Introduction. – Time-non-local (TNL) Lagrangians are generalized Lagrangians which contain integral terms involving the whole history of the system. These Lagrangians are of interest in different fields of theoretical physics, from string theory [1–3], to regularized QFT [4,5], semiclassical gravity [6], non-Markovian open quantum systems [7–9], collapse models [10–13], and recently light harvesting in the photosynthesis process [14,15].

Lagrangians with non-local coupling display some special difficulties when applied to relativistic quantum field theories [16]. A more recent approach to the problem has been proposed in [17], where Lorentz invariance breaking has been tackled by considering discretized non-local Lagrangians. In this letter, we will consider only non-relativistic theories, where this kind of problems does not arise.

A major and typical difficulty one encounters when working with TNL Lagrangians is the following. Consider for example:

\[ L[q, \dot{q}] = \frac{m}{2} \dot{q}^2 - q \int_0^t \alpha(t, s)(q(s) + \dot{q}(s))ds, \]

(1)

where \( q \) and \( \dot{q} \) are, respectively, the position and velocity of a particle, and \( \alpha(t, s) \) a memory kernel [7–9,12,13]. Suppose one wants to compute the associated Hamiltonian, and needs to transform the dependence on \( (q, \dot{q}) \) in a dependence on \( (q, p) \) (or vice versa). The first guess would be to use the standard definition given by the Lagrange formalism: \( p = \frac{dL}{d\dot{q}} \). However, one can immediately see that, since in eq. (1) the velocity \( \dot{q} \) enters the integral term, the differentiation of \( L \) with respect to \( \dot{q} \) does not make sense, and hence one does not know how to compute \( p \). One therefore needs to develop a new formalism which allows for the analysis of TNL Lagrangians. In doing so, one has always to make sure that, in the limit \( \alpha(t, s) = \delta(t - s) \), i.e. when the Lagrangian becomes local, the standard results are recovered.

Two are the main mathematical frameworks which have been developed, to deal with TNL Lagrangians: the Ostrogradsky formalism [18,19] and the one proposed by Llosa and collaborators [20,21]. The Ostrogradsky method is based on the assumption that some TNL Lagrangians can be represented as the limit of a sequence of finite higher-derivative local Lagrangians. In this way one can always make sure that, in the limit \( \alpha(t, s) = \delta(t - s) \), i.e. when the Lagrangian becomes local, the standard results are recovered.

Another interesting approach is the one proposed by Llosa and collaborators [20,21]. They introduce in the Lagrangian a new parameter, in such a way that the evolution derived by this modified Lagrangian becomes...
local in the time variable, but non-local in the new parameter. In this way one can apply the standard Lagrange formalism with respect to the time variable, and then translate the results back to the non-local case. However, unlike the standard time-local case, the authors assume the Action to be a functional of the position only. Moreover, the formalism can be applied only to those Actions for which the functional and time derivatives commute \[20,21\]. This approach allows for some reasonable mathematical treatment, but takes away from physical intuition. Other formalisms which have been proposed are less convincing or still debated \[22,23\].

The aim of this letter is to introduce a novel TNL Euler-Lagrange formalism which generalizes, in a somehow natural way, the conventional formalism, which is mathematically straightforward to handle. Moreover, such a formalism has the advantage of naturally emerging from the Hamilton’s principle, and in the time-local limit automatically reproduces the standard Euler-Lagrange formalism.

**The formalism.** – We consider a classical system described by the following TNL Lagrangian:

\[
L[q, \dot{q}, t] = L_{TL}(q, \dot{q}, t) + L_{TNL}(q, \dot{q}, t),
\]

where \(L_{TL}(q, \dot{q}, t)\) is the standard time-local (TL) Lagrangian and \(L_{TNL}(q, \dot{q}, t)\) is the time-non-local part, i.e. a functional of \(q\) and \(\dot{q}\) which accounts for the whole past history of the system:

\[
L_{TNL}(q, \dot{q}, t) = \int_0^t f(q, \dot{q}, s) \, ds,
\]

with \(f(q, \dot{q}, s)\) a generic function. Equations (2), (3) refer to a one-particle system; the generalization to many-particle systems is straightforward. Here and in the following, we denote functions with parenthesis and functionals with squared brackets. We require Hamilton’s principle to hold true, which means that the first variation \(\delta S\) of the Action functional

\[
S[q, \dot{q}] = \int_0^t L[q, \dot{q}, s] \, ds
\]

vanishes. Note that in the standard case, that is when \(L_{TNL}(q, \dot{q}, t) = 0\) so that the Action (2) becomes local, one has

\[
\delta S[q, \dot{q}] = \int_0^t \left( \frac{dL_{TL}(q, \dot{q})}{dq} \delta q + \frac{dL_{TL}(q, \dot{q})}{d\dot{q}} \delta \dot{q} \right) \, ds.
\]

Integrating this expression by parts and requiring the variation to be zero, one obtains the well known Euler-Lagrange equations \[24,25\]. However, if \(L_{TNL}(q, \dot{q}, t) \neq 0\) the standard Lagrange formalism cannot be directly applied, because the Action \(S[q, \dot{q}]\) contains a double functional. Therefore, differentiation with respect to \(q\) (or \(\dot{q}\)) at a fixed time does not make sense anymore. This is the reason why one needs to go beyond the standard formalism.

A way to circumvent this difficulty is to use the identity \[26,27\]:

\[
\delta S[q, \dot{q}] = \int_0^t \left( \frac{\delta S[q, \dot{q}]}{\delta q(s)} \delta q(s) + \frac{\delta S[q, \dot{q}]}{\delta \dot{q}(s)} \delta \dot{q}(s) \right) \, ds,
\]

where \(\delta/\delta q(s)\) denotes the functional derivative with respect to \(q(s)\). Integrating by parts eq. (6) one obtains

\[
\delta S[q, \dot{q}] = \int_0^t \left( \frac{\delta S[q, \dot{q}]}{\delta q(s)} - \frac{d}{ds} \frac{\delta S[q, \dot{q}]}{\delta \dot{q}(s)} \right) \delta q(s) \, ds = 0.
\]

Comparing this result with the local one, one then naturally defines the TNL Euler-Lagrange equations as follows:

\[
\frac{\delta S[q, \dot{q}]}{\delta q(s)} - \frac{d}{ds} \frac{\delta S[q, \dot{q}]}{\delta \dot{q}(s)} = 0, \quad \forall s \in [0, t].
\]

Some comments are in order. This equation of motion is an integro-differential equation which depends on the whole time interval \([0, t]\). As we will see in the following example, the correct interpretation of eq. (8) will require special care. Secondly, one can easily check that when the Lagrangian is local, i.e. \(L_{TNL}[q, \dot{q}, t] = 0\), eq. (8) reduces to the standard Euler-Lagrange equations. As a third comment, our formalism looks similar to the one used for (space) non-local fields in Quantum Field Theory. However, in that case fields are non-local only in the space variables and not in the time one \[27,28\]. The fact that in our case non-locality affects the evolution variable, drastically changes the physical situation. We also stress that the presence of the Action \(S[q, \dot{q}]\) instead of the Lagrangian in eq. (8) is crucial, otherwise the formalism would be inconsistent (the functional derivative would generate Dirac deltas).

We carry on the analogy with the standard formalism, introducing a generalized momentum:

\[
p(s) = \frac{\delta S[q, \dot{q}]}{\delta \dot{q}(s)}, \quad \forall s \in [0, t],
\]

and defining the following generalized Hamiltonian functional:

\[
\mathcal{H}[q, p] = -S[q, \dot{q}] + \int_0^t p(s)q(s) \, ds.
\]

The reason for introducing \(\mathcal{H}[q, p]\) is that, as in the TNL Lagrangian formalism we need to express the functional equations of motion in terms of the Action instead of the Lagrangian, in the TNL Hamiltonian formalism we need to use the generalized Hamiltonian \(\mathcal{H}[q, p]\) instead of the standard Hamiltonian, otherwise, infinities would appear from the functional derivatives. We now evaluate the first variation of \(\mathcal{H}[q, p]\), from which we will derive
the equations of motion in the TNL Hamilton formalism. Using eq. (10) one can write
\[
\delta H[q,p] = \int_0^t \left( -\frac{\delta S[q,\dot{q}]}{\delta q(s)} \delta q(s) - \frac{\delta S[q,\dot{q}]}{\delta \dot{q}(s)} \delta \dot{q}(s) 
+ \delta p(s) \dot{q}(s) + p(s) \delta \dot{q}(s) \right) \, ds 
= \int_0^t \left( -\frac{\delta S[q,\dot{q}]}{\delta q(s)} \delta q(s) + \frac{\delta H[q,p]}{\delta p(s)} \delta p(s) \right) \, ds ,
\]
where the last line comes from eq. (9). Comparing this expression with the general form of the variation of \( H[q,p] \) [27]:
\[
\frac{\delta H[q,p]}{\delta q(s)} = -\frac{\delta S[q,\dot{q}]}{\delta q(s)}, \quad \frac{\delta H[q,p]}{\delta p(s)} = \dot{q}(s) .
\]
This completes the TNL Hamiltonian formulation of the equations of motion. So far, we have developed the formalism under a rather formal point of view. We now show how this formalism applies to a specific, and very common in physics, example of TNL Lagrangian.

**Application of the formalism.** – We consider a classical particle of mass \( m \) evolving according to a general TNL second-order Lagrangian, given by eq. (2) with
\[
L_{TL}(q,\dot{q},s) = \frac{m}{2} \dot{q}^2(s) + Aq(s)\dot{q}(s) + Bq^2(s) 
+ C\dot{q}(s) + Dq(s) ,
\]
\[
L_{TNL}[q,\dot{q},s] = \int_0^s \alpha(s,r) (Eq(s)\dot{q}(r) + Fq(s)q(r) 
+ G\dot{q}(s)q(r) + H\dot{q}(s)\dot{q}(r)) \, dr ,
\]
where the coefficients \( A, \ldots, H \) are constants, and \( \alpha(t,s) \) is a two-variable function. More general expressions for the Lagrangian can be considered, and the formalism still applies. First of all, we show that such a Lagrangian, for the purposes of our formalism, can be rewritten in a simpler but equivalent way. The reason is that, since the crucial quantity is the Action \( S \), which is the integral of the Lagrangian, then \( L_{TNL} \) can be rewritten in different ways, without changing \( S \). In particular, it is convenient to integrate by parts some of the terms containing double integrals. For example, the first term of \( \int_0^t L_{TNL}[q,\dot{q},s] \) can be rewritten as follows:
\[
E \int_0^t q(s) \int_0^s \alpha(s,r) \dot{q}(r) \, dr \, ds = 
E \int_0^t q(s) \left( \alpha(s,s)q(s) - \alpha(s,0)q(0) \right. 
\left. - \int_0^s \frac{\partial \alpha(s,r)}{\partial r} q(r) \, dr \right) \, ds ,
\]
and the first two terms on the right-hand side can be absorbed in the term of \( L_{TL} \), proportional to \( q(s) \). The same happens for the third and fourth terms of \( L_{TNL}[q,\dot{q},s] \). Accordingly, without loss of generality, \( L[q,\dot{q},s] \) can be rewritten as follows:
\[
L[q,\dot{q},s] = \frac{m}{2} \dot{q}^2(s) + \tilde{A}(s)q(s)\dot{q}(s) + \tilde{B}(s)q^2(s) + \tilde{C}(s)\dot{q}(s) 
+ \tilde{D}(s)q(s) + \tilde{F}(s)q(s) \int_0^s \alpha(s,r)q(r) \, dr ,
\]
where now the coefficients \( \tilde{A}(s) \ldots \tilde{F}(s) \) are functions of time, which collect all the terms coming from the integrations by parts described here above. We can now use the formalism previously introduced, in particular eq. (8) (or, equivalently, the generalized Hamilton formalism) to find the second-order integro-differential equation of motion, which turns out to be
\[
\begin{align*}
\tilde{m}\ddot{q}(s) &= -\tilde{A}'(s)q(s) - \tilde{C}'(s) + 2\tilde{B}(s)q(s) + \tilde{D}(s) 
+ \tilde{F}(s) \int_0^s \alpha(s,r)q(r) \, dr + \int_0^t \tilde{F}(r)\alpha(s,r)q(r) \, dr .
\end{align*}
\]
This equation represents the generalization of the equation of motion of the standard local theory. The first thing one notices is that it displays a dependence on the future of \( s \), not on its past, as one would naively expect by looking at the Lagrangian. The reason is that the problem was originally formulated with boundary conditions at the initial time 0 and the final time \( t \): finding the path that minimizes the Action, among all those which take the value \( q_0 \) at time 0 and \( \dot{q} \) at time \( t \). While, in the standard time-local case, these boundary conditions can always be replaced with initial conditions on the position \( q_0 \) and velocity \( \dot{q}_0 \) at time 0, in the TNL case this is not possible anymore, in general. This does not imply that the dynamics of the system necessarily depends on the future. It could, like in the Abraham-Lorentz equation [29] for a charged particle, which takes back-reaction into account, and displays what is called pre-acceleration; or like in the Wheeler-Feynman formulation of classical electrodynamics [30]. But it could also not depend on the future, as it happens for non-Markovian stochastic processes, even if the dynamical equation formally does depend on the future. In this case, the equation does not correspond to the intuitive way of describing the dynamical evolution of the system. In general, there is no rule for discriminating which TNL equations can be rewritten in such a way to show only a past-dependence, and which really contain a dependence on the future. From a mathematical (but also physical) point of view, this is the major difference between time-local and TNL dynamics, and why TNL dynamics are by far much more complicated. We will come back on this issue. As a final note, it is easy to check that in the limit \( \tilde{F}(s) = 0 \) or \( \alpha(t,s) \to \delta(t-s) \) eq. (17) reduces to the standard equation of motion.
In order to complete our general example, we compute the
generalized Hamiltonian associated to the TNL Action
of eq. (16). In particular, computing \( p(s) \) with eq. (9),
and substituting the corresponding expression, as well as
eq. (16), in eq. (10), one finds that
\[
H[q, p] = \int_0^t \left[ \frac{1}{2m} p^2(s) - \frac{A(s)}{m} q(s)p(s) - \frac{C(s)}{m} p(s) + \frac{A(s)}{2m} - B(s) \right] q^2(s) + \frac{A(s)C(s) - D(s)}{m} \right] q(s) + \frac{C^2(s)}{2m} - \mathcal{F}(s)q(s) \int_0^s \alpha(s,r)q(r) \, dr \right] \, ds .
\] (18)

Also in this case, one can easily check that in the time-local
limit (\( s, r \) \( \rightarrow \delta(r-s) \)) this TNL Hamiltonian reduces
to the standard one. Moreover, the transformation in
(10) can be reversed and, substituting in it eq. (13),
one recovers \( S[q, \dot{q}] \) from eq. (18).

Example of a TNL system: the TNL harmonic oscillator. – We now apply the TNL formalism to a
specific physical situation in order to better understand
the effects of the TNL terms on the dynamics of the
system. In particular we consider a TNL harmonic oscillator,
described by the following Lagrangian:
\[
L[q, \dot{q}] = \frac{m}{2} \dot{q}(t)^2 - \frac{k}{2} q(t)^2 - \bar{k}(t) \int_0^t \alpha(t,s)q(s) \, ds ,
\] (19)

where an integral term, which accounts for the memory
effects, is added to the local Lagrangian. The equation of
motion becomes:
\[
m\ddot{q}(s) = -kq(s) - \bar{k} \int_0^t \alpha(s,r)q(r) \, dr .
\] (20)

Again, we can see that the equation of motion formally
depends on the future up to time \( t \), besides depending on
the past. To show that this is not necessarily a real physical
dependence on the future, let us consider an exponential
memory kernel
\[
\alpha(t,s) = \frac{\gamma}{2} e^{-\gamma(t-s)} ,
\] (21)

and show how the integro-differential equation can be transformed into an ordinary differential equation. (Note
that by taking the limit \( \gamma \rightarrow \infty \) one recovers the time-local case.) With this choice for the memory kernel, eq. (20) becomes
\[
m\ddot{q}(s) = -kq(s) - \frac{\bar{k} \gamma}{2} \int_0^t e^{-\gamma(r-s)} q(r) \, dr .
\] (22)

Note this is the same equation one obtains by applying the
formalism of Llosa et al. [20] to the Lagrangian (19).
Differentiating eq. (22) with respect to \( s \) one finds
\[
\dddot{q}(s) = -k \ddot{q}(s) - \frac{\bar{k} \gamma^2}{2m} \int_0^s e^{-\gamma(s-r)} q(r) \, dr
- \frac{\bar{k} \gamma^2}{2m} \int_s^t e^{-\gamma(r-s)} q(r) \, dr = 0 ,
\] (23)

and differentiating it once more and using eq. (22), one
can show that \( q(s) \) satisfies the following fourth-order
differential equation:
\[
\dddot{q}(s) - \left( \frac{k}{m} + \gamma^2 \right) \ddot{q}(s) + \frac{\gamma^2}{m} (k + \bar{k}) q(s) = 0 .
\] (24)

The first important message is that this is an ordinary
differential equation, with no explicit dependence on the
future (or past). This makes the point that, quite
generally, one needs special care when interpreting the
meaning of the solutions of integro-differential equations.

The space of solutions of eq. (24) is larger than that of
eq. (22). Therefore, one has to force suitable constraints on
the solutions of eq. (24) in order for them to solve eq. (22)
as well. In particular, since eq. (24) is a fourth-order differen-
tial equation, four boundary conditions are needed, to
univocally determine the solution. Two of them are given
by the boundary conditions of the variational problem,
\( q(0) = q_0 \) and \( q(t) = \bar{q} \). The dependence on the endpoint
can be removed, like in the time-local formalism, by replac-
ing it with the initial condition on velocity \( \dot{q}(0) = v_0 \).
The other two boundary conditions are given by the
“consistence conditions” of eq. (24) with eq. (22), i.e. by
asking the solutions of eq. (24) to be solutions of eq. (22)
as well. In order to find these two constraints, one eval-
uates eqs. (22), (23) at time \( s = 0 \) and then at time \( s = t \),
and compares all terms of these equations which are equal.
The result is
\[
\dddot{q}(0) - \frac{k}{m} \ddot{q}(0) = \gamma \left( \dddot{q}(0) - \frac{k}{m} \dddot{q}(0) \right) ,
\] (25)

\[
\dddot{q}(t) - \frac{k}{m} \ddot{q}(t) = -\gamma \left( \dddot{q}(t) - \frac{k}{m} \dddot{q}(t) \right) .
\] (26)

The first consistency condition can be considered as an
extra initial condition. The second consistency condition,
instead, refers to the terminal time \( t \). We are not aware of
any way to replace it with an additional initial condition
at time \( 0 \); accordingly, our original problem with boundary
conditions at \( s = 0 \) and \( s = t \) can only partly be reformu-
lated as a problem with initial conditions at \( s = 0 \). But
once again, we stress that this does not imply that the
dynamics depends on the future. It is a consequence of
the fact that the original problem was formulated with boundary
conditions both at time \( 0 \) and at time \( t \).

The solution of eq. (24) reads
\[
q(s) = a_{1,t} \sinh x_1 s + a_{2,t} \cosh x_1 s + a_{3,t} \sinh x_2 s + a_{4,t} \cosh x_2 s ,
\] (27)

where the subscript \( t \) denotes the dependence on the
endpoint, due to the condition (26). The roots \( x_1 \) and the
coefficients \( a_{i,t} \) are uniquely determined, and their explicit
expression is given in the appendix. We now discuss the
physical behavior of the solutions.

Figure 1 shows the behavior of a TNL harmonic oscillator
of mass \( m = 1 \), with initial position \( q_0 = 0 \) and initial
speed \( v_0 = 1 \). The strength of the coupling constants \( k, \tilde{k} \), and the correlation function cutoff \( \gamma \), have been chosen in such a way to highlight memory effects: \( k = 1, \tilde{k} = 10^6 \), and \( \gamma = 1 \). We see that, as expected from a dynamics with memory effects, after an initial transient of few oscillations, the trajectory stabilizes and becomes harmonic. We notice that, if also the initial velocity is \( v_0 = 0 \), the oscillator stays at rest like in the time-local case.

Figure 2 shows a comparison between the behavior of a standard harmonic oscillator (\( m = 1, v_0 = 1, \gamma = \infty \)) and those of TNL harmonic oscillators (same \( m \) and \( v_0 \)) for different values of cutoff: \( \gamma = 0.1, 0.3, 0.7 \). As we can see, for very short times, the behavior of the non-local oscillator follows that of a local one. Moreover, as expected, the bigger \( \gamma \), the closer the non-local trajectories are to the local one. We further notice that the wavelength and the amplitude of the oscillations to which the TNL oscillator stabilizes are bigger than the corresponding local oscillator.

We can then identify three main stages in the evolution of the TNL oscillator:

1. **Very-short time behavior**: The TNL harmonic oscillator behaves as the local one. The reason is that the elapsed time is so short that there cannot be any memory of the past dynamics yet. In fact, in this stage the contribution coming from the integral term of eq. (22) is negligible.

2. **Transient phase**: In this stage, the oscillator behaves in an highly non-harmonic way; this is when the TNL terms start affecting the dynamics. The length of this stage clearly depends on the non-local coupling constant \( \tilde{k} \) and memory kernel cutoff \( \gamma \).

3. **Long-time behavior**: The trajectory stabilizes to that of a local harmonic oscillator. The reason is that when the elapsed time is larger than the characteristic time of the memory kernel, the integral term of eq. (22) reaches, for all practical purposes, an asymptotic stable value, and the subsequent dynamics looks “local-like”. The long-time wavelength of the TNL oscillator is larger than that of a local one (\( \gamma = \infty \)), because the effective force to which the oscillator is subject is weaker. The reason is that when \( q(s) \) is, like in the present case, an oscillating function, the value of the integral of eq. (22) for any finite cutoff \( \gamma \) is always smaller than the local case (\( \gamma \rightarrow \infty \)).

These features of the TNL harmonic oscillator correspond to the physical intuition one would have about a standard harmonic oscillator interacting with a non-Markovian bath.

**Conclusions.** – We have proposed a TNL formalism which allows for a general analysis of TNL Lagrangians, and we have derived the corresponding Hamilton formalism. As we have discussed, to study TNL dynamics starting from their Action, one has to solve a problem with boundary conditions at time 0 and time \( t \), which in general cannot be reduced to an initial-value problem (contrary to the standard time-local case, where this can always be done). This can be embarrassing in some cases, since the solution of the problem turns out to formally depend on the future. However, in important situations —like the path integral formalism— one needs only to solve a problem with boundary conditions at the initial and final time, without any need to transform in into an initial-value problem. Therefore, the TNL formalism can be applied to such a formalism without difficulties, other than technical ones related to the evaluation of the integrals.

***

The authors wish to thank D. Dürr for illuminating discussions on this subject. They acknowledge partial financial support from MIUR (PRIN 2008), INFN, COST
coefficients

APPENDIX

We provide the explicit expressions for the roots \( x_i \) and the coefficients \( a_i \) of eq. (27), which determine the solution of eq. (24) for the given initial conditions. The roots \( x_i \) are the solutions of the characteristic polynomial associated to eq. (24) and they read \( (i = 1, 2) \):

\[
x_1 = \frac{\gamma^2}{2} - \frac{k}{m} + (-1)^{i+1} \sqrt{\left( \frac{\gamma^2}{2} - \frac{k}{m} \right)^2 + \gamma^2 \frac{k}{m}}. \quad (A.1)
\]

Since the coefficients \( a_i \) have the same denominator \( d \), it is useful to define four new coefficients \( b_i \) as follows:

\[
b_{i,t} = a_{i,t} d, \quad i = 1, \ldots, 4. \quad (A.2)
\]

The explicit expression for the denominator \( d \) and the coefficients \( b_i \) are:

\[
d = (x_1^2 - x_2^2) \left[ -2x_1 x_2 \left( \frac{k}{m} + x_1^2 \right) \gamma \cosh x_2 t + \left( \frac{k}{m} + x_1^2 \right) x_2 (2x_1 \gamma \cosh x_1 t + (x_1^2 + \gamma^2) \sinh x_1 t) - x_1 \left( \frac{k}{m} + x_2^2 \right) (x_2^2 + \gamma^2) \sinh x_2 t \right], \quad (A.3)
\]

\[
b_{1,t} = \left( \frac{k}{m} + x_2^2 \right) \left[ -x_2 \gamma \left( v_0 \left( x_2 + x_2^2 \right) - x_0 \left( -x_2^2 \right) \right) \right. \nonumber
\]

\[
+ x_0 \left( \frac{k}{m} + x_1^2 \right) \gamma \cosh x_2 t - \left( \frac{k}{m} + x_2^2 \right) x_2 (v_0 - x_0 \gamma) (\gamma \cosh x_1 t + x_1 \sinh x_1 t) \right. \nonumber
\]

\[
+ \left( v_0 x_2^2 \left( \frac{k}{m} + x_2^2 \right) - x_0 \left( \frac{k}{m} + x_1^2 \right) x_2^2 \gamma \right) \nonumber
\]

\[
+ v_0 \left( -x_1^2 + x_1^2 \right) \gamma^2 \sinh x_2 t \right], \quad (A.4)
\]

\[
b_{2,t} = x_0 d - b_{4,t}, \quad (A.5)
\]

\[
b_{3,t} = x_1 \left( \frac{k}{m} + x_1^2 \right) \left[ \left( \frac{k}{m} + x_1^2 \right) \gamma (-v_0 - x_0 \gamma) \cosh x_2 t - x_1 \gamma \left( v_0 \left( x_2 + x_1^2 \right) \right) \right. \nonumber
\]

\[
+ x_0 \left( \frac{k}{m} + x_1^2 \right) \gamma \cosh x_1 t - \left( -v_0 x_1^2 \left( \frac{k}{m} + x_1^2 \right) + x_0 x_1^2 \left( \frac{k}{m} + x_2^2 \right) \right) \gamma \right. \nonumber
\]

\[
+ v_0 \left( -x_2^2 + x_1^2 \right) \gamma^2 \sinh x_1 t \right. \nonumber
\]

\[
-x_1 x_2 \left( \frac{k}{m} + x_1^2 \right) \left( \frac{k}{m} + x_2^2 \right) \left( v_0 - x_0 \gamma \right) \sinh x_2 t \right], \quad (A.6)
\]

REFERENCES

[1] Eliezer D. A. and Woodard R. P., Nucl. Phys. B, 325 (1989) 389.
[2] Aharony O. et al., JHEP, 09 (2000) 023.
[3] Seiberg N. and Witten E., JHEP, 09 (1999) 032.
[4] Pais A. and Uhlenbeck G. E., Phys. Rev., 79 (1950) 145.
[5] Marnelius R., Phys. Rev. D, 8 (1973) 37472.
[6] Seiberg N. et al., JHEP, 09 (2000) 023.
[7] Strunz W. T., Phys. Lett. A, 224 (1996) 25.
[8] Diósi L. and Strunz W. T., Phys. Lett. A, 235 (1997) 569.
[9] Diósi L. et al., Phys. Rev. A, 58 (1998) 1699.
[10] Bassi A. and Ghirardi G. C., Phys. Rev. A, 65 (2002) 042114.
[11] Adler S. L. and Bassi A., J. Phys. A, 40 (2007) 15083.
[12] Bassi A. and Ferialdi L., Phys. Rev. A, 80 (2009) 012116.
[13] Bassi A. and Ferialdi L., Phys. Rev. Lett., 103 (2009) 050403.
[14] Roden J. et al., Phys. Rev. Lett., 103 (2009) 058301.
[15] Roden J. et al., J. Chem. Phys., 134 (2011) 034902.
[16] Lurie D., Particles and Fields (Interscience, New York, NY) 1968.
[17] Succi S., Class. Quantum Grav., 23 (2006) 1989.
[18] Ostrogradski M., Mem. Ac. St. Petersburg, 6 (1850) 385.
[19] de Urries F. J. and Julve J., J. Phys. A, 31 (1998) 6949.
[20] Llosa J. and Vives J., J. Math. Phys., 35 (1994) 2856.
[21] Gomis J. et al., Phys. Rev. D, 63 (2001) 056003.
[22] Woodard R. P., Phys. Rev. A, 62 (2000) 052105.
[23] Edelen D. G., Int. J. Eng. Sci., 7 (1969) 269.
[24] Goldstein H., Poole C. and Safko J., Classical Mechanics (Prentice Hall, Upper Saddle River, NJ) 2002.
[25] Arnold V. I., Mathematical Methods of Classical Mechanics (Springer-Verlag, New York) 1978.
[26] Gelfand I. M. and Fomin S. V., Calculus of Variations (Prentice-Hall, Englewood Cliffs, NJ) 1963.
[27] Greiner W. and Reinhardt J., Field Quantization (Springer-Verlag, Berlin) 1996.
[28] Weinberg S., The Quantum Theory of Fields, Vol. 1, Foundations (Cambridge University Press, Cambridge) 1995.
[29] Jackson J. D., Classical Electrodynamics (Wiley, New York, NY) 1999.
[30] Bauer G. et al., arXiv:1009.3103 (2010).