and Wang, Z. (2019). Provably efficient exploration in policy optimization. *arXiv preprint arXiv:1912.05830*.

Cavallo, R. (2008). Efficiency and redistribution in dynamic mechanism design. In *Proceedings of the 9th ACM conference on Electronic commerce*.

Cavallo, R. (2009). Mechanism design for dynamic settings. *ACM SIGecom Exchanges* **8** 1–5.

Cavallo, R., Parkes, D. C. and Singh, S. (2009). Efficient mechanisms with dynamic populations and dynamic types. *Harvard University Technical Report*.

Chen, M. K. and Sheldon, M. (2016). Dynamic pricing in a labor market: Surge pricing and flexible work on the Uber platform. *Ec* **16** 455.

Chen, X., Hu, J., Yang, L. F. and Wang, L. (2021). Near-optimal reward-free exploration for linear mixture MDPs with plug-in solver. *arXiv preprint arXiv:2110.03244*.

Clarke, E. H. (1971). Multipart pricing of public goods. *Public choice* **17**–33.

d’Aspremont, C. and Gérard-Varet, L.-A. (1979). Incentives and incomplete information. *Journal of Public economics* **11** 25–45.

Du, S. S., Kakade, S. M., Wang, R. and Yang, L. F. (2019). Is a good representation sufficient for sample efficient reinforcement learning? *arXiv preprint arXiv:1910.03016*.

Friedman, E. J. and Parkes, D. C. (2003). Pricing WiFi at Starbucks: issues in online mechanism design. In *Proceedings of the 4th ACM conference on Electronic commerce*.
Groves, T. (1979). Efficient collective choice when compensation is possible. *The Review of Economic Studies* **46** 227–241.

Jin, C., Krishnamurthy, A., Simchowitz, M. and Yu, T. (2020a). Reward-free exploration for reinforcement learning. In *International Conference on Machine Learning*. PMLR.

Jin, C., Yang, Z., Wang, Z. and Jordan, M. I. (2019). Provably efficient reinforcement learning with linear function approximation. *arXiv preprint arXiv:1907.05388*.

Jin, Y., Yang, Z. and Wang, Z. (2020b). Is pessimism provably efficient for offline RL? *arXiv preprint arXiv:2012.15085*.

Kandasamy, K., Gonzalez, J. E., Jordan, M. I. and Stoica, I. (2020). Mechanism design with bandit feedback. *arXiv preprint arXiv:2004.08924*.

Karlin, A. R. and Peres, Y. (2017). *Game theory, alive*, vol. 101. American Mathematical Soc.

Kaufmann, E., Ménard, P., Domingues, O. D., Jonsson, A., Leurent, E. and Valko, M. (2021). Adaptive reward-free exploration. In *Algorithmic Learning Theory*. PMLR.

Kong, D., Salakhutdinov, R., Wang, R. and Yang, L. F. (2021). Online sub-sampling for reinforcement learning with general function approximation. *arXiv preprint arXiv:2106.07203*.

Littman, M. L. (1994). Markov games as a framework for multi-agent reinforcement learning. In *Machine learning proceedings 1994*. Elsevier, 157–163.

Liu, Q., Yu, T., Bai, Y. and Jin, C. (2021). A sharp analysis of model-based reinforcement learning with self-play. In *International Conference on Machine Learning*. PMLR.

Lyu, B., Wang, Z., Kolar, M. and Yang, Z. (2022). Pessimism meets VCG: Learning dynamic mechanism design via offline reinforcement learning.

Ménard, P., Domingues, O. D., Jonsson, A., Kaufmann, E., Leurent, E. and Valko, M. (2021). Fast active learning for pure exploration in reinforcement learning. In *International Conference on Machine Learning*. PMLR.

Miryoosefi, S. and Jin, C. (2021). A simple reward-free approach to constrained reinforcement learning. *arXiv preprint arXiv:2107.05216*.

Myerson, R. B. (1989). Mechanism design. In *Allocation, Information and Markets*. Springer, 191–206.

Parkes, D. C. (2007). Online mechanisms. In *Algorithmic Game Theory* (N. Nisan, T. Roughgarden, E. Tardos and V. Vazirani, eds.). Cambridge University Press, 411–439.
Parkes, D. C. and Singh, S. (2003). An MDP-based approach to online mechanism design. In Proceedings of the 16th International Conference on Neural Information Processing Systems.

Parkes, D. C., Singh, S. and Yanovsky, D. (2004). Approximately efficient online mechanism design. In Proceedings of the 17th International Conference on Neural Information Processing Systems.

Pavan, A., Segal, I. and Toikka, J. (2014). Dynamic mechanism design: A Myersonian approach. Econometrica 82 601–653.

Pavan, A., Segal, I. R. and Toikka, J. (2009). Dynamic mechanism design: Incentive compatibility, profit maximization and information disclosure. Profit Maximization and Information Disclosure (May 1, 2009).

Qiu, S., Ye, J., Wang, Z. and Yang, Z. (2021). On reward-free rl with kernel and neural function approximations: Single-agent mdp and markov game. In International Conference on Machine Learning. PMLR.

Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. The Journal of finance 16 8–37.

Wagenmaker, A., Chen, Y., Simchowitz, M., Du, S. S. and Jamieson, K. (2022). Reward-free RL is no harder than reward-aware RL in linear Markov decision processes. arXiv preprint arXiv:2201.11206.

Wang, R., Du, S. S., Yang, L. F. and Salakhutdinov, R. (2020). On reward-free reinforcement learning with linear function approximation. arXiv preprint arXiv:2006.12274.

Wu, J., Yang, L. ET AL. (2021). Accommodating picky customers: Regret bound and exploration complexity for multi-objective reinforcement learning. Advances in Neural Information Processing Systems 34.

Yang, L. and Wang, M. (2019). Sample-optimal parametric Q-learning using linearly additive features. In International Conference on Machine Learning. PMLR.

Yang, L. and Wang, M. (2020). Reinforcement learning in feature space: Matrix bandit, kernels, and regret bound. In International Conference on Machine Learning. PMLR.

Yang, Z., Jin, C., Wang, Z., Wang, M. and Jordan, M. I. (2020). On function approximation in reinforcement learning: Optimism in the face of large state spaces. arXiv preprint arXiv:2011.04622.
ZANETTE, A., LAZARIC, A., KOCHENDERFER, M. J. and BRUNSKILL, E. (2020). Provably efficient reward-agnostic navigation with linear value iteration. Advances in Neural Information Processing Systems 33 11756–11766.

ZHANG, Z., DU, S. and JI, X. (2021). Near optimal reward-free reinforcement learning. In International Conference on Machine Learning. PMLR.

ZHOU, D., GU, Q. and SZEPESVARI, C. (2021a). Nearly minimax optimal reinforcement learning for linear mixture Markov decision processes. In Conference on Learning Theory. PMLR.

ZHOU, D., HE, J. and GU, Q. (2021b). Provably efficient reinforcement learning for discounted MDPs with feature mapping. In International Conference on Machine Learning. PMLR.

ZHOU, H., CHEN, J., VARSHNEY, L. R. and JAGMOHAN, A. (2020). Nonstationary reinforcement learning with linear function approximation. arXiv preprint arXiv:2010.04244 .
A Table of Notation

We present the following table of notations.

| Notation | Meaning |
|----------|---------|
| $R$      | summation of the reward functions of the seller and the agents |
| $R^{-i}$ | summation of the reward functions except agent $i$ |
| $\mathcal{X}$ | the log term in regret bound $\log(2ndHK/\delta)$ |
| $\nu_i, \tilde{\nu}_i, \nu^{t-1}_i$ | truthful strategy, untruthful strategy, the strategy of adopting $\tilde{\nu}_i$ for rounds $1, \ldots, t - 2$ and then switching to $\nu_i$ |
| $u_{it}, \tilde{u}_{it}, u^{t-1}_{it}$ | the utility of agent $i$ at time step $t$ when adopting strategy $\nu_i, \tilde{\nu}_i, \nu^{t-1}_i$ respectively |
| $G^1_t, G^{t-1}_t$ | the $G$ function of agent $i$ at time step $t$ when adopting strategy $\tilde{\nu}_i, \nu^{t-1}_i$ respectively |
| $F^1_t, F^{t-1}_t$ | the $F$ function of agent $i$ at time step $t$ when adopting strategy $\tilde{\nu}_i, \nu^{t-1}_i$ respectively |
| $\hat{V}, \hat{Q}$ | the value function and the action value function calculate when $\zeta_2, \zeta_3 = \text{OPT}$ |
| $\tilde{V}, \tilde{Q}$ | the value function and the action value function calculate when $\zeta_2, \zeta_3 = \text{PES}$ |

B Proof of Lemma 2.1

We include below the proof for Lemma 2.1 for completeness.

**Proof** [Proof of Lemma 2.1] The proof of Linear Markov VCG Mechanism’s properties are as follows:

1. **Truthfulness**: When agent $i$ reports the rewards untruthfully, it may change the optimal policy of $V^\pi_1(x; R)$ by altering only the reported value of $r_i$ and hence the associated value function $V^\pi_1(; r_i)$. However, agent $i$ cannot affect the value of $V^\pi_1(x; R^{-i})$, as $R^{-i}$ is independent of $r_i$. Let $\tilde{\nu}_i$ be the untruthful value function reported by agent $i$ and let $\tilde{\pi} = \arg \max_{\pi \in \Pi} V^\pi_1(x; \tilde{\nu}_i + R^{-i})$. Under VCG mechanism, agent $i$ has the following utility

$$u_i = V^\pi_1(x; r_i) - V^{\pi^{-i}}_1(x; R^{-i}) + V^\tilde{\pi}_1(x; R^{-i}) = V^\tilde{\pi}_1(x; R) - V^{\pi^{-i}}_1(x; R^{-i}).$$
In addition, we know \( u_i = V_1^*(x; R) - V_{1}^{\pi^-}(x; R^-) \) if agent \( i \) reports truthfully. Thus, we have \( u_i \geq \tilde{u}_i \), which is due to that \( \pi^* \) is the maximizer of \( V_1^*(x; R) \).

2. Individual Rationality: For any agent \( i \in [n] \), their utility is given by

\[
\begin{align*}
\quad u_i &= V_1^\pi(x; r_i) - p_i = V_1^\pi(x; R) - V_{1}^{\pi^-}(x; R^-) \\
&\geq V_{1}^{\pi^-}(x; R) - V_{1}^{\pi^-}(x; R^-) = V_{1}^{\pi^-}(x; r_i) \geq 0,
\end{align*}
\]

where we use the fact that \( r_{i,h}(s,a) \geq 0 \) for all \((i,h,s,a) \in [n] \times [H] \times S \times A\).

3. Efficiency: The chosen policy \( \pi^* \) is the maximizer of the value-function of welfare \( V_1^*(x_1; R) \) and hence is efficient.

This completes the proof.

C Proof of Theorems 4.1 and 4.2

In this section, we provide the detailed proofs of Theorem 4.1 and 4.2.

C.1 Proof of Theorem 4.1

Proof [Proof of Theorem 4.1]

Welfare Regret: Recall the definition of welfare regret in Equation (1) that \( \text{Reg}_W^T = \sum_{t=1}^{T} \text{reg}_t^W \), where \( \text{reg}_t^W = V_1^*(x; R) - \hat{V}_t^*(x; R) \). We split the instantaneous regret terms to obtain

\[
\text{Reg}_W^T = \sum_{t=1}^{K} \text{reg}_t^W + \sum_{t=K+1}^{T} \text{reg}_t^W. \tag{13}
\]

For the first summation in Equation (13), we have

\[
\sum_{t=1}^{K} \text{reg}_t^W \leq KH(n + R_{\text{max}}) = H(n + R_{\text{max}})T^{2/3} \tag{14},
\]

since we choose \( K = T^{2/3} \) and \( \text{reg}_t^W \leq H(n + R_{\text{max}}) \) for any \( t \in [K] \). The following lemma gives an upper bound on the instantaneous welfare regret \( \text{reg}_t^W \) in the exploitation phase.

Lemma C.1 With probability at least \( 1 - \delta/(c_3n) \), we have

\[
V_1^*(x_1; R) - V_{1}^{\pi^*}(x_1; R) \leq 2 \sum_{h=1}^{H} \mathbb{E}_{\pi^*} [u^*_h \mid s_1 = x_1],
\]

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where $c_\delta$ is an absolute constant and the bonus $\{u_h^i\}$ are calculated according to the corresponding trajectories and reported rewards.

With the above lemma, we have

$$\text{reg}^W_t \leq 2 \sum_{h=1}^{H} \mathbb{E}_{\pi^t}[u_h^t | s_1 = x_1] = V_1^{\pi^t}(x_1; u^t).$$

Then our goal turns into upper bound the summation of the bonus $\sum_{h=1}^{H} \mathbb{E}_{\pi^t}[u_h^t | s_1 = x_1]$, which is given by the following lemma.

**Lemma C.2** With probability at least $1 - \delta/(c_\delta n)$, for the function $u_h^i$ defined in Line 3 in Algorithm 3, we have

$$V_1^*(x_1; u^t) \leq c_\beta \sqrt{d^3 H^6 \mathcal{X}/K}$$

for some absolute constant $c_\beta$ and $\mathcal{X} = \log (2c_\delta ndHK/\delta)$.

Then, for the terms in the second summation in Equation (13), we have

$$\text{reg}^W_t = V_1^*(x_1; R) - V_1^{\pi^t}(x_1; R) \leq V_1^{\pi^t}(x_1; u^t) \leq V_1^*(x_1; u^t) \leq c_\beta \sqrt{d^3 H^6 \mathcal{X}/K},$$

where $\mathcal{X} = \log(2c_\delta ndHK/\delta)$ and the first equation is by the definition of the instantaneous regret $\text{reg}^W_t$, the first inequality is due to Lemma C.1, the second inequality is by the optimality of $V^*$, and the third inequality is by Lemma C.2. Summing the above equation form $t = K + 1$ to $T$, we have

$$\sum_{t=K+1}^{T} \text{reg}^W_t \leq c_\beta \sqrt{d^3 H^6 \mathcal{X}} \cdot T^{2/3},$$

due to our choice of $K = T^{2/3}$. And thus we upper bound the second summation in Equation (13). Combining Equation (14) and Equation (16), we obtain the upper bound of welfare regret under ETC setting.

**Agent Regret:** Recall the definition of agent regret in Equation (1) that $\text{Reg}_{iT} = \sum_{t=1}^{T} \text{reg}_{it}$ where $\text{reg}_{it} = u_{it}^s - u_{it}$. We split the instantaneous regret terms to obtain

$$\text{Reg}_{iT} = \sum_{t=1}^{K} \text{reg}_{it} + \sum_{t=K+1}^{T} \text{reg}_{it}. $$

For the first summation in Equation (17), we define $c_i^\dagger$ below to bound the instantaneous regret of agent $i$ during the exploration phase

$$c_i^\dagger = \max \{ V_1^*(x; r_i) - p_{is} - \min_{\pi} V_1^\pi(x; r_i), 0 \}.$$
Then we have
\[ \sum_{t=1}^{K} \text{reg}_{it} \leq c_{\beta}^{\frac{1}{4}} K = c_{\beta}^{\frac{1}{4}} T^{2/3}, \]
for we choose \( K = T^{2/3} \). For the second summation in Equation (17), we decompose the instantaneous regret at round \( t \) and have
\[
\text{reg}_{it} = u_{is} - u_{it} = [V_1^\pi(x_1;r_i) - V_1^{\pi^*_i}(x_1;R^-) + V_1^{\pi^*_i}(x_1;R^-)] - [V_1^\pi_t(x_1;r_i) - F_i^{-i} + G_i^{-i}]
\]
\[
= [V_1^\pi(x_1;R) - V_1^{\pi^*_i}(x_1;R)] + [F_i^{-i} - V_1^\pi_t(x_1;R^-)] + V_1^\pi_t(x_1;R^-) - G_i^{-i}]. \tag{18}
\]
For term (i), we have
\[ V_1^\pi(x_1;R) - V_1^{\pi^*_i}(x_1;R) \leq c_{\beta} \sqrt{d^3 H^6 \mathcal{X}/K}, \]
by Equation (15). The following lemma shows that \( \tilde{V}_1^{t,i}(x_1;R^-) \) is the pessimistic estimation of \( V_1^\pi(x_1;R^-) \).

**Lemma C.3** With probability at least \( 1 - \delta/(c_{\beta}^{\frac{1}{4}} n) \), we have
\[ 0 \leq V_1^\pi(x_1;R^-) - \tilde{V}_1^{t,i}(x_1;R^-) \leq 2 u_i(h,x,a), \quad \text{for all } (x,a) \in \mathcal{S} \times \mathcal{A}, h \in [H]. \tag{19} \]

For term (ii), if \( \zeta_2 = \text{PES} \), we have (ii) \( \leq 0 \) using Lemma C.3; if \( \zeta_2 = \text{OPT} \), similar to Equation (15), we have
\[ (ii) = \tilde{V}_1^{t,i}(x_1;R^-) - V_1^\pi(x_1;R^-) \leq \tilde{V}_1^{t,i}(x_1;u^i) \leq V_1^\pi(x_1;u^i) \leq c_{\beta} \sqrt{d^3 H^6 \mathcal{X}/K}. \]

For term (iii), if \( \zeta_3 = \text{OPT} \), similar to Equation (19), we have (iii) \( \leq 0 \); if \( \zeta_3 = \text{PES} \), similar to Equation (15), we have
\[ (iii) = V_1^{\pi^*_i}(x_1;R^-) - \tilde{V}_1^{t,i}(x_1;R^-) \leq V_1^{\pi^*_i}(x_1;u) \leq V_1^\pi(x_1;u) \leq c_{\beta} \sqrt{d^3 H^6 \mathcal{X}/K}. \]

Then, combining the above discussion of Equation (18) we have
\[ \sum_{t=K+1}^{T} \text{reg}_{it} \leq (T - K) c \sqrt{d^3 H^6 \mathcal{X}/K} \leq c \sqrt{d^3 H^6 \mathcal{X}} \cdot T^{2/3}, \]
where \( c = c_{\beta} \) if \( (\zeta_2, \zeta_3) = (\text{PES}, \text{OPT}) \), \( c = 3c_{\beta} \) if \( (\zeta_2, \zeta_3) = (\text{OPT}, \text{PES}) \).
**Seller Regret:** Recall the definition of agent regret in Equation (1) that \( \text{Reg}_{0t} = \sum_{t=1}^{T} \text{reg}_{0t} \) where \( \text{reg}_{0t} = u_{0*} - u_{0t} \). We can decompose the instantaneous regret terms as

\[
\text{Reg}_{0t} = \sum_{i=1}^{K} \text{reg}_{0t} + \sum_{i=K+1}^{T} \text{reg}_{0t}.
\]  

For the first summation in Equation (20), recall the individual rationality in Lemma 2.1. This leads to the following bound on the first summation

\[
\sum_{t=1}^{K} \text{reg}_{0t} \leq \sum_{t=1}^{K} u_{0*} \leq \sum_{t=1}^{K} V_1^*(x;R) \leq \sum_{t=1}^{K} H \cdot (n + R_{\max}) = H(n + R_{\max}) \cdot T^{2/3},
\]  

where the first inequality is because the mechanism do not charge the agents during the exploration phase and we have \( u_{0t} = V_1^{\pi_1}(x; r_0) \) and can infer that \( u_{0t} \geq 0 \) since the value functions \( r_{0h} \) ranging from 0 to \( R_{\max} \), the second inequality is because \( V_1^*(x; R) = u_{0*} + \sum_{i=1}^{n} u_{is} \) and \( u_{is} \geq 0 \) for all \( i \) in \([n]\) by invoking Lemma 2.1, and the third inequality is bounding \( V_1^*(x; R) \) by its upper bound \( H \cdot (n + R_{\max}) \). For the second summation in Equation (20), we decompose the instantaneous regret of the seller in the exploitation phase and have

\[
\text{reg}_{0t} = u_{0*} - u_{0t}
\]

\[
= \left[ V_1^{\pi^*}(x_1; r_0) + \sum_{i=1}^{n} p_{is} \right] - \left[ V_1^{\tilde{\pi}^*}(x_1; r_0) + \sum_{i=1}^{n} \tilde{p}_{it} \right]
\]

\[
= \left[ V_1^{\pi^*}(x_1; r_0) + \sum_{i=1}^{n} V_1^{\pi^*}(x_1; R^{-i}) - V_1^{\pi^*}(x_1; R^{-i}) \right] - \left[ V_1^{\tilde{\pi}^*}(x_1; r_0) + \sum_{i=1}^{n} (F_t^{-i} - G_t^{-i}) \right]
\]

\[
= \left[ -(n - 1)V_1^{\pi^*}(x_1; R) + \sum_{i=1}^{n} V_1^{\pi^*}(x_1; R^{-i}) \right]
\]

\[
- \left[ -(n - 1)V_1^{\tilde{\pi}^*}(x_1; R) + \sum_{i=1}^{n} [F_t^{-i} - G_t^{-i} + V_1^{\tilde{\pi}^*}(x_1; R^{-i})] \right]
\]

\[
= (n - 1) \left[ V_1^{\tilde{\pi}^*}(x_1; R) - V_1^{\pi^*}(x_1; R) \right] + \sum_{i=1}^{n} \left[ V_1^{\pi^*}(x_1; R^{-i}) - F_t^{-i} \right] + \sum_{i=1}^{n} \left[ G_t^{-i} - V_1^{\tilde{\pi}^*}(x_1; R^{-i}) \right].
\]  

For term (i) in Equation (22), we have (i) \( \leq 0 \) due to the optimality of \( V_1^* \).

For term (ii) in Equation (22), if \( \zeta_2 = \text{OPT} \), we have (ii) \( \leq 0 \) for \( \tilde{V}_1^{t,\pi}(x; R^{-i}) \) is the optimistic estimation of \( V_1^{\pi^*}(x; R^{-i}) \) invoking Lemma D.3; if \( \zeta_2 = \text{PES} \), similar to Equation (15), we have

\[
(ii) = V_1^{\pi^*}(x_1; R^{-i}) - \tilde{V}_1^{t,\pi^*}(x; R^{-i}) \leq V_1^{\pi^*}(x; u) \leq V_1^{\pi^*}(x; u) \leq c_{\beta} \sqrt{d^3 H^6 \mathcal{X}/K},
\]

where \( \pi^* \) is the greedy policy corresponding to \( V_1^{\pi^*}(x; R^{-i}) \).

For term (iii) in Equation (22), if \( \zeta_3 = \text{PES} \), we have (iii) \( \leq 0 \) for \( \tilde{V}_1^{t,\tilde{\pi}^*}(x; R^{-i}) \) is the pessimistic
estimation of $V^\hat{\pi}_1^t (x; R^{-i})$ invoking Lemma D.3; if $\zeta_3 = \text{OPT}$, similar to Equation (15), we have

$$(iii) = \hat{V}^t_1(x; R^{-i}) - V^\pi_1^t (x; R^{-i}) \leq V^\pi_1^t (x; u) \leq V^*_1(x; u) \leq c_3 \sqrt{d^3 H^6 X / K}.$$  

Then, combining the above discussion for Equation (22) we have

$$\sum_{t=K+1}^{T} \text{reg}_0 \leq \begin{cases} cn \sqrt{d^3 H^6 X} \cdot T^{2/3} & \text{if } (\zeta_2, \zeta_3) = (\text{PES}, \text{OPT}) \\ 0 & \text{if } (\zeta_2, \zeta_3) = (\text{OPT}, \text{PES}) \end{cases}$$

Combining the above equation with Equation (21), we have

$$\text{Reg}_{0T} \leq \begin{cases} \left[H(n + R_{max}) + cn \sqrt{d^3 H^6 X}\right] \cdot T^{2/3} & \text{if } (\zeta_2, \zeta_3) = (\text{PES}, \text{OPT}) \\ H(n + R_{max}) \cdot T^{2/3} & \text{if } (\zeta_2, \zeta_3) = (\text{OPT}, \text{PES}), \end{cases}$$

which concludes the proof of the seller regret.

**Individual Rationality:** We do not assume the truthfulness of other agents besides agent $i$ in this part of proof. Recall that we do not charge the agents in the exploration phase, so the utility $u_{it}$ of agent $i$ is always non-negative in the exploration phase. Thus, we just need to show the utility of agent $i$ is also non-negative in the exploitation phase. We have the following decomposition about the utility of agent $i$,

$$u_{it} = V^\hat{\pi}_1^t (x_1; r_i) - p_{it}$$

$$= V^\hat{\pi}_1^t (x; r_i) - F_t^{-i} + G_t^{-i}$$

$$= \left[V^\hat{\pi}_1^t (x; r_i) + V^\pi_1^t (x; \tilde{R}^{-i}) - F_t^{-i}\right] + \left[G_t^{-i} - V^\pi_1^t (x; \tilde{R}^{-i})\right]$$

$$= \left[V^\hat{\pi}_1^t (x_1; r_i + \tilde{R}^{-i}) - F_t^{-i}\right] + \left[G_t^{-i} - V^\pi_1^t (x_1; \tilde{R}^{-i})\right],$$

(i) \hspace{1cm} \text{(23)}

(ii)

where the greedy policy $\hat{\pi}_i$, and the $F, G$ functions are calculated using the reported rewards $\{\tilde{r}_{ih}\}_{i,h}$.

For term (i) in Equation (23), we have

$$(i) \geq V^\hat{\pi}_1^t (x_1; r_i + \tilde{R}^{-i}) - \hat{V}^{t,*}_1(x_1; \tilde{R}^{-i})$$

$$= \left[V^\hat{\pi}_1^t (x_1; r_i + \tilde{R}^{-i}) - V^*_1(x; r_i + \tilde{R}^{-i})\right] + \left[V^*_1(x; r_i + \tilde{R}^{-i}) - V^*_1(x; \tilde{R}^{-i})\right]$$

$$+ \left[V^*_1(x; \tilde{R}^{-i}) - \hat{V}^{t,*}_1(x_1; \tilde{R}^{-i})\right]$$

$$\geq - \left[V^*_1(x; r_i + \tilde{R}^{-i}) - V^\pi_1^t (x_1; r_i + \tilde{R}^{-i})\right] - \left[\hat{V}^{t,*}_1(x_1; \tilde{R}^{-i}) - V^*_1(x; \tilde{R}^{-i})\right]$$

$$\geq - c \sqrt{d^3 H^6 X / K},$$  

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for some absolute constant \(c\) and the first inequality is because \(\overline{V}_{t}^{\dagger}(x; \overline{R}^{-}) \geq \overline{V}_{t}^{\dagger}(x; \overline{R}^{-})\); the second equality is by adding and subtracting \(V_{t}^{\nu}(x; \nu_{ih} + \overline{R}^{-})\) and \(V_{t}^{\nu}(\overline{R}^{-})\); the third inequality is because \(V_{t}^{\nu}(x; \nu_{ih} + \overline{R}^{-})\) is larger that \(V_{t}^{\nu}(\overline{R}^{-})\) for they have the same transition but the former have a larger reward; and the proof of the last inequality is similar to the proof of Equation (15).

For term (ii) in Equation (23), we have

\[
(ii) \geq \overline{V}_{t}^{\dagger}(x; \overline{R}^{-}) - V_{t}^{\dagger}(x; \overline{R}^{-}) \geq -c\sqrt{d^{3}H^{6}X/K},
\]

for some absolute constant \(c\), and the proof is similar to the proof of term (i). Combing term (i) and term (ii), we have the utility of agent \(i\) in the exploitation phase is asymptotically non-negative. In other words, Algorithm 1 is *asymptotically individually rational* in ETC setting.

**Truthfulness:** We first declare some notations here. We let \(\overline{\nu}_{i}\) be a strategy used by agent \(i\) for reporting rewards when she is untruthful while \(\nu_{i}\) be the strategy used by agent \(i\) for reporting rewards when she is truthful. Furthermore, let \(\nu_{i}^{r}\) be the strategy following the reporting strategy \(\overline{\nu}_{i}\) for time steps \(1, 2, \ldots, r - 1\) and turning to the truthful reporting strategy \(\nu_{i}\) for time steps \(r, \ldots, T\). Let \(\overline{u}_{it}\) and \(u_{it}^{t-1}\) be the instantaneous utilities of agent \(i\) in round \(t\) using reporting strategy \(\overline{\nu}_{i}\) or \(\nu_{i}^{t-1}\). Let \(\overline{U}_{it}\) be the summation of utilities \(\overline{u}_{it}\) using reporting strategy \(\overline{\nu}_{i}\).

We first decompose the difference between the sum of utilities of agent \(i\) using untruthful reporting strategy \(\overline{\nu}_{i}\) and under truthfulness by adding and subtracting the term \(u_{it}^{t-1}\),

\[
\overline{U}_{iT} - U_{iT} = \sum_{t=1}^{T} \overline{u}_{it} - u_{it}^{t-1} + \sum_{t=2}^{T} u_{it}^{t-1} - u_{it}. \tag{24}
\]

Here, in the first summation, we consider the influence of utilities of agent \(i\) at time step \(t\) caused by switching from the strategy \(\overline{\nu}_{i}\) to truthful strategy \(\nu_{i}\) at time step \(t - 1\). In the second summation, we consider the influence of utilities of agent \(i\) at time step \(t\) caused by using different strategies in the first \(t - 2\) steps.

For the terms in the first summation in Equation (24), we have the following decomposition.

\[
\overline{u}_{it} - u_{it}^{t-1} = \left[ V_{1}^{\dagger}(x; r_{i}) - F_{t}^{\dagger,-i} + G_{t}^{\dagger,-i} \right] - \left[ V_{1}^{\dagger,-i}(x; r_{i}) - F_{t}^{\dagger,-i} + G_{t}^{\dagger,-i} \right]
\]

\[
= V_{1}^{\dagger}(x; r_{i}) + G_{t}^{\dagger,-i} - V_{1}^{\dagger,-i}(x; r_{i}) - G_{t}^{\dagger,-i}
\]

\[
= \left[ V_{1}^{\dagger}(x; r_{i} + \overline{R}^{-}) - \overline{V}_{t}^{\dagger}(x; r_{i} + \overline{R}^{-}) \right] + \left[ \overline{V}_{t}^{\dagger,-i}(x; r_{i} + \overline{R}^{-}) - V_{1}^{\dagger,-i}(x; r_{i} + \overline{R}^{-}) \right]
\]

\[
+ \left[ G_{t}^{\dagger,-i} - V_{1}^{\dagger,-i}(x; r_{i} + \overline{R}^{-}) \right] + \left[ V_{1}^{\dagger}(x; \overline{R}^{-}) - G_{t}^{\dagger,-i} \right],
\]

where the superscript \(\dagger\) and \(\dagger\) of the \(F, G\) functions represent that the functions are calculated when
agent $i$ using reward reporting strategy $\tilde{\nu}_i$ and $\nu_i^{t-1}$ respectively. In addition, $\pi_i^t$ and $\pi_i^{t-1}$ are the greedy policy calculated when agent $i$ using the reward reporting strategy $\tilde{\nu}_i$ and $\nu_i^{t-1}$ respectively. Here, the four difference terms can all be upper bounded by $c\sqrt{d^3H^6X/K}$ for some constant $c$ using the same technique used in the proof of Equation (15). Precisely, they are the suboptimalities of the corresponding underlying MDPs. For the terms in the second summation in Equation (24), we have the following decomposition.

$$u_{it}^{t-1} - u_{it} = \left[ V_{1}^{\pi_i^{t-1}}(x; r_i) - F_{1}^{\pi_i^{t-1}} + G_{1}^{\pi_i^{t-1}} \right] - \left[ V_{1}^{\hat{\pi}_i^t}(x; r_i) - G_{1}^{\pi_i^t} \right]$$

$$= \left[ V_{1}^{\pi_i^{t-1}}(x; r_i) - V_{1}^{\hat{\pi}_i^t}(x; r_i) \right] + \left[ V_{1}^{\pi_i^{t-1}}(x; \tilde{R}^{-i}) - F_{1}^{\pi_i^{t-1}} \right] + \left[ G_{1}^{\pi_i^t} - V_{1}^{\pi_i^{t-1}}(x; \tilde{R}^{-i}) \right]$$

Notice that the first term can be upper bounded by $V_{1}^{\pi}(x_1; r_i) - V_{1}^{\hat{\pi}_i^t}(x_1; r_i)$ where $V_{1}^{\pi}(x_1; r_i)$ is the optimal value function of the underlying MDP with reward function $r_i$. For the fourth term in the above decomposition, we can lower bounded it by $0$ if we choose $\zeta_2 = \text{OPT}$ to calculate $F_{t}^{t-1}$; we can lower bounded it by $-c\sqrt{d^3H^6X/K}$ if we choose $\zeta_2 = \text{PES}$ to calculate $F_{t}^{t-1}$. For the rest of the terms, we can upper bound them by $c\sqrt{d^3H^6X/K}$ for some constant $c$ using the same technique used in the proof of Equation (15). Combining the above two parts of analysis, we have the conclusion that Algorithm 1 is asymptotically truthful in ETC setting. Note that by the union bound, all the above results hold with probability at least $1 - \delta$. This concludes the proof.

### C.2 Proof of Theorem 4.2

In this subsection, we will show the proof of Theorem 4.2.

**Proof** When setting $\zeta_1 = \text{EWC}$, we can also decompose the regret into two parts: the regret incurred in the exploration phase and the regret incurred in the exploitation phase like what we do when $\zeta_1 = \text{ETC}$ as

$$\text{Reg}_T^W = \sum_{t=1}^{K} \text{reg}_t^W + \sum_{t=K+1}^{T} \text{reg}_t^W.$$ 

For the first part, we can bound it with $\tilde{O}(nHT^{2/3})$ using the same method as $\zeta_1 = \text{ETC}$. For the
second part, using Lemmas B.5 and B.6 in Jin et al. (2019), we have

\[
\sum_{t=K+1}^{T} \text{reg}_{t}^{W} = \sum_{t=K+1}^{T} \left[ V_{1}^{t}(x_{1}) - V_{1}^{\hat{t}}(x_{1}) \right] \\
\leq \sum_{t=K+1}^{T} \left[ V_{1}^{t}(x_{1}) - V_{1}^{\hat{t}}(x_{1}) \right] \\
\leq \sum_{t=K+1}^{T} \sum_{h=1}^{H} \left( \mathbb{E}[\xi_{h}^{t} | x_{h-1}^{t}, a_{h-1}^{t}] - \xi_{h}^{t} \right) + 2\beta \sum_{t=K+1}^{T} \sum_{h=1}^{H} \sqrt{(\phi_{h}^{t})^{\top} (A_{h}^{t})^{-1} (\phi_{h}^{t})},
\]

where \( \xi_{h}^{t} = V_{h}^{t}(x_{h}^{t}) - V_{h}^{\hat{t}}(x_{h}^{t}) \). Then we bound the two summations on the right hand side of Equation (25) respectively. For the first summation, since the computation of \( \hat{V}_{h}^{t} \) does not use the new observation \( x_{h}^{t} \) at rounds \( t \), the terms in the first summation is a martingale difference sequence bounded by \( 2(x + R_{\text{max}})H \). Then we can bound it by Azuma-Hoeffding inequality and get an \( \tilde{O}(H^{2}T^{1/2}) \) upper bound for the first summation in Equation (25). For the second summation, we can bound it using Lemma F.2 and Cauchy-Schwarz inequality,

\[
\sum_{t=K+1}^{T} \sum_{h=1}^{H} \sqrt{(\phi_{h}^{t})^{\top} (A_{h}^{t}) (\phi_{h}^{t})} \leq \sum_{t=K+1}^{T} \sum_{h=1}^{H} \sqrt{(\phi_{h}^{t})^{\top} (\tilde{A}_{h}^{t})^{-1} (\phi_{h}^{t})} \\
\leq \sum_{h=1}^{H} \left[ \sum_{t=K+1}^{T} \phi(x_{h}^{t}, a_{h}^{t})^{\top} (\tilde{A}_{h}^{t})^{-1} \phi(x_{h}^{t}, a_{h}^{t}) \right]^{1/2} \\
\leq 2H \sqrt{2dT\tilde{A}},
\]

where \( \tilde{A}_{h}^{t} = \sum_{\tau=K+1}^{t-1} \phi(x_{\tau}^{t}, a_{\tau}^{t})^{\top} + \lambda I \) is the design matrix only using the data in the exploitation phase. The first step is due to \( \tilde{A}_{h}^{t} \leq A_{h}^{t} \), the second step is by Cauchy-Schwartz inequality, and the last step uses the elliptical potential lemma in Abbasi-Yadkori et al. (2011). For the other difference terms in the proof of Theorem 4.1, we still use the same method as in the proof of Theorem 4.1 to get the \( \tilde{O}(T^{2/3}) \) bounds. Note that by the union bound, all the results in this theorem hold with probability at least \( 1 - \delta \). This concludes the proof.

D  Proof of Lemmas for Theorems 4.1 and 4.2

In this section, we present the detailed proof of the lemmas for Theorems 4.1 and 4.2. We first present the preliminaries in Subsection D.1. Then we provide the proof of Lemmas C.3, C.1 and C.2 in Subsections D.2, D.3 and D.4 respectively.
D.1 Preliminaries for the Proofs

Before stating the lemmas, we first define two operators to depict the transition of the MDP. For any function \( f(\cdot; \mathcal{R}) : \mathcal{S} \rightarrow \mathbb{R} \) with reward function \( \mathcal{R} \), we define the transition operator at each time step \( h \in [H] \) as

\[
(\mathbb{P}_h f)(x, a; \mathcal{R}) = \mathbb{E}[f(x_{h+1}) | x_h = x, a_h = a],
\]

and the Bellman operator at each time step \( h \in [H] \) as

\[
(\mathbb{B}_h f)(x, a; \mathcal{R}) = \mathbb{E}[\mathcal{R}_h(x, a) + f(x_{h+1}) | x_h = x, a_h = a] \\
= \mathbb{E}[\mathcal{R}_h(x, a) | x_h = x, a_h = a] + (\mathbb{P}_h f)(x, a).
\]

For estimated value functions \( V_{h+1}^{t, \pi} \) and corresponding action-value functions \( Q_{h+1}^{t, \pi} \). We define the model evaluation error with policy \( \pi \) in episode \( t \) at each step \( h \in [H] \) as

\[
\iota_{h} (x, a; \cdot) = (\mathbb{B}_h \hat{V}^{t, \pi}_{h+1})(x, a; \cdot) - \hat{Q}_{h+1}^{t, \pi}(x, a; \cdot),
\]

for \( \zeta_3 = \text{OPT} \) and \( \text{PES} \) respectively. In other words, \( \iota_h \) is the error in estimating the Bellman operator defined in Equation (27), based on the dataset \( \mathcal{D} \) collected in Algorithm 2.

For clarity, we define the following events to quantify the uncertainty of the estimation of the Bellman operator \( \mathbb{B}_h \) in Algorithm 3 and Algorithm 4 with different hyperparameters.

**Definition D.1** We define the following events that will occur with high probability in our proof, which are, for any \( i \in [n] \),

\[
\mathcal{E}_{i}(\mathcal{R}) = \left\{ (\phi(x, a) \mathbb{P}_h \hat{w}_h^{t, \pi}(\mathcal{R}) - \mathbb{B}_h \hat{V}_h^{t, \pi}(x, a; \mathcal{R})) \leq \mu_i(x, a), \forall(x, a) \in \mathcal{S} \times \mathcal{A}, h \in [H] \right\}, \forall \mathcal{R} \in \{R, R^{-1}, R_i + R^{-1}, \tilde{R}^{-1}\},
\]

\[
\mathcal{E}_i(\pi; R^{-1}) = \left\{ (\phi(x, a) \mathbb{P}_h \hat{w}_h^{t, \pi}(R^{-1}) - \mathbb{B}_h \hat{V}_h^{t, \pi}((x, a; R^{-1})) \leq \mu_i(x, a), \forall(x, a) \in \mathcal{S} \times \mathcal{A}, h \in [H] \right\},
\]

\[
\mathcal{E}_i(\pi; \tilde{R}^{-1}) = \left\{ (\phi(x, a) \mathbb{P}_h \hat{w}_h^{t, \pi}(\tilde{R}^{-1}) - \mathbb{B}_h \hat{V}_h^{t, \pi}((x, a; \tilde{R}^{-1})) \leq \mu_i(x, a), \forall(x, a) \in \mathcal{S} \times \mathcal{A}, h \in [H], \pi \in \{\hat{\pi}, \tilde{\pi}\} \right\},
\]

where \( \hat{w}_h^{t, \pi}(\mathcal{R}) \) denotes the learned parameter \( \hat{w}_h^{t, \pi} \) generated by Algorithm 3 corresponding to the value function \( \hat{V}_h^{t, \pi}(x, a; \mathcal{R}) \) with \( \mathcal{R} \in \{R, R^{-1}, R_i + \tilde{R}^{-1}, \tilde{R}^{-1}\} \). In addition, \( \hat{w}_h^{t, \pi}(\mathcal{R}) \) denotes the learned parameter \( \hat{w}_h^{t, \pi} \) generated by Algorithm 4 corresponding to the value function \( \hat{V}_h^{t, \pi}(x, a; \mathcal{R}) \) with \( \mathcal{R} \in \{R^{-1}, \tilde{R}^{-1}\} \) when \( \pi = \hat{\pi} \) or \( \tilde{\pi} \).

In the ETC setting, we only use the dataset \( \mathcal{D} \) collected in the exploration phase, i.e., the first \( K \) episodes before running Algorithm 3. Thus the events defined in each episode \( t \) remain the same. For example, \( \mathcal{E}_{*}(R) = \mathcal{E}_{*}(R) \) for all \( t \) in the exploitation phase. We will omit the time \( t \) for brevity in the following proof. The first two events in Definition D.1 is under the truthful reports and will be used in bounding the welfare regret and the agent and seller regret. We define the intersection of
these events as
\[
E^* = E^*(R) \bigcap \left( \bigcap_{i=1}^{n} E^*(R^{-i}) \right) \bigcap \left( \bigcap_{i=1}^{n} E(\hat{\pi}^t; R^{-i}) \right).
\] (29)

Similarly, we define the intersection of the last two events in Definition D.1 as
\[
E' = E^*(r_i h + \tilde{R}^{-i}) \bigcap \left( \bigcap_{i=1}^{n} E^*(\tilde{R}^{-i}) \right) \bigcap \left( \bigcap_{i=1}^{n} E(\hat{\pi}^t; \tilde{R}^{-i}) \right) \bigcap \left( \bigcap_{i=1}^{n} E(\hat{\pi}^t; \tilde{R}^{-i}) \right).
\]

The following lemma shows that under the appropriate choice of regularization parameter \( \lambda \) and scaling parameter \( \beta \), we can control the probability of event \( E \) and \( E' \).

**Lemma D.2 (Adaptation of Lemma 5.2 from Jin et al. (2020b))** Under the setting in Section 2, we set
\[
\lambda = 1, \quad \beta = c \cdot (n + R_{\text{max}})dH\sqrt{X}, \quad \text{where} \quad X = \log \left( \frac{2ndHK}{\delta} \right).
\]
Here \( c > 0 \) is an absolute constant and \( \delta \in (0,1) \) is the confidence parameter. It holds that
\[
P_D(E) \geq 1 - \delta/4, \quad P_D(E') \geq 1 - \delta/4.
\]

where the probability is computed under the collected dataset \( D \).

**Proof** It suffices to show that each event defined in Definition D.1 satisfies \( \Pr(E_i) \geq 1 - \delta/c_\delta \). Then we can use the union bound to prove Lemma D.2. For brevity, we just prove \( \Pr(E_i(R)) \geq 1 - \delta/c_\delta \). Unless otherwise specified, the following proof are corresponding to event \( E_i(R) \). The proof of others are similar and we will just point out the differences at the end of the proof. For simplicity, we abbreviate \( V_i^{l,*}(x,a; R) \) to \( V_i^{l,*}(x,a) \) since we just consider the value functions with reward function \( R \) in this section.

Recall the definition of the transition operator \( P_h \) and the Bellman operator \( B_h \) in Equation (26) and Equation (27). For any function \( V : S \mapsto \mathbb{R} \), Equation (2) ensures that \( P_h V \) and \( B_h V \) are linear in the feature map \( \phi \) for all \( h \in [H] \). To see this, note that Equation (2) implies
\[
(P_h V)(x,a) = \left\langle \phi(x,a), \int_S V(x') \mu_h(x') dx' \right\rangle, \quad \forall (x,a) \in S \times A, \forall h \in [H].
\]
Also, Equation (2) ensures that the reward is linear in \( \phi \) for all \( i = 0, 1, \cdots, n \) and \( h \in [H] \), which implies
\[
(B_h V)(x,a) = \left\langle \phi(x,a), \theta_h \right\rangle + \left\langle \phi(x,a), \int_S V(x') \mu_h(x') dx' \right\rangle, \quad \forall (x,a) \in S \times A, \forall h \in [H],
\]
where \( \theta_h \) represents the parameter corresponding to the reward function \( \mathcal{R} \) of \( V \). For example, if we
set $R = R$, then $\theta_h = \theta_{\theta h} + \sum_{j=1}^{n} \theta_{j h}$. Hence, there exists an unknown vector $w_h \in \mathbb{R}^d$ such that

$$\mathbb{E}_h V(x, a) = \phi(x, a)^\top w_h, \quad \forall (x, a) \in \mathcal{S} \times \mathcal{A}, \forall h \in [H]. \quad (30)$$

We then upper bound the difference between $\mathbb{E}_h \hat{V}_{h+1}^{t, *}$ and $f_h^t$. Recalling the definition of $f_h^t$ in Algorithm 3 Line 3, for all $h \in [H]$ and all $(x, a) \in \mathcal{S} \times \mathcal{A}$, we have

$$|\mathbb{E}_h \hat{V}_{h+1}^{t, *}(x, a) - f_h^t| \leq |\mathbb{E}_h \hat{V}_{h+1}^{t, *}(x, a) - \phi^\top (x, a) w_h^t|, \quad (31)$$

where the inequality is due to $0 \leq \mathbb{E}_h \hat{V}_{h+1}^{t, *} \leq H (n + R_{\text{max}})$, and the non-expansiveness of the operator $\Pi_{[0, H(n + R_{\text{max}})]} : \max\{\text{min}\{\cdot, H(n + R_{\text{max}})\}, 0\}$. Then we turn to upper bound the term $\mathbb{E}_h \hat{V}_{h+1}^{t, *} - \phi^\top w_h^t$, for all $h \in [H]$ and all $(x, a) \in \mathcal{S} \times \mathcal{A}$, we have

$$(\mathbb{E}_h \hat{V}_{h+1}^{t, *})(x, a) - \phi(x, a)^\top w_h^t = \phi(x, a)^\top (w_h - w_h^t)$$

$$= \phi(x, a)^\top w_h - \phi(x, a)^\top (\Lambda_h^t)^{-1} \left( \sum_{\tau=1}^{K} \phi(x_{\tau h}^t, a_{\tau h}^t) \cdot (R_{h+1}^t + \hat{V}_{h+1}^{t, *}(x_{\tau h+1}^t)) \right)$$

$$= \phi(x, a)^\top w_h - \phi(x, a)^\top (\Lambda_h^t)^{-1} \left( \sum_{\tau=1}^{K} \phi(x_{\tau h}^t, a_{\tau h}^t) \cdot (\mathbb{E}_h \hat{V}_{h+1}^{t, *}(x_{\tau h}^t, a_{\tau h}^t)) \right)$$

(i)

$$- \phi(x, a)^\top (\Lambda_h^t)^{-1} \left( \sum_{\tau=1}^{K} \phi(x_{\tau h}^t, a_{\tau h}^t) \cdot (R_{h+1}^t + \hat{V}_{h+1}^{t, *}(x_{\tau h+1}^t) - (\mathbb{E}_h \hat{V}_{h+1}^{t, *}(x_{\tau h}^t, a_{\tau h}^t))) \right).$$

(ii)

Here, the first equality comes from the linearity of the Bellman operator $\mathbb{E}_h$ in Equation (30) and $w_h$ is the unknown vector corresponding to $\hat{V}_{h+1}^{t, *}(x, a)$. The second equality follows from the definition of $w_h^t$ in Line 3 of Algorithm 3. The last equality comes from adding and subtracting the term $\phi(x, a)^\top (\Lambda_h^t)^{-1} \left( \sum_{\tau=1}^{K} \phi(x_{\tau h}^t, a_{\tau h}^t) \cdot (\mathbb{E}_h \hat{V}_{h+1}^{t, *}(x_{\tau h}^t, a_{\tau h}^t)) \right)$. By the triangle inequality, we have

$$| (\mathbb{E}_h \hat{V}_{h+1}^{t, *})(x, a) - \phi(x, a)^\top w_h^t | \leq |(i)| + |(ii)|.$$

In the sequel, we upper bound terms (i) and (ii) respectively. By the construction of the estimated value function $\hat{V}_{h+1}^{t, *}$ in Line 4 of Algorithm 3, we have $\hat{V}_{h+1}^{t, *} \in [0, (n + R_{\text{max}}) \cdot H]$. By the triangle inequality, we have $||\theta_h|| = || \sum_{i=0}^{n} \theta_{i h} || \leq (n + 1) \sqrt{d}$. Then, by Lemma F.4, we have

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\[ \|w_h\| \leq (n + R_{\text{max}})H\sqrt{d}. \] Hence, term (i) defined in Equation (32) is upper bounded by

\[
|\text{(i)}| = \left| \phi(x, a)^\top w_h - \phi(x, a)^\top (\Lambda_h^\dagger)^{-1}\left( \sum_{\tau=1}^K \phi(x, a)^\top \phi(x, a)^\top \right) w_h \right| \\
= \left| \phi(x, a)^\top w_h - \phi(x, a)^\top (\Lambda_h^\dagger)^{-1}(\Lambda_h^\dagger - \lambda \cdot I)w_h \right| = \lambda \cdot \left| \phi(x, a)^\top (\Lambda_h^\dagger)^{-1}w_h \right| \\
\leq \lambda \cdot \|w_h\|_{(\Lambda_h^\dagger)^{-1}} \cdot \|\phi(x, a)\|_{(\Lambda_h^\dagger)^{-1}} \leq (n + R_{\text{max}})H\sqrt{d}\lambda. 
\]

Here, the first equality comes from the linearity of the Bellman operator \( B_h \) in Equation (30). The second equality follows from the definition of \( \Lambda_h^\dagger \) in Line 3 of Algorithm 3. Also, the first inequality follows from the Cauchy-Schwarz inequality, while the last inequality follows from the fact that

\[
\|w_h\|_{(\Lambda_h^\dagger)^{-1}} = \sqrt{w_h^\top (\Lambda_h^\dagger)^{-1}w_h} \leq \|w_h\|/\lambda \leq (n + R_{\text{max}})H\sqrt{d}\lambda.
\]

It remains to upper bound term (ii). For notation simplicity, for any \( h \in [H] \), any \( \tau \in [K] \) and any function \( V : S \mapsto [0, V_{\text{max}}] \), we defined the random variable

\[
\epsilon_h^\tau(V) = R_h^\tau + V(x_{h+1}^\tau) - (B_h V)(x_h^\tau, a_h^\tau).
\]

Then by the Cauchy-Schwarz inequality, term (ii) defined in Equation (32) is upper bounded by

\[
|\text{(ii)}| = \left| \phi(x, a)^\top (\Lambda_h^\dagger)^{-1}\left( \sum_{\tau=1}^K \phi(x, a)^\top \epsilon_h^\tau(\hat{V}_{h+1}^\tau) \right) \right| \\
\leq \left\| \sum_{\tau=1}^K \phi(x, a)^\top \epsilon_h^\tau(\hat{V}_{h+1}^\tau) \right\|_{(\Lambda_h^\dagger)^{-1}} \cdot \|\phi(x, a)\|_{(\Lambda_h^\dagger)^{-1}} \\
= \left\| \sum_{\tau=1}^K \phi(x, a)^\top \epsilon_h^\tau(\hat{V}_{h+1}^\tau) \right\|_{(\Lambda_h^\dagger)^{-1}} \cdot \sqrt{\phi(x, a)^\top (\Lambda_h^\dagger)^{-1}\phi(x, a)}.
\]

In the sequel, we bound term (iii) via concentration inequalities. An obstacle to directly using concentration inequalities is that \( \hat{V}_{h+1}^\tau \) depends on \( \{(x, a)\}_{\tau \in[K]} \) via \( \{(x, a)\}_{\tau \in[K], h' > h} \). To this end, we adopt uniform concentration inequalities to upper bound

\[
\sup_{V \in \mathcal{V}_{h+1}(L, B, \lambda)} \left\| \sum_{\tau=1}^K \phi(x, a)^\top \epsilon_h^\tau(V) \right\|_{(\Lambda_h^\dagger)^{-1}}
\]

for each \( h \in [H] \), where it holds that \( \hat{V}_{h+1}^\tau \in \mathcal{V}_{h+1}(L, B, \lambda) \). For all \( h \in [H] \), we define the function

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By Lemma F.4, we have the linearity and monotonicity of conditional expectation implies because 

\( \hat{V}_{h+1}(x; \theta, \beta, \Sigma) \in \mathcal{V}(L, B, \lambda) \) with \( \|\theta\| \leq L, \beta \in [0, B], \Sigma \geq \lambda \cdot I \), where \( V_h(x; \theta, \beta, \Sigma) = \max_{a \in A} \{ \min_{x \in S} \{ \phi(x, a)^\top + \beta \cdot \sqrt{\phi(x, a)^\top \Sigma^{-1} \phi(x, a)} \} \}. \)

For all \( \epsilon > 0 \) and all \( h \in [H] \), let \( \mathcal{N}_h(\epsilon; R, B, \lambda) \) be the minimal \( \epsilon \)-cover of \( \mathcal{V}_h(R, B, \lambda) \) with respect to the distance \( \text{dist}(V, V') = \sup_{x \in S} \|V(x) - V'(x)\| \). In other words, for any function \( V \in \mathcal{V}_h(R, B, \lambda) \), there exists a function \( V^\dagger \in \mathcal{N}_h(\epsilon; R, B, \lambda) \) such that

\[
\sup_{x \in S} |V(x) - V^\dagger(x)| \leq \epsilon.
\]

By Lemma F.4, we have \( \|w^*_h\| \leq (n + R_{\text{max}})H \sqrt{Kd/\lambda} \). Hence, for all \( h \in [H] \), we have

\[
\hat{V}^*_{h+1} \in \mathcal{V}_{h+1}(L_0, B_0, \lambda), \quad \text{where} \quad L_0 = (n + R_{\text{max}})H \sqrt{Kd/\lambda}, \quad B_0 = 2\beta.
\]

Here \( \lambda > 0 \) is the regularization parameter and \( \beta > 0 \) is the scaling parameter, which are specified in Algorithm 3. For notational simplicity, we use \( \mathcal{V}_{h+1} \) and \( \mathcal{N}_{h+1}(\epsilon) \) to denote \( \mathcal{V}_{h+1}(R_0, B_0, \lambda) \) and \( \mathcal{N}_{h+1}(\epsilon; R_0, B_0, \lambda) \), respectively. Then, there exists a function \( V^\dagger_{h+1}(x) \in \mathcal{N}(\epsilon) \) such that

\[
\sup_{x \in S} |\hat{V}^*_{h+1}(x) - V^\dagger_{h+1}(x)| \leq \epsilon,
\]

because \( \hat{V}^*_{h+1} \in \mathcal{V}_{h+1} \) and \( \mathcal{N}_{h+1}(\epsilon) \) is an \( \epsilon \)-cover of \( \mathcal{V}_{h+1} \). For the same functions \( \hat{V}^*_{h+1} \) and \( V^\dagger_{h+1} \), the linearity and monotonicity of conditional expectation implies

\[
\left| (\mathbb{P}_h V^\dagger_{h+1})(x, a) - (\mathbb{P}_h \hat{V}^*_{h+1})(x, a) \right| \\
= \left| \mathbb{E} [V^\dagger_{h+1}(s_{h+1}) \mid s_h = x, a_h = a] - \mathbb{E} [\hat{V}^*_{h+1}(s_{h+1}) \mid s_h = x, a_h = a] \right| \\
= \left| \mathbb{E} [V^\dagger_{h+1}(s_{h+1}) - \hat{V}^*_{h+1}(s_{h+1}) \mid s_h = x, a_h = a] \right| \\
\leq \mathbb{E} \left[ |V^\dagger_{h+1}(s_{h+1}) - \hat{V}^*_{h+1}(s_{h+1})| \mid s_h = x, a_h = a \right] \leq \epsilon, \quad \forall (x, a) \in S \times A, \forall h \in [H].
\]

Combining Equation (37) and the definition of the Bellman operator \( \mathbb{B}_h \) in Equation (27), we have

\[
\left| (\mathbb{B}_h V^\dagger_{h+1})(x, a) - (\mathbb{B}_h \hat{V}^*_{h+1})(x, a) \right| \leq \epsilon, \quad \forall (x, a) \in S \times A, \forall h \in [H].
\]

By the triangle inequality, Equations (36) and (38) imply

\[
\left| (R^*_h(x, a) + \hat{V}^*_{h+1}(x') - (\mathbb{B}_h \hat{V}^*_{h+1})(x, a)) - (R^*_h(x, a) + V^\dagger_{h+1}(x') - (\mathbb{B}_h V^\dagger_{h+1})(x, a)) \right| \leq 2\epsilon
\]
for all \( h \in [H] \) and all \( (x, a, x') \in S \times A \times S \). Setting \( (x, a, x') = (x^*_h, a^*_h, x^*_{h+1}) \) in Equation (39), we have

\[
|\epsilon_h^*(\hat{V}^t_{h+1}) - \epsilon_h^*(V^t_{h+1})| \leq 2\varepsilon, \quad \forall \tau \in [K], \quad \forall h \in [H].
\]

Also, recall the definition of term (iii) in Equation (35). Hence for all \( h \in [H] \), we have

\[
|\text{(iii)}|^2 \leq 2 \cdot \left\| \sum_{\tau=1}^{K} \phi(x^*_h, a^*_h) \cdot \epsilon_h^*(\hat{V}^t_{h+1}) \right\|_{(A_h^*)^{-1}}^2 + 2 \cdot \left\| \sum_{\tau=1}^{K} \phi(x^*_h, a^*_h) \cdot \left( \epsilon_h^*(\hat{V}^t_{h+1}) - \epsilon_h^*(V^t_{h+1}) \right) \right\|_{(A_h^*)^{-1}}^2. \quad (40)
\]

Here we use the inequality: \( \|a + b\|^2_A \leq 2 \cdot \|a\|^2_A + 2 \cdot \|b\|^2_A \), where \( a, b \) are two vectors in \( \mathbb{R}^d \), and \( A \in \mathbb{R}^{d \times d} \) is a positive definite matrix. The second term on the right-hand side of Equation (40) is upper bounded by

\[
2 \left\| \sum_{\tau=1}^{K} \phi(x^*_h, a^*_h) \cdot \left( \epsilon_h^*(\hat{V}^t_{h+1}) - \epsilon_h^*(V^t_{h+1}) \right) \right\|_{(A_h^*)^{-1}}^2
\]

\[
= 2 \cdot \sum_{\tau=1}^{K} \sum_{\tau'\neq1}^{K} \phi(x^*_h, a^*_h)^\top (A_h^*)^{-1} \phi(x^*_h, a^*_h) \cdot \left( \epsilon_h^*(\hat{V}^t_{h+1}) - \epsilon_h^*(V^t_{h+1}) \right) \cdot \left( \epsilon_h^*(\hat{V}^t_{h+1}) - \epsilon_h^*(V^t_{h+1}) \right)
\]

\[
\leq 8\varepsilon^2 \cdot \sum_{\tau=1}^{K} \sum_{\tau'\neq1}^{K} \left\| \phi(x^*_h, a^*_h) \right\|^2 \cdot \left\| \phi(x^*_h, a^*_h) \right\| \cdot \left\| (A_h^*)^{-1} \right\|_{\text{op}} \leq 8\varepsilon^2 K^2 / \lambda,
\]

where the first step is by the definition of \( \epsilon_h^*(V) \) in Equation (34); the second step follows from Equation (39); and the last step follows from the fact that \( \|\phi(x, a)\| \leq 1 \) for all \( (x, a) \in S \times A \) by the definition of \( \phi \) and matrix operator norm \( \| (A_h^*)^{-1} \|_{\text{op}} \leq 1 / \lambda \) by the definition of \( A_h^* \) in Line 3 of Algorithm 3. Then combining Equations (40) and (41), for all \( h \in [H] \), we have

\[
|\text{(iii)}|^2 \leq 2 \cdot \sup_{V \in \mathcal{N}_{h+1}(\varepsilon)} \left\| \sum_{\tau=1}^{K} \phi(x^*_h, a^*_h) \cdot \epsilon_h^*(V) \right\|_{(A_h^*)^{-1}}^2 + 8\varepsilon^2 K^2 / \lambda. \quad (42)
\]

Note that the first term in the right-hand side of Equation (42) does not involve the estimated value function \( \hat{V}^t_{h+1} \) and \( \hat{Q}^t_{h+1} \), which are constructed based on the dataset \( D \) and depend on \( \{(x^*_h, a^*_h)\}_{\tau \in [K]} \) via \( \{(x^*_h, a^*_h)\}_{\tau \in [K], h' > h} \). Hence, we can upper bound this term via uniform concentration inequalities. We first utilize Lemma F.6 to upper bound \( \left\| \sum_{\tau=1}^{K} \phi(x^*_h, a^*_h) \cdot \epsilon_h^*(V) \right\|_{(A_h^*)^{-1}}^2 \) for any fixed function \( V \in \mathcal{N}_{h+1}(\varepsilon) \). Recall the definition of \( \epsilon_h^*(V) \) in Equation (34). Applying Lemma F.6 and
the union bound, for any fixed $h \in [H]$, we have with probability at least $1 - p \cdot |\mathcal{N}_{h+1}(\epsilon)|$,

$$
\sup_{V \in \mathcal{N}_{h+1}(\epsilon)} \left\| \sum_{\tau=1}^{K} \phi(x_h^\tau, a_h^\tau) \cdot \epsilon_h^\tau(V) \right\|^2_{(\Lambda^t_h)^{-1}} \leq (n + R_{\text{max}})^2 H^2 \cdot \left( 2 \cdot \log(1/p) + d \cdot \log(1 + K/\lambda) \right).
$$

For all $\delta \in (0, 1)$ and all $\varepsilon > 0$, we set $p = \delta / [c_3n \cdot H \cdot |\mathcal{N}_{h+1}(\epsilon)|]$. Hence, for all fixed $h \in [H]$, it holds that

$$
\sup_{V \in \mathcal{N}_{h+1}(\epsilon)} \left\| \sum_{\tau=1}^{K} \phi(x_h^\tau, a_h^\tau) \cdot \epsilon_h^\tau(V) \right\|^2_{(\Lambda^t_h)^{-1}} \leq (n + R_{\text{max}})^2 H^2 \cdot \left( 2 \cdot \log((c_3n) \cdot H \cdot |\mathcal{N}_{h+1}(\epsilon)|/\delta) + d \cdot \log(1 + K/\lambda) \right)
$$

with probability at least $1 - \delta/(c_3nH)$, which is taken with respect to the joint distribution of the data collecting process. Then, combining Equations (42) and (43), for all $h \in [H]$, it holds that

$$
\left\| \sum_{\tau=1}^{K} \phi(x_h^\tau, a_h^\tau) \cdot \epsilon_h^\tau(\hat{V}^{t}_{h+1}) \right\|^2_{(\Lambda^t_h)^{-1}} \leq (n + R_{\text{max}})^2 H^2 \cdot \left( 2 \cdot \log((c_3n) \cdot H \cdot |\mathcal{N}_{h+1}(\epsilon)|/\delta) + d \cdot \log(1 + K/\lambda) \right) + 8\varepsilon^2 K^2 / \lambda,
$$

with probability at least $1 - \delta/(c_3n)$, which follows from the union bound respect to all $h \in [H]$.

It remains to choose a proper constant $\varepsilon > 0$ and upper bound the $\varepsilon$-covering number $|\mathcal{N}_{h+1}(\epsilon)|$. In the sequel, we set $\varepsilon = dH/K$ and $\lambda = 1$. To upper bound $|\mathcal{N}_{h+1}(\epsilon)|$, we utilize Lemma F.5, which is obtained from Jin et al. (2019). Recall that, in our proof,

$$
\hat{V}^{t}_{h+1} \in \mathcal{V}_{h+1}(R_0, B_0, \lambda) \text{ where } L_0 = (n + R_{\text{max}})H \sqrt{Kd/\lambda}, \quad B_0 = 2\beta, \quad \lambda = 1,
$$

$$
\beta = c \cdot (n + R_{\text{max}})dH \sqrt{\mathcal{X}}, \quad \mathcal{X} = \log(2c_3ndHK/\delta).
$$

Here $c > 0$ is an absolute constant, $\delta \in (0, 1)$ is the confidence parameter. By setting $c > 1$ large enough, for all $h \in [H]$, it holds that

$$
|\langle (\mathbb{H}_h \hat{V}^{t}_{h+1})(x, a) - \phi(x, a) \rangle w_h^t | \leq ((n + R_{\text{max}})H \sqrt{d} + \beta/2) \cdot \sqrt{\phi(x, a)^\top (\Lambda^t_h)^{-1} \phi(x, a)}, \quad (44)
$$

with probability at least $1 - \delta/(12n + 4)$. By Equations (32), (33) and (44), for all $h \in [H]$ and all $(x, a) \in \mathcal{S} \times \mathcal{A}$, it holds that

$$
|\langle (\mathbb{H}_h \hat{V}^{t}_{h+1})(x, a) - \phi(x, a) \rangle w_h^t | \leq ((n + R_{\text{max}})H \sqrt{d} + \beta/2) \cdot \sqrt{\phi(x, a)^\top (\Lambda^t_h)^{-1} \phi(x, a)},
$$

with probability at least $1 - \delta/(c_3n)$. Then combining Equation (31) and the definition of $\beta$, we
obtain
\[ |\mathbb{E}_h \hat{V}^{t,*}_{h+1}(x, a) - f^*_h| \leq \beta \cdot \sqrt{\phi(x, a)^\top (\Lambda^*_h)^{-1} \phi(x, a)}, \]

Note that we also have
\[ |\mathbb{E}_h \hat{V}^{t,*}_{h+1}(x, a) - f^*_h| \leq H(n + R_{max}), \]
due to \( 0 \leq \mathbb{E}_h \hat{V}^{t,*}_{h+1}(x, a) \leq H(n + R_{max}) \) and \( 0 \leq f^*_h \leq H(n + R_{max}) \). Therefore, we conclude the proof of Lemma D.2.

### D.2 Proof of Lemma C.3

Based on Lemma D.2, we can show that for all \( t \) and \( h \in [H] \), the constructed Q-functions \( \hat{Q}_h^{t,*} \) and \( \hat{Q}_h^{t,\pi} \) are optimistic and pessimistic estimators of the Q-function \( Q^*_h \) based on the policy \( \pi \) respectively, which is the key to bound the suboptimality.

**Lemma D.3 (Adaptation of Lemma 5.1 from Jin et al. (2020b))** Under the setting of Lemma D.2, which satisfies \( \mathbb{P}_D(\mathcal{E}) \geq 1 - \delta/4 \) and \( \mathbb{P}_D(\mathcal{E}') \geq 1 - \delta/4 \), we have

\[
\begin{align*}
0 & \geq \tau_h^{t,*}(x, a; R) \geq -2u^*_h(x, a), \quad \text{for all } (x, a) \in \mathcal{S} \times \mathcal{A}, \ h \in [H]. \\
0 & \geq \tau_h^{t,\pi}(x, a; R^{-i}) \geq -2u^*_h(x, a), \quad \text{for all } (x, a) \in \mathcal{S} \times \mathcal{A}, \ h \in [H], \ \text{and } \pi \in \{*, \pi^t\} \quad (45) \\
0 & \leq \tau_h^{t,\pi}(x, a; R^{-i}) \leq 2u^*_h(x, a), \quad \text{for all } (x, a) \in \mathcal{S} \times \mathcal{A}, \ h \in [H], \ \text{and } \pi \in \{*, \pi^t\}.
\end{align*}
\]

and

\[
\begin{align*}
0 & \geq \tau_h^{t,*}(x, a; r_{ih} + \tilde{R}^{-i}) \geq -2u^*_h(x, a), \quad \text{for all } (x, a) \in \mathcal{S} \times \mathcal{A}, \ h \in [H]. \\
0 & \geq \tau_h^{t,\pi}(x, a; \tilde{R}^{-i}) \geq -2u^*_h(x, a), \quad \text{for all } (x, a) \in \mathcal{S} \times \mathcal{A}, \ h \in [H], \ \text{and } \pi \in \{*, \pi^t, \pi^t\} \quad (46) \\
0 & \leq \tau_h^{t,\pi}(x, a; \tilde{R}^{-i}) \leq 2u^*_h(x, a), \quad \text{for all } (x, a) \in \mathcal{S} \times \mathcal{A}, \ h \in [H], \ \text{and } \pi \in \{*, \pi^t, \pi^t\}.
\end{align*}
\]

**Proof** [Proof of Lemma D.3] The results in Lemma D.3 can split into two parts: the upper and lower bounds of \( \{\hat{v}_h\} \) and \( \{\tilde{v}_h\} \). The proof of the first part is nearly identical to the Proof of Lemma 5.1 in Jin et al. (2020b). We only prove the second part here. For brevity, we take \( \tau_h^{t,*}(x, a; R) \) as an example and the rest are similar.

We first show that on the event \( \mathcal{E} \) defined in Equation (29), the model evaluation errors \( \{\tau_h^{t,*}(x, a; R)\}_{h=1}^{H} \) are negative. In the sequel, we assume that \( \mathcal{E} \) holds. Recall the construction of \( \hat{Q}_h^{t,*} \) as in Line 3 of Algorithm 3 for all \( h \in [H] \). For all \( h \in [H] \) and all \( (x, a) \in \mathcal{S} \times \mathcal{A} \), we have

\[ \hat{Q}_h^{t,*}(x, a; R) = \min\{(f^*_h + u^*_h(x, a), (H - h + 1)(n + R_{max})\} \].

If \( f^*_h + u^*_h(x, a) \leq (H - h + 1)(n + R_{max}) \), we have \( \hat{Q}_h^{t,*}(x, a; R) = f^*_h + u^*_h(x, a) \). By the definition of
\( \hat{\gamma}^t_h(x, a; R) \) in Equation (28), we have
\[
\hat{\gamma}^t_h(x, a; R) = (\mathbb{B}_h \hat{V}^t_{h+1})(x, a) - \hat{Q}^t_h(x, a) = (\mathbb{B}_h \hat{V}^t_{h+1})(x, a) - f^t_h - u^t_h \leq 0,
\]
where the inequality follows from Lemma D.2.

If \( f^t_h + u^t_h(x, a) \geq (H - h + 1)(n + R_{\text{max}}) \), we have \( \hat{Q}^t_h(x, a; R) = (H - h + 1)(n + R_{\text{max}}) \), which implies
\[
\hat{\gamma}^t_h(x, a; R) = (\mathbb{B}_h \hat{V}^t_{h+1})(x, a) - ((H - h + 1)(n + R_{\text{max}})) \leq 0,
\]
where the inequality follows from the definition of the Bellman operator in Equation (27) and the construction of \( \hat{V}^t_{h+1} \) in Line 4 of Algorithm 3.

It remains to establish the lower bound of \( \hat{\gamma}^t_h(x, a; R) \). Combining the definition of \( \hat{\gamma}^t_h(x, a; R) \) and \( \hat{Q}^t_h(x, a; R) \), we have
\[
\hat{\gamma}^t_h(x, a; R) = (\mathbb{B}_h \hat{V}^t_{h+1})(x, a) - \hat{Q}^t_h(x, a) \geq (\mathbb{B}_h \hat{V}^t_{h+1})(x, a) - f^t_h - u^t_h \geq -2u^t_h,
\]
where the first inequality follows from the definition of \( \hat{Q}^t_h(x, a; R) \) and the second inequality follows from Lemma D.2. In summary, we conclude that on \( \mathcal{E} \),
\[
0 \geq \hat{\gamma}^t_h(x, a; R) \geq -2u^t_h(x, a), \quad \forall (x, a) \in \mathcal{S} \times \mathcal{A}, \quad \forall h \in [H].
\]
The rest of the proof is similar to the above process. Therefore, we conclude the proof of Lemma D.3.



In Equation (45) and (46), the negativity of \( \{\hat{\gamma}_h\} \) implies the optimism of \( \{\hat{Q}_h\} \), that is, \( Q^*_h \leq \hat{Q}_h \) in a pointwise manner for all \( h \in [H] \). Similarly, the non-negativity of \( \{\hat{\gamma}_h\} \) implies the pessimism of \( \{\tilde{Q}_h\} \), that is, \( \tilde{Q}_h \leq Q^*_h \) in a pointwise manner for all \( h \in [H] \).

\section{Proof of Lemma C.1}

Based on Lemma D.3 and Lemma 4.2 in Jin et al. (2020b), we can further bound the suboptimalities in the proof of Theorems 4.1 and 4.2. Now we state a generalized edition of Lemma C.1, which contains the suboptimalities that we need to bound in our proof, and give a detailed proof of it.

\textbf{Lemma D.4} Under the setting of Lemma D.2, which satisfies \( \mathbb{P}_D(\mathcal{E}) \geq 1 - \delta/4 \) and \( \mathbb{P}_D(\mathcal{E}') \geq 1 - \delta/4 \), we have
\begin{enumerate}
\item \( V^*_1(x_1; R) - V^*_{1, \pi}(x_1; R) \leq 2 \sum_{h=1}^{H} \mathbb{E}_{\pi^t} [u^t_h] \).
\item \( \hat{V}^t_1(x_1; \mathcal{R}) - V^*_{1, \pi}(x_1; \mathcal{R}) \leq 2 \sum_{h=1}^{H} \mathbb{E}_{\pi^t} [u^t_h] \) for all \( i \in [n], \pi \in \{\pi^*, \hat{\pi}^t\}, \mathcal{R} \in \{R^{-i}, \tilde{R}^{-i}\} \),
\item \( V^*_1(x_1; \mathcal{R}) - \hat{V}^t_1(x_1; \mathcal{R}) \leq 2 \sum_{h=1}^{H} \mathbb{E}_{\pi^t} [u^t_h] \) for all \( i \in [n], \pi \in \{\pi^*, \hat{\pi}^t\}, \mathcal{R} \in \{R^{-i}, \tilde{R}^{-i}\} \).
\end{enumerate}
3. \(\hat{V}_{1}^t(x_1; r_{th} + \tilde{R}^{-i}) - V_{1}^t(x_1; r_{th} + \tilde{R}^{-i}) \leq 2 \sum_{h=1}^{H} \mathbb{E}_{\pi^{t}}[u_h^i].\)

4. \(\hat{V}_{1}^t(x_1; \tilde{R}^{-i}) - V_{1}^{\pi}(x_1; \tilde{R}^{-i}) \leq 2 \sum_{h=1}^{H} \mathbb{E}_{\pi^{t}}[u_h^i]\) for all \(i \in [n],\)
\(V_{1}^{\pi}(x_1; \tilde{R}^{-i}) - \hat{V}_{1}^t(x_1; \tilde{R}^{-i}) \leq 2 \sum_{h=1}^{H} \mathbb{E}_{\pi^{t}}[u_h^i]\) for all \(i \in [n],\)

where the bonus \(\{u_h^i\}\) are calculated according to the corresponding trajectories and reported rewards. In other words, the bonus \(\{u_h^i\}\) correspond to the underlying MDPs.

**Proof** [Proof of Lemma D.4] For brevity, we just upper bound \(V_{1}^{\pi}(x; R) - \hat{V}_{1}^t(x; R)\) in this section and the rest are similar. By adding and subtracting \(\hat{V}_{1}^t\) into the difference, we can decompose it in to two terms

\[
V_{1}^{\pi}(x; R) - \hat{V}_{1}^t(x; R) = \left( V_{1}^{\pi}(x_1; R) - \hat{V}_{1}^t(x_1; R) \right) + \left( \hat{V}_{1}^t(x_1; R) - \hat{V}_{1}^t(x_1; R) \right),
\]

where \(\{\hat{V}_{1}^t\}_{h=1}^H\) are the estimated value functions constructed by Algorithm 3. Term (i) in Equation (47) is the difference between the estimated value function \(\hat{V}_{1}^t\) and the optimal value function \(V_{1}^{\pi}\), while term (ii) is the difference between \(\hat{V}_{1}^t\) and the value function \(V_{1}^{\pi}\) of \(\pi\). To further decompose terms (i) and (ii), we utilize Lemma F.3, which is obtained from Cai et al. (2019), to depict the difference between an estimated value function and the value function under a certain policy.

For term (i), we invoke Lemma F.3 with \(\pi = \hat{\pi}^t\) and \(\pi' = \pi^{*}\), we have

\[
\hat{V}_{1}^t(x; R) - V_{1}^{\pi}(x; R) = \sum_{h=1}^{H} \mathbb{E}_{\pi^{*}}[\langle \tilde{Q}_{h}^{t}(s_h, \cdot; R), \tilde{\pi}_{h}^{t} | s_h \rangle - \pi_{h}^{*} | s_h \rangle]_{A} | s_1 = x]
\]

\[
\quad + \sum_{h=1}^{H} \mathbb{E}_{\pi^{*}}[\tilde{Q}_{h}^{t}(s_h, a_h; R) - (\mathbb{E}_{h} \tilde{V}_{h+1}^{t}) (s_h, a_h; R) | s_1 = x],
\]

where \(\mathbb{E}_{\pi^{*}}\) is taken with respect to the trajectory generated by \(\pi^{*}\). By the definition of the model evaluation error \(\iota_{h}\) in Equation (28), we have

\[
V_{1}^{t}(x; R) - \hat{V}_{1}^t(x; R) = \sum_{h=1}^{H} \mathbb{E}_{\pi^{*}}[\langle \tilde{Q}_{h}^{t}(s_h, \cdot; R), \pi_{h}^{t} | s_h \rangle - \pi_{h}^{*} | s_h \rangle]_{A} | s_1 = x]
\]

\[
\quad + \sum_{h=1}^{H} \mathbb{E}_{\pi^{*}}[\iota_{h}^{t}(s_h, a_h; R) | s_1 = x].
\]
Similarly, invoking Lemma F.3 with $\pi = \pi' = \hat{\pi}^t$, for term (ii), we have

$$\hat{V}^{t, \hat{\pi}^t}(x; R) - V^{\hat{\pi}^t}(x; R) = \sum_{h=1}^{H} \mathbb{E}_{\hat{\pi}^t}[\hat{Q}^{t,*}_h(s_h, a_h; R) - \mathbb{E}_{h} \hat{V}^{t,*}_{h+1}(s_h, a_h; R) | s_1 = x]$$

$$= - \sum_{h=1}^{H} \mathbb{E}_{\hat{\pi}^t}[\hat{\iota}^{t,*}_h(s_h, a_h; R) | s_1 = x],$$

where $\mathbb{E}_{\hat{\pi}^t}$ is taken with respect to the trajectory generated by $\hat{\pi}^t$.

Combining Equations (47), (48) and (49), we have

$$V^*_1(x; R) - V^{\hat{\pi}^t}(x; R) = \sum_{h=1}^{H} \mathbb{E}_{\pi^*}[\langle \hat{Q}^{t,*}_h(s_h, \cdot; R), \pi^*_h(\cdot | s_h) \rangle | s_1 = x]$$

$$+ \sum_{h=1}^{H} \mathbb{E}_{\pi^*}[\hat{\iota}^{t,*}_h(s_h, a_h; R) | s_1 = x] - \sum_{h=1}^{H} \mathbb{E}_{\hat{\pi}^t}[\hat{\iota}^{t,*}_h(s_h, a_h; R) | s_1 = x].$$

It remains to upper bound the three terms in the right-hand side of Equation (50). For the first term, we can upper bound it by 0 following the definition of $\hat{\pi}^t$ in Line 3 of Algorithm 3. To bound the last two terms we invoke Lemma D.3, which implies

$$\sum_{h=1}^{H} \mathbb{E}_{\pi^*}[\hat{\iota}^{t,*}_h(s_h, a_h; R) | s_1 = x] \leq 0, \quad -\mathbb{E}_{\hat{\pi}^t}[\hat{\iota}^{t,*}_h(s_h, a_h; R) | s_1 = x] \leq \mathbb{E}_{\hat{\pi}^t}[u^t_h(s_h, a_h) | s_1 = x],$$

for all $x \in S$ under event $\mathcal{E}$. Therefore, we obtain that

$$V^*_1(x; R) - V^{\hat{\pi}^t}(x; R) \leq \mathbb{E}_{\hat{\pi}^t}[u^t_h(s_h, a_h) | s_1 = x].$$

Using the similar technique, we can upper bound the remaining differences, thereby concluding the proof of Lemma D.4.

Based on Lemma D.4, we just need to bound the term $\mathbb{E}_{\pi}\left[\sum_{h=1}^{H} u^t_h \mid s_1 = x_1\right]$ where $\pi \in \{\hat{\pi}^t, \pi^*, \tilde{\pi}^t\}$. Recall the definition of $u^t_h$ in Algorithm 3, we just need to bound $V^*(x_1; u)$ where $u$ represents the reward function of $u^t_h$ and $V^*$ represents the maximum value function corresponding to the reward function $u$. Actually, the exploration phase of our algorithm fully explores the underlying MDP. Thus, we can bound $V^*(x_1; u)$ with the information gathered in the exploration phase similar to Wang et al. (2020) and Qiu et al. (2021).
D.4 Proof of Lemma C.2

Proof [Proof of Lemma C.2] The proof of this lemma is similar to the proof of Lemma 3.1 and Lemma 3.2 in Wang et al. (2020), but the rewards are random instead of fixed in our setting. Using the similar technique in the previous proof of Lemma D.2 and Lemma D.4, with probability at least $1 - \delta/8$, we have

$$| (p_h \hat{v}_{h+1})(x, a) - \Pi_{[0,B]}[\phi(x,a) \top u^k_h] | \leq \min \{ \beta \cdot \sqrt{\phi(x,a) \top (\Lambda^k_h)^{-1} \phi(x,a), H(n + R_{\text{max}})} \} = u^k_h(x,a),$$

for all $h \in [H]$ and all $(x,a) \in \mathcal{S} \times \mathcal{A}$ with $B = H(n + R_{\text{max}})$, which will be used in the following proof. Then using the identical technique in the proof of Lemma 3.1 in Wang et al. (2020) and Lemma B.7 in Qiu et al. (2021), we obtain

$$V^*_1(x_1;k) \leq V^k_1(x_1) \quad \text{for all } k \in [K], \quad (51)$$

and

$$\sum_{k=1}^{K} V^k_1(x_1) \leq c_\beta \sqrt{d^3 H^4 K \log (2nd HK/\delta)}, \quad (52)$$

for some absolute constant $c_\beta$ with probability at least $1 - \delta/4$. Equation (51) and Equation (52) show that the estimation of the value function in the exploration phase is optimistic and the summation of $V^k_1(x_1)$ should be small with high probability. Then, we use the identical technique of the proof of Lemma 3.2 of Wang et al. (2020) to obtain

$$V^*_1(x_1;u^t) \leq c_\beta \sqrt{d^3 H^6 \log (2nd HK/\delta)/K},$$

which concludes the proof.

E Proof of Lower Bounds

In this section, we first state some preliminary results. The following lemma decomposes the utilities of the seller and agent $i$ according to Markov VCG mechanism.

Lemma E.1 When the actions and prices are chosen according to Markov VCG mechanism, then

$$u_{i_1} = V^*_1(x_1;R) - V^{\pi_{-i}^*}_1(x_1;R^{-i}),$$

$$u_{0_1} = \sum_{i=1}^{n} V^*_1(x_1;R^{-i}) - (n - 1)V^*_1(x_1;R).$$
We can deduce the above results just by the definition of the utilities of the agents and the seller. For the utility of agent $i$, we have

$$u_{i*} = V_{i*}^\pi(x_1; r_i) - p_{i*}$$

$$= V_{i*}^\pi(x_1; r_i) - \left[ V_{i*}^{\pi-i}(x_1; R^{-i}) - V_{i*}^\pi(x_1; R^{-i}) \right]$$

$$= V_{i*}^\pi(x_1; R) - V_{i*}^{\pi-i}(x_1; R^{-i}).$$

For the utility of the seller, we have

$$u_{0*} = V_{0*}^{\pi-i}(x_1; r_0) + \sum_{i=1}^n p_{i*}$$

$$= V_{0*}^{\pi-i}(x_1; r_0) + \sum_{i=1}^n \left[ V_{i*}^{\pi-i}(x_1; R^{-i}) - V_{i*}^\pi(x_1; R^{-i}) \right]$$

$$= \sum_{i=1}^n V_{i*}^{\pi-i}(x_1; R^{-i}) - (n - 1)V_{0*}^\pi(x_1; R).$$

This completes the proof.

We then define the estimation of $\sum_{i=1}^n V_{i}^{\pi-i}(x_1; R^{-i})$ and the error of this estimation as

$$Y_T = \frac{1}{T} \sum_{i=1}^n \sum_{t=1}^T \left( p_{it} + V_{i}^{\pi-i}(x_1; R^{-i}) \right), \quad Z_T = Y_T - \sum_{i=1}^n V_{i}^{\pi-i}(x_1; R^{-i}).$$

The next lemma states the relationships of the regret terms we defined in Equation (1), which is useful in the proof of our lower bound.

**Lemma E.2** Let $\text{Reg}^W_T, \text{Reg}_{0T}, \text{Reg}^T$ be as defined in Equation (1). Then

$$\text{Reg}^T = n\text{Reg}^W_T + TZ_T, \quad \text{Reg}_{0T} = -(n - 1)\text{Reg}^W_T - TZ_T.$$

**Proof** The proof of this lemma relies on the decomposition of these regret terms. We first define $h_{it} = p_{it} + V_{i}^{\pi-i}(x_1; R^{-i})$ for brevity. Then we have $Y_T = \frac{1}{T} \sum_{i=1}^n \sum_{t=1}^T h_{it}$. For agent $i$, we have

$$u_{it} = V_{i}^{\pi-i}(x_1; r_i) - p_{it}$$

$$= V_{i}^{\pi-i}(x_1; r_i) - \left( h_{it} - V_{i}^{\pi-i}(x_1; R^{-i}) \right)$$

$$= V_{i}^{\pi-i}(x_1; R) - h_{it}. \quad (53)$$
Combine Lemma E.1 and Equation (53), we can obtain

\[ u_i - u_{it} = \left( V^{\pi_1}_1(x_1; R) - V^{\pi_{i-1}}_1(x_1; R^{-i}) \right) - \left( V^{\pi_1}_1(x_1; R) - h_{it} \right) \]

\[ = \left( V^{\pi_1}_1(x_1; R) - V^{\pi}_1(x_1; R) \right) - \left( V^{\pi_{i-1}}_1(x_1; R^{-i}) - h_{it} \right). \]

Then by the definition of Reg\textsuperscript{\#} in Equation (1), we have

\[ \text{Reg}_T = \sum_{t=1}^{T} \sum_{i=1}^{n} (u_i - u_{it}) \]

\[ = \sum_{t=1}^{T} \sum_{i=1}^{n} \left[ \left( V^{\pi_1}_1(x_1; R) - V^{\pi}_1(x_1; R) \right) - \left( V^{\pi_{i-1}}_1(x_1; R^{-i}) - h_{it} \right) \right] \]

\[ = n \sum_{t=1}^{T} \left( V^{\pi_{i-1}}_1(x_1; R^{-i}) - V^{\pi}_1(x_1; R) \right) + T \left( V_T - \sum_{i=1}^{n} V^{\pi_{i-1}}_1(x_1; R^{-i}) \right) \]

\[ = n \text{Reg}_T + T Z_T. \]

This proves the first claim. For the seller, at time step \( t \), we have the following observation that

\[ u_{0t} = V^{\pi}_1(x_1; r_0) + \sum_{i=1}^{n} p_{it} \]

\[ = V^{\pi}_1(x_1; r_0) + \sum_{i=1}^{n} (h_{it} - V^{\pi}_1(x_1; R^{-i})) \]

\[ = \sum_{i=1}^{n} h_{it} - (n-1)V^{\pi}_1(x_1; R). \]  

(54)

Similarly, we can now combine Lemma E.1 and Equation (54) and obtain

\[ \text{Reg}_{0T} = \sum_{i=1}^{T} (u_i - u_{0t}) \]

\[ = \sum_{t=1}^{T} \left( V^{\pi_1}_1(x_1; R^{-i}) - h_{it} \right) + (n-1) \sum_{t=1}^{T} \left( V^{\pi}_1(x_1; R) - V^{\pi}_1(x_1; R) \right) \]

\[ = -T Z_T - (n-1)R_T. \]

This completes the proof of the second claim. \[\blacksquare\]

The following inequality about relative entropy will be useful in our proof of lower bound.

**Lemma E.3 (Bretagnolle-Huber Inequality)** Let \( \mathbb{P} \) and \( \mathbb{Q} \) be probability measures on the same
measurable space \((\Omega, \mathcal{F})\), and let \(A \in \mathcal{F}\) be an arbitrary event. Then,

\[ P(A) + \mathbb{Q}(A^c) \geq \frac{1}{2} \exp(-\text{KL}(P||Q)), \]  

(55)

where \(A^c = \Omega \setminus A\) is the complement of \(A\).

The proof of Theorem 4.3 is a natural extension of the proof of Theorem 1 in Kandasamy et al. (2020). We extend the result of multi-arm bandit to linear MDP setting.

**Proof** [Proof of Theorem 4.3] At the beginning of the proof, we first state a basic inequality here: for any set of real numbers \(\{x_i\}_{i \geq 1}\), and any set of non-negative real numbers \(\{a_i\}_{i \geq 1}\) such that \(\sum_{i \geq 1} a_i = 1\), we have \(\max\{x_i\}_{i \geq 1} \geq \sum_{i \geq 1} a_i x_i\). Combining the above inequality and Lemma E.2, we obtain the following two lower bounds on \(\max(n\text{Reg}_W^T, \text{Reg}_T^\sharp, \text{Reg}_{0T})\) and the first one is

\[
\max(n\text{Reg}_W^T, \text{Reg}_T^\sharp, \text{Reg}_{0T}) \geq \frac{4}{5} n\text{Reg}_W^T + \frac{1}{5} \text{Reg}_{0T} \\
= \frac{4}{5} n\text{Reg}_W^T - \frac{1}{5} (n - 1)\text{Reg}_W^T - TZ_T \\
\geq \frac{2}{5} n\text{Reg}_W^T - \frac{1}{5} TZ_T.
\]

We use Lemma E.2 in the first equality, and use the fact that \(\text{Reg}_W^T\) is non-negative in the last step. Similarly, we can obtain another lower bound as

\[
\max(n\text{Reg}_W^T, \text{Reg}_T^\sharp, \text{Reg}_{0T}) \geq \frac{2}{5} n\text{Reg}_W^T + \frac{1}{5} TZ_T.
\]

Comparing the above two lower bounds of \(\max(n\text{Reg}_W^T, \text{Reg}_T^\sharp, \text{Reg}_{0T})\), we have

\[
\max(n\text{Reg}_W^T, \text{Reg}_T^\sharp, \text{Reg}_{0T}) \geq \frac{2}{5} n\text{Reg}_W^T + \frac{1}{5} |Z_T|.
\]

For brevity, we denote \(\frac{2}{5} n\text{Reg}_W^T + \frac{1}{5} |Z_T|\) by \(S_T\) in the following passage. Our goal is to obtain a lower bound on \(\inf_{\text{Alg}} \sup_{\Theta} \mathbb{E}[S_T]\) which is also a lower bound on \(\max(n\text{Reg}_W^T, \text{Reg}_T^\sharp, \text{Reg}_{0T})\). To achieve this goal, we construct two problems in \(\Theta\) and show that no algorithm can present well on these two problems simultaneously.

We define the underlying MDP \(M_0\) of the first problem, henceforth called \(\theta_0\) as following: \(M_0\) is an episodic MDP with horizon \(H \geq 2\), state space \(S = \{x_0, x_1, x_2, \ldots, x_{n+1}, x_{n+2}\}\), and action space \(A = \{b_1, b_2, \ldots, b_A\}\) with \(|A| = A \geq n + 2\). In particular, we fix the initial state as \(s_1 = x_0\).

For the transition kernel, at the first time step \(h = 1\), we set

\[
\mathbb{P}_1(x_i|x_0, b_i) = 1, \quad \text{for all } i \in \{1, 2, \ldots, n + 1\}, \\
\mathbb{P}_1(x_{n+2}|x_0, b_i) = 1 \quad \text{for all } i \in \{n + 2, \ldots, A\}.
\]
Meanwhile, at any subsequent step \( h \in \{2, \ldots, H\} \), we set

\[
P_h(x_i | x_i, a) = 1, \quad \text{for all } a \in \mathcal{A}.
\]

In other words, state \( \{x_i\}_{i=1}^{n+2} \) are absorbing states. For the reward function, we let \( \text{Ber}(p) \) denote a Bernoulli random variable with success probability \( p \) and set

\[
\begin{align*}
    r_{0h}(s, a) &= 0, \quad \text{for all } (h, s, a) \in \{1, \ldots, H\} \times \mathcal{S} \times \mathcal{A}, \\
    r_{i1}(x_0, a) &= 0, \quad \text{for all } (i, a) \in [n + 2] \times \mathcal{A}, \\
    r_{jh}(x_i, a) &\sim \text{Ber}(1/2), \quad \text{for all } j \neq i \text{ and } (i, h, a) \in [n] \times \{2, \ldots, H\} \times \mathcal{A}, \\
    r_{ih}(x_i, a) &= 0, \quad \text{for all } (i, h, a) \in [n] \times \{2, \ldots, H\} \times \mathcal{A}, \\
    r_{jh}(x_{n+1}, a) &\sim \text{Ber}(1/2), \quad \text{for all } (j, h, a) \in [n] \times \{2, \ldots, H\} \times \mathcal{A}, \\
    r_{jh}(x_{n+2}, a) &\sim \text{Ber}(1/8), \quad \text{for all } (j, h, a) \in [n] \times \{2, \ldots, H\} \times \mathcal{A},
\end{align*}
\] (56)

which means the seller’s reward is always 0. See figure 1 for an illustration. Note that \( M_0 \) is a linear MDP with the dimension \( d = n + 2 \), to see this we set the corresponding feature map \( \phi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d \) as

\[
\begin{align*}
    \phi(x_0, b_i) &= e_i, \text{for all } i = 1, 2, \ldots, n + 1, \\
    \phi(x_0, b_i) &= e_{n+2}, \text{for all } i = n + 2, \ldots, A, \\
    \phi(x_i, b_j) &= e_i, \text{for all } i = 1, 2, \ldots, n + 1 \text{ and } j \in [A], \\
    \phi(x_i, b_j) &= e_{n+2}, \text{for all } i = n + 2, \ldots, A \text{ and } j \in [A],
\end{align*}
\]

where \( \{e_j\} \) are the canonical basis of \( \mathbb{R}^{n+2} \). Additionally, if the seller steps into state \( x_{h+1} \), the sum of agents’ utilities will be the largest. We can also obtain the following statements about problem \( \theta_0 \) directly,

\[
\begin{align*}
    V_{1,\pi^*}(x_0; R) &= Q_1(x_0, b_{n+1}; R) = \frac{1}{2} n (H - 1), \\
    V_{1,\pi^{-i}}(x_0; R^{-i}) &= Q_1(x_0, b_i; R^{-i}) = \frac{1}{2} (n - 1) (H - 1), \\
    \sum_{i=1}^{n} V_{1,\pi^{-i}}(x_0; R^{-i}) &= \frac{1}{2} n (n - 1) (H - 1).
\end{align*}
\]

The second problem, i.e., \( \theta_1 \), with the underlying MDP \( M_1 \) is nearly the same as \( \theta_0 \) but differs in reward functions at state \( x_i \) for \( i \in [n] \). Then, we define \( \theta_1 \) as

\[
\begin{align*}
    r_{jh}(x_i, a) &\sim \text{Ber}(1/2 + \delta), \quad \text{for all } j \neq i \text{ and } (i, h, a) \in [n] \times \{2, \ldots, H\} \times \mathcal{A}, \\
    r_{ih}(x_i, a) &= 0, \quad \text{for all } (i, h, a) \in [n] \times \{2, \ldots, H\} \times \mathcal{A}.
\end{align*}
\] (57)

Here we set \( \delta \in (0, 1/(2n - 2)) \). The problem \( \theta_1 \) shares the same feature maps \( \phi \) and the transition
parameters $\mu$ with problem $\theta_0$. The differences are the reward parameters. See figure 1 for an illustration. Similarly, we can obtain the following statements of problem $\theta_1$ directly,

\[
V^\pi_1(x_0; R) = Q_1(x_0, b_{n+1}; R) = \frac{1}{2} n(H - 1),
\]

\[
V^{-i}_1(x_0; R^{-i}) = Q_1(x_0, b_i; R^{-i}) = \left(\frac{1}{2} + \delta\right)(n - 1)(H - 1),
\]

\[
\sum_{i=1}^{n} V^{-i}_1(x_0; R^{-i}) = \left(\frac{1}{2} + \delta\right)n(n - 1)(H - 1).
\]

For accuracy, we denote $S_T(\theta_0)$ and $S_T(\theta_1)$ for problems $\theta_0$ and $\theta_1$ respectively. Expectations and probabilities corresponding to problem $\theta_i$ will be denoted as $\mathbb{E}, \mathbb{P}$ with subscribe $\theta_i$ respectively. Let $N_k(a) = \sum_{r=1}^{k} I\{a^r = a\}$ denote the number of times that the seller takes action $a$ at first step in the first $k$ rounds. In this notation, we rewrite the lower bound of the welfare regret in problem $\theta \in \{\theta_1, \theta_2\}$ as

\[
\mathbb{E}_\theta[\text{Reg}^W_T] = \sum_{j=1, j \neq n+1}^{n+2} (Q_1(x_0, b_{n+1}; R) - Q_1(x_0, b_j; R)) \mathbb{E}_\theta[N_K(b_j)]
\]

\[
\geq \sum_{j=1}^{n} (Q_1(x_0, b_{n+1}; R) - Q_1(x_0, b_j; R)) \mathbb{E}_\theta[N_K(b_j)]
\]

Observing that $Q_1(x_0, b_{n+1}; R) - Q_1(x_0, b_j; R) = (H - 1)/2$ in problem $\theta_0$, and that when $Y_T > [n^2/2 - n/2 + n(n - 1)\delta/2](H - 1)$, $|Z_T|$ is at least $n(n - 1)(H - 1)/2$, we get the following lower bound on $\mathbb{E}_{\theta_0}[S_T(\theta_0)]$ as

\[
\mathbb{E}_{\theta_0}[S_T(\theta_0)] \geq \frac{2}{3} n \text{Reg}^W_T + \frac{1}{5} T|Z_T|
\]

\[
\geq \frac{2}{5} n \sum_{j=1}^{n} \frac{H - 1}{2} \mathbb{E}_{\theta_0}[N_K(b_j)] + \frac{T}{5} n(n - 1)(H - 1)\delta \mathbb{P}_{\theta_0}\left(Y_T > \left[\frac{n^2}{2} - \frac{n}{2} + \frac{n(n - 1)\delta}{2}\right](H - 1)\right)
\]

\[
\geq \frac{n(H - 1)}{10} \left[\sum_{j=1}^{n} 2\mathbb{E}_{\theta_0}[N_K(b_j)] + T(n - 1)\delta \mathbb{P}_{\theta_0}(F)\right].
\]

In problem $\theta_1$, we have $|Z_T|$ is at least $n(n - 1)(H - 1)/2$ when $Y_T \leq [n^2/2 - n/2 + n(n - 1)\delta/2](H - 1)$. We drop the welfare regret, which is positive, in the analysis, and use the above statement regarding $Y_T$ under the event $F^c$ in problem $\theta_1$ to obtain

\[
\mathbb{E}_{\theta_1}[S_T(\theta_1)] \geq \frac{n(H - 1)}{10} T(n - 1)\delta \mathbb{P}_{\theta_1}(F^c).
\]
Applying Lemma E.3 to $\mathbb{P}_{\theta_0}(F) + \mathbb{P}_{\theta_1}(F^c)$, we have

$$\mathbb{P}_{\theta_0}(F) + \mathbb{P}_{\theta_1}(F^c) \geq \frac{1}{2} \exp(-\text{KL}(\theta_0^T||\theta_1^T)),$$

where $\theta_0^T$ and $\theta_1^T$ denote the probability laws of the observed rewards up to time $t$ in problem $\theta_0$ and $\theta_1$ respectively. We also notice that if the seller take action $b_{n+1}, b_{n+2}$ in the first step, then $\theta_0^T = \theta_1^T$. If the seller take action $b_i$ for $i \in \{1, 2, \cdots, n\}$ in the first step, then the reward distributions of agent $i$ are the same in both $\theta_0$ and $\theta_1$. However, for other agents $j \neq i$, the KL divergence between the corresponding distributions in the two problems is $-\log(1 - 4\delta^2)(H - 1)$ since the rewards are independent in each time step and the KL divergence between Ber$(1/2)$ and Ber$(1/2 + \delta)$ is $-\log(1 - 4\delta^2)$. Then we have

$$\text{KL}(\theta_0^T||\theta_1^T) = -(n - 1)(H - 1)\log(1 - 4\delta^2)\sum_{j=1}^{n} \mathbb{E}_{\theta_0}[N_K(b_j)] \quad (60)$$

By combining Equation (58), (59), (55), and (60), we obtain a lower bound on $\mathbb{E}_{\theta_0}[S_T(\theta_0)] + \mathbb{E}_{\theta_1}[S_T(\theta_1)]$ as

$$\mathbb{E}_{\theta_0}[S_T(\theta_0)] + \mathbb{E}_{\theta_1}[S_T(\theta_1)] \geq \frac{n(H - 1)}{10} \left[ \sum_{j=1}^{n} 2\mathbb{E}_{\theta_0}[N_K(b_j)] + T(n - 1)\delta(\mathbb{P}_{\theta_0}(F) + \mathbb{P}_{\theta_1}(F^c)) \right]$$

$$\geq \frac{n(H - 1)}{10} \left[ 2\sum_{j=1}^{n} \mathbb{E}_{\theta_0}[N_K(b_j)] ight. $$

$$\quad + \frac{1}{2} T(n - 1)\delta \exp \left( (n - 1)(H - 1)\log(1 - 4\delta^2)\sum_{j=1}^{n} \mathbb{E}_{\theta_0}[N_K(b_j)] \right) \right]$$

$$\geq \frac{n(H - 1)}{10} \min \left\{ 2x + \frac{1}{2} T(n - 1)\delta \exp \left( (n - 1)(H - 1)\log(1 - 4\delta^2)x \right) \right\}. $$

$$: = f(x).$$

We combine Equation (58) and (59) in the first step. For the second step, we combine Equation (55) and Equation 60. For the last step we substitute $\sum_{j=1}^{n} \mathbb{E}_{\theta_0}[N_K(b_j)]$ by $x$ and turn to find the minimum value of the function $f(x)$. Then, we have

$$x_0 = \frac{-1}{(n - 1)(H - 1)\log(1 - 4\delta^2)} \log \left( \frac{-T(n - 1)^2(H - 1)\delta \log(1 - 4\delta^2)}{4} \right).$$

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as the minimum of \( f(x) \). Thus, we have

\[
\mathbb{E}_{\theta_0}[S_T(\theta_0)] + \mathbb{E}_{\theta_1}[S_T(\theta_1)] \geq \frac{n(H - 1)}{10} x_0 \\
\geq \frac{-1}{5 \log(1 - 4\delta^2)} \log \left( \frac{-T(n - 1)^2(H - 1)\delta \log(1 - 4\delta^2)}{4} \right).
\]

(61)

Using the basic inequality \( x/(1 + x) \leq \log(1 + x) \leq x \) for \( x > -1 \), we have

\[
-4\delta^2 \geq \log(1 - 4\delta^2) \geq \frac{-4\delta^2}{1 - 4\delta^2} \geq -8\delta^2,
\]

when \( 0 \leq \delta^2 \leq 1/8 \). Combining Equation (61) and the above inequality, we obtain

\[
\mathbb{E}_{\theta_0}[S_T(\theta_0)] + \mathbb{E}_{\theta_1}[S_T(\theta_1)] \geq -\frac{1}{40\delta^2} \log \left( T(n - 1)^2(H - 1)\delta^3 \right).
\]

Finally, we choose \( \delta = \left( 1/(T(n - 1)^2(H - 1)) \right)^{1/3} \) to obtain the lower bound

\[
\frac{1}{2} \left( \mathbb{E}_{\theta_0}[S_T(\theta_0)] + \mathbb{E}_{\theta_1}[S_T(\theta_1)] \right) \geq c \cdot n^{4/3} H^{2/3} T^{2/3},
\]

for some absolute constant \( c \). Here \( \delta \in (0, 1/(2n - 2)) \) is satisfied when \( T \geq 8(n - 1)/(H - 1) \) and \( \delta^2 \in (0, 1/8) \) is satisfied when \( n \geq 3 \). Above all, we have the conclusion that

\[
\inf_{\text{Alg}} \sup_{\Theta} \mathbb{E} \left[ \max \left( nR^W_T, R^a_T, R_{0T} \right) \right] \geq \Omega(n^{4/3} H^{2/3} T^{2/3}).
\]

On the other hand, note that \( \max \left( nR^W_T, R^a_T, R_{0T} \right) \geq nR^W_T \) always holds, which implies

\[
\max \left( nR^W_T, R^a_T, R_{0T} \right) \geq nR^W_T = n \left[ T \cdot V^*(x_1; R) - \sum_{t=1}^{T} V_1^{\pi_t}(x_1; R) \right].
\]

(62)

Since \( R = \sum_{i=1}^{n} r_i \), we consider a simple hard instance that \( r_0 = r_1 = r_2 = \cdots = r_n = r' \) where \( r' : S \times \mathcal{A} \mapsto [0, 1] \) is some reward function. This indicates that we consider an instance with the
same reward function for all \( r_i, 0 \leq i \leq n \). Thus, under this setting, by (62), we have

\[
\max \left( nR^W_T, R^a_T, R^0_T \right) \geq n \left[ T \cdot V^\pi(x_1; R) - \sum_{t=1}^{T} V^\pi_t(x_1; R) \right] \\
= n(n + 1) \left[ T \cdot V^\pi(x_1; r') - \sum_{t=1}^{T} V^\pi_t(x_1; r') \right].
\]

The above inequality implies that the lower bound of \( \max \left( nR^W_T, R^a_T, R^0_T \right) \) can be further lower bounded by the lower bound of the regret for linear MDPs of dimension \( d \) with rewards in \([0, 1]\). Theorem 1 in Zhou et al. (2020) shows that for any algorithm, if \( d \geq 4 \) and \( T \geq 64(d - 3)^2H \), then there exists at least one linear MDP instance that incurs regret at least \( \Omega(d\sqrt{HT}) \). Therefore, we can further obtain that under the same assumptions, the minimax lower bound for \( \max \left( nR^W_T, R^a_T, R^0_T \right) \) is at least \( \Omega(n(n + 1)d\sqrt{HT}) \), i.e.,

\[
\inf_{\text{Alg}} \sup_{\Theta} \mathbb{E} \left[ \max \left( nR^W_T, R^a_T, R^0_T \right) \right] \geq \Omega(n(n + 1)d\sqrt{HT}).
\]

To sum up, we have the following lower bound of \( \max \left( nR^W_T, R^a_T, R^0_T \right) \) as

\[
\inf_{\text{Alg}} \sup_{\Theta} \mathbb{E} \left[ \max \left( nR^W_T, R^a_T, R^0_T \right) \right] \geq \Omega(n^{4/3}H^{2/3}T^{2/3} + n^2d\sqrt{HT}).
\]

Therefore, we conclude the proof of Theorem 4.3.
Figure 1: An illustration of the episodic MDPs $M_0, M_1$ with the state space $S = \{x_0, x_1, \cdots, x_{n+2}\}$ and action space $A = \{b_i\}_{i=1}^A$. Here we fix the initial state as $s_1 = x_0$, where the agent takes the action $a \in A$ and transits into the second state $s_2 \in \{x_1, \cdots, x_{n+2}\}$. In both MDPs, we have the same transition probability. At the first step $h = 1$, the transition probability satisfies $P_1(x_i|x_0, b_i) = 1$ for all $i \in \{1, 2, \cdots, n+1\}$ and $P_1(x_{n+2}|x_0, b_i) = 1$ for all $i \in \{n+2, \cdots, A\}$. Also, $x_1, x_2, \cdots, x_{n+2} \in S$ are the absorbing states. For brevity, we do not show the reward functions of $M_0, M_1$ in the caption that we have showed them in Equations (56) and (57).
F Other Supporting Lemmas

The following lemma, obtained from Abbasi-Yadkori et al. (2011), establishes the concentration of self-normalized processes.

**Lemma F.1 (Concentration of Self-Normalized Processes (Abbasi-Yadkori et al., 2011))**

Let \( \{ F_t \}_{t=0}^{\infty} \) be a filtration and \( \{ \epsilon_t \}_{t=1}^{\infty} \) be an \( \mathbb{R} \)-valued stochastic process such that \( \epsilon_t \) is \( F_t \)-measurable for all \( t \geq 1 \). Moreover, suppose that conditioning on \( F_{t-1} \), \( \epsilon_t \) is a zero-mean and \( \sigma \)-sub-Gaussian random variable for all \( t \geq 1 \), that is,

\[
\mathbb{E}[\epsilon_t | F_{t-1}] = 0, \quad \mathbb{E}[\exp(\lambda \epsilon_t) | F_{t-1}] \leq \exp(\lambda^2 \sigma^2 / 2), \quad \forall \lambda \in \mathbb{R}.
\]

Meanwhile, let \( \{ \phi_t \}_{t=1}^{\infty} \) be an \( \mathbb{R}^d \)-valued stochastic process such that \( \phi_t \) is \( F_{t-1} \)-measurable for all \( t \geq 1 \). Also, let \( M_0 \in \mathbb{R}^{d \times d} \) be a deterministic positive-definite matrix and

\[
M_t = M_0 + \sum_{s=1}^{t} \phi_s \phi_s^\top
\]

for all \( t \geq 1 \). For all \( \delta > 0 \), it holds that

\[
\left\| \sum_{s=1}^{t} \phi_s \epsilon_s \right\|_{M_t^{-1}}^2 \leq 2\sigma^2 \cdot \log \left( \frac{\det(M_t)^{1/2} \cdot \det(M_0)^{-1/2}}{\delta} \right)
\]

for all \( t \geq 1 \) with probability at least \( 1 - \delta \).

**Proof** See Theorem 1 of Abbasi-Yadkori et al. (2011) for a detailed proof.

**Lemma F.2 (Abbasi-Yadkori et al. (2011))** Let \( \{ \phi_t \}_{t=0}^{\infty} \) be a bounded sequence in \( \mathcal{R}^d \) satisfying \( \sup_{t \geq 0} \| \phi_t \| \leq 1 \). Let \( \Lambda_0 \in \mathcal{R}^{d \times d} \) be a positive definite matrix. For any \( t \geq 0 \), we define \( \Lambda_t = \Lambda_0 + \sum_{j=1}^{t} \phi_j^\top \phi_j \). Then, if the smallest eigenvalue of \( \Lambda_0 \) satisfies \( \lambda_{\min}(\Lambda_0) \geq 1 \), we have

\[
\log \left( \frac{\det(\Lambda_t)}{\det(\Lambda_0)} \right) \leq \sum_{j=1}^{t} \phi_j^\top \Lambda_{j-1}^{-1} \phi_j \leq 2 \log \left( \frac{\det(\Lambda_0)}{\det(\Lambda_0)} \right).
\]

The following lemma, obtained from Cai et al. (2019), depicts the difference between an estimated value function and the value function under a certain policy.

**Lemma F.3 (Extended Value Difference (Cai et al., 2019))** Let \( \pi = \{ \pi_h \}_{h=1}^{H} \) and \( \pi' = \{ \pi'_h \}_{h=1}^{H} \) be any two policies and let \( \{ \hat{Q}_h \}_{h=1}^{H} \) be any estimated Q-functions. For all \( h \in [H] \),
we define the estimated value function $\hat{V}_h: S \mapsto \mathbb{R}$ by setting $\hat{V}_h(x) = \langle \hat{Q}_h(x, \cdot), \pi_h(\cdot | x) \rangle_A$ for all $x \in S$. For all $x \in S$, we have

$$\hat{V}_1(x) - V_1^{\pi'}(x) = \sum_{h=1}^H \mathbb{E}_{\pi'}[\langle \hat{Q}_h(s_h, \cdot), \pi_h(\cdot | s_h) - \pi_h'(\cdot | s_h) \rangle_A | s_1 = x]$$

$$+ \sum_{h=1}^H \mathbb{E}_{\pi'}[\hat{Q}_h(s_h, a_h) - (\mathbb{B}_h \hat{V}_{h+1})(s_h, a_h) | s_1 = x],$$

where $\mathbb{E}_{\pi'}$ is taken with respect to the trajectory generated by $\pi'$, while $\mathbb{B}_h$ is the Bellman operator defined in Equation (27).

**Proof** See Section B.1 in Cai et al. (2019) for a detailed proof.

Recall the definition of $w_h^i$ in Line 3 of Algorithm 3. The following lemma upper bounds the norms of $w_h$ and $w_h^i$, respectively.

**Lemma F.4 (Bounded Weights of Value Functions (Jin et al., 2020b))** Let $V_{\text{max}} > 0$ be an absolute constant. For any function $V: S \to [0, V_{\text{max}}]$ and any $h \in [H]$, we have

$$\|w_h\| \leq \|\theta_h\| + V_{\text{max}} \sqrt{d}, \quad \|w_h^i\| \leq (n + R_{\text{max}})H \sqrt{Kd/\lambda},$$

where $w_h$ and $w_h^i$ are defined in Equation (30) and Line 3 of Algorithm 3, respectively.

**Proof** The proof is nearly identical to that of Lemma B.1 in Jin et al. (2020b). The only difference in bounding the norm of $w_h$ in our case is that the definition of $\theta_h$ is the sum of $\theta_i$ for $i = 0, 1, \cdots, n$. The difference in bounding the norm of $w_h^i$ in our case is that $|r_h^i + V_{h+1}^{i, *}| \leq 2(n + R_{\text{max}})H$. The rest of the proof follow similarly.

**Lemma F.5 (\(\varepsilon\)-Covering Number (Jin et al., 2019))** For all $h \in [H]$ and all $\varepsilon > 0$, we have

$$\log |N_h(\varepsilon; L, B, \lambda)| \leq d \cdot \log(1 + 4L/\varepsilon) + d^2 \cdot \log(1 + 8d^{1/2}B^2/(\varepsilon^2 \lambda)).$$

**Proof** [Proof of Lemma F.5] See Lemma D.6 in Jin et al. (2019) for a detailed proof.

**Lemma F.6 (Concentration of Self-Normalized Processes)** Let $V: S \mapsto [0, (n + R_{\text{max}}) \cdot (H -
that for all $p$ Equation (34) is mean-zero and $(\epsilon_h(V), a_h^\tau) \in \mathcal{V}$ from the Markov property of the process. Moreover, as it holds that $E(h, \tau \cdot \tau_{\tau+1})$ denotes the $\mathcal{F}_\tau$-algebra generated by a set of random variables and $(\tau + 1) \wedge K$ denotes $\min\{\tau + 1, K\}$. For all $\tau \in [K]$, we have $\phi(x_h^\tau, a_h^\tau) \in \mathcal{F}_{h, \tau-1}$, as $(x_h^\tau, a_h^\tau)$ is $\mathcal{F}_{h, \tau-1}$-measurable. Also, for the fixed function $V : \mathcal{S} \mapsto [0, (n + R_{\text{max}}) \cdot (H - 1)]$ and all $\tau \in [K]$, we have

$$\epsilon_h^\tau(V) = R_h^\tau + V(x_h^{\tau+1}) - (B_h V)(x_h^\tau, a_h^\tau) \in \mathcal{F}_{h, \tau},$$

as $(x_h^\tau, a_h^\tau, x_h^{\tau+1})$ is $\mathcal{F}_{h, \tau}$-measurable. Hence, $(\epsilon_h^\tau(V))_{\tau=1}^K$ is a stochastic process adapted to the filtration $\{\mathcal{F}_{h, \tau}\}_{\tau=0}^K$. Further more, we have

$$E[\epsilon_h^\tau(V) | \mathcal{F}_{h, \tau-1}] = E[R_h^\tau + V(x_h^{\tau+1}) | (x_j^j, a_j^j, x_j^{j+1})_{j=1}^{\tau-1}, (x_h^\tau, a_h^\tau)] - (B_h V)(x_h^\tau, a_h^\tau)$$

$$= E[R_h^\tau + (s_h^\tau) | s_h = x_h^\tau, a_h = a_h^\tau] - (B_h V)(x_h^\tau, a_h^\tau) = 0,$$

where the first step is because $(B_h V)(x_h^\tau, a_h^\tau)$ is $\mathcal{F}_{h, \tau-1}$-measurable and the second step follows from the Markov property of the process. Moreover, as it holds that $R_h^\tau \in (0, n + R_{\text{max}})$ and $V \in [0, (n + R_{\text{max}}) \cdot (H - 1)]$, we have $(B_h V)(x_h^\tau, a_h^\tau) \in [0, (n + R_{\text{max}}) \cdot H]$, which implies $|\epsilon_h^\tau(V)| \leq (n + R_{\text{max}}) \cdot H$. Hence, for the fixed $h \in [H]$ and all $\tau \in [K]$, the random variable $\epsilon_h^\tau(V)$ defined in Equation (34) is mean-zero and $(n + R_{\text{max}})H$-sub-Gaussian conditioning on $\mathcal{F}_{h, \tau-1}$.

We invoke Lemma F.1 with $M_0 = \lambda \cdot I$ and $M_k = \lambda \cdot I + \sum_{\tau=1}^k \phi(x_h^\tau, a_h^\tau) \phi(x_h^\tau, a_h^\tau)^\top$ for all $k \in [K]$. For the fixed function $V : \mathcal{S} \mapsto [0, (n + R_{\text{max}}) \cdot H]$ and fixed $h \in [H]$, we have

$$\Pr \left( \left\| \sum_{\tau=1}^K \phi(x_h^\tau, a_h^\tau) \cdot \epsilon_h^\tau(V) \right\|_{(\Lambda_h^\tau)^{-1}}^2 > (n + R_{\text{max}})^2 H^2 \cdot \log \left( \frac{\det(\Lambda_h^\tau)^{1/2}}{p \cdot \det(\lambda \cdot I)^{1/2}} \right) \right) \leq p$$

for all $p \in (0, 1)$. Here, we use the fact that $\Lambda_h^\tau = M_k$. To upper bound $\det(\Lambda_h^\tau)^{1/2}$, we first notice that

$$\|\Lambda_h^\tau\|_{\text{op}} = \left\| \lambda \cdot I + \sum_{\tau=1}^K \phi(x_h^\tau, a_h^\tau) \phi(x_h^\tau, a_h^\tau)^\top \right\|_{\text{op}} \leq \lambda + \sum_{\tau=1}^K \|\phi(x_h^\tau, a_h^\tau) \phi(x_h^\tau, a_h^\tau)^\top\|_{\text{op}} \leq \lambda + K,$$

where the first inequality follows from the triangle inequality of operator norm and the second
inequality follows from the fact that $\|\phi(x, a)\| \leq 1$ for all $(x, a) \in S \times A$ by our assumption. This implies $\det(\Lambda_h^t) \leq (\lambda + K)^d$. Combining with the fact that $\det(\lambda I) = \lambda^d$ and Equation (63), we have

$$Pr\left(\left\| \sum_{\tau=1}^{K} \phi(x_h^\tau, a_h^\tau) \cdot e_h^\tau(V) \right\|_{(\Lambda_h^t)^{-1}}^2 > (n + R_{\max})^2 H^2 \cdot (2 \cdot \log(1/p) + d \cdot \log(1 + K/\lambda)) \right) \leq p.$$ 

Therefore, we conclude the proof of Lemma F.6.