Topical Review

Geometric aspects of Painlevé equations

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Received 31 October 2015, revised 16 January 2016
Accepted for publication 20 January 2016
Published 12 January 2017

Abstract

In this paper a comprehensive review is given on the current status of achievements in the geometric aspects of the Painlevé equations, with a particular emphasis on the discrete Painlevé equations. The theory is controlled by the geometry of certain rational surfaces called the spaces of initial values, which are characterized by eight point configuration on $\mathbb{P}^1 \times \mathbb{P}^1$ and classified according to the degeneration of points. We give a systematic description of the equations and their various properties, such as affine Weyl group symmetries, hypergeometric solutions and Lax pairs under this framework, by using the language of Picard lattice and root systems. We also provide with a collection of basic data; equations, point configurations/root data, Weyl group representations, Lax pairs, and hypergeometric solutions of all possible cases.

Keywords: Painlevé equations, discrete Painlevé equations, space of initial values, root systems, Lax pair, hypergeometric functions, affine Weyl groups

Mathematics Subject Classification: 34M55, 14H70, 33C20, 33D15, 37K20, 39A10, 39A13

(Some figures may appear in colour only in the online journal)

1. Introduction

Today the Painlevé equations, both continuous and discrete, are well-established subjects in mathematics and mathematical physics \cite{9, 16}. In the geometric approach to the Painlevé equations, initiated by Okamoto \cite{95} and subsequently extended by Sakai \cite{112} to the discrete cases, the theory is controlled by the geometry of certain rational surfaces called the
spaces of initial values. This framework gives a systematic description of the equations, symmetries, special solutions, Lax pairs and so forth. The aim of this paper is to provide a comprehensive review on the current status of achievements in the geometric aspects of the Painlevé equations so as to serve it as a foundation of future researches in mathematics and mathematical physics. It also contains some materials which have been newly developed to complete a unified description. We put a particular emphasis on studying the discrete Painlevé equations of the second order.

Historically, the Painlevé differential equations were discovered by Painlevé and Gambier \[21, 34, 102, 103\] in the efforts for finding new transcendental functions defined by ‘good’ nonlinear ordinary differential equations. They imposed the condition that the solutions should admit only poles as movable singular points, which is now referred to as the Painlevé property. Then Fuchs \[17, 18\] formulated them as the monodromy preserving deformations of linear ordinary differential equations, and subsequently Schlesinger \[116\] and Garnier \[22, 23\] investigated their generalizations. Almost sixty years later, the Painlevé equations were found to describe the correlation function of the Ising model by Wu et al \[138\]. Inspired by this discovery, the theory of holonomic quantum field theory \[39, 115\] and a general theory of the monodromy preserving deformation of linear ordinary differential equations was established by Sato’s group in Kyoto \[37, 38, 40\].

The study of geometric aspects of the Painlevé equations, which is a main topic of this paper, has been initiated by Okamoto’s pioneering work \[95\]. For each Painlevé equation, he constructed the space of initial values which parametrizes all the solutions. Takano further found that the Painlevé equations themselves can be reproduced uniquely from the space of initial values \[73, 117\]. These works are the basis of Sakai’s approach for discrete Painlevé equations which will be mentioned below.

On the other hand, discrete integrable systems, started in 70s by the pioneering works of Ablowitz–Ladik and Hirota, have been regarded as equally or more important as the continuous integrable systems. The discrete Painlevé equations also appear in \[37, 38, 40\], and attracted attention by the discoveries of the scaling limit to the Painlevé differential equations in the context of two-dimensional quantum gravity \[7, 11, 28\]. In accordance with these studies, Grammaticos, Ramani and Papageorgiou introduced the concept of singularity confinement as a discrete counterpart of the Painlevé property and proposed to use it as an integrability detector for discrete systems \[27\]. Then Ramani, Grammaticos and Hietarinta applied this idea to obtain non-autonomous version of the two-dimensional integrable mappings known as the Quispel–Roberts–Thompson (QRT) mappings \[105, 106\] and succeeded in constructing discrete Painlevé equations systematically \[110\].

Subsequently, discrete Painlevé equations as well as their generalizations have been studied from various points of view, such as Bäcklund transformations, Lax pairs, particular solutions, \(\tau\) functions and so on. For a review of those developments, we refer to \[26, 120\]. In the meanwhile, underlying mathematical structures have gradually been clarified. Jimbo and Sakai constructed a \(q\)-difference analogue of Painlevé VI equation in the spirit of deformation theory of linear \(q\)-difference equation \[41\]. A universal symmetry structure behind the continuous and discrete Painlevé equations has been revealed in terms of the birational representations of affine Weyl groups which is also applicable to higher dimensional Painlevé type equations \[56–58, 81, 86, 88\].

In the efforts for finding a unified framework for the Painlevé type equations, Sakai proposed a class of second order discrete Painlevé equations arising from Cremona transformations of rational surfaces obtained as nine-point blow-ups of \(\mathbb{P}^2\) \[112\]. Those rational
surfaces are regarded as the spaces of initial values for discrete and continuous Painlevé equations, and are classified into 22 cases according to the configuration of nine points. The master equation of those Painlevé equations, the *elliptic Painlevé equation*, is obtained from the most generic configuration; it provides with the geometric construction of discrete Painlevé equation with affine Weyl group symmetry of type $E_8^{(1)}$ as proposed by Ohta, Ramani and Grammaticos \[92\]. Other equations are obtained from the degenerate configurations.

On the basis of this geometric approach, above-mentioned various aspects of the Painlevé equations can be investigated in a unified manner in the language of Picard lattice and root systems. For example, the hypergeometric seed solutions to all possible discrete Painlevé equations have been constructed in \[45, 50, 51\]. Lax pairs for discrete Painlevé equations have been constructed through their characterization in terms of the point configuration \[84, 140, 141\]. Geometric approach is effective particularly in the study of Painlevé equations with high symmetry such as $E$ type.

The plan of this review is as follows. In section 2, we give an overview of various aspects of the Painlevé equations to be discussed in this review. Taking the examples of the Painlevé IV equation ($\text{P}_{\text{IV}}$) and a discrete Painlevé II equation ($\text{dP}_{\text{II}}$) which arises as a Bäcklund transformation of $\text{P}_{\text{IV}}$, we introduce basic objects in the theory of Painlevé equations, such as affine Weyl group symmetry, Lax pairs, hypergeometric solutions, $\tau$ functions and the space of initial values in the sense of Okamoto and Sakai.

The space of initial values is, roughly speaking, a surface on which the solutions of the Painlevé equation in question are parametrized. For each Painlevé equation, this surface is characterized by a pair of affine root systems which represent the symmetry type and the surface type. Many properties of Painlevé equations as presented in section 2 are systematically controlled by geometry of the surface. In section 3, we provide with general frameworks of the root systems, Weyl groups and the Picard lattice relevant to the Painlevé equations. These devices will be utilized throughout subsequent sections as fundamental and powerful tools for studying the Painlevé equations.

One of the common properties of the Painlevé equations is that the space of initial values is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$, the product of two copies of the Riemann sphere, by blowing up at eight points. Therefore the configuration of the eight points, which possibly includes infinitely near points, provides with the most fundamental data of the equation. With the items obtained in section 3 in hand, we demonstrate how to associate a point configuration on $\mathbb{P}^1 \times \mathbb{P}^1$ to a given discrete equation in section 4. This provides us a practical method for determining whether it is a discrete Painlevé equation in Sakai’s class, and if so, identifying the type of the equation by its surface type and symmetry type.

If the configuration of eight points in $\mathbb{P}^1 \times \mathbb{P}^1$ is generic, the corresponding space of initial values has the largest symmetry of type $E_8^{(1)}$. Other configurations can be regarded as the degenerate cases, among all possible 22 configurations classified by Sakai. In section 5, we describe how to construct the equations and relevant characteristic features from the point configuration. In particular, we formulate a representation of affine Weyl group of type $E_8^{(1)}$ from the configuration of generic eight points in $\mathbb{P}^1 \times \mathbb{P}^1$, as well as the formalism of $\tau$ functions. We then derive a new explicit form of the three equations of type $E_8^{(1)}$, which are the elliptic, $q$- and difference Painlevé equations, from a translation of the root lattice. We also give an example demonstrating how to construct the birational representation of the affine Weyl group for a given degenerate point configuration.
Most of the Painlevé equations admit a class of particular solutions expressible in terms of the special functions of hypergeometric type for special values of parameters which correspond to reflection hyperplanes in the parameter space. In section 6, we demonstrate how to construct the hypergeometric solutions by decoupling a given equation to the Riccati equation and by linearizing it. Then we give an intrinsic formulation of this procedure by the geometric language of point configurations. The list of hypergeometric solutions associated with possible point configurations will be given in section 8.

It is a common feature of nonlinear integrable systems that they arise as the compatibility condition of certain systems of linear equations which is called a Lax pair. In section 7 we give a geometric formulation of the Lax pairs for Painlevé equations in terms of associated point configurations.

Section 8 is a comprehensive collection of data for all Painlevé equations which can be obtained by various methods discussed in this review. For each case, we provide with explicit forms of equations, point configurations/root data, Weyl group representations, Lax pairs and hypergeometric seed solutions.

In this review, we present a general geometric framework as well as algebraic tools for studying the Painlevé equations, confining ourselves to the second order equations. Even in the second order equations, there are various discrete Painlevé equations which are not directly investigated in this paper, e.g., equations arising from the translations with different directions or length in the root lattice [55, 109, 121]. Also, we do not deal with higher order or multi-variable generalizations, which are now actively studied in relation with soliton equations [19, 20, 57, 58, 87, 119, 124, 126], geometry of space of initial values [71, 114], geometry of flag varieties [89], or general theory of monodromy preserving deformations [65]. The Painlevé equations are believed to define new transcendental functions, and it was rigorously proved for the Painlevé differential equations (see, for example [128, 129]). Similar investigations for discrete Painlevé equations have been done in [80]. Asymptotic analysis for the solutions of Painlevé equations are also an important subject in view of applications [10, 13, 14, 42, 64]. Recently, applications to various areas of physics and mathematical sciences, including probability theory and combinatorics, have been actively studied based on the random matrix theory [16]. There are also other interesting relationships to various areas, such as discrete differential geometry, integrable models of quantum physics and lattice models, ultradiscrete systems, quivers and cluster algebras. We hope that the materials provided in this review will be utilized for further developments of the theory of Painlevé equations and related areas.

2. Overview of the Painlevé equations

This section is an overview of various aspects of the Painlevé equations to be discussed in this review. Taking the examples of $P_{\text{IV}}$ and $dP_{\text{II}}$, we introduce basic objects in the theory of Painlevé equations, such as affine Weyl group symmetry, Lax pairs, hypergeometric solutions, $\tau$ functions and the space of initial values.

2.1. Hamilton system and symmetry of $P_{\text{IV}}$

Let us consider $P_{\text{IV}}$

$$q'' = \frac{1}{2q} (q')^2 + \frac{3}{2} q^3 + 2tq^2 + \left( a_2 - a_0 + \frac{t^2}{2} \right) q - \frac{a_1^2}{2q},$$

(2.1)
which can be rewritten as the non-autonomous Hamiltonian system as

\[
\begin{align*}
q' &= \frac{\partial H_{IV}}{\partial p} = -a_1 + 2pq - q^2 - qt, \\
p' &= -\frac{\partial H_{IV}}{\partial q} = a_2 - p^2 + 2pq + pt,
\end{align*}
\] (2.2)

where

\[ H_{IV} = -a_0p - a_2q + pq(p - q - t), \] (2.3)

\( t \) is an independent variable and \( a_i \) (\( i = 0, 1, 2 \)) are parameters such that

\[ t = \frac{d}{dt}, \quad a_i' = 0 \quad (i = 0, 1, 2), \quad a_0 + a_1 + a_2 = 1. \] (2.4)

We introduce

\[ f_0 = -(p - q - t), \quad f_1 = -q, \quad f_2 = p. \] (2.5)

Then \( f_i \) (\( i = 0, 1, 2 \)) satisfy the following equation

\[
\begin{align*}
f_0' &= f_0(f_1 - f_2) + a_0, \\
f_1' &= f_1(f_2 - f_0) + a_1, \quad f_0 + f_1 + f_2 = t, \\
f_2' &= f_2(f_0 - f_1) + a_2,
\end{align*}
\] (2.6)

which is called the symmetric form of \( P_{IV} \) \cite{2, 81, 88}.

The following transformations \( s_i \) (\( i = 0, 1, 2 \)) and \( \pi \) on variables \( p, q \) and \( a_i \) (\( i = 0, 1, 2 \)) commute with the differentiation and are called the Bäcklund transformations:

\[
\begin{align*}
s_0 & \quad p + \frac{a_0}{p - q - t} & q + \frac{a_0}{p - q - t} & -a_0 & a_0 + a_1 & a_0 + a_2 \\
s_1 & \quad p - \frac{a_1}{q} & q & a_0 + a_1 & -a_1 & a_1 + a_2 \\
s_2 & \quad p & q + \frac{a_2}{p} & a_0 + a_2 & a_1 + a_2 & -a_2 \\
\pi & \quad -p + q + t & -p & a_1 & a_2 & a_0.
\end{align*}
\] (2.7)

For instance, \( s_0 \) is defined by the variable transformation that replaces \( p, q, a_0, a_1, a_2 \) by

\[
\begin{align*}
s_0(p) &= p + \frac{a_0}{p - q - t}, \\
s_0(q) &= q + \frac{a_0}{p - q - t}, \\
s_0(a_0) &= -a_0, \\
s_0(a_1) &= a_0 + a_1, \\
s_0(a_2) &= a_0 + a_2,
\end{align*}
\] (2.8)

respectively. Commutativity with the differentiation, \( w(f') = (w(f))^t \) for \( w = s_0, s_1, s_2, \pi \) and \( f = p, q, a_0, a_1, a_2 \), can be verified by direct calculations. Composition of those transformations are computed, for example, as

\[
s_2s_1(p) = s_2\left(p - \frac{a_1}{q}\right) = s_2(p) - \frac{s_2(a_1)}{s_2(q)} = p - \frac{a_1 + a_2}{q + \frac{a_2}{p}}. \] (2.9)
These transformations satisfy the fundamental relations
\[ s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1, \quad \pi s_i = s_{i+1} \pi \quad (i \in \mathbb{Z}/3\mathbb{Z}), \quad \pi^3 = 1, \] (2.10)
and form the extended affine Weyl group of type \( \tilde{A}_2 \) (we will give a general account of the affine Weyl groups in section 3). We define the translation \( T \) by
\[ T = s_{20} \pi^{-1}. \] (2.11)
Then the action of \( T \) is given by
\[ T \begin{pmatrix} p & q & a_0 & a_1 & a_2 \\ \bar{p} & \bar{q} & a_0 & a_1 - 1 & a_2 + 1 \end{pmatrix}, \] (2.12)
where
\[ \bar{q} = p - q - t - \frac{a_2}{p}, \]
\[ \bar{p} = \bar{q} - p + t + \frac{a_1 - 1}{\bar{q}} = -\frac{a_2}{p} - q + \frac{a_1 - 1}{p - q - t - \frac{a_2}{p}}. \] (2.13)
When the iteration of this transformation is viewed as a discrete dynamical system, (2.13) is identified as a discrete analogue of the Painlevé II equation \( (d\Pi) \) [86]. With the notation \( T^n(q) = q_n, T^n(p) = p_n \quad (n \in \mathbb{Z}) \), (2.13) is interpreted as a difference equation with respect to \( n \):
\[ q_{n+1} = p_n - q_n - t - \frac{a_2 + n}{p_n}, \]
\[ p_{n+1} = q_{n+1} - p_n + t + \frac{a_1 - (n + 1)}{q_{n+1}}. \] (2.14)

**Remark 2.1.** There are two possible ways to compute the compositions of the Bäcklund transformations. The composition defined by substitution of symbols as (2.9) is interpreted in terms of automorphisms of the field of rational functions \( K = \mathbb{C}(p, q, a_0, a_1, a_2) \). We call this convention the **symbolical composition**, since it is convenient for symbolic computations. The other way is to regard the Bäcklund transformations as the transformations of five variables \( (p, q, a_0, a_1, a_2) \). We define \( F_{s_1} \) and \( F_{s_2} \), for instance, by
\[ F_{s_1}(p, q, a_0, a_1, a_2) = \begin{pmatrix} p - \frac{a_1}{q}, q, a_0 + a_1, -a_1, a_1 + a_2 \end{pmatrix}, \] (2.15)
\[ F_{s_2}(p, q, a_0, a_1, a_2) = \begin{pmatrix} p, q + \frac{a_2}{p}, a_0 + a_2, a_1 + a_2, -a_2 \end{pmatrix}. \] (2.16)
In this convention, the composition of \( F_{s_1} F_{s_2} \), for example, is calculated as follows
\[ F_{s_1}(p, q, a_0, a_1, a_2) = \begin{pmatrix} p + \frac{a_2}{p}, a_0 + a_2, a_1 + a_2, -a_2 \end{pmatrix} = (\bar{p}, \bar{q}, \bar{a}_0, \bar{a}_1, \bar{a}_2), \] (2.17)
\[ F_{s_2}(\bar{p}, \bar{q}, \bar{a}_0, \bar{a}_1, \bar{a}_2) = \begin{pmatrix} \bar{p} - \frac{\bar{a}_1}{\bar{q}}, \bar{q}, \bar{a}_0 + \bar{a}_1, -\bar{a}_1, \bar{a}_1 + \bar{a}_2 \end{pmatrix}. \] (2.18)
Eliminating \( \dot{p}, \dot{q}, \dot{a}_0, \dot{a}_1, \dot{a}_2 \), we obtain

\[
F_x F_y (p, q, a_0, a_1, a_2) = \left( p - \frac{a_1 + a_2}{q + \frac{a_2}{p}}, q + \frac{a_2}{p}, a_0 + a_1 + 2a_2, -a_1 - a_2, a_1 \right). \tag{2.19}
\]

Therefore we have \( F_{x_1} F_{x_2} = F_{y_1}, \) where \( F_{y_1} \) is the birational transformation corresponding to \( s_{21}. \) As we will demonstrate later, this convention is convenient for numerical computations. We call this convention the \textit{numerical composition}. Note that in these two ways of computation, the order of composition is opposite to each other. This is a general phenomena as is shown schematically

\[
F \circ G(x) = F(G(x)) = F(x)|_{x ightarrow G(x)} = (x|_{x ightarrow F(x)})|_{x ightarrow G(x)}. \tag{2.20}
\]

In order to see the difference of two conventions, the following simple example may be useful. If we introduce the mappings \( f, g : \mathbb{C} \rightarrow \mathbb{C} \) by

\[
f : x \mapsto x + 1, \quad g : x \mapsto x^2, \tag{2.21}
\]

then the composition of mappings (numerical composition) is computed as

\[
f(g(x)) = f(x^2) = x^2 + 1, \quad g(f(x)) = g(x + 1) = (x + 1)^2. \tag{2.22}
\]

On the other hand, if we introduce the automorphisms \( f, g : \mathbb{C}(x) \rightarrow \mathbb{C}(x) \) by the substitutions

\[
f : x \mapsto x + 1, \quad g : x \mapsto x^2, \tag{2.23}
\]

of the variable \( x, \) then the composition of substitutions (symbolical composition) implies

\[
f(g(x)) = f(x^2) = (x + 1)^2, \quad g(f(x)) = g(x + 1) = x^2 + 1. \tag{2.24}
\]

We usually adopt the convention of symbolical composition unless otherwise stated.

2.2. Lax pair

\( P_{IV} \) (2.1) can be expressed as the compatibility condition of the following system of linear differential equations for \( \psi = \psi(x, t): \)

\[
\psi_x + \left( \frac{1 - a_1}{x} - x - t - \frac{1}{x-q} \right) \psi_x + \left( -a_2 - \frac{H_{IV}}{x} + \frac{pq}{x(x-q)} \right) \psi = 0, \tag{2.25}
\]

\[
\psi' = -\frac{pq}{x-q} \psi + \frac{x}{x-q} \psi_x, \tag{2.26}
\]

where \( \psi' = \frac{\partial}{\partial t} \) and \((q, p) = (q(t), p(t)).\) We call (2.25) and (2.26) the \textit{auxiliary linear problem} (or the \textit{Lax pair}) of \( P_{IV}. \) In general, consider the following system of differential equations

\[
\psi_{xx} + u \psi_x + v \psi = 0, \quad \psi' = a \psi + b \psi_x, \tag{2.27}
\]

where \( u, v, a, b \) are functions of \( x, t. \) From these equations we can compute \((\psi_x)_t' = -(uv\psi_x - v\psi)_x'\) and \((\psi')_x = (a\psi + b\psi_x)_x\) in the form of linear combinations of \( \psi \) and \( \psi_x, \) assuming that \((\psi_x)_t' = (\psi')_x. \) Hence we obtain \((\psi_x)_t' - (\psi')_x = P \psi + Q \psi_x, \) where \( P \) and \( Q \) are expressed in terms of \( a, b, u \) and \( v. \) We say that the system (2.27) is \textit{compatible} if the coefficients \( P \) and \( Q \) are zero, which is a natural requirement for (2.27) to have two linearly independent solutions. This implies the following equations for \( u, v, a, b: \)
which is called the compatibility condition of (2.27). In case of (2.25) and (2.26), substituting the coefficients of the system (2.25) and (2.26) into (2.28), and requiring that (2.28) holds for arbitrary \(x\), we obtain the differential equation in \(t\) which is nothing but \(P_{IV}\).

\[
dP_{II} (2.13) \text{ arises as the compatibility condition of (2.25) and the following differential-difference equation}
\]

\[
\sum_{k=0}^{n} a_k u^k = 0
\]

(2.28)

We note that (2.29) is known as a Schlesinger transformation for the system (2.25) and (2.26) [37, 38, 40]. Consider the following system of differential-difference equations

\[
\begin{align*}
\psi_{xx} + a\psi_x + b\psi &= 0, \\
\bar{\psi} &= a\psi + b\psi_x,
\end{align*}
\]

(2.30)

Then the discussion similar to the case of \(P_{IV}\) shows that the condition

\[
(\psi_{xx}) = (\bar{\psi})_{xx},
\]

(2.31)

gives

\[
(a_x - bv)a + a\bar{\tau} - (2b_x + a - bu)v - bv_x + a_{xx} = 0,
\]

(2.32)

\[
(a + b_x - bu)a + b\bar{\tau} - (a + 2b_x)u - bu_x + bu^2 - bv + 2a_x + b_{xx} = 0,
\]

which is the compatibility condition of the system (2.30). Substituting the coefficients of the system (2.25) and (2.29) into (2.32), and requiring that (2.32) holds for arbitrary \(x\), we obtain \(dP_{II}\). In this sense, (2.25) and (2.29) can be regarded as the Lax pair of \(dP_{II}\) (2.13).

### 2.3. Hypergeometric solutions

\(P_{IV}\) admits a class of special solutions expressible in terms of hypergeometric type functions. For instance, putting \(a_i = 0\) in (2.6), we find that (2.6) admits a specialization \(f_i = 0\). When \(a_0 = 0\) setting \(f_0 = -p + q + t = 0\) we have the Riccati equation

\[
q' = q^2 + tq - a_i.
\]

(2.33)

Equation (2.33) is linearized by putting \(q = -\frac{w'}{w}\)

\[
w'' - tw' - a_1 w = 0.
\]

(2.34)

Let \(H_\lambda(t)\) denote the Hermite function defined by [1]

\[
H_\lambda(t) = 2^\lambda \sqrt{\pi} \left[ \frac{1}{\Gamma\left(\frac{1-\lambda}{2}\right)} \left(\frac{1}{2}\right)^{-\lambda} - \frac{\sqrt{2} t}{\Gamma\left(\frac{-\lambda}{2}\right)} \right] \cdot \left(\frac{1-\lambda}{2}, \frac{1}{2}, t^2/2\right).
\]

(2.35)

\(H_\lambda(t)\) satisfies the differential equation

\[
H_\lambda''(t) - \lambda H_\lambda'(t) + \lambda H_\lambda(t) = 0,
\]

(2.36)

and the contiguity relation

\[
H_\lambda'(t) = \lambda H_{\lambda-1}(t).
\]

(2.37)
Note that if \( l = n \in \mathbb{N} \), \( H_n(t) \) is the Hermite polynomial
\[
H_n(t) = (-1)^n e^{t^2} \left( \frac{d}{dt} \right)^n e^{-t^2}.
\] (2.38)

Therefore \( H_{-\alpha}(t) \) solves (2.34) and corresponding \( q \) is given by
\[
q = -\frac{H'_{-\alpha}(t)}{H_{-\alpha}(t)}.
\] (2.39)

By taking this solution as a seed we can apply the Bäcklund transformations to obtain the solutions expressible by the rational functions of the Hermite functions (actually the ratio of determinants of them). We call this class of the special solutions the hypergeometric solutions to \( P_{IV} \).

Let us apply the same specialization \( a_0 = 0 \) to the \( dP_{II} \) (2.13). Then we see that it admits the specialization \( f_0 = -p + q + t = 0 \)
\[
\bar{q} = -\frac{a_2}{q + t},
\] (2.40)

which is linearized by putting \( q = -\frac{\bar{q}}{w} \) as
\[
\bar{w} - tw + (a_2 - 1)w = 0.
\] (2.41)

Since \( H_{\lambda}(t) \) satisfies the recursion relation
\[
H_{\lambda+1}(t) = tH_{\lambda}(t) + \lambda H_{\lambda-1}(t) = 0,
\] (2.42)

\( w = H_{\lambda-1}(t) = H_{-\alpha}(t) \) solves (2.41). Moreover, from the contiguity relation (2.37) \( q \) is rewritten as \( q = -\frac{H'_{-\alpha}(t)}{H_{-\alpha}(t)} \).

Thus, we have confirmed the existence of a solution which satisfies both \( P_{IV} \) (2.2) and \( dP_{II} \) (2.13) simultaneously. This fact is expected by construction, since the \( dP_{II} \) flow commutes with \( P_{IV} \) flow.

**2.4. Biquadratic pencils and autonomous \( dP_{II} \)**

In this subsection, we consider the autonomous case. In \( P_{IV} \) (2.6), the parameters are normalized in such a way that \( a_0 + a_1 + a_2 = 1 \). We here rescale the variables as \( (q, p, t, a_0, a_1, a_2)_{old} = (\delta^{-1/2}q, \delta^{-1/2}p, \delta^{-1/2}t, \delta^{-1}a_0, \delta^{-1}a_1, \delta^{-1}a_2) \) so that \( a_0 + a_1 + a_2 = \delta \). We also introduce a new independent variable \( s = t/\delta \), hence \( \frac{dt}{ds} = \delta \). Then the autonomous case is given by taking the limit \( \delta \to 0 \). The resulting equation has the same form as (2.1) or (2.2), where \( ' \) is understood as the differentiation with respect to \( s \), and \( t' = 0 \).

The Hamiltonian \( H_{IV} \) is a conserved quantity of this autonomous \( P_{IV} \). The integral curves
\[
C_\lambda : H_{IV} = -a_1 p - a_2 q + pq(p - q - t) = \lambda,
\] (2.43)

define a one-parameter family (pencil) of curves of bidegree \( (2, 2) \) on \( (q, p) \)-plane. In terms of this pencil of curves, the autonomous version of \( dP_{II} \) (2.13) is geometrically reformulated as follows. For a point \( (q_0, p_0) \) given, we choose the parameter \( \lambda \) so that \( C_\lambda \) passes through it. Then we have the following two points \( (q_1, p_0) \) and \( (q_0, p_1) \) where \( q_1 \) (resp. \( p_1 \)) is determined by solving \( H_{IV}(q_0, p_0) = H_{IV}(q_1, p_0) \) (resp. \( H_{IV}(q_1, p_1) = H_{IV}(q_1, p_0) \)) as (see figure 1)
This procedure defines a discrete dynamical system \((q_0, p_0) \rightarrow (q_1, p_1)\) on the \((q, p)\)-plane (the autonomous dP\(_I\)). Note that the mapping \((q_0, p_0) \rightarrow (q_1, p_1)\) is composed of two steps, the horizontal flip \((q_0, p_0) \rightarrow (q_0, p_0)\) followed by the vertical flip \((q_0, p_0) \rightarrow (q_1, p_0)\). In general, the class of discrete dynamical systems arising from pencils of biquadratic curves by the above procedure is called the \textit{QRT mappings}. The QRT mappings were originally obtained in \([105, 106]\) as reduction of discrete soliton equations; the above geometric formulation is due to \([123]\) (see also \([12, 49]\)).

In general, a pencil of biquadratic curves \(\lambda F(q, p) + \mu G(q, p) = 0\) is characterized by the eight points in \(\mathbb{P}^1 \times \mathbb{P}^1\) which are the intersection of \(F(q, p) = 0\) and \(G(q, p) = 0\). Such a configuration of eight points is \textit{special}, since any generic configuration of eight points determines a unique biquadratic curve passing through them. As will be discussed later, the discrete Painlevé equations (non-autonomous cases) arise from the \textit{non-special} configurations of eight points in \(\mathbb{P}^1 \times \mathbb{P}^1\).

Historically, the QRT mappings played a crucial role in the construction and development of the theory of discrete Painlevé equations. Grammaticos, Ramani and Papageorgiou [27] first observed the \textit{singularity confinement property} of the QRT mapping and noticed that it can be regarded as a discrete analogue of the Painlevé property. By using the singularity confinement property as the integrability detector, many interesting discrete Painlevé equations were found by de-autonomizing the QRT mappings [26, 110].

2.5. \(\tau\) functions

The \(\tau\) function is one of the most important objects in the theory of integrable systems. In the context of Painlevé differential equations, it is introduced as a function whose logarithmic derivative gives the Hamiltonian [37, 38, 40, 96, 101]. One can also define the Bäcklund transformations on \(\tau\) functions in such a way that they are consistent with the differential equation.
Consider the Hamiltonian (2.3) for $P_{IV}$ in terms of $f_0$, $f_1$, $f_2$

$$H_{IV} = -a_0 f_2 + a_2 f_1 + f_2 f_1 f_0.$$  

(2.45)

Applying the Bäcklund transformation $s_1$ and $s_2$ defined in (2.7), we observe that

$$s_1(H_{IV}) = H_{IV} + a_1 t, \quad s_2(H_{IV}) = H_{IV} - a_2 t.$$  

(2.46)

Hence, we slightly modify the Hamiltonian so that it is invariant with respect to $s_1$ and $s_2$:

$$h_0 = -a_0 f_2 + a_2 f_1 + f_2 f_1 f_0 + \frac{a_1 - a_2}{3} t.$$  

(2.47)

Then we see that

$$s_1(h_0) = s_2(h_0) = h_0, \quad s_0(h_0) = h_0 + \frac{a_0}{f_0}.$$  

(2.48)

Introducing two other Hamiltonians $h_1$, $h_2$ by

$$h_1 = \pi(h_0) = -a_2 f_0 + a_0 f_2 + f_0 f_2 f_1 + \frac{a_2 - a_0}{3} t,$$  

(2.49)

$$h_2 = \pi^2(h_0) = -a_0 f_1 + a_1 f_0 + f_1 f_0 f_2 + \frac{a_0 - a_1}{3} t,$$  

(2.50)

we have

$$h_2 - h_1 = f_0 - \frac{t}{3}, \quad h_0 - h_2 = f_1 - \frac{t}{3}, \quad h_1 - h_0 = f_2 - \frac{t}{3}.$$  

(2.51)

From the first equation of (2.6), (2.51) and (2.48) we have

$$\frac{f'_0}{f_0} = f_1 - f_2 + \frac{a_0}{f_0} = 2h_0 - h_1 - h_2 + \frac{a_0}{f_0} = s_0(h_0) + h_0 - h_1 - h_2.$$  

(2.52)

We now introduce the $\tau$ function $\tau_i$ ($i = 0, 1, 2$) by

$$h_1 = (\log \tau_i)' = \frac{\tau_i'}{\tau_i}.$$  

(2.53)

Substituting (2.53) into (2.52), we find that $f_0$ should be expressed as

$$f_0 = c_0 \frac{\tau_0 s_0(\tau_0)}{\tau_1 \tau_2}.$$  

(2.54)

where $c_0$ is an integration constant. Similarly, we also obtain

$$f_1 = c_1 \frac{\tau_1 s_1(\tau_1)}{\tau_2 \tau_0}, \quad f_2 = c_2 \frac{\tau_2 s_2(\tau_2)}{\tau_0 \tau_1}.$$  

(2.55)

It is a subtle question how to fix the constants $c_i$, since they may depend on the parameters $a_0$, $a_1$, $a_2$. While being aware of this point, we make the simplest possible choice by setting $c_0 = c_1 = c_2 = 1$; namely

$$f_0 = \frac{\tau_0 s_0(\tau_0)}{\tau_1 \tau_2}, \quad f_1 = \frac{\tau_1 s_1(\tau_1)}{\tau_2 \tau_0}, \quad f_2 = \frac{\tau_2 s_2(\tau_2)}{\tau_0 \tau_1}.$$  

(2.56)
Hence we introduce the Bäcklund transformations on \( \tau \) functions as follows:

\[
\begin{array}{ccc}
  s_0 & \tau_0 & \tau_1 & \tau_2 \\
  s_1 & \tau_0 & \tau_1 & \tau_2 \\
  s_2 & \tau_0 & \tau_1 & \tau_2 \\
  s_3 & \tau_0 & \tau_1 & \tau_2 \\
\end{array}
\]

One can show that this definition of the Bäcklund transformations is consistent with the differential equations (2.58) and (2.59), and that they form the extended affine Weyl group of type \( A_{2}^{(1)} \).

2.6. Space of initial values

2.6.1. Resolution of singularities by blowing-up: a simple example. Let us first consider the following simple differential equation

\[ x' = 1, \quad xy' = y. \]  
(2.58)

There exists a unique solution for generic initial value \((x(t_0), y(t_0)) = (\xi, \eta)\) with \( \xi \neq 0 \). In case of \( \xi = 0 \), (i) if \( \eta \neq 0 \) the point \((0, \eta)\) is inaccessible, namely, there is no solution passing through this point, (ii) if \( \eta = 0 \) there are infinitely many solutions. To see this, we change the variables as

\[(x_1, y_1) = \left( x, \frac{y}{x} \right), \quad \text{namely} \quad (x, y) = (x_1, x_1 y_1), \]
(2.59)

which yields a regular differential equation

\[ x_1' = 1, \quad y_1' = 0. \]  
(2.60)

The general solution to (2.60) is given by \((x_1, y_1) = (t - t_0, C)\), where \( C \) is an arbitrary constant. In terms of the variables \((x, y)\), \((x, y) = (t - t_0, C(t - t_0))\) parametrizes the solutions of (2.58) passing through \((x, y) = (0, 0)\) at \( t = t_0 \). This means that the singularity of (2.58) at \((x, y) = (0, 0)\) is resolved, and the infinitely many solutions passing through \((x, y) = (0, 0)\) are separated by the gradient variable \( y_1 = \frac{y}{x} \). The transformation (2.59) is called the blowing up at \((x, y) = (0, 0)\). By this transformation the point \((x, y) = (0, 0)\) corresponds to the line \( x_1 = 0 \), called the exceptional line (see figure 2). To be more precise,
the exceptional line should be considered as \( \mathbb{P}^1 \) including the point where the gradient variable is \( y_i = \infty \). In order to cover the whole exceptional line, we also use the variable \((\xi, \eta_i)\) such that \((x, y) = (\xi, \eta_1, \eta_i)\) as a companion to \((x_1, y_1)\). This process of blowing up replaces the point \((x, y) = (0, 0)\) by the exceptional line \( \mathcal{E} = \{x_1 = 0\} \cup \{\eta_i = 0\} \), which is graphically described in figure 3.

2.6.2. Resolution of singularities of \( P_{IV} \). Okamoto applied this type of procedures to each of the Painlevé equations to construct the *space of initial values* which parametrizes the whole set of solutions. Taking the example of \( P_{IV} \)

\[
\begin{align*}
q' &= -a_1 + 2pq - q^2 - qt, \\
p' &= a_2 - p^2 + 2pq + pt,
\end{align*}
\]

we describe how this procedure works without getting into the details. As is easily seen, there is no singularities for finite \((q, p)\). Regarding \((q, p)\) as the inhomogeneous coordinates of \( \mathbb{P}^1 \times \mathbb{P}^1 \), we investigate the singularities around the points at infinity by using three sets of local coordinates (1) \((q, 1/p)\), (2) \((1/q, p)\) and (3) \((1/q, 1/p)\).

(1) We first change the dependent variables \((q, p)\) to \((q_0, p_0) = (q, 1/p)\) to see the solutions that pass through the line \( p = \infty (p_0 = 0) \), which yields

\[
\begin{align*}
q_0' &= \frac{2q_0}{p_0} - a_1 - tq_0 - q_0^2, \\
p_0' &= 1 - (t + 2q_0)p_0 - a_2p_0^2.
\end{align*}
\]

We see that if \( p_0 = 0 \), \( q_0 = 0 \) the point \((p_0, q_0)\) is *inaccessible* (no solution can pass through the point), and \((q_0, p_0) = (0, 0)\) is the singular point at which we should apply the blowing-up: \((q_0, p_0) = (q_1p_1, p_1)\). Then we obtain

\[
\begin{align*}
q_1' &= \frac{q_1 - a_1}{p_1} + (a_2 + q_1)q_1p_1, \\
p_1' &= 1 - tp_1 - (a_2 + 2q_1)p_1^2.
\end{align*}
\]

When \( p_1 = 0 \), the point \((q_1, p_1) = (a_1, 0)\) is the only *accessible singularity*, where we need another blowing-up: \((q_1', p_1) = (a_1 + q_2p_2, p_2)\). Then we obtain a regular differential equation

\[
\begin{align*}
q_1' &= \frac{q_1 - a_1}{p_1} + (a_2 + q_1)q_1p_1, \\
p_1' &= 1 - tp_1 - (a_2 + 2q_1)p_1^2.
\end{align*}
\]
In this way, the singularity at $(q, p) = (0, \infty)$ has been resolved by two successive blowing-ups.

We introduce some notations of algebraic geometry in order to book-keep this procedure. A formal $\mathbb{Z}$-linear combination of curves on a surface is called a divisor. In $\mathbb{P}^1 \times \mathbb{P}^1$ with inhomogeneous coordinates $(q, p)$, we denote by $H_1$ and $H_2$ the classes of divisors (curves) $q = \text{const.}$ and $p = \text{const.}$, respectively. In the first blowing-up, we denote by $E_1$ the exceptional divisor $p_1 = 0$ obtained from the singularity $(q, p) = (0, \infty)$. In the blowing-up space, the divisor corresponding to $p = \infty$ ($p_0 = 0$) has two components; one is the exceptional divisor $E_1$ and the other, called the proper transform of $p = \infty$, is denoted by $H_2 - E_1$. In the second blowing-up space, we denote by $E_2$ the exceptional divisor $p_2 = 0$ obtained from the singularity $(q_1, p_1) = (a_1, 0)$, and by $E_1 - E_2$ the proper transform of $E_1$. (See figure 4).

(2) We next change the dependent variables $(q, p)$ to $(q_0, p_0) = (1/q, p)$ to see the solutions that pass through the divisor $q = \infty$ ($q_0 = 0$) which yields the differential equation

$$
\begin{align*}
q_0' &= a_1^2 + a_1 a_2 + a_2 q_0 + 2(2a_1 + a_2) q_0 p_0 + 3 q_0^2 p_0^2, \\
p_0' &= 1 - p_0^2 - (2a_1 + a_2)p_0^2 - 2q_0^3.
\end{align*}
$$

(2.64)

In this way, the singularity at $(q, p) = (0, \infty)$ has been resolved by two successive blowing-ups.

The corresponding exceptional divisors are denoted by $E_3$ and $E_4$, respectively.
We finally investigate the solution around $q = \infty, p = \infty$ by the change of coordinates $q = q_1 p_1, p = a q_2$.

\begin{align*}
\begin{cases}
q_0' &= 1 + \frac{q_0 (p_0 t - 2)}{p_0} + a_0 q_0^2, \\
p_0' &= -\frac{2 p_0}{q_0} + 1 - t p_0 - a_2 p_0^2.
\end{cases}
\end{align*}

(2.67)

In this case, we need four successive blowing-ups to resolve the singularity at $(q_0, p_0) = (0, 0)$:

(i) $(q_0, p_0) = (q_1, q_1 p_1)$,
(ii) $(q_1, p_1) = (q_2, 1 + q_2 p_2)$,
(iii) $(q_2, p_2) = (q_3, -t + q_3 p_3)$,
(iv) $(q_3, p_3) = (q_4, a_0 + t^2 + q_4 p_4)$.

(2.68)
In fact, after the fourth blowing-up, we obtain a rational differential equation with respect to 
$q_4$, but it has no singularity at $q_4 = 0$ any more. The corresponding exceptional divisors are denoted by $E_3$, $E_6$, $E_7$, and $E_8$, respectively (see figure 5).

As we have seen, all the singularities of $P_{IV}$ (2.61) were resolved by eight blowing-ups. The process of blowing-ups is read off graphically from figure 6. We denote by $X$ the surface obtained from $P^1 \times P^1$ by eight blowing-ups in this way. This surface $X$ still contains inaccessible points of $P_{IV}$ on a divisor $D$ with seven components

\[ \delta_1 = E_1 - E_2, \quad \delta_2 = H_1 - E_1 - E_5, \quad \delta_3 = E_5 - E_6, \quad \delta_4 = H_1 - E_3 - E_5, \quad \delta_5 = E_3 - E_4, \quad \delta_6 = E_6 - E_7, \quad \delta_0 = E_7 - E_8. \]

The space of initial values of Okamoto is obtained as $X \setminus D$ by removing the inaccessible divisor $D$ (sometimes called the vertical leaves) from $X$. Then $P_{IV}$ becomes a regular differential equation globally defined on this surface $X \setminus D$. In fact, $P_{IV}$ is represented as a polynomial Hamiltonian system on each chart of $X \setminus D$ constructed above. Conversely, it is also known that $P_{IV}$ is uniquely determined by this property from the surface $X \setminus D$ itself, as was shown by Takano et al [73, 117].

We also remark that the Bäcklund transformations of $P_{IV}$ as well as $dP_{II}$ are regularized on the space of initial values $X \setminus D$. The same story applies to the whole class of Painlevé differential equations [83]. One of the purposes of this paper is to pursue this philosophy in the theory of discrete Painlevé equations.

3. Root systems, Weyl groups and Picard lattice

In this section, we give a brief introduction to powerful tools that are necessary for systematically developing the geometric theory of Painlevé equations.

3.1. Root systems

The fundamental reference to the contents of this subsection is [43]. Let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix, namely

\[ a_{ii} = 2, \quad a_{ij} \in \mathbb{Z}_{\leq 0}, \quad a_{ij} = 0 \iff a_{ji} = 0 \quad (i \neq j). \]
We define the Weyl group $W(A)$ associated with the generalized Cartan matrix $A$ by the generators $s_i (i \in I)$ and the fundamental relations

\begin{align*}
s_i^2 &= 1 \quad \text{for all } i \in I, \\
s_is_j &= s_js_i \quad \text{when } (a_{ij}, a_{ji}) = (0, 0), \\
s_is_i &= s_is_i \quad \text{when } (a_{ij}, a_{ji}) = (-1, -1), \\
s_is_js_j &= s_js_is_j \quad \text{when } (a_{ij}, a_{ji}) = (-1, -2), \\
s_is_is_js_j &= s_js_is_js_j \quad \text{when } (a_{ij}, a_{ji}) = (-1, -3). \tag{3.2}
\end{align*}

As we will see below, this group $W(A)$ can be realized as a group generated by reflections acting on a vector space.

In the following, we confine ourselves to the symmetrizable cases where the matrix elements $a_{ij}$ are realized by the inner product (non-degenerated symmetric bilinear form) $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Q}$ on a $\mathbb{Q}$-vector space $V$ as

\[ a_{ij} = \langle \alpha_i, \alpha_j \rangle, \quad \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}, \tag{3.3} \]

in terms of a set of $\mathbb{Q}$-linearly independent vectors $\alpha_i (i \in I)$ such that $\langle \alpha_i, \alpha_i \rangle = 0$; $\alpha_i$ and $\alpha_i^\vee$ are called the simple roots and the simple coroots, respectively. The free $\mathbb{Z}$-submodules

\[ Q(A) = \mathbb{Z} \alpha_i, \quad Q'(A) = \mathbb{Z} \alpha_i^\vee \tag{3.4} \]

are called the root lattice and the coroot lattice, respectively.

For each element $\alpha \in V$ with $\langle \alpha, \alpha \rangle = 0$ we define the reflection $r_\alpha : V \to V$ by

\[ r_\alpha(\lambda) = \lambda - \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \alpha = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha, \quad \lambda \in V. \tag{3.5} \]

One can verify that $r_\alpha$ have the following properties:

1. $(r_\alpha)^2 = \text{id}_V$; $r_\alpha(\alpha) = -\alpha$ and $r_\alpha(\lambda) = \lambda$ if $\langle \alpha, \lambda \rangle = 0$.
2. $r_\alpha$ and $r_\beta$ commute with each other when $\langle \alpha, \beta \rangle = 0$.
3. (braid relations)

\begin{align*}
r_\alpha r_\beta r_\alpha &= r_\beta r_\alpha r_\alpha \quad \text{when } (\langle \alpha^\vee, \beta \rangle, \langle \beta^\vee, \alpha \rangle) = (-1, -1), \\
r_\alpha r_\beta r_\alpha &= r_\beta r_\alpha r_\alpha \quad \text{when } (\langle \alpha^\vee, \beta \rangle, \langle \beta^\vee, \alpha \rangle) = (-1, -2), \\
r_\alpha r_\beta r_\alpha r_\beta r_\alpha &= r_\beta r_\alpha r_\beta r_\alpha r_\beta r_\alpha \quad \text{when } (\langle \alpha^\vee, \beta \rangle, \langle \beta^\vee, \alpha \rangle) = (-1, -3). \tag{3.6}
\end{align*}

4. (isometry) $(r_\alpha(\lambda), r_\alpha(\mu)) = \langle \lambda, \mu \rangle$ for any $\lambda, \mu \in V$.
5. For any $\mathbb{Q}$-linear isometry $f : V \to V$, $f r_\alpha = r_{f(\alpha)} f$.

The reflections $r_i = r_{\alpha_i} \in \text{GL}(V)$ by the simple roots are called the simple reflections. From the properties (1), (2) and (3), we see that the correspondence $s_i \mapsto r_i (i \in I)$ defines a linear representation of $W(A)$ on $V$. We remark that each simple reflection stabilizes $Q(A)$ and $Q'(A)$ and so does $W(A)$.

The generalized Cartan matrices are classified into three types according to the signatures: (i) finite type $(+, +, \cdots)$ (ii) affine type $(0, +, +, \cdots)$ (iii) indefinite type (otherwise). We are particularly interested in the affine root systems of type A, D, E for which the inner product can be renormalized so that $\langle \alpha_i, \alpha_i \rangle = 2$ and hence $\alpha_i^\vee = \alpha_i$ for all $i \in I$: the corresponding generalized Cartan matrices $A = (a_{ij})_{i,j=0,\ldots,}$ are given by
In each case, the vector $\delta$ defined above carries a characteristic property $\langle \alpha_i, \delta \rangle = 0$ ($i = 0, \ldots, l$) and called the null root. Writing $\delta = \sum_{j=0}^{l} n_j \alpha_j$, we have $\langle \alpha_i, \delta \rangle = \sum_{j=0}^{l} \langle \alpha_i, \alpha_j \rangle n_j = 0$. This means that $[n_0, \ldots, n_l] \in \mathbb{Z}^{l+1}$ is the eigenvector of the zero eigenvalue of $A$.

The diagram illustrated in each case is called the Dynkin diagram, from which one can recover the off-diagonal entries of the generalized Cartan matrix by the following rule:
We call a permutation $\sigma$ of index set $I$ such that $a_{\sigma(i)\sigma(j)} = a_{ij}$ ($i, j \in I$) a Dynkin diagram automorphism. By abuse of terminology, the phrase of Dynkin diagram automorphism is used for a wider class of objects that are induced from such a permutation $\sigma$. For a group $G$ of Dynkin diagram automorphisms, one can define an extended Weyl group $\tilde{W}(A)$ as the group generated by the Weyl group $W(A)$ together with $\pi \sigma$ ($\sigma \in G$) such that $\pi \sigma \pi^{-1} = s_{\sigma(i)}$.

### 3.2. Kac translation

An important feature of these affine root systems is that the associated Weyl groups are of finite order and include the translations. Denoting $V_0 = \{ \lambda \in V | \langle \delta, \lambda \rangle = 0 \}$, for each $\alpha \in V_0$ we define the Kac translation $T_\alpha : V \rightarrow V$ by

$$T_\alpha (\lambda) = \lambda + \langle \delta, \lambda \rangle \alpha - \left( \frac{1}{2} \langle \alpha, \alpha \rangle \langle \delta, \lambda \rangle + \langle \alpha, \lambda \rangle \right) \delta, \quad \lambda \in V. \quad (3.14)$$

One can verify that the linear transformation $T_\alpha$ has the following properties:

1. For any $\alpha, \beta \in V_0$, $T_\alpha T_\beta = T_{\alpha + \beta}$ and hence $T_\alpha T_{\beta} = T_{\beta} T_{\alpha}$.
2. For any $w \in W(A)$, $w T_\alpha = T_{w(\alpha)} w$.
3. If $\alpha \in V_0$ and $\langle \alpha, \alpha \rangle \neq 0$, then $T_\alpha = r_\alpha^{-1} r_\alpha$.
4. For $\beta \in V_0$, $T_\alpha (\beta) = \beta - \langle \alpha, \beta \rangle \delta$.
5. (Isometry) For any $\lambda, \mu \in V$, $\langle T_\alpha (\lambda), T_\alpha (\mu) \rangle = \langle \lambda, \mu \rangle$.

As was shown in (2.13), if we have a suitable birational representation of an affine Weyl group, discrete Painlevé equations arise from its translations. Here we describe how to construct such translations as compositions of simple reflections and Dynkin diagram automorphisms.

**Example 3.1.** Let us consider the root system of type $A_2^{(1)}$ which is generated by three simple roots $\alpha_i$ ($i = 0, 1, 2$). The inner product is characterized by the generalized Cartan matrix (3.8) with $l = 2$. The corresponding Weyl group $W(A_2^{(1)})$ is generated by three reflections $r_i = e_i$ ($i = 0, 1, 2$). Their action on the simple roots $\alpha_i$ ($i = 0, 1, 2$) is computed by using (3.5) as

$$
\begin{array}{c|ccc}
\alpha_0 & \alpha_1 & \alpha_2 \\
\hline
r_0 & \alpha_0 & \alpha_0 + \alpha_1 & \alpha_0 + \alpha_2 \\
r_1 & \alpha_0 + \alpha_1 & -\alpha_1 & \alpha_1 + \alpha_2 \\
r_2 & \alpha_0 + \alpha_2 & \alpha_1 + \alpha_2 & -\alpha_2.
\end{array}
$$

(3.15)

Though the above formulae are derived as linear transformations of the vectors $\alpha_i \in V$, they can be interpreted also as a linear substitutions of variables in the field of rational functions $\mathbb{C}(\alpha_0, \alpha_1, \alpha_2)$. The action of the translation $T_\alpha$ on the simple roots is obtained from (3.14)

$$
T_{\alpha_0}(\alpha_0) = \alpha_0 + \delta, \quad T_{\alpha_1}(\alpha_1) = \alpha_1 - 2\delta, \quad T_{\alpha_2}(\alpha_2) = \alpha_2 + \delta. \quad (3.16)
$$

The translation by root vectors can be expressed as a product of simple reflections. An explicit expression for $T_{\alpha_i}$ is obtained in the following manner. We write (3.16) as
In general, suppose that we have a substitution $f$ such that

$$f[\alpha_0, \alpha_1, \alpha_2] = [f(\alpha_0), f(\alpha_1), f(\alpha_2)] = [\beta_0, \beta_1, \beta_2],$$

where $\beta_i (i = 0, 1, 2)$ are linear combinations of $\alpha_i (i = 0, 1, 2)$. Then, if $\beta_1$ has negative coefficient for instance, we consider $f_{r_1}$ to obtain

$$f_{r_1}[\alpha_0, \alpha_1, \alpha_2] = f[r_1(\alpha_0), r_1(\alpha_1), r_1(\alpha_2)] = f[\alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1] = [\beta_0 + \beta_1, -\beta_1, \beta_2 + \beta_1].$$

(We adopt the convention of symbolical compositions in the sense of remark 2.1 unless otherwise stated.) We continue this procedure until all the coefficients become positive. If negative signs appear at two or more positions, one can apply the procedure at any one of them. The results may give different but equivalent reduced (shortest) decompositions up to the fundamental relations. In case of (3.17), the procedure goes as follows

$$T_{\alpha_i} r_1[\alpha_0, \alpha_1, \alpha_2] = T_{\alpha_i}[\alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1] = [-\alpha_2, 2\alpha_0 + \alpha_1 + 2\alpha_2, -\alpha_0].$$

Therefore $T_{\alpha_i} r_1 r_2 r_0 = \text{id}$ i.e. $T_{\alpha_i} = r_0 r_2 r_0 r_1$. It is known that this procedure terminates after a finite number of steps (see, for example [43], lemma 3.11). The same procedure also applies to more general transformation such as $S[\alpha_0, \alpha_1, \alpha_2] = [\alpha_0 + m_0 \delta, \alpha_1 + m_1 \delta, \alpha_2 + m_2 \delta]$ with $m_0 + m_1 + m_2 = 0$. For example, in the case of $(m_0, m_1, m_2) = (1, -1, 0)$, namely,

$$S[\alpha_0, \alpha_1, \alpha_2] = [\alpha_0 + \delta, \alpha_1 - \delta, \alpha_2],$$

we have:

$$S[\alpha_0, \alpha_1, \alpha_2] = [2\alpha_0 + \alpha_1 + \alpha_2, -\alpha_0 - \alpha_2, \alpha_2],$$

$$S r_1[\alpha_0, \alpha_1, \alpha_2] = [\alpha_0 + \alpha_1, \alpha_0 + \alpha_2, -\alpha_0],$$

$$S r_2[\alpha_0, \alpha_1, \alpha_2] = [\alpha_1, \alpha_0, \alpha_2].$$

Hence, $S r_2 = \pi$, where $\pi(\alpha_i) = \alpha_{i+1} (i \in \mathbb{Z}/3\mathbb{Z})$ is an automorphism of the Dynkin diagram of type $A_3^{(2)}$. This example shows that the above procedure terminates possibly with a permutation of the simple roots corresponding to a Dynkin diagram automorphism. This phenomena occurs if we consider finer translations than those by root vectors.

**Example 3.2.** In the procedure explained in example 3.1 one can use arbitrary positive numbers in place of the symbols $\alpha_i$. For this purpose, the convention of numerical composition in remark 2.1 can be applied more effectively than that of symbolic composition, which reduces the complexity of computations drastically. We show such computations in case of the root system of type $D_3^{(2)}$ whose Cartan matrix is (3.9) with $l = 5$. The Dynkin diagram and the null root are given by

$$\begin{array}{c|c|c|c|c}
\alpha_0 & \alpha_5 \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\hline
\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5.
\end{array}$$

respectively. The corresponding Weyl group $W(D_3^{(2)})$ is generated by six reflections $r_i = r_{\alpha_i}$ ($i = 0, 1, \ldots, 5$). Their actions on the simple roots $\alpha_i$ are computed by using (3.5) as
In the convention of numerical composition on the parameter space, the map $f_2 = f_2 : \mathbb{C}^n \to \mathbb{C}^n$, for instance, is expressed graphically as

$$
\begin{array}{c|c|c|c|c|c|c|c}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
\hline
\alpha_0 + \alpha_1 & \alpha_1 + \alpha_2 & \alpha_1 + \alpha_3 & \alpha_1 + \alpha_4 & \alpha_1 + \alpha_5 \\
\hline
\end{array}
\xrightarrow{f_2}
\begin{array}{c|c|c|c|c|c|c|c}
\alpha_0 + \alpha_2 & \alpha_1 + \alpha_2 & \alpha_1 + \alpha_3 & \alpha_1 + \alpha_4 & \alpha_1 + \alpha_5 \\
\hline
\alpha_0 + \alpha_2 & \alpha_1 + \alpha_2 & \alpha_1 + \alpha_3 & \alpha_1 + \alpha_4 & \alpha_1 + \alpha_5 \\
\hline
\end{array}
$$

We consider a translation $S[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5] = [\alpha_0 - \delta, \alpha_1 + \delta, \alpha_2, \alpha_3, \alpha_4, \alpha_5]$, namely $F_3(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_0 - \delta, \alpha_1 + \delta, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, and try to represent it by simple reflections. Taking an initial value $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (10, 1, 2, 3, 4, 5)$ with $\delta = 30$ for instance, we start from

$$
\begin{array}{c|c|c|c|c|c|c|c}
10 & 5 & 31 & -20 & 5 \\
\hline
1 & -2 & -3 & -4 & 31 & -2 & -3 & -4 \\
\hline
\end{array}
\xrightarrow{F_3}
\begin{array}{c|c|c|c|c|c|c|c}
20 & 5 & 13 & 18 & -15 & -4 \\
\hline
2 & -10 & 31 & -18 & 3 & -4 & 13 & -18 & -15 & -4 \\
\hline
\end{array}
$$

As a result, we obtain $F_0F_1F_2F_1F_2F_3F_2F_0F_3 = F_2$, i.e. $S = \pi r_0 r_2 r_3 r_5 r_4 r_1 r_2 r_0$, where $\pi$ is a Dynkin diagram automorphism $[\alpha_0 \leftrightarrow \alpha_1, \alpha_4 \leftrightarrow \alpha_5]$.

The procedure demonstrated in examples 3.1 and 3.2 provides a practical method for expressing a given translation $\alpha_i \to \alpha_i + m_i \delta$, $m_i \in \mathbb{Z}$, such that $\sum m_i \alpha_i = 0$ ({$\delta = \sum n_i \alpha_i$}) in terms of the product of simple reflections and Dynkin diagram automorphisms.

**Remark 3.3.** In the case of the affine Weyl groups of type $A, D, E$, it is known that the translation $T_i$ by any element $\alpha \in Q(A)$ can be expressed as the product of simple reflections. Moreover, the affine Weyl group is decomposed into semi-direct product of the corresponding finite Weyl group and the group of translations [43].

### 3.3. Picard lattice

Here we introduce another fundamental tool for the geometry of Painlevé equations, the *Picard lattice* associated with the eight point blowing-up of $\mathbb{P}^3 \times \mathbb{P}^3$, which corresponds to the root system of type $E_8^{(1)}$. This tool enables us to manipulate rational maps by the language of linear algebra. We denote by
\[ \Lambda = \mathbb{Z}H_1 \oplus \mathbb{Z}H_2 \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_8, \] (3.25)

the free \( \mathbb{Z} \)-module of rank 10 generated by the symbols \( H_1, H_2 \) and \( E_1, \ldots, E_8 \). We introduce the symmetric bilinear form \( \Lambda \times \Lambda \to \mathbb{Z} : (\lambda, \mu) \mapsto \lambda \cdot \mu \) such that
\[
\begin{align*}
H_1 \cdot H_2 &= 1, \\
H_1 \cdot H_1 &= H_2 \cdot H_2 = 0, \\
E_i \cdot E_j &= -\delta_{ij}, \quad (i, j = 1, \ldots, 8; k = 1, 2).
\end{align*}
\] (3.26)

For any surface \( X \) obtained from \( \mathbb{P}^1 \times \mathbb{P}^1 \) by eight blowing-ups, \( \Lambda \) is identified with the Picard lattice \( \text{Pic} \, X \), in which \( H_1 \) and \( H_2 \) represent the divisor classes corresponding to the \( q = \text{const.} \) and \( p = \text{const.} \), respectively, and \( E_1, \ldots, E_8 \) are the exceptional divisors. We consider the ten-dimensional \( \mathbb{Q} \)-vector space
\[ V = \Lambda \otimes \mathbb{Q} = \mathbb{Q}H_1 \oplus \mathbb{Q}H_2 \oplus \mathbb{Q}E_1 \oplus \cdots \oplus \mathbb{Q}E_8, \] (3.27)
and take the inner product \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{Q} \) such that \( \langle \lambda, \mu \rangle = -\lambda \cdot \mu \) for each \( \lambda, \mu \in \Lambda \).

We define the simple roots \( \alpha_0, \alpha_1, \ldots, \alpha_8 \) by
\[
\begin{align*}
\alpha_0 &= E_1 - E_2, \\
\alpha_1 &= H_1 - H_2, \\
\alpha_2 &= H_2 - E_1 - E_2, \\
\alpha_3 &= E_2 - E_3, \\
\alpha_4 &= E_1 - E_3, \\
\alpha_5 &= E_4 - E_5, \\
\alpha_6 &= E_5 - E_6, \\
\alpha_7 &= E_6 - E_7, \\
\alpha_8 &= E_7 - E_8.
\end{align*}
\] (3.28)

Note that \( \langle \alpha_i, \alpha_j \rangle = -\delta_{ij} \cdot \delta_{ij} = 2 \) and hence \( \alpha_i^2 = \alpha_i \) for \( i = 0, 1, \ldots, 8 \). One can verify by direct calculation that \( (\langle \alpha_i, \alpha_j \rangle)_{i,j=0,\ldots,8} = (-\alpha_i \cdot \alpha_j)_{i,j=0,\ldots,8} \) gives the generalized Cartan matrix of type \( E_8^{(1)} \) (3.10). The null root is given by
\[
\delta = 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8
= 2H_1 + 2H_2 - E_1 - \cdots - E_8.
\] (3.29)

In particular, \( \alpha_i \cdot \delta = 0 \) and \( \delta \cdot \delta = 0 \). Then we obtain the root lattice
\[ Q(E_8^{(1)}) = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_8 \subset \Lambda \subset V, \] (3.30)
and the action of the affine Weyl group \( W(E_8^{(1)}) = \langle r_0, \ldots, r_8 \rangle \) \( (r_i = r_{-i}) \) on \( V \) according to the construction in section 3.1. Note also that the lattices \( \Lambda \) and \( Q(E_8^{(1)}) \) are stabilized by \( W(E_8^{(1)}) \).

The bilinear form \( \lambda \cdot \mu \) on the Picard lattice \( \Lambda \) is interpreted as the intersection form of divisors \( \lambda, \mu \). In general
\[ \lambda = d_1H_1 + d_2H_2 - m_1E_1 - \cdots - m_8E_8 \in \Lambda, \quad d_i, m_i \in \mathbb{Z}, \] (3.31)
corresponds to the class of curves on \( \mathbb{P}^1 \times \mathbb{P}^1 \) of bidegree \( (d_1, d_2) \) passing through the blowing-up points \( \mathbb{P}_i \) with multiplicity \( \geq m_i \) \( (i = 1, \ldots, 8) \). For example, the null root \( \delta \) given in (3.29), called the anti-canonical divisor, corresponds to the class of curves of bidegree \((2, 2)\) passing through the eight points.

We remark that the multiplicity \( m \) of a curve \( \phi(x, y) = 0 \) at \((x, y) = (0, 0)\) can be read from the Newton polygon which displays the exponents of \( x^iy^j \) of \( \phi \) with non-zero coefficients as in figure 7. In order to see the multiplicity at \((x, y) = (a, b)\), we change variables to \( (\xi, \eta) = (x-a, y-b) \) and construct the Newton polygon from the coefficients of \( \xi^j\eta^i \). For the multiplicity at \((\infty, b)\) and \((a, \infty)\), we take the coordinates \((\xi, \eta) = (\frac{1}{x}, y-b)\) and \((\xi, \eta) = (x-a, \frac{1}{y})\) respectively. Accordingly, for a given Newton polygon with respect to the coordinate \((x, y)\), the multiplicities at \((x, y) = (0, 0), (\infty, 0), (0, \infty), (\infty, \infty)\) can be read off from the bottom-left, bottom-right, top-left, and top-right corners respectively.

For generic configurations of eight points, it is classically known that the dimension \( d(\lambda) \) of the family and the genus \( g(\lambda) \) of generic curves of the class \( \lambda \) (3.31) are given by
and hence it follows that

\[ d(\lambda) = (d_1 + 1)(d_2 + 1) - \sum_{i=1}^{8} \frac{m_i(m_i + 1)}{2} - 1, \]

\[ g(\lambda) = (d_1 - 1)(d_2 - 1) - \sum_{i=1}^{8} \frac{m_i(m_i - 1)}{2}, \]  

(3.32)

and hence it follows that

\[ 2d(\lambda) = \lambda \cdot (\lambda + \delta), \quad 2g(\lambda) = 2 = \lambda \cdot (\lambda - \delta), \]  

(3.33)

respectively.

Noticing that each exceptional divisor \( E_i \) \((i = 1, \ldots, 8)\) satisfies \( E_i \cdot E_i = -1 \) and \( E_i \cdot \delta = 1 \), we define the subset \( M \subset \Lambda \) by

\[ M = \left\{ \lambda \in \Lambda | \lambda \cdot \lambda = -1, \lambda \cdot \delta = 1 \right\}. \]  

(3.34)

Then \( d(\lambda) = 0 \) and \( g(\lambda) = 0 \) for \( \lambda \in M \). The elements of \( M \) are sometimes called the exceptional classes. Table 1 shows some typical elements of \( M \).

**Example 3.4.** We denote by \( (f, g) \) the inhomogeneous coordinates of \( \mathbb{P}^1 \times \mathbb{P}^1 \) and consider eight blowing-up points \( P_i = (f_i, g_i) \) \((i = 1, \ldots, 8)\). For \( \lambda = H_i - E_i, H_2 - E_j \) and \( H_1 + H_2 - E_i - E_j - E_k \), the defining equations of corresponding curves are given as
respectively.

We also define the subset $R \subset \Lambda$ by

$$R = \{ \alpha \in \Lambda | \alpha \cdot \alpha = -2, \ \alpha \cdot \delta = 0 \}. \quad (3.36)$$

We simply call the elements of $R$ the roots (real roots of type $E_8^{(1)}$). Note that if $\alpha \in R$ we have $d(\alpha) = -1$ and $g(\alpha) = 0$. Table 2 shows some typical elements of $R$.

**Example 3.5.** The meaning of the dimensionality $d(\alpha) = -1$ is that, for the existence of a curve of the class $\alpha$, an extra condition is required on the configuration of points. For example, for $\lambda = H_1 - E_1 - E_2 - E_3$ and $H_1 + H_2 - E_1 - E_2 - E_3 - E_4$, such conditions are given by

$$f_i - f_j = 0, \quad g_i - g_j = 0, \quad \begin{bmatrix} 1 & f_i & g_i & f_i g_i \\ 1 & f_j & g_j & f_j g_j \\ 1 & f_k & g_k & f_k g_k \end{bmatrix} = 0, \quad (3.35)$$

respectively.

**Remark 3.6.** It is known that the set $M$ defined by (3.34) is generated by $W(E_8^{(1)})$ from one of the exceptional divisor $E_1, \ldots, E_8$. Also, the set $R$ is generated by $W(E_8^{(1)})$ from one of the simple roots $\alpha_0, \ldots, \alpha_8$.

**Remark 3.7.** By abuse of the terminology, we say that a polynomial $\phi(x, y)$ belongs to the class $\lambda \in \Lambda$ when the curve $\phi = 0$ is of the class $\lambda$. Then the dimension of the vector space of polynomials belonging to the class $\lambda$ is given by $d(\lambda) + 1$. Note also that if two polynomials $\phi(x, y)$ and $\psi(x, y)$ belong to the class $\lambda$ and $\mu$ respectively, then the product $\phi(x, y) \psi(x, y)$ belongs to the class $\lambda + \mu$. Moreover, we say that a ratio of such polynomials is a rational function of the class $\lambda$. For example, the polynomials of the class $H_1$ are expressed as $ax + b$, and the rational functions of the class $H_1$ as $(ax + b)/(cx + d)$.

**Remark 3.8.** The Painlevé equations are sometimes discussed in the framework of nine point configuration on $\mathbb{P}^2$ [53, 112] as well as eight point configuration on $\mathbb{P}^1 \times \mathbb{P}^1$. We here

| $\lambda \in R$ | Geometric meaning |
|-----------------|-------------------|
| $E_i - E_j$     | Exceptional curve passing through $P_j$ |
| $H_i - E_j - E_k$ | Line passing through $P_j$ and $P_k$ |
| $H_i + H_2 - E_i - E_j - E_k - E_l$ | Curve of bidegree (1, 1) passing through $P_i, P_j, P_k, P_l$ |

Table 2. Typical elements of $R$.
give a correspondence between these two formulations. Consider a birational mapping between \( \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \) given by

\[
(X_0 : X_1 : X_2) \rightarrow (X_0 : X_1) \times (X_0 : X_2),
\]

where \((X_0 : X_1 : \ldots : X_n)\) is a homogeneous coordinate of \( \mathbb{P}^i \). Let \( X \) be a blow-up of \( \mathbb{P}^2 \) at two points \((0 : 1 : 0)\) and \((0 : 0 : 1)\), and let \( Y \) be a blow-up of \( \mathbb{P}^1 \times \mathbb{P}^1 \) at \((0 : 1) \times (0 : 1)\), then the birational mapping gives a biholomorphic mapping between \( X \) and \( Y \) (see figure 8). The Picard lattice of \( X \) and \( Y \) are given as

\[
\begin{align*}
\text{Pic}(X) &= \mathbb{Z}\mathcal{E}_0 \oplus \mathbb{Z}\mathcal{E}_1 \oplus \mathbb{Z}\mathcal{E}_2, \\
\text{Pic}(Y) &= \mathbb{Z}H_1 \oplus \mathbb{Z}H_2 \oplus \mathbb{Z}E_i,
\end{align*}
\]

respectively, where non-vanishing intersections are given by \( \mathcal{E}_0^2 = 1, \mathcal{E}_1^2 = \mathcal{E}_2^2 = -1 \) and \( H_1 \cdot H_2 = 1, \mathcal{E}_1 \cdot \mathcal{E}_2 = -1 \) respectively. An isomorphism of these lattices are given by

\[
\begin{align*}
\mathcal{E}_0 &= H_1 + H_2 - E_1, \\
\mathcal{E}_1 &= H_1 - E_1, \\
\mathcal{E}_2 &= H_2 - E_1.
\end{align*}
\]

Further blowing up the spaces \( X \) and \( Y \), we have an isomorphism between \((n + 1)\)-point blow-up of \( \mathbb{P}^2 \) and \( n\)-point blow-up of \( \mathbb{P}^1 \times \mathbb{P}^1 \). In particular, one can use nine point blow-up of \( \mathbb{P}^2 \) instead of eight point blow-up of \( \mathbb{P}^1 \times \mathbb{P}^1 \) to describe the Painlevé equations. In the \( \mathbb{P}^2 \) formulation, the Picard lattice is given as

\[
\Lambda = \mathbb{Z}\mathcal{E}_0 \oplus \mathbb{Z}\mathcal{E}_1 \oplus \cdots \oplus \mathbb{Z}\mathcal{E}_9,
\]

with \( \mathcal{E}_0^2 = 1 \) and \( \mathcal{E}_1^2 = \cdots = \mathcal{E}_9^2 = -1 \). Then the simple roots of \( \mathcal{E}_0 \) are

\[
\alpha_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \alpha_i = \mathcal{E}_i - \mathcal{E}_{i+1} (i = 1, \ldots, 8).
\]

These corresponds to the roots in equation (3.28) in \( \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \) by the isomorphism equation (3.40) extended by \( \mathcal{E}_{i+1} = E_i (i = 2, \ldots, 8) \).
3.4. Surface type and symmetry type

It has been observed by Okamoto for all the Painlevé differential equations that there is a remarkable complementary relation between the surface type and the symmetry type in the common root lattice of type $E_8^{(1)}$, as in the following list (see table 3). The origin of this phenomena is described by the geometry of the space of initial values, as we will demonstrated below by taking $P_{IV}$ as an example [93, 95, 97–100, 112].

In the example of $P_{IV}$ shown in section 2.6.2, recall that the divisor of inaccessible points (2.69) has seven components $\delta_i$ ($i = 0, \ldots, 6$):

$$
\begin{align*}
\delta_1 &= E_1 - E_2, & \delta_2 &= H_2 - E_1 - E_5, & \delta_3 &= E_5 - E_6, & \delta_4 &= H_1 - E_3 - E_5, \\
\delta_5 &= E_3 - E_4, & \delta_6 &= E_6 - E_7, & \delta_0 &= E_7 - E_8.
\end{align*}
$$

(3.43)

Regarded as elements of $\Lambda$, they satisfy

$$
\begin{align*}
\delta_0 + \delta_1 + 2\delta_2 + 3\delta_3 + 2\delta_4 + \delta_5 + 2\delta_6 &= \delta, \\
\delta_1 \cdot \delta_i &= -2, & \delta_1 \cdot \delta = 0 & (i = 0, \ldots, 6).
\end{align*}
$$

(3.44)

Furthermore, the intersection numbers among $\delta_i$ are given by the following $7 \times 7$ matrix

$$
- [\delta_i \cdot \delta_j]_{i,j=0,\ldots,6} = \begin{bmatrix}
2 & -1 & -1 & & & & \\
-1 & 2 & -1 & & & & \\
-1 & 2 & -1 & & & & \\
-1 & 2 & -1 & & & & \\
-1 & -1 & 2 & & & & \\
& & & & \delta_0 & & \\
& & & & & \delta_6 & \\
& & & & & & \delta_5
\end{bmatrix}
$$

(3.45)

which is the Cartan matrix of type $E_6^{(1)}$. In view of the decomposition (3.44) of $\delta$ in terms of $\delta_0, \ldots, \delta_6$, the space of initial values of $P_{IV}$ is said to have the surface type of $E_6^{(1)}$.

As we mentioned before, the symmetry of $P_{IV}$ is described by the extended affine Weyl group of type $A_2^{(1)}$. This fact is explained geometrically as follows. Consider the elements $\alpha \in R$ such that

$$
\alpha \cdot \alpha = -2, & \delta_1 \cdot \alpha = 0 & (i = 0, \ldots, 6).
$$

(3.46)

For such an element $\alpha$, the reflection $r_\alpha$ satisfies $r_\alpha(\delta_i) = \delta_i$ ($i = 0, \ldots, 6$); namely, such reflections $r_\alpha$ preserve the surface type. Hence they are expected to generate the symmetry of $P_{IV}$. In fact, we can take three fundamental elements satisfying (3.46) as

$$
\alpha_0 = H_1 + H_2 - E_5 - E_6 - E_7 - E_8, & \alpha_1 = H_1 - E_3 - E_2, & \alpha_2 = H_2 - E_3 - E_4.
$$

(3.47)

Furthermore, the intersection numbers among $\alpha_0$, $\alpha_1$, $\alpha_2$ are given by the Cartan matrix of type $A_2^{(1)}$

$$
- [\alpha_i \cdot \alpha_j]_{i,j=0,1,2} = \begin{bmatrix}
2 & -1 & -1 & & & & \\
-1 & 2 & -1 & & & & \\
-1 & -1 & 2 & & & & \\
& & & & \alpha_0 & & \\
& & & & & \alpha_1 & \\
& & & & & & \alpha_2
\end{bmatrix}
$$

(3.48)

and we have

$$
\alpha_0 + \alpha_1 + \alpha_2 = \delta.
$$

(3.49)

As we will see later, the actions of the three simple reflections $r_{\alpha_i}$ ($i = 0, 1, 2$) on the Picard lattice correspond to the Bäcklund transformations $s_i$ ($i = 0, 1, 2$) for $P_{IV}$. 

26
Remark 3.9. We use the common symbols \( \alpha_i \) \((i = 0, 1, 2, \ldots, l)\) and \( \delta_j \) \((j = 0, \ldots, 8 - l)\) for the simple roots for any pair of root systems representing symmetry and surface types, respectively. Therefore \( \alpha_i \) \((i = 0, 1, 2)\) of type \( A_{2}^{1} \) used above are different from the simple roots of type \( E_{8}^{1} \) in section 3.3.

Then we see that the two sublattices \( L_1 = \mathbb{Z} \delta_0 \oplus \mathbb{Z} \delta_1 \oplus \cdots \oplus \mathbb{Z} \delta_6, \quad L_2 = \mathbb{Z} \alpha_0 \oplus \mathbb{Z} \alpha_1 \oplus \mathbb{Z} \alpha_2, \) are orthogonal complements in the root lattice \( Q(E_8^{1}) \) to each other, namely, \((L_1)^\perp = L_2,\) \((L_2)^\perp = L_1.\) In fact, if \( \lambda = d_1H_1 + d_2H_2 - \sum_{i=1}^{8} m_iE_i \in \Lambda, \) the condition \( \delta_i \cdot \lambda = 0 \) \((i = 0, \ldots, 6)\) is equivalent to

\[
\begin{align*}
\begin{cases}
m_1 = m_2, \quad d_1 = m_1 + m_5, \quad m_5 = m_6, \\
d_2 = m_3 + m_5, \quad m_3 = m_4, \quad m_6 = m_7, \quad m_7 = m_8.
\end{cases}
\end{align*}
\]

Hence, \( \lambda \in (L_1)^\perp \) if and only if

\[
\begin{align*}
\lambda &= (m_1 + m_3)H_1 + (m_3 + m_5)H_2 - m_1(E_1 + E_2) \nonumber \\
&\quad - m_5(E_3 + E_4) - m_5(E_5 + E_6 + E_7 + E_8) \nonumber \\
&= m_1(H_1 - E_1 - E_2) + m_3(H_2 - E_3 - E_4) \nonumber \\
&\quad + m_5(H_1 + H_2 - E_5 - E_6 - E_7 - E_8) \nonumber \\
&= m_1\alpha_1 + m_3\alpha_2 + m_5\alpha_0,
\end{align*}
\]

which shows that \((L_1)^\perp = L_2.\) One can verify \((L_2)^\perp = L_4\) in a similar manner.

Remark 3.10.

(1) \( Q(E_8^{1}) \) and \( \mathbb{Z} \delta \) are orthogonal complements to each other in \( \Lambda. \)

(2) The elements of \( Q(E_8^{1}) \) are not necessarily expressible as \( \mathbb{Z} \)-linear combinations of \( \delta_i \) and \( \alpha_i;\) they are expressible as linear combinations with coefficients in \( \mathbb{Q}. \) This means that \( Q(E_8^{1}) \supseteq L_1 + L_2,\) while \( \mathbb{Q} \)-vector space \( \mathbb{Q} \otimes Q(E_8^{1}) \) is generated by \( L_1 \) and \( L_2.\)

3.5. Example of PIV

Recall that the symmetry of \( \text{PIV} \) is given by the affine Weyl group \( \tilde{W}(A_{2}^{1}) = \langle s_0, s_1, s_2, \pi \rangle \) (see the last paragraph of section 3.1). Here we will describe the symmetry in terms of the Picard lattice.

The action of the reflection \( r_i = r_{a_i} \) corresponding to \( s_i \) \((i = 0, 1, 2)\) on the basis of \( \Lambda \) is computed by using (3.26), (3.5) and (3.47) as (trivial one is omitted)

\[
r_0 : \begin{cases}
H_1 \rightarrow 2H_1 + H_2 - E_5 - E_6 - E_7 - E_8, \\
E_5 \rightarrow H_1 + H_2 - E_6 - E_7 - E_8, \\
E_7 \rightarrow H_1 + H_2 - E_5 - E_6 - E_8.
\end{cases}
\]

\[
r_1 : \begin{cases}
H_2 \rightarrow H_1 + H_2 - E_1 - E_2, \\
E_1 \rightarrow H_1 - E_2, \\
E_2 \rightarrow H_1 - E_1.
\end{cases}
\]

\[
r_2 : \begin{cases}
H_1 \rightarrow H_1 + H_2 - E_3 - E_4, \\
E_3 \rightarrow H_2 - E_4, \\
E_4 \rightarrow H_2 - E_3.
\end{cases}
\]

On the other hand, the Bäcklund transformation \( \pi \) corresponds to the action of the diagram automorphism \( \pi \) on \( E_6^{1} \oplus A_{2}^{1} \).
This shows that the multiplicity \( m \) of \( \mathbb{W}(\Lambda_2^{(1)}) \) on the lattice \( \Lambda \). The relation between the Bäcklund transformation and the above action on the lattice is stated as follows.

The curve obtained from the polynomial factor arising from \( w(\tau_0), w(\tau_1), w(\tau_2), w \in \mathbb{W}(\Lambda_2^{(1)}) \) coincides with the curves in the exceptional class \( w(E_6), w(E_2), w(E_4) \), respectively, under the identification \( s_i = r_i \), where \( r_i = r_{0i} \) (\( i = 0, 1, 2 \)).

We demonstrate this relation by taking an example of \( s_2s_1s_0(\tau_0) \). Noticing (2.5), we have by using (2.57)

\[
s_{210} = s_{21} \left( \frac{\tau_2}{\tau_0} \right) = s_2 \left( \frac{f_0 - a_1}{f_1 \tau_0} \right) = \cdots = \tau_0^2 \tau_1, \tag{3.56}
\]

\[
\phi = (f_0 f_2 - a_2)(f_1 f_2 + a_2) - (a_1 + a_2)f_2^2 - a_1 p^2 - a_2^2 - 2a_2 q p - a_2^2 q^2 p^2 - q^2 p^2 - q^2 r. \tag{3.57}
\]

The curve \( \phi = 0 \) corresponds to the divisor class \( \lambda = d_1 H_1 + d_2 H_2 - \sum_{i=1}^{8} m_i E_i \in \Lambda \) as

\[
\lambda = 2H_1 + 3H_2 - E_1 - E_2 - 2E_3 - 2E_4 - E_5 - E_6 - E_7. \tag{3.57}
\]

In fact, \( \phi \) is the curve of bidegree \((2, 3)\) which implies \( d_1 = 2 \) and \( d_2 = 3 \). We compute the multiplicity \( m_3, m_4 \) at \( E_3, E_4 \), respectively, around \((q, p) = (\infty, 0)\) for instance. To this end, we apply the variable change (2.66) where the divisors \( E_3 \) and \( E_4 \) are defined. First we change variable as \((q, p) = \left( \frac{1}{q_0}, p_0 \right) \) and then \((q_0, p_0) = (q_1, q_1 p_1) \) to obtain

\[
q_0^4 \phi \left( \frac{1}{q_0}, p_0 \right) = -(a_2 q_0 + p_0)^2 - q_0 p_0 t(a_2 q_0 + p_0) + q_0 p_0^2(-a_1 q_0 + p_0 t)
= q_0^2 \left\{ -(a_2 + p_0)^2 - q_1 p_1 t(a_2 + p_1) + q_1^2 p_1^2(-a_1 + p_1 t) \right\}. \tag{3.58}
\]

This shows that the multiplicity \( m_3 \) at \((q_0, p_0) = (0, 0)\) is 2. To see the multiplicity \( m_4 \), we change variables as \((q_1, p_1) = (q_2, -a_2 + q_2 p_2) \). Then the second factor of the right-hand side of (3.58) gives

\[
q_2^2 \left[ -p_2 - a_2 + q_2 p_2 \right] t + (-a_2 + q_2 p_2)^2 \left\{ -a_1 + (-a_2 + q_2 p_2) t \right\}, \tag{3.59}
\]

which implies that \( m_4 = 2 \). Repeating the similar calculations at \((q, p) = (\infty, 0), (\infty, \infty), (\infty, 0), (\infty, \infty) \), we see that \( \phi \) belongs to the class \( \lambda \) (3.57). We note that the multiplicities \( m_i = 1, m_3 = 2 \), \( m_4 = 1 \) can also be simply read off from the pattern of the Newton polygon for \( \phi \) (figure 9) at the corners \((0, 0), (\infty, 0), (\infty, \infty) \) respectively.

On the other hand, one can verify that \( r_0r_1r_2(\tau_0) = 2H_1 + 3H_2 - E_1 - E_2 - 2E_3 - 2E_4 - E_5 - E_6 - E_7 \) by direct computation. Therefore, we have confirmed
the correspondence between the Bäcklund transformations and affine Weyl group actions on the Picard lattice. For the proof of this correspondence, we refer to [53]. We remark that \( \lambda \) corresponding to the \( \tau \) functions always belongs to the exceptional class \( M(3.34) \) with \( d(\lambda) = 0 \). This implies that the curve is uniquely determined from \( \lambda \).

This idea can be applied for analyzing the iteration of \( T \) on \( f \)-variables or equivalently \((q, p)\)-variables. Since \( f_i = \frac{\tau_k(\tau_i)}{\tau_{i+1} \tau_{i-1}} \; (i \in \mathbb{Z}/3\mathbb{Z}) \), we have

\[
w(f) = \frac{w(\tau_i)w(\tau_{i+1})}{w(\tau_{i+1})w(\tau_{i-1})},
\]

for general \( w \in \hat{W}(A_1^{(1)}) \). As seen in the above example and will be clarified in the later sections in the general setting, the polynomial factors of the rational function obtained as \( w(f) \) can be controlled by the language of the Picard lattice. This observation also suggests that the polynomial factors of the iterations \( T^n(f) \) will provide us the information of the underlying point configuration, which will be utilized in section 4.3.

### 4. Detecting point configurations in discrete Painlevé equations

In this section we demonstrate how to associate a configuration of singular points on \( \mathbb{P}^1 \times \mathbb{P}^1 \) to a given discrete equation. This provides us a method for determining whether it is a discrete Painlevé equation, and if so, identifying the type of the equation by its surface type and symmetry type.

#### 4.1. Point configuration for \( q\)-PE\(_6(1)\): an example

We start with an example of a discrete Painlevé equation from which we can easily read off the point configuration. Fixing a non-zero constant \( q \), let us consider the following rational mapping \( T \) of \((\mathbb{C}^\times)^{10} \times (\mathbb{P}^1 \times \mathbb{P}^1) \) [77, 111, 137]:

\[
T : (\kappa_1, \kappa_2, v_1, \ldots, v_8, f, g) \mapsto \left( \frac{\kappa_1}{q}, \kappa_2 q, v_1, \ldots, v_8, f, g \right).
\]

![Figure 9. Newton polygon for \( \phi \).](image-url)
where \( \mathcal{F} \) and \( \mathcal{G} \) are rational functions of \( f \) and \( g \) determined by
\[
\begin{align*}
(fg - 1)(\mathcal{F}g - 1) &= \left( g - \frac{1}{v_1} \right) \left( g - \frac{1}{v_2} \right) \left( g - \frac{1}{v_3} \right), \\
(fg - 1)(\mathcal{F}g - 1) &= \left( g - \frac{1}{v_4} \right) \left( g - \frac{1}{v_5} \right) \left( g - \frac{1}{v_6} \right) \left( g - \frac{1}{v_7} \right), \\
(fg - 1)(\mathcal{F}g - 1) &= \left( g - \frac{1}{v_8} \right) \left( g - \frac{1}{v_9} \right) \left( g - \frac{1}{v_{10}} \right) \left( g - \frac{1}{v_{11}} \right).
\end{align*}
\] (4.2)

Here the variables \( \kappa_i \) (\( i = 1, 2 \)) and \( v_i \) (\( i = 1, \ldots, 8 \)) are subject to the relation
\[
q = \frac{\kappa_1^2 \kappa_2^2}{\prod_{i=1}^{8} v_i}. \tag{4.3}
\]

With the notation \( T^n(f) = f_n, \ T^n(g) = g_n \ (n \in \mathbb{Z}) \), (4.1)–(4.3) can be interpreted as a difference equation with respect to \( n \):
\[
\begin{align*}
\left( f_n s_n - 1 \right) \left( f_{n+1} s_{n+1} - 1 \right) &= \left( g_n - \frac{1}{v_1} \right) \left( g_n - \frac{1}{v_2} \right) \left( g_n - \frac{1}{v_3} \right), \\
\left( f_n s_n - 1 \right) \left( f_{n+1} s_{n+1} - 1 \right) &= \left( g_n - \frac{1}{v_4} \right) \left( g_n - \frac{1}{v_5} \right) \left( g_n - \frac{1}{v_6} \right) \left( g_n - \frac{1}{v_7} \right), \\
\left( f_n s_n - 1 \right) \left( f_{n+1} s_{n+1} - 1 \right) &= \left( g_n - \frac{1}{v_8} \right) \left( g_n - \frac{1}{v_9} \right) \left( g_n - \frac{1}{v_{10}} \right) \left( g_n - \frac{1}{v_{11}} \right).
\end{align*}
\] (4.4)

We also use the notation \( T(x) = x, \ T^{-1}(x) = \bar{x} \) for the mapping \( T \) and its inverse, respectively. We call the discrete Painlevé equation (4.2) or (4.4) \( qPE^6 \), since it has the affine Weyl group symmetry of type \( E_{6}^{(1)} \) as will be shown later.

Since \( \mathcal{F}, \ \mathcal{G} \) are rational functions of \( f \) and \( g \), there may be some points \( (f, g) \in \mathbb{P}^1 \times \mathbb{P}^1 \) where their images \( (\mathcal{F}, \mathcal{G}) \in \mathbb{P}^1 \times \mathbb{P}^1 \) cannot be determined uniquely. To investigate such points, called the points of indeterminacy, we assume that \( \kappa_1, \kappa_2, v_1, \ldots, v_8 \) are generic and rewrite (4.2) as
\[
\begin{align*}
\mathcal{F} g - 1 &= f \left( g - \frac{1}{v_1} \right) \left( g - \frac{1}{v_2} \right) \left( g - \frac{1}{v_3} \right), \\
\mathcal{F} g - 1 &= f \left( g - \frac{1}{v_4} \right) \left( g - \frac{1}{v_5} \right) \left( g - \frac{1}{v_6} \right) \left( g - \frac{1}{v_7} \right), \\
\mathcal{F} g - 1 &= f \left( g - \frac{1}{v_8} \right) \left( g - \frac{1}{v_9} \right) \left( g - \frac{1}{v_{10}} \right) \left( g - \frac{1}{v_{11}} \right).
\end{align*}
\] (4.5)

Note that the second equation is equivalent to
\[
\begin{align*}
\mathcal{G} g - 1 &= g \left( f - \frac{1}{v_1} \right) \left( f - \frac{1}{v_2} \right) \left( f - \frac{1}{v_3} \right), \\
\mathcal{G} g - 1 &= g \left( f - \frac{1}{v_4} \right) \left( f - \frac{1}{v_5} \right) \left( f - \frac{1}{v_6} \right) \left( f - \frac{1}{v_7} \right), \\
\mathcal{G} g - 1 &= g \left( f - \frac{1}{v_8} \right) \left( f - \frac{1}{v_9} \right) \left( f - \frac{1}{v_{10}} \right) \left( f - \frac{1}{v_{11}} \right).
\end{align*}
\] (4.6)

From the first equation in (4.5) we see that \( \mathcal{F} \) is uniquely determined from \( (f, g) \) unless
\[
(f, g) = \left( 0, \frac{v_4}{\kappa_2} \right), \left( 0, \frac{v_6}{\kappa_2} \right), \left( \frac{v_1}{v_2}, 1 \right), \left( \frac{v_2}{v_3}, \frac{1}{v_2} \right), \left( \frac{v_3}{v_4}, \frac{1}{v_3} \right), \left( \frac{v_4}{v_5}, \frac{1}{v_4} \right). \tag{4.7}
\]
Also, (4.6) implies that $g$ is uniquely determined from $(f, g)$ unless

$$
(f, g) = \left( \frac{v_i}{v_7}, 0 \right), \left( \frac{v_i}{v_8}, 0 \right), \left( v_i, \frac{1}{v_1} \right), \left( v_2, \frac{1}{v_2} \right), \left( v_3, \frac{1}{v_3} \right), \left( v_4, \frac{1}{v_4} \right).
$$

(4.8)

In this way, we find that the eight points

$$
P_i : \left( v_i, \frac{1}{v_i} \right) (i = 1, 2, 3, 4), \left( 0, \frac{v_i}{\kappa_2} \right) (i = 5, 6), \left( \frac{v_i}{\kappa_1}, 0 \right) (i = 7, 8),
$$

(4.9)

are the points of indeterminacy of the mapping.

In order to investigate the behavior of the mapping around the points of indeterminacy, we next apply the blowing-up at these points. Around the point of indeterminacy $(f, g) = \left( v_1, \frac{1}{v_1} \right)$ for example, we introduce new variables $(f_i, g_i)$ by setting

$$
f = v_1 + f_1, \quad g = \frac{1}{v_1} + f_1 g_1.
$$

(4.10)

Then the indeterminate factor in (4.2) becomes

$$
\frac{g - \frac{1}{v_1}}{fg - 1} = \frac{g_1}{v_1 g_1 + \frac{1}{v_1} + f_1 g_1},
$$

(4.11)

which is regular at $f_1 = 0$, namely, $(f, g) = \left( v_1, \frac{1}{v_1} \right)$ and $T$ is determined uniquely in terms of $(f, g)$. In this way, we see that the rational mapping $T$ is promoted to a regular mapping on the surface $X$ obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing-up at the eight points of indeterminacy. This process is illustrated in figure 10.

The eight points of indeterminacy are on the curve

$$
C : \quad fg(fg - 1) = 0,
$$

(4.12)

and the points on $D = C \setminus \{P_1, \ldots, P_8\}$ are inaccessible in a similar sense in section 2.6.1, in other words, $D$ itself is stable by $T^4$. This can be verified by computing the image of the points on $D$:

4 The divisor $D$ is inaccessible in the sense that it cannot be reached from outside irrespective of parameters. For special values of parameters, it may happen that some other divisors are inaccessible, such as invariant divisors on which hypergeometric solutions exist. See section 6.
(1) \((f, 0) \mapsto (\tilde{f}, g) = \left(\frac{q_{\nu}^2}{q_{\nu+1} q_{\nu+2}} f, \frac{q_{\nu} q_{\nu+2}}{q_{\nu+1}} g \right), \tilde{f} g = 1,\)

(2) \((0, g) \mapsto (\tilde{f}, g) = \left(\frac{1}{f}, 0 \right), g = 0,\)

(3) \((f, \frac{1}{g}) \mapsto (\tilde{f}, \tilde{g}) = \left(0, \frac{q_{\nu} q_{\nu-1}}{q_{\nu+1}} g \right), \tilde{f} = 0.\)

which implies that if \(P = (f, g) \in D\), then \(\tilde{P} = (\tilde{f}, g) \in D\). Accordingly, if \(P \in X \setminus D\) then \(\tilde{P} \in X \setminus \tilde{D}\).

The inaccessible curve (divisor) \(D\) has three components \(f = 0, g = 0\) and \(fg = 1 = 0\).

In the terminology of the Picard lattice, the divisor \(D\) and its three components correspond to

\[
\delta = 2H_1 + 2H_2 - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8,
\]

\[
\delta_1 = H_1 - E_1 - E_2 - E_3 - E_4 - E_5 - E_6,
\]

\[
\delta_2 = H_2 - E_1 - E_2 - E_3 - E_4,
\]

\[
\delta_0 = H_1 + H_2 - E_1 - E_2 - E_3 - E_4,
\]

respectively, so that \(\delta = \delta_0 + \delta_1 + \delta_2\). Then the intersection numbers among \(\delta_0, \delta_1\) and \(\delta_2\) are calculated in a similar manner to section 3.3 as

\[
- [\delta_i \cdot \delta_j]_{i,j=0,\ldots,2} = \begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix},
\]

which is the Cartan matrix of type \(A^{(1)}_2\). In this sense, the surface \(X\) associated with the difference equation (4.2) is of type \(A^{(1)}_2\). As explained in section 3.4, the symmetry type of (4.2) is given by the orthogonal complement in \(\Lambda\), which is \(E^{(1)}_6\).

4.2. Point configuration for a discrete Painlevé I equation: second example

When we are given a discrete equation to study, we first try appropriate integrability criteria, such as the singularity confinement test or degree growth criterion (the algebraic entropy) \([5, 26]\). If the test suggests the equation to be possibly integrable, we may next try to detect the point configuration for further investigation. As an example of such cases, we consider the difference equation

\[
x_{n+1} + x_{n-1} = \frac{n\delta + a_0}{x_n} + b.
\]

In this case, we need more elaborate investigation compared with the previous example.

Remark 4.1. Equation (4.15) is derived in \([108]\) and it is shown to have a continuous limit to the Painlevé I equation. Generalizing the equation through the singularity confinement test \([25]\), (4.15) is interpreted as a Bäcklund transformation of the Painlevé V equation \((P_V)\) (see, for example \([90, 122]\)), which will be also demonstrated later.

Introducing the four variables \(f, g, \tilde{f}, \tilde{g}\) by \((f, g) = (x_n, x_{n+1}), (\tilde{f}, \tilde{g}) = (x_{n+1}, x_{n+2})\), we consider the rational mapping

\[
F : (a, b, f, g) \mapsto (\bar{a}, b, \tilde{f}, \tilde{g}),
\]

\[
\tilde{f} = g, \quad \tilde{g} = -f + \frac{a}{g} + b, \quad \bar{a} = a + \delta.
\]
We also consider the inverse $G = F^{-1}$ of the mapping $F$:

$$f = -g + \frac{a - \delta}{\overline{f}} + b, \quad g = \overline{f}, \quad a = \sigma - \delta. \quad (4.17)$$

Denoting by $X$ and $X'$ two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ with inhomogeneous coordinates $(f, g)$ and $(\overline{f}, \overline{g})$ respectively, we attempt to regularize two rational mappings $F : X \to X'$ and $G : X \to X$.

The points of indeterminacy of the rational mappings $(4.16)$ and $(4.17)$ can be read off easily. In fact, $P_1 : (f, g) = (\infty, 0)$ and $P'_2 : (\overline{f}, \overline{g}) = (0, \infty)$ are the points of indeterminacy of $F$ and $G$ respectively. Then we have regular mappings $F : X \setminus P_1 \to X'$ and $G : X' \setminus P'_2 \to X$. We observe from $(4.16)$ and $(4.17)$ that $F$ maps the line $\{g = 0\} \setminus P_1$ to $P'_2$, and similarly $G$ does the line $\{\overline{f} = 0\} \setminus P'_2$ to $P_1$ (See figure 11). Therefore $F$ induces a biregular mapping from $X \setminus \{g = 0\}$ to $X' \setminus \{\overline{f} = 0\}$. To summarize, we have the following correspondence:

$$X \setminus \{g = 0\} \sim X' \setminus \{\overline{f} = 0\}$$

$$\{g = 0\} \setminus P_1 \to P'_2$$

$$P_1 \leftarrow \{\overline{f} = 0\} \setminus P'_2. \quad (4.18)$$

Remark 4.2. A formal method to find the points of indeterminacy of the rational mapping $(4.16)$ and $(4.17)$ is to introduce the homogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ as $f = \frac{n}{x_0}, g = \frac{n}{y_0}$, which yields

$$F : \frac{x_1}{x_0} = \frac{y_1}{y_0}, \quad \frac{y_1}{y_0} = \frac{ax_0y_0 + bx_0y_1 - x_1y_1}{x_0y_1}, \quad (4.19)$$

$$G : \frac{x_1}{x_0} = \frac{a\overline{x}_0\overline{y}_0 + b\overline{x}_0\overline{y}_1 - \overline{x}_1\overline{y}_1}{\overline{x}_0\overline{y}_1}, \quad \frac{y_1}{y_0} = \frac{\overline{x}_1}{\overline{x}_0}. \quad (4.20)$$

The numerator and the denominator of the right-hand side of the second equation of $(4.19)$ have a common zero when $x_0 = 0, y_1 = 0$, which implies that the point of indeterminacy of $F$ is $P_1 : (f, g) = (\infty, 0)$. Similarly, the point of indeterminacy of $G$ is $P'_2 : (\overline{f}, \overline{g}) = (0, \infty)$.

To resolve the indeterminacy, we blow up $X$ at $P_1$ and $X'$ at $P'_2$ by introducing new variables $(f, g), (\overline{f}_2, \overline{g}_2)$ and their companions $(\varphi_1, \psi_1), (\varphi_2, \psi_2)$ as
We denote by \(X_{(1)}\) the surface obtained from \(X\) by blowing up at \(P_1\) and by \(E_1 = \{f_1 = 0\} \cup \{\psi_1 = 0\}\) the corresponding exceptional curve. Similarly, we denote by \(X'_{(2)}\) and \(E'_{(2)} = \{g_2 = 0\} \cup \{\varphi_2 = 0\}\) the surface obtained from \(X'\) by blowing up at \(P_2'\) and the corresponding exceptional curve respectively. In terms of the local coordinates \((f, g)\) and \((\varphi, \psi)\) we have

\[
(f, g) = \left(\frac{1}{f_1}, f_1 g_1\right) = \left(\frac{1}{\varphi_1 \psi_1}, \psi_1\right), \quad (\varphi, \psi) = \left(\varphi_2 g_2, \frac{1}{\varphi_2 g_2}\right) = \left(\varphi_2, \frac{1}{\varphi_2 \psi_2}\right).
\]

We observe that \(G\) maps all the points \((f, g) = (0, \varphi)\) on \(E_2\) except \(P_2' = (\varphi, \psi) = (0, 0)\) to \(E_1\). The points \((f, g) = (0, \varphi)\) on \(E_2\) except \(P_2' = (\varphi, \psi) = (0, 0)\) are mapped to the points \((\varphi, \psi) = (a - g_1, 0)\) except \(Q_2' = (\varphi, \psi) = (0, 0)\). It will be shown later that \(Q_2'\) is actually a regular point.

To investigate the mapping \(G\), solving (4.22) in terms of \((f_1, g_1)\) we have

\[
G : \left\{ \begin{array}{l}
\frac{f_1}{f_1} = \frac{\varphi}{a + b \varphi - fg} = \frac{\varphi_2 g_2}{a + b \varphi_2 g_2 - \varphi_2}, \\
g_1 = a + b \frac{\varphi_2 g_2}{a + b \varphi_2 g_2 - \varphi_2}.
\end{array} \right.
\]

from which we see that the mapping \(G\) has an indeterminacy at \(P_2' = (\varphi, \psi) = (a, 0)\) on \(E_2\). We observe that \(G\) maps all the points \((\varphi, \psi) = (0, \varphi)\) on \(E_2\) except \(P_2' = (\varphi, \psi) = (0, 0)\) to \(E_1\). The points \((\varphi, \psi) = (0, \varphi)\) on \(E_2\) except \(P_2' = (\varphi, \psi) = (0, 0)\) are mapped to the points \((f_1, g_1) = (0, a - \varphi)\).

In summary, at this stage the above investigation shows the following correspondence between \(X_{(1)}\) and \(X'_{(2)}\) (see figure 12):
We remark that $\mathcal{E}$ is mapped bijectively to $\mathcal{E}$.

To resolve the above indeterminacies, we take the blow-up $X_{(13)}$ of $X_1$ at $P_3$ by introducing the variables $(f_3, g_3)$ and its companion $(\varphi_3, \psi_3)$ by

$$(f_3, g_3) = (f_3, a + f_3 g_3) = (\varphi_3, \psi_3).$$

and denote by $E_3 = \{ f_3 = 0 \} \cup \{ \psi_3 = 0 \}$ the corresponding exceptional curve. Similarly, we consider the blow-up $X_{(24)}'$ of $X_{(24)}$ at $P_4'$ by introducing the variables $(\overline{f}_4, g_4)$ and $(\overline{\varphi}_4, \overline{\psi}_4)$ by

$$(\overline{f}_4, g_4) = (a + \overline{f}_4 g_4, g_4) = (\overline{\varphi}_4, \overline{\psi}_4).$$

and denote by $E_4' = \{ g_4 = 0 \} \cup \{ \varphi_4 = 0 \}$ the corresponding exceptional curve. The process of blowing-up so far is summarized in figure 13. We first investigate the image of $\{ g = 0 \} \setminus E_1$ in terms of the local coordinates $(\overline{\varphi}_4, \overline{\psi}_4)$. We have from (4.22) and (4.26)

$$F : \overline{\varphi}_4 = (b - f) g, \quad \overline{\psi}_4 = \frac{1}{(b - f)(a + bg - fg)}.$$  

which implies that $F$ maps all the points $(f, g) = (f, 0)$ ($f \neq \infty$) on $\{ g = 0 \} \setminus E_1$ to $(\overline{\varphi}_4, \overline{\psi}_4) = (0, 1/a(b - f))$ on $E_1' \setminus Q_4'$. We next investigate the image of $\{ \overline{f} = 0 \} \setminus E_2'$ by $G$ in terms of the local coordinates $(\varphi_3, \psi_3)$. We have from (4.23) and (4.25)

$$G : \varphi_3 = \frac{1}{(b - \overline{g})(a + b\overline{f} - f\overline{g})}, \quad \psi_3 = \overline{f} (b - \overline{g}),$$

Figure 13. Process of blowing up.
which implies that $G$ maps all the points $(\mathcal{J}, g) = (0, \infty)$ on $\{\mathcal{J} = 0\} \setminus E'_2$ to $(\varphi_3, \psi_3) = (1/a(b - \mathcal{J}), 0)$ on $E_3 \setminus Q_3$ where $Q_3 : (\varphi_3, \psi_3) = (0, 0)$. Now the correspondence between $X_{(13)}$ and $X'_{(24)}$ is given by

\[
\begin{array}{ccc}
X_{(13)} & \xrightarrow{f} & X'_{(24)} \\
\{g = 0\} \setminus E_1 & \xrightarrow{\sim} & E'_3 \setminus Q'_4 \\
E_1 \setminus (E_3 \cup Q_3) & \xrightarrow{\sim} & E'_2 \setminus (E'_4 \cup Q'_2) \\
E_3 \setminus Q_3 & \xrightarrow{\sim} & (\mathcal{J} = 0) \setminus E'_2
\end{array}
\tag{4.29}
\]

We investigate the mapping around $Q_3$. We write the image of $Q_3$ in terms of the local coordinates $(\mathcal{J}_2, g_2)$ on $E'_2$. We have from (4.22) and (4.25)

\[
F : \mathcal{J}_2 = (b\varphi_3(a + \psi_3) - 1)/\psi_3, \quad g_2 = \frac{\varphi_3(a + \psi_3)}{b\varphi_3(a + \psi_3) - 1},
\tag{4.30}
\]

which implies that $F$ is regular at $Q_3$ and it is mapped to $Q'_2$. We can verify by similar computations that $Q_3$ is mapped regularly to $Q'_2$ as well. The correspondence between $X_{(13)}$ and $X'_{(24)}$ is now described as (see figure 14)

\[
\begin{array}{ccc}
X_{(13)} & \xrightarrow{f} & X'_{(24)} \\
\{(g = 0) \setminus E_1\} \cup Q_1 & \xrightarrow{\sim} & E'_4 \\
(E_1 \setminus E_3) \cup Q_3 & \xrightarrow{\sim} & (E'_4 \setminus E'_3) \cup Q'_4 \\
E_3 & \xrightarrow{\sim} & (\mathcal{J} = 0) \setminus E'_2 \cup Q'_2
\end{array}
\tag{4.31}
\]

Therefore $F$ induces a biregular mapping $X_{(13)} \sim X'_{(24)}$.

In order to consider the iteration of $F$, we identify $X'$ with $X$ everytime after we apply $F : X \rightarrow X'$. Accordingly, we need to blow up $X_{(13)}$ successively at

\[
P_2 : \quad (f, g) = (0, \infty), \quad (f, g) = (f_2 g_2, \frac{1}{g_2}),
\]

\[
P_4 : \quad (f_2', g_2) = (a - \delta, 0), \quad (f_2, g_2) = (a - \delta + f_4 g_4, g_4),
\tag{4.32}
\]

where $P_2$ and $P_4$ are copies of $P'_4 \in X'$ and $P'_4 \in X'_2$, respectively. For simplicity, we show one of the two coordinate systems only for each blow-up. Note that the parameter $a$ is downshifted to $a - \delta$ in $P_4$ when identifying $X'$ with $X$. Tracing the orbit of $P_2$ by $F : X_{(13)} \rightarrow X'_{(24)}$ and applying the same procedure, we find two additional points $P_5$, $P_6$ where we need to apply blow-ups:

Figure 14. The third step of regularization.
\[
F(P_2) = P'_5 : P_5 : (f, g) = (\infty, b), (f, g) = \left( \frac{1}{g_5}, b + f_5 g_5 \right)
\]
\[
F(P_3) = P'_6 : P_6 : (f, g) = (b, \infty), (f, g) = \left( b + f_6 g_6, \frac{1}{g_6} \right)
\]
\[
F(P_4) = P'_1 : P_1 : (f, g) = (\infty, 0).
\]
(4.33)

Writing the final step in terms of the local coordinates \((f_6, g_6)\) on \(E_6\) and \((f_7, g_7)\) on \(E'_1\) as
\[
F : \overline{f}_1 = g_6, \quad \overline{g}_1 = a - f_6,
\]
we can check that \(E_6\) is mapped to \(E'_1\), which guarantees the regularity of the iterated mapping on this orbit.

We next trace the orbit of \(P_4\) on the exceptional curve \(E_2 = \{g_2 = 0\}\) to find the two points \(P_7\) on \(E_5 = \{f_5 = 0\}\) and \(P_8\) on \(E_6 = \{g_6 = 0\}\):
\[
F(P_4) = P'_7 : P_7 : (f_5, g_5) = (0, \delta), (f_5, g_5) = (f_7, \delta + f_7 g_7)
\]
\[
F(P_7) = P'_8 : P_8 : (f_6, g_6) = (-\delta, 0), (f_6, g_6) = (-\delta + f_8 g_6, g_6)
\]
\[
F(P_8) = P'_3 : P_3 : (f_1, g_1) = (0, a)
\]
(4.35)

We can verify the regularity of the iterated mapping by checking that \(E_8\) is mapped to \(E'_3\). We thus obtain an eight-point blow-up \(X_{(12563478)}\) of \(X\) by four blow-ups at \(P_1, P_3, P_5, P_6\) followed by four blow-ups at \(P_3, P_4, P_7, P_8\), on which \(F\) induces a biregular mapping.

**Remark 4.3.**

(1) In this example, only \(P_1\) and \(P_2\) arise as the indeterminacies of \(F\) and \(F^{-1}\) respectively. The other points appear as the indeterminacies of iterate \(F^4\). Such a phenomenon occurs when the mapping is finer than translations of the underlying root lattice ("projective reduction"), as shown below [55, 121].

(2) Here we have constructed the space of initial values by successive blow-ups. Some examples to which we need to apply blow-downs are investigated in [8].

Let us identify the underlying root lattice from the configuration of the eight points \(P_1, \ldots, P_8\): \(P_1\) and \(P_2\) are on the line \(f = \infty\); \(P_3\) and \(P_6\) are on the line \(g = \infty\); \(P_3, P_4, P_7, P_8\) are infinitely near point of \(P_1, P_2, P_5, P_6\), respectively. We then identify the surface type of \(X_{(12563478)}\) as follows. We note that the above set-theoretical computations tracing the orbits of points are dual to our convention of the symbolic computations on the Picard lattice (recall the difference of the symbolical composition and the numerical composition). Hence the action of discrete time evolution \(T\) on the Picard lattice is obtained as follows from the set-theoretical action of \(G\). Denoting by \(H_i\) and \(H_j\) the divisors corresponding to the lines \(f = \text{const.}\) and \(g = \text{const.}\) respectively, we have from (4.31)
\[
T : E_4 \rightarrow H_2 - E_1, \quad E_2 - E_4 \rightarrow E_1 - E_3, \quad H_1 - E_2 \rightarrow E_3.
\]
(4.36)

We also have from (4.33) and (4.35)
\[
T : E_1 \rightarrow E_6 \rightarrow E_5 \rightarrow E_2, \quad E_3 \rightarrow E_8 \rightarrow E_7 \rightarrow E_4.
\]
(4.37)

It is obvious from (4.16) that \((f = \infty) \setminus P_1\) is mapped to \((g = \infty) \setminus P_2\), which implies
\[
T : H_2 - E_2 \rightarrow H_1 - E_1.
\]
(4.38)
Combining these formulae we obtain

\[ T : \begin{align*}
H_1 & \to H_2, & H_2 & \to H_1 + H_2 - E_1 - E_3, & E_2 & \to H_2 - E_3, \\
E_4 & \to H_2 - E_1, & E_{1[135768]} & \to E_{6[02457]}. 
\end{align*} \tag{4.39} \]

In terms of the Picard lattice, the surface type is described by the following inaccessible root vectors, as illustrated in figure 15

\[ \begin{align*}
\delta_0 &= E_2 - E_4, & \delta_1 &= E_6 - E_8, & \delta_2 &= H_2 - E_2 - E_6, \\
\delta_3 &= H_1 - E_1 - E_5, & \delta_4 &= E_1 - E_3, & \delta_5 &= E_5 - E_7, 
\end{align*} \tag{4.40} \]

In fact, \( T \) induces the following transformations on the root vectors

\[ \delta_0 \to \delta_4 \to \delta_1 \to \delta_5 \to \delta_0, \quad \delta_2 \leftrightarrow \delta_3. \tag{4.41} \]

Note also that the other divisors in figure 15 are accessible which is obvious from the tracing of the orbit of indeterminacies. We remark that they generate the following infinite orbit of divisors by successive applications of \( T \):

\[ \cdots \to H_1 - E_2 \to E_3 \to E_5 \to E_7 \to E_4 \to H_2 - E_1 \to \cdots. \tag{4.42} \]

The intersection numbers among those components of inaccessible divisors are computed in a similar manner to section 3.3 as

\[ -[\delta_i \cdot \delta_j]_{i,j=0, \ldots, 5} = \begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
2 & -1 & -1 & 2 & -1 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
-1 & -1 & 2 & -1 & 0 & 0 \\
-1 & -1 & -1 & 2 & 0 & 0 \\
-1 & -1 & -1 & -1 & 2 & 0 \\
\end{bmatrix}, \tag{4.43} \]

which is the Cartan matrix of type \( D_5^{(1)} \). It is also expressed as the Dynkin diagram by

\[ \begin{array}{c}
\delta_0 \\
\delta_1 \\
\delta_2 \\
\delta_3 \\
\delta_4 \\
\delta_5 \\
\end{array} \begin{array}{c}
\| \\
\| \\
\| \\
\| \\
\| \\
\| \\
\end{array} \]

The corresponding surface is the type \( D_5^{(1)} \). Note that the basis of the orthogonal complement in the Picard lattice is given by

Figure 15. Surface of the mapping (4.16). Left: configuration of the points of indeterminacy. Right: configuration of divisors. Solid lines are inaccessible divisors. Thick lines are exceptional divisors arising in the blow-ups.
\[
\begin{align*}
\alpha_0 &= H_2 - E_5 - E_7, \quad \alpha_1 = H_1 - E_2 - E_4, \\
\alpha_2 &= H_2 - E_1 - E_3, \quad \alpha_3 = H_1 - E_0 - E_5,
\end{align*}
\] (4.45)

whose intersection numbers are given by
\[
- [\alpha_i \cdot \alpha_j]_{i,j=0, \ldots, 3} = \begin{bmatrix} 2 & -1 & -2 & -1 \\ -1 & 2 & -1 & -1 \\ -2 & 1 & 2 & -1 \\ -1 & -1 & 2 & 1 \end{bmatrix}.
\] (4.46)

This is the Cartan matrix of type \(A_4^{(1)}\) and its Dynkin diagram is given by
\[
\begin{align*}
\alpha_0 & \rightarrow \alpha_3 \\
\alpha_1 & \rightarrow \alpha_2
\end{align*}
\] (4.47)

which implies that the symmetry is type \(A_4^{(1)}\). We remark that the action of \(T\) on \(\alpha_i\) \((i = 0, 1, 2, 3)\) is given by
\[
\begin{align*}
\alpha_0 &\rightarrow \alpha_1 + \alpha_2, \quad \alpha_1 \rightarrow \alpha_3 + \alpha_2, \quad \alpha_2 \rightarrow \alpha_0, \quad \alpha_3 \rightarrow \alpha_1 + \alpha_2.
\end{align*}
\] (4.48)

The action of the simple reflections \(s_i\) \((i = 0, 1, 2, 3)\) on \(\alpha_i\) \((i = 0, 1, 2, 3)\) can be computed in the similar manner to example 3.1 as
\[
s_i(\alpha_i) = -\alpha_i, \quad s_i(\alpha_{i+1}) = \alpha_{i+1} + \alpha_i \quad (i \in \mathbb{Z}/4\mathbb{Z}),
\] (4.49)

and the Dynkin diagram automorphism \(\pi\)
\[
\pi(\alpha_i) = \alpha_{i+1} \quad (i \in \mathbb{Z}/4\mathbb{Z}).
\] (4.50)

**Remark 4.4.**

(1) Applying the similar analysis as above to the difference equation (2.13) or (2.14), one obtains the surface of type \(E_6^{(1)}\) which is exactly the same as that constructed from P_IV in sections 2.6.2 and 3.4. Namely, the differential equation P_IV (2.61) and the difference equation (2.13) or (2.14) share the same surface.

(2) We can introducing the variables \(F_i\) in the similar manner to the case of P_IV (the case of symmetry type \(A_4^{(1)}\) such that
\[
s_i(F_i \pm i) = F_i \pm i \pm \frac{\alpha_i}{F_i}, \quad \pi(F_i) = F_{i+1} \quad (i \in \mathbb{Z}/4\mathbb{Z}), \quad F_0 + F_2 = k_0, \quad F_1 + F_3 = k_1,
\] (4.51)

where \(k_0\) and \(k_1\) are constants. Then we can verify that
\[
T = \pi s_1;
\] (4.52)

under the identification \((F_1, F_2) = (f, g), \quad (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\delta, \delta - a, a, \delta), \quad (k_0, k_1) = (b, b)\). Also, it is known that this action of the affine Weyl group of type \(A_4^{(1)}\) admits a continuous flow which is nothing but the P_V. Actually, putting \((F_0, F_1, F_2, F_3) = \left(\frac{p}{\sqrt{q}}, \sqrt{q}, -\frac{p}{\sqrt{q}}, \sqrt{q}(1 - q)\right)\) under the normalization \(\delta = 1\) and \(k_0 = k_1 = \sqrt{q}\), the variables \(q\) and \(p\) obey the canonical equations with the Hamiltonian
\[
hH = q(q - 1)p(p + t) - (\alpha_1 + \alpha_3)qp + \alpha_1 p + \alpha_2 q,
\] (4.53)

which is equivalent to P_V.
\[
\frac{d^2 y}{dt^2} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) \left( \frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{(y-1)^2}{2t^2} \left( \alpha_1 y - \frac{\alpha_2}{y} \right) - \left( \alpha_2 - \alpha_0 \right) \frac{y}{t} - \frac{y^2 + 1}{2} + 1
\]  
(4.54)

for the variable \( y = 1 - \frac{1}{q} \) [68, 83, 87, 127]. Therefore, the analysis in this section reveals that the difference equation (4.15) describes a Bäcklund transformation of \( P_v \). Further, Takano’s coordinates for the space of initial values of \( P_v \) correspond to \( (f, g) \), \( (f, g) \) \((i = 3, 4, 7, 8) \) [73, 117].

**4.3. Practical method for finding point configuration**

The point configuration for a difference equation of Painlevé type, as constructed in section 4.2 for (4.15), can be obtained by an alternative ‘short-cut’ method which is suitable for computer algebra. Instead of repeating blow-ups, we may iterate the mapping, in other words, the discrete time evolution. Sufficient number of iterations will give all the point of indeterminacy.

To find the points of indeterminacy of the rational mapping (4.16), we iterate the substitution

\[
T(f) = g, \quad T(g) = -f + \frac{a}{g} + b, \quad T(a) = a + \delta.
\]  
(4.55)

Computing \( T^2(g), T^3(g), \ldots \), we have

\[
T(g) = \frac{F_1}{g}, \quad T^2(g) = \frac{F_2}{F_1}, \quad T^3(g) = \frac{F_3}{gF_2}, \ldots, T^n(g) = \frac{F_nF_{n-4}}{F_{n-1}F_{n-3}}, \ldots,
\]  
(4.56)

where

\[
F_1 = a + bg - fg,
\]
\[
F_2 = ab + b^2 g - bg^2 - f g^2 + \delta g,
\]
\[
F_3 = -a^2 b + a^2 g - ab^2 g + 2abfg + 2abg^2 - 2af g^2 + af g^2 + a\delta g + b^2 f g^2 - b f^2 g^2 - b f g^3 + 2b \delta g^2 + f g^3 - \delta g^2, \quad \ldots.
\]  
(4.57)

We observe two points \((f, g) = (b, \infty), (\infty, 0)\) appearing as the common zeros of the polynomials \( F_1, F_2 \), which give the points of indeterminacy of \( T^2(g) \). These points correspond to \( P_b \) and \( P_1 \) respectively in section 4.2. To see infinitely near points to \( P_b : (f, g) = (b, \infty) \), we introduce a parameter \( u \) by \( (f, g) = \left( b + u \epsilon, \frac{1}{\epsilon} \right) \). Then for \( \epsilon \to 0 \) we have

\[
F_2 = \frac{u + \delta}{\epsilon} + O(\epsilon^0), \quad F_3 = \frac{b(u + \delta)}{\epsilon^2} + O(\epsilon^{-1}),
\]  
(4.58)

and hence

\[
T^3(g) = \frac{F_3}{gF_2} = \frac{b(u + \delta)}{u + \delta + O(\epsilon^1)} + O(\epsilon^1),
\]  
(4.59)

which indicates another point of indeterminacy at \( u = -\delta \), namely

\[
P_b : (f, g) = \left( b - \delta \epsilon + O(\epsilon^2), \frac{1}{\epsilon} \right).
\]  
(4.60)
Similarly, near the point $P_1 (f, g) = (\infty, 0)$ we have an infinitely near singular point

$$P_3 : (f, g) = \left( \frac{1}{\epsilon}, a\epsilon + O(\epsilon^2) \right).$$

(4.61)

Analyzing $T^{-1}(f), T^{-2}(f), \cdots$ in a similar way, we have four more points:

$$P_5 : (f, g) = (\infty, b), \quad P_7 : (f, g) = \left( \frac{1}{\epsilon}, b + \delta\epsilon + O(\epsilon^2) \right),$$

$$P_2 : (f, g) = (0, \infty), \quad P_4 : (f, g) = \left( (a - \delta)\epsilon + O(\epsilon^2), \frac{1}{\epsilon} \right).$$

(4.62)

Let us give a brief account of the reason why this method works effectively for finding the point configuration. As we have suggested in section 3.5, the polynomial factors such as $F_n$ are controlled by the Picard lattice. Therefore the singularities (self-intersections) of a polynomial $F_n$ with sufficient degree, or the common zeros of $F_n$’s would give the points in the configuration. Figure 16 illustrates the behavior of curves $F_n (n = 1, \ldots, 5)$, from which we observe the four common zeros. Also, we can identify the four infinitely near points by the common tangent at each of the four common zeros.

5. Discrete Painlevé equation from point configuration

If the configuration of eight points in $\mathbb{P}^1 \times \mathbb{P}^1$ is generic, the corresponding space of initial values has the largest symmetry of type $E_8^{(1)}$, and other configurations can be regarded as the degenerate cases. In this section, we describe how to construct the equations and relevant characteristic features from the point configuration. In particular, we formulate a representation of affine Weyl group of type $E_8^{(1)}$ from the configuration of generic eight points, as well as the formalism of $\tau$ functions. We then derive a new explicit form of the three equations of type $E_8^{(1)}$, which are the elliptic, $q$- and difference Painlevé equations, from a
translation of the root lattice. We also give an example demonstrating how to construct the birational representation of the affine Weyl group for a given degenerate point configuration.

5.1. Configuration of points on \( \mathbb{P}^1 \times \mathbb{P}^1 \)

Suppose that generic \( n \) points \( P_i(\lambda_i, \mu_i) \) for \( i = 1, \ldots, n \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) are given (\( n \geq 4 \)). Since \( \mathbb{P}^1 \times \mathbb{P}^1 \) admits the action of the \( \text{PGL}(2)^2 \) group by the linear fractional transformations

\[
(f, g) = \left( \frac{(x - x_1)(x_3 - x_2)}{(x - x_1)(x_3 - x_2)}, \frac{(y - y_1)(y_3 - y_2)}{(y - y_1)(y_3 - y_2)} \right),
\]

(5.1)
of \( \mathbb{P}^1 \times \mathbb{P}^1 \) so that \( P_1(\infty, \infty), P_2(0, 0), P_3(1, 1) \) and \( P_i(f_i, g_i) \) for \( i = 4, \ldots, n \). We define birational actions \( s_0, \ldots, s_n \) on the field of rational functions in variables \( f_i, g_i \) for \( i = 4, \ldots, n \) by

\[
\begin{align*}
s_0 &: f_i \rightarrow \frac{1}{f_i}, \quad g_i \rightarrow \frac{1}{g_i}, \\
s_1 &: f_i \rightarrow g_i, \quad g_i \rightarrow f_i, \\
s_2 &: f_i \rightarrow \frac{f_i}{g_i}, \quad g_i \rightarrow \frac{1}{g_i}, \\
s_3 &: f_i \rightarrow 1 - f_i, \quad g_i \rightarrow 1 - g_i, \\
s_4 &: f_i \rightarrow \frac{1}{f_i}, \quad g_i \rightarrow \frac{1}{g_i}, \quad f_i \rightarrow \frac{f_i}{f_4}, \quad g_i \rightarrow \frac{g_i}{g_4} \quad (i \geq 5),
\end{align*}
\]

(5.2)

Then one can verify directly that \( s_i \) for \( i = 0, \ldots, n \) satisfy the fundamental relations specified by (3.2) corresponding to the following Dynkin diagram:

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

(5.3)

(e.g. \( E_8^{(1)} \) (3.12) for \( n = 8 \)). We remark that the variables \( f_i, g_i \) for \( i = 4, \ldots, n \) are regarded as the inhomogeneous coordinates of the configuration space

\[
\left\{ \left( \begin{array}{c}
x_1 \\
y_1 \\
\vdots \\
x_n \\
y_n 
\end{array} \right) \right\} / \text{PGL}(2)^2,
\]

(5.4)
of generic \( n \) points in \( \mathbb{P}^1 \times \mathbb{P}^1 \). The above transformations have simple interpretations on this configuration space except \( s_2 \). The action of \( \Sigma_n = \langle s_0, s_3, \ldots, s_n \rangle \) comes from the permutation of eight points in \((x, y)\) coordinates, and \( s_1 \) from the exchange of two coordinates of each point.

We remark that in the formulation of point configurations in \( \mathbb{P}^2 \), those transformations have different geometric meaning; \( s_0 \) is the standard Cremona transformation and other transformations correspond to exchange of points or coordinates [53, 112].

In the context of the discrete Painlevé equations, we consider the case \( n = 9 \), and the coordinates \( (f_i, g_i) \) of the points \( P_i(\lambda_i, \mu_i) \) for \( i = 1, \ldots, 8 \) play the role of parameters (or independent variables). The ninth point \( (f_9, g_9) = (f, g) \) plays different role (the dependent variable) and we do not use the action \( s_0 \).

Calculating actions for some elements \( w \in W(E_8^{(1)}) \), we observe that \( w(f) \) and \( w(g) \) are rational functions in \( f, g \) with the factorized form.
where $P_\lambda$ is the polynomial in $f, g$ corresponding to the exceptional element
\[ \lambda \in M \subset \Lambda = \mathbb{Z}H_1 \oplus \mathbb{Z}H_2 \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_8 \ (3.34), \]
which is unique up to normalization constants.

**Example 5.1.** For \( w = s_4s_3s_2s_1s_6s_5s_4s_3s_5s_4s_3s_2s_1 \), we have:

\[
\begin{align*}
w(f) &= (g_7 - g)(f_6g_4 - f_4g_6 + f_4g_6 + f_4g_6 - f_4g_6), \\
w(g) &= (g_7 - g)(f_6g_4 - f_4g_6 + f_4g_6 + f_4g_6 - f_4g_6),
\end{align*}
\]

whose factors are identified as
\[
\begin{align*}
P_{w(e_1)} &= P_{H_1 + H_2 - E_1 - E_2} = f_1g_4 - f_4g_7 + f_4g_7 + f_4g_7 - f_4g_7, \\
P_{w(e_2)} &= P_{H_1 + H_2 - E_1 - E_2} = f_6g_4 - f_4g_6 + f_4g_6 + f_4g_6 - f_4g_6. \\
P_{w(H_1 - E_2)} &= P_{H_1 - E_2} = g - g_6, \\
P_{w(H_1 - E_2)} &= P_{H_1 - E_2} = 1, \quad P_{w(E_1 - E_2)} = P_{E_1} = 1.
\end{align*}
\]

In fact, one can check that \( P_{w(e_1)} = 0 \) is a bidegree \((1, 1)\) curve which passes through \( P_1(\infty, \infty), P_2(f_g, g), \) and \( P_1(f_7, g_7) \) with multiplicity 1.

For any polynomial \( P(f, g) \) belonging to the class \( \lambda \), one can show that \( s_\lambda(P(f, g)) \) is a polynomial belonging to the class \( s_\lambda(\lambda) \) up to multiplication by a monomial factor in \( f, g \) for each \( k = 0, \ldots, 8 \). If \( \lambda = d_1H_1 + d_2H_2 - \sum_{i=1}^{8} m_iE_i \) then \( P(f, g) \) is expressed in the form

\[
\begin{align*}
P(f, g) &= \sum_{(i,j) \in S} c_{ij}f^i g^j, \\
S &= \left\{ (i, j) \mid 0 \leq i \leq d_1, 0 \leq j \leq d_2, \\
&\quad m_2 \leq i + j \leq d_1 + d_2 - m_1 \right\} \quad (5.8)
\end{align*}
\]

Applying \( s_2 \), for example, we obtain

\[
\begin{align*}
s_2(P(f, g)) &= \sum_{(i,j) \in S} s_2(c_{ij}) \left( \frac{f^i}{g^j} \right)^{\frac{1}{2}} = g^{-d_1-d_2+m_1} \sum_{(i,j) \in S} s_2(c_{ij})f^i g^{d_1+d_2-m_1-i-j} \\
&= g^{-d_1-d_2+m_1} Q(f, g),
\end{align*}
\]

where

\[
\begin{align*}
Q(f, g) &= \sum_{(i,j) \in T} s_2(c_{i,d_1+d_2-m_1-i-j}) f^i g^j, \\
T &= \left\{ (i, j) \mid 0 \leq i \leq d_1, 0 \leq j \leq d_1 + d_2 - m_1, \\
&\quad d_1 - m_1 \leq i + j \leq d_1 + d_2 - m_1 \right\} \quad (5.9)
\end{align*}
\]

Hence \( Q(f, g) \) is a polynomial of bidegree \( (d_1 + d_2 - m_1 - m_2) \) having zeros at \( P_1(\infty, \infty), P_2(0, 0) \) with multiplicity \( d_1 - m_2 \) and \( d_1 - m_1 \), respectively. Multiplicities of other zeros remain the same. This implies that \( Q(f, g) \) belongs to the class

\[
\begin{align*}
d_1H_1 + (d_1 + d_2 - m_1 - m_2)H_2 - (d_1 - m_1)E_1 - (d_1 - m_1)E_2 - m_3E_3 - \cdots - m_8E_8 \\
&= d_1(H_1 + H_2 - E_1 - E_2) + d_2H_2 - m_1(H_2 - E_2) - m_2(H_2 - E_1) - m_3E_3 - \cdots - m_8E_8 = s_2(\lambda).
\end{align*}
\]
One can verify the case of other $s_k$ in a similar manner. The factorized formulae (5.5) for $w = s_k \cdots s_2 s_1$ can be obtained by applying $s_k, s_{k-1}, \ldots$ successively to (5.5) for $w = \mathrm{id}$ with $P_{E_i} = 1, P_{H_{i-\varepsilon}} = 1, P_{H_{i+\varepsilon}} = f, P_{H_{i-\varepsilon}} = g$. In section 5.3, we will introduce the $\tau$ functions, which play an essential role in the complete description of those polynomials.

5.2. Parametrization of the eight points and the curve

Recall that the affine root system of type $E_8^{(1)}$ is realized by the simple roots

\[
\begin{align*}
\alpha_0 &= E_1 - E_2, \quad \alpha_1 = H_1 - H_2, \quad \alpha_2 = H_2 - E_1 - E_2, \\
\alpha_3 &= E_2 - E_3, \quad \alpha_4 = E_3 - E_4, \\
\alpha_5 &= E_4 - E_5, \quad \alpha_6 = E_5 - E_6, \quad \alpha_7 = E_6 - E_7, \quad \alpha_8 = E_7 - E_8,
\end{align*}
\]

in the Picard lattice $\Lambda = \mathbb{Z}H_1 \oplus \mathbb{Z}H_2 \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_8$ and that the affine Weyl group $W(E_8^{(1)}) = \langle s_0, \ldots, s_8 \rangle$ has a natural linear action on $\Lambda$ through the simple reflections $s_i$ ($i = 0, \ldots, 8$). The action of $s_j$ on the basis of $\Lambda$ is computed explicitly by using (3.26), (3.5) and (5.10) as follows:

\[
\begin{align*}
s_0 : \quad & E_1 \mapsto E_2, \\
s_1 : \quad & H_1 \mapsto H_2, \\
s_2 : \quad & E_1 \mapsto H_2 - E_2, \quad E_2 \mapsto H_2 - E_1, \quad H_1 \mapsto H_1 + H_2 - E_1 - E_2, \\
s_j : \quad & E_{j-1} \mapsto E_j \quad (j = 3, \ldots, 8).
\end{align*}
\]

Also, we introduce the set of parameters $h_1, h_2, e_1, \ldots, e_8$ and define the actions on them as

\[
\begin{align*}
s_0 : \quad & e_1 \mapsto e_2, \\
s_1 : \quad & h_1 \mapsto h_2, \\
s_2 : \quad & e_1 \mapsto h_2 - e_2, \quad e_2 \mapsto h_2 - e_1, \quad h_1 \mapsto h_1 + h_2 - e_1 - e_2, \\
s_j : \quad & e_{j-1} \mapsto e_j \quad (j = 3, \ldots, 8),
\end{align*}
\]

by identifying $H_i$ with $h_i$ ($i = 1, 2$) and $E_i$ with $e_i$ ($i = 1, \ldots, 8$), respectively. The parameter

\[
d = 2h_1 + 2h_2 - e_1 - \cdots - e_8,
\]

corresponding to the null root, denoted by the same symbol, is $W(E_8^{(1)})$-invariant and will play the role of the step-size for difference equations. The set of parameters $s_1, \kappa_2, \nu_1, \ldots, \nu_8$ used in section 4.1 and $h_1, h_2, e_1, \ldots, e_8$ are related with each other (see remark 5.8).

In order to establish a connection between the two representations (5.2), (5.11) and (5.12) of $W(E_8^{(1)})$, we introduce a parametrization of the eight points by the parameters $h_1, h_2, e_1, \ldots, e_8$. For notational convenience, we introduce a function

\[
\begin{align*}
\varphi_\kappa(s, t) &= [s - t][\kappa - s - t], \\
\varphi_\kappa(s, t) &= -\varphi_\kappa(t, s) = \varphi_\kappa(\kappa - s, t), \quad \varphi_\kappa(t, t) = 0.
\end{align*}
\]

We also require the Riemann relation

\[
\varphi_\kappa(a, b)\varphi_\kappa(c, u) + \varphi_\kappa(b, c)\varphi_\kappa(a, u) + \varphi_\kappa(c, a)\varphi_\kappa(b, u) = 0.
\]
which is equivalent to

\[ [u + a][u - a][b + c][b - c] + [u + b][u - b][c + a][c - a] + [u + c][u - c][a + b][a - b] = 0. \tag{5.17} \]

See remark 5.2 for concrete examples of the function \([u]\).

We introduce the parametrization of eight points as

\[
(\mathbf{f}, \mathbf{g}) = \left( \frac{F(e_i)}{F(e_3)}, \frac{G(e_i)}{G(e_3)} \right) \quad (i = 1, \ldots, 8),
\]

\[
F(u) = \frac{\varphi_h(e_2, u)}{\varphi_h(e_1, u)} = \frac{[e_2 - u][h_1 - e_2 - u]}{[e_1 - u][h_1 - e_1 - u]},
\]

\[
G(u) = \frac{\varphi_h(e_2, u)}{\varphi_h(e_1, u)} = \frac{[e_2 - u][h_2 - e_2 - u]}{[e_1 - u][h_2 - e_1 - u]]. \tag{5.18}
\]

By this parametrization, the representations (5.2) and (5.12) are compatible. For example, since

\[
s_2(F(u)) = s_2 \left( \frac{[e_2 - u][h_1 - e_2 - u]}{[e_1 - u][h_1 - e_1 - u]} \right) = \frac{s_2(e_2) - u[s_2(h_2) - s_2(e_2) - u]}{[s_2(e_2) - u][s_2(h_1) - s_2(e_1) - u]}
\]

\[
= \frac{[h_2 - e_1 - u][h_1 - e_2 - u]}{[h_2 - e_2 - u][h_1 - e_1 - u]} = \frac{F(u)}{G(u)},
\]

provided that \(s_2(u) = u\), we verify the consistency of the action of \(s_2\) on \(f_i\):

\[
s_2(f_i) = s_2 \left( \frac{F(e_i)}{F(e_3)} \right) = \frac{F(e_i)}{F(e_3)} \left( \frac{G(e_i)}{G(e_3)} \right) = \frac{f_i}{g_i}.
\]

Also, for the action of \(s_3\), we have

\[
s_3(f_i) = s_3 \left( \frac{F(e_i)}{F(e_3)} \right) = s_3 \left( \frac{\varphi_h(e_2, e_2)}{\varphi_h(e_1, e_2)} \right) \left( \frac{\varphi_h(e_1, e_1)}{\varphi_h(e_1, e_3)} \right) = \frac{\varphi_h(e_3, e_1)}{\varphi_h(e_1, e_3)} \left( \frac{\varphi_h(e_3, e_2)}{\varphi_h(e_3, e_1)} \right).
\]

hence the condition \(s_3(f_i) = 1 - f_i\) is equivalent to

\[
\frac{\varphi_h(e_1, e_3)}{\varphi_h(e_1, e_2)} = 1 - \frac{\varphi_h(e_2, e_2)}{\varphi_h(e_1, e_3)},
\]

which is nothing but the Riemann relation (5.16).

**Remark 5.2.** It is known that there are three classes of functions satisfying the Riemann relation (5.17):

1. rational function \([u] = u\),
2. trigonometric function \([u] = \sin \frac{u}{\omega}\),
3. elliptic function \([u] = \sigma(u; \omega_1, \omega_2)\), where \(\sigma(u; \omega_1, \omega_2)\) is the Weierstrass sigma function or an odd theta function.

For given generic eight points in \(\mathbb{P}^1 \times \mathbb{P}^1\), there exists a unique bidegree \((2, 2)\) curve

\[
C_0: \quad p(f, g) = \sum_{i,j=0}^{2} e_i f^i g^j = 0, \tag{5.19}
\]
passing through them. In fact, since we have the nine coefficients $c_{ij}$, the eight linear equations $p(f, g) = 0 (i = 1, \ldots, 8)$ uniquely determines the curve $C_0$. If the curve $C_0$ is non-singular, it is an elliptic curve and otherwise a rational curve. The three classes of the function $[u]$, (2) elliptic, (1) trigonometric and (0) rational, correspond to the cases where the curve $C_0$ is (2) smooth, (1) nodal and (0) cuspidal, respectively.

In the elliptic case, the curve $C_0$ can be identified with the complex torus $\mathbb{C}/\Omega$, $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and the rational function on $C_0$ are expressed by elliptic functions ($\Omega$-periodic meromorphic functions on $\mathbb{C}$). Since $C_0$ is a $(2, 2)$ curve, the lines $f = \text{const.}$ and $g = \text{const.}$ intersect with $C_0$ at two points, respectively. Therefore, in terms of the coordinate $u$ of the complex torus $\mathbb{C}/\Omega$, it is known that the coordinates $(f, g)$ of a point on $C_0$ can be parametrized by elliptic functions of order 2 [74, 135]:

$$ (f, g) = \left( \frac{c\sigma(u - \alpha)\sigma(u - \beta)}{\sigma(u - \gamma)\sigma(u - \delta)}, \frac{c\sigma(u - \alpha')\sigma(u - \beta')}{\sigma(u - \gamma')\sigma(u - \delta')} \right), $$

$$ \alpha + \beta = \gamma + \delta, \quad \alpha' + \beta' = \gamma' + \delta', $$

(5.20)

where $\sigma(u) = \sigma(u; \omega_1, \omega_2)$ is the Weierstrass sigma function or a theta function. From this we obtain the parametrization of the curve $C_0$

$$ C_0 : \quad (f, g) = \left( \frac{F(u)}{F(e_3)}, \frac{G(u)}{G(e_3)} \right), \quad u \in \mathbb{C}, $$

(5.21)

through the renormalization by the action of $\text{PGL}(2)^2$ and the translation of $u$. The trigonometric and rational cases are understood as degenerations of the elliptic case explained above.

Through the parametrization of the eight points (5.18), we obtain from (5.2) and (5.12) the following representation of $W(E^{(2)}_6)$ on the variables $h_1, h_2, e_1, \ldots, e_8$ and $(f, g) = (f_0, g_0)$:

$$ s_0 : \quad e_1 \leftrightarrow e_2, \quad f \rightarrow \frac{1}{f}, \quad g \rightarrow \frac{1}{g}, $$

$$ s_1 : \quad h_1 \leftrightarrow h_2, \quad f \leftrightarrow g, $$

$$ s_2 : \quad \begin{cases} e_1 \rightarrow h_2 - e_2, \\ h_1 \rightarrow h_1 - e_1 - e_2, \end{cases} \quad f \rightarrow \frac{1}{f}, \quad g \rightarrow \frac{1}{g}, $$

$$ s_3 : \quad e_2 \leftrightarrow e_3, \quad f \rightarrow 1 - f, \quad g \rightarrow 1 - g, $$

$$ s_4 : \quad e_3 \leftrightarrow e_4, \quad \begin{cases} f \rightarrow \varphi_{h_1}(e_1, e_4) \varphi_{h_2}(e_2, e_3)^{-1}f, \\ \varphi_{h_1}(e_1, e_4) \varphi_{h_2}(e_2, e_3)^{-1}g, \end{cases} $$

$$ s_5 : \quad e_{j-1} \leftrightarrow e_j, \quad f \rightarrow f, \quad g \rightarrow g \quad (j = 5, \ldots, 8). $$

(5.22)

In this representation, the variables $h_1, h_2, e_1, \ldots, e_8$ play the role of independent variables and parameters for our elliptic Painlevé equation, while $f$ and $g$ are the dependent variables.

We remark that the ninth variables $(f, g)$ are free variables independent of the curve $C_0$ (5.19). However, one may specialize the point $(f, g)$ onto $C_0$ and parametrized it as
This expression is consistent with the action of $W(E^{(1)}_8)$ if we regard $u$ as a constant (a $W(E^{(1)}_8)$-invariant parameter). Namely we have for any $w \in W(E^{(1)}_8)$, $w(f|_{C_0}) = w(f|_{C_0})$, $w(g|_{C_0}) = w(g|_{C_0})$, or, more generally
\[ w(F) = w(F|_{C_0}), \]
for any rational function $F = F(f, g)$. This implies that the points on the curve $C_0$ are confined to $C_0$ by the action of $W(E^{(1)}_8)$.

Conversely, the condition (5.24) can be used to determine the action of $w \in W(E^{(1)}_8)$ on $f$, $g$. From (5.5), we see that $w(f)$ and $w(g)$ are rational functions of the class $w(H_i)$ and $w(H_2)$, respectively (see remark 3.7). Since the dimension of polynomials of the class $w(H_i)$ ($i = 1, 2$) is 2, $w(f)$ and $w(g)$ are expressed in the form
\[ w(f) = \frac{a_1A_1(f, g) + b_1B_1(f, g)}{c_1A_1(f, g) + d_1B_1(f, g)}, \quad w(g) = \frac{a_2A_2(f, g) + b_2B_2(f, g)}{c_2A_2(f, g) + d_2B_2(f, g)}, \]
where $\{A_i(f, g), B_i(f, g)\}$ is an arbitrary basis of the vector space of polynomials in the class $w(H_i)$ ($i = 1, 2$). Then the condition (5.24), which is an identity with respect to the parameter $u$ of $C_0$, gives enough information to determine the ratios of coefficients $a_i : b_i : c_i : d_i$ ($i = 1, 2$) uniquely. We note that this method can be applied also for any inhomogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$.

**Remark 5.3.** The fact that $w(f)$ and $w(g)$ can be expressed as the rational functions of the class $w(H_i)$ and $w(H_2)$ is also expected in the degenerate cases that will be discussed later. We will use this property as a guiding principle for constructing the birational actions of affine Weyl groups on variables $f$ and $g$.

Similarly to the elliptic case described here, one can (and we will) find a compatible parametrization of the eight points in terms of parameters $h_0, e_i$ for each configuration. A mathematical basis for such parametrization is the *period mapping*, which is also a key tool of Sakai’s theory (see [112] section 5).

### 5.3. $\tau$ functions

We now introduce the $\tau$ functions and lift the representation (5.22) of $W(E^{(1)}_8)$ further to the level of them. For this purpose, we define the $\tau$ functions as a family of variables $\tau(\lambda)$ parametrized by the exceptional classes $\lambda$ in $M$ (3.34). The action of the affine Weyl group $W(E^{(1)}_8)$ on $M$ induces a natural action of $w \in W(E^{(1)}_8)$ on the $\tau$ variables as $w(\tau(\lambda)) = \tau(w(\lambda))$. In order to obtain a representation of $W(E^{(1)}_8)$ on a finite number of $\tau$ variables, we impose appropriate algebraic (bilinear) relations among the $\tau$ variables which are consistent with (5.22). In view of (5.23), we represent the variables $f, g$ in terms of the $\tau$ variables as
\[ f = \frac{\varphi_{h_0}(e_1, e_3) \tau(E_2) \tau(H_1 - E_2)}{\varphi_{h_0}(e_2, e_3) \tau(E_1) \tau(H_1 - E_1)}, \quad g = \frac{\varphi_{h_2}(e_1, e_3) \tau(E_2) \tau(H_2 - E_2)}{\varphi_{h_2}(e_2, e_3) \tau(E_1) \tau(H_2 - E_1)}. \]

It is easy to check the action (5.22) of $W(E^{(1)}_8)$ on $f, g$ is consistent if and only if the $\tau$ variables satisfy the bilinear relations
Remark 5.4. We have infinitely many bilinear relations by applying the elements of $W(E_8^{(1)})$ to (5.27). In general, such bilinear relation can be expressed as

\[
\begin{align*}
\varphi_{h_i}(e_i, e_j)\tau(E_i)\tau(H_k - E_k) + \varphi_{h_j}(e_j, e_k)\tau(E_j)\tau(H_k - E_k) = 0,
\varphi_{h_j}(e_j, e_k)\tau(H_k - E_k) + \varphi_{h_k}(e_k, e_j)\tau(E_k)\tau(H_j - E_j) = 0.
\end{align*}
\] (5.27)

Figure 17. $A, B, C, \tilde{A}, \tilde{B}, \tilde{C}$ form an octahedron in $M$.

Here, $A, B, C, \tilde{A}, \tilde{B}, \tilde{C}$ are the vertices of a regular octahedron in $M$ whose edge length is $\sqrt{2}$; $A, B, C$ and $\tilde{A}, \tilde{B}, \tilde{C}$ are antipodal to each other, respectively (see figure 17). Moreover, $a, b, c, \tilde{a}, \tilde{b}, \tilde{c}$ are the complex parameters associated with $A, B, C, \tilde{A}, \tilde{B}, \tilde{C}$, respectively. We remark that such an infinite family of bilinear relations is essentially equivalent to the bilinear equations on the root lattice of type $E_8$ obtained by Ohta et al [92].

As we have seen before, the representation of $W(E_8^{(1)})$ on $f, g$ can be specialized consistently to the curve $C_0$ as (5.23). On the level of the $\tau$ functions, this specialization is realized by

\[
\tau(E_i)|_{C_0} = [e_i - u], \quad \tau(H_k - E_i)|_{C_0} = [h_k - e_i - u]
\]
\[\times (i = 1, \ldots, 8, k = 1, 2, u \in \mathbb{C}), \] (5.29)

and for the general exceptional class $\lambda = d_1H_1 + d_2H_2 - \sum_{i=1}^{8} m_i E_i \in M$

\[
\tau(\lambda)|_{C_0} = \left[ d_1h_1 + d_2h_2 - \sum_{i=1}^{8} m_i e_i - u \right] (u \in \mathbb{C}).
\] (5.30)

The representation of $W(E_8^{(1)})$ on the $\tau$ variables with the bilinear relations (5.28) can be expressed in a closed form in terms of the twelve $\tau$ variables $\tau(E_i)$ ($i = 1, \ldots, 8$) and $\tau(H_k - E_i), \tau(H_k - E_j)$ ($k = 1, 2$) as follows:
Note that the $s_3$ action on $\tau(H_k - E_2)$ ($k = 1, 2$) comes from the bilinear relations. This representation is equivalently rewritten as

$$
\begin{align*}
&\begin{cases}
s_0 : & \tau(E_1) \leftrightarrow \tau(E_2), \quad \tau(H_k - E_1) \leftrightarrow \tau(H_k - E_2) \quad (k = 1, 2), \\
s_1 : & \tau(H_1 - E_1) \leftrightarrow \tau(H_2 - E_1), \\
s_2 : & \tau(E_1) \leftrightarrow \tau(H_2 - E_2), \quad \tau(E_2) \leftrightarrow \tau(H_2 - E_1), \\
s_3 : & \tau(E_3) \leftrightarrow \tau(E_4), \\
& \quad \tau(H_k - E_2) \rightarrow \frac{\varphi_{h_3}(e_1, e_2)\tau(E_1)\tau(H_k - E_1) + \varphi_{h_4}(e_1, e_2)\tau(E_2)\tau(H_k - E_2)}{\varphi_{h_1}(e_1, e_2)\tau(E_3)}, \\
& \quad \tau(E_{j-1}) \leftrightarrow \tau(E_j) \quad (j = 4, \ldots, 8)
\end{cases}
\end{align*}
$$

(5.31)

in terms of the twelve variables

$$
\begin{align*}
&\begin{cases}
s_0 : & \xi_1 \leftrightarrow \xi_2, \quad \eta_1 \leftrightarrow \eta_2, \quad \gamma_1 \leftrightarrow \gamma_2, \\
s_1 : & \xi_i \leftrightarrow \eta_i \quad (i = 1, 2), \\
s_2 : & \xi_1 \rightarrow \frac{\xi_1\eta_2}{\eta_1\eta_2}, \quad \xi_2 \rightarrow \frac{\xi_2\eta_1}{\eta_1\eta_2}, \quad \eta_1 \leftrightarrow \eta_2, \quad \gamma_1 \rightarrow \frac{\eta_1}{\eta_1}, \quad \gamma_2 \rightarrow \frac{\eta_2}{\eta_1}, \\
s_3 : & \left\{ \begin{array}{l}
\xi_2 \rightarrow \frac{\varphi_{h_4}(e_1, e_2)\xi_1 + \varphi_{h_5}(e_1, e_2)}{\varphi_{h_6}(e_2, e_1)\xi_1}, \\
\eta_2 \rightarrow \frac{\varphi_{h_4}(e_1, e_2)\eta_1 + \varphi_{h_5}(e_1, e_2)}{\varphi_{h_6}(e_2, e_1)\eta_1}, \\
\gamma_1 \leftrightarrow \gamma_2,
\end{array} \right.
\end{cases}
\end{align*}
$$

(5.32)

The variables $\xi_i, \eta_i$ can be regarded as homogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$, and the representation (5.22) is recovered by

$$
\begin{align*}
f = \frac{\varphi_{h_6}(e_1, e_2)\xi_2}{\varphi_{h_6}(e_2, e_1)\xi_1}, \quad g = \frac{\varphi_{h_6}(e_1, e_2)\eta_2}{\varphi_{h_6}(e_2, e_1)\eta_1}.
\end{align*}
$$

(5.34)

On the level of homogeneous coordinates $(\xi, \eta) = (\xi_1 : \xi_2 : \eta_1 : \eta_2)$, the parametrization of the curve $C_0$ and the eight points $P_1, \ldots, P_8$ are given by

$$
\begin{align*}
&\xi_i |_{C_0} = \varphi_{h_i}(e_i, u), \quad \eta_i |_{C_0} = \varphi_{h_i}(e_i, u) \quad (i = 1, 2), \\
&P_j : (\xi_1 : \xi_2 : \eta_1 : \eta_2) = (\varphi_{h_1}(e_1, e_1) : \varphi_{h_6}(e_2, e_1), \varphi_{h_2}(e_1, e_1) : \varphi_{h_6}(e_2, e_1)).
\end{align*}
$$

(5.35)

In the sequel we denote by $K$ the field of meromorphic function in the parameters, in $h_1, h_2, e_1, \ldots, e_8 \in \mathbb{C}$. For each $\lambda = d_1h_1 + d_2h_2 - \sum_{i=1}^8 m_i E_i \in \Lambda$ we introduce the $K$–vector space $L(\lambda)$ of functions of the form

$$
P(\xi, \eta) \prod_{i=1}^8 \tau_i^{-m_i}.
$$

(5.37)

Here, $P(\xi, \eta) \in K[\xi, \eta]$ is a homogeneous polynomial in $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2)$ of bidegree $(d_1, d_2)$, i.e., $P(s\xi, s\xi_2, s\eta, s\eta_2) = s^{d_1+d_2}P(\xi, \xi_2, \eta, \eta_2)$, having zeros at the eight points $P_i$ with multiplicity $\geq m_i$ ($i = 1, \ldots, 8$). For example
By a computation similar to that from (5.8) to (5.9), one can show that for each \( k = 0, \ldots, 8, s_k \) transforms \( L(\lambda) \) bijectively to \( L(s_k(\lambda)) \) for any \( \lambda \in \Lambda \). Hence, each \( w \in W(E_8^{(1)}) \) induces an isomorphism \( L(\lambda) \rightarrow L(w(\lambda)) \) for any \( \lambda \in \Lambda \). Note that for each exceptional class \( \lambda \in M \) there exists an element \( w \in W(E_8^{(1)}) \) such that \( w(E_i) = \lambda \) (see remark 3.6), which induces an isomorphism \( w : L(E_i) \rightarrow L(\lambda) \). Since \( L(E_i) = K_{\tau_i} \), we see that \( L(\lambda) \) is one-dimensional and \( L(\lambda) = K_{w(\tau_i)} = K_{\tau}(\lambda) \). This implies that for each \( \lambda = d_1h_1 + d_2h_2 - \sum l=1 \infty m_lE_l \in M \) the \( \tau \) function \( \tau(\lambda) \) can be expressed in the form

\[
\tau(\lambda) = \phi(\xi, \eta) \prod l=1 \infty \tau_i^{-m_l}.
\]  

(5.39)

where \( \phi(\xi, \eta) \) is a homogeneous polynomial of bidegree \( (d_1, d_2) \) such that the curve \( \phi(\xi, \eta) = 0 \) passes through the eight points \( P \), with multiplicity \( m_l \) \((i = 1, \ldots, 8)\). Such a homogeneous polynomial is determined uniquely up to constant multiple. By specializing (5.39) to the curve \( C_0 \) according to (5.29) and (5.30), we obtain

\[
\left[d_1h_1 + d_2h_2 - \sum l=1 \infty m_lE_l - u\right] = \phi(\xi, \eta)|_{c_0} \prod l=1 \infty -u^{-m_l}.
\]  

(5.40)

Consequently our polynomial \( \phi(\xi, \eta) \) is normalized in such a way that

\[
\phi(\xi, \eta)|_{c_0} = \left[d_1h_1 + d_2h_2 - \sum l=1 \infty m_lE_l - u\right] \prod l=1 \infty -u^{-m_l},
\]  

(5.41)

when specialized to the curve \( C_0 \). By applying \( w \) to (5.26), we have

\[
w(f) = w\left(\phi_{h_1}(e_1, e_2) \phi_{w(E_1)}(\xi, \eta) \phi_{w(H_i - E_1)}(\xi, \eta), \phi_{h_2}(e_1, e_2) \phi_{w(E_2)}(\xi, \eta) \phi_{w(H_i - E_2)}(\xi, \eta)\right),
\]

(5.42)

which give the precise form of \( w(f) \) and \( w(g) \) observed in (5.5).

**Remark 5.5.** There exists a simple geometric meaning of the bilinear relations (5.27). For instance, the functions of the form \( \tau(E_1)\tau(H_i - E_i) = \phi_{H_i - E_i}(\xi, \eta) \) \((i = 1, \ldots, 8)\) belongs to the same vector space \( L(H_i) \) of dimension 2. As a result, there exists a linear relation among any such functions, for instance

\[
c_1\tau(E_1)\tau(H_i - E_i) + c_2\tau(E_2)\tau(H_i - E_2) + c_3\tau(E_3)\tau(H_i - E_3) = 0.
\]  

(5.43)

The coefficients can be easily recovered by specializing to \( C_0 \) according to (5.29) and putting \( u = e_1, e_2, e_3 \). In general, the bilinear relations arise in such a situation that elements of a two-dimensional vector space such as \( L(H_i) \) can be obtained in several ways as products of elements associated to a exceptional class. This is a universal structure to yield bilinear relations among the \( \tau \) functions that can be observed also in the degenerate cases.
5.4. \( E_8 \)-invariant coordinates

We define convenient coordinates \( \xi(t), \eta(t) \) on \( \mathbb{P}^3 \times \mathbb{P}^3 \) in which the eight points can be treated in symmetric manner. First, we extend the coordinates \( \xi_i, \eta_i \) (\( i = 1, 2 \)) in (5.33) to

\[
\xi_i = \tau(E_i)\tau(H_1 - E_1), \quad \eta_i = \tau(E_i)\tau(H_2 - E_2), \quad (i = 1, \ldots, 8).
\]  

(5.44)

With these notations the bilinear relations for the \( \tau \) functions (5.27) are rewritten as

\[
\varphi_{h_i}(e_i, e_j)\xi_k + \varphi_{h_i}(e_j, e_i)\xi_k + \varphi_{h_i}(e_k, e_i)\xi_j = 0, \\
\varphi_{h_i}(e_i, e_j)\eta_k + \varphi_{h_i}(e_j, e_k)\eta_k + \varphi_{h_i}(e_k, e_j)\eta_j = 0,
\]  

(5.45)

for any \( i, j, k \in \{1, \ldots, 8\} \). The specialization of the coordinates \( \xi_i, \eta_i \) to the curve \( C_0 \) is given by

\[
\xi_i \mid_{C_0} = \varphi_{h_i}(e_i, u), \quad \eta_i \mid_{C_0} = \varphi_{h_i}(e_i, u) \quad (i = 1, \ldots, 8),
\]  

(5.46)

so that \( \xi_i, \eta_i \) have zero at \( u = e_i \).

For any \( W(E_8^{(1)}) \)-invariant parameter \( t \), we define a more general homogeneous coordinates \( \xi(t) \in L(H_1) = K\xi_0 \oplus K\xi_2 \) and \( \eta(t) \in L(H_2) = K\eta_0 \oplus K\eta_2 \), that have a zero at the point \( u = t \) on \( C_0 \) as

\[
\xi(t) = \frac{\varphi_{h_i}(t, e_2)\xi_i + \varphi_{h_i}(e_1, t)\xi_2}{\varphi_{h_i}(e_1, e_2)}, \quad \eta(t) = \frac{\varphi_{h_i}(t, e_2)\eta_i + \varphi_{h_i}(e_1, t)\eta_2}{\varphi_{h_i}(e_1, e_2)}. 
\]  

(5.47)

Then, from (5.16) and (5.27), we have

\[
\xi(e_i) = \xi_i \quad (i = 1, \ldots, 8), \quad \xi(t) \mid_{C_0} = \varphi_{h_i}(t, u), \\
\eta(e_i) = \eta_i \quad (i = 1, \ldots, 8), \quad \eta(t) \mid_{C_0} = \varphi_{h_i}(t, u).
\]  

(5.48)

We remark that the Riemann relation (5.16) also implies the three-term relations

\[
\varphi_{h_i}(a, b)\xi(c) + \varphi_{h_i}(b, c)\xi(a) + \varphi_{h_i}(c, a)\xi(b) = 0, \\
\varphi_{h_i}(a, b)\eta(c) + \varphi_{h_i}(b, c)\eta(a) + \varphi_{h_i}(c, a)\eta(b) = 0.
\]  

(5.49)

By the construction given above, we have \( L(H_1) = K\xi_0 \oplus K\xi_2 \) and \( L(H_2) = K\eta_0 \oplus K\eta_2 \) for generic \( a, b \). The coordinates used by Ohta et al[92] correspond to the following ones

\[
Z = \frac{\xi(b)}{\xi(a)}, \quad W = \frac{\eta(b)}{\eta(a)}.
\]  

(5.50)

where \( a \) and \( b \) are arbitrary fixed parameters (invariant under the Weyl group actions).

Then, in the inhomogeneous coordinates \( (Z, W) \), we see from (5.16) and (5.36) that the original eight points \( P_i \) (\( i = 1, \ldots, 8 \)) are parametrized as

\[
P_i : (Z_i, W_i) = (\varphi(e_i), \psi(e_i)), \quad (i = 1, \ldots, 8),
\]  

(5.51)

where

\[
\varphi(u) = \frac{\varphi_{h_1}(b, u)}{\varphi_{h_1}(a, u)}, \quad \psi(u) = \frac{\varphi_{h_2}(b, u)}{\varphi_{h_2}(a, u)}.
\]  

(5.52)

We note that \( \varphi(u) \) and \( \psi(u) \) are elliptic functions of order 2 and satisfy

\[
\varphi(h_1 - u) = \varphi(u), \quad \psi(h_2 - u) = \psi(u),
\]  

(5.53)

which follows from (5.14) and (5.52).
Advantage of the coordinates \((Z, W)\) is that the action of \(s_0, \ldots, s_8 \in W(E_8^{(1)})\) can be described simply [77]. In fact, they are invariant under the permutations \(\mathcal{S}_8 = \langle s_0, s_3, \ldots, s_8 \rangle\).

The following actions
\[
s_1(Z) = W, \quad s_1(W) = Z, \quad s_2(W) = W, \tag{5.54}
\]
are also obvious from (5.15), (5.22) and (5.36). And we also have
\[
s_2 \left( \frac{Z - Z_2}{Z - Z_1} \right) = \frac{Z - Z_2}{Z - Z_1} W - W_1, \tag{5.55}
\]
which follows from
\[
s_2 \left( \frac{\xi_2}{\xi_1} \right) = \frac{\tau(H_1 - E_2) \tau(H_2 - E_1)}{\tau(H_1 - E_1) \tau(H_2 - E_2)} = \frac{\xi_2 \eta_1}{\xi_1 \eta_2}, \tag{5.56}
\]
and
\[
Z - \varphi(t) = \frac{\varphi_{h_1}(a, b)}{\varphi_{h_2}(a, b)} \frac{\xi(t)}{\xi(a)}, \quad W - \psi(t) = \frac{\varphi_{h_1}(a, b)}{\varphi_{h_2}(a, b)} \frac{\eta(t)}{\eta(a)}. \tag{5.57}
\]

We also remark that the coordinates \((Z, W)\) are expressed in terms of the \(\tau\) functions as follows. From (5.44), (5.48), (5.50)–(5.52) and (5.57) we have
\[
\frac{\varphi_{h_1}(a, e_1)}{\varphi_{h_1}(a, e_2)} Z - Z_i = \frac{\tau(E_i) \tau(H_1 - E_1)}{\tau(E_i) \tau(H_1 - E_1)} \left( \frac{W - W_i}{W - W_j} \right), \quad \frac{\varphi_{h_2}(a, e_1)}{\varphi_{h_2}(a, e_2)} W - W_i = \frac{\tau(E_i) \tau(H_2 - E_1)}{\tau(E_i) \tau(H_2 - E_1)}. \tag{5.58}
\]

We finally give a geometric description of the action of \(W(E_8^{(1)})\) on \(\xi(t)\) and \(\eta(t)\). Since \(\xi(t) \in L(H_i)\), we have \(w(\xi(t)) \in L(w(H))\) for each \(w \in W(E_8^{(1)})\). If \(w(H_i) = d_1H_1 + d_2H_2 - \sum_{i=1}^8 m_iE_i\), \(w(\xi(t))\) is expressed in the form
\[
w(\xi(t)) = P_n(\xi, \eta; t) \prod_{i=1}^s \tau_i^{-m_i}, \tag{5.59}
\]
where \(P_n(\xi, \eta; t)\) is a homogeneous polynomial in \(\xi, \eta\) with parameter \(t\) of bidegree \((d_1, d_2)\) having zeros at \(P_n\) with multiplicity \(\geq m_i (i = 1, \ldots, 8)\). Also, by (5.29), (5.48) and (5.59) the specialization of \(P_n(\xi, \eta; t)\) to \(C_0\) is given by
\[
P_n(\xi, \eta; t) \big|_{C_0} = [w(\xi_i) - t - u][t - u] \prod_{i=1}^s \eta_i - u]^{-m_i}, \tag{5.60}
\]
where we used \(w(\xi(t)) \big|_{C_0} = w(\xi(t)) \big|_{C_0}\) (5.24). In particular, \(P_n(\xi, \eta; t)\) has a zero at \(u = t\) on \(C_0\). The polynomial \(P_n(\xi, \eta; t)\) of the class \(w(H_i)\) is determined uniquely by the normalization condition (5.60). The action of \(w\) on \(\eta(t)\) can be described in a similar way.

### 5.5. Elliptic Painlevé equation

On the basis of the materials provided in the preceding sections, we now write down the elliptic Painlevé equation explicitly.

By choosing \(w\) to be a Kac transformation \(T_\alpha\) associated with \(\alpha \in Q(E_8^{(1)})\) (3.14), we obtain the elliptic Painlevé equation of the direction \(\alpha\). Note that for any roots \(\alpha, \beta \in R\) there exists an element \(w \in W(E_8^{(1)})\) such that \(\beta = w(\alpha)\) and hence \(T_\beta = wT_\alpha w^{-1}\). This means that the elliptic Painlevé equations of two different directions \(\alpha, \beta \in R\) are transformed to each other by certain birational transformations (Bäcklund transformations).
We fix the simple root \( \alpha_1 = H_1 - H_2 \), and investigate the elliptic Painlevé equation in this direction. We see from (3.14) that the Kac translation \( T_{\alpha_1} \) acts on the Picard lattice as
\[
T_{\alpha_1}(H_i) = H_i - 2(H_1 - H_2) + \delta = H_i + 4H_2 - E_1 - \cdots - E_8, \\
T_{\alpha_1}(E_i) = E_i - (H_1 - H_2) + \delta \quad (i = 1, \ldots, 8).
\]
(5.61)
This means that \( T_{\alpha_1}(x) \) and \( T_{\alpha_1}(y) \) are rational function of bidegree \((1, 4)\) and \((4, 9)\), respectively. However, we could use
\[
T_{\alpha_1}^{-1}(H_2) = H_2 - 2(H_2 - H_1) + \delta = 4H_1 + H_2 - E_1 - \cdots - E_8,
\]
which implies that \( T_{\alpha_1}^{-1}(y) \) is a rational function of bidegree \((4, 1)\). We remark that the Kac translation \( T_{\alpha_1} \) can be expressed as a product of two involutions
\[
T_{\alpha_1} = w_2 w_1,
\]
(5.63)
where each of \( w_i \) \((i = 1, 2)\) is given as a product of eight commuting reflections as
\[
w_1 = r_{E_i} r_{H_i - E_i} : E_i \mapsto H_i - E_i, E_j \mapsto H_j - E_j, H_2 \mapsto H_1 + H_2 - E_i - E_j, \\
w_2 = r_{E_i} r_{H_i - E_i} : E_i \mapsto H_i - E_i, E_j \mapsto H_j - E_i, H_1 \mapsto H_1 + H_2 - E_i - E_j,
\]
(5.64)
Notice from (3.5), (3.26) and (3.28) that
\[
r_{H_i - E_i} : E_i \mapsto H_i - E_i, E_j \mapsto H_j - E_i, H_2 \mapsto H_1 + H_2 - E_i - E_j, \\
r_{E_i} : E_i \mapsto E_i.
\]
(5.65)
Then we find that
\[
r_{E_i} r_{H_i - E_i} : E_i \mapsto H_i - E_i, E_j \mapsto H_j - E_j, H_2 \mapsto H_1 + H_2 - E_i - E_j, \\
r_{E_i} r_{H_i - E_i} : E_i \mapsto H_2 - E_i, E_j \mapsto H_2 - E_j, H_1 \mapsto H_1 + H_2 - E_i - E_j,
\]
(5.66)
and hence these \( w_i \) \((i = 1, 2)\) act on the Picard lattice as follows:
\[
w_1(H_i) = H_i, \quad w_1(H_2) = 4H_1 + H_2 - E_1 - \cdots - E_8, \quad w_1(E_i) = H_i - E_i \quad (i = 1, \ldots, 8),
\\
w_2(H_i) = H_i + 4H_2 - E_1 - \cdots - E_8, \quad w_2(H_2) = H_2, \quad w_2(E_i) = H_2 - E_i \quad (i = 1, \ldots, 8).
\]
(5.67)
One can immediately verify that the product \( w_2 w_1 \) actually gives \( T_{\alpha_1} \) by comparing this with (5.61).

**Remark 5.6.** The elliptic Painlevé equation in the direction of \( \alpha_1 \) can be regarded as a non-autonomous version of the QRT mapping (see section 2.4). The two involutions \( w_1 \) and \( w_2 \) correspond to the vertical and horizontal flips, respectively. In fact, noticing from (5.10) that \( s_2 = r_{H_i - E_i} \), and applying permutations \( S_8 = \{ s_0, s_3, \ldots, s_8 \} \) on (5.55), we see that
\[
r_{H_i - E_i} \left( \frac{Z - Z_i}{Z - Z_j} \right) = \frac{Z - Z_i}{Z - Z_j} W - \frac{W}{W_j}, \quad r_{H_i - E_i}(W) = W.
\]
(5.68)
This implies that \( w_2(W) = W \), namely \( w_2 \) corresponds to the horizontal flip. Similarly, we see that \( r_{H_i - E_i} \) leaves \( Z \) invariant and thus \( w_1 \) corresponds to the vertical flip.

As to the description of (5.25) for \( w = T_{\alpha_1} \), several expressions are known in the literature [75, 77, 92]. Here we derive a new explicit expression based on the representation of the affine Weyl group \( W(E_8^{(1)}) \) discussed in the previous sections.
In view of (5.61) and (5.62), let us compute \( T_{n_1}(\xi(t)) \) and \( T_{n_1}^{-1}(\eta(t)) \) with a \( W(E_8^{(1)}) \)-invariant parameter \( t \). Note that we have from (5.33), (5.47) and (5.67)

\[
\begin{align*}
    w_1(\xi(t)) &= \xi(t), & w_2(\eta(t)) &= \eta(t), \\
    T_{n_1}(\xi(t)) &= w_2w_1(\xi(t)) = w_2(\xi(t)), & T_{n_1}^{-1}(\eta(t)) &= w_1w_2(\eta(t)) = w_1(\eta(t)).
\end{align*}
\]

For notational convenience, we write

\[
\mathcal{F} = T_{n_1}(F), \quad E = T_{n_1}^{-1}(F),
\]

for any function \( F \). Applying formula (5.59) with \( w = T_{n_1} \), we have from (5.61)

\[
T_{n_1}(\xi(t)) = \xi(t) = P(\xi, \eta; t)(\tau_1 \cdots \tau_8)^{-1} \in L(H_1 + 4H_2 - E_1 - \cdots - E_8),
\]

where \( P(\xi, \eta; t) = P_{T_{n_1}}(\xi, \eta; t) \). In order to obtain an explicit and compact expression for \( P(\xi, \eta; t) \), we use the linear functions (see (5.47))

\[
\xi_j = \xi(e_j), \quad \eta_j = \eta(e_j) \quad (j = 1, \ldots, 9),
\]

that have a zero at \( P_j \), where we set \( e_0 = t \) regarding \( t \) as the coordinate of \( P_9 \) on \( C_0 \). Noting that \( P(\xi, \eta; t) \) is a homogeneous polynomial of bidegree \((1, 4)\), we use \( \eta_5, \eta_6, \eta_7, \eta_8, \eta_9 \) to form the basis

\[
\eta_5 \cdots \eta_{k} \quad (k = 5, \ldots, 9),
\]

of five polynomials for the homogeneous polynomials of degree 4 in \( \eta \). We remark that instead of \( P_5, P_6, P_7, P_8 \), one may use any four points among \( P_1, \ldots, P_8 \). Since the polynomial \( P(\xi, \eta; t) \) vanishes at \( P_5, \ldots, P_8 \), it takes the form

\[
P(\xi, \eta; t) = \sum_{k=5}^{9} c_k(t)\xi_k \eta_5 \cdots \eta_{k} \eta_9. \quad (5.74)
\]

The coefficient \( c_k(t) \) is determined by the normalization condition (5.60)

\[
P(\xi, \eta; t)|_{c_0} = [T_{n_1}(h_0) - t - u][t - u] \prod_{i=1}^{8} [e_i - u] = \prod_{i=0}^{9} [e_i - u], \quad (5.75)
\]

by specializing at \( u = h_2 - e_k \), and we have

\[
c_k(t) = \frac{\prod_{i=0}^{4} [h_2 - e_i - e_k]}{[h_2 - h_1] \prod_{5 \leq j \leq 9} [e_j - e_k]} \quad (k = 5, \ldots, 9), \quad (5.76)
\]

where \( e_0 = t \) and \( e_0 = \delta - h_1 + 2h_2 - t \). A similar formula for the polynomial

\[
T_{n_1}^{-1}(\eta(t)) = \eta(t) = Q(\xi, \eta; t)(\tau_1 \cdots \tau_8)^{-1} \in L(4H_1 + H_2 - E_1 - \cdots - E_8),
\]

can be obtained by applying the simple reflection \( s_1 = s_{n_1} \) to \( T_{n_1}(\xi(t)) \) as (see (5.22) and (5.32))

\[
Q(\xi, \eta; t) = \sum_{k=5}^{9} d_k(t)\xi_k \eta_5 \cdots \eta_{k} \eta_9, \quad d_k(t) = c_k(t)|_{h_1 \cdots h_2}. \quad (5.78)
\]

Using the polynomials \( P, Q \) in (5.74) and (5.78), the elliptic Painlevé equation \( e-P(E_8^{(1)}) \) is expressed as
Equation (5.79) gives a general form of the elliptic Painlevé equation in the direction of the root $\alpha_i$,

\[
T_{\alpha_i}(h_1) = h_1 - 2(h_1 - h_2) + \delta, \quad T_{\alpha_i}(h_2) = h_2 - 2(h_1 - h_2) + 3\delta,
\]

\[
T_{\alpha_i}^{-1}(h_1) = h_1 + 2(h_1 - h_2) + \delta, \quad T_{\alpha_i}^{-1}(h_2) = h_2 + 2(h_1 - h_2) + \delta,
\]

\[
T_{\alpha_i}^{-1}(e_i) = e_i - (h_1 - h_2) + \delta, \quad T_{\alpha_i}^{-1}(e_i) = e_i + (h_1 - h_2) + \delta \quad (i = 1, \ldots, 8).
\]

It is possible to derive a simple expression of the elliptic Painlevé equation in terms of the coordinates $(Z, W)$ from (5.52), (5.53) and (5.57), one can introduce two parameters $t_w$ and $s_w = h_2 - t_w$ such that $W = \varphi(t_w) = \psi(s_w)$. By noting that $t_2$ and $W$ are $w_2$-invariant from (5.64), remark 5.6 and (5.69), these parameters are also chosen to be $w_2$-invariant. Since (5.57) implies $\eta(t_w) = \eta(s_w) = 0$, we have from (5.74)

\[
P(\xi, \eta; t_w) = c_0(t_w)\xi(\xi(t_w))\eta_1 \cdots \eta_8, \quad P(\xi, \eta; s_w) = c_0(s_w)\xi(\xi(s_w))\eta_1 \cdots \eta_8.
\]

The right-hand side of (5.79) is thus drastically simplified as

\[
w_2 \left( \frac{\xi(t_w)}{\xi(s_w)} \right) = \frac{\xi(t_w)}{\xi(s_w)} \prod_{i=1}^8 \left[ \frac{e_i - s_w}{e_i - t_w} \right],
\]

Similarly, by introducing $w_1$-invariant parameters $t_z$ and $s_z = h_1 - t_z$ such that $Z = \phi(t_z) = \phi(s_z)$, we have

\[
w_1 \left( \frac{\eta(t_z)}{\eta(s_z)} \right) = \frac{\eta(t_z)}{\eta(s_z)} \prod_{i=1}^8 \left[ \frac{e_i - s_z}{e_i - t_z} \right].
\]

Using the relations (5.57), one can rewrite (5.82) and (5.83) in terms of $(Z, W)$ coordinates as

\[
\left[ \frac{h_1 - a - t_w}{h_1 - a - s_w} \right] \frac{Z - \varphi(t_w)}{Z - \varphi(s_w)} = \left[ \frac{h_1 - a - t_w}{h_1 - a - s_w} \right] \frac{Z - \varphi(t_w)}{Z - \varphi(s_w)} \prod_{i=1}^8 \left[ \frac{e_i - s_w}{e_i - t_w} \right],
\]

\[
\left[ \frac{h_2 - a - t_z}{h_2 - a - s_z} \right] \frac{W - \psi(t_z)}{W - \psi(s_z)} = \left[ \frac{h_2 - a - t_z}{h_2 - a - s_z} \right] \frac{W - \psi(t_z)}{W - \psi(s_z)} \prod_{i=1}^8 \left[ \frac{e_i - s_z}{e_i - t_z} \right],
\]

which is an alternative form of the elliptic Painlevé equation $P(E_{8}^{(1)})$.

**Remark 5.7.** In a similar way one can show that

\[
r_{h_2, E_{n} - E_{j}} \times \left[ \frac{h_1 - a - t_w}{h_1 - a - s_w} \right] \frac{Z - \varphi(t_w)}{Z - \varphi(s_w)} = \left[ \frac{h_1 - a - t_w}{h_1 - a - s_w} \right] \frac{Z - \varphi(t_w)}{Z - \varphi(s_w)} \prod_{i=1}^8 \left[ \frac{e_i - s_w}{e_i - t_w} \right],
\]

with the parameters $t_w$ and $s_w = h_2 - t_w$ such that $W = \psi(s_w) = \psi(t_w)$. Then it is possible to derive the first equation in (5.84) by compositions of (5.85) [84].

5.6. Degeneration to q- and d-P$(E_{8}^{(1)})$ cases

5.6.1. Degeneration to q-P$(E_{8}^{(1)})$. By choosing $[\alpha]$ as trigonometric and rational functions, the bidegree $(2, 2)$-curve $C_0$ on $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the eight points $P_1, \ldots, P_8$ becomes
nodal and cuspidal curves respectively. Those cases give rise to discrete Painlevé equations \( q\)-P\( (E_8^{(1)}) \) and \( d\)-P\( (E_8^{(1)}) \) respectively.

For the description of those cases, it is convenient to change the parameters and parametrization of points. We first recall the parametrization of the curve \( C_0 \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) (5.52):

\[
(Z, W) = (\varphi(u), \psi(u)), \quad \varphi(u) = \frac{[b - u][h_1 - b - u]}{[a - u][h_2 - a - u]},
\]

\[
\psi(u) = \frac{[b - u][h_2 - b - u]}{[a - u][h_1 - a - u]},
\]

where we take \([z] = (e^{z/2} - e^{-z/2})/2\) in \( q\)-P\( (E_8^{(1)}) \) case. Then we have

\[
\varphi(u) = \frac{f(u) - f(b)}{f(u) - f(a)}, \quad \psi(u) = \frac{g(u) - g(b)}{g(u) - g(a)}, \quad f(u) = e^u + e^{h_2 - u}, \quad g(u) = e^u + e^{h_2 - u}.
\]

(5.57)

Since \( \varphi(u) \) and \( \psi(u) \) are fractional linear transforms of \( f(u) \) and \( g(u) \) respectively, it is convenient to use \( f(u) \) and \( g(u) \) for parametrization of \( C_0 \).

The above expression is in terms of ‘additive’ parameters \( h_i \) \( (i = 1, 2) \), \( u \), \( e_i \) \( (i = 1, \ldots, 8) \). For the case of \( q\)-difference Painlevé equations, we use the same symbols as ‘multiplicative’ parameters. Then the curve \( C_0 \) and the eight points \( P_i \) \( (i = 1, \ldots, 8) \) on \( C_0 \) are parametrized as

\[
(x, y) = (f(u), g(u)), \quad f(u) = u + \frac{h_1}{u}, \quad g(u) = u + \frac{h_2}{u},
\]

\[
P_i : (x_i, y_i) = (f(e_i), g(e_i)) \quad (i = 1, \ldots, 8)
\]

(5.58)

respectively, and the \((2, 2)\)-curve \( C_0 \) passing through \( P_i \) is given by

\[
(x - y)(h_2 x - h_1 y) + (h_1 - h_2)^2 = 0.
\]

(5.59)

The inhomogeneous coordinates \((Z, W)\) and \((x, y)\) are related as

\[
Z = \frac{x - f(b)}{x - f(a)}, \quad W = \frac{y - g(b)}{y - g(a)}.
\]

(5.60)

From (5.12), (5.54) and (5.55), the Weyl group \( W(E_8^{(1)}) \) acts on these multiplicative parameters as

\[
s_0 : \quad e_1 \leftrightarrow e_2,
\]

\[
s_1 : \quad h_1 \leftrightarrow h_2, \quad x \leftrightarrow y,
\]

\[
s_2 : \quad e_1 \rightarrow \frac{h_2}{e_2}, \quad e_2 \rightarrow \frac{h_2}{e_1}, \quad h_1 \rightarrow \frac{h_1 h_2}{e_1 e_2}, \quad x \rightarrow \tilde{x},
\]

\[
s_j : \quad e_{j-1} \rightarrow e_j \quad (j = 3, \ldots, 8),
\]

(5.61)

where \( \tilde{x} \) is determined by

\[
\frac{x - s_2(x_3)}{\tilde{x} - s_2(x_3)} = \frac{x - x_1 y - y_2}{x - x_2 y - y_1}.
\]

(5.62)

We also use \( q \) given by

\[
q = e^\zeta = \frac{h_2 h_2}{e_1 \cdots e_8}.
\]

(5.63)

As the multiplicative parameter for the null root \( \delta \). The Weyl group actions for the \( \tau \) variables are the same as (5.31), where the \( s_3 \) action simplifies to
By specializing (5.58) to the trigonometric case and using (5.91), we find that the inhomogeneous coordinates \((x, y)\) of \(\mathbb{P}^1 \times \mathbb{P}^1\) can be expressed in terms of the \(\tau\) variables as

\[
\begin{align*}
\tau (E_2) & \leftrightarrow \tau (E_3), \\
s_3 : \tau (H_1 - E_2) & \rightarrow \frac{x_3 - x_1 \tau (E_1) \tau (H_1 - E_2)}{x_1 - x_2 \tau (E_3)} + \frac{x_3 - x_1 \tau (E_2) \tau (H_1 - E_2)}{x_2 - x_3 \tau (E_3)}, \\
s_3 : \tau (H_2 - E_2) & \rightarrow \frac{y_3 - y_1 \tau (E_1) \tau (H_2 - E_2)}{y_1 - y_2 \tau (E_3)} + \frac{y_3 - y_1 \tau (E_2) \tau (H_2 - E_2)}{y_2 - y_3 \tau (E_3)}. 
\end{align*}
\]

(5.95)

We next write down \(q\)-P\((E_8^{(1)})\). Under the translation \(T_{\alpha_1}\) in the direction of \(\alpha_1\), the multiplicative parameters transform as (see (5.80))

\[
\begin{align*}
\bar{h}_1 & = q \frac{h_2^3}{h_1}, \\
\bar{h}_2 & = q \frac{h_3^3}{h_1}, \\
\bar{e}_i & = q e_i \frac{h_3}{h_1} (i = 1, \ldots, 8), \\
\bar{h}_1 & = q \frac{h_4^3}{h_2}, \\
\bar{h}_2 & = q \frac{h_5^3}{h_2}, \\
\bar{e}_i & = q e_i \frac{h_4}{h_2} (i = 1, \ldots, 8).
\end{align*}
\]

(5.97)

Then we obtain in a similar manner to (5.79)

\[
\begin{align*}
\frac{x - f (b)}{x - f (a)} & = \frac{P(x, y; b)}{P(x, y; a)}, \\
\frac{y - g (b)}{y - g (a)} & = \frac{Q(x, y; b)}{Q(x, y; a)},
\end{align*}
\]

(5.98)

where

\[
P(x, y; t) = \sum_{k=5}^{9} c_k (x - x_k) \prod_{5 \leq j < 9} (y - y_j), \quad c_k = \prod_{5 \leq j < 9} \left(1 - \frac{\bar{h}_j}{\bar{e}_j}\right),
\]

(5.99)

with \(e_9 = t\), \(e_0 = qh_2^2/(\bar{h}_2)\), and \(Q(x, y; t) = P(x, y; t)|_{x \rightarrow y, f \rightarrow g, h \rightarrow \bar{h}, e \rightarrow \bar{e}}\). Equation (5.98) is the \(q\)-difference Painlevé equation \(q\)-P\((E_8^{(1)})\). We also obtain an alternative form of \(q\)-P\((E_8^{(1)})\)

\[
\begin{align*}
\frac{x - f (t_x)}{x - f (t_y)} & = \frac{x - f (t_x)}{x - f (t_y)} \frac{t_y^8}{t_x^8} \prod_{5 \leq i < 9} \frac{e_i - \frac{\bar{h}_i}{\bar{e}_i}}{e_i - \frac{h_i}{e_i}} , \\
\frac{y - g (t_x)}{y - g (t_y)} & = \frac{y - g (t_x)}{y - g (t_y)} \frac{t_y^8}{t_x^8} \prod_{5 \leq i < 9} \frac{e_i - \frac{\bar{h}_i}{\bar{e}_i}}{e_i - \frac{h_i}{e_i}} .
\end{align*}
\]

(5.100)

with the parameters \(t_x, t_y\) such that \(y = g (t_x), x = f (t_x)\) respectively, by putting \(b = t_y, a = \frac{\bar{h}_2}{\bar{e}_2}\) in the first equation in (5.98), and similarly, by putting \(b = t_y, a = \frac{\bar{h}_2}{\bar{e}_2}\) in the second equation.
5.6.2. Degeneration to d-P($E_{8}^{(1)}$). We simply take $[u] = u$. Then $\varphi(u)$ and $\psi(u)$ in (5.86) become

$$
\varphi(u) = \frac{f(u) - f(b)}{f(u) - f(a)}, \quad \psi(u) = \frac{g(u) - g(b)}{g(u) - g(a)},
$$

$$
f(u) = u(u - h_1), \quad g(u) = u(u - h_2).
$$

We use the above $f(u)$ and $g(u)$ for parametrization of $C_0$ and the eight points $P_i$ ($i = 1, \ldots, 8$) on $C_0$:

$$
(x, y) = (f(u), g(u)), \quad P_i : (x_i, y_i) = (f(e_i), g(e_i)) \quad (i = 1, \ldots, 8).
$$

Then $C_0$ is given by

$$
(x - y)^2 + (h_1 - h_2)(h_2x - h_1y) = 0.
$$

The inhomogeneous coordinates $(Z, W)$ and $(x, y)$ are related as

$$
Z = \frac{x - f(b)}{x - f(a)}, \quad W = \frac{y - g(b)}{y - g(a)}
$$

The Weyl group $W(E_{8}^{(1)})$ action on the parameters is the same as (5.12). From (5.54) and (5.55), we obtain the action on $x, y$ as

$$
s_1 : x \leftrightarrow y, \quad s_2 : x \rightarrow \tilde{x},
$$

where $\tilde{x}$ is determined by

$$
\frac{\tilde{x} - s_2(x_0)}{\tilde{x} - s_2(x_2)} = \frac{x - x_1 y - y_1}{x - x_2 y - y_2}.
$$

The Weyl group actions for the $\tau$ variables have the same form as $q$-P($E_{8}^{(1)}$) case, namely they are given by (5.31) and (5.95) with $x_i = e_i(e_i - h_1), \quad y_i = e_i(e_i - h_2)$. The inhomogeneous coordinates $(x, y)$ of $\mathbb{P}^1 \times \mathbb{P}^1$ can be expressed in terms of the $\tau$ by the same formula as (5.96).

We next write down d-P($E_{8}^{(1)}$). By using (5.104) in (5.79) (or taking $q \rightarrow 1$ limit in (5.98)), we have

$$
\frac{\tau - f(b)}{\tau - f(a)} = \frac{P(x, y; b)}{P(x, y; a)}, \quad \frac{y - g(b)}{y - g(a)} = \frac{Q(x, y; b)}{Q(x, y; a)},
$$

where

$$
P(x, y; t) = \sum_{k=5}^{9} c_k (x - x_k) \prod_{5 \leq j < 9} (y - y_j), \quad c_k = \prod_{5 \leq j < 9} \prod_{j=k}^{4} (e_j - e_k),
$$

with $e_9 = t, \quad e_0 = \delta + 2h_1 - t - h_1$, and $Q(x, y; t) = P(x, y; t)|_{x \rightarrow y, f \rightarrow g, h_1 \rightarrow h_2}$. Equation (5.107) is the difference Painlevé equation d-P($E_8^{(1)}$). We also have an alternate expression of d-P($E_8^{(1)}$) as
\[
\frac{X - f(t_x)}{X - f(t_x - t_y)} = \frac{x - f(t_x)}{x - f(t_x - t_y)} \prod_{i=1}^{8} \frac{h_2 - e_i - t_y}{e_i - t_y},
\]

\[
\frac{y - g(t_x)}{y - g(t_x - t_y)} = \frac{y - g(t_x)}{y - g(t_x - t_y)} \prod_{i=1}^{8} \frac{h_1 - e_i - t_x}{e_i - t_x},
\]

with the parameters \(t_x, t_y\) such that \(y = g(t_x), x = f(t_x)\) respectively.

### 5.6.3. Relation to the ORG form

Let us finally mention on the relations between \(q\)-P\(E_8^{(1)}\) (5.98), d-P\(E_8^{(1)}\) (5.107) and the \(q\)-difference and difference Painlevé equations with \(W(E_8^{(1)})\)-symmetry derived by Ohta et al [92]. For \(x, y\) satisfying \(q\)-P\(E_8^{(1)}\) (5.98), it is possible to verify

\[
\begin{align*}
\frac{(x - y)(h_2X - \bar{h}_1Y) - (h_1 - h_2)(h_2 - \bar{h}_1)}{(x - y)(h_2X - h_1Y) - (\bar{h}_1 - h_2)(h_2 - h_1)} &= A(h_2, y), \\
\frac{(y - x)(h_1y - h_2x) - (h_2 - h_1)(h_1 - h_2)}{(y - x)(h_1y - h_2x) - (\bar{h}_1 - h_2)(h_1 - h_2)} &= A(h_1, x),
\end{align*}
\]

(5.110)

Here \(A(h, z)\) and \(B(h, z)\) are given by

\[
A(h, z) = m_0 z^4 - m_1 z^3 + (3hm_0 + m_2 - h^2 m_8)z^2
+ (2hm_1 - m_1 + h^2 m_1)z + (h^2 m_0 + h m_2 + m_4 - h^2 m_6 + h^2 m_8),
\]

\[
B(h, z) = h^2 m_8 z^4 - h^3 m_7 z^3 + (3hm_0 + 2h^2 m_6 - 3h^2 m_8)z^2
+ (hm_1 - h^2 m_5 + 2h^2 m_1)z + (h^2 m_0 + h m_2 + m_4 - h^2 m_6 + h^2 m_8),
\]

(5.111)

where \(m_i (i = 0, \ldots, 8)\) defined by \(U(z) = z^d \prod_{i=1}^{8} (e_i - z) = z^{-d} \sum_{i=0}^{8} m_i z^i\); \(m_i\) are the \(i\)-th elementary symmetric polynomials of \(e_1, \ldots, e_8\). Equation (5.110) is equivalent to the \(q\)-difference Painlevé equation with \(W(E_8^{(1)})\)-symmetry derived by Ohta et al [92 formula (4.6a) and (4.6b)]. We note that the polynomials in the left-hand sides of (5.110) are regarded as analogues of the curve \(C_0\) (5.90) passing through the eight points \(P_i\).

Equation (5.110) can be derived as follows. Solving the first equation of (5.100) in terms of \(\pi\) and substituting it into the left-hand side of the first equation of (5.110), we have

\[
\begin{align*}
\frac{(x - y)(h_2X - \bar{h}_1Y) - (h_1 - h_2)(h_2 - \bar{h}_1)}{(x - y)(h_2X - h_1Y) - (\bar{h}_1 - h_2)(h_2 - h_1)} &= \frac{uU(u) - \frac{h_2}{u}U\left(\frac{h_2}{u}\right)}{uU\left(\frac{h_2}{u}\right) - \frac{h_2}{u}U(u)},
\end{align*}
\]

(5.112)

where \(u = t_x\) is a parameter such that \(y = u + \frac{h_2}{u}\). By virtue of the symmetry with respect to interchanging \(u \leftrightarrow \frac{h_2}{u}\) of the right-hand side, there exist functions \(A(h, z)\) and \(B(h, z)\) which are polynomials in \(z\) such that

\[
\begin{align*}
\left(u - \frac{h_2}{u}\right)A\left(h_2, u + \frac{h_2}{u}\right) &= uU(u) - \frac{h_2}{u}U\left(\frac{h_2}{u}\right), \\
\left(u - \frac{h_2}{u}\right)B\left(h_2, u + \frac{h_2}{u}\right) &= uU\left(\frac{h_2}{u}\right) - \frac{h_2}{u}U(u).
\end{align*}
\]

(5.113)

By solving \(A\) and \(B\) from these equations we obtain (5.111), which gives the first equation of (5.110). The second equation is derived in a similar manner.

Similarly, d-P\(E_8^{(1)}\) (5.107) is shown to be equivalent to the difference Painlevé equation with \(W(E_8^{(1)})\)-symmetry obtained in [92].
Remark 5.8. It is sometimes convenient to change the parameters \((h_1, h_2, e_1, \ldots, e_8, u)\) to \((\kappa_1, \kappa_2, v_1, \ldots, v_8, v)\)

\[ \kappa_1 = h_1 - 2\lambda, \quad v_1 = e_1 - \lambda, \quad v = u - \lambda, \]  

by introducing a new parameter \(\lambda\) such that

\[ s_2(\lambda) = \lambda + \frac{h_1 - e_1 - e_2}{4}, \quad s_j(\lambda) = \lambda \quad (j \neq 2). \]  

Note that \(c = \lambda - \frac{h_1 + h_2}{4}\) is a central element \((W(E_8^{(1)})\)-invariant). Then (5.12) and (5.80) yield

\[ s_0 : \quad v_1 \leftrightarrow v_2, \quad s_1 : \quad \kappa_1 \leftrightarrow \kappa_2, \quad s_2 : \quad \kappa_1 \leftrightarrow \kappa_1 + 2\mu, \quad \kappa_2 \leftrightarrow \kappa_2 - 2\mu, \quad v_1 \leftrightarrow v_1 + 3\mu, \quad v_2 \leftrightarrow v_2 + 3\mu, \quad v_j \leftrightarrow v_j - \mu \quad (j = 3, \ldots, 8), \]

\[ v \leftrightarrow v - \mu, \quad \lambda \leftrightarrow \lambda + \mu, \quad \mu = \frac{\kappa_2 - v_1 - v_2}{4}, \]

\[ s_j : \quad v_{j-1} \leftrightarrow v_j \quad (j = 3, \ldots, 8), \]  

and

\[ T_{o_1}(\kappa_i) = \kappa_i - \delta, \quad T_{o_1}(\kappa_2) = \kappa_2 + \delta, \quad T_{o_1}(\lambda) = \lambda - \kappa_1 + \kappa_2 - \delta, \quad T_{o_1}(v_i) = v_i \quad (i = 1, \ldots, 8), \quad T_{o_1}(v) = v + \kappa_1 - \kappa_2 - \delta, \]

respectively. It is convenient to use \((\kappa_1, \kappa_2, v_1, \ldots, v_8, v)\) to describe \(g-P(E_8^{(1)})\) as a difference equation, while \((h_1, h_2, e_1, \ldots, e_8, u)\) is convenient for the description of underlying symmetry.

Similar change of parameters \((h_1, h_2, e_1, \ldots, e_8, u)\) to \((\kappa_1, \kappa_2, v_1, \ldots, v_8, v)\) are also used in multiplicative cases:

\[ \kappa_1 = \frac{h_i}{\lambda^2}, \quad v_1 = \frac{e_1}{\lambda}, \quad v = \frac{u}{\lambda}, \]  

where \(\lambda\) is a parameter such that \(s_2(\lambda) = \lambda \left(\frac{\kappa_1}{v_1v_2}\right)^2\) and \(s_2(\lambda) = \lambda \quad (j \neq 2).\) Note that \(c = \frac{\lambda}{(\partial h_1, \partial h_2, \partial e_1, \partial e_2)}\) is a central element. Then (5.92) and (5.97) yield

\[ s_0 : \quad v_1 \leftrightarrow v_2, \quad s_1 : \quad \kappa_1 \leftrightarrow \kappa_2, \quad s_2 : \quad \kappa_1 \leftrightarrow \kappa_1 \mu^2, \quad \kappa_2 \leftrightarrow \kappa_2 \mu^2, \quad v_1 \leftrightarrow v_1 \mu^3, \quad v_2 \leftrightarrow v_2 \mu^3, \]

\[ v_j \leftrightarrow \frac{v_j}{\mu} \quad (j = 3, \ldots, 8), \quad v \leftrightarrow \frac{v}{\mu}, \quad \lambda \leftrightarrow \lambda \mu, \quad \mu = \left(\frac{\kappa_2}{v_1v_2}\right)^2, \]

\[ s_j : \quad v_{j-1} \leftrightarrow v_j \quad (j = 3, \ldots, 8), \]  

and

\[ T_{o_1}(\kappa_i) = \frac{\kappa_i}{q}, \quad T_{o_1}(\kappa_2) = q\kappa_2, \quad T_{o_1}(\lambda) = \lambda \frac{q\kappa_2}{\kappa_1}, \quad T_{o_1}(v_i) = v_i \quad (i = 1, \ldots, 8), \quad T_{o_1}(v) = \frac{v_1}{q\kappa_2}, \]

respectively.
In this section, we discuss how to construct an explicit birational representation of the affine Weyl groups.  

5.7. Birational representation of affine Weyl groups

Under the parametrization given in remark 5.8, we rewrite $e$-$P(E_8^{(1)})$ (5.84), i.e.

$$
\begin{align*}
\varphi_{h_i}(a, t_w)Z - \varphi_{h_i}(b, t_w) & = \frac{\varphi_{h_i}(a, t_w)Z - \varphi_{h_i}(b, t_w)}{8 \prod_{i=1}^{8} [e_i - s_w]} \sum_{i=1}^{8} [e_i - s_w],  \\
\varphi_{h_j}(a, s_w)Z - \varphi_{h_j}(b, s_w) & = \frac{\varphi_{h_j}(a, s_w)Z - \varphi_{h_j}(b, s_w)}{8 \prod_{i=1}^{8} [e_i - t_w]} \sum_{i=1}^{8} [e_i - t_w],  \\
\varphi_{b_i}(a, t_z)W - \varphi_{b_i}(b, t_z) & = \frac{\varphi_{b_i}(a, t_z)W - \varphi_{b_i}(b, t_z)}{8 \prod_{i=1}^{8} [e_i - t_z]} \sum_{i=1}^{8} [e_i - t_z].
\end{align*}
$$

(5.121)

In terms of the parameters in (5.114) together with

$$
a = \alpha + \lambda, \quad b = \beta + \lambda, \quad s_w = u_w + \lambda, \\
t_w = v_w + \lambda, \quad s_z = u_z + \lambda, \quad t_z = v_z + \lambda,
$$

(5.122)

so that $u_w = \kappa_2 - v_w, s_w = \kappa_1 - v_w$ hold, we have

$$
\begin{align*}
\varphi_{h_i}(\pi, u_w)Z - \varphi_{h_i}(\beta, u_w) & = \frac{\varphi_{h_i}(\alpha, v_w)Z - \varphi_{h_i}(\beta, u_w)}{8 \prod_{i=1}^{8} [v_i - u_w]} \sum_{i=1}^{8} [v_i - u_w],  \\
\varphi_{h_j}(\alpha, u_w)Z - \varphi_{h_j}(\beta, u_w) & = \frac{\varphi_{h_j}(\alpha, v_w)Z - \varphi_{h_j}(\beta, u_w)}{8 \prod_{i=1}^{8} [v_i - u_w]} \sum_{i=1}^{8} [v_i - u_w],  \\
\varphi_{b_i}(\alpha, u_z)W - \varphi_{b_i}(\beta, u_z) & = \frac{\varphi_{b_i}(\alpha, v_z)W - \varphi_{b_i}(\beta, u_z)}{8 \prod_{i=1}^{8} [v_i - u_z]} \sum_{i=1}^{8} [v_i - u_z],  \\
\varphi_{b_j}(\alpha, u_z)W - \varphi_{b_j}(\beta, u_z) & = \frac{\varphi_{b_j}(\alpha, v_z)W - \varphi_{b_j}(\beta, u_z)}{8 \prod_{i=1}^{8} [v_i - u_z]} \sum_{i=1}^{8} [v_i - u_z],
\end{align*}
$$

(5.123)

where we have used the following relations

$$
\begin{align*}
\varphi_{h_i}(a, t_w) & = \varphi_{h_i}(\alpha, v_w), \quad \varphi_{h_j}(a, h_2 - t_w) = \varphi_{h_j}(\alpha, \kappa_2 - v_w),  \\
\varphi_{h_i}(a, t_w) & = \varphi_{h_i}(\pi, v_w), \quad \varphi_{h_i}(a, h_2 - t) = \varphi_{h_i}(\pi, v_w).
\end{align*}
$$

(5.124)

For example, the third equation in (5.124) can be verified as follows. Since $h_1 = -h_1 + 2h_2 + \delta = -\kappa_1 + 2\kappa_2 + \delta + 2\lambda$, we have

$$
\varphi_{h_1}(a, t_w) = [a - t_w][h_1 - a - t_w] = [a - v_w][\kappa_1 + 2\kappa_2 + \delta - \alpha - v_w] = \varphi_{h_i}(\pi - \kappa_2 - v_w),
$$

where we have used $\pi = \pi - \kappa = \alpha + \kappa_1 - \kappa_2 - \delta$.

5.7. Birational representation of affine Weyl groups

In this section, we discuss how to construct an explicit birational representation of the symmetry group of the surface characterized by a given point configuration, which is generated by simple reflections and lattice isomorphisms (Dynkin diagram automorphisms). We demonstrate the procedure by taking the case of the symmetry type/surface type $A_1^{(1)}/A_1^{(1)}$.

There are several methods to construct the birational representation of the affine Weyl group associated with a given symmetry type/surface type. One is to trace the procedures of blowing up and blowing down according to the transformations of the Picard lattice [112]. Another way is to consider the degeneration of the birational representation of the generic $E_8^{(1)}$/$A_1^{(1)}$ case to the given symmetry/surface type according to the scheme that is explained in section 8.3. In this section, we explain another direct way based on the principle in remark 5.3.

We use the multiplicative parameters $h_1, h_2, e_1, \ldots, e_8$. On these parameters the simple reflections act multiplicatively in the same way as they do on the basis of the Picard lattice $H_1, H_2, E_1, \ldots, E_8$. As for the variables $f$ and $g$, we construct the birational action on them according to the guiding principle as mentioned in remark 5.3; for each element $w$ of the
affine Weyl group, \( w(f) \) and \( w(g) \) should be rational functions in the class \( w(H_1) \) and \( w(H_2) \), respectively.

We consider the point configuration given by

\[
(f, g) = \left( \frac{1}{e_i}, \frac{-e_2e_3}{e}, \frac{1}{e_{i+2}} \right), (e_4, \infty), \left( 0, \frac{h_i}{e_i} \right) (i = 5, 6), \left( \frac{h_i}{e_i}, 0 \right) (i = 7, 8).
\]

(5.125)

which is illustrated graphically in figure 18. The second point is the double point at \((\infty, \infty)\) with gradient \( \frac{f}{g} = -e_2e_3 \) (see (4.60)). The pair of root bases \( \{ \alpha_i \}, \{ \delta_i \} \) representing the symmetry/surface types associated with the point configuration (5.125), and the Dynkin diagram automorphisms are given by

\[
\alpha_0 = E_7 - E_8, \alpha_1 = H_1 - E_4 - E_7, \alpha_2 = H_2 - E_1 - E_5, \\
\alpha_3 = E_5 - E_6, \alpha_4 = H_1 + H_2 - E_2 - E_3 - E_5 - E_7, \\
\delta_0 = H_1 - E_1 - E_2, \delta_1 = E_2 - E_3, \delta_2 = H_2 - E_2 - E_4, \\
\delta_3 = H_1 - E_5 - E_6, \delta_4 = H_2 - E_3 - E_6. \\
\pi_1 = \pi_{42687153}^r H_1 - H_2, \pi_2 = \pi_{42317856}^r H_1 - H_1.
\]

(5.127)

Here \( \pi_{\alpha_1\ldots\alpha_8} \) is the permutation \( E_1 \rightarrow E_8, E_2 \rightarrow E_7, \ldots, E_8 \rightarrow E_1 \) and \( r_\alpha \) is the simple reflection with respect to \( \alpha \).

From the above data, the action of affine Weyl group of type \( A_4^{(1)} \) together with the Dynkin diagram automorphisms on parameters \( e_i \) \((i = 1, \ldots, 8)\), \( h_i \) \((i = 1, 2)\) is given as follows:
Note that the Picard lattice has a trivial lattice isomorphism \( H \rightarrow H, E \rightarrow E \) which does not belong to the affine Weyl group. As to the Dynkin diagram automorphism \( \pi_2 \), we need to incorporate the corresponding transformation \( h_i \rightarrow h_i^{-1} \), \( e_i \rightarrow e_i^{-1} \) in constructing the birational representation. We also remark that \( \pi_2 \) transforms \( q = h_1^2 h_2^2 / (e_1 \cdots e_8) \) into its reciprocal \( q^{-1} \), while the other transformations \( \pi_0, \ldots, \pi_4 \) leave \( q \) invariant.

Then the action of affine Weyl group on the variables \( f \) and \( g \) can be constructed as:

\[
s_1 : \quad g \rightarrow \frac{e_1(e_4 - f)}{h_1 - e_1 f} g, \\

s_2 : \quad f \rightarrow \frac{h_2(1 - e_1 g)}{e_1(e_3 - h_2 g)} f, \\

s_3 : \quad f \rightarrow \frac{h_1 h_2}{e_2 e_3 e_7} - h_1 + e_1 f + e_2 e_3 e_7 g - f, \\

\frac{e_2 e_3 e_7}{h_1} - h_1 e_5 + e_5 e_7 f + h_1 h_2 g, \\

\frac{e_5 e_7 - e_2 e_3 e_5 + h_2 f + h_2 e_3 e_5}{e_5} g, \\

\pi_1 : \quad f \rightarrow \frac{h_1}{h_2} e_5 - h_2 g, \\

\frac{e_5}{f} g \rightarrow \frac{1}{h_1 h_2} - h_1 e_5 + e_5 e_7 f + h_1 h_2 g, \\

\pi_2 : \quad f \leftarrow g. 
\]

In the following, we discuss how to construct (5.129) from the above data. For example, we demonstrate the construction of \( s_4(f) \). Noting that \( s_4(H) = 2H + H_2 - E_2 - E_1 - E_3 - E_7 \), we first determine a basis of polynomials belonging to the divisor class of \( s_4(H) \) as (see section 8.2 for treatment of the double point)

\[
A(f, g) = -\frac{e_5}{h_2} + \frac{e_5 e_7}{h_1 h_2} f + g, \quad B(f, g) = f \left( -\frac{h_1}{e_7} + f + e_2 e_3 g \right).
\]
Then, according to our guiding principle, \( s_4(f) \) should be expressed in the form

\[
s_4(f) = \frac{aA(f, g) + bB(f, g)}{cA(f, g) + dB(f, g)}.
\]

One can determine the coefficients by investigating the image of appropriate points or divisors. In this case, \( H_1 - E_2, H_1 - E_5 \) and \( H_1 - E_7 \) are perpendicular to \( \alpha_4 = H_1 + H_2 - E_2 - E_3 - E_5 - E_7 \). Hence, the corresponding lines \( f = \infty, f = 0, \) and \( f = \frac{h}{c} \) are invariant with respect to the action of \( s_4 \). Thus we have

\[
s_4(f)|_{f=\infty} = \infty, \quad s_4(f)|_{f=0} = 0, \quad s_4(f)|_{f=\frac{h}{c}} = \frac{h}{e},
\]

first two of which yield \( d = 0 \) and \( a = 0 \), respectively. Then the third equation gives \( \frac{b}{c}e_2e_3 = 1 \), which determines \( s_4(f) \) as in (5.129). Other actions can be determined in a similar manner. One can verify that these transformations \( \{s_0, \ldots, \pi_1, \pi_2\} \) satisfy the fundamental relations of the affine Weyl group of type \( A_4^1 \) and the Dynkin diagram automorphisms

\[
\begin{align*}
  s_i^2 &= 1, \quad (s_is_{i+1})^3 = 1, \quad (s_is_j)^2 = 1 \quad (j \not\equiv i, i \pm 1 \mod 5), \\
  \pi_1^5 = \pi_2^2 = (\pi_1\pi_2)^2 &= 1, \quad \pi_1\pi_{[01234]} = \pi_{[40123]}, \quad \pi_2\pi_{[01234]} = \pi_{[32104]}\pi_2.
\end{align*}
\]

\[\text{(5.133)}\]

6. Hypergeometric solutions

Most of the Painlevé equations admit a class of particular solutions expressible in terms of hypergeometric type functions for special values of parameters which correspond to reflection hyperplanes in the parameter space. We call this class of solutions the hypergeometric solutions. In this section, taking the example of \( qP(E_6^{[1]}) \), we demonstrate how to construct the hypergeometric solutions through the Riccati equation and by linearizing it. Then we give an intrinsic formulation of this procedure by the geometric language of point configurations.

6.1. Hypergeometric solution to \( qP(E_6^{[1]}) \): an example

We have already demonstrated in section 2.3 a simple example of \( P_{IV} \) and a \( dP_{II} \), where we constructed particular solutions by means of the Hermite functions. In general, the simplest hypergeometric solutions can be constructed by looking for the special values of parameters where the equation is decoupled into Riccati equations. Then one can linearize the Riccati equation to the second order linear differential or difference equation by the standard procedure, which may be identified with the equation for a certain hypergeometric type function.

Before proceeding to the example of \( qP(E_6^{[1]}) \), we give general remarks on the linearization of (discrete) Riccati equation

\[
y = \frac{ay + b}{cy + d},
\]

where the coefficients \( a, b, c \) and \( d \) are given functions. Substituting

\[
y = \frac{G}{F},
\]

\[\text{(6.2)}\]
which may be linearized by introducing a decoupling function $h$ as

$$\bar{G} = h(aG + bF), \quad \bar{F} = h(cG + dF).$$

Then we obtain a second order linear difference equation for $F$

$$\bar{F} - \frac{h}{\xi}(ac + cd)\bar{F} + \frac{c}{\xi}h(ad - bc)E = 0, \quad G = \frac{1}{c}\left(\frac{\bar{F}}{h} - dF\right).$$

By suitable choice of decoupling function $h$, (6.5) is expected to reduce to some hypergeometric equation which typically takes the form

$$A(\bar{F} - F) + BF + C(E - F) = 0.$$  

Here, the coefficients $A, B, C$ are of factorized form. In view of this, setting $h = \frac{1}{2}$, namely

$$y = \frac{d}{c} \left(\frac{\bar{F}}{F} - F\right),$$

we see that (6.1) yields the following linear difference equation

$$cd(d)\bar{F} - \frac{bc}{d}\bar{F} + \frac{\Delta}{c}(E - F) = 0, \quad \Delta = ad - bc.$$  

Therefore, in the context of construction of hypergeometric solutions to the discrete Painlevé equation, it is a good strategy to choose the $y$ variable in such a way that the coefficients $b, c, d$ and $\Delta = ad - bc$ are factorized.

We show how this procedure works for the case of $q\cdot P(E_{6}^{(1)})$ (4.2):

$$\frac{(fg - 1)(f g - 1)}{f} = \left(g - \frac{1}{v_1}\right)\left(g - \frac{1}{v_2}\right)\left(g - \frac{1}{v_3}\right),$$

$$\frac{(fg - 1)(f g - 1)}{g} = \left(f - \frac{1}{v_1}\right)\left(f - \frac{1}{v_2}\right)\left(f - \frac{1}{v_3}\right)\left(f - \frac{1}{v_4}\right).$$

where $\kappa_1, \kappa_2, v_1, \ldots, v_8$ are parameters with $q \prod_{i=1}^{8} v_i = \kappa_1^2 \kappa_2^2$ introduced in remark 5.8, and $-$ is the time evolution corresponding to $T_{\alpha}$ such that $(\tau_1, \tau_2, \tau_3, \ldots, \tau_8) = \left(\frac{v_i}{q}, q\kappa_2, v_1, \ldots, v_8\right)$. Note that the corresponding eight points configuration

$$P_i : \ (f, g) = \left(v_i, \frac{1}{v_i}\right) \ (i = 1, 2, 3, 4), \quad \left(0, \frac{v_i}{\kappa_2}\right) \ (i = 5, 6), \quad \left(\frac{v_i}{v_j}, 0\right) \ (i = 7, 8),$$

as given in (4.9), and those points are on the reference curve $C_0 : fg(fg - 1) = 0$.

Consider the case where the parameters satisfy

$$\kappa_1 \kappa_2 = v_1 v_3 v_5 v_7, \quad \text{i.e.} \quad q^{-1}\kappa_1 \kappa_2 = v_2 v_4 v_6 v_8.$$
Then (6.9) admits the following specialization:

\[
\frac{fg - 1}{f} = \left( g - \frac{1}{v_f} \right) \left( g - \frac{1}{v_g} \right), \quad \frac{fg - 1}{g} = \left( g - \frac{1}{v_f} \right) \left( g - \frac{1}{v_g} \right), \quad \frac{fg - 1}{f} = \frac{g - \frac{1}{v_f}}{g - \frac{m}{v_g}}, \quad (6.12)
\]

\[
\frac{fg - 1}{g} = \frac{(f - v_f)(f - v_g)}{f - \frac{m}{v_f}}, \quad \frac{fg - 1}{g} = \frac{(f - v_f)(f - v_g)}{f - \frac{m}{v_f}}. \quad (6.13)
\]

This decoupling is consistent in the sense that (6.12) imply the (6.13), and vice versa. In other words, the discrete time evolution admits the specialization (6.12) under the condition (6.11). In fact, under the condition (6.11), both of the first equations of (6.12) and (6.13) imply that the point \((f, g)\) is on a \((1, 1)\)-curve passing through \(P_1, P_3, P_5\) and \(P_7\), which can be verified directly by substituting the coordinates of the points. Similarly, the second equation of (6.12) and upshift of the second equation of (6.13) mean that \((f, g)\) is on a \((1, 1)\)-curve passing through \(P_2, P_4, P_6\) and \(P_{8'} = \left( \frac{m}{v_g}, \frac{m}{v_f}, 0 \right)\). This shows the consistency of the decoupling and geometric meaning of the constraint (6.11) as well.

Equation (6.13) is a coupled Riccati equation; we obtain a Riccati equation with respect to \(f\) by eliminating \(g\) from the first equation and the upshift of the second equation, and \(g\) is determined from \(f\) by a fractional linear transformation. The Riccati equation for \(f\) is given by

\[
\alpha f + \beta = \frac{\gamma f + \delta}{\gamma f + \delta},
\]

\[
\alpha = \kappa_1^2 - \kappa_1 v_1 v_7 - \kappa_1 v_3 v_7 + q v_3 v_8 v_7 v_8, \quad \beta = \kappa_1 (v_1 v_3 v_7 - q v_3 v_8 v_8),
\]

\[
\gamma = q \kappa_1 v_7 - \kappa_1 v_7 - q v_1 v_7 v_8 + q v_2 v_7 v_8 - q v_3 v_7 v_8 + q v_4 v_7 v_8,
\]

\[
\delta = \kappa_1^2 - q \kappa_1 v_1 v_8 - q \kappa_1 v_3 v_8 + q v_1 v_3 v_8 v_8, \quad (6.14)
\]

Choosing the \(y\) and \(F\) variables as

\[
y = \frac{f - v_1}{f - \frac{m}{v_f}} = \frac{1 - \frac{v_1 v_3}{v_5}}{1 - \frac{v_1 v_3}{v_5}} \frac{F - F}{F}, \quad (6.15)
\]

we have the Riccati equation

\[
\sigma = \frac{\zeta y + \eta}{\lambda y + \mu},
\]

\[
\eta = q v_8^3 (\kappa_1 - v_1 v_7)(v_1 - v_2)(v_1 - v_4), \quad \lambda = \kappa_1 (\kappa_1 - q v_2 v_8)(\kappa_1 - q v_4 v_8)(v_7 - v_8),
\]

\[
\mu = v_8 (\kappa_1 - q v_2 v_8)(\kappa_1 - q v_4 v_8)(\kappa_1 - v_1 v_7),
\]

\[
\Delta = q v_8^2 (\kappa_1 - q v_1 v_7)(\kappa_1 - q v_2 v_8)(\kappa_1 - q v_4 v_8)(\kappa_1 - v_1 v_7)(\kappa_1 - v_3 v_7), \quad (6.16)
\]

which is linearized to

\[
A(F - F) + BF + C(F - F) = 0,
\]

\[
A = \left( 1 - \frac{\kappa_2}{v_3 v_5} \right) \left( 1 - \frac{q v_2 v_6}{\kappa_2} \right) \left( 1 - \frac{q v_4 v_6}{\kappa_2} \right), \quad B = \left( 1 - \frac{v_1 v_3}{v_5} \right) \left( 1 - \frac{v_1 v_4}{v_6} \right) \left( 1 - \frac{v_7 v_8}{v_6} \right),
\]

\[
C = q v_6^2 \left( 1 - \frac{q v_1 v_3}{v_5} \right) \left( 1 - \frac{\kappa_1}{v_1} \right) \left( 1 - \frac{v_1 v_8}{v_8} \right) \frac{1}{q v_8}, \quad (6.17)
\]
It is known that the balanced $3\phi_2$ series $\varphi = 3\phi_2 \left[ \frac{a, b, c}{d, e; q; z} \right]$, $z = \frac{de}{abc} (|q| < 1, \frac{de}{abc} < 1)$ satisfies

$$V_1(\varphi - \varphi) + V_2\varphi + V_3(\varphi - \varphi) = 0,$$

$$V_1 = \left(1 - \frac{a}{e}\right)\left(1 - \frac{b}{e}\right)(1 - c), \quad V_2 = (1 - a)(1 - b)\left(1 - \frac{c}{e}\right),$$

$$V_3 = \frac{q}{z}\left(1 - \frac{e}{q}\right)\left(1 - \frac{d}{c}\right)(1 - \frac{1}{e}),$$

where

$$\varphi = \varphi_{e^{-q'}, e^{-q'}} \quad \varphi = \varphi_{a^{-q}, e^{-q'}} \quad \varphi = \varphi_{a^{-q}, a^{-q'}/q},$$

Equation (6.18) is obtained from the following three-term relation [30, formula (2.7)]

$$U_1(\varphi - \varphi) + U_2\varphi + U_3(\varphi - \varphi) = 0, \quad \varphi = \varphi_{a^{-q}, q}, \quad \varphi = \varphi_{a^{-q}, q'/q},$$

$$U_1 = \left(1 - \frac{a}{e}\right)(1 - a), \quad U_2 = (1 - b)(1 - c), \quad U_3 = \frac{a}{z}\left(1 - \frac{d}{a}\right)(1 - \frac{e}{a}).$$

by applying the transformation [24, formula (III.10)]

$$3\phi_2 \left[ \frac{a, b, c}{d, e; q; \frac{de}{abc}} \right] = \frac{(b, de/ab, de/ab; q)_{\infty}}{(d, e, de/abc; q)_{\infty}} \frac{d/b, e/b, de/ab}{de/bc, de/ab; q; b}$$

Comparing (6.17) and (6.18), we find that (6.17) is solved by

$$F = 3\phi_2 \left[ \frac{v_1, v_1}{v_2, v_4, v_3v_5; q_v; v_2, v_5} \right]$$

Also, the numerator of $y$ is expressed by

$$F - F = \frac{\kappa_2 v_8}{v_3v_6}\left(1 - \frac{v_1v_7}{v_3}\right)\left(1 - \frac{v_1v_4}{\kappa_1}\right)\left(1 - \frac{v_1v_8}{\kappa_1}\right) 3\phi_2 \left[ \frac{q_v, q_v, \kappa_2}{v_2, v_4, v_3v_5; q_v; v_5, v_5} \right]$$

so that

$$y = \frac{\kappa_2 v_8}{v_3v_6v_7}\left(1 - \frac{q_v}{v_3}\right)\left(1 - \frac{v_1v_8}{\kappa_1}\right)\left(1 - \frac{v_1v_4}{\kappa_1}\right) 3\phi_2 \left[ \frac{q_v, q_v, \kappa_2}{v_2, v_4, v_3v_5; q_v; v_5, v_5} \right].$$
6.2. Hypergeometric solution from point configuration

The construction of the hypergeometric solutions demonstrated in section 6.1 have the geometric background, and the procedures and quantities appeared in the construction can be understood from the geometry of the point configuration. Let us describe the fundamental principle of the construction according to the example of $qE_P 6_{1−()}$. As mentioned before, the condition of the parameter (6.11) implies that the points $P_1, P_3, P_5, P_7$ is on a $(1, 1)$-curve $C_1$, and similarly, $P_2', P_4', P_6', P_8'$ are on another $(1, 1)$-curve $C_2$. Namely, we have

$$C_1 : d_{1357} = 0, \quad C_2 : d_{2468} = 0,$$

where

$$P_i = (f_i, g_i), \quad P_i' = (f_i', g_i'),$$

$$d_{ijkl} = \begin{vmatrix} 1 & f_j & g_j & f_j g_j \\ 1 & f_j & g_j & f_j g_j \\ 1 & f_k & g_k & f_k g_k \\ 1 & f_j & g_j & f_j g_j \end{vmatrix}, \quad d_{ijkl}' = \begin{vmatrix} 1 & f_j & g_j & f_j g_j \\ 1 & f_j & g_j & f_j g_j \\ 1 & f_k & g_k & f_k g_k \\ 1 & f_j & g_j & f_j g_j \end{vmatrix}. \quad (6.25)$$

Recall from examples 3.4 and 3.5 that

$$d_{ijk} = \begin{vmatrix} 1 & f & g & fg \\ 1 & f & g & fg \\ 1 & f & g & fg \\ 1 & f & g & fg \end{vmatrix} = 0, \quad P_k = (f, g),$$

represents a $(1, 1)$-curve passing through the three points $P_i, P_j, P_k$; it also passes through $P_i$ if and only if $d_{ijkl} = 0$. The Riccati equation (6.12) or (6.13) can be expressed as

$$d_{#135} = 0, \quad d_{#246} = 0, \quad P_k' = (f, g). \quad (6.26)$$

Note that the $(1, 1)$-curve $d_{#135} = 0$ corresponds to the root $\beta = H_1 + H_2 - E_4 - E_5 = R \subset \Lambda$ (see (3.36)). Consistency of those specialization with the discrete time evolution, $T_\alpha, \alpha = H_1 - H_2 \in R$, is guaranteed by the condition $\alpha \cdot \beta = 0$. In fact, the orthogonality $\alpha \cdot \beta = 0$ implies $T_\alpha(\beta) = \beta$ and hence the corresponding curve $C_\beta : d_{#135} = 0$ is preserved by $T_\alpha$, i.e., $P = (f, g) \in C_\beta \mapsto F = T_\alpha(P) \in C_\beta$. Since the genus of $C_\beta$ is 0 (see (3.36)), $C_\beta$ is isomorphic to $\mathbb{P}^1$ and thus it is natural that the resulting dynamical system on $C_\beta$ is the linear fractional transformation, i.e., the Riccati equation. We remark that the above argument applies to $T_\alpha$ and $C_\beta$ for any $\alpha, \beta \in R$ provided that $\alpha \cdot \beta = 0$.

In the context of the (discrete) Painlevé equations and their affine Weyl group symmetries, $C_\beta$ is the so-called the invariant divisor along the reflection hyperplane of the Bäcklund transformation $r_\beta$ in the Umemura theory [82, 128–134]. The condition of the parameters ((6.11) in the above example) is the defining relation of the reflection hyperplane in the parameter space. Further, classification of invariant divisors of Painlevé differential equations is a crucial step of understanding the irreducibility of Painlevé transcendents.

As to the $y$ variable (6.15), we make the choice

$$y = \frac{f - f_i}{f - f_i'}, \quad (6.29)$$
in the generic context, as motivated by the expression in terms of the \( \tau \) function (5.96). Then the Riccati equation for \( y \) takes the form

\[
\frac{\ddot{y}}{cy + d} = \frac{a y + b}{cy + d},
\]

\[
b = -f_{13}f_{15}S_{5}d_{14d8}, \quad c = f_{48}f_{68}g_{66}d_{358}, \quad d = f_{13}f_{15}f_{48}f_{68}g_{66}g_{358},
\]

\[
\Delta = ad - bc = f_{13}f_{15}f_{14}f_{48}f_{68}f_{18}g_{358}g_{48}g_{68},
\]

\[
f_{ij} = f_{i} - f_{j}, \quad f_{ij} = f_{i} - f_{j}, \quad g_{ij} = g_{i} - g_{j},
\]

(6.30)
as verified by the direct computation from (6.25) and (6.28). Accordingly, the coefficients of the linear difference equation for \( F \) variable (6.8) are expressed as factorized form. These formulae apply to any configuration of points which does not contain infinitely near points, namely to the cases of symmetry type \( E_{8}^{(1)}, E_{7}^{(1)}, E_{6}^{(1)} \) and \( D_{5}^{(1)} \). Otherwise, we need to employ appropriate limiting procedures, or it may be easier to construct the hypergeometric solutions from the equation itself directly.

We include the basic data of the fundamental hypergeometric solution of each discrete Painlevé equation in section 8.6.

Remark 6.1.

1. Applying Bäcklund transformations (birational transformations by elements of the affine Weyl group) to a known solution (seed solution), we obtain a class of solutions expressible by the rational functions of the seed solutions. Moreover, in known examples, they are always given by the ratio of determinants whose entries are the seed solutions. The determinant structure of the solutions is understood as a universal property of the Painlevé equations [48, 54, 81, 93, 97–100, 139].

2. In the class of particular solutions of hypergeometric type, the corresponding determinants are referred to as the hypergeometric \( \tau \) functions [32, 33, 46, 56, 61, 62, 68–70, 79, 81, 84, 91, 141]. Historically, the discrete Painlevé equations became familiar after they were derived as the recursion relations satisfied by the ratio of hypergeometric \( \tau \) functions [7, 11, 15, 28, 104] which appeared as the partition functions of the random matrix theory [16].

3. There is another important class of particular solutions, called the algebraic solutions. Typical examples of this class are obtained by applying Bäcklund transformations to the simple solutions characterized by the invariance with respect to the Dynkin diagram automorphisms. Many of such solutions are interpreted as simple specialization of the Schur functions or the universal characters [47, 59, 60, 63, 67, 72, 85, 88, 124].

4. Recently, the general solutions to some Painlevé equations are found to admit explicit formal series solutions [35] in the context of conformal field theory.

7. Lax pairs

It is a common feature of nonlinear integrable systems that they arise as the compatibility condition of certain systems of linear equations. The system of linear equations is called a Lax pair of the nonlinear equation. As we have seen in section 2.2, (2.25) and (2.26) constitute a Lax pair of \( P_{IV} \). See [37, 38, 40, 78, 93, 94] for Lax pairs of other Painlevé equations.

Lax pairs of discrete Painlevé equations have been discussed by many authors from various points of view. Earlier works are discussed in [26], and subsequently more systematic
approach has been used in [3, 6, 41, 76, 107, 113, 137, 140, 142]. We will explain below how one can use the geometric method for constructing Lax pairs of the discrete Painlevé equations according to the idea in [140, 142].

7.1. Lax pair for q-P$_{E6^{(1)}}$ : an example

In order to see the relation between the Lax pair and the point configuration, we consider a Lax pair for q-P$_{E6^{(1)}}$ (4.2)
\[
(fg - 1)(fg - 1) = \left( g - \frac{1}{v_1} \right) \left( g - \frac{1}{v_2} \right) \left( g - \frac{1}{v_3} \right) \left( g - \frac{1}{v_4} \right),
\]
\[
\left( g - \frac{v_5}{\kappa_2} \right) \left( g - \frac{v_6}{\kappa_2} \right) = \left( f - \frac{v_1}{\kappa_2} \right) \left( f - \frac{v_2}{\kappa_2} \right) \left( f - \frac{v_3}{\kappa_2} \right) \left( f - \frac{v_4}{\kappa_2} \right),
\]
as an example, where $\kappa_1, \kappa_2, v_1, \ldots, v_8$ are parameters with $q \prod_{i=1}^{8} v_i = \kappa_1^2 \kappa_2^2$ introduced in remark 5.8, and $-$ is the time evolution corresponding to $T_m$ such that $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \ldots, \bar{v}_8) = \left( \frac{v_i}{q}, \frac{q v_i}{Z}, v_i, \ldots, v_8 \right)$. Note that the corresponding eight points configuration is given in (4.9), and those points are on the reference curve $C_0 : fg(fg - 1) = 0$. A Lax pair is given by [142]
\[
L_1(z) = \frac{1}{q(gg - 1)gZ - 1} \left( g \prod_{i=1}^{4} (v_i - z) \right) y(z) \left[ \frac{1}{q(1 - q)} \right] y(z) - y \left( \frac{z}{q} \right)
\]
\[
+ \prod_{i=1}^{4} \left( \frac{v_i - z}{f - z} \right) \left\{ \frac{g}{1 - g} y(z) - y \left( \frac{z}{q} \right) \right\}
\]
\[
+ \prod_{i=7}^{8} \left( \frac{v_i - qz}{f(q - qz)} \right) \left\{ \frac{1}{q(1 - q)} y(z) - y(qz) \right\} = 0,
\]
\[
L_2(z) = \left( 1 - \frac{1}{z} \right) \left( z \right) + y(z) - \left( z \right) y \left( \frac{z}{q} \right) = 0.
\]

Equation (7.2) is a linear $q$-difference equation for $y(z)$ and (7.3) describes a deformation of (7.2). It turns out that the compatibility condition of the linear system (7.2) and (7.3) gives $q$-P($E_6^{(1)}$). More precise meaning of the compatibility condition is described as follows. Consider the equations $L_1(z) = 0$, $L_4(qz) = 0$, $L_2(z) = 0$, $L_2(qz) = 0$ and $L_2(q^2z) = 0$. One can eliminate four variables $y(z/q)$, $y(z)$, $y(qz)$ and $y(q^2z)$ from these five equations to obtain a linear relation among $\bar{y}(z/q)$, $\bar{y}(z)$ and $\bar{y}(z/q)$, which should coincide with $L_1(z)$. Note that one can rewrite (7.2) and (7.3) in a matrix form as
\[
Y(qz) = M_1(z) Y(z), \quad Y(z) = \left[ \begin{array}{c} y(z) \\ y(z/q) \end{array} \right], \quad M_1(z) = \left[ \begin{array}{cc} a(z) & b(z) \\ 1 & 0 \end{array} \right],
\]
\[ \mathcal{F}(z) = M_2(z)Y(z), \quad M_2(z) = \begin{bmatrix} a(z)\alpha(z)q^{-1} + \beta(qz) & b(z)\alpha(zq) \\ \alpha(z) & \beta(z) \end{bmatrix}. \quad (7.5) \]

Then the compatibility condition mentioned above is equivalently written as
\[ M_1(z)M_2(z) = M_2(qz)M_1(z), \quad (7.6) \]

which yields
\[
\sigma(z) = \frac{(a(q^2z)\alpha(qz) + \beta(q^2z)b(z) - \beta(z)\bar{B}(z))}{b(z)\alpha(qz)}, \quad \bar{B}(z) = \frac{b(z)\Delta(qz)}{\Delta(z)},
\]

\[ \Delta(z) = \det M_1(z) = a(z)\alpha(qz)b(z) - a(z)\alpha(qz)\beta(z) - \beta(z)\beta(qz). \quad (7.7) \]

More practically, one can verify the compatibility of \( L_1(z) = 0 \) and \( L_2(z) = 0 \) as follows. Eliminating \( y(qz) \) and \( y(z/q) \) from \( L_1(z) = 0 \), \( L_2(z) = 0 \) and \( L_2(qz) = 0 \), we have
\[ L_3(z) = wz(z-\bar{F})y(z) + g \sum_{i=1}^{4} (z-v_i)\phi \left( \frac{z}{q} \right) - (1 - gz) \sum_{i=7}^{8} \left( \frac{\kappa_i}{q_v} - z \right) \phi(z) = 0, \quad (7.8) \]

where
\[
w = \frac{v_1v_2v_3v_4}{f} \left( g - \frac{v_5}{\kappa_2} \right) \left( g - \frac{v_6}{\kappa_2} \right) = \frac{(1-gv_1)(1-gv_2)(1-gv_3)(1-gv_4)}{g(fg-1)}.
\]
\[
= \frac{v_1v_2v_3v_4}{f} \left( g - \frac{v_5}{\kappa_2} \right) \left( g - \frac{v_6}{\kappa_2} \right) = \frac{(1-gv_1)(1-gv_2)(1-gv_3)(1-gv_4)}{g(fg-1)}. \quad (7.9)
\]

Here we have used the first equation of (7.1). Then eliminating \( y(z) \), \( y(qz) \) from \( L_1(qz) = 0 \), \( L_2(qz) = 0 \) and \( L_2(qz) = 0 \), we get the three-term relation between \( y(qz) \), \( \mathcal{F}(z) \) and \( \mathcal{F}(z/q) \)
\[
\left( \prod_{i=1}^{4} v_i \prod_{i=5}^{6} \left( g - \frac{v_i}{\kappa_2} \right) \right) \frac{g}{f(1-gz)g} \mathcal{F}(z) + \left( \prod_{i=1}^{4} v_i \prod_{i=7}^{8} \left( \frac{\kappa_i}{q_v} - z \right) \right) \frac{1}{f(1-gz)(1-gz)g} \mathcal{F}(z) + \left( \prod_{i=1}^{4} (v_i - qz) \mathcal{F}(z) - \left( \frac{1}{g} - qz \right) \prod_{i=7}^{8} \left( \frac{\kappa_i}{q_v} - qz \right) \mathcal{F}(qz) \right) = 0. \quad (7.10)
\]

Written in terms of \( \bar{g} \) by using the second equation of (7.1), this gives exactly \( L_1(\bar{g}) \).

We next discuss the geometric characterization of the difference equation (7.2). Multiplying \( fg(fg-1)(f-z)(f-qz) \) to (7.2) yields
\[ \tilde{L}_4(z) = \frac{zf(f - z)(f - qz) \prod_{i=1}^{4} (g_{vi} - 1)}{g_{z} - 1}y(z) \]

\[ - \frac{(fg - 1)(f - z)(f - qz) \prod_{j=5}^{6} \left( \frac{g_{\kappa_2}}{v_{j}} - 1 \right)\kappa_j^2}{q_{v_{j}}v_{q}}y(z) \]

\[ + fg(fg - 1)(f - qz) \prod_{i=1}^{4} (v_{i} - z) \left\{ \frac{g}{1 - gz}y(z) - y \left( \frac{z}{q} \right) \right\} \]

\[ + fg(fg - 1)(f - z) \prod_{i=7}^{8} \left( \frac{\kappa_i}{v_{i}} - qz \right) \left\{ \left( \frac{1}{g} - qz \right)y(z) - y(qz) \right\} = 0, \quad (7.11) \]

from which we observe that the pole at \( g = \frac{1}{z} \) cancels out and \( \tilde{L}_4(z) = P(f, g) \) becomes a polynomial in \( (f, g) \) of degree \( (3, 2) \) with coefficients depending on \( \kappa_1, \kappa_2, v_1, \ldots, v_8, z, \) \( c_+ = y(qz^2) \) and \( c = y(z) \). We first observe that the \( (3, 2) \)-curve \( P(f, g) = 0 \) and the \( (2, 2) \)-curve \( C_0 : fg(fg - 1) = 0 \) intersect at the following ten points:

\[ (f, g) = \left( v_i, \frac{1}{v_i} \right) \text{ (} i = 1, \ldots, 4 \text{),} \left( 0, \frac{v_i}{\kappa_2} \right) \text{ (} i = 5, 6 \text{),} \left( \frac{\kappa_i}{v_i}, 0 \right) \text{ (} i = 7, 8 \text{),} \]

\[ \left( qz, \frac{1}{qz} \right), \quad (z, 0), \quad \text{(7.12)} \]

Further, investigating the section of \( P(f, g) = 0 \) with \( f = z, qz \), we find that \( P(f, g) \) also vanishes at the following two points (figure 19):

\[ (f, g) = \left( z, \frac{c}{c + zc} \right), \quad \left( qz, \frac{c}{c + qzc} \right). \quad (7.13) \]

We note that \( (3, 2) \)-curve \( P(f, g) \) is uniquely determined up to a scalar multiple by the vanishing property at these twelve points. It is non-trivial, however, that the polynomial \( P(f, g) \) becomes linear homogeneous in \( c_+ \) and \( c \) by the particular choice in (7.14); this is the key property for constructing the Lax pair from the point configuration.

We now turn to the example of \( q-P(E_8^{(1)}) \) discussed in section 5.6.1 and demonstrate how to construct a linear problem from the corresponding point configuration. Under the parametrization in remark 5.8, the eight blowing-up points are given by

\[ (f(v_i), g(v_i))_{i=1, \ldots, 8}, \quad \text{(7.15)} \]

where

\[ f(v) = v + \frac{\kappa_1}{v} = f \left( \frac{\kappa_1}{v} \right), \quad g(v) = v + \frac{\kappa_2}{v} = g \left( \frac{\kappa_2}{v} \right). \quad \text{(7.16)} \]

and the reference curve \( C_0 \) passing through the eight points (7.15) is written as

\[ \varphi_{22}(f, g) = (f - g)(\kappa_2 f - \kappa_1 g) + (\kappa_1 - \kappa_2)^2 = 0. \quad (7.17) \]

Supposing that the linear problem is expressed by a polynomial \( P(f, g) \) of bidegree \( (3, 2) \), we specify a set of twelve points on \( P(f, g) = 0 \) by which the polynomial is characterized. Among the twelve points, it is natural to take eight points from the blowing-up points (7.15). There are two additional points
at which the curve \( P(f, g) = 0 \) intersects with the reference curve \( C_0 \). Here, the parameters \( z \) and \( w \) should satisfy
\[
\kappa_1 \kappa_2 = z_7 w, \quad 7.19^{18}
\]
and
\[
\kappa_1 \kappa_2 = \frac{w q_1}{z}, \quad 7.20
\]
due to Abel’s theorem, or the relation between roots and coefficients for the Laurent polynomial
\[
P(f, g) = P\left(v + \frac{\kappa_1}{v}, v + \frac{\kappa_2}{v}\right) = \frac{\text{const.}}{v^5} (v - v_1) \cdots (v - v_8)(v - z)(v - w). \quad 7.21
\]
We specify the remaining two points on the \((3, 2)\)-curve \( P(f, g) = 0 \) of the form
\[
(f(z), g(z)), \quad (f(w), g(w)), \quad 7.18
\]
by choosing \( g_1 \) and \( g_2 \) so that the resulting polynomial \( P(f, g) \) becomes linear homogeneous in \( y(qz), y(z), y(z/q) \). Our choice will be made as
\[
\frac{g_1 - g(z)}{g_1 - g(\kappa_1 z)} = \frac{y(qz)}{y(z)}, \quad \frac{g_2 - g(z/q)}{g_2 - g(q \kappa_2 z)} = \frac{y(z)}{y(z/q)}. \quad 7.23
\]
The polynomials vanishing at the above ten points (effectively nine points due to the constraint (7.19)) form a three-parameter family. Since \((f - f(z)) \varphi_{22}(f, g)\) and \((f - f(w)) \varphi_{22}(f, g)\) belong to this family, we can write any member \( P(f, g) \) of this family as
\[
P(f, g) = A(f - f(z)) \varphi_{22}(f, g) + B(f - f(w)) \varphi_{22}(f, g) - CF_{32}(f, g), \quad 7.24
\]
where \( F_{32}(f, g) \) is a bidegree \((3, 2)\)-polynomial. It is convenient to choose \( F_{32}(f, g) \) by the condition that the curve \( F_{32}(f, g) = 0 \) is tangent to the lines \( f = f(z) \) and \( f = f(w) \). Putting \( f = f(z) \) we have, as a polynomial in \( g \),
\[ P(f(z), g) = B(f(z) - f(w))\varphi_{22}(f(z), g) - CF_{22}(f(z), g) \]

\[ = B'(g - g(z))(g - g(\kappa_1 z)) - C'(g - g(z))^2 \]

\[ = (g - g(z))(B'(g - g(\kappa_1 z)) - C'(g - g(z))). \] \tag{7.25}

In the second line, we have used factorization \( \varphi_{22}(f(z), g) = \kappa_1 (g - g(z))(g - g(\kappa_1 z)) \), which follows from \( \varphi_{22}(f(z), g(z)) = 0 \) and \( f(z) = f(\kappa_1 z) \). Similarly, we have

\[ P(f(w), g) = (g - g(w))\{A'(g - g(\kappa_1 w)) - C''(g - g(w))\} \]

\[ = (g - g(q\kappa_1 z))\{A'(g - g(z/q)) - C''(g - g(q\kappa_1 z))\}. \] \tag{7.26}

In view of the two relations (7.25) and (7.26), we choose the additional two points as (7.22) and (7.23). Then from (7.25) and (7.26) it follows that \( A \propto A' \propto y(z/q), B \propto B' \propto y(qz) \) and \( C \propto C' \propto C'' \propto y(z), \) namely, \( P(f, g) \) becomes linear homogeneous in \( y(qz), y(z) \) and \( y(z/q) \).

### 7.2. Lax pair for \( e-P(E_8^{(1)}) \)

#### 7.2.1. Contiguity type relations

We use the multiplicative parameters, \( \kappa_1, \kappa_2, v_1, \ldots, v_8 \), with \( \kappa_1^2 \kappa_2^2 = q \prod_{i=1}^{n} v_i \). Let \( x \) be a multiplicative odd theta function satisfying \( [x^{-1}] = [x] \) and quasi-periodicity \( [px] = x^{-p^{-1}/2}[x] \) with period \( p \), for instance, \( [x] = x^{-1}(x, \frac{p}{x}, p; p)_\infty \), where \( (a_1, \ldots, a_8; q)_\infty = \prod_{n=1}^{\infty} \prod_{i=1}^{8} (1 - a_i q^{-n}) \). We put

\[
\begin{align*}
  f_a(z) &= \left[ \frac{a}{z} \right] \left[ \frac{\kappa_1}{a z} \right], \\
  g_a(z) &= \left[ \frac{a}{z} \right] \left[ \frac{\kappa_2}{a z} \right], \\
  F(f, z) &= f_a(z) f_a(z) f_a(z) f_a(z), \quad f(z) = \frac{f_a(z)}{f_a(z)}, \\
  G(g, z) &= g_a(z) g_a(z) g_a(z) g_a(z), \quad g(z) = \frac{g_a(z)}{g_a(z)}. \end{align*} \tag{7.27}
\]

Note that for any \( a \) and \( z \) we have

\[
\begin{align*}
  f_a(z) &= -f_z(a) = f_a\left(\frac{\kappa_1}{z}\right), \\
  g_a(z) &= -g_z(a) = g_a\left(\frac{\kappa_2}{z}\right), \\
  F(f, z) &= F\left(f, \frac{\kappa_1}{z}\right), \quad G(f, z) = G\left(f, \frac{\kappa_2}{z}\right). \tag{7.28}
\end{align*}
\]

By the Riemann relation: \( g_1(b)g_a(x) + g_0(c)g_a(x) + g_a(a)g_0(x) = 0 \), we have

\[
\begin{align*}
  g(x) - g(y) &= \frac{g_a(b)g_a(y)}{g_a(x)g_a(y)}, \quad G(g(x), y) = \frac{g_a(b)g_a(y)}{g_a(x)}. \tag{7.29}
\end{align*}
\]

The time evolution \( T \) is given by \( T : \kappa_1 \mapsto \frac{\kappa_1}{p}, \kappa_2 \mapsto q\kappa_2 \). For any functions or variables \( X \), we use the notations \( \overline{X} = T(X), \overline{X} = T^{-1}(X), \) e.g.

\[
\overline{F}(f, z) = \overline{f}_a(z) f_a(z), \quad \overline{f}_a(z) = \left[ \frac{\kappa_1}{z} \right] \left[ \frac{\kappa_1}{z}\right]. \tag{7.30}
\]

We note that in the following argument we do not need to specify the discrete time evolution of \( a \) and \( b \).

The most fundamental object in the scalar Lax formulation is the linear difference equation \( L_4(z) \) among \( y(qz), y(z) \) and \( y(z/q) \). The explicit form of the equation \( L_4(z) \) is, however, rather complicated (see (7.46) below). So, it is convenient to start with the following
contiguity type equations:

\[
L_2(z) : G\left(g, \frac{\kappa_1}{z}\right)y(qz) - G(g, z)y(z) - \left[\frac{\kappa_1}{z^2}\right] F(f, z)\varphi(z) = 0, \quad (7.31)
\]

\[
L_3(z) : G\left(g, \frac{\kappa_1}{qz}\right)U(z)\varphi(z) - G(g, z)U\left(\frac{\kappa_1}{qz}\right)\varphi(qz) - \left[\frac{\kappa_1}{qz^2}\right] wF\left(\tilde{f}, z\right)y(qz) = 0, \quad (7.32)
\]

where \(U(z) = \prod_{n=1}^b \left(\frac{z}{n}\right), \tilde{f} = \tilde{f}(f, g)\) and \(w = w(f, g)\) are variables independent of \(z\) to be determined later (see (7.40) and (7.42) below).

**Remark 7.1.** The linear equations \(L_2, L_3\) given above are equivalent to those in [84]. Here, the coefficients are simplified by a gauge transformation. As a price for this, the function \(y(z)\) is no longer periodic but quasi-periodic: \(y(qz) = \lambda z y(z)\) where \(\lambda = p^2 \frac{\kappa_1}{\kappa_1}, q^\alpha = \frac{\kappa_1}{n_1}\).

As the necessary condition for the compatibility of these equations, one can easily derive \(e\)-\(P(E_k^{(1)})\) as follows. If we put \(g = g(z)\), i.e. \(G(g, z) = 0\) in \(L_2(z)\) and \(L_3(z)\), we have

\[
G\left(g, \frac{\kappa_1}{z}\right)y(qz) = \left[\frac{\kappa_1}{z^2}\right] F(f, z)\varphi(z),
\]

\[
G\left(g, \frac{\kappa_1}{qz}\right)U(z)\varphi(z) = \left[\frac{\kappa_1}{qz^2}\right] wF\left(\tilde{f}, z\right)y(qz), \quad (7.33)
\]

from which we obtain

\[
\frac{wF(f, z) F(\tilde{f}, z)}{U(z)} = \frac{G\left(g, \frac{\kappa_1}{z}\right) G\left(g, \frac{\kappa_1}{qz}\right)}{G\left(g, \frac{\kappa_1}{z}\right) G\left(g, \frac{\kappa_1}{qz}\right)} = \left[\frac{\kappa_1}{\kappa_1} \frac{\kappa_2}{q\kappa_2} \left(\frac{g(z)}{g^\alpha(z)}\right)^2\right], \quad (7.34)
\]

for \(g = g(z)\). Here we have used (7.29) to derive the second equality. Since \(g(z) = g\left(\frac{\kappa_1}{z}\right)\), we get another relation by replacing \(z\) with \(\frac{\kappa_1}{z}\). Then by taking the ratio of these two expressions, we have

\[
\frac{F\left(f, \frac{\kappa_2}{z}\right) F\left(\tilde{f}, \frac{\kappa_2}{z}\right)}{F(f, z) F\left(\tilde{f}, z\right)} = 1, \quad \text{for } g = g(z). \quad (7.35)
\]

On the other hand, putting \(f = f(z)\), i.e. \(F(f, z) = 0\) in \(L_2(z)\) and \(L_3(z)\), we have

\[
\frac{G\left(g, \frac{\kappa_1}{z}\right) G\left(g, \frac{\kappa_1}{qz}\right)}{G(g, z) G\left(g, \frac{\kappa_1}{qz}\right)} = 1, \quad \text{for } f = f(z). \quad (7.36)
\]

Equations (7.35) and (7.36) are equivalent to \(e\)-\(P(E_k^{(1)})\) (5.123), which we will verify in the rest of this section 7.2.1.

From the relations (7.35), (7.36) and (7.34), the variables \(\tilde{f}, g\) and \(w\) are determined as rational functions in \((f, g)\). For instance, substituting (7.27) into (7.35) we have
\[
\frac{\alpha (\frac{K_2}{z})}{\alpha (z)} - \frac{\beta (\frac{K_2}{z})}{\beta (z)} = \frac{F(f, z)U(\frac{K_2}{z})}{F(f, \frac{K_2}{z})U(z)} \quad \text{for } g = g(z). \tag{7.37}
\]

Solving (7.37) in terms of \( \alpha \), we obtain
\[
\alpha |_{g=g(z)} = \frac{F(f, \frac{K_2}{z})U(\frac{K_2}{z})}{F(f, \frac{K_2}{z})U(z)} \quad \text{and} \quad F(f, \frac{K_2}{z})U(z) = F(f, z)\alpha |_{g=g(z)} U(\frac{K_2}{z}), \tag{7.38}
\]

Note that
\[
F(f, \frac{K_2}{z})\alpha |_{g=g(z)} U(\frac{K_2}{z}) = F(f, z)\alpha |_{g=g(z)} U(\frac{K_2}{z}), \tag{7.39}
\]
is an elliptic function in \( z \) with poles of order 4 at \( z = a, \frac{K_2}{z} \), as is easily verified by definition (7.27). This implies that (7.39) is a rational function in \( g(z) \) and \( f(z) \), but from the symmetry with respect to \( z \mapsto \frac{K_2}{z} \) this is actually a rational function only in \( g(z) \). We also remark that the numerator is alternating with respect to \( z \mapsto \frac{K_2}{z} \), and thus the four zeros of \( \frac{K_2}{z} \) which are apparent poles of (7.39) cancel out. Further, noticing that \( g(z) \) has poles of order 1 at \( z = a, \frac{K_2}{z} \), we see that (7.39) is a polynomial of degree 4 in \( g = g(z) \). By this argument, we have
\[
\alpha = \frac{R_b(f, g)}{R_a(f, g)} \quad \text{and} \quad f = \frac{S_b(\alpha, g)}{S_a(\alpha, g)}, \tag{7.40}
\]
where \( R_r(f, g) \) and \( S_r(f, g) \) are polynomials of bidegree \((1, 4)\) in \((f, g)\). The second equation of (7.40) is obtained in a similar manner, starting from (7.37) and solving it in terms of \( f \). Here, \( S_r(\alpha, g) \) are bidegree \((1, 4)\) in \((\alpha, g)\). Also, it should be noted that \( R_a(f, g) \) and \( R_b(f, g) \) vanish at the eight points \((f(v_i), g(v_i))_{i=1, \ldots, 8}\), as is easily seen by setting \( f = f(z), \, z = v_i \) \((i = 1, \ldots, 8)\) in (7.39). Similarly, it follows from (7.36) that \( g \) is expressed as a rational function in \((f, g)\) of bidegree \((1, 4)\) having the same eight points as points of indeterminacy. These facts will be used in the next section.

We now choose the normalization of \( R_a, R_b \) as
\[
R_r(f, g) |_{g=g(z)} = \frac{F(f, \frac{K_2}{z})\alpha |_{g=g(z)} U(\frac{K_2}{z})}{F(f, z)\alpha |_{g=g(z)} U(\frac{K_2}{z})}, \tag{7.41}
\]
\begin{align*}
\text{for } r = a, b.
\end{align*}

For a degree \((1, n)\) polynomial \( P(f, g) \) vanishing at \((f(v_i), g(v_i))_{i=1}^{\geq 1, 4} \quad (\prod_{i=1}^{\geq 1, 4} v_i = \kappa_1 \kappa_2) \), we have
\[
P(f, g(z)) = \text{const.} \times \frac{F(f, \frac{K_2}{z})p(z) - F(f, z)p(\frac{K_2}{z})}{g_a(z)'^{\frac{K_2}{z}}} \quad p(z) = \prod_{i=1}^{\geq 1, 4} \frac{\kappa_1}{\kappa_1}. \tag{7.36}
\]
Then from (7.34) it turns out that \( w \) can be expressed in the form

\[
w = \frac{R_a(f, g)}{\varphi(f, g)} = \frac{S_a(\overline{f}, g)}{\psi(\overline{f}, g)},
\]

(7.42)

where \( \varphi(f, g) = 0 \) is the reference curve \( C_0 \) of bidegree \((2, 2)\) parametrized by \((f, g) = (f(z), g(z))\) (see remark 5.2). Similarly, \( \psi(\overline{f}, g) = 0 \) is a bidegree \((2, 2)\) curve parametrized by \((\overline{f}, g) = (\overline{f}(z), g(z))\). Here we normalize them as

\[
\varphi(f, g)|_{g = g(z)} = \frac{F(f, z)F\left(f, \frac{\kappa_2}{z}\right)}{\left[\begin{array}{c}
\kappa_1 \\
\kappa_2
\end{array}\right] q\kappa_2 g_a(b)^2 g_a(z)^2},
\]

\[
\psi(\overline{f}, g)|_{g = g(z)} = \frac{F(\overline{f}, z)F\left(\overline{f}, \frac{\kappa_2}{z}\right)}{\left[\begin{array}{c}
\kappa_1 \\
\kappa_2
\end{array}\right] q\kappa_2 g_a(b)^2 g_a(z)^2}.
\]

(7.43)

From (7.41) we have

\[
R_a(f, g)\overline{F}(\overline{f}, z)|_{g = g(z)} = R_a(f, g)\left(\overline{f}(z)\frac{R_b(f, g)}{R_a(f, g)} - f_b(z)\right)|_{g = g(z)}
\]

\[
= \overline{f}(z)R_b(f, g) - f_b(z)R_a(f, g)|_{g = g(z)}
\]

\[
= \frac{F\left(f, \frac{\kappa_2}{z}\right)U(z)}{\overline{f}(b)\overline{f}(\frac{\kappa_2}{z})g_a(z)^2} \left\{ f_b(z)\left(\frac{\kappa_2}{z}\right) - \overline{f}(z)\left(\frac{\kappa_2}{z}\right) f_b(z) \right\}
\]

\[
= \frac{F\left(f, \frac{\kappa_2}{z}\right)U(z)}{g_a(z)^2},
\]

(7.44)

where we have used the Riemann relation and (7.28) in the last equality. Hence

\[
\frac{R_a(f, g)}{\varphi(f, g)}|_{g = g(z)} = \left[\begin{array}{c}
\kappa_1 \\
\kappa_2
\end{array}\right] q\kappa_2 g_a(b)^2 U(z)
\]

\[
F(f, z)\overline{F}(\overline{f}, z)g_a(z)^2 = w|_{g = g(z)} \quad \text{for } g = g(z),
\]

(7.45)

as required by (7.34). The second relation of (7.42) can be derived in a similar way. From the expression of \( \overline{f} \) and \( w \) in (7.40) and (7.42), \( w\varphi = R_a(f, g) \) and \( w\varphi\overline{f} = R_b(f, g) \) are both polynomials of bidegree \((1, 4)\).

### 7.2.2. Sufficiency for compatibility

The relations (7.35), (7.36) and (7.34) are not only necessary but also sufficient for the compatibility in the sense of section 7.1. To see this, we construct the \( L_4(z) \) equation by eliminating \( \overline{f}(z) \) and \( \overline{f}(z/q) \) from \( L_2(z) \), \( L_2(z/q) \) and \( L_3(z/q) \):
\begin{align*}
L_1(z) &= \frac{\left[ q \frac{\kappa_1}{z^2} \right] w F \left( T, \frac{z}{q} \right)}{G(g, z) G\left( g, \frac{\kappa_1}{z} \right)} y(z) + \frac{U\left( \frac{z}{q} \right)}{G\left( g, \frac{\kappa_1}{z} \right)} \left\{ y\left( \frac{z}{q} \right) - \frac{G\left( g, \frac{\kappa_1}{z} \right)}{G\left( g, \frac{z}{q} \right)} y(z) \right\} \\
&+ \frac{U\left( \frac{\kappa_1}{z^2} \right) F(f, z)}{G\left( g, \frac{\kappa_1}{z} \right)} \left\{ y(qz) - \frac{G\left( g, \frac{\kappa_1}{z} \right)}{G\left( g, \frac{z}{q} \right)} y(z) \right\} = 0.
\end{align*}

Here, the variables \( w, \tilde{T} \) in (7.46) should be viewed as functions of \( (f, g) \) which are determined above in (7.40) and (7.42). Similarly, by eliminating \( y(z) \) and \( y(qz) \) from \( L_z \), we obtain

\begin{align*}
L_4(z) &= \frac{\left[ \frac{\kappa_1}{z^2} \right] w F(f, z)}{G(g, z) G\left( g, \frac{\kappa_1}{z} \right)} \tilde{y}(z) + \frac{U\left( \frac{z}{q} \right)}{G\left( g, \frac{\kappa_1}{z} \right)} \left\{ \tilde{y}\left( \frac{z}{q} \right) - \frac{G\left( g, \frac{\kappa_1}{z} \right)}{G\left( g, \frac{z}{q} \right)} \tilde{y}(z) \right\} \\
&+ \frac{U\left( \frac{\kappa_1}{qz^2} \right) F(T, z)}{G\left( g, \frac{\kappa_1}{z} \right)} \left\{ \tilde{y}(qz) - \frac{G\left( g, \frac{\kappa_1}{z} \right)}{G\left( g, \frac{z}{q} \right)} \tilde{y}(z) \right\} = 0.
\end{align*}

The compatibility means \( L_4(z) \propto T^{-1}(L_4(z)) \). We prove this compatibility assuming (7.34) and \( \varepsilon \cdot P(E_{E(1)} \cdot P) \) (7.35) and (7.36). A convenient way is to use the geometric characterizations of \( L_1(z) \) and \( L_4(z) \) as rational functions of \( f, g \) [140]. If we multiply the rational function \( L_4(z) \) by a factor \( F(f, z) F\left( \frac{f}{q}, \frac{\kappa_1}{qz} \right) \varphi(f, g) \),

\begin{equation}
\bar{L}_4(z) = F(f, z) F\left( \frac{f}{q}, \frac{\kappa_1}{qz} \right) \varphi(f, g) L_4(z),
\end{equation}

becomes a polynomial of bidegree \((3, 4)\) divided by the factor \( G\left( g, \frac{\kappa_1}{z} \right) G\left( g, \frac{z}{q} \right) \). It is actually a polynomial of bidegree \((3, 2)\) since the residues at \( g = \frac{z}{q}, \frac{\kappa_1}{z} \) vanish by (7.34), which we denote by \( P_{02}(f, g) \). This polynomial is characterized by the vanishing condition at the following 12 points:
\begin{align}
(f(v), g(v))_{v=v_h, \ldots, v_h+w,} \quad (f(x), \gamma_x)_{x=x,} \quad \text{where} \quad \frac{G(\gamma_x, \frac{k_1}{x})}{G(\gamma_x, x)} = \frac{y(x)}{y(qx)}, \quad (7.49)
\end{align}

It is directly seen that \( P_{32}(f, g) \) vanishes at the last two points by noticing \( F(f(z), z) = G(g(z), z) = 0 \). Moreover, one can verify that \( P_{32}(f, g) \) vanishes at the first eight points as follows. We have

\begin{align}
P_{32}(f, g) &= \frac{\frac{q^3k_1}{z^2}F(f, z)F\left(\frac{z}{q}, \frac{k_1}{z}\right)A(f, g, z)}{G\left(\frac{z}{q}, \frac{k_1}{z}\right)G\left(\frac{z}{q}, \frac{k_1}{z}\right)} y(z) \\
&\quad + \frac{F(f, z)\varphi(f, g)U\left(\frac{z}{q}, \frac{k_1}{z}\right)}{\frac{q^3k_1}{z^2}} \left\{ y\left(\frac{z}{q}\right) - \frac{G\left(\frac{z}{q}, \frac{k_1}{z}\right)}{G\left(\frac{z}{q}, \frac{k_1}{z}\right)} y(z) \right\} \\
&\quad + \frac{F\left(\frac{z}{q}, \frac{k_1}{z}\right)\varphi(f, g)U\left(\frac{k_1}{z}\right)}{\frac{k_1}{z^2}} \left\{ y(qz) - \frac{G(g, z)}{G\left(\frac{k_1}{z}\right)} y(z) \right\}. \quad (7.50)
\end{align}

Here we have from (7.40) and (7.42)

\begin{align}
A(f, g, z) = \varphi(f, g)wF\left(\frac{z}{q}, \frac{k_1}{z}\right) = \overline{\varphi}\left(\frac{z}{q}\right)R_f(f, g) - \overline{\varphi}\left(\frac{k_1}{z}\right)R_s(f, g), \quad (7.51)
\end{align}

which vanishes at the eight points \((f(v_i), g(v_i))_{i=1, \ldots, 8}\) as mentioned in the end of section 7.2.1. It is also obvious that the second and third terms of (7.50), and thus \( P_{32}(f, g) \), vanish at those eight points. It is also clear that \( P_{32}(f, g) \) vanishes at \((f(v), g(v))_{v=z, \ldots, z + 8}\) because of the factors \( F(f, z), F\left(\frac{z}{q}\right) = F\left(\frac{qz}{z}\right) \) (see (7.28)) in the first term (vanishing of the second and third term is obvious due to \( \varphi(f, g) \)).

Similarly, one can show that \( L_4(z) = 0 \) is a curve of bidegree \((3, 2)\) in \((\mathcal{T}, g)\) passing through

\begin{align}
(\mathcal{T}(v), g(v))_{v=v_h, \ldots, v_h+w,} \quad (\mathcal{T}(x), \gamma_x)_{x=x,} \quad \text{where} \quad \frac{G(\gamma_x', x)}{G(\gamma_x', qx)} \frac{U\left(\frac{k_1}{qx}\right)}{U\left(\frac{k_1}{x}\right)} = \frac{\varphi(x)}{\varphi(qx)}, \quad (7.52)
\end{align}

and hence, \( T^{-1}(L_4(z)) = 0 \) is a bidegree \((3, 2)\) curve in \((f, g)\) passing through

\begin{align}
(f(v), g(v))_{v=v_h, \ldots, v_h+w,} \quad (f(x), \gamma_x')_{x=x,} \quad \text{where} \quad \frac{G(\gamma_x', x)}{G(\gamma_x', qx)} \frac{U\left(\frac{k_1}{qx}\right)}{U\left(\frac{k_1}{x}\right)} = \frac{y(x)}{y(qx)}. \quad (7.53)
\end{align}

We denote by \( Q_{32}(f, g) \) a polynomial in \((f, g)\) of bidegree \((3, 2)\) defining the curve \( T^{-1}(L_4(z)) = 0 \).
Our remaining task is to express $Q_{32}(f, g)$ as a rational function in $(f, g)$ and to compare it with $P_{32}(f, g)$. To this end, we express $g$ by $g = g(f, g) = \frac{B_{41}(f, g)}{A_{41}(f, g)}$, where $A_{41}(f, g)$, $B_{41}(f, g)$ are polynomials of bidegree $(4, 1)$ vanishing at the eight points $(f(v_i), g(v_i))_{i=1, \ldots, 8}$, as explained above. We have

$$Q_{32}\left(f, \frac{B_{41}(f, g)}{A_{41}(f, g)}\right) = \frac{P_{11,2}(f, g)}{A_{41}(f, g)^2},$$

(7.54)

by the degree counting, where $P_{11,2}(f, g)$ is a polynomial of bidegree $(11, 2)$. Writing $Q_{32}(f, g)$ as

$$Q_{32}(f, g) = \alpha_3(f) g^2 + \beta_3(f) g + \gamma_3(f),$$

(7.55)

we see that $P_{11,2}(f, g)$ is expressed as

$$P_{11,2}(f, g) = \alpha_3(f) B_{41}(f, g)^2 + \beta_3(f) B_{41}(f, g) A_{41}(f, g) + \gamma_3(f) A_{41}(f, g)^2.$$

(7.56)

Hence we see that $P_{11,2}(f, g)$ has zeros with multiplicity 2 at the eight points $(f(v_i), g(v_i))_{i=1, \ldots, 8}$. We also note that if we set $z = v_i$ in (7.36) then the numerator of the left-hand side vanishes because of $U(z)$, and thus $G(g, v_i)$ in the denominator must vanish. This implies that in case of $f = f(v_i)$ it follows that $g = g(v_i)$ regardless of the generic value of $g (i = 1, \ldots, 8)$. Then we have

$$Q_{32}(f, g) |_{f=f(v_i)} = Q_{32}(f(v_i), g(v_i)) = 0,$$

(7.57)

which means that $Q_{32}(f, g)$ is divisible by $\prod_{i=1}^{8} (f - f(v_i))$. Hence we have the factorization

$$Q_{32}\left(f, \frac{B_{41}(f, g)}{A_{41}(f, g)}\right) = \prod_{i=1}^{8} \frac{f - f(v_i)}{A_{41}(f, g)^2} \hat{P}_{32}(f, g).$$

(7.58)

where $\hat{P}_{32}(f, g)$ is a certain polynomial of bidegree $(3, 2)$ and linear in $y(z), y(qz), y\left(\frac{z}{q}\right)$. Our goal is to show that $\hat{P}_{32}(f, g)$ is actually proportional to $P_{32}(f, g)$. Noticing that $P_{11,2}(f, g)$ has zeros with multiplicity 2 at $(f(v_i), g(v_i))_{i=1, \ldots, 8}$, it follows that $\hat{P}_{32}(f, g)$ also vanishes at those points. One can also observe from (7.36) that $g = g\left(\frac{z}{q}\right)$ for $(f, g) = (f(v), g(v))$ when $v = z, \frac{z}{q}$, and $g = \gamma_i^v$ for $(f, g) = (f(x), \gamma_i)$ when $x = z, \frac{z}{q}$. This implies that $Q_{32}(f, g)$ and thus $\hat{P}_{32}(f, g)$ vanish at the remaining four points as desired. Hence we have shown that $\hat{P}_{32}(f, g)$ is equal to $P_{32}(f, g)$ up to constant multiple.

7.2.3. Case of $q$-$\mathcal{P}(E_8^{(1)})$. We continuously use the parameters, $\kappa_1, \kappa_2, v_1, \ldots, v_8$ with $\kappa_1^2 \kappa_2^2 = q \prod_{i=1}^{8} v_i$, and the time evolution $T : \kappa_1 \mapsto \frac{\kappa_1}{q}, \kappa_2 \mapsto q \kappa_2$. In $q$-$E_8$ case, we put $f(z) = z + \frac{\kappa_1}{\kappa_2}, g(z) = z + \frac{\kappa_2}{\kappa_1}, U(z) = z^{-4} \prod_{i=1}^{8} (z - v_i).

We start with the contiguity type equations:

$$L_2(z) : \begin{cases} g - g\left(\frac{\kappa_1}{z}\right) y(qz) - \{g - g(z)\} y(z) - \left(z - \frac{\kappa_1}{z}\right) (f - f(z)) \gamma(z) = 0, 
\end{cases}$$

(7.59)
\[ L_3(z) : \left\{ g - g \left( \frac{\kappa_1}{qz} \right) \right\} U(z) \Phi(z) - \left\{ g - g(z) \right\} U \left( \frac{\kappa_1}{qz} \right) \Phi(qz) - w \left( z - \frac{\kappa_1}{qz} \right) \left( \Phi - \Phi(z) \right) y(qz) = 0. \] (7.60)

From the compatibility of these equations, the \( q \)-P\((E_8^{(1)}) \) Painlevé equation is derived as follows. Putting \( g = g(z) \) in \( L_2(z) \) and \( L_3(z) \), we have

\[
\frac{w \left( f - f(z) \right) \left( \Phi - \Phi(z) \right)}{U(z)} = \left\{ g - g \left( \frac{\kappa_1}{z} \right) \right\} \left\{ g - g \left( \frac{\kappa_1}{qz} \right) \right\} \left( z - \frac{\kappa_1}{z} \right) \left( z - \frac{\kappa_1}{qz} \right)
\]

\[
= \left( \frac{\kappa_1 - \kappa_2}{\kappa_1^2} \right) \left( \frac{\kappa_1 - q \kappa_2}{\kappa_1} \right), \text{ for } g = g(z). \] (7.61)

This relation holds also if \( z \) is replaced by \( \frac{\kappa_2}{z} \), and taking the ratio of these two expressions, we have

\[
\frac{\left\{ f - f \left( \frac{\kappa_2}{z} \right) \right\} \left\{ \Phi - \Phi \left( \frac{\kappa_2}{z} \right) \right\} U(z)}{\left( f - f(z) \right) \left( \Phi - \Phi(z) \right) U \left( \frac{\kappa_2}{z} \right)} = 1, \text{ for } g = g(z), \] (7.62)

along with

\[
w = \frac{\left( \frac{\kappa_1 - \kappa_2}{\kappa_1^2} \right) q \kappa_2 \left( z - \frac{\kappa_2}{z} \right)}{U(z)} \left[ \frac{U(z)}{f - f(z)} - \frac{U \left( \frac{\kappa_2}{z} \right)}{f - f \left( \frac{\kappa_2}{z} \right)} \right]
\]

\[
= \left( \frac{\kappa_1 - \kappa_2}{\kappa_1^2} \right) \left( \frac{\kappa_1 - q \kappa_2}{\kappa_1} \right) \left[ \frac{U(z)}{\Phi - \Phi(z)} - \frac{U \left( \frac{\kappa_2}{z} \right)}{\Phi - \Phi \left( \frac{\kappa_2}{z} \right)} \right]. \] (7.63)

On the other hand, putting \( f = f(z) \) in \( L_2(z) \) and \( L_3(z) \), we have

\[
\left\{ g - g \left( \frac{\kappa_1}{z} \right) \right\} \left\{ g - g \left( \frac{\kappa_1}{qz} \right) \right\} U(z)
\]

\[
\left\{ g - g(z) \right\} \left\{ \frac{\kappa_1}{qz} \right\} U \left( \frac{\kappa_1}{z} \right) \right\} U(z)
\]

\[
= 1, \text{ for } f = f(z). \] (7.64)

These relations determine the variables \( \Phi, g, w \) as rational functions in \( f, g \).\(^6\)

\(^6\) A bidegree \( (1, n) \) polynomial \( P(f, g) \) vanishing at \( (f(v), g(v)) \) is given by

\[ P(f, g) = \prod_{i=1}^{2n+2} (f(v_i) - g(v_i)) \]

\[ p(v) = v^{n+1} \prod_{i=1}^{2n+2} (v - v_i). \]
In a similar way to the elliptic case, we have

\[
L_4(z) : \begin{cases}
\frac{w}{g - g(z)} \left\{ f - f(z) \right\} \gamma(z) + \frac{U(z)}{g - g(z)} \left\{ f - f(z) \right\} \gamma(z) - \frac{g - g(z)}{g - g(z)} y(z) \\
\frac{1}{g - g(z)} \left\{ f - f(z) \right\} \gamma(z) - \frac{g - g(z)}{g - g(z)} y(z)
\end{cases}
\]

\[+ \frac{1}{g - g(z)} \left\{ f - f(z) \right\} \gamma(z) - \frac{g - g(z)}{g - g(z)} y(z) = 0.
\]

The compatibility \( L_4(z) \propto T^{-1}(L_4(z)) \) is confirmed by using the following geometric characterizations.

\( L_4(z) = 0 \): the curve of bidegree \((3, 2)\) in \((f, g)\) passing through

\[
(f(v), g(v))_{v = \ldots, y_v}, \quad (f(x), \gamma_x)_{x = \ldots, \frac{\kappa_1}{x}, 0}, \quad \frac{\gamma_x - g\left(\frac{\kappa_1}{x}\right)}{\gamma_x - g(x)} = \frac{y(x)}{y(qx)}.
\]

\( L_4(z) = 0 \): the curve of bidegree \((3, 2)\) in \(\left\{f, g\right\}\) passing through

\[
\left\{f(v), g(v)\right\}_{v = \ldots, y_v, \frac{\kappa_1}{q}}, \quad \left\{f(x), \gamma'_x\right\}_{x = \ldots, \frac{\kappa_1}{q}, \frac{\kappa_1}{q}}, \quad \frac{\gamma'_x - g\left(\frac{\kappa_1}{q}\right)}{\gamma'_x - g(x)} = \frac{\sigma(x)}{\sigma(qx)}.
\]

\( T^{-1}(L_4(z)) = 0 \): the curve of bidegree \((3, 2)\) in \((f, g)\) passing through

\[
(f(v), g(v))_{v = \ldots, y_v, \frac{\kappa_1}{x}, 0}, \quad (f(x), \gamma''_x)_{x = \ldots, \frac{\kappa_1}{x}, \frac{\kappa_1}{x}}, \quad \frac{\gamma''_x - g\left(\frac{\kappa_1}{x}\right)}{\gamma''_x - g(x)} = \frac{y(x)}{y(qx)}.
\]
8. Basic data for discrete Painlevé equations

In this section we provide with basic data for all discrete Painlevé equations of QRT type: equations, point configurations/root data, Weyl group representations, Lax pairs and hypergeometric solutions.

Remark 8.1. In the following, we also use the symbols \( E^{(1)}_1 = (A_2 + A_1)^{(1)} \) and \( E^{(1)}_2 = (A_1 + A_1)^{(1)} \) to simplify the notation. In these cases, the labels \((a)\) and \((b)\) are used to discriminate two inequivalent equations associated with different realizations of the same symmetry/surface type. These may be realized as equations with respect to two different directions (see [56] for \( q\)-PII and \( q\)-PIV of the case \( E^{(1)}_6/A^{(1)}_6 \)). In this paper, however, we formulate them in terms of different point configurations in order to represent the equations in standard form as in the literature. In table 4, we omit the case of \( A^{(1)}_0/A^{(1)}_0 \) which allows \( P_I: y'' = 6y^3 + t \) as a continuous flow, since the surface cannot be realized by eight point blowing-up of \( \mathbb{P} \times \mathbb{P} \) and there is no discrete symmetry. We also omit the case of \( A^{(1)}_0/A^{(1)}_0 \) which has no discrete flow.

8.1. Discrete Painlevé equations

In this subsection, for each point configuration in the table 4 we give an explicit form of the discrete Painlevé equation with respect to the QRT direction. We use the symbols \([e, q, d]-P(surm/\text{surface type})\) for reference to the (discrete) Painlevé equations; the first symbol represents the type of time evolution, \(e\): elliptic, \(q\): multiplicative (\(q\)-difference), \(d\): additive (difference), none: continuous (differential).

Basically we use below the parameters \( \kappa_i (i = 1, 2) \) and \( v_i (i = 1, \ldots, 8) \), and \( f, g \) denote dependent variables. Also \( \tau \) is the time evolution such that

\[
\tau_i = \frac{\kappa_i}{q}, \quad \tau_2 = q\kappa_2, \quad \tau_i = v_i (i = 1, \ldots, 8), \quad \kappa_1^2\kappa_2^2 = q \prod_{i=1}^{8} v_i, \quad (8.1)
\]

for elliptic and multiplicative cases and

\[
\tau_1 = \kappa_1 - \delta, \quad \tau_2 = \kappa_2 + \delta, \quad \tau_i = v_i (i = 1, \ldots, 8), \quad 2(\kappa_1 + \kappa_2) = \delta + \sum_{i=1}^{8} v_i \quad (8.2)
\]

for additive cases. Relation to the parameters \( h_i (i = 1, 2) \) and \( e_i (i = 1, \ldots, 8) \) used in section 5 is given in remark 5.8.

8.1.1. e-P\((E^{(1)}_6/A^{(1)}_6)\).

\[
\begin{align*}
\left\{ f - f \left( \frac{\kappa_2}{t} \right) \right\} \left\{ \tilde{f} - \tilde{f} \left( \frac{\kappa_2}{t} \right) \right\} &= \frac{f_a(t)\tilde{f}_a(t)}{f_a \left( \frac{\kappa_2}{t} \right) \tilde{f}_a \left( \frac{\kappa_2}{t} \right)} U \left( \frac{\kappa_2}{t} \right) U(t), \\
\left\{ g - g \left( \frac{\kappa_1}{s} \right) \right\} \left\{ \tilde{g} - \tilde{g} \left( \frac{\kappa_1}{s} \right) \right\} &= \frac{g_a(s)\tilde{g}_a(s)}{g_a \left( \frac{\kappa_1}{s} \right) \tilde{g}_a \left( \frac{\kappa_1}{s} \right)} U \left( \frac{\kappa_1}{s} \right) U(s). \quad (8.3)
\end{align*}
\]
where \( t \) and \( s \) are the variables such that \( g = g(t), f = f(s) \),

\[
\begin{align*}
    f_a(z) &= \left[ \frac{\alpha}{z} \right] \left[ \frac{\Gamma_1}{z} \right], \\
    g_a(z) &= \left[ \frac{\alpha}{z} \right] \left[ \frac{\Gamma_2}{z} \right], \\
    f(z) &= \frac{f_a(z)}{f_a(z)}, \\
    g(z) &= \frac{g_a(z)}{g_a(z)}, \\
    U(z) &= \prod_{i=1}^{8} \left[ \frac{v_i}{z} \right].
\end{align*}
\]  
(8.4)

\([z]\) is the multiplicative theta function given in section 7.2.1, and \( a, b \) are arbitrary.

8.1.2. \( q\)-\( P(E_6^{(1)}/A_0^{(1)}) \).

\[
\begin{align*}
    \left\{ f - f \left( \frac{\kappa_2}{l} \right) \right\} \left\{ f - \frac{\kappa_2}{l} \right\} &= U \left( \frac{\kappa_2}{l} \right), \\
    g = g(t),
\end{align*}
\]

\[
\begin{align*}
    \left\{ g - g \left( \frac{\kappa_1}{s} \right) \right\} \left\{ g - \frac{\kappa_1}{s} \right\} &= U \left( \frac{\kappa_1}{s} \right), \\
    f = f(s),
\end{align*}
\]  
(8.5)

where \( t \) and \( s \) are the variables such that \( g = g(t), f = f(s) \),

\[
U(z) = \frac{1}{z} \prod_{i=1}^{8} (z - v_i), \quad f(z) = z + \frac{\kappa_1}{z}, \quad g(z) = z + \frac{\kappa_2}{z}.
\]  
(8.6)

8.1.3. \( q\)-\( P(E_7^{(1)}/A_1^{(1)}) \).

\[
\begin{align*}
    \frac{fg - \frac{\kappa_1}{\kappa_2}}{(fg - 1)(fg - 1)} &= \prod_{i=1}^{8} \left( g - \frac{v_i}{\kappa_2} \right), \\
    \frac{fg - \frac{\kappa_1}{\kappa_2}}{fg - 1} &= \prod_{i=1}^{8} \left( f - \frac{v_i}{\kappa_2} \right).
\end{align*}
\]  
(8.7)

8.1.4. \( q\)-\( P(E_8^{(1)}/A_2^{(1)}) \).

\[
\begin{align*}
    \frac{(fg - 1)(fg - 1)}{fg} &= \prod_{i=1}^{8} \left( g - \frac{1}{v_i} \right), \\
    \frac{(fg - 1)(fg - 1)}{fg} &= \prod_{i=1}^{8} \left( f - \frac{v_i}{\kappa_2} \right).
\end{align*}
\]  
(8.8)
Table 4. List of data associated with possible point configurations.

| Symmetry/ surface | Equation | Point configuration/ root data | Weyl group representations | Lax pair | Hypergeometric solution |
|-------------------|----------|-------------------------------|---------------------------|----------|------------------------|
| Elliptic $E_8^{(1)}/A_8^{(1)}$ | Section 8.1.1 | Section 8.2.1 | (5.22) | Section 7.2 | Section 8.6.1 |
| Multiplicative $E_8^{(1)}/A_8^{(1)}$ | Section 8.1.2 | Section 8.2.2 | Section 8.4.1 | Section 8.5.1 | Section 8.6.2 |
| $E_7^{(1)}/A_7^{(1)}$ | Section 8.1.3 | Section 8.2.3 | Section 8.4.2 | Section 8.5.2 | Section 8.6.3 |
| $E_6^{(1)}/A_6^{(1)}$ | Section 8.1.4 | Section 8.2.4 | Section 8.4.3 | Section 8.5.3 | Section 8.6.4 |
| $D_5^{(3)}/A_3^{(1)}$ | Section 8.1.5 | Section 8.2.5 | Section 8.4.4 | Section 8.5.4 | Section 8.6.5 |
| $A_4^{(1)}/A_4^{(1)}$ | Section 8.1.6 | Section 8.2.6 | Section 8.4.5 | Section 8.5.5 | Section 8.6.6 |
| $E_6^{(1)}/A_3^{(1)}$ | (a) Section 8.1.7 | Section 8.2.7 | Section 8.4.6 | Section 8.5.6 | Section 8.6.7 |
| $E_6^{(1)}/A_5^{(3)}$ | (b) Section 8.1.10 | Section 8.2.10 | Section 8.4.7 | Section 8.5.7 | Section 8.6.8 |
| $E_7^{(1)}/A_6^{(1)}$ | (a) Section 8.1.8 | Section 8.2.8 | Section 8.4.6 | Section 8.5.8 | Section 8.6.9 |
| $E_7^{(1)}/A_6^{(1)}$ | (b) Section 8.1.11 | Section 8.2.11 | Section 8.4.10 | Section 8.5.10 | None |
| $A_3^{(1)}/A_1^{(1)}$ | Section 8.1.9 | Section 8.2.9 | Section 8.4.11 | Section 8.5.8 | None |
| $A_3^{(1)}/A_1^{(1)}$ | $[\alpha]^8=1$ Section 8.1.12 | Section 8.2.12 | Section 8.4.10 | Section 8.5.11 | None |
| Additive $E_8^{(1)}/A_9^{(1)}$ | Section 8.1.13 | Section 8.2.14 | Section 8.4.12 | Section 8.5.12 | Section 8.6.10 |
| $E_7^{(1)}/A_1^{(1)}$ | Section 8.1.14 | Section 8.2.15 | Section 8.4.13 | Section 8.5.13 | Section 8.6.11 |
| $E_6^{(1)}/A_1^{(1)}$ | Section 8.1.15 | Section 8.2.16 | Section 8.4.14 | Section 8.5.14 | Section 8.6.12 |
| $D_5^{(3)}/D_1^{(1)}$ | Section 8.1.16 | Section 8.2.17 | Section 8.4.15 | Section 8.5.15 | Section 8.6.13 |
| $A_3^{(1)}/D_1^{(1)}$ | Section 8.1.17 | Section 8.2.18 | Section 8.4.16 | Section 8.5.16 | Section 8.6.14 |
| $2A_1^{(1)}/D_1^{(1)}$ | Section 8.1.20 | Section 8.2.19 | Section 8.4.17 | Section 8.5.17 | Section 8.6.16 |
| $A_3^{(1)}/E_1^{(1)}$ | Section 8.1.18 | Section 8.2.22 | Section 8.4.20 | Section 8.5.20 | Section 8.6.15 |
| $A_3^{(1)}/D_1^{(1)}$ | $[\alpha]^4=1$ Section 8.1.21 | Section 8.2.20 | Section 8.4.18 | Section 8.5.18 | None |
| $A_3^{(1)}/E_1^{(1)}$ | Section 8.1.19 | Section 8.2.23 | Section 8.4.21 | Section 8.5.21 | None |
| $A_3^{(1)}/D_1^{(1)}$ | Section 8.1.22 | Section 8.2.21 | Section 8.4.19 | Section 8.5.19 | None |
\[ f = \frac{\prod_{i=5}^{8} \left(g - \frac{v_i}{\kappa_2}\right)}{v_1 v_2} \quad g = \frac{\prod_{i=3}^{4} \left(f - \frac{\kappa_1}{v_i}\right)}{v_1 v_2} \quad (8.9) \]

\[ f = -v_2 v_3 v_4 \quad g = \frac{\prod_{i=3}^{4} \left(g - \frac{v_i}{\kappa_2}\right)}{g - \frac{1}{v_1}} \quad (8.10) \]

\[ f = -v_2 v_3 v_4 \quad g = \frac{\kappa_3}{v_1 v_2 v_3 v_4} \quad (8.11) \]

\[ f = -v_2 v_3 v_4 \quad g = \frac{\kappa_3}{v_1 v_2 v_3 v_4} \quad (8.12) \]

\[ f = -v_2 v_3 v_4 \quad g = \frac{\kappa_3}{v_1 v_2 v_3 v_4} \quad (8.13) \]

\[ f = -v_2 v_3 v_4 \quad g = \frac{\prod_{i=1}^{8} \left(g - \frac{v_i}{\kappa_2}\right)}{g - \frac{1}{v_1}} \quad (8.14) \]

\[ f = -v_2 v_3 v_4 \quad g = \frac{\prod_{i=1}^{8} \left(f - \frac{\kappa_1}{v_i}\right)}{f - v_4} \quad (8.15) \]
8.1.12. q-P(\(A^{(1)}_1/A^{(1)}_2\)).

\[
\begin{align*}
\frac{f}{g} = v_1v_2v_3v_4g^2, & \quad g = -v_1v_2v_3\frac{f - \kappa_4}{v_k}. \\
\end{align*}
\]

(8.16)

The cases \(E^{(1)}_3(a), E^{(1)}_4(b)\) have the same symmetry/surface type \((A_2 + A_4)^{(1)}/A^{(1)}_2\), while the time evolutions \(T\) belong to the inequivalent directions: \(A^{(1)}_1\) for \((a)\): q-P_{IV} and \(A^{(1)}_2\) for \((b)\): q-P_{III} [44, 46, 56, 125]. Similarly, the cases \(E^{(1)}_2(a), E^{(1)}_2(b)\) have the same symmetry/surface type \((A_1 + A_1)^{(1)}/A^{(1)}_0\) with different directions \(A^{(1)}_1\) for \((a)\): q-P_{II} [33, 80] and \(A^{(1)}_1\) for \((b)\).

8.1.13. d-P(\(E^{(1)}_8/A^{(1)}_0\)).

\[
\begin{align*}
\frac{(f - f(t))(\overline{f} - f(t))}{(f - f(t))(\overline{f} - (\kappa_2 - t))} &= U(t), & \quad \overline{g} = g(t), \\
\frac{(g - g(t))((g - g(t)))}{(g - g(t))((g - \kappa_1 - s))} &= U(s), & \quad \overline{f} = f(s), \\
\end{align*}
\]

(8.17)

where \(t\) and \(s\) are the variables such that \(g = g(t), f = f(s)\).

\[
U(z) = \prod_{i=1}^{8} (z - \kappa_i), \quad f(z) = z(z - \kappa_i), \quad g(z) = z(z - \kappa_2). 
\]

(8.18)

8.1.14. d-P(\(E^{(1)}_7/A^{(1)}_1\)).

\[
\begin{align*}
\frac{(f + g - \kappa_1 + \kappa_2)(\overline{f} + g - \kappa_1 + \kappa_2 + \delta)}{(f + g)(\overline{f} + g)} &= \prod_{i=5}^{8} (g + \kappa_2 - \kappa_i), \\
\frac{(f + g - \kappa_1 + \kappa_2)(f + g - \kappa_1 + \kappa_2 - \delta)}{(f + g)(f + g)} &= \prod_{i=5}^{8} (f - \kappa_1 + \kappa_i). \\
\end{align*}
\]

(8.19)

8.1.15. d-P(\(E^{(1)}_6/A^{(1)}_2\)).

\[
\begin{align*}
(f + g)(\overline{f} + g) &= \prod_{i=1}^{4} (g + \kappa_2 - \kappa_i), \\
\frac{(f + g)(f + g)}{\prod_{i=5}^{8} (g + \kappa_2 - \kappa_i)} &= \prod_{i=5}^{8} (f - \kappa_1 + \kappa_i). \\
\end{align*}
\]

(8.20)
The following cases admit the Painlevé equations as the continuous flows commuting with the discrete time evolutions. For these equations we use the parameters $a_0, a_1, \ldots$ corresponding to the simple roots, as can be found in the literature [81, 127]. We use two types of inhomogeneous coordinates $(q, p)$ and $(f, g) = (q, qp)$ of $\mathbb{P}_1 \times \mathbb{P}_1$ depending on the situations.

### 8.1.16. **d-P(D_4^{(1)}/D_4^{(1)})** and **P(D_4^{(1)}/D_4^{(1)})** ($P_6$).

(i) Discrete Painlevé equation

$$\tau_0 = a_0 - 1, \quad \tau_2 = a_2 + 1, \quad \tau_3 = a_3 - 1, \quad \tau_4 = a_4 + 1, \quad a_0 + a_1 + 2a_2 + a_3 + a_4 = 1,$$

$$f f = \frac{t g(g - a_1)}{(g + a_2)(g + a_1 + a_2)}, \quad g + g = a_0 + a_3 + a_4 + \frac{a_3}{f - 1} + \frac{a_0}{f - t}. \quad (8.21)$$

(ii) Painlevé differential equation: $P_{V1}$ ($y = q$)

$$\frac{d^2 y}{dt^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{y - t} \right) \frac{dy}{dt} + \frac{y(y - 1)(y - t)}{t^2(t^2 - 1)} \left\{ \alpha + \frac{\beta t}{y^2} + \frac{\gamma(t - 1)}{(y - 1)^2} + \frac{\delta t(t - 1)}{(y - t)^2} \right\}$$

$$\alpha = \frac{a_1}{2}, \quad \beta = -\frac{a_2}{2}, \quad \gamma = \frac{a_3}{2}, \quad \delta = -\frac{a_3 - 1}{2}. \quad (8.22)$$

### 8.1.17. **d-P(A_3^{(1)}/D_3^{(1)})** and **P(A_3^{(1)}/D_3^{(1)})** ($P_7$).

(i) Discrete Painlevé equation

$$\tau_1 = a_1 - 1, \quad \tau_2 = a_2 + 1, \quad \tau_3 = a_3 - 1, \quad \tau_4 = a_4 + 1, \quad a_0 + a_1 + a_2 + a_3 + a_4 = 1,$$

$$q + q = 1 - \frac{a_2}{p} - \frac{a_0}{p + t}, \quad p + p = -t + \frac{a_1}{q} + \frac{a_3}{q - 1}. \quad (8.23)$$

(ii) Painlevé differential equation: $P_{V}$ ($y = 1 - \frac{1}{q}$)

$$\frac{d^2 y}{dt^2} = \left( \frac{1}{2y} + \frac{1}{y - 1} \right) \left( \frac{dy}{dt} \right)^2 - \frac{1}{t} \left( \frac{dy}{dt} \right)^2 + \frac{(y - 1)^2}{t^2} \left( a_1 y + \frac{\beta}{y} \right) + \gamma \frac{y}{t} + \delta \frac{y(y + 1)}{y - 1},$$

$$\alpha = \frac{a_1}{2}, \quad \beta = -\frac{a_2}{2}, \quad \gamma = a_0 - a_2, \quad \delta = -\frac{1}{2}. \quad (8.24)$$

### 8.1.18. **d-P(A_2^{(1)}/E_6^{(1)})** and **P(A_2^{(1)}/E_6^{(1)})** ($P_{8}$).

(i) Discrete Painlevé equation

$$\tau_1 = a_1 - 1, \quad \tau_2 = a_2 + 1, \quad a_0 + a_1 + a_2 = 1,$$

$$q + q = p - t - \frac{a_2}{p}, \quad p + p = q + t + \frac{a_1}{q}. \quad (8.25)$$
(ii) Painlevé differential equation: $P_\text{IV} (y = q)$
\[
\frac{d^2 y}{dr^2} = \frac{1}{2y} \left( \frac{dy}{dr} \right)^2 + \frac{3}{2} y^2 + 4ty^2 + 2(t^2 - \alpha) + \frac{\beta}{y},
\]
\[
\alpha = a_0 - a_2, \quad \beta = -2a_1^2.
\] (8.26)

8.1.19. $d$-$P(A_1^{(1)}/E_7^{(1)})$ and $P(A_1^{(1)}/E_7^{(1)})$ ($P_\text{II}$).

(i) Discrete Painlevé equation
\[
\xi_1 = a_1 + 1, \quad a_0 + a_1 = 1,
q = -\frac{a_1}{p}, \quad p + p = 2q^2 + t.
\] (8.27)

(ii) Painlevé differential equation: $P_\text{II} (y = q)$
\[
\frac{d^2 y}{dr^2} = 2y^3 + ty + \alpha,
\]
\[
\alpha = a_1 - \frac{1}{2}.
\] (8.28)

8.1.20. $d$-$P((2A_1)^{(1)}/D_5^{(1)})$ and $P((2A_1)^{(1)}/D_6^{(1)})$ ($P_\text{III}$).

(i) Discrete Painlevé equation
\[
\xi_0 = a_0 + 1, \quad \xi_1 = a_1 + 1, \quad a_0 + a_1 = 1,
q = -\frac{a_0}{p} - \frac{a_1}{p - 1}, \quad p + p = 1 + \frac{1 - a_0 - a_1}{q} - \frac{t}{q^2}.
\] (8.29)

(ii) Painlevé differential equation: $P_\text{III}^{(1)} (y = s, s^2 = t)$
\[
\frac{d^2 y}{ds^2} = \frac{1}{y} \left( \frac{dy}{ds} \right)^2 - \frac{1}{s} \frac{dy}{ds} + \frac{1}{s} (\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y},
\]
\[
\alpha = 4(1 + 2a_0 - 2a_1), \quad \beta = -4(1 + a_0 - a_1), \quad \gamma = 4, \quad \delta = -4.
\] (8.30)

8.1.21. $d$-$P((A_1^{(1)}/D_4^{(1)})$ and $P((A_1^{(1)}/D_4^{(1)})$ ($P_\text{III}^{(1)}$).

(i) Discrete Painlevé equation
\[
\xi_1 = a_1 + 2,
q = -q - \frac{a_1}{p} - \frac{1}{p^2}, \quad p = -p - \frac{a_1 + 1}{q} - \frac{t^2}{q^2}.
\] (8.31)
(i) Painlevé differential equation: $P_{\text{III}}^{(1)}(y = \frac{2}{s}, s^2 = t)$

\[
\frac{d^2y}{ds^2} = \frac{1}{y} \left( \frac{dy}{ds} \right)^2 - \frac{1}{s} \frac{dy}{ds} + \frac{1}{s} (\alpha y^2 + \beta) + \frac{\delta}{y}, \quad \alpha = -8, \quad \beta = -4(a_1 - 1), \quad \delta = -4.
\]

(8.32)

8.1.22. $P(A_0^{(1)}/D_8^{(1)}) (P_{\text{III}}^{(1)}).

(i) There is no discrete discrete Painlevé equation.

(ii) Painlevé differential equation: $P_{\text{III}}^{(1)}(y = \frac{2}{s}, s^2 = t)$

\[
\frac{d^2y}{ds^2} = \frac{1}{y} \left( \frac{dy}{ds} \right)^2 - \frac{1}{s} \frac{dy}{ds} + \frac{1}{s} (-8y^2 + 8).
\]

(8.33)

8.2. Point configurations and root data

In this subsection, we give the list of configurations of eight points on $\mathbb{P}^1 \times \mathbb{P}^1$ relevant to the Painlevé equations in section 8.1.

We denote by $P_{ij}$, $P_{ijk}$, … multiple points where $P_j$ is infinitely near to $P_i$, $P_k$ infinitely near to $P_j$, and so on. Moreover, $(x, y) = (A(\epsilon), B(\epsilon))_n$ in the $(x, y)$ coordinates represents an $n$th order near points around $(A(0), B(0))$. Namely, when we say a curve $F(x, y) = 0$ passes through $(A(\epsilon), B(\epsilon))_n$, it means that the first $n$ coefficients vanish in the $\epsilon$-expansion of $F(A(\epsilon), B(\epsilon))$. We attach schematic pictures of configurations of eight points and associated divisors for each case. In the pictures of configuration of divisors, we use the following notations such as $ij = H_i - E_i - E_j$, $ij = E_i - E_j$, $i = E_i$.

We also give a realization of the root basis $\{a_i\}$, configuration of the divisors of inaccessible points $\{\delta_i\}$ and the lattice isomorphisms (Dynkin diagram automorphisms) $\{\tau_i\}$ corresponding to each case. We denote by $\tau_{i_1i_2i_3\ldots i_k}$ the permutation $E_{i_1} \rightarrow E_{i_1}$, $E_{i_2} \rightarrow E_{i_2}$, …, $E_{i_k} \rightarrow E_{i_k}$.

8.2.1. $e-P(E_8^{(1)}/A_0^{(1)})$. Point configuration in $(f, g)$ coordinates:

$P_i : (f(v_i), g(v_i)) (i = 1, \ldots, 8), \quad f(z) = \begin{bmatrix} b \\ z \\ az \end{bmatrix}, \quad g(z) = \begin{bmatrix} b \\ z \\ az \end{bmatrix}.$

(8.34)

\[
\begin{array}{c}
8 \\
7 \\
6 \\
5 \\
4 \\
3 \\
2 \\
1 \\
\end{array}
\]

(8.35)
Root data:
\[
\begin{array}{c}
\alpha_0 \\
\alpha_1 \longrightarrow \alpha_2 \longrightarrow \alpha_3 \longrightarrow \alpha_4 \longrightarrow \alpha_5 \longrightarrow \alpha_6 \longrightarrow \alpha_7 \longrightarrow \alpha_8 \\
\delta_0
\end{array}
\]  
(8.36)

\[
\begin{aligned}
\alpha_0 &= E_1 - E_2, \quad \alpha_1 = H_1 - H_2, \quad \alpha_2 = H_2 - E_1 - E_2, \quad \alpha_3 = E_2 - E_3, \\
\alpha_4 &= E_3 - E_4, \quad \alpha_5 = E_4 - E_5, \quad \alpha_6 = E_5 - E_6, \quad \alpha_7 = E_6 - E_7, \quad \alpha_8 = E_7 - E_8, \\
\delta_0 &= 2H_1 + 2H_2 - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8.
\end{aligned}
\]  
(8.37)

8.2.2. q-P\(\mathcal{E}_8^{(1)}/\mathcal{A}_0^{(1)}\). Point configuration in \((f, g)\) coordinates:

\[
P_i : (f(v_i), g(v_i)) \quad (i = 1, \ldots, 8), \quad f(z) = z + \frac{\kappa_1}{z}, \quad g(z) = z + \frac{\kappa_2}{z},
\]  
(8.38)

Root data: same as section 8.2.1.

8.2.3. q-P\(\mathcal{E}_7^{(1)}/\mathcal{A}_1^{(1)}\). Point configuration in \((f, g)\) coordinates:

\[
P_i : \left(\frac{1}{v_i}, v_i\right) \quad (i = 1, \ldots, 4), \quad \left(\frac{\kappa_1}{v_i}, v_i\right) \quad (i = 5, \ldots, 8),
\]  
(8.40)

Root data:
\[
\begin{array}{c}
\alpha_0 \\
\alpha_1 \longrightarrow \alpha_2 \longrightarrow \alpha_3 \longrightarrow \alpha_4 \longrightarrow \alpha_5 \longrightarrow \alpha_6 \longrightarrow \alpha_7 \\
\delta_0 \rightarrow \delta_1
\end{array}
\]  
(8.42)

\[
\begin{aligned}
\alpha_0 &= H_1 - H_2, \quad \alpha_1 = E_3 - E_4, \quad \alpha_2 = E_2 - E_3, \quad \alpha_3 = E_1 - E_2, \\
\alpha_4 &= H_2 - E_1 - E_5, \quad \alpha_5 = E_3 - E_6, \quad \alpha_6 = E_5 - E_7, \quad \alpha_7 = E_6 - E_8, \\
\delta_0 &= H_1 + H_2 - E_3 - E_4, \quad \delta_1 = H_1 + H_2 - E_5 - E_6 - E_7 - E_8, \\
\pi &= \pi_{56781234}.
\end{aligned}
\]  
(8.43)
8.2.4. q-$\mathcal{P}(E_6^{(1)}/A_2^{(1)})$. Point configuration in $(f, g)$ coordinates:

$$P_i : \left(\frac{1}{v_i}, 0\right) (i = 1, \ldots, 4), \left(0, \frac{v_i}{\kappa_2}\right) (i = 5, 6), \left(\frac{\kappa_1}{v_i}, 0\right) (i = 7, 8), \quad (8.44)$$

Root data:

$$\alpha_0 \quad \delta_0 \quad \delta_1 \quad \delta_2 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6$$

$$\alpha_0 = E_7 - E_8, \alpha_1 = E_6 - E_5, \alpha_2 = H_2 - E_4 - E_6,$$
$$\alpha_3 = E_1 - E_2, \alpha_4 = E_2 - E_3, \alpha_5 = E_3 - E_4, \alpha_6 = H_1 - E_3 - E_7,$$
$$\delta_0 = H_1 + H_2 - E_1 - E_2 - E_3 - E_4, \delta_1 = H_1 - E_3 - E_6, \delta_2 = H_2 - E_7 - E_8,$$
$$\pi_1 = \pi_{2654378}\pi_{H_1 - E_3}, \quad \pi_2 = \pi_{2348765}\pi_{H_1 - H_2}. \quad (8.47)$$

In the following multiplicative cases, the eight points lie on the lines $f = 0, \infty, g = 0, \infty$. We also show the schematic diagram of the configuration of eight points.

8.2.5. q-$\mathcal{P}(D_6^{(1)}/A_2^{(1)})$. Point configuration in $(f, g)$ coordinates:

$$P_i : \left(\infty, \frac{1}{v_i}\right) (i = 1, 2), (v_i, \infty) (i = 3, 4), \left(0, \frac{v_i}{\kappa_2}\right) (i = 5, 6), \left(\frac{\kappa_1}{v_i}, 0\right) (i = 7, 8), \quad (8.48)$$
\[ \begin{array}{c}
\alpha_0 = E_7 - E_6, \quad \alpha_1 = E_3 - E_4, \quad \alpha_2 = H_1 - E_3 - E_7, \\
\alpha_3 = H_2 - E_1 - E_4, \quad \alpha_4 = E_5 - E_6, \quad \alpha_5 = E_5 - E_4, \\
\beta_0 = H_1 - E_4 - E_5, \quad \beta_1 = H_2 - E_3 - E_4, \quad \beta_2 = H_1 - E_5 - E_6, \quad \beta_3 = H_2 - E_7 - E_8.
\end{array} \] 

\[ \pi_1 = \pi_{2785634}, \quad \pi_2 = \pi_{78563412}r_{H_1 - H_2}. \] 

8.2.6. q-P(\(A_{10}^{(1)}/A_{4}^{(3)}\)). Point configuration in (f, g) coordinates:

\[ \begin{array}{c}
P_1 : \left( \infty, \frac{1}{v_1} \right), \quad P_{23} : \left( -\frac{v_3 v_1}{\kappa}, \frac{1}{v_2} \right), \quad P_3 : (0, \infty), \quad P_7 : \left( 0, \frac{v_1}{\kappa_2} \right) (i = 5, 6), \quad P_i : \left( 0, \frac{v_i}{v_1} \right) (i = 7, 8)
\end{array} \]
\[ \alpha_0 = E_7 - E_8, \quad \alpha_1 = H_1 - E_4 - E_7, \quad \alpha_2 = H_2 - E_1 - E_5, \]
\[ \alpha_3 = E_5 - E_6, \quad \alpha_4 = H_1 + H_2 - E_2 - E_3 - E_5 - E_7, \]
\[ \delta_0 = H_1 - E_1 - E_2, \quad \delta_1 = E_2 - E_3, \quad \delta_2 = H_2 - E_2 - E_4, \]
\[ \delta_3 = H_1 - E_5 - E_6, \quad \delta_4 = H_2 - E_3 - E_8, \]
\[ \pi_1 = \pi_42687153 r_{H_1-H_2} r_{H_2-E_3-E_5} r_{H_5-E_6}, \quad \pi_2 = \pi_42317856 r_{H_1-H_2}. \]

(8.55)

8.2.7. \( q\)-\( \mathcal{P}(E_6^{(1)}/A_5^{(1)}) \): Point configuration in \((f, g)\) coordinates:

\[
\begin{align*}
P_{18} & : \left( \frac{\kappa_1}{v_1 v_8}, \frac{1}{\epsilon} \right), \\
P_{23} & : \left( \frac{v_2 v_3}{\epsilon}, \frac{1}{\epsilon} \right), \\
P_1 & : \left( 0, \frac{v_1}{\kappa_2} \right) \quad (i = 5, 6), \\
P_7 & : \left( \frac{\kappa_1}{v_7}, 0 \right).
\end{align*}
\]

(8.56)

\[
\begin{align*}
g = \infty & & & & & & 23 & & & & & & & & & & 23 & & & & & & \quad & & & & & & \\
g = 0 & & & & & & 7 & & & & & & & & & & 18 & & & & & & \quad & & & & & & \\
f = 0 & & & & & & & & & & & & & & & & & & & & & & & & \quad & & & & & & \\
f = \infty & & & & & & & & & & & & & & & & & & & & & & & & \quad & & & & & & \\
\end{align*}
\]

(8.57)

Root data:

(8.58)

\[
\begin{align*}
\alpha_0 &= H_1 + H_2 - E_2 - E_3 - E_6 - E_7, \\
\alpha_1 &= H_1 + H_2 - E_1 - E_4 - E_6 - E_8, \\
\alpha_2 &= E_6 - E_5, \quad \alpha_3 = \delta - \alpha_4, \quad \alpha_4 = H_1 - E_4 - E_7, \\
\delta_0 &= H_1 - E_1 - E_2, \quad \delta_1 = E_2 - E_3, \quad \delta_2 = H_2 - E_2 - E_4, \\
\delta_3 &= H_1 - E_5 - E_6, \quad \delta_4 = H_2 - E_1 - E_7, \quad \delta_5 = E_1 - E_8, \\
\pi_1 &= \pi_{1234875} r_{H_1-E_4-E_6} r_{H_2-E_1-E_6}, \quad \pi_2 = \pi_{6457182} r_{H_1-H_2} r_{H_2-E_3} r_{H_5-E_6}.
\end{align*}
\]

(8.59)
8.2.8. $q$-$\mathcal{P}(E_2^{(1)}/A_0^{(1)}; a)$. Point configuration in $(f, g)$ coordinates:

\[
P_{18} : \left( -\frac{\kappa_1}{\epsilon v_1 v_8}, \epsilon \right), \quad P_{23} : \left( -\frac{v_2 v_3}{\epsilon}, \frac{1}{\epsilon} \right), \quad P_4 : (v_4, \infty),
\]

\[
P_5 : \left( 0, \frac{v_5}{\kappa_2} \right), \quad P_{67} : \left( -\frac{\kappa_1 \kappa_2}{v_6 v_7} \epsilon, \epsilon \right), \quad (8.60)
\]

Root data:

\[
\alpha_0 \longrightarrow \alpha_1 \longrightarrow \alpha_2
\]

\[
\alpha_0 = H_1 + H_2 - E_2 - E_3 - E_6 - E_7, \quad \alpha_1 = H_1 + H_2 - E_1 - E_4 - E_5 - E_8,
\]

\[
\alpha_2 = H_1 + 3H_2 - E_1 - 2E_2 - 2E_3 + E_4 - 3E_5 - E_8,
\]

\[
\delta_0 = H_1 - E_1 - E_2, \quad \delta_1 = E_2 - E_3, \quad \delta_2 = H_2 - E_2 - E_4,
\]

\[
\delta_3 = H_1 - E_5 - E_6, \quad \delta_4 = E_6 - E_7, \quad \delta_5 = H_2 - E_1 - E_6, \quad \delta_6 = E_1 - E_8,
\]

\[
\pi_1 = \pi_53281674 H_1 - E_5, \quad \pi_2 = \pi_55432187 H_1 - E_4, \quad (8.63)
\]

8.2.9. $q$-$\mathcal{P}(A_1^{(1)}/A_0^{(1)})$. Point configuration in $(f, g)$ coordinates:

\[
P_{18} : \left( -\frac{\kappa_1}{\epsilon v_1 v_8}, \epsilon \right), \quad P_{23} : \left( -\frac{v_2 v_3}{\epsilon}, \frac{1}{\epsilon} \right), \quad P_4 : (v_4, \infty), \quad P_{67} : \left( -\frac{\kappa_1 \kappa_2}{v_6 v_7} \epsilon^2, \epsilon \right), \quad (8.64)
\]
Root data:

\[ \alpha_0 \longrightarrow \alpha_1 \]

\[ \alpha_0 = H_1 + 2H_2 - 2E_2 - 2E_3 + E_4 = E_5 - E_6 - E_7, \]
\[ \alpha_1 = H_1 - E_1 + E_2 + E_3 - 2E_4 - E_8, \]
\[ \delta_0 = H_1 - E_1 - E_2, \delta_1 = E_2 - E_3, \delta_2 = H_2 - E_2 - E_4, \delta_3 = H_1 - E_3 - E_6, \]
\[ \delta_4 = E_6 - E_7, \delta_5 = E_3 - E_6, \delta_6 = H_2 - E_1 - E_5, \delta_7 = E_1 - E_8, \]
\[ \eta_1 = \pi_6723184^fH_1-E_4-E_5, \pi_2 = \pi_65432187^fH_1-E_2-E_5. \]

8.2.10. \( q-P(E_3^{(1)}/A_3^{(1)}; b) \). Point configuration in \((f, g)\) coordinates:

\[ P_1 : \left( \frac{1}{v_1}, \infty \right), \quad P_{23} : \left( \frac{v_2 \sqrt{3}}{\epsilon}, \frac{1}{\epsilon} \right), \quad P_2 : (v_3, \infty), \]
\[ P_5 : \left( 0, \frac{v_5}{\sqrt{2}} \right), \quad P_{07} : \left( \frac{k_1 \sqrt{5}}{v_6 v_7 \epsilon}, \frac{1}{\epsilon} \right), \quad P_8 : \left( \frac{k_1}{v_8}, 0 \right). \]

\[ \begin{align*}
\text{Root data:} \\
\alpha_0 \longrightarrow \alpha_1
\end{align*} \]
Remark 8.2. The two realizations of root systems $q$-$P(E_{1}^{(1)}/A_{0}^{(1)}; a)$ and $q$-$P(E_{1}^{(1)}/A_{0}^{(1)}; b)$ are transformed with each other by the reflection \( r_{E_{1}} \).

8.2.11. $q$-$P(E_{2}^{(1)}/A_{0}^{(1)}; b)$. Point configuration in \((f, g)\) coordinates:

\[
P_{1} : \left( \frac{\infty}{v_{1}}, \frac{1}{v_{1}} \right), \quad P_{23} : \left( -\frac{v_{2}v_{3}}{\varepsilon}, \frac{1}{\varepsilon_{2}} \right), \quad P_{b} : (v_{4}, \infty),
\]

\[
P_{567} : \left( \frac{\alpha_{1}\alpha_{2}^{2}}{v_{4}v_{6}v_{7}}, \varepsilon^{2} \right), \quad P_{b} : \left( \frac{\alpha_{1}}{v_{8}}, 0 \right).
\]

Root data:
Remark 8.3. The two realizations of root systems $q$-$P(E_{2}^{(1)}/A_{6}^{(1)}; a)$ and $q$-$P(E_{5}^{(1)}/A_{6}^{(1)}; b)$ are transformed with each other by the reflection $\sigma_{H_{2} - E_{3} - E_{3}}$.

8.2.12. $q$-$P(A_{3}^{(1)}/A_{6}^{(1)})$. Point configuration in $(f, g)$ coordinates:

\[ P_{123} : \left( \frac{v_{1}v_{2}v_{3}}{\epsilon^{2}}, \frac{1}{\epsilon} \right), \quad P_{1} : (v_{4}, \infty), \quad P_{567} : \left( \frac{\kappa_{1}^{2} \kappa_{2}^{2}}{\sqrt{v_{5}v_{6}v_{7}}}, \epsilon \right), \quad P_{8} : \left( \frac{\kappa_{1}}{v_{8}}, 0 \right). \]  

Root data:

\[ \alpha_{0} = H_{1} + 2H_{2} - E_{1} - E_{2} - E_{3} - E_{5} - E_{6} - E_{7}, \quad \alpha_{1} = H_{1} - E_{4} - E_{8}, \]
\[ \alpha_{2} = H_{1} + 2E_{1} - 2E_{2} - 2E_{3} + E_{4} - E_{8}, \]
\[ \delta_{0} = H_{1} - E_{1} - E_{2}, \quad \delta_{1} = E_{2} - E_{3}, \quad \delta_{2} = H_{2} - E_{2} - E_{4}, \]
\[ \delta_{3} = H_{1} - E_{5} - E_{6}, \quad \delta_{4} = E_{6} - E_{7}, \quad \delta_{5} = E_{5} - E_{6}, \quad \delta_{6} = H_{2} - E_{5} - E_{6}, \]
\[ \pi_{1} = \tau_{21435678}^{T} H_{2} - E_{3} - E_{8} \tau_{H_{2} - E_{3} - E_{3}}, \quad \pi_{2} = \tau_{35182674}^{T} H_{2} - E_{3} - E_{5}. \]  

(8.75)
8.2.13. $q$-$\mathcal{P}(\mathcal{A}_0^{(1)}/\mathcal{A}_0^{(3)})$. Point configuration in $(f, g)$ coordinates:

$$P_{18} : \left( \frac{k_1}{v_1 v_8} \right), \quad P_{224} : \left( \frac{1}{v_2 v_3 v_4 v_5} \right) \frac{1}{v_2 v_4} \epsilon^2, \quad P_{507} : \left( \frac{k_1 k_2^2}{v_5 v_6 v_7} \right) \epsilon^2 \frac{1}{v_3}, \quad (8.80)$$

Root data:

$$\alpha_0 = 2H_1 + 2H_2 = E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8, \quad \delta_0 = H_1 - E_1 - E_2, \quad \delta_1 = E_2 - E_3, \quad \delta_2 = E_3 - E_4, \quad \delta_3 = H_2 - E_2 - E_3, \quad \delta_4 = H_1 - E_5 - E_6, \quad \delta_5 = E_6 - E_7, \quad \delta_6 = E_5 - E_6, \quad \delta_7 = H_2 - E_1 - E_5, \quad \delta_8 = E_4 - E_8, \quad (8.83)$$

8.2.14. $d$-$\mathcal{P}(E_8^{(1)}/A_0^{(3)})$. Point configuration in $(f, g)$ coordinates:

$$P_i : \left( f(v_i), g(v_i) \right) (i = 1, \ldots, 8), \quad f(z) = z(z - \kappa_i), \quad g(z) = z(z - \kappa_2), \quad (8.84)$$
Root data: same as section 8.2.1.

8.2.15. $d$-$P_{E_{7}^{(1)}}/A_{1}^{(1)}$. Point configuration in $(f, g)$ coordinates:

$$P_{i} : (v_{i}, -v_{i}) \ (i = 1, \ldots, 4), \quad (\kappa_{i} - v_{i}, v_{i} - \kappa_{2}) \ (i = 5, \ldots, 8),$$

Root data: same as section 8.2.3.

8.2.16. $d$-$P_{E_{6}^{(1)}}/A_{2}^{(1)}$. Point configuration in $(f, g)$ coordinates:

$$P_{i} : (v_{i}, -v_{i}) \ (i = 1, \ldots, 4), \quad (\infty, v_{i} - \kappa_{2}) \ (i = 5, 6), \quad (\kappa_{1} - v_{i}, \infty) \ (i = 7, 8),$$

Root data: same as section 8.2.4.

8.2.17. $d$-$P_{D_{4}^{(1)}}/D_{4}^{(1)}$. Point configuration in $(f, g)$ coordinates:

$$P_{1} : (\infty, -a_{2}), \quad P_{2} : (\infty, -a_{1} - a_{2}), \quad P_{34} \left( t(1 + a_{0} \epsilon), \frac{1}{\epsilon} \right),$$

$$P_{5} : (0, 0), \quad P_{6} : (0, a_{4}), \quad P_{78} : \left( 1 + a_{3} \epsilon, \frac{1}{\epsilon} \right),$$

$$a_{0} + a_{1} + 2a_{2} + a_{3} + a_{4} = 1,$$
Root data:

\[
\begin{align*}
\alpha_0 & : \alpha_3 & \delta_0 & : \delta_3 \\
\alpha_1 & : \alpha_4 & \delta_1 & : \delta_4 \\
\delta_0 & : E_3 - E_4, & \alpha_1 & : E_1 - E_2, & \alpha_2 & : H_2 - E_1 - E_5, \\
\alpha_3 & : H_1 - E_7 - E_8, & \alpha_4 & : E_5 - E_6, & \delta_1 & : H_1 - E_1 - E_2, & \delta_2 & : H_2 - E_3 - E_7, \\
\delta_3 & : E_7 - E_8, & \delta_4 & : H_1 - E_5 - E_6, & \eta_1 & = \pi_{2348576}, & \eta_2 & = \pi_{12785634}, & \eta_3 & = \pi_{6341278}.
\end{align*}
\]  

(8.93)

**8.2.18. \(\tilde{d}^{-}(A_3^{(1)}/D_6^{(1)})\).** Point configuration in \((q, p)\) coordinates:

\[
P_{12} : \left(\frac{1}{\epsilon}, -t + (a_1 + a_2 + a_3 - 1)\epsilon\right), \quad P_{34} : \left(\frac{1}{\epsilon}, -a_2\epsilon\right),
\]  

\[
P_{56} : \left(a_1\epsilon, \frac{1}{\epsilon}\right), \quad P_{78} : \left(1 + a_3\epsilon, \frac{1}{\epsilon}\right).
\]  

(8.94)
\[ \alpha_0 = H_2 - E_1 - E_2, \ \alpha_1 = H_1 - E_3 - E_6, \ \alpha_2 = H_2 - E_3 - E_4, \ \alpha_3 = H_1 - E_7 - E_8 \]
\[ \delta_0 = E_1 - E_2, \ \delta_1 = E_3 - E_4, \ \delta_2 = H_1 - E_1 - E_3, \]
\[ \delta_3 = H_2 - E_5 - E_7, \ \delta_4 = E_3 - E_6, \ \delta_5 = E_7 - E_8, \]
\[ \pi_1 = \pi_{78563412}^{p,H_1-H_2}, \ \pi_2 = \pi_{34125678}^{p} \]

(8.97)

8.2.19. \( dP(2A_1)^{(1)}/D_6^{(1)} \). Point configuration in \((q, p)\) coordinates:

\[ P_{12} : \left( \frac{1}{\epsilon}, 1 - a_1 \epsilon \right), \ P_{34} : \left( \frac{1}{\epsilon}, -a_2 \epsilon \right), \ P_{5678} : \left( \epsilon, \frac{1 - a_1 - a_2}{\epsilon} \right), \]

(8.98)

\[ p = \infty \]
\[ q = \infty \]

(8.99)

Root data:

(8.100)

\[ \alpha_0 \longrightarrow \alpha_1 \longrightarrow \alpha_2 \longrightarrow \alpha_3 \]
\[ \delta_0 \longrightarrow \delta_2 \longrightarrow \delta_3 \longrightarrow \delta_4 \]
\[ \delta_1 \longrightarrow \delta_5 \longrightarrow \delta_6 \]

\[ \alpha_0 = 2H_1 + H_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8, \]
\[ \alpha_1 = H_2 - E_1 - E_2, \ \alpha_2 = H_2 - E_3 - E_4, \]
\[ \alpha_3 = 2H_1 + H_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8, \]
\[ \delta_0 = E_1 - E_2, \ \delta_1 = E_3 - E_4, \ \delta_2 = H_1 - E_1 - E_3, \]
\[ \delta_3 = H_2 - E_5 - E_7, \ \delta_4 = E_3 - E_6, \ \delta_5 = E_5 - E_6, \ \delta_6 = E_7 - E_8, \]
\[ \pi_1 = \pi_{34125678}^{p}, \ \pi_2 = \pi_{78563412}^{p,H_1-E_6,H_2-E_5} \]

(8.101)
8.2.20. $dP(A^{(1)}_{1}/D^{(1)}_{4})$. Point configuration in $(q,p)$ coordinates:

\[
P_{1234} : \left( -\frac{1}{\epsilon^2}, \frac{a_1}{2} \right), \quad P_{5678} : \left( -\frac{1}{\epsilon^2}, \frac{1-a_1}{\epsilon} \right),
\]

(8.102)

\[
p = \infty, \quad q = \infty
\]

(8.103)

Root data:

\[
\begin{align*}
\alpha_0 & \leftrightarrow \alpha_1 & \delta_0 & \rightarrow \delta_3 & \rightarrow \delta_4 & \rightarrow \delta_5 \\
\delta_1 & \rightarrow \delta_2 & \rightarrow \delta_5 & & & \\
\delta_6 & \rightarrow \delta_7
\end{align*}
\]

(8.104)

\[
\begin{align*}
\alpha_0 &= 2H_1 - E_5 - E_6 - E_7 - E_8, \quad \alpha_1 &= 2H_2 - E_1 - E_2 - E_3 - E_4 \\
\delta_0 &= E_1 - E_2, \quad \delta_1 = E_3 - E_4, \quad \delta_2 = E_3 - E_5, \quad \delta_3 = H_1 - E_1 - E_2, \\
\delta_4 &= H_2 - E_5 - E_6, \quad \delta_5 = E_6 - E_7, \quad \delta_6 = E_5 - E_6, \quad \delta_7 = E_7 - E_8, \\
\eta_1 &= \pi_{5678}^{1234}r_{H_1-H_2}, \quad \eta_2 = \pi_{34125678}r_{H_1-E_1}r_{H_2-E_2}r_{H_1-E_3}r_{H_1-E_4}
\end{align*}
\]

(8.105)

8.2.21. $dP(A^{(1)}_{0}/D^{(1)}_{8})$. Point configuration in $(f,g) = (q,p)$ coordinates:

\[
P_{1234} : \left( -\frac{1}{\epsilon^2}, -\frac{1}{\epsilon} + \frac{1}{2} \right), \quad P_{5678} : \left( -\frac{1}{\epsilon^2}, \frac{1}{\epsilon} \right).
\]

(8.106)

\[
f = \infty, \quad g = \infty
\]

(8.107)

Root data:

\[
\begin{align*}
\alpha_0 & \leftrightarrow \alpha_1 & \delta_0 & \rightarrow \delta_3 & \rightarrow \delta_4 & \rightarrow \delta_5 \\
\delta_1 & \rightarrow \delta_2 & \rightarrow \delta_6 & & & \\
\delta_7 & \rightarrow \delta_8
\end{align*}
\]

(8.108)
\( \alpha_0 = 2H_1 + 2H_2 - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 \)
\( \delta_0 = H_1 - E_1 - E_2, \delta_1 = E_3 - E_4, \delta_2 = E_2 - E_3, \delta_3 = E_1 - E_2, \)
\( \delta_4 = H_2 - E_1 - E_5, \delta_5 = E_5 - E_6, \delta_6 = E_6 - E_7, \delta_7 = H_1 - E_5 - E_6, \delta_8 = E_7 - E_8. \)
\( \pi = \pi_{56781234}. \)  
(8.109)

8.2.22. d-P(A\(^{(3)}\))/E\(^{(3)}\)). Point configuration in \((q, p)\) coordinates:
\[
P_{12} : \left( \frac{1}{\epsilon}, -a_2 \epsilon \right), \ P_{34} : \left( a_1 \epsilon, \frac{1}{\epsilon} \right), \ P_{5678} : \left( \frac{1}{\epsilon}, \frac{1}{\epsilon} + t + (a_1 + a_2 - 1) \epsilon \right).
\]  
(8.110)

\[
P = \infty \quad \begin{array}{c}
34 \\
5678 \\
235 \\
12112
\end{array}
\]

Root data:
\[
\alpha_0 \quad \delta_0 \\
\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4 \quad \delta_5
\]
\( \alpha_0 = H_1 + H_2 - E_1 - E_2 - E_5 - E_6 - E_7 - E_8, \alpha_1 = H_1 - E_3 - E_4, \alpha_2 = H_2 - E_1 - E_2 \)
\( \delta_0 = E_7 - E_8, \delta_1 = E_1 - E_2, \delta_2 = H_1 - E_1 - E_5, \delta_3 = E_5 - E_6, \delta_4 = H_2 - E_1 - E_5, \delta_5 = E_3 - E_4, \delta_6 = E_6 - E_7. \)
\( \pi_1 = \pi_{34125678 \Gamma H_1 - H_2}, \pi_2 = \pi_{78345612 \Gamma H_1 - E_5 - E_6}. \)  
(8.113)

8.2.23. d-P(A\(^{(3)}\))/E\(^{(3)}\)). Point configuration in \((q, p)\) coordinates:
\[
P_{12} : \left( \frac{1}{\epsilon}, -a_1 \epsilon \right), \ P_{345678} : \left( \frac{1}{\epsilon}, \frac{2}{\epsilon^2} + t + (a_1 - 1) \epsilon \right).
\]  
(8.114)
Root data:

\[ \alpha_0 \leftrightarrow \alpha_1 \]

\[ \delta_0 \]

\[ \delta_1 \rightarrow \delta_2 \rightarrow \delta_3 \rightarrow \delta_4 \rightarrow \delta_5 \rightarrow \delta_6 \rightarrow \delta_7 \]

\[ \alpha_0 = 2H_1 + H_2 - E_3 - E_4 - E_5 = E_6 = E_7 - E_8, \quad \alpha_1 = H_2 - E_1 - E_2 \]

\[ \delta_0 = H_2 - E_3 - E_4, \quad \delta_1 = E_1 - E_2, \quad \delta_2 = H_1 - E_1 - E_3, \quad \delta_3 = E_3 - E_4, \]

\[ \delta_4 = E_5 - E_8, \quad \delta_5 = E_6 - E_7 - E_8, \quad \delta_6 = E_7 - E_8, \]

\[ \pi = \prod_{8563412} f_{H_1 - E_1} f_{H_1 - E_2} \]

8.2.24. \(d-P(A_1^{(1)}/E_8^{(1)})\). This case cannot be realized by the configuration of eight points on \(\mathbb{P}^1 \times \mathbb{P}^1\). It is necessary to consider the configuration of nine points on \(\mathbb{P}^2\) \([53, 112]\).

Root data:

\[ \alpha_0 \]

\[ \delta_0 \]

\[ \delta_1 \rightarrow \delta_2 \rightarrow \delta_3 \rightarrow \delta_4 \rightarrow \delta_5 \rightarrow \delta_6 \rightarrow \delta_7 \rightarrow \delta_8 \]

\[ \alpha_0 = 2H_1 + 2H_2 - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 \]

\[ \delta_0 = E_1 - E_2, \quad \delta_1 = H_1 - H_2, \quad \delta_2 = H_2 - E_1 - E_2, \quad \delta_3 = E_2 - E_3, \]

\[ \delta_4 = E_3 - E_4, \quad \delta_5 = E_4 - E_5, \quad \delta_6 = E_5 - E_6, \quad \delta_7 = E_6 - E_7, \quad \delta_8 = E_7 - E_8 \]

8.3. Degeneration of point configurations

In this subsection, we describe the procedure of degeneration of the point configurations and the corresponding discrete Painlevé equations given in section 8.2.

We first show the multiplicative cases according to the following diagram:

\[ E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow E_5^{(1)} \rightarrow A_4^{(1)} \rightarrow E_3^{(1)}(b) \rightarrow E_2^{(1)}(b) \rightarrow A_1^{(1)} \]

\[ \rightarrow E_5^{(1)}(a) \rightarrow E_2^{(1)}(a) \rightarrow A_1^{(1)} \]

\[ \text{If } F = 8 \]
In each case, introducing a small parameter \( \varepsilon \), change the variables \( f, g, \kappa_i \) \((i = 1, 2)\) and \( v_j(i = 1, \ldots, 8)\) as indicated below and then take the limit \( \varepsilon \to 0 \) to obtain the lower case.

### 8.3.1. q-P(\( E_{1}^{(1)}/A_{3}^{(1)} \)) \( \to \) q-P(\( E_{4}^{(1)}/A_{4}^{(1)} \)).

\[
\kappa_i \to \kappa_i \varepsilon \quad (i = 1, 2), \quad v_j \to v_j \varepsilon \quad (i = 5, \ldots, 8), \quad g \to \frac{1}{g}.
\]  
(8.121)

### 8.3.2. q-P(\( E_{1}^{(1)}/A_{4}^{(1)} \)) \( \to \) q-P(\( E_{4}^{(1)}/A_{5}^{(1)} \)).

\[
v_{i} \to v_{j} \varepsilon \quad (i = 7, 8), \quad \kappa_{i} \to \kappa_{i} \varepsilon.
\]  
(8.122)

### 8.3.3. q-P(\( E_{1}^{(1)}/A_{2}^{(1)} \)) \( \to \) q-P(\( D_{5}^{(1)}/A_{3}^{(1)} \)).

\[
v_{i} \to \frac{v_{i}}{\varepsilon} \quad (i = 1, 2), \quad v_{j} \to \varepsilon v_{j} \quad (i = 3, 4), \quad \kappa_{i} \to \kappa_{i} \varepsilon,
\]
\[
\kappa_{2} \to \frac{\kappa_{2}}{\varepsilon}, \quad f \to f \varepsilon, \quad g \to g \varepsilon.
\]  
(8.123)

The degeneration from q-P(\( D_{5}^{(1)}/A_{3}^{(1)} \)) can be carried out by the simple substitution

\[
v_{i} \to v_{j} \varepsilon, \quad v_{j} \to \frac{v_{j}}{\varepsilon},
\]  
(8.124)

and the limit \( \varepsilon \to 0 \) which is denoted by \( \frac{ij}{ij} \) as shown in (8.120) (see also figure 20).

**Remark 8.4.** Here we remark on the relation between infinitely near points and the multiple point [52]. For example, consider the degeneration of \( P_{b} : (0, a \varepsilon) \) and \( P_{b} : (b \varepsilon, 0) \) by the limit \( \varepsilon \to 0 \), as in the degeneration of \( P_{0} \) and \( P_{7} \). The line connecting \( P_{a} \) and \( P_{b} \) yields \( \frac{a}{b} = -\frac{a}{b} \) in the limit \( \varepsilon \to 0 \). Therefore, it passes through the double point \( P_{a} : \left( -\frac{a}{b}, \varepsilon \right) \) in the sense of section 8.2. More generally, the condition for a curve \( F(f, g) = 0 \) to pass through the two points \( P_{a} \) and \( P_{b} \) coincides in the limit \( \varepsilon \to 0 \) with the condition that the curve passes through the double point \( P_{a} \). This condition is written as \( F = 0 \) and \( b \frac{\partial F}{\partial \varepsilon} = a \frac{\partial F}{\partial \varepsilon} \) at \( (f, g) = (0, 0) \). One can consider the degeneration of several points to a multiple point in a similar manner.

### 8.3.4. q-P(\( D_{5}^{(1)}/A_{3}^{(1)} \)) \( \to \) d-P (\( D_{5}^{(1)}/D_{4}^{(1)} \)).

For the cases admitting the Painlevé differential equations as continuous flows, we have the graphical diagram of degeneration from q-P(\( D_{5}^{(1)}/A_{3}^{(1)} \)) as shown in figure 21. We use below the root parameters as in section 8.1 for describing the point configurations. In the case of d-P(\( D_{4}^{(1)}/D_{4}^{(1)} \)), the correspondence of the parameters is given by (see (8.93))

\[
a_{0} = \kappa_{1} - v_{3} - v_{4}, \quad a_{1} = v_{1} - v_{2},
\]
\[
a_{2} = \kappa_{2} - v_{1} - v_{5}, \quad a_{3} = \kappa_{1} - v_{7} - v_{8}, \quad a_{4} = v_{5} - v_{6}.
\]  
(8.125)

We will explain the degeneration limit to some additive cases in figure 21 starting from q-P(\( D_{5}^{(1)}/A_{3}^{(1)} \)). We only show the degenerations relevant to the hypergeometric solutions in section 8.6.

We set

\[
q \to e^{\varepsilon}, \quad g \to e^{-\varepsilon \left( g + \kappa_{2} - v_{6} \right)}, \quad \kappa_{i} \to e^{\varepsilon \kappa_{i}} \quad (i = 1, 2),
\]
\[
(v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}, v_{7}, v_{8}) \to (e^{\varepsilon v_{1}}, e^{\varepsilon v_{2}}, e^{\varepsilon v_{3}}, e^{\varepsilon v_{4}}, e^{\varepsilon v_{5}}, e^{\varepsilon v_{6}}, e^{\varepsilon v_{7}}, e^{\varepsilon v_{8}}).
\]  
(8.126)
and take the limit \( \varepsilon \to 0 \). Actually, one can verify by direct calculation that \( q-P(D_5^{(1)}/A_4^{(1)}) \) (8.9) gives rise to \( d-P(D_4^{(1)}/A_3^{(1)}) \) (8.21) by the limiting procedure (8.126) under the correspondence of parameters (8.125).

**8.3.5.** \( d-P(D_4^{(1)}/D_4^{(1)}) \rightarrow d-P(A_3^{(1)}/D_4^{(1)}) \). We set

\[
\begin{align*}
f & \rightarrow e^{\varepsilon q}, \quad g \rightarrow \frac{p}{\varepsilon}, \quad t \rightarrow e^{\varepsilon}, \quad a_0 \to a_3, \\
a_1 & \rightarrow \frac{1}{\varepsilon} + a_2, \quad a_2 \rightarrow \frac{1}{\varepsilon}, \quad a_3 \rightarrow a_1, \quad a_4 \rightarrow -\frac{1}{\varepsilon} + a_0,
\end{align*}
\]

(8.127)

and take the limit \( \varepsilon \to 0 \) to obtain (8.23) from (8.21).

**8.3.6.** \( d-P(A_3^{(1)}/D_5^{(1)}) \rightarrow d-P(A_3^{(1)}/E_6^{(1)}) \). We set

\[
\begin{align*}
p & \rightarrow \frac{1}{\varepsilon^2} + \frac{p-t}{\varepsilon}, \quad q \rightarrow q\varepsilon, \quad t \rightarrow t \frac{1}{\varepsilon} - \frac{1}{\varepsilon^2}, \\
a_0 & \to a_2, \quad a_2 \rightarrow \frac{1}{\varepsilon^2} + a_0, \quad a_3 \rightarrow -\frac{1}{\varepsilon^2},
\end{align*}
\]

(8.128)

and take the limit \( \varepsilon \to 0 \) to obtain (8.25) from (8.23).

**8.4. Birational representation of affine Weyl groups**

In this subsection, for each symmetry/surface type we construct an explicit birational representation of the symmetry group generated by simple reflections \( s_0, s_1, \ldots, s_l \) and lattice isomorphisms (Dynkin diagram automorphisms) \( \tau_1, \ldots, \tau_l \), with \( \alpha_i \) and \( \eta_i \) as listed in section 8.2. We use the parameters \( h_1, h_2, e_1, \ldots, e_8 \) instead of \( \delta_1, \delta_2, v_1, \ldots, v_8 \) (see remark...
for both multiplicative and additive cases. On these parameters the simple reflections act linearly in the same way as they do on the basis of the Picard lattice, $H_1, H_2, E_1, \ldots, E_8$. Note that the Picard lattice has a trivial lattice isomorphism $H_i \mapsto -H_i, E_i \mapsto -E_i$ which does not belong to the affine Weyl group. As to the lattice automorphism $\pi_1, \ldots, \pi_r$, we need to incorporate the corresponding transformation $h_i \mapsto h_i^{-1}, e_i \mapsto e_i^{-1}$ or $h_i \mapsto -h_i, e_i \mapsto -e_i$ in constructing individual birational representations. We also give the actions on $f, g$. 

Figure 21. Degeneration of point configurations of the additive types in $\mathbb{P}^3 \times \mathbb{P}^3$ from $qP(D^1_k)/A^1_k$ case. Top: $(q, p)$ coordinates. Bottom: $(f, g)$ coordinates. Multi-indices $i_1i_2\cdots i_k$ represent the multiple points. ‘None’ means that the surface cannot be realized by eight point configuration on a $(2, 2)$-curve in the corresponding coordinates.
In the following, we use the following symbols for distinct $i, j, k, l$:

\[
\begin{align*}
\sigma_0 & \equiv e_i, \\
\sigma_{\Lambda_0} & \equiv h_l, \\
\sigma_{\Lambda_0-E_1-E_2} & \equiv e_i \to \frac{h_1}{e_i}, \quad e_j \to \frac{h_1}{e_j} + \frac{h_1}{e_i}, \\
\sigma_{\Lambda_0-E_1-E_2} & \equiv e_i \to \frac{h_2}{e_i}, \quad e_j \to \frac{h_2}{e_j}, \\
\sigma_{\Lambda_0-E_1-E_2} & \equiv e_i \to \frac{h_1 h_2}{e_i e_j e_k e_l}, \quad e_j \to \frac{h_1 h_2}{e_j e_i e_k e_l}.
\end{align*}
\]

(8.129)

8.4.1. $q$-$\mathcal{P}(E_8^1/\Lambda_0^{(1)})$. Point configuration:

\[
(f_i, g_i) = \left( e_i + \frac{h_1}{e_i}, \quad e_i + \frac{h_2}{e_i} \right) \quad (i = 1, \ldots, 8).
\]

(8.130)

Generators and actions on parameters:

\[
\begin{align*}
\sigma_0 &= \sigma_{12}, \\
\sigma_1 &= \sigma_{\Lambda_0-E_1-E_2}, \\
\sigma_2 &= \sigma_{\Lambda_0-E_1-E_2}, \\
\sigma_3 &= \sigma_{\Lambda_0-E_1-E_2}, \\
\sigma_4 &= \sigma_{\Lambda_0-E_1-E_2}, \\
\sigma_5 &= \sigma_{\Lambda_0-E_1-E_2}, \\
\sigma_6 &= \sigma_{\Lambda_0-E_1-E_2}, \\
\sigma_7 &= \sigma_{\Lambda_0-E_1-E_2}, \\
\sigma_8 &= \sigma_{\Lambda_0-E_1-E_2}.
\end{align*}
\]

(8.131)

Actions on $f, g$:

\[
\begin{align*}
\sigma_1 : f & \leftrightarrow g, \\
\sigma_2 : f & \rightarrow \frac{1}{e_i e_j (e_i e_j - h_2) f - e_i e_j (e_i e_j - h_1) g - e_i e_j (e_i + e_j) (h_1 - h_2)} \\
& \times [e_i e_j (h_1 - h_2) f g + h_2 (e_i + e_j) (e_i e_j - h_1) f] \\
& - h_1 (e_i + e_j) (e_i e_j - h_2) g + (h_1 - h_2) (e_i e_j - h_1) (e_i e_j - h_2). \\
\end{align*}
\]

(8.132)

8.4.2. $q$-$\mathcal{P}(E_7^1/\Lambda_1^{(1)})$. Point configuration:

\[
(f_i, g_i) = \left( e_i, \quad \frac{1}{e_i} \right) \quad (i = 1, \ldots, 4), \\
\left( \frac{h_1}{e_i}, \quad \frac{e_i}{h_2} \right) \quad (i = 5, \ldots, 8).
\]

(8.133)

Generators and actions on parameters:

\[
\begin{align*}
\sigma_0 &= \sigma_{\Lambda_0-E_1}, \\
\sigma_1 &= \sigma_{\Lambda_0-E_1}, \\
\sigma_2 &= \sigma_{\Lambda_0-E_1}, \\
\sigma_3 &= \sigma_{\Lambda_0-E_1}, \\
\sigma_4 &= \sigma_{\Lambda_0-E_1}, \\
\sigma_5 &= \sigma_{\Lambda_0-E_1}, \\
\sigma_6 &= \sigma_{\Lambda_0-E_1}, \\
\sigma_7 &= \sigma_{\Lambda_0-E_1}, \\
\sigma_8 &= \sigma_{\Lambda_0-E_1}, \\
\pi : h_i & \leftrightarrow h_i^{-1}, \quad e_i \leftrightarrow e_i^{-1}, \quad \sigma = \begin{pmatrix} 12345678 \\ 56781234 \end{pmatrix}.
\end{align*}
\]

(8.134)
Actions on $f, g$:

$$s_0 : f \rightarrow \frac{1}{g}, \quad g \rightarrow \frac{1}{f},$$

$$s_4 : f \rightarrow \frac{-h_2(e_1 e_5 - h_1)fg - e_5(h_1 - h_2)f + h_1(e_1 e_5 - h_2)}{e_5[-(e_1 e_5 - h_2)fg + e_1(h_1 - h_2)g + (e_1 e_5 - h_1)]},$$

$$\pi : f \rightarrow \frac{f}{h_1}, \quad g \rightarrow h_2g.$$  \hspace{1cm} (8.135)

8.4.3. $q$-$\mathcal{P}(E_8^{(1)}/A_2^{(3)})$. Point configuration:

$$(f_i, g_i) = \left( e_i, \frac{1}{e_i} \right) \quad (i = 1, \ldots, 4), \quad \left( 0, \frac{e_i}{h_2} \right) \quad (i = 5, 6), \quad \left( \frac{h_i}{e_i}, 0 \right) \quad (i = 7, 8). \hspace{1cm} (8.136)$$

Generators and actions on parameters:

$$s_0 = s_{78}, \quad s_1 = s_{35}, \quad s_2 = s_{H_6-E_8}, \quad s_3 = s_{12},$$

$$s_4 = s_{23}, \quad s_5 = s_{34}, \quad s_6 = s_{H_6-E_7},$$

$$\pi_1 : h_1 \rightarrow \frac{e_1 e_2}{h_1 h_2}, h_2 \rightarrow h_2^{-1}, e_1 \rightarrow \frac{e_2}{h_2}, e_2 \rightarrow \frac{e_1}{h_2},$$

$$e_i \rightarrow e_i^{-1}, \quad \sigma = \begin{cases} 345678, & (i = 1, 2), \\ 654378, & (i = 1, 2). \end{cases}$$

$$\pi_2 : h_1 \rightarrow h_2^{-1}, h_2 \rightarrow h_1^{-1}, e_1 \rightarrow e_1^{-1}, \quad \sigma = \begin{pmatrix} 12345678 \\ 12348765 \end{pmatrix}. \hspace{1cm} (8.137)$$

Actions on $f, g$:

$$s_2 : f \rightarrow \frac{h_2(e_1 g - 1)f}{-h_2 - e_1 e_6 fg + e_1 h_2 g - e_1 e_6},$$

$$s_6 : g \rightarrow \frac{e_7 g (e_1 - f)}{-e_7 f + (e_1 e_5 - h_1)fg + h_1},$$

$$\pi_1 : g \rightarrow h_2 g, f \rightarrow \frac{e_1 e_2 (1 - fg)}{h_2 e_1 g + f - (e_1 + e_2)fg},$$

$$\pi_2 : f \rightarrow g.$$  \hspace{1cm} (8.138)

8.4.4. $q$-$\mathcal{P}(D_6^{(1)}/A_3^{(3)})$. Point configuration:

$$(f_i, g_i) = \left( \infty, \frac{1}{e_i} \right) \quad (i = 1, 2), \quad \left( e_i, \infty \right) \quad (i = 3, 4), \quad \left( 0, \frac{e_i}{h_2} \right) \quad (i = 5, 6), \quad \left( \frac{h_i}{e_i}, 0 \right) \quad (i = 7, 8). \hspace{1cm} (8.139)$$

Generators and actions on parameters:

$$s_0 = s_{78}, \quad s_1 = s_{34}, \quad s_2 = s_{H_6-E_7}, \quad s_3 = s_{H_6-E_8}, \quad s_4 = s_{12}, \quad s_5 = s_{86},$$

$$\pi_1 : h_1 \rightarrow h_1^{-1}, e_i \rightarrow e_i^{-1}, \quad \sigma = \begin{pmatrix} 12345678 \\ 12785634 \end{pmatrix},$$

$$\pi_2 : h_1 \rightarrow \frac{1}{h_2}, h_2 \rightarrow \frac{1}{h_1}, e_i \rightarrow e_i^{-1}, \quad \sigma = \begin{pmatrix} 12345678 \\ 78563412 \end{pmatrix}. \hspace{1cm} (8.140)$$
Actions on $f$, $g$:

\[
\begin{align*}
 s_2 & : g \rightarrow g \frac{f - e_3}{f - h_1 e_1}, \\
 s_3 & : f \rightarrow f \frac{g - \frac{1}{e_1}}{g - \frac{h_1}{e_2}}, \\
 \pi_1 & : f \rightarrow \frac{f}{h_1}, \quad g \rightarrow \frac{1}{g}, \\
 \pi_2 & : f \rightarrow \frac{1}{h_2 g}, \quad g \rightarrow h_2 f.
\end{align*}
\] (8.141)

8.4.5. $q$-$P(A^{(1)}_4 / A^{(1)}_2)$. Point configuration:

\[
(f, g) = \left( \infty, \frac{1}{e_1} \right), \left( -\frac{e_2 e_3}{e}, -\frac{1}{e} \right), (e_4, \infty), \left( 0, \frac{e_1}{h_2} \right), \left( 0, \frac{h_1}{e_8} \right) (i = 5, 6), \left( \frac{h_1}{e_7}, 0 \right) (i = 7, 8). \tag{8.142}
\]

Generators and actions on parameters:

\[
\begin{align*}
 s_0 &= s_7 b, \quad s_1 = s_{h_1 - e_4 - e_5}, \quad s_2 = s_{h_2 - e_4 - e_5}, \quad s_3 = s_{h_1 + h_2 - e_4 - e_5}, \quad s_4 = s_{h_1 + h_2 - e_4 - e_5 - e_7}, \\
 \pi_1 : e_2 & \rightarrow \frac{h_2}{e_5}, \quad e_2 \rightarrow \frac{h_1 h_2}{e_2 e_5 e_7}, \quad e_1 \rightarrow e_{\sigma(0)}, \quad \sigma = \left( \begin{array}{c} 13468 \\ 26513 \end{array} \right) (i = 2, 5, 7), \\
 h_1 & \rightarrow \frac{h_1 h_2}{e_2 e_5}, \quad h_2 \rightarrow \frac{h_1 h_2}{e_2 e_7}. \\
 \pi_2 : e_1 & \rightarrow e_{\sigma(0)}^{-1}, \quad \sigma = \left( \begin{array}{c} 12345678 \\ 42317856 \end{array} \right), \quad h_1 \rightarrow \frac{1}{h_2}, \quad h_2 \rightarrow \frac{1}{h_1}. \tag{8.143}
\end{align*}
\]

Actions on $f$, $g$:

\[
\begin{align*}
 s_1 : g & \rightarrow g \frac{e_7 (e_4 - f)}{h_1 - e_1 f g}, \quad s_2 : f \rightarrow \frac{h_2 (1 - e_1 g)}{e_1 (e_5 - h_2 g) f}, \\
 s_4 : f & \rightarrow \frac{h_1 h_2}{e_2 e_3 e_7 - h_1 e_5 + e_5 e_7 f + h_1 h_2 g f}, \quad g \rightarrow \frac{e_5 e_7 e_2 e_3 e_5 + h_5 f + h_2 e_2 e_3 e_5 g}{h_2 - h_1 e_5 + e_5 e_7 f + h_1 h_2 g}, \\
 \pi_1 : f & \rightarrow \frac{h_1 e_5 - h_2 g}{e_5 f}, \quad g \rightarrow \frac{1}{h_1 h_2} \frac{-h_1 e_5 + e_5 e_7 f + h_1 h_2 g}{e_5 g}, \\
 \pi_2 : f & \rightarrow g. \tag{8.144}
\end{align*}
\]

8.4.6. $q$-$P(E^{(1)}_8 / A^{(1)}_6; b)$. Point configuration:

\[
(f, g) = \left( \infty, \frac{1}{e_1} \right), \left( -\frac{e_2 e_3}{e}, -\frac{1}{e} \right), (e_4, \infty), \left( 0, \frac{e_5}{h_2} \right), \left( -\frac{h_1 h_2}{e_6 e_7}, e \right), \left( \frac{h_1}{e_8}, 0 \right) \tag{8.145}
\]

Generators and actions on parameters:

\[ s_0 = s_{H_1 + H_2 - E_1 - E_3 - E_2}, \quad s_1 = s_{H_1 - E_1 - E_3}, \quad s_2 = s_{H_1 - E_3}, \]
\[ s_3 = s_{H_1 + H_2 - E_1 - E_3 - E_2}, \quad s_4 = s_{H_1 + H_2 - E_1 - E_3 - E_2}, \]

\[ \pi_1 : \begin{array}{c}
    h_1 \to h_2^{-1}, \quad h_2 \to h_1^{-1}, \quad e_i \to e_{i(i)}^{-1}, \quad \sigma = \left( \begin{array}{c}
        12345678 \\
        42318675 
    \end{array} \right), \\
    e_0 \to h_1 h_2, \quad e_2 \to h_2, \quad e_6 \to e_6,
\end{array} \]

\[ \pi_2 : \begin{array}{c}
    e_2 \to h_2, \quad e_6 \to e_6, \quad e_i \to e_{i(i)}^{-1} (i = 2, 6), \quad \sigma = \left( \begin{array}{c}
        134578 \\
        345781 
    \end{array} \right), \\
    h_1 \to h_2, \quad h_2 \to \frac{h_1 h_2}{e_2 e_6}.
\end{array} \]  

(8.146)

Actions on \( f, g \):

\[ s_0 : f \to \frac{f h_1 h_2 (e_2 e_3 g + f)}{e_2 e_3 (e_1 e_7 f + g h_1 h_2)}, \quad g \to \frac{e_6 e_7 g (e_2 e_3 g + f)}{e_6 e_7 f + g h_1 h_2}, \]

\[ s_1 : g \to e_6 f - h_1, \]

\[ s_2 : f \to \frac{f h_1 h_2 (e_1 g - 1)}{e_1 (g h_1 - e_5)}, \]

\[ s_3 : f \to \frac{f h_1 h_2 (e_6 (e_2 e_3 g + f) - h_1)}{e_2 e_6 e_6 (e_1 e_7 f - h_1 + g h_1 h_2)}, \quad g \to \frac{e_6 e_7 g (e_2 e_3 (g h_2 - e_5) + h_2)}{h_2 (e_6 (e_1 e_7 f - h_1 + g h_1 h_2))}, \]

\[ s_4 : f \to \frac{f h_1 h_2 (e_1 (g f - e_4) - f)}{e_1 (e_4 (e_6 e_7 f + g h_1 h_2) - f g h_1 h_2)}, \quad g \to \frac{e_6 e_7 g (e_6 (e_1 (g f - e_4) - f))}{e_6 e_7 f (e_1 g - 1) - g h_1 h_2}, \]

\[ \pi_1 : f \to g, \]

\[ \pi_2 : f \to g h_2, \quad g \to -\frac{e_2 g}{f}. \]  

(8.147)

8.4.7. \( q-P(E^{[1]}_8)/A^{[1]}_9; a \). Point configuration:

\[ (f, g) = \left( -\frac{h_1}{e_1 e_6}, \frac{e_1 e_6}{e}, \frac{1}{e}, \frac{1}{e} \right), (e_4, \infty), \left( 0, \frac{e_i}{h_2} \right) (i = 5, 6), \left( \frac{h_1}{e_7}, 0 \right) \]  

(8.148)

Generators and actions on parameters:

\[ s_0 = s_{H_1 + H_2 - E_1 - E_3 - E_2}, \quad s_1 = s_{H_1 - E_1 - E_3}, \quad s_2 = s_{H_1 - E_3}, \]
\[ s_3 = s_{H_1 + H_2 - E_1 - E_3 - E_2}, \quad s_4 = s_{H_1 + H_2 - E_1 - E_3 - E_2}, \]

\[ \pi_1 : \begin{array}{c}
    h_1 \to \frac{e_4 e_6}{h_1 h_2}, \quad h_2 \to \frac{e_1 e_6}{h_1 h_2}, \quad e_i \to e_{i(i)}^{-1}, \quad \sigma = \left( \begin{array}{c}
        23578 \\
        23875 
    \end{array} \right), \\
    e_1 \to \frac{e_6}{h_1}, \quad e_4 \to \frac{e_6}{h_2}, \quad e_6 \to \frac{e_1 e_6}{h_1 h_2},
\end{array} \]

\[ \pi_2 : \begin{array}{c}
    e_1 \to \frac{h_1 h_2}{e_1 e_6}, \quad e_2 \to \frac{h_2}{e_2}, \quad e_6 \to \frac{h_1 h_2}{e_2 e_6}, \quad e_8 \to \frac{h_2}{e_6},
\end{array} \]

\[ e_i \to e_{i(i)}, \quad \sigma = \left( \begin{array}{c}
        34578 \\
        45781 
    \end{array} \right), \quad h_1 \to \frac{h_1 h_2^2}{e_1 e_2 e_6}, \quad h_2 \to \frac{h_1 h_2}{e_2 e_6}. \]  

(8.149)
Actions on \( f, g \):

\[
s_0 : f \rightarrow \frac{h_1 h_2 f(e_1 f + e_2 e_1 e_2 g - h_1)}{e_2 e_1 f(e_1 f - e_6 h_1 + h_1 h_2 g)}, \quad g \rightarrow \frac{e_6 e_1 g(e_2 e_4 e_5 + h_2 f + e_2 e_3 h_2 g)}{h_2 (e_6 e_1 f - e_6 h_1 + h_1 h_2 g)}.
\]

\[
s_1 : f \rightarrow \frac{h_2 f(e_1 e_6 + e_6 e_0 + h_2 f g)}{e_1 e_6 (-e_2 h_2 g + e_6 e_0 + h_2 f g)}, \quad g \rightarrow \frac{e_1 e_6 e_6 g(e_4 h_2 g - e_4 e_0 - h_2 f g)}{h_2 (e_6 h_1 + e_1 e_6 e_6 g - h_1 h_2 g)}.
\]

\[
s_3 : f \rightarrow \frac{h_1 h_2^2 f(e_1 e_6 e_6 h_2 f g + (h_2 g - e_6) (e_1 e_2 e_3 e_5 e_6 g - h_1 h_2))}{e_1 e_6 e_6 e_6 g (e_2 e_3 (e_6 - h_2 g) (e_6 - h_2 g) + h_2^2 f g)} \times \frac{(e_1 e_6 e_6 h_2 f g + (h_2 g - e_6) (e_1 e_2 e_3 e_5 e_6 g - h_1 h_2))}{(h_1 (e_5 - h_2 g) (e_6 - h_2 g) + e_1 e_6 e_6 e_6 g)} \times \frac{e_1 e_6 e_6 e_6 g (e_2 e_3 (e_6 - h_2 g) (e_6 - h_2 g) + h_2^2 f g)}{h_2 (e_5 - h_2 g) (e_6 - h_2 g) + e_1 e_6 e_6 e_6 g},
\]

\[
g \rightarrow \frac{h_2^2 (h_1 (e_5 - h_2 g) (e_6 - h_2 g) + e_1 e_6 e_6 e_6 g)}{e_1 e_6 (h_2 g - e_6) + e_1 e_6 e_6 e_6 g},
\]

\[
s_4 : g \rightarrow e_7 f - h_1,
\]

\[
\pi_1 : f \rightarrow -e_4 h_2 g + e_4 e_0 + h_2 f g, \quad g \rightarrow \frac{h_2 f g}{h_2 g - e_6},
\]

\[
\pi_2 : f \rightarrow \frac{h_2 g (e_2 e_3 h_2 g - e_2 e_4 e_6 + h_2 f)}{e_2 e_3 (h_2 g - e_6)}, \quad g \rightarrow -\frac{e_2 (h_2 g - e_6)}{h_2 f}. \tag{8.150}
\]

**Remark 8.5.** The two realizations of affine Weyl groups on parameters associated with \( q \cdot P(E_6^{(1)}/A_2^{(1)}; b) \) and \( q \cdot P(E_6^{(1)}/A_2^{(1)}; a) \) are transformed with each other by the reflection \( s_{h_1 - e_7 - e_6} \). Also, the actions on \((f, g)\) variables associated with the former is transformed to that of latter by the substitution \( f \rightarrow f g^{-1} \). Conversely, the latter is transformed to the former by \( f \rightarrow f g^{-1} \).

**8.4.8.** \( q \cdot P(E_6^{(1)}/A_2^{(1)}; b) \). Point configuration:

\[
(f, g) = \left( \infty, \frac{1}{e_7}, \left( -\frac{e_2 e_3}{e}, \frac{1}{e^2}, \frac{h_1 h_2^2}{e_5 e_6 e_7 e^2} \right), \frac{h_1}{e_6}, 0 \right). \tag{8.151}
\]

Generators and actions on parameters:

\[
s_0 = s_{h_1 + 2 h_2 - e_7 - e_6 - e_5 - e_4 - e_3}, \quad s_1 = s_{h_1 - e_4 - e_3},
\]

\[
\pi_1 : h_1 \rightarrow \frac{e_1 e_2 e_3 e_6}{h_1 h_2^2}, \quad h_2 \rightarrow \frac{1}{h_2}, \quad e_1 \rightarrow e_1^{-1}, \quad e_2 \rightarrow e_2, \quad e_3 \rightarrow e_3, \quad e_4 \rightarrow e_4, \quad e_5 \rightarrow e_5, \quad e_6 \rightarrow e_6, \quad e_7 \rightarrow e_7, \quad e_8 \rightarrow e_8, \quad \sigma = \left( \begin{array}{cc} 3478 \\ 4387 \end{array} \right).
\]

\[
\pi_2 : h_1 \rightarrow \frac{1}{h_1}, \quad h_2 \rightarrow \frac{e_2 e_3}{h_1 h_2}, \quad e_2 \rightarrow \frac{e_2}{h_2}, \quad e_3 \rightarrow \frac{e_3}{h_2}, \quad e_4 \rightarrow e_4, \quad e_5 \rightarrow e_5, \quad e_6 \rightarrow e_6, \quad e_7 \rightarrow e_7, \quad e_8 \rightarrow e_8, \quad \sigma = \left( \begin{array}{cc} 134678 \\ 318674 \end{array} \right). \tag{8.152}
\]

Actions on \( f, g \):

\[
s_0 : s_0 = \pi_0 s_1 \pi_0,
\]

\[
s_1 : g \rightarrow \frac{e_8 (f - e_4) g}{e_8 f - h_1}.
\]
\[ \pi_1 : f \to -\frac{g^2e_2}{f(g - \frac{1}{e_1})}, \quad g \to gh_2, \]
\[ \pi_2 : f \to \frac{f}{h_1}, \quad g \to -\frac{f}{ge_2}. \tag{8.153} \]

**Remark 8.6.**

- The reflection corresponding to \( \alpha_2 = H_2 + 2E_1 - 2E_2 - 2E_3 + E_4 - E_8 \) (\( \alpha_2^2 = (\alpha_2, \alpha_2) = -\alpha_2 \cdot \alpha_2 = 14 \)) does not exist.
- We omitted the explicit formula of action of \( s_0 \) on \( f \) and \( g \), since their expressions are long and obtained as \( \pi s_1 \pi \).

### 8.4.9. \( q \)-P(\( E_8^{(1)}/A_9^{(1)} ; a \))

**Point configuration:**
\[ (f_0, g_0) = \left( -\frac{h_1}{e_1e_8}, \epsilon \right)_2 \left( -\frac{e_2e_3}{e}, \frac{1}{e} \right)_2, (e_4, \infty), \left( 0, \frac{e_5}{h_2} \right), \left( -\frac{h_1h_2}{e_6e_7}, \epsilon \right)_2. \tag{8.154} \]

**Generators and actions on parameters:**
\[ s_0 = s_{h_1+H_2-E_2-E_3-E_4-E_5}, \quad s_1 = s_{h_1+H_2-E_2-E_3-E_4-E_5}, \]
\[ \pi : e_i \to e_i^{-1}, \quad \sigma = \left( \begin{array}{c} 134678 \\ 643187 \end{array} \right), \quad e_2 \to \frac{e_2}{h_2}, \quad e_5 \to \frac{e_5}{h_2}, \]
\[ h_1 \to \frac{e_2e_5}{h_1h_2}, \quad h_2 \to \frac{1}{h_2}, \]
\[ \pi_2 : e_i \to e_i^{-1}, \quad \sigma = \left( \begin{array}{c} 4678 \\ 8674 \end{array} \right), \]
\[ e_1 \to \frac{e_2}{h_2}, \quad e_2 \to \frac{e_1e_2e_5}{h_1h_2}, \quad e_3 \to \frac{e_5}{h_2}, \quad e_5 \to \frac{e_2e_3e_5}{h_1h_2}, \]
\[ h_1 \to \frac{e_1e_2e_3e_5}{h_1h_2^2}, \quad h_2 \to \frac{e_2e_5}{h_1h_2}. \tag{8.155} \]

**Actions on \( f \), \( g \):**
\[ s_0 : f \to \frac{fh_1h_2(e_2e_3g + f)}{e_2e_3(e_2e_3f + gh_1h_2)}, \quad g \to \frac{e_0e_5g(e_2e_3g + f)}{e_0e_5f + gh_1h_2}, \]
\[ s_1 : f \to \frac{fh_2(e_1e_8fg - e_1e_8e_8g + h_1)}{e_1e_8(-e_1gh_2 + e_4e_5 + fgh_2)}, \quad g \to \frac{e_1e_8e_8g(egh_2 - e_4e_5 - fgh_2)}{h_2(e_1e_8e_8fg + e_0e_5 - gh_1h_2)}, \]
\[ \pi : f \to -\frac{e_2(gh_2 - e_3)}{fh_2}, \quad g \to gh_2, \]
\[ \pi_2 : f \to \frac{e_1e_5g(e_2e_3g + fgh_2 - e_2e_3e_5 + fh_2)}{h_1h_2(gh_2 - e_3)}, \quad g \to \frac{fh_2}{e_2(gh_2 - e_3)}. \tag{8.156} \]
Remark 8.7. The two realizations of affine Weyl groups on parameters associated with $q$-$P(E_8^{(1)}/A_6^{(1)}; b)$ and $q$-$P(E_8^{(1)}/A_6^{(1)}; a)$ are transformed with each other by the reflection $s_{H_i - E_i - E_i}$. Also, the actions on $(f, g)$ variables associated with the former is transformed to that of latter by the substitution $f \rightarrow f - \frac{g}{g}$. Conversely, the latter is transformed to the former by $f \rightarrow f + \frac{g}{g}$.

8.4.10. $q$-$P(A_7^{(1)}/A_6^{(1)})$. Point configuration:

$$(f, g) = \left( \frac{e_1 e_2 e_3}{e^2}, \frac{1}{e} \right)_2, \quad (e_4, \infty), \quad \left( \frac{h_1 h_2^2}{e_5 e_6 e_7}, \frac{e}{e} \right)_3, \quad \left( \frac{h_1}{e_8}, 0 \right). \quad (8.157)$$

Generators and actions on parameters:

$s_0 = s_{H_i + 2H_i - E_i - E_i - E_i - E_i - E_i}$,
$s_1 = s_{E_i - E_i - E_i}$,

$\pi_1: e_1 \rightarrow e_1^{-1}, \quad \sigma = \frac{234678}{238674}$,
$e_1 \rightarrow \frac{e_1}{h_1}, \quad e_5 \rightarrow \frac{e_5}{h_1}$,
$h_1 \rightarrow h_1^{-1}, \quad h_2 \rightarrow \frac{e_1 e_5}{h_1 h_2}$,

$\pi_2: e_1 \rightarrow e_1^{-1}, \quad \sigma = \frac{3478}{4783}$,
$e_1 \rightarrow \frac{h_1 h_2}{e_1 e_3 e_6}, \quad e_2 \rightarrow \frac{h_2}{e_1}$,
$e_5 \rightarrow \frac{h_1 h_2^2}{e_1 e_2 e_5}, \quad e_6 \rightarrow \frac{h_2}{e_5}, \quad h_1 \rightarrow \frac{h_1 h_2^2}{e_1 e_3 e_6}, \quad h_2 \rightarrow \frac{h_1 h_2^2}{e_1 e_5}. \quad (8.158)$

Actions on $f, g$:

$s_0: f \rightarrow f (f - g^2 e_1 e_2 e_3 h_1^2 h_2^2 e_7^2), \quad g \rightarrow g (g^2 e_1 e_2 e_3 - f) e_5 e_6 e_7$,

$s_1: g \rightarrow g (f - e_2) e_8$,

$\pi_1: f \rightarrow \frac{f}{h_1}, \quad g \rightarrow -\frac{g}{e_1}$,

$\pi_2: f \rightarrow \frac{g^2 h_1 h_2^2}{f e_5 e_6}, \quad g \rightarrow -\frac{g e_1}{f}. \quad (8.159)$

8.4.11. $q$-$P(A_6^{(1)}/A_6^{(1)})$. Point configuration:

$$(f, g) = \left( -\frac{h_1}{e_1 e_8}, \frac{1}{e} \right)_2, \quad \left( -\frac{e_1 e_4}{e}, \frac{1}{e} \right)_2, \quad (e_4, \infty), \quad \left( \frac{h_1 h_2^2}{e_5 e_6 e_7}, \frac{e}{e} \right)_3. \quad (8.160)$$

Generators and actions on parameters:

$\pi_1: e_1 \rightarrow e_1^{-1}, \quad \sigma = \frac{234678}{67184}$,
$e_1 \rightarrow \frac{h_2}{e_2}, \quad e_4 \rightarrow \frac{h_1 h_2}{e_2 e_3 e_5}, \quad e_5 \rightarrow \frac{h_1}{e_2}$,

$h_1 \rightarrow \frac{h_1 h_2}{e_2 e_5}, \quad h_2 \rightarrow \frac{h_1 h_2}{e_1 e_3}$.
\( \pi_2 : e_i \rightarrow e^{-1}_{\nu(i)} \), \( \sigma = \left( \begin{array}{cccc}
134678 \\
643187
\end{array} \right), \quad e_2 \rightarrow \frac{e_2}{h_2}, \quad e_3 \rightarrow \frac{e_3}{h_2}, \quad e_5 \rightarrow \frac{h_1h_2}{e_1e_2e_5}, \quad h_1 \rightarrow \frac{e_2e_5}{h_1h_2}, \quad h_2 \rightarrow h_2^{-1}. \) \hspace{1cm} (8.161)

Actions on \( f, g \):

\[ \begin{align*}
\pi_1 : f & \rightarrow -\frac{h_1h_2g}{e_5f}, \quad g \rightarrow \frac{e_2e_3}{h_2(f + e_2e_3g)}, \\
\pi_2 : f & \rightarrow -\frac{e_2e_5}{f}, \quad g \rightarrow h_2g.
\end{align*} \hspace{1cm} (8.162) \]

**Remark 8.8.**

- The reflections corresponding to \( \alpha_0 = H_1 + 2H_2 - 2E_2 - 2E_3 + E_4 - E_5 - E_6 - E_7 \) and \( \alpha_1 = H_1 - E_i + E_2 + E_3 - 2E_4 - E_8 \) (\( |\alpha_0|^2 = |\alpha_1|^2 = 8 \)) do not exist.
- The translation is given by \( \pi \).

8.4.12. \( d-P(E_1^{(1)}/A_5^{(1)}) \). Point configuration:

\( (f_i, g_i) = (e_i(e_i - h_1), e_i(e_i - h_2)) \) \( (i = 1, \ldots, 8) \). \hspace{1cm} (8.163)

Generators and actions on parameters:

\[ s_0 = s_{12}, \quad s_1 = s_{H_1 - H_2}, \quad s_2 = s_{H_1 - E_i - E_2}, \quad s_3 = s_{33}, \]
\[ s_4 = s_{34}, \quad s_5 = s_{35}, \quad s_6 = s_{36}, \quad s_7 = s_{37}, \quad s_8 = s_{38}. \] \hspace{1cm} (8.164)

Actions on \( f, g \):

\[ \begin{align*}
\pi_1 : f & \rightarrow g, \\
\pi_2 : f & \rightarrow \frac{1}{(e_1 + e_2 - h_2)f - (e_1 + e_2 - h_1)g + e_1e_2(h_1 - h_2)} \\
& \times [(h_1 - h_2)fg - (e_1 - h_2)(e_2 - h_2)(e_1e_2 - h_1)f \\
& + (e_1 - h_1)(e_2 - h_1)(e_1 + e_2 - h_2)g],
\end{align*} \hspace{1cm} (8.165) \]

8.4.13. \( d-P(E_7^{(1)}/A_3^{(1)}) \). Point configuration:

\( (f_i, g_i) = (e_i, -e_i) \) \( (i = 1, \ldots, 4) \), \( (h_1 - e_i, e_i - h_2) \) \( (i = 5, \ldots, 8) \). \hspace{1cm} (8.166)

Generators and actions on parameters:

\[ s_0 = s_{H_1 - H_2}, \quad s_1 = s_{34}, \quad s_2 = s_{23}, \quad s_3 = s_{12}, \]
\[ s_4 = s_{34}, \quad s_5 = s_{35}, \quad s_6 = s_{36}, \quad s_7 = s_{37}, \quad s_8 = s_{38}, \] \hspace{1cm} (8.167)

Actions on \( f, g \):

\[ \begin{align*}
s_0 : f & \rightarrow -g, \quad g \rightarrow -f, \\
s_4 : f & \rightarrow -(h_1 - h_2)fg - (e_1 + e_5 - h_1)(e_5 - h_2)f - (e_1 + e_5 - h_2)(e_5 - h_1)g \\
& \quad \times [(e_1 + e_5 - h_2)f + (e_1 + e_5 - h_1)g - e_1(h_1 - h_2)], \\
\pi : f & \rightarrow -h_1, \quad g \rightarrow g + h_2.
\end{align*} \hspace{1cm} (8.168) \]
8.4.14. $dPE^{(1)}_6 / A_2^{(1)}$. Point configuration:

$(f, \ g) = (e_i - e_i) \ (i = 1, \ldots, 4), \ (\infty, e_i - h_j) \ (i = 5, 6), \ (h_i - e_j, \infty) \ (i = 7, 8). \quad (8.169)$

Generators and actions on parameters:

$s_0 = s_7, \ s_1 = s_65, \ s_2 = s_{h_i-E_j}, \ s_3 = s_{h_j-E_i},$

$s_4 = s_{23}, \ s_5 = s_{34}, \ s_6 = s_{h_i-E_j},$

$e_i \rightarrow -e_i, \ \sigma = \left\{ \begin{array}{ll}
345678 & (i \neq 1, \ 2),
654378 & (i = 1, \ 2),
\end{array} \right.$

$e_i \rightarrow e_i, \ \sigma = \left\{ \begin{array}{ll}
12345678 & (i \neq 1, \ 2),
12348765 & (i = 1, \ 2),
\end{array} \right.$

(ii)

Actions on $f, \ g$:

$s_2 : f \rightarrow g(f - e) - (e_i - h_2)(f + g), \ \ g \rightarrow \frac{g}{f - e},$

$s_6 : g \rightarrow f(g + e_i + (e_i - h_2)(f + g)), \ \ f \rightarrow \frac{f}{g},$

$\pi_1 : f \rightarrow g(f - e_i) - (e_i - h_2)(f + g), \ \ g \rightarrow g + h_2,$

$\pi_2 : f \leftrightarrow g. \quad (8.170)$

8.4.15. $dP(D_4^{(1)} / D_4^{(1)})$. In the following additive cases, we use the root parameters $a_i$ instead of the parameters $h_i, \ e_i, \text{and variables} \ (q, p)$ as the dependent variables.

Point configuration:

$(f_i, g) = (q_i, q_i, p_i) = (\infty, -a_2), \ (\infty, -a_1 - a_2), \ (1(1 + a_0 e), \frac{1}{e}), \ (0, 0), \ (0, a_4), \left(1 + a_3 e, \frac{1}{e}\right)^2. \quad (8.172)$

Generators and actions on parameters:

$s_0 = s_{h_i-E_j-E_k}, \ s_1 = s_{E1-E_2}, \ s_2 = s_{h_i-E_j-E_k}, \ s_3 = s_{h_i-E_1-E_j}, \ s_4 = s_{E_1-E_2},$

$s_0 : \ a_0 \rightarrow -a_0, \ a_2 \rightarrow a_0 + a_2,$

$s_1 : \ a_1 \rightarrow -a_1, \ a_2 \rightarrow a_1 + a_2,$

$s_2 : \ a_0 \rightarrow a_0 + a_2, \ a_1 \rightarrow a_1 + a_2, \ a_2 \rightarrow a_2 + a_3, \ a_3 \rightarrow a_2 + a_4, \ a_4 \rightarrow a_2 + a_4,$

$s_3 : \ a_2 \rightarrow a_2 + a_3, \ a_3 \rightarrow -a_3,$

$s_4 : \ a_2 \rightarrow a_2 + a_4, \ a_4 \rightarrow -a_4,$

$\pi_1 : \ a_3 \rightarrow a_4, \ a_4 \rightarrow a_3,$

$\pi_2 : \ a_0 \rightarrow a_3, \ a_3 \rightarrow a_0,$

$\pi_3 : \ a_1 \rightarrow a_4, \ a_4 \rightarrow a_1. \quad (8.173)$

Actions on $p, \ q$:

$s_0 : \ p \rightarrow p - \frac{a_0}{q - t},$

$s_2 : \ q \rightarrow q + \frac{a_2}{p}. \quad (8.174)$
\[ s_3 : \ p \to p - \frac{a_3}{q - 1}, \]
\[ s_4 : \ p \to p - \frac{a_4}{q}, \]
\[ \pi_1 : \ p \to -p, \ q \to 1 - q, \ t \to 1 - t, \]
\[ \pi_2 : \ p \to pt, \ q \to \frac{q}{t}, \ t \to \frac{1}{t}, \]
\[ \pi_3 : \ p \to -q(qp + a_2), \ q \to \frac{1}{q}, \ t \to \frac{1}{t}. \]  
(8.174)

**8.4.16. \( d-P(A_3^{(1)}/D_6^{(1)}) \).** Point configuration:

\[ (q, p) = \left( \frac{1}{e}, -t - a_0 e \right)_2, \left( \frac{1}{e}, -a_2 e \right)_2, \left( a_1 e, \frac{1}{e} \right)_2, \left( 1 + a_3 e, \frac{1}{e} \right)_2, \]
\[ a_0 + a_1 + a_2 + a_3 = 1. \]  
(8.175)

Generators and actions on parameters:

- \( s_0 = s_{H_1 - E_1 - E_2}, \ s_1 = s_{H_2 - E_2 - E_3}, \ s_2 = s_{H_3 - E_3 - E_4}, \ s_3 = s_{H_4 - E_4 - E_5}, \)
- \( s_0 : a_0 \to -a_0, \ a_1 \to a_0 + a_1, \ a_3 \to a_3 + a_0, \)
- \( s_1 : a_0 \to a_0 + a_1, \ a_1 \to -a_1, \ a_2 \to a_1 + a_2, \)
- \( s_2 : a_1 \to a_1 + a_2, \ a_2 \to -a_2, \ a_3 \to a_2 + a_3, \)
- \( s_3 : a_0 \to a_0 + a_3, \ a_2 \to a_2 + a_3, \ a_3 \to -a_3, \)
- \( \pi_1 : a_0 \to a_3, \ a_1 \to a_2, \ a_2 \to a_1, \ a_3 \to a_0, \)
- \( \pi_2 : a_0 \to a_2, \ a_2 \to a_0. \)  
(8.176)

Actions on \( q, p \):

- \( s_0 : q \to q + \frac{a_0}{p + t}, \)
- \( s_1 : p \to p - \frac{a_1}{q}, \)
- \( s_2 : q \to q + \frac{a_2}{p}, \)
- \( s_3 : p \to p - \frac{a_3}{q - 1}, \)
- \( s_4 : q \to -\frac{p}{t}, \ p \to (q - 1)t, \)
- \( \pi_1 : q \to -\frac{p}{t}, \ p \to qt, \ t \to -t, \)
- \( \pi_2 : p \to p + t, \ t \to -t. \)  
(8.177)

**8.4.17. \( d-P(2A_1^{(1)}/D_6^{(1)}) \).** Point configuration:

\[ (q, p) = \left( \frac{1}{e}, 1 - a_1 e \right)_2, \left( \frac{1}{e}, -a_2 e \right)_2, \left( e, -\frac{t}{e^2} + \frac{1 - a_1 - a_2}{e} \right)_4. \]  
(8.178)
Generators and actions on parameters:
\[ s_2 = s_{H_2 - E_3 - E_4}, \ s_1 = s_{H_2 - E_1 - E_2}, \]
\[ s_1: \ a_1 \rightarrow -a_1, \]
\[ s_2: \ a_2 \rightarrow -a_2, \]
\[ \pi_1: \ a_2 \rightarrow a_1, \ a_1 \rightarrow a_2, \]
\[ \pi_2: \ a_1 \rightarrow 1 - a_1, \] (8.179)

Actions on \( q, p \):
\[ s_1: \ q \rightarrow q + \frac{a_1}{p - 1}, \]
\[ s_2: \ q \rightarrow q + \frac{a_2}{p}, \]
\[ \pi_1: \ q \rightarrow -q, \ p \rightarrow 1 - p, \ t \rightarrow -t, \]
\[ \pi_2: \ q \rightarrow \frac{t}{q}, \ p \rightarrow -\frac{q(qp + a_2)}{t}. \] (8.180)

8.4.18. \( d\)-\( PA_{4}^{(1)}/D_{4}^{(1)} \). Point configuration:
\[ (q, p) = \left(-\frac{1}{\epsilon^2}, \epsilon + \frac{a_1}{2}\right)^4, \ \epsilon = \frac{t}{\epsilon^2} + \frac{1}{\epsilon}. \] (8.181)

Generators and actions on parameters:
\[ \pi_1: \ a_1 \rightarrow 1 - a_1, \]
\[ \pi_2: \ a_1 \rightarrow -a_1. \] (8.182)

Actions on \( q, p \):
\[ \pi_1: \ q \rightarrow tp, \ p \rightarrow -\frac{q}{t}, \ t \rightarrow -t, \]
\[ \pi_2: \ q \rightarrow -q - \frac{a_1}{p} - \frac{1}{p^2}, \ p \rightarrow -p, \ t \rightarrow -t. \] (8.183)

8.4.19. \( d\)-\( PA_{6}^{(1)}/D_{6}^{(1)} \). Point configuration:
\[ (f, g) = \left(-\frac{1}{\epsilon^2}, -\frac{1}{\epsilon} - \frac{1}{2}\right)^4, \ \left(-t\epsilon^2, \frac{1}{\epsilon}\right)^4. \] (8.184)

The symmetry group of this case is a finite group \( S_2 \) generated by
\[ \pi: \ q \rightarrow \frac{t}{q}, \ p \rightarrow -\frac{q(2qp + 1)}{2t}. \] (8.185)

8.4.20. \( d\)-\( PA_{6}^{(1)}/E_6^{(1)} \). Point configuration:
\[ (q, p) = \left(\frac{1}{\epsilon}, -a_2\epsilon\right)^2, \ \left(\frac{1}{\epsilon}, \frac{1}{\epsilon} + t - a_0\epsilon\right)^4, \ \left(a_1\epsilon, \frac{1}{\epsilon}\right)^2, \]
\[ a_0 + a_1 + a_2 = \delta. \] (8.186)
Generators and actions on parameters:

\[ s_0 = s_{H_1 + H_2 - E_1 - E_3 - E_4 - E_5}, \quad s_1 = s_{H_1 - E_3 - E_2}, \quad s_2 = s_{H_2 - E_3 - E_4}, \]

\[ s_0 : \ a_0 \rightarrow -a_0, \ a_1 \rightarrow a_0 + a_1, \ a_2 \rightarrow a_0 + a_2, \]

\[ s_1 : \ a_0 \rightarrow a_0 + a_1, \ a_1 \rightarrow -a_1, \ a_2 \rightarrow a_1 + a_2, \]

\[ s_2 : \ a_0 \rightarrow a_0 + a_2, \ a_1 \rightarrow a_1 + a_2, \ a_2 \rightarrow -a_2, \]

\[ \pi_1 : \ a_0 \rightarrow -a_0, \ a_1 \rightarrow -a_1, \ a_2 \rightarrow -a_1, \]

\[ \pi_2 : \ a_0 \rightarrow -a_2, \ a_1 \rightarrow -a_1, \ a_2 \rightarrow -a_0. \]  

(8.187)

Actions on \( q, p \):

\[ s_0 : \ q \rightarrow q + \frac{a_0}{p - q - t}, \ p \rightarrow p + \frac{a_0}{p - q - t}, \]

\[ s_1 : \ p \rightarrow p - \frac{a_1}{q}, \]

\[ s_2 : \ q \rightarrow q + \frac{a_2}{p}, \]

\[ \pi_1 : \ q \rightarrow -p, \ p \rightarrow -q, \]

\[ \pi_2 : \ p \rightarrow -p + q + t. \]  

(8.188)

Note that \( \pi_1 \) and \( \pi_2 \) change the sign of \( \delta \). In section 2.1 we have shown the composition \( \pi = \pi_1 \pi_2 \) as a Bäcklund transformation which preserves the constraint \( \delta = 1 \).

8.4.21. d-P(\( A(1)^{(3)} \)/\( E_7^{(3)} \)). Point configuration:

\[ (q_1, p_1) = \left( \frac{1}{\epsilon}, -a_1 \epsilon \right), \quad \left( \frac{1}{\epsilon}, \frac{2}{\epsilon^2} + t + (a_1 - 1) \epsilon \right). \]  

(8.189)

Generators and actions on parameters:

\[ s_0 = s_{2H_1 + H_2 - E_1 - E_3 - E_4 - E_5 - E_6}, \quad s_1 = s_{H_2 - E_3 - E_4}, \]

\[ \pi : \ a_1 \rightarrow 1 - a_1. \]  

(8.190)

Actions on \( q, p \):

\[ s_1 : \ q \rightarrow q + \frac{a_1}{p}, \]

\[ \pi : \ q \rightarrow -q, \ p \rightarrow -p + 2q^2 + t. \]  

(8.191)

8.5. Lax pairs

In this section, we mainly use the parameters, \( \kappa_1, \kappa_2, \nu_1, \ldots, \nu_8 \), with \( \kappa_1^2 \kappa_2^2 \nu_1^8 = q \prod_{i=1}^{8} \nu_i \) for multiplicative cases and \( 2\kappa_1 + 2\kappa_2 = \delta + \sum_{i=1}^{8} \nu_i \) for additive cases. \( T_z \) denotes the shift operator in \( z \) such that \( T_z : z \mapsto qz \) for multiplicative cases and \( T_z : z \mapsto z + \delta \) for additive cases. Also, \( T \) stands for the shift operator of the time evolution \( T = T_{\nu_1}^{-1}T_{\nu_2} \), and we write \( \mathcal{T} = T(f) \) and \( g = \mathcal{T}^{-1}(g) \). We also include the list of points configuration characterizing the equation \( L_{\lambda}(z) = 0 \) as a curve of degree \((3, 2)\) in \((f, g)\).
8.5.1. q-P($E_8^{(1)}/A_0^{(1)}$). As given in (7.65) and (7.59), the Lax pair for q-P($E_8^{(1)}/A_0^{(1)}$) is

\[
L_1 = \frac{w(f, g)\left(\frac{z - \kappa_1}{q}\right)}{g - g\left(\frac{\kappa_1}{q}\right)}\left\{\mathcal{F} - \mathcal{F}\left(\frac{z}{q}\right)\right\} + \frac{U\left(\frac{z}{q}\right)}{g - g\left(\frac{\kappa_1}{q}\right)}\left\{\mathcal{T} - g - g\left(\frac{\kappa_1}{z}\right)\right\} - T_{c-1} - g - g\left(\frac{\kappa_1}{z}\right)
\]

\[
L_2 = \left\{g - g\left(\frac{\kappa_1}{z}\right)\right\}T - \{g - g(z)\} - \left\{z - \kappa_1\right\}\{f - f(z)\}T,
\]

where \( f(z) = z + \kappa_1, \ g(z) = z + \kappa_2, \ U(z) = z^{-k} \prod_{i=1}^{8} (z - v_i) \) and \( w(f, g) \) is a rational function in \( f, g \) (independent of \( z \)) such as

\[
w(f, g)_{g=g(z)} = \left(1 - \frac{\kappa_2}{\kappa_1}\right)\left(\frac{U(z)}{f - f(z)} - \frac{U\left(\frac{\kappa_1}{z}\right)}{f - f\left(\frac{\kappa_1}{z}\right)}\right).
\]

The linear equation \( L_1 y(z) = 0 \) is uniquely characterized as a curve of degree (3, 2) in \( (f, g) \) passing through the 12 points:

\[
(f(u), g(u))_{u=\nu_1, \ldots, \nu_6, \frac{z}{\nu_5}}, \quad (f(u), \gamma_u)_{u=\frac{z}{\nu_5}},
\]

where \( \gamma_u - g\left(\frac{z}{u}\right) = \frac{y(u)}{y(qu)} \) for \( u = z, \frac{z}{q} \).

8.5.2. q-P($E_4^{(1)}/A_1^{(1)}$).

\[
L_1 = \frac{qB_2\left(\frac{1}{cg}\right)\left(1 - \frac{1}{c}\right)}{(1-cg)f - f\left(\frac{1}{cg}\right)} + \frac{B_1\left(\frac{1}{g}\right)(1 - c)}{(1 - cg)f - f\left(\frac{1}{g}\right)} + \frac{B_1\left(\frac{1}{q}\right)}{f - f\left(\frac{1}{q}\right)} \left\{T - q - zcg\right\} - T_{c-1} - q - zcg
\]

\[
L_2 = (1 - zcg)T - (1 - zg) + z(z - f)gT,
\]

where \( B_1(z) = \frac{1}{z} \prod_{i=1}^{4} \left(1 - \frac{1}{v_i}\right), \ B_2(z) = \frac{1}{z} \prod_{i=5}^{8} \left(1 - \frac{v_i}{\kappa_i}\right) \) and \( c = \frac{\kappa_2}{\kappa_1} \).

Point configuration:

\[
\left(v_i, \frac{1}{v_i}\right)_{i=1}^{4}, \left(\frac{\kappa_1}{v_i}, \frac{\kappa_1}{v_i}\right)_{i=5}^{8}, \left(\frac{z}{q}, \frac{q\kappa_1}{z}\right), \left(\zeta, \frac{1}{z}\right), \ (z, \gamma_z), \ \left(\frac{z}{q}, \gamma_z\right),
\]

where \( \gamma_z \) is given by \( \frac{1 - \kappa_2}{1 - \kappa_1} = \frac{y(qu)}{y(u)}(u = z, \frac{z}{q}). \)
8.5.3. $q$-$P(E_6^{(1)}/A_2^{(1)})$.

\[ L_1 = \frac{\sum_{i=1}^{4} (g v_i - 1)}{g(fg - 1)} \left( \frac{1}{g} - 1 \right) + \sum_{i=1}^{6} \frac{g v_i}{fg v_i v_8} \left( \frac{g v_i}{v_i} - 1 \right) + \sum_{i=1}^{4} \frac{v_i - \frac{z}{q}}{f - \frac{z}{q}} \left( \frac{g}{1 - g \frac{z}{q}} - T_z^{-1} \right) \]

\[ + \sum_{i=1}^{8} \frac{1}{q(f - z)} \left( \frac{1}{g} - z \right) - T_z \}

\]

\[ L_2 = \left( 1 - \frac{f}{z} \right) T + T_z - \left( \frac{1}{g} - z \right). \]  

(8.197)

Point configuration:

\[ \left( \frac{v_i}{v_i}, 1 \right), \left( 0, \frac{v_i}{\kappa_2} \right), \left( \frac{v_i}{\kappa_1}, 0 \right), \left( \frac{z}{q}, 0 \right), \left( \frac{z}{q}, \frac{1}{z} \right), (z, \gamma_i), \left( \frac{z}{q}, \gamma_i \right). \]  

(8.198)

where $\gamma_i$ is given by $\frac{1}{v_i} - u = \frac{v_i(y_0, v_0)}{y_0}$ ($u = z, \frac{z}{q}$).

**Remark 8.9.** In order to obtain the Lax pair of $q$-$P(E_6^{(1)}/A_2^{(1)})$ by degeneration from $q$-$P(E_5^{(1)}/A_2^{(1)})$, we need to apply the following gauge transformation

\[ y(z) \rightarrow z^{q \log(q)} G_1(z)y(z), \quad G_1(z/q) = -zG_1(z), \]  

(8.199)

together with the limiting procedure given in section 8.3.2.

8.5.4. $q$-$P(D_6^{(1)}/A_3^{(1)})$.

\[ L_1 = \left( \frac{2}{z} \sum_{i=1}^{4} (g v_i - 1) \right) - \sum_{i=1}^{4} \frac{v_i}{v_i} \sum_{i=5}^{6} \left( g - \frac{v_i}{\kappa_2} \right) \frac{v_i}{fg v_i v_8} \left( \frac{g v_i}{v_i} - 1 \right) + \sum_{i=1}^{4} \frac{v_i - \frac{z}{q}}{f - \frac{z}{q}} \left( \frac{g}{1 - g \frac{z}{q}} - T_z^{-1} \right) \]

\[ + \sum_{i=1}^{8} \frac{1}{q(f - z)} \left( \frac{1}{g} - z \right) - T_z \}

\]

\[ L_2 = \left( 1 - \frac{f}{z} \right) T + T_z - \left( \frac{1}{g} - z \right). \]  

(8.200)

Point configuration:

\[ \left( \frac{z}{q}, 0 \right), (z, \infty), \left( \frac{z}{q}, \frac{y(z)}{y(z)} \right), \left( \frac{z}{q}, \frac{y(z)}{y(z)} \right). \]  

(8.201)
Remark 8.10. In order to take the degeneration limit from $q$-$P(E_0^{(1)}/A_2^{(1)})$ to $q$-$P(D_5^{(1)}/A_3^{(1)})$ described in section 8.3.3, we need to change variable $z \rightarrow z\varepsilon$.

8.5.5. $q$-$P(A_4^{(1)}/A_4^{(1)})$.

$$L_1 = \frac{\prod_{i=5}^{6} \left( g - \frac{v_i}{\kappa_2} \right)}{fg} + \frac{\prod_{i=1}^{4} v_i}{qg} + \frac{\prod_{i=1}^{3} v_i}{f - \frac{z}{q}} (g - T_z^{-1})$$

$$+ \frac{\prod_{i=7}^{8} \left( z - \frac{\kappa_1}{v_i} \right)}{q(f - z)} \left( T_z - \frac{1}{g} \right).$$

(8.202)

In this case and the further degenerations the $L_2$ operators are omitted, since they are the same as $q$-$P(D_5^{(1)}/A_3^{(1)})$ case (8.200). The extra four points are also the same as $q$-$P(D_5^{(1)}/A_3^{(1)})$ case and will be omitted.

Point configuration:

$$\left( \infty, \frac{1}{v_1} \right), \left( -v_2 v_1 - \frac{1}{\epsilon}, \frac{1}{\epsilon} \right), (v_4, \infty), \left( 0, \frac{v_1}{\kappa_2} \right)_{i=5}^{6}, \left( \frac{\kappa_0}{v_i}, 0 \right)_{i=7}. \quad (8.203)$$

8.5.6. $q$-$P(E_3^{(1)}/A_6^{(1)}); a$.

$$L_1 = \frac{\prod_{i=5}^{6} \left( g - \frac{v_i}{\kappa_2} \right)}{fg} + \frac{\prod_{i=1}^{4} v_i}{qg} + \frac{\prod_{i=1}^{3} v_i}{f - \frac{z}{q}} (g - T_z^{-1})$$

$$- \frac{\kappa_1}{v_8} \frac{\prod_{i=7}^{8} \left( z - \frac{\kappa_1}{v_i} \right)}{q(f - z)} \left( T_z - \frac{1}{g} \right).$$

(8.204)

Point configuration:

$$\left( -\frac{\kappa_1}{v_1 v_8 \epsilon}, \epsilon \right), \left( -v_2 v_1 - \frac{1}{\epsilon}, \frac{1}{\epsilon} \right), (v_4, \infty), \left( 0, \frac{v_1}{\kappa_2} \right)_{i=5}^{6}, \left( \frac{\kappa_0}{v_7}, 0 \right). \quad (8.205)$$

8.5.7. $q$-$P(E_2^{(1)}/A_6^{(1)}); a$.

$$L_1 = \frac{\prod_{i=5}^{6} \left( g - \frac{v_i}{\kappa_2} \right)}{f} + \frac{\prod_{i=1}^{4} v_i}{qg} + \frac{\prod_{i=1}^{3} v_i}{f - \frac{z}{q}} (g - T_z^{-1})$$

$$- \frac{\kappa_1}{v_8} \frac{\prod_{i=7}^{8} \left( z - \frac{\kappa_1}{v_i} \right)}{q(f - z)} \left( T_z - \frac{1}{g} \right). \quad (8.206)$$
Point configuration:

\[
\left( -\frac{\kappa_1}{v_1 v_8 \epsilon}, \epsilon \right)_2, \left( -v_2 v_3 \frac{1}{\epsilon}, \frac{1}{\epsilon} \right)_2, \quad (v_4, \infty), \quad \left( 0, \frac{v_5}{\kappa_2} \right), \quad \left( -\frac{\kappa_1 \kappa_2}{v_6 v_7 \epsilon}, \epsilon \right)_2. \tag{8.207}
\]

8.5.8. $q$-P($A_1^{(1)}/\Lambda_1^{(1)}$).

\[
L_1 = \frac{g \prod_{i=1}^{4} v_i}{f} + \frac{z v_1}{q} + \frac{\left( \frac{z}{q} - v_4 \right) \prod_{i=1}^{3} v_i}{f - \frac{z}{q}} (g - T_z^{-1}) - \frac{\kappa_1}{v_8 q} \frac{z}{f - z} \left( T_z - \frac{1}{g} \right). \tag{8.208}
\]

Point configuration:

\[
\left( -\frac{\kappa_1}{v_1 v_8 \epsilon}, \epsilon \right)_2, \left( -v_2 v_3 \frac{1}{\epsilon}, \frac{1}{\epsilon} \right)_2, \quad (v_4, \infty), \quad \left( \frac{\kappa_1 \kappa_2^2}{v_5 v_6 v_7 \epsilon}, \epsilon \right)_2. \tag{8.209}
\]

8.5.9. $q$-P($E_3^{(1)}/\Lambda_3^{(1)}; b$).

\[
L_1 = \frac{g \prod_{i=1}^{4} v_i}{f} + \frac{z (gv_1 - 1)}{q g} + \frac{\left( \frac{z}{q} - v_4 \right) \prod_{i=1}^{3} v_i}{f - \frac{z}{q}} (g - T_z^{-1}) + \frac{\kappa_1}{v_8 q} \frac{z}{f - z} \left( T_z - \frac{1}{g} \right). \tag{8.210}
\]

Point configuration:

\[
\left( \infty, \frac{1}{v_1} \right), \left( -v_2 v_3 \frac{1}{\epsilon}, \frac{1}{\epsilon} \right)_2, \quad (v_4, \infty), \quad \left( 0, \frac{v_5}{\kappa_2} \right), \quad \left( -\frac{\kappa_1 \kappa_2}{v_6 v_7 \epsilon}, \epsilon \right)_2, \quad \left( \frac{\kappa_1}{v_8}, 0 \right). \tag{8.211}
\]

8.5.10. $q$-P($E_2^{(1)}/\Lambda_2^{(1)}; b$).

\[
L_1 = \frac{g \prod_{i=1}^{4} v_i}{f} + \frac{z (gv_1 - 1)}{q g} + \frac{\left( \frac{z}{q} - v_4 \right) \prod_{i=1}^{3} v_i}{f - \frac{z}{q}} (g - T_z^{-1}) + \frac{\kappa_1}{v_8 q} \frac{z}{f - z} \left( T_z - \frac{1}{g} \right). \tag{8.212}
\]
8.5.11. q-P(A(1) / A(1)).

\[ L_1 = \frac{g \prod_{i=1}^{4} v_i}{f} - z \frac{q}{qg} + \left( \frac{z - v_4}{q} \right) \prod_{i=1}^{3} v_i \left( g - T_1^{-1} \right) \]

\[ + \frac{z}{q} \frac{z - \kappa_1}{v_{8}} \left( T_1 - \frac{1}{g} \right). \]  

8.5.12. d-P(E(8) / A(3)).

\[ L_1 = \frac{2z - \delta - \kappa_1}{w} \left( \frac{z - \delta}{g - g(z - \delta)} \right) \]
\[ + \frac{U(z - \delta)}{(2z - 2\delta - \kappa_1) \{ f - f(z - \delta) \}} \left[ T_1^{-1} - \frac{g - g(\kappa_1 + \delta - z)}{g - g(z - \delta)} \right] \]
\[ + \frac{U(\kappa_1 - z)}{(2z - \kappa_1) \{ f - f(z) \}} \left[ T_1 - \frac{g - g(\kappa_1 - z)}{g - g(\kappa_1 - z)} \right]. \]

\[ L_2 = (g - g(\kappa_1 - z)) T_1 - (g - g(z)) - (2z - \kappa_1)(f - f(z)) T_1^{-1}, \]  

where \( f(z) = z(z - \kappa_1), \) \( g(z) = z(z - \kappa_2), \) \( U(z) = \prod_{i=1}^{8} (z - v_i) \) and \( w = \frac{(\kappa_1 - \kappa_2 - \delta)(\kappa_1 - \kappa_2)}{(f - f(t)) \{ f - f(t) \}}. \]
8.5.13. d-P(E\(E_7^{(1)}\)/A\(2^{(1)}\)).

\[
L_1 = (\kappa_1 - \kappa_2) \left[ (f + g - \kappa_1 + \kappa_2) \prod_{i=1}^{4} (g + v_i) \left( \frac{g - \kappa_1 + \kappa_2 + z + \delta}{(f + g)(g + z)} \right) \right] + \frac{f + g - \kappa_1 + \kappa_2}{z - f} \prod_{i=1}^{4} (z - v_i) \left( T_z^{-1} - \frac{g - \kappa_1 + \kappa_2 + z + \delta}{g + z} \right) + \frac{f + g - \kappa_1 + \kappa_2}{z - f + \delta} \prod_{i=5}^{8} (z + \delta - \kappa_1 + v_i) \left( T_z - \frac{g + z + \delta}{g - \kappa_1 + \kappa_2 + z + \delta} \right).
\]

\[
L_2 = (g + z)T_z^{-1} + (-g + \kappa_1 - \kappa_2 - z) - (f - z)TT_z^{-1}.
\]  

(8.219)

Point configuration:

\[
(v_r, -v_r)_{r=1}\quad (\kappa_1 - \kappa_2)_{r=5}^{8}\quad (z, \kappa_1 - \kappa_2 - z)\quad (z + \delta, -z - \delta)\quad (z, \gamma_z)\quad (z + \delta, \gamma_z + \delta).
\]

where \(\frac{\gamma_z - \kappa_1 + \kappa_2 + \mu}{\gamma_z + \mu} = \frac{y(u - \delta)}{y(u)} \Rightarrow (u = z, z + \delta).
\]

8.5.14. d-P(E\(E_6^{(1)}\)/A\(2^{(1)}\)).

\[
L_1 = -\prod_{i=1}^{4} (g + v_i) \left( f + g \right) + \prod_{i=5}^{8} (g + \kappa_2 - v_i)
\]

\[
+ \prod_{i=1}^{4} (z - v_i) \left( \frac{-g + z + 1}{f - z} \right),
\]

\[
+ \frac{(g + z) \prod_{i=7}^{8} (z + \delta - \kappa_1 + v_i)}{f - z - \delta} \left\{ (1 + g + z) - T_z \right\}.
\]

\[
L_2 = (g + z)T_z^{-1} - \delta - (f - z)TT_z^{-1}.
\]  

(8.221)

Point configuration:

\[
(v_r, -v_r)_{r=1}\quad (\infty, \kappa_1 - \kappa_2)_{r=5}\quad (\kappa_1 - \kappa_2, \infty)_{r=7}\quad (z, \infty)\quad (z + \delta, -z - \delta),
\]

\[
\left( z, -z + \frac{y(z)}{y(z - \delta)} \right)\quad \left( z + \delta, -z - \delta + \frac{y(z + \delta)}{y(z)} \right).
\]  

(8.222)

8.5.15. d-P(D\(4^{(1)}\)/D\(4^{(1)}\)). The following cases admit both the discrete flows and the continuous flows (i.e. Painlevé differential equations). Both flows can be described as (i) deformations of a linear differential equation and (ii) deformations of a linear difference equation. We use two different coordinates \((q, p)\) and \((f, g) = (q, qp)\) depending on the surfaces and the type of flows. The point configurations on \(P^1_q \times P^1_p\) and/or \(P^1_f \times P^1_g\) are shown schematically in figure 21. The corresponding configurations in blown-up space are given in figure 22. For simplicity, we use the root parameters \(a_i\) instead of the parameters \(\kappa_i, v_i\).
The continuous flow of the case $d\cdot P(D_4^{(3)}/D_4^{(1)})$ is $P_{VI}$ given by the Hamiltonian:

$$H = \frac{q(q - 1)(q - t)}{t(t - 1)} \left\{ p^2 - \left( \frac{a_0 - 1}{q - t} + \frac{a_3}{q - 1} + \frac{a_3}{q} \right) p \right\} + \frac{(q - t)a_2(a_1 + a_2)}{t(t - 1)} , \quad (8.223)$$

where $a_0 + a_1 + 2a_2 + a_3 + a_4 = 1$. In $(f, g) = (q, qp)$ coordinates, the eight points configuration is given by

$$(f, g) = (\infty, -a_2), (\infty, -a_1 - a_2), (t(1 + a_0 e), \frac{1}{e} ), (0, 0), (0, a_4), \left( 1 + a_3 e, \frac{1}{e} \right) . \quad (8.224)$$

(i) Differential Lax form:

$$L_1 = \frac{1}{x(x - 1)} \left\{ a_2(a_1 + a_2) + \frac{q(q - 1)p}{x - q} - \frac{t(t - 1)H}{x - t} \right\} + \frac{1 - a_0}{x - t} + \frac{1 - a_3}{x - 1} + \frac{1 - a_4}{x - q} - \frac{1}{x - q} \partial_x + \partial_x^2 ,$$

$$L_2 = T_0 - \frac{1}{q - x} (x \partial_x - qp) ,$$

$$B = \partial_x - \frac{t - q}{t(t - 1)(x - q)} (x(x - 1) \partial_x - q(q - 1)p) . \quad (8.225)$$

The curve $L_1 y = 0$ is the unique curve of degree $(3, 2)$ in $(f, g)$ passing through $(8.224)$ and

$$\left( x + \epsilon, \frac{x}{\epsilon} \right) , \quad \left( x + \epsilon, (x + \epsilon) \frac{y(x + \epsilon)}{y(x + \epsilon)} \right) . \quad (8.226)$$

Compatibility of $L_1 y = L_2 y = 0$ gives the discrete flow for $T_0(=m_2 s_3 s_0 s_3 s_1 s_3 s_2)$:

$$\overline{a}_0 = a_0 - 1, \overline{a}_2 = a_2 + 1, \overline{a}_3 = a_3 - 1 ,$$

$$f \overline{f} = \frac{gl(g - a_4)}{(g + a_2)(g + a_1 + a_2)} . \quad (8.227)$$

$$g + \overline{g} = a_0 + a_3 + a_4 - 2 + \frac{t(a_0 - 1)}{f - t} + a_3 - 1 \frac{1}{f - 1} .$$

Compatibility of $L_1 y = B y = 0$ gives the $P_{VI}$ flow with Hamiltonian $(8.223)$.

(ii) Difference Lax form:

$$L_1 = f(f - 1)(f - t) \left( \frac{a_0}{f - t} + \frac{a_3}{f - 1} - \frac{z + g - a_3}{f} \right)$$

$$+ \frac{f(z - 1 + a_2)(z - a_0 - a_2 - a_3 - a_4)(f - t_2 - 1)}{z - 1 - g} + \frac{t(z - a_4)}{z - g} (1 - fT_z) ,$$

$$L_2 = T_0 T_z + \frac{1}{z - g} (1 - fT_z) ,$$

$$B = (f - 1)T_z + \frac{(z + a_2)(z + a_1 + a_2)}{z - g} (1 - fT_z) + (t(t - 1) \partial_z T_z . \quad (8.228)$$
The curve $L_{4y} = 0$ is the unique curve of degree $2, 3$ in $(f, g)$ passing through (8.224) and
\[
(\infty, z), \quad (0, z - 1), \quad \left( \frac{y(z)}{y(z) + 1}, z \right), \quad \left( \frac{y(z - 1)}{y(z)}, z - 1 \right). \tag{8.229}
\]

Compatibility of $L_{4y} = L_{2y} = 0$ gives the discrete flow for $T_3 = \pi_3 s_2 s_1 s_4 s_3 s_0$:
\[
\begin{align*}
\sigma_0 &= a_0 + 1, \quad \sigma_2 = a_2 - 1, \quad \sigma_3 = a_3 + 1, \\
\bar{f} &= \frac{\bar{g}^2 (\bar{g} - a_4)}{\bar{g} + a_2 - 1)(\bar{g} - a_0 - a_2 - a_3 - a_4), \\
g + \bar{g} &= a_0 + a_3 + a_4 + \frac{ta_0}{\bar{f} - 1} + \frac{a_3}{\bar{f} - 1}. \tag{8.230}
\end{align*}
\]

Compatibility of $L_{4y} = B y = 0$ gives the $P_{V1}$ flow with Hamiltonian (8.223).

8.5.16. $d$-$P_{A1}^{(1)}/D_9^{(1)}$). The corresponding continuous flow is the $P_{V}$ equation given by the Hamiltonian:
\[
H = \frac{1}{t} \left( q(q - 1)p(p + t) - (a_1 + a_3)qp + a_1 p + a_2 tq \right), \tag{8.231}
\]
with $a_0 + a_1 + a_2 + a_3 = 1$. The eight points configuration in $(f, g) = (q, qp)$ coordinates is given as:
\[
(f, g) = (\infty, -a_2), \quad \left( \frac{1}{e}, -\frac{1}{e} - a_0 \right), (0, 0), (0, a_1), \quad \left( 1 + a_3 e, \frac{1}{e} \right). \tag{8.232}
\]

(i) Differential Lax form:
\[
\begin{align*}
\mathcal{L}_1 &= \frac{1}{x(x - 1)} \left\{ q(q - 1)p \left( \frac{x - q}{x - q} + t(a_2 x - H) \right) \right\} \\
&\quad + \left\{ \frac{1 - a_1}{x} + t + \frac{1 - a_3}{x - 1} - \frac{1}{x - q} \right\} \partial_x + \partial_x^2, \\
\mathcal{L}_2 &= T_3 - \frac{1}{x - q} (p - \partial_x), \\
B &= \partial_t - \frac{1}{t(x - q)} (x(x - 1)\partial_x - q(q - 1)p). \tag{8.233}
\end{align*}
\]

The linear equation $\mathcal{L}_1 y = 0$ is characterized as a curve of degree $3, 2$ in $(f, g)$ passing through the points (8.232) and (8.226). Compatibility of $\mathcal{L}_1 y = \mathcal{L}_2 y = 0$ gives the discrete flow for $T_3 = (\pi_1 \pi_2)^2 s_3 s_0 s_2$:
\[
\begin{align*}
\sigma_0 &= a_0 + 1, \quad \sigma_1 = a_1 - 1, \quad \sigma_2 = a_2 + 1, \quad \sigma_3 = a_3 - 1, \\
q + \bar{q} &= 1 - \frac{a_2}{p}, \quad p + \bar{p} = -t + \frac{a_1 - 1}{\bar{q}}, \quad \frac{a_3 - 1}{\bar{q} - 1}. \tag{8.234}
\end{align*}
\]

Compatibility of $\mathcal{L}_1 y = B y = 0$ gives the $P_{V1}$ flow with Hamiltonian (8.231).
(ii) Difference Lax form:

\[ L_1 = \frac{tf(z + a_2 - 1)}{z - g - 1} (f - T_z^{-1}) + \frac{z(z - a_1)}{z - g} (fT_z - 1) - (f - 1)(z + g + tf - a_1) + a_2f, \]

\[ L_2 = T_y T_z + \frac{1}{z - g} (1 - f T_z), \]

\[ B = \frac{z + a_2}{z - g} (1 - f T_z) + \partial T_z + T_z. \] (8.235)

The curve \( L_1 y = 0 \) is the unique curve of degree (2, 3) in \((f, g)\) passing through the points (8.232) and (8.229). Compatibility of \( L_1 y = L_2 y = 0 \) gives the discrete flow for \( T_y = (n_1 n_2 s_0 s_1 s_2 s_3 s_1) \):

\[ \sigma_2 = a_2 - 1, \quad \sigma_3 = a_3 + 1, \]

\[ g + \bar{g} = a_1 + a_3 - tf + \frac{a_2}{f-1}, \quad \bar{f} = -\frac{(\bar{g} - a_1)\bar{g}}{t(g + a_2 - 1)}. \] (8.236)

Compatibility of \( L_1 y = By = 0 \) gives the PV flow with Hamiltonian (8.231).

**8.5.17. \( \sigma-P(2A_1)_{(1)}/D_6^{(1)} \).** The corresponding continuous flow is \( P_{II}^{(0)} \) with the Hamiltonian:

\[ H = \frac{1}{l} \{ p(p - 1)q^2 + (a_1 + a_2)aq + tp - a_2q \}. \] (8.237)

The eight points configuration in \((f, g)\) coordinates is given as:

\[ (f, g) = \left( \frac{1}{\epsilon}, -a_2 \right), \left( \frac{1}{\epsilon}, \frac{1}{\epsilon} - a_1 \right), (0, 0), \left( \epsilon, -\frac{1}{\epsilon} + 1 - a_2 - a_1 \right). \] (8.238)

(i) Differential Lax form:

\[ L_1 = \left\{ \frac{a_0}{x} + \frac{pq}{x(x - q)} - \frac{iH}{x^2} \right\} + \left\{ \frac{1 + a_1 + a_2}{x} - \frac{1}{x - q} + \frac{i}{x^2} - 1 \right\} \partial_1 + \partial_2^2; \]

\[ L_2 = T_n - \frac{1}{x - q} (p - \partial_n), \]

\[ B = \partial_t - \frac{q}{t(x - q)} (x \partial_t - qp). \] (8.239)

The curve \( L_1 y = 0 \) is the unique curve of degree (3, 2) in \((f, g)\) passing through the points (8.238) and (8.226). Compatibility of \( L_1 y = L_2 y = 0 \) gives the discrete flow for \( T_n = (n_1 n_2 s_0 s_1 s_2 s_3 s_1) \):

\[ \sigma_1 = a_1 + 1, \quad \sigma_2 = a_2 + 1, \]

\[ q + \bar{q} = \frac{a_2}{p} - \frac{a_1}{p - 1}, \quad p + \bar{p} = 1 - \frac{t}{\bar{q}} - \frac{a_1 + a_2 + 1}{\bar{q}}. \] (8.240)

Compatibility of \( L_1 y = By = 0 \) gives the \( P_{III}^{(0)} \) flow with Hamiltonian (8.237).
Figure 22. Point configurations corresponding to figure 21 in the blown-up spaces. We use the abbreviated notations as $ijk = H_i - E_j - E_k, ij = E_i - E_j, i = E_i$.

(ii) Difference Lax form:

\[
L_1 = \frac{z + a_2 - 1}{z - g - 1} f(T_z^{-1} - f) + f^2 + f (1 - a_1 - a_2 - g - z) - t + \frac{r_z}{g - z} (ft_z - 1),
\]

\[
L_2 = T_z T_e + \frac{1}{z - g} (1 - ft_z),
\]

\[
B = \frac{z + a_0}{z - g} (ft_z - 1) + t \partial_z T_z + z T_z.
\]
The curve $L_y = 0$ is the unique curve of degree $(2, 3)$ in $(f, g)$ passing through the points (8.238) and (8.229). Compatibility of $L_y = L_2y = 0$ gives the discrete flow for $T_2(=\pi_2 \pi_3)$. 

$$\sigma_2 = a_2 - 1, \quad g + \bar{g} = 1 + f - a_1 - a_2 - \frac{t}{f} \quad \text{ff}^f = -\frac{t \bar{g}}{g + a_2 - 1}.$$  (8.242)

Compatibility of $L, B$ gives the continuous $P^{(2)}_\beta$ flow with the Hamiltonian (8.237).

**8.5.18. $\alpha$-$\beta(A^{(1)}_1/D^{(1)}_1)$.** The corresponding continuous flow is $P^{(3)}_\beta$ with Hamiltonian:

$$H = \frac{1}{t} (p^2 q^2 + q + pt + a_1 pq).$$  (8.243)

The eight points configuration in $(f, g) = (q, qp)$ coordinates is given as:

$$(f, g) = \left( -\frac{1}{\epsilon^2}, \frac{1}{\epsilon} - \frac{a_1}{2} \right), (0, 0), \left( \epsilon, -\frac{t}{\epsilon} + 1 - a_1 \right).$$  (8.244)

(i) Differential Lax form:

$$L_1 = \left\{ \frac{1 - p}{x} + \frac{p}{x - q} - \frac{t \bar{H}}{x^2} \right\} + \left\{ \frac{a_1 + 1}{x} - \frac{1}{x - q} + \frac{t}{x^2} \right\} \partial_x + \partial_x^2,$$

$$L_2 = T_\alpha - \frac{1}{x - q} (p - \partial_x),$$

$$B = \partial_t - \frac{q}{t(x - q)} (x \partial_x - qp).$$  (8.245)

The curve $L_y = 0$ is the unique curve of degree $(3, 2)$ in $(f, g)$ passing through the points (8.244) and (8.226). Compatibility of $L_y = L_2y = 0$ gives the discrete flow for $T_\alpha(=\pi_1 \pi_2)$:

$$\sigma_1 = a_1 + 2, \quad q + \bar{q} = -\frac{1}{p^2} - \frac{a_1}{p}, \quad p + \bar{p} = -\frac{t}{\bar{q}^2} - \frac{a_1 + 1}{\bar{q}}.$$  (8.246)

Compatibility of $L_y = B = 0$ gives the $P^{(3)}_\beta$ flow with Hamiltonian (8.243).

(ii) Difference Lax form:

$$L_1 = \frac{t z}{z - g} (1 - f T_z) + \frac{f}{z - g - 1} (f - T_z^{-1}) - f(z + g + a_1 - 1) - t,$$

$$L_2 = T_3 T_z + \frac{1}{z - g} (1 - f T_z),$$

$$B = \frac{1}{z - g} (1 - f T_z) + t \partial_z T_z + z T_z.$$  (8.247)

The curve $L_y = 0$ is the unique curve of degree $(2, 3)$ in $(f, g)$ passing through the points (8.244) and (8.229). Compatibility of $L_y = L_2y = 0$ gives the discrete flow for $T_3(=\pi_2 \pi_3)$:

$$\sigma_1 = a_1 - 1, \quad g + \bar{g} = 1 - a_1 - \frac{t}{f} \quad \text{ff}^f = t \bar{g}.$$  (8.248)

Compatibility of $L_y = By = 0$ gives the $P^{(3)}_\beta$ flow with Hamiltonian (8.243).
8.5.19. $d$-$P(A^{(1)}/D_6^{(1)})$. The corresponding continuous flow is $P^{D^{(1)}_{III}}_0$ with the Hamiltonian

$$H = \frac{1}{t} \left( p^2q^2 + pq + q + \frac{t}{q} \right).$$  \hspace{1cm} (8.249)$$

The eight points configuration in $(f, g)$ coordinates is given as:

$$(f_1, g_1) = \left( -\frac{1}{\epsilon^2}, -\frac{1}{\epsilon} - \frac{1}{2} \right), \quad \left( -\epsilon^2, \frac{1}{\epsilon} \right).$$  \hspace{1cm} (8.250)$$

(i) Differential Lax form:

$$\mathcal{L}_1 = \left\{ \frac{1 - p}{x} + \frac{p}{x - q} - \frac{tH}{x^2} + \frac{t}{x^3} \right\} + \left\{ 2 - \frac{1}{x - q} \right\} \partial_x + \partial_x^2,$$

$$B = \partial_t - \frac{q}{t(x - q)}(\epsilon \partial_x - q).$$  \hspace{1cm} (8.251)$$

The curve $\mathcal{L}_1 = 0$ is the unique curve of degree $(3, 2)$ in $(f, g)$ passing through the points (8.250) and (8.226). Compatibility of $\mathcal{L}_1 y = By = 0$ gives the $P^{D^{(1)}_{III}}_0$ flow with Hamiltonian (8.249).

(ii) Difference Lax form:

$$L_1 = \frac{t}{z - g}(T_z - f^{-1}) + \frac{1}{z - g - 1}(T_{z^{-1}} - f) + g + z,$$

$$B = \frac{1}{z - g}(1 - f T_z) + t\partial_t T_z + zT_z.$$  \hspace{1cm} (8.252)$$

The curve $L_1 y = 0$ is the unique curve of degree $(2, 3)$ in $(f, g)$ passing through the points (8.250) and (8.229). Compatibility of $L_1 y = By = 0$ gives the $P^{D^{(1)}_{III}}_0$ flow with Hamiltonian (8.249). There is no discrete flow.

8.5.20. $d$-$P(A^{(1)}/E_6^{(1)})$. The corresponding continuous flow is $P^{E^{(1)}_6}$ with Hamiltonian

$$H = qp(\rho - q - t) - a_2 \rho - a_2 q.$$  \hspace{1cm} (8.253)$$

The eight points configuration in $(f, g) = (q, qp)$ coordinates is given as:

$$(f_1, g_1) = (\infty, -a_2), \quad \left( \frac{1}{\epsilon}, \frac{1}{\epsilon^2} + \frac{t}{\epsilon} - a_0 \right), \quad (0, 0), \quad (0, a_1).$$  \hspace{1cm} (8.254)$$

(i) Differential Lax form:

$$\mathcal{L}_1 = \left\{ -a_2 - \frac{H}{x} + \frac{pq}{x(x - q)} \right\} + \left\{ 1 - a_1 - t - x - \frac{1}{x - q} \right\} \partial_x + \partial_x^2,$$

$$L_2 = T_{a_1} - \frac{1}{x - q}(p - \partial_x),$$

$$B = \partial_t - \frac{1}{x - q}(\epsilon \partial_x - q).$$  \hspace{1cm} (8.255)$$

The curve $\mathcal{L}_1 y = 0$ is the unique curve of degree $(3, 2)$ in $(f, g)$ passing through the points (8.254) and (8.226). Compatibility of $\mathcal{L}_1 y = L_2 y = 0$ gives the discrete flow for $T_{a_1}(\equiv \tau_1 \delta_0 \delta_2)$:
\[ a_1 = a_1 - 1, \quad a_2 = a_2 + 1, \]
\[ q + \bar{q} = p - t - \frac{a_2}{p}, \quad p + \bar{p} = \bar{q} + t + \frac{a_1 - 1}{\bar{q}}. \] (8.256)

Compatibility of \( L_1 y = B y = 0 \) gives the P_IW flow with Hamiltonian (8.253).

(ii) Difference Lax form:

\[
L_1 = f \frac{a_2 + z - 1}{g - z + 1} (f - T^{-1}_z) - \frac{z(z - a_1)}{g - z} (1 - f T_z) + a_1 - g + f (f + t) - z, \]
\[
L_2 = T_0 T_z + \frac{1}{z - g} (1 - f T_z), \]
\[
B = \frac{a_2 + z}{z - g} (f T_z - 1) + \partial T_z + T_z. \] (8.257)

The curve \( L_1 y = 0 \) is the unique curve of degree (2, 3) in \((f, g)\) passing through the points (8.254) and (8.229). Compatibility of \( L_1 y = L_2 y = 0 \) gives the discrete flow for \( T_\beta = (\pi_0 \pi_2 \pi_4 \pi_0) \):

\[
\sigma_2 = a_2 - 1, \quad g + \bar{g} = f^2 + tf + a_1, \quad f \bar{f} = -\frac{g(g - a_1)}{\bar{g} + a_2 - 1}. \] (8.258)

Compatibility of \( L_1 y = B y = 0 \) gives the P_IW flow with Hamiltonian (8.253).

8.5.21. d-P(A(1)
\((I)/E(1)
\)). The corresponding continuous flow is P_H with the Hamiltonian

\[
H = \frac{p^2}{2} - \left( q^2 + \frac{t}{2} \right) p - aq. \] (8.259)

The eight points configuration in \((q, p)\) coordinates is given as

\[
(q, p) = \left( \frac{1}{e}, -ae \right), \quad \left( \frac{1}{e}, \frac{2}{e^2} + t + (a - 1)e \right). \] (8.260)

(i) Differential Lax form:

\[
\mathcal{L}_1 = \left\{ \frac{p}{x - q} - 2 H - 2ax \right\} - \left\{ 2x^2 + t + \frac{1}{x - q} \right\} \partial_x + \partial_x^2, \]
\[
\mathcal{L}_2 = T - \frac{1}{x - q} (p - \partial_x), \]
\[
B = \partial_t - \frac{1}{2(x - q)} (\partial_x - p). \] (8.261)

The curve \( \mathcal{L}_2 y = 0 \) is the unique curve of degree (3, 2) in \((q, p)\) passing through the eight points (8.260) and the extra four points:

\[
(q, p) = \left( x + \epsilon, -\frac{1}{\epsilon} \right), \quad \left( x + \epsilon, \frac{y(x + \epsilon)}{y(x + \epsilon)} \right). \] (8.262)

7 Space of initial values can be realized by eight points configuration only in \((q, p)\) coordinates. More points are required in \((f, g)\) coordinates.
Compatibility of $L_1 y = L_2 y = 0$ gives the discrete flow for $T_0 (= \pi s_0)$:

$$\sigma = a + 1, \quad p + \sigma = 2q^2 + t, \quad q + q = -\frac{a}{p}.$$  (8.263)

Compatibility of $L_1 y = B y = 0$ gives the $P_0$ flow with the Hamiltonian (8.259).

(ii) Difference Lax form: We put $\varphi = p - 2q^2 - t$.

$$L_2 = -p + 2qT_0 + (z + 1)T_z - 2T_z T_z^{-1},$$
$$L_3 = (-a - z + 1)T_z^{-1} + \varphi T_z T_z^{-1} - qz + zT_0,$$
$$B_2 = -p + 2q\partial_t + (z + 1)T_z - 2\partial_z T_z^{-1},$$
$$B_3 = (-a - z + 1)T_z^{-1} + \varphi\partial_z T_z^{-1} - qz + z\partial_t.$$  (8.264)

The linear difference equation $L_1 y = 0$ in $z$ obtained from $L_2 y = L_3 y = 0$ is of third order in $T_z$:

$$L_1 = 2(z + a - 1)(z + \varphi q + 1)T_z^{-1} + (2q(z + a) + (\varphi p - 2aq)(z + \varphi q + 1)$$
$$+ (z + 1)(qz + a\varphi) - \varphi) T_z - (z + 1)(z + 2)(z + \varphi q) T_z^2.$$  (8.265)

Compatibility of $L_2 y = L_3 y = 0$ gives the discrete flow for $T_0 (= \pi s_0)$. Compatibility of $B_2 y = B_3 y = 0$ gives the $P_0$ flow with Hamiltonian (8.259).

8.5.22. d-P($A_0^{(1)}/E_8^{(1)}$). The corresponding continuous flow is $P_1$ with the Hamiltonian

$$H = \frac{p^2}{2} - 2q^3 - tq.$$  (8.266)

There is no discrete flow.

(i) Differential Lax form:

$$L_1 = \left\{-4x^3 - 2tx - 2H + \frac{p}{x - q}\right\} - \frac{1}{x - q} \partial_x + \partial_x^2,$$
$$B = \partial_t - \frac{1}{2(x - q)}(\partial_x - p).$$  (8.267)

Compatibility of $L_1 y = B y = 0$ gives the $P_1$ flow with Hamiltonian (8.266).

(ii) Simple difference Lax form is not known.

8.6. Hypergeometric solutions

8.6.1. e-P($E_8^{(1)}/A_0^{(1)}$).

(i) Decoupling condition:

(1) $P_1, P_3, P_5, P_7$ are on a $(1, 1)$ curve $C_1$:

$$\kappa_1 \kappa_2 = v_1 v_3 v_5 v_7,$$  (8.268)

($f, g$) is also on $C_1$:

$$\frac{f - f\left(\frac{\kappa_2}{\ell}\right)}{f - f(t)} = f_a(t) \prod_{j=1,3,5,7} \left[\frac{v_j}{\kappa_2}\right], \quad \text{for} \quad g = g(t).$$  (8.269)
\[
\frac{g - g(\frac{\kappa_1}{s})}{g - g(s)} = \frac{g_a(s)}{g_a(\frac{\kappa_1}{s})} \prod_{j=1,3,5,7} \left[ \frac{v_j s}{\kappa_j} \right], \quad \text{for } f = f(s). \tag{8.270}
\]

(2) \( P'_2, P'_4, P'_6, P'_8 \) are on a \((1, 1)\) curve \( C_2 \) where \( P'_i = (\mathcal{J}(v_i), g(v_i)) \):
\[
\kappa_1 \kappa_2 = q v_2 v_4 v_6 v_8, \tag{8.271}
\]

\( (\mathcal{J}, g) \) is also on \( C_2 \):
\[
\frac{\mathcal{J} - \mathcal{J}(\frac{\kappa_2}{t})}{\mathcal{J} - \mathcal{J}(t)} = \frac{J_a(t)}{J_a(t)} \prod_{j=2,4,6,8} \left[ \frac{v_j t}{\kappa_j} \right], \quad \text{for } g = g(t), \tag{8.272}
\]

and \((f, g)\) is on \( C_2 \) which is a curve determined by \((f(v_i), g(v_i)) (i = 2, 4, 6, 8)\):
\[
\frac{g - g(\frac{\kappa_1}{t})}{g - g(t)} = \frac{g_a(s)}{g_a(\frac{\kappa_1}{t})} \prod_{j=2,4,6,8} \left[ \frac{v_j t}{\kappa_j} \right], \quad \text{for } f = f(s), \tag{8.273}
\]

where
\[
f_a(z) = \left[ \frac{a}{z} \frac{\kappa_1}{a z} \right], \quad g_a(z) = \left[ \frac{a}{z} \frac{\kappa_2}{a z} \right], \quad f(z) = \frac{f_a(z)}{f_a(z)}, \quad g(z) = \frac{g_a(z)}{g_a(z)}. \tag{8.274}
\]

\([z]\) is the multiplicative theta function given in section 7.2.1, and \(a, b\) are arbitrary.

(ii) Linearized equation of the Riccati equation (6.8):
\[
U_1(\mathcal{F} - F) + U_2 F + U_3(\mathcal{F} - F) = 0,
\]

\[
U_1 = \left[ \frac{v_1 v_8}{\kappa_2} \frac{q v_1 v_8}{\kappa_2} \prod_{i=3,5,7} \left[ \frac{v_i v_1}{\kappa_1} \right] \prod_{j=2,4,6} \left[ \frac{v_j v_8}{\kappa_1} \right] \right],
\]

\[
U_2 = - \prod_{i=2,4,6} \left[ \frac{v_1}{v_i} \right] \prod_{j=3,5,7} \left[ \frac{v_j}{v_8} \right],
\]

\[
U_3 = \left[ \frac{v_1 v_8}{\kappa_1} \frac{v_1 v_8}{q v_1} \prod_{i=3,5,7} \left[ \frac{q v_1 v_1}{v_1} \right] \prod_{j=2,4,6} \left[ \frac{q v_j v_8}{\kappa_2} \right] \right]. \tag{8.275}
\]

(iii) Elliptic hypergeometric integral [118]:
\[
I(t_0, t_1, \ldots, t_7|p, q) = \frac{(p; p)_{\infty}(q; q)_{\infty}}{4\pi \sqrt{-1}} \int_C \prod_{k=0}^7 \frac{\Gamma(h_k z + 1; p, q) d\tau}{\Gamma(z + 2; p, q)} \frac{d\tau}{\tau}, \tag{8.276}
\]

\[
t_0 t_1 \cdots t_7 = p^2 q^2, \tag{8.277}
\]

where
\[
\Gamma(z; p, q) = \frac{(pq/z; p, q)_\infty}{(z; p, q)_\infty},
\]

\[
(z; p, q)_\infty = \prod_{i,j=0}^{\infty} (1 - p^i q^j z) \quad (|p|, |q| < 1),
\] (8.278)

and each double sign indicates the product of two factors with different signs as \( \Gamma(az^{\pm 1}; p, q) = \Gamma(az; p, q)\Gamma(a/z; p, q) \). Moreover, the integration contour \( C \) in (8.276) is a closed curve (or a cycle) which encircles in the positive direction the sequence of poles

\[
z = p^i q^j l_k \quad (i, j \in \mathbb{N}; \ k = 0, 1, 2, \ldots, 7),
\] (8.279)

that accumulate to the origin as shown figure 23, under a certain genericity condition for simplicity of poles. For instance, if \( |t| < 1 \), then \( C \) can be chosen as the unit circle \( |z| = 1 \).

(iv) Linear difference equation for the hypergeometric function:

\[
V_1(\Phi_0 - \Phi) + V_2 \Phi + V_3(\Phi - \Phi_0) = 0,
\]

\[
\Phi(a_0, a_1, \ldots, a_7) = \Psi(u_0, u_1, \ldots, u_7)
\]

\[
= \prod_{i=0}^{6} \frac{\Gamma(qa_i/u_0; p, q)\Gamma(q/u_0; p, q)}{\Gamma(q^2/u_0; p, q)\Gamma(u_0/u_0; p, q)\prod_{1 \leq r < s \leq 6} \Gamma(u_r; p, q)} \frac{\Gamma(pu_0, u_1, \ldots, u_6, pu_1|p, q)}{\Gamma(u_0, p, q)},
\]

\[
a_i = \frac{q}{u_0u_i} \quad (i = 0, \ldots, 7), \quad q^2 a_3^2 = a_1a_2 \cdots a_7, \quad u_0u_1 \cdots u_7 = q^2,
\]

\[
\Phi = \Phi(a_0; a_1/q, a_2, a_3, \ldots, a_7), \quad \Phi_0 = \Phi(a_0; qa_1, a_2/q, a_3, \ldots, a_7),
\]

\[
V_1 = \frac{[a_2][a_0][qa_0/a_2]}{[a_2/a_0][qa_2/a_1]} \prod_{j=3}^{7} \frac{[qa_0/a_1a_j]}{[a_0/a_1]a_j}, \quad V_2 = \frac{[qa_0/a_1a_2]}{[a_0/a_1][qa_2/a_1]} \prod_{j=3}^{7} \frac{[a_j]}{[a_0/a_1]},
\]

\[
V_3 = \frac{[a_1][a_0/a_1][qa_0/a_1]}{[a_1/a_2][qa_1/a_2]} \prod_{j=3}^{7} \frac{[qa_0/a_2a_j]}{[a_0/a_2]a_j}, \quad [z] = z^{1/2}(z, p/z, p; p)_\infty.
\] (8.280)

(v) Contiguity relation \([118]\)

\[
\Phi_0 - \Phi = - \frac{[qa_0][q^2a_0][qa_2/a_1][qa_0/a_1a_2]}{[qa_0/a_1][qa_2/a_1][qa_0/a_2][a_0/a_2]} \times \prod_{j=3}^{7} \frac{[a_j]}{[qa_0/a_1]} \Phi(q^2a_0; a_1, qa_2, qa_3, \ldots, qa_7).
\] (8.281)

**Remark 8.11.** When some of the parameters \( a_i \) (\( i = 1, \ldots, 6 \)) is \( q^{-N} \) (\( N \in \mathbb{N} \)), the elliptic hypergeometric function \( \Phi \) is expressed as a finite series

\[
\Phi = 12V_{11}(a_0; a_1, \ldots, a_7) = \sum_{k=0}^{N} \frac{[q^{2k}a_0][a_0][a_0k]}{[a_0][qa_0/a_1k]} \prod_{i=1}^{7} \frac{[a_i]}{[qa_0/a_1k]},
\] (8.282)

where \( [a]_k = [a][qa] \cdots [q^{k-1}a] \) \([49, 118]\).
(vi) Hypergeometric solution (6.7):

\[
y = \frac{f - f_1}{f - f_0} = \frac{\left[ \frac{v_1 v_8}{\kappa_2} \prod_{i=3,5,7} \frac{v_1 v_i}{\kappa_i} \right] F - F}{F}
\]

\[
= \left[ \frac{q_i v_1 v_8}{v_3 v_5} \prod_{i=2,4,6} \frac{v_1 v_i}{\kappa_i} \prod_{i=3,5,7} \frac{v_1 v_i}{\kappa_i} \right] G
\]

\[
F = \Phi(a_0, a_1, \ldots, a_7), \quad G = \Phi(q^2 a_0; a_1, q a_2, q a_3, \ldots, q a_7).
\] (8.283)

(vii) Identification of parameters:

\[
a_0 = \frac{v_1 v_8}{v_3 v_5}, \quad a_1 = \frac{q \kappa_1}{v_3 v_5}, \quad a_2 = \frac{\kappa_2}{v_3 v_5}, \quad a_3 = \frac{v_1}{v_5},
\]

\[
a_4 = \frac{v_1}{v_4}, \quad a_5 = \frac{v_1}{v_6}, \quad a_6 = \frac{v_8}{v_3}, \quad a_7 = \frac{v_8}{v_5}.
\] (8.284)

8.6.2. \( q\mathcal{P}(E_8^{(1)}/\mathcal{A}_0^{(1)}) \).

(i) Decoupling condition:

1. \( P_1, P_3, P_5, P_7 \) are on a \((1, 1)\) curve \( C_1 \):

\[
\kappa_1 \kappa_2 = v_1 v_3 v_5 v_7,
\] (8.285)

\((f, g)\) is also on \( C_1 \):

\[
\frac{f - f(t)}{f - f(t)} = \frac{t^4}{\kappa_2^2} \prod_{j=1,3,5,7} \frac{\kappa_j}{t - v_j}, \quad \text{for } g = g(t),
\] (8.286)

\[
\frac{g - g(s)}{g - g(s)} = \frac{s^4}{\kappa_2^2} \prod_{j=1,3,5,7} \frac{\kappa_j}{s - v_j}, \quad \text{for } f = f(s).
\] (8.287)

2. \( P_1', P_3', P_5', P_7' \) are on a \((1, 1)\) curve \( C_2 \) where \( \mathcal{P}' = (\mathcal{F}(v_i), g(v_i)) \):

\[
\kappa_1 \kappa_2 = q v_1 v_3 v_5 v_8,
\] (8.288)

\((\mathcal{F}, g)\) is also on \( C_2 \):

\[
\frac{\mathcal{F} - \mathcal{F}(t)}{\mathcal{F} - \mathcal{F}(t)} = \frac{t^4}{\kappa_2^2} \prod_{j=2,4,6,8} \frac{\kappa_j}{t - v_j}, \quad \text{for } g = g(t),
\] (8.289)

and \((f, g)\) is on \( C_2 \).
\[
\frac{g - \frac{\kappa_1}{s}}{g - \frac{\kappa_1}{s}(s)} = \prod_{j=2,4,6,8} \frac{\kappa_1}{s - v_j}, \quad \text{for } f = f(s), \quad (8.290)
\]

where \( f(z) = z + \frac{\kappa_1}{s}, \ g(z) = z + \frac{\kappa_2}{s} \).

(ii) Linearized equation of the Riccati equation (6.8):

\[
U_1(F - F) + U_2F + U_3(F - F) = 0,
\]

\[
U_1 = \frac{\kappa_1^2(\kappa_1 - \kappa_2)(q\kappa_1 - \kappa_2)}{\kappa_1^2(\kappa_1 - \kappa_2)(q\kappa_1 - \kappa_2)} \prod_{i=3,5,7} (v_i - \kappa_1) \prod_{i=2,4,6} (v_i - v_8),
\]

\[
U_2 = -v_1v_8 \prod_{i=3,5,7} (v_i - v_8),
\]

\[
U_3 = \frac{(v_1v_8 - \kappa_1)(v_1v_8 - q\kappa_1)}{q^2(\kappa_1 - \kappa_2)(q^2\kappa_1 - \kappa_2)} \prod_{i=3,5,7} (q\kappa_1 - \kappa_2) \prod_{i=2,4,6} (q\kappa_1v_8 - \kappa_2). \quad (8.291)
\]

(iii) Linear difference equation for the hypergeometric function [31]:

\[
\Phi(a_0, a_1, \ldots, a_7) = 10W_9(a_0; a_1, \ldots, a_7; q, q) + \frac{(qa_{a_0}, a_7/a_0; q)_\infty}{(a_0/a_7, qa_{a_7}/a_0; q)_\infty} \prod_{k=1}^{6} \frac{(aq_{a_0}/a_k; q)_\infty}{(qa_{a_0}/a_k, q^2a_{a_0}/a_k; q)_\infty},
\]

\[
q^2a_0^3 = a_1a_2\cdots a_7. \quad (8.292)
\]

\[
V_4(\Phi - \Phi) + V_5\Phi + V_6(\Phi - \Phi) = 0,
\]

\[
V_4 = \frac{(1 - a_2)(1 - a_0/a_2)(1 - qa_0/a_2)}{(1 - a_2/a_0)(1 - qa_0/a_2)} \prod_{j=3}^{7} (1 - qa_0/a_j),
\]

\[
V_5 = (1 - qa_0/a_1a_2) \prod_{j=3}^{7} (1 - a_j),
\]

\[
V_6 = \frac{(1 - a_0)(1 - a_0/a_1)(1 - qa_0/a_1)}{(1 - a_1/a_2)(1 - qa_0/a_2)} \prod_{j=3}^{7} (1 - qa_0/a_2j),
\]

\[
\Phi = \Phi(a_0; a_1/q, qa_2, a_3, \ldots, a_7), \quad \Phi = \Phi(a_0; qa_1, a_2/q, a_3, \ldots, a_7). \quad (8.293)
\]

(iv) Contiguity relation [31]:

\[
\Phi - \Phi = -a_1(1 - qa_0)(1 - q^2a_0)(1 - qa_0/a_1)(1 - qa_0/a_1a_2)
\]

\[
\times \prod_{i=3}^{7} \frac{1 - a_i}{1 - qa_0/a_i} \Phi(q^2a_0; a_1, qa_2, qa_3, \ldots, qa_7). \quad (8.294)
\]
(v) Hypergeometric solution (6.7):

\[ y = \frac{f - f_1}{f - f_8} = \frac{v_8 (v_1 v_8 - \kappa_2) \prod_{i=3,5,7} (v_1 v_i - \kappa_i) F - F}{v_1^2 \kappa_1 (\kappa_3 - \kappa_2) \prod_{i=3,5,7} (v_8 - v_i) F} \]

\[ = \frac{q^2 v_3 v_5 v_2 (qv_1 v_8 - v_3 v_2) (q^2 v_1 v_8 - v_3 v_5)}{v_1 \kappa_1 (qv_1 - v_3) (qv_1 - v_5) (v_1 v_8 - \kappa_2) (qv_1 v_8 - \kappa_3) (qv_1 v_8 - \kappa_4) (qv_1 v_8 - \kappa_5) (qv_1 v_8 - \kappa_6) (qv_1 v_8 - \kappa_7)} \]

\[ \times \prod_{i=2,4,6} (v_1 - v_i) \prod_{i=3,5,7} (v_1 v_i - \kappa_i) \frac{G}{F} \]

\[ F = \Phi(a_0, a_1, \ldots, a_7), \quad G = \Phi(q^2 a_0; a_1, qa_3, qa_5, \ldots, qa_7). \]  

(8.295)

(vi) Identification of parameters:

\[ a_0 = \frac{v_1 v_8}{v_3 v_5}, \quad a_1 = \frac{q v_8}{v_3 v_5}, \quad a_2 = \frac{\kappa_2}{v_3 v_5}, \quad a_3 = \frac{v_1}{v_2}, \]

\[ a_4 = \frac{v_1}{v_4}, \quad a_5 = \frac{v_1}{v_6}, \quad a_6 = \frac{v_8}{v_3}, \quad a_7 = \frac{v_8}{v_5}. \]  

(8.296)

Remark 8.12. The difference equation (8.291) is symmetric with respect to \( v_3, v_5, v_7 \) and \( v_2, v_4, v_6 \). This means that the solution space retains this symmetry. However, if we choose a particular solution, it is not always symmetric, as in the case of (8.292) with (8.296). For example, exchanging \( v_3 \) and \( v_7 \) in this solution yields another hypergeometric solution

\[ \Phi = \Phi(\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_7), \quad \tilde{a}_0 = qa_0^2/a_1 a_2 a_7, \quad \tilde{a}_1 = qa_0/a_2 a_7, \quad \tilde{a}_2 = qa_0/a_1 a_7, \]

\[ \tilde{a}_3 = a_3, \ldots, \tilde{a}_6 = a_6, \quad \tilde{a}_7 = qa_0/a_1 a_2, \]  

(8.297)

of (8.291) which is called a Bailey transform of the original solution [24].
8.6.3. $q$-$P(E^{(1)}_1/A^{(1)}_1)$.

(i) Decoupling condition:

1. $P_1, P_3, P_7$ are on a $(1, 1)$ curve $C_1$:

$$\kappa_1\kappa_2 = v_1 v_3 v_5 v_7,$$

and $(f, g)$ is also on $C_1$:

$$\frac{f g - \kappa_1}{\kappa_2} = \left( g - \frac{v_4}{\kappa_2} \right) \left( g - \frac{v_7}{\kappa_2} \right),$$

$$\frac{f g - 1}{\kappa_2} = \left( g - 1 v_1 \right) \left( g - 1 v_3 \right).$$

(8.299)

(8.300)

2. $P'_2, P'_4, P'_6, P'_8$ are on a $(1, 1)$ curve $C_2$ where $P'_i = (\bar{\gamma}(v_i), \nu(v_i))$:

$$\kappa_1\kappa_2 = q v_2 v_4 v_6 v_8,$$

$(\bar{\gamma}, g)$ is also on $C_1$:

$$\frac{f g - \kappa_1}{q \kappa_2} = \left( g - \frac{v_4}{\kappa_2} \right) \left( g - \frac{v_8}{\kappa_2} \right),$$

$$\frac{f g - 1}{q \kappa_2} = \left( g - 1 v_2 \right) \left( g - 1 v_4 \right).$$

(8.302)

(8.303)

(ii) Linearized equation of the Riccati equation (6.8):

$$U_1(F - F) + U_2 F + U_3(E - F) = 0,$$

$$U_1 = \frac{(v_1 v_4 - \kappa_2)(q v_1 v_8 - \kappa_2)}{\kappa_1(\kappa_1 - \kappa_2)(q \kappa_1 - \kappa_2)} \prod_{j=2,4} (v_j v_8 - \kappa_j) \prod_{i=5,7} (v_1 v_i - \kappa_i),$$

$$U_2 = v_1 v_8 \prod_{j=2,4} (v_j - v_{i}) \prod_{i=5,7} (v_i - v_8),$$

$$U_3 = \frac{(v_1 v_4 - \kappa_2)(v_1 v_8 - q \kappa_1)}{q \kappa_2(q \kappa_1 - \kappa_2)(q^2 \kappa_1 - \kappa_2)} \prod_{j=2,4} (qv_j v_8 - \kappa_2) \prod_{i=5,7} (qv_1 v_i - \kappa_2).$$

(8.304)

(iii) Linear difference equation for the hypergeometric function [36]:

$$\Phi(a_0, a_1, \ldots, a_5) = s W_5(a_0; a_1, \ldots, a_5; q, q^2 a_0^2 / a_1 a_2 a_3 a_4 a_5),$$

(8.305)
$$V_1 (\Phi - \Phi) + V_2 \Phi + V_3 (\Phi - \Phi) = 0,$$

$$V_1 = \frac{(1-a_2)(1-a_0/a_2)(1-qa_0/a_1)}{a_1(a_2/a_0)(1-qa_2/a_1)} \prod_{j=3,4,5} (1 - qa_0/a_1 a_j),$$

$$V_2 = \frac{(qa_0^2/a_1 a_2 a_3 a_4 a_5)(1 - qa_0/a_1 a_2)}{a_2(1-a_2/a_0)(1-qa_1/a_2)} \prod_{j=3,4,5} (1 - qa_0/a_1 a_2 a_j),$$

$$V_3 = \frac{(1-a_1)(1-a_0/a_1)(1-qa_0/a_1)}{a_1(1-a_1/a_2)(1-qa_1/a_2)} \prod_{j=3,4,5} (1 - qa_0/a_1 a_2 a_j),$$

$$\Phi = \Phi(a_0, a_1/q, qa_2, a_3, \ldots, a_5), \quad \Phi = \Phi(a_0, qa_1, a_2/q, a_3, \ldots, a_5). \quad (8.306)$$

(iv) Contiguity relation [36]:

$$\Phi - \Phi = a_1 (1 - qa_0)(1 - q^{2}a_0)(1 - qa_2/a_1)(1 - qa_0/a_1 a_2) \prod_{i=3,4,5} \frac{1 - a_i}{1 - qa_0/a_i} \Phi(q^{2}a_0; a_1, qa_2, qa_3, qa_4, qa_5). \quad (8.307)$$

(v) Hypergeometric solution (6.7):

$$y = \frac{f - f_1}{f - f_5} = \frac{v_5(v_1 v_8 - \kappa_2)(v_1 v_5 - \kappa_1)(v_1 v_7 - \kappa_1)}{v_1 \kappa_1(\kappa_4 - \kappa_2)} \prod_{i=5,7} (v_i - v) \frac{F - F}{F}$$

$$q v_3 v_5 (qv_1 v_8 - v_1 v_5)(q^2 v_1 v_8 - v_3 v_5) \prod_{i=2,4} (v_1 - v) \prod_{i=5,7} (v_1 v_i - \kappa_1) G,$$

$$F = \Phi(a_0, a_1, \ldots, a_5), \quad G = \Phi(q^2 a_0; a_1, qa_2, qa_3, qa_4, qa_5). \quad (8.308)$$

(vi) Identification of parameters:

$$a_0 = \frac{v_1 v_4}{v_3 v_5}, \quad a_1 = \frac{q \kappa_1}{v_3 v_5}, \quad a_2 = \frac{\kappa_2}{v_3 v_5}, \quad a_3 = \frac{v_1}{v_2}, \quad a_4 = \frac{v_1}{v_4}, \quad a_5 = \frac{v_8}{v_5}. \quad (8.309)$$

8.6.4. $q$-P$(E_6^{(1)}/A_2^{(1)})$.

(i) Decoupling condition:

(1) $P_1, P_3, P_5, P_7$ are on a (1, 1) curve $C_1$:

$$\kappa_1 \kappa_2 = v_1 v_3 v_5 v_7, \quad (8.310)$$

$(f, g)$ is also on $C_1$:

$$\frac{fg - 1}{f} = \frac{g - \frac{1}{v_1}}{g - \frac{v_5}{\kappa_2}}. \quad (8.311)$$

141
\[
\frac{f'g - 1}{g} = \frac{(f - v_1)(f - v_3)}{f - \frac{\kappa_1}{v_7}}. 
\] (8.312)

(2) \( P'_2, P'_4, P'_6, P'_8 \) are on a \((1, 1)\) curve \( C_2 \) where \( P'_i = P_{i|a_1 \to a_1/q} \):
\[
\kappa_1 \kappa_2 = qv_2 v_4 v_6 v_8, 
\] (8.313)
and \( (f, g) \) is on \( C_2 \):
\[
\frac{f'g - 1}{f'} = \frac{\left( g - \frac{1}{v_2} \right) \left( g - \frac{1}{v_4} \right)}{\left( g - \frac{v_6}{\kappa_2} \right)}.
\] (8.314)

and \( (f, g) \) is on \( C_2 \):
\[
\frac{f'g - 1}{g} = \frac{(f - v_2)(f - v_4)}{f - \frac{\kappa_1}{v_8}}.
\] (8.315)

(ii) Linearized equation of the Riccati equation (6.8):
\[
U_1(F - F) + U_2 F + U_3 (F - F) = 0,
\]
\[
U_1 = \left( 1 - \frac{\kappa_2}{v_1 v_5} \right) \left( 1 - \frac{qv_2 v_6}{\kappa_2} \right) \left( 1 - \frac{q v_4 v_6}{\kappa_2} \right),
\]
\[
U_2 = \left( 1 - \frac{v_1}{v_2} \right) \left( 1 - \frac{v_1}{v_4} \right) \left( 1 - \frac{v_7}{v_8} \right),
\]
\[
U_3 = q \frac{v_8}{v_5} \left[ 1 - \frac{q v_1 v_2}{\kappa_2} \right] \left[ 1 - \frac{\kappa_1}{v_1 v_8} \right] \left[ 1 - \frac{v_1 v_8}{q \kappa_1} \right].
\] (8.316)

(iii) Linear difference equation for the hypergeometric function [30]:
\[
\Phi(a_1, a_2, a_3, b_1, b_2) = \Phi_2 \left[ \begin{array}{l}
\begin{array}{l}
a_1, a_2, a_3, b_1, b_2; q; \frac{b_1 b_2}{a_1 a_2 a_3} \\
\end{array}
\end{array}\right]
\] (8.317)
\[
V_1 (\Phi - \Phi) + V_2 \Phi + V_3 (\Phi - \Phi) = 0,
\]
\[
V_1 = \left( 1 - \frac{a_1}{b_2} \right) \left( 1 - \frac{a_1}{b_2} \right) (1 - a_3),
\]
\[
V_2 = (1 - a_3)(1 - a_2) \left( 1 - \frac{a_3}{b_2} \right).
\]
\[
V_3 = \frac{q a_1 a_2 a_3}{b_1 b_2} \left( 1 - \frac{b_2}{q} \right) \left( 1 - \frac{b_2}{a_1} \right) \left( 1 - \frac{1}{b_2} \right).
\]
\[\Phi = \Phi(a_1, a_2, q a_3, b_1, q b_2), \quad \Phi = \Phi(a_1, a_2, a_3/q, b_1, b_2/q).\] (8.318)
(iv) Contiguity relation [30]:

\[ \Phi - \Phi = \frac{b_1 b_2 (1 - a_1)(1 - a_2) \left( 1 - \frac{b_2}{a_2} \right)}{a_1 a_2 (1 - b_1)(1 - b_2)(1 - q b_2)} \Phi(q a_1, qa_2, qa_3, qb_1, q^2 b_2). \]  

(8.319)

(v) Hypergeometric solution (6.7):

\[ y = \frac{f - f_1}{f - f_k} = \frac{v_k (v_1 v_3 - \kappa_1)}{\kappa_1 (v_8 - v_7)} \Phi - \Phi = \frac{q v_8^2 (v_1 v_3 - \kappa_1) (v_1 - v_2) (v_1 - v_4)}{v_7 (q v_1 - v_3) (v_1 v_8 - \kappa_1) (v q v_8 - \kappa_1)} \frac{G}{F}, \]

\[ F = \Phi(a_1, a_2, a_3, b_1, b_2), \quad G = \Phi(q a_1, q a_2, q a_3, q b_1, q^2 b_2). \]  

(8.320)

(vi) Identification of parameters:

\[ a_1 = \frac{v_1}{v_2}, \quad a_2 = \frac{v_1}{v_4}, \quad a_3 = \frac{\kappa_2}{v_3 v_5}, \quad b_1 = \frac{q v_1}{v_3}, \quad b_2 = \frac{v_1 v_8}{\kappa_1}. \]  

(8.321)

8.6.5. q-P($D_5^{(1)}/A_3^{(1)}$)

Remark 8.13. In order to take the degeneration limit from q-P($E_6^{(1)}/A_4^{(1)}$) described in (8.120) and (8.124) consistently with the decoupling condition, we first exchange $P_1$ and $P_3$ ($v_1$ and $v_3$), $P_2$ and $P_4$ ($v_2$ and $v_4$), and then take the limit

\[ \kappa_1 \to \epsilon \kappa_1, \quad \kappa_2 \to \kappa_2/\epsilon, \quad v_j \to \epsilon v_j (j = 3, 4), \quad v_k \to v_k/\epsilon (k = 1, 2), \quad \epsilon \to 0. \]

(i) Decoupling condition:

(1) $P_1, P_3, P_5, P_7$ are on a (1, 1) curve $C_1$:

\[ \kappa_1 \kappa_2 = v_1 v_3 v_5 v_7, \]  

(8.322)

\[ (f, g) \text{ is also on } C_1: \]

\[ g = v_5 \]

\[ f = v_3 \frac{\kappa_2}{v_1}, \]  

(8.323)

\[ g = \frac{f}{v_1} - \frac{\kappa_1}{v_7}, \]  

(8.324)

(2) $P_2', P_4', P_6', P_8'$ are on a (1, 1) curve $C_2$ where $P_1' = P_{[\kappa_1/\epsilon]}$:

\[ \kappa_1 \kappa_2 = q v_1 v_4 v_6 v_8, \]  

(8.325)

\[ (f', g) \text{ is also on } C_2. \]
\( \mathcal{F} = v_4 \frac{\kappa_2}{g - \kappa_2} \) \hspace{1cm} \text{(8.326)}

and \((f, g)\) is on \( \mathcal{C}_2^1 \):

\( g = \frac{f - \kappa_1}{v_2} \) \hspace{1cm} \text{(8.327)}

(ii) Linearized equation of the Riccati equation (6.8):

\[
U_1(\mathcal{F} - F) + U_2F + U_3(\mathcal{F} - F) = 0,
\]

\[
U_1 = \left( 1 - \frac{\kappa_2}{v_1v_3} \right) \left( 1 - \frac{qv_2v_6}{\kappa_2} \right),
\]

\[
U_2 = \left( 1 - \frac{v_3}{v_4} \right) \left( 1 - \frac{v_3}{v_5} \right),
\]

\[
U_3 = \frac{qv_6}{v_5} \left( 1 - \frac{\kappa_1}{v_3v_8} \right) \left( 1 - \frac{v_3v_8}{q\kappa_1} \right).
\] \hspace{1cm} \text{(8.328)}

(iii) Linear difference equation for the hypergeometric function [30]:

\[
\Phi(a_1, a_2, b_1, z) = \phi_1 \left[ \frac{a_1, a_2}{b_1}; q; z \right]
\] \hspace{1cm} \text{(8.329)}

\[
V_1(\mathcal{F} - \Phi) + V_2\Phi + V_3(\Phi - \Phi) = 0,
\]

\[
V_1 = (1 - a_2)(a_1 - b_1),
\]

\[
V_2 = (1 - a_1)(a_2 - b_1),
\]

\[
V_3 = \frac{(1 - b_1)(q - b_1)}{z},
\]

\[
\mathcal{F} = \Phi(a_1, qa_2, qb_1, z), \quad \Phi = \Phi(a_1, a_2/q, b_1/q, z).
\] \hspace{1cm} \text{(8.330)}

(iv) Contiguity relation [30]:

\[
\Phi - \Phi = \frac{(1 - a_1)(a_2 - b_1)}{(1 - b_1)(1 - qb_1)} \Phi(qa_1, qa_2, q^2b_1, z).
\] \hspace{1cm} \text{(8.331)}

(v) Hypergeometric solution (6.7):

\[
y = \frac{f - f_0}{f - f_6} = \frac{v_6(v_3v_7 - \kappa_1)}{\kappa_1(v_8 - v_7)} \frac{\mathcal{F} - F}{F} = \frac{v_3v_6v_8(v_3v_7 - \kappa_1)(v_3v_8 - \kappa_1)}{v_4v_6v_8v_7 - \kappa_1(qv_3v_6 - \kappa_1)} \frac{G}{F},
\]

\[
F = \Phi(a_1, a_2, b_1, z), \quad G = \Phi(qa_1, qa_2, q^2b_1, z).
\] \hspace{1cm} \text{(8.332)}

(vi) Identification of parameters:

\[
a_1 = \frac{v_3}{v_4}, \quad a_2 = \frac{v_3v_7}{\kappa_1}, \quad b_1 = \frac{v_3v_8}{\kappa_1}, \quad z = \frac{v_5}{v_6}
\] \hspace{1cm} \text{(8.333)}
8.6.6. \( q\)-P(\( A_q^{(1)}/A_q^{(1)} \))

**Remark 8.14.** In order to take the degeneration limit from \( q\)-P(D\( _1^{(1)}/A_q^{(1)} \)) described in (8.120) and (8.124) consistently with the decoupling condition, we first exchange \( P_3 \) and \( P_4 \) (\( v_3 \) and \( v_4 \)), and then take the limit \( v_2 \to v_2 \epsilon, v_3 \to v_3/\epsilon, \epsilon \to 0 \).

(i) Decoupling condition:

1. \( P_1, P_3, P_5, P_7 \) are on a \((1, 1)\) curve \( C_1 \):
   \[
   \kappa_1 \kappa_2 = v_1 v_3 v_5 v_7, \tag{8.334}
   \]
   \( (f, g) \) is also on \( C_1 \):
   \[
   f = v_4 \frac{v_1}{\kappa_2}, \quad g = \frac{1}{v_1} \left( f - \frac{v_1}{v_2} \right), \tag{8.335}
   \]
   \[
   (f, g) \) is also on \( C_1 \):
   \[
   f = \frac{v_2}{\kappa_2}, \quad g = \frac{v_4}{v_1} \left( f - \frac{v_1}{v_2} \right). \tag{8.336}
   \]

2. \( P_2', P_4', P_6', P_8' \) are on a \((1, 1)\) curve \( C_2 \) where \( P_i' = P_i|_{v_0 \to v_0/q} \):
   \[
   \kappa_1 \kappa_2 = q v_2 v_4 v_6 v_8, \tag{8.337}
   \]
   \( (f, g) \) is also on \( C_2 \):
   \[
   f = -v_2 v_3 \left( g - \frac{v_6}{\kappa_2} \right), \tag{8.338}
   \]
   and \( (f, g) \) is on \( C_2 \):
   \[
   g = -\frac{1}{v_2 v_3} \left( f - \frac{v_1}{v_8} \right). \tag{8.339}
   \]

(ii) Linearized equation of the Riccati equation (6.8):
   \[
   U_1 (\mathcal{F} - F) + U_2 F + U_3 (F - F) = 0,
   \]
   \[
   U_1 = 1 - \frac{\kappa_2}{v_1 v_5}, \quad U_2 = 1 - \frac{v_7}{v_8}, \quad U_3 = q \frac{v_6}{v_5} \left( 1 - \frac{\kappa_2}{\kappa_1 v_8} \right) \left( 1 - \frac{v_4 v_8}{q v_1} \right). \tag{8.340}
   \]

(iii) Linear difference equation for the hypergeometric function:
   \[
   \Phi(a, b, z) = 2g_1 \left[ a, \frac{0}{b}; q, z \right], \tag{8.341}
   \]
   \[
   V_1 (\Phi - \Phi) + V_2 \Phi + V_3 (\Phi - \Phi) = 0,
   \]
   \[
   V_1 = b(a - 1), \quad V_2 = a - b, \quad V_3 = \frac{(b - 1)(b - q)}{z}, \tag{8.342}
   \]
   \[\Phi = \Phi(q a, q b, z), \quad \Phi = \Phi(a/q, b/q, z).\]
(iv) Contiguity relation:

$$\Phi - \Phi = \frac{(a - b)z}{(1 - b)(1 - qb)} \Phi(qa, q^2b, z).$$  \hfill (8.343)

(v) Hypergeometric solution (6.7):

$$F = \frac{v_8(v_4v_7 - \kappa_1)}{\kappa_1(v_9 - v_7)} F - F = -\frac{v_4v_5v_8(v_4v_7 - \kappa_1)}{v_9(v_4v_8 - \kappa_2)(v_5v_8 - \kappa_1)} \frac{G}{F},$$

$$F = \Phi(a, b, z), \quad G = \Phi(qa, q^2b, z).$$  \hfill (8.344)

(vi) Identification of parameters:

$$a = \frac{v_4v_7}{\kappa_1}, \quad b = \frac{v_4v_8}{\kappa_1}, \quad z = \frac{v_5}{v_6}. \hfill (8.345)$$

8.6.7. $q$-$\text{P}(E_3^{(1)}/A_9^{(1)}; b)$.

Remark 8.15. In order to take the degeneration limit from $q$-$\text{P}(A_4^{(1)}/A_9^{(1)})$ described in (8.120) and (8.124) consistently with the decoupling condition, we first exchange $P_3$ and $P_6$ ($v_5$ and $v_6$), and then take the limit $v_6 \to v_6\epsilon$, $v_7 \to v_7/\epsilon$, $\epsilon \to 0$.

(i) Decoupling condition:

1. $P_1, P_4, P_5, P_7$ are on a $(1, 1)$ curve $C_1$:

$$\kappa_1\kappa_2 = v_1v_4v_7, \hfill (8.346)$$

$$f = \frac{g}{v_1}, \quad g = \frac{1}{v_1f - v_4} \hfill (8.347)$$

2. $P'_2, P'_3, P'_4, P'_8$ are on a $(1, 1)$ curve $C_2$ where $P'_1 = P|_{v_6 \to \kappa_6/\kappa_7}$:

$$\kappa_1\kappa_2 = v_1v_3v_5v_8, \hfill (8.349)$$

$$f = -v_2v_3\left(\frac{g - \frac{v_5}{\kappa_2}}{v_3}\right), \hfill (8.350)$$

and $f$, $g$ is on $C_2$:

$$g = -\frac{1}{v_2v_3}\left(f - \frac{\kappa_1}{v_3}\right). \hfill (8.351)$$
(ii) Linearized equation of the Riccati equation (6.8):

\[
U_1(\overline{F} - F) + U_2F + U_3(F - F) = 0,
\]

\[
U_1 = -\frac{\kappa_2}{v_1}, \quad U_2 = -\frac{v_6v_7}{v_8}, \quad U_3 = qv_5\left(1 - \frac{\kappa_1}{v_4v_8}\right)\left(1 - \frac{v_4v_8}{q\kappa_1}\right).
\]  

(8.352)

(iii) Linear difference equation for the hypergeometric function:

\[
\Phi(b, z) = \psi_1\left[\frac{0}{b}, q; z\right],
\]  

(8.353)

\[
V_1(\overline{F} - \Phi) + V_2\Phi + V_3(\Phi - \Phi) = 0,
\]

\[
V_1 = bz, \quad V_2 = z, \quad V_3 = (b - 1)(b - q),
\]

\[
\Phi = \Phi(qb, z), \quad \Phi = \Phi(b/q, z/q).
\]  

(8.354)

(iv) Contiguity relation:

\[
\Phi - \Phi = \frac{z}{(1 - b)(1 - qb)}\Phi(q^2b, qz).
\]  

(8.355)

(v) Hypergeometric solution (6.7):

\[
y = \frac{f - f_4}{f - f_8} = \frac{v_4v_7}{\kappa_1} \frac{F - F}{F} = -\frac{v_4v_7v_8}{v_5(v_4v_8 - \kappa_1)(q_4v_8 - \kappa_1)} \frac{G}{\overline{F}},
\]

\[
F = \Phi(b, z), \quad G = \Phi(q^2b, qz).
\]  

(8.356)

(vi) Identification of parameters:

\[
b = \frac{v_4v_8}{\kappa_1}, \quad z = \frac{v_4v_7}{\kappa_1v_5}.
\]  

(8.357)

8.6.8. q-P(E_3^{(1)}/A_9^{(1)}; a)

**Remark 8.16.** In order to take the degeneration limit from q-P(A_9^{(1)}/A_9^{(1)}) described in (8.120) and (8.124) consistently with the decoupling condition, we first exchange P_7 and P_8 (v_7 and v_8), and then take the limit \(v_8 \to v_8\epsilon, v_1 \to v_1/\epsilon, \epsilon \to 0\).

\[
\overline{f}\overline{f} = -v_2v_3v_4 \prod_{i=5}^{6} \left(1 - \frac{v_i}{\kappa_2}\right), \quad \overline{g}\overline{g} = \frac{\kappa_1}{v_1v_2v_3v_8} \frac{f - \kappa_1}{f - v_3}.
\]  

(8.358)

(i) Decoupling condition:

1. P_1, P_4, P_5, P_8 are on a (1, 1) curve \(C_1\):

\[
\kappa_1\kappa_2 = v_1v_4v_5v_8,
\]  

(8.359)

\((f, g)\) is also on \(C_1\):
(2) $P'_2, P'_3, P'_6, P'_7$ are on a $(1, 1)$ curve $C_2$ where $P'_i = P_{i|\kappa_i = \kappa_i'/q}$:

$$\kappa_i \kappa_2 = q v_2 v_3 v_6 v_7.$$  \hfill (8.362)

$(f, g)$ is also on $C_2$:

$$f = -v_2 v_3 \left( g - \frac{v_6}{\kappa_2} \right),$$  \hfill (8.363)

and $(f, g)$ is on $C_2$:

$$g = -\frac{1}{v_2 v_3} \left( f - \frac{\kappa_1}{v_7} \right).$$  \hfill (8.364)

(ii) Linearized equation of the Riccati equation (6.8):

$$U_1 (\overline{F} - F) + U_2 F + U_3 (F - F) = 0,$$

$$U_1 = 1, \quad U_2 = 1, \quad U_3 = q \frac{v_6}{v_5} \left( 1 - \frac{\kappa_1}{v_4 v_7} \right) \left( 1 - \frac{v_4 v_7}{q \kappa_1} \right).$$  \hfill (8.365)

(iii) Linear difference equation for the hypergeometric function:

$$\Phi(a, b, z) = \begin{pmatrix} 0, 0 \\ b, q; \ z \end{pmatrix},$$  \hfill (8.366)

$$V_1 (\overline{\Phi} - \Phi) + V_2 \Phi + V_3 (\Phi - \Phi) = 0,$$

$$V_1 = -b, \quad V_2 = -b, \quad V_3 = \frac{(b - 1)(b - q)}{z},$$

$$\overline{\Phi} = \Phi(q b, z), \quad \overline{\Phi} = \Phi(b/q, z).$$  \hfill (8.367)

(iv) Contiguity relation:

$$\overline{\Phi} - \Phi = -\frac{b z}{(1 - b)(1 - q b)} \Phi(q^2 b, z).$$  \hfill (8.368)

(v) Hypergeometric solution (6.7):

$$y = \frac{f - f_4}{f - f_7} = \frac{\overline{F} - F}{F} = \frac{\kappa_1 v_4 v_7}{v_6 (v_4 v_7 - \kappa_1)(q v_7 - \kappa_1)} \Phi(b, z),$$

$$F = \Phi(b, z), \quad G = \Phi(q^2 b, z).$$  \hfill (8.369)
Identification of parameters:

\[ b = \frac{v_4 v_7}{\kappa_1}, \quad z = \frac{v_5}{v_6}. \]  

(8.370)

8.6.9. q-P(E_2^{(1)}/A_0^{(1)}; a)

**Remark 8.17.** In order to take the degeneration limit from \( q\text{-P}(E_3^{(1)}/A_3^{(1)}; a) \) described in (8.120) and (8.124) consistently with the decoupling condition, we take the limit \( v_0 \to v_0 \epsilon, \quad v_7 \to v_7 / \epsilon, \quad \epsilon \to 0 \).

\[
\begin{align*}
\bar{f} & = -v_2 v_3 v_4 \left( g - \frac{v_5}{\kappa_2} \right), \\
g & = \frac{\kappa_1}{v_1 v_2 v_3 v_8} f - v_4.
\end{align*}
\]  

(8.371)

(i) Decoupling condition:

1. \( P_1, P_2, P_5, P_8 \) are on a \((1, 1)\) curve \( C_1 \):

\[
\kappa_1 \kappa_2 = v_1 v_4 v_3 v_8. \tag{8.372}
\]

\((f, g)\) is also on \( C_1 \):

\[
\begin{align*}
&g = \frac{v_5}{\kappa_2}, \\
f & = v_4 \frac{\kappa_2}{g},
\end{align*}
\]  

(8.373)

\[
g = -\frac{\kappa_1}{v_1 v_8} \frac{1}{f - v_4}. \tag{8.374}
\]

2. \( P'_2, P'_3, P'_6, P'_7 \) are on a \((1, 1)\) curve \( C_2 \) where \( P'_0 = P_{|t = \kappa_0 / \kappa_0} \):

\[
\kappa_1 \kappa_2 = q v_2 v_3 v_6 v_7, \tag{8.375}
\]

\((\bar{f}, g)\) is also on \( C_2 \):

\[
\bar{f} = -v_2 v_3 g, \tag{8.376}
\]

and \((f, g)\) is on \( C_2 \):

\[
g = -\frac{1}{v_2 v_3} f. \tag{8.377}
\]

(ii) Linearized equation of the Riccati equation (6.8):

\[
\begin{align*}
U_1 (\bar{F} - F) + U_2 F + U_3 (F - F) &= 0, \\
U_1 &= 1, \quad U_2 = 1, \quad U_3 = -\frac{v_6 v_3 v_7}{\kappa_1 v_5}.
\end{align*}
\]  

(8.378)

(iii) Linear difference equation for the hypergeometric function:

\[
\Phi(z) = 2\Phi_0 \left[ 0, 0; q; z \right]. \tag{8.379}
\]
\[ z\overline{\Phi} + \Phi - \Phi = 0, \]
\[ \overline{\Phi} = \Phi(z/q), \quad \Phi = \Phi(qz). \]  
(8.380)

(iv) Hypergeometric solution (6.7):
\[ y = \frac{f - f_4}{f} = -\frac{F - F}{F} = -\frac{\kappa_1v_5}{qv_4v_6v_7} \frac{G}{F}, \]
\[ F = \Phi(z), \quad G = \Phi(z/q^2). \]  
(8.381)

(v) Identification of parameters:
\[ z = \frac{\kappa_1v_5}{v_4v_6v_7}. \]  
(8.382)

In the following additive cases, we put \( \delta = 1 \) for simplicity.

8.6.10. \( d\)-P(\( E_8^{(1)}/A_0^{(1)} \)).

(i) Decoupling condition:

1. \( P_1, P_3, P_5, P_7 \) are on a \( (1, 1) \) curve \( C_1 \):
\[ \kappa_1 + \kappa_2 = v_1 + v_3 + v_5 + v_7, \]  
(8.383)
\( (f, g) \) is also on \( C_1 \):
\[ \frac{f - f(\kappa_2 - t)}{f - f(t)} = \prod_{j=1,3,5,7} \frac{v_j + t - \kappa_2}{v_j - t}, \quad \text{for } g = g(t), \]  
(8.384)
\[ \frac{g - g(\kappa_1 - s)}{g - g(s)} = \prod_{j=1,3,5,7} \frac{v_j + s - \kappa_1}{v_j - s}, \quad \text{for } f = f(s). \]  
(8.385)

2. \( P'_2, P'_4, P'_6, P'_8 \) are on a \( (1, 1) \) curve \( C_2 \) where \( P'_i = (\overline{f}(v_i), g(v_i)) \):
\[ \kappa_1 + \kappa_2 = 1 + v_2 + v_4 + v_6 + v_8, \]  
(8.386)
\( (\overline{f}, g) \) is also on \( C_2 \):
\[ \frac{\overline{f} - \overline{f}(\kappa_2 - t)}{\overline{f} - \overline{f}(t)} = \prod_{j=2,4,6,8} \frac{v_j + t - \kappa_2}{v_j - t}, \quad \text{for } g = g(t), \]  
(8.387)
and \( (f, g) \) is on \( C_2 \):
\[ \frac{g - g(\kappa_1 - s)}{g - g(s)} = \prod_{j=2,4,6,8} \frac{v_j + s - \kappa_1}{v_j - s}, \quad \text{for } f = f(s), \]  
(8.388)
where
\[ f(z) = z(z - \kappa_1), \quad g(z) = z(z - \kappa_2). \]  
(8.389)
(ii) Linearized equation of the Riccati equation (6.8):

\[
U_1(F - F) + U_2 F + U_3(F - F) = 0,
\]

\[
U_1 = \frac{1}{(v_1 + v_8 - \kappa_2)(1 + v_1 + v_8 - \kappa_2)} \prod_{i=3,5,7} (v_1 + v_i - \kappa_i) \prod_{j=2,4,6} (v_j + v_8 - \kappa_j),
\]

\[
U_2 = -\prod_{i=2,4,6} (v_1 - v_i) \prod_{j=3,5,7} (v_j - v_8),
\]

\[
U_3 = \frac{(v_1 + v_8 - \kappa_2)(v_1 + v_8 - \kappa_1 - 1)}{(1 + \kappa_1 - \kappa_2)(1 + \kappa_1 - \kappa_2)} \times \prod_{i=3,5,7} (1 + v_1 + v_i - \kappa_2) \prod_{j=2,4,6} (1 + v_j + v_8 - \kappa_2).
\]  

(8.390)

(iii) Linear difference equation for the hypergeometric function:

\[
V_i(\Phi - \Phi) + V_2 \Phi + V_i(\Phi - \Phi) = 0,
\]

\[
\Phi(a_0, a_1, \ldots, a_i) = \phi(a_0; a_1, a_2, \ldots, a_i) + \tilde{\phi}(a_0; a_1, a_2, \ldots, a_i),
\]

\[
\phi(a_0; a_1, a_2, \ldots, a_i) = \phi_{\kappa} \left( \begin{array}{c}
\frac{a_0}{2} + a_1 - a_2, \ldots, a_i - a_2, a_2,
\end{array} \right),
\]

\[
\tilde{\phi}(a_0; a_1, a_2, \ldots, a_i)
\]

\[
= \frac{\Gamma(1 + 2a_1 - a_0)\Gamma(a_0 - a_2)}{\Gamma(1 + a_0 - a_2)\prod_{i=1}^{6} \Gamma(a_0 - a_i)\Gamma(a_0 - a_3)\Gamma(a_0 - a_4)\Gamma(a_0 - a_5)\Gamma(a_0 - a_6)\Gamma(a_0 - a_7)}
\]

\[
\times \phi(2a_1 - a_0; a_1 + a_2, a_0 - a_2, \ldots, a_3, a_2),
\]

\[
= 2 + 3a_0 = \sum_{i=1}^{7} a_i,
\]

\[
\Phi = \Phi(a_0; a_0 - a_1 - a_2 - a_3 - a_6 - a_7 - a_0, a_2), \quad \Phi = \Phi(a_0; a_0 - a_1 - a_2 - a_3 - a_6 - a_7 - a_0, a_2).
\]

\[
V_1 = \frac{a_2(a_0 - a_2)(1 + a_0 - a_2)}{(a_2 - a_0)(1 + a_0 - a_1)} \prod_{j=3}^{7} (1 + a_0 - a_1 - a_j),
\]

\[
V_2 = (1 + a_0 - a_1 - a_2) \sum_{j=3}^{7} a_j,
\]

\[
V_3 = \frac{a_3(a_0 - a_3)(1 + a_0 - a_3)}{(a_3 - a_0)(1 + a_0 - a_2)} \prod_{j=3}^{7} (1 + a_0 - a_2 - a_j).
\]  

(8.391)

(iv) Contiguity relation:

\[
\Phi - \Phi = \frac{(1 + a_0)(2 + a_0)(1 + a_2 - a_0)(1 + a_0 - a_1 - a_2)}{(1 + a_0 - a_1)(1 + a_0 - a_2)(1 + a_0 - a_2)}
\]

\[
\times \prod_{j=3}^{7} \frac{a_j}{1 + a_0 - a_j} \Phi(a_0 + 2; a_1, a_2, a_3 + 1, \ldots, a_7 + 1).
\]  

(8.392)
(v) Hypergeometric solution (6.7):

\[
y = \frac{f - f_1}{f - f_8} = \frac{(v_1 + v_8 - \kappa_2) \prod_{i=3,5,7} (v_1 + v_i - \kappa_i) F - F}{(\kappa_1 - \kappa_2) \prod_{i=3,5,7} (v_8 - v_i)} \\
= \frac{(1 + v_1 + v_8 - v_3 - v_5)(2 + v_1 + v_8 - v_3 - v_5)}{(1 + v_1 - v_3)(1 + v_1 - v_5)(v_1 + v_8 - \kappa_1)(1 + v_1 + v_8 - \kappa_2)} \times \prod_{i=2,4,6} (v_1 - v_i) \prod_{i=3,5,7} (v_1 + v_i - \kappa_i) G \\
F = \Phi(a_0, a_1, a_2 + 1, a_3 + 1, \ldots, a_7 + 1). \\
(8.393)
\]

(vi) Identification of parameters:

\[
a_0 = v_1 + v_8 - v_3 - v_5, \quad a_1 = 1 + \kappa_1 - v_3 - v_5, \quad a_2 = \kappa_2 - v_3 - v_5, \quad a_3 = v_1 - v_2, \\
a_4 = v_1 - v_4, \quad a_5 = v_1 - v_6, \quad a_6 = v_8 - v_3, \quad a_7 = v_8 - v_5. \\
(8.394)
\]

8.6.11. d-P\(E_7^{(1)}/A_1^{(1)}\).

(i) Decoupling condition:

(1) \(P_1, P_3, P_5, P_7\) are on a \((1, 1)\) curve \(C_1\):

\[
\kappa_1 + \kappa_2 = v_1 + v_3 + v_5 + v_7, \\
(8.395)
\]

and \((f, g)\) is also on \(C_1\):

\[
\frac{f + g - \kappa_1 + \kappa_2}{f + g} = \frac{(g + \kappa_2 - v_5)(g + \kappa_2 - v_7)}{(g + v_1)(g + v_3)}, \\
(8.396)
\]

\[
\frac{f + g - \kappa_1 + \kappa_2}{f + g} = \frac{(f - \kappa_1 + v_3)(f - \kappa_1 + v_7)}{(f - v_1)(f - v_3)}. \\
(8.397)
\]

(2) \(P'_2, P'_4, P'_6, P'_8\) are on a \((1, 1)\) curve \(C_2\) where \(P'_i = (\overline{f}(v_i), \nu(v_i))\):

\[
\kappa_1 + \kappa_2 = 1 + v_2 + v_4 + v_6 + v_8, \\
(8.398)
\]

\((\overline{f}, g)\) is also on \(C_2\):

\[
\frac{\overline{f} + g - \kappa_1 + \kappa_2 + \delta}{\overline{f} + g} = \frac{(g + \kappa_2 - v_6)(g + \kappa_2 - v_8)}{(g + v_2)(g + v_4)}, \\
(8.399)
\]

and \((f, g)\) is on \(C_2\):

\[
\frac{f + g - \kappa_1 + \kappa_2 - \delta}{f + g} = \frac{(f - \kappa_1 + v_6)(f - \kappa_1 + v_8)}{(f - v_2)(f - v_4)}. \\
(8.400)
\]
(ii) Linearized equation of the Riccati equation (6.8):

\[
U_3 (\overline{F} - F) + U_2 F + U_1 (\overline{F} - F) = 0,
\]

\[
U_1 = \frac{(v_1 + v_8 - \kappa_2)(v_1 + v_8 - \kappa_1)}{(v_1 + \kappa_1)(v_1 + \kappa_2)} \prod_{j=2,4} (v_j + v_8 - \kappa_1) \prod_{i=5,7} (v_1 + v_i - \kappa_i),
\]

\[
U_2 = \prod_{j=2,4} (v_1 - v_j) \prod_{i=5,7} (v_i - v_8),
\]

\[
U_3 = \frac{(v_1 + v_8 - \kappa_2)(v_1 + v_8 - \kappa_1 - 1)}{(1 + \kappa_1 - \kappa_2)(2 + \kappa_1 - \kappa_2)} \times \prod_{j=2,4} (1 + v_j + v_8 - \kappa_2) \prod_{i=5,7} (1 + v_1 + v_i - \kappa_2).
\]

(8.401)

(iii) Linear difference equation for the hypergeometric function [66]:

\[
\Phi(a_0, a_1, \ldots, a_5) = _7 F_6 \left(\begin{array}{c} a_0, 1 + \frac{a_0}{2}, a_1, \ldots, a_5 \\ \frac{a_0}{2}, 1 + a_0 - a_1, \ldots, 1 + a_0 - a_5 \end{array} ; 1 \right).
\]

(8.402)

\[
V_1(\overline{\Phi} - \Phi) + V_2 \Phi + V_3 (\overline{\Phi} - \Phi) = 0,
\]

\[
V_1 = \frac{a_2(a_0 - a_2)(1 + a_0 - a_2)}{(a_0 - a_1)(1 + a_2 - a_0)} \prod_{j=3,4,5} (1 + a_0 - a_1 - a_j),
\]

\[
V_2 = (1 + a_0 - a_1 - a_2) \prod_{j=3,4,5} a_j,
\]

\[
V_3 = \frac{a_1(a_0 - a_0)(1 + a_0 - a_1)}{(a_1 - a_2)(1 + a_1 - a_2)} \prod_{j=3,4,5} (1 + a_0 - a_2 - a_j),
\]

\[
\overline{\Phi} = \Phi(a_0, a_1 - 1, a_2 + 1, a_3, \ldots, a_5), \quad \overline{\Phi} = \Phi(a_0; a_1 + 1, a_2 - 1, a_3, \ldots, a_5).
\]

(8.403)

(iv) Contiguity relation:

\[
\Phi - \overline{\Phi} = -(1 + a_0)(2 + a_0)(1 + a_2 - a_1)(1 + a_0 - a_1 - a_2)
\]

\[
\times \prod_{i=3}^5 \frac{a_i}{1 + a_0 - a_i} \Phi(2 + a_0; a_1, 1 + a_2, 1 + a_3, 1 + a_4, 1 + a_5).
\]

(8.404)

(v) Hypergeometric solution (6.7):

\[
y = \frac{f - \frac{f_1}{f}}{f - \frac{f_8}{f}} = -\frac{1}{(v_1 + v_8 - \kappa_2)(v_1 + v_8 - \kappa_1)(v_1 + v_8 - \kappa_3)} \frac{F - \overline{F}}{F},
\]

\[
= -\frac{(1 + v_1 + v_8 - v_3 - v_5)(2 + v_1 + v_8 - v_3 - v_5)}{(1 + v_1 - v_3)(v_1 + v_8 - \kappa_1)(1 + v_1 + v_8 - \kappa_2)(1 + v_1 + v_8 - \kappa_2)}
\]

\[
\times \prod_{i=2,4} (v_1 - v_i) \prod_{i=5,7} (v_i v_1 - \kappa_1)
\]

\[
\prod_{i=2,4} (1 + v_1 + v_8 - v_3 - v_5) \frac{G}{F},
\]

\[
F = \Phi(a_0, a_1, \ldots, a_5), \quad G = \Phi(2 + a_0; a_1, 1 + a_2, 1 + a_3, 1 + a_4, 1 + a_5).
\]

(8.405)
(vi) Identification of parameters:

\[
\begin{align*}
a_0 &= v_1 + v_3 - v_3 - v_5, & a_1 &= \kappa_1 - v_1 - v_3, & a_2 &= \kappa_2 - v_3 - v_5, \\
a_3 &= v_1 - v_2, & a_4 &= v_1 - v_4, & a_5 &= v_8 - v_5. \\
\end{align*}
\]

(8.406)

8.6.12. d-PEG(1)/A^3

(i) Decoupling condition:

(1) \( P_1, P_3, P_5, P_7 \) are on a (1, 1) curve \( C_1 \):

\[
\kappa_1 + \kappa_2 = v_1 + v_3 + v_5 + v_7,
\]

(8.407)

\[
(f, g) \text{ is also on } C_1:
\]

\[
f + g = \frac{(g + v_l)(g + v_3)}{g + \kappa_2 - v_5},
\]

(8.408)

\[
f + g = \frac{(f - v_l)(f - v_3)}{f - \kappa_1 + v_7}.
\]

(8.409)

(2) \( P'_2, P'_4, P'_6, P'_8 \) are on a (1, 1) curve \( C_2 \) where \( P'_i = P_{l_{i+1}} \):

\[
\kappa_1 + \kappa_2 = 1 + v_2 + v_4 + v_6 + v_8,
\]

(8.410)

\[
(f, g) \text{ is on } C_2:
\]

\[
f + g = \frac{(g + v_2)(g + v_4)}{g + \kappa_2 - v_6},
\]

(8.411)

and \((f, g)\) is on \( C_2 \):

\[
f + g - 1 = \frac{(f - v_2)(f - v_4)}{f - \kappa_1 + v_8}.
\]

(8.412)

(ii) Linearized equation of the Riccati equation (6.8):

\[
U_1(\overline{F} - F) + U_2F + U_3(\overline{F} - F) = 0,
\]

\[
U_1 = (\kappa_2 - v_3 - v_5)(1 + v_2 + v_6 - \kappa_2)(1 + v_4 + v_6 - \kappa_2),
\]

\[
U_2 = (v_1 - v_2)(v_1 - v_4)(v_7 - v_8),
\]

\[
U_3 = (v_1 + v_5 + 1 - \kappa_2)(\kappa_1 - v_1 - v_8)(v_1 + v_8 - 1 - \kappa_1).
\]

(8.413)

(iii) Linear difference equation for the hypergeometric function [29]:

\[
\Phi(a_1, a_2, a_3, b_1, b_2) = \gamma F_2 \left[ \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} ; 1 \right]
\]

(8.414)

\[
V_1(\Phi - \Phi) + V_2\Phi + V_3(\Phi - \Phi) = 0,
\]

\[
V_1 = (a_1 - b_2)(a_1 - b_2)a_3,
\]

\[
V_2 = a_1a_2(a_3 - b_2),
\]

\[
V_3 = b_2(1 - b_2)(b_1 - a_3),
\]

\[
\Phi = \Phi(a_1, a_2, 1 + a_3, b_1, 1 + b_2), \quad \Phi = \Phi(a_1, a_2, -1 + a_3, b_3, -1 + b_2).
\]

(8.415)
(iv) Contiguity relation [4, 29, 136]:
\[ \Phi - \Phi = \frac{a_1 a_2 (b_2 - a_3)}{b_1 b_2 (1 + b_2)} \Phi (1 + a_1, 1 + a_2, 1 + a_3, 1 + b_1, 2 + b_2). \]  
(8.416)

(v) Hypergeometric solution (6.7):
\[ y = \frac{f - f_1}{f - f_8} = \frac{(v_1 + v_7 - \kappa_1)}{(v_8 - v_7)} \frac{F - F}{F} = \frac{(v_1 + v_7 - \kappa_1)(v_1 - v_2)(v_1 - v_3)}{(1 + v_1 - v_3)(v_1 + v_3 - \kappa_1)(1 + v_1 + v_3 - \kappa_1)} G. \]
\[ F = \Phi(a_1, a_2, a_3, b_1, b_2), \quad G = \Phi(1 + a_1, 1 + a_2, 1 + a_3, 1 + b_1, 2 + b_2). \]  
(8.417)

(vi) Identification of parameters:
\[ a_1 = v_1 - v_2, \quad a_2 = v_1 - v_4, \quad a_3 = \kappa_2 - v_3 - v_5, \]
\[ b_1 = 1 + v_1 - v_3, \quad b_2 = v_1 + v_3 - \kappa_1. \]  
(8.418)

8.6.13. d-P(D(1)/D(1))

(i) Decoupling condition:

(1) P_2, P_3, P_4, P_5 are on a (1, 1) curve C_1:
\[ a_0 + a_1 + a_2 = 0, \]
(8.419)
and \((f, g)\) is on \(C_1\)
\[ f = \frac{gt}{g + a_1 + a_2}, \]
(8.420)
\[ g = a_0 + \frac{a_0 t}{f - t}. \]
(8.421)

(2) P'_1, P'_2, P'_3, P'_4 are on a (1, 1) curve C_2 where \(P'_1 = P'_1 |_{a_0 \rightarrow a_0 - 1} = (\tau, y)\):
\[ a_2 + a_3 + a_4 = 1, \]
(8.422)
\((f, g)\) is also on \(C_2\):
\[ f = \frac{g - a_4}{g + a_2}, \]
(8.423)
and \((f, g)\) is on \(C_2\):
\[ g = \frac{a_3}{f - 1} + a_3 + a_4. \]
(8.424)

(ii) Linearized equation of the Riccati equation (6.8):
\[ U_1 (F - F) + U_2 F + U_3 (F - F) = 0, \]
\[ U_1 = a_2 (a_2 - 1) t, \quad U_2 = a_1 (1 - a_2 - a_3), \quad U_3 = (1 - a_1 - a_2) a_3. \]  
(8.425)
(iii) Linear difference equation for the hypergeometric function [1]:

\[
\Phi(\alpha_1, \alpha_2, \beta, z) = \sum_{\nu_1} \begin{pmatrix} \alpha_1, \alpha_2; \beta_1 \end{pmatrix}_{\nu_1}.
\]

\[
V_1(\Phi - \Phi) + V_2 \Phi + V_3(\Phi - \Phi) = 0,
\]

\[
V_1 = \frac{\beta(1 - \beta)}{z}, \quad V_2 = \alpha_1(\alpha_2 - \beta), \quad V_3 = \alpha_2(\alpha_1 - \beta),
\]

\[
\Phi = \Phi(\alpha_1, \alpha_2 - 1, \beta - 1, z), \quad \Phi = \Phi(\alpha_1, \alpha_2 + 1, \beta + 1, z).
\]

(iv) Contiguity relation [1]:

\[
\Phi - \Phi = \frac{\alpha_1(\alpha_2 - \beta)z}{\beta(\beta - 1)} \Phi(\alpha_1 + 1, \alpha_2, \beta + 1, z).
\]

(v) Hypergeometric solution (6.7):

\[
f = \frac{a_2 F - F}{a_1 F} = \frac{a_4}{1 - a_2} \frac{G}{F},
\]

\[
F = \Phi(\alpha_1, \alpha_2, \beta, z), \quad G = \Phi(\alpha_1 + 1, \alpha_2, \beta + 1, z).
\]

(vi) Identification of parameters:

\[
\alpha_1 = a_1, \quad \alpha_2 = a_3, \quad \beta = 1 - a_2, \quad z = \frac{1}{t}.
\]

8.6.14. d-P(A(3)/D(3)).

(i) Decoupling condition:

(1) P_3, P_6, P_7, P_8 are on a (1, 1) curve C_1:

\[
a_2 + a_3 = 0,
\]

\[a_2 \quad p
\]

and (q, p) is on C_1:

\[q = 1 - \frac{a_2}{p},
\]

\[p = \frac{a_2}{1 - q}.
\]

(2) P_1', P_2', P_3', P_6' are on a (1, 1) curve C_2 where P_1' = P_1|_a_0 = a_1 = 1 (\xi, \gamma):

\[
a_2 + a_2 = 0,
\]

(\xi, \gamma) is also on C_2:

\[\xi = \frac{a_0}{p + \gamma},
\]

and (q, p) is on C_2:

\[\xi = \frac{a_0}{p + \gamma}.
\]
(ii) Riccati equation and linearized equation:

\[
q = a_0 \frac{-q + 1}{aq - a_2 - t},
\]

\[
q = -\frac{a_2 + t}{t} \frac{F - E}{F},
\]

\[
(t + a_2)(1 + t + a_2)F - (t + a_2)(1 + t + a_0 + a_2)F + a_0 a_2 E = 0.
\]

(iii) Linear difference equation for the hypergeometric function [1]:

\[
\Phi(\alpha, \beta, z) = \frac{\Gamma(\alpha)}{\Gamma(\alpha + z)} \Gamma I_{\alpha}^{\left[\frac{\alpha}{\beta}; z\right]},
\]

\[
(z + \alpha)(z + \alpha - 1)\Phi - (z + \alpha - 1)(z + 2\alpha - \beta)\Phi + (\alpha - \beta)(\alpha - 1)\Phi = 0,
\]

\[
\Phi = \Phi(\alpha + 1, \beta, z), \quad \Phi = \Phi(\alpha - 1, \beta, z).
\]

(iv) Contiguity relation:

\[
\Phi - \phi = \frac{(\alpha - \beta)z}{\beta(z + \alpha)} \Phi(\alpha, \beta + 1, z).
\]

(v) Hypergeometric solution:

\[
q = -\frac{a_2 + t}{t} \frac{F - E}{E} = \frac{1 - a_0}{a_2 - a_0 + 1} \frac{\Gamma I_{\alpha}^{\left[\frac{a_2}{a_2 - a_0 + 2}; t\right]}}{\Gamma I_{\alpha}^{\left[\frac{a_2}{a_2 - a_0 + 1}; t\right]}}.
\]

(vi) Identification of parameters:

\[
\alpha = a_2 + 1, \quad \beta = a_2 - a_0 + 1, \quad z = t.
\]

8.6.15. $dPA_2(t) / E_0^{(1)}$.

(i) Decoupling condition:

(1) $P_5, P_6, P_7, P_8$ are on a $(1, 1)$ curve $C_1$:

\[
a_0 = 0,
\]

and $(q, p)$ is on $C_1$

\[
q = p - t,
\]
\( p = q + t. \quad (8.447) \)

(2) \( P'_1, P'_2, P'_3, P'_4 \) are on a (1, 1) curve \( C_2 \) where \( P'_i = P_{|a_{i-1} = 0, a_i = 1} = (x_i, y_i) \):

\[
a_1 + a_2 = 1, \quad (8.448)
\]

(\( q, p \)) is also on \( C_2 \):

\[
q = -\frac{a_2}{p}, \quad (8.449)
\]

and (\( q, p \)) is on \( C_2 \):

\[
p = \frac{a_1}{q}. \quad (8.450)
\]

(ii) Riccati equation and linearized equation:

\[
q = -\frac{a_2}{q + t}, \quad (8.451)
\]

\[
q = a_1 \frac{F}{F'}, \quad (8.452)
\]

\[
F - tF - a_1 F = 0. \quad (8.453)
\]

(iii) Linear difference equation for the hypergeometric function [1]:

\[\Phi(\alpha, z) = 2^\alpha \sqrt{\pi} \left[ \frac{1}{\Gamma\left(\frac{1 - \alpha}{2}\right)} \text{I}_{\frac{\alpha}{2}} \left[ \frac{1 - \alpha}{2} ; \frac{z^2}{2} \right] - \frac{\sqrt{\pi} t}{\Gamma\left(-\frac{\alpha}{2}\right)} \text{I}_{\frac{\alpha}{2}} \left[ \frac{1 - \alpha}{2} ; \frac{z^2}{2} \right] \right]. \quad (8.454)\]

\[\bar{\Phi} - z\Phi + \alpha\Phi = 0, \quad (8.455)\]

\[\Phi = \Phi(\alpha + 1, z), \quad \Phi = \Phi(\alpha - 1, z). \quad (8.455)\]

(iv) Hypergeometric solution:

\[q = a_1 \frac{\Phi(-a_1 + 1, t)}{\Phi(-a_1, t)}. \quad (8.456)\]

(v) Identification of parameters:

\[\alpha = -a_1, \quad z = t. \quad (8.457)\]

\textbf{8.6.16.} \textit{d-P} \( 2A^{(1)}_1/D_6^{(1)} \). Equation (8.29) has no hypergeometric solution as constructed by the procedure used above, since there is no (1, 1) curve preserved by the equation in the direction \((\bar{a}_0, \bar{a}_1) = (a_0 + 1, a_1 + 1) \). If we take the direction \((\bar{a}_0, \bar{a}_1) = (a_0 + 1, a_1) \) or \((\bar{a}_0, \bar{a}_1) = (a_0, a_1 + 1) \), it is known that the corresponding discrete Painlevé equation admits hypergeometric solutions expressible in terms of the Bessel functions [79].

158
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