AN INDEX OF SUMMABILITY FOR PAIRS OF BANACH SPACES

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Abstract. We introduce the notion of index of summability for pairs of Banach spaces; for Banach spaces $E, F$, this index plays the role of a kind of measure of how the $m$-homogeneous polynomials from $E$ to $F$ are far from being absolutely summing. In some cases the optimal index of summability is computed.

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1. Introduction and background

For $1 \leq q \leq p < \infty$ and Banach spaces $E, F$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we recall that a continuous linear operator $u : E \to F$ is absolutely $(p, q)$-summing if there is a constant $C \geq 0$ such that

$$\left( \sum_{k=1}^{n} \| u(x_k) \|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{E^*}} \left( \sum_{k=1}^{n} |\varphi(x_k)|^q \right)^{\frac{1}{q}}$$

for every $n \in \mathbb{N}$ and $x_1, ..., x_n \in E$. Above, and from now on the topological dual of $E$ and its closed unit ball are denoted by $E^*$ and $B_{E^*}$, respectively.

The space of absolutely $(p, q)$-summing linear operators from $E$ to $F$ is denoted by $\Pi_{(p,q)}(E; F)$. The $(p, q)$-summing norm of $u$, defined as the infimum of the constants $C$ in (1), is represented by $\pi_{p,q}(u)$. If $p = q$ the operator $u$ is simply called absolutely $p$-summing and write $\Pi_{p}(E; F)$ and $\pi_{p}(u)$ for the space of absolutely $p$-summing operators and the $p$-summing norm of $u$, respectively.

For the theory of absolutely summing operators we refer to [5].

When only sequences $(x_j)_{j=1}^{n}$ of fixed length $n$ are considered, the infimum over all $C$ satisfying (1) is denoted by $\pi_{p,q}^{(n)}(u)$ (or $\pi_{p}^{(n)}(u)$ when $p = q$). Of course, $\pi_{p,q}^{(n)}(u) \leq \pi_{p,q}(u)$. In [14] [15] the authors investigated in depth estimates of the type

$$\pi_{p,q}(u) \leq c \pi_{p,q}^{(n)}(u),$$

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where $c$ is a positive constant. These estimates show that the $(p, q)$-summing norm of an operator can be sometimes well-approximated using only "few" vectors in the definition of the $(p, q)$-summing norm. The following results of finite-dimensional nature will be crucial in this paper:

**Theorem 1.1.** (Szarek [14]) There exists a universal constant $C$ such that whenever $u : E \to F$ ($E, F$ are Banach spaces) is a finite rank linear operator (say rank$(u) = n$) and $q \geq 2$, then

$$\pi_{q,2}(u) \leq C \pi_{q,2}^{(n)}(u).$$

**Theorem 1.2.** (König, Retherford, Tomczak-Jaegermann [8]) Let $id_{X_n}$ denote the identity on a $n$-dimensional space $X_n$. For $q > 2$, we have

$$2e^{-1} n^{\frac{1}{q}} \leq \pi_{q,2}(id_{X_n}).$$

From now on, as usual, given $x_1, ..., x_n \in E$, we define

$$\| (x_k)_{k=1}^n \|_{w,p} := \sup_{x \in B_{E^*}} \left( \sum_{k=1}^n |\varphi(x_k)|^p \right)^{\frac{1}{p}}.$$

Let $m \in \mathbb{N}$ and $E_1, ..., E_m$ be Banach spaces over $\mathbb{K}$. By $\mathcal{L}(E_1, \ldots, E_m; F)$ we denote the Banach space of all bounded $m$-linear operators from $E_1 \times \cdots \times E_m$ into $F$. In the case $E_1 = \cdots = E_m = E$, we will simply write $\mathcal{L}(mE; F)$, whereas $\mathcal{L}(E; F)$ is the usual Banach space of all continuous linear operators from $E$ to $F$. For the theory of multilinear operators and polynomials between Banach spaces we refer to the excellent books of Dineen [6] and Mujica [11].

For $1 \leq q \leq p < \infty$, an $m$-linear operator $T \in \mathcal{L}(E_1, \ldots, E_m; F)$ is called multiple $(p, q)$-summing ([10] [12]) if there is a constant $C \geq 0$ such that

$$\left( \sum_{k_1, \ldots, k_m=1}^n \left\| T \left( x_{k_1}^{(1)}, \ldots, x_{k_m}^{(m)} \right) \right\|^p \right)^{\frac{1}{p}} \leq C \left\| \left( x_{k_i}^{(i)} \right)_{k_i=1}^n \right\|_{w,q}$$

for all positive integers $n$ and all $x_{k_i}^{(i)} \in E_i$, with $1 \leq k_i \leq n$ and $1 \leq i \leq m$. The vector space of all multiple $(p, q)$-summing operators is denoted by $\Pi_{(p,q)}^{\text{mult}}(E_1, \ldots, E_m; F)$. The infimum, $\pi_{(p,q)}^{\text{mult}}(T)$, taken over all possible constants $C$ satisfying (2) defines a complete norm in $\Pi_{(p,q)}^{\text{mult}}(E_1, \ldots, E_m; F)$. When $E_1 = \cdots = E_m = E$, we write $\Pi_{(p,q)}^{\text{mult}}(mE; F)$. When $k_1 = \cdots = k_m = k$, we recover the definition of the class of absolutely $(p, q)$-summing $m$-linear operators that will be denoted by $\left( \Pi_{(p,q)}(E_1, \ldots, E_m; F) ; \pi_{(p,q)}(\cdot) \right)$. For polynomials, let $\mathcal{P}(mE; F)$ denote the Banach space of $m$-homogeneous polynomials from $E$ into $F$, we recall that given $1 \leq p, q < \infty$, with $p \geq \frac{2}{m}$, a polynomial $P \in \mathcal{P}(mE; F)$ is absolutely $(p, q)$-summing if there is a constant $C \geq 0$ such that

$$\left( \sum_{k=1}^n \|P(x_k)\|^p \right)^{\frac{1}{p}} \leq C \left\| (x_k)_{k=1}^n \right\|_{w,q}^m$$
for all positive integers \( n \) and all \( x_k \in E \), with \( 1 \leq k \leq n \). We denote by \( \mathcal{P}_{(p,q)}(mE;F) \) the Banach space of all absolutely \((p,q)\)-summing polynomials from \( E \) to \( F \).

Of course, when (2) or (3) is not valid, this means that such a constant \( C \) does not exist. However it is not obvious at a first glance that there exists a constant \( C_n \) depending on \( n \) satisfying (2) or (3), since at least formally it could happen that varying the vectors \( x_1, \ldots, x_n \) the constant could tend to infinity. But it is not difficult to prove that this is not the case and when (2) or (3) fails there will exist a constant \( C \) depending on \( n \). For every \( m \in \mathbb{N} \), \( m > 1 \), \( \eta^m_{(p,q)}(E_1 \times \cdots \times E_m) \) is defined as

\[
\sup_{n} \left\{ \frac{1}{n} \sum_{i=1}^{n} |x_i|^{m} \right\}^{1/m} < \infty
\]

Definition 1.3. The \textit{multilinear} \( m \)-index of \((p,q)\)-summability of a pair \((E_1 \times \cdots \times E_m,F)\) is defined as

\[
\eta^m_{(p,q)}(E_1,\ldots,E_m;F) = \inf_{s_{m,p,q}} s_{m,p,q},
\]

where \( s_{m,p,q} \) satisfies the following:

There is a constant \( C \geq 0 \) (depending only on \( m \) and \( T \)) satisfying

\[
\left( \sum_{k_1,\ldots,k_m=1}^{n} \left\| T \left( x_{k_1}^{(1)},\ldots,x_{k_m}^{(m)} \right) \right\|^p \right)^{1/p} \leq C n^{s_{m,p,q}} \prod_{i=1}^{m} \left\| (x_{k_i}^{(i)})_i \right\|_{w,q}
\]

for every \( T \in \mathcal{L}(E_1,\ldots,E_m;F) \) and all positive integers \( n \) and \( x_{k_i}^{(i)} \in E_i \), with \( 1 \leq k_i \leq n \) and \( 1 \leq i \leq m \).

When \( E_1 = \cdots = E_m = E \), we write \( \eta^m_{(p,q)}(E;F) \) instead \( \eta^m_{(p,q)}(E,\ldots,E;F) \).

Similarly the \textit{polynomial} \( m \)-index of \((p,q)\)-summability of a pair of Banach spaces \((E,F)\) is defined as

\[
\eta^m_{(p,q)}(E,F) = \inf_{s_{m,p,q}} s_{m,p,q},
\]

where \( s_{m,p,q} \) satisfies the following:

There is a constant \( C > 0 \) (depending only on \( m \) and \( P \)) satisfying

\[
\left( \sum_{j=1}^{n} \| P(x_j) \|^p \right)^{1/p} \leq C n^{s_{m,p,q}} \left\| (x_j)_{j=1}^{n} \right\|_{w,q}
\]

for every \( P \in \mathcal{P}(mE;F) \), all positive integers \( n \) and all \( x_j \in E \), with \( 1 \leq j \leq n \).
When $m = 1$, we have $\Pi_{(p,q)}^{\text{mult}}(1; E; F) = \mathcal{P}_{(p,q)}(1; E; F) = \Pi_{(p,q)}(E; F)$ and in this case we will simply write $\eta_{(p,q)}(E; F)$.

2. Basic results

One of the cornerstones of the theory of absolutely $p$-summing linear operators is the Dvoretzky–Rogers Theorem. A weak version of this theorem asserts that if $p \geq 1$ and $E$ is a Banach space, then the identity operator on $E$, denoted by $\text{id}_E$, is absolutely $p$-summing if and only if $E$ is finite dimensional. The main goal of this section is to certify that the index of summability is always finite.

The next result provides the 2-summing norm of the identity operator when $E$ is finite dimensional and will be very important for us:

Theorem 2.1. (Pietsch [13]) If $E$ is a Banach space and $\dim E = n$, then $\pi_2(\text{id}_E) = \sqrt{n}$.

We highlight the following corollary of the above theorem for future reference. Note that below we extrapolate the notion of absolutely $p$-summing operators to $p > 0$.

Corollary 2.2. Let $0 < p < \infty$. If $E$ is a normed space and $\dim E = n$, then

\begin{equation}
\pi_p(\text{id}_E) \leq n^{\max\left\{\frac{1}{p}, \frac{1}{2}\right\}}.
\end{equation}

Proof. Let $0 < p < 2$ and $r > 0$ such that $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$. Thus, given $x_1, ..., x_n \in E$ and using Hölder’s Inequality we obtain

\begin{align*}
\left(\sum_{j=1}^{n} \|\text{id}_E(x_j)\|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{j=1}^{n} \|\text{id}_E(x_j)\|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{n} |1|^r\right)^{\frac{1}{r}} \\
&\leq \pi_2(\text{id}_E) \|\langle x_j \rangle_{j=1}^{n}\|_{w,2} n^{\frac{1}{r}}
\end{align*}

Theorem 2.1

\begin{align*}
\leq \pi_2(\text{id}_E) \|\langle x_j \rangle_{j=1}^{n}\|_{w,p}.
\end{align*}

Therefore

$$\pi_p(\text{id}_E) \leq n^{\frac{1}{r}}.$$

For the case $p \geq 2$ we use Inclusion Theorem for absolutely $p$-summing operators (see [5, Theorem 2.8]) to obtain

$$\pi_p(\text{id}_E) \leq \pi_2(\text{id}_E) = n^{\frac{1}{2}}.$$

Remark 2.3. Of course that if $X$ is a subspace of an $n$-dimensional normed space $E$, then

$$\pi_p(\text{id}_X) \leq (\dim X)^{\max\left\{\frac{1}{p}, \frac{1}{2}\right\}} \leq n^{\max\left\{\frac{1}{p}, \frac{1}{2}\right\}}.$$

Although multiple $(p, q)$-summing operators are defined for $p, q \geq 1$, the next result is also valid for $p, q > 0$. 


Proposition 2.4. Let $0 < p < \infty$ and $E_1, \ldots, E_m, F$ be Banach spaces. Then
\[
\eta_{p, p}^{m-\text{mult}}(E_1, \ldots, E_m; F) \leq \frac{m}{p} \text{ for } 0 < p \leq 2;
\]
\[
\eta_{p, p}^{m-\text{mult}}(E_1, \ldots, E_m; F) \leq \frac{m}{2} \text{ for } p \geq 2.
\]

Proof. Let $T \in \mathcal{L}(E_1, \ldots, E_m; F)$, $x_{k_i}^{(i)} \in E_i$ and $X_i = \text{span} \{ x_1^{(i)}, \ldots, x_{n_i}^{(i)} \} \subset E_i$ with $i = 1, \ldots, m$ and $k_i = 1, \ldots, n_i$. Then
\[
\left( \sum_{k_1, \ldots, k_m=1}^{n} \left\| T \left( x_{k_1}^{(1)}, \ldots, x_{k_m}^{(m)} \right) \right\|^p \right)^{\frac{1}{p}} \leq \| T \| \left( \sum_{k_1=1}^{n} \left\| x_{k_1}^{(1)} \right\|^p \right)^{\frac{1}{p}} \cdots \left( \sum_{k_m=1}^{n} \left\| x_{k_m}^{(m)} \right\|^p \right)^{\frac{1}{p}}
\]
\[
= \| T \| \left( \sum_{k_1=1}^{n} \left\| \text{id}_{X_1} \left( x_{k_1}^{(1)} \right) \right\|^p \right)^{\frac{1}{p}} \cdots \left( \sum_{k_m=1}^{n} \left\| \text{id}_{X_m} \left( x_{k_m}^{(m)} \right) \right\|^p \right)^{\frac{1}{p}}.
\]
Since $\text{id}_{X_i}$ is absolutely $p$-summing, for each $i = 1, \ldots, m$, we have
\[
\left( \sum_{k_1=1}^{n} \left\| \text{id}_{X_i} \left( x_{k_1}^{(i)} \right) \right\|^p \right)^{\frac{1}{p}} \leq \pi_p(\text{id}_{X_i}) \sup_{\psi \in B_{X_i}^*} \left( \sum_{k_1=1}^{n} \left\| \psi \left( x_{k_1}^{(i)} \right) \right\|^p \right)^{\frac{1}{p}}.
\]
By the Hahn–Banach Theorem, for each $\psi \in X_i^*$ there is an extension $\bar{\psi} \in E_i^*$ such that $\| \psi \| = \| \bar{\psi} \|$. Thus
\[
\left( \sum_{k_1=1}^{n} \left\| \text{id}_{X_i} \left( x_{k_1}^{(i)} \right) \right\|^p \right)^{\frac{1}{p}} \leq \pi_p(\text{id}_{X_i}) \sup_{\varphi \in B_{E_i}^*} \left( \sum_{k_1=1}^{n} \left\| \varphi \left( x_{k_1}^{(i)} \right) \right\|^p \right)^{\frac{1}{p}}
\]
\[
\leq \pi_p(\text{id}_{X_i}) \sup_{\varphi \in B_{E_i}^*} \left( \sum_{k_1=1}^{n} \left\| \varphi \left( x_{k_1}^{(i)} \right) \right\|^p \right)^{\frac{1}{p}}
\]
\[
= \pi_p(\text{id}_{X_i}) \left\| \left( x_{k_1}^{(i)} \right)_{k_1=1}^{n} \right\|_{w.p},
\]
and hence
\[
\left( \sum_{k_1, \ldots, k_m=1}^{n} \left\| T \left( x_{k_1}^{(1)}, \ldots, x_{k_m}^{(m)} \right) \right\|^p \right)^{\frac{1}{p}} \leq \| T \| \pi_p(\text{id}_{X_1}) \left\| \left( x_{k_1}^{(1)} \right)_{k_1=1}^{n} \right\|_{w.p} \cdots \pi_p(\text{id}_{X_m}) \left\| \left( x_{k_m}^{(m)} \right)_{k_m=1}^{n} \right\|_{w.p}
\]
\[
\leq \| T \| \prod_{i=1}^{m} \left( \pi_p(\text{id}_{X_i}) \left\| \left( x_{k_i}^{(i)} \right)_{k_i=1}^{n} \right\|_{w.p} \right).
\]
By the previous corollary, we have:
1) If $0 < p \leq 2$, then
\[
\left( \sum_{k_1, \ldots, k_m=1}^{n} \left\| T \left( x_{k_1}^{(1)}, \ldots, x_{k_m}^{(m)} \right) \right\|^p \right)^{\frac{1}{p}} \leq \| T \| \left( \prod_{i=1}^{m} \left\| x_{k_i}^{(i)} \right\|_{1, p}^{n} \right)^{\frac{1}{p}}
\]
\[
= \| T \| \prod_{i=1}^{m} \left\| x_{k_i}^{(i)} \right\|_{1, p}^{n},
\]
and
\[
\eta_{(p,p)}^{m-\text{mult}} (E_1, \ldots, E_m; F) \leq \frac{m}{p}.
\]

2) If $p \geq 2$, then, analogously,
\[
\eta_{(p,p)}^{m-\text{mult}} (E_1, \ldots, E_m; F) \leq \frac{m}{2}.
\]

The next result shows that the above estimates can not be improved, keeping its universality.

**Corollary 2.5.** $\eta_{(2,2)}^{m-\text{mult}} (\ell_2; c_0) = \frac{m}{2}$.

**Proof.** Let $t$ be a positive real number such that for each $T \in \mathcal{L}^{(m, \ell_2; c_0)}$ there is a constant $C \geq 0$ such that
\[
\left( \sum_{j_1, \ldots, j_m=1}^{n} \left\| T \left( e_{j_1}^{(1)}, \ldots, e_{j_m}^{(m)} \right) \right\|^2 \right)^{\frac{1}{2}} \leq C n^t \prod_{i=1}^{m} \left\| x_{j_i}^{(i)} \right\|_{1, 2}^{n},
\]
for all positive integers $n$ and all $x_{j_i}^{(i)} \in \ell_2$, with $1 \leq j_i \leq n$.

Now, let $T \in \mathcal{L}^{(m, \ell_2; c_0)}$ be defined by
\[
T \left( x^{(1)}, \ldots, x^{(m)} \right) = \left( x_{j_1}^{(1)} \cdots x_{j_m}^{(m)} \right)^{\frac{n}{j_1, \ldots, j_m=1}}.
\]
Of course $\| T \| = 1$ and
\[
\left( \sum_{j_1, \ldots, j_m=1}^{n} \left\| T(e_{j_1}, \ldots, e_{j_m}) \right\|^2 \right)^{\frac{1}{2}} = n^m.
\]
Since $\left\| (e_{j_i})_{j_i=1}^{n} \right\|_{1, 2} = 1$, the latter condition together with (5) imply
\[
\| e_{j_i} \|_{1, 2} = C n^t
\]
and thus $t \geq \frac{m}{2}$. The converse inequality is given by the previous proposition and the proof is done. \qed
If $q < p$ it is plain that
\[ \eta_{(p,q)}^{m-\text{mult}}(E_1,\ldots,E_m;F) \leq \eta_{(p,p)}^{m-\text{mult}}(E_1,\ldots,E_m;F). \]

The next results provide better estimates.

**Proposition 2.6.** Let $1 \leq q \leq p < \infty$ and $E_1,\ldots,E_m,F$ be Banach spaces. Then
\[ \eta_{(p,q)}^{m-\text{mult}}(E_1,\ldots,E_m;F) \leq \frac{m}{p} \text{ for } 1 \leq q \leq 2; \]
\[ \eta_{(p,q)}^{m-\text{mult}}(E_1,\ldots,E_m;F) \leq \frac{mq}{2p} \text{ for } q \geq 2. \]

**Proof.** Note that
\[ \left( \sum_{k_1,\ldots,k_m=1}^{n} \| T(x^{(1)}_{k_1},\ldots,x^{(m)}_{k_m}) \|^p \right)^{\frac{1}{p}} \leq \| T \| \left( \sum_{k_1=1}^{n} \| x^{(1)}_{k_1} \|^p \right)^{\frac{1}{p}} \cdots \left( \sum_{k_m=1}^{n} \| x^{(m)}_{k_m} \|^p \right)^{\frac{1}{p}}. \]

Let $X_i := \text{span} \{ x^{(i)}_{1},\ldots,x^{(i)}_{n_i} \} \subset E_i$ with $i = 1,\ldots,m$. Since $X_i$ is a finite dimensional Banach space it follows that $id_{X_i}$ is absolutely $q$-summing. So, by [7 Corollary 16.3.1] we have
\[ \pi_{p,q}(id_{X_i}) \leq \pi_{q}(id_{X_i})^{\frac{1}{p}}. \]

Thus, for each $i = 1,\ldots,m$, we obtain
\[ \left( \sum_{k_i=1}^{n} \| x^{(i)}_{k_i} \|^p \right)^{\frac{1}{p}} \leq \pi_{p,q}(id_{X_i}) \| (x_{k_i})_{k_i=1}^{n} \|_{w,q} \overset{(6)}{\leq} \pi_{q}(id_{X_i})^{\frac{2}{p}} \| (x_{k_i})_{k_i=1}^{n} \|_{w,q}. \]

and, for $q \geq 2$, we have
\[ \left( \sum_{k_i=1}^{n} \| x^{(i)}_{k_i} \|^p \right)^{\frac{1}{p}} \leq (n \frac{2}{p})^{\frac{2}{p}} \| (x_{k_i})_{k_i=1}^{n} \|_{w,q} = n^{\frac{4}{2p}} \| (x_{k_i})_{k_i=1}^{n} \|_{w,q}. \]

Therefore
\[ \eta_{(p,q)}^{m-\text{mult}}(E_1,\ldots,E_m;F) \leq \frac{mq}{2p}. \]

Analogously, when $1 \leq q \leq 2$ we conclude that
\[ \left( \sum_{k_1,\ldots,k_m=1}^{n} \| T(x^{(1)}_{k_1},\ldots,x^{(m)}_{k_m}) \|^p \right)^{\frac{1}{p}} = \| T \| n^{\frac{m}{p}} \prod_{i=1}^{m} \| (x^{(i)}_{k_i})_{k_i=1}^{n} \|_{w,q} \]

and
\[ \eta_{(p,q)}^{m-\text{mult}}(E_1,\ldots,E_m;F) \leq \frac{m}{p}. \]

□
It is well-known that the notion of multiple \((p; q)\)-summable operators has no sense when \(p < q\), because just the null map would satisfy the definition. But, curiously, in our context it makes sense to extrapolate the definition to \(0 < p < q\).

**Proposition 2.7.** Let \(0 < p < q < \infty\) and \(E_1, \ldots, E_m, F\) be Banach spaces. Then

\[
\eta_{(p,q)}^{m-\text{mult}} (E_1, \ldots, E_m; F) \leq \frac{m}{p} \text{ for } 0 < q \leq 2;
\]

\[
\eta_{(p,q)}^{m-\text{mult}} (E_1, \ldots, E_m; F) \leq \frac{(qp - 2p + 2q)m}{2qp} \text{ for } q \geq 2.
\]

**Proof.** Note that

\[
\left( \sum_{k_1, \ldots, k_m=1}^n \left\| T \left( x_{k_1}^{(1)}, \ldots, x_{k_m}^{(m)} \right)^p \right\| \right)^{\frac{1}{p}} \leq \| T \| \left( \sum_{k_1=1}^n \left\| x_{k_1}^{(1)} \right\| \right)^{\frac{1}{p}} \cdots \left( \sum_{k_m=1}^n \left\| x_{k_m}^{(m)} \right\| \right)^{\frac{1}{p}}.
\]

For all \(i = 1, \ldots, m\), the Hölder inequality tells us that

\[
\left( \sum_{k_1=1}^n \left\| x_{k_i}^{(i)} \right\|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k_1=1}^n \left\| x_{k_i}^{(i)} \right\|^q \right)^{\frac{1}{q}} \left( \sum_{k_m=1}^n \left\| x_{k_m}^{(m)} \right\|^q \right)^{\frac{q-p}{pq}}.
\]

Hence, for \(q \geq 2\), we have

\[
\left( \sum_{k_1, \ldots, k_m=1}^n \left\| T \left( x_{k_1}^{(1)}, \ldots, x_{k_m}^{(m)} \right)^p \right\| \right)^{\frac{1}{p}} \leq \| T \| \prod_{i=1}^m \left\| x_{k_i}^{(i)} \right\|^{\frac{q-p}{pq}}.
\]

and, for \(0 < q \leq 2\), we get

\[
\left( \sum_{k_1, \ldots, k_m=1}^n \left\| T \left( x_{k_1}^{(1)}, \ldots, x_{k_m}^{(m)} \right)^p \right\| \right)^{\frac{1}{p}} \leq \| T \| \prod_{i=1}^m \left\| x_{k_i}^{(i)} \right\|^{\frac{q-p}{pq}}.
\]

It is plain that the polynomial \(m\)-index of \((p, q)\)-summability can be estimated using the estimates for the multilinear \(m\)-index of \((p, q)\)-summability. Below we present more accurate estimates.
Proposition 2.8. Let $E, F$ be Banach spaces, $m$ be a natural number, $q > 0$ and $p < \frac{q}{m}$. Then
\[
\eta_{(p, q)}^{m-pol}(E; F) \leq \frac{1}{p} \text{ for } 0 < q \leq 2;
\eta_{(p, q)}^{m-pol}(E; F) \leq \frac{1}{p} + \frac{m(q - 2)}{2q} \text{ for } q \geq 2.
\]

Proof. For any $P \in \mathcal{P}(mE; F)$, by virtue of the Hölder inequality we have
\[
\left( \sum_{k=1}^{n} \|P(x_k)\|^p \right)^{\frac{1}{p}} \leq \|P\| \left( \sum_{k=1}^{n} \|x_k\|^{mp} \right)^{\frac{1}{p}}
\leq \|P\| \left[ \left( \sum_{k=1}^{n} \|x_k\|^{mp} \right)^{\frac{m}{mq}} \left( \sum_{k=1}^{n} |1\|^{\frac{1}{mp}} \right)^{\frac{1}{mp}} \right]^{\frac{1}{p}}
= \|P\| \left( \sum_{k=1}^{n} \|x_k\|^q \right)^{\frac{m}{mq}} \frac{q - mp}{mq}.
\]
Hence, for $0 < q \leq 2$, we have
\[
\left( \sum_{k=1}^{n} \|P(x_k)\|^p \right)^{\frac{1}{p}} \leq \|P\| \left( \sum_{k=1}^{n} \|x_k\|^q \right)^{\frac{m}{mq}} \frac{q - mp}{mq} \|x_k\|_{w, q}^n
= \|P\| \|x_k\|_{w, q}^n \|x_k\|_{w, q}^n,
\]
and
\[
\eta_{(p, q)}^{m-pol}(E; F) \leq \frac{1}{p}.
\]
For $q \geq 2$ we obtain
\[
\left( \sum_{k=1}^{n} \|P(x_k)\|^p \right)^{\frac{1}{p}} \leq \|P\| \left( \sum_{k=1}^{n} \|x_k\|^q \right)^{\frac{m}{mq}} \frac{q - mp}{mq} \|x_k\|_{w, q}^n
= \|P\| \left( \sum_{k=1}^{n} \|x_k\|^q \right)^{\frac{m}{mq}} \frac{q - mp}{mq} \|x_k\|_{w, q}^n
\]
and thus
\[
\eta_{(p, q)}^{m-pol}(E; F) \leq \frac{1}{p} + \frac{m(q - 2)}{2q}.
\]

3. Main results: vector-valued maps

We begin this section with a (simple) technical lemma.

Lemma 3.1. Let $E$ be an $n$-dimensional Banach space. If $1 \leq d \leq s \leq 2$, then there exists a constant $K > 0$ such that
\[
Kn^{\frac{2d + s(d - 2)}{2sd}} \leq \sigma_s^{(n)}(id_E).
\]
Proof. Using the Inclusion Theorem [5, Theorem 10.4] we have

\[ \pi^{(n)}_{2^{d+sd(d-2)}} (id_E) \leq \pi^{(n)}_{s,d} (id_E) \]

and by invoking Theorem 1.1 we know that there is a constant \( C > 0 \) such that

\[ \frac{1}{C} \pi_{2^{d+sd(d-2)}} (id_E) \leq \pi^{(n)}_{s,d} (id_E). \]

Theorem 1.2 assures the existence of a constant \( A > 0 \) such that

\[ An^{\frac{1}{2d+sd(d-2)}} \leq \pi_{2^{d+sd(d-2)}} (id_E). \]

Therefore

\[ Kn^{\frac{1}{2d+sd(d-2)}} \leq \pi^{(n)}_{s,d} (id_E), \]

where \( K = A/C \). \( \square \)

We recall that for \( 2 \leq q \leq \infty \), a Banach space \( E \) has cotype \( q \) if there is a constant \( C \geq 0 \) such that no matter how we select finitely many vectors \( x_1, \ldots, x_n \) from \( E \),

\[ \left( \sum_{k=1}^{n} ||x_k||^q \right)^{\frac{1}{q}} \leq C \left( \int_{0}^{1} \left( \sum_{k=1}^{n} r_k(t)x_k \right)^2 \, dt \right)^{\frac{1}{2}}, \]

where \( r_k \) denotes the \( k \)-th Rademacher function, that is, given \( k \in \mathbb{N} \) and \( t \in [0,1] \), we have \( r_k(t) = \text{sign} \left[ \sin \left( 2^k \pi t \right) \right] \). When \( q = \infty \), the left hand side will be replaced by the sup norm. It is plain that if \( q_1 \leq q_2 \), then \( E \) has cotype \( q_1 \) implies that \( E \) has cotype \( q_2 \); thus, henceforth, we will denote \( \inf \{ q : E \text{ has cotype } q \} \) by \( \cot(E) \).

Now we state and prove the main result of this section. The arguments are based in ideas taken from [4, 9]:

**Theorem 3.2.** Let \( E, F \) be infinite dimensional Banach spaces and \( r := \cot(F) \).

(a) For \( 1 \leq q \leq 2 \) and \( 0 < p \leq \frac{rq}{mr+q} \), we have

\[ \frac{m}{2} \leq \eta^{m-pol}_{(p,q)} (E;F). \]

(b) For \( 1 \leq q \leq 2 \) and \( \frac{rq}{mr+q} < p \leq \frac{2r}{mr+2} \), we have

\[ \frac{mp+2}{2p} - \frac{mr+q}{rq} \leq \eta^{m-pol}_{(p,q)} (E;F). \]

(c) For \( 2 \leq q < \infty \) and \( 0 < p \leq \frac{2r}{mr+2} \), we have

\[ \frac{m}{2} \leq \eta^{m-pol}_{(p,q)} (E;F). \]

(d) For \( 2 \leq q < \infty \) and \( \frac{2r}{mr+2} < p < r \), we have

\[ \frac{r-p}{pr} \leq \eta^{m-pol}_{(p,q)} (E;F). \]
Proof. Since \( F \) is infinite dimensional, from [5] Theorem 14.5 we have
\[
cot(F) = \sup\{2 \leq s \leq \infty : F \text{ finitely factors the formal inclusion } \ell_s \hookrightarrow \ell_\infty\},
\]
and from [5, p.304] we know that this supremum is attained. So \( F \) finitely factors the formal inclusion \( \ell_r \hookrightarrow \ell_\infty \), that is, there exist \( C_1, C_2 > 0 \) such that for every \( n \in \mathbb{N} \), there are \( y_1, \ldots, y_n \in F \) so that
\[
(7) \quad C_1 \left\| \sum_{j=1}^{n} a_jy_j \right\|_\infty \leq C_2 \left( \sum_{j=1}^{n} \left| a_j \right|^r \right)^{\frac{1}{r}}
\]
for every \( a_1, \ldots, a_n \in \mathbb{K} \).

Consider \( x_1^*, \ldots, x_n^* \in B_{E^*} \) such that \( x_j^*(x) = \|x_j\| \) for every \( j = 1, \ldots, n \). Let \( a_1, \ldots, a_n \) be scalars such that \( \sum_{j=1}^{n} |a_j|^\frac{1}{p} = 1 \) and define
\[
P_n : E \rightarrow F, \quad P_n(x) = \sum_{j=1}^{n} |a_j|^\frac{1}{p} x_j^*(x)^m y_j.
\]
Then for every \( x \in E \), by (7)
\[
\left\| P_n(x) \right\| = \left\| \sum_{j=1}^{n} |a_j|^\frac{1}{p} x_j^*(x)^m y_j \right\| \leq C_2 \left( \sum_{j=1}^{n} \left| a_j \right|^r \right)^{\frac{1}{r}} \|x\|^m = C_2 \|x\|^m,
\]
and thus
\[
(8) \quad \|P_n\| \leq C_2.
\]
Note that for \( k = 1, \ldots, n \), from (7), we have
\[
(9) \quad \left\| P_n(x_k) \right\| = \left\| \sum_{j=1}^{n} |a_j|^\frac{1}{p} x_j^*(x_k)^m y_j \right\| \geq C_1 \left\| \left( \sum_{j=1}^{n} \left| a_j \right|^\frac{1}{p} x_j^*(x_k)^m \right) \right\|_{\infty} \geq C_1 |a_k|^\frac{1}{p} x_k^*(x_k)^m = C_1 |a_k|^\frac{1}{p} \|x_k\|^m.
\]
Hence,
\[
\left( \sum_{j=1}^{n} \|x_j\|^m |a_j| \right)^{\frac{1}{p}} = \left( \sum_{j=1}^{n} \left( \|x_j\|^m |a_j|^\frac{1}{p} \right)^p \right)^{\frac{1}{p}} = \frac{1}{C_1} \left( \sum_{j=1}^{n} \left( C_1 \|x_j\|^m |a_j|^\frac{1}{p} \right)^p \right)^{\frac{1}{p}} \leq \frac{1}{C_1} \left( \sum_{j=1}^{n} \|P_n(x_j)\|^p \right)^{\frac{1}{p}}.
\]
Suppose that there exists $t \geq 0$ and $D > 0$ such that

$$
\left( \sum_{j=1}^{n} \| P_n (x_j) \|^{p} \right)^{\frac{1}{p}} \leq Dn^t \| P_n \| \| (x_j)_{j=1}^{n} \|_{w,q}^{m},
$$

hence

$$
\left( \sum_{j=1}^{n} \| x_j \|^{mp} |a_j| \right)^{\frac{1}{p}} \leq \frac{D}{C_1} \| P_n \| n^t \| (x_j)_{j=1}^{n} \|_{w,q}^{m}. \tag{10}
$$

Since this last inequality holds whenever $\sum_{j=1}^{n} |a_j|^\frac{r}{p} = 1$ and $p < r$, we have

$$
\left( \sum_{j=1}^{n} \| x_j \|^{mp} \right)^{\frac{1}{p}} = \sup \left\{ \left( \sum_{j=1}^{n} a_j \| x_j \|^{mp} \right)^{\frac{1}{p}} : \sum_{j=1}^{n} |a_j|^\frac{r}{p} = 1 \right\}
$$

$$
\leq \sup \left\{ \sum_{j=1}^{n} |a_j| \| x_j \|^{mp} : \sum_{j=1}^{n} |a_j|^\frac{r}{p} = 1 \right\}
$$

$$
\leq \left( \frac{D}{C_1} \| P_n \| n^t \| (x_j)_{j=1}^{n} \|_{w,q}^{m} \right)^{p}
$$

$$
\leq \left( \frac{DC_2}{C_1} n^t \| (x_j)_{j=1}^{n} \|_{w,q}^{m} \right)^{p}
$$

and thus, denote $\frac{DC_2}{C_1} := Q$

$$
\left( \sum_{j=1}^{n} \| x_j \|^{mp} \right)^{\frac{1}{p}} \leq n^t Q^p \| (x_j)_{j=1}^{n} \|_{w,q}^{mp}. \tag{11}
$$

Therefore

$$
\left( \sum_{j=1}^{n} \| id_X (x_j) \|^{mp} \right)^{\frac{1}{mp}} \leq n^t Q^m \| (x_j)_{j=1}^{n} \|_{w,q}^{m} \tag{12}
$$

Note that (11) is valid for any $x_1, \ldots, x_n$. So, for any $n$-dimensional subspace $X$ of $E$ we have

$$
\left( \sum_{j=1}^{n} \| id_X (x_j) \|^{mp} \right)^{\frac{1}{mp}} \leq n^t Q^m \| (x_j)_{j=1}^{n} \|_{w,q}^{m} \tag{12}
$$

for all $x_1, \ldots, x_n \in X$.

(a) Since

$$
0 < p \leq \frac{rq}{mr + q},
$$
we have
\[ mp \left( \frac{r}{p} \right)^{\ast} \leq q, \]
and
\[ \left( \sum_{j=1}^{n} \| \text{id}_X(x_j) \|^q \right)^{\frac{1}{q}} \leq n \frac{1}{m} Q^{\frac{1}{m}}. \]

So
\[ \pi_q^{(n)}(\text{id}_X) \leq n \frac{1}{m} Q^{\frac{1}{m}}. \]

Since \( q \leq 2 \), by [5, Theorem 2.8] we get
\[ \pi_2^{(n)}(\text{id}_X) \leq n \frac{1}{m} Q^{\frac{1}{m}}. \]

Now Theorem 1.1 assures us that there is a constant \( C > 0 \) such that
\[ \pi_2(\text{id}_X) \leq C \pi_2^{(n)}(\text{id}_X). \]
Using (13), (14) and Theorem 2.1 we obtain
\[ \frac{1}{C} n^{1/2} \leq n^{t/m} Q^{1/m}. \]
Thus
\[ t \geq \frac{m}{2}. \]
Therefore
\[ \eta_{(p,q)}^{m-pol}(E; F) \geq \frac{m}{2}. \]

(b) By (12), we have
\[ \pi_{mp(\frac{r}{p})}^{(n)}(\text{id}_X) \leq n \frac{1}{m} Q^{\frac{1}{m}}. \]

Since \( \frac{r q}{mr+q} \leq p \leq \frac{2r}{mr+2} \) and \( mp \left( \frac{r}{p} \right)^{\ast} = \frac{mp r}{r-p} \), we have \( q \leq mp \left( \frac{r}{p} \right)^{\ast} \leq 2 \). From Lemma 3.1 there is a constant \( K > 0 \) such that
\[ \frac{2q + mp(\frac{r}{p})^{\ast}}{2mp(\frac{r}{p})^{\ast}} \leq \pi_{mp(\frac{r}{p})}^{(n)}(\text{id}_X). \]

From (15) and (16) it follows that
\[ \frac{2q + mp(\frac{r}{p})^{\ast}}{2mp(\frac{r}{p})^{\ast}} \leq n^{t/m} Q^{1/m}. \]
Thus
\[ \frac{t}{m} \geq \frac{mp + 2}{2mp} - \frac{mr + q}{mr q} \]
and we conclude that
\[ t \geq \frac{mp + 2}{2p} - \frac{mr + q}{rq}. \]

Therefore
\[
\eta^{m-pol}_{(p,q)}(E; F) \geq \frac{mp + 2}{2p} - \frac{mr + q}{rq}.
\]

(c) Since \( q \geq 2 \), we obtain
\[
\left( \sum_{j=1}^{n} \| id_X(x_j) \|^{mp\left(\frac{1}{p}\right)} \right)^{\frac{1}{mp\left(\frac{1}{p}\right)}} \leq n^{\frac{1}{m}} Q^{\frac{1}{m}},
\]
for all \( x_1, ..., x_n \in X \). But \( \frac{2r}{mr + 2} \geq p \) implies that \( mp \left( \frac{1}{p} \right)^* \leq 2 \), and thus
\[
\left( \sum_{j=1}^{n} \| id_X(x_j) \|^{2} \right)^{\frac{1}{2}} \leq n^{\frac{1}{m}} Q^{\frac{1}{m}}.
\]

Therefore
\[
\pi^{(n)}_{2}(id_X) \leq n^{\frac{1}{m}} Q^{\frac{1}{m}}.
\]

From Theorem 1.1 it follows that
\[
\pi_{2}(id_X) \leq C \pi^{(n)}_{2}(id_X).
\]

By Theorem 2.1 we have
\[
\frac{1}{C} n^{\frac{1}{2}} = \frac{1}{C} \pi_{2}(id_X) \leq n^{t/m} Q^{\frac{1}{m}},
\]
and we conclude that
\[
t \geq \frac{m}{2},
\]
that is
\[
\eta^{m-pol}_{(p,q)}(E; F) \geq \frac{m}{2}.
\]

(d) Since \( q \geq 2 \), we have
\[
\left( \sum_{j=1}^{n} \| id_X(x_j) \|^{mp\left(\frac{1}{p}\right)} \right)^{\frac{1}{mp\left(\frac{1}{p}\right)}} \leq n^{\frac{1}{m}} Q^{\frac{1}{m}},
\]
for all \( x_1, ..., x_n \in X \). Then
\[
\pi^{(n)}_{mp\left(\frac{1}{p}\right)}(id_X) \leq n^{\frac{1}{m}} Q^{\frac{1}{m}}.
\]
But $\frac{2r}{mr+2} < p$ implies that $mp\left(\frac{r}{p}\right)^* > 2$, and from Theorem 1.1 it follows that

$$\pi_{mp\left(\frac{r}{p}\right)^*} (id_X) \leq C\pi_{mp\left(\frac{r}{p}\right)^*} (id_X).$$

By Theorem 1.2 there is a constant $A > 0$ such that

$$A \cdot n^{\frac{1}{mp\left(\frac{r}{p}\right)^*}} \leq \pi_{mp\left(\frac{r}{p}\right)^*} (id_X),$$

and thus

$$\frac{A}{C}n^{\frac{m}{mp\left(\frac{r}{p}\right)^*}} \leq n^{\frac{1}{m}\frac{Q}{m}}.$$

Finally, we obtain

$$t \geq \frac{r-p}{pr},$$

and

$$\eta_{m-pol}^m (E; F) \geq \frac{r-p}{pr}.$$

\[\square\]

**Remark 3.3.** In the above result, there is a kind of continuity. In fact, when $p = \frac{rq}{mr+q}$, from (a) we have

$$\frac{m}{2} \leq \eta_{m-pol}^m (E; F).$$

On the other hand, considering $p = \frac{mr+q}{mr+2}$ it follows from (b) that

$$\frac{mp+2}{2p} \cdot \frac{mr-q}{rq} = \frac{m}{2} \leq \eta_{m-pol}^m (E; F).$$

Now, when $p = \frac{2r}{mr+2}$, by (c) we have

$$(17)\quad \frac{m}{2} \leq \eta_{m-pol}^m (E; F).$$

Given $\epsilon > 0$ and taking $p_\epsilon = \frac{2r}{mr+2} + \epsilon$ it follows by (d) that

$$(18)\quad \frac{r-p_\epsilon}{pr} \leq \eta_{(p_\epsilon,q)}^m (E; F).$$

Again, there is a continuity between the lower estimates (17) and (18), because letting $\epsilon$ tend to zero, we have

$$p_\epsilon \to \frac{2r}{mr+2} \quad \text{and} \quad \frac{r-p_\epsilon}{pr} \to \frac{m}{2}.$$

The same behavior happens when $q = 2$.

The next two results provide optimality of $\eta_{m-pol}^m (E; F)$ in some cases:

**Corollary 3.4.** If $\frac{2}{2m+1} \leq p < \frac{2}{m+1}$, then $\eta_{m-pol}^m (\ell_1; \ell_2) = \frac{1}{p} - \frac{m+1}{2}$. 
Proof. Considering \( q = 1 \) and \( r = 2 \) in the previous theorem item (b) we have

\[
\eta_{(p,1)}^{m-pol}(\ell_1; \ell_2) \geq \frac{1}{p} - \frac{m + 1}{2}.
\]

Let us show that (19) is sharp. From [2] we know that every continuous \( m \)-homogeneous polynomial from \( \ell_1 \) to \( \ell_2 \) is absolutely \( (\frac{2}{m+1},1) \)-summing. Since \( \frac{2}{2m+1} \leq p < \frac{2}{m+1} \), let \( w > 0 \) be such that

\[
\frac{1}{p} = \frac{1}{2} + \frac{1}{w}.
\]

Given \( P \in \mathcal{P}(^{m}\ell_1; \ell_2) \), from the Hölder’s inequality we have

\[
\left( \sum_{k=1}^{n} \|P(x_k)\|^p \right)^{\frac{1}{p}} \leq n^{\frac{1}{p}} \left( \sum_{k=1}^{n} \|P(x_k)\|^{\frac{2m+1}{m+1}} \right)^{\frac{m+1}{2}} \leq Dn^{\frac{1}{p} - \frac{w-1}{2}} \|x_k\|^{w,1}_{w,1}.
\]

Corollary 3.5. Let \( K \) be a compact Hausdorff space and \( F \) be an infinite dimensional Banach space, with \( \cot(F) = r \). If \( \frac{2r}{r+2} < p < r \), then

\[
\eta_{(p,2)}(C(K); F) = \frac{1}{p} - \frac{1}{r}.
\]

Proof. By the previous theorem item (d), if \( q = 2 \) and \( \cot(F) = r \) we have

\[
\eta_{(p,2)}(C(K); F) \geq \frac{1}{p} - \frac{1}{r}.
\]

Let us show that (20) is sharp. From [3, Theorem 11.14] we know that every continuous linear operator from \( C(K) \) to \( F \), with \( \cot(F) = r \), is absolutely \( (r,2) \)-summing.

Let \( \frac{2r}{r+2} < p < r \) and let \( w > 0 \) be such that

\[
\frac{1}{p} = \frac{1}{r} + \frac{1}{w}.
\]

Given \( T \in \mathcal{L}(C(K); F) \), from the Hölder’s inequality we have

\[
\left( \sum_{k=1}^{n} \|T(x_k)\|^p \right)^{\frac{1}{p}} \leq n^{\frac{1}{w}} \left( \sum_{k=1}^{n} \|T(x_k)\|^{\frac{2m+1}{m+1}} \right)^{\frac{m+1}{2}} \leq Dn^{\frac{1}{p} - \frac{w-1}{2}} \|x_k\|^{w,1}_{w,2},
\]

i.e.,

\[
\eta_{(p,2)}(C(K); F) \leq \frac{1}{p} - \frac{1}{r}.
\]

□
4. Main results: real-valued maps

The following result complements the results of the previous section (its proof is inspired in techniques found in [3]); now we consider the case in which \( m \) is even and \( F = \mathbb{R} \).

**Theorem 4.1.** Let \( m \) be an even positive integer and \( E \) be an infinite dimensional real Banach space.

(a) If \( 1 \leq q \leq 2 \) and \( 0 < p \leq \frac{q}{m+q} \), then
\[
\frac{m}{2} \leq \eta_{(p,q)}^{m-pol}(E; \mathbb{R}) .
\]

(b) If \( 1 \leq q \leq 2 \) and \( \frac{q}{m+q} \leq p \leq \frac{2}{m+2} \), then
\[
\frac{mp + 2}{2p} - \frac{m + q}{q} \leq \eta_{(p,q)}^{m-pol}(E; \mathbb{R}) .
\]

(c) If \( 2 \leq q < \infty \) and \( 0 < p \leq \frac{2}{m+2} \), then
\[
\frac{m}{2} \leq \eta_{(p,q)}^{m-pol}(E; \mathbb{R}) .
\]

(d) If \( 2 \leq q < \infty \) and \( \frac{2}{m+2} < p < 1 \), then
\[
\frac{1 - p}{p} \leq \eta_{(p,q)}^{m-pol}(E; \mathbb{R}) .
\]

**Proof.** Let \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in E \). Consider \( x_1^*, \ldots, x_n^* \in B_{E^*} \) such that \( x_j^*(x_j) = \|x_j\| \) for every \( j = 1, \ldots, n \). Let \( a_1, \ldots, a_n \) be real numbers such that \( \sum_{j=1}^{n} |a_j|^\frac{1}{p} = 1 \) and define
\[
P_n : E \rightarrow \mathbb{R} , \quad P_n(x) = \sum_{j=1}^{n} |a_j|^\frac{1}{p} x_j^*(x)^m , \quad \text{for every } x \in E.
\]

Since \( m \) is even, it follows that \( P(x) \geq 0 \), for every \( x \in E \). Hence
\[
|P_n(x)| = P_n(x) = \sum_{j=1}^{n} |a_j|^\frac{1}{p} x_j^*(x)^m \geq |a_k|^\frac{1}{p} x_k^*(x)^m , \quad \text{for every } x \in E \text{ and } k = 1, \ldots, n,
\]
and
\[
|P_n(x_k)| = P_n(x_k) = \sum_{j=1}^{n} |a_j|^\frac{1}{p} x_j^*(x_k)^m \geq |a_k|^\frac{1}{p} x_k^*(x_k)^m = |a_k|^\frac{1}{p} \|x_k\|^m , \quad \text{for } k = 1, \ldots, n. \tag{21}
\]

Furthermore, for every \( x \in E \), we have
\[
|P_n(x)| = \left| \sum_{j=1}^{n} |a_j|^\frac{1}{p} x_j^*(x)^m \right| \leq \left( \sum_{j=1}^{n} |a_j|^\frac{1}{p} \right) \|x\|^m = \|x\|^m ,
\]
and thus
\[
\|P_n\| \leq 1 . \tag{22}
\]
Therefore,
\[
\left( \sum_{j=1}^{n} \|x_j\|^m \cdot |a_j| \right)^{\frac{1}{p}} = \left( \sum_{j=1}^{n} \left( \|x_j\|^m \cdot |a_j| \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}
\]
\[
\leq \left( \sum_{j=1}^{n} |P_n(x_j)|^{p} \right)^{\frac{1}{p}}.
\]

Suppose that there exists \( t \geq 0 \) and \( D > 0 \) such that
\[
\left( \sum_{j=1}^{n} \|P_n(x_j)\|^{p} \right)^{\frac{1}{p}} \leq D \|P_n\| \|n \cdot (x_j)_{j=1}^{n}\|_{w,q}^{m}
\]
\[
\leq Dn^{t} \|n \cdot (x_j)_{j=1}^{n}\|_{w,q}^{m}.
\]

Hence
\[
\left( \sum_{j=1}^{n} \|x_j\|^m \cdot |a_j| \right)^{\frac{1}{p}} \leq Dn^{t} \|n \cdot (x_j)_{j=1}^{n}\|_{w,q}^{m}.
\]

and since this last inequality holds whenever \( \sum_{j=1}^{n} |a_j|^{\frac{1}{p}} = 1 \) and \( p < 1 \), we have
\[
\left( \sum_{j=1}^{n} \|x_j\|^m \cdot |a_j| \right)^{1-p} = \operatorname{sup} \left\{ \sum_{j=1}^{n} a_j \|x_j\|^m : \sum_{j=1}^{n} |a_j|^{\frac{1}{p}} = 1 \right\}
\]
\[
\leq \operatorname{sup} \left\{ \sum_{j=1}^{n} |a_j| \|x_j\|^m : \sum_{j=1}^{n} |a_j|^{\frac{1}{p}} = 1 \right\}
\]
\[
\leq \left( Dn^{t} \|n \cdot (x_j)_{j=1}^{n}\|_{w,q}^{m} \right)^{p}.
\]

Therefore
\[
\left( \sum_{j=1}^{n} \|x_j\|^m \cdot \left( \frac{1}{1-p} \right)^{\frac{1}{mp}} \right)^{\frac{1}{p}} \leq D \frac{n^{t}}{m^{m}}.
\]

See that (24) is valid for any \( x_1, ..., x_n \). So, for any \( n \)-dimensional subspace \( X \) of \( E \) we have
\[
\left( \sum_{j=1}^{n} \|id_X(x_j)\|^m \cdot \left( \frac{1}{1-p} \right)^{\frac{1}{mp}} \right)^{\frac{1}{p}} \leq D \frac{n^{t}}{m^{m}}.
\]

for all \( x_1, ..., x_n \in X \).
Now we prove each item separately.

(a) Since

\[ 0 < p \leq \frac{q}{m + q}, \]

we have

\[ \frac{m p}{1 - p} \leq q \]

and thus

\[ \left( \sum_{j=1}^{n} \|id_X(x_j)\|^{q} \right)^{\frac{1}{q}} \leq D \frac{1}{m} n \frac{t}{m}. \]

So

\[ \pi_{q}^{(n)}(id_X) \leq D \frac{1}{m} n \frac{t}{m}. \]

Since \( 1 \leq q \leq 2 \) by [5, Theorem 2.8] we have

\[ \pi_{q}^{(n)}(id_X) < D \frac{1}{m} n \frac{t}{m}, \]

and from Theorem 1.4 we conclude that

\[ \pi_{2}(id_X) \leq C \pi_{2}^{(n)}(id_X). \]

By Theorem 2.1 we know that

\[ \pi_{2}(id_X) = n^{1/2} \]

and thus, from (26) and (27), it follows that

\[ \frac{1}{C} n^{1/2} \leq D \frac{1}{m} n^{t/m}. \]

Hence

\[ t \geq \frac{m}{2}, \]

i.e.,

\[ \eta_{\pi_{2}(id_X)}^{m}(E; \mathbb{R}) \geq \frac{m}{2}. \]

(b) By (25), we have

\[ \pi_{q}^{(n)}(id_X) \leq D \frac{1}{m} n \frac{t}{m}. \]

Since \( \frac{q}{m+q} \leq p \leq \frac{2}{m+2} \) we have \( q \leq \frac{mp}{1-p} \leq 2. \) From (28) and Lemma 3.1 there is a constant \( K > 0 \) such that

\[ Kn \left( \frac{2^p \cdot \frac{mp}{1-p} \cdot (q-2)}{t^{1/p}} \right) \leq n^{t/m} D \frac{1}{m}. \]

Thus

\[ \frac{t}{m} \geq \frac{mp + 2}{2mp} - \frac{m + q}{mq}. \]
and we conclude that

\[
t \geq \frac{mp + 2}{2p} - \frac{m + q}{q},
\]

that is,

\[
\eta_{m-pol}^{(p,q)}(E; \mathbb{R}) \geq \frac{mp + 2}{2p} - \frac{m + q}{q}.
\]

(c) Since \( q \geq 2 \), we have

\[
\left( \sum_{j=1}^{n} \|id_{X}(x_{j})\|^{\frac{mp}{1-p}} \right)^{\frac{1-p}{mp}} \leq D^{\frac{1}{m}} n^{\frac{t}{m}},
\]

for all \( x_{1},...,x_{n} \in X \). But \( \frac{2}{m+2} \geq p \) implies that \( \frac{mp}{1-p} \leq 2 \); hence

\[
\left( \sum_{j=1}^{n} \|id_{X}(x_{j})\|^{2} \right)^{\frac{1}{2}} \leq D^{\frac{1}{m}} n^{\frac{t}{m}},
\]

and thus

\[
\pi_{2}^{(n)}(id_{X}) \leq D^{\frac{1}{m}} n^{\frac{t}{m}}.
\]

From Theorem 1.1 we have

\[
\pi_{2}(id_{X}) \leq C\pi_{2}^{(n)}(id_{X})
\]

and from Theorem 2.1 we have

\[
\frac{1}{C} n^{\frac{1}{2}} = \frac{1}{C} \pi_{2}(id_{X}) \leq n^{t/m} D^{\frac{1}{m}}.
\]

So, we conclude that

\[
t \geq \frac{m}{2}.
\]

and

\[
\eta_{m-pol}^{(p,q)}(E; \mathbb{R}) \geq \frac{m}{2}.
\]

(d) Since \( q \geq 2 \), we obtain

\[
\left( \sum_{j=1}^{n} \|id_{X}(x_{j})\|^{\frac{mp}{1-p}} \right)^{\frac{1-p}{mp}} \leq D^{\frac{1}{m}} n^{\frac{t}{m}},
\]

for all \( x_{1},...,x_{n} \in X \).

So

\[
\pi_{(n)}^{\frac{mp}{1-p},2}(id_{X}) \leq n^{\frac{t}{m}} D^{\frac{1}{m}}.
\]
But $\frac{2}{m+2} < p$ implies that $\frac{mp}{1-p} > 2$, and from Theorem 1.1

$$\pi_{\frac{mp}{1-p}, 2}(id_X) \leq C\pi_{\frac{mp}{1-p}, 2}(id_X)$$

By Theorem 1.2, there is a constant $A > 0$ such that

$$An_{\frac{1}{mp}} \leq \pi_{\frac{mp}{1-p}, 2}(id_X),$$

thus

$$\frac{A}{C}n_{\frac{1}{mp}} \leq n^{t/m}D_{\frac{1}{m}},$$

we conclude that

$$t \geq \frac{1-p}{p},$$

so

$$\eta^{mp}_{(p,q)}(E; \mathbb{R}) \geq \frac{1-p}{p}.$$

☐

**Remark 4.2.** As in previous theorem, in this result we have a clear “continuity” in our estimates.

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