RESIDUE FORMULATION OF CHERN CHARACTER ON SMOOTH MANIFOLDS

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Abstract. The Chern character of a complex vector bundle is most conveniently defined as the exponential of a curvature of a connection. It is well known that its cohomology class does not depend on the particular connection chosen. It has been shown by Quillen [12] that a connection may be perturbed by an endomorphism of the vector bundle, such as a symbol of some elliptic differential operator. This point of view, as we intend to show, allows one to relate Chern character to a non-commutative sibling formulated by Connes and Moscovici [7]. The general setup for our problem is purely geometric. Let $\sigma$ be the symbol of a Dirac-type operator acting on sections of a $\mathbb{Z}_2$-graded vector bundle $E$. Let $\nabla$ be a connection on $E$, pulled back to $T^*M$. Suppose also that $\nabla$ respects the $\mathbb{Z}_2$-grading. The object $\nabla + \sigma$ is a superconnection on $T^*M$ in the sense of Quillen. We obtain a formula for the $H^*_c(M)$-valued Poincare dual of Quillen’s Chern character $ch(D) = \text{tr}_a e^{(\nabla + \sigma)^2}$ in terms of residues of $\Gamma(z) \text{tr}_a (\nabla + \sigma)^{-2z}$. We also compute two examples.

1. Introduction.

The historical background for noncommutative index theory has two basic parts. The first one dates back to the nineteen fifties, when Israel Gelfand pointed out to Sir Michael Atiyah that the index of a Fredholm operator was stable under small perturbations. The ultimate consequence of this remark is quite famous: the Atiyah-Singer Index Theorem [2, 3, 4]. The second part is development of noncommutative geometry by Alain Connes [6, 8].

In particular, two noncommutative versions of the Chern character were developed. There is one due to Connes [5] and a more recent one.
due to Connes and Moscovici [7]. We are interested in this more recent version, which is called the residue cocycle. Its individual terms are certain residues which are geometrically interesting, as pointed out by Higson. However, they are not very well understood.

In the present paper, we shall prove a formula which resembles the formula of Connes and Moscovici in [7], albeit in a classical setting. Suppose $M$ is a compact smooth manifold with no boundary. Let $E \to M$ be a $\mathbb{Z}_2$-graded complex vector bundle. Let $D$ be an elliptic, odd, first-order, skew-adjoint differential operator on $E$. Finally, let $\pi: T^*M \to M$ be the standard projection map of the cotangent bundle. With this setup, we will establish a formula for the Chern character of the symbol of $D$ which resembles the Connes-Moscovici Chern character in noncommutative geometry. It is comprised of finitely many residues of zeta functions constructed from the symbol of $D$ and a connection on $E$.

We shall use Quillen’s formalism in which the symbol $L$ of $D$, together with a connection on $E$, determines a superconnection on $\pi^*E$ [12]. Quillen’s superconnection $\nabla + L$ encodes all the information necessary to define the Chern character in a single object. We shall denote it by $\nabla_L$.

The supertrace of $\exp \nabla^2_L$ is a mixed differential form which enjoys the major properties of the ordinary Chern character: it is closed and its cohomology class depends only on the underlying vector bundle, not on $\nabla$ or $L$. But rather than passing to its cohomology class on $T^*M$, we take advantage of the fact that this form is rapidly decreasing on the fibers of $T^*M$. We get this convenient property by sacrificing the traditional $\frac{-1}{2\pi i}$ factor, an error which we shall also address. Thus, we define a current on $\Omega^*M$ by the Poincare Duality formula

$$PD: \eta \mapsto \int_{T^*M} \pi^*(\eta) \text{tr}_s \exp \nabla^2_L.$$  

If we expand this dual “Chern character current” in the Taylor series, we obtain:

$$PD \left[ \text{tr}_s \exp \nabla^2_L \right] = PD \left[ \text{tr}_s \left( 1 + \nabla_L + \frac{1}{2!} \nabla^2_L + \ldots \right) \right]$$

The terms on the right are closed forms and their cohomology classes depend on the isomorphism type of $\pi^*E$ only. However, they are not rapidly decreasing and we cannot form the dual currents by coupling them with the pullback of an arbitrary smooth form on $M$. 

We resolve this issue through *analytic regularization*, to be addressed in section 3, and thus obtain a new formula for the dual Chern character. Briefly, our main result can be stated as follows:

**Theorem.** For $R > 0$, let $Y_R$ be the $R$-tubular neighborhood of the zero section of $T^*M$. Then:

$$\int_{T^*M} \pi^*(\eta) \operatorname{tr}_s \exp \nabla_L^2 = \lim_{R \to 0} \sum_{z \in \mathbb{C}} \operatorname{Res}|_z \left[ \Gamma(z) \int_{T^*M \setminus Y_R} \pi^*(\eta) \operatorname{tr}_s(-\nabla_L)^{-2z} \right].$$

The limit arises due to divergence of negative powers of $L$ near zero.

We now proceed to briefly describe the Connes-Moscovici local Chern character, i.e., the noncommutative Chern character.

The Chern character in [7] is a periodic cyclic cohomology class in $HPC^*$. This cohomology is constructed from spaces of multilinear functionals on $A$. See [8], chapter 10, for construction of $HPC^*$. The Connes-Moscovici Chern character is represented by a sequence of multilinear functionals on $A$ which we proceed to describe. The $n$-th term of the sequence is zero for $n = 1, 3, 5, \ldots$; for $n = 0, 2, 4, \ldots$, let $a_0, a_1, \ldots, a_n$ be the elements of $A$. Let $\operatorname{Tr}_s$ be the supertrace and let $k$ be the variable running through all $n$-multiindices with nonnegative integer entries.

$$\phi(a_0, a_1, \ldots, a_n) = \sum_k C_{nk} \operatorname{Res}|_{z=0} \Gamma(z + |k| + \frac{n}{2})$$

$$\times \operatorname{Tr}_s \left( a_0 \delta^{(k_1)}([D,a_1]) \ldots \delta^{(k_n)}([D,a_n]) D^{-2(z+|k|+\frac{n}{2})} \right),$$

where

$$C_{nk} = \frac{(-1)^{|k|} \Gamma(\frac{n}{2} + |k|)}{k!(k_1 + 1)(k_1 + k_2 + 2) \ldots (k_1 + k_2 + \ldots k_n + n)}.$$

Also, note that the trace $\operatorname{Tr}_s(\ldots D^{-2(z+|k|+\frac{n}{2})})$ must be replaced by its meromorphic continuation in $z$, before we take the residues. Existence of such a continuation is also implied by certain axioms and is not trivial at all. Indeed, as it stands, the operators whose trace we are taking typically fail to be bounded, let alone trace class.

To us, the most important fact about the Connes-Moscovici formula is that this Chern character is a sum of residues of $\operatorname{Tr}_s(\ldots D^{-2(z+|k|+\frac{n}{2})})$ times the gamma function. Our main result expresses the Quillen’s representative of the (classical) Chern character as a sum of very similar
quantities. Much like the proof in [7], our argument hinges on the Mellin transform.

2. Superconnections and Chern Classes.

In this section, we give an overview of superconnections according to Quillen [12]. This notion shall be used to define Chern character in the spirit of Chern-Weil theory. Let $M$ be a smooth manifold with no boundary. Let $E$ be a smooth complex $\mathbb{Z}_2$-graded vector bundle over $M$. We work with the vector bundle $\Lambda^* M \otimes E$.

**Definition 2.1.** (Quillen, [12]) Let $a(E)$ be the space of smooth sections of the vector bundle $\Lambda^* M \otimes \text{End}(E)$. It naturally inherits the $\mathbb{Z}_2$-grading from the fibers.

**Definition 2.2.** Let $\omega, \nu$ be homogeneous differential forms on $M$, let $T, S \in \Gamma^\infty \text{End}(E)$ be homogeneous (purely even or purely odd) endomorphisms of $E$ and let $s \in \Gamma^\infty (\Lambda^* M \otimes E)$ be a homogeneous section. We define the graded multiplication on $a(E)$ by:

1. $$(\omega \otimes T) (\nu \otimes S) = (-1)^{\deg(\nu)\deg(T)} \omega \nu \otimes TS$$

Also, the action of $a(E)$ on $\Gamma^\infty (\Lambda^* M \otimes E)$ is defined by:

2. $$(\omega \otimes T) (\nu \otimes s) = (-1)^{\deg(\nu)\deg(T)} T \nu \otimes Ts$$

**Lemma 2.1.** The equation (1) makes $a(E)$ an associative superalgebra. Also, (2) defines an algebra action of $a(E)$ on $\Gamma^\infty (\Lambda^* M \otimes E)$. In fact, this makes $a(E)$ a subalgebra of $\Gamma^\infty \text{End}(\Lambda^* M \otimes E)$ in the sense that no nonzero element of $a(E)$ kills everything. □

The condition (2) is called $\Omega^*$-linearity. It turns out that $\Omega^*$-linear endomorphisms of $\Lambda^* M \otimes E$ are precisely the elements of $a(E)$.

**Lemma 2.2.** The algebra $a(E)$ is the $\mathbb{C}$-span of homogeneous smooth sections of $\text{End}^\pm((\Lambda^* M) \otimes E)$ which are in addition $\Omega^*$-linear. □

**Remark 2.1.** Observe that $a(E)$ is naturally a left $\Omega^* M$-module.

**Remark 2.2.** The definition 2.2 resembles the usual definition of multiplication on a tensor product of two algebras. However, it takes the $\mathbb{Z}_2$-grading into account. Such products of superalgebras are called graded tensor products and are denoted by $\hat{\otimes}$.

Next, observe that the $\mathbb{Z}_2$-grading on $E$ naturally induces one on the space of sections $\Gamma^\infty (\Lambda^* M \otimes E)$, which makes it possible to speak of even and odd linear endomorphisms of this space. The even ones preserve
the $\mathbb{Z}_2$ grading and the odd ones switch it. We are now ready to define superconnections.

**Definition 2.3.** Let $E$ be a smooth $\mathbb{Z}_2$-graded complex vector bundle. In the spirit of [12], we define a superconnection on $E$ to be an odd $\mathbb{C}$-linear map

$$\nabla: \Gamma^\infty(\Lambda^* M \otimes E) \to \Gamma^\infty(\Lambda^* M \otimes E),$$

which satisfies the so-called graded Leibniz rule:

$$\nabla(\omega s) = (d\omega)s + (-1)^{\deg(\omega)}\omega \nabla s.$$  
The curvature of $\nabla$ is defined as $\nabla^2$.

**Lemma 2.3.** The curvature is a globally well-defined, even element of $a(E)$. \hfill $\Box$

**Lemma 2.4.** Locally, any superconnection is always of the form $d \otimes \text{id} + \theta$, where $d$ is the de Rham differential and $\theta$ is a local odd section of $\Lambda^* M \otimes \text{End}(E)$. \hfill $\Box$

**Corollary 2.5.** Given a superconnection $\nabla$ and an arbitrary odd element $L$ of $a(E)$, $\nabla + L$ is also a superconnection.

**Lemma 2.6.** Any connection which respects the $\mathbb{Z}_2$-grading is a superconnection.

**Proof:** Such a connection can be locally written as $d + \theta$, where $\theta = (dx_i \Gamma^k_{ij})_{jk}$ is a matrix of 1-forms. Since the connection respects the grading, the matrices $(\Gamma^k_{ij})_{ik}$ define even endomorphisms of $E$. Presence of 1-forms, therefore, makes $\theta$ odd. \hfill $\Box$

**Definition 2.4.** We denote the superconnection $\nabla + L$ by $\nabla_L$.

**Definition 2.5.** Suppose $W$ is a finite-dimensional $\mathbb{Z}_2$-graded complex vector space and $A = \text{End}(W)$. We define the supertrace $\text{tr}_s^\mathbb{C}: A \to \mathbb{C}$ by the following equation:

$$\forall f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in A, \quad \text{tr}_s^\mathbb{C}(f) = \text{def} \text{ tr}(f_{11}) - \text{ tr}(f_{22}),$$

where the traces on the right hand side are the usual traces of operators from $W^+$ and $W^-$ into themselves. Let $R$ be another $\mathbb{Z}_2$-graded algebra. A typical simple tensor in $R \otimes A$ has the form

$$f = \begin{pmatrix} r_{11}f_{11} & r_{12}f_{12} \\ r_{21}f_{21} & r_{22}f_{22} \end{pmatrix}. $$
On simple tensors, we define \( \text{tr}_s^R : R \otimes A \to R \) by
\[
 r_{11} t r(f_{11}) - (-1)^{\text{deg}(r_{22})} r_{22} t r(f_{22}).
\]

If \( W \) is a fiber of some vector bundle \( E \), this definition extends naturally to sections.

We proceed toward the definition of Chern character. To that end, we need the graded version of the trace. Being \( \mathbb{Z}_2 \)-graded, \( \text{End}(E) \) has a supertrace \( \text{tr}_s^\mathbb{C} : \text{End}(E) \to \mathbb{C} \) which means that on \( \Lambda^*M \otimes \text{End}(E) \) the \( \Lambda^*V \)-valued supertrace \( \text{tr}^{\Lambda^*M} : \Lambda^*M \otimes (E) \to \Lambda^*M \) makes sense.

**Definition 2.6.** The 2\(k\)-th component of the Chern character form is defined as the following differential form:
\[
 ch_{2k}(E) = \frac{1}{k!} \text{tr}_s^{\Lambda^*M}(\nabla^{2k}) \in \Omega^{2k}M.
\]
The total Chern character form is
\[
 ch(E) = \sum_{k=0}^{\infty} ch_k(E).
\]
We usually write this form as \( \text{tr}_s^{\Lambda^*M}(\exp(\nabla^2)) \).

**Theorem 2.7.** (Quillen, [12]). The series defining \( ch(E) \) converges.

**Theorem 2.8.** ([12]) \( ch_k \) is closed and its cohomology class is independent on a choice of the superconnection. This class is an invariant of isomorphism classes of complex vector bundles.

### 3. Formulation of the Problem.

Here we state the main theorems of the paper. The proofs shall be given in subsequent sections.

The general geometric setup for our problem is the following. Let \( M \) be a compact, \( n \)-dimensional, smooth manifold. Let \( E \to M \) be a smooth, \( \mathbb{Z}_2 \)-graded complex vector bundle. We assume that \( E \) and \( TM \) are provided with metrics. Let \( D \) be an elliptic, odd, first-order, selfadjoint differential operator on \( E \). Finally, let \( \pi : T^*M \to M \) be the standard projection map of the cotangent bundle, and let \( \nabla \) be a superconnection on \( \pi^*E \) which arises as a pullback of some superconnection \( \nabla' \) from \( E \).

Our result is motivated by the Connes-Moscovici formula. We express the right-hand side of the Atiyah-Singer Index Theorem in a way which
resembles their residue cocycle. Namely, we shall sum over all the residues of the expression:

$$\Gamma(z) \int_{T^*M} \pi^*(\eta)\nabla_L^{-2z}.$$

The Index Theorem using Quillen’s Chern character can be stated as:

$$\text{Ind}(D) = \sum_\kappa \left( -\frac{1}{2\pi i}\right)^{2n-\kappa} \int_{T^*M} \pi^*(\text{Todd}(TM \otimes \mathbb{C}))_\kappa \text{ch}(L).$$

Here we need to prove that the integral (lemma 3.1) converges and that $\text{ch}(L)$ may indeed be used in place of the ordinary Chern character (theorem 3.3). The factor of $\left( -\frac{1}{2\pi i}\right)^{2n-\kappa}$ is to correct for the error introduced by leaving the $2\pi i$ out of Quillen’s definition of Chern character.

The right-hand side is really concerned with the Poincare Dual of $\text{ch}(L)$, i.e. with the following linear functional

$$PD[\text{ch}(L)] : \eta \mapsto \int_{T^*M} \pi^*(\eta)\text{ch}(L).$$

**Lemma 3.1.** The quantity $\|\text{ch}(L)\| = \|\text{tr}_s \exp \nabla^2_L\|$ decays exponentially along the fibers of $T^*M$. In fact, there are positive constants $C$ and $K$ such that

$$\|\text{ch}(L)\| \leq Ce^{-K\rho^2},$$

where $\rho$ is the radial coordinate on the fibers of $T^*M$ obtained from the Riemannian metric.

**Corollary 3.2.** Since $M$ is compact, the integral $\int_{T^*M} \pi^*(\eta)\text{ch}(L)$ converges.

The necessary estimates for this lemma are provided in section 3 of [12]. Essentially, it is true because $D$ is selfadjoint, so that $L$ is antiselfadjoint and $L^2$ is negative-definite, which is where the ellipticity of $D$ comes in. Also, $L^2$ increases polynomially on the fibers of $T^*M$. When we exponentiate $\nabla^2_L = \nabla^2 + [\nabla, L] + L^2$, the resulting expression decreases as $e^{-K\rho^2}$.

Next consider the Taylor expansion of $\text{ch}(L)$:

$$\text{tr}_s \exp \nabla^2_L = \text{tr}_s (1 + \nabla_L + \frac{1}{2!}\nabla^2_L + \ldots).$$

Although the Poincare dual of the left-hand side converges, the duals the individual Taylor terms do not, due to polynomial increase of $L$ along the fibers of $T^*M$. 
We get around this difficulty by analytic regularization. The general idea is that if we replace the integer power \( k \) of \( \nabla^2 L \) by a complex number \( -z \), where \( \Re(z) \gg 1 \), we also can replace the integral \( \frac{1}{k!} \int_{T^* M} \pi^*(\eta) \nabla^{2k} L \) with the following expression:

\[
(3) \quad \text{Res}_{z=-k} \Gamma(z) \int_{T^* M} \pi^*(\eta) \nabla^{-2z} L.
\]

Before we can take this residue, though, we need to pass to the meromorphic extension of

\[
(4) \quad \int_{T^* M} \pi^*(\eta) \nabla^{-2z} L.
\]

In particular, we need to prove that such an extension exists (theorem 3.4).

However, there is yet another difficulty. Let \( |\xi| \) denote the fiberwise norm on \( T^* M \). Define:

\[
Y_R = \{ \xi \in T^* M : |\xi| < R \},
\]

\[
X_R = Y_R^c.
\]

The integral in (4) would not really converge, if taken over all of \( T^* M \), because \( L^2 \) is a symbol of a second-order differential operator. Hence, \( L^2 \) is a homogeneous quadratic polynomial in the vertical coordinates \( \xi \) of the cotangent bundle. It therefore vanishes at the zero section on \( T^* M \).

The problem with divergence at infinity shall be resolved by taking \( \Re(z) > 0 \), meromorphically extending the integral to the whole complex plane and taking residues.

We deal with divergence at \( Y_R \), in the following way. First, we shall replace the integral over \( T^* M \) by that over \( X_R \). Second, we shall take the residues in \( z \). Third, we shall take the limit as \( R \to 0 \). That is, we use:

\[
\lim_{R \to 0} \sum_{z \in \mathbb{C}} \text{Res}_{z} \Gamma(z) \int_{X_R} \pi^*(\eta) \text{tr}_s(-\nabla^2_L)^{-z}.
\]

It turns out that this quantity is well-defined and is equal to the value of the current \( PD(\text{tr}_s \exp \nabla^2_L) \) on \( \eta \).

We now formally state the main results of the present work. First, we summarize the hypotheses which apply in all the theorems in the sequel: \( \text{Let } M \text{ be a compact, smooth } n\text{-manifold with no boundary. Let } \pi: T^* M \to M \text{ be the cotangent bundle and let } E \to M \text{ be a } \mathbb{Z}_2\text{-graded smooth vector bundle. Suppose also that } D \text{ is an odd, elliptic, first} \)

order selfadjoint differential operator on $E$. Thus, the symbol $L$ of $D$

is an odd endomorphism of $\pi^*E$ which is invertible everywhere but at the zero section $M \subseteq T^*M$ and pointwise anti-selfadjoint. Let $\nabla$ be the pullback of some connection $\nabla'$ on $E$ via $\pi$. Assume also that $\nabla'$, and hence $\nabla$, respects the $\mathbb{Z}_2$-grading.

**Theorem 3.3.** Let $\eta$ be a closed, smooth differential form on $M$. Consider the following integral:

$$I(\eta) = \int_{T^*M} \text{tr}_s \pi^*(\eta) \exp \nabla^2_L.$$  

a) It vanishes if $\eta$ is exact.

b) It is independent of the particular choice of $\nabla'$.

c) Let $\beta = -\int_0^\infty \text{tr}_s \exp \nabla^2_L \text{d}t$. For any $R > 0$, the following holds on the interior of $X_R$:

$$d\beta = \text{tr}_s \exp \nabla^2.$$  

Thus, the pair $(\text{tr}_s \exp \nabla^2, \beta)$ defines a relative cohomology class in $H^*(T^*M, \text{int } X_R)$.

d) $I(\eta) = \int_{T^*M \setminus X_R} \pi^*(\eta) \text{tr}_s \exp \nabla^2 - \int_{\partial(T^*M \setminus X_R)} \pi^*(\eta) \beta$.

Hence, $I(\eta)$ yields the same result as pairing of $\eta$ with the relative cohomology class defined by $(\text{tr}_s \exp \nabla^2, \beta)$.

Observe that $\pi^* : H^*M \cong H^*(T^*M)$ and that $\text{tr}_s \exp \nabla^2$ and $\pi^*(\text{ch}E^+ - \text{ch}E^-)$ are cohomologous. That is, $\text{tr}_s \exp \nabla^2$ determines the same difference Chern class.

**Theorem 3.4.** For any $R > 0$ and for any $\eta \in \Omega^* M$ and for any $z \in \mathbb{C}$ with $\text{Re}(z) \gg 0$, the following integral converges:

$$\int_{X_R} \pi^*(\eta) \text{tr}_s (-\nabla^2_L)^z.$$  

Further, it has a meromorphic extension to $\mathbb{C}$ of the form

$$\sum_K \frac{R^{K+1-2z}}{K + 1 - 2z} A_K,$$

where $A_K$ are constants.

In fact, we do have a classification of the poles. There are finitely many of them and they are located at negative integers or half-integers.

**Lemma 3.5.** Let $\eta \in \Omega^* T^*M$. Then the integral

$$\int_{X_R} \pi^*(\eta) \text{tr}_s (-\nabla^2_L)^{-2z}$$
can only have a nonzero residue at the point \((\kappa - 2n)/2\). Further, if \(n\) is even and the residue is nonzero, then \((\kappa - 2n)/2\) must be an integer. If \(n\) is odd, \((\kappa - 2n)/2\) must be a half-integer. In either case, nonzero residues occur only for even \(\kappa\).

This lemma says that at each particular point \(z \in \mathbb{C}\), the residue is a homogeneous current. That is, it vanishes on all the forms except possibly for those of some given degree.

**Theorem 3.6.** Let \(\nabla\) be a superconnection which has been pulled back from \(E\) via \(\pi\). For any \(\eta \in \Omega^{*} M\)

\[
\int_{T^{*} M} \pi^{*}(\eta) \operatorname{tr}_{s} \exp \nabla_{L}^{2} = \sum_{z \in \mathbb{C}} \lim_{R \to 0} \operatorname{Res}_{z} \Gamma(z) \int_{X_{R}} \pi^{*}(\eta) \operatorname{tr}_{s}(-\nabla_{L}^{2})^{-z}.
\]

All but finitely many residues on the right-hand side vanish as \(R \to 0\).

**Corollary 3.7.** For each \(z\), the following defines a closed current on \(\Omega^{*} M\):

\[
R_{z}: \eta \mapsto \lim_{R \to 0} \operatorname{Res}_{z} \Gamma(z) \int_{X_{R}} \pi^{*}(\eta) \operatorname{tr}_{s}(-\nabla_{L}^{2})^{-z}.
\]

**Proof:** By theorem 2.8, \(\operatorname{tr}_{s} \exp \nabla_{iL}^{2}\) is a closed form. It is also rapidly decreasing on the fibers of \(T^{*} M\), so for an exact form \(\eta = d\omega\) on \(M\), Stokes theorem yields:

\[
\int_{T^{*} M} \pi^{*} d\omega \operatorname{tr}_{s} \exp \nabla_{iL}^{2} = \pm \int_{T^{*} M} \pi^{*} \omega d \operatorname{tr}_{s} \exp \nabla_{iL}^{2} = 0.
\]

The rapid decay property assures that there is no boundary term. Thus, \(\operatorname{tr}_{s} \exp \nabla_{iL}^{2}\) induces a closed current on \(\Omega^{*} M\). By lemma 3.5, for each \(\kappa\), \(R_{z}\) either vanishes on or agrees with the current induced by \(\operatorname{tr}_{s} \exp \nabla_{iL}^{2}\). In either case, \(R_{z}\) is a closed current on \(\Omega^{n} M\).

Just as the computation in [7], our proof of theorem 3.6 hinges on Mellin transform. The simplest example of a Mellin transform is the well-known formula, valid for \(\text{Re}(\sigma) > 0\):

\[
\int_{0}^{\infty} e^{-\sigma t} t^{z-1} dt = \sigma^{-z} \Gamma(z).
\]

It says that \(\sigma^{-z} \Gamma(z)\) is the *Mellin transform* of \(e^{-\sigma t}\). (See [1] for details). Connes and Moscovici apply the same transform to the so-called *JLO cocycles* [11] in order to obtain the residue cocycle. Let
\( \Sigma_k \) be the standard \( k \)-simplex in \( \mathbb{R}^{k+1} \) with coordinates \( u_0, u_1 \ldots u_{k-1} \).

Using the notation of section 1, the JLO cocycles are comprised of multilinear functionals on a *-algebra \( A \) given by

\[
\psi_{JLO}^t(a_0, a_1, \ldots, a_k) = \text{Tr}_s \int_{\Sigma_k} a_0 e^{-u_0 tD^2} a_1 e^{-u_1 tD^2} \ldots a_{k-1} e^{-u_{k-1} tD^2} a_k \, du.
\]

We, however, apply Mellin transform to Quillen’s Chern character, which is an exponential, and obtain complex powers of the curvature.

4. Proof of Theorem 3.3.

**Theorem 3.3.** Let \( \eta \) be a closed differential form on \( M \). Under the hypotheses outlined in section 3, consider the following integral:

\[
I(\eta) = \int_{T^*M} \text{tr}_s \pi^*(\eta) \exp \nabla_L^2.
\]

a) It vanishes if \( \eta \) is exact.

b) It is independent of the particular choice of \( \nabla' \).

c) Let \( \beta = -\int_0^\infty (\exp \nabla_L^2) L \, dt \). For any \( R > 0 \), the following equation holds on the interior of \( X_R \):

\[
d\beta = \text{tr}_s \exp \nabla^2.
\]

Thus, the pair \((\text{tr}_s \exp \nabla^2, \beta)\) defines a relative cohomology class in \( H^*(T^*M, \text{int} X_R) \).

d) \( I(\eta) = \int_{T^*M \setminus X_R} \pi^*(\eta) \text{tr}_s \exp \nabla^2 - \int_{\partial(T^*M \setminus X_R)} \pi^*(\eta) \beta. \)

Hence, \( I(\eta) \) yields the same result as pairing of \( \eta \) with the relative cohomology class defined by \((\text{tr}_s \exp \nabla^2, \beta)\).

The content of this theorem is really due to [12]. For part (a), assuming \( \eta = d\omega \), we compute:

\[
\int_{Y_R} \pi^*(d\omega) \text{tr}_s \exp \nabla_L^2 = \int_{\partial Y_R} \pi^*(\omega) \text{tr}_s \exp \nabla_L^2.
\]

The right-hand side vanishes as \( R \to \infty \).

For (b), the fact that the cohomology class of \( \text{tr}_s \exp \nabla_L^2 \) is independent on \( \nabla' \) or \( L \) is not enough. We need to prove that if we replace the connection \( \nabla' \) on \( E \) with some \( \tilde{\nabla}' \), so that on \( \pi^*E \) we have \( \tilde{\nabla} = \pi^*\tilde{\nabla}' \), then there exists a differential form \( \beta_1 \), rapidly decreasing on the fibers of \( T^*M \) and such that

\[
d\beta_1 = \text{tr}_s \exp \nabla_L^2 - \text{tr}_s \exp \tilde{\nabla}_L^2.
\]

We present the so-called "homotopy" argument.
Suppose first there is some connection $\nabla_t$ on $\pi^*E$ which depends on a parameter $t$ in a differentiable way. The example we have in mind is:

$$\nabla_t = \pi^*(t\tilde{\nabla}' + (1-t)\tilde{\nabla}') + L = t\nabla' + (1-t)\tilde{\nabla}' + L.$$ 

Here, $t$ is a coordinate on the manifold $T^*M \times \mathbb{R}$. We take the pullback vector bundle $F = \text{pr}_1^*(\pi^*E)$, on $T^*M \times \mathbb{R}$ and the following defines a connection $D_t$ on $F$:

$$D_t = \nabla_t + dt\partial_t.$$ 

Then $D_t^2 = \nabla_t^2 + dt\tilde{\nabla}_t$ and more generally:

$$D_t^{2k} = \nabla_t^{2k} + \sum_{j=0}^{k-1} \nabla_t^{2j}dt\tilde{\nabla}_t\nabla_t^{2(k-j-1)} = \mu_k + dt\nu_k,$$

where $\mu_k$ and $\nu_k$ are unambiguously defined by the above equation.

Let $d'$ denote the de Rham differential on $T^*M \times \mathbb{R}$ and denote the one on $T^*M$ by simply $d$. This way, $d' = d + dt\partial_t$. By theorem 2.8, $\text{tr}_s D_t^{2k}$ is closed:

$$d' \text{tr}_s D_t^{2k} = d'\mu + d(dt\nu) = 0$$

so that $\partial_t\mu_k = d\nu_k$ and $\mu_k|_{t=1} - \mu_k|_{t=0} = d\int_0^1 \nu dt$, which means that:

$$\nabla_t^{2k} - \nabla_t^{2k} = d\int_0^1 \sum_{j=0}^{k-1} \nabla_t^{2j}\tilde{\nabla}_t\nabla_t^{2(k-j-1)} dt.$$ 

Or, taking supertraces and keeping in mind that supertraces kill supercommutators:

$$\text{tr}_s(\nabla_t^{2k} - \nabla_t^{2k}) = d\text{tr}_s \int_0^1 k\nabla_t^{2k}\tilde{\nabla}_t dt.$$ 

This implies that

$$\text{tr}_s \exp \nabla_t^2 - \text{tr}_s \exp \nabla_0^2 = d\int_0^1 \text{tr}_s \exp \nabla_t^2\tilde{\nabla}_t dt.$$ 

This equation applies to any $\nabla_t$ which depends on $t$ differentiably.

In our particular example,

$$\nabla_t = \pi^*(t\nabla' + (1-t)\tilde{\nabla}') + L,$$

so that $\nabla_0 = \tilde{\nabla}L$ and $\nabla_1 = \nabla_L$. It follows that

$$\text{tr}_s \exp \nabla_L^2 - \text{tr}_s \exp \tilde{\nabla}_L^2 = d\int_0^1 \text{tr}_s \exp \nabla_t^2\tilde{\nabla}_t dt,$$ 

This equation applies to any $\nabla_t$ which depends on $t$ differentiably.
and we may define $\beta_1$ as $\int_0^1 \text{tr}_s \exp \nabla_t^2 \nabla_t dt$.
Because $\exp \nabla_t^2$ is exponentially decreasing along the fibers of $T^*M$ for each fixed $t$, the rapid decay property of $\beta_1$ is easy to prove.
To proceed with (c), we change our definition of $\nabla_t$:
$$\nabla_t = \nabla + tL,$$
and $D_t = \nabla_t + dt\partial_t$ on $T^*M \times \mathbb{R}$. This does not affect the fact that $\partial_t \mu_k = d\nu_k$ and we have:
$$\partial_t \nabla_t^{2k} = d \sum_{j=0}^{k-1} \nabla_t^{2j} \nabla_t^{2k-j-1},$$
or, taking supertraces:
$$\text{tr}_s \partial_t \nabla_t^{2k} = d \text{tr}_s k \nabla_t^{2(k-1)} \nabla_t$$
Just as in part (b), we obtain:
$$\partial_t \text{tr}_s \exp \nabla_t^2 = d \text{tr}_s \exp \nabla_t^2 L.$$  
But as long as $L$ is invertible, $\| \exp \nabla_t^2 \| \to 0$ as $t \to 0$ (since, there are non-zeroes among the eigenvalues of $L^2$). Hence, (integrating (11)) yields:
$$\text{tr}_s \exp \nabla_L^2 = \text{tr}_s \exp \nabla_t^2$$
$$\text{tr}_s \exp \nabla_L^2 - \text{tr}_s \exp \nabla^2 = d \int_0^1 \text{tr}_s \exp \nabla_t^2 L dt,$$
we see that on $X_R$:
$$\text{tr}_s \exp \nabla^2 = -d \int_0^\infty \text{tr}_s \exp \nabla_t^2 \nabla_t dt,$$
For (d), recall how does one integrate compactly supported cohomology classes defined by pairs. If we have a pair $(\text{tr}_s \eta \exp \nabla^2, \eta \beta)$ as above, $\beta$ being equal to $\int_0^\infty \text{tr}_s \exp \nabla_t^2 L dt$, and $\eta$ being closed, then:
$$\langle [T^*M], [(\text{tr}_s \eta \exp \nabla^2, \eta \beta)] \rangle = \text{def} \int_{Y_R} \eta \text{tr}_s \exp \nabla^2 - \int_{\partial Y_R} \eta \beta.$$
Similarly, for $\beta_2 = -\int_1^\infty \text{tr}_s \exp \nabla^2_L L dt$ we already have:

\begin{equation}
\langle [T^*M], [\text{tr}_s \eta \exp \nabla^2_L, \eta \beta_2] \rangle =_{\text{def}} \int_{Y_R} \eta \text{tr}_s \exp \nabla^2_L - \int_{\partial Y_R} \eta \beta_2.
\end{equation}

Combining (12) and (14) we see that these quantities are equal. Taking $R \to \infty$, due to exponential decay, $\int_{\partial Y_R} \eta \beta \to 0$. It follows that

$$\int_{T^*M} \eta \text{tr}_s \exp \nabla^2_L = \int_{Y_R} \eta \text{tr}_s \exp \nabla^2 - \int_{\partial Y_R} \eta \beta.$$ 

5. Proof of Theorem 3.4.

**Theorem 3.4** Under the hypotheses outlined in section 3, for any $R > 0$ and for any $\eta \in \Omega^*M$ and for any $z \in \mathbb{C}$ with $\text{Re}(z) \gg 0$, the following integral converges:

\begin{equation}
\int_{X_R} \pi^*(\eta) \text{tr}_s (-\nabla^2_L)^{-z}.
\end{equation}

Further, it has a meromorphic extension to $\mathbb{C}$ of the form

$$\sum_K \frac{R^{K+1-2z}}{K+1-2z} A_K,$$

where $A_K$ are constants.

First, we set up some notation. Over each coordinate chart $U_\alpha$ of $M$, we may define a pullback chart $V_\alpha = \pi^{-1}(U_\alpha)$ of $T^*M$. It has horizontal coordinates $x = (x^1, \ldots, x^n)$, which are just the coordinates of $M$, and vertical ones $\xi = (\xi^1, \ldots, \xi^n)$. On the fibers of $T^*M$, let $\rho$ and $\Xi$ be the spherical coordinates. We can take $\rho = |\xi|$ to be the coordinate induced by the metric $g$. Let $S^*_\rho M$ denote the sphere bundle of $T^*M$ of radius $\rho$. The theorem follows by direct computation from the following proposition.

**Proposition 5.1.** Over each chart $X_R \cap V_\alpha$, there exist smooth local sections $\Theta^K(z, x, \rho, \Xi)$ of $\Lambda^*T^*M \otimes \text{End}(\pi^*E)$ whose coordinate expressions, in fact, do not depend on $\rho$ (i.e., they are pullbacks from the unit sphere bundle via the obvious map $X_R \to S^*M$), so we write $\Theta^K(z, x, \Xi)$. For all $z$ with $\text{Re}(z) \gg 0$, for all $R > 0$ and $\eta \in \Omega^*M$,
these sections $\Theta^K(z,x,\Xi)$ satisfy:

\begin{equation}
\int_{X_R \cap V_\alpha} \pi^*(\eta) \text{tr}_s(-\nabla^2_L)^{-z} = \int_\infty^R \rho^{-2z+K} d\rho \int_{s^*_M \cap V_\alpha} \pi^*(\eta) \sum_K \text{tr}_s \Theta^K(z,x,\Xi) d\Xi.
\end{equation}

The sum in the right-hand side is finite. Further, each $\Theta^K(z,x,\Xi)$ extends to an entire function in $z$ which is also smooth in $x$ and $\Xi$.

This proposition follows, essentially, by separation of powers of $\rho$ in the coordinate expression of $\nabla^2_L$.

**Proof of proposition 5.1.** The major steps in the proof are the following. First, for a suitable contour $\gamma$, we express $(-\nabla^2_L(p))^{-z}$ at each $p \in X_R$ by holomorphic functional calculus:

\begin{equation}
(-\nabla^2_L)^{-z} = \frac{1}{2\pi i} \int_\gamma \lambda^{-z}(\lambda + \nabla^2_L)^{-1} d\lambda.
\end{equation}

Convergence of the integral is obvious and the fact that $\gamma$ may be used instead of the usual counter-clockwise oriented curve follows by standard argument as in figure 1. Note that $\gamma$ does not depend on $p$.

Second, working on a single coordinate chart, $V_\alpha \cap X_R$, we shall prove that $\nabla^2_L$ is polynomial in $\rho$. Namely, for certain $G_0, G_1$ and $G_2$, which
are independent of $\rho$, we show that

$$\nabla^2_L = \rho^2 G_2 + \rho G_1 + G_0. \tag{21}$$

Indeed, if we let $\nabla = d + \theta$, then:

$$\nabla^2_L = \nabla^2 + [\nabla, L] + L^2 \tag{22}$$

$$= d\theta + \theta^2 + [d + \theta, L] + L^2$$

$$= \frac{L^2}{\rho^2 G_2} + \frac{(d\xi L + d\xi L + [\theta, L])}{\rho G_1} + \frac{(d\xi \theta + \theta^2 + d\rho L)}{G_0}.$$

Observe that $G_0$ and $G_1$ are nilpotent of degree at most $2n$, while $G_2$ is just $L^2/\rho^2$, hence invertible on $X_R$.

Third, by nilpotence of $G_0$ and $G_1$, the integrand in (20) can be expanded in a terminating geometric series:

$$(-\nabla^2_L)^{-z} = \frac{1}{2\pi i} \int_{\gamma} \frac{\lambda^{-z} (\lambda^2 + \rho^2 G_2 G_1 + G_0)^{-1}}{\lambda^{-z} (\lambda + \rho^2 G_2 G_1 + G_0)} d\lambda \tag{23}$$

$$= \frac{1}{2\pi i} \int_{\gamma} \lambda^{-z} (\lambda + \rho^2 G_2)^{-1} \sum_{k=0}^{2n} (-1)^k (\lambda + \rho^2 G_2)^{-1} (\rho G_1 + G_0)^k d\lambda.$$

Fourth, we separate the powers in the $k$-th term of these series and for certain sections $\Theta^K_i$ obtain:

$$(-1)^k (\lambda + \rho^2 G_2)^{-1} (\rho G_1 + G_0)^k = \sum_{l=0}^{k} \rho^{-2k+l} \Theta^K_i (\lambda/\rho^2) dpd\Xi + err. \tag{24}$$

The details of this computation are postponed until the end of the proof. Here, the sections $\Theta^K_i$ will depend on the quantity $\lambda/\rho^2$, but otherwise will not depend on $\rho$ explicitly. Also, $err$ represents the terms which may be ignored because they are not multiples of the vertical volume form $d\rho d\Xi$ and hence do not contribute to the integral over $X_R$ in (19). Then, ignoring those error terms, (23) becomes:

$$(-\nabla^2_L)^{-z} = \sum_{k,l} \frac{\rho^{-2(z+k)+l}}{2\pi i} \left[ \int_{\gamma/\rho^2} \left( \frac{\lambda}{\rho^2} \right)^{-z} \Theta^K_i \left( \frac{\lambda}{\rho^2} \right) \frac{d\lambda}{\rho^2} \right] d\rho d\Xi \tag{25}.$$

Since the integrals can be computed using the substitution $\sigma = \lambda/\rho^2$ we can define $\Theta^K(z, x, \Xi)$ as follows:

$$\Theta^K(z, x, \Xi) = \sum_{l-2k=K} \frac{1}{2\pi i} \int_{\gamma/\rho^2} \sigma^{-z} \Theta^K_i d\sigma.$$
Finally, observe that the above integrals do not really depend on $\rho$ because the contour $\gamma/\rho^2$ can be replaced by a certain contour $\gamma'$ independent of $\rho$. This basically finishes the proof, except we need to supply the following details:

a) The construction of $\Theta^k_l$ and the derivation of (25).

b) The exact choice of $\gamma'$.

For (a), we need to work out the details of (25). Fix $\rho > 0$. Consider some multiindex $I_{k,l} = (\iota_1, \iota_2, \ldots, \iota_k) \in \{0,1\}^k$, in which 1 appears $l$ times and 0 appears $k - l$ times. Let $G'_0 = G_0$ and $G'_1 = G_1 \rho$. Then we define $G_{I_{k,l}}$ as

$$G_{I_{k,l}} = (\lambda + L^2)^{-1} \prod_{\mu=1}^{k} \left( (\lambda + L^2)^{-1} G'_{\iota_{\mu}} \right).$$

Then, recalling that $\sigma = \lambda/\rho^2$ and $G_2 = L^2/\rho^2$:

$$G_{I_{k,l}} = (\lambda + L^2)^{-1} \prod_{\mu=1}^{k} \left( (\lambda + L^2)^{-1} G'_{\iota_{\mu}} \right)$$

$$= \rho^{-2k+l} \left( \frac{\lambda}{\rho^2} + G_2 \right)^{-1} \prod_{\mu=1}^{k} \left( \left( \frac{\lambda}{\rho^2} + G_2 \right)^{-1} G'_{\iota_{\mu}} \right)$$

$$= \rho^{-2k+l} \left( \sigma + G_2 \right)^{-1} \prod_{\mu=1}^{k} \left( \left( \sigma + G_2 \right)^{-1} G_{\iota_{\mu}} \right).$$

We define $\tilde{\Theta}^k_l$ as the sum of those $G_{I_{k,l}}$ which are multiples of the vertical volume form $d\rho d\Xi^1 \ldots d\Xi^{n-1} = \rho^{n-1} d\rho d\Xi$. (All the others do not contribute to the integral over $X_R$ in (18) and we ignore them).

Therefore we can pull both $\rho^{-2k+l}$ and $d\rho d\Xi$ out of $\tilde{\Theta}^k_l$ and define $\Theta^k_l$ through the equation:

$$\tilde{\Theta}^k_l = \rho^{-2k+l} \Theta^k_l d\rho d\Xi.$$ 

Now both (24) and (25) become clear. The only possible issue is that as we change the variable of integration from $\lambda$ to $\sigma$ in the Cauchy
integral, the contour of integration shifts:

\[ \int_{\gamma} \lambda^{-z} G_{\lambda, l} d\lambda = \int_{\gamma} \lambda^{-z} (\lambda + L^2)^{-1} \prod_{\mu=1}^{k} \left( (\lambda + L^2)^{-1} G_{\mu} \right) d\lambda \]

\[ = \int_{\gamma'} (\rho)^{-2(z+k)+l} \left( \frac{1}{\rho^2} \right)^{-z} \left( \frac{1}{\rho^2} + G_2 \right)^{-1} \prod_{\mu=1}^{k} \left( \frac{1}{\rho^2} + G_2 \right)^{-1} G_{\mu} d\rho \]

\[ = \int_{\gamma'} (\sigma)^{-2(z+k)+l} \left( \sigma + G_2 \right)^{-1} \prod_{\mu=1}^{k} \left( \sigma + G_2 \right)^{-1} G_{\mu} d\sigma. \]

Part (b) takes care of this issue. We choose \( \gamma' \) to be the vertical contour in \( \mathbb{C} \), which is parameterized by \( T - i\chi \), \( (\chi \in \mathbb{R}) \) for a suitable \( T \). We are about to show that there exists \( T \) such that

\[ 0 < T < \inf \bigcup_{p \in X_{\mathbb{R}}} \text{sp}( -\nabla^2_L(p) ). \]

Here, the notation \( \nabla^2_L(p) \) reminds us that \( \nabla^2_L \) is a section of \( \text{End}(\pi^*E) \) which depends on \( p \in T^*M \), and \( \text{sp} \) denotes the spectrum over each point \( p \).

**Lemma 5.2.** There exists \( T > 0 \) as above. In fact, there is an open subset \( U \) of \( \{ \lambda | \text{Re}(\lambda) > T \} \), such that the pointwise spectrum \( \text{sp}( -\nabla^2_L(p) ) \) is contained in \( U \) for all \( p \in X_{\mathbb{R}} \).

**Proof:** Indeed, \( \nabla^2_L \) equals \( L^2 \) plus the nilpotent term \( [\nabla, L] + \nabla^2 \). So, \( \lambda + \nabla^2_L \) is invertible whenever \( \lambda + L^2 \) is invertible. This is apparent from the geometric series (23). Thus, \( \text{sp}( -\nabla^2_L ) \subseteq \text{sp}( -L^2 ) \), so it is enough to find \( T \) such that

\[ 0 < T < \inf \bigcup_{p \in X_{\mathbb{R}}} \text{sp}( -L^2 ). \]

But by compactness of \( S^*M \), there exists \( T \) such that:

\[ 0 < T < \inf \bigcup_{p \in S^*M} \text{sp}( -L^2 ). \]

Appealing to homogeneity of \( L^2 \) in \( \rho \), one sees that \( T \) satisfies the assertion of the lemma. This finishes the proof of the lemma and the proposition. \( \square \)

Elaborating on this proof, and retaining the notation therein, we can obtain an estimate which shall be useful later:
Lemma 5.3. Suppose η is compactly supported in a single chart $U_\alpha$ of $M$. Then there exist constants $K, K'$ such that for every local section $\Theta^k$ of $\Lambda^*T^*M \otimes \text{End}(E)$ as in the proof of proposition 5.1, and for all complex $z$:

$$\left| \int_{S^*M \cap V_\alpha} \pi^*(\eta) \text{tr}_s \int_{\gamma'} \sigma^{-z} \Theta^k(p) d\sigma \right| \leq Ke^{-K'\text{Re}(z)}$$

Proof: By compactness of $S^*M$, there exists a closed loop $\gamma''$ which simultaneously surrounds all the pointwise spectra of $-L^2(p)$ for all $p \in S^*M$. Such a loop may be chosen strictly to the right of the imaginary axis. This loop can be used in place of the contour $\gamma'$ in the above integral without affecting its value. The advantage is that $\gamma''$ is compact. Then $|\sigma^{-z}| \leq Ke^{-K'\text{Re}(z)}$ for all $\sigma \in \gamma''$ and some constants $K, K'$. Also, $\pi^*(\eta)\Theta^k$ is bounded on the compact set $\gamma'' \times S^*M \cap \pi^{-1}\text{supp}(\eta)$ by some $K''$ (with the appropriate choice of charts $U_\alpha$ and $V_\alpha$, as assumed). Then, integrating out the variables $\sigma, x, \text{ and } \Xi$ over this set we see that:

$$\left| \int_{S^*M \cap V_\alpha} \pi^*(\eta) \text{tr}_s \int_{\gamma'} \sigma^{-z} \Theta^k(p) d\sigma \right|$$

$$\leq \int_{S^*M \cap V_\alpha} \pi^*(\eta) \text{tr}_s \int_{\gamma'} \sigma^{-z} \Theta^k(p) d\sigma$$

$$\leq K'' V ol(S^*M \cap V_\alpha)(\gamma'') K e^{-K'\text{Re}(z)},$$

where $K''$ comes from the supertrace.

Corollary 5.4. For all $\eta \in \Omega^*M$, there exist constants $K$ and $K'$ such that:

$$I(z, \eta) < Ke^{-K'\text{Re}(z)}.$$

Lemma 3.5. Let $\eta \in \Omega^\kappa(T^*M)$. Then $I(z, \eta)$ can only have a nonzero residue at the point $(\kappa - 2n)/2$. Further, if $n$ is even and the residue is nonzero, then $(\kappa - n)/2$ must be an integer. If $n$ is odd, $(\kappa - 2n)/2$ must be a half-integer. In either case, nonzero residues occur only for even $\kappa$.

Proof: Utilizing the proof of proposition 5.1 we reason out the case of even $n$, the odd case being treated similarly.

Let $\eta$ be a $\kappa$-form. Recall that in in the said proposition, $I(z, \eta) = \sum_{\alpha, k, l} \int_{X_R \cap V_\alpha} \pi^*(\eta) \text{tr}_s \int_{\gamma} \sigma^{-z} \rho^{-2(x+k+l)} \Theta^k d\sigma$.

We restrict our attention to a single chart $V_\alpha$. We have seen that $\Theta^k$ is expanded into the sum of terms $G_{h,k}$ using (27). The only such terms
that could possibly contribute to $I(z, \eta)$ are the ones of differential-form degree $2n - \kappa$, because $\eta$ multiplied by them must produce a $2n$-form on $T^*M$. Thus, we need to collect all the appropriate (i.e., contributing) terms of the form:

$$\pi^*(\eta) \int_{\gamma} \lambda^{-z} G_{I_{k,l}} d\lambda.$$  

(30)

The remainder of the proof is but an exercise in counting the differential form degrees. They are products of $l$ copies of $G_1$ (which is locally a matrix of 1-forms), and $k - l$ copies of $G_0$, which is $\nabla^2 + d_\rho L$. Thus, any contributing term requires $k - l \geq 1$, because at least one copy of $G_0$ is needed to supply the differential $d\rho$ for the $2n$-form. Each additional copy of $G_0$ can only supply the curvature $\nabla^2$, which is a matrix of 2-forms. Therefore, each additional copy of $G_0$ may be replaced by two copies of $G_1$ without changing the total degree of (30). So, all the contributing terms satisfy:

$$\deg(\nabla^2)(k - l - 1) + \deg(d_\rho L) + \deg(G_1)l = 2n - \kappa,$$

(31) which means that the quantity $2k - l$ is the same for all of them. But one can see from (25) that the location of the residue which arises from each contributing term of (27) depends only on that quantity. It is apparent from (27) that $I(z, \eta)$ can have at most one nonzero residue, whose location must be $\frac{1}{2}(l - 2k) = \frac{\kappa - 2n}{2}$. This location does not depend on the topological information about $M$ (other than its dimension). Neither does it depend on the vector bundle $E$, on the curvature, etc.

If $\kappa$ is even, then by (31), $l$ must be odd. Similarly, if $\kappa$ odd, then $l$ must be even. Now, suppose we equip $\Lambda^*T^*M \otimes \text{End}(\pi^*E)$ with the $\mathbb{Z}_2$-grading inherited from $\text{End}(\pi^*E)$, rather than the total one. The supertrace vanishes on the sections of $\Lambda^*T^*M \otimes \text{End}(\pi^*E)$ which are odd in that inherited grading. We say that such sections are of an odd profile. The term even profile is defined similarly.

Hence, if $l$ is even, we get a term $\int_{\gamma} \lambda^{-z} G_{I_{k,l}} d\lambda$ which involves some number of the even-profile sections $(\sigma + L^2)^{-1}$ and $\nabla^2$. Also, it involves $d_\rho L$ and an even number of copies of $G_1$. Such a term will be of odd profile and its supertrace is zero. Thus, only if $\kappa$ is even can one hope to get a non-zero residue. Combining (31) with the fact that the residue is located at $(l - 2k)/2$, we get $l - 2k = -2n + \kappa$. □
6. Proof of Theorem 3.6.

**Theorem 3.6** Under the hypotheses outlined in section 3, for any \( \eta \in \Omega^*(M) \)

\[
\int_{T^*M} \text{tr}_s \pi^*(\eta) \exp \nabla^2_L = \lim_{R \to 0} \sum_{z \in \mathbb{C}} \text{Res}|_z \Gamma(z) \int_{X_R} \text{tr}_s \pi^*(\eta)(-\nabla^2_L)^{-z}.
\]

Further, all but finitely many residues on the right-hand side vanish as \( R \to 0 \).

In fact, we shall prove that for any \( R > 0 \):

\[
\int_{X_R} \pi^*(\eta) \text{tr}_s \exp \nabla^2_L = \sum_{z \in \mathbb{C}} \text{Res}|_z \left[ \Gamma(z) \int_{X_R} \pi^*(\eta) \text{tr}_s (-\nabla^2_L)^{-z} \right].
\]

Taking limits of both sides as \( R \to 0 \), we get theorem 3.6. The outline of our proof is the following.

1) We introduce a parameter \( t \geq 0 \) and express \( \exp \nabla^2_{tL} \) using a Cauchy integral. Thus, for a suitable vertical contour \( \gamma \) in \( \mathbb{C} \),

\[
\exp \nabla^2_{tL} = \int_{\gamma} e^{-\lambda} (\lambda + \nabla^2_{tL})^{-1} d\lambda.
\]

Strictly speaking, (32) is incorrect, because the integral over \( \gamma \) does not converge. Still, the formula holds in a weak sense. That is, for all \( R > 0 \) and \( \eta \in \Omega^*M \),

\[
\int_{X_R} \pi^*(\eta) \exp \nabla^2_{tL} = \int_{X_R} \int_{\gamma} \pi^*(\eta) e^{-\lambda} (\lambda + \nabla^2_{tL})^{-1} d\lambda,
\]

where the integral over \( \gamma \) converges. This is proven using the geometric-series expansion of \( e^{-\lambda} (\lambda + \nabla^2_{tL})^{-1} \) very similar to that in (19). We show that the above integral \( \int_{\gamma} \) converges at least for those terms of the expansion which do contribute to (33). See lemma 3.5 for discussion of contributing and non-contributing terms. Thus, (33) is an equality of currents on \( \Omega^*M \) induced by \( \text{tr}_s \exp \nabla^2_{tL} \) and by \( \int_{\gamma} e^{-\lambda} (\lambda + \nabla^2_{tL})^{-1} d\lambda \) via Poincare duality, as discussed in section 3.

2) Assuming \( \text{Re}(z) \gg 1 \), we show that in the same weak sense,

\[
\int_0^\infty t^{z-1} \exp \nabla^2_{tL} dt = \Gamma(z) (-\nabla^2_L)^{-z},
\]
which means that:

$$
\int_{X_R} \pi^*(\eta) \int_0^\infty t^{z-1} \exp \nabla^2_{tL} dt = \Gamma(z) \int_{X_R} \pi^*(\eta) (-\nabla^2_L)^{-z}.
$$

This is an application of the so-called Mellin transform which is given by

$$
f \mapsto \int_0^\infty f(t) t^{z-1} \, dt.
$$

The inverse transform is given by

$$
F(z) \mapsto \frac{1}{2\pi i} \int_C t^{-z} F(z) \, dz,
$$

where $C$ is a suitable vertical contour $C$. See [1] for details. Roughly, the computation for (35) is the following:

$$
\int_0^\infty \int_{X_R} \pi^*(\eta) \exp \nabla^2_{tL} t^{z-1} dt
$$

$$
= \int_{X_R} \pi^*(\eta) tr \int_0^\infty t^{z-1} \int_\gamma e^{-\lambda} (\lambda + \nabla^2_{tL})^{-1} \, d\lambda \, dt
$$

$$
= \int_{X_R} \pi^*(\eta) tr \int_\gamma \int_0^\infty e^{-\lambda} (\lambda + \nabla^2_{tL})^{-1} t^{z-1} \, dt \, d\lambda
$$

$$
= \int_{X_R} \pi^*(\eta) \int_\gamma \Gamma(z) \lambda^{-z} (\lambda + \nabla^2_{tL})^{-1} \, d\lambda
$$

$$
= \Gamma(z) \int_{X_R} \pi^*(\eta) tr (-\nabla^2_L)^{-z}.
$$

Interchanging the integrals $\int_0^\infty \ldots dt$ and $\int_{X_R}$ is easy, because $L^2$ is negative definite and invertible, so $\exp \nabla^2_{tL}$ is rapidly decreasing and absolutely integrable on $\mathbb{R} \times X_R$. We will need to prove that we can interchange $\int_0^\infty \ldots dt$ and $\int_\gamma \ldots d\lambda$, at least for $Re(z) \gg 1$ and for the relevant terms of the geometric series expansion of $e^{-\lambda} (\lambda + \nabla^2_{tL})^{-1}$. Also, notice that the exponential is $e^{-\lambda}$, not $e^{-t\lambda}$, as one might expect from the well-known identity:

$$
\lambda^{-z} \Gamma(z) = \int_0^\infty e^{-t\lambda} t^{z-1} \, dt.
$$

The algebra behind this will be explained. Also, observe that by theorem 3.4 the integral $\int_{X_R} \ldots$ in the right-hand side has a meromorphic extension to $\mathbb{C}$ with at most simple poles.

3) Next, we simply restate (36) in terms of the inverse Mellin transform. For a certain vertical $C \subset \mathbb{C}$,

$$
\int_{X_R} tr \pi^*(\eta) \exp \nabla^2_{tL} = \frac{1}{2\pi i} \int_C t^{-z} \Gamma(z) \int_{X_R} tr \pi^*(\eta) (-\nabla^2_{tL})^{-z} \, dz.
$$

The integral $\int_{X_R} tr \pi^*(\eta) (-\nabla^2_{tL})^{-z} \, dz$ will be abbreviated by $I_t(z, \eta)$.

Since the meromorphic extension of $I_t(z, \eta)$ is defined for all $z$
except at a certain discrete set, we can choose $C$ to pass through the domain where $I_t(z, \eta)$ is defined. We will need to check the convergence of the integral $\int_C \ldots dz$. Observe that $t$ has reappeared in the subscript $tL$ on the right-hand side. This will require clarification.

4) After taking the meromorphic extension of $I_t(z, \eta)$ in the right-hand side of (37), we “collect” the residues by translating the contour $C$ to the left. We will need to prove that as the contour translates,

$$\int_C \Gamma(z) \int_{X_R} \text{tr}_s \pi^*(\eta)(-\nabla^2_{tL})^{-z}dz \rightarrow \sum_{z \in C} \text{Res} |z| \int_{X_R} \text{tr}_s \pi^*(\eta)(-\nabla^2_{tL})^{-z}.$$ 

This is an application of the residue theorem (Fig. 2).

We proceed with the proof. In order to make the equation (33) in step (1) less bulky, we introduce the following notation.

**Definition 6.1.** Let $\mu, \nu$ be smooth forms on $T^*M$. We say that they are *equal in the weak sense* and we write $\mu =_w \nu$ if for all $\eta \in \Omega^*(M)$

$$\int_{X_R} \pi^*(\eta)\mu = \int_{X_R} \pi^*(\eta)\nu.$$ 

In other words, this means that $\mu$ and $\nu$ are equal as currents on $\Omega^*(M)$.

With this kind of equivalence relation, it is possible to write $\exp(\nabla^2_{tL})$ as a Cauchy integral, similarly to (20). But first, we introduce some
more notation. We expand $\nabla^2_{tL}$, similarly to (23):

$$\nabla^2_{tL} = \nabla^2 + t[\nabla, L] + t^2L^2.$$  

Now, let $\Theta_t = \nabla^2 + t[\nabla, L]$, and let $\gamma_\tau$ denote a vertical line $\{Re(\lambda) = \tau\}$, oriented downward. (E.g., $\gamma_0$ is just the imaginary axis). Since locally, $\Theta_t$ is a matrix of differential forms of positive degree, and $T^*M$ is $2n$-dimensional, we have the following geometric series:

$$\begin{align*}
(\lambda + \nabla^2_{tL})^{-1} &= (\lambda + t^2L^2)^{-1} \sum_{k=0}^{2n} (-1)^k [(\lambda + t^2L^2)^{-1} \Theta_t]^k. 
\end{align*}$$

(38)

**Lemma 6.1.** Fix $p \in T^*M$. Choose $\epsilon \geq 0$ such that $\gamma_\epsilon$ lies to the left of the pointwise spectrum $sp(-t^2L^2(p))$ of $-t^2L^2(p)$. Then the following equality holds and the integral on the right-hand side converges:

$$\exp(\nabla^2_{tL}(p)) = \frac{1}{2\pi i} \int_{\gamma_\epsilon} e^{-\lambda} \left[ (\lambda + t^2L^2)^{-1} \sum_{k=0}^{2n} (-1)^k [(\lambda + t^2L^2)^{-1} \Theta_t]^k \right] d\lambda.$$  

(39)

In view of (38), this comes close to

$$\begin{align*}
\exp(\nabla^2_{tL}) &= \frac{1}{2\pi i} \int_{\gamma_\epsilon} e^{-\lambda} (\lambda + \nabla^2_{tL})^{-1} d\lambda.
\end{align*}$$

(40)

**Proof:** The contributing terms (see the proof of lemma 3.5) have enough negative powers of $\lambda$ to assure convergence. Note that we start the series at $k = 1$ which explains the weak equality: the integral of the term with $k = 0$ diverges. Fortunately, that term does not involve vertical differentials $d\Xi^i$ or $dp$. Thus, by the proof of lemma 3.5, it does not contribute to any current. \[\square\]

In step (2), we need to prove some version of (35) (it is not true literally). To do this, we shall:

a) Apply the geometric-series expansion (38) to

$$e^{-\lambda}(\lambda + \nabla^2_{tL})^{-1} = t^2L^2 + t[L, \nabla] + \nabla^2.$$  

b) On the $k$-th term of that expansion, perform a secondary expansion into terms which will be denoted by $\Phi_k^t$. This will separate the powers of $t$:

$$(-1)^k(\lambda + t^2L^2)^{-1} \left[ (\lambda + t^2L^2)^{-1} \Theta_t \right]^k = \sum_{l=0}^{k} t^{l-2(k+1)} \Phi_k^t.$$  

(41)
c) For each term $\Phi^k_l$, take the Mellin transform of the integral $\int_{\gamma} e^{-\lambda t - 2(k+1)} \Phi^k_l d\gamma$. Homogeneity in $t$ makes this task possible. This will yield a formula much like (35), for each of a collection of separate terms. At a certain later point, we shall reassemble the Mellin transforms of these terms into the quantity $\Gamma(z)(-\nabla^2_L)^{-z}$.

We accomplish a), b) and c) in the following proposition. Also, our earlier warning about interchanging the integrals $\int_0^\infty \ldots dt$ and $\int_{\gamma} \ldots d\lambda$ in (35) receives due attention here.

**Proposition 6.2.** There exist smooth sections $\Phi^k_l$ of $\Lambda^* T^* M \hat{\otimes} \text{End}(\pi^* E)$ such that the following expansion holds for each $k > 0$:

$$(-1)^k(\lambda + t^2 L^2)^{-1} \left[ (\lambda + t^2 L^2)^{-1} \Theta_t \right]^k = \sum_{l=0}^{k} t^{l-2(k+1)} \Phi^k_l.$$

These sections depend on the quantity $\frac{\lambda}{t^2}$, but they do not depend on $t$ in any other way. Further, there exists $\tau > 0$ such that for $\text{Re}(z) \gg 0$,

$$\int_0^\infty t^{2z-1} \text{tr}_s \exp \nabla^2_L dt$$

$$= w \int_0^\infty t^{2z-1} \text{tr}_s \left[ \int_{\gamma_0} e^{-\lambda} \sum_{k=1}^{2n} (-1)^k (\lambda + t^2 L^2)^{-1} \left[ (\lambda + t^2 L^2)^{-1} \Theta_t \right]^k d\lambda \right] dt$$

$$= \frac{1}{2} \int_{\gamma_r} \text{tr}_s \sum_{k=1}^{2n} \sum_{l=0}^{k} \Gamma(z + l/2 - k) \left( \frac{\lambda}{t^2} \right)^{-(z+l/2-k)} \Phi^k_l d\lambda t^{l/2}.$$

Here, $\gamma_0$ is the imaginary axis. Also, note that we are using $2z$ instead of $z$ in the Mellin transform.

**Proof:** This proof is very similar to that of proposition 5.1. Each term $\left[ (\lambda + t^2 L^2)^{-1} \Theta_t \right]^k$ is a non-commutative polynomial in the quantities $\nabla^2$ and $t[L, \nabla]$, which are globally well-defined sections of the vector bundle $\Lambda^* T^* M \hat{\otimes} \text{End}(\pi^* E)$. Therefore, each term can be further expanded as follows:

$$(\lambda + t^2 L^2)^{-1}(-1)^k \left[ (\lambda + t^2 L^2)^{-1} \Theta_t \right]^k = \sum_{l} \tilde{\Phi}_l^k,$$

where the quantity $\tilde{\Phi}_l^k$ is the sum of all the monomials which are products of $l$ copies of $t[L, \nabla]$, $k - l$ copies of $\nabla^2$, and $k + 1$ copies of $\Theta_t$. \end{proof}
\[
(\lambda + t^2 L^2)^{-1}. \text{ This is similar to the construction of } \Theta_l^k \text{ in the proof of proposition 5.1. But } (\lambda + t^2 L^2)^{-1} = t^{-2}(\frac{\lambda}{t^2} + L^2)^{-1}, \text{ so, we can pull } t^{-2(k+1)+l} \text{ out of } \tilde{\Phi}_l^k \text{ to obtain } \Phi_l^k:
\]
\[
(44) \quad \tilde{\Phi}_l^k = t^{l-2(k+1)}\Phi_l^k.
\]

So (41) holds and we can prove (43.) Just as in the lemma 6.1, the weak equality is there because we start the geometric series at \( k = 1 \). In what follows, by \( sp(t^2 L^2) \) we always mean the pointwise spectrum over a point \( p \in T^*M \). Define \( \tau \) by
\[
\tau = \frac{1}{2} \inf \bigcup_{p \in \mathcal{X}_R} sp(-L^2(p)).
\]
Such \( \tau \) exists by lemma 5.2. Then for each \( t > 0 \), the pointwise spectrum of \( t^2 L^2 \) is to the right of \( \gamma_{\tau t^2} \). Fixing one such \( t \) for the moment, we see that the vertical \( \gamma_{\tau t^2} \) is a suitable contour for the Cauchy integral expression of \( \exp \nabla_{tL}^2 \) (by lemma 6.1) and we can compute, for the \( k \)-th term:
\[
(45) \quad \int_{\gamma_{\tau t^2}} e^{-\lambda}(-1)^k(\lambda + t^2 L^2)^{-1}[(\lambda + t^2 L^2)^{-1}\Theta_l]^k d\lambda
\]
\[
= \int_{\gamma_{\tau}} e^{-t^2 \sigma} \sum_l t^{l-2k}\Phi_l^k d\sigma,
\]
where \( \sigma = \frac{\lambda}{t^2} \). Thus,
\[
(46) \quad \int_0^\infty t^{2z-1} \left[ \int_{\gamma_{\tau t^2}} e^{-\lambda}(-1)^k(\lambda + t^2 L^2)^{-1}[(\lambda + t^2 L^2)^{-1}\Theta_l]^k d\lambda \right] dt
\]
\[
= \int_0^\infty t^{2z-1} \left[ \int_{\gamma_{\tau}} e^{-t^2 \sigma} \sum_l t^{l-2k}\Phi_l^k d\sigma \right] dt.
\]

Since \( k > 0 \), \( \Phi_l^k \) involves at least 2 factors of \( (\sigma + L^2)^{-1} \). Therefore it is absolutely integrable with respect to \( \sigma \), while \( t^{2(z-k)+l-1}e^{-t^2 \sigma} \) is absolutely integrable in \( t \) for nonnegative \( Re(z) \). By Fubini’s theorem, we may interchange the integrals and finish the computation:
\[ (47) \quad \int_0^\infty t^{2z-1} \left[ \int_{\gamma_{\tau}} e^{-\lambda} (-1)^k (\lambda + t^2 L^2)^{-1} \left[ (\lambda + t^2 L^2)^{-1} \Theta_t \right]^k d\lambda \right] dt = \int_0^\infty t^{2z-1} \left[ \int_{\gamma_{\tau}} e^{-t^2\sigma} \sum_l t^{l-2k} \Phi_l^k d\sigma \right] dt \]

\[ = \int_{\gamma_{\tau}} \left[ \int_0^\infty t^{2z-1} e^{-t^2\sigma} \sum_l t^{l-2k} \Phi_l^k dt \right] d\sigma \]

\[ = \frac{1}{2} \int_{\gamma_{\tau}} \left[ \int_0^\infty e^{-t^2\sigma} \sum_l t^{2z+l-2k-2} \Phi_l^k dt^2 \right] d\sigma \]

\[ = \frac{1}{2} \int_{\gamma_{\tau}} \sum_l \Gamma(z + l/2 - k) \sigma^{-(z+l/2-k)} \Phi_l^k d\sigma. \quad \Box \]

This proves that

\[ \int_{X_R} \pi^* (\eta) \text{tr}_s \int_0^\infty t^{2z-1} \int_{\gamma_{\tau}} e^{-\lambda} (-1)^k (\lambda + t^2 L^2)^{-1} \left[ (\lambda + t^2 L^2)^{-1} \Theta_t \right]^k d\lambda dt \]

\[ = \int_{X_R} \pi^* (\eta) \text{tr}_s \frac{1}{2} \int_{\gamma_{\tau}} \sum_l \Gamma(z + l/2 - k) \sigma^{-(z+l/2-k)} \Phi_l^k d\sigma, \quad (48) \]

and the integrals \( \int_{X_R} \) and \( \int_0^\infty \) \( \ldots dt \) in the left-hand side can be interchanged, as remarked in our discussion after (36).

This equation is as close as we get to (36). Our next step is to apply the inverse Mellin transform to the right-hand side. Our estimate from lemma 5.3 comes in here. The inverse Mellin transform of (48) is

\[ (49) \quad \frac{1}{2\pi i} \int_C t^{-2z} \int_{X_R} \pi^* (\eta) \times \]

\[ \text{tr}_s \frac{1}{2} \int_{\gamma_{\tau}} \sum_l \Gamma(z + l/2 - k) \sigma^{-(z+l/2-k)} \Phi_l^k d\sigma d\sigma d\sigma. \]

By theorem 3.4, the vertical line \( C \) may be chosen very far to the right so the residues of \( \Gamma(z) I_l(z, \eta) \) are nowhere near. Convergence of the integral over \( C \) is assured by the estimate very similar to lemma 5.3 and by the fact that \( \Gamma(z) \) is rapidly decreasing on the vertical lines.
(50) \[
\int_{X_R} \pi^*(\eta) \operatorname{tr}_s \exp(\nabla t L)^2
\]
\[
= \sum_{k,l} \frac{1}{4\pi i} \int_C t^{-2s} \int_{X_R} \pi^*(\eta)
\]
\[
\times \Gamma(z + l/2 - k) \operatorname{tr}_s \int_{\gamma_r} \sigma^{-(z+l/2-k)} \Phi^k d\sigma d(2z)
\]
\[
= \frac{1}{2\pi i} \sum_{k,l} \int_C t^{-2s} \Gamma(z + l/2 - k)
\]
\[
\times \int_{X_R} \pi^*(\eta) \operatorname{tr}_s \int_{\gamma_r} \sigma^{-(z+l/2-k)} \Phi^k d\sigma d z.
\]

Introducing the variables \(s = z + l/2 - k\) and the verticals \(C_{l,k} = C + l/2 - k\), we may rewrite (50) as:

(51) \[
\frac{1}{2\pi i} \sum_{k,l} \int_{C_{l,k}} t^{-2(s-l/2+k)} \Gamma(s) \int_{\gamma_r} \sigma^{-s} \Phi^k d\sigma d s.
\]

In view of the next lemma, we may replace all the contours \(C_{k,l}\) with \(C\).

**Lemma 6.3.** Fix some \(p \in T^*M\). Let \(C\) and \(C'\) be two vertical lines in \(\mathbb{C}\) with the same orientation. If the expression \(\Gamma(s) \int_{\gamma_r} \sigma^{-s} \Phi^k d\sigma d s\) has no singularities between them, then:

(52) \[
\int_C t^{-2(s-l/2+k)} \Gamma(s) \int_{\gamma_r} \sigma^{-s} \Phi^k(p) d\sigma d s =
\]
\[
\int_{C'} t^{-2(s-l/2+k)} \Gamma(s) \int_{\gamma_r} \sigma^{-s} \Phi^k(p) d\sigma d s.
\]

**Proof:** First, we join \(C\) and \(C'\) by horizontal line segments \(ab\) and \(cd\), located below and above the real axis, as in Fig. 3. Then:

(53) \[
\left( \int_{a}^{b} + \int_{c}^{d} + \int_{a}^{b} + \int_{c}^{d} \right) t^{-2(s-l/2+k)} \left[ \int_{\gamma_r} \Gamma(s) \sigma^{-s} \Phi^k d\sigma \right] d s = 0.
\]

Because on the vertical lines \(\Gamma(s)\) is rapidly decreasing and the rest of the integrand is bounded in \(s\), the integrals over \(ab\) and \(cd\) vanish as those line segments move away from the real axis. The result follows.

\(\square\)

This lemma allows us to continue the computation, using \(C\) instead of \(C_{k,l}\), provided that \(C\) were originally chosen far enough to the right.
We are about to reassemble the individual terms $\int_{\gamma_r} \sigma^{-s} \Phi^k_l d\lambda$ into the quantity $(-\nabla^2_L)^{-z}$, as promised earlier.

$$\frac{1}{2\pi i} \sum_{k,l} \int_C \int_{X_R} \pi^*(\eta) \operatorname{tr}_s t^{-2(s-l/2+k)} \int_{\gamma_r} \Gamma(s) \sigma^{-s} \Phi^k_l d\sigma ds$$

$$= \frac{1}{2\pi i} \sum_{k,l} \int_C \int_{X_R} \pi^*(\eta) \operatorname{tr}_s t^{-2s} \int_{\gamma_r} \Gamma(s) \sigma^{-s} t^{-2k} \Phi^k_l d\sigma ds. \quad (54)$$

Recall that $\Phi^k_l$ depends on the quantity $\sigma = \frac{\lambda}{t^2}$. Also, recall (44):

$$\tilde{\Phi}^k_l = t^{-2(k+1)} \Phi^k_l. \quad (55)$$

So, by (45), summing the right-hand side of (55) over $l$ we obtain:

$$\frac{1}{2\pi i} \sum_l \int_C \int_{X_R} \pi^*(\eta) \operatorname{tr}_s t^{-2(s-l/2+k)} \int_{\gamma_r} \Gamma(s) \sigma^{-s} \Phi^k_l d\sigma ds$$

$$= \frac{1}{2\pi i} \sum_l \int_C \int_{X_R} \pi^*(\eta) \operatorname{tr}_s \int_{\gamma_r} \Gamma(s) \lambda^{-s} \Phi^k_l d\lambda ds$$

$$= \frac{1}{2\pi i} \int_C \int_{X_R} \pi^*(\eta) \operatorname{tr}_s \int_{\gamma_r} \lambda^{-z} (-1)^k (\lambda + t^2 L^2)^{-1}$$

$$\times \left[ (\lambda + t^2 L^2)^{-1} \Theta_l \right]^k d\lambda. \quad (56)$$

Finally, by summing this over all $k$, we recover the quantity

$$\frac{1}{2\pi i} \int_C \int_{X_R} \pi^*(\eta) \operatorname{tr}_s \Gamma(s) (-\nabla_{tL})^{-2s} ds. \quad (57)$$
We now invoke theorem 3.4 and lift our standing assumption that \( \text{Re}(z) \gg 0 \). Hence, the integral \( \int_{X_R} \ldots \) in (57) may be replaced by its meromorphic extension. We can now finish the proof, by moving the vertical \( C \) to the left and “picking up” all the residues. The procedure is explained in the following lemma.

**Lemma 6.4.** For any \( r > 0 \) such that \( (C - r) \) does not intersect the real axis at any of the residues, the following holds:

\[
\int_C \Gamma(s) \int_{X_R} \pi^*(\eta)(-\nabla^2_{tL})^{-s}ds = \\
\int_{C-r} \Gamma(s) \int_{X_R} \pi^*(\eta)(-\nabla^2_{tL})^{-s}ds + \\
\sum_{\text{Re}(s) > -r} \text{Res}_{s} \Gamma(s) \int_{X_R} \pi^*(\eta)(-\nabla^2_{tL})^{-s}.
\]

Further, substituting \( r_m = \frac{2m+1}{2} \) instead of \( r \) in the above expression, the integral on the right-hand side tends to zero as \( m \to \infty \), so that

\[
\int_C \Gamma(s) \int_{X_R} \pi^*(\eta)(-\nabla^2_{tL})^{-s}ds = \\
\sum_{s \in \mathbb{C}} \text{Res}_{s} \Gamma(s) \int_{X_R} \pi^*(\eta)(-\nabla^2_{tL})^{-s}.
\]

**Proof:** The first equation follows by the argument similar to that in lemma 6.3 (Fig. 4). Next, because of the identity \( z \Gamma(z) = \Gamma(z + 1) \), the quantity \( \sup_{y \in \mathbb{R}} |\Gamma(x + iy)| \) decays superexponentially as \( x \to -\infty \). Therefore, by our estimate in corollary (5.4) on the integral \( I_t(\eta, z) = \int_{X_R} \pi^*(\eta)(-\nabla^2_{tL})^{-s}ds \), (59) follows. \( \square \)

Finally, we need to prove that as \( R \to 0 \), only finitely many residues survive. By theorem 3.4, \( I_t(\eta, z) \) has only finitely many poles and they are at most simple. Also, \( \Gamma(z) \) has at most simple poles, at \( z = 0, -1, -2, \ldots \). So, the residues of \( \Gamma(z)I_t(\eta, z) \) are due to either one of these factors. But the theorem 3.4, provides us with some knowledge of the general algebraic form of \( I_t(\eta, z) \). It implies that for \( m >> 0 \), the residue at \( z = -m \) will be a multiple of a positive power of \( R \), so it will vanish as \( R \to 0 \). \( \square \)

7. **The de Rham operator on Riemannian surfaces.**

We consider the complexified vector bundle \( E \otimes \mathbb{C} = \Lambda^*M \otimes \mathbb{C} \) of exterior forms over a compact manifold \( M \) with no boundary. The
grading decomposition is that into differential forms of even and odd degree. Given a Riemannian metric \( g \) on \( M \), the associated de Rham operator \( D = d + d^* \) has a well-known symbol \( L = -\rho^2 \).

In order to apply the theorem 3.6, we need to compute:

\[
(-(\nabla + L)^2)^{-z} = -(L^2 + \nabla^2 + [\nabla, L])^{-z}.
\]

The argument of the function \( \nu \mapsto -\nu^{-z} \) can be viewed as \(-\rho^2\) plus some commuting perturbation which is nilpotent. It follows that we may just use the Taylor series expansion instead of the Cauchy integrals:

\[
(-(\nabla + L)^2)^{-z} = \sum_{k=0}^{2n} (\rho^2)^{-z+k} \binom{-z+k}{k} (\nabla^2 + [\nabla, L])^k,
\]

where \( \binom{z}{k} \) means \( \frac{z(z-1)\ldots(z-k+1)}{k!} \).

For example, we consider the case when \( M \) is a 2-manifold and \( \eta \equiv 1 \). Since that is exactly the todd class for any 2-surface, by the Atiyah-Singer index theorem both sides of theorem 3.6 should give us the euler characteristic. The left-hand side of theorem 3.6 yields:
We keep only these two terms because they are the only ones that can possibly contain a 4-form which can be integrated over $T^*M$. In fact, when we expand $(\nabla^2 + [L, \nabla])^3$ we see that only three terms really enter the picture, namely $\nabla^2[L, \nabla]^2$, $[L, \nabla]^2\nabla^2$, and $[L, \nabla]\nabla^2[L, \nabla]$. From $(\nabla^2 + [L, \nabla])^4$, the relevant term is $[L, \nabla]^4$.

**Lemma 7.1.** The term $[L, \nabla]^4$ vanishes as a section of $\Lambda^4 T^*M \otimes \mathrm{End}(\pi^* E)$, i.e., pointwise.

**Proof:** From section 5,

$$[L, \nabla] = d_x L + d_\xi L + [\theta, L],$$

If one uses normal coordinates on $M$ near some point $x$, then $\theta$, being comprised of Christoffel symbols is zero on the fiber of $T^*M$ over $x$. The horizontal differential $d_x L$ is also zero there. Hence on that fiber, $[L, \nabla] = d_\xi L$ which is a matrix of “vertical” forms on $T^*M$. Any power of it which is larger than $\dim(M)$ must vanish. \qed

Thus, the left-hand side of theorem 3.6 reads:

(61) \[ \int_{T^*M} \exp(-\rho^2) \frac{1}{6} (\nabla^2[L, \nabla]^2 + [L, \nabla]^2\nabla^2 + [L, \nabla]\nabla^2[L, \nabla]). \]

Similar remarks apply on the right-hand side and we obtain:

(62) \[ \lim_{R \to 0} \sum Res|_{z} \Gamma(z) \int_{X_R} \frac{-z(z + 1)(z + 2)}{6} \rho^{-2(z+3)} (\nabla^2[L, \nabla]^2 + [L, \nabla]^2\nabla^2 + [L, \nabla]\nabla^2[L, \nabla]). \]
Since \( t r_s(\nabla^2[L,\nabla]) = \omega \rho d \rho \) for some differential form \( \omega \), it is enough to see that:

\[
(63) \quad \int_0^\infty \exp(-\rho^2)\rho d\rho = \\
= \lim_{R \to 0} \sum Res[z] \Gamma(z) \int_R^\infty (-z)(z+1)(z+2)\rho^{-2(z+3)}\rho d\rho \\
= \frac{1}{2} \lim_{R \to 0} \sum Res[z] \Gamma(z+2)R^{-2(z+2)} \\
= \frac{1}{2} \lim_{R \to 0} \sum_{m=0}^\infty \frac{(-1)^m}{m!} R^{2m}.
\]

8. **The Chern character for a general spinor bundle.**

We apply our results to the Chern character of a spinor bundle \( S \to M \) associated to a vector bundle \( \pi: F \to M \), as computed by Mathai and Quillen [10]. The role of the cotangent bundle \( \pi: T^*M \to M \) is played by \( F \) in this example, and the role of \( E \to M \) is played by \( S \). So, theorem 3.6, strictly speaking does not apply, though we could have proven it in a more general setting. The reason we stated our theorem for \( T^*M \) is that we have the Atiyah-Singer index theorem in mind, for future applications. Rather than applying theorem 3.6, we will go through its proof. Namely, we shall repeat steps (2) and (3) in a simpler way.

We proceed to outline the result of [10]. Some familiarity with spin structures is assumed here. The reader can consult, e.g., *Spin Geometry* by Lawson and Michelsohn for details [9]. We also warn that the notation of [10] is quite a bit different from our own. We will briefly explain the differences in the end of this section.

Let \( \pi: F \to M \) be a complex even-dimensional vector bundle with a spin structure. In particular, this means that there is a fiberwise metric \( \mu \) on \( F \). Let \( S \to M \) be the associated spinor bundle. The assumption of spin structure implies that \( S \) can be split into a direct sum of even and odd subbundles: \( S = S^+ \oplus S^- \), where the fibers \( S^+_x \) and \( S^-_x \) of \( S^+ \) and \( S^- \) are the only two irreducible representations for the spin group of the fiber \( F_x \). Thus, the spin structure induces a \( \mathbb{Z}_2 \)-grading of \( S \).

In order to form a Chern character we need a connection \( \nabla' \) on \( S \) which respects that \( \mathbb{Z}_2 \)-grading. We also need an odd antiselfadjoint endomorphism \( L \) of \( \pi^*S \). The spinor bundle setup in [10] requires, among other things, that:
- $L$ be homogeneous of degree 1 in the radial coordinate $\rho$ of the fibers of $F$. Much as in the proof of theorems 3.4 and 3.6, $\rho$ is induced by the metric $\mu$ and is given by $\rho(p) = \sqrt{\mu(p,p)}$ for all $p$ in $F$.

- $\nabla'$ preserve the fiberwise metric $(\cdot, \cdot)$ which is induced on $S$ by the metric $\mu$ of $F$. This means that for any two sections $\alpha$ and $\beta$ of $S$,

$$d(\alpha, \beta) = (\nabla' \alpha, \beta) + (\alpha, \nabla' \beta).$$

The coordinate notation is the same as in the proof of theorem 3.4. The local coordinates on $F$ are the vertical (fiberwise) cartesian coordinates are $\xi^1, \ldots, \xi^m$, and the horizontal coordinates $x_1, \ldots, x_n$, which are also coordinates of $M$. In fact, it makes sense to choose the $\mu$-orthonormal local frame $e_1, \ldots e_m$ of $F$ and to choose coordinates $\xi^j$ associated to that frame. They may be replaced by spherical coordinates $\rho$ and $\Xi^1, \ldots, \Xi^{m-1}$ at our convenience.

To describe $L$ we recall that the spin structure of $F$ stems from the fiberwise metric $\mu$. To begin with, we have the Clifford action of $F$ on $S$ which is a fiberwise $\mathbb{R}$-linear bundle map $c: F \to \text{End}(S)$, such that for any $(x, \xi)$ in $F_x$, $c(x, \xi)^2 = -\mu_x(\xi, \xi)$. It is one of the standard axioms for a Clifford actions that $c(x, \xi)$ be fiberwise anti-selfadjoint endomorphism. Thus, each fiber $F_x$ is contained in a clifford algebra $\text{Cliff}(F_x, \mu_x)$, which is a fiber of the bundle $\text{Cliff}(F, \mu)$. Also, there is a map

$$\text{Cliff}(F, \mu) \to \text{End}(S),$$

which is an isomorphism of bundles and fiberwise an isomorphism of algebras. Now, the pullback $\pi^*F$ to the total space of $F$ is equipped with the Clifford action on $\pi^*S$ which we shall also denote $c$ instead of $\pi^*c$. Let $\tau: F \to \pi^*F$ be the tautological section. Then the endomorphism $L = c(\tau(x, \xi))$ has all the required properties. Its homogeneity in $\rho$ is obvious and it is antiselfadjoint by hypothesis.

Abbreviating $c(\tau(e_j))$ by $\gamma_j$, we may write $L = \sum_j \xi^i \gamma_j$. Since the construction of the clifford action on spinors using an orthonormal basis is completely canonical, the coordinate expression for $L$ does not involve the $x$-variables. Since,

$$\frac{1}{2}(\gamma_i \gamma_j + \gamma_j \gamma_i) = -\delta_{ij},$$

it follows that $L^2 = \mu(\xi, \xi) = -\rho^2$.

Next, to pick a connection on $\nabla'$, we start with a connection on $F$ which is locally given by $d + \theta$. The connection $\nabla'$ on $S$ is constructed
from it (see [10] and [9]). In order to describe the construction, we adopt the *summation notation*: we reserve the right to write any index as an upper or a lower index. (Since we have chosen an orthonormal local frame, there is no difference at all). Repetition of the same index on the top and on the bottom implies summation. Repetition on the top only or on the bottom only does *not*. The connection on $\pi^*S$ is given by

$$\nabla' = d + \frac{1}{4} \theta^{ij} \gamma_i \gamma_j,$$

where $\theta^{ij}$ are just the matrix entries of the endomorphism-valued1-form $\theta$. The connection $\nabla = \pi^*\nabla'$ on $\pi^*S$ therefore makes sense. Observe that since $\gamma_j$ are odd, the local endomorphism $\theta^{ij} \gamma_i \gamma_j$ of $\pi^*S$ is even. Moreover, it only depends on the variables $x_i$ and horizontal differentials $dx_i$, just as before. Therefore, the curvature of the connection $\nabla_L$ may be written as:

$$(64) \quad \nabla_L^2 = \nabla^2 + [\nabla, L] - \rho^2$$

$$= \nabla^2 + d\xi^j \gamma_j + \frac{\xi^k}{4} [\theta^{ij} \gamma_i \gamma_j, \gamma_k] - \rho^2$$

$$= \nabla^2 + (d\xi^j) \gamma_j + \frac{\xi^j}{4} \theta^{ij} \gamma_i - \rho^2$$

Here, just as in the previous example, the fact that $L^2$ is a scalar is a tremendous simplification. We may use Taylor series instead of Cauchy integrals and we have the rule $e^{a+b} = e^a e^b$.

Now, the result from [10] reads:

$$(65) \quad \text{tr}_s \exp \nabla_L^2 = (-1)^{m/2} \left( \frac{i}{2\pi} \right)^{-m/2} \det \left( \frac{\sinh(\nabla^2/2) - \nabla^2/2}{(\nabla^2/2)} \right)^{1/2}$$

$$\times \text{tr}_s \left( \sigma^{-m/2} e^{-\rho^2} \sum_{I} \epsilon(I, I') Pf(\nabla^2/2) \prod_{j \in I'} ((d\xi_j) \gamma_j + \xi^j \theta^{ij} \gamma_i) \right),$$

where:

- $I, I'$ are complementary (strictly increasing) multiindices over the set $\{1, 2, \ldots, m\}$ and $\epsilon(I, I')$ is a certain combinatorial $\pm 1$-valued function of them, which shall not be relevant here.
- $Pf(\nabla^2/2)_I$ is the Pfaffian of the submatrix of $\nabla^2/2$ determined by the multiindex $I$. For an unfamiliar reader, it suffices to know that it is a certain polynomial in the matrix entries of $\nabla^2$ which is just 1 if $I$ is the empty multiindex.

For us, (65) is greatly simplified by the fact that we are only interested in the currents induced by this Chern character on $\Omega^*M$. Therefore,
as observed in theorem 3.4 and lemma 3.5, we need only those terms of (65) which involve all the differentials $d\xi^j$, so the only relevant multiindex is $I' = (1, 2, \ldots, m)$, $I$ being empty and $Pf(\nabla^2/2)_I$ being 1. The only term of interest is therefore

$$\pi^{-m/2}e^{-\rho^2}d\xi^1 \ldots d\xi^m.$$  

If we replace $L$ by $tL$, as in theorem 3.6, (65) becomes

$$\text{tr}_{s}\exp \nabla^2_L = (-1)^{m/2}\left(\frac{i}{2\pi}\right)^{-m/2} \det \left(\frac{\sinh(\nabla^2/2)}{(\nabla^2/2)}\right)^{\frac{1}{2}}$$

$$\times \text{tr}_{s}i^{m}\pi^{-m/2}e^{-t^2\rho^2}d\xi^1 \ldots d\xi^m.$$  

Integrating this over any fiber of $F$, we see that for any $\eta \in \Omega^*M$,

$$\int_{F} \pi^*(\eta) \text{tr}_{s}\exp \nabla^2_L = \int_{F} \pi^*(\eta)(-1)^{m/2}\left(\frac{i}{2\pi}\right)^{-m/2} \det \left(\frac{\sinh(\nabla^2/2)}{(\nabla^2/2)}\right)^{\frac{1}{2}}.$$  

This allows us to understand the residue formulation of this Chern character.

A computation similar to the one in the previous example yields:

$$\exp \nabla^2_L = \sum_{k,l} \frac{1}{2\pi i} \int_{C(z)}^{z} \Gamma(z) \rho^{-2(z+k)} P_{k-l,l}(\nabla^2, [\nabla, L])dz,$$

where by $P_{\mu,\nu}(A, B)$ we denote the homogeneous non-commutative polynomial in $A$ and $B$ obtained by summing all the words comprised of $\mu$ copies of $A$ and $\nu$ copies of $B$.

We now recall (65). It involves the sum over multiindices $I'$ and the only relevant multiindex was determined to be $I' = (1, 2, \ldots, m)$, where $m$ is the fiberwise dimension of $F$. This means that in (68) only the terms with $l = m$ contribute to the current induced by Chern character on $\Omega^*M$. We have seen a special case of this in the previous example, where a normal coordinates argument was used to show that only the terms which involve two copies of $[\nabla, L]$ are relevant. (Recall from lemma 3.5 that such terms were called contributing.) In particular, it means that $[\nabla, L]$ contributes the vertical differentials and no other differentials.

Coming back to our computation, the right-hand side of (68) is readily seen to be the Taylor series for $\Gamma(z)(-\nabla^2_L)^{-2z}$. The discussion in the
previous paragraph implies that the contributing part of $P_{k-l,l}([\nabla^2, [\nabla, L]])$ is a multiple of the vertical volume form:

$$d\xi^1 \ldots d\xi^m = \rho^{m-1} d\rho d\Xi^1 \ldots d\Xi^{m-1}.$$ 

Thus, it supplies $m - 1$ powers of $\rho$. It remains to determine the residues, using the procedure from theorem 3.4. Just as in that theorem, we set

$$X_R = \text{def} \{ p \in F : \rho(p) \geq R \},$$

and integrate from $R$ to $\infty$ with respect to $\rho$. This, as we shall see, produces the residue at $(m - 2k)/2$. Let $\eta \in \Omega^* M$, and express $(-\nabla^2_L)^{-z}$ as a sum of two differential forms:

$$(-\nabla^2_L)^{-z} = \nu_z + \omega_z d\rho,$$

where neither $\nu_z$ nor $\omega_z$ involve $d\rho$. We obtain:

$$\Gamma(z) \int_{X_R} \pi^*(\eta)(-\nabla^2_L)^{-z} = \int_R^\infty \rho^{-2(z+k)+m-1} d\rho \int_{S^* M} \pi^*(\eta)\omega_z$$

$$= \frac{R^{-2(z+k)+m}}{2(z+k) + m} \int_{S^* M} \pi^*(\eta)\omega_z.$$

Counting the differential form degrees, we see that if $\deg(\eta) = \kappa$, then the only contributing term of (68) is the $(k, m)$-th term. Here $k$ satisfies $2k - m = m + n - \kappa$. This term produces a residue at $m/2 - k$ which is a current on $\kappa$-forms. But according to (67), the same current is induced by the $(n - \kappa)$-component of the differential form $\det \left( \frac{\sinh(\nabla^2/2)}{(\nabla^2/2)} \right)^{\frac{1}{2}}$, so that:

$$\int_M \eta \det \left( \frac{\sinh(\nabla^2/2)}{(\nabla^2/2)} \right)^{\frac{1}{2}}_{n-\kappa} = \lim_{R \to 0} Res \left|_{m=m-n\over 2} \left( \Gamma(z) \int_{X_R} \pi^*(\eta) \tr_n(-\nabla^2_L)^{-z} \right) \right.,$$

which agrees with lemma 3.5 if $m = n$.

**Remark 8.1.** The condition $2k - m = m + n - \kappa$ implies that we only have nonzero currents on $\kappa$ forms if $\kappa$ is of the same parity as $n$. Thus, the location of the residue is an integer if $m$ is even and a half-integer if $m$ is odd.

**Warning:** In [10], the relevant computation is in section 4, where $F$ is denoted by $E$, $\nabla^2$ is denoted by $\Omega$ and the fiberwise coordinates $\xi^j$ are denoted by $x^j$. 

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RESIDUE FORMULATION

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