All-Photonic Quantum Simulators with Spectrally Disordered Emitters

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Integrated photonics has been a promising platform for analog quantum simulation of condensed matter phenomena in strongly correlated systems. To that end, we explore the implementation of all-photonic quantum simulators in coupled cavity arrays with integrated ensembles of spectrally disordered emitters. Our model is reflective of color center ensembles integrated into photonic crystal cavity arrays. Using the Quantum Master Equation and the Effective Hamiltonian approaches, we study energy band formation and wavefunction properties in the open quantum Tavis-Cummings-Hubbard framework. We find conditions for polariton creation and (de)localization under spectral disorder and for variable ratios of emitter-cavity and cavity-cavity interactions. To quantify these properties, we introduce two metrics, the polaritonic and the nodal participation ratio, that characterize the light-matter hybridization and the node delocalization of the wavefunction, respectively. These new metrics prove to be useful tools for cavity quantum electrodynamical engineering of solid-state systems.

I. INTRODUCTION

Quantum simulation has attracted scientific attention since the early 1980s ignited by Richard Feynman’s vision on necessity of quantum mechanics in the modeling of natural phenomena [1]. Proposed implementations have included atomic, trapped ion, superconducting and photonic platforms [2–5]. We turn our focus to solid-state optical systems due to their potential for growth into large-scale commercial quantum simulators [6–9].

Nanophotonic cavities integrating quantum emitters have served as a rich playground for exploring quantum optics phenomena in solid-state systems. This includes demonstrations of weak [10] and strong [11] cavity quantum electrodynamical (QED) coupling, photon blockade and photon-induced tunneling [12], ultra-fast modulation of optical signals [13], and more. The large dipole moment of quantum emitters, paired with (sub)wavelength scale optical mode volumes in photonic crystal cavities, give rise to high optical nonlinearities and light-matter state hybridization that creates polaritons. Polaritonic interactions in nanophotonic systems can be several orders of magnitude higher than those achieved in atomic systems. Such strong interaction has been at the core of theoretical proposals for the quantum transfer [14, 15], as well as for the photonic simulation [16, 17] of Bose-Hubbard and fractional quantum Hall physics. Here, the system is made of an array of coupled cavities, each in the strong coupling regime of cavity QED, and described by the Jaynes-Cummings-Hubbard model. However, this model has been experimentally hard to achieve.

While the progress toward the realization of coupled cavity arrays (CCAs) with embedded emitters has been made with quantum dots [18, 19], the spectral disorder of these emitters has been a major roadblock to developing a large-scale resonant system. This problem is not present to such a significant extent with the color center emitters, which are atomic defects in wide band gap materials. Recently, color center integration with nanocavities in diamond [20, 21] and silicon carbide [22, 23] has been demonstrated in the weak cavity QED coupling regime. Though this regime is unsuitable for studies of polaritonic physics, proposals to demonstrate strong cavity QED regime have been presented with cavities integrating several (M) emitters, as opposed to a single emitter. Such systems are described by the Tavis-Cummings, rather than the Jaynes-Cummings model. Here, the collective coupling of emitters to the cavity effectively boosts the light-matter interaction rate by a factor of $\sqrt{M}$. Due to the small, but nonzero, spectral disorder of color centers, the collective strong coupling is possible within the cavity protection regime, if its rate overcomes the spectral disorder $\Delta$ of color centers [24, 25], i.e. $\Delta < g\sqrt{M}$. Such disordered multi-emitter cavity systems have been explored for applications in quantum light generation [26–28].

Here, we explore how all-photonic quantum simulators based on coupled cavity arrays can benefit from an increased interaction rate established in multi-emitter cavity QED. We expand the Jaynes-Cummings-Hubbard approach to the spectrally disordered Tavis-Cummings-Hubbard model (TCHM) [29] and define conditions for polariton creation utilized in all-photonic quantum simulation. Our model targets applications in technologically mature solid-state platforms and is reflective of the state-of-the-art parameters achieved in silicon carbide and diamond color center hosts.

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II. THE CCA QED MODEL

Our CCA QED model captures the single-excitation regime of the spectrally disordered Tavis-Cummings-Hubbard model comprised of emitter-cavity localizing interactions and cavity-cavity delocalizing interactions:

\[
H_{\text{TCHM}} = \sum_{n=1}^{N} \left\{ \omega_{c,n} a_{n}^\dagger a_{n} + \sum_{m=1}^{M_{c,n}} \left[ \omega_{c,m}^a \sigma_{n,m}^+ \sigma_{n,m}^- + g_{n,m} \left( a_{n}^\dagger \sigma_{n,m}^- + \sigma_{n,m}^+ a_{n} \right) \right] - J_{n,n+1} \left( a_{n+1}^\dagger a_{n+1} + a_{n}^\dagger a_{n} \right) \right\},
\]

where \(N\) is the number of cavities in the array, \(M_{c,n}\) is the number of emitters in the \(n\)-th cavity, \(\omega_{c,n}\) and \(a_{n}\) represent the angular frequency and the annihilation operator of the \(n\)-th cavity, \(\omega_{c,m}^a\), \(\sigma_{n,m}^+\) and \(g_{n,m}\) correspond to the angular frequency, the lowering operator and the emitter-cavity coupling rate of the \(m\)-th emitter in the \(n\)-th cavity, \(J_{n,n+1}\) is the photon hopping rate between the enumerated neighboring cavities. In this work we will assume \(\omega_{c,n} = \omega_{c}, g_{n,m} = g\) and \(J_{n,n+1} = J\).

A. Non-disordered CCA QED model

Before examining the spectral disorder effects in CCA QED, let us first address the energy spectrum in the fully resonant system of a linear array of coupled cavities with identical emitters. Here, the eigenergy spectrum features two CCA polariton bands with \(N\) states, each, and a degenerate set of \(N(M+1)\) subradiant states as illustrated in Figure 1. The polariton band states are parameterized by discrete momenta \(k = k_{p} = \pi p/(N+1)\), \((p = 1, 2, 3, \ldots , N)\) as

\[
E(k) = \omega_{c} - J \cos k \pm \sqrt{J^2 \cos^2 k + M g^2}.
\]

The origin of these spectral features can be decomposed to the QED and the CCA components. The resonant Tavis-Cummings model of \(M\) emitters in \(N = 1\) cavity has the spectrum of two polaritons and \(M - 1\) degenerate subradiant states, while a CCA of \(N > 1, M = 0\) cavities has a single spectral band of \(N\) photonic states. The spectrum of the resonant Tavis-Cummings-Hubbard model is a product of these components. This system has close analogs to the condensed matter models it aims to simulate.

B. The Condensed Matter analogs

For cavity-emitter arrays with translationally invariant energy scales \((\omega_{c,n} = \omega_{c}, \omega_{e,n,m} = \omega_{e}, g_{n,m} = g)\), \(J_{n,n+1} = J\), one can solve for the eigenvalues and eigenvectors analytically, both for open and periodic \((a_{1} \equiv a_{N+1})\) boundary conditions. Indeed, much of the derivation precisely parallels the calculation of the band structure of tight binding Hamiltonians commonly studied in condensed matter physics, as detailed in the Appendix. The case of no emitters \((M_{c} = 0)\) corresponds to the \(d = 1\) (Bose) Hubbard Model (HM), and the case with a single emitter in each cavity \((M_{c} = 1)\) to the Periodic Anderson Model (PAM). We should note that in the latter part of the study we will use open boundary condition due to its correspondence to experiment.

Indeed, there are a number of connections between the Tavis-Cummings-Hubbard model (TCHM), considered here in the context of CCAs, and the various Hamiltonians which are used to model strongly correlated fermion materials in condensed matter physics. For one emitter per cavity, our model has an identical geometry to the Periodic Anderson Model (PAM): the hopping of photons between cavities is analogous to the conduction electrons which hybridize between different sites, and the photon-emitter coupling maps onto the hopping between the conduction electrons and localized orbitals which, like the emitters, do not have a direct intersite (intercavity) overlap. Thus, in the single excitation sector, and for one emitter per cavity, the models are identical.

However, there are significant limitations to this analogy: If the number of excitations is greater than one, the bosonic nature of the photons in the CCA problem introduces a fundamental difference between the TCHM and the PAM. Further, the fermions in the PAM have a spin index, so there are two species of these particles. As long as the local Coulomb repulsion \(U = 0\) in the PAM, these two species are independent and hence can be considered as two independent copies of the TCHM. However this breaks down for \(U \neq 0\) and multiple excitations. Finally, the PAM does not admit multiple localized modes on different sites, so it only maps to the case of a single emitter per cavity.
It is worth noting, however, that the CM community has considered many variants of models which mix localized and itinerant quantum particles: the Kondo lattice model (KLM) replaces the two spin species of the localized electrons on each site by a single spin, thereby reducing the number of local degrees of freedom per site to two, the same value as an emitter in the TCHM. On the other hand, in the large-N limit the number of local degrees of freedom is expanded. Crudely speaking, these variants are similar to adjusting the number of emitters per cavity.

Thus, while it defies a specific mapping onto a condensed matter Hamiltonian, the TCHM provides another, novel and interesting, way of exploring the interplay of itinerant and mobile quantum particles. It is distinct from the condensed matter models, but we believe that a careful exploration of its properties, as provided in our manuscript, holds lessons for the CM model Hamiltonian community, especially in view of the many variants of models already in play there. The polaritonicity (degree of light-matter hybridization) upon which we focus in the sections to follow, holds lessons for singlet formation in which local and itinerant electrons become tightly intertwined in the PAM and KLM.

III. EFFECTS OF SPECTRAL DISORDER ON POLARITON FORMATION

A. Model parameters

The parameters of our model have been selected as representative of silicon carbide and diamond color center platforms. Recent demonstrations of emitter-cavity interaction in photonic crystal cavities support rates of approximately $g/2\pi \sim 2-7.3$ GHz [22, 30], therefore, we chose a constant value of $g/2\pi = 5$ GHz. While a variation in the coupling rate $g$ among emitters is likely to occur due to their variable positioning inside the electromagnetic mode, our prior work indicates that the collective emitter-cavity coupling still takes place [24] at a well defined rate of $g_M = \sqrt{\sum_{m=1}^{M} g_m^2}$. Therefore, keeping $g$ constant among the emitters should not take away from the overall phenomenology studied here. The experimentally demonstrated cavity loss rates reach as low as $\kappa/2\pi \sim 15-50$ GHz [22, 30], while recent modeled designs could reduce these values by at least an order of magnitude [31]. With a slight optimism, we chose cavity loss rate of $\kappa/2\pi = 10$ GHz. Our recent designs of photonic crystal molecules indicate that coupled cavity hopping rates can be straightforwardly designed in the range $1$ GHz $< J/2\pi < 200$ GHz [31], thus spanning systems from the dominant photon QED to the dominant photonic interaction character, represented in our choice of values $J/g = 0.1, 1, 10$. Fabrication imperfections may yield drifts in cavity resonant frequencies and hopping rates. The effect of this issue was studied in another platform where GaAs coupled cavity arrays were integrated with quantum dots [19] and indicates that the coupling strength is an order of magnitude higher than the frequency and hopping rate perturbations. We apply this assumption in our model, maintaining that all cavities are mutually resonant and all rates $J$ are constant. The spectral inhomogeneity of emitters in fabricated devices, the main study of our model, has been characterized as $\Delta \sim 10$ GHz for a variety of emitters in silicon carbide and diamond [32, 33]. We represent this parameter through its relation to the collective coupling rate $g_M$ in a cavity, spanning the spectral inhomogeneity across a range of values. It is worth noting that the vibronic resonances are two orders of magnitude larger than the inhomogeneous broadening, for example $8.7$ THz for the silicon vacancy in $4H$-SiC [34, 35], therefore the phonon side band is not expected to play a part in the collective emitter-cavity coupling process. Emitter lifetime in color centers is usually in the $1$-$15$ ns range [36], we select the value $\gamma/2\pi = 1/5.8$ GHz as representative. Due to $\gamma$ being the lowest rate in the system, its minimal variations among emitters [33] affect the system only marginally, therefore we assume it has a constant value in the system. Lifetime- and nearly lifetime-limited emission of color centers has been demonstrated upon photonic integration [33, 37, 38]. Due to this experimental advance, our model does not consider the dephasing terms, though such analysis may prove valuable with further development of integrated coupled cavity arrays.

B. Numerical methods

For the exact solution of the open quantum system Tavis-Cummings-Hubbard model, we have developed a software package [39] that solves the Quantum Master Equation (QME). Our code uses the Quantum Toolbox in Python (QuTiP) [40] which solves the Lindbladian:

$$L\rho(t) = -i [H_{TCHM}, \rho(t)] + \sum_{n=1}^{N} \left\{ \frac{\kappa_n}{2} D[a_n] \rho(t) + \sum_{m=1}^{M} \frac{\gamma_{n,m}}{2} D[\sigma_{n,m}] \rho(t) \right\} + PD \left[ a_{1}^{\dagger} \right] \rho(t)$$

(3)

where $D[c] \rho(t) = 2c \rho(t) c^\dagger - c^\dagger c \rho(t) - \rho(t) c^\dagger c$, $\kappa_n = \kappa$ is the cavity linewidth of the $n$-th cavity, $\gamma_{n,m} = \gamma$ is the emission rate of the $m$-th emitter in the $n$-th cavity, and $P$ is the optical (laser) pumping term. The spectral intensity reported in Figure 2 is calculated as the Fourier Transform of the correlation function $\langle A^\dagger(t + \tau) A(t) \rangle$:

$$S(\omega) = \int_{-\infty}^{\infty} \lim_{\tau \to \infty} \langle A^\dagger(t + \tau) A(t) \rangle e^{-i\omega \tau} d\tau,$$

(4)

where $A$ is replaced by the cavity annihilation and the emitter lowering operators, $a_n$ and $\sigma_{n,m}$. The QME requires the use of the full density operator because of the non-number conserving term $\rho(t) c^\dagger c$. 

The density operator requirement makes solving this system highly resource-intensive (exponential in $N \times M$), as such we have restricted our exact calculations to small systems of six elements, or two coupled cavities with two emitters per cavity.

To access modeling of larger systems we develop a software package in Python [41] that diagonalizes the Effective Hamiltonian in the approximate single-excitation regime

$$H_{\text{EFF}} = H_{\text{TCHM}} - \frac{i}{2} \sum_{n=1}^{N} \left\{ \kappa_n a_n^\dagger a_n + \sum_{m=1}^{M_n} \gamma_{n,m} \sigma_{n,m}^+ \sigma_{n,m}^- \right\},$$

(5)

which reduces the computational complexity from exponential to polynomial (cubic) in $N \times M$ for single-excitation regime. With this approximate method we numerically solve systems with hundreds of elements compared to the several using the exact QME approach.

### C. Metrics for characterizing disorder

We develop new metrics for the characterization of TCHM wavefunctions, inspired by practices in Condensed Matter Physics. The phenomenon of Anderson localization describes the loss of mobility of quantum particles due to randomness [42]. Originally studied in the context of non-interacting electrons hopping on a lattice with disordered site-energies, where all eigenstates were shown to be localized in spatial dimension less than or equal to two [43, 44], Anderson localization has subsequently been extensively investigated in many further contexts, including the effect of interactions [45], correlations in the disorder [46], and importantly, new experimental realizations from cold atomic gases [47, 48] to transport in photonic lattices [49].

A useful metric for quantifying the localization of a wavefunction $v_p$, employed in these studies, is the participation ratio, $P = \left[ \sum_p |v_p|^4 \right]^{-1}$ [50] and its generalizations [51]. Instead of measuring the participation ratio among all $N(M+1)$ vector components, we adapt $P$ to two new metrics that measure the participation among $N$ nodes (cavity-emitter sets), and two cavity- and emitter-like components. We define the nodal participation ratio, $P_N$ instead of $P$

$$P_N = \left[ \sum_{n=1}^{N} \left( \langle N_{ph,n} \rangle + \langle N_{e,n} \rangle \right)^2 \right]^{-1},$$

(6)

where $N_{ph,n} = a_n^\dagger a_n$ and $N_{e,n} = \sum_{m=1}^{M_n} \sigma_{n,m}^+ \sigma_{n,m}^-$ are the usual number operators for each state $v_p$, representing cavity excitation and the sum of all emitter excitation in a cavity. Like the classic participation ratio, $P_N$ is at a minimum (maximum) when the wavefunction is localized (delocalized). Next, we define the polaritonic participation ratio, or polartonicity,

$$P_P = \left[ \left( \sum_{n=1}^{N} \langle N_{ph,n} \rangle \right)^2 + \left( \sum_{n=1}^{N} \langle N_{e,n} \rangle \right)^2 \right]^{-1},$$

(7)

which is minimized when the wavefunction has completely cavity-like or completely emitter-like character and is maximized for an equal superposition of cavity- and emitter-like components. Note: a wavefunction can be polaritonic even when the cavity and emitter excitations do not belong to the same node.

These two new metrics allow us to seamlessly characterize multi-emitter CCAs. We normalize the metrics to $[0,1]$ range for easy comparison between the states of models. We apply these metrics to small systems to verify them against the exact solutions and then employ in larger simulations. Our analyses span three ranges of interaction quantified by the ratio $J/g = 0.1, 1, 10$ of the photon hopping to the cavity QED interaction strength.

### D. Polariton creation in small TCHM systems

In the small, exactly modeled, systems we test parameters that take a resonant Tavis-Cummings-Hubbard model from two isolated Tavis-Cummings models, to a system with delocalized polaritons, and all the way to a photonic molecule decoupled from emitters. The resonant case, which successfully benchmarks the metrics obtained in the Effective Hamiltonian approach against the exact spectral intensities obtained via QME, is detailed in the Appendix. We then expand this model system to include spectral disorder of emitter ensembles reflective of color center nanophotonic systems. The
FIG. 3. The node occupancy (top) and the participation ratios (bottom) for the lowest energy eigenstate of a system with $N = 2$ cavities and $M = 2$ emitters per cavity with random $\omega_e$ sampled from a Gaussian distribution of width $\Delta$ (single random realization for each $\Delta$). The shades of red represent wavefunction components of different emitters in the same node (cavity with emitters), while the blue represents the cavity component. Parameters and emitter frequencies are the same as those in Figure 2.

disorder is implemented by sampling emitter angular frequency, $\omega_e$, from a Gaussian distribution $P(\omega_e) = \frac{1}{\sqrt{2\pi}\Delta} \exp\left\{-\frac{(\omega_e-\omega_c)^2}{2\Delta^2}\right\}$ centered at $\omega_c$ with a width $\Delta$. Upon including spectral disorder, a number of interesting wavefunction characteristics become apparent. As shown in a set of representative plots in Figure 2, the subradiant states appear in the transmission spectrum and nodal localization emerges with an increasing disorder. A more detailed analysis of the exact solution plots for a range of parameters is available in the supporting section V.

All-photonic quantum simulators will require more than two nodes to produce noteworthy results, therefore, the role of the QME method will likely be limited to the benchmarking of approximate approaches in small Hilbert spaces. To that end, we directly compare the QME solutions in Figure 2 to the Effective Hamiltonian solutions in Fig. 3. For a vanishing disorder ($\Delta = \epsilon$, the leftmost column), the probabilities of excitation of Cavity 1 or 2 are $\frac{1}{4}$ each, and the probabilities of excitation of any of the four emitters are $\frac{1}{4}$ each, as shown in the top row. The wave function is maximally spread out both amongst the nodes (the cavity and its associated emitters) and also maximally spread between cavities and emitters. The former fact implies $P_N$ should be maximized, and the latter that $P_P$ should be maximized, as is shown in the bottom left panel. [52]

We observe the same trends in $P_N$ and $P_P$ as we do in Figure 2 namely, the increase in nodal localization and polaritonicity for increasing disorder. A detailed comparison of the transmission spectra solved using the QME to the eigenstates of the Effective Hamiltonian can be found in Figure 5 and Section V. In particular, it is shown that increasing $J$ leads the eigenvector of lowest eigenenergy to become predominantly localized in the cavities. Thus the polaritonic participation $P_P$ falls dramatically, unlike its relative insensitivity to $\Delta$ in Fig. 3.

E. Polaritonicity and localization in medium size TCHM systems

We first investigate the effects of spectral disorder on medium size systems with an open array of $N = 5$ cavities with $M = 3$ emitters per cavity on the localization and polaritonicity of the eigenstates of Eq. 5. Figure 4 explores the regime where cavity QED dominates the

FIG. 5. The nodal and the polaritonic participation ratios for the lowest energy eigenstate of a CCA with $N = 5$ and $M = 3$ for increasing $J/g$. A small amount of disorder causes the state to node localize for small $J/g$. Mean of 100 random realizations; error bars are one standard deviation.

$g$, $\kappa$, $\gamma$ are same as those in Figure 2.
FIG. 6. Changes in the energy and participation metrics as the spectral disorder, $\Delta$, is increased for the most polaritonic state (MPS) of the lower polariton band. Mean of 100 random realizations; error bars are one standard deviation. Participation metrics are normalized to $[0, 1]$. $N = 5$, $M = 3$; $g$, $\kappa$, $\gamma$ same as those in Figure 2.

photon hopping $J/g = 0.1$. For vanishing spectral disorder, the eigenspectrum has the shape resembling the features of Figure 1: two highly polaritonic delocalized CCA bands with $N = 5$ states and $N(M-1) = 10$ highly localized subradiant states, suitably characterized by the polaritonic and nodal participation ratio values. For moderate disorder $\Delta = \frac{g}{\sqrt{M}}$, the polaritonic properties of eigenstates are maintained, while the nodal delocalization somewhat decreases for polaritonic band states. The degeneracy of the subradiant states is lifted and the spectral gaps diminish with strong disorder $\Delta = g\sqrt{M}$, which is usually considered a cutoff for cavity protection. Most states become highly localized and the subradiant states gain a cavity component, as quantified by the increase of the $P_p$ value.

Regimes with an increasing rate of photon hopping to cavity QED interaction are detailed in the Appendix. While similar trends can be observed, the main difference is seen in the reduction of the number of polaritonic states in the CCA bands as the wavefunction obtains a higher cavity-like character.

This brings us to look into the formation of a polaritonic state in spectrally disordered CCA QED as a function of an increasing $J/g$ ratio. Figure 5 shows the polaritonicity and localization of the lowest energy eigenstate for an increasing $J/g$ ratio. When the photonic nature of the interaction increases, so does the cavity-like character of the wavefunction, reducing its level of polaritonicity. While this holds true for low and moderate values of disorder, in the case of high disorder, we observe an increase of $P_P$, before the decline. This is an artifact of the disorder which randomly modifies the nature of the lowest eigenstate in the system, until the interaction value increases to a level that offsets the issue. This trend is paired with the increase in the delocalization metric $P_N$ as the wavefunction loses the dominant emitter-like characteristic.

However, not all states lose polaritonicity with an increase in the photon hopping rate. Even in the case of large $J/g$ ratio (see Figures A5 and A6 for results for larger $J/g$ values), there still exist highly polaritonic states located in the middle of the lower polariton band (and an analogous one in the higher polariton band). We refer to these as the most polaritonic states (MPS). As seen in Eqs. A4 and A5 of the Appendix, where we discuss the PAM eigenvectors, the MPS is parameterized by the momentum value $k = \frac{\pi}{2}$ where the cavity and emitter bands cross for $\omega_c = \omega_e$ and hence are maximally mixed. For systems with an odd number of cavities, this state occupies every other node characterized by $P_N \approx 0.5$. Figure 6 shows how this state changes with increasing disorder. A decrease in polaritonicity and delocalization of the wavefunction take place for a range of system parameters. We note that for the case of $J/g = 1$ the wavefunction has a delayed delocalization trend. At low $J/g$ there is less variance in $P_P$ for a larger $\Delta$ compared to a higher photon hopping rate, suggesting that, as in the Tavis-Cumming model [24], the stronger cavity-emitter coupling compared to combined cavity losses provides better cavity protection against the disorder in the TCHM.

FIG. 7. Participation metrics for the most polaritonic state (MPS) of the lower polariton band for an increasing $N$. There are two distinct bands for each metric due to the parity of $N$ which converge at its large values. Mean of 100 random realizations; error bars are one standard deviation. $M = 3$; $g$, $\kappa$, $\gamma$ same as those in Figure 2.
FIG. 8. Nodal and polaritonic participation ratios as a function of number of emitters $M$ for the MPS. Increasing the number of emitters increases the cavity protection against the spectral disorder of emitter ensemble without causing node localization. Cavity protection is achieved when $P_P \geq 8$ for the MPS, which occurs when there are at least $M$ emitters per cavity for spectral disorder of $\Delta = g\sqrt{M}$ for $J/g \leq 1$. Mean of 100 random realizations; error bars are one standard deviation. $N = 5$, $M_{\text{min}} = 3$; $g$, $\kappa$, $\gamma$ same as those in Figure 2.

F. Cavity protection in large TCHM systems

Next, we look at how the the system size $(N, M)$ under open boundary condition affects localization and cavity protection by studying $P_N$ and $P_P$ trends for the MPS of the lower polariton band. To quantify the cavity protection effect, we here define the cavity protection regime as $P_P \geq 8$ for the MPS and normalized $P_P$.

We observe that the parity of the number of cavities plays a role in the properties of the MPS. For an increasing $N$, shown in Figure 7, two distinct bands emerge for each metric, but converge for large $N$. This occurs because only odd $N$ has an eigenstate at exactly $k = \pi/2$. We also observe that for nonvanishing disorder, longer CCAs localize the MPS, especially for the low $J/g$ ratio. Another way to describe this dependence on CCA size is that there is a finite localization length $\xi$, and with increasing $N \geq \xi$, it becomes more evident that the MPS occupies only a fraction of the whole array.

The number of emitters per cavity appears to have a strong influence on the formation of polaritons, as in the regular Tavis-Cummings model. Increasing the number of emitters $M$, as shown in Figure 8, effectively increases the cavity-emitter coupling $g \rightarrow g\sqrt{M}$, thus improving the polaritonic nature of the eigenstate and delocalization against disorder. For constant $\Delta$, we increase $M$ to estimate the number of emitters required to reach the cavity protection regime. Consistent with the prior studies of cavity protection, we find that at low hopping rates, $J/g \leq 1$, to reach $P_P \geq 0.8$ for $\Delta = g\sqrt{M}$ we need approximately $M$ emitters per cavity.

The Effective Hamiltonian approach and the two participation ratio metrics equip us to explore larger CCA QED systems. Studying the lower polariton band of larger systems with $N = 65$ cavities and $M = 3$ emitters per cavity, as shown in Figure 9, it is easy to visualize the effects of disorder on localization and polariton formation. We see that disorder of the emitter frequencies causes an Anderson-like localization in the nodes reflected in the reduction of the $P_N$ metrics. This localization occurs first in the emitter-like eigenstates, the states with $k > \pi/2$, and then the cavity-like states become more localized with an increasing disorder $\Delta$. Additionally, among the emitter-like eigenstates, the variance in $P_P$ increases with a growing $\Delta$. For small $J/g$ ratios, the states at $k \leq \pi/2$ become dominantly emitter-like with an increasing disorder and lose their polaritonicity. The $P_P$ values become more defined with an increasing $J/g$ ratio.

IV. DISCUSSION

In this work we introduce new metrics, inspired by condensed matter physics studies of localization and band mixing, to characterize the CCA QED eigenstates described by the Tavis-Cummings-Hubbard model. Our
goal is to provide a guiding tool for experimental implementations through the engineering of the CCA parameters.

We confirm that highly polaritonic states can be formed in coupled cavity arrays despite the presence of spectral disorder in emitter ensembles and quantify the cavity protection effect. While the systems with a dominant cavity QED interaction, relative to the photon hopping rate, support creation of numerous polaritonic states, we find that other parts of the parameter space can also be utilized to study polaritonic physics.

We suggested approximate analogies between the case of $M_n = 1$ emitter in each cavity with the periodic Anderson model where a single $f$ orbital on each site hybridizes with a conduction band, and the Kondo lattice model where the local degree of freedom is spin-$1/2$. Condensed matter systems which connect to the multi-emitter case $M_n > 1$ also have a long history, both in the investigation of multi-band materials and also as a theoretical tool providing an analytically tractable large-$N$ limit [53, 54]. Indeed, large-$N$ systems, realized for example by alkaline earth atoms in optical lattices, are also at the forefront of recent work in the atomic, molecular and optical physics community [55–57]. In short, the TCHM offers a context to explore intertwined local and itinerant quantum degrees of freedom which, while distinct from condensed matter models, might still offer insight into their behavior.

These TCHM systems can ostensibly be realized in a number of photonic frameworks, from atoms in mirrored cavities, to quantum dots in nanophotonics. It is difficult, however, to experimentally create atom-based systems that couple multiple cavities together and to create large numbers of quantum dots that emit within the relatively modest range of disorder that we have shown will recreate polariton dynamics. As such, the most likely experimental realization of our systems will be in color center based nanophotonics.

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V. APPENDIX: SUPPLEMENTARY INFORMATION

A. Condensed Matter analogies

The resonant cavity-emitter arrays ($\omega_{e,n} = \omega_c$, $\omega_{a,n,m} = \omega_a$, $g_{n,m} = g$, $J_{n,n+1} = J$), with periodic boundary conditions have a closed form solution for the eigenvalues and eigenvectors. Much of the derivation precisely parallels the calculation of the band structure of tight binding Hamiltonians. The case of no emitters ($M_n = 0$) corresponds to the $d = 1$ (Bose) Hubbard Model (HM), and the case with a single emitter in each cavity ($M_n = 1$) to the Periodic Anderson Model (PAM). Here we review those results with an emphasis on the condensed matter analogies.

In the case $M_n = 0$ one can diagonalize $H_{TCHM}$ by introducing momentum creation and destruction operators,

$$a_k^\dagger = \frac{1}{\sqrt{N}} \sum_{l=1}^{N} e^{ikl} a_1^\dagger \quad a_k = \frac{1}{\sqrt{N}} \sum_{l=1}^{N} e^{-ikl} a_l \quad (A1)$$

The transformation is canonical. $a_k^\dagger$ and $a_k$ obey the same bosonic commutation relations as the original real space operators. $H_{TCHM}$ is diagonal,

$$H_{TCHM} = \sum_k (\omega_c - 2J \cos k) a_k^\dagger a_k \quad (A2)$$

The eigenenergies are,

$$E(k) = \omega_c - 2J \cos k \quad (A3)$$

The momenta are discrete, $k = k_p = 2\pi p/N$ with $p = 1, 2, 3, \ldots N$. In the thermodynamic limit the $E(k)$ form a continuous band with a density of states that diverges at the band edges $E(k) = \omega_c \pm 2J$.

In this no-emitter limit, Eq. A2 actually provides the solution for any number of excitations. The many-excitation energies are just sums of the single particle $E(k)$ subject to the photon indistinguishability implied by the commutation relations. This solubility of the many excitation system is unique to the no-emitter limit, as discussed further below.

In the case $M_n = M = 1$ one can again solve for the eigenvalues of $H_{TCHM}$ by going to momentum space, but only in the single excitation sector. The reason is that the photon and emitter operators do not obey a consistent set of commutation relations. Thus, even though it might appear that $H_{TCHM}$ is soluble since it is quadratic in the operators, it is therefore not possible to do the same sort of canonical transformation to diagonalize. It is most straightforward to define a set of single excitation states which form a basis for the space, and then examine the matrix which arises from the application of $H_{TCHM}$ to each one. For simplicity, we focus on the resonant case where $\omega_c = \omega_c \equiv \omega_0$, but the formulae are straightforward to generalize. The energy eigenvalues are

$$E(k) = \omega_0 - J \cos k \pm \sqrt{J^2 \cos^2 k + g^2} \quad (A4)$$

The (unnormalized) eigenvectors are,

$$\Psi_{\pm}(k) = \left( \begin{array}{c} g \\ J \cos k \mp \sqrt{J^2 \cos^2 k + g^2} \end{array} \right) \quad (A5)$$

so that, while all states are polaritons in the sense of mixing photon and emitter components, the relative weights depend on momentum $k$ and band index $\pm$. 
FIG. A1. The two polariton bands of a cavity-emitter system for representative parameters $\omega_c = \omega_e = \omega_0 = 4$, cavity hopping $J = 1$, and cavity-emitter coupling $g = 0.2$. If the number of emitters $M > 1$ the gap between the two bands is enhanced from $\Delta = 2g$ to $\Delta = 2\sqrt{M}g$ and there are $M-1$ additional flat emitter bands at $E_\alpha(k) = \omega_0$ in the gap between the two polariton bands.

This $M_n = M = 1$ case easily generalizes to larger $M$. There are again two polariton bands, but with an enhanced photon-emitter hybridization $\sqrt{M}g$,

$$E(k) = \omega_0 - J \cos k \pm \sqrt{J^2 \cos^2 k + Mg^2}.$$  \hfill (A6)

with a similar $g^2 \rightarrow Mg^2$ change to the eigenvectors of Eq. A5. The remaining $M-1$ bands have purely emitter components, and are dispersionless, $E(k) = \omega_0$.

A useful approximate visualization of the eigenvalues and eigenvector weights is provided by drawing the bands as in Figure A1. For $g = 0$ the photons have $E(k) = \omega_0 - 2J \cos k$ and the emitters $E(k) = \omega_0$. When these two energy levels are hybridized by $g$ there is a level repulsion at their crossing point at $k = \pm \pi/2$ and an energy gap $\Delta = 2g$ opens. The relative photon-emitter compositions of the states can be inferred from the degree to which the polariton energy matches one of the initial ($g = 0$) photon or emitter bands. Polariton energies which are close to the original flat $E(k) = \omega_0$ emitter band are dominantly made up of emitter excitations, while those close to the original dispersing $\omega_0 - 2J \cos k$ photon band are dominantly cavity excitations.

B. Exact simulation results

We numerically solve the quantum master equation for a coupled 2 cavity system with each cavity also coupling to two emitters each. The results are shown in Figure A2. Along the top row, which shows the smallest inter-cavity coupling simulations, we see the most nodal localization regardless of emitter dispersion, as shown by only having peaks in Cavity 1 and Emitters 1 and 2 when pumping...
FIG. A3. Node occupancy of the lowest energy eigenstate of a system with $N = 2$ cavities and $M = 2$ emitters per cavity with random $\omega_e$ sampled from a Gaussian distribution of width $\Delta$. Single random realization for each $\Delta$. $g$, $\kappa$, $\gamma$, and emitter frequencies are the same as those in Figure A2. Cavity 1 and only having peaks in Cavity 2 and Emitters 3 and 4 when pumping Cavity 2. We can also see that the small inter-cavity coupling leads to highly polaritonic systems by comparing the peak heights of each cavity to those of its emitters and finding the ratios are close to unity (0.78 - 0.9). The largest inter-cavity coupling, $J/g = 10$ is heavily cavity-like as shown by cavity peak to emitter peak ratios of around 50. The middle row, $J/g = 1$ is partially polaritonic and partially photonic with peak ratios in the range 0.3-0.6. By introducing nonzero emitter detuning, we are able to access subradiant states for each of the three detuning values. These states are identified by the zero-width peaks found between the polariton peaks. Subradiant states are states that decay much slower than the polariton peaks and a number of proposals exist for their use in quantum information technologies including, light storage [58] and quantum light generation [26].

C. Benchmarking of the Effective Hamiltonian approximation and the participation ratio metrics

To benchmark the approximate Effective Hamiltonian approach against the exact Quantum Master Equation solution, we simulate the systems with same parameters in both models. The Figure A3 shows the lowest energy wavefunction calculated with the Effective Hamiltonian approach, corresponding to the excitations of the lowest energy peaks in the Figure A2. The case of the vanishing spectral disorder shows equal contribution of Cavities 1 and 2, which corresponds to the identical looking plots of the Cavity 1 and Cavity 2 excitations in the QME spectra when Cavity 1 and Cavity 2 are pumped, respectively. The asymmetry of the nodal occupations in Figure A3 for non-vanishing disorder (especially for $J/g = 0.1$) matches the non-identicality of the Cavity 1/2, as well as Emitter 1.1,1.2 and Emitter 2.1,2.2 spectra in Figure A2. Localization of the wavefunction for an increasing disorder and $J/g \leq 1$ follow the exact solution trends described in the previous section.

These parallels are closely described by the nodal $P_N$ and the polaritonic $P_P$ participation ratio shown in Figure A4. The value of $P_N$ (orange) is maximized for a fully node-delocalized wavefunction, and reduces as the wavefunction tends to increasingly excite one cavity and its emitters with an increasing disorder. The value of $P_P$ (green) is maximized for the wavefunctions that have equal excitation distribution between cavities and the emitters, and reduces with an increasing photonic interaction (high $J/g$ value) as the cavities become predominantly excited.

We conclude that the Effective Hamiltonian and the participation ratio metrics are suitable for studies of the Tavis-Cummings-Hubbard model.
FIG. A5. Effects of an increasing disorder, $\Delta$, on the energy eigenspectrum for 3 different $J/g$ values. Mean of 100 random realizations; error bars are one standard deviation. $N = 5$, $M = 3$; $g$, $\kappa$, $\gamma$ same as those in Figure A2.

D. Effects of spectral disorder on the TCHM energy spectrum

We utilize the Effective Hamiltonian approach to study medium and large TCHM systems and are especially concerned with the influence of the spectral disorder on the polaritonic and localization properties of the model wavefunctions. The Figure A6 shows the energy spectra of an $N = 5$, $M = 3$ system for various regimes of cavity QED to photon hopping ratios. For the vanishing disorder, we clearly see the three components of the spectrum illustrated in the Figure 1: two polaritonic bands with $N$ states and a subradiant band with $N(M - 1)$ degenerate states. When cavity QED interaction is significant, a band gap opens between the polariton bands and the subradiant states. With an increasing spectral disorder, the band gap closes and the subradiant state degeneracy is lifted.

Further analysis is provided by the corresponding $P_N$ and $P_P$ values shown in Figure A6. Here, we observe that the polaritonic properties of the highly hybridized states in the polariton bands and the emitter-like states in the subradiant band shift significantly only for the high levels of spectral disorder, defined by the typical cavity protection cutoff $\Delta = g\sqrt{M}$. The increasing localization trend (decreasing $P_N$) for most polariton band states with an increasing disorder is evident for all sets of parameters. In contrast, the subradiant states gain a cavity component with an increasing disorder and become more hybridized and delocalized.

While the polaritonicity of the lower (as well as the upper) polariton band reduces for most states with an increasing $J/g$ ratio, the middle state of the band remains highly polaritonic. This state, labeled the most polaritonic state (MPS) in our study, shows that even the systems with high hopping ratio can serve as testbeds for polaritonic physics explorations.
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