Remarks on M-Theory Coupling Constants
and M-Brane Tension Quantizations

J. X. Lu

Center for Theoretical Physics, Physics Department, Texas A & M University,
College Station, Texas 77843

ABSTRACT

In the absence of a complete M-theory, we gather certain quantum aspects of this
theory, namely, M-2 and M-5 brane duality and their tension quantization rule
$2\kappa^2 T_2 T_5 = 2\pi n$, the M-2 brane tension quantization $T_2 = \left(\frac{(2\pi)^2}{2\kappa^2 m}\right)^{1/3}$, su-
persymmetry, perturbative gauge and gravitational anomaly cancellations, and the
half-integral quantization of $[G^W/2\pi]$, and study the consistency among these quan-
tum effects. We find: (1) The complete determination of Hořava-Witten’s $\eta = \lambda^6/\kappa^4$
for M-theory on $R^{10} \times S^1/Z_2$ requires not only the cancellation of M-theory gauge
anomaly but also that of the gravitational anomaly, the quantization of M-2 brane
tension, and the recently recognized half-integral quantization of $[G^W/2\pi]$. (2) A
well-defined quantum M-theory necessarily requires the presence of both M-2 and
M-5 branes and allows only $n = 1$ and $m = 1$ for the respectively quantized M-2
and M-5 brane tensions. Implications of the above along with other related issues are
discussed.

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1 Introduction

Hořava and Witten [1, 2] recently proposed that the strongly coupled ten dimensional $E_8 \times E_8$ heterotic string is described by M-theory on $M^{11} = R^{10} \times S^1 / \mathbb{Z}_2$. Cancellation of the gravitational anomaly appearing on the boundary of space time in the M-theory requires the introduction of gauge fields with gauge group a copy of $E_8$ on each boundary component. The gauge fields enter via ten dimensional vector multiplets that propagate on the boundary of space time. This picture immediately implies that there must exist a supersymmetric coupling of ten-dimensional vector multiplets on the boundary of the above eleven-manifold to the eleven-dimensional supergravity multiplet propagating on the bulk. In the so-called “upstairs” approach (We will discuss the “upstairs” and “downstairs” approaches later in this section), this has been achieved to the lowest order in supersymmetry in [2] through modifying the Bianchi identity $dG = 0$ for the four-form field strength $G^{IJKL}$ to

$$dG = -\frac{\kappa^2}{\sqrt{2} \lambda^2} \delta(x^{11}) dx^{11} \text{tr} F^2,$$

(1.1)

where $\kappa$ is the $D = 11$ gravitational coupling constant while $\lambda$ is the $E_8$ gauge coupling constant. Based on the known form of the ten dimensional anomalies, Hořava and Witten extended Eq. (1.1) to

$$dG = \frac{\kappa^2}{\sqrt{2} \lambda^2} \delta(x^{11}) dx^{11} \hat{I}_4,$$

(1.2)

where

$$\hat{I}_4 = \frac{1}{2} \text{tr} R^2 - \text{tr} F^2,$$

(1.3)

with tr the trace in the fundamental representation of the corresponding group (for $E_8$, it is defined as $\text{Tr} F^2 = 30 \text{ tr} F^2$ with Tr the trace in the adjoint representation of $E_8$, we will use Hořava and Witten’s notation throughout unless stated otherwise. In their notation, $G_{IJKL} = \partial_I C_{JKL} + 23$ terms, $C = C_{IJK} dx^I dx^J dx^K$, and $dG = \frac{1}{3} G_{IJKL} dx^I dx^J dx^K dx^L$. The comparison with the notation of [3] is as follows: $C^{DKL} = 6\sqrt{2} C$ and $G^{DKL} = \sqrt{2} G$. Our indices I, J, K, L run from 1 to 11 while A, B, C, D run from 1 to 10.)
Modifying \( dG = 0 \) to the form of Eq. (1.2) implies that the three-form potential \( C \) is in general transformed under a gauge or a local Lorentz transformation (This is familiar in coupling \( N = 1 \) supergravity to super Yang-Mills theory in ten dimensions). The “Chern-Simons” interactions \( \int C \wedge G \wedge G \) present in the eleven-dimensional supergravity is therefore not invariant under either of these transformations, therefore the presence of anomalies. With the correction \( \int C \wedge X_8 \) to the eleven-dimensional supergravity action either from membrane-fivebrane duality based on M-5 brane worldvolume one loop anomalies [4] or from one-loop calculation of Type IIA superstring [3], both the above gauge and gravitational anomalies can potentially be cancelled. In the above
\[
X_8 = -\frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2.
\]
(1.4)

For a special solution of \( G \) from Eq. (1.2), this is indeed true as shown in [2, 6] even though numerical errors occurred which has been corrected recently in [7]. Requiring anomaly-freedom also fixes the ratio of \( \lambda^6 / \kappa^4 \), i.e., determining the gauge coupling constant in terms of the gravitational constant.

Not long after his above work with Hořava, Witten [8] pointed out that \( G^W / 2\pi \) should have in general a half-integral period rather than an integral one as previously used in establishing the quantization of the M-2 brane tension in terms of the gravitational constant \( \kappa \) as
\[
T_2 = \left( \frac{(2\pi)^2}{2\kappa^2 m} \right)^{1/3} \quad (m = \text{integer}).
\]
(1.5)

This is based on the observation that there is a sign ambiguity for the fermion path integral for fermions on the membrane worldvolume. This potential inconsistency in \( \tilde{G}^W \) is the one used by Witten in his paper [8]. Its relation to the present \( G \) and to the \( G_{D KL} \) given in footnote 1 is: \( G^W \equiv T_2 g^{D KL} = \sqrt{2} T_2 G \).
defining the sign of the fermion path integral is fortunately correlated to the “Chern-Simons” factor coming from the coupling of the membrane worldvolume to the three-form potential $C$. A well-defined membrane path integral can be obtained if $G^W/2\pi$ has a half-integral period in general. For the aforementioned special solution of $G$ from Eq. (1.2), it has been shown in [8] that $G^W/2\pi$ has indeed a half-integral period in general.

In this article, we intend to ask ourselves what we can learn, on a general ground, from the requirement of anomaly-freedom for M-theory on $M^{11} = R^{10} \times S^1/Z_2$ and that of a well-defined membrane path integral, i.e., $G^W/2\pi$ has a half-integral period in general. Contrary to what has been claimed in [2, 6] and recently in [7], the cancellation of pure gauge anomaly or pure gravitational anomaly or both anomalies cannot determine $\eta = \lambda^6/\kappa^4$ uniquely. To completely determine this $\eta$, we need in addition the following: (1) M-2 and M-5 brane duality and the associated quantization rule

$$2\kappa^2T_2T_5 = 2\pi n \quad (n = \text{integer}),$$

(2) $G^W/2\pi$ has a half-integral period in general, (3) M-2 brane tension $T_2$ is quantized according to Eq. (1.5). There exist also alternatives of the above which we will discuss in the next section. The other lesson of this investigation is that a well-defined quantum M-theory requires the presence of both M-2 and M-5 branes and allows only M-2 brane tension $T_2 = \left(\frac{2\pi}{2\kappa^2n}\right)^{1/3}$ and M-5 brane tension $T_5 = \left(\frac{2\pi}{(2\kappa^2)^{1/5}}\right)^{1/3}$, i.e.,

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3The final expression for $G^W/2\pi$ is correct in spite of some numerical errors in [2].

4In [4], $n$ was imposed to be a non-negative integer from the fact that a p-brane tension is the measure of the energy per unit p-brane volume and it should be non-negative. Recently, these tensions are also allowed to be negative purely from the viewpoint of classical solutions from supergravity theories. Past experiences tell us that these extended objects with negative tensions cannot be stable quantum mechanically. We therefore insist that $n$ be non-negative throughout this paper since we are considering the quantum effects of M-theory. Hence the integer $n$ in Eq. (1.5) should also be non-negative. But as we will see in the next section, this will follow automatically once $n$ is non-negative. If the non-negative condition for $n$ is dropped, the solution of $\alpha = 1$, $m = -1$ and $n = -1$ is also allowed in addition to the one obtained in the next section.
only $m = 1$ and $n = 1$ are allowed in Eqs. (1.3) and (1.6), respectively.

Two things remain to be discussed before we move on to the next section. One is about the M-2 brane tension quantization of Eq. (1.5). The other is about clarifying the role of the “upstairs” and “downstairs” approaches.

The knowledge we learned from perturbative string theories about the relations among various coupling constants indicates that there might be only one independent constant in $D = 11$ M-theory in the absence of a dilaton. This implies that there exists a relation between the M-2 brane tension $T_2$ and the gravitational constant $\kappa$ (similarly, a relation between $T_5$ and $\kappa$). For a special choice of a twelve manifold as $Q = D^4 \times D^4 \times D^4$, this relation has been established as given by Eq. (1.5) in [4, 9]. For this special choice of manifold, the $m$ in Eq. (1.5) remains still as an integer even with the recent work of Witten [8] that $G^W/2\pi$ has a half-integral period in general and $\int_Q G \wedge G \wedge G$ is half-integral in general. One cannot derive this formula for a general twelve manifold $Q$. The arguments presented above nevertheless support that Eq. (1.5) with $m = \text{integer}$ should be true in general.

In the “upstairs” approach of Hořava and Witten [2], the bulk action of M-theory on $R^{10} \times S^1/Z_2$ is given by

$$S_M^U = - \frac{1}{2\kappa^2} \int_{M_U^{11}} d^{11}x \sqrt{-g_M}(R + \cdots),$$

(1.7)

where $M_U^{11} = R^{10} \times S^1$, all fields are $Z_2$ symmetric, $Z_2$ is generated by $x^{11} \mapsto -x^{11}$ and “$\cdots$” are the terms involving three-form field and fermions. In the above, the $\kappa^2$ is the one used in the usual eleven dimensional supergravity which is believed to be the low-energy effective description of M-theory in $D = 11$. Recently, Conrad [7] claimed that the $\kappa^2$ appearing in the above “upstairs” action should be replaced by $2\kappa^2$. By this, Conrad obtained the following “downstairs” action following Hořava
and Witten\[2\]

\[
S_M^D = -\frac{1}{2\kappa^2} \int_{M^{11}_D} d^{11}x \sqrt{-g_M} (R + \cdots),
\] (1.8)

where \(M^{11}_D = R^{10} \times S^1/Z_2 = R^{10} \times I\).

In what follows, I will argue that HoŃava and Witten’s original “upstairs” action given by Eq. (1.7) is with the correct unit but their “downstairs” action given in [2] cannot be the bulk action describing the local physics observed in the bulk when the radius of the \(S^1/Z_2\) is large. On the other hand, Conrad’s above proposed “downstairs” bulk action is with the correct unit but not his “upstairs” action.

Our arguments are based on the following: (1) The “upstairs” and “downstairs” approaches each should give the same results when implemented properly. (2) The bulk M-2 and M-5 brane tensions are independent of whether any boundaries exist arbitrarily far away from a local observer or whether a dimension is compact at arbitrary large scales\[5\]. (3) The bulk Lagrangian constructed by the local observer based on local symmetries such as the local supersymmetry should be the same. (4) To the local observer, the equations of motion describing, for example, a M-2 brane with a given tension \(T_2\) in the bulk background fields, whether they are derived from the “upstairs” bulk background field action plus the M-2 brane worldvolume action or from the “downstairs” correspondent, should be the same as those from the usual eleven dimensional supergravity action plus a M-2 brane worldvolume action, e.g., those given by Duff and Stelle\[10\].

The above points, especially (4), immediately imply that the correct “upstairs” bulk action is given by Eq. (1.7) while the correct “downstairs” bulk action is given by Eq. (1.8). Otherwise, we would obtain different equations of motion describing the M-2 brane moving in the bulk from “upstairs” and “downstairs” approaches since the observer has the same worldvolume action describing the M-2 brane with a given

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\[5\]This point is borrowed from [6].
tension $T_2$ in both cases.

In the above sense, the “downstairs” bulk action can be effectively identified with either $x^{11} > 0$ or $x^{11} < 0$ component of the “upstairs” bulk action with the $Z_2$ symmetry imposed. However, as the radius of $S^1/Z_2$ approaches zero, i.e., taking the weakly coupled limit of the heterotic string, for which the sense of bulk is diminishing, two copies of the “downstairs” bulk action Eq. (1.8) have to be used such that the correct $N = 1$ $D = 10$ supergravity can be obtained, i.e., in this limit, Hořava and Witten’s “downstairs” action is with the correct normalization. Therefore, in the “downstairs” approach, studies of the aforementioned bulk properties as well as the perturbative gauge and gravitational anomalies of the M-theory need Conrad’s “downstairs” action while in obtaining the correct weakly coupled low energy effective action of the heterotic string, we need Hořava and Witten’s “downstairs” action. An independent check will be provided in section 3, based on a recent work [11], that Conrad’s “downstairs” action is needed in order to cancel both gauge and gravitational anomalies. We will never encounter the above complications if the “upstairs” approach is employed. As we will see in the next section, there is actually a subtle difference between these two approaches.

The above discussion clearly demonstrates the point of Hořava and Witten [2] that even though the “upstairs” approach does not manifest the feature of M-theory on $R^{11} \times S^1/Z_2$, it is indeed convenient for calculation. On the other hand, the “downstairs” approach is just the other way around. Carrying out the calculations properly in the “downstairs” approach is explained well in [2, 6, 7]. In the “upstairs” approach, we always begin with the action as if it is on $R^{10} \times S^1$ or simply on $R^{11}$. Only at the last step in obtaining the results we need, the $Z_2$ symmetry is imposed on the fields.
2 Analysis of Anomalies

2.1 “Upstairs” Approach

We begin with the “upstairs” approach. For all the previous studies, a specific solution of Eq. (1.2) was always chosen [2, 6, 7]. This equation actually has the following general solution

\[
G = 6dC + \alpha \frac{\kappa^2}{\sqrt{2\lambda^2}} \delta(x^{11}) dx^{11} Q_3 + \frac{(1 + \alpha)}{2} \frac{\kappa^2}{\sqrt{2\lambda^2}} \epsilon(x^{11}) \hat{I}_4, \tag{2.1}
\]

where \(\alpha\) is an as yet undetermined dimensionless constant, \(Q_3 = \frac{1}{2} \omega_{3L} - \omega_{3Y}\) and \(\epsilon(x^{11})\) is a step function such that \(\epsilon(x^{11}) = -\epsilon(-x^{11}) = 1\) if \(x^{11} > 0\) and \(d\epsilon(x^{11})/dx^{11} = 2\delta(x^{11})\). This general solution was also given recently in [12].

In terms of components, we have

\[
G_{11ABC} = 4! \partial_{[11} C_{ABC]} + \alpha \frac{\kappa^2}{\sqrt{2\lambda^2}} \delta(x^{11}) Q_{3ABC}, \tag{2.2}
\]

and

\[
G_{ABCD} = 4! \partial_{[A} C_{BCD]} + 4! \frac{(1 + \alpha)}{8} \frac{\kappa^2}{\sqrt{2\lambda^2}} \epsilon(x^{11}) \left[ \frac{1}{2} \text{tr} R_{[AB} R_{CD]} - \text{tr} F_{[AB} F_{CD]} \right], \tag{2.3}
\]

where \(C_{ABC} = 0\) at \(x^{11} = 0\).

For comparison with Hořava and Witten’s result of the pure gauge anomaly cancellation [2], we consider the gauge anomaly first. Under a gauge transformation \(\delta A = -D\epsilon(x)\), from \(\delta G = 0\) we deduce

\[
\delta C = -\frac{\alpha}{3!} \frac{\kappa^2}{\sqrt{2\lambda^2}} \delta(x^{11}) dx^{11} Q_{2Y}^1, \tag{2.4}
\]

where \(Q_{2Y}^1 = -\text{tr} \epsilon F\). In components,

\[
\delta C_{11AB} = \frac{\alpha}{3!} \frac{\kappa^2}{\sqrt{2\lambda^2}} \delta(x^{11}) \text{tr} F_{AB}, \tag{2.5}
\]
and $\delta C_{ABC} = 0$. Under this gauge transformation, the “Chern-Simons” interaction term

$$W = -\frac{\sqrt{2}}{3456\kappa^2} \int_{M^6} d^{11}x \epsilon^{M_1M_2\cdots M_1} C_{M_1M_2M_3} G_{M_4\cdots M_7} G_{M_8\cdots M_11}, \quad (2.6)$$

in the classical action of $D = 11$ supergravity is not invariant but transforms as

$$\delta W = -\frac{\alpha(1 + \alpha)^2 \kappa^4}{1536 \lambda^6} \int_{M^{10}} d^{10}x \epsilon^{A_1A_2\cdots A_{10}} \text{tr} \epsilon F_{A_1A_2} \text{tr} F_{A_3A_4} F_{A_5A_6} \text{tr} F_{A_7A_8} F_{A_9A_{10}}. \quad (2.7)$$

To cure such a gauge non-invariance of the classical theory, we have to appeal to quantum anomalies. In the present case that the gauge group is $E_8$ and the Majorana-Weyl fermions are in the adjoint representation, the anomalous variation of the effective action $\Gamma$ for the ten-dimensional fermions is

$$\delta \Gamma = \frac{1}{2} \frac{1}{(4\pi)^5} \frac{1}{6!} \int_{M^{10}} d^{10}x \epsilon^{A_1A_2\cdots A_{10}} \text{Tr}(\epsilon F_{A_1A_2} F_{A_3A_4} \cdots F_{A_9A_{10}}), \quad (2.8)$$

where $\text{Tr}$ is the trace in the adjoint representation of gauge group $E_8$. As in [2], for $E_8$ we have the identity $\text{Tr} W^6 = (\text{Tr} W^2)^3 / 7200$ (and likewise, $\text{Tr} \epsilon F^5 = \text{Tr} \epsilon F (\text{Tr} F^2)^2 / 7200$) and the relation $\text{tr} W^2 = \text{Tr} W^2 / 30$. Then we have $\text{Tr} \epsilon F^5 = (15/4) \text{tr} \epsilon F (\text{tr} F^2)^2$. With this, Eq. (2.8) can be rewritten as

$$\delta \Gamma = \frac{1}{16(4\pi)^5 4!} \int_{M^{10}} d^{10}x \epsilon^{A_1A_2\cdots A_{10}} \text{tr} \epsilon F_{A_1A_2} \text{tr} F_{A_3A_4} F_{A_5A_6} \text{tr} F_{A_7A_8} F_{A_9A_{10}}. \quad (2.9)$$

Setting $\delta W + \delta \Gamma = 0$, we have

$$\eta = \frac{\lambda^6}{\kappa^4} = \frac{\alpha(1 + \alpha)^2}{4} (4\pi)^5. \quad (2.10)$$

Unlike in [2], we can determine the $\eta$ only up to an as yet undetermined constant $\alpha$ if only the gauge anomaly cancellation is imposed. That Hořava and Witten can determine the $\eta$ uniquely at this stage is because they chose a specific solution, i.e.,

$^{6}$There seems short of a factor $\frac{1}{6}$ in Eq. (3.5) of [2] for $\delta \Gamma$ (where a $\frac{1}{6}$ factor is indeed included for Majorana-Weyl fermions). This can be easily checked if one uses the anomalous $12$-form, which should be halved for Majorana-Weyl spinors, for gauge fields from Green-Schwarz-Witten [13]. Using the standard procedure, we obtain Eq. (2.8). This was also pointed out in [7].
setting $\alpha = 1$ from the outset in Eq. (2.1), for $G_{11ABC}$ and $G_{ABCD}$. In the following, we will show that to determine the $\eta$ completely, more conditions are needed as discussed in the introduction.

We now consider both gauge and gravitational anomaly cancellations. The “Chern-Simons” interaction of Eq. (2.6) can be re-expressed as

$$W = -\frac{\sqrt{2}}{\kappa^2} \int_{M^{11}} C \wedge G \wedge G. \quad (2.11)$$

In order to determine the variation of $W$ under both a gauge transformation $\delta A = -D\epsilon$ and a local Lorentz variation $\delta \omega = -D\Theta$ with $\omega$ the spin connection, we have to determine the variation of the 3-form $C$ first. It can be deduced from $\delta G = 0$ as

$$\delta C = \frac{\alpha}{3!} \frac{\kappa^2}{\sqrt{2} \lambda^2} \delta(x^{11}) dx^{11} Q_2^1, \quad (2.12)$$

where

$$Q_2^1 = -\left(\frac{1}{2} \text{tr} \Theta R - \text{tr} \epsilon F\right). \quad (2.13)$$

Then we have

$$\delta W = -\frac{\alpha(1 + \alpha)^2}{12} \frac{\kappa^4}{\lambda^6} \int_{M^{10}} Q_2^1 \wedge \hat{I}_4 \frac{1}{4}. \quad (2.14)$$

There is an additional Green-Schwarz term which appears in M-theory whose existence can be inferred either from D = 11 membrane-fivebrane duality and the world-volume anomaly cancellation of M-5 fivebrane or from a one-loop calculation of Type IIA superstrings. It is in general

$$W_5 = \frac{1}{2\sqrt{2}(2\pi)^4} \frac{T_2}{n} \int_{M^{11}} C \wedge X_8, \quad (2.15)$$

7Actually, for $\alpha = 1$, the present $G_{11ABC}$ agrees with theirs but the present $G_{ABCD}$ are twice theirs. This short of factor 2 fortunately gives the correct expression for $G^W/2\pi$ even though their $\eta$ is short of a factor 8.

8This has been considered in for the case of $\alpha = 1$ in the “downstairs” approach. However, the variation of the quantum effective action $\delta \Gamma$ used there is for Weyl fermions but not for Majorana-Weyl fermions. In other words, a factor 2 is overcounted in the $\delta \Gamma$ there. If the correct $\delta \Gamma$ for Majorana-Weyl spinors is used, the gravitational anomaly discussed in would not be cancelled if Hofava and Witten’s $\eta = 2^7\pi^5$ is used.
where \( n \) is the non-negative integer appearing in Eq. (1.6) and \( T_2 \) is the M-2 brane tension which is quantized according to Eq. (1.5), and the 8-form \( X_8 \) is given in Eq. (1.4). Under the above gauge and local Lorentz variations,

\[
\delta W_5 = \frac{1}{24(2\pi)^4} \frac{\alpha}{n} \left( \frac{(2\pi)^2 \kappa^4}{2m \chi^6} \right)^{1/3} \int_{M^{10}} Q_2^1 \wedge X_8.
\] (2.16)

Since \( W + W_5 \) is not invariant under either gauge or local Lorentz variation, we have to appeal to quantum anomalies. The variation of quantum effective action \( \Gamma \) for ten-dimensional Majorana-Weyl fermions in the present case is

\[
\delta \Gamma = -\frac{1}{2} \frac{1}{48(2\pi)^5} \int_{M^{10}} Q_2^1 \wedge \left( -\frac{\hat{I}_2}{4} + X_8 \right),
\] (2.17)

which is the half of the \( \delta \Gamma \) used in [6].

Cancellations of both gauge and gravitational anomalies imply \( \delta W + \delta W_5 + \delta \Gamma = 0 \).

This gives

\[
\frac{\chi^6}{\kappa^4} = \frac{\alpha (1 + \alpha)^2}{4} (4\pi)^5,
\] (2.18)

and

\[
\frac{\alpha}{n} \left( \frac{(2\pi)^2 \kappa^4}{2m \chi^6} \right)^{1/3} = \frac{1}{8\pi}.
\] (2.19)

Solving the above two equations gives

\[
\alpha = \frac{\sqrt{mn^3}}{2 - \sqrt{mn^3}}.
\] (2.20)

From Eq. (2.18), we have \( \alpha > 0 \). Applying this to the above equation, we have

\[
0 < mn^3 < 4,
\] (2.21)

which has the following solutions: \( m = 1, n = 1; m = 2, n = 1; \) and \( m = 3, n = 1 \) since both \( n \) and \( m \) are non-negative integers (we could include the two cases corresponding to \( \alpha = 0 \) and \( \alpha = \infty \) in the above, one corresponding to zero gauge coupling while the other to infinity large gauge coupling). The corresponding \( \alpha \) are 1, \( \sqrt{2}/(2 - \sqrt{2}) \) and \( \sqrt{3}/(2 - \sqrt{3}) \), respectively.
If we simply stop here, we must conclude that $\eta = \frac{6}{\kappa^6}$ is quantized according to each pair of $(m, n)$ given in the above. However, the other condition, namely the half-integral period of $G^W/2\pi$, will pick the pair $(m = 1, n = 1)$, therefore, uniquely determining the $\eta$.

From Eq. (2.3), we have $G$ on the $(x^{11} = 0)$ component of the boundary as

$$G |_N = \frac{1 + \alpha}{2} \frac{\kappa^2}{\sqrt{2\lambda^2}} \hat{I}_1.$$ (2.22)

Therefore, from footnote 2, we have

$$\frac{G^W}{2\pi} = \sqrt{2} T_2 \frac{G}{2\pi},$$

$$= \left(\frac{1 + \alpha}{2\alpha m}\right)^{1/3} \frac{1}{16\pi^2} \left(\frac{1}{2} \text{tr} R^2 - \text{tr} F^2\right),$$ (2.23)

where Eq. (1.3) has been used. According to what has been discussed by Witten [8], a well-defined membrane path integral must require

$$\left(\frac{1 + \alpha}{2\alpha m}\right)^{1/3} = l \quad (l = \text{odd integer}),$$ (2.24)

such that $G^W/2\pi$ has in general a half-integral period. So we have

$$1 + \alpha = 2\alpha ml^3.$$ (2.25)

Combining this equation with Eq. (2.24), we have

$$l^3(mp)^{3/2} = 1,$$ (2.26)

which has the following unique solution

$$l = 1, \quad m = 1, \quad n = 1, \quad \text{and} \quad \alpha = 1,$$ (2.27)

since both $m$ and $n$ are non-negative integers. It is also clear from Eq. (2.26) that $l$ cannot be an even integer. This implies that even if we naively assume an integral

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9As one can see, the above conclusion is crucially based on the assumption that $m$ is an integer. If we relax $m$ to be a real non-negative number, some conclusions can still be drawn. $m$ must be bounded below $4/n^3$ and is given by $1/nl^2$ with $l$ an positive odd integer satisfying $2l > n$. Then we have $\alpha = n/(2l - n)$ which implies a quantized $\eta$ given according to Eq. (2.18).
period of $G^W/2\pi$ from the outset, the above process will force us to conclude a general 
half-integral period for $G^W/2\pi$.

The above uniquely determines
\begin{equation}
\eta = \frac{\lambda^6}{\kappa^4} = (4\pi)^5,
\end{equation}
which is eight times that of Hořava and Witten’s. This value of $\eta$ was also obtained 
recently by Conrad\cite{6}. With it, we have the needed expression
\begin{equation}
\frac{G^W}{2\pi} = \frac{1}{16\pi^2} \left( \frac{1}{2} \text{tr} R^2 - \text{tr} F^2 \right),
\end{equation}
for showing that $G^W/2\pi$ has indeed a half-integral period in general.

To summarize, the consistency of quantum M-theory must imply: (1) Both M-2 
and M-5 branes must be present. (2) Only the minimum positive integers $m = 1$ and 
n = 1 are allowed in Eq. (1.3) and Eq. (1.6), respectively. This in turn implies that a 
well-defined quantum description of M-2 brane may be possible only for a M-2 brane 
with tension $T_2 = \left( \frac{(2\pi)^2}{2\kappa^2} \right)^{1/3}$. So is for a M-5 brane with tension $T_5 = \left( \frac{2\pi}{(2\kappa^2)^2} \right)^{1/3}$.

(3) The value of $\eta = \lambda^6/\kappa^4$ can be determined uniquely to be $(4\pi)^5$.

2.2 “Downstairs” approach

To demonstrate the usefulness of different approaches, we here repeat the same pro-
cess of the previous subsection in the “downstairs” approach following the procedure 
described in \cite{5}.

The key for the present approach is Eq. (2.22), i.e., on each component of the 
boundary,
\begin{equation}
G \vert_N = \frac{1 + \alpha}{2} \frac{\kappa^2}{\sqrt{2\lambda^2}} \hat{I}_1.
\end{equation}
Under a gauge and a local Lorentz variations described in the previous subsection, 
we have standard descent equations
\begin{equation}
\hat{I}_1 = dQ_3, \quad \delta Q_3 = dQ_2^1,
\end{equation}
where $Q_3 = 1/2\omega_3 L - \omega_3 L$ and $Q_1^2$ is given by Eq. (2.13). In the “downstairs” approach, four-form field strength $G$ is always defined as $G = 6dC$ but with its boundary value given by Eq. (2.31). Using the first equation in (2.31), we therefore have (up to an irrelevant exact form)

$$C |_{N} = \frac{1 + \alpha}{12} \frac{\kappa^2}{\sqrt{2\lambda^2}} Q_3. \quad (2.32)$$

It follows using the second equation in (2.31)

$$\delta C |_{N} = \frac{1 + \alpha}{12} \frac{\kappa^2}{\sqrt{2\lambda^2}} dQ_1^2. \quad (2.33)$$

Following [6], we extend this variation to the bulk by writing

$$\delta C = \frac{1 + \alpha}{12} \frac{\kappa^2}{\sqrt{2\lambda^2}} dQ_2^1. \quad (2.34)$$

The “Chern-Simons” interaction in the “downstairs” version is again given by Eq. (2.11) but with the replacement of $M_U^{11}$ by $M_D^{11}$. Therefore we have

$$\delta W = -\frac{\sqrt{2}}{\kappa^2} \int_{M_D^{11}} \frac{1 + \alpha}{12} \frac{\kappa^2}{\sqrt{2\lambda^2}} dQ_1^2 \wedge G \wedge G,$$

$$= -\frac{1}{3} \left(\frac{1 + \alpha}{2}\right)^3 \frac{\kappa^4}{\lambda^6} \int_{M^{10}} Q_2^1 \wedge \hat{F}_4^2 \wedge \hat{I}_4^2,$$

where in reaching the second equality we have used Stokes’ theorem, $dG = 0$ and Eq. (2.34).

Similarly, in the present approach, we have the variation of $W_5$ as

$$\delta W_5 = \frac{1 + \alpha}{48\pi} \frac{1}{(2\pi)^4} \left(\frac{(2\pi)^2 \kappa^4}{2m \lambda^6}\right)^{1/3} \int_{M^{10}} Q_2^1 \wedge X_8, \quad (2.36)$$

where $T_2 = \left(\frac{(2\pi)^2}{2\kappa^2 m}\right)^{1/3}$ and Eq. (2.34) have been used.

Again $\delta W + \delta W_5$ does not vanish. But $\delta W + \delta W_5 + \delta \Gamma = 0$ are possible provided

$$\frac{\lambda^6}{\kappa^4} = 4(1 + \alpha)^3 (2\pi)^5, \quad (2.37)$$
and

\[ \frac{1 + \alpha}{n} \left( \frac{(2\pi)^2 \kappa^4}{2m \lambda^6} \right)^{1/3} = \frac{1}{4\pi} \]

(2.38)

In the above, Eq. (2.17) for quantum anomalies has been used. Combining the above two equations, we have

\[ nm^{1/3} = 1, \]

(2.39)

whose only solution is \( n = 1 \) and \( m = 1 \) since both \( n \) and \( m \) are non-negative integers\(^{10}\).

We have determined \( m \) and \( n \) completely but not for \( \alpha \). One may think that this \( \alpha \) can be fixed uniquely if we use the condition that \( G^W / 2\pi \) has a half-integral period in general as we did in the “upstairs” approach. Unfortunately, the above solutions

\[ \frac{\lambda^6}{\kappa^4} = 4(1 + \alpha)^3(2\pi)^5, \]

\[ n = 1, \quad m = 1, \]

(2.40)

(2.41)

already imply that

\[ \frac{G^W}{2\pi} = \sqrt{2} T_2 \frac{G}{2\pi}, \]

\[ = \frac{1}{16\pi^2} \left( \frac{1}{2} \text{tr} R^2 - \text{tr} F^2 \right), \]

(2.42)

i.e., \( G^W / 2\pi \) has a half-integral period in general. In other words, in the “downstairs” approach, the \( G \)-value on the boundary can be defined up to a dimensionless constant. \( G^W / 2\pi \) has always a half-integral period in general provided no gauge and gravitational anomalies.

\(^{10}\)We can also relax \( m \) to be a non-negative real number. By this, we can determine \( m = 1/n^3 \) from Eq. (2.33). So we have M-2 brane tension \( T_2 = n \left( \frac{(2\pi)^2 \kappa^4}{2\pi^2} \right)^{1/3} \) from Eq. (1.3). Applying this to Eq. (1.6), we have M-5 brane tension \( T_5 = \left( \frac{2\pi}{2\pi \kappa} \right)^{1/3} \), i.e., only the minimum M-5 brane tension is allowed.
This may make people wonder what happens. However, no contraction exists between these two approaches. One can check that the unique solutions for \( \alpha = 1, n = 1 \) and \( m = 1 \) obtained in the “upstairs” approach continue to be a special case of the present solutions. Particularly, if we set \( \alpha = 1 \) in Eq. (2.40), we obtain again \( \eta = (4\pi)^5 \).

There actually exists a subtle difference between these two approaches. In the “downstairs” approach, under a gauge and a local Lorentz variations, we deduce the variation for the three-form \( C \) all from the property of \( G_{ABCD} \) on the boundary. In the “upstairs” approach, we have an additional information from \( G_{11ABC} \). It is just this additional information that enables us to determine the aforementioned quantities uniquely.

The fact that a half-integral period of \( G^W/2\pi \) is warranted after we impose the anomaly-free condition in the “downstairs” approach may provide a new way for us to determine uniquely the values for \( \alpha, n \) and \( m \). i.e., we identify those conditions obtained from gauge and gravitational anomaly cancellations in both the “upstairs” and “downstairs” approaches.

The “downstairs” approach already determines \( m = 1 \) and \( n = 1 \). Either substituting \( mn^3 = 1 \) from Eq. (2.39) in the “downstairs” approach to Eq. (2.20) in the “upstairs” approach or identifying Eq. (2.18) in the “upstairs” approach with Eq. (2.37) in the “downstairs” approach, we have \( \alpha = 1^{[1]} \).

3 Consistency Check

We begin with a discussion showing that Conrad’s “downstairs” bulk action is indeed correct, based on the recent work by Brax and Mourad [11]. Hořava and Witten’s

\[ m = 1^{[1]} \]

\[ \text{In this new approach, we can still gain some information if we relax } m \text{ to be a non-negative real number. We can determine uniquely } \alpha = 1, \text{ therefore } \eta = (4\pi)^5, \text{ and other results given in the previous footnote.} \]
work of M-theory on $R^{10} \times S^1 / Z_2$ \cite{1,2} nevertheless suggests the existence of an open supermembrane with each of its two ends lying on each component of the boundary of spacetime. Recently, Brax and Mourad went one step further to construct a worldvolume action for an open supermembrane moving in a flat spacetime with topological defects on which the membrane can end. This action is kappa-symmetric and has global spacetime supersymmetry. To respect the gauge symmetry of the spacetime super three-form potential $C$, a spacetime super two-form potential must be introduced whose pullback contributes a new membrane boundary term, which is absent for a closed membrane, to the open membrane action. In respect to kappa symmetry, one can define two field strengths as (in our notation)

$$G = 6dC, \quad H = 6dB + 6C. \quad (3.1)$$

The above two field strengths are not independent but related to each other on a topological defect as

$$dH = G \big|_N . \quad (3.2)$$

Consider this open membrane to move in a curved spacetime with a boundary on which super Yang-Mills fields propagate. In order to preserve the kappa-symmetry and to obtain one of the heterotic strings in the weak coupling limit, one finds that the unique conclusion is Hořava and Witten’s \cite{1,2} but now from worldvolume rather than spacetime perspective. At the same time, one finds that the field strength $G$ (It is now the field strength of the background three-form potential $C$ used in the previous sections) has to be modified (in our notation) on the ($x^{11} = 0$) component of the boundary as

$$dH = G \big|_N = \frac{1}{8\sqrt{2}\pi T_2} \hat{I}_4, \quad (3.3)$$

where $T_2$ is the membrane tension and $\hat{I}_4$ is given by Eq. (1.3) (for detail, see \cite{1}).
In the case of Hořava and Witten, $G$-value on the boundary is essentially determined by the requirement of spacetime supersymmetry at the lowest quantum order (the Yang-Mills action is the quantum correction to the supergravity action) and the coefficient in front of $\hat{I}_4$ is in terms of the gauge and gravitational constants. While in the present case, $G$-value on the boundary is determined by the requirement of preserving kappa-symmetry at worldvolume one loop level and the above coefficient is in terms of the membrane tension. One therefore expects that the above two $G$-values on the boundary should agree with each other certainly in a non-trivial way, knowing the fact that kappa-symmetry links between spacetime and worldvolume supersymmetries.

As pointed out by Brax and Mourad in [11], the introduction of the two-form potential $B$ on the boundary of spacetime and its linkage to the two-form potential appearing in the low energy effective action of the heterotic string in the weak coupling limit must imply, under a gauge and a local Lorentz variations described in section 2,

$$\delta C = 0, \quad \delta B \bigg|_N = \frac{1}{3!} \frac{1}{8\sqrt{2}\pi T_2} Q^1_{\phi},$$  

(3.4)

where $Q^1_{\phi}$ is given by Eq. (2.13).

The bulk topological terms in the low energy limit of M-theory on $R^{10} \times S^1/Z_2$ in the “downstairs” approach with Hořava and Witten’s normalization as used by Brax and Mourad in [11] are

$$S_T = -\frac{2\sqrt{2}}{\kappa^2} \int_{M_{11}^{11}} C \wedge G^2 + \frac{T_2}{\sqrt{2}(2\pi)^4} \int_{M_{11}^{11}} C \wedge X_8.$$  

(3.5)

This topological action is not invariant under the gauge transformation $C \rightarrow C + d\Lambda$. Brax and Mourad then introduced a boundary action in the spirit of their worldvolume construction for the open membrane action as

$$\Delta S_T = -\frac{2\sqrt{2}}{\kappa^2} \int_{\partial M_9^{11}} B \wedge G^2 + \frac{T_2}{\sqrt{2}(2\pi)^4} \int_{\partial M_9^{11}} B \wedge X_8.$$  

(3.6)
Then the total action $S_T + \Delta S_T$ is indeed invariant under the above gauge transformation supplemented with $B \to B - \Lambda$.

However, the boundary term $\Delta S_T$ is not invariant under the variation of Eq. (3.4).

Gauge and gravitational Anomalies arise. They are on the $(x^{11} = 0)$ component of the boundary

$$\delta \Delta S_T = -\frac{4}{6\kappa^2} \left(\frac{1}{8\pi T_2}\right)^3 \int_{M^{10}} Q^1_{2} \wedge \hat{I}_4^2 + \frac{1}{48(2\pi)^5} \int_{M^{10}} Q^1_{2} \wedge X_8. \quad (3.7)$$

The striking feature of the present approach is that the last term on the right hand of the above equation is independent of any coupling constant. This will determine which bulk action one should use in the “downstairs” approach when the anomaly-free condition of M-theory is imposed.

From section 2, we have the variation of quantum effective action given by Eq. (2.17)

$$\delta \Gamma = -\frac{1}{2} \frac{1}{48(2\pi)^5} \int_{M^{10}} Q^1_{2} \wedge \left(-\frac{\hat{I}_4^2}{4} + X_8\right). \quad (3.8)$$

We expect that $\delta \Delta S_T + \delta \Gamma$ vanishes. But this is impossible since the terms involving $X_8$ do not cancel each other exactly because the last term in Eq. (3.7) is too large by a factor $2^{12}$. This indicates that Hořava and Witten’s “downstairs” bulk action is too large by a factor 2. Therefore, Conrad’s “downstairs” bulk action is correct. Using Conrad’s “downstairs” bulk action, we have $\delta \Delta S_T + \delta \Gamma = 0$ provided $T_2 = \left(\frac{(2\pi)^2}{2\kappa^2}\right)^{1/3}$, i.e., the M-2 brane tension for $m = 1$ obtained in the previous section. With this result, one can examine that the present G-value on the boundary is the same as that obtained in the previous section for $\alpha = 1, m = 1$ and $n = 1$.

The other consistency check showing the correctness of Hořava and Witten’s “upstairs” bulk action is to go to the weak coupling limit of the heterotic string. Then

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\[12\] It is clear from their paper that Brax and Mourad overcounted $\delta \Gamma$ by a factor 2 and forced themselves to agree with Hořava and Witten’s bulk “downstairs” action. Otherwise, they would obtain Conrad’s action much earlier.
we expect to obtain the low energy effective action of the $E_8 \times E_8$ heterotic string. The “upstairs” low energy effective action of M-theory on $R^{10} \times S^1/Z_2$ is

$$S = -\frac{1}{2\kappa^2} \int_{M^{11}} d^{11}x \sqrt{-g_M} R - \frac{1}{4\lambda^2} \sum_i \int_{M_i^{10}} \text{tr} F^2_i + \cdots.$$  \hspace{1cm} (3.9)

In the weak coupling limit, M-theory metric $dS_M^2 = g_{Mmn} dx^m dx^n$ can be expressed in terms of the heterotic metric as $dS_H^2 = g_H^{4/3} (dx^{11})^2 + g_H^{-2/3} g_{\mu\nu} dx^\mu dx^\nu$ with $g_H$ the coupling constant of the heterotic string. If we denote the radius of the circular 11-th dimension as $R_0$, the physics radius in terms of M-theory metric is $R_{11} = g_H^{2/3} R_0$ from the above metric relation. As in \cite{9}, $R_0$ is conventionally chosen as $\sqrt{\alpha'}$ with $\alpha'$ the string constant. With this choice and other well-established relations given in \cite{9}, the eleven dimensional gravitational constant $\kappa$, M-2 and M-5 brane tensions can all be expressed in terms of one single string constant $\alpha'$ (for detail about these derivations, see \cite{9}). Then, we have, with the $Z_2$ symmetry imposed on all fields during the dimensional reduction,

$$S = -\frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-g} e^{-2\phi} \left[ R + \frac{\alpha'}{8} \text{tr} F^2 + \cdots \right],$$  \hspace{1cm} (3.10)

where the present $\eta = (4\pi)^5$ is used. From the above action, we have the ten dimensional gravitational constant $\kappa_{10}$ satisfying $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4 g_H^2$. This is indeed the correct relation found independently in ten dimensions, lending firm support to our claim.

The relative factor $\alpha'/8$ in front of the kinetic term of gauge fields in the above action now answers once for all that one Yang-Mills instanton corresponds to one Strominger’s heterotic fivebrane \cite{14} rather than eight ones. Many physicists once argued that this factor should be $\alpha'^8$ rather than $\alpha'$ as used in \cite{14}. However, if

\footnote{To my knowledge, this was pointed out first by Paul Townsend. This relative factor $\alpha'/8$ was previously given explicitly, following the work of \cite{15,16}, in \cite{3} in the action which is essentially identical to Eq. (3.10).}
Hořava and Witten’s $\eta = 2^7 \pi^5$ is used in the above as in [9], a factor $\alpha'/4$ will be obtained instead.

It is now clear that if we take the ten dimensional action Eq. (3.10) as a standard one, one would be forced to choose $R_0 = \sqrt{\alpha'}$ if we insist that the M-theory action (3.9) be reduced to the action (3.10) in the weak coupling limit.

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