POINCARÉ POLYNOMIALS OF THE DEGENERATE FLAG VARIETIES OF TYPE C

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ABSTRACT. We give several combinatorial models for the Poincaré polynomials of the symplectic degenerate flag varieties. The models are based on the dimension counting of the explicitly described cells, extended and symplectic Dellac configurations, and on generalized Motzkin paths.

INTRODUCTION

The goal of this paper is to give several combinatorial models for the Poincaré polynomials of the symplectic degenerate flag varieties [FFL]. The Betti numbers $r_n$, $n = 0, 1, 2, \ldots$ of these singular projective varieties start with $1, 2, 10, 98, 1594$ (OEIS1). These numbers have several combinatorial interpretations [B1, FFL, FF]. Our goal is to show that the Poincaré polynomials provide rich and fruitful q-version of these numbers.

The PBW degeneration of classical flag varieties of type $A$ was introduced in [F1] in Lie theoretic terms. The resulting varieties $F^a_N$ have very explicit linear algebra description. Namely, let $W$ be an $N$-dimensional vector space with a basis $e_1, \ldots, e_N$ and let $pr_k : W \rightarrow W$ be the projection along the $k$-th basis vector to the linear span of the rest basis vectors. Then $F^a_N$ consists of collections $(V_1, \ldots, V_{N-1})$ of subspaces of $W$ such that $\dim V_k = k$ and $pr_{k+1}V_k \subset V_{k+1}$. Theses varieties are singular, but share many nice properties with their classical analogues, see [F1] [F2] [FF] [H, LS]. It was shown that the varieties $F^a_N$ can be realized as Schubert varieties [CL, CLL] and as quiver Grassmannians [CFR1]. The Betti numbers of the varieties $F^a_N$ are given by the normalized median Genocchi numbers $1, 2, 7, 38, 295, \ldots$ [Bal DR DZ B2 B3 B4 Del Dui F2]. The Poincaré polynomials of the degenerate flag varieties thus provide natural $q$-analogues of the normalized median Genocchi numbers (see [F1] [F2] [HZ1] [HZ2] [ZZ]).

The PBW degeneration procedure works for arbitrary simple Lie groups. The case of $Sp_{2n}$ was considered in [FFL]. It was shown that the corresponding degenerate flag varieties $Sp^a_{2n}$ can be explicitly described as follows. Let $W$ be a $2n$-dimensional vector space with a basis $e_1, \ldots, e_{2n}$ and a non-degenerate skew-symmetric form defined by $(e_i, e_{2n+1-i}) = 1$ for $1 \leq i \leq n$. Then $Sp^a_{2n}$ is defined as a subvariety of $F^a_{2n}$ consisting of collections $(V_1, \ldots, V_{2n-1})$ such that $V_k = V^\perp_{2n-k}$. The combinatorics of the torus fixed points counting for $Sp^a_{2n}$ is described in [B1] [FF]. In particular, it
was shown in [B1] that the number of torus fixed points in $Sp^a_{2n}$ (a.k.a. the Betti number) can be expressed in terms of the surjective pistols of [RZ].

The main objects of our paper are the polynomials $P_{Sp^a_{2n}}(q)$ – the Poincaré polynomials of $Sp^a_{2n}$.

In order to compute $P_{Sp^a_{2n}}(q)$ we first describe the cellular decomposition of $Sp^a_{2n}$. More precisely, we show that the intersections of the standard cells of $F^a_{2n}$ with the symplectic degenerate flags produce the desired cellular decomposition. In other words, the intersection of a cell in $F^a_{2n}$ with $Sp^a_{2n}$ is either empty or a cell. The resulting (symplectic) cells are labeled by the collections $(I_1, \ldots, I_n)$ of subsets of $\{1, \ldots, 2n\}$ such that $|I_k| = k$ and $I_k \subset I_{k+1} \cup \{k+1\}$ (these are the type $A$ conditions) and one extra condition: $a \in I_n$ implies $2n + 1 - a \notin I_n$. Our first result is an algorithm for computing dimension of the cell attached to a collection $(I_k)_{k=1}^n$ as above.

In [FF] the authors defined a combinatorial object called the symplectic Dellac configuration. In short, this is a $n \times 2n$ board with marked boxes satisfying certain conditions. The symplectic Dellac configurations are in one-to-one correspondence with the cells in $Sp^a_{2n}$. Our second result is an explicit formula for the dimension of the cell attached to a symplectic Dellac configuration. The formula is written in terms of the number of inversions in a configuration. As a result we obtain an explicit formula for the Poincaré polynomial $P_{Sp^a_{2n}}(q)$ as a sum over the symplectic Dellac configurations of certain powers of $q$.

Recall that in [B1] the author gave a formula for the Betti numbers $r_n = P_{Sp^a_{2n}}(1)$ expressing each number $r_n$ as a sum over the extended Dellac configurations $T_n$ of certain powers of 2. We prove the enhanced version of this formula. More precisely, we show that $P_{Sp^a_{2n}}(q)$ is equal to the sum over $T \in T_n$ of the expressions of the form $q^{inv(T)} \prod_{k \geq 1} (1 + q^k)^{n_k(T)}$ for certain statistics $inv(T)$ and $n_k(T)$ on the set of extended Dellac configurations.

Our final result is a type $C$ version of a formula from [CFR1], expressing the Poincaré polynomials of the degenerate flag varieties in type $A$ as a sum over the Motzkin paths of certain products of $q$-binomial coefficients (multiplied by certain powers of $q$). This formula is obtained by cutting the variety $F_n$ into parts labeled by the length $n$ Motzkin paths such that each path is fibered over the product of Grassmannians with a fiber being an affine space. A similar approach in the symplectic case leads to a formula for the Betti numbers of $Sp^a_{2n}$ [CFR2]. The main difference is that the Motzkin paths have to be generalized by allowing a path to terminate at any point of the upper right quadrant. We show that the symplectic degenerate flags can be cut out into parts labeled by the generalized Motzkin paths such that each part is fibered over a product of Grassmannians with a fiber being an affine space. We thus obtain a $q$-version of the $T$-fixed point counting formula from [CFR2].

Our paper is organized as follows. In §1 we recall the definition and main properties of the degenerate flag varieties. In §2 we describe cellular
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We decompose in type C and give an algorithm to compute the dimensions of the cells (Theorem 2.5). We then interpret the algorithm combinatorially in terms of the symplectic Dellac configurations (Theorem 2.7). In [3] we derive a formula for the Poincaré polynomials of the symplectic degenerate flag varieties as a sum over the extended Dellac configurations (Theorem 3.3).

In §3 we compute the Poincaré polynomials in terms of the generalized Motzkin paths (Theorem 4.3).

Finally, we give several starting examples of the Poincaré polynomials:

1. Degenerate flag varieties

We describe the definitions and basic properties of the degenerate flag varieties in types A and C.

1.1. Type A. Let us fix a basis $e_1,\ldots,e_N$ of a vector space $\mathbb{C}^N$ and projections $pr_k: \mathbb{C}^N \rightarrow \mathbb{C}^N$ such that $pr_k e_l = (1 - \delta_{k,l})e_l$. Let $Gr(k,N)$ be the Grassmannian of $k$-dimensional subspaces in $\mathbb{C}^N$. By definition, $F_a^N = \{ (V_1,\ldots,V_{N-1}) : V_k \in Gr(k,N), pr_{k+1}V_k \in V_{k+1} \}$.

The dimension of $F_a^N$ is $N(N-1)/2$ and the Euler characteristic is given by the normalized median Genocchi number. More precisely, the variety enjoys a cellular decomposition and all the cells are of even (real) dimension. The cells are labeled by the collections $I = (I_1,\ldots,I_{N-1})$ consisting of subsets of the set $\{1,\ldots,N\}$ subject to the conditions

$$|I_k| = k, \ I_k \subset I_{k+1} \cup \{k+1\}.$$ 

Each such a collection corresponds to a torus fixed point $p(I) \in F_a^N$ and is contained in a unique cell $C(I)$. In order to describe the cells we prepare some notation. Let us introduce $N-1$ different orderings $>_k$, $k = 1,\ldots,N-1$ on the set $\{1,\ldots,N\}$. Namely, for $1 \leq k \leq N-1$ we set

$$k >_k k-1 >_k \cdots >_k 1 >_k N >_k N-1 >_k \cdots >_k k + 1.$$ 

The cell $C(I)$ consists of collections of subspaces $(V_1,\ldots,V_{N-1})$ such that for all $k$ the space $V_k$ has a basis of the form (see [EI])

$$e_i + \sum_{b <_{k+1} b, b \notin I_k} x_{a,b} e_b, \ a = 1,\ldots,k, \ i_a \in I_k, \ x_{a,b} \in \mathbb{C}.$$ 

As a consequence, we obtain the following formula for the complex dimension of $C(I)$. Let us call an element $i \in I_k$ initiating, if either $i = k$ or $i \notin I_{k-1}$
(we assume that \( I_0 = \emptyset \)). Let \( I D_k \subset I_k \) be the set of initiating elements. Then one has

\[
\dim C(I) = \sum_{k=1}^{N-1} \sum_{i \in ID_k} |\{j < k : j \notin I_k\}|.
\]

**Remark 1.1.** Formula (1.3) can be easily derived from the explicit description (1.2) by computing the number of free parameters.

**Remark 1.2.** Let us consider the equioriented type \( AN_{n-1} \) quiver and its representation \( M = P \oplus I \), which is the direct sum of all indecomposable projective representations and all indecomposable injective representations (see [CFP1]). In particular, \( \dim M_k = N, 1 \leq k \leq N-1 \) and the map \( M_k \to M_{k+1} \) is given by \( pr_{k+1} \) (after fixing appropriate bases in \( M_k \)). Then each \( I \) gives rise to a subrepresentation \( S(I) \) of \( M \) of dimension \( 1, \ldots, N-1 \): the \( S(I)_k \) is spanned by \( e_a \), \( a \in I_k \). Then the initiating elements are in one-to-one correspondence with the indecomposable summands of \( S(I) \).

Let us note that all the results above generalize to the case of partial flag varieties. Namely, given a collection \( d = (d_1, \ldots, d_s) \) with \( 1 \leq d_1 < \cdots < d_s \leq N-1 \) one defines the partial degenerate flag variety \( F^a_d \) as the projection of \( F^a_N \) to the product \( \prod_{i=1}^s \text{Gr}(d_i, N) \). One easily sees that each cell \( C(I) \) in \( F^a_N \) projects to a cell \( C(I_d) \) in \( F^a_d \) thus providing a cellular decomposition of \( F^a_d \) (here \( I_d = (I_{d_1}, \ldots, I_{d_s}) \)). The dimension of the cell \( C(I_d) \) is computed by the formula (1.3) with the only reservation that the sum in the right hand side of (1.3) is taken over \( k \in \{d_1, \ldots, d_s\} \).

**Remark 1.3.** In what follows we only use the case \( N = 2n, d = (1, \ldots, n) \).

Finally, recall [F1] that a convenient way to parametrize the cells of \( F^a_N \) goes through the Dellac configurations \( DC_N \) [De]. A Dellac configuration \( D \in DC_N \) is a tableau made of \( N \) columns and \( 2N \) rows, containing \( 2N \) points such that:

- every row contains exactly one point;
- every column contains exactly two points;
- if there is a point in the box \((j, i)\) of \( D \) (i.e., in its \( j \)-th column from left to right and its \( i \)-th row from bottom to top), then \( j \leq i \leq N+j \).

If a box \((j, i)\) of a configuration contains a point \( p \), we say that \( p = (j, i) \). In [F1], the author proved that if \( D \in DC_N \) corresponds to a given cell \( C \) of \( F^a_N \), then \( \dim(C) \) is the number of inversions of \( D \), i.e., the number of pairs \((j, i), (j', i')\) of points of \( D \) such that \( j < j' \) and \( i > i' \). For example, in Figure 1 we depict the 7 elements of \( DC_3 \), whose inversions are represented by segments, computing the Poincaré polynomial \( P_{F^3_3}(q) = 1 + 2q + 3q^2 + q^3 \).

### 1.2. Type C

Now let us define the symplectic degenerate flag variety \( SpF^a_{2n} \). We fix a symplectic (skew-symmetric nondegenerate) form on \( \mathbb{C}^{2n} \)
by setting \((e_k, e_{2n+1-k}) = 1, 1 \leq k \leq n\). Let \(f_k : \mathbb{C}^{2n} \to \mathbb{C}^{2n}\) be the linear map defined by

\[
(1.4) \quad f_k e_l = \begin{cases} 
  e_l, & \text{if } l \leq k \text{ or } l \geq 2n + 1 - k, \\
  0, & \text{otherwise}.
\end{cases}
\]

In particular, the image of \(f_k\) is the \(2k\)-dimensional space

\[W_k = \text{span}(e_1, \ldots, e_k, e_{2n-k+1}, \ldots, e_{2n}).\]

We can restrict our symplectic form from \(\mathbb{C}^{2n}\) to \(W_k\) and obtain nondegenerate skew-symmetric form on \(W_k\).

**Remark 1.4.** \(W_n\) is the whole space \(\mathbb{C}^{2n}\) and \(f_n\) is the identity map.

We denote by \(SpGr^a(k, 2n) \subset \text{Gr}(k, 2n)\) the following subvariety of the Grassmannian:

\[SpGr^a(k, 2n) = \{ V \in \text{Gr}(k, 2n) : f_k V \text{ is Lagrangian in } W_k \} \]

In other words, the form evaluated on any two vectors from the image \(f_k(V)\) should vanish. In particular, \(SpGr^a(n, 2n)\) consists of all Lagrangian subspaces of \(\mathbb{C}^{2n}\).

We define the symplectic degenerate flag variety \(SpF^a_{2n}\) as follows (see [FFL]):

\[SpF^a_{2n} = \{(V_1, \ldots, V_n) : V_k \in SpGr^a(k, 2n), pr_{k+1}V_k \subset V_{k+1}\}.\]

We note that \(\dim SpF^a_{2n} = n^2\) and the Euler characteristic is given by the number of symplectic Dellac configurations \(S \in SpDC_{2n}\), defined in [FF] as the elements \(S \in DC_{2n}\) such that \(R(S) = S\) where \(R\) is the central reflexion with respect to the center of \(S\) (in other words \(SpDC_{2n}\) is the subset of \(DC_{2n}\) made of the elements that contain a point in a box \((j, i) \in \{1, \ldots, 2n\} \times \{1, \ldots, 4n\}\) if and only if the box \((2n + 1 - j, 4n + 1 - i)\) also contains a point). See for example Figure 4 in which the 10 elements of \(SpDC_4\) are depicted.

**Remark 1.5.** One can show that if \(V_n \in SpGr^a(n, 2n)\) and \(pr_{k+1}V_k \subset V_{k+1}\) for \(k = 1, \ldots, n-1\) then \(V_k \in SpGr^a(k, 2n)\) for all \(k\).

**Remark 1.6.** We have two natural embeddings of \(SpF^a_{2n}\) into the type A degenerate flag varieties. First, one can embed \(SpF^a_{2n}\) into \(F^a_{2n}\) by the map

\[(V_1, \ldots, V_n) \mapsto (V_1, \ldots, V_n, V_n^\perp, \ldots, V_1^\perp)\]
Let $I$ be a collection of subsets of the set $\{1, \ldots, n\}$. Second, $SpF^a_{2n}$ obviously sits inside $F^a_d$ for $d = (1, \ldots, n)$.

2. Dimension of cells

2.1. Algorithm. Let us consider a collection $I = (I_1, \ldots, I_n)$ consisting of subsets of the set $\{1, \ldots, 2n\}$ subject to the conditions

\[ |I_k| = k, \quad I_k \subset I_{k+1} \cup \{k + 1\}, \quad a \in I_n \text{ implies } 2n + 1 - a \notin I_n. \]

Let $G_n$ be the set of these collections. Each such a collection corresponds to a point $p(I)$ in the degenerate symplectic flag variety defined by $p(I)_k = \text{span}(e_i, i \in I_k)$. Recall (see Remark 1.6) that $SpF^a_{2n}$ sits inside $F^a_{(1, \ldots, n)}$. In what follows we denote the collection $(1, \ldots, n)$ by $d$.

Lemma 2.1. Assume that a collection $I$ satisfies (2.1) (i.e. $I \in G_n$) and let $C_A(I) \subset F^a_d$ be the corresponding cell. Then the intersection of $C(I) = SpF^a_{2n} \cap C_A(I) \subset F^a_d$ is an affine cell.

Proof. Let $I = (I_1, \ldots, I_n)$. Let $I_n$ be the complement to $I_n$ inside the set $\{1, \ldots, 2n\}$. In particular, $I_n$ contains $n$ elements and $a \in I_n$ if and only if $2n + 1 - i \notin I_n$. Let $C_A(I)$ be the cell inside $F^a_d$ corresponding to the collection $I$. In what follows we use the notation

\[ C(I) = SpF^a_{2n} \cap C_A(I). \]

We note that $C(I)$ consists of points $(V_1, \ldots, V_n) \in C_A(I)$ such that $V_n$ is Lagrangian.

Recall the ordering $<_n$ on the set $\{1, \ldots, 2n\}$ (see (1.1)). Then $V_n$ shows up as the last component of a point in $C_A(I)$ if and only if $V_n$ is the linear span of the vectors

\[ l_a = e_a + \sum_{b < a, b \in I_n} x_{a,b} e_b, \quad a \in I_n. \]

We note that the coordinates $x_{a,b}$ form a part of the coordinate system on the affine cell $C_A(I)$. Now $V_n$ is Lagrangian if and only if $(l_{a_1}, l_{a_2}) = 0$ for all $a, b \in I_n$, which is equivalent to the system of linear conditions labeled by all pairs $a_1, a_2 \in I_n$ such that $a_1 <_n a_2$:

\[ x_{a_1, 2n+1-a_2} \pm x_{a_2, 2n+1-a_1} = 0, \quad 2n + 1 - a_1 <_n a_2 \]

(recall that our symplectic form pairs nontrivially $e_a$ and $e_{2n+1-a}$). Hence $C(I)$ is cut out inside $C_A(I)$ by linear equations and hence is isomorphic to an affine space.

The proof above allows to compare the dimensions of $C_A(I)$ and that of $C(I)$.

Corollary 2.2. Let $I \in G_n$. Then

\[ \dim C_A(I) - \dim C(I) = |\{(a_1, a_2) \in I_n : a_1 <_n a_2, 2n + 1 - a_1 <_n a_2\}|. \]
Now let us give an explicit (inductive) way to compute the dimension $d(I)$ of the cell $C(I)$. We will use $n$ different orderings $\succ_k$, $k = 1, \ldots, n$ on the set $\{1, \ldots, 2n\}$ (see (1.1)). We represent $I$ graphically as follows (which reflects the latter orderings) : first we draw a parallelogram made of $2n \times n$ horizontally linked white disks as depicted in Figure 2:

![Figure 2. Skeleton of a graphical representation $P(I)$.](image)

If a disk $D$ is located in the $k$-th column (from left to right) and in the row labeled with the integer $l \in \{k + 1, \ldots, 2n, 1, \ldots, k\}$, we say that $D = [k : l]$. We then define $P(I)$ as this parallelogram in which every disk $[k : l]$ with $l \in I_k$ is painted in black.

For example, consider the collection 

$$I_0 = (\{3\}, \{3, 7\}, \{1, 4, 7\}, \{1, 4, 6, 7\}) \in I_4,$$

then $P(I_0)$ is as depicted in Figure 3.

Note that in general, for all $k \in \{1, 2, \ldots, n\}$, the $k$-th column $C_k(I)$ (from left to right) of $P(I)$ contains $|I_k| = k$ black disks, and that if a disk is black, then all the disks located on the right of it in the same row are also black. When a black disk is the first (from left to right) of its row, we say it is an initiating disk. Let $ID_k(I)$ be the set of initiating disks of $C_k(I)$. For all $D = [k : l] \in ID_k(I)$, the number of white disks of $C_k(I)$ located under $D$ is denoted by $w(D) = w(k : l)$; also, the set of black disks of $C_k(I)$ located under $D$ is denoted by $B(D)$. 

![Figure 3.](image)
Figure 3. Graph $\mathcal{P}(I_0)$.

Now, for $1 \leq k \leq n$ and $a, b \in \{1, \ldots, 2n\}$, $a + b \neq 2n + 1$ we define $s(a, b, k)$ as follows:

$$s(a, b, k) = \begin{cases} 1, & a, b \geq_k 2n - k + 1 \text{ and } 2n + 1 - b <_k a, \\ 0, & \text{otherwise}. \end{cases}$$

**Remark 2.3.** $s(a, b, k)$ is either one or zero. It is equal to 1 if both $a$ and $b$ are from the set $\{k, k-1, \ldots, 1, 2n, 2n-1, \ldots, 2n-k+1\}$ and $2n+1-b<_{k} a$ (note that if $b \geq_{k} 2n - k + 1$, then $2n + 1 - b \geq_{k} 2n - k + 1$).

**Remark 2.4.** Corollary 2.2 can be rephrased in terms of the function $s$: let $d_A(I) = \dim C_A(I)$, then $d(I) = d_A(I) - \sum_{1 \leq a < b \leq n} s(a, b, n)$.

Now let us describe the inductive algorithm for computing the dimensions $d(I) = \dim C(I)$. We do it in $n$ steps defining numbers $d_1 \leq \cdots \leq d_n = d(I)$ adding a nonnegative integer at each step.

We start with $d_1$. Let $I_1 = \{i\}$. We define $d_1$ as the number of elements $x \in \{1, \ldots, 2n\}$ such that $x <_1 i$.

Now let us define $d_2$. Let $I_1 = \{i\}$. We consider two cases. First, assume $2 \notin I_1$. Let $I_2 \setminus I_1 = \{j\}$. We define

$$d_2 = d_1 + \|\{x : x <_2 j, x \notin I_1\}\| - s(j, i, 2).$$

Second, let $2 \in I_1$. Let $I_2 = \{j_1, j_2\}$, $j_1 >_2 j_2$. Then

$$d_2 = d_1 + \|\{x : x <_2 j_2\}\| + \|\{x : x <_2 j_1\}\| - 1 - s(j_1, j_2, 2).$$

Now assume $d_{k-1}$ is already defined. Let us define $d_k$. We consider two cases. First, assume $k \notin I_{k-1}$. Let $I_k \setminus I_{k-1} = \{j\}$. We define

$$d_k = d_{k-1} + \|\{x : x <_k j, x \notin I_{k-1}\}\| - \sum_{i \in I_{k-1}} s(j, i, k).$$
Second, let $k \in I_{k-1}$ and let $I'_{k-1} = I_{k-1} \setminus \{k\}$. Let $I_k \setminus I'_{k-1} = \{j_1, j_2\}$, $j_1 > j_2$. Then

\[
d_k = d_{k-1} + |\{x : x < j_2, x \notin I'_{k-1}\}| + |\{x : x < j_1, x \notin I'_{k-1}\}| - 1
- s(j_1, j_2, k) - \sum_{i \in I'_{k-1}} (s(j_1, i, k) + s(j_2, i, k)).
\]

In terms of the graphical representation $\mathcal{P}(I)$, we obtain

\[
d_k = d_{k-1} + \sum_{D = [k : l] \in ID_k(I)} \left( w(D) - \sum_{[k : l'] \in B(D)} s(l, l', k) \right)
\]

for all $k \in \{1, \ldots, n\}$, where $d_0$ is defined as 0. For the example $I = I_0 \in \mathcal{I}_4$ of Figure 3, we obtain:

- $ID_1(I_0) = \{[1 : 3]\}$ and $d_1 = w(1 : 3) = 1$;
- $ID_2(I_0) = \{[2 : 7]\}$ and $d_2 = d_1 + (w(2 : 7) - s(7, 3)) = 1 + (3 - 0) = 4$;
- $ID_3(I_0) = \{[3 : 4], [3 : 1]\}$ and $d_3 = d_2 + w(3 : 4) + (w(3 : 1) - s(1, 7) - s(1, 4))$
  \[= 4 + 0 + (3 - 0 - 0) = 7;\]
- $ID_4(I_0) = \{[4 : 6], [4 : 4]\}$ and $d_4 = d_3 + w(4 : 6) + (w(4 : 4) - s(4, 1) - s(4, 7) - s(4, 6))$
  \[= 7 + 1 + (4 - 1 - 1 - 1) = 9.\]

**Theorem 2.5.** For all $I \in \mathcal{I}_n$ the number $d_n$ is equal to the dimension of $C(I)$.

**Proof.** Recall that $SpF^n_{2n}$ sits inside the type $A$ partial degenerate flag variety $F^n_d$ with $d = (1, \ldots, n)$ and a point $(V_1, \ldots, V_n) \in F^n_d$ belongs to $SpF^n_{2n}$ if and only if the projection $f_k V_k$ is isotropic for all $k = 1, \ldots, n$ (see (1.4) for the definition of the projections). Let $C_A(I) \subset F^n_d$ be the cell corresponding to $I$. Let $t_k : F^n_d \to \prod_{i=1}^{k} \text{Gr}(k, 2n)$ be the projecton to the first $k$ components, i.e.

\[
t_k(V_1, \ldots, V_n) = (V_1, \ldots, V_k).
\]

By construction (see (1.2)) the image $t_k C_A(I)$ is an affine space. We prove by induction on $k$ that for any $1 \leq k \leq n$ one has $\dim t_k C(I) = d_k$ (recall $C_A(I) \supset C(I)$). Note that the $k = n$ case is exactly the claim of our theorem.

The base of induction $k = 1$ is clear, since $t_1 C_A(I) = t_1 C(I)$ is an affine space of dimension $d_1$. Now assume the claim is proved for some $k - 1 < n$. We consider two cases. First, assume $k \notin I_{k-1}$. Let $I_k \setminus I_{k-1} = \{j\}$. Then

\[
\dim t_k C_A(I) = \dim t_{k-1} C_A(I) + |\{b : b < k, b \notin I_{k-1}\}|,
\]

...
since \( \dim pr_k V_{k-1} = k - 1 \) and in order to determine \( V_k \) we need to add one more vector of the form
\[
l = e_j + \sum_{b < k, b \notin I_k} x_{j,b} e_b.
\]

Now in order to ensure that the resulting point \((V_1, \ldots, V_k)\) belongs to \( t_k C_A(I) \) we need to impose \( \sum_{i \in I_{k-1}} s(j, i, k) \) linear relations on the coefficients \( x_{j,b} \). Indeed \( f_k V_k \) is Lagrangian in \( W_k \) if and only if the vector \( f_k l \) is orthogonal to the space \( f_k pr_k V_{k-1} \). Recall that \( V_{k-1} \) has a basis labeled by \( i \in I_{k-1} \) of the form \([12]\). For \( i \in I_{k-1} \), consider the product
\[
\pi = \left( f_k \left( e_j + \sum_{b < k, b \notin I_k} x_{j,b} e_b \right), f_k \left( e_i + \sum_{c < k, c \notin I_k} x_{i,c} e_c \right) \right).
\]

Since \( f_k V_k \) is Lagrangian in \( W_k \), we have \( \pi = 0 \). Let us now expand \( \pi \) in the products \((e_b, e_c)\). If \( s(j, i, k) = 0 \), then all of them equal 0. Otherwise, we obtain
\[
0 = \pi = x_{j,2n+1-i}(e_{2n+1-i}, e_i) + x_{i,2n+1-j}(e_j, e_{2n+1-j}),
\]

hence an equation of the kind \( x_{j,2n+1-i} = \pm x_{i,2n+1-j} \). Therefore,
\[
\dim t_k C(I) = \dim t_{k-1} C(I) + |\{b : b < k, j, b \notin I_{k-1}\}| - \sum_{i \in I_{k-1}} s(j, i, k),
\]

which is equal to \( d_k \). The second case \( k \in I_{k-1} \) is proved in the same way, by showing that and if \( I_k \setminus (I_{k-1} \setminus \{k\}) = \{j_1, j_2\} \), \( j_1 > k, j_2 \), then
\[
\dim t_k C_A(I) = |\{x : x < k, j_2, x \notin I_{k-1}'\}| + |\{x : x < k, j_1, x \notin I_{k-1}'\}| - 1
\]
and
\[
\dim t_k C(I) = \dim t_k C_A(I) - s(j_1, j_2, k) - \sum_{i \in I_{k-1}'} (s(j_1, i, k) + s(j_2, i, k)).
\]

\[ \square \]

2.2. Symplectic Dellac configurations.

**Definition 2.6.** Let \( S \in SpDC_{2n} \). Consider the quotient \( \hat{INV}(S) = INV(S)/\sim \), where \( INV(S) \) is the set of the inversions of \( S \), and \( \sim \) is the equivalence relation defined by \((p_1, p'_1) \sim (p_2, p'_2)\) if and only if \( p_1 = p_2 \) or \( R(p_1) = p'_2 \) and \( R(p'_1) = p_2 \) (recall that \( R \) is the central reflection with respect to the center of \( S \)). We define \( \text{inv}(S) \) as \( |\hat{INV}(S)| \).

For example, in Figure\ref{fig:example} we depict the 10 elements \( S \in SpDC_4 \), each of which has its inversions represented by segments. Equivalent inversions are drawn in a same color (the integer \( \text{inv}(S) \) is then the number of different colors used in \( S \)).

The polynomial \( \sum_{S \in SpDC_4} q^{\text{inv}(S)} = 1 + 2q + 3q^2 + 3q^3 + q^4 \) turns out to be the Poincaré polynomial of \( SpF^q_4 \). To prove the general statement (see Theorem\ref{thm:main}), let us prepare some notations.
For all $k \in \{1, \ldots, n\}$ and $l \in \{k + 1, \ldots, 2n, 1, \ldots, k\}$, we define $l_k$ as $l$ if $l \in \{k + 1, \ldots, 2n\}$, and as $2n + l$ otherwise.

For a point $p = (j, i)$ of a symplectic Dellac configuration $S \in \text{SpDC}_{2n}$, the point $(2n + 1 - j, 4n + 1 - i)$ of $S$ (symmetrical to $p$ with respect to the center of $S$) is denoted by $p_{\text{sym}}$. Note that if $p = (j, l_j)$, then $p_{\text{sym}} = (2n + 1 - j, (2n + 1 - l_j))$.

**Theorem 2.7.** Consider a collection $I = (I_1, \ldots, I_n) \in \mathcal{I}_n$, the corresponding cell $C(I)$, and the corresponding symplectic Dellac configuration $S(I)$. We have

$$\dim C(I) = \widehat{\text{inv}}(S(I)).$$

**Proof.** Recall [FL, FFL] that the bijection $I \in \mathcal{I}_n \mapsto S(I) \in \text{SpDC}_{2n}$ consists of:

(a) drawing an empty tableau $S$ made of $2n$ columns and $4n$ rows;
(b) printing the parallelogram $\mathcal{P}(I)$ in such a way that the disk $[1 : 2]$ of $\mathcal{P}(I)$ is located in the box $(1, 2_1) = (1, 2)$ of $S$;
(c) erasing all the disks, except the initiating ones (which we paint in blue);
(d) for all $i \in \{1, 2, \ldots, n\}$, if the $i$-th row (from bottom to top) of $S$ is still empty, then printing a red disk in the box $(i, i)$;
(e) finally, applying the central reflection with respect to the center of $S$ to the points of its $n$ first columns.

For the example $I = I_0 \in \mathcal{I}_4$ of Figure 3, we obtain the following:
In the computation of $d_1 \leq d_2 \leq \ldots \leq d_n = \dim(C(I))$, it is then straightforward that $d_1$ is the number of inversions whose first element is the upper point $p_1$ of the first column of $S(I)$, which is also the number of elements of $\tilde{INV}(S(I))$ containing an inversion whose first point is $p_1$.

Suppose now that for some $k \geq 2$, the integer $d_{k-1}$ is the number of elements of $\tilde{INV}(S(I))$ containing an inversion whose first point is located in one of the first $k$ columns of $S(I)$. Let $[k : l] \in ID_{k}(I)$. By definition of $S(I)$, the box $(k, l)$ of $S(I)$ contains a point $p$. Here again, it is straightforward that $w(D)$ is the number of inversions of $S(I)$ whose first point is $p$. Afterwards, let $D' = [k : l'] \in B(D)$ for some $l' < k l$, and let $k' \leq k$ such that $(k', l') \in ID_{k'}(I)$, hence the box $(k', l'_p)$ of $S(I)$ contains a point $p'$. Consider the points $p_{sym} = (2n + 1 - k, (2n + 1 - l)_k)$ and $p'_{sym} = (2n + 1 - k', (2n + 1 - l')_k)$ of $S(I)$.

The following properties are equivalent:

(i) $(p, p'_{sym}) \in INV(S(I))$;
(ii) $(p', p_{sym}) \in INV(S(I))$;
(iii) $1_k > (2n + 1 - l')_k$;
(iv) $s(l, l', k) = 1$.

The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are obvious. Now, since the dots of a Dellac configuration are located above the line $y = x$, the existence of the point $p'_{sym} = (2n + 1 - k', (2n + 1 - l')_k)$ implies

$$2n - k \leq 2n - k' < (2n + 1 - l')_k,$$

so (iii) is equivalent to

$$2n - k < k 2n + 1 - l' < l,$$

which is equivalent to (iv). In Figure 5 we illustrate a situation with 3 points $p, p', p''$ where $p$ and $p'$ have the properties (i), ..., (iv), as opposed to $p$ and $p''$. The inversions $(p, p'_{sym})$ and $(p', p_{sym})$ having the same class in $INV(S(I))$ are expressed by drawing them in red.
Figure 5. Symplectic Dellac configuration $S(I)$. 
In general, the equivalence \((i) \iff (iv)\) implies that the integer 
\[ w(D) - \sum_{D' = [k:k'] \in B(D)} s(l,l',k) \]
is the number of elements of \(\widetilde{INV}(S(I))\) that contain an inversion of \(S(I)\) whose first point is \(p\), but no inversion whose first point is located in one of the \(k - 1\) first columns of \(S(I)\). By hypothesis of induction, and in view of formula (2.4), we obtain that \(d_k\) is the number of elements of \(\widetilde{INV}(S(I))\) containing an inversion whose first point is located in one of the first \(k\) columns of \(S(I)\). By induction, we have in particular 
\[ \dim(C(I)) = d_n = |\widetilde{INV}(S(I))|. \]
\[ \square \]

3. Extended Dellac configurations

Recall [B1] that an extended Dellac configuration \(T\) is a tableau \(n \times 2n\) with the conditions of a Dellac configuration \(D \in DC_n\), except the condition \(i \leq j + n\) for all point \(p = (j, i)\) of \(T\). We denote the set of extended Dellac configurations by \(\mathcal{T}_n\).

In general, if a point of \(T \in \mathcal{T}_n\) is located in a box \((j, i)\) such that \(i \geq 2n + 1 - j\), we say that it is free and we represent it by a star instead of a dot. Let \(fr(T)\) be the number of free points of \(T\).

For example, in Figure 6 the 3 elements of \(\mathcal{T}_2\) are depicted. The numbers of free points are respectively 2, 1 and 2.

![Figure 6. The 3 elements of \(\mathcal{T}_2\).](image)

By considering the \(\binom{n+1}{2} = (n+1)n/2\) possible locations of the two points of the last column of any \(T \in \mathcal{T}_n\) (from left to right), it is straightforward that \(|\mathcal{T}_n| = \binom{n+1}{2} |\mathcal{T}_{n-1}| = (n+1)!n!/2^n\) for all \(n \geq 2\).

In [B1], we showed that \(SpDC_{2n}\) is generated by \(\mathcal{T}_n\) as follows: the idea is to partition any \(S \in SpDC_{2n}\) as

\[ S = \]

where there are no points in the blank areas, and where the areas \(*\) are obtained by applying the central reflection (with respect to the center of \(S\)) to the areas \(X_S, Y_S\) and \(Z_S\). For all area \(A\) of \(S\), the number of dots
inside $A$ is denoted by $|A|$. Consider a point $p = (j, i)$ of $S$ located in
the area $Y_S$ or $Z_S$ (in other words, such that $i \geq 2n + 1 - j$), and its
symmetrical dot $p_{\text{sym}} = (2n+1-j, 4n+1-i)$. We define a new configuration
$\varphi_p(S) \in SpDC_{2n}$ by erasing the dots $p$ and $p_{\text{sym}}$ from $S$, then by ploting
the dot $p' = (j, 4n+1-i)$ and its symmetric point $p'_{\text{sym}} = (2n+1-j, i)$.
For example, the two following elements of $SpDC_4$ are sent to one another
by $\varphi_{(2,3)}$ and $\varphi_{(2,6)}$ respectively.

By applying on $S$ the composition of all the $\varphi_p$ for the points $p$ located in
$Z_S$, we obtain a configuration $\tilde{S} \in SpDC_{2n}$ such that $Z_{\tilde{S}}$ is empty, which
means that the $n \times 2n$ following tableau

$$
Y_{\tilde{S}}
\begin{array}{c}
\vdots \\
\vdots \\
X_{\tilde{S}}
\end{array}
$$


is an extended Dellac configuration $T_S \in T_n$, with $\text{fr}(T_S) = |Y_S| = |Y_S|+|Z_S|$. It is then straightforward that the map $S \mapsto T_S$ is a surjection from $SpDC_{2n}$
to $T_n$ such that the cardinality of the set $\{S \in SpDC_{2n} : T_S = T\}$ is $2^{\text{fr}(T)}$
for all $T \in T_n$, which gives the following formula :

$$
|SpDC_{2n}| = \sum_{T \in T_n} 2^{\text{fr}(T)}.
$$

(3.1)

For example, we depict in Figure 7 how the 3 elements of $T_2$ generate the
$10 = 2^2 + 2 + 2^2$ elements of $SpDC_4$.

**Definition 3.1.** Let $T \in T_n$. Consider a free point $p$ of $T$, say $p = (j, i)$
with $i \geq 2n + 1 - j$. We define an integer $t(p) = t_1(p) + 2t_2(p)$ where

- $t_1(p) \in \{1, 2\}$ is the number of points located in the same column as
  $p$ and above $p$ (including $p$, hence $t_1(p) = 1$ if $p$ is the upper point
  of its column, otherwise $t_1(p) = 2$);
- $t_2(p)$ is the number of points of the kind $(j', i')$ with $j < j'$ and $i < i'$
  (in other words, the number of points strictly on the right of $p$ and
  above $p$).

Also, we extend the notion of inversion of a Dellac configurations to the ex-
tended Dellac configurations : we denote by $\text{inv}(T)$ the number of inversions
of $T$, i.e., of pairs of points $p_1 = (j_1, i_1)$ and $p_2 = (j_2, i_2)$ of $T$ such that
$j_1 < j_2$ and $i_1 > i_2$.

For example, consider the tableau $T \in T_5$ depicted in Figure 8, whose free
points are denoted by $a, b, c, d$. Then $(t(a), t(b), t(c), t(d)) = (1, 3, 1, 4)$. In
Figure 7. Generation of the $2^2 + 2^2$ elements of $SpDC_4$ from the 3 elements of $\mathcal{T}_2$.

In this figure, we emphasize the fact that $t(d) = 2 + 2 \times 1 = 4$ (by filling with grey the boxes located on the right of $d$ and above $d$) and $\text{inv}(T) = 11$ (by drawing segments between the points involved in every inversion).

Figure 8. Tableau $T \in \mathcal{T}_5$.

Definition 3.2. For all $T \in \mathcal{T}_n$ and $k \geq 1$, we denote by $n_k(T)$ the number of free points $p$ of $T$ such that $t(p) = k$.

Theorem 3.3. For all $n \geq 1$, the Poincaré polynomial of the symplectic degenerate flag variety $SpF_{2n}$ is

$$P_n(q) = \sum_{T \in \mathcal{T}_n} q^{\text{inv}(T)} \prod_{k \geq 1} (1 + q^k)^{n_k(T)}.$$

For the case $n = 1$, the unique element
of $T_1$ gives $P_1(q) = 1 + q$.

For the case $n = 2$, the three elements of $T_2$ give $P_2(q) = (1 + q)(1 + q^2) + q(1 + q) + q^2(1 + q)^2 = 1 + 2q + 3q^2 + 3q^3 + q^4$.

Proof of Theorem 3.3. In the proof of formula (3.1), we parted the set $SpDC_{2n}$ into $(O_T)_{T \in T_n}$, where $O_T = \{S \in SpDC_{2n} : T_S = T\}$ has the cardinality $2^{fr(T)}$. We state that

$$\sum_{S \in O_T} q^{\tilde{inv}(S)} = q^{inv(T)} \prod_{k \geq 1} (1 + q^{k})^{n_k(T)}$$

for all $T \in T_n$, which proves Theorem 3.3 in view of Theorem 2.7. Let us prove formula (3.2). Consider a configuration $S = \in O_T$.

Recall that $|Y_S| + |Z_S| = fr(T)$. Let $p = (j, i)$ be a dot of $S$ located in $Y_S$, i.e., such that $n \geq i \geq 2n + 1 - j$, and let $S'$ be the configuration $\varphi_p(S) \in SpDC_{2n}$, which is obtained by erasing $p$ and its symmetric point $p_{sym} = (2n + 1 - j, 4n + 1 - i)$ from $S$, then by plotting the points $p' = (j, 4n + 1 - i)$ and $p'_{sym} = (n + 1 - j, i)$. Note that $p' \in Z_{S'}$. We now intend to compare the dimensions of the cells $C$ and $C'$ of $SpF_{2n}$ that correspond to $S$ and $S'$ respectively.

Let $t_1$ be the integer defined as 1 if $p$ is the upper point of its column in $S$, defined as 2 otherwise. Let also $t_2$ be the number of points of $S$ of the kind $(j'', i'')$ such that $j < j'' \leq n$ and $i < i'' < 4n + 1 - i$. We compare now $\tilde{INV}(S)$ and $\tilde{INV}(S')$. We can inject $\tilde{INV}(S)$ into $\tilde{INV}(S')$ as follows (where $(p_1, p_2)$ refers to the class of an inversion $(p_1, p_2)$ modulo the equivalence relation $\sim$ defined in Definition 2.6) :

- any $(p'', p) \in \tilde{INV}(S)$ is mapped to $(p'', p_{sym}) \in \tilde{INV}(S')$;
- any $(p, p'') \in \tilde{INV}(S)$ is mapped to $(p', p'') \in \tilde{INV}(S')$;
- any other $(p'', p_{sym}) \in \tilde{INV}(S)$ also belongs to $\tilde{INV}(S')$ and is mapped to itself.

So $\tilde{inv}(S) \leq \tilde{inv}(S')$. Let us count the elements of $\tilde{INV}(S')$ not considered above:
• of course there is \((p', p'_\text{sym}) = \{(p', p'_\text{sym})\};
• if \(t_1 = 2\), let \(p^n\) be the upper point of the column of \(p\) in \(S\), it is also the lower point of the same column in \(S'\) (in which the upper point is \(p'\)), then it produces \((p^n, p'_n, p'_\text{sym})\) \(\in \tilde{\text{INV}}(S')\);
• each of the \(t_2\) points \(p'' = (i'', j'')\) of \(S\) such that \(j < j'' \leq n\) and \(i < i'' < 4n + 1 - i\), produces two elements
\[
(p', p'') = \{(p', p''), (p''', p'_\text{sym})\} \in \tilde{\text{INV}}(S'),
\]
\[
(p', p''') = \{(p', p'''), (p''', p'_\text{sym})\} \in \tilde{\text{INV}}(S').
\]
This proves that \(\tilde{\text{inv}}(S') = \tilde{\text{inv}}(S) + t\) where \(t = t_1 + 2t_2\). Moreover, if we denote
\[
T = \begin{array}{c}
Y \\
X \\
\end{array}
\]
then, as we saw when proving formula (3.1), we have \(p \in Y\) and \(t = t(p)\). It follows that
\[
\sum_{S \in \mathcal{O}_T} q^{\tilde{\text{inv}}(S)} = \left( \prod_{p \in \{\text{free points of } T\}} 1 + q^{t(p)} \right) q^{\tilde{\text{inv}}(S_T)}
\]
where
\[
S_T = \begin{array}{c}
* \\
\end{array}
\begin{array}{c}
U \\
\end{array}
\]

hence \(\tilde{\text{inv}}(S_T) = \text{inv}(T)\). \(\square\)

4. Generalized Motzkin paths

In this section we closely follow the arguments from [CFR1, Section 6]. Consider \(\mathbf{I} \in \mathcal{I}_n\). For all \(k \in \{1, \ldots, n\}\), we define
\[
f_k(\mathbf{I}) = |\{x : x \leq_k 2n, x \in I_k\}|,
\]
\[
g_k(\mathbf{I}) = k - f_k(\mathbf{I}).
\]
Graphically, the integer $f_k$ (respectively $g_k$) is the number of black disks of the column $C_k(I)$ located in the rows labeled with $k+1,\ldots,2n$ (respectively $1,2,\ldots,k$). For example, the collection $I_0 \in \mathcal{I}_4$ of Figure 3 gives

$$
(f_1(I_0), f_2(I_0), f_3(I_0), f_4(I_0)) = (1,2,2,2),
$$

$$
(g_1(I_0), g_2(I_0), g_3(I_0), g_4(I_0)) = (0,0,1,2).
$$

In general, it is easy to see that $f_k(I) - f_{k-1}(I) \in \{-1,0,1\}$ for all $k$, so the sequence of integers $(f_1(I), f_2(I), \ldots, f_n(I))$ may be assimilated into a path $((0,0), (1,f_1(I)), \ldots, (n,f_n(I))$ in an $n \times n$ grid from $(0,0)$ to the line $x = n$ using only steps $(1,1)$, $(1,0)$ and $(1,-1)$.

**Definition 4.1.** We denote by $\tilde{M}_n$ the set of paths $((0,0), (1,f_1), \ldots, (n,f_n))$ such that $f_k \geq 0$ and $f_k - f_{k-1} \in \{-1,0,1\}$ for all $k$. We call the elements of $\tilde{M}_n$ the generalized Motzkin paths [OEIS2].

Recall that the set $M_n$ of the Motzkin paths of length $n$ is the subset of $\tilde{M}_n$ made of the paths whose last point is $(n,0)$. The generating function $\sum_{n \geq 0} |M_n|t^n$ is

$$M(t) = \frac{1 - t - \sqrt{(1-t)(1-3t)}}{2t^2} = 1 + 2t + 4t^2 + \ldots$$

It is easy to see that if $\tilde{M}(t)$ is the generating function $\tilde{M}(t) = \sum_{n \geq 0} |\tilde{M}_n|t^n$, then $\tilde{M}(t) = 1 + 2tM(t) + 3t(\tilde{M}(t) - M(t))$, hence

$$\tilde{M}(t) = \frac{\sqrt{1 + 4t} - 1}{2t} = 1 + 2t + 5t^2 + 13t^3 + \ldots$$

For all $k \in \{1,\ldots,n\}$, consider

$$L_k = \text{span}(e_{k+1},\ldots,e_{2n}),$$
$$U_k = \text{span}(e_1,\ldots,e_k).$$

For $f = ((0,0), (1,f_1), \ldots, (n,f_n)) \in \tilde{M}_n$, we define

$$SpF^a_{2n}(f) = \{ (V_1,\ldots,V_n) \in SpF^a_{2n} : \forall k \dim(V_k \cap L_k) = f_k \},$$

$$SpF^a_{2n}(f) = \{ (V_1,\ldots,V_n) \in SpF^a_{2n} : \forall k \dim(V_k \cap L_k) = f_k \},$$

Let $g_k = k - f_k$ for all $k$. In view of Remark 1.5 the map

$$(V_1,\ldots,V_n) \mapsto (V_1 \cap U_1,\ldots,V_{n-1} \cap U_{n-1}, V_n \cap U_n, V_{n-1} \cap L_{n-1},\ldots,V_1 \cap L_1)$$

is an isomorphism from $SpF^a_{2n}(f)$ to the set $X$ of the collections

$$\{V_1^U,\ldots,V_{n-1}^U, V_n^U, V_{n-1}^L,\ldots,V_1^L\}$$

such that:

- $V_k^U \in \text{Gr}(g_k,U_k)$ and $V_k^U \subset V_k^{U'}$;
- $V_k^L \in \text{Gr}(f_k,L_k)$ and $pr_{k+1}V_k^L \subset V_{k+1}^L$ (where $V_n^L$ is defined as $(V_n^U)^\perp \cap L_n$).
Now, it is straightforward that
\[ X \simeq \prod_{k=1}^{n} \text{Gr}(g_k - g_{k-1}, k - g_{k-1}) \times \prod_{l=1}^{n-1} \text{Gr}(f_{n-l}, f_{n-l+1} + 1) \]
where \( g_0 = 0 \). Consequently, the Poincaré polynomial of \( \widetilde{SpF}_{2n}(f) \) is
\[ P_{\widetilde{SpF}_{2n}(f)}(q) = \prod_{k=1}^{n} \left( \frac{k - g_{k-1}}{g_k - g_{k-1}} \right) \prod_{q=1}^{n-1} \left( \frac{f_{k+1} + 1}{f_k} \right) q, \]

in view of \( g_k = k - f_k \).

**Lemma 4.2.** \( \widetilde{SpF}_{2n}(f) \) is a vector bundle over \( \widetilde{SpF}_{2n}(f) \) of rank
\[ \sum_{k=1}^{n} (g_k - g_{k-1})(2n - k - f_k) - g_n(g_n - 1)/2. \]

**Proof.** The map \( \widetilde{SpF}_{2n}(f) \to \widetilde{SpF}_{2n}(f) \) is defined by
\[ \pi : (V_1, \ldots, V_n) \mapsto (\ldots, (V_k \cap L_k) \oplus \psi V_k, \ldots), \]
where \( \psi : L_k \oplus U_k \to U_k \) is the projection to the second summand. Let us briefly recall the argument from [CFR1] proving a similar statement in type A. Fix a point \( \mathbf{V} = (V_k^L \oplus V_k^U)_{k=1}^{n} \in \widetilde{SpF}_{2n}(f) \) and consider the fiber \( \pi^{-1}(\mathbf{V}) \). Let \( (W_k)_{k=1}^{n} \in \pi^{-1}(\mathbf{V}) \). Then \( W_1 \) is naturally identified with a point from \( \text{Hom}(V_1^U, L_1/V_1^L) \), which has dimension \( g_1(2n - 1 - f_1) \). As soon as \( W_1 \) is fixed, \( W_2 \) is identified with the space \( \text{Hom}(V_2^U/V_1^L, L_2/V_2^L) \), which is \( (g_2 - g_1)(2n - 2 - f_2) \)-dimensional. We then proceed with \( W_3, \ldots, W_n \) getting the desired dimension \( \sum_{k=1}^{n} (g_k - g_{k-1})(2n - k - f_k) \). Now the correction \( g_n(g_n - 1)/2 \) comes from the condition that \( W_n \) has to be Lagrangian. In fact, since \( L_n = U_n^* \) with respect to the symplectic form and \( V_n^L = L_n \cap (V_n^U)^\perp \), the space \( W_n \) is Lagrangian if and only if the corresponding homomorphism \( j_n : V_n^U \to L_n/V_n^L \) satisfies \( v_1 + j_nv_1, v_2 + j_nv_2 \) = 0 for all \( v_1, v_2 \in V_n^U \). These are \( g_n(g_n - 1)/2 \) linearly independent linear conditions, since \( \dim V_n^U = g_n \). \( \square \)

We thus obtain the following theorem.

**Theorem 4.3.**
\[ P_{\widetilde{SpF}_{2n}(f)}(q) = \sum_{f \in \mathbb{N}_n} q^{-(n-f_n)(n-1-f_n)/2 + \sum_{k=1}^{n-1}(1-f_k + f_{k-1})(2n-k-f_k)} \times \left( \prod_{k=1}^{n-1} \left( \frac{f_k + 1}{f_{k-1}} \right) q \left( \frac{f_k + 1}{f_k} \right) q \left( \frac{f_{k-1} + 1}{f_{k-1}} \right) q \right). \]
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