SHI VARIETY CORRESPONDING TO AN AFFINE WEYL GROUP

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Abstract. In this article we show that there exists a bijection between any affine Weyl group $W_a$ and the integral points of an affine variety, denoted $\hat{X}_{W_a}$, which we call the Shi variety of $W_a$. In order to do so, we use Jian-Yi Shi’s characterization of alcoves in affine Weyl groups. We then study this variety further. We highlight combinatorial properties of irreducible components of $\hat{X}_{W_a}$ and we show how they are related to a fundamental set $P_H$ endowed with the right weak order of $W_a$. In the last section we exhibit a link between the first cohomology group of the underlying crystallographic group and some components of $\hat{X}_{W_a}$.

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1. Introduction

In this paper we introduce and investigate an affine variety $\tilde{X}_{W_a}$, called the Shi variety of $W_a$, associated to an affine Weyl group $W_a$. The construction of this variety is based on Jian-Yi Shi’s work in [19] and [20]. This development requires us to recall some basics about affine Weyl groups.

Let $V$ be an Euclidean space, $\Phi \subset V$ be an irreducible crystallographic root system, and $W$ be the corresponding Weyl group. Associated to $\Phi$ exists an infinite hyperplane arrangement, denoted $H$, known as the affine Coxeter arrangement. This hyperplane arrangement cuts out $V$ into simplices of the same volume which are called alcoves. The set of alcoves is denoted by $A$.

The associated affine Weyl group $W_a$ acts regularly on $A$, implying in particularly that there exists a one-to-one correspondence between $W_a$ and $A$ (see [14] ch.4 or Setion 3 for more details). Let $A_w$ be the corresponding alcove of $w \in W_a$.

In [19] Jian-Yi Shi characterizes any alcove $A_w$ by a $\Phi^+$-tuple of integers $(k(w,\alpha))_{\alpha \in \Phi^+}$ subject to certain conditions. From this characterization he obtains in particular that the length $\ell(w)$ is given by $\ell(w) = \sum_{\alpha \in \Phi^+} |k(w,\alpha)|$.

The initial goal of this work was to understand how left multiplication by a reflection $t \in W_a$ affects the length of an element $w \in W_a$. Identifying which reflections decrease the length of $w$ by exactly 1 was of particular interest to us. If we look at the example $W(\tilde{A}_2)$ embedded in $\mathbb{R}^3$ together with its elements of length 4 and length 5, it suggests that these elements lie on two parallel hyperplanes (see Figure 1).

![Figure 1. Elements of length 4 (red) and 5 (blue) in $W(\tilde{A}_2)$.](image)

We show in this article that the previous observation is exactly what occurs, and in the above example, the equations of the hyperplanes are $X_{13} = X_{12} + X_{23}$ and $X_{13} = X_{12} + X_{23} + 1$. 
Plan of this work. The present paper has several goals that we describe now:

The preliminary step is to recall some standard definitions and terminologies related to Coxeter groups, root systems, affine Weyl groups, Tits cone, and Shi regions. We recall this material in Sections 2 and 3.

The first goal is to introduce the necessary tools in order to see elements of $W_a$ as isometries in $\bigoplus_{\theta \in \Phi^+} \mathbb{R} \theta$. This leads us to define the $\Phi^+$-representation in Section 4. This new object has an important role in the understanding of the natural action of $W_a$ onto the variety $\hat{X}_{W_a}$. To do so, we first generalize a few formulas of [19] about the coefficients $k(w, \alpha)$. This is done in Section 4 and mainly in Proposition 4.2.

The second goal, achieved in Section 5, is to build the variety $\hat{X}_{W_a}$, whose integral points $X_{W_a}(\mathbb{Z})$ are in bijection with $W_a$. This is the main result of this article (see Theorem 5.3). To do so, we first obtain a set of equations that cuts out an affine variety denoted $X_{W_a}$, and where the $\Phi^+$-tuples of integer of $W_a$ are solutions of these equations. However, this variety is too large and we only have an injection of $W_a$ into $X_{W_a}(\mathbb{Z})$. We then see how to shrink $X_{W_a}$ in order to get a one-to-one correspondence between the integers points of a subvariety and $W_a$. This is where we introduce the notion of admissible and admitted vectors (see Definitions 5.2 and 5.4). These definitions yield, inter alia, a decomposition of $X_{W_a}$ as follows:

$$X_{W_a} := \bigsqcup_{\lambda \text{ admissible}} X_{W_a}[\lambda],$$

whereas the subvariety $\hat{X}_{W_a}$ inherits the decomposition:

$$\hat{X}_{W_a} := \bigsqcup_{\lambda \text{ admitted}} X_{W_a}[\lambda],$$

where the $X_{W_a}[\lambda]$ are the irreducible components of these two varieties. The difference between being admissible or admitted leads us to recover a polytope $P_{\Phi}$, which has already been well studied (see [3] ch VI, § 1.10 or [14] 4.9 or [15] pages 131-134), and which will play a crucial role thereafter. Indeed, thanks to it, we will give among others the number of irreducible components of $\hat{X}_{W_a}$ (see Theorem 5.3). In Section 5.4 we establish a link between the Shi regions of $W_a$ and $\hat{X}_{W_a}$. Finally, we end Section 5 by showing that the construction of $\hat{X}_{W_a}$ is functorial in the sense that it only depends of the isomorphism class of $\Phi$.

In Section 6, we show that the set of irreducible components of $\hat{X}_{W_a}$, denoted $H^0(\hat{X}_{W_a})$, has a structure of semidistributive lattice. For this purpose, we prove that the alcoves contained in $P_{\Phi}$ form a interval in the right weak order of $W_a$. This step will use the Tits cone $\mathcal{U}$ and the geometric representation of $W_a$, seen as a slice of $\mathcal{U}$. This perspective is explained beforehand in Section 2.

Our last goal is addressed in Section 7. The goal is to relate $H^1(W, \mathbb{Z}\Phi)$, the first cohomology group of $W$ with coefficients in $\mathbb{Z}\Phi$, with the irreducible components of $\hat{X}_{W_a}$ associated to the generators of $W$. 
2. Generalities about Coxeter groups

Let \((W,S)\) be a Coxeter system with \(e\) the identity element and \(S\) the set of Coxeter generators. For \(s,t \in S\) we denote \(m_{st}\) the order of \(st\). Let \(\Psi\) be an associated root system and \(\Delta = \{\alpha_1, \ldots, \alpha_n\}\) be a simple system of \(\Psi\). We denote \(\Psi^+ := \Psi \cap \text{cone}(\Delta)\). Writing \(\Psi^- = -\Psi^+\) it is known that \(\Psi = \Psi^- \sqcup \Psi^+\). For \(\alpha \in \Psi\) we denote \(s_\alpha\) the corresponding associated reflection.

The \textit{length function} \(\ell : W \rightarrow \mathbb{N}^*\) is defined as follows: \(\ell(w)\) is the smallest number \(r\) such that there exists an expression \(w = s_{i_1} \cdots s_{i_r}\) with \(s_{i_k} \in S\). By convention, \(\ell(e) = 0\). This function has been extensively studied and all basic information about it can be found in [3] or [14]. Let \(w \in W\). An expression of \(w\) is called a \textit{reduced expression} if it is a product of \(\ell(w)\) generators. The \textit{inversion set} of \(w\) is by definition

\[
N(w) := \{\alpha \in \Psi^+ \mid \ell(s_\alpha w) < \ell(w)\} = \{\alpha \in \Psi^+ \mid w^{-1}(\alpha) \in \Psi^-\}.
\]

Moreover we have \(|N(w)| = \ell(w)\). Let \(I \subset S\). The \textit{standard parabolic subgroup} \(W_I\) is defined to be the subgroup of \(W\) generated by all \(s_\alpha \in I\). It is well-known that \((W_I, I)\) is itself a Coxeter system. We denote \(\Psi_I\) its root system and \(\Delta_I\) its simple system. A subgroup \(H\) of \(W\) is called \textit{parabolic} if there exists \(w \in W\) and \(J \subset S\) such that \(H = wW_Jw^{-1}\).

2.1. Geometrical representation and Tits cone. Let \(X\) be the \(\mathbb{R}\)-vector space with basis \(\{e_s \mid s \in S\}\), and let \(B\) be the symmetric bilinear form on \(X\) defined by

\[
B(e_s, e_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right) & \text{if } m_{st} < \infty \\ -1 & \text{if } m_{st} = \infty. \end{cases}
\]

We denote by \(O_B(X)\) the orthogonal group of \(X\) associated to \(B\). For each \(s \in S\) we define \(\sigma_s : X \rightarrow X\) by \(\sigma_s(x) = x - 2B(e_s, x)e_s\). The map \(\sigma : W \hookrightarrow O_B(X)\) defined by \(s \mapsto \sigma_s\) is called the \textit{geometrical representation} of \((W,S)\) (for more information the reader may refer to [3] ch. V, § 4 or [14] ch 5.3). Since \(W\) acts on \(X\) via \(\sigma\), it also acts on \(X^*\) with the contragredient action \(\sigma^* : W \hookrightarrow GL(X^*)\) defined by \(\sigma^*(w) = ^t\sigma(w^{-1})\). The natural pairing of \(X^*\) with \(X\) is denoted by \(\langle f, x \rangle\). Therefore the action of \(W\) on \(X^*\) is characterized by \(\langle w f, wx \rangle = \langle f, x \rangle\) for \(w \in W\), \(f \in X^*\) and \(x \in X\). If \(s \in S\) and \(x \in X\) we denote

\[
Z_x = \{f \in X^* \mid \langle f, x \rangle = 0\} \\
A_s = \{f \in X^* \mid \langle f, \alpha_s \rangle > 0\}.
\]

Finally, let \(C\) be the intersection of all \(A_s\) for \(s \in S\) and let \(D = C\). The \textit{Tits cone} \(\mathcal{U}(W)\) of \(W\) is defined by \(\mathcal{U}(W) = \bigcup_{w \in W} wD\). If \(w \in W\), \(wC\) is called a \textit{chamber} of \(W\). The action of \(W\) on \(\{wC, w \in W\}\) is simply transitive (see [3] ch. V, § 4. 4). We can then identify \(W\) with the set of chambers of \(\mathcal{U}(W)\).
2.2. Canonical set of Coxeter generators. Let \((W, S)\) be a Coxeter system endowed with its length function \(\ell : W \to \mathbb{N}\). Let \(T = \{ wsw^{-1}, w \in W, s \in S \}\) be the set of reflections of \((W, S)\). Let \(W'\) be a reflection subgroup of \(W\), that is a subgroup generated by \(W' \cap T\). Denote \(\Psi_{W'} := \{ \alpha \in \Psi \mid s_\alpha \in W' \}\) and \(\Delta_{W'} := \{ \alpha \in \Psi^+ \mid N(s_\alpha) \cap \Psi_{W'} = \{ \alpha \} \}\). Matthew Dyer showed in [6] that \(\Psi_{W'}\) is a root system in \((X, B)\) with simple system \(\Delta_{W'}\). Therefore the set

\[
\chi(W') := \{ s_\alpha \mid \alpha \in \Delta_{W'} \}.
\]

is a set of Coxeter generators of \(W'\), called the canonical set of Coxeter generators of \(W'\).

Let \(f \in wD\). Then \(\text{Stab}(f)\) is a parabolic subgroup of \(W_a\). Namely \(\text{Stab}(f) = wW_jw^{-1}\) where \(W_j = \text{Stab}(w^{-1}f)\) and \(J = \{ s \in S_a \mid (f, w(\alpha_s)) = 0 \}\). Therefore, \(\text{Stab}(f)\) is a reflection subgroup of \(W_a\) with \(\chi(wW_jw^{-1})\) its canonical set of generators. Let \(u\) be the minimal coset representative of \(wW_j\). Then, by Lemma 1 in [8], one has \(\chi(wW_jw^{-1}) = u\chi(W_j)w^{-1} = uJw^{-1}\). Consequently, the simple system associated to \(\text{Stab}(f)\) is \(u(\Delta_J)\) and its root system is \(u(\Psi_I)\).

3. Affine Weyl groups and Shi parameterization

Let \((V, (-, -))\) be an Euclidean space. We denote \(\|x\| = \sqrt{(x, x)}\) for \(x \in V\). Let \(\Phi\) be an irreducible crystallographic root system in \(V\). We assume here that \(\Phi\) is essential, that is, taking the lattice \(\mathbb{Z}\Phi\), one has \(\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R} = V\). From now on, when we will say “root system” it will always mean irreducible essential crystallographic root system.

Let \(W\) be the Weyl group associated to \(\mathbb{Z}\Phi\), that is the maximal (for inclusion) reflection subgroup of \(O(V)\) admitting \(\mathbb{Z}\Phi\) as a \(W\)-equivariant lattice. Let \(\alpha \in \Phi\). We write

\[
s_\alpha : V \longrightarrow V \quad x \mapsto x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha.
\]

It is known that the Coxeter group associated to \(\Phi\), i.e the subgroup of \(O(V)\) generated by the reflections \(s_\alpha\), is actually the Weyl group \(W\). The classification of Coxeter graphs giving rise to irreducible Weyl groups is given by the Dynkin diagrams of type \(A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, \text{ and } G_2\) (see [3], Chap. VI, §4, Theorem 1). Let \(S := \{ s_{\alpha_1}, \ldots, s_{\alpha_n} \}\) be a set of Coxeter generators of \(W\).

Because of the classification of irreducible crystallographic root systems, we know that there are at most two possible root lengths in \(\Phi\). We call short root the shorter ones. To be consistent with [19], we require \(\|\alpha\| = 1\) for any short root \(\alpha \in \Phi^+\).

Let \(\alpha \in \Phi\) such that \(\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n\) with \(a_i \in \mathbb{Z}\). The height of \(\alpha\) (with respect to \(\Delta\)) is defined by the number \(h(\alpha) = a_1 + \cdots + a_n\). Height gives us an organizational principle for making inductive proofs. Height also provides a preorder on \(\Phi^+\) defined as \(\alpha \leq \beta\) if and only if \(h(\alpha) \leq h(\beta)\). We also see that \(h(\alpha + \beta) = h(\alpha) + h(\beta)\) and \(h(-\alpha) = -h(\alpha)\) for all \(\alpha, \beta, \alpha + \beta \in \Phi\). We denote \(\hat{\alpha}\) to be the highest root of \(\Phi\) and \(\alpha_0\) the highest short root.
3.1. Affine Weyl groups. From now on we will identify \( \mathbb{Z}\Phi \) and the group of its associated translations. Let \( k \in \mathbb{Z} \). Define the affine reflection as follows

\[
s_{\alpha,k} : V \rightarrow V, \quad x \mapsto x - \left( \frac{2\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} - k \right)\alpha.
\]

We consider the subgroups \( W_a \) and \( W_a^\vee \) of \( \text{Aff}(V) \) defined as follows

\[
W_a = \langle s_{\alpha,k} \mid \alpha \in \Phi, \ k \in \mathbb{Z} \rangle
\]

and

\[
W_a^\vee = \langle s_{\alpha^\vee,k} \mid \alpha \in \Phi, \ k \in \mathbb{Z} \rangle.
\]

It is known that \( W_a \cong \mathbb{Z}\Phi \rtimes W \) (see [14], Ch 4). The group \( W_a \) is called the affine Weyl group associated to \( \Phi \). The Coxeter group structure on \( W_a \) induces a Coxeter group structure on \( W_a^\vee \). The set \( S_a := \{s_{\alpha_1,1}, \ldots, s_{\alpha_n,1}\} \) is a set of Coxeter generators of \( W_a \). Throughout this article, affine Weyl group means irreducible affine Weyl group.

Let us make the following comment. In the literature about Weyl groups and affine Weyl groups it is more common to define the affine Weyl groups (associated to the root system \( \Phi \)) with the reflections associated to the hyperplanes \( \{x \in V \mid \langle x, \alpha \rangle = k\} \). Therefore, the definition of affine Weyl group for us would be in this literature the definition of the affine Weyl group associated to \( \Phi^\vee \), that is \( W_a^\vee \). Our choice to call \( W_a \) the affine Weyl group associated to \( \Phi \) is because in [19] the author gave the definition of \( W_a \) via the hyperplanes \( \{x \in V \mid \langle x, \alpha^\vee \rangle = k\} \), that is via the reflections \( s_{\alpha,k} \), and since a lot of this article is based on Jian-Yi Shi’s work it is natural to keep his conventions.

3.2. Shi parameterization. Let us write \( \alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle} \). For any \( \alpha \in \Phi \), any \( k \in \mathbb{Z} \) and any \( m \in \mathbb{R} \), we define the hyperplanes

\[
H_{\alpha,k} = \{x \in V \mid s_{\alpha,k}(x) = x\} = \{x \in V \mid \langle x, \alpha^\vee \rangle = k\},
\]

the half spaces

\[
H_{\alpha,k}^+ = \{x \in V \mid k < \langle x, \alpha^\vee \rangle\} \quad \text{and} \quad H_{\alpha,k}^- = \{x \in V \mid \langle x, \alpha^\vee \rangle < k\},
\]

and the strips

\[
H_{\alpha,k}^m = \{x \in V \mid k < \langle x, \alpha^\vee \rangle < k + m\} = H_{\alpha,k}^+ \cap H_{\alpha,k+m}^-.
\]

We denote by \( \mathcal{H} \) the set of all the hyperplanes \( H_{\alpha,k} \) with \( \alpha \in \Phi^+, k \in \mathbb{Z} \), and by \( \mathcal{H}^\vee \) the set of all the hyperplanes \( H_{\alpha^\vee,k} \) with \( \alpha \in \Phi^\vee, k \in \mathbb{Z} \). We have the relation \( H_{-\alpha,k} = H_{\alpha,-k} \). Therefore we need only to consider the hyperplanes \( H_{\alpha,k} \) with \( \alpha \in \Phi^+ \) and \( k \in \mathbb{Z} \). This last remark is quite convenient, because when we look at the collection of all the hyperplanes in \( V \), we can think of them as being indexed either by elements of \( \Phi \) or by elements of \( \Phi^+ \).

Furthermore, if we have two hyperplanes \( H_{\alpha,k} \) and \( H_{\alpha,k'} \) the geometric space strictly contained between them is

\[
H_{\alpha,\min(k,k')}^{[k-k']}
\]
The lattice $\mathbb{Z} \Phi$ acts naturally on the strips and this action is given by $\tau_{x} H_{a,k}^{m} = H_{a,k+(\alpha^{\vee},x)}^{m}$. Moreover $W$ also acts on the strips as isometries of $\mathbb{R}^{n}$ and it follows that $W_{a}$ acts also on the set of strips.

The connected components of $V \setminus \bigcup_{a \in \Phi^{+}} H_{a,k}$ are called alcoves. We denote $A$ the set of all the alcoves and $A_{e}$ the alcove defined as $A_{e} = \bigcap_{a \in \Phi^{+}} H_{a,0}^{1}$. Since $W_{a}$ acts on the strips it also acts on $A$. It turns out that this action on $A$ is regular (see [14] chapter 4). Thus, there is a bijective correspondence between the elements of $W_{a}$ and all the alcoves. This bijection is defined by $w \mapsto A_{w}$ where $A_{w} := w A_{e}$. We call $A_{w}$ the corresponding alcove associated to $w \in W_{a}$. Any alcove of $V$ can be written as an intersection of particular strips, that is there exists a $\Phi^{+}$-tuple of integers $(k(w, \alpha))_{\alpha \in \Phi^{+}}$ such that

$$A_{w} = \bigcap_{\alpha \in \Phi^{+}} H_{a,k(w,\alpha)}^{1}.$$  

In the setting of affine Weyl groups the length of any element $w \in W_{a}$ is easy to compute via the coefficients $k(w, \alpha)$. Indeed, thanks to Proposition 4.3 in [19] we have $\ell(w) = \sum_{\alpha \in \Phi^{+}} |k(w, \alpha)|$.

In [19] J.Y. Shi gave a characterization of the possible $\Phi^{+}$-tuples $(k_{\alpha})_{\alpha \in \Phi^{+}}$ over $\mathbb{Z}$ such that these tuples are the tuples of elements in $V_{a}^{1}$. In 1999 the same author gave an easier statement of this characterization. Here they are:

**Theorem 3.1** (Theorem 5.2 of [19]). Let $A = \bigcap_{a \in \Phi^{+}} H_{a,k_{a}}^{1}$ with $k_{a} \in \mathbb{Z}$. Then $A$ is an alcove, if and only if, for all $\alpha, \beta \in \Phi^{+}$ satisfying $\alpha + \beta \in \Phi^{+}$, we have the following inequality

$$||\alpha||^{2} k_{\alpha} + ||\beta||^{2} k_{\beta} + 1 \leq ||\alpha + \beta||^{2} (k_{\alpha + \beta} + 1) \leq ||\alpha||^{2} k_{\alpha} + ||\beta||^{2} k_{\beta} + ||\alpha||^{2} + ||\beta||^{2} + ||\alpha + \beta||^{2} - 1.$$  

**Theorem 3.2** (Theorem 1.1 of [21]). Let $A = \bigcap_{a \in \Phi^{+}} H_{a,k_{a}}^{1}$ with $k_{a} \in \mathbb{Z}$. Then $A$ is an alcove, if and only if, for all $\alpha, \beta \in \Phi^{+}$ satisfying $\gamma := (\alpha^{\vee} + \beta^{\vee})^{\vee} \in \Phi^{+}$ the following inequality hold

$$k_{\alpha} + k_{\beta} \leq k_{\gamma} \leq k_{\alpha} + k_{\beta} + 1.$$  

**Remark 3.1.** Notice that it is a priori not easy for a $\Phi^{+}$-tuple of integers to satisfy the inequalities (2).

We also want to warn the reader that compared to the conventions of J.Y. Shi in [19] and [20], we swap left and right multiplication. The left multiplication is defined as follows: the alcove corresponding to $w'w$ is obtained by acting by the isometry $w'$ on the alcove $A_{w}$. The right multiplication is defined as follows: the alcove corresponding to $ws$ is obtained by crossing the wall of $A_{w}$ associated to the generator $s$.

**Example 3.1.** Take $W_{a} = W(\hat{B}_{2})$ and set $\{\alpha_{1}, \alpha_{2}\}$ a simple system of $B_{2}$. We denote $s_{i} := s_{\alpha_{i}}$ for $i = 1, 2$ and $s_{3} := s_{\alpha_{0,1}}$. We also identify the alcoves with the elements of $W_{a}$. We see for example that the way to reach $s_{3}s_{2}s_{1}s_{3}s_{2}$ from the identity element $e$ is by flipping the alcove $A_{e}$ along its edges, which are associated with the generators of $W(\hat{B}_{2})$. 


Example 3.2. For $W_a = \tilde{A}_2$, the positive root system of $A_2$ is given by 3 roots, say $\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$. Shi’s parameterization is shown in the following picture:
3.3. Fundamental parallelepiped $P_H$. Let $\mathbb{Z}\Phi^\vee$ be the coroot lattice and let us write $\mathbb{Z}\Phi^\vee = \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_n^\vee$. We define its dual lattice $(\mathbb{Z}\Phi^\vee)^*$ as
$$(\mathbb{Z}\Phi^\vee)^* := \{ x \in V \mid (x, y) \in \mathbb{Z} \forall y \in \mathbb{Z}\Phi^\vee \}.$$The lattice $(\mathbb{Z}\Phi^\vee)^*$ is called the weight lattice. This lattice can be decomposed as $(\mathbb{Z}\Phi^\vee)^* = \mathbb{Z}\omega_1^\vee \oplus \cdots \oplus \mathbb{Z}\omega_n^\vee$ where $\omega_i$ is such that $(\alpha_i^\vee, \omega_j) = \delta_{ij}$. The elements $\omega_i$ are called the fundamental weights (with respect to $\Delta$). Notice that it is also possible to define the weight lattice by $(\mathbb{Z}\Phi^\vee)^* = \{ x \in V \mid (x, y) \in \mathbb{Z} \forall y \in \Phi^\vee \}$.

Since $\Phi$ is crystallographic, $\Phi^\vee$ is also crystallographic and the inclusion $\mathbb{Z}\Phi^\vee \subset (\mathbb{Z}\Phi^\vee)^*$ follows. Consequently, we are able to define the quotient group $(\mathbb{Z}\Phi^\vee)^*/\mathbb{Z}\Phi^\vee$. A good listing of these quotient groups is given in [3]. The index of connection of $\Phi$ is by definition the cardinality of this quotient group. We denote it as $f_\Phi := \left| (\mathbb{Z}\Phi^\vee)^*/\mathbb{Z}\Phi^\vee \right|$. We define $P_H := \bigcap_{\alpha \in \Delta} H^1_{\alpha,0}$ and $P_{H^\vee} := \bigcap_{\alpha \in \Delta} H^1_{\alpha^\vee,0}$. The fundamental weights $\omega_i$ are some of the vertices of $P_H$ and we have $P_H = \{ \sum_{i=1}^n c_i \omega_i \mid c_i \in [0, 1] \}$. Since $(\omega_i, \omega_j) \geq 0$ for all $i, j$, the element of maximal norm in $P_H$ is the vertex $\rho := \sum_{i=1}^n \omega_i$. Moreover, if $z \in \text{cone}(\Delta)$ we have $(z, \omega_i) \geq 0$ for all fundamental weight $\omega_i$.

It is known (see [3] ch. VI, § 1, exercice 7) that the index of connection of $\Phi$ is the determinant of the Cartan matrix associated fo $\Phi$. Moreover, the Cartan matrix associated to $\Phi^\vee$ is the transpose of the Cartan matrix associated to $\Phi$. It follows that the indexes of connection of $\Phi$ and $\Phi^\vee$ are the same. Finally, we define the two sets $\text{Alc}(P_H) := \{ w \in W_a \mid A_w \subset P_H \}$ and $\text{Alc}(P_{H^\vee}) := \{ w \in W_a^\vee \mid A_w \subset P_{H^\vee} \}$. It is well known (see [15], Section 11-6, Lemma C) that $|\text{Alc}(P_{H^\vee})| = \frac{|W(\Delta)|}{f_\Phi}$. \hfill \(\blacksquare\)

Example 3.3. Let us take $W_a = W(\tilde{B}_2)$ with simple system $\{\alpha_1, \alpha_2\}$. A short computation shows that $\omega_1 = \frac{1}{2}(2\alpha_1 + \alpha_2)$ and $\omega_2 = \alpha_1 + \alpha_2$.

![Figure 4. Fundamental parallelepiped $P_{B_2}$](image)
3.4. Tits cone and links with alcoves. Let \( \hat{V} = V \oplus \mathbb{R} \delta \) with \( \delta \) an indeterminate. The inner product \((-,-)\) has a unique extension to a symmetric bilinear form on \( \hat{V} \) which is positive semidefinite and has a radical equal to the subspace \( \mathbb{R} \delta \). This extension is also denoted \((-,-)\), and it turns out that the set of isotropic vectors associated to the form \((-,-)\) is exactly \( \mathbb{R} \delta \).

The link between \( \hat{V} \) and the geometrical representation is as follows.

Let \( \Delta = \{ \alpha_s | s \in S \} \) be the simple system associated to \( W \). We can now identify the \( X \) of section 2.1 with \( \hat{V} \), by sending \( e_s \) to \( \frac{\alpha_s}{||\alpha_s||} \). It is well known that \( (\alpha_s, \alpha_t) = ||\alpha_s|| ||\alpha_t|| \cos(\theta) \) where \( \theta \) is the angle between \( \alpha_s \) and \( \alpha_t \) in the plane generated by these two vectors. Moreover, it is also well known that \( \theta = \pi - \frac{\pi}{m_{st}} \). It follows that

\[
(\alpha_s, \alpha_t) = ||\alpha_s|| ||\alpha_t|| \cos\left(\pi - \frac{\pi}{m_{st}}\right) = -||\alpha_s|| ||\alpha_t|| \cos\left(\frac{\pi}{m_{st}}\right) = ||\alpha_s|| ||\alpha_t|| B(e_s, e_t).
\]

Furthermore we know that in the crystallographic root systems there are at most two root lengths. If \( \alpha_s \) is short we have set before that \( ||\alpha_s|| = 1 \). Therefore in the simply laced cases we have \( (\alpha_s, \alpha_t) = B(e_s, e_t) \). When \( \alpha_s \) is longer than \( \alpha_t \) we have two situations to look at: if \( m_{st} = 4 \) then \( ||\alpha_s|| = \sqrt{2} ||\alpha_t|| = \sqrt{2} \), and in particular \( (\alpha_s, \alpha_t) = \sqrt{2} B(e_s, e_t) \). If \( m_{st} = 6 \) then \( ||\alpha_s|| = \sqrt{3} ||\alpha_t|| = \sqrt{3} \) and it follows that \( (\alpha_s, \alpha_t) = \sqrt{3} B(e_s, e_t) \).

Let us denote \( \hat{V}^* = \text{Hom}(\hat{V}, \mathbb{R}) \) and \( \hat{V}^\perp = \{ x \in \hat{V} | (x, y) = 0 \ \forall y \in \hat{V} \} \). Let \( d : \hat{V} \rightarrow \mathbb{R} \) be the map defined by \( d(a + b\delta) = b \). We consider the following map

\[
i_k : \ V \rightarrow \hat{V}^*
\]

\[
v \mapsto i_k(v) : \ V \rightarrow \mathbb{R}
\]

\[
x \mapsto (v, x) + kd(x).
\]

It is well known that the linear map

\[
\Theta : \ \hat{V} \rightarrow \hat{V}^*
\]

\[
v \mapsto (v, -)
\]

is such that \( \Theta(\hat{V}) = \{ f \in \hat{V}^* | (f, \delta) = 0 \} \) and such that \( \ker(\Theta) = \hat{V}^\perp \).

Therefore \( \Theta \) induces an isomorphism denoted \( \overline{\Theta} \) between \( \hat{V}/\hat{V}^\perp \) and \( \{ f \in \hat{V}^* | (f, \delta) = 0 \} \) defined as \( \overline{f}(v) := \Theta(v) \) for all \( v \in \hat{V} \).

Since \( \hat{V}^\perp = \mathbb{R} \delta \), one has \( \hat{V}/\hat{V}^\perp \simeq V \) through the map \( \varphi \) defined as \( \varphi(x + y\delta) = x \). Thus, the map \( i_0 \) is nothing but \( \overline{\Theta} \circ \varphi^{-1} \), while \( i_1 \) is injective and such that \( i_1(V) = \{ f \in \hat{V}^* | (f, \delta) = 1 \} \). Let us define \( \nu_1 \in \text{Hom}(\hat{V}^*, \hat{V}^*) \) by \( \nu_1(f) = f + d \).

Finally, one can summarize all of this via the following commutative diagram

\[
\begin{array}{ccc}
\hat{V}/\hat{V}^\perp & \overset{\varphi^{-1}}{\longrightarrow} & \{ f \in \hat{V}^* | (f, \delta) = 0 \} \\
\overline{\Theta} \downarrow \quad & & \downarrow \nu_1 \\
V & \overset{i_0}{\longrightarrow} & \{ f \in \hat{V}^* | (f, \delta) = 1 \} \\
i_1 \downarrow \quad & & \\
\end{array}
\]
Our main goal is to see each alcove \( A_w \) of \( W_a \), as the intersection of the chamber \( wC \) and the hyperplane \( \{ f \in V^* | \langle f, \delta \rangle = 1 \} \). Of course, alcoves and chambers don’t live in the same space, but using the map \( i_1 \) we relate the notion of alcoves to the notion of chamber. More precisely, an alcove \( A_w \) (living in \( V \)) is sent via \( i_1 \) to \( wC \cap \{ f \in V^* | \langle f, \delta \rangle = 1 \} \). See for example Figure 5.

The significant thing for us is to see that the hyperplane \( H_{\alpha,k} \) in \( V \) is sent via \( i_1 \) to the affine subspace \( Z_{-\alpha^\vee + k\delta} \cap \{ f \in \hat{V}^* | \langle f, \delta \rangle = 1 \} \). Indeed for \( v \in H_{\alpha,k} \) one has \( i_1(v)(\delta) = (v, \delta) + d(\delta) = d(\delta) = 1 \). Further, we also have

\[
i_1(v)(-\alpha^\vee + k\delta) = -(v, \alpha^\vee) + d(-\alpha^\vee + k\delta) = -(v, \alpha^\vee) + k = 0.
\]

From now on we define \( E(W_a) \), or \( E \) for short, to be the set:
\[
E := \{ f \in \hat{V}^* | \langle f, \delta \rangle = 1 \}.
\]

![Figure 5. Chamber in the Tits cone of \( W(\tilde{B}_2) \) with its trace in \( E \).](image-url)

The root system of \( W_a \) is denoted \( \Phi_a \) and its simple system is denoted \( \Delta_a \). Keeping our conventions (see Section 3.1) and using [5] (Section 3.3 Definition 4 and Proposition 2) a concrete description of the affine (respectively,
positive, simple) root system of \( W_a \) is provided by:

\[
\Phi_a = \Phi^+ + \mathbb{Z}\delta = \{\alpha + k\delta \mid \alpha \in \Phi^+, \ k \in \mathbb{Z}\},
\]

\[
\Phi_a^+ = ((\Phi^+)^+ + \mathbb{N}\delta) \cup ((\Phi^-)^+ + \mathbb{N}\delta) = \{\alpha + k\delta \mid \alpha \in \Phi^+, \ k \in \mathbb{N}, \ k \in \mathbb{N}^* \text{ if } \alpha \in (\Phi^-)^-\},
\]

\[
\Delta_a = \Delta^+ \cup \{\delta - \alpha \delta\}.
\]

We end this section with the following fact:

**Proposition 3.1.** Let \( \alpha, \beta \in \Phi \). Then we have for all \( k, p \in \mathbb{Z} \) and for all \( d \in \mathbb{N} \)

\[
\text{ord}(s_\alpha s_\beta)^d = \text{ord}(s_\alpha k s_\beta p)^d).
\]

**Proof.** Let \( x, y \in \Phi_a \). In order to prevent any confusion, we denote by \( t_x \), \( t_y \) the reflections of \( V \) associated to \( x \) and \( y \), while \( s_\alpha, s_\beta, s_\alpha k, s_\beta p \) are some reflections of \( V \).

It turns out that \( t_x t_y \) is a rotation of angle \(-2\sigma\) with \( \sigma \) the oriented angle between \( x \) and \( y \). It implies in particular that \( (t_x t_y)^q \) is a rotation of angle \(-2q\sigma\). Consequently, \( \text{ord}(t_x t_y) = q \) if and only if \( \sigma = \frac{b\pi}{q} \) for some \( b \in \mathbb{Z} \).

The main point is to see that the order is entirely controlled by the angle \( \sigma \).

Therefore, when we apply this with \( x := -\alpha^\vee \), \( y := -\beta^\vee \) and \( \sigma := \theta_1 \), since \( t_\alpha = t_{-\alpha^\vee} \) and \( t_\beta = t_{-\beta^\vee} \), we have \( \text{ord}(t_\alpha t_\beta) = q \) if and only if \( \theta_1 = \frac{b\pi}{q} \) for some \( b \in \mathbb{Z} \). Let us set \( q \) to be the order of \( t_\alpha t_\beta \). Without lost of generality, since the reflections \( t_x \) are vectorial they don’t take account of the length of \( x \), and then one can assume that \( ||x|| = 1 \) for all \( x \in \Phi_a \). Notice first of all that \( \text{ord}(s_\alpha s_\beta) = \text{ord}(t_\alpha t_\beta) \) and \( \text{ord}(s_\alpha k s_\beta p) = \text{ord}(t_{-\alpha^\vee+k\delta} t_{-\beta^\vee+p\delta}) \).

Furthermore, since \( \delta \) is isotropic we have

\[
(-\alpha^\vee + k\delta, -\beta^\vee + p\delta) = (\alpha^\vee, \beta^\vee) + kp(\delta, \delta) = (\alpha^\vee, \beta^\vee).
\]

Moreover

\[
(\alpha^\vee, \beta^\vee) = ||\alpha^\vee|| ||\beta^\vee|| \cos(\theta_1) = \cos(\theta_1),
\]

and writing \( \theta_2 \) to be the angle between \(-\alpha^\vee + k\delta\) and \(-\beta^\vee + p\delta\) we have

\[
(-\alpha^\vee + k\delta, -\beta^\vee + p\delta) = ||-\alpha^\vee + k\delta|| ||-\beta^\vee + p\delta|| \cos(\theta_2) = \cos(\theta_2).
\]

It follows then that \( \cos(\theta_1) = \cos(\theta_2) \), which implies that \( \theta_2 = \theta_1 + 2\pi r \) for some \( r \in \mathbb{Z} \). Then \( \theta_2 = \frac{b\pi}{q} + 2\pi r = \frac{(b + 2pr)\pi}{q} \) and it follows that \( \text{ord}(t_{-\alpha^\vee+k\delta} t_{-\beta^\vee+p\delta}) = q \). Thus we have shown that \( \text{ord}(s_\alpha s_\beta) = \text{ord}(s_\alpha k s_\beta p) \) for all \( \alpha, \beta \in \Phi \) and for all \( k, p \in \mathbb{Z} \).

We only give the proof for \( d = 2 \), the others cases being just a tedious induction. A short computation shows that

\[
s_\alpha k s_\beta p s_\alpha k = s_{s_\alpha}(\beta)p + (ka, s_\alpha(\beta)^\vee).
\]

Therefore \( \text{ord}(s_\alpha k s_\beta p s_\alpha k s_\beta p) = \text{ord}(s_{s_\alpha}(\beta)p + (ka, s_\alpha(\beta)^\vee)s_\beta p) \) and because of what we have done before we know that \( \text{ord}(s_{s_\alpha}(\beta)p + (ka, s_\alpha(\beta)^\vee)s_\beta p) = \text{ord}(s_{s_\alpha}(\beta)p) = \text{ord}(s_\alpha s_\beta s_\alpha s_\beta) \). Thus we have shown that

\[
\text{ord}(s_\alpha s_\beta)^2 = \text{ord}(s_\alpha k s_\beta p)^2.
\]
3.5. Shi regions. Introduced by Brink and Howlett the dominance order is the partial order \( \preceq \) on \( \Psi^+ \) defined by
\[
\alpha \preceq \beta \iff \forall w \in W, \beta \in N(w) \implies \alpha \in N(w).
\]
For \( \beta \in \Psi^+ \), the dominance set of \( \beta \) is \( \text{Dom}(\beta) := \{ \alpha \in \Psi^+ \mid \alpha \prec \beta \} \).
The infinite depth on \( \Psi^+ \) is defined by \( \text{dp}_\infty(\beta) = |\text{Dom}(\beta)| \). We say that \( \beta \) is a small root if \( \text{dp}_\infty(\beta) = 0 \). The set of small roots is denoted \( \Sigma \). In the context of affine root systems the set of small roots is denoted \( \Sigma_a \). The Shi arrangement of \( W \) is defined by
\[
U(W) \setminus \bigcup_{\alpha \in \Sigma} Z_{\alpha}.
\]

The connected components of this arrangement are called the Shi regions of \( W \) (see [9] 3.1 and 3.2 for more details). We denote \( \text{Shi}(W) \) the set of Shi regions of \( W \). In particular when \( X = \hat{V} \), the Shi regions of \( W_a \) are the connected components of
\[
U(W_a) \setminus \bigcup_{\theta \in \Sigma_a} Z_{\theta}.
\]

In [20] J.Y. Shi introduced the notion of signed regions. For an integer \( k \) we define \( s_g(k) = 0 \) if \( k = 0 \), \( s_g(k) = + \) if \( k > 0 \) and \( s_g(k) = - \) if \( k < 0 \). For a \( \Phi^+ \)-tuple of integers \( (k_\alpha)_{\alpha \in \Phi^+} \) we define the map \( S_g \) as \( S_g((k_\alpha)_{\alpha \in \Phi^+}) = (s_g(k_\alpha))_{\alpha \in \Phi^+} \). We denote
\[
S_g(w) := S_g((k(w, \alpha))_{\alpha \in \Phi^+}).
\]

A signed region \( \Gamma \) of \( W_a \) is by definition a subset of \( W_a \) such that \( S_g(w) = S_g(w') \) for all \( w, w' \in \Gamma \). Therefore, being on the same signed region defines an equivalence relation on \( W_a \). We denote \( S_g(W_a) \) the set of signed regions of \( W_a \).

**Theorem 3.3.** The following map is a bijection
\[
\Lambda : \text{Shi}(W_a) \rightarrow S_g(W_a) \quad \Omega \mapsto i_1^{-1}(\Omega \cap E).
\]

**Proof.** Because of the definition of signed regions we know that the hyperplanes that separate these regions are exactly the \( H_{\alpha,0} \) and \( H_{\alpha,1} \) for \( \alpha \in \Phi^+ \). Thus, the signed regions of \( W_a \) are the connected components of \( V \setminus \bigcup_{\alpha \in \Phi^+} H_{\alpha,k} \). Moreover, it is known (see [9] Example 3.9) that for all \( \alpha \in \Phi^+ \) we have the equality
\[
\text{dp}_\infty(\alpha^\vee + k\delta) = \begin{cases} 
  k & \text{if } \alpha^\vee \in (\Phi^+) + \text{ and } k \in \mathbb{N} \\
  k - 1 & \text{if } \alpha^\vee \in (\Phi^+) - \text{ and } k \in \mathbb{N}^*.
\end{cases}
\]

Therefore, \( \text{dp}_\infty(\alpha^\vee + k\delta) = 0 \) if and only if \( k \in \{0,1\} \). It follows that the set of hyperplanes that separate the Shi regions in \( U(W_a) \) is exactly \( \{ Z_{\alpha^\vee} \mid \alpha \in \Phi^+ \} \sqcup \{ Z_{-\alpha^\vee + \delta} \mid \alpha \in \Phi^+ \} \). Hence, the connected components of \( U(W_a) \setminus \bigcup_{\alpha \in \Phi^+} Z_{\alpha^\vee} \cup \bigcup_{\alpha \in \Phi^+} Z_{-\alpha^\vee + \delta} \) are the Shi regions of \( W_a \). Moreover \( i_1^{-1}(Z_{\alpha^\vee} \cap E) = H_{\alpha,0} \) and \( i_1^{-1}(Z_{-\alpha^\vee + \delta} \cap E) = H_{-\alpha,-1} = H_{\alpha,1} \). By continuity
of $i_1^{-1}$, each Shi region intersected with $E$ is sent to a signed region of $W_a$, and since $(i_1^{-1})_E$ is a bijection, the map $\Lambda$ is also a bijection. □

Figure 6. Shi regions of $W(\tilde{A}_2)$ intersected with $E$.

Figure 7. Signed regions of $W(\tilde{A}_2)$. 
4. $\Phi^+$-representation

The idea of this section is to see the elements of $W_a$ as affine isometries in a bigger space. Since any element $w \in W_a$ is characterized by a $\Phi^+$-tuple of integers $(k(w, \alpha))_{\alpha \in \Phi^+}$ and since $W_a$ acts geometrically on itself it is natural to want to see this action in $\mathbb{R}^{|\Phi^+|}$. For a Euclidean space $E$, we identify $E$ with the space of its translation, and we denote by $\text{Isom}(E)$ the space of its affine isometries, that is the elements of $E \rtimes \text{GL}(E)$ such that the euclidean scalar product is preserved. In other words $\text{Isom}(E) = E \rtimes \text{O}(E)$.

Let us write $\Phi^+ = \{\beta_1, \beta_2, \ldots, \beta_m\}$ with $m = |\Phi^+|$. We send $\Phi^+$ to the canonical basis of $\mathbb{R}^m$ via the map $\iota$. More specifically, since an element $w \in W_a$ is identified with the $\Phi^+$-tuple $(k(w, \gamma))_{\gamma \in \Phi^+}$ we set $\iota(w) := (k(w, \gamma))_{\gamma \in \Phi^+}$. The natural action of $s_{\alpha, k} \in \text{Isom}(\mathbb{R}^{|\Delta|})$ corresponds to acting on $W_a$ by left multiplication, that is if $x \in A_w$ then $s_{\alpha, k}(x) \in A_{w'}$ where $w' = s_{\alpha, k}w$. We will denote the left multiplication by $w$ as $L_w$. The main goal of this section is to prove the following theorem:
**Theorem 4.1.** There exists an injective morphism $F : W_a \to Isom(\mathbb{R}^m)$ such that for any $w \in W_a$ the following diagram commutes. This morphism is called the $\Phi^+$-representation of $W_a$.

$$
\begin{array}{c}
W_a \xrightarrow{L_w} W_a \\
\downarrow \quad \quad \downarrow \\
\mathbb{R}^m \xrightarrow{F(w)} \mathbb{R}^m.
\end{array}
$$

We recall first of all a few results of [19] that we will use later. The following lemmas show how the coefficients $k(w, \alpha)$ behave when we act on $w$ by the available operations in $W_a$.

**Lemma 4.1** (Lemma 3.1 of [19]). Let $w$ be an element of $W \subset W_a$ and $\alpha \in \Phi^+$. Then

$$
k(w, \alpha) = \begin{cases} 
0 & \text{if } w^{-1}(\alpha) \in \Phi^+ \\
-1 & \text{if } w^{-1}(\alpha) \in \Phi^-.
\end{cases}
$$

**Lemma 4.2** (Lemma 3.2 of [19]). Let $w$ be an element of $W$ and $x \in \mathbb{Z}\Phi$. Then for all $\alpha \in \Phi^+$ one has the following formula

$$
k(\tau_x w, \alpha) = k(w, \alpha) + (x, \alpha').$$

**Corollary 4.1** (Formula (3.3.1) of [19]). Let $\alpha \in \Phi$. For all $1 \leq i \leq n$ we have

$$
k(s_i, \alpha) = \begin{cases} 
1 & \text{if } \alpha = -\alpha_i \\
0 & \text{if } \alpha \neq \pm \alpha_i \\
-1 & \text{if } \alpha = \alpha_i.
\end{cases}
$$

**Proposition 4.1** (Proposition 4.2 of [19]). Let $w \in W_a$ and $s \in S_a$. Then for all $\alpha \in \Phi^+$ one has the following formula

$$
k(sw, \alpha) = k(w, s(\alpha)) + k(s, \alpha).
$$

**Proposition 4.2.** Let $w \in W_a$ and $\alpha, \beta \in \Phi^+$. One has the following formulas

1) $k(s_\alpha w, \beta) = \begin{cases} 
k(w, s_\alpha(\beta)) & \text{if } s_\alpha(\beta) \in \Phi^+ \\
k(w, s_\alpha(\beta)) - 1 & \text{if } s_\alpha(\beta) \in \Phi^- 
\end{cases}$

2) $k(s_{\alpha, p} w, \beta) = \begin{cases} 
k(w, s_\alpha(\beta)) - p(\alpha, s_\alpha(\beta)^\vee) & \text{if } s_\alpha(\beta) \in \Phi^+ \\
k(w, s_\alpha(\beta)) - 1 - p(\alpha, s_\alpha(\beta)^\vee) & \text{if } s_\alpha(\beta) \in \Phi^- 
\end{cases}$

**Proof.** 1) Let $s_\alpha = s_1 s_2 \ldots s_n$ be a reduced expression of $s_\alpha$. Since $s_\alpha$ is a reflection one has $s_1 s_2 \ldots s_n = s_n \ldots s_2 s_1$. Using Proposition 4.1 enough times we get that $k(s_\alpha w, \beta) = k(w, s_n \ldots s_1(\beta)) + \sum_{i=1}^n k(s_i, s_{i-1} \ldots s_1(\beta))$. When $i = 1$, the empty product on the righthand side is understood as the identity element. Further, by using again Proposition 4.1 enough times we get

$$
k(s_\alpha, \beta) = \sum_{i=1}^n k(s_i, s_{i-1} \ldots s_1(\beta)).$$

Thus we have

$$
k(s_\alpha w, \beta) = k(w, s_n \ldots s_1(\beta)) + \sum_{i=1}^n k(s_i, s_{i-1} \ldots s_1(\beta)) = k(w, s_\alpha(\beta)) + k(s_\alpha, \beta).
$$

Since $s_\alpha \in W$ and $\beta \in \Phi^+$ we have by Lemma 4.1 that $k(s_\alpha, \beta) = 0$ if $s_\alpha(\beta) \in \Phi^+$ and $k(s_\alpha, \beta) = -1$ if $s_\alpha(\beta) \in \Phi^-$. The result follows.
Definition 4.1. Let \( s_\alpha(\beta) = s_\alpha(\beta^\vee) \) and
\[
k(s_\alpha w; \beta) = k(s_\alpha \tau_\beta w, \beta) = k(\tau s_\alpha(x) s_\alpha w, \beta) = k(s_\alpha w, \beta) + (s_\alpha(x), \beta^\vee).
\]
Thus it follows
\[
k(s_{\alpha,p} w; \beta) = k(\tau_{p_{\alpha,s}} s_\alpha \tau_\beta w, \beta) = k(\tau_{p_{\alpha,s}} s_\alpha w, \beta)
\]
\[
= k(s_\alpha w, \beta) + (p\alpha + s_\alpha(x), \beta^\vee)
\]
\[
= k(s_\alpha w, \beta) + p(\alpha, \beta^\vee) + (s_\alpha(x), \beta^\vee)
\]
\[
= k(s_\alpha w, \beta) - p(\alpha, s_\alpha(\beta^\vee)) + (s_\alpha(x), \beta^\vee)
\]
\[
= k(s_\alpha w, \beta) - p(\alpha, s_\alpha(\beta^\vee)).
\]

\[
\ell_{\beta_j, \beta_i}(\alpha) := \ell_{j,i}(\alpha) = \begin{cases} 1 & \text{if } s_\alpha(\beta_j) = \beta_j \\ 0 & \text{if } s_\alpha(\beta_j) \neq \pm \beta_j \\ -1 & \text{if } s_\alpha(\beta_j) = -\beta_j \end{cases}
\]

2) We define the vector \( v_{k,\alpha} \in \mathbb{R}^n \) as \( v_{k,\alpha} = (v_{k,\alpha}(\gamma))_{\gamma \in \Phi^+} \) where
\[
v_{k,\alpha}(\gamma) := \begin{cases} -k(\alpha, s_\alpha(\gamma)^\vee) & \text{if } s_\alpha(\gamma) \in \Phi^+ \\ -1 - k(\alpha, s_\alpha(\gamma)^\vee) & \text{if } s_\alpha(\gamma) \in \Phi^- \end{cases}
\]
For \( \gamma \in \Phi^+ \) we set the convention
\[
v_{k,\alpha}(-\gamma) := -v_{k,\alpha}(\gamma).
\]
3) We define \( F(s_{\alpha,k}) \) as the affine map such that for all \( x \in \mathbb{R}^m \) one has
\[
F(s_{\alpha,k})(x) := L_\alpha(x) + v_{k,\alpha}.
\]
For short we will write \( L_\alpha(w) \) instead of \( L_\alpha(\iota(w)) \). We denote \( L_\alpha(w)[\theta] \) the coefficient in position \( \theta \) of the vector \( L_\alpha(w) \). This coefficient is called the \( \theta \)-coefficient of \( L_\alpha(w) \). Similarly, the coefficient in position \( \theta \) of \( F(s_{\alpha,k})(w) \) is denoted \( F(s_{\alpha,k})(w)[\theta] \) and we have \( F(s_{\alpha,k})(w)[\theta] = L_\alpha(w)[\theta] + v_{k,\alpha}(\theta) \). We call it the \( \theta \)-coefficient of \( F(s_{\alpha,k})(w) \).

Proposition 4.3. Let \( \alpha, \theta \in \Phi^+, k \in \mathbb{Z} \) and \( w \in W_\alpha \). Then
1) \( L_\alpha^2 = id \) and \( F(s_{\alpha,k}) \in \text{Isom}(\mathbb{R}^m) \).
2) \( L_\alpha(w)[\theta] = k(w, s_\alpha(\theta)) \).
3) \( F(s_{\alpha,k}) \circ \iota(w) = \iota \circ L_{s_{\alpha,k}}(w) \).
Example 4.1. Take \(W_a = W(\tilde{B}_2)\) with positive roots
\[\Phi^+ = \{e_1 - e_2, e_2, e_1, e_1 + e_2\},\]
where \(\{e_1, e_2\}\) is the canonical basis of \(\mathbb{R}^2\). Some easy computations give us the following tables:
Table 1. Images of the positive roots by the reflections of $W(\tilde{B}_2)$.

| $s_{e_1-e_2}$ | $e_1 - e_2$ | $e_2$ | $e_1$ | $e_1 + e_2$ |
|----------------|-------------|-------|-------|-------------|
| $s_{e_2}$      | $- (e_1 - e_2)$ | $e_1$ | $e_2$ | $e_1 + e_2$ |
| $s_{e_1}$      | $e_1 + e_2$  | $- e_2$ | $e_1$ | $e_1 - e_2$ |
| $s_{e_1+e_2}$ | $e_1 - e_2$  | $- e_1$ | $- e_2$ | $- (e_1 + e_2)$ |

Table 2. Coordinates of the vectors $v_{k,\alpha}$ of $W(\tilde{B}_2)$.

| $v_{k,e_1-e_2}$ | $e_1 - e_2$ | $e_2$ | $e_1$ | $e_1 + e_2$ |
|-----------------|-------------|-------|-------|-------------|
| $v_{k,e_2}$     | $2k - 1$    | $-2k$ | $2k$  | $0$         |
| $v_{k,e_1}$     | $k - 1$     | $0$   | $2k - 1$ | $k - 1$ |
| $v_{k,e_1+e_2}$ | $2k - 1$    | $2k - 1$ | $2k - 1$ |

Figure 9. Affine maps $F(s_{\alpha,k})$ for all positive roots $\alpha \in B_2$.

Proof of Theorem 4.1. Let $w \in W_a$. Since $W_a = \langle s_{\alpha,k}, \alpha \in \Phi^+, k \in \mathbb{Z} \rangle$ we can write $w = s_{\alpha_1,k_1} \ldots s_{\alpha_q,k_q}$ such that $\alpha_i \in \Phi^+$ and $k_i \in \mathbb{Z}$. We define $F$ as $F(w) = F(s_{\alpha_1,k_1}) \ldots F(s_{\alpha_q,k_q})$. Because of Proposition 4.3 we know that $F(w) \in \text{Isom}(\mathbb{R}^m)$, and with the point 4) of this proposition we also know that this map is such that the below diagram commutes for all $w_1, w_2 \in W_a$. 

![Diagram](image_url)
Therefore, $F$ is a morphism and the commutativity of the below diagram follows for all $x \in W_a$

\[
\begin{array}{ccc}
W_g & \xrightarrow{L_x} & W_g \\
\downarrow & & \downarrow \\
\mathbb{R}^m & \xrightarrow{F(x)} & \mathbb{R}^m.
\end{array}
\]

This morphism is injective because $F(w_1) = F(w_2)$ implies for instance that $F(w_1)(\iota(e)) = F(w_2)(\iota(e))$ which means that $\iota \circ L_{w_1} = \iota \circ L_{w_2}$, that is $\iota(w_1) = \iota(w_2)$. Since $\iota(w_1) = (k(w_1, \alpha))_{\alpha \in \Phi^+}$ and $\iota(w_2) = (k(w_2, \alpha))_{\alpha \in \Phi^+}$ it follows that $w_1 = w_2$.

**Example 4.2.** Let us take again $W_a = W(\tilde{B}_2)$. Because of the previous theorem we know that $F(s_{e_1 - e_2}s_{e_2}s_{e_1 - e_2}) = F(s_{e_1 - e_2})F(s_{e_2})F(s_{e_1 - e_2})$. When we make the matrix product of the right term we obtain

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]

We know that the matrix on the right is the one associated to $s_{e_1}$. Furthermore a direct computation shows that $s_{e_1 - e_2}s_{e_2}s_{e_1 - e_2} = s_{e_1}$.

5. Structure of affine variety on $W_a$

Let $W_a$ be an affine Weyl group, $\Phi$ its crystallographic root system with $\Phi^+ = \{\beta_1, \ldots, \beta_m\}$, and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ its simple system. The aim of this section is to understand from a new geometric point of view, affine Coxeter groups based on their $\Phi^+$-uptet of integers obtained in [19]. Shi gave in [19] a characterization of elements in $W_a$ via a collection of inequalities. He also gave an easier characterization of the $\Phi^+$-tuples in Theorem 2. The goal here is to transform these inequalities into equations. From now on we will denote $A[X_\Delta] := A[X_{\alpha_1}, \ldots, X_{\alpha_n}]$ and $A[X_{\Phi^+}] := A[X_{\beta_1}, \ldots, X_{\beta_m}]$ where $A$ is a commutative ring. In particular we have $A[X_{\alpha_1}, \ldots, X_{\alpha_n}] \subset A[X_{\beta_1}, \ldots, X_{\beta_m}]$. For $w \in W_a$ and $Q \in \mathbb{R}[X_\Delta]$ we denote

\[Q(w) := Q(k(w, \alpha_1), \ldots, k(w, \alpha_n)).\]

5.1. Decomposition of the coefficients $k(w, \alpha)$.

**Theorem 5.1.** Let $w \in W_a$. Then we have

1) For all $\theta \in \Phi^+ - \Delta$ there exists a linear polynomial $P_\theta \in \mathbb{Z}[X_\Delta]$ with positive coefficients and $\lambda_\theta(w) \in [0, h(\theta^\vee) - 1]$ such that

\[k(w, \theta) = P_\theta(w) + \lambda_\theta(w).\]

Moreover, the polynomial $P_\theta$ is uniquely determined by the above equation.

2) $P_\theta = P_{\alpha} + P_{\beta}$ for all $\alpha, \beta \in \Phi^+$ such that $\theta^\vee = \alpha^\vee + \beta^\vee$.  

Proof. We proceed by induction on the height $h$ of coroots. Let $\theta \in \Phi^+ - \Delta$. Therefore $\theta^\vee \in (\Phi^\vee)^+$ and there exist $\alpha, \beta \in \Phi^+$ such that $\theta^\vee = \alpha^\vee + \beta^\vee$, notice that it is always possible to write an element of $(\Phi^\vee)^+ - \Delta^\vee$ as a sum of two others elements of $(\Phi^\vee)^+$. Thus we have $\theta = (\alpha^\vee + \beta^\vee)\in \Phi^+$ and because of Theorem 3.2 it follows $k(w, \alpha) + k(w, \beta) \leq k(w, \theta) \leq k(w, \alpha) + k(w, \beta) + 1$. Hence $k(w, \theta) = k(w, \alpha) + k(w, \beta) + \gamma_\theta(w)$ where $\gamma_\theta(w) \in [0,1]$.

If $h(\theta^\vee) = 2$ one has necessarily $\alpha^\vee, \beta^\vee \in \Delta^\vee$. Therefore by setting $P_\theta := X_\alpha + X_\beta$ and $\lambda_\theta(w) := \gamma_\theta(w)$ we have shown the base case.

Assume now that $h(\theta^\vee) = d + 1$ and assume that the formula (3) is true for all coroots of height less than $d + 1$. We also assume that for all $\gamma \in \Phi^+$ satisfying $h(\gamma^\vee) < d + 1$ the polynomial $P_\gamma$ is linear and its coefficients are positive. Since $\alpha^\vee + \beta^\vee = \theta^\vee$, we have $h(\alpha^\vee) < h(\theta^\vee)$ and $h(\beta^\vee) < h(\theta^\vee)$. It follows that $k(w, \alpha) = P_\alpha(w) + \lambda_\alpha(w)$ and $k(w, \beta) = P_\beta(w) + \lambda_\beta(w)$ with $P_\alpha, P_\beta \in \mathbb{Z}[X_\Delta]$ both linear, $\lambda_\alpha(w) \in [0, h(\alpha^\vee) - 1]$ and $\lambda_\beta(w) \in [0, h(\beta^\vee) - 1]$. It follows that:

$$k(w, \theta) = P_\alpha(w) + \lambda_\alpha(w) + P_\beta(w) + \lambda_\beta(w) + \gamma_\theta(w).$$

By setting $P_\theta := P_\alpha + P_\beta$ and $\lambda_\theta(w) = \lambda_\alpha(w) + \lambda_\beta(w) + \gamma_\theta(w)$ we have $P_\theta$ linear, its coefficients are positive since those of $P_\alpha$ and $P_\beta$ are positive, and finally since $h(\theta^\vee) = h(\alpha^\vee) + h(\beta^\vee)$ it results that $\lambda_\theta(w) \in [0, h(\theta^\vee) - 1]$. This ends the induction and the proof of points 1) and 2).

Example 5.1. Let $\Delta(B_n) := \{\alpha_1, \ldots, \alpha_n\}$ with its Dynkin diagram:

\[ \begin{array}{c}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n-1} & \alpha_n \\
\end{array} \]

FIGURE 10. Dynkin diagram of type $B_n$.

It is well-known (see, for example, [3]) that its positive root system decomposes as

$$\Phi^+(B_n) = I_1 \sqcup I_2 \sqcup I_3,$$

where $I_1 := \{ \sum_{i=1}^{j-1} \alpha_i \mid 1 \leq i < j \leq n \}$, $I_2 := \{ \sum_{i=1}^{n} \alpha_i \mid 1 \leq i \leq n \}$ and $I_3 := \{ \sum_{t=i}^{j-1} \alpha_t + 2 \sum_{t=j}^{n} \alpha_t \mid 1 \leq i < j \leq n \}$. It turns out that the short roots are those in $I_2$ and the long ones are those in $I_1 \sqcup I_3$. Moreover, the only simple root that is short is $\alpha_n$. Let $\alpha \in \Phi^+(B_n)$ and $w \in W_a$. After computation we get:

i) If $\alpha = \sum_{t=i}^{j-1} \alpha_t \in I_1$ then there exists $\lambda_\alpha(w) \in [0, j - i - 1]$ such that

$$k(w, \alpha) = \sum_{t=i}^{j-1} k(w, \alpha_t) + \lambda_\alpha(w).$$

Here $P_\alpha = \sum_{k=i}^{j-1} X_{\alpha_k}$ and it is easy to see that $j - i = h(\alpha) = h(\alpha^\vee)$.

ii) If $\alpha = \sum_{t=i}^{n} \alpha_t \in I_2$ then there exists $\lambda_\alpha(w) \in [0, 2(n - i)]$ such that

$$k(w, \alpha) = 2 \sum_{t=i}^{n-1} k(w, \alpha_t) + k(w, \alpha_n) + \lambda_\alpha(w).$$

Here $P_\alpha := \sum_{k=i}^{n-1} X_{\alpha_k} + X_{\alpha_n}$ and it is easy to see that $h(\alpha^\vee) = 2(n - i) + 1$. 

Definition 5.1. Let $J$ define the ideal $I$ is a simple root then
\[ k(w, \alpha) = \begin{cases} 
  j-1 \sum_{t=i} k(w, \alpha_t) + 2 \sum_{t=j} n-1 k(w, \alpha_t) + k(w, \alpha_n) + \gamma(\alpha(w)) & \text{if } j < n \\
  j \sum_{t=i} k(w, \alpha_t) + \gamma(\alpha(w)) & \text{if } j = n.
\end{cases} \]

Here we have
\[ P_\alpha = \begin{cases} 
  j-1 \sum_{k=i} X_{\alpha_k} + 2 \sum_{k=j} n-1 X_{\alpha_k} + X_{\alpha_n} & \text{if } j < n \\
  n \sum_{k=i} X_{\alpha_k} & \text{if } j = n,
\end{cases} \]

and it is easy to see that $h(\alpha^\vee) = 2n - (i + j) + 1$.

**Definition 5.2.** Let $\theta \in \Phi^+$. Write $I_\theta := \{0, h(\theta^\vee) - 1\}$. Notice that if $\theta$ is a simple root then $I_\theta = \{0\}$. For any root $\theta \in \Delta$ we set $P_\theta = X_\theta$ and $\lambda_0 = 0$. We denote by $P_\theta[\lambda_\alpha]$ the polynomial $P_\theta + \lambda_\alpha - X_\alpha \in A[X_\Phi^+]$. We define the ideal $J_{W_a}$ of $\mathbb{R}[X_\Phi^+]$ as $J_{W_a} := \sum_{\alpha \in \Phi^+} \langle \prod_{\lambda_\alpha \in I_\alpha} P_\alpha[\lambda_\alpha] \rangle$. We define the affine variety $X_{W_a}$ to be:

\[ X_{W_a} := V(J_{W_a}). \]

**Definition 5.3.** Let $\lambda$ be an admissible vector. We denote
\[ J_{W_a}[\lambda] := \sum_{\alpha \in \Phi^+} \langle P_\alpha[\lambda_\alpha] \rangle = \langle P_\alpha[\lambda_\alpha], \alpha \in \Phi^+ \rangle, \]

\[ X_{W_a}[\lambda] := V(J_{W_a}[\lambda]) \text{ and } X_{W_a}[0] := V(J_{W_a}[0_R^m]). \]

**Remark 5.1.** We recall that our goal is to have a bijection between $W_a$ and the integral points of a variety. $X_{W_a}$ seems to be a good candidate because the way we defined it shows that $W_a \hookrightarrow X_{W_a}(\mathbb{Z})$. However, it turns out that $X_{W_a}$ is too large. We explain this later on.

**Example 5.2.** Let us take $W_a = W(\tilde{B}_2)$ with simple system $\Delta = \{\alpha, \beta\}$ and $\alpha$ the short root of $\Delta$. The positive root system is $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$. Because of Example 5.1 we know that there exists $\lambda_{\alpha+\beta}(w) \in [0, 2]$ such that $k(w, \alpha + \beta) = k(w, \alpha) + 2k(w, \beta) + \lambda_{\alpha+\beta}(w)$, and there exists $\lambda_{2\alpha+\beta}(w) \in [0, 1]$ such that $k(w, 2\alpha + \beta) = k(w, \alpha) + k(w, \beta) + \lambda_{2\alpha+\beta}(w)$. Hence we have 6 choices of $\lambda = (\lambda_\theta)_{\theta \in \Phi^+}$. However, if we set $k(w, \alpha) = 0$ and $k(w, \beta) = 0$ we see in Figure 12 that there are only 4 alcoves satisfying this and not 6. Those are $(0, 0, 0, 0), (0, 0, 1, 0), (0, 0, 1, 1)$ and $(0, 0, 2, 1)$ with the reading direction of Figure 11.
Figure 11. Reading direction of a $\Phi^+$-tuple in $W(\tilde{B}_2)$.

Figure 12. Alcoves in $W(\tilde{B}_2)$.

5.2. Admissible, admitted vectors and components of $\tilde{X}_{W_a}$.

**Definition 5.4.** We will denote $S[W_a]$ as the system of all the inequalities coming from Theorem 3.1 or equivalently Theorem 3.2. Let $\lambda$ be an admissible vector. We say that $\lambda$ is admitted if it satisfies the system $S[W_a]$.

**Proposition 5.1.** Let $w \in W_a$. Let us write $k(w,\theta) = P_\theta(w) + \lambda_\theta(w)$ for all $\theta \in \Phi^+$. Then $(k(w,\theta))_{\theta \in \Phi^+}$ satisfies $S[W_a]$ if and only if $(\lambda_\theta(w))_{\theta \in \Phi^+}$ satisfies $S[W_a]$.

**Proof.** Let $\theta, \alpha, \beta \in \Phi^+$ such that $\theta^\vee = \alpha^\vee + \beta^\vee$. Hence by Theorem 3.2 one has $k(w,\alpha) + k(w,\beta) \leq k(w,\theta) \leq k(w,\alpha) + k(w,\beta) + 1$. Moreover because of Theorem 5.1 we know that $k(w,\alpha) = P_\alpha(w) + \lambda_\alpha(w)$ and $k(w,\beta) = P_\beta(w) + \lambda_\beta(w)$. We also know because of Theorem 5.1 that $P_\theta = P_\alpha + P_\beta$. Therefore we get $P_\theta(w) = P_\alpha(w) + P_\beta(w)$ and the previous inequalities become

$$P_\alpha(w) + \lambda_\alpha(w) + P_\beta(w) + \lambda_\beta(w) \leq P_\theta(w) + \lambda_\theta(w) \leq P_\alpha(w) + \lambda_\alpha(w) + P_\beta(w) + \lambda_\beta(w) + 1.$$
These are the same as 
\[ \lambda_\alpha(w) + \lambda_\beta(w) \leq \lambda_\delta(w) \leq \lambda_\alpha(w) + \lambda_\beta(w) + 1. \]
Thus \((k(w, \alpha))_{\alpha \in \Phi} \in S[W_a]\) if and only if \((\lambda_\alpha)_{\alpha \in \Phi} \in S[W_a].\]

**Proposition 5.2.** The affine variety \(X_{W_a}\) decomposes as \(X_{W_a} = \bigsqcup \lambda \text{ admissible} X_{W_a}[\lambda].\)
The irreducible components of \(X_{W_a}\) are the \(X_{W_a}[\lambda]\) for \(\lambda\) admissible and we have \(X_{W_a}[\lambda] = X_{W_a}[0] + \lambda.\) Moreover, each irreducible component is of codimension \(|\Delta| = n.\)

**Proof.** From the way we defined \(X_{W_a}\) it is obvious that \(X_{W_a} = \bigsqcup \lambda \text{ admissible} X_{W_a}[\lambda].\)
Moreover, being an element of \(X_{W_a}[\lambda]\), for \(\lambda\) admissible, is the same as being a solution of the system 
\[ \{ P_\alpha + \lambda_\alpha - X_\alpha = 0 \mid \alpha \in \Phi^+ \}. \]
But the solutions of this system are exactly the solutions of the system 
\[ \{ P_\alpha - X_\alpha = 0 \mid \alpha \in \Phi^+ \} + (\lambda_\alpha)_{\alpha \in \Phi^+}, \]
that is the solutions of the system \(X_{W_a}[0] + \lambda.\) Since \(X_{W_a}[0] + \lambda\) is an affine space it is clear that it is an irreducible component of \(X_{W_a}.\) As \(X_{W_a}[\lambda] := V(J_{W_a}[\lambda])\) we see that this is just the intersection of the hyperplanes associated to the polynomials \(P_\alpha[\lambda]\) for all \(\alpha \in \Phi^+.\) Since there are \(|\Phi^+| - |\Delta|\) such polynomials and since they are linearly independent as linear forms on \(\mathbb{R}^{|\Phi^+|}\), it follows that the codimension of \(X_{W_a}[\lambda]\) is \(|\Delta|\). \(\square\)

**Notation 5.1.** Let \(Y \subset \mathbb{R}^m.\) We denote by \(Y(\mathbb{Z})\) the set of integral points of \(Y.\)

**Theorem 5.2.** Let \(\gamma\) be an admissible vector. The following points are equivalent

i) \(\gamma\) is admitted.

ii) There is \(x \in X_{W_a}[\gamma]\) such that there exists \(w \in W_a\) satisfying \(x = (k(w, \alpha))_{\alpha \in \Phi^+}.\)

iii) All the integral points of \(X_{W_a}[\gamma]\) arise as in ii).

**Proof.** From Theorem 3.2 we know that a \(\Phi^+\)-tuple of integers \((k(w, \alpha))_{\alpha \in \Phi^+}\) associated to an element \(w \in W_a\) is entirely characterized by the system of inequalities \(S[W_a].\) We also know that a \(\Phi^+\)-tuple \((k_\alpha)_{\alpha \in \Phi^+} \in \mathbb{Z}^m\) satisfying the system \(S[W_a]\) provides an element \(w \in W_a\) such that \((k(w, \alpha))_{\alpha \in \Phi^+} = (k_\alpha)_{\alpha \in \Phi^+}.\)

Let \(x = (x_\alpha)_{\alpha \in \Phi^+} \in X_{W_a}(\mathbb{Z})[\gamma].\) We can write \(x_\alpha\) as \(x_\alpha = P_\alpha(x) + \gamma_\alpha.\)
Then, thanks to Proposition 5.2 if \(\gamma\) is admitted we have \(x_\alpha\) that satisfies \(S[W_a].\) However, satisfying \(S[W_a]\) implies being a \(\Phi^+\)-tuple of an element of \(W_a.\) Thus, there exists \(w \in W_a\) such that \(x = (k(w, \alpha))_{\alpha \in \Phi^+}.\) The direction \(i) \Rightarrow ii)\) follows.

Conversely, if \(x \in X_{W_a}[\gamma]\) such that \(x = (k(w, \alpha))_{\alpha \in \Phi^+}\) for some \(w \in W_a,\) then we have again \(k(w, \alpha) = P_\alpha(w) + \gamma_\alpha\) for all \(\alpha \in \Phi^+.\) Since \(w \in W_a,\)
\((k(w, \alpha))_{\alpha \in \Phi^+}\) must satisfy \(S[W_a,\]\) but once again because of Proposition
this is equivalent to say that $\gamma = (\gamma_\alpha)_{\alpha \in \Phi^+}$ is also a solution of $S[W_a]$, that is $\gamma$ is admitted. There follows the direction $ii) \Rightarrow i)$.

If we have now a point $x' = (x'_\alpha)_{\alpha \in \Phi^+} \in X_{W_a}(\mathbb{Z})[\gamma]$ with $x' \neq x$ then $x'$ decomposes as $x'_\alpha = P_\alpha(x') + \gamma_\alpha$ for all $\alpha \in \Phi^+$. However, since $x$ is coming from an element of $W_a$ we know that $\gamma$ is admitted, which means that it is a solution of $S[W_a]$. By using Proposition 5.2 it follows that $(x'_\alpha)_{\alpha \in \Phi^+}$ satisfies $S[W_a]$. Thus, $x'$ is coming from a point of $W_a$, that is there exists $w' \in W_a$ such that $x' = (k(w', \alpha))_{\alpha \in \Phi^+}$. This concludes the case $ii) \Rightarrow iii)$. The direction $iii) \Rightarrow ii)$ is clear.

**Theorem 5.3.** 1) The map $\iota : W_a \rightarrow X_{W_a}(\mathbb{Z})$ defined by $w \mapsto (k(w, \alpha))_{\alpha \in \Phi^+}$ induces by corestriction a bijective map from $W_a$ to the integral points of a subvariety of $X_{W_a}$, denoted $\tilde{X}_{W_a}$, which we call the Shi variety of $W_a$. This subvariety is nothing but $\tilde{X}_{W_a} := \bigcup_{\lambda \text{ admitted}} X_{W_a}[\lambda]$. In other words, one has the following diagram:

$$
\begin{array}{ccc}
W_a & \xrightarrow{\iota} & X_{W_a}(\mathbb{Z}) \\
\cap & & \cap \\
\tilde{X}_{W_a}(\mathbb{Z}) & \xleftarrow{} & W_a
\end{array}
$$

2) Let $\bar{\alpha} = \sum_{i=1}^{n} c_i \alpha_i$ be the highest root in $\Phi^+$. The number of irreducible components of $\tilde{X}_{W_a}$ is $n! \prod_{i=1}^{n} c_i$.

**Proof.** 1) From the way we defined $X_{W_a}$ we know that each element of $W_a$, seen as $\Phi^+$-tuple, belongs to $X_{W_a}(\mathbb{Z})$. That is $\iota$ is well defined. Moreover, we also know because of Proposition 5.1 in [19] that $\iota$ is injective. Via Theorem 5.2, one has $\iota(w) \in X_{W_a}[\lambda]$ if and only if $\lambda$ is admitted. Therefore, by deleting all the components $X_{W_a}[\gamma]$ with $\gamma$ admissible but non-admitted, $\iota$ becomes bijective.

2) The number of irreducible components of $\tilde{X}_{W_a}$ is equal to the number of admitted vectors. Moreover, by Theorem 5.2 we know that for each admitted vector $\gamma = (\gamma_\alpha)_{\alpha \in \Phi^+}$ there exists a unique $w \in W_a$ such that $\gamma_\alpha = k(w, \alpha)$ for all $\alpha \in \Phi^+$. Since $\gamma_\alpha = 0$ for all $\alpha \in \Delta$, we have a bijection between the admitted vectors and the alcoves that belong to $P_H = \bigcap_{\alpha \in \Delta} H_{\alpha,0}^\perp$. Thanks to Section 3.3 we know that $|\text{Alc}(P_{H^\vee})| = \frac{|W(\Delta)|}{f_\Phi}$. Therefore, by duality it follows that $|\text{Alc}(P_H)| = \frac{|W(\Delta^\vee)|}{f_\Phi^\vee}$. However it is clear that $|W(\Phi^\vee)| = |W(\Phi)|$, and since $f_\Phi = f_{\Phi^\vee}$ (see Section 3.3) it follows that $|\text{Alc}(P_H)| = \frac{|W(\Delta)|}{f_\Phi}$. However the number $|W(\Delta)|$ is known and is equal to $f_\Phi n! \prod_{i=1}^{n} c_i$ (see for example [3], Ch. VI, § 2, prop. 7). The result follows.

**Notation 5.2.** For $\iota(w) \in \tilde{X}_{W_a}[\lambda]$ we denote $\lambda(w) := \lambda$. For instance $X_{W_a}[\lambda(s)]$ is the irreducible component having the $\Phi^+$-tuple $(k(s, \alpha))_{\alpha \in \Phi^+}$ with $s \in S$. 

Definition 5.5. We define \( H^0(\tilde{X}_{W_a}) \) to be the set of irreducible components of \( \tilde{X}_{W_a} \). For \( \lambda = (\lambda_\alpha)_{\alpha \in \Phi^+} \), an admitted vector we denote by \( w_\lambda \) the associated element of \( \text{Alc}(P_H) \), that is \( k(w_\lambda, \alpha) = \lambda_\alpha \) for all \( \alpha \in \Phi^+ \). Here is in Table 3 the number of irreducible components of the Shi varieties of any affine Weyl group.

| Type  | Coefficients of \( \tilde{\alpha} \) | Index of connection | \(|W|\) | \(|H^0(\tilde{X}_{W_a})|\) |
|-------|----------------------------------|----------------------|--------|-----------------|
| \( A_n \) | 1,1,...,1 | \( n + 1 \) | \((n + 1)!\) | \( n!\) |
| \( B_n \) | 1,2,2,...,2 | 2 | \( 2^n n! \) | \( 2^{n-1} n! \) |
| \( C_n \) | 2,2,...,2,1 | 2 | \( 2^n n! \) | \( 2^{n-1} n! \) |
| \( D_n \) | 1,2,...,2,1,1 | 4 | \( 2^{n-1} n! \) | \( 2^{n-3} n! \) |
| \( E_6 \) | 2,2,3,4,3,2,1 | 2 | \( 2^{13} 3^5 5^7 \) | \( 2^{13} 3^5 5^7 \) |
| \( E_7 \) | 2,3,4,5,4,3,2 | 1 | \( 2^{13} 3^5 5^7 \) | \( 2^{13} 3^5 5^7 \) |
| \( F_4 \) | 3,2 | 1 | \( 2^3 \) | \( 2^3 \) |

Table 3. Number of irreducible components of the Shi varieties in any type.

Example 5.3. Let us take the group \( W(\tilde{A}_2) \). The variety \( \tilde{X}_{W(\tilde{A}_2)} \) has two components which are drawn in Figure 13 either in orange or in white. Thus, we have the decomposition as follows:

\[
\tilde{X}_{W(\tilde{A}_2)} = X_{W(\tilde{A}_2)}[(0,0,0)] \sqcup X_{W(\tilde{A}_2)}[(0,0,1)].
\]

![Figure 13. Irreducible components of \( \tilde{X}_{W(\tilde{A}_2)} \).](image-url)
Example 5.4. Let us take the group $W(\widetilde{B}_2)$. In this group there are 4 components, given by the 4 colors below. Indeed one has the following splitting according to the admitted vectors

\[
\widehat{X}_{W(\widetilde{B}_2)} = X_{W(\widetilde{B}_2)}[(0, 0, 0, 0)] \sqcup X_{W(\widetilde{B}_2)}[(0, 0, 1, 0)] \sqcup X_{W(\widetilde{B}_2)}[(0, 0, 1, 1)] \sqcup X_{W(\widetilde{B}_2)}[(0, 0, 2, 1)].
\]

The first component corresponds with the pink, the second one with the yellow, the third one with the blue and the last one with the white. Moreover we can see that all the information is contained in the parallelogram where there are the 4 alcoves associated to the 4 admitted vectors. We also see that any reflection sends a component to another one. We will see through Proposition 5.3 that it is not a coincidence. It is also possible to see that the lattice leaves stable the components. For example if we pick up the alcove $A_w = (0, 0, 0, 0)$ and if we take $x = e_1 + e_2$ then $A_{\tau_w} = (0, 2, 2, 2)$ and they are both of the same color pink.

\[
\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
Example 5.5. Let us take the group \( W(\tilde{G}_2) \). We can see all the irreducible components in the polytope \( P_{\tilde{G}_2} \). Here for example we have the 12 irreducible components of \( W(\tilde{G}_2) \). If we would color any element of the finite part \( W(G_2) \) according to the components in which they are, we would see that all the colors appear in \( W(G_2) \). The reason is that \( f_{\tilde{G}_2} = 1 \).

Figure 15. Polytope \( P_{\tilde{G}_2}(A) \) seen as set of representatives of irreducible components of \( \tilde{X}_W(G_2) \).

5.3. Action on the components.

Lemma 5.1. Let \( \theta \) be in \( \Phi^+ \). Then \( \theta^\vee = P_\theta(\alpha_1^\vee, \ldots, \alpha_n^\vee) \).

Proof. Let us write \( \theta^\vee = c_1\alpha_1^\vee + \cdots + c_n\alpha_n^\vee \) with \( c_i \in \mathbb{N} \). Then there exists \( i \in [1, n] \) such that \( \mu^\vee := c_1\alpha_1^\vee + \cdots + (c_i - 1)\alpha_i^\vee + \cdots + c_n\alpha_n^\vee \in (\Phi^\vee)^+ \). Thus, by using Theorem 5.1 we have \( P_\theta = P_\mu + P_{\alpha_i} \). Proceeding by induction the same way on \( \mu^\vee \) we get that \( P_\theta = c_1P_{\alpha_1} + \cdots + c_nP_{\alpha_n} \). However \( P_{\alpha_i} = X_{\alpha_i} \) for all \( i \in [1, n] \). It follows that \( P_\theta = c_1X_{\alpha_1} + \cdots + c_nX_{\alpha_n} \).

Lemma 5.2. Let \( w = \tau_x \bar{w} \) be an element of \( W_a \) such that \( x \in \mathbb{Z}\Phi \) and \( \bar{w} \in W \). Let \( \lambda \) be an admitted vector. Assume that \( (k(w, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\lambda] \). Then \( (k(\bar{w}, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\lambda] \). In particular \( X_{W_a}[\lambda] \) is stable under the action of \( \mathbb{Z}\Phi \).

Proof. We just have to show that for any \( \alpha \in \Phi^+ \) one has \( \lambda_\alpha(w) = \lambda_\alpha(\bar{w}) \). We know that \( k(w, \alpha) = P_\alpha(w) + \lambda_\alpha(w) \). For all \( \epsilon \in \Delta \), thanks to Lemma 4.2, we have

\[
k(w, \epsilon) = k(\bar{w}, \epsilon) + (x, \epsilon^\vee) = P_\alpha(\bar{w}) + \lambda_\alpha(\bar{w}) + (x, \epsilon^\vee).
\]

However, we also have that

\[
k(w, \alpha) = P_\alpha(w) + \lambda_\alpha(w) = P_\alpha(\{k(\bar{w}, \epsilon) + (x, \epsilon^\vee)\}_{\epsilon \in \Delta}) + \lambda_\alpha(w).
\]
Since $P_\alpha$ is a linear polynomial we can split up the sum inside the variables and we get
\[
k(w, \alpha) = P_\alpha(\{k(w, \varepsilon)\}_{\varepsilon \in \Delta}) + P_\alpha(\{(x, \varepsilon^\tau)\}_{\varepsilon \in \Delta}) + \lambda_\alpha(w)
\]
\[
= k(w, \alpha) - \lambda_\alpha(w) + \lambda_\alpha(w)
\]
\[
= k(w, \alpha) + (x, \alpha^\tau) + \lambda_\alpha(w)
\]
which is the same as $\lambda_\alpha(w) = \lambda_\alpha(w)$.

Finally it follows that $k(w, \alpha) + (x, \alpha^\tau) = k(w, \alpha) - \lambda_\alpha(w) + (x, \alpha^\tau) + \lambda_\alpha(w)$

For the second statement let us take $g \in W_a$ such that $(k(g, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[^\lambda]$ and let us take $y \in \Delta \Phi$. What we have done just before implies that $(k(\tau g, \alpha))_{\alpha \in \Phi^+}$ and $(k(\tau g, \alpha))_{\alpha \in \Phi^+}$ belong to the same component. Since $\tau g = g$ we get that $(k(g, \alpha))_{\alpha \in \Phi^+}$ and $(k(g, \alpha))_{\alpha \in \Phi^+}$ are in the same component. But once again we know that $(k(g, \alpha))_{\alpha \in \Phi^+}$ and $(k(g, \alpha))_{\alpha \in \Phi^+}$ are in the same component. Hence we have shown that $X_{W_a}[\lambda](\Delta \Phi)$ is stable under the action of $\Delta \Phi$. It follows that $X_{W_a}[\lambda]$ is also stable under $\Delta \Phi$. □

**Proposition 5.3.** Let $F : W_a \rightarrow \text{Isom}(\mathbb{R}^n)$ be the $\Phi^+$-representation of $W_a$.

Then

1) $W_a$ acts naturally on the irreducible components of $\hat{X}_{W_a}$ via the action defined as $w \circ X_{W_a}[\lambda] := F(w)(X_{W_a}[\lambda])$ for $\lambda$ admitted. Furthermore if we assume that $w \in W_a$ decomposes as $w = \tau_x w$, then $w \circ X_{W_a}[\lambda] = \tau_x w \circ X_{W_a}[\lambda]$. Finally this action is transitive.

2) The previous action induces an action on the admitted vectors by $w \circ \gamma := \gamma$ such that $w \circ X_{W_a}[\lambda] = X_{W_a}[\gamma]$. In other words we have $w \circ X[\lambda] = X[w \circ \lambda]$.

**Proof.** 1) Let $x \in W_a$ such that $(k(x, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\lambda]$. For any $w \in W_a$ there exists an admitted vector $\gamma$ such that $(k(wx, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\gamma]$. However, since we know that $F(w)(x) = (k(wx, \alpha))_{\alpha \in \Phi^+}$, it follows that $F(w)$ sends at least one element of $X_{W_a}[\lambda]$ to $X_{W_a}[\gamma]$. Moreover $F(w)$ is an isometry and then $F(w)(X_{W_a}[\lambda])$ must be an affine space of the same codimension as $X_{W_a}[\lambda]$.

Let us show now that $(F(w)(X_{W_a}[\lambda]))(\mathbb{Z}) = F(w)(X_{W_a}[\lambda](\mathbb{Z}))$.

First of all the inclusion $F(w)(X_{W_a}[\lambda](\mathbb{Z})) \subset (F(w)(X_{W_a}[\lambda]))(\mathbb{Z})$ is clear. With respect to the other inclusion, let us take $z \in (F(w)(X_{W_a}[\lambda]))(\mathbb{Z})$. Then we have $F(w)^{-1}(z) \in X_{W_a}[\lambda]$ but $F(w)^{-1} = F(w^{-1})$ and the coefficients of the matrix associated to $F(w^{-1})$ are integers together with the coordinates of the translation part. It follows that $F(w^{-1})(z) \in X_{W_a}[\lambda](\mathbb{Z})$ and thus $z \in F(w)(X_{W_a}[\lambda](\mathbb{Z}))$. The equality follows.

Finally, $F(w)(X_{W_a}[\lambda])$ is an affine space of codimension $|\Delta|$ all of whose integer points are associated to elements of $W_a$. It follows that $F(w)(X_{W_a}[\lambda])$ must be an irreducible component of $\hat{X}_{W_a}$. However $(k(x, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\gamma]$, then since the components of $X_{W_a}$ have no intersection we must have $F(w)(X_{W_a}[\lambda]) = X_{W_a}[\gamma]$. Thus, we have shown that the following map
is well defined

\[ W_a \times H^0(\tilde{X}_{W_a}) \longrightarrow H^0(\tilde{X}_{W_a})(w, X_{W_a}[\lambda]) \longrightarrow w \circ X_{W_a}[\lambda]. \]

Since \( F \) is a morphism we have

\[ e \circ X_{W_a}[\lambda] = F(e)(X_{W_a}[\lambda]) = id(X_{W_a}[\lambda]) = X_{W_a}[\lambda], \]

and for all \( w, w' \in W_a \) we also have

\[
ww' \circ X_{W_a}[\lambda] = F(ww')(X_{W_a}[\lambda])
= F(w)F(w')(X_{W_a}[\lambda]) = F(w)(F(w')(X_{W_a}[\lambda]))
= w \circ (w' \circ X_{W_a}[\lambda]).
\]

This concludes the first statement of 1). With respect to the second statement we just have to see that the action of \( F(\tau_x) \) doesn’t do anything on the components. By Lemma 5.2 we know that \( X_{W_a}[\lambda] \) is stable under \( \mathbb{Z}\Phi \). It follows that \( F(\tau_x)(X_{W_a}[\lambda]) = X_{W_a}[\lambda] \).

To show the transitivity we just have to see that we can obtain any component of \( H^0(\tilde{X}_{W_a}) \) from the component \( X_{W_a}[0] \). Let \( X_{W_a}[\gamma] \) be such a component and \((k(g, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\gamma]\). It follows that \((k(\bar{g}, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\gamma]\). As \( \bar{g} = \bar{g}e \) and since \((k(e, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[0] \) it follows that \( F(\bar{g})(X_{W_a}[0]) = X_{W_a}[\gamma], \) that is \( \bar{g} \circ X_{W_a}[0] = X_{W_a}[\gamma] \).

2) This is a straightforward consequence of the first point. □

**Proposition 5.4.** Let \( f_\Phi \) be the index of connection of \( \Phi \) and let \( \lambda \) be an admitted vector.

1) In \( W \) there are exactly \( f_\Phi \) elements belonging to \( X_{W_a}[\lambda] \).

2) \( X_{W_a}[\lambda] \) has exactly \( f_\Phi \) orbits under the action of \( \mathbb{Z}\Phi \).

3) Each orbit of \( X_{W_a}[\lambda] \) has only one element of \( W \).

**Proof.** 1) First of all we note that \( W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z})) \neq \emptyset \). Indeed, for any \( w = \tau_{xw} \in \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z})) \) we know by Lemma 5.2 that \((k(w, \alpha))_{\alpha \in \Phi^+} \) is also in \( X_{W_a}[\lambda] \), that is \( w \in W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z})) \). We need now to show that if we take another admitted vector \( \gamma \) then \(|W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z}))| = |W \cap \iota^{-1}(X_{W_a}[\gamma](\mathbb{Z}))|\). But this is true because any component can be sent to any other one via an element of \( W \). In particular there exists \( g \in W \) such that \( g \circ X_{W_a}[\lambda] = X_{W_a}[\gamma] \). Further, if \((k(w, \alpha))_{\alpha \in \Phi^+} \in X_{W_a}[\lambda] \) with \( w \in W \), then \( F(g)((k(w, \alpha))_{\alpha \in \Phi^+}) \) is also associated to an element of \( W \). The result follows since \( g \) is a bijection. By Theorem 5.3 we know that there are \( n! \prod_{i=1}^{n} c_i \) irreducible components in \( \tilde{X}_{W_a} \) where \( \bar{\alpha} = \sum_{i=1}^{n} c_i \alpha_i \) is the longest root in \( \Phi^+ \). However it is well known that \( |W| = n!(\prod_{i=1}^{n} c_i) f_\Phi \). Since all the irreducible components have same number of elements in \( W \), we deduce that this number is the quotient of \( |W| \) by the number of irreducible components, which is exactly \( f_\Phi \).

2) This is a consequence of the point above. Let us take the \( f_\Phi \) elements \( \{w_1, \ldots, w_{f_\Phi}\} \) of \( W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z})) \). Then the \( \mathbb{Z}\Phi \) orbits \( O_{w_i} \) and \( O_{w_j} \) are different for all \( i, j \in \{1, \ldots, f_\Phi\} \). Indeed, if there was an element belonging to both then it would follow that there exists \( x \in \mathbb{Z}\Phi \) such that \( w_i = \tau_x w_j \), but this is impossible because \( w_i \) and \( w_j \) are in the finite group \( W \). We
therefore have \( f_\Phi \) distinct orbits. We claim that there is no other orbit. Indeed, let \( x \in X_{W_a}[\lambda] \). Then there exists \( w \in W_a \) such that \( x = \nu(w) \). Moreover, one can write \( w = \tau_y \bar{w} \) with \( y \in \mathbb{Z}\Phi \) and \( \bar{w} \in W \). This implies that \( \nu(w) \) and \( \nu(\bar{w}) \) are in the same orbit \( O_x \). However, since \( \bar{w} \in W \cap \nu^{-1}(X_{W_a}[\lambda](\mathbb{Z})) \) it follows that \( O_x = O_{w_i} \) for some \( i = 1, \ldots, f_\Phi \).

3) Assume that we have an orbit in \( X_{W_a}[\lambda] \) with at least two elements \( w_1, w_2 \) coming from \( W \). Then there exists \( x \in \mathbb{Z}\Phi \) such that \( w_2 = \tau_x w_1 \). Since \( w_1 \) and \( w_2 \) are in \( W \) the previous equality cannot occur. It follows that each orbit has at most one element of \( W \). Moreover, by the first point above we know that \( X_{W_a}[\lambda] \) has \( f_\phi \) elements of \( W \). Since there are \( f_\phi \) orbits in \( X_{W_a}[\lambda] \) it follows that each orbit has a unique element of \( W \).

**Proposition 5.5.** Let \( W_a \) be an affine Weyl group such that \( W_a \neq W(\tilde{A}_2) \). Let \( s_i, s_j \in S \) such that \( s_i \neq s_j \). Then \( \tilde{X}_{W_a}[\lambda(s_i)] \neq \tilde{X}_{W_a}[\lambda(s_j)] \).

**Proof.** For the \( B_2 \) (= \( C_2 \)) and \( G_2 \) cases one can directly see the result on Figures 16 and 15. We assume now that the rank of \( \Phi \) is greater or equal than 3. Let us assume that \( s_i \) and \( s_j \) are in the same component \( \tilde{X}_{W_a}[\lambda(s_i)] \). We claim that there exists a root \( \theta \in \Phi^+ - \Delta \) such that \( P_{\theta}(s_i) \neq P_{\theta}(s_j) \). Assuming this claim, because of Theorem 5.1 and Corollary 4.1 we have \( k(s_i, \theta) = P_{\theta}(s_i) + \lambda_\theta(s_i) = 0 \) and \( k(s_j, \theta) = P_{\theta}(s_j) + \lambda_\theta(s_j) = 0 \), which implies that \( \lambda_\theta(s_i) \neq \lambda_\theta(s_j) \). However, since \( s_i \) and \( s_j \) are in \( \tilde{X}_{W_a}[\lambda(s_i)] \), we must have \( \lambda(s_i) = \lambda(s_j) \) and then \( \lambda_\alpha(s_i) = \lambda_\alpha(s_j) \) for all \( \alpha \in \Phi^+ - \Delta \). Therefore, \( s_i \) and \( s_j \) cannot lie in the same component.

**Proof of the claim.** Let \( \Phi \) be a root system satisfying rank(\( \Phi \)) \( \geq 3 \). This is a general fact, which can be checked case by case, that in any such a crystallographic root system there always exists a root \( \theta \in \Phi^+ - \Delta \) such that either \( \alpha_i \) is in the expression of \( \theta \) (according to \( \Delta \)) but \( \alpha_j \) is not, or \( \alpha_j \) is in the expression of \( \theta \) (according to \( \Delta \)) but \( \alpha_i \) is not. We apply this statement on the dual root system of \( \Phi \).

Let \( \theta \in \Phi^+ - \Delta \) such that \( \alpha_i^\vee \) is in the expression of \( \theta^\vee \) (according to \( \Delta^\vee \)) but \( \alpha_j^\vee \) is not, the other case being symmetric. Thus, because of Lemma 5.1 we have \( X_{\alpha_i} \) in the expression of \( P_{\theta} \), whereas \( X_{\alpha_j} \) doesn’t appear in this expression. Again because of Lemma 5.1 we know that all the coefficients of the polynomial \( P_{\theta} \) are positive. Therefore, since \( k(s_i, \alpha) = 0 \) for all \( \alpha \in \Phi^+ - \{\alpha_i\} \) and \( k(s_j, \alpha) = -1 \) we have \( P_{\theta}(s_i) \leq -1 \). Moreover, we also have \( k(s_j, \alpha) = 0 \) for all \( \alpha \in \Phi^+ - \{\alpha_j\} \) and \( k(s_j, \alpha_j) = -1 \), which implies that \( P_{\theta}(s_j) = 0 \). □

**Figure 16.** Generators \( s_2 = (0, -1, 0, 0) \) in the white component and \( s_1 = (-1, 0, 0, 0) \) in the blue component in type \( B_2 \).
Remark 5.2. The $A_2$ case fails because there are not enough roots. See Figure 13 where the generators $s_1 = (-1,0,0)$, $s_2 = (0,-1,0)$ and $s_2 = (0,0,1)$ lie in the same component. The reading direction according to Figure 4 is as follows: $(\alpha, \beta, \alpha + \beta)$.

5.4. Link between Shi regions and $\tilde{X}_{W_a}$. This subsection is a short summary of our results that we embody through Shi regions. From Section 3.5 we know that there is a bijection between the Shi regions of $W_a$ and the signed regions of $W_a$. Moreover, we have now a bijection between the integral points of an affine variety $\tilde{X}_{W_a}$ and the group $W_a$. The important fact is to see that this bijection respects the notion of signed regions. Therefore, it is clear that any signed region of $W_a$ is characterized in $\tilde{X}_{W_a}$ by the intersection of $\tilde{X}_{W_a}$ and an orthant. We know from [20] Theorem 8.1 that there are $(h + 1)^n$ signed regions in $W_a$ where $h$ is the Coxeter number of $\Phi$ and $n$ is the rank of $\Phi$. Consequently, $X_{W_a}$ meets exactly $(h + 1)^n$ orthants in $\bigoplus \mathbb{R}a$. It would be interesting to see if we can recover this result by using the equations of the variety, namely if we would be able to determine which are the orthants touched by the variety.

Finally we can state the following result:

Theorem 5.4. Let us write $\text{Ort}(W_a)$ the set of orthants in $\bigoplus_{a \in \Phi^+} \mathbb{R}a$ and $\text{Ort}(\tilde{X}_{W_a}) := \{ \tilde{X}_{W_a}(\mathbb{Z}) \cap \Upsilon \mid \Upsilon \in \text{Ort}(W_a) \}$. The following map is a bijection

$$\begin{align*}
\text{Shi}(W_a) & \longrightarrow \text{Ort}(\tilde{X}_{W_a}) \\
\Omega & \longmapsto \iota(i_1^{-1}(\Omega \cap E)).
\end{align*}$$

5.5. Invariance of $\tilde{X}_{W_a}$ up to isomorphism of root systems. Let $\Phi^{'} \subset V^{'}$ be another (irreducible essential crystallographic) root system where $V^{'}$ is an Euclidean space with inner product also denoted $(-, -)$.

We say that $\Phi$ and $\Phi^{'}$ are isomorphic, denoted $\Phi \simeq \Phi^{'}$, if there exists an isomorphism $\alpha : V \rightarrow V^{'}$ that sends $\Phi$ to $\Phi^{'}$ and preserves the crystallography, that is $f(\Phi) = \Phi^{'}$ and $2^{\langle \alpha, \beta \rangle}_{\langle \alpha, \alpha \rangle} = 2^{\langle f(\alpha), f(\beta) \rangle}_{\langle f(\alpha), f(\alpha) \rangle}$ for all $\alpha, \beta \in \Phi$.

Such an isomorphism preserves the angles between vectors and the ratio $||\alpha||/||\beta||$ for all $\alpha, \beta \in \Phi$ (see for example [15] Section 10.1). However we don’t know whether $\alpha$ preserves inner products. For instance if $V^{'} = V$ we don’t necessary have $\alpha \in O(V)$.

This notion gives rise to an equivalence relation over irreducible crystallographic root systems. A class for this relation is called an isomorphism class. We denote it by $\Phi$.

Assume that $\Phi \simeq \Phi^{'}$ via $\alpha$. Such an isomorphism induces an isomorphism $\tilde{\alpha} : W(\Phi) \rightarrow W(\Phi^{'})$, $\alpha \mapsto f \circ w \circ f^{-1}$ between Weyl groups. This isomorphism extends to an isomorphism between the associated affine Weyl groups. We also denote it by $\tilde{\alpha} : W(\Phi_a) \rightarrow W(\Phi_a^{'})$.

In the notation of the affine variety $\tilde{X}_{W_a}$ we ignored the associated crystallographic root system underlying. From now on, if $W_a = W(\Phi_a)$ we set $\tilde{X}_\phi := \tilde{X}_{W_a}$. If $\lambda \in \bigoplus_{a \in \Phi^+} \mathbb{R}a$ is an admitted vector we will also write $\tilde{X}_\phi[\lambda] := \tilde{X}_{W_a}[\lambda]$. 


We show in this section that the Shi variety variety $X_{\Phi}$ only depends on $\Phi$ and not of the choice of $\Phi$. Our goal is then to show that the construction of the variety $X_{\Phi}$ is functorial, that is $\Phi \simeq \Phi'$ implies $X_\Phi \simeq X_{\Phi'}$.

**Definition 5.6.** Let $\Phi$ and not of the choice of $\Phi$. Our goal is then to show that the construction

\[ P\Phi_\alpha := \{ (x, y) \in \Phi^+ \times \Phi^+ \mid \alpha = (x^\vee + y^\vee)^\vee \}, \]

\[ P\Phi_{f(\alpha)} := \{ (x', y') \in \Phi'^+ \times \Phi'^+ \mid f(\alpha) = (x'^\vee + y'^\vee)^\vee \}. \]

**Lemma 5.3.** Let $f : \Phi \to \Phi'$ be an isomorphism of root systems with simple systems $\Delta = \{ \alpha_1, \ldots, \alpha_n \}$ and $\Delta' = f(\Delta)$. Let $\alpha, \theta \in \Phi^+$ and let $\{ \nu_1, \ldots, \nu_p \} \subset \Phi^+$. Assume that $\theta^\vee = b_1 \nu_1^\vee + \cdots + b_p \nu_p^\vee$. Then we have

1) $f(\theta)^\vee = b_1 f(\nu_1)^\vee + \cdots + b_p f(\nu_p)^\vee$.

2) $(x, y) \in P\Phi_\alpha \iff (f(x), f(y)) \in P\Phi_{f(\alpha)}$.

**Proof.**
1) Since $\theta^\vee = \frac{20}{|\theta|^2}$, the assumption $\theta^\vee = b_1 \nu_1^\vee + \cdots + b_p \nu_p^\vee$ is the same as

\[ \theta = \frac{||\theta||^2}{||\nu_1||^2} b_1 \nu_1 + \cdots + \frac{||\theta||^2}{||\nu_p||^2} b_p \nu_p. \]

It follows that

\[ f(\theta) = \frac{||f(\theta)||^2}{||f(\nu_1)||^2} b_1 f(\nu_1) + \cdots + \frac{||f(\theta)||^2}{||f(\nu_p)||^2} b_p f(\nu_p). \]

However, since $f$ is an isomorphism of root systems we know that it preserves the ratio of lengths $|\theta|^2 / |\nu_i|^2$ for all elements $\alpha, \beta \in \Phi$. Thus, $\frac{||\theta||^2}{||\nu_i||^2} = \frac{||f(\theta)||^2}{||f(\nu_i)||^2}$ for all $i \in [1, p]$ and it follows that

\[ f(\theta) = \frac{||f(\theta)||^2}{||f(\nu_1)||^2} b_1 f(\nu_1) + \cdots + \frac{||f(\theta)||^2}{||f(\nu_p)||^2} b_p f(\nu_p). \]

Therefore

\[ f(\theta)^\vee = 2 \frac{f(\theta)}{||f(\theta)||^2} b_1 \frac{2 f(\nu_1)}{||f(\nu_1)||^2} + \cdots + b_p \frac{2 f(\nu_p)}{||f(\nu_p)||^2} = b_1 f(\nu_1)^\vee + \cdots + b_p f(\nu_p)^\vee. \]

2) By definition $(x, y) \in P\Phi_\alpha$ if and only if $\alpha^\vee = x^\vee + y^\vee$. Because of the point above this is equivalent to $f(\alpha)^\vee = f(x)^\vee + f(y)^\vee$, that is $(f(x), f(y)) \in P\Phi_{f(\alpha)}$. \hfill \Box

Let $\Psi$ be an irreducible crystallographic root system. We set $\Psi^+ = \{ \mu_1, \ldots, \mu_m \}$. We recall that $P_\alpha[\lambda_\alpha] := P_\alpha + \lambda_\alpha - X_\alpha$ for $\alpha \in \Psi^+$. Furthermore, $J_{W_\alpha}$ is the ideal of $\mathbb{R}[X_\Psi^+]$ defined by

\[ J_{W_\alpha} := \sum_{\alpha \in \Psi^+} \langle \prod_{\lambda_\alpha \in I_\alpha} P_\alpha[\lambda_\alpha] \rangle. \]

For $\lambda$ an admitted vector we also recall that
\[ J_{W_\alpha}[\lambda] := \sum_{\alpha \in \Psi^+} \langle P_\alpha[\lambda_\alpha] \rangle \text{ and } X_{W_\alpha}[\lambda] := V(J_{W_\alpha}[\lambda]). \]

Let us define $\kappa_{\Psi}$ to be the following algebras morphism

\[ \kappa_{\Psi} : \mathbb{R}[X_{\mu_1}, \ldots, X_{\mu_m}] \to \mathbb{R}[X_1, \ldots, X_m] \]

\[ X_{\mu_i} \mapsto X_i. \]
Definition 5.7. 1) We denote by $Z_\Phi$ the affine variety living in $\bigoplus_{i=1}^{m} \mathbb{R}e_i$ and defined as follows

$$Z_\Phi := V(\kappa_\Phi(J_{W_{\alpha}})).$$

2) Let $\lambda$ be an admitted vector in $\bigoplus_{\alpha \in \Phi^+} \mathbb{R}\alpha$. Then $Z_\Phi[\lambda]$ is defined to be the irreducible component of $Z_\Phi$ associated to $\lambda$, that is $Z_\Phi[\lambda] := V(\kappa_\Phi(J_{W_{\alpha}}[\lambda]))$. It follows that

$$Z_\Phi = \bigcup_{\lambda \text{ admitted in } \bigoplus_{\alpha \in \Phi^+} \mathbb{R}\alpha} Z_\Phi[\lambda].$$

3) Assume that $\lambda$ decomposes as $\lambda = \lambda_{\mu_1}1 + \cdots + \lambda_{\mu_m}m$. We define $\lambda_e$ to be the vector $\lambda_e := \lambda_{\mu_1}1 + \cdots + \lambda_{\mu_m}m$.

Theorem 5.5. Let assume that $\Phi \simeq \Phi'$ and let $f$ be this isomorphism. We denote $f(\beta) = \beta'$ and $\Phi'^+ = \{\beta'_1, \ldots, \beta'_m\}$. Then $Z_\Phi = Z_{\Phi'}$.

Proof. We know that the affine varieties $\tilde{X}_\Phi$ and $\tilde{X}_{\Phi'}$ are defined by

$$\tilde{X}_\Phi = \bigcup_{\lambda \text{ admitted in } \bigoplus_{\alpha \in \Phi^+} \mathbb{R}\alpha} \tilde{X}_\Phi[\lambda] \text{ and } \tilde{X}_{\Phi'} = \bigcup_{\gamma \text{ admitted in } \bigoplus_{\beta \in \Phi'^+} \mathbb{R}\beta} \tilde{X}_{\Phi'}[\gamma].$$

Because of Lemma 5.1, we know that the polynomials $P_\alpha$, $\alpha \in \Phi^+$, only depend on the expression of $\alpha'$ in the basis $\Delta'$, that is of the expression of $\alpha$ in the basis $\Delta$. Of course this also applies for $P_\beta$ with $\beta \in \Phi'^+$. Let $\beta \in \Phi'^+$. Then there exists $\theta \in \Phi^+ \cup \{0\}$ such that $\beta = f(\theta)$. Because of Lemma 5.1 we also know that if $\theta' = b_1\alpha_1' + \cdots + b_n\alpha_n'$ then $P_\theta = b_1X_{\alpha_1} + \cdots + b_nX_{\alpha_n}$.

Because of Lemma 5.3, the equality $\theta' = b_1\alpha_1' + \cdots + b_n\alpha_n'$ implies in particular that $\beta' = b_1f(\alpha_1') + \cdots + b_nf(\alpha_n')$. Since $\{f(\alpha_i) \mid i = 1, \ldots, n\}$ is a simple system of $\Phi'$, the polynomial $P_{\beta'}$ decomposes as $P_{\beta'} = b_1X_{f(\alpha_1)} + \cdots + b_nX_{f(\alpha_n)}$. It follows that $\kappa_\Phi(P_{\alpha}) = \kappa_{\Phi'}(P_{f(\alpha)})$ for all $\alpha \in \Phi^+$. Consequently, the components $Z_{\Phi'}[0]$ and $Z_{\Phi}[0]$ are equal. We claim now that

$$\{\lambda_e \mid \lambda \text{ admitted in } \bigoplus_{\alpha \in \Phi^+} \mathbb{R}\alpha\} = \{\gamma_e \mid \gamma \text{ admitted in } \bigoplus_{\beta \in \Phi'^+} \mathbb{R}\beta\}.$$

The key is to see that being an admitted vector in $\bigoplus_{\alpha \in \Phi^+} \mathbb{R}\alpha$ (respectively in $\bigoplus_{\beta \in \Phi'^+} \mathbb{R}\beta$) is the same thing as satisfying the system $S(W_{\Phi_{\alpha}})$ (respectively $S(W_{\Phi'_\beta})$) and $\lambda_\alpha \in I_{\alpha}$ for all $\alpha \in \Phi^+ \cup \{0\}$ (respectively $\gamma_\beta \in I_{\beta}$ for all $\beta \in \Phi'^+$).

Let us show first of all that $\lambda$ admitted if and only if $f(\lambda)$ admitted. Then we will show further what we claim above.

•) Because of the definition of admitted vectors one has the equivalences:

$$\lambda \text{ admitted } \iff \left\{ \begin{array}{l}
\lambda_\alpha = \lambda_x + \lambda_y \text{ or } \\
\lambda_\alpha = \lambda_x + \lambda_y + 1 \text{ } \forall (x,y) \in P\Phi_\alpha, \forall \alpha \in \Phi^+
\end{array} \right..$$
6. Poset structure of $H^0(\tilde{X}_W)$

**Definition 6.1.** The set $H^0(\tilde{X}_W)$ has a natural poset structure. It is defined by $X_W[\lambda] \leq X_W[\gamma]$ if and only if $\lambda_\alpha \leq \gamma_\alpha$ for all $\alpha \in \Phi^+$. There is a minimal element in this poset which is the component corresponding to the admitted vector 0. We will write either $\lambda \leq \gamma$ or $X_W[\lambda] \leq X_W[\gamma]$. The cover relation of $\leq$ is denoted by $\prec$.

**Example 6.1.** For the Coxeter group $W(A_n)$, a way to express its root system is by taking the vectors in $\mathbb{R}^{n+1}$ defined by $\Phi = \{e_i - e_j \mid 1 \leq i, j \leq n + 1 \text{ and } i \neq j\}$. The positive roots are then $\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n + 1\}$. Using Theorem 3.2 we see that in type $A_n$ an admitted vector $\lambda = (\lambda_{i,j})_{1 \leq i < j \leq n+1}$ is defined by the following conditions:

$$
\begin{cases}
\lambda_{i,j} + \lambda_{j,k} \leq \lambda_{i,k} \leq \lambda_{i,j} + \lambda_{j,k} + 1 & \text{for all } i < j < k, \\
\lambda_{i,i+1} = 0 & \text{for all } 1 \leq i < n.
\end{cases}
$$

The natural way to express a vector $v \in \bigoplus_{\alpha \in A_n^+} \mathbb{R}\alpha$ is by putting its coordinates in a triangle ordered by the height. For example in type $A_3$ the vector $v = (v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34})$ will be presented as follows:

![Figure 17. Presentation of a vector $v \in \bigoplus_{\alpha \in A_3^+} \mathbb{R}\alpha$ as a triangle. The base is the set of coordinates on the simple roots.](image)
Figure 18. Posets associated to $\tilde{X}_{W(\tilde{A}_2)}$ and $\tilde{X}_{W(\tilde{A}_3)}$. The coordinates on the simple roots are erased since they are all equal to 0. The red labels represent the natural order on $\mathbb{R}^6 = \bigoplus_{\alpha \in A_3^+} \mathbb{R}\alpha$. 
Figure 19. Poset associated to $\tilde{X}_{W(\tilde{A}_4)^+}$. See Figure 18 for the comments.
Example 6.2. The polytope $P_{B_2}$ is as follows:

Figure 20. Polytope $P_{B_2}$ seen as set of representatives of irreducible components of $\tilde{X}_{W(\tilde{B}_2)}$.

Using Figure 11, we denote the admitted vectors by dropping the two zeros corresponding to the simple roots, and by ordering the coordinates according to the height of the dual roots. Therefore, $H^0(\tilde{X}_{W(\tilde{B}_2)})$ is as follows:

Figure 21. Poset associated to $\tilde{X}_{W(\tilde{B}_2)}$.

Example 6.3. The positive roots of $B_3^\vee$ can be arranged according to their height into a shape looking like the temple of Kukulcan. Moreover the base is the set of dual simple roots. If $\lambda$ is an admitted vector, its coordinates on the dual simple roots are 0, therefore we erase the base in this presentation of $\lambda$, and we obtain the following pictures:

Figure 22. Positive roots of $B_3^\vee$, and presentation of an admitted vector $\lambda$. 
Figure 23. Poset associated to $\hat{X}_{W(\tilde{B}_3)}$. 
Proposition 6.1. Let $\lambda$ and $\gamma$ two admitted vectors. Then we have the equivalence between
\begin{enumerate}
  \item $\lambda \preceq \gamma$.
  \item There exists a unique $\alpha \in \Phi^+$ such that $s_\alpha \circ \lambda = \gamma$ and such that
    \[
    \gamma_\beta = \begin{cases} 
      \lambda_\beta + 1 & \text{if } \beta = \alpha \\
      \lambda_\beta & \text{if } \beta \neq \alpha.
    \end{cases}
    \]
\end{enumerate}

Proof. The direction ii) implies i) is obvious.

Let us prove direction i) implies ii). From the geometrical point of view, we know that two alcoves $A_x$ and $A_y$ share a common wall if and only if there exists a root $\eta \in \Phi^+$ satisfying the two following conditions
\begin{align}
(C1) \quad & k(y, \eta) = k(x, \eta) + 1, \\
(C2) \quad & k(x, \beta) = k(y, \beta) \quad \text{for all } \beta \in \Phi^+ - \{\eta\}.
\end{align}

It turns out that if $A_x \subset P_\mathcal{H}$ and $A_y \subset P_\mathcal{H}$, we have $x < y$ if and only if (C1) and (C2) are satisfied. Indeed, each admitted vector corresponds to an alcove in the polytope $P_\mathcal{H}$. Therefore, assume that $\lambda$ and $\gamma$ are admitted vectors corresponding to adjacent alcoves $A_\lambda$ and $A_\gamma$, with $A_\gamma$ covering $A_\lambda$. Thus, there exists $k \in \mathbb{Z}$ such that $F(s_{\eta,k})(\lambda) = \gamma$ and it follows that $s_{\eta,k} \circ \lambda = \gamma$. Since $s_{\eta,k} = s_{\eta}s_{\eta}$ we have $s_{\eta,k} \circ \lambda = s_{\eta}s_{\eta} \circ \lambda$. However, because of Proposition 5.3 we know that the irreducible components are invariant under translations. It follows that $\tau_1s_{\eta} \circ \lambda = s_{\eta} \circ \lambda$. Finally, we have $s_{\eta} \circ \lambda = \gamma$ with $k(w_\gamma, \eta) = k(w_\lambda, \eta) + 1$ and $k(w_\lambda, \beta) = k(w_\gamma, \beta)$ for all $\beta \in \Phi^+ - \{\eta\}$ which is exactly the condition ii).

\[\square\]

Proposition 6.2. There exists a unique alcove $A_w$ in $P_\mathcal{H}$ such that the point $x := \bigcap_{\alpha \in \Delta} H_{\alpha,1}$ is a vertex of $A_w$. Moreover, for $\alpha \in \Delta$ the hyperplanes $H_{\alpha,1}$ are some walls of $A_w$. Let $f_x \in \tilde{V}^*$ such that $i_1(x) = f_x$. We know that $Stab(f_x) = gW_\mathcal{H}g^{-1}$ where $g \in W_a$ is such that $g^{-1}f_x \in D$ and $I = \{s \in S_a \mid (f_x, g(sa)) = 0\}$. Let $u$ be the minimal coset representative of $gW_I$. Then $w = u$.

Proof. The set $\Delta_x := \{\alpha^\vee - \delta \mid \alpha \in \Delta\}$ is such that $(\alpha^\vee - \delta, \beta^\vee - \delta) = (\alpha^\vee, \beta^\vee) \leq 0$ for all $\alpha, \beta \in \Delta$. Therefore $\Delta_x$ is a simple system of the reflection subgroup $W_x := \langle s_{\alpha^\vee - \delta}, \alpha \in \Delta \rangle$ of $W_a$. In other words $\{s_{\alpha^\vee - \delta} \mid \alpha \in \Delta\} = \chi(W_x)$. Since $x \in H_{\alpha,1}$ for all $\alpha \in \Delta$, we have $f_x$ that belongs to $Z_{\alpha^\vee - \delta}$ for all $\alpha \in \Delta$. This means that $f_x$ is fixed by all the generators of the subgroup $W_x$, and then it follows that $W_x = Stab(f_x)$.

The fundamental chamber $C_{f_x}$ of $Stab(f_x)$ is by definition the set
\[
C_{f_x} := \{g \in \tilde{V}^* \mid \langle g, \alpha^\vee - \delta \rangle > 0 \; \forall \alpha \in \Delta\}.
\]

Let us write $C_x := \{y \in V \mid \langle y, \alpha^\vee \rangle > 1 \; \forall \alpha \in \Delta\}$. A quick computation shows that we have $i_1(C_x) = C_{f_x}$. However $C_x$ is nothing but the simplicial cone with vertex $x$, walls the $H_{\alpha,1}$ for $\alpha \in \Delta$ and such that $A_x \cap C_x = \emptyset$. Furthermore, the walls of $C_x$ are sent via $i_1$ to the walls of $C_{f_x}$ intersected with $E := \{g \in \tilde{V}^* \mid \langle g, \delta \rangle = 1\}$. If we would have two alcoves in $P_\mathcal{H}$ sharing $x$ as vertex then we should have a hyperplane $H_{\beta,k}$ going through $x$ and such that $P_\mathcal{H} \cap H_{\beta,k} \neq \emptyset$. Consequently, we would have a hyperplane going
through $C_x$ and then through $-C_x$. Therefore, since $i_1(H_{β,k}) = Z_{β^-,kδ} \cap E$ we would have $Z_{β^-,kδ}$ that goes through $f_x$ and it would follow that $C_{f_x}$ wouldn’t be a connected component of $U(W_a)$, which is not true by definition of the Tits cone. Thus, there exists one and only one alcove in $P_H$ with vertex $x$.

Since $Stab(f_x)$ is a parabolic subgroup of rank $n$, and since multiplying on the left by $g$ is an isometry of $V$, the alcoves corresponding to the elements of $gW_I$ are the alcoves having $x$ as a vertex. Moreover, $A_w$ has all the hyperplanes $H_{α,1}$, $α ∈ Δ$ as walls, and since $A_w ⊂ P_H$, it follows that going across a wall from $A_w$, say $H_{α,1}$ with $γ ∈ Δ$, implies that $k(s_{γ,1}w, γ) = k(w, γ) + 1$ and $k(s_{γ,1}w, θ) = k(w, θ)$ for all the other roots $θ ∈ Φ^+ − \{γ\}$. Since $ℓ(s_{γ,1}w) = \sum_{θ ∈ Φ^+} |k(s_{γ,1}w, θ)|$ and $ℓ(w) = \sum_{θ ∈ Φ^+} |k(w, θ)|$ it follows immediately that $ℓ(w) < ℓ(s_{γ,1}w)$. Thus, we must have $u = w$. □

In this paragraph we recall some basics about lattices. A lattice is a partially ordered set such that every pair $x, y$ of elements has a meet (greatest lower bound) $x \wedge y$ and a join (least upper bound) $x \vee y$. A lattice is distributive if the meet operation distributes over the join operation and the join distributes over the meet.

A lattice $L$ is join semidistributive if whenever $x, y, z ∈ L$ satisfy $x ∨ y = x ∨ z$ they also satisfy $x ∨ (y ∧ z) = x ∨ y$. This is equivalent to the following condition: If $X$ is a nonempty finite subset of $L$ such that $x ∨ y = z$ for all $x ∈ X$, then $(\bigwedge_{x ∈ X} x) ∨ y = z$. The lattice is meet semidistributive if the dual condition $(x ∧ y = x ∧ z) ⇒ (x ∧ (y ∨ z) = x ∧ y)$ holds. Equivalently, if $X$ is a nonempty finite subset of $L$ such that $x ∧ y = z$ for all $x ∈ X$, then $(\bigvee_{x ∈ X} x) ∧ y = z$. The lattice is semidistributive if it is both join semidistributive and meet semidistributive.

**Theorem 6.1.** $H^0(\tilde{X}_{W_a})$ has a structure of semidistributive lattice.

**Proof.** Let us begin by proving that $H^0(\tilde{X}_{W_a})$ is a lattice. The idea is to show that the admitted vectors, seen as alcoves in $P_H$, define an interval in the right weak order of $W_a$. In order to do so, we first have to find a maximal and minimal element. Let $λ$ be an admitted vector. Because of the way we defined it, we know that $0 ≤ k(w, α)$ for all $α ∈ Φ^+$. Moreover we have the identity element which belongs to $P_H$, and since its $Φ^+$-tuple is the vector $0_{ℝ^m}$ it follows that the admitted vector associated to the identity is lower than all the others admitted vectors in $P_H$.

We need now to have a good candidate for the maximal element of $P_H$. By Proposition 6.2 we know that there exists a unique element $w ∈ P_H$ having $x := \bigcap_{α ∈ Δ} H_{α,1}$ as vertex. We claim that $w$ is greater (in the sense of Definition 6.1) than any other element in $P_H$. If it wasn’t the case we would have a hyperplane $H_{α,k}$ with $α ∈ Φ^+ − Δ$, $k ∈ \mathbb{N}$ that would cut $P_H$ into two connected components such that $A_w$ and $A_e$ are in the same one and such that $x /∈ H_{α,k}$. Let $A_w′$ be an alcove in the connected component that does not contain $A_e$. It follows that $k(w, α) < k(w′, α)$. Let $y$ be a point of $A_w$, and $y′$ be a point of $A_w′$. Therefore, since $y$ and $y′$ are in $P_H$ there
exist $a_1, \ldots, a_n$ and $b_1, \ldots, b_n \in \mathbb{R}^+$ such that $y = a_1 \alpha_1 + \cdots + a_n \alpha_n$ and $y' = b_1 \alpha_1 + \cdots + b_n \alpha_n$.

We claim now that without loss of generality one can assume that $b_i \leq a_i$ for all $i \in [1, n]$. Let us first explain that claim. Let us write $y$ and $y'$ in the basis of fundamental weights: $y = c_1 \omega_1 + \cdots + c_n \omega_n$ and $y' = d_1 \omega_1 + \cdots + d_n \omega_n$ with $c_i$ and $d_i \in \mathbb{R}^+$. Since $y \in A_w$ and $y' \notin A_w$, and since $x$ is a vertex of $A_w$, we can take $y$ as close as we want of $x$. It follows here that there is no problem of assuming that $d_i \leq c_i$ for all $i$. Therefore we make the assumption that $d_i \leq c_i$ for all $i$. It turns out that the inverse of the Cartan matrix $C^{-1} = (h_{ij})_{i,j \in [1, n]}$ of $W$ is the change-of-basis matrix of the basis of simple roots to the basis of fundamental weights. Moreover, it is known (see [16] or [22]) that all the coefficients of $C^{-1}$ are positive. It follows that

$$C^{-1} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad C^{-1} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Thus, the $i$-th coordinate of $y$ is $\sum_{k=1}^n h_{ik} c_k$ and the $i$-th coordinate of $y'$ is $\sum_{k=1}^n h_{ik} d_k$. Since $d_i \leq c_i$ for all $i$ and since $h_{ik} \geq 0$ for all $k = 1, \ldots, n$ it follows that $\sum_{k=1}^n h_{ik} d_k \leq \sum_{k=1}^n h_{ik} c_k$. However the $i$-th coordinate of $y$ is nothing but $a_i$, and $i$-th coordinate of $y'$ is $b_i$. Finally we have shown that $b_i \leq a_i$ for all $i$.

From the way that the coefficients $k(-, -)$ are defined, one has $k(w, \alpha) = [(\alpha^\vee, y)]$ and $k(w', \alpha) = [(\alpha^\vee, y')]$ with $[ \ ]$ the floor function. Via the expressions of $y$ and $y'$ one has:

$$\begin{align*}
(\alpha^\vee, y) &= (\alpha^\vee, a_1 \alpha_1 + \cdots + a_n \alpha_n) = a_1 (\alpha^\vee, \alpha_1) + \cdots + a_n (\alpha^\vee, \alpha_n), \\
(\alpha^\vee, y') &= (\alpha^\vee, b_1 \alpha_1 + \cdots + b_n \alpha_n) = b_1 (\alpha^\vee, \alpha_1) + \cdots + b_n (\alpha^\vee, \alpha_n).
\end{align*}$$

It follows that $0 \leq (\alpha^\vee, y') \leq (\alpha^\vee, y)$ and then $[(\alpha^\vee, y')] \leq [(\alpha^\vee, y)]$, which means that $k(w', \alpha) \leq k(w, \alpha)$. This is in contradiction with the above statement. Hence, $w$ must be the maximal element of $P_H$. Thus it follows that $\text{Alg}(P_H) \subset [e, w]$.

Let us show the other inclusion. The definition of $\text{Alg}(P_H)$ in Section 3.3 shows that $h \in \text{Alg}(P_H)$ if and only if $k(h, \alpha) \geq 0$ for all $\alpha \in \Phi^+$ and $k(h, \varepsilon) = 0$ for all $\varepsilon \in \Delta$. Let $g \in [e, w]$. Since $e \leq g \leq w$ it follows that $0 \leq k(g, \alpha) \leq k(w, \alpha)$ for all $\alpha \in \Phi^+$, and in particular $0 \leq k(g, \varepsilon) \leq k(w, \varepsilon)$ for all $\varepsilon \in \Delta$. It follows that $0 \leq k(g, \varepsilon) \leq 0$ for all $\varepsilon \in \Delta$, and then $k(g, \varepsilon) = 0$ for all $\varepsilon \in \Delta$. Hence $g \in \text{Alg}(P_H)$.

Thanks to Theorem 8.1 of [17], it is known that every interval in the right weak order of any infinite Coxeter group is a semidistributive lattice. Therefore, since the elements of $P_H$ form an interval in the right weak order of $W_a$, $P_H$ inherits a structure of semidistributive lattice. However, $P_H$ is in bijection with the set of irreducible components of $\widehat{X}_{W_a}$: $H^0(\widehat{X}_{W_a})$. As the order relation on $P_H$ is the same as the order relation on $H^0(\widehat{X}_{W_a})$, we finally showed that $H^0(\widehat{X}_{W_a})$ has a structure of semidistributive lattice. $\square$
7. Cohomological link with the components of $\tilde{X}_{W_a}$

In this section we show how the irreducible components obtained before can be used to express the first group of cohomology of any Weyl group $W$ such that $W \neq W(A_2)$.

Recall our notations; $\Phi$ is an irreducible crystallographic root system associated to $W$ and lying in an Euclidean space $(V, (\cdot, -))$ with simple system $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. The set of Coxeter generators of $W$ associated to $\Delta$ is $S = \{s_1, \ldots, s_n\}$ and $m_{ij} := \text{ord}(s_i s_j)$.

We also keep the convention that $||\alpha|| = 1$ for all short roots $\alpha$. The Cartan matrix $C_{\Phi} = (h_{ij})_{i,j}$ associated to $\Phi$ is the matrix of $GL_n(\mathbb{R})$ defined by $h_{ij} = 2(\alpha_i, \alpha_j).$ It is known that $C_{\Phi}$ is the change-of-basis matrix of the basis of fundamental weights to the basis of simple roots. This implies in particular that $C_{\Phi}$ is invertible.

Let $w \in W_a$. For an admitted vector $\lambda$, $\lambda(w) = \lambda$ if and only if $\iota(w) \in \tilde{X}_{W_a}[\lambda]$. Finally, $F$ is the $\Phi^+$ representation of $W_a$ and $\iota$ is the bijection between $W_a$ and $\tilde{X}_{W_a}(\mathbb{Z})$.

We know that $W_a \simeq \mathbb{Z}\Phi \times W$. Therefore we have the short exact sequence

$$1 \rightarrow \mathbb{Z}\Phi \xrightarrow{i} \mathbb{Z}\Phi \times W \xrightarrow{\pi} W \rightarrow 1,$$

where $i$ and $\pi$ are the morphisms defined by $i(x) = \tau_x$ and $\pi(w) = \overline{w}$, where $w \in W_a$ decomposes as $w = \tau_y \overline{w}$.

7.1. Background about group cohomology. Let $A$ be a $G$-module, that is $A$ is an abelian group on which $G$ acts compatibly with the abelian group structure on $A$, or equivalently a module over the ring $\mathbb{Z}[G]$. A morphism of $G$-modules is the same thing as a morphism of $\mathbb{Z}[G]$-modules. Let $\text{Hom}_G(A, A')$ be the set of morphisms of $G$-modules between $A$ and $A'$.

We define $A^G$ to be the submodule of $G$-invariants, that is the subgroup of $A$ defined by $A^G := \{a \in A \mid ga = a \text{ for all } g \in G\}$. The functor $F : A \mapsto A^G$ from the category of $G$-modules to the category of abelian groups is covariant and left-exact. Therefore, we can define its right derived functors $R^iF$ and by definition the $i$-th cohomology group of $G$ with coefficients in $A$ is $H^i(G, A) := R^iF(A)$. Since the sets $A^G$ and $\text{Hom}_G(\mathbb{Z}, A)$ are canonically isomorphic, we can also define $H^i(G, A) = \text{Ext}^i(\mathbb{Z}, A)$ for more details the reader may refer to [11].

Let $B$ be an abelian group. Let us consider the following exact sequence:

$$1 \rightarrow B \xrightarrow{i} E \xrightarrow{p} G \rightarrow 1.$$

A section of $p$ is a morphism $s : G \rightarrow E$ such that $p \circ s = \text{id}_G$. We denote by $\text{Sec}_p(p)$ the set of sections of $p$. If $B = \mathbb{Z}\Phi$ we will just write $\text{Sec}(p)$. Two sections $s$ and $s'$ are called $B$-conjugated if there exists $b \in B$ such that for all $g \in G$ we have $s'(g) = i(b)s(g)i(b)^{-1}$.

Let $\varphi : G \rightarrow \text{Aut}(B)$ be a morphism of groups. $B$ inherits a structure of $G$-module where the action is given by $g.b := \varphi(g)(b)$. The following result is also well known and will be of particular interest to us:
Theorem 7.1 (Theorem 8.2 of [4]). The set of sections up to $B$-conjugacy of the exact sequence

$$
\begin{array}{cccccc}
1 & \longrightarrow & B & \overset{i}{\longrightarrow} & B \rtimes_{\varphi} G & \overset{p}{\longrightarrow} & G & \longrightarrow & 1
\end{array}
$$

is in bijection with $H^1(G, B)$.

7.2. Link between $H^1(W, \mathbb{Z}\Phi)$ and the components $\tilde{X}_{W_n}[\lambda(s)]$.

Definition 7.1. From Proposition 5.5 we know that two different Coxeter generators of $W$ lie in different components. Let $O_{s_i} \subset \tilde{X}_{W_n}[\lambda(s_i)]$ be the corresponding orbit of $(k(s_i, \alpha))_{\alpha \in \Phi^+}$ under the action of $\mathbb{Z}\Phi$. The elements of $O_{s_i}$ are those that can be written as $(k(\tau_{x} s_i, \alpha))_{\alpha \in \Phi^+}$ with $x \in \mathbb{Z}\Phi$. It is clear that $O_{s_i}$ has a structure of $\mathbb{Z}$-module (induced from the structure of $\mathbb{Z}\Phi$) where the scalar multiplication and the addition are defined as follows

\[
d.(k(\tau_{x} s_i, \alpha))_{\alpha \in \Phi^+} := (k(\tau_{dx} s_i, \alpha))_{\alpha \in \Phi^+},
\]

\[
(k(\tau_{x} s_i, \alpha))_{\alpha \in \Phi^+} + (k(\tau_{y} s_i, \alpha))_{\alpha \in \Phi^+} := (k(\tau_{x+y} s_i, \alpha))_{\alpha \in \Phi^+}.
\]

Example 7.1. In Figure 24 we have in orange one of the two components of $\tilde{X}_{W(\tilde{A}_2)}$. Each different filling of the orange alcoves represents an orbit in this component. The index of connection is 3 here, and we see that in the finite group there are 3 orange elements and 3 white elements. The orbits of these 3 orange different elements give us the 3 different orbits in the component $\tilde{X}_{W(\tilde{A}_2)}[(0,0,0)]$.

Figure 24. Orbits in $\tilde{X}_{W(\tilde{A}_2)}[(0,0,0)]$. 


**Definition 7.2.** The structure of \( \mathbb{Z} \)-module on each orbit induces a structure of \( \mathbb{Z} \)-module on \( \bigoplus_{i=1}^{n} O_{s_i} \). Recall that \( \iota \) is the map defined in Theorem 5.3.

We put on the set \( \bigoplus_{i=1}^{n} O_{s_i} \) an equivalence relation defined as follows

\[
(\iota(\tau_{x_1}s_1), \ldots, \iota(\tau_{x_n}s_n)) \sim (\iota(\tau_{y_1}s_1), \ldots, \iota(\tau_{y_n}s_n))
\]

\[
\iff \exists z \in \mathbb{Z} \Phi \text{ such that } \forall \alpha_i \in \Delta \text{ one has: } x_i - y_i = 2\left(\frac{z, \alpha_i}{(\alpha_i, \alpha_i)}\right)\alpha_i = (z, \alpha_i')\alpha_i.
\]

**Definition 7.3.** 1) Let \( s_i \in S \). Let \( L(s_i) \) be the subset of \( O_{s_i} \) defined by

\[
L(s_i) := \{ F(\tau_{x_i}) | x_i \in \mathbb{Z} \alpha_i \} = \{ \iota(s_i) + d\iota(\alpha_i) | d \in \mathbb{Z} \}.
\]

The set \( L(s_i) \) need to be thought of as a dashed line in \( \tilde{X}_{W_{\lambda}}(\lambda(s_i)) \).

2) Let \( A \) be a ring such that \( \mathbb{Z} \subset A \subset \mathbb{R} \). We extend the previous definition to \( A \) by

\[
L_A(s_i) := \{ \iota(s_i) + a\iota(\alpha_i) | a \in A \}.
\]

3) The relation \( \sim \) restricts naturally to \( \prod_{i=1}^{n} L(s_i) \). We define the set \( \tilde{X}^q \)

\[
\tilde{X}^q := \prod_{i=1}^{n} L(s_i)/\sim.
\]

**Remark 7.1.** Notice that we can define a similar equivalence relation, also denoted \( \sim \), on \( \prod_{i=1}^{n} L_A(s_i) \) by saying that two elements \( a, b \in \prod_{i=1}^{n} L_A(s_i) \) are equivalent if and only if there exists \( t \in A\Phi \) such that \( a_i - b_i = 2\left(\frac{t, \alpha_i}{(\alpha_i, \alpha_i)}\right)\alpha_i \) for all \( i = 1, \ldots, n \). It is also possible to define the analog of \( \tilde{X}^q \) by setting \( \tilde{X}^q_A := \prod_{i=1}^{n} L_A(s_i)/\sim \).

**Lemma 7.1.** Let \( A \) be a commutative unitary ring such that \( \mathbb{Z} \subset A \subset \mathbb{R} \). Let \( s \) be a section of the exact sequence

\[
1 \longrightarrow A\Phi \longrightarrow A\Phi \times W \overset{\pi}{\longrightarrow} W \longrightarrow 1,
\]

defined by \( s(s_i) = \tau_{x_i}s_i \) for \( s_i \in S \). Then the map \( \xi \) from \( Sec_{A\Phi}(\pi) \) to \( A\alpha_1 \times \cdots \times A\alpha_n \) defined by \( \xi(s) = (x_1, \ldots, x_n) \) is an isomorphism of \( A \)-modules. In particular \( Sec_{A\Phi}(\pi) \) is in bijection with \( L_A(s_1) \times \cdots \times L_A(s_n) \).

**Proof.** Let us first show that \( \xi \) is well defined, that is \( x_i \in A\alpha_i \) for all \( i \). We know that \( W \) is a group defined by generators and relations. The relations are given by \( s_i^2 = e \) and some braid relations. The braid relations are either of type \( st = ts \) or \( stst = tsts \) or \( ststst = tsts \) for all generators \( s, t \in S \). Since \( s \) is a morphism, it must preserve all of these relations. Therefore, the relation \( s_i^2 = e \) implies that \( \tau_{x_i}s_i\tau_{x_i}s_i = e \), that is \( \tau_{x_i+s_i(x_i)}s_i^2 = e \) and then \( x_i + s_i(x_i) = 0 \). Consequently we must have \( s_i(x_i) = -x_i \), which means that \( x_i \in H^+_s = \mathbb{R}\alpha_i \). It follows that \( (x_1, \ldots, x_n) \in \prod_{i=1}^{n} A\alpha_i \). Thus, for all \( i = 1, \ldots, n \) there exists \( k_i \in A \) such that \( x_i = k_i\alpha_i \). We claim that any choice of \( (k_1, \ldots, k_n) \in A^n \) gives rise to a section. From this claim it
is clear that the map $\xi$ is a bijection. With respect to the morphism it is obvious that the structure of $A$-module on $Sec_{A\Phi}(\pi)$ is the same as the structure of $A\Phi$.

Let us now prove the claim. Because of Proposition 3.1 we know that $\text{ord}((s\alpha_i, k_s, s\alpha_j, k_r))^d = \text{ord}((s\alpha_i, s\alpha_j))^d$ for all $d \in \mathbb{N}$. In particular for $d = 2, 3, 4$ or 6, since $s\alpha_i, k_s = \tau_{k_s} s\alpha_i, s\alpha_j = \tau_{s\alpha_i} s\alpha_j$, we see that the braid relations are preserved under the section $s$. It follows that there is no constraint on the choices of the $k_i$’s. This ends the proof. \hfill $\Box$

**Theorem 7.2.** We have a bijection between $H^1(W, Z\Phi)$ and $\tilde{X}^\pi$. 

**Proof.** $Z\Phi$ is clearly a $W$-module and then because of Theorem 7.1 we know that $H^1(W, Z\Phi)$ is in bijection with the sections, up to $Z\Phi$-conjugacy, of the group extension

$$
1 \longrightarrow Z\Phi \longrightarrow \begin{array}{c}
\end{array} Z\Phi \rtimes W \xrightarrow{\pi} W \longrightarrow 1.
$$

Let $s$ be a section of $\pi$. Since $s$ is a morphism, the images of the generators characterize entirely $s$. Thus, the data of a section is the same thing as a data of $n$ elements in $Z\Phi \rtimes W$. However, because of Lemma 7.1 we know that these $n$ elements must be written as $\tau_{\alpha_i}s_1, \ldots, \tau_{\alpha_i}s_n$ with $x_i \in Z\alpha_i$. Therefore, the data of a section is the same thing as taking one element in each $L(s_i)$ for all $i = 1, \ldots, n$. There follows the bijection between $Sec(\pi)$ and $L(s_1) \times \cdots \times L(s_n)$.

Let us take $s_1$ and $s_2$ two different sections defined as $s_1(s_i) = \tau_{x_i}s_i$ and $s_2(s_i) = \tau_{y_i}s_i$ with $x_i, y_i \in Z\alpha_i$ for all $i = 1, \ldots, n$. Since they are different there exists at least one index $j$ such that $x_j \neq y_j$. Then, $s_1$ is $\Phi$-conjugated to $s_2$ if and only if there exists $z \in Z\Phi$ such that for all $w \in W$ one has $s_2(w) = \tau_z s_1(w)\tau_{-z}$. It turns out that $\tau_z s_1(w)\tau_{-z} = \tau_{(id-w)(z)} s_1(w)$, indeed since $s_1(w) \in Z\Phi \rtimes W$ there exists $u \in Z\Phi$ such that $s_1(w) =\tau_u w$. Therefore we have $\tau_z s_1(w)\tau_{-z} = \tau_z \tau_u w\tau_{-z} = \tau_z \tau_u \tau_{-w}(z) w = \tau_z \tau_{-w}(z) \tau_u w = \tau_{(id-w)(z)} s_1(w)$. Hence, it follows

$$s_1 \sim s_2 \iff \exists z \in Z\Phi \text{ such that } \forall w \in W, s_2(w) = \tau_{(id-w)(z)} s_1(w) = \tau_{(id-s_i)}(z) s_1(s_i) \iff \exists z \in Z\Phi \text{ such that } \forall s_i \in S, y_i - x_i = (id - s_i)(z).$$

It also turns out that we have the opposite direction. Indeed, let us write $w = t_1 t_2 \cdots t_p$ with $t_i \in S$ such that it is a reduced expression. Via the following equalities we have

$$s_1(w) = s_1(t_1)s_1(t_2) \cdots s_1(t_p) = \tau_{x_1} t_1 \tau_{x_2} t_2 \cdots \tau_{x_p} t_p = \tau_{x_1 + t_1(x_2) + \cdots + t_1 t_2 \cdots t_{p-1}(x_p)} t_1 t_2 \cdots t_p,$$

$$s_2(w) = s_2(t_1)s_2(t_2) \cdots s_2(t_p) = \tau_{y_1} t_1 \tau_{y_2} t_2 \cdots \tau_{y_p} t_p = \tau_{y_1 + t_1(y_2) + \cdots + t_1 t_2 \cdots t_{p-1}(y_p)} t_1 t_2 \cdots t_p.$$

It follows that

$$s_2(w) = \tau_{(id-w)(z)} s_1(w) \iff \tau_{y_1 + t_1(y_2) + \cdots + t_1 t_2 \cdots t_{p-1}(y_p)} t_1 t_2 \cdots t_p = \tau_{(id-w)(z)} \tau_{x_1} t_1 \tau_{x_2} t_2 \cdots \tau_{x_p} t_p \iff y_1 + t_1(y_2) + \cdots + t_1 t_2 \cdots t_{p-1}(y_p) = z - w(z) + x_1 + t_1(x_2) + \cdots + t_1 t_2 \cdots t_{p-1}(x_p) \iff z - w(z) = (y_1 - x_1) + t_1(y_2 - x_2) + \cdots + t_1 t_2 \cdots t_{p-1}(y_p - x_p).$$
However, with the assumption $y_i - x_i = (id - t_i)(z)$ for all $i = 1, \ldots, p$, it follows that
\[
z - w(z) = z - t_1(z) + t_1(z) - t_1t_2(z) - t_1t_2 - \cdots - t_1t_2 \cdots t_{p-1}(z) + t_1t_2 \cdots t_{p-1}(z) - w(z) = z - t_1(z) + t_1(z - t_2(z)) + \cdots + t_1t_2 \cdots t_{p-1}(z - t_2(z)) = y_1 - x_1 + t_1(y_2 - x_2) + \cdots + t_1t_2 \cdots t_{p-1}(y_p - x_p).
\]
Therefore we have the equivalence
\[
s_1 \sim s_2 \iff \exists z \in \mathbb{Z}\Phi \text{ such that } \forall s_i \in S, \ y_i - x_i = (id - s_i)(z).
\]
However $(id - s_i)(z) = z - s_i(z) = z - (z - 2\frac{(z, x_1)}{(x_1, x_1)}\alpha_i) = 2\frac{(z, x_1)}{(x_1, x_1)}\alpha_i$. Then we have
\[
s_1 \sim s_2 \iff (\ell(\tau_{x_1, s_1}), \ldots, \ell(\tau_{x_n, s_n})) \sim (\ell(\tau_{y_1, s_1}), \ldots, \ell(\tau_{y_n, s_n}))
\]
with $\ell$ the map defined in Theorem 5.3. This proves the bijection. \qed

### 7.3. Cohomology in degree 1 with coefficients in $A\Phi$

This section is dedicated to the study of the cohomology in degree 1 when we change the lattice $\mathbb{Z}\Phi$. The main goal is to understand how the sections of $\pi$ behave up to conjugacy when the space of coefficients $\mathbb{Z}\Phi$ is replaced by $A\Phi$. We will see for instance that the more we add points in $L(s_i)$ the smaller $H^1(W, A\Phi)$ becomes.

Consequently, it is of particular interest to find a (commutative unitary) ring $A$ satisfying $\mathbb{Z} \subset A \subset \mathbb{R}$ and such that $H^1(W, A\Phi) = 0$. We also require to find $A$ minimal. We will say that there is an $A\Phi$-obstruction between two sections $s_1$ and $s_2$ if they don’t define the same class in $H^1(W, A\Phi)$.

We will see that the Cartan matrix of $W$ has a critical role in the understanding of the obstructions between sections (see Proposition 7.4). Finally, our main result in this section will be the following one:

**Theorem 7.3.** Let $A$ be a commutative unitary ring such that $\mathbb{Z} \subset A \subset \mathbb{R}$. Let $f_\Phi$ be the index of connection of $\Phi$ and $\mathbb{Z}_{f_\Phi} := \mathbb{Z}[\frac{1}{f_\Phi}]$. Then the following points are equivalent

\begin{enumerate}
    \item $H^1(W, A\Phi) = 0$ with $A$ minimal.
    \item $A = \mathbb{Z}_{f_\Phi}$.
\end{enumerate}

**Example 7.2.** Let us first illustrate this with an example. Let us take $W = W(B_2)$. The Cartan matrix of $B_2$ and its inverse are
\[
C_{B_2} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad C_{B_2}^{-1} = \begin{pmatrix} 1/2 & 2/2 \\ 1 & 2 \end{pmatrix}.
\]
Let $s_1$ and $s_2$ be two sections of $W(B_2)$. We know via Lemma 7.1 that we can identify $s_1$ with a point $(x_1, x_2) \in \mathbb{Z}\alpha_1 \times \mathbb{Z}\alpha_2$ and $s_2$ with a point $(y_1, y_2) \in \mathbb{Z}\alpha_1 \times \mathbb{Z}\alpha_2$. Denote $x_i = a_i\alpha_i$ and $y_i = b_i\alpha_i$. Moreover we know that
\[
s_1 \sim s_2 \iff \exists z \in \mathbb{Z}\alpha_1 \times \mathbb{Z}\alpha_2 \text{ such that } \begin{cases} x_1 - y_1 = 2\frac{(z, \alpha_1)}{(\alpha_1, \alpha_1)}\alpha_1 \\ x_2 - y_2 = 2\frac{(z, \alpha_2)}{(\alpha_2, \alpha_2)}\alpha_2. \end{cases}
\]
It follows that
\[ s_1 \sim s_2 \iff \exists z \in \mathbb{Z}\alpha_1 \times \mathbb{Z}\alpha_2 \text{ such that } C_{B_2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}. \]

Thus, we must have \( z_1 \) and \( z_2 \) both integral, and such that
\[
\begin{cases}
2z_1 = 2(a_1 - b_1) + 2(a_2 - b_2) \\
2z_2 = a_1 - b_1 + 2(a_2 - b_2).
\end{cases}
\]

These arithmetical conditions show how \( H^1(W(B_2), \mathbb{Z}\Phi) \) behaves compared to \( L(s_1) \times L(s_2) \), and we see in this particular situation that we do not have \( H^1(W(B_2), \mathbb{Z}\Phi) = 0 \). Indeed, by taking for example \( a_2 = b_2 = 1 \), \( a_1 = 3 \) and \( b_1 = 2 \) we have \( 2z_2 = 1 \), which is impossible since \( z_2 \) is integral. We thus have an obstruction between \( s_1 \) and \( s_2 \), which implies that \( H^1(W(B_2), \mathbb{Z}\Phi) \neq 0 \).

However, if we consider the equations in the ring \( \mathbb{Z}_2 = \mathbb{Z}[\frac{1}{2}] \) instead of \( \mathbb{Z} \), the previous obstruction no longer exists and it follows that \( H^1(W(B_2), \mathbb{Z}_2\Phi) = 0 \). Furthermore, the ring \( \mathbb{Z}_2 \) is the smallest one that satisfies the last equality.

**Theorem 7.4.** Let \( A \) be a commutative unitary ring such that \( \mathbb{Z} \subset A \subset \mathbb{R} \). Then the following points are equivalent:

1. \( H^1(W, A\Phi) = 0 \).
2. \( C_\Phi \in GL_n(A) \).

**Proof.** Let us first show the direction ii) implies i). Let \( \pi \) be the morphism from \( A\Phi \times W \) to \( W \) defined by \( \pi(\tau_w) = w \) for all \( x \in A\Phi \). We recall that \( H^1(W, A\Phi) \) is in bijection with the set \( \text{Sec}_{A\Phi}(\pi) \) up to \( A\Phi \)-conjugacy. Moreover, because of Lemma 7.1 we know that \( \text{Sec}_{A\Phi}(\pi) \) is in bijection with \( L_A(s_1) \times \cdots \times L_A(s_n) \) or again with \( A\alpha_1 \times \cdots \times A\alpha_n \). Let \( s_1, s_2 \in \text{Sec}_{A\Phi}(\pi) \) such that via the bijective correspondence above, \( s_1 \) is identified with the point \( (x_1, \ldots, x_n) \in A\alpha_1 \times \cdots \times A\alpha_n \) and \( s_2 \) with the point \( (y_1, \ldots, y_n) \). Let us denote \( x_i = a_i\alpha_i \) and \( y_i = b_i\alpha_i \) for all \( i \). Therefore, the equivalence relation between these two sections becomes in \( L_A(s_1) \times \cdots \times L_A(s_n) \) as follows:

\[
\begin{pmatrix}
x_1 - y_1 \\
\vdots \\
x_n - y_n
\end{pmatrix} = \begin{pmatrix}
2 \frac{z, \alpha_1}{(\alpha, \alpha)} \\
\vdots \\
2 \frac{z, \alpha_n}{(\alpha, \alpha)}
\end{pmatrix} \begin{pmatrix}
a_1 \\
\vdots \\
a_n
\end{pmatrix} = \begin{pmatrix}
a_1 - b_1 \\
\vdots \\
a_n - b_n
\end{pmatrix}.
\]

Writing \( z = z_1\alpha_1 + \cdots + z_n\alpha_n \) it follows that

\[
s_1 \sim s_2 \iff \exists z \in A\Phi \text{ such that } C_\Phi \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ \vdots \\ a_n - b_n \end{pmatrix}.
\]

With the assumption: \( C_\Phi \in GL_n(A) \), we also have \( C_\Phi^{-1} \in GL_n(A) \) and then there is no constraint for the choices of \( z_1, \ldots, z_n \) in \( A \). Consequently, there is no obstruction between \( s_1 \) and \( s_2 \), which implies that \( s_1 = s_2 \) in
\(H^1(W, A\Phi)\). Hence, there is an only one class in \(H^1(W, A\Phi)\), which implies the first direction.

Let us show now the other direction. Assume that \(C_\Phi \notin GL_n(A)\). Since \(C_\Phi \in GL_n(\mathbb{R})\) with its coefficients in \(\mathbb{Z}\), the previous assumption implies in particular that \(C_\Phi^{-1} \notin GL_n(A)\), and then there exists a coefficient \(q_{ij}\) of \(C_\Phi^{-1}\) such that \(q_{ij} \notin A\). Let \(s_1\) be the section with corresponding point \(x := (0, \ldots, 1, \ldots, 0) \in A\alpha_1 \times \cdots \times A\alpha_n\) where 1 is in position \(i\), and \(s_2\) be the trivial section. It turns out that \(s_1 \sim s_2\), if and only if, there exists \(z = z_1\alpha_1 + \cdots + z_n\alpha_n \in A\Phi\) such that \(C_\Phi z = x - 0 = x\). Therefore, it follows that

\[
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_n
\end{pmatrix} = C_\Phi^{-1}
\begin{pmatrix}
  0 \\
  \vdots \\
  1 \\
  0
\end{pmatrix} = \begin{pmatrix}
  q_{11} \\
  \vdots \\
  q_{ij} \\
  \vdots \\
  q_{nn}
\end{pmatrix}.
\]

Since \(z_i \in A\) and \(q_{ij} \notin A\), the sections \(s_1\) and \(s_2\) define two different elements in \(H^1(W, A\Phi)\). This is impossible since \(H^1(W, A\Phi) = 0\). Therefore, we must have \(C_\Phi \in GL_n(A)\).

**Proof of Theorem 7.3.** It is well known that the determinant of \(C_\Phi\) is \(f_\Phi\) (see [3] Ch. VI, § 1, exercice 7). Consequently \(C_\Phi^{-1} = \frac{1}{\det(C_\Phi)} D = \frac{1}{f_\Phi} D\) where \(D \in M_n(\mathbb{Z})\). Therefore, because of Theorem 7.4, if \(A = \mathbb{Z}_{f_\Phi}\) we have \(C_\Phi \notin GL_n(A)\) and it follows that \(H^1(W, \mathbb{Z}_{f_\Phi}\Phi) = 0\).

Let us show now the direction i) implies ii). Since \(H^1(W, A\Phi) = 0\), Theorem 7.3 tells us that \(C_\Phi \in GL_n(A)\), as well as its inverse. Therefore, all the coefficients of \(C_\Phi^{-1}\) are in \(A\). However, all the coefficients of \(C_\Phi^{-1}\) are of the form \(\frac{d}{f_\Phi}\) with \(d \in \mathbb{Z}\). We claim then that \(\frac{1}{f_\Phi} \in A\). Indeed, a short observation (see [22] together with its appendix for the exceptional cases) of the inverses of the Cartan matrices in type \(A, B, C, E_6, E_7, E_8, F_4\) and \(G_2\) shows that there are always a coefficient of the form \(\frac{d}{f_\Phi}\) and another one of the form \(\frac{d+1}{f_\Phi}\) in the same Cartan matrix. From this, the claim becomes obvious. In type \(D_n\) the previous statement fails but we have that \(\frac{2(n-2)}{2f_\Phi} \in A\) and \(\frac{2}{f_\Phi} \in A\). This implies in particular that \(\frac{2}{f_\Phi} \in A\). Moreover in type \(D_n\) one has \(f_\Phi = 4\). Hence \(1/2 \in A\) and then \(1/4\) as well.

Therefore, in each situation \(\mathbb{Z}\) \(\left\lfloor \frac{1}{f_\Phi} \right\rfloor \subset A\). Since \(A\) is supposed to be minimal and since \(H^1(W, \mathbb{Z}_{f_\Phi}\Phi) = 0\), one has \(A = \mathbb{Z}_{f_\Phi}\). \(\square\)

### 7.4. Concrete realization of \(H^1(W, \mathbb{Z}\Phi)\) into \(\tilde{X}_{W_n}\).

Because of Lemma 7.1 we know that the sections of \(\pi\) are in bijective correspondence with the points of \(L(s_1) \times \cdots \times L(s_n)\). We also know that the elements of \(H^1(W, \mathbb{Z}\Phi)\) are in bijection with the sections of \(\pi\) up to \(\mathbb{Z}\Phi\)-conjugacy. Furthermore, the condition of being \(\mathbb{Z}\Phi\) conjugate, when considered as a condition on pairs of elements in \(L(s_1) \times \cdots \times L(s_n)\), corresponds to the solvability of a system of linear equations defined by the Cartan matrix.

In this section we investigate in each Weyl group of type \(A, B, C, D\) how this equivalence relation behaves in \(L(s_1) \times \cdots \times L(s_n)\). First of all, notice that when the index of connection is 1, the Cartan matrix is invertible in \(\mathbb{Z}\),
which implies via Theorem 7.3 that $H^1(W, \mathbb{Z}_\Phi) = 0$. In other words, when $f_\Phi = 1$ there is no obstruction. Hence, if $\Phi$ is of type $G_2$, $F_4$ or $E_8$, the cohomology in degree 1 is trivial, and the only class of $H^1(W, \mathbb{Z}_\Phi) = 0$ is respectively sent to $((s_1), \ldots, (s_n))$ with $n = 2, 4$ or 8.

Let $s_1, s_2 \in \text{Sec}(\pi)$. Let $(x_1, \ldots, x_n) \in \mathbb{Z}_\alpha \times \cdots \times \mathbb{Z}_\alpha$ the corresponding point of $s_1$ and $(y_1, \ldots, y_n)$ the corresponding point of $s_2$. Denote $x_i = a_i\alpha_i$ and $y_i = b_i\alpha_i$. We denote by $d_i(s_1, s_2)$, or just $d_i$ if there is no confusion, the number $d_i := a_i - b_i$. We also define

$$d := d_1\alpha_1 + \cdots + d_n\alpha_n.$$

7.4.1. $A_n$. In this section $W = W(A_n)$ and $\Phi = A_n$. We denote by $C := C\_A\_n$ the Cartan matrix of $A_n$. The coefficients of $C^{-1}$ are known (see [22])

and are given by

$$(C^{-1})_{ij} = \frac{(n + 1)\min(i, j) - ij}{n + 1}.$$  

**Proposition 7.1.** The sections $s_1$ and $s_2$ define two different elements in $H^1(W, \mathbb{Z}_\Phi)$ if and only if $d_1 + 2d_2 + \cdots + nd_n \neq 0$ in $\mathbb{Z}/(n + 1)\mathbb{Z}$.

**Proof.** We know that $s_1 \sim s_2$ if and only if there exists $z \in \mathbb{Z}_\alpha \times \cdots \times \mathbb{Z}_\alpha$ such that $Cz = d$, that is if and only if $z = C^{-1}d$. Denote $z = z_1\alpha_1 + \cdots + z_n\alpha_n$. We thus obtain the following system $(S)$

$$\begin{align*}
(n + 1)z_1 &= nd_1 + (n - 1)d_2 + \cdots + d_n \\
(n + 1)z_2 &= (n - 1)d_1 + (2n - 2)d_2 + \cdots + 2d_n \\
&\vdots \\
(n + 1)z_k &= [(n + 1)\min(k, 1) - k]d_1 + [(n + 1)\min(k, 2) - 2k]d_2 + \cdots + \\
&\quad [(n + 1)\min(k, n) - nk]d_n \\
&\vdots \\
(n + 1)z_n &= d_1 + 2d_2 + \cdots + nd_n.
\end{align*}$$

Therefore, if there exists a $z$ satisfying this system, since all the $z_i$ are integral, it must also satisfy for all $k = 1, \ldots, n$ the following condition

$$\sum_{j=1}^n [(n + 1)\min(k, j) - kj]d_j \in (n + 1)\mathbb{Z},$$

which is equivalent to the following equation in $\mathbb{Z}/(n + 1)\mathbb{Z}$

$$\sum_{j=1}^n [(n + 1)\min(k, j) - kj]\overline{d_j} = 0.$$  

We claim now that if $d_1 + 2d_2 + \cdots + nd_n \in (n + 1)\mathbb{Z}$ then we have

$$\sum_{j=1}^n [(n + 1)\min(k, j) - kj]d_j \in (n + 1)\mathbb{Z}$$

for all others indexes $k$. Since $d_1 + 2d_2 + \cdots + nd_n \in (n + 1)\mathbb{Z}$ there exists $r \in \mathbb{Z}$ such that $d_1 + 2d_2 + \cdots + nd_n = (n + 1)r$, that is $d_1 = (n + 1)r - 2d_2 - \cdots - nd_n$. Therefore
\[ \sum_{j=1}^{n} [(n+1)\text{min}(k,j) - kj]d_j \]

\[ = (n+1-k)d_1 + \sum_{j=2}^{n} [(n+1)\text{min}(k,j) - kj]d_j \]

\[ = (n+1-k)((n+1)r - 2d_2 - \cdots - nd_n) + \sum_{j=2}^{n} [(n+1)\text{min}(k,j) - kj]d_j \]

\[ = r(n+1-k)(n+1) + \sum_{j=2}^{n} [(n+1)\text{min}(k,j) - kj-j(n+1-k)]d_j \]

\[ = r(n+1-k)(n+1) + \sum_{j=2}^{n} [(n+1)(\text{min}(k,j) - j)]d_j \]

\[ = (n+1)\left[r(n+1-k) + \sum_{j=2}^{n} \text{[min}(k,j) - j]\right]d_j. \]

To summarize, if there exists \( z \in \mathbb{Z}\Phi \) such that (S) is satisfied then we must have \( \sum_{j=1}^{n} [(n+1)\text{min}(k,j) - kj]d_j = 0 \) for all \( k = 1, \ldots, n \), which is equivalent to the equation \( d_1 + 2d_2 + \cdots + nd_n = 0 \). Therefore, if \( d_1 + 2d_2 + \cdots + nd_n \neq 0 \) there doesn’t exist \( z \in \mathbb{Z}\Phi \) such that \( Cz = d \), whence \( s_1 \) and \( s_2 \) define two different classes in \( H^1(W,\mathbb{Z}\Phi) \).

Conversely, if \( s_1 \) and \( s_2 \) define two different elements in \( H^1(W,\mathbb{Z}\Phi) \) we don’t have \( z \in \mathbb{Z}\Phi \) satisfying (S). However, if \( d_1 + 2d_2 + \cdots + nd_n = 0 \) it follows that \( \sum_{j=1}^{n} [(n+1)\text{min}(k,j) - kj]d_j = 0 \) for all \( k = 1, \ldots, n \). Thus, by setting \( z_k := \frac{1}{n+1} \left[ \sum_{j=1}^{n} [(n+1)\text{min}(k,j) - kj]\right]d_j \) for all \( k = 1, \ldots, n \) we have built an element in \( \mathbb{Z}\Phi \) such that \( Cz = d \), i.e. \( s_1 \) and \( s_2 \) are equivalent, which is impossible according to our assumption. \( \square \)

**Remark 7.2.** Assume that \( n+1 \) is a prime number. Then for each pair of sections \( s_1 \) and \( s_2 \) that satisfy \( d_i(s_1, s_2) = d_j(s_1, s_2) \) for all \( i, j \), we have \( s_1 = s_2 \).

Indeed Let \( s_1 \) and \( s_2 \) be two sections satisfying \( d_i(s_1, s_2) = d_j(s_1, s_2) \) for all \( i, j \). Since \( n+1 \) is a prime number, the polynomial \( \overline{X} + 2\overline{X} + \cdots + n\overline{X} \) admits all the elements of \( \mathbb{Z}/(n+1)\mathbb{Z} \) as solutions. In particular, the equality \( d_1 + 2d_2 + \cdots + nd_n = 0 \) is always true. Therefore, because of Proposition 7.1 these two sections define the same element in \( H^1(W(A_n), \mathbb{Z}\Phi) \) and then \( s_1 = s_2 \).

Note that if \( n+1 \) is not prime, the above result does not necessarily hold. Indeed, assume that \( n = 3 \) and let us take \( s_1 \) that we identify with \( (3,4,5) \) and \( s_2 \) with \( (6,7,8) \). We have here \( d_i(s_1, s_2) = 3 \) for \( i = 1, 2, 3 \). Moreover \( d_1 + 2d_2 + 3d_3 = 2 \neq 0 \). Hence \( s_1 \neq s_2 \).

See for example Figure 25.
7.4.2. $B_n$. In this section $W = W(B_n)$ and $\Phi = B_n$. We denote by $C := CB_n$ the Cartan matrix of $B_n$. The coefficients of $C^{-1}$ are known (see [22]) and are given by

$$(C^{-1})_{ij} = \frac{\min(i, j)}{1 - \min(0, n - i - 1)} = \begin{cases} \min(i, j) & \text{if } i < n \\ \frac{n}{2} & \text{if } i = n \\ 1 \leq i, j \leq n. \end{cases}$$

Proposition 7.2. Let us write $I_n := \{k \in [1, n] \mid k \text{ is odd}\}$ and $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. The sections $s_1$ and $s_2$ define two different elements in $H^1(W, \mathbb{Z}\Phi)$ if and only if $\sum_{k \in I_n} d_k = 1$ in $\mathbb{F}_2$.

Proof. The proof is almost the same as the $A_n$ case above. The corresponding system $(S)$ is in this situation as follows

$$
\begin{align*}
2z_1 &= 2d_1 + 2d_2 + \cdots + 2d_n \\
\vdots \\
2z_k &= 2\min(k, 1)d_1 + 2\min(k, 2)d_2 + \cdots + 2\min(k, n)d_n \\
\vdots \\
2z_n &= d_1 + 2d_2 + \cdots + nd_n.
\end{align*}
$$

It is obvious that $2\min(k, 1)d_1 + \cdots + 2\min(k, n)d_n \in 2\mathbb{Z}$ for all $k = 1, \ldots, n - 1$. Therefore, the only equation that matters in this system is the last one: $2z_n = d_1 + 2d_2 + \cdots + nd_n \in 2\mathbb{Z}$. This equation transferred in $\mathbb{F}_2$ becomes $\sum_{k \in I_n} k\overline{d_k} = 0$ and then $\sum_{k \in I_n} \overline{d_k} = 0$. The result follows.

□

Figure 25. Two sections giving the same class in cohomology.

These two sections define the same element in $H^1(W(A_1), \mathbb{Z}\Phi)$ since the following equation

$$X + 2X + 3X + 4X = 0$$

is satisfied for each $X \in \mathbb{Z}/5\mathbb{Z}$.
7.4.3. $C_n$. In this section $W = W(C_n)$ and $\Phi = C_n$. We denote by $C := C_{C_n}$ the Cartan matrix of $C_n$. The coefficients of $C^{-1}$ are known (see [22]) and are given by

$$(C^{-1})_{ij} = \frac{\min(i, j)}{1 - \min(0, n - j - 1)} = \begin{cases} \min(i, j) & \text{if } j < n \\ \frac{i}{2} & \text{if } j = n \\ 1 \leq i, j \leq n. \end{cases}$$

**Proposition 7.3.** The sections $s_1$ and $s_2$ define two different elements in $H^1(W, \mathbb{Z}\Phi)$ if and only if $d_n = 1$ in $\mathbb{F}_2$.

**Proof.** The corresponding system $(S)$ is in this situation as follows

$$
\begin{align*}
2z_1 &= 2d_1 + 2d_2 + \cdots + 2d_{n-1} + d_n \\
2z_2 &= 2\min(k,1)d_1 + 2\min(k,2)d_2 + \cdots + 2\min(k,n-1)d_{n-1} + kd_n \\
2z_n &= 2d_1 + 4d_2 + \cdots + 2(n-1)d_{n-1} + nd_n.
\end{align*}
$$

Consequently, we must have in particular that $2d_1 + 2d_2 + \cdots + 2d_{n-1} + d_n \in 2\mathbb{Z}$, that is $d_n \in 2\mathbb{Z}$. Therefore, when $d_n \in 2\mathbb{Z}$ the system $(S)$ is always satisfied, and it follows that $s_1 = s_2$. Since the converse direction is obvious, there follows the result. \(\square\)

7.4.4. $D_n$. In this section $W = W(D_n)$ and $\Phi = D_n$. We denote by $C := C_{D_n}$ the Cartan matrix of $D_n$. It turns out that $C$ is symmetric, this is why we just give the coefficients of $C^{-1}$ with entries $1 \leq i \leq j \leq n$

$$(C^{-1})_{ij} = \begin{cases} i & \text{if } 1 \leq i \leq j \leq n-2 \\ \frac{n-i}{2} & \text{if } i < n-1, j = n-1 \text{ or } n \\ \frac{n}{2} & \text{if } i = j = n-1 \text{ or } n. \end{cases}$$

A better visualization of this matrix is given by:

$$C^{-1} = \frac{1}{4} \begin{pmatrix}
4 & 4 & 4 & \cdots & 4 & 2 & 2 \\
4 & 8 & 8 & \cdots & 8 & 4 & 4 \\
4 & 8 & 12 & \cdots & 12 & 6 & 6 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
4 & 8 & 12 & \cdots & 4(n-2) & 2(n-2) & 2(n-2) \\
2 & 4 & 6 & \cdots & 2(n-2) & n & 2(n-2) \\
2 & 4 & 6 & \cdots & 2(n-2) & 2(n-2) & n
\end{pmatrix}$$

**Proposition 7.4.** Let us write $I_{n-2} := \{k \in [1, n-2] \mid k \text{ is odd}\}$. The sections $s_1$ and $s_2$ define two different elements in $H^1(W, \mathbb{Z}\Phi)$ if and only if we have the following points

\begin{itemize}
  \item[i)] $d_{n-1} \neq d_n$ in $\mathbb{F}_2$,
  \item[ii)] $\sum_{k \in I_{n-2}} 2d_k + n\overline{d_{n-1}} + (n-2)d_n \neq 0$ in $\mathbb{Z}/4\mathbb{Z}$,
  \item[iii)] $\sum_{k \in I_{n-2}} 2\overline{d_k} + (n-2)\overline{d_{n-1}} + nd_n \neq 0$ in $\mathbb{Z}/4\mathbb{Z}$.
\end{itemize}

**Proof.** The first line of the corresponding system $(S)$ is given by:

$$4z_1 = 4d_1 + 4d_2 + \cdots + 4d_{n-2} + 2d_{n-1} + 2d_n.$$
Thus, we must have $4d_1 + 4d_2 + \cdots + 4d_{n-2} + 2d_{n-1} + 2d_n \in 4\mathbb{Z}$, that is $2d_{n-1} + 2d_n \in 4\mathbb{Z}$, which is equivalent to $\overline{d_{n-1}} = \overline{d_n}$ in $\mathbb{F}_2$. We see then that if $\overline{d_{n-1}} = \overline{d_n}$, the $(n-2)$ first equations of $(S)$ can be satisfied.

The penultimate equation of $(S)$ is
\[ 4z_{n-1} = \sum_{k=1}^{n-1} 2k d_k + nd_{n-1} + (n-2)d_n. \]

Therefore, this previous equality compels us to have
\[ \sum_{k=1}^{n-1} 2k d_k + nd_{n-1} + (n-2)d_n \in 4\mathbb{Z}, \]
that is $\sum_{k=1}^{n-2} 2kd_k + nd_{n-1} + (n-2)d_n = 0$ in $\mathbb{Z}/4\mathbb{Z}$. If $k$ is even $2kd_k = 0$, and if $k$ is odd $2kd_k = 2d_k$. Hence we have $\sum_{k \in I_{n-2}} 2\overline{d}_k + nd_{n-1} + (n-2)d_n = 0$.

The reasoning is exactly the same for the last equation. This concludes the proof. □

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