On the Deformation Quantization Description of Matrix Compactifications

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Matrix theory compactifications on tori have associated Yang-Mills theories on the dual tori with sixteen supercharges. A noncommutative description of these Yang-Mills theories based in deformation quantization theory is provided. We show that this framework allows a natural generalization of the ‘Moyal $B$-deformation’ of the Yang-Mills theories to non-constant background $B$-fields on curved spaces. This generalization is described through Fedosov’s geometry of deformation quantization.

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1. Introduction

Matrix theory is by now the best candidate to realize the non-perturbative quantum theory underlying string theory termed $M$-theory (for recent reviews see [1]).

Compactifications of matrix theory on the tori of dimension $p \leq 4$ were shown to be equivalent to $(p + 1)$-dimensional Yang-Mills theory with 16 supercharges [2,3,4]. For $p = 5, 6$ it is known that additional degrees of freedom to those coming from $D$-branes wrapped around homology cycles of lower dimensions arise and new prescriptions have to be considered [5,6,7]. Aside from compactifications on the torus, matrix compactifications on curved manifolds have been considered in [8,9] and in a different context in [10]. Recently compactifications on Calabi-Yau threefolds have also considered in [11]. There it was shown that in DKPS limit, Calabi-Yau compactifications are simpler than $T^6$ compactifications of matrix theory. In the DLCQ description the remaining degrees of freedom are decoupled from gravity. This difference apparently has to do with the different topological properties of the Calabi-Yau and $T^6$ spaces.

In a seminal paper by Connes, Douglas and Schwarz the interconnections between matrix theory and non-commutative geometry were explored [12]. In particular it was shown that toroidal compactifications of matrix theory with non-vanishing supergravity background three form on the torus, can be described as a gauge theory on a non-commutative dual torus of the type discussed in [13]. The physical justification of this relation involves the description of matrix theory in terms of $D$-branes on backgrounds which include a tensor $C_{-ij}$ on the torus [14].

The study of compactification on non-commutative tori and orbifolds of matrix theory was discussed in [15]. In further developments an embedding of non-commutative compactifications of matrix theory in weak coupling string theory is discussed in [16].

An explicit construction of the $(p + 1)$-dimensional Yang-Mills theory with 16 supercharges, from $D0$-brane action of type IIA superstring theory in a constant $B$-field background was done very recently in [17,18]. These theories are supersymmetric Yang-Mills theories on noncommutative spaces and are non-local field theories of the type described in [19]. UV fixed points do not exist in these theories and IR and UV behaviors are disentangled by the prescription of taking the size of the torus infinite, keeping the size of the non-local scale fixed. This prescription works out also for $(2,0)$ field theories in six-dimensions in the DLCQ description. Here the noncommutativity is reflected in the resolution of ALE singularities. The description of $D$ branes in a background $B$-field has
mathematical applications as well. An example is the ADHM construction of Yang-Mills instantons on noncommutative spaces [20]. (0,2) field theory newly fit into this picture.

Thus the effect of the background $B$-field matrix compactifications on curved spaces would imply the description of these manifolds as non-commutative spaces. Here we will give an alternative description in terms of deformation quantization theory. We will be interested in compact Poisson manifolds and in particular in symplectic and Kähler manifolds as the spacetime of the underlying supersymmetric gauge theory.

Deformation quantization theory is the non-commutative geometry for Poisson manifolds (for a recent review see [21]). This theory deals with the quantization of Poisson manifolds, that is, the suitable deformation of Poisson structures on these manifolds [22]. The proof of the existence of quantization of any Poisson manifold has been found recently in [23] by using string theory techniques. Also deformation quantization can be explicitly carried over to any finite dimensional curved symplectic manifolds with a symplectic connection [24]. These spaces are called Fedosov manifolds and their differential geometry has been just recently studied [25].

The purpose of this paper is to describe some conjectures regarding the geometry of deformation quantization for matrix compactifications on the torus and on curved manifolds using Fedosov geometry of deformation quantization theory. We focus attention on the known BFSS model [2], though all constructions can be applied straightforwardly to the IKKT model [26]. The motivation of this generalization to curved manifolds is the understanding of more general compactifications and the extension from constant to non-constant background $B$-fields on the $T$-dual compact space $X$. This generalization was suggested in [12], where it is argued that the classical Lagrangian of the resulting deformed gauge theory could be constructed following the lines of [27]. In the present paper we give an alternative construction of this Lagrangian in the context of deformation quantization theory. This construction will be valid for arbitrary symplectic manifolds and indeed can be generalized to Poisson manifolds.

The organization of this paper is as follows: Sec. 2 is devoted to briefly reviewing the derivation of super Yang-Mills theory on a non-commutative $p$-dimensional dual torus [12,17,18]. It is also shown that a Weyl correspondence of Moyal quantization can be established in this framework and thus a deformed Lagrangian can be obtained in the Moyal picture. In Sec. 3 we show how the immediate generalization to non-constant background $B$-fields on the $T$-dual space involves the introduction of curved Poisson manifolds as the spacetime where the underlying gauge theory lives and it is exactly described through
the Fedosov geometry of deformation quantization. In Sec. 3 we also study the gauge theory on a Fedosov manifold and the corresponding Lagrangian is shown to describe deformations of the gauge theory on a curved manifold parametrized by the background $B$-field. The correspondence with the gauge theory on the noncommutative torus is also given. Comments about the extension to Kähler manifolds with symplectic connection and its application to ALE and $K3$ spaces and Calabi-Yau threefolds are provided. Finally in Sec. 4 we give our concluding remarks.

2. Deformation Quantization of Matrix Compactifications on Tori

In order to introduce some notation we will use in the paper we briefly review the matrix theory action. It is given by a matrix quantum mechanics action with SU($N$) gauge group and 16 supercharges [28]

$$I_M = \int dt L_M$$

with

$$L_M = \frac{1}{g_{YM}^2} \Tr_N \left( (D_t X^I)(D_t X^J) + \frac{1}{2} [X^I, X^J][X^I, X^J] + i \Theta^\alpha D_t \Theta^\alpha - \Theta^\alpha \Gamma^I_{\alpha\beta} [X_I, \Theta^\beta] \right)$$

where $X^I$ and $\Theta^\alpha$ are ($N \times N$ matrix-valued) 9 bosonic and 16 spinor coordinates of 0-brane partons ($I = 1, \cdots, 9$ are SO(9) indices with metric $\delta_{IJ}$ and $\alpha = 1, \cdots, 16$ are SO(9) spinor indices), $\Tr_N$ is the trace in the fundamental representation of SU($N$). The Majorana spinor conventions are such that the $\Gamma_I$’s are real and symmetric and obey $\{\Gamma^I, \Gamma^J\} = 2 \delta^{IJ}$, $\Gamma^1 \cdots \Gamma^9 = +1$ and $\Gamma^IJ = \frac{1}{2} [\Gamma^I, \Gamma^J]$. As $N$ goes to infinity Lagrangian (2.2) describes membranes $\Sigma$ extending along $t, x^{11}$ directions [2].

The gauge covariant derivative reads $D_t X^I \equiv \partial_t X^I - i [A_t, X^I]$ and $D_t \Theta^\alpha \equiv \partial_t \Theta^\alpha - i [A_t, \Theta^\alpha]$. These definitions ensure invariance under area-preserving diffeomorphism group $\text{SDiff}(\Sigma)$. 

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2.1. Non-commutative Torus and D0-branes with Background $B$-field

Recently it has been shown that matrix theory compactifications on $\mathbb{T}^2$ with a constant background three-form tensor field $C_{-ij}$, can be described in terms of a (2+1)-dimensional super Yang-Mills theory on a non-commutative dual torus $\tilde{T}_p^2$ $\cite{[2]}$. More recently the analogous correspondence in the context of D0-branes of type IIA superstrings compactified on $\mathbb{T}^p$ has been done $\cite{[17],[18]}$. In these works the corresponding $(p+1)$-dimensional super Yang-Mills action on the noncommutative dual $p$-torus, $\tilde{T}_p^p$ was found. This was done by placing a set of $N$ D0-branes on the space $\mathbb{T}^p$, which is a flat space $\mathbb{R}^p$ with $\mathbb{Z}^p$ periodicity.

It is known that the $X^I$ and $\Theta^\alpha$ must satisfy the known constraints of $D$-branes on tori $\cite{[3],[4]}$

$$X^A_{m,n} = X^A_{m-n,0}, \quad A = p + 1, \ldots, 9$$  \hspace{1cm} (2.3) $$X^i_{m,n} = X^i_{m-n,0} + 2\pi \ell_s E^i_a n^b \delta_{m,n}, \quad i = 1, \ldots, p$$  \hspace{1cm} (2.4) $$\Theta^\alpha_{m,n} = \Theta^\alpha_{m-n,0}$$  \hspace{1cm} (2.5)

where $n, m$ are integers and label the positions of the D0-branes or the initial and final points of the corresponding string on the lattice $\mathbb{Z}^p$, $\ell_s$ is the string scale, $E^i_a$ is the vielbein associated to the metric on the flat torus $G_{ab} = E^i_a E^j_b \delta_{ij}$.

The $T$-dual picture enables the existence of $N \times N$ hermitian matrix-valued functions through the Fourier transform $\cite{[3],[4]}$. Thus the correspondence between matrices $X^I(t)$ and matrix-valued functions on the torus, $X^I(t, \tilde{\sigma})$ is given by

$$\sigma^{-1}_N : Mat_N \rightarrow C^\infty(\tilde{T}^p) \otimes Mat_N$$  \hspace{1cm} (2.6)

where $Mat_N$ is the set of $N \times N$ non-singular matrices representing the Lie algebra $\text{su}(N)$ and $C^\infty(\tilde{T}^p)$ is the set of smooth functions on the dual torus $\tilde{T}^p$. The map $\sigma^{-1}_N$ is given by the Fourier transformation

$$X^I(t, \tilde{\sigma}) = \sigma^{-1}_N(X^I) : = \sum_{n \in \mathbb{Z}^p} X^I_{n,0}(t, N) \exp(in^j \tilde{\sigma}_j)$$  \hspace{1cm} (2.7) 

where

$$X^I_{n,0}(N, t) = \frac{1}{(2\pi)^p} \int_{\tilde{T}^p} d^p\tilde{\sigma} X^I(t, \tilde{\sigma}) \exp(-in^j \tilde{\sigma}_j)$$  \hspace{1cm} (2.8)
where $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_p)$ and $\tilde{\sigma}_j$ ($j = 1, \ldots, p$) are the local coordinates on $\tilde{T}^p$. For the fermionic counterpart there exist similar expressions. The inverse transformation $\sigma_N$ is well defined as it is given by the inverse Fourier transformation. Constraints (2.3-2.5) can be solved in the $T$-dual picture replacing the matrices $X^i(t)$ by the matrix-valued functions on the dual tori $X^i(t, \tilde{\sigma}) = i\partial^i + A^i(t, \tilde{\sigma})$ and thus the correspondence between matrix compactifications on the tori and gauge theories with 16 supercharges can be finally established [3,4].

2.2. Moyal Deformation in Toroidal Compactifications

Now we are going to repeat the same correspondence in the presence of a non-vanishing background $B$-field. It was shown in [17,18], that a non-vanishing constant background $B$-field introduces the factor

$$U = \exp\left(-i\pi B_{ij} n^i n^j\right)$$

(2.9)

for each interaction term of three strings on the lattice. This is because the closed two-form $B_{ij}$ couples to the worldsheet under the interaction $\int_{WS} B$. This coupling depends only on the homotopy type of the worldsheet embedding and is of the form $\exp(\frac{i}{2} B)\phi_{ik}^{(3)}\phi_{kj}^{(2)}\phi_{ji}^{(1)}$ where $\phi^I = (\phi^i, \phi^A)$ denotes a generic field of the theory. Generalization to interaction of $k$ strings on the lattice leads to the term $\text{Tr}(\phi^{(k)} * \phi^{(k-1)} * \ldots * \phi^{(2)} * \phi^{(1)})$ where the $*$-product is given by [17]

$$(\phi^{(2)} * \phi^{(1)})_{a_3 b_3, a_1 b_1} = \sum_{a_2, b_2} \phi^{(2)}_{a_3 b_3, a_2 b_2} \exp\left(\frac{1}{2} B \begin{vmatrix} a_3 - a_2 & a_2 - a_1 \\ b_3 - b_2 & b_2 - b_1 \end{vmatrix}\right) \phi^{(1)}_{a_2 b_2, a_1 b_1}.$$  

(2.10)

The subindices of the fields $\phi$ denote the positions of the $D0$-branes on the lattice. Solving the constraints (2.3-2.5), the $*$-product (2.10) in the dual basis $\{\tilde{\sigma}\}$ looks like [12,17,18]

$$\phi^I(\tilde{\sigma}) * \phi^J(\tilde{\sigma}) \equiv \phi^I(\tilde{\sigma}) \exp\left(i\pi \ell_s^2 B^{ij} \frac{\partial}{\partial \tilde{\sigma}^i} \frac{\partial}{\partial \tilde{\sigma}^j}\right) \phi^J(\tilde{\sigma})$$

(2.11)

$$= \sum_{k=0}^{\infty} (i\pi \ell_s^2)^k \frac{1}{k!} B^{i_1 j_1} \ldots B^{i_k j_k} \frac{\partial^k \phi^I(\tilde{\sigma})}{\partial \tilde{\sigma}^{i_1} \ldots \partial \tilde{\sigma}^{i_k}} \frac{\partial^k \phi^J(\tilde{\sigma})}{\partial \tilde{\sigma}^{j_1} \ldots \partial \tilde{\sigma}^{j_k}}.$$
for all $\phi^J(\tilde{\sigma}) \in C^\infty(\tilde{T}^p) \otimes \text{Mat}_N$. This is precisely the associative and noncommutative Moyal product of deformation quantization theory \[22\]. Thus the effect of turning on the tensor $B$-field on the lattice deform the usual product of functions on $\tilde{T}^p$ and turning it into the Moyal product, with a deformation parameter $\zeta_B$ given by the constant components of the $B$ tensor field and the constant $\pi \ell_s^2$.

From now on we leave aside the $\text{Mat}_N$ sector the right hand side of (2.6). We will consider only the $C^\infty(\tilde{T}^p)$ part. At the end we will consider the matrix-valued structure taking the trace $\text{Tr}_N$ in all relevant equations. Then all fields in the Yang-Mills theory are matrix-valued functions on the $p$-torus and they are written as

$$\phi^J(t, \tilde{\sigma}) = \sum_{n \in \mathbb{Z}^p} \Phi_n^I(t, N) L_n. \quad (2.12)$$

where $L_n \equiv \exp(in^j\tilde{\sigma}_j)$.

It is well known that the basis of the Lie algebra $\text{su}(N)$ can be seen as a two-indices infinite algebra for $p = 2$ [29,30]. The elements of this basis are denoted by $L_m, L_n, \ldots$, etc., $m = (m_1, m_2), n = (n_1, n_2), \ldots$, etc., and $m, n, \ldots \in I_N \subset \mathbb{Z} \times \mathbb{Z} - \{(0, 0)\} \mod Nq$, where $q$ is any element of $\mathbb{Z} \times \mathbb{Z}$. The basic vectors $L_m, m \in I_N$, are the $N \times N$ matrices satisfying the following commutation relations

$$[L_m, L_n] = \frac{N}{\pi} \sin\left(\frac{\pi}{N} m \times n\right) L_{m+n} \mod Nq, \quad (2.13)$$

where $m \times n := m_1n_2 - m_2n_1$. For the 2-torus $\tilde{T}^2$ the generators read $L_n \equiv \exp\left(i(n_1\tilde{\sigma}_1 + n_2\tilde{\sigma}_2)\right)$. Large $N$ limit ($N \to \infty$) of algebra (2.13) gives the area-preserving diffeomorphism algebra $\text{sdiff}(\tilde{T}^2)$.

The correspondence (2.13) can be seen as the composition of two mappings. The first one is a Lie algebra representation of $\text{su}(N)$ (for finite $N$) into a Lie algebra $\hat{\mathcal{G}}$ of self-adjoint operators acting on the Hilbert space $L^2(\mathbb{R})$, given by

$$\Psi : \text{su}(N) \to \hat{\mathcal{G}}, \quad \Phi \mapsto \Psi(\Phi^J) := \hat{\Phi}^J. \quad (2.14)$$

This map can be constructed explicitly for $N = 2$ [31]. The second mapping is a genuine Weyl correspondence $\mathcal{W}^{-1}$ which establishes a one to one correspondence between the algebra $\mathcal{B}$ of self-adjoint linear operators acting on $L^2(\mathbb{R})$ and the space of real smooth functions $C^\infty(\tilde{T}^2)$ where $\tilde{T}^2$ is seen as the classical phase-space. This correspondence $\mathcal{W}^{-1} : \mathcal{B} \to C^\infty(\tilde{T}^2)$, is given by
\begin{equation}
\phi^J(t, \bar{\sigma}^1, \bar{\sigma}^2; \zeta_B) \equiv \mathcal{W}^{-1}(\Phi^J) := \int_{-\infty}^{\infty} < \bar{\sigma}^1 - \frac{\xi}{2} | \Phi^J(t) | \bar{\sigma}^1 + \frac{\xi}{2} > \exp\left( \frac{i}{\zeta_B} \xi \bar{\sigma}^2 \right) d\xi, \quad (2.15)
\end{equation}

for all $\Phi^J \in \mathcal{B}$ and $\phi^J \in C^\infty(\mathbf{T}^2)$. Thus from the identification of $\mathcal{B}$ with $\mathcal{G}$, it follows that the correspondence $\sigma^{-1}_N$ is equal to the map composition $\sigma^{-1}_N = \mathcal{W}^{-1} \circ \Psi$ for finite $N$.

With the mapping $\Psi$ and the Weyl correspondence (2.15) it is easy to check that

\begin{equation}
\sigma^{-1}_N(\Phi^I \Phi^J) = \phi^I \star \phi^J \quad (2.16)
\end{equation}

where $\sigma^{-1}_N(\Phi^I) := \phi^I$ and $\Phi^I \in \text{Mat}_N$. One can see that $\sigma^{-1}_N$ is actually a Lie algebra isomorphism

\begin{equation}
\sigma^{-1}_N : (\text{Mat}_N, [\cdot, \cdot]) \to (\mathcal{M}_B, \{\cdot, \cdot\}_B) \quad (2.17)
\end{equation}

where $\mathcal{M}_B$ is a family of associative algebras on a fixed complex vector space denoted by $\mathcal{M} = C^\infty(\mathbf{T}^p)$. This isomorphism leads to the definition of the Moyal bracket

\begin{equation}
\sigma^{-1}_N\left( \frac{1}{i\zeta_B} [\Phi^I, \Phi^J] \right) = \frac{1}{i\zeta_B} (\phi^I(\bar{\sigma}) \star \phi^J(\bar{\sigma}) - \phi^J(\bar{\sigma}) \star \phi^I(\bar{\sigma})) \equiv \{\phi^I(\bar{\sigma}), \phi^J(\bar{\sigma})\}_B \quad (2.18)
\end{equation}

where \{\cdot, \cdot\}_B is the Moyal bracket and [\cdot, \cdot] is the usual commutator of matrices.

The algebra of quantum observables

\begin{equation}
(\mathcal{M}_B, \{\cdot, \cdot\}_B) \quad (2.19)
\end{equation}

can be defined more precisely by introducing an associative $\star$-product operation on the vector space $\mathcal{M}$ of functions $\phi^I(\bar{\sigma}, \zeta_B) = \sum_{k=0}^{\infty} \zeta_B^k \phi^I_k(\bar{\sigma})$ and $\phi^J(\bar{\sigma}, \zeta_B) = \sum_{k=0}^{\infty} \zeta_B^k \phi^J_k(\bar{\sigma})$, with $\phi^I_k(\bar{\sigma}), \phi^J_k(\bar{\sigma}) \in \mathcal{M}$. The $\star$-product is defined by $\phi^I \star \phi^J = \phi^K = \sum_{k=0}^{\infty} \phi^K_k(\bar{\sigma})$ for all $\phi^I, \phi^J, \phi^K \in \mathcal{M}$ satisfying the properties $(i)$- $\phi^K_k(\bar{\sigma})$ are polynomials in $\phi^I_k$ and $\phi^J_k$ and their derivatives. $(ii)$- $\phi^K_0(\bar{\sigma}) = \phi^I_0(\bar{\sigma}) \phi^J_0(\bar{\sigma})$. $(iii)$- $\{\phi^I, \phi^J\}_B \equiv \frac{1}{i\zeta_B} (\phi^I \star \phi^J - \phi^J \star \phi^I) = \{\phi^I_0, \phi^J_0\}_B + \ldots$, where $\{\cdot, \cdot\}_B$ stands for the Poisson bracket and the dots mean the terms of higher orders. In order to be more precise $\mathcal{M}$ is a linear space whose elements are of the form \( \phi^I = \phi^I(\bar{\sigma}, \zeta_B) = \sum_{k=0}^{\infty} \zeta_B^k \phi^I_k(\bar{\sigma}) \), where $\phi^I_k(\bar{\sigma}) \in C^\infty(\mathbf{T}^p)$.

We have seen that a background $B$-field is projected over the dual torus where it is defined as an antisymmetric tensor field. It can also be seen as a non-degenerate closed two-form on the torus $\tilde{B} = B_{ij} d\bar{\sigma}^i \wedge d\bar{\sigma}^j$. For $p$ even, $B_{ij}$ can be identified with the symplectic
two-form on the dual torus, \( i.e. \omega_{ij} = B_{ij} \). For instance for \( p = 2 \) symplectic form on \( \tilde{T}^2 \) is \( \epsilon_{ij} \), while the \( B \)-field is given by \( B_{ij} = B \epsilon_{ij} \) with \( B \) a constant value. The deformation parameter \( \zeta_B \) is identified with \( \pi \ell_s^2 B \) for fixed \( \ell_s \). Jacobi identity follows from the closeness condition of \( \tilde{B} \), \( i.e. \ d\tilde{B} = 0 \). Poisson structure on \( \tilde{T}^2 \) is given by \( \{ \phi^I, \phi^J \}_P = \frac{B_{ij}}{T^2} \partial_i \phi^I \partial_j \phi^J \).

For \( p > 2 \) the symplectic structure is identified with \( B_{ij} \). For any valued of \( p \) (even or odd) \( B_{ij} \) determines a Poisson structure on the underlying compact space.

### 2.3. Gauge Theory on the Non-commutative Torus

Finally the \( B \)-deformed Lagrangian of the gauge theory can be written as

\[
L_M = M_P \int_{\tilde{T}^p} d^p \tilde{\sigma} \text{Tr}_N \left( -\frac{1}{4T^2} F_{\mu\nu}^* F_{\mu\nu} - \frac{1}{2} (D_\mu X^A)^* (D_\mu X^A) + T^2 \{ X^A, X^B \}_B \right)
\]

\[
- \frac{i}{2} \Theta^\alpha * \Gamma_{\alpha\beta}^\mu D_\mu \Theta^\beta + \frac{T^2}{2} \Theta^\alpha * \Gamma^{A}_{\alpha\beta} \{ X_A, \Theta^\beta \}_B
\]

(2.20)

where \( \mu, \nu = 0, \ldots, p \). Fields are now \( M \)-valued fields on the \( (p + 1) \)-dimensional space time \( \mathbb{R} \times \tilde{T}^d \). Gauge fields on the dual torus are defined as \( A^i(t, \tilde{\sigma}) := \frac{1}{2\pi \ell_s^2} X^i(t, \tilde{\sigma}) \). \( A_\mu = A_\mu(t, \tilde{\sigma}) \) are composed by the \( A^0 \) and \( A^i \) fields where we redefine \( \tilde{\sigma}^\mu \equiv (\tilde{\sigma}^0, \tilde{\sigma}) \) with \( t \equiv \tilde{\sigma}^0 \).

The field strength of these fields is

\[
F_{\mu\nu}(\tilde{\sigma}) = \frac{\partial}{\partial \tilde{\sigma}^\mu} A_\nu(\tilde{\sigma}) - \frac{\partial}{\partial \tilde{\sigma}^\nu} A_\mu(\tilde{\sigma}) + \{ A_\mu(\tilde{\sigma}), A_\nu(\tilde{\sigma}) \}_B.
\]

(2.21)

The scalar fields are \( X^A(\tilde{\sigma}) \) and their coupling to the gauge fields is given through the covariant derivative

\[
D_\mu X^A(\tilde{\sigma}) = \frac{\partial}{\partial \tilde{\sigma}^\mu} X^A(\tilde{\sigma}) + \{ A_\mu(\tilde{\sigma}), X^A(\tilde{\sigma}) \}_B.
\]

(2.22)

Similar expressions hold for the Majorana spinors \( \Theta^\alpha(\tilde{\sigma}) \)

\[
D_\mu \Theta^\alpha(\tilde{\sigma}) = \frac{\partial}{\partial \tilde{\sigma}^\mu} \Theta^\alpha(\tilde{\sigma}) + \{ A_\mu(\tilde{\sigma}), \Theta^\alpha(\tilde{\sigma}) \}_B.
\]

(2.23)

We have obtained the \( (p + 1) \)-dimensional Yang-Mills theory with maximal supersymmetry on the noncommutative space \( \mathbb{R} \times \tilde{T}_B^p \). Its bosonic part was worked out some years ago in \([30,32,33,34,35]\).
One of the advantages of deformation quantization is that it permits a unified view of the mentioned correspondence between matrix compactifications and gauge theories. In the case considered in this paper, the non-commutative gauge theory Lagrangian (2.20) can be derived straightforwardly from the Moyal $B$-deformation of the matrix theory Lagrangian (2.2). The latter Lagrangian was proposed in Ref. [36]. The derivation involves the solution of the functional version of constraints (2.3-2.5). Lagrangian (2.20) is not expected to possess global anomalies (of the type of [37]) coming from the degeneracy of the Poisson bracket if $B_{ij}$ is non-degenerate.

Noncommutative gauge field theories (2.20) are non-local field theories of the type described in [19]. These theories have no renormalization group fixed point in the UV. However the theory is still well defined in the continuum. In [19] a manner to study the renormalization group by taking a limit in which the size of the torus goes to infinity, while the size of the non-locality scale is keeping fixed, was proposed. Six-dimensional (0,2) field theories in the DLCQ description do admit a generalization of this type. At the end of Sec. 4 we will return to this point.

3. Geometry of Deformation Quantization in Matrix Compactifications

The purpose of this section is to formulate the gauge theory underlying matrix compactification on tori, in terms of noncommutative geometry of the dual torus $\tilde{T}^p$ with a symplectic structure given by a $B$ field. We will use for this the deformation quantization theory given by de Wilde and Le Compte [38] and Fedosov [24]. We will find in deformation quantization theory a natural framework to generalize to non-constant background $B$-field on more general curved symplectic $T$-dual manifolds $X$. That means, $B$-fields depending on the point $x$ in $X$. For each such a point we can associate a Weyl algebra defined on the tangent bundle $TX$ to $X$, to the Moyal algebra (2.19). This gives precisely a bundle structure known in deformation quantization theory as Weyl algebra bundle [38, 24]. Gauge theory on the $T$-dual curved space $X$ will be found and we will rederive the toroidal result of Sec. 2 as an special limit when Riemannian curvature of $X$ vanishes. Finally some comments about the application of deformation quantization formalism to Kähler manifolds is also given.
3.1. Deformation Quantization

Before we start with the application of deformation quantization theory to matrix compactifications it is convenient to recall some of its terminology (for a nice review see [21]). The aim of deformation quantization program is the description of deformation of a Poisson algebra $\mathcal{A}, \mathcal{A}_\hbar$, associated to some Poisson manifold $X$ through a family of deformed product of functions $\ast_\hbar$. Here $\hbar$ is the deformation parameter. Equivalent description can be done by means of a sequence of bilinear mappings $M_k : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, k = 0, 1, \ldots$, with

$$a \ast_\hbar b = \sum_k \hbar^k M_k(a, b) \quad (3.1)$$

where $a, b \in \mathcal{A}_\hbar$.

The problem of formal deformation quantization is to classify such a families up to equivalence of the $\ast_\hbar$-product.

The most part of the realization of this program has been done in the context of algebraic structures of quantum mechanics [22]. This can be implemented with the obvious identifications $\mathcal{A} = C^\infty(X), X$ the classical phase space and the realization of $M_0(a, b)$ as the usual product of functions $ab$ and linear combinations of $M_1(a, b)$ as the Poisson bracket.

One interesting example of Poisson manifold is a symplectic manifold $(X, \omega)$. The symplectic structure is defined in terms of the Poisson structure on $X$. Locally the symplectic form can always be written as $\omega = \sum_i dq_i \wedge dp_i$, where $\{x^i\} i = 1, \ldots, 2N$ with $x^i = p^i, i \leq N$ and $x^i = q^{i-N}, i > N$, are the local coordinates on $X$.

Globally the symplectic form is defined by $\omega : TX \to T^*X$ with inverse $\omega^{-1} : T^*X \to TX$. Here $TX$ and $T^*X$ are the respective tangent and cotangent bundles to $X$. While the hamiltonian (or volume preserving) vector fields are $V_{H_a} = \omega^{-1}(dH_a)$ satisfying the $\text{sdiff}(X)$ algebra $[V_{H_a}, V_{H_b}] = V_{\{H_a, H_b\}_P}$ (for all $a \neq b$), where $\{\cdot, \cdot\}_P$ stands for the Poisson bracket with respect to $\omega$. Locally it can be written as $\{H_a, H_b\}_P = \omega^{-1}(dH_a, dH_b) = \omega^{ij} \partial_i H_a \partial_j H_b$, where $\partial_i \equiv \frac{\partial}{\partial x^i}$ and $H_i = H_i(x)$. The generators of $\text{sdiff}(X)$ are the hamiltonian vector fields $V_{H_a}$ associated to the hamiltonian functions $H_a$ and they form a Lie algebra whose Lie group is the volume-preserving diffeomorphism group $\text{SDiff}(X)$ of phase space $X$.

Moyal product for symplectic manifolds with Poisson structure given by (3.1) is characterized by the sequence
where $\omega_{ij} = \omega(x_i, x_j)$. This sequence involves only differential operators which define \(\ast\)-product only locally. The \(\ast\)-product is thus in general not globally defined on \(X\). Global \(\ast\)-product exists always for \textit{any} finite dimensional symplectic manifold [38]. Among other proofs of the existence theorem, that of [39] involves the construction of a different constant Poisson structure for each tangent space on \(X\). The tangent bundle \(TX\) becomes a Poisson manifold with fiberwise Poisson bracket and with fiberwise quantization. The quantization of \(X\) is given by the induced multiplication on \(C^\infty(X)[[\hbar]]\) from the multiplication on \(C^\infty(TX)[[\hbar]]\) or Weyl structure on \(X\). A existence proof given in [38] uses methods of Čech cohomology of \(X\). A classification of \(\ast\)-products in terms of Čech cohomology using gerbes theory was done in [40]. A treatment of deformation quantization parallel to non-commutative theory methods is given in [41]. Recently Kontsevich gave a proof of existence of the global \(\ast\)-product for every Poisson manifold using techniques of string theory and topological field theories in two-dimensions [23]. Finally a very clear perspective of the problems arising in the mathematical theory of deformation quantization was given recently by Rieffel in [42].

### 3.2. Fedosov’s Geometry of Deformation Quantization

In this subsection we discuss the elements of Fedosov’s geometry of deformation quantization theory. Some review of this subject can be found in [21, 43, 40]. We will follow the discussion for the torus but we will specify whenever general formulas for the space \(X\) be required.

**Weyl algebra bundle**

We consider \(X\) to be the dual torus \(\tilde{T}^p\) (with \(p\) even) and symplectic structure given by the tensor \(B_{ij}\). \(\tilde{T}^p\) has a natural Riemannian structure given by the flat metric \(\eta_{ij}\).

The formal Weyl algebra \(\mathcal{W}_{\tilde{\sigma}}\) associated with the tangent space \(T^p_{\tilde{\sigma}}\tilde{T}^p\) at the point \(\tilde{\sigma} \in \tilde{T}^p\) is the \textit{associative} algebra over \(\mathbb{C}\) with a unit. An element of \(\mathcal{W}_{\tilde{\sigma}}\) can be expressed by \(\tilde{\phi}^l(y) = \sum_{2k+l \geq 0} \zeta_B^k \phi^l_{k,i_1...i_l} y^{i_1} ... y^{i_l}\), where \(\zeta_B\) is the deformation parameter, \(y = (y^1, ... , y^{2n})\) is a tangent vector and the coefficients \(\phi^l_{k,i_1...i_l}\) constitute the symmetric covariant tensor of degree \(l\) at \(\tilde{\sigma} \in \tilde{T}^p\). The product on \(\mathcal{W}_{\tilde{\sigma}}\), which determines the associative algebra structure is defined by
\[ \tilde{\phi}^I \circ \tilde{\phi}^J = \tilde{\phi}^I(y, \zeta_B) \exp \left( \frac{i}{2} \zeta_B B^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right) \tilde{\phi}^J(z, \zeta_B) \mid_{z=y} \]

\[ = \sum_{k=0}^{\infty} \left( \frac{i}{2} \right)^k \frac{1}{k!} B^{i_1 i_2} \cdots B^{i_k j_k} \sum_{\sigma, y, \zeta} \frac{\partial^k \tilde{\phi}^I}{\partial y^{i_1} \cdots \partial y^{i_k}} \frac{\partial^k \tilde{\phi}^J}{\partial y^{j_1} \cdots \partial y^{j_k}}, \quad (3.3) \]

for all \( \tilde{\phi}^I, \tilde{\phi}^J \in \mathcal{W}_\sigma \). Here \( B^{ij} \) are the components of the tensor inverse to \( B_{ij} \) at \( \sigma \). Of course the product “\( \circ \)" is independent of the basis. Thus one can define an algebra bundle structure taking the disjoint union of Weyl algebras for all points \( \tilde{\sigma} \in \tilde{T}^p \). \( \tilde{\mathcal{W}} \) is the total space and the fiber is isomorphic to a Weyl algebra \( \mathcal{W}_\sigma \). Thus we have the Weyl algebra bundle structure

\[ \tilde{\mathcal{W}} \xrightarrow{\pi} \tilde{T}^p, \quad \mathcal{W}_\sigma \cong \pi^{-1}(\{\tilde{\sigma}\}), \quad (3.4) \]

where \( \pi \) is the canonical projection.

Let \( \mathcal{E}(\tilde{\mathcal{W}}) \) be the set of sections of \( \tilde{\mathcal{W}} \) which also has a Weyl algebra structure with unit. Denote by \( \tilde{\phi}^I(\tilde{\sigma}, y, \zeta_B) \) an element of \( \mathcal{E}(\tilde{\mathcal{W}}) \), it can be written as follows

\[ \tilde{\phi}^I(\tilde{\sigma}, y, \zeta_B) = \sum_{2k+l \geq 0} \zeta_B^k \tilde{\phi}_{k,i_1 \cdots i_l}(\tilde{\sigma}) y^{i_1} \cdots y^{i_l}, \quad (3.5) \]

where \( y = (y^1, \ldots, y^{2n}) \in T_{\tilde{\sigma}} \tilde{T}^p \) is a tangent vector, \( \tilde{\phi}_{k,i_1 \cdots i_l} \) are smooth functions on \( \tilde{T}^p \) and \( \tilde{\sigma} \in \tilde{T}^p \).

Fedosov’s deformation quantization theory also permits the definition of Weyl algebra-valued differential forms on \( \tilde{T}^p \). Such a \( p \)-form is defined by

\[ \tilde{\phi}^I = \sum_{2k+p \geq 0} \zeta_B^k \tilde{\phi}_{k,j_1 \cdots j_p}(\tilde{\sigma}, y) d\tilde{\sigma}^{j_1} \wedge \cdots \wedge d\tilde{\sigma}^{j_p} \quad (3.6) \]

where \( \tilde{\phi}_{k,j_1 \cdots j_p}(\tilde{\sigma}, y) = \tilde{\phi}_{k,i_1 \cdots i_l,j_1 \cdots j_p}(\tilde{\sigma}) y^{i_1} \cdots y^{i_l} \).

The set of differential forms constitutes a Grassmann - Cartan algebra \( \mathcal{C} = \mathcal{E}(\tilde{\mathcal{W}} \otimes \Lambda) = \bigoplus_{q=0}^{2n} \mathcal{E}(\tilde{\mathcal{W}} \otimes \Lambda^q) \). In this space the multiplication \( \tilde{\phi} \wedge \tilde{\phi} \) is defined by

\[ \tilde{\phi}^I \wedge \tilde{\phi}^J = \tilde{\phi}^I_{[j_1 \cdots j_p} \circ \tilde{\phi}^J_{l_1 \cdots l_q]} d\tilde{\sigma}^{j_1} \wedge \cdots \wedge d\tilde{\sigma}^{j_p} \wedge \tilde{d} \tilde{\sigma}^{l_1} \wedge \cdots \wedge \tilde{d} \tilde{\sigma}^{l_q}, \quad (3.7) \]

for all \( \tilde{\phi}^I = \sum_k \zeta_B^k \tilde{\phi}_{k,j_1 \cdots j_p}(\tilde{\sigma}, y) d\tilde{\sigma}^{j_1} \wedge \cdots \wedge d\tilde{\sigma}^{j_p} \in \mathcal{E}(\tilde{\mathcal{W}} \otimes \Lambda^p) \) and \( \tilde{\phi}^J = \sum_k \zeta_B^k \tilde{\phi}_{k,l_1 \cdots l_q}(\tilde{\sigma}, y) d\tilde{\sigma}^{l_1} \wedge \cdots \wedge d\tilde{\sigma}^{l_q} \in \mathcal{E}(\tilde{\mathcal{W}} \otimes \Lambda^q) \). The \( \circ \)-wedge product is defined by the usual wedge product on \( \tilde{T}^p \) and the \( \circ \)-product in the Weyl algebra \( \mathcal{E}(\tilde{\mathcal{W}}) \).
The correspondence between the Fedosov ∗-product and the Moyal ∗-product of Eq. (2.11) is given when the involved fields φ^I(σ, y, ζ_B) are central. That means that

\[
[φ^I, \tilde{φ}^I]_B \equiv \frac{1}{iξ_B} \left( \tilde{φ}^I ∘ φ^I - (-1)^{q_1q_2}\tilde{φ}^J ∘ φ^J \right) = 0
\]  

(3.8)

for all \( φ^I ∈ E(\tilde{W} ⊗ Λ^{q_1}) \) and \( \tilde{φ}^I ∈ E(\tilde{W} ⊗ Λ^{q_2}) \), where \( q_1 \) and \( q_2 \) are the degrees of \( φ^I \) and \( \tilde{φ}^I \) as differentiable forms respectively. The set of central forms is denoted by \( Z ⊗ Λ \). Here \( Z \) coincides with the algebra of quantum observables \( M \). Let \( φ^I(σ, y, ζ_B) \) be an element of \( E(\tilde{W}) \), we define the symbol map \( σ : E(\tilde{W}) → Z \), given by \( φ^I(σ, y, ζ_B) ↦ φ^I(σ, 0, ζ_B) \), i.e. the map \( σ \) is the projection of \( E(\tilde{W}) \) onto \( Z \). Fields \( φ^I(σ, 0, ζ_B) \) are exactly the generic fields \( φ^I(σ) \) of Sec. 2.2.

**Differential Operators**

One can define some important differential operators. The operator \( δ : E(\tilde{W}_p ⊗ Λ^q) → E(\tilde{W}_{p-1} ⊗ Λ^{q+1}) \) defined by \( δa ≡ dx^k ∧ \frac{∂a}{∂y^k} \) and its dual operator \( δ^o : E(\tilde{W}_p ⊗ Λ^q) → E(\tilde{W}_{p+1} ⊗ Λ^{q-1}) \) defined by \( δ^oa ≡ y^k \frac{∂a}{∂x^k} \) for all \( a ∈ E(\tilde{W}_p ⊗ Λ^q) \), where \( | \) stands for the contraction. The operators \( δ \) and \( δ^o \) satisfy several properties very similar to those for the usual differential and co-differential; for instance, there exists an analogue of Hodge-de Rham decomposition theorem [24].

**Symplectic Connection**

Assume the existence of a torsion-free connection on \( X \) which preserves its symplectic structure. This connection is known as symplectic connection \( \nabla_i \). This operator is a connection defined in the bundle \( \tilde{W} \) as \( \nabla : E(\tilde{W} ⊗ Λ^q) → E(\tilde{W} ⊗ Λ^{q+1}) \) and is defined in terms of the symplectic connection as \( \nabla a ≡ dx^i ∧ \nabla_i a \). In Darboux local coordinates this connection is written as \( \nabla a = da + [Γ, a]_B \) where \( Γ = \frac{1}{2}Γ_{ijk}y^iy^jd{x^k} \) is a local one-form with values in \( E(\tilde{W}) \), \( Γ_{ijk} \) are the symplectic connection’s coefficients, \( d = dx^i ∧ \frac{∂}{∂x^i} \) and \( \nabla_i \) is the covariant derivative on \( X \) with respect to \( \frac{∂}{∂x^i} \).

Following Fedosov, we define a more general connection \( D \) in the Weyl bundle \( \tilde{W} \) as follows

\[
Da = \nabla a + [\gamma, a]_B,
\]

(3.9)

where \( γ ∈ E(\mathcal{E}(\tilde{W}) ⊗ Λ^1) \) is globally defined on \( X \). The curvature of the connection \( D \) is given by \( Ω = (R + \nabla γ + \frac{1}{iξ}γ^2) \), with the normalizing condition \( γ_0 = 0 \). Here \( R \) is defined by
\[ R := \frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l \] where \( R_{ijkl} \) is the curvature tensor of the symplectic connection. It can be shown that for any section \( a \in \mathcal{E}(\mathcal{W} \otimes \Lambda) \) one has \( D^2 a = [\Omega, a]_B \).

\( X \) has two curvatures, one as Riemannian manifold \( \tilde{R}_{ijkl} \), which is defined by the linear connection on the tangent bundle of \( X \) and as Fedosov manifold the symplectic connection on the Weyl bundle has curvature \( R_{ijkl} \). These curvatures are related in the form

\[ \tilde{R}_{ijkl} = g_{ip} R^p_{jkl} = g_{ip} \omega_{pm} R_{mjkl}. \] (3.10)

where \( g_{ij} \) is the Riemannian metric on \( X \).

Since the torus is flat, Riemann curvature is thus zero \( \tilde{R}_{ijkl} = 0 \) and consequently the curvature \( R_{ijkl} \) of the symplectic connection vanishes. Something similar occurs with the coefficients of the symplectic connection \( \Gamma_{ijk} \) since they are related to Christoffel symbols of Riemannian geometry [25]. Thus \( D \) is purely gauge and it is given by

\[ D\tilde{\phi}^I = d\tilde{\phi}^I + [\gamma, \tilde{\phi}^I]_B \] (3.11)

and the curvature \( \Omega \) of \( D \) is given by

\[ \Omega = d\gamma + \frac{1}{i\xi_B} \gamma \wedge \gamma. \] (3.12)

We will see later that \( \gamma \) can be identified with the gauge fields.

**Abelian Connection**

One very important definition is that of the *Abelian connection*. A connection \( D \) is Abelian if for any section \( a \in \mathcal{E}(\mathcal{W} \otimes \Lambda) \) one has \( D^2 a = [\Omega, a]_B = 0 \). From Eq. (3.8) one immediately sees that the curvature \( \Omega \) of the Abelian connection is central. In Fedosov’s paper the Abelian connection takes the form

\[ D = -\delta + \nabla + [r, \cdot], \] (3.13)

where \( \nabla \) is a fixed symplectic connection and \( r \in \mathcal{E}(\mathcal{W}_3 \otimes \Lambda^1) \) is a globally defined one-form with the Weyl normalizing condition \( r_0 = 0 \). This connection has curvature

\[ \Omega = -\frac{1}{2} \omega_{ij} dx^i \wedge dx^j + R - \delta r + \nabla r + \frac{1}{i\hbar} r^2 \] (3.14)
with $\delta r = R + \nabla r + \frac{1}{2\pi} r^2$. This last equation has a unique solution satisfying the condition $\delta^{-1} r = 0$.

**Algebra of Quantum Observables**

Now consider the subalgebra $\mathcal{E}(\tilde{\mathcal{W}}_D)$ of $\mathcal{E}(\tilde{\mathcal{W}})$ consisting of flat sections i.e. $\mathcal{E}(\tilde{\mathcal{W}}_D) = \{ a \in \mathcal{E}(\tilde{\mathcal{W}}) | Da = 0 \}$. This subalgebra is called the algebra of Quantum Observables.

Now an important theorem [24] is: For any $a_0 \in \mathcal{Z}$ there exists a unique section $a \in \mathcal{E}(\tilde{\mathcal{W}}_D)$ such that $\sigma(a) = a_0$.

As a direct consequence of this theorem we can construct a section $a \in \mathcal{E}(\tilde{\mathcal{W}}_D)$ by its symbol $a_0 = \sigma(a)$ in the form

$$a = a_0 + \nabla_i a_0 y^i + \frac{1}{2} \nabla_i \nabla_j a_0 y^i y^j + \frac{1}{6} \nabla_i \nabla_j \nabla_k a_0 y^i y^j y^k - \frac{1}{24} R_{ijkl} \omega^{lm} \nabla_m a_0 y^i y^j y^k + \ldots \quad (3.15)$$

The last theorem states that there exists the bijective map $\sigma : \mathcal{E}(\tilde{\mathcal{W}}_D) \to \mathcal{Z}$. Therefore there exists the inverse map $\sigma^{-1} : \mathcal{Z} \to \mathcal{E}(\tilde{\mathcal{W}}_D)$. It is possible to use this bijective map to recover the Moyal product $*$ in $\mathcal{Z}$, $a_0 * b_0 = \sigma(\sigma^{-1}(a_0) \circ \sigma^{-1}(b_0))$.

In the case of the torus it is flat and Eq. (3.15) can be written as

$$\tilde{\phi}^I = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{i_1} \partial_{i_2} \ldots \partial_{i_k} \tilde{\phi}_0^I) y^{i_1} y^{i_2} \ldots y^{i_k} \quad (3.16)$$

where $\tilde{\phi}_0^I$ is equal to the generic fields $\phi^I$ of Sec. 2. Flat sections on the Weyl bundle on $\tilde{T}^p$ are described in terms of the space of closed differential forms $\mathcal{E}(\tilde{\mathcal{W}}_D) = Ker D$. Thus “physical states” are Moyal algebra-valued cohomology classes in $H^*(\mathcal{E}(\tilde{\mathcal{W}}_D))$ of the algebra of quantum observables $\mathcal{E}(\tilde{\mathcal{W}}_D)$.

**Trace on the Weyl Algebra**

In order to work with a variational principle which involves Fedosov geometry we would like to get a definition of trace. In the case $X = \mathbb{R}^{2n}$ with the standard symplectic structure $\omega = \sum_i dp_i \wedge dq_i$, the Abelian connection $D$ in the Weyl bundle is $-\delta + d$. In this case the product $\circ$ coincides with the usual Moyal $*$-product [24]. The trace in the Weyl algebra $\mathcal{E}(\tilde{\mathcal{W}}_D)$ over $\mathbb{R}^{2n}$ is the linear functional on the ideal $\mathcal{E}(\mathcal{W}_D^{\text{Comp}})$ over $\mathbb{R}^{2n}$ (which consists of the flat sections with compact support) given by

$$\tilde{\text{tr}}(a) = \int_X \sigma(a) \frac{\omega^n}{n!} \quad (3.17)$$
where $\sigma(a)$ means the projection on the center $\sigma(a(x, y, h)) := a(x, 0, h)$. This definition of the trace satisfies a series of useful properties i)-. $tr(a \circ b) = tr(b \circ a)$ and ii)-. $tr(b) = tr(A_f b)$ for all the sections $a \in \mathcal{E}(\tilde{\mathcal{W}}_D)$, $b \in \mathcal{E}(\tilde{\mathcal{W}}_D^{\text{comp}})$ where $\mathcal{E}(\tilde{\mathcal{W}}_D^{\text{comp}})$ is the Weyl algebra of sections with compact support. In last equation, $A_f$ is an isomorphism $A_f : \mathcal{E}(\tilde{\mathcal{W}}_D^{\text{comp}})(\mathcal{O}) \rightarrow \mathcal{E}(\tilde{\mathcal{W}}_D^{\text{comp}})(f(\mathcal{O}))$, where $f$ is a symplectic diffeomorphism of $\mathbb{R}^{2n}$. It is also possible to construct a trace on the algebra of sections $\mathcal{E}(\tilde{\mathcal{W}}_D)$ for arbitrary symplectic manifolds $X$. This trace satisfies properties i) and ii) but unfortunately it is too formal and does not have an explicit form.

A series of papers involving the application of the geometry of deformation quantization in the context of integrable systems [44], self-dual gravity [45] and $\mathcal{W}$-gravity [46], are found in the literature.

### 3.3. Gauge Theory on Fedosov Manifolds

In the last subsection we have reviewed some relevant facts of Fedosov’s deformation quantization theory, which we now apply to the super Yang-Mills theory coming from the matrix compactifications. We first study the general case of the gauge theory on a symplectic real manifold $X$ and its description in terms of Fedosov geometry. After that we will show how to recover the original theory with constant $B$-field on the dual torus worked out in Sec. 2. Deformation quantization allows also the description of Yang-Mills theories on Kähler spaces such as ALE and $K3$ spaces and Calabi-Yau threefolds.

**The General Case: Fedosov Manifolds**

We are going to study the general case of non-constant background $B$-field on the general Fedosov manifold $X$ with the Riemannian structure $g_{ij}$. This is described by a supersymmetric Yang-Mills theory on the curved space $X$. As we have seen before in Sec. 3.2, the Moyal algebra (2.11) has to be substituted by the Weyl algebra on the tangent bundle to $X$. Thus a field theory on a curved space parametrized by its symplectic structure enters in the analysis. $\mathcal{M}$-valued gauge fields $A_\mu(\tilde{\sigma})$ on $\mathbb{R} \times \tilde{T}^p$ are promoted to $\mathcal{E}(\tilde{\mathcal{W}}_D)$-valued connection gauge fields $\tilde{A}_\mu$ on $\mathbb{R} \times X$ given by

$$
\tilde{A} = \sum_{\mu} \tilde{A}_{\mu}(x, y, \zeta_B) dx^\mu \tag{3.18}
$$

where $x = x^\mu = (x^0, x^i)$, $x^0 = t$ and $x^i$ are the coordinates on $X$. The field components $\tilde{A}_\mu$ are written as
\[ A_\mu(x, y, \zeta_B) = \sum_{k=0}^{\infty} \frac{1}{k!} (\nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_k} A_\mu) y^{i_1} y^{i_2} \cdots y^{i_k} - \frac{1}{24} R_{ijkl} \omega^m \nabla_m A_\mu + \ldots \] (3.19)

where \( A_\mu = A_\mu(x, \zeta_B) = \tilde{A}(x, 0, \zeta_B) \).

The curvature of the gauge connection is given in geometrical terms as \( \tilde{F} = \nabla \tilde{A} + \frac{1}{i \zeta_B} \tilde{A} \wedge \tilde{A} \). In terms of the commutator it yields

\[ \tilde{F} = \nabla \tilde{A} + [\tilde{A}, \tilde{A}]_B, \] (3.20)

where \( \tilde{A} \in \mathcal{E}(\tilde{W}_D) \otimes \Lambda^1 \), \( \tilde{F} \in \mathcal{E}(\tilde{W}_D) \otimes \Lambda^2 \) and \( \nabla \) is the symplectic connection.

The scalar fields are expressed by

\[ \tilde{X}^A = \sum_{k=0}^{\infty} \frac{1}{k!} (\nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_k} X^A) y^{i_1} y^{i_2} \cdots y^{i_k} - \frac{1}{24} R_{ijkl} \omega^m \nabla_m X^A + \ldots \] (3.21)

and their interaction with the gauge fields \( \tilde{A}_\mu \) are given by the covariant derivative \( D\tilde{X}^A = \nabla \tilde{X}^A + \frac{1}{i \zeta_B} \tilde{A} \wedge \tilde{X}^A \). In terms of the commutator it yields

\[ D\tilde{X}^A = \nabla \tilde{X}^A + [\tilde{A}, \tilde{X}^A]_B \] (3.22)

where \( D \) is the connection in the Weyl bundle \( \tilde{W} \). Here we have identified the global one-form \( \gamma \) with the gauge fields \( \tilde{A}_\mu \). The Lagrangian (2.20) can be generalized to the Weyl bundle on a curved compact space \( X \) as follows

\[ L_M = \tilde{\text{Tr}} \left( -\frac{1}{4T^2} \tilde{F} \mu \nu \circ \tilde{F}^{\mu \nu} - \frac{1}{2} g^{\mu \nu} D_\mu \tilde{X}^A \circ D_\nu \tilde{X}^A + [\tilde{X}^A, \tilde{X}^B]_B \circ [\tilde{X}^A, \tilde{X}^B]_B - \frac{i}{2} \tilde{\Theta}^\alpha \circ \Gamma_{\alpha \beta}^\mu D_\mu \tilde{\Theta}^\beta + \frac{T}{2} \tilde{\Theta}^\alpha \circ \Gamma_{\alpha \beta}^A [\tilde{X}^A, \tilde{\Theta}^\beta]_B \right) \] (3.23)

where \( g_{\mu \nu} \) is the Riemannian metric on \( \mathbb{R} \times X \) and \( \tilde{\text{Tr}} \) consists of the Fedosov’s trace for the general case and the matrix trace \( \text{Tr}_N \). General Fedosov’s trace does exist but an explicit form is not known [24]. In the most general case the spinors \( \tilde{\Theta}^\alpha \) are symplectic spinors on Fedosov manifolds. They are well defined if a metaplectic structure is added on
\(X\) as was shown in [47]. Covariant derivative in general involves the gauge and symplectic connections

\[
D_\mu \tilde{\varphi}^A = \partial_\mu \tilde{\varphi}^A + [\tilde{A}_\mu + \Gamma_\mu, \tilde{\varphi}^A]_B
\]  

(3.24)

where \(\tilde{\varphi}^A\) stands for the fields \(\tilde{X}^A\) and \(\tilde{\Theta}^\alpha\). Lagrangian (3.23) corresponds to a gauge theory with Weyl algebra-valued fields in the curved space \(R \times X\) and represents deformations of a gauge theory on a curved space parametrized by a symplectic structure in \(X\) given by the \(B\)-field.

Therefore we have been able to find an alternative way of obtaining the deformed gauge theory. This deformation was suggested in [12] where was argued that it could be obtained following the paper [27]. Lagrangian (3.23) generalizes (2.20) and we shall show how to recover the latter form the former. This deformed gauge theory has been the natural generalization to Fedosov deformation quantization theory to define the procedure of quantization in a global way. This involves curved manifolds and so the gauge theory becomes defined on these spaces. Automatically is obtained the generalization to non-constant \(B\)-fields on the underlying compact space. Thus the effect of the \(B\)-field in matrix compactification on a curved space \(X\) is the introduction of a symplectic structure on \(X\) compatible with the Riemannian structure and thus deformation quantization geometry is very important to study these compactifications.

There is also some relation with the work [10], where matrix theory on a curved space is defined as the dimensional reduction the 10-dimensional Yang-Mills on curved 10-dimensional manifold \(Y\). The relation with this work is given if we identify \(Y\) with \(X \times R^{10-p}\) and the corresponding metric \(g_{IJ}\) on \(Y\) with the metric on \(X\) given by the Riemannian metric \(g_{ij}\) and on \(R^{10-p}\) the flat metric \(\eta_{AB}\). Our results might also be related to other works of matrix theory on curved spaces [8,9]. Perhaps a preliminary step in seeking this relation would be first finding a relation of our results to the paper [9]. In the general case supersymmetry would be completely broken or a part of it depending on the geometric structure of \(X\). If \(X\) is a Kähler manifold the preserved supersymmetry is \(\mathcal{N} = 1, d = 4\) and for a hyperkähler is \(\mathcal{N} = 2, d = 4\).

Finally but not least important is that the deformed gauge theory is also a non-local field theory which generalizes the non-local theories studied in [13]. The renormalization group behavior of these theories is much more involved and deserves a careful study which we will take up in a future work.
Gauge Theory on the Torus

Now we will show how to recover Lagrangian (2.20) from the general theory described by the Lagrangian (3.23). First of all we set $X = \tilde{T}^p$ and repeat the analysis of the last subsection. Thus the gauge fields can be written as

$$\tilde{A} = \sum_{\mu} \tilde{A}_{\mu}(\tilde{\sigma}, y, \zeta_B)d\tilde{\sigma}^\mu$$

(3.25)

where

$$\tilde{A}_{\mu}(\tilde{\sigma}, y, \zeta_B) = \sum_{k=0}^\infty \frac{1}{k!} (\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} A_{\mu}) y^{i_1}y^{i_2}\cdots y^{i_k}$$

(3.26)

where $A_{\mu}$ corresponds precisely to the gauge field which enters in Lagrangian (2.20). Also the symplectic connection is substituted by partial derivatives because the coefficients of the connection $\Gamma$ for the torus vanish.

The curvature of the gauge connection $\tilde{F} = d\tilde{A} + \frac{1}{\kappa_B} \tilde{A} \circ \wedge \tilde{A}$. In terms of the commutator it yields

$$\tilde{F} = d\tilde{A} + [\tilde{A}, \tilde{A}]_B,$$

(3.27)

where $\tilde{A} \in \mathcal{E}(\tilde{\mathcal{W}}_D \otimes \Lambda^1)$ and $\tilde{F} \in \mathcal{E}(\tilde{\mathcal{W}}_D \otimes \Lambda^2)$. In components it can be written as

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_{\nu} - \partial_\nu \tilde{A}_{\mu} + [\tilde{A}_{\mu}, \tilde{A}_{\nu}]_B,$$

(3.28)

The scalar fields are

$$\tilde{X}^A = \sum_{k=0}^\infty \frac{1}{k!} (\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} X^I)y^{i_1}y^{i_2}\cdots y^{i_k}$$

(3.29)

and the interaction with the gauge fields are given by the covariant derivative $D\tilde{X}^A = d\tilde{X}^A + \frac{1}{\kappa_B} \tilde{A} \circ \wedge \tilde{X}^A$ or in terms of the commutator

$$D\tilde{X}^A = d\tilde{X}^A + [\tilde{A}, \tilde{X}^A]_B.$$

(3.30)

Substituting these expressions into the general Lagrangian (3.23) we get

$$L_M = \tilde{\text{Tr}} \left( -\frac{1}{4T^2} \tilde{F}_{\mu\nu} \circ \tilde{F}^{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} D_{\mu} \tilde{X}^A \circ D_{\nu} \tilde{X}^A \right)$$
\[ +[\tilde{X}^A, X^B]_B \circ [\tilde{X}^A, X^B]_B - \frac{i}{2} \tilde{\Theta}^\alpha \circ \Gamma^\mu_{\alpha \beta} D_\mu \tilde{\Theta}^\beta + \frac{T}{2} \tilde{\Theta}^\alpha \circ \Gamma^A_{\alpha \beta} [\tilde{X}^A, \tilde{\Theta}^\beta]_B \]  

(3.31)

where

\[ D_\mu \tilde{X}^A = \partial_\mu \tilde{X}^A + [\tilde{A}_\mu, \tilde{X}^A]_B \]  

(3.32)

and

\[ D_\mu \tilde{\Theta}^\alpha = \partial_\mu \tilde{\Theta}^\alpha + [\tilde{A}_\mu, \tilde{\Theta}^\alpha]_B. \]  

(3.33)

In this case we have a flat geometry and Fedosov trace (3.17) is valid. Then substituting (3.31) into the last Lagrangian it yields

\[ L_M = \int_{\mathcal{T}^p} \frac{\omega^{p/2}}{(\frac{p}{2})!} \text{Tr}_N \sigma \left( -\frac{1}{4T^2} \tilde{F}^{\mu \nu} \circ \tilde{F}^{\mu \nu} - \frac{1}{2} \tilde{\eta}^{\mu \nu} D_\mu \tilde{X}^A \circ D_\nu \tilde{X}^A \right. \]

\[ \left. +[\tilde{X}^A, X^B]_B \circ [\tilde{X}^A, X^B]_B - \frac{i}{2} \tilde{\Theta}^\alpha \circ \Gamma^\mu_{\alpha \beta} D_\mu \tilde{\Theta}^\beta + \frac{T}{2} \tilde{\Theta}^\alpha \circ \Gamma^A_{\alpha \beta} [\tilde{X}^A, \tilde{\Theta}^\beta]_B \right) \]  

(3.34)

where \( \sigma \) is the isomorphism or symbol map which is a projection of \( \mathcal{E}(\tilde{\mathcal{W}}) \) into \( \mathcal{M}_B \) arising in (3.17). Using the fact that \( \omega^{p/2} \) is proportional to the volume form of the dual torus, \( d^p \tilde{\sigma} \) then we obtain precisely Lagrangian (2.20) for the \( \mathcal{M} \)-valued fields. Thus we have shown that general Lagrangian constitutes its natural generalization from the point of view of deformation quantization theory and it contains the gauge theory (2.20), derived from matrix theory, as a particular case.

**Gauge Theory on Kähler-Fedosov Manifolds**

Deformed gauge theory is also well defined when \( X \) is a Kähler manifold. A Fedosov \(*\)-product of the Wick type can be defined [48].

Let \( X \) be a Kähler manifold with local coordinates \((z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n)\). The symplectic form is given by \( \omega = \frac{1}{2} B_{i\overline{j}} dz^i \wedge d\overline{z}^j \) with \( B_{i\overline{j}} \) positive definite. The Poisson tensor is given by \( \Lambda = \frac{i}{2} B_{i\overline{j}} \overline{Z}_i \wedge \overline{Z}_j \) with \( Z_i = \frac{\partial}{\partial z_i} \) and \( \overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} \). A Fedosov product on the sections of the Weyl bundle on \( X \) can be defined as

\[ \tilde{\phi}^I \circ \tilde{\phi}^J = \sum_{r=0}^{\infty} \left( \frac{i^r B}{2} \right)^r M_r (\tilde{\phi}^I, \tilde{\phi}^J) \]  

(3.35)
where

\[ M_r(\tilde{\phi}^I, \tilde{\phi}^J) = \frac{1}{r!}(\frac{4}{i})^r B^{i_1 j_1} \ldots B^{i_r j_r} i_s(Z_{i_1}) \ldots i_s(Z_{i_r}) \tilde{\phi}^I i_s(\overline{Z}_{j_1}) \ldots i_s(\overline{Z}_{j_r}) \tilde{\phi}^J. \]  

(3.36)

Here \( i_s(Z_{i_r}) \tilde{\phi}^I \) is the insertion (symmetric substitution) of the vector field \( Z_{i_r} \) in the symmetric part of \( \tilde{\phi}^I \).

With this product a similar analysis of Subsec. 3.2 and 3.3 can be done with this holomorphic Fedosov product. In this context the spinors are well defined and can be treated following the formalism of [49].

In particular analysis of \( K^3 \) and Calabi-Yau manifolds is then possible. We leave the application of deformation quantization on Kähler spaces to matrix compactifications on ALE and \( K^3 \) spaces and also to Calabi-Yau threefolds for future research.

4. Concluding Remarks

In this paper we have shown that the description of matrix compactifications on noncommutative tori corresponds to a genuine Weyl correspondence of deformation quantization theory in the terms of [22]. This description leads to a natural generalization from the classical Yang-Mills theory on the noncommutative torus to the consideration of curved spaces through the Fedosov’s geometry of deformation quantization theory. In the description on curved spaces we have gained automatically the generalization to non-constant background \( B \) fields on the curved manifolds. However the application of the full formalism of deformation quantization to matrix theory still remains to be done. It is interesting to note that Fedosov deformation quantization is well defined on general Kähler manifolds [48], in particular, on ALE spaces, \( K^3 \) spaces and Calabi-Yau threefolds. It would be interesting to investigate whether deformation quantization sheds some light in ALE, \( K^3 \) and Calabi-Yau compactifications of matrix theory in a background \( B \)-field.

Lagrangian (2.20) is generalized to flat connections on the Weyl bundle on a general symplectic and Riemannian manifold \( X \). In the case of toroidal compactifications this description is rather trivial and Weyl bundle description is equivalent to the gauge theory described in Sec. 2.3. Some contact with a definition of matrix theory on curved spaces [10] has been established. This would be useful to make contact of our results with other
definitions of matrix theory on noncommutative curved spaces [8,4]. It is known that obstructions to get a global $*$-product on arbitrary symplectic manifolds may arise as global anomalies [37]. It would be interesting to check whether the general Lagrangian (3.23) is global anomaly free. Finally, very recently in Ref. [50], a relation between quantum mechanics and Yang-Mills theory has been found. This relation contains several ingredients and constructions which seems to coincide with ours in some points. It would be very interesting to find a relation between both approaches.

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