DEFORMING A HYPERSURFACE BY GAUSS CURVATURE AND SUPPORT FUNCTION

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Abstract. We study the motion of smooth, strictly convex bodies in $\mathbb{R}^n$ expanding in the direction of their normal vector field with speed depending on Gauss curvature and support function.

1. Introduction

The setting of this paper is $n$-dimensional Euclidean space, $\mathbb{R}^n$. A compact convex subset of $\mathbb{R}^n$ with non-empty interior is called a convex body. The set of convex bodies in $\mathbb{R}^n$ is denoted by $K^n$. Write $K^n_c$ for the set of origin-symmetric convex bodies and $K^n_0$ for the set of convex bodies whose interiors contain the origin. Also write respectively $F^n$, $F^n_0$, and $F^n_c$ for the set of smooth ($C^\infty$-smooth), strictly convex bodies in $K^n$, $K^n_0$, and $K^n_c$.

The unit ball of $\mathbb{R}^n$ is denoted by $B$ and its boundary is denoted by $S^{n-1}$. We write $\nu : \partial K \rightarrow S^{n-1}$ for the Gauss map of $\partial K$, the boundary of $K \in F^n$. That is, at each point $x \in \partial K$, $\nu(x)$ is the unit outwards normal at $x$.

Assume that $\varphi$ is a positive, smooth function on $S^{n-1}$. Let $F_0 : M \rightarrow \mathbb{R}^n$ be a smooth parametrization of $\partial K_0$ where $K_0 \in F^n_0$. In this paper, among other things, we study the long-time behavior of a family of convex bodies $\{K_t\} \subset F^n_0$ given by smooth maps $F : M \times [0,T) \rightarrow \mathbb{R}^n$ that satisfies the initial value problem

$$\partial_t F(x,t) = \varphi(\nu(x,t)) \frac{(F(x,t) \cdot \nu(x,t))^{2-p}}{K(x,t)} \nu(x,t), \ F(\cdot,0) = F_0(\cdot).$$

Here $F(M,t) = \partial K_t$, and $K(\cdot,t)$ is the Gauss curvature of $F(M,t)$. Moreover, $T$ is the maximal time for which the solution exists.

The support function of $K$ as a function on the unit sphere is defined by

$$h_K(u) := \nu^{-1}(u) \cdot u$$

for each $u \in S^{n-1}$. All information about the hypersurface, except for parametrization, is contained in the support function. It easy to see that as $\{K_t\}$ moves according to (1.1), then $h : S^{n-1} \times [0,T) \rightarrow \mathbb{R}$, $h(\cdot,t) := h_{K_t}(\cdot)$ evolves by

$$\partial_t h(u,t) = \varphi(u) \frac{h^{2-p}}{K(u,t)}.$$

A self-similar solution of this flow satisfies

$$h^{1-p} \det(\nabla^2 h + \text{Id} h) = \frac{c}{\varphi},$$

for some positive constant $c$. Here $\nabla$ is the covariant derivative on $S^{n-1}$ endowed with an orthonormal frame.

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When \( p = 2, \varphi \equiv 1 \), flow (1.2), among other flows, was studied by Schnürer [48] in \( \mathbb{R}^3 \), and by Gerhardt [22] in higher dimensions. Both works rely on the reflection principle of Chow and Gulliver [18], and McCoy [42]. Their result is as follows: the normalized flow evolves any smooth strictly convex body in the \( C^\infty \)-topology to an origin-centered ball. When \( p = -n, \varphi \equiv 1 \), the flow is a member of family of flows, \( p \)-centro affine normal flows, which was introduced by Stancu [46]. In \( \mathbb{R}^2 \) and for \( p = -2, \varphi \equiv 1 \), a “dual” flow to (1.2) (see Lemma 2.3) was studied by the author [29] with an application to the stability of the Busemann-Petty centroid inequality in the plane. For \( p > 2, \varphi \equiv 1 \) and in \( \mathbb{R}^2 \), it follows from Chow-Gulliver [18, Theorem 3.1] (see also Tsai [50, Example 1]) that (1.2) evolves any smooth strictly convex body in the \( C^\infty \)-topology to an origin-centered disk. See also Chow-Tsai [19–21] for expansion of convex hypersurfaces by non-homogeneous functions of principal curvatures and Gauss curvature. Moreover, in \( \mathbb{R}^2 \) the following theorems can be obtained by means of Andrews’ results. Let us set \( \tilde{K}_t := (V(B)/V(K_t))^{1/n} K_t \).

**Theorem A1.** Let \(-2 < p < \infty, p \neq 1, \varphi \equiv 1\) and assume that \( K_0 \in \mathcal{F}_2^p \) satisfies \( \int_{S^1} \frac{u}{h_{K_0}(u)} d\sigma(u) = 0 \). There exists a unique solution \( \{ K_t \} \subset \mathcal{F}_2^p \) of flow (1.2) such that \( \{ \tilde{K}_t \} \) converges in the \( C^\infty \)-topology to the unit disk. If \( p = -2 \), then \( \{ \tilde{K}_t \} \) converges in the \( C^\infty \)-topology to an origin-centered ellipse.

**Theorem A2.** Let \(-2 < p < \infty, p \neq 1\). Let \( \varphi \) be a positive, smooth even function on \( \mathbb{S}^1 \) i.e., \( \varphi(u) = \varphi(-u) \). Assume that \( K_0 \in \mathcal{F}_2^2 \). There exists a unique solution \( \{ K_t \} \subset \mathcal{F}_2^2 \) of flow (1.2) such that \( \{ K_t \} \) converges in the \( C^\infty \)-topology to an origin-symmetric strictly convex, smooth solution of (1.3).

**Theorem A3.** Let \(-2 < p \leq -1\), and \( K_0 \in \mathcal{F}_2^p \) satisfy \( \int_{S^1} \frac{u}{\varphi(u) h_{K_0}(u)} d\sigma(u) = 0 \). Then there exists a unique solution \( \{ K_t \} \subset \mathcal{F}_2^p \) of flow (1.2) such that \( \{ \tilde{K}_t \} \) converges in the \( C^\infty \)-topology to a positive strictly convex, smooth solution of (1.3).

**Remark.** These theorems can be obtained from the results of Andrews [3–7]: If \( p < 1 \), then one needs Andrews’ results about asymptotic behavior of shrinking flows by positive powers of curvature \( (\partial_t h = -\psi K^{1-p}) \), and when \( p > 1 \) one needs Andrews’ results on asymptotic behavior of expanding flows by negative powers of curvature \( (\partial_t h = \psi K^{-p}) \). We observe that the evolution equation of \( \psi K^{1-p} \) in either case satisfies, up to a positive constant, (1.2) with \( \varphi = \psi^{p-1} \). Existence of solutions to the Minkowski problem let us reverse this procedure provided \( K_0 \) satisfies the integral identity \( \int_{S^1} \frac{u}{\varphi \psi h_{K_0}} d\sigma = 0 \). This argument is invalid if \( n \geq 3 \) or \( \int_{S^1} \frac{u}{\varphi \psi h_{K_0}} d\sigma \neq 0 \). See also S. Angenent, J.J.L. Velázquez, Y.-C. Lin, T.-S. Lin, C.-C. Poon, and D.-H. Tsai [8, 9, 34–36, 38, 49] for several beautiful results about the blow-up behavior of immersed, smooth, convex, closed plane curve with rotation index \( m \geq 1 \) evolving by (1.2).

**Theorem 1.1.** Let \( n \geq 3, p = -n, \varphi \equiv 1 \) and assume that \( K_0 \in \mathcal{F}_2^n \) has its Santaló point at the origin, e.g., \( \int_{S^{n-1}} \frac{u}{h_{K_0}(u)} d\sigma(u) = 0 \). Then there exists a unique solution \( \{ K_t \} \subset \mathcal{F}_2^n \) of flow (1.2), such that \( \{(2n(T-t))^{\frac{1}{n}} K_t \} \) converges sequentially in the \( C^\infty \)-topology to the unit ball modulo \( SL(n) \).

Note that by “converges sequentially” we mean every subsequence of \( \{ \tilde{K}_t \} \) has a subsequence which converges to a limiting shape. As a corollary of this theorem, we
prove an inequality of Lutwak [40] (stronger than the Blaschke-Santaló inequality). See Theorem 10.1 for the statement.

The next theorem fills in the gap $p \neq 1$ in the statement of Theorem A2.

**Theorem 1.2.** Let $p = 1$, $\varphi$ be a positive, smooth even function on $S^1$ and $K_0 \in F_0^2$. Then there exists a unique solution $\{K_t\} \subset F_0^2$ of flow (1.2) such that $\{K_t\}$ converges in the $C^\infty$-topology to an origin-symmetric strictly convex, smooth solution of (1.3).

Let $F_0 : M \to \mathbb{R}^n$ be a smooth parametrization of $\partial K_0$ where $K_0 \in F_0^n$. Consider convex bodies $\{K_t\} \subset F_0^n$ given by the smooth embeddings $F : M \times [0,T) \to \mathbb{R}^n$ that solve the initial value problem

\[
\partial_t F(x,t) = -\frac{\mathcal{K}(x,t)}{(F(x,t) \cdot \nu(x,t))^n} \nu(x,t), \quad F(\cdot,0) = F_0(\cdot).
\]

Then as $K_t$ moves according to (1.4), $h : S^{n-1} \times [0,T) \to \mathbb{R}$, $h(\cdot,t) := h_{K_t}(\cdot)$ evolves by

\[
\partial_t h(u,t) = -\frac{\mathcal{K}}{h^n}(u,t).
\]

This flow was introduced by Stancu [46] ($p$ centro-affine normal flows for $p = \infty$). We will prove the following theorem about the asymptotic behavior of flow (1.5).

**Theorem 1.3.** Assume that $K_0 \in F_0^n$ has its centroid at the origin. Then there exists a unique solution $\{K_t\} \subset F_0^n$ of flow (1.5) such that $\{(2n(T - t))^{-\frac{n}{2n}}K_t\}$ converges sequentially in the $C^\infty$-topology to the unit ball modulo $SL(n)$.

Our proof of Theorem 1.1 with convergence in the $C^1$ topology is resulted from finding a new family of entropy functionals $B_p^\varepsilon$ (see Definition 3.9), which was inspired by definition of curvature image due to Petty [44] and an inequality of Lutwak [40], see Theorem 10.1. Our convex-geometric argument in Section 8.1 to obtain the asymptotic shapes does not rely on uniform higher order regularity estimates for the normalized solutions, and it employs only entropy functionals $B_p^\varepsilon$. We will discuss convergence in the $C^\infty$ topology in Section 9. In the course of proving our main theorems, we prove Theorems A1, A2, A3 with sequential convergence in the $C^\infty$-topology. In Section 10 we present a few applications of the flow such as a direct proof of Lutwak’s inequality 1986.

2. BACKGROUND AND NOTATION

2.1. Differential Geometry. The matrix of the radii of curvature of $\partial K$ is denoted by $\mathbf{r} = [r_{ij}]_{1 \leq i,j \leq n-1}$ and the entries of $\mathbf{r}$ are considered as functions on the unit sphere. They can be expressed in terms of the support function and its covariant derivatives as $r_{ij} := \nabla_i \nabla_j h + h \tilde{g}_{ij}$, where $[\tilde{g}_{ij}]_{1 \leq i,j \leq n-1}$ is the standard metric on $S^{n-1}$ and $\nabla$ is the standard Levi-Civita connection of $S^{n-1}$. The Gauss curvature of $\partial K$ is denoted by $\mathcal{K}$, and as a function on $\partial K$, it is also related to the support function of the convex body by

\[
\frac{1}{\mathcal{K} \circ \nu^{-1}} := S_{n-1} = \det_g [\nabla_i \nabla_j h + \tilde{g}_{ij}] := \frac{\det [r_{ij}]}{\det [\tilde{g}_{ij}]},
\]

In the sequel, for simplicity we usually denote $\mathcal{K} \circ \nu^{-1}$ by $\mathcal{K}$. The principal radii of curvature $\{\lambda_i\}_{1 \leq i \leq n-1}$ are the eigenvalues of $[r_{ij}]$ with respect to $[\tilde{g}_{ij}]$. Moreover,
we write $[f_{ij}]$ for the second fundamental form of $\partial K$ and principal curvatures are the eigenvalues of $[f_{ij}]$ with respect to $[g_{ij}]$ which we shall denote them by $\{\kappa_i\} = \{1/\lambda_i \circ \nu\}$.

2.2. Convex Geometry. We will start by definition of the polar body.

**Polar body:** The polar body, $K^*$, of convex body $K$ with the origin of $\mathbb{R}^n$ in its interior is the convex body defined as

$$K^* = \{x \in \mathbb{R}^n | x \cdot y \leq 1 \text{ for all } y \in K\}.$$

The Blaschke-Santaló inequality states that

$$\min_{x \in \text{int } K} V(K)V((K - x)^*) \leq \omega_n^2.$$ 

Equality holds exclusively for ellipsoids. The point for which the above minimum is achieved is called the Santaló point, and it will be denoted as $e_n(K)$. In what follows, we will furnish all geometric quantities associated with $K^*$ with $^*$.

**Theorem 2.1.** Let $K \in \mathcal{F}_0^n$. Suppose that $0 < a \leq h_K \leq b < \infty$ and $0 < c \leq \kappa_i \leq d < \infty$. Then

$$c_1 \leq \kappa_i^* \leq c_2,$$

for $c_1, c_2 > 0$ depending only on $a, b, c, d$.

**Proof.** In general, parameterizing $\partial K$ as a graph over the unit sphere with the corresponding radial distance function $r : S^{n-1} \to \mathbb{R}$, we can write the metric $[g_{ij}]$ and its inverse $[g^{ij}]$, the second fundamental form $[f_{ij}]$, and $[\nu^*_ij]$ in terms of $r$ and whose spatial derivatives as follows:

$$\begin{align*}
(1) \quad & g_{ij} = r^2 \delta_{ij} + \nabla_i r \nabla_j r; \\
(2) \quad & g^{ij} = \frac{1}{r^2} \left( \delta_{ij} - \frac{\nabla_i r \nabla_j r}{r^2 + |\nabla r|^2} \right); \\
(3) \quad & f_{ij} = \frac{1}{\sqrt{r^2 + |\nabla r|^2}} \left(-r \nabla_i \nabla_j r + 2 \nabla_i r \nabla_j r + r^2 \delta_{ij} \right); \\
(4) \quad & \nu^*_{ij} = \frac{\nabla_i \nabla_j}{r} + \frac{1}{r} \delta_{ij} = \frac{\sqrt{r^2 + |\nabla r|^2}}{r^2} f_{ij}.
\end{align*}$$

The proofs of (1)–(3) can be found in [53], and (4) follows from the fact that $\frac{1}{r}$ is the support function of $K^*$ (see also Oliker-Simon [37], Identities (7.6), (7.31)). We apply these formulas to $K^*$. From (2) and (4) we get $[\nu^*_{ij}] = \sqrt{r^2 + |\nabla r|^2} [f_{ij}]^*$. To prove the claim, we have only to consider points where the gradient of the radial function $r^*$ does not vanish. Around such a point we introduce an orthonormal frame $\{e_1, \cdots, e_{n-1}\}$ on $S^{n-1}$ such that $e_1 = \frac{\nabla r^*}{|\nabla r^*|}$. Then $\nabla r^* = (|\nabla r^*|, 0, \cdots, 0)$. Thus, in such a frame we may express $[\nu^*_{ij}]$ as follows

$$[\nu^*_{ij}] = [\nu_{ij}]$$

(2.1) $$\begin{pmatrix}
\frac{1}{r^2 + |\nabla r|^2} & 0 & \cdots & 0 \\
0 & \frac{1}{r^2 + |\nabla r|^2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{1}{r^2 + |\nabla r|^2}
\end{pmatrix} = AB.$$ 

The eigenvalues of $A$ are $\{\lambda_i\}$, eigenvalues of $B$ are $\{\frac{1}{r^2 + |\nabla r|^2}, \frac{1}{r^2 + |\nabla r|^2}\}$, and eigenvalues of $AB$ are $\{\frac{\sqrt{r^2 + |\nabla r|^2}}{r^2 + |\nabla r|^2} \kappa_i^*\}$. We may assume that $\kappa_1^* \leq \kappa_2^* \leq \cdots \leq \kappa_{n-1}^*$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$. It follows (for example see Corollary III.4.6 [10]) that the
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2.1

Recall that if the Radon-Nikon derivative equality holds, if and only if the Borel measure is defined as

\[ h(x) \frac{\nu(x)}{\nu(0)} = 1 \]

where \( x \in \partial K \), and \( x^* \in \partial K^* \) satisfies \( x \cdot x^* = 1 \) (this identity can be proved by taking determinant of both sides of (2.1), for non-smooth hypersurfaces see Hug [26, Theorem 2.2] for proof) it follows that \( l < K^* \) for some positive finite number depending only on \( a, b, d \). Therefore, since \( \kappa_{n-1}^* \) is bounded above, the lower bound on \( \kappa_1^* \) follows. □

Remark 2.2. Explicit equality between elementary symmetric functions of principal curvatures of \( K^* \) and principal radii of curvature of \( K \) is given by Hug [27, Theorem 5.1] in a general setting where the convex body might not be smooth. Moreover, [27, Corollary 5.1] deduces an inequality from which lower and upper bounds for the principal curvatures (and not only for their elementary symmetric functions) can be deduced.

Lemma 2.3. As \( K_t \) evolve by (1.2), their polars \( K_t^* \) evolve as follows

\[ \partial_t h^* = -\varphi \left( \frac{h^* u + \nabla h^*}{\sqrt{h^{*2} + |\nabla h^*|^2}} \right) \left( \frac{(h^{*2} + |\nabla h^*|^2)^{\frac{n}{2}}}{h^{*n}} \right) K^*, \ h^*(\cdot, t) := h_{K_t^*}(\cdot). \]

Proof. The proof is similar to the one in [28, Theorem 2.2]. □

Minkowski’s mixed volume inequality and curvature function: A convex body is said to be of class \( C_k^k \) for some \( k \geq 2 \), if its boundary hypersurface is \( k \)-times continuously differentiable, in the sense of differential geometry, and the Gauss map \( \nu : \partial K \rightarrow S^{n-1} \) is well-defined and a \( C^{k-1} \)-diffeomorphism.

Let \( K, L \) be two convex bodies and \( 0 < a < \infty \). The Minkowski sum \( K + aL \) is defined as \( h_{K+aL} = h_K + ah_L \) and the mixed volume \( V_1(K, L) \) of \( K \) and \( L \) is defined by

\[ V_1(K, L) = \frac{1}{n} \lim_{n \to 0^+} V(K + aL) - V(K). \]

A fundamental fact is that corresponding to each convex body \( K \), there is a unique Borel measure \( S_K \) on the unit sphere such that

\[ V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u)dS_K(u) \]

for each convex body \( L \). The measure \( S_K \) is called the surface area measure of \( K \).

Recall that if \( K \) is \( C^2_+ \), then \( S_K \) is absolutely continuous with respect to \( \sigma \), and the Radon-Nikon derivative \( dS_K(u)/d\sigma(u) \) defined on \( S^{n-1} \) is the reciprocal Gauss curvature of \( \partial K \) at the point of \( \partial K \) whose outer normal is \( u \). For \( K \in K^n \),

\[ V(K) = V_1(K, K) = \frac{1}{n} \int_{S^{n-1}} h_K(u)dS_K(u). \]

Of significant importance in convex geometry is the Minkowski mixed volume inequality. Minkowski’s mixed volume inequality states that for \( K, L \in K^n \),

\[ V_1(K, L)^n \geq V(K)^{n-1}V(L). \]

Equality holds, if and only if \( K \) and \( L \) are homothetic.

A convex body \( K \) is said to have a positive continuous curvature function \( f_K \), defined on the unit sphere, provided that for every convex body \( L \)

\[ V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L f_K d\sigma, \]
Lemma 3.1. **ϕ** on the boundary of **K** at most one curvature function; see [11, p. 115]. If **K** is of class **C**₂, then the curvature function is the reciprocal Gauss curvature of ∂**K** transplanted to **S**ⁿ⁻¹ via the Gauss map.

3. Entropy points and entropy functionals

Write **S**⁺ for the set of positive, smooth functions on **S**ⁿ⁻¹ and write **S**⁺ for the set of positive, smooth even function on the unit sphere. That is, **ϕ** ∈ **S**⁺, if **ϕ** ∈ **S**⁺ and **ϕ**(**u**) = **ϕ**(−**u**).

**Lemma 3.1.** There exists a unique point **e**ₚ(**K**) ∈ int **K** such that

\[
\begin{align*}
\min_{x \in K} \int_{\mathbb{S}^{n-1}} (h_K(u) - x \cdot u)^p d\sigma &= \int_{\mathbb{S}^{n-1}} (h_K(u) - e_p \cdot u)^p d\sigma & \text{if } 1 < p < \infty \\
\max_{x \in K} \int_{\mathbb{S}^{n-1}} (h_K(u) - x \cdot u)^p d\sigma &= \int_{\mathbb{S}^{n-1}} (h_K(u) - e_p \cdot u)^p d\sigma & \text{if } 0 < p < 1 \\
\min_{x \in \text{int } K} \int_{\mathbb{S}^{n-1}} -\log(h_K(u) - x \cdot u) d\sigma &= \int_{\mathbb{S}^{n-1}} -\log(h_K(u) - e_0 \cdot u) d\sigma & \text{if } p = 0 \\
\min_{x \in \text{int } K} \int_{\mathbb{S}^{n-1}} (h_K(u) - x \cdot u)^p d\sigma &= \int_{\mathbb{S}^{n-1}} (h_K(u) - e_p \cdot u)^p d\sigma & \text{if } -n < p < 0.
\end{align*}
\]

Additionally, **e**ₚ is characterized by \( \int_{\mathbb{S}^{n-1}} \frac{u}{(h_K(u) - e_p(K) \cdot u)^p} d\sigma(u) = 0 \).

Moreover, let \(-n < p < -n + 1\) and **ϕ** ∈ **S**⁺. Then there exists a unique point, \( e_p^\circ(K) \), in the interior of **K** such that

\[
\min_{x \in \text{int } K} \int_{\mathbb{S}^{n-1}} \frac{(h_K(u) - x \cdot u)^p}{\varphi(u)} d\sigma = \int_{\mathbb{S}^{n-1}} \frac{(h_K(u) - e_p^\circ \cdot u)^p}{\varphi(u)} d\sigma.
\]

Additionally, \( e_p^\circ \) is characterized by \( \int_{\mathbb{S}^{n-1}} \frac{u}{\varphi(u)(h_K(u) - e_p^\circ \cdot u)^p} d\sigma(u) = 0 \).

**Proof.** Existence and uniqueness of a point in **K** to each of the above minimization or maximization follow from the strict concavity or strict convexity of the corresponding functional and compactness of **K**. The proof of \( e_p \in \text{int } K \) follows exactly the one given by Guan and Ni [23, Lemmas 2.3, 2.4]. For completeness, we present it here. We shall consider the case \( p \neq 0 \). Suppose on the contrary that \( e_p(K) \) is on the boundary of **K** and \( \nu \) is the outer normal at \( e_p \). By Busemann’s theorem [12, Theorem 1.12], there is a rectangular coordinate system \((y_1, \cdots, y_n)\) such that \( e_p \) is the origin, \((0, \cdots, 0, 1) = \nu \), and the segment \([\nu - ty_n]\) contained in int **K** for small \( t > 0 \). In view of this fact, we may then assume that in the standard coordinate system of \( \mathbb{R}^n \) one has \( e_p = \bar{\nu}, \nu = (0, \cdots, 1) \), and **K** lies below the hyperplane \( \nu^\perp \). Take an arbitrary point \( u^+ = (u_1, \cdots, u_n) \), with \( u_n \geq 0 \), and define \( u^- = (u_1, \cdots, -u_n) \). For a fixed \( u^+ \) define \( i(u^+) \), the point on the boundary of \( K \) that \( h_K(u^+) = u^+ \cdot i(u^+) \). We have

\[
h_K(u^-) \geq u^- \cdot i(u^+) \geq u^+ \cdot i(u^+) = h_K(u^+).
\]

Moreover, \( h_K((0, \cdots, -1)) > 0 \) and \( h_K((0, \cdots, 1)) = 0 \). So the above inequality must be strict for a set of positive measure. Define \( \bar{h}(u) = h_K(u) + su_n \), and notice that \( \bar{h} \) is positive for all \( u \), provided \((0, \cdots, -s) \in \text{int } K \); which is the case if
0 < s < t. Hence, we have
\[
\frac{d}{ds}\bigg|_{s=0} \left( \int_{S^{n-1}} h^p d\sigma \right) = p \int_{S^{n-1}} h^{p-1}_K (u \cdot \nu) d\sigma \\
= p \int_{\{u_n > 0\} \cap S^{n-1}} \left( h^{p-1}_K (u^+) - h^{p-1}_K (u^-) \right) u_n d\sigma \\
= \text{sgn}(p(1 - p)).
\]

To prove that \( e_p^c \in \text{int} K \), notice that when \(-n \leq p \leq -n + 1:\)
\[
\lim_{x \to \partial K} \int_{S^{n-1}} \frac{(h_K(u) - x \cdot u)^p}{\varphi(u)} d\sigma = +\infty,
\]
while by the Blaschke-Santaló inequality the infimum is finite. \( \Box \)

**Remark 3.2.** In the sequel, we will always exclude case \( p = 1 \), unless we are working with origin-symmetric bodies.

**Definition 3.3.** For \(-n \leq p < \infty, p \neq 1, e_p(K) \) is the unique point in \( \text{int} K \) which satisfies
\[
\int_{S^{n-1}} \frac{u}{(h_K(u) - e_p \cdot u)^{1-p}} d\sigma(u) = 0.
\]

When \( n \leq p \leq -n + 1 \) and \( \varphi \in \mathcal{S}^+, e_p^c(K) \) is the unique point in \( \text{int} K \) which satisfies
\[
\int_{S^{n-1}} \frac{u}{\varphi(u) (h_K(u) - e_p \cdot u)^{1-p}} d\sigma(u) = 0.
\]

For \(-n \leq p < \infty, \varphi \in \mathcal{S}^+_c\) and \( K \in \mathcal{K}^n_c \), we define the point \( e_p^c(K) \) to be the origin.

**Remark 3.4.** In general, for \(-n + 1 < p < \infty\), a minimizing or maximizing point of
\[
\int_{S^{n-1}} \frac{(h_K(u) - x \cdot u)^p}{\varphi(u)} d\sigma
\]
may fail to be in the interior of \( K \).

**Definition 3.5.** Let \(-n \leq p < \infty. Since K satisfies \( \int \frac{u}{(h_K(u) - e_p \cdot u)^{1-p}} d\sigma = 0 \),
the indefinite \( \sigma \)-integral of \( (h_K(u) - e_p \cdot u)^{p-1} \) satisfies the sufficiency condition of Minkowski’s existence theorem in \( \mathbb{R}^n \). Hence, there exists a unique convex body (up to translations), denoted by \( \Lambda_p K \), whose surface area measure satisfies
\[
(3.1) \quad dS_{\Lambda_p K} = \left( \frac{V(K)}{\frac{1}{n} \int_{S^{n-1}} h^{p-1}_K d\sigma} \right) \frac{1}{h^{p-1}_K - e_p} d\sigma,
\]
see Theorem 4 of [15]. In addition, when \(-n \leq p \leq -n + 1 \) and \( \varphi \in \mathcal{S}^+ \), or \(-n \leq p < \infty, \varphi \in \mathcal{S}^+_c\) and \( K \in \mathcal{K}^n_c \), we define \( \Lambda_p^c K \) as a convex body with positive curvature function
\[
(3.2) \quad f_{\Lambda_p^c K} = \left( \frac{V(K)}{\frac{1}{n} \int_{S^{n-1}} h^{p-1}_K d\sigma} \right) \frac{1}{\varphi h^{p-1}_K - e_p^c}.
\]

Notice that when \( p = 1, \Lambda_p^c K \) is a ball. We point that our definition of \( \Lambda_p \) differs considerably from the usual definition of Lutwak, see [47, p. 554], but agrees when \( p = -n \) with Petty’s definition [44]. In the sequel, we will assume, after translation, that \( \Lambda_p K \) and \( \Lambda_p^c K \) have the same centroids as \( K \). That is, \( \text{cent}(K) = \text{cent}(\Lambda_p K) = \text{cent}(\Lambda_p^c K) \).
Remark 3.6. From the definition of the mixed volume we have $V_1(\Lambda_p K, K) = V(K)$. As a result, by the Minkowski mixed volume inequality $V(K) \geq V(\Lambda_p K)$, Equality holds if and only if $\Lambda_p K = K$. Using Minkowski’s mixed volume inequality once more, we get

\[ V_1(K, \Lambda_p K) \geq V(\Lambda_p K). \]

Here the equality holds if and only if $\Lambda_p K = K$. Similarly if $-n \leq p \leq -n + 1$ and $\varphi \in S^+$, or $-n \leq p < \infty$, $\varphi \in S^+_e$ and $K \in K^*_e$, we get

\[ V_1(K, \Lambda^p_\varphi K) \geq V(\Lambda^p_\varphi K), \]

and the equality holds if and only if $\Lambda^p_\varphi K = K$.

Remark 3.7. If $K \in F^n$, then by definition $K$ is of class $C^\infty_+$, so $h_K \in C^\infty$. In fact, by definition of the class $C^\infty_+$, the Gauss map $\nu$ is a diffeomorphism of class $C^\infty$ and $h_K(u) = \nu^{-1}(u) \cdot u$ is $C^\infty$. In this case, since $\Lambda^p K$, $\Lambda^p_\varphi K$ are solutions of the Minkowski problem with positive $C^\infty$ prescribed data, $\Lambda^p K$, $\Lambda^p_\varphi K$ are of class $C^\infty_+$, see Cheng-Yau [15, Theorem 1].

Definition 3.8.

\[
\mathcal{A}_p(K) := \begin{cases} 
V(K) \left( \int_{S^{n-1}} (h_K(u) - \varphi \cdot u)^p \, d\sigma \right)^{-\frac{1}{p}} & \text{if } -n \leq p < \infty \land p \neq 0 \\
V(K) \exp \left( \int_{S^{n-1}} -\frac{\log(h_K(u) - \varphi \cdot u)}{\omega_n} \, d\sigma \right) & \text{if } p = 0.
\end{cases}
\]

For $-n \leq p \leq -n + 1$, $\varphi \in S^+$, or $-n \leq p < \infty$, $p \neq 0$, $\varphi \in S^+_e$ and $K \in K^*_e$,

\[
\mathcal{A}^p_\varphi(K) := V(K) \left( \int_{S^{n-1}} \frac{(h_K(u) - \varphi \cdot u)^p}{\varphi(u)} \, d\sigma \right)^{-\frac{1}{p}}.
\]

For $p = 0$, $\varphi \in S^+_e$ and $K \in K^*_e$,

\[
\mathcal{A}^p_0(K) := V(K) \exp \left( \int_{S^{n-1}} \frac{\log h_K}{\omega_n} \, d\sigma \right).
\]

Next, we introduce a new family of entropy functionals.

Definition 3.9. $\mathcal{B}_p(K) := \frac{V(K)^{n-1} \mathcal{A}_p(K)}{V(\Lambda_p K)^{n-1}}$ and $\mathcal{B}^p_\varphi(K) := \frac{V(K)^{n-1} \mathcal{A}^p_\varphi(K)}{V(\Lambda^p_\varphi K)^{n-1}}$.

Remark 3.10. Note that we have $\mathcal{A}_p(K) \leq \mathcal{B}_p(K)$, and $\mathcal{A}^p_\varphi(K) \leq \mathcal{B}^p_\varphi(K)$. It will be shown that functionals $\frac{\log \mathcal{B}_p}{1-p}$, $\frac{\log \mathcal{B}^p_\varphi}{1-p}$, $p \neq 1$, are strictly increasing unless $K_t$ solves (in the Alexandrov sense) (1.3). From this point of view, they will play roles in deducing the asymptotic shapes under the flows, see (5.1).

4. Long-Time Existence

Lemma 4.1. Let $\{K_t\}$ be a solution of (1.2) on $[0, t_0]$. If $c_1 \leq h_{K_t} \leq c_2$ on $[0, t_0]$, then $\mathcal{K} \geq \frac{1}{a \cdot b + \delta}$ on $(0, t_0]$, where $a$ and $b$ depend only on $c_1, c_2, p, \varphi$. In particular, $\mathcal{K} \geq c_4$ on $[0, t_0]$ for some positive finite number that depends on the initial data, $c_1, c_2, p, \varphi$ and is independent of $t_0$.

Proof. Applying Tso’s trick to the evolution equation for polar bodies, Lemma 2.3, as in the proof of [28, Lemma 4.3] gives $\mathcal{K} \geq \frac{1}{a \cdot b + \delta}$ on $(0, t_0]$. It also follows from the proof of [28, Lemma 4.3] that $a$ and $b$ depend only on $c_1, c_2, p, \varphi$. The lower bound for $\mathcal{K}$ on $[0, \delta]$ for $\delta > 0$ small enough follows from the short-time
existence of the flow. The lower bound for $K$ on $[\delta, t_0]$ follows from the inequality $K \geq \frac{1}{a + b \delta}$. \qed

**Lemma 4.2.** Let $\{K_t\}$ be a solution of (1.2) on $[0, t_0]$. If $0 < c_2 \leq h_{K_t} \leq c_1 < \infty$ on $[0, t_0]$, then $K \leq c_3 < \infty$ on $[0, t_0]$. Here $c_3$ depends on the initial data, $c_1, c_2, p, \varphi$ and $t_0$.

**Proof.** We apply Tso’s trick to the speed of (1.2) similar to the argument given in the proof of [28, Lemma 4.1] to get

$$\partial_t \varphi \frac{h^{2-p}}{2c_1 - h} \geq -c' \left( \frac{\varphi h^{2-p}}{2c_1 - h} \right)^2,$$

where $c' > 0$ depends only on $c_1, c_2, p, \varphi$. Therefore,

$$\varphi \frac{h^{2-p}}{2c_1 - h}(t, u) \geq \frac{1}{c't + 1/ \min_{u \in \mathbb{S}^{n-1}} \varphi \frac{h^{2-p}}{2c_1 - h}(0, u)} \geq \frac{1}{c't_0 + 1/ \min_{u \in \mathbb{S}^{n-1}} \varphi \frac{h^{2-p}}{2c_1 - h}(0, u)}.$$

The corresponding claim for the Gauss curvature follows. \qed

These last two lemmas are enough to establish the long-time existence of solutions to (1.2) when the initial body is in $\mathcal{F}_0^2$.

**Lemma 4.3.** If $-2 \leq p < 2$ and $K_0 \in \mathcal{F}_0^2$ the lifespan of the solution to (1.2) is finite, and infinite when $p \geq 2$.

**Proof.** Let $-2 \leq p < 2$. We can put a tiny disk centered at the origin inside of $K_0$. This disk flows to infinity in finite time, so by comparison principle $K_t$ cannot exist eternally. For $p > 2$, consider an origin centered disk $B_R$, such that $K_0 \subset B_R$. Then $K_t \subset B_{R(t)}$, where $R(t) = \left( \max h_{K_0} \right)^{p-2} + t(p-2) \max \varphi \right)^{\frac{1}{p-2}}$. Thus, for any finite time $t_0$, $\{h_{K_t}\}$ remains uniformly bounded on $[0, t_0]$. So by Lemmas 4.1, 4.2 the evolution equation (1.2) is uniformly parabolic on $[0, t_0]$ and bounds on higher derivatives of the support function follow. Therefore, we can extend the solution smoothly past time $t_0$. \qed

**Proposition 4.4.** If $K_0 \in \mathcal{F}_0^2$, then the solution to (1.2) satisfies $\lim_{t \to T} \max h_{K_t} = \infty$.

**Proof.** First, let $p > 2$. In this case the flow exists on $[0, \infty)$. For this reason, we may insert a tiny disk inside of $K_0$ and use the comparison principle to prove the claim: Consider an origin centered disk $B_r$, such that $K_0 \supset B_r$. Then $K_t \supset B_{r(t)}$, where $r(t) = \left( \min h_{K_0} \right)^{p-2} + t(p-2) \max \varphi \right)^{\frac{1}{p-2}}$ and $B_{r(t)}$ expands to infinity as $t$ approaches $\infty$. When $p = 2$ the argument is similar. Second, if $p < 2$, then the flow exists only on a finite time interval. If $\max h_{K_t} < \infty$, then by Lemmas 4.1, 4.2 the evolution equation (1.2) is uniformly parabolic on $[0, T]$. Thus, the result of Krylov and Safonov [33] and standard parabolic theory allow us to extend the solution smoothly past time $T$, contradicting its maximality. \qed

To prove the long-time existence of solutions to (1.2) when $n \geq 3$ and $p = -n$, the next step is obtaining uniform lower and upper bounds on the principal curvatures. To get uniform lower and upper bounds on the principal curvatures we will use the evolution equation of $S^*_1 := \sum \lambda_i^*$. 
Lemma 4.5. Let $n \geq 3$, $\varphi \equiv 1$ and $p = -n$. Assume that $\{K_t\}$ is a solution of (1.2) on $[0, t_0]$. If $c_2 \leq h_{K_t} \leq c_1$ and $c_4 \leq K \leq c_3$ on $[0, t_0]$, then
\[
\frac{1}{C (1 + t^{-(n-2)})^{n-2}} \leq \kappa_t \leq C \left( 1 + t^{-(n-2)} \right)
\]
on $[0, t_0]$, for some $C > 0$ independent of $K_0$ and depending on $c_1, c_2, c_3, c_4, p$. In particular, $\kappa_t$ are uniformly bounded above and stay uniformly away from zero on $[0, t_0]$.

Proof. Since $c_2 \leq h_{K_t} \leq c_1$, we have $c'_2 \leq h^*(\cdot, t) = h_{K_t^*} \leq c'_1$. Moreover, since $c_4 \leq K \leq c_3$ on $[0, t_0]$, in view of the following identity we get $c'_4 \leq K^* \leq c'_3$ on $[0, t_0]$: for every $x \in \partial K$ there exists an $x^* \in \partial K^*$ such that
\[
\left( \frac{K}{h_{n+1}} \right)(x) \left( \frac{K^*}{h_{n+1}} \right)(x^*) = 1,
\]
where $x$ and $x^*$ are related by $x \cdot x^* = 1$ (This identity in $\mathbb{R}^3$ dates back at least to 1934, Salowski [45]. For proof of this identity see Guggenheimer [24, Proposition 2], Kaltenbach [32] or Oliker-Simon [37, Identity (7.33)] for smooth hypersurfaces, and Hug [26, Theorem 2.8] for non-smooth hypersurfaces.). Now we calculate the evolution equation of $r^*_i j$ using Lemma 2.3. Set $\rho := \left( \frac{|h^{n+2} + i\bar{h}^n|^2}{h_{n+1}} \right)$. Therefore,
\[
\partial_t r^*_i j = \rho S^{-2}_{n-1}(S^*_n)^{i}_{k} \bar{\nabla}_k \nabla_i r^*_j - 2 \rho S^{-3}_{n-1} \nabla_i S^{-1}_{n-1} \nabla_j S^*_n
\]
\[
+ \rho S^{-2}_{n-1}(S^*_n)^{i}_{kl} \nabla_i r^*_k \nabla_j r^*_m
\]
\[
+ (n - 2) \rho S^{-1}_{n-1} \rho_{ij} - \rho S^{-2}_{n-1}(S^*_n)^{i}_{kl} \bar{r}^*_j \bar{g}_{kl}
\]
\[
+ S^{-3}_{n-1} \bar{\nabla}_i \bar{\nabla}_j \rho + S^{-3}_{n-1} \bar{\nabla}_i \rho \bar{\nabla}_j S^*_n + S^{-3}_{n-1} \bar{\nabla}_j \rho \bar{\nabla}_i S^*_n.
\]

Thus, the evolution equation of $S^*_i$ satisfies
\[
\partial_t S^*_i = \rho S^{-2}_{n-1}(S^*_n)^{i}_{kl} \nabla_k \frac{S^*_n}{n} \nabla_i S^*_n - 2 \rho S^{-3}_{n-1} \nabla_i S^*_n
\]
\[
+ \rho g^{ij} S^{-2}_{n-1}(S^*_n)^{i}_{kl} \nabla_i r^*_k \nabla_j r^*_m
\]
\[
+ (n - 2)(n - 1) \rho S^{-1}_{n-1} - \rho S^{-2}_{n-1} S^*_1 \nabla_k \bar{g}_{kl}
\]
\[
- S^{-3}_{n-1} \bar{\nabla}_i \rho + 2 S^{-3}_{n-1} \bar{\nabla}_i \bar{\nabla}_j S^*_n.
\]

a: Estimating the terms on the first line: The first term on the first line is a good term viewed as an elliptic operator which is non-positive at the point and direction where the maximum of $S^*_i$ is achieved. The second term is a good non-positive term.

b: Estimating the term on the second line: Concavity of $\frac{1}{S^{-1}_{n-1}}$ gives
\[
\left[ \frac{(S^*_n)^{''}_{t}}{n-1} - \frac{2(n-2)}{(n-1)S^{-1}_{n-1}} S^*_n \right] \nabla_i \nabla_j S^*_n \leq 0.
\]

c: Estimating the last term on the second line: By Newton's inequality we get
\[
S^{-2}_{n-1} S^*_1 \nabla_k \bar{g}_{kl} = S^{-2}_{n-1} S^*_1 S^*_2 \geq C S^{-2}_{n-1} S^*_1 S^*_2 \geq C S^*_1.
\]

d: Estimating the terms on the last line: We apply the Bochner formula
\[
\bar{\Delta} |\bar{\nabla} f|^2 = 2 |\bar{\nabla} \bar{\nabla} f|^2 + 2 \bar{\nabla} \bar{\nabla} i \bar{\nabla} \bar{\nabla} j f + 2 \bar{\nabla} i f \bar{\nabla} j (\bar{\Delta} f)
\]
\[
= 2 |\bar{\nabla} f|^2 + 2(n-2) |\bar{\nabla} f|^2 + 2 \bar{\nabla} f \bar{\nabla} (\bar{\Delta} f)
\]
to $h^*(\cdot, t)$, here $f$ is defined on sphere. At the point where the maximum of $S^*_1$ is achieved we have $\bar{\nabla}S^*_1 = 0$. Therefore, we have

$$-\bar{\Delta}\bar{\nabla}h^* = 2\bar{\nabla}\bar{\Delta}h^* - 2\bar{\nabla}h^* \geq -C,$$

where we used boundedness of $|\nabla h^*|$ from above which follows from $h^* \leq \epsilon$. Taking into account this last inequality and that $(p + n) \geq 0$ and $(p + n - 2)(p + n) \geq 0$ imply that

$$-\bar{\Delta}\rho \leq CS^*_1 + C,$$

where we used boundedness of $|\nabla h^*|$, $c'_2 \leq h^* \leq c'_1$, and $c'_4 \leq K^* \leq c'_3$. To estimate the other term on the last line we use Young’s inequality and that $p = -n$ (this is the only place that the assumption $p = -n$ is imposed):

$$|\bar{\nabla}\rho \cdot \bar{\nabla}S^*_{n-1}| \leq \frac{1}{2}(|\bar{\nabla}S^*_{n-1}|^2 + \epsilon^{-1}|\bar{\nabla}\rho|^2) \leq C|\bar{\nabla}S^*_{n-1}|^2 + \epsilon^{-1}.$$

Combining inequalities (4.1), (4.2), (4.3), and (4.4) with the lower and upper bounds on $S^*_{n-1}$, we obtain for $\epsilon > 0$ small enough that

$$\partial_t S^*_1 \leq C' \left(1 + S^*_1 - CS^*_1 \frac{4}{n-2}\right).$$

This implies that $S^*_1 \leq C(1 + t^{-(n-2)})$ for some $C > 0$ depending on $c_1, c_2, c_3, c_4, p$. Therefore, in view of $K^* \leq c'_3$, we get

$$\frac{1}{C(1 + t^{-(n-2)})} \leq \kappa^*_i \leq C \left(1 + t^{-(n-2)}\right)^{n-2}$$

on $(0, t_0]$. Finally Theorem 2.1 yields

$$\frac{1}{C(1 + t^{-(n-2)})^{n-2}} \leq \kappa_i \leq C \left(1 + t^{-(n-2)}\right)$$

on $(0, t_0]$. Thus uniform lower and upper bounds for $\{\kappa_i\}$ on $[0, t_0]$ follow. □

**Lemma 4.6.** Let $n \geq 1$, $\varphi \equiv 1$ and $p = -n$. Assume that $\{K_i\}$ is a solution of (1.2) with $K_0 \in \mathcal{F}_0^n$. Then the lifespan of the solution is finite.

**Proof.** Taking into account Lemmas 4.1, 4.2, 4.5, the proof is similar to the one for Lemma 4.3. □

**Proposition 4.7.** Let $n \geq 3$, $\varphi \equiv 1$ and $p = -n$. Assume that $\{K_i\}$ is a solution of (1.2) with $K_0 \in \mathcal{F}_0^n$, then $\lim_{t \to T} \max h_{K_i} = \infty$.

**Proof.** Considering Lemmas 4.1, 4.2, 4.5, the proof is similar to the one for Proposition 4.4. □

**Proposition 4.8.** If $p = -n$, $\varphi \equiv 1$ and $K_0 \in \mathcal{F}_0^n$, then $\lim_{t \to T} V(K_t) = \infty$.

**Proof.** Since hypersurfaces are expanding, for each $t$ we can put a cone $C_t$ of height $\max h_{K_i}$ and with an origin-centered ball with a time-independent radius as the base of $C_t$. Lemma 4.7 shows that $\lim_{t \to T} V(C_t) = \infty$. The claim follows. □
5. Monotonicity of entropies along the flow

**Lemma 5.1.** The following statements hold:
- Let $-n \leq p < \infty$. If $\varphi \equiv 1$ and $e_p(K_0) = \bar{\sigma}$, then $e_p(K_t) = \bar{\sigma}$.
- Let $-n \leq p \leq -n + 1$ and $\varphi \in S^+$. If $e_p^e(K_0) = \bar{\sigma}$, then $e_p^e(K_t) = \bar{\sigma}$.

**Proof.** We justify the first claim.

We have

$$\frac{d}{dt} \int_{S_{n-1}} \frac{u}{h_{K_t}^{1-p}(u)} d\sigma(u) = (p - 1) \int_{S_{n-1}} \frac{u}{K_t(u)} d\sigma(u) = 0.$$ 

Therefore, $e_p(K_t) = \bar{\sigma}$ on $[0, T)$. \hfill \Box

**Lemma 5.2.** The following statements hold:
- Let $-n \leq p < \infty$ and $\varphi \equiv 1$. If $e_p(K_0) = \bar{\sigma}$, then

$$\frac{d}{dt} V(\Lambda_p K_t) = n \int_{S_{n-1}} V(\Lambda_p K_t) \frac{h_{K_t}^{2-p}}{K_t^2} d\sigma - \frac{(1 - p)n^2}{n - 1} V(K_t) V(\Lambda_p K_t) \int_{S_{n-1}} \frac{h_{K_t}^p}{K_t} d\sigma.$$

- If $-n \leq p \leq -n + 1$, $\varphi \in S^+$ and $e_p^e(K_0) = \bar{\sigma}$, or $-n \leq p < \infty$, $\varphi \in S^+_e$ and $K_0 \in F^n_e$, then

$$\frac{d}{dt} V(\Lambda_p^e K_t) = n \int_{S_{n-1}} \frac{V(\Lambda_p^e K_t)}{V(K_t)} \frac{h_{K_t}^{2-p}}{K_t^2} d\sigma - \frac{(1 - p)n^2}{n - 1} V(K_t) V(\Lambda_p^e K_t) \int_{S_{n-1}} \frac{h_{K_t}^p}{\varphi} d\sigma.$$

**Proof.** We will prove the first claim. Taking Lemma 5.1 into account, computation is straightforward:

$$\frac{d}{dt} V(\Lambda_p K_t) = \frac{1}{n - 1} \int_{S_{n-1}} h_{\Lambda_p K_t} \partial_t \left( \frac{V(K_t)}{\int \frac{h_{K_t}^p}{d\sigma} h_{K_t}^{1-p}} \right) d\sigma.$$

Note that Remark 3.7 justifies that taking time-derivative of the Gauss curvature of $\Lambda_p^e K_t$ is legitimate. \hfill \Box

**Lemma 5.3.** We have for
- $-n \leq p < \infty$ and $\varphi \equiv 1$: $\frac{d}{dt} A_p(K_t) \geq 0$, and if $e_p(K_0) = \bar{\sigma}$ and $p \neq 1$ then $\frac{d}{dt} \log B_p(K_t) \geq 0$.
- $-n \leq p \leq -n + 1$ and $\varphi \in S^+$: $\frac{d}{dt} A_p^e(K_t) \geq 0$, and if $e_p^e(K_0) = \bar{\sigma}$ then $\frac{d}{dt} B_p^e(K_t) \geq 0$.
- $-n \leq p < \infty$, $\varphi \in S^+_e$ and $K_0 \in F^n_e$: $\frac{d}{dt} A_p^e(K_t) \geq 0$, and if $p \neq 1$ then $\frac{d}{dt} \log B_p^e(K_t) \geq 0$. 

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Proof. We prove the claims for $B_p$ and $A_p$ and $p \neq 0$. Monotonicity of $A_p(K_t)$ follows from the Hölder inequality:

$$\frac{d}{dt} A_p(K_t) = \left( \int_{S^{n-1}} h^{2-p}_{K_t} \frac{d\sigma}{K_t^2} \right)^{\frac{2}{2-p}} \left( \int_{S^{n-1}} h^{p}_{K_t} d\sigma \right)^{\frac{p}{p-1}} + \frac{nV(K_t)}{\left( \int_{S^{n-1}} h^{p}_{K_t} d\sigma \right)^{\frac{p}{p-1}}} \left( \int_{S^{n-1}} \frac{u}{h^{p}_{K_t}} d\sigma \right) \frac{d}{dt} c_p(K_t)

\geq 0.

Here we used the inverse function theorem to justify that $\frac{d}{dt} c_p(K_t)$ exists. Monotonicity of $B_p(K_t)$ follows from Lemma 5.1 and inequality (3.3):

$$\frac{d}{dt} B_p(K_t) = \frac{1}{V(\Lambda_p K_t)^n} \left[ nV(K_t)^{n-1} V(\Lambda_p K_t) \left( \int_{S^{n-1}} h^{2-p}_{K_t} \frac{d\sigma}{K_t^2} \right)^{-\frac{2}{2-p}} \left( \int_{S^{n-1}} h^{p}_{K_t} d\sigma \right)^{-\frac{p}{p-1}} - nV(K_t)^{n+1} V(\Lambda_p K_t) \left( \int_{S^{n-1}} h^{p}_{K_t} d\sigma \right)^{-\frac{p}{p-1}} \right.

+ (1-p)n^2 V(K_t)^{n+1} V(\Lambda_p K_t) \left( \int_{S^{n-1}} h^{p}_{K_t} d\sigma \right)^{-\frac{p}{p-1}} + n^2 p V(K_t)^{n+1} V(\Lambda_p K_t) \left( \int_{S^{n-1}} h^{p}_{K_t} d\sigma \right)^{-\frac{p}{p-1}} \left] \right. \right] V(\Lambda_p K_t) \left( \int_{S^{n-1}} h^{p}_{K_t} d\sigma \right)^{-\frac{p}{p-1}} (V(\Lambda_p K_t) - 1) \geq 0.

\square

6. Bounding extrinsic diameter

Lemma 6.1. Fix $0 < a < \infty$. Suppose $V(K) = \omega_n$ and

$$\begin{cases}
    \int_{S^{n-1}} (h_K(u) - e_p \cdot u)^p d\sigma \leq a & \text{if } p \in (0, \infty) \\
    \exp \left( \frac{1}{n} \int_{S^{n-1}} \log(h_K(u) - e_0 \cdot u) d\sigma \right) \geq a & \text{if } p = 0 \\
    \int_{S^{n-1}} (h_K(u) - e_p(K) \cdot u)^p d\sigma \geq a & \text{if } p \in (-n, 0).
\end{cases}

Then the extrinsic diameter of $K$, $d(K)$, is bounded above by a positive number which is independent of $K$. The same statement also holds for the following cases:
• when $-n < p \leq -n+1$, $\varphi \in S^+$ and
$$
\int_{S^{n-1}} \frac{(h_K(u) - e_p^\varphi(K) \cdot u)^p d\sigma}{\varphi(u)} \geq a,
$$

• when $\varphi \in S^+_e$, $K \in K^n_e$ and
$$
\begin{cases}
\int_{S^{n-1}} \frac{h_K^p}{\varphi} d\sigma \leq a & \text{if } p \in (0, \infty) \\
\exp \left( \frac{1}{n^2} \int_{S^{n-1}} \frac{-\log h_K}{\varphi} d\sigma \right) \geq a & \text{if } p = 0 \\
\int_{S^{n-1}} \frac{h_K^p}{\varphi} d\sigma \geq a & \text{if } p \in (-n, 0).
\end{cases}
$$

Proof. We prove only the first set of claims. Proof of Guan and Ni [23, Corollary 2.5] extends to the interval $p \in (-1, \infty)$. For $-n < p < 0$, we argue as follows [16, p. 58]: Suppose on the contrary that there is a sequence of convex bodies $\{K_i\}$ satisfying the uniform lower bound, but $d(K_i) \to \infty$. Since the above inequalities are invariant under any translation, we may assume without loss of generality that $K_i$ are centered at the origin. Let $E_i$ denote John’s ellipsoid of $K_i$. That is, $\frac{1}{n} E_i \subset K_i \subset E_i$. Therefore, $\frac{h_{E_i}}{n} < h_{K_i} \leq h_{E_i}$. For any fixed $\varepsilon > 0$, we decompose $S^{n-1}$ into three sets as follows:
$$
S_1 := S^{n-1} \cap \{ h_{E_i} < \varepsilon \}, \quad S_2 := S^{n-1} \cap \{ \varepsilon < h_{E_i} < \frac{1}{\varepsilon} \}, \quad & S_3 := S^{n-1} \cap \{ h_{E_i} > \frac{1}{\varepsilon} \}.
$$

On the one hand,
$$
a \leq \int_{S^{n-1}} (h_{K_i}(u) - e_p(K_i) \cdot u)^p d\sigma \leq \int_{S^{n-1}} h_{K_i}^p d\sigma < \int_{S^{n-1}} \left( \frac{h_{E_i}}{n} \right)^p d\sigma.
$$

On the other hand, as $d(K_i) \to \infty$:
$$
\int_{S_1} \left( \frac{h_{E_i}}{n} \right)^p d\sigma \leq \left( \int_{S_1} \left( \frac{h_{E_i}}{n} \right)^{-n} d\sigma \right)^{\frac{p}{n}} |S_1|^{\frac{n}{p-n}} \\
\leq c_1 |S_1|^{\frac{n}{p-n}} \to 0,
$$
$$
|S_2| \to 0.
$$

Also, we have
$$
\int_{S_3} \left( \frac{h_{E_i}}{n} \right)^p d\sigma \leq \int_{S_3} \left( \frac{1}{n\varepsilon} \right)^p d\sigma = \left( \frac{1}{n\varepsilon} \right)^p |S_3| \leq c_3 \varepsilon^{-p}.
$$

Therefore, for any $\varepsilon > 0$ we get
$$
a \leq o(1) + c_3 \varepsilon^{-p}.
$$
Sending $\varepsilon \to 0$, we reach a contradiction. $\square$

7. Continuity of entropy map and entropy functional

Theorem 7.1 (Continuity of entropy map). The following maps are continuous.

• $e_p : (K^n, d_H) \to \mathbb{R}^n$, for $-n \leq p < \infty$,

• $e_p^\varphi : (K^n, d_H) \to \mathbb{R}^n$, for $-n \leq p \leq -n+1$ and $\varphi \in S^+$.

Here $d_H$ denotes the Hausdorff distance.
The first two claims follow from the continuity of \( A_p \). Suppose \( p \in (0,1) \). Let \( \{K_i\} \) be a family of convex bodies which converges to \( K_\infty \) as \( i \) approaches \( \infty \). Suppose for a subsequence \( i_j \) that \( \lim_{j \to \infty} e_{p}(K_{i_j}) = q \neq e_p(K_\infty) \), where \( q \in K_\infty \). Note that \( e_p(K_\infty) \) is the unique maximizer of \( \int_{S^{n-1}} (h_{K_\infty}(u) - x \cdot u)^p d\sigma \) on \( K_\infty \). Now we consider the case \(-n \leq p < 0\). Note that

\[
\limsup_{j \to \infty} \int_{S^{n-1}} (h_{K_{i_j}}(u) - e_{p}(K_{i_j}) \cdot u)^p d\sigma \\
\leq \limsup_{j \to \infty} \int_{S^{n-1}} (h_{K_{i_j}}(u) - e_{p}(K_\infty) \cdot u)^p d\sigma = \int_{S^{n-1}} (h_{K_{\infty}}(u) - e_{p}(K_{\infty}) \cdot u)^p d\sigma.
\]

This contradicts that \( e_{p}(K_{\infty}) \) is the unique maximizer of \( \int_{S^{n-1}} (h_{K_{\infty}}(u) - x \cdot u)^p d\sigma \) on \( K_{\infty} \). Now we consider the case \(-n \leq p < 0\). Note that

\[
\liminf_{j \to \infty} \int_{S^{n-1}} (h_{K_{i_j}}(u) - e_{p}(K_{i_j}) \cdot u)^p d\sigma \geq \int_{S^{n-1}} (h_{K_{\infty}}(u) - q \cdot u)^p d\sigma.
\]

Thus, we have

\[
\int_{S^{n-1}} (h_{K_{\infty}}(u) - e_{p}(K_{\infty}) \cdot u)^p d\sigma \geq \int_{S^{n-1}} (h_{K_{\infty}}(u) - q \cdot u)^p d\sigma.
\]

Contradiction! \( \square \)

**Remark 7.2.** For \( \varphi \equiv 1 \) and \( p = -n \), Theorem 7.1 was proved by Petty [43, Lemma 2.2].

**Theorem 7.3** (Continuity of \( A_p \)). The following functionals are continuous.

- \( A_p : (K^n, d_H) \to \mathbb{R}^n \), for \(-n \leq p < \infty\),
- \( A_{p}^{\varphi} : (K^n, d_H) \to \mathbb{R}^n \), for \(-n \leq p \leq -n + 1 \) and \( \varphi \in S^+ \),
- \( A_{p}^{\varphi} : (K^n, d_H) \to \mathbb{R}^n \), for \(-n \leq p < \infty \) and \( \varphi \in S^+ \).

**Proof.** The first two claims follow from the continuity of \( e_{p}(\cdot) \), \( e_{p}^{\varphi}(\cdot) \), Theorem 7.1, and that \( e_{p}, e_{p}^{\varphi} \) are interior points. The last claim is trivial in view of our agreement to set \( e_{p}^{\varphi}(K) = 0 \) whenever \( K \in K_{c}^\nu \). \( \square \)

**Theorem 7.4.** Fix \( p \) and \( 0 < a < \infty \). Define the entropy class \( S_{p,a} \) to be the set of all convex bodies \( K \) such that \( V(K) = \omega_n \) and

\[
\begin{align*}
&\left\{ \begin{array}{ll}
\text{if} \ p \in (0, \infty) \\
\text{if} \ p = 0 \\
\text{if} \ p \in (-n, 0)
\end{array} \right. \\
&\exp \left( \frac{1}{\omega_n} \int_{S^{n-1}} - \log h_K d\sigma \right) \geq a
\end{align*}
\]

Then there exist \( 0 < r, R < \infty \) depending only on \( n, p, a \) such that for any \( K \in S_{p,a} \) we have \( r \leq h_K \leq R \). Additionally, similar conclusions hold for the following sets:
• When \( \varphi \in S^+ \) and \(-n < p \leq -n + 1 \), define the entropy class \( S_{\varphi, p, a} \) to be the set of all convex bodies \( K \) such that \( V(K) = \omega_n \) and

\[
e_p^\varphi(K) = \bar{\omega}, \int_{S_n-1} \frac{h_K^p}{\varphi} d\sigma \geq a.
\]

• When \( \varphi \in S^+_e \), define \( S_{e, \varphi, p, a} \) to be the set of all origin-symmetric convex bodies \( K \) such that \( V(K) = \omega_n \) and

\[
\begin{cases}
\int_{S_n-1} \frac{h_K^p}{\varphi} d\sigma \leq a & \text{if } p \in (0, \infty) \\
\exp \left( \frac{1}{\omega_n} \int_{S_n-1} -\log h_K d\sigma \right) \geq a & \text{if } p = 0 \\
\int_{S_n-1} \frac{h_K^p}{\varphi} d\sigma \geq a & \text{if } p \in (-n, 0).
\end{cases}
\]

Proof. The last set of claims follow easily: \( S_{e, \varphi, p, a} \subset \mathcal{K}_e^n \) and \( \{d(K)\}_{K \in S_{e, \varphi, p, a}} \) is uniformly bounded by Lemma 6.1. Therefore, since volume is fixed, in-radii of convex bodies in \( S_{e, \varphi, p, a} \) are uniformly bounded below. Moreover, for \( K \in \mathcal{K}_e^n \), the ball with maximal radius enclosed by \( K \) must be centered at the origin. Next we prove remaining claims. Since volume is normalized and \( \{d(K)\}_{K \in S_{e, \varphi, p, a}} \) is uniformly bounded by Lemma 6.1, it is enough to prove there exists \( r > 0 \) such that \( h_K > r \) for any \( K \in S_{e, \varphi, p, a} \). Suppose on the contrary that there is a sequence of convex bodies \( \{K_i\} \subset S_{e, \varphi, p, a} \) such that dist(\( \bar{\omega}, \partial K_i \)) \( \to 0 \). By Lemma 6.1, \( \{d(K_i)\} \) is uniformly bounded above. Thus by Blaschke’s selection theorem \( \{K_i\} \) converges (passing to a further subsequence if necessary) to \( K_\infty \) in the Hausdorff distance and additionally \( \bar{\omega} \in \partial K_\infty \). On the other hand, by Theorems 7.1 and 7.3, \( K_\infty \in S_{e, \varphi, p, a} \). This is a contradiction.

Corollary 7.5. Under the assumptions of Theorems 1.2, A1, A2, A3 there exist \( r, R \) such that \( 0 < r < h_{K_i} \leq R < \infty \).

Proof. By Lemma 5.1 entropy points remain at the origin. By Lemma 5.3, \( K_i \) for \( t > 0 \) belongs to the same entropy class as \( K_0 \) does. Therefore, the claim follows from Theorem 7.4.

Theorem 7.6. The following statements are true:

• Let \(-n \leq p < \infty \), and assume \( \{K_i\} \subset \mathcal{K}_0^n \) with \( e_p(K_i) = \bar{\omega} \) converges in the Hausdorff distance to \( K_\infty \). Then \( \{\Lambda_p K_i\} \) converges in the Hausdorff distance to \( \Lambda_p K_\infty \).

• Let \(-n \leq p < \infty \), \( \varphi \in S^+_e \), and assume that \( \{K_i\} \subset \mathcal{K}_e^n \) converges in the Hausdorff distance to \( K_\infty \). Then \( \{\Lambda_p^e K_i\} \) converges in the Hausdorff distance to \( \Lambda_p^e K_\infty \).

• Let \(-n \leq p \leq -n + 1 \), \( \varphi \in S^+ \), and assume \( \{K_i\} \subset \mathcal{K}_0^n \) with \( e_p^{\varphi}(K_i) = \bar{\omega} \) converges in the Hausdorff distance to \( K_\infty \). Then \( \{\Lambda_p^{\varphi} K_i\} \) converges in the Hausdorff distance to \( \Lambda_p^{\varphi} K_\infty \).

Proof. We give the proof of the first statement. By Theorem 7.1, \( \bar{\omega} = e_p(K_i) \to e_p(K_\infty) \). Since \( e_p(K_\infty) = \bar{\omega} \) is an interior point of \( K_\infty \), we have \( r \leq h_{K_i} \leq R \) for some \( 0 < r, R < \infty \) independent of \( i \). On the other hand, in view of (3.1), cent(\( \Lambda_p K_i \)) = cent(\( K_i \)), and [15, Lemma 3, Lemma 4], we conclude that there exist \( 0 < r', R' < \infty \) independent of \( i \), such that \( \Lambda_p K_i \subset B_{R} \) and \( V(\Lambda_p K_i) \geq \omega_n r'^n \) (Note that in both Lemmas 3, 4 of [15] the assumption that \( K \) is of class \( C^4 \) is unnecessary). Therefore, here we do not need to know the regularity of \( \Lambda_p K_i \). Take a convergent subsequence of \( \{\Lambda_p K_i\} \) and denote it again by \( \{\Lambda_p K_i\} \). The
limiting figure must be a convex body, say \( \tilde{K} \). Choose an arbitrary convex body \( P \).

From the weak continuity of surface area measures we get

\[
\lim_{i \to \infty} \int_{S^{n-1}} h_p dS_{\Lambda_p K_i} = \int_{S^{n-1}} h_p dS_{\tilde{K}} = V_1(\tilde{K}, P).
\]

Moreover,

\[
\lim_{i \to \infty} \int_{S^{n-1}} h_p dS_{\Lambda_p K_i} = \lim_{i \to \infty} \left( \frac{V(K_i)}{n} \int_{S^{n-1}} h_p h_{K_i}^{p-1} d\sigma \right)
= \frac{V(K_\infty)}{n} \int_{S^{n-1}} h_p h_{K_\infty}^{p-1} d\sigma
= \int_{S^{n-1}} h_p dS_{\Lambda_p K_\infty} = V_1(\Lambda_p K_\infty, P).
\]

Since \( V_1(\Lambda_p K_\infty, P) = V_1(\tilde{K}, P) \) holds for any convex body \( P \), we conclude that \( \Lambda_p K_\infty \) is a translation of \( \tilde{K} \), see [47, Theorem 8.1.2]. Furthermore, note that \( \cent(K_i) = \cent(\Lambda_p K_i) \), thus \( \cent(\Lambda_p K_\infty) = \cent(K_\infty) = \cent(\tilde{K}) \). That is, no translation is needed; \( \Lambda_p K_\infty = \tilde{K} \). Finally, notice that the limit is independent of the convergent subsequence. The proof is complete. \( \square \)

**Lemma 7.7.** Under the assumptions of Theorems 1.1, 1.2, A1, A2, A3 we have

\[
\begin{cases}
\lim_{t \to T} \frac{1}{p} \log \int_{S^{n-1}} \frac{h_{K_t}^p}{\varphi} d\sigma = \infty & \text{if } p \neq 0 \\
\lim_{t \to T} \frac{1}{p} \log \int_{S^{n-1}} \frac{h_{K_t}^p}{\varphi} d\sigma = \infty & \text{if } p = 0.
\end{cases}
\]

**Proof.** We prove the claim for \( p \neq 0 \) and \( \varphi \equiv 1 \). First we consider \( p \neq -n \). Corollary 7.5 shows that \( 0 < r \leq h_{K_t} \leq R < \infty \). Thus \( \frac{\min \{h_{K_t}^{p-1} \}^{1/p}}{R} \geq \frac{1}{p} \log \int_{S^{n-1}} h_{K_t}^p d\sigma \geq \frac{1}{p} \log \int_{S^{n-1}} (\max \{h_{K_t}^{p-1} \})^p d\sigma + c \). Since \( \max h_{K_t} \to \infty \) by Propositions 4.4 and 4.7, the claim follows. Now we consider case \( p = -n \). Since \( V(K) \int_{S^{n-1}} \frac{h_{K_t}^p}{\varphi} d\sigma = GL(n) \) invariant and \( A_{-n}(K_t) \) is monotone along the flow, we get

\[
\lim_{t \to T} n A_{-n}(K_t) = \lim_{t \to T} V(K_t) \int_{S^{n-1}} \frac{1}{h_{K_t}^{n}} d\sigma = \lim_{t \to T} V(l_t K_t) \int_{S^{n-1}} \frac{1}{h_{l_t K_t}^{n}} d\sigma
\leq \limsup_{t \to T} V(l_t K_t) \int_{S^{n-1}} \frac{1}{h_{l_t K_t-p_t}^{n}} d\sigma.
\]

where \( l_t \in GL(n) \) and \( p_t \in \text{int} l_t K_t \). Note that to get this last inequality we used the fact that \( \int_{S^{n-1}} \frac{1}{h_{l_t K_t-p_t}^{n}} d\sigma \) is minimized on \( l_t K_t \) only when \( p_t = e_{-n}(l_t K_t) = l_t e_{-n}(K_t) = \tilde{a} \). (See Petty [43, Lemma 2.2] for a proof that the Santaló point mapping, \( e_{-n} \), is affinely equivariant and continuous.) Moreover, we can choose \( l_t, p_t \) such that \( V(l_t K_t) = \omega_n \), and \( 0 < r < h_{l_t K_t-p_t} < R < \infty \) for some universal constants \( r, R \). This ensures that the limit of \( n A_{-n}(K_t) \) along the flow is finite. That is, for some \( c < \infty \) we have

\[
V(K_t) \int_{S^{n-1}} \frac{1}{h_{K_t}^p} d\sigma < c.
\]
Now recall Proposition 4.8 that $\lim_{t \to T} V(K_t) = \infty$. Therefore,

$$\lim_{t \to T} \int_{S^{n-1}} \frac{1}{h_{K_t}^n} d\sigma = 0.$$ 

\[ \square \]

**Lemma 7.8.** Under each assumptions of Theorems 1.1, A1, A2, A3 we have for a subsequence of $\{K_t\}$ that

$$\lim_{t_i \to T} \frac{V_i(K_{t_i}, \Lambda_p^2 K_{t_i})}{V(\Lambda_p^2 K_{t_i})} = 1, \quad \lim_{t_i \to T} \frac{V_i(K_{t_i}, \Lambda_p K_{t_i})}{V(\Lambda_p K_{t_i})} = 1.$$ 

**Proof.** We prove the claim for $p \neq 0$. Suppose on the contrary that there exist $\varepsilon > 0$ and $t_0 > 0$, such that for any $t > t_0$ we have $\frac{V_i(K_{t_i}, \Lambda_p^2 K_{t_i})}{V(\Lambda_p^2 K_{t_i})} - 1 \geq \varepsilon$. From (5.1) it follows that

$$\frac{d}{dt} \log B_p^2(K_t) \geq n \frac{d}{dt} \left( \frac{1}{p} \log \int_{S^{n-1}} \frac{h_{K_t}^p}{\varphi} d\sigma \right).$$

On the other hand, by Lemma 7.7, $\frac{1}{p} \log \int_{S^{n-1}} \frac{h_{K_t}^p}{\varphi} d\sigma$ can be made arbitrarily large if $t$ is close enough to $T$. By integrating both sides of (7.1) on $[t_0, s]$ and then sending $s \to T$ we get

$$\lim_{t \to T} \frac{\log B_p^2(K_t)}{1 - p} = \infty.$$ 

This is a contradiction in view of Theorem 7.6: First, note that $B_p^2$ is scaling-invariant. Second, when $p \neq -n$, Corollary 7.5 shows that $0 < r \leq h_{K_t} \leq R < \infty$. Therefore, by Blaschke’s selection theorem there is a subsequence, $\{K_{t_i}\}$, that converges in the Hausdorff distance to a limiting shape, say $\tilde{K}_\infty$. Taking Theorem 7.6 into consideration, we conclude that $\{\Lambda_p^2 K_{t_i}\}$ converges to $\Lambda_p^2 \tilde{K}_\infty$. Consequently,

$$\infty \leftarrow \frac{\log B_p^2(K_{t_i})}{1 - p} = \frac{\log B_p^2(\tilde{K}_t_i)}{1 - p} \to \frac{\log B_p^2(\tilde{K}_\infty)}{1 - p}.$$ 

Since $0 < r \leq h_{K_t} \leq R < \infty$, we must have $0 < B_p^2(\tilde{K}_\infty) < \infty$. When $p = -n$, note that $\log B_p^2(K_t)$ is $GL(n)$-invariant. Therefore, we may assume $\{d(l_i \tilde{K}_t)\}$ is uniformly bounded for suitable choices of $l_i \in SL(n)$. We may now continue the previous argument for $p \neq -n, \varphi \equiv 1$ to reach a contradiction. 

\[ \square \]

8. Proofs of Theorems A1, A2, A3 with convergence in the $C^1$-topology and Proofs of Theorems 1.1, 1.3

**Remark 8.1.** Since Lemma 7.8, due to the fact $\frac{d}{dt} B_1^p \equiv 0$, is not available for case $p = 1$, we will not give the proof of Theorem 1.2 in this section. See Section 9 for its proof.

**Fact 8.2.** Let $\varphi$ be a positive function on the unit sphere of class $C^{k, \alpha}$, where $k$ is a nonnegative integer and $0 < \alpha < 1$. Assume $K \in K_0^3$ has a positive continuous curvature function such that $f_K h_{K}^{1-p} = \varphi$. Then $K$ is of class $C^{k+2, \alpha}$.

**Proof.** Since $h_K > 0$, the Gauss curvature of $K$ in the generalized sense (in the sense of Alexandrov) and thus in the viscosity sense is pinched between two positive numbers. Hence, by Caffarelli [13, Corollary 3] $K$ is strictly convex (When $n = 3$
this conclusion follows from Alexandrov’s theorem 1942 [1].). This in turn implies that \( h_K \) is \( C^1 \) (cf. [47, Corollary 1.7.3]) and so the right-hand side of \( f_K = h_K^{-1} \phi \) is positive and \( C^\alpha \). Thus, \( K \) is of class \( C^{2,\alpha} \); it follows from Caffarelli’s work [14, Theorem 4] that, for given nonnegative integer \( k \) and \( 0 < \alpha < 1 \), the solution of Minkowski’s problem is of class \( C^{k+2,\alpha} \) if the given Gauss curvature on the spherical image is of class \( C^{k,\alpha} \) (see also Jerison [31, Theorem 0.7]). So \( h_K \) is \( C^2 \). Higher order regularity up to \( C^{k+2,\alpha} \) follows by induction using [14, Theorem 4].

8.1. Proofs of Theorems A1, A2, A3 with convergence in the \( C^1 \)-topology.

We prove the statements for \( p \neq 0 \). Consider the sequence introduced in Lemma 7.8. Corollary 7.5 shows that a subsequence of \( \{ \tilde{K}_i \} \) converges in the Hausdorff distance to a limiting shape \( \tilde{K}_\infty \) with the origin in its interior. By Theorem 7.6 and Lemma 7.8 we conclude that

\[
V_1(\tilde{K}_\infty, \Lambda^p_\infty \tilde{K}_\infty) = V(\Lambda^p_\infty \tilde{K}_\infty) \Rightarrow \tilde{K}_\infty = \Lambda^p_\infty \tilde{K}_\infty.
\]

In particular, we conclude that \( \lim_{t \to T} \frac{V(\Lambda^p_t \tilde{K}_t)}{V(\tilde{K}_t)} = 1 \). Therefore, \( 0 < \lim_{t \to T} \frac{A^p_t(\tilde{K}_t)}{B^p_t(\tilde{K}_t)} = \lim_{t \to T} \frac{B^p_t(\tilde{K}_t)}{A^p_t(\tilde{K}_t)} < \infty \). So monotonicity of \( A^p_t(\tilde{K}_t) \) and \( B^p_t(\tilde{K}_t) \) yield \( \lim_{t \to T} \frac{A^p_t(\tilde{K}_t)}{B^p_t(\tilde{K}_t)} = \lim_{t \to T} \frac{B^p_t(\tilde{K}_t)}{A^p_t(\tilde{K}_t)} \). This in turn has two implications:

\[
\lim_{t \to T} \left( \frac{V(\Lambda^p_t \tilde{K}_t)}{V(\tilde{K}_t)} \right)^{n-1} = \lim_{t \to T} \frac{A^p_t(\tilde{K}_t)}{B^p_t(\tilde{K}_t)} = 1,
\]

\[
\lim_{t \to T} \int_{S^{n-1}} \frac{(h_{\tilde{K}_t}(u) - e_p \cdot u)^p}{\varphi(u)} d\sigma = \lim_{t \to T} \left( \frac{A^p_t(\tilde{K}_t)}{V(\tilde{K}_t)} \right)^{-\frac{p}{n}} = \frac{\omega_n}{c} > 0.
\]

Corollary 7.5 and equalities (8.1) and (8.2) imply that every given subsequence of \( \{ \tilde{K}_i \} \) has a convergent subsequence such that its limit satisfies

\[ V(L) = V(\Lambda^p_\infty L) \Rightarrow L = \Lambda^p_\infty L, \]

and thus \( L \) is a solution of

\[
\varphi h^1_{L^{-p}} dS_L = \lim_{t \to T} \int_{S^{n-1}} \frac{(h_{\tilde{K}_t}(u) - e_p \cdot u)^p}{\varphi(u)} d\sigma = c d\sigma.
\]

Fact 8.2 implies that \( L \in \mathcal{F}^n_\varphi \). The \( C^1 \) convergence, which is purely geometric and does not depend on the evolution equation, follows from [4, Lemma 13]. Finally, when \( p \geq 1 \) and \( p \neq n \), in view of the uniqueness theorem of Lutwak [41, Corollary 2.3, \( i=0 \)], there is only one solution to \( \varphi h^1_{L^{-p}} dS_L = c d\sigma \) in \( \mathbb{R}^n \) with volume \( \omega_n \). In particular, if \( \varphi \equiv 1 \), then \( L \) must be the unit ball. When \( p = n \) the uniqueness follows from [16, Theorem B], so if \( \varphi \equiv 1 \) and \( V(L) = \omega_n \), then \( L \) must be the unit ball. In \( \mathbb{R}^2 \) a classification result of Andrews [7] states that for \( -2 < p < 1, \varphi \equiv 1 \) the only solution with area \( \pi \) is the unit disk.

8.2. Proofs of Theorems 1.1, 1.3. A convex body \( K \) has its centroid at the origin if and only if \( K^* \) has its Santaló point at the origin [47, p. 546]. Thus, in view of Lemma 2.3, there is a one-one correspondence between solutions of (1.2) for \( p = -n \) and solutions of (1.5). In view of John’s ellipsoid lemma, there exist \( l_i \in SL(n) \), such that \( d(l_i, \tilde{K}_t) \leq 2n \). Since \( e_{-n}(l_i, \tilde{K}_t) = \tilde{\sigma} \), by the Blaschke selection theorem and by Theorem 7.1 we can show that after passing to a subsequence \( l_i, \tilde{K}_t, \to \tilde{K}_\infty \)
Moreover, arguing as in Section 8.1, $K^n$ is a weak solution of $h^{1+n}_K dS_K = c d\sigma$. It was proved by Philippis and Marini [39], and Schneider [47, Theorem 10.5.1] that origin-centered ellipsoids are the only solutions of $h^{1+n}_K dS_K = c d\sigma$ (This also follows from Fact 8.2 and the classical theorem of Pogorelov [25, Theorem 4.3.1]). Therefore, by Theorem 7.3 we get $A_n(K_i) = A_n(l_i t_i) = A_n(l_i t_i) \to n\omega_n^2$. Monotonicity of $A_n(K_i)$ then implies that $\lim_{i \to 3} A_n(K_i) = n\omega_n^2$. Consequently, for any arbitrary $\varepsilon > 0$, we can choose $N$ large enough such that $V(K_N) V(K_N^*) = \frac{1}{n}A_n(K_N) > \omega_n^2(1 + \varepsilon)$. We choose $\varepsilon > 0$ small enough so that the volume product of $K_N$ is ‘pinched’ enough in the sense of the main theorem of [30]. Hence, we can apply the argument of [29, Section 6] or [30] to prove sequential convergence in the $C^\infty$-topology.

9. CONVERGENCE IN THE $C^\infty$-TOPOLOGY IN THEOREMS 1.2, A1, A2, A3

In this section we prove convergence in the $C^\infty$-topology which was claimed in Theorems 1.2, A1, A2, A3. To do so, we will only need to obtain a uniform upper bound for the Gauss curvature of the normalized solution. In fact, in Corollary 7.5 we have established the first order regularity estimate $r < h_{K^t} < R$. On the other hand, the lower bound for the Gauss curvature given in Lemma 4.1, $K \geq 1/(a + b t^{-\frac{n-1}{n}})$, is independent of the initial data on $[t_0/2, t_0]$. It is quite standard that such a bound gives a uniform lower bound for the Gauss curvature of the normalized solution. If $n \geq 3$, using uniform lower and upper bounds on the Gauss curvature of the normalized solution, we can then employ Lemma 4.5 to obtain uniform lower and upper bounds on the principal curvatures of the normalized solution. The detailed argument is discussed in Andrews [6, Section 9] and in [28, Section 5]. Note that once we have established uniform $C^K$ estimates for $\{K_t\}$, for any $k \in \mathbb{N}$, we are able to use the monotonicity of functionals $A_p^K$ to prove that the limit satisfies $\varphi h^{1-p}_K / K = c$. Compare this argument with the argument in Section 8.1, which uses the monotonicity of functionals $B_p^\varepsilon$ and only the weak convergence of solutions in the Hausdorff distance to classify the limiting shapes provided $p \neq 1$.

Lemma 9.1. We have the following evolution equations under (1.2).

$$
\partial_t |F|^{1+n-p} = |F \cdot \nu|^{2-p}\varphi \frac{\tilde{K}^{ij}}{K^2} \nabla_i \nabla_j |F|^{1+n-p} \\
+ \varphi n(1 + n - p)\frac{|F \cdot \nu|^{3-p}}{K} |F|^{-1+n-p} \\
- \varphi (1 + n - p)|F \cdot \nu|^{2-p}\frac{\tilde{K}^{ij}}{K^2} |F|^{-1+n-p} g_{ij} \\
- \varphi (-1 + n - p)(1 + n - p)|F \cdot \nu|^{2-p}|F|^{-3+n-p}\frac{\tilde{K}^{ij}}{K^2} (F_i \cdot F)(F_j \cdot F).
$$
and
\[ \partial_t \left( \frac{(F \cdot \nu)^{2-p}}{\mathcal{K}} \right) = (F \cdot \nu)^{2-p} \frac{\partial_k}{\mathcal{K}^2} \nabla_i \nabla_j \left( \frac{(F \cdot \nu)^{2-p}}{\mathcal{K}} \right) \]
\[ + (F \cdot \nu)^{4-2p} \frac{\partial_k}{\mathcal{K}^3} \frac{f^k f^j}{i_j} \]
\[ + \varphi^2 (2-p) \frac{(F \cdot \nu)^{3-2p}}{\mathcal{K}^2} \]
\[ - \varphi (2-p) (F \cdot \nu)^{1-p} F \cdot T F \left( \nabla \left( \frac{(F \cdot \nu)^{2-p}}{\mathcal{K}} \right) \right) . \]

Proof.
\[- (F \cdot \nu)^{2-p} \frac{\partial_k}{\mathcal{K}^2} \nabla_i \nabla_j |F|^{1+n-p} = \]
\[ + \varphi(n-1)(1+n-p) \frac{(F \cdot \nu)^{3-p}}{\mathcal{K}} |F|^{-1+n-p} \]
\[ - \varphi(1+n-p) (F \cdot \nu)^{2-p} |F|^{-1+n-p} \frac{\partial_k}{\mathcal{K}^2} g_{ij} \]
\[ - \varphi(-1+n-p)(1+n-p) (F \cdot \nu)^{2-p} |F|^{-3+n-p} \frac{\partial_k}{\mathcal{K}^2} (F_i \cdot F)(F_j \cdot F) . \]

On the other hand,
\[ \partial_t |F|^{1+n-p} = \varphi(1+n-p) \frac{(F \cdot \nu)^{3-p}}{\mathcal{K}} |F|^{-1+n-p} . \]

Therefore,
\[ \partial_t |F|^{1+n-p} = (F \cdot \nu)^{2-p} \frac{\partial_k}{\mathcal{K}^2} \nabla_i \nabla_j |F|^{1+n-p} \]
\[ + \varphi n(1+n-p) \frac{(F \cdot \nu)^{3-p}}{\mathcal{K}} |F|^{-1+n-p} \]
\[ - \varphi(1+n-p) (F \cdot \nu)^{2-p} \frac{\partial_k}{\mathcal{K}^2} |F|^{-1+n-p} g_{ij} \]
\[ - \varphi(-1+n-p)(1+n-p)(F \cdot \nu)^{2-p} |F|^{-3+n-p} \frac{\partial_k}{\mathcal{K}^2} (F_i \cdot F)(F_j \cdot F) . \]

To calculate the evolution equation of \( \left( \frac{(F \cdot \nu)^{2-p}}{\mathcal{K}} \right) \), we will employ the following two evolution equations which can be obtained with straightforward computations (see for example Andrews [2, Theorem 3.7]):
\[ \partial_t h^i_j = - \nabla_i \nabla^j \left( \frac{(F \cdot \nu)^{2-p}}{\mathcal{K}} \right) - \frac{(F \cdot \nu)^{2-p}}{\mathcal{K}} f^k f^j \]
\[ \partial_t \nu = - T F \left( \nabla \left( \frac{(F \cdot \nu)^{2-p}}{\mathcal{K}} \right) \right) . \]
Therefore,
\[
\partial_t \left( \frac{(\varphi \cdot \nu)^{2-p}}{K} \right) = (\varphi \cdot \nu)^{2-p} \varphi \frac{\dot{K}^{ij}}{K^2} \nabla_i \nabla_j \left( \frac{(\varphi \cdot \nu)^{2-p}}{K} \right)
+ (\varphi \cdot \nu)^{4-2p} \varphi^2 \frac{\dot{K}^{ij}}{K^3} f^k_i f^j_k \\
+ \varphi^2 (2-p) \frac{(\varphi \cdot \nu)^{3-2p}}{K^2} \\
- \varphi (2-p) \frac{(\varphi \cdot \nu)^{1-p}}{K} F \cdot T F \left( \nabla \left( \frac{(\varphi \cdot \nu)^{2-p}}{K} \right) \right).
\]

□

**Lemma 9.2.** Assume that \( n = 2 \) and \(-\infty < p < \infty \) or \( n \geq 3 \) and \( p \leq n \). Suppose there exists \( 0 < \gamma < 1 \) such that solution \( \{K_t\} \) to (1.2) satisfies \( \gamma |F| \leq F \cdot \nu \) on \([0, t_0]\). Then there exists \( \lambda > 0 \) (independent of \( t_0 \)) such that \( \chi(\cdot, t) := |F|^{1+n-p}(\cdot, t) - \lambda \varphi \frac{(\varphi \cdot \nu)^{2-p}}{K} \) is always negative on \([0, t_0]\).

**Proof.** First we consider the case \( n \geq 3 \) and \( p \leq n \). Take \( \lambda \) such that \( \chi \) is negative at time \( t = 0 \). We will prove that \( \chi \) remains negative, perhaps for a larger value of \( \lambda \). We calculate the evolution equation of \( \chi \) and apply the maximum principle to \( \chi \) on \([0, \tau]\), where \( \tau > 0 \) is the first time that for some \( y \in \partial K_\tau \) we have \( \chi(\tau, y) = 0 \). Notice that at such a point where the maximum of \( \chi \) is achieved:

\[
\nabla \chi = 0 \Rightarrow F \cdot T F \left( \nabla \left( \lambda \varphi \frac{(\varphi \cdot \nu)^{2-p}}{K} \right) \right) = F \cdot T F \left( \nabla |F|^{1+n-p} \right) \\
= (1 + n - p)|F|^{1+n-p}|F^\top|^2,
\]

where \( F^\top(\cdot, t) \) is the tangential component of \( F(\cdot, t) \) to \( \partial K_t \). Furthermore,

\[
(F \cdot \nu)^{2-p} \varphi \frac{\dot{K}^{ij}}{K^2} \nabla_i \nabla_j \chi \leq 0,
\]

and in view of the assumption \( \gamma |F| \leq F \cdot \nu \leq |F| \) we get

\[
\gamma^{2-p} \left( \frac{\lambda}{K} \right) \nabla \chi \leq |F| \leq \gamma^{-2-p} \left( \frac{\lambda}{K} \right) \nabla \chi.
\]

Moreover, recall our assumption on \( p \) that \( p \leq n \). If \( p \leq n - 1 \), we have

\[
-\varphi (-1 + n - p)(1 + n - p)(F \cdot \nu)^{2-p}|F|^{-3+n-p} \frac{\dot{K}^{ij}}{K^2} (F_i \cdot F)(F_j \cdot F) \leq 0
\]
When \( n - 1 \leq p \leq n \), we may calculate in an orthonormal frame which \([f_{ij}]\) is diagonal to get

\[
-\varphi(1 + n - p)(F \cdot \nu)^{2-p}|F|^{-1+n-p}\frac{\dot{K}_{ij}}{K^2}g_{ij}
\]

\[
-\varphi(-1 + n - p)(1 + n - p)(F \cdot \nu)^{2-p}|F|^{-3+n-p}\frac{\dot{K}_{ij}}{K^2}(F_i \cdot F)(F_j \cdot F)
\]

\[
\leq -\varphi(1 + n - p)(F \cdot \nu)^{2-p}|F|^{-1+n-p}\frac{\dot{K}_{ij}}{K^2}g_{ij}
\]

\[
-\varphi(-1 + n - p)(1 + n - p)(F \cdot \nu)^{2-p}|F|^{-3+n-p}\frac{\dot{K}_{ij}}{K^2}g_{ij}
\]

\[
= -\varphi(n - p)(1 + n - p)(F \cdot \nu)^{2-p}\frac{\dot{K}_{ij}}{K^2}|F|^{-1+n-p}g_{ij}.
\]

Since the rest of the computations are similar in either case, let us proceed with the case \( p \leq n - 1 \). Therefore, from Lemma 9.1 it follows that at \((\tau, y)\) we have

\[
\partial_t \chi \leq \varphi n(1 + n - p)\frac{(F \cdot \nu)^{3-p}}{K}|F|^{-1+n-p} -\varphi(1 + n - p)(F \cdot \nu)^{2-p}|F|^{-1+n-p}\frac{\dot{K}_{ij}}{K^2}g_{ij}
\]

\[
-\lambda(F \cdot \nu)^{4-2p}\varphi^2\frac{\dot{K}_{ij}}{K^2}f_k^j f^k_i - \lambda \varphi^2(2 - p)\frac{(F \cdot \nu)^{3-2p}}{K^2} + \varphi(2 - p)(1 + n - p)(F \cdot \nu)^{1-p}|F|^{-1+n-p}F^\top|F|^2
\]

\[
\leq \varphi n(1 + n - p)\frac{(F \cdot \nu)^{3-p}}{K}|F|^{-1+n-p}
\]

\[
-\varphi(1 + n - p)(n - 1)(F \cdot \nu)^{2-p}|F|^{-1+n-p}K^{2+\frac{n}{n-1}}
\]

\[
-\lambda \varphi^2(n - 1)(F \cdot \nu)^{4-2p}K^{\frac{n}{n-1}-3} - \lambda \varphi^2(2 - p)\frac{(F \cdot \nu)^{3-2p}}{K^2} + \varphi(2 - p)(1 + n - p)\frac{(F \cdot \nu)^{1-p}}{K}|F|^{-1+n-p}F^\top|F|^2,
\]

where for the second inequality we used \( p \leq n \), concavity and inverse-concavity of \( \dot{K}_{ij} \):

\[
\dot{K}_{ij} \delta_{ij} \geq (n - 1)K^{\frac{n}{n-1}} \quad \text{and} \quad \dot{K}_{ij}f_k^j f^k_i \geq (n - 1)K^{\frac{n}{n-1}}.
\]

By means of \( \gamma |F| \leq F \cdot \nu \leq |F| \) and inequalities (9.1) we obtain

\[
0 \leq \partial_t \chi \leq \frac{\lambda^{\frac{2n+2p}{n+1}}}{K^{\frac{n}{n-1}}} \left( c + b \lambda^{\frac{n}{n-1}} - a \lambda^{\frac{2n}{n-1}} \right)
\]

for some \( a, b, c > 0 \) depending on \( p, \gamma, \varphi \). Therefore, taking \( \lambda \) large enough proves the claim.

Next we consider the case \( n = 2 \). We observe that

\[
-\varphi(-1 + n - p)(1 + n - p)(F \cdot \nu)^{2-p}|F|^{-3+n-p}\frac{\dot{K}_{ij}}{K^2}(F_i \cdot F)(F_j \cdot F)
\]

\[
= -\varphi(1 - p)(3 - p)(F \cdot \nu)^{2-p}|F|^{-1-p}\frac{1}{K^2}(F \cdot F)^2
\]

\[
\leq |\varphi(1 - p)(3 - p)(F \cdot \nu)^{2-p}\frac{|F|^{1-p}}{K^2}|.
\]
where $s$ is the arc-length parameter, $F_s$ is the tangent vector to the curve, and we have also used $|F_s| = 1$ to derive the last inequality. Therefore,

$$0 \leq \partial_t \chi \leq \varphi 2(3 - p) \frac{(F \cdot \nu)^{3 - p}}{K} |F|^{1 - p}$$

$$- \varphi(3 - p)(F \cdot \nu)^2|F|^{1 - p} \frac{1}{K^2}$$

$$+ |\varphi(1 - p)(3 - p)(F \cdot \nu)^2|F|^{1 - p} \frac{1}{K^2}$$

$$- \lambda \varphi^2 \frac{(F \cdot \nu)^{4 - 2p}}{K} - \lambda \varphi^2(2 - p) \frac{(F \cdot \nu)^{3 - 2p}}{K^2}$$

$$+ \varphi(2 - p)(3 - p) \frac{(F \cdot \nu)^{1 - p}}{K}|F|^{1 - p}|F^\top|^2$$

$$\leq \frac{\lambda}{K^{5 - 2p}} (c + b\lambda - a\lambda^2)$$

for some $a, b, c > 0$ depending on $\gamma, \varphi$. Thus, taking $\lambda$ large enough proves the claim. $\square$

The next corollary and our discussion in the beginning of this section completes our argument to deduce $C^\infty$ convergence in Theorem 1.2.

**Corollary 9.3.** Under each assumption of Theorems 1.2, A1, A2, A3 Gauss curvature of the normalized solution is uniformly bounded above. That is,

$$\left(\frac{V(K)}{V(B)}\right)^{\frac{p}{n + 1}} \kappa(\cdot, t) < C < \infty$$

for some $C$ depending only on $K_0, p, \varphi$.

**Proof.** Corollary 7.5 guarantees that the assumption of Lemma 9.2 is satisfied. Since the degrees of homogeneity of $|F|^{1 + n - p}$ and $\lambda \varphi \frac{(F \cdot \nu)^{2 - p}}{K}$ are equal, we conclude that the upper bound for the normalized Gauss curvature. $\square$

## 10. Applications

**Theorem 10.1** (Lutwak [40]). For any convex body $K$ we have

$$V(K)V((K - e_{-n})^*) \leq \left(\frac{V(\Lambda - nK)}{V(K)}\right)^{n-1} \omega_n^2 \leq \omega_n^2.$$

**Proof.** We may first prove the claim for $C^\infty$ convex bodies. The general case follows from the first part of Theorem 7.6 and a standard approximation argument [47, Section 3.4]. Since the inequality is translation invariant, we may assume $e_{-n}(K) = \bar{\sigma}$. We then employ (1.2) with $p = -n$ and initial data $K_0 := K$. The claim follows from Lemma 5.1, monotonicity of $\mathcal{B}_{-n}(K_\ell)$ established in Lemma 5.3, and Theorem 1.1. $\square$

**Theorem 10.2.** For any convex body $K \in \mathbb{K}^2$ we have

$$\left\{ \begin{array}{ll}
V(K) \left( \int_1 (h_K(u) - \epsilon_p \cdot u)^p d\sigma \right)^{\frac{1}{p}} \geq \pi(2\pi)^{-\frac{1}{p}} \frac{V(\Lambda_n K)}{V(K)} & \text{if } p > 1 \\
V(K) \left( \int_1 (h_K(u) - \epsilon_p \cdot u)^p d\sigma \right)^{\frac{1}{p}} \leq \pi(2\pi)^{-\frac{1}{p}} \frac{V(\Lambda_n K)}{V(K)} & \text{if } -2 < p < 1 \& p \neq 0 \\
V(K) \exp \left( \frac{\lambda \ln(h_K(u) - \epsilon_p \cdot u) d\sigma}{\pi} \right) \leq \pi \frac{V(\Lambda_n K)}{V(K)} & \text{if } p = 0.
\end{array} \right.$$
Proof. The claims follow from Lemmas 5.1, 5.3 and from Theorem A1.

As we have already mentioned, for \( p > 1 \) \( \Lambda_p K = K \) implies that \( K \) is a ball (cf. Section 8.1). In the remaining of this section, we give a stability version of this fact in \( \mathbb{R}^2 \); we prove a stronger statement: if \( \frac{V(K)}{V(\Lambda_p K)} \) is close to one, then \( K \) is close to a disk in the Hausdorff distance. To this end, we first recall Urysohn’s inequality. Let us denote the mean width of \( K \in \mathcal{K}^n \) by \( w(K) = \frac{2}{n \omega_n} \int_{S^{n-1}} h_K d\sigma \). Urysohn’s inequality states that

\[
V(K) \left( \frac{w(K)}{2} \right)^2 \leq \omega_n,
\]

and equality holds exclusively for balls. The next lemma gives a lower bound for this ratio.

**Lemma 10.3.** Assume that \( p > 1 \). For \( K \in \mathcal{K}^2 \) we have

\[
\pi \frac{V(\Lambda_p K)}{V(K)} \leq \left( \frac{V(K)}{w(K)} \right)^{\frac{p}{p-1}}.
\]

Proof. From Theorem 10 we have

\[
\pi(2\pi - \frac{4}{p}) \frac{V(\Lambda_p K)}{V(K)} \leq V(K) \left( \int_{S^1} (h_K(u) - e_p \cdot u)^p d\sigma \right)^{-\frac{2}{p}}.
\]

Applying the Hölder inequality to the right-hand side completes the proof.

**Theorem 10.4.** Assume that \( K \in \mathcal{K}^2 \) and \( p > 1 \). There exist \( \gamma, \varepsilon_0 > 0 \) such that the following holds. If \( \frac{V(K)}{V(\Lambda_p K)} \leq 1 + \varepsilon \) for \( \varepsilon \leq \varepsilon_0 \), then there exist \( x \in \text{int } K \) and an origin-centered disk \( \bar{B} \) such that

\[
d_H \left( \left( \frac{\pi}{V(K)} \right)^{\frac{1}{p}} (K - x), \bar{B} \right) \leq \gamma \varepsilon^{\frac{1}{p}}.
\]

Proof. We may assume \( V(K) = \pi \). From Lemma 10.3 we get \( w \leq 2(1 + \varepsilon)^{\frac{1}{p}} \). On the other hand, by [47, Inequality 7.31, p. 385]

\[
\left( \frac{w}{2} \right)^2 - 1 \geq \left[ \left( \frac{w}{2} \right) - \frac{1}{\rho} \right]^2 \geq \left[ 1 - \frac{1}{\rho^+} \right]^2.
\]

Consequently,

\[
\varepsilon^{\frac{1}{p}} \geq 1 - \frac{1}{\rho^+} \Rightarrow \rho^+ \leq \frac{1}{1 - \varepsilon^{\frac{1}{p}}}.
\]

Suppose the Hausdorff distance of \( K \) and a ball \( \bar{B}, K \subset \bar{B} \), of radius \( \frac{1}{1 - \varepsilon^{\frac{1}{p}}} \) and center \( x \in \text{int } K \) is \( d \). Then, the volume of the hyperspherical cap of height \( d \) inside \( \bar{B} \) is given by

\[
V_{\text{cap}} = \pi^{\frac{p+1}{2}} \frac{1}{\Gamma\left( \frac{p+1}{2} \right)} \frac{1}{1 - \varepsilon^{\frac{1}{p}}} \left( 1 - d(1 - \varepsilon^{\frac{1}{p}}) \right) \int_0^{\arccos(1-d(1-\varepsilon^{\frac{1}{p}}))} \sin^n(t) dt.
\]
Therefore,
\[
\frac{\pi^{\frac{n-1}{2}}}{\Gamma \left( \frac{n+1}{2} \right)} \int_0^\pi \sin^n(t) \, dt = V(K)
\]
\[
\leq V(\bar{B}) - V_{\text{cap}}
\]
\[
= \frac{\pi^{\frac{n-1}{2}} \left( \frac{1}{1-x_n} \right)^\frac{n}{2}}{\Gamma \left( \frac{n+1}{2} \right)} \left[ \int_0^\pi \sin^n(t) \, dt - \gamma_n' \left[ \arccos \left( 1 - d(1 - \varepsilon \frac{1}{n}) \right) \right]^\frac{n+1}{2} + \text{lower order terms} \right]
\]
\[
\leq \frac{\pi^{\frac{n-1}{2}} \left( \frac{1}{1-x_n} \right)^\frac{n}{2}}{\Gamma \left( \frac{n+1}{2} \right)} \left[ \int_0^\pi \sin^n(t) \, dt - \gamma_n' \frac{2^{\frac{n+1}{2}}}{2^{\frac{n+1}{2}}} \left( d(1 - \varepsilon \frac{1}{n}) \right)^\frac{n+1}{2} \right].
\]

This proves that the Hausdorff distance of \( K - x \) from \( \bar{B} - x \) is bounded above by \( \gamma \varepsilon \frac{1}{n} \) for some universal constant \( \gamma > 0 \). □

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