Quantum Mechanics as a Classical Theory
VI: The Classical Spin

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Abstract
In these continuation papers (VI and VII) we are interested in approaching the problem of spin from a classical point of view. In this first paper we will show that the spin is neither basically relativistic nor quantum but reflects just a symmetry property related to the Lie algebra to which it is associated. The classical approach will be paralleled with the usual quantum one to stress their formal similarities and epistemological differences. The important problem of Einstein-Bose condensation for fermions will also be addressed.

1 General Introduction
The following two papers deal with the idea of classical spin. They are intended to show that the concept of half-integral spin might be raised also in the realm of a classical theory.

This first paper begins by showing that with some supposition about the half-integer spin particle structure it is possible to recover all the results derived using quantum theory. In this case, the language of operators is substituted by a language of functions and the commutator is replaced by the Poisson bracket. Functions formally identical with creation and annihilation operators are defined and their interpretation—easier in this formalism—is described. Another function related with the 'number' operator is also derived and interpreted. When passing from the passive to the active view, the above mentioned functions become operators and we recover again all the quantum mechanical formalism in Heisenberg’s notation—only the half-spin case is explicit treated.

The Exclusion Principle of Pauli is also addressed. It is shown that the model assumed for the half-integral spin particle structure and the imposition upon the eigenfunctions to make diagonal the operators $\hat{S}_3$ and $\hat{S}_2$ are sufficient to derive the exclusion behavior; which now becomes a theorem.
As a secondary result of the picture proposed, it is shown that a Bose-Einstein condensation for fermions might be easily interpreted by this theory.

In the second paper, the phase-space expressions for the spin functions $S_3$ and $S^2$ are used to derive their equivalent quantum Schrödinger equation. This equation is then solved and we find all the possible half-integral spin eigenvalues and eigenvectors.

We then show quantitatively, why one expects fermionic Bose-Einstein condensation to take place.

We end this series (VI and VII) with our final conclusions.

### 2 Introduction

Since its manifestation as a result of the Stern-Gerlach experiment—among others—, the spin has been considered as a purely quantum mechanical manifestation of Nature that has no resemblance to any classical behavior. Many physicists defending an epistemological abyss between classical and quantum worlds (or ontologies) are now accustomed to cite this effect, the electronic spin, in support to their philosophical views.

The aim of the present paper is to show that the electronic spin could also be predicted from a purely classical point of view, without any reference to quantum mechanics. This procedure is coherent with our previous developments.

The main argument here is the fact that, when deriving the electronic spin, one needs not to make reference to any of the quantum postulates. What one needs to do is to use the Lie algebra induced by the product

$$[S_i,S_j] = i\hbar \epsilon_{ijk} S_k,$$

where the $S_i$ are operators and $[,]$ represents the commutator of these operators, together with some guess about the phenomenon itself revealed by the Stern-Gerlach experiment.

As everybody knows, the same algebra is induced by the product

$$\{S_i,S_j\} = \epsilon_{ijk} S_k,$$  \hspace{1cm} (2)

where now the $S_i$ are functions and $\{\cdot,\cdot\}$ represents the Poisson brackets.

In the next section we will see that with this product and some picture or model of the electron internal structure it is possible to derive exactly the same results already derived using the usual formalism of quantum mechanics.

### 3 The Classical Spin

We begin this section by making a picture of the electron structure. This procedure, of course, might be sustained only under a classical theory since, as is
amply known, the orthodox interpretation of quantum mechanics does not allow us to make world pictures.

This picture of the electron structure will be done based on the expected final results revealed by the Stern-Gerlach experiment, that is, space “quantization”. The resemblances and differences between this approach and the usual quantum mechanical one will be stressed whenever needed.

We want the electron to be a flat body possessing charge $e$ and mass $m$ that might have its geometry distorted in the plane perpendicular to its symmetry axis and which is rotating in this plane. This structure for the electron implies that we shall have

$$z = p_z = 0$$

if we choose the $z$-axis to be the symmetry axis fixed on the electron around which it is rigidly rotating (see fig. 1).

This geometry implies that the electron is capable of interacting with an external magnetic field $B$ with the interaction energy given \[ H = -\frac{e}{2mc} B \cdot L \]

where $L$ is the angular momentum related to the electron rotation—we forget for a moment the question about the Landé $g$-factor to which we return soon.

The three quantities characterizing the electron are:

- The angular momentum in the $z$-direction related with its rotation: $L_z$;
  $$L_z = xp_y - yp_x;$$
  with $L_x = L_y = 0$ because of (3).

- The quadrupole moments related to its possible distortions: $Q_{xy}$ and $Q_1$ where
  $$Q_{xy} = \sqrt{\frac{\alpha}{\beta}} x y + \sqrt{\frac{\beta}{\alpha}} px py$$
  and
  $$Q_1 = \frac{1}{2} \left[ \sqrt{\frac{\alpha}{\beta}} (x^2 - y^2) + \sqrt{\frac{\beta}{\alpha}} (p_x^2 - p_y^2) \right]$$
  where all other quadrupole moments vanish identically because of the condition $z = p_z = 0$ and $\alpha$ and $\beta$ are structure dimensional constants.

We now impose that the description of the electron be made in terms of the three quantities $S_i, \ i = 1, 2, 3$ related to the above defined moments as

$$S_3 = \frac{1}{2} L_z; \ S_2 = \frac{1}{2} Q_{xy}; \ S_1 = \frac{1}{2} Q_1.$$  

With this imposition it is easy to demonstrate that the three functions $S_i$ obey
\{S_i, S_j\} = \epsilon_{ijk} S_k, \quad (8)

where we will use the classical Poisson Bracket throughout.

We also see that the functions

\[ S_0 = \frac{1}{2} \left[ \sqrt{\frac{\alpha}{\beta}} (x^2 + y^2) + \sqrt{\frac{\beta}{\alpha}} (p_x^2 + p_y^2) \right] \quad (9) \]

and

\[ S^2 = S_1^2 + S_2^2 + S_3^2 = \frac{1}{16} \left[ \sqrt{\frac{\alpha}{\beta}} (x^2 + y^2) + \sqrt{\frac{\beta}{\alpha}} (p_x^2 + p_y^2) \right]^2 = \frac{S_0^2}{4} \quad (10) \]

commute with all the \( S_i \), \( i = 1, 2, 3 \) (expression (10) was expected since, as we will see, the functions \( S_i \) are the coordinate-momentum representation of the SU(2) which is of rank one).

In terms of these functions, the interaction hamiltonian becomes

\[ H = -(2\omega_0)S_3 = -(g\omega_0)S_3 = -\omega_1 S_3. \quad (11) \]

where we have put the magnetic field in the direction of the z-axis to write

\[ \omega_0 = \frac{eB_3}{2mc} \quad (12) \]

and call \( g = 2 \) the Landé factor.

The equations of motion for these functions are related with the dynamics of the problem. They might be obtained as usual by the use of the Poisson Bracket and are given by

\[ \frac{dS_1}{dt} = \{S_1, H\} = \omega_1 S_2(t), \quad (13) \]

and

\[ \frac{dS_2}{dt} = -\omega_1 S_1(t) ; \quad \frac{dS_3}{dt} = 0. \quad (14) \]

If we integrate these equations we find

\[ S_1(t) = S_1(0) \cos(\omega_1 t) + S_2(0) \sin(\omega_1 t); \]

\[ S_2(t) = -S_1(0) \sin(\omega_1 t) + S_2(0) \cos(\omega_1 t) ; \quad S_3(t) = S_3(0). \quad (15) \]

Equations (15) represent a precession taking place in the space defined by the functions \( S_i \). This precession shall not be confused with one in three-dimensional space, since in this space we postulate from the very beginning that the rotation
is taking place rigidly along the symmetry axis. Instead, we might interpret this motion as a vibration of the electron structure over the $xyp$ phase space hyperplane where it can be distorted, since the functions $S_1$ and $S_2$ are related with the quadrupole moments that take into account such a distortion (fig. 1).

If we put the magnetic field along the $z$-direction and write it as $B_3$, the equations \[ \frac{dS_i}{dt} = \{S_i, H\} = -\frac{geB_3}{2mc}\epsilon_{ijk}S_k \tag{16} \]
which can be written in vector notation as
\[ \frac{dS_i}{dt} = \frac{ge}{2mc}(S \wedge B)_i. \tag{17} \]
From this last equation we might write the dipole moment of the electron as
\[ m_s = \frac{ge}{2mc}S. \tag{18} \]
This justifies the expression (4) if we substitute $S$ for $L_z$ and the Landé $g$-factor for 2. We then expect from a purely classical argument that the structure of the half-integral spin particles be somewhat like a ring.

We now introduce the functions
\[ S_+ = S_1 + iS_2 ; \quad S_- = S_1 - iS_2, \tag{19} \]
that are given in terms of the coordinates and momenta as
\[ S_+ = \frac{1}{4} \left[ \sqrt{\frac{\alpha}{\beta}} (x + iy)^2 + \sqrt{\frac{\beta}{\alpha}} (p_x + ip_y)^2 \right], \tag{20} \]
and
\[ S_- = \frac{1}{4} \left[ \sqrt{\frac{\alpha}{\beta}} (x - iy)^2 + \sqrt{\frac{\beta}{\alpha}} (p_x - ip_y)^2 \right]. \tag{21} \]
These functions satisfy the following commutation rules
\[ \{S_+, S_-\} = -2S_3 ; \quad \{S_+, S_+\} = \{S_-, S_-\} = 0, \tag{22} \]
while for the anti-commutator, defined classically as
\[ \{f, g\}_A = \sum_{k=1}^{3} \left( \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} + \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right), \tag{23} \]
we get the anti-commutation rules
\[ \{S_+, S_-\}_A = (xp_x + yp_y) ; \quad \{S_+, S_+\}_A = \{S_-, S_-\}_A = 0. \tag{24} \]
We can easily identify the element $xp_x + yp_y$ as the unit associated with the product defined by the anti-commutation relation if we note that, for this element
\[
\{S_+, xp_x + yp_y\}_A = 2S_+ ; \{S_-, xp_x + yp_y\}_A = 2S_-
\]
and then write the first expression in (24) formally as
\[
\{S_+, S_-\}_A = 1.
\]
We might also obtain another important element of description of our problem. Defining the function $N$ as the product
\[
N = S_+ S_- = S_1^2 + S_2^2
\]
it is easy to see that
\[
N = S^2 - S_3^2 = \frac{1}{4}L_z^2 + \frac{1}{4}\left[(\frac{1}{2})\sqrt{\frac{\alpha}{\beta}}(x^2 + y^2) + (\frac{1}{2})\sqrt{\frac{\beta}{\alpha}}(p_x^2 + p_y^2)\right]^2
\]
which is nothing but the Casimir operator in this coordinate-momentum representation of the Lie group SU(3) generated by the three functions $S_i$ (with $z = p_z = 0$) and other five ones that will not concern us here.

The physical meaning of the relations (22, 24) together with the functions (20), (21) and (27) will be clarified in the next section.

Until now we were using the passive approach. In this approach we view the particle as a body with some intrinsic features moving on a fixed background or coordinate system. These features, like its rotation around the $z$-axis, are described by the functions $S_i$ (e.g. for the rotation around the $z$-axis we have $S_3$). We now pass to the active point of view.

### 4 The Active View: Operators

In the active approach, the particle is neither rotating nor being distorted by some applied magnetic field. In this case, the particle is maintained fixed in space and is viewed as generating symmetry operations upon the space itself (e.g. the electron generates the rotation along the $z$-axis and this operation is now represented by the operator $\hat{L}_z$).

Since we know that the functions $S_i$ are the generators, in the coordinate-momentum representation, of the SU(2) symmetry group, the operators related to them are automatically obtained (for the half-spin case) as the Pauli operators, written in matrix form as\(^1\)

\[
\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\(^1\)The reader may, at this point, say that we have forced this result through our choice of the $S_i$'s functional appearance and that this implies a great degree of arbitrariness. The reader is correct but we might also complain that the same is done in the usual quantum mechanical
so that our $S$-functions shall be proportional to them and might be written, in this matrix representation, as

$$
\hat{S}_3 = \frac{\hbar}{2} \hat{\sigma}_3 ; \hat{S}_2 = -i\frac{\hbar}{2} \hat{\sigma}_2 ; \hat{S}_1 = \frac{\hbar}{2} \hat{\sigma}_1,
$$

(29)

where $\hbar$ is an yet undetermined constant that will be obtained from the experiments (e.g. the Stern-Gerlach experiment) and is indeed known to be Plank’s constant $\bar{\hbar}$—which we will hereafter write instead of $h$.

This definition of the $S_i$ functions also assures that we have obeyed the usual commutation relation

$$
[\hat{S}_i, \hat{S}_j] = i\hbar \varepsilon_{ijk} \hat{S}_k.
$$

(30)

The hamiltonian related to the interaction of the particle with an external magnetic field becomes

$$
H = -\frac{ge\hbar}{2mc} B_3 \hat{\sigma}_3^2.
$$

(31)

Using expression (19) we get

$$
\hat{S}_+ = \frac{\hbar}{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
$$

(32)

and

$$
\hat{S}_- = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
$$

(33)

with the commutation relations—in terms of the matrix commutator—given by

$$
[\hat{S}_+, \hat{S}_-] = \hbar \hat{S}_3 ; [\hat{S}_+, \hat{S}_+] = [\hat{S}_-, \hat{S}_-] = 0,
$$

(34)

while the anti-commutation relations are

$$
[\hat{S}_+, \hat{S}_-]_A = \hbar \mathbf{1} ; [\hat{S}_+, \hat{S}_+]_A = [\hat{S}_-, \hat{S}_-]_A = 0.
$$

(35)

Since the operators $\hat{S}_+$ and $\hat{S}_-$ act as rotations upon the two-dimensional space defined by the operators $\hat{S}_1$ and $\hat{S}_2$ we might define, over this space, the basis

$$
|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$

(36)

chosen as to make the representation of $\hat{S}_3$ and $\hat{S}_2^2$ diagonal. In this case it is easy to see that the operators $\hat{S}_+$ and $\hat{S}_-$ are the normal modes associated with the representation when one chooses the $L_z$ operator to be $\sigma_z$, based in the results obtained in the Stern-Gerlach experiment and, after that, derive the other two Pauli operators $\sigma_x$ and $\sigma_y$. In both cases, of course, one is driven by the known behavior of the electron under the influence of a uniform magnetic field. The classical path is different from the quantum one just in the sense that it claims for a picture of the electron, something the quantum procedure does not.
with the distortions of the particle in the $xyp_zp_y$-hyperplane referred to after expression (17).

It is then easy to see that we have

$$\hat{S}_+|0\rangle = |1\rangle ; \hat{S}_+|1\rangle = 0 ; \hat{S}_-|0\rangle = 0 ; \hat{S}_-|1\rangle = |0\rangle.$$  \hspace{1cm} (37)

We might now modify our terminology of the operator $\hat{S}_\pm$ from symmetry generators to operators related with particles and occupation numbers. In this case, $|0\rangle$ is the vacuum state and $|1\rangle$ the first occupation state and $\hat{S}_+$ behaves as a particle creation operator while $\hat{S}_-$ behaves as a particle annihilation operator. This terminology, however, might be misleading. The operators $\hat{S}_+$ and $\hat{S}_-$ are only entities allowing one to get one independent mode of distortion in the $xyp_zp_y$-plane from the other. There are no particles being created nor annihilated.

By using the particle terminology and looking at expression (37) we might say that it is not possible to find any state with more than one particle. It is easy to define a number operator given by

$$\hat{N} = \hat{S}_+\hat{S}_-$$

with matrix representation written as

$$\hat{N} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \hspace{1cm} (38)$$

for which

$$\hat{N}|0\rangle = 0 ; \hat{N}|1\rangle = +1|1\rangle,$$  \hspace{1cm} (39)

justifying the name just given to it.

Returning to the expression (27), now written in operator format,

$$\hat{N} = \hat{S}_+\hat{S}_-$$

we might also interpret this operator $\hat{N}$ as representing the net ‘angular momentum’ related with the distortions. Then, the result

$$\hat{N}|0\rangle = 0$$  \hspace{1cm} (41)

is related with the fact that no distortion is present, while the result

$$\hat{N}|1\rangle = +1|1\rangle$$  \hspace{1cm} (42)

implies that the particle is being distorted. The condition that 0 and 1 are the only possible eigenvalues of the operator $\hat{N}$ means that the particle is capable
of being in only two states—one with and another without distortion in the $xyp, p_r$-hyperplane.

We can also explain why the particle symmetry group is SU(2). The functions $S_1$ and $S_2$ or their related operators are not true angular momenta but rather components of a tensor. In this case it is known that when we perform rotations upon the coordinates by an angle $\Omega$ around their symmetry axis, using the rotation group of which $\Omega$ is the sole parameter, these tensors are transformed by an angle $\Omega/2$ and so, are related with the covering group SU(2) of O(3).

5 Pauli’s Exclusion Theorem

Since the beginning of quantum mechanics the behavior of exclusion presented by electrons (and all half-integral spin particles) is introduced into non-relativistic quantum mechanics by means of a Principle, an axiom, called the Exclusion Principle due to Pauli. It is amply believed that the correct place to fit this problem is relativistic quantum mechanics where Dirac’s matrices arrive naturally and obey anticommutation relations. From the point of view of this paper this is not the case.

The spin, it was shown is not a relativistic effect although it is more correctly addressed within the realm of this theory—as all other phenomena. In relativistic quantum mechanics the spin shall have other degrees of freedom, for questions of invariance, related with the coupling of these degrees of freedom with the electric field. This is represented by the Lorentz invariant potential

$$L = \Sigma \cdot H + \Xi \cdot E$$

where $\Sigma$ and $\Xi$ are the Dirac’s matrices and $H$ and $E$ are the magnetic and electric fields respectively.

In Dirac’s theory, the matrix representation of the spin is related with 4-spinors—in contrast with Pauli’s 2-spinors—where all these degrees of freedom are made explicit. The use of spinors, however, need not be considered fundamental for the theory. It reflects just a separation of the problem into two different approaches—one in which the spatial coordinates are treated analytically, by means of the Schrödinger equation, and the other in which the intrinsic problem is treated by matrix algebra, following the Heisenberg original approach. Such an approach could be also used in the non-relativistic hydrogen atom, for example, treating the angular operators $L_3$ and $L^2$ as matrices and so obtaining, for each quantum number $\ell$ of operator $L^2$, a $(2\ell + 1)$-spinor.

The use of matrix representation for the spin is made because it is tacitly assumed among the scientific community that the spin cannot be represented by ordinary analytical functions. This follows from another general faith that the spin has no classical analogous and so, is not suitable to quantization. This
argument, however, cannot be correct if we accept that there is a total correspondence between the Schrödinger’s and Heisenberg’s calculus. In the continuation paper (VII), we will present an analytical equation accounting for the particles half-integral spin.

For the present section we are interested in showing that the results presented until now are the only ones necessary to introduce the notion of exclusion. This means that the behavior of exclusion will be consequence of the particle geometry which is also responsible for the existence of spin.

To begin with, let us consider two independent half-spin particles described by the two sets of functions

\[
\begin{align*}
S_1 &= \frac{1}{2} (x_1 y_1 + p_{x_1} p_{y_1}) \\
S_2 &= \frac{1}{2} (x_1^2 - y_1^2 + p_{x_1}^2 - p_{y_1}^2) \\
S_3 &= \frac{1}{2} (x_1 p_{y_1} + y_1 p_{x_1})
\end{align*}
\]

and

\[
\begin{align*}
R_1 &= \frac{1}{2} (x_2 y_2 + p_{x_2} p_{y_2}) \\
R_2 &= \frac{1}{2} (x_2^2 - y_2^2 + p_{x_2}^2 - p_{y_2}^2) \\
R_3 &= \frac{1}{2} (x_2 p_{y_2} + y_2 p_{x_2})
\end{align*}
\]

with the commutation rules

\[
\{S_i, S_j\} = \epsilon_{ijk} S_k \quad \text{and} \quad \{R_i, R_j\} = \epsilon_{ijk} R_k
\]

and

\[
\{S_i, R_j\} = 0 \quad \text{for all } i, j.
\]

It is easily to work in the active view with the matrix representation. In this case we are interested in the tensorial space \(S \otimes 1 + 1 \otimes R\) with the probability amplitudes defined by \(|s\rangle_S \otimes |r\rangle_R\). It has to be stressed that we are looking for a set of vectors making simultaneously diagonal the operators \(S_3 + R_3\) and \(N_{\text{total}} = S_+ S_- + R_+ R_-\).

It can be verified that we have

\[
S_3 + R_3 = \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2
\end{pmatrix}
\]

and \(N_{\text{total}} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}\)

and we shall choose some combinations of the vectors \(|s\rangle_S \otimes |r\rangle_R\) with \(s = 0, 1\) and \(r = 0, 1\) obeying

\[
\begin{align*}
S_3 |0\rangle_S |0\rangle_R &= \frac{1}{2} |0\rangle_S |0\rangle_R \\
S_3 |0\rangle_S |1\rangle_R &= \frac{1}{2} |0\rangle_S |1\rangle_R \\
S_3 |1\rangle_S |0\rangle_R &= -\frac{1}{2} |1\rangle_S |0\rangle_R \\
S_3 |1\rangle_S |1\rangle_R &= -\frac{1}{2} |1\rangle_S |1\rangle_R \\
R_3 |0\rangle_S |0\rangle_R &= \frac{1}{2} |0\rangle_S |0\rangle_R \\
R_3 |0\rangle_S |1\rangle_R &= -\frac{1}{2} |0\rangle_S |1\rangle_R \\
R_3 |1\rangle_S |0\rangle_R &= \frac{1}{2} |1\rangle_S |0\rangle_R \\
R_3 |1\rangle_S |1\rangle_R &= -\frac{1}{2} |1\rangle_S |1\rangle_R
\end{align*}
\]

that are still eigenvectors of the operators defined in expression (47).

It is easy to verify that the only two possible combinations that are still eigenvectors of the first operator in (47) are the two symmetrized vectors

\[
|\psi_1\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle_S |1\rangle_R - |1\rangle_S |0\rangle_R \right) \quad \text{and} \quad |\psi_2\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle_S |1\rangle_R + |1\rangle_S |0\rangle_R \right)
\]
and have both zero as their eigenvalue as related to operator $S_3$. The negative sign is then chosen using the anticommutation rule related with operators $S_+, S_-, R_+$ and $R_-$. The only solution for the problem becomes

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle_S |1\rangle_R - |1\rangle_S |0\rangle_R)$$

as expected. In matrix notation this vector can be written as

$$|\psi_1\rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

which is a statement of the exclusion behavior.

We then conclude that fixing the geometry of the particle structure, which induces the anticommutation relations for the operators $S_+, S_-, R_+$ and $R_-$, and the need to make diagonal the operator $S_3 + R_3$ are sufficient to obtain the exclusion behavior of entities with half-integral spin—we will prove in paper VII of this series that all half spin particles are included in our calculation.

6 The Bose-Einstein Condensation

The picture for the electron made above has other secondary, but not unimportant, consequences. One of the most striking ones is related with Einstein-Bose condensation. This phenomenon is related to the disappearance of exclusion behavior for very low temperatures.

In this paper we will address this problem qualitatively and show that we expect such phenomenon to appear. In the continuation paper (VII), where the half-integral spin Schrödinger equation will be solved exactly, the problem will be dealt with by quantitative calculations.

The most important thing to stress is that the internal energy of the electron is given if we consider the function

$$S_0 = \frac{1}{2} \left[ \sqrt{\frac{\alpha}{\beta}} (x^2 + y^2) + \sqrt{\frac{\beta}{\alpha}} (p_x^2 + p_y^2) \right]$$

and remember that it will give rise, in Schrödinger representation, to an eigenvalue equation

$$\hat{S}_0 \psi = \hbar \lambda \psi,$$

where Planck’s constant was introduced to make $\lambda$ a number without dimension.

Equation (53) might be written, after quantized, as

$$\frac{1}{2} \left[ \sqrt{\frac{\alpha}{\beta}} (\hat{x}^2 + \hat{y}^2) + \sqrt{\frac{\beta}{\alpha}} (\hat{p}_x^2 + \hat{p}_y^2) \right] \psi = \hbar \lambda \psi.$$
It might be easily checked that the ratio \( \sqrt{\alpha/\beta} \) shall have dimensions of mass/time and we can represent it as

\[
\sqrt{\frac{\alpha}{\beta}} = m\omega,
\tag{55}
\]

where \( m \) is the mass of the particle and \( \omega \) is some frequency related to the specific particle under consideration. Equation (54) becomes

\[
\left[ \frac{1}{2m} (\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m\omega^2 (\hat{x}^2 + \hat{y}^2) \right] \psi = \hbar\omega\lambda\psi,
\tag{56}
\]

with a related internal energy given by

\[
E = \lambda\hbar\omega.
\tag{57}
\]

Equation (56) is our fundamental equation. It is noteworthy that we do not expect the quantum number \( \lambda \) to have zero as one of its possible values. This is because we admitted from the very beginning that the particle is spinning around some fixed direction and so, must have some internal energy—this is a striking difference from fermions to bosons. In this case we expect

\[
\lambda \geq N > 0,
\tag{58}
\]

for some constant \( N \).

Let us suppose now that we have a confined fermion gas and we begin to drop its temperature. The fermions energy is the sum of their translation energy plus their internal energy. The energy is being extracted from the fermions translational kinetic energy. This process continues until the temperature reaches a value such that one fermion loses not only its translation energy but also its internal energy. We then say that the fermion ‘freezes’.

When the fermion freezes, its internal energy is under the minimum value \( N\hbar\omega \) and the equation (53) will not have any solution. The operator \( S^2 \) defined above will not define an eigenstate either. We might then say that the group generated by the fermion will not be SU(2) anymore. It will no longer be a fermion and condensation will take place—it is like if the possibility for the fermion to rotate and deform assures the exclusion; when the temperature drops to an exceeding low value, the internal state related with these degrees of freedom of the internal movements is no longer allowed and the fermion freezes—it is important to stress that this is not a collective behavior.

All the considerations above will be approached quantitatively in the continuation paper (VII).

7 Conclusions

In this paper we have shown how to make a model of the half-integral spin particles in the realm of a classical theory. It was also shown how the passage from
the passive to the active views might introduce all the ‘quantum mechanical’ results.

These results then show, for those who believe in some abyss between classical and quantum physics, that the half-integral spin particles are not specific of quantum mechanics.

As one consequence of the present development, it was possible to present an explanation of the intriguing phenomenon of Bose-Einstein condensation for fermions. We are now in position to ‘quantize’ the functions $S_3$ and $S^2$ to obtain a Schrödinger-like representation—eigenfunctions—for the half-integral spin particles.

In the continuation paper VII we proceed to make the quantum calculations and to treat the Bose-Einstein condensation problem in a quantitative way.

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