Algebra of Observables for Identical Particles in One Dimension

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Abstract

The algebra of observables for identical particles on a line is formulated starting from postulated basic commutation relations. A realization of this algebra in the Calogero model was previously known. New realizations are presented here in terms of differentiation operators and in terms of SU($N$)-invariant observables of the Hermitian matrix models. Some particular structure properties of the algebra are briefly discussed.

1 Introduction

The idea of the Heisenberg quantization of identical particles is to identify a fundamental \textit{algebra of observables} where the elements do not refer to individual particles, but are symmetric with respect to particle indices \[1, 2\]. Quantization amounts to choosing an irreducible representation of the algebra. The restriction to symmetric variables, in general allows for the existence of representations which are not present when non-symmetric variables are also included. This gives an algebraic way to introduce generalized particle statistics, where different statistics correspond to inequivalent representations of the same fundamental algebra.

For two identical (spinless) particles on a line, all the observables for the relative motion can be generated from the quadratic observables in the relative position $x$ and momentum $p$, which form the algebra $sl(2, \mathbb{R})$. The irreducible representations of the same algebra also classify the solutions of the Calogero model \[3\] for two particles \[4\], and the Heisenberg quantization thus suggests the interpretation of the singular $1/x^2$-potential of the Calogero model as a “statistics interaction” between the particles. This gives a specific way to introduce \textit{fractional statistics in one dimension} \[2\] (see also discussion of the many-body case in terms of the Schrödinger quantization in Ref. \[5\]). The same algebra — and, correspondingly, the notion of fractional statistics in one dimension — applies to anyons restricted to the lowest Landau level, where the dynamics of particles becomes effectively one-dimensional \[6\].
Note that these algebraic arguments for the $1/x^2$ potential as a statistics interaction have found support in statistical mechanics. The statistical distribution for fractional statistics in one dimension defined in the above way is the same as that for a more recent alternative definition of fractional statistics, the so called *exclusion* statistics, which is introduced in statistical mechanical terms (see [4, 10]). The thermodynamic quantities of anyons in the lowest Landau level represent the same statistical mechanics.

For the Calogero model it was shown some time ago, by use of the so-called exchange operator formalism, that the quadratic variables in $x$ and $p$ can be expressed in terms of linear (non-observable) variables in a way that the representations of $sl(2, \mathbb{R})$ corresponding to fractional statistics be reproduced. This amounted to introducing a modification of the fundamental commutator between (the relative) position and momentum. It was also shown that this generalized commutation relation could be extended to a set of commutation relations for the general $N$-particle system. This modified algebraic structure has been referred to as an $S_N$-extended Heisenberg algebra since it includes also permutation operators for the particles. A typical feature of the modified Heisenberg commutation relations is that they involve non-symmetric operators in $x$ and $p$ and depend explicitly on the (statistics) interaction parameter of the Calogero model [12, 13].

Motivated by these results, in Ref. [14] the question was studied whether it is possible, starting from the parameter-dependent $S_N$-extended Heisenberg algebra, to construct a closed algebra of symmetric one-particle operators which would play the role of the algebra of observables for more than two particles. It was shown that such an algebra (referred to as $G$) can indeed be constructed. It is an infinite dimensional Lie algebra which is independent of the statistics interaction parameter [14]. (One should note $G$ is a larger algebra than the related algebra $W_{1+\infty}$.) In this Letter we show how the algebra $G$ can naturally be incorporated into the Heisenberg quantization scheme, starting in Sect. 2 with a discussion of the one-particle observables for identical particles in one dimension.

In Ref. [14] the explicit construction of the elements of the algebra $G$ was based on the $S_N$-extended Heisenberg algebra, where the defining relations involve both single-particle and two-particle operators and depend on the statistics parameter. However, since the commutation relations of the algebra $G$ (which in fact define the algebra) involve only one-particle operators and do not contain the statistics parameter, it was natural to expect that there should exist a simpler way to formulate this algebra. In Sect. 3 we give such a formulation starting from a set of basic commutation relations which are more general than the canonical commutation relations and which do not depend on the statistics parameter. This formulation allows one to find new realizations of the algebra $G$. In Sect. 4 we give a brief discussion of three different representations, that of the Calogero model, a representation in the form of differentiation operators and finally a representation referring to the Hermitian matrix models. The latter two realizations are in fact much simpler for studying the structure of the algebra than the original one. In Sect. 5 we present some results of computer calculations concerning the number of independent elements of the algebra for low-order polynomials in $x$ and $p$. 

2
2 Heisenberg quantization

A system of $N$ identical particles on the line is described classically by $N$ points $(x_1, p_1), (x_2, p_2), \ldots, (x_N, p_N)$ in the one-particle phase space, taken in arbitrary order. Every observable must be a symmetric function of the $N$ points \[ [1]. \] By definition, a set of symmetric functions is \textit{complete} if it contains a set of (at least) $2N$ functions such that the $2N$ (or more) function values determine the $N$ one-particle phase space points uniquely up to permutations. A complete set of more than $2N$ functions is overcomplete.

One particular complete set of symmetric functions is the set of all one-particle observables. By definition, a one-particle observable is given by an observable $f = f(x, p)$ of the one-particle system as

\[ \langle f \rangle = N \langle f \rangle = \sum_{j=1}^{N} f(x_j, p_j) , \tag{1} \]

where $\langle f \rangle$ is the average of $f$ over the $N$ particles. We may write more compactly $\langle f \rangle = \sum_j f_j$. We call $f$ a \textit{single-particle function} (“non-observable”), as opposed to the \textit{one-particle observable} $\langle f \rangle$.

An important property of the set of one-particle observables is that it is closed under Poisson brackets, in the classical case. The mapping $f \mapsto \langle f \rangle$ preserves Poisson brackets,

\[ \{\langle f \rangle, \langle g \rangle\} = \sum_j \{f_j, g_j\} = \langle \{f, g\}\rangle . \tag{2} \]

Similar relations hold in the standard quantization, based on the canonical Heisenberg commutation relations

\[ [x_j, p_k] = i\hbar \delta_{jk} , \quad [x_j, x_k] = [p_j, p_k] = 0 . \tag{3} \]

For any given $N$ there exists a minimal complete set consisting of exactly $2N$ one-particle classical observables. For example, choose the symmetric and homogeneous polynomials

\[ \langle x^m \rangle = \sum_j x_j^m , \quad \langle x^n p \rangle = \sum_j x_j^n p_j , \tag{4} \]

with $m, n = 1, 2, \ldots, N$. This set is closed under Poisson brackets. However, it makes a fundamental distinction between coordinates and momenta, since the momentum dependence is at most linear.

The algebra to be considered here is an extension which is overcomplete, but treats coordinates and momenta on an equal footing. The commutation relations of the algebra are generic in the sense of being valid for arbitrary particle number $N$. A number of algebraic identities are necessary consequences of the commutation relations. When $N$ is fixed there will be additional algebraic relations between one-particle observables, but these $N$-dependent relations are not considered to be characteristic for the algebra itself.
3 Definition of the algebra $\mathcal{G}$

The Heisenberg commutation relations, or the corresponding classical Poisson bracket relations, are too detailed, since they involve operators, or phase space functions, that are not symmetric and therefore not observable. We want to consider here commutation relations that can be deduced from the Heisenberg relations, but are less detailed and therefore allow more general quantum theories.

The algebra we want is generated from the set of observables $\langle \langle x^m \rangle \rangle$ and $\langle \langle p^n \rangle \rangle$, with $m, n = 1, 2, \ldots$, by means of Poisson brackets in the classical case and commutators in the quantum case. We postulate the following basic commutation relations, which follow directly from the Heisenberg relations, eq. (3), but are genuinely weaker,

$$\langle \langle x^m \rangle \rangle, p_j \rangle = i\hbar x_{m-1}^j, \quad \langle \langle p^n \rangle \rangle, x_j \rangle = -i\hbar p_{n-1}^j, \quad \langle \langle x^m \rangle \rangle, x_j \rangle = [\langle \langle p^n \rangle \rangle, p_j] = 0.$$ (5)

The three examples given in Sect. 4 show that there exist non-trivial realizations.

Note that if $A$ is one of the operators $\langle \langle x^m \rangle \rangle$, $\langle \langle p^n \rangle \rangle$, and if $\langle \langle f \rangle \rangle$ is any one-particle operator, then the commutator

$$[A, \langle \langle f \rangle \rangle] = \sum_j [A, f_j]$$ (6)

is a one-particle operator, by eq. (5). For example,

$$\langle \langle x^5 \rangle \rangle, x_j^2 p_j^3 \rangle = 5i\hbar (x_j^6 p_j^2 + x_j^2 p_j x_j p_j x_j^3).$$

Hence every operator of the form

$$B_k = [A_k, [A_{k-1}, \ldots [A_2, A_1] \ldots]],$$ (7)

where each $A_j$ is one of the $\langle \langle x^m \rangle \rangle$ or $\langle \langle p^n \rangle \rangle$ operators, is a one–particle operator. It follows by repeated applications of the Jacobi identity, in the form

$$[B_k, C] = [A_k, [B_{k-1}, C]] - [B_{k-1}, [A_k, C]],$$ (8)

that the vector space spanned by all operators of the form $B_k$ is closed under commutation and hence is a Lie algebra. It is this Lie algebra we want to study here, and we will call it $\mathcal{G}$.

Note that when we use eq. (3) to compute an explicit expression for a non-zero operator of the form $B_k$ as a symmetric one-particle operator, we will in general get two different results, because we may expand the innermost commutator $[A_2, A_1]$ in two different ways. Either as

$$\langle \langle x^m \rangle \rangle, \langle \langle p^n \rangle \rangle \rangle = \sum_j \langle \langle x^m \rangle \rangle, p_j^n \rangle = i\hbar \sum_{\ell=0}^{n-1} \langle \langle p^\ell x^{m-1-\ell} p^{n-1-\ell} \rangle \rangle, $$ (9)

or as

$$\langle \langle x^m \rangle \rangle, \langle \langle p^n \rangle \rangle \rangle = \sum_j \langle \langle x^m_j \rangle \rangle, \langle \langle p^n \rangle \rangle \rangle = i\hbar \sum_{\ell=0}^{m-1} \langle \langle p^\ell x^{n-1-\ell} x^{m-1-\ell} \rangle \rangle. $$ (10)
It is a consistency condition that the two expressions for $B_k$ must be equal. Other consistency conditions follow from the Jacobi identity, for example that

$$[[\langle \langle x^k \rangle \rangle, [\langle \langle x^m \rangle \rangle, \langle \langle p^n \rangle \rangle]], [\langle \langle x^m \rangle \rangle, [\langle \langle x^k \rangle \rangle, \langle \langle p^n \rangle \rangle]]].$$

(11)

In this way we get a number of identities which can sometimes be used to reorder a product of operators $x_j$ and $p_j$ when there is a sum over the particle index $j$.

One should note that the basic commutation relations (5) can be interpreted as giving an abstract definition of the Lie algebra $G$ where the explicit expression of the elements as sums over single-particle variables is not needed. In this abstract formulation an element $\langle \langle f \rangle \rangle$ is defined by its commutators with $x_j$ and $p_j$. All elements of the algebra can be constructed by repeated use of the fundamental commutators for $\langle \langle x^m \rangle \rangle$ and $\langle \langle p^n \rangle \rangle$, and an identity between observables then simply means that they have identical commutators with both $x_j$ and $p_j$. This abstract definition is in itself a realization of the algebra in terms of differentiation operators, as discussed in some more detail in Subsect. 4.2.

4 Realizations of $G$

4.1 The $S_N$–extended Heisenberg algebra

Our first example is the extended $N$-particle Heisenberg algebra defined by the following commutation relations, containing the arbitrary real parameter $\lambda$,

$$[x_j, x_k] = [p_j, p_k] = 0, \quad [x_j, p_k] = \begin{cases} -i\lambda h K_{jk} & \text{if } j \neq k, \\ i\hbar + i\lambda h \sum_{\ell \neq k} K_{\ell k} & \text{if } j = k. \end{cases}$$

(12)

The operators $K_{jk} = K_{jk}^\dagger = K_{kj}$ are defined for $j \neq k$, and satisfy the following relations when no two of the indices $j, k, \ell, m$ are equal,

$$K_{jk}K_{kj} = 1, \quad K_{jk}K_{\ell k} = K_{\ell k}K_{k j} = K_{ij}K_{jk}, \quad [K_{jk}, K_{\ell m}] = 0,$$

$$[x_j, K_{k \ell}] = [p_j, K_{k \ell}] = 0, \quad x_j K_{jk} = K_{jk} x_k, \quad p_j K_{jk} = K_{jk} p_k.$$  

(13)

Thus they are generators of a unitary representation of the symmetric group $S_N$. The fact that explicit realizations of these relations exist, proves that they are internally consistent.

We will now prove that eq. (3) follows from the equations (12) and (13). In fact, eq. (12) gives directly that

$$[[\langle \langle x^n \rangle \rangle, p_j], [\langle \langle x^m \rangle \rangle, [\langle \langle x^k \rangle \rangle, \langle \langle p^n \rangle \rangle]]] = \begin{aligned} &\langle \langle x^n \rangle \rangle, \langle \langle x^m \rangle \rangle, \langle \langle x^k \rangle \rangle, \langle \langle p^n \rangle \rangle \rangle, \\ &\langle \langle x^n \rangle \rangle, \langle \langle x^m \rangle \rangle, \langle \langle x^k \rangle \rangle, \langle \langle p^n \rangle \rangle \rangle, \\ &\langle \langle x^n \rangle \rangle, \langle \langle x^m \rangle \rangle, \langle \langle x^k \rangle \rangle, \langle \langle p^n \rangle \rangle \rangle, \\ &\langle \langle x^n \rangle \rangle, \langle \langle x^m \rangle \rangle, \langle \langle x^k \rangle \rangle, \langle \langle p^n \rangle \rangle \rangle. \end{aligned}$$

(14)
We observe that the $\lambda$-dependent terms cancel, because we may convert any operator $x_k$ into $x_j$ by pulling it through the operator $K_{kj}$.

In a similar way we have that

$$\left[ \left< \left< p^n \right> \right>, x_j \right] = -i\hbar p_j^{n-1} - i\lambda \sum_{k \neq j} \left( K_{kj} p_j^{n-1} + p_j K_{kj} p_j^{n-2} + \cdots + p_j^{n-1} K_{kj} \right)$$

$$+ i\hbar \sum_{k \neq j} \left( K_{jk} p_k^{n-1} + p_k K_{jk} p_k^{n-2} + \cdots + p_k^{n-1} K_{jk} \right).$$

Using now also the symmetry relation $K_{jk} = K_{kj}$, we again see that the $\lambda$-dependent terms cancel. This completes the proof.

If we use the relations (12) and (13) to compute the commutator of two symmetric one-particle operators, the result will not in general be a one-particle operator. It is therefore a remarkable result, proved first in Ref. [14], that arbitrary commutators and commutators of commutators of the operators $\left< \left< x^m \right> \right>$ and $\left< \left< p^n \right> \right>$, with $m, n = 1, 2, \ldots$, can always be written as one-particle operators. As we have seen above, this result follows from eq. (5).

4.2 Differentiation operators

The index $j$ in eq. (5) is rather superfluous, and we may just as well write

$$\left[ \left< \left< x^m \right> \right>, p \right] = i\hbar x^{m-1}, \quad \left[ \left< \left< p^n \right> \right>, x \right] = -i\hbar p^{n-1}, \quad \left[ \left< \left< x^m \right> \right>, \left[ \left< \left< p^n \right> \right>, p \right] \right] = 0. \quad (16)$$

This defines immediately a natural realization of the algebra $\mathcal{G}$ as a commutator algebra of differentiation operators on the non-commutative one-particle phase space, described by the completely non-commuting variables $x$ and $p$. Remember that $x$ and $p$ here actually represent $x_j$ and $p_j$ for one arbitrary value of the index $j$, and in general we have no commutation rules for reordering monomials in $x$ and $p$.

A linear operator $A : f \mapsto A(f)$ is called a differentiation operator if it acts according to the Leibniz rule,

$$A(fg) = A(f)g + fA(g). \quad (17)$$

The commutator $C = [A, B]$ of two differentiation operators $A$ and $B$ is defined in the obvious way,

$$C(f) = A(B(f)) - B(A(f)). \quad (18)$$

It is easily verified that $C$ is again a differentiation operator.

Any operator $A$ of the form $A : f \mapsto [A, f]$ is automatically a differentiation operator. Moreover, the mapping $A \mapsto A$ preserves the commutator product. In fact, if $A(f) = [A, f]$ and $B(f) = [B, f]$, then the commutator $C = [A, B]$ is given by the Jacobi identity as

$$C(f) = [A, [B, f]] - [B, [A, f]] = [[A, B], f]. \quad (19)$$
Due to the Leibniz rule, the action of a differentiation operator on any polynomial in the (commuting or non-commuting) variables $x$ and $p$ is uniquely given by its action on $x$ and $p$. Let $X_m$ and $P_n$ be the differentiation operators representing $\langle \langle x^m \rangle \rangle$ and $\langle \langle p^n \rangle \rangle$, defined such that

$$X_m(f) = [\langle \langle x^m \rangle \rangle, f], \quad P_n(f) = [\langle \langle p^n \rangle \rangle, f].$$

(20)

Then, as an example, the commutator $A = [\langle \langle x^m \rangle \rangle, \langle \langle p^n \rangle \rangle]$ is represented by the differentiation operator $A = [X_m, P_n]$, acting in the following way,

$$A(x) = X_m(P_n(x)) = -im\hbar X_m(p^{n-1}) = m\hbar^2 \sum_{k=0}^{n-2} p^k x^{m-1} p^{n-2-k},$$

$$A(p) = -P_n(X_m(p)) = -im\hbar P_n(x^{m-1}) = -m\hbar^2 \sum_{k=0}^{m-2} x^k p^{n-1} x^{m-2-k}.$$  

(21)

A general element of the algebra $G$ can thus be represented explicitly in the form

$$A = A_x \frac{\partial}{\partial x} + A_p \frac{\partial}{\partial p},$$

(22)

showing how the operator $A$ acts on $x$ and $p$. Here $A_x = A(x)$ and $A_p = A(p)$ are polynomials in $x$ and $p$. We stress that the representation (22) is unique.

4.3 The Hermitian matrix model

In this model there are $N^2$ real coordinates arranged into an $N \times N$ complex Hermitian matrix $X_{jk}$. The $N^2$ conjugate momenta are similarly arranged into a Hermitian matrix $P_{jk}$. The Hermitian symmetry conditions for the classical variables, $X^{\dagger}_{jk} = X_{kj}$ and $P^{\dagger}_{jk} = P_{kj}$, are replaced by the conditions $X^{\dagger}_{jk} = X_{kj}$ and $P^{\dagger}_{jk} = P_{kj}$ for the operators in the quantum theory. The Poisson brackets and the corresponding commutation relations are the following,

$$\{X_{jk}, P_{\ell m}\} = \delta_{jm}\delta_{k\ell}, \quad [X_{jk}, P_{\ell m}] = i\hbar \delta_{jm}\delta_{k\ell},$$

(23)

and they are compatible with the Hermitian symmetry of the matrices $X$ and $P$.

In order to relate this matrix model to the theory of a system of identical particles, we interpret the special unitary group SU($N$) as an extension of the symmetric group $S_N$ which interchanges the particle positions. An SU($N$) matrix $U$ acts on the matrices $X$ and $P$ by conjugation, as

$$X \mapsto UXU^{-1}, \quad P \mapsto UPU^{-1}.$$  

(24)

The Poisson brackets or commutation relations, eq. (23), are preserved. The Hermitian matrix $X$ may always be diagonalized by an SU($N$) matrix $U$, and the eigenvalues $x_1, x_2, \ldots, x_N$ may be interpreted as the particle positions on the line. Different SU($N$)
matrices which diagonalize $X$ may give different orderings of the eigenvalues. The full SU($N$) group now gives a continuous way to interchange the particle positions [13].

The observables of the $N$-particle system correspond to the SU($N$) invariant quantities of the matrix model. Thus, the observable part of the $N \times N$ Hermitian matrix $X$ is its set of $N$ real eigenvalues, in arbitrary order. An equivalent complete set of SU($N$) invariants for the matrix $X$ are the symmetric one-particle observables

$$\langle \langle x^m \rangle \rangle = \sum_{j=1}^{N} x_j^m = \text{Tr} X^m,$$

with $m = 1, 2, \ldots, N$. Thus we see that the observables in the Hermitian matrix model which correspond to one-particle observables for a system of identical particles, such as $\langle \langle x^m \rangle \rangle$, $\langle \langle p^n \rangle \rangle$, $\langle \langle x^m p^n \rangle \rangle$, are traces of matrix products: $\text{Tr} X^m$, $\text{Tr} P^n$, and in general,

$$f = \text{Tr}(F_1 F_2 \cdots F_M),$$

where each matrix $F_j$ is either $X$ or $P$. We say that $f$ has order $(m, n)$ if the matrix product contains $m$ factors $X$ and $n$ factors $P$. This is a realization of the defining relations in eq. (5), in the sense that, e.g.,

$$[\text{Tr} X^m, P_{jk}] = i m \hbar (X^{m-1})_{jk}, \quad [\text{Tr} P^n, X_{jk}] = -i n \hbar (P^{n-1})_{jk}.$$

In the classical matrix model we are free to permute cyclically the factors of a matrix product inside the trace, so that $f = \text{Tr}(F_2 \cdots F_M F_1) = \ldots$. In the quantized theory the same relations hold modulo correction terms proportional to powers of $\hbar$, due to the Heisenberg commutation relations. Note, however, that since the basic commutation relations (27) hold equally well both in the classical and the quantum case, the algebra $G$ generated is the same in both cases. Therefore, we can consider for simplicity the classical case, ignoring correction terms. The classical matrix model is able to embody the full structure of the algebra $G$ (unlike the classical $N$-particle phase space), since the (quantum) non-commutativity of $x$ and $p$, postulated in (4), is replaced by the classical non-commutativity of matrices $X$ and $P$.

The set of all linear combinations of terms of the form $f$, eq. (26), is closed under Poisson brackets and is therefore a Lie algebra of symmetric one-particle observables. Let us call it $F$. Note that the element $f \in F$ is in general complex, and its complex conjugate is

$$f^* = \text{Tr}((F_1 F_2 \cdots F_M)^\dagger) = \text{Tr}(F_M \cdots F_2 F_1).$$

A subalgebra of $F$ is the set $F_R$ of all linear combinations of terms of the form $f + f^*$. We can use the observables $\text{Tr} X^m$, $\text{Tr} P^n$ to generate the algebra $G$ as a subalgebra of $F_R$. For example, considering the commutator which had an ambiguous representation in the equations (4) and (14), we have

$$\{\text{Tr} X^m, \text{Tr} P^n\} = mn \text{Tr} (X^{m-1} P^{n-1})$$

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This Poisson bracket can be computed by means of the canonical relations (23), or in two other ways by (the Poisson bracket version of) eq. (27). The result must be independent of the method, since there can be no reordering problems in the classical case. Thus there is no ambiguity in the representation of the Poisson bracket (29) as an observable of the classical Hermitian matrix model. It follows in a similar way that every element of $\mathcal{G}$ has a unique representation in this model. For this reason, the matrix model realization of $\mathcal{G}$ is perhaps the most efficient and convenient of all realizations, for computational purposes.

As already pointed out, the $N$-particle system can be considered as corresponding to the SU($N$) invariant part of the matrix model. The matrix model is larger since it also includes SU($N$)-dependent, “angular” variables. Quantum mechanically, different SU($N$) “angular momentum” representations can be given by specifying values for certain constants of motion. In this way different $N$-particle systems can be constructed, such as free fermions, the Calogero system [16], and Calogero-like systems with additional internal degrees of freedom [17]. It is of interest to note that for all these systems, $\mathcal{G}$ is the relevant algebra of SU($N$) invariant observables.

5 Remarks on the structure of $\mathcal{G}$

In this section we present results of computer (symbolic) calculations using the realizations of the algebra $\mathcal{G}$ given in the two previous subsections. We have computed the degeneracy $g(m,n) = g(n,m)$, i.e. the number of linearly independent elements of $\mathcal{G}$ of order $(m,n)$, for $m + n \leq 16$, as presented in Figure 1. In Ref. [14] the degeneracies were given for $m + n \leq 6$.

Let us outline how to compute using the classical Hermitian matrix model. One need not use all the observables $\text{Tr} \, X^m$ and $\text{Tr} \, P^n$ to generate $\mathcal{G}$ as described in Sect. 3, it is enough to use only a finite set of generators, e.g. $\text{Tr} \, P$, $\text{Tr} \, X^2$, and $\text{Tr} \, P^3$ [14]. One may generate all of $\mathcal{G}$ by starting with $\text{Tr} \, P$ and proceeding as follows. All the observables of order $(m,n)$ are obtained from those of order $(m-1,n+1)$ and $(m+1,n-2)$, by means of Poisson brackets with $\text{Tr} \, X^2$ and $\text{Tr} \, P^3$, respectively. Linear dependence of the generated observables of order $(m,n)$ has to be checked.

Defining the partial generating functions

$$G_n(q) = \sum_{m=0}^{\infty} g(n,m) q^m,$$

(30)

we recover the degeneracies $g(n,m) = g(m,n)$ listed in Fig. 1 for $n \leq 3$ with

$$G_0 = G_1 = \frac{1}{1-q},$$
$$G_2 = \frac{1}{(1-q)(1-q^2)},$$
$$G_3 = \frac{1}{(1-q)(1-q^2)(1-q^3)}.$$
This formula for the generating functions $G_0$ and $G_1$ is easily proved to all orders. For $G_2$ and $G_3$ we have no proof, only numerical verification up to the level $m + n = 16$. Eq. (31) would imply that for $n \leq 3$,

$$g(n, m) = p_n(m),$$  \hspace{1cm} (32)

the number of partitions of $m$ into at most $n$ parts. $p_n(m)$ is in fact the degeneracy for order $(m, n)$ with $n \leq 3$ of the algebra $F_R$, which was defined in Subsect. 4.3 and which contains $G$. To see this in the case $n = 3$, as an example, note that the general basis element of $F_R$ then has the form

$$\text{Tr}(X^{m_1}PX^{m_2}PX^{m_3}P + PX^{m_3}PX^{m_2}PX^{m_1}),$$  \hspace{1cm} (33)

where $m_1, m_2, m_3$ are non-negative integers and $m_1 + m_2 + m_3 = m$. This trace is invariant under any permutation of $m_1, m_2, m_3$, showing that there is a one to one correspondence between basis elements of order $(m, 3)$ and partitions of $m$ into at most three parts.

Since $G$ is a subalgebra of $F_R$, it is a remarkable fact that these two algebras are actually identical for every order $(m, n)$ with $m + n \leq 16$ and either $m \leq 3$ or $n \leq 3$. Our conjecture is that this is true for arbitrary $m$ and $n \leq 3$, or equivalently, for arbitrary $n$ and $m \leq 3$.

In contrast, for $m \geq 4$ and $n \geq 4$ we find that $F_R$ is larger than $G$. We may cite here two more generating functions,

$$G_4 = \frac{1 + q^2}{(1 - q)(1 - q^2)(1 - q^3)(1 - q^4)},$$

$$G_5 = \frac{(1 + q^2)(1 + q^3 + q^4)}{(1 - q)(1 - q^2)(1 - q^3)(1 - q^4)(1 - q^5)},$$  \hspace{1cm} (34)

which recover in a non-trivial way the degeneracies in Fig. 1 for $n = 4$ and $n = 5$, respectively. From these expressions as well as from Fig. 1 one can see that the degeneracies $g(m, n)$ of $G$ for $m \geq 4$ and $n \geq 4$ are larger than the numbers of partitions $p_n(m)$.

## 6 Conclusions

We have presented a formulation of the algebra of observables for identical particles on a line in terms of a previously studied infinite dimensional Lie algebra $G$, starting from a set of basic commutation relations which are parameter-independent. We have given two new realizations of $G$ in which the elements are represented without ambiguities. This provides a simpler way, both conceptually and practically, to study the structure of the algebra. The advantages of the new representations were demonstrated by evaluating the degeneracies of $G$ for low orders.

The algebra $G$ is overcomplete with respect to the classical coordinates and momenta of the $N$-particle system, and this is exposed by the degeneracies for given order $(m, n)$ of the algebra, which we have examined in this paper. An obvious interpretation of
these degeneracies is that the algebra, in general, involves more degrees of freedom than those present in the classical $N$-particle system. (Note, however, that for irreducible representations of $G$, the number of independent variables will be smaller than indicated by the degeneracies since the presence of Casimir operators will introduce identities between elements of the algebra.) The conceptual questions then remain: What is the most general class of particle systems for which $G$ embodies the full set of observables, and can the additional variables be related to internal degrees of freedom for such systems? Answers to these questions depend on the nature of unitary irreducible representations of $G$, which are yet to be studied.

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Figure 1: The degeneracy $g(m, n)$ of the algebra $\mathcal{G}$ as a function of the order $(m, n)$. For $\min(m, n) \leq 3$, the degeneracies equal the numbers of partitions of $\max(m, n)$ into at most $\min(m, n)$ parts. For $\min(m, n) > 3$, the degeneracies are larger than the numbers of partitions.