1++ Nonet Singlet-Octet Mixing Angle, Strange Quark Mass, and Strange Quark Condensate

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Abstract

Two strategies are taken into account to determine the $f_1(1420)$-$f_1(1285)$ mixing angle $\theta$. (i) First, using the Gell-Mann-Okubo mass formula together with the $K_1(1270)$-$K_1(1400)$ mixing angle $\theta_{K_1} = (-34 \pm 13)^\circ$ extracted from the data for $B(B \to K_1(1270)\gamma), B(B \to K_1(1400)\gamma), B(\tau \to K_1(1270)\nu_{\tau})$, and $B(\tau \to K_1(1420)\nu_{\tau})$, gave $\theta = (23^{+17}_{-23})^\circ$. (ii) Second, from the study of the ratio for $f_1(1285) \to \phi\gamma$ and $f_1(1285) \to \rho^0\gamma$ branching fractions, we have a two-fold solution $\theta = (19.4^{+4.5}_{-4.6})^\circ$ or $(51.1^{+4.5}_{-4.6})^\circ$. Combining these two analyses, we thus obtain $\theta = (19.4^{+4.5}_{-4.6})^\circ$. We further compute the strange quark mass and strange quark condensate from the analysis of the $f_1(1420)$-$f_1(1285)$ mass difference QCD sum rule, where the operator-product-expansion series is up to dimension six and to $O(\alpha_s^2, m_s^2 \alpha_s^2)$ accuracy. Using the average of the recent lattice results and the $\theta$ value that we have obtained as inputs, we get $\langle \bar{s}s \rangle/\langle \bar{u}u \rangle = 0.41 \pm 0.09$. 

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I. INTRODUCTION

The $f_1(1285)$ and $f_1(1420)$ mesons with quantum number $J^{PC} = 1^{++}$ are the members of the $1^3P_1$ states in the quark model language, and are mixtures of the pure octet $f_8$ and singlet $f_1$, where the mixing is characterized by the mixing angle $\theta$. The BaBar results for the upper bounds of $B^- \to f_1(1285)K^-$, $f_1(1420)K^-$ were available recently [1]. The relative ratio of these two modes is highly sensitive to $\theta$ [2]. On the other hand, in the two-body $B$ decay involving the $K$ meson in the final state, the amplitude receives large corrections from the chiral enhancement $a_6$ term which is inversely proportional to the strange-quark mass. The quark mass term mixes left- and right-handed quarks in the QCD Lagrangian. The spontaneous breaking of chiral symmetry from $SU(3)_L \times SU(3)_R$ to $SU(3)_V$ is further broken by the quark masses $m_{u,d,s}$ when the baryon number is added to the three commuting conserved quantities $Q_u, Q_d,$ and $Q_s$, respectively, the numbers of $q-\bar{q}$ quarks for $q = u, d,$ and $s$. The nonzero quark condensate which signals dynamical symmetry breaking is the important parameter in QCD sum rules [3], while the magnitude of the strange quark mass can result in the flavor symmetry breaking in the quark condensate. In an earlier study $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle \sim 0.8 < 1$ was usually taken. However, very recently the Jamin-Lange approach [4] together with the lattice result for $f_B/f_B$ [5] and also the Schwinger-Dyson equation approach [6] can give a central value larger than 1.

In this paper, we shall embark on the study of the $f_1(1420)$ and $f_1(1285)$ mesons to determine the mixing angle $\theta$, strange quark mass, and strange quark condensate. In Sec. II we shall present detailed discussions on the determination of the mixing angle $\theta$. Substituting the $K_1(1270)$-$K_1(1400)$ mixing angle, which was extracted from the $B \to K_1\gamma$ and $\tau \to K_1\nu_{\tau}$ data, to the Gell-Mann-Okubo mass formula, we can derive the value of $\theta$. Alternatively, from the analysis of the decay ratio for $f_1(1285) \to \phi\gamma$ and $f_1(1285) \to \rho^0\gamma$, we have a more accurate estimation for $\theta$. In Sec. III we shall obtain the mass difference QCD sum rules for the $f_1(1420)$ and $f_1(1285)$ to determine the magnitude of the strange quark mass. From the sum rule analysis, we obtain the constraint ranges for $m_s$ and $\theta$ as well as for $\langle \bar{s}s \rangle$. Many attempts have been made to compute $m_s$ using QCD sum rules and finite energy sum rules [7–13]. The running strange quark mass in the $\overline{MS}$ scheme at a scale of $\mu \approx 2$ GeV is $m_s = 101^{+29}_{-21}$ MeV given in the particle data group (PDG) average [14]. More precise lattice estimates have been recently obtained as $m_s(2\text{GeV}) = 92.2(1.3)$ MeV in [15], $m_s(2\text{GeV}) = 96.2(2.7)$ MeV in [16], and $m_s(2\text{GeV}) = 95.1(1.1)(1.5)$ MeV in [17]. These lattice results agree with strange scalar/pseudoscalar sum rule results which are $m_s \simeq 95(15)$ MeV. In the present study, we study the $m_s$ from a new frame, the $f_1(1420)$-$f_1(1285)$ mass difference sum rule, which may result in larger uncertainties due to the input parameters. Nevertheless, it can be a crosscheck compared with the previous studies. Further using the very recent lattice result for $m_s(2\text{GeV}) = 93.6 \pm 1.0$ MeV as the input, we obtain an estimate for the strange quark condensate.
II. SINGLET-OCTET MIXING ANGLE \( \theta \) OF THE \( 1^{++} \) NONET

A. Definition

In the quark model, \( a_1(1260) \), \( f_1(1285) \), \( f_1(1420) \), and \( K_{1A} \) are classified in \( 1^{++} \) multiplets, which, in terms of spectroscopic notation \( n^{2S+1}L_J \), are \( \Xi_1^0 \) \( p \)-wave mesons. Analogous to \( \eta \) and \( \eta' \), because of SU(3) breaking effects, \( f_1(1285) \) and \( f_1(1420) \) are the mixing states of the pure octet \( f_8 \) and singlet \( f_1 \),

\[
|f_1(1285)\rangle = |f_1\rangle \cos \theta + |f_8\rangle \sin \theta, \quad |f_1(1420)\rangle = -|f_1\rangle \sin \theta + |f_8\rangle \cos \theta. \tag{1}
\]

In the present paper, we adopt

\[
f_1 = \frac{1}{\sqrt{3}}(\bar{u}u + \bar{d}d + \bar{s}s), \tag{2}
\]

\[
f_8 = \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s), \tag{3}
\]

where there is a relative sign difference between the \( \bar{s}s \) contents of \( f_1 \) and \( f_8 \) in our convention. From the Gell-Mann-Okubo mass formula, the mixing angle \( \theta \) satisfies

\[
\cos^2 \theta = \frac{3m_{f_1(1285)}^2 - (4m_{K_{1A}}^2 - m_{a_1}^2)}{3(m_{f_1(1285)}^2 - m_{f_1(1420)}^2)}, \tag{4}
\]

where

\[
m_{K_{1A}}^2 = \langle K_{1A}|H|K_{1A}\rangle = m_{K_{1A}(1400)}^2 \cos^2 \theta_K + m_{K_{1A}(1270)}^2 \sin^2 \theta_K, \tag{5}
\]

with \( H \) being the Hamiltonian. Here \( \theta_K \) is the \( K_{1A}(1400) - K_{1A}(1270) \) mixing angle. The sign of the mixing angle \( \theta \) can be determined from the mass relation [14]

\[
\tan \theta = \frac{4m_{K_{1A}}^2 - m_{a_1}^2 - 3m_{f_1(1420)}^2}{3m_{f_1(1420)}^2}, \tag{6}
\]

where \( m_{f_1(1420)}^2 = \langle f_1|H|f_8\rangle \simeq (m_{a_1}^2 - m_{K_{1A}}^2)2\sqrt{2}/3 < 0 \), we find \( \theta > 0 \). Because of the strange and nonstrange light quark mass differences, \( K_{1A} \) is not the mass eigenstate and it can mix with \( K_{1B} \), which is one of the members in the \( 1^1P_1 \) multiplets. From the convention in [18] (see also discussions in [19, 20]), we write the two physical states \( K_{1A}(1270) \) and \( K_{1A}(1400) \) in the following relations:

\[
|K_{1A}(1270)\rangle = |K_{1A}\rangle \sin \theta_K + |K_{1B}\rangle \cos \theta_K, \quad |K_{1A}(1400)\rangle = |K_{1A}\rangle \cos \theta_K - |K_{1B}\rangle \sin \theta_K. \tag{7}
\]

The mixing angle was found to be \( |\theta_{K_{1A}}| \approx 33^\circ, 57^\circ \) in [18] and \( \approx \pm 37^\circ, \pm 58^\circ \) in [21]. A similar range \( 35^\circ \lesssim |\theta_{K_{1A}}| \lesssim 55^\circ \) was obtained in [22]. The sign ambiguity for \( \theta_{K_{1A}} \) is due to the fact that one can add arbitrary phases to \( |K_{1A}\rangle \) and \( |K_{1B}\rangle \). This sign ambiguity can be removed by fixing the signs of decay constants \( f_{K_{1A}} \) and \( f_{K_{1B}} \), which are defined by

\[
\langle 0|\bar{\psi}_g\gamma_\mu\gamma_5 s|K_{1A}(P, \lambda)\rangle = -if_{K_{1A}}m_{K_{1A}}\epsilon_\lambda(\lambda), \tag{8}
\]
\[ \langle 0 | \bar{\psi} \sigma_{\mu \nu} s | K_{1B}(P, \lambda) \rangle = i f_{K_{1B}}^1 \epsilon_{\mu \nu \alpha \beta}^{\alpha} \epsilon^{\beta}_{(\lambda)} P^\lambda, \]  
(9)

where \( \epsilon^{123} = -1 \) and \( \psi \equiv u \) or \( d \). Following the convention in [20], we adopt \( f_{K_{1A}} > 0 \), \( f_{K_{1B}}^1 > 0 \), so that \( \theta_{K_1} \) should be negative to account for the observable \( B(B \to K_1(1270)\gamma) \gg B(B \to K_1(1400)\gamma) \) [23, 24]. Furthermore, from the data of \( \tau \to K_1(1270)\nu_\tau \) and \( K_1(1400)\nu_\tau \) decays together with the sum rule results for the \( K_{1A} \) and \( K_{1B} \) decay constants, the mixing angle \( \theta_{K_1} = (\mp 34 \pm 13)° \) was obtained in [24]. Substituting this value into (9), we then obtain \( \theta^{\text{quad}} = (23_{-23}^{+17})° \) [25], i.e., \( \theta^{\text{quad}} = 0° \sim 40° \) [26].

B. The determination of \( \theta \)

Experimentally, since \( K^* \bar{K} \) and \( K \bar{K} \pi \) are the dominant modes of \( f_1(1420) \), whereas \( f_0(1285) \) decays mainly to the 4r states, this suggests that the quark content is primarily \( s\bar{s} \) for \( f_1(1420) \) and \( n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2} \) for \( f_1(1285) \). Therefore, the mixing relations can be rewritten to exhibit the \( n\bar{n} \) and \( s\bar{s} \) components which decouple for the ideal mixing angle \( \theta_i = \tan^{-1}(1/\sqrt{2}) \approx 35.3° \). Let \( \bar{\alpha} = \theta_i - \theta \), we rewrite these two states in the flavor basis [2]

\[
\begin{align*}
\alpha (1285) &= \frac{1}{\sqrt{2}} (\bar{u} u + \bar{d} d) \cos \bar{\alpha} + \bar{s} s \sin \bar{\alpha}, \\
\alpha (1420) &= \frac{1}{\sqrt{2}} (\bar{u} u + \bar{d} d) \sin \bar{\alpha} - s \bar{s} \cos \bar{\alpha}.
\end{align*}
\]

(10)

Since the \( f_1(1285) \) can decay into \( \phi \gamma \), we know that \( f_1(1285) \) has the \( s\bar{s} \) content and \( \theta \) deviates from its ideal mixing value. To have a more precise estimate for \( \theta \), we study the ratio of \( f_1(1285) \to \phi \gamma \) and \( f_1(1285) \to \rho^0 \gamma \) branching fractions. Because the electromagnetic (EM) interaction Lagrangian is given by

\[
\mathcal{L}_I = -A_{\text{EM}}^\mu (e_u \bar{u} \gamma_\mu u + e_d \bar{d} \gamma_\mu d + e_s \bar{s} \gamma_\mu s)
\]

\[
= -A_{\text{EM}}^\mu \left( (e_u + e_d) \bar{u} \gamma_\mu u + \bar{d} \gamma_\mu d + (e_u - e_d) \bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d \right) + e_s \bar{s} \gamma_\mu s,
\]

(11)

with \( e_u = 2/3e, e_d = -1/3e \), and \( e_s = -1/3e \) being the electric charges of \( u, d \), and \( s \) quarks, respectively, we obtain

\[
\frac{B(f_1(1285) \to \phi \gamma)}{B(f_1(1285) \to \rho^0 \gamma)} = \left( \frac{\langle \phi | e_s s \gamma_\mu s | f_1(1285) \rangle}{\langle \rho | (e_u - e_d) (\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d) / 2 | f_1(1285) \rangle} \right)^2 \left( \frac{m_{\rho}^2 - m_{\phi}^2}{m_{f_1}^2 - m_{\rho}^2} \right)^3 \text{ phase factor}
\]

\[
= \left( \frac{-e/3}{2e/3 + e/3} \right)^2 \left( \frac{\langle \phi | s \gamma_\mu s | f_1(1285) \rangle}{\langle \rho | (\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d) / 2 | f_1(1285) \rangle} \right)^2 \left( \frac{m_{f_1}^2 - m_{\rho}^2}{m_{f_1}^2 - m_{\phi}^2} \right)^3 \text{ phase factor}
\]

1 Replacing the meson mass squared \( m^2 \) by \( m \) throughout [20], we obtain \( \theta^{\text{lin}} = (23_{-23}^{+17})° \). The difference is negligible. Our result can be compared with that using \( \theta_{K_1} = -57° \) into [24], one has \( \theta^{\text{quad}} = 52° \).

2 In PDG [14], the mixing angle is defined as \( \alpha = \theta - \theta_i + \pi/2 \). Comparing it with our definition, we have \( \alpha = \pi/2 - \bar{\alpha} \).
where \( f_1 \equiv f_1(1285) \), and \( f_\phi \) and \( f_\rho \) are the decay constants of \( \phi \) and \( \rho \), respectively. Here we have taken the single-pole approximation\(^3\):

\[
\langle \phi | \bar{s} \gamma_\mu s | f_1(1285) \rangle \approx \frac{m_\phi f_\phi g_{f_1 \phi}}{m_\rho f_\rho g_{f_1 \rho}} \frac{\sin \bar{\alpha}}{\sqrt{2} \cos \bar{\alpha}} \frac{1}{\sqrt{2}} \approx \frac{m_\phi f_\phi \times 2 \tan \bar{\alpha}}{m_\rho f_\rho}.
\]

Using \( f_\rho = 209 \pm 1 \) MeV, \( f_\phi = 221 \pm 3 \) MeV\(^27\), and the current data \( \mathcal{B}(f_1(1285) \to \phi \gamma) = (7.4 \pm 2.6) \times 10^{-4} \) and \( \mathcal{B}(f_1(1285) \to \rho^0 \gamma) = (5.5 \pm 1.3)\% \)\(^14\) as inputs, we obtain \( \bar{\alpha} = (15.8^{+4.5}_{-4.0})^\circ \), i.e., two fold solution \( \theta = (19.4^{+4.5}_{-4.0})^\circ \) or \( (51.1^{+4.5}_{-4.0})^\circ \). Combining with the analysis \( \theta = (0 \sim 40)^\circ \) given in Sec. II A, we thus find that \( \theta = (19.4^{+4.5}_{-4.0})^\circ \) is much preferred and can explain experimental observables well.

### III. MASS OF THE STRANGE QUARK

We proceed to evaluate the strange quark mass from the mass difference sum rules of the \( f_1(1285) \) and \( f_1(1420) \) mesons. We consider the following two-point correlation functions,

\[
\Pi_{\mu\nu}(q^2) = i \int d^4xe^{iqx} \langle 0|T(\bar{j}_\mu(x)j_\nu^\dagger(0))|0\rangle = -\Pi_1(q^2)g_{\mu\nu} + \Pi_2(q^2)q_\mu q_\nu, \quad (14)
\]

\[
\Pi'_{\mu\nu}(q^2) = i \int d^4xe^{iqx} \langle 0|T(\bar{j}'_\mu(x)j'^\dagger_\nu(0))|0\rangle = -\Pi'_1(q^2)g_{\mu\nu} + \Pi'_2(q^2)q_\mu q_\nu. \quad (15)
\]

The interpolating currents satisfying the relations:

\[
\langle 0|j^{(5)}_{\mu}(0)|j^{(5)}_{1}(P, \lambda)\rangle = -if_{j^{(5)}_{1}}m_{j^{(5)}_{1}} \epsilon^{(5)}_{\mu}, \quad (16)
\]

are

\[
j_\mu = \cos \theta j^{(1)}_{\mu} + \sin \theta j^{(8)}_{\mu}, \quad (17)
\]

\[
j'_\mu = -\sin \theta j^{(1)}_{\mu} + \cos \theta j^{(8)}_{\mu}, \quad (18)
\]

where

\[
j^{(1)}_{\mu} = \frac{1}{\sqrt{3}}(\bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d + \bar{s} \gamma_\mu \gamma_5 s), \quad (19)
\]

\[
j^{(8)}_{\mu} = \frac{1}{\sqrt{6}}(\bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d - 2\bar{s} \gamma_\mu \gamma_5 s), \quad (20)
\]

\(^3\) The following approximation was used in\(^26\):

\[
\frac{\langle \phi | \bar{s} \gamma_\mu s | f_1(1285) \rangle}{\langle 0|\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d|2|f_1(1285)\rangle} \approx 2 \tan \bar{\alpha}.
\]
and we have used the short-hand notations for $f_1 \equiv f_1(1285)$ and $f_1' \equiv f_1(1420)$. In the massless quark limit, we have $\Pi_1 = q^2 \Pi_2$ and $\Pi_1' = q^2 \Pi_2'$ if one neglects the axial-vector anomaly\(^4\).

Here we focus on $\Pi_1^{(*)}$ since it receives contributions only from axial-vector ($^3P_1$) mesons, whereas $\Pi_2^{(*)}$ contains effects from pseudoscalar mesons. The lowest-lying $f_1^{(*)}$ meson contribution can be approximated via the dispersion relation as

$$
\frac{m^2_{f_1^{(*)}} f^2_{f_1^{(*)}}}{m^2_{f_1^{(*)}} - q^2} = \frac{1}{\pi} \int_{s_0}^{s_0'} ds \frac{\text{Im}\Pi_1^{(*)\text{OPE}}(s)}{s - q^2},
$$

where $\Pi_1^{(*)\text{OPE}}$ is the QCD operator-product-expansion (OPE) result of $\Pi_1^{(*)}$ at the quark-gluon level\(^2\), and $s_0^{(*)}$ is the threshold of the higher resonant states. Note that the subtraction terms on the right-hand side of (21), which are polynomials in $q^2$, are neglected since they have no contributions after performing the Borel transformation. The four-quark condensates are expressed as

$$
\langle 0| \bar{q} \Gamma^a \gamma_q q \bar{q} \Gamma^a \gamma_q q | 0 \rangle = -a_2 \frac{1}{16 \Lambda^2} \text{Tr}(\Gamma^a \gamma_q) \text{Tr}(\gamma_s \lambda^a) \langle \bar{q} q \rangle^2,
$$

where $a_2 = 1$ corresponds to the vacuum saturation approximation. In the present work, we have $\Gamma = \gamma_{\mu}$ and $\gamma_{\mu} \gamma_5$, for which we allow the variation $a_2 = -2.9 \sim 3.1$\(^6\),\(^2\),\(^8\),\(^9\). For $\Pi_1^{(*)\text{OPE}}$, we take into account the terms with dimension $\leq 6$, where the term with dimension=0 ($D=0$) is up to $\mathcal{O}(\alpha_s^3)$, with $D=2$ (which is proportional to $m_q^2$) up to $\mathcal{O}(\alpha_s^2)$ and with $D=4$ up to $\mathcal{O}(\alpha_s^2)$. Note that such radiative corrections for terms can read from \([30, 32]\). We do not include the radiative correction to the $D=6$ terms since all the uncertainties can be lumped into $a_2$. Note that such radiative corrections for terms with dimensions=0 and 4 are the same as the vector meson case and can read from \([30, 31]\).

Further applying the Borel (inverse-Laplace) transformation,

$$
\mathcal{B}[f(q^2)] = \lim_{\lambda \to \infty} \int_{-q^2/n^2=M^2\text{fixed}}^{\lambda} \frac{1}{n!} (-q^2)^{n+1} \left[ \frac{d}{dq^2} \right]^n f(q^2),
$$

(23)

to both sides of (21) to improve the convergence of the OPE series and further suppress the contributions from higher resonances, the sum rules thus read

$$
f_1^2 m_{f_1}^2 e^{-m_{f_1}^2/M^2} = \int_{s_0}^{s_0'} ds \frac{e^{-s/M^2}}{4\pi^2} \left[ 1 + \frac{\alpha_s(\sqrt{s})}{\pi} + \frac{F_3 \alpha_s^2(\sqrt{s})}{\pi^2} + \left( F_4 + F_4' \cos^2 \theta \right) \frac{\alpha_s^3(\sqrt{s})}{\pi^3} \right]
$$

$$
-(\cos \theta - \sqrt{2} \sin \theta)^2 [m_s(\mu_0)]^2 \int_{s_0}^{s_0'} ds \frac{1}{2\pi^2} e^{-s/M^2} \left[ 1 + \left( H_1 \ln \frac{s}{\mu_0^2} + H_2 \right) \frac{\alpha_s(\mu_0)}{\pi} \right]
$$

\(^4\) Considering the anomaly, the singlet axial-vector current is satisfied with

$$
\partial^\mu j_\mu^{(*)} = \frac{1}{\sqrt{3}} (m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s) + \frac{3\alpha_s}{4\pi} G \tilde{G}.
$$
where

\[ f_{f_1'}^2 m^2_{f_1'} e^{-m_{f_1'}^2/M^2} \]

\[ = \int_0^{s_{f_1'}} ds \frac{e^{-s/M^2}}{4\pi^2} \left[ 1 + \frac{\alpha_s(\sqrt{s})}{\pi} + F_3 \frac{\alpha_s^2(\sqrt{s})}{\pi^2} + (F_4 + F'_4 \sin^2 \theta) \frac{\alpha_s^3(\sqrt{s})}{\pi^3} \right] \]

\[ + (\sin \theta + \sqrt{2} \cos \theta)^2 |\mathbf{m}_s(\mu_0)|^2 \int_0^{s_{f_1'}} ds \frac{1}{2\pi^2} e^{-s/M^2} \left[ 1 + \left( H_1 \ln \frac{s}{\mu_0^2} + H_2 \right) \frac{\alpha_s(\mu_0)}{\pi} \right] \]

\[ + \left( H_{3a} \ln^2 \frac{s}{\mu_0^2} + H_{3b} \ln \frac{m^2_{f_1'}}{\mu_0^2} + H_{3c} - \frac{H_{3a} \pi^2}{3} \right) \left( \frac{\alpha_s(\mu_0)}{\pi} \right)^2 \]

\[ - \frac{1}{12} \left( 1 - \frac{11}{18} \frac{\alpha_s(M)}{\pi} \right) \left( \frac{\alpha_s}{\pi} G^2 \right) \]

\[ - \left[ \frac{4}{27} \frac{\alpha_s(M)}{\pi} + \left( -\frac{257}{486} + \frac{6}{9} \zeta(3) - \frac{2}{27} \beta_1 \gamma_E \right) \frac{\alpha_s^2(M)}{\pi^2} \right] \sum_{q_i=\mu,d,s} (\mathbf{m}_i q_i q_i) \]

\[ + \frac{1}{3} (\sqrt{2} \cos \theta - \sin \theta)^2 \left[ 2 a_1 \overline{m}_q \langle \bar{q} q \rangle - \frac{352 \pi \alpha_s}{81 M^2} a_2 \langle \bar{q} q \rangle \right]^2 \]

\[ + \frac{1}{3} (\sin \theta + \sqrt{2} \cos \theta)^2 \left[ 2 a_1 \overline{m}_s \langle \bar{s} s \rangle - \frac{352 \pi \alpha_s}{81 M^2} a_2 \langle \bar{s} s \rangle \right]^2, \quad (24) \]

\[ \]

where

\[ F_3 = 1.9857 - 0.1153 n_f \simeq 1.6398 \quad \text{for} \ n_f = 3, \]

\[ F_4 = -6.6368 - 1.2001 n_f - 0.0052 n_f^2 \simeq -10.2839 \quad \text{for} \ n_f = 3, \]

\[ F'_4 = -1.2395 \Delta, \]

\[ H_1 = -\frac{8}{81} \beta_1^2 = -2, \quad H_2 = \frac{2}{9} \beta_2 + 4 \beta_2 \left( \frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2} \right) - \frac{8}{9} \beta_1^2 - 4 \beta_1 \simeq 3.6667, \]

\[ H_{3a} = 4.2499, \quad H_{3b} = -23.1667, \quad H_{3c} = 29.7624, \]

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\[
\overline{m}_q \langle \bar{q}q \rangle \equiv \frac{1}{2} (\overline{m}_u \langle \bar{u}u \rangle + \overline{m}_d \langle \bar{d}d \rangle), \quad \langle \bar{q}q \rangle^2 \equiv \frac{1}{2} (\langle \bar{u}u \rangle^2 + \langle \bar{d}d \rangle^2),
\]
\[
a_1 = 1 + \frac{7}{3} \frac{\alpha_s(M)}{\pi} + \left( \frac{85}{6} - \frac{7}{6} \beta_1 \gamma_E \right) \frac{\alpha_s^2(M)}{\pi^2},
\]
with \( \beta_1 = (2n_f - 33)/6, \beta_2 = (19n_f - 153)/12, \gamma_1 = 2, \gamma_2 = 101/12 - 5n_f/18, \) and \( n_f = 3 \) being the number of flavors and \( \Delta = 1 \), and \( 0 \) for \( f_1 \) (singlet) and \( f_8 \) (octet), respectively [32]. In the calculation the coupling constant \( \alpha_s(\sqrt{s}) \) in Eqs. (24) and (25) can be expanded in powers of \( \alpha_s(M) \):
\[
\frac{\alpha_s(\sqrt{s})}{\pi} = \frac{\alpha_s(M)}{\pi} + \frac{1}{2} \beta_1 \ln \frac{s}{M^2} \left( \frac{\alpha_s(M)}{\pi} \right)^2 + \left( \frac{1}{2} \beta_2 \ln \frac{s}{M^2} + \frac{1}{4} \beta_1^2 \ln^2 \frac{s}{M^2} \right) \left( \frac{\alpha_s(M)}{\pi} \right)^3
\]
\[
+ \left( \frac{\beta_3}{2} \ln \frac{s}{M^2} + \frac{5}{8} \beta_1 \beta_2 \ln^2 \frac{s}{M^2} + \frac{1}{8} \beta_1^3 \ln^3 \frac{s}{M^2} \right) \left( \frac{\alpha_s(M)}{\pi} \right)^4 + \cdots,
\]
where \( \beta_3 \approx -20.1198 \). Using the renormalization-group result for the \( m_s^2 \) term given in [31], we have expanded the contribution to the order \( O(\alpha_s^2 m_s^2) \) at the subtraction scale \( \mu^2 = 2 \text{ GeV}^2 \) for which the series has better convergence than at the scale 1 \text{ GeV}^2; however, the convergence of the series has no obvious change if using a higher reference scale. As in the case of flavor-breaking \( \tau \) decay, the \( D = 2 \) series converges slowly; nevertheless, we have checked that this term, which intends to make the output \( m_s \) to be smaller in the fit, is suppressed due to the fact that the mass sum rules for \( f_1(1285) \) and \( f_1(1420) \) are obtained by applying the differential operator \( M^4 \partial \ln / \partial M^2 \) to both sides of [24] and [25], respectively. Nevertheless, the differential operator will instead make the \( D = 4 \) term containing \( m_s \langle \bar{s}s \rangle \) become much more important than the \( m_s^2 \) term in determining the \( f_1(1285)-f_1(1420) \) mass difference although the they are the same order in magnitude.

In the numerical analysis, we shall use \( \Lambda_{\text{QCD}}^{(3)\text{NLO}} = 0.360 \) \text{ GeV}, corresponding to \( \alpha_s(1\text{ GeV}) = 0.495, \Lambda_{\text{QCD}}^{(4)\text{NLO}} = 0.313 \) \text{ GeV}, and the following values (at the scale \( \mu = 1 \) \text{ GeV}) [4, 28, 29, 33]:
\[
\langle \bar{q}q \rangle = (0.009 \pm 0.007) \text{ GeV}^4,
\langle \bar{m}_q \bar{q}q \rangle = -f_\pi^2 m_{\pi^+}/4,
\langle \bar{q}q \rangle^2 \simeq (-0.247) \text{ GeV}^6,
\langle \bar{s}s \rangle = (0.30 \sim 1.3) \langle \bar{q}q \rangle, 
\]
\[
a_2 = -2.9 \sim 3.1,
\]
where the value of \( \langle \bar{q}q \rangle^2 \) corresponds to \( (m_u + m_d)(1\text{ GeV}) \simeq 11 \) \text{ MeV}, and we have cast the uncertainty of \( \langle \bar{q}q \rangle^2 \) to \( a_2 \) in the \( D = 6 \) term. We do not consider the isospin breaking effect between \( \langle \bar{u}u \rangle \) and \( \langle \bar{d}d \rangle \) since \( \langle \bar{d}d \rangle/\langle \bar{u}u \rangle - 1 \approx -0.007 \) is negligible in the present analysis. The threshold is allowed by \( s_f^0 = 2.70 \pm 0.15 \) \text{ GeV}^2 and determined by the maximum stability of the mass sum rule. For an estimate on the threshold difference, we parametrize in the form
\[
\left( \sqrt{s_f^0} - \sqrt{s_f^1} \right)/\sqrt{s_f^1} = \delta \times (m_{f_1} - m_{f_1})/m_{f_1}, \text{ with } \delta = 1.0 \pm 0.3.
\]
In other words, we assign a 30% uncertainty to the default value. We search for the allowed solutions for strange quark mass and the singlet-octet mixing angle \( \theta \) under the following constraints: (i) Comparing with the observables, the errors for the mass sum rule results of the \( f_1(1285) \) and \( f_1(1420) \) in the Borel
TABLE I. The fitting results in the $f_1(1284)$-$f_1(1420)$ mass difference sum rules. In fit II, we have taken the average of the recent lattice results for $m_s$, which is rescaled to 1 GeV as the input.

|       | $m_s$(1 GeV) | $(\bar{s}s)/\langle \bar{u}u \rangle$ | $((\alpha_s/\pi)G^2)$ | $a_2$ |
|-------|--------------|----------------------------------------|------------------------|-------|
| Fit I | 106.3 ± 35.1 | 0.56 ± 0.25                            | 0.0106 ± 0.0042        | 0.89 ± 0.62 |
| Fit II| $[124.7 \pm 1.3]$ | 0.41 ± 0.09                            | 0.0108 ± 0.0037        | 0.95 ± 0.45 |

window 0.9 GeV$^2 \leq M^2 \leq 1.3$ GeV$^2$ are constrained to be less than 3% on average. In this Borel window, the contribution originating from higher resonances (and the continuum), modeled by

$$\frac{1}{\pi} \int_{s_0}^{\infty} ds \ e^{-s/M^2} \text{Im}\Pi^{(OPE)}(s),$$

is about less than 40% and the highest OPE term (with dimension six) at the quark level is no more than 10%. (ii) The deviation between the $f_1(1420)-f_1(1285)$ mass difference sum rule result and the central value of the data [14] is within 1σ error: $| (m_{f_1} - m_{f_1})_{\text{sum rule}} - 144.6 \text{ MeV} | \leq 1.5 \text{ MeV}$. The detailed results are shown in Table 1. We also check that if by further enlarging the uncertainties of $s_0$ and $\delta$, e.g. 25%, the changes of results can be negligible. We obtain the strange quark mass with large uncertainty: $m_s(1 \text{ GeV}) = 106.3 \pm 35.1 \text{ MeV}$ (i.e. $m_s(2 \text{ GeV}) = 89.5 \pm 29.5 \text{ MeV}$) and $\langle \bar{s}s/\langle \bar{u}u \rangle = 0.56 \pm 0.25$ corresponding to $\theta = (19.4^{+4.5}_{-4.6})^\circ$, where the values and $m_s$ and $\langle \bar{s}s \rangle$ are strongly correlated.

Further accounting for the average of the recent lattice results [15-17]: $m_s(2 \text{ GeV}) = 93.6 \pm 1.0 \text{ MeV}$ and using the $\theta$ value that we have obtained as the inputs, we get $\langle \bar{s}s/\langle \bar{u}u \rangle = 0.41 \pm 0.09$ which is less than one and in contrast to the Schwinger-Dyson equation approach in [2] where the ratio was obtained as $(1.0 \pm 0.2)\theta$. Our prediction is consistent with the QCD sum rule result of studying the scalar/pseudoscalar two-point function in [35] where the authors obtained $\langle \bar{s}s/\langle \bar{u}u \rangle = 0.4 \sim 0.7$, depending on the value of the strange quark mass.

IV. SUMMARY

We have adopted two different strategies for determining the mixing angle $\theta$: (i) Using the Gell-Mann-Okubo mass formula and the $K_1(1270)$-$K_1(1400)$ mixing angle $\theta_{K_1} = (-34 \pm 13)^\circ$ which was extracted from the data for $B(B \rightarrow K_1(1270)\gamma), B(B \rightarrow K_1(1400)\gamma), B(\tau \rightarrow K_1(1270)\nu_\tau)$, and $B(\tau \rightarrow K_1(1420)\nu_\tau)$, the result is $\theta = (23^{+17}_{-23})^\circ$. (ii) On the other hand, from the analysis of the ratio of $B(f_1(1285) \rightarrow \phi\gamma)$ and $B(f_1(1285) \rightarrow \rho^0\gamma)$, we have $\bar{\alpha} = \theta_t - \theta = \pm (15.8^{+4.5}_{-4.6})^\circ$, i.e., $\theta = (19.4^{+4.5}_{-4.6})^\circ$. Combining these two analyses, we deduce the mixing angle $\theta = (19.4^{+4.5}_{-4.6})^\circ$.

We have estimated the strange quark mass and strange quark condensate from the analysis of the $f_1(1420)$-$f_1(1285)$ mass difference QCD sum rule. We have expanded the OPE series up to dimension six, where the term with dimension zero is up to $O(\alpha_s^3)$, with dimension=2 up to $O(m_s^2\alpha_s^2)$ and with dimension=4 terms up to $O(\alpha_s^4)$. Further using the average of the recent lattice results and the $\theta$ value that we have obtained as the inputs, we get $\langle \bar{s}s/\langle \bar{u}u \rangle = 0.41 \pm 0.09$. 

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ACKNOWLEDGMENTS

This research was supported in part by the National Center for Theoretical Sciences and the National Science Council of R.O.C. under Grant No. NSC99-2112-M-003-005-MY3.

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