Asymptotic formulae for the Lyapunov spectrum of fully-developed shell model turbulence

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Abstract

We study the scaling behavior of the Lyapunov spectra of a chaotic shell model for 3D turbulence. First, we quantify localization of the Lyapunov vectors in the wavenumber space by using the numerical results. Using dimensional arguments of Kolmogorov-type, we then deduce explicitly the asymptotic scaling behavior of the Lyapunov spectra. This in turn is confirmed by numerical results. This shell model may be regarded as a rare example of high-dimensional chaotic systems for which an analytic expression is known for the Lyapunov spectrum. Implications for the Navier-Stokes turbulence is given. In particular we conjecture that the distribution of Lyapunov exponents is not singular at null exponent.

47.27.Jv, 05.45.+b
We consider the behavior of Lyapunov spectra of a Gledzer’s type shell model of 3D turbulence [1]. In this chaotic model, a set of complex variables $u_j$, $(j = 1, 2, ..., N)$, which represents the velocity in the shell $k_j = 2^{j-10}$, are governed by the following equations of motion:

$$\left(\frac{d}{dt} + \nu k_j^2 \right) u_j = i[a_j u_{j+1} u_{j+2} + b_j u_{j-1} u_{j+1} + c_j u_{j-1} u_{j-2}]^* + f\delta_{j,4}.$$  \hspace{1cm} (1)

The coupling constants in the nonlinear terms are assumed as follows to ensure energy conservation: $a_j = k_j$, $b_j = -\frac{1}{2} k_{j-1}$, $c_j = \frac{1}{2} k_{j-2}$, $b_1 = c_1 = c_2 = a_{N-1} = a_N = b_N = 0$.

A number of remarkable properties were revealed by extensive numerical and analytical studies on this model. In particular, for large Reynolds $R (= 1/\nu)$ number the solution to this model is generally chaotic [2-3] and that its energy spectrum satisfies a Kolmogorov’s scaling law of realistic turbulence. Other aspects of this model such as intermittency, probability distributions of velocity variables [4-6] and the effects of extra helicity-like invariant are also discussed [7,8].

Because these properties can be achieved with 30-50 degrees of freedom, its detailed study is feasible to explore possible links between the conventional theory of turbulence and the chaotic dynamical systems. One of the most remarkable properties is that the distribution function of Lyapunov exponents $\lambda$ appears to diverge at $\lambda = 0$ in the limit of large Reynolds number. This possibility was pointed out before in [9] using the $\beta$- model. This suggests that the inertial subrange is connected with a large number of (almost) null exponents. Indeed, a correlation between Fourier and Lyapunov indexes was observed in the long-time average of the (squared) Lyapunov vectors. However, neither its relation to the Kolmogorov scaling is left unexplained nor the mechanism of accumulation of null exponents. In this Letter, we will see that how this characteristic Lyapunov spectrum is related with Kolmogorov’s scaling through localization of Lyapunov vectors in the wavenumber space and obtain an asymptotic formula for the Lyapunov exponents.

The equations (1) are integrated numerically by the fourth-order Runge-Kutta method,
together with \(2N\) linearized equations. Numerical parameters \(\nu = 10^{-9}, 10^{-8}, 10^{-7}, 10^{-6}\) respectively for \(N = 19, 22, 24, 27\). The forcing is fixed as \(f = 5 \times 10^{-3} \times (1 + i)\). The time step used was \(\Delta t = 5 \times 10^{-5}\) for \(N = 27\). After some transient stage, the solution apparently reaches a statistically stationary state. Below we will consider the long-time averages over this state. The energy spectrum shows about 2 decades of Kolmogorov range (not shown). The Lyapunov dimensions are \(D = 19.8476, 25.4072, 30.2088, 34.9668\) respectively for \(N = 19, 22, 24, 27\).

Let \(v_n^{(j)}\) be the \(j\)-th Lyapunov vector for the \(n\)-th Fourier mode \((j, n = 1, 2, \ldots N)\). We plot in Fig.1 the squared components of the Lyapunov vectors in time average for \(N = 27\):

\[
E^{(j)}(k_n) = \langle |v_n^{(j)}|^2 \rangle. \tag{2}
\]

Note that each Lyapunov vector is normalized as \(\sum_{n,j} |v_n^{(j)}|^2 = 1\). Several distinct features are noted.

1. Each Lyapunov vector has a support localized around a specific wavenumber.

2. The center of the support of the Lyapunov vector lies at \(n \approx D/2\) for the largest Lyapunov exponent \((j = 1)\).

3. The central wavenumber of the support decreases with \(j\), reaching \(n \approx 0\). The corresponding Lyapunov exponents are positive but those corresponding to \(\approx 0\) are small.

4. For larger \(j\), the central wavenumber increases again reaching \(n \approx D/2\) at \(j \approx D\). The Lyapunov exponents for these are negative.

5. For even larger \(j\), the central wavenumber increases beyond \(n \geq D/2\). These Lyapunov exponents asymptotically agree with the reciprocal of the viscous time scale \(-\nu k_n^2\) of the equation (1).

To summarize, for \(0 \leq n \leq D/2\), there are two Lyapunov vectors for each \(n\), one corresponding to positive exponent and the other negative. All these features are consistent with
the fact that the Lyapunov dimension measures the number of modes below the dissipation wavenumber and that each wavenumber has 2 degrees of freedom, the real and the imaginary parts of the velocity variables.

On the basis of the above observation we will introduce the following set of hypotheses regarding the Lyapunov vectors in the inertial subrange $j \leq D$ with $D \gg 1$.

1. Each Lyapunov vector in wavenumber space is supported in a localized manner around a specific wavenumber.

2. Lyapunov exponents are positive for $1 \leq j \leq D/2$ and negative for $D/2 < j$.

3. Let $n_j$ be the wavenumber for $j$–th Lyapunov vector, then
   
   \begin{align*}
   (a) \quad &n_j = D/2 - j + 1 \quad \text{for} \quad 1 \leq j \leq D/2, \\
   (b) \quad &n_j = j - D/2 \quad \text{for} \quad D/2 \leq j \leq D.
   \end{align*}

4. In the inertial subrange, the $j$–th Lyapunov exponent ($j \leq D$) is inversely proportional to the time scale $\epsilon^{-1/3}k_{n_j}^{-2/3}$, which is characteristic to wavenumber $k_{n_j} = k_02^{n_j}$.

The last hypothesis is equivalent to assume that, through the localization of the Lyapunov vectors in wavenumber space, the Lyapunov exponents can be expressed solely in terms of the energy dissipation rate $\epsilon$ and wavenumber $k$ by the dimensional arguments of Kolmogorov-type. Combining these hypotheses, we deduce the following formulae for the Lyapunov exponents $\lambda_j$:

$$\lambda_j \sim \begin{cases} 
\epsilon^{1/3}k_{n_j}^{2/3} \sim -2^{-2j/3}, & \text{for} \quad 1 \leq j \leq D/2 \\
-\epsilon^{1/3}k_{n_j}^{2/3} \sim -2^{(j-D/2+1)/3}, & \text{for} \quad D/2 \leq j \sim D
\end{cases}$$

(3)

We rescale the Lyapunov exponent as $\lambda_j / H$, where $H = \sum_{\lambda_j > 0} \lambda_j$ is the Kolmogorov-Sinai entropy. This is equivalent to choose time $\tau = at$ ($a = \sum_{j=1}^{D/2} \lambda_j$), such that $H$ is normalized. Note that this choice of time neither influences geometric structure of strange
attractors nor invariant measures on them. Noting that

\[ H = \sum_{j=1}^{D/2} \lambda_j \sim \frac{2^{D/3} - 1}{2^{2/3} - 1} \]  

we conclude that

\[
\lambda_j / H \sim \begin{cases} 
(2^{2/3} - 1) \frac{2^{D/3}}{2^{2D/3} - 1} 2^{-2j/3} & (D \to \infty), \text{ for } 1 \leq j \leq D/2 \\
\frac{2^{2(j-D/2+1)/3}(2^{2/3} - 1)}{2^{2/3}(2^{D/3} - 1)} & \text{for } D/2 \leq j \sim D 
\end{cases}
\]  

(5)

Because these expressions are free from arbitrary parameters, we can compare this phenomenological argument against the numerical results.

The Lyapunov spectra were computed for four values of viscosity. In Fig.2 we plot the positive Lyapunov exponents \( \lambda_j / H \) against \( H \) together with the theoretical curve for \( D \ll 1 \):

\[ \lambda_j / H \sim (2^{2/3} - 1) 2^{-2j/3}, \]  

(6)

which can be obtained from (5). Note that better agreement is obtained between the phenomenological theory and numerical results for larger \( N \), that is for larger Lyapunov dimensions. In Fig.4, we plot \( \sum_{i=1}^{j} \lambda_i / H \) and compare with the theoretical prediction

\[ \sum_{i=1}^{j} \lambda_i / H = 1 - 2^{-2j/3}. \]  

(7)

Again, the theoretical prediction agree well with the numerical data and this agreement is better for larger \( N \). Note that this scaling is different from the one proposed in [2] previously. In Fig.4 we replotted the cumulated Lyapunov spectra using older scaling: \( \sum_{i=1}^{j} \lambda_i / H \) vs \( j/D \). We see sizable scatters in this enlarged figure and conclude that the older scaling is premature.

For the other hand for \( j > D/2 \), a rough the agreement is seen between the theory and the numerical results. At present we do not know why the agreement is less clear.

We discuss the implications of the above results for the realistic Navier-Stokes turbulence. Suppose that the support of the Lyapunov vectors are localized in the wavenumber space in the Navier-Stokes turbulence, as was true for the case of the shell model. This implies the
existence of characteristic time scale in the motion represented by the Lyapunov vectors and 
that Lyapunov exponents are given by the reciprocal of relevant time scales. For simplicity, 
we consider Navier-Stokes flow under periodic boundary condition in a box of size $L^3$. The 
Reynolds number is assumed to be so large that the motion we consider is in the inertial 
 subrange. The total number of eddies of size $r = L/n$ is estimated as $n^3$, where $n$ is a natural 
number. This is proportional to the number of modes whose scale is larger than $r$. The 
Lyapunov exponent associated with the smallest eddies can be estimated as the reciprocal 
of the Kolmogorov time-scale, that is, $\lambda \sim r^{-2/3} \sim n^{2/3}$. In this way we find 

$$n^3 \sim \lambda^{9/2}. \tag{8}$$

This shows that the number of modes whose Lyapunov exponents do not exceed $\lambda$ is $\lambda^{9/2}$. 
Therefore, the distribution function $P(\lambda)$ of the Lyapunov exponents is 

$$P(\lambda) \sim \lambda^{7/2} \tag{9}$$

It should be noted that this distribution $P(\lambda)$ does not diverge at $\lambda = 0$, in a marked 
contrast to the case of shell model. In the latter $P(\lambda)$ has a singularity at $\lambda = 0$ like $1/\lambda$. In 
retrospect, The divergence of the distribution function of Lyapunov exponents stems from 
the condensation of modes at null wavenumber, which is a result of octave discretization 
of wavenumbers. Note that Ruelle’s argument uses the $\beta -$model, whose wavenumbers are 
also discretized in octaves. For realistic fluid turbulence, we conjecture that the distribution 
function of Lyapunov exponents is regular at $\lambda = 0$ because of no condensation at null 
wavenumber. Unfortunately, it is practically impossible to check the prediction (9) by 
the direct numerical simulations. Further studies on the accumulation of null Lyapunov 
exponents, e.g. in connection with the Anderson localization, will be reported elsewhere.
REFERENCES

[1] E. B. Gledzer, Sov. Phys. Dokl. 18, 216(1973).
[2] M. Yamada and K. Ohkitani, J. Phys. Soc. Jpn., 56, 4210(1987).
[3] K. Ohkitani and M. Yamada, Prog. Theor. Phys., 81, 329(1989).
[4] M.H. Jensen, D. Paladin and A. Vulpiani, Phys. Rev. A, 43, 798(1991).
[5] D. Pisarenko, L. Biferale, D. Courvoisier, U. Frisch and M. Vergassola, Phys. Fluids A5(1993)2533.
[6] R. Benzi, L. Biferale and G. Parisi, 'On intermittency in a cascade model for turbulence,' Physica D (Amsterdam) 65(1993)163
[7] L. Kadanoff, D. Lohse, J. Wang, and R. Benzi Phys. Fluids 7, 617(1995).
[8] L. Biferale, A. Lambert, R. Lima and G. Paladin, Physica (Amsterdam) D 80105(1995).
[9] D. Ruelle, Commun. Math. Phys. 87, 287(1982).
FIGURES

FIG. 1. Time average of squared components of Lyapunov vectors: $<|v_i^{(j)}|^2>$. Contour levels are $0.0489 \times i = 1, 2, \ldots, 10$ Two straight lines represent correspondence assumed in the hypotheses 3.(a) and (b).

FIG. 2. Distribution of Lyapunov exponents $\lambda_j/H$. Double circles ($N = 19$), double squares ($N = 22$), solid squares ($N = 24$) and solid circles ($N = 27$). The dashed line denotes the theoretical prediction: $(2^{2/3} - 1)2^{-2j/3}$.

FIG. 3. Cumulated distribution of Lyapunov exponents $\sum_j \lambda_i/H$ Symbols are the same as in Fig. 2. The dashed line denotes the theoretical prediction: $1 - 2^{-2j/3}$.

FIG. 4. Cumulated distribution of Lyapunov exponents depicted using previous scaling: $\sum_j \lambda_i/H$ against $j/D$. Symbols are the same as in Fig. 2.

FIG. 5. Distribution of negative Lyapunov exponents $|\lambda_j|/H$ against $j - D/2$. The dashed line denotes the theoretical prediction: $(2^{2/3} - 1)/(2^{2/3}(2^{D/3} - 1))2^{2(j-D/2)/3}$.