Classification of integrable hydrodynamic chains

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Received 1 March 2010
Published 12 October 2010
Online at stacks.iop.org/JPhysA/43/434027

Abstract
Using the method of hydrodynamic reductions, we find all integrable infinite (1+1)-dimensional hydrodynamic-type chains of shift 1. A class of integrable infinite (2+1)-dimensional hydrodynamic-type chains is constructed.

PACS number: 02.30.Ik
Mathematics Subject Classification: 17B80, 17B63, 32L81, 14H70

1. Introduction
We consider integrable infinite quasilinear chains of the form

\[ u_{\alpha,t} = \phi_{\alpha,1}u_{\alpha,1,x} + \cdots + \phi_{\alpha,\alpha+1}u_{\alpha+1,1,x}, \quad \alpha = 1, 2, \ldots, \phi_{\alpha,\alpha+1} \neq 0, \tag{1.1} \]

where \( \phi_{\alpha,j} = \phi_{\alpha,j}(u_1, \ldots, u_{\alpha+1}) \). Two chains are called equivalent if they are related by a transformation of the form

\[ u_{\alpha} \rightarrow \Psi_{\alpha}(u_1, \ldots, u_{\alpha}), \quad \frac{\partial \Psi_{\alpha}}{\partial u_{\alpha}} \neq 0, \quad \alpha = 1, 2, \ldots. \tag{1.2} \]

By integrability we mean the existence of an infinite set of hydrodynamic reductions [1–6].

Example 1. The Benney equations [7–9]

\[ u_{1,t} = u_{2,x}, \quad u_{2,t} = u_1u_{1,x} + u_{3,x}, \ldots, \quad u_{\alpha,t} = (\alpha - 1)u_{\alpha-1}u_{\alpha,x} + u_{\alpha+1,x}, \ldots \tag{1.3} \]

provide the most known example of integrable chain (1.1). The hydrodynamic reductions for the Benney chain were investigated in [10].

In [4–6] integrable divergent chains of the form

\[ u_{1,t} = F_1(u_1, u_2)_x, \quad u_{2,t} = F_2(u_1, u_2, u_3)_x, \ldots, \quad u_{\alpha,t} = F_{\alpha}(u_1, u_2, \ldots, u_{\alpha+1})_x, \ldots \tag{1.4} \]
were considered. In [6] some necessary integrability conditions were obtained. Namely, a nonlinear overdetermined system of PDEs for the functions $F_1, F_2$ was presented. The general solution of the system was not found. Another open problem was to prove that the conditions are sufficient. In other words, for any solution $F_1, F_2$ of the system one should find functions $F_i, i > 2$, such that the resulting chain is integrable.

Probably any integrable chain (1.1) is equivalent to a divergent chain. However, the divergent coordinates are not suitable for explicit formulas. Our main observation is that convenient coordinates are those in which the so-called Gibbons–Tsarev-type system (GT-system) related to the integrable chain is in a canonical form.

Using our version (see [11, 12]) of the hydrodynamic reduction method, we describe all integrable chains (1.1). We establish a one-to-one correspondence between integrable chains (1.1) and infinite triangular GT-systems of the form

$$
\partial_i p_j = \frac{p_i(p_i, p_j)}{p_i - p_j} \partial_i u_1,
$$

$$
\partial_i \partial_j u_1 = \frac{Q(p_i, p_j)}{(p_i - p_j)^2} \partial_i u_1 \partial_j u_1,
$$

$$
\partial_i u_m = \left( g_{m,0} + g_{m,1} p_i + \cdots + g_{m,m-1} p_i^{m-1} \right) \partial_i u_1,
$$

where $m = 2, 3, \ldots$ and $i, j = 1, 2, 3$. The functions $P, Q$ are polynomials quadratic in each of the variables $p_i$ and $p_j$, with coefficients being functions of $u_1, u_2$. The functions $p_1, p_2, p_3$, $u_1, u_2, \ldots$ in (3.1) depend on $r^1, r^2, r^3$, and $\partial_i = \partial / \partial r^i$.

**Example 1-1** (continuation of example 1). The system (1.5), (1.6) corresponding to the Benney chain has the following form:

$$
\partial_i p_j = \frac{\partial_i u_1}{p_i - p_j},
$$

$$
\partial_i \partial_j u_1 = \frac{2 \partial_i u_1 \partial_j u_1}{(p_i - p_j)^2}.
$$

$$
\partial_i u_m = \left( -(m - 2) u_{m-2} - \cdots - 2 u_2 p_i^{m-2} - u_1 p_i^{m-3} + p_i^{m-1} \right) \partial_i u_1.
$$

Equations (1.7) were firstly obtained in [10].

Given the GT-system (1.5), (1.6) the coefficients of (1.1) are uniquely defined by the following relations:

$$
\partial_i \partial_m u_m = \phi_{m,1} \partial_i u_1 + \cdots + \phi_{m,m+1} \partial_i u_{m+1}, \quad m = 2, 3, \ldots
$$

Namely, equating the coefficients at different powers of $p_i$ in (1.9), we obtain a triangular system of linear algebraic equations for $\phi_{i,j}$. Thus, the classification problem for chains (1.1) is reduced to a description of all GT-systems (1.5), (1.6). The latter problem is solved in sections 4–6.

The paper is organized as follows. Following [11, 12], we recall main definitions in section 2 (see [1–3, 11] for details). We consider only three-component hydrodynamic reductions since the existence of reductions with $N > 3$ gives nothing new [1]. In section 3 we formulate our previous results that are needed in the paper. Section 4 is devoted to a classification of the admissible polynomials $P$ and $Q$ in (1.5), (1.6). In sections 5, 6 we construct integrable chains for the generic case and for some degenerations. Section 6 also contains examples of $(2+1)$-dimensional infinite hydrodynamic-type chains integrable from the viewpoint of the method of hydrodynamic reductions. Infinitesimal symmetries of GT-systems are studied in section 7. These symmetries seem to be important basic objects in the hydrodynamic reduction approach.
2. Integrable chains and hydrodynamic reductions

According to [1–6] a chain (1.1) is called integrable if it admits sufficiently many so-called hydrodynamic reductions.

**Definition.** A hydrodynamic (1+1)-dimensional N-component reduction of a chain (1.1) is a semi-Hamiltonian (see formula (3.8)) system of the form

$$r_i^t = p_i(r^1, \ldots, r^N)r_i^x, \quad i = 1, \ldots, N$$

(2.1)

and functions $u_j(r^1, \ldots, r^N), j = 1, 2, \ldots$ such that for each solution of (2.1) functions $u_j = u_j(r^1, \ldots, r^N), i = 1, \ldots, N$ satisfy (1.1).

Substituting $u_i = u_i(r^1, \ldots, r^N), i = 1, \ldots, N$, into (1.1), calculating $t$- and $x$-derivatives by virtue of (2.1) and equating coefficients at $r_i$ to zero, we obtain

$$\partial_s u_\alpha p_\alpha = \phi_{\alpha,1} \partial_1 u_1 + \cdots + \phi_{\alpha,\alpha+1} \partial_\alpha u_{\alpha+1}, \quad \alpha = 1, 2, \ldots$$

(3.1)

It is clear from this system that

$$\partial_s u_k = g_k(p_i, u_1, \ldots, u_k) \partial_i u_1, \quad k = 2, 3, \ldots,$$

where $g_k(p, u_1, \ldots, u_k)$ is a polynomial of degree $k - 1$ in $p$ for each $k = 2, 3, \ldots$. Compatibility conditions $\partial_i \partial_j u_k = \partial_j \partial_i u_k$ give us a system of linear equations for $\partial_i u_j, \partial_j u_i, i \neq j$. This system should have a solution (otherwise we would not have sufficiently many reductions). Moreover, expressions for $\partial_i u_k, k = 2, 3, \ldots, \partial_j p_i, \partial_i \partial_j u_i, i \neq j$, should be compatible and form a so-called GT-system.

**Remark.** In what follows we assume that $N = 3$ because the case $N > 3$ gives nothing new [1].

3. GT-systems

**Definition.** A compatible system of PDEs of the form

$$\partial_i p_j = f(p_i, p_j, u_1, \ldots, u_n), \partial_j u_1, \quad j \neq i,$$

$$\partial_i \partial_j u_1 = h(p_i, p_j, u_1, \ldots, u_n) \partial_i u_1 \partial_j u_1, \quad j \neq i,$$

$$\partial_i u_k = g(p_i, u_1, \ldots, u_k) \partial_i u_1, \quad k = 1, \ldots, n - 1,$$

(3.1)

where $i, j = 1, 2, 3$ is called the n-field GT-system. Here $p_1, p_2, p_3, u_1, \ldots, u_n$ are the functions of $r^1, r^2, r^3$, and $\partial_i = \frac{\partial}{\partial r^i}$.

**Definition.** Two GT-systems are called equivalent if they are related by a transformation of the form

$$p_i \rightarrow \lambda(p_i, u_1, \ldots, u_n),$$

$$u_k \rightarrow \mu_k(u_1, \ldots, u_n), \quad k = 1, \ldots, n.$$  

(3.2)

(3.3)

**Example 2** [13]. Let $a_0, a_1, a_2$ be arbitrary constants, $R(x) = a_2 x^2 + a_1 x + a_0$. Then the system

$$\partial_i p_j = \frac{a_2 p_j^3 + a_1 p_j + a_0}{p_i - p_j} \partial_i u_1, \quad \partial_i \partial_j u_1 = \frac{2a_2 p_i p_j + a_1 (p_i + p_j) + 2a_0}{(p_i - p_j)^2} \partial_i u_1 \partial_j u_1$$

(3.4)
is a one-field GT-system. The original Gibbons–Tsarev system (1.7) corresponds to \(a_2 = a_1 = 0, a_0 = 1\). The polynomial \(R(x)\) can be reduced to one of the following canonical forms: \(R = 1\), \(R = x\), \(R = x^2\) or \(R = x(x - 1)\) by a linear transformation (3.2). A wide class of integrable 3D-systems of hydrodynamic type related to (3.4) is described in [13]. An elliptic version of this GT-system and the corresponding integrable 3D-systems were constructed in [15].

**Definition.** An additional system

\[
\partial_i u_k = g_k(p_1, u_1, \ldots, u_{n+m}) \partial_i u_n, \quad k = n + 1, \ldots, n + m, \tag{3.5}
\]

such that (3.1) and (3.5) are compatible is called an extension of (3.1) by fields \(u_{n+1}, \ldots, u_{n+m}\).

It turns out that

\[
\partial_i u_{n+1} = f(p_1, u_{n+1}, u_1, \ldots, u_n) \partial_i u_1
\]

is an extension for the GT-system (3.1). Stress that here \(f\) is the same function as in (3.1). We call this extension the regular extension by \(u_{n+1}\).

**Example 2-1.** The generic case of example 1-1 corresponds to \(R = x(x - 1)\). The regular extension by \(u_2\) is given by

\[
\partial_i u_2 = \frac{u_2(u_2 - 1)}{p_i - u_2} \partial_i u_1.
\]

If we express \(u_1\) from this formula and substitute it to (3.4), we obtain the following one-field GT-system:

\[
\partial_i p_j = \frac{p_j(p_j - 1)(p_i - u_1)}{u_1(u_1 - 1)(p_i - p_j)} \partial_i u_1,
\]

\[
\partial_i \partial_j u_1 = \frac{p_i p_j(p_i + p_j) - p_i^2 - p_j^2 + (p_i^2 + p_j^2 - 4p_i p_j + p_i + p_j)u_1}{u_1(u_1 - 1)(p_i - p_j)^2} \partial_i u_1 \partial_j u_1. \tag{3.6}
\]

The second basic notion of the hydrodynamic reduction method is the so-called GT-family of (1+1)-dimensional hydrodynamic-type systems.

**Definition.** A (1+1)-dimensional three-component hydrodynamic-type system of the form

\[
\frac{r_i}{t} = v_i(r^1, \ldots, r^N) r_{x,i}, \quad i = 1, 2, 3, \tag{3.7}
\]

is called semi-Hamiltonian if the following relation holds:

\[
\partial_i \frac{\partial_j v^k}{v^i - v^k} = \partial_j \frac{\partial_i v^k}{v^i - v^k}, \quad i \neq j \neq k. \tag{3.8}
\]

**Definition.** A Gibbons–Tsarev family associated with the Gibbons–Tsarev-type system (4.1) is a (1+1)-dimensional hydrodynamic-type system of the form

\[
\frac{r_i}{t} = F(p_1, u_1, \ldots, u_m) r_{x,i}, \quad i = 1, 2, 3, \tag{3.9}
\]

semi-Hamiltonian by virtue of (3.1).

**Example 2-2** [13]. Applying the regular extension to the generic GT-system (3.4) two times, we obtain the following GT-system:

\[
\partial_i p_j = \frac{p_j(p_j - 1)}{p_i - p_j} \partial_i w, \quad \partial_j w = \frac{2p_i p_j - p_i - p_j}{(p_i - p_j)^2} \partial_i w \partial_j w, \quad i \neq j. \tag{3.10}
\]
\[ \partial_i u_j = \frac{u_j(u_j - 1)}{p_i - u_j}, \quad j = 1, 2. \] (3.11)

Consider the generalized hypergeometric [14] linear system of the form
\[ \frac{\partial^2 h}{\partial u_j \partial u_k} = \frac{s_j}{u_j - u_k} \cdot \frac{\partial h}{\partial u_k} + \frac{s_k}{u_k - u_j} \cdot \frac{\partial h}{\partial u_j}, \quad j \neq k, \] (3.12)
\[ \frac{\partial^2 h}{\partial u_j \partial u_j} = -\left(1 + \sum_{k=1}^{n+2} s_k\right) \frac{s_j}{u_j(u_j - 1)} \cdot h + \frac{s_j}{u_j(u_j - 1)} \sum_{k \neq j}^n \frac{u_k(u_k - 1)}{u_k - u_j} \cdot \frac{\partial h}{\partial u_k} \] + \left(\sum_{k \neq j}^n \frac{s_k}{u_j - u_k} + \frac{s_j + s_{n+1}}{u_j} + \frac{s_j + s_{n+2}}{u_j - 1}\right) \cdot \frac{\partial h}{\partial u_j}. \] (3.13)

Here \( i, j = 1, 2 \) and \( s_1, \ldots, s_4 \) are the arbitrary parameters. It is easy to verify that this system is in involution and therefore the solution space is three dimensional. Let \( h_1, h_2, h_3 \) be a basis of this space. For any \( h \) we put
\[ S(p, h) = u_1(u_1 - 1)(p - u_2) \frac{hh_{1,u_1} - h_1h_1}{h_1} + u_2(u_2 - 1)(p - u_1) \frac{hh_{1,u_2} - h_2h_1}{h_1}. \]

Then the formula
\[ F = \frac{S(p, h_3)}{S(p, h_2)} \] (3.14)
defines the generic linear fractional GT-family for (3.10).

4. Canonical forms of GT-systems associated with integrable chains

For integrable chains the corresponding GT-systems involve infinite number of fields \( u_i, i = 1, 2, \ldots \) (see example 1-1). In this section we show that these GT-systems are equivalent to infinite triangular extensions of one-field GT-systems from example 2.

A compatible system of PDEs of the form
\[ \begin{align*}
\partial_i p_j &= f(p_i, p_j, u_1, \ldots, u_n)\partial_i u_1, \quad i \neq j, \\
\partial_i u_k &= g_k(p_i, u_1, \ldots, u_n)\partial_i u_1, \quad k = 1, 2, \ldots, \\
\partial_i \partial_j u_1 &= h(p_i, p_j, u_1, \ldots, u_n)\partial_i \partial_j u_1, \quad i \neq j,
\end{align*} \] (4.1)
where \( i, j = 1, 2, 3 \) is called the triangular GT-system. Here \( p_1, p_2, p_3, u_1, u_2, \ldots \) are the functions of \( r^1, r^2, r^3 \), and \( \partial_i = \frac{\partial}{\partial r^i} \).

**Definition.** A chain (1.1) is called integrable if there exists a Gibbons–Tsarev-type system of the form (4.1) and a Gibbons–Tsarev family
\[ r^i_j = F(p_i, u_1, \ldots, u_m)r^i_j, \quad i = 1, 2, 3, \] (4.2)
such that (1.1) holds by virtue of (4.1), (4.2).

Due to the equivalence transformations (3.2) we can assume without loss of generality that
\[ F(p, u_1, \ldots, u_m) = p. \] (4.3)
Under this assumption we have
\[ u_{j,t} = \sum_i \partial_i u_j r_i^t = \sum_i \partial_i u_j p_i r_i^t \]
and similarly
\[ u_{j,x} = \sum_i \partial_i u_j r_i^x. \]
Substituting these expressions into (1.1) and equating coefficients at \( r_i^t \) to zero, we obtain
\[ \partial_i u_k p_i = \phi_{k1} \partial_i u_1 + \cdots + \phi_{k\alpha} \partial_i u_{\alpha+1}, \quad \alpha = 1, 2, \ldots. \]
Using (4.1) and replacing \( p_i \) by \( p \), we obtain
\[
\begin{align*}
p &= \phi_{11} + \phi_{12} g_2, & p g_2 &= \phi_{21} + \phi_{22} g_2 + \phi_{23} g_3, \\
p g_3 &= \phi_{31} + \phi_{32} g_2 + \phi_{33} g_3 + \phi_{34} g_4, & \ldots.
\end{align*}
\]
Solving this system with respect to \( g_2, g_3, \ldots \), we obtain
\[ g_1(p) = \psi_{i,0} + \psi_{i,1} p + \cdots + \psi_{i,i-1} p^{i-1}. \]
Here \( \psi_{i,j} \) are the functions of \( u_1, \ldots, u_i \). For example,
\[ g_2 = \frac{p}{\phi_{12}} - \frac{\phi_{11}}{\phi_{12}} \quad (4.4) \]

**Remark.** Since we assume that \( \phi_{i,i-1} \neq 0 \), we have \( \psi_{i,i-1} \neq 0 \) for all \( i \). Therefore, \( g_1 = 1, g_2, \ldots \) is a basis in the linear space of all polynomials in \( p \). The coefficients \( \phi_{i,j} \) of our chain are just entries of the matrix of multiplication by \( p \) in this basis. More generally, if we do not normalize \( F = p \), then the coefficients \( \phi_{i,j} \) can be found from the equations
\[
\begin{align*}
F(p) &= \phi_{11} + \phi_{12} g_2, & F(p) g_2 &= \phi_{21} + \phi_{22} g_2 + \phi_{23} g_3, \\
F(p) g_3 &= \phi_{31} + \phi_{32} g_2 + \phi_{33} g_3 + \phi_{34} g_4, & \ldots. \quad (4.5)
\end{align*}
\]
Compatibility conditions \( \partial_i \partial_j u_{\alpha} = \partial_j \partial_i u_{\alpha}, \alpha = 2, 3, 4 \), give a system of linear equations for \( \partial_i p_j, \partial_j p_i, \partial_i \partial_j u_1 \). Solving this system, we obtain formulas (1.5), (1.6), where in principle \( P, Q \) could depend on \( u_1, u_2, u_3, u_4 \). However, it follows from compatibility conditions \( \partial_i \partial_j p_k = \partial_j \partial_i p_k \) that \( P, Q \) depend on \( u_1, u_2 \) only.

Written in (1.5) in the form
\[
\partial_i p_j = \left( \frac{R(p_j)}{p_i - p_j} + \left( z_4 p_j^2 + z_5 p_j + z_6 \right) p_i + \left( z_4 p_j^3 + z_5 p_j^2 + z_7 p_j + z_8 \right) \right) \partial_i u_1, \quad (4.6)
\]
where \( R(x) = z_4 x^4 + z_5 x^3 + z_3 x^2 + z_1 x + z_0, \) one can derive from the compatibility conditions \( \partial_i \partial_j p_k = \partial_j \partial_i p_k \) that \( \partial_i \partial_j u_1 = \partial_j \partial_i u_1 \) that equation (1.6) has the following form:
\[
\partial_i \partial_j u_1 = \left( \frac{2 z_4 p_j^2 + z_3 p_i + (p_i + p_j) + \left( z_1 (p_i + p_j) + 2 z_0 \right)}{(p_i - p_j)^2} \right) \partial_i u_1 \partial_j u_1. \quad (4.7)
\]
It is easy to verify that we can normalize \( z_9 = z_6 - z_7, g_2 = p \) by a transformation (1.2). Then the coefficients \( z_i(x, y), i = 0, \ldots, 8 \), satisfy the following pair of compatible dynamical systems with respect to \( y \) and \( x \):
\[
\begin{align*}
z_{0, y} &= 2 z_0 z_5 - z_1 z_6, & z_{1, y} &= 4 z_0 z_4 + z_1 z_5 - 2 z_2 z_6, & z_{2, y} &= 3 z_1 z_4 - 3 z_3 z_6, \\
z_{3, y} &= 2 z_2 z_4 - z_3 z_5 - 4 z_4 z_6, & z_{4, y} &= 2 z_3 z_4 - 2 z_4 z_5, & z_{5, y} &= 2 z_4 z_7 - 2 z_5 z_6 - z_3^2, \\
z_{6, y} &= 2 z_5 z_8 - z_5 z_6, & z_{7, y} &= 2 z_4 z_5 - 2 z_3 z_6 - z_5 z_6 + z_4 z_8, \\
z_{8, y} &= 2 z_4 z_4 - z_6 z_7 + z_5 z_8, \\
z_{0, x} &= 2 z_0 z_5 - z_1 z_6, & z_{1, x} &= 4 z_0 z_4 + z_1 z_5 - 2 z_2 z_6, & z_{2, x} &= 3 z_1 z_4 - 3 z_3 z_6, \\
z_{3, x} &= 2 z_2 z_4 - z_3 z_5 - 4 z_4 z_6, & z_{4, x} &= 2 z_3 z_4 - 2 z_4 z_5, & z_{5, x} &= 2 z_4 z_7 - 2 z_5 z_6 - z_3^2, \\
z_{6, x} &= 2 z_5 z_8 - z_5 z_6, & z_{7, x} &= 2 z_4 z_5 - 2 z_3 z_6 - z_5 z_6 + z_4 z_8, \\
z_{8, x} &= 2 z_4 z_4 - z_6 z_7 + z_5 z_8, \end{align*}
\]
and

\[ z_{0,x} = -z_0 z_2 - 2 z_0 z_8 + 3 z_0 z_7 - z_1 z_8, \quad z_{1,x} = -z_1 z_2 + 3 z_0 z_3 - z_1 z_6 + 2 z_1 z_7 - 2 z_2 z_8, \]
\[ z_{2,x} = -z^2 + 2 z_1 z_5 + 4 z_0 z_4 - 2 z_6 + 2 z_2 z_7 - 3 z_3 z_8, \quad z_{3,x} = 3 z_1 z_4 - z_3 z_6 - 4 z_4 z_8, \]
\[ z_{4,x} = z_2 z_4 - z_4 z_6 - z_4 z_7, \quad z_{5,x} = z_1 z_4 - z_5 z_6 - z_4 z_8, \quad z_{6,x} = z_0 z_4 - z_2^2, \]
\[ z_{7,x} = z_1 z_3 + 3 z_0 z_4 + z_1 z_5 - z_2 z_6 - z_7 z_8 + z_7^2 - z_3 z_8 = 2 z_5 z_8, \]
\[ z_{8,x} = z_0 z_3 + z_0 z_5 - z_2 z_8 - 2 z_6 z_8 + z_7 z_8. \]

These is a complete description of the GT-systems related to integrable chains (1.1).

To solve the dynamical systems we bring the polynomial \( R \) to a canonical form sacrificing to normalization (4.3).

It is obvious that linear transformations \( p_i \to a p_i + b, \) where \( a, b \) are the functions of \( u_1, u_2, \) preserve the form of the GT-system (4.6), (4.7). Moreover, there exist transformations of the form

\[ p_i = \frac{a p_i + b}{b_i - \psi}, \quad i = 1, 2, 3, \quad (4.8) \]

preserving the form of the GT-system (4.6), (4.7). Such admissible transformations are described by the following conditions:

\[ a_{ij} = z_4 (b + a \psi), \quad b_{ij} = z_4 b \psi + z_6 b - z_6 a, \quad \psi_{ij} = z_4 \psi^2 + z_5 \psi + z_6. \]

Under transformations (4.8) the polynomial \( R \) is transformed by the following simple way:

\[ R(p_i) \to (p_i - \psi)^4 R \left( \frac{a p_i + b}{p_i - \psi} \right). \]

Suppose that \( R \) has distinct roots. It is possible to verify that by an admissible transformation (4.8) we can move three of the four roots to 0, 1 and \( \infty. \) It follows from compatibility conditions for the GT-system that the fourth root \( \lambda(u_1, u_2) \) does not depend on \( u_2. \) Making transformation of the form \( u_1 \to q(u_1) \) we arrive at the canonical forms \( \lambda = u_1 \) or \( \lambda = \text{const.} \) It is straightforwardly verified that in the first case equations (4.6), (4.7) coincides with (3.6). In the second case the GT-system does not exist.

In the case of multiple roots the polynomial \( R(x) \) can be reduced to one of the following forms: \( R = 0, \) \( R = 1, \) \( R = x, \) \( R = x^2, \) or \( R = x(x - 1). \) In all these cases equations (4.6), (4.7) coincide with the corresponding equations from example 1-1.

Thus, the following statement is valid:

**Proposition 1.** There are six non-equivalent cases of GT-systems (4.6), (4.7). The canonical forms are as follows.

**Case 1:** (3.6) (generic case);

**Case 2:** (3.4) with \( R(x) = x(x - 1); \)

**Case 3:** (3.4) with \( R(x) = x^2; \)

**Case 4:** (3.4) with \( R(x) = x; \)

**Case 5:** (3.4) with \( R(x) = 1. \)

**Case 6:** (3.4) with \( R(x) = 0. \)

**Remark.** Cases 2–6 can be obtained from case 1 by appropriate limit procedures. For example, case 2 corresponds to the limit \( u_1 \to \frac{u_1}{\varepsilon}, \) \( \varepsilon \to 0. \)
It follows from (4.3), (4.4) that for any canonical form the functions \( F \) and \( g_2 \) have the following structure:

\[
g_2(p_i) = \frac{k_1 p_i + k_2}{k_3 p_i + k_4}, \quad F(p_i) = \frac{f_1 p_i + f_2}{k_3 p_i + k_4},
\]

where the coefficients are functions of \( u_1, u_2 \).

**Lemma 1.** For case 1 any function \( g_2 \) can be reduced by an appropriate transformation \( \bar{u}_2 = \sigma(u_1, u_2) \) to one of the following canonical forms:

\[
a_1: \quad g_2(p) = \frac{u_2(u_2 - 1)(p - u_1)}{u_1(u_1 - 1)(p - u_2)} \quad \text{(regular extension)};
\]

\[
b_1: \quad g_2(p) = \frac{1}{p - u_1};
\]

\[
c_1: \quad g_2(p) = \frac{u_1 - u_2}{p - \lambda} \quad \lambda = 1, 0;
\]

\[
d_1: \quad g_2(p) = \frac{u_1 - u_2}{u_1(u_1 - 1)} p + \frac{u_2 - 1}{u_1 - 1}.
\]

The GT-system from case 1 possesses a discrete automorphism group \( S_4 \) interchanging the points 0, 1, \( \infty \), \( u_1 \). The group is defined by generators

\[
s_1 : u_1 \rightarrow 1 - u_1, \quad p_i \rightarrow 1 - p_i, \quad s_2 : u_1 \rightarrow \frac{u_1}{u_1 - 1}, \quad p_i \rightarrow \frac{p_i}{p_i - 1},
\]

and

\[
s_3 : u_1 \rightarrow 1 - u_1, \quad p_i \rightarrow \frac{(1 - u_1)p_i}{p_i - u_1}.
\]

Up to this group cases \( b_1, c_1, d_1 \) are equivalent and one can take say case \( d_1 \) for further consideration. Case \( a_1 \) is invariant with respect to the group.

**Remark.** Cases \( b_1, c_1, d_1 \) are degenerations of case \( a_1 \). Namely, they can be obtained as an appropriate limit \( u_2 \rightarrow u_1, u_2 \rightarrow \lambda, u_2 \rightarrow \infty \) correspondingly.

All possible functions \( g_2 \) for cases 2–5 are described in the following.

**Lemma 2.** For the GT-system (3.4) (excluding case 6) any function \( g_2 \) can be reduced by an appropriate transformation \( \bar{u}_2 = \sigma(u_1, u_2) \) to one of the following canonical forms:

\[
a_2: \quad g_2(p) = \frac{R(u_2)}{p - u_2} \quad \text{(regular extension)};
\]

\[
b_2: \quad g_2(p) = \frac{1}{p - \lambda}, \quad \text{where } R(\lambda) = 0;
\]

\[
c_2: \quad g_2(p) = p - a_2 u_2.
\]

The discrete automorphism of the GT-system interchanges the roots of \( R \) in case \( b_2 \).

**Lemma 3.** For the GT-system (3.4) with \( R(x) = 0 \) (case 6) any function \( g_2 \) can be reduced to \( g_2(p) = p \) by an appropriate transformation \( \bar{u}_2 = \sigma(u_1, u_2) \). Furthermore, the corresponding triangular GT-system has the form

\[
\partial_t p_j = 0, \quad \partial_t \partial_t u_1 = 0, \quad \partial_t u_k = p'^{k-1}_k u_1, \quad k = 2, 3, \ldots
\]

(4.10)
5. Generic case

The next step in the classification is to find all functions $F$ of the form (4.4) for each pair consisting of a GT-system from proposition 1 and the corresponding $g_2$ from lemmas 1–3. The semi-Hamiltonian condition (3.8) yields a nonlinear system of PDEs for the functions $f_1(u_1, u_2), f_2(u_1, u_2)$. For each case this system can be reduced to the linear generalized hypergeometric system (3.12), (3.13) with a special set of parameters $s_1, s_2, s_3, s_4$ or to a degeneration of this system.

The general linear fractional GT-family for the generic case 1, $a_1$ is given by (3.14). According to (4.9), the additional restriction is that the root of the denominator has to be equal $u_2$. It is easy to verify that this is equivalent to $s_2 = 0, h_{1,a_2} = h_{2,a_2} = 0$. The latter means that $h_1(u_1), h_2(u_1)$ are linear-independent solutions of the standard hypergeometric equation

$$
u(u - 1) h(u)'' + [s_1 + s_3 - (s_1 + s_2 + 2s_4) u] h(u)' + s_1 (s_1 + s_2 + s_3 + 1) h(u) = 0.$$  

(5.1)

The function $h_3(u_1, u_2)$ is an arbitrary solution of (3.12), (3.13) with $s_2 = 0$ linearly independent of $h_1(u_1), h_2(u_1)$. Without loss of generality we can choose

$$h_3(u_1, u_2) = \int_0^{u_2} (t - u_1)^h r(t - 1)^s dt.$$  

Formula (3.14) gives

$$F(p, u_1, u_2) = \frac{f_1(u_1, u_2) p - f_2(u_1, u_2)}{p - u_2}.$$  

(5.2)

where

$$f_1 = \frac{u_2 (u_2 - 1) h_{3,u_2} + u_1 (u_1 - 1) (h_{1,u_2} - h_{3,u_2})}{u_1 (u_1 - 1) (h_{1,u_1} - h_{3,u_1})},$$

$$f_2 = \frac{u_1 u_2 (u_2 - 1) h_{3,u_2} + u_2 u_1 (u_1 - 1) (h_{1,u_2} - h_{3,u_2})}{u_1 (u_1 - 1) (h_{1,u_1} - h_{3,u_1})}.$$

Note that $h_{1,u_1} - h_{3,u_1} = \text{const}(u_1 - 1)^{i+u_1} u_1^{i,u_1}$.

For integer values of $s_1, s_3, s_4$ the hypergeometric system can be solved explicitly. For example, if $s_1 = s_2 = 0$, the above formulas give rise to $F = g_2$. If $s_4 = -2 - s_1 - s_3$, then

$$F = \frac{(u_2 - u_1)^{i+1} u_1^{i+1} (u_2 - 1)^{-1-i}}{p - u_2};$$

if $s_4 = 0$, then

$$F = \frac{(p - 1)(u_2 - u_1)^{i+1} u_1^{i+1} (u_1 - 1)^{-1-i}}{p - u_2}.$$  

Now we are to find the functions $g_3, g_4, \ldots$ in (4.1). These functions are defined up to arbitrary transformation (1.2), where $\alpha = 3, 4, \ldots$. In practice, one can look for the functions $g_3, g_4, \ldots$ linear in $u_i, i > 2$ (cf (1.8)). An extension linear in $u_i, i > 2$, is given by

$$g_i(p) = \frac{(u_1 - u_2)(u_2 - 1)p}{u_1 (u_1 - 1) (p - u_2)^2},$$

$$g_i(p) = \frac{(i - 3)(u_1 - u_2)(u_2 - 1) p u_1}{u_1 (u_1 - 1) (p - u_2)^2} - \frac{(u_1 - u_2)^i (u_2 - 1)^2 p (p - u_1) (p - 1)^{i-4}}{u_1 (u_1 - 1)^{i-2} (p - u_2)^{i-1}} - \sum_{s=1}^{i-4} \frac{(i - s - 2)(u_1 - u_2)^s (u_2 - 1)^2 p (p - u_1) (p - 1)^{i-1} u_1^{s-2}}{u_1 (u_1 - 1)^{i+1} (p - u_2)^{i+2}}.$$
The coefficients of chain (1.1) corresponding to case 1; $a_1$ are determined from (4.5), where $F$ is given by (5.2). Relations (4.5) are equivalent to a triangular system of linear algebraic equations. Solving this system, we find that for $i > 4$ coefficients of the chain read

$$
\phi_{i+1} = \frac{(u_1 - 1)(f_1u_2 - f_2)}{(u_2 - 1)(u_1 - u_2)} = Q_1, \quad \phi_i = \frac{f_2 - f_1}{u_2 - 1} = Q_2,
$$

$$
\phi_{i,4} = -u_1Q_1, \quad \phi_{i,3} = -(u_4 + i - 3)u_1 + (2 - i)u_{i+1}Q_1, \quad \phi_{i,2} = A_i,
$$

and $\phi_{i,j} = 0$ for all remaining $i, j$. For $i \leq 4$ we have

$$
\phi_{1,1} = \frac{f_1u_1 - f_2}{u_1 - u_2}, \quad \phi_{1,2} = -\frac{u_1}{u_2}Q_1,
$$

$$
\phi_{2,1} = \frac{(u_2 - 1)(f_1u_2 - f_2)}{(u_1 - 1)(u_1 - u_2)}, \quad \phi_{2,2} = \frac{f_2u_1 - f_1u_2^2}{u_2(u_1 - u_2)}, \quad \phi_{2,3} = f_1u_2 - f_2, \quad (5.3)
$$

$$
\phi_{3,1} = \phi_{3,2} = 0, \quad \phi_{3,3} = Q_2 - (u_4 - 1)Q_1, \quad \phi_{3,4} = -Q_1,
$$

$$
\phi_{4,1} = \phi_{4,2} = 0, \quad \phi_{4,3} = A_4, \quad \phi_{4,4} = Q_2 - u_4Q_1, \quad \phi_{4,5} = Q_1.
$$

The explicit formulas for other cases of proposition 1 can be obtained by limits from the above formulas. We outline the limit procedures for case 1, $d_1$. In this case the limit is given by $u_2 \to u_1 + \epsilon u_2, \epsilon \to 0$. It is easy to check that under this limit the extension $a_1$ turns to $d_1$. The limit of the system (3.12), (3.13) with $s_2 = 0$ can be easily found. The general solution of the system thus obtained is given by $h = c_1(u_2 - u_1)^{1+x_1+x_2} + h_1$, where $h_1$ is the general solution of (5.1). Let $h_1, h_2$ be the solutions of (5.1), and $h_3 = (u_2 - u_1)^{1+x_1+x_2}$. Then the limit procedure in (5.2) gives rise to

$$
F(p, u_1, u_2) = Q \times ((1 + s_1 + s_3 + s_4)h_1(p - u_1) + u_1(u_1 - 1)h_1'),
$$

where

$$
Q = (u_2 - u_1)^{1+x_1+x_2}(u_1 - 1)^{-1 - s_1}u_1^{-1 - s_3}.
$$

As usual, the most degenerate cases in classification of integrable PDEs could be interesting for applications. In our classification they are case 5, $c_2$ and case 6. The Benney chain (see examples 1 and 1-1) belongs to case 5, case $c_2$ (i.e. $g_2 = p$). Any GT-family has the form $F = f_1(u_1, u_2) + f_2(u_1, u_2)$. If $f_1 = 1$, then $F = p + k_2u_2 + k_1u_1$. The Benney case corresponds to $k_1 = k_2 = 0$. For arbitrary $k_i$ we obtain the Kupershmidt chain [16]. In the case $f_1 = A(u_1), A' \neq 0$ we obtain

$$
f_1 = k_2 \exp(\lambda u_1) + k_1, \quad f_2 = k_2k_3 \exp(\lambda u_1) + \lambda k_1(k_3u_1 - u_2).
$$

In the generic case

$$
F = \exp(\lambda u_2)(S_1(u_1)p + S_2(u_1)),
$$

where the functions $S_i$ can be expressed in terms of the Airy functions.

6. Trivial GT-system and (2 + 1)-dimensional integrable hydrodynamic chains

It was observed in [11] that (2+1)-dimensional systems of hydrodynamic type with the trivial GT-system usually admit some integrable multi-dimensional generalizations. For the chains such GT-system is defined by (4.10). That is why case 6 is of a great importance in our classification. The automorphisms of (4.10) are given by

$$
p_j \to p_j, \quad j = 1, \ldots, N, \quad u_i \to u_i + \gamma_i, \quad i = 1, 2, \ldots; \quad
$$

$$
p_j \to ap_j + b, \quad j = 1, \ldots, N, \quad
$$

$$
u_i \to a_i^{-1}u_i + (i - 1)a_i^{-2}bu_{i-2} + \cdots + b_i^{-1}u_1, \quad i = 1, 2, \ldots.
$$

(6.1)
The corresponding GT-families are of the form $F(p) = A(u_1, u_2)p + B(u_1, u_2)$, where $A(x, y), B(x, y)$ satisfies the following system of PDEs:

$$
AB_{xy} = A_x B_y, \quad AB_{xy} = A_x B_y, \quad AB_{xx} = A_x B_x,
$$

$$
AA_{yy} = A^2_y, \quad AA_{xy} = A_y A_x, \quad AA_{xx} = A^2_x + A_x B_y - A_y B_x.
$$

(6.2)

This system can be easily solved in elementary functions. For each solution formula (4.5) defines the corresponding integrable chain (1.1).

It follows from (6.2) that there are two types of $u_2$-dependence:

1. (generic case) $F(p) = \exp(\lambda u_2)(a(u_1)p + b(u_1))$,
2. $F(p) = a(u_1)p + \lambda u_2 + b(u_1)$.

In the first case there are two subcases: $b' \neq 0$ and $b' = 0$. The first subcase gives rise to

$$
a = \sigma', \quad b = k_1 \sigma \quad \sigma(x) = c_1 \exp(\mu_1 x) + c_2 \exp(\mu_2 x),
$$

where $c_1 c_2 (\lambda k_1 - \mu_1 \mu_2) = 0$.

The second subcase leads to

$$
b = c_1, \quad a(x) = c_2 \exp(\mu x) + c_3, \quad \text{where} \quad c_2 (c_1 \lambda - c_3 \mu) = 0.
$$

The same subcases for case 2 yield

$$
a = \sigma', \quad b = k_1 \sigma \quad \sigma(x) = c_1 + c_2 x + c_3 \exp(\mu x), \quad \text{where} \quad c_3 (\lambda - c_2 \mu) = 0,
$$

and

$$
b = c_1, \quad a(x) = c_2 \exp(\mu x) + c_3, \quad \text{where} \quad c_2 (\lambda - c_3 \mu) = 0.
$$

It is easy to verify that in the generic case the function $F$ can be reduced by (6.1) to the form

$$
F(p) = e^{u_2 u_1} (p - 1) + e^{u_2 - u_1} (p + 1).
$$

This case the corresponding chain reads

$$
u_{k, t} = (e^{u_2 u_1} + e^{u_2 - u_1}) u_{k+1, x} + (e^{u_2 u_1} - e^{u_2 - u_1}) u_{k, x}, \quad k = 1, 2, 3, \ldots
$$

(6.3)

As usual, this chain is the first member of an infinite hierarchy. The second flow of this hierarchy is given by

$$
u_{k, t} = (e^{u_2 u_1} + e^{u_2 - u_1}) u_{k+2, x} + (u_3 - u_1)(e^{u_2 u_1} + e^{u_2 - u_1}) u_{k+1, x} +
(e^{u_2 u_1} (u_1 - u_3 - 1) + e^{u_2 - u_1} (u_3 - u_1 - 1)) u_{k, x}, \quad k = 1, 2, 3, \ldots
$$

In case 2 with $c_3 = \lambda = 0, k_1 = 1$, we obtain the chain

$$
u_{k, t} = u_{k+1, x} + u_{1, k, x}, \quad k = 1, 2, 3, \ldots
$$

(6.4)

This chain is equivalent to the chain of the so-called universal hierarchy [17]. Chain (6.4) is a degeneration of the chain

$$
u_{k, t} = u_{k+1, x} + u_{2, k, x}, \quad k = 1, 2, 3, \ldots
$$

(6.5)

Following the line of [3, 11] it is not difficult to find (2+1)-dimensional integrable generalizations for all (1+1)-dimensional integrable chains constructed above. Some families of functions $F$ described above linearly depend on two parameters. Denote these parameters by $\gamma_1, \gamma_2$. The corresponding integrable chain

$$u_{k, t} = \gamma_1 (\phi_{k, 1} u_{1, x} + \cdots + \phi_{k, k+1} u_{k+1, x}) + \gamma_2 (\psi_{k, 1} u_{1, x} + \cdots + \psi_{k, k+1} u_{k+1, x})
$$

is also linear in $\gamma_1, \gamma_2$. We claim that the following (2+1)-dimensional chain

$$u_{k, t} = (\phi_{k, 1} u_{1, x} + \cdots + \phi_{k, k+1} u_{k+1, x}) + (\psi_{k, 1} u_{1, y} + \cdots + \psi_{k, k+1} u_{k+1, y})
$$

(6.6)
is integrable from the viewpoint of the method of hydrodynamic reductions. For each case the reductions can be easily described.

For example, in the generic case
\[ F(p) = \gamma_1 e^{\nu_1 p + \nu_2 u_1} (p + 1) \]
formula (6.6) yields the (2+1)-dimensional chain
\[ u_{k,i} = e^{\nu_1 u_1} (u_{k+1,i} - u_{k,i}) + e^{\nu_2 u_1} (u_{k+1,y} + u_{k,y}), \quad k = 1, 2, 3, \ldots \]  
(6.7)
After a change of variables of the form
\[ x \rightarrow -\frac{1}{2} x, \quad y \rightarrow \frac{1}{2} y, \quad u_1 \rightarrow \frac{1}{2} u_0, \quad u_2 \rightarrow u_1 + \frac{1}{2} u_0, \quad u_3 \rightarrow -2 u_2 + \frac{1}{2} u_0, \ldots. \]  
(6.7) can be written as
\[ u_{0,i} = e^{\nu_1 u_0 y} + e^{\nu_1 (u_{1,y} - e^\nu u_{1,x})}, \quad u_{i,i} = e^{\nu_1 u_1 x} + e^{\nu_1 (e^{\nu_1} u_{1+1,x} - u_{i+1,y})}, \quad i = 1, 2, \ldots \]
where \( i = 1, 2, \ldots \). Probably (6.8) is a first example of a (2+1)-dimensional chain integrable from the viewpoint of the hydrodynamic reduction approach.

Triangular GT-systems related to integrable (2+1)-dimensional chains with fields \( u_0, u_1, u_2, \ldots \) have the form
\[ \partial_i p_j = f_1(p_i, q_i, p_j, q_j, u_0, \ldots, u_n) \partial_i u_0, \quad \partial_i q_j = f_2(p_i, q_i, p_j, q_j, u_0, \ldots, u_n) \partial_i u_0, \]
\[ \partial_i \partial_j u_0 = h_i(p_i, q_i, p_j, q_j, u_0, \ldots, u_n) \partial_i \partial_j u_0, \]
\[ \partial_i u_k = g_k(p_i, q_i, u_0, \ldots, u_{k+1}) \partial_i u_0, \quad k = 0, 1, 2, \ldots \]  
(6.9)
Here \( i \neq j, i, j = 1, \ldots, 3, p_1, \ldots, p_3, q_1, \ldots, q_3, u_0, u_1, u_2, \ldots \) are the functions of \( r^1, r^2, r^3 \). In particular, the GT-system associated with (6.8) has the form
\[ \partial_i p_j = \partial_i \partial_j u_0 = 0, \quad \partial_i q_j = \left( \frac{p_i q_j - p_j q_i}{q_i - q_j} \right) \partial_i u_0, \quad \partial_i u_k = -\frac{p_i}{(p_i - 1) q_i} \partial_i u_0. \]
The hydrodynamic reductions of (6.8) are given by the pair of semi-Hamiltonian (1+1)-dimensional systems
\[ r^i_j = e^{\nu_1} \left( 1 - \frac{1}{q_i} \right) r^i_j, \quad r^j_i = e^{\nu_2 u_1} \left( 1 \frac{1}{(p_i - 1) q_i} + 1 \right) r^i_j. \]

Chain (6.8) is the first member of an infinite hierarchy of pairwise commuting flows where the corresponding ‘times’ are \( t_1, t_2, t_3, \ldots \). These flows and their hydrodynamic reductions can be described in terms of the generating function \( U(z) = u_1 + u_2 z + u_3 z^2 + \cdots \). The hierarchy is given by
\[ D(z) u_0 = e^{U(z)} (u_{0,y} + U(z)_y - e^{U(z)} U(z)_x), \]
\[ D(z_1) U(z_2) = e^{U(z_1)} U(z_2) + (1 + z_1) e^{U(z_1)} \left( e^{U(z_1) x} z_1 - U(z_2)_x - U(z_1)_x \right) \]
where \( D(z) = \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \cdots \). The reductions can be written as
\[ D(z) r^j_i = e^{U(z)} \left( 1 + \frac{1}{p_i - 1 - q_i} \right) r^i_j. \]

Other (2+1)-dimensional integrable chains related to two-dimensional vector spaces of solutions for the system (6.2) are degenerations of (6.8). In particular, \( F = \gamma_1 e^{\nu_1 p + \nu_2 (p + u_2)} \) leads to the following (2+1)-dimensional integrable generalization of (6.7):
\[ u_{k,i} = e^{\nu_1 u_{k+1,x}} + u_{k+1,y} + u_{2} u_{k,y}, \quad k = 1, 2, 3, \ldots \]

**Conjecture.** Any chain of the form (6.6) integrable by the hydrodynamic reduction method is a degeneration of (6.8).

We are planning to consider the problem of classification of integrable chains (6.6) in a separate paper.
7. Infinitesimal symmetries of triangular GT-systems

A scientific way to construct the functions \( g_3, g_4, \ldots \) for different cases from proposition 1 is related to infinitesimal symmetries of the corresponding GT-system. The whole Lie algebra of symmetries is one the most important algebraic structures related to any triangular GT-system. In particular, this algebra acts on the hierarchy of the commuting flows for the corresponding chain.

A vector field
\[
S = \sum_{j=1}^{N} X(p_j, u_1, \ldots, u_s) \frac{\partial}{\partial p_j} + \sum_{m=1}^{\infty} Y_m(u_1, \ldots, u_k) \frac{\partial}{\partial u_m}, \quad \frac{\partial Y_m}{\partial u_k} \neq 0
\]
is called a symmetry of the triangular GT-system if it commutes with all \( \partial_i \). Note that it follows from the definition that
\[
S(\partial_i u_1) = \partial_i(Y_1).
\]
We call (7.1) a symmetry of shift if \( k_m = m + d \) for \( m \gg 0 \). Let \( M \) be the minimal integer such that \( k_m = m + d, m > M \). If the functions \( g_i, i = 1, \ldots, M + d \), from (4.1) are known, then the functions \( X, Y_1, \ldots, Y_M \) can be found from the compatibility conditions
\[
S(\partial_i u_k) = \partial_i S(u_k), \quad S(\partial_i u_1) = \partial_i S(u_1), \quad k = 1, \ldots, M.
\]
The functions \( Y_{M+1}, Y_{M+2}, \ldots \) can be chosen arbitrarily. After that \( g_{M+d+1}, g_{M+d+2}, \ldots \) are uniquely defined by the remaining compatibility conditions.

The generic case 1, \( a_1 \). Looking for symmetries of shift 1, we find \( X = Y_1 = 0 \) and \( M = 1 \). Hence without loss of generality we can take
\[
S = \sum_{m=2}^{\infty} u_{m+1} \frac{\partial}{\partial u_m}
\]
for the symmetry. This fact gives us a way to construct all functions \( g_i, i > 3 \), in the infinite triangular extension for case 1, \( a_1 \). Indeed, it follows from the commutativity conditions
\[
S(\partial_i u_k) = \partial_i S(u_k) \quad S(\partial_i u_1) = \partial_i S(u_1), \quad k = 2, 3, \ldots.
\]
In particular,
\[
g_3 = \frac{(p_j - u_1)(2p_j u_2 - p_j - u_2^2)u_3}{u_1(u_1 - 1)(p_j - u_2)^2}.
\]
The functions \( g_i \) thus constructed are not linear in \( u_3 \). The corresponding chain is equivalent to the chain constructed in section 5 but not so simple.

It would be interesting to describe the Lie algebra of all symmetries in this case. Here we present the essential part for symmetry of shift 2:
\[
X = \frac{p_j(p_j - 1)u_3^2}{(p_j - u_2)u_2(u_2 - 1)}, \quad Y_1 = \frac{u_1(u_1 - 1)u_3^2}{(u_1 - u_2)u_2(u_2 - 1)},
\]
\[
Y_2 = -\frac{3}{2} u_4 + \frac{(2u_1 - 1)u_3^2}{u_2(u_2 - 1)} + u_3.
\]

Case 1, \( d_1 \). One can add fields \( u_3, \ldots \) in such a way that the whole triangular GT-system admits the following symmetry of shift 1:
\[
S = \frac{u_2}{u_1(u_1 - 1)} \sum_{i=1}^{N} p_i(p_i - 1) \frac{\partial}{\partial p_i} + \sum_{i=1}^{\infty} u_{i+1} \frac{\partial}{\partial u_i}.
\]
Note that these functions are not unique because of the triangular group of symmetries acting on the fields \( u_3, u_4, \ldots \).
As in the previous example, one can easily recover the whole GT-system. For example,
\[ \partial_i u_3 = \left( \frac{u_3(p_i + u_1 - 1)}{u_1(u_1 - 1)} + \frac{2u_2^2 p_i (p_i - 1)}{u_1^2 (u_1 - 1)^2} \right) \partial_i u_1. \]

Below we describe the symmetry algebra for case 5, \( c_2 \) (in particular, for the Benney chain).

**Case 5, \( c_2 \).** For the triangular GT-system (1.7), (1.8) there exists an infinite Lie algebra of symmetries \( S_i, i \in \mathbb{Z} \), where \( S_i \) is a symmetry of shift \( i \). The simplest symmetries are as follows:
\[ S_{-2} = \frac{\partial}{\partial u_1} + \sum_{i=3}^{\infty} \left( -u_{i-2} + \sum_{k+m=i-3} u_k u_m - \sum_{k+m=i-4} u_k u_m u_l + \cdots \right) \frac{\partial}{\partial u_i}, \]
\[ S_{-1} = N \sum_{j=1}^{\infty} \frac{\partial}{\partial p_j} + \sum_{i=1}^{\infty} (i - 1)u_{i-1} \frac{\partial}{\partial u_i}, \]
\[ S_0 = N \sum_{j=1}^{\infty} p_j \frac{\partial}{\partial p_j} + \sum_{i=1}^{\infty} (i + 1)u_i \frac{\partial}{\partial u_i}, \]
\[ S_1 = \sum_{j=1}^{N} (p_j^2 + 3u_1) \frac{\partial}{\partial p_j} + \sum_{i=1}^{\infty} (i + 2)u_{i+1} \frac{\partial}{\partial u_i} + \sum_{i=1}^{\infty} u_k u_m \frac{\partial}{\partial u_i} + \sum_{i=1}^{\infty} 3(i - 1)u_{i-1} \frac{\partial}{\partial u_i}, \]
\[ S_2 = \sum_{j=1}^{\infty} (p_j^3 + 4u_1 p_j + 5u_2) \frac{\partial}{\partial p_j} + \sum_{i=1}^{\infty} (i + 3)u_{i+2} \frac{\partial}{\partial u_i} + \sum_{i=1}^{\infty} 4u_1 u_i \frac{\partial}{\partial u_i} + \sum_{i=2}^{\infty} 5(i - 1)u_{2i-1} \frac{\partial}{\partial u_i} + \sum_{i=1}^{\infty} \sum_{k+m=i+1} 3u_k u_m \frac{\partial}{\partial u_i} + \sum_{i=1}^{\infty} \sum_{k+m=i} u_k u_m u_l \frac{\partial}{\partial u_i}. \]

The whole algebra is generated by \( S_1, S_2, S_{-1}, S_{-2} \). It is isomorphic to the Virasoro algebra with zero central charge. Let \( D_i \) be the vector fields corresponding to commuting flows for the Benney chain. Here \( D_{i+1} = D_x, D_{i+2} = D_t \). Then the commutator relations
\[ [S_i, D_{i+1}] = (i + 1)D_{i+1}, \]
hold. Thus, the vector field \( S_1 \) plays the role of a master symmetry for the Benney hierarchy.

**Case 6.** In this case there exist infinitesimal symmetries of form
\[ T_i = u_{i+1} \frac{\partial}{\partial u_1} + u_{i+2} \frac{\partial}{\partial u_2} + \cdots, \quad i = 0, 1, 2, \ldots, \]
\[ S_i = \sum_{j=1}^{N} p_j^{i+1} \frac{\partial}{\partial p_j} + u_{i+2} \frac{\partial}{\partial u_2} + 2u_{i+3} \frac{\partial}{\partial u_3} + 3u_{i+4} \frac{\partial}{\partial u_4} + \cdots, \quad i = -1, 0, 1, 2, \ldots. \]

Note that \([S_i, S_j] = (j - i)S_{i+j}, [T_i, T_j] = 0, [S_i, T_j] = jT_{i+j} \).

**Acknowledgments**

The authors thank M V Pavlov for fruitful discussions. VS is grateful to IHES and Brock University for hospitality. He was partially supported by the RFBR grants 08-01-464, 09-01-22442-KE, and NS 3472.2008.2.
References

[1] Ferapontov E V and Khusnutdinova K R 2004 On integrability of (2+1)-dimensional quasilinear systems Comm. Math. Phys. 248 187–206
[2] Ferapontov E V and Khusnutdinova K R 2004 The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type J. Phys. A: Math. Gen. 37 2949–63
[3] Ferapontov E V and Khusnutdinova K R 2004 Hydrodynamic reductions of multidimensional dispersionless PDEs: the test for integrability J. Math. Phys. 45 2365–77
[4] Pavlov M V 2004 Teoret. Mat. Fiz. 138 55–70 (in Russian)
Pavlov M V 2004 Classification of integrable Egorov hydrodynamic chains Theoret. Math. Phys. 138 45–58 (Engl. Transl.)
[5] Pavlov M V 2006 Classification of integrable hydrodynamic chains and generating functions of conservation laws J. Phys. A: Math. Gen. 39 10803–19
[6] Ferapontov E V and Marshal D G 2007 Differential-geometric approach to the integrability of hydrodynamic chains: the Haanties tensor Math. Ann. 339 61–99
[7] Benney D J 1973 Some properties of long nonlinear waves Stud. Appl. Math. 52 45–50
[8] Kupershmidt B A and Manin Ju I 1977 Funk. Anal. i Prilozhen 11 31–42 (in Russian)
[9] Kupershmidt B A and Manin Ju I 1977 Long wave equations with a free surface: I. Conservation laws and solutions Func. Anal. Appl. 11 31–42, 96 (Engl. Transl.)
[10] Zakharov V E 1981 On the Benney’s equations Physica 3D 193–200
[11] Gibbons J and Tsarev S P 1996 Reductions of Benney’s equations Phys. Lett. A 211 19–24
[12] Odesskii A V and Sokolov V V 2009 Systems of Gibbons–Tsarev type and integrable 3-dimensional models arXiv:0906.3509
[13] Odesskii A V and Sokolov V V 2008 Integrable pseudopotentials related to generalized hypergeometric functions arXiv:0803.0086
[14] Gelfand I M, Graev M I and Retakh V S 1992 General hypergeometric systems of equations and series of hypergeometric type Russian Math. Surveys 47 1–88
[15] Odesskii A V and Sokolov V V 2009 Integrable pseudopotentials related to elliptic curves Teoret. Mat. Fiz. 161 21–36 (arXiv:0810.3879)
[16] Kupershmidt B A 1983 Deformations of integrable systems Proc. R. Irish Acad. Sect. A 83 45–74
[17] Martines Alonso L and Shabat A B 2004 Teoret. Mat. Fiz. 140 216–29 (in Russian)
Martines Alonso L. and Shabat A B 2004 Hydrodynamic reductions and solutions of the universal hierarchy Theoret. Math. Phys. 140 1073–85 (Engl. Transl.)