A RESULT ON RELATIVE CONORMALS SPACES

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Abstract. We prove a result on the relationship between the relative conormal space of an analytic function \( f \) on affine space and the relative conormal space of \( f \) restricted to a hyperplane slice, at a point where the relative conormal space of \( f \) is "microlocally trivial".

1. Introduction

Suppose that \( \mathcal{U} \) is a connected, open subset of \( \mathbb{C}^{n+1} \); we use \((z_0, \ldots, z_n)\) as coordinates on \( \mathcal{U} \). Let \( p \in \mathcal{U} \), and let \( f : \mathcal{U} \to \mathbb{C} \) be a non-constant complex analytic function. We will assume, without loss of generality, that \( f(p) = 0 \) and \( z_0(p) = 0 \), i.e., \( p \in V(f,z_0) \).

Consider the cotangent bundle of \( \mathcal{U} \),

\[
\pi : T^* \mathcal{U} \cong \mathcal{U} \times \mathbb{C}^{n+1} \to \mathcal{U}.
\]

For all \( x \in \mathcal{U} \), we use \((d_x z_0, \ldots, d_x z_n)\) as an ordered basis for the fiber \((T^* \mathcal{U})_x := \pi^{-1}(x)\).

The (closure of the) relative conormal space of \( f \) in \( \mathcal{U} \), \( \overline{T^*_f \mathcal{U}} \subseteq T^* \mathcal{U} \), is a well-known object in the study of the singularities of the hypersurface \( V(f) \); see, for instance, [10]. The relative conormal space is given by

\[
\overline{T^*_f \mathcal{U}} := \{(x, \eta) \in T^* \mathcal{U} \mid \eta(\ker(d_x f)) \equiv 0\}.
\]

Note that we have not done what is usually done, in that we have not explicitly removed the critical locus, \( \Sigma_f \), of \( f \) before closing; this does not affect the closure. We remark that each fiber of \( \overline{T^*_f \mathcal{U}} \) over \( \mathcal{U} \) is \( \mathbb{C} \)-conic, i.e., closed under scalar multiplication, but need not be closed under addition over a point in the critical locus of \( f \).

For all \( x \in V(z_0) \), there is a canonical map

\[
\hat{r}_x : (T^* \mathcal{U})_x \to (T^*(V(z_0)))_x
\]

given by

\[
\hat{r}_x(\eta) := \eta|_{T_x V(z_0)},
\]

that is,

\[
\hat{r}_x(a_0 d_x z_0 + \cdots + a_n d_x z_n) = a_1 d_x z_1 + \cdots + a_n d_x z_n.
\]

We prove just one theorem in this paper:

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Theorem 1.1. For all $x \in V(z_0)$, if $d_x z_0 \notin (T^*_f U)_x$, then $r_x$ induces a surjection
\[ r_x : (T^*_f U)_x \rightarrow (T^*_f V(z_0))_x, \]
such that $r_x^{-1}(0) = 0$.

The proofs that $r_x^{-1}(0) = 0$ and that $r_x$ is a surjection are easy, but that everything in $(T^*_f U)_x$ maps by $r_x$ into $(T^*_f V(z_0))_x$ is difficult, and our argument heavily uses the derived category and the microsupport. Our attempts to find a more direct proof, or to find this result in the existing literature, have failed.

And so we must begin by looking at derived category definitions and results.

2. Derived Category Results

We must begin by recalling a large number of definitions and notations.

Consider a (reduced) complex analytic subspace $X$ of some open subset $W$ of some affine space $\mathbb{C}^N$.

We will look at objects in the derived category $D^b(X)$ of bounded, constructible complexes of sheaves of $\mathbb{Z}$-modules on various spaces. References for the notation and results that we will use are [4], [2], [7], [6], and [9].

For a complex submanifold $M \subseteq W$, we let $T^*_M W$ denote that conormal space of $M$ in $W$; this is the subspace of the cotangent space $T^* W$ given by
\[ T^*_M W := \{ (p, \eta) \in T^* W \mid \eta(T_p M) = 0 \}. \]
We will usually be interested in the closure $\overline{T^*_M W}$ in $T^* W$.

Suppose that $A^* \in D^b_c(X)$ and that $\mathcal{S}$ is a Whitney stratification (with connected strata) of $X$ with respect to which $A^*$ is constructible. Then, as described by Goresky and MacPherson [3], to each stratum $S$ in $\mathcal{S}$, there are an associated normal slice $N_S$ and complex link $L_S$. The isomorphism-types of the hypercohomology modules $H^*(N_S, L_S; A^*)$ are independent of the choices made in defining the normal slice and complex link; these are the Morse modules of $S$, with respect to $A^*$. We let $m^S_S(A^*) := H_{k - \dim S}(N_S, L_S; A^*)$.

The union of the closures of conormal spaces to strata with non-zero Morse modules is the microsupport, $SS(A^*)$, of $A^*$, as defined by Kashiwara and Schapira in [4], i.e.,
\[ SS(A^*) := \bigcup_{m^S_S(A^*) \neq 0} \overline{T^*_S W}. \]
(For this characterization of the microsupport, see Theorem 4.13 of [6].) The microsupport is independent of the choice of Whitney stratification.

Now we return to the notation from the introduction: $U$ is a connected, open subset of $\mathbb{C}^{n+1}$, $(z_0, \ldots, z_n)$ are coordinates on $U$, $f : U \rightarrow \mathbb{C}$ is a non-constant complex analytic function, and $p \in V(f, z_0)$. We now fix $P^*$ as being the shifted constant sheaf $\mathbb{Z}^*_U[n+1]$.

We will use the (shifted) nearby and vanishing cycles functors, $\psi_f[-1]$ and $\phi_f[-1]$, respectively, from $D^b_c(U)$ to $D^b_c(V(f))$. Let $F_{f,p}$ denote the Milnor fiber of $f$ at $p$, and let $f_0 := f|_{V(z_0)}$. Let $\xi_0 := \xi_0|_{V(f)}$. 

We define the inclusions
\[ m : V(z_0) \hookrightarrow \mathcal{U}, \quad \ell : \mathcal{U}\setminus V(z_0) \hookrightarrow \mathcal{U}, \text{ and } \tilde{m} : V(f, z_0) \hookrightarrow V(f). \]
Then, \( H^k(\psi_f[-1]\mathbf{P}^*)_p \cong H^{k+n}(F_{f,p}; \mathbb{Z}) \) and \( H^k(\phi_f[-1]\mathbf{P}^*)_p \cong \tilde{H}^{k+n}(F_{f,p}; \mathbb{Z}) \), where \( \tilde{H} \) denotes reduced cohomology. Furthermore,
\[ H^k(\psi_f[-1]\ell_!\mathbf{P}^*)_p \cong H^{k+n}(F_{f,p}, F_{f_0,p}; \mathbb{Z}) \]
and
\[ \tilde{m}^*\psi_f[-1]\mathbf{m}_*\mathbf{m}^*[−1]\mathbf{P}^* \cong \psi_{f_0}[-1]Z_{V(z_0)}[n]. \]

It is a fundamental result of Briançon, Maisonobe, and Merle that:

**Theorem 2.1.** ([1], 3.4.2) 
\[ \text{SS}(\psi_f[-1]\mathbf{P}^*) = T_f^{-1}\mathcal{U} \cap (V(f) \times \mathbb{C}^{n+1}) = T_f^{-1}\mathcal{U} \cap \pi^{-1}\pi^{-1}(0). \]

More generally, if
\[ \text{SS}(\mathcal{A}^*) = \bigcup_{S \in \mathcal{S}} T_S^{-1}\mathcal{U}, \]
then
\[ \text{SS}(\psi_f[-1]\mathbf{A}^*) = \bigcup_{S \in V(f)} T_{f_S}^{-1}\mathcal{U} \cap (V(f) \times \mathbb{C}^{n+1}). \]

(Actually, the statement above does not use the full result of [1], 3.4.2.)

Now, we are finally in a position to state and prove some results.

### 3. The Main Result

**Proposition 3.1.** Suppose that \( p \in V(f, z_0) \) and that \( d_p z_0 \notin (T_f^{-1}\mathcal{U})_p \). Then, near \( p \), we have an isomorphism
\[ \psi_{f_0}[-1]Z_{V(z_0)}[n] \cong \psi_{z_0}[-1]\psi_f[-1]\mathbf{P}^*. \]

**Proof.** We will use Theorem 3.1 of [9]. Since the set of \( x \) such that \( d_x z_0 \in (T_f^{-1}\mathcal{U})_x \) is closed, the fact that \( d_p z_0 \notin (T_f^{-1}\mathcal{U})_p \) implies that, for all \( x \) near \( p \), \( d_x z_0 \notin (T_f^{-1}\mathcal{U})_x \). This implies that the relative polar set \( \Gamma_{f,z_0} \) is empty near \( p \), which implies, by Theorem 3.1 of [9], that, for all \( x \in V(f, z_0) \) near \( p \),
\[ H^*(\psi_f[-1]\ell_!\mathbf{P}^*)_x \cong H^*(F_{f,x}, F_{f_0,x}; \mathbb{Z}) = 0, \]
that is, near \( p \),
\[ \tilde{m}^*\psi_f[-1]\ell_!\mathbf{P}^* = 0. \]

Using this last equality, and applying the functor \( \tilde{m}^*[-1]\psi_f[-1] \) to the canonical distinguished triangle
\[ \ell_!\mathbf{P}^* \to \mathbf{P}^* \to \mathbf{m}_*\mathbf{m}^*[−1]\mathbf{P}^* \xrightarrow{[1]} \ell_!\mathbf{P}^*, \]
we conclude that, near \( p \),
\[ (f) \quad \tilde{m}^*[-1]\psi_f[-1]\mathbf{P}^* \cong \tilde{m}^*\psi_f[-1]\mathbf{m}_*\mathbf{m}^*[−1]\mathbf{P}^*. \]

There is also the canonical distinguished triangle relating the nearby and vanishing cycles along \( z_0 \):
\[ \tilde{m}^*[-1]\psi_f[-1]\mathbf{P}^* \to \psi_{z_0}[-1]\psi_f[-1]\mathbf{P}^* \to \phi_{z_0}[-1]\psi_f[-1]\mathbf{P}^* \xrightarrow{[1]} \tilde{m}^*[-1]\psi_f[-1]\mathbf{P}^*. \]
Now Theorem 3.1 of [9] tells us that, near $p$, we also know that $\phi_{z_0}[-1]|\psi_f[-1]|P^* = 0$. Thus, near $p$,

\[ \hat{m}^*[-1]|\psi_f[-1]|P^* \to \psi_{z_0}[-1]|\psi_f[-1]|P^*. \]

Combining (†) and (‡) yields the desired result. \hfill \Box

We can now prove the theorem that we stated in the introduction.

**Theorem 3.2.** For all $x \in V(z_0)$, if $d_x z_0 \not\in (T_f U)_x$, then $\hat{r}_x$ induces a surjection

\[ r_x : (T_f U)_x \to (T_{f|_V(z_0)} V(z_0))_x, \]

such that $r_x^{-1}(0) = 0$.

**Proof.** The final statement is trivial to prove. Suppose that $d_x z_0 \not\in (T_f U)_x$; as $(T_f U)_x$ is $C$-conc, it follows that, if $a d_x z_0 \in (T_f U)_x$, then $a = 0$. The statement now follows from

\[ r_x^{-1}(0) = (T_f U)_x \cap \hat{r}_x^{-1}(0) = \{a d_x z_0 \in (T_f U)_x \mid a \in \mathbb{C}\}. \]

Now, suppose that $x \in V(z_0)$ and $d_x z_0 \not\in (T_f U)_x$. If $c$ is a constant, replacing $f$ by $f - c$ does not affect $T_f U$, and so we may assume that $f(x) = 0$.

We wish to look at the equality of the microsupports of the two isomorphic complexes in Proposition 3.1. While both complexes in the isomorphism in Proposition 3.1 are complexes on $V(f, z_0)$, the results that we shall use will tell us one microsupport in $T^*V(z_0)$ and the other in $T^*U$; we shall write $\text{SS}_{V(z_0)}$ and $\text{SS}_U$, respectively.

If $Y$ is an analytic subset of $V(z_0) \subset U$ and $A^* \in D^b_Y$, we may consider the microsupport $\text{SS}_{V(z_0)}(A^*)$ in $T^*V(z_0)$ or $\text{SS}_U(A^*)$ in $T^*U$. The relationship between these is trivial:

\[ \text{SS}_U(A^*) = \text{SS}_{V(z_0)}(A^*) + < d z_0 > = \{ (x, \eta + a d_x z_0) \mid (x, \eta) \in \text{SS}_{V(z_0)}(A^*), a \in \mathbb{C} \}. \]

We shall always work near $x$, and finally look at precisely the fiber over $x$.

By Theorem 2.1,

\[ \text{SS}_{V(z_0)}(\psi_{f_0}[-1]Z^*_{V(z_0)}[n]) = T_f^* V(z_0) \cap (V(f, z_0) \times \mathbb{C}^n), \]

and so

\[ \text{SS}_U(\psi_{f_0}[-1]Z^*_{V(z_0)}[n]) = (T_f^* V(z_0) \cap (V(f, z_0) \times \mathbb{C}^n)) + < d z_0 >. \]

We must do a little more work to find $\text{SS}(\psi_{z_0}[-1]|\psi_f[-1]|P^*)$. Again, by Theorem 2.1,

\[ \text{SS}_U(\psi_f[-1]|P^*) = \bigcup_{R \in \mathfrak{R}} T_R^* U = T_f^* U \cap (V(f) \times \mathbb{C}^{n+1}), \]

where $\mathfrak{R}$ is some subset of a Whitney stratification with respect to which $|\psi_f[-1]|P^*$ is constructible. Note that since are working at points $x$ such that $d_x z_0 \not\in (T_f U)_x$, if $x \in V(f)$, then, for each $R \in \mathfrak{R}$, $d_x z_0 \not\in T_R^* U$. 

Now, using the more general statement in Theorem 2.1, we have that

\[(**) \quad \SS_{\alpha} (\psi_{z_0} [-1] \psi_f [-1] P) = \left( \bigcup_{R \in \mathfrak{R}(z_0)} T_{z_0}^* U \right) \cap (V(z_0) \times \mathbb{C}^{n+1}). \]

By Proposition 3.1,

\[\SS_{\alpha} (\psi_{z_0} [-1] \psi_f [-1] P) = \SS_{\alpha} (\psi_{f_0} [-1] Z_{V(z_0)} [n]),\]

and, thus, at \(x\), we have

\[\left( T_{f}^* (V(z_0))_x \right) + < d_x z_0 > = \left( \bigcup_{R \in \mathfrak{R}(z_0)} T_{z_0}^* U \right) \]

By Lemma 5.8 of [8], since for each \(R \in \mathfrak{R}\), \(d_x z_0 \notin T_R^* U\),

\[\left( \bigcup_{R \in \mathfrak{R}(z_0)} T_{z_0}^* U \right) \subseteq \left( \bigcup_{R \in \mathfrak{R}} T_R^* U \right) + < d_x z_0 > = (T_f^* U)_x + < d_x z_0 > .\]

Therefore, we conclude that

\[\left( T_{f}^* (V(z_0))_x \right) + < d_x z_0 > = (T_f^* U)_x + < d_x z_0 > ,\]

and, hence,

\[r_x \left( T_f^* U_x \right) = r_x \left( (T_f^* U)_x + < d_x z_0 > \right) = r_x \left( T_{f}^* (V(z_0))_x + < d_x z_0 > \right) = (T_{f}^* (V(z_0)))_x .\]

\[\square\]

4. Concluding Remarks

As \(T_f^* U \subseteq U \times \mathbb{C}^{n+1}\) is \(\mathbb{C}\)-conic in the cotangent coordinates, it projectivizes to \(P(T_f^* U) \subseteq U \times \mathbb{P}^n\); in fact, for many authors, this projective object is what they mean by the relative conormal space.

As \(r_x\) preserves scalar multiplication and as \(r_x^{-1}(0) = 0\), Theorem 3.2 immediately implies a projective version of itself. We denote the projective class of \(d_x z_0\) by \([d_x z_0]\).

\[\textbf{Theorem 4.1.} \quad \text{For all } x \in V(z_0), \text{ if } [d_x z_0] \notin P(T_f^* U)_x, \text{ then } r_x \text{ induces a surjection}
\]

\[\hat{r}_x : P(T_f^* U)_x \to P(T_{f}^* (V(z_0)))_x.\]

Our primary interest in Theorem 3.2 relates to the Lê cycles and Lê numbers of \(f\); see, for instance, [5]. In many of our past works, the genericity condition that we required to guarantee the existence of the Lê cycles and numbers was that the coordinates \((z_0, \ldots, z_n)\) be prepolar. This is an inductive requirement on how the hyperplanes slices \(V(z_i)\) intersect the strata of an \(a_f\) stratification of \(V(f)\). Ideally, we would like to eliminate the need to first produce an \(a_f\) stratification. It turns out that Theorem 3.2, or its projective version, is precisely the lemma that we need.
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