Running vacuum cosmological models: linear scalar perturbations

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Abstract. In cosmology, phenomenologically motivated expressions for running vacuum are commonly parameterized as linear functions typically denoted by $\Lambda(H^2)$ or $\Lambda(R)$. Such models assume an equation of state for the vacuum given by $P_{\Lambda} = -\rho_{\Lambda}$, relating its background pressure $P_{\Lambda}$ with its mean energy density $\rho_{\Lambda} \equiv \Lambda/8\pi G$. This equation of state suggests that the vacuum dynamics is due to an interaction with the matter content of the universe. Most of the approaches studying the observational impact of these models only consider the interaction between the vacuum and the transient dominant matter component of the universe. We extend such models by assuming that the running vacuum is the sum of independent contributions, namely $\rho_{\Lambda} = \sum_i \rho_{\Lambda_i}$. Each $\Lambda_i$ vacuum component is associated and interacting with one of the $i$ matter components in both the background and perturbation levels. We derive the evolution equations for the linear scalar vacuum and matter perturbations in those two scenarios, and identify the running vacuum imprints on the cosmic microwave background anisotropies as well as on the matter power spectrum. In the $\Lambda(H^2)$ scenario the vacuum is coupled with every matter component, whereas the $\Lambda(R)$ description only leads to a coupling between vacuum and non-relativistic matter, producing different effects on the matter power spectrum.

Keywords: cosmological perturbation theory, dark energy theory

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1 Introduction

One of the most important discoveries of the 20th century was that the universe is expanding [1], and more surprisingly, that it is accelerating [2, 3]. In large scales, gravitation is the dominant physical interaction in the universe. Furthermore, in the context of General Relativity, the only way to generate a global accelerated expansion is to add an energetic component with negative pressure. This component is known as dark energy (DE), and it is thought to permeate smoothly every part of the universe, achieving a global repulsive effect and dominating the cosmic evolution in the low redshift regime.

A remarkably good and widely accepted cosmological model known as ΛCDM considers the cosmological constant (CC) on Einstein’s field equations as accounting for such an acceleration. The CC is the simplest model for DE. Another exotic component referred as cold dark matter (CDM) is required by the ΛCDM model in order to reproduce the growth of the structures in cluster and galaxy scales, as well as the Cosmic Microwave Background (CMB) spectrum [4–6]. However, this framework possesses some important theoretical problems. In the general relativity context, a bare CC requires a fine-tuning of about 100 orders of magnitude, such that when added to the expected value of the quantum vacuum energy density the sum matches with the DE density estimated from astronomical observations. This theoretical conundrum is known as the CC fine tuning problem [7–11]. Another issue is that, in spite of behaving very differently throughout the cosmic expansion history, the CDM and DE contributions — to the total energy density of the universe — are today about the same order of magnitude. This riddle is known as the cosmic coincidence problem.

Independent astronomical observations support the existence of DE [2–6, 12], but do not provide a single clue about its origin from first principles. There are several proposals for DE candidates besides the pure positive CC, like quintessence, K-essence, chameleon field, and modified gravity among others; for a review see [13, 14] and references therein.

Another formulation considers the DE as a decaying entity, whose dynamics can be modeled through an effective interaction with the matter components. It is difficult to describe such interactions from first principles due to the lack of information about the nature of the DE, therefore, they are often described phenomenologically. The most studied approaches of interacting DE are the interacting dark sector models (only considering interaction between DE and CDM), see [15–17] among many others. In this vein, the model proposed in the present work extends this idea, allowing DE to interact with all of the matter components. We also assume that DE can be decomposed as a sum of partial contributions associated to — and interacting with — the matter components (say photons, or baryons, or CDM, etc.).

In this work, we identify the DE as being a varying CC whose dynamics is due to the quantum effects of the matter fields in an evolving curved space-time, and we call it henceforth running vacuum. We will specialize on the class of models that parameterize the CC variation as a function of the Hubble parameter $H$ or the Ricci scalar $R$, with a constant gravitational coupling $G$ [11, 18, 19]. A recent motivation for this is that CC dynamics has emerged in the context of the renormalization group approach. In that scenario, the simultaneous running of the CC and $G$ due to quantum effects has been considered, see [20–24]. Those studies have shown that corrections to $G$ vary logarithmically, with a scale parameter $\mu$, while corrections to the CC evolve quadratically with $\mu$. In the cosmological context, and following [19], we identify the scale parameter as being $\mu \sim R^{1/2}$ or $\mu \sim H$. It is worth noting that previous to the renormalization group formulation of the running CC, decaying DE models were studied by several authors from the phenomenological point of view [25, 26].
Most of the work about this subject has been focused mainly on the study of density evolution, cosmological consequences and observational constraints of the running vacuum at the background level. Effects of these models at perturbative level have been studied shallowly, and have only begun to receive more attention recently [27–37]. In such perturbative studies, the running vacuum is often modeled as decaying into the dominant matter component of each cosmic era, i.e. not considering contributions from other matter components. Due to both the quantity and quality of current observational data, such an approximation may not be appropriate when modeling the evolution of linear perturbations. Unfortunately, the phenomenological expressions, $\Lambda(H^2)$ or $\Lambda(R)$, suggested by [19] for the mean vacuum energy density lack a Lagrangian origin. Therefore, the explicit form of the evolution equations for the vacuum perturbations is not univocally defined in those models. The aim of the present work is to find an auto-consistent formulation for the perturbations of the running vacuum interacting with their matter counterparts, as well as to identify their observational imprints on the cosmic background radiation (CMB) and on the matter power spectrum.

The structure of the paper is as follows. In section 2, we review two types of running vacuum models, which describe the $\Lambda$ term as linear functions $\Lambda(H^2)$ and $\Lambda(R)$, at the background level. In section 3 we show the fluid conservation equations for coupled species, where the coupling terms still remain unspecified. In sections 4 and 5, we apply the linear scalar perturbation theory to the running vacuum models described in section 2. Using the Boltzmann equation, we find the coupling terms between the running vacuum and the matter components for each model. The behavior of the vacuum perturbations for sub-horizon modes is described, and the super-horizon initial conditions are founded. In section 7 we show and discuss the result of integrating numerically the complete set of cosmological equations, for which we modified the free code CLASS [38]. Moreover, we use Planck 2015 data set [39] and the statistical analysis package MontePython [40] to derive observational constraints. Finally, in section 8 we present our conclusions and some important remarks.

2 Running vacuum from renormalization group

In the cosmological context, several works have suggested a time dependency of the DE density. Using phenomenological arguments, some authors have proposed a parametrization of it as a function of the Hubble parameter $H$, namely $\Lambda = \Lambda(H)$. These models were already confronted with observations — supernovae, baryon acoustic oscillations (BAO), CMB, and large scale structure — providing promising results [24, 36, 41–46]. In another approach, the CC issues have motivated the study of the quantum effects produced by the matter fields over the vacuum energy density in curved space-times, and their possible implications for the role of the DE in cosmology. In this sense, the renormalization group formalism is used to study and parameterize the leading quantum corrections to the vacuum energy density, laying foundations for these models in more fundamental grounds [11, 18, 19, 47].

Usually, the renormalization group approach of quantum field theory in Minkowski space provides a useful theoretical tool to investigate how the gauge coupling constants and charges run with a scale $\mu$ associated to the typical energy scale of the process. Similarly, the mean vacuum energy density $\bar{\rho}_\Lambda = \Lambda/8\pi G$ should depend on the energy scale of the gravitational processes on cosmological scales (G is the Newtonian gravitational constant). Using renormalization group arguments, it is proposed that the $\Lambda$-term on the right-hand
side of Einstein’s field equations can be expanded as an even power series of \(\mu\) \([18, 19, 48, 49]\)

\[
\frac{d \Lambda}{d \ln \mu} = m_2 \mu^2 + m_4 \mu^4 + \cdots.
\]  

(2.1)

This last expression is generic, and \(\mu\) must be selected in a way that it properly traces the gravitational energy scale of the cosmological evolution. For dynamical system analysis of covariant and non-covariant parametrizations of \(\Lambda(\mu)\) see \([50]\).

### 2.1 \(\Lambda(H^2)\) model

A natural choice for the scale parameter \(\mu\) is the total energy density of the universe

\(\mu^2 = \rho_T \propto H^2\). With that, we can rewrite eq. (2.1) as:

\[
d \Lambda/d \ln H = 2 c_2 H^2 + 4 c_4 H^4 + \cdots + n c_n H^n + \cdots
\]  

\([11, 49, 51]\), which can be integrated to obtain

\[
\Lambda(H^2) = c_0 + c_2 H^2 + c_4 H^4 + \cdots + c_n H^n + \cdots
\]

which can be rewritten as:

\[
\rho_\Lambda \equiv \rho_\Lambda^0 \frac{\Omega_\Lambda - \alpha}{1 - \alpha} + \sum_i \rho_{\Lambda i},
\]

(2.3)

where \(\rho_\Lambda^0\) is the present critical density, and we have defined

\[
\rho_{\Lambda i} \equiv \frac{\alpha}{1 - \alpha} \bar{\rho}_i
\]

(2.4)

Here we have split the vacuum energy density into a sum of independent contributions \(\bar{\rho}_{\Lambda i}\), one for each matter component \(\bar{\rho}_i\). The index \(i\) denotes the matter components, i.e., photons, massless neutrinos, massive neutrinos, CDM and baryons (\(\gamma, \nu, h, c, b\)); and the index \(\Lambda i\) denotes the vacuum partner associated to \(i\).

### 2.2 \(\Lambda(R)\) model

Another reasonable choice for the scale parameter is \(\mu^2 = R\), see \([49, 50]\). Replacing \(\mu^2 = R\) into equation (2.1) we have

\[
\Lambda(R) = c_0 + c_1 R + c_2 R^2 + \cdots + c_n R^n + \cdots
\]

In this case, the coefficients \(c_n\) have dimension \(R_\text{I}^{-n}\), where \(R_\text{I} \approx 12H_\text{I}^2\) was chosen here as being the typical curvature associated with the inflationary stage. Once again, high order terms do
not contribute efficiently to the post-inflationary stages of the universe. Setting \( c_1 = \beta \) (the model parameter for this case) we have that the vacuum energy density reduces to:

\[
\Lambda - \Lambda_0 = \beta (R - R_0) .
\]

(2.5)

Models with \( \Lambda = \Lambda(R) \) have been studied in [19]. For the flat FLRW metric the scalar curvature is given by \( R = 8\pi G \sum_i (\bar{p}_i - 3\bar{\rho}_i) + 4\Lambda \), therefore we propose to rewrite \( \bar{\rho}_\Lambda \) in the form

\[
\bar{\rho}_\Lambda \equiv \frac{\Lambda_0 - \beta R_0}{8\pi G (1 - 4\beta)} + \sum_i \bar{\rho}_{\Lambda i},
\]

(2.6)

where we have defined

\[
\bar{\rho}_{\Lambda i} \equiv \frac{\beta}{1 - 4\beta} (\bar{p}_i - 3\bar{\rho}_i).
\]

(2.7)

Here each \( \bar{\rho}_{\Lambda i} \) is interacting with the matter component \( i \), and will be treated as a partial contribution to the total vacuum energy density.

3 Linear scalar perturbations

In this section, we summarize the conservation equations for the scalar perturbations associated with a pair of fluids exchanging energy, where one of them (the vacuum) has the equation of state \( P_\Lambda = -\rho_\Lambda \), while the other one (the matter) has a non-negative pressure given by \( P = \omega \rho \) with \( \omega = \text{const.} \geq 0 \). Because of the interaction, only the energy-momentum tensor associated with the pair (vacuum plus matter) is conserved. Thus, if we split the conservation equation associated to the interacting pair we will obtain two coupled equations. The coupling term relating the matter and the vacuum conservation equations will be determined — in the next section — applying the Boltzmann equation to the matter components. The calculations and notation are based on the work of Ma and Bertschinger [57].

Let’s consider a flat FLRW spacetime in the synchronous gauge, where the perturbed line element is:

\[
ds^2 = a^2 \left[ -d\tau^2 + (\delta_{ij} + h_{ij}) dx^i dx^j \right],
\]

(3.1)

where, \( \tau \) is the conformal time, \( a = a(\tau) \) is the scale factor, and \( h_{ij} = h_{ij}(\vec{x}, \tau) \) is the metric perturbation, which can be written in Fourier space as

\[
h_{ij} \left( \vec{x}, \tau \right) = \int d^3k \ e^{i\vec{k} \cdot \vec{x}} \left[ \hat{k}_i \hat{k}_j h(\vec{k}, \tau) + \left( \hat{k}_i \hat{k}_j - \frac{\delta_{ij}}{3} \right) 6 \eta(\vec{k}, \tau) \right],
\]

(3.2)

where \( \vec{k} = k \hat{k} \), while \( h \) and \( \eta \) are the scalar functions that represent the scalar degrees of freedom of the perturbed metric. In this gauge, the unperturbed components of the Einstein field equation are:

\[
3H^2 = 8\pi Ga^2 \sum i \bar{P}, \quad -\frac{\ddot{a}}{a} + \mathcal{H}^2 = 8\pi Ga^2 \sum \bar{P},
\]

(3.3)
where $H \equiv \dot{a}/a$ is the comoving Hubble parameter, the overdot denotes derivative with respect to $\tau$, the overline denotes background averaged quantities, and the sum covers every energetic contribution (matter components $i$ as well as its vacuum counterparts $\Lambda_i$). To first order, the Einstein equations for scalar perturbations are:

\begin{align}
  k^2 \eta - \frac{1}{2} H \dot{h} &= -4\pi G a^2 \sum \bar{\rho} \delta , \\
  -k^2 \dot{\eta} &= -4\pi G a^2 \sum (\bar{\rho} + \bar{P}) \theta , \\
  \ddot{h} + 2H \dot{h} - 2k^2 \eta &= -24\pi G a^2 \sum \delta P , \\
  \left( \dot{h} + 6\dot{\eta} \right) + 2H \left( h + 6\dot{h} \right) - 2k^2 \eta &= -24\pi G a^2 \sum \left( \bar{\rho} + \bar{P} \right) \sigma ,
\end{align}

where $\delta \equiv \delta \rho/\rho$ is the density contrast, $k_i k_j \Sigma_{ij} = -k^2 (\bar{\rho} + \bar{P}) \sigma$ is the anisotropic stress, $(\bar{\rho} + \bar{P}) \theta = i k^i \delta T^0_j$, and $T^{\mu}_{\mu} = T^{\mu}_{\mu} + \delta T^{\mu}_{\mu}$ is the energy-momentum tensor, that can be written in the fluid approximation as

\begin{align}
  T^0_0 &= - \sum (\bar{\rho} + \bar{P}) , \\
  T^0_j &= \sum (\bar{\rho} + \bar{P}) v_j , \\
  T^k_j &= \sum \left( \bar{P} + \delta P \right) \delta^k_j + \Sigma^k_j ,
\end{align}

where $v_j$ is the peculiar 3-velocity of the fluid, and $\Sigma^k_j$ is the traceless anisotropic stress part of the energy-momentum tensor. In addition, we will assume an equation of state in the form $\bar{P} = \omega \bar{\rho}$ with $\omega = \text{const.}$ for each of the fluid components ($\omega = -1$ for vacuum and $\omega \geq 0$ for the matter).

### 3.1 Conservation equations for a single fluid

For a single perfect fluid, the time component of the energy-momentum tensor conservation equation ($T^{\mu}_{\mu} = 0$) becomes:

\begin{align}
  \dot{\bar{\rho}} + 3H \bar{\rho} (1 + \omega) &= 0 , \\
  \bar{p} (1 + \omega) \theta + 3H (\delta P + \bar{\rho} \delta) + \bar{\rho} \dot{\delta} + \bar{p} \delta + \bar{p} (1 + \omega) \frac{\dot{h}}{2} &= 0 ,
\end{align}

the first for the background evolution of the energy density, and the second for the density contrast evolution. In the same way, the spatial component ($T^{\mu}_{\mu} = 0$) gives us the evolution equation for $\theta$:

\begin{align}
  \dot{\bar{p}} (1 + \omega) \theta + \bar{p} (1 + \omega) \dot{\theta} + 4H \bar{p} (1 + \omega) \theta - k^2 \delta P + k^2 \bar{P} (1 + \omega) \sigma &= 0 .
\end{align}

### 3.2 Conservation equation for a vacuum-matter coupled pair

As it was stated before, we model the running vacuum (eq. (2.2) or (2.5)) as a collection of partial vacuum components, where each one of them is interacting with only one matter counterpart (eq. (2.3) or (2.6)). Each coupled pair is made up of one vacuum component $\Lambda_i$ and its associated matter source $i$. In addition, we assume that — at the background level — both the total energy density of the vacuum and each one of its partial components satisfy the equation of state: $\bar{\rho}_\Lambda + \bar{P}_\Lambda = 0$ and $\bar{\rho}_{\Lambda i} + \bar{P}_{\Lambda i} = 0$, respectively. Therefore, the
conservation equations for the coupled pair \(\{\Lambda i, i\}\) (instead eqs. (3.6) for free fluids) are

\[
\dot{\rho}_{\Lambda i} = - Q_{0i}, \quad (3.7a)
\]

\[
\dot{\rho}_i + 3H (\bar{\rho}_i + \bar{\rho}_i) = Q_{0i}, \quad (3.7b)
\]

\[
3H (\delta P_{\Lambda i} + \delta \rho_{\Lambda i}) + \bar{\rho}_{\Lambda i} \dot{\delta}_{\Lambda i} + \dot{\rho}_{\Lambda i} \delta_{\Lambda i} = - Q_{1i}, \quad (3.7c)
\]

\[
(\bar{\rho}_i + \bar{\rho}_i) \theta_i + 3H (\delta P_i + \delta \rho_i) + \bar{\rho}_i \dot{\delta}_i + \dot{\bar{\rho}}_i \delta_i + (\bar{\rho}_i + \bar{\rho}_i) \frac{h_i}{2} = Q_{1i}, \quad (3.7d)
\]

\[
-k^2 \rho P_{\Lambda i} = - Q_{2i}, \quad (3.7e)
\]

\[
(\bar{\rho}_i + \bar{\rho}_i) \theta_i + (\bar{\rho}_i + \bar{\rho}_i) \dot{\theta}_i + 4H (\bar{\rho}_i + \bar{\rho}_i) \theta_i - k^2 \delta P_i + k^2 (\bar{\rho}_i + \bar{\rho}_i) \sigma_i = Q_{2i}. \quad (3.7f)
\]

Each equation was split into two parts, one for the matter component and the other for its associated vacuum. They are linked by coupling terms \(Q_i\)'s to be determined. To solve the evolution equations for the perturbations (3.7), we should know explicitly the form of the couple terms \(Q_i\)’s.

Equation (3.7e) corresponds to a constraint for \(\delta P_{\Lambda i}\). This result is a direct consequence of the parametrization we used for the vacuum energy-momentum tensor (3.5), i.e. the fluid approximation where \(ik^i \delta T^i = (\bar{\rho} + \bar{\rho}) \theta\) is null for the vacuum components. In a more general approach, without any prescription for the \(T^{0i}\) component, equation (3.7e) would emerge as a dynamical equation for \(\delta T^{0i}\), but an additional equation for the sound speed \(\delta P_{\Lambda i}/\delta \rho_{\Lambda i}\) would be required.

In the particular case when \(\bar{\rho}_i/\rho_i = \omega_i = \text{const.}\) the equations (3.7c)-(3.7f) can be written as

\[
3H \left(\frac{Q_{2i}}{k^2} + \delta \rho_{\Lambda i}\right) + \rho_{\Lambda i} \dot{\delta}_{\Lambda i} = - Q_{1i} + Q_{0i} \delta_{\Lambda i}, \quad (3.8a)
\]

\[
\rho_i (1 + \omega_i) \theta_i + 3H (\delta P_i - \omega_i \delta \rho_i) + \bar{\rho}_i \dot{\delta}_i + \bar{\rho}_i (1 + \omega_i) \frac{h_i}{2} = Q_{1i} - Q_{0i} \delta_i, \quad (3.8b)
\]

\[
\rho_i (1 + \omega_i) \dot{\theta}_i + H \rho_i (1 + \omega_i) (1 - 3\omega_i) \theta_i - k^2 \delta P_i + k^2 \rho_i (1 + \omega_i) \sigma_i = Q_{2i} - Q_{0i} (1 + \omega_i) \theta_i. \quad (3.8c)
\]

4 \(\Lambda (H^2)\) perturbations

At this point we have assumed three vacuum features. First, the vacuum mean density is made up of a set of independent contributions interacting with each one of the matter components. Second, the energy-momentum tensor of each coupled pair \(\{i, \Lambda i\}\) (a given matter component and its vacuum partner) is conserved independently of the other pairs \(\{j, \Lambda j\}\). Third, the conservation equation for each coupled pair \(\{i, \Lambda i\}\) can be separated in two individual equations, one of them describing the evolution of the matter component \(i\) while the other one associated to its vacuum counterpart \(\Lambda i\). These two equations describing the dynamic of the split pair \(\{i, \Lambda i\}\) are not independent, but they contain a coupling term as shown in (3.7).

The goal of this section is to find the coupling terms between the vacuum and the matter perturbations associated to the \(\Lambda (H^2)\) background model described in section 2.1. We will obtain that the two none standard terms (they are not present in the context of the \(\Lambda \text{CDM}\) model) in the right-hand side of equations (3.8b) and (3.8c) cancel each other. This leaves the evolution equations for the matter perturbations of the running vacuum models exactly
like the ΛCDM equations in [57]. Therefore, the matter perturbations are only affected by
the running vacuum in an indirect fashion, through both the metric perturbation \( h \) and the
background quantities.

The background coupling term \( Q_{0i} \) associated to the \( \Lambda(H^2) \) model can be obtained as
it follows. Replacing equation (2.4) for \( \bar{\rho}_\Lambda i = \bar{\rho}_\Lambda i(\bar{\rho}_{\Lambda}) \) (which defines our split \( \bar{\rho}_\Lambda = \text{const.} + \sum_i \bar{\rho}_\Lambda i \) such that \( \Lambda - \Lambda_0 = 3\alpha(H^2 - H_0^2) \)) into the background conservation equations (3.7a)
and (3.7b) we can write them as

\[
\dot{p}_i + 3H(p_i + p_i) = 3\alpha H(p_i + p_i) \equiv Q_{0i}, \quad (4.1)
\]
which has an analytical solution when \( p_i/p_i = \omega_i = \text{const.} \):

\[
p_i = \rho_0 a^{-3(1-\alpha)(1+\omega_i)}. \quad (4.2)
\]

Up to now, neither the fluid conservation equations nor the \( \Lambda(H^2) \) background model
provide any direct information about the coupling terms \( Q_{1i} \) and \( Q_{2i} \) in equations (3.8). We
will apply the Boltzmann equation to the matter components in order to find this coupling
terms in three steps. First, we will look for the collision term of the Boltzmann equation that
reproduces the background equation (4.1). Then, we will identify the natural extension of
that background collision term as being the collision term associated with the perturbed part
of the Boltzmann equation. Finally, the comparison between the lower multipole expansion
of the perturbed Boltzmann equation and the matter conservation equations (3.7d) e (3.7f)
will allow us to identify the coupling terms \( Q_{1i} \) and \( Q_{2i} \) between the linear perturbations of
each pair \( \{\Lambda i, i\} \).

4.1 Coupling terms for massive neutrinos

The energy-momentum tensor can be expressed in terms of the phase-space distribution
function \( f(x^i, P_j, \tau) \) as:

\[
T_{\mu\nu} = \int dP_1 dP_2 dP_3 (-g)^{-1/2} \frac{P_{\mu} P_{\nu}}{P_0} f(x^i, P_j, \tau) \quad (4.3)
\]

where \( g \) is the trace of the metric, and \( P^\mu \) is the 4-momentum of the particles of a given type.
\( P^i \) is the conjugate momentum to \( x^i \), and it can be expressed in the synchronous gauge as
a function of the proper momentum \( p^i = p_i \) as \( P^i = a(\delta_{ij} + h_{ij}/2)p^j \). Additionally we can
define a comoving 3-momentum \( q_j = aP_j = n_j \), where \( n_i n_j = \delta_{ij} \).

The phase space distribution function evolves according to the Boltzmann equation:

\[
\frac{Df}{d\tau} = \frac{\partial f}{\partial \tau} + \frac{dx^i}{d\tau} \frac{\partial f}{\partial x^i} + \frac{dq}{d\tau} \frac{\partial f}{\partial q} + \frac{dn_i}{d\tau} \frac{\partial f}{\partial n_i} = \left( \frac{\partial f}{\partial \tau} \right)_C, \quad (4.4)
\]

where the right-hand side of the equation represents the collision term, such that the number
of particles, \( dN = f(x^i, P_j, \tau) dx^i dx^j dx^3 dP_1 dP_2 dP_3 \), is conserved when the collision term
vanishes.

Following the Ma and Bertschinger procedure [57], we express the phase-space distribution function
\( f(x^i, P_j, \tau) \) as a sum of a background contribution, \( f_0(q, \tau) \), plus a linear
perturbation parameterized as

\[
f(x^i, P_j, \tau) = f_0(q, \tau) \left[ 1 + \Psi(x^i, q, n_j, \tau) \right]. \quad (4.5)
\]
Replacing this expression into the Boltzmann equation (4.4), it can be split into its zero and first order contributions:

\[
\frac{\partial f_0}{\partial \tau} = \left( \frac{\partial f}{\partial \tau} \right)_{C0}, \quad (4.6a)
\]

\[
\frac{1}{f_0} \frac{\partial f_0}{\partial \tau} \Psi + \frac{\partial \Psi}{\partial \tau} + i \frac{q}{\epsilon} \left( \vec{k} \cdot \hat{n} \right) \Psi + \frac{d \ln f_0}{d \ln q} \left[ \dot{\eta} + \frac{6}{2} \left( \vec{k} \cdot \hat{n} \right)^2 \right] = \frac{1}{f_0} \left( \frac{\partial f}{\partial \tau} \right)_{C1}, \quad (4.6b)
\]

where \( \epsilon = \sqrt{q^2 + a^2 m^2} = \sqrt{P^2 + a^2 m^2} = -P_0 \), \( m \) is the mass of the particles, and the subscripts \( C0 \) and \( C1 \) correspond to the background and first order perturbation of the same collision term. Expanding the field \( \Psi \) in Legendre series

\[
\Psi \left( \vec{k}, \hat{n}, q, \tau \right) \equiv \sum_{l=0}^{\infty} \left( -i \right)^l (2l + 1) \Psi_l \left( \vec{k}, q, \tau \right) P_l \left( \vec{k} \cdot \hat{n} \right), \quad (4.7)
\]

and replacing it into the Boltzmann equation (4.6b), the differential equations for the evolution of the fields \( \Psi_l \) can be obtained. By integrating the \( q \) dependence of the energy-momentum tensor components it can be obtained

\[
(\bar{\rho}_h + \bar{P}_h) \theta_h = 4\pi a^{-4} \int q^2 dq \left( \epsilon f_0 - \epsilon \right)(q, \tau), \quad (\bar{\rho}_h + \bar{P}_h) \sigma_h = 8\pi a^{-4} \int q^2 dq \frac{q^2}{\epsilon} f_0(q, \tau) \Psi_1, \quad (4.8)
\]

In order to reproduce the background conservation equation (4.1), the zero order distribution function must satisfy the following equation:

\[
\frac{\partial f_0}{\partial \tau} = 3\alpha \left( 1 + \frac{q^2}{3\epsilon^2} \right) H f_0, \quad (4.9)
\]

whose time evolution is due to the interaction with its vacuum partner at the background level. This distribution function \( f_0(q, \tau) \) can be related to the free (non interacting with vacuum) neutrino distribution function \( \tilde{f}_0(q) \) as:

\[
f_0(q, \tau) \equiv a^{4\alpha} \left( \frac{T_0}{\epsilon} \right)^{\alpha} \tilde{f}_0(q), \quad \text{where} \quad \tilde{f}_0(q) = \frac{g_s}{e^{\epsilon/k_B T_0} \pm 1}. \quad (4.10)
\]

Here \( g_s \) is the number of spin degrees of freedom, \( T_0 \) is the temperature of the particles today, \( k_B \) is the Boltzmann constant, and \( \epsilon_0 = \epsilon(a = 1) \). Based on the CLASS code [58], which we use for numerical integration, the free neutrino distribution function, \( \tilde{f}_0(q) \), is time independent and modeled as warm dark matter. Other massive neutrino formulations in the literature include the Degenerate Fermion Gas approximation, see [59, 60], while a time-dependent distribution function was discussed by [61] among others.

The background collision term of the Boltzmann equation reproducing the \( \Lambda(H^2) \) model constraint (4.9) is then

\[
\left( \frac{\partial f}{\partial \tau} \right)_{C0} = 3\alpha \left( 1 + \frac{q^2}{3\epsilon^2} \right) H f_0. \quad (4.11a)
\]
We will use the minimal extension of this zero order collision term
\begin{equation}
\left( \frac{\partial f}{\partial \tau} \right)_{C1} = 3\alpha \left( 1 + \frac{q^2}{3\epsilon^2} \right) \mathcal{H} \delta f,
\end{equation}
as being the collision term for the first order perturbations, where \( \delta f = f_0 \Psi \) as established in (4.5). The coupling terms (for the fluid conservation equations (3.7)) associated to the Boltzmann collision term (4.11) are
\begin{align}
Q_{0h} &= 4\pi a^{-4} \int q^2 dq \epsilon \dot{f}_0(q, \tau) = 3\alpha \mathcal{H} \left( \overline{\rho}_h + \overline{P}_h \right), \\
Q_{1h} &= 4\pi a^{-4} \int q^2 dq \epsilon \dot{f}_0(q, \tau) \Psi_0 = 3\alpha \mathcal{H} \left( \delta \rho_h + \delta P_h \right), \\
Q_{2h} &= 4\pi ka^{-4} \int q^3 dq \epsilon \dot{f}_0(q, \tau) \Psi_1 = 3\alpha \mathcal{H} \left( \overline{\rho}_h + \overline{P}_h \right) \theta_h + 3\alpha \mathcal{H} 4\pi ka^{-4} \int q^3 dq \frac{q^2}{3\epsilon^2} \dot{f}_0(q, \tau) \Psi_1.
\end{align}
Replacing this coupling terms in the fluid evolution equation (3.8a), for the perturbations of the vacuum component associated to massive neutrinos, we obtain
\begin{equation}
\overline{\rho}_{\Lambda h} \left( \delta \Lambda_h + 3\mathcal{H} \delta \Lambda_h \right) + 3\mathcal{H} \frac{Q_{2h}}{k^2} = 3\alpha \mathcal{H} \left[ \left( \overline{\rho}_h + \overline{P}_h \right) \delta \Lambda_h - \left( \delta \rho_h + \delta P_h \right) \right].
\end{equation}
On the other hand, it can be noticed that replacing both the collision term (4.11b) and the background expression (4.9) into the first order Boltzmann equation (4.6b) it assumes its collisionless form
\begin{equation}
\frac{\partial \Psi}{\partial \tau} + i \frac{q}{\epsilon} (\hat{k} \cdot \hat{n}) \Psi + \frac{d \ln f_0}{d \ln q} \left[ \dot{\eta} - \frac{\dot{h}}{2} - \frac{\left( \hat{k} \cdot \hat{n} \right)^2}{2} \right] = 0.
\end{equation}
Therefore, the evolution equations for the matter perturbations remain unchanged, not only for massive neutrinos. In particular, we obtain that the evolution equations for the multipoles \( \Psi_l \) are the same as those in [57], the only difference is that for our model \( f_0(q, \tau) \) is related to the free \( \tilde{f}_0(q) \) by the eq. (4.10), and then:
\begin{equation}
\frac{\partial \ln f_0}{\partial \ln q} = \frac{\partial \ln \tilde{f}_0}{\partial \ln q} + \alpha \left( \frac{q^2}{c_0^2} - \frac{q^2}{\epsilon^2} \right),
\end{equation}
where \( \tilde{f}_0 \) was denoted as \( f_0(q) \) in [57], and also in CLASS [38] to implement non cold reliquies.

### 4.2 Coupling terms for matter components with \( \omega = \text{constant} \)

In the case in which \( \omega_i = \overline{\nu} / \overline{\rho}_i = \delta \overline{P}_i / \delta \rho_i = \text{const.} \) the analysis of the previous section reduces significantly. In order to obtain the evolution of the background energy density (4.1), for the matter components, we have that the collision term must be equal to
\begin{equation}
\left( \frac{\partial f}{\partial \tau} \right)_{C} = 3 \alpha (1 + \omega_i) \mathcal{H} f,
\end{equation}
which leads to the coupling terms
\[ Q_{0i} = 3\alpha (1 + \omega_i) H p_i, \]
\[ Q_{1i} = 3\alpha (1 + \omega_i) H \bar{p}_i \delta_i = Q_{0i} \delta_i, \]
\[ Q_{2i} = 3\alpha (1 + \omega_i)^2 H p_i \theta_i = Q_{0i} (1 + \omega_i) \theta_i. \]

(4.17a), (4.17b), (4.17c)

As it was pointed in the last section, these collision terms make that the first order Boltzmann equation (4.6b) for the matter components become the same as in [57], or (4.14), because the two terms in the right-hand side cancel in eqs. (3.8b), (3.8c); while eq. (3.8a) for the vacuum equation (4.6b) for the matter components become the same as in [57], or (4.14), because the usual matter perturbations \[ \delta_{\nu}, \delta_{c}, \delta_{b}, \theta_{\nu}, \theta_{b}, \Phi_{0}, \Phi_{1}, \Phi_{2} \] are the same that those in [57]. During the super-horizon era can be written as

\[ \partial_{\tau} (\delta_{\Lambda i} \bar{p}_{\Lambda i}) + 3H \delta_{\Lambda i} \bar{p}_{\Lambda i} + \frac{9(1 + \omega_i)^2 H^2}{k^2} \alpha \bar{p}_i \delta_i = -3\alpha (1 + \omega_i) H \bar{p}_i \delta_i. \]

(4.18a)

Equation (4.18) is identically null for \( \alpha = 0 \). In addition, for \( \alpha \neq 0 \) we can divide equation (4.18) by \( \bar{p}_{\Lambda i} \) obtaining

\[ \dot{\delta}_{\Lambda i} + 3H \delta_{\Lambda i} + \frac{9(1 + \omega_i)^2 H^2}{k^2} (1 - \alpha) \theta_i = 3(1 + \omega_i)(1 - \alpha) H (\delta_{\Lambda i} - \delta_i), \]

(4.18b)

where \( \delta_{\Lambda i} = \delta \rho_{\Lambda i} / \bar{p}_{\Lambda i} \) and \( \delta_i = \delta \rho_i / \bar{p}_i \) are the density contrasts. In this limit, \( \omega_i = \bar{p}_i / \bar{p}_\Lambda \) = const., expressions (4.12) and (4.13) reduce to (4.17) and (4.18), respectively, as expected.

### 4.3 Super-horizon initial conditions

To following expressions summarize the initial conditions for the vacuum perturbations equations, (4.13) and (4.18), as well as for the usual matter perturbations [57], during the radiation-dominated era and super-horizon scales, \( k \ll H \):

\[ \eta = \frac{2 - 11\alpha + 12\alpha^2}{1 - 2\alpha} \left( 3 - 8\alpha + 4\alpha^2 \right) + 4R_{\nu}(1 - \alpha) C_1 \]
\[ - \frac{15(1 - 2\alpha^2 + 4R_{\nu}(1 - \alpha)}{6 C_1 (k\tau)^2}, \]
\[ \theta_{\nu} = -\frac{23 + 4(1 - \alpha)R_{\nu} - 104\alpha + 108\alpha^2}{15(1 - 2\alpha^2 + 4R_{\nu}(1 - \alpha)} \frac{1}{18} C_1 (k^4\tau^3), \]
\[ \sigma_{\nu} = \frac{2 - 11\alpha + 12\alpha^2}{15(1 - 2\alpha^2 + 4R_{\nu}(1 - \alpha)} \frac{2}{3} C_1 (k\tau)^2, \]
\[ \delta_{\Lambda} = \frac{2 - 5\alpha + 3\alpha^2}{1 - 2\alpha} \frac{16}{9} C_1 (k\tau)^2, \]
\[ \delta_{\Lambda} = \frac{17 + 4R_{\nu}(1 - \alpha) - 68\alpha + 60\alpha^2}{15(1 - 2\alpha^2 + 4R_{\nu}(1 - \alpha)} \frac{2 - 5\alpha + 3\alpha^2}{1 - 2\alpha} \frac{16}{9} C_1 (k\tau)^2, \]
\[ \delta_{\Lambda} = \frac{1 - \alpha}{2 - \alpha} \frac{3}{2} C_1 (k\tau)^2, \]
\[ \delta_{\Lambda} = \frac{2 - 5\alpha + 3\alpha^2}{2 - 5\alpha + 3\alpha^2} C_1 (k\tau)^2 . \]

(4.19)

Initial conditions for \( \{ h, \delta_{\gamma}, \delta_{\nu}, \delta_{c}, \delta_{b}, \theta_{\nu}, \theta_{b}, \Phi_{0}, \Phi_{1}, \Phi_{2} \} \) are the same that those in [57]. \( C_1 \) is the only remaining integration constant. The total mean density was approximated by
\( \rho_T = \rho_\gamma + \rho_\nu + \rho_{\Lambda\gamma} + \rho_{\Lambda\nu} \) during the radiation dominated era, as well as \( \rho_\gamma = \rho_0^\gamma a^{-4(1-\alpha)} \) and \( \rho_\nu = \rho_0^\nu a^{-4(1-\alpha)} \) given by (4.2), which leads to

\[
a^{1-2\alpha} = \tau (1 - 2\alpha)^{\frac{3\pi G (\rho_0\nu + \rho_0\gamma)}{3(1-\alpha)}}, \quad \text{with} \quad \mathcal{H} = \frac{1}{\tau (1 - 2\alpha)} . \tag{4.20}
\]

\( R_\nu \equiv \frac{\rho_\nu}{\rho_\gamma} \) is a constant ratio during the radiation dominated era regardless the value of \( \alpha \). In addition, the initial conditions (4.19) reduce to the \( \Lambda \)CDM adiabatic case \([57]\) when \( \alpha = 0 \).

### 4.4 Sub-horizon evolution for the vacuum perturbations

For sub-horizon modes and \( \alpha \neq 0 \), the equation (4.18) that describe the evolution of the density contrast for vacuum components associated with the non-relativistic matter becomes:

\[
a \delta'_{\Lambda c} = -3\alpha \delta_{\Lambda c} - 3(1-\alpha)\delta_c , \tag{4.21}
\]

where \( (') \) means derivative with respect to the scale factor \( a \). Most of the geometrical constraints in cosmology for this kind of models show that \( |\alpha| \) must be in the range of \( 10^{-5} - 10^{-3} \) \([19, 44, 45]\), which allows us to estimate \( \delta_{\Lambda c} \) as being around \( \sim -3(1-\alpha)\delta_c \). This bigger amplitude of the density contrast for the vacuum components, in comparison with their matter counterparts, does not represent an observational problem for the model. That occurs by virtue of the vacuum density perturbations appear into Einstein’s equations in the form of \( \delta \rho_{\Lambda c} \equiv \rho_{\Lambda c} \delta_{\Lambda c} \approx \alpha \rho_c \delta_{\Lambda c} \), which is suppressed by a factor \( \alpha \) in comparison to the matter contribution \( \delta \rho_c \), see figure 3.

For the vacuum components associated to the ultra-relativistic matter \( \{ur = \{\gamma, \nu\} \) and the massive neutrinos \( h \) during their ultra-relativistic stage), the equation (4.18) becomes:

\[
a \delta'_{ur} \approx \delta_{ur}(1-4\alpha) - 4\delta_{ur}(1-\alpha) , \tag{4.22}
\]

where the last term in the previous equation can be neglected since the density contrast of the ultra-relativistic matter decreases considerably inside the horizon. Therefore, the vacuum perturbation evolves as \( \delta_{\Lambda ur} \propto a^{1-4\alpha} \).

For sub-horizon modes, the evolution equation for the perturbation of the vacuum component associated to baryons becomes

\[
a \delta'_{\Lambda b} = -3\alpha \delta_{\Lambda b} - 3\delta_b (1-\alpha) . \tag{4.23}
\]

During the tight coupled baryon-photon stage, the amplitude of the tightly coupled baryon and photon density contrasts decrease significantly. Furthermore, because of the small value expected for \( \alpha \), the right-hand side of the last equation can be neglected, giving \( \delta_{\Lambda b} \approx \text{const.} \). After the photon-baryon decoupling \( \delta_b \) grows quickly and becomes relevant in the last equation, leading to \( \delta_{\Lambda b} \approx -3(1-\alpha)\delta_b \).

Finally, equation (4.23) is also valid for the vacuum component associated to massive neutrinos after the non-relativistic transition, where \( |\delta_{\Lambda h}| \gg |\delta_b| \) since \( |\delta_{\Lambda h}| \) grows like \( a^{1-4\alpha} \) while \( |\delta_b| \) diminish during the ultra-relativistic stage (besides that \( \delta_{\Lambda h} \sim -5\delta_h \) before the horizon crossing, which is given by the initial conditions), and due to \( |\alpha| \ll 1 \) we obtain \( \delta_{\Lambda h} \approx \text{const.} \).
5 $\Lambda(R)$ perturbations

In this section, we apply the same procedure followed in the previous section, but now to the running vacuum model $\Lambda = \Lambda_0 + \beta(R - R_0)$. The expression (2.7), which defines our split of the $\Lambda(R)$ density, allows us to merge the background conservation equations (3.7a), (3.7b) in

$$(1 - 3\beta)\dot{\bar{\rho}}_i - 3\beta\dot{\bar{P}}_i + 3(1 - 4\beta)\mathcal{H}(\bar{\rho}_i + \bar{P}_i) = 0.$$  (5.1)

5.1 Coupling terms for massive neutrinos

In order to reproduce the evolution equation for the mean density of the massive neutrinos (5.1), the corresponding zero order distribution function must satisfy the following equation

$$\dot{f}_0 = 3\beta\mathcal{H}\frac{\varepsilon^2}{\varepsilon^2} \left(1 - \frac{q^2}{3\varepsilon^2}\right) \left(1 - \frac{q^2}{\varepsilon^2}\right) f_0.$$  (5.2)

This distribution function $f_0(q, \tau)$ can be related to the free neutrino distribution function $\tilde{f}_0(q)$ as:

$$f_0(q, \tau) \equiv \frac{\varepsilon_0}{\varepsilon}(1 - 6\beta)/(1 - 3\beta) \tilde{f}_0(q),$$  (5.3)

such that $\dot{\tilde{f}}_0(q) = 0$.

The zero order collision term for the Boltzmann equation that reproduces the background expression (5.2) is:

$$\left(\frac{\partial f}{\partial \tau}\right)_C = 3\beta\mathcal{H}\frac{\varepsilon^2}{\varepsilon^2} \left(1 - \frac{q^2}{3\varepsilon^2}\right) \left(1 - \frac{q^2}{\varepsilon^2}\right) f_0,$$  (5.5a)

and its minimal extension including linear perturbations is

$$\left(\frac{\partial f}{\partial \tau}\right)_C = 3\beta\mathcal{H}\frac{\varepsilon^2}{\varepsilon^2} \left(1 - \frac{q^2}{3\varepsilon^2}\right) \left(1 - \frac{q^2}{\varepsilon^2}\right) f,$$  (5.5b)

where $f = f_0 + \delta f$ as was defined in (4.5). The coupling terms appearing in equations (3.8) and associated to the Boltzmann collision term (5.5b) are

$$Q_{0h} = 4\pi a^{-4} \int q^2 \, dq \, \varepsilon \dot{f}_0(q, \tau),$$  (5.6a)

$$Q_{1h} = 4\pi a^{-4} \int q^2 \, dq \, \varepsilon \dot{f}_0(q, \tau) \Psi_0,$$  (5.6b)

$$Q_{2h} = 4\pi ka^{-4} \int q^3 \, dq \, \dot{f}_0(q, \tau) \Psi_1.$$  (5.6c)

Replacing these last expressions in the evolution equation for the vacuum perturbation (3.8a), we obtain

$$\bar{\rho}_{\Lambda h} \left(\delta_{\Lambda h} + 3\mathcal{H}\delta_{\Lambda h}\right) + 3\mathcal{H}\frac{Q_{2h}}{k^2} = Q_{0h}\delta_{\Lambda h} - Q_{1h},$$  (5.7a)

$$\delta\rho_{\Lambda h} + 3\mathcal{H}\delta\rho_{\Lambda h} + 3\mathcal{H}\frac{Q_{2h}}{k^2} = -Q_{1h},$$  (5.7b)

for the vacuum component associated to massive neutrinos.
Equations (5.7) are identically null for $\alpha = 0$. On the other hand, all of the $Q$’s coefficients vanish during the ultra-relativistic regime. This feature is not present in the $\Lambda(H^2)$ model described in section 4.

We emphasize that the first order Boltzmann equation for massive neutrinos reduces to the collisionless form, which can be verified replacing the expression (5.5) in equation (4.6b). Finally, the relation between $f_0(q, \tau)$ and $\tilde{f}_0(q)$ gives us

$$
\frac{\partial \ln f_0}{\partial \ln q} = \frac{\partial \ln \tilde{f}_0}{\partial \ln q} + q^2 \left( \frac{1}{\tau^2} - \frac{1}{\tau_0^2} \right) + q^2 \frac{(1 - 6\beta)(1 - 4\beta)}{(1 - 3\beta)} \left( \frac{1}{\tau^2} - \frac{1}{\tau_0^2} \right).
$$

5.2 Coupling terms for matter components with $\omega =$constant

For matter components with $P_i/\rho_i \equiv \omega_i =$const. equation (5.1) becomes

$$
\dot{\rho}_i + 3H(1 + \omega_i) \rho_i = \frac{3\beta(1 - 3\omega_i)}{1 - 3\beta(1 + \omega_i)} (1 + \omega_i) \mathcal{H} \rho_i \equiv Q_{0i},
$$

and from direct integration we obtain

$$
\rho_i = \rho_0^i a^{-3(1 - 4\beta)(1 + \omega_i)/(1 - 3\beta(1 + \omega_i))}.
$$

In order to recover eq. (5.9) we have that the collision term in the Boltzmann equation (4.4) must be equal to

$$
\left( \frac{\partial f}{\partial \tau} \right)_C = \frac{3\beta(1 + \omega_i)}{1 - 3\beta(1 + \omega_i)} (1 - 3\omega_i) \mathcal{H} f.
$$

which leads to the coupling terms

$$
Q_{0i} = \frac{3\beta(1 + \omega_i)}{1 - 3\beta(1 + \omega_i)} (1 - 3\omega_i) \mathcal{H} \rho_i \equiv Q_{0i},
$$

$$
Q_{1i} = \frac{3\beta(1 + \omega_i)}{1 - 3\beta(1 + \omega_i)} (1 - 3\omega_i) \mathcal{H} \rho_i \delta_i = Q_{0i} \delta_i,
$$

$$
Q_{2i} = \frac{3\beta(1 + \omega_i)^2}{1 - 3\beta(1 + \omega_i)} (1 - 3\omega_i) \mathcal{H} \rho_i \theta_i = Q_{0i} (1 + \omega_i) \theta_i.
$$

Remarkably, for ultra-relativistic matter ($\omega_i = 1/3$), the coupling terms (5.12) vanishes, and the mean density (5.10) becomes $\beta$-independent evolving in the form $\bar{\rho} \propto a^{-4}$, as it happen in the $\Lambda$CDM model. Therefore, in the context of the $\Lambda(R)$ model there is not any vacuum component associated with ultra-relativistic matter, which does not occur in the $\Lambda(H^2)$ model.

In addition, just as in the $\Lambda(H^2)$ model, each of the conservation equations for the standard matter perturbations remain the same as in [57]. This feature can be seen in equations (3.8b), (3.8c), where the right-hand side is identically zero due to the relationship (5.12a) between the coupling terms.

On the other hand, the right-hand side of the conservation equation for the vacuum perturbations (3.8a) does not vanish upon replacing the coupling terms (5.12). In this case, the conservation equation for each $\Lambda_i$ component (3.8a) becomes

$$
\partial_\tau (\delta \rho_{\Lambda_i}) + 3\mathcal{H} \delta \rho_{\Lambda_i} + 3\mathcal{H} \frac{\beta(1 - 3\omega_i)(1 + \omega_i)}{1 - 3\beta(1 + \omega_i)} \rho_i \left( 3\mathcal{H}(1 + \omega_i) \frac{\theta_i}{K^2} + \delta_i \right) = 0.
$$
5.3 Super-horizon initial conditions

In order to set the super-horizon ($k \ll H$) initial conditions for perturbations, we have to solve the standard equations for the matter components [57] and the additional equations for their vacuum counterparts sourced by non-relativistic matter.

During the radiation dominated era, only photons and ultra-relativistic neutrinos contribute effectively to the mean density, $\rho_T = \rho_\gamma + \rho_\nu$, because their contributions to the vacuum energy density vanishes, see eq. (2.7). Under these conditions the expansion rate is $H = \tau^{-1}$. From eq. (5.13) we have that the evolution equations for the perturbed vacuum components associated to baryons and CDM ($\omega_c = \omega_b = 0$) are:

$$\frac{9(1-4\beta)}{1-3\beta} \frac{H^2}{k^2} \delta_b + \delta\dot{\Lambda_b} + 3H\delta\Lambda_b = \frac{3(1-4\beta)}{1-3\beta} H (\delta\Lambda_b - \delta_b), \quad (5.14a)$$

$$\dot{\delta}\Lambda_c + 3H\delta\Lambda_c = \frac{3(1-4\beta)}{1-3\beta} H (\delta\Lambda_c - \delta_c). \quad (5.14b)$$

As seen in sections 5.1 and 5.2, the evolution equations for the matter components (throughout the entire cosmic history) and for metric perturbations (during the radiation dominated era) are the same as those in the $\Lambda$CDM model. Thus we can use the initial conditions found by Ma and Bertschinger (eqs. (96) in [57]) for both the matter and the metric perturbations.

During this era, the vacuum components $\Lambda_c$ and $\Lambda_b$ can be treated as test fluids, allowing us to use the Ma and Bertschinger [57] initial conditions in order to solve eqs. (5.14), giving us the solutions:

$$\delta\Lambda_b = \frac{2(1-4\beta)}{(2-3\beta)} C_1 (k\tau)^2, \quad (5.15a)$$

$$\delta\Lambda_c = \frac{3(1-4\beta)}{2(2-3\beta)} C_1 (k\tau)^2. \quad (5.15b)$$

Finally, the evolution equation for $\delta\rho_{\Lambda h}$ (5.7) becomes $\partial_\tau \delta\rho_{\Lambda h} = 0$ during the earlier stages of the radiation dominated era. In that epoch, massive neutrinos are ultra-relativistic, then $\delta\rho_{\Lambda h} = Q_{1h} = Q_{2h} = 0$, giving us the super-horizon initial condition:

$$\delta\rho_{\Lambda h} = C_h = 0, \quad (5.16)$$

where the integration constant $C_h$ must be identically null because $\delta\rho_{\Lambda h} = \delta\rho_{\Lambda h} \equiv \rho_{\Lambda h} \delta_{\Lambda h} = \beta(\rho_h - 3\overline{\rho}_h)\delta_{\Lambda h}/(1-4\beta) = 0$ early in the radiation dominated era, when $\overline{\rho}_h = \rho_h/3$.

These initial conditions complete those found by [57] for the matter and metric perturbations. Those expressions set the relative amplitude of the perturbations (and their super-horizon evolution into the radiation dominated era) among the different species for a given length scale $k$. The relative amplitude among the different $k$-modes is given by the primordial spectrum parameterized by the scalar spectral index $n_s$.

5.4 Sub-horizon evolution for the vacuum perturbations

The perturbations of the vacuum components in the context of the $\Lambda(R)$ model show a similar behavior that those in the $\Lambda(H^2)$ model. The main difference is the lack of vacuum components coupled with the ultra-relativistic matter.

Another difference is related to vacuum component associated to massive neutrinos, which appear only during the non-relativistic stage. For the non-relativistic massive neutrino case, the vacuum perturbation evolution equation (5.13) for sub-horizon modes can be
approximated by
\[ a \delta'_{\Lambda h} = -3 \delta_{\Lambda h} + 3 \frac{1 - 4\beta}{1 - 3\beta} (\delta_{\Lambda h} - \delta_h). \] (5.17)

In addition, just after the non-relativistic transition the amplitude of \( \delta_{\Lambda h} \) is effectively zero, simplifying last equation to
\[ a \delta'_{\Lambda h} = 3 \frac{1 - 4\beta}{1 - 3\beta} \delta_h, \] (5.18)
which gives us \( \delta_{\Lambda h} \approx -3 \delta_h/2 \), because \( \delta_h \propto a^r \) with \( r \sim 2 \).

6 Evolution of the density perturbations

The evolution equations for the background cosmological quantities and their linear perturbations were integrated using an adapted version of the free code CLASS [38]. We used the initial conditions shown in sections 4.3 and 5.3 for the \( \Lambda(H^2) \) and \( \Lambda(R) \) models respectively. Those initial conditions reduce to the adiabatic \( \Lambda \text{CDM} \) case when the free parameter \( \alpha \) vanishes for the \( \Lambda(H^2) \) model, and when \( \beta \) goes to zero in the context of the \( \Lambda(R) \) model. In sections 4 and 5 we concluded that the evolution equations for the matter perturbations are not modified by the running of the vacuum, in particular, the CDM equations. In consequence, the usual transformations between synchronous and Newtonian gauges remain unchanged.

Figure 1 shows the evolution of the matter density contrasts \( \delta_i \equiv \delta \rho_i / \rho_i \) for the standard \( \Lambda \text{CDM}, \Lambda(H^2), \) and \( \Lambda(R) \) models. There we display the density contrasts for CDM (red), baryons (blue), photons (green), massless neutrinos (violet) and massive neutrinos (orange). Figure 2 shows the evolution of the vacuum density contrasts, \( \delta_{\Lambda i} \equiv \delta \rho_{\Lambda i} / \rho_{\Lambda i} \), for both the \( \Lambda(H^2) \) and \( \Lambda(R) \) models. Figure 3 correspond to the evolution of the density perturbations \( \delta \rho_{\Lambda i} \) and \( \delta \rho_i \), which are sources of the metric perturbations in the running vacuum models. The matter density contrast in figure 1 is a function of time \( \tau \), otherwise the initial conditions for each model would appear to be different if expressed as a function of the scale factor because \( a(\tau) \) is model dependent. In figures 2 and 3 the vacuum perturbations evolution is
Figure 2. Evolution of density contrasts for vacuum components. Left panel: vacuum components in the model $\Lambda(H^2)$. Right panel: vacuum components in the model $\Lambda(R)$. For display purposes $\{\alpha, \beta\} = 0.01$ is shown, although, according to observational settings of similar models, it is expected that these parameters are of the order of $10^{-5} - 10^{-3}$. Matter perturbations are shown in light gray as a visual reference.

6.1 Evolution of the vacuum perturbations

For super-horizon Fourier modes during the radiation dominated era, each one of the perturbations evolves according to the expressions described in sections 4.3 and 5.3, which were used in this work as initial conditions for the numerical integration.

The sub-horizon evolution of the vacuum perturbations was broadly analyzed in sections 4.4 and 5.4. At least two additional features can be seen upon numerical integration. First, during the baryon-photon thigh coupling stage, the vacuum perturbation $\delta_{\Lambda b}$ is affected by the oscillations of the baryon density contrast $\delta_b$. Second, after the baryon-photon decoupling, the perturbation of the vacuum component interacting with baryons $\delta_{\Lambda b}$ trends to $\delta_{\Lambda c}$, since their evolution equations become almost identical.

Remarkably, the bigger amplitude of the vacuum density contrasts $\delta_{\Lambda i}$, relative to the matter components $\delta_i$ (see initial conditions in sections 4.3 and 5.3, sub-horizon evolution in section 4.4, and numerical results in figure 2), does not represent an observational problem for the running vacuum models $\Lambda(H^2)$ and $\Lambda(R)$. This is because the contribution of the vacuum and the matter perturbations to the right-hand side of the Einstein equation (3.4a) (or equation (6.4)) have the form $\delta\rho_{\Lambda i} \sim \alpha \bar{\rho}_i \delta_{\Lambda i}$ or $\delta\rho_{\Lambda i} \sim \beta \bar{\rho}_i \delta_{\Lambda i}$, and $\delta\rho_i \equiv \bar{\rho}_i \delta_i$ instead of pure $\delta_{\Lambda i}$ and $\delta_i$ contributions. Thus, the vacuum contributions $\delta\rho_{\Lambda i}$ are suppressed by $|\alpha| \ll 1$ or $|\beta| \ll 1$ factors compared with the matter contributions $\delta\rho_i$, see figures 2 and 3 or equation (6.4).

6.2 Differences with other vacuum perturbation approaches

The aim of this section is to show why the vacuum perturbations, as modeled here, have a growing mode which is not present in other analysis in the literature. Here we analyze the case where the vacuum interacts with a matter component with $\omega_i = const.$. The starting point will be eq. (3.7c), where the functional form of the vacuum-matter interaction has not
been specified yet:

\[
\dot{\delta}_i + 3 \mathcal{H} \delta_i + 3 \mathcal{H} \delta_i (1 + c_s^2) = -Q_{1i} - \delta_i \dot{\rho}_i .
\] (6.1)

In the simplest case, the background vacuum decaying or production rate is assumed to be low, i.e. \( |\dot{\rho}_i| \ll \mathcal{H}|\rho_i| \), as well as its first order counterpart \( |Q_{1i}| \ll \mathcal{H} \delta_i \dot{\rho}_i | \). In this case, the right-hand side of equation (6.1) can be neglected. Besides, it is used to demand that the vacuum sound velocity is not negative \( c_s^2 \equiv \delta P_i / \delta \rho_i \geq 0 \). Therefore, and for \( \rho_i \neq 0 \), equation (6.1) reduces to

\[
\frac{d \ln \delta_i}{d \ln a} + 3 \left( 1 + c_s^2 \right) = 0 ,
\] (6.2)

leading to the dissipation of any primordial vacuum density perturbation.

Unlike the simplified case described above, in the context of the \( \Lambda(H^2) \) and \( \Lambda(R) \) models the right-hand side of equation (6.1) is not negligible if compared with the source term on the left-hand side. In the case of the \( \Lambda(H^2) \) model with \( \alpha \neq 0 \) (then \( \rho_i \neq 0 \)) equation (6.1) becomes:

\[
\dot{\delta}_i + 3 \mathcal{H} \delta_i (1 + c_s^2) = 3 \mathcal{H} \delta_i (1 + \omega_i)(1 - \alpha) - 3 \mathcal{H} \delta_i (1 + \omega_i)(1 - \alpha) ,
\] (6.3)

where the first term on the right-hand side not only cancels out the one in the left-hand side, but sometimes the former even becomes more relevant than the latter. This occurs because \( \omega_i \geq 0 \), while that \( c_s^2 \approx 3(1 + \omega_i)^2(1 - \alpha)(i \mathbf{k} \cdot \mathbf{v}_i / \mathcal{H} / k) \) is negative in most of cases but \( |c_s^2| \ll 1 \) for sub-horizon modes, and \( |\alpha| \ll 1 \). In addition to that, \( \delta_i \) has typically the opposite sign of \( \delta_i \), thus the second term on the right-hand side becomes an additional contribution to the growth of the vacuum density perturbation.

The same conclusion applies to the \( \Lambda(R) \) model. The source terms in the right-hand side of equation (6.1) (and associated to the vacuum-matter interaction) are responsible for the growth of the vacuum density perturbations by virtue of they have the opposite effect and a bigger magnitude than the \( \Lambda CDM \) source term on the left-hand side.

### 6.3 Impact of the running vacuum over the matter perturbations

The CDM density contrast \( \delta_c \) is the dominant component driving the growth of the structures in small scales. In the context of the \( \Lambda(H^2) \) model the dynamical equation for \( \delta_c \) can be written as

\[
\delta_c' = \frac{4 \pi G}{a^2 \mathcal{H}} \int da \frac{a^2}{\mathcal{H}} \sum_i \delta_i \left[ \delta_i (1 + 3 \omega_i) + \frac{\alpha}{1 - \alpha} \delta_i \right] \left( 1 + 3 \frac{\delta P_i}{\delta \rho_i} \right) ,
\] (6.4)

where the effects of the background quantities, modified by the running of the vacuum, are mixed with the vacuum perturbations in the right-hand side. These two sources drive together the evolution of the CDM density contrast and can not be easily, if possible, separated. In that follows, we show that the vacuum perturbations can be as important as, or even more important than, the background contributions to the evolution of the small scale matter perturbations. That feature represents a strong motivation to include the vacuum perturbations when testing the \( \Lambda(H^2) \) or \( \Lambda(R) \) models against the current observational data coming from the CMB or matter spectrums.
δρ/ρ_{cr} 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Comparison between density perturbation of the matter and the vacuum components for mode \(k = 0.7 \, \text{Mpc}^{-1}\). For display purposes, density perturbations \(\delta \rho = \bar{\rho} \delta\) were normalized by the critical density \(\rho_{cr}\). The solid right colors correspond to the matter components, and the dot-dashed light colors represent their vacuum counterparts. \emph{Left panel:} \(\Lambda(H^2)\) model with \(\alpha = 0.01\). \emph{Right panel:} \(\Lambda(R)\) model with \(\beta = 0.01\).}
\end{figure}

6.3.1 Small scales

By definition, the background differences between the running vacuum and the ΛCDM models are scale independent. In particular, the relation between the horizon crossing time for the standard and \(\Lambda(H^2)\) models is 

\[ \tau_{\Lambda(H^2)}|_{H=k} \approx \tau_{\Lambda CDM}|_{H=k}/(1 - \alpha), \]

which corresponds to a percent difference of \(\sim (100\alpha)\%\).

Near the horizon crossing time the initial conditions for the matter and vacuum perturbations in section 4.3 give us

\[
\left| \frac{\delta \rho_{\Lambda ur}}{\delta \rho_{c}} \right|_{H=k} \approx 7\alpha \frac{\Omega_{ur}^{0}}{\Omega_{c}^{0}} \left( \frac{3k^{2}}{\Omega_{ur}^{0} H_{0}^{2}} \right)^{(1-\alpha)/(2-4\alpha)} \sim \frac{\alpha \, k}{3 \, H_{0}},
\]

where \(ur\) means photons, massless neutrinos, and ultra-relativistic massive neutrinos. Therefore, the vacuum perturbation \(\delta \rho_{\Lambda ur}\) represent a contribution of \(\sim (10^{5}\alpha \, k \, \text{Mpc/h})\%\) to the Einstein field equations just after the horizon crossing, i. e., when \(H = k\). Furthermore, for sub-horizon modes, the density contrasts of the vacuum component interacting with ultra-relativistic matter grows roughly as a linear function of the scale factor, \(\delta_{ur} \propto \alpha \), see section 4.4: while the CDM density contrast grows logarithmically \(\delta_{c}(a \ll a_{eq}) \propto \ln(a)\) during the radiation dominated era, and then linearly \(\delta_{c}(a \gg a_{eq}) \propto a\) during the non relativistic matter dominated era. Here \(a_{eq}\) is the scale factor when the radiation and the non relativistic matter have the same mean density \(\bar{\rho}_{m}(a_{eq}) \equiv \bar{\rho}_{ur}(a_{eq})\). In this way, during the radiation dominated era the \(|\delta \rho_{\Lambda ur}/\delta \rho_{c}|\) ratio decreases only logarithmically with the scale factor after the horizon crossing, see figure 3. For the reason that the ratio \(|\delta \rho_{\Lambda ur}/\delta \rho_{c}|_{H \leq k}\) grows with \(k\), see eq. (6.5), the vacuum perturbations seem to have a bigger/smaller impact over the matter perturbations in small/large scales (around \(k \sim 10^{-3}h/\text{Mpc}\)) than background quantities.

During the matter dominated era, see section 4.4, the density perturbations associated to matter and vacuum sub-horizon modes are related by

\[
|\delta \rho_{\Lambda n}| \gtrsim 3|\alpha \, \delta \rho_{m}|, \quad \text{and} \quad |\delta \rho_{\Lambda ur}| \gg |\alpha \, \delta \rho_{ur}|,
\]

where \(\delta \rho_{\Lambda n}\) and \(\delta \rho_{\Lambda ur}\) are the density perturbations of the vacuum and the running vacuum models, respectively.
while $\mathcal{H}_{\Lambda(H^2)} \rightarrow \mathcal{H}_{\Lambda CDM}$ when $a \rightarrow 1$. In this case, the vacuum perturbations have a role at least as important as the background quantities for the matter density contrast evolution in equation (6.4).

All of the above considerations apply to the $\Lambda(R)$ models, except the related to the vacuum component associated with ultra-relativistic matter, which does not exist in this context.

### 6.3.2 Large scales

Since the super-horizon initial conditions for the matter density contrast do not change with $\alpha$ or $\beta$ when expressed as a function of $\tau$, see sections 4.3 and 5.3, the vacuum effects over matter perturbations only become effective inside the horizon. Besides, the Fourier modes of the vacuum perturbations that fall late into the horizon remain sub-dominant. That occurs because only sub-horizon modes of the vacuum perturbations during the radiation dominated era grow faster than matter perturbations, see sections 4.4 and 6.3.1. In this way, the more relevant effects that the running vacuum has over the large scale matter perturbations come through the geometrical quantities. For example, the age of the universe (as well as the time each Fourier mode remains into the horizon until now) is bigger/smaller for $\alpha \gtrless 0$ or $\beta \gtrless 0$ allowing the matter perturbations to grow for a longer/shorter time.

### 7 Impact of the running vacuum in the CMB and matter spectrums

The constraints of the free parameters, for each model, shown at the end of this section was obtained by using the Monte Carlo Markov chain sampler MontePython [40] and the Planck 2015 data release [39].

#### 7.1 Matter power spectrum

In figure 4 the matter power spectrum for the $\Lambda(H^2)$ and $\Lambda(R)$ models is shown, in the left and right-hand side respectively. As discussed in section 6.3, the running vacuum effects over the large scale matter perturbations are dominated by the direct influence of background
quantities (and not for the indirect influence of the vacuum perturbations through its contribution to the metric perturbations). Thus, since the initial conditions for the perturbations of all the matter species are exactly the same in the $\Lambda$CDM, $\Lambda(H^2)$ and $\Lambda(R)$ models, the basic difference between them is that the age of the universe is bigger/lower than that in the standard model for $\alpha, \beta \gtrless 0$ respectively. In fact, $\tau_0|_{\Lambda(H^2)} \approx \tau_0|_{\Lambda(R)} \approx \tau_0|_{\Lambda CDM}/(1 - 2.7\alpha)$ for $\alpha \lesssim 0.3$, and $\tau_0|_{\Lambda(R)} \approx \tau_0|_{\Lambda CDM}(1 - 3\beta)/(1 - 6\beta)$ for $\beta \lesssim 0.1$ while $\tau_0|_{\Lambda(R)}/\tau_0|_{\Lambda CDM}$ goes monotonically to $\sim 7.7$ when $\beta \to 1/3$. Thus, the matter perturbations have less/more time to grow until today for positive/negative values of $\alpha$ or $\beta$, producing a decrease/increase of the matter power spectrum in large scales.

On the other hand, the small-scale matter density contrast (sourced by the metric perturbations, which are affected by the vacuum perturbation contributions) grows slower/faster for positive/negative values of $\alpha$ or $\beta$. This results in a lack/excess of the matter power spectrum in small scales, which can be seen in figure 4. This effect grows with $k$ due to the smaller scales come earlier into the horizon, where they are affected indirectly for a longer time by the running vacuum perturbations.

As we can see in figure 4, in large scales, the matter power spectrum of both running vacuum models is almost indistinguishable. A possible signature of these models should be in the small scale regime (non-linear scales), where the linear estimation of the matter power spectrum for $\Lambda(H^2)$ and $\Lambda(R)$ models are distinguishable from each other, see figure 4. In order to prove that, a non-linear analysis should be done, but that is out of the scopes of the present work.

The approach presented in this work does not lead to an instability of the matter power spectrum in small scale for negative values of $\alpha$ or $\beta$, as found by [32, 35, 37, 37], where it was used another formulation for the linear perturbations. In the present work, every cosmological observable is affected in a stable way by the small running coefficients $\alpha$ or $\beta$, and reducing to the $\Lambda$CDM case when these parameters vanish.

### 7.2 CMB power spectrum

In this section, the CMB temperature anisotropies of the $\Lambda$CDM, $\Lambda(H^2)$ and $\Lambda(R)$ models are presented. Previous parameter constraints suggest $\{\alpha, \beta\} \in (10^{-5}, 10^{-3})$ [19, 44, 45]. In figure 5 we show the $TT$ spectrum of the CMB anisotropies, for the $\Lambda(H^2)$ model on the left side, and for $\Lambda(R)$ model on the right side. Below we describe the most relevant effects of the running vacuum on the $TT$ power spectrum of the CMB, which imply some degeneracies between the basic cosmological parameters and the running parameters, $\alpha$ or $\beta$.

#### 7.2.1 Peaks position

In the case of the $\Lambda(H^2)$ model (left panel of figure 5), the main effect of the running vacuum seems to be that the position of every peak in the $TT$ spectrum is shifted to low/high $l$’s when the free parameter $\alpha$ assumes positive/negative values. The peaks shift seems to be monotonic with both the reference $\Lambda$CDM position and the value of the parameter $\alpha$. That effect is also present in the $\Lambda(R)$ running vacuum model (right panel of figure 5), but the shift is less sensitive to the value of $\beta$. In the context of the $\Lambda$CDM model, this effect can be compensated principally by the current physical densities of cold dark matter and baryons, $\Omega_c h^2$ and $\Omega_b h^2$. Besides, the bottom part of the figure 5 shows that, when comparing the running vacuum models with the standard one, the difference in the amplitude of the spectrum is not the same for odd and even peaks. This effect is similar to that produced by
the variation of $\Omega_b h^2$, which implies a degeneracy between this parameter and the parameters $\alpha$ and $\beta$ associated to the running vacuum models.

On the other hand, the change in the angular size of the sound horizon at recombination $\theta_s$ also shift the peak positions, but it also changes the amplitude of the power spectrum for $l \lesssim 10$. Since that last effect is not strong in the context of the running vacuum models (figure 5), it is not expected a big degeneracy between $\theta_s$ and the running parameters $\alpha$ or $\beta$.

7.2.2 Amplitude

The stronger effect of the running vacuum for the model $\Lambda(R)$ is the increase/decrease of the power amplitude for positive/negative values of $\beta$ (right top panel of figure 5). This change occurs in the same way for almost every scale of the $TT$ spectrum, except for the low multipoles, where the running effect is weaker. The $\Lambda(H^2)$ model shows the same kind of variation with $\alpha$, but in this case the effect is weaker. In the case of the $\Lambda(R)$ model, the variation of the optical depth during reionization, $\tau_{\text{reio}}$, should help to compensate the running vacuum effects, since it modifies the amplitude of the $TT$ spectrum after the first peak.

In addition, in figure 5, it can be seen that a global compensation in the amplitude is not enough to correct the running vacuum effects in the model $\Lambda(H^2)$ due to their discrepancy with the standard model grows with the index $l$ of the multipoles. This monotonous discrepancy in the amplitude can be countered by the variation of the scalar spectral index $n_s$.

7.2.3 Main degeneracies

The mentioned effects that the running vacuum has over the temperature spectrum of the CMB, would be the main sources of correlation between $\alpha$ and the cosmological parameters $\{\Omega_b h^2, \tau_{\text{reio}}, n_s\}$ in the case of the $\Lambda(H^2)$ running vacuum model, and between $\beta$ and $\{\tau_{\text{reio}}, \Omega_b h^2\}$ for the $\Lambda(R)$ model.

The degeneracy between $\beta$ and the baryon fraction $\Omega_b$ arises because, in the $\Lambda(R)$ running vacuum model, the baryon-photon ratio is different to that in the standard model for every redshift other than zero:

$$\left.\frac{\tilde{\rho}_b}{\tilde{\rho}_\gamma}\right|_{\Lambda(R)} = \frac{\Omega_b}{\Omega_\gamma} a^{1/(1-3\beta)} \lesssim \frac{\Omega_b}{\Omega_\gamma} a = \left.\frac{\tilde{\rho}_b}{\tilde{\rho}_\gamma}\right|_{\Lambda_{\text{CDM}}}, \quad \text{for } \beta \gtrsim 0 \quad \text{and } a < 1. \quad (7.1)$$
The $\Lambda(H^2)$ model is affected by the running vacuum in the same way, but with a weaker and opposite dependence on the sign of $\alpha$:

$$\frac{\bar{\rho}_b}{\bar{\rho}_\gamma}\bigg|_{\Lambda(H^2)} = \frac{\Omega_b}{\Omega_\gamma} a^{1-\alpha} \geq \frac{\Omega_b}{\Omega_\gamma} a = \frac{\bar{\rho}_b}{\bar{\rho}_\gamma}_{\Lambda CDM}, \quad \text{for } \alpha \geq 0 \quad \text{and } a < 1. \tag{7.2}$$

The degeneracy between $\{\tau_{\text{reio}}, n_s\}$ and the running parameters $\alpha$ or $\beta$ — related to the amplitude of the power spectrum — has an explanation outside of the background approach: the matter perturbations grow faster/slower in the context of the running vacuum model than within the standard cosmological scenario, for negative/positive values of $\alpha$ or $\beta$ (see section 6).

Finally, it is worth noticing that for low multipoles, $l \lesssim 20$, the temperature spectrum is weakly affected by the running of the vacuum. This occurs because the super-horizon evolution of the matter perturbations is the same in the standard and running vacuum models (see sections 4.3 and 5.3); and after the horizon crossing the running vacuum effects come from the background quantities that are poorly affected in low redshifts (see section 6.3). Therefore, the low multipoles of the CMB temperature spectrum are poorly affected by the running of the vacuum.

### 7.3 Planck constraints

In order to constrain the free parameters of the $\Lambda(H^2)$ and $\Lambda(R)$ models, we confront the theory with the observational data given by the CMB anisotropy spectrum. The theoretical prediction was calculated by integrating numerically the complete set of evolution equations for both the background and the scalar linear perturbations using the code CLASS [38], which was modified in order to include the vacuum perturbations and the background effects of the running vacuum. The observational data used here was the high and low $C_l$’s corresponding to the $TT$, $TE$ and $EE$ CMB spectrum given by the 2015 data release of the Planck Collaboration [39]. The Monte Carlo Markov Chains were generated using the MontePython free code [40], while the statistical analysis and plots were carried out by the GetDist package [62].

Additionally, in order to establish a fair comparison between the six-parameter cosmological standard model and the $\Lambda(H^2)$ or $\Lambda(R)$ models, the effective numbers of neutrinos was fixed as $N_{\text{eff}} = 3.046$, the neutrino mass $m_\nu = 0.06\text{eV}$ and a single massive neutrino family, as done by the Planck Collaboration [6]. The results are summarized in table 1, and figures 6, 7 and 8.

#### 7.3.1 Main parameters

The data constraints of the CMB power spectrum (table 1, figures 6 and 8) show that the cosmological parameters that are more affected by the $\Lambda(H^2)$ model are $\{n_s, \Omega_b h^2, \theta_s, \ln(10^{10} A_s), \tau_{\text{reio}}\}$, which means were shifted $\{1.2\sigma, 1\sigma, 0.5\sigma, 0.5\sigma, 0.4\sigma\}$ respectively, regarding the $\Lambda CDM$ constraints.

A larger value of the spectral index $n_s$ can offset the monotonic decrease of amplitude, as a function of $l$, generated by the running vacuum on high multipoles, in the case when $\alpha = 0$ (see figure 5). A larger value of the baryon fraction $\Omega_b h^2$ roughly generates a systematic difference in amplitude between even and odd peaks of the spectrum, in the same way as that produced by the running vacuum (see the bottom box in figure 5). At the same time, the larger value of $\Omega_b h^2$ shifts the peaks position to the higher $l$’s, increasing the discrepancy generated by the running of the vacuum. This shift in the peaks position can be offset with
According to table 1, see also figure 6, the calculated cosmological parameters \{ \Omega_\Lambda, H_0, \sigma_8 \} are shifted to higher values in the presence of the running vacuum \( \Lambda(R) \), related to the standard model. In the case of the \( \Lambda(R) \) model, the mean of these cosmological parameters \{ \Omega_\Lambda, H_0, \sigma_8 \} increases in \{ 0.6 \sigma, 0.9 \sigma, 0.6 \sigma \}, but their variances grow by a factor of \{ 2.4, 2.7, 2.3 \}, related to the \( \Lambda \)CDM values.

For the \( \Lambda(H^2) \) model, the mean of the estimated cosmological parameters \{ \Omega_\Lambda, H_0, \sigma_8 \} increases in \{ 2.1 \sigma, 3.4 \sigma, 2.5 \sigma \} while their variances grow by a factor of \{ 3.2, 4.7, 3.2 \}, related to the \( \Lambda \)CDM values.

The positive shift in the calculated cosmological parameters \{ \Omega_\Lambda, H_0, \sigma_8 \}, due to the running vacuum effects, goes in the same direction that the low-redshift constraints of the \( \Lambda \)CDM cosmological model. In this sense, the running vacuum models could reduce the tension between the high and low redshift observational constraints for \Omega_\Lambda and H_0. In addition, due to the fact that the background densities and geometric cosmological quantities of both running vacuum models have the same values as those in the standard model at redshift zero, but their evolution is affected by the running vacuum, it is expected that low redshift cosmological tests of the running vacuum models may improve the constraints of \( \Omega_\Lambda, H_0 \) and \( \alpha \) or \( \beta \) parameters.
8 Conclusions

This paper aims to advance the linear perturbations study of phenomenological models for a running cosmological constant coming up from renormalization group ideas. We studied two running vacuum models, considering the energy exchange between the vacuum and each matter component, not only with the dominant component as it is often done. This improvement in such approach seems to be necessary in order to analyze the effects of the running vacuum over the matter and the CMB power spectrums.

We considered two representative classes of models where $\Lambda$ run as a linear function of $H^2$ or $R$ with linear coefficients $3\alpha$ and $\beta$ respectively, besides $\mathcal{P}_\Lambda \equiv -\mathcal{P}_\Lambda$, for which the running of the vacuum is possible because of the energy exchanging between it and the matter components. For these models ($\Lambda(H^2)$ and $\Lambda(R)$), the vacuum energy density can be split as a constant term plus linear partial contributions, $\rho_\Lambda$, coming from the matter densities and pressures, $\rho_i$ and $\mathcal{P}_i$, where each pair $\{\Lambda_i, i\}$ exchanges energy independently of other pairs $\{\Lambda_j, j\}$ for $i \neq j$.

Although the two models analyzed in this paper lack a Lagrangian formulation, the use of the Boltzmann equation formalism for the treatment of the matter components, and a
fluid approximation for their vacuum counterparts, allows us to formulate a self-consistent treatment for the linear perturbations. As a result, the matter perturbations are only affected by its vacuum counterparts through the metric perturbations.

Unlike the matter components, the equations for the evolution the vacuum density perturbations have a nontrivial source term coming from their matter counterparts. The terms considered sub-dominant for the evolution the vacuum perturbations in intuitive treatments (but that appears in a natural form in this work) are the sources for the growth of $\delta_{\Lambda i}$, due to they exceed in magnitude, and have a different sign from the traditional terms which contribute for the dissolution of the vacuum perturbations.

The contribution of those growing vacuum perturbations to the metric ones (which behave as sources for matter perturbations throughout the entire cosmic history) allows the Fourier modes of the matter perturbations that fall earlier into the horizon to be the most affected by the running vacuum. On the other hand, the background effects are the responsible for the little differences between the standard and the running vacuum models in large scales, where the vacuum perturbations effect are sub-dominants.

The Planck constraints obtained in this work give $10^4 \alpha = -4.7 \pm 6.5$ for the $\Lambda(H^2)$ model, which is consistent with the $\Lambda$CDM model (in which $\alpha = 0$) in $\sim 0.72\sigma$, as well as

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Comparison between the CBM anisotropies constrain of the six-parameter cosmological standard model and the running vacuum model $\Lambda(H^2)$.}
\end{figure}
$10^4 \beta = -1.4 \pm 5.6$, which makes the $\Lambda(R)$ model almost indistinguishable from the $\Lambda$CDM model (given by $\beta = 0$). In addition, the constraints show that the Hubble rate, the matter fraction today, and the RMS linear matter fluctuation today are the most affected parameters by the running of the vacuum. The constraints of the Hubble parameter and the current fraction of matter are bigger and smaller, respectively, in the running vacuum models than in the standard model. As a consequence of that, the non-relativistic matter dominated era, as well as the age of the universe, are shorter than those in the standard case. That reduces the growth time of the matter perturbations, but on the other hand, the negative value of $\alpha$ or $\beta$ produces a bigger growth rate, compensating the shorter growth time due to the running of the vacuum.

Remarkably, the cosmological parameters, $\Omega_\Lambda$ and $H_0$, constrained by the CMB power spectrum, seems to alleviate the tension with low-redshift observations: they present positive shifts in the $\Lambda(H^2)$ and $\Lambda(R)$ models compared with the standard case. Even so, the shifts in these parameters — due to the running vacuum — are compatible with the $\Lambda$CDM case, because the constraints become weaker due to the degeneracies between the running parameters and the standard cosmological parameters.
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