ON EXPLICIT SOLUTIONS TO THE STATIONARY AXISYMMETRIC
EINSTEIN-MAXWELL EQUATIONS DESCRIBING DUST DISKS

C. KLEIN

ABSTRACT. We review explicit solutions to the stationary axisymmetric Einstein-Maxwell equations which can be interpreted as disks of charged dust. The disks of finite or infinite extension are infinitesimally thin and constitute a surface layer at the boundary of an electro-vacuum. The Einstein-Maxwell equations in the presence of one Killing vector are obtained by using a projection formalism. This leads to equations for three-dimensional gravity where the matter is given by a $SU(2, 1)/SU(1, 1)$ nonlinear sigma model. The $SU(2, 1)$ invariance of the stationary Einstein-Maxwell equations can be used to construct solutions for the electro-vacuum from solutions to the pure vacuum case via a so-called Harrison transformation. It is shown that the corresponding solutions will always have a non-vanishing total charge and a gyromagnetic ratio of 2. Since the vacuum and the electro-vacuum equations in the stationary axisymmetric case are completely integrable, large classes of solutions can be constructed with techniques from the theory of solitons. The richest class of physically interesting solutions to the pure vacuum case due to Korotkin is given in terms of hyperelliptic theta functions. The Harrison transformed hyperelliptic solutions are discussed. As a concrete example we study the transformation of a family of counter-rotating dust disks. To obtain algebro-geometric solutions with vanishing total charge which are of astrophysical relevance, three-sheeted surfaces have to be considered.

The matter in the disk is discussed following Bičák et al. We review the ‘cut and glue’ technique where a strip is removed from an explicitly known spacetime and where the remainder is glued together after displacement. The discontinuities of the normal derivatives of the metric at the gluing hypersurface lead to infinite disks. If the energy conditions are satisfied and if the pressure is positive, the disks can be interpreted in the vacuum case as made up of two components of counter-rotating dust moving on geodesics. In electro-vacuum the condition of geodesic movement is replaced by electro-geodesic movement. As an example we discuss a class of Harrison-transformed hyperelliptic solutions. The range of parameters is identified where an interpretation of the matter in the disk in terms of electro-dust can be given.

1. INTRODUCTION

Electromagnetic fields, especially magnetic fields play a role in astrophysics in the context of neutron stars, white dwarfs and galaxy formation. A complete relativistic understanding of such situations requires studying the coupled Einstein-Maxwell equations. Of special interest are stationary axisymmetric situations since isolated matter configurations in thermodynamical equilibrium belong within relativity to this class, see [1][2]. Since the stationary axisymmetric Einstein-Maxwell equations in vacuum in the form of Ernst [3] are completely integrable (see [4]), powerful solution generating techniques from the theory of solitons are at hand to obtain physically interesting solutions. But the equations in the matter region — which is generally approximated as an ideal fluid — do not seem
to be integrable. This makes it difficult to find global solutions which hold both in a threedimensionally extended matter region and in vacuum.

To obtain global solutions to the Einstein-Maxwell equations with these methods, one is thus limited to two-dimensionally extended matter distributions, i.e. surface layers. Infinitesimally thin disks have been discussed in Newtonian astrophysics as models for certain galaxies, see [5], and as models for the matter in accretion disks around black holes, see [6] and references therein. In this case the equations in the matter region reduce to ordinary differential equations the solutions of which determine boundary data for the vacuum equations. Alternatively surface layers can be obtained by 'cut and glue' techniques. Bičák, Lynden-Bell and Katz [7, 8] studied static spacetimes, from which they removed a strip and glued the remainder together. The non-continuous normal derivatives at the gluing plane lead to a $\delta$-type energy-momentum tensor which can be interpreted as an infinitely extended disk made up of counter-rotating dust. This method was extended in [9] to generate disk sources of the Kerr-metric. With the same techniques, disk sources for Kerr-Newman metrics [10], static axisymmetric spacetimes with magnetic fields [11] and conformastationary metrics [12] were given.

Counter-rotating disks are discussed in astrophysics as models for certain $S0$ and $Sa$ galaxies (see [13] and references given therein). These galaxies show counter-rotating matter components and are believed to be the consequence of the merger of galaxies. Recent investigations have shown that there is a large number of galaxies (see [13], the first was NGC 4550 in Virgo) which show counter-rotating streams in the disk with up to 50% counter-rotation.

By construction all disks due to 'cut and glue' techniques have an infinite extension but finite mass since the mass of the spacetime is not changed by the method. The matter in the disks can be interpreted as a two-dimensional fluid with a purely azimuthal pressure. If the energy conditions in the disk are satisfied and if the pressure in the disk is positive, the matter can alternatively be interpreted as consisting of two counter-rotating streams of pressureless matter, so-called dust. In the pure vacuum case this is best done by introducing observers rotating with the disk in a way that the energy-momentum tensor is diagonal for them. It can be shown that the corresponding dust streams move on geodesics of the inner geometry of the disk. In the electro-vacuum case, the corresponding condition on the matter is motion on electro-geodesics, i.e. solutions to the geodesic equation in the presence of a Lorentz force. It was shown in [10] that this is a more restrictive condition than in the pure vacuum case.

For vacuum so-called Riemann-Hilbert techniques (see [14, 15, 16] and references therein) were used to generate solutions for disks of finite extension. Explicit metrics could be given in terms of theta functions on hyperelliptic Riemann surfaces [14]. Since these two-sheeted surfaces are a generalization of the well-known elliptic surfaces, a powerful theory is at hand to treat hyperelliptic functions. The main advantages of this class are that it is very rich ('solitonic' solutions as the Kerr solution for a rotating black hole are contained as limiting cases), and that a whole subclass with physically interesting solutions could be identified in [17, 18] by studying the analyticity properties of the solution. The corresponding solutions for the electro-vacuum are given on three-sheeted surfaces [14] which are mathematically less well understood. Recent progress in this context was made by Korotkin [19] by considering the Riemann-Hilbert problem on multi-sheeted coverings of the complex plane. The solutions to this problem which can be used to solve the Einstein-Maxwell equations are again given in terms of theta functions.
Until now there are, however, no explicit examples for physically realistic disk solutions on three-sheeted Riemann surfaces. Therefore an intermediate step was taken in [20] where hyperelliptic solutions with charge were studied. These solutions were obtained by exploiting the $SU(2, 1)$ invariance of the stationary Einstein-Maxwell equations (see [21] to [25]). Using a so-called Harrison transformation [20], one can generate solutions with charge from solutions to the pure vacuum equations. A remarkable property of the corresponding spacetimes is the fact that their gyromagnetic ratio is always identical to 2 as in the case of the Kerr-Newman black holes. This is of interest in the context of claims in [27, 28] that this property is a hint on a deep connection between general relativity and relativistic quantum mechanics. By studying the asymptotics of Harrison-transformed pure vacuum solutions, it was shown that the thus obtained solutions will always have a non-vanishing total charge which limits their astrophysical relevance since charges seem to neutralize in our universe. This is a hint that astrophysically interesting solutions without total charge, but non-vanishing magnetic fields in terms of theta functions are to be expected only on three-sheeted surfaces.

This paper is organized as follows: In section 2 the Newtonian case is studied for illustration. The 'cut and glue' techniques of Bičák et al. [7] and disks of finite extension are presented. In section 3 we consider the Einstein-Maxwell equations in the presence of a single Killing vector. Using a projection formalism [29, 30], we perform a standard dimensional reduction of the equations. It is shown that the stationary Einstein-Maxwell equations are equivalent to three-dimensional gravity with a $SU(2, 1)/SU(1, 1) \times U(1)$ sigma model as matter. Using complex notation, one can introduce Ernst potentials [3]. We study a gauge invariant formulation of the $SU(2, 1)$ matrix of the sigma model and the related transformations of the solutions. The Harrison transformation is presented and discussed for simple examples. The asymptotic behavior of asymptotically flat solutions is studied. The stationary axisymmetric case is shown to be completely integrable. In section 4 we recall basic facts on the stationary axisymmetric pure vacuum case and on hyperelliptic disk solutions. Using the Harrison transformation on a family of counter-rotating disk solutions [31, 32], we obtain the complete transformed metric and discuss interesting limiting cases. The discussion of the energy-momentum tensor using Israel’s junction conditions [33] is presented in section 5. The case of the Harrison transformed counter-rotating disk is studied as an example. The range of the physical parameters is given where the matter in the disk can be interpreted as electro-dust. In section 6 we summarize recent results by Korotkin [19] on solutions to the Riemann-Hilbert problem on multi-sheeted Riemann surfaces in terms of Szegő kernels and solutions to the Einstein-Maxwell equations. In section 7 we add some concluding remarks.

2. Newtonian Dust Disks

To illustrate the basic concepts used in the following sections, we will briefly recall some facts on Newtonian dust disks. In Newtonian theory, gravitation is described by a scalar potential $U$ which is a solution to the Laplace equation in the vacuum region. The units in this article are chosen in a way that the Newtonian gravitational constant, the dielectric constant and the velocity of light are equal to 1. We use cylindrical coordinates $\rho$, $\zeta$ and $\phi$ and place the disk made up of a pressureless two-dimensional ideal fluid with radius $\rho_0$ in the equatorial plane $\zeta = 0$. In Newtonian theory stationary perfect fluid solutions and thus also the here considered disks are known to be equatorially symmetric.

Since we concentrate on dust disks, i.e. pressureless matter, the only force to compensate gravitational attraction in the disk is the centrifugal force. This leads in the disk to
\( \frac{\partial f}{\partial x} \) (here and in the following)

\[ U_{,\rho} = \Omega^2(\rho)\rho, \]

where \( \Omega(\rho) \) is the angular velocity of the dust at radius \( \rho \). Since all terms in \( \Omega \) are quadratic in \( \Omega \) there are no effects due to the sign of the angular velocity. The absence of these so-called gravitomagnetic effects in Newtonian theory implies that disks with counter-rotating components will behave with respect to gravity exactly as disks made up of only one component. We will therefore only consider the case of one component in this section.

Integrating \( U \) we get the boundary data \( U(\rho, \theta) \) with an integration constant \( U_0 = U(0, \theta) \) which is related to the central redshift in the relativistic case.

To find the Newtonian solution for a given rotation law \( \Omega(\rho) \), we have to construct a solution to the Laplace equation which is everywhere regular except at the disk where it has to take the boundary data \( U \). At the disk the normal derivatives of the potential will have a jump since the disk is a surface layer. Notice that one only has to solve the vacuum equations since the two-dimensional matter distribution merely leads to boundary conditions for the Laplace equation. In the Newtonian setting one thus has to determine the density for a given rotation law or vice versa, a well known problem (see e.g. [5] and references therein) for Newtonian dust disks.

There are several ways to construct Newtonian dust disks. We will only outline two possibilities which can be used with some modifications also in the relativistic case.

2.1. ‘Cut and glue’-techniques. One way to construct Newtonian disks is to start with a known equatorially symmetric solution to the Laplace equation, for instance the solution for a point mass \( m \),

\[ U = -\frac{m}{\sqrt{\rho^2 + \zeta^2}}. \]

Then a strip of width \( 2\zeta_0 \) is cut out of the space symmetrically to the equatorial plane. The solutions for positive and negative \( \zeta \) are displaced in \( \zeta \)-direction by \( \pm \zeta_0 \) and glued together at the equatorial plane. The discontinuity at this plane leads to a surface layer of infinite extension.

In the example of a point mass considered by Kuzmin [34] and Toomre [35], this leads to the solution

\[ U = -\frac{m}{\sqrt{\rho^2 + (|\zeta| + \zeta_0)^2}}. \]

The surface density \( \sigma_d \) at the equatorial plane is just given by \( 2\pi\sigma_d = U_\zeta(0^+) \), in the example

\[ U_\zeta(0^+) = \frac{m\zeta_0}{(\rho^2 + \zeta_0^2)^{3/2}}. \]

By construction the spacetime has finite mass \( m \) since the asymptotics have not been changed by the procedure of cutting and glueing. The disk is infinite, but the mass density decreases because of \( \zeta_0 \) as \( \rho^{-3} \). Since the mass density is positive, the matter in the disk can be interpreted as a disk of dust. The angular velocity as defined in \( \Omega \) is in this example given by

\[ \Omega^2 = \frac{m}{(\rho^2 + \zeta_0^2)^{3/2}}. \]
Asymptotically the angular velocity satisfies the Kepler relation \( \Omega^2 \rho^3 = m \) for test particles. Thus the matter in the disk behaves for large distances from the center as free particles. Though the gravitational field (2) can be seen as generated by the self-gravitating matter in the disk, particles in large distances from the center can be viewed as test particles since the the density tends to zero. The so constructed disks thus have physically acceptable properties: a positive mass density and a finite mass. Though they have an infinite extension, the density decreases rapidly for \( \rho \to \infty \).

Since the Laplace equation is linear, arbitrary linear combinations of potentials of the form (2) will always lead to infinite disks. Evans and de Zeeuw [36] considered disk potentials of the form

\[
U = \int \frac{\nu(\epsilon)d\epsilon}{\sqrt{\rho^2 + (|\zeta| + \epsilon)^2}}
\]

which leads to the general classical disk formula. Bičák, Lynden-Bell and Katz [7] used this technique to generate disk solutions to the static axisymmetric Einstein equations.

2.2. Disks of finite extension. To generate disks of finite extension, we use an approach which can be generalized to some extent to the relativistic case. The resulting expression will be shown to be equivalent to the Poisson integral for a distributional density. We put \( \rho_0 = 1 \) without loss of generality (we are only considering disks of finite non-zero radius) and obtain \( U \) as the solution of a so-called Riemann-Hilbert problem (see e.g. [16] and references given therein). The solution can be written in the form

\[
U(\rho, \zeta) = -\frac{1}{4\pi I} \int_{\Gamma} \frac{\ln G(K)dK}{\sqrt{(K-\zeta)^2 + \rho^2}},
\]

where \( \ln G \in C^{1,\alpha}(\Gamma) \) (Hölder continuous on \( \Gamma \)) and where \( \Gamma \) is the covering of the imaginary axis in the upper sheet of \( \mathcal{L} \) between \( -i \) and \( i \); \( \mathcal{L} \) is the Riemann surface of genus 0 given by the algebraic relation \( \mu_0^2(K) = (K-\zeta)^2 + \rho^2 \). The function \( G \) has to be subject to the conditions \( G(\bar{K}) = G(K) \) and \( G(-K) = G(K) \).

It may be checked by direct calculation that \( U \) in (7) is a solution to the Laplace equation except at the disk. The reality condition on \( G \) leads to a real potential, whereas the symmetry condition with respect to the involution \( K \to -K \) leads to equatorial symmetry. The occurrence of the logarithm in (7) is due to the Riemann-Hilbert problem with the help of which the solution to the Laplace equation was constructed, see e.g. [16].

The function \( \ln G \) is determined by the boundary data \( U(\rho, 0) \) or the energy density \( \sigma_d \) of the dust via

\[
\ln G(t) = 4 \left( U_0 + t \int_0^t \frac{U_\rho(\rho)d\rho}{\sqrt{t^2 - \rho^2}} \right)
\]

or

\[
\ln G(t) = 4 \int_t^1 \frac{\rho U_\rho d\rho}{\sqrt{\rho^2 - t^2}}
\]

respectively where \( t = -iK \). This can be seen in the following way:

At the disk the potential takes due to the equatorial symmetry the boundary values

\[
U(\rho, 0) = -\frac{1}{2\pi} \int_0^\rho \frac{\ln G(t)}{\sqrt{\rho^2 - t^2}} dt
\]
and

\[
U_\zeta(\rho, 0) = -\frac{1}{2\pi} \int_0^1 \frac{\partial_t (\ln G(t))}{\sqrt{t^2 - \rho^2}} \, dt.
\]

Both equations constitute integral equations for the ‘jump data’ \(\ln G\) of the Riemann-Hilbert problem if the respective left-hand side is known. The equations (10) and (11) are both Abelian integral equations and can be solved in terms of quadratures, i.e. (8) and (9).

To show the regularity of the potential \(U\), we prove that the integral (7) is identical to the Poisson integral for a distributional density which reads at the disk

\[
U(\rho) = -2 \int_0^1 \sigma_d(\rho') \rho' \, d\rho' \int_0^{2\pi} \frac{d\phi}{\sqrt{(\rho + \rho')^2 - 4\rho \rho' \cos \phi}} = -4 \int_0^1 \sigma_d(\rho') \rho' \, d\rho' \frac{K(k(\rho, \rho'))}{\rho + \rho'},
\]

where \(k(\rho, \rho') = 2\sqrt{\rho \rho'}/(\rho + \rho')\) and where \(K(k)\) is the complete elliptic integral of the first kind. Eliminating \(\ln G\) in (10) via (9) we obtain after interchange of the order of integration

\[
U = -\frac{2}{\pi} \left( \int_0^\rho U_\zeta(\rho') \frac{d\phi}{\rho} K(\frac{\rho'}{\rho}) \, d\rho' + \int_\rho^1 U_\zeta(\rho') \frac{d\phi}{\rho'} \, d\rho' \right)
\]

which is identical to (12) since \(K(2\sqrt{k}/(1 + k)) = (1 + k)K(k)\). Thus the integral (7) has the properties known from the Poisson integral: it is a solution to the Laplace equation which is everywhere regular except at the disk where the normal derivatives are discontinuous.

We note that it is possible in the Newtonian case to solve the boundary value problem purely locally at the disk. The regularity properties of the Poisson integral then ensure global regularity of the solution except at the disk. Such a purely local treatment will not be possible in the relativistic case.

The above considerations make clear that one cannot prescribe both \(U\) at the disk (and thus the rotation law) and the density independently. This just reflects the fact that the Laplace equation is an elliptic equation for which Cauchy problems are ill-posed. If \(\ln G\) is determined by either (8) or (9) for given rotation law or density, expression (7) gives the analytic continuation of the boundary data to the whole spacetime. In case we prescribe the angular velocity, the constant \(U_0\) is determined by the condition \(\ln G(i) = 0\) which excludes a ring singularity at the rim of the disk. For rigid rotation (\(\Omega = \text{const}\)), we get e.g.

\[
\ln G(\tau) = 4\Omega^2 (\tau^2 + 1),
\]

which leads with (7) to the well-known Maclaurin disk.

2.3. Charged static dust disks. In a Newtonian theory gravity and electromagnetism decouple. Disks of pressureless charged matter will lead to a gravitational potential \(U\) as above and to electric and magnetic fields. For disks with only one component of dust, there will be necessarily a magnetic field due to the rotating charges. Static solutions are possible if two streams of charged dust with equal densities are exactly counter-rotating. In this case the magnetic field vanishes since the effects of both streams concerning magnetic fields just compensate. This trick was used by Morgan and Morgan [37] within general relativity to describe static disks: in the case of two identical streams of counter-rotating particles the so-called ‘gravitomagnetic’ effects of relativity cancel, and the resulting spacetime is static.
In the electrostatic case, the electric potential is also a solution to the Laplace equation. If we assume the electrical density $\sigma_e$ to be proportional to the matter density $\sigma_d$ in the case of a dust disk (both densities are surface densities), i.e. $\sigma_e = Q \sigma_d$, we get from Newton’s law $F_{grav} = F_{centrifugal} + F_{el}$ the relation

$$U_{\mu}(1 - Q^2) = \Omega^2 \rho.$$  

Thus one can infer the tangential derivative at the disk for given $\Omega$ and constant $Q$ and then solve the boundary value problem for the Laplace equation as above. The electric and the gravitational potential are in this example just proportional. Similarly one can prescribe $\sigma_d$ and obtain $\Omega$ from (15). Note that $Q^2$ has to be smaller than 1. For $Q^2 = 1$, the angular velocity in the dust disk vanishes.

3. EINSTEIN-MAXWELL EQUATIONS AND GROUP STRUCTURE

In this section we study the Einstein-Maxwell equations in the presence of one Killing vector. We explore the group structure of the equations and give the Harrison transformation which generates electro-vacuum solutions from pure vacuum solutions. The solutions contain an additional real parameter related to the total charge. General properties of the transformed spacetimes as the asymptotics are discussed. In the stationary axisymmetric case, complete integrability of the equations is established.

3.1. Maxwell equations. It is instructive to consider first the Maxwell equations in the absence of gravity, i.e. on a flat background. In standard notation, the equations for the electric and respectively magnetic fields $E$ and $B$ read

$$\text{div} E = 0, \quad \text{rot} E + B, t = 0, \quad \text{div} B = 0, \quad \text{rot} B - E, t = 0.$$  

Since $\text{div} B = 0$, one can define (up to gauge freedoms) a vector potential via $B = \text{rot} A$.

It is convenient for the relativistic treatment in the following sections to introduce four-dimensional notation. We use the convention that greek indices take the values $0, 1, 2, 3$ and latin indices the values $1, 2, 3$. A four-dimensional vector potential $A_{\mu} = (A, A_a)$ is introduced which is related to the tensor $F_{\mu\nu}$ of the electromagnetic fields via

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}.$$  

In vacuum, the Maxwell equations read (indices are raised and lowered with the Minkowski metric)

$$F_{\mu\nu,\nu} = 0, \quad *F_{\mu\nu,\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta,\nu} = 0.$$  

They can be obtained from the action

$$S = \frac{1}{4} \int dx^4 F_{\mu\nu} F^{\mu\nu}.$$  

Obviously the equations are invariant under the discrete exchange $F \rightarrow *F$. In addition they are invariant under the continuous rotations

$$F + i^* F \rightarrow e^{i\theta} (F + i^* F), \quad \theta \in \mathbb{R},$$  

or in terms of the fields, $E + iB \rightarrow e^{i\theta} (E + iB)$. This is the well known $U(1)$ symmetry of the Maxwell equations in vacuum. It can be used to ‘generate’ solutions from known ones: if for instance a solution with vanishing magnetic field is given, the $U(1)$-symmetry can be applied to generate from the given electric field a solution with non-vanishing magnetic field. The transformed fields contain a new real parameter. In case there is an electric
monopole moment, there will be a magnetic monopole moment in the transformed solution. This often limits the physical relevance of the so generated solutions.

In the stationary case, the potential \( A_\mu \) can be chosen to be independent of the time coordinate \( t \). Since \( \text{rot} B = 0 \), we can define a scalar potential via \( B = \text{grad} B \), the well-known magnetic potential for stationary fields. The equation \( \text{div} B = 0 \) implies \( \Delta B = 0 \).

In four-dimensional language which will be needed in the Einstein-Maxwell case, this construction works as follows: Since \( F^{ab} \) is a three-dimensional antisymmetric tensor, it can be dualized to an axial vector by contraction with the totally antisymmetric \( \epsilon \)-tensor. The Maxwell equations \( F^{ab} = 0 \) imply that this vector must be a gradient of some potential \( B \). We can thus define the potential \( B \) via

\[
B_{\alpha c} = -\epsilon_{abc} A_{\alpha b}.
\]

The potentials can be combined to the complex potential \( \Phi = A + iB \). In this case the action is just given by

\[
S = \frac{1}{2} \int d^3x |\nabla \Phi|^2,
\]

and the Maxwell equations read

\[
\Delta \Phi = 0.
\]

In the stationary case, the Maxwell equations are thus equivalent to the Laplace equation for a single complex potential \( \Phi \).

3.2. Einstein-Maxwell equations. In the Einstein-Maxwell case, the Maxwell equations have the same form as in (24), only the partial derivatives have to be replaced by covariant derivatives since the spacetime is no longer flat,

\[
F_{\mu \nu} = 0, \quad \ast F_{\mu \nu} = 0.
\]

The tracefree energy-momentum tensor of the electromagnetic field is given by

\[
T_{\mu \nu} = F_{\mu \alpha} F_{\nu \alpha} - \frac{1}{4} g_{\mu \nu} F_{\kappa \lambda} F^{\kappa \lambda}.
\]

The Einstein equations have the form

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = T_{\mu \nu},
\]

where \( g_{\mu \nu} \) is the metric of the spacetime, \( R_{\mu \nu} \) the Ricci tensor and \( R \) the Ricci scalar. With (25) we get for (26)

\[
R_{\mu \nu} = F_{\mu \lambda} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu \nu} F_{\kappa \lambda} F^{\kappa \lambda}.
\]

Equations (24) and (27) form the Einstein-Maxwell equations. They can be derived from the action

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \left( R - \frac{1}{2} F_{\mu \nu} F^{\mu \nu} \right), \quad g = \det (g_{\mu \nu}).
\]

Since the Maxwell fields only enter the Einstein equations via the energy-momentum tensor, the \( U(1) \) symmetry (20) carries over to the Einstein-Maxwell case and can be used to generate solutions as before.

In general one would expect that the above \( U(1) \) invariance of the Einstein-Maxwell equations is the only symmetry of the equations even in the presence of Killing symmetries in the spacetime. However, it turns out that a much bigger symmetry group exists already for a single Killing vector. It is convenient to use a projection formalism which goes back
to Ehlers [29], see also [30] and [38] for additional references. The case considered here can be viewed as a special case of the standard dimensional reduction of Kaluza-Klein [38] and supergravity theories [39]. In this formalism, the metric is written in the form

$$ds^2 = -f(dt + k_adx^a)(dt + k_bdx^b) + \frac{1}{f}h_{ab}dx^adx^b.$$  

We are considering here for convenience the stationary case, i.e. we use coordinates in which the timelike Killing vector is given by \(\partial_t\), all potentials are independent of \(t\). However, the results hold with minor changes for general Killing vectors. Coordinates are introduced for convenience, the dimensional reduction can also be carried out in a coordinate-independent way. We assume that the Killing vector is timelike throughout the spacetime, i.e. that its norm \(f\) does not vanish.

The vector potential is decomposed as the metric into pieces parallel and orthogonal to the Killing vector, \(A_{\mu} = (A, A_m + k_mA)\). The Lagrangian of (28) can then be written in the form

$$L = \frac{1}{2}\sqrt{h}\left(R - \frac{1}{2}f^2h^{ab}f_{,a}f_{,b} + \frac{f^2}{4}K_{ab}K^{ab} + \frac{1}{f}h^{ab}A_{,a}A_{,b} - \frac{f}{2}(F_{ab} + AK_{ab})(F^{ab} + AK^{ab})\right)$$

where \(L\) is a three-dimensional Lagrangian density, where \(F_{ab} = A_{a,b} - A_{b,a}\), and where \(K_{ab} = k_{a,b} - k_{b,a}\). All indices are raised and lowered with \(h_{ab}\). Note that the tensor \(K_{ab}\) vanishes only if the Killing vector is hypersurface orthogonal in which case the spacetime is static.

The first part of the Maxwell equations (24) can be written in the form

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}F_{\mu\nu})_{,\nu} = 0$$

which implies \((\sqrt{h}F^{ab}/f)_{,b} = 0\). With this relation or by varying (30) with respect to \(A_a\), we obtain

$$(\sqrt{h}f(F_{ab} + AK_{ab}))_{,b} = 0.$$  

We can define as before the potential \(B\) via

$$B_{,c} = -\frac{1}{2}\epsilon_{cab}\sqrt{h}(F_{ab} + AK_{ab}).$$

Again \(A\) and \(B\) can be combined to the complex electromagnetic potential \(\Phi = A + iB\).

Similarly we get by varying (30) with respect to \(k_a\)

$$(\sqrt{h}\left(\frac{f^2}{2}K_{ab} - Af(F_{ab} + AK_{ab})\right))_{,b} = 0.$$  

This can be dualized as above by introducing the so-called twist potential \(b\) via

$$b_{,c} = \epsilon_{cab}\sqrt{h}\frac{f^2}{2}K_{ab} + BA_{,c} - AB_{,c}.$$  

The potentials \(f\) and \(b\) can be combined to the complex Ernst potential,

$$\mathcal{E} = f - \Phi\Phi + ib.$$  

The scalars \(b\) and \(B\) replace the vectors \(k_a\) and \(A_a\). The corresponding three-dimensional Lagrangian reads with \(w_a = b_{,a} - 2BA_{,a} + 2AB_{,a}\)

$$\mathcal{L} = \frac{\sqrt{h}}{2}\left(R - h_{ab}\left(\frac{1}{2}f^2(f_{,a}f_{,b} + w_aw_b) - \frac{1}{f}(A_{,a}A_{,b} + B_{,a}B_{,b})\right)\right).$$
The line element

\[ ds^2 = \frac{1}{2f^2}((df)^2 + (db + 2BdA - 2AdB)^2 - \frac{1}{f}((dA)^2 + (dB)^2)) \]

describes the invariant metric of the Riemannian symmetric space \( S = SU(2, 1)/S[U(1, 1) \times U(1)] \) in some coordinates. The stationary Einstein-Maxwell equations can thus be interpreted as three-dimensional gravity coupled to some matter model. The ‘matter’ is a \( SU(2, 1)/S[U(1, 1) \times U(1)] \) nonlinear sigma model [40, 41]. Note that sigma models are related to harmonic maps [42]. The space \( S \) can be parametrized by trigonal \( 3 \times 3 \) matrices \( V \).

\[
V = \begin{pmatrix}
\sqrt{f} & 0 & 0 \\
\frac{i\sqrt{2\Phi}}{\sqrt{f}} & 1 & 0 \\
(b + i|\Phi|^2)/\sqrt{f} & (\sqrt{2\Phi})/\sqrt{f} & 1/\sqrt{f}
\end{pmatrix}.
\]

The matrix \( V \) satisfies

\[ V^\dagger \eta V = \eta, \quad \eta = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}, \]

i.e. it is unitary with respect to the metric \( \eta \) of \( SU(2, 1) \). The action of \( G \in SU(2, 1) \) on \( V \) is

\[ V \rightarrow H(V, G)V^{-1}, \quad H(V, G) \in S[U(1, 1) \times U(1)], \]

where \( H \) restores the triangular gauge of \( V \). To obtain a gauge invariant parametrization, one introduces

\[ \chi := \Xi V^\dagger \Xi, \quad \Xi = \text{diag}(1, -1, 1), \]

on which the action of \( G \in SU(2, 1) \) is given by

\[ \chi \rightarrow \Xi G^{-1} \Xi \chi^{-1}. \]

We have

\[
\chi = \begin{pmatrix} f - 2|\Phi|^2 + (b^2 + |\Phi|^4)/f & \sqrt{2\Phi}(b - i|\Phi|^2 + if)/f & (b - i|\Phi|^2)/f \\ -\sqrt{2\Phi}(b + i|\Phi|^2 - if)/f & 1 - 2|\Phi|^2/f & -\sqrt{2\Phi}/f \\ (b + i|\Phi|^2)/f & \sqrt{2\Phi}/f & 1/f \end{pmatrix}.
\]

The \( SU(2, 1) \) symmetry can be used to generate solutions by the action of an element \( G \). We list the infinitesimal transformations and their consequences:

\[
\begin{pmatrix} 0 & 0 & 0 \\ \theta_1 & 0 & 0 \\ \theta_2 & \theta_3 & 0 \end{pmatrix}
\]

lead to gauge transformations which add physically irrelevant constants to \( \Im \mathcal{E} \) and \( \Im \Phi \),

\[
\begin{pmatrix} 0 & 0 & \theta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

is an Ehlers transformation [29] which changes \( f \rightarrow b \), i.e. which generates stationary from static solutions (if the ADM-mass of the spacetime is non-zero, the transformed solution will have a Newman-Unti-Tambourini (NUT) parameter which corresponds to a magnetic monopole and which is believed to be unphysical).
If we transform an Ernst potential of a pure vacuum solution, we are interested in solutions which are equatorially symmetric and asymptotically flat, i.e. equatorial situations. This leads to solutions to the Einstein-Maxwell equations containing one additional constant parameter which is related to the charge. For physical reasons we are interested in solutions where the metric functions have a reflection symmetry at the equatorial plane and the same is assumed to hold in a general relativistic context. A consequence of this condition is that the transformed solutions have the same asymptotic behavior, one has to use a scale transformation which changes the asymptotic conditions imply that \( \Phi \to 0 \) for \( \xi \to \infty \). Asymptotic flatness is just the mathematical formulation of the physical concept of an isolated matter distribution, e.g. a galaxy. Solutions with equatorial symmetry, i.e. a class of solutions of special physical interest. In a Newtonian setting it can be proven that perfect fluids in thermodynamical equilibrium lead to equatorially symmetric situations, and the same is assumed to hold in a general relativistic context. A consequence of this condition is that NUT-parameters are ruled out.

We assume that the pure vacuum solutions which we want to submit to a Harrison transformation satisfy these conditions. To ensure that the transformed solutions have the same asymptotic behavior, one has to use a scale transformation \((f \to 1)\) together with a transformation which changes \( \Phi \) and \( b \) by some constant \((\Phi, b \to 0)\). By exponentiating the matrices of the \(SU(2, 1)\) transformations, we thus consider a transformation of the form

\[
G = \begin{pmatrix}
1 & i\theta_1 & -i\theta_1\theta_1/2 \\
0 & 1 & -\theta_1 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & -\theta_3 & 1 \\
0 & \theta_4 + \theta_5\theta_1/2 & -\theta_5
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{-\theta_2}, 1, e^{\theta_2} \\
0 & \theta_4 + \theta_5\theta_1/2 & -\theta_5
\end{pmatrix}
\]

Since the asymptotic conditions imply that \( \chi \) and \( \chi' \) are the unit matrix at infinity, the matrix \( G \) must satisfy the condition \( \Xi G^\dagger \Xi = G^{-1} \). This leads with (45) to

\[
e^{\theta_2} = \frac{1}{1 - \theta_1\theta_1/2}, \quad \theta_3 = i\theta_1, \quad \theta_4 = 0, \quad \theta_5 = \theta_1.
\]

The matrix \( G \) thus takes the form

\[
G = \frac{1}{1 - \theta_1\theta_1/2} \begin{pmatrix}
1 & i\theta_1 & -i\theta_1\theta_1/2 \\
i\theta_1\theta_1/2 & 1 + \theta_1\theta_1/2 & -\theta_1 \\
\theta_1\theta_1/2 & -\theta_1 & 1
\end{pmatrix}
\]

If we transform an Ernst potential of a pure vacuum solution \((\Phi = 0)\), we end up with

\[
\Phi' = -\frac{\theta_1\theta_4\theta_1(f^2 + b^2)/2 - (1 + \theta_1\theta_1/2)f + 1 - ib(1 - \theta_1\theta_1/2)}{\sqrt{2}(1 - \theta_1\theta_1/2)^2 + (\theta_1\theta_1/2)^2b^2}.
\]
We are interested in transformations which preserve the equatorial symmetry, i.e. \( f(-\zeta) = f(\zeta), b(-\zeta) = -b(\zeta) \) and \( \Phi(-\zeta) = \Phi(\zeta) \). This implies for (43) that \( \theta_1 \) must be real which rules out magnetic monopoles. We put \( q = \theta_1/\sqrt{2} \) and sum up the results for the transformed potentials:

\[
\begin{align*}
(49) \quad f' &= \frac{(1 - q^2)f}{(1 - q^2)f^2 + q^4b^2}, \\
(50) \quad b' &= \frac{(1 - q^2)b}{(1 - q^2)f^2 + q^4b^2}, \\
(51) \quad \Phi' &= -q(1 - f)(1 - q^2 f) + q^2b^2 + ib(1 - q^2).
\end{align*}
\]

The real parameter \( q \) has to be in the region \( 0 < |q| < 1 \), for \( q > 1 \) the transformed spacetime would have a negative mass if the original mass was positive. The value \( q = 0 \) corresponds to the untransformed solution. The above formulas imply that the functions \( f', b' \) and \( \Phi' \) are analytic where the original functions are analytic.

A well-known example is the Harrison transformation of the Schwarzschild solution which leads to the Reissner-Nordström solution. In the Ernst picture the Schwarzschild solution reads in cylindrical Weyl coordinates with \( r_\pm = \sqrt{(\zeta \pm m)^2 + \rho^2} \)

\[
(52) \quad f = \frac{r_+ + r_- - 2m}{r_+ + r_- + 2m}, \quad b = 0,
\]

where the horizon \( (f = 0) \) is located on the axis between \(-m \) and \( m \). For the transformed solution we get with (51)

\[
(53) \quad f' = \frac{(r_+ + r_-)^2 - 4m^2}{(r_+ + r_- + 2m')^2}, \quad \Phi' = \frac{2Q}{r_+ + r_- + 2m'}, \quad m' = m\frac{1+q^2}{1-q^2}, \quad Q = -\frac{2mq}{1-q^2},
\]

and \( b' = 0 \) which is the Reissner-Nordström solution. This is a static spacetime with mass \( m' \) and charge \( Q \) subject to the relation \( m'^2 - Q^2 = m^2 \). Both \( m' \) and \( Q \) diverge for \( q \to 1 \). The extreme Reissner-Nordström solution with \( m' = Q \) is only possible in the limit \( m \to 0, |q| \to 1 \). The horizon of the solution is again located on the axis between \( \pm m \) which illustrates that the horizon degenerates in the extreme case.

3.4. Asymptotic behavior of the Harrison transformed solutions. We assume that the asymptotic behavior of the original solution, which can be read off on the axis, is of the form \( f = 1 - 2M/|\zeta|, b = -2J/\zeta^2 \) and \( \Phi = Q/|\zeta| - iJ_M/\zeta^2 \) plus terms of lower order in \( 1/|\zeta| \) where \( M \) is the Arnowitt-Deser-Misner mass, \( J \) the angular momentum, \( Q \) the electric charge and \( J_M \) the magnetic moment. The same will hold for the Harrison transformed potentials. We find (20)

\[
(54) \quad M' = M\frac{1+q^2}{1-q^2} - \frac{2q}{1-q^2}Q, \quad J' = J\frac{1+q^2}{1-q^2} - \frac{2q}{1-q^2}J_M,
\]

and

\[
(55) \quad Q' = Q\frac{1+q^2}{1-q^2} - \frac{2q}{1-q^2}M, \quad J_M' = J_M\frac{1+q^2}{1-q^2} - \frac{2q}{1-q^2}J.
\]

It is interesting to note that the quantities \( M^2 - Q^2 \) and \( J^2 - J_M^2 \) are invariants of the transformation. They are related to the Casimir operator of the \( SU(2,1) \)-group. If the original solution was uncharged, the extreme relation \( M' = \pm Q' \) is only possible in the limit \( M \to 0 \).
A consequence of the relations (55) is the presence of a non-vanishing charge if the ADM mass of the original solution is non-zero whereas the charge is. Since charges normally compensate in astrophysical settings, this limits the astrophysical relevance of Harrison transformed solutions.

A further invariant is the combination \( J_M M - J Q \) which is of importance in relation to the gyromagnetic ratio

\[
g_M = \frac{2 M J_M}{J Q}.
\]

Relation (54) implies that \( g'_M \) is equal to 2 if \( Q = J_M = 0 \) and \( q \neq 0 \). Thus all solutions which can be generated via a Harrison transformation from solutions with vanishing electromagnetic fields as the Kerr-Newman family from Kerr have a gyromagnetic ratio of 2. Due to the invariance of \( J_M M - J Q \) under Harrison transformations, a gyromagnetic ratio of 2 is not changed under the transformation.

Whether this property is an indication of a deep relation between relativistic quantum mechanics and general relativity as claimed in [27, 28] is an open question. Here it is just related to an invariant of the Harrison transformation, a subgroup of SU(2, 1). Since most of the known solutions to the Einstein-Maxwell equations can be generated via a Harrison transformation from solutions to the pure vacuum equations as the Kerr-Newman family from Kerr, a gyromagnetic ratio of 2 is well known from exact solutions. Numerical calculations of charged neutron stars [43] indicate, however, that values well below 2 are to be expected in astrophysically realistic situations.

### 3.5. The stationary axisymmetric case.

In the astrophysically important stationary axisymmetric case, the symmetry group of the equations increases again, this time to the infinite dimensional Geroch group [44, 45]. This means that the equations are completely integrable, where the notion of integrability is to be understood in a Hamiltonian sense: the equations have the same number of conserved quantities as degrees of freedom. For completely integrable systems, this number tends in a countable way to infinity. The infinite dimensional symmetry group shows up in treating the differential equation under consideration as the integrability condition of an overdetermined linear differential system for a matrix-valued function. The system contains an additional parameter, the so-called spectral parameter which reflects the infinite dimensional symmetry group.

In the presence of a second Killing vector, the metric (29) can be further specialized. The axial Killing vector \( \partial_\phi \) commutes with the timelike Killing vector \( \partial_t \). The metric \( h_{ab} \) of (29) can be chosen to be diagonal, \( h_{ab} = \text{diag}(e^{2k}, e^{2k}, \rho^2) \), and \( k_a \) can be brought into the form \( k_a = (0, 0, a) \) which leads to the Weyl-Lewis-Papapetrou metric, see e.g. [46]

\[
ds^2 = -e^{2U} (dt + a d\phi)^2 + e^{-2U} \left( e^{2k} (d\rho^2 + d\zeta^2) + \rho^2 d\phi^2 \right),
\]

where \( f = e^{2U} \). In this case the Einstein-Maxwell equations reduce to

\[
\begin{align*}
    f \Delta E &= (\nabla E + 2 \Phi \nabla \Phi) \nabla E, \\
    f \Delta \Phi &= (\nabla \Phi + 2 \Phi \nabla \Phi) \nabla \Phi,
\end{align*}
\]

where \( \Delta \) and \( \nabla \) are the standard differential operators in cylindrical coordinates, and where the potentials \( E \) and \( \Phi \) are independent of \( \phi \). The first equation generalizes the Newtonian equation \( \Delta U = 0 \) to the Einstein-Maxwell case, the second equation generalizes the
Maxwell equation (23). The duality relations (33) and (35) read

\[(3\Phi)_\xi = \frac{i}{\rho}(A_{\phi,\xi} - aA_{\xi,\xi}),\]

\[a_\xi = \frac{\rho}{f^2}(i(3E)_\xi + \Phi\bar{\Phi}_\xi - \bar{\Phi}\Phi_\xi),\]

which implies that \(a\) and \(A_\xi\) follow from \(E\) and \(\Phi\). We choose a gauge where \(A_1 = A_2 = 0\). The equations for \(R_{ab}\) of (34) are equivalent to

\[k_\xi = \frac{\xi - \bar{\xi}}{f} \left( \frac{1}{4f} (E_\xi + 2\Phi\bar{\Phi}_\xi)(\bar{E}_\xi + 2\Phi\bar{\Phi}_\xi) - \Phi_\xi\bar{\Phi}_\xi \right).\]

Thus the complete metric and the electromagnetic potential can be obtained from given potentials \(E\) and \(\Phi\) via quadratures.

The system (33) was shown to be completely integrable in [24]. In the form (47), the associated linear differential system for a \(3 \times 3\) matrix-valued function \(\Psi\) reads (we use the complex coordinate \(\xi = \zeta - i\rho\))

\[\Psi_\xi \Psi^{-1} = \begin{pmatrix} D_1 & 0 & M_1 \\ 0 & C_1 & 0 \\ -N_1 & 0 & \frac{i}{2}(C_1 + D_1) \end{pmatrix} + \frac{K - \xi}{\mu_0} \begin{pmatrix} 0 & D_1 & 0 \\ C_1 & 0 & -M_1 \\ 0 & -N_1 & 0 \end{pmatrix},\]

\[2\Psi_\xi \Psi^{-1} = \begin{pmatrix} D_2 & 0 & M_2 \\ 0 & C_2 & 0 \\ -N_2 & 0 & \frac{i}{2}(C_2 + D_2) \end{pmatrix} + \frac{K - \xi}{\mu_0} \begin{pmatrix} 0 & D_2 & 0 \\ C_2 & 0 & -M_2 \\ 0 & -N_2 & 0 \end{pmatrix},\]

where \(\Psi\) depends on the spectral parameter \(K\) which varies on the Riemann surface \(\mathcal{L}\) of genus zero given by the relation \(\mu_0^2(K) = (K - \xi)(K - \bar{\xi})\). This is a first hint on the relevance of Riemann surfaces in this context. Notice the special feature of the Ernst equation that the branch points \(\xi, \bar{\xi}\) depend on the spacetime coordinates. We denote a point on \(\mathcal{L}\) with the projection \(P = (K, \pm \mu_0(K)) = K^\pm\). On \(\mathcal{L}\) there is an involution \(\sigma\) that interchanges the sheets, i.e. with \(P = (K, \mu_0(K))\) we have \(\sigma P = P^\sigma = (K, -\mu_0(K))\). We use the notation \(\infty^\pm\) for the infinite points on different sheets of the curve \(\mathcal{L}\), namely \(\mu/K^{\sigma+1} \to \pm 1\) as \(K \to \infty^\pm\).

The expressions for \(C_i, D_i\) and \(M_i\) \((i = 1, 2)\) follow from the condition

\[\Psi(\infty^+, z, \bar{z}) = \begin{pmatrix} \mathcal{E} + 2\Phi\Phi & 1 & \sqrt{2i}\Phi \\ \mathcal{E} & -1 & -\sqrt{2i}\Phi \\ -2i\Phi e^U & 0 & \sqrt{2}\Phi e^U \end{pmatrix} = \begin{pmatrix} 1 & 0 & \sqrt{2i}\Phi \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2}\Phi \end{pmatrix} \left( \begin{pmatrix} \mathcal{E} & 1 & 0 \\ \mathcal{E} & -1 & -\sqrt{2i}\Phi \\ -\sqrt{2i}\Phi & 0 & 1 \end{pmatrix} \right)^* \]

Thus we have \(\det \Psi(\infty^+) = -2\sqrt{2}\Phi e^{3U}\). The reality conditions read

\[C_2 = \bar{D}_1, \quad C_1 = \bar{D}_2, \quad N_2 = -\bar{M}_1, \quad M_2 = -\bar{N}_2.\]

The inverse matrix takes the form

\[\Psi^{-1} = \frac{1}{2f} \begin{pmatrix} 1 & \mathcal{E} + 2\Phi\Phi & 0 \\ \mathcal{E} & -1 & -\sqrt{2i}\Phi \\ -2i\Phi e^U & 0 & \sqrt{2}\Phi e^U \end{pmatrix},\]
which implies via (62)
\[
\begin{align*}
C_1 &= (\mathcal{E}_\xi + 2\Phi\Phi_\xi)/(2f), \\
D_1 &= (\mathcal{E}_\xi + 2\Phi\Phi_\xi)/(2f), \\
M_1 &= i\Phi_\xi e^{-U}, \\
N_1 &= i\Phi_\xi e^{-U}.
\end{align*}
\]
(66)

The system (62) can be used to generate solutions to the Ernst equations. To this end one investigates the singularity structure of the matrices $\Psi_\xi\Psi^{-1}$ and $\Psi_\xi^\ast\Psi^{-1}$ with respect to the spectral parameter and infers a set of conditions for the matrix $\Psi$ which satisfies the linear system (62). These conditions can be summarized in the following theorem (see [12]):

**Theorem 3.1.** The matrix $\Psi$ is at least twice differentiable with respect to $\xi$ and $\xi$. $\Psi(P)$ is holomorphic and invertible at the branch points $\xi$ and $\xi$ such that the logarithmic derivative $\Psi_\xi\Psi^{-1}$ has a pole at $\xi$ and $\Psi_\xi^\ast\Psi^{-1}$ has a pole at $\xi$.

II. All singularities of $\Psi$ on $L$ are such that the logarithmic derivatives $\Psi_\xi\Psi^{-1}$ and $\Psi_\xi^\ast\Psi^{-1}$ are holomorphic there.

III. The matrix function $\Psi$ is subject to the reduction condition

\[
(67) \quad e\Psi(P') = \Psi(P)\gamma(P),
\]

where $\epsilon = \text{diag}(1, -1, 1)$, and where $\gamma$ is an invertible matrix independent of $\xi$, $\xi$.

IV. At $\infty^+$, the matrix function $\Psi$ is given by (63).

A proof of this theorem may be obtained by comparing the matrix $\Psi$ with the linear system (62).

**Proof.** Because of I, $\Psi$ and $\Psi^{-1}$ can be expanded in a series in the local parameters $\tau_\xi = \sqrt{K - \xi}$ and $\tau_{\bar{\xi}} = \sqrt{K - \bar{\xi}}$ in a neighborhood of $P = \xi$ and $P = \bar{\xi} \neq \xi$ respectively at all points $\xi$, $\bar{\xi}$ which are not singularities of $\Psi$. This implies that $\Psi_\xi\Psi^{-1} = \alpha_0/t + \alpha_1 + \alpha_2t + \ldots$. We recognize that, because of I and II, $\Psi_\xi^\ast\Psi^{-1} = -\alpha_0/t$ is a holomorphic function. The normalization condition IV implies that this quantity is bounded at infinity. According to Liouville’s theorem, it is a constant. Since $\Psi$, $\Psi^{-1}$ and $\Psi_\xi$ are single valued functions on $L$, they must be functions of $K$ and $\mu_0$. Therefore we have $\Psi_\xi^\ast\Psi^{-1} = \beta_0\sqrt{K - \xi} + \beta_1$. The matrix $\beta_0$ must be independent of $K$ and $\mu_0$ since $\Psi_\xi^\ast\Psi^{-1}$ must have the same number of zeros and poles on $L$. The structure of the matrices $\beta_0$ and $\beta_1$ follows from III. From the normalization condition IV, it follows that $\Psi_\xi^\ast\Psi^{-1}$ has the structure of (62). The corresponding equation for $\Psi_\xi^\ast\Psi^{-1}$ can be obtained in the same way.

The choice of the matrix $\gamma$ in (61) corresponds to a gauge freedom which is however not completely fixed. A matrix $C(K)$ with the property $[C, \gamma] = 0$ with $C(\infty) = 1$ which acts on $\Psi$ in the form $\Psi \rightarrow \Psi C(K)$ leads again to a solution of (62) for the same potentials $E$ and $\Phi$.

We note that the metric function $a$ can be directly obtained from the matrix $\Psi$ without integrating (60). We denote by $D_P F(P)$ the coefficient of the linear term in the expansion of the function $F$ in the local parameter near $P$. Using the identity $(\Psi^{-1}D_\infty^+\Psi)_\xi = \Psi^{-1}D_\infty^+ (\Psi_\xi^\ast\Psi^{-1})\Psi$, one finds with (62) to (66)

\[
(68) \quad ((\Psi^{-1}D_\infty^+\Psi)_\xi)_{12} = -\frac{i\rho}{2f}(C_1 - D_1) = -\frac{i\rho}{2f^2}(i(3E)_\xi + \Phi\Phi_\xi - \Phi_\Phi_\xi).
\]
Equation (60) leads to the expression

\[(a - a_0) e^{2U} = i D_{\infty} + (\Psi_{12} + \Psi_{22}),\]

which relates the metric function \(a\) directly to the potential \(\Psi\); the constant \(a_0\) is determined by the condition that \(a\) has to vanish on the regular part of the axis.

To generate solutions to the Ernst equation one has to construct matrices according to the above theorem. The theorem ensures that the so constructed solutions to the Ernst equation are analytic at all points where the conditions I-IV are satisfied. It also makes clear where singularities of the spacetime can be found: if the moving branch points \(\xi, \bar{\xi}\) coincide with singularities of \(\Psi\), condition I is no longer fulfilled and the theorem does not hold. These points are possible singularities, but they could be regular if the singularities of \(\Psi\) are for instance pure gauge. On the other hand, if the singularities of the spacetime are prescribed — for electro-vacuum the boundary of some matter source constitutes a singularity — one has to construct a matrix with the corresponding singularity. This means one has to solve a so-called Riemann-Hilbert problem, i.e. to find a function with prescribed singularities on \(\mathcal{L}\).

This method to construct solutions to integrable equations is also known as the inverse scattering method [48]. Explicit solutions are in general only known for scalar Riemann-Hilbert problems which lead in the case of the Ernst equation to static solutions for the pure vacuum, see [14,15,16]. In this case the Ernst equation reduces to the axisymmetric Laplace equation, the Euler-Darboux equation. In the matrix case, the solution to a Riemann-Hilbert problem is equivalent to an integral equation, see [49,15,46] for the Ernst equation. A simple special case are so-called ‘soliton’-solutions or Bäcklund transformations where the matrix \(\Psi\) has only poles and zeros of the determinant as singularities. Since the Ernst equation is an elliptic equation, it has obviously no solutions describing physical solitons, but it has solutions with the same mathematical properties. The most prominent representant of this class for the Ernst equation is the Kerr-Newman family of charged rotating black holes.

In [15] it was shown for the pure vacuum case that the Riemann-Hilbert problem can be solved explicitly for a large class of problems by exploiting the gauge freedom of the matrix \(\Psi\). This leads to solutions which are defined on certain Riemann surfaces. It is well known that large classes of solutions to non-linear integrable equations can be constructed via methods from algebraic geometry. For evolution equations like Korteweg-de Vries and Sine-Gordon, see e.g. [50], the corresponding solutions are periodic or quasi-periodic. Algebro-geometric solutions to the Ernst equation do not show any periodicity as will be discussed in the following section. A way to construct such solutions is via the so-called monodromy matrix [51,50]. For a linear system of the form

\[(\Psi_\xi = W\Psi, \quad \Psi_{\bar{\xi}} = V\Psi)\]

as (62), the monodromy matrix \(L\) can be defined as the solution of the linear differential system

\[L_\xi = [W, L], \quad L_{\bar{\xi}} = [V, L].\]

It follows from equation (71) that the characteristic polynomial

\[Q(\mu, K) = \det(L - \mu \mathbf{1})\]

is independent of the physical coordinates (the coefficients of the polynomial are ‘integrals of motion’). The equation \(Q(\mu, K) = 0\) is then the equation of a plane algebraic curve.
Since $L$ is a $3 \times 3$-matrix, relation (72) is cubic in $\hat{\mu}$ which can be always brought into normal form by a redefinition of $\hat{\mu}$:

$$\hat{\mu}^3 + P(K)\hat{\mu} + Q(K) = 0. \tag{73}$$

The functions $P$ and $Q$ are analytic in $K$. Equation (73) defines a three-sheeted Riemann surface which will in general have infinite genus. For polynomial $P$ and $Q$, the surface will be compact and will have finite genus. For a given surface the solutions to the Ernst equations can be given in terms of the theta functions on this surface which was first done in [14] for a special case.

Since the theory of these surfaces is not as well understood as the theory of hyperelliptic surfaces which occur in the pure vacuum case, we will study in the next section Harrison transformed hyperelliptic solutions. In section 6, we will come back to solutions to the Ernst equations on three-sheeted surfaces.

3.6. Conformastationary and magnetostatic solutions. There are two special cases of the above equations for which disk sources have been considered: Conformastationary metrics [52, 53] are included in the above formalism as metrics with a flat metric $h_{ab}$, there is no axial symmetry required. Writing $f = (V\bar{V})^{-1}$, the function $V$ is a complex solution to the three-dimensional Laplace equation. For asymptotically flat solutions $V$ has to tend to 1 at infinity. The electromagnetic fields are given by

$$E + iH = \text{grad} \left( \frac{1}{1 - |V|^2} \right), \quad D + iB = |V|(E + iH) + iT \times (E + iH), \tag{74}$$

where $T$ is a solution to

$$\text{rot} T = i(V\text{grad}\bar{V} - \bar{V}\text{grad}V). \tag{75}$$

Static spacetimes with a pure magnetic field can be mapped to the case of stationary axisymmetric vacuum spacetimes, see [46, 11]. Writing $A_\mu = (0, 0, 0, A(\rho, \zeta))$, the field equations reduce to the Ernst equation which is discussed in the next section. Thus all solutions to the vacuum Ernst equation can be interpreted as magnetostatic solutions by putting $f \rightarrow 1/(\Re \mathcal{E})^2$ and $A \rightarrow -\sqrt{2}3\mathcal{E}$.

4. STATIONARY AXISYMMETRIC EINSTEIN EQUATIONS, THETA FUNCTIONAL SOLUTIONS AND COUNTER-ROTATING DUST DISKS UNDER HARRISON TRANSFORMATIONS

In this section we will summarize results on the Ernst equation in the pure vacuum case, a class of solutions on hyperelliptic Riemann surfaces [14] and a member of this class describing counter-rotating dust disks [31] which were discussed in [32] and [54]. We study the action of a Harrison transformation on stationary axisymmetric solutions and discuss the transformed counter-rotating dust disk as an example.

4.1. The stationary axisymmetric vacuum. In the stationary axisymmetric vacuum, the Einstein-Maxwell equations of the previous section hold with $\Phi = 0$. Since the simplification is considerable, we list the relevant equations below. The Ernst potential $\mathcal{E} = f + ib$ is subject to the Ernst equation [53]

$$\mathcal{E}_{\xi\xi} - \frac{1}{2(\xi - \bar{\xi})}(\mathcal{E}_{\bar{\xi}} - \mathcal{E}_{\xi}) = \frac{2}{\mathcal{E} + \bar{\mathcal{E}}}\mathcal{E}_{\xi}\bar{\mathcal{E}}_{\bar{\xi}}. \tag{76}$$
If the Ernst potential is real, equation (76) is equivalent to $\Delta U = 0$. The corresponding spacetime is static and belongs to the so-called Weyl class. For a given Ernst potential the metric (57) follows from

\begin{align}
 a_\xi &= 2\rho (E - \bar{E})_\xi \\
 k_\xi &= (\xi - \bar{\xi}) \frac{E_\xi \bar{E}_\xi}{(E + \bar{E})^2}.
\end{align}

The stationary axisymmetric Einstein equations in vacuum were shown to be completely integrable in [56] and [57]. The associated linear system can be formulated for a $2 \times 2$-matrix $\Psi$ (see [58]).

\begin{align}
 \Psi_\xi \Psi^{-1} &= \begin{pmatrix} M & 0 \\
 0 & N \end{pmatrix} + \sqrt{\frac{K - \xi}{K - \bar{\xi}}} \begin{pmatrix} 0 & M \\
 N & 0 \end{pmatrix}, \\
 \Psi_{\bar{\xi}} \Psi^{-1} &= \begin{pmatrix} \bar{N} & 0 \\
 0 & \bar{M} \end{pmatrix} + \sqrt{\frac{\bar{K} - \xi}{\bar{K} - \bar{\xi}}} \begin{pmatrix} 0 & \bar{N} \\
 \bar{M} & 0 \end{pmatrix},
\end{align}

where

\begin{align}
 M &= \frac{\bar{E}_\xi}{\bar{E} + \bar{E}}, \quad N = \frac{E_\xi}{E + \bar{E}}.
\end{align}

Theorem 3.1 holds with the following changes:

**III'.** $\Psi$ is subject to the reduction condition

\begin{align}
 \Psi(P^\sigma) = \sigma_3 \Psi(P) \sigma_2,
\end{align}

where $\sigma_2, \sigma_3$ are Pauli matrices.

**IV'.** The normalization and reality condition

\begin{align}
 \Psi(P = \infty) = \begin{pmatrix} 0 & -i \\
 i & 0 \end{pmatrix}.
\end{align}

The function $E$ in (82) is then a solution to the Ernst equation (76).

We note that the choice of the normalization of $\Psi$ at infinity is different from the one in theorem 3.1. This form was chosen to implement the Harrison transformation as was done in the last section with the metric $\eta$ in (40) of $SU(2, 1)$. The choice of the matrix $\sigma_2 = \begin{pmatrix} 0 & -i \\
 i & 0 \end{pmatrix}$ in condition III is again a gauge condition, which does not fix the gauge completely, however. The remaining gauge freedom is here due to matrices $C(K) = \kappa_1(K) I + \kappa_2(K) \sigma_2$, where the $\kappa_i$ do not depend on $\xi, \bar{\xi}$ and obey the asymptotic conditions $\kappa_1(\infty) = 1$ and $\kappa_2(\infty) = 0$. The matrices $C$ act on $\Psi$ in the form $\Psi \rightarrow \Psi C(K)$. Since it is a consequence of (79) that $\det \Psi = F(K) e^{2U}$ where $F(K)$ is independent of $\xi, \bar{\xi}$ we can use this gauge freedom to choose $F(K) = 1$. The linear system (79) leads for the matrix $\chi(P) = \Psi^{-1}(\infty^-) \Psi(P)$ to one of the linear systems used in [45]. This parametrization, especially

\begin{align}
 \Psi^{-1}(\infty^-) \Psi(\infty^+) = \frac{1}{E + \bar{E}} \begin{pmatrix} 2E \bar{E} & i(E - \bar{E}) \\
 i(\bar{E} - E) & 2 \end{pmatrix},
\end{align}

reveals that the Ernst equation is an $SL(2, \mathbb{R})/SO(2)$ sigma model. For more details on the group aspect see [45] and the discussion in the previous section.
For given $\Psi(K)$, one can again directly determine the metric function $a$ without having to integrate relation (77). With (79) one finds for the matrix $S = \Psi^{-1}(\infty^+)D_{\infty^+}\Psi(\infty^+)$

$$S_\xi := (\Psi^{-1}(\infty^+)D_{\infty^+}\Psi)_\xi = \frac{\xi - \bar{\xi}}{2(\xi + \bar{\xi})^2} \begin{pmatrix} (\xi^2\bar{\xi} - \bar{\xi}^2\xi) & i(\xi - \bar{\xi})_\xi \\ i(\bar{\xi} - \xi)_\xi & -(\bar{\xi}\xi) \end{pmatrix}.$$  

This implies with (77)

$$a - a_0 = -2S_{12} = -2(\Psi^{-1}(\infty^+)D_{\infty^+}\Psi)_{12},$$

where $a_0$ is a constant which is fixed by the condition that $a$ vanishes on the regular part of the axis. The matrix $S$ will be needed to calculate the metric function $a$ after a Harrison transformation.

The monodromy matrix $L$ defined in (71) corresponding to (79) is now also a $2 \times 2$-matrix. Since this matrix can be chosen without loss of generality to be tracefree, the characteristic equation (72) takes the form

$$\tilde{\mu}^2 = P(K),$$

where $P$ is an analytic function in $K$. For polynomial $P = \sum_{i=1}^g(K - E_i)(K - F_i)$, the two-sheeted Riemann surface defined by (86) is compact. Since the spectral parameter $K$ varies on the two-sheeted surface $\mathcal{L}$, equation (85) defines a two-sheeted branched cover of $\mathcal{L}$. In other words the so defined Riemann surface $\tilde{\mathcal{L}}$ is a four-sheeted cover of the complex plane. The structure of this surface is shown in the Hurwitz diagram Fig. 1.

![Figure 1](image)

**Figure 1.** The Hurwitz diagram of $\tilde{\mathcal{L}}$ shows the Riemann surface as seen from the side.

There is an automorphism $\sigma$ of $\tilde{\mathcal{L}}$ inherited from $\mathcal{L}$ which ensures $E_i^\sigma$, $E_i$, and $F_i^\sigma$, $F_i$ have the same projection in the complex plane. The orbit space $\mathcal{L}_H = \tilde{\mathcal{L}}/\sigma$ is then, see [50], again a Riemann surface given by

$$\mu^2 = (K - \xi)(K - \bar{\xi}) \prod_{i=1}^g(K - E_i)(K - F_i).$$

The fixed points $\xi, \bar{\xi}$ of the involution $\sigma$ are additional branch points of the Riemann surface. The points $E_i = \alpha_i - i\beta_i$ have to be constant with respect to the physical coordinates. They are subject to the reality condition $E_i = F_i$ or $E_i, F_i \in \mathbb{R}$ for $i = 1, \ldots, g$. A surface of the form (87) is called hyperelliptic, since the square root of a polynomial can be considered as the straightforward generalization of elliptic surfaces. Thus it is possible to construct components of the matrix $\Psi$ on $\mathcal{L}_H$ which makes it possible to use the powerful calculus of hyperelliptic Riemann surfaces.
4.2. Solutions on hyperelliptic surfaces. Solutions to the Ernst equation on hyperelliptic surfaces were given by Korotkin in \[14\], and in the gauge (82) in \[58\]. We summarize basic facts of hyperelliptic surfaces (see e.g. \[50\] and \[60\] to \[62\]) to be able to present these solutions.

A Riemann surface has a topological invariant, the genus, which loosely speaking gives the number of holes in the surface. A surface of genus \(g > 1\) is topologically just a sphere with \(g\) handles. For genus 0, it is the Riemann sphere, in the elliptic case \(g = 1\) a torus. The hyperelliptic surface \(\mathcal{L}_H\) defined by (87) has genus \(g\). We order the branch points with \(\Im E_i < 0\) in a way that \(\Re E_1 < \Re E_2 < \ldots < \Re E_g\) and assume for simplicity that the real parts of the \(E_i\) are all different, and that there are no real branch points. On this surface we introduce a canonical basis of cycles which are non-homologous to zero, i.e. which cannot be contracted to a point. This basis consists of \(2g\) cycles \(a_i, b_i, i = 1, \ldots, g\) which do not intersect except for \(a_i, b_i\) with the same index. For the surface of genus 2 we will consider in detail later, we use the cut-system in Fig. 2. The surface resulting from cutting \(\mathcal{L}_H\) along these cycles, the fundamental polygon, is simply connected.

On a surface of genus \(g\), there are \(g\) independent holomorphic one-forms which are also called differentials of the first kind. These one-forms can be locally written as \(F(K) dK\) where \(F(K)\) is a holomorphic function. Their integrals are holomorphic functions. For general Riemann surfaces, the determination of the holomorphic differentials is a non-trivial problem. One important simplification in the theory of hyperelliptic surfaces is that these differentials are explicitly known. A basis is provided by

\[
(88) \quad d\nu_k = \left( \frac{dK}{\mu}, \frac{K dK}{\mu}, \ldots, \frac{K^{g-1} dK}{\mu} \right).
\]

The holomorphic differentials \(d\omega_k\) are normalized by the condition on the \(a\)-periods

\[
(89) \quad \int_{a_k} d\omega_k = 2\pi i \delta_{1k}.
\]

The matrix of \(b\)-periods is given by \(B_{ik} = \int_{b_i} d\omega_k\). The matrix \(B\) is a so-called Riemann matrix, i.e. it is symmetric and has a negative definite real part. The Abel map \(\omega : \mathcal{L}_H \to \text{Jac}(\mathcal{L}_H)\) with base point \(E_1\) is defined as \(\omega(P) = \int_{E_1}^P d\omega_k\), where \(\text{Jac}(\mathcal{L}_H)\) is the Jacobian.
of $\mathcal{L}_H$, $\mathbb{C}^g$ factorized with respect to the lattice of $a$- and $b$-periods. The theta function with characteristics corresponding to the curve $\mathcal{L}_H$ is given by

$$
\Theta_{pq}(x|B) = \sum_{n \in \mathbb{Z}^g} \exp \left\{ \frac{1}{2} (B(p + n), (p + n)) + (p + n, 2\pi i q + x) \right\},
$$

where $x \in \mathbb{C}^g$ is the argument and $p, q \in \mathbb{C}^g$ are the characteristics. We will mainly consider half-integer characteristics in the following. A half-integer characteristic is called odd if $4 \langle p, q \rangle$ is odd, and even if this inner product is even. The theta function with characteristics is, up to an exponential factor, equivalent to the theta function with zero characteristic (the Riemann theta function is denoted with $\Theta$) and shifted argument,

$$
\Theta_{pq}(x|B) = \Theta(x + Bp + 2\pi i q) \exp \left\{ \frac{1}{2} (Bp, p) + (p, 2\pi i q + x) \right\}.
$$

The theta function has the periodicity properties

$$
\Theta_{pq}(z + 2\pi i e_j) = e^{2\pi i p_j} \Theta_{pq}(z), \quad \Theta_{pq}(z + Be_j) = e^{-2\pi i q_j - z_j - \frac{1}{2} B_{jk} \Theta_{pq}(z)},
$$

where $e_j$ is the $g$-dimensional vector consisting of zeros except for a 1 in $j$th position.

We denote by $d\omega_{pq}$ a differential of the third kind, i.e., a one-form which is holomorphic except for two poles in $P, Q \in \mathcal{L}_H$ with residues $+1$ and $-1$ respectively. This singularity structure characterizes the differentials only up to holomorphic differentials. They can be uniquely determined by the normalization condition that all $a$-periods vanish. The differential $d\omega_{\infty^+\infty^-}$ is given up to holomorphic differentials by $-K^g dK/\mu$.

In [17][18] a physically interesting subclass of Korotkin’s solution was identified which can be written in the form

$$
\mathcal{E} = \frac{\Theta_{pq}(\omega(\infty^+) + u)}{\Theta_{pq}(\omega(\infty^-) + u)} e^I,
$$

where the characteristic is subject to the reality condition $Bp + q \in i\mathbb{R}$, where $u = (u_k) \in \mathbb{C}^g$ and where

$$
I = \frac{1}{2\pi i} \int_{\Gamma} \ln G(K) \, d\omega_{\infty^+\infty^-}(K), \quad u_k = \frac{1}{2\pi i} \int_{\Gamma} \ln G(K) \, d\omega_k.
$$

$\Gamma$ is a piece-wise smooth contour on $\mathcal{L}_H$ and $G(K)$ is a non-zero Hölder-continuous function on $\Gamma$. The contour $\Gamma$ and the function $G$ have to satisfy the reality conditions that with $K \in \Gamma$ also $\bar{K} \in \Gamma$ and $G(K) = G(\bar{K})$; both are independent of the physical coordinates. In the case of disks of radius 1 in which we are interested here, the contour $\Gamma$ is the covering of the imaginary axis in the $+$-sheet of $\mathcal{L}_H$ between $-i$ and $i$. In [13] it was shown that solutions of the above form on a Riemann surface of even genus $g = 2s$ given by $\mu^2 = (K - \xi)(K - \bar{\xi}) \prod_{i=1}^{s} (K^2 - E_i^2)(K^2 - \bar{E}_i^2)$ with a function $G$ subject to $G(-K) = \bar{G}(K)$ lead to an equatorially symmetric Ernst potential, $\mathcal{E}(\xi) = \bar{\mathcal{E}}(-\xi)$.

Notice that these solutions depend only via the branch points of $\mathcal{L}_H$ on the physical coordinates. For $g = 0$ there are no theta functions and the potential [23] takes the form

$$
\ln \mathcal{E} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\ln GdK}{\sqrt{(K - \xi)^2 + \rho^2}}.
$$

As shown in section 2, this Ernst potential is equivalent to the Poisson integral with a distributional density at the disk. The solutions can be considered as the static or Newtonian limit of the theta functional solutions [25].

In the theta functional solutions of evolution equations, the underlying Riemann surface is independent of the physical coordinates. The coordinates only occur in the argument of
the theta functions. Because of (92) the solutions are periodic or quasi-periodic. In the case of the Ernst equation, the Riemann surface itself is ‘dynamical’ since some of its branch points are parametrized by the physical coordinates. Thus the solution is given on a whole family $L_{H}(\xi, \xi')$ of surfaces. The argument of the theta functions in (93) has however no additional dependence on the physical coordinates. Here the modular dependence of the theta functions is important. Therefore the solutions show no periodicity and can be asymptotically flat.

The limit $E_i \to F_i$ in which the cut collapses corresponds to the solitonic limit for theta functional solutions to nonlinear evolution equations. The almost periodic solutions can be seen as an infinite train of solitons. In the ‘solitonic’ limit, only a finite number of solitons survive. In the case of the Ernst equation, the Kerr solution can be obtained as the corresponding limit of a genus 2 surface, see (13). In this mathematical sense, black holes can be considered as solitons. The defining equation (87) for the Riemann surface illustrates the relation between solitons and theta functional solutions: the solitons have a different monodromy property which can be obtained in the limit of a degenerate surface (collapsing cut). Zeros of first order of $\Theta$ in (96) correspond to theta functional solutions, zeros of order 2 to solitons. The $N$-solitons (57) and the Bäcklund transformations are thus contained in Korotkin’s solutions as a limiting case.

The importance of many exact solutions to Einstein’s equations (see [46]) is somewhat limited by the singularities they can have. In the stationary vacuum, the Lichnerowicz theorem [63] states that a solution is either Minkowski spacetime or it has singularities. Non-trivial solutions must have sources which show in the form of singularities in matter-free settings. For general explicit solutions to the Einstein equations, it is however difficult to localize the singularities and to show that they can be replaced by matter sources. This is also in general true for Korotkin’s solutions. An important advantage of the subclass discussed here is that general statements can be made on the regularity of the solutions:

**Theorem 4.1.** The Ernst potential (93) with $[pq]$ being a non-singular half-integer characteristic is analytic in the exterior of the disk $\zeta = 0$, $0 \leq \rho \leq 1$ where it is discontinuous iff

$$\Theta_{pq}(\omega(\infty^-) + u) \neq 0. \quad (96)$$

For a proof see [18]. Condition (96) reflects the fact that within general relativity arbitrary amounts of energy cannot be concentrated in a finite region of spacetime without forming a black hole or a singularity. This defines the range of the physical parameters where the solution is regular. A general method how to define this parameter range based on a study of the zeroes of theta functions was developed in [32].

The Ernst potential (93) follows with (82) and $J = \frac{1}{2\pi i} \int_T \ln G d\omega P^\alpha$ from a matrix $\Psi$ of the form

$$\Psi = e^{J/2} \sqrt{\det(\infty)} \sqrt{\det(K)} \left( \frac{\Theta_{pq}(u + \omega(P))}{\Theta_{pq}(u + \omega(P) + \omega(\xi))} e^{J/2} - i \frac{\Theta_{pq}(u + \omega(\infty^-))}{\Theta_{pq}(u + \omega(P) + \omega(\xi))} e^{-J/2} \right), \quad (97)$$

with $\det(K) = \Theta_{pq}(u + \omega(P))\Theta_{pq}(u - \omega(P) + \omega(\xi)) + \Theta_{pq}(u - \omega(P))\Theta_{pq}(u + \omega(P) + \omega(\xi))$.

To determine the metric function $a$ via (85), we have to calculate the matrix $S$ which leads with (27) to

$$S_{12} = \frac{1}{2iJ} D_{\infty^+} \ln \frac{\Theta_{pq}(u + \omega(\infty^-))}{\Theta_{pq}(u + \omega(\infty^-) + \omega(\xi))}. \quad (98)$$
and to determine the Harrison transformed function \( a \) in the next section,

\[
S_{21} = -\frac{\mathcal{E}}{2f} D_{\infty+} \ln \frac{\Theta_{pq}(u + \omega(\infty^+))}{\Theta_{pq}(u + \omega(\infty^+) + \omega(\xi))}.
\]

Using a degenerated version of Fay’s trisecant identity \( [62] \) (see \( [18] \) for the present case), we can write the above relations free of derivatives,

\[
S_{12} = \rho \left( \frac{\Theta_{pq}(u)\Theta_{pq}(u + 2\omega(\infty^-) + \omega(\bar{\xi}))}{L\Theta_{pq}(u + \omega(\infty^-) + \omega(\xi))\Theta_{pq}(u + \omega(\infty^-))} - 1 \right),
\]

\[
S_{21} = -\rho \left( \frac{\Theta_{pq}(u)\Theta_{pq}(u + 2\omega(\infty^+) + \omega(\xi))}{L\Theta_{pq}(u + \omega(\infty^+) + \omega(\xi))\Theta_{pq}(u + \omega(\infty^+))} - 1 \right),
\]

where

\[
L = \frac{\Theta(\omega(\infty^-))\Theta(\omega(\infty^+) + \omega(\xi))}{\Theta(0)\Theta(\omega(\xi))}.
\]

To construct the solution for the counter-rotating disks in \( [32] \), we used an algebraic approach which made it possible to establish algebraic relations between the metric functions at the disk. Let us recall that a divisor \( X \) on \( \mathcal{L}_H \) is a formal symbol \( X = n_1 P_1 + \ldots + n_k P_k \) with \( P_i \in \mathcal{L}_H \) and \( n_i \in \mathbb{Z} \). The degree of a divisor is \( \sum_{i=1}^{k} n_i \). The Riemann vector \( K_R \) is defined by the condition that \( \Theta(W + K_R) = 0 \) if \( W \) is a divisor of degree \( g - 1 \) or less.

We use here and in the following the notation \( \omega(W) = \int_{P_0}^{W} d\omega = \sum_{i=1}^{g-1} \omega(W_i) \). Note that the Riemann vector can be expressed through half-periods in the case of a hyperelliptic surface. We define the divisor \( X = \sum_{i=1}^{g} X_i \) as the solution of the Jacobi inversion problem \( (i = 1, \ldots, g) \)

\[
\omega(X) - \omega(D) = u,
\]

where the divisor \( D = \sum_{i=1}^{g} E_i \) (this corresponds to a choice of the characteristic in \( [22] \)). With the help of these divisors, we can write \( [22] \) in the form

\[
\ln \mathcal{E} = \int_{D}^{X} \frac{\tau^g d\tau}{\mu(\tau)} - \frac{1}{2\pi i} \int_{\Gamma} \ln G_{\tau} d\tau_{\tau}/\mu(\tau).
\]

Additional information follows from the reality of \( u \) which leads to \( \omega(X) - \omega(D) = \omega(\bar{X}) - \omega(\bar{D}) \). The reality condition for \( X \) implies via Abel’s theorem the existence of a meromorphic function \( R \) with poles in \( \bar{X} + \bar{D} \) and zeros in \( X + D \) (which is a rational function in the fundamental polygon),

\[
R(K) = \text{const} \prod_{i=1}^{g} \frac{(K - E_i)(K - \bar{E}_i) - Q_0(K)\mu(K)}{\prod_{i=1}^{g} (K - X_i)(K - E_i)},
\]

where \( Q_0(K) = x_0 + x_1 K + \ldots + x_{K^{g-1}} \) is a polynomial in \( K \) with purely imaginary coefficients and \( x = ibe^{-2U} \). The coefficients \( x_i \) are related to \( X \) via the relation

\[
(1 - x^2) \prod_{i=1}^{g} (K - X_i)(K - \bar{X}_i) = \prod_{i=1}^{g} (K - E_i)(K - \bar{E}_i) - Q_0(K)(K - \xi)(K - \bar{\xi}).
\]

We can use the existence of the rational function \( R \) to calculate certain integrals of the third kind as

\[
\frac{\Theta(u + \omega(P))\Theta(u + \omega(P^\sigma))}{\Theta(u + \omega(P^\sigma))\Theta(u + \omega(P) + \omega(\xi))} = \exp \left( \int_{X+D}^{\bar{X}+\bar{D}} d\omega_{P^\sigma} \right) = \frac{R(P)}{R(P^\sigma)}.
\]
This makes it possible to give an algebraic expression for $S_{12}$ and $S_{21}$. We can write $S_{12}$ in the form

$$S_{12} = \frac{1}{i(E + \mathcal{E})} D_{\infty^+} \left( \int_X \tilde{X} \, d\omega_{PQ} \right) = \frac{1}{i(E + \mathcal{E})} D_{\infty^+} \left( \int_X \tilde{X} \, d\omega_{PQ} + \int_D \, d\omega_{PQ} \right)$$

with $Q$ independent of $P$. The second integral can be reexpressed in terms of theta functions and be calculated with the help of so-called root functions: The quotient of two theta functions with the same argument but different characteristic is a root function which means that its square is a function on $\mathcal{L}_H$. Let $P_i$, $i = 1, \ldots, 2g + 2$, be the branch points of a hyperelliptic Riemann surface $\mathcal{L}_H$ of genus $g$ and $A_j = \omega(P_j)$ with $\omega(P_i) = 0$. Furthermore let \{i_1, \ldots, i_g\} and \{j_1, \ldots, j_g\} be two sets of numbers in \{1, 2, \ldots, 2g + 2\}. Then the following equality holds for an arbitrary point $P \in \mathcal{L}_H$,

$$\Theta \left[ K_R + \sum_{i=1}^{g} A_{i} \right] \left( \omega(P) \right) \Theta \left[ K_R + \sum_{j=1}^{g} A_{j} \right] \left( \omega(P) \right) = c_1 \sqrt{\frac{(K - E_{i_1}) \ldots (K - E_{i_g})}{(K - E_{j_1}) \ldots (K - E_{j_g})}}$$

where $c_1$ is a constant independent of $K$.

This implies

$$\exp \left( \int_D d\omega \right) = \pm \prod_{i=1}^{g} \sqrt{\frac{K - E_i}{K}}.$$ 

Thus we get with (106)

$$S_{12} = \frac{1}{i(E + \mathcal{E})} \left( \frac{1}{2} \sum_{i=1}^{g} (X_i - \tilde{X}_i) + \frac{1}{1 - x^2} \left( \frac{x}{2} \sum_{i=1}^{g} (E_i + \tilde{E}_i) - (\xi + \bar{\xi}) + x_{g-2} \right) \right),$$

and similarly

$$S_{21} = \frac{\mathcal{E} \mathcal{E}}{i(E + \mathcal{E})} \left( \frac{1}{2} \sum_{i=1}^{g} (X_i - \tilde{X}_i) - \frac{1}{1 - x^2} \left( \frac{x}{2} \sum_{i=1}^{g} (E_i + \tilde{E}_i) - (\xi + \bar{\xi}) + x_{g-2} \right) \right),$$

### 4.3. Counter-rotating dust disk

As an example we will discuss a class of disk solutions on a genus 2 surface where the disk can be interpreted as two counter-rotating components of pressureless matter, so-called dust. The surface energy-momentum tensor $S^{\alpha\beta}$ of these models, where $\alpha$ and $\beta$ stand for the $t$, $\rho$ and $\phi$ components, is defined on the hypersurface $\zeta = 0$. The tensor $S^{\alpha\beta}$ is related to the energy-momentum tensor $T^{\alpha\beta}$ which appears in the Einstein equations $G^{\mu\nu} = 8\pi T^{\mu\nu}$ via $T^{\alpha\beta} = S^{\alpha\beta} e^{kU \delta(\zeta)}$. The tensor $S^{\alpha\beta}$ can be written in the form

$$S^{\alpha\beta} = \sigma_+ u_+^\alpha u_+^\beta + \sigma_- u_-^\alpha u_-^\beta,$$

where $u_+ = (1, 0, \pm \Omega)$. We gave an explicit solution for disks with constant angular velocity $\Omega$ and constant relative density $\gamma = (\sigma_+ - \sigma_-) / (\sigma_+ + \sigma_-)$. A physical interpretation of the surface energy-momentum tensor $S^{\alpha\beta}$ will be given in the following section.

This class of solutions is characterized by two real parameters $\lambda$ and $\delta$ which are related to $\Omega$ and $\gamma$ and the metric potential $U_0 = U(0, 0)$ at the center of the disk via

$$\lambda = 2\Omega^2 e^{-2U_0}, \quad \delta = \frac{1 - \gamma^2}{\Omega^2}.$$
We put the radius $\rho_0$ of the disk equal to 1 unless otherwise noted. Since the radius appears only in the combinations $\rho/\rho_0$, $\zeta/\rho_0$ and $\Omega \rho_0$ in the physical quantities, it does not have an independent role. It is always possible to use it as a natural length scale unless it tends to 0 as in the case of the ultrarelativistic limit of the one component disk. The Ernst potential will be discussed in dependence of the parameters $\epsilon = z_R/(1+z_R) = 1 - e^{i\theta_0}$ and $\gamma$, where $z_R$ is the redshift of photons emitted at the center of the disk and detected at infinity.

The solution is given on a surface of genus 2 where the branch points of the Riemann surface are given by the relation $E_1 = -E_2$ and $E := E_1^2 = \alpha + i\beta$ where

$$\alpha = -1 + \frac{\delta}{2}, \quad \beta = \sqrt{\frac{1}{\lambda^2} + \delta - \frac{\delta^2}{4}}.$$  \hspace{1cm} (114)

The function $G$ in (54) reads

$$G(\tau) = \sqrt{(\tau^2 - \alpha)^2 + \beta^2 + \tau^2 + 1} / \sqrt{(\tau^2 - \alpha)^2 + \beta^2 - (\tau^2 + 1)}. \hspace{1cm} (115)$$

We note that with $\alpha$ and $\beta$ given, the Riemann surface is completely determined at a given point in the spacetime, i.e. for a given value of $\xi$.

Regularity of the solutions in the exterior of the disk restricts the physical parameters to $0 \leq \delta \leq \delta_0(\lambda) := 2 \left(1 + \sqrt{1 + 1/\lambda^2}\right)$ and $0 < \lambda \leq \lambda_c$ where $\lambda_c(\gamma)$ is the smallest value of $\lambda$ for which $\epsilon = 1$. The range of the physical parameters is restricted by the following limiting cases:

**Newtonian limit:** $\epsilon = 0$ ($\lambda = 0$), i.e. small velocities $\Omega \rho_0$ and small redshifts in the disk. The function $e^{2\Omega}$ tends independently of $\gamma$ to $1 + \lambda U_N$, where $U_N$ is the Maclaurin disk solution, and $b$ is of order $\Omega^3$.

**Ultrarelativistic limit:** $\epsilon = 1$, i.e. diverging central redshift. For $\gamma \neq 1$ it is reached for $\lambda_c = \infty$. The solution describes a disk of finite extension with diverging central redshift. For $\gamma = 1$, the limit is reached for $\lambda_c = 4.629 \ldots$. In this case the solution has a singular axis and is not asymptotically flat. This behavior can be interpreted as the limit of a vanishing disk radius. With this rescaling the solution in the exterior of the disk can be interpreted as the extreme Kerr solution (see (54) and references given therein).

**Static limit:** $\gamma = 0$ ($\delta = \delta_0(\lambda)$). In this limit, the solution belongs to the Morgan and Morgan class (57).

**One component:** $\gamma = 1$ ($\delta = 0$), i.e. no counter-rotating matter in the disk. This is the disk of (64, 65).

Analytic formulas for the complete metric in terms of theta functions are given in (54). To evaluate the hyperelliptic integrals in the expressions for the metric we use the numerical methods of (54).

At the disk the branch points $\xi, \bar{\xi}$ lie on the contour $\Gamma$ which implies that care has to be taken in the evaluation of the line integrals. The situation is however simplified by the equatorial symmetry of the solution which is reflected by the additional involution $\tilde{K} \rightarrow -\tilde{K}$ of the Riemann surface $\Sigma_2$ for $\zeta = 0$. This makes it possible to perform the reduction $K^2 \rightarrow \tau$ and to express the metric in terms of elliptic theta functions (see (18)). We denote with $\Sigma_w$ the elliptic Riemann surface defined by $\mu_w = (\tau + \rho^2)((\tau - \alpha)^2 + \beta^2)$, and let $dw$ be the associated differential of the first kind with $u_w = \frac{1}{\tau} \int_{-\rho^2}^\rho \ln G(\sqrt{\tau}) dw(\tau)$. We cut the surface in a way that the $a$-cut is a closed contour in the upper sheet around the cut $[-\rho^2, E]$ and that the $b$-cut starts at the cut $[\infty, E]$. The Abel map $w$ is defined for $P \in \Sigma_w$.
as \( w(P) = \int_{P}^{\infty} dw \). Then the real part of the Ernst potential at the disk can be written as

\[
e^{2U} = \frac{1}{Y - \delta} \left( \frac{1 - \frac{1}{\lambda} - \frac{Y}{\delta} \left( \frac{1}{\lambda^2} + \frac{\delta}{\lambda} \right)}{\sqrt{\frac{1}{\lambda^2} + \delta \rho^2}} - 1 \right)
\]

\[+ \sqrt{\frac{Y^2((\rho^2 + \alpha)^2 + \beta^2)}{1 + \delta \rho^2} - 2Y(\rho^2 + \alpha) + \frac{1}{\lambda^2} + \delta \rho^2},\]

where

\[
Y = \frac{-\frac{1}{\lambda} + \delta \rho^2 + \rho_0^2(u_w)}{\sqrt{(\rho^2 + \alpha)^2 + \beta^2}} \phi_2^2(u_w)
\]

It was shown that there exist algebraic relations between the real and imaginary parts of the Ernst potential,

\[
\frac{\delta^2}{2}(e^{4U} + b^2) = \left( \frac{1}{\lambda} - \delta e^{2U} \right) \left( \frac{1}{\lambda^2} + \delta \right) \left( \frac{1}{\lambda} - \frac{1}{\lambda^2} \right) + \delta \left( \frac{\rho^2 - 1}{2} \right),
\]

and the function \( Z := (a - a_0)e^{2U} \):

\[
Z^2 - \rho^2 + \delta e^{4U} = \frac{2}{\lambda} e^{2U}.
\]

Moreover we have

\[
i x_0 = -\frac{Z}{\delta} \left( \frac{1}{\lambda^2} \phi^2(u_w) - \frac{1}{\lambda} \right).
\]

At the disk, the normal derivatives of the metric functions are discontinuous, but they can be expressed in terms of \( \rho \)-derivatives via

\[
(e^{2U})_\zeta = \frac{Z^2 + \rho^2 + \delta e^{4U}}{2Z\rho} b_\zeta,
\]

\[
(b_\zeta = -\frac{Z^2 + \rho^2 + \delta e^{4U}}{2Z\rho} (e^{2U})_\rho + \frac{e^{2U}}{Z}).
\]

This makes it directly possible to determine all quantities in the disk in terms of elliptic functions.

4.4. Metric and Harrison transformation. If stationary axisymmetric solutions for the pure vacuum are submitted to a Harrison transformation, the complete transformed metric can be constructed. The metric function \( k \) is invariant under the action of \( SU(2,1) \) transformations. To determine the transformed metric function \( a' \), we consider the matrix \( S \) in (98). If we go over from \( 2 \times 2 \)-matrices to \( 3 \times 3 \)-matrices according to the rule

\[
\left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) \rightarrow \left( \begin{array}{ccc} A_{11} & 0 & A_{12} \\ 0 & 1 & 0 \\ A_{23} & 0 & A_{22} \end{array} \right),
\]

the matrix \( G \) acts on \( S \) as on \( \chi \). Thus we get with (47)

\[
S'_{12} = \frac{1}{(1 - q^2)^2} (S_{12} + 2iq^2 - q^4 S_{21}),
\]
which is in accordance with (60). This implies with (54) for the function \( a' \)

\[
a' - a'_0 = -\frac{2}{(1 - q^2)^2} (S_{12} - q^4 S_{21}).
\]

To determine \( a'_0 \), one has to consider \( S_{12} \) and \( S_{21} \) on the axis. In the limit \( \rho \to 0 \), there is a non-trivial contribution from the quotient of theta functions in (100) which diverges as \( 1/\rho \). Repeating the considerations of (18) in the calculation of \( a_0 \), one finds that the axis potentials can be expressed in terms of theta functions on the surface \( \Sigma \) given by \( \hat{\mu}^2 = \prod_{i=1}^g (K - E_i)(K - \hat{E}_i) \). Denoting quantities on \( \Sigma \) with a tilde, one has with (18) on the axis

\[
\frac{S_{21}}{S_{12}} = \tilde{\Theta}_{pq}(\tilde{u} + 2\tilde{\omega}(\infty^-)) / \tilde{\Theta}_{pq}(\tilde{u} + 2\tilde{\omega}(\infty^+)).
\]

In the equatorially symmetric case, this quotient is identical to one since \( 2\tilde{\omega}(\infty^+) \) is a half period on \( \tilde{\Sigma} \) (see (18)). Thus we have

\[
a'_0 = a_0 (1 + q^4) / (1 - q^2)^2.
\]

To illustrate the class of Harrison transformed hyperelliptic solutions, we will now study the transformed counter-rotating dust disks. The metric function \( f' \) in (49) is proportional to \( f \), which implies that the transformed solution vanishes exactly where the original solution has zeros. Since the set of zeros of \( f \) just defines the ergoregions, the transformed solution has the same ergoregions (if any) as the original solution. For the ergoregions of the counter-rotating dust disks see (54).

For small \( q \), the functions \( f \) and \( b \) are essentially unchanged since they are quadratic in \( q \). The electromagnetic potential \( \Phi \) is in this limit with (51) of the form

\[
\Phi' = -q(1 - f + ib).
\]

For larger \( q \), \( |f'| \) becomes smaller near the origin. Since its asymptotic values are not changed, the growth rate towards infinity increases which is reflected by the mass formula (54). In the singular limit \( q \to 1 \), the function \( f' \) is zero for all finite values of \( z \), but one at infinity. The behavior for \( b' \) is similar with the exception that \( b' \) is odd and zero at infinity. The function \( a \) also becomes singular in the limit \( q \to 1 \) which is reflected by the diverging factor \( 1/(1 - q^2)^2 \) and the constant \( a_0 \) (125) which just implies that one can no longer choose \( a \) to be zero on the axis. In the metric function \( g'_{03} = -a' f' \), the factors \( (1 - q^2)^2 \) just cancel and the function is only marginally changed with increasing \( q \). The typical behavior of \( g'_{03} \) for values of \( q \) with \( 0 < \vert q \vert < 1 \) can be seen in Fig. 3. The metric function is an even function in \( \zeta \) which vanishes on the axis and at infinity. It is analytic except at the disk where the normal derivatives have a jump.

The electromagnetic potential tends to \( -1 \) in the limit \( q \to 1 \) for finite \( \xi \), but is zero at infinity. The imaginary part is directly proportional to \( b' \) as can be seen from (50) and (51). We show a typical situation for values of \( q \) with \( 0 < \vert q \vert < 1 \) in Fig. 4 for the real part and in Fig. 5 for the imaginary part. The real part of \( \Phi \) is an even function in \( \zeta \) which vanishes at infinity and has discontinuous normal derivatives at the disk. The imaginary part of \( \Phi \) is an odd function in \( \zeta \) and has a jump at the disk. It vanishes at infinity.

Since \( f' \) has the same zeros as \( f \), the transformed solution has a diverging central redshift if the untransformed has, i.e. the ultrarelativistic limits coincide. In the case \( \gamma \neq 1 \), one has a charged disk of finite extension with diverging central redshift. For \( \gamma = 1 \), the
solution in the exterior of the disk can be interpreted as an extreme Kerr-Newman metric which is obtained as a Harrison-transformed extreme Kerr metric.
In the Newtonian limit $\lambda \to 0$, one has $f = 1 + \lambda U_N$ and $b = \lambda^2 \tilde{b}$ in lowest order. This implies with (49) to (51) for $1 - q^2 \gg \lambda$

(128) \[ f' = 1 + \frac{1 + q^2}{1 - q^2} \lambda U_N, \quad b' = \frac{1 + q^2}{1 - q^2} \lambda^2 \tilde{b}, \quad \Phi' = -\frac{q}{1 - q^2} (\lambda U_N + i\lambda^2 \tilde{b}), \]

The transformed solution thus has the same Newtonian and post-Newtonian behavior as the original metric and in addition an electromagnetic field. The magnetic field is of order $\Omega^3$ as $b'$.

Since the mass is of order $\lambda$ in the Newtonian limit, it is possible to have an extreme limit here with $M' = Q'$ as in the Reissner-Nordström solution. If we put $1 - q^2 = \kappa \lambda$ with $\kappa > 0$, we get in the limit $\lambda \to 0$ for (49) and (51)

(129) \[ f' = \frac{\kappa^2}{(\kappa - U_N)^2}, \quad \Phi' = \frac{U_N}{\kappa - U_N}, \]

a static solution similar to the extreme Reissner-Nordström solution, but with a jump in the normal derivatives of the metric functions at the disk and non-vanishing $f'$ at the origin. Since $U_N < 0$ in the whole spacetime, the solution is regular in the exterior of the disk. Thus one gets a non-singular limit in the exterior of the disk for $q \to 1$ in this case.

In the static limit one has $b' = \Im \Phi' = 0$, since both are proportional to $b$,

(130) \[ f' = \frac{(1 - q^2)f}{(1 - q^2f)^2}, \quad \Phi' = -q \frac{1 - f}{1 - q^2f}. \]

The Harrison-transformed static solution is thus again static with vanishing magnetic field but non-zero electric field.

**Figure 5.** Imaginary part of the electromagnetic potential $\Phi$ for $\epsilon = 0.85$, $\gamma = 0.95$ and $q = 0.6$. 

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**CHARGED DUST DISKS**

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5. Energy-momentum tensor

In this section we will study the energy-momentum tensor at the disk which is a surface layer. For the ‘cut and glue’ techniques, which lead to infinite disks, the energy-momentum tensor can be determined using Israel’s invariant junction conditions. In the Einstein-Maxwell case we consider the Harrison-transformed counter-rotating disks as an example and check the energy conditions. Following [10], the matter is interpreted as streams of counter-rotating electro-dust where possible.

5.1. Disk and energy-momentum tensor. To treat relativistic dust disk, it seems best to use Israel’s invariant junction conditions for matching spacetimes across non-null hypersurfaces [33]. The disk is placed in the equatorial plane and the regions \( V^\pm (\pm \zeta > 0) \) are matched at the hyperplane \( \zeta = 0 \). This is possible in Weyl coordinates since we are only considering dust i.e. vanishing radial stresses in the disk. The spacelike unit normal vector of this hypersurface in \( V^+ \) is \( n^\alpha = e^k_U(0, 0, 1, 0) \). The extrinsic curvature \( K_{AB} \) of this plane in \( V^+ \) is defined as \( K_{AB} = n_A ||B \); here capital indices take the values 0, 1, 3 corresponding to the coordinates \( t, \rho, \phi, || \). \( || \) denotes the covariant derivative with respect to \( s_{AB} \), the metric on the hypersurface. According to Israel [33] the jump \( \gamma_{AB} = K^+_{AB} - K^-_{AB} \) in the extrinsic curvature \( K_{AB} \) of the hypersurface \( \zeta = 0 \) with respect to its embeddings into \( V^\pm = \{ \pm \zeta > 0 \} \) is related to the energy momentum tensor \( S_{AB} \) of the disk via

\[
-8\pi S_{AB} = \gamma_{AB} - s_{AB} \gamma_C^C.
\]

As a consequence of the field equations the energy momentum tensor is divergence free, \( S_{AB}^{||B} = 0 \).

The relations (131) lead to

\[
-4\pi e^{(k-U)} S_{00} = (k_\zeta - 2U_\zeta) e^{2U},
\]

\[
-4\pi e^{(k-U)} (S_{03} - a S_{00}) = -\frac{1}{2} a_\zeta e^{2U},
\]

\[
-4\pi e^{(k-U)} (S_{33} - 2a S_{03} + a^2 S_{00}) = -k_\zeta \rho^2 e^{-2U}.
\]

With these formulas it is straightforward to calculate the energy-momentum tensor for a given spacetime. The discontinuity of the normal derivatives in the equatorial plane due to the ‘cut and glue’ techniques lead as in the Newtonian case to a disk like surface layer. There are purely azimuthal tensions in this case.

Whenever an energy-momentum tensor is worked out in the above way by entering with a metric into the Einstein tensor, the question has to be addressed whether the matter is physically acceptable. The usual criterion one has to check are the energy conditions, see [66]. The weak energy condition implies that the energy density must be positive for all observers, \( S_{AB} V^A V^B > 0 \) where \( V_A \) is an arbitrary timelike vector. The dominant energy condition is satisfied if the weak energy condition holds and if in addition the flux of energy is positive for any observer, i.e. that \( S^{AB} V_A \) is a non-spacelike vector for an arbitrary timelike vector \( V_A \). An energy-momentum tensor satisfying the dominant energy condition will be called physically acceptable, otherwise the matter is exotic (negative energy densities, superluminal velocities, . . . ), as Bondi called it, ‘not the cheapest material to be bought in the shops’.

A convenient way to describe the matter of the disk in the pure vacuum case was introduced by Bičák and Ledvinka in [9]: An energy-momentum tensor \( S^{AB} \) with three independent components can always be written as

\[
S^{AB} = \sigma^*_p V^A V^B + p^*_p W^A W^B,
\]
where $V$ and $W$ are the unit timelike respectively spacelike vectors $(V^A) = N_1(1, 0, \omega_\phi)$ and where $(W^A) = N_2(\kappa, 0, 1)$. This corresponds to the introduction of observers (called $\phi$-isotropic observers (FIo) in [9]) for which the energy-momentum tensor is diagonal. The condition $\mathcal{W}_AV^A = 0$ determines $\kappa$ and $\omega_\phi$ in terms of the metric,

$$\omega_\phi = g_{33}s_{00} - g_{00}s_{33} + \sqrt{(g_{33}s_{00} - g_{00}s_{33})^2 + 4(g_{03}s_{00} - g_{00}s_{03})(g_{03}s_{33} - g_{33}s_{03})}$$

and

$$\kappa = \frac{g_{03} + \omega_\phi g_{33}}{g_{00} + \omega_\phi g_{03}}.$$  

In non-static cases the FIOs rotate with respect to the locally non-rotating observers for whom the metric is diagonal. The latter rotate with the angular velocity $\omega_l$ with respect to infinity

$$\omega_l = \frac{g_{03}}{g_{33}} = \frac{ae^{\Phi}}{\rho^2 - a^2e^{\Phi}}.$$  

This quantity is a measure for the dragging of the inertial frames with respect to infinity due to the rotating matter in the disk.

If the dominant energy-condition holds and if the pressure is positive, one has $p^*/\sigma^* < 1$, and the matter in the disk can be interpreted as in [37] either as having a purely azimuthal pressure or as being made up of two counter-rotating streams of pressureless matter with proper surface energy density $\sigma^*/2$ which are counter-rotating with the same angular velocity $\sqrt{p^*/\sigma^*}$ (which is below 1, the velocity of the light),

$$S^{AB} = \frac{1}{2}\sigma^*(U_A^+U_B^+ + U_A^-U_B^-)$$

where $(U^A_\pm) = U^+(\nu^A \pm \sqrt{p^*/\sigma^*}w^A)$ is a unit timelike vector. We will always adopt the latter interpretation if the condition $p^*/\sigma^* < 1$ is satisfied which is the case in the example [31]. The energy-momentum tensor (137) is just the sum of two energy-momentum tensors for dust. Furthermore it can be shown that the vectors $U_\pm$ are geodesic vectors with respect to the inner geometry of the disk: this is a consequence of the equation $S^{AB||B} = 0$ together with the fact that $U_\pm$ is a linear combination of the Killing vectors.

Using the ‘cut and glue’ formalism, disk sources for all known stationary axisymmetric metrics can be given. Sources for static spacetimes were discussed in [7], the Kerr spacetime was considered in [2]. If the strip cut-off the spacetime includes the horizon completely in the latter case, the matter in the disk can be interpreted as dust.

In the presence of electromagnetic fields, a discontinuous electromagnetic tensor $F^{\alpha\beta}$ leads to a current density $J^A$ via the Maxwell equations

$$F^{AB} ;_\gamma = \frac{1}{\sqrt{-g}} (\sqrt{-g}F^{AB}) ,_\gamma = -4\pi J^A.$$  

Contributions to $J^A$ arise only via the normal derivatives at the disk. We define the current density $j^A$ in the disk as $s^{AB}$ by the relation $J^A = e^{-2(k-U)}j^A\delta(\zeta)$. The electromagnetic energy-momentum tensor does not produce a $\delta$-type contribution to the Einstein-Maxwell equations since $F^{\alpha\beta}$ is bounded at the disk. With (138) we get using equatorial symmetry (the derivatives are taken at $\zeta = 0^+$)

$$2\pi j_0 = -(\Re\Phi)_\zeta, \quad 2\pi(j_3 - aj_0) = -\frac{\rho}{f}(\Im\Phi)_\rho.$$
The continuity equation (the Bianchi identity) at the disk $T_{\mu\nu} = F_{\mu\nu} J^\nu = 0$ leads to the condition
\[ g_{00,\rho} s^{00} + 2 g_{03,\rho} s^{03} + g_{33,\rho} s^{33} = 2 (F_{10} j^0 + F_{13} j^3). \]

We remark that one can substitute one of the equations (132) by (140) in the same way as one replaces one of the field equations by the covariant conservation of the energy momentum tensor in the case of three-dimensional perfect fluids. This makes it possible to eliminate $k_\zeta$ from (132) and to treat the energy-momentum tensor at the disk purely on the level of the Ernst equation. It is straightforward to check the consistency of this approach with the help of (61). Thus one can solve boundary value problems of dust disks without using $k$.

The interpretation of the energy-momentum tensor is not changed with respect to the pure vacuum case. However, free particles will no longer move on geodesics but on so-called electro-geodesics where the Lorentz force is added to the geodesic equation which leads to (140) for each component of dust. It follows that the matter streams will only move on electro-geodesics in the FIO frame if $j_A w^A = 0$, i.e. if there are no currents in this frame. This is in general not the case which implies that the FIOs cannot interpret the matter in the disk as freely moving charged particles even if the energy conditions are satisfied which was always possible in the pure vacuum case. However the matter can still be interpreted as a fluid with a purely azimuthal pressure.

Since the splitting of the matter into two streams of dust is not unique, an alternative approach is to interpret the matter in the disk as two streams of particles moving on electro-geodesics in the asymptotically non-rotating frame. To this end we make the ansatz
\[ s^{AB} = \sigma_m^+ U_+^A U_+^B + \sigma_m^- U_-^A U_-^B, \quad j^A = \sigma_e^+ U_+^A + \sigma_e^- U_-^A \]
with $(U_\pm^A) = N_\pm (1, 0, \omega_\pm)$. The angular velocity follows from the electro-geodesic equation for each component,
\[ \frac{1}{2} \sigma_m^\pm (g_{00,\rho} + 2 g_{03,\rho} \omega^\pm + g_{33,\rho} \omega^2) = \sigma_e^\pm (A_{0,\rho} + A_{3,\rho} \omega^\pm). \]
Such an interpretation is possible if the angular velocities are real. Moreover the velocities $\omega_\pm \rho$ in the disk should be smaller than 1 to avoid superluminal velocities, and the energy densities have to be positive, which means we are not interested in tachyonic or otherwise exotic matter. Relation (141) leads to
\[ \sigma_e^+ N_+^2 = \frac{j^3 - \omega_- j^0}{\omega_+ - \omega_-}, \quad \sigma_e^- N_-^2 = \frac{j^3 - \omega_+ j^0}{\omega_- - \omega_+}, \]
\[ \sigma_m^+ N_+^2 = \frac{s^{03} - \omega_- s^{00}}{\omega_+ - \omega_-}, \quad \sigma_m^- N_-^2 = \frac{s^{03} - \omega_+ s^{00}}{\omega_- - \omega_+}, \]
and
\[ \omega_\pm = \frac{s^{33} - \omega_+ s^{03}}{s^{03} - \omega_- s^{00}}. \]
If we enter (142) with this, we obtain
\[ \omega_\pm = \frac{-T_2 \pm \sqrt{T_2^2 - T_1 T_3}}{T_1}. \]
The densities then follow from (141) where the continuity equation (140) guarantees that this system can be solved.

Using the above formalism, disk sources for conformastationary spacetimes [12], magnetostatic metrics [11] and the Kerr-Newman family [10] have been constructed and discussed.

5.2. Energy-momentum tensor of the Harrison-transformed counter-rotating dust disk. As an example we will now discuss the Harrison-transformed counter-rotating dust disk. It remains to be checked whether and in which range of the parameters the energy-momentum tensor (150) is physically acceptable. The above discussion of the metric indicates the extreme behavior of the metric functions for \( q \) close to one. It is plausible that the matter in the disk which is in the present example the source of such an extreme metric will in general not be physically acceptable. There can be maximal \( q \) smaller than 1 for given \( \lambda \) and \( \delta \) which limits the physical range of the parameters.

To discuss the energy-momentum tensor and the currents in the disk, it is helpful to use the algebraic relations (118) to (120) between the metric functions which exist at the disk, and which imply similar relations between the transformed potentials. With (110) and (111), we get for \( S_{21} \)

\[
S_{21} = \mathcal{E} \tilde{E} S_{12} + i x_0 f, \tag{148}
\]

and thus with (124) for the metric function \( a' \)

\[
(1 - q^2)^2 (a - a_0)' = (a - a_0) \left( 1 + q^2 \frac{2}{\delta} \left( \frac{1}{\delta \lambda} \left( \frac{1/\lambda^2 + \delta}{\sqrt{1/\lambda^2 + \delta \rho^2}} - \frac{1}{\lambda} \right) + \alpha + \frac{\rho^2}{2} \right) \right), \tag{149}
\]

where \( a_0' \) is given by (126). With this function we can calculate the angular velocity of the locally non-rotating observers \( \omega_l \) (150). The dependence of \( \omega_l \) on \( \epsilon \) and \( \gamma \) has been discussed in (54). As a function of \( q \) it is monotonically decreasing as can be seen in Fig.6. The reason for this behavior is that the function \( f' \) tends to zero in the limit \( q \to 1 \) for finite \( \rho \). \( \zeta \) whereas \( g_0^0 \) changes shape but remains finite. Thus the overall behavior of \( \omega_l \) is dominated by \( f' \). The deformation of the function \( g_0^0 \) via \( q \) has, however, the consequence that \( \omega_l \) has its maximum for large \( q \) no longer at the center at the disk but near the rim.

The energy-momentum tensor at the Harrison-transformed disk can be calculated via (132). Expressing the right-hand sides with the help of (49) and (50) via the original
functions, we get
\begin{align}
    s'_1 &= \frac{(1-q^2)^2}{N^2} \left( \frac{\rho N}{2Z} b_\rho - (1-q^4 f^2 + q^4 b^2) f_\zeta + 2q^4 b b_\zeta \right), \\
    s'_2 &= \frac{\rho}{2fN} \left( (1-q^4 f^2 + q^4 b^2) b_\rho + 2q^4 b f f_\rho \right), \\
    s'_3 &= -\frac{\rho^3 N}{2(1-q^2)^2 f^2 Z} b_\rho,
\end{align}

where \( N = (1-q^2 f)^2 + q^4 b^2 \).

This implies
\begin{align}
    2\pi j_0 &= \frac{q(1-q^2)}{N^2} \left( -((1-q^2 f)^2 - q^4 b^2) f_\zeta + 2q^2 b (1-q^2 f) b_\zeta \right) \\
    2\pi (j_3 - a j_0) &= \frac{\rho q}{(1-q^2)N f} \left( 2q^2 b (1-q^2 f) f_\rho + ((1-q^2 f)^2 - q^4 b^2) b_\rho \right).
\end{align}

We will interpret the matter in the disk as in \cite{10} in two ways: The FIOs for whom the tensor \( s^{AB} \) is diagonal rotate with angular velocity \( \omega_\phi \) with respect to infinity. We show \( \omega_\phi \) for several values of \( q \) in Fig. 7. It can be seen that \( \omega_\phi \) decreases monotonically with \( q \). For large \( q \) the maximum of the angular velocity is near or at the rim of the disk in this example.

Since the energy conditions are satisfied here, the FIOs can interpret the matter in the disk as a fluid with a purely azimuthal pressure. Alternatively they can interpret it as being made up of two streams of counter-rotating pressureless matter, where the velocity \( \Omega_c,\rho \) in the streams is below the speed of light. However it can be seen in Fig. 8 that there are in general currents in the frame of the FIOs. Thus an interpretation of the matter by the FIOs as freely moving charged particles is not possible.
FIGURE 7. Angular velocity of the FIOs with respect to infinity for $\epsilon = 0.85$, $\gamma = 0.95$, and $q = 0, \ldots, 0.9$.

FIGURE 8. Currents in the frame of the FIOs for $\epsilon = 0.85$, $\gamma = 0.95$, and $q = 0, \ldots, 0.9$.

The second interpretation in terms of two streams of electro-dust is limited by the conditions that the angular velocities have to be real and that the energy densities have to be positive. Numerically one finds in the present example that the angular velocities are real,
but in a wide range of the parameters there are negative energy densities and tachyonic behavior. Already in the uncharged case there are infinite velocities in strongly relativistic settings with negligible counter-rotation which are due to extrema of the metric function $g_{33}$ in the disk. In this case the quantity $T_1$ in (147) is zero which leads to a diverging $\omega_-$. Increasing $q$ only enhances this effect. The result is that an interpretation as non-tachyonic counter-rotating matter on electro-geodesics with positive energy densities is only possible if $q$, $\epsilon$ and $\gamma$ are not too large. In other words large values of $q$ are in this setting only possible in post-Newtonian or nearly static situations. We show plots of the angular velocities $\omega_{\pm}$ in Fig. 9 for $\epsilon = 0.36$ and $\gamma = 0.08$ where values of $q$ up to 0.75 are possible. The corresponding densities are given in Fig. 10 and the charge densities in Fig. 11. The densities vanish always at the rim of the disk, the charge densities are identically zero for $q = 0$. For larger values of $q$, the angular velocity $\omega_-$ becomes bigger and bigger at the rim of the disk and finally diverges. The density $\sigma^{-}_m$ is in this case negative in the vicinity of the rim of the disk.

6. EINSTEIN-MAXWELL EQUATIONS AND THE RIEMANN-HILBERT PROBLEM

In this section we present recent results on theta functional solutions to the Einstein-Maxwell equations which are constructed with Riemann-Hilbert techniques. We first obtain the hyperelliptic solutions starting from so-called Schlesinger equations. This method is then extended to the Einstein-Maxwell case. The keypoint in the construction is the solution of the Riemann-Hilbert problem on multi-sheeted Riemann surfaces in terms of Szegö kernels by Korotkin [19]. The material in this section is based on [67].

6.1. Riemann-Hilbert problem and Schlesinger equations. Theorem 3.1 in section 3 offers the possibility to construct matrices $\Psi$ with certain analytic properties in the spectral
parameter which lead to solutions to the Einstein-Maxwell equations. The task is to solve a so-called Riemann-Hilbert problem for $\Psi$ to obtain a matrix with prescribed singularities.

The Riemann-Hilbert problem, also known as Hilbert’s 21st problem, can be stated as follows: Consider a linear Fuchsian ordinary differential system on $\mathbb{CP}1$ for some $M \times M$
matrix $\Psi$, 

$$
\frac{d\Psi}{d\gamma} = \sum_{j=1}^{N} A_j (\gamma - \gamma_j) \Psi,
$$

(152)

where the matrices $A_j \in sl(M, \mathbb{C})$ are independent of $\gamma$, with initial condition $\Psi(\gamma = \infty) = I$. At the singularities of the equation, the solutions will not be single-valued on $\mathbb{C}P^1$. If one goes on a small loop around the singularities, a matrix $\Psi$ which is a fundamental solution of the differential system will change by multiplication with some matrix which is called the monodromy matrix. The Riemann-Hilbert problem (inverse monodromy problem) is the construction of the differential equation for a given monodromy matrix, for more details see [68].

The mathematical formulation of this problem for an $M \times M$ matrix reads: fix an $SL(M, \mathbb{C})$ monodromy representation $M$ of the fundamental group $\pi_1[\mathbb{C}P^1 \setminus \{\gamma_1, \ldots, \gamma_N\}]$. Choose the standard set of generators $l_1, \ldots, l_N$ of $\pi_1[\mathbb{C}P^1 \setminus \{\gamma_1, \ldots, \gamma_N\}]$ such that the contour $l_m$ encircles only one singularity $\gamma_m$, and the relation $l_M \ldots l_1 = I$ is satisfied. Then the monodromy representation $M$ is defined by the corresponding set of $N$ $SL(M)$ matrices $M_1, \ldots, M_N$ satisfying the relation $M_N \ldots M_1 = I$. The Riemann-Hilbert problem is to find a $SL(M, \mathbb{C})$-valued function $\Psi(P)$ defined on the universal covering $X$ of $\mathbb{C}P^1 \setminus \{\gamma_1, \ldots, \gamma_N\}$ such that $\Psi$ gains the right multiplier equal to $M_m$ being analytically continued along contour $l_m$.

Note that in this formulation the solution of the Riemann-Hilbert problem is not unique; different solutions are related by so-called Schlesinger transformations (multiplications from the left with appropriate matrix-valued rational functions). It is known that the Riemann-Hilbert problem cannot be solved explicitly in terms of known special functions in the generic case. The largest class of explicitly solvable problems known so far is in the class of problems with quasi-permutation monodromies (when each row and each column of monodromy matrix contain exactly one non-vanishing entry). The universal cover $X$ is in this case a $M$-sheeted compact Riemann surface.

A solution $\Psi \in SL(M, \mathbb{C})$ solves the equation (152). The behavior near the singularities is given (in general position, i.e. the difference between the eigenvalues is non-integer for each of the matrices $A_j$) by the asymptotic expansion of $\Psi$,

$$
\Psi(\gamma) = Q_j (I + 0(\gamma - \gamma_j))(\gamma - \gamma_j)^{T_j} C_j,
$$

(153)

where $Q_j, C_j \in SL(M, \mathbb{C})$ and where $T_j$ is a diagonal, tracefree matrix. The matrices $M_j = C_j^{-1} e^{2\pi i T_j} C_j$ are the monodromy matrices. The function $\Psi$ is thus only single-valued on the universal covering of $\mathbb{C}P^1 \setminus \{\gamma_1, \ldots, \gamma_N\}$. The so-called isomonodromy condition is satisfied if the monodromy matrices are independent of the $\gamma_j$ which implies that one can change the position of the singularities in (152) without affecting the monodromy behavior. In general position, this condition implies

$$
\frac{d\Psi}{d\gamma_j} = - \frac{A_j}{\gamma - \gamma_j} \Psi.
$$

(154)

The consistency of these equations with (152) leads to the Schlesinger system [69] for the residues $A_j$,

$$
\frac{dA_j}{d\gamma_i} = \frac{[A_i, A_j]}{\gamma_i - \gamma_j}, \quad i \neq j,
\frac{dA_i}{d\gamma_i} = \sum_{j \neq i} \frac{[A_i, A_j]}{\gamma_i - \gamma_j}.
$$

(155)

To solve the Riemann-Hilbert problem for quasi-permutation monodromies [19], we introduce two further objects on $X$: the prime form $E(P, Q)$ has the property that it vanishes
at exactly one point on $X$, namely when $P = Q$. Since the surface is compact, there can be no function with this property (the Riemann vanishing theorem states that Riemann theta functions have exactly $g$ zeros). The prime form is a $\left( -\frac{1}{\tau_a}, -\frac{1}{\tau_b} \right)$ differential on the bundle $X \times X$

$$E(a, b) = \frac{\Theta_\ast(j^a_b)}{h_\Delta(a)h_\Delta(b)}$$

where $\ast \equiv [p^* q^*]$ denotes a non-singular odd characteristic, and where the spinor $h_\Delta$ is given by $h_\Delta(a) = \sum_{a=1}^{N} \partial_{\tau_a} \Theta_\ast(0)d\omega_{\nu}(\tau_a)$ ($\tau_a$ denotes local coordinates in the vicinity of a point $a$). In local coordinates the prime form reads

$$E(a, b) = \frac{\tau_a - \tau_b}{\sqrt{d\tau_a} \sqrt{d\tau_b}} + \ldots.$$ 

The Szegö kernel $S_{pq}(P, Q)$ is defined as

$$S_{pq}(P, Q) = \frac{\Theta_{pq}(z + \int_P^Q)}{\Theta_{pq}(z)}E_0(P, Q) = \frac{\sqrt{d\tau_P} \sqrt{d\tau_Q}}{\tau_P - \tau_Q} + \ldots.$$ 

As can be seen from the representation in local coordinates, it can be viewed as a generalization of the Cauchy kernel to Riemann surfaces.

In terms of the Szegö kernel the solution to the Riemann-Hilbert problem was written by Korotkin in the form

$$\Psi_{jk}(K, \lambda) = S_{pq}(K^{(j)}, \lambda^{(k)})E_0(K, \lambda) \quad j, k = 1, \ldots, M,$$

where $K^{(j)}$ denotes a point on the $j$th sheet with projection $K$ on $\mathbb{CP}_1$, and where

$$E_0(K, \lambda) = \frac{K - \lambda}{\sqrt{dK} \sqrt{d\lambda}}$$

is the prime form on $\mathbb{CP}_1$. Obviously $\Psi$ defined in (158) is not a single-valued function on $\mathbb{CP}_1$ (it is only single-valued on $X$) since it is a solution of the Riemann-Hilbert problem.

6.2. Vacuum case. It is instructive to reconsider first the two-dimensional vacuum case. In section 4 we treated the Ernst equation as the integrability condition for the linear system (79) which had the advantage that the Ernst potential is given as one component of $\Psi(\infty^+)$. Several linear systems for the Ernst equation are known in the literature which are related through gauge transformations, see [70]. In this section we will use a linear system which makes the symmetry of the Ernst equation obvious since the matrix of the linear system is an element of the coset space $SL(2, \mathbb{R})/SO(2)$ (in an abuse of notation we will call this matrix also $\Psi$),

$$\Psi_\xi = \frac{G_sG^{-1}}{1 - \gamma} \Psi, \quad \Psi_\bar{\xi} = \frac{G_sG^{-1}}{1 + \gamma} \Psi,$$

where

$$\gamma(K, \xi, \bar{\xi}) = \frac{2}{\xi - \bar{\xi}} \left( K - \frac{\xi + \bar{\xi}}{2} + \sqrt{(K - \xi)(K - \bar{\xi})} \right),$$

and where

$$G = \frac{1}{\xi + \bar{\xi}} \left( \begin{array}{cc} 2 & i(\xi - \bar{\xi}) \\ i(\xi - \bar{\xi}) & 2\xi \bar{\xi} \end{array} \right).$$

It is straightforward to adapt theorem 3.1 to the linear system (160) and to the solution of a Riemann-Hilbert problem as is done in the following theorem by Korotkin and Nicolai.
Theorem 6.1. Let the matrices $A_j \in \mathfrak{sl}(2, \mathbb{C})$ satisfy the Schlesinger system (155), and let $\Psi(\gamma)$ be the corresponding solution of (152). Suppose that the matrix $\Psi$ satisfies the additional conditions

\begin{equation}
\Psi^T(1/\gamma)\Psi^{-1}(0)\Psi(\gamma) = I \tag{163}
\end{equation}

and

\begin{equation}
\Psi(-\bar{\gamma}) = \bar{\Psi}(\gamma). \tag{164}
\end{equation}

Let $\gamma_j = \gamma(K_j, \xi, \bar{\xi})$ with $K_j \in \mathbb{C}$ independent of $\xi, \bar{\xi}$. Then the matrix

\begin{equation}
G = \Psi(\gamma = 0, \xi, \bar{\xi}) \tag{165}
\end{equation}

is real and symmetric and satisfies the Ernst equation, and the function $\Psi$ satisfies the system (160).

The proof is analogous to the one for theorem 3.1, see [71]. The solution of the Riemann-Hilbert problem gives a $SL(2, \mathbb{C})$ matrix $\Psi$, the conditions (163) and (164) ensure that $\Psi$ is in the coset $SL(2, \mathbb{R})/SO(2)$. The involution $\sigma$ which interchanges the sheets on $L$ acts as $\gamma \rightarrow 1/\gamma$, condition (163) is thus equivalent to the reduction condition III. The reality condition (164) was previously incorporated in the normalization condition IV.

In the two-dimensional vacuum case, the covering surface $X$ is hyperelliptic given by $\hat{\mu}^2 = \prod_{j=1}^N (\gamma - \gamma_j)$ (we assume that $N$ is even). Since $\gamma$ depends on the spectral parameter $K$ via (161) which lives on the two-sheeted surface $\hat{L}$ with Hurwitz diagram Fig. I of section 4. The factorization $\hat{L}/\sigma$ leads as before to the hyperelliptic surface $L_H$. The corresponding solution of the Ernst equation can be obtained via (165). One obtains

\begin{equation}
\mathcal{E} = \frac{\Theta_{pq}(\omega(\infty^+))}{\Theta_{pq}(\omega(\infty^-))}. \tag{166}
\end{equation}

The relation to the previous form of the Ernst potential can be established as follows: Let $\tilde{g} = \tilde{g} + \tilde{n}$ and $\tilde{p}_{\tilde{g}+j} = h_j \in \mathbb{R}, \tilde{q}_{\tilde{g}+j} = 0$ for $j = 1, \ldots, n$. Consider the limit of collapsing branch cuts for $j > \tilde{g}$, i.e. $E_{\tilde{g}+j} \rightarrow F_{\tilde{g}+j}$. In this limit all quantities entering (166) can be expressed in terms of quantities (denoted with a tilde) on the surface $\hat{L}_H$ of genus $\tilde{g}$ given by $\tilde{\mu}^2 = (K - \xi)(K - \bar{\xi}) \prod_{i=1}^n (K - E_i)(K - F_i)$, the surfaces $L_H$ with the collapsing cuts removed. The holomorphic differentials have the limit, see [62],

\begin{equation}
d\omega_i \rightarrow d\tilde{\omega}_i, \quad i = 1, \ldots, \tilde{g}, \quad d\omega_i \rightarrow d\tilde{\omega}_{E_i^- F_i^+}, \quad i = \tilde{g} + 1, \ldots, n. \tag{167}
\end{equation}

In other words the holomorphic differentials of $L_H$ become holomorphic differentials on $\hat{L}_H$ and differentials of the third kind with poles at the collapsed branch cuts. Since the $b$-periods of differentials of the third kind can be expressed in terms of the Abel map of the poles, see e.g. [72],

\begin{equation}
\int_{b_j} d\omega_{pq} = \omega_j(P) - \omega_j(Q), \tag{168}
\end{equation}

one can use formula (91) to get for (166)

\[ \mathcal{E} = \frac{\Theta_{pq}(\omega(\infty^+) + \sum_{j=1}^{n} \omega_j(E_j^-) - \omega(E_j^+))}{\Theta_{pq}(\omega(\infty^-) + \sum_{j=1}^{n} \omega_j(E_j^-) - \omega(E_j^+))} \exp \left( \sum_{j=1}^{n} h_j \int_{\infty^-}^{\infty^+} \text{d} \omega_{E_j^- E_j^+} \right). \]

By taking the limit \( \sum_{j=1}^{n} \rightarrow \int_1 \) from a sum to a line integral over the \( E_j \), we get after a partial integration (we assume \( \ln G \) vanishes as the limits of integration) and the identification \( h(K) = \partial_K (\ln G) \) formula (169) where we have used \( \int_A^B \text{d} \omega_{PQ} = \int_P^Q \text{d} \omega_{AB} \). For details of the above construction see [73].

6.3. The Einstein-Maxwell case. The construction of algebro-geometric solutions to the Einstein-Maxwell equations can be carried out as above for the vacuum case. The associated linear system has the form (152) where this time the matrix \( \mathcal{G} \in SU(2,1)/SU(2) \). The matrix \( \Psi \) is constructed as the solution of a Riemann-Hilbert problem as before. This time the covering surface is three-sheeted as noted in section 3. Since the spectral parameter \( \gamma \) varies on the two-sheeted surface \( \mathcal{L} \), the covering surface \( \hat{\mathcal{L}} \) is now six-sheeted. Factorizing with respect to the involution \( \sigma \) of \( \mathcal{L} \) leads to a three-sheeted surface \( \mathcal{L}_3 \) which replaces the hyperelliptic surface \( \mathcal{L}_H \) in the vacuum case. All branch points of this surface are constant except the two branch points \( \xi \) and \( \bar{\xi} \) (the fixed points of the involution \( \sigma \)), where the first two sheets are glued. If the third sheet ‘detaches’ from the first two, one gets the vacuum solutions.

The reduction conditions to obtain \( \mathcal{G} \in SU(2,1)/SU(2) \) imply that \( \mathcal{L}_3 \) is invariant with respect to the holomorphic involution \( \sigma \), acting on every sheet of \( \mathcal{L}_3 \) as \( \gamma \rightarrow 1/\gamma \). In addition \( \mathcal{L}_3 \) has to be invariant with respect to the anti-holomorphic involution \( \tau \), acting as \( \gamma \rightarrow -\bar{\gamma} \) on the third sheet of \( \mathcal{L}_3 \); on the first and second sheets \( \tau \) acts as a superposition of the conjugation \( \gamma \rightarrow -\gamma \) with interchange of the first and second sheets.

The solution of the Riemann-Hilbert problem is again provided by (158) for \( M = 3 \). The Ernst potentials can be obtained from \( \Psi(\infty^{(1)}) \) as before. Details of the construction will be published in a forthcoming paper by Korotkin. The class of algebro-geometrical solutions of the Einstein-Maxwell system constructed in [13] is a partial case of the above construction when the third sheet is attached to the first and second sheets only via branch points of multiplicity two (i.e. all three sheets are glued together at these points).

7. Conclusion and Outlook

In this article we reviewed exact solutions to the Einstein-Maxwell equations with disk sources. The ‘cut and glue’ techniques start with known exact solution of the equations with equatorial symmetry. The hyperplanes \( \zeta_0 \) and \( -\zeta_0 \) are identified. This corresponds to removing a strip from the spacetime. At the newly formed equatorial plane, the normal derivatives of the metric functions will be discontinuous which leads via the field equations to a \( \delta \)-type energy-momentum tensor. These infinitesimally thin disks have infinite extension. For asymptotically flat spacetimes, the mass of the disks is however finite. The matter in the disk has to satisfy the energy conditions in order to be physically acceptable. If this is the case, an interpretation of the matter in the disk as two streams of counter-rotating dust is possible if the velocities in the disk are subluminal. In the absence of electromagnetic fields, the particles of the streams move on geodesics. In the non-static Einstein-Maxwell case, the electro-geodesic equations leads to additional conditions on the matter which can only be satisfied in a certain range of the physical parameters. ‘Cut and glue’-techniques thus provide a generally applicable method to find (infinite) disk sources...
to known spacetimes. They do not provide, however, a method to construct new solutions to the Einstein-Maxwell equations.

The complete integrability of the stationary axisymmetric Einstein-Maxwell equations offers powerful techniques to generate new solutions. The most efficient methods known up to day arise from algebraic geometry and lead to explicit solutions in terms of theta functions associated to certain Riemann surfaces. In the pure vacuum case, these surfaces are hyperelliptic, which makes the use of the powerful hyperelliptic calculus possible. These solutions contain the ‘solitonic’ solutions as the Kerr solution as limiting cases. An important feature of the algebro-geometric solutions is that general regularity theorems can be established. Thus the question of global regularity in the exterior of the disk source can be addressed. With these techniques, disks of finite extension can be constructed as the presented family of counter-rotating dust disks which leads to new solutions to the Einstein equations. The task is then to solve boundary value problems at the disk arising from physical models.

In the Einstein-Maxwell case, hyperelliptic disk solutions were constructed by exploiting the $SU(2, 1)$-symmetry of the field equations. This leads to charged disks. By studying the asymptotics of the so generated spacetime, it was shown that the disks always have a non-vanishing total charge and a gyromagnetic ratio of 2. To generate solutions without total charge but with non-trivial magnetic field, it seems necessary to study theta functions on three-sheeted surfaces. We gave a review on recent results on the Riemann-Hilbert problem on multi-sheeted surfaces which can be applied to the Einstein-Maxwell case.

In order to obtain disk solutions on three-sheeted surfaces, the solutions to Riemann-Hilbert problems have to be adapted to the Einstein-Maxwell equations. The general structure of the physically interesting Riemann surfaces has to be explored. An important point is to establish the analyticity properties of the solutions. If this is achieved, disk solutions can be studied. In order to construct solutions to prescribed matter models, one will have to adopt the algebraic approach to the three-sheeted case. For a complete understanding of the solution, it will be necessary to extend the numerical code for hyperelliptic theta functions to these surfaces.

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REFERENCES

[1] J. B. Hartle and D. H. Sharp, Astrophys. J., 147, 317 (1967).
[2] L. Lindblom, Astrophys. J., 208, 873 (1976).
[3] F. J. Ernst, Phys. Rev 168, 1415 (1968).
[4] D. Maison, Gen. Rel. Grav. 10, 717 (1979).
[5] J. Binney and T. Tremaine, Galactic Dynamics (Princeton Univ. Press, Princeton, 1987).
[6] O. Semeráková, Gravitation: Following the Prague Inspiration (To celebrate the 60th birthday of Jiří Bičák, ed. by O. Semeráková, T. Doležel and M. Žofka (World Scientific, Singapore, 2002), p. 111, gr-qc/0204025.
[7] J. Bičák, D. Lynden-Bell and J. Katz, Phys. Rev. D, 47, 4334 (1993).
[8] J. Bičák, D. Lynden-Bell et J. Katz, MNRAS, 265, 126 (1993).
[9] J. Bičák and T. Ledvinka, Phys. Rev. Lett., 71, 1669 (1993).
[10] T. Ledvinka, J. Bičák and M. Žofka, in Proceedings of 8th Marcel-Grossmann Meeting in General relativity, ed. by T. Piran (World Scientific: Singapore) (1999).
[11] P. Letelier, Phys. Rev. D, 60, 104042 (1999).
[12] J. Katz, J. Bičák and D. Lynden-Bell, Class. Quant. Grav., 16 , 4023 (1999).
[13] C. Struck, Physics Reports, 321, 1 (1999).
[14] D. A. Korotkin, Theor. Math. Phys. 77, 1018 (1989).
[15] C. Klein et O. Richter, Phys. Rev., D57, 857 (1998).
[16] C. Klein et O. Richter, J. Geom. Phys., 24, 53, (1997).
[17] C. Klein et O. Richter, Phys. Rev. Lett., 79, 565 (1997).
[18] C. Klein and O. Richter, Phys. Rev. D, 58, CID 124018 (1998).
[19] D. Korotkin, math-ph/0306061 (2003).
[20] C. Klein, Phys. Rev. D, 65, 084029 (2002).
[21] G. Neugebauer and D. Kramer, Ann. Phys. (Leipzig), 24, 62 (1969).
[22] W. Kinnersley, J. Math. Phys., 14, 651 (1973).
[23] Y. Tanabe, Prog. Theor. Phys., 57, 840 (1977).
[24] D. Maison, Nonlinear equations in classical and quantum field theory (Meudon/Paris, 1983/1984), Lecture Notes in Phys. 226, (Springer: Berlin) 125-139 (1985).
[25] P. Breitenlohner, G. Gibbons and D. Maison, Comm. Math. Phys., 120, 295 (1988).
[26] B. Harrison, J. Math. Phys. 9, 1744 (1968).
[27] M. King and H. Pfister, Phys. Rev. D, 65, 084033 (2002).
[28] M. King and H. Pfister, Class. Quant. Grav., 20, 205 (2003).
[29] J. Ehlers, Konstruktion und Charakterisierungen von Lösungen der Einstein’schen Gravitationsgleichungen, Dissertation (in German), Hamburg (1957).
[30] R. Geroch, J. Math. Phys., 12, 918 (1971).
[31] C. Klein and O. Richter, Phys. Rev. Lett., 83, 2884 (1999).
[32] C. Klein, Phys. Rev. D 63, 084025 (2001).
[33] W. Israel, Nuovo Cimento 44B 1 (1966); Errata: Nuovo Cimento, 48B, 463 (1967).
[34] G. Kuzmin, Astron. Zh., 33, 27 (1956).
[35] A. Toomre, Ap. J. 138, 385 (1962).
[36] N. Evans and P. de Zeeuw, MNRAS, 257, 152 (1992).
[37] T. Morgan and L. Morgan, Phys. Rev., 183, 1097 (1969); Errata: 188, 2544 (1969).
[38] D. Maison in Lecture Notes Phys. 1-126, 540 (2000).
[39] E. Cremmer and B. Julia, Nucl. Phys. B, 159, 141 (1979).
[40] H. Eichenherr and M. Forger, Nucl. Phys. B, 155, 381 (1979).
[41] D. Maison in Developments in the Theory of Fundamental Interaction, L. Turko and A. Pekalski, eds.
[42] J. Eells, Jr. and J.H. Sampson, Amer. J. Math., 86, 109 (1964).
[43] J. Novak and E. Marcq, Class. Quant. Grav., 20, 3051 (2003).
[44] R. Geroch, J. Math. Phys., 13, 394 (1972).
[45] P. Breitenlohner and D. Maison, Ann. Inst. H. Poincare, 46, 215 (1987).
[46] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, Exact Solutions of Einstein’s Field Equations, Cambridge: CUP (1980).
[47] G. Neugebauer and D. Kramer, J. Phys. A, 16, 1937 (1983).
[48] S. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, Theory of Solitons – The Inverse Scattering Method, New York: Consultants Bureau, (1984).
[49] G. Alekseev, Pis’ma Zh. Eksp. Teor. Fiz., 32, 301 (1980).
[50] E.D. Belokolos, A.I. Bobenko, V.Z. Enolskii, A.R. Its and V.B. Matveev, Algebro-Geometric Approach to Nonlinear Integrable Equations, Berlin: Springer (1994).
[51] I. M. Krichever, Nonlinear equations and elliptic curves, in Itogi nauki i techniki, Sovremenn. probl. mat., 23, 79-136, Moscow: VINITI (1983) (in Russian).
D. Korotkin, private communication.

D. Anosov and A. Bolibruch, *The Riemann-Hilbert problem*, Aspects of Mathematics, F. Vieweg: Braunschweig, Wiesbaden (1994).

L. Schlesinger, *J. Reine Angew. Math.*, **141**, 96 (1912).

C. M. Cosgrove, *J. Math. Phys.*, **21**, 2417 (1980).

D. Korotkin and H. Nicolai, *Nucl. Phys. B*, **475**, 397 (1996).

B. A. Dubrovin, Russ. Math. Surv. (Uspekhi), **36**, 11-92 (1981).

D. Korotkin et V. Matveev, *Functional Analysis and Its Applications*, **34** No.4 18-34 (2000).

C. Klein, *Theor. Math. Phys.*, **134**, 72 (2003).

MAX-PLANCK-INSTITUT FÜR PHYSIK, FÖHRINGERRING 6, 80805 MÜNCHEN, GERMANY