Low-dimensional BEC

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The Bose-Einstein condensation (BEC) temperature $T_c$ of Cooper pairs (CPs) created from a general interfermion interaction is determined for a linear, as well as the usually assumed quadratic, energy vs center-of-mass momentum dispersion relation. This explicit $T_c$ is then compared with a widely applied implicit one of Wen & Kan (1988) in $d = 2 + \epsilon$ dimensions, for small $\epsilon$, for a geometry of an infinite stack of parallel (e.g., copper-oxygen) planes as in, say, a cuprate superconductor, and with a new result for linear-dispersion CPs. The implicit formula gives $T_c$ values only slightly lower than those of the explicit formula for typical cuprate parameters.

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Bose-Einstein condensation (BEC) of Cooper pairs (CPs) can lead to a phase transition (even in 2D) in any many-fermion system dynamically capable of forming CPs. This transition could be the origin of “exotic” superconductivity in the quasi-2D cuprates and in the quasi-1D organo-metallic (Bechgaard) salts, as well as of the superfluidity in liquid $^3$He or in trapped Fermi gases in 3D.

The familiar BEC formula for the transition temperature is

$$T_c \simeq 3.31 h^2 n_B^{2/3} / m_B k_B,$$

with $n_B$ the number density of bosons of mass $m_B$ and $k_B$ the Boltzmann constant. This is a special case of the more general expression valid for any space dimensionality $d > 0$ and any boson dispersion relation $\epsilon_K = C_s K^s$
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with \( s > 0 \) and \( C_s \) constant, given by the explicit \( T_c \)-formula

\[
T_c = \frac{C_s}{k_B} \left[ \frac{s}{2 \pi^{d/2} \Gamma(d/s) g_{d/s}(1)} \right]^{s/d}. \tag{2}
\]

If \( \mu(T) \) is the boson chemical potential and \( e^{\mu(T)/k_B T} \equiv z \) the fugacity, \( g_\sigma(z) \equiv \sum_{l=1}^{\infty} z^l \Gamma(l) / l^\sigma \) are the Bose integrals. For \( z = 1 \) and \( \sigma \geq 1 \) the \( g_\sigma(1) \) is just \( \zeta(\sigma) \), the Riemann Zeta-function of order \( \sigma \) which is finite for \( \sigma > 1 \) and infinite for \( \sigma = 1 \), while the series \( g_{d/2}(1) \) diverges for all \( d/2 \leq 1 \). For \( s = 2 \), \( C_2 = \hbar^2 / 2m_B \), and since \( \zeta(3/2) \approx 2.612 \), this leads to the usual BEC \( T_c \)-formula (1). Since \( g_{d/2}(1) \) diverges for all \( d/2 \leq 1 \), \( T_c = 0 \) for all \( d \leq 2 \). This follows from the boson number equation

\[
N = N_0(T) + \sum_{K \neq 0} \left( e^{(\varepsilon_K - \mu(T))/k_B T} - 1 \right)^{-1} \tag{3}
\]

where \( N_0(T) \) is the number of bosons in the \( K = 0 \) state. At \( T = T_c \) both \( N_0(T_c) \) and the boson chemical potential \( \mu(T_c) \) virtually vanish so that replacing

\[
\sum_{k \neq 0} \rightarrow (L/2\pi)^d \int_{0^+}^{\infty} dk \int_0^{\infty} d^d k = (L/2\pi)^d \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{0^+}^{\infty} dk k^{d-1} \tag{4}
\]
in (3) eventually yields (2) where \( n_B \equiv N/L^d \).

The fact that a CP can have a linear (\( s = 1 \)), as opposed to the usual quadratic (\( s = 2 \)), dispersion relation was mentioned as far back as 1964 by Schrieffer, p. 33, for the BCS model interaction in 3D. This was recently confirmed to be the case for both 2D and 3D under a very general inter-fermion interaction for any coupling provided the fermion number-density is nonzero, i.e., in the presence of a Fermi sea. The CP dispersion relation becomes quadratic only in the extremely dilute (or vacuum) limit where the CPs are just the so-called “local pairs.” For any sizeable fermion density the nonnegative CP excitation energy \( \varepsilon_K \equiv \Delta_0 - \Delta_K \) behaves like \( \approx a(d) \hbar v_F K \), where \( \Delta_K \) (not to be confused with the BCS gap \( \Delta \)) is the (positive) binding energy of a CP of center-of-mass momentum (CMM) \( \hbar K \), \( v_F \equiv \hbar k_F / m \) and \( k_F \) the Fermi velocity and wavenumber, respectively, \( m \) the fermion effective mass, while \( a(d) \equiv 2/\pi \) and 1/2 in 2D and 3D, respectively, precisely as established previously for the BCS model interaction. For linear dispersion \( s = 1 \), \( C_1 = a(d) \hbar v_F \), \( T_c = 0 \) from (2) for all \( d \leq 1 \) only—and \( T_c > 0 \) for all \( d > 1 \), which is precisely the range of dimensionalities for all known superconductors if one includes the quasi-1D organo-metallic Bechgaard salts. Using the interpolation \( a(d) = (7/2 - 6/\pi) + (8/\pi - 13/4)d + (3/4 - 2/\pi)d^2 \),
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which correctly reduces to $1, 2/\pi$ and $1/2$ in 1D, 2D and 3D, respectively, Fig. 1 graphs (2) for $s = 1$ and 2 (in units of the Fermi temperature $T_F$) vs $d$—if one imagines all the fermions in the initial, interactionless many-fermion system paired into CPs of mass $m_B = 2m$. The particle number density of the original fermions is $n \equiv \frac{k_F^d}{2^d \pi^{d/2} d \Gamma(d/2)}$ and equals $2n_B$. These curves are upper bounds to the $T_c$ from a more realistic model where chemical equilibrium allows only a fraction of all fermions to be actually bound into pairs.

A rather general inter fermion interaction is the $S$-wave attractive separable potential whose double Fourier transform is

$$V_{pq} = -(v_0/L^2)g_pg_q.$$  \hspace{1cm} (5)

Here $L$ is the size of the “box” confining the many-fermion system, $v_0 \geq 0$ is the interaction strength and $g_p$ given, e.g. by $(1 + p^2/p_0^2)^{-1/2}$ where $p_0$ is the inverse range of the potential. Hence, $p_0 \to \infty$ implies $g_p = 1$ and corresponds to the attractive contact (or delta) potential $V(r) = -v_0\delta(r)$, while $p_0 = k_F$ implies a range of order of the average inter fermion spacing, etc. If $g_p = \theta(h\omega_D + \mu_F - p^2/2m)$, with $\theta(x)$ the unit step function, (5) becomes the BCS model interaction where $\omega_D$ is the Debye frequency and $\mu_F$ the fermionic chemical potential that becomes $E_F$ for $T = 0 = v_0$.

Using a renormalized CP equation whose coupling depends only on the two-body binding energy $B_2$ the CP excitation energy $\varepsilon_K \equiv \Delta_0 - \Delta_K$ for
zero range was obtained numerically as an exact curve that for very small \( B_2/E_F \) is virtually linear, i.e., \( \varepsilon_K \to 2\hbar v_F K/\pi \). It is only in the dilute limit (\( v_F \) or \( E_F \) → 0) that \( \varepsilon_K \) tends asymptotically to the exact quadratic \( \hbar^2 K^2 / 2(2m) \) for any coupling. Assuming \( n_B = n/2 \) and \( m_B = 2m \) to introduce the temperature scale \( T_F \) as before, Fig. 2 shows the BEC \( T_c \)'s of a pure gas of unbreakable CPs. Significantly, \( T_c \) is no longer zero in 2D—as would be predicted in a BEC picture by a quadratic relation appropriate for “local-pair” CPs in vacuum, a result that wrongly suggests that BEC cannot apply for quasi-2D cuprate superconductors.

More accurate BEC \( T_c \)'s should include refinements such as non-S-wave interactions, allowing for unpaired fermions in a more realistic binary boson-fermion mixture model—and most importantly, CP-fermion interactions that link the BEC condensate fraction temperature-dependence with that of the BCS fermionic energy gap, among other corrections.

The linear dispersion relation of a CP should not be confused with the linear dispersion of Anderson-Bogoliubov-Higgs (ABH) many-body excitation phonon-like modes. Collective modes in a superconductor were studied since the late 1950’s by several workers. A more recent treatment for 1D, 2D and 3D is available which confirms the linear ABH form \( \hbar v_F K/\sqrt{d} \) for \( d = 1, 2 \) or 3 in the zero-coupling limit. Our CPs are taken as “bosonic” even though they do not obey (Ref. [2] p. 38) Bose commutation relations. This is because for a given \( K \) they have indefinite occupation number since for fixed \( K \) there are (after the thermodynamic limit) an indefinitely large number of allowed (relative wavenumber) \( k \) values, so that—for any coupling and thus any degree of overlap between them—CPs do in fact obey the Bose-Einstein distribution from which BEC is determined. By contrast, ABH phonons (like photons or plasmons, etc.) cannot suffer a BEC as their number is always indefinite. The number of CPs, on the other hand, is fixed at half the number of (pairable) fermions if all of these are imagined paired at a given temperature and coupling.

To model cuprate superconductors consider the bosons confined to an infinite set of planes stacked along the \( z \)-direction, parallel to each other with equal spacing \( c \) between adjacent planes. The BEC transition temperature formula, for \( s = 2 \) bosons in each plane, is the implicit \( T_c \)-formula

\[
k_B T_c = \frac{2\pi n_B - 3D \hbar^2 c}{m_B \ln[M_{BC}^2 k_B T_c \nu(t_c)/\hbar^2]},
\]

and is valid in \( 2 + \epsilon \) dimensions where \( \epsilon \) is small and given by

\[
\epsilon \simeq 2[\ln(n_{B-3D} M_{BC}^3 / m_B)]^{-1}.
\]

This expressly vanishes as \( M_{BC}^3 \to \infty \), as it should. Here \( n_{B-3D} \equiv N/L^3 \) while \( \nu(t) = 1 - t + O(t^2) \simeq 1 \) if \( t \ll 1 \), where \( t \equiv \hbar^2 / M_{BC}^2 k_B T = \ldots \)
Fig. 2. 2D critical BEC temperatures $T_c$ (in units of $T_F$) for five coupling values $B_2/E_F$ for a pure boson gas of unbreakable CPs, determined by the $N_0(T_c) = 0 = \mu(T_c)$ solution of (3) inserting exact numerical CP dispersion curves, using (4). The value 0.702 marked with a square corresponds to the $T_c/T_F$ value at zero coupling (see also Ref. [1]) and contrasts with the well-known result $T_c/T_F \equiv 0$ in 2D for infinite coupling where $\epsilon_K = \hbar^2 K^2/(2m)$ exactly. Shaded areas as in Fig. 1.

$\hbar^2/2mc^2(M_B/m_B)k_BT$. Using $\hbar^2/mkB = 88,419$ K Å² with $m$ the electron mass, and $c = 12$ Å, this inequality is well satisfied for the higher cuprate transition temperatures, today ranging up to 164 K, since $T_c \gg 307K/(M_B/m_B)$ as typically $M_B/m_B$ can range from $10^2$ to $10^5$. Clearly, for $M_B \rightarrow 0$ (infinitely separated planes and/or perfect confinement to the $z$-direction in each plane) $T_c$ vanishes as it should in 2D. As state, this $T_c$-equation is implicit or transcendental, unlike the simpler explicit $T_c$ equations (1) and (2), and has been used for varied purposes by numerous authors[10, 14, 15]—though only for quadratic dispersion bosons.

We have generalized (6) for any $s > 0$ and found

$$k_BT_c = C_s \left[2\pi n_{B-3Dsc}/\Gamma(2/s)g_{2/s}(e^{-\hbar^2/M_Bc^2k_BT_c})\right]^{s/2}.$$  

Thus, an exact (again, implicit) equation for $T_c$ is obtained for any $s > 0$. For $s = 2$ and $C_2 = \hbar^2/2m_B$ we recover (8). In Fig. 1 (right) we plot results for both values of $s$ as points, for $M_B/m_B = 10^2$ and $10^5$. They can be seen to differ very slightly from the results for $s = 2$ or $s = 1$ bosons in $d = 2 + \epsilon$ dimensions, with $\epsilon$ small, that came directly from (8) which, moreover, is valid for all $d > 0$.

To compare our results, consider the Berezinskii-Kosterlitz-Thouless
transition temperature formula

\[
k_B T^{BKT}_c = \frac{\pi \hbar^2 n_B}{2 m_B}
\]

valid in 2D, and assume as before that \( n_B = n/2 \equiv k_F^2/4\pi \) and \( m_B = 2m \). This gives \( T^{BKT}_c/T_F = 1/8 = 0.125 \) and is displayed as a square in Fig. 1.

In conclusion, BEC \( T_c \)'s related to a pure gas of unbreakable composite linear-dispersion bosons were calculated in \( d = 2 \) for Cooper pairs formed via a general separable potential and whose coupling for any CMM is characterized solely by its two-body binding energy in vacuum. A \( T_c \)-formula valid for any dispersion relation of the form \( \varepsilon_K = C_s K^s \) with \( s > 0 \) and \( C_s \) constant is deduced for the BEC of a pure gas of unbreakable bosons confined to move in infinitely-many identical planes parallel to each other. It is a peculiar implicit \( T_c \)-equation valid in \( 2 + \epsilon \) where \( \epsilon \) is small and vanishes, as it must, when the inter-plane spacing or the boson mass in the direction perpendicular to the planes diverges. However, for either \( s = 2 \) or 1 we find results that are just slightly different than those of the explicit BEC \( T_c \)-formula which is valid more generally for any \( d > 0 \).

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with \( s > 0 \) and \( C_s \) constant, given by the explicit \( T_c \)-formula

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T_c = \frac{C_s}{k_B} \left[ \frac{s \Gamma(d/2) (2\pi)^d n_B}{2\pi^{d/2} \Gamma(d/s)} \right]^{s/d} .
\] (2)

If \( \mu(T) \) is the boson chemical potential and \( e^{\mu(T)/k_B T} \equiv z \) the fugacity, \( g_\sigma(z) \equiv \sum_{\ell=1}^\infty z^\ell / \ell^\sigma \) are the Bose integrals. For \( z = 1 \) and \( \sigma \geq 1 \) the \( g_\sigma(1) \) is just \( \zeta(\sigma) \), the Riemann Zeta-function of order \( \sigma \) which is finite for \( \sigma > 1 \) and infinite for \( \sigma = 1 \), while the series \( g_\sigma(1) \) diverges for all \( \sigma \leq 1 \). For \( s = 2 \), \( C_2 = \Gamma^2 / 2m_B \), and since \( \zeta(3/2) \simeq 2.612 \), this leads to the usual BEC \( T_c \)-formula (1). Since \( g_{d/2}(1) \) diverges for all \( d/2 \leq 1 \), \( T_c = 0 \) for all \( d \leq 2 \). This follows from the boson number equation

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which correctly reduces to 1, 2/π and 1/2 in 1D, 2D and 3D, respectively, Fig. 1 graphs (2) for s = 1 and 2 (in units of the Fermi temperature $T_F$) vs $d$—if one imagines all the fermions in the initial, interactionless many-fermion system paired into CPs of mass $m_B = 2m$. The particle number density of the original fermions is $n \equiv \frac{n^4_F}{2^{d-2} \pi^{d/2} d \Gamma(d/2)}$ and equals $2n_B$, and we have used $\hbar^2 k_F^2 / 2m = \frac{1}{2} m v_F^2 = E_F \equiv k_B T_F$. These curves are upper bounds to the $T_c$ from a more realistic model where chemical equilibrium allows only a fraction of all fermions to be actually bound into pairs.

![Graph](https://example.com/graph.png)

Fig. 1. **Left:** dimensionality, $d$, dependence of the critical BEC transition temperature $T_c$ according to explicit (2) (in units of the Fermi temperature $T_F$) as explained in text. Lower (full) curve is for $s = 2$; upper (dashed) curve for $s = 1$. Shaded areas refer to empirical data from Ref. [7]. **Right:** dots refer to results from implicit (6) below, as explained just after (7).

A rather general inter-fermion interaction is the $S$-wave attractive separable potential whose double Fourier transform is

$$V_{pq} = -\left(\alpha / L^2\right) g_p g_q. \quad (5)$$

Here $L$ is the size of the “box” confining the many-fermion system, $\alpha \geq 0$ is the interaction strength and $g_p$ given, e.g., by $(1 + p^2 / p_0^2)^{-1/2}$ where $p_0$ is the inverse range of the potential. Hence, $p_0 \to \infty$ implies $g_p = 1$ and corresponds to the attractive contact (or delta) potential $V(r) = -\alpha \delta(r)$, while $p_0 = k_F$ implies a range of order of the average inter-fermion spacing, etc. If $g_p = \theta(\hbar \omega_D + \mu_F - p^2 / 2m)$, with $\theta(x)$ the unit step function, (5) becomes the BCS model interaction where $\omega_D$ is the Debye frequency and $\mu_F$ the fermionic chemical potential that becomes $E_F$ for $T = 0 = \alpha$.

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zero range was obtained numerically\textsuperscript{3} as an exact curve that for very small $B_2/E_F$ is virtually linear, i.e., $\varepsilon_K \to 2\hbar v_F K/\pi$. It is only in the dilute limit ($v_F$ or $E_F \to 0$) that $\varepsilon_K$ tends asymptotically to the exact quadratic $\hbar^2 K^2/2(2m)$ for any coupling. Assuming $n_B = n/2$ and $m_B = 2m$ to introduce the temperature scale $T_F$ as before, Fig. 2 shows the BEC $T_c$'s of a pure gas of unbreakable CPs. Significantly, $T_c$ is no longer zero in 2D—as would be predicted in a BEC picture by a quadratic relation appropriate for “local-pair” CPs in vacuum, a result that wrongly suggests that BEC cannot apply for quasi-2D cuprate superconductors.

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The linear dispersion relation of a CP should not be confused with the linear dispersion of Anderson-Bogoliubov-Higgs (ABH) many-body excitation phonon-like modes. Collective modes in a superconductor were studied since the late 1950’s by several workers. A more recent treatment for 1D, 2D and 3D is available\textsuperscript{12} which confirms the linear ABH form $\hbar v_F K/\sqrt{d}$ for $d =1, 2$ or $3$ in the zero-coupling limit. Our CPs are taken as “bosonic” even though they do not obey (Ref. [2] p. 38) Bose commutation relations. This is because for a given $\tilde{K}$ they have indefinite occupation number since for fixed $\tilde{K}$ there are (after the thermodynamic limit) an indefinitely large number of allowed (relative wavenumber) $k$ values, so that—for any coupling and thus any degree of overlap between them—CPs do in fact obey the Bose-Einstein distribution from which BEC is determined. By contrast, ABH phonons (like photons or plasmons, etc.) cannot suffer a BEC as their number is always indefinite. The number of CPs, on the other hand, is fixed at half the number of (pairable) fermions if all of these are imagined paired at a given temperature and coupling.

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\[
k_B T_c = \frac{2\pi n_{B-3D} \hbar^2 c}{m_B \ln [M_{Bc^3} k_B T_c \nu(t_c)/\hbar^2]},
\]

(6)

and is valid in $2 + \epsilon$ dimensions where $\epsilon$ is small and given by

\[
\epsilon \simeq 2[\ln (n_{B-3D} M_{Bc^3}/m_B)]^{-1}.
\]

(7)

This expressly vanishes as $M_{Bc^3} \to \infty$, as it should. Here $n_{B-3D} \equiv N/L^3$ while $\nu(t) = 1 - t + O(t^2) \simeq 1$ if $t << 1$, where $t \equiv \hbar^2 / M_{Bc^3} k_B T =$
Fig. 2. 2D critical BEC temperatures \( T_c \) (in units of \( T_F \)) for five coupling values \( B_2/E_F \) for a pure boson gas of unbreakable CPs, determined by the \( N_0(T_c) = 0 = \mu(T_c) \) solution of (3) inserting exact numerical CP dispersion curves, using (4). The value 0.702 marked with a square corresponds to the \( T_c/T_F \) value at zero coupling (see also Ref. [1]) and contrasts with the well-known result \( T_c/T_F \equiv 0 \) in 2D for infinite coupling where \( \varepsilon_K = \hbar^2 K^2/2(2m) \) exactly. Shaded areas as in Fig. 1.

\[
\hbar^2/2mc^2(M_B/m_B)k_BT.
\]

Using \( \hbar^2/mk_B = 88,419 \) K Å² with \( m \) the electron mass, and \( c = 12 \) Å, this inequality is well satisfied for the higher cuprate transition temperatures, today ranging up to 164 K, since \( T_c \gg 307K/(M_B/m_B) \) as typically \( M_B/m_B \) can range from \( 10^2 \) to \( 10^5 \). Clearly, for \( M_Bc \rightarrow 0 \) (infinitely separated planes and/or perfect confinement to the z-direction in each plane) \( T_c \) vanishes as it should in 2D. As state, this \( T_c \)-equation is implicit or transcendental, unlike the simpler explicit \( T_c \) equations (1) and (2), and has been used for varied purposes by numerous authors\textsuperscript{10,14,15}—though only for quadratic dispersion bosons.

We have generalized (6) for any \( s > 0 \) and found

\[
k_BT_c = C_s \left[ 2\pi n_B - 3Dsc/\Gamma(2/s)g_{2/s}(e^{-\hbar^2/Msc^2k_BT_c}) \right]^{s/2}.
\]

Thus, an exact (again, implicit) equation for \( T_c \) is obtained for any \( s > 0 \). For \( s = 2 \) and \( C_2 = \hbar^2/2mB \) we recover (6). In Fig. 1 (right) we plot results for both values of \( s \) as points, for \( M_B/m_B = 10^2 \) and \( 10^5 \). They can be seen to differ very slightly from the results for \( s = 2 \) or \( s = 1 \) bosons in \( d = 2 + \epsilon \) dimensions, with \( \epsilon \) small, that came directly from (2) which, moreover, is valid for all \( d > 0 \).

To compare our results, consider the Berezinskii-Kosterlitz-Thouless\textsuperscript{16}
transition temperature formula

\[ k_B T_{c}^{BKT} = \frac{\pi \hbar^2 n_B}{2 m_B} \]

valid in 2D, and assume as before that \( n_B = n/2 \equiv k_F^2/4\pi \) and \( m_B = 2m \). This gives \( T_{c}^{BKT}/T_F = 1/8 = 0.125 \) and is displayed as a square in Fig. 1.

In conclusion, BEC \( T_c \)'s related to a pure gas of unbreakable composite
linear-dispersion bosons were calculated in \( d = 2 \) for Cooper pairs formed
via a general separable potential and whose coupling for any CMM is char-
acterized solely by its two-body binding energy in vacuum. A \( T_c \)-formula
valid for \( any \) dispersion relation of the form \( \varepsilon_K = C_s K^s \) with \( s > 0 \) and \( C_s \)
constant is deduced for the BEC of a pure gas of unbreakable bosons con-
fined to move in infinitely-many identical planes parallel to each other. It is
a peculiar \( implicit \) \( T_c \)-equation valid in \( 2 + \epsilon \) where \( \epsilon \) is small and vanishes,
as it must, when the inter-plane spacing or the boson mass in the direction
perpendicular to the planes diverges. However, for either \( s = 2 \) or \( 1 \) we
find results that are just slightly different than those of the \( explicit \) BEC
\( T_c \)-formula which is valid more generally for any \( d > 0 \).

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