Gas-liquid transition in the model of particles interacting at high energy

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Abstract

An application of the ideas of the inertial confinement fusion process in the case of particles interacting at high energy is investigated. A possibility of the gas-liquid transition in the gas is considered using different approaches. In particular, a shock wave description of interactions between particles is studied and a self-similar solution of Euler’s equation is discussed. Additionally, Boltzmann equation is solved for self-consistent field (Vlasov’s equation) in linear approximation for the case of a gas under external pressure and the corresponding change of Knudsen number of the system is calculated.

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1 Introduction

The possibility of the existence of a first-order phase transition which occurs between phases with different densities in QCD is a very intriguing problem for both a theoretical and experimental studying. Low energy runs at RHIC/BNL and the future facilities FAIR at GSI and NICA in Dubna will, perhaps, clarify the experimental situation with the signatures of the first-order phase transition in the low-medium energy interactions. Moreover, collisions of relativistic nuclei in the RHIC and LHC experiments at very high energies led to the revelation of a new state of matter named quark gluon plasma (QGP). At initial stages of the scattering, the dense hadron gas of the nuclei becomes almost ideal fluid, see [1]. This fluid is short-living and expanding state of strongly interacting partons at very high temperature. Details of the phase transition in such interactions are not fully clear; there are arguments for the true phase transitions as well as for the fast crossover, see discussions of the question in [1, 2, 3, 4, 5, 6].

In our paper, exploring some ”toy” model, we will try to understand, at least qualitatively, the process of the gas-liquid transition which can take place in a system of interacting relativistic particles. We assume, that the proposed ”toy” model can be applied for the case of high-energy scattering as well as for the case of scattering at low and medium energies, providing in both cases an initial conditions for further evolution of the system. The process of nuclei-nuclei scattering was widely investigated in the frameworks of different thermodynamics approaches. The statistical mechanics techniques were widely used for the description of systems of two interacting nuclei as well as for the description of the data on the multiplicities of produced particles, see [7, 8]. The possibility of applications of Boltzmann equation for the description of the scattering process was investigated as well, see [9]. Nevertheless, the attempts to describe the nuclei in the framework of Boltzmann equation with the help of global distribution functions of nuclei do not look quite satisfactorily. Nuclei scattering at high energy is highly non-equilibrium process, see for example [2, 10]. The scattering system can not be in a global equilibrium state, instead, only some local spots of the scattering area can be locally equilibrated, see [11] and also [2, 5] for more details. Therefore, in our model we consider only local spots of the equilibrated matter which, in turn, are influenced by the matter outside of the spots.

The assumption about the locally equilibrated spots of the hot matter inside the scattering region we combine with the results of papers [7] and [12]. The main idea of [12] is that non-equilibrium process can be considered as an equilibrium one but in an external field. It leads to the separation of the matter locally in the scattering area into two groups. For the first group of the matter, we use an equilibrium description whereas the second group is considered as a source of the external field. The difference between these two groups can be understood from [7], see QCD degrees of freedom separation in [13]. Following the ideas of [7], we assume that the hot drops of the hadron matter with very high density are created in the scattering region at the very first moment of the interactions. Approximately half of the total initial energy of the interacting partons is going there. The particles
inside the drops we can call as internal ("slow") or stopped particles. The initial energy of the "slow" particles in each spot is spend to heating of the dense drop. Other partons are outside of the drops of stopped particles. These outside particles we call as external or "fast", important that they remain relativistic. Therefore, the influence of the "fast" particles on the process is realized in further compression of the hot drops and can be accounted by introduction of some external field acting on the internal particles. The advantage of this approach is clear. We reduce the non-equilibrium process to the equilibrium one and, therefore, we can apply a wide variety of equilibrium kinematic approaches separately to each local spot. The whole process of the dense drop compression, therefore, is similar to the effect of inertial confinement in plasma physics, see [14]. The radiation pressure of the external area (hot shell) of the target is used in order to create a requested density of the matter, see Fig.1. Another advantage of the approach is that the relativistic dynamics will be important only for the external field description, the "slow" particles undergo a non-relativistic transport process.

The "toy" model of the problem is a gas consisting of the colliding disks which are characterized by rapidity variable. The value of the rapidity variable separate the "fast" from the "slow" particles. Namely, the rapidity of the external particles is much larger than the rapidity of the initially stopped particles. The use of this variable is justified not only by the picture of QCD high energy interactions governed by the Pomeron amplitude, see [13] and [15], but in general, by the clear energy dependence of the possible phase transition. Choosing the rapidity of the stopped particles as zero, we assume that the potential of the external field created by the external particles depends on the rapidity of these particles which is proportional to the total energy of the scattering process.

Further, in Section 2, we develop our model of the gas and introduce a pairwise potential between the particles in two dimensional space. The external field in the model depends on the rapidity variable
and it is the only characteristics of the external particles (relativistic dynamics) in this framework. For our purposes, the form of the potential is unimportant and some simple one is chosen. The following subsections of this section are dedicated to the modified Van der Vaals equation of the state of the gas, see also [5]. Section 3 is dedicated to the thermodynamical properties of the gas. In particular, basing on Enskog approach, see [16] [17], we discuss the transport characteristics of the gas. In Section 4 the shock wave description of the process of the interaction of the external and internal particles is given. There we describe the influence of the external particles on the internal ones in the form of shock wave and solve Euler’s equations following mainly by the results of paper [18]. In Section 5, we generalize the problem and consider Boltzmann equation in Vlasov’s approximation for the case of a collisionless plasma in three dimensions and describe an equilibrium state of the system. In each Section the question addressed to the issue of the gas-liquid transition is also discussed. We investigate the conditions and signs of this transition for each of the proposed frameworks. The last section of the manuscript is dedicated to the discussion of the obtained results; there the conclusion of the paper is given.

2 Equation of the state of the gas

2.1 The potential of a pairwise interactions

Our "toy" model is a system of two relativistically colliding two dimensional disk clouds. Due to the Lorentz contraction, important already at not so high energies, see [19], we restrict our consideration by two dimensional model. Therefore, similarly to high energy scattering of two nuclei, we assume that at very early stages of the collision two dimensional hot spots of stopped particles are created. For the formulation of the equation of state of these spots we need to determine the pairwise interaction potential between the particles, see Appendix A. In our task, the following potential consisting of two parts is considered:

\[ U(b) = \int f(b-x) V_1(b) f(x) \, d^2 x + V_2(b). \] (1)

Here \( V_1(b) \) is the usual non-relativistic pairwise potential arising between the particles at the mutual distance \( b \) inside the drop:

\[ V_1(b_{12}) = -\frac{A}{b} + \frac{B}{b^\alpha} \] (2)

with \( \alpha > 1 \) (in our further calculations we take \( \alpha = 2 \)) and \( A, B \) are some constants. The function \( f(x) \) is the distribution function of the particles which we take equal to

\[ f(x) = \frac{e^{-x^2/R^2}}{\sqrt{\pi R^2}}, \] (3)

An investigation of the deviation from the equilibrium state is considered in the Appendix B of the paper.

The model can be easily extended for three dimensions, but in this case we will loose an applicability of the calculations for the high-energy interactions. We plan to investigate similar 3-dimensional model ever later. Nevertheless, 3-dimensional Boltzmann equation is considered in Section 5.

We limit ourselves only by pairwise interactions.
\( R \) being a radial size, see \[21\] for justification of that choose. The potential energy term \( V_2(b) \), absent in Eq. (A.1), describes the potential energy of a particle inside of the spot in the field of the "fast" particles outside the spot. This term accounts the influence of the "fast" particles on the dynamics of "slow" particles and this is the only source of the relativistic dynamics in the problem. Therefore, similarly to high-energy relativistic model of \[21\], this external potential field \( V_2(b) \) is assumed to be depended only on the position (impact parameter) \( b \) of the particle inside of the region, and rapidity \( Y = \ln \left( \frac{s}{s_0} \right) \) of the outside particles in respect to the "slow" particles; here \( s \) is the squared total energy of the "fast" particles in c.m.f. of the "slow" process. This field creates additional pressure on the internal particles and could be considered as an analogue of the inertial confinement effect in plasma physics. For simplicity we choose the form for this term similar to what we have in the phenomenological Pomeron approach (see \[21\]):

\[
V_2(b) = -\frac{C}{2 \pi R^2} e^Y \theta \left( R^2 - b^2 \right) \tag{4}
\]

where \( \theta \) is the step function and \( C \) is a constant. With this potential, Eq. (11) for the total potential energy acquires the following form:

\[
U(b) = \frac{e^{-b^2/2R^2}}{2\pi R^2} V_1(b) + V_2(b) = \frac{e^{-b^2/2R^2}}{2\pi R^2} \left( -\frac{A}{b} + \frac{B}{b^2} \right) - \frac{C}{2 \pi R^2} e^Y \theta \left( R^2 - b^2 \right). \tag{5}
\]

Considering the integral Eq. (A.12) with the potential Eq. (5) we see that one can approximate this integral by the following expression

\[
J_2 = \int d^2 b \left( \exp \left( -\beta U_{12} \right) - 1 \right) \approx -\int_0^{b_0} d^2 b - \beta \int_{b_0}^{\infty} U_{12} d^2 b, \tag{6}
\]

where \( b_0 \) is the value of \( b \) where

\[
U_{12}(b_0) \approx 0. \tag{7}
\]

Thereby, the value of \( b_0 \) in Eq. (A.12) we find as the roots of the following simple equation:

\[
-\frac{A}{b} + \frac{B}{b^2} - C e^Y = 0 \tag{8}
\]

whose solution in the limit of large \( Y \) (high energy limit) is

\[
b_0(Y) \approx e^{-Y/2} \left( \sqrt{\frac{B}{C}} - \frac{A}{2C} e^{-Y/2} \right) = \tilde{b}_0(Y) e^{-Y/2}. \tag{9}
\]

Clearly, at very large values of final rapidity \( Y \), we obtain that \( b_0 \ll B/A \), where \( B/A \) is the inter-particle distance when the external pressure is absent. The reason of this decrease of the inter-particle distance is the presence of the additional pressure introduced in the model. The important observation

\[4\] The value of \( R \) is somehow arbitrary, it can be considered as the radius of proton for example, or as a characteristic length of hot spot inside the collision region.
of these simple calculations is that we observe here a decrease of so-called Knudsen number, see [22], which is
\[ \frac{b_0 A}{B} \ll 1. \]  
(10)
The decrease of the value of Knudsen number, in turn, indicates the transition of the gas to the liquid state under the influence of the external pressure, see again [22]. This observation, in fact, does not depend on the number of dimensions of the spot. For the case of radial symmetry the result will be the same as in the case of three dimensional spot.

2.2 Equation of the state of the gas

Now, we calculate the integral Eq. (A.12):
\[
J_2 = \int d^2 b \left( \exp(-\beta U_{12}) - 1 \right) \approx -\int_0^{b_0} d^2 b - \beta \int_{b_0}^{\infty} U(b) \, d^2 b = 
-\pi b_0^2 + \frac{A\beta}{R^2} \int_{b_0}^{\infty} e^{-b^2/2R^2} \, db - \frac{B\beta}{2R^2} \int_{b_0}^{\infty} \frac{e^{-b^2/2R^2}}{b^2} \, db^2 + \frac{C\beta \gamma}{2R^2} \int_{b_0}^{\infty} (R^2 - b^2) \, db \, db.
\]  
(11)
Performing the integration and keeping in the integrals only the leading terms at large \( Y \) \( (b_0(Y) \ll 1) \) we obtain:
\[
J_2 \approx -\pi b_0^2 + \beta \frac{A}{R} \sqrt{\frac{\pi}{2}} \left( 1 - \sqrt{\frac{2}{\pi}} \frac{b_0}{R} e^{-Y/2} \right) + \beta \frac{B}{2R^2} \left( \gamma - \frac{b_0^2}{2R^2} e^{-Y} \right) + 
+ \beta \frac{2Y + C}{2R^2} \left( R^2 - b_0^2(Y) \right).
\]  
(12)
Using shorter notations, we write the answer as
\[
J_2 \approx -\pi b_0^2 + \beta \left( \tilde{A}(Y) + \tilde{B}(Y) + \tilde{C}(Y) e^Y \right).
\]  
(13)
Therefore, the state equation of the gas in our case has the following form:
\[
P = \frac{N T}{S} - \frac{N^2 T}{S^2} \left( -\pi b_0^2(Y) + \beta \left( \tilde{A}(Y) + \tilde{B}(Y) + \tilde{C}(Y) e^Y \right) \right),
\]  
(14)
or, using the shorter notations again, we have:
\[
P = \frac{N T}{S} - \frac{N^2 T}{S^2} \left( -\pi b_0^2(Y) + \beta \, a(Y) \right).
\]  
(15)
Introducing the gas density variable
\[
\rho = \lim_{N,S \rightarrow \infty} \frac{N}{S},
\]  
(16)
we rewrite the state equation in the following form:
\[
\left( P + \rho^2 a(Y) \right) \left( 1 - \rho \pi b_0^2(Y) \right) = \rho T.
\]  
(17)
This equation of state of the gas is the modified Van der Vaals equation with the coefficients which depend on the rapidity of the process. The following critical parameters of the gas-liquid transition are found from this equation:

\[ \rho_{cr} = \frac{1}{3\pi b_0^2} e^Y, \quad P_{cr} = \frac{\hat{C}}{27\pi b_0^2} e^{5Y/2}, \quad T_{cr} = \frac{8\hat{C}}{27\pi b_0^2} e^{5Y/2}. \] (18)

An approach with a modified Van der Vaals equation in two spatial dimensions was also considered in [5] for the description of the hadron-quark phase transition in heavy ion collision. We briefly will discuss this approach in the Conclusion of the paper.

3 Transport properties of the gas

The equation of state, Eq. (17), we rewrite it in the following form:

\[ P = Z(Y, T) \rho T, \] (19)

where

\[ Z(Y, T) = \frac{1}{1 - \rho \pi b_0^2(Y)} - \beta \rho a(Y) \] (20)

This equation of state, Eq. (19), we further rewrite in the form valid in high energy limit:

\[ P = \rho T \left(1 + \eta Z(\eta)\right), \] (21)

where

\[ Z(\eta) = \frac{1}{1 - \eta}, \] (22)

In this equation, the compressibility factor \( Z(\eta) \) from Eq. (21) coincides with Enskog factor in Eq. (19). Therefore, the transport properties of the gas, which are characterizing by shear viscosity \( \zeta \), bulk viscosity \( \xi \) and heat conductivity \( \kappa \) are determined by the calculations of [16]:

\[ \zeta = \zeta_0 \eta \left(\frac{1}{\eta Z(\eta)} + 1 + 0.87 \eta Z(\eta)\right), \] (23)

\[ \xi = \xi_0 \eta \left(1.25 \eta Z(\eta)\right), \] (24)

\[ \kappa = \kappa_0 \eta \left(\frac{1}{\eta Z(\eta)} + \frac{3}{2} + 0.87 \eta Z(\eta)\right). \] (25)

The Enskog approximation, therefore, allows to find the relative changes of the transport coefficients of the system under the pressure whereas the initial values of these coefficients are given, see also calculations of [17]. In the limit of high energy, whereas \( \eta \ll 1 \), keeping only the leading in \( \eta \) terms, we obtain:

\[ \zeta \approx \zeta_0, \] (26)
\[ \xi = 1.25 \xi_0 \eta^2 Z(\eta) \approx 1.25 \xi_0 e^{-2Y} \left( \pi \tilde{b}_0^2(Y) \rho \right)^2 \left( 1 + e^{-Y} \pi \tilde{b}_0^2(Y) \rho \right), \]  
(27)

\[ \kappa = \kappa_0 \left( 1 + \frac{1}{2} \eta \right). \]  
(28)

We see, that the bulk viscosity \( \xi \) is small in the limit of large final rapidity \( Y \) compared to the case when \( Y = 0 \) for our gas.

The results obtained for the transport coefficients we can consider from the point of view of the results for the quark-gluon plasma models. Calculations of [20] demonstrated that the \( \xi / \zeta \) ratio has a maximum at some critical temperature \( T_c \). Considering the same ratio constructed from Eq. (26)-Eq. (27) values we obtain that this ratio is equal to

\[ \frac{\xi}{\zeta} = \frac{1.25 \xi_0 e^{-2Y} \left( \pi \tilde{b}_0^2(Y) \rho \right)^2}{\zeta_0} \]  
(29)

and that it is small. Moreover, this ratio has a maximum at \( \rho = \rho_{cr} \) and at critical temperature \( T_c \), corresponding to the critical parameters of the Van der Vaals equation Eq. (17):

\[ \frac{\xi}{\zeta} = \frac{1.25 \xi_0}{9 \zeta_0}, \]  
(30)

which is similar to the properties of the same ratio in the quark-gluon plasma model of [20]. Thereby our simple model demonstrates properties which are similar to the properties of much more complicated models of quark-gluon plasma. This result means that in the Enskog approximation, the bulk viscosity is proportional to the density of the gas and the maximum values of the density and bulk viscosity are achieved at the liquid state of the system. Further, when the state of incompressible liquid is achieved due the evolution, the density and the bulk viscosity of the liquid remain constant.

### 4 Shock wave point of view

In this section, we proceed following mainly by the results of paper [18]. Our flow of the compressed interacting particles can be described by the following Euler’s equations for the radial velocity, mass density

\[ \frac{\partial t \sigma + \partial_r (\sigma u)}{u} + \frac{1}{\rho u} = 0, \]  
(31)

\[ \frac{\partial t u + u \partial_r u}{\sigma} + \frac{1}{\sigma \partial_r \sigma} = 0, \]  
(32)

and, assuming an absence of dissipative processes for our gas, for the entropy

\[ \frac{\partial t s + u \partial_r s}{s} = 0. \]  
(33)

\(^5\)The critical temperatures in [20] and here have different meanings, but both of them define higher temperature comparing to initial one.
Here \( u \) is the radial velocity of the flow, \( \sigma = \rho m \) is the mass density of the gas and \( s \) is the entropy of the system:

\[
s = \ln \left( \frac{P}{\sigma^2 \exp(C(\sigma))} \right) + s_0
\]

with

\[
C(\sigma) = - \ln \left( 1 - \sigma \pi \beta_0^2(Y)/m \right)^2,
\]

see [18]. There are three partial differential equations for the main fields which characterize the flow of interacting particles under the external pressure. Solution of this system is considered in the next subsection.

### 4.1 Shock wave description of the process: self-similar solution

The process of the gas compression can be considered as the process of the propagating convergent radial shock wave described by the scaling variable

\[
\xi = \frac{r}{r_{\text{shock}}} = \frac{r}{A(t_0 - t)^\alpha}
\]

where \( A \) and \( \alpha \) are some constants to be determined later. An implosion of the shock wave occurs at time \( t = t_0 \) which corresponds to \( \xi = \infty \) and the front of the shock wave is described by the radius

\[
r_{\text{shock}} = A(t_0 - t)^\alpha.
\]

The fields of interests are assumed to have the following form:

\[
\sigma = \sigma_0 G(\xi),
\]

\[
u = \frac{\alpha}{t - t_0} V(\xi),
\]

\[
P = \frac{\alpha^2 r^2}{2(t - t_0)^2} \sigma_0 G(\xi) W(\xi),
\]

with functions \( G(\xi) V(\xi) W(\xi) \) to be found from the equations Eq. (31) - Eq. (34) which we rewrite as

\[
\frac{dV}{d\ln \xi} + (V - 1) \frac{d\ln G}{d\ln \xi} + 2V = 0,
\]

\[
(V - 1) \frac{dV}{d\ln \xi} + \frac{W d\ln G}{2 d\ln \xi} + \frac{1}{2} \frac{dW}{d\ln \xi} + W - V \left( \frac{1}{\alpha} - V \right) = 0
\]

\[
\frac{dW}{d\ln \xi} = - \frac{d\ln G}{d\ln \xi} - \frac{dC}{dG} \frac{dG}{d\ln \xi} + 2 \frac{1 - \alpha V}{\alpha \left( 1 - V \right)} = 0.
\]

The system of the differential equations Eq. (31) - Eq. (34) can be linearized by expansion of the functions \( V, W \) around \( V(\xi \to \infty) \to 0 \) and \( W(\xi \to \infty) \to 0 \) that corresponds to \( t \to t_0 \) time in
Eq. (39) - Eq. (40), see more details in [18]. The solutions of the linearized equations can be easily found:

\[ V \approx \frac{K_V}{\xi^{1/\alpha}}, \]  
\[ W \approx \frac{K_W}{\xi^{2/\alpha}}, \]

with constants \(K_V\) and \(K_W\) must be found from the full equations. Concerning the \(G\) function from Eq. (38) we note, that this function remains finite at \(\xi \to \infty\):

\[ \frac{d \ln G}{d \ln \xi} \to 0 \quad (46) \]

when \(V(\xi \to \infty) \to 0\) and \(W(\xi \to \infty) \to 0\).

In order to solve the equations Eq. (41) - Eq. (43), we need initial values of our functions \(V(\xi = 1), Z(\xi = 1), G(\xi = 1)\) which could be determined from the matching equations on the discontinuities of the flow:

\[ \sigma_1 u_1 = \sigma_2 u_2, \]  
\[ P_1 + \sigma_1 u_1^2 = P_2 + \sigma_2 u_2^2, \]  
\[ U_1 + \frac{P_1}{\rho_1} + \frac{m u_1^2}{2} = U_2 + \frac{P_2}{\rho_2} + \frac{m u_2^2}{2}. \]

with subscript 1 for the quantities before the shock and subscript 2 for the quantities after. Solutions of these equations in the limit \(P_2 \gg P_1\) were found in [18] and they are the following:

\[ \sigma_2 = \sigma_1 \left(1 + \frac{2}{Z(\eta_2)}\right), \]  
\[ P_2 = \frac{2 \sigma_1 u_1^2}{2 + Z(\eta_2)}, \]  
\[ u_2 - u_1 = -\frac{2 u_1}{2 + Z(\eta_2)}, \]  
\[ u_1 = -\frac{\dot{r}_{\text{shock}}}{r_{\text{shock}}} = \frac{\alpha r_{\text{shock}}}{t - t_0}. \]

Additional equation which relates the densities of the flow is Eq. (38):

\[ \sigma_2 = \sigma_0 G(1) = \sigma_1 G(1) \quad (54) \]

that together with Eq. (50) gives:

\[ G(1) = 1 + \frac{2}{Z(\rho_0 \pi b_0^2(Y) G(1))} = 1 + 2 \left(1 - \rho_0 \pi b_0^2(Y) G(1)\right). \]

Therefore, we obtain for \(G(1)\):

\[ G(1) = \frac{3}{1 + \rho_0 \pi b_0^2(Y)} \]
that in the high energy limit $Y \gg 1$ gives $\rho_0 \pi b_0^2(Y) \ll 1$ and

$$G(1) \approx 3$$

(57)

for any value of $\rho_0$. The functions $V(1)$ and $W(1)$ could be found as well and they have the following form:

$$V(1) = 1 - \frac{1}{G(1)}$$

(58)

and

$$W(1) = 2 \frac{G(1) - 1}{G(1)^2}.$$  

(59)

The numerical values of $\alpha = 0.8$ and $G(\infty) = 4.6$ are also calculated in [18].

4.2 Parameters of the solution and fluid state of the gas

In order to calculate the parameter $A$ from the Eq. (37) we use the condition on the front of shock wave at initial time $t = 0$:

$$r(t = 0) = At_0^\alpha = R$$

(60)

with $R$ from Eq. (3). So far we obtain

$$A = \frac{R}{t_0^\alpha}.$$  

(61)

On the other hand, the velocity of the shock wave at the initial moment may be found from Eq. (39):

$$c_{sh} = V(1) \frac{\alpha R}{t_0^\alpha},$$

(62)

that gives

$$t_0 = \alpha V(1) \frac{R}{c_{sh}}.$$  

(63)

Finally we obtain:

$$A = \frac{R}{t_0^\alpha} = \left( \frac{c_{sh}}{\alpha V(1)} \right)^\alpha R^{1-\alpha}.$$  

(64)

Now we consider the pressure, Eq. (40), achieved in the system of the interest with known solution for $G, V, W$ functions:

$$P = \frac{\alpha^2 r^2}{2 (t - t_0)^2} \sigma_0 G(\xi) W(\xi) = \frac{\alpha^2 \sigma_0 G(\xi) K_W}{2} A^{2/\alpha} r^2 - 2/\alpha.$$  

(65)

Before the compression of the gas the state equation of the gas is the usual one:

$$T_0 = \frac{P_0}{\rho_0},$$

(66)

whereas at the time of the compression Eq. (19) holds:

$$T(\xi) = \frac{P(\xi)}{\rho(\xi) Z(\eta)}.$$  

(67)
Therefore, with the pressure from Eq. (65) we obtain for the temperature ratio:

\[
\frac{T(\xi)}{T_0} = \frac{\alpha^2 \sigma_0 \rho_0}{2 \rho(\xi) P_0 Z(\eta)} G(\xi) K_W A^{2/\alpha} r^{2-2/\alpha}. \tag{68}
\]

Simplifying the expression we obtain finally

\[
T(\xi) = \frac{m c_{sh}^2}{2 Z(\eta) V(1)^2} K_W \left( \frac{R}{r} \right)^{2/\alpha - 2}. \tag{69}
\]

The maximum temperature at the center of the area is achieved when \( \xi \to \infty \) at \( r \to 0 \). Taking this limit and changing \( r \) in the expression by the minimal possible distance Eq. (9), that smooth out the divergence, we obtain the maximum temperature achieved in the system in comparison to the initial temperature \( T_0 = P_0/\rho_0 \):

\[
T(\xi = \infty) = e^{\Delta Y} \frac{m c_{sh}^2}{2 Z(\rho_0 \pi b_0^2(Y) G(\infty)) V(1)^2} K_W \left( \frac{R}{b_0(Y)} \right)^{2/\alpha - 2} \tag{70}
\]

with

\[
\Delta = \frac{1 - \alpha}{\alpha} \approx 0.25. \tag{71}
\]

The maximum temperature achieved can be also compared with the temperature \( T_0 = T(\xi = 1) \) on the edge of the shock wave. In this case we have:

\[
\frac{T(\xi)}{T(1)} = \frac{Z(\eta(1))}{Z(\eta(\xi))} G(1) \left( \frac{R}{r} \right)^{2/\alpha - 2} \tag{72}
\]

and again smoothing out the divergence we obtain:

\[
\frac{T(\infty)}{T(1)} = e^{\Delta Y} \frac{Z(\rho_0 \pi b_0^2(Y) G(1))}{Z(\rho_0 \pi b_0^2(Y) G(\infty))} \left( \frac{R}{b_0(Y)} \right)^{2/\alpha - 2}. \tag{73}
\]

The resulting expressions Eq. (70), Eq. (73) are interesting from the following point of view. Taking in Eq. (73) \( T(\infty) \) as a critical temperature of the gas-liquid transition Eq. (18) we obtain:

\[
\frac{T(\infty)}{T(1)} \propto \frac{T_{cr}}{T(Y = 0)} \approx e^{\Delta' Y/2} \tag{74}
\]

that together with Eq. (73) gives:

\[
\frac{R}{b_0(Y)} \approx e^{\Delta' Y} > 1 \tag{75}
\]

where

\[
\Delta' = \frac{7\alpha - 2}{4 - 4\alpha}. \tag{76}
\]

As mentioned above, it is well known, see [22] for example, that when the characteristic length of the system, which is \( R \) in our case, is larger than the average distance between particles, which is \( b_0(Y) \) approximately, then we can consider a fluid flow instead initial gas state.\(^6\) Thereby we obtain an important result: in our model an indication of the gas-liquid transition is given by the dynamical change of the Knudsen number of the problem. The possibility of calculation of the dynamical change of the Knudsen number is considered in the next section of the manuscript.

\(^6\)The ratio opposite to Eq. (75) ratio is an analog of Knudsen number in hydrodynamics.
5 Boltzmann equation for the self-consistent field

In this section, we generalize the model and consider a hot spot of the gas of the charged particles interacting in three dimensional space under the pressure of the external particles. The Vlasov approach to the Boltzmann equation is valid when dissipative processes are negligible, in this case a local equilibrium is achieved. We assume that this is the case for each separate drop and, thereby, we consider the following Vlasov equation for one-particle distribution function:

\[
\frac{\partial f(r, p, t)}{\partial t} + v \frac{\partial f(r, p, t)}{\partial r} + F(r) \frac{\partial f(r, p, t)}{\partial p} = 0 \tag{77}
\]

where we adopted the radial symmetry of the problem, and where

\[
F(r) = \frac{\partial V_2(r)}{\partial r} + q E = \frac{\partial V_2(r)}{\partial r} - q \frac{\partial \phi_0(r)}{\partial r} \tag{78}
\]

is a force which consists of two terms arose from the self-consistent electrical potential and some external potential. The electric field \(E = -\nabla \phi_0(r)\) in Eq. (78) is a self-consistent electric field created by our charged particles. This field is assumed to be weak enough in order to justify the linear approximation for the distribution function. We assume also that the magnetic field is small and we neglect it in our calculations.\(^8\) The Vlasov’s system of equations include Maxwell’s equations for the electric field:

\[
\text{rot} \, E = 0, \quad \text{div} \, E = 4 \pi q n \int f(r, p, t) \, d^3p, \tag{79}
\]

where \(n\) is a particle’s density.

5.1 Linear approximation: equilibrium state

We solve the Boltzmann equation Eq. (77) representing the distribution function as an equilibrium one with a small linear correction:\(^9\)

\[
f(r, p, t) = f_0(r, p) + f_1(r, p, t), \quad f_1 \ll f_0. \tag{80}
\]

The electric field has the same functional form as well:

\[
E = E^0 + E^1. \tag{81}
\]

\(^7\)We formulate the problem in 3 dimensions because of importance of longitudinal dimension in real high energy interactions. The radial symmetry of the problem in this formulation is preserved as well and in the following we assume that all vectors of interest are radial. Therefore the vector notations will be use only when it will be need in.

\(^8\)More realistic calculations will require the magnetic field inclusion for sure.

\(^9\)The corrections to the equilibrium distribution function and electric field are calculated in the Appendix B.
We are looking for an equilibrium\(^{10}\) state of a hot spot in the presence of an external pressure. The Boltzmann equation which describes the equilibrium state of the hot spot has the following form:

\[
v \frac{\partial f_0(r,p)}{\partial r} + F(r) \frac{\partial f_0(r,p)}{\partial p} = 0 ,
\]

and the solution of this equation is the Boltzmann-Maxwell distribution function:

\[
f_0(r,p) = \frac{1}{(2 \pi k_B m T)^{3/2}} e^{-\frac{r^2}{2 m k_B T}} + \frac{V_2(r)}{r k_B T} - \frac{2 \phi_0(r)}{r k_B T} .
\]

The Maxwell equation for the electric field Eq. (79), therefore, reduces to non-homogeneous Helmholtz equation which has the following form in the limit of high temperatures (large kinetic energies of the particles in comparison to the potential energy):

\[
- \Delta \phi_0(r) = 4 \pi q n \left( 1 + \frac{V_2(r)}{k_B T} - \frac{q \phi_0(r)}{k_B T} \right) ,
\]

or

\[
\Delta \phi_0(r) - 4 \pi q^2 n \frac{\phi_0(r)}{k_B T} = - 4 \pi q n \left( 1 + \frac{V_2(r)}{k_B T} \right) .
\]

We could rewrite it in the following form:

\[
\Delta \phi_0(r) - \frac{\phi_0(r)}{r_D^2} = - Q_0 - \frac{V_2(r)}{q r_D^2} ,
\]

with \( Q_0 = 4 \pi q n \) as the charge density and

\[
r_D^2 = \frac{k_B T}{4 \pi q^2 n}
\]

as the Debye length. The solution of this equation is the sum of the solutions of homogeneous and non-homogeneous Helmholtz equations:

\[
\phi_0(r) = Q_0 r_D^2 + C_0 \frac{e^{-r/r_D}}{r} + \int \frac{r'^2 \, dr' \, V_2(r') \, e^{-|r-r'|/r_D}}{q r_D^2 \, |r-r'|} ,
\]

where \( C_0 \) is some constant determined by the boundary conditions of the problem. In respect that the first term in r.h.s. of Eq. (88) is a constant, we obtain the final expression for the potential:

\[
\phi_0(r) = C_0 \frac{e^{-r/r_H}}{r} + \int \frac{r'^2 \, dr' \, V_2(r') \, e^{-|r-r'|/r_D}}{q r_D^2 \, |r-r'|} .
\]

When the external field is absent we obtain usual expression:

\[
\phi_0(r) = C_0 \frac{e^{-r/r_D}}{r}
\]

where we introduced the sign \( r_D \) for the Debye length in the case when the external pressure is absent.

\(^{10}\)In general, this state is quasi-equilibrium, the external pressure acts during a finite period of time. Nevertheless we assume that this time is long enough and we could consider this state approximately as an equilibrium one.
In order to estimate the \( r \) dependence of Eq. (89) potential we will use the following simple form of the external field potential:

\[
V_2(r) = V(Y) \theta (r_D - r)
\]  

(91)

with the coefficient \( V(Y) \) which depends only on the total rapidity of the process. The substitution of Eq. (91) into Eq. (89) gives:

\[
\phi_0(r) = C_0 e^{-r / r_D} + \frac{V(Y)}{q r_D^2} \int_0^{r_D} r'^2 dr' \frac{e^{-|r - r'|/r_D}}{|r - r'|}.
\]

(92)

In the case of the potential Eq. (91) the solutions for the electric potential Eq. (92) must be considered separately in the two different regions:

1. Region where \( r < r_D \). In this case we have:

\[
\phi_0(r) = C_0 e^{-r / r_D} + \frac{V(Y)}{q r_D^2} \left( \int_0^r r'^2 dr' \frac{e^{-|r - r'|/r_D}}{(r - r')} + \int_r^{r_D} r'^2 dr' \frac{e^{-|r - r'|/r_D}}{(r' - r)} \right).
\]

(93)

2. Region where \( r > r_D \). For these values of \( r \), we obtain:

\[
\phi_0(r) = C_0 e^{-r / r_D} + \frac{V(Y)}{q r_D^2} \left( r^2 - 2r r_D \left( e^{-1+1/r_D} - e^{-r/r_D} \right) + r_D \left( 2r_D - e^{-1+1/r_D} (2r_D - r) - e^{-r/r_D} (r + r_D) \right) \right).
\]

(94)

Integrating this expression and again keeping only leading \( r_D/r \) terms we obtain:

\[
\phi_0(r) = C_0 e^{-r / r_D} + \frac{V(Y)}{q r_D^2} \int_0^{r_D} r'^2 dr' \frac{e^{-|r - r'|/r_D}}{(r' - r)}.
\]

(95)

where \( B_0, B_1, B_2 \) are some constants.

5.2 Debye length changes

The value of the Debye length, Eq. (87), depends on the particle density \( n \) which may vary depending on the external conditions, i.e. the particle density of hot spot is not a constant. Therefore, we consider another parameter, called plasma parameter, which is defined as

\[
\mu = \frac{1}{n \int^{r_D(n)} d^3x \int d^3p f_0(r, p)} \ll 1,
\]

(97)
and which we require to stay the same in the presence and absence of the external field \( V_2 \). From Eq. (97) we see that in the case of the external pressure the Debye radius is smaller than in the case of the absence of the pressure:

\[
(r_D(Y))_{V_2 \neq 0} < (r_D)_{V_2 = 0}.
\]  

Using Eq. (90), Eq. (94) we can estimate this effect of the change in the Debye length. First of all, consider the inverse of the parameter Eq. (97) in the absence of the external field:

\[
1 \ll \frac{1}{\mu} = 4\pi n \int^{r_D(n)} r^2 dr \int d^3p f_0(r,p)
\]

where

\[
f_0(r,p) = \frac{1}{(2\pi k_B T)^{3/2}} e^{-\frac{r^2}{2m k_B T}} - \frac{2\phi_0(r)}{k_B T}.
\]

with \( \phi_0(r) \) from Eq. (90):

\[
\phi_0(r) = C_0 e^{-r/r_0D}.
\]

Simple integration gives:

\[
1 \ll \frac{1}{\mu} \approx 4\pi n \int^{r_D(n)} r^2 dr \left( 1 - \frac{q\phi_0(r)}{k_B T} \right) = N \left( 1 - \frac{C_1 q}{k_B T r_0D} \right),
\]

where \( N = \frac{4}{3} \pi r_D^3 n \) is the number of particles in the spot and \( C_1 \) is a positive constant. In the presence of the external field we have instead Eq. (90) the expression Eq. (89) with the potential given by Eq. (94). Therefore, in the leading in \( r/r_D \) order, we will obtain:

\[
1 \ll \frac{1}{\mu} \approx 4\pi n \int^{r_D(n)} r^2 dr \left( 1 - \frac{q\phi_0(r)}{k_B T} + \frac{V(Y)}{k_B T} \left( A_0 - \frac{r}{r_D} A_1 + \frac{r^2}{r_D^2} A_2 \right) \right)
\]

where \( A_0, A_1, A_2 \) are some positive constants. Integrating Eq. (103) we obtain:

\[
1 \ll \frac{1}{\mu} = N \left( 1 - \frac{C_1 q}{k_B T r_0D} + \frac{V(Y) C_2}{k_B T} \right)
\]

with some positive constant \( C_2 \). We require that both expressions Eq. (102) and Eq. (104) are equal and this gives for the Debye’s length changes:

\[
\frac{r_0D - r_D}{r_0D r_D} = C_2 \frac{V(Y)}{q}
\]

or

\[
r_D = \frac{r_0D}{1 + C_2 V(Y) r_0D / q}
\]

with \( C_2 \) as some positive constant. As it was underlined in previous Section, see also [22], the hydrodynamic description of the process is possible when the characteristic length of the system, in our case

\[\text{It is important that } C_2 \text{ is a positive. In our calculations we obtained } C_2 = \frac{41}{10\pi} - \frac{3}{4}.\]
\( r_D \), begins to decrease, i.e. Knudsen number of the system begins to decrease as well. We determine the Knudsen number as

\[
Kn = \frac{r_D}{r_{0D}} = \frac{1}{1 + C_2 V(Y) r_{0D} / q} < 1,
\]

and we indeed obtain that this number decreases when the external pressure is applied.

Basing on the particles density \( n \approx 100 \text{ fm}^{-3} \) from \cite{[24]} we can very roughly estimate the obtained value of the Knudsen number. Indeed, the external potential is proportional to the overall charge of the clouds of the "fast particles" which interact with the spot of "slow" particles:

\[
V(Y) r_{0D} \propto q N(s),
\]

where \( N(s) \) is a number of particles in the cloud, it depends on the energy of the process. Therefore, very approximately we can write:

\[
Kn \propto \frac{1}{1 + N(s)}.
\]

The estimation of the Debye length at \( T = 160 \text{ MeV} \) gives \( r_{0D} \approx 0.15 - 0.3 \text{ fm} \) for different values of \( n \), see in Fig. (2)-a.\footnote{Interesting to note, that this number is close to the size of a constituent quark obtained in \cite{[25]}.} The number of the fast particles which interact with the hot spot we can estimate as

\[
N \approx \pi r_{0D}^2 R n,
\]

where \( R \) is a size of the "fast" particles cloud in the longitudinal direction. The result of the calculations of \( Kn \) from Eq. (109) as a function of the number of particles inside the disk of the fast charged constituents is given in Fig. (2)-b. Of course, the value of \( N \) crucially depends on the microscopic description of the interactions, we can not determine \( N \) precisely in our "toy" model. Nevertheless, our results shows that we indeed traced the transition of the hot gas spot to the fluid state by the calculation of the change of the Debye radius of the spot.

\section{Conclusion}

In our paper, we considered a toy model of particles interacting at high energy and investigated the transition from gas to the liquid state of the system. The approach proposed has an analogy in the inertial confinement effect in plasma physics and, additionally, is based on the following important propositions:

\begin{itemize}
  \item Reduction of the non-equilibrium process to the equilibrium one by the separation of the degrees of freedom. The idea of \cite{[12]} is that instead accounting of several degrees of freedom we can introduce an external field which will interacts with other, different degrees of freedom.
\end{itemize}
Figure 2: The Debye’s length $r_D$ value as function of the particles density $n$ at $T = 160$ MeV in Fig. (2)-a and the value of Knudsen number $Kn$ as function of $N$ from Eq. (110) in Fig. (2)-b.

- The separation of the degrees of freedom in high-energy interactions. This is an old and well known idea, see for example [13]. In our investigation, we separate the degrees of freedom based on Landau’s paper [7]. There was assumed that at very early stages of interaction, a part of the interacting particles stopped and are in the rest. The drops of these stopped particles is the main object of our investigation. We consider the properties of these drops in the external field of the other, still relativistic particles. All relativistic dynamics holds in the description of these ”fast” particles and, therefore, in the description of the external field. The stopped particles are non-relativistic and can be described by usual techniques.

- The creation of small charged drops of liquid matter in a whole scattering region is a local process. The parameters of the drops are varied locally, but all of them quickly achieve the state of equilibrium due small sizes, which we assume to be proportional to the Debye’s length.

Our main idea, thereby, is to trace the transition of the gas spots into the liquid phase in different models based on these propositions.

For non-equilibrium system of interacting particles, it is almost impossible to write an equation of state of the system in order to trace the gas-liquid transition. Instead, especially for complex systems, this transition can be manifested by the change of so called Knudsen number, well known in hydrodynamics. We calculated this number in different models, see Eq. (10), Eq. (75) and especially Eq. (107)–Eq. (109). We see that in different models, our approach showed the liquid phase creation by the decrease of the Knudsen number. The physical picture behind all these models is simple: like in plasma physics confinement effect, the external particles create an additional pressure on the initially stopped particles compressing dense gas to the liquid state. The calculation of the Debye length changes, Eq. (107)–Eq. (109), is the most important part of our investigation. For almost ideal fluid, the Vlasov’s approximation to the Boltzmann equation found to be an appropriate one and
similar description of the effects must take the place, to our opinion, in high energy nuclei-nuclei QCD scattering as well, see [20].

Concerning the matter of the fluid’s viscosity, we have to note papers [5], where also the modified Van der Vaals equation was considered. There it was argued that the viscosity effects are not small and play a significant role in the dynamics of the phase transition. In general, it is difficult to compare our calculations with calculations of [5] because we used the ”toy” model only, in [5] much more realistic model was proposed and explored. Also, we consider the process of the drop compression whereas in [5] the system’s expansion was mostly considered. Nevertheless, we can underline an important fact that must be significant in both models. In our calculations we consider a drops of small size, \( r_D \approx 0.2 \ m \), as well as the authors of [5]. In this case the drop of the particles is not neutral anymore and it has a charge. In the case of scattering of electron beams it is an electrical one, and, what is more important, it is a color one in the case of the scattering of nuclei at high energy. Therefore, the subject of the consideration is a non-neutral plasma. The properties of this kind of plasma, including viscosity, are drastically different from the properties of the neutral one. Nor electric neither magnetic fields can not be neglected in the description of the dynamics of charged plasma. Therefore, there is a question about an applicability of usual Navier-Stokes equation in the situation when the framework similar to a magnetohydrodynamic approach may be more appropriate.

The transport properties of the system under the external pressure also were considered in our model. For this purpose, we used the well known Enskog calculation scheme, see Eq. (19) and [16, 17]. In this approach, we calculated the corrections to the transport coefficients caused by the external influence on the gas whereas the initial values of the parameters are given, see Eq. (23)-Eq. (25) and Eq. (26)-Eq. (28). Our results demonstrate that the transport properties of the system of interacting particles near the transition into the liquid phase are similar, in some sense, to the properties of the quark-gluon plasma obtained in [20]. Namely, we obtain similar rise of the bulk viscosity when the temperature rises. The explanation of this effect in the model is simple. In Enskog approximation, the bulk viscosity is proportional to the density of the state. The density grows toward some constant value whereas the system of interest undergoes the gas-liquid transition. When the state of ideal liquid is created then the density is constant due to incompressibility of the state and, therefore, the bulk viscosity as well.

There are also the following important issues arose in our framework which were designated but not fully investigated in the manuscript. The drop of ”slow”, stopped particles has a non zero charge. Definitely it must affect on the transport properties of the drop, which is located in the external magnetic and electric fields, and it must affect on the transport properties of whole bulk of scattering matter. The evolution of this mixture of liquid drops and dense quark-gluon gas is an another difficult problem, see also in [5]. We assumed, that the characteristic of the drops are varied from point to point, therefore in the system some long-range interactions must be present.\(^\text{13}\) In some sense,

\(^\text{13}\)Long-range interactions are needed in the light of the Mermin-Wagner theorem.
in our model we have a non-equilibrium system with equilibrated drops of matter inside. Perhaps the statistical description of this system is possible in the framework of so-called "super statistical" description, see[27].

Another issue is a matter of strong correlation between the particles in the QGP, see [1]. It is well known, see [28], that strong correlation means the small number of the particles in the volume determined by the Debye’s length. In turn, this contradicts to the assumption of the applicability of the Vlasov’s equation and to the assumption of the thermodynamical equilibrium of the matter in the interaction area. The possible solution of this contradiction may be found in a small size of the created liquid spots, the thermalization of them happens very fast and it is local. Perhaps, smallness of the drops will allow to describe strongly correlated equilibrated and charged particles inside the drops of hot matter. We note, that in our calculations the number of the particles inside the volume of the spot is indeed small because of the smallness of the Debye’s length.

The question about the magnetic field and additional degrees of freedom in the drop’s creation and evolution was not considered in the paper, in spite of the fact that it is very important problem, see [29]. Indeed, charged drops created in high energy interaction will rotate. The subsequent drop’s evolution dynamics, therefore, is pretty complicated, it will include also additional possible instability effects in the system of charged particles under the external pressure. All these together will influence on multiplicities of particles produced in high energy interactions. The discovering of the traces of these complicated dynamics additionally to the traces of the phase transition in the particle’s production experiments is a very interesting task for the future studying.

Finally we conclude that the main purpose in developing of our toy model was to designate some basic principles and methods for the calculations of the gas-liquid transition in the system of relativistically interacting particles. More detailed analysis of the hot drops condensation in the framework of QED and QCD is the aim of our future work, see [30].
Appendix A: gas of interacting particles

In this appendix we shortly remind the main facts concerning a virial expansion, see the detailed derivation in [23]. We consider a gas of \( N \) interacting particles each with mass \( m \) on a plane as a gas of hard disk with small thickness and radius \( r_0 \). The energy of the gas in the classical limit is given by the well known expression:

\[
E(p, q) = \sum_{i=1}^{N} \frac{p_i^2}{2m} + U(b_1, ..., b_N) \tag{A.1}
\]

where as usual the first term is the kinetic energy of \( N \) particles, \( U \) is a potential energy of their mutual interactions and \( b_1, ..., b_N \) their coordinates. The grand partition function for this Hamiltonian is

\[
Q(\mu, T, S) = e^{-\beta \Omega} = \sum_{N=0}^{\infty} \frac{(e^{\beta \mu} \lambda)^N}{N!} \frac{Z_N(S, \beta)}{N!} \tag{A.2}
\]

and

\[
Z_N(S, \beta) = \int ... \int d^2b_1, ..., d^2b_N \exp \left( -\beta U(b_1, ..., b_N) \right). \tag{A.4}
\]

Here \( \lambda \) is the De Broglie length of the quark corresponding to the average energy \( \beta \):

\[
\lambda = \left( \frac{\hbar^2 \beta}{2 \pi m} \right)^{1/2}. \tag{A.5}
\]

The potential of the problem is given by

\[
\Omega = -\frac{1}{\beta} \ln \left( 1 + e^{\beta \mu} S + e^{2 \beta \mu} \frac{2! \lambda^4}{\lambda^2} \int \int d^2b_1 d^2b_2 \exp \left( -\beta U(b_1, b_2) \right) + ... \right). \tag{A.6}
\]

Here, we used

\[
\int d^2\vec{b} = S = \pi R^2, \tag{A.7}
\]

where \( R^2 \) is the characteristic radius of the problem. In the following we will define and consider only pairwise interaction between the particles, namely we have

\[
U(b_1, b_2) = U(|b_1 - b_2|) = U(b_{12}) = U(b) = U_{12} \tag{A.8}
\]

and therefore, in the relative coordinates of the center mass we reduce the multiplicity of the integrated functions and obtain an additional \( S \) factor in the integrals:

\[
\Omega = -PS = -\frac{1}{\beta} \ln \left( 1 + S e^{\beta \mu} \lambda^2 + S e^{2 \beta \mu} \frac{2! \lambda^4}{2! \lambda^4} \int \int d^2\vec{b}_{12} \exp \left( -\beta U_{12} \right) + ... \right). \tag{A.9}
\]
Introducing variable $\zeta$

$$\zeta = \frac{e^{\beta \mu}}{\lambda^2}$$  \hspace{1cm} (A.10)

we obtain the expression for the potential in the form of the series in $\zeta$

$$\Omega = -PS = -\frac{S}{\beta} \sum_{n=1}^{\infty} \frac{J_n}{n!} \zeta^n.$$  \hspace{1cm} (A.11)

We will take into account only two first terms of this series with the following $J_1$ and $J_2$:

$$J_1 = 1, \quad J_2 = \int \int d^2 \vec{b}_{12} \left( \exp \left( -\beta U_{12} \right) - 1 \right).$$  \hspace{1cm} (A.12)

The number of particles in this gas we obtain as usual

$$N = -\left( \frac{\partial \Omega}{\partial \mu} \right)_{T,S}$$  \hspace{1cm} (A.13)

and because $\partial \zeta / \partial \mu = \beta \zeta$ we finally have:

$$N = S \sum_{n=1}^{\infty} \frac{J_n}{(n-1)!} \zeta^n.$$  \hspace{1cm} (A.14)

Excluding from Eq. (A.11) and Eq. (A.14) the variable $\zeta$, we obtain in the second order approximation the equation of state for our gas:

$$P = \frac{NT}{S} - \frac{N^2 T}{2S^2} J_2,$$  \hspace{1cm} (A.15)

where $T = 1/\beta$. 

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Appendix B: deviation from equilibrium state and non static electrical field of the gas in Vlasov’s approximation

In the linear approximation over \( f_0(r,p) \), we have the Boltzmann equation
\[
\frac{\partial f_1(r,p,t)}{\partial t} + v \frac{\partial f_1(r,p,t)}{\partial r} + \left( -\frac{\partial V_2(r)}{\partial r} + q E^0 \right) \frac{\partial f_1(r,p,t)}{\partial p} + q E^1 \frac{\partial f_0(r,p)}{\partial p} = 0 \tag{B.1}
\]
together with the Maxwell’s equations for the electric field:
\[
\text{rot} \, E^1 = 0, \quad \text{div} \, E^1 = 4\pi q n \int f_1(r,p,t) \, d^3p, \tag{B.2}
\]
where
\[
E^0 = -\text{grad} \phi_0(r) \tag{B.3}
\]
with \( \phi_0(r) \) given by Eq. (83). Solutions for the \( f_1(r,p,t) \) distribution function and \( E^1 \) field we are deriving below.

Proceeding with the equation Eq. (80) we perform a following substitution:
\[
f_1(r,p,t) = \bar{f}_1(r,p,t) f_0(r,p) \tag{B.4}
\]
with \( f_0(r,p) \) from Eq. (83). Rewriting Eq. (B.1) we obtain:
\[
\frac{\partial \bar{f}_1(r,p,t)}{\partial t} + \frac{p}{m} \frac{\partial \bar{f}_1(r,p,t)}{\partial r} + F \frac{\partial \bar{f}_1(r,p,t)}{\partial p} - \frac{q E^1}{m k_B T} p = 0, \tag{B.5}
\]
where \( F \) is given by Eq. (78). We consider a linear approximation over the equilibrium distribution, therefore, we could write the external force in the equation in following form
\[
F(r) \rightarrow \bar{F} = F(\bar{r}) = -\left( \frac{\partial V_2(r)}{\partial r} \right)_{\bar{r}=\bar{r}} + q E^0(\bar{r}). \tag{B.6}
\]
with \( r \)
\[
\bar{r} = \frac{\int r f_0(r,p) \, d^3x \, d^3p}{\int f_0(r,p) \, d^3x \, d^3p}. \tag{B.7}
\]
Thereby we have the following equation for the \( \bar{f}_1(r,p,t) \) distribution function:
\[
\frac{\partial \bar{f}_1(r,p,t)}{\partial t} + \frac{p}{m} \frac{\partial \bar{f}_1(r,p,t)}{\partial r} + \bar{F} \frac{\partial \bar{f}_1(r,p,t)}{\partial p} - \frac{q E^1}{m k_B T} p = 0 \tag{B.8}
\]
Performing Fourier transform of \( \bar{f}_1(r,p,t) \) and \( E^1(r,t) \)
\[
\bar{f}_1(r,p,t) = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{-i\omega t + i r k} \phi_0(k,p,\omega) \tag{B.9}
\]
\[
E^1(r,t) = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{-i\omega t + i r k} \tilde{E}^1(k,\omega) \tag{B.10}
\]
we obtain finally a nonlinear differential equation of the first order over momenta:

$$\frac{\partial \phi_0}{\partial p} + a p \phi_0 - b \phi_0 - c p = 0.$$  \hfill (B.11)

Here

$$a = \frac{ik}{m F}, \quad b = \frac{i \omega}{F}, \quad c = \frac{q \tilde{E}^1}{m k_B T F}.$$  \hfill (B.12)

The solution of this equation, with additional condition that at $p \to 0 \ (T \to 0)$ the solution is static, i.e. $\phi_0(k, p = 0, \omega) = 0$, is the following function:

$$\phi_0(k, p, \omega) = c e^{-a p^2 / 2 + b \int_0^p \int_0 \int_\omega E \bar{F}^1(k, \omega) e^{-i \omega (p' - p)} + i k \left( r + \frac{p'^2}{2mF} - \frac{p^2}{2mF} \right)}.$$  \hfill (B.13)

Thereby we obtain for the correction to the equilibrium state:

$$\phi_0(k, p, \omega) = q \tilde{E}^1 e^{-a p^2 / 2 + b \int_0^p \int_0 \int_\omega E \bar{F}^1(k, \omega) e^{-i \omega (p' - p)} + i k \left( r + \frac{p'^2}{2mF} - \frac{p^2}{2mF} \right)}.$$  \hfill (B.14)

We see, that this correction is suppressed in comparison to the distribution function Eq. (83) by the factor $q \tilde{E}^1$. Substituting this expression back in Eq. (B.9), we have:

$$\bar{f}_1(r, p, t) = \frac{q}{m k_B T F} \int_0^p \int_0 \int_\omega E \bar{F}^1(r + \frac{p'^2 - p^2}{2mF}, t + \frac{p' - p}{F}) \bar{f}_0(k, p).$$  \hfill (B.15)

Performing Fourier transform again, we obtain for our function in $(r, t)$ representation:

$$\bar{f}_1(r, p, t) = \frac{q}{m k_B T F} \int_0^p \int_0 \int_\omega E \bar{F}^1(r + \frac{p'^2 - p^2}{2mF}, t + \frac{p' - p}{F}) \bar{f}_0(k, p).$$  \hfill (B.16)

This correction to the static distribution function determines also the non-static electric field which we consider further.

Using Eq. (79), we find the equation for the correction $E^1$ to the electric field:

$$\text{div} E^1(r, t) = \frac{4 \pi q^2 n}{m k_B T F} \int d^3 p \int d^3 p' E^1(r + \frac{p'^2 - p^2}{2mF}, t + \frac{p' - p}{F}) f_0(r, p)$$  \hfill (B.17)

with with $f_0(r, p)$ from Eq. (83), see also Eq. (B.4). This equation is highly non-linear and non-local and, perhaps, a precise solution of the equation can be found only by numerical methods. In order to investigate approximate solutions of the equation, we perform Fourier transform of the functions in Eq. (B.17):

$$k \tilde{E}^1(k, \omega) = \frac{4 \pi q^2 n}{m k_B T F} \int d^3 p \int d^3 p' E^1(k - k', \omega) e^{-i \omega \left( \frac{p'}{mF} \right) + i (k - k') \left( \frac{p'^2 - p^2}{2mF} \right)} \tilde{f}_0(k', p),$$  \hfill (B.18)

where

$$f_0(r, p) = \frac{d^3 k}{(2 \pi)^3} e^{ir \cdot k} \tilde{f}_0(k, p).$$  \hfill (B.19)
In the right hand side of the equation the oscillating integral over $k'$ is not vanishing when

$$k' \propto \frac{2m\tilde{F}}{p^2 - p'^2} \propto \frac{\tilde{F}}{k_B T} \ll 1$$  \hspace{1cm} (B.20)

in our approximation of the weak external field. Therefore, in the region where

$$k > k'$$  \hspace{1cm} (B.21)

in the first approximation over $k'$ we have:

$$k\tilde{E}^1(k, \omega) \approx \tilde{E}^1(k, \omega) \frac{4\pi q^2 n}{m k_B T F} \int d^3 p \int_0^p dp' \int \frac{d^3 k'}{(2\pi)^3} e^{-i\omega \left( \frac{\sqrt{p^2 - p'^2}}{2m F} + i(k-k') \left( \frac{\sqrt{p^2 - p'^2}}{2m F} \right) \right)} \bar{f}_0(k', p) - \frac{\partial \tilde{E}^1(k, \omega)}{\partial k} \int d^3 p \int_0^p dp' \int \frac{d^3 k'}{(2\pi)^3} e^{-i\omega \left( \frac{\sqrt{p^2 - p'^2}}{2m F} + i(k-k') \left( \frac{\sqrt{p^2 - p'^2}}{2m F} \right) \right)} \bar{f}_0(k', p)$$  \hspace{1cm} (B.22)

or, simplifying the notations we obtain:

$$\frac{\partial \tilde{E}^1(k, \omega)}{\partial k} - \tilde{E}^1(k, \omega) (C_1(\omega, k) - C_2(\omega, k)) = 0,$$  \hspace{1cm} (B.23)

where

$$C_1(\omega, k) = \frac{\int d^3 p \int_0^p dp' \int \frac{d^3 k'}{(2\pi)^3} e^{-i\omega \left( \frac{\sqrt{p^2 - p'^2}}{2m F} + i(k-k') \left( \frac{\sqrt{p^2 - p'^2}}{2m F} \right) \right)} \bar{f}_0(k', p)}{\int d^3 p \int_0^p dp' \int \frac{d^3 k'}{(2\pi)^3} k' e^{-i\omega \left( \frac{\sqrt{p^2 - p'^2}}{2m F} + i(k-k') \left( \frac{\sqrt{p^2 - p'^2}}{2m F} \right) \right)} \bar{f}_0(k', p)},$$  \hspace{1cm} (B.24)

and

$$C_2(\omega, k) = \frac{m k_B T \tilde{F}}{4\pi q^2 n \int d^3 p \int_0^p dp' \int \frac{d^3 k'}{(2\pi)^3} k' e^{-i\omega \left( \frac{\sqrt{p^2 - p'^2}}{2m F} + i(k-k') \left( \frac{\sqrt{p^2 - p'^2}}{2m F} \right) \right)} \bar{f}_0(k', p)} \hspace{1cm} (B.25)$$

An integration of Eq. (B.22) gives

$$\tilde{E}^1(k, \omega) = C_0(\omega) e^{\int_0^k C_1(\omega, t) dt} - f_0^k C_2(\omega, t) t dt.$$  \hspace{1cm} (B.26)

We see, that the fluctuation of electric field in this region of $k$ is suppressed by the large factor $C_2$ in the power of exponent.

In the opposite limit, when

$$k \sim k' \propto \frac{2m\tilde{F}}{p^2 - p'^2} \propto \frac{\tilde{F}}{k_B T} \ll 1$$  \hspace{1cm} (B.27)

we introduce new variable $\epsilon$:

$$\epsilon = k - k'.$$

Equation Eq. (B.22) will acquire the following form:

$$k\tilde{E}^1(k, \omega) = \frac{4\pi q^2 n}{m k_B T F} \int d^3 p \int_0^p dp' \int \frac{d^3 \epsilon}{(2\pi)^3} \tilde{E}^1(\epsilon, \omega) e^{-i\omega \left( \frac{\sqrt{p^2 - p'^2}}{2m F} + i\epsilon \left( \frac{\sqrt{p^2 - p'^2}}{2m F} \right) \right)} \bar{f}_0(k - \epsilon, p).$$  \hspace{1cm} (B.28)
Fourier transform of function Eq. is

\[ \tilde{f}_0(k, p) = \int d^3 r \, e^{-i r \cdot k} f_0(r) \approx f_0(p) \left( (2\pi)^3 \delta^3(k) - \frac{\tilde{V}_2(k)}{k_B T} - \frac{q \tilde{\phi}_0(k)}{k_B T} \right) \]

with

\[ f_0(p) = \frac{1}{(2\pi k_B T)^{3/2}} e^{-\frac{p^2}{2 k_B T}} \]

therefore, in the first approximation of expansion over \( \epsilon \) we obtain:

\[ \tilde{E}_1(k, \omega) = \tilde{E}_1(k, \omega) \frac{4\pi q^2 n}{m k_B TF} \int d^3 p \, f_0(p) \int_0^p \, d p' \, p' \, e^{-i \omega \left( \frac{p'}{p} \right) + i k \left( \frac{p'^2 - p^2}{2m} \right)} - \]

\[ - \tilde{E}_1(0, \omega) \frac{4\pi q^2 n \tilde{V}_2(k)}{m (k_B T)^2 F k} \int d^3 p \, f_0(p) \int_0^p \, d p' \, p' \, e^{-i \omega \left( \frac{p'}{p} \right) + \frac{p'}{k_B T} \int \frac{d \epsilon}{2\pi^2} \frac{\epsilon}{F} - \frac{q \tilde{\phi}_0(k)}{k_B T} k} \int d^3 p f_0(p) \int_0^p \, d p' \, p' \, e^{-i \omega \left( \frac{p'}{p} \right) + \frac{p'}{k_B T} \int \frac{d \epsilon}{2\pi^2} \frac{\epsilon}{F}} \]

Finally we obtain for our electric field:

\[ \tilde{E}_1(k, \omega) \varepsilon = - \tilde{E}_1(0, \omega) \frac{2 q^2 n \tilde{F}^2}{3 \pi m (k_B T)^4 k} \left( \frac{\tilde{V}_2(k)}{k_B T} + \frac{q \tilde{\phi}_0(k)}{k_B T} \right) \int d^3 p \, f_0(p) \int_0^p \, d p' \, p' \, e^{-i \omega \left( \frac{p'}{p} \right)} \]

where

\[ \varepsilon = 1 - \frac{4\pi q^2 n}{m k_B TF k} \int d^3 p \, f_0(p) \int_0^p \, d p' \, p' \, e^{-i \omega \left( \frac{p'}{p} \right) + i k \left( \frac{p'^2 - p^2}{2m} \right)} \]

is the dielectric constant of the problem.
References

[1] E. Shuryak, Nucl. Phys. B 195 111 (2009); E. Shuryak, Prog. Part. Nucl. Phys. 62, 48 (2009).
[2] B. Berdnikov and K. Rajagopal, Phys. Rev. D 61, 105017 (2000).
[3] M. Nahrgang, C. Herold and M. Bleicher, Nucl. Phys. A904-905 2013, 899c (2013).
[4] J. Steinheimer and J. Randrup, Phys. Rev. Lett. 109, 212301 (2012).
[5] V. V. Skokov and D. N. Voskresensky, Nucl. Phys. A 828, 401 (2009); V. V. Skokov and D. N. Voskresensky, JETP Lett. 90, 223 (2009).
[6] J. Randrup, Phys. Rev. C 79, 054911 (2009).
[7] L.D.Landau, Izv. Akad. Nauk: Ser.Fiz.17, 51,(1953).
[8] D. Kharzeev, E. Levin and K. Tuchin, Phys. Rev. C 75, 044903 (2007); E. K. G. Sarkisyan and A. S. Sakharov, Eur. Phys. J. C 70, 533 (2010).
[9] U.Heinz, Phys. Rev. Lett. 51, 351 (1983); G.F.Bertsch and S.Das Gupta, Phys. Rep. Rep. 160, 4 189 (1988); T.Peter and J.Meyer-ter-Vehn, Phys. Rev. A 43, 1998 (1991).
[10] M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. D 60, 114028 (1999).
[11] G. Torrieri, arXiv:0911.5479 [nucl-th].
[12] M.A.Leontovich, ZhETF 8, 7, 844 (1938); Yu.L.Klimontovich, Sov. Phys. Usp. 26, 366 (1983).
[13] R. Kirschner, L. N. Lipatov and L. Szymanowski, Nucl. Phys. B 425, 579 (1994); L. N. Lipatov, Nucl. Phys. B 452, 369 (1995); I. Balitsky, Nucl. Phys. B 463, 99 (1996); E. N. Antonov, L. N. Lipatov, E. A. Kuraev and I. O. Cherednikov, Nucl. Phys. B 721, 111 (2005); M. A. Braun, L. N. Lipatov, M. Y. Salykin and M. I. Vyazovsky, Eur. Phys. J. C 71, 1639 (2011).
[14] S.Pfalzner, ”An introduction to inertial confinement fusion”, Taylor and Francis press.
[15] L. N. Lipatov, Sov. J. Nucl. Phys. 23, 338 (1976) [Yad. Fiz. 23 (1976) 642]; E. A. Kuraev, L. N. Lipatov and V. S. Fadin, Sov. Phys. JETP 45, 199 (1977) [Zh. Eksp. Teor. Fiz. 72, 377 (1977)]; I. I. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. 28, 822 (1978) [Yad. Fiz. 28, 1597 (1978)].
[16] S.Chapman and T.G.Cowling, ”The mathematical theory of non-uniform gases”, Cambridge University Press; D.M.Gass, J. Chem. Phys. 54,1898 (1971).
[17] M. Prakash, M. Prakash, R. Venugopalan and G. Welke, Phys. Rept. 227, 321 (1993); A. Wiranata and M. Prakash, Phys. Rev. C 85, 054908 (2012).
[18] P. Gaspard and J. Lutsko, Phys. Rev. E 70, 026306 (2004).

[19] L. P. Csernai, ”Introduction to relativistic heavy ion collisions”, Chichester, UK: Wiley (1994).

[20] D. Kharzeev and K. Tuchin, JHEP 0809, 093 (2008).

[21] E. Levin, Heavy Ion Phys. 8, 265 (1998).

[22] Yu. L. Klimontovich, Sov. Phys. Usp. 167, 23 (1997).

[23] L. D. Landau, E. M. Lifshitz and L. P. Pitaevskii, ”Statistical Physics”, Part 1 (Course Theoretical Physics, Volume 5).

[24] D. Teaney, Phys. Rev. C 68, 034913 (2003).

[25] S. Bondarenko, E. Levin and J. Nyiri, Eur. Phys. J. C 25, 277 (2002).

[26] G. Kelbg, Annalen der Physik, 7, 12 (1963); V. V. Dixit, Mod. Phys. Lett. A 5, 227 (1990); K. Dusling and C. Young, arXiv:0707.2068 [nucl-th].

[27] C. Beck, arXiv:0705.3832 [cond-mat.stat-mech]; C. Beck, Eur. Phys. J. A 40, 267 (2009).

[28] S. Ichimaru, H. Iyetomi, S. Tanaka, Physics Reports 149, Issues 2-3, 91 (1987).

[29] R. C. Davidson, ”Physics of nonneutral plasmas”, Addison-Wesley Pub. Co.: California, 1990.

[30] S. Bondarenko, K. Komoshvili, ”Modeling of changes in the Debye length for the collision processes at high energies”, in preparation.