The Higher Derivative Expansion of the Effective Action by the String Inspired Method. Part II.

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Abstract

We apply the string-inspired worldline formalism to the calculation of the higher derivative expansion of one-loop effective actions in non-Abelian gauge theory. For this purpose, we have completely computerized the method, using the symbolic manipulation programs FORM, PERL and M. Explicit results are given to sixth order in the inverse mass expansion, reduced to a minimal basis of invariants specifically adapted to the method. Detailed comparisons are made with other gauge-invariant algorithms for calculating the same expansion. This includes an explicit check of all coefficients up to fifth order.

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1. Introduction

It is a well-known fact that one-loop amplitudes in quantum field theory can often be represented by worldline path integrals over the space of closed loops. For instance, the one-loop effective action induced by a spinor loop in an Abelian or non-Abelian background gauge field may be written in the following way [1–7]:

$$
\Gamma[A] = -2 \int_0^\infty \frac{dT}{T} e^{-m^2 T} \text{tr} \int Dx D\psi \mathcal{P} \exp \left[ - \int_0^T d\tau \left( \frac{1}{4} \dot{x}^2 + \frac{1}{2} \dot{\psi} \dot{\psi} + ig A_\mu \dot{x}^\mu - ig \psi^\mu F_{\mu\nu} \psi^\nu \right) \right]. \quad (1)
$$

Here the $x^\mu(\tau)$ are the periodic functions from the circle with fixed circumference $T$ (known as the Schwinger proper-time) into $d$-dimensional Euclidean spacetime, and the $\psi^\mu(\tau)$ their antiperiodic Grassmann valued (supersymmetric) partners. $\mathcal{P}$ denotes path ordering in the non-Abelian case.

This type of path integral representation has already proven useful for various types of calculations in quantum field theory [8, 9]. In particular, for the calculation of anomalies and index densities [10–14] superparticle path integrals have shown to be a remarkably powerful alternative to heat kernel methods [15].

Recently, renewed interest in these integral representations has been triggered by the work of Bern and Kosower. In 1992 these authors established a new set of rules for one-loop calculations by representing amplitudes in ordinary quantum field theory as the infinite string tension limits of certain (super) string amplitudes [16]. Those rules are equivalent to standard Feynman rules [17], but lead to a significant reduction in the number of terms to be computed both in one-loop gauge theory [18] and quantum gravity [19] calculations.

Strassler later showed [20] that, for many cases of interest, the same integral representations can be obtained by evaluating worldline path integrals of the type eq. (1) in analogy to the Polyakov path integral, i.e. using one-dimensional perturbation theory.

This reformulation turned out to be well-suited to the calculation of one-loop effective actions in general (21–26; see also [27]), and highly efficient for the calculation of their inverse mass expansions [28–32].

The inverse mass expansion (or, more generally, the higher derivative expansion) is a standard tool for the approximative calculation of one-loop effective actions, and considerable work has gone into the determination of its coefficients. It is applied in fields as different as chiral perturbation theory [33], high temperature physics and the theory of phase transitions. In the latter context, it has been applied both to bubble nucleation during the electroweak phase transition [34] and to baryon number violation by sphaleron processes [35].

Little seems to be known, however, about the high order behaviour of this expansion [36,37], and the question of its convergence. A recent all-order calculation of the higher derivative expansion for a specific example in three-dimensional quantum electrodynamics indicates that, at least in the case considered, this expansion is an asymptotic rather than a convergent series [38].

Due to the rapidly growing capacity of computers to handle large numbers of terms in symbolic calculations, in recent times there has been growing interest in the explicit form of higher order coefficients of the inverse mass expansion [39–45].

In the first paper of this series [28], we used the ‘string-inspired method’ to calculate this expansion to order $\mathcal{O}(T^7)$ in the proper-time parameter for the simplest case of an external scalar potential.
In the present paper, we consider the more general problem of constructing the inverse mass expansion for the case of both a background gauge field and a scalar potential. Moreover, we have completed the computerization of the method, using the algebraic manipulation programs FORM [46], PERL [47] and M [48]. This allows us to push the calculation to $O(T^{12})$ in the pure scalar case, and to $O(T^6)$ in the general case (the result for the scalar case has been presented to order $O(T^8)$ in [29]). With conventional methods, the inverse mass expansion has been obtained to order $O(T^5)$ in the general case [40], and only recently to order $O(T^7)$ in the scalar case [49].

At those high orders it is, of course, essential to represent the result in the most compact form possible. In the scalar potential case, our method turns out to have the very useful property of leading to \textit{minimal} bases of operators automatically, once cyclically equivalent terms have been identified. In the general case the result for the effective Lagrangian has to be further reduced using Bianchi identities and (possibly) transposition symmetry. Usually those operations would have to be combined with partial integrations in spacetime, however this turns out not to be necessary in the present scheme.

In Chapter 2, we will explain our method of calculating the inverse mass expansion [22], which is a pure $x$–space version of the one proposed by Strassler [20, 21], made manifestly gauge invariant by using Fock–Schwinger gauge. The results of this calculation will be presented in Chapter 3, reduced to a minimal basis of invariants specifically adapted to the algorithm. We explicitly present the effective Lagrangian for the pure gauge theory case, calculated to order $O(T^6)$ in the inverse mass expansion.

Chapter 4 contains a technical comparison with previous calculations of higher derivative expansions. A considerable number of different algorithms for this type of calculation can already be found in the literature [42,50–60], and we cannot possibly discuss all of them. We will therefore pay attention mainly to those methods which have already proven suitable for higher order calculations in gauge theory. Those are:

- The method developed by Onofri [54], Fujiwara et al. [55] and Zuk [56,57] (Section 4.1). This approach is also the one most closely related to our work, a fact which becomes particularly conspicuous in the path integral formulation of [55]. We will therefore spend some effort on a detailed comparison with that method.

- A modified version of the method proposed by Nepomechie [58] (Section 4.2). This technique is not manifestly gauge invariant as it stands and thus less convenient for the present purpose. Still we will present a modified version of it, which turns out sufficiently efficient for a explicit check of our results to $O(T^5)$ completely and of $O(T^6)$ partially.

- The method invented by ’t Hooft [53] and elaborated by van de Ven [40] (Section 4.3).

Our conclusions will be offered in Chapter 5, where we will also discuss further possible generalizations.

Appendix A deals with the rather technical problem of constructing minimal bases of invariants for a background consisting of a scalar field and/or a non-Abelian gauge field. In Appendix B we discuss the impact of different choices for the Green function used in the evaluation of the path integral.
2. The inverse mass expansion from the worldline path integral

First let us set up some terminology. We refer by the ‘higher derivative expansion’ of an effective action to an expansion both in the number of external fields and the number of derivatives acting on the fields. This expansion exists in several versions, which differ by the grouping of terms. The one we will consider here is the ‘inverse mass expansion’, which is usually obtained by writing the one-loop determinant in the Schwinger proper time representation

$$\Gamma[A, V] = -\log(\det M) = -\text{Tr}(\log M) = \int_0^\infty \frac{dT}{T} \text{Tr} e^{-TM}$$ (2)

and expanding in powers of the proper-time parameter $T$. This groups together terms of equal mass dimension. Up to partial integrations in space-time, it coincides with the (diagonal part of the) ‘heat-kernel expansion’ for the second order differential operator in question. In particular, every coefficient in this expansion is separately gauge invariant.

Alternatively, one may calculate the same series up to a fixed number of derivatives, but with an arbitrary number of fields or potentials [44,56,59]. Yet another option is to keep the number of external fields fixed, and sum up the derivatives to all orders. This leads to the notion of Barvinsky-Vilkovisky form factors [22, 61].

Note that in general the proper-time integral eq. (2) need not be convergent. It has to be regularized in some way, e.g. using a simple cut-off, $\zeta$-function or dimensional regularization. We will therefore work in $d$ dimensions from the beginning. Throughout this paper we will not perform the final $T$-integration, because we are interested in the explicit form of the coefficients of the inverse mass expansion only.

In the present paper we consider the case of massive scalars in the loop and a background consisting of both a gauge field and a (possibly matrix valued) scalar potential. This corresponds to the following choice of the fluctuation operator $M$:

$$M = -D^2 + m^2 + V(x),$$ (3)

with $D_\mu = \partial_\mu + igA_\mu$.

The case of a matrix valued scalar potential is by far more general than the pure scalar case. In particular it allows to treat particles with spin in the loop and can therefore be used to calculate fluctuation determinants, e.g. around the electroweak sphaleron [35,62].

In the case of a fluctuation operator (3) the one-loop effective action can be expressed in terms of the worldline path integral (see e.g. [7])

$$\Gamma[A, V] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \text{tr} \int Dx \ P \exp \left[ -\int_0^T d\tau \left( \frac{1}{4} \dot{x}^2 + igA_\mu \dot{x}^\mu + V(x) \right) \right].$$ (4)

The path integral will be path-ordered, except if both $A$ and $V$ are Abelian. The method applied generalizes to the spinor loop case without difficulty [22,24].

To obtain an effective Lagrangian from this path integral we split it into an ordinary integral over the center of mass, and a path integral over the relative coordinate:

$$\int Dx = \int d^d x_0 \int D\tau$$

$$x^\mu(\tau) = x_0^\mu + y^\mu(\tau)$$

$$\int_0^T \,d\tau \, y^\mu(\tau) = 0.$$ (7)
This leads to an expression for the effective action in terms of an effective Lagrangian $\mathcal{L}$,

$$\Gamma[A, V] = \int d^d x_0 \mathcal{L}(x_0) ,$$

(8)

where $\mathcal{L}(x_0)$ is represented as an integral over the space of all loops with a fixed common center of mass $x_0$.

To obtain the higher derivative expansion, we Taylor-expand both $A$ and $V$ around $x_0$, and use $\dot{x}^\mu = \dot{y}^\mu$ to rewrite

$$V(x) = e^{y\theta} V(x_0)$$

$$\dot{x}^\mu A_\mu(x) = \dot{y}^\mu e^{y\theta} A_\mu(x_0) .$$

(9)

The path-ordered interaction exponential is then expanded with the result

$$\Gamma[A, V] = \int_0^\infty dT e^{-m^2 T \text{tr} \int d^d x_0 \sum_{n=0}^\infty \frac{(-1)^n}{n} T \int \mathcal{D}y \exp \left[ -\int_0^T d\tau \frac{\dot{y}^2}{4} \right]$$

\quad \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n \prod_{j=1}^n \left[ e^{y(\tau_j) \partial_{(j)} V^{(j)}(x_0)} + ig \dot{y}^{\mu_j} (\tau_j) e^{y(\tau_j) \partial_{(j)} A_{\mu_j}(x_0)} \right] .$$

(10)

Here we have labeled the background fields, and the first $\tau$–integration has been eliminated by using the freedom of choosing the 0 somewhere on the loop. This is also the origin of the factor of $\frac{T}{n}$. Our normalization is such that for the free path integral

$$\int \mathcal{D}y \exp \left[ -\int_0^T d\tau \frac{\dot{y}^2}{4} \right] = \left[ 4\pi T \right]^{-\frac{d}{2}} .$$

(11)

Individual terms in this expansion are now generated by Wick contractions in the one-dimensional worldline theory at fixed $T$, using the Green function for the second derivative on the circle,

$$G(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} .$$

(12)

Elementary fields are thus contracted by

$$\langle y^{\mu}(\tau_1) y^{\nu}(\tau_2) \rangle = -g^{\mu\nu} G(\tau_1, \tau_2) ,$$

(13)

and exponentials of fields using formulas familiar from string theory,

$$\langle e^{y(\tau_1) \partial_{(1)} e^{y(\tau_2) \partial_{(2)}}} \rangle = e^{-G(\tau_1, \tau_2) \partial_{(1)} \partial_{(2)}}$$

$$\langle \dot{y}^{\mu}(\tau_1) e^{y(\tau_1) \partial_{(1)} e^{y(\tau_2) \partial_{(2)}}} \rangle = -\dot{G}(\tau_1, \tau_2) \partial_{(2)} e^{-G(\tau_1, \tau_2) \partial_{(1)} \partial_{(2)}}$$

(14)

e etc. (a dot always denotes a derivative with respect to the first variable). We will often abbreviate $G_{ij} := G(\tau_i, \tau_j)$ etc.

To make this procedure manifestly gauge-invariant, we now take the background gauge field to be in Fock-Schwinger gauge with respect to $x_0$, imposing the gauge condition...
\[ y^\mu A_\mu (x_0 + y(\tau)) \equiv 0. \] (15)

In this gauge,
\[ A_\mu (x_0 + y) = y^\rho \int_0^1 d\eta \eta F_{\rho\mu} (x_0 + \eta y), \] (16)
and \( F_{\rho\mu} \) and \( V \) can be covariantly Taylor-expanded as (see e.g. [63])
\[ F_{\rho\mu} (x_0 + \eta y) = e^{\eta y D} F_{\rho\mu} (x_0) \]
\[ V (x_0 + y) = e^{y D} V (x_0). \] (17)

This leads also to a covariant Taylor expansion for \( A \):
\[ A_\mu (x_0 + y) = \int_0^1 d\eta \eta y^\rho e^{\eta y D} F_{\rho\mu} (x_0) = \frac{1}{2} y^\rho F_{\rho\mu} + \frac{1}{3} y^\nu y^\rho D_\nu F_{\rho\mu} + ... \] (18)

Using these formulas, we obtain the following manifestly covariant version of eq. (11):
\[ \Gamma[F,V] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \text{tr} \int d^d x_0 \frac{(-1)^n}{n!} \int dy \exp \left[ - \int_0^T \frac{dy^2}{4} \right] \int_0^{\tau_n} d\tau_2 \ldots \int_0^{\tau_n} d\tau_n \]
\[ \prod_{j=1}^n \left[ e^{y(\tau_j) D(\tau_j)} V^{(j)} (x_0) + ig y^\rho_j (\tau_j) y^\rho_j (\tau_j) \int_0^1 d\eta \eta_j \eta_j e^{\eta_j y(\tau_j) D(\tau_j)} F_{\rho_j\mu_j}^{(j)} (x_0) \right]. \] (19)

From this master formula, the inverse mass expansion of the one-loop effective action to some fixed order \( N \),
\[ \Gamma[F,V] = \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{[4\pi T]^{d/2}} \text{tr} \int d^d x_0 \sum_{n=1}^N \frac{(-1)^n}{n!} O_n[F,V], \] (20)
is obtained in three steps:

1. Wick contractions: Truncate the master formula to \( n = N \), and the covariant Taylor expansion eq. (18) accordingly. Wick contract the integrand, which is now a polynomial.

2. Integrations: Perform the \( \tau \)-integrations. The integrand is a polynomial in the worldline Green function \( G \) and its first two derivatives,
\[ \hat{G}(\tau_1, \tau_2) = \text{sign}(\tau_1 - \tau_2) - 2 \frac{(\tau_1 - \tau_2)}{T} \]
\[ \hat{G}(\tau_1, \tau_2) = 2\delta(\tau_1 - \tau_2) - 2 \frac{T}{T}. \] (21)

It is useful to first rescale all \( \tau \)-integrals to the unit circle, \( \tau_i = T u_i \), using the scaling properties \( G(Tu) = TG(u), \hat{G}(Tu) = \hat{G}(u), \tilde{G}(Tu) = \frac{1}{T} \tilde{G}(u) \). The \( \delta \)-function in \( \hat{G}(u_i, u_j) \) only contributes if \( u_i \) and \( u_j \) are neighbouring points on the loop, which also includes the case \( \hat{G}(1, u_i) \). In the non-Abelian case the coefficient 2 in front of the \( \delta \)-function has to be omitted, since only half of the \( \delta \)-function contributes to the ordered sector under consideration.
3. Reduction to a minimal basis: The result of this procedure is the effective Lagrangian at the required order, albeit in redundant form. To be maximally useful for numerical applications, it still needs to be reduced to a minimal set of invariants, using all available symmetries. Those are

- Cyclic invariance under the trace.
- Bianchi identities.
- Antisymmetry of the field strength tensor.

Usually those symmetry operations would have to be preceded by judiciously chosen partial integrations performed on the effective action. It is a remarkable property of the present calculational scheme that the reduction of our result for the effective action to a minimal basis of invariants can be achieved without any partial integrations. In particular, for the pure scalar case the reduction process amounts to nothing more than the identification of cyclically equivalent terms. We will come back to this point in Chapter 4.1. In the general case, the reduction to a minimal basis of invariants is much more involved. The method adopted here follows a proposal by Müller [64] and is explained in Appendix A.

3. Computerization and explicit results

For a computation of higher coefficients in the inverse mass expansion starting from the master formula (19) one clearly has to computerize the three steps described in the Chapter 2.

The first step (expanding the interaction exponentials, truncating them to a given order and performing all possible Wick contractions) can be done very conveniently with FORM [46] for both the pure scalar and the general case. In the pure scalar case the contraction of exponentials eq. (14) is used, for it yields the more compact intermediate expressions. The second step (integrations) is also done with FORM in the pure scalar case, where the integrations are purely polynomial. The general (gauged) case involves δ-functions stemming from the contractions yielding a second derivative of the Green function. Once the corresponding integrations are done, the remaining integrand is again purely polynomial. In the pure scalar case the second step almost completes the computation. The remaining cyclic redundancy is fixed using a PERL [47] program. In the general case the reduction algorithm described in Appendix A is used. It consists of a set of nontrivial substitution rules and therefore requires a symbolic manipulation program, which contains rule based programming and flexible pattern matching. For this purpose we chose a new system for symbolic manipulation called M [48], which turned out to be much faster than comparably flexible systems. We also used M in performing the integrations in the general case.

The coefficients were calculated to order $O(T^{12})$ in the pure scalar case (they can be found at [32]) and to order $O(T^6)$ in the general case. After the reduction into the minimal basis the results to order $O(T^5)$ read (absorbing the coupling constant $g$ into the fields, $F_{\kappa\lambda\mu\nu} \equiv D_\kappa D_\lambda F_{\mu\nu}$ etc.):

$$O_1 \ = \ V$$

$$O_2 \ = \ V^2 + \frac{1}{6} F_{\kappa\lambda} F_{\lambda\kappa}$$
\[ O_3 = V^3 + \frac{1}{2} V_\kappa V_\kappa + \frac{1}{2} V F_{\kappa \lambda} F_{\lambda \kappa} \]
\[ + \frac{1}{20} F_{\kappa \lambda \mu} F_{\kappa \mu \lambda} - \frac{2}{215} i F_{\kappa \lambda \mu} F_{\kappa \mu} \]

\[ O_4 = V^4 + 2 V V_\kappa V_\kappa + \frac{1}{5} V_\kappa V_\lambda V_\kappa + \frac{2}{5} V F_{\kappa \lambda} V F_{\lambda \kappa} + \frac{3}{5} V^2 F_{\kappa \lambda} F_{\lambda \kappa} \]
\[ - \frac{4}{5} i F_{\kappa \lambda} V_\kappa V_\lambda + \frac{1}{3} F_{\kappa \lambda} F_{\mu \lambda \kappa} V_\mu + \frac{1}{3} F_{\kappa \lambda} F_{\mu \kappa \lambda} V_\mu + \frac{1}{5} V F_{\kappa \lambda \mu} F_{\kappa \mu \lambda} \]
\[ - \frac{2}{15} F_{\kappa \lambda} F_{\mu \kappa} V_\mu - \frac{8}{15} i V F_{\kappa \lambda} F_{\mu \kappa} F_{\mu \kappa} \]
\[ - \frac{1}{21} F_{\kappa \lambda} F_{\mu \kappa} F_{\mu \kappa} + \frac{2}{35} F_{\kappa \lambda} F_{\mu \kappa} F_{\mu \kappa} F_{\mu \kappa} + \frac{4}{35} F_{\kappa \lambda} F_{\mu \kappa} F_{\mu \kappa} F_{\mu \kappa} \]
\[ - \frac{6}{35} i F_{\kappa \lambda} F_{\mu \kappa} F_{\mu \kappa} - \frac{8}{105} i F_{\kappa \lambda} F_{\mu \kappa} F_{\mu \kappa} + \frac{11}{420} F_{\kappa \lambda} F_{\mu \kappa} F_{\mu \kappa} F_{\mu \kappa} \]

\[ O_5 = V^5 + 2 V V_\kappa V_\nu V_\kappa + 3 V^2 V_\kappa V_\kappa + V V_\kappa V_\lambda V_\kappa + V^2 F_{\kappa \lambda} V F_{\lambda \kappa} \]
\[ - \frac{1}{3} i V F_{\kappa \lambda} V_\kappa + \frac{2}{3} V^3 F_{\kappa \lambda} F_{\lambda \kappa} + \frac{5}{3} V_\kappa V_\kappa V_\lambda \]
\[ - \frac{5}{3} i V V_\kappa V_\lambda V_\kappa + \frac{2}{7} V^2 F_{\kappa \lambda \mu} V_\mu \]
\[ + \frac{2}{21} V F_{\kappa \lambda \mu} F_{\mu \kappa} + \frac{2}{21} V^2 F_{\kappa \lambda \mu} F_{\mu \kappa} \]
\[ + \frac{3}{7} V F_{\kappa \lambda} F_{\mu \kappa} V_\mu + \frac{3}{7} V F_{\kappa \lambda} F_{\mu \kappa} V_\mu \]
\[ + \frac{3}{7} i F_{\kappa \lambda} V_\mu V_\nu + \frac{3}{7} i F_{\kappa \lambda} V_\mu V_\nu \]
\[ + \frac{3}{14} V F_{\kappa \lambda} V F_{\mu \kappa} \]
\[ + \frac{13}{21} V F_{\kappa \lambda} F_{\mu \kappa} + \frac{13}{21} V F_{\kappa \lambda} F_{\mu \kappa} + \frac{13}{21} V F_{\kappa \lambda} F_{\mu \kappa} \]
\[ + \frac{3}{21} V V_\kappa F_{\lambda \mu} F_{\mu \kappa} - \frac{16}{21} V V_\kappa F_{\lambda \mu} F_{\mu \kappa} \]
\[ - \frac{17}{21} i F_{\kappa \lambda} V_\mu V_\nu + \frac{17}{42} V F_{\kappa \lambda} V_\mu V_\nu \]
\[ + \frac{1}{3} V F_{\kappa \lambda} F_{\mu \kappa} F_{\nu \mu} F_{\nu \mu} - \frac{1}{3} i F_{\kappa \lambda} F_{\mu \kappa} V_\nu V_\mu + \frac{1}{6} F_{\kappa \lambda} F_{\mu \kappa} V_\mu \]
\[ + \frac{1}{6} F_{\kappa \lambda} V_\mu V_\nu F_{\nu \mu} + \frac{1}{6} F_{\kappa \lambda} F_{\mu \kappa} F_{\nu \mu} V_\nu \]
\[ - \frac{1}{6} i F_{\kappa \lambda} F_{\mu \kappa} V_\nu F_{\nu \mu} - \frac{1}{6} i V F_{\kappa \lambda} F_{\mu \kappa} F_{\nu \mu} + \frac{1}{12} V F_{\kappa \lambda} F_{\mu \kappa} F_{\nu \mu} \]
\[ + \frac{1}{14} V F_{\kappa \lambda} F_{\mu \kappa} F_{\nu \mu} + \frac{1}{14} i V F_{\kappa \lambda} F_{\mu \kappa} F_{\nu \mu} + \frac{1}{42} i V F_{\kappa \lambda} F_{\mu \kappa} F_{\nu \mu} \]
\[ + \frac{2}{7} i F_{\kappa \lambda} F_{\mu \kappa} V_\nu \]
\[- \frac{2}{21} F_{\kappa \lambda \nu} F_{\nu \mu} + \frac{2}{21} i F_{\kappa \lambda \mu} F_{\mu \rho \nu} V_{\nu} - \frac{3}{7} i F_{\kappa \lambda \mu} F_{\nu \mu \nu} V_{\nu} \]

\[- \frac{4}{7} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{4}{21} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \mu} - \frac{5}{21} V F_{\kappa \lambda \mu} F_{\nu \mu} F_{\nu \mu} \]

\[+ \frac{5}{21} V F_{\kappa \lambda \mu} V_{\nu} F_{\nu \mu} F_{\nu \mu} - \frac{5}{21} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} - \frac{5}{21} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \mu} F_{\nu \mu} \]

\[- \frac{8}{21} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} - \frac{10}{21} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} - \frac{11}{21} V F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} \]

\[- \frac{11}{42} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{11}{84} i V F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} + \frac{11}{42} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} V_{\mu} \]

\[+ \frac{17}{84} i V F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{1}{9} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} - \frac{1}{18} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} V_{\mu} \]

\[+ \frac{1}{189} \frac{1}{252} F_{\nu \nu \nu \nu} F_{\mu \mu \nu} F_{\mu \nu} + \frac{1}{378} \frac{1}{378} \frac{1}{378} F_{\nu \nu \nu \nu} F_{\mu \mu \nu} F_{\mu \nu} \]

\[- \frac{2}{21} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{2}{63} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{2}{945} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} F_{\nu \nu} \]

\[- \frac{4}{63} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{5}{63} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{5}{63} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} V_{\mu} \]

\[+ \frac{5}{63} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{1}{126} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} - \frac{5}{126} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} V_{\mu} \]

\[+ \frac{8}{63} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} - \frac{8}{189} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} - \frac{10}{189} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} F_{\nu \nu} \]

\[- \frac{10}{189} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} F_{\nu \nu} + \frac{11}{189} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{11}{189} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} V_{\mu} \]

\[- \frac{11}{378} \frac{13}{252} F_{\nu \nu \nu \nu} F_{\mu \mu \nu} F_{\mu \nu} + \frac{6}{63} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{19}{756} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} F_{\nu \nu} \]

\[- \frac{19}{756} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} F_{\nu \nu} + \frac{22}{189} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{25}{189} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} V_{\mu} \]

\[- \frac{26}{189} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{31}{378} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{34}{189} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} F_{\nu \nu} \]

\[- \frac{41}{378} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{53}{378} i F_{\kappa \lambda \mu} F_{\nu \nu} V_{\mu} + \frac{61}{756} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} V_{\mu} \]

\[+ \frac{61}{756} i F_{\kappa \lambda \mu} V_{\nu} F_{\nu \nu} V_{\mu} \]

The coefficient $O_6$ is quite lengthy and can be found in Appendix C. Even for the general case it is in principle possible to compute still higher coefficients of the inverse mass expansion. However, the calculation is practically limited by the basis reduction, since the implementation of the general rules given in Appendix A has to be extended for every new order under consideration.

4. Comparison with other methods

Finally, let us compare with other algorithms which are available for the computation of the same expansion.
Generally, methods of calculating the higher derivative expansion are either based on heat kernel [39, 41, 43, 54, 56, 57, 62] or Feynman diagram techniques [40, 50–52]. In the heat kernel approach, one evaluates the operator trace eq. (2) in $x$-space:

$$\Gamma[A, V] = \int_0^\infty \frac{dT}{T} \text{tr} \int d^d x \langle x | \exp \left[ -T(-D^2 + m^2 + V(x)) \right] | x \rangle.$$  \hspace{1cm} (22)

From the interacting heat kernel

$$\langle x | K(T) | y \rangle = \langle x | \exp \left[ -T(-D^2 + m^2 + V(x)) \right] | y \rangle$$  \hspace{1cm} (23)

one separates off the known free one,

$$\langle x | K_0(T) | y \rangle = \langle x | \exp \left[ -T(\nabla^2 + m^2) \right] | y \rangle = (4\pi T)^{-d/2} \exp\left(-\frac{(x-y)^2}{4T}\right),$$  \hspace{1cm} (24)

writing

$$\langle x | K(T) | y \rangle = (4\pi T)^{-d/2} \exp\left(-\frac{(x-y)^2}{4T}\right) H(x,y;T).$$  \hspace{1cm} (25)

$H$ is then expanded in powers of $T$,

$$H(x,y;T) = \sum_{k=0}^\infty a_k(x,y) T^k$$  \hspace{1cm} (26)

and the heat kernel coefficients $a_k$ (which are functionals of the background fields) are calculated on the diagonal $x = y$. For the calculation of those coefficients, a large variety of algorithms have been invented. Roughly, they fall into three categories:

1. Recursive $x$-space algorithms [41, 43, 49]. In our view these are too cumbersome for doing higher order calculations in gauge theory and will not be discussed here.

2. The method of Zuk [56,57], based on Onofri’s graphical representation of the heat kernel coefficients [54].

3. Nonrecursive algorithms based on the insertion of a momentum basis [58,59].

Let us first consider Zuk’s method, which is manifestly gauge invariant, and also the one most closely related to our work.

4.1 Zuk’s Method

In Onofri’s work [54], the Baker-Campbell-Hausdorff formula was employed to represent the coefficients for the pure scalar case by Feynman diagrams in a one-dimensional auxiliary field theory. Those Feynman diagrams are calculated using the Green function

$$G^{(0)}(\tau_1, \tau_2) = |\tau_1 - \tau_2| - (\tau_1 + \tau_2) + \frac{2}{T} \tau_1 \tau_2.$$  \hspace{1cm} (27)

This Green function was also used by Zuk to calculate the effective Lagrangian for the pure scalar case up to terms with four derivatives [56]. He then generalized the method to the gauge
field case, and also used Fock-Schwinger gauge to enforce manifest gauge invariance [57]. The ‘Quantum Mechanical Path Integral Method’ [27], which may be considered as an extension of the Onofri-Zuk formalism, also uses the Green function given above. As explained in detail in Appendix B, the split in the path integral eq. (5), which is necessary to extract the zero modes, is not unique. One still has the freedom of choosing a background charge \( \rho(\tau) \) on the worldline, which parametrizes the boundary conditions on the functions \( y^\mu(\tau) \). This gives us some insight into the connection of the worldline path integral approach with the Onofri-Zuk formalism: If one uses a constant background charge,

\[
\rho(\tau) = \frac{1}{T} ,
\]

one obtains the Green function

\[
G^{(c)}(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} - \frac{T}{6} ,
\]

which agrees – up to an (irrelevant) constant – with the one used in our approach, eq. (12). The effective Lagrangian \( \mathcal{L}(x_0) \) is obtained as a path integral over the space of all loops having \( x_0 \) as their common center of mass (Fig. 1).

\[
\text{Figure 1: The path integration for a uniformly distributed background charge } \rho(\tau) = 1/T .
\]

If one uses the background charge

\[
\rho(\tau) = \delta(\tau)
\]

instead, the resulting Green function turns out to be exactly the one used by Onofri, eq. (27). In this case, the boundary condition reads

\[
y(0) = y(T) = 0 ,
\]

and the effective Lagrangian \( \mathcal{L}(x_0) \) is given as path integral over the space of all loops intersecting in \( x_0 \) (Fig. 2).
The constant background charge has the special property that it is the only translationally invariant choice and therefore $G^{(c)}$ (and equivalently $G$) depend only on $\tau_1 - \tau_2$. As a consequence, cyclically equivalent terms always come with the same numerical coefficient, a fact which facilitates the cyclic identification process considerably. This does not hold true if one uses the Green function eq. (27); for example, of the three cyclically equivalent terms $V_\mu V_\mu V$, $VV_\mu V_\mu$ and $V_\mu VV_\mu$ appearing in the scalar effective action at $\mathcal{O}(T^4)$ the first two get assigned the same coefficient, while the coefficient of the third one is different.

Of all translationally invariant Green functions (which differ only by constants), $G$ has the further advantage that it has vanishing diagonal terms, i.e., $G(\tau, \tau) = 0$. This together with the symmetry of $G$ can be used to perform in the exponent the replacement

$$\frac{1}{2} \sum_{i,j=1}^{n} G(\tau_i, \tau_j) \partial_{(i)} \partial_{(j)} \rightarrow - \sum_{i<j} G(\tau_i, \tau_j) \partial_{(i)} \partial_{(j)}.$$  

(32)

For the pure scalar case the resulting effective Lagrangian is automatically in the minimal basis, i.e. it consists only of terms without box operators. In general, in the process of identifying equivalent terms one never has to integrate by parts. On the other hand one has to be cautious in comparing our results with the results of standard local heat kernel methods, for the usage of the Green function eq. (12) amounts to implicit partial integrations.

The considerations above show that the redundancies arising in Zuk's formalism can be avoided by using the translationally invariant background charge and the corresponding Green function eq. (12).

4.2 A Modified Nonrecursive Heat Kernel Method

We have also explicitly tested our results using a modification of the nonrecursive heat kernel
method mentioned above. Here one evaluates the functional trace in a plane wave basis,

$$\text{Tr } e^{-TM} = \text{tr} \int d^d x \int \frac{d^d k}{(2\pi)^d} e^{-ikx} e^{-TM} e^{ikx}.$$  \hspace{1cm} (33)$$

The net effect of commuting $e^{ikx}$ to the left is the substitution $D_\mu \rightarrow D_\mu + ik_\mu$ in $M$. After rescaling the momenta $q_\mu = \sqrt{T} k_\mu$ one finds

$$\text{Tr } e^{-TM} = \frac{e^{-m^2 T}}{(4\pi T)^{d/2}} \text{tr} \int d^d x \int \frac{d^d q}{\pi^{d/2}} e^{-q^2} \exp \left( -T(-D^2 + V) + 2i\sqrt{T} q_\mu D_\mu \right).$$  \hspace{1cm} (34)$$

The last exponential is to be expanded in powers of $T$ (note that only even numbers of momenta $q_\mu$ contribute). The $q$–integration produces totally symmetric combinations of products of the metric tensor. The intermediate result is a series in $T$, where every coefficient consists of a string of $V$’s and $D_\mu$’s and the latter are pairwise contracted. The covariant derivatives $D_\mu$ act to the right, thus a $D_\mu$ at the right end (acting on 1) can be replaced with $iA_\mu$. This would however break covariance, which we want to avoid. The first of our modifications incorporates the ‘no double derivative’ prescription discussed in Appendix A. Of each contracted pair of derivatives one moves the first one to the left and the second one to the right,

$$\ldots \bigtriangledown_\mu \ldots \bigtriangledown_\mu \ldots ,$$  \hspace{1cm} (35)$$

using the Leibniz rules

$$\bigtriangledown_\mu Y = D_\mu Y + Y \bigtriangledown_\mu ,$$  \hspace{1cm} (36)$$

$$Y \bigtriangledown_\mu = -D_\mu Y + \bigtriangledown_\mu Y ,$$  \hspace{1cm} (37)$$

where $Y$ stands for any covariant structure. In this way one obtains terms of the generic form

$$\bigtriangledown_{\mu_1} \ldots \bigtriangledown_{\mu_n} X \bigtriangledown_{\nu_1} \ldots \bigtriangledown_{\nu_n} ,$$  \hspace{1cm} (38)$$

where $X$ represents a string of covariant objects $V$, $DV$, $\ldots$, $F$, $DF$, $\ldots$ without active derivatives. Clearly the prescription (35) avoids any self-contractions. The aim is now to reduce the number of active derivatives. As an example consider terms where $X$ is antisymmetric in two of the indices $\nu_1 \ldots \nu_n$. This leads then to terms where the number of right derivatives is reduced by two, at the price of the appearance of a new field strength tensor $F$. However, even if one includes Bianchi identities within $X$, this turns out to be not enough.

According to eq. (34) there is an overall trace and thus a freedom of cyclic permutations. Cyclicity may however be exploited only after all derivatives have been executed. Nevertheless one can use this property already at this stage to ‘shuffle derivatives to the right’ in the subclass of terms which have only left derivatives. In particular one can write

$$\bigtriangledown_\mu X = igA_\mu X = XigA_\mu = X \bigtriangledown_\mu ,$$  \hspace{1cm} (39)$$

where the first equality comes from removing the total derivative $\partial_\mu X$, the second makes use of the trace (no active derivatives!), and the last one follows from the addition of $0 = \partial_\mu 1$. 

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In a similar manner one can prove the identities

\[
\begin{align*}
\hat{D}_\mu \hat{D}_\nu X &= -X \hat{D}_\nu \hat{D}_\mu + \hat{D}_\mu X \hat{D}_\nu + \hat{D}_\nu X \hat{D}_\mu , \\
\hat{D}_\lambda \hat{D}_\mu \hat{D}_\nu X &= +X \hat{D}_\nu \hat{D}_\mu \hat{D}_\lambda + \hat{D}_\lambda \hat{D}_\mu X \hat{D}_\nu + \hat{D}_\lambda \hat{D}_\nu X \hat{D}_\mu + \hat{D}_\mu \hat{D}_\nu X \hat{D}_\lambda \\
&\quad - \hat{D}_\lambda X \hat{D}_\nu \hat{D}_\mu - \hat{D}_\mu X \hat{D}_\nu \hat{D}_\lambda - \hat{D}_\nu X \hat{D}_\mu \hat{D}_\lambda ,
\end{align*}
\]

(40)

and more complicated ones for four or more left derivatives. After each of these steps one has to investigate the resulting terms again for their antisymmetry, possibly using Bianchi identities.

The algorithm comes to an end if all terms have no more than two derivatives acting at the same end. An important input at this point is the knowledge that the result can be written in manifestly covariant form. This allows one to replace the remaining derivatives according to

\[
D_\mu \to 0 , \quad D_\mu D_\nu \to \frac{i g}{2} F_{\mu \nu} .
\]

(41)

One can easily see that such a replacement rule does not exist for three (or more) derivatives: A structure \( D_\lambda D_\mu D_\nu \) can originate from \( D_\lambda F_{\mu \nu} \), but equally well from \( D_\nu F_{\mu \lambda} \), and the difference of these possibilities is nonzero (it is just \( D_\mu F_{\lambda \nu} \) via Bianchi’s identity). The further treatment of the result, namely identifying cyclic equivalent terms and reduction to the minimal basis, proceeds along the same lines as described in the Appendix A. Using the method described above we calculated all coefficients up to \( O_5 \) and all invariants in \( O_6 \) which contain two or more scalar potentials \( V \). By reduction into the minimal basis we found agreement with the results of the worldline approach.

### 4.3 The Method of ’t Hooft and van de Ven

Finally, let us comment on the calculation of the inverse mass expansion by Feynman diagrams [40, 50–52]. Here, the version most suitable to gauge theory calculations appears to be the one invented by ’t Hooft [53], and elaborated by van de Ven [40].

In this scheme, one first considers backgrounds obeying

\[
V \equiv -A_\mu A_\mu , \quad \partial_\mu A_\nu \equiv 0 .
\]

(42)

For this special case, the effective Lagrangian can be expanded in a basis consisting of strings of \( n \) \( A_\mu \)-matrices (denoted by \( \text{tr} J_j \)),

\[
L = \sum_j a_j \text{tr} J_j .
\]

(43)

The coefficients \( a_j \) can be determined from the logarithmic divergence of the one-loop diagram with \( n \) \( A_\mu \)-insertions in \( d = 2n \) spacetime dimensions.

One then chooses a minimal basis of invariants for the general background, denoted \( \text{tr} I_j \), and subjects those invariants to the conditions (42). For any fixed order in \( T \), they can then be written in terms of the basis \( \text{tr} J_j \),

\[
\text{tr} I_i = \sum_j P_{ij} \text{tr} J_j ,
\]

(44)
with a certain numerical matrix $P$. If one restricts this equation to a fixed order in $T$, the numbers of invariants on both sides turn out to match, and the matrix $P_{ij}$ to be invertible. After performing the inversion, the effective Lagrangian for the general background is obtained as

$$L = \sum_{i,j} a_j P^{-1}_{ji} \text{tr} I_i. \quad (45)$$

This method was used by van de Ven [40] to calculate the one-loop counterterms for Yang-Mills theory in even dimensions $\leq 10$, which is equivalent to calculating the order $O(T^5)$ in the inverse mass expansion. We have checked exact agreement with our result for the $O(T^5)$ by explicitly performing the necessary partial integrations and basis reduction. Since van de Ven considers the case of a real scalar field, this also involves transposition symmetry of the coefficients, i.e. the invariance (up to a sign) of the coefficients under inversion of the ordering of the simple factors, as explained in detail in [64].

For a comparison of the efficiency of both methods, one would have to computerize this method, too, which has not been done yet. Obviously, the difficulty resides in the fact that the method requires the construction of a minimal basis of invariants a priori, to ensure invertibility of the matrix $P$. Moreover, in order to compute higher coefficients one has to find the inverse of matrices of the order of the length of the minimal basis to get the prefactors of the coefficients. For the calculation of $O_6$ in the case of a complex scalar field this already amounts to a (symbolic) inversion of a $902 \times 902$-matrix with rational elements.

5. Conclusions

We have applied the string-inspired method of evaluating one-loop worldline path integrals to the calculation of inverse mass expansions of one-loop effective actions. Complete computerization of the method has allowed us to improve on existing results by one order for a background consisting of both a gauge field and a scalar potential, and by several orders for the case of only a scalar potential. Comparing with the closely related algorithm used by Onofri, Fujiwara et al. and Zuk, we have traced the difference between both approaches to the different boundary conditions imposed on the path integral. The results have been reduced to a minimal basis of invariants, and the reduction process was described in detail.

As indicated in the introduction, the formalism can easily be generalized to the case of a spin 1/2 particle in the loop coupled to external gauge bosons. A systematic investigation of the one-loop effective action induced by a fermion in a scalar background is currently being done [66, 67]. Moreover a worldline path integral formulation of the one-loop effective action of a gluon circulating in the loop was described in [24]. Finally, an extension of the present formalism to the two-loop case is under consideration, based on the construction of generalized worldline Green functions [68].

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As we have learned from A. van de Ven, he has recently also obtained the coefficient $O_6$ for the Yang-Mills case, using a novel version of the recursive heat kernel method [65]. This result has not yet been reduced to a minimal basis of invariants.
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Appendix A

As explained in Chapter 2 it is essential for practical purposes to reduce the coefficients to a minimal basis. In the pure scalar case the identification of cyclic equivalent invariants is sufficient. No partial integrations have to be performed because of the existence of a minimal basis, which does not involve any box operators. In the gauged case the situation is more complicated. Again there exists a basis without box operators, and partial integrations are not necessary, but cyclic permutations are insufficient to reach a minimal basis. Additionally one has to use several equalities, namely the Bianchi identities, the antisymmetry of the field strength tensor and the exchange of covariant derivatives:

\[ D_\mu F_\kappa\lambda + D_\kappa F_\lambda\mu + D_\lambda F_\mu\kappa = 0 , \]  
\[ F_{\mu\nu} = -F_{\nu\mu} , \]  
\[ D_\mu D_\nu X = D_\nu D_\mu X + ig[F_{\mu\nu},X] . \]

In the following we describe the basis reduction algorithm proposed by Müller, the proof of minimality can be found in [64]. Before the basis reduction any invariant in our coefficients consists of simple factors \( X \) (a \( V \), an \( F \) or covariant derivatives thereof). Like in the pure scalar case there are no self-contractions within a simple factor:

\[ X \in \{ V, F_\kappa\lambda, D_\mu_1 D_\mu_2 \ldots D_\mu_n V, D_\mu_1 D_\mu_2 \ldots D_\mu_n F_{\mu_{n+1}\mu_{n+2}} | \mu_i \neq \mu_j \} . \]

The first step of the algorithm would be the elimination of self-contractions by partial integration, which is unnecessary in our case. During the rest of the algorithm invariants with increasing number of field strength tensors are produced due to exchange of covariant derivatives. Therefore one has to start with the terms containing the maximum number of covariant derivatives and collect the corrections to invariants with smaller number of derivatives before the basis reduction. The remaining algorithm includes the following steps:

- **Removal of derivatives of ‘middle’ class**
  
The Bianchi identity exchanges an index of a derivative and the indices of a field strength tensor. This can be used for a reduction of single contractions between different simple factors. Consider the following example:

\[ \text{tr}(D_\mu D_\nu D_\rho D_\sigma F_{\kappa\lambda} \ldots X_\nu \ldots F_{\sigma\lambda} \ldots X_\rho \ldots X_\kappa \ldots X_\mu \ldots) . \]

The contractions of the derivatives of \( F_{\kappa\lambda} \) belong to different classes with respect to the contractions of \( F_{\kappa\lambda} \). This can be seen very easily in a diagrammatical picture, where the (cyclic) function \( \text{tr} \) is represented by a circle (Fig. 3).

The loop is divided into a ‘right’, a ‘middle’ and a ‘left’ sector by the contractions of \( F_{\kappa\lambda} \). Consequently the derivatives are members of a ‘right’ (\( D_\nu \)), a ‘middle’ (\( D_\rho \)) and a ‘left’ (\( D_\sigma \)) class. There are also derivatives that do not belong to any of these classes (\( D_\sigma \)).
The Bianchi identity involves all classes of derivatives and can therefore be used to eliminate one of them. It is useful to take the symmetric choice and eliminate the ‘middle’ derivative. In general the derivatives in a simple factor have to be exchanged, producing invariants with a higher number of field strength tensors, in order to apply the Bianchi identity:

\[
D_\mu D_\nu D_\rho D_\sigma F_{\kappa\lambda} = D_\mu D_\nu D_\sigma D_\rho F_{\kappa\lambda} + \text{corrections}.
\] (51)

Then the Bianchi identity is used to remove middle derivatives (\(D_\rho\) in our example):

\[
D_\mu D_\nu D_\sigma D_\rho F_{\kappa\lambda} = D_\mu D_\nu D_\sigma D_\kappa F_{\rho\lambda} + D_\mu D_\nu D_\lambda D_\rho F_{\kappa\lambda}.
\] (52)

The effect of using the Bianchi identity is a decrease of the ‘middle’ sector and correspondingly intersections between derivative contractions and this sector are removed. This procedure has to be done with all ‘middle’ derivatives in all simple factors, which corresponds to a minimization of the ‘middle’ sectors.

- Reduction of multiple contractions between simple factors

In a next step multiple contractions between simple factors are considered. The general aim of the reduction is to achieve that multiple contractions appear only between field strength tensors. To this end one applies the following rules:

\[
\text{tr}(\ldots F_{\mu\nu} \ldots D_\mu F_{\nu\kappa} \ldots) = \frac{1}{2}\text{tr}(\ldots F_{\mu\nu} \ldots D_\kappa F_{\nu\mu} \ldots),
\] (53)

\[
\text{tr}(\ldots D_\mu F_{\nu\kappa} \ldots D_\nu F_{\mu\lambda} \ldots) = \frac{1}{2}\text{tr}(\ldots D_\kappa F_{\nu\mu} \ldots D_\lambda F_{\mu\nu} \ldots) + \text{tr}(\ldots D_\mu F_{\nu\kappa} \ldots D_\mu F_{\nu\lambda} \ldots),
\] (54)

\[
\text{tr}(\ldots F_{\mu\nu} \ldots D_\mu D_\nu X \ldots) = \frac{1}{2}\text{tr}(\ldots F_{\mu\nu} \ldots i[g[F_{\mu\nu},X]]).
\] (55)
The first and the second rule use the Bianchi identity combined with antisymmetry of the field strength tensor. The second one can be considered as an exception from the first rule. In this case the reduction aim cannot be achieved completely. The third rule uses the exchange rule for derivatives and produces again invariants with a higher number of field strength tensors.

- Arrangement of indices and simple factors

After all the ordering has been done the indices can be fixed. Using our diagrammatical picture again we can always rearrange the indices in a simple factor in such a way that the contractions form the shortest possible connection on the circle. This can always be done using the exchange of derivatives and the antisymmetry of the field strength tensors and will (in general) produce terms with a higher number of field strength tensors. We illustrate the rule in the following example:

\[
\text{tr}(\ldots D_\kappa D_\lambda F_{\mu\nu} \ldots D_\kappa D_\lambda F_{\mu\nu} \ldots) = \text{tr}(\ldots D_\kappa D_\lambda F_{\mu\nu} \ldots D_\lambda D_\kappa F_{\nu\mu} \ldots) + \ldots \quad (56)
\]

After arrangement of the indices cyclic equivalent invariants are reduced by simultaneously fixing the ordering of simple factors and relabelling the indices alphabetically. This completes the algorithm.

From the algorithm described above one can read off the properties of the basis invariants:

- The basis invariants are products of simple factors, which do not contain any self-contractions.
- The basis invariants do not contain any ‘middle’ derivatives.
- In multiple contractions between simple factors, the field strength tensors are doubly contracted. There is one exception mentioned above.
- The arrangement of indices is such that the contractions form the shortest possible connection on the circle.
- The basis invariants are the lexically smallest ones among the set of possible cyclic equivalent invariants.

The following table gives the number of basis invariants as a function of the order in the proper-time parameter \(T\):

| order | total | \(v = 0\) | \(v = 1\) | \(v = 2\) | \(v = 3\) | \(v = 4\) | \(v = 5\) | \(v = 6\) |
|-------|-------|---------|---------|---------|---------|---------|---------|---------|
| 1     | 1     | 0       | 1       |         |         |         |         |         |
| 2     | 2     | 1       | 0       | 1       |         |         |         |         |
| 3     | 5     | 2       | 1       | 1       | 1       |         |         |         |
| 4     | 18    | 7       | 5       | 4       | 1       | 1       |         |         |
| 5     | 105   | 36      | 36      | 23      | 7       | 2       | 1       |         |
| 6     | 902   | 300     | 329     | 191     | 63      | 16      | 2       | 1       |

Table 1: Number of basis invariants in different orders of the expansion. \(v\) is the number of scalar background fields.
Appendix B

In this Appendix we investigate the connection of the split of the path integral with the freedom of choosing a Green function for the second derivative operator. Moreover we show that the final result (the effective action) does not depend on the actual choice of the Green function.

The Green function used in the worldline approach (eq. (12)) is in fact not the inverse of the second derivative acting on the complete space of trajectories. Partial integration on the circle yields

$$\int_0^T d\tau_1 G(\tau_1, \tau_2) \frac{1}{2} \frac{\partial^2}{\partial \tau_1^2} x(\tau_1) = x(\tau_2) - \frac{1}{T} \int_0^T d\tau_1 x(\tau_1),$$

(57)

where the second term should be absent. On the other hand we have performed the path integration only over the relative coordinates $y(\tau)$, which obey

$$\int_0^T d\tau y^\mu(\tau) = 0.$$

(58)

This shows that the Green function (12) is the inverse of the second derivative on the space of relative coordinates $y$.

This can be generalized as follows: An operator can be inverted only after extracting its zero modes. For the Laplace operator on the circle, the zero modes are just the constant functions. Therefore we perform the split

$$x(\tau) = x_0 1 + y(\tau), \quad \int \mathcal{D}x = \int dx_0 \int \mathcal{D}y.$$

(59)

The value $x_0$ is determined as weighted average

$$x_0 = \int_0^T d\tau \rho(\tau) x(\tau),$$

(60)

where the weight function $\rho$ – the so called background charge – is a periodic function on the circle with the normalization

$$\int_0^T d\tau \rho(\tau) = 1.$$

(61)

For the path integration variables $y$ this leads to the general constraint

$$\int_0^T d\tau \rho(\tau) y(\tau) = 0.$$

(62)

The defining equation for the Green function $G^{(\rho)}$ reads now

$$\mathcal{P} \frac{1}{2} \partial^2 \mathcal{P} G^{(\rho)} \mathcal{P} = \mathcal{P} G^{(\rho)} \mathcal{P} \frac{1}{2} \partial^2 \mathcal{P} = \mathcal{P},$$

(63)

where

$$\mathcal{P} = 1 - \frac{|\rho\rangle \langle \rho|}{\langle \rho | \rho \rangle}.$$

(64)
is the projector on the subspace of functions $y$ which obey the constraint (62), $P|y⟩=|y⟩$.

One can now show that for real Green functions the following conditions are sufficient to fulfill eq. (63):

$$G^{(\rho)}(\tau_1, \tau_2) = G^{(\rho)}(\tau_2, \tau_1),$$

(65)

$$\rho(\tau_1) \int_0^T \rho(\tau) G^{(\rho)}(\tau, \tau_2) d\tau = \rho(\tau_2) \int_0^T \rho(\tau) G^{(\rho)}(\tau_1, \tau) d\tau,$$

(66)

$$\frac{1}{2} \frac{\partial^2}{\partial \tau_1^2} G^{(\rho)}(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2) - \rho(\tau_1).$$

(67)

In order to solve the last equation, it is convenient to construct a generating functional $H$ as the solution to

$$\frac{1}{2} \frac{\partial^2}{\partial \tau_1^2} H(\tau_1, \tau_2; \sigma) = \delta(\tau_1 - \tau_2) - \delta(\tau_1 - \sigma).$$

(68)

$G^{(\rho)}$ is then obtained from a convolution of $H$ with $\rho$:

$$G^{(\rho)}(\tau_1, \tau_2) = \int_0^T d\sigma \rho(\sigma) H(\tau_1, \tau_2; \sigma).$$

(69)

The requirements of symmetry (65) and periodic boundary conditions determine $H$ uniquely as

$$H(\tau_1, \tau_2; \sigma) = |\tau_1 - \tau_2| - |\tau_1 - \sigma| - |\tau_2 - \sigma| + \frac{2}{T}(\tau_1 - \sigma)(\tau_2 - \sigma) + h(\sigma)$$

(70)

up to a function $h(\sigma)$. With the special choice

$$h(\sigma) = \int_0^T \rho(\tau)|\sigma - \tau| d\tau - \frac{2\sigma}{T} \left( \sigma - \int_0^T \rho(\tau) \tau d\tau \right),$$

(71)

we can achieve

$$\int_0^T \rho(\tau_1) G^{(\rho)}(\tau_1, \tau_2) d\tau_1 = 0,$$

(72)

such that eq. (66) is trivially fulfilled.

Any $G^{(\rho)}$ constructed from (69) can be used as Green function for the evaluation of the path integral. Different choices of $\rho$ lead to different effective Lagrangians, but to the same effective action. Let us shortly verify this assertion for the scalar potential case, where the integrand is just the universal exponential

$$\exp \left[ -\frac{1}{2} \sum_{i,j=1}^n G^{(\rho)}(\tau_i, \tau_j) \partial_{i\beta} \partial_{\beta j} \right]$$

(73)

(note that for a general $G^{(\rho)}$ diagonal terms have to be included). Using the observation

$$H(\tau_1, \tau_2; \sigma) = G(\tau_1, \tau_2) - G(\tau_1, \sigma) - G(\tau_2, \sigma) + h(\sigma),$$

(74)

where $G$ is our original Green function eq. (12), we may rewrite the exponent as
\[
-\frac{1}{2} \sum_{i,j=1}^{n} G^{(\rho)}(\tau_i, \tau_j) \partial_{(i)} \partial_{(j)} = -\frac{1}{2} \sum_{i,j=1}^{n} G(\tau_i, \tau_j) \partial_{(i)} \partial_{(j)} \\
+ \sum_{i=1}^{n} \partial_{(i)} \int_{0}^{T} d\sigma \rho(\sigma) \sum_{j=1}^{n} G(\tau_j, \sigma) \partial_{(j)} \\
- \frac{1}{2} \sum_{i,j=1}^{n} \partial_{(i)} \partial_{(j)} \int_{0}^{T} d\sigma \rho(\sigma) h(\sigma) .
\]

(75)

This shows that all \(\rho\)- and \(h\)-dependent terms in the effective Lagrangian carry at least one free factor of \(\sum \partial_{(i)}\) and therefore are total derivatives. This argument can be easily generalized to the gauge theory case. In particular, by performing the shift \(h(\sigma) \rightarrow h(\sigma) + c\) we see from eq. (75) that two Green functions that differ only by a constant \(c\) lead to effective Lagrangians that differ only by total derivatives and thus give the same effective action.
Appendix C

The coefficient $O_6$ written in the minimal basis reads

\[
O_6 = V^6 + 6V^2V_\kappa VV_\kappa + 4V^3V_\kappa V_\kappa \\
+ \frac{2}{7} i V^2F_{\kappa\lambda}V_\lambda V_\kappa + \frac{5}{7} V^4F_{\kappa\lambda}F_{\lambda\kappa} + \frac{8}{7} V^3F_{\kappa\lambda}VF_{\lambda\kappa} \\
- \frac{8}{7} i VF_{\kappa\lambda}V_\lambda VV_\kappa + \frac{9}{7} VF_{\kappa\lambda}VV_\kappa + \frac{9}{7} V_{\kappa\lambda}V_{\lambda\kappa}V_\lambda \\
+ \frac{9}{14} V_\mu V_\nu V_\lambda V_\kappa - \frac{10}{7} i VF_{\kappa\lambda}VV_\kappa + \frac{12}{7} V^2F_{\kappa\lambda}V_\lambda V_\kappa \\
+ \frac{17}{14} V_\mu V_\nu V_\lambda + \frac{18}{7} V_\mu V_\lambda V_\kappa - \frac{18}{7} V^2F_{\kappa\lambda}V_\lambda V_\kappa \\
- \frac{24}{7} i V^2F_{\kappa\lambda}F_{\lambda\kappa} + \frac{26}{7} VV_{\kappa\lambda}V_\lambda V_\kappa + \frac{26}{7} VV_{\kappa\lambda}V_\lambda V_\kappa \\
- \frac{26}{7} i VV_{\kappa\lambda}V_\lambda F_{\lambda\kappa} - VF_{\kappa\lambda}V_\lambda F_{\nu\mu} - 2VF_{\kappa\lambda}F_{\mu\nu}F_{\lambda\kappa} \\
+ V_{\kappa\lambda\mu}V_{\mu\lambda} + V_{\kappa\lambda\mu}V_{\mu\lambda\kappa} + \frac{1}{2} VF_{\kappa\lambda}F_{\lambda\mu}V_{\mu\nu} \\
+ \frac{1}{2} VV_{\kappa\lambda}F_{\mu\lambda}F_{\mu\lambda} + \frac{1}{2} V^2F_{\kappa\lambda}F_{\mu\lambda\kappa}V_{\mu\nu} + \frac{1}{2} V^2F_{\kappa\lambda}F_{\nu\mu}F_{\lambda\kappa} \\
- \frac{1}{7} VF_{\kappa\lambda}V_\mu F_{\mu\lambda}V_\kappa + \frac{7}{7} VV_{\kappa\lambda}F_{\mu\lambda}F_{\mu\kappa}V_{\lambda\kappa} - \frac{2}{7} VF_{\kappa\lambda}VF_{\mu\lambda}F_{\mu\kappa} \\
- \frac{2}{7} i VF_{\kappa\lambda\mu}V_{\mu\kappa}V_\lambda - \frac{2}{7} i VF_{\kappa\lambda\mu}V_\mu V_{\mu\kappa} - \frac{2}{21} i VF_{\kappa\lambda}V_\mu V_{\mu\kappa}F_{\mu\kappa} \\
+ \frac{3}{7} VV_{\kappa\lambda\mu}V_{\mu\lambda\kappa} + \frac{3}{7} V^2F_{\kappa\lambda}F_{\mu\lambda}F_{\mu\kappa} + \frac{3}{7} V^2F_{\kappa\lambda}VF_{\mu\lambda}F_{\mu\kappa} \\
- \frac{3}{7} i F_{\kappa\lambda}V_{\mu\lambda}V_{\mu} - \frac{4}{7} VF_{\kappa\lambda}F_{\mu\lambda}VV_{\mu\kappa} - \frac{4}{7} VF_{\kappa\lambda}F_{\mu\lambda}V_{\mu}V_\kappa \\
- \frac{4}{7} VF_{\kappa\lambda}VV_{\mu\lambda}F_{\mu\kappa} - \frac{4}{7} VF_{\kappa\lambda}VF_{\mu\lambda}F_{\mu\kappa} - \frac{4}{7} VF_{\kappa\lambda}VF_{\mu\lambda}V_{\mu}F_{\mu\kappa} \\
- \frac{4}{7} i VF_{\kappa\lambda}V_{\mu\lambda}V_{\mu\kappa} - \frac{4}{7} i V^2F_{\kappa\lambda}F_{\mu\lambda}F_{\mu\kappa} - \frac{4}{7} i V^2F_{\kappa\lambda}VF_{\mu\lambda}F_{\mu\kappa} \\
+ \frac{5}{14} V^3F_{\kappa\lambda\mu}F_{\kappa\lambda\mu} + \frac{6}{7} V^2F_{\kappa\lambda\mu}F_{\mu\lambda\kappa} + \frac{6}{7} V^2F_{\kappa\lambda\mu}V_{\mu}V_\kappa \\
+ \frac{6}{7} V^2F_{\kappa\lambda\mu}F_{\mu\lambda\kappa} + \frac{6}{7} V^2F_{\kappa\lambda\mu}F_{\mu\kappa\lambda} + \frac{8}{7} VF_{\kappa\lambda}F_{\mu\kappa}V_\nu V_\mu \\
- \frac{8}{7} VV_{\kappa\lambda}F_{\mu\lambda}V_{\mu\kappa} - \frac{9}{7} i VF_{\kappa\lambda}V_{\mu\lambda}F_{\mu\kappa} - \frac{9}{7} i VF_{\kappa\lambda}V_{\mu\lambda}F_{\mu\kappa} \\
+ \frac{9}{14} V^2F_{\kappa\lambda\mu}VF_{\kappa\lambda\mu} - \frac{10}{7} i V^3F_{\kappa\lambda}F_{\mu\lambda\kappa}F_{\mu\kappa} - \frac{11}{7} VV_{\kappa\lambda}F_{\mu\lambda}V_{\mu\kappa} \\
- \frac{11}{7} i F_{\kappa\lambda}V_{\mu\lambda}V_{\mu\kappa} - \frac{11}{7} i F_{\kappa\lambda}V_{\mu\lambda}V_{\mu\kappa} + \frac{11}{14} VF_{\kappa\lambda}F_{\mu\lambda}VV_{\kappa} \\
+ \frac{11}{14} VF_{\kappa\lambda}F_{\mu\lambda\kappa}V_{\mu\kappa} + \frac{11}{14} VF_{\kappa\lambda}V_{\mu\lambda\kappa}V_{\mu\kappa} + \frac{11}{14} VF_{\kappa\lambda}V_{\mu\lambda\kappa}V_{\mu\kappa} \\
+ \frac{11}{14} VF_{\kappa\lambda}V_{\mu\lambda\kappa}V_{\mu\kappa} + \frac{11}{14} VF_{\kappa\lambda}F_{\mu\lambda\kappa}V_{\mu\kappa} + \frac{11}{21} V_{\kappa\lambda}V_{\mu\lambda}V_{\mu\kappa}
\]
\[-\frac{12}{7} V^2 V_\kappa \lambda F_{\lambda \mu} F_{\mu \kappa} - \frac{15}{7} V V_\kappa \lambda F_{\lambda \mu} F_{\mu \kappa} - \frac{15}{7} i V V_{\kappa \lambda} F_{\lambda \mu} V_{\mu \kappa}\]

\[-\frac{15}{7} i V V_\kappa \lambda V_\lambda F_{\mu \kappa} - \frac{17}{7} i F_{\kappa \lambda} V_\mu V_{\mu \kappa} + \frac{17}{14} V F_{\kappa \lambda \mu} V V_\kappa F_{\mu \lambda}\]

\[+ \frac{17}{14} V F_{\kappa \lambda} V_{\mu \kappa} V_{\mu} + \frac{17}{14} V F_{\kappa \lambda} V_{\mu \lambda} V_{\mu \kappa} + \frac{17}{14} V V_\kappa F_{\lambda \mu} V F_{\mu \lambda}\]

\[-2 i F_{\kappa \lambda \mu} V_\nu V_{\lambda \kappa} - 2 i F_{\kappa \lambda \mu} V_\nu V_{\lambda} - 2 i F_{\kappa \lambda \nu} V_{\lambda \mu} V_{\kappa}\]

\[-i V V_\kappa \lambda F_{\lambda \mu \kappa} V_{\nu} - i V V_\kappa \lambda F_{\lambda \nu} V_{\mu \kappa} + \frac{1}{2} V F_{\kappa \lambda \mu} V_{\mu \nu} F_{\nu \mu \lambda}\]

\[+ \frac{1}{3} V^2 F_{\kappa \lambda} F_{\mu \nu} F_{\nu \lambda} F_{\mu \kappa} - \frac{1}{3} i V^2 F_{\kappa \lambda} F_{\mu \nu} F_{\nu \lambda} F_{\mu \kappa} - \frac{1}{6} i V F_{\kappa \lambda} V F_{\lambda \mu} V F_{\mu \kappa}\]

\[-\frac{1}{6} i V^2 F_{\kappa \lambda \mu} F_{\nu \lambda} F_{\mu \kappa} - \frac{1}{7} V F_{\kappa \lambda} V F_{\lambda \mu} F_{\mu \nu} - \frac{1}{9} i V F_{\kappa \lambda} V F_{\lambda \nu} V F_{\mu \kappa}\]

\[-\frac{1}{4} i V F_{\kappa \lambda} F_{\lambda \mu \nu} V F_{\kappa \mu \nu} + \frac{1}{18} i V F_{\kappa \lambda} F_{\lambda \mu} F_{\nu \lambda} V_{\mu} - \frac{1}{21} V F_{\kappa \lambda \mu} V F_{\nu \lambda} V_{\mu}\]

\[+ \frac{1}{21} V F_{\kappa \lambda \mu} V_{\nu \lambda} V_{\nu \kappa} - \frac{1}{21} V F_{\kappa \lambda \mu} V_{\nu \lambda} V_{\nu \kappa} + \frac{1}{21} V F_{\kappa \lambda \mu} V_{\nu \lambda} V_{\nu \kappa}\]

\[-\frac{1}{21} i V F_{\kappa \lambda} F_{\lambda \mu \nu} V_{\nu} F_{\mu \kappa} + \frac{1}{21} i V V_\kappa F_{\lambda \mu} F_{\nu \lambda} F_{\mu \kappa} - \frac{1}{21} i V V_\kappa F_{\lambda \mu} F_{\nu \lambda} F_{\mu \kappa}\]

\[+ \frac{1}{42} V_{\kappa \lambda \mu \nu} V_{\nu \lambda \kappa} - \frac{2}{7} i F_{\kappa \lambda \mu \nu} V_{\nu \lambda} V_{\mu \kappa} - \frac{2}{7} i V F_{\kappa \lambda \mu} V_{\nu \lambda} F_{\mu \kappa}\]

\[-\frac{2}{9} i V F_{\kappa \lambda} F_{\lambda \mu} V_{\nu \lambda} F_{\nu \kappa} - \frac{2}{9} i V F_{\kappa \lambda} F_{\lambda \mu} V_{\nu \lambda} F_{\nu \kappa} + \frac{2}{21} F_{\kappa \lambda} V_{\mu \lambda} V_{\nu} V_{\mu} V_{\nu}\]

\[+ \frac{2}{21} F_{\kappa \lambda \mu \nu} V_{\nu} F_{\mu \kappa} + \frac{2}{21} V F_{\kappa \lambda \mu \nu} V F_{\lambda \kappa \mu} + \frac{2}{21} i V F_{\kappa \lambda} F_{\lambda \nu} V_{\mu} V_{\kappa}\]

\[-\frac{2}{21} V F_{\kappa \lambda} F_{\lambda \mu \nu} F_{\nu \lambda} F_{\mu \kappa} + \frac{2}{63} i V F_{\kappa \lambda} F_{\lambda \mu} F_{\nu \lambda} V_{\nu} - \frac{3}{7} i V F_{\kappa \lambda} F_{\lambda \mu} V_{\nu} V_{\mu \kappa}\]

\[-\frac{2}{7} i V F_{\kappa \lambda} F_{\lambda \mu} V_{\nu} V_{\nu} F_{\mu \kappa} - \frac{2}{7} i V F_{\kappa \lambda} F_{\lambda \mu} V_{\nu} V_{\nu} F_{\mu \kappa}\]

\[-\frac{3}{7} i V F_{\kappa \lambda} V_{\mu \nu} V_{\mu \kappa} - \frac{3}{7} i V F_{\kappa \lambda} F_{\mu \nu} F_{\mu \nu} F_{\mu \kappa}\]

\[-\frac{3}{7} V F_{\kappa \lambda} V_{\mu \nu} F_{\mu \kappa} V_{\mu \nu} - \frac{4}{7} V V_\kappa F_{\lambda \mu} F_{\nu \lambda} F_{\nu \kappa} + \frac{4}{7} V V_\kappa F_{\lambda \mu} F_{\nu \lambda} F_{\nu \kappa}\]

\[-\frac{4}{7} i F_{\kappa \lambda} V_{\mu \nu} V_{\mu \nu} F_{\nu \lambda} F_{\nu \kappa} + \frac{4}{21} i V F_{\kappa \lambda} F_{\lambda \mu} V_{\mu \kappa} F_{\nu \kappa}\]

\[-\frac{4}{21} i V F_{\kappa \lambda} F_{\lambda \mu \nu} F_{\mu \kappa} - \frac{4}{21} i V F_{\kappa \lambda} F_{\lambda \nu} V_{\mu \kappa} F_{\mu \kappa} - \frac{4}{21} i V F_{\kappa \lambda} F_{\lambda \nu} V_{\mu \kappa} F_{\mu \kappa}\]

\[-\frac{4}{63} V F_{\kappa \lambda} F_{\lambda \mu \nu} V_{\nu} V_{\mu \kappa} + \frac{4}{63} i V F_{\kappa \lambda} F_{\lambda \mu \nu} V_{\nu} V_{\mu \kappa} - \frac{5}{7} i V F_{\kappa \lambda} V_{\nu} V_{\nu} F_{\mu \lambda}\]

\[+ \frac{5}{9} i V F_{\kappa \lambda} F_{\lambda \mu \nu} V_{\nu} V_{\mu \kappa} + \frac{5}{9} F_{\kappa \lambda} F_{\nu \mu \lambda} V_{\nu} V_{\mu \kappa} + \frac{5}{9} i V F_{\kappa \lambda} V_{\nu} V_{\nu} F_{\mu \lambda}\]

\[+ \frac{5}{9} i V F_{\kappa \lambda} F_{\lambda \mu \nu} V_{\nu} V_{\mu \kappa} + \frac{5}{9} F_{\kappa \lambda} F_{\nu \mu \lambda} V_{\nu} V_{\mu \kappa} - \frac{5}{9} i V F_{\kappa \lambda} V_{\nu} V_{\nu} F_{\mu \lambda}\]

\[-\frac{5}{9} i V F_{\kappa \lambda} F_{\lambda \mu \nu} F_{\mu \kappa} + \frac{5}{14} V F_{\kappa \lambda \mu} F_{\lambda \mu \nu} V_{\kappa} + \frac{5}{14} V F_{\kappa \lambda \mu} V_{\kappa} F_{\lambda \mu \nu}\]
\[ + \frac{5}{14} VF_{\kappa \lambda} V_{\mu \nu} F_{\kappa \nu \lambda \kappa} + \frac{5}{14} V V F_{\nu \mu \nu} F_{\kappa \nu \lambda \kappa} - \frac{5}{18} \ i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \nu \lambda \nu} \]
\[ + \frac{5}{21} F_{\kappa \lambda \nu} F_{\kappa \mu \nu} V_{\nu \nu} + \frac{5}{21} F_{\kappa \lambda \nu} V_{\nu \kappa \nu} + \frac{5}{21} V F_{\kappa \lambda \nu} F_{\nu \nu \nu} \]
\[ + \frac{5}{21} V F_{\kappa \lambda \nu} F_{\nu \nu \nu} V_{\nu \kappa} + \frac{5}{21} V V F_{\nu \nu \nu} F_{\nu \nu \nu} + \frac{5}{21} V F_{\kappa \lambda \nu} F_{\nu \nu \nu} \]
\[ + \frac{5}{21} i F_{\kappa \lambda \nu} F_{\nu \nu \nu} V_{\nu \nu} + \frac{5}{42} V^{2} F_{\kappa \lambda \nu} F_{\nu \nu \nu} + \frac{5}{63} V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu} \]
\[ + \frac{5}{63} V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} + \frac{5}{126} V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} - \frac{7}{9} \ i V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} \]
\[ - \frac{8}{9} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \kappa} - \frac{8}{9} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \kappa} - \frac{8}{9} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \kappa} \]
\[ - \frac{8}{9} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \kappa} - \frac{8}{9} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \kappa} + \frac{8}{9} \ i F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \kappa} \]
\[ + \frac{8}{9} \ i V V F_{\nu \nu \nu} F_{\nu \nu \nu} F_{\nu \nu \nu} - \frac{8}{21} \ i V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} F_{\nu \nu \nu} - \frac{8}{21} \ i V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} F_{\nu \nu \nu} \]
\[ - \frac{8}{9} i F_{\kappa \lambda \nu} F_{\nu \nu \nu} V_{\nu \nu} F_{\nu \nu \nu} - \frac{8}{9} i F_{\kappa \lambda \nu} F_{\nu \nu \nu} V_{\nu \nu} F_{\nu \nu \nu} + \frac{10}{21} V F_{\kappa \lambda \nu} F_{\nu \nu \nu} F_{\nu \nu \nu} \]
\[ - \frac{10}{21} V F_{\kappa \lambda \nu} F_{\nu \nu \nu} F_{\nu \nu \nu} + \frac{10}{21} i F_{\kappa \lambda \nu} F_{\nu \nu \nu} V_{\nu \nu} F_{\nu \nu \nu} - \frac{10}{21} i F_{\kappa \lambda \nu} F_{\nu \nu \nu} V_{\nu \nu} F_{\nu \nu \nu} \]
\[ + \frac{10}{63} F_{\kappa \lambda \nu} F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{10}{63} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ - \frac{10}{63} V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} F_{\nu \nu \nu} - \frac{10}{63} i F_{\kappa \lambda \nu} V F_{\nu \nu \nu} F_{\nu \nu \nu} - \frac{10}{63} i F_{\kappa \lambda \nu} V F_{\nu \nu \nu} F_{\nu \nu \nu} \]
\[ + \frac{10}{63} i F_{\kappa \lambda \nu} V F_{\nu \nu \nu} F_{\nu \nu \nu} - \frac{10}{63} i F_{\kappa \lambda \nu} V F_{\nu \nu \nu} F_{\nu \nu \nu} - \frac{10}{63} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} F_{\nu \nu \nu} \]
\[ - \frac{10}{21} V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ - \frac{10}{21} V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ - \frac{10}{21} \ i F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} - \frac{11}{21} F_{\kappa \lambda \nu} F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ - \frac{11}{21} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ - \frac{11}{21} \ i F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} - \frac{11}{21} \ i F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ + \frac{11}{21} \ i F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{11}{21} \ i F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ - \frac{11}{21} i V V F_{\nu \nu \nu} F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} - \frac{11}{21} i V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} - \frac{11}{21} i V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ - \frac{11}{42} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{11}{42} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{11}{42} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ + \frac{11}{84} V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{11}{84} V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{11}{252} V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ - \frac{13}{18} i V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} - \frac{13}{18} i V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{13}{63} i V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ + \frac{13}{63} i V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{13}{63} i V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ + \frac{16}{21} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{16}{21} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{16}{21} F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ + \frac{17}{21} V V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{17}{21} V V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} + \frac{17}{21} V V F_{\kappa \lambda \nu} V F_{\nu \nu \nu} V_{\nu \nu} V_{\nu \nu} \]
\[ + \frac{17}{42} V F_{\kappa\lambda\mu} V_{\lambda} F_\nu \nu_{\nu} + \frac{17}{42} V F_{\kappa\lambda\mu} V_{\nu} F_\nu \nu_{\lambda} + \frac{17}{42} V V_{\nu} \lambda F_{\mu\nu} F_\lambda \nu_{\mu} \\
- \frac{17}{42} i V F_{\kappa\lambda\mu} V_{\lambda} F_\nu \nu_{\nu} + \frac{17}{42} V F_{\kappa\lambda\mu} V F_{\nu} \nu_{\nu} F_\lambda - \frac{19}{21} i V F_{\kappa\lambda\mu} V F_\nu \nu_{\lambda} \\
- \frac{19}{21} V V_{\kappa} F_{\mu\nu} V_{\nu} F_\lambda + \frac{19}{21} i F_{\kappa\lambda\mu} V_{\nu} F_{\nu} V_{\mu} - \frac{19}{21} i V V_{\kappa} F_{\mu\nu} F_\lambda F_{\nu} \\
- \frac{20}{63} i V F_{\kappa\lambda\mu} F_{\nu} \nu_{\nu} F_\nu - \frac{20}{63} i F_{\kappa\lambda\mu} V_{\nu} F_{\nu} F_\nu - \frac{20}{63} i V F_{\kappa\lambda\mu} F_{\nu} \nu_{\nu} V_{\nu} \\
- \frac{22}{21} F_{\kappa\lambda} F_{\mu\nu} V_{\nu} \nu_{\mu} - \frac{22}{21} F_{\kappa\lambda} V_{\nu} \nu_{\nu} F_{\nu} F_{\nu} \nu_{\mu} - \frac{22}{21} F_{\kappa\lambda} V_{\nu} \nu_{\nu} F_{\nu} F_{\nu} \nu_{\mu} \\
+ \frac{22}{21} i F_{\kappa\lambda} F_{\mu\nu} V_{\nu} F_{\nu} V_{\kappa} + \frac{23}{21} V V_{\nu} \lambda F_{\mu\nu} F_{\nu} V_{\kappa} - \frac{23}{21} i V V_{\nu} \lambda F_{\mu\nu} F_{\nu} V_{\kappa} \\
+ \frac{23}{42} V F_{\kappa\lambda\mu} F_{\mu\nu} V_{\nu} + \frac{23}{42} V F_{\kappa\lambda\mu} V_{\nu} F_{\nu} F_{\mu} - \frac{23}{42} i F_{\kappa\lambda\mu} V_{\nu} F_{\nu} F_{\nu} V_{\mu} \\
+ \frac{23}{84} V^2 F_{\kappa\lambda} F_{\mu\nu} F_{\nu} V_{\kappa} - \frac{23}{126} i V F_{\kappa\lambda} F_{\mu\nu} F_{\nu} V_{\kappa} - \frac{23}{126} i V F_{\kappa\lambda} V_{\nu} F_{\mu} F_{\nu} V_{\kappa} \\
- \frac{25}{42} i V F_{\kappa\lambda\mu} V_{\nu} \nu_{\nu} F_{\nu} - \frac{26}{63} i F_{\kappa\lambda\mu} V_{\nu} F_{\nu} V_{\lambda} - \frac{26}{63} i V F_{\kappa\lambda\mu} V_{\nu} F_{\nu} V_{\kappa} \\
+ \frac{26}{63} V V_{\kappa} F_{\mu\nu} F_{\nu} V_{\mu} - \frac{29}{63} F_{\kappa\lambda} V_{\nu} \nu_{\nu} F_{\nu} V_{\mu} - \frac{29}{63} F_{\kappa\lambda} V_{\nu} \nu_{\nu} F_{\nu} V_{\mu} \\
- \frac{29}{63} F_{\kappa\lambda} V_{\nu} \nu_{\nu} F_{\nu} V_{\mu} - \frac{31}{63} i V F_{\kappa\lambda} F_{\mu\nu} V_{\nu} F_{\mu} + \frac{31}{126} V F_{\kappa\lambda} V V_{\nu} F_{\mu} F_{\nu} F_{\kappa} \\
- \frac{31}{126} i V V_{\kappa} F_{\mu\nu} F_{\nu} V_{\mu} + \frac{32}{126} V F_{\kappa\lambda} F_{\nu} \nu_{\nu} F_{\nu} V_{\kappa} - \frac{34}{63} F_{\kappa\lambda} F_{\mu\nu} V_{\nu} V_{\mu} \\
+ \frac{37}{252} F_{\kappa\lambda\mu} V_{\nu} F_{\mu} \nu_{\nu} V_{\lambda} + \frac{37}{252} F_{\kappa\lambda\mu} V_{\nu} F_{\nu} \nu_{\nu} V_{\lambda} + \frac{38}{63} F_{\kappa\lambda\mu} V_{\nu} \nu_{\nu} V_{\nu} V_{\kappa} \\
+ \frac{43}{63} i F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\lambda} + \frac{43}{126} F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\lambda} + \frac{43}{126} F_{\kappa\lambda} V_{\nu} \nu_{\nu} V_{\lambda} \\
- \frac{43}{126} i F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\lambda} + \frac{43}{126} i F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\lambda} - \frac{43}{126} i V F_{\kappa\lambda} F_{\mu\nu} V_{\nu} F_{\mu} \\
+ \frac{46}{63} i V F_{\kappa\lambda\mu} V_{\nu} F_{\nu} V_{\kappa} - \frac{50}{63} i V F_{\kappa\lambda\mu} V_{\nu} F_{\nu} V_{\kappa} - \frac{50}{63} i V F_{\kappa\lambda\mu} V_{\nu} F_{\nu} V_{\kappa} \\
- \frac{53}{63} i V F_{\kappa\lambda\mu} F_{\nu} \nu_{\nu} V_{\nu} - \frac{61}{63} i V F_{\kappa\lambda\mu} F_{\nu} \nu_{\nu} V_{\nu} - \frac{61}{63} i V F_{\kappa\lambda\mu} F_{\nu} \nu_{\nu} V_{\nu} \\
- \frac{64}{63} i V F_{\kappa\lambda\mu} F_{\nu} \nu_{\nu} V_{\nu} - \frac{64}{63} i V F_{\kappa\lambda\mu} F_{\nu} \nu_{\nu} V_{\nu} - \frac{64}{63} i V F_{\kappa\lambda\mu} F_{\nu} \nu_{\nu} V_{\nu} \\
- \frac{74}{63} i V V_{\kappa} F_{\nu} \nu_{\nu} F_{\nu} V_{\vee} + \frac{97}{126} F_{\kappa\lambda\mu} V_{\nu} \nu_{\nu} V_{\nu} + \frac{97}{126} F_{\kappa\lambda\mu} V_{\nu} \nu_{\nu} V_{\nu} \\
+ \frac{97}{126} F_{\kappa\lambda\mu} V_{\nu} \nu_{\nu} V_{\nu} - \frac{97}{126} F_{\kappa\lambda\mu} V_{\nu} \nu_{\nu} V_{\nu} - \frac{97}{126} F_{\kappa\lambda\mu} V_{\nu} \nu_{\nu} V_{\nu} \\
+ \frac{1}{5} F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} - \frac{1}{5} F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} + \frac{1}{5} F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} \\
+ \frac{1}{5} i F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} - \frac{1}{5} i F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} + \frac{1}{5} i F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} \\
+ \frac{1}{7} F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} + \frac{1}{7} F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} + \frac{1}{7} F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} \\
+ \frac{1}{7} i F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} + \frac{1}{7} i F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} - \frac{1}{7} i F_{\kappa\lambda} F_{\nu} \nu_{\nu} V_{\nu} F_{\nu} F_{\rho} \\
\]
\[
\begin{align*}
+ \frac{3}{35} F_{\kappa \lambda} F_{\mu \nu} F_{\rho \kappa} F_{\nu \mu} & - \frac{3}{35} F_{\kappa \lambda} F_{\mu \nu} V_{\nu} F_{\nu \rho \kappa} & - \frac{3}{35} V F_{\kappa \lambda} F_{\kappa \mu} F_{\nu \rho \kappa} F_{\nu \\
- \frac{3}{35} V F_{\kappa \lambda} F_{\mu \nu} F_{\nu \rho \kappa} & - \frac{3}{35} V F_{\kappa \lambda} F_{\mu \nu} F_{\nu \rho \kappa} & + \frac{3}{35} i F_{\kappa \lambda} F_{\kappa \mu} F_{\nu \lambda} V_{\rho} \\
+ \frac{3}{35} i F_{\kappa \lambda} F_{\mu \nu} V_{\nu \mu} & - \frac{3}{35} i F_{\kappa \lambda} F_{\mu \nu} V_{\nu \rho \kappa} & + \frac{3}{35} i V F_{\kappa \lambda} F_{\kappa \mu} F_{\nu \rho \kappa} F_{\nu} \\
+ \frac{3}{70} F_{\kappa \lambda} F_{\mu \nu} F_{\nu \rho \kappa} V_{\mu} & + \frac{3}{70} F_{\kappa \lambda} F_{\mu \nu} V_{\nu} F_{\nu \rho \kappa} & + \frac{3}{70} V F_{\kappa \lambda} F_{\kappa \mu} F_{\nu \rho \kappa} F_{\nu} \\
+ \frac{3}{70} V F_{\kappa \lambda} F_{\mu \nu} F_{\nu \rho \kappa} & - \frac{4}{7} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} V_{\rho} & - \frac{4}{15} F_{\kappa \lambda} F_{\mu \nu} V_{\nu} F_{\nu \mu} \\
+ \frac{4}{15} F_{\kappa \lambda} F_{\mu \nu} V_{\nu \rho \kappa} & - \frac{4}{21} V F_{\kappa \lambda} F_{\nu \rho \kappa} F_{\nu \rho \kappa} & + \frac{4}{21} V F_{\kappa \lambda} F_{\nu \rho \kappa} F_{\nu \rho \kappa} \\
+ \frac{4}{21} V F_{\kappa \lambda} F_{\mu \nu} F_{\nu \rho \kappa} & - \frac{4}{21} i F_{\kappa \lambda} F_{\mu \nu} V_{\nu \mu} F_{\nu \kappa} & - \frac{4}{21} i V F_{\kappa \lambda} F_{\mu \nu} F_{\nu \lambda} F_{\nu \kappa} \\
- \frac{4}{35} i F_{\kappa \lambda} F_{\mu \nu} V_{\nu \rho \kappa} & - \frac{4}{35} i F_{\kappa \lambda} F_{\mu \nu} V_{\nu \mu} F_{\nu \kappa} & - \frac{4}{35} i V F_{\kappa \lambda} F_{\kappa \mu} F_{\nu \rho \kappa} F_{\nu} \\
+ \frac{4}{35} i V F_{\kappa \lambda} F_{\mu \nu} V_{\nu \rho \kappa} F_{\nu \mu} & + \frac{4}{105} F_{\kappa \lambda} F_{\mu \nu} V_{\mu} F_{\rho \nu \kappa} & + \frac{4}{105} F_{\kappa \lambda} F_{\mu \nu} V_{\nu} F_{\nu \rho \kappa} F_{\nu} \\
- \frac{4}{315} V F_{\kappa \lambda} F_{\mu \nu} F_{\nu \rho \kappa} & + \frac{4}{315} i F_{\kappa \lambda} F_{\mu \nu} V_{\nu \lambda} F_{\nu \rho \kappa} & + \frac{4}{315} i V F_{\kappa \lambda} F_{\mu \nu} V_{\nu \kappa} F_{\nu \rho \kappa} \\
- \frac{5}{21} i F_{\kappa \lambda} F_{\mu \nu} V_{\nu} F_{\rho \mu \kappa} & + \frac{5}{21} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \rho \kappa} & + \frac{5}{21} i V F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \kappa} \\
+ \frac{5}{28} V F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \rho \kappa} & + \frac{5}{63} F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} V_{\nu \rho \kappa} & + \frac{5}{63} F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \rho \kappa} \\
+ \frac{5}{63} i V F_{\kappa \lambda} F_{\mu \nu} F_{\nu \rho \kappa} F_{\nu \mu} & + \frac{5}{63} i F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} F_{\nu \rho \kappa} & + \frac{5}{126} i V F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} F_{\nu \rho \kappa} F_{\nu} \\
- \frac{5}{126} V F_{\kappa \lambda} F_{\nu \mu} F_{\nu \rho \kappa} & + \frac{5}{126} i F_{\kappa \lambda} F_{\kappa \nu \mu} F_{\nu \mu} F_{\nu \rho \kappa} & - \frac{6}{35} V F_{\kappa \lambda} F_{\kappa \nu \mu} F_{\nu \mu} F_{\nu \rho \kappa} F_{\nu} \\
+ \frac{6}{35} i F_{\kappa \lambda} F_{\kappa \nu \mu} V_{\nu} F_{\rho \mu} & + \frac{6}{35} i F_{\kappa \lambda} F_{\nu \mu} V_{\nu \kappa} F_{\rho \mu} & + \frac{6}{35} i V F_{\kappa \lambda} F_{\kappa \nu \mu} F_{\kappa \nu \mu} F_{\rho \mu} \\
- \frac{7}{18} i F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} V_{\nu} & - \frac{8}{21} i F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} V_{\nu \kappa} & - \frac{8}{21} i F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} F_{\nu \rho \kappa} F_{\nu} \\
- \frac{8}{35} i F_{\kappa \lambda} F_{\nu \mu} F_{\nu \rho \kappa} & - \frac{8}{35} i F_{\kappa \lambda} F_{\nu \mu} V_{\nu \mu} F_{\nu \rho \kappa} & - \frac{8}{35} i V F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} F_{\nu \rho \kappa} F_{\nu} \\
- \frac{9}{35} i F_{\kappa \lambda} F_{\nu \mu} F_{\nu \rho \kappa} & - \frac{9}{35} i F_{\kappa \lambda} F_{\nu \mu} F_{\nu \rho \kappa} F_{\nu \mu} & - \frac{9}{70} F_{\kappa \lambda} F_{\nu \mu} F_{\nu \kappa} F_{\kappa \nu \mu} \\
- \frac{10}{35} V F_{\kappa \lambda} F_{\kappa \nu \mu} F_{\rho \mu} & - \frac{10}{63} V F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} F_{\rho \mu} & - \frac{10}{63} V F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} F_{\rho \mu} F_{\nu} \\
- \frac{10}{63} V F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} F_{\kappa \rho \nu} & - \frac{10}{63} i V F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} F_{\nu \rho \kappa} F_{\nu} & - \frac{11}{35} V F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} F_{\nu \rho \kappa} F_{\nu} \\
- \frac{11}{35} i F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} F_{\nu \rho \kappa} & - \frac{11}{42} F_{\kappa \lambda} F_{\nu \mu} V_{\nu \mu} F_{\nu \rho \kappa} & - \frac{11}{42} V F_{\kappa \lambda} F_{\nu \mu} F_{\nu \mu} F_{\nu \rho \kappa} F_{\nu} \\
\end{align*}
\]
\[
\begin{align*}
&+ \frac{23}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{23}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{23}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&+ \frac{26}{315} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{26}{315} V F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&+ \frac{31}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{31}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{31}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&+ \frac{34}{315} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{34}{315} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{34}{315} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&- \frac{37}{210} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{37}{210} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{37}{210} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&- \frac{43}{126} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{43}{126} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{43}{126} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&+ \frac{46}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{46}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{46}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&+ \frac{47}{420} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{47}{420} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{47}{420} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&- \frac{53}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{53}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{53}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&+ \frac{58}{315} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{58}{315} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{58}{315} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&+ \frac{61}{252} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{61}{252} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{61}{252} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&+ \frac{67}{630} i V F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{67}{630} i V F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{67}{630} i V F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&- \frac{71}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{71}{315} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{71}{315} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&- \frac{73}{315} V F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{73}{315} V F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{73}{315} V F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&- \frac{73}{630} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{73}{630} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{73}{630} i F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&- \frac{83}{630} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{83}{630} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{83}{630} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&- \frac{89}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{89}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{89}{315} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&+ \frac{1260}{1260} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{1260}{1260} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{1260}{1260} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&+ \frac{1260}{1260} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{1260}{1260} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} + \frac{1260}{1260} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} \\
&- \frac{1260}{1260} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{1260}{1260} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho} - \frac{1260}{1260} F_{\kappa \lambda} F_{\mu \nu} F_{\kappa \rho \nu} V_{\rho}
\end{align*}
\]
\[-\frac{7}{188} F_{\kappa \lambda} F_{\mu \nu} F_{\lambda \kappa} F_{\rho \sigma \rho} F_{\rho \sigma \mu} - \frac{7}{396} F_{\kappa \lambda \mu} F_{\nu \kappa \rho} F_{\kappa \lambda \sigma} F_{\sigma \rho \nu} - \frac{7}{396} F_{\kappa \lambda \mu} F_{\nu \kappa \rho} F_{\sigma \mu \lambda} F_{\nu \sigma \rho} \]

\[-\frac{8}{87} i F_{\kappa \lambda} F_{\mu \nu} F_{\mu \rho} F_{\nu \kappa} F_{\mu \rho} - \frac{7}{231} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\rho \sigma} F_{\mu \rho} F_{\rho \sigma} + \frac{8}{231} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\rho \sigma} F_{\mu \rho} + \frac{8}{231} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\rho \sigma} F_{\mu \rho} \]

\[-\frac{9}{55} i F_{\kappa \lambda} F_{\mu \nu} F_{\sigma \rho \rho} F_{\sigma \nu} F_{\sigma \nu} - \frac{10}{231} i F_{\kappa \lambda} F_{\nu \mu} F_{\nu \rho} F_{\mu \sigma} F_{\rho \nu} \]

\[-\frac{11}{315} i F_{\kappa \lambda} F_{\mu \nu} F_{\lambda \rho} F_{\nu \mu} F_{\kappa \sigma} + \frac{13}{165} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \rho} F_{\nu \mu} F_{\kappa \sigma} - \frac{13}{420} \]

\[-\frac{13}{462} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{13}{693} i F_{\kappa \lambda} F_{\mu \nu} F_{\mu \rho} F_{\rho \mu} F_{\sigma \mu} F_{\nu \sigma} \]

\[-\frac{19}{77} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \rho} F_{\nu \sigma} F_{\kappa \sigma} - \frac{19}{198} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \rho} F_{\nu \sigma} F_{\nu \sigma} \]

\[-\frac{19}{315} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\sigma \mu} F_{\nu \sigma} + \frac{19}{462} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \sigma} \]

\[-\frac{19}{924} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{19}{1155} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} \]

\[-\frac{19}{231} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{19}{693} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} \]

\[-\frac{23}{126} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{23}{198} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} \]

\[-\frac{23}{231} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{23}{252} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} \]

\[-\frac{23}{462} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{23}{693} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} \]

\[-\frac{23}{3465} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{25}{154} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{25}{231} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} \]

\[-\frac{25}{1386} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{26}{385} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{26}{495} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} \]

\[-\frac{26}{495} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{26}{693} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} \]

\[-\frac{26}{693} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} + \frac{27}{154} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} - \frac{27}{385} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} \]

\[-\frac{27}{770} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} - \frac{27}{495} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} \]

\[-\frac{27}{770} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} - \frac{29}{495} i F_{\kappa \lambda} F_{\mu \nu} F_{\nu \mu} F_{\nu \mu} F_{\nu \mu} \]
\[\begin{align*}
+ \frac{398}{3465} F_{\kappa\lambda} F_{\mu\nu} F_{\kappa\nu} F_{\mu\rho} F_{\kappa\sigma} F_{\rho\sigma} &+ \frac{404}{3465} i F_{\kappa\lambda} F_{\mu\lambda} F_{\kappa\rho} F_{\mu\sigma} F_{\nu\sigma} + \frac{404}{3465} i F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\rho} F_{\sigma\nu} F_{\rho\mu} \\
- \frac{421}{3465} F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\rho} F_{\kappa\nu} F_{\rho\mu} &- \frac{428}{3465} F_{\kappa\lambda\mu} F_{\kappa\rho} F_{\mu\rho} F_{\lambda\sigma} F_{\kappa\sigma} + \frac{452}{3465} i F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\rho} F_{\kappa\sigma} F_{\rho\sigma} F_{\kappa\mu} \\
- \frac{458}{3465} F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\rho} F_{\kappa\nu} F_{\rho\mu} F_{\kappa\sigma} &- \frac{458}{3465} F_{\kappa\lambda\mu} F_{\mu\rho} F_{\nu\rho} F_{\kappa\sigma} F_{\sigma\rho} + \frac{479}{3465} i F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\rho} F_{\kappa\nu} F_{\kappa\mu} F_{\sigma\rho} \\
- \frac{499}{6930} F_{\kappa\lambda} F_{\mu\nu} F_{\rho\sigma} F_{\nu\rho} F_{\kappa\sigma} &- \frac{512}{3465} F_{\kappa\lambda} F_{\mu\nu} F_{\kappa\rho} F_{\nu\rho} F_{\kappa\sigma} F_{\sigma\rho} + \frac{541}{3465} i F_{\kappa\lambda} F_{\mu\nu} F_{\rho\sigma} F_{\kappa\nu} F_{\sigma\mu} F_{\rho\kappa} \\
- \frac{541}{3465} F_{\kappa\lambda} F_{\mu\nu} F_{\rho\sigma} F_{\nu\rho} F_{\kappa\sigma} &- \frac{541}{3465} F_{\kappa\lambda} F_{\mu\nu} F_{\mu\nu} F_{\nu\rho} F_{\kappa\sigma} F_{\sigma\mu} - \frac{541}{3465} i F_{\kappa\lambda} F_{\mu\nu} F_{\rho\sigma} F_{\mu\nu} F_{\kappa\rho} F_{\sigma\kappa} \\
+ \frac{554}{6930} F_{\kappa\lambda} F_{\mu\nu} F_{\mu\rho} F_{\nu\sigma} F_{\kappa\rho} &+ \frac{554}{3465} i F_{\kappa\lambda} F_{\mu\nu} F_{\nu\rho} F_{\nu\sigma} F_{\kappa\rho} + \frac{569}{3465} i F_{\kappa\lambda} F_{\mu\nu} F_{\mu\nu} F_{\nu\sigma} F_{\nu\sigma} F_{\kappa\rho} F_{\sigma\kappa} \\
+ \frac{13860}{6930} F_{\kappa\lambda} F_{\mu\nu} F_{\mu\rho} F_{\mu\rho} F_{\nu\sigma} &+ \frac{13860}{3465} i F_{\kappa\lambda} F_{\mu\nu} F_{\nu\rho} F_{\mu\rho} F_{\nu\sigma} F_{\kappa\rho} + \frac{743}{3465} i F_{\kappa\lambda} F_{\nu\rho} F_{\rho\mu} F_{\sigma\kappa} F_{\sigma\nu} F_{\nu\lambda} \\
- \frac{743}{3465} F_{\kappa\lambda} F_{\nu\rho} F_{\kappa\rho} F_{\nu\sigma} F_{\kappa\nu} &- \frac{761}{3465} i F_{\kappa\lambda} F_{\nu\rho} F_{\nu\rho} F_{\kappa\rho} F_{\kappa\sigma} F_{\sigma\rho} - \frac{793}{3465} i F_{\kappa\lambda} F_{\nu\rho} F_{\nu\rho} F_{\rho\mu} F_{\sigma\kappa} F_{\sigma\kappa} \\
- \frac{794}{3465} F_{\kappa\lambda} F_{\nu\rho} F_{\kappa\rho} F_{\nu\rho} F_{\kappa\sigma} &- \frac{838}{3465} i F_{\kappa\lambda} F_{\nu\rho} F_{\nu\rho} F_{\kappa\rho} F_{\nu\sigma} F_{\kappa\sigma} - \frac{851}{6930} i F_{\kappa\lambda} F_{\nu\rho} F_{\nu\rho} F_{\nu\sigma} F_{\mu\kappa} \\
+ \frac{949}{13860} F_{\kappa\lambda} F_{\nu\rho} F_{\kappa\rho} F_{\sigma\mu} F_{\nu\sigma} &+ \frac{949}{3465} i F_{\kappa\lambda} F_{\nu\rho} F_{\nu\rho} F_{\sigma\mu} F_{\sigma\nu} F_{\nu\sigma} + \frac{953}{13860} i F_{\kappa\lambda} F_{\nu\rho} F_{\mu\nu} F_{\kappa\rho} F_{\sigma\nu} \\
+ \frac{10395}{1718} F_{\kappa\lambda} F_{\nu\rho} F_{\nu\rho} F_{\mu\sigma} F_{\sigma\nu} &+ \frac{10395}{1718} i F_{\kappa\lambda} F_{\nu\rho} F_{\nu\rho} F_{\mu\sigma} F_{\sigma\nu} F_{\nu\sigma}
\end{align*}\]
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