Rigorous Dynamics of Expectation-Propagation-Based Signal Recovery from Unitarily Invariant Measurements

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Abstract—This paper investigates sparse signal recovery based on expectation propagation (EP) from unitarily invariant measurements. A rigorous analysis is presented for the state evolution (SE) of an EP-based message-passing algorithm in the large system limit, where both input and output dimensions tend to infinity at an identical speed. The main result is the justification of an SE formula conjectured by Ma and Ping in 2016.

I. INTRODUCTION

Consider the $N$-dimensional signal recovery from compressed, linear, and noisy measurements $y \in \mathbb{C}^M \, (N \geq M)$,

$$y = Ax + w, \quad w \sim \mathcal{CN}(0, \sigma^2 I_M),$$

where $x \in \mathbb{C}^N$ and $A \in \mathbb{C}^{M \times N}$ denote a sparse signal vector and a measurement matrix, respectively. The goal is to estimate the unknown signals $x$ from the knowledge about the measurement vector $y$ and matrix $A$, as well as about the statistics of all random variables. Throughout this paper, we postulate the following mild assumptions:

Assumption 1: The signal vector $x$ has independent and identically distributed (i.i.d.) zero-mean non-Gaussian elements with unit variance and finite fourth moments.

Assumption 2: The Gram matrix $AA^H$ is unitarily invariant. Furthermore, the empirical eigenvalue distribution of $AA^H$ converges almost surely to a deterministic distribution $\rho(\lambda)$ with finite fourth moments in the large system limit, where both $M$ and $N$ tend to infinity while the compression rate $\delta = M/N \in (0, 1]$ is kept constant.

For an i.i.d. Gaussian matrix $A$—satisfying Assumption 2— the approximate message passing (AMP) was proved in [3] to be asymptotically Bayes-optimal when the compression rate $\delta$ is larger than the so-called belief-propagation (BP) threshold $\delta^\text{BP}$. However, it has been recognized that the original AMP fails to converge for non-i.i.d. measurement matrices [4], [5]. To solve this limitation, several message-passing algorithms have been proposed on the basis of expectation propagation (EP) [7], the expectation-consistent approximation [8], [9], or of the turbo principle [10]–[12]. These algorithms are essentially the same as each other, with the only exception of [8]. In this paper, the algorithms are referred to as EP-based algorithms, since Céspedes et al. [2] first proposed this type of algorithms, to the best of author’s knowledge.

The main advantage of EP-based algorithms is that they are asymptotically Bayes-optimal for unitarily invariant measurement matrices. This claim was conjectured in [11], by proposing state evolution (SE) equations of the EP-based algorithms based on two heuristic assumptions, and by investigating the properties of the SE equations. However, the rigorous justification of the conjecture is still open. The purpose of this paper is to prove the conjecture by presenting a rigorous derivation of the SE equations.

Notation: The notation $o(1)$ denotes a vector with almost surely vanishing Euclidean norm in the large system limit. For a matrix $M \in \mathbb{C}^{M \times N}$, the singular-value decomposition (SVD) of $M$ is written as $M = \Phi_M(\Sigma_M, O)\Psi_M^H$ for $M \leq N$. The unitary matrix $\Psi_M = (\Psi_M, \Phi_M^\perp)$ is divided into two parts that correspond to the non-zero and zero singular values, respectively. For $M > N$, we have $M = \Phi_M(\Sigma_M, O)^T \Psi_M^H$ and $\Phi_M = (\Phi_M, \Phi_M^\perp)$.

When $M$ is full rank, the pseudo-inverse of $M$ is denoted by $M^\dagger = M^H(MM^H)^{-1}$ for $M \leq N$ and by $M^\dagger = (MM^HM)^{-1}M^H$ for $M > N$. Furthermore, $P_M^\perp = M^\dagger M$ for $M > N$ represents the orthogonal projection onto the space spanned by the columns of $M$, while $P_M = I_M - P_M^\perp$ denotes the projection onto the orthogonal complement.

II. EXPECTATION PROPAGATION

We start with an EP-based algorithm [7], [11], which is message passing between two modules—called modules $A$ and $B$. Module $A$ computes the extrinsic mean $x_{A \rightarrow B}^t$ and covariance $v_{A \rightarrow B}^t I_N$ of $x$ in iteration $t$,

$$x_{A \rightarrow B}^t = x_{B \rightarrow A}^t + \gamma(v_{B \rightarrow A}^t)W^t(y - Ax_{A \rightarrow B}^t),$$

$$v_{A \rightarrow B}^t = \gamma(v_{B \rightarrow A}^t) - v_{B \rightarrow A}^t \equiv \varphi_{A \rightarrow B}(v_{B \rightarrow A}^t),$$

where $x_{A \rightarrow B}^t$ and $v_{B \rightarrow A}^t I_N$ denote prior mean and covariance of $x$ provided from module $B$, while $x_{B \rightarrow A}^0 = 0$ and $v_{B \rightarrow A}^0 = 1$ are used in the initial iteration. In (4), the linear minimum mean-square error (LMMSE) filter $W^t$ is given by

$$W^t = A^H(\sigma^2 I_M + v_{B \rightarrow A}^t AA^H)^{-1}.$$
Furthermore, the function $\gamma(v_{B-A}^t) \equiv \gamma_t$ is defined as
\[
1 \gamma(v_{B-A}^t) = \lim_{M = \delta N \to \infty} \frac{\text{Tr}(W^t A)}{N} \Rightarrow \int \delta \lambda d \rho(\lambda), \quad (5)
\]
due to Assumption 2. As proved in Section IV, $\gamma_t$ eliminates dependencies between estimation errors in the two modules.

On the other hand, module B computes the posterior mean $\tilde{\eta}_t(x_{A-B}^t)$ and variance MMSE($v_{A-B}^t$) $= N^{-1} E[\|x - \tilde{\eta}_t(x_{A-B}^t)||^2] x_{A-B}^t$ of $x$, by regarding the message $x_{A-B}^t$ as the additive white Gaussian noise (AWGN) observation of $x$.

\[
x_{A-B}^t = x + \omega^t, \quad \omega^t \sim \mathcal{CN}(0, v_{A-B}^t I_N). \quad (6)
\]

If a termination condition is satisfied, module B outputs $\tilde{\eta}_t(x_{A-B}^t)$ as an estimate of $x$. Otherwise, the extrinsic mean $x_{B-A}^{t+1}$ and covariance $v_{B-A}^{t+1}$ are fed back to module A.

\[
x_{B-A}^{t+1} = v_{B-A}^{t+1} \left( \frac{\tilde{\eta}_t(x_{A-B}^t)}{\text{MMSE}(v_{A-B}^t)} - x_{A-B}^t \right) = \eta_t(x_{A-B}^t),
\]

\[
v_{B-A}^{t+1} = \frac{1}{\text{MMSE}(v_{B-A}^t)} - v_{B-A}^t = \varphi_{B-A}(v_{B-A}^t). \quad (7)
\]

The following lemma is used to prove that $\gamma_t$ eliminates dependencies between estimation errors in the two modules.

**Lemma 1 (Ma and Ping [11]):** Let $z^t \sim \mathcal{CN}(0, v_{A-B}^t I_N)$ denote an independent circularly symmetric complex Gaussian random vector with covariance $v_{A-B}^t$. Then,

\[
\lim_{\|z\| \to 0} \mathbb{E} [z^H \tilde{\eta}_t(x + \epsilon + z^t)] = 0, \quad (9)
\]

\[
\lim_{\|z\| \to 0} \mathbb{E} [z^H \tilde{\eta}_t(\tilde{x}_{A-B}^t + z^t)] = \text{NMSE}(v_{A-B}^t). \quad (10)
\]

### III. MAIN RESULT

The following theorem is the main result of this paper, which describes the rigorous dynamics of the mean-square error (MSE) for the estimate $\tilde{\eta}_t(x_{A-B}^t)$ of the EP-based algorithm in the large system limit.

**Theorem 1:** Define $\text{mse}_{A-B} = \varphi_{A-B}(\text{mse}_{A-B}^t)$ and $\text{mse}_{B-A} = \varphi_{B-A}(\text{mse}_{B-A}^t)$. Then, the instantaneous MSE $\text{mse}_t = \lim_{M = \delta N \to \infty} N^{-1} \|x - \tilde{\eta}_t(x_{A-B}^t)||^2$ for the EP-based algorithm converges almost surely to MMSE($\text{mse}_{A-B}^t$) in iteration $t$.

Theorem 1 was originally conjectured in [11], and implies that the EP-based algorithm predicts the exact dynamics of the extrinsic variances in the large system limit. The fixed-points (FPs) of the SE equations were proved in [11] to correspond to those of an energy function that describes the Bayes-optimal performance—derived in [13] via a non-rigorous tool in statistical physics. Thus, the Bayes-optimal performance derived in [13] is achievable when the SE equations have a unique FP, or equivalently when the compression rate $\delta$ is larger than the BP threshold [4].

Let us prove Theorem 1. We first formulate the estimation errors of the EP-based algorithm. Let $h_t = x - x_{A-B}^t$ and $q_t = x - x_{B-A}^t$ denote the estimation errors in modules A and B, respectively. From [1], [2], [7], and the SVD $A = U(\Sigma, O)V^H$, we obtain the error recursions with the initial condition $q_0 = x$.

\[
m_t = b_t - \gamma_t \tilde{W}_t \{O, b_t + \hat{w}\}, \quad b_t = V^H q_t, \quad (11)
\]

\[
q_{t+1} = q_0 - \eta_t(q_0 - h_t), \quad h_t = V m_t, \quad (12)
\]

with $\hat{w} = U^H w$. In (11), the linear filter $\tilde{W}_t$ is given by

\[
\tilde{W}_t = (\Sigma, O)^H (\sigma^2 I_M + v_{B-A}^t \Sigma)^{-1} \quad (13)
\]

We next introduce several notations to present a general theorem, of which a corollary is Theorem 1. Let $Q_t = (q_0, \ldots, q_{t-1}) \in \mathbb{C}^{N \times t}$. The matrices $B_t, M_t \in \mathbb{C}^{N \times t}$, and $H_t \in \mathbb{C}^{N \times t}$ are defined in the same manner. The dynamics of the set $X_{t,t'} = (Q_{t+1}, B_t, M_t, H_t)B_t^H M_t = Q_t^H H_t M_t = G_t(B_t, Q_{t+1} = F_t(H_t, q_0))$ conditioned on $\Theta = (\Sigma, \hat{w})$ is investigated for $t' = t$ or $t' = t + 1$, in which the $\tau$th columns of $G_t(B_t)$ and $F_t(H_t, q_0)$ are equal to the right-hand sides (RHS) on the first equations in (11) and (12) with $t = \tau$, respectively. The condition $B_t^H M_t = Q_t^H H_t$ imposes the unitary property on $V$, and is obtained from the second equations in (11) and (12). The set $X_{t,t'}$ represents the history of errors in all preceding iterations just before updating (11), while $X_{t,t+1}$ does just before updating (12).

We define $m_t = P_M m_t = M_t a_t$, $\alpha_t = M_t^+ m_t$, and $m_t = m_t - m_t$. See the end of Section II for the notations. The vectors $q_t, q_t^{-1}$, and $\beta_t = Q_t^+ q_t$ are defined in the same manner. For notational convenience, we define $\alpha_0 = 0$, $\beta_0 = 0$, $Q_0 = 0$, $B_0 = O$, $M_0 = O$, $H_0 = O$, $M_0^+ = O$, and $Q_0^+ = O$, implying $P_{M_0} = I_N$ and $P_{Q_0^+} = I_N$.

**Theorem 2:** For any iteration $\tau = 0, 1, \ldots$,

(a) Each element in $q_{\tau+1}$ has finite fourth moments. Furthermore, the following limit exists for all $\tau' \leq \tau + 1$:

\[
\zeta_{\tau+1, \tau'} = \lim_{M = \delta N \to \infty} \frac{1}{N}^\tau \nu_{\tau}^t = \frac{1}{N}^\tau \nu_{\tau}^t.
\]

In particular, the properties $\zeta_{\tau+1, \tau+1} = v_{B-A}^t$, and $\text{mse}_t \equiv \text{MMSE}(v_{B-A}^t)$ hold. The minimum eigenvalues of $N^{-1}M_{\tau+1}^t M_{\tau+1}$ and $N^{-1}Q_{\tau+2}^H Q_{\tau+2}$ are strictly positive in the large system limit.

(b) Let $\{z_{\tau} \sim \mathcal{CN}(0, I_N)\}$ denote a sequence of independent standard complex Gaussian vectors that are independent of $V$. Let $\mu_{\tau} = \lim_{M = \delta N \to \infty} N^{-1} |z_{\tau}^4|$, and define

\[
b_{\tau} = B_{\tau} \beta_{\tau} + M_{\tau} a_{\tau} + B_{\tau} \alpha_{\tau} + B_{\tau} a_{\tau} + \mu_{\tau}^{1/2} z_{\tau}, \quad (15)
\]

\[
\hat{h}_{\tau} = H_{\tau} a_{\tau} + Q_{\tau+1} a_{\tau} + H_{\tau} + \mu_{\tau}^{1/2} z_{\tau}. \quad (16)
\]

Then, for any $k \in \mathbb{N}$, all $k$-tuples of the elements in $b_{\tau}$ conditioned on $\Theta$ and $X_{\tau-\tau+1}$ and in $h_{\tau}$ conditioned on $\Theta$ and $X_{\tau-\tau+1}$ converge in distribution to the corresponding $k$-tuples for $b_{\tau}$ and $h_{\tau}$ in the large system limit.

(c) Let $\omega \in \mathbb{C}^N$ denote any vector that is independent of $V$, and satisfies $\lim_{N \to \infty} N^{-1} |\omega|^2 = 1$. Suppose that $D$ is any $N \times N$ Hermitian matrix such that $D$ depends only
on $\Sigma$, and that $N^{-1} \text{Tr}(D^2)$ is almost surely convergent as $N \to \infty$. Then, for all $\tau' \leq \tau$ and $\tau'' \leq \tau + 1$

$$\lim_{M=\delta N \to \infty} \frac{1}{N} b^H_{\nu} \omega_{\alpha} = 0,$$ (17)

$$\lim_{M=\delta N \to \infty} \frac{1}{N} b^H_{\nu} D b_{\eta} = \frac{\zeta_{\tau', \tau'}}{N} \lim_{N \to \infty} \frac{1}{N} \text{Tr}(D),$$ (18)

$$\lim_{M=\delta N \to \infty} \frac{1}{N} b^H_{\nu} m_{\tau} = \frac{\zeta_{\tau', \tau'}}{N} \lim_{N \to \infty} \frac{1}{N} \text{Tr}(D),$$ (19)

$$\lim_{M=\delta N \to \infty} \frac{1}{N} m^H_{\nu} m_{\tau} = \frac{\zeta_{\tau', \tau'}}{N} \lim_{N \to \infty} \frac{1}{N} \text{Tr}(D),$$ (20)

$$\lim_{M=\delta N \to \infty} \frac{1}{N} h^H_{\nu} h_{\tau} = \frac{\zeta_{\tau', \tau'}}{N} \lim_{N \to \infty} \frac{1}{N} \text{Tr}(D),$$ (21)

$$\lim_{M=\delta N \to \infty} \frac{1}{N} h^H_{\nu} q_{\tau''} = \frac{\zeta_{\tau', \tau''}}{N} \lim_{N \to \infty} \frac{1}{N} \text{Tr}(D).$$ (22)

with

$$\gamma_{\tau', \tau'} = \gamma_{\tau', \tau'} \int \frac{\delta(\sigma^2 + 2v^H_{\nu} A \lambda)}{(\sigma^2 + v^H_{\nu} A \lambda)} d\nu(\lambda).$$ (23)

Theorem 1 follows immediately from Theorem 2. A sketch for the proof of Theorem 2 is presented in the next section. See [14] for the detailed proof.

IV. PROOF OF THEOREM 2

A. Technical Lemmas

The proof strategy is based on a conditioning technique used in [3]. A challenging part in the proof is to evaluate the distributions of the estimation errors in each iteration conditioned on the estimation errors in all preceding iterations. Bayati and Montanari [3] evaluated the conditional distributions via the conditional distribution of the measurement matrix $A$. Since the LMMSE filter is used in module $A$, the conditional distribution of $A$ can be regarded as the posterior distribution of $A$ given linear, noiseless, and compressed observations of $A$, determined by the estimation errors in all preceding iterations. For i.i.d. Gaussian measurement matrices, it is well known that the posterior distribution is also Gaussian. The proof in [3] heavily relies on this well-known fact.

The main contribution of this paper is to extend the argument in [3] to the case of the unitary matrix $V$. Assumption 2 implies that $V$ is independent of $U$ and $\Sigma$, and a Haar matrix [11]—uniformly distributed on the space of all possible $N \times N$ unitary matrices. Under coordinate rotations in the row and column spaces of $V$, it is possible to show that the linear, noiseless, and compressed observation of $V$ is equivalent to observing part of the elements in $V$. Since any Haar matrix is bi-unitarily invariant [11], the distribution of $V$ after the coordinate rotations is the same as the original one. Thus, evaluating the conditional distribution of $V$ reduces to analyzing the conditional distribution of a Haar matrix given part of its elements. This argument was implicitly used in [3].

Evaluation of this conditional distribution is a technically challenging part in this paper, while this part is not required for i.i.d. Gaussian measurements. We use further coordinate rotations to reveal the statistical structure of the conditional Haar matrix. We know that a Haar matrix has similar properties to an i.i.d. Gaussian matrix as $N \to \infty$. In particular, a finite number of linear combinations of the elements in a Haar matrix were proved to converge in distribution to jointly Gaussian-distributed random variables as $N \to \infty$ [15]. Note that the classical central limit theorem cannot be used, since the elements of a Haar matrix are not independent. Using this asymptotic similarity between Haar and i.i.d. Gaussian matrices, we arrive at the following two lemmas:

**Lemma 2:** For $t \geq 0$, $t' > 0$, and $N - t - t' > 0$, suppose that $\tilde{V}$ is an $(N - t - t') \times (N - t - t')$ Haar matrix and independent of $V$, and that both $Q_{t', t} \in \mathbb{C}^{N \times t'}$ and $M_t \in \mathbb{C}^{N \times t}$ are full rank for $t > 0$. Let $\epsilon_{t, 0} = \|q_0\|^{-2} b^H_{\nu} m_0 q_0$, and

$$\epsilon_{t, \tau} = \frac{1}{N} H^H_{t, \tau} q_{t', \tau} \quad \epsilon_{t, \tau'} = \frac{1}{N} H^H_{t, \tau'} q_{t, \tau'}$$

for $t > 0$, with $\Gamma_t = M_t^H - M_t^H B_t (B_t^H P_{t, M_t} B_t) - B_t^H P_{t, M_t}$, and $\Delta_t = Q_{t, t' + 1} - Q_{t, t' + 1} H_t (H_t^H P_{t, M_t} H_t) - H_t^H P_{t, M_t}$. Then, the following properties hold:

$$b_t \sim B_t^H q_t + \epsilon_{t, 0} + \epsilon_{t, \tau} + \epsilon_{t, \tau'} + \epsilon_{t, \tau''}$$

for $t > 0$, with $\gamma_{t, \tau'} = \gamma_{t, \tau'} \int \frac{\delta(\sigma^2 + 2v^H_{\nu} A \lambda)}{(\sigma^2 + v^H_{\nu} A \lambda)} d\nu(\lambda).$ (24)

Then conditioned on $\Theta$ and $X_t$, for $t > 0$, and for all $\tau \leq \tau' + 1$

$$V^H q_{t' + 1} \sim \dot{V}_{M_t^H} \Psi_{t', t'} \tilde{V}_{t', t'}^H (\epsilon_{t', \tau} + \epsilon_{t', \tau'}) + \epsilon_{t', \tau''}.$$ (25)

with $V_{t', t'}^0 = b^H_{\nu} / \|q_0\|$. See the end of Section II for the other notations, as well as $\Phi_{t, M_t} = I_N$ and $\Phi_{t, 0} = I_{N-1}$.

**Lemma 3:** For $t' > 0$, $t > 0$, and $N - t - t' > 0$, suppose that $\tilde{V}$ is the Haar matrix defined in Lemma 2. Let $a \in \mathbb{C}^{N - t - t'}$ denote a vector that are independent of $\tilde{V}$ and satisfies $\lim_{N \to \infty} N^{-1} |a|^2 \lesssim 1$. Suppose that $\tilde{V}$ is a vector such that, for all $k \in \mathbb{N}$, every $k$-tuple of the elements in $z$ follows $CN(0, I_k)$ as $N \to \infty$.

- If the minimum eigenvalues of $N^{-1} M^H_t M_t$ and $N^{-1} B_t^H P_{t, M_t} B_t$ are strictly positive in the large system limit, then the convergence in distribution holds

$$\Phi_{M_t, t', 0} \Psi_{t', t'}^0 \tilde{V}_{t', t'}^H a \overset{d}{\to} z + M_t o(1) + P_{M_t} B_t o(1)$$

conditioned on $a$, $\Theta$, and $X_t$ in the large system limit.

- If the minimum eigenvalues of $N^{-1} \Psi_{t', t'}^0 \tilde{V}_{t', t'}^H a \overset{d}{\to} z + Q_{t, t'} o(1) + P_{M_t} H_t o(1)$

conditioned on $a$, $\Theta$, and $X_t$ in the large system limit.
In order to prove Theorem 2, we need the strong law of large numbers for the elements of a Haar matrix, which are dependent random variables.

**Theorem 3 (Etzemdi [16]):** Let \( \{X_n\}_{n=1}^{\infty} \) denote a sequence of complex random variables with finite second moments, and define \( S_n = \sum_{i=1}^{n} X_i \). The strong law of large numbers for \( T_n = (S_n - E[S_n])/n \) holds, i.e. \( \lim_{n \to \infty} T_n = 0 \), if the following assumptions hold:

\[
\sup_i \mathbb{E}(|X_i|) < \infty, \quad \sup_i \mathbb{E}(|S_i|) < \infty, \quad \limsup_{n \to \infty} \frac{1}{n^\gamma} \mathbb{V}[S_n] < \infty \quad \text{for some } \gamma < 2. \tag{33}
\]

**Lemma 4:** Suppose that \( V \) is an \( N \times N \) Haar matrix. Let \( a \in \mathbb{C}^N \) and \( b \in \mathbb{C}^N \) denote random vectors that are independent of \( V \) and satisfy \( \lim_{N \to \infty} N^{-1/2} ||a||^2 = 1 \), \( \lim_{N \to \infty} N^{-1} ||b||^2 = 1 \), and \( \lim_{N \to \infty} N^{-1} ||b||^2 H a = C \). Furthermore, we define a Hermitian matrix \( D \in \mathbb{C}^{N \times N} \) such that \( D \) is independent of \( V \), and that \( N^{-1} \text{Tr}(D^2) \) is almost surely convergent as \( N \to \infty \). Then,

\[
\lim_{N \to \infty} \frac{1}{N} b^H V a \overset{a.s.}{\rightarrow} 0, \tag{34}
\]

\[
\lim_{N \to \infty} \frac{1}{N} b^H V D V a \overset{a.s.}{\rightarrow} C \lim_{N \to \infty} \frac{1}{N} \text{Tr}(D). \tag{35}
\]

**B. Sketch of Proof by Induction**

We are ready to prove Theorem 2. The proof is by induction. We omit the proof for the case \( \tau = 0 \), and only present a sketch of the proof for a general case, because of space limitation.

We assume that Theorem 2 is correct for all \( \tau < t \), and prove that Theorem 2 holds for \( \tau = t \). Note that we can use Lemma 2 since the induction hypothesis \( \{a\} \) for \( \tau < t \) implies that \( M_t \) and \( Q_t \) are full rank for \( t' = t \) and \( t' = t + 1 \).

**Convergence of \( b_t \) to \( \beta_t \) for \( \tau = t \):** We first prove \( \epsilon_{1,t} \overset{a.s.}{\rightarrow} o(1) \) in (25), given by (24). We use the submultiplicative property of the Euclidean norm to obtain the upper bound \( \|\epsilon_{1,t}\|^2 \leq ||N\Gamma_t||^2 ||(N^{-1} H_t^H q_t)||^2 \).

Let us note that \( N^{-1} H_t^H q_t \) converges almost surely to zero in the large system limit. By definition,

\[
\frac{1}{N} H_t^H q_t = \frac{H_t^H q_t - H_t^H Q_t}{N} \left( Q_t H_t \right)^{-1} Q_t H_t. \tag{36}
\]

The induction hypothesis (22) for \( \tau < t \) implies that \( N^{-1} H_t^H q_t \) and \( N^{-1} H_t^H Q_t \) converge almost surely to zero. Furthermore, the induction hypothesis (41) for \( \tau < t \) implies that \( \|\langle N^{-1} Q_t H_t \rangle Q_t^H q_t^H \| \) is bounded. Thus, \( N^{-1} H_t^H q_t \overset{a.s.}{\rightarrow} 0 \) holds in the large system limit.

In order to complete the proof of \( \epsilon_{1,t} \overset{a.s.}{\rightarrow} o(1) \), we need to prove that \( ||N\Gamma_t||^2 \) is bounded. The boundedness can be proved in the same manner, although the details are omitted. Thus, \( \epsilon_{1,t} \overset{a.s.}{\rightarrow} o(1) \).

Next use Lemma 3 to evaluate the last term on the RHS of (25). It is possible to confirm that the last term on the RHS of (30) reduces to \( P_{\hat{M}_t} B_t \), which is determined by \( M_t o(1) + B_t o(1) \). Thus, we use (25) and Lemma 3 to find that, for all \( k \in \mathbb{N} \), any \( k \)-tuple of the elements in \( b_t \), conditioned on \( \Theta \) and \( X_{t+1} \), converges to distribution in the corresponding \( k \)-tuple for (13), when \( \mu_t \) in (15) is defined as

\[
\mu_t \overset{a.s.}{\rightarrow} \lim_{M = cN \to \infty} \frac{1}{N} q_t^H \Phi_t \mathcal{P}^t \Phi_t^H q_t. \tag{37}
\]

In order to complete the proof, we shall evaluate (37). Since we can show \( \Phi_t \mathcal{P}^t \Phi_t^H q_t = \mathcal{P}^t q_t - \Phi_t^H \mathcal{P}^t q_t \), we have

\[
\mu_t \overset{a.s.}{\rightarrow} \lim_{M = cN \to \infty} \frac{1}{N} \left( ||q_t^H||^2 - q_t^H \mathcal{P}^t q_t \right). \tag{38}
\]

It is possible to prove that the second term converges almost surely to zero, by repeating the proof of \( \epsilon_{1,t} = o(1) \). Thus, we have \( \mu_t \overset{a.s.}{\rightarrow} \lim_{M = cN \to \infty} ||q_t^H||^2. \tag{39} \)

Eqs. \((17) - (20)\) for \( \tau = t \): We first prove (18) for \( \tau = t \). We use (25), \( \epsilon_{1,t} = o(1) \), and Lemma 4 to have

\[
\lim_{M = cN \to \infty} \frac{1}{N} b_t^H D b_t \overset{a.s.}{\rightarrow} \lim_{M = cN \to \infty} \frac{1}{N} b_t^H D b_t \beta_t \tag{40}
\]

conditioned on \( \Theta \) and \( X_{t+1} \) for \( \tau' < t \). Using the induction hypothesis (18) for \( \tau < t, q_t = Q_t \beta_t, \) and \( q_t^H q_t = 0 \) yields (18) for \( \tau' < t = t \).

For \( \tau' = t \), (25) and Lemma 4 imply

\[
1 \overset{a.s.}{\rightarrow} \frac{1}{N} b_t^H D b_t \rightarrow 1 \overset{H_t^H B_t^H D b_t \beta_t}{N} \tag{41}
\]

conditioned on \( \Theta \) and \( X_{t+1} \) in the large system limit. The induction hypothesis (18) for \( \tau < t \) implies that the first term converges almost surely to \( \lim_{M = cN \to \infty} ||q_t^H||^2 ||Q_t^H D||^2 \).

Furthermore, it is possible to prove that the second term converges almost surely to \( \mu_t ||N^{-1} D||^2 \) in the large system limit, since \( N^{-1} ||D||^2 \) is assumed to be bounded as \( N \to \infty \). Thus, (18) holds.

To prove (17) for \( \tau = t \), we repeat the same proof to obtain

\[
\lim_{M = cN \to \infty} \frac{1}{N} b_t^H \omega \overset{a.s.}{\rightarrow} \lim_{M = cN \to \infty} \frac{1}{N} b_t^H \omega \overset{a.s.}{\rightarrow} 0 \tag{42}
\]

conditioned on \( \Theta \) and \( X_{t+1} \), where we have used the induction hypothesis (17) for \( \tau < t \).

Let us prove (19) and (20) for \( \tau = t \). Using (5), (13), (17), and (18), we obtain

\[
\frac{\gamma_t}{N} b_t^H \tilde{W}_t \left\{ (\Sigma, O) b_t + \tilde{w} \right\} \overset{a.s.}{\rightarrow} \frac{1}{N} b_t^H b_t \tag{43}
\]

in the large system limit. From (11) and (42), (19) holds.

Similarly, we use (11), (13), and (42) to obtain

\[
\frac{m_t^H \gamma_t}{N} \tilde{W}_t \left\{ (\Sigma, O) b_t + \tilde{w} \right\} \tag{44}
\]

in the large system limit. Using (13), (17), (18), and Assumption 2 we find that the second term reduces to (23) for \( t' = t' \). Thus, (20) holds for \( \tau = t \).

**Convergence of \( h_t \) to \( \beta_t \) for \( \tau = t \):** The proof for the convergence of \( h_t \) is omitted, since it is the same as for the convergence of \( b_t \) to (15).
Eq. (22) for \( \tau = t \): The proof of (22) for \( \tau = t \) is omitted, since it is the same as the proof for (18) with \( D = I_N \).

Eq. (21) for \( \tau = t \): We only prove the existence of (14) for \( \tau' \leq \tau = t \), since the case \( \tau' = t + 1 \) can be proved in the same manner. Using (12) yields

\[
\frac{1}{N}q_{t+1}^\alpha q_t = \frac{q_{t+1}^\alpha q_t - q_{t+1}^\alpha \eta_t(q_0 - h_t)}{N}. \tag{44}
\]

The induction hypothesis (13) \( \tau < t \) implies that the first term is convergent in the large system limit.

In order to prove the existence of (14) for \( \tau' \leq \tau = t \), it is sufficient to confirm

\[
\frac{1}{N}q_{t+1}^\alpha q_t \rightarrow \tilde{\zeta}_{t,\tau'} - \frac{1}{N}E_z_t \left[ q_{t+1}^\alpha \eta_t(q_0 - h_t)^1 \right], \tag{45}
\]

in the large system limit. In (45), the expectation is over independent standard complex Gaussian vectors \( \tilde{Z}_t = \{z_\tau : \tau = 0, \ldots, t \} \). Furthermore, \( h_t^G \) is recursively defined as

\[
h_t^G = H_t^G \alpha_t + Q_{t+1}^t \eta_t(1) + H_t^G \eta_t(1) + v_{t+1}z_t, \tag{46}
\]

with \( H_t^G = (h_0^G, \ldots, h_{t-1}^G) \).

Checking that the assumptions in Theorem 3 hold, we use Theorem 3 and the property (b) for \( \tau = t \) to have

\[
\frac{q_{t+1}^G q_t}{N} \rightarrow \frac{1}{N}E_z_t \left[ q_{t+1}^G \eta_t(q_0 - h_t)^1 \right], \tag{47}
\]

in the large system limit, where \( h_t \) is given by (16). Repeating the same argument in the order \( \tau = t - 1, \ldots, 0 \), we arrive at (45). Thus, (14) exists for \( \tau' \leq \tau = t \).

Eq. (22) for \( \tau = t \): We shall prove (22) for \( \tau = t \). From (12) and (20), we find

\[
\lim_{M \rightarrow \infty} \frac{1}{M} h_t^H q_t \rightarrow \lim_{M \rightarrow \infty} \frac{1}{M} m_t^H b, \tag{48}
\]

conditioned on \( \Theta \) and \( \chi_{t+1}^t \) for \( \tau' \leq t \), which is almost surely equal to zero, because of (19) for \( \tau = t \). Thus, (22) holds for \( \tau' \leq t = t \).

We use (12) and (22) for \( \tau' = 0 = \tau = t \) to have

\[
\lim_{M \rightarrow \infty} \frac{1}{M} h_t^H q_t^t \rightarrow \lim_{M \rightarrow \infty} \frac{1}{M} h_t^H \eta_t(q_0 - h_t), \tag{49}
\]

for \( \tau' = t + 1 \). It is possible to prove

\[
\frac{1}{N}q_{t+1}^G q_t \rightarrow \frac{1}{N}E_z_t \left[ h_t^G \eta_t(q_0 - h_t) \right], \tag{50}
\]

in the large system limit, by repeating the proof of (45).

Let us prove that the RHS of (50) is equal to zero. From (5), (20), (21), and (23), and the induction hypothesis \( \zeta_{t,t} = v_{t+1} \rightarrow A \), we find that the random vector \( h_t^G \) induced from the randomness of \( Z_t \) has i.i.d. proper complex Gaussian elements with vanishing mean in the large system limit and variance \( v_{t+1} \rightarrow A \), given by (3). We use Lemma 1 to find that the RHS of (50) is equal to zero. Thus, (22) holds for \( \tau = t \).

Property (a) for \( \tau = t \): The proof for the existence of the fourth moments is omitted. See [3] Lemma 1(g) for evaluating the minimum eigenvalues of \( N^{-1}M_{t+1}^HM_{t+1} \) and \( N^{-1}Q_{t+1}^tQ_{t+2}^t \) for \( \tau = t \).

We repeat the proof of (45) to obtain

\[
mse_t \rightarrow \lim_{M \rightarrow \infty} \frac{1}{M}E_{z_t} \left[ \|q_0 - \hat{\eta}_t(q_0 - h_t)\|^2 \right], \tag{51}
\]

which reduces to \( \text{MMSE}(v_{t+1}^t \rightarrow A) \).

Let us prove \( \zeta_{t+1,t+1} = v_{t+1}^t \rightarrow A \). Applying (7) to (12), and using (8), we have

\[
q_{t+1} = \frac{v_{t+1}^t}{\text{MMSE}(v_{t+1}^t \rightarrow A)} - \frac{v_{t+1}^t}{v_{t+1}^t \rightarrow A} h_t, \tag{52}
\]

We use Lemma 1, and (22) to evaluate (14) as

\[
\zeta_{t+1,t+1} \rightarrow \frac{(v_{t+1}^t)^2}{\text{MMSE}(v_{t+1}^t \rightarrow A)} - \frac{(v_{t+1}^t)^2}{v_{t+1}^t \rightarrow A} = v_{t+1}^t \rightarrow A. \tag{53}
\]

Thus, property (a) holds.

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