From $p$-Values to Posterior Probabilities of Hypothesis

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Abstract

Minimum Bayes factors are commonly used to transform two-sided $p$-values to lower bounds on the posterior probability of the null hypothesis, as in Pericchi et al. (2017). In this article, we show posterior probabilities of hypothesis by transforming the commonly used $-e \cdot p \cdot \log(p)$, proposed by Vovk (1993) and Sellke et al. (2001). This is achieved after adjusting this minimum Bayes factor with the information to approximate it to an exact Bayes factor, not only when $p$ is a $p$-value but also when $p$ is a pseudo $p$-value in the sense of Casella and Berger (2001). Additionally we show the fit to a refined version to linear models.
1 Pseudo $P$-Values

Under the null hypotheses, $p$-values are well known to have Uniform(0,1), in Casella and Berger (2001) a more general definition is given

**Definition 1.** A *$p$-value* $p(X)$ is a statistic satisfying $0 \leq p(x) \leq 1$ for every sample point $x$. Small values of $p(X)$ give evidence that $H_1$ is true. A $p$-value is valid if, for every $\theta \in \Theta_0$ and every $0 \leq \alpha \leq 1$,

$$P_{\theta}(p(X) \leq \alpha) \leq \alpha.$$  

**Remark 1.** We consider any $p$-value complying the Definition 1 without equality for all $\alpha$ a pseudo $p$-value.

The “Robust Lower Bound” (RLB) as is called in Pericchi et al. (2017) and proposed by Sellke et al. (2001) is:

$$B_{L}(p) = \begin{cases} 
-e \cdot p \cdot \log(p) & p < e \smallskip 
1 & \text{otherwise}
\end{cases}$$

when under $H_0$ $p$ is Uniform(0,1) and the density of $p$ under $H_1$ is $Beta(\xi,1)$ for $0 < \xi < 1$. Note that this calibration has been proposed already in Vovk (1993). Another class of decreasing densities is $Beta(1,\xi)$ with $\xi > 1$. This leads to the "$-e \cdot q \cdot \log(q)$" calibration, where $q = 1 - p$ see Held and Ott (2018).

In contrast with the Remark 1 if we consider $p(x)$ a pseudo $p$-value under $H_0$, that is,

$$p \sim Beta(\xi_0,1) \quad \text{with} \quad \xi_0 > 1, \text{ fixed but arbitrary},$$

under the test

$$H_0 : p \sim Beta(\xi_0,1) \quad \text{vs} \quad H_1 : p \sim f(p|\xi)$$
with \( f(p|\xi) \sim Beta(\xi, 1) \) for \( 0 < \xi < 1 \), the RLB for

\[
B_L(p, \xi_0) = \begin{cases} 
-e \cdot \xi_0 \cdot p^{\xi_0} \log(p) & p < e^{-1} \\
1 & \text{otherwise}
\end{cases}
\]  

(1)

where \( \xi_0 \) has to be estimated or calculated theoretically, but we known
that \( \xi_0 = 1 \) when the \( p \)-value is not pseudo \( p \)-value.

On the other hand, since \( f(p|\xi) = \xi p^{\xi-1} \) has its maximum in \( \xi = -\frac{1}{\log(p)} < 1 \)
with \( p < e^{-1} \) then \( f(p|\xi) \) is decreasing for \( \xi > -\frac{1}{\log(p)} \), thus for any Bayes Factor \( B_{01} \)

\[
B_{01} \geq B_L(p) > B_L(p, \xi_0) \quad \text{con} \quad \xi_0 > 1
\]

see Figure 1.

![Graph of RLB for different \( \xi_0 \).](image)

Figure 1: Graph the \( RLB_{\xi_0} \) for different \( \xi_0 \).

In the sequel, we want to calibrate \( RLB_{\xi_0} \) such that \( RLB_{\xi_0} \approx B_{01} \)

**Lemma 1.** \( B_L(p_{val}, \xi) = -e \cdot \xi \cdot p_{val}^{\xi} \cdot \log(p_{val}) \geq e \cdot \xi \cdot p_{val}^{\xi} > p_{val}^{\xi} \), for,

\( 0 < p_{val} < e^{-1} \) and \( \xi \geq 1 \). Note that \( B_L(p_{val}, 1) = B_L(p_{val}) \)

**Proof.** Let \( h(p_{val}) = -e \cdot \xi \cdot \log(p_{val}) \), then 

\[
\frac{d[h(p_{val})]}{dp_{val}} = -\frac{e \cdot \xi}{p_{val}} < 0,
\]

thus \( h \) is decreasing with minimum at \( \xi = e^{-1} \). So, \( h(p_{val}) \geq h(e^{-1}) = e \cdot \xi \) the which
implies \( B_L(p_{val}, \xi)/p_{val}^{\xi} = h(p_{val}) \geq e \cdot \xi \), so \( B_L(p_{val}, \xi) \geq e \cdot \xi \cdot p_{val}^{\xi} > p_{val}^{\xi} \)
\section*{Theorem 1.}

The RLB $\xi$ is a valid $p$-value, for $\xi \geq 1$, that is,

$$P(B_L(p, \xi) \leq \alpha|p \sim f(p|\xi)) \leq \alpha, \text{ for each } 0 \leq \alpha \leq 1.$$ 

\textit{Proof.} First of all it can be seen that $B_L(p, \xi) = -e \cdot \xi \cdot p^\xi \cdot \log(p)$ is well defined, since $0 \leq B_L(p, \xi) \leq 1$.

Let $\alpha \in [0, 1]$, denote for $D_B$ the subset of $R_p$ (range of $p$) such that

$$-e \cdot \xi \cdot p^\xi \cdot \log(p) \leq \alpha,$$

then

$$(B_L(p, \xi) \leq \alpha) = [-e \cdot \xi \cdot p^\xi \cdot \log(p) \leq \alpha] = (p \in D_B)$$

where $(p \in D_B)$ is the event that consists of all the result $x$ such that the point $p(x) \in D_B$. Therefore,

$$F_B(\alpha) = P(B_L(p, \xi) \leq \alpha|p \sim f(p|\xi)) = P(-e \cdot \xi \cdot p^\xi \cdot \log(p) \leq \alpha|p \sim f(p|\xi)) = P(p \in D_B|p \sim f(p|\xi)) = \int_{D_B} f_p(p) dp = \int_0^\rho \xi p^{\xi-1} dp = \rho^\xi$$

where $\rho$ is determined such that

$$0 < \rho < \frac{1}{e} \text{ and } \alpha = -e \cdot \xi \cdot \rho^\xi \cdot \log(\rho)$$

as shown in the figure 2 in the case $\xi = 1$

now, by Lemma 1 $F_B(\alpha) = \rho^\xi < -e \cdot \xi \cdot \rho^\xi \cdot \log(\rho) = \alpha$. \qed
Figure 2: Plot of $B_L(p)$ indentifying $\rho$ such that $-e \cdot \rho \cdot \log(\rho) = \alpha$.

2 Adaptive $\alpha$ with Strategy PBIC

An adaptive $\alpha$ allows us to adapt the statistical significance with the information, but more importantly, it allows us to arrive at equivalent results with a Bayes factor. In Pérez and Pericchi (2014) this adaptive $\alpha$ based in BIC is presented as:

$$
\alpha_n(q) = \left[ \chi^2_\alpha(q) + q \log(n) \right]^\frac{q}{2} - 1 \times n^\frac{q}{2} \cdot \Gamma \left( \frac{q}{2} \right) 
$$

and in Vélez et al. (2022) a version to nested linear models based in PBIC (Prior-based Bayesian Information Criterion, see Bayarri et al. (2019)) is presented as:

$$
\alpha_{(b,n)}(q) = \frac{[g_{n,\alpha}(q) + \log(b) + C]^\frac{q}{2} - 1}{b^{\frac{n-j}{2(n-j-1)}} \cdot \left( \frac{2(n-1)}{n-j} \right)^{\frac{q}{2}} \Gamma \left( \frac{q}{2} \right)} \times \exp \left\{ -\frac{n-j}{2(n-1)} (g_{n,\alpha}(q) + C) \right\}, 
$$

where $b = \frac{|X_i^T X_j|}{|X_i^T X_i|}$ and $X_i, X_j$ are design matrix and

$$
C = 2 \sum_{m_i=1}^{q_i} \log \left( 1 - e^{-v_{mi}} \right) - 2 \sum_{m_j=1}^{q_j} \log \left( 1 - e^{-v_{mj}} \right),
$$

and

$$
\alpha_{(b,n)}(q) = \frac{[g_{n,\alpha}(q) + \log(b) + C]^\frac{q}{2} - 1}{b^{\frac{n-j}{2(n-j-1)}} \cdot \left( \frac{2(n-1)}{n-j} \right)^{\frac{q}{2}} \Gamma \left( \frac{q}{2} \right)} \times \exp \left\{ -\frac{n-j}{2(n-1)} (g_{n,\alpha}(q) + C) \right\},
$$

where $b = \frac{|X_i^T X_j|}{|X_i^T X_i|}$ and $X_i, X_j$ are design matrix and

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C = 2 \sum_{m_i=1}^{q_i} \log \left( 1 - e^{-v_{mi}} \right) - 2 \sum_{m_j=1}^{q_j} \log \left( 1 - e^{-v_{mj}} \right),
$$

and

$$
\alpha_{(b,n)}(q) = \frac{[g_{n,\alpha}(q) + \log(b) + C]^\frac{q}{2} - 1}{b^{\frac{n-j}{2(n-j-1)}} \cdot \left( \frac{2(n-1)}{n-j} \right)^{\frac{q}{2}} \Gamma \left( \frac{q}{2} \right)} \times \exp \left\{ -\frac{n-j}{2(n-1)} (g_{n,\alpha}(q) + C) \right\},
$$

where $b = \frac{|X_i^T X_j|}{|X_i^T X_i|}$ and $X_i, X_j$ are design matrix and

$$
C = 2 \sum_{m_i=1}^{q_i} \log \left( 1 - e^{-v_{mi}} \right) - 2 \sum_{m_j=1}^{q_j} \log \left( 1 - e^{-v_{mj}} \right),
$$
\[ v_{ml} = \frac{\hat{\xi}_{ml}}{d_{ml}(1+n_{ml}^{e})} \] with \( l = i, j \) corresponding to each model. Here \( n_{ml}^{e} \), with \( l = i, j \), refers to The Effective Sample Size (called TESS) corresponding to that parameter, see \( \text{Bayarri et al. (2019)} \).

If we adjust (2) replacing the constant \( C_\alpha \) with the PBIC strategy the following expression is obtained

\[
\alpha_n(q) = \frac{[\chi^2_\alpha(q) + q \log(n) + C]^2 - 1}{n^2 2^{q-1} \Gamma \left( \frac{q}{2} \right)} \times \exp \left\{ -\frac{1}{2} \left( \chi^2_\alpha(q) + C \right) \right\}. \tag{4}
\]

Note that this adaptive \( \alpha \) is still of BIC structure.

### 2.1 Binomial Models

Consider comparing two binomial models \( S_1 \sim \text{binomial}(n_1, p_1) \) and \( S_2 \sim \text{binomial}(n_2, p_2) \) via the test

\[
H_0 : p_1 = p_2 \quad \text{vs} \quad H_1 : p_1 \neq p_2.
\]

Defining \( n = n_1 + n_2 \) and \( \hat{p} \) the MLE from \( p_1 - p_2 \), then the equation (4) is

\[
\alpha_n = \left[ \frac{2}{n\pi(\chi^2_\alpha(1) + \log(n) + C)} \right]^{1/2} \times \exp \left\{ -\frac{1}{2} \left( \chi^2_\alpha(1) + C \right) \right\}, \tag{5}
\]

here \( \chi^2_\alpha(1) \) is the quantile \( \alpha \) from chi-square with \( df = 1 \), \( C = -2 \log \left( \frac{1 - e^{-v}}{\sqrt{2v}} \right) \),

\[ v = \hat{p}^2/[d(1+n^e)], \quad d = \left( \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \right), \quad n^e = \max \left\{ \frac{n_1^2}{\sigma_1^2}, \frac{n_2^2}{\sigma_2^2} \right\} \]

The Table II shows the behavior \( \alpha_n \) when \( \alpha = 0.5 \) and \( n_1 \) and \( n_2 \) take different values.
Adaptive $\alpha$ via PBIC ($\alpha_n$) in equation 5 for testing equality of two proportions.

3 Adjusting $RLB_\xi$ with Adaptive $\alpha$

In this section, we use the equation (1) with the adaptive $\alpha$ in equation (3) and in equation (4) for obtaining an approximation to an objective Bayes Factor calibrating the $RLB_{\xi_0}$ by The Effective Sample Size (Berger et al. (2014)) and the parameters involved, according to what is established in Pericchi et al. (2017).

Using these ideas, a calibration of (1) when evaluated in (4) results in the following Bayes Factor, which has a simple expression.

$$B(\alpha,q,n,\xi_0) = -\alpha^{\xi_0} \log(\alpha) \Gamma(q/2)^{\xi_0} n^{\xi_0/2} \left[ \frac{2}{\chi^2_\alpha(q) + q \cdot \log(n) + C} \right]^{\xi_0/2 - (\xi_0 - 1)}.$$ (6)

When it comes to a p-value that is not a pseudo p-value $\xi_0 = 1$ and the Bayes factor simplifies to

$$B(\alpha,q,n) = -\alpha \log(\alpha) \Gamma(q/2)n^{q/2} \left[ \frac{2}{\chi^2_\alpha(q) + q \cdot \log(n) + C} \right]^{\frac{q}{2}}.$$ (7)

The refined version to linear models, for this calibration is obtained when evaluated in (3).
\[ B(\alpha, q, n, b) = -\alpha \log(\alpha) \Gamma(q/2) b^{\frac{n-j}{2(q-n)}} \left[ \frac{2(n-1)}{(g_{n,\alpha}(q) + \log(b) + C)(n-j)} \right]^{\frac{q}{2}} \]  

(8)

in this case only we consider \( \xi_0 = 1 \) since \( \alpha \) take value that are not pseudo p-value.

3.1 Balanced One Way Anova

Suppose we have \( k \) groups with \( r \) observations each, for a total sample size of \( kr \) and let \( H_0 : \mu_1 = \cdots = \mu_k = \mu \) vs \( H_1 : \) At least one \( \mu_i \) different.

Then the design matrices for both models are:

\[
X_1 = \begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix},
X_k = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix},
\]

and the adaptive \( \alpha \) for linear model in accordance with what was presented in [Vélez et al. 2022] is

\[
\alpha(k, r) = \frac{[g_{r,\alpha}(k-1) - \log(k) + (k-1) \log(r) + C]^{\frac{k-3}{2}}}{(k-1)^{r-1}(r-1)^{r-1} \Gamma \left( \frac{k-1}{2} \right)} \times \exp \left\{ -\frac{r-1}{2(r-1/k)} \left( g_{r,\alpha}(k-1) + C \right) \right\}.
\]

Here, the number of replicas \( r \) is The Effective Sample Size (TESS). Therefore, the Bayes factor for this test with respect to equation [8] is:
\[ B(\alpha, k, r) = -\alpha \log(\alpha) \Gamma((k-1)/2) (k^{-1} r^{k-1})^{\frac{r-1}{2k-1}} \left[ \frac{2(r-1/k)}{(g_{r,\alpha}(k-1) - \log(k) + (k-1) \log(r) + C)(r-1)} \right]^{\frac{k-1}{2}} \]

A very important case arises when \( k = 2 \). For this situation, simplifies to

\[ B(\alpha, r) = -\alpha \log(\alpha) \left( \frac{r}{2} \right)^{\frac{r-1}{2}} \left[ \frac{2(r-1)\pi}{(g_{r,\alpha}(1) - \log\left( \frac{r}{2} \right) + C)(r-1)} \right]^{\frac{1}{2}} \]

4 Calibrating \( P \)-Values

In this section we will use (6) and (8) to determine posterior probabilities for the null hypothesis. Since for any Bayes factor \( B_{01} \)

\[ B_{01} \geq B_L(p, \xi_0) \quad \text{con} \quad \xi_0 \geq 1, \text{ fixed but arbitrary}, \]

a lower bound for the posterior probability of the null hypothesis can be obtained as:

\[ \min P(H_0|\text{Data}) = \left[ 1 + \frac{1}{B_L(p, \xi_0)} \right]^{-1}. \quad (9) \]

The Figure 3 shows these posterior probabilities (called \( P_{RLB_{\xi_0}} \)) for different values of \( \xi_0 \)

4.1 Testing Equality of Two Means with Unequal Variances

Consider comparing two normal means via the test

\[ H_0 : \mu_1 = \mu_2 \quad \text{versus} \quad H_1 : \mu_1 \neq \mu_2, \]
Figure 3: Lower bound for posterior probability for the null hypothesis $H_0$ for $\xi_0 = 1, \xi_0 = 1.1, \xi_0 = 1.2, \xi_01.3$.

where the associated known variances, $\sigma^2_1$ and $\sigma^2_2$ are not equal.

$$Y = X\mu + \epsilon = \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{2n_2} \end{pmatrix},$$

$$\times \epsilon \sim N(0, \text{diag}\{\sigma^2_1, \ldots, \sigma^2_1, \sigma^2_2, \ldots, \sigma^2_2\})$$

Defining $\alpha = (\mu_1 + \mu_2)/2$ and $\beta = (\mu_1 - \mu_2)/2$ places this in the linear model comparison framework,

$$Y = B\begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \epsilon.$$
with

\[ \mathbf{B} = \begin{pmatrix}
1 & 1 \\
\vdots & \vdots \\
1 & 1 \\
1 & -1 \\
\vdots & \\
1 & -1
\end{pmatrix} \]

where we are comparing \( M_0 : \beta = 0 \) versus \( M_1 : \beta \neq 0 \).

So for \( \mathbf{B} \),

\[ C = -2 \log \left( \frac{1 - e^{-v}}{\sqrt{2}v} \right) \]

\[ v = \frac{\bar{\beta}^2}{\sigma^2 (1 + n e^2)} , d = \left( \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2} \right) , n^e = \max \left\{ \frac{n_1^2}{\sigma^1_n^2} , \frac{n_2^2}{\sigma^2_n^2} \right\} \left( \frac{\sigma^2_n^2}{n_1^2} + \frac{\sigma^2_n^2}{n_2^2} \right) . \]

A special case is the standard test of equality of means when \( \sigma^2 = \sigma^2_2 = \sigma^2 \).

Then

\[ n^e = \min \left\{ n_1 \left( 1 + \frac{n_1}{n_2} \right) , n_2 \left( 1 + \frac{n_2}{n_1} \right) \right\} . \]

For other hand, considering \( \mu = \mu_1 - \mu_2 \) with \( \sigma^2 = \sigma^2_2 = \sigma^2 \)

- \( H_0 : \mu_1 = \mu_2 \iff \mu = 0 \)
- \( H_0 : \mu_1 \neq \mu_2 \iff \mu \neq 0 \)

Assuming priors

- \( \mu | \sigma^2, H_1 \sim Normal(0, \sigma^2 / \tau_0) , \tau_0 \in (0, \infty) \)
- \( \pi(\sigma^2) \propto 1/\sigma^2 \) for both \( H_0 \) and \( H_1 \).

The Bayes factor is:

\[ BF_{01} = \left( \frac{n + \tau_0}{\tau_0} \right)^{1/2} \left( \frac{t^2 \frac{\tau_0}{n + \tau_0} + l}{t^2 + l} \right)^{\frac{l+1}{2}} \]

where

\[ t = \frac{|\mathbf{Y}|}{s/\sqrt{n}} \]
t-statistic with degrees of freedom \( l = n - 1 \) and \( n = n_1 + n_2 \) see Roger et al. (2018).

The Figure 4 shows the posterior probability for the null hypothesis \( H_0 \) when \( n = 50 \) and \( n = 100 \) for the Robust Lower Bound with \( \xi_0 = 1 \) (called \( P_{RLB} \)), the Bayes factor of the equation (6) (called \( P_{PL} \)), the Bayes factor of the equation (6) (called \( P_{PGL} \) with \( \xi_0 = 1 \)) and for the Bayes factor \( BF_{01} \) (called \( P_{BF_{0,1}} \)). Note that the posterior probability with \( BF_{01} \) when \( \tau_0 = 6 \) looks very similar to the result obtained using the Bayes factors of the equations (6) and (8).

**Figure 4:** Posterior probability for the null hypothesis \( H_0 \) for \( n = 50 \) and \( n = 100 \) using the Bayes factor \( RLB_{\xi_0} \) with \( \xi_0 = 1 \), the Bayes factor \( BF_{01} \), the Bayes factor of the equation (7) and equation (8).

### 4.2 Fisher’s Exact Test

This is an example where the \( p \)-value is a pseudo \( p \)-value (see the example 8.3.30 in Casella and Berger (2001)). Let \( S_1 \) and \( S_2 \) be independent observations with \( S_1 \sim binomial(n_1, p_1) \) and \( S_2 \sim binomial(n_2, p_2) \). Consider testing \( H_0 : p_1 = p_2 \) vs \( H_1 : p_1 \neq p_2 \).
Under $H_0$, if we let $p$ common value of $p_1 = p_2$, the joint pmf of $(S_1, S_2)$ is

$$f(s_1, s_2|p) = \binom{n_1}{s_1} \binom{n_2}{s_2} p^{s_1+s_2} (1-p)^{n_1+n_2-(s_1+s_2)}$$

and the conditional pseudo $p$-value is

$$p(s_1, s_2) = \min_{j=s_1} \left\{ \sum_{j=s_1}^{\min\{n_1, s\}} f(j|s), \right\}$$ (10)

the sum of hypergeometric probabilities, with $s = s_1 + s_2$.

It is important to note that in Bayesian tests with point null hypothesis it is not possible to use continuous prior densities because this distributions (as well as posterior distributions) will grant zero probability to $p = (p_1 = p_2)$. A reasonable approximation will be to give $p = (p_1 = p_2)$ a positive probability $\pi_0$ and to $p \neq (p_1 = p_2)$ the prior distribution $\pi_1 g_1(p)$ where $\pi_1 = 1 - \pi_0$ and $g_1$ proper. One can think of $\pi_0$ as the mass that would be assigned to the real null hypothesis, $H_0 : p \in ((p_1 = p_2) - b, (p_1 = p_2) + b)$, if it had not been preferred to approximate by the null point hypothesis. Therefore, if

$$\pi(p) = \begin{cases} 
\pi_0 & p = (p_1 = p_2) \\
\pi_1 g_1(p) & p \neq (p_1 = p_2)
\end{cases}$$

then

$$m(s) = \int_{\Theta} f(s|p) \pi(p) dp$$

$$= f(s|(p_1 = p_2)) \pi_0 + \pi_1 \int_{p \neq (p_1 = p_2)} f(s|p) g_1(p) dp$$

$$= f(s|(p_1 = p_2)) \pi_0 + (1 - \pi_0) m_1(s)$$

where $m_1(s) = \int_{p \neq (p_1 = p_2)} f(s|p) g_1(p) dp$ is the marginal density of $(S = S_1 + S_2)$ with respect to $g_1$. 

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So,
\[ \pi((p_1 = p_2)|s) = \frac{\pi_0 f(s|(p_1 = p_2))}{m(s)} \]
thus

\[
\text{odds posterior} = \frac{\pi((p_1 = p_2)|s)}{1 - \pi((p_1 = p_2)|s)}
\]
\[
= \frac{f(s|(p_1 = p_2))\pi_0}{m(s)(1 - f(s|(p_1 = p_2))\pi_0)}
\]
\[
= \frac{f(s|(p_1 = p_2))\pi_0}{m(s) - f(s|(p_1 = p_2))\pi_0}
\]
\[
= \frac{f(s|(p_1 = p_2))\pi_0}{(1 - \pi_0)m_1(s)}
\]
\[
= \frac{\pi_0 f(s|(p_1 = p_2))}{\pi_1 m_1(s)}
\]
\[
= \text{odds prior} \cdot \frac{f(s|(p_1 = p_2))}{m_1(s)}
\]

and the Bayes Factors is
\[ B_{01} = \frac{f(s|(p_1 = p_2))}{m_1(s)}. \]

Now, if we take \( g_1(p) = Beta(a,b) \) such that \( E(p) = \frac{a}{a+b} = (p_1 = p_2) \), then
\[ BF_{Test} = \frac{B(a,b)}{B(s+a, n_1 + n_2 - s + b)} p^s(1 - p)^{n_1 + n_2 - s}. \]

the Figure shows the posterior probability for the null hypothesis \( H_0 \) when \( n = n_1 + n_2 = 50 \) and 100 for the Robust Lower Bound with \( \xi_0 = 1 \), the Bayes factor of the equation \( (6) \) (called \( P_{PG_\xi_0} \)) and for the Bayes factor \( BF_{Test} \) (called \( P_{BF_{Test}} \)). We can note that all the \( P_{PG_\xi_0} \) are comparable even though in the case \( \xi_0 = 1 \) it is a \( p \)-value and not a pseudo \( p \)-value.
4.3 Linear Regression Models

Consider comparing two nested linear models

\[ M_3 : y_l = \beta_1 + \beta_2 x_{l2} + \beta_3 x_{l3} + \epsilon_l \]

with \( M_2 : y_l = \beta_1 + \beta_2 x_{l2} + \epsilon_l \) via the test

\[ H_0 : M_2 \text{ versus } H_1 : M_3, \]

with \( 1 \leq l \leq n \) and the errors \( \epsilon_l \) are assumed to be independent and normally distributed with unknown residual variance \( \sigma^2 \). According with the equation (3), in Vélez et al. (2022) and in Bayarri et al. (2019)

\[ b = (n - 1)s_3^2(1 - \rho_{x3}^2), \]

Figure 5: Posterior probability for the null hypothesis \( H_0 \) for \( n = 50 \) and \( n = 100 \) using the Bayes factor \( RLB_{\xi_0} \) with \( \xi_0 = 1 \), the Bayes factor \( BF_{Test} \) and the Bayes factor of the equation (6).
where $s^2_3$ is the variance $x_{v3}$ and $\rho_{23}$ is the correlation between $x_{v2}$ and $x_{v3}$, and

$$C = 2 \log \left( \frac{1 - e^{-v_2}}{\sqrt{2}v_2} \right) - 2 \log \left( \frac{1 - e^{-v_3}}{\sqrt{2}v_3} \right),$$

where $v_2 = \beta^2_2/[d_2(1 + n_2^2)]$, $d_2 = \sigma^2/s^2_{x_{12}}$, $n_2^2 = s^2_{x_{12}}/\max_i\{(x_{i2} - \bar{x}_2)^2\}$ and $v_3 = \beta^2_3/[d_3(1 + n_3^2)]$, $d_3 = \sigma^2(X \tilde{X})^{-1}$, $n_3^2 = X \tilde{X} / \max_i\{| \tilde{X}_i |^2\}$ with $\tilde{X} = (I_n - X^*(X^*X^*)^{-1}X^*)x_{13}$ and $X^* = (I_n|x_{12})$.

As an example, we analyze a data set taken from Acuna (2015) which can be accessed at [http://academic.uprm.edu/eacuna/datos.html](http://academic.uprm.edu/eacuna/datos.html). We want to predict the average mileage per gallon (denoted by mpg) of a set of $n = 82$ vehicles using four possible predictor variables: cabin capacity in cubic feet ($vol$), engine power ($hp$), maximum speed in miles per hour ($sp$) and vehicle weight in hundreds of pounds ($wt$).

Through the Bayes factors in (7) and (8) we want to choose the best model to predict the average mileage per gallon by calculating the posterior probability of the null hypothesis of the following test

$$H_0 : M_2 : mpg=\beta_1+\beta_2wt+\epsilon_l \text{ vs } H_1 : M_3 : mpg=\beta_1+\beta_2wt+\beta_3sp+\epsilon_l$$

with $\alpha = 0.05$, $q = 1$, $j = 3$, the posterior probabilities for the null hypothesis $H_0$ are:

$$P_{PL} = 0.9253192, P_{PG_1} = 0.7209449$$

where $P_{PL}$ is the posterior probability associated to Bayes factor in equation (8) and $P_{PG_1}$ is the posterior probability associated to Bayes factor in equation (7). The use of this posterior probability in both cases will change the inference, since the p-value the F test is $p = 0.0325$ whose is smaller than 0.05.

### 4.3.1 Findley’s Counterexample

Consider the following simple linear model [Findley] (1991)

$$Y_i = \frac{1}{\sqrt{i}} \cdot \theta + \epsilon_i, \quad \text{where } \epsilon_i \sim N(0,1), i = 1, 2, 3, \ldots, n$$
and we are comparing the models $H_0 : \theta = 0$ and $H_1 : \theta \neq 0$. This is a Classical and challenging counter example against BIC and the Principle of Parsimony. In [Bayarri et al. (2019)] it is shown the inconsistency of BIC but the consistency of PBIC in this problem. Here we will show the posterior probabilities of the null hypothesis for this test using the Bayes factors from equations (7) and (8) when $n$ grows and $\alpha = 0.05$ and $\alpha = 0.01$, we will also show the posterior probabilities when $n$ fixed and $0 < \alpha < 0.05$. For calculations

$$C = -2 \log \left( \frac{1 - e^{-v}}{\sqrt{2v}} \right), v = \frac{\hat{\theta}^2}{d(1 + n^e)}, d = \left( \sum_{i=1}^{n} \frac{1}{i} \right)^{-1}, n^e = \sum_{i=1}^{n} \frac{1}{i}$$

The Figure 6 and Figure 7 shows through posterior probability of the null hypothesis $H_0$ the consistency of Bayes factor based in PBIC (equation (8)), and the inconsistency of Bayes factor based in BIC (equation (7)).

5 Discussion and Final Comments

1. It will be possible to estimate the appropriate $\xi_0$ that best fits the pseudo p-value in (10).

2. The Bayes factors (6) and (8) are simple to use and provides results equivalent to the sensitive Bayes factors of hypothesis tests whose p-value may be a pseudo p-value. We hope that this development will give tools to the practice of Statistics.
Figure 6: Posterior probability for the null hypothesis $H_0$ for $n = 100$, $n = 1000$ and $n = 10000$ using the Bayes factor of the equation (7) and (8).

Figure 7: Posterior probability for the null hypothesis $H_0$ for $\alpha = 0.05$ and $\alpha = 0.01$ using the Bayes factor of the equation (7) and (8) when $n$ grows.
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