A GLOBAL SPACE-TIME ESTIMATE FOR DISPERSIVE OPERATORS
THROUGH ITS LOCAL ESTIMATE

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Abstract. We will show that a local space-time estimate implies a global space-time estimate
for dispersive operators. In order for this implication we consider a Littlewood-Paley type
square function estimate for dispersive operators in a time variable and a generalization of
Tao’s epsilon removal lemma in mixed norms. By applying this implication to the fractional
Schr¨ odinger equation in \( \mathbb{R}^{2+1} \) we obtain the sharp global space-time estimates with optimal
regularity from the previous known local ones.

1. Introduction

Let us consider a Cauchy problem of a dispersive equation in \( \mathbb{R}^{n+1} \)
\[
\begin{cases}
i \partial_t u + \Phi(D)u = 0, \\ u(0) = f,
\end{cases}
\]
where \( \Phi(D) \) is the corresponding Fourier multiplier to the function \( \Phi \). We assume that \( \Phi \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) is a real-valued function satisfying the following conditions:

**Condition 1.1.**

- \( |\nabla \Phi(\xi)| \neq 0 \) for all \( \xi \neq 0 \).
- There is a constant \( \mu \geq 1 \) such that \( \mu^{-1} \leq |\Phi(\xi)| \leq \mu \) for any \( \xi \) with \( |\xi| = 1 \).
- There is a constant \( m \geq 1 \) such that \( \Phi(\lambda \xi) = \lambda^m \Phi(\xi) \) for all \( \lambda > 0 \) and all \( \xi \neq 0 \).
- The Hessian \( H_\Phi(\xi) \) of \( \Phi \) has rank at least 1 for all \( \xi \neq 0 \).

The solution \( u \) to (1.1) becomes the Schr¨ odinger operator \( e^{-it\Delta} f \) if \( \Phi(\xi) = |\xi|^2 \) and the wave operator \( e^{it\sqrt{-\Delta}} f \) if \( \Phi(\xi) = |\xi| \). When \( \Phi(\xi) = |\xi|^m \) for \( m > 1 \), the solution is called the fractional
Schr¨ odinger operator \( e^{it\sqrt{-\Delta}} f \).

Let \( e^{it\Phi(D)} f \) denote the solution to (1.1). Our interest is to find suitable pairs \( (q, r) \) which
satisfy the global space-time estimate
\[
\|e^{it\Phi(D)} f\|_{L^q_x(\mathbb{R}^n; L^r_t(\mathbb{R}))} \leq C \|f\|_{\dot{H}^s(\mathbb{R}^n)},
\]
where \( \dot{H}^s(\mathbb{R}^n) \) denotes the homogeneous \( L^2 \) Sobolev space of order \( s \). By scaling invariance the
regularity \( s = s(r, q) \) should be defined as
\[
s = n(\frac{1}{2} - \frac{1}{q}) - \frac{m}{r}.
\]

This problem for \( \mu = 1 \) has been studied by many researchers. For the Schr¨ odinger operator,
Planchon [13] conjectured that the estimate (1.2) is valid if and only if \( r \geq 2 \) and \( \frac{n+1}{q} + \frac{1}{r} \leq \frac{n}{2} \).
Kenig–Ponce–Vega [10] showed the conjecture is true for \( n = 1 \). In higher dimensions \( n \geq 2 \),
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\|e^{it\Phi(D)} f\|_{L^q_x(\mathbb{R}^n; L^r_t(\mathbb{R}))} \leq C \|f\|_{\dot{H}^s(\mathbb{R}^n)},
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it was proven by Vega [21] that (1.2) holds for $q \geq \frac{2(n+2)}{n}$ and $\frac{n+1}{q} + \frac{1}{r} \leq \frac{3}{2}$. When $n = 2$ Rogers [15] showed it for $2 \leq r < \infty$, $q > \frac{10}{3}$ and $\frac{3}{q} + \frac{1}{r} < 1$, and later the excluded endline $\frac{3}{q} + \frac{1}{r} = 1$ was obtained by Lee–Rogers–Vargas [17]. When $n \geq 3$, Lee–Rogers–Vargas [17] improved the previous known result to $r \geq 2$, $q > \frac{2(n+3)}{n+1}$ and $\frac{n+1}{q} + \frac{1}{r} = \frac{2}{3}$. Recently it is shown by Du–Kim–Wang–Zhang [9] that the estimate (1.2) with $r = \infty$, that is, the maximal estimate fails for $n \geq 3$. For a case of the wave operator it is known that (1.2) holds for $(r, q)$ pairs $2 \leq r \leq q$, $q \neq \infty$ and $\frac{3}{r} + \frac{n-1}{2q} \leq \frac{n-1}{3}$ (see [8, 9, 13, 18]). Particularly, when $r = \infty$, Rogers–Villarroya [10] showed that (1.2) with regularity $s > n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{2}$ is valid for $q \geq \frac{2(n+1)}{n-1}$.

For the fractional Schrödinger operator the known range of $(r, q)$ for which the estimates hold is that $2 \leq r \leq q$, $q \neq \infty$ and $\frac{3}{r} + \frac{1}{q} \leq \frac{2}{3}$ (see [12, 19, 20]).

The case of $\mu > 1$ has an interesting in its own right. The solution $u$ is formally written as

$$u(t, x) = e^{i\Phi(\xi)} f(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x - \xi + i\Phi(\xi))} \hat{f}(\xi) d\xi.$$  

From this form we see that the space-time Fourier transform of $u$ is supported in the surface $S = \{(\xi, \Phi(\xi))\}$. It is known that the operator $u$ is related to the curvature of $S$ such as the sign of Gaussian curvature and the number of nonvanishing principle curvature. The Schrödinger operator corresponds to a paraboloid which has a positive Gaussian curvature, and the wave operator corresponds to a cone whose Gaussian curvature is zero. We are also interested in operators corresponding to a surface with negative Gaussian curvature. When $\mu > 1$ there is a surface with negative Gaussian curvature. For instance, the surface $\{(\xi_1, \xi_2, \xi_1^4 + 2\xi_3^2\xi_2 - 2\xi_1\xi_3^2 + \xi_3^4)\}$ has negative Gaussian curvature on a neighborhood of the point $(1, 0, 1)$.

In this paper we will establish a local-to-global approach as follows.

**Theorem 1.2.** Let $I = (0, 1)$ be a unit interval and $B = B(0, 1)$ a unit ball in $\mathbb{R}^n$. Let $q_0, r_0 \in [2, \infty)$, $s(r, q)$ defined as (1.3) and $\Phi$ satisfy Condition (1.3). Suppose that the local estimate

$$\|e^{it\Phi(\xi)} f\|_{L^q_B(\mathbb{R}^n; L^r_B(I))} \leq C \|f\|_{H^s(r_0; \mathbb{R}^n)}$$

holds for all $\epsilon > 0$. Then for any $q > q_0$ and $r > r_0$, the global estimate

$$\|e^{it\Phi(\xi)} f\|_{L^q_B(\mathbb{R}^n; L^r_B(B))} \leq C \|f\|_{H^s(r_0; \mathbb{R}^n)}$$

holds, where $H^s(\mathbb{R}^n)$ denotes the inhomogeneous $L^2$-Sobolev space of order $s$ and $\dot{H}^s(\mathbb{R}^n)$ denotes homogeneous one.

The maximal estimate, which is (1.3) with $r_0 = \infty$, is related to pointwise convergence problems. When $n = 2$ it was proven that the maximal estimates with $m > 1$ and $\mu = 1$ are valid for $q_0 = 3$ and $s > \frac{1}{m}$ (see [3, 5]). By interpolating with a Strichartz estimate

$$\|e^{it\Phi(\xi)} f\|_{L^q_T(\mathbb{R}^2; L^r_T(\mathbb{R}))} \leq \|e^{it\Phi(\xi)} f\|_{L^q_T(\mathbb{R}^2; L^2_T(\mathbb{R}))} \leq C \|f\|_{H^{2-m/4}(\mathbb{R}^2)},$$

we have (1.5) for the line $\frac{3}{q} + \frac{1}{r} = 1$ with $r \geq 2$. By Theorem 1.2 we can obtain the following global space-time estimates which is the Planchon conjecture for $n = 2$ except the endline.

**Corollary 1.3.** Let $m > 1$ and $\mu = 1$. For $2 \leq r < \infty$ and $\frac{3}{q} + \frac{1}{r} < 1$, the global estimate

$$\|e^{it\Phi(\xi)} f\|_{L^q_T(\mathbb{R}^2; L^r_T(\mathbb{R}))} \leq C \|f\|_{H^{1-\frac{4}{m}}(\mathbb{R}^2)}$$

**Notation.** Throughout this paper let $C > 0$ denote various constants that vary from line to line, which possibly depend on $n$, $q$, $r$, $m$ and $\mu$. We use $A \lesssim B$ to denote $A \leq CB$, and if $A \lesssim B$ and $B \lesssim A$ we denote by $A \sim B$. 


2. Proof of Theorem 1.2

In this section we prove Theorem 1.2 by using two propositions. In subsection 2.1 we consider a Littlewood–Paley type inequality by which the initial data \( f \) can be assumed to be Fourier supported in \( \{1/2 \leq |\xi| \leq 2\} \). In subsection 2.2 we prove a mixed norm version of Tao’s \( \varepsilon \)-removable lemma by which the global estimates with a compact Fourier support are reduced to a local ones. In subsection 2.3 we show the two propositions imply Theorem 1.2.

2.1. A Littlewood-Paley type inequality. We discuss a Littlewood-Paley type inequality for the operator \( e^{it\Phi(D)} \) in a time variable.

Let a cut-off function \( \phi \in C^\infty_0 \left( \left[\frac{1}{2}, 2\right] \right) \) satisfy \( \sum_{k \in \mathbb{Z}} \phi(2^{-k}x) = 1 \). We define Littlewood-Paley projection operators \( P_k \) and \( \tilde{P}_k \) by

\[
\tilde{P}_k f(\xi) = \phi(2^{-k}|\xi|) \hat{f}(\xi) \quad \text{and} \quad \tilde{P}_k f(\tau) = \phi(2^{-mk}|\tau|) \hat{f}(\tau)
\]

for \( \xi \in \mathbb{R}^n \) and \( \tau \in \mathbb{R} \), respectively.

**Lemma 2.1.** Suppose that \( \Phi \) satisfies Condition 1.1. Then for \( 1 < r < \infty \),

\[
\left\| e^{it\Phi(D)} f(x) \right\|_{L^r_t(\mathbb{R})} \leq C_{m,r} \left( \sum_{j,k \in \mathbb{Z}} \left\| \tilde{P}_j e^{it\Phi(D)} P_k f(x) \right\|^2 \right)^{1/2} \left\| \right\|_{L^r_t(\mathbb{R})}
\]

for all functions \( f \) and all \( x \in \mathbb{R}^n \).

**Proof.** For simplicity,

\[
F(t) := e^{it\Phi(D)} f(x) \quad \text{and} \quad F_k(t) := e^{it\Phi(D)} P_k f(x).
\]

Since the projection operators are linear, we have an identity

\[
F(t) = \sum_{j \in \mathbb{Z}} \tilde{P}_j F(t) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \tilde{P}_j F_k(t).
\]

For any test function \( \psi \in C^\infty_0 \left( \left[-2, 2\right] \right) \) with \( \psi = 1 \) in \([-1, 1]\), the Fourier transform \( \hat{f} \) of \( f \) is defined by

\[
\hat{f}(\tau) = \lim_{R \to \infty} \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\tau} \psi \left( \frac{t}{R} \right) f(t) dt
\]

in the distributional sense.

We claim that \( \tilde{P}_j F_k(t) = 0 \) if

\[
|k - j| > \frac{\log_2 \mu}{m} + 2.
\]

Indeed, using the above definition of the Fourier transform we can write

\[
\tilde{P}_j F_k(\tau) = \frac{1}{(2\pi)^{n+1}} \phi \left( \frac{|\tau|}{2^{mj}} \right) \lim_{R \to \infty} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left( \int_{\mathbb{R}} e^{it\tau} e^{it\Phi(\xi)} \psi \left( \frac{t}{R} \right) dt \right) \phi \left( \frac{2^k |\xi|}{2^j} \right) \hat{f}(\xi) d\xi.
\]

In the right side of the above equation, we see that the range of \( (\tau, \xi) \) is contained in

\[
2^{m(j-1)} \leq |\tau| \leq 2^{m(j+1)} \quad \text{and} \quad 2^{k(j-1)} \leq |\xi| \leq 2^{k(j+1)}.
\]

From Condition 1.1 we have a bound

\[
\mu^{-1} 2^m \leq |\Phi(\xi)| \leq \mu 2^m.
\]

Then it follows that for \( k \) and \( j \) satisfying (2.1),

\[
|\tau + \Phi(\xi)| > 0.
\]
By the integration by parts it implies that there exists a constant $C_0 > 0$ such that
\[ \left| \int_{\mathbb{R}} e^{it\xi} e^{it\Phi(\xi)} \left( \frac{f}{R} \right) dt \right| \leq \frac{1}{C_0 R}. \]

From this estimate and the Lebesgue dominated convergence theorem we obtain $\tilde{P}_j F_k = 0$, which implies the claim.

By the claim, the Littlewood-Paley theory and the Cauchy-Schwarz inequality,
\[ \left\| e^{it\Phi(D)} f(x) \right\|_{L^p_t(\mathbb{R}^n)} = \left\| \sum_{j \in \mathbb{Z}} \tilde{P}_j \left( \sum_{k \in \mathbb{Z}} F_k(\cdot, x) \right) \right\|_{L^p_t(\mathbb{R}^n)} \]
\[ \leq C \left\| \left( \sum_{j \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z} : |k-j| \leq \log^2 \mu m + 2} \tilde{P}_j F_k(\cdot, x) \right| \right)^2 \right\|_{L^p_t(\mathbb{R}^n)}^{1/2} \]
\[ \leq C_{m, \mu} \left( \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z} : |k-j| \leq \log^2 \mu m + 2} \left| \tilde{P}_j F_k(\cdot, x) \right|^2 \right)^{1/2} \left\| \right\|_{L^p_t(\mathbb{R}^n)}.
\]

This is the desired inequality. \qed

Using the above lemma we can have the following proposition.

**Proposition 2.2.** Let $2 \leq q, r < \infty$. Suppose that $\Phi$ satisfies Condition (3.1). If the estimate
\[ \left\| e^{it\Phi(D)} f \right\|_{L^2_t(\mathbb{R}^n; L^p_t(\mathbb{R}^n))} \leq C \left\| f \right\|_{L^2(\mathbb{R}^n)} \quad (2.2) \]
holds for all $f$ with $\text{supp} \hat{f} \subset \{ 1/2 \leq |\xi| \leq 2 \}$, then the estimate
\[ \left\| e^{it\Phi(D)} f \right\|_{L^2_t(\mathbb{R}^n; L^p_t(\mathbb{R}^n))} \leq C_{m, \mu} \left\| f \right\|_{H^{\frac{n}{2} - \frac{n}{q} - \frac{n}{p}}(\mathbb{R}^n)} \]
holds for all $f$.

**Proof.** The Minkowski inequality and Lemma 2.1 allow that
\[ \left\| e^{it\Phi(D)} f \right\|_{L^2_t(\mathbb{R}^n; L^p_t(\mathbb{R}^n))} \leq C_{m, \mu} \left( \sum_{k \in \mathbb{Z}} \left\| e^{it\Phi(D)} P_k f \right\|_{L^2_t(\mathbb{R}^n)}^2 \right)^{1/2} \left\| \right\|_{L^2_t(\mathbb{R}^n)}.
\]

Since $\tilde{P}_j$ is bounded in $L^p$, it is bounded by
\[ C_{m, \mu} \left( \sum_{k \in \mathbb{Z}} \left\| e^{it\Phi(D)} P_k f \right\|_{L^2_t(\mathbb{R}^n)}^2 \right)^{1/2} \left\| \right\|_{L^2_t(\mathbb{R}^n)}.
\]

By the Minkowski inequality we thus have
\[ \left\| e^{it\Phi(D)} f \right\|_{L^2_t(\mathbb{R}^n; L^p_t(\mathbb{R}^n))} \leq C_{m, \mu} \left( \sum_{k \in \mathbb{Z}} \left\| e^{it\Phi(D)} P_k f \right\|_{L^2_t(\mathbb{R}^n; L^p_t(\mathbb{R}^n))}^2 \right)^{1/2}.
\]

Apply (2.2) to the right side of the above estimate after parabolic rescaling. Then we obtain
\[ \left\| e^{it\Phi(D)} f \right\|_{L^2_t(\mathbb{R}^n; L^p_t(\mathbb{R}^n))} \leq C_{m, \mu} \left( \sum_{k \in \mathbb{Z}} \left( \frac{2k}{2} - \frac{\mu}{2} - \frac{m}{2} \right) \left\| P_k f \right\|_2^2 \right)^{1/2}
\]
\[ = C_{m, \mu} \left\| f \right\|_{H^{\frac{n}{2} - \frac{n}{q} - \frac{n}{p}}(\mathbb{R}^n)}.
\]

\qed
2.2. Local-to-global arguments. We will show that the global estimate (2.2) is obtained from its local estimate. Adopting the arguments in [19], we consider the dual estimate of (2.2).

Let \( S = \{(\xi, \Phi(\xi)) \in \mathbb{R}^n \times \mathbb{R} : 1/2 \leq |\xi| \leq 2\} \) be a compact hypersurface with the induced (singular) Lebesgue measure \( d\sigma \). We define the Fourier restriction operator \( \mathcal{R} \) for a compact surface \( S \) by the restriction of \( \hat{f} \) to \( S \), i.e.,

\[
\mathcal{R} f = \hat{f}|_S.
\]

Its adjoint operator \( \mathcal{R}^* f = \overline{\hat{f}d\sigma} \) can be viewed as \( e^{i\mu(\mathcal{D})} \hat{g} \), where the Fourier transform \( \hat{g}(\xi) \) of \( g \) corresponds to \( f(\xi, \Phi(\xi)) \).

Let \( \rho > 0 \) be the decay of \( d\sigma \), i.e.,

\[
|d\sigma(x)| \lesssim (1 + |x|)^{-\rho}, \quad x \in \mathbb{R}^{n+1}.
\]

It is known that \( \rho \) is determined by the number of nonzero principal curvatures of the surface \( S \), which is equal to the rank of the Hessian \( H_\Phi \). Specifically, if \( H_\Phi \) has rank at least \( k \) then

\[
\rho = k/2,
\]

see [17] subsection 5.8, VIII]. From Condition [14] we have \( k \geq 1 \).

When a function \( f \) has a compact Fourier support, the \( \hat{f}d\sigma \) decays away from the support of \( \hat{f} \) because of the decay of \( d\sigma \). Thus if \( f \) and \( g \) are compactly Fourier supported and their supports are far away from each other then the interaction between \( \hat{f}d\sigma \) and \( gd\sigma \) is negligible.

**Definition 2.3.** A finite collection \( \{Q(z_i, R)\}_{i=1}^N \) of balls in \( \mathbb{R}^{n+1} \) with radius \( R > 0 \) is called \((N,R)\)-sparse if the centers \( \{z_i\} \) are \((N R)^{\gamma}\)-separated where \( \gamma := n/\rho \geq 2 \).

From the definition of \((N,R)\)-sparse we have a kind of orthogonality as follows. Let \( \phi \) be a radial Schwartz function which is positive on the ball \( B(0,3/2) \) and \( \phi = 1 \) on the unit ball \( B(0,1) \) and whose Fourier transform is supported on the ball \( B(0,2/3) \).

**Lemma 2.4 ([19] in the proof of Lemma 3.2).** Let \( \{Q(z_i, R)\}_{i=1}^N \) be a \((N,R)\)-sparse collection and \( \phi_i(z) = \phi(R^{-1}(z - z_i)) \) for \( i = 1, \cdots, N \). Then there is a constant \( C \) independent of \( N \) such that

\[
\left\| \sum_{i=1}^N f_i * \hat{\phi}_i \right\|_2 \leq CR^{1/2} \left( \sum_{i=1}^N \|f_i\|_2^2 \right)^{1/2}
\]

for all \( f_i \in L^2(\mathbb{R}^{n+1}) \).

A proof of the above lemma is given in Appendix.

Let \( \mathbb{I}_R = (0, R) \) denote an \( R \)-interval and \( \mathbb{B}_R \) the ball of radius \( R \) centered at the origin in \( \mathbb{R}^n \). Using Lemma 2.4 we have an intermediate result.

**Proposition 2.5.** Let \( R > 0 \) and \( 1 < q, r \leq 2 \). Suppose that there is a constant \( A(R) \) such that

\[
\|\mathcal{R}(\chi_{\mathbb{I}_R} \times \mathbb{B}_R f)\|_{L^2(\mathcal{R}^2)} \leq A(R) \|f\|_{L^q(\mathbb{R}^n; L^r(\mathbb{R}))}
\]

for all \( f \in L^q(\mathbb{R}^n; L^r(\mathbb{R})) \). Then for any \((N,R)\)-sparse collection \( \{Q(z_i, R)\}_{i=1}^N \) there is a constant \( C \) independent of \( N \) such that

\[
\|\mathcal{R} f\|_{L^2(\mathcal{R}^2)} \leq CA(R) \|f\|_{L^q(\mathbb{R}^n; L^r(\mathbb{R}))}
\]

for all \( f \) supported in \( \bigcup_{i=1}^N Q(z_i, R) \).

**Proof.** Let \( f_i = f \chi_{Q(z_i, R)} \). Then, \( \mathcal{R} f_i = \hat{f}_i|_S = \overline{\hat{\phi}_i}|_S = (\hat{f}_i * \hat{\phi}_i)|_S \).
where $\phi_i(z)$ is defined as in Lemma 2.4. Since $\hat{\phi}_i$ is supported on the ball $B(0, \frac{2}{3R})$, we may restrict the support of $\hat{f}_i$ to a $O(1/R)$-neighborhood of the surface $S$ and write

$$\mathfrak{R}f_i = (\hat{f}_i|_{N_{1/R}(S)} * \hat{\phi}_i)|_S$$

where $N_{1/R}(S)$ is a $O(1/R)$-neighborhood of the surface $S$. Let $\mathfrak{R}$ be another restriction operator defined by $\mathfrak{R}f = \hat{f}|_{N_{1/R}(S)}$. If $f$ is supported in $\cup_{i=1}^N Q(z_i, R)$, we write

$$\mathfrak{R}f = \sum_{i=1}^N (\mathfrak{R}f_i * \hat{\phi}_i)|_S.$$ 

By Lemma 2.4,

$$\|\mathfrak{R}f\|_{L^2(dx)} \leq CR^{1/2} \left( \sum_{i=1}^N \|\mathfrak{R}f_i\|_{L^2(N_{1/R}(S))}^2 \right)^{1/2}.$$ 

Since the estimate (2.5) is translation invariant, by a slice argument we have

$$\|\mathfrak{R}f_i\|_{L^2(N_{1/R}(S))} \leq CR^{-1/2} A(R) \|f_i\|_{L^q_2(\mathbb{R}^n; L^q_1(\mathbb{R}))}.$$ 

By combining the previous two estimates,

$$\|\mathfrak{R}f\|_{L^2(dx)} \leq CA(R) \left( \sum_{i=1}^N \|f_i\|_{L^q_2(\mathbb{R}^n; L^q_1(\mathbb{R}))}^2 \right)^{1/2}.$$ 

If $1 \leq r \leq q \leq 2$ then by $\ell^r \subset \ell^q \subset \ell^2$,

$$\left( \sum_{i=1}^N \|f_i\|_{L^2_2(\mathbb{R}^n; L^q_1(\mathbb{R}))}^2 \right)^{1/2} \leq \left( \sum_{i=1}^N \|f_i\|_{L^q_2(\mathbb{R}^n; L^q_1(\mathbb{R}))}^q \right)^{1/q}\left( \int_{\mathbb{R}^n} \left( \sum_{i=1}^N \|f_i\|^q_{L^q_1(\mathbb{R})} dx \right)^{1/q} \right)^{1/q} \leq \left( \int_{\mathbb{R}^n} \left( \sum_{i=1}^N \|f_i\|^q_{L^q_1(\mathbb{R})} \right)^{q/r} dx \right)^{1/q} = \|f\|_{L^2_2(\mathbb{R}^n; L^q_1(\mathbb{R}))}.$$ 

If $1 \leq q \leq r \leq 2$ one can use the embedding $\ell^r \subset \ell^2$ and the Minkowski inequality to get

$$\left( \sum_{i=1}^N \|f_i\|_{L^2_2(\mathbb{R}^n; L^q_1(\mathbb{R}))}^q \right)^{1/q} \leq \left( \sum_{i=1}^N \|f_i\|_{L^q_2(\mathbb{R}^n; L^q_1(\mathbb{R}))}^q \right)^{1/r}\left( \int_{\mathbb{R}^n} \left( \sum_{i=1}^N \|f_i\|^r_{L^q_1(\mathbb{R})} dx \right)^{1/q} \right)^{1/q} \leq \left( \int_{\mathbb{R}^n} \left( \sum_{i=1}^N \|f_i\|^r_{L^q_1(\mathbb{R})} \right)^{r/q} dx \right)^{1/q} = \|f\|_{L^2_2(\mathbb{R}^n; L^q_1(\mathbb{R}))}.$$ 

Therefore we have (2.6). $\square$

We now extend the $(N, R)$-sparse sets to the whole space. For this we need the following decomposition lemma.
Lemma 2.6 ([19]). Let $E$ be a subset in $\mathbb{R}^n$ with $|E| > 1$. Suppose that $E$ is a finite union of finitely overlapping cubes of side-length $c \sim 1$. Then for each $K \in \mathbb{N}$, there are subsets $E_1, E_2, \cdots, E_K$ of $E$ with

$$E = \bigcup_{k=1}^K E_k$$

such that each $E_k$ has $O(|E|^{1/K})$ number of $(O(|E|), |E|^{O(\gamma^{-1})})$-sparse collections

$$S_1, S_2, \cdots, S_{O(|E|^{1/K})}$$

of which the union $S_1 \cup S_2 \cup \cdots \cup S_{O(|E|^{1/K})}$ is a covering of $E_k$.

This lemma is a precise version of Lemma 3.3 in [19]. A detailed proof can be found in Appendix.

Using the above lemma we have the following proposition.

Proposition 2.7. Let $1 < q_0, r_0 < \infty$. Suppose that for any $\epsilon > 0$ and any $(N, R)$-sparse collection $\{Q(z_i, R)\}_{i=1}^N$ in $\mathbb{R}^{n+1}$, the estimate

$$\|\mathfrak{R}f\|_{L^2(d\nu)} \leq C_N R^{\epsilon} \|f\|_{L^{p_0}(\mathbb{R}^n; L^q_0(\mathbb{R}))}$$

(2.7)

holds for all $f$ supported in $\bigcup_{i=1}^N Q(z_i, R)$. Then for any $1 \leq q < q_0$ and $1 \leq r < r_0$, the estimate

$$\|\mathfrak{R}f\|_{L^2(d\nu)} \leq C \|f\|_{L^q_0(\mathbb{R}^n; L^r_0(\mathbb{R}))}$$

holds for all $f \in L^q_0(\mathbb{R}^n; L^r_0(\mathbb{R}))$.

Proof. By interpolation (see [7]), it suffices to show that for $1 \leq q < q_0$ and $1 \leq r < r_0$, the restricted type estimate

$$\|\mathfrak{R}\chi_E\|_{L^2(d\nu)} \leq C \|\chi_E\|_{L^q(\mathbb{R}^n; L^r(\mathbb{R}))}$$

for all subset $E$ in $\mathbb{R}^{n+1}$. We may assume $|E| > 1$, otherwise the estimate is trivial. Since the set $S$ is compact, $\chi_E$ can be replaced with $\chi_E \ast \varphi$, where $\varphi$ is a bump function supported on a cube of sidelen cath $c \sim 1$ such that $\varphi$ is positive on $S$. Thus, we may further assume that $E$ is the union of $c$-cubes.

We denote by proj$(E)$ the projection of $E$ onto the $x$-plane. For each grid point $x \in c\mathbb{Z}^n \cap$ proj$(E)$, we define $E_x$ to be the union of $c$-cubes in $E$ that intersect $\mathbb{R} \times \{x\}$. Let $E^j$ be the union of $E_x$ which satisfies

$$2^{j-1} < \text{ the number of } c \text{-cubes contained in } E_x \leq 2^{j+1}$$

for $j \in \mathbb{N}$, (see Figure [1]). Then,

$$E = \bigcup_{j \geq 1} E^j.$$

By using Lemma 2.6 with

$$K := \frac{\log(1/\epsilon)}{2 \log \gamma} + 1,$$

the $E^j$ is decomposed into $E^j_k$’s which are covered by $O(|E^j|^{1/K})$ number of $(O(|E^j|), |E^j|^{C\gamma^{-1}})$-sparse collections. We apply (2.7) to these sparse collections and obtain

$$\|\mathfrak{R}\chi_{E^j_k}\|_{L^2(d\nu)} \leq C \|E^j_k\|^{1/K}(|E^j_k|^{C\gamma^{-1}})^{\epsilon} \|\chi_{E^j_k}\|_{L^{p_0}(\mathbb{R}^n; L^q_0(\mathbb{R}))}.$$ 

Summing over $k$, we have

$$\|\mathfrak{R}\chi_{E^j}\|_{L^2(d\nu)} \leq \sum_{k=1}^K \|\mathfrak{R}\chi_{E^j_k}\|_{L^2(d\nu)}$$

$$\leq C \|E^j\|^{1/K}(|E^j|^{C\gamma^{-1}})^{\epsilon} \|\chi_{E^j}\|_{L^{p_0}(\mathbb{R}^n; L^q_0(\mathbb{R}))}.$$
Figure 1. The sets $E$, proj$E$, $E_x$ and $E^j$ in the proof of Proposition 2.7.

where $K$ is absorbed into $C_\epsilon$. Since $|E^j| \leq 2^{j+1}|\text{proj}(E^j)|$, we have

$$\|\mathcal{R}\chi_{E^j}\|_{L^2(d\sigma)} \leq C_\epsilon 2^{j(\frac{1}{q_0}+\delta(\epsilon))}|\text{proj}(E^j)|^{\frac{1}{q_0}}+\delta(\epsilon),$$

where

$$\delta(\epsilon) := \frac{1}{K} + C\gamma K^{-1}\epsilon.$$

Since $\lim_{\epsilon\to 0}\delta(\epsilon) = 0$, we can take $\epsilon > 0$ such that

$$0 < \delta(\epsilon) + \epsilon \leq \min\left(\frac{1}{q} - \frac{1}{q_0}, \frac{1}{r} - \frac{1}{r_0}\right).$$

Thus,

$$\|\mathcal{R}\chi_E\|_{L^2(d\sigma)} \leq \sum_{j \geq 1} \|\mathcal{R}\chi_{E^j}\|_{L^2(d\sigma)}$$

$$\leq C_\epsilon \sum_{j \geq 1} 2^{j(\frac{1}{q_0}+\delta(\epsilon))}|\text{proj}(E^j)|^{\frac{1}{q_0}}+\delta(\epsilon)$$

$$\leq C \sum_{j \geq 1} 2^{-\epsilon j}2^{\frac{1}{2}j}|\text{proj}(E^j)|^{\frac{1}{q}}$$

$$\leq C \sum_{j \geq 1} 2^{-\epsilon j}\|\chi_E\|_{L_{q}^{2}(\mathbb{R}^n;L_{r}^{1}(\mathbb{R}))}$$

$$\leq C \|\chi_E\|_{L_{q}^{2}(\mathbb{R}^n;L_{r}^{1}(\mathbb{R}))}.$$

Combining Proposition 2.5 and Proposition 2.7, we obtain an extension of Tao’s epsilon removal lemma as follows.

**Proposition 2.8.** Let $1 < q_0, r_0 \leq 2$. Suppose that

$$\|\mathcal{R}(\chi_{I_{R} \times \mathbb{R}}f)\|_{L^2(d\sigma)} \leq C_i R^\epsilon\|f\|_{L_{q_0}^{2}(\mathbb{R}^n;L_{r_0}^{1}(\mathbb{R}))}$$
for all $\epsilon > 0$, $R > 1$ and all $f \in L^q_t(\mathbb{R}^n; L^r_x(\mathbb{R}))$. Then for any $1 \leq q < q_0$ and $1 \leq r < r_0$,
\[ \| \mathcal{R}f \|_{L^2(\mathbb{R}^n; L^r_x(\mathbb{R}))} \leq C \| f \|_{L^q_t(\mathbb{R}^n; L^r_x(\mathbb{R}))} \]
for all $f \in L^q_t(\mathbb{R}^n; L^r_x(\mathbb{R}))$.

Now we are ready to prove Theorem 1.2. The theorem follows from Proposition 2.2 and Proposition 2.8 as follows.

2.3. Proof of Theorem 1.2. Let $P_0$ be the Littlewood-Paley projection operator as in subsection 2.1. By rescaling $x \mapsto 2^{-k}x$ and $t \mapsto 2^{-m}t$, the estimate (1.3) implies
\[ \| e^{it\Phi(D)}P_0 f \|_{L^q_t(\mathbb{R}; L^r_x(\mathbb{R}))} \leq C \epsilon 2^{k\epsilon} \| P_0 f \|_{L^2(\mathbb{R}^n)} \]
for all $k \geq 1$ and $\epsilon > 0$. Since $m \geq 1$, we have
\[ \| e^{it\Phi(D)}P_0 f \|_{L^q_t(\mathbb{R}; L^r_x(\mathbb{R}))} \leq C \epsilon 2^{k\epsilon} \| P_0 f \|_{L^2(\mathbb{R}^n)}. \]
By Proposition 2.8 and duality,
\[ \| e^{it\Phi(D)}P_0 f \|_{L^q_t(\mathbb{R}^n; L^r_x(\mathbb{R}))} \leq C \| P_0 f \|_{L^2(\mathbb{R}^n)}. \]
By Proposition 2.2 we obtain the desired estimate. \hfill \Box

3. Appendix

3.1. Proof of Lemma 2.4. We divide the left side of (2.4) into two parts
\[ \| \sum_{i=1}^N f_i \ast \phi_i |s| \|_2^2 = \sum_i \| f_i \ast \phi_i |s| \|_2^2 + \sum_{i \neq j} f_i \ast \phi_i f_j \ast \phi_j d\sigma. \]
By a basic restriction estimate we have $\| f_i \ast \phi_i |s| \|_2 \lesssim R^{1/2} \| f_i \|_2$. Thus,
\[ \sum_{i=1}^N \| f_i \ast \phi_i |s| \|_2^2 \lesssim R \sum_{i=1}^N \| f_i \|_2^2. \] (3.1)
By Parseval’s identity,
\[ \int f_i \ast \phi_i f_j \ast \phi_j d\sigma = \int \hat{f}_j \phi_j((\hat{f}_i \phi_i) \ast d\sigma), \]
where the $\ast$ denotes the inverse Fourier transform. It is bounded by
\[ (\sup_{z,w} |\phi_j^{1/2} \ast (\phi_j^{1/2} \ast (\hat{f}_i \phi_i) \ast d\sigma)(z-w)|) \| f_i \phi_i^{1/2} \|_1 \| f_j \phi_j^{1/2} \|_1. \]
By the Cauchy-Schwarz inequality and Plancherel’s theorem,
\[ \| \hat{f}_i \phi_i^{1/2} \|_1 \lesssim R^{(n+1)/2} \| f_i \|_2. \]
By (2.3),
\[ \sup_{z,w} |\phi_j^{1/2} \ast (\phi_j^{1/2} \ast (\hat{f}_i \phi_i) \ast d\sigma)(z-w)| \lesssim |z_i - z_j - 2R|^{-\rho}. \]
Since $|z_i - z_j| \geq (NR)^\gamma$ and $\gamma \geq 2$, we have that $|z_i - z_j - 2R|$ is comparable to $|z_i - z_j|$. Thus,
\[ \sup_{z,w} |\phi_j^{1/2} \ast (\phi_j^{1/2} \ast (\hat{f}_i \phi_i) \ast d\sigma)(z-w)| \lesssim |z_i - z_j|^{-\rho}. \]
Combining these estimates we have
\[
\sum_{i \neq j} \int f_i * \varphi_i f_j * \varphi_j d\sigma \lesssim R^{n+1} \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{\tilde{N}} |z_i - z_j|^{-\rho} \|f_i\|_2 \|f_j\|_2
\]
\[
\lesssim R^{n+1} N \max_{i,j} |z_i - z_j|^{-\rho} \sum_{i=1}^{\tilde{N}} \|f_i\|_2^2.
\]
Since \(|z_i - z_j| \geq (NR)^{-\gamma} \geq N^{1/\rho} R^{n/\rho}\), it follows that
\[
\sum_{i \neq j} \int f_i * \varphi_i f_j * \varphi_j d\sigma \lesssim R \sum_{i=1}^{\tilde{N}} \|f_i\|_2^2.
\]
From the above estimate and (3.1) we obtain (2.4).

3.2. Proof of Lemma 2.6. Fix \(K \in \mathbb{N}\). We define \(R_0 = 1\) and \(R_k\) for \(k = 1, 2, \cdots, K\) recursively by
\[
R_k = |E|^{1/k} R_{k-1}^1.
\]
From this definition we have \(R_k = |E|^{k+1}/k!\). Let \(E_0 = \emptyset\). We define \(E_k\) for \(k = 1, 2, \cdots, K\) to be the set of all \(x \in E \setminus \bigcup_{j=0,1,2,\cdots,k-1} E_j\) such that
\[
|E \cap B(x, R_k)| \leq |E|^{k/K}.
\]
Then, \(E = \bigcup_{k=1}^{K} E_k\). From this construction it follows that that for \(x \in E_k\), \(k = 2, 3, \cdots, K\),
\[
|E \cap B(x, R_{k-1})| \geq |E|^{(k-1)/K}.
\]
We cover \(E_k\) with finitely overlapping \(R_k\)-balls \(C_{E_k} := \{B_i = B(x_i, R_k) : x_i \in E_k\}\). Since \(E\) is a finite union of cubes of side-length \(c \sim 1\), it is obvious that \(\# C_{E_k} \lesssim |E|\). For each \(B_i \in C_{E_k}\) we cover \(E_k \cap B_i\) with finitely overlapping \(R_{k-1}\)-balls \(C_{E_k \cap B_i} := \{B_{ij}' = B'(y_j, R_{k-1}) : y_j \in E_k \cap B_i\}\), that is,
\[
E_k \cap B_i = \bigcup_{B_{ij}' \in C_{E_k \cap B_i}} E_k \cap B_{ij}'.
\]
Since \((E \setminus E_k) \cap B_{ij}' \subset ((E \setminus E_k) \cap B_i)\) for all \(j\), we have
\[
(E_k \cap B_i) \cup ((E \setminus E_k) \cap B_i) \supset \bigcup_{B_{ij}' \in C_{E_k \cap B_i}} (E_k \cap B_{ij}') \cup ((E \setminus E_k) \cap B_{ij}'),
\]
thus
\[
E \cap B_i \supset \bigcup_{B_{ij}' \in C_{E_k \cap B_i}} E \cap B_{ij}'.
\]
By finitely overlapping,
\[
\# C_{E_k \cap B_i} \lesssim \max_{B_{ij}' \in C_{E_k \cap B_i}} \frac{|E \cap B_i|}{|E \cap B_{ij}'|}.
\]
By (3.3) and (3.4) the above is bounded by \(C \|E\|^{1/K}\), and we have \(\# C_{E_k \cap B_i} \leq C \|E\|^{1/K}\) for all \(i\). Thus,
\[
E_k \subset \bigcup_{i=1}^{\tilde{N}} \bigcup_{j=1}^{\tilde{N}} B_{ij}'.
\]
We choose \(O(R_k)\)-separated balls \(\{B_{ij}'(x_{ij})\}_{i=1}^{O(|E|)}\). Then it becomes a \((O(|E|), R_{k-1})\)-sparse collection because of (3.2). Since \(R_{k-1} = |E|^{O(n^{-k-1})}\) and every \(B_i \in C_{E_k}\) has the covering
$C_{E_k \cap B_i}$ of cardinality $O(|E|^{1/K})$, there are $O(|E|^{1/K})$ number of $(O(|E|), |E|^O(\gamma^{-1}))$-sparse collections $S_1, S_2, \cdots, S_{O(|E|^{1/K})}$ such that

$$E_k \subset \bigcup_{j=1}^{O(|E|^{1/K})} \bigcup_{B' \in S_j} B'.$$

□

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