$\mathcal{N} = 2$ Quiver Gauge Model
and
Partial Supersymmetry Breaking

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Abstract

We construct an action of $\mathcal{N} = 2$ affine $A_n$ quiver gauge model having non-canonical kinetic terms and equipped with electric and magnetic FI terms. $\mathcal{N} = 2$ supersymmetry is shown to be broken to $\mathcal{N} = 1$ spontaneously and $\mathcal{N} = 1$ multiplets realized on the vacua are given. We also mention the models with different gauge groups. It is argued that the affine $A_1$ quiver gauge model provides a dynamical realization to approach the Klebanov-Witten $\mathcal{N} = 1$ fixed point.

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1 Introduction

Supersymmetry has become one of the most remarkable and attractive ideas in theoretical physics. In particular, various investigations beginning with [1, 2] have been made on $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in four dimensions, taking advantage of its powerful properties. Furthermore, we can extract the important information of $\mathcal{N} = 1$ super Yang-Mills theory, such as the low energy effective superpotential, breaking $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ by a superpotential [2, 3, 4].

On the other hand, in view of the fact that superstring theories produce, in some backgrounds, extended supersymmetry in four dimensions and have no adjustable parameter, it is natural to consider spontaneous breaking of the extended supersymmetry so as to obtain more realistic $\mathcal{N} = 1$ supersymmetric models. Although it had been argued that, based on the supercharge algebra, rigid $\mathcal{N} = 2$ supersymmetry is not spontaneously broken to $\mathcal{N} = 1$, a loophole has been first pointed out in [5] by the argument based instead on the supercurrent algebra which has been modified by an additional space-time independent term. In [6] and [7, 8, 9], $\mathcal{N} = 2$, $U(1)$ and $U(N)$ gauge models with $\mathcal{N} = 2$ vector multiplet only have been constructed, establishing this modification of the algebra by introducing magnetic Fayet-Iliopoulos (FI) term. It was shown that the partial breaking of $\mathcal{N} = 2$ supersymmetry indeed occurs in such models. (See also [10, 11, 12] for related discussions and [13] for supergravity.)

In the $U(N)$ gauge theory which contains only the $\mathcal{N} = 2$ vector multiplet, the magnetic FI term which causes the partial breaking can be easily introduced in the harmonic superspace formalism [14](see [15] for a review) as a constant shift of the auxiliary field [9, 16]. In addition, it was shown that partial supersymmetry breaking can occur even in the presence of hypermultiplets in the adjoint representation. However, the addition of hypermultiplets in fundamental representation makes it difficult, as pointed out in [17, 18].

In this paper we overcome this difficulty by considering a model with hypermultiplets in bi-fundamental representation, whose matter content is described by a quiver diagram. This model is, therefore, a quiver gauge model. We will show that, in addition to electric FI term, it is possible to introduce a magnetic FI term for any $\mathcal{N} = 2$ quiver gauge theory. This statement leads to the conclusion that in generic $\mathcal{N} = 2$ quiver gauge theory with these terms, $\mathcal{N} = 2$ supersymmetry can be broken to $\mathcal{N} = 1$ spontaneously. As an illustration, we will describe this explicitly in a specific model, affine $A_1$ quiver gauge model, focusing on the Coulomb branch.

This model may seem reminiscent of the one discussed in [19]: a flow, by a mass deformation, from the $\mathcal{N} = 2$ affine $A_1$ theory on the world volume of the D3-branes at
\( \mathbb{C}^2/\mathbb{Z}_2 \) orbifold singularity to \( \mathcal{N} = 1 \) quiver gauge theory on that at conifold singularity. Indeed, we can show that in special points of the Coulomb branch the mass spectrum is the same as that of the theory at the conifold singularity. The remarkable point of our model is that the masses are produced dynamically, and thus we can dynamically approach the theory on conifold, namely conifold geometry.

The organization of this paper is as follows. In section 2, we construct \( \mathcal{N} = 2 \) affine \( A_{n-1} \) quiver gauge model equipped with electric and magnetic FI terms. The necessary condition to introduce the magnetic FI term without breaking \( \mathcal{N} = 2 \) supersymmetry in the action is examined. We also mention the cases of different types of quiver gauge theories. As an illustration, we mainly consider one of the simplest models, affine \( A_1 \) quiver gauge model, in the subsequent sections. In section 3, we derive the scalar potential which is needed in the analysis of the vacua. We show, in section 4, that \( \mathcal{N} = 2 \) supersymmetry is broken to \( \mathcal{N} = 1 \) spontaneously on the Coulomb branch by observing the mass spectrum and the appearance of the Nambu-Goldstone fermion. As an application of our model, in section 5, we consider the dynamical realization of \( \mathcal{N} = 1 \) quiver gauge theory on the world volume of the D3-branes at the conifold singularity which has been considered in [19]. The notations on the harmonic superspace used in this paper are collected in appendix.

2 \( \mathcal{N} = 2 \) quiver gauge model

When we place \( N \) D3-branes at \( A_{n-1} \) orbifold \( \mathbb{C}^2/\mathbb{Z}_n \), the gauge theory realized on the world volume is \( \mathcal{N} = 2 \) affine \( A_{n-1} \) quiver gauge theory [20], namely \( U(N)^n \) gauge theory composed of \( n \) vector multiples \( V_{i}^{++} \) in adjoint representation and \( n \) hypermultiplets \( q_{i}^{+} \) in bi-fundamental representation, where \( I = 1, \cdots, n \). In this section, we will construct \( \mathcal{N} = 2 \) affine \( A_{n-1} \) quiver gauge model, which has the same matter content as above, but with non-canonical kinetic terms and electric and magnetic FI terms. We examine a necessary condition to introduce the magnetic FI term without breaking \( \mathcal{N} = 2 \) supersymmetry in the action. This magnetic FI term causes the partial spontaneous breaking of \( \mathcal{N} = 2 \) supersymmetry as will be seen in the next section.

In the case with additional D5-branes wrapping on non-trivial \( S^2 \)s, as in [20, 21], the gauge group of the world volume theory is \( \prod_i U(N_i) \) and the rank of gauge group of each node is in general different. Although we will not consider the gauge model with this matter content explicitly, we will mention the condition necessary to introduce the magnetic FI term at the end of this section.
The action for the vector multiplet of the $U(N)^n$ gauge symmetry is given by

$$S_V = -\frac{i}{4} \int d^4x \sum_{I=1}^n [(D)^4 F_I(W_I) - (\bar{D})^4 \bar{F}_I(\bar{W}_I)]$$

(2.1)

where $W_I$ is the curvature of $V_I^{++}$ and $(D)^4 = \frac{1}{16} (D^+)^2 (D^-)^2$. See appendix for the concrete form. $\mathcal{F}_I$ is the prepotential for the $I$-th $U(N)$ which is denoted by $U(N_I)$. $V_I^{++}$ is composed of a complex scalar $\phi_I$, $SU(2)$ doublet Weyl fermions $\lambda_I^A (A = 1, 2$ labels $SU(2)$ automorphism of $\mathcal{N} = 2$ supersymmetry) and a real auxiliary field $D_I^{AB}$, which transform as adjoint representation of $U(N_I)$. The $U(N_I)$ gauge group is generated by $t_a^I$ ($a = 0, 1, \cdots, N^2_I - 1$), and $t_a^I$ represents the overall $U(1)$ part of $U(N_I)$. As was done in [6], (2.1) reduces in components to

$$S_V = \sum_{I=1}^n \left[ -\text{Im} \mathcal{F}_{I|ab} D^m \phi_I D_m \bar{\phi}_I^b - \frac{1}{2} \bar{\mathcal{F}}_{I|ab} \bar{\phi}_I^a \sigma^m D_m \phi_I^b + \frac{1}{2} \mathcal{F}_{I|ab} \lambda_I^A \sigma^m D_m \bar{\lambda}_I^b ight.$$

$$+ \frac{1}{4} \text{Im} \mathcal{F}_{I|ab} D_I^{AB} D_{IAB}^b + \frac{i}{4} \mathcal{F}_{I|abc} \lambda_I^A \lambda_I^B D_{IAB}^c - \frac{i}{4} \bar{\mathcal{F}}_{I|abc} \bar{\lambda}_I^A \bar{\lambda}_I^B D_{IAB}^c \

- \frac{1}{4} \text{Im} \mathcal{F}_{I|ab} v^a_{Imn} v^b_{I} - \frac{1}{8} \text{Re} \mathcal{F}_{I|abc} \varepsilon^{mnq}_{I} v^a_{Imn} v^b_{Ipq} \

- \frac{i}{4} (\mathcal{F}_{I|abc} \lambda_I^A \sigma^{mn} \lambda_I^b - \bar{\mathcal{F}}_{I|abc} \bar{\lambda}_I^A \sigma^{mn} \bar{\lambda}_I^b) v^c_{Imn} \

- \frac{i}{12} \mathcal{F}_{I|abcd} (\lambda_I^A \lambda_I^B) (\lambda_I^C \lambda_I^D) + \frac{i}{12} \bar{\mathcal{F}}_{I|abcd} (\bar{\lambda}_I^A \bar{\lambda}_I^B) (\bar{\lambda}_I^C \bar{\lambda}_I^D) \

+ \frac{1}{2} \text{Im} \mathcal{F}_{I|ab} \left[ \bar{\lambda}_I^A f_{cd}^b (i \sqrt{2} \phi_I^c) \lambda_I^d + \lambda_I^A f_{cd}^b (-i \sqrt{2} \phi_I^c) \lambda_I^d \right] \

$$+ \frac{1}{2} \text{Im} \mathcal{F}_{I|ab} f_{cd}^e \bar{\phi}_I^f \phi_I^e f_{ef}^g \bar{\phi}_I^g \phi_I^f \right].$$

(2.2)

where the symbol $\mathcal{F}_{I|ab...}$ denotes the derivative of $\mathcal{F}_I$ with respect to $\phi_I^a, \phi_I^b, \ldots$.

Let us introduce the $\mathcal{N} = 2$ hypermultiplets $q_i^{+j}$ ($i = 1, \ldots, N_I, \ j = \bar{1}, \ldots, \bar{N}_{I+1}$) transforming as $(N_I, \bar{N}_{I+1})$ under $U(N_I) \times U(N_{I+1})$:

$$U(N_1) \ U(N_2) \ U(N_3) \cdots \ U(N_n)$$

$$q_1^+ \ N_1 \ N_2 \ 1 \cdots 1$$

$$q_2^+ \ 1 \ N_2 \ \bar{N}_3 \cdots 1$$

$$\vdots$$

$$q_n^+ \ \bar{N}_1 \ 1 \ 1 \ \cdots \ N_n.$$  

The matter part of the action is

$$S_q = - \int d\bar{\zeta} d\zeta (-4)^n \sum_{i=1}^n (\bar{q}_i^+)^{(\bar{i})} (\mathcal{D}^{++} q_i^+)_{\bar{i}}$$

(2.3)

\footnote{From now on we use this notation. But keep in mind that we are considering $U(N)^n$ gauge model with the same ranks}
where $d\zeta(-4) = d^4y d^4\theta$ and

$$
(\mathcal{D}^+ q^+_I)^i_j = D^+ q^+_I + iV^+ I^a(t^a_I)^i_j q^+_I + iV^+ I^{+a} (I^a)^i_j q^+_I.
$$

(2.4)

As explained in appendix, after eliminating infinitely many auxiliary fields, a hypermultiplet $q^+_I$ contains $SU(2)$ doublet complex scalars $Q_{IA}$, and a pair of Weyl fermions, $\psi_I$ and $\kappa_I$. In components, the matter action $S_q$ reduces to

$$
S_q = \int d^4 x \sum_{I=1}^n \left[ - \bar{Q}_{IA} \gamma^\alpha D^\alpha m Q_{IA} - \frac{i}{2} \bar{\psi}_I \gamma^\alpha D^\alpha m \psi_I - \frac{i}{2} \kappa_I \gamma^\alpha D^\alpha m \kappa_I \right. \\
+ i\bar{Q}_{IA} j D^A j Q_{IB} \left. - i\bar{Q}_{IA} j \bar{Q}_{IB} \right] - (\bar{Q}_{IA} j \phi_I - \bar{Q}_{IA} j \phi_{I+1}) ((\bar{\phi}_I k Q_{k I} - \bar{\phi}_{I+1} \bar{k} Q_{k I})) \\
- (\bar{Q}_{IA} j \bar{\phi}_I - \bar{Q}_{IA} j \bar{\phi}_{I+1}) ((\phi_I k Q_{k I} - \phi_{I+1} \bar{k} Q_{k I})) \\
+ i\bar{\psi}_I j \lambda_{I+1} j Q_{IA} \left. - i\bar{\psi}_I j \lambda_{IA} j Q_{IA} \right] + i\bar{Q}_{IA} j \phi_{I+1} j Q_{IA} + i\bar{Q}_{IA} j \phi_{IA} \\
+ i\kappa_I j \lambda_{IA} j Q_{IA} \left. - i\kappa_I j \lambda_{IA} j Q_{IA} \right] + i\bar{Q}_{IA} j \kappa_{I+1} j Q_{IA} + i\bar{Q}_{IA} j \kappa_{IA} \\
+ \frac{1}{\sqrt{2}} i\kappa_I j \phi_I j Q_{IA} + \frac{1}{\sqrt{2}} i\bar{\phi}_I j \phi_{I+1} j Q_{IA} \right].
$$

(2.5)

By construction the action $S_V + S_q$ is invariant under the $\mathcal{N} = 2$ supersymmetry transformation law:

$$
\delta \eta \phi_I = -i \sqrt{2} \eta \lambda^B, \\
\delta \eta \psi_{im} = i \lambda^A \sigma^m \lambda^B, \\
\delta \eta \lambda^A_{I \alpha} = \frac{1}{2} \eta \lambda^A \sigma^m \psi_{im} + \sqrt{2} \eta \lambda^A \sigma^m m \phi - i \eta \lambda^A \phi_I, \\
\delta \eta \lambda^A_{I \alpha} = \eta \lambda^A \psi_{im} + \sqrt{2} \eta \lambda^A m \phi - i \eta \lambda^A \phi_I, \\
\delta \eta \lambda^A_{I \alpha} = 2i \eta \lambda^A m \phi - i \eta \lambda^A \phi_I, \\
\delta \eta \lambda^A_{I \alpha} = 2i \eta \lambda^A \psi_{im} + \sqrt{2} \eta \lambda^A m \phi + i \eta \lambda^A \phi_I.
$$

(2.6) - (2.12)

electric and magnetic FI terms

We introduce the electric FI term

$$
S_e = \int dud\zeta(-4) \sum_{I=1}^n \left[ \text{Tr}_{U(N_I)} \Xi_I^{++} V_I^{++} + h.c. \right] = \int d^4x \sum_{I=1}^n \left[ \xi_I^{AB} D^{0}_{IAB} + h.c. \right]
$$

(2.13)

where $\Xi_I^{++} = \xi_I^{AB} u_I^+ u_I^+$ and $\xi_I^{AB}$ is the electric FI parameter of $U(N_I)$ gauge group. In the three vector notation, the electric FI parameter can be written as $\xi_I^{A} = i\xi^a_I (\tau^a)^A_B$.
where $\tau_\alpha$ ($\alpha = 1, 2, 3$) are the Pauli matrices. $S_c$ causes a constant shift of the auxiliary fields in the dual vector multiplets and thus the magnetic FI term $S_m$ is introduced to shift the auxiliary field in the original vector multiplets by a constant \[6, 17, 18, 9\]. We shall shift the auxiliary field as

$$D_{i}^{AB} = D_{i}^{AB} + 4i\xi_{D}^{AB}\delta_{0}^{a}, \quad \bar{D}_{i}^{AB} = D_{i}^{AB} - 4i\bar{\xi}_{D}^{AB}\delta_{0}^{a}, \quad (2.14)$$

so that the supersymmetry transformation law \[2.8\] changes to

$$\delta\lambda_{i}^{Aa} = (D_{i})^{A}_{B}\eta^{B} + \ldots, \quad \delta\bar{\lambda}_{i}^{Aa} = -(\bar{D}_{i})^{A}_{B}\eta^{B} + \ldots. \quad (2.15)$$

It is easy to see that the action with the shift \[2.14\]

$$S_{V}|_{D\rightarrow \bar{D}} = -\frac{i}{4} \int d^{4}x \sum_{I=1}^{n}[(D)^{4}\mathcal{F}_{i}(\bar{W}_{I}) - (\bar{D})^{4}\bar{\mathcal{F}}_{i}(\bar{W}_{I})] \quad (2.16)$$

where $\bar{W} \equiv W|_{D\rightarrow \bar{D}}$ is invariant under the $\mathcal{N} = 2$ supersymmetry transformations with \[2.13\]. In addition obviously $S_{c}|_{D\rightarrow \bar{D}}$ is $\mathcal{N} = 2$ superinvariant. As we will see in the next section, the magnetic FI term introduced above causes partial spontaneous supersymmetry breaking.

Next we examine $S_{q}$ in \[2.5\]. It is known \[17, 18\] that there is a difficulty in introducing the magnetic FI term in the presence of hypermultiplets in fundamental representation. However we find that the magnetic FI term can be introduced without breaking $\mathcal{N} = 2$ supersymmetry in the case with hypermultiplets in bi-fundamental representation. Let us show this explicitly. We examine the following terms contained in $S_{q}$

$$i\bar{Q}_{IAj}^{i}D_{i}^{ABj}Q_{IB}^{i} - i\bar{Q}_{IA}^{j}D_{i}^{AB}Q_{IB}^{i} + i\bar{\psi}_{j}^{i}A_{j}^{i}Q_{IA}^{i} - i\bar{\psi}_{j}^{i,A_{j}^{i}}Q_{IA}^{i} - i\bar{Q}_{IA}^{i}\lambda_{IA}^{j}i\psi_{i}^{j} + i\bar{Q}_{IA}^{i}\lambda_{IA}^{j}i\psi_{i}^{j} + i\bar{\psi}_{j}^{i,A_{j}^{i}}Q_{IA}^{i} + i\bar{\psi}_{j}^{i,A_{j}^{i}}Q_{IA}^{i} - i\bar{Q}_{IA}^{i}\lambda_{IA}^{j}i\bar{k}_{i}^{j} - i\bar{Q}_{IA}^{i}\lambda_{IA}^{j}i\bar{k}_{i}^{j} \quad (2.17)$$

Under the shift \[2.14\], \[2.17\] acquires additional terms

$$-2\bar{Q}_{IA}^{i}(\xi_{D}^{I} - \xi_{D}^{I+1})^{AB}(t_{0}^{I})_{j}^{i}Q_{IB}^{j} - 2\bar{Q}_{IA}^{i}(\xi_{D}^{I+1} - \xi_{D}^{I+2})^{AB}(t_{0}^{I+1})_{j}^{i}Q_{IB}^{j} \quad (2.18)$$

Now for the $\mathcal{N} = 2$ invariance of the action with the replacement \[2.14\], the following terms have to vanish

$$-2\delta\bar{Q}_{IA}^{i}(\xi_{D}^{I} - \xi_{D}^{I+1})^{AB}(t_{0}^{I})_{j}^{i}Q_{IB}^{j} - 2\bar{Q}_{IA}^{i}(\xi_{D}^{I} - \xi_{D}^{I+1})^{AB}(t_{0}^{I})_{j}^{i}\delta Q_{IB}^{j}$$

$$+2\delta\bar{Q}_{IA}^{i}(\xi_{D}^{I+1} - \xi_{D}^{I+2})^{AB}(t_{0}^{I+1})_{j}^{i}Q_{IB}^{j} + 2\bar{Q}_{IA}^{i}(\xi_{D}^{I+1} - \xi_{D}^{I+2})^{AB}(t_{0}^{I+1})_{j}^{i}\delta Q_{IB}^{j}$$

$$-4\bar{\psi}_{j}^{i}(\xi_{D}^{I}B\eta^{B})(t_{0}^{I})_{j}^{i}Q_{IA}^{j} + 4\bar{\psi}_{j}^{i}(\xi_{D}^{I+1}B\eta^{B})(t_{0}^{I+1})_{j}^{i}Q_{IA}^{j}$$
\[\begin{align*}
&+4\bar{Q}^A_i(\xi_{IAB}^D)(t^I_0)^i_j \psi^j \bar{\psi}^i - 4\bar{Q}^A_i(\xi_{I+1AB}^D)(t^{I+1}_0)^i_j \bar{\psi}^i_j \\
&-4\kappa_i(\xi_{IAB}^D)(t^I)_i^j\tilde{Q}_I\bar{A}_i^j + 4\kappa_i(\xi_{I+1AB}^D)(t^{I+1}_0)^i_j\bar{Q}_I\bar{A}_i^j \\
&-4\bar{Q}^A_i(\xi_{IAB}^D)(t^I_0)^i_j \bar{\psi}^i_j + 4\bar{Q}^A_i(\xi_{I+1AB}^D)(t^{I+1}_0)^i_j \bar{\psi}^i_j. \tag{2.19}
\end{align*}\]

We find that this is achieved if we choose the magnetic FI parameters such that

\[\xi^D_I = \xi^{I+1}_D \tag{2.20}\]

where \((t^I_0)^i_j = \delta^i_j/\sqrt{2N}\). A bi-fundamental hypermultiplet interacts with two different gauge sectors, and thus we can introduce the magnetic FI terms such that the effect from the shift of the auxiliary field of one gauge sector and that of the other sector cancel out with each other. We can also see that the matter part does not contribute to the magnetic FI term as the additional terms (2.18) cancels out for (2.20).

Summarizing the action of the \(\mathcal{N} = 2\) quiver gauge model is given by

\[S = S_V + S_q + S_e + S_m = [S_V + S_q + S_e] |_{D \rightarrow D}. \tag{2.21}\]

Each part is given in (2.1), (2.3) and (2.13), and \(\xi_D\) is subject to (2.20). We have seen that this is invariant under the \(\mathcal{N} = 2\) supersymmetry transformation with the replacement (2.14).

We comment on the case with hypermultiplets in fundamental representation. As pointed out in [17, 18], it is hard to introduce the magnetic FI term which causes the partial spontaneous supersymmetry breaking. \footnote{In [9], the magnetic FI term is introduced even in the presence of hypermultiplets in fundamental representation. However, as explained below, the magnetic FI parameter is imaginary, and thus \(\mathcal{N} = 2\) supersymmetry remains unbroken in the vacua.} This can be seen as follows. In this case, terms with \(\xi^{I+1}_D = \xi^D_I = 0\) in (2.19) have to be deleted for the \(\mathcal{N} = 2\) superinvariance. This forces us to set the magnetic FI parameter \(\xi^{IAB}_D\) to be imaginary. However, the real part of the magnetic FI parameter causes partial spontaneous supersymmetry breaking [6], and thus for imaginary magnetic FI parameter \(\mathcal{N} = 2\) supersymmetry remains unbroken in the vacua. In other words, when we introduce the real part of the magnetic FI parameter in the presence of hypermultiplets in fundamental representation, the action is no longer invariant under the \(\mathcal{N} = 2\) supersymmetry transformation.

In this paper we mainly consider affine \(A_{n-1}\) quiver gauge models. However, it is now obvious that we can introduce the magnetic FI term in any \(\mathcal{N} = 2\) quiver gauge model with any number of nodes (gauge sectors) and any number of arrows (bi-fundamental hypermultiplets). Let us comment on two generalizations of our model among them. The
first one is the case when the rank of each gauge group is different. Such an $\mathcal{N} = 2$ quiver gauge model is obtained by considering the additional D5-branes wrapping on non-trivial $S^2$'s (though we have non-canonical kinetic terms) and is also interesting because, in contrast to the superconformal case above, we have running gauge couplings. Even in this case, the argument on the magnetic FI term is similar to that given above. The only difference is that the condition of the magnetic FI parameter (2.20) is changed as

$$
\frac{\xi_D^I}{\sqrt{N_I}} = \frac{\xi_D^{I+1}}{\sqrt{N_{I+1}}}.
$$

(2.22)

This change comes from the normalization of the generator: $(t_0^I)^i_j = \delta^i_j/\sqrt{2N_I}$. So we have no difficulty in adding the magnetic FI term. The second one is the case when the gauge group is different from $A_{n-1}$. In fact, we can also construct $\mathcal{N} = 2$, $D_n$, $E_6$, $E_7$ and $E_8$ quiver gauge models. Even in these cases, all we have to do is to relate the FI parameters in accordance with (2.20) or (2.22).

### 3 The minimal model

We will show that in our model $\mathcal{N} = 2$ supersymmetry is partially broken to $\mathcal{N} = 1$ spontaneously. As an illustration we will focus on the affine $A_1$ quiver gauge model to which we refer as the minimal model in the following sections.

The minimal model is composed of a pair of hypermultiplets $q_i^+$ and a pair of vector multiplets $V_{I}^{++}$ ($I = 1, 2$). $q_i^+$ and $q_i^+$, respectively, transform as bi-fundamental, $(N_1, \bar{N}_2)$ and $(N_1, N_2)$, and $V_{I}^{++}$ transform as adjoint under $U(N_I)$. The action of this model is given by (2.21), with summing only over $I = 1, 2$.

Let us write down the scalar potential in component. The scalar potential is

$$
V = \sum_{I=1,2} [V_I^{(1)} + V_I^{(2)}],
$$

(3.1)

$$
V_I^{(1)} = \frac{1}{2} g_{Iab} P_I^a P_I^b + \frac{1}{4} g_{Iab} D_I^{aAB} |D_I^{bAB}| - 2i \xi_D^{I AB} \xi_D^{I} \mathcal{F}_{I[0]} + 2i \xi_D^{I+1 AB} \xi_D^{I+1} \mathcal{F}_{I[0]} - 4i \xi_D^{AB} \xi_D^{I} \mathcal{F}_{IAB}.
$$

(3.2)

$$
V_I^{(2)} = (\bar{Q}_{I A} \bar{\phi}_j^i \phi^{k}_{I+1} - \bar{Q}_{I A} \bar{\phi}_j^i \phi^{k}_{I+1} \bar{\phi}^j_{I+1}) (\bar{Q}_{I k} Q_{I+1}^{A} \phi^{k}_{I+1} - \bar{Q}_{I k} \bar{Q}_{I+1}^{A} \phi^{k}_{I})
$$

$$
+ (\bar{Q}_{I A} \bar{\phi}_j^i \phi^{k}_{I+1} - \bar{Q}_{I A} \bar{\phi}_j^i \phi^{k}_{I+1} \bar{\phi}^j_{I+1}) (\bar{Q}_{I k} Q_{I+1}^{A} \phi^{k}_{I+1} - \bar{Q}_{I k} \bar{Q}_{I+1}^{A} \phi^{k}_{I}).
$$

(3.3)

where $P_I^a = -i f_{abc} \bar{\phi}_c^a \phi_b^c$, $\mathcal{F}_{I[ab...]}$ represents $\mathcal{F}_{I[ab...]}$ evaluated at $\theta^\pm = \bar{\theta}^\pm = 0$, and $g_{I[ab]} = \text{Im} \mathcal{F}_{I[ab]}$ is the Kähler metric. $D_I^{aAB}$ is obtained by solving the equation of motion as

$$
D_I^{aAB} = -2 g_I^{ab} [(\xi_I + \xi_I^I)^{AB} \delta^0_0 + \xi_I^{AB} \mathcal{F}_{I[0b]} + \bar{\xi}_I^{AB} \bar{\mathcal{F}}_{I[0b]} + \mathcal{Q}_{I[1b]} + \mathcal{Q}_{I[2b]}].
$$

(3.4)
where \( \mathcal{D}_{1I} \) and \( \mathcal{D}_{2I} \) are the contributions from the hypermultiplets \( q^+_1 \) and \( q^+_2 \) respectively:

\[
\mathcal{D}_{11|b} = \frac{i}{2} [Q^A_{1j} (t^i_b)^j Q^B_{1j} + (A \leftrightarrow B)], \quad \mathcal{D}_{21|b} = \frac{i}{2} [Q^A_{2j} (t^i_b)^j Q^B_{2j} + (A \leftrightarrow B)],
\]

\[
\mathcal{D}_{12|b} = \frac{i}{2} [Q^A_{1j} (t^i_b)^j Q^B_{1j} + (A \leftrightarrow B)], \quad \mathcal{D}_{22|b} = \frac{i}{2} [Q^A_{2j} (t^i_b)^j Q^B_{2j} + (A \leftrightarrow B)]. \tag{3.5}
\]

We can rewrite \( V^{(1)}_I \) in (3.1) as

\[
V^{(1)}_I = \frac{1}{2} g_{Ia} \mathcal{P}^a_P | \mathcal{D}^a_{IAB} | - 2i (\xi_I^{AB} - \bar{\xi}_I^{AB}) (\xi_D^{IAB} + \bar{\xi}_D^{IAB}) \tag{3.6}
\]

where

\[
| \mathcal{D}^a_{IAB} | = D^a_{IAB} + 4i \xi_D^{IAB} \delta_0^a
\]

\[
= -2g_I^{ab} \left[ (\xi_I + \bar{\xi}_I)^{AB} \delta_0^b + (\xi_D + \bar{\xi}_D)^{AB} \mathcal{F}_{I|0b} + \mathcal{D}^{AB}_{I|1b} + \mathcal{D}^{AB}_{I|2b} \right]. \tag{3.7}
\]

## 4 Vacua of the minimal model

In this section, we will find the \( \mathcal{N} = 1 \) supersymmetric vacua in the Coulomb branch \( \langle Q_I \rangle = 0 \) by analyzing the condition stabilizing the scalar potential derived in the previous section.

The constraint, \( \langle g_{Ia} \mathcal{P}^a_P \rangle = 0 \), can be satisfied by vanishing non-diagonal components of the vacuum expectation value of \( \phi_I \), that is, \( \langle \phi_I^r \rangle = 0 \) where \( t^I_r \) represent non-Cartan generators of the gauge group \( U(N_I) \). Then, we consider the condition to stabilize the scalar potential. While the derivative of the scalar potential \( V \) with respect to the hypermultiplet scalar \( Q \) is trivially zero in the Coulomb branch \( \langle Q_I \rangle = 0 \), the non-trivial vacuum condition is derived from the derivative with respect to \( \phi_I \) in the vector multiplet

\[
0 = \langle \frac{\partial}{\partial \phi^q_I} V \rangle = \frac{i}{4} \sum_{\alpha} \langle \mathcal{F}_{I|abc} | D_I^{ba} D_I^{ca} \rangle, \tag{4.1}
\]

where \( D_I^{ab} = i D_I^{aa}(\tau_a)^A_{B}. \) Note that the index \( I \) is not summed over here.

Let us examine the case with the single trace prepotential of degree \( n_I + 2 \)

\[
\mathcal{F}_I = \sum_{k=1}^{n_I+1} \frac{g_k}{(k+1)!} \text{Tr} W^{k+1} \tag{4.2}
\]

for concreteness. Let \( E^I_{ij} \), \( i = 1, \ldots, N_I \), be the fundamental matrix of gauge group \( U(N_I) \) which has 1 at the \( (i, j) \) component and 0 otherwise. Cartan generators can be written as \( t^I_i = E^I_{ii} \). We have \( \langle \partial V / \partial \phi^r_I \rangle = 0 \) because \( \langle \mathcal{F}_{I|ij} \rangle = \langle D_I^I \rangle = 0. \) Noting that the
points specified by $\langle F_i | i \rangle = 0$ correspond to the unstable vacua, we derive the vacuum conditions as follows,

$$\sum_\alpha \langle D_i^{\alpha} D_i^{\alpha} \rangle = 0, \text{ for all } i \text{ and } I. \quad (4.3)$$

As in [9], we can choose the FI parameters by using $SU(2)$ rotation as

$$\xi_I + \bar{\xi}_I = (0, e_I, \xi_I), \quad (\xi_D^I + \bar{\xi}_D^I) = (0, m_I, 0). \quad (4.4)$$

Furthermore, we can set $m_I/\xi_I < 0$, without loss of generality. In these choices of the FI parameters, we obtain the following vacuum condition,

$$\langle F_i | i \rangle = -2 \left( \frac{e_I}{m_I} + i \frac{\xi_I}{m_I} \right). \quad (4.5)$$

Note that the minus sign in front of $i \xi_I/m_I$ has been excluded by the positivity criterion of the Kähler metric: $\langle g_i | i \rangle = \text{Im} \langle F_i | i \rangle > 0$. In the original bases, this means

$$\langle F_I | 00 \rangle = - \left( \frac{e_I}{m_I} + i \frac{\xi_I}{m_I} \right). \quad (4.6)$$

The vacuum expectation values of the diagonal components of $\phi_I$ are determined from the above equations of degree $n_I$. Thus, the gauge symmetry $U(N_I)$ is broken to $\prod_{i=1}^{n_I} U(N_{I|i})$ with $N_I = \sum_{i=1}^{n_I} N_{I|i}$.

We can easily evaluate the vacuum energy

$$\langle V \rangle = \sum_{I=1,2} \left( -4m_I \xi_I - 4i \sum_\alpha (\xi_I - \bar{\xi}_I)^\alpha (\xi_D^I + \bar{\xi}_D^I)^\alpha \right) \quad (4.7)$$

which comes from the last two terms in [3.6]. As pointed out in [6], using the freedom to choose the imaginary part of $\xi_I^\alpha$, we can obtain the vanishing vacuum energy; if we set $(\xi_I - \bar{\xi}_I)^2 = i \xi_I$, then the vacuum energy is zero. The vanishing vacuum energy may indicate that $\mathcal{N} = 1$ supersymmetry remains in the vacuum.

In the subsequent subsections, we will show that the mass spectrum on the vacuum can be written in terms of $\mathcal{N} = 1$ multiplets, and that a linear combination of fermions becomes the Nambu-Goldstone fermion associated with the partial supersymmetry breaking.
4.1 Mass spectrum

The masses of the component fields contained in the vector multiplets are similar to those in the pure $U(N)$ Yang-Mills case \cite{11, 12}. The $\mathcal{N} = 2$, $U(N_I)$ vector multiplet decomposes into three types of $\mathcal{N} = 1$ multiplets: $\mathcal{N} = 1$, $\prod_{i=1}^{n_I} U(N_{I|i})$ massless vector multiplet, $\mathcal{N} = 1$ massive chiral multiplet in adjoint representation with mass $M_I = m_I \langle g_F^a \mathcal{F}_{I|000} \rangle$ where $t^I_\alpha$ represent unbroken generators, and $\mathcal{N} = 1$ massive vector multiplets which correspond to the broken generators when the gauge symmetry is broken.

Let us turn to the matter part. The mass of the scalar components $Q^i_{1A\bar{j}}$ is easily obtained by evaluating the second derivative of $V$. First we examine $V_I^{(1)}$ in (3.6). We observe its vacuum expectation value vanishes as follows. Since the first and the last terms in (3.6) do not contain the scalar $Q_I$, only the second term can contribute to the mass. However, it vanishes as

$$\left\langle \partial_{Q^i_{1A\bar{j}}} \partial_{Q^j_{K\bar{B}}} \sum_I g_{I|ab} D^{iCD}_I D^{\bar{j}CD}_{I}\right\rangle = -4 \sum_I \left\langle \left(\text{Re} D^{iCD}_I \right) \partial_{Q^i_{1A\bar{j}}} \partial_{Q^j_{K\bar{B}}} \left( D^{\bar{j}CD}_{I} + D^{\bar{j}CD}_{I}\right) \right\rangle$$

$$=-4i \delta^1_{\bar{j}} \delta^K_{\bar{A}} \left\langle (t^I_a)^{i}_{j} \delta^1_{\bar{J}} \text{Re} D^{a}_{1AB} - (t^2_a)^{i}_{j} \delta^1_{\bar{J}} \text{Re} D^{a}_{2AB} \right\rangle$$

$$=-4i \delta^2_{\bar{j}} \delta^K_{\bar{A}} \left\langle -(t^1_a)^{i}_{j} \delta^1_{\bar{J}} \text{Re} D^{a}_{1AB} + (t^2_a)^{i}_{j} \delta^1_{\bar{J}} \text{Re} D^{a}_{2AB} \right\rangle$$

$$=0. \quad (4.8)$$

In the last equality, we have used $\left\langle \text{Re} D^{a}_{I AB} \right\rangle = -2i(\tau_2)_{AB} m_{\bar{J}} \sigma^a_0$ and the relation (2.20). Next, we examine $V_I^{(2)}$ in (3.6). It is easy to see that the scalar components $Q^i_{1A\bar{j}}$ and $Q^i_{2A\bar{j}}$ have the same mass $m_{\bar{j}}^2 = 2|a_{1i} - a_{2j}|^2$ where

$$\left\langle \phi_I \right\rangle = \text{diag}(a_{11}, \ldots, a_{IN_I}). \quad (4.9)$$

The masses of the fermions $\kappa^I$ and $\psi^I$ can be seen from the following terms of $S_q$ (2.6)

$$\frac{1}{\sqrt{2}} \kappa^i_{1A\bar{j}}(\phi^i_{1\bar{j}} \delta^1_{\bar{J}} - \phi^i_{2\bar{j}} \delta^2_{\bar{J}}) \psi^i_{1\bar{j}} + \frac{1}{\sqrt{2}} \kappa^i_{2A\bar{j}}(\phi^i_{1\bar{j}} \delta^1_{\bar{J}} - \phi^i_{2\bar{j}} \delta^2_{\bar{J}}) \psi^i_{2\bar{j}} + h.c. \quad (4.10)$$

Let us examine the first term from which masses of $\kappa_1$ and $\psi_1$ are determined. As $\left\langle \phi_I \right\rangle$ is diagonal (4.9), we can rewrite it as

$$\frac{1}{2\sqrt{2}} \begin{pmatrix} \kappa^i_{11\bar{j}} & \psi^i_{1\bar{j}} \\ \psi^i_{1\bar{j}} & \psi^i_{1\bar{j}} \end{pmatrix} \begin{pmatrix} 0 & a_{1i} - a_{2j} \\ a_{1i} - a_{2j} & 0 \end{pmatrix} \begin{pmatrix} \kappa^i_{11\bar{j}} \\ \psi^i_{1\bar{j}} \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} \psi_{1\bar{j}} & \psi_{1\bar{j}} \\ \psi_{1\bar{j}} & \psi_{1\bar{j}} \end{pmatrix} \begin{pmatrix} a_{1i} - a_{2j} & 0 \\ 0 & a_{1i} - a_{2j} \end{pmatrix} \begin{pmatrix} \psi^i_{1\bar{j}} \\ \psi^i_{1\bar{j}} \end{pmatrix} \quad (4.11)$$
where we have defined $\psi_\pm \equiv (\kappa \pm \psi)/\sqrt{2}$. By taking the normalization of the kinetic terms into account, one sees that the masses of $\psi^i_{+1} \, \bar{\psi}^j$ and $\psi^i_{-1} \, \bar{\psi}^j$ can be evaluated as $\sqrt{2|a_{1i} - a_{2j}|^2}$. In the same way, by examining the second term in (4.10), the mass of $\psi^i_{+2} \, \bar{\psi}^j$ and $\psi^i_{-2} \, \bar{\psi}^j$ is found to be the same as that of $\psi^\pm_1$.

Thus in the vacuum, $\mathcal{N} = 2$ hypermultiplet $q^+_i$ in $(N_1, \bar{N}_2)$ representation is decomposed into various massive multiplets according to the branching rule and the massive multiplets with mass $\sqrt{2|a_{1i} - a_{2j}|^2}$ transform as $(N_1|_i, \bar{N}_2|_j)$. The same is true for another hypermultiplet $q^+_2$ in $(\bar{N}_1, N_2)$.

4.2 Nambu-Goldstone fermion

In the case of the $\mathcal{N} = 2$, $U(N)$ gauge model with/without hypermultiplets in adjoint representation [7, 8, 9], a linear combination of the overall $U(1)$ fermions in the $\mathcal{N} = 2$ vector multiplet becomes the Nambu-Goldstone fermion. One might think that as there are two gauge sectors, $U(N)^2$, two Nambu-Goldstone fermions would emerge. However, this is not correct. We show that only one combination of these fermions becomes the Nambu-Goldstone fermion.

The vacuum expectation values of the supersymmetry transformations of component fields vanish except for $\langle \delta \lambda^A_I \rangle$. In the choice of the FI parameters (4.4), we obtain

$$\langle D^{q^A}_{I AB} \rangle = im_I \delta^a_0 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

with $m_1 = m_2$ which follows from the condition (2.20). Thus, letting $\lambda^a_I \equiv \frac{1}{\sqrt{2}}(\lambda^1_I \pm \lambda^2_I)$,

$$\langle \delta \lambda^a_I^{+} \rangle = i \sqrt{2} m_I \delta^a_0 (\eta^1 - \eta^2), \quad \langle \delta \lambda^a_I^{-} \rangle = 0 .$$

Furthermore, combining $\lambda^{0+}_1$ and $\lambda^{0+}_2$, we find

$$\langle \delta (\lambda^{0+}_1 - \lambda^{0+}_2) \rangle = i \sqrt{2} (m_1 - m_2) (\eta^1 - \eta^2) = 0 ,$$

$$\langle \delta (\lambda^{0+}_1 + \lambda^{0+}_2) \rangle = i \sqrt{2} (m_1 + m_2) (\eta^1 - \eta^2) \neq 0$$

where in the last equality we have used $m_1 = m_2$. So, the fermion $\lambda^{0+}_1 + \lambda^{0+}_2$ can be the Nambu-Goldstone fermion. In order to conclude that this is the Nambu-Goldstone fermion, we have to show this fermion is exactly massless.

The mass term of $\lambda^0$ can be read off from (2.2) as

$$\frac{i}{4} \langle \mathcal{F}_{I000} D^{0AB}_I \rangle \lambda^0_I \lambda^0_B ,$$

12
with the replacement (2.14). It is easy to see that this is proportional to \( im_{I} \lambda_{I}^{0+} \lambda_{I}^{0-} \), and thus we conclude that \( \lambda_{I}^{0+} + \lambda_{I}^{0+} \) is massless as \( \lambda_{I}^{0+} \) are massless. As a result, we can identify \( \lambda_{I}^{0+} + \lambda_{I}^{0+} \) as the Nambu-Goldstone fermion associated with the partial spontaneous supersymmetry breaking.

Let us comment on the affine \( A_{n-1} \) quiver gauge model with \( n > 2 \). In this case \( \delta \lambda_{I}^{0+} \) are independent of \( I \) because of the condition (2.20). Therefore, only the vacuum expectation value of the supersymmetry transformation of the combination, \( \tilde{\lambda} = \sum_{I} \lambda_{I}^{0+} \), is not zero. As \( \lambda_{I}^{0+} \) are massless and so \( \tilde{\lambda} \) are, we may identify \( \tilde{\lambda} \) with the Nambu-Goldstone fermion associated with partial supersymmetry breaking.

5 A dynamical realization of Klebanov-Witten model

In [19], Klebanov and Witten considered the \( \mathcal{N} = 2 \) affine \( A_{1} \) quiver gauge theory realized on the world volume on D3-branes at orbifold singularity (dual to type IIB superstring in \( AdS_{5} \times S^{5}/\mathbb{Z}_{2} \) [22]), and discussed a flow to \( \mathcal{N} = 1 \) fixed point by adding the mass operator [23] of the chiral multiplet \( \Phi_{I} \) in adjoint representation, which breaks \( \mathcal{N} = 2 \) supersymmetry to \( \mathcal{N} = 1 \). It was shown that the superpotential of the effective theory at the fixed point can be regarded as that of the world volume theory at the conifold singularity (dual to type IIB superstring in \( AdS_{5} \times T^{1,1} \)).

As our minimal model is \( \mathcal{N} = 2 \) affine \( A_{1} \) quiver gauge theory with non-canonical kinetic terms and electric and magnetic FI terms, it is expected that our model might describe the \( \mathcal{N} = 1 \) quiver gauge theory dual to type IIB superstring in \( AdS_{5} \times T^{1,1} \) in some points of vacua. If so, the minimal model may provide a dynamical realization of the statement of [19] because the \( \mathcal{N} = 1 \) chiral multiplet \( \Phi_{I} \) in adjoint representation becomes massive dynamically. We show that this is the case.

First, we examine the matter content realized on the vacuum. Let us consider the following point of vacua

\[
\langle \phi_{1} \rangle = \langle \phi_{2} \rangle = \text{diag}(a, \ldots, a).
\]  

(5.1)

The \( U(N)^{2} \) gauge symmetry is unbroken in this vacuum, while \( \mathcal{N} = 1 \) chiral multiplet in the adjoint representation becomes massive with mass \( M_{I} = m_{I} \langle g_{I}^{aa} \mathcal{F}_{I} |_{0a} \rangle \) [8]. As was seen in section 4.1, the \( \mathcal{N} = 1 \) chiral multiplets \( (A_{I}, \bar{B}_{I}) \) in bi-fundamental representation acquire masses \( m_{ij}^{2} = 2|a_{1i} - a_{2j}|^{2} \), but in the vacuum (5.1) these become massless. Thus, the massless multiplets in the vacuum are \( \mathcal{N} = 1, U(N)^{2} \) vector multiplet, \( \mathcal{N} = 1 \) chiral multiplets \( A_{I} \) and \( B_{I} \) \((I = 1, 2)\) in \((N, \bar{N})\) and \((\bar{N}, N)\) representations, respectively. This
is exactly the matter content of the $\mathcal{N} = 1$ quiver gauge theory dual to IIB superstring in $AdS_5 \times T^{1,1}$.

Next, we examine the superpotential realized in the vacuum. Expanding each field around its vacuum expectation value, we obtain the fluctuation action in the $\mathcal{N} = 1$ vacuum. The matter part of the action can be written in terms of $\mathcal{N} = 1$ multiplets. The superpotential term which contains the chiral multiplets $A_I$ and $B_I$ in bi-fundamental representation is

$$W_{\text{matter}} = \text{Tr}[(A_1 B_1 - A_2 B_2) \Phi_1 - (B_1 A_1 - B_2 A_2) \Phi_2]$$

(5.2)

up to an overall numerical coefficient. Here we have not written the gauge index explicitly. Of course, this is the ordinary superpotential of $\mathcal{N} = 2$ supersymmetric gauge theory with hypermultiplets in $\mathcal{N} = 1$ superspace formalism [19, 23], because we are considering the Coulomb branch. On the other hand, the mass term of the chiral multiplets $\Phi_I$ in the superpotential is found to be

$$W_\Phi = M_1 \text{Tr} \Phi_1^2 + M_2 \text{Tr} \Phi_2^2.$$  

(5.3)

The effective theory is described by the massless fields, and the massive chiral multiplets $\Phi_I$ in adjoint representation should be integrated out. To achieve this, we note that the mass matrix $M_I$ is determined by the vacuum expectation value $\langle \phi \rangle$ and the prepotential $\mathcal{F}_I$. Though we are working in special points of vacua, $M_I$ still depends on the choice of the prepotential function. Suppose that the prepotentials are set to satisfy $M_1 = -M_2$. By integrating out $\Phi_I$ using the equation of motion, we obtain the following superpotential up to an overall numerical coefficient

$$\text{Tr}(A_1 B_1 A_2 B_2 - B_1 A_1 B_2 A_2)$$

(5.4)

which is the well-known superpotential of the world volume theory on the D3-branes at the conifold singularity.

It is interesting to take the higher order terms in $\Phi_I$ into account, and examine the deformations of the superpotential (5.4).

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Appendix

A Superfields in Harmonic Superspace

Harmonic superspace \cite{14,15} is an extension of the $\mathcal{N} = 2$ superspace \cite{24} by including harmonic variables $u_A^\pm (A = 1, 2)$

\[(u_A^+, u_A^-) \in SU(2), \quad u_A^+ u_A^- = 1, \quad u_A^+ = u_A^- \quad \text{A.1}\]

where the bar “−” means the complex conjugation

\[\overline{Q}_A = \bar{Q}_A, \quad \overline{Q}_A \epsilon_{AB} Q_B = \epsilon_{AB} \epsilon_{BC} \bar{Q}_C = -\bar{Q}_A, \quad \overline{\xi}_{AB} = \bar{\xi}_{AB}, \quad \overline{\xi}_{AB} \epsilon_{AC} \epsilon_{BD} \xi_{CD} = \epsilon_{AC} \epsilon_{BD} \epsilon_{CE} \epsilon_{DF} \bar{\xi}_{EF} = \bar{\xi}_{AB}. \quad \text{A.2}\]

We introduce an $\mathcal{N} = 2$ vector multiplet $V^{++} = V^{++}_a t_a$ transforming as adjoint representation under the gauge group: hermitian matrices $(t_a)^i_j$ generate $[t_a, t_b] = i f_{ac}^b t_c$. $V^{++}$ is the analytic superfield satisfying $D^+ V^{++} = \bar{D}^+ V^{++} = 0$. In the analytic basis

\[(y^m, \theta^\pm, \bar{\theta}^\pm, u_A^\pm) = (x^m - 2i \theta^A \sigma^m \bar{\theta}^B u_A^+(u_B^+), \theta^A u_A^+, \bar{\theta}^A u_A^-, u_A^\pm) \quad \text{A.3}\]

$D^\pm$ and $\bar{D}^\pm$ are given as

\[D^\pm_\alpha = \frac{\partial}{\partial \theta^\pm_\alpha}, \quad D^\pm_= = -\frac{\partial}{\partial \theta^\pm_+} + 2i(\sigma^m \bar{\theta}^-)_{\alpha} \frac{\partial}{\partial y^m}, \quad \text{A.4}\]

and thus $V^{++}$ is a superfield in analytic subspace $(y, \theta^+, \bar{\theta}^+, u_A^\pm)$. In the Wess-Zumino gauge $V^{++}$ is given as

\[V^{++} = -2i \theta^+ \sigma^m \bar{\theta}^+ v_m(y) - i \sqrt{2}(\theta^+)^2 \phi(y) + i \sqrt{2}(\bar{\theta}^+)^2 \phi(y) + 4(\bar{\theta}^+)^2 \theta^+ \lambda^A(y) u_A^- - 4(\theta^+)^2 \bar{\theta}^+ \bar{\lambda}^A(y) u_A^- + 3(\theta^+)^2 (\bar{\theta}^+)^2 D^{AB}(y) u_A^- u_B^- \quad \text{A.5}\]
where \( v_m, \phi, \lambda^A \) and \( D^{AB} \) are vector, complex scalar, \( SU(2) \) doublet Weyl spinors and auxiliary field, respectively. The curvature \( \bar{W} \) of \( V^{++} \) defined by

\[
\bar{W} = -\frac{1}{4}(D^+)^2 \sum_{n=1}^{\infty} \int dv_1 \cdots dv_n (-i)^{n+1} \frac{V^{++}(v_1) \cdots V^{++}(v_n)}{(u^+v_1^+)(v_1^+v_2^+) \cdots (v_n^+u^+)}
\]

is evaluated to give

\[
\bar{W} = \bar{\theta}^A \sigma^{mn} \bar{\theta}_A \bar{v}_{mn} - i \sqrt{2} \bar{\phi} + i \sqrt{2}(\bar{\theta})^4 \eta^{mn} D_mD_n \phi + \frac{4}{3} i(\bar{\theta}^A \bar{\theta}^B) D_m \lambda_A \sigma^m \bar{\theta}_B - 2 \bar{\theta}^A \lambda_A
\]

\[
+ \bar{\theta}^A \bar{\theta}_m D_{AB} - \frac{2}{3} \sqrt{2}(\bar{\theta}^A \bar{\theta}^B)[\bar{\phi}, \bar{\theta}_A \lambda_B] + i(\bar{\theta})^4 \varepsilon_{AB}[\lambda^A, \lambda^B] + i \sqrt{2}(\bar{\theta})^4[\bar{\phi}, [\bar{\phi}, \bar{\theta}]]
\]

\[
- 2i \bar{\theta}^+ \bar{\theta}^- \bar{\eta} + \cdots
\]

where \( v_{mn} = \bar{\partial}_m v_n - \partial_n v_m + i[v_m, v_n] \) and \( D_m \circ = \partial_m \circ + i[v_m, \circ] \). The ellipsis represents terms which do not contribute to the action \( (2.1) \).

Next we introduce an \( \mathcal{N} = 2 \) hypermultiplet \( q^+_{iJ} \) transforming as bi-fundamental representation under the gauge group. The \( q^+ \) hypermultiplet is an analytic superfield satisfying \( D^+ q^+ = \bar{D}^+ q^+ = 0 \), and can be expanded as

\[
q^+ = F^+(y, u) + \theta^+ \psi(y, u) + \bar{\theta}^+ \bar{\kappa}(y, u) + (\theta^+)^2 M^-(y, u) + (\bar{\theta}^+)^2 N^-(y, u)
\]

\[
+ i \theta^+ \sigma^m \bar{\theta} m(y, u) + (\theta^+)^2 \theta^+ \gamma^{(-2)}(y, u) + (\bar{\theta}^+)^2 \bar{\theta}^+ \chi^{(-2)}(y, u)
\]

\[
+ (\theta^+)^2 (\bar{\theta}^+)^2 P^{(-3)}(y, u)
\]

in the analytic basis. \( q^+ \) contains infinitely many auxiliary fields which are eliminated by solving the equations of motion derived from \( (2.3) \): \( \mathcal{D}^{++} q^+ = 0 \) where \( \mathcal{D}^{++} \) is given in \( (2.4) \) and

\[
D^{++} = \partial^{++} - 2i \theta^+ \sigma^m \bar{\theta} m \frac{\partial}{\partial y^m} + \theta^+ \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^+}, \quad \partial^{++} = u^+ \frac{\partial}{\partial u^-}
\]

We find that the auxiliary fields are eliminated by

\[
F^{+i}_{+i} = Q^{A}_{i} (y)^i_{j} u^+_A,
\]

\[
\psi^i_{+i} = \psi(y)^i_{i}, \quad \bar{\kappa}^i_{+i} = \bar{\kappa}(y)^i_{i},
\]

\[
M^{+i}_{+i} = - \sqrt{2} (\bar{\phi}^i_{j} Q^{A}_{i} - \bar{\phi}_i^{+j} Q^{A}_{i}) u^-_A,
\]

\[
N^{+i}_{+i} = \sqrt{2} (\phi^i_{j} Q^{A}_{i} - \phi_i^{-j} Q^{A}_{i}) u^-_A
\]

\[
A^{+i}_{+i} = 2 D_m Q^{A}_{i} u^-_m,
\]

\[
\gamma^{(-2)i}_{+i} = 2i (\bar{\lambda}^A_{i} Q^{B}_{j} - \bar{\lambda}^{A}_{i} Q^{B}_{j}) u^-_A u^-_B,
\]

\[
\chi^{(-2)i}_{+i} = 2i (\lambda^A_{i} Q^{B}_{j} - \lambda^A_{i} Q^{B}_{j}) u^-_A u^-_B
\]

\[
P^{(-3)i}_{+i} = -i (D^{AB}_{i} Q^{C}_{j} - D^{AB}_{i} Q^{C}_{i}) u^-_A u^-_B u^-_C
\]

The physical fields are \( SU(2) \) doublet complex scalars \( Q^A \) and a pair of \( SU(2) \) isosinglet spinors, \( \psi \) and \( \kappa \).
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