Operators invariant relative to a completely nonunitary contraction

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Abstract
Given a contraction $A$ on a Hilbert space $\mathcal{H}$, an operator $T$ on $\mathcal{H}$ is said to be $A$-invariant if $\langle Tx, x \rangle = \langle TAx, Ax \rangle$ for every $x \in \mathcal{H}$ such that $\|Ax\| = \|x\|$. In the special case in which both defect indices of $A$ are equal to 1, we show that every $A$-invariant operator is the compression to $\mathcal{H}$ of an unbounded linear transformation that commutes with the minimal unitary dilation of $A$. This result was proved by Sarason under the additional hypothesis that $A$ is of class $C_{00}$, leading to an intrinsic characterization of the truncated Toeplitz operators. We also adapt to our more general context other results about truncated Toeplitz operators.

Mathematics Subject Classification Primary 47A45; Secondary 47B35

1 Introduction

Suppose that $A$ is a completely nonunitary contraction acting on a Hilbert space $\mathcal{H}$ and $U$ is the minimal unitary dilation of $A$ acting on $\mathcal{K} \supset \mathcal{H}$. Thus, $A^n = P_\mathcal{H} U^n|_\mathcal{H}$ is the compression of $U^n$ to $\mathcal{H}$ for every positive integer $n$. It is of interest to consider, more generally, operators of the form $P_\mathcal{H} X|_\mathcal{H}$, where $X$ is in the commutant $\{U\}'$ of $U$. The commutant lifting theorem [12,17] shows that every element of $\{A\}'$ is of this form. When $A$ is the unilateral shift on the Hardy space $H^2$, the collection $\{P_\mathcal{H} X|_\mathcal{H} : X \in \{U\}'\}$ consists precisely of the Toeplitz operators on $H^2$. When $A$ is an operator of class $C_{00}$ with defect indices equal to 1, the collection $\{P_\mathcal{H} X|_\mathcal{H} : X \in \{U\}'\}$ is hard to characterize intrinsically. However, a larger collection, obtained by considering closed unbounded linear transformations $X$ that commute with $U$, has been identified in [14] with the class of those bounded operators $Y$ on $H^2$.
that are $A$-invariant in the sense that they satisfy the identity

$$\langle YA x, Ax \rangle = \langle Y x, x \rangle$$

for every vector $x \in \mathcal{H}$ such that $\|Ax\| = \|x\|$. Of course, operators $A$ of the type just described can be identified up to unitary equivalence with compressions of the unilateral shift to co-invariant subspaces, and the class of operators $Y$ described above is in that case the class of truncated Toeplitz operators [14].

Our purpose in this paper is to consider arbitrary operators $A$ with defect indices equal to 1 and the class of bounded operators on $\mathcal{H}$ that can be obtained as compressions of (possibly) unbounded linear transformations that commute with $U$. We call these operators truncated multiplication operators and we show, in particular, that operators in this class are characterized by the fact that they are $A$-invariant. Operators $A$ with defect indices equal to 1 are always complex symmetric and, in the $C_{00}$ case, it is known [14] that the corresponding $A$-invariant operators satisfy the same complex symmetry. This result no longer persists if $A$ is not of class $C_{00}$. In this case, the complex symmetric truncated $A$-invariant operators belong, roughly speaking, to the linear space generated by $\{A\}'$ and $\{A^*\}'$.

The remainder of the paper is organized as follows. Section 2 contains a description of the functional models of contractions with defect indices equal to one, as well as the definition of truncated multiplication operators and their symbols in this context. In Sect. 3, we characterize the class of truncated multiplication operators by $A$-invariance. The main result of Sect. 4 establishes the extent to which the symbol of an $A$-invariant operator is uniquely determined. In Sect. 5 we describe some useful and explicit unitary equivalences between model spaces. Finally, in Sect. 6 we discuss complex symmetries, in particular the decomposition of $A$-symmetric operators into complex symmetric and complex skew symmetric summands.

2 Preliminaries

We denote by $\mathbb{C}$ the complex plane, by $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ the open unit disk, by $T = \partial \mathbb{D}$ the unit circle, and by $\chi$ the identity function $\chi(\lambda) = \lambda$. Normalized arclength defines a Borel probability measure $m$ on $T$, $L^p$ stands for the corresponding space $L^p(T, m)$, and $H^p \subset L^p$ is the Hardy space for $p \in [1, +\infty]$. We recall that an element $h \in H^p$ can also be considered to be an analytic function on $\mathbb{D}$, and the values of $u$ on $T$ can be recovered as radial limits almost everywhere with respect to $\mu$.

As noted in the introduction, we focus on contractions $A$ acting on a Hilbert space $\mathcal{H}$ with the property that the operators $I_{\mathcal{H}} - A^* A$ and $I_{\mathcal{H}} - AA^*$ have rank equal to one. In a different terminology, $T$ has defect indices 1 and 1, where the defect indices are a measure of how far $A$ and $A^*$ are from being isometric. In addition, we impose the condition that $A$ has no nonzero reducing subspace $K$ with the property that the restriction $A|_K$ is a unitary operator. In other words, $A$ is supposed to be completely nonunitary.

Sz.-Nagy and Foias have developed a functional model for completely nonunitary contractions, showing for instance that such a contraction $A$ is uniquely determined, up to unitary equivalence, by a purely contractive analytic function $\Theta_A$ whose values are operators between two Hilbert spaces with dimensions equal to the defect indices of $A$. The function $\Theta_A$ is called the characteristic function of $A$, and it plays an analogous role to that of the characteristic matrix of a linear operator on a finite dimensional space. In our case, the defect indices are both equal to 1, so the characteristic function of $A$ can be thought of simply as a function $u \in H^\infty$. 

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such that \( \|u\|_\infty \leq 1 \). Such a function is purely contractive precisely when \(|u(0)| < 1\), that is, when \( u \) is not identically equal to a constant of modulus one. Thus, throughout this paper, we work with a purely contractive function \( u \in H^\infty \). When the characteristic function of \( A \) is an inner function in \( H^\infty \), the minimal unitary dilation of \( A \) is a bilateral shift, and this allows for the construction of a particularly simple functional model for \( A \). In our more general setting, this dilation is a unitary operator with spectral multiplicity at most 2.

We now describe the functional model associated to a given purely contractive function in \( H^\infty \). Fix \( u \in H^\infty \) such that \( \|u\|_\infty \leq 1 \) and \(|u(0)| < 1\), that is, when \( u \) is not identically equal to a constant of modulus one. Thus, throughout this paper, we work with a purely contractive function \( u \in H^\infty \). When the characteristic function of \( A \) is an inner function in \( H^\infty \), the minimal unitary dilation of \( A \) is a bilateral shift, and this allows for the construction of a particularly simple functional model for \( A \). In our more general setting, this dilation is a unitary operator with spectral multiplicity at most 2.

We now describe the functional model associated to a given purely contractive function in \( H^\infty \). Fix \( u \in H^\infty \) such that \( \|u\|_\infty \leq 1 \) and \(|u(0)| < 1\), and define the function \( \Delta \in L^\infty \) by

\[
\Delta(\xi) = (1 - |u(\xi)|^2)^{1/2}, \quad \xi \in \mathbb{T}.
\]

Using this function, we construct spaces

\[
K = L^2 \oplus (\Delta L^2)^-, \quad K_+ = H^2 \oplus (\Delta L^2)^-, \quad G = \{uf \oplus \Delta f : f \in H^2\},
\]

and finally,

\[
H^u = K_+ \ominus G.
\]

Note for further use that a function \( f \oplus g \in K_+ \) belongs to \( H \) if and only if

\[
\pi f + \Delta g \in L^2 \ominus H^2.
\]

We define now operators \( U \in B(K) \), \( U_+ \in B(K_+) \), and \( S_u \in B(H^u) \) by

\[
U(f \oplus g) = \chi f \oplus \chi g, \quad f \oplus g \in K, \quad U_+ = U|_{K_u},
\]

and

\[
S_u = P_{H^u} U|_{H^u} = (U^*_+ | H^u)^*.
\]

Then the operator \( S_u \) is completely nonunitary, it has defect indices equal to 1, and its characteristic function coincides with \( u \). Moreover \( U_+ \) is the minimal isometric dilation of \( S_u \), and \( U \) is the minimal unitary dilation of \( S_u \). We refer to [17] or [11] for an exposition of these facts.

Observe that the operator \( S_u \) is of class \( C_{00} \), that is,

\[
\lim_{n \to \infty} \|S_u^n h\| = \lim_{n \to \infty} \|S_u^{*n} h\| = 0, \quad h \in \mathcal{H},
\]

if and only if \( u \) is an inner function, that is, \( \Delta = 0 \). In this case \( H^u = H^2 \ominus uH^2 \). In this paper we concern ourselves primarily with the case in which \( u \) is not inner. All of the arguments in the paper, with the exception of the proof of Proposition 4.2, work equally well if \( u \) is an inner function. However, these results were already known in the inner case. We refer to [14] for a detailed discussion.

We record for further use the formula

\[
U^*_+(f \oplus g) = \bar{\chi}(f - f(0)) \oplus \bar{\chi} g, \quad f \oplus g \in K_+.
\]

We use the linear manifolds

\[
K^\infty = \{f \oplus g : f \in L^\infty, \ g \in L^\infty \cap (\Delta L^2)^-\}, \quad K^\infty_+ = K_+ \cap K^\infty,
\]

and

\[
H^u_\infty = H^u \cap K^\infty.
\]
It is clear that $K^\infty$ is dense in $K$ and $K^\infty_+$ is dense in $K_+$. To show that $H^\infty_u$ is dense in $H_u$, we consider the vectors $\chi_n^\prime + 0$ and $\chi^{-n} u \oplus \chi^{-n} \Delta$, $n \in \mathbb{Z}$. These elements of $K^\infty$ span a dense linear manifold in $K$, and therefore their orthogonal projections onto $H_u$ span a dense linear manifold in $H_u$. These orthogonal projections are again bounded functions. In fact, $P_{H_u}(\chi_n^\prime + 0) = 0$ for $n < 0$, and
\[
P_{H_u}(\chi_n^\prime + 0) = \chi_n^\prime + 0 - P_G(\chi_n^\prime + 0).
\]
The second projection is easily calculated as
\[
P_G(\chi_n^\prime + 0) = \sum_{j=0}^{n} \alpha_{n-j}(\chi_j u \oplus \chi^\prime j \Delta), \quad \alpha_{n-j} = \langle u, \chi^{n-j} \rangle, \quad j = 0, \ldots, n.
\]
Similarly, $P_{H_u}(\chi^{-n} u \oplus \chi^{-n} \Delta) = 0$ for $n \leq 0$, and
\[
P_{H_u}(\chi^{-n} u \oplus \chi^{-n} \Delta) = P_{H^2}(\chi^{-n} u) \oplus \chi^{-n} \Delta, \quad n > 0,
\]
where $P_{H^2} : L^2 \to H^2$ denotes the orthogonal projection, so
\[
P_{H^2}(\chi^{-n} u) \equiv \chi^{-n} u - \sum_{j=0}^{n-1} \alpha_j \chi^{-n-j}, \quad \alpha_j = \langle u, \chi^j \rangle, \quad j = 0, \ldots, n - 1.
\]
Two particularly important vectors in $H^\infty_u$ are defined by
\[
k_0 = \tilde{k}_0^u = P_{H_u}(1 \oplus 0) = (1 - u(0)u) \oplus (-\overline{u(0)}\Delta)
\]
and
\[
\tilde{k}_0 = \tilde{k}_0^u = P_{H_u}(\overline{u} u \oplus \overline{u} \Delta) = (\overline{u} - u(0)) \oplus (\overline{u} \Delta).
\]
The operator $S_u$ maps $H_u \ominus \overline{\tilde{k}_0}$ isometrically onto $H_u \ominus \overline{k}_0$, $S_u h = U h$ for $h \in H_u \ominus \overline{\tilde{k}_0}$, and
\[
S_u \tilde{k}_0 = -u(0)k_0, \quad S_u^* k_0 = -\overline{u(0)}k_0.
\]
Using these facts and the equalities $\|k_0\|^2 = \|\tilde{k}_0\|^2 = 1 - |u(0)|^2$, it is easy to verify the identities
\[
I_{H_u} - S_u S_u^* = k_0 \otimes k_0, \quad I_{H_u} - S_u^* S_u = \tilde{k}_0 \otimes \overline{\tilde{k}_0}, \quad (2.2)
\]
where we use the notation $v \otimes \tilde{w}$ for the rank one operator $h \mapsto \langle h, w \rangle v$.

The linear manifolds $K^\infty$ and $K^\infty_+$ are clearly invariant under $U$. The linear manifold $H^\infty_+$ is also invariant under $S_u^*$, as seen from the formula
\[
S_u(f \oplus g) = \chi f \oplus \chi g - \langle f \oplus g, \tilde{k}_0 \rangle (u \oplus \Delta), \quad f \oplus g \in H_u.
\]
Similarly, $H^\infty_+$ is invariant under $S_u^*$ because
\[
S_u^*(f \oplus g) = \overline{\chi} f \oplus \overline{\chi} g - \langle f \oplus g, k_0 \rangle (\overline{1} \oplus 0), \quad f \oplus g \in H_u.
\]

It is well known that the commutant $\{U\}'$ consists of multiplication operators by matrix functions
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]

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where $a, b, c, d \in L^\infty$. We require a larger class consisting of unbounded linear transformations that commute with $U$. Suppose that we are given functions $a, b, c, d \in L^2$. We consider the matricial function
\[
F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
and the linear transformation $M_F : \mathbb{K}^\infty \to \mathbb{K}$ given by
\[
M_F(f \oplus g) = (af + bg) \oplus (cf + dg), \quad f \oplus g \in \mathbb{K}^\infty.
\]
In other words, $M_F$ is the operator of multiplication by $F$. (Observe that modifying the values of $b, c, d$ on $\{\zeta \in \mathbb{T} : \Delta(\zeta) = 0\}$ does not alter the operator $M_F$. It is useful however to allow for arbitrary $b, c, d \in L^2$.) Generally, $M_F$ is not continuous but it is closable, as can be seen from the inclusion $M_F^* \subset (M_F)^*$, where
\[
F^* = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix}.
\]
The equality $M_F Uv = UM_F v$ holds for every $v \in \mathbb{K}_{u}^\infty$. The operator $M_F$ is bounded if and only if $a \in L^\infty$ and the functions $b, c, d$ are essentially bounded on $\{\zeta \in \mathbb{T} : \Delta(\zeta) \neq 0\}$.

We define a linear transformation $A_F : \mathbb{H}^\infty \to \mathbb{H}$ by
\[
A_F v = P_{\mathbb{H}_u} M_F v, \quad v \in \mathbb{H}^\infty.
\]
We also have $A_F^* \subset (A_F)^*$, so $A_F$ is always closable. In particular, $A_F$ is bounded if and only if $A_F^*$ is bounded. If $M_F$ is bounded then, of course, $A_F$ is bounded as well, but not conversely [2,3].

**Definition 2.1** A bounded linear operator $T \in \mathcal{B}(\mathbb{H}_u)$ is called a truncated multiplication operator if there exists a function $F$ as above such that $Tv = A_F v$ for every $v \in \mathbb{H}^\infty$. The collection of all truncated multiplication operators is denoted $\mathcal{T}_u$.

The collection $\mathcal{T}_u$ is a linear space, closed under taking adjoints. In other words, $\mathcal{T}_u$ is an operator system.

**Remark 2.2** In the preceding definition, it seems natural to view $F$ as the symbol of the truncated Toeplitz operator $T$. Note however that there are nonzero functions $F$ such that $A_F = 0$, and thus a given operator in $\mathcal{T}_u$ may have more than one symbol. The symbols $F$ with the property that $A_F = 0$ are described in Proposition 4.2.

**Example 2.3** The operator $S_u$ itself belongs to $\mathcal{T}_u$. One symbol for $S_u$ is the function
\[
\begin{bmatrix} \chi & 0 \\ 0 & \chi \end{bmatrix},
\]
as can be seen directly from (2.1).

**Example 2.4** The function
\[
F = \begin{bmatrix} \chi u & \chi \Delta \\ 0 & \chi \end{bmatrix}
\]
is the symbol of an operator $T \in \mathcal{T}_u$ with the property that $Tx = 0$ for every $x \in \mathbb{H}_u \ominus \mathbb{C}k_0$ and $T\bar{k}_0 = (1 - |u(0)|^2)k_0$. To see this, consider an arbitrary vector $x = h \oplus g \in \mathbb{H}_u$, so
\[
Tx = P_{\mathbb{H}_u}(\chi (\bar{u}h + \Delta g) \oplus 0) = P_{\mathbb{H}_u}(P_{H^2}(\chi (\bar{u}h + \Delta g)) \oplus 0).
\]
It was noted earlier that \( \overline{\mu}h + \Delta g \in L^2 \ominus H^2 \), and therefore \( P_{H^2}(\chi(\overline{\mu}h + \Delta g)) = P_{H^2}(\rho \oplus 0) \) for some \( \rho \in \mathbb{C} \). The constant \( \rho \) is equal to zero if \( x \perp \overline{\mathbb{C}}k_0 \). For \( x = k_0 \), we have

\[
\chi(\overline{\mu}h + \Delta g) = \overline{\mu}(u - u(0)) + \Delta^2 = 1 - u(0)\overline{\mu},
\]

and thus \( P_{H^2}(\chi(\overline{\mu}h + \Delta g)) = 1 - |u(0)|^2 \).

There is a special class of matrix functions \( F \) with the property that \( A_F \) commutes with \( S_u \) on the space \( H_u^\infty \). These functions are of the form

\[
F = \begin{bmatrix}
a & 0 \\
\Delta c & a - uc \end{bmatrix},
\]

where \( a \in H^2 \) and \( c \in L^2 \). It is easily seen that functions of this form satisfy \( M_F(K_+^\infty) \subseteq K_+ \) and \( M_F(K_+^\infty \cap G) \subseteq G \). Thus, if \( x \in H_u^\infty \), we have \( P_{H_u}(M_F P_G Ux) = 0 \) and \( P_{H_u}(U P_G M_F x) = 0 \) and therefore

\[
A_F S_u x = P_{H_u} M_F S_u x = P_{H_u} (M_F Ux - M_F P_G Ux) = P_{H_u} M_F Ux
= P_{H_u} U M_F x = P_{H_u} (U A_F x) + P_{H_u} (U P_G M_F x) = S_u A_F x.
\]

In the case in which \( u \neq 0 \), the commutant lifting theorem implies that every bounded operator \( T \in \{S_u\}' \) is of the form \( A_F \), where \( F \) is a function of the form 2.3 with \( a \in H^\infty \) and \( c \in L^\infty \); see [16, Lemma 2.1].

**Lemma 2.5** Suppose that the function \( u \) is not identically zero. Then the commutant \( \{S_u\}' \) is commutative.

**Proof** Suppose that the operators \( T, T' \in \{S_u\}' \) are determined by the functions

\[
F = \begin{bmatrix}
a & 0 \\
\Delta c & a - uc \end{bmatrix}, \quad F' = \begin{bmatrix}
a' & 0 \\
\Delta c' & a' - uc' \end{bmatrix},
\]

respectively, for some \( a, a' \in H^\infty \) and \( c \in L^\infty \). A calculation shows that

\[
FF' = F'F = \begin{bmatrix}
a'' & 0 \\
\Delta c'' & a'' - uc'' \end{bmatrix},
\]

where \( a'' = aa' \) and \( c'' = ac' + a'c - ucc' \). Suppose that \( x \) is an arbitrary vector in \( H_u \). Then

\[
TT' x = P_{H_u} (FP_{H_u} (F'x)) = P_{H_u} (F F' x) - P_{H_u} (FP_{G} (F'x)) = P_{H_u} (F F' x),
\]

since \( FG \subseteq G \subseteq H_u^{-1} \). It follows that \( A_{F''} \) is a symbol for \( TT' \). Similarly, \( A_{F''} \) is a symbol for \( T'T \), and thus \( TT' = T'T \).

The commutant \( \{S_u\}' \) is not commutative if \( u \equiv 0 \); see Example 4.3.

### 3 Characterization of truncated multiplication operators by invariance

In this section, we show that truncated multiplication operators are characterized intrinsically by their properties as operators, without reference to a symbol. Fix a function \( u \in H^\infty \) such that \( \|u\|_\infty \leq 1 \) and \( |u(0)| < 1 \).
Definition 3.1 Suppose that $A$ is a contraction on a Hilbert space $\mathcal{H}$. A bounded linear operator $T \in B(\mathcal{H})$ is said to be $A$-invariant if the equality

$$\langle Tx, y \rangle = \langle TAx, Ay \rangle$$

holds for every pair of vectors $x, y \in \ker(I_A - A^*A)$.

Lemma 3.2 Suppose that $A \in B(\mathcal{H})$ is a contraction and $T \in B(\mathcal{H})$ is an arbitrary operator. Denote by $\mathcal{D}_A = [(I - A^*A)\mathcal{H}]^\perp$ and $\mathcal{D}_A^\perp = [(I - AA^*)\mathcal{H}]^\perp$ the defect spaces of $A$, and by $P_{\mathcal{D}_A}$ and $P_{\mathcal{D}_A^\perp}$ the corresponding orthogonal projections. Then the following conditions are equivalent:

1. $T$ is $A$-invariant.
2. $T$ is $A^*$-invariant.
3. There exists operators $X, Y \in B(\mathcal{H})$ such that $T - ATA^* = XP_{\mathcal{D}_A^\perp} + P_{\mathcal{D}_A}Y$.
4. There exists operators $X, Y \in B(\mathcal{H})$ such that $T - A^*TA = XP_{\mathcal{D}_A} + P_{\mathcal{D}_A^\perp}Y$.

Proof The operator $A$ maps the space $\ker(I - A^*A) = \mathcal{D}_A^\perp$ isometrically onto $\ker(I - AA^*) = \mathcal{D}_A$. Thus, given arbitrary vectors $u, v \in \ker(I - AA^*)$, there exist unique $x, y \in \ker(I - A^*A)$ such that $Ax = u$, $Ay = v$ $A^*u = x$, and $A^*v = y$. If $T$ is $A$-invariant, we see that

$$\langle Tu, v \rangle = \langle TAx, Ay \rangle = \langle Tx, y \rangle = \langle T^*u, A^*v \rangle,$$

and this shows that $T$ is $A^*$-invariant. This establishes that (1) implies (2) and the equivalence of (1) and (2) follows by symmetry.

Suppose now that $T$ is $A$-invariant and observe that

$$\langle Tx, y \rangle - \langle TAx, Ay \rangle = \langle (T - ATA^*)x, y \rangle, \quad x, y \in \ker(I - A^*A) = \mathcal{D}_A^\perp.$$

It follows that $(I - P_{\mathcal{D}_A})(T - ATA^*)(I - P_{\mathcal{D}_A}) = 0$, and thus (4) is satisfied with

$$X = T - ATA^*,$$

$$Y = (T - ATA^*)(I - P_{\mathcal{D}_A}).$$

Conversely, if (4) is satisfied, the identity $(I - P_{\mathcal{D}_A})(T - ATA^*)(I - P_{\mathcal{D}_A}) = 0$ follows immediately, thus showing that $T$ is $A$-invariant. We conclude that (1) is equivalent to (4). The equivalence of (2) and (4) is proved the same way, replacing $A$ by $A^*$.

Remark 3.3 In the special case of the operator $S_u$, (2.2) shows that $\ker(I_{H_u} - S_u^*S_u) = H_u \oplus \mathbb{C}k_0$. Moreover, given $x \in H_u$, we have $Ux = S_u x \in H_u$ precisely when $x \in H_u \oplus \mathbb{C}k_0$. Thus an operator $T \in B(H_u)$ is $S_u$-invariant if and only if

$$\langle Tx, y \rangle = \langle TUX, Uy \rangle, \quad x, y \in H_u \oplus \mathbb{C}k_0.$$

The polarization identity shows that an operator $T \in B(H_u)$ is $S_u$-invariant if and only

$$\langle Tx, x \rangle = \langle TUX, Ux \rangle, \quad x \in H_u \oplus \mathbb{C}k_0. \quad (3.1)$$

The invariance condition can be written equivalently as

$$\langle Tx, x \rangle = \langle TUX^*, U^*x \rangle, \quad x \in H_u \oplus \mathbb{C}k_0. \quad (3.2)$$

We now state the main result in this section.

Theorem 3.4 The following four conditions on an operator $T \in B(H_u)$ are equivalent:

$$\langle Tx, y \rangle = \langle TUX, Uy \rangle, \quad x, y \in H_u \oplus \mathbb{C}k_0.$$
(1) \( T \in \mathcal{T}_u \).
(2) \( T \) is \( S_u \)-invariant.
(3) There exist vectors \( v, w \in \mathcal{H}_u \) such that \( T - S_u T S_u^* = v \otimes k_0 + k_0 \otimes w \).
(4) There exist vectors \( \tilde{v}, \tilde{w} \in \mathcal{H}_u \) such that \( T - S_u^* T S_u = \tilde{v} \otimes \tilde{k}_0 + \tilde{k}_0 \otimes \tilde{w} \).

**Proof** The equations (2.2) show that \( P_{D_S^n} \) and \( P_{D_S} \) are constant multiples of \( k_0 \otimes k_0 \) and \( \tilde{k}_0 \otimes \tilde{k}_0 \), respectively. Since \( X(k_0 \otimes k_0) = (Xk_0) \otimes k_0 \) and \( (k_0 \otimes k_0)Y = k_0 \otimes (Y^* k_0) \) for every \( X, Y \in \mathcal{B}(\mathcal{H}_u) \), the equivalence of (2), (3), and (4) follows immediately from Lemma 3.2.

Suppose now that (1) holds, and thus \( T v = A_F v, v \in \mathcal{H}_u^\infty \), for some matrix \( F \). Since \( \tilde{k}_0 \in \mathcal{H}_u^\infty \), it follows that \( \mathcal{H}_u^\infty \cap (\mathcal{H}_u \ominus \mathbb{C} k_0) \) is dense in \( \mathcal{H}_u \ominus \mathbb{C} \tilde{k}_0 \). Thus, it suffices to verify (3.1) for \( v \in \mathcal{H}_u^\infty \cap (\mathcal{H}_u \ominus \mathbb{C} \tilde{k}_0) \). For such a vector \( v \) we have

\[
\langle T U v, U v \rangle = \langle P_{H_u} M_F U v, U v \rangle = \langle M_F U v, U v \rangle = \langle U M_F v, U v \rangle = \langle M_F v, v \rangle = \langle P_{H_u} M_F v, v \rangle = \langle T v, v \rangle,
\]

where we used the facts that \( U \) is unitary and \( M_F \) commutes with \( U \). We conclude that (1) implies (2).

We come now to the heart of the proof by showing that (3) implies (1). Suppose that (3) holds for some vectors \( v = a_1 \oplus c \) and \( w = a_2 \oplus b \) in \( \mathcal{H}_u \). We define a matrix function \( F \) by

\[
F = \begin{bmatrix} a_1 + \overline{a_2} & b \\ c & 0 \end{bmatrix}.
\]

We show first that:

(i) the operator \( A_F \) is bounded,
(ii) the sequence \( \{S_u^n T S_u^* \}_{n \in \mathbb{N}} \) converges in the weak operator topology to an operator \( T' \) such that \( T' = S_u T' S_u^* \), and
(iii) \( T = A_F + T' \) on \( \mathcal{H}_u^\infty \).

To do this, fix a vector \( x = g \oplus h \in \mathcal{H}_u^\infty \) and iterate the relation \( T - S_u T S_u^* = v \otimes k_0 + k_0 \otimes w \) to obtain

\[
Tx = S_u^n T S_u^* x + \sum_{j=0}^{n-1} [S_u^j v \otimes S_u^j k_0 + S_u^j k_0 \otimes S_u^j w] x, \quad n \in \mathbb{N}.
\]

(3.3)

We show that the sum above converges weakly to \( A_F x \). We calculate first

\[
\sum_{j=0}^{n-1} [S_u^j k_0 \otimes S_u^j w] x = \sum_{j=0}^{n-1} \langle x, S_u^j w \rangle S_u^j k_0 = P_{H_u} \sum_{j=0}^{n-1} \langle x, \chi^j w \rangle (\chi^j \oplus 0),
\]

where

\[
\langle x, \chi^j w \rangle = \langle \overline{\alpha_2} g + \overline{b} h, \chi^n \rangle.
\]

Therefore, the sum \( \sum_{j=0}^{n-1} \langle x, \chi^j w \rangle \chi^j \) converges in \( L^2 \) to \( P_{H^2} (\overline{\alpha_2} g + \overline{b} h) \) and thus \( \sum_{j=0}^{n-1} [S_u^j k_0 \otimes S_u^j w] x \) converges in norm to \( P_{H_u} (P_{H^2} (\overline{\alpha_2} g + \overline{b} h) \oplus 0) = P_{H_u} ((\overline{\alpha_2} g + \overline{b} h) \oplus 0) \). Similarly,

\[
\sum_{j=0}^{n-1} [S_u^j v \otimes S_u^j k_0] x = \sum_{j=0}^{n-1} \langle x, S_u^j k_0 \rangle S_u^j v = \sum_{j=0}^{n-1} \langle x, \chi^j \oplus 0 \rangle S_u^j v
\]
Moreover, since
\[
\sum_{j=0}^{n-1} [S^j u v \otimes S^j k_0] x = T x - S^n u T S^{n+1} u x - \sum_{j=0}^{n-1} [S^j u k_0 \otimes S^j w] x,
\]
it follows that the vectors on the left hand side of this equation are bounded in \( H_u \). To show that they have a weak limit in \( H_u \), it suffices to consider their scalar product with another element \( x' = g' + h' \in H_u^\infty \). We have
\[
\left\langle \sum_{j=0}^{n-1} [S^j u k_0 \otimes S^j w] x, x' \right\rangle = \left\langle \sum_{j=0}^{n-1} (g, \chi^j) \chi^j, x', x' \right\rangle,
\]
and the functions \( \sum_{j=0}^{n-1} (g, \chi^j) \chi^j \) converge to \( g \) in \( H^2 \) as \( n \to \infty \). Since \( v \in H_u \) and \( x' \) is bounded, the scalar products above tend to \( \langle g v, x' \rangle = \langle P_{H_u} (a_1 g \oplus cg), x' \rangle \) as \( n \to \infty \). We conclude that the sum
\[
\sum_{j=0}^{n-1} [S^j u v \otimes S^j k_0 + S^j u k_0 \otimes S^j w] x
\]
converges weakly to \( P_{H_u} \((a_1 + \overline{a_2}) g + \overline{b} h) \oplus cg) = A_F x \). The identity (3.3) shows that \( \| A_F x \| \leq 2 \| T \| \), thus proving (i). Rewriting (3.3) as
\[
S^n u T S^{n+1} u x = T - \sum_{j=0}^{n-1} [S^j u v \otimes S^j k_0 + S^j u k_0 \otimes S^j w],
\]
we see that \( S^n u T S^{n+1} u x \) converges weakly to \( T x - A_F x \) for \( x \in H_u^\infty \), so the weak convergence of \( \{ S^n u T S^{n+1} u \}_{n \in \mathbb{N}} \) follows from the fact that the sequence \( \{ \| S^n u T S^{n+1} u \| \}_{n \in \mathbb{N}} \) is bounded. Also, \( S^n u T^n S^{n+1} u \) is the weak limit of the sequence \( \{ S^{n+1} u T S^{n+1} u \}_{n \in \mathbb{N}} \), so it is equal to \( T' \). This proves (ii) and (iii).

To conclude the proof of (1), it suffices to show that \( T' \in T_u \). To do this, we observe that for every \( n \in \mathbb{N} \), we have \( U^{n+1} T \in H_u \) and \( U^n T \in H_u \). We define an operator \( T_n \in B(H_{\chi^{n+1} u}) \) by
\[
T_n x = U^n T' U^{n+1} x, \quad x \in H_{\chi^{n+1} u}, n \in \mathbb{N}.
\]
Given \( n \in \mathbb{N} \) and \( x \in H_{\chi^{n+1} u} \subset H_{\chi^{n+1+1} u} \), we have
\[
T_{n+1} x = U^n U_T' S^n u U^{n+1} x
= T_n x + U^n (U_T - S_u) T' S^n u U^{n+1} x.
\]
The vector \( U^n (U_T - S_u) T' S^n u U^{n+1} x \) belongs to \( H_{\chi^{n+1+1} u} \), and thus
\[
T_n x = P_{H_{\chi^{n+1} u}} T_{n+1} x, \quad x \in H_{\chi^{n+1} u}.
\]
In particular, \( T' = P_{H_u} T_n \big| H_u \) for every \( n \in \mathbb{N} \). Since \( \| T_n \| \leq \| T \|, n \in \mathbb{N} \), it follows that there exists an operator \( X \in B(K) \) with the property that \( T_n = P_{H_{\chi^{n+1} u}} X \big| H_{\chi^{n+1} u} \) for every \( n \in \mathbb{N} \). In fact, \( \bigcup_{m \in \mathbb{N}} H_{\chi^{m+1} u} \) is dense in \( K \), and
\[
X x = \lim_{n \to \infty} T_n x
\]
if \( x \in \mathcal{H}_{x^{\mu}} \) for some \( m \in \mathbb{N} \). The operator \( X \) satisfies the identity \( X = U_+ X U_+^* \). This implies that
\[
X(g \oplus 0) = \lim_{n \to \infty} U_+^n X U_+^n (g \oplus 0) = 0, \quad g \in H^2.
\]
Analogously, the equality \( X^* = U_+ X^* U_+^* \) yields \( X^*(g \oplus 0) = 0 \) for \( g \in H^2 \). Thus, \( X \) is of the form \( X = 0_{H^2} \oplus Y \), where \( Y \in \mathcal{B}(\mathcal{D}(\Delta L^2)^-) \). Since \( X \) commutes with \( U_+ \), it follows that \( Y \) commutes with multiplication by \( \chi \). Thus \( Y \) must be the operator of multiplication by some bounded measurable function \( d \), and therefore
\[
X(g \oplus h) = 0 \oplus d h, \quad g \oplus h \in K_+.
\]
The equality \( T' = P_{\mathcal{H}_{u}} X |_{\mathcal{H}_{u}} \) shows that \( T' \) is a truncated multiplication operator with symbol
\[
\begin{bmatrix}
0 & 0 \\
0 & d
\end{bmatrix}.
\]
Putting these facts together, we have shown that
\[
T = A_F + T' = A_{F'},
\]
where
\[
F' = \begin{bmatrix}
a_1 + \bar{a}_2 & \bar{b} \\
c & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & d
\end{bmatrix}.
\]
We have thus established the equivalence of (1) and (3), thus concluding the proof. \( \square \)

**Corollary 3.5** For every purely contractive function \( u \in H^{\infty} \), the operator system \( T_u \) is closed in the weak operator topology.

**Proof** By Theorem 3.4, membership of an operator \( T \) in \( T_u \) is characterized by the system of equations
\[
\langle TS_u x, S_u x \rangle = \langle Tx, x \rangle, \quad x \in \mathcal{H}_u \oplus \mathbb{C} \tilde{k}_0.
\]
Each of these equations is given by a continuous linear functional in the weak operator topology. \( \square \)

**Example 3.6** Given an arbitrary scalar \( \mu \in \mathbb{C} \), we define a bounded linear operator \( X_\mu \in \mathcal{B}(\mathcal{H}_u) \) by
\[
X_\mu x = \begin{cases}
U x = S_u x, & x \in \mathcal{H}_u \oplus \mathbb{C} \tilde{k}_0, \\
\mu \tilde{k}_0, & x = \tilde{k}_0.
\end{cases}
\]
It is easily verified using Theorem 3.4(2) that \( X_\mu \) is a truncated multiplication operator. We show in Corollary 3.9 that the commutant of \( X_\mu \) consists entirely of truncated multiplication operators. These rank one perturbations of \( S_u \) have been considered earlier in [5] (when \( u \) is inner) and [1] (see also [7,9]). The following result follows from [1].

**Proposition 3.7** Fix \( \mu \in \mathbb{C} \), a purely contractive function \( u \in H^{\infty} \), and let \( X_\mu \) be defined as in Example 3.6. Then:

1. For \( |\mu| < 1 \), the operator \( X_\mu \) is a completely nonunitary contraction with defect indices equal to 1.
(2) For $|\mu| > 1$, the operator $X_\mu$ is invertible and $X_\mu^{-1}$ is a completely nonunitary contraction with defect indices equal to 1.

(3) For $|\mu| = 1$, the operator $X_\mu$ is unitary with spectral multiplicity equal to 1.

**Corollary 3.8** With the notation of Proposition 3.7, the commutant of the operator $X_\mu$ is commutative for all $\mu \in \mathbb{C}\backslash\{0\}$. The commutant of $X_0$ is also commutative if $u$ is not a constant function.

**Proof** If $|\mu| \neq 1$, the corollary follows from parts (1) and (2) of Proposition 3.7 and from Lemma 2.5. The case $|\mu| = 1$ is a consequence of the general description of commutants of normal operators. The case $\mu = 0$ follows from the fact that the characteristic function of $X_0$ is zero precisely when $u$ is a constant function. \(\square\)

**Corollary 3.9** Let $\mu \in \mathbb{C}$, and let $X_\mu \in \mathcal{B}(\mathcal{H}_u)$ be the operator defined in Example 3.6. Then every operator $T \in \mathcal{B}(\mathcal{H}_u)$ that commutes with either $X_\mu$ or with $X_\mu^*$ is a truncated multiplication operator.

**Proof** We observe first that $\langle X_\mu h, Uk \rangle = \langle h, k \rangle$ if $h, k, Uk \in \mathcal{H}_u$. This is immediate if $Uh \in \mathcal{H}_u$ as well. On the other hand, if $h = \tilde{k}$, then $\langle X_\mu h, Uk \rangle = \langle h, k \rangle = 0$. Suppose now that $TX_\mu = X_\mu T$ and $k, Uk \in \mathcal{H}_u$. Then

$$\langle TUk, Uk \rangle = \langle TX_\mu k, Uk \rangle = \langle X_\mu Tk, Uk \rangle = \langle Tk, k \rangle,$$

by the preceding observation applied to $h = Tk$. Thus $T$ is $S_u$-invariant and $T \in \mathcal{T}_u$ by Theorem 3.4. If $TX_\mu^* = X_\mu^* T$ then the above argument shows that $T_\mu^* \in \mathcal{T}_u$ and thus $T \in \mathcal{T}_u$ because $\mathcal{T}_u$ is a selfadjoint space. \(\square\)

**Remark 3.10** In the case in which $u$ is an inner function, it was shown in [15] that every algebra contained $\mathcal{T}_u$ is contained either in $\{X_\mu\}'$ or in $\{X_\mu^*\}'$ for some $\mu \in \mathbb{C}$. It would be interesting to see whether this result remains true if $u$ is not inner. Note, incidentally, that $\mathcal{T}_u$ does contain a noncommutative algebra if $u$ is a constant function, namely the commutant of $X_0$ (see Example 4.3).

**Remark 3.11** In case $u$ is an extreme point of the unit ball of $H^\infty$, it is known (see, for instance, [13, Chapter IV]) that the projection onto the first component yields a unitary operator $J : \mathcal{H}_u \to \mathcal{H}(u)$, where $\mathcal{H}(u)$ is the de Branges–Rovnyak space associated to $u$. The operator $X = JS_u^*J^*$ is precisely the restriction to $\mathcal{H}(u)$ of the backward shift $f \mapsto \overline{\tau}(f - f(0))$. Therefore, Theorem 3.4 yields a characterization of those operators in $\mathcal{B}(\mathcal{H}(u))$ that are $X$-invariant.

### 4 Nonuniqueness of the symbol of a truncated multiplication operator

As noted earlier, the symbol of an operator in $\mathcal{T}_u$ is not unique. The proof of Theorem 3.4 shows that a certain sequence related with an operator $T \in \mathcal{T}_u$ converges in the weak operator topology. The following result identifies that limit in terms of an arbitrary symbol for $T$.

**Proposition 4.1** Suppose that $T \in \mathcal{B}(\mathcal{H}_u)$ is a truncated multiplication operator with symbol

$$
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
$$

\(\square\)
Then $d$ is essentially bounded on $\{\zeta \in \mathbb{T} : \Delta(\zeta) \neq 0\}$, and the sequence $\{S_n^* T S_n\}_{n \in \mathbb{N}}$ converges in the weak operator topology to the truncated multiplication operator with symbol
\[
\begin{bmatrix}
0 & 0 \\
0 & d
\end{bmatrix}.
\]

In particular, the function $d$ is uniquely determined almost everywhere on $\{\zeta \in \mathbb{T} : \Delta(\zeta) \neq 0\}$.

**Proof** Let $x = g \oplus h$ and $x' = g' \oplus h'$ be two vectors in $H_u$. We have
\[
\langle S_n^* T S_n x, x' \rangle = \langle T S_n^* S_n x, x' \rangle, \quad n \in \mathbb{N},
\]
and $S_n x = P_+(\overline{\chi}^n g) \oplus \overline{\chi}^n h$, where $P_+ : L^2 \to H^2$ denotes the orthogonal projection. By the M. Riesz theorem, $P_+$ also defines a bounded operator on $L^p$ for $p \in (2, +\infty)$. We have $g \in L^\infty \subset L^6$, $\lim_{n \to \infty} \|P_+(\overline{\chi}^n g)\|_2 = 0$, and
\[
\|P_+(\overline{\chi}^n g)\|_4 \leq \|P_+(\overline{\chi}^n g)\|_2^{1/4} \|P_+(\overline{\chi}^n g)\|_6^{3/4}, \quad n \in \mathbb{N}.
\]

We deduce that $\lim_{n \to \infty} \|P_+(\overline{\chi}^n g)\|_4 = 0$. Similarly, $\lim_{n \to \infty} \|P_+(\overline{\chi}^n g')\|_4 = 0$. Expand now
\[
\langle T S_n^* S_n x, x' \rangle = \langle a P_+(\overline{\chi}^n g), P_+(\overline{\chi}^n g') \rangle + \langle b \overline{\chi}^n h, P_+(\overline{\chi}^n g') \rangle + \langle c P_+(\overline{\chi}^n g), \overline{\chi}^n h' \rangle + \langle d \overline{\chi}^n h, \overline{\chi}^n h' \rangle.
\]
The fourth term on the right hand side is equal to $\langle dh, h' \rangle = \langle P_{H_u}(0 \oplus dh), x' \rangle$ for every $n \in \mathbb{N}$, and we show that the remaining three terms converge to zero as $n \to \infty$. The Hölder inequality yields
\[
\langle a P_+(\overline{\chi}^n g), P_+(\overline{\chi}^n g') \rangle \leq \|a\|_2 \|P_+(\overline{\chi}^n g)\|_4 \|P_+(\overline{\chi}^n g')\|_4,
\]
\[
\langle b \overline{\chi}^n h, P_+(\overline{\chi}^n g') \rangle \leq \|b\|_2 \|h\|_4 \|c P_+(\overline{\chi}^n g')\|_4,
\]
\[
\langle c P_+(\overline{\chi}^n g), \overline{\chi}^n h' \rangle \leq \|c\|_2 \|c P_+(\overline{\chi}^n g)\|_4 \|h'\|_4,
\]
and the sequences in the right hand side tend to zero, as shown above. We also see that $\|\langle dh, h' \rangle\| \leq \|T\| \|h\|_2 \|h'\|_2$.

To conclude the proof, we deduce from this inequality that $d$ is essentially bounded on $\{\zeta \in \mathbb{T} : \Delta(\zeta) \neq 0\}$. We observe that
\[
U^{sn}_+(u \oplus \Delta) = P_+(\overline{\chi}^{sn} u) \oplus \overline{\chi}^n \in H_u^\infty,
\]
and thus $\|\langle dh, h' \rangle\| \leq \|T\| \|h\|_2 \|h'\|_2$ if $h$ and $h'$ are of the form
\[
h = \sum_{j=1}^n \alpha_j \overline{\chi}^j, \quad h' = \sum_{j=1}^n \alpha'_j \overline{\chi}^j
\]
for some $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_n \in \mathbb{C}$. Moreover, since $\langle dh, h' \rangle = \langle d \overline{\chi}^m h, \overline{\chi}^m h' \rangle$ for every $m \in \mathbb{N}$, we must have $\|\langle dh, h' \rangle\| \leq \|T\| \|h\|_2 \|h'\|_2$ whenever $h = \Delta$ and $h' = \Delta q$ for some trigonometric polynomials $p$ and $q$. Since the trigonometric polynomials form a dense linear manifold in $L^2$, we see that
\[
\|\langle df\Delta, g\Delta \rangle\| \leq \|T\| \|f\Delta\|_2 \|g\Delta\|_2 \tag{4.1}
\]
for every pair $f, g$ of functions in $L^2$. This finally implies that $|d| \leq \|T\| \epsilon$ almost everywhere on $\{\zeta \in \mathbb{T} : \Delta(\zeta) \neq 0\}$. Indeed, in the contrary case, there exist $\epsilon, M > 0$ such that $\|T\| + \epsilon \leq |d| \leq M$ on a set $\sigma \subset \{\zeta \in \mathbb{T} : \Delta(\zeta) \neq 0\}$ of positive arclength. Then the choice $f = \overline{d} \chi_\sigma, g = 1$ contradicts (4.1). \qed

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We can now describe all the symbols associated to the zero operator.

**Proposition 4.2** Suppose that $T \in \mathcal{B}(\mathcal{H}_a)$ is a truncated multiplication operator with symbol

$$
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix},
$$

where $a, b, c \in L^2$ and $d \in L^\infty$. Then $T = 0$ if and only the following two conditions are satisfied:

1. $d = 0$ almost everywhere on the set $\{ \zeta \in \mathbb{T} : \Delta(\zeta) \neq 0 \}$.
2. There exist functions $f_1, f_2 \in H^2$ such that:
   
   a. $a = uf_1 + \overline{uf_2}$,
   
   b. $c = \Delta f_1$ and $b = \Delta \overline{f}_2$ almost everywhere on the set $\{ \zeta \in \mathbb{T} : \Delta(\zeta) \neq 0 \}$.

**Proof** The case in which $u$ is an inner function is proved in [14, Theorem 3.1]. We may therefore assume that $u$ is not inner, and thus the set $\{ \zeta \in \mathbb{T} : \Delta(\zeta) \neq 0 \}$ has positive arclength.

Suppose first that conditions (1) and (2) are satisfied and set

$$
F_1 = \begin{bmatrix}
  uf_1 & 0 \\
  \Delta f_1 & 0
\end{bmatrix}, \quad F_2 = \begin{bmatrix}
  uf_2 & 0 \\
  \Delta \overline{f}_2 & 0
\end{bmatrix},
$$

so $F = F_1 + F_2^*$. If $x = h \oplus g$ is an arbitrary element of $\mathbb{H}_u^\infty$, we have

$$
A_{F_1}x = uf_1 h \oplus \Delta f_1 h \in \mathcal{G},
$$

and thus $A_{F_1} = 0$. Similarly, $A_{F_2}^* = 0$, and therefore $T|\mathbb{H}_u^\infty = A_{F} = 0$.

Conversely, suppose that $T = 0$. Condition (1) follows from Proposition 4.1. In addition, we have $Tk_0 = T^* k_0 = 0$. Since $k_0 = (1 - \overline{u(0)}u) \oplus (-\overline{u(0)}\Delta)$, the vectors

$$
Fk_0 = [a(1 - \overline{u(0)}u) - \overline{bu(0)}\Delta] \oplus [c(1 - \overline{u(0)}u)],
$$

$$
F^* k_0 = [\overline{a}(1 - \overline{u(0)}u) - \overline{cu(0)}\Delta] \oplus [\overline{b}(1 - \overline{u(0)}u)],
$$

must belong to $\mathbb{H}_u^\perp = [H^2 \perp \{0\}] + \mathcal{G}$, that is,

$$
Fk_0 = (g_1 + uh_1) \oplus (\Delta h_1),
$$

$$
F^* k_0 = (g_2 + uh_2) \oplus (\Delta h_2),
$$

for some $g_1, g_2 \in H^2 \perp$ and $h_1, h_2 \in H^2$. Equating the second components, we see that

$$
c = \frac{\Delta h_1}{1 - \overline{u(0)}u} = \Delta f_1, \quad \overline{b} = \frac{\Delta h_2}{1 - \overline{u(0)}u} = \Delta f_2,
$$

where $f_j = h_j/(1 - \overline{u(0)}u) \in H^2$ for $j = 1, 2$. Define now $F_1$ and $F_2$ by (4.2) and set

$$
F_0 = F - F_1 - F_2^* = \begin{bmatrix}
  a_0 & 0 \\
  0 & 0
\end{bmatrix},
$$

where $a_0 = a - uf_1 - \overline{uf}_2$. The hypothesis and the first part of the proof show that $F_0$ is also a symbol of the zero operator, and thus the vectors

$$
F_0 k_0 = a_0 (1 - \overline{u(0)}u) \oplus 0, \quad F_0^* k_0 = \overline{a_0} (1 - \overline{u(0)}u),
$$

by Springer.
must belong to $H_u^\perp$. Observe that, given $f \in H^2$, the equality $\Delta f = 0$ implies that $f$ vanishes almost everywhere on the set $\{ \xi : \Delta(\xi) \neq 0 \}$, and thus $f = 0$ by the F. and M. Riesz theorem. Therefore there exist functions $g_1, g_2 \in H^2_u$ such that

$$a_0(1 - u(0)u) = g_1, \quad \overline{a_0}(1 - u(0)u) = g_2.$$  

We have then

$$a_0 = \frac{g_1}{1 - u(0)u} = \frac{g_2}{1 - u(0)u},$$

so

$$g_1(1 - u(0)\overline{u}) = \overline{g_2}(1 - u(0)u).$$

The left hand side of this equality has a Fourier series with no analytic terms, while the right hand side has only analytic terms. We conclude that $g_1 = g_2 = 0$, thus establishing that $F_0 = 0$. The proposition follows.

**Example 4.3** We examine the special case $u = 0$. In this case, $K_+ = H^2 \oplus L^2$, $G = \{ 0 \} \oplus H^2$, and $H_u = H^2 \oplus (L^2 \ominus H^2)$, and thus the operator $S_u$ is of the form $A \oplus B$, where $A$ is the forward shift on $H^2$ and $B$ is the forward (co-isometric) shift on $L^2 \ominus H^2$. (In other words, $S_u$ is unitarily equivalent to $A \oplus A^*$.) A symbol

$$F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

represents the zero operator in $\mathcal{T}_u$ precisely when $a = d = 0$ and $\overline{b}, c \in H^2$. In particular, the $(1,1)$ and $(2,1)$ entries of the symbol of an operator $T \in \mathcal{T}_u$ are uniquely determined by $T$. The operators that commute with $S_u$ are described, using the commutant lifting theorem, as the truncated multiplication operators with a symbol of the form

$$F = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix},$$

where $a, d \in H^\infty$ and $b \in L^\infty$. The commutant of $S_u$ is not commutative, as illustrated by the operators $T_1$ and $T_2$ with symbols

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } F_2 = \begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix},$$

for which $T_1T_2 = 0$ and $T_2T_1 = T_2 \neq 0$.

**5 An analog of the Crofoot operator**

Suppose that $u \in H^\infty$ satisfies $\|u\|_\infty \leq 1$ and $|u(0)| < 1$. The operator $X_\mu$ introduced in Example 3.6 is a completely nonunitary contractions with defect indices equal to 1 provided that $|\mu| < 1$. The characteristic function of $X_\mu$ is equal to

$$u_\alpha = \frac{u - \alpha}{1 - \alpha u} \in H^\infty,$$

where $\alpha \in \mathbb{D}$ is chosen such that $u_\alpha(0) = -\mu$. Thus, there exists a unitary operator in $\mathcal{B}(H_u, H_u^\alpha)$ that intertwines $X_\mu$ and $S_{u_\alpha}$. Such a unitary operator was first written explicitly by Crofoot [6] for the case in which $u$ is inner, and thus $u_\alpha$ is inner as well. He showed that
it is the restriction to $H_u$ of the multiplication operator by a function in $H^\infty$. We prove an analogous result for arbitrary purely contractive functions $u \in H^\infty$. To begin with, a simple calculation shows that the function $\Delta_\alpha = (1 - |\alpha|^2)^{1/2}$ satisfies

$$\Delta_\alpha = \frac{(1 - |\alpha|^2)^{1/2}}{|1 - \overline{\alpha}u|} - \Delta,$$

so $(\Delta_\alpha L^2)^\perp = (\Delta L^2)^\perp$, and therefore $H_u$ and $H_{u_\alpha}$ are both subspaces of $K$. More precisely,

$$H_{u_\alpha} = K_+ \ominus G_\alpha,$$

where

$$G_\alpha = \{u_\alpha f \ominus \Delta_\alpha f : f \in H^2\}.$$

We consider the bounded measurable function $F_\alpha$ defined by

$$F_\alpha = \begin{bmatrix}
(1 - |\alpha|^2)^{1/2}(1 - \overline{\alpha}u)^{-1} & 0 \\
\overline{\alpha}\Delta|1 - \overline{\alpha}u|^{-1} & (1 - \overline{\alpha}u)|1 - \overline{\alpha}u|^{-1}
\end{bmatrix} \begin{bmatrix}
(1 - |\alpha|^2)^{1/2}(1 - \overline{\alpha}u)^{-1} & 0 \\
\overline{\alpha}(1 - |\alpha|^2)^{-1/2}\Delta_\alpha & (1 - \overline{\alpha}u)|1 - \overline{\alpha}u|^{-1}
\end{bmatrix}.$$

Since the $(1, 1)$ entry of $F_\alpha$ belongs to $H^\infty$, it follows that $M_{F_\alpha}$ leaves $K_+$ invariant.

**Proposition 5.1** The operator $M_{F_\alpha}$ maps $H_u$ isometrically onto $H_{u_\alpha}$.

**Proof** Suppose that $f \ominus g \in H_u$ and thus $uf + \Delta g \in L^2 \ominus H^2$. As noted above, the vector $f_\alpha \ominus g_\alpha = M_{F_\alpha}(f \ominus g)$ belongs to $K_+$. A direct calculation shows that

$$\overline{uf_\alpha} + \Delta g_\alpha = (1 - |\alpha|^2)^{1/2}(\overline{uf} + \Delta g)(1 - \alpha \overline{u})^{-1},$$

and this function belongs to $L^2 \ominus H^2$ because $(1 - \alpha \overline{u})^{-1}$ is conjugate analytic and bounded. We conclude that $f_\alpha \ominus g_\alpha \in H_{u_\alpha}$. In order to calculate the norm of $f_\alpha \ominus g_\alpha$ we observe that

$$w = (\overline{u}(1 - |\alpha|^2)^{-1/2}u_\alpha f) \oplus (\overline{u}(1 - |\alpha|^2)^{-1/2}\Delta_\alpha f) \in G_\alpha$$

and thus

$$\|f_\alpha \ominus g_\alpha\|^2 = \|(f_\alpha \ominus g_\alpha) - w\|^2 + \|w\|^2 = \|(f_\alpha \ominus g_\alpha) - w\|^2 + |\alpha|^2(1 - |\alpha|^2)^{-1}\|f\|^2.$$

Since

$$(f_\alpha \ominus g_\alpha) - w = ((1 - |\alpha|^2)^{-1/2}f) \oplus ((1 - \overline{\alpha}u)|1 - \overline{\alpha}u|^{-1}g),$$

it follows that $\|(f_\alpha \ominus g_\alpha) - w\|^2 = (1 - |\alpha|^2)^{-1}\|f\|^2 + \|g\|^2$ and hence that $\|f_\alpha \ominus g_\alpha\|^2 = \|f \ominus g\|^2$. The fact that $M_{F_\alpha}$ maps $H_u$ onto $H_{u_\alpha}$ follows from the above considerations applied to the operator $M_{F_\alpha^{-1}}$ because

$$F_{\alpha^{-1}} = \begin{bmatrix}
(1 - |\alpha|^2)^{1/2}(1 + \overline{\alpha}u_\alpha)^{-1} & 0 \\
\overline{\alpha}\Delta_\alpha|1 - \overline{\alpha}u_\alpha|^{-1} & (1 - \overline{\alpha}u_\alpha)|1 - \overline{\alpha}u_\alpha|^{-1}
\end{bmatrix},$$

and $u = (u_\alpha + \alpha)(1 + \overline{\alpha}u_\alpha)^{-1}$. $\square$

We denote by $V_\alpha \in B(H_u, H_{u_\alpha})$ the unitary operator defined by $V_\alpha x = M_{F_\alpha} x, x \in H_u$. In the case in which $u$ is inner, $V_\alpha$ is precisely the operator constructed in [6].

**Proposition 5.2** An operator $T \in B(H_u)$ is a truncated multiplication operator if and only if $V_\alpha TV_\alpha^* \in B(H_{u_\alpha})$ is a truncated multiplication operator. Thus, $T_{u_\alpha} = \{V_\alpha TV_\alpha^* : T \in T_u\}$.
Proof Fix $T \in B(H_u)$ and define $T_\alpha = V_\alpha TV_\alpha^*$, so
\[
(T_\alpha V_\alpha x, V_\alpha x) = (Tx, x), \quad x \in H_u.
\]
Since $M_{F_\alpha}U = UM_{F_\alpha}$, it follows that $Ux \in H_u$ if and only if $UV_\alpha x \in H_{u_\alpha}$. We conclude from the preceding identity that $T$ is $U$-invariant if and only if $T_\alpha$ is $U$-invariant. The proposition follows from Theorem 3.4.

6 Complex symmetries

Suppose that $H$ is a (complex) Hilbert space. A map $C : H \to H$ is called a conjugation if it is conjugate linear, isometric, and $C^2 = I_H$. A bounded operator $T \in B(H)$ is said to be $C$-symmetric (respectively $C$-skew-symmetric) if $C^*T = T^*$ (respectively, $C^*T = -T^*$). The operator $T$ is said to be complex symmetric if it is $C$-symmetric for some conjugation $C$. An operator $T$ can be complex symmetric relative to several conjugations. For instance, suppose that $u \in B(L^2)$ is the bilateral shift, that is, $Uf = \chi f$, $f \in L^2$. Given an arbitrary function $v \in L^\infty$ such that $|v| = 1$ almost everywhere, the formula
\[
C_v f = v\overline{f}, \quad f \in L^2,
\]
defines a conjugation on $L^2$ such that $U$ is $C_v$-symmetric. (It easy to see that these are all the conjugations relative to which $U$ is symmetric.)

Proposition 6.1 Suppose that $T \in B(H)$, and $C$ and $D$ are two symmetries such that $T$ is both $C$-symmetric and $D$-symmetric. Then at least one of the following is true:

1. There exists a constant $\gamma \in \mathbb{T}$ such that $D = \gamma C$.
2. There exists a proper reducing subspace $K$ for $T$ such that both $T|K$ and $T|K^\perp$ are complex symmetric.

Proof Suppose that (1) is not true, and therefore the operator $V = DC$ is not a scalar multiple of $I_H$. The operator $V$ is unitary and
\[
VT = DCT = DT^*C = TDC = TV.
\]
Moreover, we have
\[
CVC = CD = (DC)^{-1} = V^{-1} = V^*,
\]
so $V$ is $C$-symmetric. If $E_V$ denotes the spectral measure of $V$, it follows that $E_V(\omega)$ is also $C$-symmetric for every Borel set $\omega \subset \mathbb{T}$, and therefore $E(\omega)TE(\omega)$ is also $C$-symmetric. To show that (2) is true, simply choose $\omega$ such that $0 \neq E(\omega) \neq I_H$ and set $K = E(\omega)H$. Then $T|K$ is $C|K$-symmetric and $T|K^\perp$ is $C|K^\perp$-symmetric.

Given a function $u \in H^\infty$ such that $|u|_\infty \leq 1$ and $|u(0)| < 1$, the operator $S_u$ does not have any nontrivial reducing subspaces unless $u = 0$. For $u = 0$, $S_u$ has exactly one pair of complementary nontrivial reducing subspaces, and the restrictions of $S_u$ to these spaces are a unilateral shift and the adjoint of a unilateral shift, neither of which is complex symmetric. It follows that, up to a constant multiple of modulus one, there is at most one conjugation $C$ such that $S_u$ is $C$-symmetric. If $u$ is inner or, more generally, if $u$ is an extreme point of the unit ball of $H^\infty$, it follows from [10] that $S_u$ is complex symmetric (see also [8]). More general results about functional models [4] show that $S_u$ is always complex symmetric. We describe below the essentially unique conjugation $C_u$ such that $S_u$ is $C_u$-symmetric.

The spaces $K$, $G$, and $H_u$ in the following statement are as defined in Sect. 2.
Proposition 6.2 Let $u \in H^\infty$ be such that $\|u\|_\infty \leq 1$ and $|u(0)| < 1$. Then the operator $C : K \to K$ defined by

$$C(f \oplus g) = (\overline{u} f + \overline{u} \Delta \overline{g}) \oplus (\overline{u} \Delta \overline{f} - \overline{u} \Delta \overline{g}), \quad f \oplus g \in K,$$

is a conjugation such that $U$ is $C$-symmetric. Moreover, we have $C H_u = H_u$ and the operator $C_u = C|H_u$ is a conjugation such that $S_u$ is $C_u$-symmetric.

Proof The operator $C$ is simply complex conjugation followed by multiplication by the matrix function

$$\chi \begin{bmatrix} u & \Delta \\ \Delta & -\overline{u} \end{bmatrix}.$$ 

It is easily seen that the matrix

$$\begin{bmatrix} u(\xi) & \Delta(\xi) \\ \Delta(\xi) & -u(\xi) \end{bmatrix}$$

is unitary for $\xi \in \mathbb{T}$, and thus $C$ is an isometry. The operator $C^2$ is the multiplication operator by the matrix function

$$\begin{bmatrix} \overline{u} & \Delta \\ \Delta & -\overline{u} \end{bmatrix} \begin{bmatrix} u & \Delta \\ \Delta & -\overline{u} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and thus $C^2 = I_{H_u}$. The identity $U^* C = C U$ is also immediate. Observe next that

$$C(u f \oplus \Delta f) = \overline{u} \Delta f \oplus 0, \quad f \in H^2,$$

which shows that $C(G) = H^2 \oplus \{0\}$ and thus $C(H^2 \oplus \{0\}) = G$ as well. We conclude that $C(H_u^+) = H_u^+$, $C(H_u) = H_u$, and $C_u$ is indeed a conjugation on $H_u$. Finally,

$$S_u C_u = P_{H_u} U C|H_u = P_{H_u} C U^*|H_u = C P_{H_u} U^*|H_u = C_u S_u^*,$$

showing that $S_u$ is $C_u$-symmetric. \(\square\)

We note for further use the equality

$$C_u k_0 = \tilde{k}_0. \tag{6.1}$$

The linear manifold $K^\infty$ is invariant under the conjugation $C$. It is not the case that every multiplication operator $M_F$ satisfies the equation $M_F v = C M_{F^*} C v$ for every $v \in K^\infty$.

Proposition 6.3 The multiplication operator $M_F$ by the matrix function

$$F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

satisfies the equation $M_F = C M_{F^*} C|K^\infty$ if and only if the equality

$$\Delta(d - a) = -uc - \overline{u} b \tag{6.2}$$

holds almost everywhere on $\{\xi \in \mathbb{T} : \Delta(\xi) \neq 0\}$.

Proof The operator $C M_{F^*} C|K^\infty$ is the operator of multiplication by the matrix

$$\begin{bmatrix} u & \Delta \\ \Delta & -\overline{u} \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} \overline{u} & \Delta \\ \Delta & -u \end{bmatrix},$$

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and a calculation shows that

\[
\begin{bmatrix}
    u & \Delta \\
    \Delta & -\bar{u}
\end{bmatrix}
\begin{bmatrix}
    a & c \\
    b & d
\end{bmatrix}
\begin{bmatrix}
    \bar{u} & \Delta \\
    \Delta & -u
\end{bmatrix}
- F =
\begin{bmatrix}
    \Delta h & -uh \\
    -\bar{u}h & -\Delta h
\end{bmatrix}
\]

where

\[
h = \Delta(d - a) + uc + \bar{ub}.
\]

The proposition follows. \(\square\)

**Corollary 6.4** If the matrix

\[
F = \begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
\]

satisfies the equality \(\Delta(a - d) = uc + \bar{ub}\) almost everywhere on \(\{\zeta \in \mathbb{T} : \Delta(\zeta) \neq 0\}\), then \(A_F = C_u A_{F^*} C_u |H_\infty^u\).

In the particular case in which \(u\) is an inner function, the function \(\Delta\) is equal to zero almost everywhere. Thus, the preceding corollary shows that every operator in \(T_u\) is \(C_u\)-symmetric. This result [14, Section 2.3] plays an important role in the study of truncated Toeplitz operators. If \(u\) is not inner, there are operators in \(T_u\) that are not \(C_u\)-symmetric. For instance, the operator with symbol

\[
\begin{bmatrix}
    0 & 0 \\
    0 & 1
\end{bmatrix}
\]

is not \(C_u\)-symmetric.

Suppose that \(T \in B(H_u)\) is a truncated multiplication operator. Then the operator \(C_u T^* C_u\) is also a truncated multiplication operator. It follows that \(T\) can be written in a unique way as a sum \(T = T_1 + T_2\), where \(T_1 = (1/2)(T + C_u T^* C_u)\) is a \(C_u\)-symmetric truncated multiplication operator and \(T_2 = (1/2)(T - C_u T^* C_u)\) is a \(C_u\)-skew-symmetric operator. The above calculations allow us to show that the operators \(T_1\) and \(T_2\) have symbols of a special form.

**Proposition 6.5** Suppose that \(T \in T_u\). Then:

(1) If \(T\) is \(C_u\)-symmetric then it has a symbol of the form

\[
\begin{bmatrix}
    a & b \\
    c & a - (\bar{ub} + uc)/\Delta
\end{bmatrix}
\]

for some \(a, b, c \in L^2\).

(2) If \(T\) is \(C_u\)-skew symmetric then it has a symbol of the form

\[
\begin{bmatrix}
    -\Delta f & uf \\
    \bar{uf} & \Delta f
\end{bmatrix}
\]

for some \(f \in L^2\).

**Proof** By Theorem 3.4, \(T\) has a symbol of the form

\[
G = \begin{bmatrix}
    \alpha & \beta \\
    \gamma & \delta
\end{bmatrix}
\]

with \(\alpha, \beta, \gamma \in L^2\) and \(\delta \in L^\infty\). If \(T\) is \(C_u\) symmetric, the function \(F\) such that \(M_F = (1/2)(M_G + C M_G^* C)\) is again a symbol for \(T\) and it has the form specified in (1) by Proposition
of 6.3. If $T$ is $C_u$-skew-symmetric, we use instead the operator $M_H = (1/2)(M_G - CM_G^*C)$. The proof of Proposition 6.3 shows that $H$ has the form specified in (2). □

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