A NOTE ON THE TRANSPORT METHOD FOR HYBRID INVERSE PROBLEMS

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Abstract. There are several hybrid inverse problems for equations of the form \( \nabla \cdot D \nabla u - \sigma u = 0 \) in which we want to obtain the coefficients \( D \) and \( \sigma \) on a domain \( \Omega \) when the solutions \( u \) are known. One approach is to use two solutions \( u_1 \) and \( u_2 \) to obtain a transport equation for the coefficient \( D \), and then solve this equation inward from the boundary along the integral curves of a vector field \( X \) defined by \( u_1 \) and \( u_2 \). It follows from an argument given by Bal and Ren in [3] that for any nontrivial choices of \( u_1 \) and \( u_2 \), this method suffices to recover the coefficients on a dense set in \( \Omega \). This short note presents an alternate proof of the same result from a dynamical systems point of view.

1. Introduction

Suppose \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n \), and \( f \in C^\infty(\partial\Omega) \). Let \( D \) be a uniformly positive function on \( \Omega \), and \( \sigma \) be a nonnegative function on \( \Omega \), and consider the problem

\[
\nabla \cdot D \nabla u - \sigma u = 0 \quad \text{on } \Omega \nonumber
\]

\[
\quad u |_{\partial\Omega} = f. \quad (1.1)
\]

For these notes we will consider \( D \in C^1(\Omega) \) and \( \sigma \in C(\Omega) \).

In several hybrid inverse problems involving equations of this type, we can take advantage of physical phenomena to recover the solution \( u \) to (1.1) for a given boundary condition \( f \), without a priori knowledge of \( D \) and \( \sigma \) [2, 3, 4, 8, 10]. To complete these problems, we need a method of recovering \( D \) and \( \sigma \) from the solutions \( u \).

One approach [3, 6, 11] is to note that the equation in (1.1) can be written out as

\[
D \Delta u + \nabla D \cdot \nabla u - \sigma u = 0. \quad (1.2)
\]

If \( u \) is known, this can be viewed as a transport equation for \( D \) with coefficients determined by \( u \). Indeed, if we have two known solutions \( u_1 \) and \( u_2 \) to (1.2), we can multiply the equation for \( u_1 \) by \( u_2 \) and vice versa, and subtract the two to obtain

\[
D( u_2 \Delta u_1 - u_1 \Delta u_2 ) + \nabla D \cdot (u_2 \nabla u_1 - u_1 \nabla u_2) = 0. \quad (1.3)
\]

This eliminates \( \sigma \) to provide a transport equation for \( D \) with known coefficients. Assuming we can measure \( D |_{\partial\Omega} \), then it follows from the basic theory of transport equations [9, Ch. 3] that we can measure (1.3) to obtain \( D \) on all of the integral curves of the vector field

\[
X := u_2 \nabla u_1 - u_1 \nabla u_2 \quad (1.4)
\]

that intersect the boundary of \( \Omega \). Once \( D(x) \) is known, we can solve for \( \sigma(x) \) using (1.2). Note that the maximum principle implies that if \( u \) is positive on the boundary, then \( u \) must be positive inside the domain, eliminating the possibility of difficulties if \( u(x) = 0 \).
The major potential problem with this transport method is the possibility that not every point in $\Omega$ can be reached from the boundary by following an integral curve of $X$. In [6], the authors use the existence of complex geometrical optics (CGO) solutions to (1.2) to show that there exist boundary conditions $f_1$ and $f_2$ for which the corresponding solutions $u_1$ and $u_2$ give rise to a vector field $X$ whose boundary-intersecting integral curves cover $\Omega$. However, the rapid exponential decay of CGOs can be difficult to work with in practice.

Fortunately, it turns out that any non-trivial positive boundary conditions yield a pair of solutions $u_1, u_2$ whose corresponding vector field $X$ lets us recover the coefficients on a dense set in $\Omega$. This follows from the argument given in the proof of Theorem 2.2 in [3], which actually shows that we can recover the coefficients almost everywhere; a version of this same argument is used to analyze the stability of the reconstruction in [7]. This article presents an alternate proof for the density result by considering the flow on $\Omega$ generated by $X$ and applying dynamical systems point of view. More precisely, we prove the following.

**Theorem 1.1.** Suppose $f_1, f_2 \in C(\partial \Omega)$ with $f_2$ positive and $f_1/f_2$ not constant. Let $u_1$ and $u_2$ be the solutions to (1.2) with $u_1 = f_1$ and $u_2 = f_2$ on $\partial \Omega$, and let $X$ be the vector field defined by (1.4). Then the union of the integral curves of $X$ that intersect the boundary of $\Omega$ is dense in $\Omega$.

In other words, given continuous, positive, linearly independent boundary conditions, we can get arbitrarily close to any point in $\Omega$ from the boundary by following an integral curve of $X$. It follows that the transport method allows us to recover $D$ and $\sigma$ on a dense set without special care in selecting the boundary conditions $f_1$ and $f_2$. Note that if $D$ and $\sigma$ are a priori continuous, then we can recover $D$ and $\sigma$ on all of $\Omega$ by continuity.

### 2. Proof of Theorem 1.1

To begin, we will fix some notation. Let $u_1, u_2$, and $X$ be as in the statement of Theorem 1.1 and make the following definitions.

**Definition 2.1.** Let $x, y \in \bar{\Omega}$. We say that $x \sim y$ if there exists an integral curve $\gamma : [0, b] \rightarrow \bar{\Omega}$ defined by $\dot{\gamma}(t) = X(\gamma(t))$ such that both $x$ and $y$ lie in the image of $\gamma$.

**Definition 2.2.** For a set $A \subset \bar{\Omega}$, define

$$\Sigma_A = \{y \in \bar{\Omega} | y \sim x \text{ for some } x \in A\}.$$ 

In other words, $\Sigma_A$ is the union of all integral curves of $X$ that intersect $A$.

With this notation, the statement of Theorem 1.1 is that the closure of $\Sigma_{\partial \Omega}$ is the same as the closure of $\Omega$; i.e.

$$\bar{\Sigma}_{\partial \Omega} = \Omega.$$

Before beginning the proof of Theorem 1.1, we make the following remark: since $D$ is uniformly positive, we can replace $X$ by $DX$ in Definition 2.1. In other words, the following definition is equivalent to Definition 2.1.

**Definition 2.3.** Let $x, y \in \bar{\Omega}$. We say that $x \sim y$ if there exists an integral curve $\gamma : [0, b] \rightarrow \bar{\Omega}$ defined by $\dot{\gamma}(t) = DX(\gamma(t))$ such that both $x$ and $y$ lie in the image of $\gamma$. 
Indeed, if we have an integral curve \( \gamma : [0, b] \rightarrow \Omega \) defined by the equation \( \dot{\gamma}(t) = X(\gamma(t)) \), we can define a function \( g \) by the ODE
\[
\dot{g}(t) = D(\gamma(g(t))) \quad \text{and} \quad g(0) = 0.
\]
Since \( D \) is uniformly positive, \( g \) is increasing, so there exists \( b' \) such that \( g(b') = b \). Then we can define a new curve \( \tilde{\gamma} : [0, b'] \rightarrow \bar{\Omega} \) by reparametrizing \( \gamma \) with \( g \):
\[
\tilde{\gamma}(t) = \gamma(g(t)).
\]
Now \( \tilde{\gamma}([0, b']) = \gamma([0, b]) \) and
\[
\dot{\tilde{\gamma}}(t) = DX(\tilde{\gamma}(t)).
\]
Therefore if \( x \sim y \) according to Definition 2.1 then \( x \sim y \) according to Definition 2.3 and the converse follows similarly. With this in mind, we turn to the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Suppose that \( \Omega \setminus \Sigma \) contains an open set \( U \). Then no integral curve of the vector field \( DX \) joins any point of \( U \) to \( \partial \Omega \), so it follows that \( \Sigma_U \) is disjoint from \( \partial \Omega \), and therefore \( \Sigma_U \subset \Omega \).

Now the vector field \( DX \) gives a flow on \( \Sigma_U \), defined for all time, that maps \( \Sigma_U \) to itself. Moreover,
\[
\nabla \cdot DX = \nabla \cdot D(u_2 \nabla u_1 - u_1 \nabla u_2) = 0,
\]
so the vector field \( DX \) is divergence free. This means that the flow of \( DX \) preserves volume, so the Poincaré Recurrence Theorem applies to maps defined by this flow. This gives us the following result, (see e.g. [1], p71-72):

**Proposition 2.1 (Poincaré Recurrence Theorem).** Let \( W \subset \Sigma_U \) be open. For \( x \in W \) and \( k \in \mathbb{N} \), define
\[
x_k = \gamma_x(k),
\]
where \( \gamma_x \) is the integral curve defined by \( \dot{\gamma}_x(t) = DX(\gamma_x(t)) \), with the initial condition \( \gamma_x(0) = x \). Then for almost every \( x \in W \), \( x_k \in W \) for infinitely many \( k \).

The basic idea of the proof of Theorem 1.1 is as follows. A short calculation shows that (2.1)
\[
X = u_2^2 \nabla u,
\]
where \( u = u_1/u_2 \). The maximum principle, together with the positivity of \( f \), guarantees that \( u_2 \) is uniformly positive, so \( u \) is well defined. Moreover the integral curves of \( X \) and \( DX \) are the same as the integral curves of \( \nabla u \), by the same logic used in the discussion of Definition 2.3. If any integral curve of \( X \) were closed, we could integrate \( \nabla u \) along that curve and obtain two different values of \( u \), which would be a contradiction. The main idea of the proof is to apply the Poincaré Recurrence Theorem to a well chosen subset \( W \subset \Sigma_U \), to provide us with a trajectory that approximates a closed curve well enough to force a contradiction.

To obtain this subset \( W \), define \( u = u_1/u_2 \). Since \( u \) is not constant at the boundary, unique continuation guarantees that \( u \) is not constant on \( \Sigma_U \). Therefore there exists some point \( y \) in \( \Sigma_U \) such that
\[
|\nabla u(y)| > 0.
\]
Then the regularity of $u_1$ and $u_2$ guarantees that there exists an open set $V \subset \Sigma_U$ containing $y$ and a positive constant $c$ such that $|\nabla u| > c$ on $V$.

Now consider an open set $W$ which contains $y$ and is compactly contained in $V$. Applying the Poincaré Recurrence Theorem to $W$, we see that there exists $x_0 \in W$ such that $x_k \in W$ for infinitely many $k$.

Let $\{x_k\}$ denote the subsequence of $\{x_k\}$ such that $x_{k_j} \in W$, and let $\gamma^j : [k_j, k_{j+1}] \to \Omega$ be the integral curve of $DX$ joining $x_{k_j}$ to $x_{k_{j+1}}$. We can obtain $u(x_{k_{j+1}})$ from $u(x_{k_j})$ by integrating $\nabla u$ over $\gamma^j$; in other words

$$u(x_{k_{j+1}}) - u(x_{k_j}) = \int_{k_j}^{k_{j+1}} \nabla u \cdot \dot{\gamma}^j \, dt. \tag{2.2}$$

For each $j$, one of the following two things must happen:

- Case I: the image of $\gamma^j$ is entirely contained in $V$, or
- Case II: the image of $\gamma^j$ contains points outside $V$.

In Case I, we can parametrize (2.2) to get

$$u(x_{k_{j+1}}) - u(x_{k_j}) = \int_{k_j}^{k_{j+1}} \nabla u(\gamma^j(t)) \cdot \dot{\gamma}^j(t) \, dt = \int_{k_j}^{k_{j+1}} \nabla u(\gamma^j(t)) \cdot DX(\gamma^j(t)) \, dt.$$

Then (2.1) implies that

$$u(x_{k_{j+1}}) - u(x_{k_j}) = \int_{k_j}^{k_{j+1}} Du_2^2(\gamma^j(t)) |\nabla u(\gamma^j(t))|^2 \, dt.$$

Since the image of $\gamma^j$ is entirely contained in $V$, and $k_{j+1} - k_j \geq 1$, we have

$$u(x_{k_{j+1}}) - u(x_{k_j}) \geq \min_{\Omega} Du_2^2 \cdot c^2 > 0.$$

In Case II, the length of the portion of $\gamma^j$ contained in $V$ must be at least twice the distance from $W$ to the exterior of $V$, so (2.2) tells us that

$$u(x_{k_{j+1}}) - u(x_{k_j}) \geq 2c \text{ dist}(W, \text{ext } V) > 0.$$

In both cases, $u(x_{k_{j+1}}) - u(x_{k_j})$ is bounded below uniformly in $j$. By setting $q$ to be the minimum of the bounds in both cases, we see that $u(x_{k_{j+1}}) - u(x_{k_j}) \geq q$ for each $j \in \mathbb{N}$, and therefore $u$ is unbounded in $W$. But this contradicts the continuity of $u$, which is guaranteed by the continuity and positivity of $u_1$ and $u_2$, and so our initial supposition is false. Therefore $\Sigma_{\partial \Omega} = \Omega$ as claimed.

As a final remark, note that if $\sigma \equiv 0$, we can take $u_2$ to be the identity function. Then (2.1) implies that $X = \nabla u_1$, and Theorem 1.1 gives us a neat corollary:
Corollary 2.2. Suppose \( u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \), and
\[
\nabla \cdot D\nabla u = 0
\]
in \( \Omega \). Then the set of integral curves of \( \nabla u \) that intersect the boundary of \( \Omega \) is dense in \( \Omega \).

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