Abstract

In this paper, a constrained attack-resilient estimation algorithm (CARE) is developed for stochastic cyber-physical systems. The proposed CARE can simultaneously estimate the compromised system states and attack signals. It has improved estimation accuracy and attack detection performance when physical constraints and operational limitations are available. In particular, CARE is designed for simultaneous input and state estimation that provides minimum-variance unbiased estimates, and these estimates are projected onto the constrained space restricted by inequality constraints subsequently. We prove that the estimation errors and their covariances from CARE are less than those from unconstrained algorithms and confirm that this property can further reduce the false negative rate in attack detection. We show that estimation errors of CARE are practically exponentially stable in mean square. Finally, an illustrative example of attacks on a vehicle is given to demonstrate the improved estimation accuracy and detection performance compared to an existing unconstrained algorithm.

Key words: Detection, Kalman filtering, Recursive estimation, Stability, State estimation

1. Introduction

Cyber-Physical Systems (CPS) play a vital role in the metabolism of applications from large-scale industrial systems to critical infrastructures, such as smart grids, transportation networks, precision agriculture, and industrial control systems [1]. Recent developments in CPS and their safety-critical applications have led to a renewed interest in CPS security. The interaction between information technology and the physical system has made control components of CPS vulnerable to malicious attacks [2]. Recent cases of CPS attacks have clearly illustrated their susceptibility and raised awareness of the security challenges in these systems. These include attacks on large-scale critical infrastructures, such as the German steel mill cyber attack [3], and Maroochy Water breach [4]. Similarly, malicious attacks on avionics and automotive vehicles have been reported, such as the U.S. drone RQ-170 captured in Iran [5], and disabling the brakes and stopping the engine on civilian cars [6, 7].

Related work. Traditionally, most works in the field of attack detection had only focused on monitoring the cyber-space misbehavior [8]. With the emergence of CPS, it becomes vitally important to monitor physical misbehavior as well because the impact of the attack on physical systems also needs to be addressed [9]. In the last decade, attention has been drawn from the perspective of the control theory that exploits some prior information on the system dynamics for detection and attack-resilient control. For instance, a unified modeling framework for CPS and attacks is proposed in [10]. A typical control architecture for the networked system under both cyber and physical attacks is proposed in [11]; then attack scenarios, such as Denial-of-Service (DoS) and false-data injection (FDI) are analyzed using this control architecture in [12].

In recent years, model-based detection has been tremendously studied. Attack detection has been formulated as an $\ell_0/\ell_\infty$ optimization problem, which is NP-hard [13]. A convex relaxation has been studied in [14]. Furthermore, the worst-case estimation error has been analyzed in [15]. Multirate sampled data controllers have been studied to guarantee detectability in [16] and to detect zero-dynamic attacks in [17]. A residual-based detector has been designed for power systems against false-data injection attacks, and the impact of attacks has been analyzed in [18].

In addition, some papers have studied active detection, such as [19, 20], where the control input is watermarked with a pre-designed scheme that sacrifices optimality. The aforementioned methods have the problem that the state estimate is not resilient concerning the attack signal, and incorrect state estimates make it more challenging for defenders to react to malicious attacks consequently.

Attack-resilient estimation and detection problems have been studied to address the above challenge in [21, 22, 23], where attack detection has been formulated as a simultane-
ous input and state estimation problem, and the minimum-variance unbiased estimation technique has been applied. More specifically, the approach has been applied to linear stochastic systems in [21], stochastic random set methods in [22], and nonlinear systems in [23]. These detection algorithms rely on statistical thresholds, such as the \( \chi^2 \) test, which is widely used in attack detection [20, 21]. Since the detection accuracy improves when the covariance decreases, a smaller covariance is desired.

On top of the minimum-variance estimation approach, the covariance can be further reduced when we incorporate the information of the input and state in terms of constraints. There have been several investigations on Kalman filtering with state constraints [25, 26, 27, 28]. The state constraints are induced by unmodeled dynamics and operational processes. Some of these examples include vision-aided inertial navigation [29], target tracking [30] and power systems [31]. Constraints on inputs are also considered, such as avoiding reachable dangerous states under the assumption that the attack input is constrained [32] and designing a resilient controller based on the partial knowledge of the attacker in terms of inequality constraints [33]. The methods in [32, 33] can efficiently be used to maneuver a class of attacks when input inequality constraints are available but cannot resiliently address the estimation problem due to the false-data injection. This problem remains to be solved with a stability guarantee in the presence of inequality constraints. In the current paper, we aim to solve the resilient estimation problem and investigate the stability and performance of the algorithm design that integrates with information aggregation. To the best of our knowledge, this is the first investigation that considers both state and input inequality constraints for attack-resilient estimation with guaranteed stability.

Contributions. Our main contributions of this work can be summarized as follows. i) We propose a constrained attack-resilient estimation algorithm (CARE) that can estimate the compromised system states and the attack signals simultaneously. CARE first provides minimum-variance unbiased estimates, and then they are projected onto the constrained space induced by information aggregation. ii) The proposed CARE has better estimation performance. We show that the projection strictly reduces the estimation errors and covariances. iii) We are the first to investigate the stability of the estimation algorithm with inequality constraints and prove that the estimation errors are practically exponentially stable in mean square. iv) The proposed CARE has better attack detection performance. We provide rigorous analysis that the false negative rate is reduced by using the proposed algorithm. v) The proposed algorithm is compared with the state-of-the-art method to show the improved estimation and attack detection performance.

Paper organization. The rest of the paper is organized as follows. Section 2 introduces notations, \( \chi^2 \) test for detection, and problem formulation. The high-level idea of the proposed algorithm is presented in Section 3.1. Section 3.2 gives a detailed algorithm derivation. Section 4 demonstrates the performance improvement and investigates the stability analysis of the proposed algorithm. Section 5 presents an illustrative example of vehicle attacks. Finally, Section 6 draws the conclusion.

\( \chi^2 \) Test for Detection. Given a sample of Gaussian random variable \( \bar{\sigma}_k \) with unknown mean \( \sigma_k \) and known covariance \( \Sigma_k \), the \( \chi^2 \) test provides statistical evidence of whether \( \sigma_k = 0 \) or not. In particular, the sample \( \bar{\sigma}_k \) is being normalized by \( \bar{\sigma}_k^\top \Sigma_k^{-1} \bar{\sigma}_k \), and we compare the normalized value with \( \chi^2_{df}(\alpha) \), where \( \chi^2_{df}(\alpha) \) is the \( \chi^2 \) value with degree of freedom \( df \) and statistical significance level \( \alpha \). We reject the null hypothesis \( H_0 \): \( \sigma_k = 0 \), if \( \bar{\sigma}_k^\top \Sigma_k^{-1} \bar{\sigma}_k > \chi^2_{df}(\alpha) \), and accept alternative hypothesis \( H_1 \): \( \sigma_k \neq 0 \), i.e., there is significant statistical evidence that \( \sigma_k \) is non-zero. Otherwise, we accept \( H_0 \), i.e., there is no significant evidence that \( \sigma_k \) is non-zero.

False negative rate. Given a set of vectors \( \{\sigma_k\} \), the false negative rate of the \( \chi^2 \) test is defined as the ratio of the number of false negative test results \( N_{neg} \) and the number of non-zero vectors in the given set \( N_{\sigma \neq 0} \).

\[
F_{neg}(\{\sigma_k\}, \{\Sigma_k\}) \triangleq \frac{N_{neg}}{N_{\sigma \neq 0}} = \frac{\sum_k(1_k)}{N_{\sigma \neq 0}},
\]

where

\[
1_k \triangleq \begin{cases} 
1, & \text{if } \bar{\sigma}_k^\top \Sigma_k^{-1} \bar{\sigma}_k \leq \chi^2_{df}(\alpha) \text{ and } \sigma_k \neq 0 \\
0, & \text{otherwise}
\end{cases}
\]
Problem Formulation. Consider the following linear time-varying (LTV) discrete-time stochastic system:

\[
\begin{align*}
    x_{k+1} &= A_k x_k + B_k u_k + G_k d_k + w_k, \\
    y_k &= C_k x_k + v_k,
\end{align*}
\]

where \( x_k \in \mathbb{R}^m \), \( u_k \in \mathbb{R}^n \) and \( y_k \in \mathbb{R}^p \) are the state, the control input and the sensor measurement, respectively. The attack signal is modeled as a simultaneous input \( d_k \in \mathbb{R}^d \), which is unknown to the defender. System matrices \( A_k, B_k, C_k \) and \( G_k \) are known and bounded with appropriate dimensions. We assume that \( \text{rank}(C_k G_{k-1}) = n_d \), \( 0 \leq n_d \leq n_y \). This is typical assumption as in \cite{31, 35, 36}. The interpretation of this assumption is that the impact of the attack \( d_{k-1} \) on the system dynamics can be observed by \( y_k \). The process noise \( w_k \) and the measurement noise \( v_k \) are assumed to be i.i.d. Gaussian random variables with zero means and covariances \( Q_k \triangleq \mathbb{E}[w_k w_k^\top] \geq 0 \) and \( R_k \triangleq \mathbb{E}[v_k v_k^\top] > 0 \). Moreover, the measurement noise \( v_k \), the process noise \( w_k \), and the initial state \( x_0 \) are uncorrelated with each other.

The adopted attack model in \cite{37, 25, 27} is known as the FDI attack that is a very general type of attack, and includes physical attacks, Trojans, replay attacks, overflow bugs, packet injection, etc. \cite{36, 38, 41}. Because of this generality, this attack model has been widely used in CPS security literature (e.g., \cite{10, 12, 21}).

In the cyber-space, digital attack signals could be unconstrained, but their impact on the physical world is restricted by physical and operational constraints (i.e., \( x_k \) and \( d_k \) are constrained). For example, a vehicle has a limit on acceleration, velocity, steering angle, and change of steering angle. Any physical constraints and ability limitations on attack signals and states are presented by the inequality constraints

\[
A_k d_k \leq b_k, \quad B_k x_k \leq c_k,
\]

where matrices \( A_k, B_k, c_k \) are known and bounded with appropriate dimensions. Throughout this paper, we assume that the feasible sets of the constraints in \cite{1} are non-empty.

Remark 1. Gaussian noise in \cite{4} is one of the general ways to model physical systems so that the filtering algorithms use this model to track the level of uncertainties. Therefore, many pieces of work consider Gaussian noise even in the presence of bounded constraints \cite{34, 22, 27, 35}.

Problem statement. Given the stochastic system in \cite{3}, we aim to design an attack-resilient estimation algorithm that can simultaneously estimate the compromised state \( x_k \) and the attack signal \( d_k \). In addition, we seek to improve estimation accuracy and detection performance with a stability guarantee when incorporating the information of the input and state in terms of constraints in \cite{1}.

3. Algorithm Design

To address the problem statement described in Section 2, we propose a constrained attack-resilient estimation algorithm (CARE), as sketched in Fig. 1, which consists of a minimum-variance unbiased estimator (MVUE) and an information aggregation step via projection. In particular, the optimal estimation provides minimum-variance unbiased estimates, and these estimates are projected onto the constrained space eventually in the information aggregation step. We outline the essential steps of CARE in Section 3.1 and provide a detailed derivation of the algorithm in Section 3.2.

3.1. Algorithm Statement

The proposed CARE can be summarized as follows:

\begin{align*}
\text{prediction: } \hat{x}_{k-1} &= A_{k-1} \hat{x}_{k-1} + B_{k-1} u_{k-1}; \quad \text{(5)} \\
\text{attack estimation: } \hat{d}_{k-1}^u &= M_k (y_k - C_k \hat{x}_{k-1}); \quad \text{(6)} \\
\text{time update: } \hat{x}_k &= \hat{x}_{k-1} + G_k \hat{d}_{k-1}^u; \quad \text{(7)} \\
\text{measurement update: } \hat{x}_k^u &= \hat{x}_k^t + L_k (y_k - C_k \hat{x}_k^t); \quad \text{(8)} \\
\text{projection update: } \\
\hat{d}_{k-1} &= \arg \min_d (d - \hat{d}_{k-1}^u)^\top (P_k^{d,u})^{-1} (d - \hat{d}_{k-1}^u) \\
&\quad \text{subject to } A_{k-1} d \leq b_{k-1}; \quad \text{(9)} \\
\hat{x}_k &= \arg \min_x (x - \hat{x}_k^t)^\top (P_k^{x,u})^{-1} (x - \hat{x}_k^t) \\
&\quad \text{subject to } B_k x \leq c_k. \quad \text{(10)}
\end{align*}

Given the previous state estimate \( \hat{x}_{k-1} \) and its error covariance \( P_{k-1}^{x,x} \triangleq \mathbb{E}[(\hat{x}_{k-1} - x_{k-1})^\top] \), the current state can be predicted by \( \hat{x}_k \) in \cite{2} under the assumption that the attack signal \( d_{k-1} \) is absent. The unconstrained attack estimate \( \hat{d}_{k-1}^u \) can be obtained by comparing the difference between the predicted output \( C_k \hat{x}_k \) and the measured output \( y_k \) in \cite{6}, where \( M_k \) is the optimal filter gain that can be obtained by applying Gauss-Markov theorem, as shown in Proposition 3 later. The state prediction \( \hat{x}_k^t \) can be updated incorporating the unconstrained attack estimate \( \hat{d}_{k-1}^u \) in \cite{7}. The output \( y_k \) is used to correct the current
state estimate in \([5]\), where \(L_k\) is the filter gain that is obtained by minimizing the state error covariance \(P_k^{x, u}\). In the information aggregation step (projection update), we apply the input constraint in \([9]\) by projecting \(\hat{d}_{k-1}^u\) onto the constrained space and obtain the constrained attack estimate \(\hat{d}_{k-1}\). Similarly, the state constraint in \([10]\) is applied to obtain the constrained state estimate \(\hat{x}_k\). The complete algorithm is presented in Algorithm 1.

**Algorithm 1** Constrained Attack-Resilient Estimation:

\[
\hat{d}_{k-1}, P_{k-1}^{x, u}, \hat{x}_k, P_k^{x, u} = \text{CARE}(\hat{x}_{k-1}, P_{k-1}^{x, u})
\]

**1. Prediction**

\[\hat{x}_{k-1} = A_{k-1} \hat{x}_{k-1} + B_{k-1} u_{k-1}; \]

**2.** \(P_{k-1}^{x, u} = A_{k-1} P_{k-1}^{x, u} A_{k-1}^\top + Q_{k-1}; \)

**3. Attack estimation**

\[\hat{R}_k = (C_k P_k^{x, u} C_k^\top + R_k)^{-1};\]

\[M_k = (G_k^\top C_k^\top \hat{R}_k C_k G_k - I)^{-1} G_k^\top C_k \hat{R}_k;\]

**5.** \(\hat{d}_{k-1}^u = M_k (y_k - C_k \hat{x}_k);\)

**6.** \(P_{k-1}^{d, u} = (G_k^\top C_k^\top R_k C_k G_k - I)^{-1};\)

**7.** \(P_{k-1}^{x} = P_{k-1}^{x, u} A_{k-1}^\top + A_{k-1} P_{k-1}^{d, u} G_k^\top C_k^\top R_k C_k G_k - I)^{-1};\)

**8.** \(\hat{d}_k = \hat{x}_k + G_k^\top \hat{R}_k;\)

**9.** \(P_{k-1}^{d, u} = (I - L_k C_k)^\top G_k^\top C_k^\top R_k (I - L_k C_k)^\top + R_k;\)

**10.** Measurement update

\[L_k = P_k^{x, u} C_k^\top - G_k^\top M_k R_k;\]

**11.** \(\hat{x}_k = \hat{x}_k + L_k (y_k - C_k \hat{x}_k);\)

**12.** \(P_k^{x} = (I - L_k C_k)^\top G_k^\top C_k^\top R_k (I - L_k C_k)^\top + L_k R_k L_k^\top;\)

**13.** \(P_k^{x, u} = (I - L_k C_k)^\top G_k^\top C_k^\top R_k (I - L_k C_k)^\top + L_k R_k L_k^\top;\)

**14.** \(\hat{d}_{k-1} = \hat{d}_{k-1}^u - \gamma_{k-1}(\hat{x}_{k-1} - \hat{d}_{k-1} - B_{k-1} u_{k-1});\)

**15.** \(P_{k-1}^{d, u} (I - \gamma_{k-1} \hat{d}_{k-1}^u)\)

**16.** \(\gamma_k = P_k^{x, u} B_k^\top (B_k P_k^{x, u} B_k^\top)^{-1};\)

**17.** \(\hat{x}_k = \hat{x}_k - \gamma_k (B_k \hat{x}_k - B_k \hat{x}_k^u);\)

**18.** \(P_k^{x} = (I - \gamma_k B_k) P_k^{x, u} (I - \gamma_k B_k)^\top;\)

which is a linear function of the attack signal \(d_{k-1}\). Under the assumption that there is no projection update, i.e., the state and attack estimates are unconstrained, we design the optimal gain matrix \(M_k\) such that the estimate becomes the best linear unbiased estimate (BLUE) by the following two propositions.

**Proposition 2.** Assume that there is no projection update and \(E[\hat{x}_0] = E[\hat{x}_0^u] = 0\). The state estimates \(\hat{x}_k\) and the unconstrained attack estimates \(\hat{d}_{k}^u\) are unbiased for all \(k\), i.e., \(E[\hat{x}_k] = E[\hat{d}_{k-1}^u] = 0\), if and only if \(M_k C_k G_{k-1} = I\).

**Proof:** Sufficiency: Assume that \(M_k C_k G_{k-1} = I\), the statement can be proved by induction. First, we will show the statement holds when \(k = 0\) as a base case. By the definition, the errors of the time update and the measurement update in \([7]\) and \([8]\) are given by

\[
\hat{x}_{k-1} = \hat{x}_k - \hat{d}_{k-1} = A_{k-1} \hat{x}_{k-1} + G_{k-1} \hat{d}_{k-1} + \hat{w}_{k-1}
\]

and the error of the unconstrained attack estimate is

\[
\hat{d}_{k-1} \triangleq d_{k-1} - \hat{d}_{k-1} = d_{k-1} - M_k (C_k A_k \hat{x}_{k-1} + C_k w_{k-1}) + C_k G_{k-1} \hat{w}_{k-1} + \hat{v}_{k-1}
\]

under the assumption that \(E[\hat{x}_0] = E[\hat{x}_0^u] = 0\) and the process noise and measurement noise are zero-mean Gaussian, i.e., \(E[w_k] = E[v_k] = 0\), the expectation of the term \([14a]\) is zero at \(k = 1\). Since \(d_{k-1}\) is deterministic for all \(k\), i.e., \(E[d_{k-1}] \neq 0\), we have \(E[\hat{d}_{k-1}^u] = 0\) if \(I - M_k C_k G_{k-1} = 0\), i.e., the expectation of the term \([14a]\) is zero at \(k = 1\). Then we have \(E[\hat{x}_1] = E[\hat{x}_1^u] = 0\) by applying expectation operation on \([12]\) and \([13]\). In the inductive step, suppose \(E[\hat{x}_k] = E[\hat{x}_k^u] = 0\); then \(E[\hat{d}_{k-1}^u] = 0\) if \(M_k C_k G_{k-1} = I\). Then, similarly, we have \(E[\hat{x}_{k+1}] = E[\hat{x}_{k+1}^u] = 0\) by \([12]\) and \([13]\). Since there is no projection update, we have \(E[\hat{x}_k] = E[\hat{x}_k^u] = 0\) for all \(k\).

**Necessity:** Assume that \(E[\hat{x}_k] = E[\hat{d}_{k-1}^u] = 0\) for all \(k\), or equivalently \(E[\hat{x}_k] = E[\hat{d}_{k-1}] = 0\), the state estimate also can be proved by induction. In \([14a]\), if \(E[\hat{d}_{k-1}] = 0\) for any \(d_0\), we have \(M_k C_k G_{k} = I\). Therefore, following a similar procedure, we can show that the necessity holds.

**Proposition 3.** Assume that there is no projection update and \(E[\hat{x}_0] = E[\hat{x}_0^u] = 0\). The unconstrained attack estimates \(\hat{d}_{k}^u\) are BLUE if

\[
M_k = (G_k^\top C_k^\top \hat{R}_k C_k G_k - I)^{-1} G_k^\top C_k^\top \hat{R}_k,
\]

where \(\hat{R}_k \triangleq (C_k P_k^{x, u} C_k^\top + R_k)^{-1}\).

**Proof:** Substituting \([3a]\) into \([3b]\), we have

\[
y_k = C_k G_{k-1} d_{k-1} + C_k (A_{k-1} \hat{x}_{k-1} + B_{k-1} u_{k-1} + \hat{w}_{k-1}) + \hat{v}_k.
\]
Subtraction of $C_k \hat{x}_{k-1}$ on the both sides of \[16\] yields
\[
y_k - C_k \hat{x}_{k-1} = C_k G_{k-1} d_{k-1} + C_k \left(A_k \hat{x}_{k-1} + w_{k-1}\right) + v_k.
\]
(17)

Since the covariances of the process noise $w_{k-1}$ and the measurement noise $v_k$ are known, with \[11\], the covariance of the error term in \[17\] can be expressed as $C_k P_{x_k}^{d,u} C_k^\top + R_k$. Applying the Gauss-Markov theorem (see Appendix \[A\]), we can get the minimum-variance-unbiased linear estimator (BLUE) of $d_{k-1}$ in \[6\] with $M_k = (G_{k-1} C_k^\top \hat{R}_k C_k G_{k-1})^{-1} G_{k-1} C_k^\top \hat{R}_k$, where $\hat{R}_k \triangleq (C_k P_{x_k}^{d,u} C_k^\top + R_k)^{-1}$.

**Remark 4.** The rank condition $\text{rank}(C_k G_{k-1}) = n_d$ is the sufficient condition of $M_k C_k G_{k-1} = I$ needed in Proposition \[2\] if $M_k$ is found by \[15\] in Proposition \[4\].

The error covariance can be found by $P_{k-1}^{d,u} = M_k \hat{R}_k M_k^\top = (G_{k-1} C_k^\top \hat{R}_k C_k G_{k-1})^{-1}$. The cross error covariance of the state estimate and the attack estimate is $P_{k-1}^{x,d} = -P_{k-1}^{x} A_k C_k^\top M_k G_{k-1}$.

**Time update.** Given the unconstrained attack estimate $\hat{d}_{k-1}$, the state prediction $\hat{x}_{k}^{*}$ can be updated as in \[7\]. We derive the error covariance of $\hat{x}_{k}^{*}$ as
\[
P_{k}^{x,u} \triangleq \mathbb{E}[(\hat{x}_{k}^{*}) (\hat{x}_{k}^{*})^\top] = A_{k-1} P_{k-1}^{d,u} A_{k-1}^\top + A_{k-1} P_{k-1}^{d,d} G_{k-1} + G_{k-1} P_{k-1}^{d,u} A_{k-1}^\top + Q_{k-1} - G_{k-1} M_k C_k Q_{k-1} - Q_{k-1} C_k^\top M_k^\top G_{k-1},
\]
where $P_{k-1}^{d,d} = (P_{k-1}^{d,u})^\top$.

**Measurement update.** In this step, the measurement $y_k$ is used to update the propagated estimate $\hat{x}_{k}^{*}$ as shown in \[8\]. The covariance of the state estimation error is
\[
P_{k}^{x,u} \triangleq \mathbb{E}[(\hat{x}_{k}^{*}) (\hat{x}_{k}^{*})^\top] = (I - L_k C_k) G_{k-1} M_k R_k L_k^\top + L_k R_k L_k^\top + L_k R_k M_k^\top G_{k-1} (I - L_k C_k)^\top + (I - L_k C_k) P_{k-1}^{x,u} (I - L_k C_k)^\top.
\]

The gain matrix $L_k$ is obtained by minimizing the trace of $P_{k}^{x,u}$, i.e., $\min_{L_k} \text{tr}(P_{k}^{x,u})$. The solution is given by $L_k = (P_{k-1}^{x,u} C_k^\top - G_{k-1} M_k R_k \hat{R}_k)$, where $\hat{R}_k \triangleq C_k P_{x_k}^{d,u} C_k + R_k - C_k G_{k-1} M_k R_k - R_k M_k^\top G_{k-1} C_k^\top$.

**Projection update.** We are now in the position to project the estimates onto the constrained space. Apply the first constraint in \[4\] to the unconstrained attack estimate $\hat{d}_{k-1}$, and the attack estimation problem can be formulated as the following constrained convex optimization problem
\[
\hat{d}_{k-1} = \arg \min_d (d - \hat{d}_{k-1})^\top W_{k-1}^d (d - \hat{d}_{k-1})
\]
subject to $A_{k-1} d \leq b_{k-1}$.

(18)

where $W_{k-1}^d$ can be any positive definite symmetric weighting matrix. In the current paper, we select $W_{k-1}^d = (P_{k-1}^{d,u})^{-1}$ which results in the smallest error covariance as shown in \[25\]. From Karush-Kuhn-Tucker (KKT) conditions of optimality, we can find the corresponding active constraints. We denote $\bar{A}_k$ and $\bar{b}_k$ the rows of $A_k$ and the elements of $b_k$ corresponding to the active constraints of $A_{k-1} d \leq b_{k-1}$. Then \[18\] becomes
\[
\hat{d}_{k-1} = \arg \min_d (d - \bar{d}_{k-1}^u)^\top (P_{k-1}^{d,u})^{-1} (d - \bar{d}_{k-1}^u)
\]
subject to $\bar{A}_{k-1} d = \bar{b}_{k-1}$.

(19)

The solution of \[19\] can be found by $\hat{d}_{k-1} = \bar{d}_{k-1} - \gamma_{k-1} (\bar{A}_{k-1} \bar{d}_{k-1} - b_{k-1})$, where
\[
\gamma_{k-1} = P_{k-1}^{d,u} A_{k-1}^\top (\bar{A}_{k-1} P_{k-1}^{d,u} A_{k-1})^{-1}.
\]

(20)

The attack estimation error is
\[
\hat{d}_{k-1} = (I - \gamma_{k-1} \bar{A}_{k-1}) \bar{d}_{k-1} + \gamma_{k-1} (\bar{A}_{k-1} \bar{d}_{k-1} - b_{k-1}) = \bar{d}_{k-1} - \gamma_{k-1} (\bar{A}_{k-1} \bar{d}_{k-1} - b_{k-1}).
\]

(21)

The error covariance can be found by
\[
P_{k-1}^{d} \triangleq \mathbb{E}[(\hat{d}_{k-1}) (\hat{d}_{k-1})^\top] = (I - \gamma_{k-1} \bar{A}_{k-1}) (I - \gamma_{k-1} \bar{A}_{k-1})^\top
\]
under the assumption that $\gamma_{k-1} (\bar{A}_{k-1} \bar{d}_{k-1} - b_{k-1}) = 0$ holds. Notice that this assumption holds when the ground truth $\bar{d}_{k-1}$ satisfies the active constraint $\bar{A}_{k-1} \bar{d}_{k-1} = \bar{b}_{k-1}$. From \[22\], it can be verified that $\gamma_{k-1} \bar{A}_{k-1} P_{k-1}^{d,u} (\gamma_{k-1} \bar{A}_{k-1})^\top = \gamma_{k-1} \bar{A}_{k-1} P_{k-1}^{d,u} (\gamma_{k-1} \bar{A}_{k-1})^\top$. Therefore, from \[22\] we have
\[
P_{k-1}^{d} = P_{k-1}^{d,u} - \gamma_{k-1} \bar{A}_{k-1} P_{k-1}^{d,u} \gamma_{k-1} \bar{A}_{k-1}^\top
\]

\[
+ \gamma_{k-1} \bar{A}_{k-1} P_{k-1}^{d,u} (\gamma_{k-1} \bar{A}_{k-1})^\top = (I - \gamma_{k-1} \bar{A}_{k-1}) P_{k-1}^{d,u}.
\]

(23)

Similarly, applying the second constraint in \[4\] to the unconstrained state estimate $\hat{x}_{k}^{*}$, we formalize the state estimation problem as follows:
\[
\hat{x}_{k} = \arg \min_x (x - \hat{x}_{k}^{*})^\top W_{k}^x (x - \hat{x}_{k}^{*})
\]
subject to $B_k x \leq c_k$.

(24)

where we select $W_{k}^x = (P_{k}^{x,u})^{-1}$ for the smallest error covariance. We denote $B_k$ and $c_k$ the rows of $B_k$ and the elements of $c_k$ corresponding to the active constraints.
of $B_k \mathbf{x} \leq c_k$. Using the active constraints, we reformulate (24) as follows:

$$\hat{x}_k = \arg \min_{\mathbf{x}} (\mathbf{x} - \hat{x}_k^u)^\top (P_k^{-1} (\mathbf{x} - \hat{x}_k^u))$$

subject to $B_k \mathbf{x} = c_k$. (25)

The solution of (25) is given by $\hat{x}_k = \hat{x}_k^u - \gamma_k^x (B_k \hat{x}_k^u - c_k)$, where

$$\gamma_k^x \triangleq P_k^{-1} B_k^\top (B_k P_k^{-1} B_k^\top)^{-1}. (26)$$

Under the assumption that $\gamma_k^x (B_k \hat{x}_k^u - c_k) = 0$ holds, the state estimation error covariance can be expressed as

$$P_k^e = \Gamma_k P_k^{-1} \Gamma_k^\top, (27)$$

where $\Gamma_k \triangleq I - \gamma_k^x B_k$. Notice that this assumption holds when the ground truth $x_k$ satisfies the active constraint $B_k \mathbf{x} = c_k$.

4. Performance and Stability Analysis

In Section 4.1, we show that the projection induced by inequality constraints improves attack-resilient estimation accuracy and detection performance by decreasing estimation errors and the false negative rate in attack detection. Notice that the estimate $d_{k-1}$ and the ground truth $d_{k-1}$ satisfy the active constraint $A_{k-1} d_{k-1} - b_{k-1} = 0$ in (19) and the inequality constraint $A_k d_{k-1} \leq b_{k-1}$ in (4), respectively. However, it is uncertain whether the ground truth satisfies the active constraints or not. In this case, from (21) we have

$$\mathbb{E}[d_{k-1}] = \gamma_k^{-1} (\bar{A}_{k-1} d_{k-1} - b_{k-1}) \neq 0. (28)$$

A similar statement holds for the state estimation error:

$$\mathbb{E}[\hat{x}_k] = \gamma_k^x (B_k \hat{x}_k - c_k) \neq 0. (29)$$

These considerations indicate that the projection potentially induces biased estimates, rendering the traditional stability analysis for unbiased estimation invalid. In this context, we will prove that estimation errors of the CARE are practically exponentially stable in mean square which will be proven in Section 4.2.

4.1. Performance Analysis

For the analysis of the performance through the projection, we first decompose the state estimation error $\tilde{x}_k$ into two orthogonal spaces as follows:

$$\tilde{x}_k = (I - \gamma_k^x B_k)\tilde{x}_k + \gamma_k^x B_k \tilde{x}_k. (30)$$

We will show that the errors in the space $I - \gamma_k^x B_k$ remain identical after the projection, while the errors in the space $\gamma_k^x B_k$ reduce through the projection, as in Lemma 5.

Lemma 5. The decomposition of $\tilde{x}_k$ in the space $I - \gamma_k^x B_k$ is equal to that of $\tilde{x}_k^u$, and the decomposition of $\tilde{x}_k$ in the space $\gamma_k^x B_k$ is equal to that of $\tilde{x}_k^u$ scaled by $\alpha_k$, i.e.

$$(I - \gamma_k^x B_k)\tilde{x}_k = (I - \gamma_k^x B_k)\tilde{x}_k^u, (31)$$

$$\gamma_k^x B_k \tilde{x}_k = \alpha_k \gamma_k^x B_k \tilde{x}_k^u, (32)$$

where $\alpha_k = \text{diag}(\alpha_1, \ldots, \alpha_n)$, and

$$\alpha_k \triangleq (\gamma_k^x (B_k \tilde{x}_k^u - c_k))((\gamma_k^x B_k \tilde{x}_k^u(i))(\gamma_k^x B_k \tilde{x}_k^u(i))\top) \in [0, 1)$$

for $i = 1, \ldots, n$. Similarly, it holds that $(I - \gamma_k^x A_k) d_k = (I - \gamma_k^x A_k) d_k^u$ and $\gamma_k^x A_k d_k = \kappa_k \gamma_k^x (A_k d_k^u)$, where $\kappa_k = \text{diag}(\kappa_1, \ldots, \kappa_n)$, and $\kappa_k \triangleq (\gamma_k^x (A_k d_k(i))(A_k d_k(i))\top) \in [0, 1)$ for $i = 1, \ldots, n$.

Proof: The relationship in (31) can be obtained by applying $B_k \tilde{x}_k = c_k$ to $\tilde{x}_k = x_k - \hat{x}_k = x_k - \gamma_k^x (B_k \hat{x}_k^u - c_k)$$

$$\tilde{x}_k^u + \gamma_k^x (B_k \hat{x}_k^u - c_k)$$

$$\tilde{x}_k^u + \gamma_k^x (B_k \hat{x}_k^u - c_k)$$

$$\tilde{x}_k^u + \gamma_k^x (B_k \hat{x}_k^u - c_k).$$

which implies (31). The solution of $B_k \mathbf{x} \leq c_k$ defines a closed convex set $C_k$. The point $\tilde{x}_k^u$ is not an element of the convex set. The point $\tilde{x}_k$ has the minimum distance from $\tilde{x}_k^u$ with metric $d(a, b) \triangleq \|a - b\|_W^2$ in the convex set $C_k$ by (25). Since the solution $\tilde{x}_k$ is in the closed set $C_k$, and $\gamma_k^x B_k$ is a weighted projection with weight $W_k^\top$, the relationship (32) holds. The statements for attack estimation errors can be proven by a similar procedure, which is omitted here.

With the results from Lemma 5, we can show that the projection reduces the estimation errors and the error covariances, as formulated in Theorem 6.

Theorem 6. CARE reduces the state and attack estimation errors and their error covariances from the unconstrained algorithm, i.e.

$$\|\tilde{x}_k\| \leq \|\tilde{x}_k^u\| \text{ and } \|d_k\| \leq \|d_k^u\|, \quad P_k^e \leq P_k^{e,u} \quad \text{and} \quad P_k^d \leq P_k^{d,u}. \quad \text{Strict inequality holds if rank}(B_k) \neq 0, \quad \text{and rank}(A_k) \neq 0, \quad \text{respectively.}$$

Proof: The statement for $\|\tilde{x}_k\| \leq \|\tilde{x}_k^u\|$ is the direct result of Lemma 7 where strict inequality holds if $\alpha_k \neq 0$ for some $i$. The statement for $\|d_k\| \leq \|d_k^u\|$ can be proved by a similar procedure. To show the rest of the properties, we first identify the equality

$$(I - \gamma_k^x B_k)^\top \gamma_k^x B_k = 0. (33)$$

Since we have $B_k \gamma_k^x = I$ by (28), it holds that $\gamma_k^x B_k \gamma_k^x = \gamma_k^x$, and $B_k \gamma_k^x B_k = B_k$, i.e.

$$\gamma_k^x = B_k^\top. \quad \text{Then, we have} \quad B_k^\top (\gamma_k^x)^\top B_k = 0, \quad \text{which implies} \quad \tilde{x}_k^u (I - \gamma_k^x B_k)^\top \gamma_k^x B_k \tilde{x}_k = 0.$$

Notice that (33) holds for $\tilde{x}_k^u (I - \gamma_k^x B_k)^\top \gamma_k^x B_k \tilde{x}_k = 0$ as well. Similar to (23), we have $P_k^e = (I - \gamma_k^x B_k) P_k^{e,u} = P_k^{e,u} - \gamma_k^x B_k P_k^{e,u}$. Given that $\gamma_k^x B_k P_k^{e,u} = P_k^e - B_k^\top (B_k P_k^{e,u})^\top B_k P_k^{e,u} > 0$ is positive definite, we have the desired result $P_k^e < P_k^{e,u}$. The relation for $P_k^d$ can be obtained by a similar procedure.
The properties in Theorem 3 are desired for accurate estimation as well as attack detection. More specifically, since the false negative rate of a $\chi^2$ attack detector is a function of the estimate $\hat{\sigma}_k$ and the covariance $\Sigma_k$ as in (1), more accurate estimations can reduce the false negative rate under the following assumption.

**Assumption 7.** In the presence of the attack ($d_k \neq 0$), the following two conditions hold: (i) $\|d_k\| < \frac{1}{2}\|d\|$ and (ii) the ground truth $d_k$ satisfies the condition $d_k^\top (P_k^{d,u})^{-1} d_k > \chi^2_{df}(\alpha)$.

**Remark 8.** Assumption 7 implies that the unconstrained attack estimation error $\hat{d}_k$ is small with respect to the ground truth $d_k$, and the normalized ground truth attack signal is larger than $\chi^2_{df}(\alpha)$; otherwise, it cannot be distinguished from the noise. Notice that Assumption 7 is only considered for small false negative rates (Theorem 3), but not for the estimation performance (Theorem 6) and stability analysis in Section 4.3 where we will show the stability of the attack estimation error $\hat{d}_k$ (Theorem 13) which renders the stability of $\hat{d}_k^u$.

According to (1), we denote the false negative rates of the proposed CARE and the unconstrained algorithm as $F_{neg} \{\{d_k\}, \{P_k^d\}\}$ and $F_{neg} \{\{\hat{d}_k\}, \{P_k^{d,u}\}\}$, respectively. The following Theorem 9 demonstrates that the false negative rate of CARE is less or equal to that of the unconstrained algorithm.

**Theorem 9.** Under Assumption 7 given a set of attack vectors $\{d_k\}$, the following inequality holds

$$F_{neg} \{\{d_k\}, \{P_k^d\}\} \leq F_{neg} \{\{\hat{d}_k\}, \{P^{d,u}_k\}\}.$$  \hspace{1cm} (34)

**Proof:** To prove (34) is equivalent to showing that the number of false negative test results of CARE is less or equal to that of the unconstrained algorithm

$$\sum_{k} (1_k) \leq \sum_{k} (1^*_k),$$  \hspace{1cm} (35)

If there is no projection ($\gamma^d_k = 0$), it holds that $d_k = \hat{d}_k$ and $P_k^d = P_k^{d,u}$. And, if there is no attack ($d_k = 0$), it holds that $1_k = 1^*_k = 0$. Therefore, we have

$$\sum_{k \in \mathcal{K}_0} (1_k) = \sum_{k \in \mathcal{K}_0} (1^*_k),$$  \hspace{1cm} (36)

where $\mathcal{K}_0 \triangleq \{k \mid \gamma^d_k = 0 \text{ or } d_k = 0\}$. In the rest of the proof, we consider the case for $k \in \mathcal{K} \triangleq \{k \mid \gamma^d_k \neq 0 \text{ and } d_k \neq 0\}$. Rewriting the normalized test result from CARE by substituting $P_k^{d,u}$ with $(I - \gamma^d_k A_k) P_k^{d,u} (I - \gamma^d_k A_k)^\top$ according to (22), we have the following:

$$
\hat{d}_k^\top (P_k^{d,u})^{-1} \hat{d}_k = \hat{d}_k^\top ((I - \gamma^d_k A_k) P_k^{d,u} (I - \gamma^d_k A_k)^\top)^{-1} \hat{d}_k
= ((I - \gamma^d_k A_k)^{-1} \hat{d}_k)^\top (P_k^{d,u})^{-1} ((I - \gamma^d_k A_k)^{-1} \hat{d}_k)
= (d_k^\top + (I - \gamma^d_k A_k)^{-1} \gamma^d_k b_k)^\top (P_k^{d,u})^{-1}
\times (d_k^\top + (I - \gamma^d_k A_k)^{-1} \gamma^d_k b_k),
$$

where $(I - \gamma^d_k A_k)^{-1} d_k = d_k^u + (I - \gamma^d_k A_k)^{-1} \gamma^d_k b_k$ has been applied. Now we expand and rearrange (37), resulting in the following:

$$
\hat{d}_k^\top (P_k^{d,u})^{-1} \hat{d}_k = (d_k^u)^\top (P_k^{d,u})^{-1} d_k^u + ((I - \gamma^d_k A_k)^{-1} \gamma^d_k b_k)^\top (P_k^{d,u})^{-1} ((I - \gamma^d_k A_k)^{-1} \gamma^d_k b_k)
+ 2 (d_k^\top)^\top (P_k^{d,u})^{-1} (d_k^\top + (I - \gamma^d_k A_k)^{-1} \gamma^d_k b_k)
= (d_k^u)^\top (P_k^{d,u})^{-1} d_k^u + (\gamma^d_k b_k)^\top ((I - \gamma^d_k A_k) P_k^{d,u} (I - \gamma^d_k A_k)^\top)^{-1} \gamma^d_k b_k
+ 2 (d_k^\top)^\top (d_k^\top + (I - \gamma^d_k A_k)^{-1} \gamma^d_k b_k)^\top (P_k^{d,u})^{-1} \gamma^d_k b_k.
$$

(38)

Applying (22) and (23) to (38), we have

$$
\hat{d}_k^\top (P_k^{d,u})^{-1} \hat{d}_k = (d_k^u)^\top (P_k^{d,u})^{-1} d_k^u
+ (\gamma^d_k b_k)^\top (I - \gamma^d_k A_k) P_k^{d,u} (I - \gamma^d_k A_k)^\top + 2 (d_k^\top)^\top (P_k^{d,u})^{-1} \gamma^d_k b_k.
$$

(39)

\[ \hat{\text{residue (res.)}} \]

Since $\hat{d}_k$ satisfies the input active constraint, we can substitute $b_k$ with $A_k d_k$. Then the residue defined in (39) can be written as follows:

$$
\text{res.} = (\gamma^d_k A_k d_k)^\top (P_k^{d,u})^{-1} \gamma^d_k A_k d_k
+ 2 (d_k^\top)^\top (P_k^{d,u})^{-1} \gamma^d_k b_k.
$$

(40)

Expanding and rearranging (40), we have the following:

$$
\text{res.} = 2 d_k^u P_k^d d_k - 2 d_k^u P_k^d d_k - 2 (d_k^\top)^\top P_k^d d_k
+ 2 d_k^u P_k^d d_k + \|\gamma^d_k A_k\|_2 d_k^\top (P_k^{d,u})^{-1} d_k
+ \|\gamma^d_k A_k\|_2^2 d_k^\top (P_k^{d,u})^{-1} d_k
- 2 \|\gamma^d_k A_k\|_2^2 d_k^\top (P_k^{d,u})^{-1} d_k,
$$

(41)

where $P_k^d \triangleq (\gamma^d_k A_k)^\top (P_k^{d,u})^{-1} > 0$. Using the result $\|d_k\| < \|d_k^u\|$ from Theorem 6 and the first inequality in Assumption 7, we obtain $\|d_k\| < \|d_k^u\| < \frac{1}{2}\|d\|$. Then we have $\text{res.} > 0$, since (11) to (13) are positive, respectively. Therefore, from (39), we have

$$
(d_k^u)^\top (P_k^{d,u})^{-1} d_k^u > \hat{d}_k^\top (P_k^{d,u})^{-1} d_k.
$$

(44)

Considering the condition in (44), we can divide the set $\mathcal{K} = \bigcup_{i=1}^3 \mathcal{K}_i$ into three partitions as follows:

$$
\mathcal{K}_1 \triangleq \{k \mid \|\gamma^d_k A_k\|^2 d_k^\top (P_k^{d,u})^{-1} d_k < \chi^2_{df}(\alpha)\}
\mathcal{K}_2 \triangleq \{k \mid \chi^2_{df}(\alpha) < \|\gamma^d_k A_k\|^2 d_k^\top (P_k^{d,u})^{-1} d_k \}
\mathcal{K}_3 \triangleq \{k \mid \|\gamma^d_k A_k\|^2 d_k^\top (P_k^{d,u})^{-1} d_k \leq \chi^2_{df}(\alpha)\}.
$$

According to (2), we have

$$
\sum_{k \in \mathcal{K}_i} (1_k) = \sum_{k \in \mathcal{K}_i} (1^*_k) \text{ for } i = 1, 2 \text{ and } 3
$$

(45)

$$
\sum_{k \in \mathcal{K}_i} (1_k) < \sum_{k \in \mathcal{K}_i} (1^*_k).
$$

(46)

Therefore, from (36), (45) and (46), we conclude that (35) holds, which completes the proof.
4.2. Stability Analysis

Although the projection reduces the estimation errors and their error covariances as shown in Theorems 9 and 10, it trades the unbiased estimation off according to (28) and (29). In the absence of the projection, Algorithm 1 reduces to the algorithm in [24], which is an unbiased estimation, while the traditional stability analysis for unbiased estimation becomes invalid after the projection is applied.

To prove the recursive stability of the biased estimation, it is essential to construct a recursive relation between the current estimation error \( \bar{x}_k \) and the previous estimation error \( \bar{x}_{k-1} \). However, the construction is not straightforward compared to that in filtering with equality constraints [35] or filtering without constraints [39]. Especially, it is difficult to find the exact recursive relation between \( \bar{x}_k \) and \( \bar{x}_{k-1} \) since \( \bar{x}_k \) is also a function of \( \bar{x}_{k-1} \), i.e., \( \bar{x}_k = \bar{x}_k - \gamma_k (\bar{x}_{k-1}^2 - c_k) \). Then, we have \( \bar{x}_k \neq (I - \gamma_k B_k) \bar{x}_{k-1} \), since the inequality \( \bar{x}_k \bar{x}_{k-1} \leq c_k \) holds.

To address this issue, we decompose the estimation error \( \bar{x}_k \) into two orthogonal spaces as in [40] and Lemma 5 [39] becomes \( \bar{x}_k = A_k \bar{x}_k \), where \( A_k \triangleq (I - \gamma_k B_k) + \alpha_k \gamma_k B_k \).

Note that \( \alpha_k \) is an unknown matrix and thus cannot be used for the algorithm. We use it only for analytical purposes. Now under the following assumptions, we present the stability analysis of the proposed Algorithm [1].

**Assumption 10.** We have \( \text{rank}(B_k) < n \) \( \forall k \). There exist \( \hat{a}, \hat{c}_y, \hat{g}, \hat{g}, \hat{m}, \hat{g}, \hat{b}, \hat{b} > 0 \), such that the following holds for all \( k \geq 0 \):

\[
\begin{align*}
    &\|A_k\| \leq \hat{a}, \\
    &\|C_k\| \leq \hat{c}_y, \\
    &\|M_k\| \leq \hat{m}, \\
    &Q_k \geq g I.
\end{align*}
\]

**Remark 11.** In Assumption 12, it is assumed that \( \text{rank}(B_k) < n \) \( \forall k \), i.e., the number of the state constraints are less than the number of state variables. The rest of Assumption 12 is widely used in the literature on extended Kalman filtering [39] and nonlinear input and state estimation [35].

To show the boundedness of the unconstrained state error covariance \( P_k^{x,u} \), we first define the matrices \( A_{k-1} \triangleq (I - G_{k-1} M_k C_k) A_{k-1} \) and \( A_{k-1} \triangleq (I - G_{k-1} M_k (C_k G_{k-1} M_k)^{-1} C_k) A_{k-1} \).

**Theorem 12.** Let the pair \((C_k, A_{k-1})\) be uniformly detectable, then the unconstrained state error covariance \( P_k^{x,u} \) is bounded, i.e., there exist non-negative constants \( p \) and \( \bar{p} \) such that \( p I \leq P_k^{x,u} \leq \bar{p} I \) for all \( k \).

**Proof:** The unconstrained state estimation error can be found by

\[
\begin{align*}
\bar{x}_k &= (I - \bar{L}_k C_k) A_{k-1} \bar{x}_{k-1} + (I - \bar{L}_k C_k) \bar{w}_{k-1} + \bar{L}_k \bar{v}_k,
\end{align*}
\]

where \( \bar{w}_{k-1} \triangleq (I - \bar{G}_{k-1} M_k C_k) \bar{w}_{k-1} \), and \( \bar{L}_k \triangleq L_k C_k G_{k-1} M_k - L_k - G_{k-1} M_k \). Therefore, the update law of unconstrained covariance is calculated from (47) and (27) as follows:

\[
\begin{align*}
P_k^{x,u} &= (I - \bar{L}_k C_k) A_{k-1} \bar{P}_{k-1}^{x,u} A_{k-1}^\top + (I - \bar{L}_k C_k) \bar{Q}_{k-1} (I - \bar{L}_k C_k)^\top,
\end{align*}
\]

where \( \bar{Q}_{k-1} \triangleq \mathbb{E}[\bar{w}_{k-1}^\top \bar{w}_{k-1}] \). The covariance update law [18] is identical to the covariance update law of the Kalman filtering solution of the transformed system

\[
\begin{align*}
x_k &= A_{k-1} \bar{x}_{k-1} + \bar{w}_{k-1} - \bar{v}_k,
\end{align*}
\]

where \( \bar{w}_{k-1} \triangleq -G_{k-1} M_k C_k \bar{w}_{k-1} - G_{k-1} M_k \bar{v}_k + \bar{w}_{k-1} \). However, in the transformed system, the process noise and measurement noise are correlated, i.e., \( \mathbb{E} [\bar{w}_{k-1}^\top \bar{v}_k] = -G_{k-1} M_k R_k \neq 0 \). To decouple the noises, we add a zero term \( \bar{Z}_k \) to the state equation in [49], and obtain the following:

\[
\begin{align*}
x_k &= A_{k-1} \bar{x}_{k-1} + \bar{u}_{k-1} + \bar{w}_{k-1},
\end{align*}
\]

where \( \bar{A}_{k-1} \triangleq (I - \bar{Z}_k C_k) A_{k-1} \), \( \bar{u}_{k-1} \triangleq \bar{Z}_k y_k \) is the known input, and \( \bar{w}_{k-1} = (I - \bar{Z}_k C_k) \bar{w}_{k-1} - \bar{Z}_k \bar{v}_k \) is the new process noise. The new process noise and the measurement noise could be decoupled by choosing the gain \( \bar{Z}_k \) such that \( \mathbb{E} [\bar{w}_{k-1}^\top \bar{v}_k] = 0 \). The solution can be found by \( \bar{Z}_k = G_{k-1} M_k (C_k C_{k-1} M_k)^{-1} \bar{Z}_k A_{k-1} \).

Then, the system becomes

\[
\begin{align*}
x_k &= \bar{A}_{k-1} \bar{x}_{k-1} + \bar{u}_{k-1} + \bar{w}_{k-1} - \bar{v}_k,
\end{align*}
\]

Since the pair \((C_k, \bar{A}_{k-1})\) is uniformly detectable, by Theorem 5.2 in [28], the statement holds.

**Theorem 12** shows that the uniform detectability of the transformed system is one of the sufficient conditions of boundedness of \( P_k^{x,u} \). Under the assumption of boundedness of \( P_k^{x,u} \) from Theorem 12, we show the constrained estimation errors \( \bar{x}_k \) and \( \bar{d}_k \) are practically exponentially stable in mean square as in Theorem 13.

**Theorem 13.** Consider Assumption 10 and assume that there exist non-negative constants \( \hat{a}, \hat{b}, \hat{c}_x, \hat{c}_d, \hat{c}_y, \hat{c}_d, \hat{c}_y \) such that for all \( k \)

\[
\begin{align*}
\mathbb{E} [\|\bar{x}_k\|^2] &\leq \hat{a} e^{-\hat{b} k} \mathbb{E} [\|\bar{x}_0\|^2] + c_x, \\
\mathbb{E} [\|\bar{d}_k\|^2] &\leq \hat{a} e^{-\hat{b} k} \mathbb{E} [\|\bar{d}_0\|^2] + c_d.
\end{align*}
\]

---

2Please refer to [35] for the definition of uniform detectability.
Proof: Consider the Lyapunov function $V_k = (\hat{x}_k^u)^T(P_{k}^{-u}u - 1)(\hat{x}_k^u)$. After substituting \((17)\) into the Lyapunov function, we obtain

\[
\begin{align*}
V_k &= (\hat{x}_k^u)^T A_k^{-1} (I - L_k C_k)^T (P_u^{-u})^{-1} \\
&\times (I - L_k C_k) A_k^{-1} (I - L_k C_k)^T \\
&+ 2 (\hat{x}_k^u)^T \Gamma_k^{-1} A_k^{-1} (I - L_k C_k)^T \\
&\times (P_u^{-u})^{-1} (I - L_k C_k) w_{k-1} \\
&+ 2 (\hat{x}_k^u)^T \Gamma_k^{-1} A_k^{-1} (I - L_k C_k)^T (P_u^{-u})^{-1} \tilde{L}_k v_k \\
&+ w_{k-1}^T (I - L_k C_k)^T (P_u^{-u})^{-1} (I - L_k C_k) w_{k-1} \\
&+ w_k^T (I - L_k C_k)^T (P_u^{-u})^{-1} \tilde{L}_k v_k \\
&+ v_k^T L_k (P_u^{-u})^{-1} L_k v_k.
\end{align*}
\]

By the uncorrelated property \((19)\) of $w_{k-1}$, $v_k$ and $\hat{x}_{k-1}$, the Lyapunov function \((50)\) becomes

\[
\begin{align*}
\mathbb{E}[V_k] &= \mathbb{E}((\hat{x}_k^u)^T A_k^{-1} (I - L_k C_k)^T (P_u^{-u})^{-1} \\
&\times A_k^{-1} (I - L_k C_k) A_k^{-1} (I - L_k C_k)^T \\
&+ w_k^T (I - L_k C_k)^T (P_u^{-u})^{-1} L_k v_k).
\end{align*}
\]

The following statements are formulated to deal with each term in \((51)\).

Claim 14. There exists a constant $\delta \triangleq \left(\frac{q^2}{d^2} + 1\right)^{-1} \in (0, 1)$, such that $\Gamma_k^{-1} A_k^{-1} (I - L_k C_k)^T (P_u^{-u})^{-1} (I - L_k C_k) A_k^{-1} \Gamma_k^{-1} < \delta (P_u^{-u})^{-1}$.

Proof: Since $\text{rank}(B_k) < n \forall k$, it holds that $\text{rank}(B_k) < n \forall k$ and thus $\Gamma \neq 0$. Therefore, \(\|\Gamma_k^{-1}\| = 1\) because $\gamma_k^{-1} B_k - 1$ is a projection matrix. From Assumption 10 and Theorem 12, we have $Q_k^{-1} \geq q' I$, and $P_u^{-u} \preceq p' I$. Since $\|A_k^{-1}\|$ is upper bounded by $a' \triangleq \frac{1}{2} (1 + \gamma_0 m_0)$, we can have $A_k^{-1} A_k^{-1} \leq a'^2 I$. Then we have

\[
\begin{align*}
Q_k^{-1} &\geq \frac{q' A_k^{-1} \Gamma_k^{-1}}{a'^2} \\
&\geq \frac{q'}{a'^2} \frac{\Gamma_k^{-1} A_k^{-1} \Gamma_k^{-1}}{a'^2} \\
&\geq \frac{q'}{a'^2 p} A_k^{-1} \Gamma_k^{-1} P_u^{-u} A_k^{-1} \Gamma_k^{-1}.
\end{align*}
\]

Substitution of \((52)\) into \((46)\) yields

\[
\begin{align*}
P_k^{-u} &= (1 + \frac{q'}{a'^2 p}) (I - L_k C_k) A_k^{-1} \Gamma_k^{-1} P_k^{-u} A_k^{-1} \Gamma_k^{-1} \\
&\times (I - L_k C_k)^T > 0,
\end{align*}
\]

where the inequality holds because $R_k > 0$. As $1 + \frac{q'}{a'^2 p} P_u^{-u} > 0$, the inverse of the left hand side of \((53)\) exists and is symmetric positive definite. By the matrix inversion lemma \([12]\), it follows that

\[
\begin{align*}
1 + \frac{q'}{a'^2 p} (P_k^{-u})^{-1} - \frac{\Gamma_k^{-1} A_k^{-1} (I - L_k C_k)^T}{(I - L_k C_k) A_k^{-1} \Gamma_k^{-1} > 0}.
\end{align*}
\]

Since $\gamma_k^{-1} B_k - 1$ is a positive definite matrix, and $\|A_k^{-1}\| \leq 1$, we have

\[
I - \gamma_k^{-1} B_k^{-1} \leq \Gamma_k^{-1} \leq I - \gamma_k^{-1} B_k^{-1} + \alpha_k^{-1} \gamma_k^{-1} B_k^{-1} \leq I,
\]

which implies $\|\Gamma_k^{-1}\| \leq 1$. Since $\|\Gamma_k^{-1}\| = 1$ and $\|\Gamma_k^{-1}\| \leq 1$, inequality \((54)\) proves the claim.

Claim 15. There exists a positive constant $c \triangleq p(1 + l_0) (1 + \gamma_0 m_0)^2 \hat{q} \text{ rank}(Q_k^{-1}) + p(l_0 m_0 - l + \gamma_0)^2 \|v_k\| \|w_k\| \|r_2 \| \text{ rank}(R_k)$, such that

\[
\mathbb{E}[\|L_k C_k\| (P_k^{-u})^{-1} (I - L_k C_k)] \|w_k\| \|v_k\| \|w_{k-1}\| \leq c.
\]

Proof: The first term is bounded by

\[
\begin{align*}
\mathbb{E}[\|L_k C_k\| (P_k^{-u})^{-1} (I - L_k C_k)] \|w_k\| \|v_k\| \|w_{k-1}\| &\leq \hat{p}(l_0 m_0 + l + \gamma_0)^2 \hat{q} \text{ rank}(Q_k^{-1}),
\end{align*}
\]

where we apply $\|w_{k-1}\| \leq \|w_k\|^2 = \text{ tr}(w_k w_k^T) \leq \hat{q} \text{ rank}(Q_k^{-1})$. Likewise, the second term is bounded by

\[
\begin{align*}
\mathbb{E}[\|L_k C_k\| (P_k^{-u})^{-1} L_k \|v_k\| \|w_{k-1}\| &\leq \hat{p}(l_0 m_0 + l + \gamma_0)^2 \|r_2 \| \text{ rank}(R_k).
\end{align*}
\]

These complete the proof.

Through Claims 14 and 15, \((51)\) becomes

\[
\mathbb{E}[V_k] \leq \delta \mathbb{E}[V_{k-1}] + c.
\]

By recursively applying the above relation, we have

\[
\begin{align*}
\mathbb{E}[V_k] &\leq \delta^k \mathbb{E}[V_0] + \sum_{i=0}^{k-1} \delta^i c \leq \delta^k \mathbb{E}[V_0] + \sum_{i=0}^{\infty} \delta^i c \\
&= \delta^k \mathbb{E}[V_0] + \frac{c}{1 - \delta},
\end{align*}
\]

which implies practical exponential stability of the estimation error

\[
\begin{align*}
\mathbb{E}[\|\hat{x}_k^u\|^2] &\leq \frac{\delta^k}{p} \delta^k \mathbb{E}[\|\hat{x}_k^u\|^2] + \frac{cp}{(1 - \delta)} \\
&= a'_x e^{-\delta k} \mathbb{E}[\|\hat{x}_0^u\|^2] + c'_x,
\end{align*}
\]

where $(\hat{x}_k^u)^T (P_k^{-u})^{-1} (\hat{x}_k^u) \geq \min \left(\|P_k^{-u}\|^{-1}\|\hat{x}_k^u\|^2 \geq \frac{1}{p} \|\hat{x}_k^u\|^2\right)$ and $(\hat{x}_k^u)^T (P_k^{-u})^{-1} (\hat{x}_k^u) \leq \max \left(\|P_k^{-u}\|^{-1}\|\hat{x}_k^u\|^2 \leq \frac{1}{p} \|\hat{x}_k^u\|^2\right)$ have been applied. Constants are defined by

\[
\begin{align*}
a'_x &\triangleq \frac{\delta^k}{p} \delta^k \mathbb{E}[\|\hat{x}_0^u\|^2] \quad b'_x \triangleq \ln(1 + \frac{q'}{h^2 a'^2 p}) \\ c'_x &\triangleq \frac{cp}{(1 - \delta)}.
\end{align*}
\]

Since $\hat{x}_k$ is a linear transformation of $\hat{x}_k^u$, the same stability holds for $\hat{x}_k$. Likewise, the same stability holds for $\hat{d}_k$ in \((21)\) because it is a linear transformation of $\hat{x}_k$. We omit its details.
5. Illustrative Example

In this example, we test Algorithm 1 on a vehicle model with input and state constraints and compare the estimation accuracy and the detection performance with an unconstrained algorithm.

5.1. Experimental Setup

![Figure 2: Kinematic Bicycle Model.](image)

We consider a kinematic bicycle model (Fig. 2) in [42]. The nonlinear continuous-time model is given as

\[
\begin{align*}
\dot{x} &= v \cos(\psi + \beta) \\
\dot{y} &= v \sin(\psi + \beta) \\
\dot{\psi} &= \frac{v}{l_r} \sin(\beta) \\
\dot{\beta} &= \arctan\left(\frac{l_r}{l_f + l_r} \tan(\delta)\right),
\end{align*}
\]

where \(x\) and \(y\) are the coordinates of the center of mass, \(v\) is the velocity of the center of mass, \(\beta\) is the angle of the velocity \(v\) with respect to the longitudinal axis of the vehicle, \(\mu\) is the acceleration, \(\psi\) is the heading angle of the vehicle, \(\delta\) is the steering angle of the front wheel, and \(l_f\) and \(l_r\) represent the distance from the center of mass of the vehicle to the front and rear axles, respectively.

Since the proposed algorithm is for linear discrete-time systems, we perform the linearization and discretization as in [43] with sampling time \(T_s = 0.01\) s. We rewrite the system in the form of (5), where \(x_k = [x_k, y_k, \psi_k, v_k]^T\) is the state vector, \(u_k = [\beta^u_k, \alpha^u_k]^T = \left[\arctan\left(\frac{l_r}{l_f + l_r} \tan(\delta^u_k)\right), \alpha^u_k\right]^T\) is the input vector, and \(d_k = [\beta^d_k, \alpha^d_k]^T = \left[\arctan\left(\frac{l_r}{l_f + l_r} \tan(\delta^d_k)\right), \alpha^d_k\right]^T\) is the attack input vector. We consider the scenario that attack input is injected into the input, i.e., \(G_k = B_k\). The system matrices are given as follows:

\[
A_k = \begin{bmatrix}
1 & 0 & 0 & T_s \\
0 & 1 & v_k T_s & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad B_k = G_k = \begin{bmatrix}
0 & 0 \\
v_k T_s & 0 \\
0 & 0 \\
0 & 0 & 0 & T_s
\end{bmatrix},
\]

and \(C_k = I\). The noise covariances \(Q_k\) and \(R_k\) are considered as diagonal matrices with \(\text{diag}(Q_k) = [0.1, 0.1, 0.001, 0.0001]\) and \(\text{diag}(R_k) = [0.01, 0.01, 0.001, 0.00001]\).

The vehicle is assumed to have state constraints on the location \(0 \leq x_k \leq 20, 0 \leq y_k \leq 5\) and the velocity \(0 \leq v_k \leq 22\), and input constraints on the steering angle \(|\delta| \leq 1.0472\) and the acceleration \(|\alpha| \leq 3.5\).

The unknown attack signals are

\[
\begin{align*}
\delta^d_k &= \begin{cases}
0, & 0 \leq k < 100 \\
1.1 \sin(0.05k), & 0 \leq k < 100
\end{cases} \\
\alpha^d_k &= \begin{cases}
0, & 0 \leq k < 100 \\
3.5, & 100n \leq k < 100(n + 1) \\
-3.5, & 100(n + 1) \leq k < 100(n + 2)
\end{cases},
\end{align*}
\]

where \(n = 1, 2, \cdots, 5\).

The constraints on the vehicle can be formulated by inequality constraints as in [44].

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\delta^d_k \\
\alpha^d_k \\
x_k \\
y_k \\
\psi_k \\
v_k
\end{bmatrix}
\leq
\begin{bmatrix}
1.0472 - \delta^d_{k-1} \\
1.0472 + \delta^d_{k-1} \\
3.5 - \alpha^d_{k-1} \\
3.5 + \alpha^d_{k-1}
\end{bmatrix}.
\]

To reduce the effect of instantaneous noises, the cumulative sum algorithm (CUSUM) is adopted [44]. The \(\chi^2\) test is utilized in a cumulative form. The \(\chi^2\) CUSUM detector is characterized by the detector state \(S_k \in \mathbb{R}_+\):

\[
S_k = \phi S_{k-1} + \tilde{d}_k^T P_k^{-1} \tilde{d}_k, \quad S_0 = 0,
\]

where \(0 < \phi < 1\) is the pre-determined forgetting rate. At each time \(k\), the CUSUM detector [55] is used to update the detector state \(S_k\) and detect the attack. In particular, we conclude that the attack is presented if

\[
S_k > \sum_{i=0}^{\infty} \phi^i \chi^2_f(\alpha) = \frac{\chi^2_f(\alpha)}{1 - \phi}.
\]

All values are in standard SI units: \(m\) (meter) for \(l_f, l_r, x_k,\) and \(y_k\); \(rad\) for \(\delta^u_k, \beta^u_k, \delta^d_k,\) and \(\psi_k\); \(m/s\) for \(v_k\); \(m/s^2\) for \(\alpha^u_k\) and \(\alpha^d_k\).

5.2. Results

We show a comparison of the proposed algorithm (CARE) and the unified linear input and state estimator (ISE) introduced in [44]. Figure 3 shows the estimation errors of the constrained states \((x_k, y_k)\) and the traces of the state error covariances, and Fig. 4 shows the unknown attack signals and their estimates and traces of the attack estimation error covariances. As expected, CARE produces
smaller state estimation error and lower covariance. When the attack happens after $k = 100$, the estimates obtained by CARE are closer to the true values and have lower error covariances (cf. Table 1). The estimates are used to calculate the detector state $S_k$ in (55). The statistical significance of the attack is tested using the CUSUM detector. The threshold is calculated by $\chi^2_{df}(\alpha)$, where $\alpha = 0.01$ and the forgetting rate $\phi = 0.15$. The detector states and the threshold are plotted in log-scale (Fig. 5). When the attack is present, CARE can detect the attack by producing high detector state values above the threshold, while the detector state values from ISE are oscillating around the threshold, suffering from a high false negative rate of 66.44%.

6. Conclusion

In this paper, we presented a constrained attack-resilient estimation algorithm (CARE) of linear stochastic cyber-physical systems. The proposed algorithm produces minimum-variance unbiased estimates and utilizes physical constraints and operational limitations to improve estimation accuracy and detection performance via projection. In particular, CARE first provides minimum-variance unbiased estimates, and then these estimates are projected onto the constrained space. We formally proved that estimation errors and their covariances from CARE are less than those from unconstrained algorithms and showed that CARE had improved false negative rate in attack detection. Moreover, we proved that the estimation errors of the proposed estimation algorithm are practically exponentially stable. A simulation of attacks on a vehicle demonstrates the effectiveness of the proposed algorithm and reveals better attack-resilient properties compared to an existing algorithm.

A. Gauss-Markov Theorem

Theorem 16 (Gauss-Markov Theorem [45]). Given the linear model $y = Hx + v$, where $v$ is a zero-mean random variable with positive-definite covariance matrix $R_v$, and $H$ is full rank $m \times n$ matrix with $m \geq n$, the minimum-variance-unbiased linear estimator of $x$ given $y$ is $\hat{x} = (H^T R_v^{-1} H)^{-1} H^T R_v^{-1} y$.

Acknowledgements

This work has been supported by the National Science Foundation (ECCS-1739732 and CMMI-1663460).

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