ESTIMATES OF CONJUGATE HARMONIC FUNCTIONS WITH GIVEN SET OF SINGULARITIES AND APPLICATION

I. CHYZHYKOV\textsuperscript{1,*} and Y. KOSANIAK\textsuperscript{2}

\textsuperscript{1}School of Mathematical Sciences, Guizhou Normal University, Guiyang, Guizhou, 550025, China
e-mail: chyzhykov@yahoo.com

\textsuperscript{2}Faculty of Mechanics and Mathematics, Lviv Ivan Franko National University,
Universytets’ka 1, 79000, Lviv, Ukraine
e-mail: yulia.kosaniak@ukr.net

(Received September 9, 2020; revised December 2, 2020; accepted December 14, 2020)

Abstract. Let $E$ be an arbitrary closed set on the unit circle $\partial \mathbb{D}$ and $u$ be a harmonic function on the unit disk $\mathbb{D}$ satisfying $|u(z)| \lesssim (1 - |z|)^\gamma \rho^{-q}(z)$ where $\rho(z) = \text{dist}(z, E)$, $\gamma, q$ are some real constants, $\gamma \leq q$. We establish an estimate of the conjugate $\tilde{u}$ of the same type which is sharp in some sense, and in the case $E = \partial \mathbb{D}$ coincides with known estimates. As an application we describe growth classes defined by the non-radial condition $|u(z)| \lesssim \rho^{-q}(z)$ in terms of smoothness of the Stieltjes measure associated to the harmonic function $u$.

1. Introduction and main results

1.1. Classes of analytic functions defined by a non-radial growth condition. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\partial \mathbb{D}$ denote its boundary. In the classical theory of analytic and harmonic functions in the unit disk (see, e.g. [14], [27]) dominates the approach where the unit circle is considered as ‘one collective singularity’, and main characteristics of a function, such as the maximum modulus $M(r, f)$, integral means $M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}}$, $0 < p < \infty$, convergence exponent $\inf\{\mu > 0 : \sum_n (1 - |a_n|)^{\mu+1} < \infty\}$ of a sequence $(a_n)$ in $\mathbb{D}$, etc. depend on $r$ or $|a_n|$, or in other words, on the distance $1 - r$ to the unit circle only. In the language of weighted function spaces, it means that a weight $\omega$ is radial, i.e. $\omega(z) = \omega(|z|)$ (see [20], [22], [28]). In the frame of this approach one is

\*Corresponding author.

\textit{Key words and phrases:} conjugate harmonic function, analytic function, unit disk, modulus of continuity.

\textit{Mathematics Subject Classification:} primary 31A05, secondary 30J99, 30E20, 31A10, 31A20.

0133-3852 © 2021 Akadémiai Kiadó, Budapest
not able to use the structure of the set $E \subset \partial \mathbb{D}$ of singular points of an analytic function $f$. However, some problems of the spectral theory naturally lead us to spaces with non-radial weights (see [2], [3], [5], [15]). Some recent achievements in the theory of function spaces with non-radial weights can be found in [2], [3], [4], [23], though the topic is not well-understood in the general case.

It is worth to note that almost one hundred years ago, V. Golubev proposed another approach, published later in his work [18].

Given an arbitrary compact set $E \subset \mathbb{C}$ of singularities of an analytic function $f$, Golubev studied growth, decrease, and zero distribution of $f$ using characteristics in terms of the quantity $\rho(z) = \rho_E(z) = \text{dist}(z, E)$. In particular, he constructed a theory of canonical products, which are now called Golubev’s products. As an example we cite one of his results.

**Theorem A** [18, §3]. If $(a_n)$ is a sequence in $\mathbb{C}$ such that $p = \inf\{\alpha > 0 : \sum_n \rho^\alpha(a_n) < \infty\}$ is finite, then there exists an analytic function $\varphi(z)$ vanishing exactly on $(a_n)$ such that, for an arbitrary $\varepsilon > 0$,

$$\log |\varphi(z)| < \rho^{-p-\varepsilon}(z)$$

in some neighborhood of $E$.

Golubev’s results remained unknown for a long time. Recently there has been a renewed interest in his approach, which was adopted in a series of works by A. Borichev, L. Golinski and S. Kupin [5,6], S. Favorov and L. Golinski [15–17], and others. In particular, in [15] Favorov and Golinski have found optimal Blaschke-type condition for zero set for the class of analytic functions in the unit disk satisfying

$$\log |f(z)| \leq D \rho_E^{-q}(z)$$

where $E = \overline{E} \subset \partial \mathbb{D}$, $D$, $q$ are nonnegative constants. It is noticeable (see [1]) that the notion of a type of a set $E$ characterizing ‘sparseness’ plays an important role. In [5], Borichev, Golinski and Kupin consider classes of analytic functions in $\mathbb{D}$ defined by the distances to two disjoint finite sets on the unit circle. Results of such type effectively apply to the study of complex perturbation of certain self-adjoint operators.

### 1.2. Conjugate harmonic functions.

Let $u$ be a harmonic function in the unit disk $\mathbb{D}$. We denote $M_\infty(r, u) = \sup\{|u(z)| : |z| = r\}$, $0 \leq r < 1$. Estimates of the growth of the harmonic conjugate $\tilde{u}(z)$ have many applications in the function theory (see, e.g. [7], [14], [19], [29]). In particular, for $1 < p < \infty$, by M. Riesz theorem ([29]) $M_p(r, u) \leq \psi\left(\frac{1}{1-r}\right)$ implies $M_p(r, \tilde{u}) = O\left(\psi\left(\frac{1}{1-r}\right)\right)$, where $\psi$ is a positive increasing function on $[1, \infty)$, $r \in [0, 1)$. The cases $p = 1$ and $p = \infty$ are more delicate.

*Analysis Mathematica 47, 2021*
Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a nondecreasing function. Suppose that there exist positive constants $a$ and $c$ such that

$$
\frac{\varphi(x_2)}{x_2^a} \leq c \frac{\varphi(x_1)}{x_1^a}, \quad x_2 > x_1 \geq \frac{1}{2}.
$$

Let

$$
\tilde{\varphi}(x) = \int_{\frac{1}{2}}^{x} \frac{\varphi(t)}{t} \, dt, \quad x \geq \frac{1}{2}.
$$

**Theorem 1.1** [26]. Suppose that $u$ is harmonic in $\mathbb{D}$, $\varphi$ is an increasing function on $\mathbb{R}_+$, and (1.2) holds. Then

1. $M_\infty(r, u) = O(\varphi(\frac{1}{1-r}))$ implies $M_\infty(r, \tilde{u}) = O(\tilde{\varphi}(\frac{1}{1-r}))$, $r \rightarrow 1-$;

2. $M_1(r, u) = O(\varphi(\frac{1}{1-r}))$ implies $M_1(r, \tilde{u}) = O(\tilde{\varphi}(\frac{1}{1-r}))$, $r \rightarrow 1-$.

Theorem 1.1 is sharp, but it does not give us particular information on local growth of $u$ and $\tilde{u}$ in a neighborhood of a particular point on $\partial \mathbb{D}$.

In the sequel, the relation $A \lesssim B$ means that there exists a constant $C$ such that $A \leq CB$, and the relation $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$.

**Example 1.2.** Let $f(z) = \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}}, \ z \in \mathbb{D}$, where the branch of the square root is chosen such that $f(0) = 1$. We have

$$
f(z) = \left(\frac{1 - |z|^2}{|1 - z|^2} + \frac{2i \Im z}{|1 - z|^2}\right)^{\frac{1}{2}} = \left|\frac{1 + z}{1 - z}\right|^{\frac{1}{2}} \left(\cos \frac{\gamma}{2} + i \sin \frac{\gamma}{2}\right)
$$

where $\gamma = \arg \frac{1+z}{1-z} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\tg \gamma = \frac{2 \Im z}{1 - |z|^2}, \ z \in \mathbb{D}$.

It is clear that both $\Re f$ and $\Im f$ are bounded outside a neighborhood of the point 1. Moreover, $\cos \frac{\gamma}{2} \asymp 1$, because $\frac{\gamma}{2} \in (-\frac{\pi}{4}, \frac{\pi}{4})$. Using elementary trigonometry we get for $z \in \mathbb{D}$

$$
\sin^2 \frac{\gamma}{2} = \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + \tg^2 \gamma}}\right)
$$

$$
= \frac{1}{2} \left(1 - \frac{1 - |z|^2}{\sqrt{(1 - |z|^2)^2 + 4(\Im z)^2}}\right) \asymp \frac{|\Im z|^2}{|1 - z|^2}
$$

as $|z| \rightarrow 1-$. Therefore

$$
\Re f(z) \asymp \frac{1}{|1 - z|^{\frac{1}{2}}}, \quad \Im f(z) \asymp \frac{|\Im(1 - z)|}{|1 - z|^\frac{3}{2}}.
$$

We see that conjugate harmonic functions have the same majorant $(\rho_E(z))^{-\frac{1}{2}}, \ E = \{1\}$. This is essentially better than the radial majorant $\left(1 - r\right)^{-\frac{1}{2}}$. 

*Analysis Mathematica 47, 2021*
In view of Theorem 1.1 and Example 1.2 the problem of estimating the harmonic conjugates for harmonic functions from classes of the form

\begin{equation}
|u(z)| \leq \frac{D}{\rho_E^q(z)}, \quad z \in \mathbb{D},
\end{equation}

where \( D \) and \( q \) are some constants, arises naturally.

The following theorem is the main result of this paper.

**Theorem 1.3.** Let \( u \) be a harmonic function in the unit disk and \( \tilde{u} \) be its harmonic conjugate function. Let \( E = \overline{E} \subset \partial \mathbb{D} \), \( q \) and let \( \gamma \) be real constants satisfying \( q > 0, \gamma \leq q \). If, for some \( C_0 > 0 \),

\begin{equation}
|u(z)| \leq \frac{C_0(1 - |z|)^\gamma}{\rho^q(z)},
\end{equation}

then there exists \( C_1 = C_1(C_0, q, \gamma) > 0 \) such that

\begin{equation}
|\tilde{u}(z)| \leq C_1 \begin{cases} 
\frac{(1 - |z|)^\gamma}{\rho^q(z)}, & q > 0 > \gamma; \\
\log \frac{1 - |z|}{\rho^q(z)}, & q > \gamma \geq 0; \\
\log \frac{1}{\rho(z)}, & q = \gamma > 0,
\end{cases}
\end{equation}

for \( \frac{1}{2} \leq |z| < 1 \).

**Remark 1.4.** If \( E = \partial \mathbb{D} \), that is \( \rho(z) = 1 - |z| \), and \( p > 0 \), the hypothesis \( |u(z)| \lesssim (1 - |z|)^{-p} \), \( z \in \mathbb{D} \), by Theorem 1.3 with \( q = p/2, \gamma = -p/2 \) implies \( |\tilde{u}(z)| \lesssim (1 - |z|)^{-p} \). Similarly, \( |u(z)| \lesssim 1 \) implies \( |\tilde{u}(z)| \lesssim \log \frac{1}{1 - |z|} \) which are known to be sharp bounds.

**Remark 1.5.** The restriction \( \gamma \leq q \) is natural, because otherwise \( |u(z)| \lesssim (1 - |z|)^{\eta} \), \( \eta = \gamma - q > 0 \), which yields \( u \equiv 0 \).

**Example 1.6.** Consider the Poisson kernel \( P_0(z) = \frac{1 - |z|^2}{|z|^2} \), which is the real part of the Schwarz kernel. Then \( E = \{1\} \), and the assumptions of the theorem hold with \( q = 2 \) and \( \gamma = 1 \). It is clear that for the conjugate function \( Q(z) = \frac{2\text{Im}(z-1)}{|z|^2} \) we have \( |Q(z)| \asymp \frac{1}{|1-z|^2} \) as \( 1 - z = te^{i\pi} \), \( t \to 0^+ \). It means that the estimate given by (1.6) in the case \( q > \gamma \geq 0 \) is sharp up to the factor \( \log \frac{1}{1-r} \).

**Remark 1.7.** We do not know whether the factor \( \log \frac{1}{1-r} \) in (1.6) is necessary.

### 1.3. An application

We give an application to the growth of Poisson-type integrals. In order to proceed we need the definition of the Riemann–Liouville fractional integral. The fractional integral \( D^{-\alpha}h \) and the fractional

\begin{equation}
\text{Analysis Mathematica 47, 2021}
\end{equation}
derivative \( D^\alpha h \) of order \( \alpha > 0 \) for an integrable function \( h: (0,1) \rightarrow \mathbb{R} \) are defined (see, e.g. [13]) by the following formulas:

\[
D^{-\alpha} h(r) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-x)^{\alpha-1} h(x) \, dx, \quad D^0 h(r) \equiv h(r),
\]

\[
D^\alpha h(r) = \frac{d^p}{d r^p} \{ D^{-(p-\alpha)} h(r) \}, \quad \alpha \in (p-1;p], \ p \in \mathbb{N}.
\]

We write \( u_\alpha(re^{i\varphi}) = r^{-\alpha} D^{-\alpha} u(re^{i\varphi}) \), where the operator \( D^{-\alpha} \) is taken with respect to the variable \( r \). The reason of this definition is that \( u_\alpha \) is harmonic in \( \mathbb{D} \) ([13, Ch. IX]).

Let

\[
S_\alpha(z) = \Gamma(1+\alpha) \left( \frac{2}{(1-z)^{\alpha+1}} - 1 \right), \quad P_\alpha(z) = \text{Re} \, S_\alpha(z).
\]

**Remark 1.8.** Note that \( S_0(z) \) are \( P_0(z) \) the Schwartz kernel and the Poisson kernel, respectively, and \( P_\alpha(re^{i\varphi}) = D^\alpha(r^\alpha P_0(re^{i\varphi})) \).

Our starting point is the following two theorems. The first one is a representation theorem.

**Theorem 1.9** (M. Djrbashian, [13]). Let \( u \) be harmonic in \( \mathbb{D} \), \( \alpha > -1 \). The equality

\[
(1.7) \quad u(re^{i\varphi}) = \int_{\partial \mathbb{D}} P_\alpha(z\bar{\zeta}) \, d\mu(\zeta),
\]

where \( \mu \) is a real finite Borel measure on \( \partial \mathbb{D} \), holds true if and only if

\[
\sup_{0 < r < 1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})| \, d\varphi < +\infty.
\]

The idea of the next result goes back to Hardy and Littlewood [19] and consists in the observation that there is an interplay between the growth of Poisson–Stieltjes integral and the smoothness of the Stieltjes measure. Growth of \( M_\infty(r,u) \) where \( u \) is of the form (1.7) was described in [9], [10].

Let \( \mu \) is a complex finite Borel measure on \( \partial \mathbb{D} \). Let \( U_\delta(\xi) = \{ \zeta \in \partial \mathbb{D} : |\zeta - \xi| < \delta \}, \ \xi \in \partial \mathbb{D} \), be the \( \delta \)-neighborhood of \( \xi \) in \( \partial \mathbb{D} \), \( \delta > 0 \). We define the modulus of continuity of \( \mu \) on a set \( E \subset \partial \mathbb{D} \) by

\[
\omega_E(\delta,\mu) = \sup\{|\mu(l)| : l \text{ is an arc, } l \subset U_\delta(\xi), \ \xi \in E\}
\]

and write \( \omega(\delta,\mu) = \omega_{\partial \mathbb{D}}(\delta,\mu) \) for short.
THEOREM 1.10 [9]. Let $u$ be a harmonic function in $\mathbb{D}$. Let $\alpha \geq 0$, $0 < \gamma < 1$ or $\gamma = 1$ and $\alpha = 0$. Then $u(z)$ has form (1.7) where $\mu$ is a real finite Borel measure on $\partial \mathbb{D}$, and $\omega(\delta, \mu) = O(\delta^\gamma)$, $\delta \to 0+$, if and only if

$$M_\infty(r, u) = O((1 - r)^{\gamma - \alpha - 1}), \quad r \to 1-$$

and

$$\sup_{0 < r < 1} \int_0^{2\pi} |u_\alpha(re^{i\phi})| d\phi < +\infty.$$ 

Asymptotic behavior of $M_p(r, u)$ where $1 < p < \infty$ is described in [8] (see also [12]). The disadvantage of the mentioned results is that they do not describe local growth of a harmonic function.

THEOREM 1.11. Let $0 < \gamma < 1$, $\alpha \geq 0$, and $u(z)$ have the form (1.7) where $\mu$ is a complex finite Borel measure on $\partial \mathbb{D}$, supp $\mu = E \subset \partial \mathbb{D}$.

(i) If $\omega_E(\delta, \mu) = O(\delta^\gamma)$, $\delta \to 0+$, then

$$|u(z)| \lesssim \frac{1}{\rho^{\alpha+1-\gamma}(z)}, \quad z \in \mathbb{D};$$

(ii) If (1.8) holds, then

$$\omega_E(\delta, \mu) = O\left(\delta^\gamma \log \frac{1}{\delta}\right), \quad \delta \to 0+.$$ 

We say that a complex finite Borel measure $\mu$ on $\partial \mathbb{D}$ belongs to the class $\Lambda_\gamma(E)$, $\gamma \in [0, 1]$, if for arbitrary $\varepsilon > 0$ we have $\omega_E(\delta, \mu) = O(\delta^{\gamma-\varepsilon})$, $\delta \to 0+$. In particular, an arbitrary complex finite Borel measure on $\partial \mathbb{D}$ belongs to $\Lambda_0$, and measures absolutely continuous with respect to the Lebesgue measure belong to $\Lambda_1$.

COROLLARY 1.12. Under assumptions Theorem 1.11 $\mu \in \Lambda_\gamma(E)$ if and only if for all $q > \alpha + 1 - \gamma$ (1.4) holds.

2. Some lemmas

Given an arc $l$ on the unit circle, a point $w \in \mathbb{D}$ and a positive constant $\lambda$ we define the measure $\nu_w^\lambda(l)$ by

$$\nu_w^\lambda(l) := \int_{l} \frac{|d\zeta|}{|\zeta - w|^\lambda}.$$ 

REMARK 2.1. For $\lambda = 2$ the quantity $\nu_w^\lambda(l)(1 - |w|^2)$ equals the harmonic measure of $l$. The measure $\nu_w^\lambda$ can be continued on all Borel subsets of $\partial \mathbb{D}$ in the standard way.
The following lemma plays an important role in our arguments.

**Lemma 2.2.** Let $E$ be a finite union of closed (open) arcs which are pair-wisely disjoint, $\lambda > 0$, and $w \in \mathbb{D}$. Then there exists a constant $C = C(\lambda)$ such that

$$
(2.1) \quad \nu_w^\lambda(E) \leq C \begin{cases}
(\rho(w))^{1-\lambda} - (\rho(w) + \frac{|E|}{2})^{1-\lambda}, & \lambda > 1; \\
\log(1 + \frac{|E|}{\rho(w)}), & \lambda = 1; \\
(\rho(w) + |E|/2)^{1-\lambda} - (\rho(w))^{1-\lambda}, & \lambda < 1
\end{cases}
$$

holds for all $w \in \mathbb{D}$.

**Proof of Lemma 2.2.** Let $l_k, k \in \{1, \ldots, N\}$, denote the arcs that form the set $E$, i.e. $E = \bigcup_{k=1}^N l_k$. We may assume that $e^{i\alpha_k}$ and $e^{i\beta_k}$ are the endpoints of $l_k$, and $\alpha_k \leq \beta_k \leq \alpha_{k+1}, k \in \{1, \ldots, N-1\}, \beta_N \leq \alpha_1 + 2\pi$. We write $\zeta = e^{it}, w = se^{i\theta}, a_k = e^{i\alpha_k}, b_k = e^{i\beta_k}, k \in \{1, \ldots, N\}$. Let $\rho(w) = \min_{\zeta \in E} |\zeta - w| = |\zeta_0 - w|$, and $\zeta_0 = e^{i\theta_0}$. Then

$$
(2.2) \quad \nu_w^\lambda(E) = \sum_{k=1}^N \int_{\alpha_k}^{\beta_k} dt = \sum_{k=1}^N \int_{\alpha_k}^{\beta_k} dt,
$$

where $\tilde{\alpha}_k = \alpha_k - \theta, \tilde{\beta}_k = \beta_k - \theta$. Dividing some segments $[\tilde{\alpha}_j, \tilde{\beta}_j]$ onto two segments, renumerating and shifting them on a multiple value of $2\pi$ we may achieve that

$$
\bigcup_{k=1}^{N_0} [\tilde{\alpha}_k, \tilde{\beta}_k] \subset [0, \pi], \quad \bigcup_{k=N_0+1}^N [\tilde{\alpha}_k, \tilde{\beta}_k] \subset [\pi, 2\pi]
$$

for some integer number $N_0$, $0 \leq N_0 < N, \tilde{\beta}_k \leq \tilde{\alpha}_{k+1}, k \in \{1, \ldots, N-1\}$.

Since $\frac{1}{|e^{it} - s|^\lambda}$ is decreasing as a function of $t$ on $[0, \pi]$ for $s \in (0, 1)$, we have

$$
(2.3) \quad \sum_{k=1}^{N_0} \int_{\tilde{\alpha}_k}^{\tilde{\beta}_k} \frac{dt}{|e^{it} - s|^\lambda} \leq \int_{\tilde{\alpha}_1}^{\tilde{\alpha}_1 + \sum_{k=1}^{N_0} (\tilde{\beta}_k - \tilde{\alpha}_k)} \frac{dt}{|e^{it} - s|^\lambda}.
$$

Similarly, we obtain

$$
(2.4) \quad \sum_{k=N_0+1}^N \int_{\tilde{\alpha}_k}^{\tilde{\beta}_k} \frac{dt}{|e^{it} - s|^\lambda} \leq \int_{\tilde{\beta}_N - \sum_{k=N_0+1}^N (\tilde{\beta}_k - \tilde{\alpha}_k)}^{\tilde{\beta}_N} \frac{dt}{|e^{it} - s|^\lambda}.
$$
Due to our hypothesis \( \theta_0 - \theta = \tilde{\alpha}_1 \) (mod \( 2\pi \)) or \( \theta_0 - \theta = \tilde{\beta}_N \) (mod \( 2\pi \)). We may assume that the first equality holds. Then, taking into account that \( |e^{it} - s| = |e^{i(2\pi - t)} - s| \), we obtain from (2.2)–(2.4)

\[
(2.5) \quad \nu_w^\lambda(E) \leq \left( \int_{\tilde{\alpha}_1}^{\tilde{\alpha}_0 + \frac{1}{2} \sum_{k=1}^N (\tilde{\beta}_k - \tilde{\alpha}_k)} \int_{\beta_N - \sum_{k=N_0 + 1}^N (\tilde{\beta}_k - \tilde{\alpha}_k)} dt \right) \frac{dt}{|e^{it} - s|^\lambda}
\]

We denote \( \delta = \frac{1}{2} \sum_{k=1}^N (\tilde{\beta}_k - \tilde{\alpha}_k) \leq \frac{1}{2} |E| \). Using the elementary inequality \( x^2 + y^2 \leq (x + y)^2 \leq 2(x^2 + y^2) \), \( x, y \geq 0 \), and standard estimates we deduce for \( \frac{1}{2} \leq s < 1 \) that

\[
(2.6) \quad \int_{\tilde{\alpha}_0}^{\tilde{\alpha}_0 + \delta} \frac{dt}{|e^{it} - s|^\lambda} = \int_{\tilde{\alpha}_0}^{\tilde{\alpha}_0 + \delta} \frac{dt}{(4s \sin^2 t/2 + (1 - s)^2)^{\frac{\lambda}{2}}}
\]

The latter integral can be computed explicitly depending on \( \lambda \). We start with the case \( \lambda > 1 \). Then (2.5) and (2.6) imply

\[
(2.7) \quad \nu_w^\lambda(E) \leq 2 \int_{\tilde{\alpha}_0}^{\tilde{\alpha}_0 + \delta} \frac{dt}{|e^{it} - s|^\lambda}
\]

\[
\leq C(\lambda) \left( \frac{1}{(\tilde{\alpha}_0 + 1 - s)^{\lambda-1}} - \frac{1}{(\tilde{\alpha}_0 + \delta + 1 - s)^{\lambda-1}} \right)
\]

\[
= \frac{C(\lambda)}{(\tilde{\alpha}_0 + 1 - s)^{\lambda-1}} \left( 1 - \frac{1}{(1 + \frac{\delta}{\tilde{\alpha}_0 + 1 - s})^{\lambda-1}} \right)
\]

\[
\leq \frac{C(\lambda)}{\rho(w)^{\lambda-1}} \left( 1 - \frac{1}{(1 + \frac{|E|}{2\rho(w)})^{\lambda-1}} \right) = \frac{C(\lambda)}{\rho(w)^{\lambda-1}} \left( 1 - \frac{C(\lambda)}{\rho(w) + \frac{|E|}{2})^{\lambda-1}} \right)
\]

as required.

Let \( \lambda \in (0, 1) \). Similarly, we deduce

\[
(2.8) \quad \nu_w^\lambda(E) \leq C(\lambda) \left( (\tilde{\alpha}_0 + \delta + 1 - s)^{1-\lambda} - (\tilde{\alpha}_0 + 1 - s)^{1-\lambda} \right)
\]

\[
= C(\lambda) (\tilde{\alpha}_0 + 1 - s)^{1-\lambda} \left( 1 + \frac{\delta}{\tilde{\alpha}_0 + 1 - s} \right)^{1-\lambda} - 1
\]

Analysis Mathematica 47, 2021
\[ C(\lambda)\rho(w)^{1-\lambda}\left(1 + \frac{|E|}{2\rho(w)}\right)^{1-\lambda} - 1 \]
\[ = C(\lambda)((\rho(w) + |E|/2)^{1-\lambda} - (\rho(w))^{1-\lambda}) \]

Finally, for \( \lambda = 1 \) we get
\[ \nu_{w}^{1}(E) \lesssim \log\left(\frac{\tilde{\alpha}_{0} + \delta + 1 - s}{\tilde{\alpha}_{0} + 1 - s}\right) \lesssim \log\left(1 + \frac{|E|}{2\rho(w)}\right). \]

The assertion of the lemma follows from (2.7), (2.8) and the latter estimate. \( \square \)

### 3. Proofs of the theorems

**Proof of Theorem 1.3.** We start with the case \( \gamma < q \). We write \( F(z) := u(z) + i\tilde{u}(z) \), so \( F \) is analytic in \( \mathbb{D} \). Then

\[
F(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{Re^{i\theta} + z}{Re^{i\theta} - z} u(Re^{i\theta}) \, d\theta + i \text{Im} F(0)
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 + \frac{z}{R}e^{-i\theta}}{1 - \frac{z}{R}e^{-i\theta}} u(Re^{i\theta}) \, d\theta + i \text{Im} F(0).
\]

We are going to apply the Djrbashian operator \( D^{\alpha}(r^{\alpha}F(re^{i\varphi})) \) where \( D^{\alpha} \) is the Riemann–Liouville operator of order \( \alpha > 0 \). It is known ([13, p. 577]) that \( D^{\alpha}(r^{\alpha}S_{0}(re^{i\varphi})) = S_{\alpha}(re^{i\varphi}) \), where \( S_{\alpha}(z) = \Gamma(1 + \alpha)(\frac{2}{(1-z)e^{i\gamma} - 1}) \). The same arguments show that \( D^{\alpha}(r^{\alpha}S_{0}(\frac{r}{R}e^{i\varphi})) = S_{\alpha}(\frac{r}{R}e^{i\varphi}) \).

Taking in account the equality \( D^{\alpha}(r^{\alpha}) = \Gamma(1 + \alpha), \alpha > 0 \), we get

\[
F_{\alpha}(re^{i\varphi}) := D^{\alpha}(r^{\alpha}F(re^{i\varphi}))
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} S_{\alpha}\left(\frac{z}{R}e^{-i\theta}\right) u(Re^{i\theta}) \, d\theta + i \text{Im} F(0)\Gamma(1 + \alpha).
\]

Then

\[
|F_{\alpha}(z)| \leq C_{0}\Gamma(\alpha + 1) \frac{1}{2\pi} \int_{0}^{2\pi} \left(\frac{2}{1 - \frac{z}{R}e^{-i\theta}}\right)^{\gamma} \frac{1}{\rho^{\eta}(Re^{i\theta})} \, d\theta + |\text{Im} F(0)|\Gamma(\alpha + 1) \leq C(1 - R)^{\gamma} \int_{0}^{2\pi} \frac{d\theta}{|1 - \frac{z}{R}e^{-i\theta}|^{\alpha+1}\rho^{\eta}(Re^{i\theta})} + C.
\]

We write

\[
\nu_{w}^{\lambda}(\theta) := \nu_{w}^{\lambda}(\{e^{it} : t \in [0, \theta]\}) = \int_{0}^{\theta} \frac{dt}{|e^{it} - w|^{\lambda}}.
\]
Due to the ‘layer cake representation’ ([21, Theorem 1.13]) we have

\[
\int_0^{2\pi} \frac{d\theta}{|1 - \frac{z}{R}e^{-i\theta}|^{\alpha+1} \rho^{q}(Re^{i\theta})} = \int_0^{2\pi} \frac{dv_{z/R}^{\alpha+1}(\theta)}{\rho^{q}(Re^{i\theta})} = q \int_0^{\infty} y^{q-1} \nu_{z/R}^{\alpha+1}\left(\left\{e^{i\theta} : \frac{1}{\rho(Re^{i\theta})} > y\right\}\right) dy = q \int_0^{\frac{1}{2}} y^{q-1} \nu_{z/R}^{\alpha+1}\left(\left\{e^{i\theta} : \rho(Re^{i\theta}) < \frac{1}{y}\right\}\right) dy.
\]

Note that \(\{e^{i\theta} : \rho(Re^{i\theta}) < t\}\) is a finite union of open arcs when \(t > 1 - R\) (c.f. [15, p. 39]). Applying Lemma 2.2 we obtain

\[
(3.3) \quad \int_0^{2\pi} \frac{d\theta}{|1 - \frac{z}{R}e^{-i\theta}|^{\alpha+1} \rho^{q}(Re^{i\theta})} \leq C(q, \alpha) \int_0^{\frac{1}{2}} y^{q-1} \rho^{\alpha}(z/R) dy 
\]

\[
\leq \frac{C}{(1 - R)^q \rho^{\alpha}(z/R)}, \quad \frac{R - 1}{2} \leq |z| < R.
\]

Substituting this estimate into (3.2) we get

\[
|F_{\alpha}(z)| \leq \frac{C}{(1 - R)^q \gamma \rho^{\alpha}(z/R)}, \quad \frac{R - 1}{2} \leq |z| < R.
\]

We choose \(R = (1 + r)/2, |z| = r\), and suppose that \(r \in [1/2, 1)\). Then we have

\[
(3.4) \quad |F(re^{i\varphi})| = \left|r^{-\alpha} D^{-\alpha} F_{\alpha}(re^{i\varphi})\right| = \frac{r^{-\alpha}}{\Gamma(\alpha)} \left|\int_0^r (r - t)^{\alpha-1} F_{\alpha}(te^{i\varphi}) dt\right|
\]

\[
\leq \int_0^r \frac{(r - t)^{\alpha-1} dt}{\left(\frac{1}{2} - \frac{1}{2}\right)^{q-\gamma} \rho^{\alpha}(\frac{2e^{i\varphi}}{1+r})} \leq \frac{1}{\rho^{\alpha}(\frac{2e^{i\varphi}}{1+r})} \int_0^r \frac{(r - t)^{\alpha-1} dt}{(1 - t)^{q-\gamma}},
\]

where we used the estimate \(\rho(z) \leq 2\rho(\tau z), \tau \in (0, 1), z \in \mathbb{D}\) ([15, p. 41]).

We now distinguish two subcases. First, let \(q > \gamma \geq 0\). We then choose \(\alpha = q - \gamma > 0\). It follows from (3.4) that

\[
|F(re^{i\varphi})| \lesssim \frac{1}{\rho^{q-\gamma}(\frac{2e^{i\varphi}}{1+r})} \int_0^r \frac{(r - t)^{q-\gamma-1} dt}{(1 - t)^{q-\gamma}} \lesssim \frac{\log \frac{1-\tau}{1-r}}{\rho^{q-\gamma}(\frac{2e^{i\varphi}}{1+r})}.
\]

Next, let \(q > 0 > \gamma\). Then we take \(\alpha = q\). Estimate (3.4) yields

\[
|F(re^{i\varphi})| \lesssim \frac{1}{\rho^{\alpha}(\frac{2e^{i\varphi}}{1+r})} \int_0^r \frac{(r - t)^{q-1} dt}{(1 - t)^{q-\gamma}} \lesssim \frac{(1 - r)^{\gamma}}{\rho^{\alpha}(\frac{2e^{i\varphi}}{1+r})}.
\]

Analysis Mathematica 47, 2021
Finally, we consider the case $q = \gamma > 0$. Now we estimate $F(z)$ using (3.1) as follows:

$$|F(z)| \lesssim \int_0^{2\pi} \left( \frac{1}{|1 - \frac{z}{R}e^{-i\theta}|} + 1 \right) |u(Re^{i\theta})| d\theta + 1$$

$$\lesssim \int_0^{2\pi} \left( \frac{1}{|1 - \frac{z}{R}e^{-i\theta}|} \frac{(1 - R)^q}{\rho^q(Re^{i\theta})} \right) d\theta + 1 \lesssim (1 - R)^q \int_0^{2\pi} \frac{d\nu_{z/R}(\theta)}{\rho^q(Re^{i\theta})} + 1$$

$$= (1 - R)^q \int_0^{\infty} y^{q-1} \nu_{z/R}(\{ e^{i\theta} : 1 - \rho(Re^{i\theta}) > y \}) dy + 1$$

$$\lesssim (1 - R)^q \int_0^{1 - \frac{\pi}{2}} y^{q-1} \log \left( 1 + \frac{2\pi}{\rho(z/R)} \right) dy + 1 \lesssim \log \frac{2\pi + 2}{\rho(z/R)}, \quad R/2 \leq |z| < R.$$
Hence, using (3.5), we obtain as $r \to 1-$

\begin{equation}
(3.6) \quad |u(re^{i\varphi})| \leq O(1) + \int_{[-\pi, \pi] \cap \{\theta : e^{i\theta} \in E\}} \frac{\mu(\theta)}{|re^{i\varphi} - e^{it}|^{2+\alpha}} d\theta
\end{equation}

\begin{align*}
\lesssim 1 + \int_{|\varphi - \theta| \leq \rho(re^{i\varphi})} \frac{\omega_E(|\theta - \varphi|, \rho(re^{i\varphi}))}{\rho(re^{i\varphi})^{2+\alpha}} d\theta + \int_{\rho(re^{i\varphi}) < |\theta - \varphi| \leq \pi} \frac{\omega_E(|\theta - \varphi|, \mu)}{|\theta - \varphi|^{2+\alpha}} d\theta
\end{align*}

\begin{align*}
\lesssim 1 + \frac{1}{\rho(re^{i\varphi})^{2+\alpha}} \int_0^{\frac{\pi}{\rho(re^{i\varphi})}} \tau^{-\gamma} d\tau + \frac{1}{\rho(re^{i\varphi})} \int_{\rho(re^{i\varphi})}^{\pi} \frac{1}{\tau^{2+\alpha-\gamma}} d\tau \lesssim (\rho(re^{i\varphi}))^{-\alpha + \gamma - 1}.
\end{align*}

(ii) Our arguments are similar to that from [9]. We start with the case $\alpha = 0$.

**Remark 3.1.** By Nevanlinna’s Theorem, at any point $e^{i\theta} \in \partial \mathbb{D}$, $0 \leq \theta \leq 2\pi$, such that $\mu(\{e^{i\theta}\}) = 0$, for some sequence $(r_n)$ (see [13], [24, p. 57])

\begin{equation}
(3.7) \quad \mu_0(\theta) = \lim_{r_n \uparrow 1} \int_0^\theta u(r_ne^{i\phi}) d\phi.
\end{equation}

Let $F(z) = u(z) + iv(z)$ be an analytic function in $\mathbb{D}$. By Theorem 1.3, $|v(re^{i\varphi})| = O((\rho(re^{i\varphi}))^{\gamma-1} \log \frac{1}{1-r})$ as $r \to 1-$.

Define the analytic function $\Phi(z) = \int_0^z F(\zeta) d\zeta$, $z \in \mathbb{D}$. For any fixed $\varphi \in [0, 2\pi]$ and $0 < r' < r'' < 1$ we have

\[ |\Phi(r''e^{i\varphi}) - \Phi(r'e^{i\varphi})| \lesssim \int_{r'}^{r''} (1-t)^{\gamma-1} \log \frac{1}{1-t} dt \leq (1-r')^{\gamma} \log \frac{1}{1-r'} \]

Thus, there exists $\lim_{r \uparrow 1} \Phi(re^{i\varphi}) \equiv \Phi(e^{i\varphi})$ uniformly in $\varphi$. Therefore, $\Phi(\varphi) \overset{\text{def}}{=} \Phi(e^{i\varphi})$ is continuous on $[0, 2\pi]$.

First, we suppose that $h \in (0, 1)$,

\[ z_0 = e^{i\varphi}, \quad z_1 = (1-h)e^{i\varphi}, \quad z_2 = (1-h)e^{i(\varphi + h)}, \quad z_3 = e^{i(\varphi + h)}, \]

and $e^{i\varphi} \in E$. Then by the Cauchy integral theorem we have

\[ \Phi(z_3) - \Phi(z_0) = \left( \int_{[z_0, z_1]} + \int_{z_1}^{z_2} + \int_{[z_2, z_3]} \right) F(z) dz. \]

If $h > 0$ sufficiently small, we obtain

\begin{equation}
(3.8) \quad \left| \int_{[z_0, z_1]} F(z) \right| \lesssim \int_{1-h}^1 \frac{\log \frac{1}{1-r}}{(\rho(re^{i\varphi}))^{1-\gamma}} dr
\end{equation}
\[
\leq \int_{1-h}^{1} \frac{\log \frac{1}{1-r}}{(1-r)^{1-\gamma}} \, dr \lesssim h^\gamma \log \frac{1}{h}, \quad h \to 0^+.
\]

Analogously, \( \left| \int_{z_2}^{z_3} F(z) \, dz \right| \lesssim h^\gamma \log \frac{1}{h} \). It is clear that
\[
\left| \int_{z_1}^{z_2} F(z) \, dz \right| \lesssim \log \frac{1}{h}, \quad h \to 0^+.
\]

If \( h < 0 \), similar estimates hold. Therefore,
\[
\left| \Phi(e^{i(\varphi+h)}) - \Phi(e^{i\varphi}) \right| \lesssim h^\gamma \log \frac{1}{h}, \quad h \to 0^+.
\]

For \( R \in (0, 1) \) we write
\[
\lambda_R(\theta) = \int_{0}^{\theta} F(Re^{i\sigma}) \, d\sigma = \int_{0}^{\theta} \frac{d\Phi(Re^{i\sigma})}{iRe^{i\sigma}}
\]
\[
= \frac{\Phi(Re^{i\theta})}{iRe^{i\theta}} - \frac{\Phi(R)}{iR} + \frac{1}{R} \int_{0}^{\theta} \Phi(Re^{i\sigma})e^{-i\sigma} \, d\sigma.
\]

Since \( \Phi(z) \) is continuous on \( \overline{D} \), we have \( \Phi(Re^{i\sigma}) \xrightarrow{\theta} \Phi(e^{i\sigma}) \) as \( R \to 1^- \). Therefore,
\[
\lambda_R(\theta) \Rightarrow -\Phi(e^{i\theta})ie^{-i\theta} + i\Phi(1) + \int_{0}^{\theta} \Phi(e^{i\sigma})e^{-i\sigma} \, d\sigma \equiv \lambda(\theta), \quad R \to 1^-,
\]
where \( \lambda \in C[0, 2\pi] \). Combining the latter relationship with (3.9), we deduce
\[
(3.10) \quad \left| \lambda(e^{i(\varphi+h)}) - \lambda(e^{i\varphi}) \right| \lesssim h^\gamma \log \frac{1}{h}, \quad h \to 0^+.
\]

On the other hand, \( u(Re^{i\varphi}) = \int_{\partial D} P_0(Re^{i\varphi}\zeta)d\mu(\zeta) \), where, by (3.7) and the definition of \( \lambda \),
\[
\mu_0(\theta) = \lim_{r \to 1^-} \int_{0}^{\theta} \Re F(Re^{i\phi}) \, d\phi = \lambda(\theta).
\]

Thus, \( \mu \) satisfies required smoothness properties in the case \( \alpha = 0 \).

Let now \( \alpha > 0 \). We need the following lemma.

**Lemma 3.2** [11, Lemma 14]. Let \( 0 \leq \gamma < \alpha < \infty \). Then there exists a constant \( C(\gamma, \alpha) > 0 \) such that
\[
(3.11) \quad D^{-\gamma} \frac{1}{|1-r\zeta|^\alpha} \leq \frac{C(\gamma, \alpha)}{|1-r\zeta|^{\alpha-\gamma}}, \quad \zeta \in \overline{D}, \; 0 < r < 1.
\]
Applying this lemma, we obtain that \( u_\alpha(z) = \int_{\partial D} P_0(z\zeta) d\mu(\zeta) \) satisfies
\[
|u_\alpha(z)| \leq C \rho^{1-\gamma}(z), \quad z \in \mathbb{D}.
\]

Applying the proved assertion of (ii) for \( \alpha = 0 \) completes the proof. \( \square \)

References

[1] P. Ahern and D. Clark, On inner functions with \( B^p \) derivative, *Michigan Math. J.*, **23** (1976), 107–118.

[2] A. Aleman and O. Constantin, Spectra of integration operators on weighted Bergman spaces, *J. Anal. Math.*, **109** (2009), 199–231.

[3] A. Aleman, S. Pott and M. C. Reguera, Characterizations of a limiting class \( B_{\infty} \) of Békollé–Bonami weights, *Rev. Mat. Iberoam.*, **35** (2019), 1677–1692.

[4] G. Bao, H. Wulan and K. Zhu, A Hardy–Littlewood theorem for Bergman spaces, *Ann. Acad. Sci. Fenn.*, **43** (2018), 807–821.

[5] S. Borichev, L. Golinskii and S. Kupin, A Blaschke-type condition and its application to complex Jacobi matrices, *Bull. Lond. Math. Soc.*, **41** (2009), 117–123.

[6] S. Borichev, L. Golinskii and S. Kupin, On zeros of analytic functions satisfying non-radial growth conditions, *Rev. Mat. Iberoam.*, **34** (2018), 1153–1176.

[7] M. L. Cartwright, On analytic functions regular in the unit circle. I, *Quart. J. Math.*, **4** (1933), 246–257.

[8] I. Chyzhykov, A generalization of Hardy–Littlewood’s theorem, *Mat. Metodi Fiz.-Mekh. Polya*, **49** (2006), 74–79 (in Ukrainian).

[9] I. Chyzhykov, Growth and representation of analytic and harmonic functions in the unit disk, *Ukr. Math. Bull.*, **3** (2006), 31–44.

[10] I. Chyzhykov, On a complete description of the class of functions without zeros analytic in a disk and having given orders, *Ukrainian Math. J.*, **59** (2007), 1088–1109.

[11] I. Chyzhykov, Argument of bounded analytic functions and Frostman’s type conditions, *Illinois J. Math.*, **53** (2009), 515–531.

[12] I. Chyzhykov, Asymptotic behaviour of \( p \)th means of analytic and subharmonic functions in the unit disc and angular distribution of zeros, *Israel J. Math.*, **236** (2020), 931–957.

[13] M. Djrbashian, *Integral Transforms and Representations of Functions in the Complex Domain*, Nauka (Moscow, 1966) (in Russian).

[14] P. L. Duren, *Theory of \( H^p \) Spaces*, Academic Press (New York, London, 1970).

[15] S. Favorov and L. Golinskii, A Blaschke-type condition for analytic and subharmonic functions and application to contraction operators, in: *Linear and Complex Analysis*, Amer. Math. Soc. Transl. Ser. 2, 226, Adv. Math. Sci., 63, Amer. Math. Soc. (Providence, RI, 2009), pp. 37–47.

[16] S. Favorov and L. Golinskii, Blaschke-type conditions for analytic and subharmonic functions in the unit disk: local analogs and inverse problems, *Comput. Methods Funct. Theory*, **12** (2012), 151–166.

[17] S. Favorov and L. Golinskii, On a Blaschke-type condition for subharmonic functions with two sets of singularities on the boundary, in: *Complex Function Theory, Operator Theory, Schur Analysis and Systems Theory*, D. Alpay, B. Fritzsche, B. Kirstein, Eds., Springer Internat. Publishing (2020).

[18] V. V. Golubev, Study on the theory of singular points of single-valued analytic functions, in *Single Valued Analytic Functions, Automorphic Functions*, Izdat. Fiz. Mat. Lit. (Moscow, 1961), pp. 197–370 (in Russian).
[19] G. H. Hardy and J. Littlewood, Some properties of fractional integrals. II, *Math. Z.*, **34** (1931/32), 403–439.

[20] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces*, Graduate Texts in Mathematics, vol. 199, Springer (New York, Berlin, 2000).

[21] E. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, vol. 14, Amer. Math. Soc. (Providence, RI, 1997).

[22] J. Peláez and J. Rättyä, Weighted Bergman spaces induced by rapidly increasing weights, *Mem. Amer. Math. Soc.*, **227** (2014).

[23] J. Peláez and J. Rättyä, Weighted norm inequalities for derivatives on Bergman spaces, preprint.

[24] I. I. Privalov, *Boundary Values of Single Valued Functions*, MGU (Moscow, 1941).

[25] M. M. Sheremeta, On the asymptotic behaviour of Cauchy–Stieltjes integrals, *Mat. Stud.*, **7** (1997), 175–178.

[26] A. L. Shields and D. L. Williams, Bounded projections and the growth of harmonic conjugates in the unit disc, *Michigan Math. J.*, **29** (1982), 3–25.

[27] M. Tsuji, *Potential Theory in Modern Function Theory*, reprinting of the 1959 original, Chelsea Publishing Co. (New York, 1975).

[28] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics, vol. 199, Springer (New York, Berlin, 2005).

[29] A. Zygmund, *Trigonometric series*, vol. 1, Cambridge Univ. Press (New York, 1959).