MORSE-NOVIKOV COHOMOLOGY OF LOCALLY
CONFORMALLY KÄHLER SURFACES

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ABSTRACT. We review the properties of the Morse-Novikov cohomology and compute it for all known compact complex surfaces with locally conformally Kähler metrics. We present explicit computations for the Inoue surfaces $S^0$, $S^+$, $S^-$ and classify the locally conformally Kähler (and the tamed locally conformally symplectic) forms on $S^0$. We prove the non-existence of LCK metrics with potential on these Inoue surfaces.

Keywords: Morse-Novikov cohomology, tamed, locally conformally symplectic, locally conformally Kähler, Lee form, Inoue surfaces, Kato surfaces, solvmanifold, mapping torus.

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1. Introduction

The Morse-Novikov cohomology of a manifold $M$ refers to the cohomology of the complex of smooth real forms $\Omega^\bullet(M)$, with the differential operator perturbed with a closed one-form $\eta$, defined as follows
\begin{equation}
\text{d}_\eta := \text{d} - \eta \wedge \cdot
\end{equation}
Indeed, the closedness of $\eta$ implies $d_\eta^2 = 0$, whence $d_\eta$ produces a cohomology, which we denote by $H_\eta^\bullet(M)$.

Throughout this paper, we shall use the name Morse-Novikov for the cohomology $H_\eta^\bullet(M)$, although the name Lichnerowicz cohomology is also used in the literature (see [BK], [HR]). Its study began with Novikov ([Nov1], [Nov2]) and was independently developed by Guedira and Lichnerowicz ([GL]).

The Morse-Novikov cohomology has more than one description. To begin with, consider the following exact sequence of sheafs:
\begin{equation}
0 \rightarrow \text{Ker} \text{d}_\eta \rightarrow \Omega^0_M(\cdot) \xrightarrow{\text{d}_\eta} \Omega^1_M(\cdot) \xrightarrow{\text{d}_\eta} \Omega^2_M(\cdot) \xrightarrow{\text{d}_\eta} \cdots
\end{equation}
where we denote by $\Omega^k_M(\cdot)$ the sheaf of smooth real $k$-forms on $M$. In fact, the sequence above is an acyclic resolution for $\text{Ker} \text{d}_\eta$, as each $\Omega^k_M(\cdot)$ is soft, [D, Proposition 2.1.6 and Theorem 2.1.9]. Thus, by taking global sections in (1.2), we compute the cohomology groups of $M$ with values in the sheaf $\text{Ker} \text{d}_\eta$, $H^i(M, \text{Ker} \text{d}_\eta)$. What we obtain is actually the Morse-Novikov cohomology.

The sheaf $\text{Ker} \text{d}_\eta$ has the property that there exists a covering $\{U_i\}_i$ of $M$, such that it is constant when restricted to each $U_i$. In order to see this, one simply takes a contractible covering $\{U_i\}_i$ for which $\eta = df_i|_{U_i}$, then by considering the map $g \mapsto e^{-f_i}g$, one gets an isomorphism $\text{Ker} \text{d}_\eta|_{U_i} \simeq \mathbb{R}$.

Moreover, the covering $\{U_i\}_i$ and the isomorphisms above associate to $\text{Ker} \text{d}_\eta$ a line bundle $L_\eta$, which is trivial on this covering and whose transition maps are $g_{ij} = e^{f_i-f_j}$. It is immediate that $(U_i, e^{-f_i})$ defines a global nowhere vanishing section $s$ of $L_\eta^*$, which is the dual of $L_\eta$ and by means of $s$, $L_\eta^*$ is isomorphic to the trivial bundle. We define a flat connection $\nabla$ on $L_\eta^*$ by $\nabla s = -\eta \otimes s$. Then $H^i_\eta(M)$ can also be computed as the cohomology of the following complex of forms with values in $L_\eta^*$:
\begin{equation}
0 \rightarrow \Omega^0(M, L_\eta^*) \xrightarrow{\nabla} \Omega^1(M, L_\eta^*) \xrightarrow{\nabla} \Omega^2(M, L_\eta^*) \xrightarrow{\nabla} \cdots
\end{equation}

Remark 1.1: Unlike de Rham cohomology, Morse-Novikov cohomology $H^i_\eta$, is not a topological invariant, it depends on $[\eta] \in H^1_{dR}$. Also, Riemannian properties involving this one-form can be important. For instance, it was shown in [LLMP] that if on a compact manifold $M$ there exists a Riemannian
metric $g$ and a closed one-form $\eta$ such that $\eta$ is parallel with respect to $g$, then for any $i \geq 0$, $H^i_\eta(M) = 0$.

Some properties verified by the Morse-Novikov cohomology, important for this paper, are summarized in the following:

**Proposition 1.2:** Let $M$ be an $n$-dimensional manifold and $\eta$ a closed one-form. Then:

1. if $\eta' = \eta + df$, for any $i \geq 0$, $H^i_{\eta'}(M) \cong H^i_{\eta}(M)$ and the isomorphism is given by the map $[\alpha] \mapsto [e^{-f}\alpha]$.
2. (HR, GL) if $\eta$ is not exact and $M$ is connected and orientable, $H^0_{\eta}(M)$ and $H^n_{\eta}(M)$ vanish.
3. (BK) the Euler characteristic of the Morse-Novikov cohomology coincides with the Euler characteristic of the manifold, as a consequence of the Atiyah-Singer index theorem, which implies that the index of the elliptic complex $(\Omega^k(M), d_{\eta})$ is independent of $\eta$.

Motivated by the natural setting that locally conformally symplectic and locally conformally Kähler manifolds provide for the Morse-Novikov cohomology, the aim of this paper is to present some explicit examples and computations on the Inoue surfaces $S^0$, $S^+$ and $S^-$. Moreover, regarding the recent results in [AD], we also draw some consequences involving the locally conformally Kähler metrics or more general, tamed locally conformally symplectic forms the on the surface $S^0$, and we prove that the Inoue surfaces cannot bear LCK metrics with potential.

The paper is organized as follows. Section 2 is devoted to introducing locally conformally symplectic and locally conformally Kähler manifolds. In Section 3, we compute the Morse-Novikov cohomology of the Inoue surface $S^0$ and classify the locally conformally Kähler metrics on $S^0$. In Section 4 we consider the Inoue surfaces $S^+$ and $S^-$, and prove the non-existence of LCK metrics either with potential or with $d\theta$-exact fundamental form on all the Inoue surfaces. In Section 5 we give a brief overview of the Morse-Novikov cohomology of LCK surfaces in class VI and VII.

2. **Locally conformally symplectic and locally conformally Kähler manifolds**

**Definition 2.1:** Locally conformally symplectic manifolds (shortly LCS) are smooth real (necessarily even-dimensional) manifolds endowed with a nondegenerate two-form $\omega$ which satisfies the equality

$$d\omega = \theta \wedge \omega$$

for some closed one-form $\theta$, called the Lee form.
Equivalently, this means there exists a non-degenerate two-form $\omega$, an open covering $\{U_i\}$, of the manifold, and smooth functions $f_i$ on $U_i$ such that $e^{-f_i}\omega$ are symplectic, which literally explains their name.

Equality (2.1) rewrites as $d\theta \omega = 0$, hence the problem of studying on an LCS manifold the Morse-Novikov cohomology associated to the Lee form of an LCS structure is natural.

**Definition 2.2:** On a complex manifold $X = (M, J)$, a Hermitian metric $g$ is called locally conformally Kähler (shortly LCK) if there exists a closed one-form $\theta$ such that the fundamental two-form $\omega$ associated to $g$ satisfies $d\omega = \theta \wedge \omega$.

Equivalently, a (complex) manifold $M ((M, J))$ is LCS (LCK) if it admits a symplectic (Kähler) cover $(\tilde{M}, \Omega)$ such that the deck group acts by homotheties with respect to the symplectic (Kähler) form $\Omega$. The pull-back $\tilde{\omega}$ of the LCS (LCK) form of $M$ is then conformal to the symplectic (Kähler) form of the covering.

There are many examples of LCS manifolds coming from LCK geometry (see [DO]), but the relation between LCS and LCK is interesting and similar to the relation between symplectic and Kähler manifolds. LCS manifolds which are not LCK were constructed in [BM]. Examples of LCK manifolds include the Hopf manifolds $S^1 \times S^{2n-1}$, the Inoue surfaces $S^0$, $S^-$ and some wide subclasses of the Inoue surface $S^+$ (see [Tr]) and some higher dimensional analogues of $S^0$ called Oeljeklaus-Toma manifolds (see [OT]).

Among the LCK metrics, the following two are of special interest and were intensively studied:

**Definition 2.3:** An LCK metric $g$ on a complex manifold $(X, J)$ is called Vaisman if the fundamental two-form $\omega$ of $g$ is parallel with respect to the Levi-Civita connection of $g$.

The prototype of Vaisman manifolds is $S^1 \times S^{2n-1}$, but there are compact LCK manifolds which do not admit Vaisman metrics, such as the LCK Inoue surfaces (see [B]). Since the Lee form is parallel for Vaisman manifolds, the result in [LLMP] (see Remark 1.1) applies and the Morse-Novikov cohomology with respect to this form vanishes.

**Definition 2.4:** An LCK metric with potential $g$ on a manifold $X = (M, J)$ is an LCK metric such that there exists a covering $\tilde{X}$ on which the pull-back $\tilde{\omega}$ of its fundamental form $\omega$ satisfies $\tilde{\omega} = d^c \tilde{f}$, where $d^c = JDJ^{-1}$, for a plurisubharmonic function $f : \tilde{X} \to \mathbb{R}^+$, such that $\gamma^*f = e^{c_\gamma}f$ ($c_\gamma \in \mathbb{R}$), for any deck transformation $\gamma \in \pi_1(X)$.
In other words, the Kähler metric on the cover $\tilde{X}$ has global, positive and automorphic potential, see [OV1].

Vaisman manifolds and non-diagonal Hopf manifolds provide examples of LCK manifolds with potential. For more details, see e.g. [OV1].

Until F. Belgun showed in [B] that there exists no LCK metric on a subclass of Inoue surfaces $S^+$, it was generally believed that all complex surfaces with odd first Betti number carry an LCK metric. In this context, the characterization of LCK metrics on complex surfaces is of particular interest. A weaker condition than LCK was considered in [AD], namely locally conformally symplectic forms which tame the complex structure $J$ (this parallels symplectic forms taming a complex structure):

**Definition 2.5:** ([AD]) A locally conformally symplectic form $\omega$ on a complex surface $X = (M, J)$ tames $J$ if $\omega(X, JX) > 0$ for any non-zero vector field $X$ on $M$.

It was proved in [AD] that any compact complex surface $X = (M, J)$ with odd first Betti number admits a locally conformally symplectic form which tames $J$. Moreover, in the same paper, the following subsets of $H^1_{dR}(M)$ are introduced:

\[ C(X) = \{ [\theta] \in H^1_{dR}(M) \mid \text{there exists } \omega \in \Omega^{1,1}(X), \omega > 0, d\theta \omega = 0 \} \]

\[ T(X) = \{ [\theta] \in H^1_{dR}(M) \mid \text{there exists } \omega \in \Omega^2(X), \omega^{1,1} > 0, d\theta \omega = 0 \} \]

as cohomological invariants similar to the Kähler cone in the Kähler setting. For the Inoue surfaces $S^+$ and $S^-$, the authors characterize the above sets. We here give similar characterizations for $S^0$.

### 3. Morse-Novikov cohomology of the Inoue surface $S^0$

#### 3.1. Description of the LCS manifold $S^0$

In [I], M. Inoue introduced three types of complex compact surfaces, which are traditionally referred to as the Inoue surfaces $S^0$, $S^+$ and $S^-$. In [Tr], Tricerri endowed the Inoue surfaces $S^0$, $S^-$ and some subclasses of $S^+$ with locally conformally Kähler metrics, in particular, by forgetting the complex structure, with locally conformally symplectic structures.

We review the construction of $S^0$ and insist on its description as mapping torus of the 3-dimensional torus $\mathbb{T}^3$.

Let $A$ be a matrix from $\text{SL}_3(\mathbb{Z})$ with one real eigenvalue $\alpha > 1$ and two complex eigenvalues $\beta$ and $\overline{\beta}$. We denote by $(a_1, a_2, a_3)^t$ a real eigenvector of $\alpha$ and by $(b_1, b_2, b_3)^t$ a complex eigenvector of $\beta$. Let $G_A$ be the group of affine transformations of $\mathbb{C} \times \mathbb{H}$ generated by the transformations:

\[ (z, w) \mapsto (\beta z, \alpha w), \]

\[ (z, w) \mapsto (z + b_i, w + a_i). \]
for all $i = 1, 2, 3$, where $\mathbb{H}$ stands for the Poincaré half-plane.

As a complex manifold, $S^0$ is $(\mathbb{C} \times \mathbb{H})/G_A$, where the complex structure, which we shall denote by $J$, is the one inherited from $\mathbb{C} \times \mathbb{H}$.

We now explain its structure as a mapping torus. Denote by $\mathbb{T}^3$ the standard 3-dimensional torus, namely $\mathbb{T}^3 = \mathbb{R}^3/\langle f_1, f_2, f_3 \rangle$, where $f_1$ (resp. $f_2$, $f_3$) is the translation with $(1, 0, 0)$ (resp. $(0, 1, 0)$, $(0, 0, 1)$).

Let $\mathbf{x} := (x, y, z)^t$, and consider the automorphism $\varphi$ of $\mathbb{R}^3$ with matrix $A^t$ in the canonical basis.

It clearly descends to an automorphism $\varphi$ of $\mathbb{T}^3$, since $A^t$ belongs to $\text{SL}_3(\mathbb{Z})$. Let $\tilde{\mathbf{x}}$ denote a point of $\mathbb{T}^3$. We define the manifold $\mathbb{T}^3 \times_\varphi \mathbb{R}^+ := (\mathbb{T}^3 \times \mathbb{R}^+) / (\tilde{\mathbf{x}}, t) \sim (\varphi(\tilde{\mathbf{x}}), \alpha t)$ which has the structure of a compact fiber bundle over $S^1$ by considering $p : \mathbb{T}^3 \times_\varphi \mathbb{R}^+ \to S^1$, $[(\tilde{\mathbf{x}}, t)] \mapsto e^{2\pi i \log \alpha t}$

Here we denote by $[,]$ the equivalence class with respect to $\sim$.

In order to write explicitly a diffeomorphism between $\mathbb{T}^3 \times_\varphi \mathbb{R}^+$ and $S^0$, let

$B := \begin{pmatrix} \text{Re} b_1 & \text{Re} b_2 & \text{Re} b_3 \\ \text{Im} b_1 & \text{Im} b_2 & \text{Im} b_3 \\ a_1 & a_2 & a_3 \end{pmatrix}$

Now the requested diffeomorphism acts as:

$[\tilde{\mathbf{x}}, t] \mapsto [(B \cdot \mathbf{x}, t)]$,

where $x + iy$ and $z + it$ are coordinates on $\mathbb{C} \times \mathbb{H}$ and $[(x + iy, z + it)]$ denotes the equivalence class of $(x + iy, z + it)$, under the action of $G_A$. It is straightforward to check this map is well defined and indeed an isomorphism.

The LCK structure given by Tricerri in $\mathbb{T}^3 \times_\varphi \mathbb{R}^+$ is given as a $G_A$-invariant globally conformally Kähler structure on $\mathbb{C} \times \mathbb{H}$ and in the coordinates $(z, w)$, the expressions for the metric and the Lee form, respectively, are:

$g = -i \left( \frac{dw \otimes d\overline{w}}{w_2^2} + w_2 dz \otimes d\overline{z} \right)$

$\theta = \frac{dw_2}{w_2}$

where $w_2 = \text{Im}(w)$. For our description as fiber bundle and real coordinates $(x, y, z, t)$, the Lee form $\theta$ is $\frac{dt}{\alpha}$.

We denote by $\vartheta$ the volume form of the circle of length 1.

A simple computation shows that

$\theta = \ln \alpha \cdot \vartheta$. 

3.2. Explicit computation of the Morse-Novikov cohomology. To compute by hand the Morse-Novikov cohomology groups of $S^0$, we shall use the following twisted version of the Mayer-Vietoris sequence:

**Lemma 3.1:** ([HHR, Lemma 1.2]) Let $M$ be the union of two open sets $U$ and $V$ and $\theta$ a closed one-form. Then there exists a long exact sequence

\[
\cdots \to H^i_\theta(M) \xrightarrow{\alpha} H^i_\theta(U) \oplus H^i_\theta(V) \xrightarrow{\beta_*} H^i_{\theta|U \cup V}(U \cap V) \xrightarrow{\delta} H^{i+1}_\theta(M) \to \cdots
\]

where for some partition of unity $\{\lambda_U, \lambda_V\}$ subordinated to the covering $\{U, V\}$, the above morphisms are:

\[
\delta(\sigma) = [d\lambda_U \wedge \sigma] = -[d\lambda_V \wedge \sigma], \\
\alpha(\sigma) = (\sigma|_U, \sigma|_V), \\
\beta(\sigma, \tau) = \sigma|_{U \cap V} - \tau|_{U \cap V}.
\]

We first choose the open sets $U_1$ and $U_2$ which cover the circle:

\[
U_1 := \{e^{2\pi it} \mid t \in (0, 1)\}, \quad U_2 := \{e^{2\pi it} \mid t \in (\frac{1}{2}, \frac{3}{2})\},
\]

and take as open sets $U := p^{-1}(U_1)$ and $V := p^{-1}(U_2)$, representing a covering of $S^0$. The sets $U$ and $V$ are the trivializations of $S^0$ as fiber bundle over $S^1$. Therefore, we have

\[
\varphi_{U_1} : U \longrightarrow U_1 \times \mathbb{T}^3, \quad (x, t) \mapsto (e^{2\pi i\alpha t}, \hat{x}), \ t \in (1, \alpha), \\
\varphi_{U_2} : V \longrightarrow U_2 \times \mathbb{T}^3, \quad (x, t) \mapsto (e^{2\pi i\alpha t}, \hat{x}), \ t \in (\alpha \frac{1}{2}, \alpha \frac{3}{2}).
\]

Since $U_1 \cap U_2$ is disconnected, the transition maps $g_{U_1U_2} := \varphi_{U_1} \circ \varphi_{U_2}^{-1}$ are given by:

\[
g_{U_1U_2} : U_1 \cap U_2 \times \mathbb{T}^3 \to U_1 \cap U_2 \times \mathbb{T}^3, \\
g_{U_1U_2}(m, \hat{x}) = \begin{cases} (m, \hat{x}), & \text{if } m = e^{2\pi it}, \text{ with } t \in (\frac{1}{2}, 1) \\
(m, (A^t)^{-1} \cdot \hat{x}), & \text{if } m = e^{2\pi it}, \text{ with } t \in (1, \frac{3}{2})
\end{cases}
\]

As $\theta$ is not exact, we already know that $H^0_\theta(S^0)$ and $H^3_\theta(S^0)$ vanish (see [HHR]). Concerning the other Morse-Novikov cohomology groups, we prove the following result:

**Theorem 3.2:** On $S^0$, for the Lee form $\theta$ given by Tricerri, $H^1_\theta(S^0)$ vanishes, $H^2_\theta(S^0) \simeq \mathbb{R}$ and $H^3_\theta(S^0) \simeq \mathbb{R}$.

**Proof.** The proof is algebraic and the key is to explicitly write the morphism $\beta_*$. 

Consider the functions $f : U_1 \to (0, 1)$, $f(e^{2\pi i t}) = t$ and $g : U_2 \to (\frac{1}{2}, \frac{3}{2})$, $g(e^{2\pi i t}) = t$. Then on $U_1$, $\vartheta = df$ and on $U_2$, $\vartheta = dg$. Moreover, we observe that on $W_1$, $f$ and $g$ coincide and on $W_2$, $g = f + 1$. Therefore, 
\[ \theta = \log \alpha \cdot dp^* f \text{ on } U, \quad \text{and } \theta = \log \alpha \cdot dp^* g \text{ on } V, \]
and hence $\theta$ is exact on these two open sets.

We have the following diagram:
\[
\begin{array}{ccc}
H^0_{\theta|U}(U) \oplus H^0_{\theta|V}(V) & \xrightarrow{\beta_*} & H^0_{\theta|U \cap V}(U \cap V) \\
\Phi \downarrow & & \downarrow \Psi \\
\mathbb{R}^2 & \xrightarrow{\gamma} & \mathbb{R}^2
\end{array}
\]
where $\Phi$ and $\Psi$ are the isomorphisms defined as
\[
\Phi([\sigma], [\eta]) = (e^{-\text{inop}^*f} \sigma, e^{-\text{inop}^*g} \eta),
\]
\[
\Psi([\omega]) = (e^{-\text{inop}^*f} \omega|_{\partial^{-1}(W_1)}, e^{-\text{inop}^*f} \omega|_{\partial^{-1}(W_2)}),
\]
and $\gamma$ makes the diagram commutative, hence $\gamma(a, b) = (a - b, a - ab)$.

As $\alpha \neq 1$, $\gamma$ is an isomorphism, and hence $\beta_*$ is an isomorphism, too. Consequently, the connecting morphism $\delta : H^0_{\theta|U \cap V}(U \cap V) \to H^1_{\theta}(S^0)$ is 0 and we can start the Mayer-Vietoris from $H^3_{\theta}(S^0)$:
\[
0 \to H^1_{\theta}(S^0) \to H^1_{\theta|U}(U) \oplus H^1_{\theta|V}(V) \to H^1_{\theta|U \cap V}(U \cap V) \to \cdots \to H^3_{\theta|U \cap V}(U \cap V) \to 0
\]

We look now at the other morphisms $\beta_*$ linking cohomology groups of degree $i \geq 1$.
\[
\begin{array}{ccc}
H^i_{\theta|U}(U) \oplus H^i_{\theta|V}(V) & \xrightarrow{\beta_*} & H^i_{\theta|U \cap V}(U \cap V) \\
\Phi \downarrow & & \downarrow \Psi \\
H^i_{dR}(T^3) \oplus H^i_{dR}(T^3) & \xrightarrow{\gamma} & H^i_{dR}(T^3) \oplus H^i_{dR}(T^3)
\end{array}
\]
Using the fact that $\theta$ is exact when restricted to $U$ and $V$, the isomorphism $\Phi$ is obtained by the following composition of isomorphisms:
\[
H^i_{\theta|U}(U) \overset{f_1}{\to} H^i_{dR}(U) \overset{f_2}{\to} H^i_{dR}(U_1 \times T^3) \overset{f_3}{\to} H^i_{dR}(T^3),
\]
where $f_1([\sigma]) = [e^{-f} \sigma]$, $f_2([\eta]) = [\varphi_U \cdot \eta]$, $f_3([\omega]) = [i^* \omega]$ and $i : T^3 \to U_1 \times T^3$ is defined as $i(t) = (m, t)$, for some point $m$ in $U_1$. 

From now on we denote by $W_1$ and $W_2$ the two connected components of $U_1 \cap U_2$, namely
\[
W_1 = \{e^{2\pi i t} | t \in (\frac{1}{2}, 1)\}, \quad W_2 = \{e^{2\pi i t} | t \in (1, \frac{3}{2})\}.
\]
The same holds for $V$, the only difference being that $f'_1 : H'_{\theta U}(V) \rightarrow H'_{dR}(V)$ is given by $[\sigma] \mapsto [e^{-g}\sigma]$ and $f'_2 : H'_{dR}(V) \rightarrow H'_{dR}(U_2 \times T^3)$ is given by $[\eta] \mapsto [(\varphi U_2)\ast \eta]$. Thus:

$$\Phi = f_3 \circ f_2 \circ f_1 \oplus f'_3 \circ f'_2 \circ f'_1.$$ 

As for $\Psi$, there is a similar sequence, consisting of isomorphisms:

$$H'_{\theta U \cap V}(U \cap V) \xrightarrow{g_1} H'_{dR}(U \cap V) \xrightarrow{g_2} H'_{dR}(U \cap V \times T^3) \xrightarrow{g_3} H'_{dR}(T^3) \oplus H'_{dR}(T^3).$$

Here, the isomorphisms $g_1$, $g_2$ and $g_3$ are given by $[\sigma] \mapsto [e^{-f}\sigma]$, $[\eta] \mapsto [(\varphi U)\ast \eta]$ and $[\omega] \mapsto (i_1^*[\omega]|_{W_1}, i_2^*[\omega]|_{W_2})$, where $i_1 : T^3 \rightarrow W_1 \times T^3$ denotes the injection $t \mapsto (m, t)$ for some $m$ in $W_1$ and $i_2 : T^3 \rightarrow W_2 \times T^3$, $i_2(t) = (n, t)$ for some point $n$ in $W_2$. We define $\Psi = g_3 \circ g_2 \circ g_1$.

A straightforward computation shows that $\gamma = \Psi \circ \beta_\ast \circ \Phi^{-1}$ is given by:

$$\begin{pmatrix} [a], [b] \end{pmatrix} \mapsto \begin{pmatrix} [a - b], [a - \alpha \cdot i_2^*(g_{U_1U_2})|W_2\ast \pi^*b) \end{pmatrix},$$

where $\pi : V \times T^3 \rightarrow T^3$ is the projection on the second factor.

We investigate now the map $i_2^*((g_{U_1U_2})|W_2\ast \pi^*) : H'_{dR}(T^3) \rightarrow H'_{dR}(T^3)$ for $i = 1, 2, 3$. It is an easy observation that

$$i_2^*((g_{U_1U_2})|W_2\ast \pi^*) = (\pi \circ (g_{U_1U_2})|W_2 \circ i_2)_\ast.$$ 

Since $\pi \circ (g_{U_1U_2})|W_2 \circ i_2 : T^3 \rightarrow T^3$ is given by the matrix $(A^i)^{-1}$, the map induced in homology, $(\pi \circ (g_{U_1U_2})|W_2 \circ i_2)_\ast : H_1(T^3) \rightarrow H_1(T^3)$ has the matrix $(A^i)^{-1}$ in the canonical basis. Therefore, the map of the matrix induced by the pushforward $((\pi \circ (g_{U_1U_2})|W_2 \circ i_2)_\ast : H^1_{dR}(T^3) \rightarrow H^1_{dR}(T^3)$ in the canonical basis $\{[dx], [dy], [dz]\}$ is $((A^i)^{-1})^{-1} = A$.

As a consequence, we obtain the matrix of $\gamma : H^1_{dR}(T^3) \oplus H^1_{dR}(T^3) \rightarrow H^1_{dR}(T^3) \oplus H^1_{dR}(T^3)$ to be the following:

$$\begin{bmatrix} I_3 & -I_3 & -I_3 & -I_3 \\ I_3 & -I_3 & -I_3 & -I_3 \\ I_3 & -I_3 & -I_3 & -I_3 \\ I_3 & -I_3 & -I_3 & -I_3 \end{bmatrix}$$

By performing a transformation which keeps the rank constant, namely adding the first three columns to the last three, the matrix above has the same rank as:

$$\begin{bmatrix} I_3 & O_3 \\ I_3 & -I_3 \end{bmatrix}$$

Moreover, this further implies that the rank is controlled by the block $I_3 - \alpha \cdot A$, which would be a nonsingular matrix if and only if $\frac{1}{\alpha}$ were an eigenvalue of $A$, which is not the case. Hence, $\gamma$ and implicitly $\beta_\ast$ is an isomorphism, whence from the Mayer-Vietoris sequence, $H^1_\theta(S^0)$ has to vanish.

Since we already know the matrix of $(\pi \circ (g_{U_1U_2})|W_2 \circ i_2)_\ast : H^1_{dR}(T^3) \rightarrow H^1_{dR}(T^3)$ is $A$ in the basis $\{[dx], [dy], [dz]\}$, we can easily compute the matrix of $(\pi \circ (g_{U_1U_2})|W_2 \circ i_2)_\ast : H^2_{dR}(T^3) \rightarrow H^2_{dR}(T^3)$ in the basis $\{[dy \wedge dz], [dz \wedge}$
\[dx_1, \{dx_1, dx_2\}\) to be \((A^*)^t\). Therefore, the matrix of \(\gamma : H^2_{dR}(\mathbb{T}^3) \oplus H^2_{dR}(\mathbb{T}^3) \rightarrow H^2_{dR}(\mathbb{T}^3) \oplus H^2_{dR}(\mathbb{T}^3)\) is:

\[
\begin{bmatrix}
I_3 & -I_3 \\
I_3 & -\alpha \cdot (A^*)^t
\end{bmatrix}
\]

which by the same arguments as above has the same rank as:

\[
\begin{bmatrix}
I_3 & O_3 \\
I_3 & I_3 - \alpha \cdot (A^*)^t
\end{bmatrix}
\]

Since \(A^* = A^{-1}\) (because \(A\) lives in \(SL_3(\mathbb{Z})\)) and a matrix and its transpose have the same eigenvalues, \((A^*)^t\) has the same eigenvalues as \(A^{-1}\), thus \(\frac{1}{\alpha}\) is one of them. Therefore, the rank of the block \(I_3 - \alpha \cdot (A^*)^t\) is 2, because \(\frac{1}{\alpha}\) is an eigenvalue of \((A^*)^t\) of multiplicity 1. We infer that the matrix of \(\gamma : H^2_{dR}(\mathbb{T}^3) \oplus H^2_{dR}(\mathbb{T}^3) \rightarrow H^2_{dR}(\mathbb{T}^3) \oplus H^2_{dR}(\mathbb{T}^3)\) has rank 5, forcing \(\ker \gamma\) to be 1-dimensional and from the Mayer-Vietoris sequence, we obtain \(H^2_\partial(S^0) \simeq \mathbb{R}\).

For the final case, when \(i = 3\), it is straightforward that \((\pi \circ (g_{U_1}U_2)|_{W_2} \circ i_2) : H^3_{dR}(\mathbb{T}^3) \rightarrow H^3_{dR}(\mathbb{T}^3)\) is given by the multiplication with the determinant of the matrix of \((\pi \circ (g_{U_1}U_2)|_{W_2} \circ i_2) : H^3_{dR}(\mathbb{T}^3) \rightarrow H^3_{dR}(\mathbb{T}^3)\). In this case, the determinant is 1, hence \(\gamma : H^3_{dR}(\mathbb{T}^3) \oplus H^3_{dR}(\mathbb{T}^3) \rightarrow H^3_{dR}(\mathbb{T}^3) \oplus H^3_{dR}(\mathbb{T}^3)\) is given by the 2 \(\times 2\) matrix:

\[
\begin{bmatrix}
1 & -1 \\
1 & -\alpha
\end{bmatrix}
\]

and thus it defines an isomorphism. By the Mayer-Vietoris sequence, we obtain:

\[\dim \mathbb{R} H^3_\partial(S^0) = 6 - \dim \mathbb{R} \text{Im}(\beta) : H^2_{\partial U}(U) \oplus H^2_{\partial V}(V) \rightarrow H^2_{\partial U\cup V}(U \cup V) = 1\]

In conclusion, \(H^3_\partial(S^0) \simeq \mathbb{R}\), \(H^2_\partial(S^0) \simeq \mathbb{R}\) and the rest of the Morse-Novikov cohomology groups vanish.

**Remark 3.3:** Since \(H^1_{dR}(S^0) \simeq \mathbb{R}\) (see [1]), \(H^1_{dR}(S^0) = \mathbb{R}[\partial]\), hence every closed, but not exact one-form is, up to adding an exact one-form, a multiple of \(\partial\). Let \(\theta_1 = t \cdot \partial\) with \(t \neq 0\). By applying the same method as above and replacing \(\ln \alpha\) with \(t\), we can compute the Morse-Novikov cohomology groups \(H^i_{\theta_1}(S^0)\). Moreover, we observe that for \(t \neq \ln \alpha\) and \(t \neq -\ln \alpha\), the morphisms \(\gamma : H^i_{dR}(\mathbb{T}^3) \oplus H^i_{dR}(\mathbb{T}^3) \rightarrow H^i_{dR}(\mathbb{T}^3) \oplus H^i_{dR}(\mathbb{T}^3)\) are in fact isomorphisms, and thus we obtain:

**Corollary 3.4:** For \(\theta_1 = t \cdot \partial\) and \(t \neq \ln \alpha, -\ln \alpha\), \(H^i_{\theta_1}(S^0)\) vanish for all \(i \geq 0\).

We now treat the above two exceptions.

The case \(t = \ln \alpha\) is the one discussed above.
For $t = -\ln \alpha$, we apply the following version of Poincaré duality:

**Proposition 3.5:** ([HR Proposition 1.5]) On a compact oriented $n$-dimensional manifold $M$, we have the following isomorphism:

$$H^{n-k}_\eta(M) \simeq H^k_\eta(M)$$

for any closed one-form $\eta$.

Therefore, when $t = -\ln \alpha$, we have $\theta_1 = -\theta$, $H^1_{\theta_1}(S^0) \simeq \mathbb{R}$, $H^2_{\theta_1}(S^0) \simeq \mathbb{R}$ and the rest of the cohomology groups vanish. Thus, we computed the Morse-Novikov cohomology of $S^0$ with respect to any closed one-form.

This result will be useful in Subsection 3.4.

3.3. Finding generators for $H^2_\theta(S^0)$ and $H^3_\theta(S^0)$. Denote by

$$\Omega := -i \left( \frac{dw \wedge d\bar{w}}{w^2} + w_2 dz \wedge d\bar{z} \right)$$

the global conformally symplectic two-form on $\mathbb{C} \times \mathbb{H}$, in the coordinates $(z, w)$, which descends to a two-form $\omega$ on $S^0$. Notice that $\Omega_1 := -i \frac{dw \wedge d\bar{w}}{w^2}$ and $\Omega_2 := -iw_2 dz \wedge d\bar{z}$ are two-forms which are invariant with respect to the factorization group $G_A$. They descend to $S^0$ to two forms which we shall denote by $\omega_1$ and $\omega_2$ and we have $\omega = \omega_1 + \omega_2$. Tricerri showed that $\omega$ is an LCK form and it is the fundamental two-form of the metric induced by

$$g = -i \left( \frac{dw \otimes d\bar{w}}{w^2} + w_2 dz \otimes d\bar{z} \right)$$

on $S^0$, which we shall denote by $g_1$. Then we have the following:

**Proposition 3.6:** Let $\omega$ be the above defined LCS form of $S^0$ and $\theta = \frac{dw_2}{w_2}$ its Lee form, as in **Theorem 3.2**. Then:

$$H^2_\theta(S^0) = \mathbb{R}[\omega]$$

$$H^3_\theta(S^0) = \mathbb{R}[\theta \wedge \omega].$$

Before proving these equalities, we define the notion of twisted Laplacian. Namely, by extending the metric $g_1$ to the space of $k$-forms $\Omega^k(S^0)$, we consider the Hodge star operator $*: \Omega^k(S^0) \to \Omega^{4-k}(S^0)$, given by $u \wedge *v = g_1(u, v)d\text{vol}$. Note that the real dimension of $S^0$ is 4. Then the following operators depending on $\theta$ can be defined (they indeed make sense on any manifold $M$ endowed with a closed one-form $\theta$, although we shall treat specifically the case of $S^0$):

$$\delta_\theta : \Omega^{k+1}(S^0) \to \Omega^k(S^0), \quad \delta_\theta = -*_{-\theta}*$$

$$\Delta_\theta : \Omega^k(S^0) \to \Omega^k(S^0), \quad \Delta_\theta = \delta_\theta d_\theta + d_\theta \delta_\theta$$
ward computation shows that $\Omega_1$ does not contain harmonic and the fundamental two-form of the metric $k (3.2) \Omega_1 = \omega$.

Remark 3.7: $\delta_\theta$ is the adjoint of $d_\theta$ with respect to the inner product on $\Omega^k (S^0)$ given by $\langle \eta, \varphi \rangle = \int_{S^0} \eta \wedge *\varphi$. Observe that $\delta_\theta$ and $\Delta_\theta$ are perturbations of the usual codifferential and Laplacian operators, which are recovered by replacing $\theta$ with $0$. The motivation for introducing the operators twisted with $\theta$ is to develop Hodge theory in the context of working with $d_\theta$ instead of $d$. They were first considered in [Va1] for locally conformally Kähler manifolds and later in [GL] in the LCS setting.

The following analogue of Hodge decomposition holds:

Theorem 3.8: ([GL]) Let $M$ be a compact manifold, $\theta$ a closed one-form, $d_\theta$ and $\Delta_\theta$ defined as above. Then we have an orthogonal decomposition:

$$\Omega^k (M) = \mathcal{H}_\theta^k (M) \oplus d_\theta \Omega^{k-1} (M) \oplus \delta_\theta \Omega^{k+1} (M)$$

(3.2)

where $\mathcal{H}_\theta^k (M) = \{ \eta \in \Omega^k (M) | \Delta_\theta \eta = 0 \}$. Moreover,

$$H^k_\theta (M) \cong \mathcal{H}_\theta^k (M).$$

Thus, we observe that important properties of the Hodge-de-Rham theory for the operator $d$ are shared by the same theory applied to $d_\theta$.

We now give the proof of Proposition 3.6. Since we proved in Theorem 3.2 that $H^2_\theta$ and $H^3_\theta$ are isomorphic to $\mathbb{R}$, it is enough to show that $\omega$ and $\theta \wedge \omega$ are $d_\theta$-closed, but not $d_\theta$-exact.

We shall prove that with respect to the Hodge decomposition (3.2), the harmonic and the $d_\theta$-exact parts of $\omega$ do not vanish. Indeed, a straightforward computation shows that $\Omega_1 = d_{d_{\omega_2}} \frac{d\omega_1}{\omega_2}$. Since $-\frac{d\omega_1}{\omega_2}$ is $G_A$-invariant, it descends to a one-form $\eta$ on $S^0$ and we have $w_1 = d_\theta \eta$. As $\omega$ is the fundamental two-form of the metric $g_1$, which is Hermitian with respect to the complex structure of $S^0$ induced form the standard one on $\mathbb{C} \times \mathbb{H}$, an easy linear algebra computation (see [GH, p. 31]) shows that the Riemannian volume form $d \text{vol}$ equals $\frac{1}{n!} \omega^n$. In the general case of complex dimension $n$, the volume form $d \text{vol}$ is $\frac{1}{n!} \omega^n$. This further implies that $\ast \omega_2 = \omega_1$. Consequently,

$$d_{d_{\omega_1}} \ast \omega_2 = d_{d_{\omega_1}} \omega_1 = d \omega_1 + \theta \wedge \omega_1.$$
we obtain, as in the case of $\omega$, that $\theta \wedge \omega$ is not $d_{\theta}$-exact, since its harmonic part is not zero. This implies $H^3_{\theta}(M) = \mathbb{R}[\theta \wedge \omega]$.

**Remark 3.9:** We notice that the alternate sum of the dimensions of the Morse-Novikov cohomology $H^i_{\theta}(S^0)$ groups is 0, which equals indeed the Euler characteristic of $S^0$.

**Remark 3.10:** The Lee form $\theta$ with respect to which we computed the Morse-Novikov cohomology has important properties: it is nowhere vanishing and it is harmonic (and hence the metric we worked with is a Gauduchon metric).

Moreover, the Novikov Betti numbers $b^{Nov}_i$ of $\theta$ vanish, since it has no zeros (see for more details [Nov1], [Nov2], [F]). It is proven in [P, Lemma 2] that if $\eta = a \cdot \tau$, where $\tau$ is an integer closed one-form and $e^a$ is transcendental, then $b^{Nov}_i = \dim_{\mathbb{R}} H^i_{\eta}$. However, here is not the case, since $\theta = \ln \alpha \cdot \vartheta$, with $\vartheta$ an integral one-form and $e^{\ln \alpha}$ an algebraic number.

We note again that the Inoue surface $S^0$ is a mapping torus of the 3-dimensional $\mathbb{T}^3$, which is a contact manifold. However, the diffeomorphism that defines this mapping torus does not preserve the standard contact structure of $\mathbb{T}^3$. In general, if $(M, \alpha)$ is a contact manifold and $\varphi : M \to M$ is a diffeomorphism preserving $\alpha$, one can consider the mapping torus of $M$ with respect to $\varphi$, $M_{\varphi} := M \times [0, 1]/(x, 0) \sim (\varphi(x), 1)$. Then $M_{\varphi}$ admits the LCS form $\omega := d\alpha - \vartheta \wedge \alpha$, where $\vartheta$ is the integer volume form of the circle. The Morse-Novikov cohomology of $M_{\varphi}$ with respect to $\vartheta$ vanishes, as a consequence of [P, Lemma 2].

### 3.4. Classification of LCK structures on $(S^0, J)$

Using our previous explicit computation of the Morse Novikov cohomology with respect to all the closed one-forms on $S^0$ ([Remark 3.3] [Corollary 3.4]), we are able to describe all the possible Lee forms of an LCK metric on $S^0$ and classify the LCK metrics. We prove the following result:

**Theorem 3.11:** On the complex surface $(S^0, J)$, the only possible Lee class for LCK metrics is $[\theta] \in H^1(S^0)$, where $\theta$ is the Lee form of Tricerri’s metric.

For the proof, we use the structure of solvmanifold of $S^0$ that we now describe following [S] (see also [Kam]).

We consider the following coordinates on $\mathbb{H} \times \mathbb{C} = \{(x + it, z) \mid x \in \mathbb{R}, t > 0, z \in \mathbb{C}\}$. The group structure on $\mathbb{H} \times \mathbb{C}$ is:

$$(x + it, z) \cdot (x' + it', z') = (tx' + x + it \cdot t', \beta \log a \cdot z' + z)$$
Thus, as a group, \( \mathbb{H} \times \mathbb{C} \) can be expressed as a group of matrices as:

\[
G = \left\{ \begin{bmatrix}
t & 0 & 0 & x \\
0 & \beta^{\log_{\alpha} t} & 0 & z \\
0 & 0 & \beta^{\log_{\alpha} t} & \bar{z} \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \mid x \in \mathbb{R}, t > 0, z \in \mathbb{C} \right\}
\]

The group \( G \) is solvable. Consider the following lattice:

\[
\Gamma = \left\{ \begin{bmatrix}
\alpha^s & 0 & 0 & x_1 \\
0 & \beta^s & 0 & x_2 \\
0 & 0 & \beta & x_3 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \mid s \in \mathbb{Z} \right\}
\]

where

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \text{ and } w_1, w_2, w_3 \text{ are integers.}
\]

Then \( S^0 \) identifies with \( G/\Gamma \).

The solvable Lie algebra corresponding to \( G \) is:

\[
g = \text{span}\{ A, X, Y_1, Y_2 \mid [A, X] = -2rX, \\
[A, Y_1] = rY_1 + sY_2, [A, Y_2] = rY_2 - sY_1 \},
\]

where \( r = -\frac{\ln \alpha}{2} \) and \( \beta = e^{r+i\ell} \).

One can define a left invariant complex structure \( J_0 \) on \( G/\Gamma \) by \( JA = X \), \( JY_1 = Y_2 \).

The manifolds \( (S^0, J) \) and \( (G/\Gamma, J_0) \) are biholomorphic. Via this biholomorphism, the dual base of \( \{ A, X, Y_1, Y_2 \} \) consisting of left invariant one-forms is \( \{ \vartheta, x, y_1, y_2 \} \), where \( \vartheta \) was defined in Subsection 3.1, and \( x, y_1 \) and \( y_2 \) satisfy:

\[
(3.3) \quad dx = 2r\vartheta \wedge x, dy_1 = -r\vartheta \wedge y_1 + s\vartheta \wedge y_2, dy_2 = -r\vartheta \wedge y_2 - s\vartheta \wedge y_1
\]

The LCK form given by Tricerri can also be written as:

\[
\omega = -\vartheta \wedge x - y_1 \wedge y_2
\]

with the Lee form \( \theta = -2r\vartheta = \ln \alpha \cdot \vartheta \).

**Proof of Theorem 3.11.** We prove that \( \theta \) and its cohomologous one-forms are the only possible Lee forms for LCK metrics on the Inoue surface \( S^0 \). Indeed, let \( \theta_1 \) be another possible Lee form for an LCK metric. As \( b_1(S^0) = 1 \), \( \theta_1 \) is, up to a an exact one-form, in one of the following three cases:

1. \( \theta_1 = t\vartheta \), with \( t \) different from \( \ln \alpha \) or \( -\ln \alpha \).
2. \( \theta_1 = \ln \alpha \cdot \vartheta \) and thus coincides with the Lee form of Tricerri’s LCK metric.
3. \( \theta_1 = -\ln \alpha \cdot \vartheta \).
Before separately discussing these three cases, we need two general results:

**Claim 3.12:** In all of the three cases, $\theta_1$ is left invariant.

Indeed, this follows from the left invariance of $\vartheta$.

**Claim 3.13:** If $\omega_1$ is an LCK form with the Lee form $\theta_1$ (no matter in which of the three possibilities above), then it cannot be $d\theta_1$-exact.

*Proof.* Indeed, by contradiction, assume that $\omega_1 = d\theta_1 \eta = d\eta - \theta_1 \wedge \eta$. It was proven in [S, Proposition 1.2] that one can further find a left invariant form $\eta_0$ such that $\omega_0 := d\theta_1 \eta_0$ is still an LCK form. We obtain the following:

$$
\omega_0(Y_1, Y_2) = d\eta_0(Y_1, Y_2) - \theta_1 \wedge \eta_0(Y_1, Y_2).
$$

However, since $\theta_1 = t\vartheta$ and $\vartheta(Y_1) = \vartheta(Y_2) = 0$, we have

$$
\omega_0(Y_1, Y_2) = d\eta_0(Y_1, Y_2) = Y_1(\eta_0(Y_2)) - Y_2(\eta_0(Y_1)) = \eta_0([Y_1, Y_2]).
$$

But since $\eta_0$, $Y_1$, and $Y_2$ are left invariant, the first two terms in the right hand side vanish and thus,

$$
\omega_0(Y_1, Y_2) = -\eta_0([Y_1, Y_2]).
$$

By relations (3.3), we obtain $[Y_1, Y_2] = 0$, and hence $\omega_0(Y_1, Y_2) = 0$. This contradicts the fact that $\omega_0$ is the Kähler form of a Hermitian metric $g_0$, since $\omega_0(Y_1, Y_2) = g_0(JY_1, Y_2) = g_0(Y_2, Y_2) \neq 0$.

Therefore, an LCK form on $S^0$ with Lee form $\theta_1$ is not allowed to be $d\theta_1$-exact. ■

Now, for $\theta_1$ as in **Case 1** above, by Corollary 3.4 the Morse-Novikov cohomology vanishes, so the LCK metric would be exact, which we saw it is impossible. So, **Case 1** is excluded.

**Case 2**, $\theta_1 = \theta$ and we showed that $H^2_{\theta} (S^0) = \mathbb{R}[\omega]$. Therefore, we see that any other LCK metric $\omega_1$ has to be of the type $r\omega + d\theta_1 \eta$. Note that $r \neq 0$, since by Claim 3.13 an LCK form on $S^0$ cannot be $d\theta_1$-exact.

**Case 3** is a bit more involved. Now $\theta_1 = -\theta$ and we already shown in Subsection 3.2 that $H^1_{\theta_1} (S^0) \simeq \mathbb{R}$, $H^2_{\theta_1} (S^0) \simeq \mathbb{R}$ and the other groups vanish. However, there is still no LCK metric with $\theta_1$ as Lee form in order to argue that, we use the results in [AD].

Let $\eta$ be a closed one-form on a complex compact surface $(M, J)$ and let $\mathcal{L}_\eta := L_\eta \otimes \mathbb{C}$, where $L_\eta$ is the line bundle associated to $\eta$, as presented in the first section. Let $g$ be a Gauduchon metric with the corresponding Lee form denoted by $\theta^g$.

**Definition 3.14:** The degree of $\mathcal{L}_\eta$ with respect to the Gauduchon metric
$g$ is defined to be:

$$\deg_g L_\eta = -\frac{1}{2\pi} \int_M g(\theta^g, \eta) v_g$$

where $v_g$ is the volume form of $g$.

It was shown in [AD, Lemma 4.1] that on a compact complex surface $(M, J)$ with $b_1(M) = 1$, the sign of $\deg_g L_\eta$ does not depend on the choice of the Gauduchon metric on $M$. In the case of $(S^0, J)$, we can choose as Gauduchon metric Tricerri’s metric $\omega$, with its corresponding Lee form $\theta$ (see also [Remark 3.10]). Therefore, the sign of $L_\eta$ is the sign of $$-\frac{1}{2\pi} \int_M g(\theta, \theta_1) v_g$$ and it is positive, since $\theta_1 = -\theta$. Nevertheless, in [AD, Proposition 4.3], it is proved that on a compact complex surface $(M, J)$ with $b_1(M) = 1$, if $\eta$ is the Lee form of an LCK metric or more general, the Lee form of an LCS form which tames $J$, then the degree of $L_\eta$ is negative, thus excluding $\theta_1$ from the possible Lee forms.

Hence $C(S^0) = \{[\theta]\}$ and the proof is complete. \qed

**Remark 3.15:** With the same proof, we obtain $T(S^0) = \{[\theta]\}$, since we only use the $d\theta_1$-closedness of $\omega_1$ and the positiveness $\omega_1(X, JX) > 0$ for any non-zero $X$.

**Remark 3.16:** There are LCK metrics in $S^0$ which are not left invariant. For instance, consider the form

$$\Omega_1 = -i(e^{\sin(2\pi \log_\alpha w_2)} \frac{dw \wedge d\overline{w}}{w_2^2} + w_2 dz \wedge d\overline{z}).$$

It is straightforward that $\Omega_1$ is LCK with the Lee form $\theta = \frac{dw}{w_2}$ and it is not left invariant. Note that also $\omega$ and $\Omega_1$ are two LCK metrics with the same Lee form, but which are not conformal.

### 4. Morse-Novikov cohomology of the Inoue surfaces $S^+$ and $S^-$

We describe the Inoue surfaces $S^+$ and $S^-$ and compute their Morse-Novikov cohomology groups.

#### 4.1. The complex surface $S^+$

Let $N = (n_{ij}) \in SL_2(\mathbb{Z})$ be a matrix with real eigenvalues $\alpha > 1$ and $1/\alpha$ and $(a_1, a_2)^t, (b_1, b_2)^t$ real eigenvectors corresponding to $\alpha$ and $1/\alpha$. Let us fix some integers $p, q, r$ with $r \neq 0$ and a complex number $z$. Let $e_1, e_2$ be defined as

$$e_1 = \frac{1}{2} n_{11}(n_{11} - 1)a_1 b_1 + \frac{1}{2} n_{12}(n_{12} - 1)a_2 b_2 + n_{11} n_{12} b_1 a_2$$

and $c_1, c_2$ defined by

$$(c_1, c_2) = (c_1, c_2) \cdot N^t + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q).$$
We denote by $G^{+}_{N,p,q,r,z}$ the group of affine transformations of $\mathbb{R}^3 \times \mathbb{R}^+$ (we consider here the coordinates $x, y, w, t$, where $t > 0$) generated by the following:

\[
g_0(x, y, w, t) = (x + \text{Re } z, y + \text{Im } z, \alpha w, \alpha t)\]
\[
g_i(x, y, w, t) = (x + b_i w + c_i, y + b_i t, w + a_i, t), \quad i = 1, 2\]
\[
g_3(x, y, w, t) = (x + \frac{b_1 a_2 - b_2 a_1}{r}, y, w, t).
\]

We define $S^{+}_{N,p,q,r,z}$ to be $\mathbb{R}^3 \times \mathbb{R}^+ / G^{+}_{N,p,q,r,z}$ and denote by $[x, y, w, t]$ the class of $(x, y, w, t)$.

The transformations $g_0, g_1, g_2$ and $g_3$ satisfy the relations:

\[
g_3 g_i = g_i g_3, \quad \text{for} \quad i = 0, 1, 2,
\]
\[
g_1^{-1} g_2^{-1} g_1 g_2 = g_3
\]
\[
g_0 g_1 g_0^{-1} = g_1^{n_{11}} g_2^{n_{12}} g_3^p,
\]
\[
g_0 g_2 g_0^{-1} = g_1^{n_{21}} g_2^{n_{22}} g_3^q.
\]

**Remark 4.1:** Tricerri showed in [Tr] that for $z \in \mathbb{R}$, the complex surface $S^{+}_{N,p,q,r,z}$ carries an LCK metric given by the two-form

\[
\omega = 2 \frac{1 + y^2}{t^2} dt \wedge dw - 2 \frac{y}{t} (dt \wedge dx + dy \wedge dw) + 2dy \wedge dx.
\]

In this case, the Lee form is $\theta = \frac{dt}{t}$. Belgun proved in [Be] that for $z \in \mathbb{C} \setminus \mathbb{R}$, $S^{+}_{N,p,q,r,z}$ does not carry an LCK metric. We shall work with the parameter $z = 0$, since $S^{+}_{N,p,q,r,z}$ analytically deforms to $S^{+}_{N,p,q,r,0}$. Moreover, we shall use the more convenient notations $S^+$ and $G^+$ from now on.

We consider $p : \mathbb{R}^3 \times \mathbb{R}^+ \to S^1$, $p(x, y, w, t) = e^{2\pi i \log_\alpha t}$. This map descends to a submersion $\pi : S^+ \to S^1$, which endows $S^+$ with the structure of a fiber bundle with fiber $F$. Therefore, $S^+$ is mapping torus of $F$. The fiber over the point $s = e^{2\pi i \log_\alpha t} \in S^1$ is described by

\[
F \simeq \pi^{-1}(s) = \{[x, y, w, t] \mid x, y, w \in \mathbb{R}\}
\]

Following [I], we denote by $\Gamma_t$ the normal subgroup of $G^+$ generated by $g_1, g_2$ and $g_3$ with $t$ fixed. Then $\pi^{-1}(s) \simeq \mathbb{R}^3 / \Gamma_t$. We trivialize $S^+$ with the open sets:

\[
U = \{[x, y, w, t] \mid x, y, w \in \mathbb{R}, t \in (1, \alpha)\}
\]
\[
V = \{[x, y, w, t] \mid x, y, w \in \mathbb{R}, t \in (\sqrt{\alpha}, \alpha \sqrt{\alpha})\}.
\]

The generic fiber is $F = \{[x, y, w, \sqrt[4]{\alpha}] \mid x, y, w \in \mathbb{R}\} \simeq \mathbb{R}^3 / \Gamma_{\sqrt[4]{\alpha}}$. The trivialization of $S^+$ over $U$ and $V$ is given by the diffeomorphisms:

\[
\varphi_U : U \to U_1 \times F
\]
\[ \varphi_U([x, y, w, t]) = (\epsilon^{2\pi \text{ilog}_{\alpha}} t, [x, \frac{\dot{y}}{\alpha}, w, \sqrt{\alpha}]) \]
\[ \varphi_V : V \to U_2 \times F \]
\[ \varphi_V([x, y, w, t]) = (\epsilon^{2\pi \text{ilog}_{\alpha}} t, [x, \frac{\dot{y}}{\alpha}, w, \sqrt{\alpha}]) \]

The transition function \( g_{UV} = \varphi_U \circ \varphi_V^{-1} : U \cap V \times F \to U \cap V \times F \) is given by:
\[
g_{UV}(m, \{x, y, w, \sqrt{\alpha}\}) = \begin{cases} 
(m, \{x, y, w, \sqrt{\alpha}\}), & \text{for } m \in W_1 \\
(m, \{x, \alpha y, \frac{1}{\alpha} w, \sqrt{\alpha}\}), & \text{for } m \in W_2
\end{cases}
\]

Differentiably, \( F \) is circle bundle over the two-torus \( \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \). Indeed, let \( p : F \to \mathbb{T}^2 \), defined by
\[
p((x, y, w)) = \left( \frac{a_2 y - b_2 \sqrt{\alpha} w}{\sqrt{\alpha}(b_1 a_2 - a_1 b_2)}, \frac{-a_1 y + b_1 \sqrt{\alpha} w}{\sqrt{\alpha}(b_1 a_2 - a_1 b_2)} \right)
\]

Then \( p \) is a well defined submersion onto \( \mathbb{T}^2 \), whose fiber is the circle \( \mathbb{R}/(x \sim x + \frac{b_1 a_2 - a_1 b_2}{p}) \).

We shall compute the Morse-Novikov cohomology groups of \( S^+ \) with respect to the Lee form of Tricerri’s metric:
\[
\theta = \frac{dt}{t} = \ln \alpha \cdot \pi^* \vartheta.
\]

Since \( b_1(S^+) = 1 \), every closed one-form is, up to an exact factor, a multiple of \( \theta \).

By applying the Mayer-Vietoris sequence to the open sets \( U \) and \( V \), the following is an exact sequence:
\[
0 \to H^0_\theta(S^+) \to H^0_{\theta|U}(U) \oplus H^0_{\theta|V}(V) \to H^0_{\theta|U \cap V}(U \cap V) \to \\
\quad \to \cdots \to H^3_{\theta|U \cap V}(U \cap V) \to 0
\]

As in the case of \( S^0 \), we shall be interested in the morphisms \( H^i_{\theta|U}(U) \oplus H^i_{\theta|V}(V) \xrightarrow{\delta} H^i_{\theta|U \cap V}(U \cap V) \), where \( i = 0, 1, 2, 3 \), which further yield the morphisms \( \gamma_i : H^i_{dR}(F) \oplus H^i_{dR}(F) \to H^i_{dR}(F) \oplus H^i_{dR}(F) \). The case \( i = 0 \) is identical to the one in \( S^0 \), yielding an isomorphism which allows us to consider the Mayer-Vietoris sequence from \( H^0_{\theta}(S^+) \). We want to write down explicitly the morphisms \( ((g_{UV})|W_2) : H^i_{dR}(F) \to H^i_{dR}(F) \).

According to [FGG], for any circle bundle \( F \to \mathbb{T}^2 \), except for the trivial one, \( H^1_{dR}(F) \simeq \mathbb{R}^2, H^2_{dR}(F) \simeq \mathbb{R}^2 \). Moreover, the generators are given by:
\[
H^1_{dR}(F) = \mathbb{R}\langle p^*[\eta_1], p^*[\eta_2] \rangle
\]
\[
H^2_{dR}(F) = \mathbb{R}\langle [\eta \wedge p^*\eta_1], [\eta \wedge p^*\eta_2] \rangle
\]
where \( \eta_1 \) and \( \eta_2 \) are integral closed one-forms on \( \mathbb{T}^2 \), such that \( [\eta_1 \wedge \eta_2] \) generates \( H^2(\mathbb{T}^2) \) and \( \eta \) is the curvature form of \( F \), satisfying \( d\eta = p^*\eta_1 \wedge p^*\eta_2 \).

Let us take \( \eta_1 = dy \) and \( \eta_2 = dw \) written in the coordinates on \( \mathbb{R}^2 \). Since \( (g_{UV}|_{W_2}) : \mathbb{R}^3/\Gamma \sqrt{\alpha} \rightarrow \mathbb{R}^3/\Gamma \sqrt{\alpha} \) is given by

\[
(g_{UV}|_{W_2}(x, y, z)) = (x, \alpha \cdot y, \frac{1}{\alpha} \cdot w),
\]

we obtain the following:

**Claim 4.2:** The matrices of \( (g_{UV}|_{W_2})_* : H^1_{dR}(F) \rightarrow H^1_{dR}(F) \), for \( i = 1, 2 \) in the basis \( \{p^*\eta_1, p^*\eta_2\} \) and \( \{[\eta \wedge p^*\eta_1], [\eta \wedge p^*\eta_2]\} \) are both equal to:

\[
A = T^{-1} \begin{bmatrix} \frac{1}{\alpha} & 0 \\ 0 & \alpha \end{bmatrix} T
\]

where \( T = \begin{bmatrix} b_1 \sqrt{\alpha} & b_2 \sqrt{\alpha} \\ a_1 & a_2 \end{bmatrix} \).

Similar to the computation for \( S^0 \), we can prove that the morphism \( \beta_* : H^1_{\theta|U}(U) \oplus H^1_{\theta|V}(V) \rightarrow H^1_{\theta|U \cap V}(U \cap V) \) yields a linear application

\[
\gamma_i : H^1_{dR}(F) \oplus H^1_{dR}(F) \rightarrow H^1_{dR}(F) \oplus H^1_{dR}(F),
\]

whose matrix in the case \( i = 1, 2 \) is

\[
\begin{bmatrix} I_2 & -I_2 \\ -I_2 & -\alpha \cdot A \end{bmatrix},
\]

whilst for \( i = 3 \) it is

\[
\begin{bmatrix} 1 & -1 \\ 1 & -\alpha \end{bmatrix}.
\]

Since \( \frac{1}{\alpha} \) is an eigenvalue of \( A^{-1} \), we conclude that

**Claim 4.3:** \( \text{Ker} \gamma_1 \) and \( \text{Ker} \gamma_2 \) are both one-dimensional and \( \gamma_3 \) is an isomorphism.

Now, applying the Mayer-Vietoris sequence, we obtain:

**Proposition 4.4:** For \( S^+ = S^+_{N, p, q, r, 0} \) and Tricerri’s Lee form \( \theta = \frac{dt}{t} \), the following holds:

\[
H^1_\theta(S^+) \cong \mathbb{R}, \quad H^2_\theta(S^+) \cong \mathbb{R}^2, \quad H^3_\theta(S^+) \cong \mathbb{R},
\]
and both $H^0_\theta(S^+) \text{ and } H^4_\theta(S^+)$ vanish.

**Remark 4.5:** Note that the result above holds also for $S^+_{N,p,q,r,z}$ with parameter $z \in \mathbb{C}$, but with respect to the one-form which corresponds to $\frac{dt}{t}$ via the diffeomorphism between $S^+_{N,p,q,r,z}$ and $S^+_{N,p,q,r,0}$.

**Remark 4.6:** For $\theta_1 = -\theta$, we obtain by Poincaré duality the cohomology groups

$$H^1_{\theta_1}(S^+) \simeq \mathbb{R}, \quad H^2_{\theta_1}(S^+) \simeq \mathbb{R}^2, \quad H^3_{\theta_1}(S^+) \simeq \mathbb{R}.$$  

Since $b_1(S^+) = 1$, by the same argument used in Corollary 3.4, $H^i_{\theta_1}(S^+) = 0$, if $\theta_1 \neq \pm \theta$.

4.2. Finding generators for $H^2_\theta(S^+_{N,p,q,r,0})$. As in the case of $S^0$, we shall apply the twisted Hodge decomposition.

Let $\zeta = \frac{y dt}{t^2} - dy$. Then $\zeta$ is an invariant form on $\mathbb{C} \times \mathbb{H}$ with respect to the action of the group $G^+$ and defines a one-form on $S^+$. We prove that $\zeta$ is $\Delta_\theta$-harmonic. Indeed, since with respect to Tricerri’s metric:

$$d\text{vol} = \omega \wedge \omega = \frac{dy \wedge dx \wedge dt \wedge dw}{w^2},$$

we obtain:

$$\delta_\theta \zeta = -\ast d_{-\theta} \ast \zeta = -\ast d_{-\theta} \frac{dx \wedge dw \wedge dt}{t^2} = -\ast 0 = 0$$

Moreover, it is easy to see that also $d_\theta \zeta = 0$ holds, therefore $\zeta$ is harmonic with respect to $\Delta_\theta$, hence its class $[\zeta] \in H^1_\theta(S^+)$ doesn’t vanish and thus

$$H^1_\theta(S^+) = \mathbb{R}[\zeta].$$

We observe that $\omega = 2\frac{dt \wedge dw}{t^2} + h$, where

$$h := 2\frac{y^2}{t^2} dt \wedge dw - 2\frac{y}{t}(dt \wedge dx + dy \wedge dw) + 2dy \wedge dx.$$  

Then $h$ is a well defined form on $S^+$, which is $d_\theta$-closed. Moreover,

$$\delta_\theta h = -\ast d_{-\theta} \ast h = -2\ast d_{-\theta} \frac{dt \wedge dw}{t^2} = -2 \ast 0 = 0.$$  

Therefore, $h$ is harmonic and since

$$\frac{dt \wedge dw}{t^2} = d_{\theta}(-\frac{dw}{t}),$$

$\omega = d_{\theta}(-\frac{dw}{t}) + h$ is the twisted Hodge decomposition of $\omega$, it implies that $H^2(S^+ + \omega = [h] \neq 0$.

Let now $\tau := \frac{dy \wedge dt}{t}$. This is a well defined form on $S^+$, which is $d_\theta$-closed.

$$\delta_\theta \tau = -\ast d_{-\theta} \ast \tau = -\ast d_{-\theta} \frac{dx \wedge dw}{t} = -\ast 0 = 0.$$
So \( \tau \) is also harmonic and since it is not a multiple of \( h \), we get

\[
H^2_\theta(S^+) = \mathbb{R}([\tau], [h]).
\]

Lastly, we know from [G], that \( \theta \wedge \omega \) is harmonic, hence \( 0 \neq [\theta \wedge \omega] \in H^3_\theta(S^+) \) and

\[
H^3_\theta(S^+) = \mathbb{R}([\theta \wedge \omega]).
\]

Note that Remark 4.5 applies here too.

4.3. The complex surface \( S^- \). Let \( N \in \text{GL}_2(\mathbb{Z}) \) with \( \det N = -1 \) and eigenvalues \( \alpha > 1 \) and \( -\frac{1}{\alpha} \). We consider \((a_1, a_2)^t\) and \((b_1, b_2)^t\) real eigenvectors corresponding to \( \alpha \) and \( -\frac{1}{\alpha} \) and let \( c_1, c_2 \) be defined by

\[
-(c_1, c_2) = (c_1, c_2) \cdot N + (c_1, c_2) + \frac{b_1a_2 - b_2a_1}{r}(p, q),
\]

where \( e_1 \) and \( e_2 \) are defined as in the case of \( S^+ \) and \( p, q, r \) \( (r \neq 0) \) are integers. We denote by \( G^-_{N, p, q, r} \) the group of affine transformations of \( \mathbb{R}^3 \times \mathbb{R}^+ \) generated by:

\[
g_0(x, y, w, t) = (-x, -y, \alpha w, \alpha t) \quad g_i(x, y, w, t) = (x + b_iw + c_i, y + b_it + w + a_i, t), \quad i = 1, 2 \
g_3(x, y, w, t) = (x + \frac{b_1a_2 - b_2a_1}{r}, y, w, t).
\]

The complex surface \( S^-_{N, p, q, r} \) is defined to be \( \mathbb{R}^3 \times \mathbb{R}^+ / G^-_{N, p, q, r} \). Since \( \langle g_0^2, g_1, g_2, g_3 \rangle = G^+_{N2, p_1, q_2, r, 0} \) for some integer numbers \( p_1 \) and \( p_2 \), the following is immediate:

**Claim 4.7:** (1) \( S^+_{N2, p_1, q_2, r, 0} \) is a double unramified covering of \( S^-_{N, p, q, r} \).

Let

\[
\omega_1 = 2 \frac{1 + y^2}{t^2} dt \wedge dw - 2 \frac{y}{t}(dt \wedge dx + dy \wedge dw) + 2dy \wedge dx.
\]

Then \( \omega_1 \) defines an LCK metric on \( S^-_{N, p, q, r, 0} \) with Lee form \( \theta_1 = \frac{\omega_1}{4\pi} \). We are interested in the Morse-Novikov cohomology of \( S^-_{N, p, q, r, 0} \) with respect to \( \theta_1 \).

Let \( \pi : S^+_{N2, p_1, q_2, r, 0} \to S^-_{N, p, q, r} \) be the covering map given by \([x, y, w, t] \mapsto [[x, y, w, t]]\), where we make distinctions between the equivalence classes with respect to the two factorizations. In other words,

\[
S^-_{N, p, q, r} \simeq S^+_{N2, p_1, q_2, r, 0} / (\text{id}, \sigma),
\]

where \( \sigma : S^+_{N2, p_1, q_2, r, 0} \to S^+_{N2, p_1, q_2, r, 0} \) is the involution

\[
\sigma([x, y, w, t] = [-x, -y, \alpha w, \alpha t]).
\]
Since $\pi^* \theta_1 = \theta$, like in the case of de Rham cohomology, there is an injection $H^i_{\theta_1}(S^+_{N,p,q,r}) \hookrightarrow H^i_{\theta}(S^+_{N^2,p,q,r})$. Therefore, the following inequalities hold:

$$\dim_{\mathbb{R}} H^1_{\theta_1}(S^-_{N,p,q,r}) \leq 1, \quad \dim_{\mathbb{R}} H^2_{\theta_1}(S^-_{N,p,q,r}) \leq 2, \quad \dim_{\mathbb{R}} H^3_{\theta_1}(S^-_{N,p,q,r}) \leq 1.$$

As the other Inoue surfaces, $S^-_{N,p,q,r}$ is also a fiber bundle over $S^1$. Let $\pi : S^-_{N,p,q,r} \to S^1$ be the submersion given by $[x, y, w, t] \mapsto e^{2\pi \log_\alpha t}$, which endows $S^-_{N,p,q,r}$ with the structure of a fiber bundle over $S^1$ whose fiber is $F = \mathbb{R}^3 / \Gamma \sqrt[4]{\alpha}$ and transition function $g_{UV} : U \cap V \times F \to U \cap V \times F$ is given by:

$$g_{UV}(m, [x, y, w, \sqrt[4]{\alpha}]) = \begin{cases} (m, [x, y, w, \sqrt[4]{\alpha}]), & m \in W_1 \\ (m, [-x, -\alpha y, \frac{1}{\alpha} w, \sqrt[4]{\alpha}]), & m \in W_2 \end{cases}.$$

We need to check as in the other two examples of Inoue surfaces the matrices of $((g_{UV})_{W_2})_* : H^i_{dR}(F) \to H^i_{dR}(F)$. For $i = 1$, in the base $\{p^* \eta_1, p^* \eta_2\}$, $(g_{UV})_{W_2})_*$ corresponds to the matrix

$$A = T^{-1} \begin{bmatrix} -\frac{2}{\alpha} & 0 \\ 0 & \alpha \end{bmatrix} T$$

where $T$ is the same as in the case of $S^+$ and for $i = 2$, in the base $\{[\eta \wedge p^* \eta_1], [\eta \wedge p^* \eta_2]\}$, $(g_{UV})_{W_2})_*$ corresponds to the matrix:

$$B = T^{-1} \begin{bmatrix} \frac{1}{\alpha} & 0 \\ 0 & -\alpha \end{bmatrix} T.$$

They further yield in the Mayer-Vietoris sequence the linear applications $\gamma_i : H^i_{dR}(F) \oplus H^i_{dR}(F) \to H^i_{dR}(F) \oplus H^i_{dR}(F)$, whose matrices are for $i = 1$:

$$\begin{bmatrix} I_2 & -I_2 \\ I_2 & -\alpha \cdot A \end{bmatrix},$$

for $i = 2$:

$$\begin{bmatrix} I_2 & -I_2 \\ I_2 & -\alpha \cdot B \end{bmatrix}$$

and for $i = 3$:

$$\begin{bmatrix} 1 & -1 \\ 1 & -\alpha \end{bmatrix}.$$

As $\alpha$ is an eigenvalue of $B^{-1}$, but not of $A^{-1}$, $\gamma_1$ and $\gamma_3$ are isomorphisms and $\ker \gamma_2$ is one-dimensional. From the Mayer-Vietoris sequences, we obtain:

**Proposition 4.8:** For every $S^- = S^-_{N,p,q,r}$ and $\theta = \frac{4t^3}{t}$, the following holds:

$$H^2_\theta(S^-) \simeq \mathbb{R}, \quad H^3_\theta(S^-) \simeq \mathbb{R},$$

$$H^1_\theta(S^-) \simeq \mathbb{R}, \quad H^2_\theta(S^-) \simeq \mathbb{R}, \quad H^3_\theta(S^-) \simeq \mathbb{R}.$$
and \( H^i_\theta(S^-) \) vanish, for all \( i = 0, 1, 4 \).

Since the same proof works as for \( S^+ \) to show that \( \omega_1 \) is not \( d_\theta \)-exact and we know from [G] that \( \theta \wedge \omega \) is harmonic, we get:

**Corollary 4.9:** \( H^2_\theta(S^-) = \mathbb{R}[\omega_1] \) and \( H^3_\theta(S^-) = \mathbb{R}[\theta \wedge \omega_1] \).

**Remark 4.10:** Another way of computing the Morse-Novikov cohomology groups of Inoue surfaces is provided in [M], where this cohomology is considered for solvmanifolds. More precisely, the author shows that by multiplying an invariant Lee form with any real number, one obtains vanishing Morse-Novikov cohomology, except for some values which are eigenvalues of a certain operator, which closely resembles our computations.

### 4.3.1. Nonexistence of LCK metrics with potential.

**Remark 4.11:** According to [AD], the set of Lee classes of LCK metrics on \( S^+ \) and \( S^- \) has at most one element, namely for \( S = S^\pm_{N,p,q,r} \), \( S^\pm_{N,p,q,r} \) with \( z \in \mathbb{R} \), \( C(S) = \mathcal{T}(S) = \{[\theta]\} \) and for \( z \in \mathbb{C} \setminus \mathbb{R} \), \( C(S^\pm_{N,p,q,r,z}) = \emptyset \) and \( \mathcal{T}(S^\pm_{N,p,q,r,z}) = \{[\theta]\} \), where \( \theta \) is the Lee form of Tricerri’s metric.

From the remark above and [Theorem 3.11] we now derive:

**Corollary 4.12:** The Inoue surfaces have no LCK metric with potential.

**Proof.** In [AD] Lemma 3.7] it is shown that on a compact complex manifold, if \( g \) is an LCK metric with potential with the Lee form \( \theta \), then for any \( t \geq 1 \), \( t\theta \) is also the Lee form of an LCK metric with potential. However, by [Theorem 3.11] and the results in [AD], on the Inoue surfaces, \( t\theta \), where \( \theta \) is the one-form considered above, cannot be the Lee form of an LCK metric for any \( t \in \mathbb{R} \setminus \{1\} \), therefore there is no LCK metric with potential on all \( S^0, S^+, S^- \).

We recall that for \( S^0 \) we proved in [Claim 3.13] a stronger result, namely that it cannot admit LCK structures \((g, \omega, \theta)\) \( d_\theta \)-exact two-form \( \omega \). By using a similar argument we can prove more:

**Proposition 4.13:** On the surfaces \( S^\pm_{N,p,q,r,0} \) and \( S^- \) there exist no LCK metrics which are \( d_\theta \)-exact.

**Proof.** We shall use the solvmanifold structure of \( S^\pm \) and \( S^- \). In [H], \( S^\pm \) are described as solvmanifolds \( G/\Gamma \), where \( G \) is a solvable Lie group with Lie algebra generated by \( \{e_1, e_2, e_3, e_4\} \) satisfying:
\[ [e_2, e_3] = -e_1, \quad [e_2, e_4] = -e_2, \quad [e_3, e_4] = e_3. \]

The standard complex structure \( J \) is \( G \)-left-invariant and satisfies:
\[
J e_1 = e_2, \quad J e_2 = -e_1, \quad J e_3 = e_4 - ae_2, \quad J e_4 = -e_3 - ae_1,
\]
where \( a \in \mathbb{R} \). The form \( \theta \) is the dual of the left-invariant vector field \( e_4 \).

By contradiction, assume there exists \( \omega = d\theta \eta \) an LCK form on \( S^+ \) or \( S^- \). By [S Proposition 1.2], we may choose a left-invariant form \( \eta_0 \) such that \( d\theta \eta_0 \) is still LCK. Then
\[
d\theta \eta_0(e_1, Je_1) = -\eta([e_1, e_2]) - \theta \wedge \eta_0(e_1, e_2) = 0,
\]
contradicting, thus, the fact that \( d\theta \eta_0 \) is the fundamental form of a Hermitian metric.

**Remark 4.14:** The nonexistence of \( d\theta \)-exact LCK metrics on some manifolds which cannot admit LCK metrics with potential is related to [OVV, Conjecture 1.5], which states that on a compact manifold, a \( d\theta \)-exact LCK form is actually with potential.

5. Morse-Novikov cohomology of other LCK surfaces

We briefly discuss the Morse-Novikov cohomology of other compact complex surfaces which are known to admit LCK metrics.

Since the blow-up of a manifold at a point is LCK if and only if the manifold itself is LCK (see [Vu]), we are only interested in the minimal model of the surface (i.e. not containing smooth rational curves of self-intersection -1).

LCK metrics have been found in both classes of non-Kähler surfaces, VI and VII (see [Kod]), whose minimal models are denoted by VI\(_0\) and VII\(_0\). They are the only classes of surfaces in which LCK metrics may exist.

The known examples of LCK surfaces among class VI\(_0\) are properly elliptic surfaces and Kodaira surfaces and are actually Vaisman (see [Be]). Therefore, by Remark 1.1 the Morse-Novikov cohomology with respect to the Lee forms of Vaisman metrics vanishes.

Class VII\(_0\) consists of minimal complex surfaces with \( b_1 = 1 \) and Kodaira dimension \(-\infty\). It further divides into two subclasses, namely, with \( b_2 = 0 \) and \( b_2 > 0 \). In the first case, the classification is complete (see [B3]). They are either Inoue surfaces (for which we computed the Morse-Novikov cohomology in this paper) or Hopf surfaces, for which we can conclude:

**Proposition 5.1:** The Hopf surfaces have vanishing Morse Novikov cohomology with respect to any closed one-form.
Proof. Recall that the Hopf surfaces are finitely covered by $S^1 \times S^3$ and the Morse-Novikov cohomology reflects the topology and not the complex structure of a manifold. Moreover, as $b_1 = 1$, all closed one-forms are proportional (with a real multiplicative factor), up to an exact one-form, and thus they are parallel with respect to the natural product metric, and hence Remark 1.1 yields the vanishing of the twisted cohomology. ■

As regards class VII$_0$ with $b_2 > 0$, the only known examples are the Kato surfaces. They were introduced in [Kat] and in [N1] it was proven that they can be deformed as complex surfaces to a blow-up at finitely many points of the Hopf surface $S^1 \times S^3$. In particular, they are diffeomorphic to $(S^1 \times S^3)^n \# n\mathbb{CP}^1$, where $n$ is the number of blown-up points. In fact, a stronger result was proved by Nakamura in [N2], where it is shown that any surface from class VII$_0$ with a cycle of rational curves is a complex deformation of a blow-up of a Hopf surface.

By [Br], all Kato surfaces carry LCK metrics. Since $b_1 = 1$, as above, all the closed one-forms are proportional (with a real factor), up to an exact one-form, and identify with the pullback on $(S^1 \times S^3)^n \# n\mathbb{CP}^1$ of the multiples of the volume form of the circle $S^1$.

It was shown in [FP, Lemma 4.2] that for any closed one-form $\theta$ on a Kato surface $S$ (and more generally on a surface of class VII$_0$ with a cycle of rational curves), we have $H^2_\theta(S) \simeq \mathbb{R}^{b_2(S)}$ and the rest of the Morse-Novikov cohomology groups vanish.

Remark 5.2: The result in [FP, Lemma 4.2] also follows from the following relation proven in [YZ], between the Morse-Novikov cohomology groups of a compact surface and its blow-up at a point:

$$H^2_{\pi^*\theta}(\tilde{M}_p) \simeq H^2_\theta(M) \oplus \mathbb{R},$$

$$H^i_{\pi^*\theta}(\tilde{M}_p) \simeq H^i_\theta(M), \quad i \neq 2.$$

where $\pi : \tilde{M}_p \to M$ is the blow-up of $M$ at the point $p$ (which was, in fact, proven in the more general case of a $n$-dimensional manifold and for the blow-up along a submanifold). One now takes $M$ to be $S^1 \times S^3$ and $\theta$ any real multiple of $\vartheta$ (which denotes the volume form of the circle). Since for any $i \geq 0$, $H^i_\vartheta(S^1 \times S^3) = 0$, one reobtains the cited result.

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REFERENCES

[AD] V. Apostolov, G. Dloussky, Locally Conformally Symplectic Structures on Compact Non-Kähler Complex Surfaces, Int. Math. Res. Not., 9 (2016), 2717-2747.

[B] A. Banyaga, On the geometry of locally conformal symplectic manifolds, Infinite Dimensional Lie Groups in Geometry and Representation Theory, 79-91, World Scientific Publishing, 2002.

[Be] F.A. Belgun, On the metric structure of non-Kähler complex surfaces, Math. Ann. 317 (2000), 1–40.

[Bo] F. A. Bogomolov, Classification of surfaces of class VII0 with b2 = 0, Math. USSR-Izv., 10 (1976), 255–269.

[Br] M. Brunella, Locally conformal Kähler metrics on Kato surfaces, Nagoya Math. J. 202 (2011), 77-81.

[BK] G. Bande, D. Kotschick, Moser stability for locally conformally symplectic structures, Proc. Amer. Math. Soc. 137 (2009), 2419–2424.

[BM] G. Bazzoni, J. Marrero, Locally conformal symplectic nilmanifolds with no locally conformal Kähler metrics, arXiv:1407.5510.

[DG] G. Bande, D. Kotschick, Moser stability for locally conformally symplectic structures, Proc. Amer. Math. Soc. 137 (2009), 2419–2424.

[Be] F.A. Belgun, On the metric structure of non-Kähler complex surfaces, Math. Ann. 317 (2000), 1–40.

[Bo] F. A. Bogomolov, Classification of surfaces of class VII0 with b2 = 0, Math. USSR-Izv., 10 (1976), 255–269.

[Br] M. Brunella, Locally conformal Kähler metrics on Kato surfaces, Nagoya Math. J. 202 (2011), 77-81.

[BK] G. Bande, D. Kotschick, Moser stability for locally conformally symplectic structures, Proc. Amer. Math. Soc. 137 (2009), 2419–2424.

[Be] F.A. Belgun, On the metric structure of non-Kähler complex surfaces, Math. Ann. 317 (2000), 1–40.

[Bo] F. A. Bogomolov, Classification of surfaces of class VII0 with b2 = 0, Math. USSR-Izv., 10 (1976), 255–269.

[Br] M. Brunella, Locally conformal Kähler metrics on Kato surfaces, Nagoya Math. J. 202 (2011), 77-81.

[BM] G. Bazzoni, J. Marrero, Locally conformal symplectic nilmanifolds with no locally conformal Kähler metrics, arXiv:1407.5510.

[D] A. Dimca, Sheaves in Topology, Springer Verlag, 2004.

[DO] S. Dragomir, L. Ornea, Locally conformal Kähler geometry, Birkhäuser, 1998.

[F] M. Farber, Topology of closed one-forms, Amer. Math. Soc. vol 108, 2004.

[FGG] M. Fernández, M. Gotay, A. Gray, Compact parallelizable four dimensional symplectic and complex manifolds, Proc. Amer. Math. Soc. 103 (1988), 1209-1212.

[FP] A. Fujiki, M. Pontecorvo, Bi-Hermitian metrics on Kato surfaces, preprint arXiv:1607.00192.

[G] R. Goto, On the stability of locally conformal Kähler structures, J. Math. Soc. Japan 66 (2014), no. 4, 1375–1401.

[GH] P. Griffiths, J. Harris, Principles of algebraic geometry, Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. 1978

[GL] F. Guedira, A. Lichnerowicz, Géométrie des algèbres de Lie locales de Kirilov, J.Math. Pures et Appl. 63 (1984), 407–484.

[HR] S. Haller, T. Rybicki, On the group of diffeomorphisms preserving a locally conformal symplectic structure Ann. Global Anal. Geom. 17 (1999), no. 5, 475–502.

[H] K. Hasegawa, Complex and Kähler structures on compact solvmanifolds, J. Symplectic Geom. 3 (2005), no. 4, 749–767.

[I] M. Inoue, On surfaces of class VII0, Invent. Math., 24 (1974), 269-310.

[Kam] Y. Kamishima, Note on locally conformal Kähler surfaces, Geom. Dedicata, 84 (2001), 115–124.

[Kat] M. Kato (1978), Compact complex manifolds containing “global” spherical shells. I, Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), pp. 45–84, Kinokuniya Book Store, Tokyo, 1978.

[Kod] K. Kodaira, On the structure of compact complex analytic spaces, Amer. J. Math. : I, 86 (1964), 751–798; II, 88 (1966), 682–721; III, 90 (1969), 55–83; IV, 90 (1969), 1048–1066.

[LLMP] M. de León, B. López, J.C. Marrero, E. Padrón, On the computation of the Lichnerowicz-Jacobi cohomology, J. Geom. Phys. 44 (2003), 507–522.

[M] D.V. Millionshchikov, Cohomology of solvable Lie algebras, and solvmanifolds (Russian) Mat. Zametki 77 (2005), no. 1, 67–79; translation in Math. Notes 77 (2005), no. 1-2, 61–71.
[N1] I. Nakamura, Classification of non-Kähler complex surfaces (Japanese) Translated in Sugaku Expositions 2 (1989) 209–229. Sugaku 36 (1984), no. 2, 110–124.
[N2] I. Nakamura, On surfaces of class VII$_0$ with curves II, Tohoku J. Math. 42 (1990), 475–516.
[Nov1] S. P. Novikov, Multi-valued functions and functionals. An analogue of Morse theory, Soviet Math. Doklady, 24 (1981), 222–226.
[Nov2] S. P. Novikov, The Hamiltonian formalism and a multi-valued analogue of Morse theory, Russian Math. Surveys, 37 (1982), 1–56.
[OG] L. Ornea, P. Gauduchon, Locally conformal Kähler metrics on Hopf surfaces, Annales de l’Institut Fourier, 48 (1998), 1107-1127.
[OV1] L. Ornea, M. Verbitsky, LCK rank of locally conformally Kähler manifolds with potential, J. Geom. Phys., 107 (2016), 92–98.
[OVV] L. Ornea, M. Verbitsky, V. Vuletescu, Weighted Bott-Chern and Dolbeault cohomology for LCK manifolds with potential, preprint arXiv:1504.01501
[OT] ] K. Oeljeklaus, M. Toma, Non-Kähler compact complex manifolds associated to number fields, Ann. Inst. Fourier, Grenoble, 55 (2005), no. 1, 161–171.
[P] A. V. Pazhitnov, An analytic proof of the real part of Novikov’s inequalities, Soviet Mat. Dokl. 35 (1987), 456–457.
[S] H. Sawai, Locally conformal Kähler structures on compact solvmanifolds, Osaka J. Math. 49 (2012), 1087-1102.
[Tr] F. Tricerri, Some examples of locally conformal Kähler manifolds, Rend. Dem. Mat. Univers. Politecn. Torino 40 (1) (1982), 81–92.
[Va1] I. Vaisman, Remarkable operators and commutation formulas on locally conformally Kähler manifolds, Compos. Math., 40 (1980), no. 3, 277–289.
[Va2] I. Vaisman, Locally conformal symplectic manifolds, Int. J. Math. Math. Sci. 8 (3) (1985), 521–536.
[Vu] V. Vuletescu, Blowing-up points on locally conformally Kähler manifolds, Bull. Math. Soc. Sci. Math. Roumanie 52(100) (2009), 387–390.
[YZ] X. Yang, G. Zhao, A note on the Morse-Novikov cohomology of blow-ups of locally conformal Kähler manifolds, Bull. Aust. Math. Soc. 91 (2015) 155–166.