Eigenvalue repulsion estimates and some applications for the one-dimensional Anderson model

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Abstract
We show that the spacing between eigenvalues of the discrete 1D Hamiltonian with \textit{arbitrary potentials} which are bounded and with Dirichlet or Neumann boundary conditions is bounded away from zero. We prove an explicit lower bound, given by $C e^{-bN}$, where $N$ is the lattice size, and $C$ and $b$ are some finite constants. In particular, the spectra of such Hamiltonians have no degenerate eigenvalues. As applications we show that to a leading order in the coupling, the solution of a nonlinearly perturbed Anderson model in one dimension (on the lattice) remains exponentially localized in probability and average sense for initial conditions given by a unique eigenfunction of the linear problem. We also bound the derivative of the eigenfunctions of the linear Anderson model with respect to a potential change.

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1. Introduction
We consider the one-dimensional Anderson model on the lattice, $\Lambda$,

$$H^\Lambda u_n (x) = u_n (x + 1) + u_n (x - 1) + \varepsilon_x u_n (x) = E_n u_n (x),$$

(1.1)

with $x, n \in \{1, 2, \ldots, |\Lambda|\}, \omega = \{\varepsilon_x\}$ is the realization of the potential and $H^\Lambda$ is the Hamiltonian on the domain $\Lambda$, with normalized eigenfunctions $\{u_n (x)\} \in l^2 (\Lambda)$ and eigenvalues $E_n$. We will also denote by $N \equiv |\Lambda|$ the size of the domain. Furthermore, $H^\Lambda$ satisfies some boundary conditions to be specified later, which include both the Dirichlet
and Neumann boundary conditions (b.c.). Our first result applies to arbitrary uniformly bounded potential,
\[ \sup_{x \in \Lambda} |\epsilon_x| \equiv W < \infty. \]  
We will show in the following section that the minimal distance between the eigenvalues of \( H_{\omega}^\Lambda \) is bounded below by a constant of order \( e^{-bN} \) for every \( \omega \) and as long as \( W < \infty \) and the boundary conditions defining \( H_{\omega}^\Lambda \) are of the allowed class. Note, that this result holds for all bounded potentials. Our proof, while not necessarily the simplest one, is instructive and may be of more general interest.

Then, in the following section, we show two applications, motivated by the study of the Anderson localization problem, both linear and nonlinear. In particular, in \cite{13, 14} we have shown that for the nonlinearly perturbed Anderson model,
\[ i \partial_t \psi = H_{\omega}^\Lambda \psi + \beta |\psi|^2 \psi, \]  
with the initial condition of \( \psi (x, 0) = u_0 (x) \), the first-order nonlinear correction to the solution is given by
\[ \psi^{(1)} (x, t) = \beta \sum_n c_n^{(1)} (t) u_n (x) e^{-i E_n t}, \]  
with
\[ c_n^{(1)} (t) = \frac{V_0^{000}}{E_n - E_0} (1 - e^{i(E_n - E_0)t}). \]  
Since \( H_{\omega}^\Lambda \) depends on the realization of the potential, \( \omega \), so is \( u_0 (x) \). We show here that on average, the fractional power of the solution of (1.3) to the first order in \( \beta \) remains exponentially bounded for all times. In particular, a probabilistic bound implying exponential localization with high probability is found. We also show that the ordinary average is exponentially bounded at least for times which are exponential in \( N \). This implies Anderson localization with probability 1 (for sufficiently small \( \beta \)) and for finite time exponential in \( N \). The expansion in \( \beta \) is conjectured to be asymptotic \cite{15}, therefore \( \psi^{(1)} \) describes the dynamics generated by the nonlinear Schrödinger equation (NLSE) for sufficiently small \( \beta \). It is argued in \cite{10, 21} that the quantity \( |c_n^{(1)} (t)| \) characterizes the states that are most effective in a possible spreading mechanism for model (1.3). It is possibly related also to the analysis of \cite{9}. Higher order corrections in the perturbative analysis of (1.3) involve products of \( c_n^{(1)} \) and other combinations of energies. Estimates on energy combinations and their correlations with wavefunctions were recently proven \cite{4}.

In the second application, we control the averages of fractional powers of the derivative of the eigenfunctions of \( H_{\omega}^\Lambda \) with respect to some \( \epsilon_x \), and show that they are exponentially small in the distance between \( x \) and the localization center of the eigenfunction. We also bound the averages by some power of the volume of the system and interpolate between the fractional and ordinary averages.

The proof of the eigenvalue repulsion is based on the transfer matrix representation of the solutions of the one-dimensional problem, and study the dependence of the eigenfunctions on the energy \cite{8}. By studying the properties of the matrices as transformations of the hyperbolic space, in terms of the complex energy as a parameter, the presence and absence of continuous spectrum for classes of random Schrödinger operators on graphs can be naturally analyzed \cite{6}. This is close to our approach. We then show that the condition for the energy parameter to have a value that corresponds to an eigenvalue requires a path to return to the starting point, in some sense. Next, we prove monotonicity of a rotation number/angle associated with the path, as a function of the energy parameter. Monotonicity with respect to the energy parameter
is also used in the hyperbolic space representation; there, it appears as a basic property of the Möbius transformation [5]. By bounding the rate of rotation from above, as a function of the energy parameter, a minimal distance between the eigenvalues follows. There has been a lot of work proving the simplicity of the spectrum for a rather general class of multidimensional Anderson models. These results are proven in the probabilistic sense, that is, the result is proven to hold for almost all potentials. See the theorem in [12], and another simple argument in theorem 1 of [20].

The applications mentioned above use the exponentially small minimal distance between the eigenvalues in a crucial way. We decompose dyadically the space of potentials, \( \omega \), to subsets where the minimal distance between eigenvalues is in a dyadic interval, \( I_m \in [2^{-m-1}, 2^{-m}] \). Then, the sum over \( m \) is bounded up to \( m \leq \bar{b}N \) due to the eigenvalue repulsion. We estimate each term by a combination of two probabilistic estimates: first, the Minami estimate for the probability of finding at least two eigenvalues in an interval (see the theorem and its proof in [16] or [5]):

\[
\Pr \left( \operatorname{Tr} P^{(\Lambda)}_{H_{\omega}}(I) \geq 2 \right) \leq (\pi \|\rho\|_\infty I N)^2, \tag{1.6}
\]

where \( \|\rho\|_\infty \) is the supremum of \( \rho \), the probability distribution of the potential, \( I \) is some energy interval while \( P^{(\Lambda)}_{H_{\omega}}(I) \) is the spectral projection on that interval and \( H_{\omega}^{(\Lambda)} \) is the Hamiltonian corresponding to a one-dimensional Anderson problem with Dirichlet b.c. on a domain \( \Lambda \) (the Minami estimate is more general, see theorem 9 or the theorem and its proof in [16]). The second bound we use is the fractional moment bound of Aizenman (see theorem 1.2 in [1]) (see also related bounds in [2], theorem 1, and [7], theorems 2 and 3),

\[
\left\langle \sum_n |u_n(x) u_n(y)| \right\rangle \leq D e^{-\mu|x-y|}, \tag{1.7}
\]

where \( \mu > 0 \) and \( D > 0 \) are some constants.

2. Lower bound on level spacings

2.1. Main result

The main result is the following theorem:

**Theorem 1** (Eigenvalue repulsion). *Given the tight binding model:

\[
H u(x) = u(x - 1) + \varepsilon_x u(x) + u(x + 1), \quad 1 \leq x \leq N,
\]

with Dirichlet b.c. \( u(0) = 0, u(N + 1) = 0 \) or Neumann b.c. \( u(0) = u(1), u(N + 1) = u(N) \) and with \( 0 \leq \varepsilon_n \leq W < \infty \) for all \( n \), there exists a constant \( 0 < \eta(W) < 1 \) such that

\[
|E_i - E_j| \geq \frac{\pi(1/\eta(W) - 1)}{(1/\eta(W))^N - 1}
\]

for all \( i \neq j \) and eigenvalues \( E_i, E_j \).

2.2. Setup

For simplicity, we first prove the main theorem for Dirichlet b.c. and then describe the modifications needed for the Neumann case in a separate subsection. Obviously, \( E \) is an eigenvalue of \( H \) if and only if there exists a non-trivial vector \( \tilde{u} = \{u(x)\}_{x=0}^N \) so that \( H \tilde{u} = E \tilde{u} \) and

\[
u(0) = 0 \tag{2.1}
\]
\[ u(N + 1) = 0. \]  

(2.2)

Since \( u(0) = u(1) = 0 \) implies \( \vec{u} \equiv 0 \), we can set (without loss of generality)

\[ u(1) = 1. \]  

(2.3)

For arbitrary \( E \), given (2.1) and (2.3), we can calculate all the components \( u(x) \) of \( \vec{u} \) by a recursive formula:

\[ u(x + 1, E) = (E - \varepsilon_x)u(x, E) - u(x - 1, E). \]  

(2.4)

\( E \) is an eigenvalue of \( H \) iff (2.2) holds.

**Definition 2.** Let \( \alpha(x, E) \) be the angle between the 2D vector \( (u(x - 1, E), u(x, E)) \) and the positive direction of abscissa at the corresponding Cartesian plane. \( \psi(x, E) \) is said to be a version of angular ratio between \( u(x - 1, E) \) and \( u(x, E) \) iff \( \exists m \in \mathbb{Z} \), so that

\[ \alpha(x, E) + 2m\pi = \psi(x, E). \]  

(2.5)

which implies that \( E \) is an eigenvalue and vice versa. Also

1. if \( u(x - 1) \neq 0 \), then \( \tan(\psi(x)) := \frac{u(x - 1)}{u(x)}, \)
2. if \( u(x) \neq 0 \), then \( \cot(\psi(x)) := \frac{u(x - 1)}{u(x)}. \)

**Remark 3.** Here and in what follows, we use \( u(x) \) as a shorthand for \( u(x, E) \) and \( \phi(x) \) as a shorthand for \( \phi(x, E) \), etc.

By the recursive formula (2.4), we have

\[ \psi(x + 1) = \begin{cases} \arctan(E - \varepsilon_x - \cot(\psi(x))) + (k + 2m)\pi & \text{if } k\pi < \psi(x) < (k + 1)\pi \\ \left(k - \frac{1}{2}\right)\pi + 2m\pi & \text{if } \psi(x) = k\pi, \end{cases} \]  

(2.6)

where \( k, m \in \mathbb{Z} \) and \( \arctan: \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \).

The upper row of (2.6) can be obtained by dividing (2.4) by \( u(x) \), identifying \( \cot(\psi(x)) := \frac{u(x - 1)}{u(x)} \), taking \( \arctan \) of both sides, and then adding an arbitrary integer number of full rotations \( m \). (In what follows, \( m = 0 \) will be chosen so that \( \psi(x) \) are differentiable as functions of \( E \).)

The lower row of (2.6) refers to the case when \( \psi(x, E) = k\pi \iff u(x) = 0 \iff u(x + 1) = -u(x - 1) \). Then, it is obvious by considering the directions of 2D vectors \( (u(x - 1), 0) \) and \( (0, -u(x + 1)) \). Once again, an arbitrary integer number of full rotations is added.

By (2.6), the sequence \( \{\psi(x)\} \) is well defined up to addition of \( 2m\pi \), \( m \in \mathbb{N} \). Therefore, any version of (2.6) can be used to verify if condition (2.5) holds for an energy \( E \). In particular, we can set \( m = 0 \) and choose \( \{\psi(x, E)\}_{x=1}^{N-1} \) to be

\[ \psi(1) \equiv \frac{\pi}{2}, \]

\[ \psi(x + 1) = \begin{cases} \arctan(E - \varepsilon_x - \cot(\psi(x))) + k\pi & \text{if } k\pi < \psi(x) < (k + 1)\pi \\ \left(k - \frac{1}{2}\right)\pi & \text{if } \psi(x) = k\pi. \end{cases} \]  

(2.7)

The version (2.7) of (2.6) is especially convenient for our further use because \( \psi(N + 1, E) \) turns out to be a continuously differentiable function of \( E \) (see proposition 4).

The angle variable \( \psi(x, E) \) is known as the Prüfer angle [11].

\[ \text{We thank Michael Aizenman for bringing this to our attention.} \]
2.3. Proof

2.3.1. Proof for Dirichlet b.c. Eigenvalues satisfy (2.5) as explained in section 2.3. We will show that $\varphi(N+1, E)$ rotates monotonically counterclockwise (proposition 4), and that there is no degeneracy (proposition 5).

We then show that the rotation speed $\varphi'(N+1, E)$ is bounded from above (proposition 6). But $\varphi(N+1, E)$ must change by the angle of $\pi$ between every pair of eigen-energies (see (2.5)), and its rotation speed is bounded from above; therefore, the spacing between eigenvalues is bounded from below, that is $|E_{i+1} - E_i| \geq \frac{\pi}{\varphi(N+1)_{\text{max}}}$. 

**Proposition 4.** $\varphi(N+1, E)$ is a continuously differentiable and a strictly increasing function of $E$.

**Proof.** $\varphi(2, E) = \arctan(E - \varepsilon_1)$ is continuously differentiable and $\varphi'(2, E) > 0$. Next we use induction in $n$.

If $\varphi(x, E) \neq k\pi$, then $\varphi(x+1, E)$ is continuously differentiable and increasing, according to the definition, since $\arctan(\cdot)$ is a strictly increasing differentiable function of its argument. The argument of $\arctan(\cdot)$, $E - \cot(\varphi(x))$, is strictly increasing and continuously differentiable (by induction assumption on $\varphi(x)$ starting from $\varphi(2)$).

If $\varphi(x, E) = k\pi$ and is continuously differentiable and increasing (with respect to $E$), then $\varphi(x+1, E)$ is continuous because the single side limits (2.8) and (2.9) are equal to each other as follows from definition (2.7):

$$\lim_{\varphi(x) \to k\pi^-} \varphi(x+1, E) = \pi(k-1) + \lim_{\varphi(x) \to k\pi^-} \arctan(-\cot(\varphi(x))) = \pi(k-1) + \arctan(-(-\infty)) = k\pi - \frac{\pi}{2} \quad (2.8)$$

$$\lim_{\varphi(x) \to k\pi^+} \varphi(x+1, E) = k\pi + \lim_{\varphi(x) \to k\pi^+} \arctan(-(-\infty)) = k\pi + \arctan(-(+\infty)) = k\pi - \frac{\pi}{2}. \quad (2.9)$$

Left and right ‘single-sided’ derivatives of $\varphi(x+1, E)$ exist and are equal. That is because, for any $\varphi(x) \neq k\pi$ point, the following holds:

$$\varphi'(x+1, E) = \frac{d\varphi(x+1, E)}{dE} = \frac{1 + \frac{\varphi'(x)}{\sin^2(\varphi(x))}}{1 + (E - \varepsilon_x - \cot(\varphi(x))^2}. \quad (2.10)$$

Taking the single side limits at $\varphi(x) = k\pi$, and recalling that $\varphi(x+1)$ is continuous, one obtains

$$\lim_{\varphi(x) \to k\pi^-} \varphi'(x+1, E) = \lim_{\varphi(x) \to k\pi^-} \varphi'(x+1, E) = \varphi'(x, E)$$

$$\Rightarrow \varphi'(x+1, E) = \varphi'(x, E). \quad (2.11)$$

Hence derivative exists and (by induction) is positive as required. □

**Proposition 5.** The spectrum of $H$ is simple.

The general proof of the simplicity of the spectrum is given in theorem 8. The simplicity of the spectrum can also be shown using the ‘$\varphi$’ (Prüfer angle [11]) formalism used here.

**Proof.** To ensure that no degeneracy occurs (that is $|E_i - E_j| \neq 0$), we need to show that the solutions of $\varphi(x+1, E) = k\pi$ are simple. It is sufficient to show that $\varphi'(x+1, E) \neq 0$. 

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We saw that \( \phi'(2, E) > 0 \) consequently.

For \( \phi(x, E) \neq 0 \), we have that \( \phi'(x + 1, E) = \frac{1 + \frac{\phi'(x)}{\sin \phi(x)}}{1 + (E - \xi_x - \cot \phi(x))^2} > 0 \) (by induction assumption).

For \( \phi(x, E) = 0 \), \( \phi'(x + 1, E) = \phi'(x, E) > 0 \) (by induction assumption).

Therefore, \( \phi'(N + 1, E) > 0 \) and there is no degeneracy.

Another way to ensure the absence of degenerate eigenvalues of \( H \) is by successive use of (2.7), and considering the limits of \( \phi(N + 1, E) \) at \( E \to \pm \infty \):

\[
\lim_{E \to -\infty} \phi(2, E) = -\frac{\pi}{2} \Rightarrow \lim_{E \to -\infty} \phi(3, E) = -\frac{3\pi}{2} \cdots \Rightarrow \lim_{E \to -\infty} \phi(N + 1, E) = -\frac{(2N - 1)\pi}{2},
\]

(2.12)

\[
\lim_{E \to +\infty} \phi(2, E) = \frac{\pi}{2} \Rightarrow \lim_{E \to +\infty} \phi(3, E) = \frac{\pi}{2} \cdots \Rightarrow \lim_{E \to +\infty} \phi(N + 1, E) = \frac{\pi}{2}.
\]

(2.13)

By continuity and monotonicity of \( \phi(N + 1, E) \), there exist exactly \( N \) different solutions of \( \phi(N + 1, E) = k\pi \) for \( E \in (-\infty, +\infty) \). □

**Proposition 6.** The ratio of derivatives \( \frac{\phi'(x+1, E)}{\phi'(x, E)} \) is bounded above.

**Proof.** By (2.10)

\[
\phi'(x + 1, E) = \frac{d\phi(x + 1, E)}{dE} = \frac{1}{1 + (E - \xi_x - \cot \phi(x))^2} + \frac{\phi'(x)}{\sin \phi(x)}.
\]

\[
\leq 1 + \frac{\phi'(x)}{1 + (E - \xi_x - \cot \phi(x))^2} = 1 + \frac{\phi'(x)}{-2(E - \xi_x) \sin \phi(x) \cos \phi(x) + (E - \xi_x)^2 \sin^2 \phi(x)}.
\]

(2.14)

It is left to find a lower bound for the denominator:

\[
q = 1 - 2(E - \xi_x) \sin \phi(x) \cos \phi(x) + (E - \xi_x)^2 \sin^2 \phi(x).
\]

(2.15)

For convenience, we define \( h := E - \xi_x \); then

\[
q(h, \phi(x)) = 1 - 2h \sin \phi(x) \cos \phi(x) + h^2 \sin^2 \phi(x).
\]

(2.16)

We are only interested in cases where \( |h| = |E - \xi_x| \leq W + 2 \), because otherwise \( E \) is out of the spectrum interval and cannot be in an interval between any pair of eigenstates. Furthermore, (2.15) is \( \pi \)-periodic in \( \phi(x) \). Therefore, we are looking at a bound on the \( q \), on the compact set (closed rectangle):

\[
A = \{|h| \leq W + 2 \} \times \{0 \leq \phi(x) \leq \pi \}.
\]

(2.17)

Due to the continuity of \( q \) in both \( h \) and \( \phi(x) \), it has minimum in \( A \). Therefore, to show that \( q \) is bounded away from zero, it is sufficient to prove that \( q \) is positive in \( A \) (the formal continuity of \( q \) in arguments \( (h, \phi(x)) \) is obvious, and not to be confused with the continuity of functions \( \phi(x), \phi'(x) \) with respect to \( h \) or \( E \)).

Given expression (2.16) with fixed \( \phi(x) \), it can be evaluated as a quadratic function in \( h \) with minimum value:

\[
\min_h [1 - 2h \sin \phi(x) \cos \phi(x) + h^2 \sin^2 \phi(x)] = \sin^2 \phi(x).
\]

(2.18)

Therefore, \( q(h, \phi(x)) \) is positive for any \( \phi(x) \neq k\pi \), but also positive for \( \phi(x) = k\pi \), since \( \phi(x) = k\pi \Rightarrow q = 1 \). We hence have

\[
\eta(W) := \min_{|E - \xi_x| \leq W + 2, \phi(x)} |q| > 0.
\]

(2.19)
(Calculation of an analytic expression for $\eta$ is given in appendix B. It demonstrates the dependence of the bound on $W$.) Substituting into the derivative inequality (2.14), we obtain a recursive inequality:

$$\psi'(x + 1, E) \leq 1 + \frac{\psi'(x, E)}{\eta}. \tag{2.20}$$

\[\square\]

**Remark 7.** No separate argument is required for $\psi(x) = k\pi$, since limit (2.11) exists, and is a particular case of (2.14).

**Proof (Proof of theorem 1 for Dirichlet b.c.)** Evaluating inequality (2.20) recursively, from $\psi_2(E) \leq 1$ to $\psi(x + 1, E)$ (recall $\psi_2(E) = \arctan(E - \epsilon_1)$), one obtains

$$\psi'(x + 1, E) \leq \left(1 + \frac{1}{\eta} \left(1 + \frac{1}{\eta} \left(1 + \frac{1}{\eta} \left(1 + \frac{1}{\eta} \left(1 + \frac{\psi_2'}{\eta} \right) \right) \right) \right) \cdots \right)^{N-1} \leq \left(1 + \frac{1}{\eta} \right)^{N-1} = \eta^{-N} - 1.$$

(2.21)

Since no degeneracy occurs (by proposition 5) and since eigenvalues satisfy (2.5), and the derivative $\psi'(x + 1, E)$ satisfies (2.21) for any $-2 \leq E \leq W + 2$, we have

$$|E_i - E_j| \geq \frac{\pi(1/\eta(W) - 1)}{(1/\eta(W))^N - 1}$$

for any pair of eigenstates $E_i, E_j$. \[\square\]

2.3.2. **Proof for Neumann b.c.** Neumann b.c. imply

$$u(1, E) = u(1, E) \neq 0 \Rightarrow \psi(1, E) \equiv \frac{\pi}{4}. \tag{2.22}$$

The (2.5) is modified to be

$$\psi(N + 1) = (k + 1/4)\pi \Leftrightarrow u(N + 1) = u(N). \tag{2.23}$$

This implies that $E$ is an eigenvalue and vice versa. The derivative $\psi'(2, E) > 0$, and therefore proposition 4 holds. Limits in proposition 5 hold starting from $\psi(2, E)$. Proposition 6 holds without modifications. Thus theorem 1 holds for Neumann b.c. as well.

2.3.3. **Simplicity of the spectrum for general boundary conditions.**

**Theorem 8.** Let $H$ be defined as in theorem 1 with boundary conditions such that the normalization determines the value of the eigenfunction at two adjacent points. Then the spectrum of $H$ is simple.

**Proof.** Let $\tilde{v}$ and $\tilde{u}$ be eigenvectors of $H$ with eigenvalue $E_0$. Without loss of generality, $\tilde{v}$ and $\tilde{u}$ can be normalized so that $u(1) = v(1) = a$ and $u(0) = v(0) = b$. But form this point all the remaining elements of $\tilde{v}$ and $\tilde{u}$ can be determined using (2.4). Therefore $\tilde{v} \equiv \tilde{u}$. \[\square\]
2.4. Remarks on level repulsion

(1) As expected, the limit on spacing vanishes when $W$ approaches infinity. That is, when nearly infinite barriers are allowed (see (B.6), (B.7)).

(2) The typical sensitivity to the energy of the angle variable $\varphi(N + 1)$ is exponentially large in $N$ at the proximity of an eigenvalue (see appendix A).

(3) The result does not hold for periodic boundary conditions. For periodic boundary conditions, degeneracy might occur, and there is no lower bound on the energy spacing between non-degenerate states. For example consider $\varepsilon_j \equiv 0$. The eigenvalues

$$E_j = 2 \cos \frac{2\pi j}{N}, \quad j = 1, \ldots, N,$$

are pairwise degenerate, except $j = N$ and $j = N/2$ (in case of even $N$).

(4) This work does not prove that the proposed limit is optimal. However, when considering exponential localization in disordered potentials, it appears that optimal bound on inter-level spacings is indeed exponential in the chain length $N$.

3. Applications

3.1. Bound on first-order term in perturbation theory

In this section, we will show some applications of theorem 1. Our main interest will be in problems related to the still open question of whether there is localization for the Anderson model perturbed by a small nonlinearity. The numerical results so far are inconclusive, see e.g. [10, 17–19, 21]. The only rigorous result that applies to the full nonlinear system is the finite time result of [22]. In recent works [13–15], we developed a renormalized perturbation expansion for the nonlinear problem. The first-order term can now be controlled rigorously, as we will show now. The correction is given by the following term (see (1.4) and (1.5)):

$$c_n^{(1)} = V_n^{000} \left( \frac{1 - \omega(E_n - E_0)}{E_n - E_0} \right),$$

where

$$V_n^{000} = \sum_y u_n(y) u_3^0(y).$$

We define

$$\psi^{(1)}(x, t) = \sum_n c_n^{(1)}(t) u_n(x).$$

This is the correction to the wavefunction in first-order perturbation theory [13] (in this section, lattice sites are denoted by $x$ and $y$, while eigenstates by $n$). Following [3], we are interested in bounding the fractional moments of $|\psi^{(1)}|$:

$$|\psi^{(1)}(x, t)|^s \leq \sum_n \left| c_n^{(1)}(t) u_n(x) \right|^s,$$

for $0 < s < 1$. Namely,

$$\langle |\psi^{(1)}(x, t)|^s \rangle \leq \sum_n \left( \left| c_n^{(1)}(t) u_n(x) \right|^s \right) \leq 2 \sum_{y,n} \int d\mu(\omega) \left| \frac{u_n(x) u_n(y) u_3^0(y)}{E_n - E_0} \right|^s,$$
where $\langle \cdot \rangle$ is the average with respect to the measure $\mu(\omega)$ which is defined on the random potentials.

The Anderson model. We spell out the class of models which are considered here. Let $\omega = (\omega_x)_{x \in \mathbb{Z}}$ be a set of independent, identically distributed positive valued random variables indexed by $x \in \mathbb{Z}$. We will designate the probability distribution of $\omega_x$ by $\rho$. We will furthermore assume that the random variables take value in $[0, W]$, $W$ finite. We will denote by $\mu(\omega)$ the induced measure on the product space of all the $\omega_x$. We now define the random potential to be

$$\epsilon_\omega(x) := \omega_x.$$ Let

$$H_\omega u(x) := (H_0 + \epsilon_\omega(x)) u(x) = u(x + 1) + u(x - 1) + \epsilon_\omega(x)u(x).$$

The spectrum of $H_0$ is $[-2, 2]$ and is purely absolutely continuous. For each $\omega$, $\epsilon_\omega$ is a uniformly bounded function of $x$, with a bound $W$, independent of $\omega$. It follows that since each $\epsilon_\omega$ is uniformly bounded, $H_\omega$ is a self-adjoint operator on $l^2(\mathbb{Z})$, with spectrum in the interval $[-2, 2] + [0, W]$. We denote by $H_\Lambda$ the restriction of $H_\omega$ to the domain $\Lambda(\mathbb{Z})$, with Dirichlet b.c. Then, we have the following theorem of Minami [5, 16].

**Theorem 9.** For the Anderson model defined above, suppose the density of states, $n(E)$, exists at energy $E$ and is positive. Suppose that the fractional moment for Green’s function, of the finite volume Hamiltonian, at energy $E$ is exponentially decaying in the sense of [3]. Then we have

$$\text{Pr}(\text{Tr} E_\Lambda(J) \geq 2) \leq \frac{\pi^2}{2} \|\rho\|_\infty^2 |J|^2 |\Lambda|^2,$$

where $\|\rho\|_\infty$ is the supremum of $\rho$, the probability distribution of the potential, $J$ is some energy interval and $E_\Lambda(J)$ is the spectral projection to it.

It is known that the above conditions on the density of states and fractional moments hold for our Anderson model, with the restriction on the measure $\mu$.

**Lemma 10.** For any Hamiltonian in a finite box $\Lambda$ of size $N$, with orthonormal eigenfunctions, $\sum_x u_n(x) u_m(x) = \delta_{n,m}$. For any $x$,

$$\exists n, \quad |u_n(x)| \geq N^{-1/2}. \quad (3.6)$$

**Proof.** Assume that $\exists x$ such that $\forall n, |u_n(x)| < N^{-1/2}$. Due to orthonormality of the eigenfunctions of a Hamiltonian

$$1 = \sum_n |u_n(x)|^2 < N^{-1} \sum_n 1 = 1,$$

which is in a contradiction to the assumption. \qed

**Theorem 11.** For the one-dimensional Anderson model, and for $0 < s < \frac{1}{5}$,

$$\langle |\psi_1^{(1)}(x, t)|^2 \rangle \leq C_s N^{9/4 - s/2} e^{-\mu_3 x}, \quad (3.7)$$

where $C_s$ is a constant which depends only on $s$.

**Proof.** Let us define a set of potentials with the help of the dyadic decomposition,

$$\mathcal{V}_n(m) = \{ \omega : |E_n - E_0| \in I_m \}, \quad (3.8)$$
where
\[ I_m = [2^{-m-1}, 2^{-m}]. \]  
(3.9)

The denominators of (3.5) cannot be arbitrarily small by theorem 1 (eigenvalue repulsion),
\[ |E_n - E_0| \geq C e^{-bN}, \]  
(3.10)

where \( C \) and \( b \) are constants. Therefore, combining the decomposition (3.8) with (3.5) and (3.10) yields
\[
\langle |\psi^{(1)}(x, t)|^{s} \rangle \leq 2 \sum_{m=0}^{M} 2^{(m+1)} \sum_{y,n} \int d\mu(\omega) \chi(V_m(n)) \left| u_n(x) u_n(y) u_0^2(y) \right|^s,
\]  
(3.11)

with
\[
\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1.
\]  
(3.12)

and \( \chi(V_m(n)) \) is the characteristic function of the set of potentials, \( V_m(n) \). Applying the generalized Hölder inequality, one finds
\[
\langle |\psi^{(1)}(x, t)|^{s} \rangle \leq \sum_{n,y} \sum_{m=0}^{M} 2^{(m+1)} \left( \int d\mu(\omega) \chi(V_m(n))^{p_1} \right)^{1/p_1}
\times \left( \int d\mu(\omega) \left| u_n(x) u_n(y) \right|^{p_2} \right)^{1/p_2}
\times \left( \int d\mu(\omega) \left| u_0(y) u_0(0) u_0^{-1}(0) \right|^{3p_3} \right)^{1/p_3},
\]  
(3.13)

with \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \). To estimate
\[
J_1 = \left( \int d\mu(\omega) \chi(V_m(n))^{p_1} \right)^{1/p_1},
\]  
(3.14)

we use the fact that \( \chi^{p_1} = \chi \) and Minami estimate for the probability of finding at least two eigenvalues in an interval \( I \) (see (1.6) or the theorem and its proof in [16]):
\[
\Pr(\text{Tr} P_{\mu}^{A_1}(I) \geq 2) \leq (\pi \|\rho\|_{\infty} I N)^2.
\]  
(3.15)

Since we are not interested in a particular energy interval, we will cover the energy band with \( I^{-1} \) intervals of size \( I = |I_m| = 2^{-m-1} \), which gives
\[
J_1 \leq (\pi \|\rho\|_{\infty} N)^2 I^{-1/2} 2^{-(m+1)/p_1}.
\]  
(3.16)

To bound
\[
J_2 = \left( \int d\mu(\omega) \left| u_n(x) u_n(y) \right|^{p_2} \right)^{1/p_2}
\]  
(3.17)

and
\[
J_3 = \left( \int d\mu(\omega) \left| u_0(y) u_0(0) u_0^{-1}(0) \right|^{3p_3} \right)^{1/p_3},
\]  
(3.18)

we choose \( 1/p_2 = s \) and \( 1/p_3 = 3s \), which sets \( 1/p_1 = 1 - 4s \) and \( s < \frac{1}{4} \). Then, we proceed by combining lemma 10 to bound, \( |u_0(0)|^{-1} \), with the result of Aizenman (theorem 1.2 in [11]):
\[
\sum_n \left( \int d\mu(\omega) \left| u_n(x) u_n(y) \right| \right) \leq D e^{-\mu|x-y|},
\]  
(3.19)
where $\mu > 0$ and $D > 0$ are some constants. This yields
\[ J_2 \leq D^s e^{-\mu |x-y|} \] (3.20)
and
\[ J_3 \leq N^{3s/2} D^{3s} e^{-3\mu |y|}. \] (3.21)

Plugging (3.16), (3.20) and (3.21) back into (3.13) gives
\[ \langle |\psi(1)(x, t)|^{s} \rangle \leq D^{ds} (\pi \|\rho\|_{\infty})^{2(1-4s)} \sum_{m=0}^{M} (2(5s-1)(m+1)) \sum_{y} e^{-\mu |x-y|} e^{-3\mu |y|}. \] (3.22)

Setting $s < \frac{1}{2}$, and using the triangle inequality for the last sum, we obtain
\[ \sum_{m=0}^{M} 2^{(5s-1)(m+1)} < \frac{1}{1 - 2^{5s-1}}, \] (3.23)
and
\[ \sum_{y} e^{-\mu |x-y|} e^{-3\mu |y|} \leq e^{-\mu |x|} \sum_{y} e^{-2\mu |y|} \leq e^{-\mu |x|} \frac{1}{1 - e^{-2\mu s}}. \] (3.24)

Therefore, for $0 < s < \frac{1}{2}$:
\[ \langle |\psi(1)(x, t)|^{s} \rangle \leq C_{r,\delta} N^{3-13s/2} e^{-\mu |x|}, \] (3.25)

with
\[ C_{r,\delta} = \frac{D^{4s}(\pi \|\rho\|_{\infty})^{2(1-4s)}}{(1 - 2^{5s-1})(1 - e^{-2\mu s})}. \]

\[ \square \]

**Corollary 12.** For $\nu > 0$ and $0 < s < \frac{1}{2}$,
\[ \Pr \left( |\psi(1)(x, t)|^{s} \right) \geq C_{r,\delta} N^{3-13s/2} e^{-(\mu - \nu s)|x|} \leq e^{-\nu |x|}. \] (3.26)

**Proof.** Using the Chebychev inequality
\[ \Pr(|\psi(1)(x, t)|^{s} \geq A) \leq A^{-1} C_{r} N^{3-13s/2} e^{-\mu |x|}, \] (3.27)
and choosing
\[ A = C_{r} N^{3-13s/2} e^{-(\mu - \nu s)|x|} \] (3.28)
gives
\[ \Pr(|\psi(1)(x, t)|^{s} \geq C_{r} N^{3-13s/2} e^{-(\mu - \nu s)|x|} \leq e^{-\nu |x|}. \] (3.29)
or
\[ \Pr(|\psi(1)(x, t)|^{s} \geq C_{r} N^{3-13s/2} e^{-(\mu - \nu s)|x|} \leq e^{-\nu |x|}. \] (3.30)
\[ \square \]
3.2. Bound on the derivative of an eigenfunction

An important object in the study of the properties of eigenfunctions is the sensitivity to a change of the potential at some point of an eigenfunction. We have by direct computation:

$$\frac{\partial u_0(x)}{\partial \varepsilon y} = u_0(y) \sum_{n \neq 0} \frac{u_n(x) u_n(y)}{E_0 - E_n}.$$  \hspace{1cm} (3.31)

The above analysis could be extended to obtain bounds on this derivative.

**Theorem 13.** For a one-dimensional Anderson problem, and $0 < s < \frac{1}{3}$,

$$E_s \equiv \left\| \frac{\partial u_0(x)}{\partial \varepsilon y} \right\|_s \leq K_s N^{3-7s/2} e^{-\mu s|x-y|} e^{-\mu s|y|},$$  \hspace{1cm} (3.32)

**Proof.** Proceeding in a similar manner to the previous subsection,

$$E_s \leq \sum_{n \neq 0} \frac{\left| \frac{u_0(y) u_n(x) u_n(y)}{E_0 - E_n} \right|^s}{E_0 - E_n^s} \leq \sum_{n \neq 0} \sum_{m=0}^M 2^{s(m+1)} \int d\mu(\omega) \chi(V_n(m)) \left| \frac{u_0(y) u_n(x) u_n(y)}{E_0 - E_n} \right|^s.$$  \hspace{1cm} (3.33)

Now use the generalized Hölder inequality,

$$E_s \leq \sum_{n \neq 0} \sum_{m=0}^M 2^{s(m+1)} \left( \int d\mu(\omega) \chi(V_n(m)) \right)^{1/p_1} \left( \int d\mu(\omega) \left| \frac{u_0(y) u_n(x) u_n(y)}{E_0 - E_n} \right|^s \right)^{1/p_2} \times \left( \int d\mu(\omega) \left| \frac{u_0(y) u_n(x) u_n(y)}{E_0 - E_n} \right|^s \right)^{1/p_3},$$  \hspace{1cm} (3.34)

with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. Setting $1/p_2 = 1/p_3 = s$, we have $1/p_1 = 1 - 2s$. Then using (3.16),

$$E_s \leq (\pi \|\rho\|_\infty N)^{2(1-2s)} \left( \sum_{m=0}^M 2^{(3s-1)(m+1)} \right) \times \sum_{n \neq 0} \left( \int d\mu(\omega) \left| \frac{u_0(y) u_n(x) u_n(y)}{E_0 - E_n} \right|^s \right) \left( \int d\mu(\omega) \left| \frac{u_0(y) u_n(x) u_n(y)}{E_0 - E_n} \right|^s \right)^s.$$  \hspace{1cm} (3.35)

Bounding the sum for $s < \frac{1}{3}$, utilizing the result of Aizenman (3.19) and lemma 10 for the last term gives

$$E_s \leq K_s N^{3-7s/2} e^{-\mu s|x-y|} e^{-\mu s|y|},$$

with

$$K_s = \frac{D^{2s} (\pi \|\rho\|_\infty)^{2(1-2s)}}{1 - 2^{1-3s}}.$$  \hspace{1cm} \□
3.3. Ordinary averages, \( s = 1 \)

Note that in the previous bounds, the results of theorem 1 were not used, since \( s \) was selected such that the sums (3.23) and (3.35) were convergent even for unbounded \( M \). In the following, we will calculate the bounds on the ordinary averages, namely for \( s = 1 \), and then interpolate between those two results.

**Theorem 14.** For the one-dimensional Anderson model on a box \( \Lambda \) of size \( N \), and \( \psi^{(1)}(x, t) \) defined in (1.4):

\[
\langle |\psi^{(1)}(x, t)| \rangle \leq AN^{13/2}. \tag{3.36}
\]

**Proof.** For the first-order correction of the wavefunction we obtain (substituting \( s = 1 \) in (3.13))

\[
\langle |\psi^{(1)}(x, t)| \rangle \leq \sum_{n, y} \sum_{m=0}^{M} 2^m \left( \int d\mu(\omega) \chi(V_m(n)) \right)^{1/p_1} \times \left( \int d\mu(\omega) |u_n(x) u_n(y)|^{p_2} \right)^{1/p_2} \times \left( \int d\mu(\omega) |u_0(y) u_0(0) u_0^{-1}(0)|^{3p_3} \right)^{1/p_3}. \tag{3.37}
\]

Using the bounds (3.16), (3.20) and (3.21) gives

\[
\langle |\psi^{(1)}(x, t)| \rangle \leq D^{1/p_1 + 1/p_1} (\pi \|\rho\|_\infty)^{2/p_1} N^{5/2 + 2/p_1} \times \left( \sum_{m=0}^{M} 2^{(1-1/p_1)(m+1)} \right) \sum_y e^{-\mu|x-y|/p_1} e^{-\mu|y|/p_1}. \tag{3.38}
\]

Setting \( 1/p_1 = 1 - \epsilon \) and \( 1/p_2 = 1/p_3 = \epsilon/2 \) and using the triangle inequality for the last sum yields

\[
\langle |\psi^{(1)}(x, t)| \rangle \leq \frac{D^\epsilon (\pi \|\rho\|_\infty)^{2(1-\epsilon)}}{(1 - e^{-\mu/2})(1 - 2 - \epsilon)} N^{5/2 + 2(1-\epsilon)} 2M \epsilon. \tag{3.39}
\]

Since \( M = \tilde{b}N \) (3.12), we will set \( \epsilon = 1/N \) to remove the exponential dependence on \( N \); this gives

\[
\langle |\psi^{(1)}(x, t)| \rangle \leq D^{1/N} 2^\epsilon (\pi \|\rho\|_\infty)^{2(1-1/N)} (1 - e^{-\mu/2N})(1 - 2 - 1/N) N^{5/2 + 2(1-1/N)}, \tag{3.40}
\]

or

\[
\langle |\psi^{(1)}(x, t)| \rangle \leq AN^{11/2}. \tag{3.41}
\]

with

\[
A = \frac{2^b + \pi^2 \|\rho\|_\infty^2 D}{\mu \ln 2}. \tag{3.42}
\]

**Theorem 15.** For the one-dimensional Anderson model on a box \( \Lambda \) of size \( N \),

\[
\left| \frac{\partial u_0(x)}{\partial \epsilon_y} \right| \leq BN^{11/2}. \tag{3.43}
\]
Proof. Similarly, for the bound on the derivative, found from (3.34) by substituting $s = 1$,

$$E_1 \leq \sum_{n \neq 0} \sum_{m=0}^M 2^{(m+1)} \left( \int \frac{d\mu (\omega)}{V_n (m)} \right)^{1/p_1} \left( \int \frac{d\mu (\omega)}{u_n (x) u_n (y)} \right)^{1/p_2} \left( \int \frac{d\mu (\omega)}{u_0 (y) u_0 (0)} \right)^{1/p_1} \left( \int \frac{d\mu (\omega)}{|u_0 (0)|} \right)^{1/p_2},$$

we obtain

$$E_1 \leq D^{1/p_2+1/p_1} (\pi \| \rho \|_{\infty})^{2/p_1} N^{5/2+2/p_2} \left( \sum_{m=0}^M 2^{(1-1/p_1)(m+1)} \right) e^{-\mu|x-y|/\rho_1} e^{-\mu y/\rho_2};$$

(3.44)

setting as before $1/p_1 = 1 - \epsilon$ and $1/p_2 = 1/p_3 = \epsilon/2$ gives

$$E_1 \leq \frac{D^\epsilon (\pi \| \rho \|_{\infty})^{2(1-\epsilon)}}{(1 - 2^{-\epsilon})^2} N^{9/2-2\epsilon} 2^{M} e^{-\mu y/2} e^{-\mu y/2}.$$

(3.46)

Since $M = \tilde{b}N$ (3.12), we will set $\epsilon = 1/N$ to remove the exponential dependence on $N$:

$$E_1 \leq \frac{D^{1/N} 2^{\tilde{b}} (\pi \| \rho \|_{\infty})^{2(1-1/N)}}{(1 - 2^{-1/N})^2} N^{9/2-2/N} e^{-\mu y/2} e^{-\mu y/2},$$

(3.47)

or

$$\left\| \frac{\partial u_0 (x)}{\partial y} \right\| \leq BN^{11/2}.$$

(3.48)

with

$$B = D^{2^b} (\pi \| \rho \|_{\infty})^2.$$

(3.49)

Remark 16. The result above is not optimal; in fact it can be shown, using a different argument, that (3.43) can be improved to be linear in $N$. This will be shown elsewhere.

Corollary 17. For the one-dimensional Anderson model on a box $\Lambda$ of size $N$, and $0 < s \leq 1$,

$$\| \psi (x, t) \| \leq A_s N^{(11s+3)/2} e^{-\mu (1-x)/|x|/9}$$

(3.50)

and

$$\left\| \frac{\partial u_0 (x)}{\partial y} \right\| \leq B_s N^{5s+1} e^{-2\mu (1-x)|x|/5} e^{-2\mu (1-x)y/5}.$$

(3.51)

Proof. Since $f (s) \equiv \| \cdot \|_s$ is a holomorphic and bounded function for $0 < s \leq 1$, we utilize Hadamard three-line interpolation theorem. For the first-order correction to the wavefunction using theorem 11 for $s = 2/11$ and theorem 14 gives

$$\| \psi (x, t) \|^2 \leq (C_2/11) N^{95/44} e^{-2\mu (1-x)|x|/11^6} \left( AN^{1/2} \right)^{1-s},$$

(3.52)

with

$$\theta = \frac{11}{9} (1 - s),$$

(3.53)

leading to

$$\| \psi (x, t) \|^2 \leq A_s N^{(191s+43)/36} e^{-2\mu (1-x)|x|/9} \leq A_s N^{(11s+3)/2} e^{-2\mu (1-x)|x|/9}.$$

(3.54)
Similarly, for the derivative of the eigenfunction combining theorem 13 for \( s = 2/7 \) and theorem 15 gives
\[
\left\langle \left( \frac{\partial u_0(x)}{\partial \varepsilon_y} \right)^2 \right\rangle \leq \left( K_2/7 \right) N^2 e^{-2\mu(x-y)/7} e^{-2\mu|y|/7} \vartheta (BN^{11/2})^{1-\theta},
\]
with
\[
\theta = \frac{2}{3} (1-s).
\]
(3.55)

Or
\[
\left\langle \left( \frac{\partial u_0(x)}{\partial \varepsilon_y} \right)^2 \right\rangle \leq B_4 N^{(49s+6)/10} e^{-2\mu(1-s)|x-y|/5} e^{-2\mu(1-s)|y|/5}
\leq B_5 N^{5s+1} e^{-2\mu(1-s)|y|/5}.
\]
(3.56)

□

Remark 18. This collyory suggests exponential bounds for both the first-order correction of the wavefunction and the derivative of the eigenfunction for the whole range \( 0 < s < 1 \).

3.4. Time-dependent bound

In this subsection, we will eliminate the exponential dependence on the volume (see (3.39)) of the bound on the average first-order correction to the wavefunction, \( \psi^{(1)} \), by using the a priori bound
\[
\left| 1 - e^{(E_n - E_0)\varepsilon} \right| \leq t.
\]
(3.57)

Theorem 19. For the one-dimensional Anderson model on a box \( \Lambda \) of size \( N \), and \( \varepsilon_0(t) < \varepsilon < 1 \), such that \( \varepsilon_0(t) \to 0 \) as \( t \to \infty \), and \( t \leq 2^N \) with \( \vartheta \) given by (3.12),
\[
\left| \psi^{(1)}(x, t) \right| \leq K_4 N^{5/2(1-\varepsilon)} t^{(2\varepsilon)/3} \log_2 t e^{-\varepsilon|x|/2}.
\]
(3.58)

Proof. We start with
\[
\left| \psi^{(1)}(x, t) \right| \leq D^{1/p_1+1/p_2} (\pi \| \varrho \|_{\infty})^{2/p_1} N^{5/2+2/p_1}
\times \left( \sum_{m=0}^M 2^{1-\varepsilon} \right) \sum_{y} e^{-\mu(x-y)/p_2} e^{-\mu|y|/p_2}
\]
and set \( 1/p_2 = 1/2 - 2\varepsilon \), \( 1/p_3 = (1/2 - \varepsilon) \) and \( 1/p_1 = 3\varepsilon \) for \( 0 < \varepsilon < 1/3 \). If one is not interested in the \( t \) dependence of the bound, one calculates
\[
\left| \psi^{(1)}(x, t) \right| \leq 2D^{1-3\varepsilon} (\pi \| \varrho \|_{\infty})^{6\varepsilon} N^{5/2+6\varepsilon} \left( \sum_{m=0}^M 2^{(m+1)(1-3\varepsilon)} \right) e^{-\mu|x|(1/2-2\varepsilon)}.
\]
(3.60)

To obtain the time-dependent bound, we split the sum into two parts:
\[
S \equiv \sum_{m=0}^M 2^{1-3\varepsilon(m+1)} \leq \sum_{m=0}^{M_1} 2^{1-3\varepsilon(m+1)} + t \sum_{m=M_1+1}^M 2^{-3\varepsilon(m+1)}
\]
(3.61)

where we have used the fact that
\[
\left| 1 - e^{(E_n - E_0)\varepsilon} \right| \leq \min(t, 2^m),
\]
(3.62)
which defines \( M_1 = \log_2 t \). Therefore, for sufficiently large \( t \)

\[
S \leq 2t^{1-3\epsilon} \log_2 t + t \left( \frac{2^{-3\epsilon}M_1 + 1}{1 - 2^{-3\epsilon}} \right) \leq 2t^{1-3\epsilon} \left( \log_2 t + \frac{1}{1 - 2^{-3\epsilon}} \right) \leq 3t^{1-3\epsilon} \log_2 t \tag{3.63}
\]

and

\[
(|\psi^{(1)}(x, t)|) \leq K_\epsilon N^{5/2 + 6\epsilon} t^{1-3\epsilon} (\log_2 t) e^{-\mu |x|/(1 - 2\epsilon)},
\]

with

\[
K_\epsilon = 6D^{1-3\epsilon} (\pi \|\rho\|_\infty)^{6\epsilon} 1 - e^{-\epsilon},
\]

and \( t \leq 2bN \).

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\section*{Appendix A. Relation between \( \varphi(N+1) \) and the normalized amplitude}

\textbf{Definition 20.} For \( x \in [1, ..N+1] \) and \( u(x-1, E) \neq 0 \), we define \( z(x, E) := \frac{u(x)}{u(x-1)} \).

\textbf{Corollary 21.} For \( \varphi(x, E) \neq \pi/2 + k\pi \) by definitions 2 and 20, \( z(x, E) = \tan(\varphi(x, E)) \). Furthermore, for fixed \( 2 \leq x \leq N+1 \) there are at most \( N \) singularities with \( \varphi(x, E) = \pi/2 + k\pi \), where \( z(x, E) \) is undefined. Hence, singularities are not dense, and if \( z(x, E) \) is well defined at \( E = E_0 \), then it is also well defined in a small neighborhood of \( E_0 \).

\textbf{Lemma 22.} Let \( E_0 \) be an eigenvalue of \( H \) with corresponding normalized eigenvector \( \{v(x)\}_{x=1}^{N} \); then \( v(N)^2 = \left( \frac{\partial z(N+1, E)}{\partial E} \bigg|_{E=E_0} \right)^{-1} \).

\textbf{Proof.} Since \( v(N, E_0) \neq 0 \), \( z(N+1, E) \) is well defined in a small neighborhood of \( E = E_0 \). Hence, by differentiability of \( \varphi(N+1, E) \), \( \frac{\partial z(N+1, E)}{\partial E} \bigg|_{E=E_0} \) exists. By (2.4) and (2.2), the following holds:

\[
0 = z(N+1, E_0) = E_0 - \varepsilon_x - \frac{1}{z(N, E_0)}. \tag{A.1}
\]

(For Neumann b.c. \( z(N+1, E_0) = 1 \), the rest of the argument still applies.)

Differentiating the LHS of (A.1) w.r.t. \( E \) and \( \varepsilon_x \), one obtains

\[
\frac{\partial z(N+1)}{\partial E} = \frac{\partial^2 z(N+1)}{\partial E \partial \varepsilon_x} \bigg|_{E=E_0} = 0. \tag{A.2}
\]

Differentiating the RHS of (A.1), one obtains \( \frac{\partial z(N+1)}{\partial \varepsilon_x} = -1 \). Observing that under the constraint \( z(N+1, E) = 0 \) (as in (A.1)) \( E \) is a function of \( \varepsilon_x \) and using \( \frac{\partial^2 z(N+1)}{\partial E \partial \varepsilon_x} \bigg|_{E=E_0} = v_N^2 \) (resulting from the Feynman–Hellman theorem) one obtains

\[
\frac{dz(N+1, E)}{dE} \bigg|_{E=E_0} = \frac{d\varepsilon_x}{dE} \bigg|_{E=E_0} = v_N^2. \tag{A.3}
\]

\textbf{Corollary 23.} For a random potential with Anderson localization, the derivative \( \varphi'(x) \) typically takes exponentially large (in \( N \)) values.
Appendix B. Expression for $\eta(W)$

The minimum of $q$ (as in (2.16)) can be either at local extremum point or at the boundaries of $A$ (defined by (2.17)); we will check both cases:

1. Local extrema in inner points: vanishing of the gradient $(\frac{\partial q}{\partial h}, \frac{\partial q}{\partial \phi}) = (0, 0)$ implies

$$ \sin \phi(x) \cos \phi(x) = h \sin^2 \phi(x) \quad \Longrightarrow \quad \cos \phi(x) = h \sin \phi(x) \quad (\phi(x) \neq 0, \pi) $$

$$ \cos 2\phi(x) = h^2 \sin 2\phi(x) \quad \Longrightarrow \quad 2 \cos 2\phi(x) = h \sin 2\phi(x) \quad (\phi(x) \neq 0, \pi). $$

(B.1)

The case $h = 0$ is not interesting: if indeed extrema is obtained for $h = 0$, then $q(h = 0, \phi(x)) \equiv 1$. Elsewhere (B.1) implies $\sin^2 \phi = 0 (\phi = k\pi)$; consequently, it is not an inner point of $A$, that is no local extrema in $A$ with $q(h, \phi(x)) \neq 1$.

2. Boundaries of $A$:

$$ \phi(x) = 0, \pi \quad \Rightarrow \quad q = 1 $$

(B.2)

$$ h = W + 2 \quad \Rightarrow \quad \phi(x)^{\text{extr}} = \frac{1}{2} \arctan \frac{2}{(W + 2)} \left( \frac{\pi}{2} \right) $$

(B.3)

$$ h = -(W + 2) \quad \Rightarrow \quad \phi(x)^{\text{extr}} = -\frac{1}{2} \arctan \frac{2}{(W + 2)} \left( \frac{\pi}{2} \right) $$

(B.4)

(with superscript ‘extr’ standing for an extremum value on the appropriate boundary).

Substituting (B.3) and (B.4) into (2.16) gives the following four options:

$$ q^{\text{extr}} = \left\{ \begin{array}{ll}
1 - (W + 2) \sin \left( \arctan \frac{2}{W + 2} \right) + (W + 2)^2 \sin^2 \left( \frac{1}{2} \arctan \frac{2}{W + 2} \right) \\
1 - (W + 2) \sin \left( \arctan \frac{2}{W + 2} \right) + (W + 2)^2 \cos^2 \left( \frac{1}{2} \arctan \frac{2}{W + 2} \right) \\
1 - (W + 2) \sin \left( \arctan \frac{2}{W + 2} \right) + (W + 2)^2 \sin^2 \left( \frac{1}{2} \arctan \frac{2}{W + 2} \right) \\
1 - (W + 2) \sin \left( \arctan \frac{2}{W + 2} \right) + (W + 2)^2 \cos^2 \left( \frac{1}{2} \arctan \frac{2}{W + 2} \right). 
\end{array} \right. $$

(B.5)

By the arguments that lead to (2.19), all the entries of (B.5) are positive. The minimum of $q$ is hence

$$ \eta(W) = \min \left\{ q \right\} $$

$$ = 1 - (W + 2) \sin \left( \arctan \frac{2}{W + 2} \right) + (W + 2)^2 \min \left\{ \sin^2 \left( \frac{1}{2} \arctan \frac{2}{W + 2} \right), \cos^2 \left( \frac{1}{2} \arctan \frac{2}{W + 2} \right) \right\}. $$

(B.6)

Using (2.18) combined with (B.3) and (B.4), one can obtain a simpler, yet weaker bound on $\eta$:

$$ \eta(W) \geq \min \left\{ \sin^2 \left( \frac{1}{2} \arctan \frac{2}{W + 2} \right), \cos^2 \left( \frac{1}{2} \arctan \frac{2}{W + 2} \right) \right\}. $$

(B.7)
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