Abstract

Multi-particle scattering states are constructed for massive Wigner particles in the general operator-algebraic setting of wedge-local quantum field theory. The apparent geometrical restriction of the conventional wedge-local Haag-Ruelle argument to two-particle scattering states is overcome with a swapping symmetry argument based on wedge duality.

1 Introduction

Wedge locality has become an increasingly prominent concept in mathematical physics ever since wedge duality was established in the Wightman framework by Bisognano and Wichmann [BW75]. In particular, while interacting local quantum field theories (QFT) in four dimensions are still missing, non-trivial wedge-local QFT have emerged in recent years [GL07, BLS11]. This provides strong motivation to develop $N$-particle scattering theory in the wedge-local setting, which is the goal of the present paper.

The classical Wigner particle concept can still be consistently formulated in wedge-local theories as it does not depend on any notion of localization in configuration space. Accordingly we may define massive single particle states $\Psi_1 \in H$ as eigenvectors corresponding to positive eigenvalues of the relativistic mass operator $M := \sqrt{H^2 - P^2}$.

Two-particle scattering states were then constructed in [GL07, BS08] along the lines of Haag-Ruelle, using that two particles can be separated by two wedge regions [BBS01]. Scattering states with a larger number of particles however appeared inaccessible or even unnatural in the wedge-local setting as a result of a simple geometric consideration: it is impossible to write down three or more wedge-local operators whose localization regions are space-like separated.

In this paper we give a construction of scattering states for an arbitrary number of massive Wigner particles in the general wedge-local setting. Underlying our arguments is a simple swapping symmetry, which follows from wedge duality and augments cyclicity of the vacuum $\Omega$ for wedge algebras. It states that for a given wedge-local bounded operator $A \in \mathfrak{A}(W) \subset \mathcal{B}(H)$ localized in a wedge $W \subset \mathbb{R}^d$ there exists $A^\perp \in \mathfrak{A}(W^\perp)$ such that

$$A \Omega = A^\perp \Omega,$$

where $A^\perp$ is localized in a translate $W^\perp := W' + x$, $x \in \mathbb{R}^d$, of the causal complement $W'$ in Minkowski space of dimension $d = s + 1$. The symmetry (1) itself has been known for

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1 up to technical points discussed in Section 3.1
some time in the context of integrable models\textsuperscript{2}, but its utility for the construction of scattering states seems to have so far escaped the attention of the experts. In fact, its application in scattering theory appears very natural from the perspective of the causal geometry of wedge regions.

Let us now explain the role of the swapping relation (1) for scattering theory by sketching the convergence argument as an example. Let us recall the standard definitions of Haag-Ruelle theory by selecting $A_k \in \mathfrak{A}(\mathcal{W}) \ (1 \leq k \leq n)$ with non-vanishing projection $\Psi_1 = E_{\{M=m\}}A_k\Omega$ onto one-particle space of mass $m > 0$ and smear their space-time translates $\alpha_x(A_k) := U(x)A_kU(x)^*$ first with an auxiliary Schwartz function $\chi \in \mathcal{S}(\mathbb{R}^d)$ and afterwards with a positive-energy Klein-Gordon solutions $f_k$ (also for mass $m$) to obtain creation-operator approximants

$$B_k := A_k(\chi) := \int d^d x \chi(x)\alpha_x(A_k),$$

$$B_{kr}(f_k) := \int d^d x f_k(\tau, x)\alpha_{(\tau, x)}(B_k), \quad (\tau \in \mathbb{R}).$$

The smearing operation (2) suitably restricts the energy-momentum transfer, while (3) may be understood as a comparison dynamics in the sense of scattering theory. More precisely due to mass gaps we may arrange $B_{kr}(f_k)\Omega \in E_{\{M=m\}}\mathcal{I}^c$ for suitable $\chi$ (supported in a sufficiently small neighbourhood of the mass shell) and then $B_{kr}(f)\Omega = f_k(P)B_k\Omega$ is a one-particle state created from the vacuum, which is independent of the parameter $\tau \in \mathbb{R}$. Scattering states are now to be constructed as limits

$$\Psi^+ := \lim_{\tau \to \infty} \Psi_\tau, \quad \Psi_\tau := B_{1\tau}(f_1)B_{2\tau}(f_2)\ldots B_{n\tau}(f_n)\Omega,$$

whose existence can be reduced to the one-particle convergence if the norm of pairwise commutators is sufficiently decaying in $\tau$. However, even if the Klein-Gordon solutions $f_k$ describe wave packets which separate for large enough $\tau \to \pm \infty$, we should not expect $B_{k\tau}(f_k)$ to commute in a general wedge-local model.

Here the swapping relation (1) enters and yields a second family of creation operators defined analogously in terms of $A_1^\perp$ which satisfy

$$B_{k\tau}^+(f_k)\Omega = B_{k\tau}(f_k)\Omega.$$ \hspace{1cm} (5)

Across the two operator families we now obtain for suitably propagating wave packets $f_k$ an asymptotic decay

$$\left\| B_{j\tau}(f_j), B_{k\tau}^+(f_k) \right\| \leq C_N(1 + \tau)^{-N} \text{ for } 1 \leq j < k \leq n, \tau > 0.$$ \hspace{1cm} (6)

To establish convergence of (4) we estimate via Cook’s method ($0 < \tau_1 < \tau_2$)

$$\| \Psi_{\tau_2} - \Psi_{\tau_1} \| = \left\| \int_{\tau_1}^{\tau_2} d\tau \partial_\tau \Psi_\tau \right\| \leq \int_{\tau_1}^{\tau_2} d\tau \| \partial_\tau \Psi_\tau \|,$$

where the integrand on the right hand side is expanded using the product rule. To estimate the resulting terms we make use of (5) to write

$$B_{1\tau}(f_1)\ldots(\partial_\tau B_{k\tau}(f_k))\ldots B_{n\tau}(f_n)\Omega$$

$$= B_{1\tau}(f_1)\ldots(\partial_\tau B_{k\tau}(f_k))\ldots B_{n-1\tau}(f_{n-1})B_{n\tau}^+(f_n)\Omega$$

$$= B_{n\tau}^+(f_n)B_{1\tau}(f_1)\ldots(\partial_\tau B_{k\tau}(f_k))\ldots B_{n-1\tau}(f_{n-1})$$

$$+ \text{ (commutators),}$$

\textsuperscript{2}Swapping relations are mentioned e.g. in [BS08] above Thm. 3.2 for bounded operators, in [Le03] below (3.13) for wedge-local fields, and indirectly in even earlier works of Schroer. The general connection to wedge-duality has been investigated in depth by Borchers [Bor95], Rem. 1.1 and subsequent comments.
where commutator terms vanish rapidly as $\tau \to \infty$ by (6), $\|B_{j\tau}(f_j)\| \leq C(1 + |\tau|^s/2)$ and $\|B_{j\tau}(f_j)\| \leq C(1 + |\tau|^s/2)$. Iterating a total of $n - k$ times, the derivative term will act directly on the vacuum so that we can make use of $\partial_\tau B_{k\tau}(f_k)\Omega = 0$ as in standard Haag-Ruelle theory. Altogether (6) and polynomial norm growth of $B_{j\tau}(f_j), B_{j\tau}^\perp(f_j)$ yield for $\tau > 0$ the rapid decay 

$$\|B_1(\tau) \ldots (\partial_\tau B_k(\tau)) \ldots B_n(\tau)\Omega\| \leq C'_N(1 + \tau)^{-N}.$$ 

Summing up these terms, we obtain convergence of outgoing scattering states $\Psi^+$ from Cook’s method (7). A similar swapping argument yields the Fock structure of these scattering states for any number of particles $n \geq 0$. For $n \leq 2$ swapping is strictly speaking not necessary, as scattering states can be directly constructed via $\lim_{\tau \to \infty} B_{\tau}(f)B_{\tau}^\perp(f^\perp)\Omega$ as in [BBS01, GL07]. Lastly it is important to point out that beyond swapping, it is also necessary that all operators $A_k$ entering in (4) are localizable in a common wedge $W$. Further, the propagation velocities of $f_k$ must be suitably restricted to match the wedge geometry and be in correspondence with the fixed ordering of creation-operator approximants in (4), as will be made precise in Sections 3 and 4.

Our construction applies in particular to the model of Grosse and Lechner [GL07]. This model originated from a proposed quantum field theory on a non-commutative space-time, which may be motivated from gravitational considerations [DFR95]. Only later a reinterpretation as wedge-local quantum field theory on ordinary Minkowski space-time was discovered and it was shown that this model exhibits non-trivial 2-particle scattering [GL07]. The curious message of [GL07] was that the model itself is Poincaré-covariant, while Lorentz symmetry is broken at the level of scattering states. To clarify this effect, which is impossible in local quantum field theories, we give a general analysis of Poincaré covariance of the scattering states in Section 5. We intend to apply these results to extend the pioneering analysis of Grosse and Lechner to the multi-particle scattering data in a subsequent publication.

This paper is structured as follows. In Section 2 we introduce the wedge-local variant of the Haag-Kastler framework providing the standing assumptions of our construction. The wedge-local Haag-Ruelle theorem is established in Section 3 under certain geometrical restrictions allowing for a streamlined proof. These restrictions are lifted in Section 4, where we also obtain residual Lorentz covariance properties and pave the ground for a general discussion of wave operators and S-matrices in Section 5.

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2 Wedge-Local Quantum Field Theories

Our results are valid for Quantum Field Theory models defined on general Minkowski space-time $\mathbb{R}^d$, whose metric we take in the mainly-minus convention and whose spatial dimension we denote by $s := d - 1$. The family of wedge regions is defined as the orbit $\mathcal{P}W_r := \{\lambda W_r = \Lambda W_r + x, \lambda = (x, \Lambda) \in \mathcal{P}\}$ of the conventional right wedge $W_r := \{(t, x) \in \mathbb{R}^d : |t| < x^1\}$ under the action of the Poincaré group $\mathcal{P}$ [BW75].

A wedge-local quantum field theory model in operator-algebraic formulation is specified by mathematical objects $(\mathfrak{A}, \alpha, \mathcal{H}, \Omega)$, where $\mathcal{H}$ is the Hilbert space of pure
states containing the \textit{vacuum} as a distinguished unit vector $Ω \in \mathcal{H}$. The \textit{wedge-local net} $\mathfrak{A}$ is a mapping from the family wedge regions $\mathcal{P}W_i \ni \mathcal{W}$ to von Neumann algebras $\mathfrak{A}(\mathcal{W}) \subset \mathcal{B}(\mathcal{H})$, which serves to describe Einstein causality at the quantum mechanical level. Poincaré symmetry acts on the wedge-local net $\mathfrak{A}$ by a given group of isomorphisms\(^3\) $\alpha_\lambda$ and we denote by $\lambda = (x, \Lambda) \in \mathcal{P}_+^\times = \mathbb{R}^d \times \mathcal{L}_+^\times$ the elements of the proper orthochronous Poincaré group.

Guided by physical intuition we ask that these objects satisfy wedge-local variants of the Haag-Kastler postulates, which are concerned with the algebraic and representation-theoretic properties of $\mathfrak{A}$. Firstly, for any choice of wedge regions $\mathcal{W}, \mathcal{W}_1, \mathcal{W}_2$ we have

\begin{align*}
\text{Isotony} \quad &\mathfrak{A}(\mathcal{W}_1) \subset \mathfrak{A}(\mathcal{W}_2) \quad \text{for} \quad \mathcal{W}_1 \subset \mathcal{W}_2, \tag{HK1} \\
\text{Locality} \quad &\mathfrak{A}(\mathcal{W}_1) \subset \mathfrak{A}(\mathcal{W}_2)' \quad \text{for} \quad \mathcal{W}_1 \subset \mathcal{W}_2', \tag{HK2} \\
\text{Wedge-Duality} \quad &\mathfrak{A}(\mathcal{W}') = \mathfrak{A}(\mathcal{W})', \tag{HK2'} \\
\text{Translation-Covariance} \quad &\alpha_x(\mathfrak{A}(\mathcal{W})) = \mathfrak{A}((\mathcal{W} + x), \ x \in \mathbb{R}^d), \tag{HK3} \\
\text{Poincaré-Covariance} \quad &\alpha_\lambda(\mathfrak{A}(\mathcal{W})) = \mathfrak{A}((\lambda \mathcal{W}), \ \lambda \in \mathcal{P}_+^\times). \tag{HK3'}
\end{align*}

Here the Minkowski causal complement $\mathcal{W}' = -\Lambda \mathcal{W}_r + x$ of $\mathcal{W}$ is also a wedge region and $\mathfrak{A}(\mathcal{W})'$ denotes the commutant of $\mathfrak{A}(\mathcal{W})$ relative to $\mathcal{B}(\mathcal{H})$.

On the representation-theoretic side we further assume that translations are unitarily implemented on the vacuum Hilbert space $\mathcal{H}$ by a strongly continuous $s+1$-parameter group, $\alpha_x(A) = U(x)AU(x)^*$. The representing unitaries are generated by the \textit{energy-momentum operators} via $U(x) = U(t, \vec{x}) = e^{it\mathcal{H} - i\vec{x} \cdot \vec{P}}$, whose joint spectral resolution in terms of projection-operator-valued measures will be denoted by $\Delta \mapsto E(\Delta)$. Focusing also in particular on the analysis of scattering theory it will be convenient to further impose the following standard assumptions concerned with the vacuum representation and its one-particle spectrum,

\begin{align*}
\text{Uniqueness of } \Omega \quad &E(\{0\}) \mathcal{H} = \mathbb{C} \Omega, \tag{HK4} \\
\text{Cyclicity of } \Omega \quad &\mathfrak{A}(\mathcal{W})\Omega = \mathcal{H}, \tag{HK5} \\
\text{Mass Gap} \quad &H_m \subset \text{supp } E \subset \{0\} \cup H_m \cup H_M \subset \tilde{V}^+, \tag{HK6}
\end{align*}

for some $M > m > 0$, where $H_m := \{ (\omega_m(p), p) \in \mathbb{R}^d \}, \omega_m(p) := \sqrt{p^2 + m^2}$, is the (positive) hyperboloid of mass $m > 0$ and $H_M := \{ (\omega, p) \in \mathbb{R}^d, \omega \geq \omega_M(p) \}$ denotes the convex hull of $H_M$. Note that (HK6) implies in particular that the one-particle subspace $\mathcal{H}_1$ and the corresponding orthogonal projection $E_m := E(H_m)$ are non-trivial. We may extended any given wedge-local net also to regions obtained as sum of a given wedge and any open bounded region $O \subset \mathbb{R}^{d+1}$ by setting $\mathfrak{A}(O+W) := (\cup_{x \in O} \mathfrak{A}(\{W+x\}))'$. For later convenience we will also introduce some refined terminology for wedge regions concerning their geometry in the case of more than two dimensions. Recalling that any wedge region can be written as $\mathcal{W} = \Lambda \mathcal{W}_r + x$, we may define the corresponding centered wedge as $\mathcal{W}_c := \Lambda \mathcal{W}_r$. $\mathcal{W}_c$ is uniquely characterized by the coordinate origin being contained in its edge, and we will call such wedges \textit{centered}. This concept may be motivated heuristically by noting that scattering situations are concerned with phenomena at very large distances, making finite translation by $x \in \mathbb{R}^d$ in a sense

\footnote{The formulation of our main results requires only space-time translations. With some abuse of notation we denote translation automorphisms by the same letter $\alpha$, or $\alpha_x$, where $x \in \mathbb{R}^d$ is identified with $\lambda_x = (x, 1) \in \mathcal{P}_+^\times$. In particular the basic version of the framework given by (HK1)–(HK6) suffices for multi-particle scattering provided a suitable swapping assumption holds, and we will state explicitly when the strengthened variants (HK2') or (HK3') are required.}
negligible. Centered wedges $W$ are convex cones in the sense that $W + W \subset W$. This assures that the causal ordering given via the precursor relation \cite{BBS01}

$$O_1 \prec_W O_2 \iff O_2 - O_1 \subset W_c$$  \hspace{1cm} (8)

for regions $O_1, O_2 \subset \mathbb{R}^d$ is transitive and anti-symmetric (in the stronger sense that $O_1 \prec_W O_2$ and $O_2 \prec_W O_1$ imply $O_1 = O_2 = \emptyset$). Thus the precursor relation is a partial order, which is in fact Poincaré covariant.

**Proposition 1.** For any $\lambda = (x, \Lambda) \in \mathcal{P}$, any wedge $W$ and any sets $O_1, O_2 \subset \mathbb{R}^{s+1}$ we have

$$O_2 \prec_W O_1 \iff \lambda O_2 \prec_{\Lambda W} \lambda O_1.$$  \hspace{1cm} (9)

**Proof.** Follows from the elementary computation

$$O_2 \prec_W O_1 \iff O_1 - O_2 \subset W_c \iff \Lambda O_1 - \Lambda O_2 \subset \Lambda W_c \iff \Lambda O_1 + x - \Lambda O_2 - x \subset (\Lambda W)_c \iff \lambda O_2 \prec_{\Lambda W} \lambda O_1.$$  \hspace{1cm} (10)

It is clear that the causal complement $W'$ of any wedge region $W$ is also a wedge-region, and that $(W_c)' = (W')_c$. We say that $W'$ is the complementary wedge to $W$. More generally we will say that a wedge $W'$ is opposite to a given wedge $W$ if $W'$ can be translated into the complement of $W$, i.e. if for some $x \in \mathbb{R}^d$ we have $W' + x \subset W$. Lastly we will see that the construction of scattering states is most convenient for the geometrical situation of a given wedge whose edge is parallel to the time-zero hyperplane. This is equivalent to $W = RW_i + x$ for $x \in \mathbb{R}^d$ and some spatial rotation $R \in SO(s) \subset L_+^s$, and we will call such wedges $W$ upright or non-tilted. This is relevant as for upright $W$ the restriction of $\prec_W$ to certain hyperplanes behaves almost like a total relation, which will be helpful for establishing the Fock structure of scattering states in Section 3.2.

**Lemma 2** ("quasi-totality" of $\prec_W$ for velocity supports). Let $W$ be an upright wedge and let $V_k, V'_k \subset \mathbb{R}^{s+1}$, $(k = 1, 2)$, be sets of the form ("velocity supports")

$$V_k = \{1\} \times V_k, \quad V_k \subset \mathbb{R}^s, \quad (\text{similarly for } V'_k)$$  \hspace{1cm} (11)

satisfying

$$V_2 \prec_W V_1, \quad \text{and} \quad V'_2 \prec_W V'_1.\hspace{1cm} (12)$$

Then necessarily at least one of the two relations

$$V'_2 \prec_W V_1, \quad \text{or} \quad V_2 \prec_W V'_1$$

must be satisfied as well.

**Proof.** Let $\Lambda$ be s.t. $\Lambda W = W_i$ and note that as $W$ is upright we can choose $\Lambda$ as a spatial rotation. We obtain by Proposition 1 that

$$\Lambda V_2 \prec_W \Lambda V_1, \quad \text{and} \quad \Lambda V'_2 \prec_W \Lambda V'_1.$$ (13)

Due to the choice as spatial rotation, the sets $V_k := \Lambda V_k$ are still of the form (10), and analogously for $V'_k$. Dropping bars, the two assumptions (11) for $W = W_i$ translate to inequalities

$$e_1 \cdot (g_1 - g_2) > 0 \quad \text{and} \quad e_1 \cdot (g'_1 - g'_2) > 0 \quad \forall g_k \in V_k, \quad g'_k \in V'_k, \quad (k = 1, 2),$$ (14)
where \( e_1 \in \mathbb{R}^s \) denotes the spatial unit-vector in 1-direction. Assuming that \( V'_2 \prec_W V_1 \) is false, there must be \( g'_2 \in V'_2, g'_1 \in V_1 \) forming an ordering “obstruction”. Namely,
\[
\neg (V'_2 \prec_W V_1) \iff \neg (\forall g_1 \in V_1 \forall g'_2 \in V'_2 : e_1 \cdot (g_1 - g'_2) > 0) \\
\iff \exists g'_1 \in V_1 \exists g'_2 \in V'_2 : e_1 \cdot (g'_1 - g'_2) \leq 0.
\]
(14)

For any given \( g_2 \in V_2 \) and \( g'_1 \in V'_1 \) we can now estimate by transitivity
\[
e_1 \cdot (g'_1 - g_2) = e_1 \cdot (g'_1 - g'_2') + e_1 \cdot (g'_2 - g'_1) + e_1 \cdot (g'_1 - g_2) > 0,
\]
where we used that the first and last term on the right are strictly positive for any \( g_2 \in V_2 \) and \( g'_1 \in V'_1 \) as particular instances of (13) and the middle term is non-negative due to (14). By definition this implies \( V'_2 \prec_W V'_1 \).

Finally, let us remark that given any Haag-Kastler net of von Neumann algebras \( \mathcal{O} \mapsto A(\mathcal{O}) \) defined for open bounded regions \( \mathcal{O} \subset \mathbb{R}^d \), there exists a canonical associated wedge-local net. On the other hand starting from a wedge-local net the question of existence or non-existence of local observables can be highly non-trivial, as explained in the recent review of Lechner [Le15]. While previously the existence of suitable localized operators was always regarded as essential for going beyond two-particle scattering states, cf. [BBS01] or [Le06] Section 6, we will see in the following that scattering theory in most wedge-local models can be studied in reasonable generality without any reference to local observables.

3 Construction of Scattering States

3.1 Swapping Relations for Opposite Wedge Algebras

At the core of our subsequent arguments to establish convergence and Fock structure of scattering states will be certain swapping identities, such as (1). Due to the mass gap assumption and with our desired application to the construction of scattering states it will be in fact sufficient to impose (1) only after projection to the one-particle subspace.

**Definition 3** (swapping symmetry of single-particle states). We say that a single-particle vector \( \Psi_1 \in E_m \mathcal{H} \) of mass \( m > 0 \) is swappable with respect to a given wedge \( W \) if there exist operators \( A \in A(W) \), \( A^\perp \in A(W^\perp) \) localized in \( W \) and an opposite wedge \( W^\perp = W' + x \) such that
\[
\Psi_1 = E_m A \Omega = E_m A^\perp \Omega.
\]
(15)

As a matter of fact, swapping relations (1), (15) can be obtained as a consequence of wedge duality (HK2\(^\sharp\)), which is a basic and well-established structural property in quantum field theory.

**Lemma 4** (D. Buchholz, private communications (2017)). In the vacuum representation of a net \( A \) satisfying wedge-duality (HK2\(^\sharp\)) there exist nontrivial vectors satisfying the swapping relation, i.e. for suitable \( A \in A(W) \) and \( A^\perp \in A(W^\perp) \) we have
\[
\Psi = A \Omega = A^\perp \Omega.
\]
(16)

Moreover under the same assumptions, the subspaces \( \mathcal{H}^W \subset \mathcal{H} \) of swappable vectors \( \Psi \) associated to each wedge \( W \) are dense.
Let us briefly motivate and introduce the mathematical preliminaries necessary for the proof of Lemma 4 by noting that in Wightman quantum field theories there is a natural relation between oppositely localizable operators provided by the TCP-Operator. In the general operator-algebraic framework, a similar mapping is accessible abstractly by invoking Tomita-Takesaki theory. In the following we will only give the basic definitions and refer to [BR87, Sec. 2.5] for proofs and further details.

The Tomita-Takesaki construction starts from the observation that the adjoint operation \( A \mapsto A^* \) can encode non-trivial information about the structure of a general von Neumann algebra \( \mathcal{M} \). In particular if \( \mathcal{M} \) has a cyclic and separating vector \( \Omega \) one may define the closable Tomita operator \( S : \mathcal{M} \Omega \rightarrow \mathcal{M} \Omega \) by setting

\[
SA\Omega := A^*\Omega, \quad S = J\Delta^{1/2},
\]

where the positive self-adjoint modular operator \( \Delta \) and the anti-unitary modular conjugation \( J \) are obtained by polar decomposition. The Tomita-Takesaki theorem [BR87, Thm. 2.5.14] states that

\[
J\mathcal{M}J = \mathcal{M}' \quad \text{and} \quad \Delta^{i\tau}\mathcal{M}\Delta^{-i\tau} = \mathcal{M}, \quad (\tau \in \mathbb{R}).
\]

In our case we take \( \mathcal{M} = \mathfrak{A}(\mathcal{W}) \), so that the modular objects \( S_W, J_W \) and \( \Delta_W \) will depend on the wedge \( \mathcal{W} \). It is clear that \( S_W \Omega = S_W \Omega = \Omega = S_W' \Omega \) and one has further [BR87, Prop. 2.5.11]

\[
\Delta_W \Omega = \Omega, \quad J_W \Omega = \Omega.
\]

In this notation the basic idea for the proof of Lemma 4 is that for given self-adjoint \( A = A^* \in \mathfrak{A}(\mathcal{W}) \), \( S_W \) acts trivially on \( A\Omega \) so that (17) and (19) yield

\[
A\Omega = A^*\Omega = S_W A\Omega = J_W \Delta_W^{1/2} A\Omega = J_W \Delta_W^{1/2} A \Delta_W^{-1/2} J_W \Omega.
\]

Here a candidate for \( A^\perp \) can be extracted up to domain questions, assuming that (18) applies here also for imaginary \( \tau = -i/2 \). Technically it remains to show that \( \tilde{A} := \Delta_W^{1/2} A \Delta_W^{-1/2} \) makes sense as bounded operator which is in \( \mathfrak{A}(\mathcal{W}) \), because then (18) and wedge-duality give \( A^\perp := J_W \tilde{A} J_W \in \mathfrak{A}(\mathcal{W}') = \mathfrak{A}(\mathcal{W}') \).

Before proceeding to the proof of Lemma 4, we should point out that the swapping relation (16) is established exactly for “touching” wedges \( \mathcal{W}^\perp = \mathcal{W}' \). Then \( \mathcal{W}^\perp \cap \mathcal{W} \) is empty, so that in this case (16) is non-trivial also for local theories. Let us recall that wedge duality (HK2\textsuperscript{♯}) can be proven in the Wightman framework via the Bisognano-Wichmann property [BW75]

\[
U(\Lambda_{\mathcal{W}}(2\pi \tau)) = \Delta_{\mathcal{W}}^{-i\tau}.
\]

Here \( U \) denotes the unitary implementation of Poincaré symmetry, and \( \Lambda_{\mathcal{W}}(2\pi \tau) \) is the oriented one-parameter group of boosts preserving the given wedge \( \mathcal{W} \). In the setting of standard subspaces, wedge-duality (HK2\textsuperscript{♯}) and (21) have recently been established for any finite or infinite multiplets of massive or massless scalar irreducible unitary representations of the Poincaré group [Mo17].

**Proof of Lemma 4.** We follow the argument of Buchholz [Bu17]. As we keep \( \mathcal{W} \) fixed, we drop wedge indices on the modular objects. To establish existence we consider vectors of the form \( \Psi = A\Omega \) with \( A^* = A \in \mathfrak{A}(\mathcal{W}) \). Rigorous control over (20) is then obtained
by passing to operators $A_\delta$, $(\delta > 0)$, which are “regularized” with respect to the adjoint action of the modular group by setting

$$A_\delta := \int \frac{d\tau}{\sqrt{2\pi \delta}} e^{-\frac{\tau^2}{2\delta}} \Delta^{it} A \Delta^{-it}.$$ \hspace{1cm} (22)

From the Tomita-Takesaki theorem (18) we see that the integrand is pointwise in $\mathfrak{A}(W)$ so that $A_\delta \in \mathfrak{A}(W)$ as wedge-algebras are weakly closed. Secondly we obtain from strong continuity of $\Delta^{it}$ that $A_\delta \to A$ in the weak operator topology, so that by modular invariance of $\Omega$ we have $A_\delta \Omega \to A\Omega$ in norm as $\delta \to 0$. Further due to (22) the adjoint action of the modular group on $A_\delta$ may be computed explicitly as

$$\Delta^{it} A_\delta \Delta^{-it} = \int \frac{d\tau}{\sqrt{2\pi \delta}} e^{-\frac{\tau^2}{2\delta}} \Delta^{it} A \Delta^{-it} = \int \frac{d\tau'}{\sqrt{2\pi \delta}} e^{-\frac{(\tau'-\tau)^2}{2\delta}} \Delta^{it'} A \Delta^{-it'}.$$ \hspace{1cm} (23)

Returning to (20) we now define $\tilde{A}_\delta := \Delta^{1/2} A_\delta \Delta^{-1/2}$ as a quadratic form on a suitable domain. It will be convenient to restrict to $D_\omega(\Delta^\pm) := \{E_\Delta([k, K])\psi, \psi \in \mathcal{H}, 0 < k < K\}$, which is dense in $\mathcal{H}$ by spectral calculus. For $\Psi_1, \Psi_2 \in D_\omega(\Delta^\pm)$ the function $t \mapsto \langle \Psi_1, \Delta^{it} A_\delta \Delta^{-it} \Psi_2 \rangle$ is entire analytic. It further coincides for $t \in \mathbb{R}$ with the entire function defined by the right hand side of (23). By analyticity these two entire functions coincide for all $t \in \mathbb{C}$ so that

$$\langle \Psi_1, \Delta^{1/2} A_\delta \Delta^{-1/2} \Psi_2 \rangle = \int \frac{d\tau}{\sqrt{2\pi \delta}} e^{-\frac{\tau^2}{2\delta}} \langle \Psi_1, \Delta^{it+1/2} A \Delta^{-it-1/2} \Psi_2 \rangle$$ \hspace{1cm} (24)

$$= \int \frac{d\tau'}{\sqrt{2\pi \delta}} e^{-\frac{(\tau'+i/2)^2}{2\delta}} \langle \Psi_1, \Delta^{it'} A \Delta^{-it'} \Psi_2 \rangle.$$ \hspace{1cm} (25)

From (25) we see firstly that (24) in fact defines a bounded bilinear form, so that $A_\delta$ extends to a bounded operator on all of $\mathcal{H}$, and secondly that $\tilde{A}_\delta \in \mathfrak{A}(W)$ by repeating the argument below (22). Thus the swapping partner may be obtained as in (20) by setting $A_\delta^\pm := J\tilde{A}_\delta J$, and noting that $A_\delta^\pm \in \mathfrak{A}(W')$ due to (18) and wedge duality (HK2$^5$).

To establish density of swappable vectors let $\Psi \in \mathcal{H}$ and $\epsilon > 0$. By cyclicity of $\Omega$ there exists $A \in \mathfrak{A}(W)$ such that $\|\Psi - A\Omega\| \leq \epsilon/2$. We may then decompose $A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) =: A_1 + iA_2$ such that the above argument applies to $A_\delta \Omega$, $k = 1, 2$, and the swapping partner of $A_\delta$ is then given by $A_\delta^\pm := (A_1)^\dagger + i(A_2)^\dagger$. Choosing $\delta > 0$ sufficiently small yields $\|\Psi - A_\delta\Omega\| \leq \|\Psi - A\Omega\| + \|A_1\Omega - (A_1)\delta\Omega\| + \|A_2\Omega - (A_2)\delta\Omega\| \leq \epsilon$ so that we obtain density. \hfill $\square$

**Corollary 5.** Assuming (HK2$^5$), single-particle vectors satisfying the swapping relation (15) w.r.t. any given wedge $W$ are dense in the single-particle space $\mathcal{H}_1 := E_m\mathcal{H}$.

We note that for space-time dimension $d \geq 2 + 1$ the dense sets of swappable vectors constructed in Lemma 4 in general have a non-trivial dependence on $W$. Interestingly certain wedge-local models also admit a dense subspace of vectors which are swappable in the sense of Definition 3 for all wedges simultaneously, as can be seen from the results of [BLS11] in the class of deformed local theories.

A simple and immediate consequence of the swapping relation is the consistency of our definition of scattering states (4) with previous discussions of two-particle scattering in wedge-local models [GL07, BS08], where the physically obvious opposite-localization prescription $\Psi^\pm := \lim_{\tau \to \infty} B_\tau(f) B_\tau^\dagger (f^\dagger) \Omega$ has been used. With the swapping relation as main technical tool at hand, we may in fact swap

$$\Psi^+ = \lim_{\tau \to \infty} B_\tau(f) B_\tau^\dagger (f^\dagger) \Omega = \lim_{\tau \to \infty} B_\tau(f) B_\tau^\dagger (f^\dagger) \Omega,$$ \hspace{1cm} (26)
where \( B_\tau(f^+) \) is defined in terms of \( \bar{A} \in \mathfrak{A}(\mathcal{W}) \) with \( \bar{A}\Omega = A^\perp\Omega \). The new prescription from (26) with all operators localized in the same wedge \( \mathcal{W} \) now generalizes to \( N \)-particle scattering theory, as will be seen in the next section.

### 3.2 Wedge-local Haag-Ruelle Theorem

As comparison dynamics for the construction of scattering states we may restrict to regular positive-energy Klein-Gordon solutions \( f_k \), which are of the form

\[
f_k(t, x) = \int \frac{d^d k}{(2\pi)^d} e^{ikx - \omega_m(k)t} \tilde{f}_k(k),
\]

\[
\omega_m(k) := \sqrt{k^2 + m^2}, \quad \tilde{f}_k \in C_0^\infty(\mathbb{R}^d).
\]

**Definition 6** (Haag-Ruelle creation operator approximants). For \( A \in \mathfrak{A}(\mathcal{W}) \), \( \chi \in \mathcal{S}(\mathbb{R}^{s+1}) \), and \( f \) a regular positive-energy Klein-Gordon solution we set for \( \tau \in \mathbb{R} \)

\[
B := A(\chi) = \int d^{s+1}x \chi(x)\alpha_x(A),
\]

\[
B_\tau(f) := \int d^s x f(\tau, x)\alpha(\tau, x)(A).
\]

For our main result (Theorem 7) and in the following we will always assume \( \chi \) to be chosen as in Lemma 8 below, in accordance with the mass gap (HK6). The restrictions on propagation of wave packets mentioned in the introduction are made precise using the precursor relation (8), to constrain the velocity supports

\[
\mathcal{V}_{f_k} := \{(1, k/\omega_m(k)), \ k \in \text{supp } \tilde{f}_k\}.
\]

Basic intuition for handling localizations of creation-operator approximants comes from the fact that regular \( f_k \) are rapidly decreasing outside the cone \( \mathcal{Y}_\delta^{\mathcal{W}} := \mathbb{R}\mathcal{V}_{f_k}^{\mathcal{W}} \) generated by any \( \delta \)-neighbourhood \( \mathcal{V}^{\mathcal{W}}_{f_k} \supset \mathcal{V}_{f_k} \), as seen from standard non-stationary phase estimates\(^4\).

**Theorem 7.** Fix a wedge \( \mathcal{W} \) and let \( \Psi_k \in \mathcal{H} \) \((1 \leq k \leq n)\) be single-particle vectors isolated from the remaining energy-momentum spectrum which satisfy the swapping relation \( \Psi_k = E_m A_k \Omega = E_m A_k^\perp \Omega \), \( A_k \in \mathfrak{A}(\mathcal{W}) \), \( A_k^\perp \in \mathfrak{A}(\mathcal{W}^\perp) \).

(i) For any family of regular positive-energy Klein-Gordon solutions \( f_k \) satisfying

\[
\mathcal{V}_{f_n} \prec \mathcal{W} \mathcal{V}_{f_{n-1}} \prec \mathcal{W} \ldots \prec \mathcal{W} \mathcal{V}_{f_1},
\]

\[
\Psi_\tau := B_{1\tau}(f_1)B_{2\tau}(f_2)\ldots B_{n\tau}(f_n)\Omega \quad (\tau \in \mathbb{R})
\]

converges in norm for \( \tau \to \infty \).

(ii) Let \( \Psi^+ := \lim_{\tau \to \infty} \Psi_\tau \), \( \Psi'^+ := \lim_{\tau \to \infty} \Psi'_\tau \) be scattering states as in (i), constructed from operators localizable with respect to the same wedge \( \mathcal{W} \). Then for upright \( \mathcal{W} \) their scalar products can be computed using the Fock prescription

\[
\langle \Psi^+, \Psi'^+ \rangle = \delta_{n^2^\prime} \prod_{k=1}^n \langle B_{k\tau}(f_k)\Omega, B'_{k\tau}(f'_k)\Omega \rangle,
\]

where the right-hand side is independent of \( \tau \).

\(^4\)The velocity support estimates for regular Klein-Gordon solutions are due to Ruelle [Ru62], for details see e.g. [A, Thm. 5.3]. Via such estimates, disjointness \( \mathcal{V}_k \cap \mathcal{V}_j = \emptyset \), \((k \neq j)\) is sufficient for local QFT to control equal-time commutators [Hep65], and to some limited extent also non-equal time-commutators [Du17].
Analogous statements hold for the convergence and Fock structure of any two incoming scattering states \((\tau \rightarrow -\infty)\) defined using the reversed ordering of wave packets

\[
\mathcal{V}_{f_n} \succ_{\mathcal{W}} \mathcal{V}_{f_{n-1}} \succ_{\mathcal{W}} \cdots \succ_{\mathcal{W}} \mathcal{V}_{f_1}.
\]  

(34)

We should point out that the ordering prescription (31) is not new. Such relations are well known in the form-factor programme and related constructive work, see e.g. [Smi92, p. 8] and references therein, or [Le06, Sec. 6]. However in contrast to the results from [BBS01, Le06], we note that our arguments require neither the existence of local observables, nor temperateness of suitable polarization-free generators.

**Lemma 8** (Haag-Ruelle Lemma, wedge-local version). Let \(A \in \mathfrak{A}(\mathcal{W})\) and \(K \subset K' \subset H_m\) be compact subsets of the mass shell, such that \(K\) can be separated from \(H_m \setminus K'\) by a smooth function. Then there exists a suitable \(\chi \in \mathcal{S}(\mathbb{R}^{s+1})\) (with \(\hat{\chi}\) supported in a sufficiently small neighbourhood of the mass shell as dictated by the mass gaps (HK6)) such that \(B := A(\chi)\) satisfies

(i) \(B\Omega \in E(K')\mathcal{H} \subset E(H_m)\mathcal{H}\),

(ii) \(E(K)B\Omega = E(K)A\Omega\),

(iii) \(B^*\Omega = 0\),

(iv) \(B^*\Psi_1 = E_{\Omega}B^*\Psi_1\) for all \(\Psi_1 \in E(K' \cap H_m)\mathcal{H}\), where \(E_{\Omega} := |\Omega\rangle\langle\Omega|\).

(v) \(B\) is almost wedge-local (w.r.t. \(\mathcal{W}\)), i.e. for any \(r > 0\) there exists \(B_r \in \mathfrak{A}(\mathcal{W} + \mathcal{C}_r)\) so that for any \(N \in \mathbb{N}\) we have for a suitable \(C_N > 0\) that

\[
\|B - B_r\| \leq \frac{C_N}{1 + r^N}.
\]  

(35)

Here \(\mathcal{C}_r := \{x \in \mathbb{R}^{s+1} : |x^0| + |x| < r\}\) denotes the double cone of radius \(r\).

Lemma 8 has a well-known counterpart in strictly local theories [Ha58] [Ru62], which allows us to skip the proof. In particular the main spectral statements (i), (iii) and (iv) may be understood by noting that the smearing operation \(B := A(\chi)\) restricts the Arveson spectrum\(^5\) \(\text{Sp}_B \alpha \subset \text{supp} \hat{\chi}\). The only modification appears in (v), where the statement of the lemma needs to be adapted for the wedge-local case. From Lemma 8 we immediately obtain analogous properties for the creation-operator approximants \(B_{\tau}(f)\) defined by the standard LSZ prescription (29).

**Proposition 9** (elementary properties of \(B\) and \(B_{\tau}\)).

(i) \(B_{\tau}(f)\Omega = \hat{f}(\mathcal{P})B\Omega\) for all \(\tau \in \mathbb{R}\).

(ii) If \(A\Omega = A^+\Omega\), the corresponding Haag-Ruelle operators satisfy \(B_{\tau}(f)\Omega = B_{\tau}^+(f)\Omega\).

(iii) \(\partial_\tau B_{\tau}(f)\Omega = 0\).

(iv) \(\|B_{\tau}(f)\| \leq C(1 + |\tau|^{s/2})\).

(v) \(\partial_\tau B_{\tau}(f)\) exists in norm and \(\|\partial_\tau B_{\tau}(f)\| \leq C'(1 + |\tau|^{s/2})\).

(vi) \(B_{1\tau}(f_1)^* B_{2\tau}(f_2)\Omega = E_{\Omega}B_{1\tau}(f_1)^* B_{2\tau}(f_2)\Omega\), independently of velocity supports and operators possibly associated to different wedges \(\mathcal{W}_1, \mathcal{W}_2\), where \(E_{\Omega} := |\Omega\rangle\langle\Omega|\).

\(^5\)See e.g. [Arv82] or [BDN15], Sec. 3.
Having adapted the statements of Lemma 8 and Proposition 9 as required by wedge-locality, we will skip the proofs which carry over literally from standard Haag-Ruelle theory (up to weakened localization) and refer to [A, Sec. 5] or [Du17] for further details. Still the most significant consequence of wedge-locality for Haag-Ruelle theory is contained in the following localization and commutator estimates, whose proofs will be sketched for the convenience of the reader in Appendix A.

**Lemma 10.** Let $A \in \mathcal{A}(\mathcal{W})$. For any $\tau \in \mathbb{R}$ and $\delta > 0$ the corresponding $B_\tau := B_\tau(f)$ can be approximated by $B_\tau^\delta \in \mathcal{A}(\mathcal{W}_f + \mathcal{C}_\delta \mathcal{V}_f + \mathcal{W})$, $(\delta > 0)$, such that for any $N \in \mathbb{N}$

$$
\left\| B_\tau^\delta - B_\tau \right\| \leq \frac{C_N^\delta}{1 + |\tau|^N},
$$

where the constants $C_N^\delta$ depend on $f, A$ and $\chi$, but are independent of $\tau$.

For later use in Section 4 we note that analogous approximants $B_\tau^\delta$ exist if $f$ is replaced by the pointwise product $\tilde{f} := fh$ with a polynomially bounded measurable function $h : \mathbb{R}^d \to \mathbb{C}$.

**Corollary 11** (commutators with ordered velocity support). Let $B, B^\perp$ be as in Lemma 8 for a pair of opposite wedges $\mathcal{W}, \mathcal{W}^\perp$, respectively, and let $f, f^\perp$ be ordered by $\mathcal{V}_f \prec \mathcal{W} \mathcal{V}_f^\perp$. Then for any $\tau > 0$,

$$
\left\| \left[ B_\tau^\perp (f^\perp), B_\tau (f) \right] \right\| \leq \frac{C_N}{1 + |\tau|^N},
$$

where $C_N$ depend on operators and smearing functions as in Lemma 10. For $\tau < 0$ estimate (37) holds under the reversed ordering assumption $\mathcal{V}_f \prec \mathcal{W} \mathcal{V}_f^\perp$. The commutator estimate extends to the cases that one or both of the operators in (37) are replaced by their adjoints or $\tau$-derivatives.

**Proof of Theorem 7.** Ad (i) Setting $\Psi_\tau := \Psi^{(n)}_\tau := B_1(f_1)B_2(f_2) \ldots B_n(f_n)\Omega$ we would like to establish convergence for $\tau \to \infty$. Due to Proposition 9 (v) and (iv), Cook’s method is applicable and we can write for $0 < \tau_1 < \tau_2$

$$
\left\| \Psi_{\tau_2} - \Psi_{\tau_1} \right\| = \left\| \int_{\tau_1}^{\tau_2} d\tau \, \partial_\tau \Psi_\tau \right\| \leq \int_{\tau_1}^{\tau_2} d\tau \left\| \partial_\tau \Psi_\tau \right\|. \quad (38)
$$

Convergence will follow from the rapid decay estimate $\left\| \partial_\tau \Psi_\tau \right\| \leq C_N \tau^{-N}$ for $\tau > 0$.

The latter is obtained by induction with respect to the number of particles $n$, with starting case $n = 1$ given by $\partial_\tau \Psi_\tau = 0$ as seen in Proposition 9 (iii). For the induction step we write

$$
\partial_\tau \Psi_\tau = \partial_\tau (B_1(f_1)B_2(f_2) \ldots B_{n-1}f_{n-1}B_n(f_n)\Omega
+ B_1(f_1) \ldots B_{n-1}f_{n-1} \partial_\tau B_n(f_n)\Omega
= \partial_\tau (B_1(f_1)B_2(f_2) \ldots B_{n-1}f_{n-1}) B_n^\perp(f_n)\Omega,
$$

where we first used Proposition 9 (iii) to drop the term with derivative operator acting directly on the vacuum and used that the swapping relation (15) implies $B_nf(f_n)\Omega = B_n^\perp(f_n)\Omega$. Now there are oppositely wedge-localized pairs of HR-operators whose commutators can be controlled using Corollary 11, and may estimate for $\tau > 0$

$$
\left\| \partial_\tau \Psi_\tau \right\| \leq \left\| B_n^\perp(f_n) \right\| \left\| \partial_\tau \Psi^{(n-1)} \right\|
+ \left\| \partial_\tau (B_1(f_1) \ldots B_{n-1}f_{n-1}, B_n^\perp(f_n)\Omega \right\| \Omega. \quad (40)
$$

11
Here the first summand is rapidly decreasing for \( \tau \to \infty \) by the induction assumption and Proposition 9 (iv). The second summand can be generously bounded from above by expanding the derivative and commutator as

\[
\sum_{k=1}^{n-1} \sum_{j=1}^{n-1} B_{1\tau}(f_1) \cdots (\partial_{\tau} B_{k\tau}(f_k)) \cdots [B_{j\tau}(f_j), B_{n\tau}^+(f_n)] \cdots B_{n-1\tau}(f_{n-1}).
\] (41)

Estimating the corresponding operator norm in (40) by expanding in terms of \( \|B_{k\tau}\| \leq C_k(1 + |\tau|^{3/2}), \|\partial_{\tau} B_{k\tau}\| \leq C_k(1 + |\tau|^{3/2}), \|B_{j\tau}(f_j), B_{n\tau}^+(f_n)\| \leq C_N(1 + \tau)^{-N} \), and \( \|\partial_{\tau} B_{k\tau}(f_k), B_{n\tau}^+(f_n)\| \leq C_N(1 + \tau)^{-N} \) yields an overall rapid decay. Here we used that Corollary 11 applies due to transitivity of the precursor ordering. Together we obtain that (40) decays faster than any polynomial, and thus convergence of outgoing scattering states follows from (38). The existence of incoming states follows analogously for opposite operator ordering.

\textbf{Ad (ii)} Letting \( \Psi^+ := \lim_{\tau \to \infty} B_{1\tau}(f_1) \cdots B_{n\tau}(f_n)\Omega \) and another scattering state \( \Psi'^+ := \lim_{\tau \to \infty} B_{1\tau}'(f_1') \cdots B_{n\tau}'(f_n')\Omega \) defined with respect to the same wedge \( \mathcal{W} \), we denote the minimum number of particles by \( N := \min(n, n') \). We will assume instead of \( \mathcal{W} \) only the following weaker technical ordering condition: adjacent pairs of velocity supports are precursor-comparable from the rear also across the two families, in the sense that

\[
\forall 0 \leq j < N : \mathcal{V}_{f_{n-j}} \prec_{\mathcal{W}} \mathcal{V}_{f'_{n'-j-1}} \text{ or } \mathcal{V}_{f'_{n'-j}} \prec_{\mathcal{W}} \mathcal{V}_{f_{n-j}}.
\] (42)

For upright wedges (42) follows from Lemma 2, but the argument based on (42) can be also applied for non-upright \( \mathcal{W} \), e.g. to compute \( \|\Psi^+\|^2 = \langle \Psi^+, \Psi'^+ \rangle \).

The proof of the Fock relation (33) is now by induction on the minimum number of particles \( N \). By continuity of the scalar product we may write

\[
\langle \Psi^+, \Psi'^+ \rangle = \lim_{\tau \to \infty} \langle \Omega, B_{n\tau}(f_n)^* \cdots B_{1\tau}(f_1)^* B_{1\tau}'(f_1') \cdots B_{n\tau}'(f_n')\Omega \rangle.
\] (43)

For \( N = 0 \) the Fock identity (33) follows from \( \|\Omega\| = 1 \) or Lemma 8 (iii), in the respective cases vacuum-vacuum or for a non-zero number of creation operators. Assuming (33) holds for \( N \) particles, we now distinguish the two cases \( \mathcal{V}_{f_n} \prec_{\mathcal{W}} \mathcal{V}_{f'_{n'-1}} \) or \( \mathcal{V}_{f'_{n'}} \prec_{\mathcal{W}} \mathcal{V}_{f_{n-1}} \), determining on which side of (43) the swapping should be performed. Let us proceed for the case \( \mathcal{V}_{f'_{n'}} \prec_{\mathcal{W}} \mathcal{V}_{f_{n-1}} \), by swapping

\[
\langle \Psi^+, \Psi'^+ \rangle = \langle \Omega, B_{n\tau}(f_n)^* \cdots B_{1\tau}(f_1)^* B_{1\tau}'(f_1') \cdots B_{n\tau}'(f_n')\Omega \rangle
\]

\[
= \langle \Omega, B_{n\tau}(f_n)^* \cdots B_{1\tau}(f_1)^* B_{1\tau}'(f_1') \cdots B_{n\tau}'(f_n')\Omega \rangle
\]

\[
= \langle \Omega, B_{n\tau}^* B_{n-1\tau}^* \cdots B_{1\tau}^* B_{1\tau}' \cdots B_{n\tau}' \Omega \rangle
\]

\[
+ \langle \Omega, B_{n\tau}^* B_{n-1\tau}^* \cdots B_{1\tau}^* B_{1\tau}' \cdots B_{n\tau}' \Omega \rangle,
\]

where in the last step and below we suppress obvious wave packet dependences. Expanding the commutator gives

\[
\sum_{k=1}^{n-1} B_{k\tau}^* B_{n\tau}^+ \cdots B_{1\tau}^* B_{1\tau}' \cdots B_{n\tau}'
\]

\[
+ B_{n-1\tau}^* \cdots B_{1\tau}^* \sum_{k=1}^{n'-1} B_{1\tau}' \cdots B_{k\tau}^* B_{n\tau}^+ \cdots B_{n'-1\tau}.
\]
Here Corollary 11 applies due to $\mathcal{V}_{f_{n'}} \sim_W \mathcal{V}_{f_{n-1}}$, the assumed orderings (31) of the velocity supports of $f_k$ and $f_k'$ within each family, and transitivity of the precursor ordering. This yields $\|B_{\tau_k}(f_k)^* B_{\tau_{n'}}(f_n)\| \leq C_N(1+\tau)^{-N}$ and $\|B_{\tau_k}(f_k), B_{\tau_{n'}}(f_n)\| \leq C_N(1+\tau)^{-N}$, so that together with $\|B_{\tau_k}^*\| \leq C_j(1+|\tau|^{s/2})$ and $\|B_{\tau_k}'\| \leq C_j'(1+|\tau|^{s/2})$ from Proposition 9 (iv), we can estimate for $\tau > 0$

$$
\left| \langle \Psi, \Psi' \rangle - \left( \Omega, B_{\tau_k}^* B_{\tau_{n'}}^* B_{\tau_{n'-1}} \ldots B_{\tau_1}^* B_{\tau_1}^* \ldots B_{\tau_{n'-1}} \Omega \right) \right| \leq C_N \tau^{-N}. \quad (44)
$$

As $\lim_{\tau \to \infty} \langle \Psi, \Psi' \rangle$ exists by part (i) of this theorem, which was established above,

$$
\lim_{\tau \to \infty} \langle \Psi, \Psi' \rangle = \lim_{\tau \to \infty} \left( \Omega, (B_{\tau_k}^*)^* B_{\tau_{n'}}^* \ldots B_{\tau_1}^* B_{\tau_1}^* \ldots B_{\tau_{n'-1}} \Omega \right)
$$

where the right hand side was rewritten using the clustering identity from Proposition 9 (vi). The existence of the limit on the right-hand side now follows for the one-particle matrix element in the first factor from Proposition 9 (iii), and for the second factor from the induction assumption, respectively. By induction we obtain finally the Fock formula (33).

For the complementary ordering $\mathcal{V}_{f_{n'}} \sim_W \mathcal{V}_{f_{n-1}}$ we swap instead on the opposite side of (43), making use of $B_{\tau_k}(f_n)\Omega = B_{\tau_k}(f_n)\Omega$. Following otherwise the same chain of arguments we obtain the limit (33) also in this case. □

To conclude this section let us recall that in dimension $1 + 1$ all wedges are upright in a trivial sense. In higher dimension the restriction to upright wedges seems to be unphysical as it singles out a non-Poincaré-covariant family of localization wedges. We will later see that the uprightness restriction is of a technical nature arising due to the a priori Lorentz-frame dependent formulation of Haag-Ruelle theory. Consequently it can be lifted by passing to a variant of the Haag-Ruelle creation operator approximants (3) adapted to the reference frame of a given (non-upright) operator localization wedge $W$.

### 4 Localization in General Wedges

The goal of this section is to remove the assumption of localization of operators in upright wedges from Theorem 7 (ii), as will be needed for a physically satisfactory discussion of the known Poincaré-covariant wedge-local models (e.g. as in [BLS11]). We recall that these additional considerations are specific to the case of spatial dimension $s > 1$. The following simple example illustrates the causal restrictions in the non-upright case which invalidate Lemma 2 and allows to visualize how these are resolved below.

**Remark 12** (canonical non-upright wedge). A non-upright wedge can be obtained by boosting the right wedge $\mathcal{W}_t = \{ x \in \mathbb{R}^d, |x^0| < x^1 \}$, $d \geq 3$, in $x^2$-direction with rapidity $\beta \in \mathbb{R} \setminus \{0\}$, yielding

$$
\mathcal{W} := A^{(2)}_{\beta} \mathcal{W}_t = \{ x \in \mathbb{R}^d, |\cosh(\beta)x^0 - \sinh(\beta)x^2| < x^1 \}. \quad (45)
$$

For concreteness we may take $d = 3$. The relevant part determining the precursor ordering of velocity supports $\mathcal{V}_1 \sim_W \mathcal{V}_2 \iff \mathcal{V}_2 - \mathcal{V}_1 \subset \mathcal{W}$ is the restriction of $\mathcal{W}$ to $\{ x^0 = 0 \}$. For the upright case $\beta = 0$ this restriction is a half plane, and the opposite ordering $\mathcal{V}_2 \sim_W \mathcal{V}_1$ corresponds to inclusion in the complementary open half-plane. Exactly this special geometrical situation is necessary for the validity of Lemma 2.
Further this means physically that the scattering states constructed in Theorem 7 cover the entire 2-particle velocity space up to a set of measure zero.\(^6\)

However for the non-upright case \(\beta \neq 0\) the restriction of \(W\) to \(\{x_0 = 0\}\) yields merely a cone \(C := \{x \in \mathbb{R}^{d-1}, |\sinh(\beta)x_2^2| < x_1^2\}\). Hence there is a non-trivial region of the two-particle velocity space which cannot be decomposed into ordered configurations. For example we may take the corresponding velocity supports concentrated in sufficiently small neighbourhoods of points \(v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2\) for which

\[
v_1 \not\in W v_2 \quad \text{and} \quad v_2 \not\in W v_1 \iff v_2 - v_1 \in \mathbb{R}^2 \setminus (C \cup (-C)) =: \Xi, \tag{46}\]

where a “causally forbidden” region \(\Xi\) appears, which has vanishing measure only if \(\beta = 0\).

### 4.1 Haag-Ruelle Theorem with Adapted Lorentz Frame

Difficulties as in (46) result from the implicit Lorentz-frame dependence of the Haag-Ruelle operators \(B_\tau(f)\). Nevertheless the latter were well suited for the case of upright \(W\), which motivates us to adapt the construction from Theorem 7 by passing to a suitable reference frame.\(^7\)

**Definition 13** (adapted Haag-Ruelle operators). For a general (possibly non-upright) wedge \(W\), \(A \in \mathcal{A}(W)\), \(B = A(\chi)\) as before and regular positive-energy Klein-Gordon solutions \(f\), we set for \(\tau \in \mathbb{R}\)

\[
B^\Lambda_\tau(f) := \int d^d x f(\Lambda(\tau, x)) \alpha^{(\Lambda(\tau, x))}(B), \tag{47}
\]

where \(\Lambda \in \mathcal{L}^*(W) := \{\Lambda \in \mathcal{L}^+_1 \mid \Lambda W = W\} \) or more generally \(\Lambda \in \mathcal{L}^+_1\).

In fact, such \(B^\Lambda_\tau(f)\) appear naturally in the discussion of Lorentz covariance in standard Haag-Ruelle theory. Here we just introduce them in an ad-hoc manner, even if the wedge-local net may not be Lorentz covariant. In the following we will see that they can equally well serve as creation-operator approximants, which will turn out suitable for our cause. We should emphasize that no Lorentz transformation is applied to \(B\) — only the hyperplane used for smearing the translates \(\alpha_x(B)\) is modified. Fortunately it is not necessary to repeat our arguments from Section 3.2. We will instead infer the existence of the limits

\[
\Psi^\Lambda := \lim_{\tau \to \infty} B^\Lambda_1(f_1) B^\Lambda_2(f_2) \cdots B^\Lambda_n(f_n) \Omega \tag{48}
\]

and their Fock structure for suitably ordered wave packets from a redefinition of the wedge-local net and the results of Section 3.2. The basic observation is that the modification of passing from \(f\) to \(f^\Lambda(x) := f(\Lambda x)\) and from translation by \(\alpha_x\) to modified translation automorphisms \(\alpha^\Lambda_x := \alpha_{\Lambda x}\) entering in (47) are both compatible with the underlying structures in a sense to be made precise now.

**Lemma 14.** Let \(\Lambda \in \mathcal{L}^+_1\).

(i) \(f^\Lambda(x) := f(\Lambda x)\) defines a regular positive-energy Klein-Gordon solution iff \(f\) is a regular positive-energy Klein-Gordon solution.

---

\(^6\)Underlying this simple picture is of course the intuition of conventional (e.g. bosonic) particle statistics, which may be misleading in the general wedge-local setting as illustrated by recent examples of Longo, Tanimoto and Ueda [LTU17].

\(^7\)Constructions using Lorentz-covariant creation-operator approximants (e.g. [Her13]) face similar problems as in (46) when applied in a wedge-local setting.
(ii) Setting $\alpha_x^A := \alpha_x A_x$ and $\mathfrak{A}^A(\mathcal{W}) := \mathfrak{A}(\Lambda \mathcal{W})$, $(\mathfrak{A}^A, \alpha^A, \Omega)$ is a wedge-local quantum field theory satisfying (HK1)–(HK6), and possibly (HK2)', (HK3)', iff the corresponding assumptions hold for $(\mathfrak{A}, \alpha, \Omega)$.

If (HK3') holds, we set further $\alpha^A_\lambda(A) := \alpha_{(\Lambda x, \Lambda_{\lambda} \Lambda^{-1} x)}(A)$ for $\lambda = (x, \Lambda_1) \in \mathcal{P}_+^\downarrow$.

Proof. Lorentz invariance of the Klein-Gordon equation (i) is standard, so let us only comment that the restriction to orthochronous Lorentz transformation is essential for preserving the positive-energy property, and that the regularity property can be concluded via the representation (27) and standard (non-)stationary phase estimates.

Statement (ii) follows from elementary computations which we illustrate for the example of (HK3'). Letting $\lambda = (x, \Lambda_1) \in \mathcal{P}_+^\downarrow$ we obtain

$$\alpha^A_\lambda \mathfrak{A}^A(\mathcal{W}) = \alpha_{(\Lambda x, \Lambda_{\lambda} \Lambda^{-1} x)} \mathfrak{A}(\Lambda \mathcal{W}) = \mathfrak{A}(\Lambda \Lambda_{\lambda}^{-1} \Lambda \mathcal{W} + \Delta x) = \mathfrak{A}^A(\Lambda_1 \mathcal{W} + x)$$

where we used that (HK3') holds for the original net $\mathfrak{A}$.

It should be noted that Lemma 14 (ii) applies also to wedge-local nets which are not Poincaré covariant (HK3'). In particular the basic definitions $\alpha^A_x := \alpha_x A_x$ and $\mathfrak{A}^A(\mathcal{W}) := \mathfrak{A}(\Lambda \mathcal{W})$, do not make use of Lorentz-transformation isomorphisms, they are only a passive redefinition on the level of the wedge-local net.

To establish the Haag-Ruelle theorem for the adapted scattering state approximants in (48) we rewrite the adapted Haag-Ruelle operators in terms of the boosted net of Lemma 14 as

$$B^\lambda_{\tau}(f) = \int \! d^d x \, f^{\lambda}(\tau, x) \alpha^{\lambda}_{\tau, x}(B), \text{ and similarly}$$

$$B = A(\chi) = \int \! d^d x' \chi(x') \alpha^x_{x'}(A)$$

$$= \int \! d^d x \chi(\Lambda x) \alpha^\Lambda_{\Lambda x}(A) = \int \! d^d x \chi^\Lambda(x) \alpha^\Lambda_{\Lambda x}(A),$$

where we used Lorentz invariance of $d^d x$. Due to $\chi^\Lambda(x) := \chi(\Lambda x) \in \mathcal{S}(\mathbb{R}^d)$ we know that $B$ is almost wedge-local also for the redefined net $\mathfrak{A}^A$. Therefore Theorem 7 may be applied to the rewritten operators (49). It remains to rephrase the statement of Theorem 7 from the boosted net $(\mathfrak{A}^A, \alpha^A, \Omega)$ to return to the terminology of the original theory $(\mathfrak{A}, \alpha, \Omega)$.

Let $\mathcal{W}$ be any wedge, $\Psi_j = E_m A_j \Omega = E_m A_j^\perp \Omega$, $A_j \in \mathfrak{A}(\mathcal{W})$, $A_j^\perp \in \mathfrak{A}(\mathcal{W}^\perp)$, and $\Lambda \in \mathcal{L}^+_{\perp}$. Then $\Psi_j$ are obviously also swappable with respect to the boosted net and in particular $A_j \in \mathfrak{A}^A(A^{-1} \mathcal{W})$. For $\Lambda \in \mathcal{L}^*(\mathcal{W})$ we get $A_j \in \mathfrak{A}^A(\mathcal{W}_t)$, where $A^{-1} \mathcal{W}_t = \mathcal{W}_t$ is upright. Hence assuming uprightness is redundant for the adapted Haag-Ruelle construction with $\Lambda \in \mathcal{L}^*(\mathcal{W})$. Secondly we see from (49) that applying Theorem 7 to outgoing scattering-state approximants interpreted via the boosted net now requires the ordering

$$\mathcal{V}_{f^n_A} \prec \omega, \mathcal{V}_{f^n_{A-1}} \prec \omega, \ldots \prec \omega, \mathcal{V}_{f^1_A},$$

with $\mathcal{V}_{f^n_A}$ as in (30), denoting the velocity support of $f^n_A(x) := f_j(\Lambda x)$. In terms of the original net, (50) is by covariance of the ordering relation (Proposition 1) equivalent to

$$\Lambda \mathcal{V}_{f^n_A} \prec \omega \Lambda \mathcal{V}_{f^n_{A-1}} \prec \omega \Lambda \mathcal{V}_{f^{n-1}_A} \prec \omega \Lambda \mathcal{V}_{f^{n-1}_{A-1}} \prec \omega \Lambda \mathcal{V}_{f^1_A}.$$ 

This is also consistent with a corresponding localization of the adapted Haag-Ruelle operators (49) similarly as in Lemma 10, but with respect to adapted velocity supports

$$\mathcal{V}_{f^A_j} := \Lambda \mathcal{V}_{f^A_j}.$$

The result of this discussion will be summarized in Theorem 15.
Theorem 15. Let \( \Lambda \in \mathcal{L}^\uparrow_+ \) and \( \Psi_j = E_m A_j \Omega = E_m A^\uparrow_j \Omega \) with \( A_j \in \mathfrak{A}(\mathcal{W}) \), \( A^\uparrow_j \in \mathfrak{A}(\mathcal{W}^\uparrow) \) (as in Theorem 7).

(i) For regular positive-energy Klein-Gordon solutions \( f_j \) satisfying
\[
\mathcal{V}_f \prec \mathcal{V}_f \prec \cdots \prec \mathcal{V}_f,
\]
the scattering state approximants \( \Psi^\Lambda_\tau := B^\Lambda_{\mathcal{W}}(f_1)B^\Lambda_{\mathcal{W}}(f_2)\ldots B^\Lambda_{\mathcal{W}}(f_n)\Omega \) converge in norm for \( \tau \to \infty \).

(ii) For \( \Lambda \in \mathcal{L}^\ast(\mathcal{W}) \) scalar products of \( \Psi^\Lambda_\tau := \lim_{\tau \to \infty} B^\Lambda_{\mathcal{W}}(f_1)\ldots B^\Lambda_{\mathcal{W}}(f_n)\Omega \), \( \Psi^\Lambda_{\Lambda} := \lim_{\tau \to \infty} B^\Lambda_{\mathcal{W}}(f_1)\ldots B^\Lambda_{\mathcal{W}}(f_n)\Omega \) constructed w.r.t. the same wedge \( \mathcal{W} \) satisfy
\[
\langle \Psi^\Lambda_\tau, \Psi^\Lambda_{\Lambda} \rangle = \delta_{nm} \prod_{j=1}^n \langle B^\Lambda_{\mathcal{W}}(f_j)\Omega, B^\Lambda_{\mathcal{W}}(f_j')\Omega \rangle.
\]

Analogous statements hold for incoming scattering states assuming opposite ordering.

4.2 Lorentz-Frame Independence and Residual Covariance

For the adapted creation-operator approximants \( B^\Lambda_{\mathcal{W}}(f_j) \), convergence of approximants \( \Psi^\Lambda_\tau := B^\Lambda_{\mathcal{W}}(f_1)\ldots B^\Lambda_{\mathcal{W}}(f_n)\Omega \) has now been established for general wedges, i.e. upright or tilted. The new ordering restrictions (53) appear optimal in the context of Remark 12, and the Fock structure follows without additional assumptions. However, as in standard Haag-Ruelle theory, the choice HR-operators \( B^\Lambda_{\mathcal{W}}(f_j) \) creating a given one-particle vector \( \Psi_j = B^\Lambda_{\mathcal{W}}(f_j)\Omega \) is not unique. Fock structure (Theorem 15 (ii)) implies only for fixed \( \Lambda \), that resulting scattering states do not depend on this freedom of choosing \( B^\Lambda_{\mathcal{W}}(f_j) \). In the following we will exclude also any unphysical dependence on \( \Lambda \in \mathcal{L}^\ast(\mathcal{W}) \), for which one has to handle the non-trivial dependence of localization of \( B^\Lambda_{\mathcal{W}}(f_j) \) on \( \Lambda \). We begin by considering the \( \Lambda \)-dependence of one-particle vectors, to be followed by discussing the influence on ordering conditions and finally on scattering states.

Lemma 16. Let \( \Lambda \in \mathcal{L}^\uparrow_+ \) and \( f \) a regular positive-energy Klein-Gordon solution.

(i) The wave packet of \( f^\Lambda(x) := f(\Lambda x) \) as defined in (27) is given by
\[
\tilde{f}^\Lambda(k) = \frac{\omega_m(\Lambda m(k))}{\omega_m(k)} \tilde{f}(\Lambda m(k)),
\]
where \( \tilde{f} \) is the wave packet of \( f \) and \( \Lambda m(k) \) denotes the spatial part of \( \Lambda \cdot (\omega_m(k), k) \). In particular, \( \text{supp} \tilde{f}^\Lambda = \Lambda^{-1}_m(\text{supp} \tilde{f}) \).

(ii) The \( \Lambda \)-dependence of one-particle vectors is
\[
B^\Lambda_{\mathcal{W}}(f)\Omega = \frac{\omega_m(P)}{\omega_m(\Lambda^{-1}_m(P))} \tilde{f}(P) E(H_m) B\Omega.
\]

These one-particle covariance formulas are well-known from the discussion of Lorentz-covariance in the local case and we will only briefly sketch the computations in Appendix A. They are important for the present discussion, as (56) suggests a non-trivial dependence of \( \lim_{\tau \to \infty} \Psi^\Lambda_\tau \) on the auxiliary boost \( \Lambda \). However the dependence can be
absorbed by passing to Klein-Gordon solutions \( f_j^{(A)} \) defined via modified wave packets \( \tilde{f}_j^{(A)}(p) := \frac{\omega_m(A_m^{-1}(p))}{\omega_m(p)} f_j(p) \), which have identical velocity supports and give via (56) that

\[
B_{\Lambda_j}^{\pm}(f_j^{(A)})\Omega = \tilde{f}_j(P)E(H_m)B_j\Omega, \text{ for any } \Lambda \in \mathcal{L}_\pm^1.
\]  

(57)

While the above argument coincides with the familiar result from local QFT, the discussion of scattering-state dependence requires additional care in the wedge-local case due to additional ordering requirements. For brevity reasons we shall focus on \( \Lambda \)-dependence only within the preferred class of reference frames for a given localization wedge \( W \) defined by \( \mathcal{L}^*(W) := \{ \Lambda \in \mathcal{L}_+^1 : \Lambda W_t = W_t \} \) as in Theorem 15.\(^8\)

Remark 17. Clearly any \( \Lambda, \Lambda' \in \mathcal{L}^*(W) \) are related by an element \( \bar{\Lambda} := \Lambda^{-1}\Lambda' \) from the stabilizer \( \text{Stab}_{\mathcal{L}_+^1} W_t := \{ \Lambda \in \mathcal{L}_+^1 : \Lambda W_t = W_t \} \cong O(1,1)_+ \times SO(d-2) \), where the first factor is generated by boosts \( \Lambda_\beta \) in \( x^1 \)-direction (\( \beta \in \mathbb{R} \)), and the second by rotations fixing \( x^1 \). In particular we note for later reference that \( \text{Stab}_{\mathcal{L}_+^1} W_t \) is path connected, and that we may smoothly interpolate between any \( \Lambda, \Lambda' \in \mathcal{L}^*(W) \) via arbitrarily often differentiable maps \( \Lambda^\gamma : [0,1] \to \mathcal{L}^*(W) \) such that \( \Lambda^0 = \Lambda, \Lambda^1 = \Lambda' \).

Proposition 18 (\( \mathcal{L}^*(W) \)-invariance of velocity ordering). For regular Klein-Gordon solutions \( f_1, f_2 \) and any \( \Lambda, \Lambda' \in \mathcal{L}^*(W) \) we have

\[
\mathcal{V}_f^{\Lambda} \prec_{\mathcal{W}} \mathcal{V}_f^{\Lambda'} \iff \mathcal{V}_{f_1}^{\Lambda} \prec_{\mathcal{W}} \mathcal{V}_{f_2}^{\Lambda'},
\]

(58)

Proof. By Proposition 1 we have \( \mathcal{V}_f^{\Lambda} \prec_{\mathcal{W}} \mathcal{V}_f^{\Lambda'} \iff \mathcal{V}_{f_1}^{\Lambda} \prec_{\mathcal{W}} \mathcal{V}_{f_2}^{\Lambda} \) and similarly for \( \Lambda' \), allowing us to reduce (58) to the case \( \mathcal{W} = W_t \) up to boosts acting on \( f_j \). Thus (58) amounts to a property of the relativistic velocity transformation law. Let us assume that \( \mathcal{V}_{f_1}^{\Lambda} \prec_{W_t} \mathcal{V}_{f_2}^{\Lambda} \). By Remark 17 we may write \( \Lambda' = \bar{\Lambda} \Lambda, \Lambda = \Lambda \beta R_1 \) with a boost \( \Lambda_\beta \) in \( x^1 \)-direction of rapidity \( \beta \in \mathbb{R} \) and a spatial rotation \( R_1 \) preserving \( x^1 \). Hence from \( f_j^{\Lambda'} = f_j^{\Lambda \beta R_1} = (f_j^{\Lambda \beta}) R_1, (j = 1,2) \), we obtain for the spatial projection \( \mathcal{V}_{f_j}^{\Lambda'} \) of \( \mathcal{V}_{f_j}^{\Lambda} \) that

\[
\mathcal{V}_{f_j}^{\Lambda'}(k) = \left\{ \frac{k}{\omega_m(k)}, k \in R_1^{-1}(\text{supp } f_j^{\Lambda \beta}) \right\}\left\{ \frac{R_1^{-1}k}{\omega_m(k)}, k \in \text{supp } f_j^{\Lambda \beta} \right\} = R_1^{-1}\mathcal{V}_{f_j}^{\Lambda \beta}.
\]

Here we used Lemma 16 (i), that \( R \) from the rotation subgroup of \( \mathcal{L}_+^1 \) act on \( H_m \) by \( R_m(k) = Rk \), and \( \omega_m(R_1^{-1}k) = \omega_m(k) \). By covariance (Proposition 1)

\[
\mathcal{V}_{f_j}^{\Lambda} \prec_{W_t} \mathcal{V}_{f_j}^{\Lambda'} \iff R_1^{-1}\mathcal{V}_{f_j}^{\Lambda \beta} \prec_{W_t} R_1^{-1}\mathcal{V}_{f_j}^{\Lambda \beta} \iff \mathcal{V}_{f_j}^{\Lambda \beta} \prec_{W_t} \mathcal{V}_{f_j}^{\Lambda \beta},
\]

where we used that \( R_1 W_t = W_t \), as \( R_1 \) is also a rotation preserving \( x_1 \). The remaining \( x^1 \)-boost gives

\[
\mathcal{V}_{f_j}^{(\Lambda \beta) R_1} = \left\{ \frac{k}{\omega_m(k)}, k \in (\Lambda \beta)^{-1}\text{supp } f_j \right\}\left\{ \frac{(\Lambda \beta)_m(k)}{\omega_m((\Lambda \beta)_m(k))}, k \in \text{supp } f_j \right\}
\]

\[
= \left\{ \frac{(\sinh(-\beta)\omega_m(k) + \cosh(-\beta)k^1, k^2, \ldots, k^d)}{\cosh(-\beta)\omega_m(k) + \sinh(-\beta)k^1}, k \in \text{supp } f_j \right\},
\]

\(^8\)Preliminary computations suggest that Theorem 15 also extends to all \( \Lambda \in \mathcal{L}_+^1 \) as long as the ordering (53) holds for \( \Lambda \in \mathcal{L}^*(W) \).
where we used the group action property \((\Lambda_{\beta})^{-1}_m(k) = (\Lambda_{-\beta})^{-1}_m(k) = (\Lambda_{-\beta})_m(k)\). From this we obtain \(\mathcal{V}_{f_1}^{\Lambda_{\beta} \Lambda} \prec \omega, \mathcal{V}_{f_2}^{\Lambda_{\beta} \Lambda} \prec \omega \iff \forall k_2 \in \text{supp} f_2^\Lambda, k_1 \in \text{supp} f_1^\Lambda:\)

\[
- \sinh(\beta)\omega_m(k_2) + \cosh(\beta)k_2^1 = \frac{-\sinh(\beta)\omega_m(k_1) + \cosh(\beta)k_1^1}{\cosh(\beta)\omega_m(k_1) - \sinh(\beta)k_1^1} > 0.
\]

Passing to the common denominator and using \(\cosh(\beta)^2 - \sinh(\beta)^2 = 1\), this is equivalent to \(k_2^1/\omega_m(k_2) - k_1^1/\omega_m(k_1) > 0\). As the equivalence holds for all \(k_2 \in \text{supp} f_2^\Lambda, k_1 \in \text{supp} f_1^\Lambda\), we have shown that \(\mathcal{V}_{f_1}^{\Lambda_{\beta} \Lambda} \prec \omega, \mathcal{V}_{f_2}^{\Lambda_{\beta} \Lambda} \prec \omega\).

This establishes that all choices \(\Lambda \in \mathcal{L}^*(\mathcal{W})\) are equivalent with respect to the ordering restriction. That is a prerequisite for the following commutator estimate, which extends Corollary 11 and will be required for comparing scattering states defined for distinct \(\Lambda \in \mathcal{L}^*(\mathcal{W})\).

**Lemma 19** (commutator decay). Let \(A \in \mathfrak{A}(\mathcal{W}), A^\perp \in \mathfrak{A}(\mathcal{W}^\perp)\), and \(f, f^\perp\) s.t. \(\mathcal{V}_{f_1}^{\Lambda} \prec \omega, \mathcal{V}_{f_2}^{\Lambda_{\beta} \Lambda} \prec \omega\) for some \(\Lambda \in \mathcal{L}^*(\mathcal{W})\). Then for any compact continuously differentiable curve \(\Lambda^\gamma \in \mathcal{L}^*(\mathcal{W}), \gamma \in [0, 1]\), and \(\tau > 0\), we have \(\|[(\partial_\gamma B^\Lambda_\tau(f(\Lambda^\gamma)), B^\perp_\tau f(\Lambda^\gamma))]| \leq C_N \tau^{-N}\) uniformly in \(\gamma\), with \(f(\Lambda)\) as in (57).

**Proof.** As before, \(B^\perp_\tau f(\Lambda^\gamma)\) may be understood as a creation operator with \(\Lambda = 1\) with respect to the family of boosted theories \((\mathfrak{A}^{\Lambda^\gamma}, \alpha^{\Lambda^\gamma}, \Omega)\) from Lemma 14. Therefore Lemma 10 applies and yields wedge-local approximants \((B^\perp_\tau)^{(\delta)} := (B^\perp_\tau f(\Lambda^\gamma))^{(\delta)}\) \((\delta > 0)\) such that for any \(N \in \mathbb{N}\), \(\|(B^\perp_\tau)^{(\delta)} - B^\perp_\tau f(\Lambda^\gamma)\| < C_N/\tau^N\) with \((B^\perp_\tau)^{(\delta)} \in \mathfrak{A}(\mathcal{W}^\perp + \tau \mathcal{V}_{f}^{\Lambda_{\beta} \Lambda} + C_\delta |\tau|)\). We already used that \(\mathcal{V}_{f_2}^{\Lambda_{\beta} \Lambda} = \mathcal{V}_{f_2}^{\Lambda_{\beta} \Lambda}\) holds for all \(\gamma\), as the supports of the packets of \(f^\perp\) and \(f(\Lambda^\gamma)\) coincide by definition. Additionally due to compactness and continuous \(\gamma\)-dependence of \(f(\Lambda^\gamma)\) we can in fact chose \(C_N = C_N^\gamma\) uniformly in \(\gamma \in [0, 1]\). For the second operator we similarly note that

\[
\partial_\gamma B^\Lambda_\tau(f(\Lambda^\gamma)) = B^\Lambda_\tau(\partial_\gamma f(\Lambda^\gamma)) + \int d^\gamma x \left( (\partial_\mu B)(\Lambda^\gamma(\tau, x)) f(\Lambda^\gamma(\tau, x)) \right.
\]

\[
B(\Lambda^\gamma(\tau, x))(\partial_\mu f(\Lambda^\gamma))(\Lambda^\gamma(\tau, x)) \right) w^\mu_{(\tau, x, \gamma)},
\]

with implied summation over \(\mu\), and where \(w^\mu_{(\tau, x, \gamma)} := (\partial_\gamma \Lambda^\gamma(\tau, x))^\mu\) satisfies by continuous differentiability and compactness of \(\gamma \rightarrow \Lambda^\gamma\) the bound \(|w^\mu_{(\tau, x, \gamma)}| \leq C(|\tau| + |x|)\).

Therefore Lemma 10 applies and yields \((\partial_\gamma B^\perp_\tau)^{(\delta)} \in \mathfrak{A}(\mathcal{W}^\perp + \tau \mathcal{V}_{f}^{\Lambda_{\beta} \Lambda} + C_\delta |\tau|)\) such that \(\|((\partial_\gamma B^\perp_\tau)^{(\delta)} - \partial_\gamma B^\perp_\tau f(\Lambda^\gamma))\| < C_\Delta/\tau^N\) with \(C_\Delta\) are uniform in \(\gamma\). Finally, the commutator estimate follows from the proof of Corollary 11 in Appendix A.

**Theorem 20** (\(\Lambda\)-independence of scattering states). Assume that for some \(\Lambda_0 \in \mathcal{L}^*(\mathcal{W})\) \(\mathcal{V}_{f_n}^{\Lambda_0} \prec \omega, \mathcal{V}_{f_{n-1}}^{\Lambda_0} \prec \omega \ldots \prec \omega \mathcal{V}_{f_1}^{\Lambda_0}\). Then for any \(\Lambda' \in \mathcal{L}^*(\mathcal{W})\) the scattering states

\[
\Psi_{(\Lambda')} := \lim_{\tau \rightarrow \infty} B_1^\Lambda(f_1^{(\Lambda')}) \ldots B_1^\Lambda(f_n^{(\Lambda')}) \Omega
\]

are well-defined and the limit is independent of \(\Lambda'\).

**Proof.** Convergence follows from Proposition 18 and Theorem 15. Using the above preparations we can establish \(\Lambda\)-independence by generalizing the arguments familiar from the local case. Due to Remark 17 we can interpolate between the two reference frames specified by \(\Lambda^0 = \Lambda_0\) and \(\Lambda^1 = \Lambda'\) with a differentiable curve \(\Lambda^\gamma \in \mathcal{L}^*(\mathcal{W})\),
\[ \gamma \in [0, 1]. \] Now we estimate for \( \tau > 0 \) inductively with respect to the particle number \( n \) that
\[ \left\| \Psi^{(A)}_\tau - \Psi^{(A')}_\tau \right\| \leq \int_0^1 d\gamma \left\| \partial_\gamma \Psi^{(A\gamma)}_\tau \right\| \leq C_N \tau^{-N}. \]
For \( n = 1 \) this follows from (57) with \( C_N = 0 \). The induction step is established by expanding \( \left\| \partial_\gamma \Psi^{(A\gamma)}_\tau \right\| \leq \left\| (\partial_\gamma (B^\gamma_{1\tau} \ldots B^\gamma_{n-1\tau})) B^\gamma_{n\tau} \Omega \right\| + \left\| B^\gamma_{1\tau} \ldots B^\gamma_{n-1\tau}, \partial_\gamma B^\gamma_{n\tau} \Omega \right\| \) where we abbreviated \( B^\gamma_{\tau} := B^\gamma_N (f_j^{(A\gamma)}) \). Here the second term vanishes due to (57) and the first term may be estimated by swapping
\[ \left\| (\partial_\gamma (B^\gamma_{1\tau} \ldots B^\gamma_{n-1\tau})) B^\gamma_{n\tau} \Omega \right\| = \left\| (\partial_\gamma (B^\gamma_{1\tau} \ldots B^\gamma_{n-1\tau})) B^\gamma_{n\tau} \Omega \right\| \]
\[ \leq \left\| B^\gamma_{n\tau} \right\| \left\| (\partial_\gamma (B^\gamma_{1\tau} \ldots B^\gamma_{n-1\tau})) \Omega \right\| + \left\| \partial_\gamma (B^\gamma_{1\tau} \ldots B^\gamma_{n-1\tau}), B^\gamma_{n\tau} \right\| \]
where both terms are rapidly decreasing in \( \tau \). For the first term this is obtained from the induction assumption and \( \left\| B^\gamma_{n\tau} \right\| \leq C (1 + |\tau|^{s/2}) \) (uniformly in \( \gamma \in [0, 1] \)). The second term is estimated by expansion of the commutator similarly as in (41), using Lemma 19 and polynomial bounds including \( \left\| \partial_\gamma B^\gamma_{n\tau} \right\| \leq C (1 + |\tau|^{s/2 + 1}). \)

5 Wave Operators, S-Matrix, and Wedge Transitions

We have now sufficient understanding of the construction from Section 4.1 to begin with a general and model-independent analysis of the multi-particle scattering data in wedge-local models. In particular we propose a formalism for wave operators and S-matrices, which emphasizes the potential physical peculiarities of multi-particle scattering in the wedge-local setting. These considerations will provide the foundation for the study of the multi-particle structure of the Grosse-Lechner model and related wedge-local theories in subsequent work.

Guided by conventional Haag-Ruelle theory we additionally need to address restrictions of our construction regarding swapping and ordering conditions. Regarding the former it will be convenient to introduce in addition to the one-particle space \( \mathcal{H}_1 := E_m \mathcal{H} \) the (non-closed) subspaces
\[ \mathcal{H}_1^W := \{ \Psi_1 \in \mathcal{H}_1, \Psi_1 \text{ swappable w.r.t. } W + x \text{ for some } x \in \mathbb{R}^d \}, \]
\[ \mathcal{H}_{1c}^W := \{ \tilde{f}(P) \Psi_1, \Psi_1 \in \mathcal{H}_1^W, \tilde{f} \in C_c^\infty(\mathbb{R}^d) \}. \]
(60)

It is clear that \( \mathcal{H}_1^W = \mathcal{H}_1^W + y = U(y) \mathcal{H}_1^W = \mathcal{H}_1^{W+y} \) for any \( y, y' \in \mathbb{R}^d \) by symmetry of the definition, and if covariance (HK3) applies \( U(\Lambda) \mathcal{H}_1^W = \mathcal{H}_1^{\Lambda W} \). Lastly Lemma 4 shows that wedge-duality (HK2) yields \( \mathcal{H}_1^W = E(H_m) \mathcal{H} \) for any wedge \( W \). Further independent of duality \( \mathcal{H}_{1c}^W \subset \mathcal{H}_1^W \) is dense by spectral calculus, but one should not expect \( \mathcal{H}_{1c}^W \) to be a subspace of \( \mathcal{H}_1^W \), cf. [BBS01] Lemma 3.4. It is clear by definition that for any one particle vector \( \Psi_k \in \mathcal{H}_{1c}^W \) we can find associated creation operators such that \( \Psi_k = B^\gamma_{k\tau} (f_k) \Omega = B^\gamma_{k\tau} \Lambda (f_k) \Omega \), so that we can proceed to the corresponding ordered scattering states. The basic conceptual issue to be addressed in the passage from the Haag-Ruelle construction to the wave operators and the S-matrix concerns the potential implicit dependence of scattering states on the choice of creation-operator approximants \( \tilde{B}_{k\tau}^\gamma (f_k) \).
Lemma 21. Let $A_k, A'_k \in \mathcal{A}(W)$ together with KG-solutions $f_k, f'_k$ and auxiliary functions $\chi, \chi' \in \mathcal{S}(\mathbb{R}^d)$ (cf. Lemma 8) such that $B_{kT}^A(f_k)\Omega = B_{kT}^{A'}(f'_k)\Omega$ with $V_n \prec_W V_{n-1} \prec_W \ldots \prec_W V_1$ where $V_k := \mathcal{V}_k^A$ and analogously for $V_k' := \mathcal{V}_k^{A'}$, $A, A' \in \mathcal{L}^*(W)$. Then the outgoing limits $\Psi^+, \Psi'^+$ of $\Psi_\tau := B_{kT}^A(f_1) \ldots B_{kT}^A(f_n)\Omega$ and $\Psi'_\tau := B_{kT}^{A'}(f'_1) \ldots B_{kT}^{A'}(f'_n)\Omega$ coincide. The same holds for incoming limits with ordering assumptions replaced by $V_1 \prec_W V_2 \prec_W \ldots \prec_W V_n$.

Proof. For $\Lambda = \Lambda'$ we find directly $||\Psi^+ - \Psi'^+||^2 = ||\Psi^+||^2 - 2 \Re\langle \Psi^+, \Psi'^+ \rangle + ||\Psi'^+||^2$. This vanishes, as due to Fock structure (Theorem 15 (ii)) and coinciding one-particle vectors we obtain $\langle \Psi^+, \Psi'^+ \rangle = ||\Psi^+||^2 = ||\Psi'^+||^2$. The case of general $\Lambda, \Lambda' \in \mathcal{L}^*(W)$ follows from the above via Theorem 20. \hfill \Box

Further one can make sense of velocity supports and the corresponding ordering assumptions without reference to Klein-Gordon solutions. For a single-particle state $\Psi_1 \in \mathcal{H}_1$ the classical propagation region and the corresponding $\Lambda$-velocity support ($\Lambda \in \mathcal{L}_+^\Lambda$) are given in terms of the energy-momentum spectral measure $E_{(H, P)}(\Delta)$ ($\Delta \subset \mathbb{R}^{+\perp}$ Borel) by

$$\Upsilon_{\Psi_1} := \{ t \cdot (\omega, k), \, t \in \mathbb{R}, \, (\omega, k) \in \text{supp}(E_{(H, P)}(\Psi_1))\},$$

$$\mathcal{V}_1^\Lambda := \Upsilon_{\Psi_1} \cap \Lambda T_1, \quad T_1 := \{(1, x), \, x \in \mathbb{R}^4\}.$$  \hfill \(61\)

The precursor ordering is lifted to a relation on one-particle vectors $\Psi_1, \Psi_2 \in \mathcal{H}_1$ by setting for $\Lambda \in \mathcal{L}^*(W)$

$$\Psi_2 \prec_W \Psi_1 :\iff \mathcal{V}_2^\Lambda \prec_W \mathcal{V}_{\Psi_1}^\Lambda\quad \hfill \(62\)$$

which is well-defined as a consequence of Proposition 18.

The multi-particle configurations accessible via our wedge-local Haag-Ruelle construction can be conveniently expressed by the following notion of ordered Fock spaces replacing the conventional definition based on bosonic or fermionic statistics.

Definition 22. The ordered tensor products over one-particle Hilbert space $\mathcal{H}_1$ with respect to a partial order $\prec$ on $\mathcal{H}_1$ are defined as closure $\otimes_{\prec}^n \mathcal{H}_1 := \overline{\otimes_{\prec}^n \mathcal{H}_1}$ of the finite linear spans

$$\otimes_{\prec}^n \mathcal{H}_1 := \text{span}\{\Psi_1 \otimes \ldots \otimes \Psi_n, \, \Psi_k \in \mathcal{H}_1, \Psi_1 \prec \Psi_2 \prec \ldots \prec \Psi_n\}.$$  \hfill \(63\)

Using the conventions $\otimes_0^0 \mathcal{H}_1 := C\Omega, \otimes_1^1 \mathcal{H}_1 := \mathcal{H}_1$ we obtain corresponding ordered Fock spaces $\Gamma^\prec(\mathcal{H}_1) := \bigoplus_{n=0}^\infty \otimes_{\prec}^n \mathcal{H}_1$. The subspace of finite linear combinations of ordered tensor product vectors with $\Psi_k \in \mathcal{H}_1 \subset \mathcal{H}_1$ will be denoted by $\Gamma_0^\prec(\mathcal{H}_1) := \bigoplus_{n=0}^\infty \otimes_{\prec}^n \mathcal{H}_1$.

To proceed to the scattering data note that $\Gamma_0^\prec(\mathcal{H}_1)^W \subset \Gamma^\prec(W, \mathcal{H}_1)$ is dense and the wave operators are defined by linear extension of the isometries obtained from the wedge-local Haag-Ruelle construction of Theorem 15. Just as for ordinary bosonic and fermionic statistics, unsymmetrized Fock space $\Gamma^u(\mathcal{H}_1) := \bigoplus_{n=0}^\infty \mathcal{H}_1^{\otimes n}$ provides a common enveloping space into which ordered tensor products and Fock spaces embed naturally. The possible dependence of scattering states on a given wedge of reference $W$, noted by Grosse and Lechner [GL07], extends also to multi-particle scattering states and is most conveniently expressed on the level of wave operators.
Definition 23 (wave operators). For any given centered wedge $\mathcal{W}$ we set

$$
\mathcal{W}^+_\mathcal{W} : \left\{
\begin{array}{l}
\Gamma^+_{0}^W(\mathcal{H}^W_{1c}) \rightarrow \mathcal{H}, \\
\Psi_1 \otimes \ldots \otimes \Psi_n \mapsto \lim_{\tau \rightarrow \infty} B^\Lambda_{1\tau}(f_1) \ldots B^\Lambda_{n\tau}(f_n)\Omega,
\end{array}
\right.
$$

(64)

$$
\mathcal{W}_-^\Lambda : \left\{
\begin{array}{l}
\Gamma^+_{0}^W(\mathcal{H}^W_{1c}) \rightarrow \mathcal{H}, \\
\Psi_1 \otimes \ldots \otimes \Psi_n \mapsto \lim_{\tau \rightarrow \infty} B^\Lambda_{1\tau}(f_1) \ldots B^\Lambda_{n\tau}(f_n)\Omega,
\end{array}
\right.
$$

(65)

where for $\Lambda \in \mathcal{L}^*(\mathcal{W})$ suitable $B_{k\tau}^\Lambda(f_k)\Omega = \Psi_k$ with $B_k$ swappable and almost wedge-local w.r.t. the given wedge $\mathcal{W}$.

Proposition 24. Assuming wedge-duality $(\text{HK}2^5)$, the wave operators (64), (65) are well-defined and extend to bounded linear isometries $\mathcal{W}^+_\mathcal{W} : \Gamma^+\mathcal{W}(\mathcal{H}_1) \rightarrow \mathcal{H}$, and $\mathcal{W}_-^\Lambda : \Gamma^-\mathcal{W}(\mathcal{H}_1) \rightarrow \mathcal{H}$.

Proof. Well-definedness of $\mathcal{W}^+_\mathcal{W}$ on $\Gamma^+_{0}^W(\mathcal{H}^W_{1c})$ follows by noting that the computation from the proof of Lemma 21 extends to linear combinations of $\Psi^+$. As the Fock structure also implies isometry of $\mathcal{W}^+_\mathcal{W}$ the wave operators further extend to the closures $\Gamma^+\mathcal{W}(\mathcal{H}_1) = \overline{\Gamma^+_{0}^W(\mathcal{H}^W_{1c})}$ by continuity and using that $\overline{\mathcal{H}^W_{1c}} = \mathcal{H}_1$ (Lemma 4). The construction of $\mathcal{W}_-^\Lambda$ is analogous on the oppositely ordered spaces. \(\square\)

Due to translation covariance it is sufficient to consider $\mathcal{W}^+_\mathcal{W}$ for centered wedges $W = \Delta W$. In other words we will now see that the wave operators in fact depend on the wedge $\mathcal{W}$ only modulo translations. Given $(\text{HK}3^\dagger)$ this symmetry consideration in fact extends to the full Poincaré group, whose action $U_0(\lambda)$ on $\Gamma^\nu(\mathcal{H}_1)$ is defined by

$$
U_0(\lambda)(\Psi_1 \otimes \Psi_2 \otimes \ldots \otimes \Psi_n) := (U(\lambda)\Psi_1) \otimes (U(\lambda)\Psi_2) \otimes \ldots \otimes (U(\lambda)\Psi_n).
$$

(66)

While $U_0(x)$ preserves velocity-ordered Fock spaces, boosts act in general non-trivially. Explicitly, Proposition 1 shows that $U_0(\lambda)\Gamma^\nu(\mathcal{H}_1) = \Gamma^\nu(\mathcal{H}_1)\lambda$, $U_0(\lambda)\Gamma^\nu(\mathcal{H}_1) = \Gamma^\nu(\mathcal{H}_1)\lambda$, and analogously for the subspaces $\Gamma^\nu(\mathcal{H}_1)\lambda$.

Theorem 25. For $\lambda = (a, \Lambda) \in \mathcal{P}^+_+$ we have $\mathcal{W}^+_\mathcal{W} = \mathcal{W}^+_\mathcal{W}$ and $U(\lambda)\mathcal{W}^+_\mathcal{W} = \mathcal{W}^+_\mathcal{W} U_0(\lambda)$.

Proof. The first statement follows trivially from translation symmetry of Definition 23. For the second statement let us consider only the outgoing case, and note that it is sufficient to establish the identities for special $\Psi^+$ of ordered tensor product form

$$
\Psi^+ = \mathcal{W}^+_\mathcal{W}(B^N_{1\tau}(f_1)\Omega \otimes \ldots \otimes B^N_{n\tau}(f_n)\Omega) = \lim_{\tau \rightarrow \infty} B^N_{1\tau}(f_1) \ldots B^N_{n\tau}(f_n)\Omega.
$$

(67)

with auxiliary boost $\Lambda' \in \mathcal{L}^*(\mathcal{W})$ and velocity supports ordered correspondingly, that is by $V^N_{f_1} \succ \mathcal{W} V^N_{f_2} \succ \mathcal{W} \ldots \succ \mathcal{W} V^N_{f_n}$. From continuity of $U(\lambda)$, we obtain

$$
U(\lambda)\Psi^+ = \lim_{\tau \rightarrow \infty} U(\lambda)B^N_{1\tau}(f_1)U(\lambda)^*U(\lambda) \ldots U(\lambda)^*U(\lambda)B^N_{n\tau}(f_n)\Omega.
$$

(67)

Using $U(\lambda)U(x) = U(\Delta x)U(\lambda)$, the adjoint action of $U(\lambda)$ yields due to

$$
U(\lambda)B^N_{1\tau}(f_j)U(\lambda)^* = \int d^nx f_j(\Lambda'(\tau, x)) U(\lambda)\alpha_{\Lambda'_{(\tau, x)}}(B^N_j)U(\lambda)^* = \int d^nx f_j(\Lambda'_{(\tau, x)}) \alpha_{\Lambda'_{(\tau, x)}}(B^N_j) = B^N_{1\tau}(f^\prime_j).
$$

(68)
again a Haag-Ruelle operator with $B_j' := U(\Lambda, a)B_jU(\Lambda, a)^*$ from the class of almost-
 wedge local operators considered in Lemma 8 (with respect to the transformed wedge $\Lambda W$) and $f_j'(x) := f_j(\Lambda^{-1}x)$. Starting from (67) covariance of $\mathcal{W}^\pm_\mathcal{V}$ is now obtained via

$$U(\lambda)\Psi = \lim_{r \to \infty} B^{\Lambda N}_j(f_j'(1))B^{\Lambda N}_j(f_j'(2)) \ldots B^{\Lambda N}_j(f_j'(n))\Omega$$

$$= \mathcal{W}^+_\mathcal{V}(B^{\Lambda N}_j(f_j'(1))\otimes \ldots \otimes (B^{\Lambda N}_j(f_j'(n))))$$

$$= \mathcal{W}^+_\mathcal{V}((U(\Lambda)B^{N}_j(f_j(1))\otimes \ldots \otimes (U(\Lambda)B^{N}_j(f_j(n))))$$

$$= \mathcal{W}^+_\mathcal{V}U(\Lambda)(B^{N}_j(f_j(1))\otimes \ldots \otimes B^{N}_j(f_j(n)).$$

Here we first used (68), well-definedness of the wave-operators (Proposition 24), then again (68), and lastly (66). Finally we extend by linearity and continuity to all of $\Gamma^W_\mathcal{V}$, whereby we obtain the covariance identity.

For local theories $\mathcal{W}^\pm_\mathcal{V}$ are equivalent to the conventional Haag-Ruelle wave operators as a consequence of Lemma 21. Therefore in local theories they must be $W$-independent and Lorentz-covariant up to suitable identification of ordered Fock spaces by standard arguments. In the general wedge-local setting on the other hand, a non-trivial dependence of $\mathcal{W}^\pm_\mathcal{V}$ on the wedge $W$ should be expected, as noticed in [GL07]. The resulting asymptotic breaking of Lorentz symmetry in higher dimensions will be strongly model dependent, so that it is beyond the scope of our present general analysis. The lesson to be learned is that there must be a residual Lorentz covariance with respect to the stabilizer of $\mathcal{W}_c$ in any wedge-local theory.

Finally let us note that also the $S$-matrix in wedge-local theories, as accessible via our construction with suitable ordering restrictions, will inherit the wedge-dependence of the wave operators.

**Definition 26** (S-matrix and wedge-transition maps). The $S$-matrices and wedge-
 transition maps between final and initial states are defined as

$$S^{W_i}_i := (\mathcal{W}^+_i)^*\mathcal{W}^-_i, \quad S^{W_i}_i := (\mathcal{W}^+_i)^*\mathcal{W}^-_i, \quad S^{W_i}_i := (\mathcal{W}^+_i)^*\mathcal{W}^-_i. \quad (69)$$

depending on centered wedges $W_i, W_i, W_i, W_i$ entering in the Haag-Ruelle construction.

**Theorem 27.** $S$-matrices and wedge transition maps (69) are Poincaré-covariant in the sense that for $\lambda = (a, \Lambda) \in \mathcal{P}_+^1$ we have

$$U_0(\lambda)S^{W_i}_i U_0(\lambda)^* = S^{AW_i}_i, \quad U_0(\lambda)S^{W_i}_i U_0(\lambda)^* = S^{AW_i}_i, \quad U_0(\lambda)S^{W_i}_i U_0(\lambda)^* = S^{AW_i}_i.$$ 

If the wave operators are asymptotically complete (i.e. have dense range in $\mathcal{H}$) we have additional transition identities such as $S^{W_i}_i = S^{W_i}_i S^{W_i}_i S^{W_i}_i.$

**Proof.** Covariance identities follow from Theorem 25. The wedge-transition formula is a consequence of (69) using that asymptotic completeness and isometry of $\mathcal{W}^+_i$ imply $\mathcal{W}^+_i(\mathcal{W}^+_i)^* = 1$ and analogously for $\mathcal{W}^-_i.$

It is important to highlight that in our construction the localization wedge $W$ must agree among all creation operators used to define a scattering state. Additionally even if there is a non-trivial overlap between two distinct ordered Fock spaces, for non-vanishing $\Psi \in \Gamma^W_\mathcal{V}_i \cap \Gamma^W_\mathcal{V}_i$ one will in general have $\mathcal{W}^+_i \Psi \neq \mathcal{W}^+_i \Psi.$ The analysis of this localization-dependence can be carried much further in models with stronger (e.g. string-like) localization. In this case also scattering states can be constructed for mixed string-directions and the dependence on these directions can be taken into account on the level of the asymptotic Fock spaces [FGR96].
6 Concluding Remarks

We developed $N$-particle scattering theory for general wedge-local quantum field theories with isolated mass shells. In particular we constructed scattering states for arbitrarily many particles, even with reduced localization information available from wedge-locality. This implies also that the asymptotic particle structure of wedge-local models with isolated mass shells must be as rich as for strictly local theories.

This brings us to the problem of asymptotic completeness (AC) which, in spite of recent progress [Le06, DT11, DG14], is largely open both in the local and wedge-local setting. Using our construction of $N$-particle scattering states, we intend to establish AC in the wedge-local model of Grosse and Lechner [GL07]. This will give the first example of a relativistic theory in 4-dimensional space-time, which is interacting and asymptotically complete. Furthermore we expect that the non-trivial $S$-matrix of this model will be factorizing, which is an unusual feature in higher dimensions. On the other hand also interesting counterexamples to two-particle asymptotic completeness have recently been constructed in wedge-local setting [LTU17], which ought to be instructive also at the multi-particle level.

It is not known whether the existence of an interpolating wedge-local net has any consequences on the properties of an $S$-matrix beyond the basic symmetry principles discussed in Section 5. As a first step one may ask whether there is any meaningful generalization of the LSZ reduction formula for the wedge-local setting, which is the conventional point of departure for investigating analyticity properties of the $S$-matrix. Phrased differently, one may ask in which generality the inverse scattering problem is solvable within the class of wedge-local models. Here some positive related results are known for non-local models [BW84], or for a certain class of field theories formulated on Krein spaces [AG01].

Lastly let us point out that a general scattering theory for massless particles in the wedge-local setting curiously appears to require new ideas. In particular many of the conventional technical tools may fail without mass gaps, including energy bounds and clustering estimates which are indispensable in all previous constructions of scattering states in the local setting without mass gaps, see e.g. [Bu77, Dy05, AD17, Du17].

A Some Technical Arguments

For the convenience of the reader we will briefly explain how the standard proof of commutator estimates for Haag-Ruelle operators also yields the corresponding results in the wedge-local setting. Due to the covariance arguments from Section 4.1 it is sufficient to consider the case of non-adapted HR-operators corresponding to $B^\Lambda (f)$ with $\Lambda = 1$.

Lemma 28. Let $f$ be a regular Klein-Gordon solution of mass $m > 0$.

(i) $|f(t, x)| \leq C/(1 + |t|^{s/2})$ for any $(t, x) \in \mathbb{R}^{s+1}$,

(ii) $|f(t, x)| \leq C_{c,N}/(1 + |t|^N + |x|^N)$ for $(t, x) \in \mathbb{R}^d \setminus \mathcal{V}_f$,

(iii) $\|f_t\|_{L^1(\mathbb{R}^s)} \leq C(1 + |t|^{s/2})$, where $f_t(x) := f(t, x)$,

where $\epsilon > 0$, and $N \in \mathbb{N}$ are arbitrary, $C > 0$, $C_{c,N} > 0$ are suitable constants depending on $f$, and $\mathcal{V}_f := \mathbb{R}\mathcal{V}_f$ is the cone generated by the $\epsilon$-enlarged velocity support $\mathcal{V}_f := \{ (1, v) \in \mathbb{R}^d, \exists (1, v') \in \mathcal{V}_f, |v - v'| < \epsilon \}$.

Proof. See [A, Thm. 5.3].

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Proof of Lemma 10. Let $\delta > 0$ be given and $B_\tau \in \mathfrak{A}(\mathcal{W} + \mathcal{C}_\tau)$, $\|B - B_\tau\| \leq C_N/(1 + |\tau|^N)$ as in Lemma 8. Suitable wedge-local approximants may then be obtained by restricting the integration in the definition of $B_\tau(f)$ to the asymptotically dominant part $f^\tau(x) := f(x)1_{W^\tau/2}(x)$ (Lemma 28) and setting $r(\tau) := \delta |\tau|/2$ to obtain

$$B_\tau^{(\delta)} := (B_{r(\tau)})_r(f^\tau) = \int \partial^d x \ f^\tau(\tau, x) \alpha(\tau, x) \ (B_{r(\tau)}) \in \mathfrak{A}(\mathcal{W} + \mathcal{C}_\delta|\tau|/2 + \tau V^{\delta/2})$$

where the localization was computed for given $\tau \in \mathbb{R}$ by covariance, isotony and noting that $\mathcal{Y}_f^{\delta/2} \cap \{x \in \mathbb{R}^d, \ |x| = \tau\} = \tau \mathcal{Y}_f^{\delta/2} \subset \tau \mathcal{Y}_f + \mathcal{C}_\delta|\tau|/2$ and $\mathcal{C}_\delta|\tau|/2 + \mathcal{C}_\delta|\tau|/2 \subset \mathcal{C}_\delta|\tau|$. The approximation in norm is established by $\|B_\tau(f) - B_\tau^{(\delta)}\| \leq \|B_\tau(f) - (B_{r(\tau)})_r(f^\tau)\| + \|(B_{r(\tau)})_r(f - f^\tau)\| \leq \|B - B_{r(\tau)}\| \|f\|_{L^1(\mathbb{R}^d)} + \|B_{r(\tau)}\| \|f^\tau - f^\tau\|_{L^1(\mathbb{R}^d)}$ using $\|B_\tau\| \leq \|B\| + C_1$, $\|f - f^\tau\|_{L^1(\mathbb{R}^d)} \leq C_N/(1 + |\tau|^N)$ due to Lemma 28 and that $\|B - B_{r(\tau)}\| \leq C_N/(1 + \delta^N|\tau|^N)$ is sufficient to compensate the polynomial growth in Lemma 28 (iii) and obtain overall $\|B_\tau(f) - B_\tau^{(\delta)}\| \leq C_{\delta,N}/(1 + |\tau|^N)$. □

Proof of Corollary 11. To estimate $\|B_\tau^{(\delta)}(f^\perp, B_\tau(f))\|$, let $\delta > 0$ and $B_\tau^{(\delta)}$, $B_\tau^{(\perp)}$ corresponding approximants as from Lemma 10, i.e. $B_\tau^{(\delta)} \in \mathfrak{A}(\tau \mathcal{Y}_f + \mathcal{C}_\delta|\tau| + \mathcal{W}),$ s.t. $\|B_\tau^{(\delta)} - B_\tau(f)\| \leq C_N/(1 + |\tau|^N)$, and let analogously $B_\tau^{(\perp)} \in \mathfrak{A}(\tau \mathcal{Y}_f + \mathcal{C}_\perp |\tau| + \mathcal{W}^\perp),$ s.t. $\|B_\tau^{(\perp)} - B_\tau^{(\delta)}(f\perp)\| \leq C_{\delta,N}/(1 + |\tau|^N)$.

Choosing $\delta > 0$ sufficiently small the localization regions of $B_\tau^{(\delta)}$ and $B_\tau^{(\perp)}$ will be space-like separated for any large enough $\tau > 0$: By assumption we have $\mathcal{Y}_f - \mathcal{Y}_f^\perp \subset \mathcal{W}_c$ with $\mathcal{V}_f - \mathcal{V}_f^\perp$ compact and $\mathcal{W}_c$ open. Thus there exists $\epsilon > 0$ such that $\mathcal{V}_f - \mathcal{V}_f^\perp + \mathcal{C}_\epsilon \subset \mathcal{W}_c$, where $\mathcal{C}_\epsilon := \{x \in \mathbb{R}^d, \ |x| = |x| + \epsilon < \epsilon\}$ and as $\mathcal{W}_c$ is a convex cone this implies also $\tau (\mathcal{V}_f - \mathcal{V}_f^\perp + \mathcal{C}_\epsilon) \subset \mathcal{W}_c$ for any $\tau > 0$. To obtain space-like separation recall that $\mathcal{W} = \mathcal{W}_c + x_1, \mathcal{W}^\perp = \mathcal{W}_c^\perp + x_2$, for $x_1, x_2 \in \mathbb{R}^d$. Thus we get for $\delta < \epsilon/3$ and $\tau > 3(|x_1_c| + |x_2_c|)/\epsilon$ and any $|x_1'|_c < \delta, |x_2'|_c < \delta$ that

$$\tau (\mathcal{V}_f - \mathcal{V}_f^\perp + \frac{x_1 - x_2}{\tau} + x_1' - x_2') \subset \mathcal{W}_c \subset \mathcal{W}_c = (\mathcal{W}_c^\perp)' \implies \tau \mathcal{V}_f + x_1 + \tau x_1' + \mathcal{W}_c \subset (\mathcal{W}_c^\perp + \tau \mathcal{V}_f^\perp + x_2 + \tau x_2')',$$

where we used $\mathcal{W}_c + \mathcal{W}_c = \mathcal{W}_c$ and that $\mathcal{O}_1 + \mathcal{O}_2 \subset \mathcal{O}_3 \iff \mathcal{O}_1 \subset (\mathcal{O}_3 - \mathcal{O}_2)'$ for arbitrary $\mathcal{O}_k \subset \mathbb{R}^d$. Due to $\tau \mathcal{C}_\delta = \mathcal{C}_\tau$ this is equivalent to $\mathcal{W} + \tau \mathcal{V}_f + \mathcal{C}_\delta \subset (\mathcal{W}^\perp + \tau \mathcal{V}_f^\perp + \mathcal{C}_\delta)'$ for $\delta < \epsilon/3$ and $\tau > 3(|x_1|_c + |x_2|_c)/\epsilon$, as claimed.

For such $\tau$, $\delta$ we now obtain from locality that $[B_\tau^{(\delta)}, B_\tau^{(\perp)}] = 0$, which implies the commutator estimate by expanding

$$\|B_\tau^{(\delta)}(f^\perp, B_\tau(f))\| \leq \|B_\tau^{(\delta)}(f^\perp) - B_\tau^{(\perp)}(B_\tau^{(\delta)}(f^\perp, B_\tau(f))), B_\tau(f) - B_\tau^{(\delta)}(B_\tau(f))\| + \|B_\tau^{(\delta)}(B_\tau(f) - B_\tau^{(\delta)}(f))\| + \|B_\tau^{(\delta)}(f^\perp, B_\tau(f))\|$$

where $\|B_\tau^{(\delta)}(f^\perp) - B_\tau^{(\perp)}(B_\tau(f))\| \leq 2\|B_\tau^{(\perp)}(f^\perp) - B_\tau^{(\delta)}(B_\tau(f))\| \|B_\tau(f)\| \leq 2C_N^c C/(1 + |\tau|^N') \cdot (1 + |\tau|)^{s/2} \leq C_N^c |\tau|^{-N}$ due to Lemma 10 and Proposition 9 (iv) and analogously for the second non-vanishing commutator. □
Proof of Lemma 16.  

Ad (i) The wave packet $\tilde{f}_\tau^\Lambda$ of $f_\tau^\Lambda$ can be computed via Fourier inversion theorem by noting that

$$f_\tau^\Lambda(x) = \int \frac{dk}{(2\pi)^d} e^{-i(\omega_m(k),k)\cdot(x)} \hat{f}(k)$$

where we substituted $k' := \Lambda^{-1}(k)$ after rewriting with respect to the standard Lorentz-invariant measure $d^4k/\omega_m(k)$ (more precisely $\Lambda$-invariant, see e.g. [RS2] Thm. IX.37) and used that $(\Lambda(\omega_m(k),k))^0 = \omega_m(\Lambda_m(k))$ due to $(\omega_m(k),k) \in H_m$ and Lorentz-invariance of the mass hyperboloid $H_m$.

Ad (ii) We obtain

$$B_\tau^\Lambda(f)\Omega = \int d\tau \int d^4x f^\Lambda(\tau,x) e^{i\Lambda^{-1}(\tau,x)^\mu x_\mu} B\Omega = e^{iH_\lambda}\int d\tau \int d^d x f^\Lambda(\tau,x) e^{-iP_\lambda\cdot x} B\Omega = e^{iH_\lambda}\int d\tau \int d^d x \mathcal{P}_\lambda(p) f^\Lambda(\tau,x) e^{-iP_\lambda\cdot x} B\Omega = e^{iH_\lambda}\int d\tau \tilde{f}_\tau^\Lambda(\mathcal{P}_\lambda) B\Omega.$$  

Here we first used translation-invariance of $\Omega$, $P_\mu(\Lambda x)_\mu = (\Lambda^{-1})^\mu_\nu x_\nu$, and then we abbreviated $(H_\lambda, \mathcal{P}_\lambda) := \Lambda^{-1}(H,P)$, $f_\Lambda^\Lambda(\tau,x) := f(\Lambda^{-1}(\tau,x))$. Further due to (55), $f_\Lambda^\Lambda(\tau,x) = \omega_m(\Lambda_m(k)) \hat{f}(\Lambda_m(k)) e^{-i\omega_m(k)t}$, and therefore $e^{-i\omega_m(\mathcal{P}_\lambda)t} B\Omega = e^{-i\omega_m(\mathcal{P}_\lambda)t} E(H_m) B\Omega = e^{-iH_\lambda t} E(H_m) B\Omega$, so that $\tau$-dependent terms cancel in (70). Finally (56) is obtained by inserting $\mathcal{P}_\lambda = \Lambda^{-1}_m(\mathcal{P})$.  

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