Dynamical symmetries of periodically-driven quantum systems and their spectroscopic signatures

Georg Engelhardt\textsuperscript{1} and Jianshu Cao\textsuperscript{1}\textsuperscript{*}

\textsuperscript{1}Beijing Computational Science Research Center, Beijing 100193, People’s Republic of China
\textsuperscript{2}Department of Chemistry, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139, USA

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Spatial symmetries of quantum systems leads to important effects in spectroscopy, such as selection rules and dark states. Motivated by the increasing strength of light-matter interaction achieved in recent experiments, we investigate a set of dynamically-generalized symmetries for quantum systems, which are subject to a strong periodic driving. Based on Floquet response theory, we study rotational, particle-hole, chiral and time-reversal symmetries and their signatures in spectroscopy, including symmetry-protected dark states (spDS), a Floquet band selection rule (FBSR), and symmetry-induced transparency (siT). Specifically, a dynamical rotational symmetry establishes dark state conditions, as well as selection rules for inelastic light scattering processes; a particle-hole symmetry introduces dark states for symmetry related Floquet states and also a transparency effect at quasienergy crossings; chiral symmetry and time-reversal symmetry alone do not imply dark state conditions, but can be combined to the particle-hole symmetry. Our predictions reveal new physical phenomena when a quantum system reaches the strong light-matter coupling regime, important for superconducting qubits, atoms and molecules in optical or plasmonic fields cavities, and optomechanical systems.

\textbf{Introduction.} In nature, the interaction between light and matter is of the order of $10^{-6} \sim 10^{-9}$ on the scale of the photon energy, such that low-order perturbation theory is a powerful tool to predict optical signals \cite{1}. Over the last few decades, the interaction strength has been pushed to the strong coupling regime in opto-mechanical systems \cite{2}, quantum dot systems \cite{3, 4}, and superconducting quantum qubits \cite{5, 6}. As standard nonlinear perturbation theory becomes unfeasible under these conditions, Floquet response theory has been developed recently, describing systems which are subject to a possible strong, but time-periodic driving field (of frequency $\Omega$), and a weak, but arbitrary probe field \cite{7, 8}. For a monochromatic probe of frequency $\omega_p$, system observables generate response frequencies $\omega_p \pm n\Omega$ termed Floquet bands, as has been shown for the optical conductivity in Ref. \cite{9}. Floquet response theory has been also applied to calculate fluorescence spectra \cite{10}.

Spatial symmetries give rise to appealing physical properties. Inversion symmetry results in selection rules for dipole transitions; particle-hole, chiral and time-reversal symmetry establishes the so-called periodic table, a classification scheme for topological insulators \cite{11}; the hybridization of molecular orbitals follows group theoretical principals \cite{12}; and symmetries have an essential impact on transport properties in molecules \cite{13, 14}. For periodically-driven systems, these spatial symmetries can be generalized to dynamical symmetries, which can give rise to a generalized periodic table for topological insulators \cite{15}, and new control mechanism \cite{16, 17}. A time-parity can induce coherent destruction of tunneling \cite{18}.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Symmetry & Effect & Example \\
\hline
RS & spDS & benzene-ring (Fig. 1) \\
PHS & FBSR & (Fig. 2) \\
$2 \times$ PHS & siT & two-level sys. (Fig. 3) \\
TRS & none & (Fig. 4) \\
CS & none & (Fig. 5) \\
none & aDS & all (Figs. 1-3) \\
\hline
\end{tabular}
\caption{Overview of the spectroscopic signatures of dynamical rotational symmetry (RS), particle-hole symmetry (PHS), and time-reversal symmetry (TRS). The signatures include symmetry-protected dark states (spDS), Floquet band selection rule (FBSR), and accidental dark states (aDS). The right column lists example models.}
\end{table}

In this letter we investigate the impact of dynamical symmetries on the spectroscopic properties of periodically-driven quantum systems, as described by the Floquet response theory. Investigating $N$-fold dynamical rotational symmetries and particle-hole symmetries, we establish the Floquet-band selection rule (FBSR), dark state conditions and a symmetry-induced transparency (siT) condition. The relation to time-reversal symmetry and chiral symmetry is also explained. Dark states can appear by two distinct mechanisms: (i) accidentally, due to a specific combination of parameters, termed accidental dark states (aDS); (ii) or because of symmetry conditions, termed symmetry-protected dark states (spDS). Similar to symmetry-protected excitation of topological band structures, spDS do not disappear under symmetry-conserving parameter variations. An overview of the symmetries and their optical effects is given in table I.

\text{* jianshu@mit.edu}
Floquet response theory. We apply a semiclassical approach based on the general Hamiltonian
\[
\hat{H}(t) = \hat{H}_0(t) + \int_0^\infty d\omega \left[ \lambda \hat{V} (\hat{a}_\omega + \hat{a}^\dagger_\omega) + \omega \hat{a}^\dagger_\omega \hat{a}_\omega \right],
\]
where \(\hat{H}_0(t) = \hat{H}_0(t + \tau)\) denotes the system driven by a time-periodic classical electromagnetic field of frequency \(\Omega = 2\pi/\tau\). The perturbation field is given by a continuum of photonic operators \(\hat{a}_\omega\) with frequencies \(\omega\) which are coupled via the dipole transition operator \(\hat{V}\) with strength \(\lambda\) to the driven system. The physical properties of \(\hat{H}_0(t)\) are determined by the Floquet equation
\[
\left[ \hat{H}_0(t) - i \frac{d}{dt} \right] |u_\mu(t)\rangle = \epsilon_\mu |u_\mu(t)\rangle,
\]
where \(|u_\mu(t)\rangle = |u_\mu(t + \tau)\rangle\) and \(\epsilon_\mu\) are the associated Floquet states and quasienergies, which generalize the concept of eigenstates and eigenenergies of time-independent systems. It is implicitly assumed that \(\hat{H}_0(t)\) describes a weakly dissipative system, such that for long times it approaches the stationary state
\[
\rho(t) = \sum_\mu p_\mu |u_\mu(t)\rangle \langle u_\mu(t)|,
\]
which is diagonal in the Floquet basis. Eq. (3) is in agreement with recent investigations for closed and open driven quantum systems. In the model calculations, we assume a Floquet-Gibbs distribution with \(p_\mu = e^{-\beta \epsilon_\mu}\), but all our predictions hold even if the Floquet-Gibbs distributions breaks down.

The interaction of \(\hat{H}_0(t)\) with the probe field is treated using the input-output formalism and a perturbation expansion for small \(\lambda\). The input field consists of a probe beam (of frequency \(\omega_p\)) and a signal beam (of frequency \(\omega_s = \omega_p + n\Omega\), integer \(n\)). As shown separately, the intensity change of the signal field is proportional to the probe-signal coherence \(\langle \hat{a}^\dagger_{\omega_p} \hat{a}_{\omega_s}\rangle\) and is given by
\[
\Delta I_{coh}(\omega_s) = -i \tilde{\chi}_n(\omega_p) \langle \hat{a}^\dagger_{\omega_s} \hat{a}_{\omega_p} \rangle + \text{c.c.},
\]
where the susceptibility \(\tilde{\chi}_n(\omega_p)\) can be evaluated using Floquet theory and reads
\[
\tilde{\chi}_n(\omega_p) = i \lambda^2 \sum_{\nu,\mu,m} \frac{V^{\nu\mu}_m}{\epsilon_\nu - \epsilon_\mu + m\Omega - \omega_p - i \gamma_{\nu,\mu}}. \tag{4}
\]

The index \(n\) denotes the Floquet band, which describes non-elastic scattering of the probe field, and the dynamical dipole elements read
\[
V^{(0)}_{\nu\mu} = \frac{1}{\tau} \int_0^\tau \langle u_\lambda(t) | \hat{V} | u_\mu(t) \rangle \ e^{-i n\Omega t} \ dt. \tag{5}
\]

\[
V^{(1,0)}_{\nu\mu} = \frac{1}{\tau} \int_0^\tau \langle u_\lambda(t) | \hat{V} | u_\mu(t) \rangle \ e^{-i n\Omega t} \ dt.
\]

The special case of \(n = 0\) in Eq. (19) has been previously considered \cite{11} and predicts a periodic structure of the absorption spectrum related to the summation over \(m\). For a weak driving, this term reduces to the linear response function of the standard nonlinear spectroscopy \cite{6}. The relationship between Floquet response theory and standard response theory is non-trivial and will be investigated separately. In the derivation of Eq. (19) we have used Eq. (3) as the stationary state. For an arbitrary density matrix, one has to incorporate the non-diagonal elements of the density matrix as well.

Dark states are defined by the condition \(V^{(m)}_{\nu\mu} = 0\), such that the corresponding resonances in Eq. (19) vanish. This condition can be fulfilled by special combinations of parameters, which we denote as aDS, or as a consequence of a symmetry, which we denote as spDS. As we will explain, the spDS can even give rise to the complete vanishing of a Floquet band \(\tilde{\chi}_n(\omega_p) = 0\) for specific \(n\). Moreover, sIT can appear due to a destructive interference of two transitions with \(V^{(m)}_{\nu\mu} \neq 0\). We consider the following class of symmetry operations \cite{21}
\[
\Sigma \left[ \hat{H}_0(t_s + \beta_s t) - i \frac{d}{dt} \right] \Sigma^{-1} = \alpha_S \left[ \hat{H}_0(t) - i \frac{d}{dt} \right], \tag{6}
\]
where \(\Sigma\) is a time-independent spatial operator. By specifying \(\Sigma_s, t_s\), and \(\alpha_S, \beta_s = \pm 1\) one can define a set of dynamical symmetries with unique spectroscopic signatures.

**Dynamical rotational symmetry.** With \(\alpha_S = \beta_S = 1\),
metry, which gives rise to the eigen equation

\[ \hat{N} = 0 \] with eigenvalues \[ \pi \] and \[ \Delta = 0 \].

As shown in detail in the SI, for a dipole transition operator with \[ \hat{R} \hat{V} \hat{R} = \alpha_V^{(R)} \hat{V} \] with \( \alpha_V^{(R)} = \pm 1 \), Eq. (8) establishes a sufficient condition

\[ \hat{V}_{\nu,\mu}^{(m)} \propto \begin{cases} 1 & \text{if } e^{i \frac{2\pi}{N} (m\nu - m\mu + m)} \alpha_V^{(R)} = 1, \\ 0 & \text{else,} \end{cases} \] (9)

which defines spDSs. Applying Eq. (25) to evaluate the susceptibility in Eq. (19), we find

\[ \chi_0(\omega) = \begin{cases} 1 & \text{if } e^{i \frac{2\pi}{N} \nu} = 1, \\ 0 & \text{else,} \end{cases} \] (10)

which is the condition for the complete disappearance of Floquet band \( n \), i.e., the FBSR. We emphasize that these relations are only valid for the stationary state Eq. (3).

Physically, this effect is caused by the synchronization of the system with the driving field, such that the density matrix adopts the dynamical rotational symmetry, i.e., \( \rho(t + n/N \tau) = \hat{R}^n \rho(t) \hat{R}^n \).

As an example, we consider a benzene ring driven by circular-polarized light sketched in Fig. 1(a), which is described by a tight-binding Hamiltonian

\[ \hat{H}_0(t) = \sum_{j,j' = 1}^6 J_{j,j'} |e_j\rangle \langle e_{j'}| + \sum_{j = 1}^6 [i f_j(t) |e_j\rangle \langle e_{j+1}| + h.c.], \]

where \( |e_j\rangle \) denotes the excitation on site \( j \) (defined modulo 6), \( J_{j,j'} = E_0 \) is the onsite energy, \( J_{j,j'} = \delta_{j,j' \pm 1} \) \( J_0 \) is the tunneling constant, and \( f_j(t) = f_0 \cos(\Omega t + 2\pi j/6) \) is the time-dependent tunneling strength with the driving amplitude \( f_0 \). The driving terms are motivated by the Peierls substitution describing a vectorial current-gauge-field coupling \( j \cdot \mathbf{A}(t) \) with a circularly rotating vector potential \( \mathbf{A}(t) \). The dipole transition operator is a transition dipole exciting the ground state \( |g\rangle \) to the single-excitation manifold \( \hat{V} = \sum_{j = 1}^N d_0 |e_j\rangle \langle g| \), whose quasienergies (depicted in Fig. 1(b)) and Floquet-states determine the spectroscopic signatures. The stationary state is \( \rho_\tau(t) = |g\rangle \langle g| \) in agreement with Eq. (3), i.e., a Floquet-Gibbs state for low temperatures. A rotational symmetry is fulfilled for \( N = 6 \) and \( \hat{R} = \sum_{j = 1}^n |e_j\rangle \langle e_{j+1}| \).

In Fig. 1(c) we depict the susceptibility \( \chi_0(\omega) \) of the benzene model. The resonances of the dark states defined by Eq. (25) are marked by dashed lines (optically invisible), and only the transitions, \( \hat{V}_{0,1}^{(1)} \hat{V}_{1,0}^{(1)} \) and \( \hat{V}_{3,0}^{(1)} \hat{V}_{0,3}^{(1)} \) are visible. An aDS can be found for \( \hat{V}_{0,1}^{(1)} \hat{V}_{1,0}^{(1)} \) at \( f_0 = 1.5 \Omega \). As a consequence of the FBSR in Eq. (10), only Floquet bands \( \chi_n(\omega) \) with \( n \mod 6 = 0 \) appear.

Particle-hole symmetry. A particle-hole symmetry is defined for \( -\alpha_s = -\beta_s = 1, t_s = t_p = \tau N_1 / 2N_2 \) with integers \( N_1 \in \{0, 1\}, N_2 \geq 1 \), and \( \hat{\Sigma} = \hat{P} \hat{\kappa} \hat{P} \) with a unitary \( \hat{P} \) and the complex conjugation operator \( \hat{\kappa} \), such that

\[ \hat{P} \hat{H}^*(t + t_p) \hat{P} = -\hat{H}(t). \] (11)
The particle-hole symmetry establishes a symmetry between excitation and deexcitation processes, and has its origin in fermionic systems, where adding and removing quasiparticles results in physically equivalent behaviors. Here we use the particle-hole symmetry in a general context. Using the particle-hole symmetry in Eq. (17), we find that each Floquet state \( |u_\mu(t)\rangle \) with quasienergy \( \epsilon_\mu \) has its symmetry-related partner

\[
|u_{\mu'}(t)\rangle = c_{\mu'}^{\dagger}(t)\hat{P}|u_\mu(t + t_P)\rangle^*
\]

with energy \( \epsilon_{\mu'} = -\epsilon_\mu \) and a gauge-dependent \( c_{\mu'}^{(P)} \). For \( t_P = \tau/(2N_2) \) the particle-hole symmetry gives rise to a rotational symmetry defined by \( \hat{R} = \hat{P}\hat{P} \) and \( t_R = \tau/N_2 \), such that the dark state rules of the rotational symmetry apply. Besides the rotational symmetry, the particle-hole symmetry can give rise to a distinct dark state condition.

For a dipole transition operator with \( \hat{P}^\dagger\hat{V}^*\hat{P} = \alpha_\nu^*(\hat{V}) \), \( \alpha_\nu^{(P)} = \pm 1 \), \( t_P = 0, \tau/2 \), and \( \hat{P}^*\hat{P} = 1 \), the particle-hole symmetry results in \( V_{\mu,\mu'}^{(m)} = \alpha_\nu^{(P)} e^{im\Omega t_P} V_{\mu,\mu'}^{(m)} \) for symmetry related states \( \mu, \mu' \), so that

\[
V_{\mu,\mu'}^{(m)} \propto \begin{cases} 0 & \text{if } \alpha_\nu^{(P)} e^{im\Omega t_P} = -1; \mu, \mu' \text{ sym. rel.} \\ 1 & \text{else,} \end{cases}
\]

as shown in detail in the SI. In contrast to Eq. (25), where each transition can vanish for an appropriate \( m \), only transitions between symmetry-related states are affected by Eq. (13).

To illustrate Eq. (13), we use the dimer model sketched in Fig. 2(a), with the Hamiltonian given by

\[
\hat{H}_0(t) = \Delta \left( \hat{A}_{f,f} - \hat{A}_{g,g} \right) + J_0 \hat{A}_{e_1,e_2} + h_1(t) \left[ \hat{A}_{e_1,f} + \hat{A}_{g,e_1} + \tau \hat{A}_{e_1,e_2} \right],
\]

(14)

where \( \hat{A}_{\alpha,\beta} \equiv |\alpha\rangle \langle \beta| + \text{h.c.} \), and \( g, e_1, e_2 \) and \( f \) label the ground state, two single-excitation states and the double excitation state, respectively. \( \Delta \) is the excitation gap, \( J_0 \) is the tunneling constant, and \( h_1(t) = f_0 \cos(\Omega t) \) is the driving field. The \( \tau \) term enhances higher-order dipole elements \( V_{\mu,\mu'}^{(m)} \). The particle-hole symmetry is defined by \( \hat{P} = \hat{A}_{g,f} + \hat{A}_{e_1,e_2} - \hat{A}_{f,e_2} \) and \( t_P = 0 \). The quasienergy spectrum in Fig. 2(b) is symmetric with respect to \( E = 0 \). The dipole transition operator is \( \hat{V} = \hat{A}_{e_1,f} + \hat{A}_{g,e_1} \), such that \( \hat{P}^\dagger\hat{V}^*\hat{P} = -\hat{V} \). In Fig. 2(c), we depict the susceptibility in Eq. (19). According to the above considerations, the transitions between the particle-hole symmetry related pairs vanish, i.e., \( V_{1,4}^{(m)} = V_{4,1}^{(m)} = V_{2,3}^{(m)} = V_{3,2}^{(m)} = 0 \) for all \( m \). These resonances are marked by dashed lines. The other transitions not affected by the symmetry constraint remain visible in Fig. 2(c).

**Symmetry-induced transparency.** The particle-hole symmetry can also give rise to a siT at a quasienergy crossing \( \epsilon_\mu = \epsilon_{\mu'} = 0 \) of symmetry related Floquet states \( \mu, \mu' \). In particular, we require that the system exhibits two particle-hole symmetries \( \hat{P}_1 \neq \pm \hat{P}_2, \hat{P}_1^2 = 1 \) and \( [\hat{P}_1, \hat{P}_2] = 0 \). In this case, there exist a basis such that \( \hat{P}_1|u_\mu(t + t_P)\rangle = |u_{\mu'}(t)\rangle \) and \( \hat{P}_2|u_\mu(t + t_P)\rangle = |u_{\mu'}(t)\rangle \). Using these relations in Eq. (10), we find \( V_{\mu,\mu'}^{(m)} = \alpha_\nu V_{\mu,\mu'}^{(m)} e^{im\Omega t_P} = \alpha_\nu V_{\mu,\mu'}^{(m)} \). Inserting this relation into the susceptibility \( \tilde{\chi}_n(\omega) \) in Eq. (19) for \( \omega = \Omega \), we obtain the transparency condition

\[
\tilde{\chi}_n(\Omega) \propto \begin{cases} 1 & \text{if } \epsilon_{\mu} = \epsilon_{\mu'} = 0 \\ 0 & \text{else.} \end{cases}
\]

Details are given in the SI. Even though both effects are symmetry induced, the spDS and the transparency effect are based on different mechanisms. While a spDS is generated by a vanishing dipole element, the siT is generated by a destructive interference of two transitions with non-vanishing dipole elements.

For illustration, we consider an ac-driven two-level system (TLS) sketched in Fig. 3(a) and described by the Hamiltonian

\[
\hat{H}_0(t) = \hbar \frac{1}{2} \hat{\sigma}_x + \frac{f_0}{2} \cos(\Omega t) \hat{\sigma}_z,
\]

(16)

where \( \hat{\sigma}_x, \hat{\sigma}_z \) are the Pauli matrices, \( \hbar \) is the TLS tunneling amplitude, and \( f_0 \) the driving strength. The TLS is weakly dissipative, as in the spin-boson model, such that it reaches the stationary state in Eq. (3). The dipole transition operator in Eq. (1) is \( \hat{V} = \hat{\sigma}_x \). For \( \hat{R} = \hat{\sigma}_z \) and \( t_R = \tau/2 \), the two-level system exhibits a rotational symmetry, which is equivalent to a parity symmetry in this case. It gives rise to an exact quasienergy crossing, depicted in the spectrum in Fig. 3(b) at \( f_0 \approx 2.4\Omega \), which is the location for the coherent destruction of tunneling effect \( [29, 38] \), and enables the siT in the current context.

For the TLS, a particle-hole symmetry is defined for \( \hat{P}_1 = \hat{\sigma}_x \) and \( t_{R1} = \tau/2 \), for \( h_\times = 0 \) a second particle hole symmetry is given for \( \hat{P}_2 = 1 \) and \( t_{R2} = \tau/2 \). As in this case \( \epsilon_\mu = 0 \) and \( \hat{P}_2 \hat{\sigma}_z \hat{P}_1 = (-1)\hat{\sigma}_x \), siT with \( \chi_n(\omega) = 0 \) appears according to Eq. (15) and the response \( \chi_n(\omega) \) is complete suppressed for all \( f_0 \) and \( \eta \). In Fig. 3(c) we consider \( \chi_n(\omega) \) for a finite, but small \( \hbar \ll \Omega \), such that the quasienergy degeneracy is lifted except of the crossing, and the particle-hole symmetry \( \hat{P}_2 \) is slightly broken. As a consequence, the siT is not complete, but scales as \( \chi_n(\omega) \propto \hbar \omega/\Omega \) at the crossing. This is related to a small asymmetry of the response peaks of \( \chi_n(\omega) \). Instead of the Floquet-Gibbs distribution, we choose \( p_0 = 0.6 \) and \( p_1 = 0.4 \) in Eq. (3) in Fig. 3(c) to highlight the siT.

**Time-reversal and chiral symmetries.** A time-reversal symmetry (chiral symmetry) is defined by Eq. (15) for \( \alpha_S = -\beta_S = 1 \) (\( \alpha_S = \beta_S = -1 \)), arbitrary \( t_S \), and \( \Sigma = \hat{T}\hat{K}, (\Sigma = \hat{C}) \), where \( \hat{T} (\hat{C}) \) is a unitary operator. As shown in the SI, neither time-reversal symmetry nor chiral symmetry alone implies spDSs. How-
ever, the combination of time-reversal symmetry and chiral symmetry defines a particle-hole symmetry with $\hat{P} = \hat{C}\hat{T}$, and $t_P = t_T - t_C$. When they further fulfill $t_T - t_C \in \{0, \pi/2\}$, $\hat{C}^*\hat{C} = 1$, $\hat{T}^*\hat{T} = 1$, and $[\hat{C}, \hat{T}] = 0$, such that $P^*\hat{P} = 1$, spDSs appear because of the particle-hole symmetry. In general, the presence of any two symmetries out of particle-hole symmetry, chiral symmetry, and time-reversal symmetry implies the existence of the third one. We thus establish a sufficient condition for dark states by a particle-hole symmetry alone, or by a combination of any two of the three symmetries.

Conclusions. We have predicted spectroscopic signature in periodically-driven quantum systems, namely symmetry-protected dark states, a Floquet band selection rule, symmetry-induced transparency, and accidental dark states. The first two effects are protected by symmetries, such that variations of symmetry preserving parameters do not destroy them. The symmetry-induced transparency results from the destructive interference of two transitions between degenerate states. These effects are difficult to observe in the weak driving regime as the Floquet band $n = 0$ does not contain the full information of strongly-driven systems. One has to include the non-zero Floquet bands to ensure that a system possesses a particular dynamical symmetry, thus these effects are expected in the strong driving regime.

Our theoretical results are experimentally observable in systems that can reach the strong-light matter coupling regime such as cold-atom experiments [8]. Superconducting circuits [9,10] are an ideal platform to test our predictions. For experiments with molecules, strong driving fields are necessary to generate high-order Floquet bands, but in cavity QED or plasmonic fields the strong driving interaction condition can be relaxed for molecules ensembles interacting collectively with the light field [13,14].

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In this supplemental information, we provide details to establish the dark state conditions, which are sketched in the main text. The starting point is the Floquet equation
\[
\left[ \hat{H}_0(t) - i \frac{d}{dt} \right] |u_\mu(t)\rangle = \epsilon_\mu |u_\mu(t)\rangle ,
\] (17)

\[\sum_j \sum_m f_j \left( t - m + \frac{\tau}{T} \right) \hat{A}_j^{(m)}, \]
where \( \hat{A}_j^{(m)} = \hat{R}^{(m)} \hat{A}_j^{(0)} \hat{R}^{(m)} \) for arbitrary Hermitian operators \( \hat{A}_j^{(0)} \), and the arbitrary functions \( f_j (t + \tau) = f_j (t) \) shall be \( \tau \) periodic.

One can construct a Hamiltonian fulfilling a time-rotational symmetry defined by \( \hat{H}_0(t) = \sum_j \sum_m f_j \left( t - m + \frac{\tau}{T} \right) \hat{A}_j^{(m)} \), where \( \hat{A}_j^{(m)} = \hat{R}^{(m)} \hat{A}_j^{(0)} \hat{R}^{(m)} \) for arbitrary Hermitian operators \( \hat{A}_j^{(0)} \), and the arbitrary functions \( f_j (t + \tau) = f_j (t) \) shall be \( \tau \) periodic.

Here, we provide the details to establish the dark state conditions, which are sketched in the main text. The starting point is the Floquet equation
\[
\left[ \hat{H}_0(t) - i \frac{d}{dt} \right] |u_\mu(t)\rangle = \epsilon_\mu |u_\mu(t)\rangle ,
\] (17)

Supplementary information

I. FLOQUET EQUATION AND TIME-SPATIAL SYMMETRY

In this supplemental information, we provide details to establish the dark state conditions, which are sketched in the main text. The starting point is the Floquet equation
\[
\left[ \hat{H}_0(t) - i \frac{d}{dt} \right] |u_\mu(t)\rangle = \epsilon_\mu |u_\mu(t)\rangle ,
\] (17)
and the time-spatial symmetry relation for a Hamiltonian in the Floquet space
\[ \hat{\Sigma} \left( \hat{H}_0(t_s + \beta_S t) - i \frac{d}{dt} \right) \hat{\Sigma}^{-1} = \alpha_S \left( \hat{H}_0(t) - i \frac{d}{dt} \right), \]
(18)
where the symmetry is specified by the spatial operator \( \hat{\Sigma} \), the time-shift \( t_S \), \( \alpha_S = \pm 1 \) and \( \beta_S = \pm 1 \).

## II. DIPOLE ELEMENTS

The susceptibility, introduced in the main text
\[ \tilde{\chi}_n(\omega) = i\lambda^2 \sum_{\nu,\mu,m} V_{\nu,\mu}^{(-n-m)} V_{\mu,\nu}^{(m)} \left( \frac{p_{\nu} - p_{\mu}}{\epsilon_{\nu} - \epsilon_{\mu} + m\Omega - \omega - i\gamma_{\nu,\mu}} \right), \]
(19)
is expressed in terms of the dynamical dipole elements
\[ V_{\nu,\mu}^{(n)} = \frac{1}{\tau} \int_0^\tau \langle u_{\nu}(t) | \hat{V}(t) | u_{\mu} \rangle e^{-in\Omega t} dt \]
(20)
which fulfill
\[ V_{\nu,\mu}^{(n)} = \frac{1}{\tau} \int_0^\tau \langle u_{\nu}(t) | \hat{V} | u_{\mu}(t) \rangle \ e^{-in\Omega t} dt, \]
\[ = \frac{1}{\tau} \int_0^\tau \left( \langle u_{\nu}(t) | \hat{V} | u_{\mu}(t) \rangle \right)^* e^{-in\Omega t} dt, \]
\[ = \left( \frac{1}{\tau} \int_0^\tau \langle u_{\nu}(t) | \hat{V} | u_{\mu}(t) \rangle e^{in\Omega t} dt \right)^*, \]
(21)
This relation will be used to prove the dark state condition based on a time-spatial particle-hole symmetry in Sec. IV.

## III. ROTATIONAL SYMMETRY

Here we present the derivation of the dark state condition imposed by a rotational symmetry. For a unitary operator \( \hat{\Sigma} = \hat{R} \), \( t_S = t_P = \tau/N \) with a positive integer \( N \), and \( \alpha_S = \beta_S = 1 \), the general time-spatial symmetry condition Eq. (18) specifies to a rotational symmetry
\[ \hat{R} \hat{H} \left( t + \frac{\tau}{N} \right) \hat{R}^\dagger = \hat{H}(t), \]
(22)
Applying Eq. (18) to Eq. (17), we find
\[ \left[ \hat{H}_0(t) - i \frac{d}{dt} \right] \hat{R} | u_{\mu}(t + t_R) \rangle = \epsilon_{\mu} \hat{R} | u_{\mu}(t + t_R) \rangle. \]
(23)
This implies that every Floquet state is also an eigenstate of the rotational symmetry operator, such that
\[ \hat{R} | u_{\mu}(t + t_R) \rangle = \pi_{\mu} | u_{\mu}(t) \rangle, \]
(24)
with eigenvalues \( \pi_{\mu} = e^{i2\pi m_{\mu}/N} \) and integers \( m_{\mu} = \{0, N - 1\} \). We require that the transition dipole operator obeys \( \hat{R}^\dagger \hat{V} \hat{R} = \alpha_{\nu}^{(R)} \hat{V} \) with \( \alpha_{\nu}^{(R)} = \pm 1 \). Combining with Eq. (24), the dynamical dipole element fulfills
\[ V_{\nu,\mu}^{(n)} = \frac{1}{\tau} \int_0^\tau \langle u_{\nu}(t) | \hat{V} | u_{\mu}(t) \rangle e^{-in\Omega t} dt, \]
Recalling that $\alpha^{(R)} = \pm 1$, the last line establishes the dark state condition for a rotational-symmetry

$$V^{(n)}_{\nu,\mu} \propto \begin{cases} 1 & \text{if } e^{i2\pi(m\nu-m\mu+n)}\alpha^{(R)}_{\nu} = 1, \\ 0 & \text{else,} \end{cases} \quad (25)$$

which is presented in the main text.

### IV. PARITCLE-HOLE SYMMETRY

Here, we provide details to establish the dark state condition induced by a dynamical particle-hole symmetry

$$\hat{P}\hat{H}^*(t_P + t)\hat{P}^t = -\hat{H}(t), \quad (26)$$

where $t_P$ assumes the values $t_P = N_1\tau/2N_2$ for integers $N_1 = 0, 1$ and $N_2 \geq 1$ and depends on the specific system. We obtain this from the general definition of the dynamical symmetries Eq. (18) for $\hat{\Sigma} = \hat{P}\hat{k}$, with the complex conjugation operator $\hat{k}$ and the unitary operator $\hat{P}$, $\alpha_S = -\beta_S = -1$ and $t_S = t_P$. Applying the definition in Eq. (26) to the Floquet equation Eq. (17), we find

$$\left[\hat{H}_0(t) - i\frac{d}{dt}\right] \hat{P}|u_\mu(t_P + t)\rangle^* = -\epsilon_\mu \hat{P}|u_\mu(t_P + t)\rangle^*, \quad (27)$$

which indicates that for every Floquet state $|u_\mu(t)\rangle$ with quasienergy $\epsilon_\mu$, there is a symmetry-related partner

$$|u_{\mu'}(t)\rangle = c^{(P)}_\mu \hat{P}|u_\mu(t_P + t)\rangle^* \quad (28)$$

with quasienergy $\epsilon_{\mu'} = -\epsilon_\mu$, and a gauge-dependent phase factor $c^{(P)}_\mu$. The phase factor cannot be removed by a simple gauge transformation as two Floquet states $\mu, \mu'$ are coupled. However, we can apply Eq. (28) twice and obtain

$$|u_\mu(t)\rangle = c^{(P)}_\mu c^{(P)*}_{\mu'} \hat{P}\hat{P}^* |u_\mu(2t_P + t)\rangle. \quad (29)$$

In general, $c^{(P)}_\mu \neq c^{(P)}_{\mu'}$, and we cannot find a gauge transformation such that $c^{(P)}_\mu c^{(P)*}_{\mu'} = 1$. Only if $\hat{P}\hat{P}^* = 1$ and $t_P \in \{0, \tau/2\}$, we have $c^{(P)}_\mu c^{(P)*}_{\mu'} = 1$, which will be used in the evaluation of the dynamical dipole elements.

Furthermore, we require $\hat{P}^t\hat{V}\hat{P} = \alpha V^*\hat{V}^*$ with $\alpha^* = \pm 1$. Using Eq. (28) and Eq. (29), we find that the dynamical dipole element obeys

$$V^{(n)}_{\nu,\mu'} = \frac{1}{\tau} \int_0^\tau \langle u_\mu(t) | \hat{V} | u_{\mu'}(t) \rangle e^{-i\Omega t} dt$$

$$= \frac{1}{\tau} \int_0^\tau c^{(P)}_{\mu'} c^{(P)}_\mu \langle u_{\mu'}(t + t_P) | \hat{P}^t\hat{V}\hat{P} | u_\mu(t) \rangle e^{-i\Omega t} dt$$

$$= \frac{1}{\tau} \int_0^\tau \alpha^{(P)}_\nu \langle u_{\mu'}(t)^* | \hat{V}^* | u_\mu(t) \rangle^* e^{-i\Omega(t-t_P)} dt.$$
Thus, we have established symmetry-related pairs with respect to $\hat{\omega}$ of dimension two. As $\hat{\omega}$ requires $\omega$ which requires $\tau$.

This transparency effect appears when the two corresponding terms in the susceptibility in Eq. (19) cancel each other, where we have used Eq. (21) in the last line. From line two to line three we have used the $\tau$-periodicity of the integrand. This proves the particle-hole symmetry induced condition for dark states

\[ \hat{V}^{(m)}_{\mu,\mu'} \propto \begin{cases} 0 & \text{if } \alpha^{(P)}_{\mu,\mu'} e^{-i\Omega t} = -1, \quad \mu, \mu' \text{ sym. rel.} \\ 1 & \text{else,} \end{cases} \]

which is presented in the main text.

\section{Symmetry-Induced Transparency}

As explained in the main text, a transparency effect can be induced by two distinct particle-hole symmetries, if two symmetry-related resonances corresponding to the quasienergies $\epsilon_\mu = -\epsilon_{\mu'}$ become degenerate, i.e., $\epsilon_\mu = \epsilon_{\mu'} = 0$. This transparency effect appears when the two corresponding terms in the susceptibility in Eq. (19) cancel each other, which requires

\[ V^{(-n-m)}_{\mu',\mu} V^{(m)}_{\mu,\mu'} = V^{(-n-m)}_{\mu',\mu} V^{(m)}_{\mu,\mu'} , \]

which will be investigated in the following, for a given resonance $\omega = m\Omega$.

We denote the two particle-hole symmetries with $\hat{P}_1 = \hat{P}_1^\dagger$, $\hat{P}_2 = \hat{P}_2^\dagger$ with corresponding time shifts $t_{P_1}, t_{P_2}$, and require $[\hat{P}_1, \hat{P}_2] = 0$ as well as $\hat{P}_1 \neq \pm \hat{P}_2$ (i.e., they are distinct). The Floquet states $\mu$ with $\epsilon_\mu = 0$ form a subspace of dimension two. As $\hat{P}_1$ and $\hat{P}_2$ commute, and $\hat{P}_1 = \hat{P}_1^\dagger$, there is a basis of this subspace such that

\[ \begin{align*}
\hat{P}_1 |v_\mu(t_{P_1} + t)\rangle^* &= |v_\mu(t)\rangle, \\
\hat{P}_1 |v_{\mu'}(t_{P_1} + t)\rangle^* &= |v_{\mu'}(t)\rangle, \\
\hat{P}_2 |v_\mu(t_{P_2} + t)\rangle^* &= |v_\mu(t)\rangle, \\
\hat{P}_2 |v_{\mu'}(t_{P_2} + t)\rangle^* &= -|v_{\mu'}(t)\rangle.
\end{align*} \]

In this basis, there are no pairs of symmetry related partners as introduced in Sec. IV. From Eq. (33) we obtain a basis with a symmetry-related pairs by

\[ \begin{align*}
|u_\mu(t)\rangle &= \frac{1}{\sqrt{2}} (|v_\mu(t)\rangle + |v_{\mu'}(t)\rangle), \\
|u_{\mu'}(t)\rangle &= \frac{1}{\sqrt{2}} (|v_\mu(t)\rangle - |v_{\mu'}(t)\rangle).
\end{align*} \]

which are related by the symmetry operations as

\[ \begin{align*}
\hat{P}_1 |u_{\mu'}(t_{P_1} + t)\rangle^* &= |u_\mu(t_{P_1} + t)\rangle, \\
\hat{P}_1 |u_\mu(t_{P_1} + t)\rangle^* &= |u_{\mu'}(t)\rangle, \\
\hat{P}_2 |u_\mu(t_{P_2} + t)\rangle^* &= |u_{\mu'}(t)\rangle, \\
\hat{P}_2 |u_{\mu'}(t_{P_2} + t)\rangle^* &= |u_\mu(t)\rangle.
\end{align*} \]

Thus, we have established symmetry-related pairs with respect to $\hat{P}_1$. Please note that the gauges phases are fixed by the procedure, so we do not consider them in the following calculations.

Using the symmetry $\hat{P}_2$ to evaluate the dynamical dipole elements, we find

\[ V^{(n)}_{\mu,\mu'} = \frac{1}{\tau} \int_0^\tau \langle u_{\mu'}(t) | \hat{V} | u_\mu(t) \rangle e^{-i\Omega t} dt \\
= \frac{1}{\tau} \int_0^\tau \langle u_\mu(t_{P_2} + t) | \hat{P}_2^\dagger \hat{V} \hat{P}_2 | u_{\mu'}(t_{P_2} + t) \rangle^* e^{-i\Omega t} dt \]
\[ \int_0^\tau \alpha_V^{(P)} \langle u_{\mu}(t) | \hat{V}^* | u_{\mu'}(t) \rangle e^{-i\Omega(t_{P_1} + t)} dt \]
\[ = \alpha_V^{(P)} e^{-i\Omega P_1} \left( \frac{1}{\tau} \int_0^\tau \langle u_{\mu}(t) | \hat{V} | u_{\mu'}(t) \rangle e^{i\Omega dt} \right)^* \]
\[ = \alpha_V^{(P)} e^{-i\Omega P_1} V_{\mu,\mu'}^{(-n)*}, \]  
(36)

Using \( \hat{P}_1 \) to evaluate the dipole elements based on Eq. \([30]\), we find \( V_{\mu,\mu'}^{(n)} = \alpha_V^{(P_2)} e^{-i\Omega P_2} V_{\mu,\mu'}^{(n)*} \). Combining both results, we obtain

\[ V_{\mu,\mu'}^{(n)} = \alpha_V^{(P_2)} \alpha_V^{(P_1)} e^{-i\Omega P_1 - P_2}. \]  
(37)

Inserting Eq. \([37]\) into the susceptibility in Eq. \([32]\), we find the transparency condition

\[ \chi_n(m\Omega) \propto \begin{cases} 0 & \text{if } e^{-i\Omega (P_1 - P_2)} = 1; \epsilon_\mu = \epsilon_{\mu'} = 0 \\ 1 & \text{else}, \end{cases} \]  
(38)

which is the condition presented in the main text. Noteworthy, for \( e^{-i\Omega (P_1 - P_2)} = -1 \) the two resonances can also add up constructively.

VI. CHIRAL SYMMETRY

Here we derive a constrain for dynamical dipole elements under a chiral symmetry, and explain why the chiral symmetry on its own does not imply a dark state condition. For a unitary \( \hat{\Sigma} = \hat{C} \), \( \alpha_S = \beta_S = -1 \) and an arbitrary \( t_S = t_C \), the general time-spatial symmetry relation Eq. \([15]\) defines a chiral symmetry

\[ \hat{C} \hat{H}(t_C - t) \hat{C}^\dagger = -\hat{H}(t). \]  
(39)

Applying this definition to the Floquet equation Eq. \([17]\), we find

\[ \left[ \hat{H}_0(t) - \frac{i}{\tau} \frac{d}{dt} \right] \hat{C} | u_\mu(t_C - t) \rangle = -\epsilon_\mu \hat{C} | u_\mu(t_C - t) \rangle. \]  
(40)

Similar to the particle-hole symmetry, Eq. \([40]\) implies that for every Floquet state \( | u_\mu(t) \rangle \) with quasienergy \( \epsilon_\mu \), there is a symmetry-related partner

\[ | u_{\mu'}(t) \rangle = c^{(C)}_\mu \hat{C} | u_\mu(t_C - t) \rangle, \]  
(41)

with quasienergy \( \epsilon_{\mu'} = -\epsilon_\mu \) and a gauge-dependent phase factor \( c^{(C)}_\mu \). The \( c^{(C)}_\mu, c^{(C)}_{\mu'} \) can not be arbitrarily changed by a gauge transformation. Chiral symmetry is not sufficient to determine a constrain for the dynamical dipole elements. To see this we conjugate Eq. \([41]\) and insert it into itself, and obtain

\[ | u_\mu(t) \rangle = c^{(C)}_\mu c^{(C)}_{\mu'} \hat{C}^* \hat{C}^\dagger | u_\mu(t) \rangle^*. \]  
(42)

When requiring \( \hat{C}^* \hat{C} = 1 \), we can find a gauge transformation, for which \( c^{(C)}_\mu c^{(C)}_{\mu'} = 1 \). Using this and \( \hat{C}^\dagger \hat{V} \hat{C} = \alpha_V^{(C)} \hat{V} \) with \( \alpha_V^{(C)} = \pm 1 \) to evaluate the dynamical dipole elements, we find

\[ V_{\mu,\mu'}^{(n)} = \frac{1}{\tau} \int_0^\tau \langle u_{\mu}(t) | \hat{V} | u_{\mu'}(t) \rangle e^{-i\Omega t} dt \]
\[ = \frac{1}{\tau} \int_0^\tau c^{(C)}_{\mu'} c^{(C)}_{\mu} \langle u_{\mu'}(t_C - t) | \hat{C}^\dagger \hat{V} \hat{C} | u_\mu(t_C - t) \rangle e^{-i\Omega t} dt \]
\[ = \frac{1}{\tau} \int_0^\tau \alpha_V^{(C)} \langle u_{\mu'}(t) | \hat{V} | u_\mu(t) \rangle e^{-i\Omega (t_C - t)} dt \]
reversal symmetry in Eq. (18) is defined for $\hat{\Sigma} = \hat{\kappa}$, to guarantee this. Similar to the chiral symmetry, this does not establish a dark state condition. A time-reversal symmetry in Eq. (18) is defined for $\hat{\Sigma} = \hat{\kappa}$, with a unitary operator $\hat{T}$, the complex conjugation operator $\hat{\kappa}$, $\alpha_S = -\beta_S = 1$, and an arbitrary $t_S = t_T$, so that

$$\hat{T}\hat{H}^*(t_T - t)\hat{T}^\dagger = \hat{H}(t).$$

(44)

Applying this definition to the Floquet equation, we find

$$\left[\hat{H}_0(t) - i\frac{d}{dt}\right] \hat{T} |u_\mu(t_T - t))|^* = \epsilon_\mu \hat{T} |u_\mu(t_T - t))^*, \quad (45)$$

Thus, all Floquet states fulfill

$$|u_\mu(t)) = c^{(T)}_\mu \hat{T} |u_\mu(t_T - t))^*$$

(46)

with a phase factor $c^{(T)}_\mu$, which can be transformed away by a gauge transformation. Using this to evaluate the dipole elements, we find

$$V^{(n)}_{\mu,\nu} = \frac{1}{\tau} \int_0^\tau \langle u_\mu(t)|\hat{V}|u_\nu(t)\rangle e^{-i\Omega t} dt,$$

$$= \frac{1}{\tau} \int_0^\tau \langle u_\mu(t_T - t)|^* \hat{T}^\dagger \hat{T} |u_\nu(t_T - t))^* e^{-i\Omega t} dt,$$

$$= \frac{1}{\tau} \int_0^\tau \alpha^{(T)}_V \langle u_\mu(t)|^* \hat{V}^* |u_\nu(t))^* e^{-i\Omega (t_T - t)} dt,$$

$$= e^{-i\Omega t_T} \alpha^{(T)}_V \left( \frac{1}{\tau} \int_0^\tau \langle u_\mu(t)|\hat{V}|u_\nu(t)\rangle e^{-i\Omega t} dt \right)^*,$$

$$= e^{-i\Omega t_T} \alpha^{(T)}_V \langle \nu_\mu,\nu \rangle,$$

(47)

Similar to the chiral symmetry, this does not establish a dark state condition.

VIII. COMBINED CHIRAL SYMMETRY AND TIME REVERSAL SYMMETRY

Importantly, the chiral-symmetry relation Eq. (43) and the time-reversal symmetry relation Eq. (47) together do provide a sufficient condition for a dark state. In the derivations in Sec. [VI] and Sec. [VII] we could choose the gauge fields independently, such that $c^{(C)}_\mu c^{(C)*}_\mu = 1$ and $c^{(T)}_\mu c^{(T)*}_\mu = 1$. Yet, these gauges conditions can not be simultaneous fulfilled in general. However, the combination of a time-reversal symmetry and a chiral symmetry defines a particle-hole symmetry with $\hat{P} = \hat{C}\hat{T}$ and $t_P = t_T - t_C$. When we additionally require that $\hat{C}\hat{C}^* = 1$, $\hat{T}\hat{T}^* = 1$, $[\hat{C},\hat{T}] = 0$, such that $\hat{P}^*\hat{P} = 1$, and $t_T - t_C = 0, \tau/2$, all requirements for dark states in Sec. [IV] are fulfilled.