Relativistic Effects in Electromagnetic Meson–Exchange Currents for One–Particle Emission Reactions

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Abstract

Following recent studies of non–relativistic reductions of the single–nucleon electromagnetic current operator, here we extend the treatment to include meson exchange current operators. We focus on one–particle emission electronuclear reactions. In contrast to the traditional scheme where approximations are made for the transferred momentum, transferred energy and momenta of the initial–state struck nucleons, we treat the problem exactly for the transferred energy and momentum, thus obtaining new current operators which retain important aspects of relativity not taken into account in the traditional non–relativistic reductions. We calculate the matrix elements of our current operators between the Fermi sphere and a particle–hole state for several choices of kinematics. We present a comparison between our results using approximate current operators and those obtained using the fully–relativistic operators, as well as with results obtained using the traditional non–relativistic current operators.

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1. Introduction

In recent work [1, 2, 3] an improved version of the single–nucleon electromagnetic current has been studied. There the so–called “on–shell form” of the current was derived as a non-relativistic expansion in terms of the dimensionless parameter $\eta \equiv p/m_N$, where $p$ is the three–momentum of the struck nucleon (the one in the initial nuclear state to which the virtual photon in electron scattering reactions is attached) and $m_N$ is the nucleon mass. Generally (that is, for nucleons in typical initial–state nuclear wave functions) $p$ lies below a few hundred MeV/c and thus $\eta$ is characteristically of order 1/4. Barring some extreme choice of kinematics such as the selection of extremely large missing momenta in $(e,e'N)$ reactions — conditions for which presently no approach can be guaranteed to work — an expansion in powers of $\eta$ is well motivated. Similar arguments do not however apply equally for other dimensionless scales in the problem. Indeed, a specific goal of this past work has been to obtain current operators which are not expanded in either $\kappa \equiv q/2m_N$ or $\lambda \equiv \omega/2m_N$, where $q$ is the three–momentum and $\omega$ the energy transferred in the scattering process, since one wishes the formalism to be applicable at GeV energies where these dimensionless variables are clearly not small.

Traditionally, many studies have indeed been undertaken assuming that $\kappa \ll 1$ and $\lambda \ll 1$ aimed of course at treatments where non–relativistic wave functions are employed [4]–[7]. For high–energy conditions the current operators so obtained are bound to fail, whereas our past work on the single–nucleon current provides a way to incorporate classes of relativistic corrections into improved, effective operators for use with the same non–relativistic wave functions.

Not only the single–nucleon (one–body) current, but also the two–body meson exchange currents (MEC) have frequently been evaluated using similar traditional non–relativistic expansions [8]–[21] in which $\kappa$ and $\lambda$ are both treated as being small, together with the assumptions that all nucleon three–momenta in the problem are small compared with $m_N$. In other work [22]–[23], relativistic currents have been used directly in cases where the nuclear modeling permitted.

Our goal in the present work is to extend our previous approach for the single–nucleon current operators now to include a treatment of pion–exchange MEC. We make expansions only in $\eta_i \equiv p_i/m_N$, where $\{p_i\}$ are the initial–state nucleon three–momenta, whereas we treat the dependences in the on–shell form exactly for $\kappa$, $\lambda$ and any high–energy nucleon momenta, specifically for any nucleons in the final state not restricted to lie within the Fermi sea. Such new MEC operators may straightforwardly be employed in place of previous non–relativistic expansions using the same non–relativistic initial and
final nuclear wave functions employed in the past, since our effective operators incorporate specific classes of relativistic effects (see the discussions of the single–nucleon current referred to above).

In the present work, as a first step, we focus on the general form of the MEC matrix elements for pionic diagrams (the so–called seagull and pion–in–flight contributions) and plan to extend our treatment to other diagrams and other meson exchanges in future work. We do not present any results for electromagnetic response functions, postponing such discussions until the corresponding correlation effects have been brought under control (also work in progress). Finally, in the present work we focus on specific classes of matrix elements, namely those with one high–energy nucleon in the final state, i.e. one–particle–one–hole (1p–1h) matrix elements; in subsequent work we shall extend the scope to include 2p–2h configurations.

The organization of the paper is as follows: After reviewing the treatment of the single–nucleon current in Sect. 2.1, in Sect. 2.2 we discuss the new approach for the electromagnetic meson–exchange currents that treats the problem exactly for the transferred energy and transferred momentum. We check the quality of our expansions in powers of the bound nucleon momenta divided by $m_N$ by calculating the matrix elements of the current operators between the Fermi sphere and a particle–hole state. We compare with the matrix elements obtained using the full current operators as well as with the results for the traditional non–relativistic expansions. These results are presented in Sect. 3 (with reference to specific details that are covered in an appendix), together with a brief discussion of the high–$q$ limits reached by the currents. Finally in Sect. 4 we summarize our main conclusions.

2. Current Operators

2.1. The Electromagnetic Current Operator

We start our discussion with the single–nucleon on–shell electromagnetic current operator and its non–relativistic reduction. Although this case has been already treated in detail in Refs. [1, 2, 3], here we provide a simpler derivation that will be also applied to the case of the MEC operators for one–particle emission reactions. The single–nucleon electromagnetic current reads

$$J^\mu(P's';Ps) = \bar{u}(p',s') \left[ F_1(Q^2)\gamma^\mu + \frac{i}{2m_N} F_2(Q^2)\sigma^{\mu\nu}Q_\nu \right] u(p,s), \quad (1)$$
where \( P^\mu = (E, p) \) is the four–momentum of the incident nucleon, \( P'^\mu = (E', p') \) the four–momentum of the outgoing nucleon and \( Q^\mu = P'^\mu - P^\mu = (\omega, q) \) the transferred four–momentum. The spin projections for incoming and outgoing nucleons are labeled \( s \) and \( s' \), respectively. We follow the conventions of Bjorken and Drell \[26\] for the \( u \)–spinors.

For convenience in the discussions that follow of the scales in the problem we introduce the dimensionless variables:

\[
\eta = \frac{p}{m_N}, \quad \varepsilon = \frac{E}{m_N} = \sqrt{1 + \eta^2}, \quad \lambda = \frac{\omega}{2m_N}, \quad \kappa = \frac{q}{2m_N} \quad \text{and} \quad \tau = -\frac{Q^2}{4m_N^2} = \kappa^2 - \lambda^2.
\]

For the outgoing nucleon, \( \eta' \) and \( \varepsilon' \) are defined correspondingly.

For any general operator whose \( \gamma \)-matrix form is given by

\[
\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}
\]

one has \( \bar{\pi}(p', s') \Gamma u(p, s) = \chi_{s'}^\dagger \chi_s \), with the current operator \( \Gamma \) given by

\[
\Gamma = \frac{1}{2} \sqrt{(1 + \varepsilon)(1 + \varepsilon')} \left( \Gamma_{11} + \Gamma_{12} \frac{\sigma \cdot \eta}{1 + \varepsilon} - \frac{\sigma \cdot \eta'}{1 + \varepsilon'} \Gamma_{21} - \frac{\sigma \cdot \eta'}{1 + \varepsilon'} \Gamma_{22} \frac{\sigma \cdot \eta}{1 + \varepsilon} \right).
\]

This general result will be used throughout this work in discussing the non–relativistic reductions of the various current operators.

An important point in our approach is that we expand only in powers of the bound nucleon momentum \( \eta \), not in the transferred momentum \( \kappa \) or the transferred energy \( \lambda \). This is a very reasonable approximation as the momentum of the initial nucleon is relatively low in most cases, since the typical values of \( \eta \) lie below \( \eta_F \equiv k_F/m_N \), where \( k_F \) is the Fermi momentum (\( \eta_F \) is typically about 1/4). However, for those cases corresponding to short–range properties of the nuclear wave functions it will be necessary to be very careful with the approximations made. Indeed, for large values of \( \eta \) a fully–relativistic approach will likely prove necessary. Expanding up to first order in powers of \( \eta \) we get \( \varepsilon \simeq 1 \) and \( \varepsilon' \simeq 1 + 2\lambda \).

Thus, the non–relativistic reductions of the time and space components of the single–nucleon electromagnetic current operator can be evaluated in a rather simple form.

Let us consider first the case of the time component. We have

\[
J^0(P's'; Ps) = \bar{\pi}(p', s') J^0 u(p, s) = \chi_{s'}^\dagger J^0 \chi_s ,
\]

with the current operator \( J^0 = F_1 \gamma^0 + i F_2 \sigma^0 q / 2m_N \). Using the general result given by Eq. (3) and expanding up to first order in \( \eta \), it is straightforward to get the relation

\[
J^0 \simeq \frac{\kappa}{\sqrt{\tau}} G_E + \frac{i}{\sqrt{1 + \tau}} \left( G_M - \frac{G_E}{2} \right) (\kappa \times \eta) \cdot \sigma ,
\]
where we have introduced the Sachs form factors $G_E = F_1 - \tau F_2$ and $G_M = F_1 + F_2$, and have used the relations
\begin{align}
\lambda &\simeq \tau + \kappa \cdot \eta \\
\kappa^2 &\simeq \tau (1 + \tau + 2 \kappa \cdot \eta).
\end{align}

The expression (5) coincides with the leading–order expressions already obtained in previous work [1, 3]; in those studies a different approach was taken which, while more cumbersome, does yield terms of higher order than the ones considered in the present work. It is important to remark again that no expansions have been made in terms of the transferred energy and transferred momentum; indeed, $\kappa$, $\lambda$ and $\tau$ may be arbitrarily large in our approach.

Let us consider now the case of space components. Thus, we have $J(P's'; Ps) = \pi(p', s') u(p, s) = \chi^1_s \mathcal{J} \chi_s$. Introducing the matrix form of the vector component for the single–nucleon electromagnetic current operator in the general relation (3), one can finally write
\begin{equation}
\mathcal{J} \simeq \frac{1}{\sqrt{1 + \tau}} \left\{ i G_M (\sigma \times \kappa) + \left( G_E + \frac{\tau}{2} G_M \right) \eta + G_E \kappa \\
- \frac{G_M}{2(1 + \tau)} (\kappa \cdot \eta) \kappa - \frac{i G_E}{2(1 + \tau)} (\sigma \times \kappa) \cdot \eta \\
- i \tau (G_M - G_E/2) (\sigma \times \eta) + \frac{i (G_M - G_E)}{2(1 + \tau)} (\kappa \times \eta) \sigma \cdot \kappa \right\},
\end{equation}
where we have used the relations given by Eqs. (6,7).

In order to compare with the previous work [3], we write the expression for the transverse component of the current, i.e., $\mathcal{J}^\perp = \mathcal{J} - \frac{\mathcal{J} \cdot \kappa}{\kappa^2} \kappa$. After some algebra we get the final result
\begin{equation}
\mathcal{J}^\perp \simeq \frac{1}{\sqrt{1 + \tau}} \left\{ i G_M (\sigma \times \kappa) + \left( G_E + \frac{\tau}{2} G_M \right) \left( \eta - \frac{\kappa \cdot \eta}{\kappa^2} \kappa \right) \\
- i G_M (\sigma \times \kappa) \cdot \eta + \frac{i G_M}{2(1 + \tau)} (\eta \times \kappa) \sigma \cdot \kappa \right\}.
\end{equation}
It is straightforward to prove that this expression coincides with the result given by Eq. (25) in Ref. [3] for an expansion in powers of $\eta$ up to first order.

Therefore, as can be seen from Eqs. (5,9), at linear order in $\eta$ we retain the spin–orbit part of the charge and one of the relativistic corrections to the transverse current, the first–order convective spin–orbit term. It is also important to remark here that the current operators given by Eqs. (5,8) satisfy the property of current conservation $\lambda J_0 = \kappa \cdot J$. 

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Finally, it is also interesting to quote the results obtained in the traditional non–relativistic reduction [3],[7], where it is assumed that $\kappa << 1$ and $\lambda << 1$:

\[
\mathbf{J}^0_{\text{nonrel}} = G_E \\
\mathbf{J}^\perp_{\text{nonrel}} = -iG_M[\kappa \times \sigma] + G_E \left[\eta - \left(\frac{\kappa \cdot \eta}{\kappa^2}\right)\kappa\right].
\]

(10)

Note that this traditional non–relativistic reduction contains both terms of zeroth and first order in $\eta$, i.e., the convection current, and is therefore not actually of lowest order in $\eta$.

The present expansion for the electromagnetic current operator of the nucleon was first checked in Ref. [1], where the inclusive longitudinal and transverse responses of a non–relativistic Fermi gas were found to agree with the exact relativistic result within a few percent if one uses relativistic kinematics when computing the energy of the ejected nucleon. Recently the same expansion has been tested with great success by comparing with the relativistic exclusive polarized responses for the $^2\text{H}(e,e'p)$ reaction at high momentum transfers [3]. This relativized current has also been applied to the calculation of inclusive and exclusive responses that arise in the scattering of polarized electrons from polarized nuclei [2, 27].

We see that the expansion of the current to first order in the variable $\eta = p/m_N$ yields quite simple expressions; moreover the various surviving pieces of the relativized current (i.e., charge and spin–orbit in the longitudinal and magnetization and convection in the transverse) differ from the traditional non–relativistic expressions only by multiplicative ($q, \omega$)-dependent factors such as $\kappa/\sqrt{\tau}$ or $1/\sqrt{1 + \tau}$, and therefore are easy to implement in already existing non–relativistic models. In the next section we perform a similar expansion for the MEC and later return to check our results through direct comparisons with the exact relativistic matrix elements.

2.2. Meson–Exchange Currents

Once the procedures for expanding the single–nucleon electromagnetic current are fixed, it is clear how to proceed to obtain relativistic expansions for the meson–exchange currents. In the present work we begin by focusing on pion exchange MEC effects, leaving the treatment of other mesons to future work. Following the ideas and methods developed in the previous section, our main aim here is to get new non–relativistic reductions for MEC treating the problem of the transferred energy and transferred momentum as above, namely in an un-expanded form while expanding only in the initial nucleon momenta. In this way the expressions obtained will retain important aspects of relativity not included
in the traditional non–relativistic MEC used throughout the literature.

Let us consider the meson exchange current operator $J^{MEC}_\mu$. For definiteness we focus on one–particle emission reactions where the matrix element of $J^{MEC}_\mu$ taken between the Fermi sphere and a particle–hole state, namely

$$\langle ph^{-1}|J^{MEC}_\mu|F\rangle = \sum_{s',t',h'<k_F} \left[ \langle ph'|J^{MEC}_\mu|h'h'\rangle - \langle ph'|J^{MEC}_\mu|h'h\rangle \right],$$  \hspace{1cm} (11)

$k_F$ being the Fermi momentum, is the relevant one. Since, as seen below when the pion–exchange operators are given, the currents being considered all have isospin dependences of the form $\tau_a(1)\tau_b(2)$, the first term (direct term) vanishes,

$$\sum_{t'} \langle tp'|\tau_a(1)\tau_b(2)|th\rangle = \langle tp|\tau_a|th\rangle \sum_{t'} \langle t'|\tau_b|t'\rangle = 0.$$  \hspace{1cm} (12)

Therefore, the only remaining term is the exchange term, and we can simply write

$$\langle ph^{-1}|J^{MEC}_\mu|F\rangle = -\sum_{s',t',h'<k_F} \langle ph'|J^{MEC}_\mu|h'h\rangle.$$  \hspace{1cm} (13)

In what follows we will be interested in the evaluation of the particle–hole matrix elements $\langle ph'|J^{MEC}_\mu|h'h\rangle$ and their new non–relativistic expressions. As in our treatment of the single–nucleon current, it is convenient to express the results in terms of spin matrix elements of particular operators:

$$\langle ph'|J^{S,P}_\mu|h'h\rangle \equiv \chi^\dagger_{s_1'}\chi^\dagger_{s_2'}\mathcal{J}^{S,P}_\mu(1,2)\chi_{s_1}\chi_{s_2},$$  \hspace{1cm} (14)

where $S(P)$ denotes the seagull (pion–in–flight) contributions to the MEC, as shown in Fig. 1.

2.2.1. Seagull Current Operator

The relativistic seagull current operator is given by

$$J^{S}_\mu(Q) = \frac{f^2}{V^2m^2_\pi} \epsilon_{zab} \bar{u}(p_1',s_1')\tau_a\gamma_5 K_1 u(p_1,s_1) \frac{F^V_1}{K_1^2 - m^2_\pi} \bar{u}(p_2',s_2')\tau_b\gamma_5\gamma_\mu u(p_2,s_2) + (1 \leftrightarrow 2),$$  \hspace{1cm} (15)

where $V$ is the volume enclosing the system and the different nucleon kinematic variables are: $P_1 = (E_1,p_1)$, $P_1' = (E_1',p_1')$, $P_2 = (E_2,p_2)$ and $P_2' = (E_2,p_2')$ (see Fig. 1). The four–momentum of the exchanged pion is $K_1 = (E_{k_1},k_1)$ (with $K_2$ likewise) and its mass is $m_\pi$. The terms $f$ and $F^V_1$ represent the pion–nucleon coupling and pseudovector form
factor, respectively. Note that the following kinematic relations are satisfied for the two diagrams involved

\[ \eta'_1 = \eta_2 + 2\kappa \]  
\[ \eta'_2 = \eta_1 \]  
\[ \zeta_1 = \eta'_1 - \eta_1 = \eta_2 - \eta_1 + 2\kappa \]  
\[ \zeta_2 = \eta'_2 - \eta_2 = \eta_1 - \eta_2 \]

where we follow the general notation introduced in Section 2.1 and have also defined \( \zeta_{1,2} \equiv k_{1,2}/m_N \) with \( k_{1,2} \) the three–momenta of the exchanged pions.

The particle–hole matrix element of the seagull current is then given by

\[ \langle p' h' | J^S_{\mu} | h' h \rangle = \frac{f^2}{V^2 m^2} \epsilon_{a b} \langle t_p | \tau_a | t_{h'} \rangle \langle t_{h'} | \tau_b | t_h \rangle \]

\[ \times \left\{ \bar{u}(p'_1, s'_1) \gamma_5 K_1 u(p_1, s_1) \frac{F^V_1}{K_1^2 - m^2} \bar{u}(p'_2, s'_2) \gamma_5 \gamma_\mu u(p_2, s_2) \right. \]

\[ - \left. \bar{u}(p'_2, s'_2) \gamma_5 K_2 u(p_2, s_2) \frac{F^V_1}{K_2^2 - m^2} \bar{u}(p'_1, s'_1) \gamma_5 \gamma_\mu u(p_1, s_1) \right\}, \]  

where now \( p'_1 = p \) is the momentum of the ejected particle above the Fermi sea, which can be large for large momentum transfer, while \( p_2 = h \) is the momentum of the bound nucleon before the interaction (related to the missing momentum in \((e, e'p)\) reactions), which can be considered small compared to the nucleon mass. Finally, \( p'_2 = p_1 = h' \) is the intermediate momentum of the bound nucleon interacting with the ejected nucleon by pion exchange. Therefore \( p'_2 \) and \( p_1 \) are small compared with the nucleon mass. In dimensionless terms, we can safely expand in \( \eta_1, \eta_2 \) and hence \( \eta'_2 \), whereas we cannot in general expand in \( \eta'_1 \). Moreover, \( \zeta_2 \) is small, whereas \( \zeta_1 \) can be large. By analogy with the single–nucleon electromagnetic current, let us now proceed to the evaluation of the time and space components separately.

\[ \text{Time Component} \]

Using the general result given by Eq. (3) and the matrices \( \gamma_5 K_1 \) and \( \gamma_5 \gamma_0 \) we can write

\[ \langle p' h' | J^S_{\mu} | h' h \rangle = - \frac{f^2}{4 V^2 m^2} \epsilon_{a b} \langle t_p | \tau_a | t_{h'} \rangle \langle t_{h'} | \tau_b | t_h \rangle \sqrt{(1 + \epsilon_1)(1 + \epsilon_2)(1 + \epsilon'_1)(1 + \epsilon'_2)} \]

\[ \times \left\{ \chi^\dagger_{s_1} \left[ m_N \sigma \cdot \zeta_1 - E_{k_1} \frac{\sigma \cdot \eta_1}{1 + \epsilon_1} - E_{k_1} \frac{\sigma \cdot \eta'_1}{1 + \epsilon'_1} + m_N \frac{\sigma \cdot \eta'_1}{1 + \epsilon'_1} (\sigma \cdot \zeta_1) \right] \chi_{s_1} \right. \]

\[ \times \left. \frac{F^V_1}{K_1^2 - m^2} \chi^\dagger_{s_2} \left[ \frac{\sigma \cdot \eta_2}{1 + \epsilon_2} + \frac{\sigma \cdot \eta'_2}{1 + \epsilon'_2} \right] \chi_{s_2} \right\} \]
\[ - \chi_{s_1}^\dagger \left[ \frac{\sigma \cdot \eta_1}{1 + \varepsilon_1} + \frac{\sigma \cdot \eta_1'}{1 + \varepsilon_1'} \right] \chi_{s_1} \frac{F^V}{K_1^2 - m_\pi^2} \]
\[ \times \chi_{s_2}^\dagger \left[ m_N \sigma \cdot \zeta_2 - E \frac{\sigma \cdot \eta_2}{1 + \varepsilon_2} - E \frac{\sigma \cdot \eta_2'}{1 + \varepsilon_2'} + m_N \frac{\sigma \cdot \eta_2'(\sigma \cdot \zeta_2)(\sigma \cdot \eta_2)}{1 + \varepsilon_2} \right] \chi_{s_2} \right) , \quad (21) \]

where \( E_k(\varepsilon_k) \) is, as mentioned previously, the energy of the pion.

We note that the resulting expansion for the MEC should be used together with the single–nucleon current, which has been developed to first order in \( \eta \). Therefore, in order to be consistent, we want to perform the expansion of the MEC also to first order in the corresponding small quantities \{\( \eta_1, \eta_2, \eta_2', \zeta_2 \}\}, whereas \{\( \eta_1', \kappa, \zeta_1 \)\} are treated exactly. Using the kinematic relations given in Eqs. (16–19), the following non–relativistic kinematic reductions are involved (up to first order in the small quantities)

\[ \varepsilon_1 \simeq \varepsilon_2 \simeq \varepsilon_2' \simeq 1 \quad (22) \]
\[ \varepsilon_1' \simeq 1 + 2\lambda \quad (23) \]
\[ E_{k_1} \simeq 2m_N\lambda \quad (24) \]
\[ E_{k_2} \simeq 0 \quad (25) \]

where we follow the notation introduced in Section 2.1. Using these non–relativistic expansions and the kinematic relations given by Eqs. (16–19), it is straightforward to obtain

\[ \mathcal{J}_0^S(1, 2) = \frac{\mathcal{F}}{2\sqrt{1 + \lambda}} \left\{ \frac{\sigma_1 \cdot (\zeta_1 - \lambda \eta_1) \sigma_2 \cdot (\eta_2 + \eta_2')}{K_1^2 - m_\pi^2} \right. \]
\[ \left. - \frac{\sigma_1 \cdot [(1 + \lambda) \eta_1 + \eta_1'] \sigma_2 \cdot \zeta_2}{K_2^2 - m_\pi^2} \right\} , \quad (26) \]

having defined

\[ \mathcal{F} = -\frac{f^2 m_N}{V^2 m_\pi^2} i\epsilon_{zab} \langle t_p|\tau_a|t_h'\rangle \langle t_h'|\tau_b|t_h \rangle F^V_1 . \quad (27) \]

We can still simplify this expression by using the relation \( \lambda \simeq \tau + \kappa \cdot \eta_2 \) (valid up to first order in powers of \( \eta_2 \)): the result is

\[ \mathcal{J}_0^S(1, 2) = \frac{\mathcal{F}}{2\sqrt{1 + \tau}} \left\{ \frac{\sigma_1 \cdot (\zeta_1 - \tau \eta_1) \sigma_2 \cdot (\eta_2 + \eta_2')}{K_1^2 - m_\pi^2} \right. \]
\[ \left. - \frac{\sigma_1 \cdot [(1 + \tau) \eta_1 + \eta_1'] \sigma_2 \cdot \zeta_2}{K_2^2 - m_\pi^2} \right\} . \quad (28) \]

Examining this result, we see that we have retained some terms that should actually be neglected because they give 2\textsuperscript{nd}–order contributions: this is the case for the factor \( \tau(\sigma_1 \cdot \eta_1) \) which is multiplied by \( \sigma_2 \cdot (\eta_2 + \eta_2') \) and \( \sigma_2 \cdot \zeta_2 \). When these contributions to the current
are omitted, then the comparison between our result and the traditional non–relativistic expression \[13\], \[17\] becomes more straightforward. Moreover note that if one neglects the term \( \tau (\sigma_1 \cdot \eta_1) \), then one recovers for the time component of the seagull current an expression that is similar to the traditional non–relativistic reduction \[13\] except for the common factor \( 1/\sqrt{1+\tau} \), which accordingly incorporates important aspects of relativity not considered in the traditional non–relativistic reduction. This result is similar to the discussion given in Ref. \[1\] for the case of the single–nucleon electromagnetic current. Finally, note that, strictly speaking, in order to be consistent with the non–relativistic expansion, the contribution of \( \sigma_1 \cdot \eta_1 \) should be also neglected. Therefore, our final expression for the non–relativistic reduction of the seagull current can be written in the form

\[
J^S_0(1, 2) = \frac{\mathcal{F}}{2\sqrt{1+\tau}} \left\{ \frac{(\sigma_1 \cdot \zeta_1) \sigma_2 \cdot (\eta_2 + \eta'_2)}{K_1^2 - m_\pi^2} - \frac{(\sigma_1 \cdot \eta'_1) (\sigma_2 \cdot \zeta_2)}{K_2^2 - m_\pi^2} \right\} .
\]

(29)

In order to obtain a truly first–order expansion of the current it is convenient to reexpress the momenta involved in Eq. (28) in terms of the momentum transfer \( \kappa \), which can in principle be large, and the nucleon momenta \( \eta_1, \eta_2 \), which lie below the Fermi surface and are kept as the parameters of the expansion. By inserting Eqs. (16–19) into Eq. (28) and keeping only terms linear in \( \eta_1, \eta_2 \), one gets (note that we keep the full pion propagators)

\[
J^S_0(1, 2) = \mathcal{F} \frac{\sigma_1 \cdot \kappa}{\sqrt{1+\tau}} \left[ \frac{\sigma_2 \cdot (\eta_1 + \eta_2)}{K_1^2 - m_\pi^2} - \frac{\sigma_2 \cdot (\eta_1 - \eta_2)}{K_2^2 - m_\pi^2} \right] .
\]

(30)

Finally, it is also interesting to examine the limit \( \eta_F \to 0 \), since this will provide some understanding of how the MEC effects are expected to evolve in going from light (\( \eta_F \) very small) to heavy nuclei (\( \eta_F \approx 0.29 \)). Obviously, in this case the following relations are satisfied

\[
\eta_1 = \eta_2 = \eta'_2 = \zeta_2 = 0 \\
\zeta_1 = \eta'_1 = 2\kappa
\]

(31)\hspace{1cm}(32)

and therefore the seagull current simply reduces to

\[
\left[ J^S_0(1, 2) \right]_{\eta_F \to 0} = 0 .
\]

(33)

This is a consequence of the fact that the time component of the seagull current is of first order in the small variables involved or, in other words, it is of \( O(\eta_F) \).

\underline{Space Components}
The particle–hole matrix element is given by

\[
\langle \phi' | J^S | h'h \rangle = \frac{f^2}{V^2 m^2} \epsilon \bar{\epsilon} \langle t_p | \tau_a | t_{h'} \rangle \langle t_{h'} | \tau_b | t_h \rangle
\]

\[
\times \left\{ \bar{u}(p_1', s_1') \gamma_5 K_1 u(p_1, s_1) \frac{F^V}{K_1^2 - m^2} \bar{u}(p_2', s_2') \gamma_5 \gamma u(p_2, s_2) - \bar{u}(p_2', s_2') \gamma_5 K_2 u(p_2, s_2) \frac{F^V}{K_2^2 - m^2} \bar{u}(p_1', s_1') \gamma_5 \gamma u(p_1, s_1) \right\} .
\]

(34)

Using again the general relation (12) for the matrix forms for \( \gamma_5 \gamma \) and \( \gamma_5 K_{1,2} \), one can write

\[
\langle \phi' | J^S | h'h \rangle
\]

\[
= - \frac{f^2}{4V^2 m^2} \epsilon \bar{\epsilon} \langle t_p | \tau_a | t_{h'} \rangle \langle t_{h'} | \tau_b | t_h \rangle \sqrt{(1 + \epsilon_1)(1 + \epsilon_2)(1 + \epsilon_1')(1 + \epsilon_2')}
\]

\[
\times \left\{ \chi_1^\dagger \left[ m_N \sigma \cdot \zeta_1 - E_{k_1} \left( \frac{\sigma \cdot \eta_1}{1 + \epsilon_1} + \frac{\sigma \cdot \eta_1'}{1 + \epsilon_1'} \right) + m_N \frac{\sigma \cdot \eta_1}{1 + \epsilon_1} \frac{\sigma \cdot \eta_1}{1 + \epsilon_1} \right] \chi_{s_1}
\]

\[
\times \frac{F^V}{K_1^2 - m^2} \chi_2^\dagger \left[ \sigma + \frac{\sigma \cdot \eta_2}{1 + \epsilon_2} \frac{\sigma \cdot \eta_2}{1 + \epsilon_2} \right] \chi_{s_2}
\]

\[
- \chi_1^\dagger \left[ m_N \sigma \cdot \zeta_2 - E_{k_2} \left( \frac{\sigma \cdot \eta_2}{1 + \epsilon_2} + \frac{\sigma \cdot \eta_2'}{1 + \epsilon_2'} \right) + m_N \frac{\sigma \cdot \eta_2}{1 + \epsilon_2} \frac{\sigma \cdot \eta_2}{1 + \epsilon_2} \right] \chi_{s_1}
\]

\[
\times \frac{F^V}{K_2^2 - m^2} \chi_2^\dagger \left[ \sigma + \frac{\sigma \cdot \eta_1}{1 + \epsilon_1} \frac{\sigma \cdot \eta_1}{1 + \epsilon_1} \right] \chi_{s_2} \right\} .
\]

(35)

We can now make use of the non–relativistic reductions as explained in the previous section in connection with the time component. Then, considering only terms up to first order in the small quantities in which we are expanding, after some algebra one finally gets

\[
\mathcal{J}^S(1, 2) \simeq \mathcal{F} \left\{ \frac{1}{\sqrt{1 + \lambda}} \frac{\sigma_1 \cdot (\zeta_1 - \lambda \eta_1)}{K_1^2 - m^2} - \sqrt{1 + \lambda} \frac{\sigma_1 \cdot (\zeta_2 - \lambda \eta_2)}{K_2^2 - m^2} \right\} .
\]

(36)

This expression can be further simplified by using the relation \( \lambda \simeq \tau + \kappa \cdot \eta_2 \) and expanding the terms \( 1/\sqrt{1 + \lambda} \) and \( \sqrt{1 + \lambda} \) in powers of \( \eta_2 \) (up to first order). The final result is

\[
\mathcal{J}^S(1, 2) \simeq \mathcal{F} \left\{ \frac{1}{\sqrt{1 + \tau}} \frac{\sigma_1 \cdot \left[ (1 - \frac{\kappa \cdot \eta_2}{2(1 + \tau)}) \zeta_1 - \tau \eta_1 \right]}{K_1^2 - m^2} - \sqrt{1 + \tau} \frac{\sigma_1 \cdot (\zeta_2 - \lambda \eta_2)}{K_2^2 - m^2} \right\} .
\]

(37)

It is important to note that neglecting the term \( (\kappa \cdot \eta_2)/[2(1 + \tau)] \) compared to 1, and \( \tau(\sigma_1 \cdot \eta_1) \) compared to \( \sigma_1 \cdot \zeta_1 \) (good approximations — see the next section) one simply recovers the traditional non–relativistic expression \[15\] except for the factors \( 1/\sqrt{1 + \tau} \) and \( \sqrt{1 + \tau} \) that multiply the contributions given by the two diagrams involved. As in the case of the time component, this result indicates that important relativistic effects can
be simply accounted for by these multiplicative terms. In next section we shall present results for a wide choice of kinematics showing the validity of the obtained expressions. By inserting Eqs. (16–19) into Eq. (37) and expanding up to first order in \( \eta_1, \eta_2 \), one gets

\[
\mathcal{J}^S(1,2) = \mathcal{F} \left\{ \frac{2\sigma_1 \cdot \kappa}{\sqrt{1+\tau}} \left( 1 - \frac{\kappa \cdot \eta_2}{2(1+\tau)} \right) + \frac{\sigma_1 \cdot \eta_2}{\sqrt{1+\tau}} - \sqrt{1+\tau} \sigma_1 \cdot \eta_1 \right\} \frac{\sigma_2}{K^2 - m^2_\pi} - \frac{\sigma_1}{K^2 - m^2_\pi} \sigma_2 \cdot (\eta_1 - \eta_2) \right\}.
\]

Finally, in the limit \( \eta_F \to 0 \), one obtains

\[
\left[ \mathcal{J}^S(1,2) \right]_{\eta_F \to 0} \simeq \frac{2\mathcal{F} (\sigma_1 \cdot \kappa) \sigma_2}{\sqrt{1+\tau} Q^2 - m^2_\pi},
\]

which shows that the space components of the seagull current are of \( O(1) \) and contribute even for nucleons at rest, as do the charge and magnetization pieces of the one–body current.

### 2.2.2. Pion–in–flight Current Operator

The relativistic pion–in–flight current operator reads

\[
J^V_{\mu}(Q) = \frac{f^2 F_\pi}{V^2 m^2_\pi} i\epsilon_{zab} \frac{K_1 - K_2}_\mu}{(K^2_1 - m^2_\pi)(K^2_2 - m^2_\pi)} \bar{\psi}(p'_1) \gamma_5 \gamma_\mu K_1 u(p_1) \bar{\psi}(p'_2) \gamma_5 \gamma_\mu K_2 u(p_2),
\]

where the kinematic variables are defined in Fig. 1 and where the kinematic relationships given in Eqs. (16) (19) for the seagull diagram are again satisfied. In order to preserve gauge invariance, we choose the electromagnetic pion form factor to be \( F_\pi = F_1^V \) (see Ref. [3]).

The exchange particle–hole matrix element is given by

\[
\langle ph'|J^V_{\mu}|h'h \rangle = \frac{f^2 F_1^V}{4 V^2 m^2_\pi} i\epsilon_{zab} \langle p'|t_\tau a|t_{h'} \rangle \langle t_h|\tau_b|t_h \rangle \frac{K_1 - K_2}_\mu}{(K^2_1 - m^2_\pi)(K^2_2 - m^2_\pi)}
\times \bar{\psi}(p'_1) \gamma_5 K_1 u(p_1) \bar{\psi}(p'_2) \gamma_5 K_2 u(p_2).
\]

Again using Eq. (2) and the matrix forms of \( \gamma_5 K_1 (\gamma_5 K_2) \) we can write

\[
\langle ph'|J^V_{\mu}|h'h \rangle = \frac{f^2 F_1^V}{4 V^2 m^2_\pi} i\epsilon_{zab} \langle p'|t_\tau a|t_{h'} \rangle \langle t_h|\tau_b|t_h \rangle \times \sqrt{(1+\varepsilon_1)(1+\varepsilon'_1)(1+\varepsilon_2)(1+\varepsilon'_2)} \frac{K_1 - K_2}_\mu}{(K^2_1 - m^2_\pi)(K^2_2 - m^2_\pi)}
\times \chi^\dagger_{s_1} \left[ m_N \sigma \cdot \gamma_1 - E_{k_1} \left( \frac{\sigma \cdot \eta_1}{1+\varepsilon_1} + \frac{\sigma \cdot \eta'_1}{1+\varepsilon'_1} \right) + m_N \sigma \cdot \eta'_1 \left( \sigma \cdot \gamma_1 \right) \right] \chi_{s_1}
\times \chi^\dagger_{s_2} \left[ m_N \sigma \cdot \gamma_2 - E_{k_2} \left( \frac{\sigma \cdot \eta_2}{1+\varepsilon_2} + \frac{\sigma \cdot \eta'_2}{1+\varepsilon'_2} \right) + m_N \sigma \cdot \eta'_2 \left( \sigma \cdot \gamma_2 \right) \right] \chi_{s_2}.
\]
Following the arguments discussed for the seagull current we expand up to first order in powers of the variables \(\{\eta_1, \eta_2, \eta_2', \zeta_2\}\), whereas \(\{\eta_1', \kappa, \zeta_1\}\) are treated exactly. The non–relativistic reductions for the kinematic variables are given by Eqs. (22–25) and after some algebra one obtains

\[
\mathcal{J}_P(1,2) \simeq -\frac{F}{\sqrt{1 + \lambda}} \frac{m_N(K_1 - K_2)_\mu}{(K_1^2 - m_\pi^2)(K_2^2 - m_\pi^2)} \sigma_1 \cdot \zeta_1 \sigma_2 \cdot \zeta_2 .
\] (43)

Using again the first–order relation \(\lambda \simeq \tau + \kappa \cdot \eta_2\) and expanding \(1/\sqrt{1 + \lambda}\) in powers of \(\eta_2\), the pion–in–flight matrix element may finally be cast into the form

\[
\mathcal{J}_P(1,2) \simeq -\frac{F}{\sqrt{1 + \tau}} \frac{m_N(K_1 - K_2)_\mu}{(K_1^2 - m_\pi^2)(K_2^2 - m_\pi^2)} \sigma_1 \cdot \zeta_1 \sigma_2 \cdot \zeta_2 .
\] (44)

This expression is similar to the traditional non–relativistic current [15] except for the common factor \(1/\sqrt{1 + \tau}\), which should again include important aspects of relativity not taken into account in the traditional non–relativistic reduction.

Once more we can express this matrix element in terms of \(\kappa, \eta_1\) and \(\eta_2\) and keep only linear terms in the small momenta, obtaining

\[
\mathcal{J}_0^P(1,2) = -\frac{4Fm_N^2}{(K_1^2 - m_\pi^2)(K_2^2 - m_\pi^2)} \frac{\tau}{\sqrt{1 + \tau}} \sigma_1 \cdot \kappa \sigma_2 \cdot (\eta_1 - \eta_2)
\] (45)

\[
\mathcal{J}_P(1,2) = -\frac{4Fm_N^2}{(K_1^2 - m_\pi^2)(K_2^2 - m_\pi^2)} \frac{\sigma_1 \cdot \kappa \sigma_2 \cdot (\eta_1 - \eta_2)}{\sqrt{1 + \tau}} \kappa .
\] (46)

Note that the space component of the pionic current is, in leading order, purely longitudinal; its transverse components are in fact of second order in \(\eta_F\).

Finally the limit \(\eta_F \to 0\) reduces to \(\langle J^P_\mu \rangle = 0\), and we see that all the components of the pionic current are of \(O(\eta_F)\) in the expansion.

3. Results

We present here a discussion of the numerical results obtained for the MEC matrix elements. In the following we compare the fully–relativistic results with those obtained using two different expansions for the MEC: i) the traditional non–relativistic approach (TNR), where \(\kappa, \lambda\) and all nucleon three–momenta \(\eta\) are treated as being small and ii) our new non–relativistic (also referred to as “relativized”) approach (NR) where we expand only in powers of bound nucleon three–momenta, not in \(\kappa\) or \(\lambda\). In order to check the validity of our expansions we compute the transition matrix element of the current between the Fermi sea and a p-h excitation, i.e. \(\langle ph^{-1}|J^MEC|F_\rangle\). Furthermore, to assess the quality
of the expansions performed for the MEC in the last section, we show here comparisons of matrix elements for the relativistic MEC taken between Dirac spinors with matrix elements of the expanded MEC taken between Pauli spinors. In what follows we use the notation of the lower diagrams shown in Fig. 1. The connection with the general terminology introduced in the previous section is given by the following relations:

\[ P = P'_1 \]

is the four–momentum of the ejected nucleon and

\[ H = P_2, \quad H' = P'_2 = P_1 \]

are the initial and intermediate four–momenta of the bound nucleons. The corresponding three–momenta are denoted by: \( p = |p| = |p'_1|, \quad h = |h| = |p_2| \) and \( h' = |h'| = |p'_2| = |p_1| \).

### 3.1. Seagull current

In order to simplify our analysis we first extract from the currents the factors which are common to both relativistic and non–relativistic currents, namely, coupling constants, form factors and isospin matrix elements. Accordingly, for the seagull current \( J^S_\mu \), we define a dimensionless function \( K^S_\mu(q, \omega, h) \) as follows:

\[
\sum_{h'} \langle ph'|J^S_\mu|h'h\rangle = \frac{f^2}{V m_n^2} i([\tau_1 \times \tau_2]_z) F_1^V k_F^2 K^S_\mu(q, \omega, h),
\]

(47)

where \( ([\tau_1 \times \tau_2]_z) \) stands for the corresponding matrix element and implies a summation over isospin. Using the definition of the seagull current, the expression for the function \( K^S_\mu \) is

\[
K^S_\mu(q, \omega, h) = \frac{1}{k_F^2} \sum_{s_{h'}} \int \frac{d^3h'}{(2\pi)^3} \left[ \frac{\bar{u}(p) \gamma^5 (P - H') u(h')}{(p - h')^2 - m^2} \right] \pi(h') \gamma_5 \gamma_\mu u(h)
\]

\[
- \frac{\pi(h') \gamma^5 (H' - H) u(h)}{(H' - H)^2 - m^2} \bar{u}(p) \gamma_5 \gamma_\mu u(h'),
\]

(48)

where the sum runs over the third spin component of the spinor \( u(h') \equiv u(h', s_{h'}) \) and the integral over \( h' \) is performed below the Fermi sea \( |h'| \leq k_F \). This expression for \( K^S_\mu \) is the exact relativistic result. Note that we have divided by the squared Fermi momentum \( k_F^2 \) in order to obtain a dimensionless function; a factor \( k_F^2 \) is correspondingly included in Eq. (47). In the previous section we have performed an expansion in powers of the small quantities \( h/m_N \) and \( h'/m_N \). Therefore we define for the components of the function \( K^S_\mu \) the following “non–relativistic” approximations:

\[
K^{S,NR}_0(q, \omega, h) = - \frac{1}{2 m_N k_F^2} \frac{1}{\sqrt{1 + \tau}} \sum_{s_{h'}} \int \frac{d^3h'}{(2\pi)^3} \left[ \chi_{s_{h'}}^\dagger \frac{\sigma \cdot (p - h' - \tau h') \chi_{s_{h'}}}{(P - H')^2 - m^2} \chi_{s_{h'}}^\dagger \frac{\sigma \cdot (h' + h) \chi_{s_{h'}}}{(P - H')^2 - m^2} \right]
\]

(47)
where \( \text{S,NR1} \) is meant to denote non–relativistic approximation number 1 for the seagull contributions. These expressions should be compared with the traditional non–relativistic seagull current that can be obtained by taking the limit \( \kappa \to 0 \) and \( \tau \to 0 \), namely

\[
K_{0}^{\text{S,TNR}}(q, \omega, h) = -\frac{1}{2m_{N}k_{F}^{2}} \sum_{s_{h'}} \int \frac{d^{3}h'}{(2\pi)^{3}} \left\{ \frac{\chi_{s_{h'}}^{\dagger} \sigma \cdot (p - h') \chi_{s_{h'}}}{(P - H')^{2} - m_{\pi}^{2}} \chi_{s_{h}}^{\dagger} \sigma \cdot (h' + h) \chi_{s_{h}} \right\} \]

(51)

where \( \text{S,TNR} \) denotes the traditional non–relativistic approximation for the seagull contributions. In addition one would like to find approximations to the currents where the relativistic effects are accounted for as corrective factors consisting of simple functions of \( (q, \omega) \) such as the combination of \( \kappa, \tau \) and \( \sqrt{1 + \tau} \) previously found for the case of the single–nucleon current (see Sect. 2.1). One could thus easily implement the relativistic corrections in existing models of traditional non–relativistic MEC. For these reasons we define a second approximation for the seagull current in which we neglect the factors \( \kappa \cdot h/[2m_{N}(1 + \tau)] \) and \( \tau h' \) in the function \( K^{\text{S,NR1}} \), thereby yielding a second non–relativistic approximation for the seagull current, \( K_{0}^{\text{S,NR2}} \), which differs from the traditional current only by the factor \( \sqrt{1 + \tau} \):

\[
K_{0}^{\text{S,NR2}}(q, \omega, h) = -\frac{1}{2m_{N}k_{F}^{2}} \sum_{s_{h'}} \int \frac{d^{3}h'}{(2\pi)^{3}} \left\{ \frac{\chi_{s_{h'}}^{\dagger} \sigma \cdot (p - h') \chi_{s_{h'}}}{(P - H')^{2} - m_{\pi}^{2}} \chi_{s_{h}}^{\dagger} \sigma \cdot (h' + h) \chi_{s_{h}} \right\} \]

(53)

\[
= \frac{1}{\sqrt{1 + \tau}} K_{0}^{\text{S,TNR}}(q, \omega, h) \]

(54)
$$K^{S,NR2}(q, \omega, h) = -\frac{1}{k_F} \frac{1}{\sqrt{1 + \tau}} \sum_{s,h^\prime} \int \frac{d^3 h^\prime}{(2\pi)^3} \left[ \frac{\chi_{s_p}^\dagger \sigma \cdot (p - h^\prime) \chi_{s_{h'}}^s}{(P - H')^2 - m_p^2} \chi_{s_{h'}}^s \sigma X_{sh} ight. \\
- \left. \frac{1}{1 + \tau} \frac{\chi_{s_{h'}}^s \sigma \cdot (h^\prime - h) \chi_{s_h}^s}{(H' - H)^2 - m_N^2} \chi_{s_{h'}}^s \sigma X_{sh} \right],$$

(55)

where “S,NR2” denotes non–relativistic approximation number 2 for the seagull contributions. In what follows we check the validity of the various approximations to the full current introduced above by performing numerical calculations of the functions $K^{S}(q, \omega, h)$ for several choices of the kinematical variables.

First we notice that, for fixed momentum and energy transfer, $(q, \omega)$, there are restrictions on the values of the momentum of the hole $h$, since our nucleons are on–shell and the momentum of the ejected particle $P^\mu = H^\mu + Q^\mu$ must satisfy $P^\mu P^\mu = m_N^2$. In particular, the restriction

$$2h \cdot q = \omega^2 - q^2 + 2E_h\omega,$$

(56)

fixing the angle between the hole momentum $h$ and the momentum transfer $q$ is seen to hold. Therefore the functions $K^{S}_\mu$ depend only on the variables $(q, \omega, h, \phi_h)$, where $h = |h|$ is the magnitude of the hole momentum and $\phi_h$ is the azimuthal angle of $h$ in a coordinate system with the $z$-axis in the direction of $q$. The angle between $h$ and $q$ is then given by

$$\cos \theta_h = \frac{\omega^2 - q^2 + 2E_h\omega}{2hq}.$$

(57)

As this must lie between $-1$ and $1$, one obtains a restriction on the values of $\omega$ for which a contribution to the on–shell matrix element exists.

For fixed values of $(q, \omega)$, there is another restriction generated by the condition that the particle momentum $p$ must lie above the Fermi sea; in this work, however, we are only interested in the high–momentum region where relativistic corrections are expected to be important and there such Pauli–blocking effects can be ignored.

In Figs. 2–4 we show the dominant (real or imaginary) parts of the four vector components of the seagull function $K^{S}_\mu$. The components not shown in the figures are found to be negligible in our calculations, as a result of cancellations occurring among different pieces: in order to understand the reasons for these cancellations we have explored the symmetries of the integrals involved in the various components. A summary of that study is given in the Appendix for the particular case of the pion-in-flight current; a similar procedure can be followed for the seagull current.

We choose the typical value $k_F = 250$ MeV/c for the Fermi momentum and in all of the figures the hole kinematics correspond to $h = 175$ MeV/c and $\phi_h = 0^\circ$. Because of the
above mentioned symmetries in the currents it is possible to relate the matrix elements corresponding to other choices of the angle $\phi_h$ to the ones calculated here — for example the symmetry between the $\phi_h = 90^\circ$ and $\phi_h = 0^\circ$ cases is discussed for the pion-in-flight current in the Appendix. We have checked that the same connection obtains for the values $\phi_h = 90^\circ$ and $\phi_h = 180^\circ$, with the exception that in this case the roles of some of the components are switched. However, since no new information concerning the validity of the expansion is obtained from values of $\phi_h$ different from zero, we show results only for the latter case.

The function $K^S_\mu$ is displayed for three values of the momentum transfer, namely $q = 500, 1000$ and $2000$ MeV/c, as a function of $\omega$. For each $q$, the allowed values of $\omega$ are restricted to the intervals displayed in the figures. Note that the $\omega$-values in the figures have been chosen to lie in a region around the approximate quasielastic peak position, $\omega = \sqrt{q^2 + m_N^2} - m_N$, which for the selected momenta occurs for $\omega \sim 125, 433$ and $1271$ MeV, respectively (approximately at the center of the $\omega$–region displayed in each situation).

In each of the panels in Figs. 2–4 we plot four curves, corresponding to the fully–relativistic $K^S_\mu$ (solid lines) and to the non–relativistic approximation $K^{S,NR1}_\mu$ (dashed lines); moreover the results obtained with the traditional non–relativistic current $K^{S,TNR}_\mu$ (dot-dashed lines) and with our simplified, non–relativistic approximation $K^{S,NR2}_\mu$ (dotted lines) are also shown. The functions $K^S_\mu$ are spin–matrices, i.e., they depend on the spin projections $s_p, s_h$ of the particle and of the hole, respectively; accordingly we write

$$K^\mu = \begin{pmatrix} K^\mu_{11} & K^\mu_{12} \\ K^\mu_{21} & K^\mu_{22} \end{pmatrix}.$$ (58)

For the sake of brevity, in the figures we show results only for the spin components $K_{11}$ and $K_{12}$, corresponding to $(s_p, s_h) = (1/2, 1/2)$ and $(1/2, -1/2)$ respectively. Similar results are found for the remaining components of the currents. As for the dependence upon the angles $\phi_h$, relationships between the spin components displayed in the figures and the remaining ones can be established and the same comments made with respect to the dependence on $\phi_h$ are valid here as well.

Looking at Figs. 2–4, we first note that the imaginary parts of $K^S_0, K^S_1$ and $K^S_3$ and the real part of $K^S_2$ are not shown. Indeed, as mentioned before, they turn out to be negligible in comparison with the other components for all of the situations we have explored and therefore in the following we shall focus only on the remaining four larger contributions to the current shown in the figures.

Second, in all calculations we use relativistic kinematics and the full pion propagator
even when computing the traditional matrix element. The importance of using relativistic kinematics is crucial in the evaluation of the angle between $h$ and $q$ arising from the on–shell condition given in Eq. (56). Non–relativistic kinematics would lead instead to the relationship $2h \cdot q = 2m_N \omega - q^2$, where a factor $\omega^2$, which is clearly important for the high values of $q$ considered here, does not appear. In fact, as pointed in Refs. [1, 15] in discussing the one–body responses, our approximation to the relativistic current is accurate only if the proper relativistic kinematics are used in computing the energy of the particle with momentum $p$. In fact the plane waves are then solutions of the free Klein-Gordon equation rather than of the Schrödinger equation, thus automatically accounting for relativity in the kinetic energy operator. The additional relativistic dynamics incorporated here, arising from the Dirac spinology, enter as modifications of the current operator. For high momentum transfers, both ingredients (relativistic kinematics and current corrections due to spinology) are of the same level of importance.

For moderate momentum transfers, say $q = 500$ MeV/c, the relativistic kinematics alone allow one to obtain agreement between the relativistic $K_S^\mu$ and traditional $K_S^{S,TNR}^\mu$ functions shown in Fig. 2 (compare solid with dot-dashed lines). Although the agreement becomes better for approximations $NR1$ and $NR2$, all of the curves turn out to be close enough to each other to allow the conclusion that for moderate values of $q$ it is sufficient to use the traditional seagull current (but including relativistic kinematics) for computing one–particle knock–out matrix elements.

The situation changes at higher $q$-values. As shown in Figs. 3 and 4, the traditional approximation (dot-dashed line) for the time and longitudinal components, Re$K_S^0$ and Re$K_S^3$, does not agree with the exact relativistic result (solid line). This reflects the approximations made in the derivation of this current, where in particular the factors $\kappa$ and $\lambda$ are (incorrectly) treated as of higher order. In contrast, our approximations $NR1$ and $NR2$ (dashed and dotted lines) are both very close to the fully–relativistic result for all of the $q$-values considered. Therefore the relativistic corrections included in these components of the current in our approximation NR1 (or in its simplified version NR2) appear to be sufficient for a proper description of the relativistic effects.

With regard to the transverse components of the seagull current in the $s_p = 1/2$, $s_h = -1/2$ case, namely Re$K_S^1$ and Im$K_S^2$, we first note that for low momentum transfers they dominate over the longitudinal components to the left of the quasielastic peak. As functions of $\omega$ these transverse components are nearly linear and cross the $\omega$-axis somewhere to the right of the quasielastic peak. This change of sign accounts for the negative interference between the seagull and one–body current contributions in the transverse electromagnetic response to the right of the quasielastic peak [13, 18]. The value of $\omega$
where these functions vanish decreases with increasing $q$: for $q = 2000$ MeV/c the position of the zero almost coincides with the center of the quasielastic peak. The figures also show that at low energy (below the quasielastic peak) the exact result almost coincides with the other three curves. On the other hand, for high energy (above the quasielastic peak) discrepancies occur between the exact and traditional currents: thus at the end of the allowed $\omega$-region, the traditional current (dot-dashed lines) accounts for only about one-half of the exact result (solid lines) in absolute value. On the other hand, our two non–relativistic approximations NR1 and NR2 (dashed and dotted lines) are much closer to the exact result. Hence our results show that the relativized, simplified approximation NR2 to the seagull current is a valid representation of the exact relativistic current for all of the values of the momentum transfer considered here.

### 3.2. Pion–in–flight current

Now we perform a similar analysis for the pion–in–flight (or pionic) current $J_{\mu}^{P}$. First we define dimensionless functions $K_{\mu}^{P}$ for this current as we did for the seagull case; thus in the matrix element

$$\sum_{h'}\langle ph'|J_{\mu}^{P}|h'h\rangle = \frac{f^2}{V m_{\pi}^2} i\langle[\tau_1 \times \tau_2]|_{z}\rangle F_{1}^{V} k_{\mu}^{2} K_{\mu}^{P}(q, \omega, h)$$

(59)

the pionic function $K_{\mu}^{P}$ reads

$$K_{\mu}^{P}(q, \omega, h) = \frac{1}{k_{\mu}^{2}} \sum_{s_{h}, h'} \int \frac{d^3h'}{(2\pi)^3} (P + H - 2H')_{\mu} \frac{\pi(p)\gamma_{5}(P - H')u(h')\pi(h')\gamma_{5}(H' - H)u(h)}{[(P - H')^2 - m_{\pi}^2][(H' - H)^2 - m_{\pi}^2]}.$$  

(60)

We also introduce the traditional non–relativistic function, denoted “P,TNR”, for the pionic contributions

$$K_{\mu}^{P,TNR}(q, \omega, h) = \frac{1}{k_{\mu}^{2}} \sum_{s_{h}, h'} \int \frac{d^3h'}{(2\pi)^3} (P + H - 2H')_{\mu} \frac{\chi_{sp}^{\dagger}\sigma \cdot (p - h')\chi_{sh'}\chi_{sp}^{\dagger}\sigma \cdot (h' - h)\chi_{sh}}{[(P - H')^2 - m_{\pi}^2][(H' - H)^2 - m_{\pi}^2]}.$$  

(61)

Finally, our approximated pionic function, introduced in Sect. 2, is simply given by

$$K_{\mu}^{P,NR}(q, \omega, h) = \frac{1}{\sqrt{1 + \tau}} K_{\mu}^{P,TNR}(q, \omega, h),$$

(62)

where “P,NR” stands for non–relativistic approximation for the pionic contributions.

In Figs. 5–7 we display the various components of the pionic function $K_{\mu}^{P}$ for the same kinematics as employed for the seagull current. The meaning of the curves is the same as in Figs. 2–4, except that now only one relativistic approximation is suggested, since the
simplicity of our result does not require additional assumptions. The results obtained are similar to the ones already found for the seagull current, and we can summarize them in the following points:

- **The important contributions arise from the real parts of** $K_P^0$, $K_P^1$ and $K_P^3$, and imaginary part of $K_P^2$. The reasons for this dominance are discussed in the Appendix, where we are able to inter-relate the general behavior of the curves shown in Figs. 5–7 and we explore the symmetries of the various pieces of this current.

- **For low momentum transfers** ($q$ below some “moderate” value of about 500 MeV/c) the traditional approach, the exact matrix elements and, of course, our present approximation are all very close if one takes into account relativistic kinematics. This justifies the use of the traditional MEC for low to moderate momentum transfers [15, 18, 19].

- **For high momentum transfers** ($q \geq 1000$ MeV/c) the traditional expression (dot-dashed line) clearly disagrees with the fully relativistic result (solid line). From Figs. 6 and 7 it is apparent that the major differences between these two functions disappear if we use our approximated current $P,NR$ (dashed lines). In fact, it is a particularly gratifying result that, apart from using relativistic kinematics, the simple factor $1/\sqrt{1 + \tau}$ applied to the traditional current $K_{P,TNR}^\mu$ is able to reproduce the exact relativistic matrix element remarkably well. Although some disagreements between our approximation and the relativistic current for the case of Im $K_P^2$ exist, it is however clear that the traditional result is much worse. Moreover, the disagreement is found only away from the quasielastic peak where the corresponding one–particle emission response is small, the two matrix elements (solid and dashed lines) being equal at the peak where the approximations associated with the factor $1/\sqrt{1 + \tau}$ are expected to work better.

In conclusion we see that the new pionic current obtained by multiplying the traditional non–relativistic one with the spinology factor $1/\sqrt{1 + \tau}$ significantly improves the relativistic content of the current and hence one can use this current for computing one–particle emission responses for high momentum transfers within non–relativistic models, at least near the quasielastic peak.

### 3.3. The large–$q$ limit

Let us end this section with a brief discussion of the behavior of the currents in the large–$q$ limit. We start from the non–relativistic reductions of the currents as given in
Eqs. (5,9,30,38,45,46) and consider the limit $\kappa \to \infty$. For these conditions

$$\lambda = \frac{1}{2} \left[ \sqrt{(2\kappa + \eta)^2 + 1} - \sqrt{\eta^2 + 1} \right] \simeq \kappa$$

and

$$\tau = (\kappa + \lambda)(\kappa - \lambda) \simeq \kappa(1 - \hat{\kappa} \cdot \eta) \ .$$

By inserting Eqs. (63,64) into the currents we obtain then for the single–nucleon current

$$\mathcal{J}^0 \simeq \sqrt{\kappa} \left[ G_E(1 + \frac{1}{2} \hat{\kappa} \cdot \eta) + i \left( G_M - \frac{G_E}{2} \right) (\hat{\kappa} \times \eta) \cdot \sigma \right]$$

$$\mathcal{J}^\parallel = \frac{\lambda}{\kappa} \mathcal{J}^0 \kappa \simeq \mathcal{J}^0 \tilde{\kappa}$$

$$\mathcal{J}^\perp \simeq i\sqrt{\kappa} G_M \left\{ (\sigma \times \hat{\kappa}) - \frac{1}{2} \left[ (\sigma \cdot \hat{\kappa})(\hat{\kappa} \times \eta) + (\sigma \times \hat{\kappa})(\hat{\kappa} \cdot \eta) \right] \right\}$$

and likewise, for the meson–exchange currents,

$$\mathcal{J}^S_0(1,2) \simeq \mathcal{F} \sqrt{\frac{\kappa}{m^2_\pi}} (\sigma_1 \cdot \hat{\kappa}) \sigma_2 \cdot (\eta_1 - \eta_2)$$

$$\mathcal{J}^S(1,2) \simeq \mathcal{F} \sqrt{\frac{\kappa}{m^2_\pi}} \sigma_1 \cdot \sigma_2 \cdot (\eta_1 - \eta_2)$$

$$\mathcal{J}^P_0(1,2) \simeq -\mathcal{F} \sqrt{\frac{\kappa}{m^2_\pi}} (\sigma_1 \cdot \hat{\kappa}) \sigma_2 \cdot (\eta_1 - \eta_2)$$

$$\mathcal{J}^P(1,2) \simeq -\mathcal{F} \sqrt{\frac{\kappa}{m^2_\pi}} \hat{\kappa}(\sigma_1 \cdot \hat{\kappa}) \sigma_2 \cdot (\eta_1 - \eta_2) ,$$

where the inverse of the pion propagator has been expanded to leading order in the parameters $\eta$ and $1/\kappa$:

$$K_1^2 - m^2_\pi = m^2_\pi \left[ (\varepsilon_1 - \varepsilon_1)^2 - (\eta_1 - \eta_1)^2 \right] - m^2_\pi \simeq -4m^2_N\kappa(1 - \hat{\kappa} \cdot \eta_1) \quad (72)$$

$$K_2^2 - m^2_\pi = m^2_\pi \left[ (\varepsilon_2 - \varepsilon_2)^2 - (\eta_2 - \eta_2)^2 \right] - m^2_\pi \simeq -m^2_\pi . \quad (73)$$

At large $q$, it is of significance that:

- All of the currents grow asymptotically as $\sqrt{\kappa}$. This result is supported by our numerical results, which show that the currents at $q = 2000$ MeV/c are roughly twice as large as those at $q = 500$ MeV/c.
• The time components of the seagull and pion-in-flight currents tend to cancel each other. The same happens for the longitudinal components. Hence only the transverse components of the seagull current survives in this limit.

• The longitudinal and time components of both the seagull and pion-in-flight currents become equal as \( q \to \infty \). Since in this limit \( \lambda \simeq \kappa \), it follows that in the large-\( q \) limit the seagull and pionic currents are separately gauge invariant. Moreover, in this kinematical regime, the correlations among nucleons are not expected to play a significant role, thus implying the separate realization of gauge invariance in each sector of the nuclear response. By extension, the current carried by each individual meson should be expected to be separately conserved.

• Finally, if the form factors are neglected, the single–nucleon current and the MEC display the same asymptotic behavior in \( q \). Of course the inclusion of form factors will change the \( q \)-dependences of the currents.

4. Summary and Conclusions

In this work we have found new approximations to pionic electromagnetic meson–exchange currents using an approach which parallels recent work involving expansions of the electroweak single–nucleon current in powers of the momentum of the initial bound nucleon \( \eta = p/m_N \). Our goal here and in that previous work has been to obtain current operators that can be implemented in computing response functions for high momentum transfers in quasielastic kinematics using non–relativistic models. Our approach allows features of relativity to be taken into account through the use of relativistic kinematics and the Dirac-spinology content implicit in the new currents.

In this paper we have first illustrated the basic procedure by reviewing the simpler case involving the expansion of the single–nucleon current. We have then turned to our main focus in the present work and applied the expansion ideas to a study of the pion–exchange seagull and pion–in–flight MEC. A distinguishing feature of the recently–obtained single–nucleon current is that it incorporates relativistic effects through multiplicative factors involving the dimensionless variables \( \kappa \), \( \tau \) and \( \sqrt{1 + \tau} \) (arising from the Dirac spinology) — factors that are easy to implement in traditional non–relativistic models. Accordingly, in our expansion of the MEC we have sought to identify corresponding factors which can embody the essential features of the relativistic MEC. We have also examined the behavior of the currents in the asymptotic limit where \( q \to \infty \) and found that only the
seagull transverse current survives, and moreover that the latter and the pionic current are separately conserved.

Finally, we have tested the quality of our approximations by computing the transition matrix elements $\langle ph\,^{-1}|J^{MEC}_\mu|F\rangle$ for the various components of the currents, *i.e.*, for matrix elements taken between the ground state of a Fermi gas and a particle–hole excitation. We have compared the exact relativistic matrix elements with our non–relativistic approximations and with the traditional non–relativistic expressions. The differences between our newly–obtained currents and the exact ones are small even for very high momentum transfers, whereas the traditional expressions fail at high $q$. Due to the quality of our results, we believe that these currents can very safely be used in non–relativistic models for computing MEC effects in one–particle emission nuclear responses.

Appendix. Symmetries and relevance of the various components of the currents

In this appendix we study the structure of the integrals $K^a_\mu$ involved in the calculations of the MEC matrix elements to assess the relative weight of the different pieces into which they may be decomposed for a variety of kinematical conditions. In fact, in Figs. 2–7 we have only shown the dominant among the four components $K^a_\mu$ (real or imaginary part). The components not shown in the figures have been also computed and found to be small; here we present arguments to help in understanding why they are so.

For this purpose it is sufficient to consider only the traditional non–relativistic currents, since they are modified simply by multiplicative factors in our various approximations and thus do not change our conclusions as far as the issue at stake here is concerned. For illustration, we restrict ourselves to the analysis of the pion-in-flight current; the same arguments can be applied to an analysis of the seagull current as well.

Our analysis basically amounts to studying the behavior of the pionic function defined by Eq. (61). To this end, we first extract the uninteresting constant factor $1/k_F^2(2\pi)^3$ and define the following integral appearing in the non–relativistic pionic current

$$K^\mu = \sum_{s_{h'}} \int d^3h'(P + H - 2H')^\mu \chi^\dagger_{s_{h'}} \sigma \cdot (p - h')\chi_{s_{h'}} \chi^\dagger_{s_{h'}} \sigma \cdot (h' - h)\chi_{s_{h}},$$

(74)

which depends upon the dimensionless momenta $\kappa$, $\eta$ and $\eta_F$. We aim to identify the leading order in the expansion in $\eta$ and $\eta_F$ of the terms into which this integral splits. Since $\eta < \eta_F$, we shall identify $O(\eta) = O(\eta_F)$. From this investigation we shall see how
the occurrence of cancellations rendering some matrix elements smaller than expected emerges.

We start by performing the summation over the spin of the intermediate hole using the completeness relation 
\[ \sum_{s,h'} \chi_{s,h'} \chi_{s,h'}^\dagger = 1 \]
and by defining the spin-matrix
\[ \Gamma \equiv \sum_{s,h'} \sigma \cdot (p - h') \chi_{s,h'} \chi_{s,h'}^\dagger \sigma \cdot (h' - h) \]
\[ = q \cdot (h' - h) - (h - h')^2 + i[q \times (h' - h)] \cdot \sigma . \] (75)

We shall work in the coordinate system where \( q = q e_3 \) and study the spin components \( K^{\mu}_{11} \) and \( K^{\mu}_{12} \) for which
\[ \Gamma_{11} = q \cdot (h' - h) - (h - h')^2 = q(h' - h)_3 - (h - h')^2 \] (76)
\[ \Gamma_{12} = q(h' - h)_1 - iq(h' - h)_2 . \] (77)

**Spin component \( K^{\mu}_{11} \).** In this case we obtain
\[ K^{\mu}_{11} = \int d^3 h' (P + H - 2H')^\mu \frac{q(h' - h)_3 - (h - h')^2}{([P - H']^2 - m^2)(H' - H)^2 - m^2} . \] (78)

As a consequence, at the non–relativistic level, we have
\[ \text{Im } K^{\mu}_{11} = 0 \] (79)

This is no longer true for the relativistic pionic contribution, although the corresponding imaginary parts of the relativistic pionic matrix elements remain very small.

Next we examine the real components for \( \mu = 0, \ldots, 3 \).

**Re \( K^{3}_{11} \) component**

First, the longitudinal component turns out to read
\[ \text{Re } K^{3}_{11} = \int d^3 h' (q + 2h_3 - 2h'_3) \frac{q(h' - h)_3 - (h - h')^2}{([P - H']^2 - m^2)(H' - H)^2 - m^2} . \] (80)

where \( O(\eta_F) \) means that the associated integrand is linear in \( h \) or \( h' \). Similarly \( O(\eta_F^2) \) means that the integrand is proportional to \( h^2 \), \( (h')^2 \) or \( hh' \) and so on. The leading terms in Eq.(80) are clearly of order \( \eta_F \) and \( \eta_F^2 \) and their expressions are given by
\[ O(\eta_F) = \int d^3 h' \frac{q^2(h'_3 - h_3)}{([P - H']^2 - m^2)(H' - H)^2 - m^2} \] (81)
\[ O(\eta_F^2) = -\int d^3 h' \frac{2q(h'_3 - h_3)^2 + q(h - h')^2}{([P - H']^2 - m^2)(H' - H)^2 - m^2} . \] (82)
In comparing these two pieces, we see that the integrand in the term $O(\eta_F)$ is proportional to $h'_3 - h_3$, which can be positive or negative, thus potentially leading to cancellations. In fact, this term is found to be close to zero near the quasielastic peak (QEP). The reason is the following: at the QEP $p = q$, and, for $q$ large, $h$ is almost perpendicular to $q$ and thus for the case $\phi_h = 0^\circ$ at the QEP we can set $h \sim h e_1$. The corresponding integral accordingly vanishes to first order, namely
\[
\int d^3h' \frac{(h'_3 - h_3)}{[(P - H')^2 - m_N^2][(H' - H)^2 - m_N^2]} \approx \int d^3h' \frac{h'_3}{[Q^2 - m_N^2][(H' - H)^2 - m_N^2]} = 0 ,
\]
because the denominator is invariant with respect to the inversion of $h'_3$. As a consequence, for the longitudinal component of the pionic current $K_{11}$, we have
\[
\text{Re } K^3_{11} \approx 0 \quad \text{near the QEP},
\]
as we have checked numerically. For instance, in Fig. 6 we see that Re $K^3_{11}$ crosses the $\omega$-axis somewhat short of the middle of the $\omega$-allowed domain (the approximate position of the QEP) because of the approximation made in the denominator and of the presence of the other term $O(\eta_F^2)$. This is found to be negative in this region and not entirely negligible; hence the total matrix element reaches zero slightly to the left of the QEP.

Therefore, we see in this case that, although a priori this current if of $O(\eta_F)$, its actual weight depends upon the value of the coefficient that multiplies $\eta_F$ in the expansion, which is $\omega$-dependent and in some cases may be small.

On the other hand, we see in the same figure that in the regions far from the QEP (especially for high $\omega$) this component is large: indeed here the cancellations are much weaker and then the behavior of Re $K^3_{11}$ is again of $O(\eta_F)$.

\underline{Re $K^0_{11}$ component}

The same conclusions obtained for the third component are valid as well for the Re $K^0_{11}$. In this case:
\[
\text{Re } K^0_{11} = \int d^3h' \left( \omega + 2 \frac{h^2}{2m_N} - 2 \frac{h'^2}{2m_N} \right) \frac{q(h'_3 - h_3) - (h - h')^2}{[(P - H')^2 - m_N^2][(H' - H)^2 - m_N^2]} = O(\eta_F) + O(\eta_F^2) + O(\eta_F^3) ,
\]
with
\[
O(\eta_F) = \int d^3h' \frac{\omega q(h'_3 - h_3)}{[(P - H')^2 - m_N^2][(H' - H)^2 - m_N^2]},
\]
\[
O(\eta_F^2) = - \int d^3h' \frac{\omega (h - h')^2}{[(P - H')^2 - m_N^2][(H' - H)^2 - m_N^2]} .
\]
Now, for the same reasons as before, the term $O(\eta_F)$ almost vanishes near the QEP. Then

$$\text{Re} K_1^0 \simeq 0 \quad \text{near the QEP.} \quad (88)$$

Hence $\text{Re} K_1^0$ behaves like $\text{Re} K_1^3$, as also found in our calculations (see Fig. 6). Even finer details of the results can be interpreted in the same manner. For instance:

- The zero of $K_1^0$ occurs slightly to the right of the zero of $K_1^3$, because, in the present case, a piece proportional to $2q(h_3 - h_3')$ is missing in the term $O(\eta_F^2)$. Hence the $O(\eta_F^2)$ term is less negative than in the $K_1^3$ case.

- Also, to second order, the following relation

$$K_1^3 = \frac{q}{\omega} K_1^0 - \int d^3h' \frac{2q(h_3 - h_3')^2}{[(P - H')^2 - m_\pi^2][(H' - H)^2 - m_\pi^2]} \quad (89)$$

is seen to hold. For instance, for $q = 1000$ MeV/c, $\phi_h = 0^\circ$ and $\omega = 550$ MeV, it turns out that $K_1^0 = -15$, $\frac{q}{\omega} K_1^0 = -27$, while $K_1^3 = -40$. Hence an estimate of $-13$ follows for the magnitude of the second–order term, represented by the integral on the right–hand side of Eq. (89).

**Re $K_1^1$, Re $K_1^2$ components**

Concerning the transverse part we have

$$\text{Re} K_{11}^T = \int d^3h'2(h - h')_T \frac{q(h'_3 - h_3) - (h - h')^2}{[(P - H')^2 - m_\pi^2][(H' - H)^2 - m_\pi^2]} = O(\eta_F^2) + O(\eta_F^3). \quad (90)$$

The major contribution is expected to arise from the second–order term, namely

$$O(\eta_F^2) = \int d^3h'2(h - h')_T \frac{q(h'_3 - h_3)}{[(P - H')^2 - m_\pi^2][(H' - H)^2 - m_\pi^2]} . \quad (91)$$

Now, for $\phi_h = 0^\circ$ and near the QEP, we have $h \simeq h_1 e_1$ and for the components 1 and 2 we obtain

$$\text{Re} K_{11}^1 \simeq \int d^3h' \frac{2q(h_1 - h'_1)h'_3}{[Q^2 - m_\pi^2][(H' - H)^2 - m_\pi^2]} \simeq 0 \quad (92)$$

$$\text{Re} K_{11}^2 \simeq -\int d^3h' \frac{2qh'_2h'_3}{[Q^2 - m_\pi^2][(H' - H)^2 - m_\pi^2]} \simeq 0 . \quad (93)$$

In the first case exact cancellations occur when $h'_3 \to -h'_3$, but in the second case we actually have double cancellations when $h'_3 \to -h'_3$ and $h'_2 \to -h'_2$. Accordingly it should be expected that:
One has
\[ |\text{Re} K_{11}^2| \ll |\text{Re} K_{11}^1| \, . \] (94)

Actually the cancellations in \( \text{Re} K_{11}^2 \) are so strong that this component is even smaller than \( \text{Im} K_{11}^2 \) for the relativistic current (hence \( \text{Re} K_{11}^2 \) is not displayed).

- Both \( \text{Re} K_{11}^2 \) and \( \text{Re} K_{11}^1 \) vanish around the QEP.

- If \( \phi_h = 90^\circ \) then \( h \simeq h_2 e_2 \): accordingly the roles of \( \text{Re} K^1 \) and \( \text{Re} K^2 \) switch, i.e.
\[ |\text{Re} K_{11}^1| \ll |\text{Re} K_{11}^2| \, . \] (95)

All of these properties have been checked in our numerical results.

Since \( \text{Re} K_{11}^1 \) is zero around the QEP for \( \phi_h = 0^\circ \), we infer that this piece is actually of \( O(\eta_F^3) \). Moreover, as the approach to zero of \( \text{Re} K_{11}^2 \) is faster than in the case of \( \text{Re} K_{11}^1 \), \( \text{Re} K_{11}^2 \) is likely of \( O(\eta_F^4) \) (very small). Clearly a precise determination of the actual order would require a more detailed analysis of the integrals, or to compute analytically the integrals in the static limit. However, we believe that the arguments given above are enough for reaching an adequate understanding of the results.

**Spin component** \( K_{12}^{\mu} \). For the \( s_p = 1/2, s_h = -1/2 \) component we have
\[
K_{12}^{\mu} = \int d^3 h' (P + H - 2H')^\mu \frac{q(h' - h) - iq(h' - h)\omega}{[(P - H')^2 - m_N^2][(H' - H)^2 - m_N^2]} \, . \] (96)

Hence the real and imaginary parts of this matrix element are given by
\[
\text{Re} K_{12}^{\mu} = \int d^3 h' (P + H - 2H')^\mu \frac{q(h' - h)\omega}{[(P - H')^2 - m_N^2][(H' - H)^2 - m_N^2]} \, , \quad (97)
\]
\[
\text{Im} K_{12}^{\mu} = \int d^3 h' (P + H - 2H')^\mu \frac{q(h' - h)\omega}{[(P - H')^2 - m_N^2][(H' - H)^2 - m_N^2]} \, . \] (98)

**\( K_{12}^0 \) component**

For the time component we have
\[
K_{12}^0 = \int d^3 h' \left( \omega + 2 \frac{k^2}{2m_N} - 2 \frac{k^2}{2m_N} \right) \frac{q(h' - h) - iq(h' - h)\omega}{[(P - H')^2 - m_N^2][(H' - H)^2 - m_N^2]} \, . \]

\[ = O(\eta_F) + O(\eta_F^3) \, . \] (99)

To first order we obtain
\[
K_{12}^0 \simeq \int d^3 h' \omega \frac{q(h' - h) - iq(h' - h)\omega}{[(P - H')^2 - m_N^2][(H' - H)^2 - m_N^2]} \, . \] (100)
For \( \phi_h = 0^\circ \) and near the QEP again \( h \simeq h_1 e_1 \), and hence

\[
K_{12}^0 \simeq \int d^3 h' \omega \frac{q(h' - h)_1 - i q h'_2}{(Q^2 - m_\pi^2)((H' - H)^2 - m_\pi^2)}. \tag{101}
\]

As in the case of the \( K_{11} \) component, cancellations occur in a way that yields

\[
\text{Im} K_{12}^0 \simeq 0 \quad \text{at the QEP.} \tag{102}
\]

Hence \( \text{Re} K_{12}^0 \) is expected to be the dominant component.

### \( K_{12}^3 \) component

A similar result obtains for the longitudinal component, namely

\[
K_{12}^3 \simeq \int d^3 h' q \frac{q(h' - h)_1 - i q(h' - h)_2}{[(P - H')^2 - m_\pi^2][(H' - H)^2 - m_\pi^2]} \tag{103}
\]

and, for the same reasons as before,

\[
\text{Im} K_{12}^3 \simeq 0 \quad \text{at the QEP.} \tag{104}
\]

Also the following relationship

\[
K_{12}^3 \simeq \frac{q}{\omega} K_{12}^0 \tag{105}
\]

is found to hold to leading order, as can be verified in the figures. For instance, from Fig. 6, for \( q = 1000 \text{ MeV/c} \) and \( \phi_h = 0^\circ \), at the maximum (\( \omega \sim 450 \text{ MeV} \)) we get \( \text{Re} K_{12}^0 \sim -11 \), \( \text{Re} K_{12}^3 \sim -25 \) and \( \frac{q}{\omega} K_{12}^0 = -24.4 \).

### \( K_{12}^1, K_{12}^2 \) components

For the transverse components we have

\[
K_{12}^1 = \int d^3 h' 2(h - h')_1 \frac{q(h' - h)_1 - i q(h' - h)_2}{[(P - H')^2 - m_\pi^2][(H' - H)^2 - m_\pi^2]} \tag{106}
\]

\[
K_{12}^2 = \int d^3 h' 2(h - h')_2 \frac{q(h' - h)_1 - i q(h' - h)_2}{[(P - H')^2 - m_\pi^2][(H' - H)^2 - m_\pi^2]}, \tag{107}
\]

and for the real and imaginary parts of \( K^1 \)

\[
\text{Re} K_{12}^1 = - \int d^3 h' \frac{2q(h'_1 - h_1)^2}{[(P - H')^2 - m_\pi^2][(H' - H)^2 - m_\pi^2]} \tag{108}
\]

\[
\text{Im} K_{12}^1 = \int d^3 h' \frac{2q(h'_1 - h_1)(h'_2 - h_2)}{[(P - H')^2 - m_\pi^2][(H' - H)^2 - m_\pi^2]} \tag{109}
\]
First we see that the integrand in $\text{Re} K_{12}^1$ is proportional to $2(h'_1 - h_1)^2 > 0$, and hence no cancellations occur. Accordingly this integral, although of $O(\eta_F^2)$, turns out to be of the same order of magnitude as $K_{12}^0$ (of $O(\eta_F)$, but with cancellations), as is clearly seen in our results. On the other hand, cancellations do occur in $\text{Im} K_{12}^1$; hence, as before,

$$\text{Im} K_{12}^1 \simeq 0 \quad \text{at the QEP.}$$

We thus expect $\text{Re} K_{12}^1$ to be the dominant part.

In connection with $K_{12}^2$ we obtain

$$\text{Re} K_{12}^2 = - \int d^3 h' \frac{2q(h'_1 - h_1)(h'_2 - h_2)}{[(P - H')^2 - m^2][(H' - H)^2 - m^2]}$$

$$\text{Im} K_{12}^2 = \int d^3 h' \frac{2q(h'_2 - h_2)^2}{[(P - H')^2 - m^2][(H' - H)^2 - m^2]}.$$ (111)

First we see that, as found in our calculations, the following relation

$$\text{Re} K_{12}^2 = - \text{Im} K_{12}^1$$

holds, and hence the real part of $K_{12}^2$ is negligible with respect to the imaginary part, which is thus the dominant one. Also, since the integrand for $\text{Im} K_{12}^2$ is proportional to $(h_2 - h'_2)^2 > 0$, no direct relationship between $\text{Im} K_{12}^2$ and $\text{Re} K_{12}^1$ exists.

Again, for $\phi_h = 90^\circ$, the roles of $K^1$ and $K^2$ are switched, and from the above expressions we find

$$\text{Re} K_{12}^1(\phi_h = 0^\circ) = - \text{Im} K_{12}^2(\phi_h = 90^\circ)$$

$$\text{Im} K_{12}^2(\phi_h = 0^\circ) = - \text{Re} K_{12}^1(\phi_h = 90^\circ).$$ (114)

$$\text{Im} K_{12}^1(\phi_h = 0^\circ) = - \text{Re} K_{12}^1(\phi_h = 90^\circ).$$ (115)

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Figure 1: \textit{MEC diagrams considered in this work.}
\( q = 500 \text{ MeV}/c, \phi_R = 0^\circ \)

Figure 2: Seagull current matrix element \( K^1_\nu \) for \( q = 500 \text{ MeV}/c \). The kinematics for the hole \( h \) is \( h = 175 \text{ MeV}/c \) and \( \phi_h = 0^\circ \). First column: spin (\( \frac{1}{2}, \frac{1}{2} \)) component. Second column: spin (\( \frac{1}{2}, -\frac{1}{2} \)) component. Solid: relativistic, dashed: our approximation NR1, dot-dashed: traditional non-relativistic TNR, dotted: our approximation NR2.
\[ q = 1000 \text{ MeV/c}, \phi_h = 0^\circ \]

Figure 3: The same as Fig. 2 for \( q = 1000 \text{ MeV/c} \)
\( q = 2000 \text{ MeV}/c, \phi_h = 0^\circ \)

Figure 4: The same as Fig. 2 for \( q = 2000 \text{ MeV}/c \).
Figure 5: Pion-in-flight current matrix element $K^p_{\mu}$ for $q = 500$ MeV/c. The kinematics for the hole $h$ is $h = 175$ MeV/c and $\phi_h = 0^\circ$. First column: spin $(\frac{1}{2}, \frac{1}{2})$ component. Second column: spin $(\frac{1}{2}, -\frac{1}{2})$ component. Solid: relativistic, dashed: our approximation NR, dot-dashed: traditional non-relativistic TNR.
Figure 6: The same as Fig. 5 for $q = 1000 \text{ MeV}/c$, $\phi_h = 0^\circ$
Figure 7: The same as Fig.5 for $q = 2000 \text{ MeV}/c$. 

$q = 2000 \text{ MeV}/c$, $\phi_h = 0^\circ$