PERIODIC ORBITS IN MAGNETIC FIELDS IN DIMENSIONS GREATER THAN TWO

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ABSTRACT. The Hamiltonian flow of the standard metric Hamiltonian with respect to the twisted symplectic structure on the cotangent bundle describes the motion of a charged particle on the base. We prove that under certain natural hypotheses the number of periodic orbits on low energy levels for this flow is at least the sum of Betti numbers of the base.

The problem is closely related to the existence question for periodic orbits on energy levels of a proper Hamiltonian near a Morse–Bott non-degenerate minimum. In this case, when some extra requirements are met, we also give a lower bound for the number of periodic orbits. Both of these questions are very similar to the Weinstein conjecture but differ from it in that the energy levels may fail to have contact type.

We give a simple proof of the fact that bounded sets in the cotangent bundle to the torus, with a twisted symplectic structure, have finite Hofer–Zehnder capacity. As a consequence, we obtain the existence of periodic orbits on almost all energy levels for magnetic fields on tori.

1. Introduction

The motion of a charge in a magnetic field on a manifold is given by the Hamiltonian vector field of the standard metric Hamiltonian on the cotangent bundle with respect to a non-standard symplectic structure. This so-called twisted symplectic structure is the sum of the standard one and the pull-back of the magnetic field two–form.

One of the objectives of the present paper is to prove the existence of periodic orbits on low energy levels for such Hamiltonian systems, provided that the metric and the magnetic field two–form satisfy a certain condition. The condition is that the magnetic field two–form is symplectic and compatible with the metric. For example, this requirement is met when the manifold, with these structures, is Kähler. More specifically, we show (Theorem 2.3) that under this condition the number of periodic orbits on every low energy level, when the orbits are non-degenerate, is no less than the sum of Betti numbers of the manifold. In the degenerate case, a similar lower bound is obtained in [Ke] in terms of the minimal number of critical points of a function on the manifold.

The problem of existence of periodic orbits for symplectic magnetic fields can be generalized as follows. Consider a proper function on a symplectic manifold and assume that this function has a Morse-Bott non-degenerate minimum along a symplectic submanifold. Then the problem is to find a lower bound for the number
of periodic orbits of the Hamiltonian flow on the levels near the minimum. For example, when the submanifold is just a point the answer is given by Weinstein’s theorem, [We2]. We provide a lower bound (Theorem 2.1) when the orbits are non-degenerate and again a certain compatibility condition is satisfied. The case of degenerate orbits is treated in [Ke] using a method relying on Moser’s proof, [Mc, Be], of Weinstein’s theorem.

Both of these problems are similar to the Weinstein conjecture, [We2], on the existence of periodic orbits on contact type hypersurfaces. (See, e.g., [FHV, HV, HZ, MS, Vi1, Vi2] for more information and further references.) However, the essential difference is that the energy levels in question may fail to have contact type. For example, in the case of the magnetic field, the twisted symplectic form is never exact on the energy level when the magnetic field is not exact and the dimension of the base is greater than two. This fact makes the problem difficult to solve by standard symplectic topology techniques.

The problem of existence of periodic orbits on almost all levels is accessible by making use of symplectic capacities. For example, if one can show that bounded sets in the ambient manifold have finite Hofer–Zehnder capacity, it follows that a proper Hamiltonian has periodic orbits on almost all energy levels, [HZ, Section 4.2]. In Section 3, we prove that the capacity of bounded sets is finite for a twisted symplectic form on the cotangent bundle to a torus for any closed magnetic field two–form (Theorem 3.1). This implies the “almost existence” of periodic orbits for magnetic fields on tori (Corollary 3.3).

This paper is one of very few (see also [Bi, BT, Ke, Lu, Po]) focusing on magnetic fields on manifolds of dimension greater than two. Much more is known about the existence of periodic orbits for magnetic fields on surfaces. The reader interested in the review of these results from the symplectic topology perspective should consult [Gi2]. The necessary symplectic geometry material can be found in [HZ, MS].

2. Periodic orbits on low energy levels

2.1. Periodic orbits near a minimum. Let \((W, \omega)\) be a symplectic manifold and let \(H: W \to \mathbb{R}\) be a smooth function. Assume that \(H\) has a Bott non-degenerate minimum along a compact symplectic submanifold \(M \subset W\). The normal bundle \(\nu\) to \(M\) in \(W\) is a symplectic vector bundle of dimension \(2m = \text{codim} M\). The Hessian \(d^2H: \nu \to \mathbb{R}\) is a positive–definite fiberwise quadratic form on \(\nu\). Thus at every point \(x \in M\) we have \(m\) canonically defined eigenvalues of the Hessian \(d^2H\) on the fiber of \(\nu\) over \(x\) with respect to the linear symplectic form \(\omega_x^+\) on this fiber. The eigenvalues of \(d^2H\) are equal at every point \(x \in M\) if and only if the linear Hamiltonian flow of \(d^2H\) with respect to \(\omega_x^+\) is periodic with all its orbits having the same period. In this case, the flow gives rise to a free \(S^1\)-action on \(S\nu = \{d^2H = 1\}\) and hence to the principal \(S^1\)-bundle \(pr: S\nu \to S\nu/S^1\).

Recall that a closed integral curve of a vector field (or a line field) is said to be non-degenerate if its Poincaré return map does not have unit as an eigenvalue. Denote by \(\text{SB}(N)\) the sum of Betti numbers of a manifold \(N\).

**Theorem 2.1.** Assume that for every \(x \in M\) all eigenvalues of \(d^2H\) with respect to \(\omega_x^+\) are equal. Then for a sufficiently small \(\epsilon > 0\), the number of periodic orbits of the Hamiltonian flow of \(H\) on the level \(\{H = \epsilon\}\) is at least \(\text{SB}(S\nu/S^1)\) if the \(S^1\)-bundle \(pr\) is trivial and at least \(\text{SB}(S\nu)/2 + 1\) otherwise, provided that all of the periodic orbits are non-degenerate.
The theorem will be proved in Section 2.3. Note that the bundle \( pr \) is always non-trivial when \( m > 1 \). The reason is that its restriction to a fiber \( \mathbb{C}P^{m-1} \) of the bundle \( S\nu/S^1 \to M \) is the Hopf fibration.

**Remark 2.2.** The fiberwise Hamiltonian flow of \( d^2H \) can alternatively be described as the Hamiltonian flow of \( dH \) with respect to the Poisson structure on the total space of \( \nu \) given by the family of fiberwise symplectic forms \( \omega_x^+ \), \( x \in M \).

As follows from the results of [Ke], when the periodic orbits are not required to be non-degenerate, the number of periodic orbits is still greater than or equal to \( \text{CL}(S\nu/S^1) = \text{CL}(M) + m \). Here \( \text{CL} \) stands for the cup–length and the action of \( S^1 \) is given, after suitable rescaling, by the linear Hamiltonian flow of \( d^2H \), described above. For \( \nu = \mathbb{R}^{2n} \) and \( M \) a point, this is a particular case of Weinstein’s theorem [We1]; see also [Mo].

It seems to be likely that the bound of Theorem 2.1 can be significantly improved. (For example, as stated, this bound is not necessarily higher than the bound from below for the degenerate case mentioned above, [Ke].) We conjecture that under the hypothesis of the theorem, the number of periodic orbits is at least \( \text{SB}(S\nu/S^1) = m \text{SB}(M) \). (The equality follows from [Hu, Theorem 2.5, p. 233].)

**2.2. Periodic trajectories of a charge in a magnetic field.** Let \( M \) be a Riemannian manifold and let \( \sigma \) be a symplectic form on \( M \) (a magnetic field). Then \( \omega = d\lambda + \pi^*\sigma \) is symplectic on \( W = T^*M \). Here \( d\lambda \) is the standard symplectic form on \( T^*M \) and the map \( \pi : T^*M \to M \) is the natural projection. Take \( H : T^*M \to \mathbb{R} \) to be the standard metric Hamiltonian. More explicitly, let us identify \( TM \) and \( T^*M \) by means of the Riemannian metric on \( M \). Then \( H(X) = g(X,X)/2 \), where \( g \) is the metric. The Hamiltonian flow of \( H \) with respect to \( \omega \) describes the motion of a charge on \( M \) in the magnetic field \( \sigma \). (The reader interested in more details should consult, e.g., [Gi2].) We say that a symplectic form \( \sigma \) and a metric \( g \) are compatible if there exists an almost complex structure \( J \) such that \( g(X,Y) = \sigma(X,JY) \) for all \( X \) and \( Y \) and \( J \) is \( g \)-orthogonal (cf., [MS, Section 4.1]). This condition is equivalent to that at every point all eigenvalues of \( g \) with respect to \( \sigma \) are equal. In Section 2.3 we will prove the following

**Theorem 2.3.** Assume that \( \sigma \) is compatible with the Riemannian metric on \( M \), e.g., \( M \) is Kähler with these structures. For a sufficiently small \( \epsilon > 0 \), the number of periodic orbits on the energy level \( \{ H = \epsilon \} \) is at least \( \text{SB}(M) \), provided that all of the orbits are non-degenerate.

**Remark 2.4.** Similarly to Theorem 2.1, the lower bound given by Theorem 2.3 is probably far from sharp. In the non-degenerate case there should be conjecturally at least \( \text{SB}(STM/S^1) = m \text{SB}(M) \) periodic orbits on every low energy level, where \( 2m = \dim M \). When \( M \) is a surface, this is exactly the statement of the Theorem 2.3 which in this case was originally proved in [Gi1]. Note also that under the hypothesis of Theorem 2.3 there are at least \( \text{SB}(M) + 1 \) periodic orbits, when \( \chi(M) = 0 \) but \( M \neq \mathbb{T}^2 \). This can be seen easily from the proof of the theorem.

If the periodic orbits are not assumed to be non-degenerate, the lower bound of Theorem 2.3 should be replaced by \( \text{CL}(M) + m \), [Ke].

When the energy value \( \epsilon > 0 \) is not small, the level \( \{ H = \epsilon \} \) may fail to carry a periodic orbit. See [Gi3] for \( m = 1 \) and [Gi4, Example 4.2] for \( m \geq 2 \).
2.3. The proofs of theorems 2.1 and 2.3. The theorems will follow from a more general result, proved in [Gi1], which we now state.

Let $E$ be a compact odd-dimensional manifold with a free circle action. Denote the quotient $E/S^1$ by $B$ and the natural projection $E \to B$ by $pr$. Recall that a two–form $\eta$ on $E$ is said to be maximally non-degenerate if it has a one–dimensional null–space, denoted in what follows by ker $\eta_x$, at every point $x \in E$. Equivalently, this means that $\eta$ has maximal possible rank (equal to $\dim E - 1$) at every point of $E$. Recall also that a characteristic of $\eta$ is an integral curve of the line field ker $\eta$.

**Theorem 2.5.** Let $\eta$ be a closed maximally non-degenerate two–form on $E$ such that the cohomology class $[\eta]$ lies in the image $pr^*(H^2(B))$. Assume that the field of directions ker $\eta$ is $C^1$-close to the fibers of $pr$ and that all of the closed characteristics of $\eta$ are non-degenerate. Then $\eta$ has at least $SB(B)$ closed characteristics if the $pr$ principle $S^1$-bundle $pr$ is trivial and at least $SB(E)/2 + 1$ closed characteristics otherwise.

**Remark 2.6.** We conjecture that in the non-degenerate case, regardless of whether $pr$ is trivial or not, the number of closed characteristics is greater than or equal to $SB(B)$, which would give a much higher lower bound than that of Theorem 2.5. Note that this conjecture would imply the hypothetical lower bound $m SB(M)$ mentioned in Remarks 2.2 and 2.4. The conjecture is proved in [Gi1] for a surface $B$ and in [Gi3] for the case where $\eta$ is a $C^0$-small perturbation of the pull–back of a symplectic form on $B$. The latter result indicates that the requirement that ker $\eta$ is $C^1$-close to the fibers can perhaps be replaced by $C^0$-closeness.

Note in this connection that the above conjecture is erroneously claimed to be proved by the first of the authors in [Gi3] (Corollary 3.8). In fact, Corollary 3.8 does not follow from Theorem 2.7 of that paper for the reasons outlined in Remark 2.7 below. At present, the conjecture (and hence Corollary 3.8 of [Gi3]) appears to be an open problem.

**Remark 2.7.** Under the hypothesis of Theorem 2.5, the “horizontal component” of the average of $\eta$ is the pull-back of a symplectic form on $B$. If $\eta$ were close to this pull-back, Theorem 2.5, with an improved lower bound, would follow from [Gi3] Theorem 2.7]. However, the form $\eta$ may fail to be close to this pull-back and as a consequence [Gi3] Theorem 2.7] does not apply. In fact, the main point of Theorem 2.5 is that the form $\eta$ is not assumed to be close to the pull–back of any two-form on $B$. In other words, one may think of $\eta$ as a “Hamiltonian perturbation” of a non-Hamiltonian vector field generating the $S^1$-action. This is exactly what makes Theorem 2.5, in contrast with the results of [Gi3], applicable to magnetic flows in higher dimensions.

**Remark 2.8.** It is easy to see that the cohomology condition on $\eta$ cannot be omitted – without this condition $\eta$ may fail to have closed characteristics. (See [Gi1] for a more detailed discussion.)

If the closed characteristics of $\eta$ are not required to be non-degenerate, the low bound of Theorem 2.5 should be replaced by the minimal number of critical points of a smooth function on $B$, $K_4$. In both of these lower bounds only the integral curves close to fibers and winding only once along the fibers are counted. In contrast with the lower bound of Theorem 2.7, the lower bound of [Ke] is apparently sharp for the number of periodic orbits in the class specified.
Proof of Theorem 2.5. When $B$ is a surface, the theorem is proved in [Gi1]. Hence, in what follows we will assume that $2k + 1 = \dim E \geq 5$.

Furthermore, as is shown in [Gi1], the number of closed characteristics is at least $\frac{\text{SB}(E)}{2}$. Since $\frac{\text{SB}(E)}{2} = \text{SB}(B)$ when the bundle $\text{pr}$ is trivial, we will assume from now on that $\text{pr}$ is not trivial.

We need to recall some details of the argument from [Gi1]. Consider closed characteristics of $\eta$ which are close to the orbits of the $S^1$-action and wind only once along the orbits. There exists a Morse–Bott function $f : E \to \mathbb{R}$ such that the closed characteristics of $\eta$ of this type comprise exactly the critical manifolds of $f$. Hence it is sufficient to prove that $f$ has at least $\frac{\text{SB}(E)}{2} + 1$ critical manifolds.

Denote by $b_i$ the Betti numbers of $E$ and by $\mu_i$ the number of critical manifolds of $f$ of index $i$. All (co)homology groups will be taken with real coefficients and we will use the convention that $\mu_i = 0$ whenever $i$ is negative.

Since the critical manifolds of $f$ are circles, the Morse–Bott inequalities for $f$ turn into

(1) \[ \mu_i + \mu_{i-1} \geq b_i \] for $i = 0, \ldots, 2k + 1$.

The inequalities (1) can be refined when $i = 1$ and $i = 2k$. Namely, we claim that

(2) \[ \mu_1 \geq b_1 \] and \[ \mu_{2k-1} \geq b_{2k}, \]

Let us prove the first of the inequalities (2) (cf., [Gi1]). Denote by $L \subset H_1(E)$ the subspace generated by the critical manifolds of index zero and $L' \subset H_1(E)$ a complementary subspace to $L$, i.e., $L \oplus L' = H_1(E)$. The dimension of this space does not exceed the number of critical circles of index one:

(3) \[ \mu_1 \geq \dim L'. \]

The critical manifolds of $f$ are close to the orbits of the $S^1$-action. As a consequence, the projections of critical manifolds to $B$ are contained in small balls. In particular, these projections are contractible and $\text{pr}_*(L) = 0$ in $H_1(B)$. Thus (3) implies that $\mu_1 \geq \dim L' \geq \dim \text{pr}_*(H_1(E))$. However, $\text{pr}_*(H_1(E)) = H_1(B)$ and so

\[ \mu_1 \geq \dim H_1(B). \]

Finally, since $\text{pr}$ is a non-trivial bundle, $b_1 = \dim H_1(B)$. Hence $\mu_1 \geq b_1$ which proves the first inequality in (3). The second inequality follows from the first one with $f$ replaced by $-f$.

Adding up the Morse–Bott inequalities (1) for all $i = 0, \ldots, 2k + 1$ except $i = 1$ and $i = 2k$ and the refined Morse–Bott inequalities (2), we obtain

\[ 2 \sum_{i=0}^{2k} \mu_i - (\mu_0 + \mu_{2k}) \geq \sum_{i=0}^{2k+1} b_i. \]

As a consequence,

\[ \sum \mu_i \geq \text{SB}(M) + \frac{1}{2}(\mu_0 + \mu_{2k}). \]

Since $f$ must have a local minimum and a local maximum, the second term in the latter formula is greater than or equal to one. Therefore, $\sum \mu_i \geq \frac{\text{SB}(E)}{2} + 1$ which completes the proof. \qed
Proof of Theorem 2.1. Throughout the proof we will keep the notations and conventions of Section 2.1. Using the symplectic neighborhood theorem, let us identify a neighborhood of $M$ in $W$ with a neighborhood $U$ of the zero section in the total space of the normal bundle $\nu$ so that the symplectic structure on $U$ is linear and equal to $\omega^\perp$ on the fibers of $\nu$. Without loss of generality we may assume that $U$ contains the level $E = \{d^2H = 1\}$.

Denote by $\varphi_\epsilon$ the fiberwise dilation $y \mapsto \epsilon y$ in the fibers on $\nu$. Let $E_\epsilon$ be the level $\varphi_\epsilon^* H/\epsilon^2 = 1$ and $\psi_\epsilon: E \to E_\epsilon$ be the fiberwise central projection. Consider the vector field $X_\epsilon$ on $E$ obtained by pushing forward the Hamiltonian vector field of $H/\epsilon^2$ on the level $\{H = \epsilon^2\}$ to $E_\epsilon$ by means of $(\psi_\epsilon \varphi_\epsilon)^{-1}$.

Clearly, the assertion of the theorem is equivalent to $X_\epsilon$ having the required number of periodic orbits.

It is not hard to show (see Remark 2.9 below) that $X_\epsilon \to X_0$ as $\epsilon \to 0$, where $X_0$ is the fiberwise Hamiltonian vector field of $d^2H$ with respect to $\omega^\perp$. In other words, $X_0$ is the Hamiltonian vector field of $d^2H$ for the Poisson structure which has the fibers of $\nu$ as its leaves and is given by $\omega^\perp$. Furthermore, the vector field $X_\epsilon$ generates the null-space line field of the two–form $\eta_\epsilon = (\psi_\epsilon \varphi_\epsilon)^* \omega$.

By the hypothesis on the eigenvalues of $d^2H$, the flow of $X_0$ gives rise to a free $S^1$-action on $E$. The form $\eta_\epsilon$ satisfies the cohomology condition of Theorem 2.3. Thus, by Theorem 2.3, when $\epsilon > 0$ is small enough the form $\eta_\epsilon$ has at least $\text{SB}(E/S^1)$ or $\text{SB}(E)/2 + 1$ closed characteristics, depending on whether $pr: E \to E/S^1$ is trivial or not and provided that the closed characteristics are non-degenerate. This completes the proof of Theorem 2.1.

Remark 2.9. Set $H_\epsilon = \varphi_\epsilon^* H/\epsilon^2$. Then $H_\epsilon \to d^2H$ for any $k \geq 0$ as $\epsilon \to 0$ on a small neighborhood $U$ of the zero section. Furthermore, consider the family of symplectic forms $\omega_\epsilon = \varphi_\epsilon^* \omega$. This family of forms does not converge as $\epsilon \to 0$, but the corresponding Poisson structures do. To be more specific, denote by $\omega^{-1}$ the Poisson structure on $U$ corresponding to the symplectic structure $\omega$, and by $\omega_0^{-1}$ the Poisson structure on $U$ whose leaves are the fibers of $\nu$ and which is equal to $\omega^\perp$ on the fibers. One can verify (see [K]) that $\omega^{-1}_\epsilon \to \omega_0^{-1}$ on $U$ as $\epsilon \to 0$ for any $k \geq 0$. Therefore, the Hamiltonian vector field of $H_\epsilon$ with respect to $\omega_\epsilon$ $C^k$-converges to the Hamiltonian vector field of $d^2H$ with respect to $\omega_0^{-1}$.

Proof of Theorem 2.3. The normal bundle $\nu$ to $M$ is canonically isomorphic to $T^*M$. Furthermore, let us identify the tangent and cotangent bundles to $M$ by means of $\sigma$. Then $\omega_\perp^x = \sigma_x$ for all $x \in M$. The compatibility condition of the corollary is equivalent to that for every $x \in M$ the eigenvalues of the metric on $T_xM$ with respect to $\sigma_x$ are equal. Thus the hypothesis of Theorem 2.1 holds under the compatibility condition.

As a consequence, Theorem 2.1 gives a lower bound for the number of periodic orbits on the energy level $\{H = \epsilon\}$ in terms of the sum or Betti numbers of $E = ST^*M$ or $E/S^1$.

The $S^1$-bundle $pr: E \to E/S^1$ is trivial only when $M = T^2$. In this case $\text{SB}(E/S^1) = \text{SB}(M)$. Thus let us assume that $pr$ is non-trivial.

If $\chi(M) = 0$, we have $\text{SB}(E)/2 = \text{SB}(M)$ and Theorem 2.3 follows from Theorem 2.1 with an even higher lower bound.
Assume that $\chi(M) \neq 0$. We claim that

\[
\frac{\text{SB}(E)}{2} + 1 \geq \text{SB}(M).
\]

(4)

Denote by $b_i$ the Betti numbers of $E$ and by $\beta_i$ the Betti numbers of $M$. The condition $\chi(M) \neq 0$ implies that

\[
b_i = \begin{cases} 
\beta_i & \text{if } 0 \leq i \leq n - 1 \\
\beta_{i-n+1} & \text{if } n \leq i \leq 2n - 1,
\end{cases}
\]

where $n = \dim M$. Adding up these equalities for all $i = 0, \ldots, 2n - 1$, we obtain

\[
\text{SB}(E) \geq 2\text{SB}(M) - (\beta_0 + \beta_n).
\]

(5)

Without loss of generality we can assume that $M$ is connected. Then $\beta_0 = \beta_n = 1$ and (4) follows from (5). This completes the proof of Theorem 2.3.

3. Magnetic fields on tori

In this section we give a simple proof of the fact that for a charge on a torus there are periodic orbits on almost all energy levels. Let $\sigma$ be a closed two-form on a torus $T^n$. (Note that $n$ can be odd.) As in Section 2.2, consider the twisted symplectic structure $\omega = d\lambda + \pi^*\sigma$ on $W = T^*T^n$.

**Theorem 3.1.** Every bounded set in $W$ has finite Hofer–Zehnder capacity.

**Remark 3.2.** The reader interested in the definition and properties of the Hofer–Zehnder capacity should consult [HZ]. Theorem 3.1 is not new. In fact, the theorem has been known to experts for quite some time. When the cohomology class $[\sigma]$ is rational, the theorem follows from [HZ, Theorem 1.2]. For $\sigma$ symplectic, the theorem is proved in [Gi2, Lemma 5.3]. When $\sigma$ is the pull-back of a two-form under a projection $T^n \to T^{n-1}$, the theorem becomes a particular case of a result of G. Lu, [Lu, Theorem E]. Finally, for $[\sigma] \neq 0$, Theorem 3.1 follows from [Lu, Theorem C].

**Proof.** Note first that to prove the theorem for $\sigma$ it suffices to prove the theorem for any form in the cohomology class $[\sigma]$. Indeed, the fiberwise shift by a one-form $\alpha$ sends bounded sets to bounded sets and transforms the twisted symplectic form $\omega = d\lambda + \pi^*\sigma$ on $W = T^*T^n$ into the form $\omega - \pi^*d\alpha$.

Hence without loss of generality, we may assume that $\sigma$ is a translation–invariant form $T^n$. In other words, in some coordinates $x_1, \ldots, x_n$ on $T^n$, we have

\[
\sigma = \sum a_{ij}dx_i \wedge dx_j,
\]

(6)

for some constants $a_{ij}$.

For an exact form $\sigma$, i.e., when $a_{ij} = 0$, the theorem is proved in [Li]. This fact is very easy to see, [Li, Proposition 4, p. 136]: A bounded set in $T^*S^1$ is contained in an annulus and the latter can be embedded into a disc. By taking the product, we conclude that a bounded set in $T^*T^n$ can be symplectically embedded into a polydisc. A polydisc has finite capacity and, as a consequence of monotonicity, a bounded set in $T^*T^n$ has finite capacity.

Thus we may assume that $\sigma$ given by (6) is non-zero.

We claim that $(W, \omega)$ is symplectomorphic to the product $\mathbb{R}^{2k} \times W_1$ with $k \geq 1$, where $\mathbb{R}^{2k}$ is equipped with the standard symplectic form and $W_1 = \mathbb{R}^{n-2k} \times T^n$ is given a translation–invariant symplectic form.
Let us prove the claim. Consider the universal covering \( \tilde{W} = \mathbb{R}^{2n} \) of \( W \) with the pull-back linear symplectic form \( \tilde{\omega} \). Let \( L \) be the inverse image of \( T^n \) in \( \tilde{W} \) and let \( L^\perp \) be the symplectic orthogonal complement to \( L \) with respect to \( \tilde{\omega} \). Pick a linear subspace \( E \) in \( L^\perp \) which is transversal to \( L^\perp \cap L \). The space \( E \) is symplectic because the null–space of \( \tilde{\omega}|_{L^\perp} \) is exactly \( L^\perp \cap L \). Moreover, \( \dim E > 0 \). This follows from the fact that \( \text{rk} \tilde{\omega}|_{L^\perp} > 0 \), and so \( L \) is not Lagrangian and \( L^\perp \neq L \). Fix a symplectic subspace \( \tilde{W}_1 \) in \( \tilde{W} \) which contains \( L \) and is transversal to \( E \). Thus \( \tilde{W} \) is symplectomorphic to \( E \times \tilde{W}_1 \).

The decomposition \( \tilde{W} = E \times \tilde{W}_1 \) induces the required direct product decomposition of \( W \). To see this, note that \( W = \tilde{W}/\Gamma \), where \( \Gamma \) is a discrete subgroup in \( L \). Thus \( W = E \times W_1 \), where \( W_1 = \tilde{W}_1/\Gamma \). It is clear that \( E \) is symplectomorphic to \( (\mathbb{R}^{2k}, \omega_0) \) with \( 2k > 0 \) and that the resulting symplectic structure on \( W_1 \) is translation–invariant.

Observe now that every bounded subset of \( W_1 \) can be symplectically embedded into \( T^{2(n-k)} \) with some translation–invariant symplectic structure. As a result, every bounded open set in \( W \) is symplectomorphic to a bounded open set in \( \mathbb{R}^{2k} \times T^{2(n-k)} \), where \( 2k > 0 \). The latter open sets have finite capacity as proved in [FHV] and [Ma]; see also [Lu] for further generalizations. (It is essential that \( 2k > 0 \) and hence the space \( E \) is non-trivial: the torus \( T^{2(n-k)} \) may have infinite capacity, [HZ, Section 4.5].)

As in Section 2.2 let \( H : T^*T^n \to \mathbb{R} \) be a metric Hamiltonian. Finiteness of capacity implies (see [HZ, Section 4.2, Theorem 4]) “almost existence” of periodic orbits:

**Corollary 3.3.** Almost all, in the sense of measure theory, levels of \( H \) carry a periodic orbit.

To put this general benchmark result in perspective, let us state some recently proved more subtle theorems concerning the existence of periodic orbits in the magnetic problem on higher–dimensional tori. (See, e.g., [Gi2] for a review of results for surfaces.)

According to the first result, due to Polterovich, [P2], when \( \sigma \neq 0 \), there exists a sequence of positive energy values \( c_k \to 0 \) such that on every level \( \{H = c_k\} \) there exists a contractible periodic orbit. This is a rather deep theorem: Corollary 3.3 guarantees the existence of periodic orbits on almost all levels of \( H \), but does not guarantee that these orbits are contractible. For instance, if \( \sigma = 0 \) periodic orbits are just closed geodesics and a metric on \( T^n \) can easily fail to have contractible closed geodesic (e.g., a flat metric). This result also holds for any compact manifold \( M \) with \( \chi(M) = 0 \) in place of \( T^n \). Note also that as follows from a theorem of G. Lu, [L1, Theorem C], almost all levels of \( H \) carry a contractible periodic orbit when \( [\sigma] \neq 0 \).

The second theorem, a version of Hopf’s rigidity, is due to Bialy, [B]. By Bialy’s theorem, every energy level of \( H \) carries an orbit with conjugate points, provided that the metric is conformally flat and again \( \sigma \neq 0 \). This fact is related to the question of existence of contractible periodic orbits because every such orbit (with non-zero Maslov index) would have conjugate points. Thus Bialy’s theorem serves as indirect evidence in favor of the affirmative answer to the existence question.

In conclusion note that when \( \sigma \) is exact, all high energy levels have contact type and thus carry periodic orbits, [IV]. (See [Gi2, Remark 2.3] for more details.)
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