Zero-energy Corner States in a Non-Hermitian Quadrupole Insulator

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Abstract: We find a zero-energy corner state in a non-Hermitian quadrupole insulator without an Hermitian counterpart, and the response of the system near zero energy can be drastically different from Hermitian systems. © 2020 The Author(s)

The quadrupole insulator (QI) is an interesting type of higher-order topological insulators that has a non-vanishing bulk quadrupole moment, and also supports zero-energy corner states [1]. Non-Hermitian physics is another field that attracts considerable interests in recent years. Notable phenomena include “exceptional points”, where the Hamiltonian becomes defective, and parity-time (PT) symmetry, where the spectra can be completely real. This paper combines these two fields, following earlier researches [2, 3]. We pay special attention to the location and field distribution of the zero-energy corner states, and whether the Hamiltonian becomes defective. We also discuss how to excite these corner states with external sources using “partial Jordan decomposition”.

The non-Hermitian QI model studied in this paper is pictorially shown in Fig. 1(a), where all parameters are real. It is a natural non-Hermitian generalization of the QI model [1], while maintaining the sublattice symmetry. This model can also be viewed as a two-dimensional generalization of the non-Hermitian Su-Schrieffer-Heeger (SSH) model [2]. It is important to note that in non-Hermitian systems, the open-boundary spectrum can differ significantly from the periodic-boundary spectrum [2], since bulk modes in an open-boundary system carry a spatially grow/decay factor. In our non-Hermitian QI, the gapless condition is $t^2 = γ^2 + λ^2$. This system also has PT symmetry. When $|t/γ| > 1$, the open boundary spectrum is purely real, indicating the system has unbroken PT symmetry; otherwise the system has broken PT symmetry.

Now we proceed to see if a large array ($N \times N$ unit cells, $N \gg 1$) of non-Hermitian QI with open boundary condition supports any zero-energy corner states. We are mostly interested in the system with unbroken PT symmetry, so without generality we assume $t > γ > 0, λ > 0$ from now on. When the intercell hopping strength $λ$ is large, namely $t + γ < λ$, it’s straightforward to verify that the four corner states in the Hermitian QI still exist (in the thermodynamic limit), just with modified spatial decay rates. Each corner state is localized at one corner of the array, and has support on only one sublattice. We call this regime “near-Hermitian regime”.

When $λ$ decreases such that $\sqrt{t^2 - γ^2} < λ < t + γ$, among the above four corner states, only the one at the top-left corner survives. Additionally, there is a new corner state, also localized at the top-left corner, but having

![Fig. 1. A non-Hermitian QI. (a) Tight binding model on a square lattice. The dashed black square represents a unit cell, consisting four sites. Red and blue lines with arrows represent asymmetric intracell hopping, and yellow lines represent symmetric intercell hopping. Dashed lines represent hopping terms with negative signs. There is no on-site potential. (b) The phase diagram of a large non-Hermitian QI with open boundary condition. Green: near-Hermitian regime; Cyan: intermediate regime; White: trivial regime. Solid black lines: the system is gapless. Between two dashed orange lines ($|t/γ| < 1$), the system has broken PT symmetry; elsewhere the system has unbroken PT symmetry. (c) Spectrum in the near-Hermitian regime. Parameters: $t = 0.6, γ = 0.4, λ = 0.7, N = 20$.](image)
support on two sublattices (2 and 3), with field distribution
\[
\phi(x, y, 2) = -\phi(y, x, 3) = [(1 - \gamma t - \lambda)_x]^{y} \text{ and } [(-1 + \gamma t / \lambda)_x]^{y}, \phi(x, y, 1) = -\phi(y, x, 4) = 0. \tag{1}
\]

(we use \(x\) and \(y\) also as integer-valued coordinates of unit cells in the \(x\) and \(y\) directions, respectively.) We call this regime “intermediate regime”. The surviving state has a decay rate larger than that of the bulk states, and \(\phi\) necessarily decays slower in space than bulk states do. Thus, we can refer them as “supralocalized mode” and “sublocalized mode”, respectively. Although numerical study still gives four zero-energy states (Fig. 1(c)), these two are the only linearly independent ones. When \(\lambda < \sqrt{t^2 - \gamma^2}\) there are no more zero energy states. The phase diagram of our non-Hermitian QI with open boundary condition is summarized in Fig. 1(b).

We also study how to excite these corner states. We work with a non-Hermitian QI with unbroken PT symmetry (i.e. \(|r| > |\gamma|\)). Since the spectrum is purely real, adding an overall infinitesimal loss to the system will ensure all transients eventually die down when the system is excited, and one only needs to consider the driving equation \((H - E)\psi = \xi\), where \(\psi\) is the external source with driving frequency \(E\). Fig. 2 shows the response of the system in intermediate regime to external sources on different sites with \(E\) close to 0 obtained from numerical studies. Surprisingly, we found that placing the source at the bottom-right corner gives the strongest excitation of the corner states at the top-left corner. This contradicts the common sense in Hermitian systems where one usually places the source where the target state locates. Moreover, we found that by placing the source on sublattice 1 or 2 or 3, the supralocalized state is predominantly excited, while if the source is on sublattice 4 the sublocalized state will be predominant.

To explain the above results we study Jordan decomposition of the Hamiltonian \(H = PJP^{-1}\), where \(P\) is invertible but no longer unitary. Since we are mostly interested in zero-energy corner states, we only need to focus on Jordan blocks for eigenvalue 0. It turns out the Jordan blocks for eigenvalue 0 are
\[
J_0 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{2}
\]
and the associated four columns of the matrix \(P\) and rows of \(P^{-1}\) are states localized at the top-left corner and bottom-right corner, respectively. Thus we conclude the geometric multiplicity of eigenvalue 0 is 2, and \(H\) is defective at zero energy. The transition between the intermediate regime and the near-Hermitian regime is not induced by bulk gap closure. The specific forms of those columns and rows (how they distribute on each sublattice) also perfectly explain the phenomenon in Fig. 2.

This system can be implemented by an array of ring resonators, where the asymmetric intracell coupling can be achieved by having gain and loss at two halves of the anti-resonance link rings.

References

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