A LOG-MOTIVIC COHOMOLOGY FOR SEMISTABLE VARIETIES AND ITS $p$-ADIC DEFORMATION THEORY

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ABSTRACT. We construct log-motivic cohomology groups for semistable varieties and study the $p$-adic deformation theory of log-motivic cohomology classes. Our main result is the deformational part of a $p$-adic variational Hodge conjecture for varieties with semistable reduction: a rational log-motivic cohomology class in bidegree $(2n, n)$ lifts to a continuous pro-class if and only if its Hyodo-Kato class lies in the $n$-th step of the Hodge filtration. This generalises [BEK14, Theorem 8.5] which treats the good reduction case. In the case $n = 1$ the lifting criterion is the one obtained by Yamashita for the logarithmic Picard group [Yam11, Theorem 3.1]. Along the way, we relate log-motivic cohomology to logarithmic Milnor $K$-theory and the logarithmic Hyodo-Kato Hodge-Witt sheaves.

1. INTRODUCTION

In the present work we construct a variant $\mathbb{Z}_{\log}(n)$ of the motivic complexes of Suslin-Voevodsky [SV00a] suitable for semistable varieties. Our approach relies on a definition of finite correspondences due to Suslin-Voevodsky [SV00b, §3] which also includes singular varieties (see also [MVW06, Appendix 1A] and [CD19, §8 and §9]). Then the complexes $\mathbb{Z}_{\log}(n)$ are defined analogously as simplicial sheaves associated to a certain sheaf with transfers and coincides with the usual motivic complexes on the smooth locus.

In the case $n = 1$, in order to get a geometric interpretation as $\mathbb{Z}_{\log}(1)$, we will modify the given logarithmic structure $M$ on the semistable variety and define a log-structure $N$ which is trivial on the smooth locus of the variety. By considering the image $N^{\text{gp}}$ of $N^{\text{gp}}$ under the structure morphism, we can then relate its first cohomology to the diagonal log-motivic cohomology. Then we define logarithmic Milnor $K$-groups by applying the Milnor functor to the group $N^{\text{gp}}$, and prove that the corresponding sheaf is the cohomology sheaf $H^n(\mathbb{Z}_{\log}(n))$, in analogy to the smooth case which was proved by Kerz [Ker09]. We also relate the modulo $p^n$ residue of the log-Milnor $K$-group to modified logarithmic Hyodo-Kato Hodge-Witt sheaves, making precise an old result of Hyodo [Hyo88].

Let $k$ be a perfect field of characteristic $p > 0$, and let $K = \text{Frac} W(k)$. Let $X$ be a $W(k)$-scheme with semistable reduction, with special fibre $Y$ and generic fibre $X_K$. For each $m \in \mathbb{N}$, let $X_m$ be the reduction of $X$ modulo $p^m$, so $X_1 = Y$. Our main motivation comes from the problem of constructing $K$-cohomology classes (or cycles) on $X_K$. One strategy is to attempt to lift classes from the special fibre (this strategy is especially appealing if the reduction $Y$ is highly degenerate and thus has an abundance of easily accessible cycles). In the second half of the paper we

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state and prove an analogue of the $p$-adic variational Hodge conjecture [BEK14] for semistable varieties, which yields a lifting criterion for motivic cohomology classes in terms of their logarithmic Chern class in Hyodo-Kato cohomology. For $n = 1$, the lifting criterion coincides with the lifting criterion of the logarithmic Picard group considered by Yamashita [Yam11]. It general it uses a semistable version of the pro-complexes $\mathbb{Z}_X(n)$ of Bloch-Esnault-Kerz by gluing the complexes $\mathbb{Z}_{\log,Y}(n)$ and the log-syntomic complex of Kato-Tsuji along the modified logarithmic Hyodo-Kato de Rham-Witt sheaf. We will also use a construction of the log-syntomic complex by Nekovář-Nizioł [NN16]. The pro-complexes $\mathbb{Z}_{\log,X}(n)$ enjoy some of the nice properties of the pro-complexes $\mathbb{Z}_X(n)$ in the smooth case stated in [BEK14] §7. For example, the top cohomology sheaf is the log-Milnor $K$-group as pro-sheaf $K_\log^\text{Mil}(X,n)$ and, at least rationally, it is an extension of the log-motivic complex $\mathbb{Z}_{\log,Y}(n)$ by a truncated de Rham complex. In contrast to the smooth case ([BEK14 Proposition 7.3]) we do not expect this property to hold integrally since there is no integral truncated de Rham complex. In the good reduction case, the main theorem [BEK14, Theorem 1.3] is concerned with deforming classes of vector bundles and its proof has two parts. The first is [BEK14] Theorem 8.5 which concerns lifting algebraic cycle classes to the continuous Chow group – our Theorem 1.1 is a generalisation of this to the semistable reduction case. The second part is the Chern character comparison isomorphism [BEK14 Theorem 11.1] between continuous $K$-theory and continuous Chow groups; here a restriction on the dimension of the special fibre is needed. We do not give a semistable analogue of this second theorem here, but we consider it to be an interesting problem to investigate the relationship between our logarithmic Chow groups and a logarithmic incarnation of $K$-theory for log-smooth schemes.

(ii) We are aware that the assumptions of unramifiedness and on the dimension in [BEK14 Theorem 1.3] have been removed in the recent work [AMMN20 Theorem D], which uses topological cyclic homology as a new suitable tool in $p$-adic deformation theory. They also prove in [AMMN20 Theorem E] a more
general result on lifting classes in higher $K$-theory to continuous $K$-theory by using $p$-adic derived de Rham cohomology. The subject of this note is different in the sense that we lift log-motivic cohomology classes by considering their Chern classes in log-crystalline cohomology.

Finally, let us point out that motives and motivic complexes have been constructed for singular varieties in a series of papers, notably by Kahn-Miyazaki-Saito-Yamazaki [KMSY21a], [KMSY21b], [KMSY21c] and Binda-Park-Østvær [BPØ20]. In the “Motives with modulus” series, Kahn-Miyazaki-Saito-Yamazaki construct a triangulated tensor category of motives with modulus $\text{MDM}^{\text{eff}}_{\text{gm}}$ in the same way as Voevodsky constructed his category $\text{DM}^{\text{eff}}_{\text{gm}}$ in [Voe00], starting from the category $\text{Cor}_k$ of smooth varieties with finite correspondences as morphisms. A motive with modulus is a pair $(M, M_\infty)$ where $M$ is a $k$-variety and $M_\infty$ is an effective Cartier divisor on $M$ such that $M - M_\infty$ is smooth. The category $\text{Cor}_k$ is replaced by $\text{MCor}_k$ of finite correspondences between $M - M_\infty$ and $N - N_\infty$ (for two modulus pairs $(M, M_\infty)$, $(N, N_\infty)$) that satisfy a certain condition on the Cartier divisors. One of their main results is a characterisation of Bloch’s higher Chow groups and Voevodsky’s motivic cohomology in terms of a derived internal Hom between two motives with modulus in $\text{MDM}^{\text{eff}}_{\text{gm}}$. A crucial difference to the construction of Voevodsky is that $A^1$-invariance is replaced by $□$-invariance, where $□ = (\mathbb{P}^1, \infty)$ is the motive with modulus where $\infty$ is the reduced divisor on $\mathbb{P}^1$. The theory of Kahn-Miyazaki-Saito-Yamazaki is then extended and translated into the language of logarithmic geometry by Binda-Park-Østvær. In fact, they construct a triangulated tensor category $\text{logDM}^{\text{eff}}_{\text{gm}}(k)$ of effective log-motives starting from the category $\text{lSM}/k$ of fine and saturated (fs) log-schemes that are log-smooth over $\text{Spec} k$ equipped with the trivial log-structure, and where the category $\text{MCor}_k$ is replaced by the category $\text{lCor}/k$ of finite log-correspondences. Any fs log-scheme $X \in \text{lSM}/k$ gives rise to a log-motive $M(X) \in \text{logDM}^{\text{eff}}_{\text{gm}}(k)$. Their construction generalises Voevodsky’s category of effective motives. For example, if $X$ and $Y$ are fs log-schemes in $\text{lSM}/k$ such that $X - \partial X$ and $Y - \partial Y$ are smooth subschemes, where the log-structure is trivial, then

$$\text{Hom}_{\text{logDM}^{\text{eff}}_{\text{gm}}(k)}(M(Y)[i], M(X)) \cong \text{Hom}_{\text{MDM}^{\text{eff}}_{\text{gm}}}(M(Y - \partial Y)[i], M(X - \partial X)).$$

In both works, the main example is the motive associated to a toroidal embedding $j : U \rightarrow X$ of a smooth $k$-variety into a normal variety $X$, with $M$ the log-structure defined by $\mathcal{O}_X \cap j_* \mathcal{O}_U$. In the present paper, we consider the category $\text{SemiStab}_k$ of semistable varieties. These are normal crossing divisors inside $\text{W}^\text{log}(k)$-schemes that are log-syntomic, but not log-smooth, over $\text{Spec} W(k)$ equipped with the trivial log-structure. In analogy to [BPØ20], we define a category $\text{SemiStabCor}_k$ with objects the semistable varieties and morphisms finite log-correspondences. This leads to the notion of sheaves with transfer and allows us to define the log-motivic complexes $Z_{\text{log}}(r)$ in an ad-hoc fashion using the simplicial approach [SV00a]. We hope to construct, in a future project, a derived category $\mathcal{C}$ of effective log-motives such that a semistable variety $X$ gives rise to a log-motive $M(X)$ in $\mathcal{C}$, complimentary to the works of Binda-Park-Østvær and Kahn-Miyazaki-Saito-Yamazaki.

1.3. Conventions. All schemes are assumed to be separated and of finite type over the base.
\section{Log-motivic cohomology}

\subsection{Finite log-correspondences}

For a morphism of fine log-schemes $f : (X, M_X) \to (Y, M_Y)$, the trivial locus of $f$ is the locus of points $x \in X$ such that $(f^* M_Y)_{\text{triv}} \to M_X$. We shall abusively write the trivial locus of $f$ simply as $X^{\text{triv}}$ without reference to $f$, since the morphism will always be clear from the context (it will be the structure morphism). Note that $X^{\text{triv}} \subset X$ is open by \cite[Proposition 2.3.1]{Shi00}.

Recall that if $(X, M_X)$ and $(Y, M_Y)$ are morphisms of fs (fine and saturated) log-schemes, then the log-structure on the fibre product $(X, M_X) \times_{(B, M_B)} (Y, M_Y)$ taken in the category of log-schemes is coherent but not necessarily fs. Instead, we may take the fibre product in the category of fs log-schemes, which we denote by $(X, M_X) \times_{(B, M_B)}^f (Y, M_Y)$. Note that the underlying scheme of $(X, M_X) \times_{(B, M_B)} (Y, M_Y)$ is $X \times_B Y$, but this is not case for $(X, M_X) \times_{(B, M_B)}^f (Y, M_Y)$ in general. There is however a natural morphism

$$(X, M_X) \times_{(B, M_B)}^f (Y, M_Y) \to (X, M_X) \times_{(B, M_B)} (Y, M_Y)$$

which is a finite morphism on the underlying schemes \cite[Remark 12.2.36(i)]{GRT1}, and is an isomorphism over the trivial locus $(X \times_B Y)^{\text{triv}}$.

Let $(\text{Spec } k, L)$ be the standard log-point, i.e. $L$ is the log-structure on $\text{Spec } k$ associated to $\mathbb{N} \to k, 1 \mapsto 0$. We shall abusively write $\text{Spec } k$ denote to denote the log-scheme whose underlying scheme is $\text{Spec } k$, and the log-structure is the trivial log-structure.

\begin{definition}
An fs log-scheme $(X, M_X)$ over $(\text{Spec } k, L)$ is called a semistable variety if étale locally on $X$ the structure morphism $(X, M_X) \to (\text{Spec } k, L)$ factors as

$$(X, M_X) \xrightarrow{u} (\text{Spec } k[T_1, \ldots, T_n]/(T_1 \cdots T_b), P) \xrightarrow{\delta} (\text{Spec } k, L)$$

for some $a \geq b$, where $P$ is the log-structure associated to $\mathbb{N}^b \to k[T_1, \ldots, T_n]/(T_1 \cdots T_b)$, $e_i \mapsto T_i$, where $u$ is strict and étale, and $\delta$ is the morphism induced by the diagonal.

\end{definition}

\begin{definition}
For a semistable variety $(X, M_X)$ we will use the log-structure $M_X$ to define an alternative log-structure $N_X$ on $X$ which will be very important in this paper. Let $U = \text{Spec } A \subset X$ be an affine and let the structure morphism $\alpha : M_X \to \mathcal{O}_X$ be locally defined on $U$ by the homomorphism of monoids $\mathbb{N}^r \to \mathcal{O}(U) = A$, $e_i \mapsto \pi_i$. We define a new log-structure $N_X$ locally on $U$ by the homomorphism

$$\beta : \mathbb{N}^r \to A, \quad e_i \mapsto g_i := \pi_i + \prod_{j \neq i}^r \pi_j.$$ 

It is easy to see that $g_i \in \mathcal{O}(U) \cap j_* \mathcal{O}(U^\text{sm})^*$ where $j : U^\text{sm} \hookrightarrow U$ is the open immersion of the smooth part. Then, evidently, $X^{\text{triv}} = X^\text{sm}$, and we get a homomorphism of sheaves of monoids

$$\beta : N_X \to O_X \cap j_* O_{X^\text{sm}}^*$$

where $O_X \cap j_* O_{X^\text{sm}}^*$ is considered as a sheaf of monoids with respect to multiplication. Let

$$\beta^{\text{gp}} : N_X^{\text{gp}} \to (O_X \cap j_* O_{X^\text{sm}}^*)^{\text{gp}}$$
be the associated homomorphism of sheaves of abelian groups and let $N^{gp}_{X} = \text{Im}(N_{X}) \subseteq (O_{X} \cap j_{!}O_{X,\text{sm}})^{gp}$. Write SemiStab$_k$ for the category of semistable varieties equipped with the log-structure $N_{X}$. We will consider $(X, N_{X})$ as a log-scheme over Spec $k$ (equipped with the trivial log-structure).

We wish to enlarge SemiStab$_k$ into an additive category SemiStabCor$_k$ by including the notion of finite log-correspondence, analogously to the classical smooth setting of Suslin-Voevodsky [SV00a §1], [MVW06 Lecture 1].

**Definition 2.4.** Let $(X, N_{X})$ be an object of SemiStab$_k$ and let $(Y, M_{Y})$ be any fs log-scheme over Spec $k$. A finite log-correspondence from $(X, N_{X})$ to $(Y, M_{Y})$ is a finite correspondence $Z \in \text{Cor}(X, Y)$ (see [SV00b §3], [MVW06 Appendix 1A] and [CD19 §8 and §9] for finite correspondences between possibly singular schemes), such that the restriction $Z_{X/Y}$ of $Z$ to $X^{triv} \times_{k} Y$ has support in $X^{triv} \times_{k} Y^{triv}$. The group of finite log-correspondences from $(X, N_{X})$ to $(Y, M_{Y})$ is denoted by Cor$((X, N_{X}), (Y, M_{Y}))^{*}$, or simply Cor$(X, Y)^{*}$ when the log-structures are clear from the context.

For example, let $f : (X, N_{X}) \to (Y, M_{Y})$ be a morphism of fs log-schemes over Spec $k$ where $(X, N_{X})$ is an object of SemiStab$_k$. Let $\Gamma_{f}$ be the graph of the underlying morphism $f : X \to Y$. Then $\Gamma_{f} \subseteq X \times_{k} Y$ is closed because $Y$ is separated over Spec $k$. Moreover, the projection $\text{pr}_{X} : \Gamma_{f} \to X$ is an isomorphism, so $\Gamma_{f}$ is a universally integral relative cycle by [MVW06 Theorem 1A.6 & Theorem 1A.10], and hence $\Gamma_{f} \in \text{Cor}(X, Y)$. By [Ogu18 III. Proposition 1.2.8], we have $f(X^{triv}) \subseteq Y^{triv}$. Hence $\Gamma_{f}$ restricted to $X^{triv} \times_{k} Y$ has support in $X^{triv} \times_{k} Y^{triv}$.

Let $(X, N_{X}), (Y, N_{Y}), (Z, N_{Z})$ be objects of SemiStab$_k$, and let $V \in \text{Cor}(X, Y)^{*}$, $W \in \text{Cor}(Y, Z)^{*}$. Let $W \circ V \in \text{Cor}(X, Z)$ be the composition of $V$ and $W$ as defined in [MVW06 Definition 1A.11], so $W \circ V$ is the pushforward of $W_{V}$ along the projection $X \times_{k} Y \times_{k} Z \to X \times_{k} Z$, where $W_{V}$ is the relative cycle given by pulling back $W$ along the map $V \to Y$ [MVW06 Theorem 1A.8]. Since the restriction of $W_{V}$ to $X^{triv} \times_{k} Y^{triv} \times_{k} Z$ is the relative cycle $(W_{X^{triv}})_{Y^{triv}}$, we have that $(W \circ V)_{Y^{triv}} = W_{X^{triv}} \circ V_{Y^{triv}}$, and $W_{X^{triv}} \circ V_{Y^{triv}} \in \text{Cor}(X^{triv}, Y^{triv})$ because $W$ and $V$ are finite log-correspondences. The composition of finite correspondences therefore gives a well-defined composition

$$\text{Cor}(X, Y)^{*} \times \text{Cor}(Y, Z)^{*} \to \text{Cor}(X, Z)^{*}$$

$$(V, W) \mapsto W \circ V$$

for finite log-correspondences. If $f : (X, N_{X}) \to (Y, N_{Y})$ is a morphism, then $W \circ \Gamma_{f}$ is the relative cycle $W_{X}$. In particular, $id_{X} := \Gamma_{id} \in \text{Cor}(X, X)^{*}$ is the identity with respect to composition.

**Definition 2.5.** Let SemiStabCor$_k$ be the category whose objects are the same as those of SemiStab$_k$ and whose morphisms from $(X, N_{X})$ to $(Y, N_{Y})$ are the elements of Cor$(X, Y)^{*}$.

Then SemiStabCor$_k$ is an additive category and there is a faithful functor SemiStab$_k \to$ SemiStabCor$_k$ given by

$$(X, N_{X}) \mapsto (X, N_{X}), \quad (f : (X, N_{X}) \to (Y, N_{Y})) \mapsto \Gamma_{f}.$$  

**Definition 2.6.** A presheaf with transfers is a contravariant additive functor $F :$ SemiStabCor$_k \to Ab$ to the category of abelian groups.
An important source of presheaves with transfers is as follows. If \((Y,M_Y)\) is an fs log-scheme over \(\text{Spec} \, k\), the presheaf
\[
Z_{tr}(Y)^* : \text{SemiStab}_k \to \text{Ab}
\]
\[
(X,N_X) \mapsto \text{Cor}(X,Y)^*
\]
is a presheaf with transfers by virtue of the composition product of finite log-correspondences.

We shall say that a presheaf of abelian groups \(F : \text{SemiStab}_k \to \text{Ab}\) is a Zariski sheaf if the restriction of \(F\) to each \((X,N_X)\) in \(\text{SemiStab}_k\) is a Zariski sheaf on \(X\). That is, if \(i_1 : (U_1,N_{U_1}) \hookrightarrow (X,N_X)\) and \(i_2 : (U_2,N_{U_2}) \hookrightarrow (X,N_X)\) are open immersions such that \(X = U_1 \cup U_2\), then the sequence
\[
0 \to F(X,N_X) \xrightarrow{\text{diag}} F((U_1,N_{U_1}) \amalg (U_2,N_{U_2})) \xrightarrow{(i_1,-)} F \left( (U_1,N_{U_1}) \times^F_{(X,N_X)} (U_2,N_{U_2}) \right)
\]
is exact. Notice that the underlying scheme of \((U_1,N_{U_1}) \times^F_{(X,N_X)} (U_2,N_{U_2})\) is \(U_1 \cap U_2\) because \(i_1\) and \(i_2\) are strict.

**Lemma 2.7.** Let \((Y,M_Y)\) be an fs log-scheme over \(\text{Spec} \, k\). Then \(Z_{tr}(Y)^*\) is a Zariski sheaf. In particular, \(C_* (Z_{tr}(Y)^*)\) is a chain complex of Zariski sheaves, where \(C_* (-)\) is the simplicial construction given in [SV00a §0] and [MVW06 §2].

**Proof.** Let \((X,N_X)\) be an object of \(\text{SemiStab}_k\) and let \((U_1,N_{U_1}) \hookrightarrow (X,N_X)\), \((U_2,N_{U_2}) \hookrightarrow (X,N_X)\) be open immersions such that \(X = U_1 \cup U_2\). The map \(Z_{tr}(Y)^*(X,N_X) \xrightarrow{\text{diag}} Z_{tr}(Y)^*((U_1,N_{U_1}) \amalg (U_2,N_{U_2}))\) is the pullback of cycles along the surjective morphism \((U_1 \amalg U_2) \times Y \to X \times Y\), and is therefore injective. To see that \(Z_{tr}(Y)^*\) is a Zariski sheaf, it remains to show that if \(Z_1\) and \(Z_2\) are finite log-correspondences from \((U_1,N_{U_1})\) (resp. \((U_2,N_{U_2})\)) to \((Y,M_Y)\) that coincide on \((U_1 \cap U_2) \times Y\), then there is a finite log-correspondence \(Z\) from \((X,N_X)\) to \((Y,M_Y)\) whose restriction to \(U_i \times Y\) is \(Z_i\) for each \(i = 1, 2\). By definition, \(Z_1 = \sum_{i=1}^s \lambda_j Z_{1,j}\) and \(Z_2 = \sum_{i=1}^t \mu_j Z_{2,j}\) are finite linear combinations, where the \(Z_{1,j}\) (resp. \(Z_{2,j}\)) are universally integral relative cycles of \(U_1 \times Y\) (resp. \(U_2 \times Y\)) which are finite and surjective over \(U_1\) (resp. over \(U_2\)). For each \(i = 1, 2\), let \(\iota_i : (U_1 \cap U_2) \times Y \hookrightarrow U_i \times Y\) be the obvious open immersion. Then, by assumption, we have
\[
\sum_{j=1}^s \lambda_j \iota_1^{-1}(Z_{1,j}) = \sum_{j=1}^t \mu_j \iota_2^{-1}(Z_{2,j}).
\]
We see then that \(s = t\). Re-labelling, we may assume that \(\lambda_j = \mu_j\) and \(\iota_1^{-1}(Z_{1,j}) = \iota_2^{-1}(Z_{2,j})\) for all \(j = 1, \ldots, s\). But then the cycle
\[
Z = \sum_{j=1}^s \lambda_j (Z_{1,j} \cup Z_{2,j})
\]
is a finite correspondence from \(X\) to \(Y\) whose restriction to \(U_i \times Y\) is \(Z_i\) for each \(i = 1, 2\). Moreover, it is clear that \(Z\) is a finite log-correspondence. This proves that \(Z_{tr}(Y)^*\) is a Zariski sheaf.

Now let \(\Delta^\bullet\) be the cosimplicial \(k\)-scheme given by
\[
\Delta^i = \text{Spec} \, k[X_0, \ldots, X_i]/(X_0 + \cdots + X_i - 1)
\]
with the \(j\)-th face map \(\partial^j : \Delta^i \to \Delta^{i+1}\) given by setting \(X_0 = 0\). We consider \(\Delta^\bullet\) as a cosimplicial fs log-scheme over \(\text{Spec} \, k\) by endowing each \(\Delta^i\) with trivial
Definition 2.10. Let \((X, N_X)\) be an fs log-scheme over Spec \(k\). Since each \(\Delta^i\) is (classically) smooth and \(\mathbb{Z}_{tr}(Y)^*\) is a Zariski sheaf, the presheves

\[ C_i(\mathbb{Z}_{tr}(Y)^*) : (X, N_X) \mapsto \mathbb{Z}_{tr}(Y)^*((X \times \Delta^i, \text{pr}_X^*N_X)) \]

are also Zariski sheaves for each \(i\), and thus \(C_*(\mathbb{Z}_{tr}(Y)^*)\) is a complex of Zariski sheaves.

\[ \square \]

2.8. Log-motivic cohomology.

For \(n \geq 1\), let \((\mathbb{A}_k^n, D_n)\) be the log-scheme over Spec \(k\) whose underlying scheme is \(\mathbb{A}_k^n\), and whose log-structure is the log-structure associated to the divisor

\[ D_n = \{0\} \times \mathbb{A}_k^{n-1} + \mathbb{A}_k^1 \times \{0\} \times \mathbb{A}_k^{n-2} + \cdots + \mathbb{A}_k^{n-1} \times \{0\}. \]

Following [SV09a, §3], define \(\mathbb{Z}_{tr}(\mathbb{A}_k^n)^*\) to be the presheaf with transfers \(\mathbb{Z}_{tr}(\mathbb{A}_k^n, D_n)^*/\mathcal{E}_n\) where \(\mathcal{E}_n\) is the sum of the images of the maps \(\mathbb{Z}_{tr}(\mathbb{A}_k^{n-1}, D_{n-1})^* \to \mathbb{Z}_{tr}(\mathbb{A}_k^n, D_n)^*\) induced by the embeddings \(\mathbb{A}_k^{n-1} \to \mathbb{A}_k^n\) given by \((x_1, \ldots, x_{n-1}) \mapsto (x_1, 1, \ldots, x_{n-1})\).

In the definition of the log-motivic complex \(\mathbb{Z}_{\log}(n)\) we need to work with a more restrictive class of finite log-correspondences in \(\text{Cor}(X, \mathbb{A}_k^n)^*\). Let \(Z \in \text{Cor}(X, \mathbb{A}_k^n)^\ast \subset \text{Cor}(X, \mathbb{P}^n)\). Consider \(Z_i := \text{pr}_i(Z)\), where \(\text{pr}_i : \mathbb{P}^n_k \to \mathbb{P}^1_k\) is the \(i\)-th projection, which is finite over \(Z\) and \(Z_i \in \text{Cor}(X, \mathbb{A}_k^1)^* \subset \text{Cor}(X, \mathbb{P}^1_k)\). Then \(Z_i\) defines an element in \(\text{Pic}(X \times \mathbb{P}^1_k) = \text{Pic}(X) \times Z\) and there exists a rational function \(f_i\) on \(X \times \mathbb{P}^1_k\) such that for \(f_i|_{X \times \mathbb{A}_k^1}\) we have \(Z_i = D(f_i)\). Define

\[ \text{Cor}_0(X, \mathbb{A}_k^n)^* = \{ Z \in \text{Cor}(X, \mathbb{A}_k^n)^* \mid f_i(0) \in N_X^{\mathbb{P}}(X) \text{ for all } i \}. \]

Note that the condition on \(f_i\) is compatible with the general assumption that we deal with finite log-correspondences, names \(f_i(0)|_{X^m} \in \mathcal{O}(X^m)^*\) if and only if \(Z_i|_{X^m} \in \text{Cor}(X^m, \mathcal{G}_m)\). For \(Y = (\mathbb{A}_k^n, D_n)\), consider the modified presheaf with transfer, also denoted by \(\mathbb{Z}_{tr}(\mathbb{A}_k^n)^*\):

\[ \text{SemiStab}_k \to \text{Ab} \]

\[ (X, N_X) \mapsto \text{Cor}_0(X, \mathbb{A}_k^n)^* \]

From now on, whenever we write \(\mathbb{Z}_{tr}(Y)^*\) for \(Y = (\mathbb{A}_k^n, D_n)\) we shall always mean this restricted presheaf with transfer. As in Lemma 2.7, \(\mathbb{Z}_{tr}(Y)^*\) is in fact a Zariski sheaf.

Definition 2.9. The log-motivic complex \(\mathbb{Z}_{\log}(n)\) of weight \(n\) is the complex of sheaves with transfers \(C_*(\mathbb{Z}_{tr}(\mathbb{A}_k^n)^*)[-n]\).

Since \(\mathbb{Z}_{\log}(n)[n]\) is a direct summand of \(C_*(\mathbb{Z}_{tr}(\mathbb{A}_k^n, D_n)^*\)), the log-motivic complex \(\mathbb{Z}_{\log}(n)\) is a complex of Zariski sheaves. If \((X, N_X)\) is object of \(\text{SemiStab}_k\), then \(\mathbb{Z}_{\log, X}(n)\) denotes the restriction of \(\mathbb{Z}_{\log}(n)\) to the Zariski site of \(X\).

Definition 2.10. Let \((X, N_X)\) be an object of \(\text{SemiStab}_k\). We define the log-motivic cohomology of \((X, N_X)\) to be the hypercohomology of \(\mathbb{Z}_{\log}(n)\) with respect to the Zariski topology:

\[ H^i_{\log} - M_t(X, \mathbb{Z}(n)) := H^i_{\text{Zar}}(X, \mathbb{Z}_{\log}(n)). \]
Notice that if $X$ is a smooth scheme over Spec $k$, considered as a log-scheme by endowing it with the trivial log-structure, then the log-motivic cohomology of $X$ coincides with the motivic cohomology of $X$ as defined by Suslin-Voevodsky.

Remark 2.11. Of course, it would be desirable to work with the “full” monoid sheaf $\mathcal{O}_Y \cap j_* \mathcal{O}_{Y^{sm}}$ in the definition of the log-motivic complex. The main reason why we use the possibly smaller monoid sheaf $N_Y$ is a comparison between logarithmic Milnor $K$-theory and the modified logarithmic Hyodo-Kato sheaf which provides a semistable version of the Bloch-Gabber-Kato theorem (Theorem 3.10).

The comparison map uses explicitly the elements $g_i \in N_Y(Y)^{sp}$ and is – a priori – not defined for $\mathcal{O}_Y \cap j_* \mathcal{O}_{Y^{sm}}$. Moreover, the $p$-adic deformation theory carried out in Section 3 relies on a gluing argument along the logarithmic Hyodo-Kato sheaf, hence only makes sense for a log-motivic cohomology defined by using the more restrictive class of finite log-correspondences.

In the following we will construct an example where we can work with the full monoid sheaf $\mathcal{O}_Y \cap j_* \mathcal{O}_{Y^{sm}}$ and all the results in the paper from Theorem 3.10 onwards will hold. In particular, in this special situation we can work with the full logarithmic Milnor class group (see Definition 2.3) lies in $N_Y(U)$ and is bigger than 2 then we get an isomorphism $K_{mil, n}(U) / p^s \simeq K_{mil, n}(U) / p^s$ and all results in the paper from Theorem 3.10 onwards will hold.

2.12. The log-motivic complex of weight one.

Let $\mathcal{M}^*(\mathbb{P}^1_k; 0, \infty) : \text{SemiStab}_k \to \text{Ab}$ be the functor which sends a semistable variety $(X, N_X)$ to the group of rational functions on $X \times \mathbb{P}^1_k$ which are regular in a neighbourhood of $X \times \{0, \infty\}$ and equal to 1 on $X \times \{0, \infty\}$. Then $\mathcal{M}^*(\mathbb{P}^1_k; 0, \infty)$ is a sheaf for the Zariski topology on SemiStab$_k$.

Let $(Y, N_Y)$ be a semistable variety over $k$. Then the trivial locus $Y^{triv}$ of the structure morphism $(Y, N_Y) \to \text{Spec } k$ coincides with the smooth locus $Y^{sm}$ of $Y$. Let $j : Y^{sm} \to Y$ be the open immersion. We have a short exact sequence of abelian groups

$$0 \to \mathcal{M}^*(\mathbb{P}^1_k; 0, \infty)(Y^{sm}) \to \mathbb{Z}^{tr{\times}}(\mathbb{G}_m)(Y^{sm}) \to \mathbb{Z} \oplus \mathcal{O}_Y(Y^{sm}) \to 0$$

by [MVW06] Lemma 4.4. We shall extend this exact sequence over $Y$ as follows:
Recall that Cor$(Y, \mathbb{A}^1_k) \subset Cor(Y, \mathbb{P}^1_k)$ and Pic$(Y \times \mathbb{P}^1_k) = \text{Pic}(Y) \times \mathbb{Z}$, so to any $Z \in \text{Cor}(Y, \mathbb{A}^1_k)$ we can associate a unique rational function $f$ on $Y \times \mathbb{P}^1_k$ such that the Weil divisor $D(f)$ is $Z$, and such that there exists $n \in \mathbb{Z}$ with $f/t^n = 1$ on $Y \times \{\infty\}$. Define

$$
\text{Cor}_0(Y, \mathbb{A}^1_k)^* := \{Z = D(f) \mid f(0) \in N^{\text{gp}}(Y), \text{ and } Z_{Y, \text{sm}} \in \text{Cor}(Y^{\text{sm}}, \mathbb{G}_m)\}.
$$

In particular, if $Z = D(f) \in \text{Cor}_0(Y, \mathbb{A}^1_k)^*$ then $f(0)|_{Y, \text{sm}} \in \mathcal{O}^*(Y^{\text{sm}})$. Define

$$
\lambda : \text{Cor}_0(Y, \mathbb{A}^1_k)^* \to \mathbb{Z} \oplus N_Y(Y)^{\text{gp}}
$$

$$Z \mapsto (n, (-1)^n f(0)).$$

Then $\lambda$ is surjective and we can rewrite $\lambda$ as a surjective map

$$Z_{tr}(\mathbb{A}^1_k)^*((Y, N_Y)) \to \mathbb{Z} \oplus N_Y(Y)^{\text{gp}}$$

(see the construction of $\lambda$ in the proof of [MVW06 Lemma 4.4]). The kernel of $\lambda$ is exactly $\mathcal{M}^*(\mathbb{P}^1_k; 0, \infty)((Y, N_Y))$, so we get a short exact sequence

$$0 \to \mathcal{M}^*(\mathbb{P}^1_k; 0, \infty)((Y, N_Y)) \to Z_{tr}(\mathbb{A}^1_k)^*((Y, N_Y)) \to \mathbb{Z} \oplus N_Y(Y)^{\text{gp}} \to 0.
$$

Since $\lambda$ respects transfers [MVW06 Lemma 4.5] we can apply the functor $C_*$ to the exact sequence of sheaves with transfers

$$0 \to \mathcal{M}^*(\mathbb{P}^1_k; 0, \infty) \to Z_{tr}(\mathbb{A}^1_k)^* \to \mathbb{Z} \oplus N_Y^{\text{gp}} \to 0$$

to get an exact sequence of complexes of sheaves with transfer

$$0 \to C_*(\mathcal{M}^*(\mathbb{P}^1_k; 0, \infty)) \to C_*(Z_{tr}(\mathbb{A}^1_k)^*) \to C_*(\mathbb{Z} \oplus N_Y^{\text{gp}}) \to 0
$$
on $Y$. Splitting off $0 \to C_*(\mathcal{Z}) \to C_*(\mathcal{Z}) \to 0$ yields an exact sequence

$$0 \to C_*(\mathcal{M}^*(\mathbb{P}^1_k; 0, \infty)) \to Z_{\log Y}(1)[1] \to C_*(\mathbb{G}_m^{\text{log}}) \to 0.
$$

But $C_*(\mathbb{G}_m^{\text{log}}) = N_Y^{\text{gp}}$ because $N_Y^{\text{gp}}(U \times \Delta^n) = N_Y^{\text{gp}}(U)$. By [MVW06 Lemma 4.6] (which applies to $Y$ since the smoothness assumption is not used in the proof, nor in [MVW06 Lemma 2.18]) the complex $C_*(\mathcal{M}^*(\mathbb{P}^1_k; 0, \infty))$ is an acyclic complex of sheaves. Then we have shown the following:

**Proposition 2.13.** Let $(Y, N_Y)$ be a semistable variety over $k$. Then

$$Z_{\log Y}(1) \cong N_Y^{\text{gp}}[-1] =: \mathbb{G}_m^{\text{log}}[-1].$$

This generalises the smooth case considered in [SV00a Lemma 3.2].

**Corollary 2.14.**

$$H^i_{\log \mathcal{M}}(Y, \mathbb{Z}(1)) \cong \begin{cases} 
H^{i-1}_{\text{Zar}}(Y, \mathbb{G}_m^{\text{log}}) & \text{if } i = 1, 2, \\
0 & \text{if } i \neq 1, 2.
\end{cases}
$$

**Remark 2.15.** Note that our definition of $H^2_{\log \mathcal{M}}(Y, \mathbb{Z}(1))$ does not reproduce the logarithmic Picard group considered in [Yam11]. We have equipped the semistable variety $Y$ with a modified log-structure in order to obtain a geometric interpretation which generalises to higher codimension, whereas we do not have such a geometric interpretation for $\text{Pic}^{\text{log}}(Y) := H^1(Y, \mathcal{M}^{\text{gp}})$. On the other hand we will see that the $p$-adic deformation theory of $H^1(Y, N^{\text{gp}})$ is very similar to that of the usual logarithmic Picard group. See Remark 4.3 Proposition 4.9 and Remark 4.11.
3. Logarithmic Milnor $K$-groups

We are going to define logarithmic Milnor $K$-groups and relate them to the cohomology of the complexes $\mathbb{Z}_\log(n)$ in analogy to the smooth case proven by Kerz [Ker09, Theorem 1.1].

Let $(Y, N_Y)$ be a semistable variety. According to [Kat96, Proposition 11.3] $Y$ has a covering by open affines $U$ such that $U = \text{Spec} A/\pi_1 \cdots \pi_r$ when $A$ is a smooth $k$-algebra and each $A/\pi_i$ is smooth. Let $U^{\text{sm}}$ be the smooth locus of $U$ and $j : U^{\text{sm}} \to U$ the open immersion. Define $N_Y(U)$ as in Definition 2.3 and define the functions

$$g_i := \pi_i \prod_{j \neq i} \pi_j \in N_Y(U).$$

**Definition 3.1.** For $U \subset Y$ as above, define

$$K_{\log,n}(U) := \left( \frac{(N_Y(U)^{gp}) \otimes n}{I} \right)$$

where $I$ is the subgroup generated by elements of the form $a \otimes (1 - a)$ with $a, 1 - a \in N_Y(U)^{gp}$, those of the form $a \otimes (-a)$ with $a \in N_Y(U)^{gp}$, and those of the form $g_i^n \otimes (1 - \pi_i^{n_i} x)$ ranging over subsets $I \subset \{1, \ldots, r\}$, where $g_i := \prod g_i^{n_i}$ with $n_i \geq 0$, $\pi_i^{n_i} := \prod \pi_i^{n_i}$ with $n_i \geq 0$, and $x \in N_Y(U)$ such that $1 - \pi_i^{n_i} x \in N_Y(U)^{gp}$.

The elements of $I$ are called (as they are for the usual Milnor $K$-groups) Steinberg relations. The residue class of $a_1 \otimes \cdots \otimes a_n$ in $K_{\log,n}(U)$ is denoted by the symbol $\{a_1, \ldots, a_n\}$.

**Proposition 3.2.** Let $U \subset Y$ be open and let $V = U^{\text{sm}}$. Then there is a canonical map

$$\mu : K_{\log,n}(U) \to K_n(Y) := \bigoplus_{\eta \in U^0} K_n^{\text{Mil}}(V_\eta)$$

induced by a canonical map

$$N_Y^{gp}(U) \to N_Y^{gp}(V) = \bigoplus_{\eta \in U^0} \mathcal{O}_Y(V_\eta)$$

where $V_\eta := U_\eta \cap V$ and $U_\eta$ is the component of $U$ with generic point $\eta$. The map $\mu$ is injective.

**Proof.** The canonical map

$$N_Y^{gp}(U) \to N_Y^{gp}(V) = \bigoplus_{\eta \in U^0} \mathcal{O}_Y(V_\eta)$$

induces maps

$$N_Y^{gp}(U)^{\otimes n} \to \bigoplus_{\eta \in U^0} \mathcal{O}_Y(V_\eta)^{\otimes n}$$

and

$$K_{\log,n}(U) \to \bigoplus_{\eta \in U^0} K_n^{\text{Mil}}(V_\eta).$$

It is easy to check that the Steinberg relations in Definition 3.1 vanish in $\bigoplus_{\eta \in U^0} K_n^{\text{Mil}}(V_\eta)$. Since $K_n^{\text{Mil}}(V_\eta) \to K_n^{\text{Mil}}(k(\eta))$ is injective by [Ker09, Theorem...
6.1], it is enough to consider the map $K^{\text{Mil}}_{\log,n}(U) \rightarrow \prod_{\eta \in U_0} K^{\text{Mil}}_n(k(\eta))$. The injectivity for $n = 1$ trivially follows from the inclusions

$$N^{\text{gp}}_{\eta}(U) \hookrightarrow \mathcal{O}_Y(U) \cap j_*\mathcal{O}_Y(U_{\text{sm}})^* \hookrightarrow \mathcal{O}_Y(U_{\text{sm}})^* \hookrightarrow \prod_{\eta \in U_0} k(\eta)^*.$$ 

Now let $n \geq 2$. Assume that a symbol $\{a_1, \ldots, a_n\}$ maps to $1$ in $\bigoplus_{\eta \in U_0} K^{\text{Mil}}_n(V(\eta))$. Then for all $\eta_i$ corresponding to $\pi_i$ (where $\eta_i$ is the generic point of Spec $A/\mathfrak{m}_i$) there exists $a_j \in N_{\mathcal{O}_Y}(U)^{\text{gp}}$ such that $a_j = 1 + \pi_i x, x \in N_{\mathcal{O}_Y}(U)$, or elements $a_j, a_j' \in N_{\mathcal{O}_Y}(U)^{\text{gp}}$ such that $a_j = a + \pi_i x, a_j' = 1 - a$ with $a \in N_{\mathcal{O}_Y}(U)^{\text{gp}}, x \in N_{\mathcal{O}_Y}(U)$, or such that $a_j = a + \pi_i x, a_j' = -a$ with $a \in N_{\mathcal{O}_Y}(U)^{\text{gp}}, x \in N_{\mathcal{O}_Y}(U)$, or such that $a_j = g_i x + \pi_i x, a_j' = 1 - \pi_i y$ with $y, x \in N_{\mathcal{O}_Y}(U)$. In all of these cases we can write $a_j' = a(1 + \pi_i x)$ for $a \in N_{\mathcal{O}_Y}(U)^{\text{gp}}$ and $(1 + \pi_i x) \in N_{\mathcal{O}_Y}(U)^{\text{gp}}$. Using the vanishing of the Steinberg relations in Definition 3.1 it suffices to prove the following statement: the symbol $\{1 + \pi_i x, 1 + \pi_i y, 1 + \pi_i z\}$ vanishes in $K^{\text{Mil}}_{\log,n}(U)$ for $x, y, z \in N_{\mathcal{O}_Y}(U)$, and $\{1 + \pi_i x, 1 + \pi_i y, 1 + \pi_i z\} = \{1 + \pi_i x, 1 + \pi_i y, 1 + \pi_i z\}$ for $x, y, z \in N_{\mathcal{O}_Y}(U)$.

Let $\{x, y, z\} \in N_{\mathcal{O}_Y}(U)^{\text{gp}}$, so it suffices to show the claim for the elements of the form $\{1 + \pi_i x, 1 + \pi_i y, 1 + \pi_i z\}$ where $x, y, z \in N_{\mathcal{O}_Y}(U)$. But

$$\{1 + \pi_i x, 1 + \pi_i y, 1 + \pi_i z\} = \{1 + \pi_i x, 1 + \pi_i y, 1 + \pi_i z\}$$

where $x = \frac{xy}{1 + \pi_i x}$ and $y = -g_1 x$. This proves the claim.

Now let $\sum z$ be a finite sum of symbols $z = \{z_1, \ldots, z_n\}$ in $K^{\text{Mil}}_{\log,n}(U)$ that vanishes in $K^{\text{Mil}}_n(k(\eta))$ for all $i$. Let us start with the case $i = 1$. Let $T_1$ be the subgroup of $K^{\text{Mil}}_{\log,n}(U)$ that is generated by symbols where at least one entry is of the form $1 + \pi_1 x, 1 + \pi_1 y$. We show that $\sum z \in T_1$. Without loss of generality we can assume that $\sum z$ decomposes as a sum of triples $z_1 + z_2 + z_3$ which become a bilinear relation mod $\pi_1$ and a sum of individual symbols $z_0$ that become a Steinberg relation mod $\pi_1$. Let $z_0 = \{z_1, \ldots, z_n\}$. As before, we see that $z_0$ is in $K^{\text{Mil}}_{\log,n}(U)$ equivalent to a symbol which contains an entry $1 + \pi_1 x_1$, hence $z_0 \in T_1$. Then let us consider a triple $z_1 + z_2 + z_3$ which becomes a bilinear relation mod $\pi_1$. Then

$$z_1 + z_2 + z_3 = \{c_1, \ldots, c_n\} - \{a_1, \ldots, a_n\} - \{b_1, \ldots, b_n\}$$

with $\pi_i = \pi_i \mathfrak{B}_i$ mod $\pi_1$ and $\pi_i = \pi_i \mathfrak{B}_i$ mod $\pi_1$ for $i \geq 2$ (without loss of generality). Since any two liftings of $\pi_i, \mathfrak{B}_i, \pi_i \mathfrak{B}_i$ differ by a factor of $(1 + \pi_1 x_1)$, we conclude that $z_1 + z_2 + z_3$ is a sum of symbols which all contain an entry of the type $(1 + \pi_1 x_1)$, hence is in $T_1$. 
Now let $T_2$ be the subgroup of $K_{\log,n}^\text{Mil}(U)$ generated by symbols that contain two entries of the forms $(1 + \pi_1 \lambda_1)$, $(1 + \pi_2 \lambda_2)$ or an entry of the form $(1 + \pi_1 \pi_2 \lambda_{12})$. By repeating the above argument in the $T_1$-case, we conclude that the condition “$\sum z \mod \pi_i$ vanishes in $K_n^\text{Mil}(k(\eta_i))$ for $i = 1, 2$” implies that $\sum z \in T_2$. By induction we conclude that our element $\sum z$ is a sum of symbols with entries $1 + \pi_1 \eta_i x_1, \ldots, 1 + \pi_1 \eta_i x_s$ such that $I_1 \cup \cdots \cup I_s = \{1, \ldots, r\}$. It follows from the first part of the proof that such a symbol vanishes in $K_{\log,n}^\text{Mil}(U)$. This proves the proposition. \hfill \Box

Remark 3.3. Define

$$K_{\log,n}^\text{Mil}(U) := \frac{(\mathcal{O}_Y(U) \cap j_* \mathcal{O}_Y\text{sm}(U)^{\text{gp}})^{\otimes n}}{I}$$

where $I$ is the subgroup generated by the elements of the form $a \otimes (1 - a)$, $a \otimes (-a)$ for $a, 1 - a \in (\mathcal{O}_Y(U) \cap j_* \mathcal{O}_Y\text{sm}(U)^{\text{gp}})$, and by elements of the form $g_f \otimes (1 - n_i f)$ as in Definition 3.1. One can then check that Proposition 3.2 still holds for this larger Milnor $K$-group, the proof easily passes over. Note that this group is only used in a special example discussed in Remark 2.11.

We construct a map

$$\mathcal{H}^n(\mathcal{Z}_{\log}(n))(U) \to K_{\log,n}^\text{Mil}(U)$$

as follows. Let $Z \in \text{Cor}(U, A^n_k) \subset \text{Cor}(U, \mathbb{P}^n_k)$. Consider $\text{pr}_i(Z) = Z_i$, where $\text{pr}_i : \mathbb{P}^n_k \to \mathbb{P}^1_k$ is the $i$-th projection. Note that $\text{pr}_i$ is finite over $Z$ and $Z_i \in \text{Cor}(U, A^1_k) \subset \text{Cor}(U, \mathbb{P}^1_k)$. In analogy to the proof of [MVW06, Lemma 4.4] there exist integers $n_i$ and rational functions $f_i$ on $U \times \mathbb{P}^1$ such that $f_i/\eta_i^n \equiv 1$ on $U \times \{\infty\}$ and $D(f_i) = Z_i$. Let

$$\text{Cor}_0(U, A^n_k)^* = \{ Z \in \text{Cor}(U, A^n_k) | f_i(0) \in \Delta^\text{gp}(U) \ \forall i \}$$

In particular $f_i(0) |_V \in \mathcal{O}(V)^*$ for all $i$.

For each $i \in \{1, \ldots, n\}$, we obtain a map

$$\text{Cor}_0(U, A^n_k)^* \to \mathbb{Z} \oplus \Delta^\text{gp}(U)$$

$$Z \mapsto (n_i), ((-1)^{n_i} f_i(0))$$

The collection of $n_i$ extend to yield a linear map

$$\lambda : \text{Cor}_0(U, A^n_k)^* \to \mathbb{Z}^n \oplus K_{\log,n}^\text{Mil}(U)$$

$$Z \mapsto ((n_i), \epsilon(-1)^{n_i} f_i(0)) \ldots, ((-1)^{n_i} f_i(0))$$

Consider the subgroup $E_n$ in $\text{Cor}_0(U, A^n_k)^*$ defined by the sum of the images of the inclusions $k_i : A^{n-1}_k \to A^n_k$, $(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_1, 1, x_i, \ldots, x_{n-1})$. Then $\lambda$ maps $E_n$ onto $\mathbb{Z}^n$. Indeed, let $Z = \kappa_i(h)$ for $h \in \text{Cor}_0(U, A^{n-1}_k)^*$. Then the corresponding function $f_i$ for $\text{pr}_i(Z)$ is $f_i = t_i - 1$ such that $D(f_i) = pr_i(Z) = U \times \{1\}$ and $\lambda(Z) = (\epsilon, 1)$ for some $\epsilon \in \mathbb{Z}^n$.

Cancelling out $\mathbb{Z}^n$ as in the case $n = 1$ yields a map also denoted by

$$\lambda : \text{Cor}_0(A^n_k)^* \to K_{\log,n}^\text{Mil}(U)$$

If $Z$ lies in the image of $\partial_1 - \partial_0$ (where $\partial_0$ and $\partial_1$ are induced by the face maps $\partial_0, \partial_1 : \Delta^0 \rightarrow \Delta^1$) then the restriction $Z_V \in \text{Cor}(V, G_m^n)$ is in the image of $\partial_1 - \partial_0$, hence becomes zero under the map

$$\text{Cor}(V, G_m^n) \to K_n^\text{Mil}(V) = \mathcal{H}^n(\mathcal{Z}_{\log}(n)|_V)$$.
Then by definition of $\mathcal{K}_{\log, n}^{\text{Mil}}(U)$, this implies that the image $\partial_1 - \partial_0$ in $\text{Cor}(U, \mathbb{A}^n_k)^*$ maps to zero in $\mathcal{K}_{\log, n}^{\text{Mil}}(U)$. This defines a map

$$\lambda : \mathcal{H}^n(\mathbb{Z}_{\log}(n))(U) \rightarrow \mathcal{K}_{\log, n}^{\text{Mil}}(U).$$

Conversely, let $z = \{z_1 \otimes \cdots \otimes z_n\} \in \mathcal{K}_{\log, n}^{\text{Mil}}(U)$ where $z_i \in \mathbb{A}^{\text{gp}}^n(U) \simeq \mathbb{Z}_{\log}(1)[1](U)$. There is a product map

$$\mathbb{Z}_{\log}(1)[1] \otimes \cdots \otimes \mathbb{Z}_{\log}(1)[1] \rightarrow \mathbb{Z}_{\log}(n)[n]$$

defined in [SV00a] page 141 or [MVW06] Construction 3.11. This defines a class $[z] \in \mathcal{H}^n(\mathbb{Z}_{\log}(n))(U)$. The following proposition implies that $z \mapsto [z]$ factors through $\mathcal{K}_{\log, n}^{\text{Mil}}(U)$.

**Proposition 3.4.** For each $i, n \geq 0$, the restriction map of the homotopy-invariant sheaf with transfers $\mathcal{H}^i(\mathbb{Z}_{\log}(n))(U)$ to $V = U^{\text{sm}}$ is injective. For $i = n = 1$, we have the injection $\mathbb{A}^{\text{gp}}_n(U) \hookrightarrow \mathcal{O}_{V, \mathbb{Z}_{\log}}^\ast(V) = \mathcal{H}^1(\mathbb{Z}_{\log}(1))(V)$.

**Proof.** Consider the commutative diagram

$$
\begin{array}{ccc}
\text{Cor}_0(\Delta^{n+1-i} \times U, \mathbb{A}^n_k) & \xrightarrow{\partial_0^{-i}} & \text{Cor}_0(\Delta^n \times U, \mathbb{A}^n_k)^* \\
\downarrow{\iota_{n+1-i}} & & \downarrow{\iota_{n-i}} \\
\text{Cor}(\Delta^{n+1-i} \times V, \mathbb{G}_m^n) & \xrightarrow{\partial_0^{-i}} & \text{Cor}(\Delta^n \times V, \mathbb{G}_m^n)^* \\
\end{array}
$$

with vertical maps $\iota_j$ sending $\alpha$ to $\alpha|_{V \times \mathbb{G}_m^n}$. Take two elements $Z, Z' \in \ker \left( \sum_{j=0}^{n-i} (-1)^j \partial_j^{-1-i} \right) \subset \text{Cor}_0(\Delta^{n-i} \times U, \mathbb{A}^n_k)^*$ and assume that there exists $Y \in \text{Cor}(\Delta^{n+1-i} \times V, \mathbb{G}_m^n)$ such that

$$\iota_{n-i}(Z) - \iota_{n-i}(Z') = \sum_{j=0}^{n-i} (-1)^j \partial_j^{-1-i}(Y).$$

Define $\overline{Y}$ to be the closure of $Y$ in $\Delta^{n+1-i} \times U \times \mathbb{P}^n_k$. We claim that $\overline{Y} \subset \text{Cor}_0(\Delta^{n+1-i} \times U, \mathbb{A}^n_k)^*$. Indeed, suppose that the support of $\overline{Y}$ is not contained in $\Delta^{n+1-i} \times U \times \mathbb{A}^n_k$. Then there exists a $t \in \{1, \ldots, n\}$ such that $\text{pr}_t(\overline{Y}) \subset \Delta^{n+1-i} \times U \times \mathbb{P}^1$ is not contained in $\Delta^{n+1-i} \times U \times \mathbb{A}^1_k$ (where $\text{pr}_t$ is induced by the $t$-th projection $\text{pr}_t : \mathbb{P}^n_k \rightarrow \mathbb{P}^1_k$). On the other hand, $\text{pr}_t(\overline{Y})$ is the closure of $\text{pr}_t(Y) \in \text{Cor}(\Delta^{n+1-i} \times V, \mathbb{G}_m^n)$. The analogous commutative diagram to
\[
\begin{array}{ccc}
\text{Cor}_0(\Delta^{n+1-i} \times U, \mathbb{A}^1_k) & \xrightarrow{\partial_0^{n-i}} & \text{Cor}_0(\Delta^{n-i} \times U, \mathbb{A}^1_k) \\
\vdots & & \vdots \\
\text{Cor}(\Delta^{n+1-i} \times V, \mathbb{G}_m) & \xrightarrow{\partial_0^{-n-i}} & \text{Cor}(\Delta^{n-i} \times V, \mathbb{G}_m)
\end{array}
\]

shows that
\[
(3.4.2) \quad \sum_{j=0}^{n+1-i} (-1)^j \partial_j^{n-i}(\text{pr}_t(\overline{Y})) = \text{pr}_t(Z - Z') = \text{pr}_t(Z) - \text{pr}_t(Z')
\]

because \(\text{pr}_t(Z)\) and \(\text{pr}_t(Z')\) are the closures of \(\text{pr}_t(t_{n-i}(Z))\) and \(\text{pr}_t(t_{n-i}(Z'))\). Now, for \(n = i\) the injectivity of \(\overline{\Delta^n}^\text{log}(U) \to \text{Cor}_0^*(V)\) implies that \(\text{pr}_t(Z)\) and \(\text{pr}_t(Z')\) in \(\text{Cor}_0(U, \mathbb{A}^1_k)^*\) define the same cohomology class, and since \((\partial_1^1 - \partial_0^1)(\text{pr}_t(\overline{Y})) = \text{pr}_t(Z) - \text{pr}_t(Z')\) we conclude that \(\text{pr}_t(\overline{Y}) \in \text{Cor}_0(\Delta^1 \times U, \mathbb{A}^1_k)^*\). For \(i < n\) the cohomology of both complexes (for \(\mathbb{A}^1_k\) and \(\mathbb{G}_m\)) vanishes because the complexes \(Z_{\text{log}}(1)\) and \(Z(1)\) are acyclic in degrees \(< 1\) (Proposition 2.13). Hence \(\text{pr}_t(Z)\) and \(\text{pr}_t(Z')\) in \(\text{Cor}_0(\Delta^{n-i} \times U, \mathbb{A}^1_k)^*\) vanish in the cohomology and the formula (3.4.2) then implies that \(\text{pr}_t(\overline{Y}) \in \text{Cor}_0(\Delta^{n+1-i} \times U, \mathbb{A}^1_k)^*\). Hence we conclude that \(\overline{Y}\) is closed in \(\Delta^{n+1-i} \times U \times \mathbb{A}^n_k\) and hence proper over \(\Delta^{n+1-i} \times U\), because the projection \(\Delta^{n+1-i} \times U \times \mathbb{P}^n_k \to \Delta^{n+1-i} \times U\) is proper. Since all \(\text{pr}_t(\overline{Y})\) are quasi-finite over \(\Delta^{n+1-i} \times U\) for all \(t\), \(\overline{Y}\) is itself quasi-finite over \(\Delta^{n+1-i} \times U\), hence finite. This shows the claim. The above argument also shows that, given an element \(Z\) in \(\text{Cor}(\Delta^{n+1-i} \times U, \mathbb{A}^1_k)^*\), \(Z\) is the closure of its restriction to \(\Delta^{n+1-i} \times V \times \mathbb{G}_m\) and therefore the vertical maps in the diagram are injective.

The commutative diagram (3.4.1) and the injectivity of the map \(t_{n-1-i}\) imply that
\[
Z - Z' = \sum_{j=0}^{n-i} (-1)^j \partial_j^{n-i}(\overline{Y}).
\]

This shows that the cohomology of the upper complex in (3.4.1) injects into the cohomology of the lower complex. Since the complexes \(Z_{\text{log}}(n)\) resp. \(Z(n) = C_* \mathbb{Z}_{\text{ur}}(\mathbb{G}_m^\text{nr})[-n]\) are direct summands of these complexes, the proposition follows.

\[\square\]

**Remark 3.5.** Proposition 3.4 shows that the map
\[
\mathcal{H}^i(Z_{\text{log}}^Y(n) \otimes^L \mathbb{Z}/p^r) \to u_* \mathcal{H}^i(Z_{Y^\text{un}}(n) \otimes^L \mathbb{Z}/p^r)
\]
is an injection for all \(i, n, r \geq 0\). Indeed, since all of the terms of \(Z_{\text{log}}^Y(n)\) and \(Z_{Y^\text{un}}(n)\) are free abelian groups, the complexes \(Z_{\text{log}}^Y(n) \otimes \mathbb{Z}/p^r\) and \(Z_{Y^\text{un}}(n) \otimes \mathbb{Z}/p^r\) represent the derived tensor products \(Z_{\text{log}}^Y(n) \otimes^L \mathbb{Z}/p^r\) and \(Z_{Y^\text{un}}(n) \otimes^L \mathbb{Z}/p^r\). At one point in the proof of Proposition 3.4 we argue by projecting down to \(\mathbb{P}^1\) and use acyclicity of \(Z_{\text{log}}(1)\) and \(Z(1)\) for \(i < n - 1\) and injectivity for \(n = i - 1\). The
short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^r \to 0$ shows that this remains true after tensoring with $\mathbb{Z}/p^r$. The rest of the proof remains the same.

As a corollary, we obtain

**Theorem 3.6.** If $k$ is infinite then there is a canonical isomorphism

$$
\mathcal{K}_{\log,n}^{\text{Mil}}(U) \to \mathcal{H}^n(\mathbb{Z}_{\log}(n))(U).
$$

**Proof.** We consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}^n(\mathbb{Z}_{\log}(n))(U) & \xrightarrow{\lambda} & \mathcal{K}_{\log,n}^{\text{Mil}}(U) \\
\downarrow & & \downarrow \circ \\
\mathcal{H}^n(\mathbb{Z}_{\log}(n))(V) & \xrightarrow{\lambda} & \mathcal{K}_{n}^{\text{Mil}}(V) \\
\end{array}
\]

Propositions 3.2 and 3.4 imply that all vertical maps in the diagram are injective, and hence the upper right map $h$ is well-defined. The upper maps $\lambda$ and $h$ are the restrictions of the corresponding homomorphisms $\lambda$ and $h$ on the smooth part, which are proven to be isomorphisms by [Ker09, Theorem 1.1] when the field $k$ is infinite. Hence $h \circ \lambda = \text{id}$ on $\mathcal{H}^n(\mathbb{Z}_{\log}(n))(U)$ and $\lambda \circ h = \text{id}$ on $\mathcal{K}_{\log,n}^{\text{Mil}}(U)$ when $k$ is infinite.

**Remark 3.7.**

(i) Proposition 3.4 should be compared with [MVW06, Theorem 11.3]: If $U$ is dense open in a smooth $X$, and $F$ is a homotopy-invariant sheaf with transfer, then the restriction map $F(X) \to F(U)$ is injective. One might expect that this also holds for $Y_{\text{sm}} \subset Y$, $Y$ semistable, and any homotopy-invariant sheaf $F$ with transfer on $Y$.

(ii) When $k$ is finite, one should consider a refined version of $\mathcal{K}_{\log,n}^{\text{Mil}}$ analogous to the improved Milnor $K$-theory of [Ker10]. Indeed, define the improved logarithmic Milnor $K$-theory $\mathcal{K}_{\log,n}^{\text{Mil}}(U)$ to be the image of $\mathcal{K}_{\log,n}^{\text{Mil}}(U)$ under the composition

$$
\mathcal{K}_{\log,n}^{\text{Mil}}(U) \hookrightarrow \mathcal{K}_{n}^{\text{Mil}}(U_{\text{sm}}) \to \mathcal{K}_{n}^{\text{Mil}}(U_{\text{sm}})
$$

where the first arrow is the inclusion in Proposition 3.2 and the second arrow is the canonical map to improved Milnor $K$-theory $\mathcal{K}_{n}^{\text{Mil}}(U_{\text{sm}})$. This second map is an isomorphism when $k$ is infinite [Ker09, Proposition 10(5)], so $\mathcal{K}_{\log,n}^{\text{Mil}}(U) \simeq \mathcal{K}_{\log,n}^{\text{Mil}}(U)$ in this case too. With this refinement, Proposition 3.6 holds when $k$ is finite by the same proof. Notice that Theorem 3.6 holds in weight $n = 1$ even when $k$ is finite by Proposition 2.13; this is because $\mathcal{K}_{1}^{\text{Mil}} = \mathcal{K}_{1}^{\text{Mil}}$ by [Ker10 Proposition 10(1)], so there is no difference between logarithmic Milnor $K$-theory and improved logarithmic Milnor $K$-theory in weight one.

Let $W_{s,\omega}_{Y/k,\log}^n$ denote the modified logarithmic Hyodo-Kato Hodge-Witt sheaf on $Y_{\text{et}}$. It is defined as follows: For $Y$ the closed fibre of a regular $W(k)$-scheme $X$ with semistable reduction, let $M_Y := i^* M_X = i^* j_* O_{X_K}$ be the usual log-structure on $Y$ (where $i : Y \hookrightarrow X$ is the closed immersion and $j : X_K \hookrightarrow X$ is the open immersion of the generic fibre). Let $u : Y_{\text{sm}} \hookrightarrow Y$ be the open immersion of the smooth part. Then the modified (extended) Hyodo-Kato complex $W_{s,\omega}^\bullet_{Y/k}$ is the
$W_s(\mathcal{O}_Y)$-subalgebra of $\mathcal{A}^* := u_s W_s \Omega_{\text{sm}/k}^1[\theta]/\theta^2$, where $\theta$ is an indeterminate in degree one satisfying $\theta a = (-1)^q a \theta$ for $a \in u_s W_s \Omega_{\text{sm}/k}^1$ and $d \theta = 0$, generated by $d W_s(\mathcal{O}_Y)$ and the image of $d \log : M_Y \to \mathcal{A}^1$ defined on $u^{-1}i^{-1}(\mathcal{O}_X)$ by the composition

$$u^{-1}j^{-1}(\mathcal{O}_X) \to \mathcal{O}_{\text{sm}} \xrightarrow{d \log[-1]} W_s \Omega_{\text{sm}/k}^1$$

and on $K^*$ by $a \mapsto \text{ord}_K(a) \theta$ (see [HK94, 1.4]). Then we recall [HK94] Proposition 1.5:

**Proposition 3.8.** The sequence

$$0 \to W_s \omega_{Y/k}^1[-1] \to W_s \tilde{\omega}_{Y/k} \to W_s \omega_{Y/k}^1 \to 0$$

is exact.

The map $d \log : M_Y^{\text{gp}} \to W_s \tilde{\omega}_{Y/k}^1$ induces a map $d \log : (M_Y^{\text{gp}})^{\otimes n} \to W_s \tilde{\omega}_{Y/k}^n$. Write $W_s \tilde{\omega}_{Y/k, \log}$ for the image. As a corollary of Proposition 3.8 using $1 - \varphi$ on $W_s \tilde{\omega}_{Y/k}$ we obtain

**Proposition 3.9.** There is an exact sequence

$$0 \to W_s \omega_{Y/k, \log}^{n-1} \to W_s \tilde{\omega}_{Y/k, \log}^n \to W_s \omega_{Y/k, \log}^n \to 0$$

In the next section we will glue the log-motivic complex $\mathcal{Z}_{\log}(n)$ through its top cohomology $H^n(\mathcal{Z}_{\log}(n))(U) \simeq \mathcal{K}_{\log,n}^{\text{Mil}}(U)$ with the log-syntomic complex of Kato-Tsuiji via the modified logarithmic Hyodo-Kato sheaf $W_s \tilde{\omega}_{Y/k, \log}$ in order to achieve a semistable analogue of the deformational part of the main result of [BEK14]. We construct a canonical map

$$d \log : \mathcal{K}_{\log,n}^{\text{Mil}}(U) \to W_s \tilde{\omega}_{Y/k, \log}^n(U)$$

as follows. For $n = 1$, the map

$$d \log : \bigoplus_Y^{M_Y^{\text{gp}}}(U) \to W_s \tilde{\omega}_{Y/k, \log}^1(U)$$

is given by the assignments

$$x \in \mathcal{O}_Y(U)^* \mapsto d \log[x] \in W_s \omega_{Y/k, \log}^1,$$

$$g_i = \beta(e_i) \mapsto d \log(\pi_i) \in W_s \tilde{\omega}_{Y/k, \log}^1,$$

where $\pi_i = \alpha(e_i)$ for the structure map $\alpha : M_Y \to \mathcal{O}_Y$, and $d \log$ is the canonical map on $M_Y^{\text{gp}}$ that we recalled above (so $\prod_{i=1}^r g_i \mapsto \theta$). Note that $d \log(\pi_i)$ in $\bigoplus_{n \in U^n} W_s \Omega_{k(n)/k}^1$ has $j$-component $d \log[\pi_i]$ for $j \neq i$ and $i$-component $-\sum_{j \neq i} d \log[\pi_j]$. The above is extended to a map $d \log : \mathcal{K}_{\log,n}^{\text{Mil}}(U) \to W_s \tilde{\omega}_{Y/k, \log}^n(U)$ by taking exterior products.

Since the $d \log$ map is surjective on $M_Y^{\otimes n}$ by definition, it is also surjective on $\mathcal{K}_{\log,n}^{\text{Mil}}(U)$. By composing $d \log$ with the canonical injective map

$$W_s \tilde{\omega}_{Y/k, \log}^n \hookrightarrow \bigoplus_{n \in U^n} W_s \Omega_{k(n)/k}^1$$
it is clear that symbols \{a, -a\} and \{a, 1 - a\}, \ a \in \mathcal{N}_V(U)^{\mathbb{P}} and \{1 - \pi_i, g_i x\} vanish in \bigoplus_{g \in U} W_n \Omega^\alpha_{k(\eta)}/k^j since they vanish in \bigoplus_{g \in U} K_n^{\text{Mil}}(k(\eta)), hence \ d\log is well-defined on \ K_n^{\text{Mil}}(U). We will prove the following semistable analogue of the Bloch-Kato-Gabber theorem:

**Theorem 3.10.** We have an isomorphism

\[ K_n^{\text{Mil}}(U)/p^s \cong W_n\tilde{\omega}_Y/k_{\log}(U). \]

**Proof.** Let \( V = U^{\text{sm}} \). Since \( K_n^{\text{Mil}}(V) \) is \( p \)-torsion-free by [Izh91, Proposition 3.4] that \( K_n^{\text{Mil}}(U) \) is \( p \)-torsion-free as well. Hence we have an isomorphism

\[ K_n^{\text{Mil}}(U)/p \cong p^{s-1}K_n^{\text{Mil}}(U)/p^s. \]

Then consider the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & K_n^{\text{Mil}}(U)/p & \times_{p^{s-1}} & K_n^{\text{Mil}}(U)/p^s & \longrightarrow & K_n^{\text{Mil}}(U)/p^{s-1} & \longrightarrow & 0 \\
& & \downarrow d\log & & \downarrow d\log & & \downarrow d\log & & \\
0 & \longrightarrow & \tilde{\omega}_Y/k_{\log}(U) & \times_{p^{s-1}} & W_n\tilde{\omega}_Y/k_{\log}(U) & \longrightarrow & W_{n-1}\tilde{\omega}_Y/k_{\log}(U) & \longrightarrow & 0
\end{array}
\]

By induction, it suffices to show that the left vertical arrow is an isomorphism. Since it is surjective by definition, we need to show injectivity. The proof of Proposition [3.4] implies that the map \( K_n^{\text{Mil}}(U)/p \rightarrow K_n^{\text{Mil}}(V)/p \) is injective as well. Indeed, if a symbol \( \{a_1, \ldots, a_n\} \) vanishes in \( \prod_{\eta \in U^0} K_n^{\text{Mil}}(k(\eta))/p \) then for each \( \eta_i \) there exists \( j \in \{1, \ldots, n\} \) such that \( a_j = b_{j}^{p} + \pi_i x = b_{j}^{p}(1 + \pi_i \tilde{\pi}_j) \). One then follows the proof of Proposition [3.4] to conclude.

Let \( K_n^{\text{Mil}}(U) \) be the image of \((\mathcal{O}(U)^*)^n \) in \( K_n^{\text{Mil}}(U) \). Since the composite map \((\mathcal{O}(U)^*)^n \rightarrow \tilde{\omega}_Y/k_{\log}(U)\) factors through the injection \( j_* K_n^{\text{Mil}}(V)/p \rightarrow j_* \Omega^n_{Y^{\text{sm}}}/k_{\log}(V) \) (which is an isomorphism by the Bloch-Kato-Gabber theorem [BK89, Corollary 2.8]), we see that \( d\log \) restricted to \( K_n^{\text{Mil}}(U)/p \) is injective. Using the exact sequence

\[ 0 \rightarrow \omega_n^{n-1} \rightarrow \omega_n^{\alpha} \rightarrow \tilde{\omega}_Y/k_{\log}(U) \rightarrow \omega_n^{\alpha} \rightarrow 0 \]

we will conclude the proof as below.

Consider the composite map (which is surjective)

\[ K_n^{\text{Mil}}(U)/p \rightarrow \tilde{\omega}_Y/k_{\log}(U) \rightarrow \omega_n^{\alpha}(U). \]

For \( g_i = \pi_i + \sum_{j \neq i} \pi_j \) the image \( d\log(g_i) \) in \( \omega_n^{\alpha}/k_{\log} \) has \( j \)-component (in \( \Omega^1_{k(\eta)/k} \)) \( d\log \pi_i \) for \( j \neq i \) and \( i \)-component (in \( \Omega^1_{k(\eta)/k} \)) \( -\sum_{j \neq i} d\log \pi_j \). It is then clear that the kernel of the map

\[ (\mathcal{N}_V^{\mathbb{P}}(U)/p)/\mathcal{O}(U)^{\mathbb{P}} \rightarrow \omega_n^1/U^{\text{sm}}(U)/\text{image} (\mathcal{O}(U)*) \]

is generated by \( \left( \prod_{i=1}^r g_i \right) \pm 1 \).
Since for all $a \in \mathbb{N}^p(U)$, the symbol $\{a, a\}$ vanishes in $\mathcal{K}_{\log,n}(U)/p$, because $\{a, -1\}$ is $p$-divisible, we see that the $\mathbb{F}_p$-rank of the kernel of the map
\[
(K_{\log,n}(U)/p)/(K_{n}(U)/p) \rightarrow \omega_{Y/k,\log}^n(U) / \text{image}(\mathcal{O}(U)^* \otimes n)
\]
is equal to the $\mathbb{F}_p$-rank of $\omega_{Y/k,\log}^{n-1}/\text{image}(\mathcal{O}(U)^* \otimes n-1)$. But this is also the $\mathbb{F}_p$-rank of $\omega_{Y/k,\log}^{n-1}/\text{image}(\mathcal{O}(U)^* \otimes n-1) \wedge \theta$. Hence the $d \log$ map
\[
\mathcal{K}_{\log,n}(U)/p \rightarrow \tilde{\omega}_{Y/k,\log}^n(U),
\]
which is already known to be surjective, must be an isomorphism. □

Define $\mathcal{K}_{\log,Y,*}^n$ to be the Zariski sheafification of the presheaf $U \mapsto \mathcal{K}_{\log,Y,*}^n(U)$. Then we have the following semistable analogue of [GL00, Theorem 8.5]:

**Proposition 3.11.** For each $n, s \geq 0$ there is a quasi-isomorphism

\[
\mathcal{Z}_{\log,Y}(n) \otimes^L \mathbb{Z}/p^s \simeq W_s \tilde{\omega}_{Y/k,\log}^n[-n]
\]
in $D(Y_{\text{Zar}})$.

**Proof.** Recall from Remark 3.5 that

\[
\mathcal{H}^i(\mathcal{Z}_{\log,Y}(n) \otimes^L \mathbb{Z}/p^s) \rightarrow u_* \mathcal{H}^i(\mathcal{Z}_{Y^{\text{sm}}}(n) \otimes^L \mathbb{Z}/p^s)
\]
where $u : Y^{\text{sm}} \hookrightarrow Y$ is the inclusion of the smooth locus. By [GL00] Theorem 8.3 we have

\[
\mathcal{H}^i(\mathcal{Z}_{Y^{\text{sm}}}(n) \otimes^L \mathbb{Z}/p^s) \simeq \begin{cases} 0 & \text{if } i \neq n \\ W_s \Omega^{n}_{Y^{\text{sm}}/k,\log} & \text{if } i = n \end{cases}
\]
so we deduce that $\mathcal{Z}_{\log,Y}(n) \otimes^L \mathbb{Z}/p^s$ is acyclic outside of cohomological degree $n$. It therefore suffices to show that $\mathcal{H}^n(\mathcal{Z}_{\log,Y}(n) \otimes^L \mathbb{Z}/p^s) \simeq W_s \tilde{\omega}_{Y/k,\log}^n$. To see this, the above vanishing and the exact triangle

\[
\mathcal{Z}_{\log,Y}(n) \rightarrow \mathcal{Z}_{\log,Y}(n) \rightarrow \mathcal{Z}_{\log,Y}(n) \otimes^L \mathbb{Z}/p^s \rightarrow \mathcal{Z}_{\log,Y}(n) \otimes^L \mathbb{Z}/p^{s+1}
\]
gives a short exact sequence

\[
0 \rightarrow \mathcal{H}^n(\mathcal{Z}_{\log,Y}(n)) \rightarrow \mathcal{H}^n(\mathcal{Z}_{\log,Y}(n)) \rightarrow \mathcal{H}^n(\mathcal{Z}_{\log,Y}(n) \otimes^L \mathbb{Z}/p^s) \rightarrow 0
\]
fitting into the following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{K}_{\log,Y,n}^n & \longrightarrow & \mathcal{K}_{\log,Y,n}^n & \longrightarrow & \mathcal{K}_{\log,Y,n}^n/p^s & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}_{\log,Y,n}^n & \longrightarrow & \mathcal{K}_{\log,Y,n}^n & \longrightarrow & \mathcal{K}_{\log,Y,n}^n/p^s & \longrightarrow & 0
\end{array}
\]

where the isomorphisms $\mathcal{K}_{\log,Y,n}^n \simeq \mathcal{H}^n(\mathcal{Z}_{\log,Y}(n))$ are by Theorem 3.6 (when $k$ is finite use the improved logarithmic $K$-theory as in Remark 3.7(ii)). The map $p^s : \mathcal{K}_{\log,Y,n}^n \rightarrow \mathcal{K}_{\log,Y,n}^n$ in the lower sequence is injective because $\mathcal{K}_{\log,Y,n}^n$ is $p$-torsion free. Indeed, $\mathcal{K}_{\log,Y,n}^n$ injects into $u_* \mathcal{K}_{\log,Y,n}^n$ by Proposition 3.2 and $\mathcal{K}_{\log,Y,n}^n$ is $p$-torsion free. Hence the lower sequence is also exact and we conclude that there is an induced isomorphism $\mathcal{K}_{\log,Y,n}^n/p^s \simeq \mathcal{H}^n(\mathcal{Z}_{\log,Y}(n) \otimes^L \mathbb{Z}/p^s)$. Then the proposition follows from Theorem 3.10 □
4. Log-syntomic cohomology and the p-adic variational Hodge conjecture

Let $k$ be a perfect field of characteristic $p > 2$, and let $K = \text{Frac} W(k)$. In this section we fix a natural number $n < p$. Let $X$ be a scheme over $W(k)$ with semistable reduction, that is étale locally on $X$ the structure morphism factors as

$$X \xrightarrow{\delta} \text{Spec } W(k)[t_1, \ldots, t_a]/(t_1 \cdots t_b - p) \xrightarrow{\phi} \text{Spec } W(k)$$

for some $a \geq b$, where $u$ is a smooth morphism and $\delta$ is induced by the diagonal map. Then the generic fibre $X_K$ is smooth and the special fibre $Y$ is a reduced normal crossings divisor on $X$. If $Y$ is endowed with the inverse image $M_Y$ of the divisorial log-structure $M_X$ associated to $Y \hookrightarrow X$, then $(Y, M_Y)$ is a semistable variety in the sense of (2.1). For each $m \in \mathbb{N}$, set $X_m = X \times_{W(k)} W_m(k)$ and let $M_{X_m}$ be the pullback (in the sense of log-structures) of $M_X$ along the closed immersion $X_m \hookrightarrow X$. Then $(X_m, M_{X_m})$ is a log-scheme over $(\text{Spec } W_m(k), L_m)$ where $L_m$ is the log-structure associated to $\mathbb{N} \to W_m(k)$, $1 \mapsto p$. In the case $m = 1$ we have $(X_1, M_{X_1}) = (Y, M_Y)$.

In order to construct a log-motivic complex $Z_{\log X, \cdot}(n)$ as a co-pro-complex in the derived category in the sense of [BEK14], we need a good definition of log-syntomic complexes. By this we mean a complex that allows us to glue the log-motivic complex $Z_{\log Y}(n)$ defined in [2.8] along a logarithmic (Hyodo-Kato) Hodge-Witt sheaf, using Theorem 3.6 and Theorem 3.10. In [NN16 §3] a complex $\Gamma^\bullet(X, s_{\log}(n))$ is defined and is identified with the homotopy limit of the diagram

$$\begin{array}{ccc}
\Gamma^\bullet_{HK}(X) & \xrightarrow{1 - \varphi_{n-1}} & \Gamma^\bullet_{HK}(X) \\
\downarrow_{(N,0)} & & \downarrow_{(N,0)} \\
\Gamma^\bullet_{HK}(X) & \xrightarrow{1 - \varphi_{n-1}} & \Gamma^\bullet_{HK}(X)
\end{array}$$

(4.0.1)

where $\Gamma^\bullet_{HK}(X)_{Q}$ is the Hyodo-Kato cohomology, $\Gamma_{\text{DR}}$ is induced by the Hyodo-Kato isomorphism and $\varphi_{n}$ is the divided Frobenius $\varphi \frac{p^n}{p^n}$. We will give an equivalent description of $\Gamma^\bullet(X, s_{\log}(n))$ using the logarithmic Hyodo-Kato sheaves. We can reconstruct the commutative diagram (4.0.1) by applying $\Gamma^\bullet$ to a commutative diagram of pro-sheaves in the category $\mathbb{Q} \otimes D_{\text{pro}}(Y_{\text{et}})$, namely

$$\begin{array}{ccc}
\mathbb{Q} \otimes W_{\omega}^{\bullet}_{Y/k} & \xrightarrow{(1 - \varphi_{n-1}, \iota_{\text{DR}})} & \mathbb{Q} \otimes W_{\omega}^{\bullet}_{Y/k} \oplus \omega_{X/W(k)}^{\bullet} \otimes \mathbb{Q}/\text{Fil}^{n} \\
\downarrow_{(N,0)} & & \downarrow_{(N,0)} \\
\mathbb{Q} \otimes W_{\omega}^{\bullet}_{Y/k} & \xrightarrow{1 - \varphi_{n-1}} & \mathbb{Q} \otimes W_{\omega}^{\bullet}_{Y/k}
\end{array}$$

(4.0.2)

Here $\iota_{\text{DR}} : \mathbb{Q} \otimes W_{\omega}^{\bullet}_{Y/k} \to \omega_{X/W(k)}^{\bullet} \otimes \mathbb{Q}$ is the Hyodo-Kato isomorphism [HK94 5.4], where $\omega_{X/W(k)}^{\bullet}$ is the logarithmic de Rham pro-complex induced by $\omega_{X/W(k)}^{\bullet}$ with locally free components $\omega_{X/W(k)}^{i} = \bigwedge^{i} \omega_{X/W(k)}^{1}$, where $\omega_{X/W(k)}^{1}$ is generated by $dt_i/t_i$ for $1 \leq i \leq b$ and $dt_i$ for $i > b$, subject to the relation $\sum_{i=1}^{b} dt_i/t_i = 0$. Using the Hyodo-Kato exact sequence [HK94 Proposition 1.5]

$$0 \to W_{\omega}^{\bullet}_{Y/k}[-1] \xrightarrow{\wedge \theta} W_{\omega}^{\bullet}_{Y/k} \to W_{\omega}^{\bullet}_{Y/k} \to 0$$

(4.0.3)
we can redefine the homotopy limit of (10.2) as
\[(4.0.4)\]
\[\mathcal{S}_{\log,X_i}(n)_{\text{ét}} = \Cone(W_s\tilde{\omega}^\bullet_{Y/k} \otimes \mathbb{Q} \xrightarrow{(1-\varphi_s, \text{id})} W_s\tilde{\omega}^\bullet_{Y/k} \otimes \mathbb{Q} \oplus \omega_X/\mathbb{Q}/\text{Fil}^n \otimes \mathbb{Q}).\]

Here \(\text{Fil}^n\) is the Hodge filtration and \(\text{id}\) is the composite map

\[W_s\tilde{\omega}^\bullet_{Y/k} \otimes \mathbb{Q} \xrightarrow{} W_s\omega^\bullet_{Y/k} \otimes \mathbb{Q} \xrightarrow{\sim} \omega_{X/W(k)} \otimes \mathbb{Q} \xrightarrow{} \omega_{X/W(k)}/\text{Fil}^n \otimes \mathbb{Q}.\]

We can further simplify the construction by introducing the Nygaard complexes on the level of \(W_s\omega^\bullet_{Y/k}\); for each \(s \geq 0\) they are defined via an exact sequence
\[(4.0.5)\]
\[0 \rightarrow N^{-1-s}W_s\omega^s_{Y/k} \rightarrow N^{-s}W_s\omega^s_{Y/k} \rightarrow W_s\omega^s_{Y/k} \rightarrow 0.\]

with relations \(\varphi(\theta) = p\theta, d\theta = \theta d = 0\) and \(V(\theta) = \theta.\)

\[\textbf{Lemma 4.1.}\] For each \(s \geq 0\) there is an exact sequence of pro-complexes

\[0 \rightarrow W_s\omega^s_{Y/k, \log}[s] \rightarrow N^{-s}W_s\omega^s_{Y/k} \xrightarrow{1-\varphi_s} W_s\omega^s_{Y/k} \rightarrow 0\]
on \(Y_{\text{ét}}.\)

\[\textbf{Proof.}\] Consider the following commutative diagram

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 \rightarrow W_s\omega^s_{Y/k, \log}[-s] & \rightarrow N^{-s}W_s\omega^s_{Y/k}[-1] & \xrightarrow{1-\varphi_s} & W_s\omega^s_{Y/k}[-1] \rightarrow 0 \\
\downarrow{\wedge \theta} & \downarrow{\wedge \theta} & \downarrow{\wedge \theta} & \\
0 \rightarrow W_s\omega^s_{Y/k, \log}[-s] & \rightarrow N^{-s}W_s\omega^s_{Y/k} & \xrightarrow{1-\varphi_s} & W_s\omega^s_{Y/k} \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 \\
\end{array}
\]

The vertical sequences are exact; the rightmost sequence is (4.0.3), the middle sequence is (4.0.3) and the leftmost sequence is exact by the definition of \(W_s\omega^s_{Y/k, \log}\) and \(W_s\omega^s_{Y/k, \log}\). The statement of the lemma is therefore equivalent to the exactness of the sequence

\[0 \rightarrow W_s\omega^s_{Y/k, \log}[-s] \rightarrow N^{-s}W_s\omega^s_{Y/k} \xrightarrow{1-\varphi_s} W_s\omega^s_{Y/k} \rightarrow 0.\]

for each \(s \geq 0\). To see this, first note that \(1-\varphi_s : W_s\omega^{s+i}_{Y/k} \rightarrow W_s\omega^{s+i}_{Y/k}\) is an isomorphism for all \(i > 0\) and \(s \geq 0\) by the same proof as ([Il79] I. Lemme 3.30). Next, observe that \(1-\varphi_s : \tau_{<s}N^{-s}W_s\omega^s_{Y/k} \rightarrow \tau_{<s}W_s\omega^s_{Y/k}\) is an isomorphism. Indeed, let \(i \leq s-1\). Then for \(\beta\) a local section of \(W_s\omega^i_{Y/k}\) we have \(\beta = (p^{s-1-i}V - \text{id})\alpha\) where

\[\alpha = -(p^{s-1-i}V) \sum_{m=0}^{\infty} (p^{s-1-i}V)^m \beta,\]

so \(1-\varphi_s\) is surjective. On the other hand, if \(\alpha\) is a local section of \(W_s\omega^i_{Y/k}\) such that \(\alpha = p^{s-1-i}V\alpha\), we get \(\alpha \in (p^{s-1-i}V)^m W_s\omega^i_{Y/k}\).
for all $m \geq 0$, and hence $\alpha = 0$ so $1 - \varphi_s$ is injective. Finally, we must show that the sequence

\[(4.1.1) \quad 0 \to W_r \omega^s_{Y/k, \log} \to W_r \omega^s_{Y/k} / dV W_r - 1 \omega^s_{Y/k} / dV - 1 \omega^s_{Y/k} \to 0\]

is exact for each $r \geq 1$. To see this, consider the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & W_r \omega^s_{Y/k, \log} & \to & W_r \omega^s_{Y/k} & \to & W_r \omega^s_{Y/k} / dV W_r - 1 \omega^s_{Y/k} / dV - 1 \omega^s_{Y/k} & \to 0 \\
\downarrow & & \downarrow & 1 - \varphi_s & \downarrow & & \downarrow & 1 - \varphi_s & \downarrow & & \downarrow & 1 - \varphi_s & \downarrow & \to & 0 \\
0 & \to & W_r \omega^s_{Y/k, \log} & \to & W_r \omega^s_{Y/k} & \to & W_r \omega^s_{Y/k} / dV W_r - 1 \omega^s_{Y/k} / dV - 1 \omega^s_{Y/k} & \to 0 \\
\end{array}
\]

The two vertical sequences are obviously exact, and the middle horizontal sequence is exact by [Lor02] Corollary 2.13. Therefore (4.1.1) is exact if and only if $1 - \varphi_s : dV W_r - 1 \omega^s_{Y/k} / dV - 1 \omega^s_{Y/k}$ is an isomorphism. The map $V : dW_r \omega^s_{Y/k} / W_r + 1 \omega^s_{Y/k}$ factors through $p : W_r \omega^s_{Y/k} \to W_r + 1 \omega^s_{Y/k}$, as

\[
dW_r \omega^s_{Y/k} / \psi \to W_r + 1 \omega^s_{Y/k}
\]

and since $V d = pdV$, the map $\psi$ has image contained in $dV W_n - 1 \omega^s_{Y/k}$. The map $\psi : dV W_r - 1 \omega^s_{Y/k} / dV - 1 \omega^s_{Y/k}$ is an isomorphism. Hence

\[
N^n W_r \omega^s_{Y/k} \otimes \mathbb{Q} \cong W_r \omega^s_{Y/k} \otimes \mathbb{Q}
\]

we get that (4.0.4) is quasi-isomorphic to

\[
\text{Cone}(W_r \omega^s_{Y/k, \log}[-n] \otimes \mathbb{Q} \to \omega^s_{X_r/W(k)} \otimes \mathbb{Q})
\]

We can then modify the definition of $\mathcal{G}_{\log, X_r(n)_{\text{et}}}$ again to get the following interpretation

\[(4.1.2) \quad \mathcal{G}_{\log, X_r(n)_{\text{et}}} = \text{Cone}(W_r \omega^s_{Y/k, \log}[-n] \otimes \mathbb{Q} \to \omega^s_{X_r/W(k)} \otimes \mathbb{Q})
\]

where the map is defined by the composition

\[
W_r \omega^s_{Y/k, \log}[-n] \otimes \mathbb{Q} \to W_r \omega^s_{Y/k}[-n] \otimes \mathbb{Q} \to N^n W_r \omega^s_{Y/k} \otimes \mathbb{Q} \cong W_r \omega^s_{Y/k} \otimes \mathbb{Q} \cong \omega^s_{X_r/W(k)} \otimes \mathbb{Q}
\]

where $W_r \omega^s_{Y/k} \otimes \mathbb{Q} \cong \omega^s_{X_r/W(k)} \otimes \mathbb{Q}$ is the Hyodo-Kato isomorphism. Then we still have that $R \Gamma(X, \mathcal{G}_{\log, X_r(n)_{\text{et}}})$ is quasi-isomorphic to the Nekovář-Niziol complex.
$\Gamma(X, s_{\log}(n))$. By definition, we have an exact triangle

\[(4.1.3) \quad \omega^\geq_{X/k(W(k)} \otimes \mathbb{Q}[-1] \to \mathfrak{S}_{\log, X_\geq(n)} \otimes W_\log Y/k, \log \otimes \mathbb{Q}[n] \to \mathbb{Q}^{1\geq \geq} \to \]

in $\mathbb{Q} \otimes D_{\text{pro}}(Y_{\text{et}})$. Define $\mathfrak{S}_{\log, X_\geq(n)} := \tau_{\leq n} \mathcal{R}_{\epsilon^*} \mathfrak{S}_{\log, X_\geq(n)}$ where $\epsilon : (X_\geq)_{\text{et}} \to (X_\geq)_{\text{Zar}}$ is the morphism of sites. Since we have an isomorphism

$$\epsilon_* \omega^\geq_{X/k(W(k)} \otimes \mathbb{Q}[-1] \cong \mathcal{R}_{\epsilon^*} \omega^\geq_{X/k(W(k)} \otimes \mathbb{Q}[-1]$$

in $D_{\text{pro}}(Y_{\text{Zar}})$, the complex $\mathcal{R}_{\epsilon^*} \omega^\geq_{X/k(W(k)} \otimes \mathbb{Q}[-1]$ has cohomological support in degrees $[1, n]$. By [BEK14] Lemma A.1, applying $\tau_{\leq n} \circ \mathcal{R}_{\epsilon^*}$ to (4.1.3) therefore gives an exact triangle

\[(4.1.4) \quad \omega^\geq_{X/k(W(k)} \otimes \mathbb{Q}[-1] \to \mathfrak{S}_{\log, X_\geq(n)} \otimes W_\log Y/k, \log \otimes \mathbb{Q}[n] \to \mathbb{Q}^{1\geq \geq} \to \]

we need an integral version of the complex $\mathfrak{S}_{\log, X_\geq(n)}$. Let $M_X$ be the divisorial log-structure associated to $Y \to X$, and for each $m \in \mathbb{N}$ let $X_m$ be the pullback log-structure on $X_m$. Let $(\Spec W_m(k), W_m(L)) \to (\Spec W_m(k)[T], \mathcal{L})$ be the closed immersion with log-structure $\mathcal{L}$ on $\Spec W_m(k)[T]$ associated to $\mathbb{N} \to W_m(k)[T]$.

In order to glue a log-syntomic complex with the log-motivic complex $Z_{\log, Y}(n)$ along the logarithmic Hyodo-Kato sheaf $W_\log Y/k, \log$ using the canonical map

$$Z_{\log, Y}(n) \to H^2(\mathfrak{S}_{\log, X_\geq(n)}[n]) \simeq K^\text{Mil}_{\text{log}, X_\geq(n)} \to W_\log Y/k, \log$$

we may therefore use it for the integral definition of the log-syntomic complex due to Kato [Kat94a] and Tsuji [Tsu99], which we now recall. Note that $(Z^*, N^*)$ is smooth over $(\Spec W(k)[T], \mathcal{L})$ and we can assume that

$$\begin{array}{c}
\xymatrix{ (X^*, M^*) \ar[r] & (Z^*, N^*) } \\
\Spec W(k, L) \ar[u] \ar[r]^T & (\Spec W(k)[T], \mathcal{L}) \ar[u]
\end{array}$$

is cartesian. Let $X^m = X^i \otimes \mathbb{Z}/p^m\mathbb{Z}$ and $Z^m = Z^i \otimes \mathbb{Z}/p^m\mathbb{Z}$, with induced log-structures $M^m$ and $N^m$, respectively. We assume that there exists a lifting of Frobenius $F : (Z^*, N^*) \to (Z^*, N^*)$ of the absolute Frobenius on $(Z^*, N^*)$. Let $(D^*_m, M_{D^*_m}, N_{D^*_m}) \to (Z^*_m, N^*_m)$ be the PD-envelope of $(X^i_m, M^i_m) \to (Z^*_m, N^*_m)$, and let $J^m_{D^*_m} \subseteq \mathcal{O}_{D^*_m} = \mathcal{O}_{Z^*_m}$ be the $n$-th divided power of $J^m_{D^*_m} := \ker(\mathcal{O}_{D^*_m} \to \mathcal{O}_{X^*_m})$. Let $J^m_{D^*_m, X^*_m} \subseteq \mathcal{O}_{X^*_m}$ be the complex on the etale site of $X^*_m$ which on each $X^i$ is the complex

$$\begin{array}{c}
J^m_{D^*_m} \otimes \mathcal{O}_{X^*_m} \to J^m_{D^*_m} \otimes \mathcal{O}_{X^*_m} \otimes \mathbb{Z}/p^m \to \cdots \to J^m_{D^*_m} \otimes \mathcal{O}_{X^*_m} \otimes \mathbb{Z}/p^m \to \cdots.
\end{array}$$

Let $\varphi : \mathcal{O}_{D^*_m} \to \mathcal{O}_{D^*_m}$ be the Frobenius induced by $F$. Then we have $\varphi(J^m_{D^*_m} \subseteq p^n \mathcal{O}_{D^*_m}$. Define $-n \varphi : J^m_{D^*_m} \to \mathcal{O}_{D^*_m}$ by $p^n \varphi(a \mod p^m) = b \mod p^m$ for $a \in J^m_{D^*_m}$ and $b \in \mathcal{O}_{D^*_m}$ such that $\varphi(a) = p^n b$. This induces a homomorphism of
complexes $p^{-n}\varphi : j_{m,X}^\log (n) \to j_{m,X}^\log (0)$ which is $p^q\varphi$ on $J_{m,X}^{[n-q]}$ and $p^{-q}\varphi$ on $\omega_{Z/W_m(k)}^q$. We make the assumption that there exist sections $T_1, \ldots, T_d$ of $M_Z$ such that $d\log T_i (1 \leq i \leq d)$ form a basis of $\omega_{Z/W(k)}^1$ and $F^*(T_i) = T_i^p (1 \leq i \leq d)$ (see [Tsu99 (2.1.1)]). Define $s_{m,X}^\log (n)$ to be the mapping fibre of $1-p^{-n}\varphi : j_{m,X}^\log (n) \to j_{m,X}^\log (0)$, and set $s_{m,X}^\log (n) = R\theta_* s_{m,X}^\log (n)$ where $\theta : (X')_{\et} \to (X)_{\et}$ is the morphism induced by the hypercovering $X^\bullet \to X$.

Next we shall recall Tsuji’s definition of log-syntomic regulators [Tsu99 §2.2]. Let $C_m$ be the complex (which is quasi-isomorphic to $\theta^\ast M_{X_m}^{gp} [-1]$) given by
\[
1 + JD_m \to M_{D_m}^{gp} \\
\deg 0 \quad \deg 1
\]
Define a homomorphism $C_{m+1} \to s_{m,X}^\log (1)$ given by
\[
1 + J_{D_{m+1}} \to s_{m,X}^\log (1) = JD_m \\
a \mapsto \log a \mod p^m
\]
in degree 0 and
\[
M_{D_{m+1}}^{gp} \to s_{m,X}^\log (1) = O_{D_m} \otimes_{\mathcal{O}_{D_m}} \omega_{Z/W_m(k)}^1 \oplus O_{D_m} \\
b \mapsto (d\log b \mod p^m, p^{-1}\log b^p \varphi(b)^{-1})
\]
in degree 1. Note that $\log(b^p \varphi(b)^{-1})$ is in $pO_{D_{m+1}} \subset O_{D_m}$ because $b^p \varphi(b)^{-1} \in 1 + pO_{D_{m+1}}$. By composing with $R\theta_*$ we get a map
\[
(4.1.5) \quad M_{X_{m+1}}^{gp} \to s_{m,X}^\log (1)[1].
\]
For any $0 \leq n, n', n + n' \leq p - 1$ there is a product structure
\[
s_{m,X}^\log (n) \otimes s_{m,X}^\log (n') \to s_{m,X}^\log (n + n')
\]
[Tsu99 §2.2]. Applying $R\theta_*$ gives
\[
(4.1.6) \quad s_{m,X}^\log (n) \otimes s_{m,X}^\log (n') \to s_{m,X}^\log (n + n').
\]
Together, (4.1.5) and (4.1.6) induce symbols maps
\[
(M_{X_{m+1}}^{gp})^{\otimes q} \to \mathcal{H}^q(s_{m,X}^\log (q))
\]
for each $q \geq 0$ [Tsu99 (2.2.1)]. These constructions are independent of the choice of embedding system and lifting of Frobenius. We have the following

**Proposition 4.2.** [Tsu99 Lemma 3.4.11, Proposition 2.4.1] The symbol map $(M_{X_{m+1}}^{gp})^{\otimes q} \to \mathcal{H}^q(s_{m,X}^\log (q))$ is surjective.

It follows from [NN16 Proposition 3.8] that the complex $R\Gamma(X, s_{m,X}^\log (n) \otimes \mathbb{Q})$ is isomorphic to the complex $R\Gamma(X, \mathcal{O}_{log,X}(n))$ which we defined before. A crucial point for this comparison is the existence of an isomorphism
\[
R\Gamma(X/W(k), \mathcal{O}_{cris}/J_X^{[n]}_{X/W(k)} ) \cong R\Gamma_{dR}(X_K)/\text{Fil}^n
\]
which links the log-crystalline cohomology of $(X,M_X)$ over $W(k)$ equipped with the trivial log-structure to the de Rham cohomology of the generic fibre. This is
proven in [NN16, Corollary 2.4] and is a consequence of Beilinson’s comparison ([NN16, Theorem 2.1]) using derived log de Rham complexes ([Bei13, (1.9.2)]). It was also proven in [Lan99, Lemma 2.7] based on the original proof of Kato-Messing [KM92, Lemma 4.5] for syntomic schemes in the absence of log-structures.

For a $W(k)$-scheme $X$ with semistable reduction let $X_m = X \times_{\text{Spec} W(k)} \text{Spec} W_m(k)$, and write $i_m : X_m \hookrightarrow X$ for the closed immersion. Then $X_m$ is equipped with the log-structure $M_{X_m} = i_m^* M_X$ locally defined by

$$N^r \to \mathcal{O}_{X_m}$$

$$\pi_i \mapsto \pi_i^{(m)}$$

if $X$ is locally given by $\text{Spec} W(k)[T_1, \ldots, T_n]/(\pi_1 \cdots \pi_r - p)$ and where $\pi_i^{(m)} = \pi_i \mod p^m$. Let $j : X_{sm}^m \hookrightarrow X_m$ be the open subscheme of $X_m$ such that $X_{sm}^m \to \text{Spec} W_m(k)$ is smooth. We consider the log-structure $N_{X_m}$ associated to

$$N^r \to \mathcal{O}_{X_m}$$

$$\pi_i \mapsto g_i^{(m)} := \pi_i^{(m)} + \prod_{j \neq i} \pi_j^{(m)}$$

Then $N_{X_m} := \text{Im}(N_{X_m}) \subset \mathcal{O}_{X_m} \cap j_*(\mathcal{O}_{X_m}^\text{sm})$.

Remark 4.3. We have an exact sequence (with $X_1 = Y$)

$$0 \to U_1 \to \Lambda_{X_m}^{\text{gp}} \to \Lambda_Y^{\text{gp}} \to 0$$

where $U_1 := (1 + px \mid x \in \mathcal{O}_{X_m})$.

Define $\mathcal{K}^{\text{Mil}}_{\log, X_m, I}$ to be the Zariski sheafification of the presheaf on $X_m$ given by

$$U \mapsto (\Lambda_{X_m}(U)^\text{gp})^\otimes \mathcal{O}_{X_m} / I_m$$

where $I_m$ is the subgroup generated by elements of the form $a \otimes (1 - a)$ with $a, 1 - a \in \Lambda_{X_m}(U)^\text{gp}$, those of the form $a \otimes (-a)$ with $a \in \Lambda_{X_m}(U)^\text{gp}$, and those of the form $g_i^{(m) \mid I} x \otimes (1 - \pi_i^{(m) \mid I})$ ranging over subsets $I \subset \{1, \ldots, r\}$, where $g_i^{(m) \mid I} := \prod_{r \in I} g_i^{(m) \mid r}$ with $n_i \geq 0$, $\pi_i^{(m) \mid I} := \prod_{r \in I} \pi_i^{(m) \mid r}$ with $n_i \geq 0$, and $x \in \mathcal{O}_{X_m}(U)^\text{*.}$ Consider again the surjective symbol map

$$(M_{X_{m+1}}^{\text{gp}})^{\otimes q} \to \mathcal{H}^q(s_{m, X}^{\log}'(q))$$

of [Tsu99] and [Kat87]. For $x \in \mathcal{O}_{X_{m+1}}(U)^*$, $R_{\log-\text{syn}}(x) \in H^1(s_{m, X}^{\log}'(1))$ is defined as for $M_{X_{m+1}}^{\text{gp}}$. For $g_i^{(m+1)} \in \Lambda_{X_{m+1}}(U)$, define $R_{\log-\text{syn}}(g_i^{(m+1)}) := R_{\log-\text{syn}}(e_i)$ where $e_i \in M_{X_{m+1}}$ is the element mapping to $\pi_i^{(m+1)}$ under $M_{X_{m+1}} \to \mathcal{O}_{X_{m+1}}$. This extends to a map

$$R_{\log-\text{syn}} : (\Lambda_{X_{m+1}}^{\text{gp}})^{\otimes q} \to \mathcal{H}^q(s_{m, X}^{\log}'(q)(\mathcal{E}))$$

which factors through

$$(4.3.1) R_{\log-\text{syn}} : K_{\log, X_{m+1}, \mathcal{E}}^{\text{Mil}} \to \mathcal{H}^q(s_{m, X}^{\log}'(q)(\mathcal{E})).$$

Proposition 4.4. We have an exact sequence

$$0 \to \frac{p^2 d\omega_{X_m/W_m(k)}^{-1}}{p d\omega_{X_m/W_m(k)}^{q-2}} \to \mathcal{H}^q(s_{m, X}^{\log}'(q)(\mathcal{E})) \to W_m \omega_{Y/k, \log}^q \to 0$$
where the second map is defined by

\[(d \log b_1 \wedge \cdots \wedge d \log b_q, s_{\varphi,q}([b_1, \ldots, b_q])) \mapsto d \log b_1 \wedge \cdots \wedge d \log b_q\]

for \(b_i \in M^{p,\mathrm{gp}}_{X_{m+1}}\), where \(s_{\varphi,q}([b_1, \ldots, b_q])\) is defined as in \([\text{Kur}98, 2.7 \text{pg 208}]\) (see also \([\text{Lsu}99, \text{Lemma 2.4.6}]\)), where \(b_i\) is the image of \(b_i\) in \(M^{p,\mathrm{gp}}_{X}\) and \(d \log\) is the Hyodo-Kato map \([\text{HK}94, (1.1)]\). The first map is defined as follows: take a lifting of \(z = p\omega \in p\omega^{q-1}_{X_m/W_{m(k)}}\) in \(p\mathcal{O}_{D_m} \otimes_{\mathcal{O}_m} \omega^{q-1}_{Z_m/W_{m(k)}}\), say \(px\omega\) with \(\omega = d \log b_2 \wedge \cdots \wedge d \log b_q\). Then the image of \(z\) under the first map is the class of

\[(d \log (\exp px)\omega, s_{\varphi,q}([\exp(px), b_2, \ldots, b_q])).\]

It is clear that the class of this element is well-defined in \(\mathcal{H}^q(s_{m,X}^{\log, \prime}(q))\). Note that, in order to simplify the notation, we omit the index and work with an embedding \(X_m \hookrightarrow D_m\).

**Proof.** We recall that there is an isomorphism between the cohomology of the original log-syntomic complex of Kato and Tsuji and the sheaf of \(p\)-adic vanishing cycles:

\[\mathcal{H}^q(s_{m,X}^{\log, \prime}(q)) \cong i^* R^j j_* \mathbb{Z}/p^m(q)\]

for \(q < p\) (see \([\text{Lsu}99, \text{Theorem 3.2.2}]\)) and the sheaf \(i^* R^j j_* \mathbb{Z}/p^m(q)\) is generated by symbols, that is the map

\[i^* j_* \mathcal{O}_{X_K}^s \otimes \cdots \otimes i^* j_* \mathcal{O}_{X_K}^s \to i^* R^j j_* \mathbb{Z}/p^m(q)\]

defined by taking the cup-product of the boundary map

\[i^* j_* \mathcal{O}_{X_K}^s \to i^* R^j j_* \mathbb{Z}/p^m(1)\]

arising from the Kummer sequence, is surjective. Moreover, \(i^* R^j j_* \mathbb{Z}/p^m(q)\) is equipped with a filtration \(U^0 \supset U^1 \supset \cdots\) such that \(U^0/U^1\) is isomorphic to \(W_m \omega^{q-1}_{X_{m}/k, \log}\), the sheaf defined in \([\text{Lsu}99, \text{Lemma 4.1}]\) (see \([\text{Hyd}88, \text{Theorem 1.6}]\)), and \(U^1\) is generated by symbols \(\{i^*(1 + px), x_2, \ldots, x_q\}\) with \(z \in \mathcal{O}_X\) and \(x_i \in i^* j_* \mathcal{O}_{X_K}^s, i = 2, \ldots, q\). This shows that the kernel of

\[\mathcal{H}^q(s_{m,X}^{\log, \prime}(q), \mathcal{E}) \to W_m \omega^{q-1}_{X_{m}/k, \log}\]

consists of classes of elements where the first component is of the form

\[d \log (1 + px) \wedge d \log b_2 \wedge \cdots \wedge d \log b_q\]

with \(x \in \mathcal{O}_{D_i}\) and \(b_i \in M^{p,\mathrm{gp}}_{D_{m+1}}, i = 2, \ldots, q\). This element is the image of \(\log(1 + px) \wedge d \log b_2 \wedge \cdots \wedge d \log b_q\), (where \(x\) is the image of \(x\) in \(\mathcal{O}_{X_m}\) and \(b_i\) is the image of \(b_i\) in \(M^{p,\mathrm{gp}}_{X_m}\)), which is an element of \(p\omega^{q-1}_{X_m/W_{m(k)}}\).

We show that the kernel of \(p\omega^{q-1}_{X_m/W_{m(k)}} \to \mathcal{H}^q(s_{m,X}^{\log, \prime}(q), \mathcal{E})\) contains \(p^2 d \omega^{q-2}_{X_m/W_{m(k)}}\). If \(p\omega\) is a lifting of \(p\omega\), then a necessary condition for the image of \(p\omega\) to vanish is that \(p\omega\) is closed. If \(p\omega = p\log b_1 \wedge \cdots \wedge d \log b_{q-1}\) then

\[s_{\varphi,q}([\exp(p), b_1, \ldots, b_{q-1}]) = \left(\frac{\varphi}{p^q} - 1\right) p\omega\]

\[= \frac{\varphi(p)}{p} \cdot \frac{\varphi}{p^{q-1}} (d \log b_1 \wedge \cdots \wedge d \log b_{q-1}) - p \log b_1 \wedge \cdots \wedge d \log b_{q-1}\]

\[= (1 - p)(d \log b_1 \wedge \cdots \wedge d \log b_{q-1}) \text{ modulo an exact form} \]
hence is not exact. The same argument holds for any other multiple \( c \hat{\omega}, c \in \mathcal{W}_m(k) \).

Hence for \( p \hat{\omega} \) to vanish in \( \mathcal{H}^q(s_{\log, X}^{\log, \prime}(q)_{\etale}) \) it is necessary that

\[
p\hat{\omega} = pdz = pdb_1 \wedge \frac{db_2}{b_2} \wedge \cdots \wedge \frac{db_q}{b_q-1} = pdb_1 \wedge d \log b_2 \wedge \cdots \wedge d \log b_q-1.
\]

The second component of the image of \( p\hat{\omega} \) is then \( s_{\varphi, q}(\{\exp(pb_1), b_1, \ldots, b_{q-1}\}) \). In order to decide whether it is a boundary of an element in \( \mathcal{O}_D \otimes_{\mathcal{O}_m} \omega_{Z_m/W_m(k)}^{q-2} \) it suffices to consider the case \( q = 2 \) (the proof shows that the general case follows from this using the formula for \( s_{\varphi, q} \) in [Kur98]). Then

\[
s_{\varphi, 2}(\{\exp(pb_1), b_1\}) = \frac{1}{p} \log \left( \frac{\exp(\varphi(pb_1))}{\exp(p^2b_1)} \right) \left( \frac{1}{p} d \log \varphi(b_1) \right) - \frac{1}{p} \log \frac{\varphi(b_1)}{b_1^q} d(pb_1)
\]

\[
= (\varphi(b_1) - pb_1) \frac{1}{p} d \log \varphi(b_1) - \frac{1}{p} \log \frac{\varphi(b_1)}{b_1^q} d(pb_1)
\]

\[
= \frac{1}{p} d\varphi(b_1) - b_1 d \log \varphi(b_1) - \log \frac{\varphi(b_1)}{b_1^q} db_1
\]

Let \( \varphi(b_1) = b_1^q + px \). Then the above continues as

\[
s_{\varphi, 2}(\{\exp(pb_1), b_1\}) = \frac{1}{p} d\varphi(b_1) - b_1 d \log b_1^{q-1} \left( 1 + \frac{px}{b_1^q} \right) - \log \left( 1 + \frac{px}{b_1^q} \right) db_1
\]

\[
= \frac{1}{p} d\varphi(b_1) - pb_1 - b_1 d \log \left( 1 + \frac{px}{b_1^q} \right) - \log \left( 1 + \frac{px}{b_1^q} \right) db_1
\]

\[
= \frac{1}{p} d\varphi(b_1) - pb_1 - db_1 \log \left( 1 + \frac{px}{b_1^q} \right)
\]

\[
= \frac{1}{p} d\varphi(b_1) \text{ modulo an exact form}.
\]

Therefore \( s_{\varphi, 2}(\{\exp(pb_1), b_1\}) \) is exact if \( b_1 = pb_1 \) for some \( b_1 \), which gives \( p \hat{\omega} = p^2dz' \), hence \( p\hat{\omega} = p^2 \hat{\omega} \) for \( \omega \in \omega_{Z_m/W_m(k)}^{q-2} \).

We have shown that \( s_{\varphi, 2}(\{\exp(pb_1), b_1\}) \) vanishes in \( \mathcal{H}^q(s_{\log, X}^{\log, \prime}(q)) \) if and only if \( b_1 = pb_1 \), yielding an injection \( p^q \omega_{X_m/W_m(k)}^{q-1}/p^2 \omega_{X_m/W_m(k)}^{q-2} \hookrightarrow \mathcal{H}^q(s_{\log, X}^{\log, \prime}(q)) \) in analogy to the good reduction case considered in [BEK14]. This completes the proof of Proposition 4.4. □

We define \( s_{X, \log}^{\log, \prime}(n) := \tau_{\leq n} R\epsilon_* s_{X, \log}^{\log, \prime}(n)_{\etale} \) where \( \epsilon : (X, \epsilon)_{\etale} \rightarrow (X, \epsilon)_{\mathcal{Zar}} \) is the morphism of sites.

**Definition 4.5.** Since \( s_{X, \log}^{\log, \prime}(n) \) is acyclic in degrees \( > n \), we can define the logarithmic pro-complex \( \mathbb{Z}_{\log, X}(n) \) in \( D_{pro}(Y_{\mathcal{Zar}}) \) via the homotopy cartesian diagram

\[
\begin{array}{ccc}
\mathbb{Z}_{\log, X}(n) & \longrightarrow & \mathbb{Z}_{\log, Y}(n) \\
\downarrow & & \downarrow d \log \\
\mathbb{Z}_{\log, X}^{\log, \prime}(n) & \longrightarrow & \mathcal{H}^n(s_{X, \log}^{\log, \prime}(n)_{\etale} \hookrightarrow W_* \hat{\omega}_{Y/k, \log}^{n} \rightarrow n)
\end{array}
\]
where “dlog” is defined using that \( \mathbb{Z}_{\log,Y}(n) \) is acyclic in degrees \( > n \) by definition, the map \( \lambda : \mathcal{H}^n(\mathbb{Z}_{\log,Y}(n)) \to K^\text{Mil}_{\log,Y,n} \), and the map dlog defined after Proposition 3.9.

We do not quite have a semistable analogue of the fundamental triangle in [BEK14] Theorem 5.4. In any case, we have such a triangle by considering 

\[ R\Gamma(X, s_{\log}^1(n)_{\mathbb{Q}}(n)) \]

namely we have an exact triangle

\[ (4.5.1) \quad R\Gamma(X, \omega_{X,W(k)} \otimes \mathbb{Q}[-1]) \to R\Gamma(X, s_{\log}^1(n)_{\mathbb{Q}}(n)) \to R\Gamma(Y, W, \omega_{Y/k,\log}^n[-n] \otimes \mathbb{Q}) \to +1 \]

which is sufficient for proving our main result Theorem 4.10. The point is that we have a corresponding triangle for the Nekovář-Nizioł complex \( R\Gamma(X, \mathcal{G}_{\log,Y,n}) \).

As in the smooth case we have

**Lemma 4.6.** The log-motivic pro-complex \( \mathbb{Z}_{log,X}(n) \) is acyclic in degrees \( > n \).

**Proof.** Note that \( s_{\log}^1(n)_{\mathbb{Q}}(n) \) and \( \mathbb{Z}_{\log,Y}(n) \) are acyclic in degrees \( > n \). By the definition of \( \mathbb{Z}_{log,X}(n) \) we have a long exact sequence

\[ \cdots \to \mathcal{H}^i(\mathbb{Z}_{log,X}(n)) \to \mathcal{H}^i(s_{\log}^1(n)_{\mathbb{Q}}(n)) \oplus \mathcal{H}^i(\mathbb{Z}_{\log,Y}(n)) \to \mathcal{H}^i(W, \omega_{Y/k,\log}^n[-n]) \to \cdots . \]

Since \( \mathcal{H}^n(s_{\log}^1(n)_{\mathbb{Q}}(n)) \to W, \omega_{Y/k,\log}^n \) is surjective, \( \mathbb{Z}_{log,X}(n) \) is acyclic in degrees \( > n \). □

**Proposition 4.7.** Suppose that \( k \) is infinite. For each \( n \geq 0 \) (with \( n < p \)) there is a canonical isomorphism

\[ \mathcal{H}^n(\mathbb{Z}_{log,X}(n)) \cong K^\text{Mil}_{log,X,n} \]

in \( Sh_{pro}(\mathcal{Y}_{zar}) \).

**Proof.** The exact sequences

\[ 0 \to p\omega_{X,W(k)}^{n-1}/p^2d\omega_{X,W(k)}^{n-2} \to \mathcal{H}^n(s_{\log}^1(n)_{\mathbb{Q}}(n)) \to W, \omega_{Y/k,\log}^n \to 0 \]

and

\[ 0 \to \mathcal{H}^n(\mathbb{Z}_{log,X}(n)) \to \mathcal{H}^n(s_{\log}^1(n)_{\mathbb{Q}}(n)) \oplus \mathcal{H}^n(\mathbb{Z}_{\log,Y}(n)) \to W, \omega_{Y/k,\log}^n \to 0 \]

induce the exact sequence at the bottom of the following commutative diagram

\[ \begin{array}{cccccc}
0 & \to & U^{1}K_{log,X,n}^\text{Mil} & \to & K_{log,X,n}^\text{Mil} & \to & K_{log,Y,n}^\text{Mil} & \to & 0 \\
& & \text{Rlog-syn} & \downarrow & \text{Rlog-syn} & \downarrow & \text{Rlog-syn} & \downarrow & \\
0 & \to & p\omega_{X,W(k)}^{n-1}/p^2d\omega_{X,W(k)}^{n-2} & \to & \mathcal{H}^n(\mathbb{Z}_{log,X}(n)) & \to & \mathcal{H}^n(\mathbb{Z}_{\log,Y}(n)) & \to & 0 \\
\end{array} \]

The right vertical map is the isomorphism in Proposition 3.6 and the map \( R_{\text{log-syn}} \) is induced by the log-syntomic regulator [1.3.1]. We shall show the map \( (*) \) is an isomorphism. Note that \( (*) \), which is the restriction of \( R_{\text{log-syn}} \), turns out to be the inverse of the exponential map

\[ (4.7.1) \quad \text{Exp} : \frac{p\omega_{R_m,W_m(k)}^{n-1}/p^2d\omega_{R_m,W_m(k)}^{n-2}}{R_m,W_m(k)} \to U^{1}K_{log,n}^\text{Mil} (R_m) \]
induced by \( pad \log b_1 \wedge \cdots \wedge d \log b_{r-1} \mapsto \{ \exp(pa), b_1, \ldots, b_{r-1} \} \), where \( R_m \) is a local ring on the syntomic scheme \( X_m/W_m(k) \) which is flat. Indeed, the following facts cited in [BEK14: §12] also hold for the ring \( R_m \):

- \( k^\text{Mil}_{\log,Y,n} \) is \( p \)-torsion free. Indeed, \( k^\text{Mil}_{\log,Y,n} \) injects into \( k^\text{Mil}_{Y,x,m,n} \) which is \( p \)-torsion free.
- \( U^1k^\text{Mil}_{\log,n}(R_m) \) is \( p \)-primary torsion of finite exponent. The proof using pointy bracket symbols for \( K_2(R,pR) \) passes over verbatim.

The existence of the exponential map also holds more generally for rings satisfying the assumption 2.1 in [Kur98], so we have

\[
\text{Exp} : \frac{p\omega_{R_m/W_m(k)}^{n-1}}{p^2d\omega_{R_m/W_m(k)}^{n-2}} \rightarrow (k^\text{Mil}_{\log,n}(R_m))^\wedge
\]

into the \( p \)-adic completion. Then steps 1 and 2 in the proof of [BEK14 Theorem 12.3] carry over to show the existence of \( \{1,1\} \). Since [Kur98 Corollary 1.3] holds for more general rings including \( R_m \), \( \text{Exp} \) vanishes on \( p^2d\omega_{R_m/W_m(k)}^{n-2} \). It is clear that \((\ast)\) composed with \( \text{Exp} \) is the identity on \( p\omega_{R_m/W_m(k)}^{n-1}/p^2d\omega_{R_m/W_m(k)}^{n-2} \), so it remains to show that \( \text{Exp} \) is surjective.

Define \( G_n = p\omega_{R_m/W_m(k)}/p^2d\omega_{R_m/W_m(k)}^{n-2} \) and define a filtration \( U^iG_n \) by defining \( U^iG_n \) to be the image of \( p\omega_{R_m/W_m(k)}^{n-1} \) in \( G_n \). Inductively define subsheaves

\[
0 = B_0 \subset B_1 \subset \cdots \subset Z_2 \subset Z_1 \subset Z_0 = \omega_{R_1/k}^q
\]

using the inverse Cartier operator \( C^{-1} \) by the formulae

\[
B_1 = d\omega_{R/k}^{q-1}
\]

\[
Z_1 = \ker \left( d : \omega_{R/k}^q \rightarrow \omega_{R/k}^{q+1} \right)
\]

\[
C^{-1} : B_s \simto B_{s+1}/B_1
\]

\[
C^{-1} : Z_s \simto Z_{s+1}/B_1
\]

as in [Hy08 (1.5)]. Then the analogue of [Hil79 I. Proposition 2.2.8] holds: \( B_i \) is locally generated by sections of the form \( x_1^{r'} \cdot d\log x_1 \wedge \cdots \wedge d\log x_n, x_j \in \mathcal{N}_Y \), \( 0 \leq r \leq i-1 \). Define a filtration \( U^i(k^\text{Mil}_{\log,n}(R_m)) \) of \( k^\text{Mil}_{\log,n}(R_m) \) by setting \( U^i(k^\text{Mil}_{\log,n}(R_m)) \) to be the subgroup generated by symbols of the form \( \{1+p^i x_1, x_2, \ldots, x_n\} \) where \( x_1 \in R_m \) and \( x_2, \ldots, x_n \in N_{R_m} \). Then \( U^1k^\text{Mil}_{\log,n}(R_m) = \ker \left( k^\text{Mil}_{\log,n}(R_m) \rightarrow k^\text{Mil}_{\log,n}(R_1) \right) \).

For each \( i \geq 1 \), the analogue of [Kur88 Lemma 2.3.2] holds: the map

\[
\lambda_i : \omega_{R_1/k}^{n-1} \rightarrow \text{gr}^i k^\text{Mil}_{\log,n}(R_m)
\]

\[
ad \log b_1 \wedge \cdots \wedge d \log b_{n-1} \mapsto \{1+p^i \hat{a}, \hat{b}_1, \ldots, \hat{b}_{n-1} \}
\]

(where \( \hat{a} \) and the \( \hat{b}_i \) are liftings of \( a \) and the \( b_i \) to \( R_m \)) annihilates \( B_{i-1} \), hence induces a map

\[
\omega_{R_1/k}^{n-1}/B_{i-1} \rightarrow \text{gr}^i k^\text{Mil}_{\log,n}(R_m).
\]

By the obvious semistable analogue of [Hil79 I. Corollaire 2.3.14 (b)] (see also [Hy08 (2.6)]) we have an isomorphism

\[
\omega_{R_1/k}^{n-1}/B_{i-1} \simeq \text{gr}^i G_n.
\]
On the other hand, consider the composite map
\[ \omega_{R^1/k/B_i}^{n-1} \xrightarrow{\lambda_i} \text{gr}^i K_{\log,n}(R_m) \to \text{gr}^i G_n \xrightarrow{\sim} \omega_{R^1/k/B_{i+1}}^{n-1} \]
which coincides with the inverse Cartier operator, which is injective. The second arrow is by definition surjective. Since the first map is also surjective, the second map is an isomorphism. Hence Exp is an isomorphism between \( \frac{p^\infty R_m/W_m(k)}{p^\infty d\omega_{R_m/W_m(k)}} \) and \( U^1 K_{\log,n}(R_m) \). This completes the proof of Proposition 4.7.

\[ \square \]

**Remark 4.8.** Without the assumption that \( k \) is infinite, we should replace the logarithmic Milnor \( K \)-theory pro-sheaf \( K_{\log,X,n}^\text{Mil} \) with the improved logarithmic Milnor \( K \)-theory pro-sheaf \( \tilde{K}_{\log,X,n}^\text{Mil} \) along the lines of Remark 3.7(ii). With this modification, Proposition 4.7 also holds when \( k \) is finite by the same proof. Notice that Proposition 4.7 holds in weight \( n = 1 \) without modification, because \( K_{\log,X,1}^\text{Mil} = K_{\log,X,1}^\text{Mil} \).

**Proposition 4.9.** The log-motivic pro-complex of weight one \( Z_{\log,X}(1) \) is quasi-isomorphic to \( N^\text{gp} X [-1] \), hence

\[ H^2_{\text{cont}}(Y, Z_{\log,X}(1)) \cong H^1_{\text{Zar}}(X, N^\text{gp} X). \]

If \( X \) is proper over Spec \( W(k) \) then we have \( H^2_{\text{cont}}(Y, Z_{\log,X}(1)) \cong H^1_{\text{Zar}}(X, N^\text{gp} X) \) where \( N^\text{gp} X \) will be defined in the proof.

**Proof.** We have a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & 1 & + & p^\infty_O X [-1] & \xrightarrow{\log} & N^\text{gp} X [-1] & \xrightarrow{\cong} & N^\text{gp} Y [-1] & \to & 0 \\
& & & \downarrow & & & \downarrow & & \cong & & \\
0 & \to & p^\infty_O X [-1] & \xrightarrow{\cong} & Z_{\log,X}(1) & \to & Z_{\log,Y}(1) & \to & 0
\end{array}
\]

where the left vertical arrow is the \( p \)-adic logarithm, which is an isomorphism, and the right vertical arrow is from Proposition 2.13. By Lemma 4.6 we have \( H^i(Z_{\log,X}(1)) = 0 \) for all \( i \geq 2 \). The first map \( (d, \bar{\omega}) : J_{D_m} \to \hat{\omega}_{D_m/W_m(k)} \otimes O_{D_m} \) in the definition of \( s_{m,X}^\text{log} \) is injective, so \( s_{m,X}^\text{log} (1) \) is acyclic in degrees \( \neq 1 \). Since \( H^0(Z_{\log,Y}(1)) = 0 \) by Corollary 2.14 we conclude from the sequence in the proof of Lemma 4.6 that \( H^0(Z_{\log,X}(1)) = 0 \). The middle vertical arrow is induced from the canonical map (compatible with \( W_{n} \omega_{1,Y_{\log}[\log][-1]} \)) \( N^\text{gp} X [-1] \to s_{m,X}^\text{log} (1) \) (defined in the same way as for \( M^\text{gp} X [-1] \)) and the reduction map \( N^\text{gp} X \to N^\text{gp} Y \). This proves the first statement of the proposition.

For the second statement, we first need to define \( N^\text{gp} X \). We will do this locally, so let \( U \subset X \) be an open such that \( U = \text{Spec} W(k)[T_1, \ldots, T_n]/(f_1 \cdots f_r - p) \) and such that the log-structure \( M_X = j_* O_{X/k} \) is associated to

\[
\begin{align*}
\mathbb{N} & \to O_X(U) \\
e_i & \mapsto f_i
\end{align*}
\]
Define the log-structure \( N_X \) by
\[
\mathbb{N}^r \to \mathcal{O}_X(U) \\
e_i \mapsto f_i + \prod_{j=1 \atop j \neq i} f_j.
\]

Since \( X \) is regular, it is integral, so \( \mathcal{O}_X(U) \) is an integral domain, and \( \mathcal{O}_X(U) \setminus \{0\} \) is a multiplicative monoid. We obtain a homomorphism of monoids
\[
N_X(U) \to (\mathcal{O}_X(U) \setminus \{0\})\text{gp}.
\]

Define \( N_X^{\text{gp}} \) as the image of \( N_X \) inside \( (\mathcal{O}_X(U) \setminus \{0\})\text{gp} \). Then we have the canonical reduction map for each \( m \)
\[
N_X^{\text{gp}} \rightarrow N_X^{\text{gp}}_m.
\]

Note that \( N_X^{\text{gp}} \) is, in general, not contained in \( j_* \mathcal{O}_{X_K} \), hence is very different from \( M_X^{\text{gp}} \).

Now consider the short exact sequence associated to taking continuous cohomology of pro-sheaves:
\[
0 \to \lim_{\leftarrow m} \mathbb{H}^1_{\text{Zar}}(Y, \mathbb{Z}_{\log, X_m}(1)) \to \mathbb{H}^2_{\text{cont}}(Y, \mathbb{Z}_{\log, X}(1)) \to \lim_{\leftarrow m} \mathbb{H}^2_{\text{Zar}}(Y, \mathbb{Z}_{\log, X_m}(1)) \to 0.
\]

By the first part of the proposition, the middle entry of the sequence is \( H^1_{\text{cont}}(Y, N_X^{\text{gp}}) \). Applying the first part of the proposition to the first and final entries in the sequence yields \( \lim_{\leftarrow m} \mathbb{H}^1_{\text{Zar}}(Y, \mathbb{Z}_{\log, X_m}(1)) \simeq \lim_{\leftarrow m} H^0(Y, N_X^{\text{gp}}) = 0 \) (because the system \( \{H^0(Y, N_X^{\text{gp}})\}_m \) is Mittag-Leffler), and \( \lim_{\leftarrow m} \mathbb{H}^2_{\text{Zar}}(Y, \mathbb{Z}_{\log, X_m}(1)) \simeq \lim_{\leftarrow m} H^1_{\text{Zar}}(Y, N_X^{\text{gp}}_m) \). In particular, we have \( H^1_{\text{cont}}(Y, N_X^{\text{gp}}) \simeq \lim_{\leftarrow m} H^1_{\text{Zar}}(Y, N_X^{\text{gp}}_m) \).

Now consider the following commutative diagram with exact rows
\[
H^0(N_X^{\text{gp}}) \rightarrow H^1(1 + p\mathcal{O}_X) \rightarrow \lim_{\leftarrow m} H^1(N_X^{\text{gp}}_m) \rightarrow H^1(N_X^{\text{gp}}) \rightarrow H^2(1 + p\mathcal{O}_X)
\]

where \( \hat{X} \) is the formal completion of \( X \) along the special fibre. If \( X \) is proper over \( \text{Spec } W(k) \) then the second and fifth vertical arrows in the diagram are isomorphisms by formal GAGA, so the middle arrow is also an isomorphism. That is,
\[
\mathbb{H}^2_{\text{cont}}(Y, \mathbb{Z}_{\log, X}(1)) \cong H^1_{\text{cont}}(Y, N_X^{\text{gp}}) \cong \lim_{\leftarrow m} H^1(Y, N_X^{\text{gp}}_m) \cong H^1(X, N_X^{\text{gp}}).
\]

□

We now have enough to obtain our main result: a generalisation to the semistable case of “the formal part” of the \( p \)-adic variational Hodge conjecture à la [BEK14]. In the following we use of the continuous cohomology of pro-complexes, see [Jan88] and [BEK14 Appendix B].
Theorem 4.10. Let \( n < p \). Let \( X \) be a proper regular flat scheme over \( \text{Spec} \, W(k) \) with semistable reduction. Let \( z \in H^n_{dR}(X/W(k))Q \). Then its log-crystalline Chern class \( c_{\text{cryst}}(z) \in H^n(Y, W_\omega_{Y,k,log}^n) \otimes Q \to H^n_{dR}(X/W(k))Q \). This follows from \([\text{Nee01, Lemma 1.4.4}]\). From this we have the top two rows of the following commutative diagram

\[
\begin{array}{c}
\text{Theorem 6.1.} \\
\text{We see from this diagram that } z \in H^n(Y, W_\omega_{Y,k,log}^n) \text{ if and only if its Chern class } c_{\text{cryst}}(z) \text{ is in } \text{Fil}^n \text{ under the Hyodo-Kato isomorphism.} \tag{4.11} \\
\text{Remark 4.11. Although we do not reprove Yamashita’s result for the logarithmic Picard group \([\text{Yam11 \S 3}]\), we point out that the } p\text{-adic deformation theory of both } H^1(Y, N_{\text{log}}^p) \text{ and } \text{Pic}^0(Y) \text{ coincide. We have exact sequences}
\end{array}
\]
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