Splitting indistinguishable photons: Using linear optics to exceed the limit of photon blockade

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Photon-photon interactions are an essential requirement of quantum photonic information processing. One way to generate these interactions is to utilize an atom strongly coupled to an optical cavity. This system exhibits the photon blockade effect which enables single photon switching and creation of non-classical light. But the nonlinear effects enabled by this system suffer from a fundamental time-bandwidth constraint. For the the simple case of splitting an input pulse of two indistinguishable photons, this constraint imposes a limit on the efficiency of routing photons to different output ports. We show that this limit can be exceeded by combining the strongly-coupled atom with linear optics. By optimizing the unitary of the linear optical transformation, we achieve improved splitting efficiency for both un-entangled and entangled photons. Our results suggest that it may be possible to improve the efficiency of nonlinear optical processes at the single photon level by making suitable use of linear optics. These results could have implications for quantum information processing with photons.

I. INTRODUCTION

Photon-photon interactions are essential for photonic quantum information processing [1, 2]. They enable two qubit gates between photons, which are a key requirement for quantum computation [3]. They also play an important role in photonic quantum simulation [4]. But generating these interactions is challenging because they require optical nonlinearities at the single photon level.

One way to generate optical nonlinearities at the single photon level is to exploit strong light-matter interactions between a two-level atom and an optical cavity [5]. These systems can exhibit photon blockade [6], which allows a single photon to strongly modify the cavity reflectivity and thus control the transmission and reflection of subsequent photons [6–8]. If photon blockade shown by an atom-cavity system were ideal, two indistinguishable photons incident on the system would be split to different output ports, something which cannot be done with linear optics with probability greater than 50% [9]. But all previous realizations of photon blockade failed to achieve a complete splitting of two indistinguishable photons due to a fundamental time-bandwidth limit, originally analyzed by Rosenblum et al. [9]. This limit imposes a maximum splitting efficiency of 66% for an input pulse containing unentangled photons, and 77% when time-energy entanglement is introduced between input photons. This limit constrains the achievable single photon nonlinearity using a two-level atom in a cavity [10].

In this work, we show that this limit in splitting efficiency can be exceeded by adding linear optics after the atom-cavity system. These linear optics create a unitary transformation between the output ports of the atom-cavity system. By optimizing the unitary transformation we achieve a splitting efficiency of greater than 75% for unentangled input photons, and 90% for entangled input photons. The improvement in splitting efficiency is enabled by two-photon interference between the two output ports of the atom-cavity system, which we exploit through the optimized unitary. Our results suggest that it may be possible to extract stronger nonlinear effects from a two-level atom than previously believed.

II. SPLITTING EFFICIENCY FOR UNENTANGLED PHOTONS

Fig. 1 shows the experimental setup we analyze. The photon enters the input port of a microtoroidal cavity, defined in the diagram as $\hat{a}_{in}$. The microtoroidal cavity has both a clockwise and counter-clockwise propagating mode, each of which couple to two-level atom with equal cavity-atom coupling constants $g$. The counterclockwise mode decay to output mode $\hat{a}_{out}$ with a decay rate of $2\kappa_{ex}$, while the clockwise mode decays to mode bout with the same rate. This part of the system is identical to the one originally studied by Rosenblum et. al., and is known to have a limited routing efficiency.

In order to improve the splitting efficiency we inject the output ports of the cavity into an arbitrary linear optical unitary operation, as shown in Fig. 1. We restrict our attention specifically to unitaries with two-input and two-output modes to preserve the same photonic Hilbert space. In this case, the Mach-Zehnder interferometer shown in Fig. 1 can apply the most general linear optical unitary on the photonic states. The Mach-Zehnder interferometer transforms the modes $\hat{a}_{out}$ and $\hat{b}_{out}$ into modes $\hat{c}_{out}$ and $\hat{d}_{out}$ according to the transformation:

$$
\begin{pmatrix}
\hat{e}_{out} \\
\hat{d}_{out}
\end{pmatrix} =
\begin{pmatrix}
e^{i\phi}\sin(\omega) & \cos(\omega) \\
e^{i\phi}\cos(\omega) & -\sin(\omega)
\end{pmatrix}
\begin{pmatrix}
\hat{a}_{out} \\
\hat{b}_{out}
\end{pmatrix}
$$

(1)
Detecting one photon each in the output modes with efficiency. We define this figure as the probability of improving photon-photon splitting.

The figure of merit that quantifies how close the system is to the ideal two-photon splitter is the splitting efficiency. We calculate the splitting efficiency from the correlations we calculate in Eqs. (2)–(5) have the field of two indistinguishable photons that are unentangled in time and frequency. Appendix A provides the full derivation of the splitting efficiency after the usual transformation. The splitting efficiency is given by:

$$S = P_{cd} + P_{dc},$$

The correlation function $\Gamma^{cd}$ is the probability density of detecting a photon in mode $c$ at time $t$, and then a second photon in mode $d$ at time $t + \tau$, while $\Gamma^{dc}$ is similarly defined with the two output modes reversed. The probability densities of detecting the two photons at the same port, $\Gamma^{cc}$ and $\Gamma^{dd}$, can be defined in the same way. We integrate the correlations in Eqs. (4) and (5) over both $t$ and $\tau$ to obtain the total probabilities:

$$P_{cd} = \int_{0}^{\infty} \int_{0}^{\infty} \Gamma^{cd}(t, \tau) \, dt \, d\tau$$
$$P_{dc} = \int_{0}^{\infty} \int_{0}^{\infty} \Gamma^{dc}(t, \tau) \, dt \, d\tau$$

The splitting efficiency, denoted $S$, is the sum of these two probabilities: $S = P_{cd} + P_{dc}$.

Fig. 2 illustrates the numerical model we use to analyze the system in Fig. 1. We perform our analysis in the fast cavity regime which allows us to adiabatically eliminate the cavity modes and approximate the atom-cavity system as a 1D atom that couples to the modes of the fiber with an enhanced decay rate $2\gamma = 2g^2/\kappa$ [11]. To model a quantum light source, we employ a feeder cavity at the input to the atom-cavity system. We define $\hat{a}$ as the bosonic operator for the feeder cavity, and $\sigma$ the atomic lowering operator for the effective 1D two-level system.

In the above equation the unitary is controlled by two phase parameters, $\omega$ and $\phi$. The phase $\omega$ controls the amplitude splitting ratio of the Mach Zehnder interferometer, and physically corresponds to the path length difference between the two interferometer arms. The other phase parameter, $\phi$, applies an additional phase shift between the two input modes of the interferometer. Our goal is to optimize the two parameters of the unitary to improve photon-photon splitting.

The figure of merit that quantifies how close the system behaves to an ideal single photon splitter is the splitting efficiency. We define this figure as the probability of detecting one photon each in the output modes $\hat{c}_{out}$ and $\hat{d}_{out}$, given that the input mode $\hat{a}_{in}$ contains two photons initially. We calculate the splitting efficiency from the two-time correlation functions.

$$\Gamma^{cc} = \langle \hat{c}_{out}^\dagger(t) \hat{c}_{out}^\dagger(t + \tau) \hat{c}_{out}(t + \tau) \hat{c}_{out}(t) \rangle$$
$$\Gamma^{dd} = \langle \hat{d}_{out}^\dagger(t) \hat{d}_{out}^\dagger(t + \tau) \hat{d}_{out}(t + \tau) \hat{d}_{out}(t) \rangle$$
$$\Gamma^{cd} = \langle \hat{c}_{out}^\dagger(t) \hat{d}_{out}^\dagger(t + \tau) \hat{d}_{out}(t + \tau) \hat{c}_{out}(t) \rangle$$
$$\Gamma^{dc} = \langle \hat{d}_{out}^\dagger(t) \hat{c}_{out}^\dagger(t + \tau) \hat{c}_{out}(t + \tau) \hat{d}_{out}(t) \rangle$$

$$H_0 = -i \left( \kappa \hat{a}^\dagger \hat{a} + 2\gamma \hat{\sigma}^\dagger \hat{\sigma} + 2\sqrt{\kappa \gamma} \hat{\sigma}^\dagger \hat{\sigma} \right)$$

This non-Hermitian Hamiltonian, combined with quantum jumps, fully describes the evolution of the quantum system [13, 14]. Here, the quantum jumps are generated by the collapse operators $\hat{a}_{out} = \sqrt{2\kappa} \hat{\sigma} + \sqrt{2\gamma} \hat{\sigma}^\dagger$ and $\hat{b}_{out} = \sqrt{2\kappa} \hat{\sigma} - \sqrt{2\gamma} \hat{\sigma}^\dagger$ [12, 15, 17].

Each of these collapse operators corresponds to a photon detection event at the respective output port, which removes a quantum excitation from the system. We don’t need to consider contributions from vacuum noise operators $\hat{a}_{in}$ and $\hat{b}_{in}$ to these collapse operators because the correlations we calculate in Eq. 2–5 have the field operators in normal order [9].

We first analyze the splitting efficiency for the case where the feeder cavity initially contains a two-photon Fock state. This initial condition generates an output field of two indistinguishable photons that are unentangled in time and frequency. Appendix A provides the full derivation of the splitting efficiency after the unitary transformation. The splitting efficiency is given by:
Fig. 2. Schematic of the model we analyse. A feeder cavity is the source of a 2 photon pulse. Atom-cavity system is replaced with an enhanced atom after adiabatic elimination of cavity modes. The box with the letter U denotes a unitary transformation.

III. SPLITTING EFFICIENCY FOR TIME-ENERGY ENTANGLLED PHOTONS

The routing efficiency calculated in the previous analysis could potentially be improved by introducing time-energy entanglement between the two photons. Indeed, Rosenblum et. al. showed that a time-energy entangled input state could improve the optical routing efficiency from 64% to 77%. By introducing an optimized linear-optical transformation we could potentially improve this routing efficiency even further.

To incorporate time-energy entanglement into our analysis, we utilize the model introduced by Rosenblum et. al. which is illustrated in Figure 4. Instead of initializing the feeder cavity to a two-photon fock state, we instead excite the cavity with a three-level atom with equally spaced energy levels. The intermediate level has a short lifetime, resulting in a pair of photons with time-energy entanglement. The Hamiltonian for this cascaded system is given by:

$$H_1 = -i(\hat{\kappa}\hat{a}^\dagger\hat{a} + 2\gamma\hat{\sigma}^\dagger\hat{\sigma} + 2\sqrt{\kappa}\hat{\sigma}^\dagger\hat{a} + 2\delta\hat{\sigma}^\dagger\hat{s}_{a} + 2\sqrt{2\chi}\hat{d} (\hat{a}^\dagger)^2 \hat{\sigma}_{a})$$

(10)
FIG. 3. (a) Contour plot of routing efficiency for an input of 2 unentangled photons. (b) Slices of the contour plot corresponding to the splitting efficiencies for the optimal unitary transformation at $\omega = 0.303$ and the identity matrix.

FIG. 4. Model for calculating the splitting efficiency for a pair of time-energy entangled photons. A three level atom with two cascaded transformations with identical frequencies generated the entangled photon pair which is input to the feeder cavity. The rest of the model is the same as in fig. 2.

In the above equation, $\hat{\sigma}_s$ is the lowering operator of the three-level atom from the upper state $|e\rangle$ to the ground state $|g\rangle$ and it decays with a rate of $4\delta$. $\chi$ is the bandwidth of the left mirror of the feeder cavity and is assumed to be very small [9].

We employ the same approach as for the unentangled photons to analyze the routing efficiency. We initialize the three-level atom in state $|e_0g\rangle$, which contains two excitations, while the cavity is empty, and the two-level atom starts in the ground state. We follow the same steps as in Appendix A to obtain an analytical expression for splitting efficiency and equation C1 gives the expression for splitting efficiency in this case.

Figure 5 plots the resulting splitting efficiencies for a time-energy entangled two-photon state. We plot the splitting efficiencies for two unitary transformations corresponding to $\omega = 0.283$, which results in the maximum routing efficiency, and $\omega = 0$, which gives the splitting efficiency for when the interferometer is absent from the setup in Fig. 1. The two results are plotted in the limit $\kappa/\delta \to \infty$, which corresponds to maximum time-energy entanglement between input photons and gives the highest splitting efficiency in both cases. When the unitary transformation corresponds to the optimal case, the maximum splitting efficiency is 90%. If we don’t perform a
unitary transformation of the modes $\hat{a}_{\text{out}}$ and $\hat{b}_{\text{out}}$, the maximum achievable splitting efficiency is 77%. Thus, by tuning the unitary transformation, we improve the splitting efficiency for a pair of time-energy entangled photons.

In summary, we showed that linear optics can improve on the splitting efficiency possible with photon blockade using an atom-cavity system. We improved the splitting efficiency from 64% to 75% for unentangled photons, and from 77% to 90% for photons that are time-energy entangled. We showed how 2 photon interferences created by the unitary transformation realized by a Mach Zehnder Interferometer enables this increase in the splitting efficiency. By changing the temporal nature of the interference, i.e. by varying the unitary transformation in time could lead to a further improvement. 2 photon controlled phase gate implemented with the nonlinearity of a two-level atom also suffers from the time-bandwidth limit in its fidelity [4]. Incorporating interference of the output modes of the two-level atom with a general linear optical transformation may lead to an increase in the gate fidelity. Our results highlight the potential to generate improved quantum optical interaction by combining single-photon nonlinearities with linear optical systems, with potential applications in photon quantum information processing and quantum simulation.

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Appendix A: Derivation of splitting efficiency for unentangled photons

The system in figure 2 is initialized to the state $|2g\rangle$, with two photons in the feeder cavity and the atom in the ground state. Before the detection of the first photon, this state evolves under the Hamiltonian $H_0$ [9] to

$$|\phi(t)\rangle = a(t) |2g\rangle + \beta(t) |1e\rangle$$  (A1)

with

$$a(t) = e^{-2\kappa t}$$
$$\beta(t) = -\frac{2\sqrt{2}\kappa}{2\gamma - \kappa} (e^{-\kappa t} - e^{-2\gamma t}) e^{-\kappa t}$$  (A2)

After the first photon is detected, the state of the system collapses to $|1g\rangle$ or $|0e\rangle$. In the first case, the state of the system evolves to

$$|\phi(t)\rangle = a(t) |1g\rangle + b(t) |0e\rangle$$  (A3)

with

$$a(t) = e^{-\kappa t}$$
$$b(t) = -\frac{2\sqrt{\kappa}}{2\gamma - \kappa} (e^{-\kappa t} - e^{-2\gamma t})$$  (A4)

Substituting these expressions into Eqs. 4 and 5 gives the following second order correlations.

$$\Gamma^{cd}(t,\tau) = 4|c(\tau)|^2 \exp(-i\phi) \exp(-i\phi) \cos(\omega) - \sin(\omega) | \sin(\omega) \sqrt{\kappa \gamma} \beta(t) +$$
$$\left( b(\tau) \exp(-i\phi) \cos(\omega) - \sin(\omega) \right) \sqrt{\gamma} + a(\tau) \cos(\omega) \exp(-i\phi) \sqrt{\kappa}$$
$$\left( \exp(-i\phi) \sin(\omega) \sqrt{2\kappa a(t)} + (\cos(\omega) + \exp(-i\phi) \sin(\omega)) \sqrt{\gamma} \beta(t) \right)^2$$  (A5)

$$\Gamma^{cd}(t,\tau) = 4|c(\tau)|^2 \cos(\omega) \exp(-i\phi) \cos(\omega) + \exp(-i\phi) \sin(\omega) | \sqrt{\kappa \gamma} \beta(t) +$$
$$\left( b(\tau) \cos(\omega) + \exp(-i\phi) \sin(\omega) \right) \sqrt{\gamma} + a(\tau) \exp(-i\phi) \sin(\omega) \sqrt{\kappa}$$
$$\left( \cos(\omega) \exp(-i\phi) \sqrt{2\kappa a(t)} + (\exp(-i\phi) \cos(\omega) - \sin(\omega)) \sqrt{\gamma} \beta(t) \right)^2$$  (A6)

Integrating these second order correlations over $t$ and $\tau$ and summing the two results gives the splitting efficiency.

$$S = \frac{8(\frac{\gamma}{\kappa})^3 + 16(\frac{\gamma}{\kappa})^2 (\sin^2(2\omega) \cos(2\phi) + 2 \sin(4\omega) \cos(\phi)) + 44(\frac{\gamma}{\kappa})^2 + (2^2 (2^2 (2^2 (2^2 (5 - 2^2) + 5) - 3) \cos(4\omega) + 38^2 + 3}{4(2^2 + 1)^2 (2^2 + 3)}$$  (A7)

$S$ is periodic in $\omega$ and $\phi$ and is a convex function within one period. Using standard calculus, we can obtain a
global maximum of $S$ and it occurs at $\phi = 0$, $\omega = 0.303$ and $\frac{\gamma}{\omega} = 0.92$.

**Appendix B: Two photon interference**

A general single mode input to the Mach Zehnder Interferometer (Figure 2) is given by:

$$|\phi\rangle_{ab} = d|11\rangle_{ab} + e|20\rangle_{ab} + f|02\rangle_{ab} \quad \text{(B1)}$$

To obtain the state at the output of the interferometer for this input, we need to use the inverse of the unitary transformation given by eq. 2., which is

$$\left( \begin{array}{c} \hat{a}_{\text{out}}^{\dagger} \\ \hat{b}_{\text{out}}^{\dagger} \end{array} \right) = \left( \begin{array}{cc} e^{i\phi}\sin(\omega) & e^{i\phi}\cos(\omega) \\ \cos(\omega) & -\sin(\omega) \end{array} \right) \left( \begin{array}{c} \hat{c}_{\text{out}}^{\dagger} \\ \hat{d}_{\text{out}}^{\dagger} \end{array} \right) \quad \text{(B2)}$$

We first express the input state $|11\rangle$ as $\hat{a}_{\text{out}}^{\dagger}\hat{b}_{\text{out}}^{\dagger}|\text{vac}\rangle$ where $|\text{vac}\rangle$ is the vacuum state of the electromagnetic field. Transforming this state as per eq. B1, we get the output state

$$\left( e^{i\phi}\sin(\omega)\hat{c}_{\text{out}}^{\dagger} + e^{i\phi}\cos(\omega)\hat{d}_{\text{out}}^{\dagger} \right) \left( \cos(\omega)\hat{c}_{\text{out}}^{\dagger} - \sin(\omega)\hat{d}_{\text{out}}^{\dagger} \right) |\text{vac}\rangle = e^{i\phi}\left( \frac{\sin(2\omega)}{\sqrt{2}} |20\rangle - \frac{\cos(2\omega)}{\sqrt{2}} |02\rangle + \frac{\cos(2\omega)}{\sqrt{2}} |11\rangle \right) \quad \text{(B3)}$$

Following the same steps, it can be shown that the basis states $|20\rangle_{ab}$ and $|02\rangle_{ab}$ transform as follows.

$$|20\rangle_{ab} = e^{2i\phi}(\sin^2(\omega) |20\rangle_{cd} + \cos^2(\omega) |02\rangle_{cd} + \frac{\sin(2\omega)}{\sqrt{2}} |11\rangle_{cd}) \quad \text{(B4)}$$

$$|02\rangle_{ab} = e^{2i\phi}(\cos^2(\omega) |20\rangle_{cd} + \sin^2(\omega) |02\rangle_{cd} - \frac{\sin(2\omega)}{\sqrt{2}} |11\rangle_{cd}) \quad \text{(B5)}$$

If in eq. B1, we have $e + f = 0$ or in other words, $e = -f = g/\sqrt{2}$, and the input to the interferometer takes the form $|\phi\rangle_{ab} = d|11\rangle_{ab} + g|20\rangle_{ab} - |02\rangle_{ab}$, the probability amplitude of $|11\rangle_{cd}$ at the output of the interferometer is

$$s = e^{i\phi}(d \times \cos(2\omega) + g \times \sin(2\omega)\cos(\phi)) \quad \text{(B6)}$$

If $\phi = 0$ and the phase difference between the complex probability amplitudes $d$ and $g$ is $\Delta$ the probability of finding the two input photons split into different output modes of the interferometer, i.e. the splitting efficiency $S$ is

$$S = \frac{1}{2} + \frac{|d|^2 - |g|^2}{\cos(4\omega) + 2|d||h\cos(\Delta)|\sin(4\omega)} \quad \text{(B7)}$$

Let $|d|^2 - |g|^2 = x$ and $2|d||h\cos(\Delta) = y$. Then the maximum value of the above expression for $S$ is

$$S_{\text{max}} = \frac{1}{2} + \sqrt{\frac{x^2 + y^2}{2}} \quad \text{(B8)}$$

$S_{\text{max}}$ is 1, or in other words the splitting efficiency is a 100%, when $x^2 + y^2 = 1$, which occurs when $\cos(\Delta) = 1$. With this constraint, we get $x^2 + y^2 = (|d|^2 + |g|^2)^2$, which evaluates to 1 because of the normalization condition for the wavefunction (eq. B6). $\cos(\Delta) = 1$ is equivalent to saying that $d$ and $g$ are real.

**Appendix C: Derivation of splitting efficiency for entangled photons**

To generate a pair of entangled photons, we first initialize the system in figure 4 to state $|e0\rangle$, with the three level atom in the excited state, feeder cavity containing zero photons and the two level atom in its ground state. We follow the same steps as in Appendix A1 to calculate the splitting efficiency with one difference. In this case, we need to calculate the probability of detecting photons conditioned on the event that the two photon pulse emitted by the three level atom enters the feeder cavity [9]. Following these steps, we obtain the following expression for splitting efficiency

$$S = \frac{32\gamma^2(2\gamma + 2\delta + 3)\sin(4\omega) + (-4\gamma(4\gamma^2 + \gamma - 2) + 2\gamma(2\gamma - 3)\delta - 5\delta - 7) - 6\delta - 3)\cos(4\omega) + 4\gamma(2\gamma(2\gamma + 13)\delta + 4\gamma(\gamma + 5) + 8) + 19\delta + 17) + 6\delta + 3}{4(2\gamma + 1)^2(2\gamma + 3)(2\gamma + 2\delta + 1)} \quad \text{(C1)}$$

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