Cusp and $b_1$ growth for ball quotients and maps onto $\mathbb{Z}$ with finitely generated kernel

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Abstract
Let $M = \mathbb{B}^2/\Gamma$ be a smooth ball quotient of finite volume with first betti number $b_1(M)$ and let $E(M) \geq 0$ be the number of cusps (i.e., topological ends) of $M$. We study the growth rates that are possible in towers of finite-sheeted coverings of $M$. In particular, $b_1$ and $E$ have little to do with one another, in contrast with the well-understood cases of hyperbolic 2- and 3-manifolds. We also discuss growth of $b_1$ for congruence arithmetic lattices acting on $\mathbb{B}^2$ and $\mathbb{B}^3$. Along the way, we provide an explicit example of a lattice in $\text{PU}(2,1)$ admitting a homomorphism onto $\mathbb{Z}$ with finitely generated kernel. Moreover, we show that any cocompact arithmetic lattice $\Gamma \subset \text{PU}(n,1)$ of simplest type contains a finite index subgroup with this property.

1 Introduction
Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$ with its Bergman metric and $\Gamma$ be a torsion-free group of isometries acting with finite covolume. Then $M = \mathbb{B}^n/\Gamma$ is a manifold with a finite number $E(M) \geq 0$ of cusps. Let $b_1(M) = \dim H_1(M;\mathbb{Q})$ be the first betti number of $M$. One purpose of this paper is to describe possible behavior of $E$ and $b_1$ in towers of finite-sheeted coverings. Our examples are closely related to the ball quotients constructed by Hirzebruch [19] and Deligne–Mostow [15, 27].

For $n = 1$, $M$ is better-known as a hyperbolic 2-manifold, and the behavior of $b_1$ and $E$ is very well-understood via basic topology of Riemann surfaces. For 3-manifolds, particularly finite volume hyperbolic 3-manifolds, growth of $b_1$ in finite covering spaces has been a subject of immense interest [11, 12, 40, 5, 1]. For both hyperbolic 2- and 3-manifolds, what is very well-known is the contribution

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of the cusps to $b_1$: for Riemann surfaces $b_1 \geq \mathcal{E} - 1$, and for finite volume hyperbolic 3-manifolds $b_1 \geq \mathcal{E}$ (which is known by ‘half lives, half dies’, and follows easily from Poincaré duality). The first result in this paper shows that no such theorem holds for two-dimensional ball quotients.

**Theorem 1.** There exist infinite towers \( \{A_j\}, \{B_j\}, \{C_j\}, \{D_j\} \) of distinct smooth finite volume noncompact quotients of \( \mathbb{B}^2 \) such that:

1. \( b_1(A_j) \) is uniformly bounded for all \( j \) and \( \mathcal{E}(A_j), \text{vol}(A_j) \to \infty \) as \( j \to \infty \);
2. \( b_1(B_j) \) and \( \mathcal{E}(B_j) \) are both uniformly bounded but \( \text{vol}(B_j) \to \infty \) as \( j \to \infty \);
3. \( b_1(C_j), \mathcal{E}(C_j), \text{and} \text{vol}(C_j) \) all grow linearly with \( j \);
4. \[
\text{vol}(D_j)^{\frac{2}{3}} \ll b_1(D_j) \\
\text{vol}(D_j)^{\frac{5}{8}} \sim \mathcal{E}(D_j).
\]

Moreover, all these manifolds can be taken as quotients of \( \mathbb{B}^2 \) by a neat and/or arithmetic lattice.

This leads to the following question: *For which nonnegative integers \( \alpha, \beta \) does there exist a noncompact finite volume ball quotient with \( b_1(M) = \alpha \) and \( \mathcal{E}(M) = \beta \)?* It would be interesting to find infinitely many ball quotient manifolds with \( b_1 = 0 \) and arbitrarily many cusps, if such examples exist, especially in a tower of finite coverings; it seems that we do not know a single example for which the associated lattice is neat. See [36] for more discussion of this question and the first examples of a manifold \( M \) with \( \mathcal{E}(M) = 1 \).

Our methods also require proving the following result, for which no example exists in the literature.

**Theorem 2.** There exists a torsion-free lattice \( \Gamma \) in \( \text{PU}(2,1) \) and a homomorphism \( \rho : \Gamma \to \mathbb{Z} \) with finitely generated kernel.

Our example is nonuniform. In joint work with Catanese, Keum, and Toledo [6], we will show that the fundamental group of the so-called Cartwright–Steger surface also has this property (in fact, we knew of that example first and the proof of Theorem 2 is modeled on the argument from [6]).

Examples of lattices in \( \text{PU}(2,1) \) admitting homomorphisms onto \( \mathbb{Z} \) with infinitely generated kernel are well-known. Indeed, many quotients of \( \mathbb{B}^2 \) are known to admit a holomorphic fibration over a hyperbolic 2-manifold (see [4]), and this fibration induces the homomorphism to \( \mathbb{Z} \). Since the kernel of the homomorphism from the hyperbolic 2-manifold group to \( \mathbb{Z} \) has infinitely generated kernel, the lattice in \( \text{PU}(2,1) \) inherits the same property. In fact, it follows from work of Napier and Ramachandran [26] that a kernel of a homomorphism to \( \mathbb{Z} \) is infinitely generated if and only if, perhaps after passing to a finite-sheeted covering, there is an associated fibration over a hyperbolic 2-manifold. Studying this condition more carefully, we will show the following.
Theorem 3. Let $\Gamma \subset PU(n, 1)$ be a cocompact arithmetic lattice commensurable with $G(\mathbb{Z})$, where $G$ is the algebraic group defined by a hermitian form on $\ell^{n+1}$ for a CM field $\ell$. Then there exists $\Gamma' \subset \Gamma$ of finite index and $\rho : \Gamma' \to \mathbb{Z}$ with finitely generated kernel.

The proof of Theorem 3 is inspired by an alternate approach to Theorem 2 via Albanese mappings suggested by a referee. We use a result of Clozel [10] to ensure that $\mathbb{B}^n/\Gamma'$ is a manifold with holomorphic 1-forms $\eta, \sigma$ with $\eta \wedge \sigma \neq 0$. This implies via [29] that the subspaces of $H^1_{\text{et}}$ arising from homomorphisms to $\mathbb{Z}$ with finitely generated kernels are a finite union of proper subspaces. The existence of $\rho$ follows, though we cannot say much about finding a specific $\rho$. In contrast, Theorem 2 has a completely explicit proof that studies a fibration over an elliptic curve.

Furthermore, Theorem 3 has the following immediate consequence, for which we could not find a reference for $n \geq 4$.

Corollary 4. For every $n \geq 2$, there exists a discrete, finitely generated, infinite covolume (i.e., geometrically infinite) subgroup of $PU(n, 1)$ with limit set the entire ideal boundary $\partial \mathbb{B}^n$ of $\mathbb{B}^n$.

When $M = \mathbb{B}^2/\Gamma$ is closed (that is, $\mathcal{L}(M) = 0$) it is well-known to experts that there are towers $\{M_j\}$ for which $b_1(M_j)$ is identically zero and towers for which $b_1(M_j)$ grows linearly in $j$, thus linearly in volume. Examples of towers with $b_1$ identically 0 follow from work of Rogawski [34], and are quotients of the ball by a certain family of so-called congruence arithmetic lattices (see §2.2). For quotients of $\mathbb{B}^n$ for $n \geq 3$, a similar result follows from work of Clozel [9]. On the other hand, Marshall [24] showed that $b_1(M_j) \ll \text{vol}(M_j)$ for all principal congruence arithmetic quotients of $\mathbb{B}^2$ and exhibited towers, incommensurable with Rogawski’s, with precisely this growth type. We will prove the following, which for $n = 2$ reproduces Marshall’s lower bound by geometric methods and proves the $b_1$ part Theorem 1(4).

Theorem 5. For $n = 2, 3$, there exists an arithmetic quotient $M_0$ of $\mathbb{B}^n$ and a tower of congruence coverings $\{M_j\}$ of $M_0$ such that

$$\text{vol}(M_j)^{\frac{n}{n+1}} \ll b_1(M_j).$$

One can take $M_0$ to be compact or noncompact.

We prove this result by constructing towers of retractions of ball quotients onto totally geodesic complex curves. This method would imply that Theorem 5 holds for all $n$ if there exist examples of retractions of this kind for $n > 3$, but we know no examples. For $n = 3$, our result is not optimal, as Cossutta has a lower bound of $\text{vol}(M_j)^{\frac{n}{n+2}}$ [14], and it would be interesting to achieve that lower bound geometrically. Our retractions are the ones described by Deraux [17]. Simon Marshall informed us that the correct upper bound from endoscopy should be $b_1(M_j) \ll \text{vol}(M_j)^{\frac{n}{n+1}}$ for all $n \geq 2$, and it would be very interesting to find the optimal growth rate and give a geometric interpretation.
We now describe the organization of the paper. In §2 we briefly give some preliminaries on ball quotients and arithmetic groups. In §3 we describe Hirzebruch’s ball quotient [19] and build the families \{A_j\} and \{B_j\} from Theorem 1. In the process, we prove Theorem 2. We then prove Theorem 3 in §4. In §5 we construct the families \{C_j\} and \{D_j\} and prove Theorem 5. In §6 we make some closing remarks and raise some questions.

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2 Preliminaries

2.1 The ball and its quotients

Let B^n denote the unit ball in \mathbb{C}^n with its Bergman metric. See [18] for more on the geometry of B^n. The holomorphic isometry group of B^n is PU(n, 1), and finite volume manifold quotients of B^n are quotients by torsion-free lattices in PU(n, 1). It is well-known that closed (resp. finite volume noncompact) manifold quotients of B^n are smooth projective (resp. quasiprojective) varieties. We also note that B^1 is the Poincaré disk, and so the manifold quotients of B^1 are precisely the finite volume hyperbolic 2-manifolds.

For what follows, let \Gamma \subset PU(n, 1) be a torsion-free lattice and \bar{M} = B^n/\Gamma. Since \mathbb{B}^n has negative sectional curvature (in fact, constant holomorphic sectional curvature −1), \bar{M} has a finite number E(\bar{M}) \geq 0 of topological ends. When \bar{M} is noncompact, each end of \bar{M} is homeomorphic to N × [0, \infty), where N is an infranil (n − 1)-manifold (see e.g., [25]).

The lattice \Gamma is called neat if the subgroup of \mathbb{C}^* generated by its eigenvalues is torsion-free. In particular, a neat lattice is torsion-free. It is well-known that every lattice contains a neat subgroup of finite index. Suppose that \Gamma is a neat lattice and \bar{M} = \mathbb{B}^n/\Gamma is noncompact. Then \bar{M} admits a smooth toroidal compactification by (n − 1)-dimensional abelian varieties generalizing the usual compactification of a punctured Riemann surface by a finite set of points [2] [20].

2.2 Arithmetic lattices

One method of constructing lattices in PU(n, 1) (in fact, the only method that is known to work for n \geq 4) is via arithmetic subgroups of algebraic groups. Let G be a \mathbb{Q}-algebraic group with

\[ G(\mathbb{R}) \cong SU(n, 1) \times SU(n + 1)^r \]

and \Gamma the image in PU(n, 1) of G(\mathbb{Z}) after projection onto the SU(n, 1) factor of G(\mathbb{R}) followed by projection of SU(n, 1) onto PU(n, 1). Then \Gamma is a lattice.
in \( \operatorname{PU}(n, 1) \). It is known that \( \mathbb{B}^n/\Gamma \) is noncompact if and only if \( G \) is the special unitary group of a hermitian form of signature \((n, 1)\) over an imaginary quadratic field.

For any natural number \( N \), we have the usual reduction homomorphisms
\[
r_N : G(\mathbb{Z}) \to G(\mathbb{Z}/N\mathbb{Z})
\]
arising from inclusion of \( G(\mathbb{Z}) \) into \( \text{GL}_m(\mathbb{Z}) \) for some \( m \) and reducing matrix entries modulo \( N \). Let \( K_N \) be the kernel of \( r_N \) and \( \Gamma(N) \) the image of \( K_N \) in \( \Gamma = \Gamma(1) \). We call \( \Gamma(N) \) a principal congruence kernel. Any lattice \( \Delta \) in \( \operatorname{PU}(n, 1) \) containing some \( \Gamma(N) \) as a subgroup of finite index is called a congruence arithmetic lattice. We also note that \( \Gamma(N) \) is neat for all sufficiently large \( N \).

It is well-known that there exist arithmetic lattices in \( \operatorname{PU}(n, 1) \) that are not congruence arithmetic. Given an arbitrary arithmetic lattice \( \Lambda \), commensurable with \( \Gamma = \Gamma(1) \) as above, we define congruence subgroups of \( \Lambda \) by
\[
\Lambda(N) = \Lambda \cap \Gamma(N).
\]
In particular, we can talk of the family of congruence covers of any arithmetic quotient of \( \mathbb{B}^n \).

**Example 6.** We recall the construction of the arithmetic lattices of \( \operatorname{PU}(n, 1) \) of ‘simplest type’. In fact, all lattices appearing in this paper are of this kind. Let \( k \) be a totally real field and \( \ell \) a totally imaginary quadratic extension of \( k \). We call \( \ell \) a CM field. We denote the nontrivial element of \( \text{Gal}(\ell/k) \) by \( z \mapsto \overline{z} \), since this action extends to complex conjugation at any extension of a real embedding of \( k \) to a complex embedding of \( \ell \).

Let \( h \) be a nondegenerate hermitian form on \( \ell^{n+1} \) with respect to the \( \text{Gal}(\ell/k) \)-action. We obtain a \( \mathbb{Q} \)-algebraic group \( G \) such that
\[
G(\mathbb{Q}) = \{ g \in \text{PGL}_{n+1}(\ell) : \overline{g}hg = h \},
\]
where \( \overline{g} \) is the \( \text{Gal}(\ell/k) \)-conjugate transpose. For every real embedding \( \nu : k \to \mathbb{R} \), \( h \) extends to a hermitian form \( h'_\nu \) on \( \mathbb{C}^{n+1} \) whose signature is independent of the two complex embeddings of \( \ell \) that extend \( \nu \), and
\[
G(\mathbb{R}) \cong \prod_{\nu : k \to \mathbb{R}} \operatorname{PU}(h'_\nu).
\]
We assume that there is a fixed \( \nu_1 \) such that \( \operatorname{PU}(h'_{\nu_1}) \cong \operatorname{PU}(n, 1) \) and that \( \operatorname{PU}(h'_\nu) \cong \operatorname{PU}(n + 1) \) for all \( \nu \neq \nu_1 \). Then projection onto the \( \operatorname{PU}(n, 1) \) factor embeds \( G(\mathbb{Z}) \) as a lattice which is cocompact if and only if \( k \neq \mathbb{Q} \).

### 3 Hirzebruch’s ball quotient

The following construction is due to Hirzebruch \[19\]. Let \( \zeta = e^{\pi i/3} \) and \( \Lambda = \mathbb{Z}[\zeta] \). Then \( E = \mathbb{C}/\Lambda \) is the elliptic curve of \( j \)-invariant 0, and we let \( S \) denote the
abelian surface $E \times E$. Take coordinates $[z, w]$ on $S$ and consider the following elliptic curves on $S$:

- $C_0 = \{w = 0\}$
- $C_{\infty} = \{z = 0\}$
- $C_1 = \{z = w\}$
- $C_\zeta = \{w = \zeta z\}$

Then $C^2_\alpha = 0$ for each $\alpha$ and $C_\alpha \cap C_\beta = \{(0, 0)\}$ for every $\alpha \neq \beta$.

Consider the blowup $\tilde{S}$ of $S$ at $[0, 0]$ with exceptional divisor $D$, and let $\tilde{C}_\alpha$ be the proper transform of $C_\alpha$ to $\tilde{S}$. Then $\tilde{C}_\alpha$ is an elliptic curve on $\tilde{S}$ of self-intersection $-1$. If $C = \bigcup_{\alpha \in \{0, \infty, 1, \zeta\}} \tilde{C}_\alpha$, define $M = \tilde{S} \setminus C$. Hirzebruch conjectured and Holzapfel proved [20] that $M$ is the quotient of $B^2$ by an arithmetic lattice $\Gamma$. It is noncompact and $\Gamma$ is neat. In other words, $\tilde{S}$ is the smooth toroidal compactification of the quotient of $B^2$ by a neat arithmetic lattice. It appears as the third example in the appendix to [37] and in [33]. It is also one of the five complex hyperbolic manifolds of Euler number one that admits a smooth toroidal compactification [8].

We need a few more facts before constructing the families $\{A_j\}$ and $\{B_j\}$ from Theorem 1. Since the fundamental group is a birational invariant, $\pi_1(\tilde{S}) \cong \pi_1(S) \cong \mathbb{Z}^4$.

In particular, every étale cover $\pi : S' \to S$ induces an étale cover $\tilde{\pi} : \tilde{S}' \to \tilde{S}$, and more specifically, $\tilde{S}'$ is the blowup of $S'$ at the points over $[0, 0] \in S$. Every such cover is Galois (i.e., regular) with covering group $G$, a quotient of $\mathbb{Z}^4$.

Let $\tilde{\pi} : \tilde{S}' \to \tilde{S}$ be a finite étale cover with Galois group $G$ and degree $d$, so $\tilde{S}'$ is an abelian surface blown up at $d$ points. The inclusion $M \hookrightarrow \tilde{S}$ induces a surjection $\pi_1(M) \to \pi_1(\tilde{S})$ [23, Prop. 2.10], and in fact $H^1_{L^2}(M, \mathbb{C}) \cong H^1(\tilde{S}, \mathbb{C})$ (see [28]), where $H^1_{L^2}$ denotes $L^2$ cohomology. The inverse image $N = \tilde{\pi}^{-1}(M)$ of $M$ in $\tilde{S}'$ is a étale cover $N$ of $M$, and the number of connected components of $\tilde{\pi}^{-1}(M)$ is the index of $\Gamma = \pi_1(M)$ in $G$ under the induced homomorphism $\pi_1(M) \to \pi_1(\tilde{S}) \to G$.

This homomorphism is surjective, so $N$ is a connected covering of $M$. Lastly, if $D = \tilde{\pi}^{-1}(C)$ is the pullback of $C$ to $\tilde{S}'$, then $N = \tilde{S}' \setminus D$ and $E(N)$ is the number of connected components of $D$. In other words, $N$ is the quotient of the ball by a neat lattice for which the associated smooth toroidal compactification is an abelian surface blown up at $d$ points.

In order to compute $b_1$ for these coverings, we need the following two lemmas. The first is due to Nori.
Lemma 7 (Nori [31]). Let \( X \) and \( Y \) be smooth connected varieties over \( \mathbb{C} \) and \( f : X \to Y \) an arbitrary morphism. Then:

A. there is a nonempty Zariski-open \( U \subset Y \) such that \( f^{-1}(U) \to U \) is a fiber bundle in the usual topology.

B. if \( f \) is dominant, the image of \( \pi_1(X) \) has finite index in \( \pi_1(Y) \).

C. if the general fiber \( F \) of \( f \) is connected and there is a codimension two subset \( S \) of \( Y \) outside which all the fibers of \( f \) have at least one smooth point (i.e., \( f^{-1}(p) \) is generically reduced on at least one irreducible component of \( f^{-1}(p) \), for all \( p \notin S \)), then

\[
\pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to 1
\]

is exact.

We also need the following easy lemma from group theory.

Lemma 8. Let \( \Gamma \) be a finitely generated group, \( r \geq 1 \), and \( \rho : \Gamma \to \mathbb{Z}^r \) a homomorphism with finitely generated kernel \( K \). If \( K \) can be generated by \( n \) elements, then every finite index subgroup \( \Gamma' \subseteq \Gamma \) containing \( K \) has abelianization of rank at most \( n + r \).

Proof. Let \( \sigma_1, \ldots, \sigma_n \) be generators for \( K \). Then \( \rho(\Gamma') \subseteq \mathbb{Z}^r \) is a free abelian subgroup, and is hence generated by at most \( r \) elements. Let \( t_1, \ldots, t_s \) (\( s \leq r \)) be generators for \( \rho(\Gamma') \). Then we can lift \( t_i \) to \( \tilde{t}_i \in \Gamma' \), and the \( \tilde{t}_i \) along with \( \sigma_1, \ldots, \sigma_n \) generate \( \Gamma' \), hence \( \Gamma' \) can be generated by \( n + r \) elements. The cardinality of a generating set is an obvious upper bound for the rank of the abelianization of \( \Gamma' \), so the lemma follows. \( \square \)

Finally, we need some notation for \( \pi_1(S) \). Write \( \pi_1(S) \) as the group of translations of \( \mathbb{C}^2 \) generated by:

\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} & v_3 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
v_2 &= \begin{pmatrix} \zeta \\ 0 \end{pmatrix} & v_4 &= \begin{pmatrix} 0 \\ \zeta \end{pmatrix}
\end{align*}
\]

Under the natural inclusions we then have:

\[
\begin{align*}
\pi_1(C_0) &= \langle v_1, v_2 \rangle \\
\pi_1(C_\infty) &= \langle v_3, v_4 \rangle \\
\pi_1(C_1) &= \langle v_1 + v_3, v_2 + v_4 \rangle \\
\pi_1(C_\zeta) &= \langle v_1 + v_4, v_2 - v_3 + v_4 \rangle
\end{align*}
\]

We now have all the tools necessary to build the families \( \{A_j\} \) and \( \{B_j\} \). Notice that, since \( M \) is the quotient of the ball by a neat arithmetic lattice, any finite coverings will also have these properties.
The towers \( \{ A_j \} \)

Fix a prime \( p \) and consider the homomorphisms

\[
\rho_j : \pi_1(S) \cong \mathbb{Z}^4 \to P_j = \mathbb{Z}/p^j \mathbb{Z}
\]

such that \( \rho_j(v_1) \) is a generator \( \delta_j \) for \( P_j \) and \( \rho_j(v_k) \) is trivial for \( k \neq 1 \). By the above discussion, this induces a connected \( p^j \)-fold cyclic cover \( A_j \) of \( M \).

The restriction of \( \rho_j \) to \( \pi_1(C_{\alpha}) \) is surjective for \( \alpha \in \{ 0, 1, \zeta \} \), so the cusp of \( M \) associated with \( C_{\alpha} \) lifts to a single cusp of \( A_j \). However, \( \pi_1(C_{\infty}) \) is contained in the kernel of \( \rho_j \), so the cusp of \( M \) associated with \( \hat{C}_{\infty} \) lifts to \( p^j \) distinct cusps. Therefore, \( E(A_j) = p^j + 3 \).

We now must show that the betti numbers of the \( A_j \) are uniformly bounded. Notice that each \( \rho_j \) factors through the surjective homomorphism \( \psi : \pi_1(M) \to \mathbb{Z}^2 \) induced by the map \( \tilde{S} \to E \) given by blowdown followed projection of \( S \) onto its first factor. By Lemma 8 it suffices to show that \( \psi \) has finitely generated kernel. In other words, we must prove the following.

**Proposition 9.** Let \( E = \mathbb{C}/\mathbb{Z}[e^{2\pi i/3}] \), \( S = E \times E \), \( \tilde{S} \) be the blowup of \( S \) at \( [0, 0] \), and \( M \subset \tilde{S} \) be Hirzebruch’s noncompact ball quotient. Consider the holomorphic map \( M \to E \) induced by blowdown \( \tilde{S} \to S \) followed by projection of \( S \) onto its first factor. Then the induced surjective homomorphism

\[
\psi : \pi_1(M) \to \pi_1(E) \cong \mathbb{Z}^2
\]

has finitely generated kernel.

**Proof.** Notice that the general fiber \( F \) of the map \( M \to E \) is a reduced 3-punctured elliptic curve. Indeed, \( F \) intersects each \( C_\alpha \) exactly once for \( \alpha \in \{ 0, 1, \zeta \} \). The only fiber not of this type is the fiber above 0, which is a 4-punctured \( \mathbb{P}^1 \), namely the intersection of \( M \) with the exceptional divisor \( D \) of the blowup \( \tilde{S} \to S \). See Figure 1. This satisfies the assumptions of Lemma 7 with \( X = M \) and \( Y = E \), so there is an exact sequence

\[
\pi_1(F) \to \pi_1(M) \xrightarrow{\psi} \pi_1(E) \to 1.
\]

In particular, \( \pi_1(F) \) maps onto \( \ker(\psi) \). Since \( \pi_1(F) \) is finitely generated, the proposition follows.

Choosing the \( \delta_j \) to be compatible with a family of homomorphisms \( P_{j+1} \to P_j \), the \( \{ A_j \} \) form a tower of coverings, which proves the first part of Theorem 1. The associated lattices in \( PU(2,1) \) are neat and arithmetic. Note that the \( A_j \) all have \( L^2 \) first betti number exactly 4. The above also proves Theorem 2.

**Proof of Theorem 2.** Combine Proposition 9 with Lemma 8.

8
The towers \( \{B_j\} \)

Let \( p \) be an odd prime, and fix a generator \( \delta_j \) for \( P_j = \mathbb{Z}/p^j\mathbb{Z} \) (compatibly for a family of homomorphisms \( P_{j+1} \to P_j \)). Since \( p \) is odd, notice that \( \delta_j^2 \) is also a generator for \( P_j \). We then define homomorphisms \( \rho_j : \pi_1(S) \to P_j \) by

\[
\rho_j(v_i) = \delta_j, \quad i = 1, 2, 3, 4.
\]

Let \( \{B_j\} \) be the associated tower of \( p^j \)-fold cyclic covers of \( M \). Here, the restriction of \( \rho_j \) to \( \pi_1(C_\alpha) \) is surjective for every \( \alpha \), which implies each \( \tilde{C}_\alpha \) has a unique lift to the associated covering, and hence \( \mathcal{E}(B_j) = 4 \).

Each \( \rho_j \) factors through homomorphism \( \pi_1(S) \to \mathbb{Z}^2 \) induced by the holomorphic mapping

\[
f([z, w]) = z + w
\]

from \( S \) onto \( E \). The general fiber of \( f \) above \( w \in E \) is the curve

\[
F_w = C_{-1} + [0, w]
\]

where \( C_{-1} \) is the curve in coordinates \([z, w]\) on \( S \) described by \( \{w = -z\} \). Taking the proper transform \( \tilde{F}_w \) of \( F_w \) to \( \tilde{S} \), we see that for \( w \neq 0 \), the induced map \( M \to E \) has fiber above \( w \) a punctured elliptic curve, i.e., \( \tilde{F}_w \) minus its points of intersection with each \( \tilde{C}_\alpha \), \( \alpha \in \{0, \infty, 1, \zeta\} \). Above 0, the fiber is the union of \( D \cap M \) and \( \tilde{F}_0 \) minus its intersection with each of the \( \tilde{C}_\alpha \). Each fiber is again reduced.

Lemma\(^7\) again applies, so the kernel of the induced homomorphism from \( \pi_1(M) \) to \( \mathbb{Z}^2 \) has finitely generated kernel. Then \( b_1(B_j) \) is uniformly bounded above by Lemma\(^8\). This proves the second part of Theorem\(^1\) and again the fundamental groups of our examples are neat and arithmetic.
4 The proof of Theorem \textcircled{3}

We begin with the following proposition, which is well-known to experts.

**Proposition 10.** Suppose that $V$ is a compact complex manifold that admits two holomorphic $1$-forms with nonzero wedge product. Given $\Lambda \subset \text{PSL}_2(\mathbb{R})$ a cocompact lattice with $\mathcal{O} = \mathbb{H}^2/\Lambda$ the associated closed hyperbolic $2$-orbifold and $f : V \to \mathcal{O}$ a holomorphic mapping with connected fibers, define $W_f = f^*(H^{1,0}(\mathcal{O})) \subset H^{1,0}(V)$ to be the pull-backs to $V$ of holomorphic $1$-forms on $\mathcal{O}$. Finally, define

$$Z = \{W_f \subset H^{1,0}(M) : f : V \to \mathcal{O} \text{ as above}\}.$$ 

Then $Z$ is a finite union of proper linear subspaces of $H^{1,0}(V)$.

**Proof.** It is clear that $Z$ is a union of linear subspaces that are maximal isotropic subspaces of $H^{1,0}(V)$ for the wedge product. A version of the Castelnuovo–de Franchis theorem due to Catanese \cite{7, Thm. 1.10} implies that every maximal isotropic subspace $Z \in Z$ of dimension at least two determines a holomorphic mapping $f$ from $V$ onto a curve $C_Z$ of genus $g \geq 2$ and $Z = f^*(H^{1,0}(C))$. In particular, since $V$ supports two 1-forms with nonzero wedge product, every such $Z \in Z$ is a proper subspace. The $Z_f \in Z$ associated with mappings onto hyperbolic $2$-orbifolds of genus $1$ are lines, and hence are also proper subspaces. The proposition follows immediately from the finiteness of the set of maps $f$ from $V$ onto a hyperbolic $2$-orbifold \cite{13, 16}. \hfill \Box

We now prove Theorem \textcircled{3}

**Proof of Theorem \textcircled{3.**} Let $\Gamma \subset \text{PU}(n, 1)$ be a lattice satisfying the assumptions of the theorem. It is well-known theorem of Kazhdan that we can replace $\Gamma$ with a torsion-free subgroup of finite index with infinite abelianization \cite{21}. See also \cite{4}. Then $\mathbb{B}^n/\Gamma$ is an $n$-dimensional smooth complex projective variety (also a compact Kähler manifold) that supports a nontrivial holomorphic $1$-form $\eta$.

Using the action of Hecke operators, Clozel showed that there is $\Gamma' \subset \Gamma$ of finite index and nontrivial holomorphic $1$-forms $\eta, \sigma$ on $V = \mathbb{B}^n/\Gamma'$ such that $\eta \wedge \sigma \neq 0$ \cite{10, Prop. 3.2}. In particular, Proposition \textcircled{10} applies to $V$.

Associated with every nonzero $\eta \in H^{1,0}(V)$ there is a homomorphism

$$\rho_{\eta} : \Gamma' \to \mathbb{Z}.$$ 

If $\ker(\rho_{\eta})$ is not finitely generated, it follows from work of Napier–Ramachandran that there exists a holomorphic map $f : V \to C$ onto a curve of genus $g \geq 1$ such that $\eta \in f^*(H^{1,0}(C))$ \cite{29, Thm. 4.3}. Indeed, if $g \geq 2$, then $\eta$ clearly lies in the set $Z$ defined in the statement of Proposition \textcircled{10}. However, this still holds when $C$ has genus $1$ since the induced homomorphism

$$f_* : \Gamma' \to \pi_1(C) \cong \mathbb{Z}^2$$
factors through a surjective homomorphism $\Gamma' \to \Lambda$, where $\Lambda$ is a cocompact lattice in $\text{PSL}_2(\mathbb{R})$ (cf. [30]), so again $\eta \in Z$.

In particular, if $\eta \notin Z$ then $\rho_\eta$ has finitely generated kernel. Since $Z$ is a finite union of proper subspaces of $H^{1,0}(V)$, the existence of such an $\eta$ is immediate. This completes the proof.

**Remark.** Note that the 1-forms $\eta$ and $\sigma$ with $\eta \wedge \sigma \neq 0$ might not be a 1-form associated with a finitely generated kernel. Indeed, both $\eta$ and $\sigma$ might lie in distinct linear subspaces contained in $Z$.

**Remark.** Using quasiprojective analogues of the above arguments, we expect that Theorem 3 also holds for nonuniform arithmetic lattices in $\text{PU}(n,1)$.

## 5 Deligne–Mostow orbifolds

To build the towers $\{C_j\}$ and $\{D_j\}$ and prove Theorem 5, we use the ball quotient orbifolds constructed by Deligne and Mostow [15, 27]. Given an $(n+3)$-tuple $\mu = (\mu_i)$ of rational numbers, we consider the following condition:

\[
(1 - \mu_i - \mu_j)^{-1} \in Z \text{ when } i \neq j \text{ and } \mu_i + \mu_j < 1 \tag{INT}
\]

For any such $\mu$, Deligne and Mostow produced a finite volume ball quotient orbifold $O_\mu$ by a (sometimes partial) compactification of the space of $n + 3$ distinct points on $\mathbb{P}^1$. In other words, $O_\mu$ is the quotient of $\mathbb{B}^n$ by a lattice $\Lambda_\mu$ that contains elements of finite order. Mostow then showed that one can relax (INT) to a half-integral condition $\frac{1}{2}(\text{INT})$ that produces new orbifolds. We refer to [15, 27, 22, 39, 17] for more on their geometry.

First, we need to recall when there are totally geodesic inclusions $O_\nu \hookrightarrow O_\mu$ between $m$- and $n$-dimensional Deligne–Mostow orbifolds. Let $\mu = (\mu_i)$ be an $(n+3)$-tuple satisfying (INT), and (after reordering the $\mu_i$ if necessary), suppose that

\[
\sum_{j=m+3}^{n+3} \mu_j < 1.
\]

We then define the **hyperbolic contraction**

\[
\nu = (\mu_1, \ldots, \mu_{m+2}, \sum_{j=m+3}^{n+3} \mu_j),
\]

which also satisfies (INT). The associated inclusion of the space of $m+3$ distinct points on $\mathbb{P}^1$ into the space of $n+3$ distinct points then induces a totally geodesic embedding

\[
O_\nu = \mathbb{B}^m / \Lambda_\nu \hookrightarrow O_\mu = \mathbb{B}^n / \Lambda_\mu.
\]

There is a similar statement for $\frac{1}{2}(\text{INT})$. See [15 §8] or [17].

Crucial to our construction are so-called **forgetful maps**, which were studied in detail in [17]. Rather than inclusions of moduli spaces, we now consider
when the surjective holomorphic map of moduli spaces given by forgetting some number of points induces a surjective holomorphic mapping between Deligne–Mostow orbifolds. Deraux [17] gave a complete treatment of how one determines the pairs $\mu, \nu$ for which this induces a surjective holomorphic mapping $O_\mu \rightarrow O_\nu$, and we refer the reader there for details.

What we exploit is that there are many instances where one can find a pair $(\mu, \nu)$ such that $O_\nu$ is both a geodesic suborbifold and a quotient space of $O_\mu$, and the associated composition

$$O_\nu \hookrightarrow O_\mu \rightarrow O_\nu$$

is the identity. In other words, $O_\mu$ admits a holomorphic retraction onto $O_\nu$.

Let $X$ be a connected manifold, $Y$ a submanifold, and $f : X \rightarrow Y$ be a retraction. The induced map on fundamental groups $f_\ast$ then has $f_\ast|_{\pi_1(Y)} = \text{id}_{\pi_1(Y)}$. From this, one can easily prove the following by elementary covering space theory.

**Lemma 11.** Let $X$ be a connected manifold, $Y$ a submanifold, and $f : X \rightarrow Y$ a continuous retraction. Given a finite sheeted covering $X' \rightarrow X$, let $Y'$ be the finite covering of $Y$ associated with the finite index subgroup $f_\ast(\pi_1(X'))$ of $\pi_1(Y)$. Then $Y'$ lifts to a submanifold of $X'$, and $X'$ admits a retraction onto $Y'$.

Notice that it follows immediately from the lemma that $b_1(X') \geq b_1(Y')$. We now explain how one constructs the families $\{C_j\}$ and $\{D_j\}$.

**The towers $\{C_j\}$**

Let $O_\mu = \mathbb{B}^2/\Gamma_\mu$ be a cusped Deligne–Mostow orbifold and $f : O_\mu \rightarrow O_\nu$ be a surjective holomorphic mapping onto $O_\nu = \mathbb{B}^1/\Gamma_\nu$. For example, one can take

$$\mu = \left(\begin{array}{cccc}
2 & 2 & 3 & 4 \\
6 & 6 & 6 & 6
\end{array}\right)$$

$$\nu = \left(\begin{array}{cccc}
1 & 3 & 4 & 4 \\
6 & 6 & 6 & 6
\end{array}\right)$$

While $O_\mu$ has cusps, we note that $O_\nu$ is compact (as it turns out, the target is always compact when the dimensions differ).

Then, we can find a neat subgroups $\Gamma \subset \Gamma_\mu$ and $\Delta \subset \Gamma_\nu$ of finite index for which we have a surjective holomorphic mapping

$$h : C_0 = \mathbb{B}^2/\Gamma \rightarrow \Sigma = \mathbb{B}^1/\Delta$$

for which the induced map $h_* : \Gamma \rightarrow \Delta$ is also surjective. Note that $\Sigma$ is a compact Riemann surface of genus $g \geq 2$.

Fix a cusp of $C_0$ and let $P \subset \Gamma$ be a representative for the conjugacy class of maximal parabolic subgroups of $\Gamma$ associated with this cusp. We claim that $h_*(P)$ is a (possibly trivial) cyclic subgroup of $\Sigma$. Indeed, $P$ is nilpotent, so
$h_\ast(P)$ is also nilpotent. However, $\Sigma$ is a hyperbolic surface group, so every nilpotent subgroup is cyclic.

Given a finite covering $\Sigma' \to \Sigma$, Galois with group $G$, recall that the number of cusps of $C'$ over our chosen cusp is exactly the index of $P$ in $G$ under the associated surjection $\Gamma \to G$. Since $b_1(\Sigma) \geq 4$ and $h_\ast(P)$ is cyclic, we can find a homomorphism $\rho : \Delta \to \Z$ with $h_\ast(P) \subset \ker(\rho)$. Let $\rho_j$ be the composition of $\rho$ with reduction modulo $j$ and $\Delta_j$ the kernel of $\rho_j$.

Set:

$$\Sigma_j = \Bbb{B}^1/\Delta_j$$
$$C_j = \Bbb{B}^2/h_\ast^{-1}(\Delta_j)$$

Then $\Sigma_j$ (resp. $C_j$) is a covering of $\Sigma$ (resp. $C_0$) of degree $j$. Furthermore, $b_1(\Sigma_j)$ grows linearly in $j$, hence $b_1(C_j)$ does as well. Since $P \subset h_\ast^{-1}(\Delta_j)$, we see that the cusp associated with $P$ lifts to $j$ cusps of $C_j$. In particular, $E(C_j)$ also grows linearly in $j$. Therefore $\{C_j\}$ has the required properties.

**Remark.** Taking

$$\mu = \left( \begin{array}{cccc}
2 & 2 & 3 & 1 \\
6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 \\
\end{array} \right)$$
$$\nu = \left( \begin{array}{cccc}
1 & 3 & 4 & 4 \\
6 & 6 & 6 & 6 \\
\end{array} \right)$$

one also sees that there exist quotients of $\Bbb{B}^3$ for which Theorem 13 holds.

**Theorem 5 and the towers $\{D_j\}$**

Let $D_0 = C_0 = \Bbb{B}^2/\Gamma$ and $\Sigma = \Bbb{B}^1/\Delta$ be as above, and notice that our map is, in fact, a retraction onto a geodesic submanifold. We now let $D_j = \Bbb{B}^2/\Gamma(j)$ be the congruence covering of $D_0$ of level $j$ as in §2.2. As is well-known, $\Delta$ is also arithmetic and

$$\Delta(j) = \Delta \cap \Gamma(j).$$

It follows from Lemma 11 that $b_1(D_j) \geq b_1(\Sigma_j)$, where $\Sigma_j$ is now defined to be the congruence covering $\Bbb{B}^1/\Delta(j)$ of $\Sigma$.

Since $\# \text{PSL}_2(\Z/j\Z) \sim j^3$, using strong approximation for Zariski-dense subgroups [32], one can show that $b_1(\Sigma_j)$ grows like $j^3$, hence $b_1(D_j)$ grows at least like $j^3$. However, $G(\Z/j\Z)$ has order $j^8$, so a similar strong approximation argument shows that

$$\text{vol}(D_j) \sim [\Gamma : \Gamma(j)] \sim \# G(\Z/j\Z) \sim j^8,$$

and we see that

$$\text{vol}(D_j)^{\frac{1}{2}} \ll b_1(D_j).$$

Notice that if we take $D_j$ to be the quotient of $\Bbb{B}^2$ by a congruence arithmetic lattice, an upper bound with the same exponent follows from [24].
To calculate $E(D_j)$, we need to compute the index in $G(\mathbb{Z}/j\mathbb{Z})$ of a given maximal parabolic subgroup of $\Gamma$. If $P$ is a maximal parabolic subgroup of $\Gamma$ then its image in $\Gamma/\Gamma(j)$ is contained in some Borel subgroup, but since the diagonal entries of parabolic matrices in $\Gamma$ are units of a fixed imaginary quadratic number field, we see that their images in $G(\mathbb{Z}/j\mathbb{Z})$ are, up to a fixed multiplicative factor, contained in the subgroup of strictly upper-triangular matrices. Elementary counting implies that this index is of order $j^5$. As argued above for $b_1$, it follows that $E(D_j) \sim \text{vol}(D_j)^{5/8}$. This completes the proof for noncompact quotients when $n = 2$.

Applying the same argument to

\[
\mu = \left( \begin{array}{cccc}
2 & 2 & 3 & 3 \\
6 & 6 & 6 & 6 \\
1 & 1 & 1 & 1
\end{array} \right)
\]

\[
\nu = \left( \begin{array}{cccc}
1 & 3 & 4 & 4 \\
6 & 6 & 6 & 6
\end{array} \right)
\]

gives noncompact examples when $n = 3$, where now $[\Gamma : \Gamma(j)]$ grows like $j^{15}$, so $b_1$ grows at least as fast as $j^{1/5}$. We note, however, that $E$ now grows like $j^{2/5}$.

For compact examples, consider:

\[
\mu_1 = \left( \begin{array}{cccc}
3 & 3 & 3 & 7 \\
8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8
\end{array} \right)
\]

\[
\mu_2 = \left( \begin{array}{cccc}
3 & 3 & 3 & 4 \\
8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8
\end{array} \right)
\]

\[
\mu_3 = \left( \begin{array}{cccc}
1 & 3 & 3 & 3 \\
8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8
\end{array} \right)
\]

By [17] Thm. 3.1(v), there is a holomorphic retraction $f: O_{\mu_3} \to O_{\mu_1}$ induced by a forgetful map. Clearly the restriction of $f$ to $O_{\mu_2}$ is a retraction of it onto $O_{\mu_1}$. These give all the desired examples.

**Remark.** Notice that our lattice $\Gamma$ is not necessarily congruence arithmetic, since the initial Deligne–Mostow lattice $\Gamma_\mu$ might not be congruence arithmetic. One can often show that the quotient $O_{\Sigma_\mu}$ of $O_{\mu}$ by a certain symmetric group is the quotient of the ball by a maximal arithmetic lattice, which is then congruence arithmetic (see [33]). However, we were not able to find an example for which the map $O_{\mu} \to O_{\mu}$ descends to $O_{\Sigma_\mu}$. In particular, for $n = 3$, we do not know that the families $\{M_j\}$ in Theorem 5 are necessarily congruence arithmetic, though we suspect that such examples exist.

**Remark.** Unfortunately, this result is not optimal for $n = 3$, as Cossetta proved the lower bound $\text{vol}(M_j)^{1/4}$ for the growth of $b_1$ [14]. It would be interesting to see if a more subtle use of fibrations over curves could meet, or beat, that bound.

6 Closing questions and remarks

We begin reiterating a question from the introduction.
**Question 1.** Fix $n \geq 2$. For which pairs $(\alpha, \beta)$ is there a smooth finite volume quotient $M = \mathbb{B}^n/\Gamma$ with $b_1(M) = \alpha$ and $\mathcal{E}(M) = \beta$? Can we always assume $\Gamma$ is arithmetic and/or neat?

Particularly interesting is the case $(0, \beta)$ for any $\beta \geq 1$. We do not know of an infinite family of examples, though [37] contains examples for which the lattice is arithmetic but not neat. With the assumption of neatness we apparently do not know a single example.

**Question 2.** Does there exist a manifold quotient $M$ of $\mathbb{B}^n$ with $\mathcal{E}(M) = 1$? For which $n$ can $M$ be the quotient by a neat and/or arithmetic lattice?

When $n = 2$, see [36] for an example of a one-cusped manifold. This manifold is arithmetic, but the lattice is not neat. In an earlier paper [38], we showed that for any $k \geq 1$, there exists a constant $n_k$ such that $\mathcal{E}(M) > k$ for every arithmetic quotient of $\mathbb{B}^n$ with $n \geq n_k$. In particular, for $n$ sufficiently large, if one-cusped quotients of $\mathbb{B}^n$ exist, they cannot be arithmetic.

**Question 3.** Let $M$ be a finite volume quotient of $\mathbb{B}^n$ and $\{M_j\}$ a tower of finite-sheeted coverings. What are the possible growth rates for $b_1(M_j)$?

As mentioned in the introduction, we know examples for all $n \geq 2$ where $b_1(M_j)$ is identically zero [34, 9]. There are also examples where the growth is nontrivial [21, 40]. In both families, the lattices can be chosen arithmetic.

**Question 4.** Let $M$ be the quotient of $\mathbb{B}^n$ by a congruence arithmetic lattice, $n \geq 2$, and $\{M_j\}$ a family of congruence coverings. What are the possible growth types for $b_1(M_j)$ as a function of the covering degree?

When $M$ is noncompact, one can always determine the growth rate of $\mathcal{E}(M_j)$ by elementary counting methods. Simon Marshall informed us that endoscopy should give a bound $b_1(M_j) \ll \text{vol}(M_j)^{\frac{n+2}{n+1}}$ for principal congruence lattices. Cossutta has the upper and lower bounds $\frac{n+2}{(n+1)^2}$ and $\frac{n-2}{(n+1)^2}$, respectively. If $M$ retracts onto a holomorphically embedded geodesic submanifold $\mathbb{B}^1/\Lambda$, then the methods in this paper give towers with a lower bound of the form $\frac{3}{n+2}$ (and could do more if the number of distinct retractions increases with $j$), but we do not know a single example of a quotient of $\mathbb{B}^n$, $n \geq 4$, that retracts onto a geodesic suborbifold of any codimension.

**Question 5.** Let $M$ be a quotient of $\mathbb{B}^n$, and suppose that $M$ contains a totally geodesic quotient of $\mathbb{B}^m$ for some $1 \leq m < n$. Does $M$ admit a finite sheeted covering $N \to M$ such that $N$ retracts onto one of its $m$-dimensional geodesic suborbifolds?

For example, the analogous result is always true for hyperbolic 3-manifolds and totally geodesic hyperbolic 2-manifolds [1]. There are also a number of known examples of hyperbolic $n$-manifolds retracting onto totally geodesic hyperbolic $m$-submanifolds [3], though the general case for $n \geq 4$ remains open.
References

[1] I. Agol. The virtual Haken conjecture. Doc. Math., 18:1045–1087, 2013.

[2] A. Ash, D. Mumford, M. Rapoport, and Y. Tai. Smooth compactification of locally symmetric varieties. Math. Sci. Press, 1975. Lie Groups: History, Frontiers and Applications, Vol. IV.

[3] N. Bergeron, F. Haglund, and D. Wise. Hyperplane sections in arithmetic hyperbolic manifolds. J. Lond. Math. Soc. (2), 83(2):431–448, 2011.

[4] Nicolas Bergeron and Laurent Clozel. Spectre automorphe des variétés hyperboliques et applications topologiques. Astérisque, (303), 2005.

[5] F. Calegari and N. Dunfield. Automorphic forms and rational homology 3-spheres. Geom. Topol., 10:295–329 (electronic), 2006.

[6] F. Catanese, J.H. Keum, M. Stover, and D. Toledo. The geometry of the Cartwright–Steeger surface. In progress.

[7] Fabrizio Catanese. Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations. Invent. Math., 104(2):263–289, 1991.

[8] L. F. Di Cerbo and M. Stover. Classification and arithmeticity of smooth toroidal compactifications with $3c_2^2 = c_1^2 = 3$. arXiv:1505.01414.

[9] L. Clozel. On the cohomology of Kottwitz’s arithmetic varieties. Duke Math. J., 72(3):757–795, 1993.

[10] L. Clozel. Produits dans la cohomologie holomorphe des variétés de Shimura. II. Calculs et applications. J. Reine Angew. Math., 444:1–15, 1993.

[11] D. Cooper, D. D. Long, and A. W. Reid. Essential closed surfaces in bounded 3-manifolds. J. Amer. Math. Soc., 10(3):553–563, 1997.

[12] D. Cooper, D. D. Long, and A. W. Reid. On the virtual Betti numbers of arithmetic hyperbolic 3-manifolds. Geom. Topol., 11:2265–2276, 2007.

[13] Kevin Corlette and Carlos Simpson. On the classification of rank-two representations of quasiprojective fundamental groups. Compos. Math., 144(5):1271–1331, 2008.

[14] Mathieu Cossutta. Asymptotique des nombres de Betti des variétés arithmétiques. Duke Math. J., 150(3):443–488, 2009.

[15] P. Deligne and G. D. Mostow. Monodromy of hypergeometric functions and nonlattice integral monodromy. Inst. Hautes Études Sci. Publ. Math., (63):5–89, 1986.
[16] Thomas Delzant. Trees, valuations and the Green-Lazarsfeld set. *Geom. Funct. Anal.*, 18(4):1236–1250, 2008.

[17] M. Deraux. Forgetful maps between Deligne-Mostow ball quotients. *Geom. Dedicata*, 150:377–389, 2011.

[18] W. Goldman. *Complex hyperbolic geometry*. Oxford Mathematical Monographs. Oxford University Press, 1999.

[19] F. Hirzebruch. Chern numbers of algebraic surfaces: an example. *Math. Ann.*, 266(3):351–356, 1984.

[20] R.-P. Holzapfel. Chern numbers of algebraic surfaces—Hirzebruch’s examples are Picard modular surfaces. *Math. Nachr.*, 126:255–273, 1986.

[21] D. Kazhdan. Some applications of the Weil representation. *J. Analyse Mat.*, 32:235–248, 1977.

[22] F. C. Kirwan, R. Lee, and S. H. Weintraub. Quotients of the complex ball by discrete groups. *Pacific J. Math.*, 130(1):115–141, 1987.

[23] J. Kollár. *Shafarevich maps and automorphic forms*. M. B. Porter Lectures. Princeton University Press, 1995.

[24] S. Marshall. Endoscopy and cohomology growth on $U(3)$. *Compos. Math.*, 150(6):903–910, 2014.

[25] D. B. McReynolds. Peripheral separability and cusps of arithmetic hyperbolic orbifolds. *Algebr. Geom. Topol.*, 4:721–755 (electronic), 2004.

[26] N. Mok. Projective algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume. In *Perspectives in analysis, geometry, and topology*, volume 296 of *Progr. Math.*, pages 331–354. Birkhäuser, 2012.

[27] G. D. Mostow. Generalized Picard lattices arising from half-integral conditions. *Inst. Hautes Études Sci. Publ. Math.*, (63):91–106, 1986.

[28] V. Murty and D. Ramakrishnan. The Albanese of unitary Shimura varieties. In *The zeta functions of Picard modular surfaces*, pages 445–464. Univ. Montréal, 1992.

[29] T. Napier and M. Ramachandran. Hyperbolic Kähler manifolds and proper holomorphic mappings to Riemann surfaces. *Geom. Funct. Anal.*, 11(2):382–406, 2001.

[30] Terence Napier and Mohan Ramachandran. Filtered ends, proper holomorphic mappings of Kähler manifolds to Riemann surfaces, and Kähler groups. *Geom. Funct. Anal.*, 17(5):1621–1654, 2008.
[31] M. V. Nori. Zariski’s conjecture and related problems. *Ann. Sci. École Norm. Sup. (4)*, 16(2):305–344, 1983.

[32] M. V. Nori. On subgroups of $\text{GL}_n(F_p)$. *Invent. Math.*, 88(2):257–275, 1987.

[33] J. Parker. On the volumes of cusped, complex hyperbolic manifolds and orbifolds. *Duke Math. J.*, 94(3):433–464, 1998.

[34] J. Rogawski. *Automorphic representations of unitary groups in three variables*, volume 123 of *Annals of Mathematics Studies*. Princeton University Press, 1990.

[35] M. Stover. Moduli of abelian varieties and Deligne–Mostow orbifolds. In preparation.

[36] M. Stover. One-cusped complex hyperbolic 2-manifolds exist. In preparation.

[37] M. Stover. Volumes of Picard modular surfaces. *Proc. Amer. Math. Soc.*, 139(9):3045–3056, 2011.

[38] M. Stover. On the number of ends of rank one locally symmetric spaces. *Geom. Topol.*, 17(2):905–924, 2013.

[39] D. Toledo. Maps between complex hyperbolic surfaces. *Geom. Dedicata*, 97:115–128, 2003.

[40] T. N. Venkataramana. Virtual Betti numbers of compact locally symmetric spaces. *Israel J. Math.*, 166:235–238, 2008.

[41] B. Weisfeiler. Strong approximation for Zariski-dense subgroups of semisimple algebraic groups. *Ann. of Math. (2)*, 120(2):271–315, 1984.