Frequency Detection and Change Point Estimation for Time Series of Complex Oscillation

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Abstract

We consider detecting the evolutionary oscillatory pattern of a signal when it is contaminated by non-stationary noises with complexly time-varying data generating mechanism. A high-dimensional dense progressive periodogram test is proposed to accurately detect all oscillatory frequencies. A further phase-adjusted local change point detection algorithm is applied in the frequency domain to detect the locations at which the oscillatory pattern changes. Our method is shown to be able to detect all oscillatory frequencies and the corresponding change points within an accurate range with a prescribed probability asymptotically. This study is motivated by oscillatory frequency estimation and change point detection problems encountered in physiological time series analysis. An application to spindle detection and estimation in sleep EEG data is used to illustrate the usefulness of the proposed methodology. A Gaussian approximation scheme and an overlapping-block multiplier bootstrap methodology for sums of complex-valued high dimensional non-stationary time series without variance lower bounds are established, which could be of independent interest.

1 Introduction

A major task in physiological time series analysis is to detect and estimate the complex oscillatory pattern of the observed stochastic process over time. In the past century, researchers have established various physiological knowledge about the complex oscillatory signals and its clinical applications; see, for example, dynamics in the breathing signal (Benchetrit 2000) and electrocardiogram (Malik 1996) for a far from exhaustive list of reference. However, there are still a lot left unknown when we encounter a physiological time series, probably due to its complicated characteristic structure. Specifically, common characteristics shared by physiological time series include but not exclusively the following. First, the time series is usually composed of multiple oscillatory components, and each component usually

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oscillates with time-varying frequency, amplitude, or even oscillatory morphology. Second, the signal is usually contaminated by non-stationary noise, and various artifacts. Moreover, the frequency or amplitude of an oscillatory component might abruptly jump from one to another.

There have been quite a few analysis tools developed in the time-frequency (TF) analysis society (Daubechies (1992), Flandrin (1998)) toward studying this kind of time series with various clinical applications. However, TF analysis tools, particularly those nonlinear-type tools, are not widely considered in the time series society and their statistical properties are largely unknown, except few current efforts; for example, Adak (1998); Nason et al. (2000); Chen et al. (2014); Bruna et al. (2015); Sourisseau et al. (2019). It is interesting to ask if it is possible to apply existing TF analysis tools and the underlying ideas, to design more suitable statistical analysis and inferential tools. In this study, motivated by the clinical needs of detecting oscillatory components and quantifying its dynamics, we focus on two critical problems with a direct clinical interest. Specifically, under the oscillatory signal model with non-stationary noise model that we will introduce soon, we design a statistic to determine if there is an oscillatory component in a given physiological signal, and provide a strategy applying the wavelet analysis to decide if an oscillatory component has a change point behavior in its frequency over time.

### 1.1 Challenges in statistical analysis of complex oscillation

Let \( \{X_{i,n} := \mu_{i,n} + \epsilon_{i,n}\}_{i=1}^{n} \), where \( \mathbb{E}(\epsilon_{i,n}) = 0 \), be an observed non-stationary time series and \( \mu_{i,n} \) is a deterministic signal. A commonly used tool for oscillatory frequency detection is the periodogram defined as \( I_{n,X}(\omega) = \frac{1}{n} \left| \sum_{j=1}^{n} X_{j,n} e^{-i\omega j} \right|^2 \), \( \omega \in [0, \pi] \). In principle, \( I_{n,X}(\omega) \) should be large if \( X_{i,n} \) is oscillatory at frequency \( \omega \). However, non-stationarity in the noises \( \epsilon_{i,n} \) as well as possible change points in the oscillation bring great challenges to the rigorous statistical analysis of complex oscillatory signals. We will discuss these challenges in detail in the next two subsections.

#### 1.1.1 Spectral dependency and changes in phase

Let \( \omega_{j}^* = 2\pi j/n, \ j = 1, 2, \cdots, \lfloor n/2 \rfloor \) be the canonical frequencies. A fundamental result for classic oscillatory frequency detection is that \( I_{n,X}(\omega_{j}^*), \ j = 1, 2, \cdots, \lfloor n/2 \rfloor \), are asymptotically independent when \( \{X_{i,n}\} \) is stationary. See for instance Davis and Mikosch (1999) and Lin and Liu (2009). Consequently a Gumbel-type limiting distribution can be derived for the maximum deviation of the periodogram on the canonical frequencies under the null hypothesis that there is no oscillation. Nevertheless, the periodograms on the canonical frequencies are no longer asymptotically independent for non-stationary time series. See, for instance, Dwivedi and Subba Rao (2011) and Zhou (2014) for detailed calculations and discussions. As a consequence, it has been a difficult and open problem to derive the maximum deviation
of the periodogram on a dense set of frequencies for non-stationary time series. Without such results, it is difficult to distinguish peaks of the periodogram which reflect the oscillation from those caused by the random noise.

Another difficulty in frequency detection under complex oscillation is that changes in the phase of the oscillation may dampen the periodogram. For instance, consider the case $\mu_{i,n} = \cos(2\pi \omega i)$ for $1 \leq i \leq n/2$ and $\mu_{i,n} = \cos(2\pi \omega i + \pi)$ for $i > n/2$ where we assume that $\omega n/2$ is an integer. Clearly $\mu_{i,n}$ is oscillating at frequency $2\pi \omega$. However, the periodogram will not have a peak at $2\pi \omega$ as the Fourier transforms of $\mu_{i,n}$ before and after the change of phase cancel each other. We point out that the classic frequency detection literature in statistics typically assumes that there is no abrupt change in the phase of the oscillation (cf. e.g. Hannan (1961), Lin and Liu (2009)).

1.1.2 Spectral energy leak

One major challenge in oscillation change point detection is the spectral energy leak phenomenon. For $\omega \in [0, \pi]$, let $\{L_n(i,\omega) := \sum_{k=1}^{i} X_{k,n} e^{\sqrt{-1} \omega k}\}_{i=1}^{n}$ be the partial sum process of the Fourier transform of $X_{i,n}$ at frequency $\omega$. One of the most popular change point detection algorithms is the cumulative sum (CUSUM) test, which utilizes $C_n(i,\omega) := [L_n(i,\omega) - i/nL_n(n,\omega)]/\sqrt{n}$. In principle $C_n(i,\omega)$ should be small uniformly across $i$ if there is no oscillation change point at frequency $\omega$. Now let us investigate a simple example where $X_{i,n} = \cos(\omega^0 i) + \epsilon_i$ with $\epsilon_i$ i.i.d. standard normal. Clearly $X_{i,n}$ is oscillating at $\omega^0$ without any change points. Figure 1 plots the heat map of $|C_n(i,\omega)|$ for this example. We observe large values of $|C_n(i,\omega)|$ in a frequency band around $\omega^0$, although $C_n(i,\omega)$ are indeed uniformly small at frequency $\omega^0$. Therefore, the CUSUM test fails in this case as it produces strong false positive information near an oscillatory frequency. This problem persists if we apply other change point detection algorithms such as those based on binary segmentation or dynamic programming to the spectral domain. The cause of the problem is spectral energy leak in the sense that $\sum_{k=1}^{i} \cos(\omega^0 k)e^{\sqrt{-1} \omega k}$ is a nonlinear function of $i$ with magnitude $O(|\omega - \omega^0|^{-1})$ if $i$ is large and $\omega$ is close (but not too close) to $\omega^0$.

The spectral energy leak problem persists if we estimate the oscillatory frequencies first and then apply change point detection algorithms directly to the Fourier transforms at the estimated frequencies. It is well-known that the parametric rate for oscillatory frequency estimation is $O_p(n^{-3/2})$ (cf. Genton and Hall (2007)). Even at this very fast convergence rate, it can be shown that $|L_n(i,\omega) - L_n(i,\hat{\omega})| = O_p(\sqrt{n})$ for sufficiently large $i$, where $\omega$ is a true oscillatory frequency and $\hat{\omega}$ is its estimate; see Lemma D.10 and Remark 3 of the supplementary material for more detailed calculations and discussion. The estimation error is not negligible asymptotically and change point detection algorithms will behave erratically if we simply plug-in the estimated oscillatory frequencies.
Figure 1: Heat map of CUSUM statistics $|C_n(i, \omega)|$ with $n = 1000$, mean function $\mu_{i,n} = \cos(0.6\pi i)$ and $\epsilon_{i,n}$ i.i.d. $N(0, 1)$. Plot shows energy leakage around frequency $\omega = 0.6\pi$.

1.2 Proposed methodology and its theoretical property

In this paper, we devise a two-stage methodology for oscillatory frequency detection and change point estimation respectively which addresses the aforementioned challenges.

1.2.1 The dense progressive periodogram test

One limitation of the classic oscillatory frequency detection algorithms is that only the canonical frequencies are considered. Hence the estimation accuracy is at most $O_P(1/n)$ which is slower than the parametric rate $O_P(n^{-3/2})$ for oscillatory frequency estimation (Genton and Hall, 2007). To address the latter issue as well as phase cancellation, we propose to investigate the progressive periodogram defined as $\{|L_n(i, \omega)|/\sqrt{n}\}_{i=1}^n$ on a dense grid of frequencies with mesh size no larger than $O(n^{-3/2})$ for the oscillatory frequency detection. Note that in principle $\max_{1 \leq i \leq n} |L_n(i, \omega)|/\sqrt{n}$ should be large if there are oscillations of sufficient length with possible occasional abrupt phase changes at frequency $\omega$.

The key to the successful implementation of the above-mentioned dense progressive periodogram test (DPPT) is to investigate the maximum deviation

$$F(W) := \max_{\omega \in W} \max_{1 \leq i \leq n} |L_n(i, \omega)|/\sqrt{n}$$

(1)

under the null hypothesis that there is no oscillation, where $W$ is a dense collection of frequencies with cardinality $p$. As we mentioned in Section 1.1, the latter is a difficult problem due to spectral dependency caused by non-stationarity of $\{\epsilon_{i,n}\}$. In this paper, we tackle this problem by viewing (1) as the maximum of a complex-valued high-dimensional dependent random vector of dimensionality $pn$. Then we utilize Stein’s method of expectation approximation (Stein, 1986) to show that the law of $F(W)$ can be well approximated by that of the DPPT of a non-stationary Gaussian process which preserves the covariance structure of $\{X_{i,n}\}$. A high-dimensional extension to the overlapping-block multiplier bootstrap (OBMB) in Zhou (2013) is proposed to approximate the behaviour of the latter non-stationary Gaussian DPPT. We will show that the DPPT with the OBMB is able to detect oscillatory...
frequencies within an $O_p(n^{-1})$ range if there are changes in the phases of the oscillation. For oscillatory frequencies whose phases do not change over time, the DPPT is able to detect them within a nearly optimal $O_p(n^{-3/2} \log n)$ range. In the special case where there is no oscillation, the DPPT will be shown to be able to accept the null hypothesis of no oscillation with a prescribed probability asymptotically.

In the literature, Gaussian approximations to the maximum of high-dimensional sums using Stein’s method were investigated by, among others, Chernozhukov et al. (2013) for independent data and Zhang and Wu (2017) and Zhang and Cheng (2018) for time series. An important assumption in the latter papers is that the variances of the vector components should be bounded from both above and below and hence be balanced. Consequently, their results cannot be used directly for the DPPT since the variances of $L_n(i, \omega)$ are proportional to $i$ and hence are highly unbalanced across time. In this paper, we generalize Nazarov’s anti-concentration inequality (Nazarov, 2003) to the unbalanced variance case and extend the results of the aforementioned papers to complex-valued high-dimensional time series without variance lower bound. As many high-dimensional problems are without balanced variances among the vector components, our result may be of separate interest.

1.2.2 The phase-adjusted local change point detection algorithm

As mentioned in Section 1.1.2, the spectral energy leak phenomenon has to be carefully addressed in order to perform oscillation change point detection. One remedy to the latter phenomenon is to reduce error amplification of the Fourier transforms. This observation inspires us to consider a local change point detection algorithm. Specifically, for any oscillatory frequency $\hat{\omega}_k$ estimated from stage 1 and time point $i$, we utilize the norm of the difference between phase-adjusted local Fourier transforms

$$T(i) = \frac{1}{\sqrt{2m}} \left| \sum_{l=i-m}^{i} e^{-\sqrt{-1}\hat{\omega}_k(l-i)} X_l - \sum_{l=i+1}^{i+m+1} e^{-\sqrt{-1}\hat{\omega}_k(l-i)} X_l \right|$$

to test whether there is an oscillation change point at time $i$ and frequency $\omega_k$, where $m$ is a bandwidth controlling the size of the local neighborhood. Note that the global phase $e^{-\sqrt{-1}\hat{\omega}_ki}$ has modulation 1 and can be removed, but it is kept to simplify the design of the subsequent bootstrap. $T(i)$ should be large if the oscillatory pattern at frequency $\omega_k$ changes at time $i$. Since $T(i)$ performs Fourier transforms only in a radius $m$ neighborhood of $i$, the angles of the Fourier transforms are amplified at most $m$ times uniformly over time. As we will require $m/n \to 0$, the energy leak problem is greatly reduced and will be shown to be asymptotically negligible. We adopt an extension of the OBMB with adjusted phases to approximate the maximum deviation of $T(i)$ uniformly across time $i$. We will show that the local change point detection algorithm has the correct Type-I error rate asymptotically if there is no change point at the oscillatory frequency. If there are oscillation change points, the latter algorithm is able to detect all change points within an $O(\log m)$ range with a
pre-specified probability asymptotically, where the $O(\log \tilde{m})$ rate is almost the parametric $O(1)$ rate for change point detection except a factor of logarithm.

1.3 Literature review

The statistics literature of unknown periodicity detection dates back at least to [Fisher (1929)] who considered testing the existence of a sinusoidal signal under i.i.d. Gaussian noise. Fisher’s test was based on the maximum of the periodogram over the canonical frequencies and was later generalized to accommodate stationary and dependent noises and multiple oscillatory frequencies; see for instance [Hannan (1961), Chiu (1989) and Lin and Liu (2009)]. See also [Paraschakis and Dahlhaus (2012)] for the estimation of time-varying oscillatory frequencies and phases. To our knowledge, none of the previous spectral domain literature on periodicity detection considered non-stationary noises due to the difficulties mentioned above. On the other hand, there are a few statistics papers, see for instance [Oh et al. (2004)] and [Genton and Hall (2007)], considering time domain estimation of unknown periodicity with application to light curve estimation of variable stars. See also the related astronomy literature cited therein. These papers typically assume that there is an oscillatory signal and aim at estimating the period and the corresponding periodic function. Another interesting contribution is [Dahlhaus et al. (2017)] where the authors consider estimating a generalized state-space model with an unknown periodic pattern function and a hidden stochastic and integrated phase process. It is worth mentioning that classic time series analysis of seasonal processes typically assumes that the periodicity is known and then removes the periodic trend by differencing the process at an appropriate order [Box et al. (2015)].

To our knowledge, there exists no statistics literature on oscillation change point detection. The huge recent statistics literature on change point detection typically focuses on problems in the time domain and hence are free from the energy leak phenomenon in the spectral domain. For reviews of recent advances in change point detection, we refer the readers to [Aue and Horváth (2013)] and [Niu et al. (2016)]. On the other hand, in the past decades, time-frequency (TF) analysis has been widely studied in the applied mathematics and application fields due to its flexibility to handle complicated and nonstationary time series. TF tools can be roughly classified into three types – linear, bilinear and nonlinear [Flandrin (1998)]. Since the linear-type TF analysis are directly related to this work, we focus on it. The basic idea is dividing the signal into segments and evaluating the spectrum for each segment, where how the signal is partitioned distinguishes different methods. When a fixed window is chosen and the Fourier transform is applied, it is the short-time Fourier transform (STFT) [Flandrin (1998)]; when the segments depend on a dilated mother wavelet, it is the continuous wavelet transform (CWT) [Daubechies (1992)]. Due to the fundamental difference of their associated group structure (Heisenberg group for STFT and the affine group for the CWT), their fundamental differences manifest in various aspects. What concerns us in this work is their capability to study different functional spaces [Daubechies (1992)]. Particularly, the
CWT can be applied to characterize local regularity, and hence has been widely applied to study singularities in the signal processing (Jaffard and Meyer, 1996). We remark that the upcoming oscillatory component detection statistic can be understood as applying the STFT with the 0-1 kernel, and the local change point detection statistic can be viewed as detecting discontinuity by the Haar wavelet.

The rest of the paper is organized as follows. Section 2 introduces a flexible class of non-stationary time series models for the noises \( \{ \epsilon_{i,n} \} \). In Section 3 we introduce the two-stage methodology in detail. Section 4 investigates the consistency and accuracy of the proposed methodology. In particular, some optimality properties of the methodology are established. Section 5 performs numerical experiments to investigate the finite sample property of the two-stage methodology. In Section 6 we apply our methods to a sleep EEG dataset for spindle detection and estimation. Finally, additional simulation results, Gaussian approximation and comparison schemes, and proofs of the theoretical results are put in the online supplementary material.

### 2 Piecewise locally stationary time series model

Motivated by the aforementioned complex oscillatory pattern detection and estimation problems, consider an observed time series \( X = \{X_{i,n}\}_{i=1}^{n} \) which follows the model

\[
X_{i,n} = \mu_{i,n} + \epsilon_{i,n},
\]

where \( \{ \epsilon_{i,n} \}_{i=1}^{n} \) is a centred non-stationary noise process whose data generating mechanism evolves both smoothly and abruptly over time. The mean \( \mu_{i,n} \) is assumed to have the form

\[
\mu_{i,n} = \sum_{k=1}^{M_k} \sum_{r=0}^{M_k} (A_{r,k} \cos(\omega_k i) + B_{r,k} \sin(\omega_k i)) \mathbb{I}(b_{k,r} < i \leq b_{k,r+1}) + f(i/n), \quad \omega_k \in \Omega,
\]

where \( \Omega \) is a finite set of unknown oscillatory frequencies, \( b_{k,1} < \cdots < b_{k,M_k} \) are the unknown change points corresponding to the oscillatory frequency \( \omega_k \) with the convention \( b_{k,0} = 0 \) and \( b_{k,M_k+1} = n \) and \( f \) is assumed to be a smooth function. \( f \) is usually understood as the trend or baseline wandering in biomedical signal processing. Here we assume that all \( \omega_k \in \Omega \) are sufficiently high in the sense that \( \min_{1 \leq k \leq |\Omega|} \omega_k \geq \delta_0 \) for some positive constant \( \delta_0 \) to distinguish the oscillation from the smooth trend \( f \). The purpose of this paper is to test whether \( \Omega \) is empty and if not, we would like to accurately estimate all \( \omega_k \) and then test and locate the corresponding change points \( \{b_{k,r}\}_{r=1}^{M_k} \).

This section is devoted to the modelling of \( \{ \epsilon_{i,n} \} \). Many physiological time series, like breathing signal and photoplethysmogram, are oscillatory and contaminated by non-stationary noises with complex generating mechanisms. Signals like surface electroencephalogram (EEG) is usually understood as stochastic, and the non-stationarity can be seen clearly.
from their time series plots, while spindles showing up during deep sleep can be modelled by oscillatory components. See Section 6. For the sake of better modelling these time series, it is desirable to have a flexible non-stationary time series model for the stochastic process \( \{ \epsilon_{i,n} \} \), either noise or EEG or others, which allows the underlying data generating mechanism to change both smoothly and abruptly over time. To this end, we shall adopt the piecewise locally stationary (PLS) time series model proposed in Zhou (2013).

We say \( \{ \epsilon_{i,n} \} \) is PLS with \( r \) break points (PLS(\( r \))) if there exist constants \( 0 = s_0 < s_1 < \ldots < s_r < s_{r+1} = 1 \) and nonlinear filters (measurable functions) \( \mathcal{G}_0, \ldots, \mathcal{G}_r \) such that

\[
\epsilon_{i,n} = \mathcal{G}_j(t_i, F_i), \quad \text{if} \quad s_j < t_i \leq s_{j+1},
\]

\( j = 0, 1, \ldots, r \), where \( t_i = i/n, F_i = (\ldots, e_0, \ldots, e_i) \) and \( e_i \) are i.i.d. random variables. The function \( \mathcal{G}_j \) is assumed to be smooth between \( s_j \) and \( s_{j+1} \) in some appropriate sense. Hence the time series is locally stationary between \( s_j \) and \( s_{j+1} \) in the sense that the data generating mechanism changes slowly. And the data generating mechanism changes abruptly from \( \mathcal{G}_{j-1} \) to \( \mathcal{G}_j \) at \( s_j \), \( j = 1, 2, \ldots, r \).

A locally stationary model is more general compared with a stationary model as shown in the versatility of the generating function. A PLS(\( r \)) model allows for additional abrupt changes in the mechanism of data generation adding to a more generalized model assumption.

Assume that \( \max_{1 \leq i \leq n} \| \epsilon_{i,n} \|_q \leq C_q \) for some finite constant \( C_q \) and some \( q \geq 4 \), where \( \| \cdot \|_q := (\mathbb{E} | \cdot |^q)^{1/q} \) is the \( L^q \) norm of a random variable. We define the physical dependence measures for PLS time series as

\[
\delta_q(k) = \max_{0 \leq j \leq r} \sup_{s_j < t \leq s_{j+1}} \| \mathcal{G}_j(t, F_i) - \mathcal{G}_j(t, F_{i,i-k}) \|_q,
\]

where \( F_{i,i-k} \) is defined as \( F_{i,i-k} := (\ldots, \hat{e}_{i-k}, \ldots, e_i) \), and \( \hat{e}_{i-k} \) is an identically distributed copy of \( e_{i-k} \) and is independent of \( \{ e_j \}_{j \in \mathbb{Z}} \). Using the idea of coupling, \( \delta_q(k) \) measures the influence of the innovations of the underlying data generating mechanism \( k \) steps ahead on the current time series observation uniformly across time.

An example of a PLS model is the PLS linear processes which is defined as

\[
\mathcal{G}_j(t, F_i) = \sum_{k=0}^{\infty} a_{j,k}(t)e_{i-j}, \quad s_j < t \leq s_{j+1},
\]

where \( a_{j,k}(\cdot) \) are Lipschitz continuous functions on \( [s_j, s_{j+1}] \). Between \( s_j \) and \( s_{j+1} \), the system is linear with smoothly varying coefficients. The dependence measures of the system can be shown to be

\[
\delta_q(k) = O \left( \max_{0 \leq j \leq r} \sup_{s_j < t \leq s_{j+1}} |a_{j,k}(t)| \right).
\]

We refer to Zhou (2013) and Zhou (2014) for more discussions and examples of PLS processes and the associated dependence measures.
3 The two-stage methodology

3.1 First stage: oscillatory frequency detection

Given time series data $X = \{X_{i,n}\}_{i=1}^n$, our first goal is to detect and then estimate the oscillatory frequencies. Recall that $W$ is a dense set of possible oscillatory frequencies with mesh size no greater than $O(n^{-3/2})$. Denote $p := |W|$ as the size of the set of potential frequencies $W$. Throughout this article we assume that $p$ is proportional to $n^{3/2} \log n$. In practice, one could pick $W = \{\delta_0 n^{3/2} \log(n), \delta_0 n^{3/2} \log(n) + 1, \ldots, \delta_0 n^{3/2} \log(n) + [(\pi - \delta_0)n^{3/2} \log(n)]/(n^{3/2} \log(n))\}$, where $\delta_0$ is some small positive constant. The set of potential frequencies $W$ has its lowest frequency at $\delta_0$ as a rule of thumb in order to avoid small frequencies as the oscillation at the latter frequencies represents the smooth trend $f(\cdot)$.

The dense progressive periodogram test (DPPT) statistic $F(W)$ should be large if there is an oscillation at some $\omega \in W$. In particular, $\tilde{F}(\omega) := \max_{1 \leq k \leq n} |L_n(k, \omega)| / \sqrt{n}$ will show peaks at the oscillatory frequencies. Figure 2 shows a typical plot of $\tilde{F}(\omega)$ for a time series with two oscillatory frequencies contaminated by PLS noises.

![First Stage Statistics vs. Frequency](image)

Figure 2: Example of $\tilde{F}(\omega)$ with $n = 2000$, $\mu_{i,n} = 2 \cos(2\pi i 0.07) + 1.5 \cos(2\pi i 0.3)$ and $\epsilon_{i,n} = 0.5 \cos(i/n)\epsilon_{i-1,n} + \epsilon_i$, $\epsilon$ i.i.d. standard normal.

As we mentioned in the introduction, spectral dependency caused by the high density of $W$ as well as the non-stationarity of $\{\epsilon_{i,n}\}$ makes it difficult to investigate the limiting distribution of $F(w)$. As a solution to this conundrum we will use an extension of the OBMB [Zhou, 2013] to approximate the critical values of the DPPT under the null hypothesis of no oscillation. The bootstrap is simple to implement and will be shown to be able to detect all oscillatory frequencies accurately with a prescribed probability asymptotically.

Define $S_{j,m}(w) = \sum_{i=j}^{j+m-1} \sin(iw)X_{i,n}$, $C_{j,m}(w) = \sum_{i=j}^{j+m-1} \cos(iw)X_{i,n}$ and $E_{j,m}(w) = C_{j,m}(w) + \sqrt{-1}S_{j,m}(w)$ for an integer bandwidth $m < n$. The OBMB statistic is defined as

$$\tilde{F}_{m,l}(W) = \max_{w \in W} \tilde{F}_{m,l}(\omega), \text{ where } \tilde{F}_{m,l}(\omega) := \max_{1 \leq k \leq n-m} \left\{ \left| \sum_{j=1}^{k} E_{j,m}(w)G_{j,l} \right| \right\} / \sqrt{m(n-m)},$$
where $G_{j,l}$, $j = 1, 2, \ldots, n-m$, $l = 1, 2, \ldots$ are i.i.d. standard normal random variables independent of $\{X_{i,n}\}_{i=1}^n$. A heuristic reason that multiplier bootstrap statistic works is because with high probability the conditional covariance structure of $\tilde{F}_{m,l}(W)$ can approximate that of the DPPT. See Theorem 4.1 in Section 4 for a rigorous treatment.

The DPPT is performed by simulating the distribution of $F(W)$ by that of the multiplier bootstrap statistic $\tilde{F}_{m,l}(W)$. Let $\{\tilde{F}_{m,l}(W)\}_{l=1}^K$ be $K$ (say, $K = 1,000$) multiplier bootstrap statistics that we generate. Under a pre-specified significance level $\alpha \in (0, 1)$ we estimate the $\alpha$-th critical value for $F$ using the empirical $(1-\alpha)$-th quantile

$$
crit_{\alpha,1}(W) := \text{Quantile}(\{\tilde{F}_{m,l}(W)\}_{l=1}^K, 1-\alpha).
$$

If $F(W) \leq \text{crit}_{\alpha,1}(W)$, then we claim that there is no oscillation in the series. Otherwise, the first potential oscillatory frequency can be estimated by taking the frequency that maximizes $F(W)$:

$$
\hat{\omega}_1 := \{\omega \in W | \tilde{F}(\omega) = F(W)\}.
$$

If there are multiple maximizers, let $\hat{\omega}_1$ be the smallest among them. Take $W_2 := W \setminus [\hat{\omega}_1 - \log(m)/(4m^{1/2}), \hat{\omega}_1 + \log(m)/(4m^{1/2})]$. The second frequency estimate is taken to be

$$
\hat{\omega}_2 := \{\omega \in W_2 | \tilde{F}(\omega) = F(W_2)\}
$$

if $F(W_2) > \text{crit}_{\alpha,2}(W_2)$, where $\text{crit}_{\alpha,2}(W_2)$ is defined in the same way. Repeat the previous steps until $F(W_k) \leq \text{crit}_{\alpha,k}(W_k)$ for some integer $k$. The detailed algorithm is described as follows:

**Algorithm 1: Frequency Estimation**

**Result:** estimated frequencies $\hat{\Omega}$

Let $W_1 = W$, $k = 1$ and $\hat{\Omega}_0 = \emptyset$;

Compute $F(W_1)$ and $\text{crit}_{\alpha,1}(W_1)$;

while $F(W_k) > \text{crit}_{\alpha,k}(W_k)$ do

$\hat{\omega}_k = \arg\max_{\omega \in W_k} \tilde{F}(\omega)$;

$\hat{\Omega}_k = \hat{\Omega}_{k-1} \cup \{\hat{\omega}_k\}$;

$W_{k+1} = W_k \setminus [\hat{\omega}_k - \log(m)/(4m^{1/2}), \hat{\omega}_k + \log(m)/(4m^{1/2})]$;

Compute $F(W_{k+1})$ and $\text{crit}_{\alpha,k+1}(W_{k+1})$;

Increase $k$ by 1;

end

In Algorithm 1, we remove a $\log m/(4m^{1/2})$ neighborhood from each estimated oscillatory frequency before we perform the next iteration. The reason is that it can be shown that $\tilde{F}_{m,l}(\omega) \gg Q_{1-\alpha}(\max_{\omega \in W} \max_{1 \leq k \leq n} |\sum_{j=1}^k \epsilon_{j,n} e^{\sqrt{-1} \theta_j}|/\sqrt{m})$ with high probability if $|\omega - \omega_k| \ll m^{-1/2}$ for some $\omega_k \in \Omega$, where $Q_{1-\alpha}(Z)$ is the $(1-\alpha)$ quantile of a random variable $Z$. Here $a_n \gg b_n$ for positive sequences $a_n$ and $b_n$ means that $a_n/b_n \to \infty$ as $n \to \infty$. $a_n \ll b_n$ is defined similarly. Therefore, the bootstrap critical values will be too large if $\omega$ is within an
For \( m \) nearly 1, \( m \) is the number of oscillatory frequencies. Hence, we remove a log \( m/(4m^{1/2}) \) neighborhood of the estimated frequencies to avoid reduced sensitivity for oscillatory frequency detection. On the other hand, we need the oscillatory frequencies to be well separated in order for Algorithm 1 to detect all of them. In particular, we require that \(|\omega_i - \omega_j| \gg m^{-1/2} \log m\) for all \( \omega_i \), \( \omega_j \in \Omega \), \( \omega_i \neq \omega_j \).

### 3.2 Second stage: oscillation change point testing and estimation

Given the set of estimated frequencies \( \hat{\Omega} \) (not empty), the second stage aims at testing the existence and then estimating the locations of any change points at each oscillatory frequency \( w \in \Omega \). To this end, we propose a phase-adjusted local change point detection algorithm that compares the phase-adjusted Fourier transforms at frequency \( \hat{\omega} \) before and after each time point. Specifically, let \( \hat{w} \in \hat{\Omega} \) and let \( B_k \) be a set of potential change points at step \( k, k = 1, 2, \cdots \). Let \( B_1 = \{ \hat{m} + m', \hat{m} + m' + 1, \cdots, n - \hat{m} - m' \} \), where \( \hat{m}, m' \in \mathbb{N} \). The statistic for the second stage is defined as

\[
T(B_1, \hat{w}) = \max_{i \in B_1} \left| \sum_{l=1}^{i} \exp(\sqrt{-1}\hat{w}(l - i))X_{l,n} - \sum_{l=i+1}^{i+m' + 1} \exp(\sqrt{-1}\hat{w}(l - i))X_{l,n} \right| \sqrt{2m},
\]

where \( \hat{m} \) is a bandwidth satisfying \( \hat{m} \to \infty \) with \( \hat{m}/n \to 0 \).

A phase-adjusted OBMB is employed here to approximate the critical values of \( T(B_1, \hat{w}) \). Specifically, for \( j \in B_1 \), define

\[
\Phi_i(j, \hat{w}) = \left( \sum_{l=1}^{i} \cos(\hat{w}(l - j))X_{l,n} - \sum_{l=i+1}^{i+m' + 1} \cos(\hat{w}(l - j))X_{l,n} \right) / \sqrt{2m'},
\]

\[
\Psi_i(j, \hat{w}) = \left( \sum_{l=1}^{i} \sin(\hat{w}(l - j))X_{l,n} - \sum_{l=i+1}^{i+m' + 1} \sin(\hat{w}(l - j))X_{l,n} \right) / \sqrt{2m'},
\]

and \( \Upsilon_i(j, \hat{w}) = \Phi_i(j, \hat{w}) + \sqrt{-1}\Psi_i(j, \hat{w}) \), where \( m' \) satisfies \( m' \to \infty \) with \( m'/\hat{m} \to 0 \). Now let \( G_{i,j} \) be i.i.d standard Gaussian random variables independent of the data, where \( i, j \in \mathbb{Z} \). We define the bootstrap statistic as

\[
\hat{T}_l(B_1, \hat{w}) = \max_{j \in B_1} \left| \sum_{i=j-\hat{m}}^{j} \Upsilon_i(j, \hat{w})G_{i,l} - \sum_{i=j+1}^{j+m' + 1} \Upsilon_i(j, \hat{w})G_{i,l} \right| \sqrt{2m} \quad \text{where} \quad l \in \mathbb{N},
\]

and simulate the distribution of \( T(B_1, \hat{w}) \) by generating \( K_0 \) bootstrap statistics \( \{\hat{T}_l(B_1, \hat{w})\}_{l=1}^{K_0} \). Next find the estimated critical value under \( \beta \) significance level for the first change point:

\[
crit_{\beta,1}(B_1) = \text{Quantile} \left( \{\hat{T}_l(B_1, \hat{w})\}_{l=1}^{K_0}, 1 - \beta \right).
\]
We claim that there is no change point at frequency \( \omega \) at level \( \beta \) if \( T(B_1, \hat{\omega}) \leq \text{crit}_{\beta,1}(B_1) \). Define the first change point estimator as

\[
\hat{b}_1 := \arg\max_{i \in B_1} \left| \sum_{l=i-m}^{i} \exp(-1\hat{\omega}(l-i))X_{l,n} - \sum_{l=i+1}^{i+m+1} \exp(-1\hat{\omega}(l-i))X_{l,n} \right| / \sqrt{2m} \tag{7}
\]

provided that \( T(B_1, \hat{\omega}) > \text{crit}_{\beta,1}(B_1) \). If there are multiple maximizers, then let \( \hat{b}_1 \) be the smallest among them. Define \( B_2 := B_1 \setminus [\hat{b}_1 - \tilde{m}, \hat{b}_1 + \tilde{m}] \), which is the second potential set of change point. We iterate the aforementioned procedure until \( T(B_k, \hat{\omega}) \leq \text{crit}_{\beta,k}(B_k) \) for some \( k \). The detailed algorithm is listed as follows.

**Algorithm 2: Change Point Estimation**

Result: estimated change points \( \hat{B} \) at oscillatory frequency \( \omega \).

Let \( B_1 = \{\hat{m} + m', \hat{m} + m' + 1, \ldots, n - \hat{m} - m'\} \), \( k = 1 \) and \( \hat{B}_0 = \emptyset \);

Compute \( T(B_1, \hat{\omega}) \) and \( \text{crit}_{\beta,1}(B_1) \);

while \( T(B_k, \hat{\omega}) > \text{crit}_{\beta,k}(B_k) \) do

obtain \( \hat{b}_k \) using (7);

\( \hat{B}_k = \hat{B}_{k-1} \cup \hat{b}_k \);

\( B_{k+1} = B_k \setminus [\hat{B}_k - \tilde{m}, \hat{B}_k + \tilde{m}] \);

Compute \( T(B_{k+1}, \hat{\omega}) \) and \( \text{crit}_{\beta,k+1}(B_{k+1}) \);

Increase \( k \) by 1;

end

### 3.3 Tuning parameter selection

To implement our methodology, one needs to select three tuning parameters: \( m, \tilde{m} \) and \( m' \). In this article, we suggest using the minimum volatility (MV) method proposed in Politis et al. (1999) as a data-driven way to select those parameters. The MV method takes advantage of the fact that the (conditional) covariances of the test and bootstrap statistics become stable when the tuning parameters are chosen in an appropriate range. Specifically, in order to choose \( m \) in Stage 1, we observe that the accuracy of the bootstrap is determined by how well the conditional covariance of \( \sum_{j=1}^{k} C_{j,m}(\omega) + \sqrt{-1}S_{j,m}(\omega) \) approximates the covariance of \( L_n(k, \omega) / \sqrt{n} \). The block size \( m \) determines the latter accuracy.

To utilize the MV method, we observe that the latter conditional covariance is expected to behave stably as a function of \( m \) if \( m \) is in an appropriate range. Therefore, we define \( V_m^{(0)}(k, \omega) := \sum_{j=1}^{k} [C_{j,m}(\omega) + S_{j,m}^2(\omega)] / (m(n-m)) \). For a sequence of candidate block sizes \( m_1 < m_2 < \cdots < m_l \), calculate, for each \( i = 1, 2, \cdots, l \),

\[
\tilde{V}_{m_i}^{(0)} = \sum_{k=1}^{n} \sum_{\omega \in W} \text{Var} \{ \{ V_{m_j}^{(0)}(k, \omega) \}_{j=i-3}^{i+3} \},
\]
where "\( \hat{\text{Var}} \)" stands for sample variance and \( m_j, j \leq 0 \) or \( j \geq l + 1 \) are defined as linear extrapolations of the arithmetic sequence \( \{m_i\}_{i=1} \). We select the block size that minimizes \( \hat{\text{Var}}^{(0)} \).

For Stage 2, one can use the MV method and select the block size \( \tilde{m} \) that minimizes the volatility of

\[
\hat{\text{Var}}^{(1)} := \frac{1}{2\tilde{m}(n-2\tilde{m}-1)} \sum_{i=\tilde{m}+1}^{n-\tilde{m}-1} \left( \sum_{l=i-\tilde{m}}^{i+\tilde{m}+1} e^{\sqrt{-1} \omega(l-i)} X_{l,n} - \sum_{l=i+1}^{i+\tilde{m}+1} e^{\sqrt{-1} \omega(l-i)} X_{l,n} \right)^2.
\]

Observe that \( \hat{\text{Var}}^{(1)} \) is the average of \( \{T^2(i, \hat{w})\}_{i=\tilde{m}+1}^{n-\tilde{m}-1} \). The rationale is that, under the null hypothesis of no change point, the test statistic \( T(i, \hat{w}) \) as a function of \( \tilde{m} \) should behave stably if \( \tilde{m} \) is in an appropriate range. Once \( \tilde{m} \) is chosen, \( m' \) can be selected to minimize the volatility of

\[
\hat{\text{Var}}^{(2)} := \frac{1}{2\tilde{m}(n-2\tilde{m}-1)} \sum_{j=\tilde{m}+1}^{n-\tilde{m}-1} \sum_{i=j-\tilde{m}}^{j+\tilde{m}+1} [\Phi^2_i(j, \hat{w}) + \Psi^2_i(j, \hat{w})].
\]

The detailed implementation of the MV method for \( \tilde{m} \) and \( m' \) is similar to that of Stage 1 and is omitted here.

4 Theoretical results

In this section, we shall demonstrate the asymptotic consistency of the OBMB for both stages. Those results are key to the theoretical justification of our methodology as they establish that the OBMB can well approximate the probabilistic behaviour of the DPPT and the phase-adjusted local change point detection algorithms asymptotically under the corresponding null hypotheses of no oscillation and no change points. In the literature, \( L^\infty \) Gaussian approximations to high dimensional time series are typically realized by the non-overlapping multiplier bootstrap (cf. Jirak (2015) and Dette and Gösmann (2018)). The non-overlapping multiplier bootstrap could suffer from a relatively small number of blocks for time series of moderate length and hence may be numerically unstable under such circumstances. To remedy the latter problem, the OBMB is implemented and theoretically justified.

4.1 Bootstrap consistency under the null

Recall the definitions of \( F(W) \) and \( \tilde{F}(W) \) in Section 3.1 where \( \tilde{F}(W) \) is short for \( \tilde{F}_{m,l}(W) \). Observe that both \( F(W) \) and \( \tilde{F}(W) \) can be viewed as the maximum of the coordinate-wise sums of a high-dimensional vector. Define \( n \)-dimensional vectors \( C^{(\epsilon)}(\omega) \) and \( S^{(\epsilon)}(\omega) \) with
the $k$-th vector coordinate

\[ C_k^{(e)}(\omega) := \sum_{j=1}^{k} \cos(j\omega)\epsilon_j \quad \text{and} \quad S_k^{(e)}(\omega) := \sum_{j=1}^{k} \sin(j\omega)\epsilon_j, \]

where $k = 1, 2, \ldots, n$ respectively. Let

\[ \Theta^{(e)}(W) := [C^{(e)}(\omega_1)^\top, S^{(e)}(\omega_1)^\top, \ldots, C^{(e)}(\omega_p)^\top, S^{(e)}(\omega_p)^\top]^\top / \sqrt{n} \in \mathbb{R}^{2np}, \]

where $W = \{\omega_1, \ldots, \omega_p\}$. For any given $w \in W$, define the 2$n$-dim vector $S(w)$ with $k$-th vector coordinate

\[ S_k(w) := \begin{cases} C_{1,m}(w)G_1 & \text{if } k \leq m \\ \sum_{j=1}^{k-m+1} C_{j,m}(w)G_j & \text{if } m < k \leq n \\ S_{1,m}(w)G_1 & \text{if } n+1 \leq k \leq n+m+1 \\ \sum_{j=1}^{k-m+1} S_{j,m}(w)G_j & \text{if } n+m+2 \leq k \leq 2n. \end{cases} \]

Here we omit the subscript $l$ in $G_{j,l}$ to simplify notation. Finally, we define $S(W) = [S(w_1)^\top, \ldots, S(w_p)^\top]^\top / \sqrt{m(n-m)} \in \mathbb{R}^{2np}$. Define the following measure of difference in covariance structure between the two statistics $F(W)$ and $\hat{F}(W)$ as

\[ \Delta := \max_{1 \leq i,j \leq 2np} |[\text{Cov}(\Theta^{(e)}(W)) - \text{Cov}(S(W)|X)]_{ij}|, \]

where Cov$(A)$ denotes the covariance matrix of $A$, and for a matrix $D$, $D_{ij}$ denotes the entry of $D$ at its $i$-th row and $j$-th column. Now we are ready to present the first main theorem of our findings.

**Theorem 4.1.** Suppose Assumptions[17 in Section D] of the supplementary material hold true and $\mu_{i,n} = f(i/n)$ in (3). Further assume that

\[ m \asymp n^{\theta} \text{ with } 0 < \theta < 1 \quad \text{and} \quad q > 8 \log p/((1-\theta) \log n). \tag{8} \]

Then, we have $\Delta = O_p(p^{4/q}\sqrt{m/n + 1/m})$. Define the sequence of events $A_n = \{\Delta \leq (p^{4/q}\sqrt{m/n + m^{-1}})h_n\}$ where $h_n > 0$ is a sequence diverging at an arbitrarily slow rate. Then $\mathbb{P}(A_n) = 1 - o(1)$. On the event $A_n$, we have

\[ \sup_{|x| > d_{n,p}^*} \left| \mathbb{P}(F(W) \leq x) - \mathbb{P}(\hat{F}(W) \leq x|X) \right| \]

\[ \lesssim (p^{4/q}\sqrt{m/n + 1/m})^{1/3}h_n^{1/3}\log^2(pn) + G^*(n,np), \quad \text{where} \]

\[ d_{n,p}^* = C[(p^{4/q}\sqrt{m/n + 1/m})^{1/3}h_n^{1/3}\log^{1/6}(pn) + \log^{-1/2}(np) + d_{n,np}^*] \]

with some finite constant $C$ that does not depend on $n$, and $G^*(n,np)$ as well as $d_{n,np}^*$ are defined in Proposition[18 in the supplementary material with $h$ therein replaced by $np$.

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In \((9)\), the term \(G^*(n, np)\) corresponds to the Gaussian approximation error of \(F(W)\) by \(\max_{\theta \in W} \max_{1 \leq k \leq n} |\sum_{j=1}^{k} y_{j,n} e^{-i\theta_j y_j}|/\sqrt{n}\), where \(\{y_{i,n}\}\) is a centered Gaussian time series that preserves the covariance structure of \(\{\epsilon_{i,n}\}\). The term \((p^{1/q} \sqrt{m/n} + 1/m)^{1/3}h^{1/3}(n) \log^2(pn)\) corresponds to the bootstrap error when approximating the distribution of \(\max_{\theta \in W} \max_{1 \leq k \leq n} |\sum_{j=1}^{k} y_{j,n} e^{-i\theta_j y_j}|/\sqrt{n}\) by that of \(\tilde{F}(W)\). The requirement \(q > 4 \log p/((1 - \theta) \log n)\) in Theorem 4.1 is to ensure that \(p^{1/q} \sqrt{m/n} + 1/m\) converges to 0 polynomially fast. For the DPPT, note that we set \(p \asymp n^{3/2} \log n\), in which case the above requirement is equivalent to \(q > 12/(1 - \theta)\). As we will discuss after the proof, \(G^*(n, np)\) and \(d_{n,p}^0\) converge to 0 when \(n \to \infty\).

As a result, Theorem 4.1 asserts that under the null hypothesis of no oscillation the conditional cumulative distribution function of the bootstrap well approximates that of \(F(W)\) with high probability if \(|x| > d_{n,p}^0\) as the right hand side of \((9)\) converges to 0. The restricted range \(|x| > d_{n,p}^0\) is due to the unbalanced variances of \(L_n(i, \omega)\) across \(i\) and hence there is no positive lower bounds for the latter variances. The unbalanced variances of \(L_n(i, \omega)\) lead to the possible failure of the Gaussian approximation when \(|x|\) is very small. On the other hand, Lemma D.5 in the supplementary material assures that, for \(\alpha \in (0, 1 - \alpha_0]\) where \(\alpha_0\) is any positive constant, the \((1 - \alpha)\) quantile of \(F(W)\) is no less than \(c_{\alpha}\sqrt{\log n}\) for sufficiently large \(n\) and some positive constant \(c_{\alpha}\). Hence \(d_{n,p}^0\) is dominated by the latter quantile and Theorem 4.1 can be used for the DPPT.

The next theorem shows the validity of the phase-adjusted OBMB for the second stage statistics under the null hypothesis of no change points at an oscillatory frequency \(\omega\). We first need to introduce some notation. Let \(\Theta^{(2)} \in \mathbb{R}^{2(n - \bar{m} - m')}\) be the vectorized stage 2 statistics at the true oscillatory frequency \(w\); that is, for \(m + m' \leq i \leq n - \bar{m} - m'\) we define the coordinate of \(\Theta^{(2)}\) as

\[
\Theta_{i-(\bar{m}+m')+1}^{(2)} = \left( \sum_{k=i-\bar{m}}^{i} \cos(w(k - i))\epsilon_k - \sum_{k=i+1}^{i+\bar{m}+1} \cos(w(k - i))\epsilon_k \right) / \sqrt{2\bar{m}}
\]

and

\[
\Theta_{i+n-2(\bar{m}+m')+1}^{(2)} = \left( \sum_{k=i-\bar{m}}^{i} \sin(w(k - i))\epsilon_k - \sum_{k=i+1}^{i+\bar{m}+1} \sin(w(k - i))\epsilon_k \right) / \sqrt{2\bar{m}}.
\]

We can define the vectorized multiplier bootstrap statistics at the true oscillatory frequency \(w\) in a similar way. Let \(\tilde{S}^{(2)} \in \mathbb{R}^{2(n - \bar{m} - m')}\). Similarly, for \(m + m' \leq i \leq n - \bar{m} - m'\), we define the coordinate of \(\tilde{S}^{(2)}\) as

\[
\tilde{S}_{i-(\bar{m}+m')+1}^{(2)} = \left( \sum_{k=i-\bar{m}}^{i} \Phi_k(i, w)G_k - \sum_{k=i+1}^{i+\bar{m}+1} \Phi_k(i, w)G_k \right) / \sqrt{2\bar{m}}
\]

and

\[
\tilde{S}_{i+n-2(\bar{m}+m')+1}^{(2)} = \left( \sum_{k=i-\bar{m}}^{i} \Psi_k(i, w)G_k - \sum_{k=i+1}^{i+\bar{m}+1} \Psi_k(i, w)G_k \right) / \sqrt{2\bar{m}}.
\]
Lastly, define a measure of difference in covariance structure for the second stage statistics as

$$\Delta' := \max_{1 \leq i,j \leq 2(n-\tilde{m}-m')} |\text{Cov}(\Theta^{(2)}) - \text{Cov}(\tilde{\Theta}^{(2)}|X)|_{ij}|. \quad (10)$$

**Theorem 4.2.** Assume that $\Omega \neq \emptyset$, $\omega \in \Omega$ and there is no change point at frequency $\omega$. Suppose that Assumptions 3 to 8 in Section D of the supplementary material hold true. Further assume that $\tilde{m} \approx n^{\gamma}$ with $16/29 < \gamma_1 < 1$, $m' \approx n^\eta$ with $0 < \eta < \gamma_1$ and $q > 4/(\gamma_1 - \eta)$. Then, one can find a sequence of events with probability at least $1 - C/\log^{q/2} n$, where $C$ is a finite positive constant which does not depend on $n$, such that on the latter events, we have $\Delta' \leq 1/m' + n^{2/q} \sqrt{m'/\tilde{m}} \log n$ and

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(T(B_1, \hat{w}) \leq x) - \mathbb{P}(\hat{T}(B_1, \hat{w}) \leq x|X) \right| = o(1). \quad (11)$$

Theorem 4.2 implies that the phase-adjusted OBMB achieves the correct Type-I error rate $\beta$ asymptotically if there is no change point at frequency $\omega$. The assumption $q > 4/(\gamma_1 - \eta)$ ensures that $n^{2/q} \sqrt{m'/\tilde{m}} \log n$, hence $\Delta'$, converges to 0 algebraically fast. The $o(1)$ error in (11) is composed of three parts. First, we show that $\sup_{x \in \mathbb{R}} |\mathbb{P}(T(B_1, \hat{w}) \leq x) - \mathbb{P}(T(B_1, w) \leq x)|$ converges to 0 and, conditional on the data, $\sup_{x \in \mathbb{R}} \left| \mathbb{P}(\hat{T}(B_1, \hat{w}) \leq x) - \mathbb{P}(\hat{T}(B_1, w) \leq x) \right|$ converges to 0 algebraically fast with high probability. Second, we derive that

$$\sup_{x \in \mathbb{R}} |\mathbb{P}(T(B_1, w) \leq x) - \mathbb{P}(\hat{T}(y)(B_1, w) \leq x)| = o(1),$$

where $T(y)(B_1, w)$ is the version of $T(B_1, w)$ with $\{\tilde{X}_{t,n}\}$ therein replaced by a centered Gaussian process $\{y_{t,n}\}$ with the same covariance structure. Then utilizing comparison of distribution results for complex Gaussian random vectors, we show that, with high probability,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(T(y)(B_1, w) \leq x) - \mathbb{P}(\hat{T}(B_1, w) \leq x|X) \right| = O((\Delta')^{1/3} \log^{7/6} n).$$

**4.2 Estimation accuracy**

The accuracy of our methodology is composed of two parts. First, for given probabilities $1 - \alpha$ and $1 - \beta$, we hope that our methodology is able to estimate the correct number of oscillatory frequencies with probability $1 - \alpha$ and the correct number of change points with probability $1 - \beta$ asymptotically. When there is no oscillation or change point, it is equivalent to the requirement that our methodology achieves the correct Type-I error rates asymptotically. Second, we would like our methodology to estimate the oscillatory frequencies and change points, if they exist, within an accurate range. The following Theorem 4.3 and Proposition 1 establish the desired result for our stage 1 methodology.
The rate $n^{-1}h_n$ is slower than the parametric rate $n^{-3/2}$ for oscillatory frequency estimation (cf. Genton and Hall (2007)). A further investigation reveals that this is caused by possible changes in the phase of the oscillation. When there is an abrupt change in the phase, the Fourier transformation of the oscillation curve is not maximized at the true oscillatory frequency. Instead, it will be maximized in an $O(1/n)$ neighborhood of the oscillatory frequency. The following is a detailed example.

Example 1. Let $\mu_i = C_1 \cos(\omega_0 i + \alpha_1)$, $i = 1, 2, \cdots , n_1$, and $\mu_i = C_2 \cos(\omega_0 i + \alpha_2)$, $i = n_1 + 1, \cdots , n$, where $C_1, \alpha_1 > 0$, $i = 1, 2$, $0 < \omega_0 < \pi/2$, $0 < \alpha_1 - \alpha_2 < \pi/2$. Write $n_2 = n - n_1$ and let the noises $\epsilon_{i,n} = 0$. Assume that $n_1 = c_1 n$ for some $c_1 \in (0, 1)$. Then $F(\omega_0) = C_1^2 n_1^2 + C_2^2 n_2^2 + 2C_1 C_2 n_1 n_2 \cos(\alpha_1 - \alpha_2) + O(1)$. For any $\omega'$ such that $|\omega' - \omega_0| \approx 1/(nh_n)$, where $h_n$ is diverging at an arbitrarily slow rate, elementary but tedious calculations yield that

$$F(\omega') - F(\omega_0) = C_1 C_2 \sin(\alpha_1 - \alpha_2) n_1 n_2^2 (\omega' - \omega_0)(1 + o(1)).$$

Hence, if $\omega' - \omega_0 > 0$, then $F(\omega') > F(\omega_0)$ for sufficiently large $n$. Clearly, if $|\omega' - \omega_0| \gg 1/n$, $F(\omega') < F(\omega_0)$. Therefore, we conclude that for this example, $F(W)$ is maximized at a point $\omega^*$ such that $1/(nh_n) \leq |\omega^* - \omega_0| \leq h_n/n$ for sufficiently large $n$. This example also shows that the rate $n^{-1}h_n$ in Theorem 4.3 cannot be improved (except for a factor of an arbitrarily slowly diverging function) for the DPPT if there exist changes in the phase of the oscillation.

The following Proposition shows that if the phase of the oscillation does not change over time; that is, if only the amplitude of the oscillation is allowed to change over time, then the DPPT can detect the oscillatory frequencies at the $n^{-3/2}$ parametric rate except for a factor of logarithm.

**Proposition 1.** Suppose that the assumptions of Theorem 4.3 and (8) hold true. Further assume that $\Omega \neq \emptyset$, and for each $\omega_k \in \Omega$ and all $r = 0, 1, \cdots , M_k$, we have

$$A_{r,k}/\sqrt{A_{r,k}^2 + B_{r,k}^2} = c_k, B_{r,k}/\sqrt{A_{r,k}^2 + B_{r,k}^2} = d_k$$

for some constants $c_k, d_k$.

If $A_{r,k}^2 + B_{r,k}^2 \neq 0$. Then we have $P \left( \max_k |\hat{w}_k - w_k| \lesssim n^{-3/2} \log(n), |\hat{\Omega}| = |\Omega| \right) \rightarrow 1 - \alpha$.

If we write $A_{r,k} + \sqrt{-1}B_{r,k} = C_{r,k} \exp(\sqrt{-1} \theta_{r,k})$, $C_{r,k} \geq 0$, $0 \leq \theta_{r,k} < 2\pi$, then (12) is equivalent to $\theta_{r,k} = \theta_k$ for some $\theta_k$; that is, there is no change in the phase of the oscillation.
On the other hand, note that $C_{r,k}$, the magnitude of the oscillation, is allowed to change with respect to $r$. For the sleep EEG data analyzed in this paper, we are interested in detecting spindles which could be modeled as an oscillation at a fixed frequency that occurs for a short period of time and then vanishes. It can be easily seen that (12) is suitable to model such short-term oscillations.

Remark 1. The level $\alpha$ in Theorem 4.3 and Proposition 1 can be chosen as $\alpha = \alpha_n \rightarrow 0$ as long as $\alpha_n$ converges to 0 slower than the right hand side of (9) and $1 - P(A_n)$. In this case, it can be shown that the probabilities in Theorem 4.3 and Proposition 1 equal $1 - \alpha_n(1 + o(1))$ asymptotically.

The following theorem investigates the asymptotic accuracy of the phase-adjusted local change point detection algorithm. Observe that the established $O_P(\log(\tilde{m}))$ rate of abrupt change point estimation is nearly the parametric $O_P(1)$ rate except a factor of logarithm.

**Theorem 4.4** (Accuracy of change point estimation). Write $\Omega = \{\omega_1, \ldots, \omega_K\}$ and let $D_k := \{b_{1,k}, \ldots, b_{M_k,k}\}$ be the set of change points associated with $\omega_k$ and $\hat{D}_k$ be the set of all estimated change points by Algorithm 2 using $\hat{\omega}_k$, $k = 1, 2, \ldots, K$. For each $k = 1, \ldots, K$, suppose that $\gamma_1 < 2/3$ if the oscillatory phase changes at frequency $\omega_k$. Further assume that all assumptions of Theorem 4.3 and Assumption 9 in the supplementary material hold true. Then for each $k$, $k = 1, \ldots, K$, we have

1. If $D_k = \emptyset$ then $\mathbb{P}(|\hat{D}_k| = 0) \rightarrow 1 - \beta$.

2. If $|D_k| > 0$, then $\mathbb{P}\left(\max_r |\hat{b}_{r,k} - b_{r,k}| \leq \log(\tilde{m})h_n, |\hat{D}_k| = |D_k|\right) \rightarrow 1 - \beta$ for any sequence $h_n > 0$ that diverges to infinity arbitrarily slowly.

5 Simulation study

We shall perform our simulation studies under various models for the non-stationary noise $\{\epsilon_{i,n}\}$ listed as follows.

(M1) : The first model is a locally stationary model with $\epsilon_{k,n} := 0.5 \cos(k/n)\epsilon_{k-1,n} + \epsilon_k$, where $\epsilon_k$ are i.i.d standard normal.

(M2) : The second model is piece-wise locally stationary with

$$\epsilon_{k,n} := [0.5 \cos(k/n)\mathbb{1}_{(k/n)<0.75} + (k/n - 0.5)\mathbb{1}_{(k/n)\geq0.75}]\epsilon_{k-1,n} + \epsilon_k,$$

where $\epsilon_k$ are i.i.d standard normal.

(M3) : The third model is piece-wise locally stationary with multiple breaks with

$$\epsilon_{k,n} := [0.5 \cos(k/n)\mathbb{1}_{(k/n)<0.3} + (k/n - 0.3)^2\mathbb{1}_{0.3\leq(k/n)<0.75} + 0.3 \sin(k/n)\mathbb{1}_{(k/n)\geq0.75}]\epsilon_{k-1,n} + \epsilon_k,$$

where $\epsilon_k$ are i.i.d standard normal.
(M4) : The last model is locally stationary with heavy tails: \( \epsilon_{k,n} := 0.6 \cos(k/n)\epsilon_{k-1,n} + \epsilon_k \), where \( \epsilon_k \) are i.i.d \( t \) distributed with 5 degrees of freedom.

Throughout our simulations, the tuning parameters are selected according to the MV method described in Section 3.3. The simulations are performed with 1000 repetitions and for each repetition, the OBMB is performed using 1000 pseudo samples.

### 5.1 Stage 1 Null Simulation

The simulated data sets are generated according to \( X_{k,n} = \mu_{k,n} + \epsilon_{k,n} \), where \( \mu_k = k/n \). Observe that \( X_{i,n} \) has a smoothly time-varying mean and there is no oscillation in \( \{X_{i,n}\} \). The noises \( \epsilon_{i,n} \) are generated according to (M1) - (M4) described in the beginning of this section. The simulated rejection rates are reported in Table 1. From Table 1 we see that the DPPT has reasonably accurate rejection rates under the null hypothesis of no oscillation for various kinds of non-stationary noises \( \epsilon_{i,n} \).

| Model | \( n = 1000 \) | \( n = 2000 \) | \( n = 1000 \) | \( n = 2000 \) |
|-------|----------------|----------------|----------------|----------------|
| M1    | 0.046          | 0.0545         | 0.1035         | 0.1205         |
| M2    | 0.058          | 0.0525         | 0.1235         | 0.11           |
| M3    | 0.0455         | 0.053          | 0.1075         | 0.111          |
| M4    | 0.0445         | 0.0465         | 0.1055         | 0.1125         |

Table 1: Simulated rejection rates under Stage 1 null conditions when \( \mu_{k,n} = k/n \).

### 5.2 Stage 2 Null Simulation

The second stage Type-I error simulation is performed under the null conditions a): \( \mu_{k,n} = 2 \sin(\omega k) \) with \( \omega = \pi/15 \) and b): \( \mu_{k,n} = 2.5 \sin(\omega_1 k) + 2 \sin(\omega_2 k) \) with \( \omega_1 = 0.17(2\pi) \) and \( \omega_2 = 0.3805(2\pi) \). The noises \( \epsilon_{k,n} \) are generated from model (M1) - (M4). Observe that the mean function \( \mu_{k,n} \) does not have change points in its oscillatory behaviour. The simulated rejection rates for a) and b) are recorded in Tables 2 and 3, respectively. Based on the results in Tables 2 and 3 the simulated rejection rates are reasonably close to the nominal level \( \beta \).

### 5.3 Estimation accuracy for short-term oscillations

We are concerned of short-term oscillations since spindles in sleep EEG signal could be modeled as short-term oscillations within a given frequency band. In this subsection we
Table 2: Simulated rejection rates for the proposed stage 2 algorithm when $\mu_{k,n} = 2 \sin(\omega k)$ with $\omega = \pi/15$.

|          | Simulated Rejection Rates |          |          |
|----------|---------------------------|----------|----------|
|          | $\beta = 0.05$            | $\beta = 0.10$ |
| Model    | $n = 1000$                | $n = 2000$ | $n = 1000$ | $n = 2000$ |
| M1       | 0.0462                    | 0.051     | 0.101     | 0.126     |
| M2       | 0.0275                    | 0.044     | 0.0725    | 0.102     |
| M3       | 0.0462                    | 0.056     | 0.0975    | 0.111     |
| M4       | 0.065                     | 0.047     | 0.144     | 0.116     |

Table 3: Simulated rejection rates for the proposed stage 2 algorithm when $\mu_{k,n} = 2.5 \sin(\omega_1 k) + 2 \sin(\omega_2 k)$ with $\omega_1 = 0.17(2\pi)$, $\omega_2 = 0.3805(2\pi)$ and $n = 1000$.

|          | Simulated Rejection Rates |          |          |
|----------|---------------------------|----------|----------|
|          | $\beta = 0.05$            | $\beta = 0.10$ |
| Model    | $\omega_1$               | $\omega_2$ | $\omega_1$ | $\omega_2$ |
| M1       | 0.048                     | 0.032     | 0.117     | 0.0987    |
| M2       | 0.036                     | 0.0307    | 0.101     | 0.0753    |
| M3       | 0.048                     | 0.0363    | 0.133     | 0.0888    |
| M4       | 0.038                     | 0.0275    | 0.102     | 0.075     |

would like to investigate the accuracy of our methodology in this situation. Specifically, we would like to emulate short-term oscillations at two oscillatory frequencies and two different time locations where $n = 2000$,

$$
\mu_k = 2 \cos(\omega_1 k)I_{0.1n \leq k \leq 0.45n} + 2.5 \cos(\omega_2 k)I_{0.55n \leq k \leq 0.8n}
$$

(13)

with $\omega_1 = 0.17007(2\pi)$ and $\omega_2 = 0.38007(2\pi)$. The noises $\epsilon_{k,n}$ are generated from models $(M1) - (M4)$. Tables 4 and 5 report the accuracy of the estimators by computing their mean squared errors (MSE) and the probability of estimating the accurate number of oscillatory frequencies and change points. It can be seen that the simulated probabilities of estimating the correct number of oscillatory frequencies and change points are high. Furthermore, since there is no change in phase, Proposition 1 implies that the estimation precision $|\hat{w} - w| \approx n^{-3/2} \log(n) \approx 10^{-4}$ which implies that $MSE(\hat{w}) \approx 10^{-8}$. It can be seen that the results from Table 4 are consistent with this theoretical accuracy. Similarly, comparing the result from Table 5 with the theoretical accuracy of Theorem 4.4, we have $MSE(\hat{b}_i) \approx \log(n)^2 \approx 10^1$, the two results stay consistent. Finally, additional simulation results on the power performance as well as the estimation accuracy of our methodology can be found in Section A of the supplementary material.
| Model | $MSE(\hat{w}_1)$ | $MSE(\hat{w}_2)$ | $P(|\Omega| = 2)$ | $MSE(\hat{w}_1)$ | $MSE(\hat{w}_2)$ | $P(|\Omega| = 2)$ |
|-------|-----------------|-----------------|-------------------|-----------------|-----------------|-------------------|
| M1    | 5.26e-09        | 1.14e-07        | 0.99              | 5.27e-09        | 1.13e-07        | 0.969             |
| M2    | 4.9e-09         | 1.02e-07        | 0.996             | 4.9e-09         | 1.02e-07        | 0.986             |
| M3    | 4.9e-09         | 9.6e-08         | 0.992             | 4.9e-09         | 9.57e-08        | 0.983             |
| M4    | 4.9e-09         | 1.24e-07        | 0.987             | 4.9e-09         | 1.24e-07        | 0.968             |

Table 4: Simulated stage 1 estimation accuracy for $\mu_{k,n}$ specified in (13).

6 Example: detecting sleep spindles

We demonstrate how the proposed two-stage algorithm can be applied to studying the electroencephalogram (EEG) signal. Sleep spindles are bursts of neural oscillatory activity that are captured by the surface EEG during sleep. They are generated by the complicated interplay of the thalamic reticular nucleus and other thalamic nuclei [De Gennaro and Ferrara, 2003] during N2 sleep stage. N2 sleep stage is defined based on the AASM sleep stage classification system [Iber et al., 2007]. Spindles oscillate in a frequency range of about 11 to 16 Hz with a duration of 0.5 seconds or greater (usually 0.5-1.5 seconds). The EEG signal during N2 sleep stage serves a good example for the change point detection problem. The spindle might exist from time to time, and there might be multiple spindles during the N2 stage. The dynamics of spindles encode important physiological information [De Gennaro and Ferrara, 2003]. While it is possible to have experts reading it, it might not be feasible if the data size is large. We thus need an automatic detection algorithm to achieve this goal.

To apply our proposed two-stage algorithm we check if the model (3) is reasonable. Although the spindle frequency might change from time to time, the frequency changes slowly and the spindles exist only for a relatively short period, so we can reasonably assume that the frequency $\omega_k$ in (3) is fixed. On the other hand, the appearance of spindle can be well captured by the amplitude $A_{r,k}$ and $B_{r,k}$ in (3). Moreover, the EEG signal other than the spindle is non-stationary, and we assume that it can be well captured by the PLS model. We emphasize that how well the PLS model captures this non-stationarity is out of the scope of this paper.

In this section, the EEG time series was recorded from the standard polysomnogram (PSG) signals on patients suspicious of sleep apnea syndrome at the sleep center in Chang Gung Memorial Hospital (CGMH), Linkou, Taoyuan, Taiwan. under the approval of the Institutional Review Board of CGMH (No. 101-4968A3). All recordings were acquired on the Alice 5 data acquisition system (Philips Respironics, Murrysville, PA). The EEG signal is sampled at 200Hz. The sleep stages, including wake, rapid eyeball movement (REM) and stage 1, stage 2 and stage 3 of NREM, were annotated by two experienced sleep specialists according to the AASM 2007 guidelines [Iber et al., 2007], and a consensus was reached. According to the protocol, the sleep specialists provide annotation for 30-seconds long epochs.
Below, we focus on those epochs labeled as the N2 stage. A 10-sec segment of the EEG signal from channel C4A1 and its analysis are shown in Figure 3. In the two-stage algorithm, we first want to find and estimate the frequencies of existing oscillations in the signal. The first-stage, the DPPT statistic with $m = 16$, is shown in the second panel against a grid of test frequencies in Hz, where the horizontal line shows the 90th simulated quantile value of the first iteration step by using the OBMB. Clearly, there is a peak around 14Hz, which suggests that there exists an oscillatory component inside the EEG signal. Note that the size of the peaks is affected by the amplitude of the oscillatory pattern and sample size of the time series data. Under significant level $\alpha = 0.1$, we are confident that there is one oscillatory component with the estimated frequency 14.1Hz. For the detected oscillatory component, next we estimate if there is any change point. The third panel shows the second-stage statistics with $m' = 50$ and $m' = 8$ plotted against time position. All tuning parameters stated above are selected by the MV method. The horizontal line indicates the 90-th percent simulated quantile of the first iteration step under estimated frequency 14.1Hz. The estimated break points are at 698-th and 1135-th units under $\beta = 0.1$. Again, the size and shape of each peak in the second-stage statistics are affected by the choice of the bandwidth parameter $m$ and amplitude of the oscillatory pattern existing in the data. In the forth panel, the detected spindle is colored in red, which coincides with the experts’ annotation.

Another segment of the EEG signal from channel C4A1 without any annotated spindle is shown in the first panel in Figure 4. The first-stage, the DPPT statistic, is shown in the second panel against a grid of test frequencies in Hz, where the horizontal line shows 90th
simulated quantile value by using the OBMB. While there seem to be peaks around 20Hz, 26Hz and 32Hz, they are not significant under significant level $\alpha = 0.1$. This finding suggests that there does not exist an oscillatory component inside the EEG signal that is sufficiently strong or long.

Figure 3: First panel: The recorded EEG signal. Second panel: the DPPT statistics $\tilde{F}(\cdot)$. Third panel: The second-stage statistics $T(\cdot, 14.1)$. Fourth panel: the EEG signal under analysis with the detected spindle superimposed in red.

Figure 4: First panel: The recorded EEG signal. Second panel: the DPPT statistics $\tilde{F}(\cdot)$. 

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The above preliminary analysis results indicate the potential of the proposed two-stage algorithm. A systematic application of the proposed algorithm to physiological signals for clinical applications will be reported in our future work.

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Abstract

Section A of this supplementary material demonstrates additional simulation results. Section B contains Gaussian approximation results for the sums of real- and complex-valued high dimensional non-stationary time series without variance lower bounds. In Section C, Gaussian comparison results for real- and complex-valued high dimensional vectors without variance lower bounds are established. All proofs of the theoretical results of the paper can be found in Section D.

A Additional simulation results

A.1 Stage 1 power simulation

The first stage power simulation is performed under the setting where $\mu_{k,n} = A \cos(\omega k)$ for $\omega = 0.1(2\pi)$ and $\epsilon_{k,n} = 0.5 \cos(k/n)\epsilon_{k-1,n} + \epsilon_k$ i.i.d. standard normal. The significance level is set at $\alpha = 0.05$ and 0.1 and $n = 1000$. Figure 5 shows the simulated rejection rates plotted against the amplitude $A$. The variance of the generated time series data is approximately one and the signal strength is quantified by the oscillatory amplitude $A$. Thus $A$ is approximately the signal to noise ratio and Figure 5 show that the simulated rejection rates increase quite fast and approach 1 when $A$ is larger than 0.35.

![Figure 5: Simulated rejection rates for $\mu_{k,n} = A \cos(\omega k)$ for various values of $A$.](image-url)
A.2 Additional simulations on estimation accuracy for short-term oscillations

In this subsection, we would like to emulate a short-term oscillatory pattern with one oscillatory frequency where

\[ \mu_{k,n} = 3 \cos(\omega_k)I_{0.1n < k \leq 0.25n} \]

for \( \omega = 0.1(2\pi) \). The signal is contaminated by non-stationary noises \( \epsilon_{k,n} \) defined by models \((M1) - (M4)\).

Tables 1 and 2 report the accuracy of the estimators by computing their mean squared errors (MSE) and the probability of estimating the accurate number of oscillatory frequencies and change points. It can be seen that the simulated probabilities of estimating the correct number of oscillatory frequencies and change points are high. Furthermore, since there is no change in phase, Proposition 1 implies that the squared estimation precision

\[ |\hat{w} - w|^2 \approx n^{-3} \log^2(n) \approx 10^{-7} \]

when \( n = 1000 \). Similar calculations can be performed for \( n = 2000 \).

It can be seen that the results from Table 1 are consistent with this theoretical accuracy. Similarly, comparing the result from Table 2 with the theoretical accuracy of Theorem 4.4, we have for \( n = 1000 \), \( MSE(\hat{b}_i) \approx \log(n)^2 \approx 10^4 \), the two results stay consistent.

| n = 1000 | \( \alpha = 0.05 \) | \( \alpha = 0.1 \) |
| Model | \( MSE(\hat{w}) \) | \( P(|W| = 1) \) | \( MSE(\hat{w}) \) | \( P(|W| = 1) \) |
|-------|-----------------|--------------|-----------------|--------------|
| M1    | 7.17e-07        | 0.990        | 7.20e-07        | 0.972        |
| M2    | 7.70e-07        | 0.983        | 7.71e-07        | 0.974        |
| M3    | 7.23e-07        | 0.986        | 7.29e-07        | 0.978        |
| M4    | 6.63e-07        | 0.984        | 6.55e-07        | 0.967        |

| n = 2000 | \( \alpha = 0.05 \) | \( \alpha = 0.1 \) |
| Model | \( MSE(\hat{w}) \) | \( P(|W| = 1) \) | \( MSE(\hat{w}) \) | \( P(|W| = 1) \) |
|-------|-----------------|--------------|-----------------|--------------|
| M1    | 1.07e-07        | 0.986        | 1.04e-07        | 0.954        |
| M2    | 1.25e-07        | 0.993        | 1.25e-07        | 0.980        |
| M3    | 1.23e-07        | 0.991        | 1.23e-07        | 0.975        |
| M4    | 1.24e-07        | 0.988        | 1.25e-07        | 0.965        |

Table 6: Simulated stage 1 estimation accuracy for \( \mu_{k,n} = 3 \cos(\omega_k)I_{0.1n < k \leq 0.25n} \).

B Gaussian approximation without variance lower bounds

This section of the appendix deals with approximations to the sum of an \( h \)-dimensional non-stationary time series \( \{x_i\} \) (either real or complex) by a centered Gaussian time series \( \{y_i\} \) with the same covariance structure without assuming there is a lower bound for the coordinate-wise variances of the normalized sum of \( \{x_i\} \).

First we need the following Lemma B.1 which extends Nazarov’s Inequality (Nazarov, 2003) to Gaussian random vectors without variance lower bound.
Table 7: Simulated stage 2 estimation accuracy for $\mu_{k,n} = 3 \cos(\omega k) \mathbb{I}_{0.1n<k\leq0.25n}$, where $b_1 = 0.1n$ and $b_2 = 0.25n$.

Lemma B.1 (An Extended Version of Nazarov’s Inequality). Let $Y = (Y_1, \ldots, Y_h)^\top$ be a centred Gaussian random vector, where $h > 1$. Let $y \in \mathbb{R}^h$ satisfy $y_i > c$ or $y_i + a < -c$ for some $c > 0$ and $a > 0$. Then, for any $b > 0$,

$$
\mathbb{P}(Y \leq y + a) - \mathbb{P}(Y \leq y) \leq 4(a/b)^{\log b} \frac{bh}{c^2/2} \exp(-(c/b)^2/2).
$$

where $Y \leq y$ means $Y_i \leq y_i$ for $i = 1, \ldots, h$.

Proof. For some constant $b > 0$, we first separate the coordinates whose variances are greater than $b$ and the coordinates whose variances are smaller than $b$. First note that

$$
\mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \cap B^c) \geq \mathbb{P}(A) - \mathbb{P}(B^c).
$$

Therefore, by taking $A = \{Y_i \leq y_i, \forall i$ where $\sigma_i > b\}$ and $B = \{Y_i \leq y_i, \forall i$ where $\sigma_i \leq b\}$, we have

$$
\mathbb{P}(Y_i \leq y_i) \geq \mathbb{P}(Y_i \leq y_i, \forall i$ where $\sigma_i > b) - \mathbb{P}(Y_i > y_i, \exists i$ where $\sigma_i \leq b).
$$

We can further expand using the above fact:

$$
\mathbb{P}(Y \leq y + a) - \mathbb{P}(Y \leq y) \leq \mathbb{P}(Y_i \leq y_i + a, \forall i$ where $\sigma_i \leq b$ and $Y_i \leq y_i + a, \forall i$ where $\sigma_i > b) - \mathbb{P}(Y_i \leq y_i, \forall i$ where $\sigma_i \leq b$ and $Y_i \leq y_i, \forall i$ where $\sigma_i > b) \leq \mathbb{P}(Y_i \leq y_i + a, \forall i$ where $\sigma_i > b) - \mathbb{P}(Y_i \leq y_i, \forall i$ where $\sigma_i > b) + \mathbb{P}(Y_i > y_i, \exists i$ where $\sigma_i \leq b).
$$
Then by Nazarov’s inequality [Nazarov, 2003], we can bound the part where the condition \( \sigma_i > b \) holds. Specifically,

\[
\mathbb{P}(Y_i \leq y_i + a, \forall i \text{ where } \sigma_i > b) - \mathbb{P}(Y_i \leq y_i, \forall i \text{ where } \sigma_i > b) \leq (a/b)(\sqrt{2 \log h} + 2) \leq 4(a/b)\sqrt{\log h}.
\]

To evaluate \( \mathbb{P}(Y_i > y_i, \exists i \text{ where } \sigma_i \leq b) \), when the assumption \( y_i > c \) for some \( c > 0 \) is satisfied, we need to bound the tail probability under the condition \( \sigma_i \leq b \). Let \( n(b) \) be the number of coordinates satisfy \( \sigma_i \leq b \). Then

\[
\mathbb{P}(Y_i > y_i, \exists i \text{ where } \sigma_i \leq b) = \mathbb{P}(Y_i/\sigma_i > y_i/\sigma_i, \exists i \text{ where } \sigma_i \leq b) \leq \sum_{k=1}^{n(b)} \int_{y_i/\sigma_i}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)dt \quad \text{(Union bound)}
\]

\[
\leq \frac{n(b)}{\sqrt{2\pi}} \int_{c/b}^{\infty} \exp(-t^2/2)dt \leq \frac{bh}{c\sqrt{2\pi}} \exp(-(c/b)^2/2).
\]

Similarly, for the case where there exists an \( y_i \) such that \( y_i + a < -c \) and \( \sigma_i \leq b \), we have

\[
\mathbb{P}(Y \leq y + a) - \mathbb{P}(Y \leq y) \leq \mathbb{P}(Y \leq y + a) \leq \mathbb{P}(Y_i \leq y_i + a) \leq \frac{bh}{c\sqrt{2\pi}} \exp(-(c/b)^2/2).
\]

The following corollary is a direct application of Lemma B.1 by picking proper constants \( b \) and \( c \).

**Corollary B.1.** Let \( Y = (Y_1, ..., Y_h)^\top \) be a centred Gaussian random vector, where \( h > 1 \). Take \( \delta > 0 \). Then for \( c = 2\sqrt{2 \log(h)}^{1-\delta} \) and \( a > 0 \), we have

\[
\sup_{|x| > c+a} \mathbb{P}(\max_{1 \leq j \leq h} Y_j - x \leq a) \leq 4a(\log h)^{1+\delta} + \frac{1}{2\sqrt{\pi}(\log h)^{1/2}h}.
\]

**Proof.** Pick \( b = \log(h)^{-1-\delta} \). Then, for \( y \in \mathbb{R}^h \) so that \( y_i > c \) or \( y_i < -c - a \),

\[
\frac{h}{\sqrt{2\pi}c} \frac{b}{\sqrt{2\pi}} \exp(-(c/b)^2/2) = \frac{h}{2\sqrt{\pi}(\log h)^{1/2}} \exp(-2(\log h)) \leq \frac{1}{2\sqrt{\pi}(\log h)^{1/2}h}.
\]

Combining the previous results we get

\[
\mathbb{P}(Y \leq y + a) - \mathbb{P}(Y \leq y) \leq 4a(\log h)^{1+\delta} + \frac{1}{2\sqrt{\pi}(\log h)^{1/2}(h)h}.
\]

We thus get the proof by writing the above quantities entrywisely. \( \square \)
The next proposition is an extension of Proposition A.1 in Zhang and Cheng (2018). We shall first introduce some notation used in the latter paper.

Let \( \{x_i\} = \{(x_{i1}, \cdots, x_{ih})^\top\} \) be a centered \( h \)-dimensional \( M \)-dependent times series. Take any truncation levels \( M_x > 0 \). For \( N \geq M \) and \( N, M, r \to +\infty \) as \( n \to +\infty \), define block sums:

\[
A_{ij} := \sum_{l=iN+(i-1)M-N+1}^{iN+(i-1)M} x_{l,j}, \quad B_{ij} := \sum_{l=i(N+M)-M+1}^{i(N+M)} x_{l,j},
\]

the block sum for truncated \( \chi_{i,j} := (x_{i,j} \wedge M_x) \lor (-M_x) \):

\[
\tilde{A}_{ij} := \sum_{l=iN+(i-1)M-N+1}^{iN+(i-1)M} \tilde{x}_{l,j}, \quad \tilde{B}_{ij} := \sum_{l=i(N+M)-M+1}^{i(N+M)} \tilde{x}_{l,j},
\]

and the block sum for the truncated and centered \( \tilde{x}_{i,j} := (x_{i,j} \wedge M_x) \lor (-M_x) - \mathbb{E}(x_{i,j} \wedge M_x) \lor (-M_x) \):

\[
\bar{A}_{ij} := \sum_{l=iN+(i-1)M-N+1}^{iN+(i-1)M} \bar{x}_{l,j}, \quad \bar{B}_{ij} := \sum_{l=i(N+M)-M+1}^{i(N+M)} \bar{x}_{l,j}.
\]

Let \( \varphi(M_x) \) be the smallest finite constant which satisfies

\[
\mathbb{E}(A_{ij} - \bar{A}_{ij})^2 \leq N\varphi^2(M_x), \quad \mathbb{E}(B_{ij} - \bar{B}_{ij})^2 \leq M\varphi^2(M_x)
\]

uniformly for \( i \) and \( j \). Also, let \( \phi(M_x) \) be a constant which satisfies

\[
\max_{1 \leq j, k \leq h} \frac{1}{n} \sum_{i=1}^{n} \sum_{l=i-N+1}^{iN} \left| \mathbb{E}x_{ij}x_{lk} - \mathbb{E}\tilde{x}_{ij}\tilde{x}_{lk} \right| \leq \phi(M_x).
\]

Let \( \{y_i\} \) be the Gaussian counterpart of \( \{x_i\} \) with the same covariance structure of \( \{x_i\} \), and take any truncation level \( M_y \). Define \( \varphi(M_y) \) and \( \phi(M_y) \) similarly based on \( \{y_{ij}\} \). Set \( \phi(M_x, M_y) := \phi(M_x) + \phi(M_y) \), and set \( \varphi(M_x, M_y) := \varphi(M_x) \lor \varphi(M_y) \).

If \( \{x_i = G_{i,n}(\mathcal{F}_i)\} \) is not \( M \)-dependent, let \( \{x_i^{(M)} := \mathbb{E}(x_i|e_i, e_{i-1}, \cdots, e_{i-M})\} \) be the \( M \)-dependent approximation to \( \{x_i\} \). Recall that \( \mathcal{F}_i = (\cdots, e_{i-1}, e_i) \) and \( \{e_i\}_{i \in \mathbb{Z}} \) are i.i.d. random variables. Define \( \{y_i^{(M)}\} \) as the \( M \)-dependent sequence of Gaussian random variables which preserves the covariance structure of \( \{x_i^{(M)}\} \). Similarly we can define \( A_{ij}^{(M)}, \tilde{A}_{ij}^{(M)}, \bar{A}_{ij}^{(M)}, \tilde{B}_{ij}^{(M)}, \bar{B}_{ij}^{(M)} \) and \( \phi^{(M)}(M_x, M_y) \) are defined similarly based on \( \{x_i^{(M)}\} \) and \( \{y_i^{(M)}\} \).

**Proposition B.1.** Let \( \{x_i\} = \{(x_{i1}, \cdots, x_{ih})^\top\} \) be a centered \( h \)-dimensional \( M \)-dependent times series. Let \( \{y_i\} \) be its Gaussian counterpart. Define \( M_{xy} = \max\{M_x, M_y\} \). Suppose \( 2\sqrt{\beta}(6M + 1)M_{xy}/\sqrt{n} \leq 1 \), where \( \beta > 0 \) is a constant. Also, suppose \( M_x > u_x(\gamma) \) and
$M_y > u_y(\gamma)$ for some $\gamma \in (0, 1)$, where $u_x(\gamma)$ is the $(1 - \gamma)$-quantile of $\max_{i,j} |x_{ij}|$ with $u_y(\gamma)$ defined similarly. Further, assume that $\max_{1 \leq i \leq n} |\sum_{k,l=1}^h \text{Cov}(x_{i,k}, x_{i,l})|/n \leq a_1$ for some finite constant $a_1$. Define $\bar{m}_{x,k} := \max_{1 \leq i \leq n} \sum_{k,l=1}^h \mathbb{E}|x_{ij}|^{1/k}/n$, $k = 1, 2, \ldots$, $T_X := \max_{1 \leq i \leq h} \sum_{i=1}^n x_{i,j}/\sqrt{n}$ and $\bar{m}_{y,k}$ and $T_Y$ are defined similarly. Then, for any $\psi > 0$,

$$\sup_{|t| > d_h} |\mathbb{P}(T_X \leq t) - \mathbb{P}(T_Y \leq t)| \lesssim (\psi^3 + \psi^2 \beta + \psi \beta^2) (2M + 1)^2 \left(\bar{m}_{x,3}^3 + \bar{m}_{y,3}^3\right) \psi \varphi(M_x, M_y) \sqrt{\log(h/\gamma)} + \gamma + (\beta^{-1} \log(h) + \psi^{-1}) (\log h)^{1+\delta} + h^{-1} (\log h)^{-1/2},$$

where $d_h := \beta^{-1} \log(h) + \psi^{-1} + 2\sqrt{2} \log(h)^{-\delta}$.

**Proof.** Let

$$g_0(x) = \begin{cases} 0, & x \geq 1, \\ 30 \int_1^x s^2(1-s)^2 ds, & 0 < x < 1, \\ 1, & x \leq 0. \end{cases}$$

and pick $g(s) = g_0(\psi(s - t - e_\beta))$ with $e_\beta = \beta^{-1} \log h$ and $t$ to be chosen later. Denote

$$G_k := \sup_{x \in \mathbb{R}} \partial^k g(x)/\partial x^k, \quad k = 1, 2, \ldots$$

We have $G_0 \lesssim 1$, $G_1 \lesssim \psi$, $G_2 \lesssim \psi^2$ and $G_3 \lesssim \psi^3$. Moreover, the function also satisfies

$$\mathbb{I}(x \leq t + e_\beta) \leq g(x) \leq \mathbb{I}(x \leq t + e_\beta + \psi^{-1}), \forall x \in \mathbb{R},$$

where $\mathbb{I}$ is the indicator function. Define $m(y) = g \circ F_\beta(y)$, where

$$F_\beta(y) = \beta^{-1} \sum_{i=1}^h \exp(\beta y_i), \quad y = (y_1, \ldots, y_h)^\top \in \mathbb{R}^h.$$

Denote $X := (X_1, X_2, \ldots, X_h)^\top = \sum_{i=1}^n x_i/\sqrt{n}$ and $Y := (Y_1, Y_2, \ldots, Y_h)^\top$ is defined similarly. Based on Equation (39) in the proof of Proposition A.1 of [Zhang and Cheng (2018)](#), we have the following

$$|\mathbb{E}[m(X) - m(Y)]| \lesssim (\psi^2 + \psi \beta) \phi(M_x, M_y) + (\psi^3 + \psi^2 \beta + \psi \beta^2) (2M + 1)^2 \left(\bar{m}_{x,3}^3 + \bar{m}_{y,3}^3\right) \psi \varphi(M_x, M_y) \sqrt{\log(h/\gamma)} + \gamma := B. \quad (3)$$
Now assume $|t| > e_\beta + \psi^{-1} + 2\sqrt{2}\log^{-\delta} h$, we have
\[
\mathbb{P}(\max_{1 \leq j \leq h} X_j \leq t) \leq \mathbb{P}(F_\beta(X) \leq t + e_\beta) \leq \mathbb{E}[g(F_\beta(X))] \\
\leq \mathbb{E}[g(F_\beta(Y))] + B \\
\leq \mathbb{P}(F_\beta(Y) \leq t + e_\beta + \psi^{-1}) + B \\
\leq \mathbb{P}\left(\max_{1 \leq j \leq h} Y_j \leq t + e_\beta + \psi^{-1}\right) + B.
\]

Then, by Lemma B.1 and assuming $|t| > e_\beta + \psi^{-1} + 2\sqrt{2}(\log h)^{-\delta}$, we have
\[
\mathbb{P}\left(\max_{1 \leq j \leq h} Y_j \leq t + e_\beta + \psi^{-1}\right) - \mathbb{P}\left(\max_{1 \leq j \leq h} Y_j \leq t\right) \leq (e_\beta + \psi^{-1})(\log h)^{1+\delta} + h^{-1}(\log h)^{-1/2}.
\]

Thus, since $T_X = \max_{1 \leq j \leq h} X_j$ and $T_Y = \max_{1 \leq j \leq h} Y_j$, we conclude
\[
\mathbb{P}(T_X \leq t) - \mathbb{P}(T_Y \leq t) \leq B + (e_\beta + \psi^{-1})(\log h)^{1+\delta} + h^{-1}(\log h)^{-1/2}.
\]

The opposite direction can be proved similarly by noting that
\[
\mathbb{P}\left(\max_{1 \leq j \leq h} X_j \leq x\right) \geq \mathbb{P}\left(\max_{1 \leq j \leq h} Y_j \leq x - e_\beta - \psi^{-1}\right) - B
\]
and
\[
\mathbb{P}\left(\max_{1 \leq j \leq h} Y_j \leq x - e_\beta - \psi^{-1}\right) - \mathbb{P}\left(\max_{1 \leq j \leq h} Y_j \leq x\right) \geq -(e_\beta + \psi^{-1})(\log h)^{1+\delta} - h^{-1}(\log h)^{-1/2}.
\]

Let $\{x_i\}_{i=1}^n$ be a general centered $h$-dim non-stationary time series satisfying

(A0) $x_i = \mathcal{G}_{i,n}(\cdots, e_{i-1}, e_i) \in \mathbb{R}^h$, $i = 1, \cdots, n$ (triangular array), where $e_i$, $i \in \mathbb{Z}$, are i.i.d. random elements and $\mathcal{G}_{i,n}$ are $h$-dim vector-valued Borel-measurable functions. For an integer $k \geq 0$ and a positive real number $q \geq 1$, define the physical dependence measures of $\{x_i\}$

$$
\theta_{j,k,q} = \max_{1 \leq i \leq n} \| \mathcal{G}_{i,j,n}(\cdots, e_{i-1}, e_i) - \mathcal{G}_{i,j,n}(\cdots, \hat{e}_{i-k}, e_{i-k+1}, \cdots, e_{i-1}, e_i) \|_q,
$$

where $\mathcal{G}_{i,j,n}(\cdot)$ is the $j$-th component function of $\mathcal{G}_{i,n}$, $j = 1, 2, \cdots, h$, and $\hat{e}_{i-k}$ is identically distributed as $e_{i-k}$ and is independent of $\{e_i\}_{i \in \mathbb{Z}}$.

We make the following three assumptions for the time series $\{x_i\}$ which corresponds to Assumptions (2.1) to (2.3) of Zhang and Cheng (2018) but without assuming lower variance bounds.
(A1) Assume that \( \max_{1 \leq i \leq n} \max_{1 \leq j \leq h} \mathbb{E} x_{i,j}^4 < c_1 \) for some finite \( c_1 > 0 \) and there exists \( D_n > 0 \) such that one of the following two conditions holds:

\[
\max_{1 \leq i \leq n} \mathbb{E} \exp\left( \max_{1 \leq j \leq h} |x_{i,j}|/D_n \right) \leq 1, \tag{4}
\]

or

\[
\max_{1 \leq i \leq n} \mathbb{E} g\left( \max_{1 \leq j \leq h} |x_{i,j}|/D_n \right) \leq 1 \tag{5}
\]

for some strictly increasing and convex function \( g \) defined on \( [0, \infty) \) satisfying \( g(0) = 0 \).

(A2) Assume that there exist \( M = M(n) > 0 \) and \( \gamma = \gamma(n) \in (0, 1) \) such that

\[
n^{3/8} M^{-1/2} l_n^{-5/8} \geq C_2 \max\{D_n l_n, l_n^{1/2}\} \text{ under Condition } (4) \tag{6}
\]

\[
n^{3/8} M^{-1/2} l_n^{-5/8} \geq C_1 \max\{D_n g^{-1}(n/\gamma), l_n^{1/2}\} \text{ under Condition } (5) \tag{7}
\]

for \( C_1, C_2 > 0 \), where \( l_n = \log(hn/\gamma) \lor 1 \). In both cases, suppose \( n^{7/4} M^{-2} l_n^{-9/4} \geq C_3 > 0 \).

(A3) Assume that \( \max_{1 \leq i \leq h} |\sum_{k,l=1}^n \text{Cov}(x_{k,i}, x_{l,i})/n| \leq a_1 \) and

\[
\sum_{j=0}^{\infty} j \max_{1 \leq k \leq h} \theta_{j,k,3} < a_2
\]

for some finite constants \( a_1 \) and \( a_2 \), where \( \theta_{j,k,q} \) is the \( j \)-th physical dependence measure of the \( k \)-th coordinate process of \( \{x_i\} \) with respect to the \( L^q \) norm.

Recall that \( \{x^{(M)}_i := \mathbb{E}(x_i|e_i, e_{i-1}, \ldots, e_{i-M})\} \) is the \( M \)-dependent approximation to \( \{x_i\} \) and \( \{y^{(M)}_i\} \) is an \( M \)-dependent sequence of Gaussian random variables which preserves the covariance structure of \( \{x^{(M)}_i\} \).

**Lemma B.2.** Let \( \{x_j\}_{j=1}^n \) be a \( h \)-dim time series satisfy Assumptions (A0)-(A3). Then

\[
\phi^{(M)}(M_x, M_y) \leq C'(1/M_x + 1/M_y^2) \text{ and } \varphi^{(M)}(M_x, M_y) \leq C''(1/M_x^{5/6} + \sqrt{N}/M_x^3 + 1/M_y^2)
\]

for some finite constants \( C' \) and \( C'' \), where we recall that \( \phi^{(M)}(M_x, M_y) \) and \( \varphi^{(M)}(M_x, M_y) \) are the versions of \( \phi(M_x, M_y) \) and \( \varphi(M_x, M_y) \) defined based on \( \{x^{(M)}_i\} \) and \( \{y^{(M)}_i\} \).

**Proof.** The results follow from steps 2 and 3 in proof of Theorem 2.1 in Zhang and Cheng (2018). From Zhang and Cheng (2018, step 2 in proof of Theorem 2.1), we get \( \phi^{(M)}(M_x) \leq C'/M_x \) and \( \varphi^{(M)}(M_x) \leq C''(1/M_x^{5/6} + \sqrt{N}/M_x^3) \). From Zhang and Cheng (2018, step 3 in proof of Theorem 2.1), we get \( \phi^{(M)}(M_y) \leq C'/M_y^2 \) and \( \varphi^{(M)}(M_y) \leq C''/M_y^2 \) for \( C', C'' > 0 \).

\( \square \)

S.8
Theorem B.1. Let \( \{x_i\}_{i=1}^n \) be a time series satisfy Assumptions (A0)-(A3). Let \( \{y_j \in \mathbb{R}^h\} \) be Gaussian random vectors with the same covariance structure as \( \{x_j\} \). For any integer \( M > 0 \) and positive real number \( q \geq 1 \), let \( \Xi_M := \max_{1 \leq k \leq M} \sum_{j=M}^{\infty} j \theta_{k,j,2} \) and \( \Theta_{M,i,q} := \sum_{j=M}^{\infty} \theta_{i,j,q} \), where \( i = 1, \ldots, h \). Assume that \( q \geq 2 \) and \( \max_{1 \leq j \leq h} \Theta_{0,j,q} < \infty \). Then

\[
\sup_{|x| > d_{n,h}} \left| \mathbb{P}(T_X \leq x) - \mathbb{P}(T_Y \leq x) \right| \lesssim G(n, h),
\]

where \( T_X := \max_{1 \leq j \leq h} \sum_{i=1}^{n} x_{ij} / \sqrt{n}, \ T_Y := \max_{1 \leq j \leq h} \sum_{i=1}^{n} y_{ij} / \sqrt{n}, \)

\[
d_{n,h} := n^{-1/8} M^{1/2} l_n^{11/7} + l_\gamma + \Xi_{M}^{1/3} \delta/3.
\]

and

\[
G(n, h) := n^{-1/8} M^{1/2} l_n^{11/7} + \gamma + (n^{1/8} M^{-1/2} l_n^{-3/8} \gamma^{q/1+q}) \left( \sum_{j=1}^{h} \Theta_{M,j,q}^{q} \right)^{1/(1+q)} + \Xi_{M}^{1/3} \delta/3.
\]

Proof. Follow the same \( M \)-dependent sequence construction in Lemma B.2 and get \( x_i^{(M)} \) and \( y_i^{(M)} \). By construction, \( x_{1,j} \) and \( x_{i+1,j,k}^{(l-1)} \) are independent for any \( 1 \leq j, k \leq h \). Denote \( X^{(M)} := \sum_{i=1}^{n} x_{i}^{(M)} / \sqrt{n} \) and \( Y^{(M)} := \sum_{i=1}^{n} y_{i}^{(M)} / \sqrt{n} \). The triangular inequality and (16) in Zhang and Cheng (2018) imply that

\[
|\mathbb{E}[m(X) - m(Y^{(M)})]| \lesssim |\mathbb{E}[m(X^{(M)}) - m(Y^{(M)})]| + (G_0 G_1)^{1/(1+q)} \left( \sum_{j=1}^{h} \Theta_{M,j,q}^{q} \right)^{1/(1+q)},
\]

where we recall the definitions of \( G_0 \) and \( G_1 \) from (2). Following the arguments in the proof of Proposition A.1 in Zhang and Cheng (2018) and using Proposition B.1 if all conditions are satisfied, we have

\[
\sup_{|x| > d_{n,h}} \left| \mathbb{P}(T_X^{(M)} \leq x) - \mathbb{P}(T_Y^{(M)} \leq x) \right|
\]

\[
\lesssim (\psi^2 + \psi \beta) \phi(M_x, M_y) + (\psi^3 + \psi^2 \beta + \psi \beta^2) \frac{(2M + 1)^2}{\sqrt{n}} (\bar{m}_{x,3} + \bar{m}_{y,3})
\]

\[
+ G_1 \psi(M_x, M_y) \sqrt{8 \log(h/\gamma)} + G_0 \gamma + (\psi)^{1/(1+q)} \left( \sum_{j=1}^{h} \Theta_{M,j,q}^{q} \right)^{1/(1+q)}
\]

\[
+ (\beta^{-1} \log(h) + \psi^{-1}) (\log(h))^{1+\delta} + h^{-1} (\log h)^{-1/2}.
\]

By Assumption (A2) where \( n^{7/4} M^{-1/4} l^{-9/4} > C_3 M \), we have

\[
(\psi^2 + \psi \beta) \phi(M_x, M_y) \lesssim n^{-1/8} M l_n^{-9/4},
\]

(11)

\[
(\psi^3 + \psi^2 \beta + \psi \beta^2) \frac{(2M + 1)^2}{n^{1/2}} \lesssim n^{-1/8} M l_n^{-9/4},
\]

(12)

\[
\psi \phi(M_x, M_y) \sqrt{8 \log(h/\gamma)} \lesssim n^{-1/8} M^{1/2} l_n^{7/8},
\]

(13)

\[
(\beta \log(h) + \psi^{-1}) (\log(h))^{1+\delta} \lesssim \frac{5^{5/2} M u}{\sqrt{n}} \lesssim n^{-1/8} M l_n^{15/8}.
\]

S.9
Finally, by Step 5 in the proof of Theorem 2.1 in [Zhang and Cheng (2018)] and Theorem C.1 it follows that

$$
\sup_{|t|>d'_h} |\mathbb{P}(T_Y \leq t) - \mathbb{P}(T_{Y^{(M)}} \leq t)| \leq \Xi^{1/3}_M (\log h)^{1+\delta} + \frac{1}{(\log h)^{1/2}h},
$$

(15)

where $d'_h = C(\Xi^{1/3}_M \log(h)^{\delta/3} + \log(h)^{-\delta})$. The result follows from equations (11)-(15).

The left is verifying the conditions in Proposition B.1, since (10) is based on $x_i^{(M)}$ and $y_i^{(M)}$. Consider $g$ in Assumption (A1). We have

$$
\mathbb{P}\left(\max_{1 \leq i \leq n} \max_{1 \leq j \leq h} |x_{ij}^{(M)}| > u\right) \leq \sum_{i=1}^{n} \sum_{j=1}^{h} \mathbb{P}\left(g\left(\max_{1 \leq j \leq h} |x_{ij}^{(M)}|/\mathcal{D}_n\right) > g(u/\mathcal{D}_n)\right)
$$

$$
\leq n \max_{1 \leq i \leq n} \mathbb{E}g\left(\max_{1 \leq j \leq h} |x_{ij}^{(M)}|/\mathcal{D}_n\right)/g(u/\mathcal{D}_n) \leq nC_1/g(u/\mathcal{D}_n).
$$

The last inequality follows from Jensen’s inequality and Assumption (A1). By setting the above equation to $\gamma$, we get $u \leq \mathcal{D}_ng^{-1}(n/\gamma)$. Similarly, we have

$$
\mathbb{P}(\max_{1 \leq i \leq n} \max_{1 \leq j \leq h} |y_{ij}^{(M)}| > u) \leq \sum_{i=1}^{n} \sum_{j=1}^{h} \mathbb{P}\left(|y_{ij}^{(M)}| > u\right) \leq nh \exp(-u^2/(2\sigma^2)),
$$

where $\sigma^2 = \max_{1 \leq i \leq n} \max_{1 \leq j \leq h} \mathbb{E}\left(y_{ij}^{(M)}\right)^2 < \infty$. Then, by setting the above equation to $\gamma$, we get $u \leq \sqrt{2\sigma^2}\sqrt{\log(nh/\gamma)} = C'n^{1/2}$, where $C = \sqrt{2\sigma^2}$. Therefore, $u_x(\gamma) \leq \mathcal{D}_ng^{-1}(n/\gamma)$ and $u_y(\gamma) \leq l_n^{1/2}$. Then, by the assumption $n^{3/8}M^{-1/2}l_n^{-5/8} \geq C_1\max\{\mathcal{D}_nh^{-1}(n/\gamma),l_n^{1/2}\}$, we can choose $u \approx n^{3/8}M^{-1/2}l_n^{-5/8}$, which leads to

$$
\mathbb{P}(\max_{1 \leq i \leq n} \max_{1 \leq j \leq h} |x_{ij}^{(M)}| \leq u) \geq 1 - \gamma, \quad \mathbb{P}(\max_{1 \leq i \leq n} \max_{1 \leq j \leq h} |y_{ij}^{(M)}| \leq u) \geq 1 - \gamma.
$$

(16)

Next we will quantify $\phi(M_x, M_y)$ and $\phi(M_x, M_y)$. From Lemma B.2 we have $\phi(M_x, M_y) \leq C'(1/M_x + 1/M_y^2)$, $\phi(M_x, M_y) \leq C''(1/M_x^{5/6} + \sqrt{N}/M_x^3 + 1/M_y^2)$ and $\tilde{m}_{x,3} + \tilde{m}_{y,3} < \infty$. (Observe that $\mathbb{E}(|y_{ij}^{(M)}|^3) \leq \mathbb{E}(|x_{ij}|^3)$ and $\max_{i,j} \mathbb{E}|x_{ij}|^3 < \infty$ by Assumption (A1). Hence $\tilde{m}_{x,3} + \tilde{m}_{y,3} < \infty$.) We can set

$$
\psi \approx n^{1/8}M^{-1/2}l_n^{-3/8}, \quad M_x = M_y = u \approx n^{3/8}M^{-1/2}l_n^{-5/8}.
$$

Let $\beta \approx \sqrt{\mathbb{E}(x_M)}$, which implies $2\sqrt{5}\beta(6M + 1)M_{xy}/\sqrt{n} \approx 1$. Thus, the conditions in Proposition B.1 are satisfied, and we have finished the proof.

\[ \square \]

**Corollary B.2.** Let $x_i$ and $y_i$ be random vectors satisfying the conditions in Theorem B.1. Then, for $q \geq 2$ and

$$
T_x := \max_{1 \leq j \leq h} \left| \sum_{i=1}^{n} x_{ij} \right|/\sqrt{n} \quad \text{and} \quad T_y := \max_{1 \leq j \leq h} \left| \sum_{i=1}^{n} y_{ij} \right|/\sqrt{n},
$$

S.10
we get the following Gaussian approximation bound

\[
\sup_{|x| > d_{n,2h}} |\mathbb{P}(T_x \leq x) - \mathbb{P}(T_y \leq x)| \lesssim G(n, 2h),
\]

where \(d_{n,2h}\) is defined in (8) and \(G(n, 2h)\) is defined in (9).

**Proof.** Note that for \(X \in \mathbb{R}^h\) and let \(\tilde{X} := \begin{bmatrix} X \\ -X \end{bmatrix} \in \mathbb{R}^{2h}\)

\[
\mathbb{P}(\max_{1 \leq j \leq h} |X_j| \leq x) = \mathbb{P}(\{X_j \leq x, \text{ for all } j\}) = \mathbb{P}(\{\max_{1 \leq j \leq 2h} \tilde{X}_j \leq x\}).
\]

The result follows from Theorem B.1. \(\square\)

The following proposition establishes a Gaussian approximation result for sums of high dimensional complex-valued non-stationary time series under the classic norm of complex numbers without assuming that the variances of the normalized sums have a positive lower bound. Observe that the results of Theorem B.1 are for Gaussian approximations on hype\-rcubes of a high-dimensional Euclidean space. Suppose we take \(|z| = \sqrt{\Re(z)^2 + \Im(z)^2}\) to be the norm for a complex number \(z\), the region \(\max_{1 \leq k \leq h} |z_k| \leq x\) for an \(h\)-dimensional complex vector \(\vec{z} = (z_1, z_2, \cdots, z_h)^\top\) is not a hypercube when \(\vec{z}\) is viewed as a vector on \(\mathbb{R}^{2h}\). Hence the results of Theorem B.1 cannot be used directly here. To tackle this problem, we adopt the idea of simple convex set approximation used in Chernozhukov et al. (2017). In particular, we shall approximate a circle on the plane by regular convex polygons from both inside and outside.

**Proposition B.2.** Let \(\{z_i = z_{i,1} + \sqrt{-1}z_{i,2}\}\) be a centered \(h\)-dimensional complex-valued time series such that the \(2h\)-dimensional real-valued time series \(\{\tilde{z}_i = (z_{i,1}^\top, z_{i,2}^\top)^\top\}\) satisfies the assumptions of Theorem B.1. Take the centered \(h\)-dimensional complex Gaussian time series \(\{g_i\}\) that has the same covariance and pseudo-covariance structures as those of \(\{z_i\}\). Define

\[
\tilde{T}_z = \max_{1 \leq j \leq h} \left| \sum_{i=1}^n z_{ij} \right| / \sqrt{n} \quad \text{and} \quad \tilde{T}_g = \max_{1 \leq j \leq h} \left| \sum_{i=1}^n g_{ij} \right| / \sqrt{n}.
\]

Then, we have

\[
\sup_{x > d_{n,h}^*} |\mathbb{P}(\tilde{T}_z \leq x) - \mathbb{P}(\tilde{T}_g \leq x)| \lesssim G(n, 2nh) + \frac{h}{n^{3q/2}} + n^{-1/2} \log^{1+\delta}(nh) := G^*(n, h),
\]

where \(d_{n,h}^* = d_{n,2nh}(1 + \pi^2/(4n^2))\) and \(d_{n,2nh}\) and \(G(n, 2nh)\) are defined in (8) and (9), respectively.
Proof. We treat \( \{ \tilde{z}_i \} \) as a centered \( 2h \) dimensional non-stationary time series. Observe that, by the assumption that \( \max_{1 \leq j \leq 2h} \Theta_{0,j,q} < \infty \) for \( \{ \tilde{z}_i \} \), we have that
\[
\left\| \sum_{i=1}^{n} z_{ik,r} \right\|_q \leq C, \quad r = 1,2
\]
uniformly for all \( k = 1, 2, \cdots, h \) for some finite constant \( C \). Therefore,
\[
\left\| \max_{1 \leq k \leq h} \left| \sum_{i=1}^{n} z_{ik} \right| \right\|_q \leq Ch^{1/q}
\]
by a simple maximum inequality. Hence, if \( x \geq n^{3/2} \), by Markov’s inequality, we have
\[
\mathbb{P}(\max_{1 \leq k \leq h} \left| \sum_{i=1}^{n} z_{ik} \right| > x) \leq \frac{h}{n^{3q/2}}.
\]
Similarly, \( \mathbb{P}(\max_{1 \leq k \leq h} \left| \sum_{i=1}^{n} g_{ik} \right| > x) \leq \frac{h}{n^{3q/2}}. \) Therefore, we obtain that
\[
\left| \mathbb{P}(\max_{1 \leq k \leq h} \left| \sum_{i=1}^{n} z_{ik} \right| \leq x) - \mathbb{P}(\max_{1 \leq k \leq h} \left| \sum_{i=1}^{n} g_{ik} \right| \leq x) \right| \leq \frac{h}{n^{3q/2}}. \quad (18)
\]
if \( x \geq n^{3/2} \). Now if \( d_{n,h}^* < x < n^{3/2} \), we apply the regular polygon approximation technique. Define
\[
z_{ijl} := z_{ij,1} \cos(\pi l/n) + z_{ij,2} \sin(\pi l/n), \quad j = 1, 2, \cdots, h, l = 0, 1, \cdots, n - 1.
\]
Observe that \( \cap_{l=1}^{n-1}\{|z_{ijl}| \leq x \cos(\pi/(2n)) \} \) is a subset of \( \{|z_{ij}| \leq x \} \); Meanwhile, \( \{|z_{ij}| \leq x \} \) is a subset of \( \cap_{l=1}^{n-1}\{|z_{ijl}| \leq x \} \). Therefore,
\[
\mathbb{P}\left(\max_{1 \leq j \leq h, 0 \leq l \leq n-1} |z_{ijl}| \leq x \cos(\pi/(2n)) \right) \leq \mathbb{P}(\tilde{T}_z \leq x) \leq \mathbb{P}\left(\max_{1 \leq j \leq h, 0 \leq l \leq n-1} |z_{ijl}| \leq x \right). \quad (19)
\]
The same result holds with \( z \) in (19) replaced by \( g \), where \( g_{ijl} \) are defined analogously. By the inequality that \( 1/\cos(x) \leq 1 + x^2 \) for \( x \in [0, \pi/4] \), we have that \( x \cos(\pi/(2n)) \geq d_{n,2nh} \) for \( n \geq 2 \). Hence by Corollary B.2
\[
\sup_{x > d_{n,h}^*} \left| \mathbb{P}\left(\max_{1 \leq j \leq h, 0 \leq l \leq n-1} |z_{ijl}| \leq x \cos(\pi/(2n)) \right) - \mathbb{P}\left(\max_{1 \leq j \leq h, 0 \leq l \leq n-1} |g_{ijl}| \leq x \cos(\pi/(2n)) \right) \right| \lesssim G(n,2nh). \quad (20)
\]
Similarly,
\[
\sup_{x > d_{n,h}^*} \left| \mathbb{P}\left(\max_{1 \leq j \leq h, 0 \leq l \leq n-1} |z_{ijl}| \leq x \right) - \mathbb{P}\left(\max_{1 \leq j \leq h, 0 \leq l \leq n-1} |g_{ijl}| \leq x \right) \right| \lesssim G(n,2nh). \quad (21)
\]
Note that \( |x - x \cos(\pi/(2n))| \lesssim xn^{-2} \lesssim n^{-1/2} \) since \( x \in [d_{n,h}^*, n^{3/2}] \). Therefore by Corollary B.1 we have that

\[
\sup_{n^{3/2} > x > d_{n,h}^*} \left| \mathbb{P} \left( \max_{1 \leq j \leq h, 0 \leq l \leq n-1} |g_{ijl}| \leq x \right) - \mathbb{P} \left( \max_{1 \leq j \leq h, 0 \leq l \leq n-1} |g_{ijl}| \leq x \cos(\pi/2n) \right) \right| \lesssim n^{-1/2} \log^{1+\delta}(nh) + \frac{1}{nh \log^{1/2}(nh)}. \tag{22}
\]

The proposition follows by (18) to (22) as we notice that \( \frac{1}{nh \log^{1/2}(nh)} \) is dominated by \( G(n, 2nh) \).

\[\square\]

### C Gaussian comparison without variance lower bounds

We will be modifying the comparison of distribution theorem in Chernozhukov et al. (2015) by dropping the assumption of lower variance bound in the anti-concentration inequality. We will also extend such results to complex-valued high-dimensional Gaussian vectors. The results established in this section are crucial for the theoretical investigation of the multiplier bootstrap proposed in this paper.

**Theorem C.1.** *(Comparison of distribution without variance lower bound).* Let \( X = (X_1, \ldots, X_h)^\top \) and \( Y = (Y_1, \ldots, Y_h)^\top \) be centered Gaussian random vectors in \( \mathbb{R}^h \). Let \( \Sigma_X = (\sigma_{i,j}^X)_{i,j=1}^h \) with \( \sigma_{i,j}^X = \text{Cov}(X_i, X_j) \), and define \( \Sigma_Y \) and \( \sigma_{i,j}^Y \) similarly. Suppose that \( h \geq 2 \) and there exists a finite constant \( c \) such that \( c > \sigma_{i,j}^Y > 0 \) for all \( 1 \leq j \leq h \). Define

\[
\Delta = \max_{i,j \leq h} |\Sigma_X - \Sigma_Y|_{i,j}.
\]

Then,

\[
\sup_{|x| > d_h} \left| \mathbb{P} \left( \max_{1 \leq j \leq h} X_j \leq x \right) - \mathbb{P} \left( \max_{1 \leq j \leq h} Y_j \leq x \right) \right| = O \left( \Delta^{1/3} \log(h)^{1+4\delta/3} + \frac{1}{(\log h)^{1/2}h} \right),
\]

where \( d_h = 2\Delta^{1/3} \log(h)^{\delta/3} + 2\sqrt{2} \log(h)^{-\delta} \).

**Proof.** Let

\[
g_0(x) = \begin{cases} 
0, & x \geq 1, \\
30 \int_0^1 s^2(1-s)^2 ds, & 0 < x < 1, \\
1, & x \leq 0.
\end{cases}
\]

and pick \( g(s) = g_0(\psi(s - t - e_\beta)) \) with \( e_\beta = \beta^{-1} \log h \), where \( \psi, \beta > 0 \) and \( t \) will be picked later. Denote \( G_k := \sup_{x \in \mathbb{R}} \partial^k g(x)/\partial x^k \), \( k = 1, 2, \ldots \). We have \( G_0 \lesssim 1 \), \( G_1 \lesssim \psi \), \( G_2 \lesssim \psi^2 \) and \( G_3 \lesssim \psi^3 \). Moreover, the function also satisfies

\[\mathbb{I}(x \leq t + e_\beta) \leq g(x) \leq \mathbb{I}(x \leq t + e_\beta + \psi^{-1}), \forall x \in \mathbb{R}.\]

S.13
Now assume $|t| > e_\beta + \psi^{-1} + \log(h)^{-\delta}$,

$$P(\max_{1 \leq j \leq h} X_j \leq t) \leq P(F_\beta(X) \leq t + e_\beta)$$

$$\leq \mathbb{E}[g(F_\beta(X))]$$

$$\leq \mathbb{E}[g(F_\beta(Y))] + C_0(\psi^2 + \beta\psi)\Delta$$

$$\leq P(F_\beta(Y) \leq t + e_\beta + \psi^{-1}) + C_0(\psi^2 + \beta\psi)\Delta$$

$$\leq P\left(\max_{1 \leq j \leq h} Y_j \leq t + e_\beta + \psi^{-1}\right) + C_0(\psi^2 + \beta\psi)\Delta,$$

for some absolute constant $C_0 > 0$, where we utilized Theorem 1 of Chernozhukov et al. (2015) in the third inequality above. Then, by Lemma B.1 and assume $|x| > e_\beta + \psi^{-1} + 2\sqrt{2}(\log h)^{-\delta}$,

$$P\left(\max_{1 \leq j \leq h} Y_j \leq t + e_\beta + \psi^{-1}\right) - P\left(\max_{1 \leq j \leq h} Y_j \leq t\right) \leq (e_\beta + \psi^{-1})(\log h)^{1+\delta} + h^{-1}(\log h)^{-1/2}.$$

Thus, we conclude that when $|t| > e_\beta + \psi^{-1} + 2\sqrt{2}(\log h)^{-\delta}$,

$$P\left(\max_{1 \leq j \leq h} X_j \leq t\right) - P\left(\max_{1 \leq j \leq h} Y_j \leq t\right) \leq C_0(\psi^2 + \beta\psi)\Delta + (e_\beta + \psi^{-1})(\log h)^{1+\delta} + h^{-1}(\log h)^{-1/2}.$$

The opposite direction can be proven similarly by noting that

$$P\left(\max_{1 \leq j \leq h} X_j \leq t\right) \geq P\left(\max_{1 \leq j \leq h} Y_j \leq t - e_\beta - \psi^{-1}\right) - C(\psi^2 + \beta\psi)\Delta.$$

and

$$P\left(\max_{1 \leq j \leq h} Y_j \leq t - e_\beta - \psi^{-1}\right) - P\left(\max_{1 \leq j \leq h} Y_j \leq t\right) \geq -(e_\beta + \psi^{-1})(\log h)^{1+\delta} - h^{-1}(\log h)^{-1/2}.$$

Finally, pick $\beta = \psi \log(h)$ and $\psi^{-1} = \Delta^{1/3}\log(h)^{\delta/3}$ (note that $e_\beta = \beta^{-1}\log(h) = \psi^{-1}$). Then, for some constant $C > 0$, the following inequality holds

$$C_0(\psi^2 + \beta\psi)\Delta + (e_\beta + \psi^{-1})(\log h)^{1+\delta} = C_0\psi^2(1+\log(h))\Delta + 2\psi^{-1}\log(h)^{1+\delta} \leq C\Delta^{1/3}\log(h)^{1+4\delta/3}.$$

The lower bound can be derived similar to the previous steps. $\square$

The following proposition establishes a comparison result for complex-valued Gaussian random vectors.

**Proposition C.3.** Let $Z = (z_1, \cdots, z_h)^\top$ and $W = (w_1, \cdots, w_h)^\top$ be centered complex-valued Gaussian random vectors in $\mathbb{C}^h$. Write $z_i = z_{i1} + \sqrt{-1}z_{i2}$ and $w_i = w_{i1} + \sqrt{-1}w_{i2}$, $i = 1, 2, \cdots, h$. Let $\tilde{Z} = (z_{11}, z_{12}, \cdots, z_{h1}, z_{h2})^\top$, $\tilde{W} = (w_{11}, w_{12}, \cdots, w_{h1}, w_{h2})^\top$ and $\Delta =$
Hence the proposition follows by noting that $h_{\text{max}}$. Suppose that $c \geq \max(\text{Var}(z_{ik}), \text{Var}(w_{ik})) > 0$, $i = 1, 2, \ldots, h$, $k = 1, 2$, for some positive and finite constant $c$. Then we have

$$\sup_{x>d_h^*}[\mathbb{P}(\max_{1 \leq j \leq h} |z_j| \leq x) - \mathbb{P}(\max_{1 \leq j \leq h} |w_j| \leq x)] \leq \tilde{\Delta}^{1/3} \log^{1+2\delta}(h^2) + h^{-1} \log^{1+\delta}(h^2),$$

where $d_h^* = C(\tilde{\Delta}^{1/3} \log(h^2)^{5/3} + \log(h^2)^{-\delta})(1 + \pi^2/(4h^2))$ for some absolute constant $C$.

**Proof.** The proof of this Proposition is similar to that of Proposition B.2. We shall only outline the proofs here. First of all, if $x \geq h$, then we have $\|z_{i,k}\|_{\psi_2} \leq C$ and $\|w_{i,k}\|_{\psi_2} \leq C$ for some finite constant $C$, where $\| \cdot \|_{\psi_2} := \inf\{c > 0 : \mathbb{E}\psi_2(\cdot |/c) \leq 1\}$ with $\psi_2(x) = e^{x^2} - 1$ is the Orcliz norm. A simple maximum inequality of the Orcliz norm yields that

$$\mathbb{P}(\max_{1 \leq k \leq h} |z_k| > x) \lesssim h \exp(-h^2).$$

Similarly,

$$\mathbb{P}(\max_{1 \leq k \leq h} |\sum_{i=1}^n w_k| > x) \lesssim h \exp(-h^2).$$

If $d_h^* < x < h$, then by the same regular-polygon-approximation technique used in Proposition B.2, define $\bar{z}_{i,l} = z_{i1} \cos(\pi l/h) + z_{i2} \sin(\pi l/h)$, $i, l = 1, 2, \ldots, h$. Define $\bar{w}_{i,l}$ similarly. Then we have

$$\mathbb{P}(\max_{1 \leq i, l \leq h} |\bar{z}_{i,l}| \leq x \cos(\pi/(2h))) \leq \mathbb{P}(\max_{1 \leq i, l \leq h} |z_{i,l}| \leq x) \leq \mathbb{P}(\max_{1 \leq i, l \leq h} |\bar{z}_{i,l}| \leq x).$$

(24)

The same inequality holds with $z$ in (24) replaced by $w$. Now Theorem C.1 implies that

$$\sup_{x>d_h^*}[\mathbb{P}(\max_{1 \leq i, l \leq h} |\bar{z}_{i,l}| \leq x \cos(\pi/(2h))) - \mathbb{P}(\max_{1 \leq i, l \leq h} |\bar{w}_{i,l}| \leq x \cos(\pi/(2h)))] \lesssim A(\tilde{\Delta}, h),$$

(25)

where $A(\tilde{\Delta}, h) = \tilde{\Delta}^{1/3} \log^{1+2\delta}(2h^2) + (\log 2h^2)^{-1/2}(2h^2)^{-1}$. Similarly,

$$\sup_{x>d_h^*}[\mathbb{P}(\max_{1 \leq i, l \leq h} |\bar{z}_{i,l}| \leq x) - \mathbb{P}(\max_{1 \leq i, l \leq h} |\bar{w}_{i,l}| \leq x)] \lesssim A(\tilde{\Delta}, h).$$

(26)

Note that $|x - x \cos(\pi/(2h))| \lesssim |x|h^{-2} \lesssim h^{-1}$ if $|x| \leq h$. Therefore by Corollary B.1 we have that

$$\sup_{h>x>d_h^*}[\mathbb{P}(\max_{1 \leq i, l \leq h} |z_{i,l}| \leq x) - \mathbb{P}(\max_{1 \leq i, l \leq h} |z_{i,l}| \leq x \cos(\pi/2h))] \lesssim h^{-1} \log^{1+\delta}(h^2) + \frac{1}{h^2 \log^{1/2}(h^2)},$$

(27)

Hence the proposition follows by noting that $h^{-1} \log^{1+\delta}(h^2)$ dominates both $\frac{1}{h^2 \log^{1/2}(h^2)}$ and $h \exp(-h^2)$. □
D Proof of the theoretical results of the paper

In the sequel, we shall omit the subscript \( n \) in \( X_{i,n} \) and \( \epsilon_{i,n} \) in the time series model (2) to simply notation. Meanwhile, the symbol \( C \) denotes a generic positive and finite constant which may vary from place to place. Some assumptions are placed in order.

**Assumption 1.** Assume that \( \max_{1 \leq i \leq n} \mathbb{E}X_i^4 < c_1 \) for \( c_1 > 0 \) and there exists \( D_n > 0 \) such that one of the following two conditions holds:

\[
\max_{1 \leq j \leq n} \mathbb{E}\exp(|X_j|/D_n) \leq 1, \tag{28}
\]

or

\[
\max_{1 \leq j \leq n} \mathbb{E}g(|X_j|/D_n) \leq 1, \tag{29}
\]

for some strictly increasing convex function \( g \) defined on \([0, \infty)\) satisfying \( g(0) = 0 \).

**Remark 2.** Observe that under the assumption that \( \max_{1 \leq i \leq n} \mathbb{E}(g(|X_i|)) \leq c \) or \( \max_{1 \leq i \leq n} \mathbb{E}(\exp(|X_i|)) \leq c \) for some finite constant \( c \), \( D_n \) can be chosen as a finite constant that does not depend on \( n \).

**Assumption 2.** Assume there exist \( M = M(n) > 0 \) and \( \gamma = \gamma(n) \in (0, 1) \) such that

\[
n^{3/8}M^{-1/2}l_n^{-5/8} \geq C_1 \max\{D_nl_n^{1/2}\} \text{ under Condition (28)} \tag{30}
\]

\[
n^{3/8}M^{-1/2}l_n^{-5/8} \geq C_2 \max\{D_n^{-1}(n/\gamma), l_n^{1/2}\} \text{ under Condition (29)} \tag{31}
\]

for \( C_1, C_2 > 0 \), where \( D_n \) is given in Assumption 1, and \( l_n = \log(pn/\gamma_n) \vee 1 \) with \( p = |W| \).

In both cases, suppose \( n^{7/4}M^{-2}l_n^{-9/4} \geq C_3 > 0 \).

**Assumption 3.** Assume that

\[
\max_{1 \leq i \leq n} \|\epsilon_i\|_q \leq C_q \quad \text{and} \quad \delta_q(k) = O((k + 1)^{-d}) \tag{32}
\]

for some finite constants \( C_q, q \geq 4 \) and \( d \geq 5 \). Define \( \Theta_{k,q} = \sum_{l=k}^{+\infty} \delta_q(l) \). Further assume that, for some finite constant \( C \),

\[
\|G_j(t, \mathcal{F}_0) - G_j(s, \mathcal{F}_0)\|_4 \leq C|t - s|, \quad t, s \in [s_j, s_{j+1}], \quad j = 0, 1, \ldots, r. \tag{33}
\]

**Assumption 4.** \( \min_{\omega \in W} \omega = \delta_0 \) for some positive constant \( \delta_0 \).

**Assumption 5.** \( f(\cdot) \) is twice differentiable on \([0, 1]\) with Lipschitz continuous second derivatives.
Assumption 6. Let \( v(t, \omega) = \sum_{k=-\infty}^{\infty} \text{Cov}(G_j(t, F_0), G_j(t, F_k)) \exp(\sqrt{-1} k \omega) \), where \( t \in (s_j, s_{j+1}) \) for any \( j = 0, 1, \ldots, r \), be the spectral density function of \( \{\epsilon_i\} \) at time \( t \) and frequency \( \omega \). Assume that there is a positive constant \( \delta_1 \) such that \( v(t, \omega) \geq \delta_1 \) for all \( t \in [0, 1] \) and \( \omega \in (0, \pi) \).

Assumption 7. For \( k = 0, 1, \ldots, r \), and \( j = 0, 1, \ldots, r \), let \( \gamma_{j,k}(t) = \text{Cov}(G_j(t, F_0), G_j(t, F_k)) \). Assume that, for each \( j \) and \( k \), \( \gamma_{j,k}(t) \) is twice continuous differentiable on \( [s_j, s_{j+1}] \) with Lipschitz continuous second derivatives.

Assumption 8. If \( \Omega \) is not empty, then there exist a positive constant \( \delta_2 \) such that \( \delta_0 \leq \omega_i \leq \pi - \delta_2, \forall \omega_i \in \Omega \). And there exists a positive constant \( c \) such that \( |\omega_i - \omega_j| \geq c / \log n \) for any \( \omega_i \), \( \omega_j \in \Omega \) and \( \omega_i \neq \omega_j \).

Assumption 9. If \( \Omega \) is not empty, then \( \forall \omega_k \in \Omega \), assume that \( |A_{0,k}| + |B_{0,k}| \geq \delta_3 > 0 \) if there is no change point at frequency \( \omega_k \). If there are change points at frequency \( \omega_k \), then we assume that \( b_{k,r} = c_{k,r} n \), where \( 0 < c_{k,1} < \cdots < c_{k,M_k} < 1 \) are constants, and \( |A_{r,k} - A_{r-1,k}|^2 + |B_{r,k} - B_{r-1,k}^2| \geq \delta_4, r = 1, 2, \ldots, M_k + 1 \), where \( \delta_4 > 0 \) is a constant.

Some discussions of these assumptions are in order. Assumptions 1 and 2 are regularity conditions for the high dimensional Gaussian approximation which correspond to Assumptions (A1) and (A2) in Section B. Typical choice of \( g \) in (29) is the power function \( g = x^q \), \( q \geq 1 \). \( M \) in Assumption 2 is the dependence truncation constant used in the Gaussian approximation proof. Assumption 2 puts a relatively weak constraint on \( M(n), \gamma(n) \) and \( l_n \) relative to the data length \( n \). Assumption 3 puts constraints on the moments and dependence of the noise sequence \( \{\epsilon_i\} \). It requires that \( \epsilon_i \) has finite \( q \)-th moment for \( q \geq 4 \) and weak dependence which decays at a sufficiently fast algebraic rate. Equation (33) is a piece-wise stochastic Lipschitz continuity condition which guarantees that the data generating mechanism of the noise process is smooth between adjacent jump points. Assumption 4 requires that our candidate frequencies to be separated from 0 in order to avoid detecting variations caused by the smooth trend \( f(\cdot) \). Observe that \( v(t, \omega) \) in Assumption 6 is the instantaneous spectral density of the noise process \( \{\epsilon_i\} \) at time \( t \) and frequency \( \omega \). Hence Assumption 4 is a mild condition that requires that the instantaneous spectral density of the noise process \( \{\epsilon_i\} \) is uniformly positive. Assumption 7 requires that the auto-covariances of the noise process are piece-wise twice differentiable with respect to time with piece-wise Lipschitz continuous derivatives. Assumption 8 puts a positive lower bound on the oscillatory frequencies to distinguish the oscillation from the smooth trend \( f(\cdot) \). Meanwhile, a mild technical assumption is put in Assumption 8 to separate the oscillatory frequencies from \( \pi \). Assumption 8 also requires that the oscillatory frequencies to be well separated and are at least \( O(1 / \log n) \) away from each other. Assumption 9 requires that the amplitudes of the oscillation have a positive lower bound. Meanwhile, the oscillation change points, when they exist, are required to be well separated with jump sizes larger than a positive constant.

Now we would like to summarize how to implement the Gaussian approximation results in Section B to the DPPT. Let \( X_i \) be a centered univariate PLS(\( r \)) noise, \( i = 1, 2, \ldots, n \).
Define

\[ \tilde{x} := [\tilde{x}_1, \ldots, \tilde{x}_n] = \begin{pmatrix} X_1 & X_2 & \cdots & X_{n-1} & X_n \\ X_1 & X_2 & \cdots & X_{n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \]

Now, let \( w_i \in W \) and \( e_j = [\exp(\sqrt{-1} j w_1), \ldots, \exp(\sqrt{-1} j w_p)]^\top \in \mathbb{C}^p \). Then we define the vectorized data as

\[ x_i := \tilde{x}_i \otimes e_i \in \mathbb{C}^{np}. \quad (34) \]

Here, \( x_i \) are complex random vectors satisfying

\[ \max_{1 \leq j \leq pn} \left| \sum_{i=1}^{n} x_{ij} \right| / \sqrt{n} = F(W). \quad (35) \]

Thus, we can write \( F(W) \) in a form matching Proposition B.2. If \( X_i \) satisfies Assumptions 1-3, it implies that \( \{x_i\} \) satisfies conditions of Proposition B.2. By Proposition B.2, we get that, for \( np \)-dim centered complex Gaussian random vectors \( \{y_i\} \) having the same covariance and pseudo-covariance structures of \( \{x_i\} \),

\[ \sup_{|x| > d_{n, pn}^*} \left| \mathbb{P}(F(W) \leq x) - \mathbb{P}\left( \max_{1 \leq j \leq pn} \left| \sum_{i=1}^{n} y_{ij} / \sqrt{n} \right| \leq x \right) \right| \lesssim G^*(n, pn). \quad (36) \]

We now briefly discuss when the magnitudes of \( d_{n, pn}^* \) as well as the right hand side of (36) will converge to 0. Under Assumption 3, we have that \( \max_{1 \leq i \leq n} |X_i|^4 < \infty \). Hence we can choose \( q = 4, g(x) = x^4 \) and let \( D_n \) be a constant. Observe that Assumption 3 implies that

\[ \Theta_{M, j, q} \leq \sum_{k=M} C(k + 1)^{-d} \leq C \int_{M}^{\infty} y^{-d} dy \leq C M^{-d+1} \]

uniformly in \( j \). Choose \( M = n^{1/4} \log^{-6}(n) \) and \( \gamma = \log^{-1}(n) \). We have that Assumption 2 is satisfied for the above choices of \( M \) and \( \gamma \). Note that for our test \( p = O(n^{3/2} \log(n)) \). We obtain that the right hand side of (36) is of the order \( O(\log^{-1}(n)) \) which converges to 0. Meanwhile, simple calculations yield that \( d_{n, pn}^* = O(\log^{-10/7}(n) + \log^{-\delta}(n)) \), which goes to 0. As shown in Lemma D.5 below, the critical values of the DPPT under the null hypothesis of no oscillation is of the order \( O(\sqrt{\log n}) \) which is asymptotically larger than \( d_{n, pn}^* \). Hence, under Assumptions 1, 3, the Gaussian approximation result established in Proposition B.2 is sufficient for the DPPT.

Observe that faster convergence of the Gaussian approximation error can be obtained under stronger moment and dependence assumptions. For instance, in the best scenario where (28) holds and the dependence of \( \{X_i\} \) is of exponential decay, one can choose \( M = O(\log n) \) and \( \gamma = O(1/n) \). In this case the convergence rate of the Gaussian approximation is of the order \( O(n^{-1/8} \log^{29/14}(n)) \).
D.1 First Stage proof

D.1.1 Consistency

Denote by $W$ the set of candidate frequencies and denote $p = |W|$. Define a random vector $\Theta(W)$ by

$$\Theta(W) := [\Theta(w_1)^T, \ldots, \Theta(w_p)^T]^T / \sqrt{n} \in \mathbb{R}^{2np}$$

and the vector $\Theta(w) \in \mathbb{R}^{2n}$ is defined coordinate wise

$$\Theta_k(w) := \begin{cases} \sum_{j=1}^k \cos(w_j)X_j & \text{for } k \leq n \\ \sum_{j=1}^{k-n} \sin(w_j)X_j & \text{for } n < k \leq 2n. \end{cases}$$

Note we can write

$$F(W) = \max_{1 \leq i \leq n, 1 \leq j \leq p} |\Theta_i(\omega_j) + \sqrt{-1}\Theta_{i+n}(\omega_j)| / \sqrt{n}.$$ 

Moreover, if we put $\bar{\Theta}_k(W) := [\sum_{l=1}^k e^{\sqrt{-1}\omega_l}X_l, \ldots, \sum_{l=1}^k e^{\sqrt{-1}\omega_p}X_l]^T \in \mathbb{C}^p$, we have

$$F(W) = \max_{1 \leq k \leq n} \|\bar{\Theta}_k(W)\|_\infty / \sqrt{n}.$$ 

Recall that for the fixed integer bandwidth $m$, we defined

$$S_{j,m}(w) = \sum_{i=j}^{j+m-1} \sin(2\pi iw)X_i$$

and

$$C_{j,m}(w) = \sum_{i=j}^{j+m-1} \cos(2\pi iw)X_i.$$ 

For an i.i.d. standard Gaussian random variables $G_i$, define $S(w) \in \mathbb{R}^{2n}$ by

$$S_k(w) := \begin{cases} S_{1,m}(w)G_1 & \text{if } k \leq m \\ \sum_{j=1}^{k-m+1} S_{j,m}(w)G_j & \text{if } m < k \leq n \\ C_{1,m}(w)G_1 & \text{if } n < k \leq n + m \\ \sum_{j=1}^{k-n-m+1} C_{j,m}(w)G_j & \text{if } n + m < k \leq 2n, \end{cases}$$

where $S_k(w)$ denotes the $k$-th coordinate of $S(w)$. Define a $n$-dim complex vector $SZ(w)$ such that $SZ_k(w) = \sqrt{-1}S_k(w) + S_{k+n}(w)$, $k = 1, 2, \ldots, n$. Let

$$S(W) := [S^T(w_1), \ldots, S^T(w_p)]^T / \sqrt{m(n-m)} \in \mathbb{R}^{2np},$$

and

$$SZ(W) := [SZ^T(w_1), \ldots, SZ^T(w_p)]^T / \sqrt{m(n-m)} \in \mathbb{C}^{np}.$$ 

Then we have

$$\tilde{F}(W) = \max_{1 \leq j \leq p} \max_{1 \leq i \leq n} |\sqrt{-1}S_i(w_j) + S_{i+n}(w_j)| / \sqrt{m(n-m)} = \max_{1 \leq i \leq n} \|SZ_i(W)\|_\infty / \sqrt{m(n-m)}.$$
Lemma D.1. Assume the same conditions and notation as in Theorem [C.1] \( \max_{1 \leq j \leq h} X_j = \max_{1 \leq j \leq h} \epsilon_j + O(a_n) \) and \( \max_{1 \leq j \leq h} Y_j = \max_{1 \leq j \leq h} \xi_j + O(b_n) \). Then
\[
\sup_{|x| > h, \epsilon_j + O(a_n) + O(b_n)} \left| P\left( \max_{1 \leq j \leq h} X_j \leq x \right) - P\left( \max_{1 \leq j \leq h} Y_j \leq x \right) \right|
\]
\[
= \sup_{|x| > h, \epsilon_j + O(a_n) + O(b_n)} \left| P\left( \max_{1 \leq j \leq h} \epsilon_j \leq x \right) - P\left( \max_{1 \leq j \leq h} \xi_j \leq x \right) \right|
\]
\[
+ (O(a_n) + O(b_n)) \log(h)^{1+\delta} + \frac{1}{2\sqrt{\pi} (\log h)^{1/2} h}.
\]

Proof. The result follows directly from Corollary [B.1]. \( \square \)

Lemma D.2. Let \( \{x_i \in \mathbb{R}^h\} \) be a centered PLS time series. Suppose that
\[
\max_{1 \leq j \leq h, 1 \leq i \leq n} \|x_{i,j}\|_q < \infty
\]
for some \( q \geq 2 \), where \( x_{i,j} \) is the \( j \)-th component of \( x_i \). Further assume that \( \max_{1 \leq j \leq h} \delta_{j,q}(k) = O((k + 1)^{-d}) \) for some \( d > 1 \), where \( \delta_{j,q}(k) \) is the physical dependence measure of the \( j \)-th component process of \( \{x_i\} \). Define \( S_{k,m} = \sum_{i=k}^{k+m-1} x_i \) and
\[
(A_{r,s})_{k,l} = \frac{1}{m(n-m)} \sum_{j=r}^{s} (S_{j,m})_k (S_{j,m})_l,
\]
where \( 1 \leq r \leq s \leq n-m+1 \). Then for \( 1 \leq k, l \leq h \),
\[
\max_{1 \leq r \leq s \leq n-m+1} \|(A_{r,s})_{k,l} - E(A_{r,s})_{k,l}\|_{q'} = O(\sqrt{m/n}),
\]
where \( q' = q/2 \).

Proof. Observe that \( S_{j,m} \) can be written in a physical representation \( S_{j,m} = R_{j,n}(F_{j+m}) \), where \( R_{j,n} \) is some filter function. For any \( t \leq i + m \), let \( S_{j,m}^{(t)} = R_{j,n}(F_{j+m}^{(t)}) \), where \( F_{j+m}^{(t)} \) is a coupled version of \( F_{j+m} \) such that the innovation \( \epsilon_i \) is replaced by an i.i.d. copy. Let \( (S_{j,m}^{(t)})_l \) be the \( l \)-th coordinate of the vector \( S_{j,m}^{(t)} \). Note that
\[
\|(S_{i,m})_k - (S_{i,m}^{(t)})_k\|_{q'} = \|(S_{i,m}_t - (S_{i,m}^{(t)})_t)(S_{i,m})_k - (S_{i,m}^{(t)})_k\|
\]
\[
+ ((S_{i,m})_t + (S_{i,m}^{(t)})_t)(S_{i,m})_k - (S_{i,m}^{(t)})_k\|/2
\]
\[
\leq (\|(S_{i,m})_k\|_{q} + \|(S_{i,m}^{(t)})_k\|_{q})\|(S_{i,m})_t - (S_{i,m}^{(t)})_t\|_q/2
\]
\[
+ (\|(S_{i,m})_t\|_q + \|(S_{i,m}^{(t)})_t\|_q)\|(S_{i,m})_k - (S_{i,m}^{(t)})_k\|_q/2.
\]
Then, the result follows by the proof of [Zhou, 2013] Lemma 1. \( \square \)

Lemma D.3. Let \( \{\epsilon_i\} \) be a centered PLS(r) time series satisfying Assumptions [1]-[3]. Then for a bounded sequence \( |a_i| \leq C \) where \( C > 0 \) we have
\[
\max_{1 \leq k \leq n} \sum_{i=1}^{k} a_i \epsilon_i = O_p(\sqrt{n \log(n)}).
\]
Proof. By Gaussian approximation results in Theorem B.1 for $y_i$ normally distributed with same covariance structure as $\epsilon_i$, we have
\[
\sup_{t > d_n} \left| \mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_i \epsilon_i / \sqrt{n} \right| > t \right) - \mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_i y_i / \sqrt{n} \right| > t \right) \right| \to 0,
\]
where $d_n \to 0$.

Note that if $X_i \sim N(0, \sigma_i^2)$ and $\sigma_i \leq \sigma$, then $\mathbb{E} \left[ \max_{1 \leq i \leq n} |X_i| \right] \leq \sigma \sqrt{\log(2n)}$ by a simple Orliz norm maximum inequality. Hence
\[
\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_i y_i \right| = O_p \left( \sqrt{\log(2n)} \max_{1 \leq i \leq n} \sqrt{\Var \left( \sum_{j=1}^{i} a_j y_j \right)} \right).
\]

Note that we also have
\[
\Var \left( \sum_{i=1}^{n} a_i y_i \right) \leq 4 \sum_{i=1}^{n} \Var(y_i) + 4 \sum_{i \neq j} \Cov(y_i, y_j)
\]
\[
\leq 4n\sigma^2 + 4C \sum_{k=1}^{n-1} (n-k)k^{-d} = O(n),
\]
where we utilized Lemma 6 of Zhou (2014) which guarantees that $\Cov(y_i, y_j) \leq C|i-j|^{-d}$ under Assumption 3. It implies that
\[
\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_i y_i \right| = O_p(\sqrt{n \log(n)}).
\]
Combining with the Gaussian approximation results, we get
\[
\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_i \epsilon_i \right| = O_p(\sqrt{n \log(n)}).
\]

The next lemma controls the smooth trend of the first stage statistics under the null; that is, $\mu_i = f(i/n)$.

**Lemma D.4.** Suppose that function $f$ is twice differentiable on $[0,1]$ with Lipschitz continuous second derivatives. Then, for $w \in [cn^{-\theta}, \pi]$ for some constant $c > 0$ and $\theta \in [0,1/4)$, we have
\[
\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} f(i/n) \exp(wi\sqrt{-1}) \right| = O(n^{1/4+\theta}).
\]
Proof. We shall only prove the case where \( j = n \) as the other cases follow by the similar arguments. Choose an integer \( m = \frac{n^{3/4}}{} \). First, group \( \sum_{i=1}^{n} f(i/n) \exp(\sqrt{1} w) \) into

\[
\sum_{j=0}^{\lfloor n/m \rfloor - 1} \sum_{i=jm+1}^{(j+1)m} f(i/n) \exp(\sqrt{1} w) + \sum_{i=\lfloor \frac{n}{m} \rfloor m+1}^{n} f(i/n) \exp(\sqrt{1} w).
\]

Below we only show the control of \( \sum_{j=0}^{\lfloor n/m \rfloor - 1} \sum_{i=jm+1}^{(j+1)m} f(i/n) \exp(\sqrt{1} w) \), since the term \( \sum_{i=\lfloor \frac{n}{m} \rfloor m+1}^{n} f(i/n) \exp(\sqrt{1} w) \) can be handled in the same way. By Taylor expansion, we have

\[
\sum_{j=0}^{n/m-1} \sum_{i=jm+1}^{(j+1)m} f(i/n) \exp(\sqrt{1} w) = \sum_{j=0}^{n/m-1} \sum_{i=jm+1}^{(j+1)m} \frac{(f(k)(i/n - jm/n)^k)}{k!} + O((i/n - jm/n)^3) \exp(\sqrt{1} w).
\]

The first term in the expansion can be computed as

\[
\left| \sum_{i=\lfloor \frac{n}{m} \rfloor m+1}^{n} f(i/n) \exp(\sqrt{1} w) \right| = |f(jm)| \left| \frac{\exp(\sqrt{1} mw) - 1}{\exp(\sqrt{1} w) - 1} \right| \leq |\pi f(jm)|/|w|.
\]

The second order terms can be simplified as

\[
\left| \sum_{i=\lfloor \frac{n}{m} \rfloor m}^{(j+1)m-1} f'(jm)(i/n - jm/n) \exp(\sqrt{1} wi) \right| \leq C \left| \frac{m \exp(\sqrt{1}(m + 2)w) - (m + 1) \exp(\sqrt{1}(m + 1)w) + \exp(\sqrt{1} w)}{n(\exp(\sqrt{1} w) - 1)^2} \right| = O(1).
\]

The third term is tedious to compute but the simplified results show that

\[
\left| \sum_{i=\lfloor \frac{n}{m} \rfloor m}^{(j+1)m-1} f''(jm)(i/n - jm/n)^2 \exp(\sqrt{1} wi) \right| \leq C \left| \frac{m^2 e^{\sqrt{1}(m + 3)w} + (-2m^2 - 2m + 1) e^{\sqrt{1}(m + 1)w} + (m + 1)^2 e^{\sqrt{1}(m + 1)w} - e^{\sqrt{1} w} e^{\sqrt{1} w}}{n^2(\exp(\sqrt{1} w) - 1)^3} \right| = O(1).
\]

Using a less precise bound for the remaining term, we have

\[
\left| \sum_{i=\lfloor \frac{n}{m} \rfloor m}^{(j+1)m-1} (i/n - jm/n)^3 \exp(\sqrt{1} wi) \right| \leq \sum_{i=\lfloor \frac{n}{m} \rfloor m}^{(j+1)m-1} |(i/n - jm/n)^3| = O \left( \frac{m^4}{n^3} \right)
\]
By combining the previous results, we get
\[ \sum_{i=1}^{n} f(i/n) \exp(wi\sqrt{-1}) = \sum_{j=0}^{n/m} O(1/|w| + m^4/n^3) = O(n/(m|w|) + m^3/n^2) = O(n^{1/4+\theta}). \]

Next we shall prove the first main result of this paper. Define
\[ \Theta_k(\omega) = \sum_{i=1}^{k} X_i \exp(\omega i\sqrt{-1}), \quad \Theta_k^{(\mu)}(\omega) = \sum_{i=1}^{k} \mu_i \exp(\omega i\sqrt{-1}), \quad \Theta_k^{(\epsilon)}(\omega) = \sum_{i=1}^{k} \epsilon_i \exp(\omega i\sqrt{-1}). \]

Clearly, we have
\[ \Theta_k(\omega) = \Theta_k^{(\mu)}(\omega) + \Theta_k^{(\epsilon)}(\omega). \]

**Proof of Theorem 4.1.** First note that
\[ \sup_{x} \left| P(F(W) \leq x) - P(\tilde{F}(W) \leq x|X) \right| = \sup_{x} \left| P(\max_k \|n^{-1/2}\Theta_k(W)\|_{\infty} \leq x) - P(\|SZ(W)\|_{\infty} \leq x|X) \right|. \]

By Assumption 4 and Lemma D.4, we have, under the null hypothesis of no oscillation,
\[ \max_{1 \leq k \leq n} |\Theta_k^{(\mu)}(W)| \leq \max_{1 \leq k \leq n} \max_{w \in W} \left\{ \left| \sum_{i=1}^{k} f(i/n) \cos(\omega i) \right|, \left| \sum_{i=1}^{k} f(i/n) \sin(\omega i) \right| \right\} \]
\[ = O(n^{1/4}), \]
which implies that
\[ \max_k |n^{-1/2}\Theta_k(W)| - \max_k |n^{-1/2}\Theta_k^{(\epsilon)}(W)| \leq \max_k |n^{-1/2}\Theta_k^{(\mu)}(W)| = O(n^{-1/4}). \]

Write it as
\[ \max_k |n^{-1/2}\Theta_k(W)| = \max_k |n^{-1/2}\Theta_k^{(\epsilon)}(W)| + O(n^{-1/4}). \]

Now, let \( \{y_i\} \) be centred Gaussian random variables with the same covariance structure as \( \{\epsilon_i\} \). Let \( \Theta_k^{(Y)}(W) \) be defined the same way as \( \Theta_k^{(\epsilon)}(W) \) by replacing \( \epsilon_i \) with \( Y_i \). Then by Proposition B.2, we get
\[ \sup_{|x| > d_{n, np}} \left| P(\max_k \|n^{-1/2}\Theta_k^{(\epsilon)}(W)\|_{\infty} \leq x) - P(\max_k \|n^{-1/2}\Theta_k^{(Y)}(W)\|_{\infty} \leq x) \right| \lesssim G^*(n, np). \]
Note that $G^*(n, np)$ is the Gaussian approximation bound in Proposition B.2 By (39), (40)
and similar proofs as those of Lemma D.1 and Proposition C.3, we have
\[
\sup_{|x| > d_{n, np}^* + O(n^{-1/4})} \left| \mathbb{P}(\max_k \|n^{-1/2} \Theta_k(Y)(W)\|_\infty \leq x) - \mathbb{P}(\max_k \|n^{-1/2} \Theta_k(W)\|_\infty \leq x) \right| \\
\lesssim G^*(n, np) + \log(np)^{1+\delta} n^{-1/4} + 1/(np \log^{1/2}(np)) \lesssim G^*(n, np).
\] (41)

Note that $d_{n, np}^* + O(n^{-1/4}) \leq 2d_{n, np}$ for sufficiently large $n$.

By (41), the rest is to control the distance between $\mathbb{P}(\max_k \|n^{-1/2} \Theta_k(Y)(W)\|_\infty \leq x)$
and $\mathbb{P}(\|SZ(W)\|_\infty \leq x|X)$, which boils down to controlling $\Delta$ by Proposition C.3
Since $\text{Cov}(\Theta(Y)(W)) = \text{Cov}(\Theta(\epsilon)(W))$, we consider
\[
\Delta := \sup_{i,j} |\text{Cov}(\Theta(\epsilon)(W)) - \text{Cov}(S(W))|X|_{ij}
\]
in Proposition C.3. The rest of the proof is to bound $\Delta$. Observe that the $(s, t)$-th entry of
$\text{Cov}(S(W)|X)$, $\text{Cov}(S(W)|X)_{st}$, where $1 \leq s, t \leq 2np$, can be represented as
\[
\text{Cov}(S(W)|X)_{st} = \frac{1}{m(n-m)} \sum_{i=1}^{k-1} S_{i,m}(w)S_{i,m}(w'),
\] (42)
or $\frac{1}{m(n-m)} \sum_{i=1}^{k-1} S_{i,m}(w)C_{i,m}(w')$, or $\frac{1}{m(n-m)} \sum_{i=1}^{k-1} C_{i,m}(w)C_{i,m}(w')$
for some $k, \omega$ and $\omega'$ depending on $s, t$ when $m < k \leq n$, or other format when $1 \leq k \leq m$.
Without loss of generality, we will focus on the case (42) with $m < k \leq n$ instead of other combinations
of $S_{i,m}$ and $C_{i,m}$ since it would not affect the rest of proof beside cosmetic reasons.

We further break down $S_{i,m}(w)$ into
\[
S_{i,m}(w) = S_{i,m}(\epsilon) + S_{i,m}(\mu),
\] (43)
where $S_{i,m}(\epsilon)(w)$ is the stochastic part and $S_{i,m}(\mu)(w)$ is the deterministic part. Observe that,
under the null hypothesis of no oscillation, $S_{i,m}(\omega) = O(1)$ uniformly over $i$ and $\omega \in W$.
Therefore
\[
S_{i,m}(w)S_{i,m}(w') = (S_{i,m}(\epsilon)(w) + S_{i,m}(\mu)(w))(S_{i,m}(\epsilon)(w') + S_{i,m}(\mu)(w')) \\
= (S_{i,m}(\epsilon)(w) + O(1))(S_{i,m}(\epsilon)(w') + O(1)) \\
= S_{i,m}(\epsilon)S_{i,m}(\epsilon) + S_{i,m}(\epsilon)S_{i,m}(\mu) + S_{i,m}(\mu)S_{i,m}(\epsilon) + S_{i,m}(\mu)S_{i,m}(\mu) + O(1).
\]

Note that $\sum_{i=1}^{k-1} S_{i,m}(\epsilon)'S_{i,m}(\epsilon) + S_{i,m}(\mu)S_{i,m}(\mu)'$ is a linear combination of $\{\epsilon_{i,j}\}_{i=1}^n$. Using
the result that $S_{i,m}(\omega) = O(1)$ and Lemma 6 of Zhou [2013], we have
\[
\left\| \max_k \left| \sum_{i=1}^{k-1} S_{i,m}(\epsilon)'S_{i,m}(\epsilon) + S_{i,m}(\mu)S_{i,m}(\mu) \right| \right\|_q = O(m\sqrt{n}).
\]
Hence
\[
\left\| \max_{k} \max_{w,w'} \sum_{i=1}^{k-m+1} S^{(w')}_{i,m}(w)S^{(e)}_{i,m}(w) + S^{(w)}_{i,m}(w)S^{(e)}_{i,m}(w') \right\|_q = O(m\sqrt{n}p^{2/q}). \tag{44}
\]

By Lemma D.2 for fixed \(w, w' \in W\)
\[
\frac{1}{m(n-m)} \left\| \max_{1 \leq k \leq n-m} \sum_{i=1}^{k-m+1} S^{(e)}_{i,m}(w)S^{(e)}_{i,m}(w') - \sum_{i=1}^{k-m+1} \mathbb{E}S^{(e)}_{i,m}(w)S^{(e)}_{i,m}(w') \right\|_{q'} = O(\sqrt{m/n}).
\]
Recall that \(q' = q/2\). Therefore,
\[
\frac{1}{m(n-m)} \left\| \max_{w,w' \in W} \max_{1 \leq k \leq n-m} \sum_{i=1}^{k-m+1} S^{(e)}_{i,m}(w)S^{(e)}_{i,m}(w') - \sum_{i=1}^{k-m+1} \mathbb{E}S^{(e)}_{i,m}(w)S^{(e)}_{i,m}(w') \right\|_{q'} = O(p^{1/q} \sqrt{m/n}). \tag{45}
\]

Recall the definition of \(S^{(e)}_{k}(w)\) and \(C^{(e)}_{k}(w)\) in Section 4.1. Observe that, if \(\text{Cov}(S(W)|X)_st\) can be written in the form of (42), then the corresponding \((s, t)\) entry of \(\text{Cov}(\Theta^{(e)}(W))\) is of the form \(n^{-1}\mathbb{E}[S^{(e)}_{k}(w)S^{(e)}_{l}(w')]\) or \(n^{-1}\mathbb{E}[S^{(e)}_{k}(w')S^{(e)}_{l}(w)]\) for some \(l \geq k\). We shall only focus on the first case due to symmetry.

Observe that, for all possible \(k, l\) so that \(k \leq l\) and \(k \geq m\), and \(w, w'\), we have
\[
\left| \frac{1}{m(n-m)} \sum_{i=1}^{k-m+1} \mathbb{E}S^{(e)}_{i,m}(w)S^{(e)}_{i,m}(w') - \frac{1}{n} \mathbb{E}S^{(e)}_{k}(w)S^{(e)}_{l}(w') \right| \leq \left| \frac{1}{m(n-m)} \sum_{i=1}^{k-m+1} \mathbb{E}S^{(e)}_{i,m}(w)S^{(e)}_{i,m}(w') - \frac{1}{n} \mathbb{E}S^{(e)}_{k}(w)S^{(e)}_{l}(w') \right|
\]
\[+ \left| \frac{1}{n} \mathbb{E} \left( S^{(e)}_{k}(w) \sum_{i=k+1}^{l} \sin(2\pi w' i) \epsilon_i \right) \right|.
\]

Note that
\[
\left| \frac{1}{n} \mathbb{E} \left( S^{(e)}_{k}(w) \sum_{i=k+1}^{l} \sin(2\pi w' i) \epsilon_i \right) \right| \leq \frac{1}{n} \sum_{i=k+1}^{l} \sum_{j=1}^{k} |\text{Cov}(\epsilon_i, \epsilon_j)| \leq C \frac{1}{n} \sum_{i=k+1}^{l} \sum_{j=1}^{k} |i-j|^{-d} = O(1/n),
\]
where we utilized \(|\text{Cov}(\epsilon_i, \epsilon_j)| \leq C|i - j|^{-d}\) by Lemma 6 of Zhou (2014). Recall \(m = n^q\) and
\( \theta < 1 \). Hence,

\[
\frac{1}{m(n-m)} \sum_{i=1}^{k-m+1} \left| \mathbb{E} S_{i,m}^{(e)}(w) S_{i,m}^{(e)}(w') - \frac{1}{n} \mathbb{E} S_{k}^{(e)}(w) S_{k}^{(e)}(w') \right|
\]

\[
\leq \sum_{|i-j| \leq m} \frac{|m - |i-j||}{m(n-m)} - \frac{1}{n} \left| \mathbb{E} \epsilon_i \epsilon_j \right| + \frac{1}{n} \sum_{|i-j| > m} \left| \mathbb{E} \epsilon_i \epsilon_j \right| \quad \text{(the indices } i, j \text{ satisfy } 1 \leq i, j \leq k)
\]

\[
\leq \sum_{|i-j| \leq m} \left( \frac{m}{n^2} + \frac{|i-j|}{mn} \right) |i-j|^{-d} + \frac{1}{n} \sum_{|i-j| > m} |i-j|^{-d}
\]

\[
\leq \sum_{s=0}^{m-1} \left( \frac{m}{n^2} + \frac{s}{mn} \right) (k-s)s^{-d} + \frac{1}{n} \sum_{t=m}^{k-1} (k-t)t^{-d}
\]

\[
= O \left( \frac{m}{n} \right) + O \left( \frac{1}{m} \right) + O(m^{-d-1}) = O(m/n) + O(1/m).
\]

Therefore,

\[
\frac{1}{m(n-m+1)} \sum_{i=1}^{k-m+1} \left| \mathbb{E} S_{i,m}^{(e)}(w) S_{i,m}^{(e)}(w') - \frac{1}{n} \mathbb{E} S_{k}^{(e)}(w) S_{k}^{(e)}(w') \right| = O(m/n) + O(m^{-1}).
\]

uniformly for all \( 1 \leq k \leq l \leq n - m \) and \( w, w' \in W \). Together with (44) and (45), we have that

\[
\Delta = O(1/m) + O(p^{2/q}/\sqrt{n}) + O(p^{4/q}/\sqrt{m/n}) + O(m/n) + O(1/m) = O(p^{4/q}/\sqrt{m/n} + 1/m).
\]

Finally, by Markov’s inequality, (44) and (45), we have

\[
\mathbb{P}(A_n) \geq 1 - ||\Delta||_{q'}^p/[h_0^{q'}(p^{4/q}/\sqrt{m/n} + 1/m)^q] \geq 1 - C/h_0^{q'}
\]

with right hand side converging to 1. Combing with Proposition C.3 with \( \delta = 0.5 \), we have the theorem. \( \square \)

### D.1.2 Estimation accuracy

The following lemma establishes that, for any given constant \( c \in (0, 1) \), the \((1-c)\) quantile of \( F(W) \geq \sqrt{\log(Cn^{1/2+\theta})} \) for any \( \theta \in (0, 1/7) \). Observe that the threshold \( d_{n,np}^* \) goes to 0 in Theorem 4.1. Hence, Theorem 4.1 can be applied to the DPPT.

**Lemma D.5.** Under Assumptions 1 to 7 and the null hypothesis that \( \mu_{i,n} = f(i/n) \), we have that for any \( \theta \in (0, 1/7) \) and \( c \in (0, 1) \), there exists a positive constant \( C < \infty \) which does not depend on \( n \) such that

\[
\mathbb{P}(F(W) \geq \sqrt{\log(Cn^{1/2+\theta})}) \geq c
\]

for sufficiently large \( n \).
Proof. Denote

\[ F^*(W) = \max_{\omega \in W} |L_n(n, \omega)| / \sqrt{n}. \]

Obviously \( F(W) \geq F^*(W) \). Let \( \{y^*_i\} \) be a centered Gaussian process that preserves the covariance structure of \( \{x_i\} \). Define

\[ T_y^* = \max_{\omega \in W} |L_{n,y}(n, \omega)| / \sqrt{n}, \]

where

\[ L_{n,y}(n, \omega) = \sum_{k=1}^{n} y_k^* e^{\sqrt{-1} \omega_k}. \]

By Proposition B.2 and Lemma D.4, we have that

\[ \sup_{|x| > d^*_{n,p}} |\mathbb{P}(F^*(W) \leq x) - \mathbb{P}(T_y^* \leq x)| \lesssim G^*(n, p) \to 0. \]  

Observe that \( d^*_{n,p} \to 0 \). Consider an equally-spaced subset \( W^* \) of \( W \) such that the mesh size of \( W^* \) is proportional to \( n^{-1/2 - \theta} \) for some \( \theta \in (0, 1/7) \). Let \( CL_y(\omega) = \sum_{k=1}^{n} y_k^* \cos(\omega k) / \sqrt{n} \) and \( T_{y}^{**} = \max_{\omega \in W^*} |CL_y(\omega)| \). Then clearly \( T_y^* \geq T_{y}^{**} \). By (48) and the fact that \( \sqrt{\log(C n^{1/2 + \theta})} \gg 1 \) when \( n \) is sufficiently large, it suffices to show that

\[ \mathbb{P}(T_{y}^{**} \geq \sqrt{\log(C n^{1/2 + \theta})}) \geq c \]  

for sufficiently large \( n \). Define \( \Gamma(i, j) = \text{Cov}(y_i^*, y_j^*) \). Then, by Lemma 6 of Zhou (2014) and Assumption 3, we have \( |\Gamma(i, j)| \leq C(|i - j| + 1)^{-d} \). Hence, for any \( \omega, \omega' \in W^* \), we have

\[ \text{Cov}(CL_y(\omega), CL_y(\omega')) = \frac{1}{n} \sum_{|k| < n} \sum_{1 \leq i, i+k \leq n} \Gamma(i, i+k) \cos(\omega_i) \cos(\omega'(i+k)). \]

By Assumption 7 and the proof of Lemma D.4, we have that, for each \(|k| \leq n^{1/4} \),

\[ \left| \sum_{1 \leq i, i+k \leq n} \Gamma(i, i+k) \cos(\omega_i) \cos(\omega'(i+k)) \right| = O(n^{5/8 + 3\theta/4}). \]

For \(|k| > n^{1/4} \), we have

\[ \left| \sum_{1 \leq i, i+k \leq n} \Gamma(i, i+k) \cos(\omega_i) \cos(\omega'(i+k)) \right| \leq n \max_{i} |\Gamma(i, i+k)| \leq nC|k|^{-d}. \]

Hence

\[ \text{Cov}(CL_y(\omega), CL_y(\omega')) \leq Cn^{-\alpha} \]  

for some \( \alpha > 0 \). For any \( \omega \in W^* \), let \( CL'_y(\omega) \) be a centered Gaussian random variable with the same variance as \( CL_y(\omega) \) and \( CL'_y(\omega) \) and \( CL'_y(\omega') \) are independent for \( \omega \neq \omega' \).
Let \( T^{***} = \max_{\omega \in W} |CL_y'(\omega)| \). By Assumption 6, we have that \( \text{Var}(CL_y'(\omega)) \geq \delta_1/2 \) for sufficiently large \( n \). Hence by Theorem 2 of Chernozhukov et al. (2015), we have

\[
\sup_x \left| \mathbb{P}(T^{**} \leq x) - \mathbb{P}(T^{***} \leq x) \right| \leq C n^{-\alpha/3} \log^{2/3}(n). 
\]

Since \( T^{***} \) is the maximum of an independent Gaussian vector of length \( O(n^{1/2+\theta}) \) and each component's variance is bounded from below. It is straightforward to derive that

\[
\mathbb{P}(T^{***} \geq \sqrt{\log(C n^{1/2+\theta})}) \geq c
\]

for some finite constant \( C \) that does not depend on \( n \). Therefore (49) holds by (51). The lemma follows.

**Lemma D.6.** Suppose that the assumptions of Theorem 4.3 hold true. Then on a sequence of events \( D_n \) with \( P(D_n) \geq 1 - 2h_n^{-q'} \), where \( h_n \) is a positive sequence of real numbers which diverges to infinity at an arbitrarily slow rate, we have that, conditional on \( X = \{X_i\}_{i=1}^n \),

(a) if \( \Omega = \emptyset \), then we have \( \mathbb{P}(\tilde{F}_{m,l}(W) \geq \log(pn)) \lesssim n^{-c} \); (b) if \( \Omega \neq \emptyset \), then \( \mathbb{P}(\tilde{F}_{m,l}(W) \geq \sqrt{m} \log(pn)) \lesssim n^{-c} \), where \( c \) is any positive and finite constant.

Recall that \( \tilde{F}_{m,l}(W) \) is the multiplier bootstrap statistic and \( q' = q/2 \). Hence Lemma D.6 implies that the critical values of the bootstrap are bounded by \( \log n \) with high probability if \( \Omega = \emptyset \). If \( \Omega \) is not empty, then the critical values are bounded by \( \sqrt{m} \log n \) with high probability.

**Proof.** Recall the definition of \( S_{j,n}(\omega), S_{j,n}^{(\mu)}(\omega) \) and \( S_{j,n}^{(\epsilon)}(\omega) \) in (43). Decompose \( C_{j,n}(\omega) \) into \( C_{j,n}^{(\mu)}(\omega) \) and \( C_{j,n}^{(\epsilon)}(\omega) \) in a similar way. Since \( G_i \) are i.i.d standard Gaussian, we observe that, conditional on \( X \),

\[
\frac{1}{m(n - m)} \left[ \text{Var} \left( \sum_{j=1}^{n-m} \sum_{l=1}^{j+m} \cos(wl)X_l G_j \right) + \text{Var} \left( \sum_{j=1}^{n-m} \sum_{l=1}^{j+m} \sin(wl)X_l G_j \right) \right] = \frac{1}{m(n - m)} \left[ \sum_{j=1}^{n-m} \left( \sum_{l=1}^{j+m} \cos(wl)X_l \right)^2 + \sum_{j=1}^{n-m} \left( \sum_{l=1}^{j+m} \sin(wl)X_l \right)^2 \right] = \frac{1}{m(n - m)} \left[ \sum_{j=1}^{n-m} (C_{j,n}(\omega))^2 + \sum_{j=1}^{n-m} (S_{j,n}(\omega))^2 \right].
\]

For the stochastic part, based on similar arguments as those for the proof of (45), we have that

\[
\left\| \frac{1}{m(n - m)} \max_{1 \leq k \leq n - m} \max_{\omega \in W} \sum_{j=1}^{k} \left[ (S_{j,m}^{(\epsilon)}(w))^2 - \mathbb{E}(S_{j,m}^{(\epsilon)}(w))^2 \right] \right\|_{q'} \lesssim p^{2/q} \sqrt{\frac{m}{n}},
\]

(52)
and
\[
\frac{1}{m(n-m)} \max_{k} \max_{w} \left| \sum_{i=1}^{k} S_{i,m}^{(\mu)}(w)S_{i,m}^{(c)}(w) \right| \lesssim p^{1/q}/\sqrt{n}. \tag{53}
\]

Note that \( p^{1/q}/\sqrt{n} \ll p^{2/q}\sqrt{\frac{m}{n}} \) and \( p^{2/q}\sqrt{\frac{m}{n}} \) converges to zero polynomially fast. Therefore, if we define a sequence of events \( D_n := D_{n,1} \cap D_{n,2} \), where
\[
D_{n,1} := \left\{ \frac{1}{m(n-m)} \max_{1 \leq k \leq n-m} \max_{\omega \in W} \sum_{j=1}^{n-m} \left[ (S_{j,m}^{(c)}(w))^2 - \mathbb{E}(S_{j,m}^{(c)}(w))^2 \right] \leq p^{2/q}/\sqrt{m/nh_n} \right\},
\]
\[
D_{n,2} := \left\{ \frac{1}{m(n-m)} \max_{1 \leq k \leq n-m} \max_{\omega \in W} \sum_{j=1}^{n-m} S_{i,m}^{(\mu)}(w)S_{i,m}^{(c)}(w) \leq p^{1/q}/\sqrt{nh_n} \right\},
\]
then \( P(D_n) \geq 1 - 2/h_n^{q} \) by Markov’s inequality. Meanwhile, it is easy to show that
\[
\frac{1}{m(n-m)} \max_{1 \leq k \leq n-m} \max_{\omega \in W} \sum_{j=1}^{n-m} \mathbb{E}(S_{j,m}^{(c)}(w))^2 \lesssim 1. \tag{54}
\]

For the deterministic part, due to Assumption \( \mathcal{A} \), it is easy to see that
\[
\max_{w \in W} \max_{1 \leq j \leq n-m} \left| \sum_{l=j}^{j+m} \sin(wl)\mu_{l} \right| = O(m) \tag{55}
\]
if \( \Omega \neq \emptyset \). Otherwise, by Assumption \( \mathcal{A} \), the bound in \( \text{(55)} \) becomes \( O(1) \).

Similar result holds for the cosine terms. Hence, by a simple Orlicz norm maximum inequality, we have that, conditional on \( X \) and on the event \( D_n \),
\[
P\left( \max_{w \in W} \max_{1 \leq k \leq n-m} \left| \sum_{j=1}^{k} \sum_{l=j}^{j+m} e^{\sqrt{-\mathbb{E}w}} X_{t,G_{j}}/\sqrt{m(n-m)} \right| \geq \sqrt{m \log(pn)} \right) \lesssim n^{-c} \tag{56}
\]
for any positive and finite constant \( c \) if \( \Omega \neq \emptyset \). If \( \Omega = \emptyset \), then by Lemma \( \text{D.3} \) we have that, conditional on \( X \) and on the event \( D_n \),
\[
P\left( \max_{w \in W} \max_{1 \leq k \leq n-m} \left| \sum_{j=1}^{k} \sum_{l=j}^{j+m} e^{\sqrt{-\mathbb{E}w}} X_{t,G_{j}}/\sqrt{m(n-m)} \right| \geq \log(pn) \right) \lesssim n^{-c} \tag{57}
\]
for any positive and finite \( c \). The lemma follows.

\( \Box \)

**Lemma D.7.** Assume \( \Omega \neq \emptyset \). Let \( \Omega = \{\omega_1, \ldots, \omega_K\} \). Then under the assumptions of Proposition \( \text{[2]} \) and \( \text{[8]} \) held true, we have \( \max_{1 \leq k \leq K} |\hat{\omega}_k - \omega_k| = o_p(n^{-3/2} \log n) \) and
\[
P( \max_{1 \leq k \leq K} |\hat{\omega}_k - \omega_k| \geq n^{-3/2} \log n) \lesssim n^{(\theta - 1)/8} + G^*(n, pn). \]
Proof. For the simplicity of presentation, we shall only prove the case where \( \Omega = \{ \omega_0 \} \); that is, there is only one oscillatory frequency. The general case follows by similar arguments since the number of oscillatory frequencies is assumed to be bounded and the frequencies are well separated by Assumption 8.

Under the above assumption, the mean function is

\[
\mu_j = \sum_{0 \leq r \leq M_0} (A_r \cos(\omega_0 j) + B_r \sin(\omega_0 j))\mathbb{I}(b_r \leq j \leq b_{r+1}) + f(j/n).
\]

By Lemma D.4 and Assumption 4, the contribution of \( f(\cdot) \) is negligible. Hence, without loss of generality, we set \( f(\cdot) = 0 \) in the sequel. Recall the definitions of \( \bar{\Theta}_k^{(\mu)}(\omega) \), \( \bar{\Theta}_k^{(\epsilon)}(\omega) \), and \( \bar{\Theta}_k(\omega) \) in (37). Define

\[
\bar{\Theta}^{(\mu)}(\omega) := \max_{1 \leq k \leq n} |\bar{\Theta}_k^{(\mu)}(\omega)| \quad \text{and} \quad \bar{\Theta}(\omega) := \max_{1 \leq k \leq n} |\bar{\Theta}_k(\omega)|.
\]

Note that \( F(W) = \max_{\omega \in \mathbb{W}} \bar{\Theta}(\omega) / \sqrt{n} \). It is easy to see that \( \bar{\Theta}^{(\mu)}(\omega) = \max_{1 \leq i \leq M_0 + 1} |\bar{\Theta}_i^{(\mu)}(\omega)| \). Elementary but tedious calculations using sums of trigonometric series yield that there exist finite and positive constants \( c \) and \( C \) such that

\[
cn \leq \bar{\Theta}^{(\mu)}(\omega_0) \leq Cn
\]

Note that by (52) to (55) in Lemma D.6 and the Markov’s inequality, the critical values of the DPPT is no larger than \( n/\log n \) with probability at least \( 1 - C \log^q n \exp(\log p - q(1 - \theta) \log n/4) \geq 1 - C n^{(\theta - 1)q/8} \). Note that, we used assumption (8) in the above inequality. Therefore the critical values will be surpassed with probability at least \( 1 - C n^{(\theta - 1)q/8} \). If \( |\omega - \omega_0| \geq 1/n \), then

\[
\bar{\Theta}^{(\mu)}(\omega)/\bar{\Theta}^{(\mu)}(\omega_0) \leq 1 - c_0
\]

for some \( c_0 > 0 \). If \( |\omega - \omega_0| < 1/n \), then using the formula for sums of trigonometric series, the assumption that there is no phase jump, as well as Taylor expansion, we have

\[
\bar{\Theta}^{(\mu)}(\omega_0) - \bar{\Theta}^{(\mu)}(\omega) \geq C_0 n^2 |\omega - \omega_0|^2
\]

for some finite and positive constant \( C_0 \).

Recall that \( \bar{\Theta}_k^{(\gamma)}(\omega) = \sum_{i=1}^k y_i \exp(\omega_i \sqrt{-1}) \) by replacing \( \epsilon_i \) in \( \bar{\Theta}_k^{(\epsilon)}(\omega) \) by \( y_i \), where \( \{y_i\} \) is a centered Gaussian sequence having the same auto-covariance structure as that of \( \{\epsilon_i\} \). By the proof of Lemma D.3 we have that

\[
P(\max_{1 \leq k \leq n} \max_{\omega \in \Omega} |\bar{\Theta}^{(\gamma)}_k(\omega)| \geq \sqrt{n} \log n) = P(\max_{1 \leq k \leq n} \max_{\omega \in \Omega} |\bar{\Theta}^{(\gamma)}_k(\omega)| \geq \sqrt{n} \log n) + G^*(n, pn),
\]

Using an Orcliz norm maximum inequality (note that \( p \geq n^{3/2} \log n \)), we have

\[
P(\max_{1 \leq k \leq n} |\bar{\Theta}^{(\gamma)}_k(\omega)| \geq \sqrt{n} \log n) \leq n^{-c}
\]
for any positive and finite constant $c$ if $n$ is sufficiently large. Define the sequence of events

$$A_n = \{ \max_{1 \leq k \leq n} \max_{\omega \in \Omega} |\tilde{\Theta}_k^{(\epsilon)}(\omega)| \geq \sqrt{n \log n} \}. $$

Since $G^*(n, pn)$ dominates $n^{-c}$, we have $\mathbb{P}(A_n) \lesssim G^*(n, pn)$ which converges to 0. On the event $A_n^c$, we have by (59) and (60) that, if $|\omega - \omega_0| \geq n^{-1}/\log n$, then

$$\tilde{\Theta}(\omega) - \tilde{\Theta}(\omega_0) \leq -C \min(n, n^3|\omega - \omega_0|^2) + 2 \max_{1 \leq k \leq n} \max_{\omega \in \Omega} |\tilde{\Theta}_k^{(\epsilon)}(\omega)| < 0. \tag{63}$$

On the other hand, if $|\omega - \omega_0| < n^{-1}/\log n$, then note that

$$|\tilde{\Theta}_k(\omega) - \tilde{\Theta}_k(\omega_0)| \leq \int_{\omega_0}^{\omega} \left| \sum_{j=1}^k \epsilon_j \exp(tj\sqrt{-1}) \right| dt := \int_{\omega_0}^{\omega} |\tilde{\Theta}_k^{(\epsilon)}(t)| dt. $$

Similarly to the arguments above and by a simple chaining technique, we have that

$$\mathbb{P} \left( \max_{1 \leq k \leq n} \sup_{|t - \omega_0| < n^{-1}/\log n} |\tilde{\Theta}_k^{(\epsilon)}(t)| \geq C_0 n^{3/2} \log n \right) \lesssim G^*(n, pn) + n^{-1} \lesssim G^*(n, pn). \tag{64}$$

Define the sequence of events

$$B_n = \{ \max_{1 \leq k \leq n} \sup_{|t - \omega_0| < n^{-1}/\log n} |\tilde{\Theta}_k^{(\epsilon)}(t)| \geq C_0 n^{3/2} \log n \}. $$

Then we have, on the event $B_n^c$, if $n^{-1}/\log n > |\omega - \omega_0| > n^{-3/2} \log n$,

$$\tilde{\Theta}(\omega) - \tilde{\Theta}(\omega_0) \leq -C_0 n^3|\omega - \omega_0|^2 + \max_{1 \leq k \leq n} \max_{|\omega - \omega_0| < n^{-1}/\log n} |\tilde{\Theta}_k^{(\epsilon)}(\omega_0) - \tilde{\Theta}_k^{(\epsilon)}(\omega)|$$

$$< -C_0 n^3|\omega - \omega_0|^2 + C_0 n^{3/2} \log n|\omega - \omega_0| < 0. \tag{65}$$

By (63) and (65), we have that, on the event $A_n^c \cap B_n^c$, $|\hat{\omega}_0 - \omega_0| \leq n^{-3/2} \log n$. Note that $1 - \mathbb{P}(A_n \cap B_n) = O(G^*(n, pn))$. Similarly, for any fixed $\delta > 0$, we can derive

$$\mathbb{P}( |\hat{\omega}_0 - \omega_0| > \delta n^{-3/2} \log n ) = O(G^*(n, pn)). $$

Hence the lemma follows.

\[ \square \]

**Lemma D.8.** Suppose $\Omega \neq \emptyset$. Let $\Omega = \{\omega_1, \ldots, \omega_K\}$. Then under the assumptions of Theorem 4.3, we have that $\max_{1 \leq k \leq K} |\hat{\omega}_k - \omega_k| = O_P(n^{-1})$ and

$$\mathbb{P}( \max_{1 \leq k \leq K} |\hat{\omega}_k - \omega_k| \geq n^{-1}h_n ) \lesssim n^{(\theta-1)/8} + G^*(n, pn)$$

for any sequence $h_n > 0$ that diverges to infinity arbitrarily slowly.
Proof. This lemma follows from similar and simpler arguments as those of Lemma D.7. We shall briefly outline the proof here. First of all, elementary calculations using sums of trigonometric series yield that there exist finite and positive constants $c$ and $C$ such that $cn \leq \tilde{\Theta}(\omega_k) \leq Cn$ for any $\omega_k \in \Omega$. Note that by Lemma D.6, the critical values of the DPPT is no larger than $n/\log n$ with probability at least $1 - Cn^{(\theta - 1)/8}$. Therefore the critical values will be surpassed with probability as least $1 - Cn^{(\theta - 1)/8}$ for the first $K$ steps of the DPPT. Furthermore, elementary calculations yield that $\tilde{\Theta}(\omega') = O(n/h(n))$ if $h(n)/n \leq |\omega' - \omega_k|$ for all $\omega_k \in \Omega$. Note that by Assumption 8, two oscillatory frequencies are at least $O(1/\log n)$ away. On the other hand, we have by (61) and (62) that

$$P( \hat{A}_n ) \lesssim G^*(n, np),$$

where

$$\hat{A}_n = \{ \max_{1 \leq k \leq n} \max_{\omega \in \Omega} |\Theta_k^{(c)}(\omega)| \geq \sqrt{n \log n} \}. \quad (67)$$

On the event $\hat{A}_n^c$, we have $\Theta(\omega') < \Theta(\omega_k)$ for all $\omega_k \in \Omega$ if $h(n)/n \leq |\omega' - \omega_k|$. Hence the lemma follows.

Proof of Theorem 4.3. Part 1 of Theorem 4.3 follows directly from Theorem 4.1. Hence, we only need to prove Part 2. By Lemma D.8, we have that

$$P(\{ \max_{1 \leq k \leq K} |\omega_k - \omega_k| \geq n^{-1}h_n \} \lesssim n^{(\theta - 1)/8} + G^*(n, pn).$$

Observe that $G^*(n, pn)$ converges to 0. Hence, to prove Part 2, we only need to prove that

$$P(|\tilde{\Omega}| = |\Omega|) \to 1 - \alpha. \quad (68)$$

By the proof of Lemma D.8, we have that with probability at least $1 - Cn^{(\theta - 1)/8}$, the critical values will be surpassed for the first $K$ steps of the DPPT. That is, $P(|\tilde{\Omega}| < |\Omega|) \lesssim n^{(\theta - 1)/8}$. Thus, to prove (68), it suffices to prove

$$P(|\tilde{\Omega}| > |\Omega|) \to \alpha. \quad (69)$$

Define the sets $V = \bigcup_{i=1}^K [w_i - [h_n/n + \log(m)/(4m^{1/2})], w_i + [h_n/n + \log(m)/(4m^{1/2})]] \cap W$ and $V' = \bigcup_{i=1}^K [\tilde{w}_i - \log(m)/(4m^{1/2}), \tilde{w}_i + \log(m)/(4m^{1/2})] \cap W$. Observe that $V$ is a fixed set and $V'$ is a random set. Recall the definition of $\hat{A}_n$ in (67). On the event $\hat{A}_n^c$, by Lemma D.8, $V'$ is a subset of $V$ and the cardinality of the set difference $|V \setminus V'| = O(\sqrt{n}h_n \log n)$ (note that $p \asymp n^{3/2}/\log n$).

Recall the definition of $L_{n,y}$ in (47). Note that, for any $\omega \in V \setminus V'$, we have that $E(\max_{1 \leq i \leq n} |L_{n,y}(i, \omega)|^2) = O(n)$. Hence by the proof of Lemma D.5, it follows that

$$P\left( F(V \setminus V') \geq \sqrt{\log(Cn^{1/2 + \theta_1})} \right) \leq G^*(n, n^{3/2}h_n \log n) + n^{-c} \lesssim G^*(p, pn)$$

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for any positive and finite constant \( c \in (1/8, \theta_1) \), where \( \theta_1 \) is any constant in \((1/8, 1/7)\). Note that \( n^{-c} \) is bounded by \( G^*(p, pn) \) when \( c \) is larger than \( 1/8 \). Furthermore, \( G^*(n, n^{3/2}h_n \log n) \) is bounded by \( G^*(n, pn) \). Define the events

\[
E_n = \left\{ F(V\setminus V') \geq \sqrt{\log(Cn^{1/2+\theta_1})} \right\},
\]

which by the above argument satisfies \( \mathbb{P}(E_n) \to 0 \) when \( n \to \infty \). By Lemma D.5, we have that there exists constants \( C_\alpha > 0 \) and \( \theta_2 \in (\theta_1, 1/7) \) such that

\[
\mathbb{P}\left( F(W\setminus V) \geq \sqrt{\log(C_\alpha n^{1/2+\theta_2})} \right) > \alpha.
\]

Hence, on the event \( E_n^c \), we have that the \((1 - \alpha)\)-quantile of \( F(W\setminus V) \) and the \((1 - \alpha)\)-quantile of \( \tilde{F}(W\setminus V') \) are asymptotically equivalent. In particular, asymptotically the \((1 - \alpha)\)-quantile of \( F(W\setminus V) \) is at least as large as \( O(\sqrt{\log n}) \).

Next, we discuss the multiplier bootstrap statistics \( \tilde{F}_{m,l} \). We have that, on the event \( \tilde{A}_n^c \) and for any \( \omega \in W\setminus V' \),

\[
\sum_{j=1}^{k} \sum_{i=j}^{j+m-1} \exp(\sqrt{-1}i\omega)\mu_{i,n}^2/(m(n-m)) \lesssim \log^{-1} m
\]

uniformly in \( k \) since \( \omega \) is a least \( O(\log m/m^{1/2}) \) away from an oscillatory frequency. Therefore, with similar arguments as above, we have that the \((1 - \alpha)\)-quantile of \( \tilde{F}(W\setminus V) \) and \( \tilde{F}(W\setminus V') \) are asymptotically equivalent with probability at least \( 1 - O(h_n^{-d'}) \). Now, by the proof of Theorem 4.1 we have that, with probability as least \( 1 - O(h_n^{-d'}) \),

\[
|\mathbb{P}(F(W\setminus V) \leq x) - \mathbb{P}(\tilde{F}(W\setminus V) \leq x|X)| \to 0 \quad (70)
\]

uniformly for all \( x > d^\circ_{n,|W\setminus V|} \). Note that \( d^\circ_{n,|W\setminus V|} \) converges to 0 and the \( 1 - \alpha \) quantile of \( F(W\setminus V) \) is at least as large as \( O(\sqrt{\log n}) \). Hence by (70), we have \( \mathbb{P}(|\hat{\Omega}| > |\Omega|) \to \alpha \). The theorem follows.

\[ \Box \]

**Proof of Proposition 4** The proof of this proposition follows from Lemma D.7 and the essentially the same arguments as those in the proof of Theorem 4.3. Details are omitted. \[ \Box \]

### D.2 Proof of results in stage 2

#### D.2.1 Consistency

The following Lemmas D.9, D.10 and D.11 show that the error in estimating the oscillatory frequency has asymptotically negligible impact on the local change point detection algorithm.
Lemma D.9. Let \( \{\epsilon_k\} \) be a zero mean PLS time series satisfying Assumption 3 and \( m^\circ \) satisfy \( m^\circ \asymp n^{\gamma}, \gamma \in (0,1) \). Define

\[
T^{(e)}_n(w) := \sup_{m^\circ < l < n - m^\circ} \left| \sum_{k=l-m^\circ}^{l} e^{-lw(k-l)}\epsilon_k - \sum_{k=l+1}^{l+m^\circ+1} e^{-lw(k-l)}\epsilon_k \right|/\sqrt{2m^\circ}.
\]

Suppose \( \hat{w} \) is an estimator of the oscillatory frequency \( w \). If \( \mathbb{P}(|w - \hat{w}| \geq a_n) = O(b_n) \) for some sequences \( a_n \) and \( b_n \), then

\[
\mathbb{P}\left(|T^{(e)}_n(w) - T^{(e)}_n(\hat{w})| \geq a_n m^\circ n^{1/q} \log(n) \right) \lesssim b_n + 1/\log^q n.
\]

Proof. Define the events \( \tilde{B}_n = \{|w - \hat{w}| \geq a_n\} \) and

\[
H^{(e)}_{l,m^\circ}(\omega) = \sum_{k=l-m^\circ}^{l} e^{-lw(k-l)}\epsilon_k - \sum_{k=l+1}^{l+m^\circ+1} e^{-lw(k-l)}\epsilon_k.
\]

For any \( w' \) such that \( |w' - w| < a_n \), we have

\[
H^{(e)}_{l,m^\circ}(\omega) - H^{(e)}_{l,m^\circ}(\omega') = \int_{\omega}^{\omega'} \left[ \sum_{k=l-m^\circ}^{l} \sqrt{-1(k-l)}e^{\sqrt{-1}\theta(k-l)}\epsilon_k - \sum_{k=l+1}^{l+m^\circ+1} \sqrt{-1(k-l)}e^{\sqrt{-1}\theta(k-l)}\epsilon_k \right] d\theta
\]

\[
:= \int_{\omega}^{\omega'} (H^{(e)}_{l,m^\circ})'(\theta) d\theta.
\]

By Lemma 6 of Zhou (2013), we have that

\[
\max_{1 \leq l \leq n} \sup_{\theta \in [0,\pi]} \|(H^{(e)}_{l,m^\circ})'(\theta)\|_q = O((m^\circ)^{3/2}). \tag{71} \]

Therefore, by (71) and a simple \( L^q \) maximum inequality, we have

\[
\| \max_{1 \leq l \leq n} \sup_{|w' - w| < a_n} |H^{(e)}_{l,m^\circ}(\omega) - H^{(e)}_{l,m^\circ}(\omega')| \|_q \leq \int_{\omega-a_n}^{\omega+a_n} \| \max_{1 \leq l \leq n} \|(H^{(e)}_{l,m^\circ})'(\theta)\|_q d\theta
\]

\[
\lesssim a_n (m^\circ)^{3/2} n^{1/q}. \tag{72} \]

The above inequality implies that, on the event \( \tilde{B}_n^c \), we have

\[
\| \max_l |H^{(e)}_{l,m^\circ}(\omega) - H^{(e)}_{l,m^\circ}(\hat{w})| \|_q \lesssim a_n (m^\circ)^{3/2} n^{1/q}. \tag{73} \]

Hence, the lemma follows by (73) and the Markov’s inequality. \( \square \)

Note that the probability bound can be decreased to \( b_n + n^{-\theta q} \) for some \( \theta > 0 \) if we increase the threshold to \( a_n m^\circ n^{1/q + \theta} \). Lemma D.9 implies that if \( a_n = n^{-3/2} \log(n) \), then

\[
|T^{(e)}_n(w) - T^{(e)}_n(\hat{w})| = O_p(n^{1/q-3/2} m^\circ \log n).
\]

And if \( a_n = n^{-1} h_n \), then

\[
|T^{(e)}_n(w) - T^{(e)}_n(\hat{w})| = O_p(n^{1/q-1} m^\circ h(n)).
\]

Both bounds can converge to 0 for sufficiently large \( q \) when \( m^\circ \asymp n^{\gamma}, \gamma \in (0,1) \).
Lemma D.10. Let $\mu_k = \mu_{k,n}$ be the mean function defined in $[3]$ and $n^\circ$ satisfy $n^\circ \asymp n^\gamma$, $\gamma \in (0,1)$. Assume that $\Omega \neq \emptyset$, $\omega \in \Omega$ and there is no change point at frequency $\omega$. Define

$$T^{(\mu)}_n(\omega) = \sup_{m^\circ < l < n - m^\circ} \left| \sum_{k=l-m^\circ}^{l} e^{\sqrt{-1}w(k-l)} \mu_k - \sum_{k=l+1}^{l+m^\circ+1} e^{\sqrt{-1}w(k-l)} \mu_k \right| / \sqrt{2m^\circ}.$$ 

Suppose $\hat{w}$ is an estimator of the oscillatory frequency $w$. If $\mathbb{P}(|w - \hat{w}| \geq a_n) = O(b_n)$ for some sequences $a_n$ and $b_n$ with $a_n m^\circ \to 0$, then

$$\mathbb{P}\left( |T^{(\mu)}_n(\omega) - T^{(\mu)}_n(\hat{w})| > C(m^\circ)^{3/2} a_n \right) \lesssim b_n$$

for some finite constant $C$ which does not depend on $n$.

Proof. Recall the events $\bar{B}_n = \{|w - \hat{w}| \geq a_n\}$. Define

$$H^{(\mu)}_{l,m^\circ}(\omega) = \sum_{k=l-m^\circ}^{l} e^{\sqrt{-1}w(k-l)} \mu_k - \sum_{k=l+1}^{l+m^\circ+1} e^{\sqrt{-1}w(k-l)} \mu_k.$$ 

For any $\omega'$ such that $|\omega' - w| < a_n$, we have

$$H^{(\mu)}_{l,m^\circ}(\omega) - H^{(\mu)}_{l,m^\circ}(\omega') = \int_{\omega'}^{\omega} \left[ \sum_{k=l-m^\circ}^{l} e^{\sqrt{-1}(k-l)} e^{\sqrt{-1}\theta(k-l)} \mu_k - \sum_{k=l+1}^{l+m^\circ+1} e^{\sqrt{-1}(k-l)} e^{\sqrt{-1}\theta(k-l)} \mu_k \right] d\theta,$$

$$:= \int_{\omega'}^{\omega} H^{(\mu)}_{l,m^\circ}(\theta) d\theta. \quad (74)$$

If $|\omega - \omega'| < a_n$ and for any oscillatory part in $\mu_k$ with frequency different from $\omega$, we have by Assumption $[8]$ that the contribution of such oscillation to $H^{(\mu)}_{l,m^\circ}(\theta)$ is $O(m^\circ \log n)$ uniformly in $l$ when $n$ is sufficiently large.

Similarly, by similar arguments as those in Lemma D.4, we have that the contribution of the smooth function $f(\cdot)$ to $H^{(\mu)}_{l,m^\circ}(\theta)$ is $O((m^\circ)^{5/4})$ uniformly in $l$. Hence by (74), we have the contributions of the oscillatory frequencies other than $\omega$ and the smooth function $f(\cdot)$ to $|H^{(\mu)}_{l,m^\circ}(\omega) - H^{(\mu)}_{l,m^\circ}(\omega')|$ is of the order $a_n(m^\circ)^{5/4}$ uniformly over $l$ if $|\omega - \omega'| < a_n$. On the other hand, elementary but tedious calculations yield that, when $\mu_k = A\cos(\omega k + \theta)$, $A > 0$, $0 \leq \theta < 2\pi$, and $|\omega' - \omega| \leq a_n$,

$$|H^{(\mu)}_{l,m^\circ}(\omega) - H^{(\mu)}_{l,m^\circ}(\omega')| = A \frac{\sin^2(m^\circ(\omega' - \omega)/2)}{|\sin((\omega' - \omega)/2)|} + O(m^\circ a_n), \quad (75)$$

where the $O(m^\circ a_n)$ term is uniformly in $l$. Therefore,

$$|H^{(\mu)}_{l,m^\circ}(\omega) - H^{(\mu)}_{l,m^\circ}(\omega')| \lesssim (m^\circ)^2 |w - w'| + m^\circ a_n \lesssim (m^\circ)^2 a_n$$
uniformly in $l$ if $|\omega' - \omega| \leq a_n$. Since $\mathbb{P}(\bar{B}_n^\circ) = O(b_n)$, the lemma follows. \hfill \square
Remark 3. In order for the bound \((m^o)^{3/2} a_n\) to converge to 0, we need \(m^o \ll n / \log^{2/3} n\) if \(a_n = n^{-3/2} \log n\) and \(m^o \ll n^{2/3}\) if \(a_n = n^{-1}\), where \(n^{-3/2} \log(n)\) and \(n^{-1}\) come from Lemmas \[D.7\] and \[D.8\]. In fact, Lemma \[D.10\] demonstrates the effectiveness of the local change point test in relieving the energy leak problem. Observe that in classic global change point detection algorithms such as those based on the CUSUM statistic, binary segmentation or dynamic programming, typically the detection window length \(m^o\) in some steps are proportional to \(n\). In this case \((m^o)^{3/2} a_n \geq C\) for some positive constant \(C\) even if \(a_n = n^{-3/2}\). Therefore, one cannot plug-in the estimated oscillatory frequency \(\hat{\omega}\) for change point detection without appropriate adjustments for the estimation error. Unfortunately, estimating the error in frequency estimation under complex oscillation is a very difficult problem.

Lemma D.11. Let \(\omega \in \Omega\) and assume that there is no change point at frequency \(\omega\). Take \(B\) to be the set of potential change points. Recall (5) and (6), and

\[
\Upsilon_k(i, \omega) = \left( \sum_{l = k - m'}^k \exp(-1w(l - i))X_l - \sum_{l = k + 1}^{k + m' + 1} \exp(-1w(l - i))X_l \right) / \sqrt{2m'}.
\]

For \(\hat{m} + m' \leq i \leq n - \hat{m} - m'\), define

\[
\hat{T}(i, w) = \left( \sum_{k = i - \hat{m}}^i \Upsilon_k(i, w)G_k - \sum_{k = i + 1}^{i + \hat{m} + 1} \Upsilon_k(i, w)G_k \right) / \sqrt{2\hat{m}}
\]

and \(\hat{T}(B, \omega) = \max_i |\hat{T}(i, w)|\), where \(G_k\) are i.i.d. standard normal random variables independent of the data. If \(P(|w - \hat{w}| \geq a_n) = O(b_n)\) for some sequences \(a_n\) and \(b_n\), \(\hat{m} \asymp n^{\gamma_1}\) with \(1/6 < \gamma_1 < 1\), \(m' \asymp n^{\gamma}\) with \(0 < \gamma < 1\), \(a_n m' \to 0\), and Assumptions 3 to 8 hold true, then one can construct a sequence of events \(H_n\) with \(P(H_n) \geq 1 - C(b_n + 1/\log^3 n)\), such that on the events \(H_n\) and conditional on the data,

\[
\sup_{x \in \mathbb{R}} \left| P(\hat{T}(B, w) \leq x) - P(\hat{T}(B, \hat{w}) \leq x) \right| \lesssim [a_n m' (\sqrt{m'} + n^{1/4} \log n)]^{1/3} \log^2 n + n^{-1} \log^{1.5} n.
\]

Proof. Take

\[H_n = \{|T_n^{(c)}(w) - T_n^{(c)}(\hat{w})| \leq a_n m' n^{1/4} \log(n)\} \cap \{|T_n^{(\mu)}(w) - T_n^{(\mu)}(\hat{w})| \leq C(m')^{3/2} a_n\}.
\]

This lemma follows from Lemmas \[D.9\] and \[D.10\] and Proposition \[C.3\]. Note that there is no need to put constraint on the range of \(x\) here since the conditional variances of the real and imaginary parts of \(\hat{T}(i, w)\) have positive lower bounds uniformly in \(i\) for sufficiently large \(n\) by Assumption \[6\] (see also \[83\] below with \(i = k\) therein).

The following Lemma \[D.12\] establishes a uniform Gaussian approximation result for the local change point detection algorithm.
Lemma D.12. Assume Assumptions 3 to 8 hold, and \( \tilde{m} \asymp n^{\gamma_1} \) with \( \gamma_1 > 16/29 \). Take \( B \) to be the set of potential change points. Define

\[
T^{(c)}(B, \omega) := \max_{i \in B} \left| \sum_{k=i}^{i} \exp(-1\omega(k-i))\epsilon_k - \sum_{k=i+1}^{i+\tilde{m}+1} \exp(-1\omega(k-i))\epsilon_k \right| / \sqrt{2\tilde{m}}.
\]

And let \( T^{(y)}(B, \omega) \) be the version of \( T^{(c)} \) with \( \epsilon_k \) therein replaced by \( y_k \), where \( \{y_k\} \) is a centered Gaussian time series preserving the covariance structure of \( \{\epsilon_k\} \). Then, for any \( \omega \in \Omega \), we have that

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}(T^{(c)}(B, \omega) \leq x) - \mathbb{P}(T^{(y)}(B, \omega) \leq x) \right| \to 0. \tag{76}
\]

Proof. Let \( U_k = [\cos(\omega_k)\epsilon_k, \sin(\omega_k)\epsilon_k]^\top, k = 1, 2, \cdots, n \). For any \( j \in \mathbb{Z} \), define the projection operator \( \mathcal{P}_j(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_j) - \mathbb{E}(\cdot|\mathcal{F}_{j-1}) \). Let

\[
D_i = \sum_{j=i}^{\infty} \mathcal{P}_i(U_j),
\]

\( i = 1, 2, \cdots, n \), where we let \( U_j = 0 \) if \( j > n \). Let

\[
\Sigma_k = \mathbb{E}(D_kD_k^\top),
\]

\( k = 1, 2, \cdots, n \). Now, by the proof of Lemma 5 in Zhou (2014), we have that

\[
\max_{1 \leq k \leq \tilde{m}} \max_i \left| \sum_{j=i-k}^{i} [\Sigma_j - \text{Diag}(v(j/n, \omega), v(j/n, \omega))] / \tilde{m} \right| = O(n^{3/8}/\tilde{m}) = O(n^{-\alpha_4}) \tag{77}
\]

for some \( \alpha_4 > 0 \). Recall that \( v \) is defined in Assumption 3. By Assumptions 3 to 8 and the proofs of Theorem 1 and Corollary 2 of Wu and Zhou (2011), we have that, on a possibly richer probability space from a proper construction procedure, there exist i.i.d. standard normal random vectors \( Y_i, i = 1, 2, \cdots \), such that

\[
\max_{1 \leq i \leq n} |S_{U,i} - S_{Y,i}| = O_p(n^{8/29}\log^{35/29} n), \tag{78}
\]

where

\[
S_{U,i} := \sum_{j=1}^{i} U_j \quad \text{and} \quad S_{Y,i} := \sum_{j=1}^{i} \Sigma_j^{1/2} Y_j.
\]

Let \( \tilde{U}_i = (U_i)_1 + \sqrt{-1}(U_i)_2 \) and \( \tilde{Y}_i = (\Sigma_i^{1/2} Y_i)_1 + \sqrt{-1}(\Sigma_i^{1/2} Y_i)_2 \), where \( v_j \) denotes the \( j \)-th entry of a vector \( v \). Then, (78) implies that

\[
\max_{1 \leq i \leq n} |\tilde{S}_{U,i} - \tilde{S}_{Y,i}| = O_p(n^{8/29}\log^{35/29} n), \tag{79}
\]

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where \( \tilde{S}_{U,i} = \sum_{j=1}^{i} \tilde{U}_j \), and \( \tilde{S}_{Y,i} \) is defined similarly. Note that
\[
\left| \sum_{k=i-\tilde{m}}^{i} \exp(\sqrt{-1} \omega(k-i)) \epsilon_k - \sum_{k=i+1}^{i+\tilde{m}+1} \exp(\sqrt{-1} \omega(k-i)) \epsilon_k \right| = |\tilde{S}_{U,i+\tilde{m}+1} + \tilde{S}_{U,i-\tilde{m}-1} - 2\tilde{S}_{U,i}|.
\]
Therefore, by combining this with (79), we have that
\[
T^{(c)}(B, \omega) = T^{(\tilde{Y})}(B, \omega) + O_p(n^{8/29} \log^{35/29} n/\sqrt{\tilde{m}}) = o_p(n^{-\alpha_3})
\] (80)
for some \( \alpha_3 > 0 \), where
\[
T^{(\tilde{Y})}(B, \omega) := \max_i \left| \sum_{k=i-\tilde{m}}^{i} \tilde{Y}_k - \sum_{k=i+1}^{i+\tilde{m}+1} \tilde{Y}_k \right| / \sqrt{2\tilde{m}}.
\]
By (80), a similar regular convex polygon approximation argument as that in the proof of Proposition C.3 and Nazarov’s inequality \cite{Nazarov2003}, we have
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}(T^{(c)}(B, \omega) \leq x) - \mathbb{P}(T^{(\tilde{Y})}(B, \omega) \leq x) \right| \to 0.
\] (81)
Note that the range of \( x \) does not have to be constrained since the eigenvalues of \( \sum_{j=i-\tilde{m}}^{i} \Sigma_j / \tilde{m} \) are uniformly bounded away from 0 for sufficiently large \( n \) by Assumption 6 and (77) above.

Define
\[
\tilde{Y}_k = \text{Diag}(v^{1/2}(k/n, \omega), v^{1/2}(k/n, \omega)) Y_k.
\]
and
\[
T^{(D)}(\omega) := \max_i \left| \sum_{k=i-\tilde{m}}^{i} \tilde{Y}_k - \sum_{k=i+1}^{i+\tilde{m}+1} \tilde{Y}_k \right| / \sqrt{2\tilde{m}}.
\]
By (77) and Proposition C.3, we have that
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}(T^{(\tilde{Y})}(B, \omega) \leq x) - \mathbb{P}(T^{(D)}(B, \omega) \leq x) \right| \to 0.
\] (82)
Note that the range of \( x \) does not need to be constrained in (82) since all \( v(j/n, \omega) \) are bounded away from 0. Finally, again by the proof of Lemma 5 in \cite{Zhou2014}, we obtain that,
\[
\max_{i \leq k} \left| \mathbb{E} [S_{y,i} S_{y,k}^\top] - \sum_{j=k-\tilde{m}}^{i+\tilde{m}+1} \text{Diag}(v(j/n, \omega), v(j/n, \omega)) / (2\tilde{m}) \right| = O(n^{-\alpha_7}),
\] (83)
where \( S_{y,i} = (\sum_{j=i-\tilde{m}}^{i} \cos(\omega_j) y_j, \sin(\omega_j) y_j)^\top - \sum_{j=i+1}^{i+\tilde{m}+1} \cos(\omega_j) y_j, \sin(\omega_j) y_j)^\top / \sqrt{2\tilde{m}} \) and \( \alpha_5 > 0 \) is a constant. Hence, we obtain by Proposition C.3 that
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}(T^{(c)}(B, \omega) \leq x) - \mathbb{P}(T^{(D)}(B, \omega) \leq x) \right| \to 0.
\]
By combining the above bounds, the lemma follows. \( \square \)
The following lemma establishes that the covariance structure of the real part of the phase-adjusted OBMB is asymptotically close to that of the target Gaussian process. The same result can be established for the covariance structure of the imaginary part and the covariance between the real and imaginary parts using the same arguments.

**Lemma D.13.** Assume that assumptions 3 to 8 hold true. Take \( \omega \in \Omega \). Further, assume that there is no change point at frequency \( \omega \). For any \( k \in [\tilde{m} + m', n - \tilde{m} - m'] \), let \( k^* = k - \tilde{m} - m' + 1 \). Define

\[
\Delta'': = \max_{\tilde{m} + m' \leq i, j \leq n - \tilde{m} - m'} \left| \text{Cov}(\tilde{S}^{(2)}_i, \tilde{S}^{(2)}_j | X) - \text{Cov}(\Theta^{(2)}_i, \Theta^{(2)}_j) \right|.
\]

Under the assumption that \( m' \to \infty \) and \( m'/\tilde{m} \to 0 \), we have

\[
P(\Delta'' \lesssim 1/m' + m'/\tilde{m} + n^{1/q'} \sqrt{m'/\tilde{m} \log n}) \geq 1 - C/ \log q' n
\]

for some finite positive constant \( C \). Recall \( q' = q/2 \).

**Proof.** Recall the definitions of \( \Theta^{(2)}_i \) and \( \tilde{S}^{(2)}_i \) in Section 4.1. We first separate \( \Theta^{(2)}_i \) into individual summations. Denote

\[
E_{i,\tilde{m}}^+ = \sum_{k=i-\tilde{m}}^i \cos(w(k - i + \tilde{m}))\epsilon_k
\]

and

\[
E_{i,\tilde{m}}^- = \sum_{k=i+1}^{i+\tilde{m}+1} \cos(w(k - i + \tilde{m}))\epsilon_k.
\]

Then

\[
\Theta^{(2)}_i = (E_{i,\tilde{m}}^+ - E_{i,\tilde{m}}^-)/\sqrt{2\tilde{m}}
\]

and

\[
2\tilde{m}\text{Cov}(\Theta^{(2)}_i, \Theta^{(2)}_j) = \text{Cov} \left( E_{i,\tilde{m}}^+, E_{i,\tilde{m}}^-; E_{j,\tilde{m}}^+, E_{j,\tilde{m}}^- \right) = \text{Cov} \left( E_{i,\tilde{m}}^+, E_{j,\tilde{m}}^+ \right) - \text{Cov} \left( E_{i,\tilde{m}}^-, E_{j,\tilde{m}}^- \right) - \text{Cov} \left( E_{i,\tilde{m}}^+, E_{j,\tilde{m}}^- \right) + \text{Cov} \left( E_{i,\tilde{m}}^-, E_{j,\tilde{m}}^+ \right).
\]

If \( i \leq j \), \( E_{i,\tilde{m}}^+ \) and \( E_{j,\tilde{m}}^- \) do not overlap, and we have

\[
\text{Cov} \left( E_{i,\tilde{m}}^+, E_{j,\tilde{m}}^- \right) \leq \sum_{i-\tilde{m} \leq k \leq j \leq j+\tilde{m}} \left| \mathbb{E}\epsilon_k\epsilon_l \right| \leq \tilde{m} \sum_{k=0}^{\tilde{m}} (\tilde{m} - k)|j - i + \tilde{m} + k|^{-q} + \sum_{k=1}^{\tilde{m}} k|j - i + k|^{-q} \leq \sum_{k=0}^{\tilde{m}} (\tilde{m} - k)k^{-q} + \sum_{k=1}^{\tilde{m}} k^{-q+1} \leq \tilde{m}^{-q+2} = O(1)
\]

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since \( q \geq 4 \) by Assumption [3]. For other terms, we will control them one by one below.

For the multiplier bootstrap part we have

\[
2\tilde{m}\text{Cov}(\tilde{S}^{(2)}_{i*, \tilde{S}^{(2)}_{j*}|X}) = \text{Cov} \left( \sum_{k=i-\tilde{m}}^{i+\tilde{m}+1} \Phi_k(i, w) G_k, \sum_{k=j-\tilde{m}}^{j+\tilde{m}+1} \Phi_k(j, w) G_k \right)
\]

\[
= \begin{cases} 
0 & \text{if } |i - j| > 2\tilde{m} + 2 \\
-\sum_{k=i+\delta_1}^{j-\delta_1} \Phi_k(i, w) \Phi_k(j, w) & \text{if } \tilde{m} + 1 < |i - j| \leq 2\tilde{m} + 2 \\
\sum_{k=i+\delta_1}^{j-\delta_1} \Phi_k(i, w) \Phi_k(j, w) & \text{if } 0 \leq |i - j| \leq \tilde{m} + 1
\end{cases}
\]

where \( \delta_1 = |i - j| - \tilde{m} \).

Case 1, \(|i - j| > 2\tilde{m} + 2\). By the above analysis, since both summations do not overlap, we have

\[
\text{Cov}(\Theta^{(2)}_{i*}, \Theta^{(2)}_{j*}) = O(1/\tilde{m})
\]

and

\[
\text{Cov}(\tilde{S}^{(2)}_{i*}, \tilde{S}^{(2)}_{j*}|X) = 0.
\]

Thus,

\[
|\text{Cov}(\Theta^{(2)}_{i*}, \Theta^{(2)}_{j*}) - \text{Cov}(\tilde{S}^{(2)}_{i*}, \tilde{S}^{(2)}_{j*}|X)| = O(1/\tilde{m}).
\]

Case 2, \( \tilde{m} + 1 < |i - j| \leq 2\tilde{m} + 2 \). Without loss of generality, assume \( i \leq j \). Note that \( \text{Cov}(\tilde{E}_{i, \tilde{m}}, \tilde{E}_{j, \tilde{m}}^+|X) \) is the only term with overlapping entries. Thus,

\[
\text{Cov}(\Theta^{(2)}_{i*}, \Theta^{(2)}_{j*}) = -\text{Cov}(\tilde{E}_{i, \tilde{m}}, \tilde{E}_{j, \tilde{m}}^+|X) / (2\tilde{m}) + O(1/\tilde{m}).
\]

We can further expand overlapping and non-overlapping parts and note the covariance between the non-overlapping part is well controlled:

\[
\text{Cov}(\tilde{E}_{i, \tilde{m}}, \tilde{E}_{j, \tilde{m}}^+) = \text{Cov} \left( \sum_{k=i+\delta_1}^{i+\tilde{m}+1} \cos(w(k-i))\epsilon_k, \sum_{k=j-\delta_1}^{j-\tilde{m}} \cos(w(k-j))\epsilon_k \right)
\]

\[
= \text{Cov} \left( \sum_{k=i+\delta_1}^{i+\tilde{m}+1} \cos(w(k-i))\epsilon_k, \sum_{k=j-\tilde{m}}^{j-\tilde{m}} \cos(w(k-j))\epsilon_k \right)
\]

\[
+ \text{Cov} \left( \sum_{k=i+\delta_1}^{i+\tilde{m}} \cos(w(k-i))\epsilon_k, \sum_{k=j-\delta_1}^{j-\delta_1} \cos(w(k-j))\epsilon_k \right)
\]

\[
+ \text{Cov} \left( \sum_{k=i+\delta_1}^{j-\delta_1} \cos(w(k-i))\epsilon_k, \sum_{k=j-\tilde{m}}^{j-\tilde{m}} \cos(w(k-j))\epsilon_k \right)
\]

\[
= \text{Cov} \left( \sum_{k=i+\delta_1}^{j-\delta_1} \cos(w(k-i))\epsilon_k, \sum_{k=j-\tilde{m}}^{j-\tilde{m}} \cos(w(k-j))\epsilon_k \right) + O(1).
\]
Now we shall move on to the multiplier bootstrap part. Note that under our assumption about the trend,
\[
\left( \sum_{l=k-m'}^{k} \cos(w(l-i)) \mu_l - \sum_{l=k+1}^{k+m'+1} \cos(w(l-i)) \mu_l \right) / \sqrt{2m'} = O(1).
\]
Thus, we can break down
\[
\Phi_k(i, w) = \left( \sum_{l=k-m'}^{k} \cos(w(l-i)) \epsilon_l - \sum_{l=k+1}^{k+m'+1} \cos(w(l-i)) \epsilon_l \right) / \sqrt{2m'} + O(1)
\]
\[
= \Phi_k^{(c)}(i, w) + O(1).
\]
Breaking down the covariance structure of the bootstrap statistics in the same way like that in [4], we get
\[
\text{Cov}(\tilde{S}_{i,w}^{(2)}, \tilde{S}_{j,w}^{(2)} | X) = - \sum_{k=i+\delta_1}^{j-\delta_1} \Phi_k(i, w) \Phi_k(j, w) / (2\tilde{m})
\]
\[
= - \sum_{k=i+\delta_1}^{j-\delta_1} (\Phi_k^{(c)}(i, w) + O(1)) (\Phi_k^{(c)}(j, w) + O(1)) / (2\tilde{m})
\]
\[
= \sum_{k=i+\delta_1}^{j-\delta_1} \Phi_k^{(c)}(i, w) \Phi_k^{(c)}(j, w) + a_n \sum_{k=i+\delta_1}^{j-\delta_1} \Phi_k^{(c)}(i, w) / (2\tilde{m}) + b_n \sum_{k=i+\delta_1}^{j-\delta_1} \Phi_k^{(c)}(j, w) / (2\tilde{m}) + O(1/\tilde{m}),
\]
where \(a_n\) and \(b_n\) are some bounded sequence. Since (85) holds for all \(j\) satisfying \(\tilde{m} + 1 < |i - j| \leq 2\tilde{m} + 2\),
\[
\left\| \max_{j \in [i+\tilde{m}+2,i+2\tilde{m}+2]} \sum_{k=i+\delta_1}^{j-\delta_1} \Phi_k^{(c)}(i, w) \right\|_q \leq \left\| m' \max_{j \in [i+\tilde{m}+2,i+2\tilde{m}+2]} \left\| \sum_{k=i+\delta_1-m'}^{j-\delta_1} \epsilon_k / \sqrt{2m'} \right\|_q = O(\tilde{m}m').
\]
We thus have
\[
\left\| \max_{i \in B, j \in [i+\tilde{m}+2,i+2\tilde{m}+2]} \sum_{k=i+\delta_1}^{j-\delta_1} \Phi_k^{(c)}(i, w) \right\|_q = O(\sqrt{\tilde{m}m'n^{1/q})}.
\]
Then, the following holds uniformly for all \(\tilde{m} + 1 < |i - j| \leq 2\tilde{m} + 2\)
\[
\text{Cov}(\tilde{S}_{i,w}^{(2)}, \tilde{S}_{j,w}^{(2)} | X) = - \sum_{k=i+\delta_1}^{j-\delta_1} \Phi_k^{(c)}(i, w) \Phi_k^{(c)}(j, w) / (2\tilde{m}) + O_{[q]}(n^{1/q} \sqrt{m'/\tilde{m}}),
\]
where the asymptotic notation \(X_n = O_{[q]}(a_n)\) means that the random variable \(X_n\) and scalar \(a_n\) satisfies \(\|X_n/a_n\|_q = O(1)\). Define
\[
S_{i,k,m'} = \sum_{l=k}^{k+m'} \cos(w(l-i)) \epsilon_l.
\]
Expand the quadratic term, and we have

$$
\sum_{k=\delta_1}^{j-\delta_1} \Phi_k^{(e)}(i, w) \Phi_k^{(e)}(j, w) / (2\bar{m})
$$

$$
= \sum_{k=\delta_1}^{j-\delta_1} (S_{i,k-m',m'} - S_{i,k+1,m'}) (S_{j,k-m',m'} - S_{j,k+1,m'}) / (4\bar{m}m')
$$

$$
= \sum_{k=\delta_1}^{j-\delta_1} S_{i,k-m',m'} S_{j,k-m',m'} / (4\bar{m}m') + \sum_{k=\delta_1}^{j-\delta_1} S_{i,k+1,m'} S_{j,k+1,m'} / (4\bar{m}m')
$$

$$
- \sum_{k=\delta_1}^{j-\delta_1} S_{i,k-m',m'} S_{j,k+1,m'} / (4\bar{m}m') - \sum_{k=\delta_1}^{j-\delta_1} S_{i,k+1,m'} S_{j,k-m',m'} / (4\bar{m}m').
$$

By the proof of Lemma 1 of Zhou (2013), we have for a fixed $j \in [\bar{m} + m', \ldots, n - \bar{m} - m']$

$$
\left\| \max_{|i-j| \leq \bar{m}} \left( \sum_{k=\delta_1}^{j-\delta_1} \Phi_k^{(e)}(i, w) \Phi_k^{(e)}(j, w) - \mathbb{E} \sum_{k=\delta_1}^{j-\delta_1} \Phi_k^{(e)}(i, w) \Phi_k^{(e)}(j, w) \right) \right\|_{q'} = O(\sqrt{m'/\bar{m}}).
$$

This implies that for all $i, j$,

$$
\left\| \max_{\bar{m}+m' \leq j \leq n-\bar{m}-m'} \max_{|i-j| \leq \bar{m}} \left( \sum_{k=\delta_1}^{j-\delta_1} \Phi_k^{(e)}(i, w) \Phi_k^{(e)}(j, w) - \mathbb{E} \sum_{k=\delta_1}^{j-\delta_1} \Phi_k^{(e)}(i, w) \Phi_k^{(e)}(j, w) \right) \right\|_{q'} = O(n^{1/q'} \sqrt{m'/\bar{m}}).
$$

Lastly, note that the non-overlapping sums satisfy

$$
\mathbb{E} \sum_{k=\delta_1}^{j-\delta_1} S_{i,k-m',m'} S_{j,k+1,m'} / (4\bar{m}m') = 2 \sum_{k=\delta_1}^{j-\delta_1} \text{Cov} (S_{i,k-m',m'}, S_{j,k+1,m'}) / (4\bar{m}m') = O(1/m').
$$

So we have

$$
\sum_{k=\delta_1}^{j-\delta_1} \Phi_k^{(e)}(i, w) \Phi_k^{(e)}(j, w) / (2\bar{m})
$$

$$
= \mathbb{E} \sum_{k=\delta_1}^{j-\delta_1} S_{i,k-m',m'} S_{j,k-m',m'} / (4\bar{m}m') + \mathbb{E} \sum_{k=\delta_1}^{j-\delta_1} S_{i,k+1,m'} S_{j,k+1,m'} / (4\bar{m}m') + O(1/m').
$$
Combining the previous results, we have

\[
\left| \text{Cov}(\tilde{S}_i^{(2)}, \tilde{S}_j^{(2)} | X) - \text{Cov}(\Theta_i^{(2)}, \Theta_j^{(2)}) \right|
\]

\[
= \mathbb{E} \left[ \sum_{k=i+\delta_1}^{j-\delta_1} S_{i,k-m',m'} S_{j,k-m',m'} + S_{i,k,m'} S_{j,k,m'}/(4\tilde{m}m') \right]
\]

\[
- \text{Cov} \left( \sum_{k=i+\delta_1}^{j-\delta_1} \cos(w(k-i)) \epsilon_k, \sum_{k=i+\delta_1}^{j-\delta_1} \cos(w(k-j)) \epsilon_k \right)/(2\tilde{m}) \right| \]

\[
+ O(1/\tilde{m}) + O(1/m') + O_{n'q'}(n^{1/q'} \sqrt{m'/\tilde{m}})
\]

\[
\leq \mathbb{E} \left[ \sum_{k=i+\delta_1}^{j-\delta_1} S_{i,k,m'} S_{j,k,m'}/(2\tilde{m}m') - \frac{1}{2\tilde{m}} \sum_{j-\delta_1}^{j-\delta_1} \sum_{k=i+\delta_1}^{j-\delta_1} \cos(w(k-i)) \cos(w(l-j)) \mathbb{E} \epsilon_k \epsilon_l \right]
\]

\[
+ O(1/\tilde{m}) + O(1/m') + O_{n'q'}(n^{1/q'} \sqrt{m'/\tilde{m}})
\]

\[
\leq \sum_{|k-l| \leq m'} \left| \frac{m' - |k - l|}{2\tilde{m}m'} - \frac{1}{2\tilde{m}} \right| \mathbb{E} \epsilon_k \epsilon_l + \frac{1}{2\tilde{m}} \sum_{|k-l| > m'} \mathbb{E} \epsilon_k \epsilon_l
\]

\[
+ O(1/\tilde{m}) + O(1/m') + O_{n'q'}(n^{1/q'} \sqrt{m'/\tilde{m}}) + O(m'/\tilde{m})
\]

\[
= O(1/\tilde{m}) + O(1/m') + O_{n'q'}(n^{1/q'} \sqrt{m'/\tilde{m}}) + O(m'/\tilde{m}).
\]

**Case 3, |i - j| \leq \tilde{m} + 1.** Assume \( i \leq j \). Then,

\[
2\tilde{m}\text{Cov}(\Theta_i^{(2)}, \Theta_j^{(2)}) = \text{Cov}(E_{i,\tilde{m}}^+, E_{j,\tilde{m}}^+) - \text{Cov}(E_{i,\tilde{m}}^-, E_{j,\tilde{m}}^+) + \text{Cov}(E_{i,\tilde{m}}^-, E_{j,\tilde{m}}^-) + O(1)
\]

and

\[
2\tilde{m}\text{Cov}(\tilde{S}_i^{(2)}, \tilde{S}_j^{(2)} | X) = \left[ \sum_{k=i+\delta_1}^{j} \sum_{k=i+\delta_2}^{j} \sum_{k=i}^{j} \Phi_k(i, w)\Phi_k(j, w) \right].
\]

Similarly to Case 2, we get for all |i - j| \leq \tilde{m} + 1 the following bounds hold simultaneously.

\[
\left| \text{Cov}(E_{i,\tilde{m}}^+, E_{j,\tilde{m}}^+) / (2\tilde{m}) - \sum_{k=i+\delta_2}^{j} \Phi_k(i, w)\Phi_k(j, w) / (2\tilde{m}) \right|
\]

\[
= O(1/\tilde{m}) + O(1/m') + O_{n'q'}(n^{1/q'} \sqrt{m'/\tilde{m}}) + O(m'/\tilde{m}),
\]

\[
\left| \text{Cov}(E_{i,\tilde{m}}^-, E_{j,\tilde{m}}^-) / (2\tilde{m}) - \sum_{k=i}^{j} \Phi_k(i, w)\Phi_k(j, w) / (2\tilde{m}) \right|
\]

\[
= O(1/\tilde{m}) + O(1/m') + O_{n'q'}(n^{1/q'} \sqrt{m'/\tilde{m}}) + O(m'/\tilde{m}),
\]

S.43
Theorem follows.

Note that the range of $\omega$ with high probability. Note that under the hypothesis that there is no change point at frequency with asymptotically negligible errors and, conditional on the change point, $T(B, \omega)$ can be well approximated by $\hat{T}(B, \omega)$ with asymptotically negligible errors with high probability. Note that under the hypothesis that there is no change point at frequency $\omega$, we have $T(B, \omega) = T(\omega)(B, \omega) + O(\tilde{m}^{-1/2})$. By Lemma D.12, we have

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(T(\omega)(B, \omega) \leq x) - \mathbb{P}(T(\omega)(B, \omega) \leq x) \right| \to 0,$$

where $\{y_k\}$ is a centered Gaussian time series preserving the covariance structure of $\{\epsilon_k\}$. On the other hand, by Lemma D.13 and Proposition C.3, we have, on an event with probability at least $1 - C/\log q n$,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(\hat{T}(B, \omega) \leq x | X) - \mathbb{P}(T(\omega)(B, \omega) \leq x) \right| \leq \left( \frac{1}{m'} + n^{1/q} \sqrt{\frac{m'}{\tilde{m}}} \log n \right)^{1/3} \log^{7/6} n + \frac{\log^{13/12} n}{n}.$$ 

Note that the range of $x$ does not need to be constrained here as the marginal variances of the components in $T(\omega)(\omega)$ are bounded away from 0 by (83) with $i = k$ therein. The theorem follows.

D.2.2 Estimation accuracy

Lemma D.14. Under the assumptions of Theorem 4.4, we have that

$$\mathbb{P}(\max_{1 \leq r \leq M_k} |\hat{b}_{r,k} - b_{r,k}| \geq h_n \log \tilde{m}) \to 0$$
for any $\omega_k \in \Omega$ such that $M_k \neq 0$ and any sequence $h_n$ that diverges to infinity at an arbitrarily slowly rate.

Proof. Without loss of generality, we assume that $M_k = 1$ since other cases follow by essentially the same arguments. We shall omit the subscript $k$ in the sequel for simplicity. Write the mean function as essentially the same arguments. We shall omit the subscript $k$

Recall the definitions of $H$ for any $\omega$ have that

On the other hand, by the same argument for Lemmas D.9 and D.12, we have that

Hence, with probability converging to 1, which implies that

First of all, elementary calculations and Lemma D.10 show that

On the other hand, by the same argument for Lemmas D.9, D.10 and Assumption 9, we have that

Hence, with probability converging to 1,

which implies that $\mathbb{P}(|b_1 - b_1| > \tilde{m}) \to 0$.

Now, let $a_n$ be a diverging sequence which is dominated by $\tilde{m}$. Elementary but tedious calculations and the proof of Lemma D.10 yield that, uniformly for all $l$ such that $a_n < |l - b_1| < \tilde{m}$ and $\omega'$ such that $|\omega' - \omega| = O(g_n/n)$ for some $g_n > 0$ diverging at an arbitrarily slowly rate,

where

$\gamma_{1,n} = \frac{C_2}{4} \left| \frac{\sin(R_n b/2)}{\sin(b/2)} \right|^2,$

$\gamma_{2,n} = \frac{C_2}{4} \left| \frac{\sin(\tilde{m}b/2)}{\sin(b/2)} \right|^2$.
Part 1 of the Theorem follows by Theorem 4.2. By Lemma D.14, we just need to show that
\begin{equation}
|H_{l,m}^{(\mu)}(\hat{\omega})|^2 - |H_{l,m}^{(\mu)}(\hat{\omega})|^2 \leq -\bar{m}r_n + O(r_n\bar{m}^2 g_n/n) \leq -\bar{m}r_n
\end{equation}
uniformly for all \(l\) satisfying \(|l - b_1| \in [a_n, \bar{m}]\), were we utilized the fact that \(g_n\) can approach infinity arbitrarily slowly and \(\bar{m} \ll n\) and hence \(\bar{m}^2 g_n/n \ll \bar{m}\). Furthermore, by the same argument for Lemmas D.9 and D.12, we have that
\begin{equation}
\max_{|l-b_1| \leq \bar{m}} |H_{l,m}^{(\epsilon)}(\hat{\omega})| = O_{\epsilon}((\bar{m} \log \bar{m})^{1/2}).
\end{equation}
Choose \(a_n = h_n \log \bar{m}\), where \(h_n > 0\) is diverging at an arbitrarily slow rate. Then, we find that uniformly for all \(l\) such that \(|b_1 - l| \in [a_n, \bar{m}]\),
\begin{equation}
|H_{l,m}(\hat{\omega})|^2 - |H_{b_1,m}(\hat{\omega})|^2 \leq |H_{l,m}^{(\mu)}(\hat{\omega})|^2 - |H_{l,m}^{(\mu)}(\hat{\omega})|^2 |H_{l,m}^{(\epsilon)}(\hat{\omega})|^2 - |H_{b_1,m}^{(\epsilon)}(\hat{\omega})|^2 + 2I + 2II,
\end{equation}
where
\begin{equation}
I = |e^{\sqrt{-1}l\omega} H_{l,m}^{(\mu)}(\hat{\omega}) - e^{\sqrt{-1}l\omega} H_{b_1,m}^{(\mu)}(\hat{\omega})||H_{l,m}^{(\epsilon)}(\hat{\omega})|,
\end{equation}
and
\begin{equation}
II = |e^{\sqrt{-1}l\omega} H_{l,m}^{(\epsilon)}(\hat{\omega}) - e^{\sqrt{-1}l\omega} H_{b_1,m}^{(\epsilon)}(\hat{\omega})||H_{l,m}^{(\mu)}(\hat{\omega})|.
\end{equation}
Following the arguments above, it is easy to show that \(|e^{\sqrt{-1}l\omega} H_{l,m}^{(\epsilon)}(\hat{\omega}) - e^{\sqrt{-1}l\omega} H_{b_1,m}^{(\epsilon)}(\hat{\omega})| = O_{\epsilon}(r_n \log \bar{m})\) and \(|e^{\sqrt{-1}l\omega} H_{l,m}^{(\mu)}(\hat{\omega}) - e^{\sqrt{-1}l\omega} H_{b_1,m}^{(\mu)}(\hat{\omega})| = O(r_n)\), where the bounds are uniform across \(l\) satisfying \(|b_1 - l| \in [a_n, \bar{m}]\). Hence, by (86), (88), and (89), uniformly for all \(l\) such that \(|b_1 - l| \in [a_n, \bar{m}]\), we have, with probability approaching 1,
\begin{equation}
|H_{l,m}(\hat{\omega})|^2 - |H_{b_1,m}(\hat{\omega})|^2 \leq -C_2^2 \bar{m}r_n/20 + O(\bar{m} \log \bar{m} + \sqrt{\log \bar{m}} + r_n \sqrt{\log \bar{m}}).
\end{equation}
Observe that (90) is negative for sufficiently large \(n\). Hence the lemma follows.

**Proof of Theorem 4.4.** Part 1 of the Theorem follows by Theorem 4.2. By Lemma D.14, we just need to show that
\begin{equation}
P(|\hat{D}_k| = D_k) \to 1 - \beta.
\end{equation}
Note that, by (86) in Lemma D.14, \(H_{b_1,k,m} \geq C\bar{m}\) with probability approaching 1. On the other hand, the critical values for the first \(M_k\) steps are at most \(O_{\epsilon}(\max(1, m'/\sqrt{\bar{m}}) \sqrt{\log \bar{m}}),\) which is dominated by \(\bar{m}\). Hence \(P(|\hat{D}_k| < |D_k|) \to 0.\)

We now show that \(P(|\hat{D}_k| > |D_k|) \to \beta.\) By the similar arguments as those in the proof of Theorem 4.3, we have that, after \(M_k\) steps, the \((M_k + 1)\)-th step of change point estimation is asymptotically equivalent to performing change point test on the set \(\hat{B}_{M_k+1} :=\)
\[ B_1 - \bigcup_{j=1}^{M_k} [b_{j,k} - h_n \log \tilde{m} - \tilde{m}, b_{j,k} + h_n \log \tilde{m} + \tilde{m}] \]. Note that there is no change point on the set \( \tilde{B}_{M_{k+1}} \) and each point in \( \tilde{B}_{M_{k+1}} \) is at least \( \tilde{m} + h_n \log \tilde{m} \) away from a change point. By the proof of Theorem 4.2, we have that

\[
\mathbb{P}(T(\tilde{B}_{M_{k+1}}) > \text{crit}_{\beta,M_{k+1}}(\tilde{B}_{M_{k+1}})) \rightarrow \beta.
\]

The Theorem follows.