EXACT SMOOTH PIECEWISE POLYNOMIAL SEQUENCES ON ALFELD SPLITS

GUOSHENG FU, JOHNNY GUZMÁN AND MICHAEL NEILAN

Abstract. We develop exact piecewise polynomial sequences on Alfeld splits in any spatial dimension and any polynomial degree. An Alfeld split of a simplex is obtained by connecting the vertices of an n-simplex with its barycenter. We show that, on these triangulations, the kernel of the exterior derivative has enhanced smoothness. Byproducts of this theory include characterizations of discrete divergence-free subspaces for the Stokes problem, commutative projections, and simple formulas for the dimensions of smooth polynomial spaces.

1. Introduction

An Alfeld split (or refinement) of an n-dimensional simplex is obtained by connecting each vertex of the simplex with its barycenter [1, 19, 12]. This is also known as a barycenter refinement in some communities [18, 5, 14]. Such meshes are useful in several areas of computational mathematics. For example, one can construct relatively low-order $C^1$ finite elements on Alfeld splits. This is the idea behind the famous (cubic) Clough-Tocher finite elements in two dimensions [8], and the (quintic) Alfeld elements in three dimensions [1]. This family of triangulations have also been used to develop simple low-order, inf-sup stable, and divergence-free yielding finite elements for the Stokes and Navier-Stokes problems; see [2] for two dimensional case and [21] for the three dimensional case.

In this paper we will show that these $C^1$ finite elements and Stokes finite element pairs are connected via an exact sequence consisting of piecewise polynomial spaces. The sequence is a de Rham complex, but where the finite element spaces have extra smoothness as compared to the canonical Whitney-Nédélec spaces; see [3, 16, 17].

As a first step to prove these results, we take a single non-degenerate n-dimensional simplex $T$ ($n \geq 2$), and consider the split (mesh) $T^z$ which is obtained by adjoining the vertices of $T$ to its barycenter $z$. We study $k$-forms with piecewise polynomial coefficients on these (local) meshes, and show that the kernel of the exterior derivative has enhanced smoothness properties. In particular, if $\omega$ is a piecewise polynomial $k$-form on $T^z$ with vanishing exterior derivative, then there exists a continuous piecewise polynomial $(k+1)$-form $\rho$ such that $\omega = d\rho$ (cf. Theorems 3.1–3.2). The case $k = n - 1$ has been recently established in [11]. This result allows us to develop $n$ new (local) de Rham complexes consisting of piecewise polynomials. In addition, a simple byproduct of this result is a dimension formula of the space of continuous, piecewise polynomial $k$-forms with continuous exterior derivative. For example, we are able to recover the dimension of local $C^1$ spaces on Alfeld splits by taking $k = 0$ in this framework [12]. Another instance ($k = 1$, $n = 3$) is the dimension of the space consisting of continuous piecewise polynomial vector-fields whose curl is continuous.

We then develop unisolvent sets of degrees of freedom for the spaces in the complexes in three dimensions. These degrees of freedom induce projections onto these spaces that commute with the differential operator, and allow us to formulate the global finite element spaces and three global discrete complexes with varying level of regularity. One of the sequences connects the $C^1$ finite element space of Alfeld [1] to the inf-sup stable Stokes pair of Zhang [21] globally. This is done by introducing a new $H^1(\text{curl})$-conforming finite element space that may be useful for fourth order curl problems [22]. Finally, we show that the complexes are exact on contractible domains.

We mention that Christiansen and Hu [6] have recently studied smoothed discrete de Rham sequences in any dimension. Their triangulations have different splits, and they only considered low-order polynomial approximations in higher dimensions.
The paper is organized as follows: In Section 2 we give preliminary results on differential forms on one simplex. In Section 3 we define finite element spaces on an Alfeld split of a single simplex. Important surjectivity properties of the exterior derivative are established. In Section 4 we focus on the three dimensional case. We provide degrees of freedom of several finite element spaces that induce projections that satisfy commuting diagrams. In Section 5 we define the corresponding global finite element complexes. We show exactness properties on contractible domains. Finally in Section 6 we summarize our results and state possible future directions.

2. Polynomial differential forms on a simplex

Let \( T = [x_0, \ldots, x_n] \) be an \( n \)-simplex with vertices \( \{x_i\}_{i=0}^n \). We denote by \( \Delta_s(T) \) the set of \( s \)-simplices of \( T \). We note that the cardinality of \( \Delta_s(T) \) is \( \binom{n+1}{s+1} \). We let \( t_i = x_i - x_0 \) for \( 1 \leq i \leq n \) and assume that the determinant of the matrix \( [t_1, \ldots, t_n] \) is positive. We let \( \lambda_i \) for \( 0 \leq i \leq n \) be the barycentric coordinates for \( T \), that is, \( \lambda_i \) is the unique linear function such that \( \lambda_i(x_j) = \delta_{ij}, \ 0 \leq i,j \leq n \). We denote by \( F_i \) the face of \( T \) opposite to \( x_i \), that is \( F_i = [x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \), where \( \cdot \) represents omission. Note that \( \lambda_i \) vanishes on \( F_i \). The differential \( d\lambda_i : \mathbb{R}^n \to \mathbb{R} \) is given by \( d\lambda_i(r) = \text{grad} \lambda_i \cdot r \). For integer \( k \in [1, n] \), and \( 0 \leq \sigma(1) < \sigma(2) < \cdots < \sigma(k) \leq n \), we define the \( k \)-form \( d\lambda_{\sigma(1)} \wedge d\lambda_{\sigma(2)} \wedge \cdots \wedge d\lambda_{\sigma(k)} \) as follows:

\[
(d\lambda_{\sigma(1)} \wedge d\lambda_{\sigma(2)} \wedge \cdots \wedge d\lambda_{\sigma(k)})(v_1, v_2, \ldots, v_k) := \det[d\lambda_{\sigma(i)}(v_j)],
\]

where \( v_1, \ldots, v_k \in \mathbb{R}^n \).

We define the space of polynomials on \( T \) with respect to the barycentric coordinates:

\[
\mathcal{P}_r(T) := \left\{ \sum_{|\alpha| \leq r} a_\alpha \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} : a_\alpha \in \mathbb{R} \right\},
\]

and we use the convention \( \mathcal{P}_r(T) = \{0\} \) if \( r \) is negative. If \( f = [x_{r(0)}, x_{r(1)}, \ldots, x_{r(s)}] \in \Delta_s(T) \) is a \( s \) sub-simplex of \( T \) where \( \tau : \{0, 1, \ldots, s\} \to \{0, 1, \ldots, n\} \) is increasing, then we define

\[
\mathcal{P}_r(f) := \left\{ \sum_{|\alpha| \leq r} a_\alpha \lambda_{\tau(1)}^{\alpha_1} \cdots \lambda_{\tau(s)}^{\alpha_s} : a_\alpha \in \mathbb{R} \right\}.
\]

Using the short-hand notation \( d\lambda_\sigma = d\lambda_{\sigma(1)} \wedge d\lambda_{\sigma(2)} \wedge \cdots \wedge d\lambda_{\sigma(k)} \), we define the space of \( k \)-forms with polynomial coefficients on \( T \) as follows:

\[
\mathcal{P}_r \Lambda^k(T) := \left\{ \sum_{\sigma \in \Sigma(k,n)} a_\sigma d\lambda_\sigma : a_\sigma \in \mathcal{P}_r(T) \right\},
\]

where \( \Sigma(k,n) \) is the set of increasing maps \( \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\} \). If \( f \in \Delta_s(T) \) with \( s \geq k \), then

\[
\mathcal{P}_r \Lambda^k(f) := \left\{ \sum_{\sigma \in \Sigma(k,s)} a_\sigma d\lambda_{\tau \circ \sigma} : a_\sigma \in \mathcal{P}_r(f) \right\},
\]

where here we used the notation \( \tau \circ \sigma = \{\tau(\sigma(1)), \tau(\sigma(2)), \ldots, \tau(\sigma(k))\} \).

For a polynomial \( a \in \mathcal{P}_r(T) \), we see that the 1-form \( da \) is given as

\[
da(\lambda_1, \ldots, \lambda_n) = \sum_{j=1}^n \frac{\partial a}{\partial \lambda_j}(\lambda_1, \ldots, \lambda_n) d\lambda_j.
\]

If \( \omega = \sum_\sigma a_\sigma d\lambda_\sigma \in \mathcal{P}_r \Lambda^k(T) \) then

\[
d\omega = \sum_\sigma da_\sigma \wedge d\lambda_\sigma,
\]

and therefore \( d\omega \in \mathcal{P}_{r-1} \Lambda^{k+1}(T) \).

The Koszul operator can be defined using barycentric coordinates:

\[
k\omega = \sum_\sigma \sum_{i=1}^k (-1)^{i+1} a_\sigma \lambda_{\sigma(i)} d\lambda_{\sigma(1)} \wedge \cdots \wedge d\lambda_{\sigma(i)} \wedge \cdots \wedge d\lambda_{\sigma(k)}.
\]
Hence, \( \kappa \omega \in \mathcal{P}_{r+1} \Lambda^{k-1}(T) \).

Suppose that \( \omega = \sum_{\sigma} a_{\sigma} d\lambda_{\sigma} \in \mathcal{P}_{r} \Lambda^{k}(T) \) and suppose that \( f = [\tau(0), \tau(1), \tau(2), \ldots, \tau(s)] \) is an \( s \)-simplex of \( T \). Then the trace of \( \omega \) on \( f \) is given by

\[
\text{tr}_f \omega = \sum_{\sigma \subset \tau} \text{tr}_f a_{\sigma} d\lambda_{\sigma} \in \mathcal{P}_{r} \Lambda^{k}(f),
\]

where \( \text{tr}_f a_{\sigma} := a_{\sigma} | f \) is simply the restriction of \( a_{\sigma} \) to \( f \). We say that \( \sigma \subset \tau \) if \( \{\sigma(1), \ldots, \sigma(k)\} \subset \{\tau(0), \tau(1), \ldots, \tau(s)\} \).

We define the space

\[
\mathcal{P}_r \Lambda^{k}(f) = \mathcal{P}_{r-1} \Lambda^{k}(f) + \kappa\mathcal{P}_{r+1} \Lambda^{k+1}(f).
\]

The following result is contained in [3, Theorem 4.8].

**Proposition 2.1.** Let \( \omega \in \mathcal{P}_r \Lambda^{k}(T) \). Then if \( r \geq 1 \), \( \omega \) is uniquely determined by

\[
\int_f \text{tr}_f \omega \wedge \eta \quad \text{for all } \eta \in \mathcal{P}_{r+k-s} \Lambda^{s-k}(f), \ f \in \Delta_s(T), \ s \geq k.
\]

We also need a result in the case \( r = 0 \). To do this, we first state a result from Arnold et al. [3, Lemma 4.6].

**Proposition 2.2.** Let \( \omega \in \mathcal{P}_r \Lambda^{k}(T) \). Suppose that \( \text{tr}_F \omega = 0 \), for \( 1 \leq i \leq n \) and

\[
\int_T \omega \wedge \eta \quad \text{for all } \eta \in \mathcal{P}_{r-n+k} \Lambda^{n-k}(T).
\]

Then, \( \omega = 0 \). In particular, if \( \omega \in \mathcal{P}_0 \Lambda^{k}(T) \) with \( k \leq n-1 \) satisfies \( \text{tr}_F \omega = 0 \) for \( 1 \leq i \leq n \), then \( \omega = 0 \).

**Lemma 2.3.** Define the set of \( k \)-simplices that have \( x_0 \) as a vertex:

\[
S_k(T, x_0) := \{ f \in \Delta_k(T) : x_0 \in \Delta_0(f) \}.
\]

Then any \( \omega \in \mathcal{P}_0 \Lambda^{k}(T) \) is uniquely determined by

\[
(2.1) \quad \int_f \text{tr}_f \omega \quad \text{for all } f \in S_k(T, x_0).
\]

**Proof.** We have that \( \dim \mathcal{P}_0 \Lambda^{k}(T) = \binom{n}{k} \) which is exactly the cardinality of \( S_k(T, x_0) \). Thus to prove the result, we show that if \( \omega \in \mathcal{P}_0 \Lambda^{k}(T) \) vanishes on \( (2.1) \), then \( \omega = 0 \). The result is clearly true if \( k = n \) by Proposition 2.2 and so we assume that \( k \leq n-1 \).

Suppose that \( \omega \in \mathcal{P}_0 \Lambda^{k}(T) \) vanishes on \( (2.1) \), so that \( \text{tr}_f \omega = 0 \) for \( f \in S_k(T, x_0) \). For any \( f \in S_{k+1}(T, x_0) \) it is easy to see that the cardinalities of the sets \( \Delta_k(f) \) and \( \Delta_k(f) \cap S_k(x_0, T) \) are \( (k+2) \) and \( (k+1) \), respectively. Therefore, using Proposition 2.2, we have \( \text{tr}_f \omega = 0 \) for all \( f \in S_{k+1}(T, x_0) \). We continue by induction to conclude that \( \text{tr}_f \omega = 0 \) for any \( f \in S_{n-1}(T, x_0) \). Finally we apply Proposition 2.2 once more to get \( \omega = 0 \).

We will also need the following two lemmas.

**Lemma 2.4.** Suppose that \( \omega \in \mathcal{P}_r \Lambda^{k}(T) \) satisfies \( \text{tr}_F \omega = 0 \) for some \( i \in \{0, 1, \ldots, n\} \). Then,

\[
\omega = d\lambda_i \wedge v + \lambda_i w,
\]

where \( v \in \mathcal{P}_r \Lambda^{k-1}(T) \) and \( w \in \mathcal{P}_{r-1} \Lambda^{k}(T) \).

**Proof.** Without loss of generality we assume that \( i = n \).

We note that we can write \( \omega \) is the following form:

\[
\omega = \sum_{\sigma \in \Sigma(k,n)} a_{\sigma} d\lambda_{\sigma} = \sum_{\sigma \in \Sigma(k,n-1)} a_{\sigma} d\lambda_{\sigma} + \sum_{\sigma \in \Sigma(k,n)} a_{\sigma} d\lambda_{\sigma(1)} \wedge \cdots \wedge d\lambda_{\sigma(k-1)} \wedge d\lambda_n
\]

where
with $a_\sigma \in \mathcal{P}_r(T)$. We then have $0 = \text{tr}_{\mathcal{P}_n} \omega = \sum_{\sigma \in \Sigma(k,n-1)} (\text{tr}_{\mathcal{P}_n} a_\sigma) d\lambda_\sigma$, and so $\text{tr}_{\mathcal{P}_n} a_\sigma = 0$ for all $\sigma \in \Sigma(k,n-1)$. Therefore $a_\sigma = \lambda_n b_\sigma$ for some $b_\sigma \in \mathcal{P}_{r-1}(T)$. The desired result now follows upon setting

$$v = (-1)^{k-1} \sum_{\sigma \in \Sigma(k,n)} a_\sigma d\lambda_{\sigma(1)} \wedge \cdots \wedge d\lambda_{\sigma(k-1)}, \quad w = \sum_{\sigma \in \Sigma(k,n-1)} b_\sigma d\lambda_\sigma.$$

\[ \square \]

**Lemma 2.5.** Suppose that $\omega = d\lambda_i \wedge v$ with $v \in \mathcal{P}_r \Lambda^{k-1}(T)$ for some $i \in \{0, 1, \ldots, n\}$. Then if $\text{tr}_{\mathcal{P}_n} v = 0$, there holds $\omega |_{F_i} = 0$, and so $\omega = \lambda_i w$ for some $w \in \mathcal{P}_{r-1} \Lambda^k(T)$. Conversely, if $\omega |_{F_i} = 0$ then we have $\text{tr}_{\mathcal{P}_n} v = 0$.

**Proof.** Without loss of generality, we assume that $i = n$. Let $t_j = x_j - x_0$ for $1 \leq j \leq n$. Then $\{t_j\}_{j=1}^n$ forms a basis of $\mathbb{R}^n$, and $\{t_j\}_{j=1}^{n-1}$ is a basis of the tangent space of $F_n$. We also have $d\lambda_n(t_n) = 1$ and $d\lambda_n(t_j) = 0$ for $1 \leq j \leq n-1$.

Suppose that $\omega = d\lambda_n \wedge v$ with $v \in \mathcal{P}_r \Lambda^{k-1}(T)$ and $\text{tr}_{\mathcal{P}_n} v = 0$. For vectors $r_i \in \mathbb{R}^n$ ($1 \leq i \leq k$), we can write $r_i = \sum_{j=1}^n a_{ij} t_j$ where $a_{ij} \in \mathbb{R}$. Then, if $x \in F_n$, we have

$$\omega_x(r_1, r_2, \ldots, r_k) = \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n a_{1j_1} a_{2j_2} \cdots a_{kj_k} \omega_x(t_{j_1}, t_{j_2}, \ldots, t_{j_k})$$

$$= \sum_{1 \leq j_1, \ldots, j_k \leq n} a_{1j_1} a_{2j_2} \cdots a_{kj_k} (d\lambda_n \wedge v)_x(t_{j_1}, t_{j_2}, \ldots, t_{j_k})$$

$$= \sum_{1 \leq j_1, \ldots, j_k \leq n} a_{1j_1} a_{2j_2} \cdots a_{kj_k} \sum_{\ell=1}^k (-1)^{\ell-1} d\lambda_n(t_{j_\ell}) v_x(t_{j_1}, \ldots, \widehat{t_{j_\ell}}, \ldots, t_{j_k})$$

$$= \sum_{1 \leq j_1, \ldots, j_k \leq n} a_{1j_1} a_{2j_2} \cdots a_{kj_k} \sum_{\ell=1}^k (-1)^{\ell-1} \text{tr}_{\mathcal{P}_n} v_x(t_{j_1}, \ldots, \widehat{t_{j_\ell}}, \ldots, t_{j_k})$$

$$= 0.$$

Since $x \in F_n$ and $r_1, \ldots, r_k \in \mathbb{R}^n$ where arbitrary, we conclude that $\omega |_{F_n} = 0$.

Now assume that $\omega |_{F_n} = 0$. Then for any $x \in F_n$ we have

$$0 = \omega_x(t_{j_1}, t_{j_2}, \ldots, t_{j_{k-1}}) = \text{tr}_{\mathcal{P}_n} v_x(t_{j_1}, t_{j_2}, \ldots, t_{j_{k-1}})$$

for any $1 \leq j_1 < \ldots < j_{k-1} \leq n - 1$. This implies that $\text{tr}_{\mathcal{P}_n} v = 0$.

\[ \square \]

### 3. Polynomial differential forms on an Alfeld split

Here, we apply the results of the previous section to derive some exactness properties of polynomial differential forms on an Alfeld split simplex. As before, we let $T = [x_0, \ldots, x_n]$ be an $n$-simplex. We set $z = \frac{1}{n+1} \sum_{i=0}^n x_n$ to be the barycenter of $T$, and we subdivide $T$ into $(n+1)$ $n$-simplices by adjoining the vertices of $T$ with $z$. Namely, we set $T_i = [z, x_0, \ldots, \widehat{x_i}, \ldots, x_n]$ so that $T = \cup_{i=0}^n T_i$. We set $T^z = \{T_0, \ldots, T_n\}$ to be the mesh of this sub-division. We denote the set of $s$-dimensional simplices in $T^z$ as $\Delta_s(T^z) = \{f \in \Delta_s(T_i) : T_i \in T^z\}$. The cardinality of this set is given by

$$\#\Delta_s(T^z) = \begin{cases} \binom{n+2}{s+1} & \text{for } s \leq n-1, \\ n+1 & \text{for } s = n. \end{cases}$$

We let $\mu$ be the hat function associated with the barycenter $z$, that is, $\mu$ is uniquely determined by the conditions $\mu|_{T_i} \in \mathcal{P}_1(T_i)$, $\mu \in H^0_0(T)$, and $\mu(z) = 1$. We denote by $\mu_i$ the restriction of $\mu$ to $T_i$, and we note that $\mu_i = (n+1)\lambda_i$ for $i = 0, 1, \ldots, n$. 


We define the following local spaces:

\[\begin{align*}
V^k(T^z) &:= \{ \omega \in L^2(\mathcal{A}^k(T)) : \omega|_T \in \mathcal{P}_r\mathcal{A}^k(T_i) \text{ for } 0 \leq i \leq n \}, \\
V^k_{d,r}(T^z) &:= \{ \omega \in V^k(T^z) : d\omega \in L^2(\mathcal{A}^{k+1}(T)) \}, \\
M^k_r(T^z) &:= \{ \omega \in C^0(\mathcal{A}^k(T)) : \omega|_T \in \mathcal{P}_r\mathcal{A}^k(T_i) \text{ for } 0 \leq i \leq n \}, \\
M^k_{d,r}(T^z) &:= \{ \omega \in M^k_r(T^z) : d\omega \in C^0(\mathcal{A}^{k+1}(T)) \}.
\end{align*}\]

We also define the analogous spaces with homogenous boundary conditions for \(k \leq n-1\):

\[\begin{align*}
\hat{V}^k_{d,r}(T^z) &:= \{ \omega \in V^k_{d,r}(T^z) : \text{tr}_F \omega = 0 \text{ for all } F \in \Delta_{n-1}(T) \}, \\
\hat{M}^k_r(T^z) &:= \{ \omega \in M^k_r(T^z) : \omega|_{\partial T} = 0 \}, \\
\hat{M}^k_{d,r}(T^z) &:= \{ \omega \in M^k_{d,r}(T^z) : \omega|_{\partial T} = 0, d\omega|_{\partial T} = 0 \}.
\end{align*}\]

In the case \(k = n\), we will also set average zero constraints:

\[\begin{align*}
\hat{V}^n_{d,r}(T^z) &:= \{ \omega \in V^n_{d,r}(T^z) : \int_T \omega = 0 \}, \\
\hat{M}^n_{d,r}(T^z) &:= \{ \omega \in M^n_{d,r}(T^z) : \omega|_{\partial T} = 0, \int_T \omega = 0 \}.
\end{align*}\]

It is well known (see, for example, [3]) that \(\text{tr}_f \omega\) is single-valued for \(\omega \in V^k_{d,r}(T^z)\) for all \(f \in \Delta_s(T^z)\) and \(s \geq k\). Moreover, if \(\omega \in V^k_{d,r}(T^z)\) (resp., \(\omega \in \hat{V}^k_{d,r}(T^z)\)) with \(d\omega = 0\), then there exists a \(\rho \in V^{k-1}_{d,r+1}(T^z)\) (resp., \(\rho \in \hat{V}^{k-1}_{d,r+1}(T^z)\)) such that \(d\rho = \omega\). The goal of this section is to prove the following related results.

**Theorem 3.1.** Suppose that \(\omega \in \hat{V}^k_{d,r}(T^z)\) and \(d\omega = 0\). Then there exists a \(\rho \in \hat{M}^{k-1}_{d,r+1}(T^z)\) such that \(d\rho = \omega\).

**Theorem 3.2.** Suppose that \(\omega \in V^k_{d,r}(T^z)\) and \(d\omega = 0\). Then there exists a \(\rho \in M^{k-1}_{d,r+1}(T^z)\) such that \(d\rho = \omega\).

**Remark 3.3.** The case \(k = n-1\) in Theorem 3.1, which corresponds to local finite element pairs for the Stokes problem, has been established in [2] [21] [11].

The next corollaries easily follow.

**Corollary 3.4.** Suppose that \(\omega \in \hat{M}^k_{d,r}(T^z)\) and \(d\omega = 0\). Then there exists a \(\rho \in \hat{M}^{k-1}_{d,r+1}(T^z)\) such that \(d\rho = \omega\).

**Corollary 3.5.** Suppose that \(\omega \in \hat{M}^k_{d,r}(T^z)\) and \(d\omega = 0\). Then there exists a \(\rho \in \hat{M}^{k-1}_{d,r+1}(T^z)\) such that \(d\rho = \omega\).

**Corollary 3.6.** There holds

\[\begin{align*}
\dim M^k_{d,r}(T^z) &= \dim M^{k+1}_{d,r-1}(T^z) + \dim M^k_r(T^z) - \dim V^{k+1}_{d,r-1}(T^z), \\
\dim \hat{M}^k_{d,r}(T^z) &= \dim \hat{M}^{k+1}_{d,r-1}(T^z) + \dim \hat{M}^k_r(T^z) - \dim \hat{V}^{k+1}_{d,r-1}(T^z).
\end{align*}\]

**Remark 3.7.** Using Corollary 3.6, one can obtain explicit formulas for the dimensions of \(M^k_{d,r}(T^z)\) and \(\hat{M}^k_{d,r}(T^z)\) in terms of \(r, k,\) and \(n\). To this end, we can easily show that

\[
\dim M^n_{d,r}(T^z) = \sum_{s=0}^{n} \# \Delta_s(T^z) \dim \mathcal{P}_{r-s-1} \Lambda^s(\mathbb{R}^s) = (n+1) \binom{r-1}{n} + \sum_{s=0}^{n-1} \binom{n+2}{s+1} \binom{r-1}{s},
\]

where we used that

\[
\dim \mathcal{P}_{r-s-1} \Lambda^s(\mathbb{R}^s) = \binom{r-1}{s}.
\]
Hence, since \( \dim M^k_r(T^s) = \binom{n}{k} \dim M^0_r(T^s) \), we have

\[
\dim M^k_r(T^s) = \binom{n}{k} \left[ (n+1) \binom{r-1}{n} + \sum_{s=0}^{n-1} \binom{n+2}{s+1} \binom{r-1}{s} \right] \\
= \binom{n}{k} \left[ \binom{r+n+1}{n+1} - \binom{r}{n+1} \right].
\]

Likewise, Proposition 2.4 implies that (see [3] for details)

\[
\dim V^k_{d,r}(T^s) = \sum_{s=k}^{n} \# \Delta_s(T^s) \dim \mathcal{P}_{r+k-s} \mathcal{A}^{s-k}(\mathbb{R}^s) \\
= \binom{n}{k} \left[ \binom{r+k}{n+k} \binom{r-1}{n} + \sum_{s=k}^{n-1} \binom{n+2}{s+1} \binom{r-1}{s} \binom{r+k}{n+k} \right] \\
= \binom{r+k}{n+k} \left[ \binom{r+n+1}{n+k} - \binom{r}{n+k} \right].
\]

We then find

\[
\dim M^k_{d,r}(T^s) = \binom{n}{k+1} \left[ \binom{r+n}{n+1} - \binom{r-1}{n+1} \right] + \binom{n}{k} \left[ \binom{r+n+1}{n+1} - \binom{r}{n+1} \right] \\
- \binom{r+k}{n+k} \left[ \binom{r+n+1}{n+k} - \binom{r-1}{n+k} \right].
\]

Similar arguments also show that

\[
\dim \tilde{M}^k_{d,r}(T^s) = \binom{n}{k+1} \left[ \binom{r+n}{n+1} - \binom{r-2}{n+1} \right] + \binom{n}{k} \left[ \binom{r+n+1}{n+1} - \binom{r-1}{n+1} \right] \\
- \binom{r+k}{n+k} \left[ \binom{r+n-1}{n-k} - \binom{r-2}{n-k} \right].
\]

Remark 3.8. Corollary 3.6 gives, for example, the local dimension of \( C^1 \) elements on an Alfeld split. Namely, taking \( k = 0 \) in the dimension count yields

\[
\dim M^0_{d,r}(T^s) = n \left[ \binom{r+n}{n+1} - \binom{r-1}{n+1} \right] + \binom{r+n+1}{n+1} - \binom{r}{n+1} \right] - r \left[ \binom{r+n}{n} - \binom{r-1}{n} \right] \\
= \binom{r+n}{n} + n \binom{r-1}{n}.
\]

This dimension count has also been established in [12] (also see [19] using different arguments. Note that, since \( \dim \mathcal{P}_r(T) = \binom{n}{r} \), \( M^0_{d,r}(T^s) = \mathcal{P}_r(T) \) for \( r \leq n \).

3.1. Preliminary results. Before proving the main results in this section, we need some preliminary results. We start with a well-known result stating that the traces of forms in \( V^k_{d,r}(T^s) \) are single-valued.

Proposition 3.9. If \( \omega \in V^k_{d,r}(T^s) \), then \( \text{tr}_f \omega \) is single valued for any sub-simplex \( f \in \Delta_s(T^s) \) for \( s \geq k \). In particular, let \( T_1, T_2 \subset T^s, \omega_i = \omega|_{K_i} \), and suppose that \( f \subset \Delta_s(T_1), \Delta_s(T_2) \). Then if \( r_1, r_2, \ldots, r_k \in \mathbb{R}^n \) are tangent to \( f \), then

\[
(\omega_1)_x(r_1, \ldots, r_k) = (\omega_2)_x(r_1, \ldots, r_k) \quad \text{for all } x \in f.
\]

Next, we prove an analogue of Lemma 2.4 on an Alfeld split.

Lemma 3.10. Any \( \omega \in \tilde{V}^k_{d,r}(T^s) \) satisfies

\[
\omega = d\mu \wedge v + \mu w
\]

for some \( v \in V^{k-1}_r(T^s) \) and \( w \in V^k_{r-1}(T^s) \). Moreover, \( \text{tr}_f v \) is single-valued for all \( f \in \Delta_s(T) \) and \( s \geq k-1 \).
Proof. Applying Lemma 3.10 to each $T_i \in T^z$ and recalling that $\mu_i = (n + 1)\lambda_i$, we get the representation (3.3). Moreover, the value $tr_f v$ is clearly single-valued for $F \in \Delta_{n-1}(T)$.

Let $f \in \Delta_s(T)$ with $k - 1 \leq s \leq n - 2$. Let $K_1, K_2 \in T^s$ such that $f \in \Delta_s(K_1)$ and $f \in \Delta_s(K_2)$, and set $v_i = v|_{K_i}$. Writing $f = [x_T(0), x_T(1), \ldots, x_T(s)]$, we define $f' = [z, x_T(0), x_T(1), \ldots, x_T(s)]$, and note that $f' \in \Delta_{s+1}(K_1)$ and $f' \in \Delta_{s+1}(K_2)$.

Let $\{r_i\}_{i=1}^{k-1} \subset \mathbb{R}^n$ be linearly independent vectors that are tangent to $f$, and set $t = z - x_T(0)$. Fix an arbitrary point $x \in f$, and note that $\mu(x) = 0$ because $f \subset \partial T$ and $\mu \in H^1_0(T)$. It then follows from the representation (3.3) that $\omega_x = (d\mu \land v)_x$. We also note that $\{r_i\}_{i=1}^{k-1}$ and $t$ are tangent to $f'$, and it thus follows that the quantity $\omega_z(t, r_1, \ldots, r_{k-1})$ is single-valued because $\omega \in \tilde{V}^k_{d,r}(T^z)$. Using these two properties and the identities $d\mu(t) = 1$ and $d\mu(r_i) = 0$, we find that

$$tr_f(v_1)_x(r_1, \ldots, r_{k-1}) = (v_1)_x(r_1, \ldots, r_{k-1})$$

$$= (d\mu \land v_1)_x(t, r_1, \ldots, r_{k-1})$$

$$= \omega_x(t, r_1, \ldots, r_{k-1})$$

$$= (d\mu \land v_2)_x(t, r_1, \ldots, r_{k-1})$$

$$= (v_2)_x(r_1, \ldots, r_{k-1})$$

$$= tr_f(v_2)_x(r_1, \ldots, r_{k-1}).$$

Thus, $tr_f v$ is single-valued. \(\square\)

The following lemma will be crucial.

Lemma 3.11. Let $\omega \in \tilde{V}^k_{d,r}(T^z)$ and let $\ell \geq 0$ be an integer. If $r \geq 1$, then there exists $\gamma \in \mathcal{P}_r \Lambda^{k-1}(T)$ and $\psi \in \tilde{V}^k_{d,r-1}(T^z)$ such that

$$\mu^{\ell} \omega = d(\mu^{\ell+1} \gamma) + \mu^{\ell+1} \psi.$$ 

Let $r = 0$ and in addition if $k = n$ assume that $\int_T \mu^\ell \omega = 0$. Then there exists a $\gamma \in \mathcal{P}_0 \Lambda^{k-1}(T)$ such that

$$\mu^\ell \omega = d(\mu^{\ell+1} \gamma).$$

Proof. Let us first consider the case $r \geq 1$. By Lemma 3.10 we have

(3.4) $\omega = d\mu \land v + \mu w$, 

where $v \in V^{k-1}_r(T^z)$ and $w \in \tilde{V}^k_{r-1}(T^z)$. Moreover, $tr_f v$ is single valued for all $f \in \Delta_s(T)$ with $s \geq k - 1$. According to Proposition 2.1 there exists a unique $\gamma \in \mathcal{P}_r \Lambda^{k-1}(T)$ such that

$$(\ell + 1) \int_f tr_f \gamma \land \eta = \int_f tr_f v \land \eta \quad \text{for all } \eta \in \mathcal{P}^r_{r+k-1-s} \Lambda^{s-k+1}(f), \quad f \in \Delta_s(T), \quad k-1 \leq s \leq n-1,$$

and

$$\int_T \gamma \land \eta = 0 \quad \text{for all } \eta \in \mathcal{P}^r_{r+k-1-n} \Lambda^{n-k+1}(T).$$

It then follows from Proposition 2.1 that $(\ell + 1) tr_f \gamma = tr_f v$ for all $F \in \Delta_{n-1}(T)$. Hence, by Lemma 2.5 we have that

$$d\mu \land v = (\ell + 1) d\mu \land \gamma + \mu \phi$$

for some $\phi \in V^{k-1}_{r-1}(T^z)$. Using this identity and the Leibniz rule

$$d(\mu^{\ell+1} \gamma) = (\ell + 1) \mu^{\ell} d\mu \land \gamma + \mu^{\ell+1} d\gamma,$$

we have

$$\mu^\ell (d\mu \land v) = (\ell + 1) \mu^{\ell} d\mu \land \gamma + \mu^{\ell+1} \phi = d(\mu^{\ell+1} \gamma) - \mu^{\ell+1} d\gamma + \mu^{\ell+1} \phi.$$

It then follows from (3.3) that

$$\mu^\ell \omega = d(\mu^{\ell+1} \gamma) + \mu^{\ell+1} \psi.$$
with $\psi = -d\gamma + \phi + w \in V^{k}_{d,r-1}(T^z)$. Finally, since $\text{tr}_f(\mu^s\omega)$ and $\text{tr}_f(d(\mu^{s+1}\gamma))$ are single-valued on $f \in \Delta_s(T^z)$ for $s \geq k$, we conclude that $\text{tr}_f(\mu^{s+1}\psi)$ is single-valued. Therefore $\psi \in V^{k}_{d,r-1}(T^z)$. This proves the result in the case $r \geq 1$.

For the case $r = 0$, we have (3.4) with $w = 0$. Applying Lemma 2.3 we uniquely determine $\gamma \in \mathcal{P}_0\Lambda^k(T)$ by the conditions

$$\int_{f} \text{tr}_f \gamma = \int_{f} \text{tr}_f v \quad \text{for all } f \in S_{k-1}(T,x_0).$$

In this way, applying Proposition 2.4 we have $(\ell + 1)\text{tr}_f \gamma = \text{tr}_f v$ for all $F \in S_{n-1}(T,x_0)$. Hence, by Lemma 2.5 we have that

$$\xi = 0 \quad \text{on } T_i, 1 \leq i \leq n,$$

where $\xi := \omega - (\ell + 1)d\mu \wedge \gamma \in V^{k}_{d,0}(T^z)$. Hence, $\text{tr}_f \xi = 0$ for all $F \in \Delta_{n-1}(T_0)$. If $k \leq n - 1$, we apply Proposition 2.4 to get $\xi = 0$ on $T_0$, and therefore $\xi = 0$ on $T$. If $k = n$, we use the assumption that $\int_T \mu^1 \omega = 0$ to obtain

$$0 = \int_T (\mu^s \omega - d(\mu^{s+1}\gamma)) = \int_T \mu^s \xi = \int_{T_0} \mu^{s+1} \xi.$$

This implies that $\xi = 0$ on $T_0$ and hence $\xi = 0$ on all of $T$. Finally, we finish the proof by applying the product rule:

$$d(\mu^{s+1}\gamma) = (\ell + 1)\mu^s d\mu \wedge \gamma = \mu^s \omega.$$ $\square$

3.2. **Proof of Theorem 3.1** Let $\omega \in \hat{V}^{k}_{d,r}(T^z)$ and $d\omega = 0$. Assume we have found $\gamma_{r-1}, \ldots, \gamma_{r-j}$ with $\gamma_i \in \mathcal{P}_r\Lambda^{k-1}(T)$ and $\omega_{r-(j+1)} \in V^{k}_{d,r-(j+1)}(T^z)$ such that

$$\omega = d(\mu\gamma_r + \mu^2 \gamma_{r-1} + \cdots + \mu^j \gamma_{r-j} + \mu^{j+1} \omega_{r-(j+1)}).$$

Then, we see that

$$0 = d(\mu^{j+1} \omega_{r-(j+1)}) = \mu^j (\mu d\omega_{r-(j+1)} + (j + 1) d\mu \wedge \omega_{r-(j+1)}),$$

which implies that $\mu d\omega_{r-(j+1)} + (j + 1) d\mu \wedge \omega_{r-(j+1)} = 0$ on $T$. Hence, we have that $d\mu \wedge \omega_{r-(j+1)} = 0$ on $\partial T$. Using Lemma 2.5 we have $\text{tr}_F \omega_{r-(j+1)} = 0$ for all $F \in \Delta_{n-1}(T)$. Or in other words, $\omega_{r-(j+1)} \in V^{k}_{d,r-(j+1)}(T^z)$. We then apply Lemma 3.11 to get

$$\mu^{j+1} \omega_{r-(j+1)} = d(\mu^{j+2} \gamma_{r-(j+1)}) + \mu^{j+2} \omega_{r-(j+2)},$$

where $\gamma_{r-(j+1)} \in \mathcal{P}_{r-(j+1)}\Lambda^{k-1}(T)$ and $\omega_{r-(j+2)} \in V^{k}_{d,r-(j+2)}(T^z)$. It follows that

$$\omega = d(\mu\gamma_r + \mu^2 \gamma_{r-1} + \cdots + \mu^j \gamma_{r-j} + \mu^{j+1} \gamma_{r-(j+1)} + \mu^{j+2} \omega_{r-(j+2)}).$$

Continuing by induction we have

$$\omega = d(\mu\gamma_r + \mu^2 \gamma_{r-1} + \cdots + \mu^r \gamma_1) + \mu^r \omega_0,$$

Note that if $k = n$, then we have $\int_T \omega = 0$ and so $\int_T \mu^r \omega_0 = 0$. We can apply Lemma 3.11 to write

$$\mu^r \omega_0 = d(\mu^{r+1} \gamma_0),$$

for some $\gamma_0 \in \mathcal{P}_0\Lambda^k(T)$. This completes the proof. $\square$
3.3. Proof of Theorem 3.2 Let $\omega \in V_{d,r}(T^z)$ with $d\omega = 0$. We will consider the case $r \geq 1$ first. Define $\Pi \omega \in P_r \Lambda^k(T)$ such that
\[
\int_f \text{tr}_f \Pi \omega \wedge \eta_f = \int_f \text{tr}_f \omega \wedge \eta_f \quad \text{for all } \eta_f \in P_{r+k-s} \Lambda^{s-k}(f), \ f \in \Delta_0(T), \ s \geq k.
\]
This is the canonical projection; see [3]. Then using standard results in [3] it holds that $d(\Pi \omega) = 0$ since $d\omega = 0$, and moreover, if $k = n$, $\int_f \Pi \omega = \int_f \omega$. Therefore, $\xi := \omega - \Pi \omega$ satisfies $d\xi = 0$. If $k \leq n - 1$, then there holds $\text{tr}_f \xi = 0$ for all $F \in \Delta_{n-1}(T)$, whereas if $k = n$, then $\int_f \xi = 0$. Thus $\xi \in \tilde{V}_{d,r}(T^z)$, and so, by Theorem 3.1 there exists $\varphi \in M_{r+1}^{k-1}(T^z)$ such that $d\varphi = \xi$. Using the exact sequence property of $\{P_r \Lambda^k(T)\}$ there exists $\psi \in P_{r+1} \Lambda^{k-1}(T)$ such that $d\psi = \Pi \omega$. Setting $\rho = \varphi + \psi \in M_{d,r+1}^{k-1}(T^z)$, we have $d\rho = \omega$.

Now consider the case $r = 0$. Applying Lemma 2.3 we define $\Pi \omega \in P_0 \Lambda^k(T)$ uniquely by the conditions
\[
\int_f \text{tr}_f \Pi \omega = \int_f \text{tr}_f \omega \quad \text{for all } f \in S_k(T, x_0).
\]
Let $\xi = \omega - \Pi \omega$ then by Lemma 2.3 have that $\text{tr}_f \xi = 0$ for $1 \leq i \leq n$. Consider the case, $k \leq n - 1$. Consider an arbitrary $k$ sub-simplex $f \in \Delta_k(F_j)$ then it also belongs to another $f \in \Delta_k(F_j)$ for another $j \neq 0$. Hence, $\text{tr}_f \xi = 0$. Applying Lemma 2.3 once more we have that $\text{tr}_f \xi = 0$. Hence, we have that $\text{tr}_f \xi = 0$ for all $F \in \Delta_{n-1}(T)$. Moreover, if $k = n$ we see that $\int_T \xi = 0$. Therefore, $\xi \in \tilde{V}_{d,r}(T^z)$. Hence, using Theorem 3.1 we have a $\rho \in \tilde{M}_{d,r+1}^{k-1}(T^z)$ such that $d\rho = \xi$. Again, using the exact sequence property of $\{P_r \Lambda^k(T)\}$ there exists $\psi \in P_{r+1} \Lambda^{k-1}(T)$ such that $d\psi = \Pi \omega$, and hence $d(\rho + \psi) = \omega$. □

3.4. Proof of Corollary 3.4 Let $\omega \in \tilde{M}_{d,r}^k(T^z) \subset \tilde{V}_{d,r}(T^z)$ and $d\omega = 0$. Theorem 3.1 gives a $\rho \in \tilde{M}_{d,r}^{k-1}(T^z)$ such that $d\rho = \omega$ and therefore $d\rho$ is continuous and vanishes on $\partial T$. In other words, $\rho \in \tilde{M}_{d,r+1}^{k-1}(T^z)$. □

3.5. Proof of Corollary 3.5 The result follows from Corollary 3.4 by noting that $\tilde{M}_{d,r}^k(T^z) \subset \tilde{M}_{d,r}^k(T^z)$. □

3.6. Proof of Corollary 3.6 We consider the following sequences:
\[
\begin{align*}
\cdots & \quad \overset{d}{\rightarrow} \overset{d}{\rightarrow} \overset{d}{\rightarrow} \overset{d}{\rightarrow} \overset{d}{\rightarrow} \quad \cdots \\
M_{d,r+1}^{k-1}(T^z) & \quad M_{d,r}^k(T^z) & \quad M_{r+1}^{k-1}(T^z) & \quad V_{d,r+2}^{k+2}(T^z) & \quad \cdots \\
\cdots & \quad \overset{d}{\rightarrow} \overset{d}{\rightarrow} \overset{d}{\rightarrow} \overset{d}{\rightarrow} \overset{d}{\rightarrow} \quad \cdots
\end{align*}
\]
Theorem 3.2 and the results in [3] show that both sequences are exact, i.e., the range of each map is the kernel of the succeeding map.

Denote by
\[
\begin{align*}
\ker M_{d,r}^k(T^z) & = \{ \omega \in M_{d,r}^k(T^z) : d\omega = 0 \}, \\
\text{range } M_{d,r}^k(T^z) & = \{ d\omega : \omega \in M_{d,r}^k(T^z) \}.
\end{align*}
\]
The rank-nullity theorem shows that
\[
\dim M_{d,r}^k(T^z) = \dim \ker M_{d,r}^k(T^z) + \dim \text{range } M_{d,r}^k(T^z),
\]
and the exactness of the first sequence gives
\[
\dim \text{range } M_{d,r}^k(T^z) = \dim \ker M_{r+1}^{k-1}(T^z)
\]
\[
= \dim M_{r+1}^{k-1}(T^z) - \dim \text{range } M_{r+1}^{k-1}(T^z)
\]
\[
= \dim M_{r+1}^{k+1}(T^z) - \dim \ker V_{d,r+2}^{k+2}(T^z).
\]
On the other hand, we have, by the exactness of the second sequence,
\[
\dim \ker M_{d,r}^k(T^z) = \dim \ker M_{d,r}^k(T^z) = \dim M_{d,r}^k(T^z) - \dim \text{range } M_{d,r}^k(T^z)
\]
The spaces with homogenous boundary conditions are given by
\[
\begin{align*}
\dim M^k_r(T^z) &= \dim M^k_r(T^z) - \dim \ker V^k_{d,r-1}(T^z), \\
\dim M^k_r(T^z) &= \dim M^k_r(T^z) - (\dim V^k_{d,r-1}(T^z) - \dim V^k_{d,r-1}(T^z)), \\
\end{align*}
\]
where the differential operators appearing in these definitions are understood to be in the weak sense.

We further set
\[
\begin{align*}
M^0_r(T^z) &= \dim \ker V^0_{d,r-1}(T^z) + \dim V^0_{d,r-1}(T^z), \\
M^1_r(T^z) &= \dim \ker V^1_{d,r-1}(T^z) + \dim V^1_{d,r-1}(T^z), \\
M^2_r(T^z) &= \dim \ker V^2_{d,r-1}(T^z) + \dim V^2_{d,r-1}(T^z), \\
M^3_r(T^z) &= \dim \ker V^3_{d,r-1}(T^z) + \dim V^3_{d,r-1}(T^z).
\end{align*}
\]

Applying these identities to (3.5), we find that
\[
\begin{align*}
\dim M^2_{d,r}(T^z) &= \dim M^2_{d,r}(T^z) - \dim \ker V^2_{d,r-1}(T^z) \\
&\quad + \dim M^2_r(T^z) - \dim V^2_{d,r-1}(T^z) + \dim \ker V^2_{d,r-1}(T^z) \\
&= \dim M^2_{d,r}(T^z) + \dim M^2_r(T^z) - \dim V^2_{d,r-1}(T^z).
\end{align*}
\]

The dimension count (3.2) is obtained similarly. This concludes the proof.

\[\square\]

4. Local Smooth Finite Element de Rham Complexes in three dimensions

In this section we translate some of the results of Section 3 in three dimensions (n = 3) using vector proxies. Namely, we reprove the results using vector notation and standard differential operators for the benefit of the readers that are more comfortable with vector calculus notation. Moreover, we define local de Rham complexes in three dimension with enhanced smoothness and provide unisolvent sets of degrees of freedom. The last two spaces in one of the sequences correspond to the divergence-free velocity and pressure Stokes elements developed by Zhang [21]. The complex we propose characterize the divergence-free subspace of the discrete velocity space as well as show the relationship between the Stokes pair and the C1 Clough-Tocher element [1].

We start by translating our spaces using vector notation by identifying 0- and 3-forms with scalar functions, and 1- and 2-forms with vector-valued functions. With a slight abuse of notation, we set
\[
\begin{align*}
V^3_r(T^z) &= V^0_r(T^z) = \{\omega \in L^2(T) : \omega|_{T_i} \in P_r(T_i) \text{ for } 0 \leq i \leq 3\}, \\
V^2_r(T^z) &= V^2_r(T^z) = [V^0_r(T^z)]^3.
\end{align*}
\]

We then define
\[
\begin{align*}
V^0_{d,r}(T^z) &= \{\omega \in V^0_r(T^z) : \text{grad } \omega \in [L^2(T)]^3\}, \\
V^1_{d,r}(T^z) &= \{\omega \in V^1_r(T^z) : \text{curl } \omega \in [L^2(T)]^3\}, \\
V^2_{d,r}(T^z) &= \{\omega \in V^2_r(T^z) : \text{div } \omega \in L^2(T)\}, \\
V^3_{d,r}(T^z) &= V^3_r(T^z),
\end{align*}
\]
where the differential operators appearing in these definitions are understood to be in the weak sense.

We further set
\[
\begin{align*}
M^0_r(T^z) &= M^0_r(T^z) = \{\omega \in C^0(T) : \omega|_{T_i} \in P_r(T_i) \text{ for } 0 \leq i \leq 3\}, \\
M^1_r(T^z) &= M^1_r(T^z) = [M^0_r(T^z)]^3,
\end{align*}
\]
and
\[
\begin{align*}
M^0_{d,r}(T^z) &= \{\omega \in M^0_r(T^z) : \text{grad } \omega \in [C^0(T)]^3\}, \\
M^1_{d,r}(T^z) &= \{\omega \in M^1_r(T^z) : \text{curl } \omega \in [C^0(T)]^3\}, \\
M^2_{d,r}(T^z) &= \{\omega \in M^2_r(T^z) : \text{div } \omega \in C^0(T)\}, \\
M^3_{d,r}(T^z) &= M^3_r(T^z).
\end{align*}
\]

The spaces with homogenous boundary conditions are given by
\[
\begin{align*}
\bar{V}^0_{d,r}(T^z) &= \{\omega \in V^0_{d,r}(T^z) : \omega|_F = 0 \text{ for all } F \in \Delta_2(T)\}, \\
\bar{V}^1_{d,r}(T^z) &= \{\omega \in V^1_{d,r}(T^z) : \omega \times n_F|_F = 0 \text{ for all } F \in \Delta_2(T)\}, \\
\bar{V}^2_{d,r}(T^z) &= \{\omega \in V^2_{d,r}(T^z) : \omega \cdot n_F|_F = 0, \text{ for all } F \in \Delta_2(T)\},
\end{align*}
\]
show that the following sequences are exact:
\[ \tilde{V}_{d,r}^3(T^2) = \{ \omega \in V_{d,r}^3(T^2) : \int_T \omega \, dx = 0 \} \]

Here, \( n_F \) is unit normal pointing out of \( F \).

For \( 0 \leq k \leq 2 \) we define
\[ \tilde{M}_k^3(T^2) = \{ \omega \in M_k^3(T^2) : \omega|_F = 0 \text{ for all } F \in \Delta_2(T) \}, \]

and for \( k = 3 \),
\[ \tilde{M}_3^3(T^2) = \{ \omega \in M_3^3(T^2) : \omega|_F = 0 \text{ for all } F \in \Delta_2(T), \int_T \omega = 0 \}. \]

Finally, we define
\[
\begin{align*}
\tilde{M}_{d,r}^0(T^2) & = \{ \omega \in M_{d,r}^0(T^2) : \text{grad } \omega|_F = 0 \text{ for all } F \in \Delta_2(T) \}, \\
\tilde{M}_{d,r}^1(T^2) & = \{ \omega \in M_{d,r}^1(T^2) : \text{curl } \omega|_F = 0 \text{ for all } F \in \Delta_2(T) \}, \\
\tilde{M}_{d,r}^2(T^2) & = \{ \omega \in M_{d,r}^2(T^2) : \text{div } \omega|_F = 0 \text{ for all } F \in \Delta_2(T) \}, \\
\tilde{M}_{d,r}^3(T^2) & = \tilde{M}_3^3(T^2).
\end{align*}
\]

Note that \( \tilde{V}_{d,r}^3(T^2) \) is the \( H_0^1 \)-conforming Lagrange finite element space, \( \tilde{V}_{d,r}^1(T^2) \) is the \( H_0(\text{curl}) \)-conforming Nedelec space of second type, and \( \tilde{V}_{d,r}^2(T^2) \) is the \( H_0(\text{div}) \)-conforming Nedelec space of the second type. The space \( \tilde{V}_{d,r}^3(T^2) \) is simply the space of piecewise polynomials with vanishing mean, but without any continuity restrictions. Together, these canonical finite element spaces form an exact discrete de Rham complex:

\[
(4.1) \quad 0 \rightarrow \tilde{V}_{d,r}^0(T^2) \xrightarrow{\text{grad }} \tilde{V}_{d,r}^1(T^2) \xrightarrow{\text{curl }} \tilde{V}_{d,r}^2(T^2) \xrightarrow{\text{div }} \tilde{V}_{d,r}^3(T^2) \rightarrow 0.
\]

Theorems 3.1–3.2 and Corollaries 3.4–3.6 hold with \( d \) being one of the differential operators grad, curl, div. Essentially these results show that any discrete space in (4.1) can be replaced by its continuous analogue, and the exactness property will still be preserved provided that the spaces to the left of the replacement are modified accordingly. In particular, Theorems 3.1, 3.3 and Corollaries 3.5, 3.6 show that the following sequences are exact:

\[
\begin{align*}
(4.2a) & \quad 0 \rightarrow \tilde{M}_{d,r}^0(T^2) \xrightarrow{\text{grad }} \tilde{M}_{d,r}^1(T^2) \xrightarrow{\text{curl }} \tilde{M}_{d,r}^2(T^2) \xrightarrow{\text{div }} \tilde{M}_{d,r}^3(T^2) \rightarrow 0, \\
(4.2b) & \quad 0 \rightarrow \tilde{M}_{d,r}^0(T^2) \xrightarrow{\text{grad }} \tilde{M}_{d,r}^1(T^2) \xrightarrow{\text{curl }} \tilde{M}_{d,r}^2(T^2) \xrightarrow{\text{div }} \tilde{V}_{d,r}^3(T^2) \rightarrow 0, \\
(4.2c) & \quad 0 \rightarrow \tilde{M}_{d,r}^0(T^2) \xrightarrow{\text{grad }} \tilde{M}_{d,r}^1(T^2) \xrightarrow{\text{curl }} \tilde{V}_{d,r}^2(T^2) \xrightarrow{\text{div }} \tilde{V}_{d,r}^3(T^2) \rightarrow 0.
\end{align*}
\]

For example, the exactness of the third sequence means:

\[
\begin{align*}
(4.3a) & \quad \text{If } \omega \in \tilde{M}_{d,r}^1(T^2) \text{ and curl } \omega = 0, \text{ there exists } \rho \in \tilde{M}_{d,r}^0(T^2) \text{ such that grad } \rho = \omega. \\
(4.3b) & \quad \text{If } \omega \in \tilde{V}_{d,r}^2(T^2) \text{ and div } \omega = 0, \text{ there exists } \rho \in \tilde{M}_{d,r}^1(T^2) \text{ such that curl } \rho = \omega. \\
(4.3c) & \quad \text{If } \omega \in \tilde{V}_{d,r}^3(T^2), \text{ there exists } \rho \in \tilde{V}_{d,r}^2(T^2) \text{ such that div } \rho = \omega.
\end{align*}
\]

Note that (4.3a) is the main contribution among the three: Property (4.3a) follows from the exactness of (4.1), and if \( \omega \in \tilde{M}_{d,r}^1(T^2) \subset \tilde{V}_{d,r}^1(T^2) \) satisfies curl \( \omega = 0 \), then the exactness of (4.1) implies that \( \omega = \text{grad } \rho \) for some \( \rho \in V_{d,r}^0(T^2) \). By definition, we get \( \rho \in \tilde{M}_{d,r}^0(T^2) \), i.e., property (4.3a).

Although we have proved these results in the previous section using the language of differential forms, we will give a sketch a proof (4.3b) using vector notation for the benefit of those readers that feel more comfortable with vector notation. We start by giving an instance of Lemma 2.4.
Lemma 4.1. Let $T = [x_0, x_1, x_2, x_3]$. Suppose that $\omega \in [\mathcal{P}_r(T)]^3$ with $\omega \cdot n_{F_1} = 0$ on $F_1$. Then $\omega = \text{grad} \lambda_i \times v + \lambda_i w$, where $v \in [\mathcal{P}_r(T)]^3$ and $w \in [\mathcal{P}_{r-1}(T)]^3$.

Proof. With out loss of generality we assume that $i = 3$. Then it is easy to see that
\begin{equation}
\omega = a_1 \text{grad} \lambda_2 \times \text{grad} \lambda_3 + a_2 \text{grad} \lambda_1 \times \text{grad} \lambda_3 + a_3 \text{grad} \lambda_1 \times \text{grad} \lambda_2,
\end{equation}
where $a_1, a_2, a_3 \in \mathcal{P}_r(T)$. Since $\omega \cdot n_{F_3} = 0$ on $F_3$ and $\text{grad} \lambda_3$ is parallel to $n_{F_3}$, we have $\omega \cdot \text{grad} \lambda_3 = 0$ on $F_3$. Applying this identity to (4.4), we have
\begin{equation}
a_3 (\text{grad} \lambda_1 \times \text{grad} \lambda_2) \cdot \text{grad} \lambda_3 = 0 \quad \text{on } F_3.
\end{equation}
This implies that $a_3 = 0$ on $F_3$, or equivalently, that $a_3 = \lambda_3 b$ for some $b \in \mathcal{P}_{r-1}(T)$. The result now follows if we let $v = -a_1 \text{grad} \lambda_2 - a_2 \text{grad} \lambda_1$ and $w = b \text{grad} \lambda_1 \times \text{grad} \lambda_2$.

Next we state an instance of Lemma 3.10

Lemma 4.2. Any $\omega \in \mathcal{V}^2_{d,r}(T^2)$ satisfies
\begin{equation}
\omega = \text{grad} \mu \times v + \mu w
\end{equation}
for some $v \in \mathcal{V}^1_r(T^2)$ and $w \in \mathcal{V}^2_{r-1}(T^2)$. Moreover, $v \cdot t_e$ is single-valued on all edges of $e$ of $T$, where $t_e$ is a unit tangent vector to $e$.

Proof. First by Lemma 4.1 we see that $\omega$ has the form (4.5). Thus, to complete the proof, we must show that $v \cdot t_e$ is single-valued on all edges of $e$ of $T$.

To this end, let $e$ be an edge of $T$. Let $T_1, T_2 \in T^2$ such that they have a common (internal) face $F$ and that $e$ is an edge of the face $F$. Let $n$ be a unit normal vector to the face $F$. Since the tangential components of $\text{grad} \mu$ on $F$ are single-valued we have that
\begin{equation}
\text{grad} \mu|_{T_1} = \text{grad} \mu|_{T_2} + an
\end{equation}
for a constant $a$. Since $\omega \in \mathcal{V}^2_{d,r}(T^2)$ it must be that
\begin{equation*}
\omega|_{T_1} \cdot n = \omega|_{T_2} \cdot n \quad \text{on } F.
\end{equation*}
In particular, if we use that $\mu = 0$ on $e$, we have
\begin{equation*}
(\text{grad} \mu|_{T_1} \times v|_{T_1}) \cdot n = (\text{grad} \mu|_{T_2} \times v|_{T_2}) \cdot n \quad \text{on } e.
\end{equation*}
Therefore,
\begin{equation*}
(\text{grad} \mu|_{T_1} \times v|_{T_1}) \cdot v|_{T_1} = (\text{grad} \mu|_{T_2} \times v|_{T_2}) \cdot v|_{T_2} \quad \text{on } e.
\end{equation*}
By (4.4), $(\text{grad} \mu|_{T_1} \times n) = (\text{grad} \mu|_{T_2} \times n)$ which is parallel to $t_e$. This proves the result.

We now prove an instance of Lemma 3.11

Lemma 4.3. Let $\omega \in \mathcal{V}^2_{d,r}(T^3)$ and let $t \geq 0$ be an integer. There exists $\gamma \in [\mathcal{P}_r(T)]^3$ and $\psi \in \mathcal{V}^2_{d,r-1}(T^3)$ (in the case $r = 0$, $\psi \equiv 0$) such that
\begin{equation}
(\ell + 1) \int_{\partial T} (\gamma \cdot t_e) \eta \, ds = \int_{\partial T} (v \cdot t_e) \eta \, ds, \quad \forall \eta \in \mathcal{P}_r(e), \quad \forall e \in \Delta_1(T),
\end{equation}
\begin{equation}
(\ell + 1) \int_{\partial T} (v \times nF) \cdot \eta \, dA = \int_{\partial T} (v \times nF) \cdot \eta \, dA, \quad \forall \eta \in \mathcal{D}_{r-1}(T), \quad \forall F \in \Delta_2(T),
\end{equation}
where $v \in \mathcal{V}^1_r(T^3)$ and $w \in \mathcal{V}^2_{r-1}(T^3)$. Moreover, $v \cdot t_e$ is single-valued on all edges of $e$ of $T$. Applying Proposition 2.4, we uniquely define $\gamma \in [\mathcal{P}_r(T)]^3$ such that it satisfies
\begin{equation}
\mu^\ell \omega = \text{curl} (\mu^{\ell+1} \gamma) + \mu^{\ell+1} \psi.
\end{equation}

Proof. We prove the case $r \geq 1$ and leave the case $r = 0$ to the reader. By the previous lemma we have $\omega = \text{grad} \mu \times v + \mu w$ for some $v \in \mathcal{V}^1_r(T^3)$ and $w \in \mathcal{V}^2_{r-1}(T^3)$. Moreover, $v \cdot t_e$ is single-valued on all edges of $e$ of $T$. Applying Proposition 2.4, we uniquely define $\gamma \in [\mathcal{P}_r(T)]^3$ such that it satisfies
Here, $D_s(F) = P_{s-1}(F) + x_F P_{s-1}(F)$ is the local Raviart-Thomas space on $F$, and $D_s(T) = P_{s-1}(T) + x P_{s-1}(T)$ is the local Raviart-Thomas space on $T$. Using (4.8a)–(4.8c) and Stokes Theorem, we easily find that $(\ell + 1)\gamma \times n_F = v \times n_F$ on $F$ for all faces $F$ of $T$. Because grad $\mu$ is parallel to $n_F$, we have

$$\text{grad } \mu \times v = (\ell + 1)\text{grad } \mu \times \gamma + \mu \phi,$$

for some $\phi \in V^2_{d,r-1}(T^2)$. Using the product rule we get

$$\text{curl } (\mu^{\ell+1} \gamma) = (\ell + 1)\mu^{\ell} \text{grad } \mu \times \gamma + \mu^{\ell+1} \text{curl } \gamma,$$

and hence

$$\mu^\ell \text{grad } \mu \times v = \text{curl } (\mu^{\ell+1} \gamma) + \mu^{\ell+1} (\text{curl } \gamma + \phi).$$

If we let $\psi = -\text{curl } \gamma + \phi + w \in V^2_{d,r}(T^2)$ we arrive at the equation (4.3b). We know that $\mu^\ell \omega \cdot n$ and $d(\mu^{\ell+1} \gamma) \cdot n$ are single valued across all interior face of $T^2$, and therefore, $\mu^{\ell+1} \psi \cdot n$ is also single valued. Hence, $\psi \in V^2_{d,r-1}(T^2)$. \hfill \Box

Proof of (4.3b). For readability, we prove (4.3b) with $r - 1$ replaced by $r$.

Let $\omega \in V^2_{d,r}(T^2)$ and div $\omega = 0$. Assume we have found $\gamma_r, \ldots, \gamma_{r-j}$ with $\gamma_\ell \in [P_\ell(T)]^3$ and $\omega_{r-(j+1)} \in V^2_{d,r-(j+1)}(T^2)$ such that

$$\omega = \text{curl } (\mu \gamma_r + \mu^2 \gamma_{r-1} + \cdots + \mu^{j+1} \gamma_{r-j}) + \mu^{j+1} \omega_{r-(j+1)}.$$

Then, we see that

$$0 = \text{div } (\mu^{j+1} \omega_{r-(j+1)}) = \mu^j (\text{div } \omega_{r-(j+1)} + (j + 1) \text{grad } \mu \cdot \omega_{r-(j+1)}),$$

which implies that $\mu \text{div } \omega_{r-(j+1)} + (j + 1) \text{grad } \mu \cdot \omega_{r-(j+1)} = 0$ on $T$. Hence we have that grad $\mu \cdot \omega_{r-(j+1)}$ is constant on $\partial T$. This shows that $\omega_{r-(j+1)} \cdot n_F = 0$ for all $F \in \Delta_2(T)$. Or in other words, $\omega_0 \in V^2_{d,r-(j+1)}(T^2)$. We then apply Lemma 4.3 to get

$$\mu^{j+1} \omega_{r-(j+1)} = \text{curl } (\mu^{j+1} \gamma_{r-(j+1)}) + \mu^{j+1} \omega_{r-(j+2)},$$

where $\gamma_{r-(j+1)} \in [P_{r-(j+1)}(T)]^3$ and $\omega_{r-(j+2)} \in V^2_{d,r-(j+2)}(T^2)$. It follows that

$$\omega = \text{curl } (\mu \gamma_r + \mu^2 \gamma_{r-1} + \cdots + \mu^{j+1} \gamma_{r-j} + \mu^{j+2} \gamma_{r-(j+1)}) + \mu^{j+2} \omega_{r-(j+2)}.$$

Continuing by induction we have

$$\omega = \text{curl } (\mu \gamma_r + \mu^2 \gamma_{r-1} + \cdots + \mu^r \gamma_1 + \mu^{r+1} \gamma_0).$$

This completes the proof. \hfill \Box

4.1. Degrees of Freedom. Our goal is to develop degrees of freedom (DOFs), and in turn, to construct analogous global versions of the spaces appearing in (4.2). However, to develop DOFs, special care must be taken to ensure that the induced finite element spaces satisfy the same exactness properties as (4.2) due to the intrinsic smoothness of the spaces. In particular, it is a simple exercise (cf. [11]) to show that functions in $M^0_{d,r}(T^2)$ are $C^2$ at the vertices in $T^2$, and this influences the construction of DOFs and global finite element spaces. For example, if we consider the global analogue of the third sequence in (4.2), then natural choices would be to take $M^1_{d,r-1}$ as the vector-valued Lagrange space, $V^2_{d,r-2}$ the $H$(div)-conforming Nedelec space, and $V^3_{d,r-3}$ the space of piecewise polynomials. This selection would indeed form a discrete (global) complex, but a simple counting argument shows that the resulting sequence is not exact on general contractible domains.

To construct the desired global spaces, it seems necessary to consider finite element spaces with additional smoothness at the vertices. In particular, guided by the $C^1$ Clough-Tocher space, we consider the subspaces of $M^k_{d,r-k}(T^2)$, $M^k_{r-k}(T^2)$, and $V^k_{d,r-k}(T^2)$ that have $C^{2-k}$ continuity on $\Delta_0(T^2)$, and formulate the global finite element spaces using these subspaces (in the case $k = 3$, no additional continuity is added). This framework is also adopted in [7] on general meshes, where finite element
spaces are constructed to form a subsequence of the de Rham complex with minimal $L^2$ smoothness. Here, we show that, on Alfeld splits, this framework yields finite element spaces with greater global smoothness.

However, it turns out that these additional smoothness constraints at the vertices are redundant in many cases as the next lemma shows. Its proof is given in the appendix.

**Lemma 4.4.** Any $\omega \in M^k_{d,r}(T^z)$ is $C^{2-k}$ on $\Delta_0(T^z)$ for $k = 0, 1, 2$.

We introduce the local spaces with added continuity at the vertices as
\[
\tilde{M}^{1}_{c,r-1}(T^z) := \{ \omega \in M^{1}_{r-1}(T^z) : \omega \text{ is } C^1 \text{ on } \Delta_0(T^z) \},
\]
\[
\tilde{V}^{2}_{c,r-2}(T^z) := \{ \omega \in V^{2}_{d,r-2}(T^z) : \omega \text{ is } C^0 \text{ on } \Delta_0(T^z) \},
\]
and set
\[
M^{1}_{c,r-1}(T^z) = M^{1}_{c,r-1}(T^z) \cap \tilde{M}^{1}_{c,r-1}(T^z) \quad \text{and} \quad \tilde{V}^{2}_{c,r-2}(T^z) = V^{2}_{c,r-2}(T^z) \cap \tilde{V}^{2}_{c,r-2}(T^z).
\]

**Remark 4.5.** The space $V^{2}_{c,r-2}(T^z)$ corresponds to the nodal $H(\text{div})$ finite element introduced in [20, 7], and the space $M^{1}_{c,r-1}(T^z)$ is a vector-valued Hermite finite element space. Using [7, Lemma 10], we have for $r \geq 4$
\[
\dim V^{2}_{c,r-2}(T^z) = 3(\#\Delta_0(T^z)) + \left( \frac{1}{2} r(r - 1) - 3 \right)(\#\Delta_2(T^z)) + \frac{1}{2} r(r - 3)(r - 1)(\#\Delta_3(T^z)) \right]
\]
\[= (2r - 5)(r^2 + r + 3),
\]
\[
\dim \tilde{V}^{2}_{c,r-2}(T^z) = 3(\#\Delta_0(T^z) \setminus \Delta_0(T)) + \left( \frac{1}{2} r(r - 1) - 3 \right)(\#\Delta_2(T^z) \setminus \Delta_2(T))
\]
\[+ \frac{1}{2} r(r - 3)(r - 1)(\#\Delta_3(T^z)) \right]
\[= 2r^3 - 5r^2 + 3r - 15.
\]
Furthermore, using [7, Lemma 5] we find that ($r \geq 4$)
\[
\dim M^{1}_{c,r-1}(T^z) = 3 \left[ 4(\#\Delta_0(T^z)) + (r - 4)(\#\Delta_1(T^z)) + \frac{1}{2} (r - 2)(r - 3)(\#\Delta_2(T^z))
\right]
\]
\[+ \frac{1}{6} (r - 2)(r - 3)(r - 4)(\#\Delta_3(T^z)) \right]
\[= (r - 2)(2r^2 + r + 9),
\]
\[
\dim \tilde{M}^{1}_{c,r-1}(T^z) = 3 \left[ 4(\#\Delta_0(T^z) \setminus \Delta_0(T)) + (r - 4)(\#\Delta_1(T^z) \setminus \Delta_1(T)) + \frac{1}{2} (r - 2)(r - 3)(\#\Delta_2(T^z) \setminus \Delta_2(T))
\right]
\]
\[+ \frac{1}{6} (r - 2)(r - 3)(r - 4)(\#\Delta_3(T^z)) \right]
\[= (r - 3)(2r^2 - 3r + 10).
\]

**Lemma 4.6.** If $r \leq 3$, then $M^{1}_{c,r-1}(T^z) = [P_{r-1}(T)]^3$ and $V^{2}_{c,r-2}(T^z) = M^{2}_{r-2}(T^z)$. In particular, the above dimension counts for $M^{1}_{c,r-1}(T^z)$ and $V^{2}_{c,r-2}(T^z)$ are valid in the case $r = 3$ as well.

**Proof.** We consider the case $r = 3$, as the other cases are considerably simpler.

If $r = 3$, then $V^{2}_{c,r-2}(T^z)$ consists of vector-valued piecewise linear polynomials that are continuous at the vertices. This implies that the functions are continuous on $T$, and thus $V^{2}_{c,r-2}(T^z) = M^{2}_{r-2}(T^z)$.

Next, we write
\[
M^{2}_{r}(T^z) = [P_{2}(T)]^3 + \mu [P_{1}(T)]^3 + \mu^2 [P_{0}(T)]^3,
\]
and note the inclusion $M^{1}_{c,2}(T^z) \subset M^{2}_{r}(T^z)$. Let $\omega \in M^{1}_{c,2}(T^z)$ and write $\omega = \omega^{(2)} + \mu \omega^{(1)} + \mu^2 \omega^{(0)}$ with $\omega^{(i)} \in [P_{i}(T)]^3$. Let $T_1, T_2 \in T^z$ and set $F = \partial T_1 \cap \partial T_2$. We then have
\[
\text{(4.9)} \quad \text{grad } (\omega_1 - \omega_2) = \omega^{(1)} \otimes (\text{grad } \mu_1 - \text{grad } \mu_2) + 2\mu \omega^{(0)} \otimes (\text{grad } \mu_1 - \text{grad } \mu_2) \text{ on } F,
\]
where $\omega_j$ and $\mu_j$ is the restriction of $\omega$ and $\mu$ to $T_j$, respectively. Restricting this identity to the boundary vertex $a \in \Delta_0(F) \cap \Delta_0(T)$, we obtain

$$0 = \text{grad} (\omega_1 - \omega_2)(a) = \omega^{(1)}(a) \otimes (\text{grad} \mu_1 - \text{grad} \mu_2),$$

which implies that $\omega^{(1)}(a) = 0$. Thus, $\omega^{(1)}$ vanishes on all vertices of $T$, and therefore $\omega^{(1)} \equiv 0$.

Next, we restrict (4.9) to the barycenter of $T$ to get

$$0 = \text{grad} (\omega_1 - \omega_2)(z) = 2\omega^{(0)}(z) \otimes (\text{grad} \mu_1 - \text{grad} \mu_2).$$

We then conclude that $\omega^{(0)} \equiv 0$, and so $\omega = \omega^{(2)} \in [\mathcal{P}_2(T)]^3$. Thus, $M_{c,2}^1(T^z) = [\mathcal{P}_2(T)]^3$. \hfill \Box

We study finite element spaces in the sequence, where $C^{2-k}$ continuity on $\Delta_0(T^z)$ is added to (4.10) and without boundary conditions:

(4.10a) \hspace{1cm} \mathbb{R} \longrightarrow M_{d,r}^0(T^z) \xrightarrow{\text{grad}} M_{d,r-1}^1(T^z) \xrightarrow{\text{curl}} M_{d,r-2}^2(T^z) \xrightarrow{\text{div}} M_{d,r-3}^3(T^z) \longrightarrow 0,

(4.10b) \hspace{1cm} \mathbb{R} \longrightarrow M_{d,r}^0(T^z) \xrightarrow{\text{grad}} M_{d,r-1}^1(T^z) \xrightarrow{\text{curl}} M_{d,r-2}^2(T^z) \xrightarrow{\text{div}} V_{d,r-3}^3(T^z) \longrightarrow 0,

(4.10c) \hspace{1cm} \mathbb{R} \longrightarrow M_{d,r}^0(T^z) \xrightarrow{\text{grad}} M_{c,r-1}^1(T^z) \xrightarrow{\text{curl}} V_{c,r-2}^2(T^z) \xrightarrow{\text{div}} V_{d,r-3}^3(T^z) \longrightarrow 0.

Note that, compared to (4.10), only the second and third spaces in (4.10c) have been altered.

Theorem 4.7. The sequences (4.10) are exact for $r \geq 1$.

Proof. It has already been established that the first two sequences (4.10a) and (4.10b) are exact. We now show that (4.10c) is exact as well.

(i) The surjectivity $\text{div} : V_{c,r-2}^2(T^z) \rightarrow V_{d,r-3}^3(T^z)$ follows from the surjectivity of $\text{div} : M_{d,r-2}^2(T^z) \rightarrow V_{d,r-3}^3(T^z)$ and the fact that $M_{d,r-2}^2(T^z) \subset V_{d,r-3}^3(T^z)$.

(ii) Similarly the surjectivity of $\text{grad} : M_{d,r}^0(T^z) \rightarrow \ker M_{c,r-1}^1(T^z)$ follows from the surjectivity of $\text{grad} : M_{d,r}^0(T^z) \rightarrow \ker M_{r-1}^1(T^z)$ and the inclusion $M_{c,r-1}^1(T^z) \subset M_{r-1}^1(T^z)$.

(iii) Let $r \geq 3$. By the rank-nullity theorem, part (i), and Remark 4.5

$$\dim \ker V_{c,r-2}^2(T^z) = \dim V_{c,r-2}^2(T^z) - \dim V_{d,r-3}^3(T^z)$$

$$= (2r - 5)(2^2 + r + 3) - \frac{4}{6}r(r - 1)(r - 2)$$

$$= \frac{4}{3}r^3 - r^2 - \frac{1}{3}r - 15.$$

On the other hand, we have, by part (ii), Remark 4.5 and Corollary 3.6

$$\dim \text{range} M_{c,r-1}^1 = \dim M_{c,r-1}^1(T^z) - \dim \text{grad} M_{d,r}^0(T^z)$$

$$= \dim M_{c,r-1}^1(T^z) - \dim M_{d,r}^0(T^z) + 1$$

$$= (r - 2)(2r^2 + r + 9) - \left(\frac{2}{3}r^3 - 2r^2 + \frac{22}{3}r - 2\right) + 1$$

$$= \frac{4}{3}r^3 - r^2 - \frac{1}{3}r - 15.$$

Thus, $\dim \ker V_{c,r-2}^2(T^z) = \dim \text{range} M_{c,r-1}^1(T^z)$, and since range $M_{c,r-1}^1 \subset \ker V_{c,r-2}^2(T^z)$ we conclude that $\text{range} M_{c,r-1}^1(T^z) = \ker V_{c,r-2}^2(T^z)$. We then conclude that the sequence is exact for $r \geq 3$.

If $r \leq 2$, then $M_{d,r}^0(T^z) = \mathcal{P}_r(T)$, $M_{c,r-1}^1(T^z) = [\mathcal{P}_{r-1}(T)]^3$, and $V_{c,r-2}^2(T^z) = [\mathcal{P}_{r-2}(T)]^3$, and so the sequence is clearly exact in this case. \hfill \Box

We now present unisolvent sets of degrees of freedom for the three-dimensional spaces in the previous section, and show that the DOFs naturally induce a set of commutative projections. First, we consider the family spaces with the largest amount of smoothness, $M_{d,r-k}^k(T^z)$ ($k = 0, 1, 2, 3$).
Applying Corollary 3.8, we find that the dimension of these spaces are
\[
\dim M_{d,r}(T^z) = \frac{2}{3}r^3 - 2r^2 + \frac{22}{3}r - 2, \quad \dim M_{d,r-1}(T^z) = (r-1)(2r^2 - 7r + 18),
\]
\[
\dim M_{d,r-2}(T^z) = \max\{2r^3 - 12r^2 + 32r - 30, 0\}, \quad \dim M_{d,r-3}(T^z) = \max\{\frac{2}{3}r^3 - 5r^2 + \frac{43}{3}r - 15, 0\}.
\]
Likewise, we have
\[
\dim \tilde{M}_{d,r}(T^z) = \max\{\frac{2}{3}(r-2)(r-3)(r-4), 0\}, \quad \dim \tilde{M}_{d,r-1}(T^z) = \max\{(2r-5)(r-3)(r-4), 0\},
\]
\[
\dim \tilde{M}_{d,r-2}(T^z) = \max\{(2r-4)(r^2 - 6r + 10), 0\}, \quad \dim \tilde{M}_{d,r-3}(T^z) = \max\{\frac{1}{3}(2r-7)(r^2 - 7r + 15) - 1, 0\}.
\]

We start with the DOFs of the $C^1$ finite element space; the proof of the lowest order case ($r = 5$) is found in [14].

**Lemma 4.8.** Let $r \geq 5$. Then, a function $\omega \in \tilde{M}_{d,r}^0(T^z)$ is uniquely determined by the following degrees of freedom (DOFs):

\begin{align*}
(4.11a) \quad & D^\alpha \omega(a), & \forall |\alpha| \leq 2, & \forall a \in \Delta_0(T) & (40 \text{ DOFs}), \\
(4.11b) \quad & \int_e \omega \sigma ds, & \forall \sigma \in P_{r-6}(e), & \forall e \in \Delta_1(T) & (6(r-5) \text{ DOFs}), \\
(4.11c) \quad & \int_e \frac{\partial \omega}{\partial n_\sigma} \sigma ds, & \forall \sigma \in P_{r-5}(e), & \forall e \in \Delta_1(T) & (12(r-4) \text{ DOFs}), \\
(4.11d) \quad & \int_F \omega \sigma dA, & \forall \sigma \in P_{r-6}(F), & \forall F \in \Delta_2(T) & \left(\frac{4}{2}(r-5)(r-4)\right) \text{ DOFs}, \\
(4.11e) \quad & \int_F \frac{\partial \omega}{\partial n_F} \sigma dA, & \forall \sigma \in P_{r-4}(F), & \forall F \in \Delta_2(T) & \left(\frac{4}{2}(r-3)(r-2)\right) \text{ DOFs}, \\
(4.11f) \quad & \int_T \text{grad} \omega \cdot \text{grad} \sigma dx, & \forall \sigma \in \tilde{M}_{d,r}^0(T^z), & & \left(\frac{2}{3}(r-4)(r-3)(r-2)\right) \text{ DOFs}.
\end{align*}

Here, $n_\epsilon$ are two orthonormal normal vectors that are orthogonal to the edge $e$. In the case $r = 6$, the sets listed in (4.11b) and (4.11d) are omitted.

**Proof.** By a simple dimension count, we have the number of total DOFs in (4.11) equals the dimension of the space $M_{d,r}^0(T^z)$.

Let $\omega \in \tilde{M}_{d,r}^0(T^z)$ be such that the DOFs (4.11) vanish. The DOFs in (4.11b)–(4.11d) implies that $\omega|_e = 0$ and $\text{grad} \omega|_e = 0$ for all $e \in \Delta_1(T)$. Combining these results with the DOFs on (4.11d)–(4.11e), we conclude that $\omega|_F = 0$ and $\text{grad} \omega|_F = 0$ for all $F \in \Delta_2(T)$. Hence, $\omega \in \tilde{M}_{d,r}^0(T^z)$. Taking $\sigma = \omega$ in (4.11f), we get $\text{grad} \omega = 0$. Hence $\omega = 0$. This completes the proof. \hfill \Box

**Remark 4.9.** The set of DOFs is not unique. For example, we can obtain another set of DOFs by simply changing the internal DOFs (4.11) in the set (4.11) to be
\[
\int_T \omega \sigma dx, & \forall \sigma \in \tilde{M}_{d,r}^0(T^z).
\]
The reason for our choice of DOFs (4.11) will be clear in the next section when we discuss commutative projections.

**Remark 4.10.** The proof of Lemma 4.8 shows that if $\omega \in \tilde{M}_{d,r}^0(T^z)$ vanishes at the DOFs (4.11b)–(4.11d) restricted to a single face $F \in \Delta_2(T)$, then $\omega|_F = 0$ and $\text{grad} \omega|_F = 0$. Thus, the DOFs induce a global $C^1$ finite element space.

**Lemma 4.11.** Let $r \geq 5$, then a function $\omega \in M_{d,r-1}^1(T^z)$ is uniquely determined by the following degrees of freedom:

\begin{align*}
(4.12a) \quad & D^\alpha \omega(a), & \forall |\alpha| \leq 1, & \forall a \in \Delta_0(T) & (48 \text{ DOFs}),
\end{align*}
(4.12b) \[ \int_{e} \omega \cdot \kappa \, ds \quad \forall \kappa \in [\mathcal{P}_{r-5}(e)]^3, \quad \forall e \in \Delta_1(T) \quad (18(r-4) \text{ DOFs}), \]

(4.12c) \[ \int_{e} (\text{curl} \omega) \cdot \kappa \, ds \quad \forall \kappa \in [\mathcal{P}_{r-4}(e)]^3, \quad \forall e \in \Delta_1(T) \quad (18(r-3) \text{ DOFs}), \]

(4.12d) \[ \int_{f} (\omega \cdot n_F) \kappa \, dA \quad \forall \kappa \in \mathcal{P}_{r-4}(F), \quad \forall F \in \Delta_2(T) \quad (2(r-2)(r-3) \text{ DOFs}) \]

(4.12e) \[ \int_{F} (n_F \times \omega \times n_F) \cdot \kappa \, dA \quad \forall \kappa \in D_{r-5}(F), \quad \forall F \in \Delta_2(T) \quad (4(r-3)(r-5) \text{ DOFs}), \]

(4.12f) \[ \int_{F} (\text{curl} \omega \times n_F) \cdot \kappa \, dA \quad \forall \kappa \in [\mathcal{P}_{r-5}(F)]^3, \quad \forall F \in \Delta_2(T) \quad (4(r-3)(r-4) \text{ DOFs}), \]

(4.12g) \[ \int_{T} \omega \cdot \kappa \, dx \quad \forall \kappa \in \text{grad} M^{(2)}_{d,r}(T^z), \quad (\frac{2(r-4)(r-3)(r-2)}{3} \text{ DOFs}), \]

(4.12h) \[ \int_{T} \text{curl} \omega \cdot \kappa \, dx \quad \forall \kappa \in \text{curl} M^{(2)}_{d,r-1}(T^z), \quad (\frac{(r-4)(r-3)(4r-11)}{3} \text{ DOFs}), \]

where we recall that \( D_{r-5}(F) \) is the local Raviart–Thomas space on the face \( F \).

**Proof.** The number of conditions is \((r-1)(2r^2-7r+18)\) which equals the dimension of \( M^{(2)}_{d,r-1}(T^z) \). We show that if \( \omega \in M^{(2)}_{d,r-1}(T^z) \) vanishes at \((1,1,2)\), then \( \omega \equiv 0 \). In this case, it is easy to see that \( \omega = \text{curl} \omega = 0 \) on all edges, and that \( \text{curl} \omega \times n_F = 0 \) and \( \omega \cdot n_F = 0 \) on all faces.

To simplify notation, we use the following standard surface differential operators on a face \( F \), with normal direction \( n_F \) and tangential direction \( t_F \) on its boundary \( \partial F \): For a smooth scalar field \( \phi \), we denote

\[
\text{grad}_F \phi = n_F \times \text{grad} \phi \times n_F, \quad \text{rot}_F \phi = \text{grad} \phi \times n_F,
\]

and for a smooth vector field \( \psi \), we denote

\[
\text{curl}_F \psi = n_F \times \text{curl} \psi, \quad \text{div}_F \psi = n_F \cdot \text{curl} (n_F \times \psi).
\]

We also denote the tangential trace of a smooth vector field \( \psi \) on \( F \) as

\[
\psi_F = n_F \times \psi \times n_F.
\]

Stokes theorem on a face \( F \subset \partial T \) yields

\[
\int_{F} (\text{curl}_F \omega)q \, dA - \int_{F} (\text{rot}_F q) \cdot \omega \, dA = \int_{\partial F} \omega \cdot tq \, ds = 0.
\]

For any \( q \in \mathcal{P}_{r-5}(T) \), \( \text{grad}_F q \in D_{r-5}(F) \) and hence using (4.12c), we have

\[
\int_{F} (\text{curl}_F \omega)q \, dA = \int_{F} (\text{rot}_F q) \cdot \omega \, dA = \int_{F} \text{grad} q \cdot (n_F \times \omega) \, dA = \int_{F} (\text{grad} q \times n_F) \cdot (n_F \times \omega \times n_F) \, dA = 0, \quad \forall q \in \mathcal{P}_{r-5}(T).
\]

Since \( \text{curl}_F \omega \in \mathcal{P}_{r-2}(F) \) and vanishes on the \( \partial F \) we conclude that \( \text{curl}_F \omega = 0 \) on \( F \) and therefore \( \text{curl} \omega \cdot n_F = \text{curl}_F \omega = 0 \) on \( F \). Since \( \text{curl} \omega \times n_F = 0 \) on \( F \) we have that \( \text{curl} \omega = 0 \) on \( F \) or that \( \text{curl} \omega = 0 \) on \( \partial F \).

It then follows that \( \omega_F = \text{grad}_F p \) for some \( p \in \mathcal{P}_r(F) \). Since \( \omega \) vanishes on \( \partial F \), we may assume that \( p \) and its derivatives vanish on \( \partial F \) as well. That is, \( \omega_F = \text{grad}_F (b^2_F w) \) for some \( w \in \mathcal{P}_{r-6}(F) \), where \( b_F \in \mathcal{P}_3(F) \) is the cubic face bubble corresponding to \( F \). It then follows from Stokes Theorem that, for all \( \kappa \in D_{r-5}(F) \),

\[
0 = \int_{F} \omega_F \cdot \kappa \, dA = \int_{F} \text{grad}_F (b^2_F w) \cdot \kappa \, dA = -\int_{F} b^2_F w (\text{div}_F \kappa) \, dA.
\]
Since $\text{div}_F : D_{r-5}(F) \rightarrow \mathcal{P}_{r-6}(F)$ is surjective, we conclude that $w = 0$, and so $\omega_F = 0$. Therefore $\omega|_{\partial T} = 0$ and $\omega \in M_{d,r-1}(T^2)$. Finally, the DOFs \[4.12î\] implies that $\text{curl} \omega = 0$ on $T$, and the DOFs \[4.12g\] then give $\omega \equiv 0$. □

Remark 4.12. The proof of Lemma \[4.11\] shows that if $\omega \in M_{d,r-1}(T^2)$ vanishes on \[4.12a–4.12î\] restricted to a single face $F \in \Delta_2(T)$, then $\omega|_F = \text{curl} \omega|_F = 0$.

Lemma 4.13. A function $\omega \in M_{d,r-2}^2(T^2)$ ($r \geq 5$) is uniquely determined by the values
\[
\begin{align*}
(4.13a) \quad & \omega(a), \quad \text{div} \omega(a) \\
(4.13b) \quad & \int_e \omega \cdot \kappa \, ds \quad \forall \kappa \in [\mathcal{P}_{r-4}(e)]^3, \\
(4.13c) \quad & \int_e (\text{div} \omega) \kappa \, ds \quad \forall \kappa \in \mathcal{P}_{r-5}(e), \\
(4.13d) \quad & \int_e \omega \cdot \kappa \, da \quad \forall \kappa \in [\mathcal{P}_{r-5}(F)]^3, \\
(4.13e) \quad & \int_e (\text{div} \omega) \kappa \, da \quad \forall \kappa \in \mathcal{P}_{r-6}(F), \\
(4.13f) \quad & \int_T \omega \cdot \kappa \, dx \quad \forall \kappa \in \text{curl} M_{d,r-1}^1(T^2), \\
(4.13g) \quad & \int_T (\text{div} \omega) \kappa \, dx \quad \forall \kappa \in \text{curl} M_{d,r-3}^2(T^2),
\end{align*}
\]

Proof. The number of degrees of freedom equals the dimension of $M_{d,r-2}^2(T^2)$. If $\omega$ vanishes at the DOFs, then standard arguments show that $\omega = 0$ and $\text{div} \omega = 0$ on $\partial T$ by using \[4.13a–4.13c\]. Therefore $\text{div} \omega \in M_{d,r-3}^2(T^2)$, and so \[4.13g\] implies that $\text{div} \omega = 0$. The exactness of the first sequence in \[4.2\] shows that $\omega = \text{curl} \rho$ for some $\rho \in M_{d,r-1}^1(T^2)$, and therefore, using \[4.13î\], we obtain $\omega \equiv 0$. □

Lemma 4.14. Any $\omega \in M_{d,r-3}^3(T^2)$ is uniquely determined by the degrees of freedom
\[
\begin{align*}
(4.14a) \quad & \omega(a) \\
(4.14b) \quad & \int_e \kappa \, ds \quad \forall \kappa \in \mathcal{P}_{r-5}(e), \\
(4.14c) \quad & \int_F \kappa \, da \quad \forall \kappa \in \mathcal{P}_{r-6}(F), \\
(4.14d) \quad & \int_T \omega \, dx \\
(4.14e) \quad & \int_T \kappa \, dx \quad \forall \kappa \in M_{d,r-3}^3(T^2),
\end{align*}
\]

Proof. The boundary degrees of freedom \[4.14î\] are simply the Lagrange degrees of freedom, and so if $\omega \in M_{d,r-3}^3(T^2)$ vanishes on \[4.14a–4.14e\], then we easily conclude that $\omega|_{\partial T} = 0$. If, in addition, $\omega$ vanishes on \[4.14a–4.14a\], then we easily obtain that $\omega = 0$, and that the degrees of freedom are unisolvent. □

With the degrees of freedom for $M_{d,r-k}^k(T^2)$ established, we turn our attention to the continuous finite element spaces, $M_{c,r-1}^1(T^2)$ and $M_{c,r-2}^2(T^2)$. The degrees of freedom of the former is given in the next lemma.

Lemma 4.15. Let $r \geq 5$. A function $\omega \in M_{c,r-1}^1(T^2)$ is uniquely determined by the values
\[
\begin{align*}
(4.15a) \quad & D^a \omega(a) \quad |a| \leq 1 \\
(4.15b) \quad & \int_T \omega \, dx \quad (a \in \Delta_0(T)),
\end{align*}
\]
Lemma 4.16. Let

\( F \)

and therefore by (4.15f),

\[ (4.16d) \]

where we recall that \( D_{r-5}(F) \) is the local Raviart–Thomas space on the face \( F \).

\[ (4.15e) \]

where we have used (4.15e) in the last equality. Because curl

\[ (4.15b) \]

Applying Stokes Theorem, we find that, for any \( \omega \in \mathcal{P}_{r-5}(F) \),

\[ (4.15c) \]

In this case, it is easy to see that \( \omega|_e = 0 \) on all \( e \in \Delta_1(T) \), \( (\text{curl} \omega|_F) \cdot n_F|_e = 0 \) on all \( e \in \Delta_1(F) \) and \( F \in \Delta_2(T) \), and \( \omega \cdot n_F|_F = 0 \) on all \( F \in \Delta_2(T) \).

Proof. The total number of conditions in (4.15) is \( (r-2)(2r^2 + r + 9) \) which equals the dimension of \( M^1_{c,r-1}(T^z) \). Suppose that \( \omega \in M^1_{c,r-1}(T^z) \) vanishes on the DOFs. We show that \( \omega \equiv 0 \). This is done by adopting similar arguments as the proof of Lemma 4.11.

By using the same arguments as in Lemma 4.11, we conclude that \( \omega_F = \text{grad}_F(b^2_F \omega) \) for some \( w \in \mathcal{P}_{r-6}(F) \). Consequently, we have

\[ 0 = \int_F \omega_F \cdot \kappa \, dA = - \int_F b^2_F w(\text{div}_F \kappa) \, dA \quad \forall \kappa \in D_{r-5}(F), \]

where we have used (4.15e) in the last equality. Because \( \text{curl}_F \omega = (\text{curl} \omega) \cdot n_F \), and \( (\text{curl} \omega \cdot n_F) \) vanishes on the edges of \( F \), we conclude that \( \text{curl}_F \omega = 0 \) on \( F \).

By using the same arguments as in Lemma 4.11 we conclude that \( \omega_F = \text{grad}_F(b^2_F \omega) \) for some \( w \in \mathcal{P}_{r-6}(F) \). Consequently, we have

\[ 0 = \int_F \omega_F \cdot \kappa \, dA = - \int_F b^2_F w(\text{div}_F \kappa) \, dA \quad \forall \kappa \in D_{r-5}(F), \]

and therefore \( w = 0 \) and \( \omega_F|_F = 0 \). Thus, \( \omega = 0 \) on \( F \). Finally, it follows from (4.15g) that \( \text{curl} \omega = 0 \), and therefore, by (4.15), \( \omega = 0 \).

\[ (4.16g) \]

Lemma 4.16. Let \( r \geq 5 \), then a function \( \omega \in M^2_{r-2}(T^z) \) is uniquely determined by the following degrees of freedom:

\[ (4.16a) \]

\[ (4.16b) \]

\[ (4.16c) \]

\[ (4.16d) \]

\[ (4.16e) \]

\[ (4.16f) \]
Lemma 4.17. Let \( r \geq 5 \), then a function \( \omega \in V_{c,r-2}^{2}(T^{2}) \) is uniquely determined by the values
\[
\begin{align*}
(4.17a) & \quad \omega(a) \quad \forall a \in \Delta_{0}(T) \quad (12 \text{ DOFs}), \\
(4.17b) & \quad \int_{\epsilon} (\omega \cdot n_{F}) \kappa \, ds \quad \forall \kappa \in \mathcal{P}_{r-4}(\epsilon) \quad \forall e \in \Delta_{1}(F), \forall F \in \Delta_{2}(T) \quad (12(r-3) \text{ DOFs}), \\
(4.17c) & \quad \int_{F} (\omega \cdot n_{F}) \kappa \, dA \quad \forall \kappa \in \mathcal{P}_{r-5}(F) \quad \forall f \in \Delta_{2}(T) \quad (2(r-3)(r-4) \text{ DOFs}), \\
(4.17d) & \quad \int_{T} \omega \cdot \kappa \, dx \quad \forall \kappa \in \text{curl} \, \tilde{M}_{c,r-1}^{1}(T^{2}) \quad \frac{(r-3)(4r^{2}+3r+14)}{3} \text{ DOFs}, \\
(4.17e) & \quad \int_{T} (\text{div} \, \omega) \kappa \, dx \quad \forall \kappa \in \tilde{V}_{d,r-3}^{3}(T^{2}) \quad \frac{2(r-2)(r-1)r}{3} - 1 \text{ DOFs}.
\end{align*}
\]

Lemma 4.18. Let \( r \geq 5 \), then a function \( \omega \in V_{d,r-3}^{3}(T^{2}) \) is uniquely determined by the following degrees of freedom:
\[
\begin{align*}
(4.18a) & \quad \int_{T} \kappa \, dx, \quad (1 \text{ DOFs}), \\
(4.18b) & \quad \int_{T} \omega \cdot \kappa \, dx \quad \forall \kappa \in \tilde{V}_{d,r-3}^{3}(T^{2}), \quad \frac{2(r-2)(r-1)r}{3} - 1 \text{ DOFs}.
\end{align*}
\]

Proof. Trivial. \( \square \)

4.2. Commuting Projections. In this section we show that the degrees of freedom given in the previous section yield projections that commute with the differential operators. We first consider the sequence with highest smoothness.

Theorem 4.19. Let \( \Pi_{d,0} : C^{\infty}(T) \to M_{d,r}^{0}(T^{2}) \) be the projection induced by the DOFs (4.11), that is,
\[
\phi(\Pi_{d,0}p) = \phi(p), \quad \forall \phi \in \text{DOFs in (4.11)}.
\]
Likewise, let \( \Pi_{d,1} : [C^{\infty}(T)]^{3} \to M_{d,r-1}^{1}(T^{2}) \) be the projection induced by the DOFs (4.12), \( \Pi_{d,2} : [C^{\infty}(T)]^{3} \to M_{d,r-2}^{2}(T^{2}) \) be the projection induced by the DOFs (4.13), and \( \Pi_{d,3} : C^{\infty}(T) \to M_{d,r-3}^{3}(T^{2}) \) be the projection induced by the DOFs (4.14). Then for \( r \geq 5 \) the following diagram commutes
\[
\begin{align*}
\mathbb{R} & \to C^{\infty}(T) & \text{grad} & \to [C^{\infty}(T)]^{3} & \text{curl} & \to [C^{\infty}(T)]^{3} & \text{div} & \to C^{\infty}(T) & \to 0 \\
\downarrow \Pi_{d,0} & & \downarrow \Pi_{d,1} & & \downarrow \Pi_{d,2} & & \downarrow \Pi_{d,3} & & & \\
\mathbb{R} & \to M_{d,r}^{0}(T^{2}) & \text{grad} & \to M_{d,r-1}^{1}(T^{2}) & \text{curl} & \to M_{d,r-2}^{2}(T^{2}) & \text{div} & \to M_{d,r-3}^{3}(T^{2}) & \to 0.
\end{align*}
\]
More specifically, we have
\[
\begin{align*}
(4.19a) & \quad \text{grad} \, \Pi_{d,0}p = \Pi_{d,1} \text{grad} \, p, \quad \forall p \in C^{\infty}(T) \\
(4.19b) & \quad \text{curl} \, \Pi_{d,1}p = \Pi_{d,2} \text{curl} \, p, \quad \forall p \in [C^{\infty}(T)]^{3}, \\
(4.19c) & \quad \text{div} \, \Pi_{d,2}p = \Pi_{d,3} \text{div} \, p, \quad \forall p \in [C^{\infty}(T)]^{3}.
\end{align*}
\]
Proof. (i) Proof of (4.12a). We take \( p \in C^\infty(T) \). Since \( \rho := \text{grad} \, \Pi_0 p - \Pi_1 \text{grad} \, p \in M_{d,r-1}(T^z) \), we only need to prove that \( \rho \) vanishes at the DOFs (4.11).

For the vertex-based terms, we have, for all \( |\alpha| \leq 1 \) and \( a \in \Delta_0(T) \),
\[
D^a \rho(a) = D^a(\text{grad} \, \Pi_{d,0} p(a) - \Pi_{d,1} \text{grad} \, p(a)) = 0,
\]
by the definition of \( \Pi_{d,0} \), \( \Pi_{d,1} \) and the DOFs (4.11a), (4.12a).

For the edge-based terms, we have, for all \( \kappa \in [P_{r-4}(e)]^3 \),
\[
\int_e (\text{curl} \, \rho) \cdot \kappa \, ds = \int_e (\text{curl} \, (\text{grad} \, \Pi_{d,0} p - p)) \cdot \kappa \, ds = 0 \quad \forall \kappa \in [P_{r-4}(e)]^3
\]
by the definition of \( \Pi_{d,1} \) and (4.12c).

For the face-based terms, we have, for all \( \kappa \in P_{r-4}(F) \),
\[
\int_F (\rho \cdot n_F) \kappa \, dA = \int_F (\text{grad} \, \Pi_{d,0} p - \Pi_{d,1} \text{grad} \, p) \cdot n_F \kappa \, dA = 0
\]
by the definitions of \( \Pi_{d,0} \), \( \Pi_{d,1} \) and the DOFs (4.11a), (4.12d). We also have, for all \( \kappa \in D_{r-5}(F) \),
\[
\int_F (n_F \times \rho \times n_F) \cdot \kappa \, dA = \int_F (n_F \times (\text{grad} \, \Pi_{d,0} p - p) \times n_F) \cdot \kappa \, dA
\]
\[
= \int_F \text{grad}_F (\Pi_{d,0} p - p) \cdot \kappa \, dA
\]
\[
= - \int_F (\Pi_{d,0} p - p) \text{div}_F \kappa \, dA + \int_{\partial F} (\Pi_{d,0} p - p) \kappa \cdot n_{\partial F} \, ds,
\]
where \( n_{\partial F} \) is unit normals tangent to \( F \) and perpendicular to the edges of \( F \). Since for \( \kappa \in D_{r-5}(F) \), \( \text{div}_F \kappa \in P_{r-6}(F) \), and \( \kappa \cdot n_{\partial F} |e \in P_{r-6}(e) \) for all three edges \( e \) of \( F \), the right hand side of the above expression vanishes by the DOFs (4.11b) and (4.11d). Moreover, we have by the definition of \( \Pi_{d,1} \) and (4.12d),
\[
\int_F (\text{curl} \, \rho \times n_F) \cdot \kappa \, dA = \int_F (\text{curl} \, \text{grad} \, (\Pi_{d,0} p - p) \times n_F) \cdot \kappa \, dA = 0 \quad \forall \kappa \in [P_{r-5}(F)]^3.
\]

For the cell-based terms, we have, for \( \kappa \in \text{grad} \, M_{d,r-1}(T^z) \),
\[
\int_T \rho \cdot \kappa \, dx = \int_T (\text{grad} \, \Pi_{d,0} p - \Pi_1 \text{grad} \, p) \cdot \kappa \, dx = 0
\]
using the definitions of \( \Pi_{d,0}, \Pi_{d,1} \) and the DOFs (4.11a) and (4.12g). We also have using definition of \( \Pi_{d,1} \) and (4.12h),
\[
\int_T \text{curl} \, \rho \cdot \kappa \, dx = \int_T (\text{curl} \, \text{grad} \, (\Pi_{d,0} p - p)) \cdot \kappa \, dx = 0.
\]
Combining the above results, we conclude that \( \rho = \text{grad} \Pi_{d,0}p - \Pi_{d,1} \text{grad} p = 0 \). This completes the proof for the identity (4.19a).

(ii) Proof of (4.19b): Let \( p \in [C^\infty(T)]^3 \) and set \( \rho = \text{curl} \Pi_{d,1}p - \Pi_{d,2} \text{curl} p \in M^3_{d,r-2}(T^\ast) \). We show that \( \rho \) vanishes at the DOFs (4.13).

First, we have for all \( a \in \Delta_0(T) \),
\[
\rho(a) = (\text{curl} \Pi_{d,1}p)(a) - (\Pi_{d,2} \text{curl} p)(a) = 0
\]
by (4.12a) and (4.13a). Furthermore, we have
\[
\text{div} \rho(a) = -\text{div} \Pi_{d,2} \text{curl} p(a) = -\text{div} \text{curl} p(a) = 0
\]
by (4.13a). Similar arguments show that
\[
\int_e (\text{div} \rho) \kappa \, ds = 0 \quad \forall \kappa \in \mathcal{P}_{r-5}(e), \quad \forall e \in \Delta_1(T),
\]
\[
\int_F (\text{div} \rho) \kappa \, dA = 0 \quad \forall \kappa \in \mathcal{P}_{r-6}(F), \quad \forall F \in \Delta_2(T),
\]
\[
\int_T (\text{div} \rho) \kappa \, dx = 0 \quad \forall \kappa \in \tilde{M}_{d,r-3}(T^\ast)
\]
by using (4.13c), (4.13e), and (4.13g).

Next we have, for \( \kappa \in [\mathcal{P}_{r-4}(e)]^3 \),
\[
\int_e \rho \cdot \kappa \, ds = \int_e (\text{curl} \Pi_{d,1}p - \text{curl} p) \cdot \kappa \, ds = 0
\]
by (4.12a) and (4.13b).

Let \( F \in \Delta_2(T) \) and \( \kappa \in [\mathcal{P}_{r-5}(F)]^3 \). We then have
\[
\int_F (\rho \times n_F) \cdot \kappa \, dA = \int_F (\text{curl} \Pi_{d,1}p - \text{curl} p) \times n_F \cdot \kappa \, dA = 0
\]
by (4.12b) and (4.13d). We also have, for \( \kappa \in \mathcal{P}_{r-5}(F) \),
\[
\int_F (\rho \cdot n_F) \kappa \, dA = \int_F (\text{curl} \Pi_{d,1}p - \text{curl} p) \cdot n_F \kappa \, dA
\]
\[
= \int_F (\text{curl} F \Pi_{d,1}p - \text{curl} F p) \kappa \, dA
\]
\[
= \int_F (\text{rot} F \kappa \cdot (\Pi_{d,1}p - p)) \, dA + \int_{\partial F} (\Pi_{d,1}p - p) \cdot t_{\partial F} \kappa \, ds = 0
\]
by (4.13d), (4.12c), and (4.12b).

Finally, we have for all \( \kappa \in \text{curl} \tilde{M}^1_{d,r-1}(T^\ast) \),
\[
\int_T \rho \cdot \kappa \, dx = \int_T (\text{curl} \Pi_{d,1}p - \text{curl} p) \cdot \kappa \, dx = 0
\]
by (4.12a) and (4.13b). Thus, \( \rho \) vanishes on (4.13), and so \( \rho \equiv 0 \).

(iii) Proof of (4.19c): Let \( p \in [C^\infty(T)]^3 \) and set \( \rho = \text{div} \Pi_{d,2}p - \Pi_{d,3} \text{div} p \in M^3_{d,r-3}(T^\ast) \). Similar to parts (i)–(ii), we show that \( \rho \) vanishes on (4.14).

First, we clearly have \( \rho(a) = 0 \) by (4.14a) and (4.13a). We further have
\[
\int_e \rho \kappa \, ds = 0 \quad \forall \kappa \in \mathcal{P}_{r-5}(e), \quad \forall e \in \Delta_1(T)
\]
by (4.14b) and (4.13c):
\[
\int_F \rho \kappa \, dA = 0 \quad \forall \kappa \in \mathcal{P}_{r-6}(F), \quad \forall F \in \Delta_2(T)
\]
by (4.14c) and (4.13c); and
\[ \int_T \rho \kappa \, dx = 0 \quad \forall \kappa \in \hat{M}^3_{d,r-3}(T^z) \]
by (4.14c) and (4.13g). Finally, we have
\[ \int_T \rho \, dx = \int_T (\text{div} \Pi_{d,2} p - \text{div} p) \, dx \]
\[ = \int_{\partial T} (\Pi_{d,2} p - p) \cdot n \, dA = 0 \]
by (4.14d) and (4.13d). Therefore, \( \rho \) vanishes on (4.11), and so \( \rho \equiv 0 \).

By using similar arguments as those given in Theorem 4.19, we obtain commutative projections for the second sequence in (4.10). The proof is given in the appendix.

**Theorem 4.20.** Let \( \Pi_{d,0} : C^\infty(T) \to M^0_{d,r}(T^z) \) be the projection induced by the DOFs (4.11), \( \Pi_{d,1} : \left[ C^\infty(T) \right]^3 \to M^1_{d,r-1}(T^z) \) be the projection induced by the DOFs (4.12), \( \Pi_2 : \left[ C^\infty(T) \right]^3 \to M^2_{r-2}(T^z) \) be the projection induced by the DOFs (4.16), and \( \Pi_3 : C^\infty(T) \to V^3_{d,r-3}(T^z) \) be the projection induced by the DOFs (4.18). Then for \( r \geq 5 \) the following diagram commutes
\[
\begin{array}{cccc}
\mathbb{R} & \to & C^\infty(T) & \xrightarrow{\text{grad}} \left[ C^\infty(T) \right]^3 & \xrightarrow{\text{curl}} [C^\infty(T)]^3 & \xrightarrow{\text{div}} C^\infty(T) & \to 0 \\
\downarrow \Pi_{d,0} & & \downarrow \Pi_{d,1} & & \downarrow \Pi_2 & & \downarrow \Pi_3 \\
\mathbb{R} & \to & M^0_{d,r}(T^z) & \xrightarrow{\text{grad}} M^1_{d,r-1}(T^z) & \xrightarrow{\text{curl}} M^2_{r-2}(T^z) & \xrightarrow{\text{div}} V^3_{d,r-3}(T^z) & \to 0.
\end{array}
\]
More specifically, we have
\begin{align*}
(4.20a) & \quad \text{grad} \Pi_{d,0} p = \Pi_{d,1} \text{grad} p, \quad \forall p \in C^\infty(T) \\
(4.20b) & \quad \text{curl} \Pi_{d,1} p = \Pi_2 \text{curl} p, \quad \forall p \in [C^\infty(T)]^3, \\
(4.20c) & \quad \text{div} \Pi_2 p = \Pi_3 \text{div} p, \quad \forall p \in [C^\infty(T)]^3.
\end{align*}

Finally, we state the commutative projections for the third sequence in (4.10). The proof is given in the appendix.

**Theorem 4.21.** Let \( \Pi_{d,0} : C^\infty(T) \to M^0_{d,r}(T^z) \) be the projection induced by the DOFs (4.11), \( \Pi_{c,1} : \left[ C^\infty(T) \right]^3 \to M^1_{c,r-1}(T^z) \) be the projection induced by the DOFs (4.15), \( \Pi_{c,2} : \left[ C^\infty(T) \right]^3 \to V^2_{c,r-2}(T^z) \) be the projection induced by the DOFs (4.17), and \( \Pi_3 : C^\infty(T) \to V^3_{d,r-3}(T^z) \) be the projection induced by the DOFs (4.18). Then, the following diagram commutes
\[
\begin{array}{cccc}
\mathbb{R} & \to & C^\infty(T) & \xrightarrow{\text{grad}} \left[ C^\infty(T) \right]^3 & \xrightarrow{\text{curl}} [C^\infty(T)]^3 & \xrightarrow{\text{div}} C^\infty(T) & \to 0 \\
\downarrow \Pi_{d,0} & & \downarrow \Pi_{c,1} & & \downarrow \Pi_{c,2} & & \downarrow \Pi_3 \\
\mathbb{R} & \to & M^0_{d,r}(T^z) & \xrightarrow{\text{grad}} M^1_{c,r-1}(T^z) & \xrightarrow{\text{curl}} V^2_{c,r-2}(T^z) & \xrightarrow{\text{div}} V^3_{d,r-3}(T^z) & \to 0.
\end{array}
\]
More specifically, we have
\begin{align*}
(4.21a) & \quad \text{grad} \Pi_{d,0} p = \Pi_{c,1} \text{grad} p, \quad \forall p \in C^\infty(T) \\
(4.21b) & \quad \text{curl} \Pi_{c,1} p = \Pi_{c,2} \text{curl} p, \quad \forall p \in [C^\infty(T)]^3, \\
(4.21c) & \quad \text{div} \Pi_{c,2} p = \Pi_3 \text{div} p, \quad \forall p \in [C^\infty(T)]^3.
\end{align*}
5. Global Smooth Finite Element de Rham Complexes in Three Dimensions

In this section we study the global finite element spaces induced by the degrees of freedom in Section 4. To this end, we suppose that $\Omega \subset \mathbb{R}^3$ is a polyhedral domain. Let $T_h$ be a simplicial triangulation of $\Omega$, and let $T_h^i$ be the simplicial triangulation obtained by adding connecting each bar ycenter of $T \in T_h$ with its vertices, i.e., $T_h^i$ is obtained by performing an Affeld split to each $T \in T_h$.

The degrees of freedom in Lemmas 4.8, 4.11, 4.13–4.18 (cf. Remarks 4.10 and 4.12) lead to the following global spaces, for $r \geq 5$,

$$M^0_{d,r}(T_h^i) = \{ \omega \in C^1(\Omega) : \omega|_T \in M^0_{d,r}(T^i) \ \forall T \in T_h, \ \omega \text{ is } C^2 \text{ at vertices}\},$$

$$M^1_{d,r-1}(T_h^i) = \{ \omega \in [C^0(\Omega)]^3 : \text{curl } \omega \in [C^0(\Omega)]^3, \ \omega|_T \in M^1_{d,r-1}(T^i) \ \forall T \in T_h, \ \omega \text{ is } C^1 \text{ at vertices}\},$$

$$M^2_{d,r-2}(T_h^i) = \{ \omega \in [C^0(\Omega)]^3 : \text{div } \omega \in C^0(\Omega), \ \omega|_T \in M^2_{d,r-2}(T^i) \ \forall T \in T_h\},$$

$$M^3_{d,r-3}(T_h^i) = \{ \omega \in C^0(\Omega) : \omega|_T \in M^3_{d,r-3}(T^i) \ \forall T \in T_h\}.$$

Clearly the following sequences of spaces

$$(5.1a) \quad \mathbb{R} \rightarrow M^0_{d,r}(T_h^i) \rightarrow M^1_{d,r-1}(T_h^i) \rightarrow M^2_{d,r-2}(T_h^i) \rightarrow M^3_{d,r-3}(T_h^i) \rightarrow 0,$$

$$(5.1b) \quad \mathbb{R} \rightarrow M^0_{d,r}(T_h^i) \rightarrow M^1_{d,r-1}(T_h^i) \rightarrow M^2_{d,r-2}(T_h^i) \rightarrow V^3_{d,r-3}(T_h^i) \rightarrow 0,$$

$$(5.1c) \quad \mathbb{R} \rightarrow M^0_{d,r}(T_h^i) \rightarrow M^1_{d,r-1}(T_h^i) \rightarrow V^2_{d,r-2}(T_h^i) \rightarrow V^3_{d,r-3}(T_h^i) \rightarrow 0$$

forms a complex. In addition, we can define commuting projections. For example, for the first sequence we can define $\pi_{d,i}$ such that $\pi_{d,i} \omega|_T = \Pi_{d,i}(\omega|_T)$ for all $T \in T_h$, and by using Theorem 4.10 we get the following commuting diagram for the second sequence (5.1b):

$$\begin{array}{cccccc}
\mathbb{R} & \rightarrow & C^\infty(\Omega) & \rightarrow & [C^\infty(\Omega)]^3 & \rightarrow & C^\infty(\Omega) \\
\downarrow \pi_{d,0} & & \downarrow \pi_{d,1} & & \downarrow \pi_{d,2} & & \downarrow \pi_{d,3} \\
M^0_{d,r}(T_h^i) & \rightarrow & M^1_{d,r-1}(T_h^i) & \rightarrow & M^2_{d,r-2}(T_h^i) & \rightarrow & M^3_{d,r-3}(T_h^i) \\
\n\end{array}$$

Similar results hold for the other two sequences in (5.1) as well.

Note that the top row is an exact sequence if $\Omega$ is contractible; see for example [9]. In the next result we will show that the bottom row is also exact on contractible domains. Unfortunately, the projections by themselves do not prove the discrete exactness property because they require extra smoothness. However, the exactness of the first and last mapping can be proved easily, and the exactness of the second mapping will follow from a counting argument.

**Theorem 5.1.** Suppose that $\Omega$ is contractible. Then the complexes in (5.1) are exact.

**Proof.** We prove exactness of the second sequence (5.1b). The other two can be proved by similar arguments.

(i) Let $\omega \in M^1_{d,r-1}(T_h^i)$ with curl $\omega = 0$. Then using a standard result from [9] (see also [10]) there exists a $\rho \in H^2(\Omega)$ such that grad $\rho = \omega$. Since $\omega$ is $C^1$ at the vertices, $\rho$ is $C^2$ at the vertices. Also on each $T \in T_h$, $\omega \in M^1_{d,r-1}(T^i)$, and using that grad $\rho = \omega$, we have $\rho \in M^0_{d,r}(T^i)$. Hence, $\rho \in M^0_{d,r}(T_h^i)$.

(ii) Next, it is shown in [21] that div $: M^2_{d,r-2}(T_h^i) \rightarrow V^3_{d,r-3}(T_h^i)$ is a surjection for $r \geq 5$.

(iii) Finally, we show that curl $: M^1_{d,r-1}(T_h^i) \rightarrow M^2_{d,r-2}(T_h^i)$ is a surjection for $r \geq 5$ using a counting argument. Let $V$, $E$, $F$, and $T$ denote the number of vertices, edges, faces, and tetrahedron in $T_h$, respectively.
respectively. We then set
\[ \ker M_{r-2}^2(\mathcal{T}_h^n) := \{ \omega \in M_{r-2}^2(\mathcal{T}_h^n) : \text{div} \omega = 0 \}. \]

By the rank-nullity theorem, part (i), and Lemmas 4.8 and 4.11 we have that
\[
\dim \text{curl } M_{d,r-1}^1(\mathcal{T}_h^n) = \dim M_{d,r-1}^1(\mathcal{T}_h^n) - \dim \text{grad } M_{d,r}^0(\mathcal{T}_h^n)
= \dim M_{d,r-1}^1(\mathcal{T}_h^n) - \dim M_{d,r}^0(\mathcal{T}_h^n) + 1
= \left( 12V + [3(r-4) + 3(r-3)]E + \frac{1}{2}((r-2)(r-3) + (r-3)(r-5) + (r-3)(r-4))F \right.
+ (r-3)(2r-5)(r-4)T - \left( 10V + [(r-5) + 2(r-4)]E + \frac{1}{2}(r-5)(r-4) \right.
+ \frac{1}{2}(r-3)(r-2)F + \frac{2}{3}(r-4)(r-3)(r-2)T ) + 1
+ 2V + (3r-8)E + \left( \frac{3}{2}r^2 - \frac{21}{2}r + 17 \right)F + \left( \frac{4}{3}r^3 - 13r^2 + \frac{125}{3}r - 44 \right)T + 1.
\]

Likewise, by the rank-nullity theorem, part (ii), and Lemmas 4.10 and 4.18 we obtain
\[
\dim \ker M_{r-2}^2(\mathcal{T}_h^n) = \dim M_{r-2}^2(\mathcal{T}_h^n) - \dim V_{d,r-3}^3(\mathcal{T}_h^n)
= \left( 3V + 3(r-3)E + \frac{3}{2}(r-3)(r-4)F \right.
+ \left. 3\left[ \frac{1}{4}(r-3)(r-3)(r-4) + \frac{4}{6}(r-5)(r-4)(r-3) \right]T \right)
- \left( \frac{4}{6}r(r-1)(r-2)T \right)
+ 3V + (3r-3)E + \left( \frac{3}{2}(r-3)(r-4)F + \left[ \frac{4}{3}r^3 - 13r^2 + \frac{125}{3}r - 45 \right]T. \right.
\]

We then find that
\[
\dim \ker M_{r-2}^2(\mathcal{T}_h^n) - \dim \text{curl } M_{d,r-1}^1(\mathcal{T}_h^n) = V - E + F - T - 1 = 0,
\]
by an Euler relation. Since \( \text{curl } M_{d,r-1}^1(\mathcal{T}_h^n) \subseteq \ker M_{r-2}^2(\mathcal{T}_h^n) \), we have \( \ker M_{d,r-1}^1(\mathcal{T}_h^n) = \ker M_{r-2}^2(\mathcal{T}_h^n) \), and therefore the complex 5.11 is exact.

6. Concluding Remarks

In this paper we have developed several new local discrete de Rham complexes with varying level of smoothness on Alfeld splits. These results lead, e.g., to characterizations of discrete divergence-free subspaces for the Stokes problem and local dimension formulas of smooth piecewise polynomial spaces. We have also constructed analogous global complexes in two dimensions and projections that commute with the differential operators. In the future, we plan on using our techniques to study different types of splits (e.g., Powell-Sabin, Worsey-Frain) as done in 6 for low-order approximations. In addition, we plan to construct degrees of freedom for the spaces in any spatial dimension.

References

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It is shown in [1] that functions in $M^2_{d,r-1}(T^2)$ are $C^2$ on $\Delta_0(T)$. Moreover, it is clear that the result is true in the cases $k = 2, 3$. Thus, it suffices to prove the result $k = 1$. For readability, we prove the result with $r$ replaced by $r - 1$.

Define $\tilde{M}^1_{d,r-1}(T^2) = \{ \omega \in M^1_{d,r-1}(T^2) : \omega \text{ is } C^1 \text{ on } \Delta_0(T^2) \}$. We show that $\tilde{M}^1_{d,r-1}(T^2) = M^1_{d,r-1}(T^2)$.

Let $\kappa \in M^2_{d,r-2}(T^2) \subset V^2_{d,r-2}(T^2)$ satisfy $\text{div } \kappa = 0$. Using Theorem 4.7 there exists $\omega \in M^1_{d,r-1}(T^2)$ such that $\kappa = \text{curl } \omega$. Because $\kappa$ is continuous, we have $\omega \in M^1_{d,r-1}(T^2)$, and so $\omega \in \tilde{M}^1_{d,r-1}(T^2)$. Consequently, we easily deduce that

$$\mathbb{R} \longrightarrow M^0_{d,r}(T^2) \xrightarrow{\text{grad}} \tilde{M}^1_{d,r-1}(T^2) \xrightarrow{\text{curl}} M^2_{d,r-2}(T^2) \xrightarrow{\text{div}} V^3_{d,r-3}(T^2) \longrightarrow 0$$

is exact. This implies that

$$\dim \tilde{M}^1_{d,r-1}(T^2) = \dim \text{grad } M^0_{d,r}(T^2) + \dim M^2_{d,r-2}(T^2) - \dim V^3_{d,r-3}(T^2).$$

On the other hand, the exactness of (4.10) yields

$$\dim M^1_{d,r-1}(T^2) = \dim M^0_{d,r}(T^2) + \dim M^2_{d,r-2}(T^2) - \dim V^3_{d,r-3}(T^2),$$

and therefore $\dim \tilde{M}^1_{d,r-1}(T^2) = \dim M^1_{d,r-1}(T^2)$. Since $\tilde{M}^1_{d,r-1}(T^2) \subset M^1_{d,r-1}(T^2)$, we conclude that $\tilde{M}^1_{d,r-1}(T^2) = M^1_{d,r-1}(T^2)$. \hfill \Box

\footnote{Note that the proof of Theorem 4.7 does not depend on Lemma 4.4.}
APPENDIX B. PROOF THEOREM 4.20

Property (4.20a) is the same as (4.19a), so we only need to prove (4.20b) and (4.20c).

(i) Proof of (4.20b). Let \( p \in [C^\infty(\Omega)]^3 \) and set \( \rho = \Pi_{d,2} \text{curl} \rho - \Pi_2 \text{curl} \rho \in M^2_{r-2}(T^*) \). Then using the definitions of \( \Pi_{d,2} \) and \( \Pi_2 \), and by using the DOFs (4.16a), (4.16c), (4.13a), (4.13b), (4.13d), (4.13f), (4.13g), and (4.19a), we conclude that \( \rho \) vanishes on the DOFs of \( M^2_{r-2}(T^*) \). Thus, applying Lemma 4.16, we get \( \rho = 0 \). Using (4.19b), we have
\[ \text{curl} \Pi_{d,1} p = \Pi_{d,2} \text{curl} p = \Pi_2 \text{curl} p, \]
and so (4.20b) is satisfied.

(ii) Proof of (4.20c). Let \( p \in [C^\infty(T)]^3 \) and set \( \rho = \text{div} \Pi_{d,1} p - \Pi_3 \text{div} \rho \in V^3_{d,r-3}(T^*) \). We show that \( \rho \) vanishes on the DOFs (4.18). First, by (4.18a), the divergence theorem, and (4.16c), we have
\[ \int_T \rho \, d\mathbf{x} = \int_T (\text{div} \Pi_{d,1} p - \text{div} \rho) \, d\mathbf{x} = \int_{\partial T} (\Pi_{d,1} p - \rho) \cdot \mathbf{n} \, d\mathbf{A} = 0. \]

Next, we apply the definitions of \( \Pi_2 \), \( \Pi_3 \), and the DOFs (4.16c), (4.18) to obtain
\[ \int_T \rho \kappa \, d\mathbf{x} = 0 \quad \forall \kappa \in \tilde{V}^3_{d,r-3}(T^*). \]
Finally, applying Lemma 4.18 we conclude that \( \rho = 0 \). This concludes the proof.

APPENDIX C. PROOF THEOREM 4.21

Proof. (i) Proof of (4.21c). Set \( \rho = \text{div} \Pi_{c,2} p - \Pi_3 \text{div} \rho \in V^3_{d,r-3}(T^*) \). We show that \( \rho \) vanishes on (4.18). We easily find that
\[ \int_T \rho \kappa \, d\mathbf{x} = 0 \quad \forall \kappa \in \tilde{V}^3_{d,r-3}(T^*) \]
by (4.18a) and (4.17c). We also have
\[ \int_T \rho \, d\mathbf{x} = \int_T (\text{div} \Pi_{c,2} p - \text{div} \rho) \, d\mathbf{x} = \int_{\partial T} (\Pi_{c,2} p - \rho) \cdot \mathbf{n} \, d\mathbf{A} = 0 \]
by (4.17c). Thus \( \rho \) vanishes on (4.18), and so \( \rho = 0 \).

(ii) Proof of (4.21b). Set \( \rho = \text{curl} \Pi_{c,1} p - \Pi_{c,2} \text{curl} \rho \in V^2_{c,r-3}(T^*) \). We show that \( \rho \) vanishes on (4.17). We clearly have \( \rho(a) = 0 \) for all \( a \in \Delta_0(T) \) by (4.17a) and (4.15a). Furthermore,
\[ \int_T (\rho \cdot \mathbf{n}_F) \kappa \, ds = 0 \quad \forall \kappa \in \Delta_{T-4}(e), \quad \forall e \in \Delta_1(F), \quad \forall F \in \Delta_2(T), \]
by (4.17a) and (4.15a). Next, we apply Stokes Theorem and (4.17b), (4.15b), (4.15c) to get
\[ \int_F (\rho \cdot \mathbf{n}_F) \kappa \, d\mathbf{A} = \int_F (\text{curl}_F \Pi_{c,1} p - \text{curl}_F \rho) \kappa \, d\mathbf{A} = \int_F (\Pi_{c,1} p - p) \cdot \text{rot}_F \kappa \, d\mathbf{A} + \int_{\partial F} (\Pi_{c,1} p - p) \cdot \mathbf{t} \kappa \, ds = \int_F (n_F \times (\Pi_{c,1} p - p) \times n_F) \cdot (\text{grad} \kappa \times n_F) \, d\mathbf{A} + \int_{\partial F} (\Pi_{c,1} p - p) \cdot \mathbf{t} \kappa \, ds = 0 \]
for all \( \kappa \in \mathbb{P}_{r-5}(F) \). Finally, using (4.17d) and (4.15g), we have
\[ \int_T \rho \cdot \kappa \, d\mathbf{x} = 0 \quad \forall \kappa \in \text{curl}_F \mathbb{P}^1_{c,r-1}(T^*), \]
and by (4.17d), we have
\[ \int_T (\text{div} \rho) \kappa \, d\mathbf{x} = 0 \quad \forall \kappa \in \tilde{V}^3_{d,r-3}(T^*). \]
Thus, $\rho$ vanishes on all the DOFs (4.17), and therefore $\rho \equiv 0$.

(iii) Proof of (4.21a). Set $\rho = \text{grad} \Pi_{d,0} p - \Pi_{c,1} \text{grad} p \in M_{c,r-1}^1(T^z)$. We show that $\rho$ vanishes on (4.15).

We have $D^\alpha \rho(a) = 0$ for all $|\alpha| \leq 1$ and $a \in \Delta_0(T)$ by (4.15a) and (4.11a), and

$$\int_e \rho \cdot \kappa \, ds = \int_e (\text{grad} \Pi_{d,0} p - \text{grad} p) \cdot \kappa \, ds = 0 \quad \forall \kappa \in [P_{r-5}(e)]^3$$

by (4.15b) and (4.11a)-(4.11c). Furthermore, we have

$$\int_e (\text{curl} \rho |_F \cdot n_F) \kappa \, ds = 0$$

by (4.15c).

Let $\kappa \in P_{r-4}(F)$. Then

$$\int_F (\rho \cdot n_F) \kappa \, dA = \int_F (\text{grad} \Pi_{d,0} p - \text{grad} p) \cdot n_F \kappa \, dA = 0$$

by (4.15d) and (4.11c). Moreover, we have

$$\int_F (n_F \times \rho \times n_F) \cdot \kappa \, dA = 0 \quad \forall \kappa \in D_{r-5}(F)$$

by using the exact same arguments as those found in the proof of Theorem 4.19.

Finally, we apply (4.15f) and (4.11f) to get

$$\int_T \rho \cdot \kappa \, dx = 0 \quad \forall \kappa \in \text{grad} M_{d,r}^0(T^z),$$

and use (4.15g) to get

$$\int_T \text{curl} \rho \cdot \kappa \, dx = 0 \quad \forall \kappa \in \text{curl} M_{c,r-1}^1(T^z).$$

Thus, $\rho$ vanishes on the DOFs (4.15), and thus $\rho \equiv 0$. \qed