Quantitative robustness of regularity for 3D Navier-Stokes system in $\dot{H}^\alpha$-spaces

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Abstract. We present stability and regularity results for the 3D Navier-Stokes system in a periodic box in $\dot{H}^\alpha$ spaces, with $\alpha \in [1/2, 1]$. This note is the analytical part of our forthcoming detailed analysis of schemes for numerical verification of regularity of the 3D Navier-Stokes system. Therefore we pay a special attention to obtaining quantitative results, i.e. ones with explicit constants.

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1. Introduction

We consider stability of a regular solution \( u \) of a 3D Navier-Stokes system in the periodic cube \( Q_L := [0, L]^3 \). Namely, we take an \( L \)-periodic pair \((u, p)\) solving\(^{(1)}\) in \( Q_L \times [0, T] \)

\[
\begin{aligned}
&u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f \\
&\text{div} u = 0 \\
&u(0) = u_0
\end{aligned}
\]

where \( \nu \) is a positive viscosity parameter and \( f \) is a given function (forcing). We assume that \( u \) is more regular than a weak solution. More precisely, we assume that it is additionally an \( \alpha \)-strong solution, i.e. \( u \in L^\infty(\dot{H}^\alpha) \cap L^2(\dot{H}^{1+\alpha}) \) for \( \alpha \in [1/2, 1] \). Next, we take a weak solution \( v \) to \((NS_g, v_0)\) and ask what are the conditions on differences of data of \( u \) and \( v \), i.e. on \(|f - g|\) and \(|u_0 - v_0|\), that allow \( v \) to inherit \( \alpha \)-strong regularity of \( u \).

This problem is referred to as a problem of stability of strong solutions or, in a more debonair manner, as a problem of robustness of regularity. It will play a central role in our forthcoming detailed analysis of the schemes for the numerical verification of regularity that are based on the idea presented in [CCRT] and further refined in [M-RRS]. Our idea is to enhance the result of [M-RRS] threefold.

(i) We plan to substitute the step \( \dot{H}^{1/2} \mapsto \dot{H}^1 \) there with \( \dot{H}^\alpha \mapsto \dot{H}^{\alpha + \delta} \) for the sake of numerics.

(ii) We want to take into account phenomena that are connected with scaling of \( Q_L \).

(iii) Finally, we plan to provide tangible numerical results.

In order to fulfill this program, we need a stability result for the Navier-Stokes system in \( \dot{H}^\alpha(Q_L) \) spaces with explicit constants. As we were unable to find such a result and, at the same time, we find it interesting by itself, we present it in this note (see Theorem 1, subsection 1.2). As a byproduct (or rather, as a needed ingredient) we obtain also a global-in-time regularity result for small data and a regularity result for small time (respectively, Theorems 2 and 3 in subsection 1.2). In the former, in addition to the standard blowup characterization of the maximal time of existence we provide also a caloric characterization. These regularity results also are quantitative in the sense of explicitly computed constants and smallness parameters.

Let us remark, that having the \( \dot{H}^\alpha \) stability result (qualitatively it is stated as Theorem 3.5 in [CDGG] ) we can immediately state a (qualitative) stability result in \( \dot{H}^\alpha, \alpha \geq 1/2 \) via the Ladyzhenskaya-Prodi-Serrin-type condition. However this is not satisfactory for us.

\(^{(1)}\) For rigorous definition of the solution and for presentation of underlying function spaces we refer to Section 2.
1.1. Current state of research in stability of the Navier-Stokes system

A numerous variants of the stability problem problem have been a subject of an intensive studies. It is common to deal with stability issues separately for each class of special regular solutions. Let us recall results concerning

(i) stability of two-dimensional solutions (in cylindrical domains, under the slip boundary conditions, see [Za3–Za7])

(ii) stability of axially-symmetric solutions without swirl, where the perturbed solution is either swirlless [Za8] or have small swirl near the axis of symmetry [Za10].

There are also results that combine (i) and (ii), for instance in [Za9] one considers the stability problem of linear combinations of two-dimensional and axially symmetric solutions.

(iii) stability of helicoidal solutions.

1.2. The results

Recall that $\nu > 0$ is a viscosity parameter and that $\alpha$-strong solution to a Navier-Stokes system is such weak solution, that additionally belongs to $L^\infty(\dot{H}^\alpha) \cap L^2(\dot{H}^{1+\alpha})$.

We denote the Fourier-series-based norm in $\dot{H}^\alpha(Q_L)$ with $| \cdot |_{\alpha,L}$. All the needed (standard) definitions has been shifted to Section 2. Let us take

$$K_2 := \sqrt{2}C_S(1-\alpha)C_S(1)C_S\left(\alpha - \frac{1}{2}\right)\frac{2\pi}{L},$$

$$K_3 := \varepsilon^{-3}_{1}\frac{27}{128}C^4_S(1-\alpha)C^4_S(1)\left(\frac{2\pi}{L}\right)^{-4\alpha + 6}\left[1 + \left(\frac{2\pi}{L}\right)^\alpha C_S\left(\alpha - \frac{1}{2}\right)L^{d(\alpha-2)}\right]^4,$$

$$K_4 := \frac{1}{4\varepsilon^2}\left(\frac{4\pi^2}{L^2}\right)^{-1},$$

where $C_S(\beta)$ denotes a numerical constant of the optimal normalized three-dimensional Sobolev-Poincaré inequality

$$|f|_{L^{\beta^*}(Q_{2\pi})} \leq C_S(\beta)|f|_{\beta,2\pi},$$

where $\beta \in [0,2)$, $\beta^* := \frac{6}{3-2\beta}$ (compare subsection 2.5).
1.2.1 The stability result

We refine and generalize Theorem 1 of [M-RRS] by

**Theorem 1.** (Robustness of regularity). Take $T_*>0$, $\alpha \in [1/2; 1]$ and data $u_0, v_0 \in \dot{H}^\alpha_{\text{div}}(Q_L)$, $f, g \in L^2(0, T_*; \dot{H}_{\text{div}}^{\alpha-1}(Q_L))$.

Assume that $u$ is an $\alpha$-strong solution to $(NS_{f, u_0})$ with its time of existence $T_*$. Take any positive $\tilde{\nu}, \varepsilon_1, \varepsilon_2$ such that

$$\tilde{\nu} + \varepsilon_1 + \varepsilon_2 < \nu.$$

Then every Leray-Hopf weak solution $v$ that starts close to $u$ and that has a similar forcing is also an $\alpha$-strong solution. More precisely, let us fix any $T < T_*$. Under the proximity assumption for the data

$$\left( |u_0 - v_0|_{\alpha,L}^2 + K_4 \int_0^T |f-g|_{\alpha-1,L}^2(t) \, dt \right) \frac{K_3}{2} \frac{\int_0^T \left| \nabla u(t) \right|^4_{L^{2/\alpha}(Q_L)} \, dt}{\nu} < \left( \frac{\tilde{\nu}}{K_2} \right)^2 \quad (A1)$$

$v$ is an $\alpha$-strong solution to $(NS_{g,v_0})$ with its time of existence $T_*(g, v_0) > T$. Moreover, $v$ is close to $u$ according to the following formula

$$\sup_{t \in [0,T]} |u-v|_{\alpha,L}^2(t) + (\nu - (\tilde{\nu} + \varepsilon_1 + \varepsilon_2)) \int_0^T |u-v|_{\alpha+1,L}^2(t) \, dt \leq \left( \frac{\tilde{\nu}}{K_2} \right)^2 \quad (1.1)$$

The proof of Theorem 1 is given in Section 3.

Observe that $\int_0^T \left| \nabla u(t) \right|^4_{L^{2/\alpha}(Q_L)} \, dt$, used in the proximity assumption (A1), is given automatically by the fact that $u$ is the $\alpha$-strong solution, as by interpolation

$$L^\infty(\dot{H}^\alpha) \cap L^2(\dot{H}^{1+\alpha}) \hookrightarrow L^4(W^{1,\frac{3}{2-\alpha}}).$$

To clarify this point quantitatively, let us state

**Corollary 1.** For the proximity assumption (A1) suffices

$$\left( |u_0 - v_0|_{\alpha,L}^2 + K_4 \int_0^T |f-g|_{\alpha-1,L}^2(t) \, dt \right) \exp \left( K_3 \int_0^T \left| \nabla u(t) \right|^4_{L^{2/\alpha}(Q_L)} \, dt \right) < \left( \frac{\tilde{\nu}}{K_2} \right)^2 \quad (A2)$$

where $C_I(\alpha, T)$ comes from Definition 2 in subsection 2.5.

The proof of Corollary 1 is given in Section 3.
1.2.2 The regularity results

In order to show Theorem 1, we need the following theorem on local existence of strong solutions and on their uniqueness

**Theorem 2.** (local-in-time $\alpha$-strong solutions). Take $T > 0$, $\alpha \in [1/2, 1]$ and data

$$u_0 \in \dot{H}_\text{div}^\alpha(Q_L), \ f \in L^2(0, T; \dot{H}_\text{div}^{\alpha-1}(Q_L)).$$

Then there is $T^*_*(u_0, f) \in (0, T]$ such that there exists a

$$C((0, T^*_*(u_0, f)); \dot{H}_\text{div}^\alpha(Q_L)) \cap L^2(0, T^*_*(u_0, f); \dot{H}_\text{div}^{1+\alpha}(Q_L))$$

solution to $(\text{NS}_{f, u_0})$. Moreover such solution is unique among Leray-Hopf weak solutions to $(\text{NS}_{f, u_0})$.

$T^*_*(u_0, f)$ can be characterized by the blowup, i.e.

$$|u(t)|_{\alpha, L}^2 + \int_0^t |u(t)|_{\alpha+1, L}^2 \, dt \xrightarrow{t \to T^*_*(u_0, f)} \infty.$$

Observe that in the above theorem we get continuity in time of the $\dot{H}_\text{div}^\alpha$ norm and not only boundedness in time, that is in the definition of an $\alpha$-strong solution. The proof of Theorem 2 is standard. For clarity we present it in subsection 4.2. We obtain there also the following characterizations of the time $T^*_*(u_0, f)$.

In the following theorem, we denote by $u^{L_0}$ the solution of the homogeneous heat system that emanates from $P^k_0 u_0$, compare (4.1). Recall that $K_2$ and $K_3$ come from Theorem 1.

**Lemma 1.** (Caloric characterization of $T_*$). Take any positive $\varepsilon_1, \varepsilon_2$ and $k_0 \in \mathbb{Z}^d$ so large that

$$\mu := \nu - \left(\varepsilon_1 + \varepsilon_2 + K_2 \left(\frac{1}{\sqrt{2}} |u_0 - P^k_0 u_0|^\alpha_{\alpha, L}\right)\right) > 0.$$

Take any $\delta > 0$ and $\sigma \in (0, 1)$. The time $T_0$ that yields

$$\left(\frac{K_2}{\nu - \varepsilon_1 - \varepsilon_2 - \sigma \mu}\right)^2 \frac{1}{2\delta} \left(\frac{4\pi^2}{L^2}\right)^{2\alpha} \int_0^{T_0} |u^{L_0}(\tau) \otimes u^{L_0}(\tau)|_{1+\alpha, L}^2 \, d\tau$$

$$- K_3 \left(\int_0^{T_0} |\nabla u^{L_0}(\tau)|^4_{L^4} \frac{\dot{H}_\text{div}^\alpha(Q_L)}{L^{2}} \, d\tau + \delta T_0\right)$$

$$= \epsilon \left(\frac{1}{\nu - \varepsilon_1 - \varepsilon_2 - \sigma \mu} - 1\right)^2$$

is a lower bound for $T_*(f, u_0)$, i.e. $T_*(f, u_0) \geq T_0$. Moreover

$$\sup_{t \leq T_0} \frac{1}{2} |u(t)|_{\alpha, L}^2 + \frac{4\pi^2}{L^2} \int_0^{T_0} |u(t)|_{\alpha+1, L}^2 \, dt \leq \left(\frac{\nu - \varepsilon_1 - \varepsilon_2 - \sigma \mu}{K_2}\right)^2.$$

Finally, we obtain also the following global-in-time $\alpha$-strong solutions for small data.
Theorem 3. Take any $T \in (0, \infty]$ and any positive $\bar{\nu}, \varepsilon_2$ such that

$$\bar{\nu} + \varepsilon_2 < \nu.$$  

Assume that data $f$, $u_0$ satisfy the following smallness condition

\[ \left( |u_0|^2_{0,L} + K_4 \int_0^T |f(t)|^2_{0-1,L} \, dt \right) < \left( \frac{\bar{\nu}}{K_2} \right)^2. \tag{A4} \]

Then $(NS_{f,u_0})$ has the $\alpha$-regular solution on $[0,T]$ with the estimate

\[ \sup_{t \in [0,T]} |u|^2_{0,L} + (\nu - \bar{\nu} - \varepsilon_2) \int_0^T |u|^2_{0+1,L} \leq \left( \frac{\bar{\nu}}{K_2} \right)^2 \]

For the proof, see subsection 4.3.

2. Preliminaries

Here we present the detailed setting of our problem. It is standard and based on [CF], Chapter 4 and [Tem].

2.1. Function spaces.

**Homogeneous Sobolev spaces.** Let us introduce the Fourier basis

$$\omega_j^j = e^{2\pi i k \cdot x} b_j, \quad \omega_L,k = (\omega_1^1, \ldots, \omega_N^N)$$

where $k \in \mathbb{Z}^d$, $x \in Q_L = [0, L]^d$ and $b_j$ is the $j$-th canonical vector of $\mathbb{R}^N$. We introduce $\hat{\mathbb{Z}}^d := \mathbb{Z}^d \setminus \{0\}$. The space

$$\hat{H}^s(Q_L) := \left\{ u = \sum_{k \in \hat{\mathbb{Z}}^d} u_k \cdot \omega_L,k \mid u_k \in \mathcal{C}^N, u_k = \bar{u}_{-k}, u_0 = 0, \sum_{k \in \hat{\mathbb{Z}}^d} |k|^{2s} |u_k|^2 < +\infty \right\},$$

where $s \in \mathbb{R}$, becomes the Hilbert space with the product

$$\langle u, w \rangle_{s,L} := \sum_{k \in \hat{\mathbb{Z}}^d} |k|^{2s} u_k \bar{w}_k$$

that generates the norm $| \cdot |_{s,L}$. We will also use the generalized scalar product (duality formula)

$$\langle u, w \rangle_{\alpha,\beta,L} := \sum_{k \in \hat{\mathbb{Z}}^d} (|k|^\alpha u_k)(|k|^\beta w_k)$$
for $\alpha, \beta \in \mathbb{R}$. For $\alpha + \beta = 2s$ one has $\langle u, w \rangle_{\alpha, \beta, L} = \langle u, w \rangle_{s, L}$.

We call $\dot{H}^s(Q_L)$ a real $(u_l = \bar{u}_l - k)$, zero average ($u_0$ plays no role, as we sum over $\hat{Z}_d$), fractional ($s \in \mathbb{R}$), homogenous Sobolev space of periodic functions ($u^l(x + Le_j) = u^l(x)$ thanks to the Fourier-series-based definition). The homogeneity of $\dot{H}^s(Q_L)$ follows from absence of lower-order terms in its norm.

Observe that we have the following scaling-invariance. Let us define for $u : Q_L \rightarrow \mathbb{R}^N$ its dilation $u_\delta : Q_{\delta L} \rightarrow \mathbb{R}^N$ by $u_\delta(x) = u(\delta x)$. The Fourier coefficients of $u$ and $u_\delta$ are identical, because $\omega_{L,k}(x) = \omega_{\delta L,k}(\delta x)$.

One of advantages of working with homogeneous Sobolev spaces is that for any $s \in \mathbb{R}$ $$(\dot{H}^s(Q_L))^* \simeq \dot{H}^{-s}(Q_L),$$ see [Tem].

The sequential norm $| \cdot |_{s,L}$ agrees with the integro-differential Sobolev norm

$$|u|_{\dot{H}^s(Q_L)} := \left( \int_{Q_L} |\nabla^s u|^2 \right)^{\frac{1}{2}}$$

up to a scaling parameter. More precisely

$$|u|_{\dot{H}^s(Q_L)} = L^{\frac{d}{2}} \left( \frac{4\pi^2}{L^2} \right)^{\frac{2}{2}} |u|_{s,L}.$$ The above formula will become clear after we define $\nabla^s$ in the next subsection.

We will use also the zero-divergence subspace of $\dot{H}^s(Q)$, i.e.

$$\dot{H}^s_{\text{div}}(Q) := \{ u \in \dot{H}^s(Q) | k \cdot u_k = 0, \ k \in \hat{Z}_d \}$$

closed under the norm $| \cdot |_s$.

From now on we work with domain and target dimensions equal 3, i.e. $d = N = 3$. Lebesgue spaces. We will need also $L^p(\Omega)$ spaces with the integro-differential norm $|f|_{L^p(\Omega)} := \left( \int_{\Omega} |f|^p \right)^{\frac{1}{p}}$.

2.2. Stokes operator

The (stationary) Stokes problem in a periodic cube $Q_L$, i.e. the problem of finding for a certain $f \in H^{-1}(Q_L)$ a pair $(u, p) \in \dot{H}^1_{\text{div}}(Q_L) \times L^2(Q_L)$ that satisfies

$$-\Delta u + \nabla p = f, \quad \text{div} u = 0$$
in $Q_L$ admits in our periodic setting the following explicit solution

$$u = \sum_{k \in \hat{Z}_3} u_k \cdot \omega_{L,k}, \quad p = \sum_{k \in \hat{Z}_3} p_k e^{\frac{2\pi i}{L} k \cdot x}.$$
where
\[ u^j_k = -\frac{L^2}{4\pi^2 |k|^2} \left( f_k - \frac{(k \cdot f_k)k}{|k|^2} \right), \quad \text{for } j = 1, 2, 3, \quad p_k = \frac{Lk \cdot f_k}{2i\pi |k|^2}. \]

Under assumption of divergent-free forcing the pressure vanishes and the ”solution mapping” \( f \mapsto u \) from \( \dot{H}^0_{\text{div}}(Q_L) \) to \( \dot{H}^2_{\text{div}}(Q_L) \) is bijective. Hence its inverse is meaningful. We denote it by \( A : \dot{H}^2_{\text{div}}(Q_L) \mapsto \dot{H}^0_{\text{div}}(Q_L) \) and refer to as the Stokes operator. In our case it degenerates to \(-\Delta\). On the level of Fourier coefficients \( A \) is the multiplication with \(-\frac{4\pi^2 |k|^2}{L^2}\), i.e.
\[ (Au)_k = -u_k \frac{4\pi^2 |k|^2}{L^2}. \]

Hence we have
\[ Au = -\sum_{k \in \mathbb{Z}^d} \lambda_k u_k \cdot \omega_{L,k} \quad (2.1) \]
with \( \lambda_k := \frac{4\pi^2 |k|^2}{L^2} \). Consequently
\[ |Au|_{0,L} = \frac{4\pi^2}{L^2} |u|_{2,L} \]
The formula (2.1) admits a generalization that defines powers of the operator \( A \). Namely \( A^\alpha : \dot{H}^{2\alpha}_{\text{div}}(Q_L) \mapsto \dot{H}^0_{\text{div}}(Q_L) \) is given by
\[ A^\alpha u := \sum_{k \in \mathbb{Z}^d} \lambda_k^\alpha u_k \cdot \omega_{L,k}. \]
The formula (2.1) is thus generalized to
\[ |A^\alpha u|_{0,L} = \left( \frac{4\pi^2}{L^2} \right)^\alpha |u|_{2\alpha,L} \quad \alpha \in \mathbb{R}. \quad (2.2) \]
Now, observing \( A^\alpha u = \nabla^{2\alpha} u \), we can justify also the formula
\[ |u|_{\dot{H}^s(Q_L)} = L^\frac{d}{2} \left( \frac{4\pi^2}{L^2} \right)^\frac{s}{2} |u|_{s,L} \]
from subsection 2.1. This identity and (2.1) can be together further generalized to
\[ |A^\alpha u|_{\beta,L} = \left( \frac{4\pi^2}{L^2} \right)^{\alpha-\gamma} |A^\gamma u|_{\delta,L} = L^{-\frac{d}{2}} \left( \frac{4\pi^2}{L^2} \right)^{\alpha-\frac{\delta}{2}} |u|_{\dot{H}^\delta(Q_L)} \quad (2.3) \]
for
\[ 2\alpha + \beta = 2\gamma + \delta, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}. \]
In view of the definition of the operator \( A \) we see that \( A^{\alpha+\beta} = A^\alpha \circ A^\beta \).
2.4. Weak solution to Navier-Stokes system

In this subsection we drop the precise control over constants, as it is superfluous here.

Let us denote by $B(a, b) = a \cdot \nabla b$. Take

$$f \in L^2(0, T; \dot{H}^{-1}_{\text{div}}(Q_L)), \quad u_0 \in \dot{H}^0_{\text{div}}(Q_L).$$

The first energy inequality motivates that $u$ solving $(NS_{f,u_0})$ belongs to

$$L^\infty(0, T; \dot{H}^0_{\text{div}}(Q_L)) \cap L^2(0, T; \dot{H}^1_{\text{div}}(Q_L)).$$

In particular $u(t) \in \dot{H}^1_{\text{div}}(Q_L)$ for a.a. $t \in [0, T]$. Consequently we have

$$A u(t) = -\Delta u(t) \in \dot{H}^{-1}_{\text{div}}$$

and

$$B(u(t), u(t)) \in \dot{H}^1_{\text{div}},$$

(2.4)

because (suboptimally)

$$\langle u(t) \cdot \nabla u(t), \varphi \rangle_{0,L} \leq C |u(t)|_{L^3(Q_L)} |u(t)|_{H^1(Q_L)} |\varphi|_{L^6(Q_L)}$$

(2.5)

for a sufficiently smooth, time-independent $\varphi$. Hence, taking $f(t) \in \dot{H}^{-1}_{\text{div}}(Q_L)$ and testing formally $(NS_{f,u_0})$ with divergence-free, smooth, $Q_L$-periodic $\varphi$, we get for a.a. $t \in [0, T]$

$$\langle u_{,t}(t), \varphi \rangle_{-1;1;L} = \langle f(t) - B(u(t), u(t)) - \nu A(u(t)), \varphi \rangle_{-1;1;L}$$

(2.6)

Therefore for a.a. $t \in [0, T]$ we have $u_{,t}(t) \in \dot{H}^{-1}_{\text{div}}(Q_L)$ thanks to (2.4). This motivates the following definition

**Definition 1.** (weak solution) Take

$$f \in L^2(0, T; \dot{H}^{-1}_{\text{div}}(Q_L)), \quad u_0 \in \dot{H}^0_{\text{div}}(Q_L).$$

We call

$$u \in L^\infty(0, T; \dot{H}^0_{\text{div}}(Q_L)) \cap L^2(0, T; \dot{H}^1_{\text{div}}(Q_L))$$

with the distributional time derivative $u_{,t} \in L^1(0, T; \dot{H}^{-1}_{\text{div}}(Q_L))$ the (variational) weak solution to $(NS_{f,u_0})$ iff the formula (2.6) holds for every test function $\varphi \in \dot{H}^1_{\text{div}}(Q_L)$ at almost every $t \in [0, T]$. The initial condition is attained in the $C_\omega(0, T; \dot{H}^0_{\text{div}}(Q_L))$ sense, namely

$$\langle u(t), \xi \rangle_{0,L} \xrightarrow{t \to 0} \langle u_0, \xi \rangle_{0,L} \quad \text{for any} \quad \xi \in \dot{H}^0_{\text{div}}(Q_L)$$

Let us comment the way in which the initial condition is met. The fact that time derivative $u_{,t} \in L^1(0, T; \dot{H}^{-1}_{\text{div}}(Q_L))$ implies that $u$ has a representative in $C([0, T], \dot{H}^0_{\text{div}}(Q_L))$. This information together with $u \in L^\infty(0, T; \dot{H}^0_{\text{div}}(Q_L))$ implies, in turn, $u \in C_\omega(0, T; \dot{H}^0_{\text{div}}(Q_L))$. For details, see for instance Lemmas 2.2.3 and 2.2.5 of [Pok]. This motivates the way in which we require the attainment of the initial condition in Definition 1.

It holds
Lemma 2. For any $T > 0$ there is a weak solution to $(NS_{f,u_0})$ that satisfies $u,t \in L^4(0,T; \dot{H}^{-1}_\text{div}(Q_L))$ and

$$\frac{1}{2} \frac{d}{dt} |u(t)|^2_{0,L} + \nu |u(t)|^2_{1,L} \leq \langle f(t), u(t) \rangle_{-1,1,L}$$

(2.7)

The proof can be found in Chapter 3 of [Tem]. Observe that higher time regularity of $u,$ provided by Lemma 2 can be justified by an optimal estimate of the l.h.s. of in (2.4), that uses the parabolic embedding that gives $u \in L^4(L^3).$ Inequality (2.7) is referred to as the energy inequality and a weak solution that obeys it is called a Leray-Hopf weak solution. It is not known if it is unique. If it is additionally in $L^r(L^s)$ with $\frac{3}{s} + \frac{2}{r} \leq 1,$ $s \in [3,\infty]$ (the Ladyzhenskaya-Prodi-Serrin class), it becomes unique and regular, see [Ser, Gal, ESŠ].

2.5. Imbeddings and interpolations

First we show a result needed in Section 4 to estimate the nonlinear term. Let us define

$$K_{(2.8)}(\alpha, L) = \frac{2\pi}{L} C_S(1 - \alpha) C_S(1),$$

$$K_{(2.9)}(\alpha, L) = \left( \frac{2\pi}{L} \right)^{\alpha-2} C_S \left( \alpha - \frac{1}{2} \right),$$

where we denote by $C_S(\beta)$ a numerical constant of the optimal $2\pi$-normalized three-dimensional Sobolev-Poincaré inequality for $\dot{H}^\beta(Q_{2\pi}),$ i.e.

$$|f|_{L^{\beta^*}(Q_{2\pi})} \leq C_S(\beta) |f|_{\beta,2\pi},$$

where $\beta \in [0,1]$, $\beta^* := \frac{6}{3-2\beta}$. (Recall that working here with null-mean-value functions, we do not need the lower order terms in the r.h.s. of the inequality above.)

Proposition 1. Take $\alpha \in [0,1]$. Then

$$\langle B(a,b), A^\alpha w \rangle_{0,L} \leq K_{(2.8)}(\alpha, L) |a|_{1,L} |\nabla b|_{L^{\frac{3}{\alpha}}(Q_L)} |A^{\frac{\alpha+1}{\alpha}} w|_{0,L},$$

(2.8)

$$|v|_{L^{\frac{3}{\alpha^*}}(Q_L)} \leq K_{(2.9)}(\alpha, L) |v|^{\frac{3}{\alpha-1,L}} |v|^{\frac{3}{2\alpha,L}},$$

(2.9)

provided the r.h.s.’s are meaningful.

Proof. First we perform the estimates for $L = 2\pi$ (where we drop the dependence on $Q_{2\pi}$) and next we rescale.

Step 1. (case $L = 2\pi$) The Hölder inequality gives

$$\langle B(a,b), A^\alpha w \rangle_{0,2\pi} \leq |a|_{L^6} |\nabla b|_{L^{\frac{3}{\alpha}}} |A^\alpha w|_{L^{\frac{3}{\alpha^*}}} \quad \text{for} \quad \alpha \in [0,2].$$
For \( \alpha \in [0, 1] \) we have
\[
|A^\alpha \omega|_{L^{\frac{4}{1+2\alpha}}} \leq C_S(1-\alpha)|A^\alpha \omega|_{1-\alpha,2\pi} = C_S(1-\alpha)|A^{\frac{\alpha+1}{2}}\omega|_{0,2\pi},
\] (2.10)
where the equality in (2.10) follows from (2.2). Combine the above two estimates to get via a Sobolev-Poincaré inequality
\[
\langle B(a, b), A^\alpha w \rangle_{0,2\pi} \leq C_S(1-\alpha)C_S(1)|a|_{1,2\pi} |\nabla b|_{L^{\frac{4}{1+2\alpha}}(Q_{2\pi})} |A^{\frac{\alpha+1}{2}}\omega|_{0,2\pi}
\] (2.11)
for \( \alpha \in [0, 1] \). Estimate (2.11) is the \( Q_{2\pi} \)-case of (2.8). Similarly we get \( Q_{2\pi} \)-case of (2.9), namely
\[
|v|_{L^{\frac{4}{1+2\alpha}}} \leq C_S \Big( \alpha - \frac{1}{2} \Big) |v|_{\alpha-\frac{1}{2},2\pi} = C_S \Big( \alpha - \frac{1}{2} \Big) |v|_{\alpha-1,2\pi} |v|_{\alpha,2\pi},
\] (2.12)
where the equality follows from an interpolation, where we have constant 1 in view of the definition of the norm \( | \cdot |_{s,L} \).

**Step 2.** (a general \( L \) by rescaling) For \( h : Q_L \rightarrow \mathbb{R}^3 \) denote its dilation \( h_{nL} : Q_{2\pi} \rightarrow \mathbb{R}^3 \) with \( \bar{h} \). Recall from subsection 2.1 that the Fourier coefficients of \( h \) and \( \bar{h} \) are identical. Hence \( |a|_{\beta,L} = |\bar{a}|_{\beta,2\pi} \), \( A^\beta w = \left( \frac{4\pi^2}{L^2} \right)^\beta \bar{A}^\beta \omega \). As \( \nabla = A^\frac{1}{2} \), we can write
\[
\langle B(a, b), A^\alpha w \rangle_{0,L} = \left( \frac{4\pi^2}{L^2} \right)^{\alpha+\frac{3}{2}} \langle B(\bar{a}, \bar{b}), A^\alpha \bar{w} \rangle_{0,2\pi} \leq \left( \frac{4\pi^2}{L^2} \right)^{\alpha+\frac{1}{2}} C_S(1-\alpha)C_S(1) |\bar{a}|_{1,2\pi} \nabla \bar{b}|_{L^{\frac{4}{1+2\alpha}}(Q_{2\pi})} |A^{\frac{\alpha+1}{2}}\omega|_{0,2\pi} =: \\
\left( \frac{4\pi^2}{L^2} \right)^{\alpha+\frac{1}{2}} C_S(1-\alpha)C_S(1) I,
\]
where the inequality follows from step 1. In order to scale back \( I \) to \( Q_L \), we need to know how Lebesgue norms behave under scaling. It holds
\[
|\nabla b|_{L^p(Q_L)} = \left( \frac{L}{2\pi} \right)^{\frac{3}{p}-1} |\nabla \bar{b}|_{L^p(Q_{2\pi})}.
\]
Taking this into consideration, we get
\[
I = \left( \frac{4\pi^2}{L^2} \right)^{-\alpha} |a|_{1,L} |\nabla b|_{L^{\frac{3}{p-1}}(Q_L)} |A^{\frac{\alpha+1}{2}}\omega|_{0,L}.
\]
Altogether the formulas that share \( I \) yield
\[
\langle B(a, b), A^\alpha w \rangle_{0,L} \leq \frac{2\pi}{L} C_S(1-\alpha)C_S(1) |a|_{1,L} |\nabla b|_{L^{\frac{3}{p-1}}(Q_L)} |A^{\frac{\alpha+1}{2}}\omega|_{0,L},
\]
which is (2.8). Analogously we get (2.9), as
\[
|v|_{L^{\frac{3}{p-1}}(Q_L)} = \left( \frac{L}{2\pi} \right)^{2-\alpha} |\bar{v}|_{L^{\frac{3}{p-1}}(Q_{2\pi})} \leq \left( \frac{L}{2\pi} \right)^{2-\alpha} C_S \Big( \alpha - \frac{1}{2} \Big) |\bar{v}|_{\alpha-\frac{1}{2},2\pi} |\bar{v}|_{\alpha,2\pi}.
\]

\( \Box \)

Next we present a result that facilitates the desired scaling-invariance of constants in Corollary 1. To formulate it, we use
Definition 2. \( C_I(\alpha, T) \) is the numerical constant of the following \( 2\pi \)-normalized interpolation inequality

\[
|\nabla f|_{L^4(0, T; L^\frac{3}{2\alpha}(Q_{2\pi}))} \leq C_I(\alpha, T) \|f\|_{L^\infty(0, T; |\cdot|_{|\alpha, 2\pi})}^{\frac{3}{4}} \|f\|_{L^2(0, T; |\cdot|_{1+\alpha, 2\pi})}^{\frac{1}{4}}
\]

The above interpolation holds for

\[
f \in L^\infty(0, T; \dot{H}^\alpha(Q_{2\pi})) \cap L^2(0, T; \dot{H}^{1+\alpha}(Q_{2\pi}))
\]

in view of

(i) The standard interpolation inequality for integro-differential norms.

(ii) The Poincaré inequality that allows us to write homogeneous Sobolev integro-differential seminorms in the r.h.s. of the interpolation inequality from (i). (Elements of \( \dot{H}^\beta \) can be identified with those of \( H^\beta \) that have null mean value).

(iii) The equivalence of integro-differential and Fourier-based norms, compare (2.3).

Rescaling the above interpolation formula, we obtain

Proposition 2. Take \( u \in L^\infty(0, T; \dot{H}^\alpha(Q_L)) \cap L^2(0, T; \dot{H}^{1+\alpha}(Q_L)) \). Then

\[
|\nabla u|_{L^4(0, T; L^\frac{3}{2\alpha}(Q_{2\pi}))} \leq C_I(\alpha, T) \left( \frac{4\pi^2}{L^2} \right)^{\frac{\alpha-1}{2}} |u|_{L^\infty(0, T; |\cdot|_{|\alpha, L})}^{\theta(\alpha)} |u|_{L^2(0, T; |\cdot|_{1+\alpha, L})}^{1-\theta(\alpha)}
\]

3. Stability

In this section we prove Theorem 1. Recall that we work with the given \( T_* > 0 \) and \( \alpha \)-strong solution \( u \) to \((\text{NS}_f, u_0)\) that exists on \([0, T_*)\) as well as a Leray-Hopf weak solution \( v \) to \((\text{NS}_g, v_0)\). The system for the difference \( w := u - v \) reads

\[
\begin{align*}
\begin{cases}
w_t + \nu Aw + B(w, u) + B(u, w) - B(w, w) = h \\
\text{div}w = 0 \\
w(0) = u_0 - v_0
\end{cases}
\end{align*}
\]

(3.1)

In the subsection 3.1 we will derive higher order estimates for (3.1) (more precisely, \( \alpha \)-order estimates). Next, we conclude the proof via a blowup argument in subsection 3.2.
3.1. Energy estimates

We are going to test (3.1) with $A^\alpha w$. Let us first comment on rigorousness of our estimates. We restrict ourselves to the time interval $[0, T^*(g, v_0) \wedge T_*)$ where $T^*(g, v_0)$ denotes the blowup time of the $\alpha$-strong solution to $(NS_{g,v_0})$, given by Theorem 2. This solution coincides on the interval $[0, T^*(g, v_0))$ with the interesting for us Leray-Hopf weak solution $v$, again thanks to Theorem 2 (its uniqueness part). Hence

$$w = u - v \in L^2(0, T^*(g, v_0) \wedge T_*; \dot{H}^{1+\alpha}(Q_L)).$$

Consequently

(i) $A^\alpha w$ is admissible as a test function to $\nu A w$ in (3.1).

We have also that

$$A^{\frac{\alpha}{2}}w, t = \nabla^\alpha w, t \in L^2(0, T^*(g, v_0) \wedge T_*; \dot{H}^{-1}_{\text{div}}(Q_L))$$

for $\alpha \geq 1/2$, compare (4.11). This and the already known

$$A^{\frac{\alpha}{2}}w = \nabla^\alpha w \in L^2(0, T^*(g, v_0) \wedge T_*; \dot{H}^1(Q_L))$$

allows us to write

$$\langle A^{\frac{\alpha}{2}}w, t (t), A^{\frac{\alpha}{2}}w(t) \rangle_{-1,1;L} = \frac{1}{2} \frac{d}{dt} |A^{\frac{\alpha}{2}}w|_{0,L}^2(t).$$

Identity (3.2) is the Fourier-series version of the known integro-differential formula for a generalized differentiation of a product, compare for instance Lemma 2.2.5 of [Pok].

(ii) By (3.2) we have justified the admissibility of $A^\alpha w$ as a test function to the evolutionary part of (3.1).

Let us remark that the above justification works well only for $\alpha$-order estimates for $\alpha \geq 1/2$. Otherwise we do not have enough information on $w, t$ to use the duality formula (3.2). For instance for $\alpha = 0$ one has $u, v \in L^2(\dot{H}^1)$ and $u, t, v, t \in L^4(\dot{H}^{-1})$ (see Lemma 1). In order to have the duality formula (3.2) with such low regularity of the time derivative, we would need to assume $u, v \in L^4(\dot{H}^1)$ (which is already well within the Ladyzhenskaya-Prdii-Serrin class). One can in this case justify the estimates differently, see [CDGG], proof of Theorem 3.3, especially pages 61–63.

(iii) Finally, testing the nonlinear and forcing terms of (3.1) with $A^\alpha w$ is admissible. It can be seen directly in the estimates (3.4)–(3.7) below.

Now we proceed with estimates. Testing (3.1) with $A^\alpha w$ we get at a.a. $t \in [0, T^*(g, v_0) \wedge T_*)$

$$\frac{1}{2} \frac{d}{dt} |A^{\frac{\alpha}{2}}w|_{0,L}^2 + \nu |A^{\frac{\alpha+1}{2}}w|_{0,L}^2 \leq |\langle h, A^\alpha w \rangle_{-1,1;L}|$$

$$+ |\langle B(w, u) + B(u, w) - B(w, w), A^\alpha w \rangle_{-1,1;L}|$$

$$\leq \frac{1}{2} \frac{d}{dt} |A^{\frac{\alpha}{2}}w|_{0,L}^2 + \nu |A^{\frac{\alpha+1}{2}}w|_{0,L}^2.$$
We estimate the force term as follows

\[ |\langle h, A^\alpha w \rangle_{-1,1;L} | = |\langle A^\frac{\alpha-1}{2} h, A^\frac{\alpha+1}{2} w \rangle_{0,0;L} | \leq \varepsilon_2 |A^\frac{\alpha+1}{2} w|^2_{0,L} + \frac{1}{4\varepsilon_2} |A^\frac{\alpha-1}{2} h|^2_{0,L}. \]  

(3.4)

To control the nonlinear terms we use (2.8) of Proposition 1 and get

\[ I := |\langle B(w,u), A^\alpha w \rangle_{-1,1;L} | \leq K(2.8)(\alpha, L)|w|_{1,L} |\nabla u|_{L^2,\nu}(Q_L) |A^\frac{\alpha+1}{2} w|^2_{0,L} \]

\[ \leq K(2.8)(\alpha, L)|w|^\frac{1}{2}_{1-\alpha,L} |w|^\frac{1}{2}_{1+\alpha,L} |\nabla u|_{L^2,\nu}(Q_L) |A^\frac{\alpha+1}{2} w|^2_{0,L} \]

\[ \leq K(2.8)(\alpha, L) \left( \frac{4\pi^2}{L^2} \right)^{\frac{3-\alpha}{2}} |w|^\frac{1}{2}_{1-\alpha,L} |\nabla u|_{L^2,\nu}(Q_L) |A^\frac{\alpha+1}{2} w|^2_{0,L} \]  

(3.5)

In (3.5) we use also interpolation of \(| \cdot |_{1} \) between \(| \cdot |_{1-\alpha} | \cdot |_{1+\alpha} \) with constant 1 (which follows from the definition of \(| \cdot |_{s,L} \) and the Cauchy-Schwarz inequality) and (2.2). Observe that the term containing \(u\) is finite for a.e. \( t \) thanks to Proposition 2. Similarly

\[ II := |\langle B(u,w), A^\alpha w \rangle_{-1,1;L} | \]

\[ \leq K(2.8)(\alpha, L)K(2.9)(\alpha, L)|w|_{1,L} |\nabla w|_{\alpha-1,L} |\nabla w|_{\alpha,L} |A^\frac{\alpha+1}{2} w|^2_{0,L} \]

\[ = K(2.8)(\alpha, L)K(2.9)(\alpha, L) \left( \frac{4\pi^2}{L^2} \right)^{\frac{3-\alpha}{2}} |w|^\frac{1}{2}_{1,L} |\nabla w|_{\alpha,L} |A^\frac{\alpha+1}{2} w|^2_{0,L} \]

\[ \leq K(2.8)(\alpha, L)K(2.9)(\alpha, L) \left( \frac{4\pi^2}{L^2} \right)^{\frac{1-\alpha}{2}} |w|^\frac{1}{2}_{1-\alpha,L} |w|_{\alpha,L} |A^\frac{\alpha+1}{2} w|^2_{0,L} \]  

(3.6)

for the equality above we use \( A^{\frac{1}{2}} = \nabla \) and (2.2) and again (2.2) for the last inequality.

We begin the estimate of the last nonlinear part by using (3.6) with \( u := w \)

\[ III := |\langle B(w,w), A^\alpha w \rangle_{-1,1;L} | \]

\[ \leq K(2.8)(\alpha, L)K(2.9)(\alpha, L) \left( \frac{4\pi^2}{L^2} \right)^{\frac{3-\alpha}{2}} |w|^\frac{1}{2}_{1,L} |w|^\frac{1}{2}_{\alpha,L} |A^\frac{\alpha+1}{2} w|^2_{0,L} \]  

(3.7)

where for the second inequality, we interpolate \(|w|_{1,L} \leq |w|^\frac{1}{2}_{1-\alpha,L} |w|^\frac{1}{2}_{1+\alpha,L} \) and use (2.2).

Estimates (3.4)–(3.7) in (3.3) give

\[
\frac{1}{2} \frac{d}{dt} |A^{\frac{\alpha}{2}} w|^2_{0,L} + \nu |A^{\frac{\alpha+1}{2}} w|^2_{0,L} \leq \varepsilon_2 |A^{\frac{\alpha+1}{2}} w|^2_{0,L} + \frac{1}{4\varepsilon_2} |A^{\frac{\alpha-1}{2}} h|^2_{0,L} \\
+ K(2.8)(\alpha, L) |A^{\frac{\alpha+1}{2}} w|^3_{0,L} \left[ \left( \frac{4\pi^2}{L^2} \right)^{\frac{3-\alpha}{2}} |w|^\frac{1}{2}_{1-\alpha,L} |\nabla u|_{L^2,\nu}(Q_L) \right] \\
+ K(2.9)(\alpha, L) \left( \frac{4\pi^2}{L^2} \right)^{\frac{3-\alpha}{2}} |u|^\frac{1}{2}_{1,L} |w|^\frac{1}{2}_{\alpha,L} \\
+ K(2.9)(\alpha, L) \left( \frac{4\pi^2}{L^2} \right)^{\frac{1-\alpha}{2}} |w|^\frac{1}{2}_{1-\alpha,L} |w|^\frac{1}{2}_{\alpha,L} |A^{\frac{\alpha+1}{2}} w|^2_{0,L} 
\]

(3.8)
In (3.8) we need the restriction $\alpha \in [1/2, 1]$, because we have used Proposition 1. Observe that the last summand of (3.8) gives the critically growing term $|A^{\alpha+1} w|_{0,L}^2$. Therefore it may seem more natural to stop estimate (3.7) for $\|w\|_{\alpha,L} \leq 1$, but we prefer to keep the energy estimate in the form (3.8) and work for the entire range $\alpha \in [1/2, 1]$.

In the last-but-one term on the r.h.s. of (3.8) we use

$$|u|_{1,L} = \left| L^{-\frac{d}{2}} \nabla u \right|_{L^2(Q_L)} \leq \left| L^{-\frac{d}{2}} L^{\frac{d(2\alpha-1)}{6}} \nabla u \right|_{L^{2-\alpha}(Q_L)},$$

which follows from (2.3) and the Hölder inequality. We use also $|f|_{1-\alpha,L} \leq |f|_{\alpha,L}$ holding for $\alpha \geq 1/2$ to obtain from (3.8) via (2.2)

$$\frac{d}{2} |A^{\frac{d}{2}} w|_{0,L}^2 + \nu |A^{\frac{\alpha+1}{2}} w|_{0,L}^2 \leq \varepsilon_2 |A^{\frac{\alpha+1}{2}} w|_{0,L}^2 + \frac{1}{4 \varepsilon_2} |A^{\frac{\alpha+1}{2}} h|_{0,L}^2$$

$$+ K_{(2.8)}(\alpha, L) |A^{\frac{\alpha+1}{2}} w|_{0,L}^\frac{3}{4} |w|_{\alpha,L} \nabla u \left|_{L^{2-\alpha}(Q_L)} \right|$$

Expressing above all the norms of fractional derivatives with the norms of a respective homogenus Sobolev space via (2.2) we arrive at

$$\frac{d}{2} |w|_{\alpha,L}^2 + \nu |A^{\frac{\alpha}{2}} w|_{\alpha+1,L}^2 \leq \varepsilon_2 |A^{\frac{\alpha}{2}} w|_{\alpha+1,L}^2 + \frac{1}{4 \varepsilon_2} \left( \frac{4 \pi^2}{L^2} \right)^{\frac{\alpha}{4}} |h|_{\alpha-1,L}^2$$

$$+ \left( \frac{4 \pi^2}{L^2} \right)^{\frac{3-\alpha}{4}} |w|_{\alpha+1,L}^\frac{1}{2} |w|_{\alpha,L}^\frac{1}{2} \nabla u \left|_{L^{2-\alpha}(Q_L)} \right|$$

$$\times K_{(2.8)}(\alpha, L) \left[ \left( \frac{4 \pi^2}{L^2} \right)^{\frac{1-\alpha}{4}} + K_{(2.9)}(\alpha, L) L^{\frac{d(\alpha-2)}{4}} \left( \frac{4 \pi^2}{L^2} \right)^{\frac{\alpha}{4}} \right]$$

$$+ K_{(2.8)}(\alpha, L) \times K_{(2.9)}(\alpha, L) \left( \frac{4 \pi^2}{L^2} \right)^{2-\frac{\alpha}{4}} |w|_{\alpha,L}^2 |w|_{1+\alpha,L}^2.$$
The Young inequality
\[ Y^{\frac{3}{4}} c X^{\frac{1}{4}} \leq \varepsilon_1 Y + \varepsilon_1^{-3} \frac{27}{256} c^4 X \]
used in the third term of the preceding inequality allows us to write
\[ \dot{X} + (\nu - \varepsilon_1 - \varepsilon_2) Y \leq K_4 H + K_3 X U + K_2 X^{\frac{1}{2}} Y, \tag{3.9} \]
where
\[ K_2 = \sqrt{2} K_{(2,8)}(\alpha, L) K_{(2,9)}(\alpha, L) \left( \frac{4\pi^2}{L^2} \right)^{1-\frac{4}{7}}, \]
\[ K_3 = \varepsilon_1^{-3} \frac{27}{128} K_{(2,8)}(\alpha, L) \left( \frac{4\pi^2}{L^2} \right)^{-2\alpha + 1} \left[ 1 + K_{(2,9)}(\alpha, L) L^{d(\alpha - 2) - 4} \left( \frac{4\pi^2}{L^2} \right)^4 \right], \]
\[ K_4 = \frac{1}{4\varepsilon_2} \left( \frac{4\pi^2}{L^2} \right)^{-1}. \]
The above choices agree with the definition of \( K_2, K_3, K_4 \) in subsection 1.2. To see this, consider the formulas for \( K_{(2,8)}(\alpha, L), K_{(2,9)}(\alpha, L) \) as in subsection 2.5. The ODI (3.9) will give us stability via a blowup argument.

### 3.2. Proof of Theorem 1 via the blowup argument

Recall that assumptions of Theorem 1 fix a positive \( T \) that satisfies \( T < T_* \), where \( T_* \) is the given time of existence of the reference \( \alpha \)-strong solution \( u \). The proximity assumption (A1) reads
\[ \left( |u_0 - v_0|^2_{\alpha, L} + K_4 \int_0^T |f - g|^2_{\alpha - 1, L}(t) dt \right) e^{K_3 \int_0^T |\nabla u(t)|^4_{L^{\frac{4}{\alpha - \alpha}}(\Omega_L)} dt} < \left( \frac{\bar{\nu}}{K_2} \right)^2, \]
where \( \bar{\nu} \) is any positive number that satisfies \( \bar{\nu} < \nu - \varepsilon_1 - \varepsilon_2 \).

**Step 1.** (lower bound for \( T_*(g, v_0) \)) We show that
\[ T_*(g, v_0) > T. \]
Assume the contrary \( T_*(v) \leq T \ (\ < T_*(u)) \). We define
\[ \mu := \nu - \varepsilon_1 - \varepsilon_2 - \bar{\nu}, \]
positive by our assumptions. Hence the proximity assumption (A1) gives
\[ X(0) < \left( \frac{\bar{\nu}}{K_2} \right)^2 = \left( \frac{\nu - \varepsilon_1 - \varepsilon_2 - \mu}{K_2} \right)^2. \]
We have the following alternative

(i) either $X(t) \leq \left(\frac{\nu - \varepsilon_1 - \varepsilon_2 - \mu}{K_2} \right)^2$ for $t \in [0, T_*(g, v_0))$

(ii) or $X(t)$ exceeds $\left(\frac{\nu - \varepsilon_1 - \varepsilon_2 - \mu}{K_2} \right)^2$ on $[0, T_*(g, v_0))$. Thanks to continuity of $X$ on $[0, T_*(v))$ and fact that it starts below $\left(\frac{\nu - \varepsilon_1 - \varepsilon_2 - \mu}{K_2} \right)^2$, there exists the smallest, positive time $\bar{t} \in (0, T_*(v))$ such that $X(\bar{t}) = \left(\frac{\nu - \varepsilon_1 - \varepsilon_2 - \mu}{K_2} \right)^2$.

Keeping this in mind, observe that ODI (3.9) reads

$$\dot{X} + \left(\nu - \varepsilon_1 - \varepsilon_2 - K_2X^2\right)Y \leq K_3XU + K_4H.$$  (3.10)

It gives for almost any $t \leq T_*(g, v_0)$ (case (i)) or for almost any $t \leq \bar{t}$ (case (ii)) that $\dot{X} + \mu Y \leq K_3XU + K_4H$. Consequently

$$X(t) + \mu \int_0^t Y \leq \left( X(0) + K_4 \int_0^t H \right) e^{K_3 \int_0^t U(s) ds} \leq \left( X(0) + K_4 \int_0^T H \right) e^{K_3 \int_0^T U(s) ds} < \left( \frac{\tilde{\nu}}{K_2} \right)^2,$$

where the third inequality comes from our assumption (A1).

In the case (i) we drop the first summand of the l.h.s. of (3.10), so $\mu \int_0^t Y < \left( \frac{\tilde{\nu}}{K_2} \right)^2$ for any $t < T_*(g, v_0)$, hence

$$\int_0^{T_*(g,v_0)} Y \leq \frac{1}{\mu} \left( \frac{\tilde{\nu}}{K_2} \right)^2 < +\infty.$$ As also in case (i) one assumes that the continuous $X(t) \leq \left( \frac{\tilde{\nu}}{K_2} \right)^2$ on $[0, T_*(v))$, then $T_*(v)$ can not be a blowup time.

In the case (ii) we have $X(\bar{t}) < \left( \frac{\tilde{\nu}}{K_2} \right)^2$ from (3.10), but $X(\bar{t}) = \left( \frac{\tilde{\nu}}{K_2} \right)^2$ here, which is a contradiction.

As neither (i) nor (ii) can hold, we have contradicted $T_*(g, v_0) \leq T$.

**Step 2.** (proximity estimate) We already know that $T_*(g, v_0) > T$. Therefore we rewrite the alternative from the previous step, writing there $T$ in place of $T_*(g, v_0)$. Case (ii) is again a contradiction, so (3.10) holds, for any $t < T$. We know that $T < T^*(g, v_0)$, so we can take $t \to T$ in (3.10). This gives (1.1).}

### 3.2. Proof of Corollary 1

It follows from Proposition 2 in subsection 2.5 and (A1).

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4. Appendix

Here we prove our theorems on existence of strong solutions to \((NS_{f,u_0})\).

4.1. Proof of Theorem 2

This theorem serves as an auxiliary result for our main Theorem 1, therefore its proof has been postponed until now. Nevertheless, the approach for proving both Theorem 1 and Theorem 2 is similar. In the former case we had the blowup argument basing on the reference solution \(u\). Now, the approximate solution \(u^m\) plays the role of a reference. As we are already familiar with the method and we follow closely the proof of Theorem 3.5 in [CDGG] (except for the step 5), we omit some details in the considerations below.

**Step 1.** (splitting the initial data) We denote by

\[ P_{k_0} : \dot{H}^0_{\text{div}}(Q) \mapsto \text{span}\{e^{2\pi ik \cdot x} : k \leq k_0\} \]

the projection on the low-frequency space; \(k_0\) will be fixed later. Let us decompose the initial datum \(u_0\) into \(u_0^{Lo} = P_{k_0}u_0\) and \(u_0^{Hi} = u_0 - P_{k_0}u_0\). Let the former evolve with the homogenous Stokes, which degenerates in our setting to the homogenous heat (compare subsection 2.2), i.e. to

\[
\begin{cases}
  u^{Lo}_t + \nu Au^{Lo} = 0, \\
  u^{Lo}(0) = u_0^{Lo}.
\end{cases}
\]  

(4.1)

Observe that for a low frequency data \(u_0^{Lo}\) the above Stokes problem admits an exact finite Fourier series solution \(u^{Lo}\) that belongs to \(P_{k_0}(\dot{H}^0_{\text{div}})\). Take any \(m \geq k_0\) and \(u^m\) to be the \(m\)-th Fourier series approximation of \((NS)_{f,u_0}\). Then \(u^m - u^{Lo} := u^{m,Hi}\) solves

\[
\begin{cases}
  u^{m,Hi}_t + \nu \Delta u^{m,Hi} + P_m[(B(u^{m,Hi},u^{m,Hi}) + B(u^{m,Hi},u^{Lo}) + B(u^{Lo},u^{m,Hi})] = F \\
  \nabla \cdot u^{m,Hi} = 0 \\
  u^{m,Hi}(0) = P_m(u_0^{Hi}) = P_m u_0 - P_{k_0} u_0.
\end{cases}
\]  

(4.2)

with \(F = P_m f - P_m[B(u^{Lo},u^{Lo})]\).

**Step 2.** (derivation of ODI) System (4.2) is formally equivalent to (3.1) with \(P_m B\) in place of \(B\) and \(w := u^{m,Hi}, \ u := u^{Lo}, \ f := F\). Our ”eigenvalue” definition of \(A^\alpha\) reduces testing (4.2) with \(A^\alpha u^{m,Hi}\) to multiplication a system of ODEs by

\[
\sum_{k \in \mathbb{Z}^d, |k| \leq m} \lambda_k^m u^{m,Hi}_k \cdot \omega_{L,k},
\]
hence the estimates of subsection 3.1 are justified also for (4.2) Consequently, along lines of Section 3 we obtain an analogue of the ODI (3.9)

$$\dot{X}_m + (\nu - \varepsilon_1 - \varepsilon_2)Y_m \leq K_2X^2_mY_m + K_3X_mU + K_4H_m$$

with

$$X_m := \frac{1}{2}|u^{m,H_i}|^2_{\alpha,L}, \quad Y_m := \frac{4\pi^2}{L^2}|u^{m,H_i}|^2_{\alpha+1,L},$$

$$U := |\nabla u^{Lo}|^4_{L^{\frac{4}{3}}(Q_L)}, \quad H_m := |P_m f|^2_{-1,L} + K_5(u^{Lo})X^\frac{1}{2}_m,$$

with

$$K_5(u^{Lo}) = \sqrt{2}\left(\frac{4\pi^2}{L^2}\right)^\alpha |u^{Lo} \otimes u^{Lo}|_{1+\alpha,L},$$

where the $K_5(u^{Lo})$ term of the forcing $H_m$ follows from $P_m[B(u^{Lo}, u^{Lo})]$ part of $F$. Namely, we have

$$|\langle P_m[B(u^{Lo}, u^{Lo})], A^{\alpha}u^{m,H_i}\rangle| \leq |\text{div}(u^{Lo} \otimes u^{Lo})|_{\alpha,L}|A^{\alpha}u^{m,H_i}|_{-\alpha,L}$$

We use above the fact that div is the (scalar) multiplication with $k$ and (2.2) to get

$$|\langle P_m[B(u^{Lo}, u^{Lo})], A^{\alpha}u^{m,H_i}\rangle| \leq \left(\frac{4\pi^2}{L^2}\right)^\alpha |u^{Lo} \otimes u^{Lo}|_{1+\alpha,L}|u^{m,H_i}|_{\alpha,L},$$

hence the $C(u^{Lo})$ term in $H_m$. We rewrite our ODI as

$$\dot{X}_m + (\nu - \varepsilon_1 - \varepsilon_2 - K_2X^\frac{1}{2}_m)Y_m \leq X_m[K_3U + \delta] + \frac{1}{4\delta}K^2_5(u^{Lo}) \tag{4.3}$$

**Step 3.** (gaining the smallness) Observe that $u^{Lo}$ solves the linear heat system, so we have the full control on $U$ and $C(u^{Lo})$ in terms of $u^{Lo}$. Thanks to splitting $u^m$ into $u^{m,H_i}$ and $u^{Lo}$ we can now gain smallness of $X_m(0) = \frac{1}{2}[P_m u_0 - P_{k_0} u_0]^2_{\alpha,L}$. Namely, let us fix a positive $\mu < \nu - \varepsilon_1 - \varepsilon_2$ and choose $k_0$ large enough that for any $m \geq k_0$ holds

$$\nu - \varepsilon_1 - \varepsilon_2 - K_2X^\frac{1}{2}_m(0) \geq \mu \Leftrightarrow X_m(0) \leq \left(\frac{\nu - \varepsilon_1 - \varepsilon_2 - \mu}{K_2}\right)^2 \tag{4.4}$$

**Step 4.** $(m$-uniform $\alpha$-regularity bound) Fix any $\sigma \in (0, 1)$. By the time continuity of $X_m$ there exists $T_m$ such that $\nu - \varepsilon_1 - \varepsilon_2 - K_2X^\frac{1}{2}_m(t) \geq \sigma \mu$ for $t \in [0, T_m]$. It gives in (4.3)

$$\dot{X}_m + \sigma Y_m \leq X_m[K_3U + \delta] + \frac{1}{4\delta}K^2_5(u^{Lo}),$$

for $t \leq T_m$, i.e.

$$X_m(t) + \sigma \int_0^t Y_m \leq \left(X_m(0) + \frac{1}{4\delta} \int_0^t K^2_5(u^{Lo}) d\tau\right) e^{\int_0^t (K_3U(t) + \delta) d\tau} \tag{4.5}$$
As \( X_m(0) \leq \left( \frac{\nu - \varepsilon_1 - \varepsilon_2 - \mu}{K_2} \right)^2 \), there exists \( T_0 > 0 \) such that r.h.s. of (4.5) stays below \( \left( \frac{\nu - \varepsilon_1 - \varepsilon_2 - \sigma\mu}{K_2} \right)^2 \) for small enough \( t \leq T_0 \). Consequently

\[
X_m(t) \leq \left( \frac{\nu - \varepsilon_1 - \varepsilon_2 - \sigma\mu}{K_2} \right)^2 \Leftrightarrow \nu - \varepsilon_1 - \varepsilon_2 - K_2 X_m^\frac{1}{2}(t) \geq \sigma\mu \quad (4.6)
\]

for \( t \leq T_0 \) and independently from \( m \).

We use (4.6) in (4.3) to conclude that (4.5) holds for \( t \leq T_0 \) uniformly in \( m \). Thus we have an additional \( L^\infty(\dot{H}^\alpha) \cap L^2(\dot{H}^{1+\alpha}) \) estimate for \( u_m^{\dot{H}} \), hence \( u_m \), uniform for \( m \geq k_0 \).

**Step 5.** (time-continuity \( C(\dot{H}^\alpha) \)). In this step we do not need the precise control over constants. We divert from [CDGG] and use the following duality estimate for the weak solution (2.5) to (NS)

\[
\left| \int_0^T \langle u_{,t}, \varphi \rangle_{-1,1;L} \right| \leq C \int_0^T \left( |f|_{\alpha-1,1} \varphi_{1-\alpha,L} + |\nabla u|_{\alpha,L} |\nabla \varphi|_{-\alpha,L} + |\text{div}(u \otimes u)|_{\alpha-1,1} \varphi_{1-\alpha,L} \right). \quad (4.7)
\]

We estimate the last summand above by

\[
\int_0^T |u \otimes u|_{\alpha,L} \varphi_{1-\alpha,L} \leq \left( \int_0^T \int_{Q_L} |\nabla^\alpha u|^{\frac{10}{5}} \right)^\frac{3}{5} \left( \int_0^T \int_{Q_L} |u|^5 \right)^\frac{2}{5} |\varphi|_{L^2(|\cdot|_{1-\alpha})} \quad (4.8)
\]

Plugging (4.8) into (4.7) gives

\[
\left| \int_0^T \langle u_{,t}, \varphi \rangle_{-1,1;L} \right| \leq |\varphi|_{L^2(|\cdot|_{1-\alpha})} \left[ |f|_{L^2(|\cdot|_{\alpha-1})} + |u|_{L^2(|\cdot|_{1+\alpha})} + |\nabla^\alpha u|_{L^{\frac{10}{5}}(\dot{H}^\frac{10}{5})} \right] |\varphi|_{L^5(L^5)}. \quad (4.9)
\]

The norms on the r.h.s. of (4.9) are finite thanks to our assumptions (for \( f \)) and \( L^\infty(\dot{H}^\alpha) \cap L^2(\dot{H}^{1+\alpha}) \) estimate from previous steps (for \( u \)). Specifically, the last summand in (4.9) is finite by parabolic embedding (\( L^{\frac{10}{5}} \) norm) and by interpolation

\[
L^\infty(\dot{H}^\alpha) \cap L^2(\dot{H}^{1+\alpha}) \hookrightarrow L^5(\dot{H}^{\alpha+\frac{2}{5}}) \hookrightarrow L^5(L^5)
\]

for \( \alpha \geq 1/2 \). Hence (4.9) means that

\[
u_{,t} \in (L^2(\dot{H}^{1-\alpha}_{\text{div}}))^* = L^2(\dot{H}^{\alpha-1}_{\text{div}}).
\]

This information interpolated (in the sense of ‘espaces des traces’, see for instance Lemma 2.2.4 in [Pok]) with \( u \in L^2(\dot{H}^{\alpha+1}_{\text{div}}) \) gives \( u \in C(\dot{H}^\alpha) \).
**Step 6.** (uniqueness) The $L^\infty(\dot{H}^\alpha)$ regularity implies for $\alpha \geq 1/2$ that we are in the Prodi-Serrin class, where Leray-Hopf solutions are unique.

**Step 7.** (blowup criterion) Take any $T$ such that the $\alpha$-energy norm, as formulated in Theorem 2, stays finite, i.e.

$$u \in C([0, T]; \dot{H}^\alpha) \cap L^2([0, T]; \dot{H}^{1+\alpha}).$$

Then in view of steps 1–6 there exists the unique (in Leray-Hopf class)

$$C([T, T_1], \dot{H}^{\alpha}_{\text{div}}) \cap L^2([T, T_1], \dot{H}^{1+\alpha}_{\text{div}}),$$

solution to $(NS_{f, u(T)})$ and it satisfies for a.a. $t \in [0, T_1]$ the weak formulation (2.6). Therefore it is a weak solution on $[0, T_1)$ to $(NS_{f, u_0})$. Hence $T$ is not the maximal existence time. □

4.2. Proof of Lemma 1

Here we prove the caloric characterization of $T^*_L(f, u_0)$. From (4.5) in the step 4 of the proof of Theorem 2 we see that any $T_0 > 0$ that yields

$$\left(\left(\frac{\nu - \varepsilon_1 - \varepsilon_2 - \mu}{K_2}\right)^2 + \frac{1}{4\delta} \int_0^{T_0} K^2_3(u^{L_0}(\tau))d\tau\right)e^{\int_0^{T_0}(K_3^0(U(\tau)+\delta)d\tau} \leq e^{\int_0^{T_0}(K_3^0(U(\tau)+\delta)d\tau} \leq \left(\frac{\nu - \varepsilon_1 - \varepsilon_2 - \sigma\mu}{K_2}\right)^2$$

(4.10)

which clarifies (A3), after one considers the formulas

$$K^2_3(u^{L_0}) = 2\left(\frac{4\pi^2}{L^2}\right)^{2\alpha} |u^{L_0} \otimes u^{L_0}|^2_{1+\alpha, L},$$

$$U := |\nabla u^{L_0}(\tau)|^4_{L^{2+\alpha}(Q_L)},$$

as in the step 4 of the proof of Theorem 2. The remaining to prove bound follows from (4.11). □
4.3. Proof of Theorem 3

Here we one in easier situation than when proving Theorem 2, because we do not have to split the initial data to gain smallness, as it is already assumed. Hence we get for the Fourier approximations $u^m$

$$\dot{X}_m + (\nu - \varepsilon_1 - \varepsilon_2 - K_2 X_m^\frac{1}{2}) Y_m \leq K_4 H$$

with

$$X_m := \frac{1}{2} |u^m|_{\alpha,L}^2, \quad Y_m := \frac{4\pi^2}{L^2} |u^m|_{\alpha+1,L}^2, \quad H := |f|_{\alpha-1,L}^2,$$

compare the computations that provided (4.3) in step 2 of the proof of Theorem 2. We finish the proof as the proof of our stability result via the blowup argument, compare subsection 3.1. In fact Theorem 3 can be seen also as a stability result with null initial data and null reference solution $u$ in (A1).

Concluding remarks

References

[CDGG] Chemin J. Desjardins B., Gallagher I., Grenier E.; Mathematical geophysics, Oxford 2006.

[CCRT] Chernyshenko S., Constantin P., Robinson J., Titi E., A priori regularity of the three-dimensional Navier-Stokes from numerical computations, Math. Phys. 48 (2007), no. 6, 065204.

[CF] Constantin P., Foias C., Navier-Stokes Equations, Chicago 1988.

[ESS] Escauriaza L.; Seregin G., Šverak V.; $L^{3, \infty}$-solutions of Navier-Stokes equations and backward uniqueness, Uspekhi Mat. Nauk 58 (2003), no 2 (350), 3–44.

[Gal] Galdi G.; An introduction to the Navier-Stokes initial boundary value problem, Birkhäuser, 2000.

[M-RSS] Marin-Rubio P., Robinson J., Sadowski W.; Solutions of the 3D Navier-Stokes equations for initial data in $H^{\frac{1}{4}}$; robustness of regularity for bounded sets of initial data in $H^{1}$, J.M.A.A. 400 (2013), no 1, 76–85.

[Pok] Pokorný M.; Navier-Stokes system.

[Ser] Serrin J.; The initial value problem for the Navier-Stokes system.

[Tem] Temam R.; Navier-Stokes Equations and Nonlinear Functional Analysis, Second edition, SIAM, 1995.
[Za1] Zajączkowski W. M., *Global special regular solutions to the NSE in cylindrical domains under boundary slip conditions*, Gakuto Internat. Ser. Math. Sc. Appl. 21 (2004).

[Za2] Zajączkowski W. M., *Global special regular solutions to the NSE in axially symm domains under boundary slip conditions*.

[Za3] Zajączkowski W. M., *Long time existence of regular solutions to NSE in cylindrical domains under boundary slip condition*, Studia Math. 169 (2005), 243–285.

[Za4] Zajączkowski W. M., *Global regular nonstationary flow for the NSE in a cylindrical pipe*, TMNA 26 (2005), 221–286.

[Za5] Zajączkowski W. M., *Global regular solutions to the NSE in a cylinder*, Banach Center Publ. 74 (2006), 235–255.

[Za6] Zajączkowski W. M., Nowakowski B., *Global existence of solution for NSE in cylindrical domains*, Appl. Math. 36 (2009), 169–182.

[Za7] Zajączkowski W. M., *On global regular solutions to the NSE in cylindrical domains*, TMNA 37 (2001), 55–85.

[Za8] Zajączkowski W. M., Wiegner M., *On stability of axially symmetric solutions to the NSE in cylindrical domains and with boundary slip conditions*.

[Za9] Zajączkowski W. M., *Some global regular solutions to NSE*, Math. Meth. Appl. Sc. 30 (2006), 123–151.

[Za10] Zajączkowski W. M., Zadrzyńska E., *Global regular solutions with large swirl to the NSE in a cylinder*, J. Math. Fluid Mech. 11 (2009), 126–169.