Semi-device independent random number expansion without entanglement

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By testing the classical correlation violation between two systems, the random number can be expanded and certified without applying classical statistical method. In this work, we propose a new random number expansion protocol without entanglement, and the randomness can be guaranteed only by the 2-dimension quantum witness violation. Furthermore, we only assume that the dimensionality of the system used in the protocol has a tight bound, and the whole protocol can be regarded as a semi-device independent black-box scenario. Comparing with the device independent random number expansion protocol based on entanglement, our protocol is much easier to implement and test.

I. INTRODUCTION

True random numbers have significant applications in numerical simulation, lottery games, biological systems and cryptography \cite{1}. More particularly, security of the quantum key distribution (QKD) protocol \cite{2} is based on the random selection of the state preparation and measurement, if the state preparation and measurement are known by the eavesdropper precisely, she can apply the man-in-the-middle attack \cite{3} to get all of the secret information without being discovered. True random numbers should be unpredictable for the third party, so most of true random number generation protocols are based on unpredictable physical processes \cite{4–11}. Unfortunately, the true random number generated by these protocols can only be characterized with the classical statistical method, such as the Statistical Test Suite from NIST et al. \cite{12,13}. Inspired by the device independent quantum information processing based on non-local correlations of entanglement particles \cite{14–16}, Colbeck et al. \cite{17,18} have proposed the true random number generation protocol based on the GHZ test, Pironio et al. \cite{19,20} have proposed the true random number generation protocol certified by the Bell inequality violation, they also have given a proof of concept experimental demonstration of their protocol by approximately one metre distance. The true random number generation protocol based on entanglement, which require no assumption about the internal working of the device both in two states measurement sides, thus the true random number can not be generated with only the classical method, and the randomness of their experimental result can only be certified by the Bell inequality violation. Since the protocol require the pre-established true random number to select the measurement basis, it also can be called device independent random number expansion protocol correspondingly. Comparing with the random number generation protocol certified by the classical statistical method, the device independent random number expansion protocol offers a new method to quantify unequivocally the observed random numbers.

Both of the two device independent random number expansion protocols strongly suggest that only entanglement based protocols are suitable for establishing the quantified true random numbers \cite{17,20}. However, the entanglement based protocol has much more complicated experimental setups comparing with the one-way system, where the first black box prepares an arbitrary quantum state, sends it to the other black box for performing an arbitrary measurement. Furthermore, most commercial true random number generation systems are based on one-way protocols. Inspired by the method of device-independent test of the classical and quantum dimensions given by Gallego et al. \cite{21}, Pawlowski et al. \cite{22} have proposed a semi-device independent one-way QKD protocol with 4 input states and 2 measurement bases, security of which was based on the 2-dimensional quantum witness and the quantum random access code. Here, we propose the one-way semi-device independent random number expansion protocol without entanglement, the randomness of which can be quantified with 2-dimension quantum witness, and the experimental demonstration can be established by combining the commercial QKD setup with different modulation protocols, the randomness of which can be proved in the following section by applying the numerical calculation method. Similar to Colbeck and Pironio’s models, our protocol require no assumption about the internal working of the state preparation and measurement device, except that the 2-dimensional quantum system and collective attacks are bounded. However, we need the quantum state to be prepared and measured in the same safe area, the quantum state and classical information should not be divulged to the eavesdropper in the unsafe area.

II. MODEL DESCRIPTION

In this section, we give the semi-device independent random number expansion protocol, where only two black boxes should be considered. The two black boxes can be used for illustrating the state preparation and measurement respectively, detailed scenario can be de-
picted precisely in Fig. 1.

![Diagram](image)

**FIG. 1:** Semi-device independent random number expansion protocol. The protocol require the state preparation black box and the state measurement black box respectively, both of the two black boxes are in the same safe area.

In the semi-device independent random number expansion protocol, we randomly select four classical input bits \( a \in \{00, 01, 10, 11\} \) in the first black box, when pressing the button \( a \), the first black box will emit the classical or quantum state \( \rho_a \), then the prepared state \( \rho_a \) will be sent to the second black box correspondingly. When pressing the button \( y = \{0, 1\} \), the second black box will emit the measurement outcome \( b = \{0, 1\} \). We suppose that only the 2-dimensional system will be considered in this protocol, that is \( \rho_a \in \mathbb{C}^2 \).

Formally, we can estimate the probability distribution by repeating this procedure many times, which can be illustrated precisely as the following equation,

\[
P(b|ay) = \text{tr}(\rho_a M_y^b),
\]

where, \( M_y^b \) is the measurement operator acting on two dimension Hilbert space with the input parameter \( y \) and the output parameter \( b \) by considering the prepared state \( \rho_a \). In this protocol, the true random number can be produced by only considering the data table \( P(b|ay) \). More precisely, we do not require any assumption on how the probability was obtained with two black boxes, except that the state preparation and measurement can be guaranteed with 2-dimension quantum witness.

We will use the following expectation value to illustrate the probability distribution for the convenient analysis in the following section,

\[
E_{ay} = P(b = 0|ay),
\]

\[
P(b = 0|ay) + P(b = 1|ay) = 1,
\]

where the set of probability distributions \( E_{ay} \) can be used for illustrating the quantum dimension witness. In the theoretical side, two types of 2-dimension quantum witness have been proposed \cite{21, 22}, we will apply the following tight 2-dimension classical witness in our security analysis

\[
T = E_{000} + E_{001} + E_{010} - E_{011} - E_{100} + E_{101} - E_{110} - E_{111} \leq 2.828,
\]

where we only consider the 4 state preparation and 2 measurement basis case in this inequation. The other similar expression with 3 state preparation and 2 measurement bases case has also been given in Ref. \cite{21}, but the 2-dimension quantum witness in this case can not be used for expanding the true random number.

More precisely, the tight 2-dimension quantum witness can be given as the following inequation (More detailed information about this inequation can also be found in Ref. \cite{22})

\[
T = E_{000} + E_{001} + E_{010} - E_{011} - E_{100} + E_{101} - E_{110} - E_{111} \leq 2.828,
\]

the maximal value of the two-dimension quantum witness can be calculated numerically. More interestingly, it also can be analyzed by applying the 2-to-1 quantum random access code protocol \cite{22, 25}, where Alice receives two uniformly distributed bits and sends the encoded physical system to Bob, Bob is asked to guess one of Alice’s bits randomly. This 2-dimension quantum witness is the main tool to analyze the proposed random number expansion protocol, and our main result is to establish the relationship between the randomness of the measurement outcome and its expected 2-dimension quantum witness violation.

We quantify the randomness of the measurement outcome \( b \) conditioned on the input values \( a, y \) by the min-entropy \cite{24}

\[
H_\infty(B|A, Y) = -\log_2(\max_{b,a,y} P(b|a, y)).
\]

From this equation, we can find that the purpose of this paper is to obtain the upper bound of the conditional probability distribution \( P(b|a, y) \) for a given 2-dimension quantum witness \( T \). More precisely, the maximal probability distribution \( \max_{b,a,y} P(b|a, y) \) denotes the solution to the following optimization problem:

\[
\max_{b,a,y} P(b|a, y)
\]

subject to :

\[
E_{000} + E_{001} + E_{010} - E_{011} - E_{100} + E_{101} - E_{110} - E_{111} = T
\]

\[
E_{ay} = \text{tr}(\rho_a M_y^b)
\]

where the optimization is carried arbitrary quantum states \{\( \rho_0, \rho_01, \rho_{10}, \rho_{11} \)\} and measurement operators \{\( M_0^0, M_0^1 \)\} and \{\( M_1^0, M_1^1 \)\}, where \( M_0^0 + M_1^1 = M_1^0 + M_1^1 = I \). Fortunately, Masanes \cite{27} has proved that only the projective measurement should be considered in case of 2-observable and 2-measurement outcomes has been considered. Since \( T \) is the linear expression of the probabilities, we can only consider pure states \cite{21} preparation in our numerical calculation. Without loss of generality, the state preparation and measurement in our numerical calculation can be illustrated precisely with the following equations respectively,

\[
\rho_a = |\varphi(a)\rangle \langle \varphi(a)|,
\]
\[ |\varphi(a)\rangle = \left( \begin{array}{c}
\cos\left(\frac{\theta}{2}\right) \\
\frac{1}{2}e^{i\eta\sin(\frac{\theta}{2})}
\end{array} \right), \quad (8)
\]
\[ M_0^0 = \left( \begin{array}{cc}
1 & 0 \\
0 & 0
\end{array} \right), \quad (9)
\]
\[ M_1^0 = \left( \frac{1}{2}e^{i\eta\sin(\theta)} \begin{array}{c}
\cos^2\left(\frac{\theta}{2}\right) \\
\frac{1}{2}e^{-i\eta\sin(\theta)}
\end{array} \right), \quad (10)
\]

where \( a \in \{00, 01, 10, 11\} \), \( 0 \leq \theta, \theta \leq \pi \), \( 0 \leq \eta, \eta \leq 2\pi \).

By applying the maximization problem, we get the min-entropy bound of the measurement outcome for given 2-dimension quantum witness \( T \), detailed expression of the relationship between the 2-dimension quantum witness and the min-entropy bound can be depicted precisely in Fig. 2.

FIG. 2: The relationship between the 2-dimension quantum witness and the the min-entropy bound. The min-entropy starts at zero in the 2-dimension classical witness case, systems that violate the 2-dimension quantum witness 2.64 have a positive min-entropy.

The calculation result shows that if the the violation of the 2-dimension quantum witness is larger than 2.64, the semi-device independent true random number can be expanded correspondingly. The maximal value of the min-entropy bound in our numerical calculation is 0.206, which can be satisfied in case of the 2-dimension quantum witness violation is 2.828.

III. EXAMPLE DESCRIPTION

In this section, we give a particular protocol to illustrate the semi-device independent random number expansion protocol. This protocol is equal to the (2,1,0.85) quantum random access code protocol [22, 23]. In this particular protocol, the state preparation in the first black box can be illustrated precisely as the following equations

\begin{align*}
|\varphi(00)\rangle &= \cos\left(\frac{\theta}{8}\right)|0\rangle + \sin\left(\frac{\theta}{8}\right)|1\rangle, \\
|\varphi(01)\rangle &= \cos\left(\frac{\theta}{8}\right)|0\rangle + \sin\left(\frac{\theta}{8}\right)|1\rangle, \\
|\varphi(10)\rangle &= \cos\left(\frac{\theta}{8}\right)|0\rangle + \sin\left(\frac{\theta}{8}\right)|1\rangle, \\
|\varphi(11)\rangle &= \cos\left(\frac{\theta}{8}\right)|0\rangle + \sin\left(\frac{\theta}{8}\right)|1\rangle. \quad (11)
\end{align*}

For the state measurement in the second black box, we will apply the two projective measurements with the following bases

\begin{align*}
\{M_0^0 = |0\rangle\langle 0|, & \quad M_1^0 = |1\rangle\langle 1|\}, \\
\{M_0^1 = |+\rangle\langle +|, & \quad M_1^1 = |--\rangle\langle --|\}. \quad (12)
\end{align*}

where, \(|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |\rangle\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\). The 2-dimension quantum witness in this protocol is 2.828, which is the maximal 2-dimension quantum witness violation. Combining this state preparation and measurement protocol with the true random number extraction analysis result, we can numerically calculate min-entropy bound of the expanded random bit is 0.206. Note that we only need true random numbers \( a \) and \( y \) to estimate dimension witness value, no more random numbers should be pre-established by two black boxes, thus our random number expansion protocol only need few random number seed.

Since the BB84 protocol is also based on the 4 input states and 2 measurement bases case, one natural question is considering whether the BB84 protocol can be used for generating the true random number in our randomness analysis. Unfortunately, it can not be used for generating the true random number because of it does not violate the 2-dimension quantum witness 2.64. More precisely, the state preparation in the BB84 protocol can be illustrated as

\begin{align*}
|\tilde{\varphi}(00)\rangle &= |0\rangle, \quad |\tilde{\varphi}(01)\rangle = |\rangle, \\
|\tilde{\varphi}(10)\rangle &= |+\rangle, \quad |\tilde{\varphi}(11)\rangle = |1\rangle. \quad (13)
\end{align*}

the measurement bases are equal to the (2,1,0.85) random access code case. Then the dimension witness achieves \( T = 2 \), which indicates that no true random number can be generated by considering the semi-device independent random number expansion protocol.

IV. DISCUSSION

We have proposed a new true random number expansion protocol in this paper, we can quantify the randomness with 2-dimension quantum witness violation, not based on the classical statistical method.

Comparing with the quantified random number expansion protocol based on entanglement, we give a much simpler method, and our protocol does not need any entanglement. Unfortunately, since the maximal ratio of the expanded random number is 0.206, our protocol has a much lower random number expansion efficiency. However, since our semi-device independent protocol is much
easier to implement than the full device independent protocol based on entanglement, thus the semi independent protocol will generate much more random numbers than the full device independent protocol in the same period of time. It is an open question to discuss if people can find a much higher efficiency random number expansion protocol in the future research with the similar method. Similar to the security analysis given by Pironio et al. [20], it also will be very interesting to analytically prove the min-entropy bound by considering the quantum dimension witness violation.

Device-independent quantum information has attracted much attentions for its higher level security comparing with the protocol based on trusted devices. Combining the semi-device independent random number expansion protocol with the device independent QKD protocol, we hope to get a much higher security than the QKD protocol based solely on some mathematical methods certified random numbers.

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