Feasibility of constraining the curvature parameter of the symmetry energy using elliptic flow data

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Abstract. A QMD transport model that employs a modified momentum dependent interaction (MDI2) potential, supplemented by a phase-space coalescence model fitted to FOPI experimental multiplicities of free nucleons and light clusters is used to study the density dependence of the symmetry energy above the saturation point by a comparison with experimental elliptic flow ratios measured by the FOPI-LAND and ASYEOS collaborations in $^{197}$Au+$^{197}$Au collisions at 400 MeV/nucleon impact energy. A previous calculation using the same model has proven that neutron-to-proton and neutron-to-charged particles elliptic flow ratios probe on average different densities allowing in principle the extraction of both the slope $L$ and curvature $K_{\text{sym}}$ parameters of the symmetry energy. To make use of this result a Gogny interaction inspired potential is modified by the addition of a density-dependent, momentum-independent term, while enforcing a close description of the empirical nucleon optical potential, allowing independent modifications of $L$ and $K_{\text{sym}}$. Comparing theoretical predictions with experimental data for neutron-to-proton and neutron-to-charged particles elliptic flow ratios the following constraint is extracted:

$$L = 85^{+22}_{-20}(\text{exp})^{+20}_{-12}(\text{th})^{+16}_{-12}(\text{sys}) \text{ MeV} \quad \text{and} \quad K_{\text{sym}} = 96^{+315}_{-170}(\text{exp})^{+170}_{-166}(\text{th}) \text{ MeV.}$$

Theoretical errors include effects due to uncertainties in the isoscalar part of the equation of state, value of the isovector neutron-proton effective mass splitting, in-medium effects on the elastic nucleon-nucleon cross-sections, Pauli blocking algorithm variants and scenario considered for the conservation of the total energy of the system. Systematical uncertainties are generated by the inability of the transport model to reproduce experimental light-cluster-to-proton multiplicity ratios. A value for $L$ free of systematical theoretical uncertainties can be extracted from the neutron-to-proton elliptic flow ratio alone:

$$L = 84^{+30}_{-19}(\text{exp})^{+19}_{-12} \text{ MeV.}$$

It is demonstrated that elliptic flow ratios reach a maximum sensitivity on the $K_{\text{sym}}$ parameter in heavy-ion collisions of about 250 MeV/nucleon impact energy, allowing a reduction of its experimental component of uncertainty to about 150 MeV.

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1 Introduction

The density dependence of the isospin dependent part of the equation of state (asy-EoS) of nuclear matter, commonly known as the symmetry energy (SE), represents one of the remaining open questions in nuclear physics. Its impact on the structure of rare isotopes, dynamics of heavy-ion collisions, properties of astrophysical objects such as neutron stars and explosions of supernovae has motivated a large number of both theoretical and experimental investigations. Experimental studies of isospin diffusion, Pygmy and giant dipole resonances, neutron skin thickness and other phenomena has made possible the extraction of constraints with satisfactory accuracy for the density dependence of the SE in the vicinity of or below the saturation point $\rho_0$. Similarly, recent advances in theoretical many-body simulations of nuclear matter has allowed increasingly more accurate predictions for the asy-EoS up to densities close to the saturation point. The density dependence far above saturation, $\rho \geq 2\rho_0$, has remained up to now extremely uncertain. Its knowledge is mandatory for a proper understanding of properties of neutron stars such as radius and maximum allowed mass and may provide the key for a simple solution of the so called hyperon puzzle.

It is common practice to perform a Taylor expansion in density of the SE around the saturation point $\rho_0$:

$$S(\rho) = S_0 + \frac{L}{3} \frac{\rho - \rho_0}{\rho} + \frac{K_{\text{sym}}}{18} \left( \frac{\rho - \rho_0}{\rho} \right)^2 + \cdots,$$  \hspace{1cm} (1)
and express constraints for its density dependence in terms of allowed ranges for the value of SE at saturation ($S_0$), its slope $L$ and curvature $K_{sym}$ parameters. The results of the majority of the numerous available studies are compatible with $S_0=32\pm2$ MeV and $L=60\pm20$ MeV. In contrast, studies aimed at an explicit determination of the asymmetric part of the compressibility modulus $K_\tau$, allowing an indirect determination of $K_{sym}$, lead to biased values for the latter as a consequence of potentially unphysical correlations between $L$ and $K_{sym}$ induced in $K_\tau$ by the choice of the interaction $\tau$.

These correlations can be traced back to the particular expressions for the employed Gogny and Skyrme type interactions which include minimal density dependent terms introduced as effective approximations to the 3-body nucleon - nucleon force [30,31]. Their strength has been fixed by requiring that empirical saturation properties of nuclear matter are reproduced. Microscopical model calculations using realistic two- and three-nucleon interactions have however demonstrated the sensitivity of both the value of SE at saturation and the maximum possible mass of neutron stars to the yet insufficiently constrained part of the 3-body interaction: intermediate range (3 pion loops) and the spatial and spin structure of the short-range 3-neutron terms [32,33]. Empirical information on the density dependence of the SE above saturation should therefore be extracted using models that lift these constraints appearing in early studies as result of using simple parameterizations for the interaction.

Intermediate energy heavy-ion collisions (HIC) provide the unique opportunity to create and study in terrestrial laboratories chunks of nuclear matter in the vicinity of twice saturation density [34]. Several promising observables have been identified for this purpose: the ratio of neutron-to-proton yields of squeezed out nucleons [35], light cluster emission [36], $\pi^-/\pi^+$ multiplicity ratio in central collisions [37], differential transverse flow [38] and others.

Elliptic flow ratios (EFR) and differences (EFD) of isospin partners have shown to be sufficiently sensitive to probe the superproton density dependence of SE [39,40] and by making use of the FOPI-LAND [41,42] and, more recently, ASYEOS [43] experimental data constraints for the asy-EoS stiffness have been extracted [39,44,45]. All these studies have made use of EoS parameterizations that allow adjustments of the SE stiffness by modifying the value of a single parameter. Consequently only the slope of the SE averaged over the probed density region could be extracted. This may in general be different from the slope at saturation $L$, but well within the quoted uncertainties for the mentioned studies.

Using an upgraded version of the Tübingen QMD transport model it has been shown that the neutron-to-proton elliptic flow ratio (nPnEFR) and neutron-to-hydrogen elliptic flow ratio (nHnEFR) probe on average different density regimes, 1.4-1.5$^*$ and 1.0-1.1$^*$ respectively [43]. The case of neutron-to-charged particles elliptic flow ratio (nCHnEFR) is similar to that of nHnEFR. It may thus be possible to extract constraints for both the slope $L$ and curvature $K_{sym}$ parameters from a comparison of transport model predictions with combined experimental data for npnEFR and nHnEFR, or alternatively npnEFR and nCHnEFR.

The present study aims at extracting constraints for both the slope $L$ and curvature $K_{sym}$ from the FOPI-LAND npnEFR and ASYEOS nCHnEFR experimental data.

Heavy-ion collision dynamics is simulated using an upgraded version [45,46] of the Tübingen quantum molecular dynamics model (QMD) transport model [52,53] which provides a semi-classical framework for the description of such reactions and accounts for relevant quantum aspects such as stochastic scattering and Pauli blocking of nucleons. It includes the production of all nucleonic resonances with masses below 2 GeV, in total 11 N$^*$ and 10 $\Delta$ resonances.

QMD-type transport models provide a solution for the time dependence of the density matrix of the system by the method of the Weyl transformation applied to the many-body Schrödinger equation. Generally, the expectation values for the position and momentum operators can be shown to satisfy the classical Hamiltonian equations of motion [54,55]. These can be factorized to each particle by approximating the total wave-function of the system as the product of individual nucleon wave functions, represented by Gaussian wave packets of finite spread in phase space,

$$\frac{d\hat{r}_i}{dt} = \frac{\partial U_{\hat{r}_i}}{\partial \hat{p}_i} + \frac{\hat{p}_i}{m}, \quad \frac{d\hat{p}_i}{dt} = -\frac{\partial U_{\hat{r}_i}}{\partial \hat{r}_i}. \quad (2)$$

The model

2.1 Transport model

The article ends with a section devoted to summary and conclusions.
The average of the potential operator is understood to be taken over the entire phase-space and weighted by the Wigner distribution of particle \( i \). The potential operator \( U_j \) is in this case the sum of the Coulomb and strong interaction potential operators. In all kinematic equations the relativistic relation between mass, energy and momentum is used.

Description of pion production in heavy-ion collisions close to threshold requires transport models that enforce the conservation of the energy of the system (locally or globally) by taking into account the potential energies of hadrons in nuclear matter \([50,51,56,57]\). In the present model this is achieved by including potential energies in the total energy conservation constraint imposed when determining the final state of a 2-body scattering, decay or absorption process,

\[
\sum_j \sqrt{p_j^2 + m_j^2} + U_j = \sum_i \sqrt{p_i^2 + m_i^2} + U_i,
\]

both indexes running over all particles present in the system and corresponding, from left to right, to the final and initial states of an elementary reaction. This scenario has been referred to as the “global energy conservation” (GEC) scenario in \([50]\). Their impact on flow ratio observables has been shown to be within the uncertainty induced by the experimental data \([50]\). Most of the results presented in this article have therefore been obtained by neglecting these effects, corresponding to the standard so-called “vacuum energy conservation” (VEC) scenario in the mentioned reference. It is nevertheless important to present results also for this scenario in order to assess the impact on the curvature parameter \( K_{sym} \) and to confirm compatibility of the extracted constraints with a similar study of pionic observables with a similar description of experimental data \([51]\).

### 2.2 Initialization of nuclei

Following the first findings the Code Comparison Project \([55]\), the initialization part of the TuQMD model has been modified to better reproduce nuclear density profiles \([51]\). In previous versions of the model \([10,14,50]\), the radius mean square (rms) of initialized nuclei was determined solely from the position of the centroids of the wave function of nucleons. This is however inaccurate for the case of Gaussian-type nucleon wave functions of finite width, as used in QMD transport models, leading to an effective value grows with increasing wave function width, reaching about 10% for values customarily used in transport models in connection with heavy nuclei. Additionally the distribution used to sample nucleon centroids was not leading to good enough density profiles and had to be fine tuned to the nucleus of interest.

In Fig. 1 density profiles for the \(^{197}\)Au nucleus produced by the initialization routines of the transport model used in Ref. \([10,14,50]\) (“QMD old”) and that of Ref. \([51]\) and present version (“QMD new”) for two values of the wave function width \( L_N \) that correspond to those customarily employed in the literature for light \( L_N=4.33 \text{ fm}^2 \) and heavy \( L_N=8.66 \text{ fm}^2 \) nuclei \([55]\). For the higher shown value of \( L_N \) it is impossible to describe the value of the central density and rms simultaneously. Consequently two cases are shown that were built to describe only one of these quantities. The realistic density profile \([50,60]\) corresponding to a rms value of \( \sqrt{< r^2 > }=5.43 \text{ fm} \) and diffuseness \( a=0.6 \text{ fm} \) is also shown.

**Fig. 1.** Nucleon density profiles for the \(^{197}\)Au nucleus produced by the initialization routines of the transport model used in Ref. \([10,14,50]\) (“QMD old”) and that of Ref. \([51]\) and present version (“QMD new”) for two values of the wave function width \( L_N \) that correspond to those customarily employed in the literature for light \( L_N=4.33 \text{ fm}^2 \) and heavy \( L_N=8.66 \text{ fm}^2 \) nuclei \([55]\). For the higher shown value of \( L_N \) it is impossible to describe the value of the central density and rms simultaneously. Consequently two cases are shown that were built to describe only one of these quantities. The realistic density profile \([50,60]\) corresponding to a rms value of \( \sqrt{< r^2 > }=5.43 \text{ fm} \) and diffuseness \( a=0.6 \text{ fm} \) is also shown.

The choice \( L_N=4.33 \text{ fm}^2 \) exhibits the same problem, its severity is however much smaller. Consequently a reasonable description of the realistic density profiles can be achieved in this case, see Fig. 1. The choice of the value for the wave function width \( L_N \) amounts to a compromise between a good description of empirical density profiles and high stability of the initialized nucleus, as they cannot be achieved simultaneously. The former is without argument more important for the study of elliptic flow observ-
ables for impact energies above 100 MeV/nucleon. Consequently, we use $L_N=4.33$ fm$^2$ throughout this study.

Neutrons and protons are initialized using the same coordinate-space probability distribution. Consequently, possible neutron skin effects are neglected in this study. Momenta of nucleons are initialized randomly by taking into account the value of the local Fermi momentum with the constraint that the nucleon in question is bound and that it is not Pauli blocked. Only initializations of nuclei with a binding energy per nucleon in the range $7.0$ MeV $\leq E_B\leq 9.0$ MeV are accepted.

The impact of the improvement of density profiles on pion multiplicities in central collisions is small, leaving the results of Ref. [50] unchanged. In contrast the impact on flow observables in mid-central and, especially, peripheral collisions is non-negligible as it will be shown in Section 4.1.

### 2.3 Pauli blocking

The Pauli blocking algorithm is one of the sources of model dependence of transport models [58] as there is no universally accepted method for the calculation of occupancy probability once the unavoidable approximations that make transport models solvable are introduced and the fermionic nature of nucleons is lost. The wide variety of Pauli blocking algorithms implemented in existing transport models lead to Pauli blocking factors that can differ even by a factor of two [58].

The Pauli blocking algorithm implemented in TuQMD is the same as that of an earlier version of the QMD model [61]. Every potential two-nucleon collision is blocked with a probability

$$P_{\text{blocked}} = 1 - (1 - P_1)(1 - P_2),$$

where $P_1$ and $P_2$ are the occupation fractions of phase-space around nucleons 1 and 2 respectively.

The determination of the occupation fraction makes use of the property of the Wigner distribution (the Weyl transform of the wave function) of having a magnitude less than $(2/\hbar)^3$ in absolute terms [54]. This implies that it must take values different from zero in a volume in phase-larger than $(\hbar/2)^3$. Consequently the value $V_0=2(\hbar/2)^3$ is adopted, the factor of 2 accounting for spin degeneracy. Two spheres of radii $R_r$ and $R_p$ are associated to each nucleon in $r$- and $p$-space respectively. The occupancy fraction is determined as the ratio between the sum of volumes of spheres overlaps of the nucleon of interest with nearby nucleons and the volume $V_0$

$$P_i^{(1)} = 2 \sum_j \delta_{\tau_i,\tau_j} \frac{V_i^{(r)}(r) V_j^{(p)}(p)}{V_0}.$$  

The isospin dependence of the occupancy fraction is introduced by adding contributions only from nucleons $j$ with the same isospin as that of nucleon $i$ ($\tau_i = \tau_j$). The factor of 2 in front of the sum ensures that in the case of isospin symmetric matter similar blocking probabilities as those of the original isospin independent TuQMD Pauli blocking algorithm are recovered. More importantly, with this choice the initial Fermi distribution of nucleon momenta are preserved for durations larger than 200 fm/c when isospin symmetric nuclear matter is simulated in a box. Setting $R_r=3.0$ fm, the value of $R_p$ is determined from the relation

$$V_0 = \frac{4\pi R_r^3}{3} \frac{4\pi R_p^3}{3}. $$

For the case of a nucleon close to the surface only part of the classically available phase-space is allowed. To correct for this effect the occupancy fraction due to nucleons that find themselves in a direction, both in coordinate and momentum space, within a cone of 45$^\circ$ opening angle around the direction nucleon-center of mass and nucleon momentum - center of mass momentum respectively. The location of the center of mass and the total momentum are determined considering only nucleons for which there is a non-zero overlap of the phase-space spheres with those of the nucleon in question. The result is extrapolated to 4$\pi$ solid angle by multiplying it with the factor $4\sqrt{2} \approx 6.83$ and denoted $P_i^{(2)}$. The final occupancy fraction is determined as

$$P_i = \text{Max}\{P_i^{(1)}, P_i^{(2)}\}. $$

This algorithm has been developed at times when the evaluation of transcendental functions was computationally expensive and misses contributions from nucleons further away in $r$ and $p$-space than $2R_r$ and $2R_p$ respectively. This approximation grows worse with increasing wave function width $L_N$.

To correct for this, a second Pauli algorithm, that takes into account the proper overlap of the gaussian Wigner functions associated to each nucleon, has been implemented. Starting from the expression for the Wigner function of a nucleon of momentum $p_i$ and centroid position $r_i$,

$$f_i(r,p) = \frac{1}{\pi^2 \hbar^3} e^{-|r-r_i|^2/\hbar} e^{-|p-p_i|^2/\hbar},$$

the occupancy fraction of the phase-space of nucleon $i$ by that of nucleon $j$ is readily found to be equal to

$$P_{ij} = \text{Erfc} \left( \frac{|r_i - r_j|}{\sqrt{2} L_N} \right) \text{Erfc} \left( \frac{|p_i - p_j|}{\sqrt{2} L_N} \right),$$

with $\text{Erfc}(x)$ being the complementary error function. In this case the total occupancy fraction of nucleon $i$ is defined to be

$$P_i = \frac{1}{2} \sum_j \delta_{\tau_i,\tau_j} P_{ij},$$

omitting thus possible surface corrections. The factor 1/2 takes into account spin degeneracy. For the case of central Au+Au collisions at an impact energy of 400 MeV/nucleon the new blocking algorithm leads to the decrease of the number of successful collisions by about 15% as compared to the original one. The two Pauli blocking algorithms (PBA) will be referred in the following as sPBA (“spheres”) and gPBA (“gaussians”) respectively.
2.4 Equation of state

A modified version of the MDI Gogny-inspired parametrization of the equation of state of nuclear matter [46], named MD12 in the following, has been selected for the present study. Its potential component reads

\[ E_N^{\rho,\beta} = A_u(x,y) \frac{\rho(1-\beta^2)}{4\rho_0} + A_t(x,y) \frac{\rho(1+\beta^2)}{4\rho_0} \]

\[ + \frac{B}{\sigma+1} \rho_0^\sigma (1-x\beta^2) + \frac{D}{3} \rho_0^2 (1-y\beta^2) \]

\[ + \frac{1}{\rho_0} \sum_{\tau,\tau'} C_{\tau,\tau'} \int d^3p d^3p' f_r(p,\tau)f_r(p',\tau') 1 + (p-p')^2/A^2. \]

The difference resides in the extra term proportional to the parameter, that has been introduced in order to allow independent variations of the slope \(L\) and curvature \(K_{sym}\) parameters of the SE, while keeping the neutron-proton isovector effective mass difference constant. This is a mandatory feature given the present uncertainties on the value of this parameter and its potential impact on the extracted constraints on the SE stiffness. A similar modified Gogny-inspired parametrization of the EoS has been recently used to study thermal properties of asymmetric nuclear matter [47], however in this case the extra parameters introduced lead to correlated modifications of the slope \(L\), curvature \(K_{sym}\) and isovector effective mass difference.

The corresponding single-particle potential is given by

\[ U_\tau(\rho,\beta,p) = A_u(x,y) \frac{\rho_\tau}{\rho_0} + A_t(x,y) \frac{\rho_\tau}{\rho_0} \]

\[ + B \left( \frac{\rho}{\rho_0} \right)^\sigma (1-x\beta^2) - 4\tau x \frac{B}{\sigma+1} \rho_0^\sigma \beta \rho_\tau \]

\[ + D \left( \frac{\rho}{\rho_0} \right)^2 (1-y\beta^2) - 4\tau y \frac{D}{3} \rho_0^2 \beta \rho_\tau \]

\[ + \frac{2C_{\tau,\tau'}}{\rho_0} \int d^3p' \frac{f_r(p,\tau)p'}{1 + (p-p')^2/A^2} \]

\[ + \frac{2C_{\tau,\tau'}}{\rho_0} \int d^3p' \frac{f_r(p',\tau')}{1 + (p-p')^2/A^2}. \]

In the above expressions \(\rho, \beta\) and \(p\) denote the density, isospin asymmetry and momentum variables respectively. The label \(\tau\) designates the isospin component of the nucleon or resonance and takes the value \(\tau=+1\) (=1) for neutrons (protons). For cold nuclear matter it holds \(f_r(p,\tau) = (2/h^3)\Theta(p_F - p)\), with \(p_F\) the Fermi momentum of nucleons with isospin \(\tau\).

The dependence of the \(A_u\) and \(A_t\) parameters on the stiffness parameters \(x\) and \(y\) is required to be such as to lead to an expression of the symmetry energy that is independent of them for a particular value of the density, denoted here as \(\tilde{\rho}\). This can be chosen to be different from the saturation density \(\rho_0\) as opposed to Ref. [46] with

\[ A_u(x,y) = A_u^0 - \frac{2B(x-1)\tilde{\rho}^{\sigma-1}}{\sigma+1} \frac{2D(y-1)\tilde{\rho}}{\rho_0} \]

\[ = \tilde{A}_u - \frac{2B\tilde{\rho}^{\sigma-1}}{\sigma+1} \frac{2yD}{3} \tilde{\rho} \rho_0 \] (14)

\[ A_t(x,y) = A_t^0 + \frac{2B(x-1)\tilde{\rho}^{\sigma-1}}{\sigma+1} \frac{2D(y-1)\tilde{\rho}}{\rho_0} \]

\[ = \tilde{A}_t + \frac{2B\tilde{\rho}^{\sigma-1}}{\sigma+1} + \frac{2yD}{3} \tilde{\rho}. \] (15)

with obvious definitions for \(\tilde{A}_u\) and \(\tilde{A}_t\) that include all terms independent of \(x\) or \(y\).

The analytical expressions for the integrals appearing in Eq. (13) and Eq. (15) and the coefficients of their Taylor expansion in isospin asymmetry are needed in the process of fixing model parameters. They are listed, for convenience, in Appendix A.

Consistency with the transport model requires the use of the relativistic expression for kinetic energy in the process of fixing the parameters of the potential. The contribution of the kinetic term to the equation of state is given by

\[ E_{kin}^{\rho,\beta}(\rho,\beta) = \sum_{\tau=1}^{2} \frac{3(1+\tau\beta)}{2p_F^2(\tau)} \left[ \frac{p_F^2(\tau)}{4} \sqrt{p_F^2(\tau) + m^2} \right] \]

\[ + \frac{m}{8} \ln \left[ \frac{p_F^2(\tau) + m^2}{m} \right]. \] (16)

Expanding it in powers of the isospin asymmetry \(\beta\), the contribution of the kinetic term to the symmetry energy is found to take a simple form

\[ S_{kin}^{\rho,\beta} = \frac{p_F^2}{6\sqrt{p_F^2 + m^2}}. \] (17)

As before, \(p_F\) is the Fermi momentum of symmetric nuclear matter of density \(\rho\). Also the non-relativistic expressions for the effective isoscalar mass and the isovector neutron-proton mass difference have to be replaced by their relativistic counterparts. For the former the following expression is found

\[ m_s^2(\rho,\beta) = \frac{m^2 - 2E_F \frac{\partial U}{\partial \rho} + E_s^2 \left( \frac{\partial U}{\partial \beta} \right)^2}{1 + \frac{2E_F \frac{\partial U}{\partial \rho}}{m^2}}. \] (18)

The expression of the latter can then be derived in a straightforward manner. For the case when second and higher powers of \(\partial U/\partial \rho\) can be neglected one arrives at an approximation that resembles the corresponding non-relativistic one [42,63], with the exception of kinematical
factors
\[ \delta m^*_{n-p}(\rho, \beta, p) \equiv \frac{m^*_n - m^*_p}{m} \]
\[ \approx \frac{E_p}{\rho} \left( 1 + \frac{p^2}{2m^*} \right) \left( \frac{\partial U}{\partial \rho} + \frac{\partial U}{\partial p} \right) \left( 1 + \frac{p^2}{2m^*} \right). \]

In the above relations, the relativistic energy is denoted by \( E = \sqrt{p^2 + m^2} \), while \( U, U_n \) and \( U_p \) stand for the isoscalar, neutron and proton single-particle potentials of Eq. (13), respectively. In the actual calculations the exact relativistic expressions for effective masses have been used.

With all the needed ingredients in place the parameters appearing in the potential part of the EoS, Eq. (12), can be fixed. The 11 unknown parameters \( A, C_I = C_{1,1} = C_{-1,-1}, C_u = C_{1,-1} = C_{-1,1}, \tilde{A}_u, \tilde{A}_l, B, \sigma, D, \tilde{\rho}, x \) and \( y \) are determined from a non-linear system of equations that use as input the following quantities: value of the optical potential, small deviations from it being visible, and the slope \( \rho \) at saturation. The choice made for each of these quantities will be discussed in the following.

The values of the \( A, C_I \) and \( C_u \) parameters are determined by optimally reproducing the momentum dependent part of the optical potential and the value of neutron-proton effective mass difference. The first constraint is compatible with an effective isoscalar nucleon mass \( m^*_n = 0.7 m \) for both the Gogny inspired MDI interaction [66] and the empirical nucleon optical potentials [65,67,68]. The energy dependence of these two classes of potentials is however different above 200 MeV/nucleon kinetic energy, the former being attractive while latter is repulsive. Given the well known impact of the momentum dependent part of the optical potential on heavy-ion dynamics in general and flow observables in particular, model parameters were fixed such as to reproduce as closely as possible the momentum dependence of the empirical optical potential. To that end the value of the optical potential at infinite incident momentum is required to be \( U_0(\rho_0, 0, \infty) = 75 \) MeV [17]. This choice leads to a good description of the empirical energy dependence of the optical potential, small deviations from it being visible only at very low momenta. It has been verified that these imperfections impact the value of elliptic flow ratios only marginally. Alternative choices, such as requiring the value of the empirical potential at low kinetic energies to be reproduced, lead to potentials that resemble the energy dependence of the MDI interaction and thus deviate strongly from the empirical ones above 200 MeV/nucleon kinetic energy.

The extraction of the value of the neutron-proton effective mass difference from experimental data has been the aim of several recent investigations. For the present study a value at saturation density, \( \delta m^*_{n-p}(\rho_0, \beta, p = p_F) = 0.33 \beta \), in close agreement with the average of currently undisputed results [62,65,67,68], has been adopted. The precise value of this quantity is however still the subject of ongoing debates spurred by a few controversial lower (or even of opposite sign) extracted values [69,70], that still require further confirmation or possibility alternative explanations. Consequently the impact of this quantity on the extracted constraints for the stiffness for the SE will be studied by varying its magnitude between the lowest and highest allowed values, at 1σ level, as reported in these studies.

Four parameters, \( \tilde{A}_u + \tilde{A}_l, \tilde{A}_l, B \) and \( D \) are determined from the chosen density dependence of the EoS of symmetric nuclear matter, namely saturation density \( \rho_0 = 0.16 \text{ fm}^{-3} \), binding energy per nucleon at saturation \( E_B = 16 \text{ MeV} \), the values of the incompressibility modulus \( K_0 \) and skewness parameter \( J_0 \). The value of the third one is set to \( K_0 = 245 \text{ MeV} \), in agreement with recent extractions from nuclear structure and heavy-ion collision experimental data [11,12,72,73]. The value of the skewness parame-

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### Table 1. Input quantities and their values (first and second columns) together with the model parameters appearing in Eq. (12) and their determined values (third and fourth columns).

| Input | Parameters |
|-------|------------|
| \( \rho_0 \text{ [fm}^{-3}] \) | 0.16 | \( A \text{ [MeV]} \) | 708.001 |
| \( E_B \text{ [MeV]} \) | -16.0 | \( C_I \text{ [MeV]} \) | -13.183 |
| \( m^*_n/m \) | 0.70 | \( C_u \text{ [MeV]} \) | -140.405 |
| \( \delta m^*_{n-p}(\rho_0, \beta = 0.5) \) | 0.165 | \( B \text{ [MeV]} \) | 137.305 |
| \( K_0 \text{ [MeV]} \) | 245.0 | \( \sigma \) | 1.2516 |
| \( J_0 \text{ [MeV]} \) | -350.0 | \( \tilde{A}_l \text{ [MeV]} \) | -130.495 |
| \( \tilde{\rho} \text{ [fm}^{-3}] \) | 0.10 | \( \tilde{A}_u \text{ [MeV]} \) | -8.828 |
| \( S(\tilde{\rho}) \text{ [MeV]} \) | 25.5 | \( D \text{ [MeV]} \) | 7.357 |

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### Table 2. Values of the \( x \) and \( y \) parameters for selected values for \( L \) and \( K_{sym} \). The other parameters of the model take the values listed in Table 1. In the upper half \( L \) and \( K_{sym} \) combinations corresponding to the MDI [66] (for the listed values of \( x_{MDI1} \)) and cMDI2 potentials are presented.

| Input | Parameters |
|-------|------------|
| \( L \text{ [MeV]} \) | \( K_{sym} \text{ [MeV]} \) | \( x \) | \( y \) | \( x_{MDI1} \) |
| 151.0 | 349.0 | 1.179 | -14.459 | -2.0 |
| 106.0 | 135.0 | 1.028 | -9.021 | -1.0 |
| 60.5 | -81.0 | 0.879 | -3.543 | 0.0 |
| 15.0 | -298.0 | 0.721 | 1.990 | 1.0 |
| -31.0 | -512.0 | 0.571 | 7.429 | 2.0 |
| 60.0 | 600.0 | 6.715 | -41.836 | N/A |
| 60.0 | 300.0 | 4.152 | -24.994 | N/A |
| 60.0 | 0.0 | 1.589 | -8.161 | N/A |
| 60.0 | -300.0 | -0.973 | 8.672 | N/A |
| 60.0 | -600.0 | -3.536 | 25.504 | N/A |
ter is not known accurately at present. Its extraction from earth-based laboratory measurements \(^{75,76,77}\) has been impacted up to now by the same potentially unphysical correlations with the lower order Taylor coefficients \(^{75}\) as for the case of the symmetry energy parameters. Determination from astrophysical observables is still affected by large uncertainties \(^{79}\). Due to the lack of more trustworthy information, the value \(J_0=\approx350\) MeV has been chosen for this study, in agreement with the ranges put forward in the mentioned references. This choice for \(K_0\) and \(J_0\) reproduces closely the density dependence of the nuclear matter incompressibility \(K_0\) around the so-called crossing density \(\rho_c=0.10\) fm\(^{-3}\). The stiffness of the isoscalar equation of state has an important impact on the magnitude of collective flows. By taking the difference or ratio of observables corresponding to isospin partners this sensitivity is greatly suppressed \(^{10}\). Consequently, uncertainties in the isoscalar EoS will have a limited, though finite, impact on the extraction of the slope and curvature of the symmetry energy.

The remaining four parameters, \(\bar{A}_n-\bar{A}_\bar{p}, \bar{p}, x\) and \(y\) determine the density dependence of the symmetry energy. The parameter \(\bar{p}\) defines the density at which the magnitude of the symmetry energy is supposed to be known from other sources and used as input to the model. In the context of a Taylor expansion of the symmetry energy around the saturation point it is natural and customary to make the choice \(\bar{p}=\rho_0\). The value of the symmetry energy at saturation is however not accurately known at present. In contrast, it has been possible to extract precise values at sub-saturation densities from experimental data of static properties of nuclei. The study of properties of doubly-magic nuclei (binding energies, rms charge radii and single-particle energies) using a few carefully selected Skyrme energy density functionals has resulted in a value for the symmetry energy \(S(\bar{\rho})=25.5\pm1.0\) MeV at \(\bar{\rho}=0.10\) fm\(^{-3}\). Similarly, the binding energy difference of heavy isotope pairs has allowed the extraction of an even more precise value \(S(\bar{\rho})=26.65\pm0.22\) MeV at a slightly higher value for the density \(\bar{\rho}=0.11\) fm\(^{-3}\). These empirical findings are in good agreement with many body microscopic N\(^3\)LO chiral perturbation theory effective potentials \(^{17,18,19}\). The empirical value extracted in Ref. \(^{12}\) will be used as input to the model. As it will be shown in the results section, elliptic flow ratios at impact energies of collective flows. By taking the difference or ratio of observables corresponding to isospin partners this sensitivity is greatly suppressed \(^{10}\). Consequently, uncertainties in the isoscalar EoS will have a limited, though finite, impact on the extraction of the slope and curvature of the symmetry energy.

The values of input quantities and majority of model parameters appearing in Eq. \(^{12}\) are summarized in Table \(^{12}\). Values for \(x\) and \(y\) model parameters corresponding to selected combinations for \(L\) and \(K_{\text{sym}}\) are listed in Table \(^{2}\). The combinations in the upper half of the table correspond to \(L\) and \(K_{\text{sym}}\) values that the MDI potential \(^{46}\) leads to for integer values of \(x\) (denoted \(x_{\text{MDI}}\) in Table \(^{2}\)) and that have been used in previous studies \(^{40,44,50,51}\). They will be mimicked in the present study by a constrained version of the full MDI2 potential referred to as cMDI2 in Section \(^{4}\).

### Table 3. Values of \(r\)-space coalescence parameters determined from a fit of FOPI experimental multiplicities of free nucleons and light clusters in central Au+Au collisions (\(b\leq2\) fm) at 400 MeV/nucleon \(^{15}\) for three values of \(\delta p\). The asy-EoS parameters have been set to \(L=80.0\) MeV and \(K_{\text{sym}}=0.0\) MeV.

| \(\delta p\) [GeV/c] | \(\delta r_{pp}\) [fm] | \(\delta r_{np}\) [fm] | \(\delta r_{nn}\) [fm] |
|-------------------|------------------|------------------|------------------|
| 0.15              | 6.25             | 6.78             | 7.03             |
| 0.20              | 3.75             | 4.22             | 4.25             |
| 0.25              | 2.25             | 3.25             | 2.75             |

### 2.5 Coalescence model

A minimum spanning-tree (MST) algorithm is employed to generate the final state spectra of intermediate energy heavy-ion collisions. Within such a model, nucleons that are located closer than a predefined range in both coordinate and momentum space are assumed to be part of a cluster. To take into account possible isospin effects and alleviate some of the shortcomings of semi-classical transport models that lead to underprediction of light fragment multiplicities, three cut-off parameters, that fix the maximum allowed separation of a nucleon belonging to a fragment from nucleons of the same cluster, have been introduced in coordinate space, \(\delta r_{nn}, \delta r_{np}\) and \(\delta r_{pp}\). In momentum space a nucleon is considered to belong to a cluster if its momentum in the rest frame of the (potential) cluster is smaller than a certain cut-off value. Consequently, two cut-off parameters, \(\delta p_n\) and \(\delta p_p\) have been considered. To avoid the possibility of final spectra depending on the sequence in which nucleons are tested in the coalescence algorithm the momentum cut-off parameters are increased in steps from 0 to the desired value \(^{52}\). Within this approach clusters identified at an earlier step are used as input for the following one and can grow or evaporate nucleons depending on the momenta of the extra nucleons that may fulfill the coalescence criterion at the current step. This procedure also leads to clusters with nucleons having minimum momenta in the cluster’s rest frame and thus reduce the dependence of the final spectra on the moment at which the coalescence algorithm is applied. In the actual calculations a number of steps equal to 5 has been used.

The coalescence algorithm is applied to identify all clusters with \(A\leq15\). Also a number of 23 additional clusters with \(A>15\) that correspond to known stable or unstable isotopes of \(B, C, N\) and \(O\) are identified. Clusters with lifetimes larger than 1 ms \(^{34,85}\) are considered as stable while the rest are decayed until clusters stable against strong interaction decays are reached (ex. \(^3\)Li→\(p^++\)\(^3\)He). For clusters with \(A\leq15\) that do not correspond to a known stable or unstable isotope, protons or neutrons are evaporated for proton and neutron rich clusters respectively until a known nucleus is reached (ex. \(^{1}p+^{6}n^{+}+\)\(^3\)He). Nucleons belonging to clusters with other masses are discarded from final spectra.

The values of the five coalescence parameters are determined from a fit of cluster and free nucleon multiplicities
to experimental data. For the present case of Au+Au collisions at an impact energy of 400 MeV/nucleon the FOPI cluster multiplicity experimental data for central collisions ($b \leq 2.0$ fm) have been used [18], namely the multiplicities of $p, n, ^2H, ^3H, ^3He, ^4He, Li, Be$ and C have been fitted. The neutron multiplicity has not been measured experimentally. Its value has been deduced by subtracting the number of neutrons bound in experimentally detected clusters from the total of 236. The quality of the fit is impacted only marginally by the incorporation of this information in the multiplicity fit.

A correlation between the values of the $r$ and $p$-space coalescence parameters has been observed. Lower values for the latter lead to higher values for the former. Acceptable values for the two should be of the order of the range of nuclear forces and close to the Fermi momentum of nucleons in nuclei. It has also been noticed that the quality of the fit is not altered if the constraint $\delta p_p = \delta p_n = \delta p$ is enforced.

By performing tests with $\delta p$ in the range 0.1-0.3 GeV/c the middle-ground value $\delta p = 0.2$ GeV/c has been selected for the results presented in this paper. It allows a compromise between the best possible description of experimental cluster and free nucleons multiplicities and values of $r$-space coalescence parameters comparable to the range of the strong force for coalescence times ($t_C$) in the range 100-150 fm/c. For values $t_C \geq 125$ fm/c the independence of theoretical elliptic flow ratios values on this parameter is also achieved. Consequently all results reported in this paper have been derived by setting $t_C = 150$ fm/c. It is worth mentioning that the impact of varying $\delta p$ in the specified range on the extracted values of $L$ and $K_{sym}$ is significantly smaller than the uncertainty induced by the inaccuracy of presently available experimental data for elliptic flow ratios.

A comparison between experimental multiplicities for clusters with $Z \leq 6$ in central ($b \leq 2$ fm) $^{197}$Au+$^{197}$Au collisions at 400 MeV/nucleon impact energy and theoretical results of the fitted model for three different values of the $p$-space coalescence parameter $\delta p = 0.15, 0.20$ and 0.25 GeV/c is presented in the left panel of Fig. (2). It is readily observed that multiplicities of neutrons, protons and helium isotopes depend rather strongly on the value of this parameter, the opposite being true for deuterons and tritons. Also, for all $Z \leq 2$ isotopes smaller values lead to closer agreement to experimental data. In fact, for even smaller values ($\delta p = 0.10$ GeV/c), not shown here, the model is able to reproduce experimental data at 2σ confidence level, at the expense of unrealistically large values for the $\delta r$ parameters.

Another noteworthy feature of the results presented in Fig. (2) is that the experimental deuteron multiplicity is closely reproduced, in opposition to similar models reported in the literature [29]. The same holds true for Be, B and C nuclei. The multiplicities of Li, He isotopes and triton are under-predicted, but the discrepancy to experimental data is smaller as compared to results reported elsewhere. The reason behind this improvement has been identified to be the momentum dependence of the MDI2 interaction. In contrast, the MDI interaction leads to deuteron multiplicities that under-predict the experimental values by at 30-50%, depending on the stiffness of the symmetry energy.

Fig. 2. (Left panel) Multiplicities of free nucleons and clusters with $Z \leq 6$ for three values of the momentum space coalescence parameter ($\delta p$=0.15, 0.20 and 0.25 GeV/c) compared to the experimental FOPI data [25] (full squares with error bars). The ay-EoS stiffness parameters have been set to $L = 80.0$ MeV and $K_{sym} = 0.0$ MeV. The values of the coalescence parameters are listed in Table (3). (Right panel) Dependence of free nucleon and cluster multiplicities on the stiffness of the symmetry energy.
The right panel of Fig. 2 displays the dependence of free nucleon and cluster multiplicities on the stiffness of the symmetry energy for five values of the slope parameter $L$. For each case the value of the curvature $K_{\text{sym}}$ corresponds to the one shown in the upper half part of Table 2. While the $p$-space coalescence parameter has been kept fixed to $\delta p = 0.20$ GeV/$c$, the $r$-space coalescence parameters have been determined in each case from fits to the FOPI experimental data. Such an approach is legitimate since the stiffness of the symmetry energy is not a priori known. The quality of the fit is however not significantly affected if the $r$-space coalescence parameters were considered as independent of the SE stiffness, taking them, for example, as the average of the values obtained from the stiffness dependent fits. The dependence of multiplicities on the stiffness of the asy-EoS is not monotonic and for certain cluster species is negligible ($^3\text{H}$ and $^3\text{He}$) suggesting that additional physics input is needed in the transport model in order to allow a precise description. The sensitivity of cluster multiplicities to the stiffness of the symmetry energy is of similar magnitude as to the value of the $\delta p$ parameter.

3 Model validation

3.1 Transverse and Elliptic Flows

Predictions of the model described in the previous section have been compared with relevant available experimental data to test its validity. A comparison with transverse and elliptic flow data due to the FOPI [49], FOPI-LAND [44] and ASYEOS [43] collaborations will be presented in this section. The following setting of model parameter has been used, if not otherwise stated: the MDI2 potential has been used with the stiffness parameters of the isoscalar EoS as mentioned in Table 1, the stiffness parameter of the asy-EoS have been set to $L = 80.0$ MeV and $K_{\text{sym}} = 0.0$ MeV (close to the central values of these parameters extracted in this work, see Section 4.3), vacuum parametrizations for elastic $NN$ scattering cross-sections and sPBA Pauli blocking algorithm.

Model predictions for transverse and elliptic flows compared to experimental FOPI data are shown in Fig. 3 and Fig. 4 respectively. The left panel of each figure presents proton flows for three centrality ranges $b_0 < 0.25$, $0.25 < b_0 < 0.45$ and $0.45 < b_0 < 0.55$ where $b_0$ is the reduced impact parameter. Model predictions generally agree reasonably well with experimental values, some deviations are however observed in mid-peripheral collisions for the transverse flow $v_1$ and mid-central collisions for the elliptic flow $v_2$ for lower values of the reduced transverse momentum. An improved description of $v_1$ may be obtained by a finer tuning of the compressibility modulus $K_0$ and skewness parameter $J_0$ as the transverse flow is sensitive to them [71]. This is however outside the scope of the present study.

The right panels of Fig. 3 and Fig. 4 present elliptic flows of protons and light clusters in mid-central collisions compared to published FOPI data. Model predictions for transverse flow of light clusters (deuteron, $A = 3$ and $\alpha$ particle) describe experimental values visibly better than in the case of protons. For the case of elliptic flow the model generally predicts a stronger effect than measured experimentally for every particle species (protons, deuterons, tritons). This observable is however affected by important uncertainties due to in-medium effects on elastic cross-sections and the Pauli blocking algorithm used. By using either empirical FU3FP4 in-medium cross-sections parametrizations [73, 86] or the gPBA Pauli blocking algorithm as opposed to the standard choices a better description of the FOPI experimental values for $v_2$ can be achieved. These modifications change the predictions for $v_1$ only very slightly.

A comparison of model predictions and experimental FOPI-LAND data is shown in Fig. 5. Predictions for elliptic flows of neutrons, protons and hydrogen for different stiffnesses of the asy-EoS are presented. It is seen that while theoretical predictions for the standard choice of model parameters do agree with the experimental values of flow for a certain value of the stiffness parameter, this value of the stiffness parameter depends strongly on the particle species. This can be traced back to a rather strong dependence of $v_2$ values on the compressibility modulus of symmetric nuclear matter, in-medium effect on elastic cross-sections and the Pauli blocking algorithm used. This sensitivity is illustrated by presenting model predictions for three values of the compressibility modulus, $K_0 = 210$, 245 and 280 MeV and by switching between vacuum and empirical in-medium elastic cross-sections [73, 86] parametrizations or between sPBA and gPBA Pauli blocking algorithms. The impact on the magnitude of $v_2$ can amount to as much as 40% of the value obtained using the standard choice of model parameters.

In Fig. 6 a comparison between model predictions and the recent experimental data of the ASYEOS collaboration [13] for elliptic flow of neutrons and charged particles is shown. Only calculations for the standard choice of model parameters ($K_0 = 245$ MeV, vacuum cross-sections and sPBA Pauli blocking algorithm) are presented. As before, the stiffness of the SE has been adjusted by changing the value of the $x_{MDI}$ parameter. Elliptic flow of neutrons is more sensitive to the density dependence of the SE, as compared to elliptic flow of clusters.

By comparing Fig. 6 and Fig. 5 its is evident that the description of experimental data elliptic flow of neutrons due to the ASYEOS and FOPI-LAND collaborations is of similar quality, requiring a similar value for $K_0$ for a proper description. Surprisingly this is not the case for the elliptic flow of hydrogen (FOPI-LAND) and charged particles (ASYEOS). The latter would require a sensibly stiffer compressibility modulus than the standard $K_0 = 245$.

An obvious source of systematic uncertainties for model predictions of hydrogen and charged particles flows is the inability of coalescence models to reproduce the experimental values of proton-to-clusters multiplicity ratios. To partially account for this effect theoretical corrected
Fig. 3. (Left panel) Transverse flow of protons as a function of center-of-mass reduced rapidity ($y_P$ stands for the projectile’s rapidity) for three impact parameter ranges. (Right panel) Transverse flow of protons and light clusters in mid-central collisions as a function of center-of-mass relative rapidity. In both cases the corresponding FOPI experimental data [49] for $^{197}$Au+$^{197}$Au at an impact energy of 400 MeV/nucleon are shown for comparison. The kinematical cut $p_T/p_P > 0.4$ has been applied to results, with $p_P$ the projectile’s momentum in the center-of-mass frame. The reduced parameter $b_0 = b/b_{\max}$ with $b_{\max} = 13.4$ fm has been used to express centrality ranges.

Fig. 4. (Left panel) Elliptic flow of protons as a function of relative transverse momentum $p_T/p_P$ for three impact parameter ranges. (Right panel) Elliptic flow of protons, deuterons and tritons in mid-central collisions as a function of the relative transverse momentum. For each particle species calculations with the indicated choices for elastic cross-sections, vacuum (“vac cs”) or in-medium (“med cs”), and Pauli blocking algorithm (sPBA or gPBA) are shown, see text for details. In each case the kinematical cut $|y/y_P| < 0.4$ has been applied. The same observation for the experimental data set and reduced impact parameter $b_0$ as for Fig. (3) are in order.
Fig. 5. Theoretical predictions for elliptic flow of neutrons (bottom), protons (middle) and hydrogen (top) as a function of the stiffness parameter $x_{MDI}$ (see Table 2 for the corresponding values of $L$ and $K_s$). Calculations using the indicated combinations for elastic cross-sections and Pauli blocking algorithms are shown. For the case “vac cs+sPBA” calculations for three values of the compressibility modulus $K_0$ have been performed, while for the rest only the standard $K_0=245$ MeV has been chosen. The FOPI-LAND set B integrated impact parameter [44] data, multiplied by a factor 1.15 to include the omitted reaction plane dispersion correction factor [57], are shown for comparison. Theoretical spectra have been filtered accordingly.

Values for elliptic flow of hydrogen ($v_2^n$) and charged clusters ($v_2^{ch}$) can be determined by employing the theoretical value of elliptic flow and the experimental multiplicity for each cluster species

$$v_2^n = \frac{M_p^{exp} v_2^n + M_d^{exp} v_2^n + M_t^{exp} v_2^n}{M_p^{MDI} + M_d^{MDI} + M_t^{MDI}}$$

$$v_2^{ch} = \frac{M_p^{exp} v_2^{ch} + \sum_{Z_i \geq 1, N_i \geq 1} M_{Z_i, N_i}^{exp} v_2^{ch}}{M_p^{MDI} + \sum_{Z_i \geq 1, N_i \geq 1} M_{Z_i, N_i}^{exp}}$$

This approach is justifiable since theoretical elliptic flows of individual cluster species can describe experimental data for mid-central collisions reasonably well, see Fig. 4. In the above expressions the available experimental multiplicities of clusters in central collisions [18] have been used as a first approximation, since values for the needed $b < 7.5$ fm case are not available in the literature. The correction factors $f_2^{corr} = \frac{v_2^n}{v_2^n}$ and $f_2^{ch} = \frac{v_2^{ch}}{v_2^{ch}}$ depend on the various model parameters, in particular asy-EoS stiffness. Their average values were determined to be $f_2^{corr} = 1.075$ and $f_2^{ch} = 1.10$. The exact values, determined using the appropriate multiplicities, may be somewhat higher, particularly for $f_2^{ch}$, since multiplicities of intermediate mass fragments increase with impact parameter up to $b=8.0$ fm [SS]. Similar correction factors can be determined for the case $b < 2.0$ fm, obtaining the moderately higher values $f_2^{corr} = 1.11$ and $f_2^{ch} = 1.14$. Conservative ranges for these parameters may thus be given by $f_2^{corr} = 1.075 \pm 0.05$ and $f_2^{ch} = 1.10 \pm 0.05$. A potentially superior approach would be to determine, both theoretically and experimentally, coalescence invariant elliptic flows of neutrons and protons, similar to the case of coalescence invariant neutron and proton multiplicity spectra [4] or develop transport models that account directly for light cluster degrees of freedom [89].

Multiplying theoretical $v_2^{ch}$ values for the ASYEOS case in Fig. 3 by $f_2^{ch}$ shifts predictions toward the experimental data, a discrepancy of about 20 – 25% persists for lower $p_T$ values. For the case of the FOPI-LAND data a
scattering of $v^H_1$ with $f^{H\,\text{cor}}_1$, makes a stronger case for $K_0=210\,\text{MeV}$, see Fig. 7.

4 Density Dependence of Symmetry Energy

4.1 Previous Version of the Model

A previous version of the model \cite{40,44} has been used to study the density dependence of the symmetry energy using elliptic flow observables. Constraints for the slope of the symmetry energy have been extracted from the FOPI-LAND experimental data for neutron-to-proton EFR and EFD: $L=118^{+6}_{-57}\,\text{MeV (npEFR)}$ and $L=129^{+36}_{-80}\,\text{MeV (npEFD)}$ respectively \cite{13}. As already noted in that study the central values are significantly larger than the ones extracted from an analysis of neutron skin thickness and isospin diffusion at lower energies, $L=70\,\text{MeV}$ \cite{11}. A smaller difference with respect to similar analyses of the transverse momentum dependent version of the FOPI-LAND data using the UrQMD transport model \cite{39,45}, which arrive at the constraint $L=89\pm45\,\text{MeV (at 2\sigma CL)}$, has been found. In these last cases part of the discrepancy could be understood as originating from the different methods that where used to analyze experimental data: ERAT + multiple centrality bins \cite{44} and multiplicity + one centrality bin \cite{39,45}. In this context it is worth mentioning that the more precise value of the slope $L$ extracted from the recent ASYEOS collaboration data \cite{43} is in full agreement with the previous studies of the same group \cite{39}.

The study in Ref. \cite{44} has made use of a density cut-off algorithm to determine final spectra of neutrons and protons, which were considered as free if they were located in a region of density less than $\rho_0/8$ at clusterring time. Moreover, the value of the cut-off density has not been adjusted to reproduce as closely as possible measured free nucleon multiplicities. To make use of existing data for EFR of neutron-to-hydrogen and neutron-to-charged particles the coalescence model described in Section 2.2 has been developed. As described in Section 2.2 the initialization part of the model has been improved to describe nuclear density profiles more accurately. In the context of the study of pion production in heavy-ion collisions \cite{51} the strength of the Coulomb interaction has been slightly modified to fit its contribution to binding energy as determined from nuclear mass formulae. Additionally, the isospin independent Pauli blocking algorithm employed previously has been replaced by an isospin dependent one.

The impact of these modifications on the value of the npEFR and the extracted constraint for the SE stiffness is presented in Fig. 7. For this calculation the new MDI potential together with the Cugnon parametrization of cross-sections and a value of the compressibility modulus $K_0=210\,\text{MeV}$ have been employed. Starting from the old calculation (dashed-double dotted curve) the model modifications described above are switched on incrementally, first the coalescence algorithm (dash-dotted curve) followed by the improved initialization of nuclei (dashed curve), the modified Coulomb interaction (dotted curve) and finally the isospin dependence in the Pauli blocking algorithm (full curve). The before last modification has little impact on EFR and the extracted value for the slope $L$. The other three improvements have a comparable and non-negligible impact, the first two leading to a softer constraint for the SE stiffness while the last one leads to a stiffer asy-EoS. The combined effect of these model modifications leads to the new constraints: $L=108^{+27}_{-27}\,\text{MeV (npEFR)}$ and $L=91^{+39}_{-24}\,\text{MeV (npEFD)}$. Similar conclusions hold for neutron-to-hydrogen EFR and EFD. The difference with respect to the results presented in Sec.\ref{sec:4.2} is mostly due to the softer compressibility modulus, the impact of the different parametrization used for elastic cross-sections and modification of the optical potential at high momenta are of secondary importance.

It is worth noting that values of the same observables determined using different methods to analyze experimental data lead to constraints for the value of $L$ that differ by 12 MeV (softer) if neutron-to-proton observables are used. It has been noted previously \cite{15} that the impact is even larger, of the order 25-35 MeV, if neutron-to-hydrogen observables are used.
4.2 Constrained MDI2 potential

It is customary to extract the stiffness of the symmetry energy using parametrizations that employ a single free parameter for that purpose. The MDI potential belongs to that category, a parameter denoted $x_{MDI}$ had been introduced to allow changes of the asy-EoS stiffness. As already pointed out, this leads to a constraint between $L$ and $K_{sym}$ and the two-dimensional parameter space associated to these quantities is reduced to a one-dimensional one (curve). Integer values of $x$ have been used to simulate heavy-ion reactions with different stiffnesses for the SE. The corresponding values for $L$ and $K_{sym}$ are shown in Table 4 together with the values of $x_{MDI}$ to distinguish it from the parameter $x$ appearing in the MDI2 potential. By adjusting the values of the parameters $x$ and $y$ appearing in MDI2 to the values listed in Table 4, a potential with the same $L$-$K_{sym}$ combinations as for MDI, labeled cMDI2, can be obtained. The momentum dependence of cMDI2 and MDI potentials is however different.

The potential cMDI2 has been employed to simulate $^{197}$Au+$^{197}$Au collisions with an impact energy of 400 MeV per nucleon. Theoretical results for elliptic flow of neutrons, protons, hydrogen and charged particles have been compared to the corresponding experimental results obtained by the FOPI-LAND [44] and ASYEOS [43] collaborations in Section 3.1. A comparison of the theoretical and experimental elliptic flow ratios relevant for the extraction of the SE stiffness are presented in Fig. 8 and Fig. 9 for the FOPI-LAND and ASYEOS cases respectively. The extracted values for the slope parameter $L$ using all experimentally measured observables and certain combinations of them are presented in Table 4.

Firstly, it is noted that the extracted value of $L$ is impacted by different methods employed to analyze the experimental FOPI-LAND data. Analyzing experimental data using multiple centrality bins and the ERAT observable to determine impact parameter and alternatively one centrality bin and multiplicities lead to slope parameter values that differ by $\delta L=39$ MeV if the nhEFR is used. For npEFR the discrepancy is reduced to $\delta L=9$ MeV, well below the quoted uncertainty due to other systematical and statistical error effects. The culprit was found to be primarily related to the number of centrality bins used, and only marginally due to the observable used to extract the value of the impact parameter [87].

Secondly, constraints for $L$ extracted using ASYEOS data are systematically below those obtained using FOPI-LAND data. For the integrated nhchEFR the difference with respect to npEFR result is even slightly outside the $1\sigma$ CL compatibility region. This discrepancy can be understood if theoretical EFR are computed by using the correct values $\tilde{\nu}_{2}^{H}$ and $\tilde{\nu}_{2}^{ch}$, introduced in Eq. (20), for elliptic flow of hydrogen and charged particles respectively. The new extracted constraints for $L$, labeled as $L_{corr}$ in the rightmost column of Table 4, are now in better agreement with each other. However the extracted value for $L_{corr}$ using nhEFR is now barely compatible with that corresponding to npEFR. Systematical uncertainties affecting theoretical values for hydrogen and charged particles flows due to under-predicted values for proton-to-cluster multiplicity ratios will thus have to be better understood and eliminated before a fully consistent picture.
for the extracted stiffness of SE using npEFR, nhEFR and nchEFR observables can be achieved and the available experimental data sets can be used to their full potential.

Lastly, values for the extracted values of \( L \) and \( L_{\text{corr}} \) using certain combinations of observables are presented at the bottom of Table (4). Since npEFR and nchEFR probe, on average, different densities, npEFR + nchEFR and alternatively npEFR + \( p_T \) dependent nchEFR can be used to obtain independent constraints for both \( L \) and \( K_{\text{sym}} \). The value of the slope \( L \), together with the corresponding \( K_{\text{sym}} \), extracted using the cMDI2 potential will be used in Section 4.3 as a benchmark for those results. The same observations as above apply when comparing to results obtained using different experimental data sets or the impact of systematic uncertainties of theoretical cluster multiplicities.

The values of npEFR and nhEFR measured by the FOPI-LAND collaboration have been compared in Fig. (9) with theoretical predictions for different choices of certain model parameters or ingredients: three values for the compressibility modulus \( K_0 \) (210, 245 and 280 MeV), vacuum versus in-medium elastic \( NN \) cross-sections and sPBA versus gPBA Pauli blocking algorithms. The impact of these parameters on elliptic flow is sizable, see Fig. (9). By constructing elliptic flow ratios the model dependence is reduced substantially, particularly for npEFR (Fig. (9)). Its impact on the extracted values of \( L \) is presented in Table (5) for both FOPI-LAND and ASYEOS observables. Additionally, the impact of enforcing the conservation of total energy (GEC scenario) is shown to lead to a stiffer value of \( L \) by 10-20 MeV, with the exception of \( p_T \) dependent nchEFR. In total the estimated residual model dependence on \( L \) amounts to 18, 29, 22 and 17 MeV when extracted from npEFR, nhEFR, nchEFR and \( p_T \) dependent nchEFR respectively.

In view of the results presented above it is concluded that the most reliable constraint for the density dependence of the symmetry energy can currently be extracted by using the npEFR observable. The following values for the slope and curvature of the SE are obtained by taking the average of the two values listed in Table (4) and also taking into account the residual model dependence.
summarized in Table 5:

$$L = 84 \pm 30\text{(exp)} \pm 18\text{(th)} \text{ MeV}$$ (21)

$$K_{\text{sym}} = 30 \pm 142\text{(exp)} \pm 85\text{(th)} \text{ MeV}.$$

The experimental error includes the uncertainty due to the method used to analyze experimental data. The theoretical error has been determined by adding in quadrature the deviations observed by modifying, within reasonable limits, model parameters from their standard values (see Table 5). Systematical uncertainties due to the correlations between $L$ and $K_{\text{sym}}$ have not been included and will be estimated in Section 4.3. Additionally, possible contributions to the theoretical uncertainty due to the failure of the model to accurately describe experimental multiplicities cannot be excluded. Their magnitude should be however small given the good description of experimental flows of neutrons and protons.

4.3 MDI2 potential

In this section we investigate the possibility of a simultaneous extraction of the values of $L$ and $K_{\text{sym}}$ from a comparison of model predictions with available experimental data for elliptic flow ratios. For that purpose the full MDI2 potential, that allows independent adjustments of these two parameters, will be used in simulations of HIC.

Neutron-to-proton and neutron-to-hydrogen EFR have been shown to probe, on average, nuclear matter with different values for density [13]. Consequently, the two observables are most sensitive to the density dependence of SE in regions of density close to $1.5 \rho_0$ and $\rho_0$ respectively.

It is expected that their sensitivity to changing the value of $K_{\text{sym}}$, while keeping the slope parameter $L$ fixed, is different.

The results of such a calculation are shown in Fig. 10 for nhEFR and npEFR. Several simulations have been performed by keeping the value of $L$ fixed while the curvature parameter has been varied in the interval $-600 \leq K_{\text{sym}} \leq 600$ MeV. The slopes of the nhEFR and npEFR dependence on $K_{\text{sym}}$ are evidently different, the former being negative while the latter is positive. The results are in agreement with the expectations from the results on the average densities probed by these observables. Consequently, the slope and curvature $K_{\text{sym}}$ parameters can be determined from simultaneously comparing theoretical npEFR and nhEFR to experimental data.

In the case of the MDI2 potential the symmetry energy has been constrained to take a fixed value at sub-saturation, $S(0.10 \text{ fm}^{-3})=25.5$ MeV, a calculation for which the simulations presented in Fig. 10. For the case $L=80$ MeV one has $31.8 \leq S_0 \leq 39.6$ MeV, the lower limit corresponding to $K_{\text{sym}}=600$ MeV. This range of variation for $S_0$ includes the average favored value extracted from various terrestrial and astrophysical measurements (see Refs. [63, 90] for a list of constraints extracted from various sources) but it is several times wider than the reported uncertainty. To test for a possible dependence of EFR on the value of symmetry energy at sub-saturation, a simulation with $S_0=30.5$ MeV and $L=80$ MeV has been performed. The result is shown in Fig. 10 for both nhEFR and npEFR (curve interpolating triangle points) and is found to be nearly identical to the standard calculation.
Fig. 11. (Left panel) Constraint for the slope $L$ and curvature $K_{sym}$ extracted from FOPI-LAND npEFR and ASYEOS $p_T$ dependent nchEFR. Curves for the 1, 2, 3 and 4 $\sigma$ confidence levels are shown labeled by the corresponding $\chi^2$/point. For this case the comparison model-experiment yields a minimum value for the goodness of fit parameter equal to $\chi^2$/point=0.14. (Right panel). Dependence of the extracted values for the $(L,K_{sym})$ pair on the combination of observables used, (npEFR,nchEFR) versus (npEFR, nchEFR($p_T$)). For each case the impact of the parameter $f_{corr}$ used to correct for the systematical under-prediction of cluster-to-proton multiplicity ratios is shown. The constraint extracted using the GEC scenario for total energy conservation is also presented. Contour curves correspond to 1 sigma confidence levels.

Fig. 12. (Left panel) Dependence of the extracted values for $L$ and $K_{sym}$ on the compressibility modulus $K_0$, in-medium effects on elastic $NN$ cross-sections and Pauli blocking algorithm. (Right Panel) Impact of the value of the isovector nucleon-proton effective mass splitting on the extracted values for $L$ and $K_{sym}$. The results correspond to the comparison theory-experiment using $\{\text{npEFR, nchEFR($p_T$)}\}$ set of observables. Only 1-$\sigma$ confidence limit contour curves are shown.
for \( L = 80 \text{ MeV} \). It is concluded that EFR in HIC of impact energies close to 400 MeV/nucleon are nearly insensitive to the value of the SE at saturation and consequently this quantity cannot be constrained from such studies.

An additional consequence of the noted sensitivity of npEFR to \( K_{\text{sym}} \) is a systematic uncertainty that affects the extracted value of \( \Delta K \) from FOPI-LAND npEFR experimental data using the cMDI2 potential in the previous section. The cMDI2 potential differs from MDI2 by a potentially unrealistic constraint between \( L \) and \( K_{\text{sym}} \). Taking this effect into account the constraint of Eq. (21) for the slope is corrected to

\[
\Delta L = 84 \pm 30(\text{exp}) \pm 19(\text{theor}) \text{ MeV} \quad (22)
\]

The additional uncertainty has been added in quadrature to the theoretical component of the error and has been estimated to amount to 6 MeV by assuming that the realistic value of the curvature parameter lies in the range -300 \( \leq K_{\text{sym}} \leq 300 \text{ MeV} \). It is evident from Fig. (10) that this component of the uncertainty scales nearly linear with the assumed width of this range.

It turns out that the combination of experimental data for npEFR and nhEFR measured by the FOPI-LAND collaboration is not precise enough to yield a constraint for \( K_{\text{sym}} \) with a 1 \( \sigma \) CI within the range \(-600 \leq K_{\text{sym}} \leq 600 \text{ MeV} \). This can be achieved by replacing the FOPI-LAND nhEFR data by either ASYEOS nhEFR or \( p_T \) dependent nchEFR. The result for the latter choice is shown in the left panel of Fig. (11). The extracted values for the slope and curvature parameters read \( L = 81 \pm 24 \text{ MeV} \) and \( K_{\text{sym}} = 188 \pm 307 \text{ MeV} \) at 1 \( \sigma \) CI. The 1, 2, 3, and 4 \( \sigma \) CL contour curves are plotted, labeled by the corresponding value of \( \chi^2/\text{point} \). The minimum value for this quantity is reached for \( L = 72 \text{ MeV} \) and \( K_{\text{sym}} = 78 \text{ MeV} \).

The favored values for the stiffness parameters for the combination of observables npEFR+nchEFR can be read from the right panel of Fig. (11): \( L = 66 \pm 20 \text{ MeV} \) and \( K_{\text{sym}} = 285 \pm 315 \text{ MeV} \), (the upper limit for \( K_{\text{sym}} \) is obviously underestimated). The values for the slope parameter \( L \) extracted using the two combination of observables are in very good agreement with each other. The curvature \( K_{\text{sym}} \) extracted using npEFR + nchEFR is stiffer than the one obtained using the npEFR + nchEFR\( (p_T) \) data set, the two are however compatible at 1 \( \sigma \) confidence level. Results for npEFR+nchEFR by using the GEC scenario for total energy conservation in the simulation of HIC are also presented. They are compatible at 1 \( \sigma \) level with the corresponding VEC case, the extracted slope and curvature parameters being stiffer by \( \Delta L = 17 \text{ MeV} \) and \( \Delta K_{\text{sym}} = 192 \text{ MeV} \), respectively.

Additionally, the impact of correcting \( \varepsilon_2^{ch} \) due to systematic under-prediction of cluster-to-proton multiplicity ratios, as prescribed by Eq. (20), on the extracted values for \( L \) and \( K_{\text{sym}} \) is also shown in the right panel of Fig. (11). Results for three values of the correction factor are presented: \( f_{\text{corr}} = 1.05, 1.10 \) and 1.15. Together with the uncorrected case, \( f_{\text{corr}} = 1.0 \), clear trends for the both npEFR + nchEFR and npEFR + nchEFR\( (p_T) \) are evidenced. The magnitude and sign of the changes of \( L \) and \( K_{\text{sym}} \) with \( f_{\text{corr}} \) depend strongly on the combinations of observable used. Close values, well within the 1 \( \sigma \) level of uncertainty, for \( L \) and \( K_{\text{sym}} \) are extracted for values of the correction factor in the range \( 1.05 \leq f_{\text{corr}} \leq 1.10 \). The central values of the two SE parameters determined by averaging the results for the two combinations of observables and the most probable value for the correction parameter, \( f_{\text{corr}} = 1.10 \) (see Section 3.1), read: \( L = 85 \text{ MeV} \) and \( K_{\text{sym}} = 95 \text{ MeV} \).

In Fig. (12) the model dependence of the extracted constraint for the SE from npEFR+nchEFR\( (p_T) \) combination of observables is studied. In the left panel the impact of variation of the compressibility modulus in the range \( 210 \leq K_0 \leq 280 \text{ MeV} \), vacuum versus in-medium elastic \( NN \) cross-sections and sPBA versus gPBA Pauli blocking algorithms is presented.

A correlation between the extracted value of \( L \) with the used value of \( K_0 \) is observed, a stiffer isoscalar EoS leading to a softer slope of the SE. The value of \( K_{\text{sym}} \) appears as uncorrelated to the compressibility modulus of symmetric nuclear matter or the slope \( L \). This contrast the correlation with positive slope between \( L \) and \( K_{\text{sym}} \) built in the original MDI potential \[16\] (and mimicked by cMDI2 potential) and other potentials based on the Skyrme interaction \[78,91,92\]. It provides a posteriori support for the introduction of the extra parameter \( y \) in the expression of the MDI2 potential and can be used to constrain the lesser known terms of the short-range \( NN \) interaction. Nevertheless, this latter finding bears a rather small statistical significance.

The impact on the extracted values of these two parameters by varying \( K_0 \) in the mentioned interval amounts to \( \Delta L \approx -30 \text{ MeV} \) and \( \Delta K_{\text{sym}} \approx 75 \text{ MeV} \). It induces uncertainties in the extrapolated values of the SE of 8, 19 and 35 MeV at 2, 3 and 4\( \sigma \), respectively. The trends for \( L \) are the same as in the case of the cMDI2 potential, see Table 5, but larger in magnitude. A constraint of the SE extracted from heavy-ion data that allows precise extrapolations at densities encountered at the center of neutron stars will therefore require also precise quantitative determination of the density dependence of the isoscalar part of the EoS, namely the compressibility \( K_0 \) and the skewness parameter \( J_0 \).

In-medium effects on the elastic \( NN \) cross-sections favor, on average, a moderately softer values for \( L \) and \( K_{\text{sym}} \). Choosing the gPBA Pauli blocking algorithm leads to a slightly stiffer \( L \) and \( K_{\text{sym}} \) as compared to the standard sPBA option.

The impact of changing the value of the isovector neutron-proton mass difference \( \delta m_{n-p} \) is presented in the right panel of Fig. (12). A correlation between \( \delta m_{n-p} \) and the stiffness of SE is noticed only for the \( K_{\text{sym}} \), a higher value of former quantity favors a stiffer \( K_{\text{sym}} \) albeit with a small statistical significance.

The results presented in this section are summarized by the following constraints for the slope \( L \) and curvature \( K_{\text{sym}} \) parameters

\[
L = 85 \pm 22(\text{exp}) \pm 20(\text{th}) \pm 12(\text{sys}) \text{ MeV} \quad (23)
\]

\[
K_{\text{sym}} = 96 \pm 315(\text{exp}) \pm 170(\text{th}) \pm 166(\text{sys}) \text{ MeV} \quad (23)
\]
The indicated uncertainties are of experimental, theoretical (model dependence) and systematical (underprediction of cluster-to-proton multiplicity ratios) origin. The theoretical uncertainty has been determined by adding in quadrature all model dependence illustrated in Fig. 11 and Fig. 12. The quoted value for the systematical error for $K_{\text{sym}}$ has been estimated as the half difference between the maximum and minimum average values for $K_{\text{sym}}$ when the $f_{\text{corr}}$ parameter is varied in the range 1.05-1.15, since the case $f_{\text{corr}}=1.0$ is clearly unrealistic. The value of $L$ is in good agreement with the corrected value $L_{\text{corr}}$ extracted using the cMDI2 potential and the same combination of observables (see bottom lines of Table 1).

5 Discussion and Outlook

The slope $L$ and curvature $K_{\text{sym}}$ of the SE enter in the expression of the isospin dependent component of nuclear matter compressibility,

$$K_{\tau} = K_{\text{sym}} - 6L - \frac{J_0}{K_0} L,$$

where corrections of order $\delta^2$ or higher have been neglected. This quantity is usually extracted from experimental data of neutron skin sizes [8], location of the centroid of isoscalar giant resonances [23,28,29], isospin diffusion in HIC [5] or determined theoretically using as input effective potentials of the Skyrme or Gogny type constrained by nuclear data. The determined values for $K_{\tau}$ generally agree with each other but the claimed accuracies vary significantly: $K_{\tau}=-500\pm50$ MeV [25], $K_{\tau}=-500\pm100$ MeV [28], $K_{\tau}=-370\pm120$ MeV [27], $K_{\tau}=-500\pm150$ MeV [8] and $K_{\tau}=-600\pm250$ MeV [29]. In Ref. [23] the authors determine directly $K_{\text{sym}}$, the obtained values and accuracies depend strongly on the data set and fitting procedure.

The constraints derived in Section 4.2 and Section 4.3 for the cMDI2 and MDI2 potentials lead to the following values for $K_{\tau}$ respectively

$$K_{\tau} = -354 \pm 228 \text{ MeV(cMDI2)}$$

$$K_{\tau} = -290 \pm 421 \text{ MeV(MDI2)}.$$ 

The result extracted using the cMDI2 potential is almost the same as that of Ref. [27]. This is due to a very similar and potentially unconstrained parameter between $L$ and $K_{\text{sym}}$ enforced by both the cMDI2 and MDI Gogny potentials. Their energy dependence is however different at high values of momentum. This, together with the different values for $K_0$, induces modifications of the extracted value for $L$ of about 10 MeV, as can be seen by comparing the results in Section 4.4 and Section 4.2, leaving $K_{\tau}$ almost unaffected. The uncertainty of cMDI2 extracted $K_{\tau}$ is at the upper limit of those quoted in the literature.

The density dependence of SE favored by the full MDI2 is somewhat stiffer and consequently the value of $K_{\tau}$ is closer to zero, the central value lying outside of the 3\(\sigma\) CL interval favored by the most accurate extractions in the literature quoted above. The MDI2 constraint is two times less accurate than the cMDI2 one. Clearly, this is entirely due to lifting the constraint between $L$ and $K_{\text{sym}}$. This conclusion casts some doubt at least on the validity of the reported accuracies for some of the extracted values of $K_{\tau}$ in the literature. More careful analyses need to be performed on whether the so far employed observables are in fact sensitive to $K_{\tau}$ (not just $L$) and that the reported results are not biased by the parametrization used for the symmetry energy.

Up to 1.5\(\rho_0\) the density dependence of the SE is dominated by the slope term. Consequently the constraints extracted using cMDI2 and MDI2 potentials are of comparable accuracies in this region. However, the allowed range for $K_{\text{sym}}$ extracted in this work is clearly not precise enough for the purpose of extrapolating the SE at densities above 1.5\(\rho_0\). The largest contribution to the determined uncertainty originates from experimental data, particularly the FOPI-LAND npeEFR. An improvement of the experimental relative accuracy for this observable, comparable to that achieved by the ASYEOS collaboration for nchEFR, would lead to a decrease of the experimental uncertainty of $K_{\text{sym}}$ to an estimated value of 200 MeV. Further improvements may be possible if heavy-ion collisions are studied experimentally at bombarding energies different from 400 MeV/nucleon.

To investigate for this possibility, the sensitivity of npeEFR and nHEFR to $L$ and $K_{\text{sym}}$ has been determined for $^{197}\text{Au} + ^{197}\text{Au}$ collisions at projectile energies between 150 and 1200 MeV/nucleon and impact parameter $b \leq 7.5$ fm. The results, restricted to the FOPI-LAND geometry and filter, are presented in Fig. 13. The sensitivity of npeEFR to $L$ reaches a maximum around 600 MeV/nucleon and decreases towards both higher and lower energies. In the case of $K_{\text{sym}}$, the maximum of sensitivity of the npeEFR observable is reached close to 250 MeV/nucleon impact energy and then it decreases as $\nu_2$ changes sign at lower impact energies [32]. For nHEFR the sensitivity is seen to increase monotonically towards lower impact energies. This is due to the fact of probing, on average, ever lower subsaturation densities. In this context the correlations of possible non-negligible higher order terms in the Taylor expansion of the SE (see Eq. (1)) with $L$ and $K_{\text{sym}}$ induced by the parametrization chosen for the EoS may lead to a bias in the extracted values for these two parameters. As experimental data will become more accurate this issue will need to be investigated. A similar observation is in order if densities far above saturation are probed as is the case of the npeEFR observable at the upper end of the interval of impact energies.

It has been investigated how this conclusion is affected if, instead of determining the coalescence model parameters from fits to experimental multiplicities, independent values of impact energy and asy-EoS stiffness are chosen for them. To that end the relation $\delta r_{pp}=\delta r_{pp}=\delta r_{nn}=\delta r$ has been enforced. Calculations with values in the ranges $\delta r=2.0-4.0$ fm and $\delta p=0.2-0.3$ GeV/c have been performed. A non-negligible dependence of the sensitivity of EFR to
The definition \( \text{npEFR}(L) \) is the energy for the slope difference of predictions of the chosen observable (npEFR or nhEFR) between a stiff and a soft asy-EoS. For projectile energies, it is defined as \( \text{npEFR}(L) = 80; K_{\text{sym}} = 0 \) scaled by a factor of 0.25, is also shown. Quoted values of \( L \) and \( K_{\text{sym}} \) are in units of MeV.

Fig. 13. (Left panel) Sensitivity of the npEFR (top) and nhEFR (bottom) to the slope \( L \) and curvature \( K_{\text{sym}} \) as a function of projectile kinetic energy for \(^{197}\text{Au} + ^{197}\text{Au} \) collisions. The plotted quantity is defined as the difference of predictions of the chosen observable (npEFR or nhEFR) between a stiff and a soft asy-EoS. For the slope \( L \) it is defined as \( \text{np(h)EFR}(L=120; K_{\text{sym}} = 0) - \text{np(h)EFR}(L=40; K_{\text{sym}} = 0) \). Similarly for the curvature \( K_{\text{sym}} \), the definition \( \text{np(h)EFR}(L=80; K_{\text{sym}} = 600) - \text{np(h)EFR}(L=80; K_{\text{sym}} = -600) \) has been used. The energy dependence of \( \text{np(h)EFR}(L=80; K_{\text{sym}} = 0) \), scaled by a factor of 0.25, is also shown. Quoted values of \( L \) and \( K_{\text{sym}} \) are in units of MeV.

(Right panel) The sensitivity of npEFR (top) and nhEFR (bottom) to the slope \( L \) and curvature \( K_{\text{sym}} \) as a function of projectile kinetic energy for \(^{197}\text{Au} + ^{197}\text{Au} \) collisions defined as ratios of the mentioned observables for a stiff and a soft asy-EoS. The same combinations of \( L \) and \( K_{\text{sym}} \) as for the left panel have been used to determine the plotted ratios.

\( L \) and \( K_{\text{sym}} \) on the values of the coalescence parameters has been observed in the low energy region. However, the conclusions stated above remain qualitatively the same. It thus becomes apparent that experimental measurements of EFR at projectile energies in the neighborhood of 250 MeV/nucleon are best suited for a more precise determination of \( K_{\text{sym}} \). Based on the calculated sensitivities in Fig. 13 and those described in the paragraph above it is estimated that \( K_{\text{sym}} \) could be extracted with an accuracy in the neighborhood of 150 MeV.

This result challenges the expectation that using higher projectile energies is better suited for the extraction of the density dependence of symmetry energy using flow observables. This view is based on the well known fact that the maximum density probed in heavy-ion collisions increases with impact energy, namely from 1.75\( \rho_0 \) at 200 MeV/nucleon to 3.25\( \rho_0 \) at 2.0 GeV/nucleon for the case of Sn+Sn mid-central collisions [33]. The lifetime of the high-density fireball does however decrease quite rapidly towards higher impact energies. The sensitivity of an observable to the high density EoS is therefore the result of these two competing features that have opposite effects.

6 Summary

A QMD transport model that employs a modified momentum dependent interaction (MDI2) potential, supplemented by a phase-space coalescence model fitted to experimental multiplicities has been used to study the density dependence of the symmetry energy above the saturation point by a comparison with experimental neutron-to-proton, neutron-to-hydrogen and neutron-to-charged particles elliptic flow ratios (EFR) in \(^{197}\text{Au} + ^{197}\text{Au} \) heavy-ion collisions at 400 MeV/nucleon impact energy. It has been recognized that for a trustworthy extrapolation of symmetry energy stiffness constraints to high density, both the slope \( L \) and curvature \( K_{\text{sym}} \) parameters need to be determined accurately. Such a study has been suggested to be possible by a calculation, using the same transport model, that has revealed that neutron-to-proton and neutron-to-hydrogen (or alternatively neutron-to-charged particles) EFR probe on average different densities [13] at this impact energy.

To that end, a Gogny interaction inspired MDI potential [16] has been modified, by the addition of an extra density dependent - momentum independent term, to allow for independent variations of the slope \( L \) and curvature \( K_{\text{sym}} \) parameters. The particular choice for this extra term also allows independent modifications of the isovector neutron-proton effective mass difference, whose value is currently a topic of hot debate. The momentum dependent part of the potential has been adjusted to reproduce the empirical nucleon optical potential [64,65,66] following the procedure outlined in Ref. [17].
The final state spectra of heavy-ion collisions have been determined by making use of a minimum spanning tree coalescence algorithm that recognizes all clusters with \( A \leq 15 \) and a few additional ones of higher mass and \( Z \leq 8 \). All clusters with lifetimes larger than 1 ms are considered as stable, the rest are decayed via strong-interaction channels until a stable daughter is reached. The \( r \)-space coalescence parameters are assumed to be isospin dependent and are determined from fits to experimental light cluster multiplicities for each impact energy of interest, up to \( Z \leq 6 \) where available \[48\]. It is observed that multiplicities of free neutrons and protons are systematically overpredicted, those for \(^3\)H, \(^4\)He, \(^{14}\)N, \(^{20}\)Ne, \(^{20}\)O, \(^{27}\)Ar, \(^{36}\)Ar are underpredicted while \(^2\)H, Be, B, C nuclei are generally in good agreement with experiment.

Theoretical transverse and elliptic flows of protons and light clusters (taken separately) generally reproduce well the corresponding FOPI experimental data \[19\]. It is however noticed that theoretical elliptic flows of charged particles systematically under-predict the ASYEOS experimental data. This can be traced back to the underprediction of experimental light-cluster-to-proton multiplicity ratios, a systematic effect that also impacts elliptic flow of hydrogen values. Multiplicative correction factors that amount to approximately 1.1 and 1.075 respectively, with a 5% uncertainty, have been estimated for these two cases. In principle such effects can be avoided by developing or using existing transport models that include light-cluster degrees of freedom \[39\] or compare theoretical and experimental coalescence invariant results \[5\].

The different sensitivity of neutron-to-proton and neutron-to-hydrogen EFR to the value of \( K_{\text{sym}} \) has been proven by fixing the value of the slope \( L \) and varying the curvature in the interval \(-600 \text{ MeV} \leq K_{\text{sym}} \leq 600 \text{ MeV} \). Consequently the following constraint for the density dependence of the symmetry energy has been extracted from a comparison with experimental FOPI-LAND neutron-to-proton and ASYEOS neutron-to-charged particles EFR data

\[
L = 85 \pm 22(\text{exp}) \pm 20(\text{th}) \pm 12(\text{sys}) \text{ MeV} \\
K_{\text{sym}} = 96 \pm 315(\text{exp}) \pm 170(\text{th}) \pm 166(\text{sys}) \text{ MeV} .
\]

Theoretical errors include effects due to uncertainties in the isoscalar part of the equation of state, value of the isovector neutron-proton effective mass splitting, in medium effects on the elastic nucleon-nucleon cross-sections, Pauli blocking algorithm variants and scenario considered for the conservation of the total energy of the system. Systematical uncertainties are generated by the inability of the transport model to reproduce light-cluster-to-proton multiplicity ratios. The extracted value for \( L \) is in agreement with constraints extracted from other studies and of comparable total uncertainty. The value of \( K_{\text{sym}} \) is however imprecise and the extracted symmetry energy cannot be extrapolated accurately above 1.5 saturation density.

It has been shown that a much more precise constraint for \( K_{\text{sym}} \) and the isospin dependent component of the nuclear matter compressibility \( K_{s} \) can falsely be reported if potentials that include potentially unphysical correlations between their parameters are employed. Such existing constraints for the latter quantity should be revisited by studying in more depth the model dependence induced by these type of correlations. The correlation between \( L \) and \( K_{\text{sym}} \), extracted from models based on the Skyrme or Gogny interactions \[20\] is not favoured by the findings of the present study. The difference is however of a rather small statistical significance.

A constraint for \( L \), free of the mentioned systematical uncertainties, can be extracted from the FOPI-LAND neutron-to-proton EFR alone. After correcting for the sensitivity of this observable to \( K_{\text{sym}} \) and considering the same possible model dependence sources as before, the constraint

\[
L = 84 \pm 30(\text{exp}) \pm 19(\text{theor}) \text{ MeV}
\]

is extracted. The experimental error includes also uncertainties due to different possible approaches to analyze the experimental data. The difference with respect to constraints for \( L \) extracted using a previous version of the model \[44\] and the same experimental data have been shown to be the result of an unsatisfactory description of nuclear density profiles, the use of a density cut-off method to determine free nucleon spectra in that study and the isospin dependence of the Pauli blocking algorithm.

Finally, an analysis of the dependence of the sensitivity of EFR to \( K_{\text{sym}} \) on the energy of the projectile nucleus has been performed. It has been shown that, contrary to expectations, a lower impact energy of about 250 MeV/nucleon for \(^{197}\)Au+\(^{197}\)Au heavy-ion collisions is most suitable for constraining the curvature parameter \( K_{\text{sym}} \). This is the result of two competing effects with opposite impact: higher impact energies lead to higher probed central densities but with a lower lifetime of the fireball \[34\]. The maximum sensitivity for \( L \) is reached around 600 MeV/nucleon impact energies, but in this case the energy dependence is weaker than for \( K_{\text{sym}} \).

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### Appendix A

The analytical expressions of the integrals over one and two Fermi spheres that appear in the formulae for the single-particle potential and equation of states respectively for the case of cold nuclear matter are provided in this Appendix. They are needed for fixing the parameters appearing in Eq. \[12\]. The derivations can be performed using
only elementary methods, but are rather tedious. The result for the integral appearing in Eq. (13) reads
\[ I_1(p_F(\tau)) = \int \frac{d^3p'}{1 + (p - p')^2/A^2} \]
\[ = \frac{2\pi}{h^2} A^3 \left[ \frac{A^2 + 2p_F(\tau) - p_F^2(\tau)}{2Ap} \ln \frac{A^2 + [p + p_F(\tau)]^2}{A^2 + [p - p_F(\tau)]^2} + \frac{2p_F(\tau)}{A} + 2 \left( \arctan \frac{p - p_F(\tau)}{A} - \arctan \frac{p + p_F(\tau)}{A} \right) \right]. \] (27)

The analytical expression for the two-Fermi-spheres integral appearing in Eq. (12) is
\[ I_2(p_F(\tau), p_F'(\tau')) = \int \frac{d^3p}{h^2} \frac{d^3p'}{h^2} \frac{f_\tau(r, p) f_\tau'(r', p')}{1 + (p - p')^2/A^2}, \]
\[ = 16\pi^2 A^3 \left[ \frac{1}{2} \frac{p_F(\tau)p_F'(\tau')}{A} \left( \frac{p_F^2(\tau) + p_F^2(\tau') - 1}{A^2} \right) \right] \]
\[ + \frac{2}{3} \frac{p_F^2(\tau) - p_F^2(\tau')}{A} \arctan \frac{p_F(\tau) - p_F(\tau')}{A} \]
\[ - \frac{2}{3} \frac{p_F^2(\tau) + p_F^2(\tau')}{A} \arctan \frac{p_F(\tau) + p_F(\tau')}{A} \]
\[ + \left\{ \frac{1}{8} \frac{[p_F^2(\tau) - p_F^2(\tau')]^2}{A^2} + \frac{1}{4} \frac{[p_F^2(\tau) - p_F^2(\tau')]^2}{A^2} \right\} \ln \frac{A^2 + [p_F^2(\tau) - p_F^2(\tau')]^2}{A^2 + [p_F(\tau) - p_F(\tau')]^2}. \] (28)

Owing to the symmetry with respect to the interchange of isospin labels of the integral in Eq. (28), it can easily be shown that there are no contributions to the equation of state from terms proportional to odd powers of the isospin asymmetry \( \beta \) provided that the interaction is charge symmetric \((C_1,1=C_{-1},-1)\) since the following identities hold true
\[ S^{(2n+1)}_{1,1} = 0, \]
\[ S^{(2n+1)}_{-1,-1} = 0, \]
\[ S^{(2n+1)}_{1,-1} + S^{(2n+1)}_{-1,1} = 0. \] (29)

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