OUTER ACTIONS OF A DISCRETE AMENABLE GROUP ON APPROXIMATELY FINITE DIMENSIONAL FACTORS I, GENERAL THEORY

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Abstract. To each factor \( M \), we associate an invariant \( \text{Ob}_m(M) \) to be called the intrinsic modular obstruction as a cohomological invariant which lives in the “third” cohomology group:

\[
\text{H}^3_{\text{out}}(\text{Out}(M) \times \mathbb{R}, \text{H}^1_{\theta}(\mathbb{R}, \text{U}(\mathcal{C})), \text{U}(\mathcal{C}))
\]

where \( \{\mathcal{C}, \mathbb{R}, \theta\} \) is the flow of weights on \( M \). If \( \alpha \) is an outer action of a countable discrete group \( G \) on \( M \), then its modulus \( \text{mod}(\alpha) \in \text{Hom}(G, \text{Aut}_\theta(\mathcal{C})) \), \( N = \alpha^{-1}(\text{Cnt}_r(M)) \) and the pull back

\[
\text{Ob}_m(\alpha) = \alpha^*(\text{Ob}_m(M)) \in \text{H}^3_{\text{out}}(G \times \mathbb{R}, N, \text{U}(\mathcal{C}))
\]

to be called the modular obstruction of \( \alpha \) are invariants of the outer conjugacy class of the outer action \( \alpha \).

We prove that if the factor \( M \) is approximately finite dimensional and \( G \) is amenable, then the invariants uniquely determine the outer conjugacy class of \( \alpha \) and the every invariant occurs as the invariant of an outer action \( \alpha \) of \( G \) on \( M \). In the case that \( M \) is a factor of type \( \text{III}_\lambda \), \( 0 < \lambda \leq 1 \), the modular obstruction group \( \text{H}^3_{\text{out}}(G \times \mathbb{R}, N, \text{U}(\mathcal{C})) \) and the modular obstruction \( \text{Ob}_m(\alpha) \) take simpler forms. These together with examples will be discussed in the forthcoming paper, [KtT2].

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§0. Introduction

With the successful completion of the cocycle conjugacy classification of amenable discrete group actions on AFD factors by many hands over more than two decades, [C3, J1, JT, O, ST1, ST2, KwST, KtST1], it is only naturally to consider the outer conjugacy classification of amenable discrete group outer actions on AFD factors. In fact, the work on the program has been already started by the pioneering works of Connes, [Cnn 3, 4, 6], Jones [J1] and Ocneanu [Ocn]. In this article, we complete the outer conjugacy classification of discrete amenable group outer actions on AFD factors. The cases of type I, II$_1$ and II$_\infty$ with additional technical assumption were already completed by Jones, [J1], and Ocneanu, [Ocn], so the case of type III will be mainly considered although the technical assumption in the case of type II$_\infty$ placed in the work of Ocneanu [Ocn] must be removed.

As in the case of the cocycle conjugacy classification, we first associate invariants which are intrinsic to any factor $M$, the flow of weights, the modulus, the characteristic square and the modular obstruction $\text{Ob}_m(M)$. Then the outer conjugacy invariants are given by the pull back of these intrinsic quantities of the factor by the outer action. To be more precise, let $M$ be a separable factor. Associated with $M$ is the characteristic square:

\[
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which is equivariant under Aut(M) × R. The middle vertical exact sequence is the source of the intrinsic invariant:

$$\Theta(M) \in \Lambda_{\text{mod} \times \theta}(\text{Aut}(M) \times \mathbb{R}, \text{Cnt}_r(M), \mathcal{U}(\mathcal{C})).$$

To avoid heavy notations and to see the essential mechanism governing the above exact characteristic square, let us consider the situation that a group H equipped with a distinguished pair of normal subgroups M ⊂ L ⊂ H which acts on the ergodic flow {C, R, θ}, i.e., the action α of H on C commutes with the flow θ. Assume that the normal subgroup L does not act on C, i.e., L ⊂ Ker(α), so that the action α factors through the both quotient groups Q = H/L and G = H/M. In the case that H = Aut(M), the groups L and M stand for Cnt_r(M) and Int(M), therefore Q = Out_r,θ(M) and G = Out(M).

Let \( \tilde{H}, \tilde{G} \) and \( \tilde{Q} \) denote respectively the product groups H × R, G × R and Q × R. We denote the unitary group \( \mathcal{U}(\mathcal{C}) \) simply by A for the simplicity.

In the case that H = Aut(M), then we require appropriate Borelness for mappings. But Q can fail to have a reasonable Borel structure, so we treat Q as a discrete group. On the product group \( \tilde{Q} = Q \times \mathbb{R} \), we consider the product Borel structure as well as the product topology.

In this circumstance, we will see that each characteristic cocycle \( (\lambda, \mu) \in Z_\alpha(\tilde{H}, L, A) \) gives rise to an \( \tilde{H} \)-equivariant exact square:

\[
\begin{array}{ccccccc}
1 & 1 & 1 & \downarrow & & & \\
\downarrow & & & & \downarrow & & \\
1 & \rightarrow & T & \rightarrow & A & \overset{\partial}{\rightarrow} & B & \rightarrow & 1 \\
\downarrow & & & \downarrow & & & & & \\
1 & \rightarrow & U = E^\theta & \rightarrow & E & \overset{\partial_\theta}{\rightarrow} & Z & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & K & \rightarrow & L & \overset{\partial}{\rightarrow} & H & \rightarrow & 1 \\
\downarrow & & & \downarrow & & & \downarrow & & \\
1 & 1 & 1 & & & & & \\
\end{array}
\]

with \( E = A \times_\mu L \). The subgroup K of L is normal in H and depends on the characteristic invariant \( \chi = [\lambda, \mu] \in \Lambda_\alpha(\tilde{H}, L, A) \). We denote it by \( K(\chi) \)
or \( K(\lambda, \mu) \) to indicate the dependence of \( K \) on \( \chi \) or \( (\lambda, \mu) \). We then define a subgroup \( Z_\alpha(\tilde{H}, L, M, A) \) of \( Z_\alpha(\tilde{H}, L, A) \) to be the subgroup consisting of those \( (\lambda, \mu) \in Z_\alpha(\tilde{H}, L, A) \) such that \( M \subset K(\lambda, \mu) \) and \( \Lambda_\alpha(\tilde{H}, L, M, A) \) to be

\[
\Lambda_\alpha(\tilde{H}, L, M, A) = \{ \chi \in \Lambda_\alpha(\tilde{H}, L, A) : M \subset K(\chi) \}.
\]

In order to study the outer conjugacy class of an outer action \( \alpha \) of \( G \) on a factor \( M \), we need to fix a cross-section \( s : Q \to G \) of the quotient map \( \pi : G \to Q \) with kernel \( N = L/M = \text{Ker}(\pi) \) and also to restrict the group of \( A \)-valued 3-cocycles on \( \tilde{Q} \) to the group \( Z^3_{3,s}(\tilde{Q}, A) \) of standard cocycles and a smaller coboundary group:

\[
B^3_{3,s}(\tilde{Q}, A) = \partial_{\tilde{Q}}(B^3_3(Q, A))
\]

and to form the quotient group:

\[
H^3_{3,s}(\tilde{Q}, A) = Z^3_{3,s}(\tilde{Q}, A) / B^3_{3,s}(\tilde{Q}, A).
\]

The cross-section \( s \) gives rise to a link between the group \( H^3_{3,s}(\tilde{Q}, A) \) and the group \( \text{Hom}_G(N, H^1_\theta) \) of equivariant homomorphisms which in turn allows us to define the fiber product:

\[
H^3_{3,s}(G \times \mathbb{R}, N, A) = H^3_{3,s}(\tilde{Q}, A) \ast s \text{Hom}_G(N, H^1_\theta).
\]

We then show that this group falls in the modified Huebshmann - Jones - Ratcliffe exact sequence which sits next to the Huebshmann - Jones - Ratcliffe exact sequence:

\[
\begin{array}{c}
\text{1} & \text{1} \\
\downarrow & \downarrow \\
H^1(Q, \mathbb{T}) & \longrightarrow & H^1(G, \mathbb{T}) \\
\inf & \longrightarrow & \longrightarrow \\
\downarrow & \downarrow & \downarrow \\
\text{Hom}(H, \mathbb{T}) & \longrightarrow & \text{Hom}(H, \mathbb{T}) \\
\text{res} & \longrightarrow & \text{res}
\end{array}
\]
An action $\alpha$ of $H$ on a factor $M$ with $M = \alpha^{-1}(\text{Int}(M))$ and $L = \alpha^{-1}(\text{Cnt}_r(M))$ gives rise naturally to the modular characteristic invariant $\chi_m(\alpha) \in \Lambda_\alpha(\tilde{H}, L, M, A)$ and

$$\text{Ob}_m(\alpha) = \delta(\chi_m(\alpha)) \in H^\text{out}_{\alpha,s}(G \times \mathbb{R}, N, A) = H^3_{\alpha,s}(\tilde{Q}, A) \ast_s \text{Hom}_G(N, H^1_\theta).$$

The cohomology element $\text{Ob}_m(\alpha)$ will be called the modular obstruction of the outer action $\alpha$ of $G$.

In the original setting, $H = \text{Aut}(M)$, the corresponding $\text{Ob}_m(\alpha)$ will be denoted by $\text{Ob}_m(M)$ and called the intrinsic modular obstruction of $M$.

In this article, we will prove the following outer conjugacy classification:

**Theorem.** i) If $\alpha$ is an outer action of a group $G$ on a factor $M$, then the pair of the modulus $\text{mod}(\alpha) \in \text{Hom}(G, \text{Aut}_\theta(\mathcal{C}))$ of $\alpha$ and the modular obstruction:

$$\text{Ob}_m(\alpha) \in H^\text{out}_{\alpha,s}(G \times \mathbb{R}, N, A)$$

is an outer conjugacy invariant of $\alpha$ with $N = \alpha^{-1}(\text{Cnt}_r(M))$. 
ii) If \( G \) is a countable discrete amenable group and \( \mathcal{M} \) is an approximately finite dimensional factor, then the pair \((\text{Ob}_m(\alpha), \text{mod}(\alpha))\) is a complete invariant for the outer conjugacy class of \( \alpha \).

ii) With a countable discrete amenable group \( G \) and an AFD factor \( \mathcal{M} \) fixed, every triplet occurs as the invariant of an outer action of \( G \) on the \( \mathcal{M} \).

Contrary to the case of the cocycle conjugacy classification, the outer conjugacy classification of outer actions of a countable discrete amenable group on an AFD factor will be carried out by a unified approach without splitting the case base on the type of the base factor. Indeed, the theory is very much cohomological and therefore algebraic. Nevertheless, our classification does not fall in the traditional classification doctrine of Mackey, we will follow the strategy proposed in an earlier work of Katayama - Sutherland - Takesaki, [KtST1]. Namely, we first introduce a standard Borel structure to the space of outer actions of a countable discrete group \( G \) on a separable factor \( \mathcal{M} \) and associate functorially invariants in the Borel fashion.

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§1. Preliminary and Notations

Let \( \mathcal{M} \) be a separable factor and \( G \) a separable locally compact group. We mean by an outer action \( \alpha \) of \( G \) on \( \mathcal{M} \) a Borel map from \( G \) into the group \( \text{Aut}(\mathcal{M}) \) of automorphisms of \( \mathcal{M} \) such that

\[
\alpha_g \circ \alpha_h \equiv \alpha_{gh} \mod \text{Int}(\mathcal{M}), \quad g, h \in G, \tag{1.1}
\]

where \( \text{Int}(\mathcal{M}) \) means the group of inner automorphisms. If in addition the following happens

\[
\alpha_g \not\equiv \text{id} \mod \text{Int}(\mathcal{M}) \quad \text{unless} \quad g = 1,
\]
then the outer action $\alpha$ is called free.

Remark. One should not confused an outer action with a free action of $G$ on $\mathcal{M}$. A free action of a discrete group $G$ is by definition that a homomorphism $\alpha : g \in G \mapsto \alpha_g \in \text{Aut}(\mathcal{M})$ such that $\alpha_g \notin \text{Int}(\mathcal{M}), g \neq 1$. There is no good definition for the freeness of an action $\alpha$ of a continuous group $G$. Although one might take the triviality $\mathcal{M}' \cap \mathcal{M} \rtimes_{\alpha} G = \mathcal{C}$ of the relative commutant of the original factor $\mathcal{M}$ in the crossed-product as the definition of the freeness of $\alpha$, which is an easy consequence of the freeness of $\alpha$ in the discrete case.

Let $\{\mathcal{M}, \mathcal{R}, \theta, \tau\}$ be the non-commutative flow of weights on $\mathcal{M}$ in the sense of Falcone - Takesaki, [FT1], and $\{\mathcal{C}, \mathcal{R}, \theta\}$ be the Connes - Takesaki flow of weights, [CT], so that $\mathcal{C}$ is the center of $\mathcal{M}$ and the flow $\{\mathcal{C}, \mathcal{R}, \theta\}$ is the restriction of the non-commutative flow of weights. The von Neumann algebra $\tilde{\mathcal{M}}$ is generated by $\mathcal{M}$ together with one parameter unitary groups $\{\varphi^{it} : \varphi \in \mathfrak{W}_0(\mathcal{M}), t \in \mathbb{R}\}$, where $\mathfrak{W}_0(\mathcal{M})$ means the set of all faithful semi-finite normal weights on $\mathcal{M}$ and $\{\varphi^{it}\}$'s are related by the Connes cocycle derivatives:

$$\varphi^{it}\psi^{-it} = (D\varphi : D\psi)_t, \quad \varphi, \psi \in \mathfrak{W}_0(\mathcal{M}), \ t \in \mathbb{R}.$$ (1.2)

The non-commutative flow $\theta$ then acts on $\tilde{\mathcal{M}}$ by

$$\theta_s(x) = x, \quad x \in \mathcal{M};$$
$$\theta_s(\varphi^{it}) = (e^{-s}\varphi)^{it}, \quad \varphi \in \mathfrak{W}_0(\mathcal{M}), \ s, t \in \mathbb{R}.$$ (1.3)

Associated with the non-commutative flow of weights is the extended unitary group $\tilde{\mathcal{U}}(\mathcal{M}) = \{u \in \mathcal{U}(\tilde{\mathcal{M}}) : u\mathcal{M}u^* = \mathcal{M}\}$. Each $u \in \tilde{\mathcal{U}}(\mathcal{M})$ gives rise to an automorphism $\tilde{\text{Ad}}(u) = \text{Ad}(u)|_\mathcal{M}$ of $\mathcal{M}$. The set of such automorphisms will be denoted by $\text{Cnt}_r(\mathcal{M})$ and it is a normal subgroup of $\text{Aut}(\mathcal{M})$.

An important property of the non-commutative flow of weights is the relative commutant of $\mathcal{M}$ in $\tilde{\mathcal{M}}$:

$$\mathcal{M}' \cap \tilde{\mathcal{M}} = \mathcal{C}.$$ 

A continuous one parameter family $\{u_s \in \mathfrak{U}(\tilde{\mathcal{M}}) : s \in \mathbb{R}\}$ is called $\theta$-one cocycle if

$$u_{s+t} = u_s \theta_s(u_t), \quad s, t \in \mathbb{R}.$$. 
The set of all $\theta$-one cocycles in $\mathcal{C}$ form a group relative to the pointwise product in $\mathcal{C}$ and is denoted by $\mathbb{Z}_1^{\theta}$.

The action $\theta$ on $\tilde{M}$ is known to be stable in the sense that every $\theta$-one cocycle $\{u_s\}$ is coboundary, i.e., there exists $v \in \mathcal{U}(\tilde{M})$ such that

$$u_s = \theta_s(v)v^* = (\partial v)_s, \quad s \in \mathbb{R}.\$$

The set $\{\partial v : v \in \mathcal{U}(\mathcal{C})\}$ of coboundaries is a subgroup of $\mathbb{Z}_1^{\theta}$ and denoted by $\mathcal{B}_1^{\theta}$. The quotient group $H_1^{\theta} = \mathbb{Z}_1^{\theta}/\mathcal{B}_1^{\theta}$ is an abelian group which is the first cohomology group of the flow of weights. The elements of extended unitary group $\tilde{\mathcal{U}}(M)$ are then characterized by the fact that for $u \in \mathcal{U}(\tilde{M})$:

$$u \in \tilde{\mathcal{U}}(M) \iff (\partial u)_t \in \mathcal{C}, \quad t \in \mathbb{R}.\$$

Therefore the map $\partial : v \in \tilde{\mathcal{U}}(M) \mapsto \partial v \in \mathbb{Z}_1^{\theta}$ is surjective. An important fact about this map is that the exact sequence

$$1 \to \mathcal{U}(M) \to \tilde{\mathcal{U}}(M) \to \mathbb{Z}_1^{\theta} \to 1$$

splits equivariantly as soon as a faithful semi-finite normal weight $\varphi$ is fixed, i.e., to each faithful semi-finite normal weight $\varphi$ there corresponds a homomorphism $b_\varphi : c \in \mathbb{Z}_1^{\theta} \mapsto b_\varphi(c) \in \tilde{\mathcal{U}}(M)$ such that

$$\tilde{\text{Ad}}(b_\varphi(c)) = \sigma_c^\varphi \quad \text{if } \varphi \text{ is dominant, } c \in \mathbb{Z}_1^{\theta};$$

$$b_{\alpha(\varphi)}(c) = \alpha \circ b_\varphi \circ \alpha^{-1}(c), \quad c \in \mathbb{Z}_1^{\theta}, \quad \alpha \in \text{Aut}(M); \tag{1.4}$$

$$(D\varphi : D\psi)_c = b_\psi(c)b_\varphi(c^*), \quad \varphi, \psi \in \mathcal{W}_0(M).$$

This was proven by Falcone - Takesaki [FT2] among other things.$^1$

The group $\text{Cnt}_r(M)$ of “extended modular” automorphisms is a normal subgroup of $\text{Aut}(M)$ but not closed in the case of type $\mathbb{III}_0$. Nevertheless it is a Borel subgroup so that its inverse image $N(\alpha) = \alpha^{-1}(\text{Cnt}_r(M))$, denoted

---

$^1$The coboundary operation in [FT2] was defined differently as $\partial u_s = u \theta_s(u^*)$, so in our case the map $b_\varphi : c \in \mathbb{Z}_1^{\theta} \mapsto b_\varphi(c) \in \tilde{\mathcal{U}}(M)$ behaves as described here.
simply by $N$ in the case that $\alpha$ is fixed, is a normal Borel subgroup of the original group $G$. Thus the quotient group $Q = G/N$ cannot be expected to be a good topological group in general unless $G$ is discrete. Thus we consider mainly discrete groups. Other than the definition of the invariants of $\alpha$ we do not have any substantial result on continuous groups any way at the moment. Interested readers are challenged to go further in the direction of the cocycle conjugacy problem of one parameter automorphism groups: clearly the very first step toward the continuous group actions on a factor.

§2. Modified Huebschmann Jones Ratcliffe Exact Sequence

We recall the Huebschmann - Jones - Ratcliffe exact sequence, [Hb, J1, Rc]:

$$1 \longrightarrow H^1(Q, A) \xrightarrow{\pi^*} H^1(G, A) \longrightarrow H^1(N, A)^G \longrightarrow$$

$$\longrightarrow H^2(Q, A) \longrightarrow H^2(G, A) \longrightarrow \Lambda(G, N, A) \xrightarrow{\delta} H^3(Q, A) \xrightarrow{\pi^*} H^3(G, A),$$

where either i) $G$ is a separable locally compact group acting on a separable abelian von Neumann algebra $\mathcal{C}$ with $A = \mathcal{U}(\mathcal{C})$ and $N$ a Borel normal subgroup, or ii) $G$ is a discrete group and $N$ a normal subgroup. We need the second case because the automorphism group $\text{Aut}(M)$ of a separable factor $M$ and the normal subgroup $\text{Cnt}_r(M)$ will be taken as the groups $G$ and $N$. If $\text{Cnt}_r(M)$ is not closed as in the case of an AFD factor $M$, then the quotient group $\text{Aut}(M)/\text{Cnt}_r(M)$ does not have a good topological property beyond the discrete group structure.

We are interested in the exactness at $H^3_\alpha(Q, A)$. In particular, we need an explicit construction of $[\lambda, \mu] \in \Lambda(G, N, A)$ such that $\delta[\lambda, \mu] = [c]$ for those $c \in Z^3_\alpha(Q, A)$ with $\pi^*(c) \in B^3_\alpha(G, A)$ in terms of a cochain $\mu \in C^2(G, A)$ with $\pi^*(c) = \partial^2_C(\mu)$. In the situation where Polish topologies are available on $G, N$ and $Q$, we assume or demand that all cocycles and cochains are Borel and when it is appropriate equalities are considered modulo null sets relative to the relevant measures. This kind of restrictions requires us nailing down several objects explicitly rather than relying on mere existence of the required objects through abstract mechanism.

Given a cocycle $(\lambda, \mu) \in Z_\alpha(G, N, A)$, we have a $G$-equivariant exact se-
Choose a cross-section $s_\pi$ of $\pi:

\begin{align*}
1 & \longrightarrow N \longrightarrow G \stackrel{\pi}{\longleftarrow} Q \longrightarrow 1,
\end{align*}

which generates the cocycle $n_N \in Z(Q, N):

\begin{align*}
s_\pi(p) s_\pi(q) &= n_N(p, q) s_\pi(pq), \quad p, q \in Q.
\end{align*}

Then we get the associated three cocycle $c_E \in Z_3^\alpha(Q, A)$ given by the following:

\begin{align*}
c_E(p, q, r) &= (\partial_Q(s_\pi \cdot n_N))(p, q, r) \\
&= \alpha_{s_\pi(p)}(s_\pi(n_N(q, r))) s_\pi(n_N(p, qr)) s_\pi(n_N(p, q)) s_\pi(pq, r))^{-1},
\end{align*}

which is expressed in terms of $(\lambda, \mu)$ and $n_N$ directly:

\begin{align*}
c^{\lambda, \mu}(p, q, r) &= \lambda(s_\pi(p) n_N(q, r) s_\pi(p)^{-1}, s_\pi(p)) \\
&\quad \times \mu(s_\pi(p) n_N(q, r) s_\pi(p)^{-1}, n_N(p, qr)) \\
&\quad \times \mu(n_N(p, q), n_N(p, qr))^{-1}. \tag{2.2}
\end{align*}

This can be shown by a direct computation from the definition, which we leave to the reader. We denote the cohomology class $[c_E] \in H_3^\alpha(Q, A)$ of $c_E$ by $\delta([\lambda, \mu])$, which does not depends on the choices of the cross-sections $s_\pi$ and $s_\pi$ but only on the cohomology class of $(\lambda, \mu)$. 
Lemma 2.1. The image $\delta(\Lambda(G,N,A))$ in $H^3(Q,A)$ consists of precisely those $[\xi] \in H^3_a(Q,A)$ such that $\pi^*([\xi]) = 1$ in $H^3_a(G,A)$. More precisely if a cochain $\mu \in C^2_a(G,A)$ gives $\partial G \mu = \pi^* (\xi)$, then

$$\lambda(m,g) = \mu(g,g^{-1}mg)\mu(m,g)^{-1}, \quad m \in N, g \in G. \quad (2.3)$$

together with the restriction $(i_N)_*(\mu)$ gives an element of $Z(G,N,A)$ such that $[\xi] = \delta[\lambda,\mu]$ where $i_N$ is the embedding of $N$ into $G$, i.e.,

$$1 \longrightarrow N \xrightarrow{i_N} G \xrightarrow{\pi} Q \longrightarrow 1.$$ 

The cochain $f \in C^2_a(Q,A)$ given by

$$f(p,q) = \mu(s_\pi(p),s_\pi(q))\mu(n_N(p,q),s_\pi(pq))^{-1} \in A, \quad (2.4)$$

relates the original cocycle $\xi \in Z^3_a(Q,A)$ and the new cocycle $c^{\lambda,\mu}$ in the following way:

$$\xi = (\partial_Q f)c^{\lambda,\mu}.$$ 

Proof. First we construct a $G$-equivariant exact sequence:

$$E: \quad 1 \longrightarrow A \longrightarrow E \longrightarrow N \longrightarrow 1$$

from the data $\pi^*(\xi) = \partial G(\mu) \in B^3_a(G,A)$ with $\mu \in C^2_a(G,A)$. Let $B = A^Q$ be the abelian group of all $A$-valued, (Borel if applicable), functions on $Q$ on which $Q$ acts by:

$$(\alpha_p(b))(q) = \alpha_p(bqp), \quad p,q \in Q, \quad b \in B = A^Q.$$ 

Viewing $A$ as the subgroup of $B$ consisting of all constant functions, we get an exact sequence:

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 1.$$ 

By the cocycle identity, we have

$$\xi(q,r,s) = \alpha_p^{-1} \left( \xi(pq,r,s)\xi(p,qr,s)^{-1}\xi(p,q,rs)\xi(p,q,r)^{-1} \right)$$
gives the following with \( \eta(p, q, r) = \alpha_p^{-1}(\xi(p, q, r)) \) viewed as an element of \( C^2(Q, B) \) as a function of \( p \):

\[
\xi(q, r, s) = \alpha_q(\eta(pq, r, s))\eta(p, qr, s)^{-1}\eta(p, qrs)\eta(p, q, r)^{-1}
\]

= Constant in \( p \).

Hence \( j_*(\partial Q \eta) = \partial Q(j_*(\eta)) = 1 \), so that \( j_*(\eta) \in Z^2(Q, C) \) and therefore we get an exact sequence based on \( j_*(\eta^{-1}) \):

\[
1 \longrightarrow C \longrightarrow D \xleftarrow{\sigma} Q \longrightarrow 1,
\]

with \( D = C \rtimes_{\alpha, j_*(\eta^{-1})} Q \) and a cross-section \( s_\sigma : p \in Q \mapsto (1, p) \in D \) such that

\[
s_\sigma(p)s_\sigma(q) = j(\eta(p, q)^{-1})s_\sigma(pq), \quad p, q \in Q.
\]

With \( \mu \in C^2(G, A) \) such that \( \pi^*(\xi) = \partial G \mu \), we get

\[
\xi(\pi(g), \pi(h), \pi(k)) = \alpha_g(\mu(h, k))\mu(gh, k)^{-1}\mu(g, hk)\mu(g, h)^{-1}
\]

= \( \alpha_g(\eta(p\pi(g), \pi(h), \pi(k)))\eta(p, \pi(gh), \pi(k))^{-1} \times \eta(p, \pi(g), \pi(hk))\eta(p, \pi(g), \pi(h))^{-1} \).

Thus, \( \zeta = i_*(\mu)\pi^*(\eta^{-1}) \in Z^2(G, B) \), which allows us to create an exact sequence:

\[
1 \longrightarrow B \longrightarrow F \xleftarrow{\tilde{\sigma}} G \longrightarrow 1
\]

with \( F = B \rtimes_{\alpha, \zeta} G \) and the cross-section \( s_{\tilde{\sigma}} \), given by \( s_{\tilde{\sigma}}(g) = (1, g), g \in G \), such that

\[
s_{\tilde{\sigma}}(p)s_{\tilde{\sigma}}(q) = j(\zeta(p, q))s_{\tilde{\sigma}}(pq).
\]

Since \( j_*(\zeta) = j_*(\eta^{-1}) \), the next diagram is commutative:

\[
1 \xrightarrow{j} B \longrightarrow F \xleftarrow{\tilde{\sigma}} G \longrightarrow 1
\]

\[
1 \xrightarrow{\pi} C \longrightarrow D \xleftarrow{\sigma} Q \longrightarrow 1
\]
With $E = \text{Ker}(j \times \pi)$, we get the expanded commutative diagram:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & \rightarrow & A & \rightarrow & E & \rightarrow & N & \rightarrow & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & \rightarrow & B & \rightarrow & F & \rightarrow & G & \rightarrow & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & \rightarrow & C & \rightarrow & D & \rightarrow & Q & \rightarrow & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & 1 & 1
\end{array}
\]

The construction of the above diagram came equipped with cross-sections, $s_\tilde{\sigma}$ and $s_\sigma$ such that

\[s_\sigma \circ \pi = (j \times \pi) \circ s_\tilde{\sigma}.
\]

We now look at the extension $E$, which is the kernel $\text{Ker}(j \times \pi)$. An element $(b, g) \in F$ belongs to $E$ if and only if $j(b) = 1$ and $\pi(g) = 1$; if and only if $(b, g) \in A \times N$. For $m, n \in N$ we have

\[s_\tilde{\sigma}(m)s_\tilde{\sigma}(n) = \mu(m, n)\eta(\pi(m), \pi(n))^{-1}s_\tilde{\sigma}(mn) = \mu(m, n)s_\tilde{\sigma}(mn),
\]

so that we get

\[E = A \times_\mu N.
\]

As $E$ is a normal subgroup of $F$, each $s_\tilde{\sigma}(g), g \in G$, normalizes $E$. Since the value of the two-cocycle $\zeta$ belongs to $B$ each element of which commutes with $A$ and $s_\tilde{\sigma}(N)$, the restriction of $\alpha_g = \text{Ad}(s_\tilde{\sigma}(g))$ to $E$ gives rise to an
honest action of $G$ which is consistent with the original action of $G$ on $A$. Thus we obtain a $G$-equivariant exact sequence:

$$
E : 1 \longrightarrow A \longrightarrow E \xrightarrow{\tilde{\sigma}|_E} N \longrightarrow 1.
$$

Now we compare the original $[\xi]$ and $[c_E]$ in $H^3(Q, A)$. The cross-section $s_{\tilde{\sigma}}$ takes $N$ into $E$ so that its restriction $s_{\tilde{\sigma}}|_N$ is a cross-section for $\tilde{\sigma}|_E$. The associated three cocycle $c_E \in Z^3(Q, A)$ is obtained by:

$$
c_E(p, q, r) = \partial_Q(s_{\tilde{\sigma}} \circ n_N)(p, q, r).
$$

Consider the map $s = s_{\tilde{\sigma}} \circ s_\pi : Q \mapsto E$ and compute

$$(\partial_Q s)(p, q) = s_{\tilde{\sigma}}(s_\pi(p))s_{\tilde{\sigma}}(s_\pi(q))s_{\tilde{\sigma}}(s_\pi(pq))^{-1}
= \mu(s_\pi(p), s_\pi(q))\eta(p, q)^{-1}s_{\tilde{\sigma}}(s_\pi(p)s_\pi(q))s_{\tilde{\sigma}}(s_\pi(pq))^{-1}
= \mu(s_\pi(p), s_\pi(q))\eta(p, q)^{-1}s_{\tilde{\sigma}}(n_N(p, q)s_\pi(pq))s_{\tilde{\sigma}}(s_\pi(pq))^{-1}
= \mu(s_\pi(p), s_\pi(q))\eta(p, q)^{-1}\mu(n_N(p, q), s_\pi(pq))^{-1}\eta(\pi(n_N(p, q)), pq)
\times s_{\tilde{\sigma}}(n_N(p, q))s_{\tilde{\sigma}}(s_\pi(pq))s_{\tilde{\sigma}}(s_\pi(pq))^{-1}
= \mu(s_\pi(p), s_\pi(q))\mu(n_N(p, q), s_\pi(pq))^{-1}\eta(p, q)^{-1}s_{\tilde{\sigma}}(n_N(p, q)).
$$

Thus with

$$
f(p, q) = \mu(s_\pi(p), s_\pi(q))\mu(n_N(p, q), s_\pi(pq))^{-1} \in A,
$$

we get

$$
1 = (\partial_Q \partial_Q s)(p, q, r) = \partial_Q f(p, q, r)\partial_Q \eta(p, q, r)^{-1}\partial_Q(s_{\tilde{\sigma}} \circ n_N)(p, q, r)
= \partial_Q f(p, q, r)\xi(p, q, r)^{-1}c_E(p, q, r),
$$

so that

$$
[\xi] = [c_E] \in \delta(\Lambda(G, N, A)).
$$
Now we compute the associated characteristic cocycle $(\lambda, \mu) \in Z(G, N, A)$:
\[
\lambda(m, g)s_\sigma(m) = \alpha_g(s_\sigma(g^{-1}mg)) = s_\sigma(g)s_\sigma(g^{-1}mg)s_\sigma(g)^{-1} \\
= \mu(g, g^{-1}mg)\eta(\pi(g), \pi(g^{-1}mg))^{-1}s_\sigma(mg)s_\sigma(g)^{-1} \\
= \mu(g, g^{-1}mg)\mu(m, g)^{-1}\eta(\pi(m), \pi(g))s_\sigma(m)s_\sigma(g)^{-1} \\
= \mu(g, g^{-1}mg)\mu(m, g)^{-1}s_\sigma(m),
\]
which proves (2.3).

As we will need only the construction of a $G$-equivariant short exact sequence from the cochain $\mu \in C_\alpha^2(G, A)$ with $\pi^*(\xi) = \partial_G \mu$, we leave the proof for the converse to the reader. It is a direct computation.

In the sequel, the group $G$ appears as the quotient group of another group $H$ by a normal subgroup $M$, i.e., $G = H/M$. Let $\pi_G$ be the quotient map $\pi_G : H \mapsto G$. Set $L = \pi^{-1}(N)$ and

\[
\tilde{H} = H \times \mathbb{R}, \quad \tilde{G} = G \times \mathbb{R}, \quad \text{and} \quad \tilde{Q} = Q \times \mathbb{R}. \tag{2.5}
\]

Whenever an action $\alpha$ of $\tilde{H}$ on a group $E$ is given, we denote the restriction of $\alpha$ to $\mathbb{R}$ by $\theta$. When an action $\alpha$ of the group $H$ is given and the cross-sections $s_\alpha : g \in G \mapsto s_\alpha(g) \in H$ for $\pi_G$, $s : p \in Q \mapsto s(p) \in G$ for the quotient map $\pi : g \in G \mapsto \pi(g) = gN \in Q$ and $\tilde{s} : p \in Q \mapsto \tilde{s}(p) = s_\alpha(s(p)) \in H$ for the map $\tilde{\pi} = \pi \circ \pi_G$ are specified, we use the abbreviated notations:

\[
\alpha_g = \alpha_{s_\alpha(g)}, \quad g \in G; \quad \alpha_p = \alpha_{\tilde{s}}(p), \quad p \in Q,
\]

which satisfy:

\[
\alpha_g \circ \alpha_h = \alpha_{n_M(g, h)} \circ \alpha_g, \quad g, h \in G; \\
\alpha_p \circ \alpha_q = \alpha_{n_L(p, q)} \circ \alpha_q, \quad p, q \in Q, \tag{2.6}
\]

where

\[
n_M(g, h) = s_\alpha(g)s_\alpha(h)s_\alpha(gh)^{-1} \in M, \quad g, h \in G; \\
n_L(p, q) = \tilde{s}(p)\tilde{s}(q)\tilde{s}(pq)^{-1} \in L, \quad p, q \in Q. \tag{2.7}
\]

We examine the last half of the HJR-exact sequence:

\[
H^2_\alpha(\tilde{H}, A) \to \text{res} \to \Lambda_\alpha(\tilde{H}, L, A) \overset{\delta_{\text{HJR}}}{\to} H^3(\tilde{Q}, A) \overset{\text{inf}}{\to} H^3_\alpha(\tilde{H}, A).
\]

First we show:
Lemma 2.2. For each \( \mu' \in Z^2_\alpha(\tilde{H}, A) \), there is an element \( \mu \in Z^2_\alpha(\tilde{H}, A) \) such that \( \mu' \) and \( \mu \) are cohomologous and \( \mu \) satisfies the condition:

\[
\mu(\tilde{h}, \tilde{k}) = \mu_H(h, k)\alpha_h(d_\mu(s; k)), \quad \tilde{h} = (h, s), \tilde{k} = (k, t) \in \tilde{H} = H \times \mathbb{R}, \quad (2.8)
\]

where

\[
\mu_H \in Z^2_\alpha(H, A), \quad d(\cdot ; h) \in Z^1_\alpha(\mathbb{R}, A);
\]

\[
\theta_s(\mu_H(h, k))\mu_H(h, k)^* = d(s; h)\alpha_h(d(s; k))d(s; hk)^*;
\]

equivalently

\[
\partial_\theta \mu_\iota = \partial_H d. \quad (2.9')
\]

Proof. We recall \( H^2_\alpha(\mathbb{R}, A) = \{1\} \). So we may and do assume that \( \mu'(s, t) = 1, s, t \in \mathbb{R} \). Consider the group extension:

\[
1 \longrightarrow A \xrightarrow{i} F = A \times \mu' \xrightarrow{j} \tilde{H} \longrightarrow 1.
\]

The assumption on the restriction \( \mu'|_{\mathbb{R} \times \mathbb{R}} \) allows us to find a one parameter subgroup \( \{u(s) : s \in \mathbb{R}\} \) of \( F \) with \( j(u(s)) = s, s \in \mathbb{R} \). Choose a cross-section \( s'_j : h \in \tilde{H} \mapsto s'_j(h) \in F \) of the map \( j \) such that

\[
s'_j(h)s'_j(k) = \mu'(h, k)s'_j(hk), \quad h, k \in H.
\]

Now set

\[
s_j(h, s) = s'_j(h)u(s), \quad (h, s) \in \tilde{H}.
\]

Now we compute the associated 2-cocycle \( \mu' \):

\[
\mu(h, s; k, t) = s_j(h, s)s_j(k, t)s_j(hk, s + t)^{-1}
\]

\[
= s'_j(h)u(s)s'_j(k)u(t)s'_j(hk)u(s + t)^{-1}
\]

\[
= s'_j(h)\mu'(s; k)s'_j(k, s)u(t)s'_j(hk)u(s + t)^{-1}
\]

\[
= s'_j(h)\mu'(s; k)\mu'(k; s)^{-1}s'_j(k)u(s)u(t)s'_j(hk)u(s + t)^{-1}
\]

\[
= \alpha_h\left(\mu'(s; k)\mu'(k; s)^{-1}\right)s'_j(h)s'_j(k)s'_j(hk)^{-1}
\]

\[
= \alpha_h\left(\mu'(s; k)\mu'(k; a)^{-1}\right)\mu'(h; k)
\]
for each \((h, s), (k, t) \in \tilde{H}\). Setting

\[
\mu_H = \mu'|_H \quad \text{and} \quad d(s; h) = \mu'(s; h)\mu'(h; s)\]

we obtain the first formula and also

\[
d(s + t; h) = d(s; h)\theta_s(d(t; h)), \quad s, t \in \mathbb{R}, h \in H.
\]

We next check the second identity which follows from the cocycle identity for \(\mu\) as seen below:

\[
1 = \alpha_\tilde{g}(\mu(\tilde{h}, \tilde{k}))\mu(\tilde{g}\tilde{h}, \tilde{k})^*\mu(\tilde{g}, \tilde{h})\mu(\tilde{g}, \tilde{h})^*,
\]

\[
\tilde{g} = (g, s), \tilde{h} = (h, t), \tilde{k} = (k, u) \in \tilde{H},
\]

\[
= \alpha_\tilde{g}\left(\alpha_h(d_\mu(s; k))\mu_t(h, k)\right)\alpha_g(h(s + t; k)^*\times \mu_t(gh, k)^*\alpha_g(d_\mu(s; hk))\mu_t(g, hk)\alpha_g(d_\mu(s; h)^*)\mu_t(g, h)^*
\]

\[
= \alpha_\tilde{g}\left(\theta_s(\alpha_h(d_\mu(s; k))\mu_t(h, k))\alpha_h(d_\mu(s; h)^*\theta_s(d_\mu(t; k)^*))\mu_t(gh, k)^*
\times \alpha_g(d_\mu(s; h))\mu_t(g, h)\alpha_g(d_\mu(s; h)^*)\mu_t(g, h)^*
\]

\[
= \alpha_\tilde{g}\left(\theta_s(\mu_t(h, k))\alpha_h(d_\mu(s; k)^*)\mu_t(gh, k)^*
\times \alpha_g(d_\mu(s; h))\mu_t(g, h)\alpha_g(d_\mu(s; h)^*)\mu_t(g, h)^*
\]

\[
= \alpha_\tilde{g}\left(\theta_s(\mu_t(h, k))\mu_t(h, k)^*\alpha_h(d_\mu(s; k)^*)\mu_t(gh, k)^*
\times \alpha_g(d_\mu(s; h))\mu_t(g, h)\alpha_g(d_\mu(s; h)^*)\mu_t(g, h)^*
\]

\[
= \alpha_\tilde{g}\left(\theta_s(\mu_t(h, k))\mu_t(h, k)^*\alpha_h(d_\mu(s; k)^*)\mu_t(gh, k)^*
\times \alpha_g(d_\mu(s; h))\mu_t(g, h)\alpha_g(d_\mu(s; h)^*)\mu_t(g, h)^*
\]

This proves the lemma.

**Definition 2.3.** A cocycle \(\mu \in Z^2_\alpha(\tilde{H}, A)\) of the form (2.8) will be called standard and \(d_\mu\) and \(\mu_H\) in (1) will be called naturally the \(\mathbb{R}\)-part and the \(H\)-part of the cocycle \(\mu\).
Lemma 2.4. i) If $\mu \in Z_2^2(\tilde{H}, A)$ is standard, then the $(\lambda, \mu) = \text{res}(\mu) \in Z_\alpha(\tilde{H}, L, A)$ is given by the following:

$$
\lambda_\mu(m; \tilde{g}) = \alpha_g(d_\mu(s; g^{-1}mg))\mu_H(g, g^{-1}mg)\mu_H(m; g)^*, \quad \tilde{g} = (g, s) \in \tilde{H}, \ m \in L.
$$

(ii) If $(\lambda, \mu) \in Z_\alpha(\tilde{H}, L, A)$, then $c = c_{\lambda, \mu} = \delta_{\text{HJR}}(\lambda, \mu)$ is given by:

$$
c(\tilde{p}, \tilde{q}, \tilde{r}) = \alpha_p \left( \lambda(n_L(q, r); s) \lambda(\hat{s}(p)n_L(q, r))\hat{s}(p)^{-1}, \hat{s}(p) \right)
\times \mu(\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}, n_L(p, qr))
\times \left\{ \mu(n_L(p, q), n_L(pq, r)) \right\}^*
$$

(2.2')

for each triplet $\tilde{p} = (p, s), \tilde{q} = (q, t), \tilde{r} = (r, u) \in \tilde{Q}$.

(iii) If $(\lambda, \mu) = (\lambda_\mu, \mu) = \text{res}(\mu)$ with $\mu \in Z_3^2(\tilde{Q}, A)$ standard, then the 3-cocycle $c = c_\mu = \delta_{\text{HJR}}(\lambda_\mu, \mu)$ is cobounded by $f \in C_3^2(\tilde{Q}, A)$ given by:

$$
f(\tilde{p}, \tilde{q}) = \mu(\hat{s}(\tilde{p}); \hat{s}(\tilde{q}))^*\mu(n_L(p, q); \hat{s}(pq))
= \mu(\hat{s}(p), s; \hat{s}(q), t)^*\mu(n_L(p, q); \hat{s}(pq), s + t)
= \alpha_p(d_\mu(s; \hat{s}(q))^*\mu_H(\hat{s}(p), \hat{s}(q))^*\mu_H(n_L(p, q); \hat{s}(pq)) \in A,
$$

where $\hat{s}$ is a cross-section of the quotient homomorphism $\hat{\pi}: H \mapsto Q = H/L$.

Proof. i) The 2-cocycle $\mu \in Z_2^2(\tilde{H}, A)$ gives rise to the following commutative diagram of exact sequences equipped with cross-sections $s_H$ and $s_J$:

$$
\begin{array}{ccc}
1 & \longrightarrow & A \\
\| & & \| \\
\| & & \| \\
1 & \longrightarrow & A \\
\end{array}
\quad
\begin{array}{ccc}
F = A \times_\mu \tilde{H} & \longrightarrow & \tilde{H} \\
\quad & \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
E = A \times_\mu L & \longrightarrow & L \\
\end{array}
\quad
\begin{array}{ccc}
1 & \longrightarrow & 1 \\
\| & & \| \\
1 & \longrightarrow & 1 \\
\end{array}
$$

\text{Graph}
The action $\alpha$ of $\tilde{H}$ on $E$ is given by $\alpha_{\tilde{g}} = \text{Ad}(s_{\alpha}(\tilde{g}))|_E$, $\tilde{g} = (g, s) \in H$, viewing $E$ as a submodule of $F$, where the cross-section $s_{\alpha}$ is given by

$$s_{\alpha}(g) = (1, g) \in F = A \times_{\mu} \tilde{H}, \quad g \in H.$$  

The action $\theta$ of $\mathbb{R}$ on $E$ is given by:

$$\theta_s(a, m) = (\theta_s(a) \lambda_{\mu}(m; s), m), \quad s \in \mathbb{R}, (a, m) \in E = A \times_{\mu} L.$$  

Hence $\theta_s(s_j(m)) = \lambda_{\mu}(m; s)s_j(m), m \in L, s \in \mathbb{R}$. Now the cocycle

$$\text{res}(\mu) = (\lambda_{\mu}, \mu) \in Z(\tilde{H}, L, A)$$

is given by

$$\lambda_{\mu}(m, \tilde{g})s_j(m) = \alpha_{\tilde{g}}(s_j(\tilde{g}^{-1}m\tilde{g})) = s_{\alpha}(\tilde{g})s_j(\tilde{g}^{-1}m\tilde{g})s_{\alpha}(\tilde{g})^{-1}$$

$$= s_{\alpha}(\tilde{g})s_{\alpha}(\tilde{g}^{-1}m\tilde{g})s_{\alpha}(\tilde{g})^{-1} = \mu(\tilde{g}, \tilde{g}^{-1}m\tilde{g})s_{\alpha}(m\tilde{g})s_{\alpha}(\tilde{g})^{-1}$$

$$= \mu(\tilde{g}, \tilde{g}^{-1}m\tilde{g})s_{\alpha}(m\tilde{g})s_{\alpha}(\tilde{g})^{-1} = \mu(\tilde{g}, \tilde{g}^{-1}m\tilde{g})s_{\alpha}(m)\mu(\tilde{g}, \tilde{g}^{-1}m\tilde{g})^{-1}$$

$$= \mu(\tilde{g}, \tilde{g}^{-1}m\tilde{g})s_{\alpha}(m)\mu(\tilde{g}, \tilde{g}^{-1}m\tilde{g})^{-1} = \lambda_{\mu}(m, \tilde{g})$$

for $m \in L, \tilde{g} = (g, s) \in \tilde{H}$. As $\mu$ is standard, we get further simplification:

$$\lambda_{\mu}(m; g, s) = \alpha_{\tilde{g}}(d(s; g^{-1}mg))\mu_{\alpha}(g, g^{-1}mg)\mu_{\alpha}(m, g)^*, (g, s) \in \tilde{H}, m \in L.$$  

ii) Now suppose $(\lambda, \mu) \in Z_{\alpha}(\tilde{H}, L, A)$. The associated $A$-valued 3-cocycle $c = c_{\lambda, \mu}$ is given by (2.2) and the formula (2.2') follows from (2.2) and the cocycle identity, see [ST2: (1.7), page 411] for $\lambda$:

$$\lambda(\hat{s}(\tilde{p})n_L(q, r)\hat{s}(\tilde{p})^{-1}; \hat{s}(\tilde{p})) = \lambda(\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}; \hat{s}(p), s)$$

$$= \alpha_{\hat{s}(p)}(\lambda(n_L(q, r)); s))\lambda(\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}; \hat{s}(p))$$

for $\tilde{p} = (p, s), \tilde{q} = (q, t), \tilde{r} = (r, u) \in \tilde{Q}$ because $\hat{s}(\tilde{p}) = (\hat{s}(p), s)$. 
iii) Now assume \((\lambda, \mu) = (\lambda_\mu, \mu)\) with \(\mu \in Z^2_\alpha(\bar{H}, A)\) standard. First we compute the associated cocycle \(c = c_\mu:\)

\[
c_\mu(\bar{p}, \bar{q}, \bar{r}) = \alpha_\mu(\lambda_\mu(n_L(q, r); s))\lambda_\mu(\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}, \hat{s}(p))
\times \mu(\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}, n_L(p, qr))
\times \left\{ \mu(n_L(p, q), n_L(pq, r)) \right\}^* \\
= \alpha_\mu\left( \mu(s; n_L(q, r))\mu(n_L(q, r); s)^* \right)
\times \mu(\hat{s}(p)\hat{s}(p)^{-1}\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}\hat{s}(p))
\times \mu(\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}; \hat{s}(p))^*
\times \mu(\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}; n_L(p, qr))
\times \left\{ \mu(n_L(p, q); n_L(pq, r)) \right\}^* \\
= \alpha_\mu\left( d_\mu(s; n_L(q, r)) \right)\mu_\mu(\hat{s}(p); n_L(q, r))
\times \mu_\mu(\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}; \hat{s}(p))^*
\times \mu_\mu(\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}; n_L(p, qr))
\times \left\{ \mu_\mu(n_L(p, q); n_L(pq, r)) \right\}^*.
\]

We now compute the coboundary of \(f:\)

\[
\partial_Q f(\bar{p}, \bar{q}, \bar{r}) = \alpha_\bar{p}(f(\bar{q}, \bar{r}))f(\bar{p}, \bar{q}, \bar{r})\{ f(\bar{p}, \bar{q})f(\bar{p}, \bar{r}) \}^*
\]

\[
= \alpha_\bar{p}\left( \mu(\hat{s}(\bar{q}); \hat{s}(\bar{r}))^*\mu(n_L(q, r); \hat{s}(\bar{q}, \bar{r})) \right)
\times \mu(\hat{s}(\bar{p}); \hat{s}(\bar{q}, \bar{r}))^*\mu(n_L(p, qr); \hat{s}(\bar{q}, \bar{r}))
\times \left\{ \mu(\hat{s}(\bar{p}); \hat{s}(\bar{q}))^*\mu(n_L(p, q); \hat{s}(\bar{q})) \right\}^*
\times \mu(\hat{s}(\bar{p})\hat{s}(\bar{q})\hat{s}(\bar{r}))\mu(\hat{s}(\bar{p})\hat{s}(\bar{q})\hat{s}(\bar{r}))^*\mu(\hat{s}(\bar{p})\hat{s}(\bar{q}); \hat{s}(\bar{r}))^*
\times \mu(\hat{s}(\bar{p}); n_L(q, r)\hat{s}(\bar{q}, \bar{r}))^*. 
\]
We compute some terms below:

\[
\times \left\{ \mu(\hat{s}(\hat{p}); n_L(q, r))\mu(\hat{s}(\hat{p})n_L(q, r); \hat{s}(\hat{q}\hat{r})) \right\}
\]

\[
\times \mu(\hat{s}(\hat{p}); \hat{s}(\hat{q}\hat{r}))^* \mu(n_L(p, qr); \hat{s}(\hat{p}q\hat{r}))
\]

\[
\times \left\{ \mu(\hat{s}(\hat{p}); \hat{s}(\hat{q}))^* \mu(n_L(p, q); \hat{s}(\hat{p}q)) \right\}
\]

\[
\times \mu(\hat{s}(pq); \hat{s}(r))^* \mu(n_L(pq, r); \hat{s}(pqr)) \right\}^*
\]

\[
= \mu(\hat{s}(\hat{p}); n_L(q, r))\mu(\hat{s}(\hat{p}); \hat{s}(\hat{q})))^* \mu(\hat{s}(\hat{p})\hat{s}(\hat{q}); \hat{s}(\hat{r}))^* \times \mu(\hat{s}(\hat{p}); n_L(q, r))\mu(\hat{s}(\hat{p})n_L(q, r); \hat{s}(\hat{q}\hat{r}))
\]

\[
\times \mu(\hat{s}(\hat{p}); \hat{s}(\hat{q}\hat{r}))^* \mu(n_L(p, qr); \hat{s}(\hat{p}q\hat{r}))
\]

\[
\times \left\{ \mu(n_L(p, q); \hat{s}(\hat{p}q))^* \mu(\hat{s}(\hat{p}q); \hat{s}(\hat{r}))^* \mu(n_L(pq, r); \hat{s}(\hat{p}q\hat{r})) \right\}^*
\]

We compute some terms below:

\[
\mu(\hat{s}(\hat{p})n_L(q, r); \hat{s}(\hat{q}\hat{r})) = \mu(\hat{s}(\hat{p})n_L(q, r)\hat{s}(\hat{p})^{-1}\hat{s}(\hat{p}); \hat{s}(\hat{q}\hat{r}))
\]

\[
= \mu(\hat{s}(\hat{p})n_L(q, r)\hat{s}(\hat{p})^{-1}; \hat{s}(\hat{p}))^* \times \mu(\hat{s}(\hat{p})n_L(q, r)\hat{s}(\hat{p})^{-1}; \hat{s}(\hat{p})\hat{s}(\hat{q}\hat{r}))\mu(\hat{s}(\hat{p}); \hat{s}(\hat{q}\hat{r}))
\]

\[
= \mu(\hat{s}(\hat{p})n_L(q, r)\hat{s}(\hat{p})^{-1}; \hat{s}(\hat{p}))^* \mu(\hat{s}(\hat{p}); \hat{s}(\hat{q}\hat{r}))
\]

\[
\times \mu(\hat{s}(\hat{p})n_L(q, r)\hat{s}(\hat{p})^{-1}; n_L(p, qr)\hat{s}(\hat{p}q\hat{r}))
\]

\[
= \mu(\hat{s}(\hat{p})n_L(q, r)\hat{s}(\hat{p})^{-1}; \hat{s}(\hat{p}))^* \mu(\hat{s}(\hat{p}); \hat{s}(\hat{q}\hat{r})) \mu(n_L(p, qr); \hat{s}(\hat{p}q\hat{r}))^*
\]

\[
\times \mu(\hat{s}(\hat{p})n_L(q, r)\hat{s}(\hat{p})^{-1}; n_L(p, qr))
\]

\[
\times \mu(\hat{s}(\hat{p})n_L(q, r)\hat{s}(\hat{p})^{-1}; n_L(q, r); \hat{s}(\hat{p}q\hat{r}))
\]

\[
\mu(\hat{s}(\hat{p})\hat{s}(\hat{q}); \hat{s}(\hat{r})) = \mu(n_L(p, q)\hat{s}(\hat{p}q); \hat{s}(\hat{r}))
\]
\[ = \mu(n_L(p, q); \dot{s}(\tilde{p}\tilde{q}))^* \mu(n_L(p, q); \dot{s}(\tilde{p}\tilde{q}) \dot{s}(\tilde{r})) \mu(\dot{s}(\tilde{p}\tilde{q}); \dot{s}(\tilde{r})) \]
\[ = \mu(n_L(p, q); \dot{s}(\tilde{p}\tilde{q}))^* \mu(\dot{s}(\tilde{p}\tilde{q}); \dot{s}(\tilde{r})) \mu(n_L(p, q); n_L(pq, r) \dot{s}(\tilde{p}\tilde{q}\tilde{r})) \]
\[ = \mu(n_L(p, q); \dot{s}(\tilde{p}\tilde{q}))^* \mu(\dot{s}(\tilde{p}\tilde{q}); \dot{s}(\tilde{r})) \mu(n_L(pq, r); \dot{s}(\tilde{p}\tilde{q}\tilde{r}))^* \]
\[ \times \mu(n_L(p, q); n_L(pq, r)) \mu(n_L(p, q)n_L(pq, r); \dot{s}(\tilde{p}\tilde{q}\tilde{r})). \]

We then substitute the above expression in the original calculation:
\[ \partial_Q f(\tilde{p}, \tilde{q}, \tilde{r}) = \mu(\dot{s}(\tilde{p}) \dot{s}(\tilde{q}), \dot{s}(\tilde{r}))^* \mu(\dot{s}(\tilde{p}), n_L(q, r)) \]
\[ \times \mu(\dot{s}(\tilde{p}) n_L(q, r), \dot{s}(\tilde{q}\tilde{r})) \mu(\dot{s}(\tilde{p}), \dot{s}(\tilde{q}\tilde{r}))^* \mu(n_L(p, qr), \dot{s}(\tilde{p}\tilde{q}\tilde{r})) \]
\[ \times \mu(n_L(p, q), \dot{s}(\tilde{p}\tilde{q}))^* \mu(\dot{s}(\tilde{p}\tilde{q}), \dot{s}(\tilde{r})) \mu(n_L(pq, r), \dot{s}(\tilde{p}\tilde{q}\tilde{r}))^* \]
\[ = \mu(n_L(p, q), n_L(pq, r))^* \mu(\dot{s}(\tilde{p}), n_L(q, r)) \]
\[ \times \mu(\dot{s}(\tilde{p}) n_L(q, r) \dot{s}(\tilde{p})^*, \dot{s}(\tilde{p})) \]
\[ \times \mu(\dot{s}(\tilde{p}) n_L(q, r) \dot{s}(\tilde{p})^*, \dot{s}(\tilde{p})) \]
\[ = \alpha_p \left( d_\mu(s; n_L(q, r)) \right) \mu_\mu(\dot{s}(p); n_L(q, r)) \]
\[ \times \mu_\mu(\dot{s}(p)n_L(q, r)\dot{s}(p)^*; n_L(p, qr)) \]
\[ \times \mu_\mu(\dot{s}(p)n_L(q, r)\dot{s}(p)^*; n_L(p, qr)) \]
\[ \times \left\{ \mu_\mu(n_L(p, q); n_L(pq, r)) \right\}^* \]
\[ = c_\mu(\tilde{p}, \tilde{q}, \tilde{r}) \]

for each triplet \( p = (p, q), \tilde{q}, \tilde{r} \in Q. \) This completes the proof. \( \bigstar \)
Lemma 2.5. i) Every cohomology class \([c] \in H^3_\alpha(\tilde{Q}, A)\) can be represented by a cocycle \(c\) of the form:

\[
c(\tilde{p}, \tilde{q}, \tilde{r}) = \alpha_p(d_c(s; q, r))c_Q(p, q, r),
\]

\[
\tilde{p} = (p, s), \tilde{q} = (q, t), \tilde{r} = (r, u) \in \tilde{Q},
\]

where \(c_Q \in Z_3^3(\tilde{Q}, A)\) and \(d_c(\cdot, q, r) \in Z^1_\theta\).

ii) Given a function \(d : \mathbb{R} \times Q^2 \to A\) and \(c_Q \in Z_3^3(\tilde{Q}, A)\), the function \(c\) given by:

\[
c(\tilde{p}, \tilde{q}, \tilde{r}) = \alpha_p(d(s; q, r))c_Q(p, q, r)
\]

is an element of \(Z_3^3(\tilde{Q}, A)\) if and only if

a) for each fixed \(q, r \in Q\), \(d(\cdot, q, r)\) is an \(\mathbb{R}\)-cocycle, i.e.,

\[
d(s + t, q, r) = d(s, q, r)\theta_s(d(t, q, r)), \quad s, t \in \mathbb{R}, q, r \in Q;
\]

b) \(c_Q\) and \(d\) are linked by the following formula:

\[
\theta_s(c_Q(p, q, r))c_Q(p, q, r)^* = \alpha_p(d(s; q, r))d(s; p, qr)\{d(s; p, q)d(s; pq, r)\}^*
\]

for each \(s \in \mathbb{R}, p, q, r \in Q\), i.e.,

\[
\partial_Qd = \partial_\theta c_Q.
\]

iii) For a cocycle \(c \in Z_3^3(\tilde{Q}, A)\) of the form (2.11) the following are equivalent:

a) There exists \(a \in C_\alpha^2(Q, A)\) such that

\[
c = \partial_Q a;
\]

b) There exists \(a \in C_\alpha^2(Q, A)\) such that

\[
d_c(s; q, r) = \theta_s(a(q, r))a(q, r)^*, q, r \in Q, s \in \mathbb{R}; \quad c_Q = \partial_Q a.
\]
Proof. The assertion (i) follows from the fact that the additive real line \( \mathbb{R} \) has trivial second and third cohomologies. Every 3-cocyle we encounter in this paper will be of this form without perturbation anyway. So we omit the proof.

ii) This follows directly from the cocycle identity for \( c \). We omit the detail.

iii) This equivalence again follows from a direct easy computation.

Definition 2.6. A cocycle \( c \in Z^3_\alpha(\tilde{Q}, \mathbb{R}) \) of the form \( (2.11) \) will be called standard. We will concentrate on the subgroup \( Z^3_{\alpha,s}(\tilde{Q}, A) \) of all standard cocyles in \( Z^3_\alpha(\tilde{Q}, A) \). The index “s” stands for “standard”. We then set

\[
H^3_{\alpha,s}(\tilde{Q}, A) = Z^3_{\alpha,s}(\tilde{Q}, A) / \partial \tilde{Q}(C^2_\alpha(Q, A)).
\]

The coboundary group \( B^3_{\alpha,s}(\tilde{Q}, A) = \partial \tilde{Q}(C^2_\alpha(Q, A)) \) is a subgroup of the usual third coboundary group \( B^3_\alpha(\tilde{Q}, A) = \partial \tilde{Q}(C^2_\alpha(\tilde{Q}, A)) \), so that we have a natural surjective homomorphism:

\[
H^3_{\alpha,s}(\tilde{Q}, A) \twoheadrightarrow H^3_\alpha(\tilde{Q}, A)
\]

The fixed cross-section \( s: Q \mapsto G \) allows us to consider the fiber product

\[
H^3_{\alpha,s}(\tilde{Q}, A) \times s \text{Hom}_G(N, H^1_{\theta})
\]

consisting of those pairs \( ([c], \nu) \in H^3_{\alpha,s}(\tilde{Q}, A) \times \text{Hom}_G(N, H^1_{\theta}) \) such that

\[
[d_c(\cdot, q, r)] = \nu(n_N(q, r)) \quad \text{in} \; H^1_{\theta}, \quad q, r \in Q.
\]

The group \( H^3_{\alpha,s}(\tilde{Q}, A) \ast s \text{Hom}_G(N, H^1_{\theta}) \) will be denoted by \( H^{out}_{\alpha,s}(G \times \mathbb{R}, N, A) \) for short. The suffix “s” is placed to indicate that this fiber product depends heavily on the cocyle \( n_N \) hence on the cross-section \( s \). As mentioned earlier, the invariant for outer actions of \( G \) must respect the cross-section \( s \) because a change in the cross-section results an alteration on the outer conjugacy class from the original outer conjugacy class. Before stating the main theorem of the section, we still need some preparation.
Theorem 2.7. Suppose that \( \{ \mathcal{C}, \mathbb{R}, \theta \} \) is an ergodic flow and a homomorphism \( \alpha: g \in H \mapsto \alpha_g \in \text{Aut}_\theta(\mathcal{C}) \), the group of automorphisms of \( \mathcal{C} \) commuting with \( \theta \). Assume the following:

i) a pair of normal subgroup \( M \subset L \subset H \) is given;
ii) the subgroup \( L \), hence \( M \) as well, acts trivially on \( \mathcal{C} \), i.e., \( L \subset \text{Ker}(\alpha) \);
iii) with \( G = H/M, N = L/M \) and \( Q = H/L \), let \( \pi_G: H \mapsto G, \pi: G \mapsto Q \) and \( \hat{\pi} = \pi \circ \pi_G: H \mapsto Q \) be the quotient maps such that

\[
\text{Ker}(\pi_G) = M, \quad \text{Ker}(\pi) = N \quad \text{and} \quad \text{Ker}(\hat{\pi}) = L;
\]

iv) Fix a cross-section \( s: Q \mapsto G \) of the map \( \pi \) and choose cross-sections \( s_H: G \mapsto H \) and \( \dot{s}: Q \mapsto H \) in such a way that

\[
\dot{s} = s_H \circ s.
\]

Set \( \tilde{H} = H \times \mathbb{R}, \tilde{G} = G \times \mathbb{R} \) and \( \tilde{Q} = Q \times \mathbb{R} \). Let \( A \) denote the unitary group \( \mathfrak{U}(\mathcal{C}) \) of \( \mathcal{C} \). Under the above setting, there is a natural exact sequence which sits next to the Huebschmann-Jones-Ratcliffe exact sequence:

\[
\begin{array}{cccccc}
H^2(H, \mathbb{T}) & \longrightarrow & H^2(H, \mathbb{T}) \\
\text{Res} & & \text{Res} & & \\
\Lambda(\tilde{H}, L, M, A) & \longrightarrow & \Lambda_\alpha(H, M, \mathbb{T}) \\
\delta & & \delta_{\text{HJR}} & & \\
H^\text{out}_\alpha(G \times \mathbb{R}, N, A) & \longrightarrow & H^3(G, \mathbb{T}) \\
\text{Inf} & & \text{Inf} & & \\
H^3(H, \mathbb{T}) & \longrightarrow & H^3(H, \mathbb{T})
\end{array}
\]

(2.13)

We need some preparation.
Lemma 2.8. To each characteristic cocycle \((\lambda, \mu) \in Z_\alpha(\tilde{H}, L, A)\), there corresponds uniquely an \(\tilde{H}\)-equivariant exact square:

\[
\begin{array}{c}
1 \quad 1 \quad 1 \\
\downarrow \quad \downarrow \quad \downarrow \\
1 \overset{}{\longrightarrow} T \overset{}{\longrightarrow} A \overset{\partial_\theta}{\longrightarrow} B \overset{}{\longrightarrow} 1 \\
\downarrow \quad \downarrow j \quad \downarrow j \\
1 \overset{}{\longrightarrow} U = E^\theta \overset{}{\longrightarrow} E \overset{\bar{\partial}_\theta}{\longrightarrow} Z \overset{}{\longrightarrow} 1 \\
\downarrow i \quad \downarrow i \quad \downarrow i \\
1 \overset{}{\longrightarrow} K \overset{}{\longrightarrow} L \overset{\partial}{\longrightarrow} H \overset{}{\longrightarrow} 1 \\
\downarrow j \quad \downarrow j \quad \downarrow j \\
1 \quad 1 \quad 1
\end{array}
\]

(2.14)

with \(E = A \times_\mu L\).

Proof. The cocycle \((\lambda, \mu)\) gives an \(\tilde{H}\)-equivariant exact sequence:

\[E : 1 \overset{}{\longrightarrow} A \overset{i}{\longrightarrow} E = A \times_\mu L \overset{j}{\longrightarrow} L \overset{}{\longrightarrow} 1.\]

With \(U = E^\theta\), the fixed point subgroup of \(E\) under the action \(\theta\) of \(\mathbb{R}\), we set

\[B = A/\mathbb{T} \cong B^1_\theta(\mathbb{R}, A); \quad Z = E/U; \quad K = K(E) = j(U) \cong U/\mathbb{T}; \quad H = Z/B.\]

As the real line \(\mathbb{R}\) does not act on the group \(H\), we have

\[j(\theta_s(x)x^{-1}) = j(x)j(x)^{-1} = 1, \quad x \in E, \ s \in \mathbb{R}; \]

\[\theta_s(x)x^{-1} = (\partial_\theta x)_s \in A,\]

and \(a : s \in \mathbb{R} \mapsto a_s = (\partial_\theta x)_s \in A\) is a cocycle, a member of \(Z^1_\theta(\mathbb{R}, A)\). Thus \(Z \subset Z^1_\theta(\mathbb{R}, A)\) and naturally \(H \subset H^1_\theta(\mathbb{R}, A)\). The map \(\partial_\theta\) can be viewed either as the quotient map: \(E \mapsto Z\) or the coboundary map described above. Now
it is clear that these groups \( \mathbb{T}, A, \cdots, H \) form a commutative exact square of (2.14) on which \( \widetilde{H} \) acts.

We will denote the subgroup \( K \) of \( L \) in (2.14) by \( K(\lambda, \mu) \) or \( K(\chi) \) to indicate the dependence of \( K \) on the cocycle \( (\lambda, \mu) \in Z_\alpha(\widetilde{H}, L, A) \) or the characteristic invariant \( \chi = [\lambda, \mu] \in \Lambda_\alpha(\widetilde{H}, L, A) \). We then define the subgroups:

\[
Z_\alpha(\widetilde{H}, L, M, A) = \{ (\lambda, \mu) \in Z_\alpha(\widetilde{H}, L, A) : K(\lambda, \mu) \supset M \}; \\
\Lambda_\alpha(\widetilde{H}, L, M, A) = \{ \chi \in \Lambda_\alpha(\widetilde{H}, L, A) : K(\chi) \supset M \}.
\] (2.15)

A cocycle \( (\lambda, \mu) \in Z(\widetilde{H}, L, A) \) belongs to the subgroup \( Z_\alpha(\widetilde{H}, L, M, A) \) if and only if the cocycle \( (\lambda, \mu) \) satisfies the conditions:

\[
(\lambda|_{M \times \widetilde{H}}, \mu|_{M \times M}) \in Z(\widetilde{H}, M, \mathbb{T}); \quad \lambda(m; s) = 1, \quad s \in \mathbb{R}, m \in M.
\]

Let \( \text{pr}_H : \widetilde{H} \mapsto H \) be the projection map from \( \widetilde{H} = H \times \mathbb{R} \) to the \( H \)-component and \( i_{A, \mathbb{T}} : \mathbb{T} \mapsto A \) be the canonical embedding of \( \mathbb{T} \) to \( A \). Finally, let \( i_{L, M} : M \mapsto L \) be the embedding of \( M \) into \( L \). Then we have naturally:

\[
\Lambda_\alpha(\widetilde{H}, L, A) \xrightarrow{i_{L,M}^*} \Lambda(\widetilde{H}, M, A) \xrightarrow{\text{id}} \Lambda(\widetilde{H}, M, A)
\]

\[
\Lambda(H, M, \mathbb{T}) \xrightarrow{\text{pr}_H^*} \Lambda(\widetilde{H}, M, \mathbb{T}) \xrightarrow{(i_{A, \mathbb{T}})^*} \Lambda(\widetilde{H}, M, A)
\]

In terms of these maps, we can restate the subgroup \( \Lambda_\alpha(\widetilde{H}, L, M, A) \) in the following way:

\[
\Lambda_\alpha(\widetilde{H}, L, M, A) = (i_{L,M}^*)^{-1}((i_{A, \mathbb{T}})^* \circ \text{pr}_H^*)(\Lambda(H, M, \mathbb{T})).
\]

The above maps also generates the following chain:

\[
H^2(H, \mathbb{T}) \xrightarrow{\text{pr}_H^*} H^2(\widetilde{H}, \mathbb{T}) \xrightarrow{(i_{A, \mathbb{T}})^*} H^2_\alpha(\widetilde{H}, A) \xrightarrow{\text{res}} \Lambda(\widetilde{H}, L, A)
\]
and the range of the composed map

\[
\text{Res} = \text{res} \circ (i_A, T) \circ \text{pr}_H^* : H^2(H, \mathbb{T}) \mapsto \Lambda_\alpha(\tilde{H}, L, A)
\]

is contained in the group \(\Lambda_\alpha(\tilde{H}, L, M, A)\) defined above, which generates the maps:

\[
H^2(H, \mathbb{T}) \xrightarrow{\text{Res}} \Lambda_\alpha(\tilde{H}, L, M, A).
\]

Coming back to the original situation that \(H = \text{Aut}(M), M = \text{Int}(M)\) and \(L = \text{Cnt}_r(M)\), we know that each element of \(\text{Res}(H^2(H, \mathbb{T}))\) gives a perturbation of the action of \(\text{Aut}(M)\) on \(M\) differ by \(\text{Int}(M)\). Hence we must be concerned with the quotient group

\[
\Lambda_\alpha(\tilde{H}, L, M, A)/\text{Res}(H^2(H, \mathbb{T})).
\]

The map \(\delta_{\text{HJR}} = \delta\) in the the Huebschmann - Jones - Ratcliffe exact sequence:

\[
1 \longrightarrow H^1(\tilde{Q}, A) \xrightarrow{\pi^*} H^1(\tilde{H}, A) \longrightarrow H^1(L, A)^\tilde{H} \longrightarrow \quad \\
H^2(\tilde{Q}, A) \longrightarrow H^2(\tilde{H}, A) \longrightarrow \Lambda(\tilde{H}, L, A) \xrightarrow{\delta} H^3(\tilde{Q}, A) \xrightarrow{\pi^*} H^3(\tilde{H}, A),
\]

to be abbreviated the \textbf{HJR-exact sequence}, \([\text{Hb, J1, Rc}]\), gives a natural map \(\delta_{\text{HJR}} : \Lambda_\alpha(\tilde{H}, L, M, A) \mapsto H^3(\tilde{Q}, A)\). The map \(\delta_{\text{HJR}}\) will be called the \textbf{HJR map} and the modified HJR map \(\delta\) relevant to our discussion will be constructed along with the other two maps:

\[
\partial : H^3_{\alpha, s}(G \times \mathbb{R}, N, A) \longrightarrow H^3(G, \mathbb{T});
\]
\[
\text{Inf} : H^3_{\alpha, s}(G \times \mathbb{R}, N, A) \longrightarrow H^3(H, \mathbb{T}).
\]

**Construction of the modified HJR-map \(\delta\):** First we fix a cocycle \((\lambda, \mu) \in \Lambda_\alpha(\tilde{H}, L, M, A)\) and consider the corresponding crossed extension \(E\):

\[
1 \longrightarrow A \xrightarrow{i} E \xrightarrow{j} L \longrightarrow 1.
\]
As \( M \subset K(\lambda, \mu) \), with \( V = M \times_\mu \mathbb{T} \) and \( F = E/V \) we have an \( \tilde{H} \)-equivariant exact square:

\[
\begin{array}{cccc}
1 & 1 & 1 & \\
\downarrow & \downarrow & \downarrow & \\
1 \rightarrow & T \rightarrow & A \rightarrow & B \rightarrow 1 \\
\downarrow & i & \downarrow & \\
1 \rightarrow & V \rightarrow & E \rightarrow & F \rightarrow 1 \\
\downarrow & j \uparrow \sigma_j & \pi_N & \\
1 \rightarrow & M \rightarrow & L \rightarrow & N \rightarrow 1 \\
\downarrow & \downarrow & \downarrow & \\
1 & 1 & 1 & \\
\end{array}
\]

As \( M \subset K(\lambda, \mu) \), we get a \( G \)-equivariant homomorphism \( \nu_\chi : N \rightarrow H \subset H^1_\theta(\mathbb{R}, A) \), where \( G = H/M \), i.e., \( \nu_\chi \in \text{Hom}_G(N, H^1_\theta(\mathbb{R}, A)) \).

**Lemma 2.9.** Fix \((\lambda, \mu) \in \mathbb{Z}_\alpha(\tilde{H}, L, M, A)\).

i) For the cross-section \( \sigma_j : m \in L \mapsto (1, m) \in E = A \times_L L \) of the map \( j : E \rightarrow L \) associated with the cocycle \((\lambda, \mu)\), the cocycle \( c = c^{\lambda, \mu} \) given by (2.2') is standard with

\[
d_c(s; q, r) = \lambda(n_L(q, r); s), \quad q, r \in Q, s \in \mathbb{R};
\]

\[
c_Q(p, q, r) = \lambda(\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}; \hat{s}(p))\mu(\hat{s}(p)n_L(q, r)\hat{s}(p)^{-1}, n_L(p, qr))
\times \{\mu(n_L(p, q), n_L(pq, r))\}^*, \quad p, q, r \in Q.
\]

ii)\( ([\chi], \nu_\chi) \in H^3_{\alpha,s}(\tilde{Q}, A) \ast_s \text{Hom}_G(N, H^1_\theta(\mathbb{R}, A)) \);

**Proof.** i) This is obvious from the formula (2.2').
ii) The cross-section $s_H$ gives an $M$-valued 2-cocycle:

$$n_M(g, h) = s_H(g)s_H(h)s_H(gh)^{-1} \in M, \quad g, h \in G,$$

which allows us to relate $s_H(n_N(q, r))$ and $n_L(q, r)$:

$$\pi_G(n_L(q, r)) = \pi_G\left(s(q)s(r)s(qr)^{-1}\right) = s(q)s(r)s(qr)^{-1} = n_N(q, r).$$

Hence $n_L(q, r) \equiv s_H(n_N(q, r)) \mod M$. As for each $m \in M, \ell \in L$ and $s \in \mathbb{R}$ we have

$$\lambda(m\ell; s) = \theta_s(\mu(m; \ell)^*)\mu(m; \ell)\lambda(m; s)\lambda(\ell; s) = \theta_s(\mu(m; \ell)^*)\mu(m; \ell)\lambda(\ell; s),$$

we get

$$[\lambda(m\ell; \cdot)] = [\lambda(\ell; \cdot)] \text{ in } H^1_G(\mathbb{R}, A) \text{ for every } m \in M, \ell \in L.$$

Thus $[\lambda(n_L(q, r); \cdot)] = \nu_\chi(n_N(q, r)) \in H^1_G(\mathbb{R}, A)$, which precisely means that

$$(\left[ c^{\lambda, \mu}, \nu \right], \nu) \in \text{H}^\text{out}_{\alpha, s}(G \times \mathbb{R}, N, A).$$

Thus we obtain an element:

$$\delta(\chi) = (\left[ c^{\lambda, \mu}, \nu \right], \nu) \in \text{H}^\text{out}_{\alpha, s}(G \times \mathbb{R}, N, A),$$

and therefore the map

$$\delta : \Lambda_\alpha(\tilde{H}, L, M, A) \mapsto \text{H}^\text{out}_{\alpha, s}(G \times \mathbb{R}, N, A).$$

We will call $\delta$ the modified HJR-map. To distinguish this modified HJR map from the original HJR map, we denote the original one by $\delta_HJR$ and the modified one simply by $\delta$. Note that the map $\delta$ does not depend on the choice of the section $s_H: G \mapsto H$, but depends on the choice of the section $s: Q \mapsto G$. We now begin the proof of Theorem 2.7.
**Ker(δ) = Im(Res):** First assume that
\[ \mu \in Z^2(H, \mathbb{T}) \quad \text{and} \quad \chi = \text{Res}(\mu) \in \Lambda(\tilde{H}, L, M, A). \]

Then we have a commutative diagram of exact sequences:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{T} & \longrightarrow & F = \mathbb{T} \times_{\mu} H & \overset{j_H}{\longrightarrow} & H & \longrightarrow & 1 \\
& & \downarrow & & \cap & & \parallel & & \\
1 & \longrightarrow & A & \longrightarrow & \tilde{F} = A \times_{\mu} H & \overset{j_{\tilde{F}}}{\longrightarrow} & H & \longrightarrow & 1 \\
& & \parallel & & \cup & & \parallel & & \\
1 & \longrightarrow & A & \longrightarrow & E = A \times_{\mu} L & \overset{j_{\tilde{F}}}{\longrightarrow} & L & \longrightarrow & 1 \\
\end{array}
\]

The action \( \alpha \) of \( H \) on \( E \) is given by \( \alpha_h = \text{Ad}(s_H(h))|_E \), \( h \in H \), viewing \( E \) as a submodule of \( \tilde{F} \), where the cross-section \( s_H \) is given by
\[
s_H(h) = (1, h) \in \tilde{F} = A \times_{\mu} H, \quad h \in H.
\]

The action \( \theta \) of \( \mathbb{R} \) on \( E \) is given by:
\[
\theta_s(a, m) = (\theta_s(a), m), \quad s \in \mathbb{R}, (a, m) \in E = A \times_{\mu} L.
\]

Hence \( \theta_s(s_j(m)) = s_j(m), m \in L, s \in \mathbb{R} \). Now \( \text{res}(\mu) = (\lambda_{\mu}, \mu) \in Z^2(\tilde{H}, L, A) \) is given by (2.10). As \( \mu \) takes values in \( \mathbb{T} \), we have \( \mu = \mu_H \), i.e., \( d_\mu = 1 \).

Consequently, \( \lambda_{\mu}(m; s) = 1, m \in L, s \in \mathbb{R} \) which entails
\[
\lambda_{\mu}(n_L(p, q); s) = 1, p, q \in Q = H/L, s \in \mathbb{R}.
\]

By Lemma 2.4.(iii), the associated 3-cocyle \( c_{\mu} = c_{\lambda_{\mu}, \mu} \in Z_3^\alpha(\tilde{Q}, A) \) is co-bounded by \( f \) of the form:
\[
f(p, q) = \mu(\dot{s}(p), \dot{s}(q))^* \mu(n_L(p, q), \dot{s}(pq)) \in \mathbb{T}.
\]
This shows that $\text{Im } (\text{Res}) \subset \text{Ker}(\delta)$.

We are now moving to show the reversed inclusion: $\text{Im } (\text{Res}) \supset \text{Ker}(\delta)$. We first compare the original HJR-exact sequence and our modified HJR sequence. To this end, we recall that the cohomology group $H^3_{\alpha,s}(\tilde{Q}, A)$ is obtained as the quotient group of a subgroup $Z^3_{\alpha,s}(\tilde{Q}, A)$ of $Z^3_{\alpha}(\tilde{Q}, A)$ by a subgroup $B^3_{\alpha,s}(\tilde{Q}, A)$ of $B^3_{\alpha}(\tilde{Q}, A)$. Thus we have a natural map: $H^3_{\alpha,s}(\tilde{Q}, A) \mapsto H^3_{\alpha}(\tilde{Q}, A)$. Consequently, the above HJR-exact sequence applied to our context yields the following commutative diagram:

\[
\begin{array}{cccccc}
H^2_{\alpha}(\tilde{H}, A) & \xrightarrow{\text{res}} & \Lambda_{\alpha}(\tilde{H}, L, A) & \xrightarrow{\delta_{\text{HJR}}} & H^3_{\alpha}(\tilde{Q}, A) & \xrightarrow{\text{inf}} & H^3_{\alpha}(\tilde{H}, A) \\
\uparrow^{(i_{\alpha,T})_*} & & \uparrow & & \uparrow & & \\
H^2(H, \mathbb{T}) & \xrightarrow{\text{Res}} & \Lambda_{\alpha}(\tilde{H}, L, M, A) & \xrightarrow{\delta} & H^3_{\alpha,s}(\tilde{Q}, A) & & 
\end{array}
\]

Suppose $\chi = [\lambda, \mu] \in \text{Ker}(\delta) \subset \Lambda_{\alpha}(\tilde{H}, L, M, A)$. Then

\[1 = \delta(\chi) = ([c_{\chi}], \nu_{\chi}) \text{ in } H^3_{\alpha,s}(\tilde{Q}, A) \ast_s \text{Hom}_G(N, H^1_\theta).
\]

The above assumption also means $\delta_{\text{HJR}}(\chi) = 1$. Hence the HJR-exact-sequence guarantees that the 2-cocyle $\mu$ on $L$ can be extended to $\tilde{H}$ as an $A$-valued 2-cocyle over $\tilde{H}$ which we denote by $\tilde{\mu}$ again so that $\lambda = \lambda_{\tilde{\mu}}$. To proceed further, we need the following:

**Lemma 2.10.** If a 2-cocycle $\mu \in Z^2_{\alpha}(\tilde{H}, A)$ is standard, and if

\[\chi = \text{res}(\mu) = [\lambda_{\mu}, \mu] \in \Lambda_{\alpha}(\tilde{H}, L, A)
\]

generates trivial $\nu_{\chi} = 1$ of $\text{Hom}_G(N, H^1_\theta(\mathbb{R}, A))$, then there exists a standard $\tilde{\mu} \in Z^2_{\alpha}(\tilde{H}, A)$ such that

i) $\mu \equiv \tilde{\mu} \mod B^2_{\alpha}(\tilde{H}, A)$;

ii) $\lambda_{\tilde{\mu}}(m; s) = 1$, \quad $m \in L$, $s \in \mathbb{R}$, i.e., $d_{\tilde{\mu}}(s; m) = 1$.

**Proof.** Let $\pi_2$ be the quotient map: $c \in \mathbb{Z} \mapsto [c] \in H = \mathbb{Z}/B$. The condition $\nu_{\chi}(n) = 1, n \in N$, implies that

\[\pi_2(\partial_{\theta}(s_j(m))) = \hat{\partial}_{\theta}(m) = \nu_{\chi}(\pi_{\theta}(m)) = 1 \in H^1_\theta, \quad m \in L;
\]

\[\partial_{\theta}(s_j(m)) \in B^1_\theta,
\]
so that for each $m \in L$ there exists $a(m) \in A$ such that
\[
\lambda(m; s) = \theta_s(s_j(m))s_j(m)^{-1} = \partial_\theta(s_j(m))_s = \theta_s(a(m))a(m)^*.
\]
Extending the function $a : L \mapsto A$ to the entire $\tilde{H}$ in such a way that
\[
a(g, s) = a(g), \quad (g, s) \in \tilde{H} = H \times \mathbb{R},
\]
we define a new 2-cocycle:
\[
\tilde{\mu}(g, h) = a(g)^*\alpha_\tilde{g}(a(h)^*)\mu(\tilde{g}, h)a(gh), \quad \tilde{g}, \tilde{h} \in \tilde{H},
\]
where $g$ and $h$ are the $H$-component of $\tilde{g}$ and $\tilde{h}$ respectively. We then examine if $\tilde{\mu}$ remains standard:
\[
\tilde{\mu}(g, s; h, t) = a(g)^*\theta_s(a(h)^*)\alpha_\tilde{g}(d_\mu(s; h))\mu_t(g, h)a(gh)
\]
\[
= \alpha_g(\theta_s(a(h)^*)a(h))\alpha_\tilde{g}(d_\mu(s; h))a(g)^*\alpha_g(a(h)^*)\mu_t(g, h)a(gh)
\]
\[
= \alpha_g(\theta_s(a(h)^*)a(h)d_\mu(s; h))a(g)^*\alpha_g(a(h)^*)\mu_t(g, h)a(gh).
\]
Therefore with
\[
d_\tilde{\mu}(s; h) = \theta_s(a(h)^*)a(h)d_\mu(s; h), \quad s \in \mathbb{R}, h \in H;
\]
\[
\tilde{\mu}(g, h) = a(g)^*\alpha_g(a(h)^*)\mu_t(g, h)a(gh), \quad g, h \in H,
\]
we confirm that $\tilde{\mu}$ is standard. Now the corresponding characteristic cocycle have the form:
\[
\lambda_{\tilde{\mu}}(m; g, s) = \tilde{\mu}(g, s; g^{-1}mg)\tilde{\mu}(m; g, s)^*
\]
\[
= \alpha_g(d_\tilde{\mu}(s; g^{-1}mg))\tilde{\mu}_H(g; g^{-1}mg)\tilde{\mu}_H(m; g)^*
\]
\[
= \alpha_g(\theta_s(a(g^{-1}mg)^*)a(g^{-1}mg)d_\mu(s; g^{-1}mg))
\]
\[
\times a(g)^*\alpha_g(a(g^{-1}mg)^*)a(mg)a(m)a(g)a(mg)^*
\]
\[
\times \mu_t(g; g^{-1}mg)\mu_t(m; g)^*
\]
\[
= \alpha_g(\theta_s(a(g^{-1}mg)^*)d_\mu(s; g^{-1}mg))
\]
\[
\times a(m)\mu_t(g; g^{-1}mg)\mu_t(m; g)^*
\]
\[
= \alpha_g, s(a(g^{-1}mg)^*)a(m)\lambda_{\mu}(m; g, s).
\]
With $g = 1$, we get
\[ \lambda_{\tilde{\mu}}(m; s) = \theta_s(a(m)^*)a(m)\lambda(m; s) = 1, \quad m \in L, g \in H, s \in \mathbb{R}. \]

This completes the proof.

So we replace the original characteristic cocycle $(\lambda, \mu)$ by the modified one $(\lambda_{\tilde{\mu}}, \tilde{\mu})$ by Lemma 2.10 so that
\[ \lambda = \lambda_{\mu} \quad \text{and} \quad d_{\mu}(s; m) = 1, \quad m \in L, s \in \mathbb{R}, \]
and $\mu \in Z^2_\alpha(\tilde{H}, A)$ is standard.

Now we use the fact that the HJR map $\delta_H$ pushes $(\lambda_{\mu}, \mu)$ to $c_{\mu} \in Z^3_\alpha, s(\tilde{Q}, A) \subset Z^3_\alpha(\tilde{Q}, A)$ which is cobounded by $f$ of Lemma 2.4 (iii):
\[ f(\tilde{p}, \tilde{q}) = \alpha_p(d_{\mu}(s; \hat{s}(q))^*)\mu_{\tilde{H}}(\hat{s}(p), \hat{s}(q))^*\mu_{\tilde{H}}(n_{L}(p, q); \hat{s}(pq)) \in A. \]

We examine $\partial_\theta(f|_Q)$ by making use of the relation between $d_{\mu}$ and $\mu_{\tilde{H}}$ in the formula (2.9):
\[
\theta_s(f(q, r))f(q, r)^* \\
= \theta_s\left(\mu_{\tilde{H}}(\hat{s}(q), \hat{s}(r))^*\mu_{\tilde{H}}(n_{L}(q, r); \hat{s}(qr))\right) \\
\times \left\{\mu_{\tilde{H}}(\hat{s}(q), \hat{s}(r))^*\mu_{\tilde{H}}(n_{L}(q, r); \hat{s}(qr))\right\}^* \\
= \theta_s\left(\mu_{\tilde{H}}(\hat{s}(q), \hat{s}(r))^*\mu_{\tilde{H}}(n_{L}(q, r); \hat{s}(r))\right) \\
\times \theta_s\left(\mu_{\tilde{H}}(n_{L}(q, r); \hat{s}(qr))\mu_{\tilde{H}}(n_{L}(q, r); \hat{s}(qr))\right)^* \\
= d_{\mu}(s; \hat{s}(q))\alpha_q(d_{\mu}(s; \hat{s}(r)))d_{\mu}(s; \hat{s}(q)\hat{s}(r))^* \\
\times \left\{d_{\mu}(s; n_{L}(q, r))d_{\mu}(s; \hat{s}(qr))d_{\mu}(s; n_{L}(q, r)\hat{s}(qr))^*\right\}^* \\
= d_{\mu}(s; \hat{s}(q))\alpha_q(d_{\mu}(s; \hat{s}(r)))d_{\mu}(s; n_{L}(q, r))(d_{\mu}(s; \hat{s}(qr)))^* \right\}^*.
\]

Next we compare this with $\partial_Q f$ computed in the proof of Lemma 2.4. (iii). Substituting $p, q, r$ in place of $\tilde{p}, \tilde{q}, \tilde{r}$ in the last expression of $\partial_Q f$, we obtain
\[
(\partial_Q f)(p, q, r) \\
= \mu_{\tilde{H}}(\hat{s}(p); n_{L}(q, r))\mu_{\tilde{H}}(\hat{s}(p)n_{L}(q, r)\hat{s}(p)^{-1}; \hat{s}(p))^* \\
\times \mu_{\tilde{H}}(\hat{s}(p)n_{L}(q, r)\hat{s}(p)^{-1}; n_{L}(p, qr))\left\{\mu_{\tilde{H}}(n_{L}(p, q); n_{L}(pq, r))\right\}^*.
\]
Combining the above two coboundary calculations, we obtain:

\[
(\partial_{\tilde{Q}}(f|Q))(\tilde{p}, \tilde{q}, \tilde{r}) = \alpha_p(\theta_s(f(q, r))f(q, r)^*)(\partial_Q(f|Q))(p, q, r)
\]

\[
= \alpha_p\left(d_\mu(s; \hat{\mathbf{s}}(q))\alpha_q(d_\mu(s; \hat{\mathbf{s}}(r))) \times \{d_\mu(s; n_L(q, r))d_\mu(s; \hat{\mathbf{s}}(qr))\}^* \times \right.
\]

\[
\times \mu_n(\hat{\mathbf{s}}(p); n_L(q, r))\mu_n(\hat{\mathbf{s}}(p)n_L(q, r)\hat{\mathbf{s}}(p)^{-1}; \hat{\mathbf{s}}(q)) \times 
\]

\[
\times \mu_n(\hat{\mathbf{s}}(p)n_L(q, r)\hat{\mathbf{s}}(p)^{-1}; n_L(p, qr)) \times 
\]

\[
\times \left\{\mu_n(n_L(p, q); n_L(pq, r))\right\}^* \times 
\]

\[
\times \left\{\mu_H(\hat{\mathbf{s}}(p); n_L(q, r)\hat{\mathbf{s}}(p)^{-1}; \hat{\mathbf{s}}(q)) \times 
\]

\[
\times \left\{\mu_H(\hat{\mathbf{s}}(p); n_L(p, q); n_L(pq, r))\right\}^* \times 
\]

\[
\right).
\]

Comparing this with \(c_\mu\), we conclude

\[
(\partial_{\tilde{Q}}(f|Q))(\tilde{p}, \tilde{q}, \tilde{r}) = \alpha_p\left(d_\mu(s; \hat{\mathbf{s}}(q))\alpha_q(d_\mu(s; \hat{\mathbf{s}}(r))) \times \right.
\]

\[
\times \{d_\mu(s; n_L(q, r))d_\mu(s; \hat{\mathbf{s}}(qr))\}^* \times 
\]

\[
\times \mu_n(\hat{\mathbf{s}}(p); n_L(q, r))\mu_n(\hat{\mathbf{s}}(p)n_L(q, r)\hat{\mathbf{s}}(p)^{-1}; \hat{\mathbf{s}}(q)) \times 
\]

\[
\times \mu_n(\hat{\mathbf{s}}(p)n_L(q, r)\hat{\mathbf{s}}(p)^{-1}; n_L(p, qr)) \times 
\]

\[
\times \left\{\mu_n(n_L(p, q); n_L(pq, r))\right\}^* \times 
\]

\[
\times \mu_H(\hat{\mathbf{s}}(p); n_L(q, r)\hat{\mathbf{s}}(p)^{-1}; \hat{\mathbf{s}}(q)) \times 
\]

\[
\times \left\{\mu_H(\hat{\mathbf{s}}(p); n_L(p, q); n_L(pq, r))\right\}^* \times 
\]

\[
\right) c_\mu(\tilde{p}, \tilde{q}, \tilde{r}).
\]

Now we use the assumption that \(\delta(\lambda, \mu) = c_\mu \in B^3_{\alpha, \mu}(\tilde{Q}, A)\), which means the existence of a new cochain \(\xi \in C^2_\alpha(Q, A)\) such that

\[
c_\mu(\tilde{p}, \tilde{q}, \tilde{r}) = \alpha_{\tilde{p}}(\xi(q, r))\xi(p, qr)\{\xi(p, q)\xi(pq, r)\}^*.
\]

Therefore, we get

\[
\alpha_{\tilde{p}}((\xi^* f)(q, r))f(p, qr)\{f(p, q)f(pq, r)\}^*
\]

\[
= \alpha_p\left(d_\mu(s; \hat{\mathbf{s}}(q))\alpha_q(d_\mu(s; \hat{\mathbf{s}}(r))) \times \right.
\]

\[
\times \alpha_{\tilde{p}}(\xi(q, r))\xi(p, qr)\{\xi(p, q)\xi(pq, r)\}^*,
\]

equivalently

\[
\alpha_{\tilde{p}}((\xi^* f)(q, r))((\xi^* f)(p, qr)\{f(p, q)(\xi^* f)(pq, r)\})^*
\]

\[
= \alpha_p\left(d_\mu(s; \hat{\mathbf{s}}(q))\alpha_q(d_\mu(s; \hat{\mathbf{s}}(r))) \times \right.
\]

\[
\times \alpha_{\tilde{p}}(\xi(q, r))\xi(p, qr)\{\xi(p, q)\xi(pq, r)\}^*.
\]
Setting \( s = 0 \), we obtain \( \partial_Q (\xi^* f|_Q) = 1 \). With \( p = 1 \), we get

\[
\theta_s((\xi^* f)(q, r))(\xi^* f)(q, r))^* = d_\mu(s; \hat{s}(q))\alpha_q(d_\mu(s; \hat{s}(r)))d_\mu(s; \hat{s}(qr))^*. 
\] (2.16)

We now use the formula (2.9), which states that \( d_\mu \) gives rise to an element \([d_\mu] \in \mathbb{Z}_1^\alpha(H, H_\theta)\). The assumption \( \nu_\chi = 1 \) entails that the cocycle \([d_\mu] \) factors through \( Q \), i.e., there exists a map \( a: (m, h) \in L \times H \mapsto a(m, h) \in A \) such that

\[
d_\mu(s; mh) = \theta_s(a(m, h))a(m, h)^*d_\mu(s; h), \quad m \in L, h \in H. \] (2.17)

We write \( H \) in term of the cross-section \( \hat{s} \) and the cocycle \( n_L: H = L \rtimes n_L Q \). Writing \( g = m_L(g)\hat{s}(\hat{\pi}(g)), h \in H \), with

\[
b(g) = a(m_L(g), \hat{s}(\hat{\pi}(g)) \in A, \] (2.18)

we obtain

\[
d_\mu(s; g) = \theta_s(b(g))b(g)^*d_\mu(s; \hat{s}(\hat{\pi}(g))), \quad g \in H. \] (2.19)

Then the right hand side of the formula (2.9) becomes:

\[
d_\mu(s; g)\alpha_g(d_\mu(s; h))d_\mu(s; gh)^* \\
= \theta_s(b(g))b(g)^*d_\mu(s; \hat{s}(\hat{\pi}(g)))\alpha_g\left(\theta_s(b(h))b(h)^*d_\mu(s; \hat{s}(\hat{\pi}(h)))\right) \\
\times \left(\theta_s(b(gh))b(gh)^*d_\mu(s; \hat{s}(\hat{\pi}(gh)))\right)^* \\
= \theta_s(b(g))b(g)^*\alpha_g\left(\theta_s(b(h))b(h)^*\left(\theta_s(b(gh)^*)b(gh)\right) \right) \\
\times \theta_s(\left((\xi^* f)(\hat{\pi}(g), \hat{\pi}(h))\right)(\xi^* f)(\hat{\pi}(g), \hat{\pi}(h)))^* \\
= \theta_s\left(b(g)\alpha_g(b(h))(\xi^* f)(\hat{\pi}(g), \hat{\pi}(h))b(gh)^*\right) \\
\times \left((b(g)\alpha_g(b(h))(\xi^* f)(\hat{\pi}(g), \hat{\pi}(h))b(gh)^*)^*. 
\]
Equating this to the left hand side of (2.9), we get

\[
\theta_s(\mu_0(g, h))\mu_0(g, h)^* = \theta_s\left(b(g)\alpha_g(b(h))(\xi^* f)(\hat{\pi}(g), \hat{\pi}(h))b(gh)^*\right)
\times \left(b(g)\alpha_g(b(h))(\xi^* f)(\hat{\pi}(g), \hat{\pi}(h))b(gh)^*\right)^* \quad g, h \in H.
\]

Hence \(\mu_0 = \hat{\pi}^*(\xi f^*)(\partial_H b^*)\mu_0 \in Z^2(H, \mathbb{T})\). Finally we compare

\[
\text{Res}(\mu_0) = (\lambda_{\mu_0}, \mu_0|_L)
\]

and \((\lambda_{\mu}, \mu)\). First, we compare the \(\mu\)-components of the characteristic cocycle and obtain

\[
\mu_0(m, n) = b(m)^*b(n)^*b(mn)\mu_0(m, n), \quad m, n \in L,
\]

since \((\xi f^*)(\hat{\pi}(m), \hat{\pi}(n)) = 1\). Second, we also get

\[
\lambda_{\mu_0}(m; g, s) = \alpha_g(d_{\mu_0}(s; g^{-1}mg))\mu_0(g; g^{-1}mg)\mu_0(m; g)^*
\]

\[
= \mu_0(g; g^{-1}mg)\mu_0(m; g)^*
\]

\[
= (\partial_H b^*)(g; g^{-1}mg)\mu_0(g; g^{-1}mg)(\partial_H b^*)(m; g)^*\mu_0(m; g)^*
\]

\[
= b(g)^*\alpha_g(b(g^{-1}mg)^*)b(mg)b(m)b(g)b(mg)^*\mu_0(g; g^{-1}mg)\mu_0(m; g)^*
\]

\[
= \alpha_g(b(g^{-1}mg)^*)b(m)\mu_0(g; g^{-1}mg)\mu_0(m; g)^*
\]

\[
= \alpha_g(b(g^{-1}mg)^*)b(m)\lambda_{\mu}(m; g, s).
\]

Therefore we conclude that

\[
\text{Res}([\mu_0]) = [\lambda_{\mu}, \mu] = \chi \in \Lambda_\alpha(\tilde{H}, L, M, A).
\]

This completes the proof of \(\text{Ker}(\delta) \subset \text{Im}(\text{Res})\) and so \(\text{Ker}(\delta) = \text{Im}(\text{Res})\).
Lemma 2.11. There is a natural commutative diagram of exact sequences:

\[
\begin{array}{ccc}
\Lambda(\tilde{H}, L, M, A) & \xrightarrow{\delta} & H^3_{\alpha,s}(\tilde{Q}, A) *_{\ast} \text{Hom}_G(N, H^1_A(\mathbb{R}, A)), \\
\downarrow i_L & & \downarrow i \\
\Lambda(H, M, T) & \xrightarrow{\delta_{H,R}} & H^3(G, T) \xrightarrow{\text{Inf}} H^3(H, T)
\end{array}
\]

Proof. Map \(\delta\): Fix a cross-section \(\delta_L : H^1_0(\mathbb{R}, A) \hookrightarrow Z^1_0(\mathbb{R}, A)\) and set

\[
\zeta_\nu(s; n) = (\delta_L(\nu(n)))_s \in A, \quad s \in \mathbb{R}, n \in N, \quad \nu \in \text{Hom}_G(N, H^1_A(\mathbb{R}, A)).
\]

Choose \([c], \nu) \in H^3_{\alpha,s}(\tilde{Q}, A) *_{\ast} \text{Hom}_G(N, H^1_A(\mathbb{R}, A)))\) so that

\[
\partial_2[c] = \nu \cup n_s,
\]
i.e.,

\[
[d(\cdot; q, r)] = \nu(n_s(q, r)) \quad \text{in} \quad H^1_0(\mathbb{R}, A), \quad q, r \in Q.
\]

Hence there exists \(f \in C^2_\alpha(Q, A)\) such that

\[
d_c(s; q, r) = \theta_s(f(q, r)) f(q, r)^* \zeta_\nu(s; n(q, r)), \quad q, r \in Q, \quad s \in \mathbb{R}, \quad (2.20)
\]

and therefore,

\[
c(\tilde{p}, \tilde{q}, \tilde{r}) = c_Q(p, q, r) \alpha_p \left( \theta_s(f(q, r)) f(q, r)^* \zeta_\nu(s; n(q, r)) \right),
\]

\[
\tilde{p} = (p, s), \tilde{q}, \tilde{r} \in \tilde{Q}. \quad (2.21)
\]

The necessary condition for \(c \in Z^3_{\alpha,s}(\tilde{Q}, A)\) in (2.12) gives the following:

\[
\theta_s(c_Q(p, q, r)) c_Q(p, q, r)^* = \alpha_p \left( \theta_s(f(q, r)) f(q, r)^* \zeta_\nu(s; n(q, r)) \right)
\]

\[
\times \theta_s(f(p, qr)) f(p, qr)^* \zeta_\nu(s; n(p, qr))
\]

\[
\times \{ \theta_s(f(p, q)) f(p, q)^* \zeta_\nu(s; n(p, q))
\]

\[
\times \theta_s(f(pq, r)) f(pq, r)^* \zeta_\nu(s; n(pq, r)) \}^* \quad (2.22)
\]
for each \( p, q, r \in Q \) and \( s \in \mathbb{R} \).

Now we are going to consider the pull back \( \tilde{\pi}^*(c) \in Z_3^3(\tilde{G}, A) \) with

\[
\tilde{\pi}(g, s) = (\pi(g), s), \quad \tilde{g} = (g, s) \in \tilde{G} = G \times \mathbb{R},
\]

But we first check the pull back \( \tilde{\pi}^*(\nu \cup n_N) \in Z_2^2(G, H_1^0) \). To this end, with

\[
m_N(g) = gs(\pi(g))^{-1} \in N \quad \text{and} \quad n_N(g) = s(\pi(g))g^{-1} \in N,
\]

we observe first

\[
n_N(\pi(g), \pi(h)) = s(\pi(g))s(\pi(h))s(\pi(gh))^{-1}, \quad g, h \in G,
\]

\[
= n_N(g)g n_N(h)h\{n_N(gh)gh\}^{-1}
\]

\[
= n_N(g)gn_N(h)g^{-1}n_N(gh)^{-1};
\]

and that

\[
\nu(n_N(\pi(g), \pi(h))) = \nu(n_N(g))\alpha_g(\nu(n_N(h)))\nu(n_N(gh))^{-1} \text{ in } H_1^0, \quad g, h \in G.
\]

Hence we can choose \( a(g, h) \in A, g, h \in G \), such that

\[
\zeta_\nu(s; n_N(\pi(g), \pi(h))) = \theta_s(a(g,h))a(g,h)^*\zeta_\nu(s; n_N(g))
\]

\[
\times \alpha_g(\zeta_\nu(s; n_N(h)))\zeta_\nu(s; n_N(gh))^*
\]

(2.23)

We apply now this to the pull back of the above (2.22) and obtain for each \( g, h, k \in G \):

\[
\theta_s(c_Q(\pi(g), \pi(h), \pi(k)))c_Q(\pi(g), \pi(h), \pi(k))^*
\]

\[
= \alpha_g\left(\theta_s(\pi^*(f)(h, k)\pi^*(f)(h, k)^*\theta_s(a(h, k))a(h, k)^*\zeta_\nu(s; n_N(h))
\times \alpha_h(\zeta_\nu(s; n_N(k)))\zeta_\nu(s; n_N(hk))^*
\times \theta_s(\pi^*(f)(g, hk)\pi^*(f)(g, hk)^*\theta_s(a(g, hk))a(g, hk)^*)
\right)
\]
\[ \times \zeta_\nu(s; n_N(g)) \alpha_g(\zeta_\nu(s; n_N(hk))) \zeta_\nu(s; n_N(ghk))^* \]
\[ \times \left\{ \theta_s(\pi^*(f)(g, h)) \pi^*(f)(g, h)^* \theta_s(a(g, h)) a(g, h)^* \right\} \]
\[ \times \zeta_\nu(s; n_N(g)) \alpha_g(\zeta_\nu(s; n_N(h))) \zeta_\nu(s; n_N(gh))^* \]
\[ \times \theta_s(\pi^*(f)(gh, k)) \pi^*(f)(gh, k)^* \theta_s(a(gh, k)) a(gh, k)^* \]
\[ \times \zeta_\nu(s; n_N(gh)) \alpha_{gh}(\zeta_\nu(s; n_N(k))) \zeta_\nu(s; n_N(ghk))^* \right\}^* \]
\[ = \alpha_g \left( \theta_s(\pi^*(f)(h, k)) \pi^*(f)(h, k)^* \theta_s(a(h, k)) a(h, k)^* \right) \]
\[ \times \theta_s(\pi^*(f)(gh, k)) \pi^*(f)(gh, k)^* \theta_s(a(gh, k)) a(gh, k)^* \]
\[ \times \left\{ \theta_s(\pi^*(f)(g, h)) \pi^*(f)(g, h)^* \theta_s(a(g, h)) a(g, h)^* \right\} \]
\[ \times \theta_s(\pi^*(f)(gh, k)) \pi^*(f)(gh, k)^* \theta_s(a(gh, k)) a(gh, k)^* \right\}^* \]

and hence, for each each \( g, h, k \in G \),
\[ \theta_s \left( c_Q(\pi(g), \pi(h), \pi(k)) \partial_G(\pi^*(f)^* a^*)(g, h, k) \right) \]
\[ = c_Q(\pi(g), \pi(h), \pi(k)) \partial_G(\pi^*(f)^* a^*)(g, h, k). \]

The ergodicity of the flow \( \theta \) yields that
\[ \pi^*(c_Q) \partial_G(\pi^*(f)a)^* \in Z^3(G, T). \]

Now we change the cocycle \( c \) to \( c' \) within the cohomology class, i.e., \( c' = (\partial_Q b)c \) with \( b \in C^2_c(Q, A) \) which gives:
\[ d'(s; q, r) = \theta_s(b(q, r)) b(q, r)^* d(s; q, r), \quad s \in \mathbb{R}, q, r \in Q. \]

We also change the cross-section \( \mathfrak{s}_Z : H^1_\theta \rightarrow Z^1_\theta \) to \( \mathfrak{s}'_Z : H^1_\theta \rightarrow Z^1_\theta \). Then there exists a map \( n \in N \mapsto e(n) \in A \) such that
\[ \zeta'_\nu(s; n) = \mathfrak{s}'_Z(n) = \theta_s(e(n)) e(n)^* \zeta_\nu(s; n), \quad n \in N. \]

Thus we obtain, for each \( s \in \mathbb{R}, g, h \in G \),
\[ d'(s; q, r) = \theta_s \left( (f(q, r)e(n_f(q, r))^* b(q, r) \right) \]
\[ f(q, r)^* b(q, r)^* e(n_f(q, r)) \zeta'_\nu(s; n_f(q, r)); \]
Therefore, the cochains $f$ and $a$ are transformed to the following $f'$ and $a'$:

$$
f'(p, q) = f(p, q)e(n_N(p, q))^*b(p, q), \quad p, q \in Q;
$$

$$
a'(g, h) = e(n_N(p, g), \pi(h)))a(g, h)e(n_N(g))^*
\times \alpha_g(e(n_N(h))^*)e(n_N(gh)), \quad g, h \in G.
$$

Thus we get

$$
\pi^*(c'_Q)\partial_G(\pi^*(f')a')^* = \pi^*(c_Q)\pi^*(\partial_Q b)\partial_G\left(\pi^*(fb)\pi^*(e^*n_N)^*a\right)^*
\times \partial_G(\pi^*(e^*n_N))\partial_G^2(\pi^*(e^*n_N)^*)
= \pi^*(c_Q)\partial_G(\pi^*(f^*)a^*).
$$

Finally, in the choice of $f$ and $a$ we have precisely the ambiguity of $C^2(Q, \mathbb{T})$ and $C^2(G, \mathbb{T})$ which result the change on $\pi^*(c_Q)\partial_G(\pi^*(f^*)a^*) \in Z^3(G, \mathbb{T})$ by $B^3(G, \mathbb{T})$. Thus we have a well-defined homomorphism

$$
\partial : H^3_{\alpha, s}(\tilde{Q}, A) * s \text{ Hom}_G(N, H^1_{\theta}(\mathbb{R}, A)) \rightarrow H^3(G, \mathbb{T}),
$$

which depends on the choice of the section $s : Q \rightarrow G$.

Now fix $\chi = [\lambda, \mu] \in \Lambda_\alpha(\tilde{H}, L, M, A)$ and set

$$
([c^\lambda, \mu], \nu_\chi) = \delta(\chi) \in H^3_{\alpha, s}(\tilde{Q}, A) * s \text{ Hom}_G(N, H^1_{\theta}(\mathbb{R}, A)).
$$
Associated with \((\lambda, \mu)\) is an \(\tilde{H}\)-equivariant exact square:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & \longrightarrow & T & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 1 \\
\downarrow & i \downarrow & \downarrow \\
1 & \longrightarrow & V & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 1 \\
\downarrow & j \downarrow & \downarrow & \uparrow s_j \\
1 & \longrightarrow & M & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 
\end{array}
\]

The far left column exact sequence corresponds to the restriction

\[
(\lambda, \mu)|_{M \times H} = i^*_{L,M}(\lambda, \mu) \in \Lambda(H, M, T).
\]

To go further, we need the following:

**Sublemma 2.12.** In the above context, we have

\[
\delta_{\text{HIR}}(i^*_{L,M}(\chi)) = \partial(\delta(\chi)) \in H^3(G, \mathbb{T}), \quad \chi \in \Lambda_\alpha(\tilde{H}, L, M, A).
\]

**Proof.** First we arrange the cross-sections in the following way:

- \(s_h(n s(p)) = s_h(n) s_h(s(p)), \quad n \in N, p \in Q;\)
- \(s_j(m s_h(n)) = s_j(m) s_j(s_h(n)), \quad m \in M, n \in N.\)

We further arrange the cross-sections \(s_h\) on \(N\), \(s_j\) and \(s_z\) on \(H\), so that they
satisfy the following composition rules:

\[
\begin{array}{c}
E \xrightarrow{\partial_\theta} Z \\
\downarrow \quad \downarrow \\
\uparrow s_j \quad \uparrow s_Z \\
\end{array}
\]

\[
\begin{array}{c}
L \xrightarrow{\partial} H \\
\downarrow \quad \downarrow \\
\uparrow s_{\mathcal{H}} \\
\end{array}
\]

\[
\begin{array}{c}
\pi_G \xrightarrow{\nu} \uparrow s_{\mathcal{H}} \\
\end{array}
\]

As \( s_{\mathcal{H}} \neq s_\delta \circ \nu \) if \( M \neq K(\lambda, \mu) \), i.e., if \( \nu \) is not injective, the second composition rule needs to be justified. For each \( n \in N \), set \( m = s_\delta \circ \nu(n) s_{\mathcal{H}}(n)^{-1} \in M \) so that \( m s_{\mathcal{H}}(n) = s_\delta(\nu(n)) \). Then we have

\[
\begin{align*}
\mathcal{S}_j(\mathcal{S}_\delta(\nu(n))) &= \mathcal{S}_j(m) \mathcal{S}_j(s_{\mathcal{H}}(n)); \\
\mathcal{S}_Z(\nu(n)) &= \partial_\theta(\mathcal{S}_j(\mathcal{S}_\delta(\nu(n)))) = \partial_\theta(\mathcal{S}_j(m) \mathcal{S}_j(s_{\mathcal{H}}(n))) \\
&= \partial_\theta(\mathcal{S}_j(s_{\mathcal{H}}(n))),
\end{align*}
\]

which justifies the second composition rule.

For each \( \ell \in L \), we write

\[
\ell = m_M(\ell) s_\delta(\hat{\partial}(\ell)) = m_M(\ell) s_\delta(\nu(\pi_G(\ell)))
\]

and obtain

\[
\begin{align*}
\mathcal{S}_j(\ell) &= \mathcal{S}_j(m_M(\ell)) s_j \circ s_\delta(\nu(\pi_G(\ell))), \quad \ell \in L; \\
\partial_\theta(\mathcal{S}_j(\ell)) &= \partial_\theta(s_j \circ s_\delta(\nu(\pi_G(\ell)))) = s_Z(\nu(\pi_G(\ell))).
\end{align*}
\]

Each \( g \in G \) is uniquely written in the form:

\[
g = m_N(g) s(\pi(g)), \quad g \in G,
\]

with \( m_N(g) \in N \). Therefore we have

\[
\mathcal{S}_{\mathcal{H}}(g) = \mathcal{S}_{\mathcal{H}}(m_N(g)) s(\pi(g)), g \in G, s_{\mathcal{H}}(m_N(g)) \in L.
\]
Then the product $gh$ of each pair $g, h \in G$ gives:

$$m_N(gh)s(\pi(gh)) = gh$$

$$= m_N(g)s(\pi(g))m_N(h)s(\pi(h))$$

$$= m_N(g)s(\pi(g))m_N(h)s(\pi(g))^{-1}s(\pi(g))s(\pi(h))$$

$$= m_N(g)s(\pi(g))m_N(h)s(\pi(g))^{-1}n_N(\pi(g), \pi(h))s(\pi(gh))$$

$$1 = m_N(g)s(\pi(g))m_N(h)s(\pi(g))^{-1}n_N(\pi(g), \pi(h))m_N(gh)^{-1}.$$ 

We observe the following relation between the cocycles $n_L$ and $n_N$.

$$\pi_G(n_L(p, q)) = \pi_G(s(p)s(q)s(pq))^{-1}$$

$$= s(p)s(q)s(pq)^{-1} = n_N(p, q), \quad p, q \in Q.$$ 

We then further compute:

$$n_M(g, h) = s_H(g) s_H(h) s_H(gh)^{-1}, \quad g, h \in G,$$

$$= s_H(m_N(g)) s_H(\pi((g))) s_H(m_N(h)) s_H(\pi((h)))$$

$$\times s_H(m_N(gh)) s_H(\pi((gh)))^{-1}$$

$$= s_H(m_N(g)) s_H(s(\pi((g)))) s_H(m_N(h)) s_H(s(\pi(h)))$$

$$\times \{s_H(m_N(gh)) s_H(s(\pi(gh)))\}^{-1}$$

$$= s_H(m_N(g)) s_H(s(\pi(g))) s_H(m_N(h)) s_H(s(\pi(g))^{-1}s(\pi(g))) s_H(s(\pi(h)))$$

$$\times \{s_H(m_N(gh)) s_H(s(\pi(gh)))\}^{-1}$$

$$= s_H(m_N(g)) s_H(s(\pi(g))) s_H(m_N(h)) s_H(s(\pi(g))^{-1}n_L(\pi(g)), \pi(h))$$

$$\times \{s_H(m_N(gh))\}^{-1}.$$ 

We now take the cross-section $s_j$ and choose $b(g, h) \in A$ so that the following computation is valid:

$$s_J(n_M(g, h)) = s_J(s_H(g) s_H(m_N(h)) s_H(g)^{-1}$$

$$\times s_H(m_N(g)) n_L(\pi(g), \pi(h)) s_H(m_N(gh))^{-1})$$
\[ b(g, h)\alpha_{\sigma_t(g)}(s_j(s_t(m_N(h))))s_j(s_t(m_N(g))) \times s_j(n_L(\pi(g), \pi(h)))s_j(s_t(m_N(gh)))^{-1} = b(g, h)\alpha_{\sigma_t(m_N(g))}\delta(\pi(g))(s_j(s_t(m_N(h))))s_j(s_t(m_N(g))) \times s_j(n_L(\pi(g), \pi(h)))s_j(s_t(m_N(gh)))^{-1} = b(g, h)s_j(s_t(m_N(g)))\alpha_{\delta(\pi(g))}(s_j(s_t(m_N(h)))) \times s_j(n_L(\pi(g), \pi(h)))s_j(s_t(m_N(gh)))^{-1}. \]

We summarize this here for later use:
\[ s_j(n_M(g, h)) = b(g, h)s_j(s_t(m_N(g)))\alpha_{\delta(\pi(g))}(s_j(s_t(m_N(h)))) \times s_j(n_L(\pi(g), \pi(h)))s_j(s_t(m_N(gh)))^{-1} \tag{2.24} \]

We then apply the coboundary operator \( \partial_\theta \) to the both side to obtain:
\[ 1 = \partial_\theta(b(g, h))\partial_\theta(s_j(s_t(m_N(g))))\partial_\theta(\alpha_{\delta(\pi(g))}(s_j(s_t(m_N(h)))) \times \partial_\theta(s_j(n_L(\pi(g), \pi(h))))\partial_\theta(s_j(s_t(m_N(gh)))^{-1}) \]

and use the composition rules among cross-sections to drive:
\[ s_\xi(\nu(n_N(\pi(g), \pi(h)))) = \partial_\theta(b(g, h)^{-1})s_\xi(\nu(m_N(g))^{-1}) \times \alpha_g(s_\xi(\nu(m_N(h))^{-1}))s_\xi(\nu(m_N(gh))) \].

Since \( n_N(g) = m_N(g)^{-1}, g \in G \), we have
\[ s_\xi(\nu(n_N(\pi(g), \pi(h)))) = \partial_\theta(b(g, h)^{-1})s_\xi(\nu(n_N(g)) \times \alpha_g(s_\xi(\nu(n_N(h))))s_\xi(\nu(n_N(gh))^{-1})) \]
equivalently
\[ \zeta_\nu(s; n_N(\pi(g), \pi(h))) = \theta_s(b(g, h)^{-1})b(g, h)\zeta_\nu(s; n_N(g)) \times \alpha_g(\zeta_\nu(s; n_N(h)))\zeta_\nu(s; n_N(gh))^*, \quad g, h \in G. \]

Therefore the elements \( b(g, h)^{-1} \in A \) serves as \( a(g, h) \) of (2.23) in the construction of \( \partial(\delta(\chi)) \).
Since the cochain $f$ above computation shows that the third cohomology class: we apply the coboundary operation $\partial_G$ to the both side of (2.24) relative to the outer action $\alpha_{s_1}$ of $G$ on $E$ to obtain:

$$c^\lambda_{G}(g, h, k) = \alpha_{s_1}(g) (s_j(n_M(h, k))) s_j(n_M(g, hk))$$

$$\times \{s_j(n_M(g, h)) s_j(n_M(gh, k))\}^{-1}$$

$$= (\partial_G a)(g, h, k)^{-1} \alpha_{s_1}(g) (u(h) \alpha_{\hat{\delta}(\pi(h))} (u(k))) w(h, k) u(hk)^{-1}$$

$$\times u(g) \alpha_{\hat{\delta}(\pi(g))} (u(hk)) w(g, hk) u(hk)^{-1}$$

$$\times \{u(g) \alpha_{\hat{\delta}(\pi(g))} (u(h)) w(g, h) u(gh)^{-1} u(gh) \}^{-1}$$

$$\times \alpha_{\hat{\delta}(\pi(gh))} (u(k)) w(gh, k) u(ghk)^{-1}$$

$$= (\partial_G a)(g, h, k)^{-1} \alpha_{\hat{\delta}(\pi(g))} (u(hk)) w(h, k)$$

$$\times w(g, hk) \{w(g, h) \alpha_{\hat{\delta}(\pi(gh))} (u(k)) w(gh, k)\}^{-1}$$

$$= (\partial_G a)(g, h, k)^{-1} w(g, h) \alpha_{\hat{\delta}(\pi(gh))} (u(k))$$

$$\times w(g, h)^{-1} \alpha_{\hat{\delta}(\pi(g))} (w(h, k))$$

$$\times w(g, hk) \{w(g, h) \alpha_{\hat{\delta}(\pi(gh))} (u(k)) w(gh, k)\}^{-1}$$

$$= (\partial_G a)(g, h, k)^{-1} \alpha_{\hat{\delta}(\pi(g))} (w(h, k)) w(g, hk) \{w(g, h) w(gh, k)\}^{-1}$$

$$= (\partial_G a)(g, h, k)^{-1} c_{G}^\lambda (\pi(g), \pi(h), \pi(k)).$$

Since the cochain $f \in C^2_\alpha(Q, A)$ of (2.20) is taken to be 1 in our case, the above computation shows that the third cohomology class:

$$[\partial_G (a^{-1}) \pi^* (c_{G}^\lambda)] \in H^3(G, \mathbb{T})$$
is indeed the 3-cohomology class associated with the far left column \( H \)-equivariant exact sequence of the exact square before the lemma:

\[
1 \longrightarrow T \longrightarrow V \longrightarrow M \longrightarrow 1
\]

which corresponds to the characteristic invariant \( i_{L,M}^* (\chi) \in \Lambda(H, M, T). \)

**Im(\( \delta \)) \subset Ker(\( \text{inf}\-\partial \)).** As seen above, we have \( \partial(\delta(\chi)) = \delta_{\text{HJR}}(i_{L,M}^*(\chi)) \in H^3(G, T) \). Hence we conclude

\[
1 = \text{inf}\-\partial(\delta(\chi)) = \text{inf}\-\partial(\delta(\chi)), \quad \chi \in \Lambda(H, L, M, A).
\]

**Im(\( \delta \)) \supset Ker(\( \text{inf}\-\partial \)).** First, we compare our sequence with HJR-exact sequence:

\[
\Lambda_\alpha(H, L, A) \xrightarrow{\delta_{\text{HJR}}} H^3_\alpha(Q, A) \xrightarrow{\text{inf}} H^3_\alpha(N, s) \xrightarrow{\iota_{A, T}*} H^3(H, T)
\]

Now suppose that \( \text{Inf}([c], \nu) = 1 \) in \( H^3(H, T) \). The 3-cocycle \( c \in Z^3_{\alpha, s}(\tilde{Q}, A) \) is naturally an element of \( Z^3_{\alpha, s}(\tilde{Q}, A) \). We denote this element by \( \tilde{c} \) and its cohomology class \( [\tilde{c}] \in H^3(\tilde{Q}, A) \). First, the image \( \partial([c], \nu) \in H^3(G, T) \) is obtained as the class of \( \partial_G(\pi^* (f) a^{-1}) \pi^* (c_Q) \) where \( f \in C^2_\alpha(Q, A) \) and \( a \in C^2_\alpha(G, A) \) are obtained subject to the following conditions:

\[
d_c(s; q, r) = \theta_s(f(q, r)) f(q, r)^* \zeta_\nu(s; n_N(q, r)), \quad s \in \mathbb{R}, q, r \in Q;
\]

\[
\zeta_\nu(s; n(\pi(g), \pi(h))) = \theta_s(a(g, h)) a(g, h)^* \zeta_\nu(s; n_N(g)), \quad g, h \in G, \quad (2.25)
\]

\[
\times \alpha_g(\zeta_\nu(s; n_N(h))) \zeta_\nu(s; n_N(gh))^*,
\]

where \( \zeta_\nu(s; n) = \sharp_c(\nu(n)) s, n \in N, s \in \mathbb{R} \). The image \( \text{Inf}([c], \nu) \) is obtained as the class of

\[
\pi_H^* (\partial_G(\pi^* (f) a^{-1}) \pi^* (c_Q)) = \partial_H(\tilde{\pi}_H^* (f) \pi_H^* (a))^{-1} \tilde{\pi}_H^* (c_Q) \in Z^3(H, T).
\]
The assumption that \( \text{Inf}([c], \nu) = 1 \) means that \( \partial_H(\hat{\pi}^*(f)\pi_H^*(a))^{-1} \hat{\pi}^*(c_Q) \in B^3(H, \mathbb{T}) \), i.e., there exists \( b \in C^2(H, \mathbb{T}) \) such that
\[
\partial_H(\hat{\pi}^*(f)\pi_H^*(a))^{-1} \hat{\pi}^*(c_Q) = \partial_H b
\]
Hence for each triple \( g, h, k \in H \) we have
\[
c_Q(\hat{\pi}(g), \hat{\pi}(h), \hat{\pi}(k)) = \alpha_g \left( b(h, k)f(\hat{\pi}(h), \hat{\pi}(k))a(\pi_G(h), \pi_G(k)) \right)
\times b(g, hk)f(\hat{\pi}(g), \hat{\pi}(hk))a(\pi_G(g), \pi_G(hk))
\times \left\{ b(g, h)f(\hat{\pi}(g), \hat{\pi}(k))a(\pi_G(g), \pi_G(k)) \right\}^*.
\]
With \( u(g, h) = a(\pi_G(g), \pi_G(h))b(g, h)f(\hat{\pi}(g), \hat{\pi}(h)) \), we get
\[
\hat{\pi}^*(c_Q) = \partial_H u,
\]
and
\[
c(\hat{\pi}(\tilde{g}), \hat{\pi}(\tilde{h}), \hat{\pi}(\tilde{k})) = \alpha_g \left( d_c(s; \hat{\pi}(h), \hat{\pi}(k)) \right)c_Q(\hat{\pi}(g), \hat{\pi}(h), \hat{\pi}(k))
= \alpha_g(d_c(s; \hat{\pi}(h), \hat{\pi}(k)))\alpha_g \left( u(h, k) \right)u(g, hk)\{u(g, h)u(gh, k)\}^*.
\]
The identities (2.25) yields the following computations, for each \( g, h, k \in H \),
\[
d_c(s; \hat{\pi}(h), \hat{\pi}(k)) = \theta_s(f(\hat{\pi}(h), \hat{\pi}(k)))f(\hat{\pi}(h), \hat{\pi}(k))^*\zeta_\nu(s; n(\hat{\pi}(h), \hat{\pi}(k)))
\times z_\nu(s; n_G(\pi_G(h)))\alpha_g(\zeta_\nu(s; n_G(\pi_G(h))))\zeta_\nu(s; n_G(\pi_G(gh)))^*;
\]
\[
d_c(s; \hat{\pi}(h), \hat{\pi}(k)) = \theta_s(f(\hat{\pi}(h), \hat{\pi}(k)))f(\hat{\pi}(h), \hat{\pi}(k))^*
\times \theta_s(a(\pi_G(h), \pi_G(k)))a(\pi_G(h), \pi_G(k))^* \times \theta_s(a(\pi_G(gh), \pi_G(gh)))a(\pi_G(gh), \pi_G(gh))^*.
\]
With \( v(s; g) = \zeta_\nu(s; n_G(\pi_G(g))) \), we get
\[
d_c(s; \hat{\pi}(h), \hat{\pi}(k)) = \theta_s(u(h, k))u(h, k)^*v(s; g)\alpha_g(v(s; h))v(s; gh)^*
= (\partial_g(u(h, k)))_s(\partial_H v)(s; h, k)
\]
Substituting this to the above computation of $\hat{\pi}^*(c)$ and setting

$$w(g, s; h, t) = u(g, h)\alpha_g(v(s; h)^*),$$

we obtain:

$$(\hat{\pi}^*c)(\tilde{g}, \tilde{h}, \tilde{k}) = \alpha_{g}\left(\theta_s(u(h, k))u(h, k)^*v(s; h)\alpha_h(v(s; k))v(s; hk)^*\right)
\times \alpha_g(u(h, k))u(g, hk)\{u(g, hu(gh, k))\}^*;$$

$$(\partial_{\hat{H}}w)(\tilde{g}, \tilde{h}, \tilde{r}) = \alpha_{\tilde{g}}(w(\tilde{h}; \tilde{k}))w(\tilde{h}; \tilde{hk})\{w(\tilde{g}; \tilde{h})w(\tilde{g}h; \tilde{k})\}^*$$
$$= \alpha_{g}\theta_s\left(u(h, k)\alpha_h(v(t; k))^*)\right)
\times \alpha_g(u(h, k))u(g, hk)\{u(g, hu(gh, k))\}^*$$
$$= \alpha_{g}\theta_s\left(u(h, k)\alpha_h(v(t; k))^*)\right)
\times \{u(g, h)\alpha_g(v(s; h))^*u(gh, k)\alpha_{gh}(v(s + t; k))^*)\}^*$$
$$= \alpha_g\theta_s\left(u(h, k)\alpha_h(v(t; k))^*)\right)
\times \{u(g, h)\alpha_g(v(s; h))^*u(gh, k)\alpha_{gh}(v(s + t; k))^*)\}^*$$
$$= (\hat{\pi}^*c)(\tilde{g}, \tilde{h}, \tilde{k}).$$

Therefore, we conclude

$$\hat{\pi}^*(c) = \partial_{\hat{H}}w.$$  \hfill (2.26)

Hence the element $(\lambda, \mu)$ given by

$$\lambda(\ell; g, s) = w(g, s; g^{-1}\ell g)w(\ell; g, s)^*, \quad \ell \in L, g \in H, s \in \mathbb{R};$$
$$\mu(m, n) = w(m, n), \quad m, n \in L,$$
is a characteristic cocycle in $Z_\bar{\alpha}(\tilde{H}, L, A)$. In terms of the original $a, b$ and $f$, we get

$$
\lambda(m; g, s) = a(\pi_c(g), \pi_c(g^{-1}mg))b(g, g^{-1}mg)f(\dot{\pi}(g), \dot{\pi}(g^{-1}mg))
\times \alpha_g(\zeta_\nu(s; n_N(\pi_c(g^{-1}mg))))^*a(\pi_c(m), \pi_c(g))^*
\times b(m, g)^*f(\dot{\pi}(m), \dot{\pi}(g))^*;
\mu(m; n) = a(\pi_c(m), \pi_c(n))b(m, n).
$$

(2.27)

Now observe that if $m, n \in M$, then both $\lambda$ and $\mu$ takes values in $\mathbb{T}$, so that $\chi = [\lambda, \mu] \in \Lambda_\alpha(\tilde{H}, L, M, A)$.

We are now going to compare the new cocycle $c^{\lambda, \mu}$ and the original $c$ in the next lemma to complete the proof of Lemma 2.12 and therefore Theorem 2.7:

**Lemma 2.13.** The cochain $W \in C^2_\alpha(\tilde{Q}, A)$ defined by

$$
W(\tilde{p}, \tilde{q}) = w(\tilde{s}(\tilde{p}), \tilde{s}(\tilde{q}))w(n_L(p, q), \tilde{s}(\tilde{pq}))^*, \quad \tilde{p} = (p, s), \tilde{q} = (q, t) \in \tilde{Q},
$$

falls in $C^2_\alpha(Q, A)$ and its coboundary $\partial_{\tilde{Q}}W$ bridges the difference between $(c^{\lambda, \mu}, \nu_\chi)$ and the original $(c, \nu)$, i.e., $([c], \nu) = \delta(\chi)$. Therefore

$$
\text{Ker(Inf)} \subset \text{Im}(\delta).
$$

**Proof.** First we observe that for any pair $\tilde{p} = (p, s), \tilde{q} = (q, t) \in \tilde{Q}$

$$
W(\tilde{p}, \tilde{q}) = w(\tilde{s}(\tilde{p}), \tilde{s}(\tilde{q}))w(n_L(p, q), \tilde{s}(\tilde{pq}))^*
= u(\tilde{s}(\tilde{p}), \tilde{s}(\tilde{q}))\alpha_p(v(s; \tilde{s}(\tilde{q})))u(n_L(p, q), \tilde{s}(\tilde{pq}))^*
= a(s(p), s(q))b(\tilde{s}(\tilde{p}), \tilde{s}(\tilde{q}))f(p, q)\alpha_p(\zeta_\nu(s; n_N(s(q)))
\times a(n(p, q), s(pq))^*b(n_L(p, q), \tilde{s}(\tilde{pq}))^*f(p, q)^*
= a(s(p), s(q))b(\tilde{s}(\tilde{p}), \tilde{s}(\tilde{q}))f(p, q) \quad \text{(as } n_N(s(q)) = 1)
\times a(n(p, q), s(pq))^*b(n_L(p, q), \tilde{s}(\tilde{pq}))^*f(p, q)^*
= W(p, q).
$$
Thus $W$ is constant on $\mathbb{R}$-variables, so that it belongs to $C^2_\alpha(Q, A)$.

By Lemma 2.1, we have

$$c = (\partial_Q W) c^{\lambda,\mu}$$

and therefore

$$c^{\lambda,\mu} \equiv c \mod B^3_{\alpha,s}(\tilde{Q}, A), \text{ i.e., } [c^{\lambda,\mu}] = [c] \text{ in } H^3_{\alpha,s}(\tilde{Q}, A).$$

Setting $g = 1$ in (2.27), we obtain for each $m \in L$

$$\lambda(m; s) = a(1, \pi_G(m)) b(1, m) f(1, \tilde{\pi}(m))$$

$$\times \zeta_\nu(s; n_N(\pi_G(m)))^* a(\pi_G(m), 1)^*$$

$$\times b(m, 1)^* f(\tilde{\pi}(m), 1)^*$$

$$= \zeta_\nu(s; n_N(\pi_G(m)))^* = \zeta_\nu(s; \pi_G(m)^{-1}))^*$$

since the cochains $a, b$ and $f$ can be chosen such a way that whenever $1$ appears in the arguments they take value $1$. As

$$\zeta_\nu(\cdot; \pi_G(m)^{-1})^* \equiv \zeta_\nu(\cdot; \pi_G(m)) \mod B^1_{\tilde{g}},$$

i.e., $\nu(\pi_G(m)^{-1})^{-1} = \nu(\pi_G(m)), m \in L$, we conclude that $\nu = \nu_{[\lambda,\mu]}$. Therefore we conclude that $([c], \nu) = \delta([\lambda,\mu])$. This completes the proof of the inclusion $\text{Ker(Inf)} \subset \text{Im}(\delta)$. ◇

**Lemma 2.14.** Let $A$ denote the unitary group $U(\mathcal{C})$ of an abelian separable von Neumann algebra $\mathcal{C}$ or the torus group $\mathbb{T}$. Let $\alpha$ be an action of a countable discrete group $G$ on $\mathcal{C}$. To each $c \in Z^3_\alpha(G, A)$, there corresponds a countable group $H = H(c)$ and a normal subgroup $M = M(c)$ such that:

i) the group $G$ is identified with the quotient group $H/M$;

ii) there corresponds a characteristic cocycle

$$(\lambda, \mu) = (\lambda_c, \mu_c) \in Z_\alpha(H, M, A)$$

such that

$$[c] = \delta_{\text{HJR}}([\lambda, \mu])$$

in the HJR-exact sequence relative to $\{H, M, A\}$;

iii) the group $M$ is abelian.
Proof. First extend the coefficient group $A$ to the unitary group

$$B = \mathcal{U}(\mathcal{C} \ltimes \ell^\infty(G))$$

on which $G$ acts by $\alpha \otimes \rho$ with $\rho$ the right translation action of $G$ on $\ell^\infty(G)$, which will be denoted by $\alpha$ again whenever it will not cause any confusion, and obtain an exact sequence:

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{s} C \longrightarrow 1,$$

where $i(a) = a \otimes 1 \in B, a \in A$, and $s_j$ a cross-section which can be fixed without reference to the cocycle $c$. Then set

$$u(x, g, h) = u_c(x, g, h) = \alpha_x^{-1}(c(x, g, h)) \in A, \quad x, g, h \in G,$$

and view $u(\cdot, g, h)$ as an element of $B = \mathcal{U}(\ell^\infty(G) \otimes \mathbb{C}) = \text{Map}(G, A)$. The cocycle identity gives that $c = \partial_\alpha^G u \in B^2_\alpha(G, C)$. Since $j(A) = \{1\}$ in $C$, $\mu = \mu_c = j_*(u)$ is in $Z^2_\alpha(G, C)$. Let $M$ be the subgroup of $C$ generated by the saturation $\{\alpha_g(\mu(h, k)) : g, h, k \in G\}$ of the range of $\mu$, so that $\mu$ belongs to $Z^2_\alpha(G, M)$. Now consider the twisted semi-direct product:

$$H = H(c) = M \rtimes_{\alpha, \mu} G$$

and obtain an exact sequence:

$$1 \longrightarrow M \longrightarrow H \xrightarrow{\pi_G} G \longrightarrow 1.$$ 

Set $E = j^{-1}(M)$ to obtain a crossed extension $E \in \text{Ext}_\alpha(H, M, A)$. With $\mathfrak{s}_H$ the cross-section given by

$$\mathfrak{s}_H(g) = (1, g) \in H, \quad g \in G,$$

we obtain

$$\mu(g, h) = \mathfrak{s}_H(g)\mathfrak{s}_H(h)\mathfrak{s}_H(gh)^{-1}, \quad g, h \in G.$$
Thus $\mu \in Z^2(\alpha, G, M)$. Now observe that

$$f(g, h) = s_j(\mu(g, h))^{-1}u(g, h) \in i(A)$$

and that

$$(\partial_G f)(g, h, k) = \partial_G(s_j \circ \mu)(g, h, k)^{-1}c(g, h, k), \quad g, h, k \in G.$$ 

Thus we conclude that $\delta_{HJR}(\chi_E) = [\partial_G(s_j \circ \mu)] = [c] \in H^3_{\alpha}(G, A)$. 

**Remark 2.15.** The last lemma shows that if $G$ is a countable discrete amenable group, then so is $H$ because $M$ is abelian and countable, and the quotient $G = H/M$ is amenable. Another important fact is that the groups $H$ and $M$ depend heavily on the choice of the cocycle $c$. Two cohomologous cocycles $c, c' \in Z^3_{\alpha}(G, A)$ need not produce isomorphic $H(c)$ and $H(c')$. In fact, the subgroups $M(c)$ and $M(c')$ are not isomorphic. We will address this inconvenience later. If we use the entire $C$ in place of $M$, then the resulting groups are isomorphic in a natural way. But in this way, we will lose the countability of $H$.

**Definition 2.16.** We call the group $H(c)$ the resolution group of the cocycle $c \in Z^3_{\alpha}(G, A)$ and the characteristic cocycle $(\lambda_c, \mu_c) \in Z^2(\alpha, H, M, A)$ a resolution of the cocycle $c$. We also call the map $\pi_G : H(c) \mapsto G$ resolution map and the pair $\{H(c), \pi_c\}$ a resolution system.

**Corollary 2.17.** Let $\{\mathcal{C}, \mathbb{R}, \theta\}$ be an ergodic flow and $G$ a discrete countable group acting on the flow $\{\mathcal{C}, \mathbb{R}, \theta\}$ via $\alpha$. Let $N$ be a normal subgroup of $G$ such that $N \subset \text{Ker}(\alpha)$. Then with $Q = G/N$ the quotient group of $G$ by $N$ and $s : Q \mapsto G$ a cross-section of the quotient map $\pi : G \mapsto Q$, for any pair

$$([c], \nu) \in H^3_{\alpha,s}(Q \times \mathbb{R}, \mathcal{U}(\mathcal{C})) \ast s \text{Hom}_{G}(N, H^1_{\mathbb{R}}(\mathbb{R}, \mathcal{U}(\mathcal{C})))$$

there exist a countable discrete group $H$ and a surjective homomorphism $\pi_G : H \mapsto G$ and $\chi \in \Lambda_{\pi^*_G(\alpha)}(H \times \mathbb{R}, L, M, A)$ such that

$$([c], \nu) = \delta(\chi)$$
where \( L = \pi_G^{-1}(N) \), \( M = \text{Ker}(\pi_G) \) and \( \delta \) is the modified HJR-map in Lemma 2.11 associated with the exact sequence:

\[
1 \longrightarrow M \longrightarrow L \xrightarrow{\pi_G} G \xrightarrow{\pi} Q \longrightarrow 1.
\]

Moreover, the kernel \( M = \text{Ker}(\pi_G) \) is chosen to be abelian. Hence if \( G \) is amenable in addition, then \( H \) is amenable.

**Proof.** Let \( \partial \) be the map in Lemma 2.11:

\[
\partial : H^3_{\alpha,s}(Q \times \mathbb{R}, \mathcal{U}(\mathcal{C})) \ast_s \text{Hom}_G(N, H^1_{\theta}(\mathbb{R}, \mathcal{U}(\mathcal{C}))) \mapsto H^3(G, \mathbb{T}).
\]

Set \([c_G] = \partial([c], \nu) \in H^3(G, \mathbb{T})\) and choose a cocycle \( c_G \in Z^3(G, \mathbb{T}) \) which represents the cohomology class \([c_G]\). Let \( H = H(c_G) \) be the resolution group of \( c_G \) in Lemma 2.14, i.e., the group \( H \) is equipped with a surjective homomorphism \( \pi_G : H \mapsto G \) such that \( \pi_G^*([c_G]) = 1 \) in \( H^3(H, \mathbb{T}) \). Thus with \( L = \pi_G^{-1}(N) \triangleleft H \) and \( M = \text{Ker}(\pi_G) \triangleleft H \), we have an exact sequence:

\[
1 \longrightarrow M \longrightarrow L \xrightarrow{\pi_G} G \xrightarrow{\pi} Q \longrightarrow 1,
\]

with specified cross-section \( s \) of \( \pi : G \mapsto Q \). This generates the associated modified HJR-exact sequence of (2.13) in Theorem 2.7. As

\[
\text{Inf}([c], \nu) = \pi_G^*(\partial([c], \nu)) = \pi_G^*([c_G]) = 1,
\]

there exists \( \chi \in \Lambda_{\alpha}((\bar{Q}, L, M, A) \ast_s \text{Hom}_G(N, H^1_{\theta}(\mathbb{R}, \mathcal{U}(\mathcal{C}))) \mapsto H^3(G, \mathbb{T}).
\]

**Change on the Cross-Section \( s : Q \mapsto G \):** As mentioned repeatedly, the group \( H^3_{\alpha,s}(Q \times G, A) \ast_s \text{Hom}_G(N, H^1_{\theta}) \) depends heavily on the cross-section \( s : Q \mapsto G \). So we are going to examine what change occurs if we change the cross-section from \( s : Q \mapsto G \) to another \( s' : Q \mapsto G \). The change does not alter the groups \( H^3_{\alpha,s}(Q \times G, A) \ast_s \text{Hom}_G(N, H^1_{\theta}) \), but the fiber product \( H^3_{\alpha,s}(Q \times G, A) \ast_s \text{Hom}_G(N, H^1_{\theta}) \) changes to \( H^3_{\alpha,s}(Q \times G, A) \ast_{s'} \text{Hom}_G(N, H^1_{\theta}). \)
Proposition 2.18. In the setting as above, if \( s' : Q \to G \) is another cross-section of the homomorphism \( \pi : G \to Q = G/N \), then there is a natural isomorphism

\[
\sigma_{s', s} : \mathcal{H}^\text{out}_{\alpha, s}(G \times \mathbb{R}, N, A) \to \mathcal{H}^3_{\alpha, s}(\tilde{Q}, A) \ast_{s'} \text{Hom}_{G}(N, H^1_0),
\]

(2.28)

where \( \tilde{Q} = Q \times \mathbb{R} \) as before. Furthermore, if \( s'' : Q \to G \) is the third cross-section of \( \pi \), then the isomorphisms satisfy the following chain rule:

\[
\sigma_{s'', s} = \sigma_{s'', s'} \circ \sigma_{s', s}
\]

(2.29)

Proof. The new cross-section \( s' : Q \to G \) generates a new \( N \)-valued 2-cocycle:

\[
n'_N(p, q) = s'(p)s'(q)s'(pq)', \quad p, q \in Q.
\]

Then the 2-cocycle \( n'_N \) is written in terms of \( n_N \) and \( n_{s', s} \) as follows:

\[
n'_N(p, q) = n_{s', s}(p)s(p)n_{s', s}(q)s(q)\{n_{s', s}(pq)s_{s}(pq)\}^{-1} \quad p, q \in Q.
\]

Hence for each \( \nu \in \text{Hom}_{G}(N, H^1_0) \) we have

\[
\nu(n'_N(p, q)) = \nu(n_{s', s}(p))\alpha_p(\nu(n_{s', s}(q)))\nu(n_N(p, q))\nu(n_{s', s}(pq))^{-1}.
\]

For each \( ([c], \nu) \in \mathcal{H}^\text{out}_{\alpha, s}(G \times \mathbb{R}, N, A) \), we set

\[
d'_{c'}(s; q, r) = d_c(s; q, r)\zeta_{\nu}(s; n_{s', s}(q))\alpha_q(\zeta_{\nu}(s; n_{s', s}(r))\zeta_{\nu}(s; n_{s', s}(qr))^*;
\]

\[
c'_Q(p, q, r) = c_Q(p, q, r), \quad s \in \mathbb{R}, p, q, r \in Q,
\]

where \( \zeta_{\nu}(s; n) = s_{s}(n)s, n \in N, s \in \mathbb{R} \). As \( \partial_Q d_{c'} = \partial_Q d_c \), the new pair \( (d_{c'}, c'_Q) \) gives a standard 3-cocycle \( c' \in \mathcal{Z}^3_{\alpha, s}(\tilde{Q}, A) \) which is not congruent to \( c = (d_c, c_Q) \in \mathcal{Z}^3_{\alpha, s}(\tilde{Q}, A) \) modulo \( B^3_{\alpha, s}(\tilde{Q}, A) \) in general although they are
congruent modulo $B^3_\alpha(\bar{Q}, A)$. Now we define the map $\sigma'_{s', s}$ in the following way:
$$\sigma'_{s', s}([c], \nu) = ([c'], \nu), \quad ([c], \nu) \in H^3_{\alpha, s}(G \times \mathbb{R}, N, A).$$
Then as
$$[d\sigma' \cdot q, r]) = \nu(n'_N(q, r)) \in H^3_\theta, \quad q, r \in Q,$$
the pair $([c'], \nu)$ belongs to $H^3_{\alpha, s}(\bar{Q}, A) *_{s'} \text{Hom}_{\mathcal{C}}(N, H^1_{\theta}).$
To check the multiplicativity of $\sigma'_{s', s}$, for each pair $h, k \in H^1_\theta$ we choose
$$a(h, k) \in A$$
such that
$$\sigma_z(h) \sigma_z(k) = \partial_\theta(a(h, k)) \sigma_z(hk).$$
Then for each pair $([c], \nu), ([\bar{c}], \bar{\nu}) \in H^3_{\alpha, s}(G \times \mathbb{R}, N, A)$, we have
$$d\sigma' \cdot q, r) = d\sigma \cdot q, r) \sigma_z(\nu(n'_{s', s}(q))) \alpha_q(\sigma_z(\nu(n'_{s', s}(r)))) \times \sigma_z(\nu(n'_{s', s}(qr)))^{-1};$$
$$d\bar{\sigma} \cdot q, r) = d\bar{\sigma} \cdot q, r) \sigma_z(\bar{\nu}(n'_{s', s}(q))) \alpha_q(\sigma_z(\bar{\nu}(n'_{s', s}(r)))) \times \sigma_z((\bar{\nu}(n'_{s', s}(qr)))^{-1};$$
$$(d\sigma' d\bar{\sigma}) \cdot q, r = (d\sigma \cdot q, r) \sigma_z(\nu(n'_{s', s}(q))) \alpha_q(\sigma_z(\nu(n'_{s', s}(r)))) \times \sigma_z(\nu(n'_{s', s}(qr)))^{-1};$$
$$= (d\sigma \cdot q, r) \sigma_z(\nu(\bar{\nu}(n'_{s', s}(q)))) \alpha_q(\sigma_z(\nu(\bar{\nu}(n'_{s', s}(r)))) \times \sigma_z((\nu(\bar{\nu}(n'_{s', s}(qr))))^{-1};$$
$$\times \partial_\theta(a(\nu(n'_{s', s}(q))), \bar{\nu}(n'_{s', s}(q)))) \partial_\theta(a(\nu(n'_{s', s}(r)), \bar{\nu}(n'_{s', s}(r)))) \times \partial_\theta(a(\nu(n'_{s', s}(qr)), \bar{\nu}(n'_{s', s}(qr))))^{-1};$$
$$= d(\sigma \bar{\sigma}) \cdot q, r)(\partial_\theta \partial_Q b)(q, r),$$
where $b \in C^1_\alpha(Q, A)$ is given by
$$b(q) = a(\nu(n'_{s', s}(q)), \bar{\nu}(n'_{s', s}(q))) \in A.$$ 
Also we have
$$(c\bar{c})'_Q = (c\bar{c})_Q = c_Q \bar{c}_Q (\partial_Q \partial_Q b) = c'_Q \bar{c}'_Q.$$
Therefore, we get \([c'][\bar{c}'] = [(c \bar{c})']\) in \(H^3_{\alpha,s}(\widetilde{Q}, A)\) by Lemma 2.5, i.e., \(\sigma_{s',s}\) is multiplicative.

The chain rule follows from the definition of \(\sigma_{s',s}\). We leave it to the reader. The chain rule also gives that the map \(\sigma_{s',s}\) is an isomorphism.

The chain rule (2.29) allows us to define the cohomology group independent of the cross-section in the following way: first let \(S\) be the set of all cross sections \(s: Q \mapsto G\) of the homomorphism \(\pi\) and set:

\[
H^\text{out}_{\alpha}(G, N, A) = \left\{ ([c], \nu)_s : s \in S \right\} \in \prod_{s \in S} H^\text{out}_{\alpha,s}(G \times \mathbb{R}, N, A) : (\nu, [c])_{s'} = \sigma_{s',s}(\nu, [c]), \quad s', s \in S \right\}.
\]

(2.30)

Definition 2.19. The group \(H^\text{out}_{\alpha}(G, N, A)\) will be called the modular obstruction group. Each pair \((c, \zeta) \in Z^3_{\alpha,s}(\widetilde{Q}, A) \times \text{Map}(N, Z^1_\theta)\) which gives rise to an element \(([c], [\zeta]) \in H^\text{out}_{\alpha}(G, N, A)\) will be called a modular obstruction cocycle.

§3. Outer Actions of a Discrete Group on a Factor

Let \(\mathcal{M}\) be a separable factor. Associated with \(\mathcal{M}\) is the characteristic square:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & \mathbb{T} & \mathbb{A} \\
\downarrow & \downarrow & \downarrow \\
\mathbb{1} & \mathcal{U}(\mathcal{M}) & \mathcal{U}(\mathcal{M}) \\
\downarrow & \downarrow & \downarrow \\
\mathcal{A}d & \mathcal{A}d & \mathcal{A}d \\
\downarrow & \downarrow & \downarrow \\
\mathcal{I}nt(\mathcal{M}) & \mathcal{C}nt_r(\mathcal{M}) & \mathcal{H}^1_\theta(\mathbb{R}, A) \\
\downarrow & \downarrow & \downarrow \\
1 & 1 & 1
\end{array}
\]

(3.1)
where $A = \mathcal{U}(\mathcal{C})$ is the unitary group of the flow $\{\mathcal{C}, \mathbb{R}, \theta\}$ of weights on $M$, which is $\text{Aut}(M) \times \mathbb{R}$-equivariant. Applying the previous section to the groups

$$H = \text{Aut}(M), \ M = \text{Int}(M), \ G = \text{Out}(M), \ N = \text{Cnt}_r(M)$$

$$Q = \text{Out}_{r, \theta}(\tilde{M}) = \text{Out}(M)/H_1^1(\mathbb{R}, A) = \text{Aut}(M)/\text{Cnt}_r(M),$$

we obtain the intrinsic invariant and the intrinsic modular obstruction:

$$\Theta(M) \in \Lambda_{\text{mod} \times \theta}(\text{Aut}(M) \times \mathbb{R}, \text{Cnt}_r(M), A);$$

$$\text{Ob}_m(M) \in H_{\text{out} \mod \times \theta, s}^3(\text{Out}(M), H_1^1(\mathbb{R}, A), A).$$

Choosing a cross-section: $g \in \text{Out}(M) \mapsto \alpha_g \in \text{Aut}(M)$, we obtain an outer action of $\text{Out}(M)$ on $M$, i.e.,

$$\alpha_g \circ \alpha_h \equiv \alpha_{gh} \mod \text{Int}(M), \ g, h \in \text{Out}(M);$$

$$\alpha_{id} = \text{id}; \ \alpha_g \not\in \text{Int}(M) \ \text{if} \ g \not= \text{id}.$$}

Choosing $\{u(g, h) \in \mathcal{U}(M) : g, h \in \text{Out}(M)\}$ so that

$$\alpha_g \circ \alpha_h = \text{Ad}(u(g, h))\alpha_{gh}, \ \ g, h \in \text{Out}(M),$$

we obtain a 3-cocycle $c \in Z^3(\text{Out}(M), \mathbb{T})$:

$$c(g, h, k) = \alpha_g(u(h, k))u(g, h)\{u(g, h)u(gh, k)\}^*, \ \ g, h, k \in \text{Out}(M).$$

Its cohomology class $[c] \in H^3(\text{Out}(M), \mathbb{T})$ does not depend on the choice of the cross-section $\alpha : g \in \text{Out}(M) \mapsto \alpha_g \in \text{Aut}(M)$ nor on the choice of $\{u(g, h)\}$. The intrinsic obstruction $\text{Ob}(M) = [c]$ of $M$ is, by definition, the cohomology class $[c] \in H^3(\text{Out}(M), \mathbb{T})$.

**Proposition 3.1.** The intrinsic obstruction $\text{Ob}(M)$ of the factor $M$ is the image $\partial(\text{Ob}_m(M))$ of the intrinsic modular obstruction $\text{Ob}_m(M)$ of $M$ under the map

$$\partial : H_{\text{out} \mod \times \theta, s}^3(\text{Out}(M), H_1^1(\mathbb{R}, A)) \mapsto H^3(\text{Out}(M), \mathbb{T})$$
given by Lemma 2.11.

**Proof.** In the notations in the last section, we take \( \text{Aut}(\mathcal{M}) \) for \( H \), \( \text{Out}(\mathcal{M}) \) for \( G \), \( \text{Out}_{\tau,\theta}(\tilde{\mathcal{M}}) \) for \( Q \), \( \text{Cnt}_{\tau}(\mathcal{M}) \) for \( L \), \( \text{Int}(\mathcal{M}) \) for \( M \) and \( N \) for \( H_{\theta}^1(\mathbb{R}, A) \). Then with

\[
\bar{\chi} = \Theta(\mathcal{M}) \in \Lambda_{\text{mod} \times \theta}(\text{Aut}(\mathcal{M}) \times \mathbb{R}, \text{Cnt}_\tau(\mathcal{M}), A)
\]

the characteristic square (3.1) gives that \( M = K(\bar{\chi}) \). The characteristic invariant \( \chi \in \Lambda(\text{Aut}(\mathcal{M}), \text{Int}(\mathcal{M}), \mathbb{T}) \) associated with the \( \text{Aut}(\mathcal{M}) \)-equivariant exact sequence:

\[
1 \longrightarrow \mathbb{T} \longrightarrow U(\mathcal{M}) \longrightarrow \text{Int}(\mathcal{M}) \longrightarrow 1
\]

is precisely \( \chi = i_{\text{Cnt}_{\tau}(\mathcal{M}), \text{Int}(\mathcal{M})}^*(\bar{\chi}) \) the pull back in Lemma 2.11. Then the obstruction \( \text{Ob}(\mathcal{M}) = \delta_{\text{HJR}}(\chi) \) is \( \partial(\text{Ob}_m(\mathcal{M})) = \partial(\delta(\bar{\chi})) \) by Lemma 2.11. \( \diamond \)

Therefore, the modular obstruction \( \text{Ob}_m(\mathcal{M}) \) contains the information carried by the obstruction \( \text{Ob}(\mathcal{M}) \).

Let \( G \) be a countable group. Fix a free outer action \( \alpha \) of \( G \) on \( \mathcal{M} \). If \( \dot{\alpha}_g \) is the class of \( \alpha_g \) in \( \text{Out}(\mathcal{M}) \), then the map \( \dot{\alpha} : g \in G \mapsto \dot{\alpha}_g \in \text{Out}(\mathcal{M}) \) is an injective homomorphism. With \( N = \dot{\alpha}^{-1}(H_{\theta}^1(\mathbb{R}, A)) \triangleleft G \), the quotient map \( \pi : g \in G \mapsto \pi(g) = gN \in Q = G/N \) and a cross-section \( s : Q \rightarrow G \) of \( \pi \), we get the modular obstruction

\[
\text{Ob}_m(\alpha) \in H_{\text{out}, s}^1(G \times \mathbb{R}, N, A).
\]

Two outer actions \( \alpha \) and \( \beta \) of \( G \) on the same factor \( \mathcal{M} \) are, by definition, **outer conjugate** if there exists an automorphism \( \sigma \in \text{Aut}(\mathcal{M}) \) such that

\[
\dot{\beta}_g = \dot{\sigma} \dot{\alpha}_g \dot{\sigma}^{-1}, \quad g \in G,
\]

where \( \dot{\sigma} \in \text{Out}(\mathcal{M}) \) is the class of \( \sigma \) in \( \text{Out}(\mathcal{M}) \), i.e., \( \dot{\sigma} = \sigma \text{Int}(\mathcal{M}) \in \text{Out}(\mathcal{M}) \).

**Theorem 3.2.** Let \( G \) be a countable discrete group and \( \mathcal{M} \) a separable infinite factor with flow of weights \( \{ \mathcal{C}, \mathbb{R}, \theta \} \). Suppose that \( \alpha : g \in G \mapsto \alpha_g \in \mathcal{M}. \)
\textbf{Outer Conjugacy}

\textit{Aut}(\mathcal{M}) \text{ is a free outer action of } G \text{ on } \mathcal{M}. \text{ Set } N = N(\alpha) = \alpha^{-1}(\text{Cnt}_\tau(\mathcal{M})), \text{ } \mathcal{Q} = G/N \text{ and fix a cross-section } \varsigma: \mathcal{Q} \hookrightarrow G \text{ of the quotient map } \pi: G \hookrightarrow \mathcal{Q}.

i) The modular obstruction:

\[ \text{Ob}_m(\alpha) \in \mathbb{H}^3_{\text{mod}(\alpha) \times \theta}(\mathcal{Q} \times \mathbb{R}, \mathbb{U}(\mathcal{C})) \ast \text{Hom}_G(N, H^1_{\theta}(\mathbb{R}, \mathbb{U}(\mathcal{C}))) \]

is an invariant for the outer conjugacy class of \( \alpha \).

\[ \sigma \circ \text{mod}(\alpha_g) \circ \sigma^{-1} = \text{mod}(\beta_g), \quad g \in G; \quad \sigma_*(\text{Ob}_m(\alpha)) = \text{Ob}_m(\beta), \]

then the automorphism \( \sigma \) of \( \mathcal{C} \) can be extended to an automorphism denoted by \( \sigma \) again to the non-commutative flow of weights \( \{ \tilde{\mathcal{M}}, \mathbb{R}, \theta, \tau \} \) such that

\[ \sigma \circ \alpha_g \circ \sigma^{-1} \equiv \beta_g \text{ mod Int}(\mathcal{M}), \quad g \in G. \]

\textbf{Proof.} We continue to denote the unitary group \( \mathbb{U}(\mathcal{C}) \) by \( A \) for short. Let \([c^\alpha] \in \mathbb{H}^3(G, \mathbb{T})\) be the obstruction \( \text{Ob}(\alpha) \) and \( c^\alpha \in Z^3(G, \mathbb{T}) \) represent \([c]\) which is obtained by fixing a family \( \{u(g, h) \in \mathcal{U}(\mathcal{M}) : g, h \in G\} \) such that

\[ \alpha_g \circ \alpha_h = \text{Ad}(u(g, h)) \circ \alpha_{gh}, \quad g, h \in G, \]

and by setting

\[ c^\alpha(g, h, k) = \alpha_g(u(h, k))u(g, hk)\{u(g, h)u(gh, k)\}^* \in \mathbb{T}, \quad g, h, k \in G. \]

Let \( \pi_G: H = H(c^\alpha) \hookrightarrow G \) be the resolution group of the cocycle \( c^\alpha \in Z^3(G, \mathbb{T}) \) and the resolution map, i.e., \( \pi^*_{G}(c^\alpha) \in B^3(H, \mathbb{T}). \) Choose \( b: h \in H \hookrightarrow b(h) \in \mathbb{T} \) such that

\[ c^\alpha(\pi_G(g), \pi_G(h), \pi_G(k)) = b(h, k)b(g, hk)\{b(g, h)b(gh, k)\}^*, \quad g, h, k \in H. \]
OUTER CONJUGACY

Setting
\[ \bar{u}_H(g, h) = b(g, h)^* u(\pi_G(g), \pi_G(h)), \quad g, h \in H, \]
we obtain
\[ \alpha_{\pi_G(g)}(\bar{u}_H(h, k)) \bar{u}_H(g, hk) \{ \bar{u}_H(g, h) \bar{u}_H(gh, k) \}^* = 1. \]
Hence \( \{ \alpha_{\pi_G}, \bar{u}_H \} \) is a cocycle twisted action of \( H \). Then by [ST1: Theorem 4.13, page 156], there exists a family \( \{ v_H(g) \in \mathcal{U}(M) : g \in H \} \) such that
\[ \bar{u}_H(g, h) = \alpha_{\pi_G(g)}(v_H(h)^*) v_H(g)^* v_H(gh), \quad g, h \in H, \]
so that the map
\[ g \in H \mapsto \beta_g = \text{Ad}(v_H(g)) \circ \alpha_{\pi_G(g)} \in \text{Aut}(M) \]
is an action of \( H \) on \( M \) as seen below:
\[
\begin{align*}
\beta_g \circ \beta_h &= \text{Ad}(v_H(g)) \circ \alpha_{\pi_G(g)} \circ \text{Ad}(v_H(h)) \circ \alpha_{\pi_G(h)} \\
&= \text{Ad}(v_H(g) \alpha_{\pi_G(g)}(v_H(h))) \circ \alpha_{\pi_G(g)} \circ \alpha_{\pi_G(h)} \\
&= \text{Ad}(v_H(g) \alpha_{\pi_G(g)}(v_H(h))) \circ \text{Ad}(\bar{u}_H(g, h)) \circ \alpha_{\pi_G(gh)} \\
&= \text{Ad}(v_H(gh)) \circ \alpha_{\pi_G(gh)} = \beta_{gh}, \quad g, h \in H.
\end{align*}
\]
Therefore, the outer action \( \alpha_{\pi_G} \) is perturbed to an action \( \beta \) by inner automorphisms. With \( s_H \) a cross-section of \( \pi_G \), the map \( \hat{\beta} : g \in G \mapsto \beta_{s_H(g)} \in \text{Aut}(M) \) is an outer action of \( G \) on \( M \) which is an inner perturbation of the original outer action \( \alpha \) because
\[ \hat{\beta}_g = \beta_{s_H(g)} = \text{Ad}(v_H(s_H(g))) \circ \alpha_{\pi_G(s_H(g))} \]
\[ = \text{Ad}(v_H(s_H(g))) \circ \alpha_g, \quad g \in G. \]
Hence we may and do replace the outer action \( \alpha \) by \( \hat{\beta} \). Then the outer action \( \alpha \) is given by an action \( \beta \) of \( H \) in the following way:
\[ \alpha_g = \beta_{s_H(g)}, \quad g \in G. \]
The action $\beta$ of $H$ gives rise to the characteristic invariant $\chi(\beta) \in \Lambda(H, M, \mathbb{T})$ with $M = \text{Ker}(\pi_G) = \alpha^{-1}(\text{Int}(M))$, so that the obstruction $\text{Ob}(\alpha)$ becomes the image $\delta_{\text{HJR}}(\chi(\beta))$ of $\chi(\beta)$ under the HJR-map $\delta_{\text{HJR}}$.

i) We only need to prove that the modular obstruction is unchanged under the perturbation by inner automorphisms. Choose $\{w(p, q) : p, q \in Q\} \subset \widehat{\mathcal{U}(M)}$ so that

$$\alpha_{p}\circ\alpha_{q} = \tilde{\text{Ad}}(w(p, q))\circ\alpha_{pq}, \quad p, q \in Q,$$

where $\alpha_{p}$ means $\alpha_{s(p)}$ for short. We write $\alpha_{\tilde{p}} \in \text{Aut}(\widehat{M})$ for $\alpha_{p}\circ\theta_{s}, \tilde{p} = (p, s) \in \tilde{Q} = Q \times \mathbb{R}$. Then for each triple $\tilde{p} = (p, s), \tilde{q} = (q, t), \tilde{r} = (r, u) \in \tilde{Q}$, the cocycle $c = c^\alpha$ representing $\text{Ob}_m(\alpha)$ is given by:

$$c^\alpha(\tilde{p}, \tilde{q}, \tilde{r}) = \alpha_{\tilde{p}}(w(q, r))w(p, qr)\{w(p, q)w(pq, r)\}^*$$

$$= \alpha_{p}(\theta_{s}(w(q, r))w(q, r)^*)\alpha_{p}(w(q, r))w(p, qr)\{w(p, q)w(pq, r)\}^*$$

$$= \alpha_{p}(d(s; q, r)c_{Q}(p, q, r),$$

where

$$d(s; q, r) = \theta_{s}(w(q, r))w(q, r)^*;$$

$$c_{Q}(p, q, r) = \alpha_{p}(w(q, r))w(p, qr)\{w(p, q)w(pq, r)\}^*.$$

The $G$-equivariant homomorphism $\nu : N \mapsto H^1_{\theta}(\mathbb{R}, A)$ is given by $\nu_{\alpha}(m) = \partial_{\theta}(\alpha_{m}) \in H^1_{\theta}(\mathbb{R}, A), m \in N$. Let $\{v(g) : g \in G\} \subset \mathcal{U}(M)$ and set

$$\beta_{g} = \text{Ad}(v(g))\circ\alpha_{g}, \quad g \in G.$$
Therefore, we get

\[ c^\beta(p, q, r) = \beta_p \left( v(s(q))\alpha_q(v(s(r)))w(q, r)v(s(qr))^* \right) \]
\[ \times v(s(p))\alpha_p(v(s(qr))w(p, qr)v(s(pqr))^* \]
\[ \times \left( v(s(p))\alpha_p(v(s(q))w(p, q)v(s(pq))^* \right) \]
\[ \times v(s(pq))\alpha_{pq}(v(s(r))w(pq, r)v(s(pqr))^* \right) \]
\[ = \beta_p \left\{ \theta_s \left( v(s(q))\alpha_q(v(s(r)))w(q, r)v(s(qr))^* \right) \right\} \]
\[ \times \left( v(s(q))\alpha_q(v(s(r)))w(q, r)v(s(qr))^* \right)^* \}
\[ \times \beta_p \left( v(s(q))\alpha_q(v(s(r)))w(q, r)v(s(qr))^* \right) \]
\[ \times v(s(p))\alpha_p(v(s(qr))w(p, qr)v(s(pqr))^* \]
\[ \times \left( v(s(p))\alpha_p(v(s(q))w(p, q)v(s(pq))^* \right) \]
\[ \times v(s(pq))\alpha_{pq}(v(s(r))w(pq, r)v(s(pqr))^* \right) \]
\[ = v(s(p))\alpha_p \left( \theta_s(w(q, r))w(q, r)^* \right) v(s(p))^* \]
\[ \times v(s(p))\alpha_p \left( v(s(q))\alpha_q(v(s(r)))w(q, r)v(s(qr))^* \right) v(s(p))^* \]
\[ \times v(s(p))\alpha_p(v(s(qr))w(p, qr)v(s(pqr))^* \]
\[ \times \left( v(s(p))\alpha_p(v(s(q))w(p, q)v(s(pq))^* \right) \]
\[ \times v(s(pq))\alpha_{pq}(v(s(r))w(pq, r)v(s(pqr))^* \right) \]
\[ = \alpha_p \left( \theta_s(w(q, r))w(q, r)^* \right) \alpha_p \left( \alpha_q(v(s(r)))w(q, r) \right) \]
\[ \times w(p, qr) \left( w(p, q)\alpha_{pq}(v(s(r))w(pq, r) \right) \]
\[ = \alpha_p \left( \theta_s(w(q, r))w(q, r)^* \right) w(p, q)\alpha_{pq}(v(s(r)))w(p, q) \]
\[ \times \alpha_p(w(q, r))w(p, qr) \left( w(p, q)\alpha_{pq}(v(s(r))w(pq, r) \right)^* \]
Therefore, the inner perturbation $\beta$ of the outer action $\alpha$ of $G$ does not change the modular obstruction cocycle $c^\alpha$, i.e., $c^\alpha = c^\beta$ as seen above. Hence $\text{Ob}_m(\alpha) = \text{Ob}_m(\beta)$. This proves the assertion (i).

ii) Assume that $M$ is an approximately finite dimensional factor with non-commutative flow $\{\tilde{M}, \mathbb{R}, \theta, \tau\}$ of weights and the flow $\{\mathcal{C}, \mathbb{R}, \theta\}$ of weights on $M$, and suppose that $G$ is a countable discrete amenable group. Let $\hat{\alpha}$ and $\hat{\beta}$ be outer actions of $G$ on $M$ such that

a) $N = N(\hat{\alpha}) = \hat{\alpha}^{-1} (\text{Cnt}(M)) = N(\hat{\beta}) = \hat{\beta}^{-1} (\text{Cnt}(M))$;

b) $\text{mod}(\hat{\alpha}_g) = \text{mod}(\hat{\beta}_g), \ g \in G$;

c) with $Q = G/N$ and $A = \mathcal{U}(\mathcal{C})$

$$(\text{Ob}_m(\hat{\alpha}), \nu_{\hat{\alpha}}) = (\text{Ob}_m(\hat{\beta}), \nu_{\hat{\beta}}) \in H^3_{\alpha, s}(Q \times \mathbb{R}, A) \ast \text{Hom}_G(N, H^1_\theta(\mathbb{R}, A)).$$

We want to conclude from this data that the outer actions $\hat{\alpha}$ and $\hat{\beta}$ of $G$ are outer conjugate. The assumption (c) implies that

$$\text{Ob}(\hat{\alpha}) = \partial((\text{Ob}_m(\hat{\alpha}), \nu_{\hat{\alpha}})) = \partial((\text{Ob}_m(\hat{\beta}), \nu_{\hat{\beta}})) = \text{Ob}(\hat{\beta}) \in H^3(G, \mathbb{T}).$$

Therefore, we may and do choose the same obstruction cocycle $c = c^{\hat{\alpha}} = c^{\hat{\beta}}$, which allows us to pick up the common resolution system $\pi_{c}: H = H(c) \mapsto G$ and actions $\alpha$ and $\beta$ of $H$ on $M$ which give $\hat{\alpha}$ and $\hat{\beta}$ respectively:

$$\hat{\alpha}_g = \alpha_{\text{aut}(g)}, \ \hat{\beta}_g = \beta_{\text{aut}(g)}, \ g \in G.$$ 

First, the resolution group $H$ is amenable by Lemma 2.14 and the actions $\alpha$ and $\beta$ of $H$ give rise to the following invariants:

$$L = \pi_{G}^{-1}(N) = \alpha^{-1}(\text{Cnt}(M)) = \beta^{-1}(\text{Cnt}(M)), \quad \chi_{\text{m}}(\alpha), \ \chi_{\text{m}}(\beta) \in \Lambda_{\text{mod}(\alpha) \times \theta}(H \times \mathbb{R}, L, A).$$

Since $M = \text{Ker}(\pi_{c}) = \alpha^{-1}(\text{Int}(M)) = \beta^{-1}(\text{Int}(M))$, we have

$$M = K(\chi_{\text{m}}(\alpha)) = K(\chi_{\text{m}}(\beta)).$$
Therefore, the modular characteristic invariant $\chi_m(\alpha)$ and $\chi_m(\beta)$ are both members of $\Lambda_{\alpha \times \theta}(\tilde{H}, L, M, A)$ with $\tilde{H} = H \times \mathbb{R}$, where we are now going to use $\alpha$ for $\text{mod}(\alpha) = \text{mod}(\beta)$. The resolution system $\{H, \pi_G\}$ generates the following modified HJR-exact sequence:

$$
\cdots \longrightarrow H^2(H, \mathbb{T}) \xrightarrow{\text{Res}} \Lambda_{\alpha \times \theta}(\tilde{H}, L, M, A) \xrightarrow{\delta} H^3(\tilde{Q}, A) \xrightarrow{\partial} H^3(H, \mathbb{T}),
$$

such that

$$
\delta(\chi_m(\alpha)) = \text{Ob}_m(\hat{\alpha}) = \text{Ob}_m(\hat{\beta}) = \delta(\chi_m(\beta)).
$$

With this, our assertion (ii) follows immediately from the next theorem.

**Theorem 3.3.** Let $\alpha$ and $\beta$ be two actions of a countable discrete group $H$ on an infinite factor $M$ with $L = \alpha^{-1}(\text{Cnt}_r(M)) = \beta^{-1}(\text{Cnt}_r(M))$ and $M = \alpha^{-1}(\text{Int}(M)) = \beta^{-1}(\text{Int}(M))$. Let $G = H/M$ and $\pi_G : H \mapsto G$ be the quotient map. Suppose that $s_H : G \mapsto H$ is a cross-section and set

$$
\hat{\alpha}_g = \alpha_{s_H(g)}, \quad \hat{\beta}_g = \beta_{s_H(g)}, \quad g \in G,
$$

to obtain outer actions $\hat{\alpha}$ and $\hat{\beta}$ of $G$ on $M$.

i) The two outer actions $\hat{\alpha}$ and $\hat{\beta}$ of $G$ are outer conjugate if and only if the two original actions $\alpha$ and $\beta$ of $H$ are outer conjugate.

ii) If the two actions $\alpha$ and $\beta$ of $H$ on $M$ are outer conjugate, then there exists an automorphism $\sigma \in \text{Aut}_\theta(\mathcal{C})$ such that

a) $\text{mod}(\alpha) = \sigma \circ \text{mod}(\beta) \circ \sigma^{-1}$;

b) their modular characteristic invariants $\chi_m(\alpha) \in \Lambda_{\alpha \times \theta}(\tilde{H}, L, M, A)$ and $\sigma_*(\chi_m(\beta)) \in \Lambda_{\alpha \times \theta}(\tilde{H}, L, M, A)$, where $\tilde{H} = H \times \mathbb{R}$, have the same image:

$$
\text{Ob}_m(\hat{\alpha}) = \delta(\chi_m(\alpha)) = \delta(\sigma_*(\chi_m(\beta))) = \text{Ob}_m(\sigma \hat{\beta} \sigma^{-1})
$$

in the group $H^3_{\alpha \times \theta}(\tilde{Q}, A) \ast \text{Hom}_G(N, H^1_\theta(\mathbb{R}, A))$ under the modified HJR-map $\delta$, where $N = L/M$, $Q = H/L = G/N$ and $\tilde{Q} = Q \times \mathbb{R}$. 
iii) If $M$ is an approximately finite dimensional infinite factor and $H$ is amenable in addition, then the existence of an automorphism $\sigma \in \text{Aut}_\theta(\mathcal{C})$ such that:

a) $\text{mod}(\alpha) = \sigma \circ \text{mod}(\beta) \circ \sigma^{-1}$;

b) their modular characteristic invariants $\chi_m(\alpha) \in \Lambda_{\alpha \times \theta}(\tilde{H}, L, M, A)$ and $\sigma_*(\chi_m(\beta)) \in \Lambda_{\alpha \times \theta}(\tilde{H}, L, M, A)$, where $\tilde{H} = H \times \mathbb{R}$, have the same image:

$$\delta(\chi_m(\alpha)) = \delta(\sigma_*(\chi_m(\beta))) \in H^3_{\alpha,s}(\tilde{Q}, A) \ast \text{Hom}_G(N, \mathcal{H}^1_\theta(\mathbb{R}, A)),$$

is sufficient for $\alpha$ and $\beta$ to be outer conjugate.

Proof. i) It is obvious that the outer conjugacy of the outer actions $\dot{\alpha}$ and $\dot{\beta}$ of $G$ follows from that of the original actions $\alpha$ and $\beta$ of $H$. So suppose that the outer actions $\dot{\alpha}$ and $\dot{\beta}$ of $G$ are outer conjugate, which means the existence of an automorphism $\sigma \in \text{Aut}(M)$ and a family $\{u(g) : g \in G\} \subset U(M)$ of unitaries such that

$$\text{Ad}(u(g))\circ \dot{\alpha}_g = \sigma \circ \dot{\beta}_g \circ \sigma^{-1}, \quad g \in G.$$

Writing each $h \in H$ in the form:

$$h = m_M(h)s_H(\pi_G(h)), \quad h \in H, \quad m_M(h) \in M,$

we have

$$\alpha_h = \alpha_{m_M(h)} \circ \dot{\alpha}_{\pi_G(h)};$$

$$\beta_h = \beta_{m_M(h)} \circ \dot{\beta}_{\pi_G(h)}, \quad h \in H.$$

As $\alpha_m$ and $\beta_m$ are inner for each $m \in M$, they are in the following form:

$$\alpha_m = \text{Ad}(v(m)), \quad \beta_m = \text{Ad}(w(m)), \quad m \in M,$$

for some $v(m), w(m) \in U(M)$. Therefore, we have, for each $h \in H$,

$$\sigma \circ \beta_h \circ \sigma^{-1} = \sigma \circ \beta_{m_M(h)} \circ \dot{\beta}_{\pi_G(h)} \circ \sigma^{-1} = \sigma \circ \beta_{m_M(h)} \circ \dot{\beta}_{\pi_G(h)} \circ \sigma^{-1}$$

$$= \sigma \circ \text{Ad}(w(m_M(h))) \circ \sigma^{-1} \circ \dot{\beta}_{\pi_G(h)} \circ \sigma^{-1}$$

$$= \text{Ad}(\sigma(w(m_M(h)))) \circ \text{Ad}(u(\pi_G(h))) \circ \dot{\alpha}_{\pi_G(h)}$$

$$= \text{Ad}(\sigma(w(m_M(h)))) \circ \text{Ad}(u(\pi_G(h))) \circ \dot{\alpha}_{m_M(h)} \circ \alpha_h$$

$$= \text{Ad}(\sigma(w(m_M(h)))) \circ \text{Ad}(u(\pi_G(h))) \circ \text{Ad}(v(m_M(h))) \circ \dot{\alpha}_{m_M(h)} \circ \alpha_h$$

$$= \text{Ad}(u(h)) \circ \dot{\alpha}_h,$$
where \( u(h) = \sigma(w(m_M(h)))u(\pi_G(h))v(m_M(h))^* \). Hence the actions \( \alpha \) and \( \beta \) of \( H \) are outer conjugate.

ii) Assume that the two actions \( \alpha \) and \( \beta \) of \( H \) on \( M \) are outer conjugate. Then there exist \( \sigma \in \text{Aut}(M) \) and a family \( \{ u(h) : h \in H \} \subset \mathcal{U}(M) \) such that \( u(1) = 1 \) and

\[
\sigma \circ \beta_h \circ \sigma^{-1} = \text{Ad}(u(h)) \circ \alpha_h, \quad h \in H.
\]

Since \( \text{Int}(M) \) acts on the flow of weights trivially, we have

\[
\text{mod}(\sigma) \circ \text{mod}(\beta_h) \circ \text{mod}(\sigma)^{-1} = \text{mod}(\alpha_h), \quad h \in H,
\]

and conclude that \( \text{mod}(\sigma) \) conjugates \( \text{mod}(\alpha) \) and \( \text{mod}(\beta) \), i.e., the assertion (a). Replacing \( \beta_g, g \in H \), by \( \sigma \circ \beta_g \circ \sigma^{-1}, g \in H \), we may and do assume from now on for short that \( \text{mod}(\alpha) = \text{mod}(\beta) \) and

\[
\beta_g = \text{Ad}(u(h)) \circ \alpha_h, \quad h \in H.
\]

As \( \alpha \) and \( \beta \) are both actions, we have

\[
\text{Ad}(u(gh)) \circ \alpha_{gh} = \beta_{gh} = \beta_g \circ \beta_h = \text{Ad}(u(g)) \circ \alpha_g \circ \text{Ad}(u(h)) \circ \alpha_h
\]

\[
= \text{Ad}(u(g)\alpha_g(u(h))) \circ \alpha_g \circ \alpha_h
\]

\[
= \text{Ad}(u(g)\alpha_g(u(h))) \circ \alpha_{gh}, \quad g, h \in H.
\]

Thus we get

\[
\mu(g, h) = u(g)\alpha_g(u(h))u(gh)^* \in \mathbb{T}, \quad g, h \in H,
\]

and \( \mu \in Z^2(H, \mathbb{T}) \). Each \( \alpha_m, m \in L \), falls in \( \text{Cnt}_r(M) \), so that it is of the form:

\[
\alpha_m = \widetilde{\text{Ad}}(v(m)), \quad v(m) \in \widetilde{\mathcal{U}}(M).
\]

As \( \beta_m = \text{Ad}(u(m)) \circ \alpha_m \), we may choose \( w(m) = u(m)v(m), m \in L \). The unitary families \( \{ v(m) : m \in L \} \) and \( \{ w(m) : m \in L \} \) generate the corresponding modular characteristic cocycles:

\[
\alpha_{\tilde{g}}(v(g^{-1}mg)) = \lambda_\alpha(m; g, s)v(m); \quad \beta_{\tilde{g}}(w(g^{-1}mg)) = \lambda_\beta(m; g, s)w(m);
\]

\[
v(m)v(n) = \mu_\alpha(m, n)v(mn); \quad w(m)w(n) = \mu_\beta(m, n)w(mn),
\]
with $\tilde{g} = (g, s) \in \tilde{H}$ and $m, n \in L$. Now we take a closer look at $(\lambda_\beta, \mu_\beta) \in Z_\alpha(\tilde{H}, L, A)$:

$$
\mu_\beta(m, n) = w(m)w(n)w(mn)^*, \quad m, n \in L,
$$
$$
= (u(m)v(m))(u(n)v(n))(u(mn)v(mn))^*
$$
$$
= u(m)\alpha_m(u(n))v(m)v(n)v(mn)^*u(mn)^*
$$
$$
= u(m)\alpha_m(u(n))\mu_\alpha(m, n)u(mn)^*
$$
$$
= \mu_\alpha(m, n)u(m)\alpha_m(u(n))u(mn)^*
$$
$$
= \mu_\alpha(m, n)\mu(m, n),
$$

and for $(m, \tilde{g}) = (m, g, s) \in L \times \tilde{H}$

$$
\lambda_\beta(m; \tilde{g})u(m)v(m) = \lambda_\beta(m; \tilde{g})w(m)
$$
$$
= \beta_{\tilde{g}}(w(g^{-1}mg))
$$
$$
= \Ad(u(g))\alpha_{\tilde{g}}\left(u(g^{-1}mg)v(g^{-1}mg)\right)
$$
$$
= u(g)\alpha_g(u(g^{-1}mg))\lambda_\alpha(m; \tilde{g})v(m)u(g)^*
$$
$$
= \mu(g, g^{-1}mg)u(mg)\lambda_\alpha(m; \tilde{g})v(m)u(g)^*
$$
$$
= \lambda_\alpha(m; \tilde{g})\mu(g, g^{-1}mg)\mu(m, g)^*u(m)\alpha_m(u(g))v(m)u(g)^*
$$
$$
= \lambda_\alpha(m; \tilde{g})\mu(g, g^{-1}mg)\mu(m, g)^*u(m)v(m),
$$

Therefore the characteristic cocycles $(\lambda_\beta, \mu_\beta) \in Z_{\alpha \times \theta}(\tilde{H}, L, A)$ is of the form:

$$
\mu_\beta(m, n) = \mu(m, n)\mu_\alpha(m, n), \quad m, n \in L;
$$
$$
\lambda_\beta(m; \tilde{g}) = \mu(g, g^{-1}mg)\mu(m, g)^*\lambda_{\alpha}(m; \tilde{g}), \quad \tilde{g} = (g, s) \in \tilde{H}.
$$

Thus we conclude that $\chi_m(\chi(\beta)) = \text{Res}([\mu])\chi_m(\chi(\alpha))$ in $\Lambda_{\alpha \times \theta}(\tilde{H}, L, M, A)$. In virtue of Theorem 2.7, this is equivalent to the fact that

$$
\text{Ob}_m(\hat{\alpha}) = \text{Ob}_m(\hat{\beta}) \in H^3_{\alpha, s}(\tilde{Q}, A) \ast_{\ast} \text{Hom}_G(N, H^1_{\theta}).
$$

iii) Suppose that $\mathcal{M}$ is an infinite AFD factor and $H$ is amenable. The automorphism $\sigma \in \text{Aut}_\theta(\mathcal{C})$ can be extended to an automorphism in $\text{Aut}_{\tau, \theta}(\tilde{M})$
by \cite{ST3} which will be denoted again by $\sigma$. Replacing $\{\beta_g : g \in H\}$ by $\{\sigma \circ \beta_g \circ \sigma^{-1} : g \in H\}$, we may and do assume that $\text{mod}(\alpha) = \text{mod}(\beta)$ and $\delta(\chi_m(\alpha)) = \delta(\chi_m(\beta))$ in the invariant group $H^3_{\alpha,s}(\widetilde{Q}, A) \ast_s \text{Hom}_G(N, H^1_{\theta})$. The modified HJR-exact sequence of Theorem 2.7 yields the existence of a cohomology class $[\mu] \in H^2(H, T)$ such that 

$$\chi_m(\beta) = \text{Res}([\mu])\chi_m(\alpha) \in \Lambda_{\alpha \times \theta}(\widetilde{H}, L, M, A).$$

A cocycle perturbation of $\alpha$, denoted by $\alpha$ again, leaves a subfactor $B$ of type $\text{I}_\infty$ pointwise invariant. Let $u : g \in H \mapsto u(g) \in \mathcal{U}(B)$ be a projective unitary representation of $H$ in $B$ with the multiplier $\mu \in Z^2(H, \mathbb{T})$ representing $[\mu]$ such that

$$u(g)u(h) = \mu(g, h)u(gh), \quad g, h \in H.$$

Set $u_\alpha g = \text{Ad}(u(g)) \circ \alpha_g, g \in H$. Then it is a straightforward calculation to show that $\chi_m(u_\alpha) = \text{Res}([\mu])\chi_m(\alpha)$. Therefore, the characteristic invariant $\chi_m(u_\alpha)$ is precisely $\chi_m(\beta)$ of $\beta$. Hence the cocycle conjugacy classification theorem, \cite{KtST1}, guarantees the concycle conjgacy of $u_\alpha$ and $\beta$. Therefore, the original actions $\alpha$ and $\beta$ are outer conjugate.

\section*{§4. Model Construction}

As laid down in \cite{KtST1}, the construction of a model from a set of invariants is an integral part of the classification theory. It is particularly important here because the invariants associated with outer actions do not form a standard Borel space. For example, the classification functor cannot be Borel in the case of type $\text{III}_0$. So we have to begin with a desingularization of the space of invariants.

We fix an ergodic flow $\{\mathcal{C}, \mathbb{R}, \theta\}$ to begin with. An action $\alpha$ of a group $G$ on the flow $\{\mathcal{C}, \mathbb{R}, \theta\}$ means a homomorphism $g \in G \mapsto \alpha_g \in \text{Aut}_\theta(\mathcal{C})$, where \n
$$\text{Aut}_\theta(\mathcal{C}) = \{\sigma \in \text{Aut}(\mathcal{C}) : \sigma \circ \theta_s = \theta_s \circ \sigma, s \in \mathbb{R}\}.$$

As before, we denote the unitary group $\mathcal{U}(\mathcal{C})$ of $\mathcal{C}$ by $A$ for short. The first cohomology group $H^1_\theta = H^1_\theta(\mathbb{R}, A)$ can not be a standard Borel space if the flow $\theta$ is properly ergodic. So we have to consider the first cocycle group $Z^1_\theta = Z^1_\theta(\mathbb{R}, A)$ instead together with the coboundary subgroup $B^1_\theta = \cdots$
Next we fix a countable discrete amenable group $G$ and an exact sequence:

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1$$

together with a cross-section $s$ which will be fixed throughout as in the previous section and therefore the $N$-valued cocyle $n_N$:

$$n_N(p, q) = s(p)s(q)s(pq)^{-1}, \quad p, q \in Q,$$

is also fixed. Let $\text{Hom}_R(Q, \text{Aut}(\mathcal{C}))$ be the set of all homomorphisms $\alpha : p \in Q \mapsto \alpha_p \in \text{Aut}_\theta(\mathcal{C})$ from $Q$ into the group of all automorphisms of $\mathcal{C}$ commuting with the flow $\theta$. It is easily seen to be a Polish space. Each $\alpha \in \text{Hom}_R(Q, \text{Aut}(\mathcal{C}))$ can be identified with an action of $G$ whose kernel contains $N$. So we view $\alpha$ as an action of $G$ on $\mathcal{C}$ freely whenever necessary. We also use the notations $\tilde{Q} = Q \times \mathbb{R}$ and $\tilde{G} = G \times \mathbb{R}$ freely. We fix the action $\alpha$ of $Q$ and consequently of $G$ on the flow $\{\mathcal{C}, \mathbb{R}, \theta\}$ throughout this section and the joint action $\alpha \times \theta$ will be denoted by the single character $\alpha$ for short. With these data, we have the group of modular obstructions: $H^{\text{out}}_{\alpha, s}(G \times \mathbb{R}, N, A)$ with $A = \mathcal{U}(\mathcal{C})$ which will be fixed throughout this section. The group $H^{\text{out}}_{\alpha, s}(G \times \mathbb{R}, N, A)$ is not a standard Borel group in general, in particular it will never be standard except trivial cases if the flow $\theta$ is properly ergodic. Also there is no way to construct a model directly from an element $([c], \nu) \in H^{\text{out}}_{\alpha, s}(G \times \mathbb{R}, N, A)$ either. We must desingularize the group of invariants first. To this end, we first consider the group $Z^{\text{out}}_{\alpha, s}(G, N, A)$ of modular obstruction cocycles $(c, \zeta)$. However, being an obstruction cocycle, $(c, \zeta)$ does not allow us to construct an outer action of $G$ directly. We recall Corollary 2.17 to find a resolution system:

$$1 \longrightarrow M \longrightarrow H \xrightarrow{\pi_G} G \longrightarrow 1$$

with $H$ a countable discrete group such that

i) the normal subgroup $M$ is abelian;
ii) relative to the modified HJR-exact sequence:

\[
\begin{array}{c}
\cdots \rightarrow H^2(H, \mathbb{T}) \xrightarrow{\text{Res}} \Lambda_\alpha(\tilde{H}, L, M, A) \\
\downarrow \delta \quad \downarrow \partial \\
H^3(G, \mathbb{T}) \xrightarrow{\text{Inf}} H^3(G, \mathbb{T})
\end{array}
\]

where \( L = \pi_G^{-1}(N) \) the Inf-image of \([c], \nu\) vanishes, i.e.,

\[\text{Inf}([c], \nu) = 1 \in H^3(H, \mathbb{T}).\]

Hence there exists a modular characteristic invariant

\[\chi \in \Lambda_\alpha(\tilde{H}, L, M, A)\]

such that \( \delta(\chi) = ([c], \nu) \).

We also recall that in this resolution procedure we need several extra data. For example the map \( \partial : H^3_{\alpha, s}(G \times \mathbb{R}, N, A) \rightarrow Z_3^3(G, \mathbb{T}) \) requires a choice of \((a, f) \in C^2_\alpha(G, A) \times Z^2(Q, A)\) so that \( c_G = \pi^*(c_Q) \partial_G(\pi^*(f)a)^* \in Z^3(G, \mathbb{T}) \).

But in any case we do have a resolution system \( \{H, \pi_G, L, M\} \) of \([c], \nu\).

So instead of going through all steps of desingularizations starting from the cocycle \((c, \zeta) \in Z^3_{\alpha, s}(G, N, A)\), we move directly to \( \{H, \pi_G, L, M\} \) and call \((\lambda, \mu) \in \Lambda_\alpha(\tilde{H}, L, M, A)\) a resolution of the modular obstruction \([c], \nu \in H^3_{\alpha, s}(G \times \mathbb{R}, N, A)\) if

\[\delta([\lambda, \mu]) = ([c], \nu).\]

If we begin with \((\lambda, \mu) \in Z^3_{\alpha}(\tilde{H}, L, M, A)\), it is easy to see the corresponding obstruction cocycle \((c, \zeta) \in Z^3_{\alpha, s}(G, N, A)\):

a) First we fix a cross-section \( s_H : G \rightarrow H \) of the map \( \pi_G; \)
b) With \( \dot{s}(p) = s_{i\ast} \sigma(p), p \in Q \), and \( n_L(p,q) = \dot{s}(p) \dot{s}(q) \dot{s}(pq)^{-1} \in L \) we have

\[
\zeta(s; n) = \lambda(s_{i}(n); s), \quad n \in N;
\]

\[
d_c(s; q, r) = \lambda(n_L(q, r); s), \quad q, r \in Q, s \in \mathbb{R};
\]

\[
c_Q(p, q, r) = \lambda(\dot{s}(p) n_L(q, r) \dot{s}(p)^{-1}; \dot{s}(p)) \mu(\dot{s}(p) n_L(q, r) \dot{s}(p)^{-1}, n_L(p, qr))
\times \{\mu(n_L(p, q), n_L(pq, r))\}^*, \quad p, q, r \in Q.
\]

We will write \((c, \zeta) = \partial_\delta(\lambda, \mu)\). Let \( \text{Rsn}(H, \pi_G, ([c], \nu)) \) be the set of all \((\lambda, \mu) \in Z_\alpha(\tilde{H}, L, M, A)\) such that \( \delta([\lambda, \mu]) = ([c], \nu) \). On the space \( Z_{\alpha, s}^{\text{out}}(G, N, A) \) of modular obstruction cocycles, the group \( C^2_\alpha(Q, A) \) acts in the following way:

\[
(c, \zeta) \mapsto ((\partial_Q b) c, \zeta), b \in C^2_\alpha(Q, A),
\]

which does not change the cohomology class of \((c, \zeta)\). Also the group

\[
Z^2(H, \mathbb{T}) \times C^1(N, A)
\]

acts on \( Z_\alpha(\tilde{H}, L, M, A) \) without changing the cohomology class of \( \delta_\sigma(\lambda, \mu)\):

\[
(\lambda, \mu) \mapsto ((\partial_1 a) \lambda_\xi, (\partial_2 a) \xi_L \eta), \quad (\xi, a) \in Z^2(H, \mathbb{T}) \times C^1(N, A),
\]

where \((\lambda_\xi, \xi_L)\) is the characteristic cocycle given by (2.10):

\[
\lambda_\xi(m; g, s) = \xi(g, g^{-1} mg) \xi(m, g), \quad m \in L, (g, s) \in \tilde{H};
\]

\[
\xi_L = \text{the restriction of } \xi \text{ to } L \times L.
\]

Now as soon as we have a characteristic cocycle \((\lambda, \mu)\), we have a covariant cocycle \( \{M, H, \alpha^{\lambda, \mu}\} \) equipped with a map \( u : m \in L \mapsto u(m) \in \tilde{U}(M) \) such that

\[
u(m)u(n) = \mu(m, n)u(mn), \quad m, n \in L;
\]

\[
\alpha_m^{\lambda, \mu} = \tilde{\text{Ad}}(u(m));
\]

\[
\alpha_g^{\lambda, \mu} \text{th}_s(u(g^{-1} mg)) = \lambda(m; g, s)u(m), \quad (g, s) \in \tilde{H};
\]
which therefore gives:

\[ \dot{\alpha}_{\lambda,\mu}^g = \alpha_{\lambda,\mu}^g, \quad g \in G, \]

whose modular obstruction cocycle is precisely \( \delta_{\eta H}(\lambda, \mu) \). The action of \( a \in C^1(N, A) \) on \((\lambda, \mu)\) does not change the action \( \alpha_{\lambda,\mu}^g \) itself, but on the unitary family \( \{u(m) : m \in L\} \) which is perturbed to \( \{(au)(m) = a(m)u(m) : m \in L\} \). So this does not cause any interesting change. The perturbation by \( \xi \in Z^2(H, \mathbb{T}) \) gives somewhat non trivial change on \( \alpha_{\lambda,\mu}^g \). Namely, what we need is to consider the left regular \( \xi \)-projective representation, say \( v_\xi : g \in H \mapsto v_\xi(g) \in U(\ell^2(H)) \), so that

\[ v_\xi(g)v_\xi(h) = \xi(g, h)v_\xi(gh), \quad g, h \in H. \]

Now the new action \( g \in H \mapsto \alpha_{\lambda,\mu}^g \otimes \text{Ad}(v_\xi(g)) \in \text{Aut}(M \otimes \mathcal{L}(\ell^2(H))) \) has the modular characteristic cocycle \((\lambda_\xi \lambda, \xi L \mu)\), which is of course does not change the outer conjugacy class of the outer action \( \dot{\alpha}_{\lambda,\mu}^g \) of \( G \). The change caused by the action of \( b \in C^2_\alpha(Q, A) \) is again absorbed by changing the unitary family \( \{u(n_L(p, q)) : p, q \in Q\} \) to \( \{b(p, q)u(n_L(p, q)) : p, q \in Q\} \), which does not change the outer action \( \dot{\alpha}_{\lambda,\mu}^g \) itself. Therefore the scheme of model constructions looks like:

\[
\begin{array}{ccc}
(\lambda, \mu) \in Z_\alpha(\tilde{H}, L, M, A) & \longrightarrow & \delta_{\eta H}(\lambda, \mu) \in Z_{\alpha, s}^\text{out}(G, N, A) \\
\downarrow & & \uparrow \\
\alpha_{\lambda,\mu}^g \in \text{Act}(H, M) & \longrightarrow & \dot{\alpha}_{\lambda,\mu}^g = \alpha_{\eta H} \in \text{Oct}(G, M)
\end{array}
\]

where \( \text{Act}(G, M) \) and \( \text{Oct}(G, M) \) are respectively the spaces of actions and outer actions of \( G \) on \( M \). Summarizing the discussion, we get the following:

**Theorem 4.1.** Let \( G \) be a countable discrete amenable group and \( N \) a normal subgroup. Let \( \{\mathcal{C}, \mathbb{R}, \theta\} \) be an ergodic flow and \( \alpha \) an action of \( G \) on the flow \( \{\mathcal{C}, \mathbb{R}, \theta\} \) with \( \text{Ker}(\alpha) \supset N \), i.e., \( \alpha \) is a homomorphism of \( G \) into the group \( \text{Aut}_\theta(\mathcal{C}) \) of automorphisms commuting with the flow \( \theta \) with \( \alpha_m = \text{id}, m \in N \). Let \( A \) denote the unitary group \( \mathcal{U}(\mathcal{C}) \).

For every modular obstruction cocycle \((c, \zeta) \in Z_{\alpha,s}^\text{out}(G, N, A)\), there exists an amenable resolution system \( \{H, L, M, \pi_G, \lambda, \mu\} \) with \((\lambda, \mu) \in Z_\alpha(\tilde{H}, L, M, A)\).
A) and a cross-section $\mathcal{s}_H : G \mapsto H$ of the map $\pi_G$ such that

$$\delta_{\mathcal{s}_H}(\lambda, \mu) \equiv (c, \zeta) \mod B_{\alpha, s}^\text{out}(G, N, A).$$

Consequently, the action $\alpha_{\lambda, \mu}$ associated with the characteristic cocycle $(\lambda, \mu)$ gives an outer action $\hat{\alpha}_{\lambda, \mu} = \alpha_{\mathcal{s}_H}$ of $G$ on the approximately finite dimensional factor $M$ with flow of weights $\{\mathcal{C}, \mathbb{R}, \theta\}$ such that

$$\text{Ob}_m(\hat{\alpha}_{\lambda, \mu}) = ([c], [\zeta]) \in H_{\alpha, s}^\text{out}(G \times \mathbb{R}, N, A).$$

The homomorphism $\nu = [\zeta] \in \text{Hom}(N, H_0^1)$ is injective if and only if $\hat{\alpha}$ is free.

§5. Non-Triviality of the Exact Sequence:

$$1 \longrightarrow H_0^1 \longrightarrow \text{Out}(M) \longrightarrow \text{Out}_{\tau, \theta}(\widetilde{M}) \longrightarrow 1$$

**Theorem 5.1.** Let $\alpha$ be an outer action of a countable discrete group $G$ on a separable factor $M$ with $N = \alpha^{-1}(\text{Cnt}_r(M))$ with modular obstruction

$$\text{Ob}_m(\alpha) = ([c], \nu) \in H_{\alpha, s}^\text{out}(G, N, \mathcal{U}(\mathcal{C})).$$

Let $Q$ be the quotient group $Q = G/N$ and $s$ be a cross-section of the quotient map $\pi : G \mapsto Q$. Then the map $\alpha_s : p \in Q \mapsto \alpha_{s(p)} \in \text{Aut}(M)$ can be perturbed by $\text{Cnt}_r(M)$ to an action of $Q$ if and only if the modular obstruction

$$\text{Ob}_m(\alpha) = ([c], \nu) \in H_{\alpha, s}^\text{out}(G, N, A) = H_{\alpha, s}^\text{out}(\widetilde{Q}, A) *_s \text{Hom}_G(N, H_0^1)$$

has trivial

$$[c \cdot \alpha_p(\partial_Q(b))] = [c(\bar{p}, \bar{q}, \bar{r})\alpha_p(\partial_Q(b)(s; q, r))] = 1$$

for some $b(\cdot, q) \in Z_1^0(\mathbb{R}, A)$, which implies $\nu \cup n_N \in B_2^\alpha(Q, H_0^1)$. 
Proof. Suppose \([c \cdot \alpha_p (\partial_Q(b))] = 1\) for some \(b(.,q) \in \mathbb{Z}_0^1(\mathbb{R},A)\). Choose \(\{u(p,q) \in \mathbb{U}(\mathcal{M}) : p,q \in \mathcal{Q}\}\) so that

\[
\alpha_{\tilde{p}} \circ \alpha_{\tilde{q}} = \text{Ad}(u(p,q)) \circ \alpha_{\tilde{pq}}, \quad \tilde{p}, \tilde{q} \in \tilde{\mathcal{Q}}.
\]

The associated modular obstruction cocycle \((c,\nu) \in \mathbb{Z}^\text{out}_{\alpha,s}(\tilde{G},N,A)\) is given by:

\[
c(\tilde{p}, \tilde{q}, \tilde{r}) = \alpha_{\tilde{p}}(u(q,r))u(p,qr)\{u(p,q)u(pq,r)\}^*;
\]

\[
\nu(m) = [\alpha_m] \in H^1_\theta(\mathbb{R},A), \quad m \in \mathbb{N}.
\]

The triviality of \([c \cdot \alpha_p (\partial_Q(b))]\) means the existence of \(f \in \mathbb{C}^2(Q,A)\) such that \(c \cdot \alpha_p (\partial_Q(b)) = \partial_{\tilde{Q}}f\). Setting

\[
v(p,q) = f(p,q)^*w(p)\alpha_p(w(q))u(p,q)w(pq)^*,
\]

where \(w(p) \in \tilde{\mathcal{M}}\) with \(w(p)^*\theta_t(w(p)) = b(t,p)\), we get

\[
\text{Ad}(w(p)) \circ \alpha_p \circ \text{Ad}(w(q)) \circ \alpha_q = \text{Ad}(v(p,q)\circ \text{Ad}(pq) \circ \alpha_{\tilde{pq}}) \quad \text{and} \quad \partial_{\tilde{Q}}v = 1.
\]

Since \(v(q,r)^*\theta_t(v(q,r)) = 1\) for \(t \in \mathbb{R}\), the unitaries \(v(q,r)\) are elements of \(\mathcal{M}\). Setting

\[
w\alpha_p = \tilde{\text{Ad}}(w(p)) \circ \alpha_p,
\]

obtain a cocycle crossed action \(\{w\alpha, v\}\) of \(Q\) on \(\mathcal{M}\). As the fixed point algebra \(\mathcal{M}^{w\alpha}\) can be assumed to be properly infinite without loss of generality, we can find a family \(\{a(p) \in \mathbb{U}(\mathcal{M}) : p \in \mathcal{Q}\}\) such that

\[
1 = a(p)\alpha_p(a(q))v(p,q)a(pq)^*
\]

\[
f(p,q)^*a(p)w(p)\alpha_p(a(q))u(p,q)w(pq)^*a(pq)^* = f(p,q)^*a(p)w(p)\alpha_p(a(q)w(q))u(p,q)\alpha_p(a(pq)w(pq))^*;
\]

\[
f(p,q) = a(p)w(p)\alpha_p(a(q)w(q))u(p,q)\alpha_p(a(pq)w(pq))^*.
\]

Hence \(\beta = a\cdot w\alpha : \tilde{p} \in \tilde{\mathcal{Q}} \mapsto a\cdot w\alpha_{\tilde{p}} = \text{Ad}(a(p)w(p)) \circ \alpha_{\tilde{p}} \in \text{Aut}(\tilde{\mathcal{M}})\) is an action of \(\tilde{Q}\) on \(\tilde{\mathcal{M}}\). The restriction of \(\beta\) to \(\mathcal{M}\) is precisely a \(\text{Cnt}_r(\mathcal{M})\)-perturbation of
the original action $\alpha$ since $\widetilde{\text{Ad}}(a(p)w(p)) \in \text{Cnt}_r(M)$. Now we have
\[
d_c(s, q, r) = \theta_s(u(q, r))u(q, r)^* \\
= \theta_s\left(f(q, r)\alpha_q((a(r)w(r))^*)a(q)w(q))^*a(pq)w(qr)\right) \\
\times \left(f(q, r)\alpha_q((a(r)w(r))^*)(a(q)w(q))^*a(pq)w(qr)\right)^* \\
= (\partial_\theta f(q, r))_s\alpha_q(b(s, r)^*)b(s, q)^*b(s, pq) \\
\]
where $\theta_s(a(p)w(p))(a(p)w(p))^* = \theta_s(w(p))(w(p))^* = b(s, p) \in A$. Hence we have
\[
\nu(n_N(q, r)) = [d_c(\cdot, q, r)] = [\partial_\widetilde{Q}(b(\cdot, \cdot)^*)(q, r)] \text{ in } H^1_\theta.
\]
Thus we conclude that $\nu \cup n_N \in B^2_\theta(\widetilde{Q}, H^1_\theta)$.

Conversely, suppose that $\alpha_s$ is perturbed to an action of $Q$ by $\text{Cnt}_r(M)$. Choose \( \{w(p) \in \tilde{U}(M) : p \in Q\} \) so that
\[
\widetilde{\text{Ad}}(w(p))\circ \alpha_p \circ \widetilde{\text{Ad}}(w(p))\circ \alpha_q = \widetilde{\text{Ad}}(w(pq))\circ \alpha_{pq}, \quad p, q \in Q.
\]
Let $\{u(p, q) \in \tilde{U}(M) : p, q \in Q\}$ be a family such that
\[
\alpha_p \circ \alpha_q = \text{Ad}(u(p, q))\circ \alpha_{pq}.
\]
Then we have
\[
f(p, q) = w(p)\alpha_p(w(q))u(p, q)w(pq)^* \in A,
\]
and that
\[
(\partial_\widetilde{Q}f)(\tilde{p}, \tilde{q}, \tilde{r}) = \alpha_{\tilde{p}}\left(w(q)\alpha_q(w(r))u(q, r)w(qr)^*\right) \\
\times \left(w(p)\alpha_p(w(q))u(p, qr)w(pqr)^*\right) \\
\times \left\{w(p)\alpha_{pq}(w(q))u(p, q)w(pq)^*\right\}^* \\
\times \left\{w(p)\alpha_{pq}(w(q))u(p, q)w(pq)^*\right\}^* \\
\times \left\{w(p)\alpha_{pq}(w(q))u(p, q)w(pq)^*\right\}^* \\
\times \left\{w(p)\alpha_{pq}(w(q))u(p, q)w(pq)^*\right\}^* \\
= c(\tilde{p}, \tilde{q}, \tilde{r})\alpha_p(b(s, q)\alpha_q(b(s, r)b(s, qr)^*),
\]
where \( b(s, p) = w(p)^* \theta_s(w(p)) \in \mathbb{Z}_0^1(\mathbb{R}, A) \). Thus we conclude
\[
[c(\tilde{p}, \tilde{q}, \tilde{r}) \alpha_p(\partial_Q(b)(s; q, r))] = 1.
\]

This characterization has an immediate consequence:

**Theorem 5.2.** If \( M \) is an approximately finite dimensional factor of type \( \text{II} \) with flow of weights \( \{ \mathcal{C}, R, \theta \} \), then the exact sequence:
\[
1 \longrightarrow \mathbb{H}_0^1(\mathbb{R}, \mathcal{U}(\mathcal{C})) \longrightarrow \text{Out}(M) \longrightarrow \text{Out}_{\tau, \theta}(\tilde{M}) \longrightarrow 1
\]
does not split.

**Proof.** Let \( G \) be the discrete Heisenberg group:
\[
G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}
\]
and
\[
N = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : c \in \mathbb{Z} \right\}
\]
be the center of \( G \) as in Example 7.1 on [KtST1]. We write an element of \( G \) as \((a, b, c) \in \mathbb{Z} \) with the multiplication rule:
\[(a, b, c)(a', b', c') = (a + a', b + b', c + c' + ab').\]

We then form the quotient group \( Q = G/N \) and obtain an exact sequence:
\[
1 \longrightarrow N \longrightarrow G \overset{\pi_Q}{\longrightarrow} Q \longrightarrow 0.
\]
The quotient group \( Q \) is isomorphic to \( \mathbb{Z}^2 \). Define a cross-section \( s \) of \( \pi_Q \) in the following way:
\[
s(a, b) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad (a, b) \in \mathbb{Z}^2 = Q,
\]
and compute
\[ n_N(a, b; a', b') = ab' \in \mathbb{Z} \] (5.1)

where \( N \) is identified with \( \mathbb{Z} \). Choose \((\lambda, \mu) \in Z(G, N, \mathbb{T})\) to be trivial, i.e., \( \mu(m, n) = 1, m, n \in N \) and \( \lambda(m, g) = 1, m \in N, g \in G \). But choose \( \nu = e^{iT} \in \text{Hom}(N, \mathbb{T}) = \hat{N} = \mathbb{T} \) with \( T > 0 \) to be determined, so that the characteristic cocycle \((\lambda, \mu) \in Z(G \times \mathbb{R}, N, \mathbb{T})\) is given by:

\[
\begin{align*}
\mu(m, n) &= 1, \; m, n \in N; \\
\lambda(m, (g, s)) &= \exp(iT' ms), \; s \in \mathbb{R}, g \in G,
\end{align*}
\]

where \( T' = 2\pi/T \). Let \( \mathcal{M} \) be an AFD factor of type III with flow of weights \( \{e, \mathbb{R}, \theta\} \). Viewing the torus \( \mathbb{T} \) as the subgroup \( \mathcal{U}(\mathbb{C}, \mathbb{R}) \) of the unitary group \( \mathcal{A} = \mathcal{U}(\mathbb{C}) \), we view the cocycle \((\lambda, \mu)\) as an element of \( Z_{\alpha}(\tilde{G}, N, \mathcal{A}) \). Now choose \( T > 0 \) such that \( \sigma^T_n \not\in \text{Int}(\mathcal{M}) \) for every \( n \in \mathbb{Z}, n \neq 0 \), with \( \varphi \) a preassigned faithful semi-finite normal weight on \( \mathcal{M} \). Such a \( T \in \mathbb{R} \) exists because \( \{t \in \mathbb{R}, \sigma^T_t \in \text{Int}(\mathcal{M})\} \) must be a meager subgroup of \( \mathbb{R} \). Let \( \alpha = \alpha^{\lambda, \mu} \) be the action of \( G \) on \( \mathcal{M} \) associated with the cocycle \((\lambda, \mu)\) and mod(\( \alpha_g \)) = id. The construction yields that the action \( \alpha \) is free and it enjoys the following property:

\[ \alpha_m = \sigma^\varphi_{mT}, \; m \in N, \]

with \( \varphi \) a dominant weight on \( \mathcal{M} \). We can assume the invariance \( \varphi = \varphi^* \alpha_g, g \in G \) for \( \alpha \). The freeness of \( \alpha \) shows that the map: \( \check{\alpha} : g \in G \mapsto \check{\alpha}_g = [\alpha_g] \in \text{Out}(\mathcal{M}) \) is an injective homomorphism such that \( \hat{\alpha}_m = \sigma_{mT} \in H^1(\mathbb{R}, \mathcal{A}) \subset \text{Out}(\mathcal{M}) \). We are now going to compute the modular obstruction cocycle \((c, \nu) \in Z_{\alpha,s}(\tilde{Q}, \mathcal{A}) \ast_s \text{Hom}(N, H^1_0)\). Since

\[
\begin{align*}
\alpha_{\check{p}^* \check{q}} &= \alpha_{s(p)s(q), t} = \alpha_{s(p)s(q)} \ast \theta_{s+t} \\
&= \alpha_{n_N(p, q)} \ast \theta_{s+t} \\
&= \text{Ad}(\varphi^{iT'n_N(p, q)}) \ast \alpha_{\check{p} \check{q}},
\end{align*}
\]
with \( u(p, q) = \varphi^{iT'qN(p,q)} \) we get

\[
c(\tilde{p}, \tilde{q}, \tilde{r}) = \alpha_p(u(q, r))u(p, qr)\{u(p, q)u(pq, r)\}^* \\
= \theta_s(u(q, r))u(q, r)\alpha_p(u(q, r))u(p, qr)\{u(p, q)u(pq, r)\}^* \\
= \exp(-iT'sn_N(q, r))\varphi^{iT'(n_N(q,r)+n_N(p,q)-n_N(p,q))} \\
= \exp(-iT'sn_N(q, r))
\]

and \( c_Q = 1 \). In order for \( c \cdot \alpha_p(\partial Q(b)) \) with some \( b(\cdot, q) \in Z^1_\theta(\mathbb{R}, A) \) to be trivial, it is necessary and sufficient that there exists \( f \in C^2(Q, A) \) such that \( \partial_Q f = c \cdot \alpha_p(\partial Q(b)) \). The function \( f \) satisfies the equations:

\[
\exp(-iT'sn_N(q, r)) = \alpha_q(b(s, r)^*)b(s, q)^*b(s, qr)f(q, r)^*\theta_s(f(q, r))
\]

which means that \( [\exp(-iT'sn_N(q, r))] \in B^2(Q, H^1_\theta) \). As \( \text{mod}(\alpha_p) = \text{id}, p \in Q \), and \( Q \) is a free abelian group, the second cohomology group \( H^2(Q, H^1_\theta) \) is isomorphic to the group \( X(Q^2, H^1_\theta) \) of all \( H^1_\theta \)-valued skew symmetric bihomomorphisms. We have

\[
\exp\left(-iT'sn_N(q, r)\right)\exp\left(iT'sn_N(r, q)\right) \\
= \left(\alpha_q(b(s, r)^*)b(s, q)^*b(s, qr)f(q, r)^*\theta_s(f(q, r))\right) \\
\times \left(\alpha_r(b(s, q)^*)b(s, r)^*b(s, rq)f(r, q)^*\theta_s(f(r, q))\right)^* \\
= f(r, q)f(q, r)^*\theta_s(f(r, q)^*f(q, r)).
\]

By (5.1), we have

\[
\exp(-iT'sn_N(q, r) + iT'sn_N(r, q)) = \exp(-iT's(ab' - a'b))
\]

where \( q = (a, b), r = (a', b') \) Thus it follows from (5.2) that the modular automorphisms \( \sigma^\varphi_{T(ab' - a'b)} \) are inner, which contradicts to the choice of \( T \). Therefore \( [c(\tilde{p}, \tilde{q}, \tilde{r})\alpha_p(\partial Q(b)(s; q, r))] \neq 1 \) in \( H^3_{\alpha_s}(\tilde{Q}, A) \). Theorem 5.1 says that \( \alpha_s \) cannot be perturbed into an action of \( Q \) by \( \text{Cnt}_r(M) \).
Since we have a commutative diagram of exact sequences:

\[
\begin{array}{cccccc}
1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\nu & & \alpha & & \tilde{\alpha} & & \tilde{\alpha} & & \nu \\
1 & \longrightarrow & H^1_\theta & \longrightarrow & \text{Out}(\tilde{M}) & \longrightarrow & \text{Out}_{\tau,\theta}(\tilde{M}) & \longrightarrow & 1
\end{array}
\]

if the second sequence splits via cross-section \( s_\pi \), then the associated injection \( \tilde{\alpha} \) of \( Q \) into \( \text{Out}_{\tau,\theta}(\tilde{M}) \) is composed with the cross-section \( s_\pi \) to be an outer action of \( Q \), say \( \beta \). But \( H^3(Q, \mathbb{T}) = 1 \), so that \( \beta \) can be perturbed into an action of \( Q \), denoted by \( \beta \) again. Then we have \( \beta_p \equiv \alpha_{s(p)} \mod \text{Cnt}_r(\tilde{M}) \). Therefore the \( \alpha_s \) is perturbed to an action by \( \text{Cnt}_r(\tilde{M}) \), which contradicts to the fact \([c(\tilde{p}, \tilde{q}, \tilde{r})\alpha_p(\partial_Q(b)(s; q, r))] \neq 1 \) for any \( b(\cdot, p) \in Z^1(\mathbb{R}, H^1_\theta) \) as seen above.

\[ \heartsuit \]

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