Abstract

Linear systems under the influence of nonlinear and random linear perturbations, and with random initial and boundary conditions, are discussed. The notion of states of a system is substituted by the notion of the generating vectors for n-point information (n-pi), n=0,1,2,..., characterizing a system at different space-time points. A complete system of equations for physical n-pi are derived and certain universal expansions are proposed.

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1 Introduction

When the external and/or internal, multiplicative and/or additive random perturbations of a dynamical system appear, denoted symbolically by $\zeta$, then it is assumed that the initial conditions (IC) cease to have physical value, see [1]. In this situation, we are looking for such approximations to solutions of considered equations in which such dependence on the IC would be suppressed, or, at least, this would be the case, for appropriately large times, see again [1] and [2, 3].

In this study we realize the above premise by the demand that IC are also random variables. IC, however, differ from the $\zeta$ that do not appear in considered equations explicitly. This fact significantly facilitates the derivation of the equations for the so-called n-point information (n-pi), see [5, 4]. In these equations, perturbations $\zeta$ of a system are treated as parameters according to which some sort of averaging or smoothing will be done when equations for n-pi will be appropriately transformed. For example, equations for n-pi will be successfully closed (the closure problem). Not to be underestimated is the fact that the equations for n-pi are linear. This means that after solving the closure problem and after going to the stochastical equations ($\zeta$ are treated as random variables) - all the time - we are dealing with linear equations. Such is the advantage of using equations in which the IC are treated as random variables. But that's not all: the linear equation for n-pi can be formulated in the free Fock space, and this makes the individual operators appearing in the derived equations to have interesting property: they are explicitly right-or left-reversible. This opens up access to the Algebraic Analysis, which provides a number of useful statements and formulas, see [10] and [6]. To clarify confusions caused by double meaning of that term, see [11]. It is conceivable that the ease of building a number of inverse operators is related to the fact that we use generating vectors for multitime n-pi. This may mean that the use of such vectors is a more appropriate tool than the use of the concept of the state vectors associated with a given time.

It is remarkable that, for the averaged quantities which are the n-pi, the linear equations are obtained in the case of linear as well as nonlinear original Eq.7. In other words, less detailed description of the system is expressed by means of linear equations. A similar situation can be found in quantum mechanics, see Schrödinger equations. In the paper, we propose a modification of derived equations for n-pi by adding a 'quantum term', see Sec.4. It turns out that such a term can also be obtained in the case of essentially non-linear original theory [7], if in Eq.8 an appropriate definition of the operator-valued functions is used, see [6]. In a quantum description of systems this operator is responsible for existing nontrivial perturbative solutions even in the case of non-singular operators $\hat{L}$. 


2 All possible states of a considered system

Let us assume that all possible states of the system are described by some function-functional \( f-f \) \( \varphi \) depending on the three kinds of quantities: \( \tilde{x}, \zeta, \alpha \):

\[
\varphi = \varphi[\tilde{x}; \zeta, \alpha]
\]  

(1)

In Eq. (1) the components of \( \tilde{x} \) describe different points in the space-time and different components of the field \( \varphi, \zeta \) - represents different perturbations acting on the system and \( \alpha \) - expresses IC, or/and boundary conditions imposed on the system. With this interpretation of \( \alpha \) we can think that Eq. (1) expresses the causality relation especially in the situation in which the \( \varphi \) and \( \alpha \) belong to the same type of entities. We used here square brackets to express possible functional dependence of \( \varphi \) on \( \zeta \) and \( \alpha \).

What can you say about the system without precise specification of variables \( \tilde{x}, \zeta, \alpha \)? We can not say that it is a discrete or continuous system, but we can say that the system is subjected to some 'unwanted' and possibly random perturbations (forces), which are denoted by \( \zeta \). If that being the case, the IC (initial conditions) (and probably the boundary conditions) no longer have physical meaning, and the same can be said about the various states of the system. As a particular case of (1) is \( \varphi \) which does not depend on \( \zeta \):

\[
\varphi = \varphi[\tilde{x}; \alpha]
\]  

(2)

But even in this case, the IC, \( \alpha \), can also be non-physical entities when the states described by f-f \( \varphi \) are very sensitive to small changes of \( \alpha \). The same can be said when \( \zeta \) is not random, but the solutions are very sensitive to small changes of initial conditions (IC).

In both cases IC, \( \alpha \), are treated as a random variable(s) and one can derive similar equations upon a more physical entities as the averages, or correlation functions, or other quantities which we call n-pi. So, in general case, we will consider n-pi:

\[
V[\tilde{x}(n); \zeta] \equiv \int \varphi[\tilde{x}_1; \zeta, \alpha] \cdots \varphi[\tilde{x}_n; \zeta, \alpha] W[\zeta, \alpha] \delta[\alpha]
\]  

(3)

for \( n = 1, 2, \ldots, \infty \), where \( W \) is a weighting functional, or density of probability that the random variable \( A \) takes a value \( x \) and \( \Delta \) takes a value \( \zeta \). (They can be functions and so \( \int \) can means the functional integral). \( \delta[\alpha] \) is a generalization of differential \( dx \) to the case of function \( \alpha \), see \[13\]. In fact, one can use a more general integrals like this:

\[
V[\tilde{x}(n); \zeta] = < \varphi[\tilde{x}_1; \zeta, \alpha] \cdots \varphi[\tilde{x}_n; \zeta, \alpha] > \equiv \int \varphi[\tilde{x}_1; \zeta, \alpha] \cdots \varphi[\tilde{x}_n; \zeta, \alpha] d\mu[\zeta, \alpha]
\]  

(4)

in which the appropriately defined measure, or pseudo measure, \( \mu \), depends on \( \zeta \).
Now we say something about the equations satisfied by $\varphi$: The basic idea that we take, and that comes from Newton’s is that their form does not depend on $\alpha$. This which the random variable $\zeta$ differs from $\alpha$ is that $\zeta$ explicitly enters the equations, which satisfies $\varphi$. This facts make possible to derive a complete system of equations for $V[\tilde{x}(n); \zeta]$, and then we can use the formula

$$V(\tilde{x}(n)) = \int V[\tilde{x}(n); \zeta]d\mu(\zeta)$$

(5)

to obtain quantities reflecting in some way our observations. In fact, sometimes it is assumed that $\zeta$ is a generalized function. In this case, we will assume that $\zeta$ are the usual functions that depend on parameter(s) of which we go to the limes after these integrations. In fact, remarks of App.4 show that such a trick is not necessary.

So, we will assume that IC, $\alpha$, not appear in explicit way in the considered equation:

$$L[\tilde{x}, \zeta; \varphi] + \lambda N[\tilde{x}, \zeta; \varphi] + G[\tilde{x}, \zeta] = 0$$

(6)

Here, $L$ depends linearly on $\varphi$, $N$ - nonlinear, and $G$ does not depend on $\varphi$ at all. $\lambda$ is an expansion parameter describing the strength of nonlinear perturbation of the linear (kinematic) theory.

We assume that dependence on the random perturbations $\zeta$ in Eq.6 is linear. Therefore, we consider equation:

$$L[\tilde{x}; \varphi] + \lambda N[\tilde{x}; \varphi] + G(\tilde{x}) + \zeta(\tilde{x}')\Delta[\tilde{x}'; \varphi] = 0$$

(7)

In the last term we understand summation with respect to certain components of the vector $\tilde{x}'$. We remind you that in the previous author’s papers, to save space, we put in $\tilde{x}$ the discrete indexes of the field $\varphi$. We can do the same, for the perturbation field $\zeta$. The $\tilde{x}'$ means that there are additional indexes in comparison with $\tilde{x}$ and the summation in the last part of the Eq.7 applies to these indexes.

For $\Delta = \text{const}$ in $\varphi$, we have additive perturbations. In other case, we have multiplicative perturbations. In fact, $\Delta$ can be proportional to $L, N, G$ and to another quantity. In the paper, we will assume, for $\Delta$, a linear dependence on $\varphi$ and certain quantum mechanical generalization.

Eq.7 describes micro-properties of the system, which are too detailed. Less detailed information about the system are described by the linear equation

$$\left(\hat{L} + \lambda \hat{N} + \hat{G} + \zeta \cdot \hat{\Delta} \right)| V; \zeta > = |0; \zeta >_{\text{info}} = \hat{P}_0|0; \zeta >_{\text{info}}$$

(8)

for the vector $| V; \zeta >$ generating n-pi given by the formulas (3) or (4), see also Eq.9 and Eq.13. A very simple derivation of this equation and explicit forms of operators $\hat{L}, \hat{N}, \hat{G}$, and definition of ‘vacuum’ vector $|0; \zeta >_{\text{info}}$, can be found in [7]. See, however, App.1 where it is shown that in a certain cases an appropriate set of ‘less detailed information’ about the system may be used to retrieve detailed information.
Let us notice the difference between meaning of symbols $N$ and $\hat{N}!!$. The first means nonlinear functional, the second means a linear operator which is closely related to the first.

Uncontrolled perturbations $\zeta$ appearing in this equation can be treated as a fixed quantity under which we will make averaging when we find a solution to Eq.\ref{eq:8} represented by the series:

$$|V;\zeta> = \sum_{n=1}^{\infty} \int d\tilde{x}_{(n)} V[\tilde{x}_{(n)};\zeta]|\tilde{x}_{(n)}> + |0;\zeta>_{info} \tag{9}$$

with the n-point functions $V[\tilde{x}_{(n)};\zeta]$ called the $n$-point information (n-pi) about the system, which have interpretation given by Eqs \ref{eq:3} or \ref{eq:4}. The vectors $|\tilde{x}_{(n)}>$, for $n=1,2,...$, are linearly independent orthonormal vectors explicitly expressed by Eqs \ref{eq:13}. The vector $|0;\zeta>_{info}$ describes so called the local information vacuum, see \ref{eq:12} and \ref{eq:8}, and as the only basis vectors does not depend on the $\tilde{x}$, see also App.2.

The operator $\hat{L}$ is a right invertible operator, which in the case of classical statistical field theory is a diagonal operator with respect to the projectors $\hat{P}_n; n=1,2,...$:

$$\hat{P}_n \hat{L} = \hat{L} \hat{P}_n \tag{10}$$

where the project $\hat{P}_n$ projects on the n-th term in the expansion \ref{eq:9}. The projector $\hat{P}_0$ projects on the subspace generated by the vector $|0;\zeta>_{info}$.

In the case of quantum field theory, the operator $\hat{L}$ is an invertible or right invertible diagonal, plus a lower triangular operator related to the commutation relations of the canonical conjugate operator variables, with respect to the same set of projectors $\hat{P}_n$.

In the case of polynomial nonlinearity, the operator $\hat{N}$ is an upper triangular operator in a classical as well as in quantum field theory:

$$\hat{P}_n \hat{N} = \sum_{n\leq m} \hat{P}_n \hat{N} \hat{P}_m \tag{11}$$

see \ref{eq:8}.

The operator $\hat{G}$, in the both cases, is a left invertible operator, which is lower triangular operator:

$$\hat{P}_n \hat{G} = \sum_{m<n} \hat{P}_n \hat{G} \hat{P}_m \tag{12}$$

Similar property has the operator $\hat{\Delta}$ discussed in Sec.4.

All these operators are linear operators considered in the free Fock space (FFS) constructed by means of the vectors like \ref{eq:9} in which

$$|\tilde{x}_{(n)}> = \hat{\eta}^*(\tilde{x}_1)\cdots \hat{\eta}^*(\tilde{x}_n)|0> \tag{13}$$

and the operators $\hat{\eta}^*$ satisfy the Cuntz relations:
\[ \hat{\eta}(\tilde{y})\hat{\eta}^*(\tilde{x}) = \delta(\tilde{y} - \tilde{x}) \cdot \hat{I} \]  \hspace{1cm} (14)

where \((\hat{\eta}^*(\tilde{x}))^* = \hat{\eta}(\tilde{x})\), \(\hat{I}\) is the unit operator in FFS and other relations take place:

\[ \hat{\eta}(\tilde{y})|0 >= 0, \quad <0|\hat{\eta}^*(\tilde{y}) = 0 \]  \hspace{1cm} (15)

see [8]. Moreover, we will assume that all components of vectors \(\tilde{x}, \tilde{y}\) are discrete so that \(\delta(\tilde{y} - \tilde{x})\), in fact, is Kronecker delta. The operator \(\zeta \cdot \Delta\) will be described in Secs 4 and 5.

3 A dominant role of kinematic term \(\hat{L}\)

In this case a solution is sought in the form of series:

\[ |V; \zeta > = \sum_{j=0}^{\infty} \lambda^j |V; \zeta >^{(j)} \]  \hspace{1cm} (16)

To get subsequent approximations, \(\lambda^j |V; \zeta >^{(j)}\), we transform Eq.8 as follows: we will assume that the kinematic term \(\hat{L}\) is a right invertible operator:

\[ \hat{L}L^{-1}_R = \hat{I} \]  \hspace{1cm} (17)

We get then the equivalent equation to the Eq.8

\[ \left( \hat{I} + \hat{L}^{-1}_R \left( \lambda \hat{N} + \zeta \cdot \hat{\Delta} \right) + \hat{L}^{-1}_R \hat{G} \right) |V; \zeta >= \] 

\[ \hat{L}^{-1}_R |0 >_{info} + \hat{P}_L |V; \zeta > \]  \hspace{1cm} (18)

where a projector

\[ \hat{P}_L = \hat{I} - \hat{L}^{-1}_R \hat{L} \]  \hspace{1cm} (19)

Taking into account the permutation symmetry condition:

\[ |V; \zeta > = \hat{S}|V; \zeta >, \]  \hspace{1cm} (20)

(for an explicit form of the projector \(\hat{S}\) see [14]), we can project Eq.18 as follows:

\[ \left( \hat{I} + \hat{S}L^{-1}_R \left( \lambda \hat{N} + \zeta \cdot \hat{\Delta} \right) + \hat{S}L^{-1}_R \hat{G} \right) |V; \zeta >= \] 

\[ \hat{S}L^{-1}_R |0 >_{info} + \hat{S}\hat{P}_L |V; \zeta > \]  \hspace{1cm} (21)

This equation differs from Eq.18 that an arbitrary part of the solution \(|V; \zeta >, \hat{S}\hat{P}_L |V; \zeta >,\) is projected on the smaller part of FFS than in the case of Eq.18. We will identify this part with the zero-th approximation \((\lambda = 0)\):
To get higher approximation terms in the expansion (16), it is recommended to transform Eq. (21) as follows:

\[
\left\{ i + \lambda \left[ i + S \hat{L}_R^{-1} \left( \hat{G} + \zeta \cdot \hat{\Delta} \right) \right]^{-1} \hat{S} \hat{L}_R^{-1} \hat{N} \right\} \mid V; \zeta > = \\
\left[ i + \hat{S} \hat{L}_R^{-1} \left( \hat{G} + \zeta \cdot \hat{\Delta} \right) \right]^{-1} \left( \hat{S} \hat{L}_R^{-1} \mid 0; \zeta >_{inf} + \hat{S} \hat{P}_L \mid V; \zeta > \right)
\]  

(23)

Are the equations (21) and (23) equivalent to the previous equations? The answer may be positive, if the previous equations are overdetermined equations. There are two reasons for this: considered previous equations have identical shape in the free (FFS) as well as in the symmetrical Fock space (SFS), and another more fundamental and surprising reason is such that Eq. (8) can be overdetermined even with respect to the micro-equation (7), see App.1.

For lower triangular perturbation operators \( \hat{\Delta} \), see Sec.4, the inverse appearing in Eq. (23) is not difficult to calculate, for any values of the \( \zeta \). There is another problem related to these operators, namely - we can not be sure that the positivity conditions considered in Sec.5 are satisfied in the case when the term (25) occurs in the perturbation operator \( \hat{\Delta} \). However, the functional integration representation of solutions convinces us that it is not the case.

The expansion (16) is useful and in some sense obligatory, for an infinite system of branching equations. However, for essential nonlinearities considered in [6], obtained equations for n-pi, are closed. In this case, one can find a more effective (e.g. numerical) methods of approximations to the generating vectors \( \mid V; \zeta > \). Moreover, equations for \( \mid V > \) can be easy derived, see App.4.

4 Random perturbations \( \zeta \cdot \hat{\Delta} \)

Until now we did not say anything about the random perturbation operator, \( \zeta \cdot \hat{\Delta} \). We know that in FFS n-pi are described by Eq. (8) in which external forces acting on the system are described by the lower triangular operator

\[
\hat{G}_{\text{ext}} = \sum_{n=1}^{\infty} \hat{P}_n \hat{G}_{\text{ext}} \hat{P}_{n-1}
\]

(24)

see [9]. On the other hand, we know that the symmetric Green’s functions of quantum field theory, which codify the causality condition, are described by equations of the type (8), where \( \hat{G}_{\text{QFT}} \) is the operator with property:

\[
\hat{G}_{\text{QFT}} = \sum_{n=2}^{\infty} \hat{P}_n \hat{G}_{\text{QFT}} \hat{P}_{n-2}
\]

(25)
The latter operator describes disorders of the system caused by attempting simultaneous measurement of canonically conjugate variables. So, we propose the following linear combination

\[ \zeta \cdot \Delta = \zeta_{exe} \cdot \hat{G}_{ext} + \zeta_{QFT} \cdot \hat{G}_{QFT} \]  

for the operator \( \zeta \cdot \Delta \). In this way we can hope that the above formula gives a general frame to describe a system under influence (perturbations) of the random external fields created by the entities simultaneously subjected to quantum fluctuations. In the case of the system of Brownian particles (dust particles), the random perturbations (fluctuations) are caused by fast moving atoms in the gas or liquid. In the case of the economic system, the 'dust particles' can be large companies and atoms may be substituted by smaller units like customers.

5 Positivity conditions and the term \( \zeta_{QFT} \cdot \hat{G}_{QFT} \)

Eq.8 derived by means of definitions of n-pi [11] and Eq.7 allows operators which are diagonal, upper triangular and very specific lower triangular operators as [24]. But it seems the operator \( \zeta_{QFT} \cdot \hat{G}_{QFT} \) does not belong to these classes of operators. This can cause a conflict with the positivity conditions for 2-pi \( V[\tilde{x}, \tilde{y}; \zeta] \) resulting from the definitions [13] or [14]:

\[ \sum_{i,j=1}^{n} V[\tilde{x}_i, \tilde{x}_j; \zeta] \eta(\tilde{x}_i) \eta(\tilde{x}_j) \geq 0 \]  

for an arbitrary choice of \( n \), functions \( \eta, \zeta \) and points \( \tilde{x}_i, \tilde{y}_j \). (We assume that considered fields \( \varphi, \eta, \zeta \) are real-valued functions). Please do not confuse \( \eta \) without the hat with \( \hat{\eta} \) with the hat, which is the operator satisfying the Cuntz relations. Similar restrictions also occur for higher even n-pi. See also [12], Sec.3. After integration of inequality (27), the same inequality is satisfied by the functions [15].

One can prove at least formally, by means of the functional integral representation, see e.g. [13], that positivity conditions are also satisfied in the case of Eq.8 with terms given by Eq.25. To see this let us define the n-pi with the help of generating functional integral:

\[ V[\eta; \zeta] = \int \delta^2 \beta e^{-H[\beta; \zeta]} e^{i\beta \eta} \]  

where \( \eta, \beta, \zeta \) are real functions and \( \beta \eta \) in the second exponent - a scalar product, see [13]. It is easy to see that n-pi defined by the n-th order functional derivatives:

\[ - (i)^n \frac{\delta^n}{\delta \eta(\tilde{y}_1) \cdots \delta \eta(\tilde{y}_n)} V[\eta; \zeta] \big|_{\eta=0} \equiv V[\tilde{x}(n); \zeta] \]  

satisfy the above positivity conditions if \( H \) is a real function. One can also show that the generating functional (28) satisfies an equation similar to Eq.8, see [13].
In the case of highly nonlinear interaction \( N \), in Eq.7, (by this we mean a non-polynomial dependence of the \( N \) functional on the field \( \varphi \) one can define the linear operator \( \hat{N} \) of Eq.8, which in addition to the diagonal part contains the lower triangular part, see \[6\]; Sec.2. **One can use positivity conditions** (27), and similar relations, to justify for certain values of parameters, at least, proposed definitions of operator-valued functions.

6 App.1 What can you say about solutions to an equation with the knowledge of their averages and n-point information (n-pi)?

We want to give a simple example that the knowledge of averages of 'fields' \( \varphi[\tilde{x}; \alpha] \) and their n-pf (e.g., correlation functions) allows to reconstruct the fields. We consider extremely simple case in which \( \tilde{x}, \alpha \) take only two values: 1 and 2. We take arithmetic averages, which correspond to a constant probability equal in this case to 1/2. So, we have:

\[
< \varphi[\tilde{x}; \cdot] > = \frac{1}{2} (\varphi[\tilde{x}; 1] + \varphi[\tilde{x}; 2])
\]

(30)

for , (Two equations). For correlation functions (2-pi), we get additional 4 equations:

\[
< \varphi[\tilde{x}_i; \cdot] \varphi[\tilde{y}_j; \cdot] > = \frac{1}{2} (\varphi[\tilde{x}_i; 1] \varphi[\tilde{y}_j; 1] + \varphi[\tilde{x}_i; 2] \varphi[\tilde{y}_j; 2])
\]

(31)

by means of which one can calculate the field \( \varphi[\tilde{x}; \alpha] \). We assume here that the l.h.s. of the above equations are known; The equation which satisfies the field together with the additional conditions allow to find them in a unique way.

In other words, it is possible that there are situations in which averaged formulas are used because they - in contrary to unaveraged quantities - have got the physical meanings.

7 App.2 About vectors \( |0; \zeta >_{inf,o} \) and \( |0> \)

Why in the Eq.8 appears the vectors \( |0; \zeta >_{inf,o} \)? This vector appears in Eq.8 because we want to have equations for the generating vector \( |V> \) with right inverse operators defined in the whole FFS, see \[12\]. Of course, appropriate modifications of Eq.8 for generating vector \( |V> \), is made in accordance with the definitions of n-pi and micro- (local) Eq.7. It results, from construction of vector \( |0>_{inf,o} \) in \[9\], that

\[
|0; \zeta >_{inf,o} \sim |0>
\]

(32)

In \[8\] we assumed that \( |0; \zeta >_{inf,o} \) represents the local vacuum (no local information about the system). It depends on the global characteristics of the system and perturbations \( \zeta \):

9
The system and random perturbations acting on the system, both form a quasi-isolated system, for which the rest of the world, with a good approximation, can be treated as the whole Universe.

In [8] we have identified the basic vector $|0 >$ of FFS with a vector $|U >$ describing the whole Universe:

$$| 0 > = | U >$$

(34)

The reason may be this that with the help of this vector, for the most complex systems, a whole Fock space can be created. For example - the Universe. With the help of this vector and creation operators $\hat{\eta}^*(\tilde{x})$ one can retrieve all local information about the system.

$$| V > = \int | V; \zeta > d\mu[\zeta] = \sum_n \int d\tilde{x}(n) \left\{ (\int V[\tilde{x}(n); \zeta]d\mu[\zeta]) \hat{\eta}^*(\tilde{x}_1) \cdots \hat{\eta}^*(\tilde{x}_n)|U > \right\} + | 0; V >_{info}$$

(35)

where

$$| 0; V >_{info} = \int d\mu[\zeta]| 0; \zeta >_{info} \equiv | 0 >_{info}$$

(36)

8. App.3 About creation and annihilation operators $\hat{\eta}^*(\tilde{x})$ and $\hat{\eta}(\tilde{x})$

These operators can depend on considered systems, but in calculating the n-pi we use only Cuntz relations [13] and [15]. You do not even need to use an involutorial property: xxx

$$(\hat{\eta}^*)^* = \hat{\eta}$$

(37)

9. App.4 About equations for n-pi $V(\tilde{x}(n))$ generated by the vector $| V >$

By straight integration, it is seen from Eq.23 that the j-th order approximation to n-pi $V(\tilde{x}(n))$ is given by the lower order approximations to m-pi $V[\tilde{x}(m); \zeta]$ generated by the vector $| V; \zeta >$.

For closed equations for n-pi, obtained in the case of essentially nonlinear theories, the n-pi $V(\tilde{x}(n))$ is expressed by m-pi $V[\tilde{x}(m); \zeta]$, with $m < n$. In other words, even when we do not have exclusive (complete) equations for physical n-pi $V(\tilde{x}(n))$, there calculations are realized by simplier, in the above sense, m-pi $V[\tilde{x}(m); \zeta]$. 

10
In general case, by integrating Eq.8 with respect to variable $\zeta$, we get the following equation for generating vector $|V>$:

$$
\left[ (\hat{L} + \lambda \hat{N} + \hat{G}) |V> + \int d\mu[\zeta] \ast \zeta \cdot \hat{\Delta}|V; \zeta> \right] = \int d\mu[\zeta]|0; \zeta >_{info} = \hat{P}_0 \int d\mu[\zeta]|0; \zeta >_{info}
$$

(38)

see (3)-(5). Of course, in this equation we have to know the generating vector $|V; \zeta>_{info}$ and the local vacuum vector $|0; \zeta >_{info}$, for a set of all possible perturbations $\zeta$. However, we do not need to know these quantities in a very precise manner because they are smoothed out by integration. We can go back to Eq.8 and seek solutions, for $|V; \zeta >$, in the form of a Volterra series, e.g.:

$$
|V; \zeta> = \sum_{j=0}^{\infty} \frac{1}{j!} \int d\tilde{y}_{(j)}|V(\tilde{y}_{(j)}) > \zeta(\tilde{y}_1) \cdots \zeta(\tilde{y}_j)
$$

(39)

see (39). Here

$$
|V(\tilde{y}_{(j)}) >_{pertur} = \frac{\delta^j}{\delta \zeta(\tilde{y}_1) \cdots \delta \zeta(\tilde{y}_j)}|V; \zeta > |_{\zeta=0}
$$

(40)

Equations for vectors $|V(\tilde{y}_{(j)}) >$ are complete, see Eq.8. We must note here that if the perturbation field $\zeta$ contains any sub-indices, then the same sub-indices are included in vectors $|V(\tilde{y}_{(j)}) >$ and then in the formula (39) we use Einstein’s summation convention.

We have the following relations:

$$
|V; \zeta> = \sum_{j=0}^{\infty} \frac{1}{j!} \int d\tilde{y}_{(j)}|V(\tilde{y}_{(j)}) >_{pertur} \zeta(\tilde{y}_1) \cdots \zeta(\tilde{y}_j) = \sum_{n=1}^{\infty} \int dx_{(n)}|V[x_{(n)}; \zeta] \hat{n}^*(\tilde{x}_1) \cdots \hat{n}^*(\tilde{x}_n)|0 > + |0; \zeta >_{info}
$$

(41)

From the Cuntz relations (14)-(15), we get:

$$
|V[x_{(n)}; \zeta] = = \int < \hat{n}(\tilde{x}_1) \cdots \hat{n}(\tilde{x}_n)|V; \zeta> = \sum_{j=0}^{\infty} \frac{1}{j!} \int d\tilde{y}_{(j)} < \hat{n}(\tilde{x}_1) \cdots \hat{n}(\tilde{x}_n)|V(\tilde{y}_{(j)}) >_{pertur} \zeta(\tilde{y}_1) \cdots \zeta(\tilde{y}_j)
$$

(42)

and from (15):

$$
|V[x_{(n)}] = \int V[x_{(n)}; \zeta]d\mu[\zeta] = \sum_{j=0}^{\infty} \frac{1}{j!} \int d\tilde{y}_{(j)} < \hat{n}(\tilde{x}_1) \cdots \hat{n}(\tilde{x}_n)|V(\tilde{y}_{(j)}) >_{pertur} \int d\mu[\zeta]\zeta(\tilde{y}_1) \cdots \zeta(\tilde{y}_j)
$$

(43)

where, in this and other equations, we freely have changed the orders of appropriate operations.

In the case of small perturbations $\zeta$, only few terms of the Volterra series (43) has to be taken into account:-) to get physical n-pi $V(x_{(n)})$. Hence, we claim that equations for $V(x_{(n)})$ are also complete. We do not take into account the $\zeta$-dependence of the vacuum vector $|0; \zeta >_{info}$ which indeed does not appear, for certain, perturbative type calculations.
10 App.5 Difference between completeness and closeness of equations

Equations, for a given set of quantities are called complete if by means of them, with the help of reasonable set of additional conditions, one can solve them.

Equations, for a given subset of quantities are called close if by means of them, with the help of reasonable set of additional conditions, one can solve them. In this way the closure problem is solved.

References

[1] van Kampen, N.G. 1981. *Stochastic processes in physics and chemistry*. Elsevier Science Publishers B.V. 1981

[2] Lin, Y.K. and G.Q. Cai. 1995. *Probabilistic Structural Dynamics*. McGraw-Hill, New York.

[3] Sobczyk, K. 1991. *Stochastic Differential Equations with Applications to Physics and Engineering*. Kluwer Academic Publishers B.V. 1991

[4] Monin, A.S. and A.M. Jaglom. 1967. *Statistical Hydromechanics, vol. 2*, MIR Publishers, Moscow.

[5] Hanckowiak, J. 2010. *Models of the 'Universe' and a Closure Principle*. arXiv: 1010.3352 physics.gen-ph

[6] Hanckowiak, J. 2013. *Nonlinearity and linearity, friends or enemies? Algebraic Analysis of Science?*. arXiv:1304.3453v1 [physics.gen-ph] 11 Apr 2013

[7] Hanckowiak, J. 2011. *Unification of some classical and quantum ideas*, arXiv:1107.1365v1

[8] Hanckowiak, J. 2012. *Metaphysics of the free Fock space with local and global information*. arXiv:1206.4589v1[physics.gen-ph]

[9] Hanckowiak, J. 2011’. *Local and global information and equations with left and right invertible operators in the free Fock space*. arXiv:1112.1870v1 [physics.gen-ph]

[10] Przeworska-Rolewicz, D. 1988. *Algebraic Analysis*. PWN Polish Science Publishers, Warsaw and D. Reidel Publishing Company. Dordercht..., Tokyo

[11] Przeworska-Rolewicz, D. 2000. *Two centuries of the term 'Algebraic Analysis'*. PAS, Warsaw (Internet)
[12] Hanckowiak, J. 2010. *Free Fock space and functional calculus approach to the n-point information about the 'Universe'.* arXiv:1011.3250v1 [physics.gen-ph]

[13] Rzewuski, J. 1969. *Field Theory, part II.* Ilife Books LTE, London

[14] Hanckowiak, J. 2007. *Reynolds’ dream? A description of random field theory within a framework of algebraic analysis and classical mechanics.* Nonlinear Dyn. (2007). 50. 191-211