Beyond Level Planarity*

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Abstract. In this paper we settle the computational complexity of two open problems related to the extension of the notion of level planarity to surfaces different from the plane. Namely, we show that the problems of testing the existence of a level embedding of a level graph on the surface of the rolling cylinder or on the surface of the torus, respectively known by the name of CYCLIC LEVEL PLANARITY and TORUS LEVEL PLANARITY, are polynomial-time solvable.

Moreover, we show a complexity dichotomy for testing the SIMULTANEOUS LEVEL PLANARITY of a set of level graphs, with respect to both the number of level graphs and the number of levels.

1 Introduction and Overview

The study of level drawings of level graphs has spanned a long time; the seminal paper by Sugiyama et al. on this subject [20] dates back to 1981, well before graph drawing was recognized as a distinguished research area. This is motivated by the fact that level graphs naturally model hierarchically organized data sets and level drawings are a very intuitive way to represent such graphs.

Formally, a level graph \((V, E, \gamma)\) is a directed graph \((V, E)\) together with a function \(\gamma : V \to \{1, \ldots, k\}\), with \(1 \leq k \leq |V|\). The set \(V_i = \{v \in V : \gamma(v) = i\}\) is the \(i\)-th level of \((V, E, \gamma)\). A level graph \((V, E, \gamma)\) is proper if for each \((u, v) \in E\), either \(\gamma(u) = \gamma(v) + 1\), or \(\gamma(u) = k\) and \(\gamma(v) = 1\). Let \(l_1, \ldots, l_k\) be \(k\) horizontal straight lines on the plane ordered in this way with respect to the \(y\)-axis. A level drawing of \((V, E, \gamma)\) maps each vertex \(v \in V_i\) to a point on \(l_i\) and each edge \((u, v) \in E\) to a curve monotonically increasing in the \(y\)-direction from \(u\) to \(v\). Observe that a level graph \((V, E, \gamma)\) containing an edge \((u, v) \in E\) with \(\gamma(u) > \gamma(v)\) does not admit any level drawing. A level graph is level planar if it admits a level embedding, i.e., a level drawing without crossings; see Fig. 1(a). The LEVEL PLANARITY problem asks to test whether a given level graph is level planar.

The LEVEL PLANARITY problem has been studied for decades [11,16,17,19,13], starting from a characterization of the single-source level planar graphs [11] and culminating in a linear-time decision algorithm for general level graphs [17]. A complete characterization of level planarity in terms of “minimal” forbidden subgraphs is still missing [12,15]. Level planarity has also been studied to take into account further requirements such as a clustering of the vertices (CLUSTERED LEVEL PLANARITY [13]) and consecutivity constraints for the vertex orderings on the levels (\(T\)-LEVEL PLANARITY [12]).

Differently from the standard notion of planarity, the concept of level planarity does not immediately extend to representations of level graphs on surfaces different from the plane.\(^1\) When considering the surface \(\mathbb{S}^2\) of a sphere, level drawings are usually defined as follows: The vertices have to be placed on the \(k\) circles given by the intersection of \(\mathbb{S}^2\) with \(k\) parallel planes, and each edge is a curve on \(\mathbb{S}^2\) that is monotone in the direction orthogonal to these planes. The notion of level planarity in this setting goes by the name of RADIAL LEVEL PLANARITY and is known to be decidable

\(^1\) All surfaces considered in this paper are connected and orientable; the genus of any such surface is the maximum number of cuttings along non-intersecting closed simple curves that do not disconnect the surface.

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which multiple level graphs are considered and the goal is to find a simultaneous level embedding of them. The problem expressed in Observation 1 and Lemmata 3-6) that, starting from any proper instance of $T^L$ and $C^L$, produces an equivalent instance of the so-called $S^L$ problem. In particular, motivated by the growing interest in simultaneous embedding of multiple planar graphs, which allow to display several relationships on the same set of entities in a unified representation, we define a new notion of level planarity in which multiple level graphs are considered and the goal is to find a simultaneous level embedding of them. The problem $S^L$ (see the seminal paper on the topic \cite{10} and the recent survey by Bläsius et al. \cite{6}) takes as input $k$ planar graphs $(V, E_1), \ldots, (V, E_k)$ and asks whether planar drawings of these graphs exist mapping each vertex to the same point of the plane. In analogy with this definition, we introduce the problem $S^L$.
We show that this positive result cannot be extended (unless P=NP), as the problem becomes NP-complete even for two levels (see Theorem 5), hence SIMULTANEOUS LEVEL PLANARITY is polynomial-time solvable for this family of instances. We show that this positive result cannot be extended (unless P=NP), as the problem becomes NP-complete even for two graphs on two levels and for three graphs on two levels (see Theorem 4). Altogether, this establishes a tight border of tractability for SIMULTANEOUS LEVEL PLANARITY.

2 Preliminaries

A tree $T$ is a connected acyclic graph. The degree-1 vertices of $T$ are the leaves of $T$, denoted by $L(T)$, while the remaining vertices are the internal vertices of $T$.

A directed graph $G = (V, E)$ without directed cycles is a directed acyclic graph (DAG). Let $(u, v)$ be an arc in $E$, that is an edge directed from $u$ to $v$; vertex $u$ is a parent of $v$ and $v$ is a child of $u$ in $G$. A vertex $v$ of $G$ is a source (sink) if it does not have parents (children) in $G$.

Embeddings on levels. An embedding of a graph on a surface $\mathbb{Q}$ is a mapping $\Gamma$ of each vertex $v$ to a distinct point $\Gamma(v)$ on $\mathbb{Q}$ and of each edge $e = (u, v)$ to a simple Jordan curve $\Gamma(e)$ on $\mathbb{Q}$ connecting $\Gamma(u)$ and $\Gamma(v)$, such that no two curves cross except at a common endpoint.

Let $I$ denote the unit interval and let $S^3$ denote the boundary of the unit disk. We can define the surface $S$ of the standing cylinder, the surface $R$ of the rolling cylinder, and the surface $T$ of the torus as $S = I \times S^1$, as $R = S^1 \times I$, and as $T = S^1 \times S^1$, respectively. Coordinates on $S$, $R$, and $T$ are hence points $(x \in I, y \in S^1)$, $(x \in S^1, y \in I)$, and $(x \in S^1, y \in S^1)$, respectively.

A curve $c$ from $p = (e^{2\pi i \alpha}, y)$ to $p’ = (e^{2\pi i \alpha’}, y’)$ on $R$ or $T$ with $\alpha < \alpha’$ is monotone if, for each two points $w = (e^{2\pi i \alpha w}, y_w) \in c$ and $z = (e^{2\pi i \alpha z}, y_z) \in c$ such that $p, w, z, p’$ appear in this order along $c$, it holds that $\alpha < \alpha_w < \alpha_z < \alpha’$. A curve $c$ from $p = (x, e^{2\pi i \alpha})$ to $p’ = (x’, e^{2\pi i \alpha’})$ on $S$ with $x < x’$ is monotone if, for each two points $w = (x_w, e^{2\pi i \alpha_w}) \in c$ and $z = (x_z, e^{2\pi i \alpha_z}) \in c$ such that $p, w, z, p’$ appear in this order along $c$, it holds that $x < x_w < x_z < x’$.

The $j$-th level of the standing cylinder, of the rolling cylinder, and of the torus with $k$ levels is defined as $l_j = \{j\} \times S^1$, $l_j = \{e^{2\pi i \frac{j}{k}}\} \times I$, and $l_j = \{e^{2\pi i \frac{j}{k}}\} \times S^1$, respectively; refer to Fig. 2.

Problems RADIAL LEVEL PLANARITY, CYCLIC LEVEL PLANARITY, and TORUS LEVEL PLANARITY take as input a level graph $G = (V, E, \gamma)$ and ask to find an embedding $\Gamma$ of $G$ on $S$, on $R$, and on $T$, respectively, in which, for each $v \in V$, we have $\Gamma(v) \in l_1(v)$, and for each edge $(u, v) \in E$, curve $\Gamma(e)$ is monotone. Embedding $\Gamma$ is called a radial level embedding, a cyclic level embedding, a torus level embedding of $G$, respectively. A level graph admitting a radial, cyclic, or torus level embedding is called radial, cyclic, or torus level planar, respectively.
Further, we call linear ordering where the strip of degree larger than one to illustrate that the edge ordering on $E$ in $I^\prime$ is $v$-consecutive.

In the following we show that **Torus Level Planarity** is a generalization of **Radial Level Planarity** and **Cyclic Level Planarity**, which both generalize **Level Planarity**.

**Lemma 1.** Every positive instance of **Radial Level Planarity** is a positive instance of **Torus Level Planarity**. Further, **Radial Level Planarity** reduces in linear time to **Torus Level Planarity**.

**Proof.** The first part of the statement can be easily proved by observing that any level embedding on $S$ is also a level embedding on $T$. We prove the second part of the statement. Given an instance $G = (V = \bigcup_{i=1}^{k} V_i, E, \gamma)$ of **Radial Level Planarity** we construct an instance $G' = (V = \{a, b, c, d\}, E \cup \{(a, b), (b, c), (c, d), (d, a)\}, \gamma')$ of **Torus Level Planarity**, where $\gamma'(v) = \gamma(v)$, for each $v \in V$, $\gamma'(a) = \gamma'(c) = k$, and $\gamma'(b) = \gamma'(d) = 1$. Suppose that $G$ admits a radial level embedding $I^\prime$ on $S$. Consider the corresponding torus level embedding $I''$ of $G$ on $T$, which exists by the first part of the statement. Since $G$ does not contain any edge $(u, v) \in E$ such that $\gamma(u) > \gamma(v)$ (as otherwise, $G$ would not be radial level planar), the strip of $T$ between $l_k$ and $l_1$ does not contain any edge. Hence, cycle $(a, b, c, d)$ can be added to $I''$ to obtain a torus level embedding of $G'$ on $T$. Suppose that $G'$ admits a torus level embedding $I''$ on $T$. A radial level embedding of $G$ on $S$ can be obtained by removing the drawing of cycle $(a, b, c, d)$ in $I''$.

**Lemma 2.** Every positive instance of **Cyclic Level Planarity** is a positive instance of **Torus Level Planarity**. Further, **Cyclic Level Planarity** reduces in linear time to **Torus Level Planarity**.

**Proof.** The first part of the statement can be proved by observing that any level embedding on $R$ is also a level embedding on $T$. We prove the second part of the statement. Given an instance $G = (V = \bigcup_{i=1}^{k} V_i, E, \gamma)$ of **Cyclic Level Planarity**, we construct an instance $G' = (V = \{w_1, \ldots, w_k\}, E \cup \bigcup_{i=1}^{k-1} \{(w_i, w_{i+1})\}, \gamma')$ of **Torus Level Planarity**, where $\gamma'(v) = \gamma(v)$ for each $v \in V$, and $\gamma'(w_i) = i$. Suppose that $G$ admits a cyclic level embedding $I^\prime$ on $R$. Add a drawing of cycle $(w_1, \ldots, w_k, w_1)$ to $I^\prime$ along the boundary of one of the two bases of $R$, thus obtaining a cyclic level embedding $I''$ of $G'$ on $R$. From the first part of the statement there exists a torus level embedding of $G'$ on $T$. Suppose that $G'$ admits a torus level embedding $I''$ on $T$. A cyclic level embedding of $G$ on $R$ can be obtained by removing the drawing of cycle $(w_1, \ldots, w_k, w_1)$ in $I''$.

Lemmas 1 and 2 imply that the torus surface combines the power of representation of the standing cylinder and of the rolling cylinder. To strengthen this fact, we provide in Fig. 3 an example of a level graph that is neither radial level planar nor cyclic level planar, yet it allows for a torus level embedding; notice that the underlying (non-level) graph in the example is also planar.

**Orderings and PQ-trees.** Let $A$ be a finite set. Then any permutation of $A$ determines a total ordering on $A$, which we call **linear ordering**. When considering the first and the last elements of the permutation as consecutive, we talk...
about circular orderings. Let \( O \) be a circular ordering on \( A \) and let \( O' \) be the circular ordering on \( A' \subseteq A \) obtained by restricting \( O \) to the elements of \( A' \). Then \( O' \) is a suborder of \( O \) and \( O \) is an extension of \( O' \). Let \( A \) and \( S \) be finite sets, let \( O' = s_1, s_2, \ldots, s_{|S|} \) be a circular ordering on \( S \), let \( \phi : S \to A \) be an injective map, and let \( A' \subseteq A \) be the image of \( S \) under \( \phi \); then \( \phi(O') \) denotes the circular ordering \( \phi(s_1), \phi(s_2), \ldots, \phi(s_{|S|}) \). By an overload of definitions, we also say that a circular ordering \( O' \) on \( S \) is a suborder of a circular ordering \( O \) on \( A \) (and \( O \) is an extension of \( O' \)) if \( \phi(O') \) is a suborder of \( O \).

An unrooted PQ-tree is a tree whose leaves are the elements of a ground set \( A \). We are interested in the existence of circular orderings on \( A \) subject to a set of consecutivity constraints on \( A \), each of which specifies that a subset of the elements of \( A \) has to appear consecutively in all the sought circular orderings on \( A \). An unrooted PQ-tree \( T \) can be used to represent all and only the circular orderings in \( O(T) \) on \( A \) satisfying a given set of consecutivity constraints on \( A \). The orderings in \( O(T) \) are called PQ-representable. The internal nodes of \( T \) are either P-nodes or Q-nodes. The orderings in \( O(T) \) are all and only the circular orderings on the leaves of \( T \) obtained by arbitrarily ordering the neighbours of each P-node and by arbitrarily selecting for each Q-node a given circular ordering on its neighbours or its reverse ordering. Observe that possibly \( O(T) = \emptyset \), if \( T \) is the empty tree, or \( O(T) \) represents all possible circular orderings on \( A \), if \( T \) is a star centered at a P-node. In the latter case, we call \( T \) the universal PQ-tree on \( A \).

We now illustrate three fundamental linear-time operations which are defined on PQ-trees (see also \([8]\)). Let \( T \) and \( T' \) be PQ-trees on \( A \) and let \( X \subseteq A \):

**Reduction.** The reduction of \( T \) by \( X \), denoted by \( T \oplus X \), constructs a new PQ-tree on \( A \) representing the circular orderings in \( O(T) \) in which the elements of \( X \) appear consecutively.

**Projection.** The projection of \( T \) to \( X \), denoted by \( T|_X \), constructs a new PQ-tree on \( X \) representing the circular orderings on \( X \) that are suborders of circular orderings in \( O(T) \).

**Intersection.** The intersection of \( T \) and \( T' \), denoted by \( T \cap T' \), constructs a new PQ-tree on \( A \) representing the circular orderings in \( O(T) \cap O(T') \).

**Simultaneous PQ-Ordering.** Let \( D = (N, Z) \) be a DAG with nodes \( N = \{T_1, \ldots, T_k\} \), where \( T_i \) is a PQ-tree, and each arc \((T_i, T_j; \phi) \in Z \) consists of a source \( T_i \), of a target \( T_j \), and of an injective map \( \phi : L(T_j) \to L(T_i) \). Given an arc \( a = (T_i, T_j; \phi) \in Z \) and circular orderings \( O_i \in O(T_i) \) and \( O_j \in O(T_j) \), we say that arc \( a \) is satisfied by \((O_i, O_j)\) if \( O_i \) extends \( \phi(O_j) \). The **Simultaneous PQ-Ordering** problem asks to find circular orderings \( O_1 \in O(T_1), \ldots, O_k \in O(T_k) \) on \( L(T_1), \ldots, L(T_k) \), respectively, such that each arc \((T_i, T_j; \phi) \in Z \) is satisfied by \((O_i, O_j)\).

Let \((T_i, T_j; \phi) \) be an arc in \( Z \). An internal node \( \mu_i \) of \( T_i \) is fixed by an internal node \( \mu_j \) of \( T_j \) (and \( \mu_j \in T_j \) fixes \( \mu_i \) in \( T_i \)) if there exist leaves \( x, y, z \in L(T_j) \) and \( \phi(x), \phi(y), \phi(z) \in L(T_i) \) such that (i) removing \( \mu_j \) from \( T_j \) makes leaves \( x, y, \) and \( z \) pairwise disconnected in \( T_j \), and (ii) removing \( \mu_i \) from \( T_i \) makes leaves \( \phi(x), \phi(y), \) and \( \phi(z) \) pairwise disconnected in \( T_i \). Observe that by (i) the three paths connecting \( \mu_j \) with \( x, y, \) and \( z \) in \( T_j \) share no node other than \( \mu_j \), while by (ii) those connecting \( \mu_i \) with \( \phi(x), \phi(y), \) and \( \phi(z) \) in \( T_i \) share no node other than \( \mu_i \). Since any ordering \( O_j \) determines a circular ordering on the paths connecting \( \mu_j \) with \( x, y, \) and \( z \) in \( T_j \), we have that any ordering \( O_i \) extending \( \phi(O_j) \) must determine the same circular ordering on the paths connecting \( \mu_i \) with \( \phi(x), \phi(y), \) and \( \phi(z) \) in \( T_i \); this is the reason why we say that \( \mu_i \) is fixed by \( \mu_j \).

Theorem 1 below will be one key ingredient of the algorithms we will present in the next section. However, in order to exploit such a result we need to consider instances \( D = (N, Z) \) of Simultaneous PQ-Ordering with the property that, for each arc \((T_i, T_j; \phi) \in Z \) and for each internal node \( \mu_j \in T_j \), tree \( T_i \) contains exactly one node \( \mu_i \) that is fixed by \( \mu_j \). This property can be guaranteed by an operation, called *normalization*, defined in \([7]\) as follows. Consider each arc \((T_i, T_j; \phi) \in Z \) and replace \( T_j \) with \( T_i |_{\phi(L(T_j))} \cap T_j \) in \( D \), that is, we replace tree \( T_j \) with its intersection with the projection of its parent \( T_i \) to the set of leaves of \( T_j \) obtained by applying mapping \( \phi \) to the leaves \( L(T_j) \) of \( T_j \). Instances produced by such an operation are said to be normalized.

Consider a normalized instance \( D = (N, Z) \) of Simultaneous PQ-Ordering. Let \( \mu \) be a P-node of a PQ-tree \( T \) with parents \( T_1, \ldots, T_r \) and let \( \mu_i \in T_i \) be the unique node in \( T_i \), with \( 1 \leq i \leq r \), fixed by \( \mu \). The fixedness \( f_{\text{fixed}}(\mu) \) of \( \mu \) is defined as \( f_{\text{fixed}}(\mu) = \omega + \sum_{i=1}^{r} (f_{\text{fixed}}(\mu_i) - 1) \), where \( \omega \) is the number of children of \( T \) fixing \( \mu \). A P-node \( \mu \) is \( k \)-fixed if \( f_{\text{fixed}}(\mu) \leq k \). Also, instance \( D \) is \( k \)-fixed if all the P-nodes of any PQ-tree \( T \in N \) are \( k \)-fixed.

**Theorem 1** (Bläsius and Rutter \([7]\), Theorem 3.2 and Theorem 3.3). 2-fixed instances of Simultaneous PQ-Ordering can be tested in quadratic time.
3 Torus Level Planarity

In this section we provide a polynomial-time testing and embedding algorithm for Torus Level Planarity that is based on the following simple observation.

**Observation 1** A proper level graph \( G = (\bigcup_{i=1}^{k} V_i, E, \gamma) \) is torus level planar if and only if there exist circular orderings \( \mathcal{O}_1, \ldots, \mathcal{O}_k \) on \( V_1, \ldots, V_k \) such that, for each \( 1 \leq i \leq k \) with \( k+1 = 1 \), there exists a radial level embedding of the level graph \( (V_i \cup V_{i+1}, (V_i \times V_{i+1}) \cap E, \gamma) \) in which the circular orderings on \( V_i \) along \( l_i \) and on \( V_{i+1} \) along \( l_{i+1} \) are \( \mathcal{O}_i \) and \( \mathcal{O}_{i+1} \), respectively.

In view of Observation 1, we now focus on a level graph \( G = (V_1 \cup V_2, E, \gamma) \) on two levels. We denote by \( V_1^+ \) and \( V_2^- \) the subsets of \( V_1 \) and \( V_2 \) that are incident to edges in \( E \), respectively. Let \( \Gamma \) be a radial level embedding of \( G \). Consider a closed curve \( c \) separating levels \( l_1 \) and \( l_2 \) and intersecting all the edges in \( E \) exactly once. The **edge ordering on \( E \) in \( \Gamma \)** is the circular ordering in which the edges in \( E \) intersect \( c \) according to a clockwise orientation of \( c \) on the surface \( S \) of the standing cylinder; refer to Fig. 3b. Further, let \( \mathcal{O} \) be a circular ordering on the edge set \( E \). Such an ordering is called **vertex-consecutive (v-consecutive)** if, for each vertex in \( V_1 \cup V_2 \), its incident edges are consecutive in \( \mathcal{O} \). Notice that the edge ordering on \( E \) in the embedding in Fig. 3b is v-consecutive.

Let \( \mathcal{O} \) be a v-consecutive ordering on \( E \). We define orderings \( \mathcal{O}_1^+ \) on \( V_1^+ \) and \( \mathcal{O}_2^- \) on \( V_2^- \) induced by \( \mathcal{O} \), as follows. Consider the edges in \( E \) one by one as they appear in \( \mathcal{O} \). Append the end-vertex in \( V_1^+ \) of the currently considered edge to a list \( L_1^+ \). Since \( \mathcal{O} \) is v-consecutive, the occurrences of the same vertex appear consecutively in \( L_1^+ \), when such a list is regarded as circular. Hence, \( L_1^+ \) can be turned into a circular ordering \( \mathcal{O}_1^+ \) on \( V_1^+ \) by removing all repetitions but one of the same vertex. Circular ordering \( \mathcal{O}_2^- \) can be constructed analogously. We have the following.

**Lemma 3.** Let \( \mathcal{O} \) be a circular ordering on \( E \) and let \( (\mathcal{O}_1, \mathcal{O}_2) \) be a pair of circular orderings on \( V_1 \) and \( V_2 \). There exists a radial level embedding of \( G \) whose edge ordering is \( \mathcal{O} \) and the circular orderings on \( V_1 \) and \( V_2 \) along \( l_1 \) and \( l_2 \) are \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \), respectively, if and only if \( \mathcal{O} \) is v-consecutive, and \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) extend the orderings \( \mathcal{O}_1^+ \) and \( \mathcal{O}_2^- \) on \( V_1^+ \) and \( V_2^- \) induced by \( \mathcal{O} \), respectively.

**Proof.** The necessity is trivial. For the sufficiency, suppose that \( \mathcal{O} \) is v-consecutive and that \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) extend the orderings \( \mathcal{O}_1^+ \) and \( \mathcal{O}_2^- \) on \( V_1^+ \) and \( V_2^- \) induced by \( \mathcal{O} \), respectively. We construct a radial level embedding \( \Gamma \) of \( G \) with the desired properties, as follows. Let \( \Gamma^* \) be a radial level embedding consisting of \( |E| \) (non-crossing) monotone curves, each connecting a distinct point on \( l_1 \) and a distinct point on \( l_2 \). We associate each of these curves with a distinct edge in \( E \), so that the edge ordering of \( \Gamma^* \) is \( \mathcal{O} \). Note that, since \( \mathcal{O} \) is v-consecutive, all the occurrences of the same vertex in \( V_1^+ \) and of \( V_2^- \) appear consecutively along \( l_1 \) and \( l_2 \), respectively. Hence, we can transform \( \Gamma^* \) into a radial level embedding \( \Gamma' \) of \( G' = (V_1^+ \cup V_2^-, E, \gamma) \), by continuously deforming the curves in \( \Gamma^* \) incident to occurrences of the same vertex in \( V_1^+ \) (in \( V_2^- \)) so that their end-points on \( l_1 \) (on \( l_2 \)) coincide. Since the circular orderings on \( V_1^+ \) and on \( V_2^- \) along \( l_1 \) and \( l_2 \) are \( \mathcal{O}_1^+ \) and \( \mathcal{O}_2^- \), respectively, we can construct \( \Gamma \) by inserting the isolated vertices in \( V_1^+ \backslash V_2^- \) and \( V_2^- \backslash V_1^+ \) at suitable points along \( l_1 \) and \( l_2 \), so that the circular orderings on \( V_1 \) and on \( V_2 \) along \( l_1 \) and \( l_2 \) are \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \), respectively.

We construct an instance \( I(G) \) of **Simultaneous PQ-Ordering** starting from a level graph \( G = (V_1 \cup V_2, E, \gamma) \) on two levels as follows; refer to the dashed box in Fig. 4, where \( I(G) = I(G_{i+1}) \) and \( i = 1 \). We define the level trees \( T_1 \) and \( T_2 \) of the first and second level as the universal PQ-trees on \( V_1 \) and \( V_2 \), respectively. Also, we define the layer tree \( T_{1,2} \) as the PQ-tree on \( E \); then, for each vertex \( v \in V_1 \cup V_2 \), we replace the current PQ-tree with its reduction by \( E(v) \) (that is, we let \( T_{1,2} = T_{1,2} \oplus E(v) \)), where \( E(v) \) is the set of edges in \( E \) incident to \( v \). We define the consistency trees \( T_1^+ \) and \( T_2^+ \) of the first and of the second level as the universal PQ-trees on \( V_1^+ \) and \( V_2^- \), respectively. Instance \( I(G) \) contains the level trees \( T_1 \) and \( T_2 \), the layer tree \( T_{1,2} \), and the consistency trees \( T_1^+ \) and \( T_2^- \), together with the arcs \( (T_1, T_1^+, \iota) \), \( (T_2, T_2^-, \iota) \), \( (T_{1,2}, T_1^+, \phi_1^+) \), and \( (T_{1,2}, T_2^-, \phi_2^-) \), where \( \iota \) denotes the identity map, \( \phi_1^+ \) assigns to each vertex in \( V_1^+ \) an incident edge in \( E \), and \( \phi_2^- \) assigns to each vertex in \( V_2^- \) an incident edge in \( E \). We have the following.
we have that \( O_1 \) and the circular ordering on \( V_2 \) along \( l_2 \) is \( O_2 \) if and only if instance \( I(G,i+1) \) of SIMULTANEOUS PQ-ORDERING admits a solution in which the circular ordering on the leaves of \( T_1 \) is \( O_1 \) and the circular ordering on the leaves of \( T_2 \) is \( O_2 \).

**Lemma 4.** Level graph \( G \) admits a radial level embedding in which the circular ordering on \( V_1 \) along \( l_1 \) is \( O_1 \) and the circular ordering on \( V_2 \) along \( l_2 \) is \( O_2 \) if and only if instance \( I(G,i+1) \) of SIMULTANEOUS PQ-ORDERING admits a solution in which the circular ordering on the leaves of \( T_1 \) is \( O_1 \) and the circular ordering on the leaves of \( T_2 \) is \( O_2 \).

**Proof.** We prove the necessity. Let \( \Gamma \) be a radial level embedding of \( G \). We construct an ordering on the leaves of each tree in \( I(G) \) as follows. Let \( O_1, O_2, O_1^+, O_2^- \) be the circular orderings on \( V_1 \) along \( l_1 \), on \( V_2 \) along \( l_2 \), on \( V_1^+ \) along \( l_1 \), and on \( V_2^- \) along \( l_2 \) in \( \Gamma \), respectively. Let \( \mathcal{O} \) be the edge ordering on \( E \) in \( \Gamma \). Note that \( \mathcal{O} \in \mathcal{O}(I(T_{1,2})) \) since \( \mathcal{O} \) is \( v \)-consecutive by Lemma 3. The remaining trees are universal, hence \( O_1 \in \mathcal{O}(T_1), O_2 \in \mathcal{O}(T_2), O_1^+ \in \mathcal{O}(T_1^+), \) and \( O_2^- \in \mathcal{O}(T_2^-) \).

We prove that all arcs of \( I(G) \) are satisfied. Arc \( (T_1, T_1^+, \iota) \) is satisfied if and only if \( O_1 \) extends \( O_1^+ \). This is the case since \( \iota \) is the identity map, since \( V_1^+ \subseteq V_1 \), and since \( O_1 \) and \( O_1^+ \) are the circular orderings on \( V_1^+ \) along \( l_1 \). An analogous proof shows that arc \( (T_2, T_2^-, \iota) \) is satisfied. Arc \( (T_{1,2}, T_1^+, \phi_1^+) \) is satisfied if and only if \( O_1 \) extends \( O_1^+ \). This is due to the fact that \( \phi_1^+ \) assigns to each vertex in \( V_1^+ \) an incident edge in \( E \) and to the fact that, by Lemma 3, ordering \( O \) is \( v \)-consecutive and \( O_1^+ \) is induced by \( O \). An analogous proof shows that arc \( (T_{1,2}, T_2^-, \phi_2^-) \) is satisfied.

We now prove the sufficiency. Suppose that \( I(G) \) is a positive instance of SIMULTANEOUS PQ-ORDERING, that is, there exist orderings \( O_1, O_2, O_1^+, O_2^- \), and \( \mathcal{O} \) of the leaves of the trees \( T_1, T_2, T_1^+, T_2^- \), and \( T_{1,2} \), respectively, that satisfy all arcs of \( I(G) \). We prove that there exists a radial level embedding \( \Gamma \) of \( G \) in which the circular orderings on \( V_1 \) and \( V_2 \) are \( O_1 \) and \( O_2 \), respectively. Since \( \iota \) is the identity map and since arcs \( (T_1, T_1^+, \iota) \) and \( (T_2, T_2^-, \iota) \) are satisfied, we have that \( O_1^+ \) and \( O_2^- \) are restrictions of \( O_1 \) and \( O_2 \) to \( V_1^+ \) and \( V_2^- \), respectively. Also, since \( (T_{1,2}, T_1^+, \phi_1^+) \) and \( (T_{1,2}, T_2^-, \phi_2^-) \) are satisfied, we have that \( \mathcal{O} \) extends both \( O_1^+ \) and \( O_2^- \). By the construction of \( T_{1,2} \), ordering \( O \) is \( v \)-consecutive. By Lemma 4, a radial level embedding \( \Gamma \) of \( G \) exists in which the circular ordering on \( V_1 \) along \( l_1 \) is \( O_1 \) and the circular ordering on \( V_2 \) along \( l_2 \) is \( O_2 \).

We are now ready to show how to construct an instance \( I^*(G) \) of SIMULTANEOUS PQ-ORDERING starting from a proper level graph \( G = (\bigcup_{i=1}^k V_i, E, \gamma) \) on \( k \) levels; refer to Fig. 4. For each \( i = 1, \ldots, k \), let \( I(G, i+1) \) be the instance of SIMULTANEOUS PQ-ORDERING constructed as described above starting from the level graph on two levels \( G_{i,i+1} = (V_i \cup V_{i+1}, (V_i \times V_{i+1}) \cap E, \gamma) \) (in the construction \( V_i \) takes the role of \( V_1 \) and \( V_{i+1} \) takes the role of \( V_2 \), and \( k + 1 = 1 \)). Note that two instances \( I(G_{i-1,i}) \) and \( I(G_{i,i+1}) \) share exactly the level tree \( T_i \), whereas non-adjacent instances are disjoint. We define \( I^*(G) = \bigcup_{i=1}^k I(G_{i,i+1}) \) and obtain \( I^*(G) \) by applying normalization to \( I^u(G) \). We now present two lemmata about properties of instance \( I^*(G) \).

**Lemma 5.** \( I^*(G) \) is 2-fixed, has size \( |I^*(G)| \in O(|G|) \), and can be constructed in \( O(|G|) \) time.

**Proof.** Every PQ-tree \( T \) in \( I^u(G) \) is either a source with exactly two children or a sink with exactly two parents; also, the normalization of \( I^u(G) \) to obtain \( I^*(G) \) does not alter this property. Thus every P-node in a PQ-tree \( T \) in \( I^*(G) \) is at most 2-fixed. In fact, recall that for a P-node \( \mu \) of a PQ-tree \( T \) with parents \( T_1, \ldots, T_r \), we have that \( \text{fixed}(\mu) = \omega + \sum_{i=1}^r (\text{fixed}(\mu_i) - 1) \), where \( \omega \) is the number of children of \( T \) fixing \( \mu \), and \( \mu_i \in T_i \) is the unique node in \( T_i \), with \( 1 \leq i \leq r \), fixed by \( \mu \). Hence, if \( T \) is a source PQ-tree, it holds that \( \omega = 2 \) and \( r = 0 \); whereas, if \( T \) is a sink PQ-tree, it holds that \( \omega = 0 \), \( r = 2 \), and \( \text{fixed}(\mu_i) = 2 \) for each parent \( T_i \) of \( T \). Therefore \( I^*(G) \) is 2-fixed.
Since every internal node of a PQ-tree in $I^*(G)$ has degree greater than 2, to prove the bound on $|I^*(G)|$ it suffices to show that the total number of leaves over all PQ-trees in $I^*(G)$ is linear in $|G|$. Since $\mathcal{L}(T_i) = V_i$ and $\mathcal{L}(T_i^-), \mathcal{L}(T_i^+) \subseteq V_i$, the number of leaves over all level and consistency trees is at most $3 \sum_{i=1}^{k} |V_i| = O(|G|)$. Also, since $\mathcal{L}(T_{i,i+1}) \subseteq (V_i \times V_{i+1}) \cap E$, the number of leaves over all layer trees is at most $\sum_{i=1}^{k} |(V_i \times V_{i+1}) \cap E| = O(|G|)$. Thus $|I^*(G)| = O(|G|)$.

We already discussed how every layer tree $T_{i,i+1}$ can be constructed in $O(|G|)$ time; also, layer trees and consistency trees are stars, hence they can be trivially constructed in linear time in the size of their leaf sets. Finally, the normalization of each arc $(T_i, T_j; \phi)$ can be performed in $O(|T_i| + |T_j|)$ time. Hence, the whole construction can be performed in $O(|G|)$ time.

**Lemma 6.** Level graph $G$ admits a torus level embedding if and only if $I^*(G)$ is a positive instance of SIMULTANEOUS PQ-ORDERING.

**Proof.** Suppose that $G$ admits a torus level embedding $\Gamma$. For each $i = 1, \ldots, k$, denote by $O_i$ the circular ordering on $V_i$ along $l_i$. By Observation 7 embedding $\Gamma$ determines a radial level embedding $T_{i,i+1}$ of $G_{i,i+1}$. By Lemma 4 for each $i = 1, \ldots, k$, there exists a solution for the instance $I(G_{i,i+1})$ in which the circular ordering on the leaves of $T_i$ is $O_i$ and the circular ordering on the leaves of $T_{i,i+1}$ is $O_{i+1}$. Since the circular ordering on the leaves of $T_i$ is $O_i$ both in $I(G_{i-1,i})$ and $I(G_{i+1,i+1})$, since each arc of $I^*(G)$ is satisfied as it belongs to exactly one instance $I(G_{i,i+1})$, and since $I(G_{i,i+1})$ is a positive instance of SIMULTANEOUS PQ-ORDERING, it follows that the circular orderings deriving from the instances $I(G_{i,i+1})$ define a solution for $I^*(G)$.

Suppose that $I^*(G)$ admits a solution. Let $O_1, \ldots, O_k$ be the circular orderings on the leaves of the level trees $T_1, \ldots, T_k$ in this solution. By Lemma 4 for each $i = 1, \ldots, k$ with $k + 1 = 1$, there exists a radial level embedding of level graph $G_{i,i+1}$ in which the circular orderings on $V_i$ along $l_i$ and $V_{i+1}$ along $l_{i+1}$ are $O_i$ and $O_{i+1}$, respectively. By Observation 1 $G$ is torus level planar.

We thus get the main result of this paper.

**Theorem 2.** TORUS LEVEL PLANARITY can be tested in quadratic (quartic) time for proper (non-proper) instances.

**Proof.** Consider any instance $G$ of TORUS LEVEL PLANARITY. Assume first that $G$ is proper. By Lemma 5 it is possible to construct a 2-fixed instance $I^*(G)$ of SIMULTANEOUS PQ-ORDERING with $|I^*(G)| = O(|G|)$ in linear time. By Lemma 4 instance $I^*(G)$ is equivalent to $G$. Finally, by Theorem 1 instance $I^*(G)$ can be tested in quadratic time.

If $G$ is not proper, then subdivide every edge $(u, v)$ that spans $h > 2$ levels with $h - 2$ vertices, which are assigned to levels $\gamma(u) + 1, \gamma(u) + 2, \ldots, \gamma(v) - 1$. This increases the size of the graph quadratically in the worst case, hence the quartic running time follows.

We are now ready to state our result about CYCLIC LEVEL PLANARITY.

**Theorem 3.** CYCLIC LEVEL PLANARITY can be solved in quadratic (quartic) time for proper (non-proper) instances.

**Proof.** The statement comes from Theorem 2 and Lemma 2.

We remark that the testing algorithms of Theorems 2 and 3 can be turned into embedding algorithms with the running time, as the algorithm [7] to test 2-fixed instances of SIMULTANEOUS PQ-ORDERING also provides the orderings on the leaf sets of the PQ-trees, if the instance is positive.

We also remark that our techniques allow us to solve an even more general problem, that we call TORUS $\mathcal{T}$-LEVEL PLANARITY, in which a level graph $G = (\bigcup_{i=1}^{k} V_i, E, \gamma)$ is given together with a set of PQ-trees $T = \{T_1, \ldots, T_k\}$ such that $\mathcal{L}(T_i) = V_i$, where each tree $T_i$ encodes consecutivity constraints on the ordering on $V_i$ along $l_i$. The goal is then to test the existence of a level embedding of $G$ on $T$ in which the circular ordering on $V_i$ along $l_i$ belongs to $O(T_i)$.

This problem has been studied in the plane [12] under the name of $\mathcal{T}$-LEVEL PLANARITY, and has been shown to be NP-hard in general and polynomial-time solvable for proper instances. While the former result implies the NP-hardness of TORUS $\mathcal{T}$-LEVEL PLANARITY, the techniques of this paper show that TORUS $\mathcal{T}$-LEVEL PLANARITY can also be solved in polynomial time for proper instances. Namely, in the construction of instance $I^*(G)$ of SIMULTANEOUS PQ-ORDERING performed in this section, it suffices to replace layer tree $T_i$ with PQ-tree $T_i$. Analogous considerations allow us to extend this result to RADIAL $\mathcal{T}$-LEVEL PLANARITY and CYCLIC $\mathcal{T}$-LEVEL PLANARITY.
4 Simultaneous Level Planarity

In this section we prove that Simultaneous Level Planarity is NP-complete for three graphs on three levels and for three graphs on two levels, while it is polynomial-time solvable for two graphs on two levels. Both NP-hardness proofs rely on a reduction from the NP-complete problem BETWEENNESS [18], which asks for a ground set $S$ and a set $X$ of ordered triplets of $S$, with $|S| = n$ and $|X| = k$, whether a linear order $\prec$ of $S$ exists such that, for any $(\alpha, \beta, \gamma) \in X$, it is $\alpha \prec \beta \prec \gamma$ or $\gamma \prec \beta \prec \alpha$. Both proofs exploit the following two gadgets.

The ordering gadget is a pair $[G_1, G_2]$ of level graphs on levels $l_1$ and $l_2$, where the bottom level $l_1$ contains $nk$ vertices $u_1,1, \ldots, u_{i,n}$, and the top level $l_2$ contains $n(k-1)$ vertices $v_1,1, \ldots, v_{i,n}, v_{k-1,1}, \ldots, v_{k-1,n}$. For $i = 1, \ldots, k - 1$ and $j = 1, \ldots, n$, $G_1$ contains edge $(u_{i,j}, v_{i,j})$ and $G_2$ contains edge $(u_{i+1,j}, v_{i,j})$. See $G_1$ and $G_2$ in Fig. 5(a). Consider any simultaneous level embedding $I'$ of $\langle G_1, G_2 \rangle$ and assume, w.l.o.g. up to a renaming, that $u_1,1, \ldots, u_{i,n}$ appear in this left-to-right order along $l_1$. We have the following.

Lemma 7. For every $i = 1, \ldots, k$, vertices $u_{i,1}, \ldots, u_{i,n}$ appear in this left-to-right order along $l_1$; also, for every $i = 1, \ldots, k - 1$, vertices $v_{i,1}, \ldots, v_{i,n}$ appear in this left-to-right order on $l_2$.

Proof. Suppose, for a contradiction, that the statement does not hold. Then let $k^*$ be the smallest index such that either:

(A) for every $i = 1, \ldots, k^* - 1$, vertices $u_{i,1}, \ldots, u_{i,n}$ appear in this left-to-right order along $l_1$; for every $i = 1, \ldots, k^* - 1$, vertices $v_{i,1}, \ldots, v_{i,n}$ appear in this left-to-right order along $l_2$; and vertices $u_{k^*,1}, \ldots, u_{k^*,n}$ do not appear in this left-to-right order along $l_1$; or

(B) for every $i = 1, \ldots, k^*$, vertices $u_{i,1}, \ldots, u_{i,n}$ appear in this left-to-right order along $l_1$; for every $i = 1, \ldots, k^*-1$, vertices $v_{i,1}, \ldots, v_{i,n}$ appear in this left-to-right order along $l_2$; and vertices $v_{k^*,1}, \ldots, v_{k^*,n}$ do not appear in this left-to-right order along $l_2$.

Suppose that we are in Case (A), as Case (B) is analogous. Then $v_{k^*-1,1}, \ldots, v_{k^*-1,n}$ appear in this left-to-right order along $l_2$, while $u_{k^*,1}, \ldots, u_{k^*,n}$ do not appear in this left-to-right order along $l_1$, Hence, there exist indices $i$ and $j$ such that $v_{k^*-1,i}$ is to the left of $v_{k^*,1,j}$ along $l_1$, while $u_{k^*,i}$ is to the right of $u_{k^*-1,j}$ along $l_1$. Hence, edges $(u_{k^*,i}, u_{k^*-1,j})$ and $(u_{k^*,i}, v_{k^*,1,j})$ cross, thus contradicting the assumption that $I'$ is a simultaneous level embedding, as they both belong to $G_2$.

The triplet gadget is a path $T = (w_1, \ldots, w_5)$ on two levels, where $w_1, w_3$, and $w_5$ belong to the same level $l_1$ and $w_2$ and $w_4$ belong to the same level $l_2$. See $G_3$ in Fig. 5(a). We have the following.

Lemma 8. In every level embedding of $T$, vertex $w_3$ is between $w_1$ and $w_5$ along $l_i$.

Proof. Suppose, for a contradiction, that $w_3$ is to the left of $w_1$ and $w_5$ along $l_i$; the case in which it is to their right is analogous. Also assume that $l_i$ is below $l_j$, as the other case is symmetric. If $w_2$ is to the left of $w_3$ along $l_j$, then edges $(w_1, w_2)$ and $(w_3, w_4)$ cross, otherwise edges $(w_3, w_2)$ and $(w_5, w_4)$ cross. In both cases we have a contradiction.

We are now ready to prove the claimed NP-completeness results.

Theorem 4. Simultaneous Level Planarity is NP-complete for three graphs on two levels and for two graphs on three levels.

Proof. Both problems clearly are in $NP$. We prove the $NP$-hardness only for three graphs on two levels (see Fig. 5(a)), as the other proof is analogous (see Fig. 5(b)). From an instance $\langle S = \{u_{1,1}, \ldots, u_{1,n}\}, X = \{u_{1,a_i}, u_{1,b_i}, u_{1,c_i} : i = 1, \ldots, k\} \rangle$ of BETWEENNESS, we construct an instance $\langle G_1(V, E_1, \gamma), G_2(V, E_2, \gamma), G_3(V, E_3, \gamma) \rangle$ of Simultaneous Level Planarity as follows: Pair $\langle G_1, G_2 \rangle$ contains an ordering gadget on levels $l_1$ and $l_2$, where the vertices $u_{1,1}, \ldots, u_{1,n}$ of $G_1$ are the elements of $S$. Graph $G_3$ contains $k$ triplet gadgets $T_i(u_{i,a_i}, u_{i,b_i}, y_i, u_{i,c_i})$, for $i = 1, \ldots, k$. Vertices $x_j, y_1, \ldots, x_k, y_k$ are all distinct and are on $l_2$. Clearly, the construction can be carried out in linear time. We prove the equivalence of the two instances.

($\Rightarrow$) Suppose that a simultaneous level embedding $I'$ of $\langle G_1, G_2, G_3 \rangle$ exists. We claim that the left-to-right order of $u_{1,1}, \ldots, u_{1,n}$ along $l_1$ satisfies the betweenness constraints in $X$. Suppose, for a contradiction, that $u_{1,a_i}, u_{1,b_i}, u_{1,c_i} \in$
X exists with $u_{1,b}$ not between $u_{1,a_1}$ and $u_{1,c_1}$ along $l_1$. By Lemma $7$ $u_{i,b_i}$ is not between $u_{i,a_i}$ and $u_{i,c_i}$. By Lemma $8$ $T_i(u_{i,a_i}, x_i, u_{i,b_i}, y_i, u_{i,c_i})$ is not planar in $\Gamma$, a contradiction.

($\Leftarrow$) Suppose that $(S, X)$ is a positive instance of BETWEENNESS, and assume, w.l.o.g. up to a renaming, that $u_1, u_2, \ldots, u_n$ is a solution for $(S, X)$, that is, the ordering $u_{1,1}, \ldots, u_{1,n}$ satisfies all the betweenness constraints in $X$. Construct a straight-line simultaneous level planar drawing of $(G_1, G_2, G_3)$ with:

(i) $u_{1,1}, \ldots, u_{1,n}, \ldots, u_{k,1}, \ldots, u_{k,n}$ in this left-to-right order along $l_1$,

(ii) $v_{1,1}, \ldots, v_{1,n}, \ldots, v_{k-1,1}, \ldots, v_{k-1,n}$ in this left-to-right order along $l_2$,

(iii) $x_i$ and $y_i$ to the left of $x_{i+1}$ and $y_{i+1}$, for $i = 1, \ldots, k-1$, and

(iv) $x_i$ to the left of $y_i$ if and only if $u_{1,a_i} < u_{1,c_i}$.

Properties (i) and (ii) guarantee that, for any two edges $(u_{i,j}, v_{i,j})$ and $(u_{i',j'}, v_{i',j'})$, vertex $u_{i,j}$ is to the left of $u_{i',j'}$ along $l_1$ if and only if $v_{i,j}$ is to the left of $v_{i',j'}$ along $l_2$, which implies the planarity of $G_1$ in $\Gamma$. The planarity of $G_2$ in $\Gamma$ is proved analogously. Properties (i) and (iii) imply that no two paths $T_i$ and $T_j$ cross each other, while Property (iv) guarantees that each path $T_i$ is planar. Hence, the drawing of $G_3$ in $\Gamma$ is planar.

We remark that the graphs in the proof of Theorem $4$ can be made connected, via new vertices and edges, at the expense of using an additional level. Also, the proof remains valid even if the simultaneous embedding is with fixed edges or geometric (see $10$ for proper definitions).

In contrast to the NP-hardness results, a reduction to Cyclic Level Planarity allows us to establish the following.

**Theorem 5.** Simultaneous Level Planarity is quadratic-time solvable for two graphs on two levels.

**Proof.** Let $\langle G_1(V, E_1, \gamma), G_2(V, E_2, \gamma) \rangle$ be an instance of the SIMULTANEOUS LEVEL PLANARITY problem, where each of $G_1$ and $G_2$ is a level graph on two levels $l_1$ and $l_2$. We define a proper instance $(V, E, \gamma)$ of Cyclic Level Planarity as follows. The vertex set $V$ is the same as the one of $G_1$ and $G_2$, as well as the function $\gamma : V \to \{1, 2\}$; further, $E$ contains an edge $(u, v)$ for every $(u, v) \in E_1$ and an edge $(v, u)$ for every $(u, v) \in E_2$. We prove that $\langle G_1(V, E_1, \gamma), G_2(V, E_2, \gamma) \rangle$ is simultaneous level planar if and only if $(V, E, \gamma)$ is cyclic level planar.

($\Rightarrow$) Consider a simultaneous level embedding of $G_1$ and $G_2$, map it to the surface $\mathbb{R}$ of the rolling cylinder, and wrap the edges of $G_2$ around the part of $\mathbb{R}$ delimited by $l_1$ and $l_2$ and not containing the edges of $G_1$; this results in a cyclic level embedding of $(V, E, \gamma)$.

($\Leftarrow$) Conversely, consider a cyclic level embedding of $(V, E, \gamma)$ on $\mathbb{R}$, reroute the edges of $G_2$ so that they lie in the part of $\mathbb{R}$ delimited by $l_1$ and $l_2$ and containing the edges of $G_1$, and map this drawing to the plane; this results in a simultaneous level embedding of $G_1$ and $G_2$.

The statement of the theorem then follows from Corollary $5$ and from the fact that the described reduction can be performed in linear time.
5 Conclusions and Open Problems

In this paper we have settled the computational complexity of two of the main open problems in the research topic of level planarity. Namely, we have shown that the CYCLIC LEVEL PLANARITY and the TORUS LEVEL PLANARITY problems, which are natural extensions of the LEVEL PLANARITY problem to embeddings on the surfaces of the cylinder and of the torus, are polynomial-time solvable.

Our algorithms run in quartic time in the graph size; it is hence an interesting challenge to design new techniques to improve this time bound. In particular, both the quest for a linear-time algorithm for proper instances and the quest for an algorithm for non-proper instances that does not include a transformation to equivalent proper instances of quadratic size are worth future research efforts.

An intriguing research direction is the one of extending the concept of level planarity to surfaces with genus larger than one. While the definitions of level embeddings on the sphere, on the cylinder, and on the torus quite naturally follow from the one on the plane, there seems to be more than one meaningful way to arrange \( k \) levels on a high-genus surface. A reasonable choice would be the one shown in Fig. 6 in which the arrangement of the levels is a set of parallel paths between two distinguished levels \( l_s \) and \( l_t \) (and edges only connect vertices on two levels on the same path). Note that RADIAL LEVEL PLANARITY and TORUS LEVEL PLANARITY can be regarded as special cases of this setting (with only one and two paths of levels between \( l_s \) and \( l_t \), respectively). It seems plausible that our techniques could be strengthened in order to derive an FPT algorithm for LEVEL PLANARITY in this setting, where the parameter is the genus of the surface.

We also introduced a notion of simultaneous level planarity for level graphs and we established a complexity dichotomy for the corresponding decision problem. It would be interesting to study how to embed multiple level graphs on surfaces other than the plane. While the complexity of the SIMULTANEOUS RADIAL LEVEL PLANARITY problem seems to be the same as the complexity of SIMULTANEOUS LEVEL PLANARITY, it is not hard to see that the SIMULTANEOUS CYCLIC LEVEL PLANARITY problem is intractable already for two graphs on two levels.
References

1. P. Angelini, G. Da Lozzo, G. Di Battista, F. Frati, and V. Roselli. The importance of being proper (in clustered-level planarity and T-level planarity). *Theor. Comp. Sci.*, 571:1–9, 2015.
2. C. Auer, C. Bachmaier, F. Brandenburg, and A. Gleißner. Classification of planar upward embedding. In M. J. van Kreveld and B. Speckmann, editors, *Graph Drawing (GD ’11)*, volume 7034 of *LNCS*, pages 415–426. Springer, 2012.
3. C. Bachmaier and W. Brunner. Linear time planarity testing and embedding of strongly connected cyclic level graphs. In D. Halperin and K. Mehlhorn, editors, *ESA ’08*, volume 5193 of *LNCS*, pages 136–147. Springer, 2008.
4. C. Bachmaier, W. Brunner, and C. König. Cyclic level planarity testing and embedding. In S. Hong, T. Nishizeki, and W. Quan, editors, *GD ’07*, volume 4875 of *LNCS*, pages 50–61, 2007.
5. O. Bastert and C. Matuszewski. Layered drawings of digraphs. In M. Kaufmann and D. Wagner, editors, *Drawing Graphs: Methods and Models*, volume 2025 of *LNCS*, pages 87–120. Springer, 2001.
6. T. Bläsius, S. G. Kobourov, and I. Rutter. Simultaneous embeddings of planar graphs. In R. Tamassia, editor, *Handbook of Graph Drawing and Visualization*, Discrete Mathematics and Its Applications, chapter 11, pages 349–382. Chapman and Hall/CRC, 2013.
7. T. Bläsius and I. Rutter. Simultaneous pq-ordering with applications to constrained embedding problems. *ACM Trans. Algorithms*, 12(2):16, 2016.
8. K. S. Booth and G. S. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using pq-tree algorithms. *J. Comput. Syst. Sci.*, 13(3):335–379, 1976.
9. F. Brandenburg. Upward planar drawings on the standing and the rolling cylinders. *Comput. Geom.*, 47(1):25–41, 2014.
10. P. Brass, E. Cenek, C. A. Duncan, A. Efrat, C. Erten, D. P. Ismailescu, S. G. Kobourov, A. Lubiw, and J. S. Mitchell. On simultaneous planar graph embeddings. *Comput. Geom. Theory Appl.*, 36(2):117–130, 2007.
11. G. Di Battista and E. Nardelli. Hierarchies and planarity theory. *IEEE Trans. Syst. Man Cybern.*, 18(6):1035–1046, 1988.
12. A. Estrella-Balderrama, J. J. Fowler, and S. G. Kobourov. On the characterization of level planar trees by minimal patterns. In D. Eppstein and E. R. Gansner, editors, *Graph Drawing (GD ’09)*, volume 5849 of *LNCS*, pages 69–80. Springer, 2010.
13. M. Forster and C. Bachmaier. Clustered level planarity. In P. van Emde Boas, J. Pokorný, M. Bieliková, and J. Stuller, editors, *SOFSEM*, volume 2932 of *LNCS*, pages 218–228, 2004.
14. R. Fulek, M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. *Thirty Essays on Geometric Graph Theory*, chapter Hanani–Tutte, Monotone Drawings, and Level-Planarity, pages 263–287. Springer New York, New York, NY, 2013.
15. P. Healy, A. Kuusik, and S. Leipert. A characterization of level planar graphs. *Discrete Mathematics*, 280(1-3):51–63, 2004.
16. L. S. Heath and S. V. Pemmaraju. Recognizing leveled-planar dags in linear time. In F. Brandenburg, editor, *Graph Drawing, Symposium on Graph Drawing, GD ’95, Passau, Germany, September 20-22, 1995, Proceedings*, volume 1027 of *Lecture Notes in Computer Science*, pages 300–311. Springer, 1995.
17. M. Jünger, S. Leipert, and P. Mutzel. Level planarity testing in linear time. In S. Whitesides, editor, *GD ’98*, volume 1547 of *LNCS*, pages 224–237. Springer, 1998.
18. J. Opatrny. Total ordering problems. *SIAM J. Comput.*, 8(1):111–114, 1979.
19. B. Randerath, E. Speckenmeyer, E. Boros, P. L. Hammer, A. Kogan, M. Makino, B. Simeone, and O. Cepek. A satisfiability formulation of problems on level graphs. *Electronic Notes in Discrete Mathematics*, 9:269–277, 2001.
20. K. Sugiyama, S. Tagawa, and M. Toda. Methods for visual understanding of hierarchical system structures. *IEEE Trans. Syst. Man Cybern.*, 11(2):109–125, 1981.
21. A. Wotzlaw, E. Speckenmeyer, and S. Porschen. Generalized k-ary tanglegrams on level graphs: A satisfiability-based approach and its evaluation. *Discrete Applied Mathematics*, 160(16-17):2349–2363, 2012.