The limit of a Stanley-Wilf sequence is not always rational, and layered patterns beat monotone patterns

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Abstract

We show the first known example for a pattern \( q \) for which \( L(q) = \lim_{n \to \infty} \sqrt[n]{S_n(q)} \) is not an integer. We find the exact value of the limit and show that it is irrational, but algebraic. Then we generalize our results to an infinite sequence of patterns. We provide further generalizations that start explaining why certain patterns are easier to avoid than others. Finally, we show that if \( q \) is a layered pattern of length \( k \), then \( L(q) \geq (k - 1)^2 \) holds.

1 Introduction

Let \( S_n(q) \) be the number of permutations of length \( n \) (or, in what follows, \( n \)-permutations) that avoid the pattern \( q \). For a brief introduction to the area of pattern avoidance, see [4]; for a more detailed introduction, see [5]. A recent spectacular result of Marcus and Tardos [8] shows that for any pattern \( q \), there exists a constant \( c_q \) so that \( S_n(q) < c_n^q \) holds for all \( n \). As pointed out by Arratia in [11], this is equivalent to the statement that \( L(q) = \lim_{n \to \infty} \sqrt[n]{S_n(q)} \) exists. Let us call the sequence \( \sqrt[n]{S_n(q)} \) a Stanley-Wilf sequence. It is a natural and intriguing question to ask what the limit \( L(q) \) of a Stanley-Wilf sequence can be, for various patterns \( q \).

The main reason this question has been so intriguing is that in all cases where \( L(q) \) has been known, it has been known to be an integer. Indeed, the results previously known are listed below.

1. When \( q \) is of length three, then \( L(q) = 4 \). This follows from the well-known fact [10] that in this case, \( S_n(q) = \binom{2n}{n}/(n + 1) \).

2. When \( q = 123 \cdots k \), or when \( q \) is such that \( S_n(q) = S_n(12 \cdots k) \), then \( L(q) = (k - 1)^2 \). This follows from an asymptotic formula of Regev [9].

3. When \( q = 1342 \), or when \( q \) is such that \( S_n(q) = S_n(1342) \), then \( L(q) = 8 \). See [3] for this result and an exact formula for the numbers \( S_n(1342) \).

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In this paper, we show that $L(q)$ is not always an integer. We achieve this by proving that $14 < L(12453) < 15$. Then we compute the exact value of this limit, and see that it is not even rational; it is the number $9 + 4\sqrt{2}$. We compute the limit of the Stanley-Wilf sequence for an infinite sequence of patterns, and see that as the length $k$ of these patterns grows, $L(q)$ will fall further and further below the largest known possible value, $(k - 1)^2$. Finally, we show that while for certain patterns, our methods provide the exact value of the limit of the Stanley-Wilf sequence, for certain others they only provide a lower bound on this limit. This starts explaining why certain patterns are easier to avoid than others. Among other results, we will confirm a seven-year old conjecture by proving that in the sense of logarithmic asymptotics, a layered pattern $q$ is always easier to avoid than the monotone pattern of the same length.

2 Proving an upper bound

Let $p = p_1p_2\cdots p_n$ be a permutation. Recall that $p_i$ is called a left-to-right minimum of $p$ if $p_j > p_i$ for all $j < i$. In other words, a left-to-right minimum is an entry that is smaller than everything on its left. Note that $p_1$ is always a left-to-right minimum, and so is the entry 1 of $p$. Also note that the left-to-right minima of $p$ always form a decreasing sequence. For the rest of this paper, entries that are not left-to-right minima are called remaining entries.

Now we are in a position to prove our promised upper bound for the numbers $S_n(12453)$.

Lemma 2.1 For all positive integers $n$, we have

$$S_n(12453) < (9 + 4\sqrt{2})^n < 14.66^n.$$ 

Proof: Let $p$ be a permutation counted by $S_n(12453)$, and let $p$ have $k$ left-to-right minima. Then we have at most \( \binom{n}{k} \) choices for the set of these left-to-right minima, and we have at most \( \binom{n}{k} \) choices for their positions. The string of the remaining entries has to form a 1342-avoiding permutation of length $n - k$. Indeed, if there was a copy $acdb$ of 1342 among the entries that are not left-to-right minima, then we could complete it to a 12453 pattern by simply prepending it by the closest left-to-right minimum that is on the left of $a$. The number of 1342-avoiding permutations on $n - k$ elements is less than $8^{n-k}$ as we know from [3]. This shows that

$$S_n(12453) < \sum_{k=1}^{n} \binom{n}{k} \cdot 8^{n-k} < \sum_{k=1}^{n} \left( \frac{n}{k} \cdot \sqrt{8}^{n-k} \right)^2 \leq \left( \sum_{k=1}^{n} \binom{n}{k} \cdot \sqrt{8}^{n-k} \right)^2 < (1 + \sqrt{8})^{2n} = (9 + 4\sqrt{2})^n,$$

and the proof is complete. \( \diamond \)

Corollary 2.2 We have

$$L(12453) < 14.66.$$
3 Proving a lower bound

We have seen in Corollary 2.2 that $L(12453) = 9 + 4\sqrt{2} < 14.66$. In order to prove that this limit is not an integer, it suffices to show that it is larger than 14. In what follows, we are going to work towards a good lower bound for the numbers $S_n(12453)$, and thus the number $L(12453)$.

Where is the waste in the proof of the upper bound in the previous section? The waste is that there are some choices for the left-to-right minima that are incompatible with some choices for the 1342-avoiding permutation of the remaining entries. This is a crucial concept of the upcoming proof, so we will make it more precise.

We have mentioned in the previous section, that determining the left-to-right minima of a permutation $p$ means to determine the set $T$ of positions these minima will be, and to determine the set $Z$ of entries that are the left-to-right minima. In other words, the ordered pair $(T, Z)$ of equal-sized subsets of $[n] = \{1, 2, \ldots, n\}$ describes the left-to-right minima of $p$.

**Definition 3.1** Let $n$ be a positive integer, and let $m \leq n$ be a positive integer. Let $T$ and $Z$ be two $m$-element subsets of $[n]$. Finally, let $S$ be a permutation of the elements of the set $[n] - Z$. If there exists an $n$-permutation $p$ so that its left-to-right minima are precisely the elements of $Z$, they are located in positions belonging to $T$, and its string of remaining entries is $S$, then we say that the triple $(T, Z, S)$ is compatible. Otherwise, we say that the triple $(T, Z, S)$ is incompatible.

Clearly, if $(T, Z, S)$ is compatible, then there is exactly one permutation $p$ satisfying all criteria specified by $(T, Z, S)$.

**Example 3.2** If $n = 4$, and $T = \{1, 3\}$, $Z = \{1, 2\}$, and $S = 43$, then $(T, Z, S)$ is compatible as shown by the permutation 2413.

**Example 3.3** If $n = 4$, and $T = \{1, 3\}$, $Z = \{1, 3\}$, and $S = 24$, then $(T, Z, S)$ is incompatible. Indeed, the only permutation allowed by $T$ and $S$ is 3214, but for this permutation $Z = \{1, 2, 3\}$, not $\{1, 3\}$.

Returning to the method by which we proved our upper bound for $L(12453)$, we will show that in a sufficient number of cases, our triples $(T, Z, S)$ are compatible. This will show that the upper bound is quite close to the precise value of $L(12453)$.

What is a good way to check that a particular choice $(T, Z)$ of left-to-right minima is compatible with a particular choice of $S$? For shortness, let us call the procedure of putting together $S$ and a string $(T, Z)$ of left-to-right minima merging. One has to check that in the permutation obtained by merging our left-to-right minima with $S$, the left-to-right minima are indeed the entries in $Z$. That is, there are no additional left-to-right minima, and there the entries in $Z$ are indeed all left-to-right minima. This is achieved exactly when any remaining entry is larger than the closest left-to-right minimum on its left.
In our efforts to find a good lower bound on $L(12453)$, we will only consider a special kind of permutations. Let $N$ be a positive integer so that $S_n(1342) > 7.99^n$ for all $n > N$. (We know from [3] that such an $N$ exists as $L(1342) = 8$.)

Consider permutations whose string $S$ of remaining entries has the following property. If we cut $S$ into $\lfloor |S|/N \rfloor$ blocks of consecutive entries of length $N$ each (the last block can be of size between $N$ and $2N - 1$), then the entries of any given block $B$ are all smaller than the entries on any block on the left of $B$, and larger than the entries of any block on the right of $B$. Let us call these strings $S$ block-structured. See Fig. 1 for the generic diagram of a block-structured string in the (unrealistic) case of $N = 2$.

![Figure 1: A block-structured string.](image)

The number of such strings $S$ is obviously more than $7.99^\lfloor |S|/N \rfloor$. It is obvious that they are all 1342-avoiding as a 1342-pattern cannot start in a block and end in another one. We claim that a sufficient number of these strings $S$ will be compatible with a sufficient number of the choices $(T, Z)$ of left-to-right minima.

First, look at the very special case when $S$ is decreasing. In this case, we will write $S^{\text{dec}}$ instead of $S$. Now our permutation $p$ consists of two decreasing sequences (so it is 123-avoiding), namely the left-to-right minima and $S^{\text{dec}}$. The following Proposition is very well-known.

**Proposition 3.4** Let $1 \leq m \leq n$. Then the number of 123-avoiding $n$-permutations having exactly $m$ left-to-right minima is

$$A(n, m) = \frac{1}{n} \binom{n}{m} \binom{n}{m+1},$$

(1)

a Narayana number.

For a proof, see [11] or [5].

The significance of this result for us is the following. If we just wanted to merge $(T, Z)$ and $S$ together, with no regard to the existing constraints, the total number of ways to do that would be of course at most $\binom{n}{m} \cdot \binom{n}{m}$. The above formula shows that roughly $\frac{1}{n}$ of these mergings will actually be good, that is, they will not violate any constraints, they will lead to compatible triples $(T, Z, S)$. The factor $\frac{1}{n}$ is not a significant loss from our point of view, since $\lim_{n \to \infty} \sqrt[n]{1/n} = 1$. 

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Now let us return to the general case of block-structured strings $S$. In other words, take a 123-avoiding $n$-permutation $(T, Z, S^{\text{dec}})$, and replace its string $S^{\text{dec}}$ by a block-structured string $S$ taken on the entries that belong to $S^{\text{dec}}$. We claim that after this replacement, a sufficient number of triples $(T, Z, S)$ will be compatible.

Here is the outline of the proof of that claim. Because of the structure of a block-structured $S$, it is true that every entry in $S$ is at most $N$ positions away from the position it was $S^{\text{dec}}$. Therefore, if we merge $(T, Z)$ and $S^{\text{dec}}$ together so that each remaining entry $x$ is not simply larger than the left-to-right minimum that is closest to and preceding the position $j$ of $x$, but also larger than the left-to-right minimum closest to and preceding position $j - N$, then we will be done. Indeed, in this case replacing $S^{\text{dec}}$ by any block-structured string $S$ will not violate any constraints.

Therefore, we will have a lower bound for the number of compatible triples $(T, Z, S)$ if we find a lower bound for the number of compatible triples $(T, Z, S^{\text{dec}})$ in which each remaining entry has the mentioned stronger property.

In order to find such a lower bound, take a 123-avoiding permutation $p'$ which is of length $n - N$. Let $p'$ have $m$ left-to-right minima. Denote $(T', Z')$ the string of the left-to-right minima of $p'$, and let $S^{\text{dec}}'$ denote the decreasing string of remaining entries of $p'$. Now prepend $p'$ with the decreasing string taken on the $N$-element set \{\(n - N + 1, n - N + 2, \ldots, n\}\}, to get an $n$-permutation. In this $n$-permutation, move each of the original $m$ left-to-right minima of $p'$ to the left by $N$ positions. Let us call the obtained $n$-permutation $p''$.

It is then clear that the left-to-right minima of $p''$ are the same as the left-to-right minima of $p'$. Furthermore, because of the translation we used to create our new permutation, $p''$ has the property that if $x$ is a remaining entry of $p''$ and is in position $j$, then $x$ is larger than the left-to-right minimum that is closest to, and preceding, position $j - N$.

Now we can use the argument that we outlined four paragraphs ago. For easy reference, we sketch that argument again. If $S^{\text{dec}}$ is replaced by any block-structured permutation of the same size taken on the same set of elements, (resulting in the $n$-permutation $p^*$) then each remaining entry $x$ will move within its block only, that is, $x$ will move at most $N$ positions from its original position. Therefore, $x$ will still be larger than the left-to-right minimum closest to it and preceding it.

This shows that if $p'$ and $(T', Z')$ lead to a compatible triple, then so too will $p^*$ and $(T, Z)$, where $(T, Z)$ describes the left-to-right minima of $p^*$. Proposition \ref{prop:compatible_triples} implies that the number of compatible triples $(T', Z', p')$ is $\frac{1}{m} \binom{n - N}{m} \binom{n - N + 1}{m + 1}$. As $N$ is a constant, we have

$$\lim_{n \to \infty} \sqrt[n]{\frac{1}{m} \binom{n - N}{m} \binom{n - N + 1}{m + 1}} \tag{2}$$

Now restrict our attention to the particular case when $m = \lfloor n/3 \rfloor$. We claim that permutations of this particular type are sufficiently numerous to provide the lower bound we need. Using Stirling’s formula, a routine computation yields that in this case, we have

$$\lim_{n \to \infty} \sqrt[n]{\frac{n}{m}} \left( \frac{n}{m} \right) \left( \frac{n}{m} \right) = \left( \frac{3^n}{2^{2n/3}} \right)^2 \geq 1.88^{2n}.$$
Besides, we have more than $7.99^{2n/3}$ choices for the block-structured string $S$ by which we replace $S_{\text{dec}}$. Therefore, we have proved the following lower bound.

**Lemma 3.5** For $n$ sufficiently large, the number of $n$-permutations of length $n$ that avoid the pattern $12453$ is larger than

$$1.88^{2n} \cdot 7.99^{2n/3} \geq 14.12^n.$$ 

Lemma 3.5 and Corollary 2.2 together immediately yield the following.

**Theorem 3.6** We have

$$14.12 \leq L(12453) \leq 14.66.$$ 

In particular, $L(12453) = \lim_{n \to \infty} \sqrt[n]{S_n(12453)}$ is not an integer.

### 4 The exact value of $L(12453)$

If we are a little bit more careful with our choice of $m$ in the argument of the previous section, we can find the exact value of $L(12453)$. It turns out to be the upper bound proved in Corollary 2.2.

**Theorem 4.1** We have $L(12453) = (1 + \sqrt{8})^2 = 9 + 4\sqrt{2}$.

**Proof:** The above argument works for any $m < n$ instead of $m = \lfloor n/3 \rfloor$, and for any positive real number $8 - \epsilon < 8$ instead of $7.99$.

In order to find the best possible lower bound for $L(12453)$, set $m = \alpha n$. Repeating the argument of the proof of Lemma 3.5, we see that for $n$ sufficiently large, we have $S_n(12453) \geq (\frac{n}{\alpha n})^2 (8 - \epsilon)^{n - \alpha n}$. Then the function $f(\alpha) = \lim_{n \to \infty} (\frac{n}{\alpha n})^2 (8 - \epsilon)^{n - \alpha n}$ has a maximum on the compact interval $[0, 1]$. Choose the $\alpha$ providing that maximum. We claim that for that optimal $\alpha$, we must have $f(\alpha) = 9 + 4\sqrt{2}$. Indeed, we have

$$(1 + \sqrt{8 - \epsilon})^{2n} = \left( \sum_{m=0}^{n} \binom{n}{m}^2 (8 - \epsilon)^{n-m} \right)^2 \leq (n+1)^2 \frac{n}{\alpha n}^2 (8 - \epsilon)^{n - \alpha n},$$

because the square of an $(n+1)$-term sum has $(n+1)^2$ terms. Taking $n$th roots, and then taking limits as $n$ goes to infinity, we see that

$$(1 + \sqrt{8 - \epsilon})^2 \leq f(\alpha)$$

for any positive $\epsilon$, proving our claim. $\Diamond$
5 Some generalizations

In this Section, we will provide some interesting generalizations of our results. We will need the following simple recursive properties of pattern avoiding permutations.

**Proposition 5.1** Let \( q \) be a pattern of length \( k \) that starts with 1, and let \( q' \) be the pattern of length \( k + 1 \) that is obtained from \( q \) by adding 1 to each entry of \( q \) and prepending it with 1. Let \( p \) be a permutation whose string of remaining entries is \( S \). Then the following hold.

1. If \( S \) avoids \( q \), then \( p \) avoids \( q' \).
2. If \( q \) itself starts with 1, then \( p \) avoids \( q' \) if and only if \( S \) avoids \( q \).

Iteratively applying part 2 of Proposition 5.1 and the method explained in the previous sections, we get the following theorem.

**Theorem 5.2** Let \( k \geq 4 \), and let \( q_k \) be the pattern \( 12 \cdots (k - 3)(k - 1)k(k - 2) \). So \( q_4 = 1342 \), \( q_5 = 12453 \), and so on. Then we have

\[
L(q_k) = (k - 4 + \sqrt{8})^2.
\]

**Proof:** Induction on \( k \). For \( k = 4 \), the result is proved in [3], and for \( k = 5 \), we have just proved it in the previous section. Assuming that the statement is true for \( k \), we can prove the statement for \( k + 1 \) the very same way we proved it for \( k = 5 \), using the result for \( k = 4 \), and part 2 of Proposition 5.1.

The method we used to prove Lemma 2.1 can also be used to prove the following recursive result.

**Lemma 5.3** Let \( q \) be a pattern of length \( k \) that starts with 1, and let \( q' \) be the pattern of length \( k + 1 \) that is obtained from \( q \) by adding 1 to each entry of \( q \) and prepending it with 1. Let \( c \) be a constant so that \( S_n(q) < c^n \) for all \( n \). Then we have

\[
S_n(q') < (1 + \sqrt{c})^{2n} = (1 + c + 2\sqrt{c})^n.
\]

This is an improvement of the previous best result [3], that only showed \( S_n(q') < (4c)^n \).

The following generalization of Theorem 4.1 can be proved just as that Theorem is.

**Theorem 5.4** Let \( q \) and \( q' \) be as in Lemma 5.3. Then we have

\[
L(q') = 1 + L(q) + 2\sqrt{L(q)}.
\]
In a sense, this result generalizes Regev’s result [9] that showed that $L(12 \cdots k) = (k - 1)^2$. Our result shows that this particular growth rate, that is, that $\sqrt{L(q)}$ grows by one as the pattern grows by one, is not limited to monotone patterns.

An interesting consequence of this Theorem is that if $q$ is as above, and $L(q) < (k - 1)^2$, in other words, $q$ is harder (or easier, for that matter) to avoid than the monotonic pattern of the same length, then repeatedly prepending $q$ with 1 will not change this. That is, the obtained new patterns will still be more difficult to avoid than the monotonic pattern of the same length.

Are the methods presented in this paper useful at all if the pattern $q$ does not start in the entry 1? We will show that for most patterns $q$, the answer is in the affirmative, as far as a lower bound is concerned. Let us say that the pattern $q$ is indecomposable if it cannot be cut into two parts so that all entries on the left of the cut are larger than all entries on the right of the cut. For instance, 1423 and 3142 are indecomposable, but 3412 is not as we could cut it after two entries. Therefore, we call 3412 decomposable. It is routine to verify that as $k$ grows, the ratio of indecomposable patterns among all $k!$ patterns of length $k$ goes to 1.

**Theorem 5.5** Let $q$ be an indecomposable pattern of length $k$, and let $L = \lim_{n \to \infty} \sqrt[n]{S_n(q)}$. Let $q'$ be defined as in Lemma 5.3. Then we have

$$\lim_{n \to \infty} \sqrt[n]{S_n(q')} \geq 1 + L + 2\sqrt{L}.$$  

**Proof:** This Theorem can be proved as Lemma 3.5 and Theorem 4.1 are. Indeed, as $q$ is indecomposable, any block-structured string $S$ will avoid $q$ if each block does. Now apply part 1 of Proposition 5.1 to see that our argument will still provide the required lower bound. ◇

If $q = q_1 q_2 \cdots q_k$ is an indecomposable pattern, then so is its reverse complement, that is, the pattern $q^c$ whose $i$th entry is $k + 1 - q_{k+1-i}$ for all $i$. This leads to the following Corollary.

**Corollary 5.6** Let $q$ be an indecomposable pattern, and let $q''$ be the pattern obtained from $q$ by prepending $q$ with a 1, and appending the entry $k + 2$ to the end of $q$. Then we have

$$L(q'') \geq \left(2 + \sqrt{L(q)}\right)^2.$$  

Our methods will not provide an upper bound for $\lim_{n \to \infty} \sqrt[n]{S_n(q'')}$ because the string $S$ of remaining entries of a $q''$-avoiding permutation does not have to be $q$-avoiding. (Only part 1 and not part 2 of Proposition 5.1 applies.) That condition is simply sufficient, but not necessary, in this general case. Nevertheless, Theorem 5.5 is interesting. It shows that for almost all patterns $q$, if we prepend $q$ by the entry 1, the limit of the corresponding Stanley-Wilf sequence will grow at least as fast as for monotone $q$. If $q$ started in 1, then this growth will be the same as for monotone $q$.

Now it is a little easier to understand why, in the case of length 4, the patterns that are the hardest to avoid, are along with certain equivalent ones, 1423 and 1342. Indeed, removing the starting 1 from
them, we get the decomposable patterns 423 and 342. As these patterns are decomposable, Theorem 5.5 does not hold for them, so the limit of the Stanley-Wilf sequence for the patterns 1423 or 1342 does not have to be at least $1 + 4 + 4 = 9$, and in fact it is not.

A particularly interesting application of Theorem 5.4 is as follows. Recall that a layered pattern is a pattern that consists of decreasing subsequences (the layers) so that the entries increase among the layers. For instance, 3217654 is a layered pattern. In 1997, several people (including present author) have observed, using numerical evidence computed in [12], that if $q$ is a layered pattern of length $k$, then for small $n$, the inequality $S_n(12\ldots k) \leq S_n(q)$ seems to hold. We will now show that this is indeed true in the sense of logarithmic asymptotics.

**Theorem 5.7** Let $q$ be a layered pattern of length $k$. Then we have

$$L(q) \geq (k-1)^2.$$ 

Equivalently, $L(q) \geq L(12\ldots k)$.

In order to prove Theorem 5.7, we need the following powerful Lemma, due to Backelin, West, and Xin.

**Lemma 5.8** Let $v$ be any pattern of length $k - r$. Then for all positive integers $n$ and $r < k$, we have

$$S_n(12\ldots rv) = S_n(r(r-1)\ldots 21v),$$

where $v$ is taken on the set \{r + 1, r + 2, \ldots, k\}.

Now we are in position to prove Theorem 5.7.

**Proof:** (of Theorem 5.7) Induction on $k$. If $q$ has only one layer, then $q$ is the decreasing pattern, and the statement is obvious. Now assume $q$ has at least two layers, and that we know the statement for all layered patterns of length $k - 1$. As $q$ is layered, it is of the form $r(r-1)\ldots 21v$ for some $r$, and some layered pattern $v$. Therefore, Lemma 5.8 applies, and we have $S_n(q) = S_n(12\ldots rv)$. If this last pattern is denoted by $q^*$, then we obviously also have $L(q) = L(q^*)$. We further denote by $q^{*-}$ the pattern obtained from $q^*$ by removing its first entry. Note that $q^{*-}$ is still a layered pattern, just its first several layers may have length 1.

Assume first that $r > 1$. Then note that $q^{*-}$ starts with its smallest entry. Therefore, Theorem 5.4 applies, and by the induction hypothesis we have

$$L(q) = L(q^*) = 1 + L(q^{*-}) + 2\sqrt{L(q^{*-})} \geq 1 + (k - 2)^2 + 2(k - 2) = (k-1)^2,$$

which was to be proved.

Now assume that $r = 1$. Then $q$ is a layered pattern that starts with a layer of length 1. Therefore, instead of applying Theorem 5.4 we need to, and almost always can, apply Theorem 5.5 for the pattern
Indeed, \( q^* \) is a layered pattern, and as such, is indecomposable, except when it has only one layer, that is, it is the decreasing permutation. Therefore, Theorem 5.5 implies

\[
L(q) \geq 1 + L(q^*) + 2\sqrt{L(q^*)} \geq 1 + (k - 2)^2 + 2(k - 2) = (k - 1)^2.
\]

Finally, if \( r = 1 \), and \( q^* \) is the decreasing pattern, then we simply have \( q = 1k(k - 1) \cdots 2 \). In that case, our statement is just a special case of Lemma 5.8. Indeed, choosing \( v \) to be the decreasing pattern, Lemma 5.8 shows \( S_n(q) = S_n(12 \cdots k) \).

This completes the proof. \( \Diamond \)

Here is another way in which our results start explaining why certain patterns are easier to avoid than others. We formulate our observations in the following Corollary.

**Corollary 5.9** Let \( q_1 \) and \( q_2 \) be patterns so that \( L(q_1) \leq L(q_2) \). Let \( q'_i \) be the pattern obtained from \( q_i \) by prepending \( q_i \) by a 1. Furthermore, let \( q_1 \) start with the entry 1, and let \( q_2 \) be indecomposable. Then we have

\[
L(q'_1) = 1 + L(q_1) + 2\sqrt{L(q_1)} \leq 1 + L(q_2) + 2\sqrt{L(q_2)} \leq L(q'_2).
\]

For instance, if we set \( q_1 = 123 \) and \( q_2 = 213 \), we get the well-known statement weakly comparing the limits of the Stanley-Wilf sequences of 1234 and 1324, first proved in [6].

### 6 Further Directions

Our results raise two interesting kinds of questions. We have seen that the limit of a Stanley-Wilf sequence is not simply not always an integer, but also not always rational. Is it always an algebraic number? If yes, can its degree be arbitrarily high? Can it be more than two? Is it always an algebraic integer, that is, the root of a monic polynomial with integer coefficients? The results so far leave that possibility open.

The second question is related to the size of the limit of \( \sqrt[n]{S_n(q)} \) if \( q \) is of length \( k \). The largest value that this limit is known to take is \( (k - 1)^2 \), attained by the monotonic pattern. Before present paper, the smallest known value, in terms of \( k \), for this limit was \( (k - 1)^2 - 1 = 8 \), attained by \( q = 1342 \). As Theorem 5.2 shows, the value \( (k - 4 + \sqrt{8})^2 \) is also possible. As \( k \) goes to infinity, the difference of the assumed maximum \( (k - 1)^2 \) and this value also goes to infinity, while their ratio goes to 1. Is it possible to find a series of patterns \( q_k \) so that this ratio does not converge to 1? We point out that it follows from a result of P. Valtr (published in [7]) that for any pattern \( q \) of length \( k \), we have \( \lim_{n \to \infty} \sqrt[n]{S_n(q)} \geq e^{-3}k^2 \), so the mentioned ratio cannot be more then \( e^3 \).

Finally, now that the Stanley-Wilf conjecture has been proved, and we know that the limit of a Stanley-Wilf sequence always exists, we can ask what the largest possible value of this limit is, in terms of \( k \). In [1], R. Arratia conjectured that this limit is at most \( (k - 1)^2 \), and, following the footsteps
of Erdős, he offered 100 dollars for a proof or disproof of the conjecture $S_n(q) \leq (k - 1)^{2n}$, for all $n$ and $q$. Our results provide some additional support for this conjecture as they show that there is a wide array of patterns $q$ for which $\sqrt{L(q)}$ grows by one when $q$ is prepended by the entry 1. In fact, numerical evidence suggests that even the following stronger version of Arratia’s conjecture could be true.

**Conjecture 6.1** Let $q$ be a pattern of length $k$. Then $L(q) \leq (k - 1)^2$, where equality holds if and only if $q$ is layered, or the reverse of $q$ is layered.

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