Compact difference schemes for weakly-nonlinear parabolic and Schrödinger-type equations and systems

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Abstract

The implicit compact finite-difference scheme was developed for evolutionary partial differential parabolic and Schrödinger-type equations and systems with a weak nonlinearity. To make a temporal step of the compact implicit scheme we need to solve a non-linear system. We use for this step a simple explicit difference scheme and then Newton – Raphson iterations, which are implemented by the double-sweep method. Numerical experiments confirm the 4-th order of an algorithm. The Richardson extrapolation improves it up to the 6-th order.

Key words: compact difference scheme, parabolic equation, Schrödinger-type equation, high-order scheme, weak non-linearity, Newton – Raphson iterations, Richardson extrapolation

1 Introduction

Weakly nonlinear partial differential equations and systems (parabolic and Schrödinger-like types) may describe a wide spectrum of physical, physiological, ecological, genetic, etc phenomena [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

We propose here an implicit compact finite-difference scheme for the approximation and numerical solution of such kind of models. The numerical scheme is effective for the mixed initial-boundary problems and provides the 4-th order. Earlier we considered such compact difference schemes for the finite-difference approximation of the linear differential equations with a constant [16] and variable [17, 18, 19] coefficients.

Here we develop an approach to the approximation of weak non-linear partial differential equations and systems

\[ \partial_t u = P(\partial_x)u + F(x, u), \]

where \(P(\partial_x)\) is a linear differential operator, and \(F\) is a given function, which is smooth with respect to both arguments. There are scalar and vector versions of the model, where the unknown function \(u\) is a scalar or vector function and \(P\) is a scalar or a matrix operator. For the scalar version we consider here the case

\[ P = \alpha \partial_x^2. \]

We are based here on the coefficients of the compact schemes for the diffusion equation and Schrödinger equation [16]. However, such kind of compact scheme (with high approximation

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order) for other linear equations and systems, which are correct in the sense of I.G. Petrovskii, may be found, and then they may be modified for the weak non-linear models.

We consider here only the Dirichlet boundary conditions on both ends of the segment. However, the construction of compact scheme can be modified weakly for other boundary conditions without order loss, see [19].

The compact scheme is implicit, and we need to inverse a matrix for every temporal step. To modify the approach for the case of non-linear equations and systems, where we need to solve at any temporal step a non-linear system, we first use a simple explicit scheme and obtain a first guess of its solution. Then we do several Newton–Raphson iterations to solve the original non-linear system with a good accuracy. The considered scheme is economical, because the Jacobi matrix is tridiagonal, and we can use the double-sweep method for its inversion.

Our numerical experiments confirm a high accuracy of such approach. The Richardson extrapolation method further improves the results and provides the 6-th order error decrease.

## 2 Fisher–Kolmogorov–Petrovsky–Piskunov (FKPP) model

The Fisher–Kolmogorov–Petrovsky–Piskunov equation

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + \phi(u), \quad D = \text{const} > 0, \quad \phi \in C^2
\]  

(1)

describes gene propagation [1, 2, 5]. Here we take non-linear function: \( \phi(u) = u(1-u) \) with the values of initial data \( u(0, x) = u_0(x) \) with values on the segment \( [0,1] \), e.g.: \( u_0(x) = \cos(x), x \in [0, \pi/2] \). We will approximate non-linear partial differential equation (1) on the grid \( x_j = jh, \ h = \pi/2N, \ j = 0, \ldots, N; \ t = n\tau, \ \tau \) is a temporal step; \( u^n_j \approx u(n\tau, x_j) \), \( u^n_0 = u^n_N = 0 \).

First, we approximate this equation by using the following implicit one-layer 4-th order compact difference scheme for the non-homogeneous linear diffusion equation:

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(x), \quad D = \text{const} > 0,
\]

(2)

for which we have the following representation [16, 19]:

\[
a_0(u^{n+1}_{j-1} + u^{n+1}_{j+1}) + b_0u^{n+1}_j + a_1(u^n_{j-1} + u^n_{j+1}) + b_1u^n_j = p_0(f^{n+1}_{j-1} + f^{n+1}_{j+1}) + q_0f^{n+1}_j + p_1(f^n_{j-1} + f^n_{j+1}) + q_1f^n_j, \quad j = 1, \ldots, N - 1.
\]

(3)

Here we use the following coefficients: \( a_0 = 2(6\nu - 1); b_1 = -2(6\nu + 1); b_0 = -4(6\nu + 5); b_1 = 4(6\nu - 5); p = p_0 = p_1 = -\tau; q = q_0 = q_1 = -10\tau \) to provide the 4-th order.

If we substitute the function \( \phi(u) \) into equation (2) instead of \( f(x) \), then we will get the following system of non-linear equations:

\[
a_0(u^{n+1}_{j-1} + u^{n+1}_{j+1}) + b_0u^{n+1}_j = a_1(u^n_{j-1} + u^n_{j+1}) + b_1u^n_j + p(\phi(u^n_{j-1}) + \phi(u^n_{j+1}) + \phi(u^{n+1}_j) + \phi(u^{n+1}_{j+1})) + q_0(\phi(u^{n+1}_{j-1}) + \phi(u^n_j)).
\]

(4)

To make a temporal step one need to solve the system of non-linear equations with respect to unknown values \( \{u^{n+1}_j\}_{j=1}^{N-1} \).
2.1 The Explicit Euler scheme as the first guess

To solve numerically non-linear system (4) we can use the following algorithm:

- Use an explicit Euler scheme to obtain a rough estimate of unknown values \( u_j^{n+1} \):
  \[
  \tilde{u}_j^{n+1} = u_j^n + \nu(u_{j-1}^n - 2u_j^n + u_{j+1}^n) + \tau \phi(u_j^n).
  \]

- Apply Newton – Raphson method and linearize system (4). We represent the solution in the form \( u_j^{n+1} = \tilde{u}_j^{n+1} + \delta_j \) and neglect by the terms \( O(\delta_j^2) \) in the Taylor formula for \( \phi(u) \) for any \( j \). Thus we obtain the system of linear algebraic equations with respect to \( \{\delta_j\}_{j=1}^{N-1} \):

  \[
  \begin{align*}
  & [a_0 - p\phi'(\tilde{u}_j^{n+1})] \delta_{j+1} + [b_0 - q\phi'(\tilde{u}_j^{n+1})] \delta_j + [a_0 - p\phi'(\tilde{u}_{j-1}^{n+1})] \delta_{j-1} = \\
  & [a_1 u_j^{n+1} + b_1 u_j^n + a_1 u_{j-1}^n] - [a_0 \phi(\tilde{u}_j^{n+1}) + b_0 \phi(\tilde{u}_j^{n+1}) + a_0 \phi(\tilde{u}_{j-1}^{n+1})] + \\
  & p[\phi(\tilde{u}_j^{n+1}) + \phi(u_{j+1}^n) + \phi(u_{j-1}^n) + \phi(u_j^n)] + q[\phi(\tilde{u}_j^{n+1}) + \phi(u_{j-1}^{n+1})].
  \end{align*}
  \]

- If for a given small \( \delta > 0 \) the inequality \( |\delta_j| \leq \delta \) is fulfilled for all \( j \), we finish this temporal step by putting:
  \[
  u_j^{n+1} = \tilde{u}_j^{n+1} + \delta_j.
  \]

Otherwise we define the following guess

\[
\tilde{u}_j^{n+1} = u_j^{n+1} + \delta_j
\]

and repeat the previous step of the algorithm.

2.2 Adams explicit scheme as the first guess

We can use another approach for the first part of the algorithm:

- Use an explicit Euler scheme to obtain a rough estimate of the values \( u_j^{n+1/2} \):
  \[
  \tilde{u}_j^{n+1/2} = u_j^n + \nu(u_{j-1}^n - 2u_j^n + u_{j+1}^n)/2 + \tau \phi(u_j^n)/2.
  \]

- Then use an explicit central difference scheme to obtain a rough estimate of the values \( u_j^{n+1} \):
  \[
  \tilde{u}_j^{n+1} = u_j^n + \nu(\tilde{u}_{j-1}^{n+1/2} - 2\tilde{u}_j^{n+1/2} + \tilde{u}_{j+1}^{n+1/2}) + \tau \phi(u_j^{n+1/2}).
  \]

- Then we apply the Newton – Raphson method to linearize system (4), and do several iterations (5) until the values \( \{\delta_j\}_{j=1}^{N-1} \) will be small enough.
3 Parabolic weakly-nonlinear system

The described method could be generalized for solution of weak non-linear parabolic systems [5, 6, 7, 8, 9]. If the matrix $\alpha$ has a simple structure, we can reduce such system to the following quasi-diagonal form:

\[
\begin{cases}
\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + \phi_1(u, w); \\
\frac{\partial w}{\partial t} = D_2 \frac{\partial^2 w}{\partial x^2} + \phi_2(u, w),
\end{cases}
\]

$D_1, D_2 > 0$.  

As one of the examples of a weak non-linear parabolic quasi-diagonal system, we consider here the FitzHugh – Nagumo model of biological neuron [14]. Here $u$ is a membrane voltage and $w$ is a recovery variable:

\[
\phi_1(u, w) = \epsilon(w - \alpha u - \beta), \quad \phi_2(u, w) = -(u - \mu w + w^3), \quad \alpha, \beta, \epsilon > 0; \mu \in \mathbb{R}
\]

We solve this system with the Dirichlet boundary conditions. We can obtain the first guess for the solution on the $(n + 1)$-th time step by using the Euler scheme. Fixing the unitless parameter $\nu$ (e.g. taking $\nu = \tau h^{-2} \max(D_1, D_2)$), we have:

\[
\begin{align*}
\tilde{u}_{n+1}^0 &= u^n + \nu Mu^n + \tau \phi_1(u^n, w^n); \\
\tilde{w}_{n+1}^0 &= u^n + \nu Mw^n + \tau \phi_2(u^n, w^n),
\end{align*}
\]

where $M$ is a tridiagonal matrix with $-2$'s on a main diagonal and 1's on side ones. Then we use the compact difference scheme and apply the Newton – Raphson method (here $A_{new}, A_{old}, B$ are the tridiagonal matrices of the compact difference scheme (3)):

\[
\begin{align*}
F(u^{n+1}, w^{n+1}) &= A_{new}u^{n+1} + A_{old}u^n - B(\phi_1(u^{n+1}, w^{n+1}) + \phi_1(u^n, w^n)) = 0; \\
G(u^{n+1}, w^{n+1}) &= A_{new}w^{n+1} + A_{old}w^n - B(\phi_2(u^{n+1}, w^{n+1}) + \phi_2(u^n, w^n)) = 0,
\end{align*}
\]

Let us represent the solution in the form $u^{n+1} = \tilde{u}^{n+1} + \delta_u$, $w^{n+1} = \tilde{w}^{n+1} + \delta_w$, and linearize the system:

\[
\begin{align*}
F(\tilde{u}^{n+1}, \tilde{w}^{n+1}) + \delta_u \frac{\partial F}{\partial u} + \delta_w \frac{\partial F}{\partial w} &= 0; \\
G(\tilde{u}^{n+1}, \tilde{w}^{n+1}) + \delta_u \frac{\partial G}{\partial u} + \delta_w \frac{\partial G}{\partial w} &= 0.
\end{align*}
\]

After the linearization and a simplification, we obtain the following liner system:

\[
\begin{align*}
\Sigma_u + (A_{new} - \frac{\partial \phi_1}{\partial u} |_{(\tilde{u}^{n+1}, \tilde{w}^{n+1})})B\delta_u - \frac{\partial \phi_1}{\partial w} |_{(\tilde{u}^{n+1}, \tilde{w}^{n+1})}B\delta_w &= 0; \\
\Sigma_w + (A_{new} - \frac{\partial \phi_2}{\partial w} |_{(\tilde{u}^{n+1}, \tilde{w}^{n+1})})B\delta_w - \frac{\partial \phi_2}{\partial u} |_{(\tilde{u}^{n+1}, \tilde{w}^{n+1})}B\delta_u &= 0,
\end{align*}
\]

where

\[
\begin{align*}
\Sigma_u &= A_{new}u^{n+1} + A_{old}u^n - B(\phi_1(u^{n+1}, w^{n+1}) + \phi_1(u^n, w^n)); \\
\Sigma_w &= A_{new}w^{n+1} + A_{old}w^n - B(\phi_2(u^{n+1}, w^{n+1}) + \phi_2(u^n, w^n)).
\end{align*}
\]

4 Nonlinear Schrödinger equation

This is one of most famous non-linear partial differential equations, see e.g. [4]. NLSE describes many physical phenomena, e.g. in the plasma physics and the non-linear optics. The equation can be interpreted as a infinite dimensional Hamiltonian system, and the system is fully
integrable; there are explicit soliton-like solutions of NLSE, see e.g. [10]. The non-linearity in the equation is smooth in the real sense, but it is not analytical with respect to the unknown complex-valued function $\psi(t, x)$:

$$i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + \beta |\psi|^2 \psi = 0. \quad (9)$$

It can be rewritten as the following system of PDEs, which is similar to system (6):

$$u = \text{Re}(\psi), \; w = \text{Im}(\psi), \; \phi_1(u, w) = -\beta(w^3 + wu^2), \; \phi_2(u, w) = \beta(w^3 + uw^2).$$

We will use the following tentative solution (soliton) for equation (9):

$$\psi = \sqrt{2\alpha\beta^{-1}} \cosh \sqrt{\alpha}(x - Ut) \exp(-0.5iUx + i(r^2 - \alpha)t)), \quad (10)$$

where $\alpha$ and $U$ are the soliton’s parameters.

We repeat here an aforementioned compact scheme approach for the NLSE: a simple explicit scheme is used to obtain a first guess on a temporal step and then the iterations according to the Newton – Raphson method that are implemented by the double-sweep algorithm. There is a deep difference with system (6): the spectrum of the operator (1) is imaginary here against real and negative spectrum of system (6). However, our numerical approach works for both systems.

This approach also may be developed similarly for the Ginzburg – Landau equation [10, 11] and the Gross – Pitaevsky equation [3, 15].

5 Numerical experiments

In order to examine the properties of the general approach for solving the problems (1, 6, 9), we conducted a series of numerical experiments. Note that for two first problems we do not have analytic solutions, so we compared obtained solutions with ones calculated on a very fine mesh.

Numerical experiments confirm the 4-th order of the approach, see Fig. 2, 4 and Table 2. The Richardson extrapolation method improve the order up to the 6-th, see Fig. 3, 6.

5.1 Accuracy and approximation order

We will use below for the evaluations the Chebyshev norm

$$\|u(x)\|_C = \max_{x \in [0, L]} |u(x)|,$$

but the similar estimations were obtained in the $L^2$-norm.

We integrate evolutionary equations and systems and compare exact and approximate solutions of the boundary problem in a time moment $T$. We choose the moment by the following criteria:

- the difference between solutions of the considered problem in the moments $t = 0$ and $t = T$ is essential;
- in the moment $t = T$ our solution is far from stationary one yet.
5.1.1 FKPP

Numerical experiments on the tentative solutions of FKPP equation (1) at \( u_0 = \cos^2(x) \), \( \phi(u) = u(1 - u) \) are represented for several time moments \( T \) on the Fig.1. They confirmed 4-th order of the compact scheme (3) for both Euler- and Adams-based versions, see Fig.2.

Adams-based algorithm provides more accurate solutions without Newton – Raphson iterations, see Table 1; it also more preferable for the case of small number of spatial grid nodes, see 2. However, both first guess versions demonstrate fast convergence to an accurate solution, see the results of experiments with a fixed threshold \( \delta \) from (5), see Table 2.

\[
\begin{align*}
\text{Initial guess} & \quad \text{Initial condition} & \quad \text{0 iter.} & \quad \text{1 iter.} & \quad \text{2 iter.} \\
\text{Euler} & \quad \cos^2(x) & \quad 6.46 \times 10^{-5} & \quad 7.97 \times 10^{-5} & \quad 7.36 \times 10^{-5} \\
\text{Adams} & \quad \cos^2(x) & \quad 3.19 \times 10^{-5} & \quad 8.00 \times 10^{-5} & \quad 7.36 \times 10^{-5} \\
\text{Euler} & \quad 0.19 + (0.9 - x)^2 \cos(x) & \quad 8.57 \times 10^{-5} & \quad 8.76 \times 10^{-5} & \quad 7.57 \times 10^{-5} \\
\text{Adams} & \quad 0.19 + (0.9 - x)^2 \cos(x) & \quad 3.66 \times 10^{-5} & \quad 6.97 \times 10^{-5} & \quad 7.57 \times 10^{-5} \\
\end{align*}
\]

Figure 1: Solution of FKPP eq. (1) for various integration times \( T \), \( D = 0.01, \nu = 0.3, N = 128, u_0 = \cos^2(x), \phi(u) = u(1 - u) \).

Table 1: Error in C-norm for a different number of iterations for FKPP eq. (1), \( D = 1, \nu = 0.5, N = 64, T = 0.0771, \phi(u) = u(1 - u) \).

\[
\begin{align*}
\delta & \quad \text{Init. guess} & \quad N = 8 & \quad N = 16 & \quad N = 32 & \quad N = 64 & \quad N = 128 & \quad N = 256 & \quad \text{Order} \\
10^{-2} & \quad \text{Euler} & \quad 3.38 \times 10^{-4} & \quad 2.23 \times 10^{-5} & \quad 1.40 \times 10^{-6} & \quad 8.76 \times 10^{-8} & \quad 5.48 \times 10^{-9} & \quad 3.41 \times 10^{-10} & \quad 3.975 \\
10^{-2} & \quad \text{Adams} & \quad 2.62 \times 10^{-4} & \quad 1.77 \times 10^{-5} & \quad 1.11 \times 10^{-6} & \quad 6.97 \times 10^{-8} & \quad 4.36 \times 10^{-9} & \quad 2.72 \times 10^{-10} & \quad 3.983 \\
10^{-6} & \quad \text{Euler} & \quad 2.93 \times 10^{-4} & \quad 1.93 \times 10^{-5} & \quad 1.21 \times 10^{-6} & \quad 8.44 \times 10^{-8} & \quad 5.46 \times 10^{-9} & \quad 3.41 \times 10^{-10} & \quad 4.008 \\
10^{-6} & \quad \text{Adams} & \quad 2.93 \times 10^{-4} & \quad 1.93 \times 10^{-5} & \quad 1.21 \times 10^{-6} & \quad 7.07 \times 10^{-8} & \quad 4.37 \times 10^{-9} & \quad 2.72 \times 10^{-10} & \quad 3.942 \\
10^{-10} & \quad \text{Euler} & \quad 2.93 \times 10^{-4} & \quad 1.93 \times 10^{-5} & \quad 1.21 \times 10^{-6} & \quad 7.57 \times 10^{-8} & \quad 4.73 \times 10^{-9} & \quad 2.95 \times 10^{-10} & \quad 3.984 \\
10^{-10} & \quad \text{Adams} & \quad 2.93 \times 10^{-4} & \quad 1.93 \times 10^{-5} & \quad 1.21 \times 10^{-6} & \quad 7.57 \times 10^{-8} & \quad 4.73 \times 10^{-9} & \quad 2.95 \times 10^{-10} & \quad 3.984 \\
\end{align*}
\]
Figure 2: Logarithm of the error in C-norm for both versions of the algorithm and for the various numbers (0 2) of iterations on the tentative solution of the FKPP equation (1) \( D = 1, \nu = 0.5, T = 0.0771, u_0 = 0.19 + (0.9 - x)^2\cos(x), \phi(u) = u(1 - u) \). For this tentative solution, 2 iterations are sufficient for convergence.

We can also use Richardson extrapolation technique to improve the order by calculating the solution on a finer grid. If we obtain a family of approximate solutions \( u = u_h(t, x) \) at \( t = T \) and \( \tau = h^2|\nu^j|/\max_j \theta_j \), and use the representation

\[
u_h(T, x) = u(T, x) + h^4u_*(T, x) + o(h^4), \tag{11}
\]

then we can calculate \( u_h \) at \( h = h_* \) and at \( h = h_*/2 \). After that we substitute the approximation into (11), neglect by terms \( o(h^4) \) and obtain the following algebraic system for two functions \( u \) and \( u_* \):

\[
u_{h_*} = u + h_*^4u_*, u_{h_*/2} = u + h_*^4u_*/16 \rightarrow u = u(T, x) \approx [16u_{h_*/2} - u_{h_*}] / 15.
\]

Numerical experiments show approximation order increase together with dramatic error decrease, see Fig. 3. Thus, the Euler-based compact scheme with 1 Newton Raphson iteration and with Richardson extrapolation gives us a very exact and economical algorithm for FKPP equation.

5.1.2 FitzHugh – Nagumo system

Numerical experiments on FitzHugh – Nagumo system (7) show the 4-th order of the Euler-based compact difference scheme (3), see Fig. 4. We measured the order for the both components of solutions \( u \) and \( w \).

In case of a system (7) our algorithm converges slower because here we have block tridiagonal matrix used in the Newton – Raphson iterations, and the system size is 2\( N \) instead of \( N \) in case of the FKPP equation (1). It also requires more iterations as the unitless Courant parameter \( \nu \) grows.
Figure 3: Logarithm of the error in C-norm for various algorithms and number of iterations (0 – 3) on tentative solution $D = 1, \nu = 0.5, T = 0.0771, u_0 = 0.19 + (0.9 - x)^2 \cos(x), \phi(u) = u(1 - u)$ of FKPP equation (1) using Richardson extrapolation. High order enables error to hit the precision of $10^{-12}$ for $N = 64$.

Figure 4: Logarithm of the error in C-norm on tentative solution $u(0, x) = \sin(x), w(0, x) = \sin^2(x), x \in [0, 2\pi], \mu = \alpha = \epsilon = 2, \beta = D_1 = D_2 = 1, T = 2, \nu = 0.1$ of FitzHugh – Nagumo system (7). At every time step, we did sufficient for convergence ($\max|\delta| < 10^{-12}$) number of Newton – Raphson method iterations; however, one may relax the restrictions on $\max|\delta|$ up to $10^{-4}$, since it do not affect the solution much.

### 5.1.3 Nonlinear Schrödinger equation

Compact difference scheme (3) may be used to solve non-linear Schrödinger equation (9), too. Even when it is rewritten as the system of PDEs, this problem is completely different from (6), because the spectrum of the problem is imaginary compared to spectrum of parabolic system.
It is possible to overcome the complexity of $\psi$ in (9) with real $u$ and $v$: one should split the matrices in (8) into their real and imaginary parts and apply them to $u$ and $w$, correspondingly.

Numerical experiments confirm the 4-th order for NLSE equation, see Fig.5. The Richardson extrapolation technique is also applicable here, and it improves the order up to the 6-th, see Fig.6.

5.2 Stability and Efficiency

A criterion of stability is very important for various finite-difference schemes usage and comparison. Usually the criterion bounds its unitless Courant number $\nu = D\tau h^{-2}$. Our compact scheme is unconditionally stable for linear diffusion equation [16]. During our numerical experiments, we did not experienced stability problems even for large values of the Courant parameter $\nu \approx 100$.

However, one may experience the convergence issues for the large values of $\tau$: for instance, for the problem (7), there was no convergence after 10000 Newton – Raphson iterations for typical parameters of the problem: $\epsilon = \alpha = \mu = 2$, $\beta = D_1 = D_2 = 1$ for $\tau \approx 0.77$. This corresponds to the large values of $\nu^*$: for $N = 64$, the critical value of $\nu^* \approx 78$. This behaviour does not depend on the version of the algorithm (Adams- and Euler-based). When we use the Courant number about the critical value $\nu^*$ we need in the large number of Newton – Raphson iterations, i.e. it is not economical approach.

But what is an optimal relation between steps $\tau$ and $h$? We conducted a series of numerical experiments to estimate it. We obtained an empirical restriction on the unitless parameter of the problem:

$$\Lambda = \frac{\tau^2 D}{h^2 T} \approx 0.15 - 0.20,$$

where $T$ is typical integration time.

Let us estimate the number of arithmetic operations (multiplications only) per node of the temporal-spatial grid. Let $N$ be the number of spatial nodes of the mesh, $M \gg 1$ - the number of temporal steps, $I$ - the number of Newton – Raphson iterations (see sections 2.1, 2.2). Then, in assumption that we can calculate both $\phi(u), \partial_u \phi(u)$ using $N$ multiplications, we can estimate the number of calculations (multiplications) $C$ with an accuracy up to $O(N)$. The number of operations needed per one point of the mesh in given in a Table 3.
Table 3: Number of calculations $C^*$ per temporal-spatial grid node for various algorithms for FKPP eq. (1). Number of Newton – Raphson iterations $I$ per node does not depend on the first guess choice.

| Algorithm / First guess | Euler | Adams |
|------------------------|-------|-------|
| No iterations          | 4     | 8     |
| $I$ iterations         | $10 + 16I$ | $14 + 16I$ |

6 Summary and discussion

We developed an approach for the high-order approximation of weakly non-linear PDEs and systems. The approach was tested on the non-linear Schrödinger equation, the Fisher – Kolmogorov – Petrovsky – Piskunov equation and the FitzHugh – Nagumo model. Numerical experiments confirmed the 4-th order of the approach. We also show that our algorithm may be combined with the Richardson extrapolation technique, further improving the order up to the 6-th.

During the numerical testing of our approach, we did not experienced stability issues, i.e. it seems to be unconditionally stable. However, one may experience the convergence problems for the Newton – Raphson iterations while using large temporal steps. The recommendation on efficient use of our algorithm is also included into the text.

We have considered here the Dirichlet boundary conditions. However, the compact schemes may be applied for other boundary conditions. The function $f$ and its derivatives must be included into the difference boundary conditions to avoid loss of the order.

The article was prepared within the framework of the Academic Fund Program at the National Research University Higher School of Economics (HSE) in 2016-2017 (grant No. 16-05-0069) and by the Russian Academic Excellence Project ”5-100”.

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