Anyon and Loop Braiding Statistics in Field Theories with a Topological $\Theta$–term

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We demonstrate that the anyon statistics and three-loop statistics of various 2d and 3d topological phases can be derived using semiclassical nonlinear sigma model field theories with a topological $\Theta$-term. In our formalism, the braiding statistics has a natural geometric meaning: The braiding process of anyons or loops leads to a nontrivial field configuration in the space-time, which will contribute a braiding phase factor due to the $\Theta$-term.

Introduction

One of the key properties of topological states is that, the gapped topological excitations above the ground state can have nontrivial braiding statistics. In both 2d and 3d, all discrete lattice gauge theories have a deconfined topological phase \cite{1}. 2d discrete gauge theories have point particle topological excitations, while 3d discrete gauge theories have both particle excitations and loop excitations which correspond to gauge charge and gauge flux loop respectively. The simplest lattice discrete gauge theory (which we call "plain gauge theory") already has nontrivial braiding statistics \cite{8–11}. Also, if we couple a 2d $p+ip$ topological superconductor to a $\mathbb{Z}_2$ gauge field, then the vison of the gauge field would acquire a Majorana fermion zero mode, which will grant the vison a nonabelian statistics \cite{4,5,6,7}. More exotic gauge theories can be constructed by coupling the plain gauge theory to matter fields, and drive the matter fields into certain nontrivial short range entangled (SRE) state or symmetry protected topological (SPT) phase \cite{3,4}. For example, once we couple a 2d $p+ip$ topological superconductor to a $\mathbb{Z}_2$ gauge field, then the vison of the gauge field would acquire a Majorana fermion zero mode, which will grant the vison a nonabelian statistics \cite{5,6,7}. Also, if we couple a 2d bosonic SPT phase with $\mathbb{Z}_2$ symmetry to a $\mathbb{Z}_2$ lattice gauge theory, the lattice gauge theory will have both semion and anti-semion excitations \cite{7}, which is different from a plain lattice gauge theory.

Recently these results have been generalized to 3d systems. It was demonstrated that once a 3d lattice discrete gauge theory is coupled to a 3d SPT state, the loop excitations (fluctuating gauge flux loops) would acquire nontrivial multi-loop braiding statistics \cite{8–11}, in addition to the standard particle-loop statistics of the plain gauge theory. For example when loop-B and loop-C are both linked to loop-A, namely none of the loops is contractible, the system wave function could acquire a universal phase angle after braiding loop-C through loop-B as shown in Fig.1(a). These braiding statistics can be used as a diagnostics for SPT phases \cite{8}.

Besides the standard group cohomology description of SPT phases introduced in Ref.\cite{3,4} it was pointed out in Ref.\cite{12,13} that the bosonic SPT phases can also be described by semiclassical nonlinear sigma model (NLSM) field theories with a topological $\Theta$-term. In this theory all the field variables are fluctuating Landau order parameters that transform nontrivially under global symmetry. The goal of this work is to demonstrate that the nontrivial statistics between topological excitations after coupling the SPT phases to a discrete gauge theory can also be described and calculated using this NLSM field theory. Basically the braiding phase factor comes from the $\Theta$–term in the field theory, as long as we carefully analyze the field configuration in the space-time which corresponds to the braiding process. The NLSM field theory with a topological term can be viewed as the continuum limit field theory description for these braiding statistics.

2d Anyon statistics

We will first look at 2d systems, and as an example let us start with the 2d SPT state with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, which can be described by the following $(2+1)d$ O(4) NLSM with a $\Theta$-term at $\Theta = 2\pi$ \cite{13}:

\[
S = \int d^2x d\tau \left( \frac{1}{g} (\partial_{\mu} \bm{n})^2 + \frac{\Theta}{\Omega_3} \epsilon_{abcd} n^a \partial_{\tau} n^b \partial_\phi n^c \partial_{\tau} n^d \right),
\]

where $\bm{n}$ is a four component vector with unit length, and $\Omega_3 = 2\pi^2$ is the volume of a three dimensional sphere with unit radius. Under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, the vector $\bm{n}$ transforms as

\[
\begin{align*}
\mathbb{Z}_2^A : n^1, n^2 \rightarrow -n^1, -n^2, & \quad n^3, n^4 \rightarrow n^3, n^4; \\
\mathbb{Z}_2^B : n^1, n^2 \rightarrow n^1, n^2, & \quad n^3, n^4 \rightarrow -n^3, -n^4.
\end{align*}
\]

Now let us couple the vector $\bm{n}$ to a $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$ gauge field. The excitations that will have nontrivial braiding statistics are the vison excitations ($\pi$-gauge flux) of gauge fields.
Let us consider the following braiding process: one pair of $Z_2^A$ visons and one pair of $Z_2^B$ visons are created in space at one instance in time, then they are annihilated at another later instance after braiding one $Z_2^A$ vison with one $Z_2^B$ vison. In the $(2+1)d$ space-time, this process corresponds to one linking between $Z_2^A$ and $Z_2^B$ vison loops, as shown in Fig. 1(b). Because the $Z_2$ gauge fields are coupled to the four-component vector $n$, the $Z_2^A$ vison is bound with a $\pm 1/2$ vortex of $(n^1, n^2)$, while $Z_2^B$ vison is bound with a $\pm 1/2$ vortex of $(n^3, n^4)$. Then the braiding process in the space-time can be viewed as a linking configuration between $(n^1, n^2)$ half-vortex loop and $(n^3, n^4)$ half-vortex loop. Due to the $\Theta$-term in Eq. 1, this configuration will contribute a phase factor $\exp(\pm i \pi/2) = \pm i$ to the action, which implies the mutual braiding statistics between the $Z_2^A$ vison and $Z_2^B$ vison.

To calculate this phase factor explicitly, let us first consider a finite segment of $Z_2^A$ vison loop along the $\hat{t}$ direction. A vison is always bound with either a $1/2$ vortex or $1/2$ vortex of $(n_1, n_2)$. Around this segment, the $O(4)$ vector $n$ has the following configuration with cylindrical coordinate $(r, \phi, \tau) (x = r \cos \phi, y = r \sin \phi$, see Fig. 1(b) inset):

$$
\begin{align*}
  n^1 &= \sin \alpha(r) \cos f(\phi), \\
  n^2 &= \sin \alpha(r) \sin f(\phi), \\
  n^3 &= \cos \alpha(r) N^1(\tau), \\
  n^4 &= \cos \alpha(r) N^2(\tau),
\end{align*}
$$

where $N = (N_1, N_2)$ is an $O(2)$ unit vector $|N|^2 = 1$. $N$ is a function of $r$ only. $\alpha(r)$ is a nonnegative continuous function that satisfies $\alpha(0) = 0$, $\alpha(\infty) = \pi/2$. Along the $\hat{t}$ axis, i.e., $r = 0$, we have $(n^3, n^4) = N$. Using this configuration, we can compute the $\Theta$-term:

$$
\int d^2x d\tau \frac{2\pi i}{\Omega_3} \epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_{\tau} n^d = 2\pi \int_0^{2\pi} d\phi \partial_\phi f \int d\tau \frac{2\pi i}{\Omega_3} \epsilon_{ab} N^a \partial_{\tau} N^b.
$$

If $n^1$ and $n^2$ form a full vortex loop along the $\hat{t}$ axis, namely $f(\phi) \sim \phi$, the $O(4)\Theta$-term reduces to a $1d$ $O(2)$ NLSM with $\Theta = 2\pi$. If there is a $Z_2^A$ vison line along the $\hat{t}$ axis, i.e., $n^1$ and $n^2$ form a $1\pm 1/2$ vortex line along $\hat{t}$ axis, namely $f(\phi) = \mp \phi/2$, then the $(2+1)d$ $O(4)$ NLSM reduces to a $1d$ $O(2)$ NLSM of vector $N$ with $\Theta = \pm \pi$. Now let us consider two linked vison loops, and in Eq. (4) $\tau$ becomes the parameter along the $Z_2^A$ vison loop. Since the two loops are linked, vector $N$ will have a $\pm 1/2$ vortex winding along $Z_2^A$ vison loop:

$$
\int d\tau \epsilon_{ab} N^a \partial_{\tau} N^b = \pm \pi.
$$

Combining Eq. (4) and Eq. (5), together, we conclude that this linking configuration (which corresponds to a braiding process in the space-time) would contribute factor $\pm i$ to the action. In other words, the linking configuration in Fig. 1(b) corresponds to $\pm 1/4$ instanton of the four component vector $n$ in the $(2+1)d$ space-time.

Now let us consider a $2d$ SPT state with $Z_2$ global symmetry only, and couple it to a $Z_2$ gauge field. This SPT state can be described by the same field theory Eq. (1), and under the $Z_2$ symmetry $n \rightarrow -n$. A vison of this $Z_2$ gauge field can be viewed as a bound state between the $Z_2^A$ vison and $Z_2^B$ vison discussed previously. Then the linking configuration in Fig. 1(b) can be interpreted as creating a pair of visons, self-twisting one vison by $2\pi$, then annihilating them. The phase $\pm i$ corresponds to topological spin-$\pm 1/4$ of the vison, which is consistent with the semion and anti-semion statistics of the vison proved in Ref. [2].

All the analysis above can be straightforwardly generalized to $Z_N$ gauge theory coupled to a $2d$ $Z_N$ SPT state. The $2d$ $Z_N$ SPT state is described by the same field theory Eq. (1) [13], where $\Theta = 2\pi k, k = 0, 1, \ldots (N-1)$. The same analysis above leads to the result that the topological spin of the $2\pi/N$ flux excitations can be $k/N^2$, namely self-twisting such excitation will grant its wave function a phase $\exp(2\pi i k/N^2)$.

### 3d loop statistics

Now we consider 3d bosonic SPT states with $Z_2^A \times Z_2^B \times Z_2^C$ symmetry. In terms of field theory, one of these SPT states is described by the following $(3+1)d$ $O(5)$ NLSM:

$$
S = \int d^3x d\tau \frac{1}{g} (\partial_\tau \epsilon)^2 + i \Omega_4 \epsilon_{abcde} n^a \partial_x n^b \partial_y n^c \partial_{\tau} n^d n^e.
$$

where $\Omega_4 = 8\pi^2/3$ is the volume of a four dimensional sphere with unit radius. Under the $Z_2^A \times Z_2^B \times Z_2^C$ symmetry, the five component vector $n$ transforms as

$$
\begin{align*}
  Z_2^A : n^1, n^2 &\rightarrow -n^1, -n^2, & n^{3, 4, 5} &\rightarrow n^{3, 4, 5}, \\
  Z_2^B : n^3, n^4 &\rightarrow -n^3, -n^4, & n^{1, 4, 5} &\rightarrow n^{1, 4, 5}, \\
  Z_2^C : n^5, n^5 &\rightarrow -n^5, -n^5, & n^{1, 2, 3} &\rightarrow n^{1, 2, 3}.
\end{align*}
$$

Now let us couple this SPT state to $Z_2^A \times Z_2^B \times Z_2^C$ gauge field, and consider the statistics between the three loops in Fig. 1(a), in which the base loop is a vison loop of $Z_2^A$ gauge field, and it is linked with vison loops of both $Z_2^B$ and $Z_2^C$ gauge fields.

A vison loop can be bound with either a $+1/2$ vortex or $-1/2$ vortex, both cases exist in the system, and they correspond to different excitations. As an example let us study the braiding statistics of vison loops bound with $+1/2$ vortex. The choice of $+1/2$ vortex gives each vison loop an orientation, as marked out in Fig. 1(a). Let us first look at the $Z_2^B$ vison loop. Following the same calculation as Eq. (4), because $Z_2^B$ vison loop is bound with a half-vortex loop of $(n_2, n_3, n_4)$, the $O(5)$ NLSM with $\Theta = 2\pi$ is reduced to an $O(3)$ NLSM with $\Theta = \pi$ in the $(1+1)d$ world-sheet of the $Z_2^B$ vison loop, and the three
component vector on this world sheet is \( \mathbf{N} \sim (n_1, n_4, n_5) \):

\[
S_{1d,B} = \int d\tau dx \frac{1}{g} (\partial_{\mu} \mathbf{N})^2 + \frac{i\pi}{4\pi} \epsilon_{abc} N^a \partial_x N^b \partial_{\tau} N^c. \tag{8}
\]

On the \((1+1)\)d world sheet of \( \mathbb{Z}^B_2 \) vison loop, the braiding between \( \mathbb{Z}^B_2 \) and \( \mathbb{Z}^C_2 \) vison loops corresponds to the space-time configuration \( \mathbf{N}(x, \tau) \) in Fig.2 and this configuration carries \( 1/2 \) \( \text{O}(3) \) instanton number, thus it will contribute a factor \( i \) to the action. This implies that the three-loop braiding statistics angle is \( \theta_{BC,A} = \pi/2 \). The statistics angle \( \theta_{AC,B} \) can be calculated in the same way after interchanging \( n_1 \) and \( n_3 \) in the \( \text{O}(5) \) vector, which will lead to factor \(-1\) due to the antisymmetrization in the \( \Theta \)-term in Eq. (9). Thus \( \theta_{AC,B} = -\pi/2 \).

The loop braiding statistics can also be understood in a different way. Ref. [10] pointed out that the three-loop braiding in Fig.4(a) can also be viewed as a link of the \( \mathbb{Z}^B_2 \) and \( \mathbb{Z}^C_2 \) vison loops braiding with the \( \mathbb{Z}^A_2 \) vison loop, as illustrated in Fig.4(a). This link-loop braiding statistics can be described by the NLSM as well. As the vison link braid through the vison loop, the space-time configuration of the \( \text{O}(5) \) vector \( \mathbf{n} \) around the vison link can be described as following:

\[
\begin{align*}
n^1 &= \cos \alpha(\tau), \\
n^2 &= \sin \alpha(\tau) N^1(x, y, z), \\
n^3 &= \sin \alpha(\tau) N^2(x, y, z), \\
n^4 &= \sin \alpha(\tau) N^3(x, y, z), \\
n^5 &= \sin \alpha(\tau) N^4(x, y, z),
\end{align*}
\tag{9}
\]

where \( \mathbf{N} = (N^1, N^2, N^3, N^4) \) is an \( \text{O}(4) \) unit vector \( |\mathbf{N}|^2 = 1 \) that describes the configuration of the (linked) half-vortex loops bound to the vison loops of \( \mathbb{Z}^B_2 \) and \( \mathbb{Z}^C_2 \). The time \( \tau \) (running from 0 to 1) parameterizes a full braiding of the \( \mathbb{Z}^B_2 \times \mathbb{Z}^C_2 \) vison link with the \( \mathbb{Z}^A_2 \) vison loop. Suppose the \( n^1 \) component is energetically more favored, then the \( \mathbb{Z}^A_2 \) branch cut disconnection of the \( \mathbb{Z}^C_2 \) vison loop will be bound with a \( n^1 \) domain wall. Let the braiding of the \( \mathbb{Z}^B_2 \times \mathbb{Z}^C_2 \) vison link initiates from one side of the domain wall, and ends up at the other side of the domain wall, then \( \alpha(\tau) \) will be a continuous function satisfying \( \alpha(0) = \pi, \alpha(1) = 0 \). Plugging the configuration Eq. (9) into the NLSM, we have the \( \text{O}(5) \) \( \Theta \)-term of \( \mathbf{n} \) is reduced to an \( \text{O}(4) \) \( \Theta \)-term of \( \mathbf{N} \) at \( \Theta = 2\pi \):

\[
\begin{align*}
-\int_0^1 d\tau \int d^3x \frac{2\pi i}{\Omega_4} \epsilon_{abcd} N^a \partial_x N^b \partial_y N^c \partial_z N^d \\
= \int d^3x \frac{2\pi i}{\Omega_3} \epsilon_{abcd} N^a \partial_x N^b \partial_y N^c \partial_z N^d. \tag{10}
\end{align*}
\]

According to our previous calculation, the linking configuration between \((N_1, N_2)\) half-vortex loop and \((N_3, N_4)\) half-vortex loop corresponds to the 1/4 \( \text{O}(4) \) soliton in the \( 3d \) space, so the above \( \text{O}(4) \) \( \Theta \)-term in Eq. (10) will result in a \( \pi/2 \) phase angle accumulated in the loop-loop braiding, which equals to the three-loop braiding angle \( \theta_{BC,A} \) calculated already in our paper.

The non-trivial link-loop braiding statistics implies that the \( \mathbb{Z}^B_2 \times \mathbb{Z}^C_2 \) vison link must carry the charge of the \( \mathbb{Z}^A_2 \) gauge field. Let us denote the \( \mathbb{Z}^A_2 \) charge carried by the \( \mathbb{Z}^B_2 \times \mathbb{Z}^C_2 \) vison link as \( q_{BC}^A \). It is related to the braiding angle by \( \theta_{BC,A} = -\pi q_{BC}^A \). The minus sign is due to the reversed link-loop braiding direction as shown in Fig.5(a) (which corresponds to the positive three-loop braiding direction). As shown in Fig.5(b), the torus traced out by the \( \mathbb{Z}^C_2 \) vison loop through braiding with the \( \mathbb{Z}^C_2 \) vison loop (in the linking with the \( \mathbb{Z}^B_2 \) vison loop) actually forms a Gaussian surface enclosing the \( \mathbb{Z}^C_2 \) vison loop. So the three-loop braiding statistics angle \( \theta_{AC,B} \) measures the \( \mathbb{Z}^A_2 \) charge carried by the \( \mathbb{Z}^C_2 \) vison loop in the \( \mathbb{Z}^B_2 \times \mathbb{Z}^C_2 \) link, denoted \( q_{AC}^B \), and \( \theta_{AC,B} = \pi q_{AC}^B \). Similarly from Fig.5(c), the three-loop braiding statistics angle \( \theta_{AB,C} \) measures the \( \mathbb{Z}^A_2 \) charge carried by the \( \mathbb{Z}^B_2 \) vison loop in the same \( \mathbb{Z}^B_2 \times \mathbb{Z}^C_2 \) link, denoted \( q_{AB}^C \), and
\[ \theta_{AB,C} = \pi q_1^A. \] Obviously, \( q_{BC}^A = q_2^A + q_3^A \), thus
\[ \theta_{AB,C} + \theta_{BC,A} + \theta_{AC,B} = 0, \tag{11} \]
which is precisely the cyclic relation [8,10], and it implies that \( \theta_{BC,A} = 0 \) (given \( \theta_{BC,A} = \pi/2 \) as previously calculated).

Our NLSM Eq. (6), the statistics of the \( 2 \) symmetry. There are in total three different nontrivial \( 3 \) symmetries, thus after gauging the symmetries, \( \theta_{AB,A} = \pm \pi/2, \theta_{AA,B} = \pi. \) The third type of SPT state is equivalent to the two SPT states discussed above weakly coupled together, thus
\[ \theta_{AB,A} = \theta_{AB,B} = \pm \pi/2, \theta_{AA,B} = \theta_{BB,A} = \pi. \tag{14} \]

In summary, we have computed the anyon braiding statistics, and three-loop statistics of \( 2d \) and \( 3d \) topological phases constructed by coupling plain gauge theories to bosonic SPT states. Our calculation is based on semiclassical field theories, and all the braiding phases naturally come from the topological \( \Theta \)–term in the field theory.

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Here $\theta_{BB,A}$ stands for the full braiding statistics angle between two $\mathbb{Z}_2^B$ vison loops while they are both linked with a $\mathbb{Z}_2^A$ vison loop.