REFLECTION SUBGROUPS OF FINITE AND AFFINE WEYL GROUPS

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Abstract. We discuss the classification of reflection subgroups of finite and affine Weyl groups from the point of view of their root systems. A short case free proof is given of the well known classification of the isomorphism classes of reflection subgroups using completed Dynkin diagrams, for which there seems to be no convenient source in the literature. This is used as a basis for treating the affine case, where finer classifications of reflection subgroups are given, and combinatorial aspects of root systems are shown to appear. Various parameter sets for certain types of subsets of roots are interpreted in terms of alcove geometry and the Tits cone, and combinatorial identities are derived.

1. Introduction

1.1. Let \( V \) be a Euclidean space, that is, a real vector space with a positive definite symmetric bilinear form which we write \( \langle -,- \rangle : V \times V \to \mathbb{R} \). Let \( \Phi \) be a finite reduced root system in \( V \), in the sense of [2]. The following basic facts may be found in [2] or [18]. For \( \alpha \in \Phi \), we have the corresponding coroot \( \alpha^\vee = \frac{2\alpha}{\langle \alpha,\alpha \rangle} \). The set \( \Phi^\vee := \{ \alpha^\vee | \alpha \in \Phi \} \) is again a reduced root system in \( V \), called the root system dual to \( \Phi \).

For \( \alpha \in \Phi \) the corresponding reflection \( s_\alpha : V \to V \) is defined by

\[
s_\alpha (v) = v - \langle \alpha^\vee, v \rangle \alpha \quad (v \in V).\n\]

The Weyl group \( W = W(\Phi) \) of \( \Phi \) is the group generated by the reflections \( s_\alpha \), \( \alpha \in \Phi \), and it is well known that any reflection in \( W \) is of the form \( s_\alpha \) for some \( \alpha \in \Phi \). In this work we shall assume that \( W \) (or \( \Phi \)) is crystallographic; this means that \( \langle \alpha,\beta^\vee \rangle \in \mathbb{Z} \) for all \( \alpha,\beta \in \Phi \).

A subset \( \Phi_+ \subseteq \Phi \) is a positive system if it consists of the roots of \( \Phi \) which are positive relative to some vector space total ordering of \( V \). Such an ordering is defined by its cone \( V_+ \subseteq V \) of ‘positive elements’, which is closed under positive linear combinations, and satisfies \( V = V_+ \cup -V_+ \cup \{0\} \) (disjoint union). A subset \( \Lambda \subseteq \Phi \) is a simple system if \( \Lambda \) is linearly independent and every root of \( \Phi \) is a linear combination of \( \Lambda \) in which all nonzero coefficients have the same sign, i.e. are either all positive or all negative. It is well known that each positive system contains a unique simple system \( \Lambda \subseteq \Phi_+ \), and conversely. Moreover the positive systems (and hence the simple systems) of \( \Phi \) are in bijection with \( W \), which acts on them regularly; that is, given two simple systems \( \Lambda_1, \Lambda_2 \) of \( \Phi \), there is a unique element \( w \in W \) such that \( \Lambda_2 = w\Lambda_1 \).

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1.2. A root subsystem of $\Phi$ is a subset $\Psi \subseteq \Phi$ which is itself a root system. This is equivalent to the requirement that for $\alpha, \beta \in \Psi$, $s_\alpha(\beta) \in \Psi$.

Positive subsystems and simple subsystems of $\Phi$ are respectively defined as the positive systems and simple systems of a root subsystem of $\Phi$.

A reflection subgroup $W'$ of $W$ is a subgroup which is generated by a subset of the reflections $s_\alpha$ ($\alpha \in \Phi$). Any such subgroup has an associated set $\Psi = \Psi(W')$ of roots of $\Phi$, where $\Psi = \{\alpha \in \Phi \mid s_\alpha \in W'\}$. Then $\Psi$ is clearly a root subsystem of $\Phi$, and the correspondence $W' \leftrightarrow \Psi(W')$ is a bijection between reflection subgroups of $W$ and root subsystems of $\Phi$. In this work, we shall discuss the classification of root subsystems under various types of equivalence (isomorphism, $W$-conjugacy or equality), for both the finite cases and the associated affine root systems. Our treatment therefore includes a classification of reflection subgroups under the corresponding equivalence.

1.3. A root subsystem $\Psi \subseteq \Phi$ is closed if, whenever $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Psi$.

The closed subsystems of a finite root system $\Phi$ were classified by Borel and de Siebenthal in [1], using the connection between crystallographic Weyl groups and Lie algebras. The classification of all subsystems may be deduced from this. This was also done in [9], and a treatment may be found in [15]. A general classification for all unitary (complex) reflection groups may be found in [16]. In [3], the classification of reflection subgroups of finite Weyl groups $W$ is used in the determination of the conjugacy classes of $W$, and the reference given for this result is to [1]. It therefore seems reasonable to begin with a clear, concise and self-contained treatment of the finite case. We do this in §2.

Our results in the case of the affine Weyl groups have some slight overlap with those of [10], although the methods are different. Background on affine Weyl groups is collected in §3. We give two bijective parameterisations of the reflection subgroups of affine Weyl groups. Their parameterisation in §4 (by specifying their “canonical simple system”) generalizes the classification of reflection subgroups of the infinite dihedral group by specifying their set of generating reflections of minimal total length. Their parameterisation in §5 (by explicitly describing their root system) generalizes the classification of non-trivial reflection subgroups of the infinite dihedral group by specifying their reflections by a coset in $\mathbb{Z}$ of a subgroup of $\mathbb{Z}$. One or the other parameterisation may be more advantageous for a particular application (e.g. computing the index of a reflection subgroup, or a conjugate of it); the relation between the parameterisations is considered in detail in §6.

2. The case of a finite root system

We begin with the following well known characterisation of the simple subsystems of $\Phi$.

**Lemma 1.** Let $\Phi$ be a finite root system in the Euclidean space $V$. A subset $\Gamma \subseteq \Phi$ is a simple subsystem of $\Phi$ (i.e. a simple system in a root subsystem of $\Phi$) if and only if

(i) $\Gamma$ is linearly independent.

(ii) For any pair $\alpha, \beta \in \Gamma$, $\langle \alpha, \beta^\vee \rangle \leq 0$.

**Proof.** If $\Gamma$ is a simple subsystem of $\Psi \subseteq \Phi$, then by definition $\Gamma$ is linearly independent, and the property (ii) is standard (cf. [18, Appendix, Lemma (10)]).
Conversely, let $\Gamma \subseteq \Phi$ satisfy the conditions (i) and (ii) and write $W'$ for the group generated by the reflections $s_\gamma$ ($\gamma \in \Gamma$). Let $\Psi$ be the set of roots $W'T \subseteq \Phi$. Then $\Psi$ is a crystallographic root system with Weyl group $W'$, for if $\alpha \in \Psi$, $\alpha = w\gamma$ for some $w \in W'$ and $\gamma \in \Gamma$; hence $s_\alpha = s_{w\gamma} = ws_\gamma w^{-1} \in W'$, and so $\Psi$ is precisely the set of roots $\alpha \in \Phi$ such that $s_\alpha \in W'$. Replacing $W'$ with $W$ and $\Psi$ with $\Phi$, we may therefore suppose that the reflections $s_\gamma$ ($\gamma \in \Gamma$) generate $W$, and that $\Phi = W\gamma$.

Let $\Phi_+$ be the roots of $\Phi$ which are positive linear combinations of $\Gamma$. If $\dot{\alpha}$ denotes the unit vector in the direction of $\alpha$, the assumptions imply that if $\gamma_1, \gamma_2 \in \Gamma$, then $\langle \gamma_1, \gamma_2 \rangle \in \{-\cos \frac{\pi}{2}, -\cos \frac{\pi}{4}, -\cos \frac{\pi}{4}, -\cos \frac{\pi}{2}\}$. It now follows from [7, Lemma (4.2)] that $\Phi = \Phi_+ \sqcup -\Phi_+$. □

We shall also require the following result.

**Lemma 2.** Let $\Phi$ be an indecomposable finite crystallographic root system in the Euclidean space $V$, and let $\Gamma \subseteq \Phi$ be a simple subsystem of $\Phi$. Let $C$ be the corresponding (closed) fundamental chamber for the action of $W = W(\Phi)$ on $V$. Then $\alpha \in C \cap \Phi$ if and only if $\alpha$ is the highest root $\omega$ of $\Phi$ or $\alpha = \omega^*\gamma$, where $\omega^*$ is the highest root of $\Phi^\vee$.

**Proof.** It follows from arguments based on the fact that for any two roots $\alpha_1, \alpha_2 \in \Phi$, we have $\langle \alpha_1, \alpha_2 \rangle = 4 \cos^2 \phi$, where $\phi$ is the angle between $\alpha_1$ and $\alpha_2$, that there are at most two possibilities for the length of a root in $\Phi$ (see [2, Ch. VI, §1.3]).

It is also standard that $W$ acts transitively on the set of roots of a given length. Now it is clear that the roots $\omega, \omega^*\gamma$ of the statement lie in $C$, and by the regularity of the action of $W$ on the set of chambers, they are the only roots in their $W$-orbit in $C$. But the highest root is always long, whence the $W$-orbit of $\omega$ and $\omega^*\gamma$ include all roots of $\Phi$. This proves the Lemma. □

Note that in the above lemma, $\omega$ and $\omega^*\gamma$ have the same length (and hence coincide) if and only if $\Phi$ has just one root length. In that case, we declare all roots of $\Phi$ to be both short and long.

The root subsystem $\Psi \subseteq \Phi$ is said to be parabolic if $\Phi$ has a simple system $\Pi$ such that there is a subset $\Gamma \subseteq \Pi$ which is a simple system for $\Psi$. The parabolic subgroups (i.e. the Weyl groups of parabolic subsystems) are known to be precisely the stabilisers in $W$ of points of $V$. In general $W$ will have reflection subgroups other than parabolic ones; the next definition provides a convenient way of describing them.

**Definition 1.** Let $\Psi \subseteq \Phi$ be a root subsystem. We say that $\Phi$ is an elementary extension of $\Psi$ if there is a simple system $\Pi$ and an indecomposable component $\Sigma$ of $\Phi$ such that a simple system for $\Psi$ is obtained by deleting one element of $\Sigma$ from $\Pi \cup \{-\theta\}$, where $\theta$ is a root of $\Sigma$ which lies in the chamber $C$ defined by the positive system corresponding to $\Pi$.

By Lemma [2] $\theta$ is the highest root of $\Sigma$ or $\theta^\vee$ is the highest root of $\Sigma^\vee$. We shall prove

**Theorem 1.** Let $\Phi$ be a finite crystallographic root system in the Euclidean space $V$, and let $\Psi \subseteq \Phi$ be a root subsystem. Then there is a root subsystem $\hat{\Psi}$ with $\Psi \subseteq \hat{\Psi} \subseteq \Phi$, such that either $\Psi$ is parabolic in $\hat{\Psi}$ or $\hat{\Psi}$ is an elementary extension of $\Psi$. 
Proof. Let \( \Gamma \) be a simple system for \( \Psi \), with corresponding positive system \( \Psi^+ \subseteq \Psi \). Denote the Weyl group of \( \Psi \) by \( W' \) and let \( C = \{ v \in V \mid \langle \gamma, v \rangle \geq 0 \text{ for } \gamma \in \Gamma \} \) be the fundamental chamber for the action of \( W' \) corresponding to the simple system \( \Gamma \).

Since \( \Psi \neq \Phi \), there is a root \( \alpha \in \Phi \setminus \Psi \). There is a \( W' \)-transform \( w\alpha \in -C \) \((w \in W')\), and since clearly \( w\alpha \notin \Psi \), we may assume that \( \alpha \in -C \).

It follows that \( \langle \alpha, \gamma \rangle \leq 0 \) for all \( \gamma \in \Gamma \).

Let \( \tilde{\Gamma} = \Gamma \cup \{ \alpha \} \). If \( \tilde{\Gamma} \) is linearly independent, then by Lemma 1, \( \tilde{\Gamma} \) is a simple subsystem of \( \Phi \), and taking \( \hat{\Psi} \) to be the root system corresponding to \( \tilde{\Gamma} \), clearly \( \Psi \) is a parabolic subsystem of \( \hat{\Psi} \).

Suppose then that \( \tilde{\Gamma} \) is linearly dependent. As with any set of roots, \( \tilde{\Gamma} \) has a unique partition \( \tilde{\Gamma} = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k \), where the \( \Gamma_i \) are mutually orthogonal (i.e. if \( \gamma_i \in \Gamma_i \) and \( i \neq j \) then \( \langle \gamma_i, \gamma_j \rangle = 0 \)), and cannot be further decomposed as in \( \mathbb{2} \).

Evidently \( \alpha \) lies in precisely one of the sets \( \Gamma_i \), and we may suppose \( \alpha \in \Gamma_1 \). Now the matrix of pairwise inner products of the elements of \( \Gamma_1 \) satisfy the conditions of \( \mathbb{2} \) Ch. V, \( \S 3 \), Lemma 5, hence there are positive real numbers \( c_\gamma \) such that \( \sum_{\gamma \in \Gamma_1} c_\gamma \gamma = 0 \), and the \( c_\gamma \) are uniquely determined, up to a common multiple, and for all \( \gamma, c_\gamma \neq 0 \). It follows from Lemma 1 that every proper subset of \( \Gamma_1 \) is a simple subsystem of \( \Phi \).

Now the matrix \( (\langle \gamma, \delta \rangle)_{\gamma, \delta \in \Gamma_1} \) is an indecomposable generalised Cartan matrix of affine type in the sense of \( \mathbb{4} \) \( \S 15.3 \). These have been completely classified (see, e.g. op. cit. \( \S \S 15.3, 17.1 \)), and the following statement is an easy consequence of this classification (for an alternative proof using affine Weyl groups, and independent of the classification, see Corollary \( \mathbb{3} \) (b)).

There is an element \( \theta \) \( \in \Gamma_1 \) such that if \( \Gamma'_1 = \Gamma_1 \setminus \{ \theta \} \), then

(i) \( \Gamma'_1 \) is a simple system for an indecomposable root system \( \Psi'_1 \), and
(ii) \( \theta \in \Psi'_1 \).

By Lemma \( \mathbb{2} \) \( \theta \) is the negative of either the highest root of \( \Psi_1 \) or of the coroot of the highest root of \( \Psi'_1 \).

Let \( \Psi = \Psi'_1 \cup \Psi_2 \cup \cdots \cup \Psi_k \), where \( \Psi'_1 \) is as above, and for \( i = 2, \ldots, d \), \( \Psi_i \) is the root subsystem with simple system \( \Gamma_i \). Then \( \Psi \) is evidently an elementary extension of \( \Psi \) with the desired properties. \( \square \)

Corollary 1. Let \( \Phi \) be a finite crystallographic root system in the Euclidean space \( V \). The Dynkin diagram of any root subsystem of \( \Phi \) is obtained by a sequence of operations of the following type on Dynkin diagrams, beginning with the Dynkin diagram of \( \Phi \):

(1) Delete one or more nodes.
(2) Replace some indecomposable component \( \Sigma \) of the Dynkin diagram by the completed Dynkin diagram of \( \Sigma \) or \( \Sigma^\vee \), and delete at least one node from the resulting affine component.

Proof. If \( \Psi \) is a parabolic subsystem of \( \Phi \), its Dynkin diagram is obtained from that of \( \Phi \) by deleting nodes. By Theorem 1, we need only relate the Dynkin diagram of a subsystem \( \Psi \) of \( \Phi \) to that of \( \Phi \) when \( \Phi \) is an elementary extension of \( \Psi \). In that case, taking into account Lemma \( \mathbb{2} \), the result follows from the nature of an elementary
Let Ψ be any root system in $V$. Necessary background on affine Weyl groups, most of which may be found in [2, Ch. VI, §3.1].

For the remainder of §3.2, for any root system in $V$, as in [11]. For simplicity, assume $\Phi$ spans $V$. Write $P = P(\Phi)$ and $Q = Q(\Phi)$. For $v \in V$, denote by $\tau_v : V \rightarrow V$ the translation $x \mapsto x + v$. For any subgroup $B$ of $V$, let $\mathcal{B} := \{ \tau_\alpha | \alpha \in B \}$ denote the group of translations of $V$ by elements of $B$.

For $\alpha \in \Phi$ and $m \in \mathbb{Z}$, define the affine reflection

$$s_{\alpha,m}(v) = v - ((\alpha, v) - m)\alpha^\vee = s_{\alpha}(v) + m\alpha^\vee.$$ 

We let $H_{\alpha,m} := \{ v \in V | s_{\alpha,m}(v) = v \} = \{ v \in V | (v, \alpha) = m \}$ denote the reflecting hyperplane of $s_{\alpha,m}$.

Definition 2. The affine Weyl group $W^a$ associated to $\Phi$ is the group generated by $\{ s_{\alpha,m} | \alpha \in \Phi, m \in \mathbb{Z} \}$. The extended affine Weyl group $W^e$ is the group generated by $W \cup \tau_\alpha$. The alcoves of $W^a$ are the connected components (for the standard topology of $V$) of $V \setminus \bigcup_{\alpha \in \Phi, n \in \mathbb{Z}} H_{\alpha,n}$. A special point of $W^a$ is a point of $V$ which for each $\alpha \in \Phi$, lies in some reflecting hyperplane $H_{\alpha,m}$ ($m \in \mathbb{Z}$).
Proposition 1. (a) The group $W^a$ is the semidirect product $W \ltimes \tau_Q$ of $W$ and $\tau_Q$, while $\tilde{W}^a$ is the semidirect product $W \ltimes \tau_P$ of $W$ and $\tau_P$. Moreover $W^a$ is a normal subgroup of $\tilde{W}^a$ (i.e. $W^a \leq \tilde{W}^a$) such that $\tilde{W}^a/W^a \simeq P/Q$.

(b) Let $\Pi$ be a simple system of $\Phi$. Then $W^a$ is a Coxeter group, with Coxeter generators $S^a := \{s_\alpha \mid \alpha \in \Pi\} \cup \{s_{\theta,1} \mid \theta \in H\}$, where $H$ is the set of highest roots of the indecomposable components of $\Phi$ (with respect to their simple systems contained in $\Pi$).

(c) Let $\Pi := \{v \in V \mid \langle v, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Pi, \quad \langle v, \theta \rangle \leq 1 \text{ for all } \theta \in H\}$. Then $\Pi$ is the closure of the fundamental alcove $A^0$ for $(W^a, S^a)$ on $V$. In particular, $\Pi$ is the closure of $A^0$, $A^0$ is the interior of $A$, and $S^a$ is the set of reflections in the walls $\{H_{\alpha,0} \mid \alpha \in \Pi\} \cup \{H_{\theta,1} \mid \theta \in H\}$ of $A$.

(d) Each closed alcove contains at least one special point of $W^a$. The set of all special points coincides with the coweight lattice of $\Phi$.

(e) The indecomposable components of $W^a$ are the affine Weyl groups associated to the indecomposable components of $\Phi$, and the group $\tilde{W}^a$ is the direct product of the extended affine Weyl groups of the indecomposable components of $\Phi$.

Remark 1. See [2] Ch V, §1–3 for more details on alcoves, walls, special points etc. For indecomposable $\Phi$, the above facts can be found in [2]. For decomposable $\Phi$, the statements are either in [2] or follow readily from the results there about decomposable affine Weyl groups. It is well known that any closed alcove is a fundamental domain for the action of $W^a$ on $V$. In this paper, by a fundamental domain for a group acting on a set, we shall simply mean a set of orbit representatives, though our fundamental domains will usually satisfy additional properties such as connectedness.

3.3. Linear realisation. The groups $W^a$ and $\tilde{W}^a$ appear above as groups of affine transformations of $V$. They may also be viewed as groups of linear transformations of a space $\hat{V}$ which contains $V$ as a subspace of codimension 1. In particular we shall see that there is an affine root system in $\hat{V}$ on which both $W^a$ and $\tilde{W}^a$ act naturally.

Let $\hat{V} = V \oplus \mathbb{R}\delta$, where $\delta$ is an indeterminate. The inner product $\langle -,- \rangle$ on $V$ has a unique extension to a symmetric bilinear form on $\hat{V}$ which is positive semidefinite, and has radical equal to the subspace $\mathbb{R}\delta$. This extension to $\hat{V}$ will also be denoted by $\langle -,- \rangle$. For any non-isotropic element $v \in \hat{V}$, let $t_v : \hat{V} \to \hat{V}$ denote the $\mathbb{R}$-linear map $x \mapsto x - \langle x, v^\vee \rangle v$ where $v^\vee = \frac{\langle x, v \rangle}{\langle v, v \rangle} v$. For any $v \in V$, let $t_v : \hat{V} \to \hat{V}$ denote the $\mathbb{R}$-linear map $x \mapsto x + \langle x, v \rangle \delta$.

Let $\Pi$ be a simple system of $\Phi(\subset V \subset \hat{V})$ with corresponding set $\Phi_+$ of positive roots.

Definition 3. Define the following subsets of $\hat{V}$:

$$\Phi^a := \Phi + \mathbb{Z}\delta$$

$$\hat{\Pi} := \Pi \cup \{-\theta + \delta \mid \theta \in H\},$$

$$\Phi_+^a := \{\alpha + n\delta \mid \alpha \in \Phi, n \in \mathbb{Z}_{\geq 0}, n > 0 \text{ if } \alpha \in -\Phi_+\}.$$
Proof. Let $W_1^a$ (resp., $\tilde{W}_1^a$) denote the group of linear endomorphisms of $\tilde{V}$ defined in (a).

The proof will use the following elementary formulae concerning the action of these groups on $\tilde{V}$, which are listed here for future reference.

\begin{align}
(6) & \quad \hat{s}_\alpha t_\gamma = t_{\hat{s}_\alpha(\gamma)} \hat{s}_\alpha, \quad \text{for } \alpha \in \Phi^a, \gamma \in \tilde{V} \\
(7) & \quad \hat{s}_{\alpha+n\delta} = t_{n\alpha^\vee} \hat{s}_\alpha, \quad \text{for } \alpha \in \Phi, n \in \mathbb{Z} \\
(8) & \quad \hat{s}_{\alpha+n\delta} \hat{s}_{\alpha+m\delta} = t_{(n-m)\alpha^\vee}, \quad \text{for } \alpha \in \Phi, n, m \in \mathbb{Z} \\
(9) & \quad \hat{s}_{\alpha+n\delta}(\beta + m\delta) = s_\alpha(\beta) + (m - n(\beta, \alpha^\vee))\delta \quad \text{for } \alpha, \beta \in \Phi \text{ and } n, m \in \mathbb{Z} \\
(10) & \quad wt_\gamma(\alpha + m\delta) = w(\alpha) + (m + (\alpha, \gamma))\delta \quad \text{for } w \in W, \gamma \in V, \alpha \in \Phi, m \in \mathbb{Z}.
\end{align}

Consider the contragredient action of $\tilde{W}_1^a$ and $W_1^a$ on the dual space

$$\tilde{V}^* := \text{Hom}_R(\tilde{V}, \mathbb{R}),$$

defined by $wf = f \circ w^{-1}$ for $f \in \tilde{V}^*$ and $w \in \tilde{W}_1^a$. Define $d \in \tilde{V}^*$ by $d(v + c\delta) = c$ for $v \in V$, $c \in \mathbb{R}$. The affine map $i: V \rightarrow \tilde{V}^*$ defined by $i(v)(x) = (v, x) + d(x)$ for $v \in V$ and $x \in \tilde{V}$, is injective, and has image equal to the affine hyperplane $E = \{ f \in \tilde{V}^* \mid f(\delta) = 1 \}$ of $\tilde{V}^*$. This is $\tilde{W}_1^a$-stable since $\tilde{W}_1^a$ fixes $\delta$. Since $E$ spans $\tilde{V}^*$ as real vector space, restriction to $E$ defines an isomorphism of $\tilde{W}_1^a$ (and hence of $W_1^a$) with a group of affine transformations of $E$. But from the definitions, one readily checks that

\begin{align}
(11) & \quad t_\gamma i = i\tau_{-\gamma} \quad \text{for } \gamma \in V \\
\text{and} & \\
(12) & \quad \hat{s}_\alpha i = is_\alpha \quad \text{for } \alpha \in \Phi
\end{align}

The first assertion of (a) follows by noting that by (11), (12), $i : V \rightarrow E$ is an isomorphism of $W^a$-modules. We observe also that under the above map, the element $s_{\alpha,m} = \tau_{m\alpha^\vee} \hat{s}_\alpha$ of $W^a$ is mapped to $t_{-m\alpha^\vee} \hat{s}_\alpha = \hat{s}_{m\delta-\alpha}$ in $\tilde{W}_1^a$. This completes the proof of (a).

It is not difficult to prove (b) using the above formulae and well known facts about root systems of Weyl groups. Instead, we simply observe that the proof of (b) reduces easily to the special case in which $\Phi$ is indecomposable, in which case $\Phi^a$ may be identified with the system of real roots of an untwisted affine Kac-Moody Lie algebra [14, Prop. 6.3(a)], and given that $\Phi^a$ is $\tilde{W}^a$-stable, the claims of (b)
are well known (see [14] Ch 6–7). To see that Φ is \( \hat{W}^a \)-stable, it is clear that Φ is stable under \( W^a \), and to prove stability under \( \hat{W}^a \), we apply (10) with γ in the coweight lattice and note that \( (\alpha, \gamma) \in \mathbb{Z} \).

3.4. Let \( R \) be a fixed subgroup of \( P \) containing \( Q \). It is well-known that \( R \) is \( W \)-stable, so one may form the semidirect product \( \hat{W} := W \rtimes \tau_R \) which satisfies \( W^a \subseteq \hat{W} \subseteq \hat{W}^a \) and \(|\hat{W}:W^a| = [R:Q] \). Many of our results for \( \hat{W}^a \) apply equally to any of the subgroups \( \hat{W} \).

For \( v \in V \subseteq \hat{V} \), we have \( \hat{s}_v = s_v + 0 \oplus \text{id}_\delta = s_v + \text{id}_\delta \). Henceforth, we shall therefore write \( s_\alpha \) instead of \( \hat{s}_\alpha \) for any \( \alpha \in \Phi \). We shall regard \( \hat{W}^a, W^a \) and \( \hat{W} \) as groups of linear transformations of \( \hat{V} \), or as groups of affine transformations of \( V \), as convenience dictates.

Let \( \Delta \subseteq V \setminus \mathbb{R}^\delta \). The (indecomposable) components of \( \Delta \) are the inclusion-minimal non-empty subsets \( \Delta' \) of \( \Delta \) which are orthogonal to their complement in \( \Delta \). Say that \( \Delta \) is indecomposable if it has only one component. If \( \Delta \) is a simple subsystem or a root subsystem of \( \Phi \), these coincide with the usual notions of indecomposable components and indecomposability.

3.5. Linear versus affine. This subsection and the next two describe relationships between \( \Phi^a \) and root systems of affine Weyl groups, and the realisation of \( W^a \) as a group of affine transformations in a smaller space. It will turn out that comparison of the two realisations of affine Weyl groups as linear groups and as affine linear groups can be effectively exploited.

Recall ([2] Ch IV, §1.9) that for any set \( I \), an \( I \)-indexed Coxeter matrix is any symmetric matrix of the form \( M = (m_{ij})_{i,j \in I} \), where \( m_{ij} \in \mathbb{Z}_{>0} \amalg \{\infty\} \), and \( m_{ij} = 1 \) if and only if \( i = j \). Let \( M \) be such a Coxeter matrix. The Coxeter system \((W_M, S_M)\) with Coxeter matrix \( M \) has a standard faithful representation, the “Tits representation”, as a reflection group of isometries of a space \( U \) with basis \( e_i \) for \( i \in I \). This representation is described in [2] Ch V, §4 and [12] 5.3–5.4. Denote by \( B_M \) the matrix \( B_M = (-\cos \frac{\pi}{m_{i,j}})_{i,j \in I} \). The space \( U \) has a symmetric \( \mathbb{R} \)-valued bilinear form \( \langle \cdot, \cdot \rangle \) satisfying \( \langle (e_i, e_j) \rangle_{i,j \in I} = B_M \). The simple reflections \( S_M \) are the orthogonal reflections \( r_i \) on \( U \) whose fixed hyperplanes are the \( e_i^\perp \) \( (i \in I) \); \( W_M \) is then the group of linear transformations of \( U \) generated by \( S_M \). There is an associated root system \( \Phi_M \) for \((W_M, S_M)\) defined by \( \Phi_M = W_M \Pi_M \) where \( \Pi_M = \{e_i | i \in I\} \).

3.6. Now take \( \hat{V} = V \oplus \mathbb{R}^\delta \) as above, and note that with respect to the form \( \langle \cdot, \cdot \rangle \), the set of isotropic elements of \( \hat{V} \) is \( \mathbb{R}^\delta \). For any non-isotropic \( \alpha \in \hat{V} \), we set \( \hat{\alpha} := \frac{1}{|\alpha|^2} \alpha \) where \( |\alpha|^2 := \langle \alpha, \alpha \rangle^{1/2} \). Define the map \( \iota : \hat{V} \setminus \mathbb{R}^\delta \to \hat{V} \setminus \mathbb{R}^\delta \) by \( \iota(\alpha) = \hat{\alpha} \). For any subset \( \Gamma \) of \( \hat{V} \setminus \mathbb{R}^\delta \), we set \( \Gamma^\vee = \{ \alpha^\vee | \alpha \in \Gamma \} \) and \( \hat{\Gamma} := \iota(\Gamma) = \{ \hat{\alpha} | \alpha \in \Gamma \} \).

Remark 2. If \( M \) is the Coxeter matrix of \((W^a, S^a)\) and \( W^a \) is indecomposable, then there is an identification \( U = \hat{V} \) (as real vector spaces with symmetric bilinear forms) such that \( \Phi_M = \hat{\Phi}, \Pi_M = \hat{\Pi} \) and \((W^a, S^a) = (W_M, S_M) \). This is not true if \( W^a \) is decomposable, since \( \hat{\Pi} \) is then linearly dependent, and \( \hat{V} \) is obtained from \( U \) by taking the quotient by an appropriate subspace of the radical of \( U \). The advantage of the root system \( \Phi^a \) is that its root subsystems may be treated in a
It is evident from the proof of Proposition \[8\] that the indecomposable components of the root system \(\Phi^a\) may be identified with the real root systems of untwisted affine Kac-Moody Lie algebras associated to the components of \(\Phi\) \[3]. If \(\Phi\) is not indecomposable, then \(\tilde{\Pi}\) is not linearly independent. However, \(\tilde{\Pi}\) is positively independent in the following sense: a subset \(\Sigma\) of a \(\mathbb{R}\)-vector space is said to be positively independent if \(\sum_{\gamma \in \Sigma} c_{\gamma} \gamma = 0\) with all \(c_{\gamma} \geq 0\) and almost all \(c_{\gamma} = 0\) implies that all \(c_{\gamma} = 0\). It follows that \((\tilde{\mathcal{V}}, \langle \cdot, \cdot \rangle, \tilde{\Pi})\) is a based root datum with root system \(\Phi^a\) in the sense of \[8\] with positive roots \(\Phi^a_+ = \iota(\Phi^a_+)\) and associated Coxeter system \((W^a, S^a)\). This is enough to ensure that standard facts about root systems of Coxeter groups apply to \(\Phi^a\); see \[8\] for properties of based root systems and \[8, 17\] for discussion of root systems closely related to those considered here.

The fact proved below will play an important role in this paper. Let \(M = (m_{\alpha, \beta})_{\alpha, \beta \in \Gamma}\) be Coxeter matrix where \(\Gamma\) is a finite set, \(B_M := \left( -\cos \frac{\pi}{m_{\alpha, \beta}} \right)_{\alpha, \beta \in \Gamma}\) be the matrix of the invariant form on the space of the Tits representation of the corresponding Coxeter group.

**Proposition 3.** Suppose that \(\Gamma \subseteq V \setminus \{0\}\) is indecomposable and linearly dependent, and that the matrix \(B := (\langle \alpha, \beta \rangle)_{\alpha, \beta \in \Gamma}\) is equal to \(B_M\) for some \(\Gamma\)-indexed Coxeter matrix \(M\) (cf. \[3, 5\]). Then there exists \(\theta \in \Gamma\) and scalars \(c_{\gamma} \in \mathbb{R}_{>0}\) for \(\gamma \in \Gamma\) with the following property: the set \(\Lambda := \{ c_{\gamma} \gamma \mid \gamma \in \Gamma \setminus \{\theta\}\}\) is the set of simple roots of an indecomposable, reduced crystallographic root system \(\Psi\) and \(-c_0 \theta\) is the highest root of \(\Psi\) with respect to \(\Lambda\). Moreover, the affine Weyl group of \(\Psi\) is naturally isomorphic to \(W_M\) and the set \(\{ c_{\gamma} \gamma \} \gamma \in \Gamma\) is uniquely determined up to multiplication by a positive scalar.

**Proof.** Consider the Coxeter matrix \(M = (m_{\alpha, \beta})_{\alpha, \beta \in \Gamma}\) and the associated reflection representation of \((W_M, S_M)\) in \(U\), a \(\mathbb{R}\)-vector space with basis \(e_\alpha\) for \(\alpha \in \Gamma\), as described in \[3, 5\] Set \(\Delta := \{ e_\alpha \mid \alpha \in \Gamma\}\). We have by construction that

\[(e_\alpha, e_\beta)_{\alpha, \beta \in \Gamma} = B.\]

If \(\alpha \neq \beta\), then \((e_\alpha, e_\beta) = -\cos \frac{\pi}{m_{\alpha, \beta}} \leq 0\). It follows from \[2,\ Ch\ V, \S 3,\ Lemme\ 5\], the subspace \(\mathbb{R}\Gamma\) has dimension \(|\Gamma| - 1\) and there is a linear relation \(\sum_{\alpha \in \Gamma} c_\alpha \alpha = 0\) which is unique up to multiplication by a non-zero scalar; moreover this relation may be uniquely chosen so that \(c_\alpha > 0\) and \(\sum_{\alpha \in \Gamma} c_\alpha = 1\). Hence in particular our assumptions imply that the matrix \(B\) is positive semidefinite of corank 1.

The linear map \(L: U \rightarrow V\) defined by \(e_\alpha \mapsto \alpha\) for \(\alpha \in \Gamma\) satisfies \(\langle L(u), L(u') \rangle = (u, u')\). The form \(\langle \cdot, \cdot \rangle\) on \(U\) is positive semidefinite and clearly \(y := \sum_{\alpha \in \Delta} c_\alpha e_\alpha \in U\) spans its radical. Following \[2,\ Ch\ V, \S 4.9\], let \(U^*\) denote the dual space of \(U\) and \(F\) denote the affine subspace \(F = \{ f \in U^* \mid f(y) = 1 \}\). This is a coset of the subspace \(T := \{ f \in U^* \mid f(y) = 0 \}\) of \(U^*\), so \(T\) acts simply transitively by translation on \(F\). Clearly \(T\) is canonically isomorphic to \((U/\mathbb{R}y)^*\). The form \(\langle \cdot, \cdot \rangle\) on \(U\) induces a natural positive definite inner product on the quotient \(U/\mathbb{R}y\), which may thus be naturally identified with \(T := (U/\mathbb{R}y)^*\), which therefore has a natural induced structure of finite-dimensional real inner product space. Note that since \(\langle y, e_\alpha \rangle = 0\) for all \(\alpha \in \Gamma\), \(W_M\) fixes \(y\), and hence acts on \(F\) as a group of
affine isometries. Let \( C := \{ f \in U^* \mid f(\Delta) \subseteq \mathbb{R}_{\geq 0} \} \) denote the (closed) standard fundamental chamber for \((W_M, S_M)\) acting on \(U^*\); it is a closed simplicial cone, with walls \( \{ f \in U^* \mid f(e_\alpha) = 0 \} \) for \( \alpha \in \Gamma \). The group \( W \) acts faithfully by restriction as a group of affine transformations of \( F \), and we shall therefore identify \( W \) with a group of affine transformations of \( F \) (cf. \[2\] Ch V, §4, Prop. 10]). Then \( W \) is a faithful, effective, indecomposable group generated by affine reflections. The intersection \( A := C \cap F \) is a closed fundamental alcove for \( W \) acting on \( F \). Note that \( A \) is a closed simplex in \( F \). Its walls are the intersections of \( F \) with the walls of \( C \) and hence are also indexed naturally by \( \Gamma \). Consider the unit normals (in \( T \)) to the walls of \( C \), in the direction of \( C \) from the wall. The matrix of inner products of these normals is \((e_\alpha, e_\beta)_{\alpha, \beta \in \Gamma}\) by \[2\] Ch V, §4, Prop. 10].

By \[2\] Ch V, §3, Prop. 12], we may chose a special point \( O \) of \( W_M \) on \( F \) such that \( O \in A \). Taking origin to be \( 0 = O \), the affine space \( F \) acquires the structure of a real Euclidean space, which is mapped isomorphically to \( T \) by translation by \(-O\). Denote the inner product on \( F \) as \((- \mid -\)) . It follows from \[2\] Ch VI, §2, Prop. 8], that \( W_M \) acting on \( F \) is the affine Weyl group of a reduced crystallographic root system \( \Psi \) in \( F \), as in Proposition \[1\] with Weyl group \( W_0 \). Since \( W_M \) is indecomposable, \( \Psi \) is indecomposable. Every closed alcove of \( W_M \) on \( F \) is contained in a closed chamber of \( W_0 \) on \( F \). Choose a set of simple roots \( \Lambda \) for \( \Psi \) such that the corresponding closed fundamental chamber \( C' \) for \( W_0 \) contains \( A \). Then \( A \) is the unique alcove of \( W_M \) in \( F \) which is contained in \( C' \) and which contains 0 in its closure. Hence \( A \) is the closed fundamental alcove for \((W_M, S_M)\). Let \( \omega_0 \) denote the highest root of \( \Psi \) with respect to \( \Lambda \). Then we have by Proposition \[1\] that

\[
A = \{ v \in F \mid (v \mid \beta) \geq 0 \text{ for all } \beta \in \Gamma \text{ and } (v \mid \omega_0) \leq 1 \}.
\]

It follows that the unit normals to the walls of \( A \) are \( \{ \hat{\beta} \mid \beta \in \Lambda' \} \) where \( \Lambda' := \Lambda \cup \{-\omega_0\} \).

This shows that the matrix \((\langle \hat{\alpha} \mid \hat{\beta} \rangle)_{\alpha, \beta \in \Lambda'} \) coincides (up to permutations of its rows and columns) with the matrix \((\langle \alpha, \beta \rangle)_{\alpha, \beta \in \Gamma} \). Since \( \Lambda' \) spans \( F \) and \( \Gamma \) spans \( \mathbb{R}\Gamma \), this implies that there is a unique identification of \( F = \mathbb{R}\Gamma \) as inner product spaces such that \( \{ \hat{\alpha} \mid \alpha \in \Gamma \} = \{ \beta \mid \beta \in \Lambda' \} \). Let \( \hat{\theta} \in \Gamma \) with \( \theta = \hat{\theta}_0 \). We may write \( \Lambda' = \{ c_\alpha \alpha \mid \alpha \in \Gamma \} \) for uniquely determined \( c_\alpha \in \mathbb{R}_{>0} \). All assertions of the Proposition are now clear.

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3.9. The following provides a Corollary-free proof of the claim at the end of the proof of Theorem \[1\].

**Corollary 3.** Let \( \Phi \) be a crystallographic root system, and let \( \Gamma \subseteq \Phi \) be an indecomposable subset, which is linearly dependent over \( \mathbb{R} \) and satisfies \( \langle \alpha, \beta \rangle \leq 0 \) for all distinct \( \alpha, \beta \in \Gamma \). Then

1. Let \( B := (\langle \hat{\alpha}, \hat{\beta} \rangle)_{\alpha, \beta \in \Gamma} \) and \( M = (m_{\alpha, \beta})_{\alpha, \beta \in \Pi} \) where \( m_{\alpha, \beta} \) denote the order of \( s_\alpha s_\beta \) if \( \alpha \neq -\beta \), and \( m_{\alpha, \beta} := \infty \) if \( \alpha = -\beta \). Then \( M \) is the Coxeter matrix of an indecomposable affine Weyl group, and \( B = B_M \).

1. Let \( W' := \langle s_\alpha \mid \alpha \in \Gamma \rangle \). There is an element \( \theta \in \Gamma \) such that \( \Gamma' := \Gamma \setminus \{ \theta \} \) is a simple system for the root system \( \Phi' \subseteq \Phi \) of \( W' \), and \( \theta \in \Phi' \).

**Proof.** Clearly \( M \) is a Coxeter matrix. Since \( \Phi \) is crystallographic, rank two considerations \[2\] Ch VI, §1.3 imply that \( \langle \hat{\alpha}, \hat{\beta} \rangle = -\cos \frac{\pi}{m_{\alpha, \beta}} \) for all \( \alpha, \beta \in \Gamma \). Hence \( B = B_M \), and thus we are in the situation of Proposition \[3\]. It follows from that
Proposition that $M$ is the Coxeter matrix of an indecomposable affine Weyl group, whose root system we denote by $W$. Let $\theta$ be as in the conclusion of Proposition 3.

Since $\Gamma' := \Gamma \setminus \{\theta\}$ is linearly independent, it is a simple system of some root subsystem $\Phi'$ of $\Phi$, by Lemma 1 and evidently $\Phi' := W'\Gamma'$ where $W' := \langle s_\alpha \mid \alpha \in \Gamma' \rangle$. Moreover since some multiple $c_0\theta \in \Phi'$, the reflection $s_\theta \in W'$ and hence $W' = \langle s_\alpha \mid \alpha \in \Gamma \rangle$. It follows also that there is some $\alpha \in \Phi'$ with $s_\alpha = s_\theta$. But $\alpha \in \Phi'$ and $s_\alpha = s_\theta$ implies $\alpha = \pm \theta$, whence $\theta \in \Phi'$. □

4. Subsystems of affine root systems

In this section, we classify the reflection subgroups of $W^a$ (a) up to equality and (b) up to isomorphism. The classification (b) was conjectured (in equivalent form) and partly proved by Coxeter [5, 6], and proved in [7]. The proof here via the stronger result (a) is along similar lines, but is much simpler and independent of the classification of affine Weyl groups. The classification in (a) is obtained in conjunction with a classification all simple subsystems of $\Phi^a$ (up to equality); the reflection subgroups are in bijective correspondence with those simple subsystems which are contained in a fixed positive subsystem of the ambient affine root system.

4.1. Let $\Phi$ be a crystallographic root system in $V$ and $\Phi^a$ the corresponding affine root system (in $\widehat{V}$, see Definition 3). For non-isotropic $v \in \widehat{V}$, we henceforth write $s_v$ for the orthogonal reflection in $v^\perp$, so that for $x \in \widehat{V}$, $s_v(x) = x - \langle x, \frac{2v}{(v,v)} \rangle v$. Note that this transformation $s_v$ is the 'canonical' set of Coxeter generators of $W^a$, see Definition 3. A root subsystem $\Sigma$ of $\Phi^a$ is a subset of $\Phi^a$ with the property that $s_\alpha(\beta) \in \Sigma$ for all $\alpha, \beta \in \Sigma$. The root subsystems of $\Phi^a$ are in natural bijective correspondence with the reflection subgroups of $W^a$ by the map which to a reflection subgroup $W'$ associates the set of all roots $\alpha$ in $\Phi^a$ such that $s_\alpha \in W'$; the inverse map associates to a root subsystem $\Sigma$ the subgroup $\langle s_\alpha \mid \alpha \in \Sigma \rangle$ (we use here simple consequences of properties of reflection subgroups of Coxeter systems from [7]).

The notion of a simple subsystem of $\Phi^a$ is more delicate than the corresponding notion for $\Phi$ (see [11, 6]). Consider the following set of real numbers.

$$\text{COS} := \{-\cos \frac{\pi}{m} \mid m \in \mathbb{N}_{\geq 2}\} \cup (-\infty, -1] \subseteq \mathbb{R}.$$ 

Let $\Sigma$ be a root subsystem of $\Phi^a$ with corresponding reflection subgroup $W'$. A subset $\Gamma$ of $\Phi^a$ is called a simple system for $\Sigma$ if $W' = \langle s_\alpha \mid \alpha \in \Gamma \rangle$. $\Gamma$ is positively independent and $\langle \tilde{\alpha}, \tilde{\beta} \rangle \in -\text{COS}$ for all distinct $\alpha, \beta \in \Gamma$. Then $\Sigma = W'\Lambda$.

Remark 3. In [6] the notion of abstract simple system is defined by replacing positive independence of $\Gamma$ by the weaker condition that each component of $\Gamma$ be positively independent. In this work we shall not make use of this concept.

For any simple system $\Lambda$ of $\Sigma$, $S_\Lambda := \{ s_\alpha \mid \alpha \in \Lambda \}$ is a set of Coxeter generators of $W'$. If $\Sigma \subseteq \Phi$, its simple systems in the above sense coincide with those in the sense of [11].

Note that any subset of $\Phi^a_+$ is positively independent. It therefore follows from the general theory of reflection subgroups of Coxeter systems ([7, 8]) that $\Sigma$ has a unique simple system $\Lambda$ contained in $\Phi_+$; we call this the canonical simple system of $\Sigma$ (or of the corresponding reflection subgroup). We write $\Lambda = \Delta_{W'}$. Then $S_\Lambda$ is the ‘canonical’ set of Coxeter generators of $W'$ (in the sense of [7]), with respect to the simple reflections $S^a$ of $W^a$. If $\Sigma$ is irreducible, then any two simple systems
of Σ are in the same \( W' \times \{ \pm 1 \} \)-orbit, as follows from [17], from [5.7] and [11], or from [6].

If \( \Delta \) is a simple system of some root subsystem \( \Sigma \) of \( \Phi^a \), we refer to \( \Delta \) as a simple subsystem of \( \Phi^a \). If \( \Delta \subseteq \Phi \), we write \( W_\Delta := \langle s_\alpha \mid \alpha \in \Delta \rangle \) for the subgroup of \( W \) generated by reflections corresponding to elements of \( \Phi \) and \( \Phi_\Delta = W_\Delta \Delta := \{ \alpha \in \Phi \mid s_\alpha \in W_\Delta \} \) for the corresponding root subsystem of \( \Phi \).

4.2. Let \( \pi \) denote the projection \( \hat{V} \to V \) with \( \delta \mapsto 0 \). Note that
\[ \pi(\Phi^a) \subseteq \Phi, \quad \langle \alpha, \beta \rangle = \langle \pi(\alpha), \pi(\beta) \rangle \text{ for } \alpha, \beta \in \Phi^a. \]
The next result follows immediately from this by well known properties [2, Ch VI, §2, no. 3] of rank two root subsystems of \( \Phi \).

**Lemma 3.**

(a) A subset \( \Delta \) of \( \Phi \) (resp. \( \Phi^a \)) is a simple subsystem of \( \Phi \) if and only if \( \Delta \) is positively independent, and \( \langle \alpha, \beta \rangle \leq 0 \) for all distinct \( \alpha, \beta \in \Delta \).

(b) A subset \( \Delta \) of \( \Phi^a_+ \) (resp., \( \Phi^a_0 \)) is the canonical simple system of some reflection subgroup \( W' \) of \( W \) (resp. \( W^a \)) if and only if \( \langle \alpha, \beta \rangle \leq 0 \) for all distinct \( \alpha, \beta \in \Delta \).

**Remark 4.** More generally, this Lemma applies mutatis mutandis to real root systems of Kac-Moody Lie algebras e.g. see [17] 5.5, Theorem 1.

4.3. **Negative product subsets.**

**Definition 4.** We say that a subset \( \Delta \) of \( \hat{V} \) is a np subset (negative product subset) if \( \langle \alpha, \beta \rangle \leq 0 \) for all distinct \( \alpha, \beta \in \Delta \).

**Lemma 4.**

(a) \( \Delta \subseteq \Phi^a \) is a np subset if and only if the restriction of \( \pi \) to \( \Delta \) is injective and \( \pi(\Delta) \) is a np subset of \( \Phi \).

(b) Given a np subset \( \Delta' \) of \( \Phi \), the np subsets \( \Delta \) of \( \Phi^a \) with \( \pi(\Delta) = \Delta' \) are the sets of the form \( \Delta = \{ \alpha + f(\alpha)\delta \mid \alpha \in \Delta' \} \) where \( f: \Delta' \to \mathbb{Z} \) is an arbitrary function.

(c) For \( \Delta \) and \( f \) as in (b), We have \( \Delta \subseteq \Phi^a_+ \) if and only if \( f(\alpha) \geq 0 \) for \( \alpha \in \Phi^a_+ \) and \( f(\alpha) > 0 \) for \( \alpha \in -\Phi^a_+ \).

**Proof.** This follows directly from (13). \( \square \)

4.4. As an immediate Corollary of Lemmas 3 and 4, we obtain a parameterisation of the reflection subgroups of \( W^a \) (or equivalently, root subsystems of \( \Phi^a \)) in terms of np subsets of \( \Phi \).

For any additive subgroup \( Z \) of \( \mathbb{R} \), let \( \mathcal{R}(\Phi, Z) \) denote the set of all pairs \( (\Gamma, f) \) where \( \Gamma \) is a np subset of \( \Phi \) and \( f: \Gamma \to Z \cap \mathbb{R}_{\geq 0} \) is a function such that \( f(\alpha) > 0 \) if \( \alpha \in -\Phi^a_+ \cap \Gamma \).

**Theorem 2.** The reflection subgroups of the affine Weyl group \( W^a \) are in natural bijective correspondence with \( \mathcal{R}(\Phi, Z) \). The correspondence attaches to \( (\Gamma, f) \in \mathcal{R}(\Phi, Z) \) the unique reflection subgroup \( W' \) of \( W^a \) with \( \Delta_{W'} = \{ \alpha + f(\alpha)\delta: \alpha \in \Gamma \} \).

We denote the reflection subgroup \( W' \) of \( W^a \) attached to the pair \( (\Gamma, f) \in \mathcal{R}(\Phi, Z) \) of the Theorem by \( W^a(\Gamma, f) \), its corresponding root system in \( \Phi^a \) by \( \Phi^a(\Gamma, f) \), and its canonical simple system by \( \Delta(\Gamma, f) \subseteq \Phi^a_+ \). A particular case is \( W^a = W^a(\Gamma_0, f_0) \), where \( (\Gamma_0, f_0) \in \mathcal{R}(\Phi, Z) \) is defined by \( \Gamma_0 = \pi(\Pi) \) and
\[ f_0(\alpha) := \begin{cases} 0, & \text{if } \alpha \in \Phi^a_+ \\ 1, & \text{if } \alpha \in -\Phi^a_+ \end{cases} \]
4.5. Classification of np subsets. The np subsets of Φ are described in the following Lemma.

**Lemma 5.** Let Γ′ be a simple subsystem of Φ with indecomposable components Γ′_i for i = 1, . . . , n. For each i, let Γ_i denote either Γ′_i or Γ′_i ∪ {α_i} where either −α_i is the highest root of ΦΓ_i′ or −α_i is the highest root of ΦΓ′_i (or both). Then Γ := ∪_i Γ_i is a np subset of Φ whose components are the Γ_i. Moreover any np subset Γ of Φ is obtained in this way from some simple subsystem Γ′ of Φ.

**Proof.** Well known properties of highest roots [2 Ch VI, §1. 8] imply that Γ as in the Lemma is a np subset of Φ. The converse follows by applying Proposition 5 to the indecomposable components Γ_i of Γ and using Lemma 2.

**Remark 5.** Suppose Γ is a np subset of Φ. The components Γ_i are uniquely determined by Γ, but Γ_i need not uniquely determine α_i and Γ′_i as above if Γ_i′ ≠ Γ_i. However, W_i := W_i Γ_i′, and any two choices of Γ′ (for fixed Γ) are in the same W_i orbit. See also Remark 13.

4.6. We now obtain the following known description (7) of the reflection subgroups of W^a, up to isomorphism as Coxeter systems.

**Theorem 3.** Consider a root subsystem Ψ of Φ with indecomposable components Ψ_i for i = 1, . . . , n. For each i, let W_i denote either the Weyl group of Ψ_i, the affine Weyl group corresponding to Ψ_i, or the affine Weyl group corresponding to Ψ_i′. Each W_i has a standard realisation as a Coxeter group, with simple reflections determined up to conjugacy.

The affine Weyl group W^a has a reflection subgroup isomorphic to W′ := W_1 × . . . × W_n as a Coxeter system. Conversely, any reflection subgroup of W^a is isomorphic to a Coxeter system W′ arising in this way.

**Proof.** Using Lemma 5, the isomorphism type of the reflection subgroup attached to (Γ, f) in Theorem 2 may be described as follows. Suppose Γ is obtained as in Lemma 5 from the simple subsystem Γ′ of Φ, and let Γ_i, Γ_i′ be as the sets of roots defined there. Then W′ has indecomposable components W_i′ with simple roots Δ_i = { α + f(α)δ : α ∈ Γ_i }, for i = 1, . . . , n. If Γ_i = Γ_i′, then W_i′ is isomorphic to the Weyl group of ΦΓ_i′. If Γ_i = Γ_i′ ∪ {α_i} where −α_i is the highest root of ΦΓ_i′, then W_i′ is isomorphic to the affine Weyl group of ΦΓ_i′. Finally, if Γ_i = Γ_i′ ∪ {α_i} where −α_i is the highest root of ΦΓ_i′, then W_i′ is isomorphic to the affine Weyl group of ΦΓ_i′.

4.7. The next Lemma gives necessary and sufficient conditions for a np subset Δ ⊆ Φ^a to be positively independent in terms of its parameterisation in Lemma 4(b).

**Lemma 6.** Let Γ be a np subset of Φ and f : Γ → ℤ be a function. Let Γ_i for i = 1, . . . , n denote the components of Γ. Let J denote the set of indices i with 1 ≤ i ≤ n and Γ_i linearly dependent. For each i ∈ J there are unique integers c_γ ∈ ℤ≥0 for γ ∈ Γ_i such that ∑_γ∈Γ_i c_γ γ = 0 and gcd({ c_γ | γ ∈ Γ_i }) = 1; we set d_i := ∑_γ∈Γ_i c_γ f_γ. Let Δ := c_γ f_γ | γ ∈ Γ }

(a) Δ is positively independent if and only if either d_i > 0 for all i ∈ J or d_i < 0 for all i ∈ J.
(b) each component of $\Delta$ is positively independent if and only if $d_i \neq 0$ for all $i \in J$.

Proof. Each $\Gamma_i$ is either linearly independent or there is a (real) linear relation among its elements, which is unique up to a scalar by [2] Ch V, §3, Lemma 5]. Since $\Gamma$ is contained in the lattice $\mathbb{Z}\Phi$, the coefficients of these linear relations can be taken in $\mathbb{Q}_{>0}$ by loc. cit., and then by rescaling, the existence and uniqueness of the $c_\gamma$ for $\gamma \in \Gamma_i$, $i \in J$ satisfying the conditions of the Lemma follows.

For any set of scalars $a_\gamma \in \mathbb{R}_{>0}$ ($\gamma \in \Gamma$), we have $\sum_\gamma a_\gamma \gamma + f(\gamma) \delta = 0$ if and only if $\sum_\gamma a_\gamma \gamma = 0$ and $\sum_\gamma a_\gamma f(\gamma) = 0$. Now by [2], $\sum_\gamma a_\gamma \gamma = 0$ if and only if there are non-negative scalars $b_i$ for $i \in J$, such that $a_\gamma = b_i c_\gamma$ if $\gamma \in \Gamma_i$ for some $i \in J$, and $a_\gamma = 0$ otherwise. For such scalars $a_\gamma$, we have $\sum_\gamma a_\gamma f(\gamma) = \sum_{i \in J} b_i d_i$. Hence $\Delta$ is positively independent precisely if, for all non-negative scalars $b_i$ ($i \in J$) which are not all zero, $\sum_{i \in J} b_i d_i \neq 0$. The assertion (a) follows immediately from this. The components of $\Delta$ are $\Delta_i := \{ \alpha + f(\alpha) \delta : \alpha \in \Gamma_i \}$ for $i = 1, \ldots, n$, so (b) follows from (a) applied to each $\Gamma_i$.

\[ \square \]

Definition 5. If the condition in (a) (resp. (b)) holds, we say that $(\Gamma, f)$ is compatible (resp. strongly compatible).

4.8. The following parameterisation of simple subsystems of $\Phi^a$ follows immediately from Lemma 6 and the definitions, in the same way as Theorem 2.

Proposition 4. The simple subsystems of the root system $\Phi^a$ of $W^a$ are in natural bijective correspondence with pairs $(\Gamma, f)$ where $\Gamma$ is a np subset of $\Phi$ and $f : \Gamma \rightarrow \mathbb{Z}$ is a function such that $(\Gamma, f)$ is compatible. The correspondence attaches to $(\Gamma, f)$ the simple subsystem $\{ \alpha + f(\alpha) \delta : \alpha \in \Gamma \}$ of $\Phi^a$.

Remark 6. We note that rank 1 simple subsystems of $\Phi^a$ correspond naturally to roots in $\Phi^a$. Theorem 2 and Proposition 4 directly generalize the descriptions ([5], Proposition 2) of the positive roots and roots of $\Phi^a$ in terms of those of $\Phi$.

4.9. Short and long roots. For any non-isotropic $v \in \hat{V}$, let $k_v = \langle v, v \rangle \in \mathbb{R}_{>0}$. If $\Psi$ is a subset of some indecomposable crystallographic root system in $V$, then $\{ k_v : v \in \Psi \}$ has either 1 or 2 elements. If this set has two elements, we denote them by $k_{\Psi, +}$ and $k_{\Psi, -}$ where $k_{\Psi, +} > k_{\Psi, -}$. Otherwise, we define $k_{\Psi, +} := k_{\Psi, -} := k_v$ for any $v \in \Psi$. Set $k_\Psi := k_{\Psi, +} / k_{\Psi, -}$. It is known from [2] that $k_\Psi \in \{1, 2, 3\}$. Write $\Psi_{\text{short}} := \{ v \in \Psi \mid \langle v, v \rangle = k_{\Psi, -} \}$ and $\Psi_{\text{long}} := \{ v \in \Psi \mid \langle v, v \rangle = k_{\Psi, +} \}$.

Using the above notation and that of Lemma 5 we now define $k_{\beta, \Gamma} \in \{1, 2, 3\}$ for any np subset $\Gamma$ and element $\beta \in \Gamma$. There is a unique component $\Gamma_i$ in which $\beta$ lies. If $\Gamma_i = \Gamma_i \setminus \beta$, we set $k_{\beta, \Gamma} := 1$. Otherwise, $\Gamma_i \neq \Gamma_i \setminus \beta$, and we set

\[ k_{\beta, \Gamma} := \begin{cases} 1, & \text{if } k_{\alpha_i} = k_{\Gamma_i, +} \\ \frac{k_{\alpha_i}}{k_{\alpha_i}} \in \{1, k_{\Gamma_i}\} & \text{if } k_{\alpha_i} = k_{\Gamma_i, -}. \end{cases} \]

4.10. The technical lemma below describes the root coefficients in certain root subsystems of $\Phi^a$. Let $\Gamma$ be an indecomposable np subset of $\Phi$. Let $\Gamma' := \Gamma$ if $\Gamma$ is linearly independent. Otherwise, write $\Gamma = \Gamma' \cup \{-\alpha\}$ where $\Gamma'$ is an indecomposable simple subsystem of $\Phi$ and $\langle \alpha, \Gamma' \rangle \subseteq \mathbb{R}_{\geq 0}$. Let $\Sigma$ be the root subsystem of $\Phi$ with simple system $\Gamma'$.
Lemma 7. Maintain the above notation. If \( \Gamma = \Gamma' \), set \( \Delta' := \Gamma' \) otherwise, set \( \Delta' := \Gamma' \cup \{ \omega \} \subseteq \Phi^\alpha \) where \( \omega := \delta - \alpha \). Let \( W' := \langle s_\gamma \mid \gamma \in \Delta' \rangle \). Then \( W' \) is a reflection subgroup of \( W^\alpha \) having \( \Delta' \) as a simple system. Let \( \Psi' := W' \Delta \) denote the indecomposable root subsystem of \( \Phi^\alpha \) corresponding to \( \Delta' \).

(a) Define coefficients \( a_{\beta,\gamma} \), for \( \beta \in \Sigma \) and \( \gamma \in \Gamma' \), by \( \beta = \sum_{\gamma \in \Gamma'} a_{\beta,\gamma} \gamma \). If \( \Gamma \) is linearly dependent, then

\[
\beta + n\delta = n\omega + \sum_{\gamma \in \Gamma'} (a_{\beta,\gamma} + na_{\alpha,\gamma})\gamma
\]

for \( \beta \in \Sigma \) and \( n \in \mathbb{Z} \).

(b) If \( \Gamma \) is linearly independent, then \( \Psi' = \Sigma \).

(c) If \( \Gamma \) is linearly dependent then \( \Psi' = \{ \beta + nk_{\beta,\Gamma}\delta \mid \beta \in \Sigma, n \in \mathbb{Z} \} \).

\[\text{Proof.} \] Parts (a) and (b) are trivial. In (c), the case \( \Gamma \neq \Gamma' \) with \( k_\alpha = k_{\Gamma,+} \) is a special case of Proposition 2 since then all \( k_{3,\Gamma} = 1 \). The remaining case is that in which \( \Delta \neq \Gamma' \) and \( k_\alpha = k_{\Gamma,-} \). Set \( b_\beta = \frac{2}{k_{\beta,\Gamma}} \) for \( \beta \in \Sigma \), so \( \beta' = b_\beta \beta \). We have \( k_{\alpha'} = k_{\alpha,\Gamma} \). Consider \( \Delta'' = \Gamma^\prime \cup \{ b_\alpha \delta - \alpha \} \). Here, \( \delta' := b_\alpha \delta \) plays the analogous role for \( \Delta'' \) to that played by \( \delta \) for \( \Delta' \). By the case of (c) previously considered, the root system \( \Psi'' \) corresponding to \( \Delta'' \) is

\[\Psi'' = \{ \beta + n\delta' \mid \beta \in \Sigma', n \in \mathbb{Z} \} = \{ \beta' + nb_\alpha \delta \mid \beta \in \Sigma, n \in \mathbb{Z} \} \]

Hence

\[\Psi' = \{ (\beta' + nb_\alpha \delta)'' \mid \beta \in \Sigma, n \in \mathbb{Z} \} = \{ \beta + nb_\alpha \delta \mid \beta \in \Sigma, n \in \mathbb{Z} \} \]

which is as required since \( b_\alpha = \frac{k_\alpha}{k_\beta,\Gamma} \).

\[
4.11 \text{ Let } \Gamma \text{ be a np subset of } \Phi \text{ and let } f : \Gamma \to \mathbb{Z} \text{ such that } (\Gamma, f) \text{ is compatible. We assume that } \Gamma \text{ arises as in Lemma 3 from a simple system } \Gamma' \subseteq \Gamma \text{ and use the notation } \Gamma_i, \Gamma_i', \alpha_i \text{ introduced there. For each } i \text{ with } \Gamma_i' \neq \Gamma_i, \text{ there is a unique linear relation } \sum_{\gamma \in \Gamma_i} c_\gamma \gamma = 0 \text{ with } c_\alpha = 1, \text{ and in this relation, we have } c_\gamma \in \mathbb{N}_{>0} \text{ for all } \alpha \in \Gamma_i. \text{ Let } \Sigma = \Phi_{\Gamma_i} \text{ and } \Sigma_i = \Phi_{\Gamma_i'} \text{; thus } \Sigma \text{ is a root system with indecomposable components } \Sigma_i \text{ of which } \Gamma' \text{ is a simple system, and } \Gamma' \subseteq \Sigma \subseteq \Sigma_i. \text{ As in Lemma 3, for each } \beta \in \Sigma, \text{ write } \beta = \sum_{\gamma \in \Gamma,} a_{\beta,\gamma} \gamma. \text{ Then } a_{\beta,\gamma} \in \mathbb{Z}, a_{\beta,\gamma} = 0 \text{ unless } \beta \in \Sigma_i \text{ and } \gamma \in \Gamma_i' \text{ for some } i, \text{ and for fixed } \beta \in \Sigma, \text{ either all } a_{\beta,\gamma} \text{ are non-negative or they are all non-positive. By Proposition 2, } \Delta := \{ \gamma + f(\gamma)\delta \mid \gamma \in \Gamma \} \text{ is a simple subsystem of } \Phi^\alpha. \text{ Let } W'' := \langle s_\gamma \mid \gamma \in \Delta \rangle \text{ be the associated Coxeter group, and } \Psi := W'' \Delta \text{ be the corresponding root system. Define } r_\beta \in \mathbb{Z} \text{ for all } \beta \in \Sigma \text{ by } r_\beta := \sum_{\gamma \in \Gamma'} a_{\beta,\gamma} f(\gamma) = -r_{-\beta}. \text{ We also define } K_i \in \mathbb{Z} \text{ for } i = 1, \ldots, n \text{ as follows. If } \Gamma_i = \Gamma_i', \text{ we set } K_i = 0. \text{ Otherwise, set } K_i := \sum_{\gamma \in \Gamma_i} c_\gamma f(\gamma) = f(\alpha_i) + r_{-\alpha_i}. \]

Proposition 5. Let \( \Gamma \) be a np subset of \( \Phi \) and let \( f : \Gamma \to \mathbb{Z} \) a function such that \( (\Gamma, f) \) is compatible. Then the root system \( \Psi \) of which \( \Delta := \{ \alpha + f(\alpha)\delta \mid \alpha \in \Gamma \} \) is a simple system is given by

\[\Psi = \{ \beta + (r_\beta + mK_i k_{3,\Gamma})\delta \mid i = 1, \ldots, n, \beta \in \Sigma_i, m \in \mathbb{Z} \} \]
Remark 7. The above result applies in particular in the case that $\Delta \subseteq \Phi^a_\Gamma$, when $\Psi = \Phi^a(\Gamma, f)$. In that case, we have $r_\beta \geq 0$ for all $\beta \in \Sigma$ which are positive with respect to $\Gamma'$. If, further, $\Gamma_i' \neq \Gamma_i$, then using [2] Ch VI, §1, Prop. 25] for $\Sigma$ and $\Sigma'_\Gamma$, we find that $0 \leq r_\beta \leq K_i k_{\beta, \Gamma}$ for all $\beta \in \Sigma_i$ which are positive with respect to $\Gamma_i'$.

Proof. The proof easily reduces to the case where $\Gamma$ is indecomposable, which we henceforth assume. Let $\Delta'$, $\Psi'$ be as in Lemma 7. There is an isometry $i : \mathbb{R}\Delta' \to \mathbb{R}\Delta$, mapping the basis $\Delta'$ of the left side to the basis $\Delta$ of the right side; more precisely, $\gamma \mapsto \gamma + f(\gamma)\delta$ for $\gamma \in \Gamma'$, and, if $\Gamma' \neq \Gamma$, $\delta - \alpha \mapsto f(-\alpha)\delta - \alpha$. Clearly, this isometry induces an isomorphism $W' : W(\Gamma, f)$ of Coxeter systems and maps $\Psi'$ bijectively to $\Psi$. From the expressions for elements of $\Psi'$ as linear combinations of the simple roots $\Delta'$ in Lemma 7 we obtain corresponding expressions for elements of $\Psi$ as linear combinations of simple roots $\Delta$. One readily checks that expanding the latter linear combinations gives the desired result, using the definitions of $r_\beta$, $K_i$ and $k_{\beta, \Gamma}$.

4.12. Weyl data, alcoves and compatible pairs. For the remainder of this section, fix $(\Gamma, f) \in \mathcal{R}(\Phi, \mathbb{Z})$. We shall show how to explicitly describe in terms of the data $(\Gamma, f)$ the fundamental alcove of $W' : W(\Gamma, f)$, the volume of an alcove, the index $[W^a : W']$, and the minimal length (say, left) coset representatives of $W'$ in $W^a$ to describe these results, we maintain all the notation from the last subsection.

It is clear that $\Phi^a(\Gamma, f)$ is finite (equivalently, $W^a(\Gamma, f)$ is finite) if and only if $\Gamma = \Gamma'$. Also, $W^a(\Gamma, f)$ has only affine type components (i.e. no finite components) if and only if $\Gamma_i \neq \Gamma_i'$ for all $i$. Finally, $W^a(\Gamma, f)$ is of finite index in $W^a$ if and only if its translation subgroup is of finite index in that of $W^a$, which is the case if and only if $\Gamma_i \neq \Gamma_i'$ for all $i$ and $\mathbb{R}\Phi = \mathbb{R}\Gamma'$.

4.13. The fundamental coweights $\omega_\alpha := \omega_\alpha(\Gamma')$ for $\alpha \in \Gamma'$ are defined as the basis of $\mathbb{R}\Gamma'$ which is dual to $\Gamma'$ with respect to $\langle -, - \rangle$. Recall that $f_\Gamma$ denotes the index of connection of $\Phi\Gamma'$ (= $[\sum_{\gamma \in \Gamma'} \mathbb{Z} \omega_\gamma : \sum_{\gamma \in \Gamma'} \mathbb{Z} \gamma]$) and let $r_i := |\Gamma_i'|$ denote the rank of $\Phi\Gamma_i'$.

Lemma 8. (a) The closed fundamental alcove for $W^a(\Gamma, f)$ on $V$ is

$$C = C(\Gamma, f) := \{ v \in V \mid \langle v, \gamma \rangle + f(\gamma) \geq 0 \text{ for all } \gamma \in \Gamma \}.$$

(b) $C = \bigoplus_{i=0}^n C_i$ (orthogonal direct sum of subsets) where $C_0$ is the subspace $C_0 := \{ v \in V \mid \langle v, \Gamma \rangle = 0 \}$ and for $i = 1, \ldots, n$,

$$C_i = \{ v \in \mathbb{R}\Gamma_i' \mid \langle v, \gamma \rangle + f(\gamma) \geq 0 \text{ for all } \gamma \in \Gamma_i \}.$$

(c) For $i = 1, \ldots, n$ let $v_{i,0} := -\sum_{\alpha \in \Gamma_i'} f(\alpha)\omega_\alpha$. If $\Gamma_i' = \Gamma_i$, then $C_i$ is the simplicial cone $v_{i,0} + \sum_{\alpha \in \Gamma_i} \mathbb{R}_{\geq 0} \omega_\alpha$ with vertex $v_{i,0}$, while if $\Gamma_i' \neq \Gamma_i$, then $C_i$ is the simplex with vertices $v_{i,0}$ and $v_{i,0} + K_i \omega_\alpha$ for $\alpha \in \Gamma_i'$.

Proof. We refer to [13] Proposition 3.12], [2] Ch V, §4, no. 4–6], [17] 5.6] and [8] for basic facts concerning fundamental chambers and Tits cones used below. The closed fundamental chamber for $W^a(\Gamma, f)$ acting on $\tilde{V}$ is

$$\tilde{C} := \{ \phi \in \tilde{V}^* \mid \phi(\alpha + f(\alpha)\delta) \geq 0 \text{ for all } \alpha \in \Gamma \}.$$
assumed we are in this situation. Let 

\[ \mu \]

be the measure normalised in a similar way to orthogonal. Consider the full-dimensional parallelepiped \( P \) in \( \mathbb{R}^m \), which is a vertex of the change of variables formula in multiple integrals, its volume is equal to

\[ \mu(P) = \sqrt{\det(\langle \omega_\alpha, \omega_\beta \rangle)_{\alpha,\beta \in \Gamma_i}} \]

where \( \mu \) is the measure on \( \mathbb{R}^m \) with sides of unit length (using the metric arising from \( \langle -,- \rangle \)) is 1, where \( m = \dim(V) \). Recall that \( (\Gamma, f) \) is a fixed element of \( \mathcal{R}(\Phi, \mathbb{Z}) \), which defines a root subsystem of \( \Phi^a \) as in Proposition 1. By Lemma 8 the chamber \( C \) is a fundamental domain for the action of \( W(\Phi^a(\Gamma, f)) \) on \( V \).

4.14. Volume. We maintain the notation of the above section and in addition, consider Lebesgue measure \( \mu \) on \( V \), normalised so that the measure of an \( m \)-dimensional hypercube in \( V \) with sides of unit length (using the metric arising from \( \langle -,- \rangle \)) is 1, where \( m = \dim(V) \). Recall that \( (\Gamma, f) \) is a fixed element of \( \mathcal{R}(\Phi, \mathbb{Z}) \), which defines a root subsystem of \( \Phi^a \) as in Proposition 1. By Lemma 8, the chamber \( C \) is a fundamental domain for the action of \( W(\Phi^a(\Gamma, f)) \) on \( V \).

Corollary 4. (a) If the measure \( \mu(C) \) is finite, then

\[ \mu(C) = \frac{1}{\prod_{\beta \in \Gamma_i} 2^{\langle \beta, \beta \rangle}} \prod_{i} K_i \prod_{c \in \Phi^a(\Gamma_i)} c \alpha \]

(b) \( \mu(C) \) is finite if and only if the index \([W^a : W^a(\Gamma, f)]\) is finite.

(c) Suppose also \((\Gamma', f') \in \mathcal{R}(\Phi, \mathbb{Z}) \) with \( W^a(\Gamma, f) \subseteq W^a(\Gamma', f') \) and that both \( \mu(C(\gamma, f)) \) and \( \mu(C(\Gamma', f')) \) are finite. Then

\[ [W^a(\Gamma', f') : W^a(\Gamma, f)] = \frac{\mu(C(\Gamma, f))}{\mu(C(\Gamma', f'))} \]

Remark 8. Since \( W^a = W^a(\Gamma_0, f_0) \), the measure \( \mu(A) \) of the fundamental alcove of \( W^a \) is finite and may be computed as a special case of the formula in (a)

Proof. If the measure \( \mu(C) \) is finite, then \( \mathbb{R}\Phi = \mathbb{R}\Gamma \) and for all \( i \), \( \Gamma_i' \neq \Gamma_i \). We assume we are in this situation. Let \( \mu_i \) denote Lebesgue measure on \( \mathbb{R}\Gamma_i \), normalised in a similar way to \( \mu \). We have \( \mu(C) = \prod_{i} \mu(C_i) \) since the \( C_i \) are pairwise orthogonal. Consider the full-dimensional parallelepiped \( P_i \) in \( \mathbb{R}\Gamma_i \) with 0 as one vertex and with the edges incident with 0 given by endpoints \( \omega_\alpha \) for \( \alpha \in \Gamma_i' \). By the change of variables formula in multiple integrals, its volume is

\[ \mu_i(P_i) = \sqrt{\det(\langle \omega_\alpha, \omega_\beta \rangle)_{\alpha,\beta \in \Gamma_i'}} \]
(this is also a well-known formula of Hadamard). By expressing the elements of \( \Gamma' \) and the corresponding coroots and fundamental coweights in terms of an orthonormal basis of \( \mathbb{R} \Gamma_1 \) and using (3), one checks that the determinant under the square root sign is equal to

\[
\frac{1}{f_{\Gamma_1'}} \prod_{\beta \in \Gamma_1'} \frac{2}{\langle \beta, \beta \rangle}
\]

On the other hand, it is easy to see using Lemma 8(c) and the change of variable formula that

\[
\mu_i(C_i) = \mu_i(P_i) \prod_i \frac{K_{n_i}}{n_i!} \prod_{\alpha \in \Gamma_1'} c_{n_i}.
\]

Combining the above facts, using \( f_{\Gamma_1'} = \prod_i f_{\Gamma_1'} \), gives (a). Parts (b) and (c) follow readily from [2, Ch VI, §2, Lemma 1].

4.15. From [7], the set \( \{ w \in W^a \mid w\Delta(\Gamma, f) \subseteq \Phi^+_a \} \) is known to be the set of minimal length left coset representatives of \( W^a(\Gamma, f) \) in \( W^a \) (where length is with respect to the standard length function of \( (W^a, S^a) \)). The next result explicitly describes a set of left coset representatives of \( W^a(\Gamma, f) \) in \( \tilde{W} = W \ltimes R \) (with \( R \) as in 3.4), reducing to the above set if \( \tilde{W} = W^a \) i.e. if \( R = Q \).

**Proposition 6.** Let \( G := \{ w \in \tilde{W} \mid w\Delta(\Gamma, f) \subseteq \Phi^+_a \} \)

(a) For \( w \in W \) and \( \gamma \in R \), we have \( wt \gamma \in G \) if and only if for all \( \alpha \in \Gamma \),

\[
\begin{cases}
\langle \alpha, \gamma \rangle \geq -f(\alpha), & \text{if } w(\alpha) \in \Phi_+
n\langle \alpha, \gamma \rangle \geq 1 - f(\alpha), & \text{if } w(\alpha) \in \Phi_-
\end{cases}
\]

(b) For \( x \in \tilde{W} \), we have \( x \in G \) if and only if \( x^{-1}(A) \subseteq C \). Further, we have

\( C = \cup_{x \in G} x^{-1}(A) \).

(c) \( G \cong \tilde{W}/W^a(\Gamma, f) \) is a set of left coset representatives of \( W^a(\Gamma, f) \) in \( \tilde{W} \).

(d) \( |G| = [W^a : W^a(\Gamma, f)] \cdot [R : Q] \).

**Proof.** Part (a) follows easily from the definition of \( \Phi^a(\Gamma, f) \) and [10]. Note that \( W^a \) acts simply transitively on the set of its alcoves and \( \tilde{W} = W \ltimes \tau_R \) acts on the set of alcoves. It follows that \( \tilde{W} \) is the semi-direct product of its normal subgroup \( W^a \) with the subgroup

\[
\{ w \in \tilde{W} \mid w(A) = A \} = \{ w \in \tilde{W} \mid w(\Pi) = \Pi \} \cong R/Q,
\]

and this reduces the proofs of (b)–(d) to the case \( \tilde{W} = W^a \) where the statements are standard or trivial. \( \square \)

5. **Subsystems of affine root systems II**

In the previous section, root subsystems of \( \Phi^a \) were classified using the concept of negative product (np) set, which arises naturally in considering simple subsystems of affine root systems. In this section we give an alternative classification in terms of collections of subsets of \( \mathbb{Z} \).
5.1. For subsets $A$, $B$ of any abelian group, define $A \pm B := \{ a \pm b \mid a \in A, b \in B \}$ and $nA = \{ na \mid a \in A \}$ for $n \in \mathbb{Z}$. Note that $2A \neq A + A$ in general.

Any subset $\Psi$ of $\Phi^a$ may be described as $\Psi := \{ \alpha + n\delta \mid \alpha \in \Phi, n \in Z_{\alpha} \}$ where, for each $\alpha \in \Phi$, $Z_{\alpha}$ is a subset of $\mathbb{Z}$. Given subsets $Z_{\alpha}$ for $\alpha \in \Phi$, the next lemma gives necessary and sufficient conditions on the $Z_{\alpha}$ in order that the corresponding set $\Psi$ be a root subsystem of $\Phi^a$.

Lemma 9. Let $\{ Z_{\alpha} \mid \alpha \in \Phi \}$ be a family of (possibly empty) subsets of $\mathbb{Z}$. Then $\Psi := \{ \alpha + n\delta \mid \alpha \in \Phi, n \in Z_{\alpha} \}$ is a root subsystem of $\Phi^a$ if and only if, for all $\alpha, \beta \in \Phi$, we have

$$Z_{\beta} - \langle \beta, \alpha^\vee \rangle Z_{\alpha} \subseteq Z_{s_{\alpha}(\beta)}.$$

Proof. This follows directly from (9) and the definition of a root subsystem. \(\square\)

5.2. The root subsystems of $\Phi^a$ may be described in terms of solutions of (16) as follows.

Corollary 5. The root subsystems of $\Phi^a$ are in bijective correspondence with pairs $(\Psi, \{ Z_{\alpha} \}_{\alpha \in \Psi})$ where $\Psi$ is a root subsystem of $\Phi$ and $\{ Z_{\alpha} \}_{\alpha \in \Psi}$ is a family of non-empty subsets of $\mathbb{Z}$ satisfying (10) for all $\alpha, \beta \in \Psi$. The correspondence attaches to $(\Psi, \{ Z_{\alpha} \}_{\alpha \in \Psi})$ the root subsystem $\Phi^a (\Psi, \{ Z_{\alpha} \}_{\alpha \in \Psi}) := \{ \alpha + n\delta \mid \alpha \in \Phi, n \in Z_{\alpha} \}$ of $\Phi^a$.

Proof. Observe that for any solution $\{ Z_{\alpha} \}_{\alpha \in \Phi}$ of (16), $\{ \alpha \in \Phi \mid Z_{\alpha} \neq \emptyset \}$ is clearly a root subsystem $\Psi$ of $\Phi$; hence the corresponding root subsystem of $\Phi^a$ is of the required form. Conversely, if $\Psi$ is a subsystem of $\Phi$ and $\{ Z_{\alpha} \}_{\alpha \in \Psi}$ satisfy (10) for all $\alpha, \beta \in \Psi$, a solution of (16) is obtained by taking $Z_{\alpha} = \emptyset$ for $\alpha \in \Phi \setminus \Psi$. The corresponding root subsystem of $\Phi^a$ is then $\Phi^a (\Psi, \{ Z_{\alpha} \}_{\alpha \in \Psi})$. \(\square\)

Note that under the above bijection, the root subsystems of $\Phi$ correspond to the case where each non-empty subset $Z_{\alpha} = \{ 0 \}$.

5.3. The above description of root subsystems is convenient for studying the inclusion relations in the poset of root subsystems of $\Phi^a$. In fact, it is obvious that

$$\Phi^a (\Psi, \{ Z_{\alpha} \}_{\alpha \in \Psi}) \subseteq \Phi^a (\Psi', \{ Z'_{\alpha} \}_{\alpha \in \Psi'}) \iff \Psi \subseteq \Psi' \text{ and } Z_{\alpha} \subseteq Z'_{\alpha} \text{ for all } \alpha \in \Psi'.$$

The next lemma describes the natural action of the extended affine Weyl group on the root subsystems of $\Phi^a$. This corresponds to the conjugation action of $W^a$ on the set of reflection subgroups of its normal subgroup $W^a$. The action is described for reflections in $W$, and translations, which together generate $W^a$.

Lemma 10. Let $w \in W^a$ and write $w (\Phi^a (\Psi, \{ Z_{\alpha} \}_{\alpha \in \Psi})) = \Phi (\Psi', \{ Z'_{\alpha} \}_{\alpha \in \Psi'})$.

(a) If $w \in W$, then $\Psi' = w (\Psi)$ and $Z'_{\alpha} = Z_{w^{-1} (\alpha)}$ for $\alpha \in \Psi'$.

(b) If $w = t_\gamma$, where $\gamma \in P (\Phi)$ (the coweight lattice), then $\Psi' = \Psi$ and $Z'_{\alpha} = Z_{\alpha} + \langle \alpha, \gamma \rangle$ for all $\alpha \in \Psi'$.

Proof. This follows directly from the definitions and (10). \(\square\)
5.4. For any indecomposable subset $\Gamma$ of $\Phi$, define another subset $\Gamma^o$ of $V$ as follows:

$$\Gamma^o = \{ \alpha \mid \alpha \in \Gamma_{\text{short}} \} \cup \{ \frac{1}{k_\Gamma} \alpha \mid \alpha \in \Gamma_{\text{long}} \}$$

(see [19] for the notation). If $\Psi$ is an indecomposable root system in $V$ with simple system $\Gamma$, then $\Psi^o$ is a root system in $V$ (homothetic to $\Psi^\vee$, and hence isomorphic to $\Psi^\vee$) with simple system $\Gamma^o$, and $W_\Phi = W_{\Psi^o}$.

5.5. Fix an additive subgroup $Z$ of $\mathbb{R}$. Let $\Psi$ be a root subsystem of $\Phi$, with components $\Psi_1, \ldots, \Psi_n$.

A subgroup $X$ of $P_Z(\Psi)$ is said to be a $Z$-admissible coweight lattice for $\Psi$ if it is of the form $X = \bigoplus_{i=1}^n X_i \subseteq P(\Psi)$ where either $X_i = 0$, in which case we set $m_i = 0$, or $X_i = m_i P(\Psi_i)$ for some $m_i \in Z \cap \mathbb{R}_{>0}$.

We shall require some facts about the elements $n_\alpha \in \mathbb{N}$ defined by $\langle \alpha, X \rangle = n_\alpha Z$ for all $\alpha \in \Psi$. Choose a simple system $\Delta_i$ for $\Psi_i$ and set $\Delta := \cup_i \Delta_i$. Let $\omega_\alpha = \omega_\alpha(\Delta)$ for $\alpha \in \Delta$ denote the corresponding fundamental coweights. We have $X = \bigoplus_{\alpha \in \Delta} n_\alpha Z_\alpha$ where for $\alpha \in \Delta_i$, we set $n_\alpha = m_i k_{\Psi_i}$ if $X_i = m_i P(\Psi_i)$ and $\alpha$ is long in $\Psi_i$, and otherwise set $n_\alpha = m_i$. It follows that if $\alpha \in \Delta$, we have $n_\alpha = n'_\alpha$. Since $n_\alpha = n_w(\alpha)$ for $w \in W_\Psi$, by the $W_\Psi$-invariance of $P(\Psi_i)$ and $P(\Psi_i^o)$, it follows that

$$n_\alpha = m_i k_{i,\alpha}$$

for all $\alpha \in \Psi_i$ where $k_{i,\alpha} = k_{\Psi_i}$ if $\alpha$ is long in $\Psi_i$ and $X_i = m_i P(\Psi_i^o)$, and $k_{i,\alpha} = 1$ otherwise.

It follows also that

$$X = \{ p \in P(\Psi) \mid \langle p, \alpha \rangle \leq n_\alpha Z \text{ for all } \alpha \in \Psi \}.$$ 

For the left hand side is clearly contained in the right. Conversely, if an element $x = \sum_{\alpha \in \Delta} a_\alpha \omega_\alpha$ of $P(\Psi)$ is in the right hand side, then for $\alpha \in \Delta_i$ we have $a_\alpha = \langle x, \alpha \rangle \in n_\alpha Z = n_\alpha Z$ and so $x \in X$. We shall also require the following divisibility property, which is a consequence of (17).

$$\langle \beta, \alpha^\vee \rangle n_\alpha Z \subseteq n_\beta Z$$

for $\alpha, \beta \in \Psi$. This is trivial except in the situation where $\alpha, \beta$ are in the same component $\Psi_i$, are not orthogonal, $X_i = m_i P(\Psi_i^o)$ and $\beta$ is long and $\alpha$ is short in $\Psi_i$; but then $\langle \beta, \alpha^\vee \rangle = k_{\Psi_i}$ by rank two considerations and the result remains valid.

The $Z$-admissible coweight lattice (for $\Psi$) corresponding to $X$ is defined to be the group $Y = \sum_i Y_i$ where $Y_i = 0$ if $X_i = 0$, and $Y_i = m_i Q(\Psi_i)$ (resp., $Y_i = m_i Q(\Psi_i^o)$ if $X_i = P(\Psi_i)$ (resp., $X_i = P(\Psi_i^o)$). This gives a natural bijective correspondence between $Z$-admissible coweight lattices and $Z$-admissible coweight lattices of $\Psi$.

5.6. Second parameterisation. The next theorem gives our second main parameterisation of the root subsystems of $\Phi^o$. To formulate it, let $R'(\Phi, Z)$ denote the set of all pairs $(\Psi, X)$ where $\Psi$ is a root subsystem of $\Phi$ and $X \in P_Z(\Psi)/X'$ is a coset $X = a + X'$, where $a \in P_Z(\Psi)$, of some $Z$-admissible coweight lattice $X'$ of $\Psi$. 
**Theorem 4.** There is a natural bijection between $\mathcal{R}'(\Phi, Z)$ and the set of root subsystems of $\Phi^\alpha$. The bijection attaches to $(\Psi, X) \in \mathcal{R}'(\Phi, Z)$ the root subsystem $\Phi^\alpha(\Psi, X) := \Phi^\alpha(\Psi, \{Z_\alpha\}_{\alpha \in \Psi})$ where $Z_\alpha := \{\{\alpha, x\} \mid x \in X\} \subseteq Z$ and $\Phi^\alpha(\Psi, \{Z_\alpha\}_{\alpha \in \Psi}) := \{\alpha + n\delta \mid \alpha \in \Psi, n \in Z_\alpha\}$.

**Proof.** First we show that there is a map $g: (\Psi, X) \mapsto \Phi^\alpha(\Psi, X)$ as indicated from $\mathcal{R}'(\Phi, Z)$ to the set of root subsystems of $\Phi^\alpha$. Fix $(\Psi, X) \in \mathcal{R}'(\Phi, Z)$ and define $Z_\alpha := \{\alpha, X\} \subseteq Z$ for $\alpha \in \Psi$. Write $X = a + X'$ where $X'$ is an admissible coweight lattice for $\Psi$ and $a \in P(\Psi)$. Note that $Z_\alpha = \{a, \alpha\} + Z_{\alpha a}$ for the unique integer $n_\alpha \geq 0$, such that $n_\alpha Z = \langle X', \alpha \rangle$. It is easy to check from (17) and (19) that the pair $(\Psi, \{Z_\alpha\}_{\alpha \in \Psi})$ satisfies the condition of Corollary 5 and so determines a root subsystem $g(\Psi, X) = \Phi^\alpha(\Psi, \{Z_\alpha\}_{\alpha \in \Psi})$ as in Corollary 5.

We now show that $g$ is injective. Fix $(\Psi, X) \in \mathcal{R}'(\Phi, Z)$ and define $Z_\alpha := \{\alpha, X\} \subseteq Z$ for $\alpha \in \Psi$ as above. Since

$$\Psi = \{\alpha \in \Phi \mid \alpha + n\delta \in \Phi^\alpha(\Psi, X) \text{ for some } n \in Z, \}$$

it will suffice to show that $Z_\alpha \subseteq \Phi^\alpha(\Psi, X)$ uniquely determines $(\Psi, X)$. It follows from (18) that

$$X' = \{p \in P(\Psi) \mid \langle p, \alpha \rangle \subseteq n_\alpha Z \text{ for all } \alpha \in \Psi \}$$

From this, we obtain $X = \{p \in P(\Psi) \mid \langle p, \alpha \rangle \subseteq Z_\alpha \text{ for all } \alpha \in \Psi \}$. The injectivity of the map $g$ follows.

Now to show that $g$ is surjective, it will suffice to show that for $(\Gamma, f)$ as in Theorem 2 there is some pair $(\Psi, X)$ with $\Phi^\alpha(\Gamma, f) = \Phi^\alpha(\Psi, X)$. In the following argument, we use the notation of (4.11) concerning $\Gamma$. Choose $\Psi = \Sigma$ and $P \in P(\Sigma)$ such that $\langle p, \alpha \rangle = r_\alpha$ for all $\alpha \in \Gamma'$ and define $X' = \oplus_i X_i \subseteq P(\Sigma)$ where

$$X'_i := \begin{cases} 0, & \text{if } \Gamma' = \Gamma_i \\ K_i P(\Gamma_i), & \text{if } \Gamma' \neq \Gamma_i \text{ and } \alpha_{i,0} \text{ is long} \\ K_i P(\Gamma_i^{\alpha}), & \text{if } \Gamma' \neq \Gamma_i \text{ and } \alpha_{i,0} \text{ is short}. \end{cases}$$

Then $X' = \oplus X'_i$ is a Z-admissible coweight lattice of $P(\Sigma)$ and from Proposition 5 and (17), we see that $\Phi^\alpha(\Sigma, P + X') = \Phi^\alpha(\Gamma, f)$. This completes the proof. \(\square\)

5.7. Evidently there is a unique bijection $j = j_\Sigma: \mathcal{R}(\Phi, Z) \mapsto \mathcal{R}'(\Phi, Z)$ such that, setting $j(\Gamma, f) = (\Psi, X)$, we have $\Phi^\alpha(\Psi, X) = \Phi^\alpha(\Gamma, f)$. The above proof indicates how to compute $j(\Gamma, f)$ explicitly from $(\Gamma, f)$. Computing $j^{-1}(\Psi, X)$ amounts to determining the canonical simple system of $\Phi^\alpha(\Psi, X)$. We describe below one simple way in which this may be done; a more natural procedure using alcove geometry will be described in $[\text{??}]$.

Suppose $(\Psi, X) \in \mathcal{R}'(\Phi, Z)$. We may find $(\Gamma, f) \in \mathcal{R}(\Phi, Z)$ with $\Phi^\alpha(\Gamma, f) = \Phi^\alpha(\Psi, X)$ as follows. For $\alpha \in \Psi$, write $Z_\alpha = \{X, \alpha\} \subseteq Z$ and $Z'_\alpha := \{n \in Z_\alpha \mid \alpha + n\delta \in \Phi^\alpha(\Psi, X)\}$. Let $\Psi' := \{\alpha \in \Psi \mid Z'_\alpha \neq 0\}$ and for $\alpha \in \Psi'$, let $r'_{\alpha} := \min(Z'_\alpha)$. Let $\Delta$ denote the simple system contained in $\Phi^\alpha_+$ of $\Phi^\alpha(\Psi, X)$ and $\Delta' := \{\alpha + r'_{\alpha} \alpha \mid \alpha \in \Psi'\}$. One then checks that $\Delta \subseteq \Delta'$. It is well known that $\Delta$ is the unique inclusion-minimal subset $\Delta'$ of $\Phi^\alpha_+ \cap \Phi^\alpha(\Psi, X)$ such that $\Phi^\alpha(\Psi, X) \cap \Phi^\alpha_+ \subseteq N\Delta$. It follows that $\Delta$ is the unique inclusion-minimal subset $\Delta'$ of the finite set $\Delta'$ such that $\Delta'' \subseteq N\Delta'$. Thus, $\Delta$ in $\Phi^\alpha_+$ may be effectively determined. Set $\Gamma = \pi(\Delta)$ (a np subset in $\Phi$) and define $f: \Gamma \rightarrow Z_{\geq 0}$ by $f(\alpha) = r'_{\alpha}$. Then $\Phi^\alpha(\Gamma, f) = \Phi^\alpha(\Psi, X)$. 


5.8. We may now use the action of $\tilde{W}^\alpha$ on $V$ to describe the $\tilde{W}^\alpha$-action on root subsystems of $\Phi^\alpha$ in terms of their parameterisation in Theorem 3. For any subset $\Sigma$ of $\Phi$, we let $p_\Sigma: V \to \mathbb{R}\Sigma$ denote the orthogonal projection of $V$ on $\mathbb{R}\Sigma$.

**Corollary 6.** Let $w = t_\gamma x \in \tilde{W}^\alpha$ where $x \in W$ and $\gamma \in P(\Phi)$. Let $(\Psi, X) \in \mathcal{R}'(\Phi, Z)$. Then $\Psi' := x(\Psi)$ is a root subsystem of $\Phi$ and $X'' := p(\tau_\gamma x(X))$ is a coset in $P(\Psi')$ of some admissible coweight lattice of $P(\Psi')$, where $p = p_{\Psi'}$. Further, $w\Phi^\alpha(\Psi, X) = \Phi^\alpha(\Psi', X'')$.

**Remark 9.** Note that the isomorphism $t_\gamma x \leftrightarrow \tau_\gamma x$ between the affine and linear versions of $\tilde{W}^\alpha$ used implicitly above is different from the one $(t_\gamma x \leftrightarrow \tau_{-\gamma} x)$ used in Proposition 2.

**Proof.** Write $X = a + X'$ where $a \in P(\Psi)$ and $X'$ is an admissible coweight lattice of $P(\Psi)$. Clearly, $\Psi'$ is a root subsystem of $\Phi$ and $X''' := x(X')$ is an admissible coweight lattice of $P(\Psi')$. We have $\tau_\gamma x(X) = \gamma + x(a) + X''$ where clearly $x(a) \in P(\Psi')$. Hence $p\tau_\gamma x(X) = p(\gamma) + x(a) + X''$. To prove the assertion about $X'''$, it will suffice to show that $p(\gamma) \in P(\Psi')$. But this holds since for any $\alpha \in \Psi'$, we have $\langle \alpha, p(\gamma) \rangle = \langle \alpha, \gamma \rangle \in \mathbb{Z}$. This proves the first claim. The final claim follows from Lemma 10 as follows. Write $Z_\alpha = \langle X, \alpha \rangle$ for $\alpha \in \Psi$. Then by Lemma 10, $w\Phi^\alpha(\Psi, X) = w\Phi^\alpha(\Psi, \{Z_\alpha\}_{\alpha \in \Psi}) = \Phi^\alpha(\Psi', \{Z'_{\alpha} \}_{\alpha \in \Psi'})$ where for $\alpha \in \Psi'$, $Z'_{\alpha} = Z_{x^{-1}\alpha} + \langle \alpha, \gamma \rangle = \langle x^{-1}\alpha, X \rangle + \langle \alpha, \gamma \rangle$. But from the above, $X'' = p\tau_\gamma x(X) = p(\gamma) + x(a) + X'(\gamma) = p(\gamma) + x(X)$, and so

$$\langle \alpha, X'' \rangle = \langle \alpha, p(\gamma) + x(X) \rangle = \langle \alpha, p(\gamma) \rangle + \langle \alpha, x(X) \rangle = \langle \alpha, \gamma \rangle + \langle x^{-1}\alpha, X \rangle.$$ 

Hence $\Phi^\alpha(\Psi', \{Z'_{\alpha} \}_{\alpha \in \Psi'}) = \Phi^\alpha(\Psi', X')$ as required. $\Box$

5.9. **Affine versus finite.** Now fix $(\Psi, X) \in \mathcal{R}'(\Phi, Z)$ and write $X = b + X'$, where $b \in P(\Psi)$ and $X'$ is an admissible coweight lattice of $P(\Psi)$. Let $\{\Psi_i\}$ be the components of $\Psi$, and write $X' = \sum_i X'_i$ where for each $i$, either $X'_i = m_i P(\Psi_i)$, $X'_i = m_i P(\Psi_i^\circ)$ with $m_i \in \mathbb{Z}_{>0}$ or $X'_i = 0$ and $m_i := 0$. Set $Y_i := m_i Q(\Psi_i)$. $Y_i := m_i Q(\Psi_i^\circ)$ or $Y_i := 0$ accordingly, and define $Y = \sum_i Y_i$. It is clear that $\Phi^\alpha(\Psi, X)$ is finite (equivalently, $W^\alpha(\Psi, X)$ is finite) if and only if $X' = \{0\}$. Also, $W^\alpha(\Psi, X)$ has only affine type components (i.e. no finite components) if and only if $X'_i \neq \{0\}$ for all $i$. Finally, $W^\alpha(\Psi, X)$ is of finite index in $W^\alpha$ if and only if its translation subgroup is of finite index in that of $W^\alpha$, which is the case if and only if $X'_i \neq 0$ for all $i$ and $\mathbb{R}\Psi = \mathbb{R}\Psi$.

5.10. We now describe a decomposition of $W^\alpha(\Psi, X)$ as the semidirect product of its translation subgroup and a finite reflection group, and explicitly describe its elements.

**Proposition 7.** Let $X = b + X'$ be a coset in $P(\Psi)$ of an admissible coweight lattice $X'$ for $\Psi$, and let $Y$ be the admissible coweight lattice corresponding to $X'$. Then

- (a) $W^\alpha(\Psi, X)$ is a semidirect product $W^\alpha(\Psi, X) = W'' \rtimes t_Y$ of the finite reflection subgroup $W'' := \{ t_{b-w(b)}w \mid w \in W_\Psi \}$ by the translation group $t_Y := \{ t_\gamma \mid \gamma \in Y \}$.

- (b) $W^\alpha(\Psi, X) = \{ t_{b-w(b)+\gamma}w \mid w \in W_\Psi, \gamma \in Y \}$.

**Proof.** A typical root of $\Phi^\alpha(\Psi, X)$ is of the form $\alpha + (\alpha, b + z)\delta$ where $\alpha \in \Psi$ and $z \in X'$, with corresponding reflection $s_{\alpha + (\alpha, b + z)\delta} = t_{(\alpha, b + z)\alpha + s_{\alpha}}$. For each $\alpha \in \Psi$, we have $s'_\alpha := s_{\alpha + (\alpha, b)\delta} = t_{(\alpha, b)\alpha + s_{\alpha}} = t_{b - s_{\alpha} b} \in W^\alpha(\Psi, X)$. It is easy to check
that the reflection subgroup of $W^\alpha$ generated by $s_\alpha'$ for $\alpha \in \Psi$ is $W''$. For any affine Weyl group, the group of translations is generated by the translations which are products of two reflections. Therefore, the subgroup $T'$ of translations of $W^\alpha(\Psi, X)$ is generated by elements $s_{\alpha+(a,b,z)}\delta s_{\alpha+(\alpha,b)} = t_{(\alpha,z)\alpha}$ for $z \in X'$ and $\alpha \in \Psi$. Write $(\alpha, X') = n_{\alpha Z}$, so $T'$ is generated by $t_{n_{\alpha},\alpha}$ for $\alpha \in \Psi$. But it follows easily from (17) that $T' = t_Y$. The rest of (a) follows readily, and (b) follows immediately from (a). □

5.11. The next result describes the pointwise stabiliser in $\widetilde{W}$ of a root subsystem and the centraliser in $\widetilde{W}$ of the corresponding reflection subgroup, where $\widetilde{W}$ is as in [5,4]

**Proposition 8.** (a) The pointwise stabiliser of $\Phi^a(\Psi, X)$ in $\widetilde{W}$ is the semidirect product $W'' \rtimes T$ where $W''$ is the parabolic subgroup $W'' = W_{\Phi \cap L^+}$ of $W$ and $T$ is the group of translations $T = t_{R \cap \Psi} = \{ t_\gamma \mid \gamma \in R \cap \Psi \}$. (b) For $w \in W$ and $\gamma \in R$, $wt_\gamma$ centralises $W^a(\Psi, X)$ if and only if for each component $\Psi_i$ of $\Psi$, one of the two conditions (i) or (ii) below hold:

(i) for all $\alpha \in \Psi_i$, $w(\alpha) = \alpha$ and $\langle \gamma, \alpha \rangle = 0$

(ii) $X_i = 0$ and for all $\alpha \in \Psi_i$, $w(\alpha) = -\alpha$ and $\langle \gamma, \alpha \rangle = -2\langle \alpha, \alpha \rangle$

where $\alpha$ is a representative of the coset $X$.

Proof. Part (a) follows from (10), using the fact that the centraliser in $W$ of any subset $L$ of $V$ is the parabolic subgroup of $W$ generated by $s_\alpha$ for $\alpha \in \Phi \cap L^+$. For part (b), one notes first that $x \in \widetilde{W}$ centralises $W^a(\Psi, X)$ if and only if for each $\beta \in \Phi^a(\Psi, X)$, we have $x \beta = \pm \beta$. It is readily verified that this suffices to conclude that $wt_\gamma$ is in the centraliser.

To prove necessity in (b), one notes first that if $x \in \widetilde{W}$ centralises $W^a(\Psi, X)$, then for each $\beta \in \Phi^a(\Psi, X)$, we have $x \beta = \pm \beta$. Writing $x = wt_\gamma$ and $\beta = \alpha + n\delta$, (10) implies that $w\alpha = \pm \alpha$ for each $\alpha \in \Psi$. By irreducibility, for each component $\Psi_i$ of $\Psi$, we then have either $w\alpha = \alpha$ for all $\alpha \in \Psi_i$ or $w\alpha = -\alpha$ for all $\alpha \in \Psi_i$. A further application of (10) now shows that either (i) or (ii) holds. □

5.12. We describe the $\widetilde{W}$-orbit of $\Phi^a(\Psi, X)$ and the setwise stabiliser of $\Phi^a(\Psi, X)$ in $\widetilde{W}$ (which coincides with the normaliser in $\widetilde{W}$ of $W^a(\Psi, X)$). For this, we consider a second root system $\Phi^p(\Psi_1, X_1)$, for which we use similar notation as for $\Phi^a(\Psi, X)$. In particular, write $X = a + X'$ where $a$ is a representative of $X$ and $X_1 = a_1 + X_1'$, etc.

**Proposition 9.** Let $w \in W$ and $\gamma \in R$. Then

(a) $t_\gamma w(\Phi^a(\Psi_1, X_1)) = \Phi^a(\Psi, X)$ if and only if $w(\Psi_1) = \Psi$, $w(X_1') = X'$ and $a \in w(a_1) + p_\Psi(\gamma) + X'$.

(b) $\Phi^a(\Psi_1, X_1)$ and $\Phi^p(\Psi, X)$ are in the same $\widetilde{W}$-orbit if there exists $w' \in W$ with $w'(\Psi_1) = \Psi$, $w'((X_1')'' = X'$ and $a - w'(a_1) = p_\Psi(R') + X'$.

(c) $t_\gamma w$ normalises $W^a(\Psi, X)$ if and only if $w(\Psi) = \Psi$, $w(X') = X'$ and $p_\Psi(\gamma) = a - w(a) + X'$.

Proof. Part (a) follows from Corollary 6 while (b) and (c) follow readily from (a). We omit the details. □
6. Comparison of the parameterisations

6.1. Consider the bijection (discussed in 5.7) \( j = j_\mathbb{Z} : \mathcal{R}(\Phi, \mathbb{Z}) \to \mathcal{R}'(\Phi, \mathbb{Z}) \), where both sides are parameter sets for the set of reflection subgroups of \( W^a \). We indicate below how \( j \) arises as the restriction (to “integer points”) of a bijection \( j_\mathbb{R} : \mathcal{R}(\Phi, \mathbb{R}) \to \mathcal{R}'(\Phi, \mathbb{R}) \) where both sides correspond naturally to the discrete reflection subgroups of the semidirect product \( W \rtimes \tau_v \).

Let \( \Phi := \Phi + \mathbb{R} \delta \) and

\[
\Phi_+ := \{ \alpha + n\delta \mid \alpha \in \Phi, n \in \mathbb{Z}_{\geq 0}, n > 0 \text{ if } \alpha \notin \Phi_+ \}.
\]

The group \( \tilde{W}^a := \langle s_\alpha \mid \alpha \in \Phi \rangle \) is a (generally non-discrete) group of linear transformations of \( \hat{V} \). Note that \( \tilde{W}^a \) is naturally identified as the group of affine transformations of \( V \) generated by \( W \) and the group of translations \( \tau_v = \{ \tau_v \mid v \in V \} \) of \( V \); under this identification, \( \tilde{W}^a \) is a semidirect product \( \tilde{W}^a = W \rtimes \tau_v \).

We define root subsystems of \( \Phi^a \) as for \( \Phi^a \). A root subsystem \( \Psi \) of \( \Phi^a \) is discrete (as a subset of \( \hat{V} \) in the usual topology) if and only if for each \( \alpha \in \Phi \), the set \( Z_\alpha := \{ n \in \mathbb{R} \mid n + n\delta \in \Psi \} \) is a discrete subset of \( \mathbb{R} \). It is easy to see that this holds precisely when the corresponding reflection subgroup \( \langle s_\alpha \mid \alpha \in \Psi \rangle \) is a discrete subgroup of \( \text{GL}(\hat{V}) \). We define simple subsystems of \( \Phi^a \) as positively independent subsets of \( \Phi_+ \). There are natural bijective correspondences between discrete root subsystems of \( \Phi^a \), discrete reflection subgroups of \( \tilde{W} \), and simple subsystems of \( \Phi \) which are contained in \( \Phi_+ \). The simple subsystem of \( \Phi \) corresponding to a discrete reflection subgroup \( W' \) of \( W \) and contained in \( \Phi_+ \) will be written \( \Delta_{W'} \).

6.2. The main results (Theorems 2 and 4) hold in this more general context, with essentially the same proofs, but modified as follows: one replaces “reflection subgroups of the affine Weyl group \( W^a \)” by “discrete reflection subgroups of \( \tilde{W}^a \),” “root subsystems of \( \Phi^a \)” by “discrete root subsystems of \( \Phi^a \),” “\( Z = \mathbb{Z} \)” by “\( Z = \mathbb{R} \)” (and correspondingly “\( \mathcal{R}(\Phi, \mathbb{Z}) \)” by “\( \mathcal{R}(\Phi, \mathbb{R}) \)” etc) and “\( \Phi^a \)” by \( \Phi \) (and correspondingly “\( \Phi^a_\beta \)” by \( \Phi_\beta \) etc).

This gives a natural bijection \( j_\mathbb{R} : \mathcal{R}(\Phi, \mathbb{R}) \to \mathcal{R}(\Phi, \mathbb{R}) \) which extends \( j = j_\mathbb{Z} \), since both sides are naturally in bijection with discrete root subsystems of \( \Phi^a \). The bijection \( j_\mathbb{R} \) has similar properties to those of \( j_\mathbb{Z} \) as discussed in 5.7.

Remark 10. In fact, most of the theory (with obvious exceptions such as Corollary 3(b)) established in the previous Sections 4–5 for reflection subgroups of \( W^a \) extends mutatis mutandis to discrete reflection subgroups of \( \tilde{W}^a \).

6.3. Let \( k \subseteq \mathbb{R} \) denote either \( \mathbb{Z} \) or \( \mathbb{R} \). For a root subsystem \( \Psi \) of \( \Phi \) and a \( k \)-admissible subgroup \( X' = \sum_i X_i' \) of \( P_k(\Psi) \), we define

\[
\mathcal{R}'_{(\Psi, X')}(\Phi, k) := \{ (\Psi, X) \in \mathcal{R}'(\Phi, k) \mid X \in P_k(\Psi)/X' \}.
\]

and

\[
\mathcal{R}_{(\Psi, X')}(\Phi, k) := j_k^{-1}\left( \mathcal{R}'_{(\Psi, X')}(\Phi, k) \right).
\]

Clearly, \( j_k \) restricts to a bijection

\[
j_{k, \Psi, X} : \mathcal{R}_{(\Psi, X')}(\Phi, k) \to \mathcal{R}'_{(\Psi, X')}(\Phi, k).
\]

We have \( \mathcal{R}'(\Phi, k) = \bigcup_{(\Psi, X')} \mathcal{R}'_{(\Psi, X')}(\Phi, k) \) (and similarly with \( \mathcal{R}' \) replaced by \( \mathcal{R} \)) where the union is over pairs \((\Psi, X')\) such that \( \Psi \) is a root subsystem of \( \Phi \) and \( X' \) is a \( k \)-admissible coweight lattice in \( P_k(\Psi) \).
A preliminary discussion of the bijections \( j^\pm_\mathbb{Z} \) is given in §[5.7] The description of \( j^\pm_\mathbb{R} \) there is reasonably natural and explicit. Here we wish to provide a more natural interpretation of \( j^\pm_\mathbb{Z} \) than was available previously. Now \( j^\pm_\mathbb{R} \) is obtained by restriction of \( j^\pm_\mathbb{Z} \), so it sufficient to describe the maps \( j^\pm_\mathbb{Z} \). The main result of this section is Theorem \[5\] which express \( j^\pm_\mathbb{Z} \) as a composite of maps which have natural interpretations in terms of the alcove geometry of a certain affine Weyl group \( W' = W'_\Psi, X' \) attached to \( (\Psi, X') \).

6.4. Fix a root subsystem \( \Psi \) of \( \Phi \) and an \( \mathbb{R} \)-admissible subgroup \( X' = \sum_j X'_j \) of \( P_\Psi(\Psi) \). Let \( Y' \) be the \( \mathbb{R} \)-admissible coroot lattice of \( \Psi \) corresponding to \( X' \), and let \( W' = W'_\Psi, X' \) be the group of affine transformations of \( V \) generated by \( W_\Psi \cup \tau_Y \). Note that \( W' \) acts faithfully by restriction as a discrete affine reflection group on \( \mathbb{R}\Psi \); we shall think of \( W' \) in this way. It is isomorphic to Coxeter group to \( W^\alpha(\Psi, X') \) for any \( X \in P_\Psi(\Psi)/X' \).

**Remark** 11. For fixed \( \Psi \) and varying \( X \), the subgroups \( W'_\Psi, X' \) are precisely the discrete reflection groups acting in \( \mathbb{R}\Psi \) whose linear parts are (the restriction to \( \mathbb{R}\Psi \) of) \( W_\Psi \) and which have 0 as a special point.

6.5. **Cosets and a fundamental domain.** Denote by \( \Gamma'_0 \) the simple system of \( \Psi \) with \( \Gamma'_0 \subseteq \Phi^+ \) and define the corresponding coweights \( \omega'_\alpha := \omega_\alpha(\Gamma'_0) \) for \( \alpha \in \Gamma'_0 \); thus \( \langle \alpha, \omega'_\beta \rangle = \delta_{\alpha, \beta} \) if \( \alpha, \beta \in \Gamma'_0 \). The group \( X' \) acts by translation on \( P_\Psi(\Psi) = \mathbb{R}\Psi \). Since \( X' = \sum_{i,j \neq 0} \sum_{\alpha \in \Psi} \mathbb{Z}m_i k_i, \omega_\alpha \), it follows that the region

\[
D := \{ \sum_{\alpha \in \Gamma'_0} e_\alpha \omega'_\alpha \mid e_\alpha \in \mathbb{R}, \ 0 \leq e_\alpha < K_i k_i, \alpha \text{ if } X'_j \neq 0 \text{ and } \alpha \in \Psi \} \subseteq \mathbb{R}\Psi
\]

is a fundamental domain for the action of \( X' \) by translation on \( \mathbb{R}\Psi \). This gives a bijection \( h: P_\Psi(\Psi)/X' \cong D \) such that \( h^{-1}(d) = d + X' \) for all \( d \in D \).

6.6. Let \( B \) denote the set of alcoves of \( W' \) acting on \( \mathbb{R}\Psi \), i.e. the connected components of the complements of reflecting hyperplanes of \( W' \) in \( \mathbb{R}\Psi \). Any alcove \( B \) is a direct sum of simplices and simplicial cones lying in the pairwise orthogonal subspaces of \( \mathbb{R}\Psi \) given by the linear spans of the components \( \Psi_i \) of \( \Psi \). Each alcove \( B \) may be written as

\[
B = \{ v \in \mathbb{R}\Psi \mid \langle \alpha, v \rangle > c_\alpha \text{ for all } \alpha \in \Sigma \}
\]

where each \( c_\alpha \in \mathbb{R}, \Sigma = \Sigma_B \subseteq \Psi \) and \( \Sigma \) is minimal under inclusion; the hyperplanes \( \{ v \in \mathbb{R}\Psi \mid \langle \alpha, v \rangle = c_\alpha \} \) for \( \alpha \in \Sigma_B \) are the walls of \( B \). We define the lower closure \( B_0 \) of \( B \) as

\[
B_0 = \{ v \in \mathbb{R}\Psi \mid \langle \alpha, v \rangle \geq c_\alpha \text{ for all } \alpha \in \Sigma \} \cap -\Phi^+ \}
\]

Thus \( B_0 \) is obtained by removing all points of some of the walls of \( B \) from the topological closure \( \overline{B} = \{ v \in \mathbb{R}\Psi \mid \langle \alpha, v \rangle \geq c_\alpha \text{ for all } \alpha \in \Sigma \} \) of \( B \); the walls removed are precisely those whose unit normal (outward from \( B \) in \( \mathbb{R}\Psi \)) is a positive scalar multiple of a positive root (i.e one in \( \Psi \cap \Phi^+ \)).
6.7. **Alcoves.** Basic properties of alcoves and their lower closures are recorded in the following Lemma.

**Lemma 11.**

(a) \( R \Psi = \bigcup_{B \in \mathcal{B}} B_0 \) (disjoint union).

(b) If \( B \in \mathcal{B} \) and \( \lambda \in X' \), then \( C := B + \lambda \in \mathcal{B} \) and \( C_0 = B_0 + \lambda \).

(c) Every \( v \in X' \) is contained in \( B_0 \) for a unique \( B \in \mathcal{B} \).

(d) Every \( v \in Y' \) is contained in \( \overline{B} \) for a unique \( B \in \mathcal{B} \).

(e) Let \( B \) be an alcove for \( W' \) on \( R \Psi \) with \( 0 \in \overline{B} \). Let \( G := \{ C \in \mathcal{B} \mid 0 \in C \} \).

Then \( G = \{ wB \mid w \in W \} \).

(f) The region \( D' = \bigcup_{C \in G} C_0 \) is a fundamental domain for the action on \( R \Psi \) of the translation subgroup \( T = \{ x \to x + v \mid x \in X' \} \) of \( W' \).

**Proof.** The notion of lower closure of an alcove is related in an obvious way to the notion of upper closure of a facet as discussed in [13, 6.1–6.2]. By simple modification of the discussion in [13, 6.2], one sees that (a)–(c) hold. For (d)–(e), see [2, Ch V, §3, no. 10]. Finally, (f) follows from (a), (b), (e) and Proposition II(a) applied to \( W' \).

\( \square \)

6.8. By definition, \( D' \) is a union of lower closures of alcoves. We observe that the same holds for \( D \).

**Lemma 12.** We have \( D = \bigcup_{B \subseteq D} B_0 \).

**Proof.** Observe that

\[ D := \{ v \in R \Psi \mid 0 \leq \langle v, \alpha \rangle < K_i k_i, \alpha \text{ if } X'_i \neq 0 \text{ and } \alpha \in \Phi_i \cap \Gamma_0 \} \]

It is easy to check that the walls of \( D \) are reflecting hyperplanes of \( W' \). The desired conclusion follows from Lemma 11(a) since the definition of lower closure and the form of the above inequalities defining \( D \) imply that if \( v \in D \) is in \( B_0 \) for some \( B \in \mathcal{B} \), then \( B_0 \subseteq D \).

\( \square \)

6.9. Since \( D' \) is a fundamental domain for \( Y' \) acting by translation on \( R \Psi \), there is a unique map \( k : D \to D' \) such that \( k(d) - d \in Y' \) for all \( d \in D \). For any alcove \( B \in \mathcal{B} \) with \( B \subseteq D \), there is a unique \( \gamma_B \in Y' \) such that \( B_0 + \gamma_B \subseteq D' \). The map \( k \) is determined by the condition that \( k(d) = d + \gamma_B \) for all \( B \in \mathcal{B} \) with \( B \subseteq D \) and all \( d \in B_0 \). Obviously \( k \) is injective, and the image of \( k \) is the union of the lower closed alcoves in \( D \) which are translates by elements of \( Y' \) of lower closed alcoves in \( D \).

**Remark 12.** Let \( B \) be some alcove of \( W' \) with \( 0 \in \overline{B} \). Consider the subgroup \( G = \{ w \in W \mid wB = \tau \gamma B \text{ for some } \gamma \in X' \} \). It is well known (and follows from Lemma 11 and Proposition II(a)) that \( G \cong X'/Y' \), under the map taking each \( w \in G \) to the coset \( \gamma + Y' \) where \( \gamma \in X' \) with \( wB = \tau \gamma B \). We have \( \text{Im}(k) \subseteq D' = \bigcup_{w \in W \} \text{Im}(k) \} \). It is not difficult to see that \( M \) is a set of coset representatives for \( W'/G \).

6.10. Recall that \( \mathcal{R}_{\langle \Phi, X' \rangle}(\Phi, \mathbb{R}) := \mathcal{R}(\Phi, \mathbb{R}) \). We shall require the following detailed description of this subset of \( \mathcal{R}(\Phi, \mathbb{R}) \).

**Lemma 13.** Let \((\Gamma, f) \in \mathcal{R}(\Phi, \mathbb{R})\). Then \((\Gamma, f) \in \mathcal{R}_{\langle \Phi, X' \rangle}(\Phi, \mathbb{R})\) if and only if the following conditions (i)–(iv) hold:

(i) there is some simple subsystem \( \Gamma' \) of \( \Psi \) with \( \Psi \supseteq \Gamma \supseteq \Gamma' \).

(ii) if \( X'_i = 0 \) then \( \Gamma \cap \Psi_i = \Gamma' \cap \Psi_i \).
(iii) If \( X'_i = m_i P(\Psi_i) \) where \( m_i > 0 \) then \( (\Gamma \setminus \Gamma') \cap \Psi_i = \{ \alpha_{i,0} \} \) where \( -\alpha_{i,0} \) is the highest root of \( \Psi_i \) with respect to its simple system \( \Psi_i \setminus \Gamma' \), and \( K_i = m_i \).

(iv) If \( X'_i = m_i P(\Psi_i) \), \( m_i > 0 \), then \( (\Gamma \setminus \Gamma') \cap \Psi_i = \{ \alpha_{i,0} \} \) where \( -\alpha_{i,0}^{\vee} \) is the highest root of \( \Psi_i^{\vee} \) with respect to its simple system \( (\Psi_i \setminus \Gamma')^{\vee} \) and \( K_i = m_i \).

Here, if \( X'_i \neq 0 \) and \( \sum_{\alpha \in \Psi_i \cap \Gamma} c_\alpha \alpha = 0 \) with all \( c_\alpha \in \mathbb{R} \) and \( c_{\alpha_{i,0}} = 1 \), we write \( K_i := \sum_{\alpha \in \Psi_i \cap \Gamma} c_\alpha f(\alpha) \).

**Proof.** When \( \mathbb{R} \) is replaced by \( \mathbb{Z} \), this is implicit in the proof of Theorem \([3]\) see especially \((20)\). The proof for \( \mathbb{R} \) is essentially the same. \( \square \)

#### 6.11.

Let \( \widetilde{R}(\Phi, X') (\Phi, \mathbb{R}) \) denote the set of all triples \( x = (\Gamma, \Gamma', f) \) such that \( (\Gamma, f) \in R(\Phi, \mathbb{R}) \) and the conditions (i)–(iv) above hold. To \( x = (\Gamma, \Gamma', f) \in \widetilde{R}(\Phi, X') (\Phi, \mathbb{R}) \), we attach a point \( v_x := \sum_{\alpha \in \Psi_i \cap \Gamma} c_\alpha f(\alpha) \in P_\mathbb{R}(\Psi) = \mathbb{R} \).

**Lemma 14.** Let \( B \) be the (unique) alcove of \( W' \) in \( \mathbb{R} \) such that \( 0 \in \overline{B} \) and \( B \) is in the fundamental chamber for \( W_\Phi \) with respect to its positive system \( \Gamma' \). Then

(a) We have \( j_2(\Gamma, f) = (\Psi, v_x + X') \).

(b) The coset \( v_x + X' \in \mathbb{R} \Psi/X' \) depends only on \( (\Gamma, f) \) and not on \( \Gamma' \).

(c) For fixed \( (\Gamma, \Gamma') \), the map \( x = (\Gamma, \Gamma', f) \mapsto v_x \) defines a bijection from the set of triples \( (\Gamma, \Gamma', f) \in \widetilde{R}(\Phi, X') (\Phi, \mathbb{R}) \) to the lower closure \( B_0 \) of \( B \).

**Proof.** For (a), observe that \( \tilde{\Phi}^o(\Gamma, f) = \tilde{\Phi}^o(\Psi, v_x + X') \). Then (b) follows from (a).

For (c), note that the only restrictions on the function \( f \) are that \( f(\alpha) \geq 0 \) for all \( \alpha \in \Gamma \), that \( f(\alpha) > 0 \) if \( \alpha \in -\Phi_+ \cap \Gamma \), and that \( \sum_{\alpha \in \Gamma} c_\alpha f(\alpha) = m_i \) if \( \Gamma'_i \neq \Gamma_i \). One checks using this that the set of points \( v_x \) arising as in (b) is the set of all solutions \( v \) of the following system of simultaneous inequalities for \( v \in \mathbb{R} \Psi \):

\[
\begin{cases}
\langle v, \alpha \rangle \geq 0, & \text{for } \alpha \in \Gamma' \cap \Psi_+ \\
\langle v, \alpha \rangle > 0, & \text{for } \alpha \in \Gamma' \cap -\Psi_+ \\
\langle v, \alpha \rangle \geq -m_i, & \text{for } \alpha \in (\Gamma \setminus \Gamma') \cap \Psi_{i,+} \\
\langle v, \alpha \rangle > -m_i, & \text{for } \alpha \in (\Gamma \setminus \Gamma') \cap -\Psi_{i,+},
\end{cases}
\]

where \( \Psi_+ = \Psi \cap \Phi_+ \), etc. and \( i \) in the last two inequalities is such that \( \Gamma_i \neq \Gamma'_i \). On the other hand, the alcove \( B \) is defined by the set of inequalities above with each non-strict inequality replaced by strict inequality; the equations obtained by replacing inequalities by equalities define the walls of \( B \). From the definition of lower closure, the set of inequalities above also defines \( B_0 \). \( \square \)

#### 6.12.

There is a map \( q_k : \widetilde{R}(\Phi, X') (\Phi, k) \to R(\Phi, X') (\Phi, k) \) defined by \( q(\Gamma, \Gamma', f) = (\Gamma, f) \). Write \( q = q_{\mathbb{Z}} \). Next define a function \( g : D' \to \widetilde{R}(\Phi, X') (\Phi, \mathbb{R}) \) as follows.

Let \( \Delta \in D' \). There is a unique alcove \( B \in \mathcal{B} \) such that \( 0 \in \overline{B} \) and \( d \in B_0 \). Let \( \Gamma' \) be the unique simple system for \( \Psi \) such that \( B \) is contained in the fundamental chamber of \( W_\Phi \) on \( \mathbb{R} \Psi \) with respect to \( \Gamma' \). There is a unique subset \( \Gamma' \) of \( \Psi \) such that \( (\Gamma, \Gamma') \) satisfies the conditions of Lemma \([13] \)–(iv) with the condition \( K_i = m_i \) omitted in (iii)–(iv). By Lemma \([13]\) there is a unique function \( f : \Gamma \to \mathbb{R} \) such that \( x := (\Gamma, \Gamma', f) \in \widetilde{R}(\Phi, X') (\Phi, \mathbb{R}) \) and \( v_x = d \). We define \( g(d) = x \).

It is a simple consequence of Lemmas \([14] \) and \([15]\) that the map \( q \) is surjective and \( g \) is a bijection.
6.13. We now come to the main result of this section.

**Theorem 5.** The bijection \( R'_{(\Phi, X)}(\Phi, \mathbb{R}) \rightarrow R'_{(\Phi, X)}(\Phi, \mathbb{R}) \) given by restriction of \( j^{-1}_\mathbb{R} \) is given by \( j^{-1}_\mathbb{R}(\Psi, X) = gqkh(X) \) for \( X \in P(\Psi)/X' \), where \( g \) and \( h \) are the maps defined in \( \ref{6.12} \) and \( k \) is defined in \( \ref{6.8} \).

**Proof.** Write \( h(X) = d \), so \( X = d + X' \). Set \( g(d) = d' \), so \( d - d' \in X' \) and \( X = d' + X' \). Set \( g(d') = (\Gamma, \Gamma', f) = x \). By definition of \( g \), we have \( v_x = d' \) in Lemma \( \ref{14} \). Then \( j_Xgqkh(X) = j_X(\Gamma, f) = (\Psi, v_x + X') = (\Psi, d' + X') = (\Psi, X) \), and the Theorem follows.

6.14. If \( k = \mathbb{Z} \) and \( X' \) has finite index in \( P(\Psi) \), then evidently

\[
|\mathcal{R}'_{(\Phi, X)}(\Phi, \mathbb{Z})| = |\mathcal{R}'_{(\Phi, X)}(\Phi, \mathbb{Z})| \in \mathbb{N}.
\]

This equality implies various identities in \( \mathbb{Z} \) by counting the points of \( R'_{(\Phi, X')} (\Phi, \mathbb{Z}) \) and \( R'_{(\Phi, X)} (\Phi, \mathbb{Z}) \) according to the definitions. We shall formulate these identities explicitly below in the special case that \( \Psi = \Phi \) is irreducible; in general, the identities easily reduce to those in this special case, by considering indecomposable components of \( \Psi \).

Assume that \( \Psi = \Phi \) is indecomposable and \( X' \neq \emptyset \). We have either \( X' = MP(\Phi) \) or \( X' = MP(\Phi^\circ) \) for some \( M \in \mathbb{N}_{>0} \), so \( |P(\Phi) : X'| \) is finite. Fix the np subset \( \Gamma := \Pi \cup \{-\omega\} \) of \( \Phi \) where \( \omega \) is the highest root of \( \Phi \) with respect to \( \Pi \) if \( X' = P(\Phi) \), and \( -\omega \sigma \) is the highest root of \( \Psi \sigma \) with respect to \( \Pi \sigma \) if \( X' = P(\Phi^\circ) \).

**Lemma 15.** Let \( W \) act on np subsets \( \Sigma \subseteq \Phi \) by \( (w, \Sigma) \mapsto w(\Sigma) \). Then the setwise stabiliser of \( \Gamma \) in \( W \) satisfies \( \text{Stab}_W(\Gamma) \cong P(\Phi)/Q(\Phi) \) and in particular \( |\text{Stab}_W(\Gamma)| = f_\Phi \).

**Proof.** Suppose that \( \omega \) is the highest root of \( \Phi \) (the other case follows from this one by considering the np subset \( \Gamma' \subseteq \Phi' \)). Let \( a : \tilde{W}\sigma \rightarrow W \) denote the natural projection. Using that \( \pi(\tilde{\Pi}) = \Gamma \), it follows that \( a \) restricts to a map \( a' : \text{Stab}_{\tilde{W}\sigma}(\tilde{\Pi}) \rightarrow \text{Stab}_W(\Gamma) \). Using the definition of \( \tilde{W}\sigma \), one readily checks that \( a' \) is surjective. The group \( \text{Stab}_{\tilde{W}\sigma}(\tilde{\Pi}) = \text{Stab}_{\tilde{W}\sigma}(A) \) is discussed in detail (to the extent that its elements are explicitly described) in [2, Ch. VI, §3, no. 3]. From the results there, one has in particular that \( \text{Stab}_{\tilde{W}\sigma}(\tilde{\Pi}) \cong P(\Phi)/Q(\Phi) \) and the restriction of \( a \) to \( \text{Stab}_{\tilde{W}\sigma}(\tilde{\Pi}) \) is injective. Hence

\[
\text{Stab}_W(\Gamma) = a(\text{Stab}_{\tilde{W}\sigma}(\tilde{\Pi})) \cong \text{Stab}_{\tilde{W}\sigma}(\tilde{\Pi}) \cong P(\Phi)/Q(\Phi)
\]

and the result follows.

**Remark 13.** By considering irreducible components of \( \Sigma \), one can determine (using the proof of the above Lemma) the setwise stabiliser in \( W_\Sigma \) of any np subset \( \Sigma \) of \( \Phi \). We do not go into the more delicate question of the stabiliser of \( \Sigma \) in \( W \).

6.15. Consider the integers \( c_\alpha \in \mathbb{N}_{\geq 0} \) which are defined by \( \sum_{\alpha \in \Gamma} c_\alpha \alpha = 0 \) and \( c_\alpha = 1 \). Define a generalized “ordered partition function” \( \psi^{(\Gamma)} : \mathbb{Z} \rightarrow \mathbb{N} \) by

\[
\psi^{(\Gamma)}(M) := \{ f : \Gamma \rightarrow \mathbb{N} \mid \sum_{\alpha \in \Gamma} c_\alpha f(\alpha) = M \} = \{ f : \Gamma' \rightarrow \mathbb{N} \mid \sum_{\alpha \in \Gamma'} c_\alpha f(\alpha) \leq M \}.
\]
Define some non-standard descent statistics on \(W\) as follows. For \(w \in W\), define \(d(w) = \{ \alpha \in \Gamma \mid w(\alpha) \in \Phi_+ \}\). For \(i \in \mathbb{N}\), let \(d_i := |\{ w \in W \mid \sum_{\alpha \in d(w)} c_\alpha = i \}|\).

Set \(h = \sum_{\alpha \in \Gamma} c_\alpha\) (\(h\) is the Coxeter number of \(\Phi\) if \(\omega\) is the highest root of \(\Phi\), by \([2]\) Ch VI, §1, no 11, Prop. 31). Then \(d_i = 0\) if \(i = 0\) or \(i \geq h\).

**Proposition 10.** Let \(N' = |\Pi_{\text{long}}|\) and \(N := |\Pi|\). For any \(M \in \mathbb{N}_{>0}\),
\[
\sum_{i=0}^{h} d_i \cdot \psi^{(\Gamma)}(M - i) = \begin{cases} M^N, & \text{if } \omega \in \Phi_{\text{long}} \\ M^N k_\Phi^{N'}, & \text{if } \omega \in \Phi_{\text{short}}. \end{cases}
\]

where \(\frac{d_i}{f_\Phi} \in \mathbb{N}\) for all \(i\).

**Proof.** For any \(\Delta \subseteq \Gamma\), define \(d_\Delta := |\{ w \in W \mid d(w) = \Delta \}|\). Note that \(d_\Phi = d_\Gamma = 0\). To prove the formula, it will suffice to show that
\[
\frac{1}{f_\Phi} \sum_{\Delta \subseteq \Gamma} d_\Delta \cdot \psi^{(\Gamma)}(M - \sum_{\alpha \in \Delta} c_\alpha) = \begin{cases} M^N, & \text{if } \omega \in \Phi_{\text{long}} \\ M^N k_\Phi^{N'}, & \text{if } \omega \in \Phi_{\text{short}}. \end{cases}
\]

The right hand side is \(|D \cap P(\Phi)| = |R^{(\Phi, \chi)}(\Phi, \mathbb{Z})|\). On the other hand, consider the set \(\tilde{R}^{(\Phi, \chi)}(\Phi, \mathbb{Z})\). From Lemma [13] one sees that a typical element of this set is of the form \((w(\Gamma), w(\Gamma'), f \circ w^{-1}) : w(\Gamma) \rightarrow \mathbb{N}\) for (unique) \(w \in W\) and \(f : \Gamma \rightarrow \mathbb{N}\) such that \(f(\alpha) > 0\) if \(w(\alpha) \in \Phi_+\) and \(\sum_{\alpha \in \Gamma} f(\alpha)c_\alpha = M\). For fixed \(w\), the number of possible functions \(f\) is therefore \(\psi^{(\Gamma)}(M - \sum_{\alpha \in \Delta} c_\alpha)\) where \(\Delta = \{ \alpha \in \Gamma \mid w(\alpha) \in \Phi_- \}\). Hence the left hand side is equal to \(\frac{1}{f_\Phi} |\tilde{R}^{(\Phi, \chi)}(\Phi, \mathbb{Z})|\). But since \(W\) acts simply transitively on the simple systems \(\Gamma'\) of \(\Phi\), we get from Lemma [13] that each fiber of the map \(q_\Phi^* : \tilde{R}^{(\Phi, \chi)}(\Phi, \mathbb{Z}) \rightarrow \tilde{R}^{(\Phi, \chi)}(\Phi, \mathbb{Z})\) given by \((\Gamma, \Gamma', f) \mapsto (\Gamma, f)\) has cardinality \(f_\Phi\) and so
\[
|\tilde{R}^{(\Phi, \chi)}(\Phi, \mathbb{Z})| = f_\Phi |\tilde{R}^{(\Phi, \chi)}(\Phi, \mathbb{Z})| = f_\Phi |\tilde{R}^{(\Phi, \chi)}(\Phi, \mathbb{Z})|.
\]
This completes the proof of the formula.

It remains to show that \(\frac{d_i}{f_\Phi} \in \mathbb{N}\) for all \(i\). Let \(G := \text{Stab}_W(\Gamma)\). One readily checks that for \(w \in W, g \in G\) and \(\alpha \in \Gamma\), we have \(d(wg) = g^{-1}d(w)\) and \(c_\alpha = c_{g(\alpha)}\).

It follows readily from this that \(d' := \{ w \in W \mid \sum_{\alpha \in d(w)} c_\alpha = i \}\) is a union of left cosets of \(G\) (i.e. satisfies \(d'(i) = d'(i)G\)) and so its order \(d_i\) is divisible by \(|G| = f_\Phi\).

\[\square\]

**Remark 14.** Using the longest element of \(W\), one easily shows that the sequence \((d_0, d_1, \ldots, d_h)\) is symmetric i.e \(d_i = d_{h_i - i}\) for \(i = 0, \ldots, h\). One might conjecture that this sequence is also strictly unimodal i.e. \(d_0 < d_1 < \ldots < d_k\) where \(k\) is the largest integer satisfying \(2k \leq h - 1\).

6.16. We discuss the identity in the previous Lemma in the case \(\Phi\) is of type \(A_n\), \(n \geq 1\). We identify \(W\) with the symmetric group \(S_{n+1}\) on \(\{1, \ldots, n+1\}\). Consider the sequence \((i_0, i_1, i_2, \ldots, i_{n+1}) := (n+1, 1, 2, \ldots, n+1)\). For \(\sigma \in W\), let
\[d_\sigma := |\{ k \in \mathbb{N} \mid 1 \leq k \leq n+1, \sigma(i_{k-1}) > \sigma(i_k) \}|\]
denote the number of cyclically consecutive descents of \(\sigma\). For \(i = 0, \ldots, n+1\), let \(d_i := |\{ \sigma \in W \mid d_\sigma = i \}|\), so \(d_0 = d_{n+1} = 0\). The identity becomes
\[
M^n = \sum_{i=1}^{n} \frac{d_i}{n+1} \binom{M+i-1}{n}
\]
for $M \in \mathbb{N}_{\geq 1}$. For $n = 2$, this is the well known identity expressing square numbers as a sum of two consecutive triangular numbers.

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