ON CONTRACTIBLE EDGES IN CONVEX DECOMPOSITIONS

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Abstract. Let \( \Pi \) be a convex decomposition of a set \( P \) of \( n \geq 3 \) points in general position in the plane. If \( \Pi \) consists of more than one polygon, then either \( \Pi \) contains a deletable edge or \( \Pi \) contains a contractible edge.

1. Introduction

Let \( P \) be a set of \( n \geq 3 \) points in general position in the plane. A convex decomposition of \( P \) is a set \( \Pi \) of convex polygons with vertices in \( P \) and pairwise disjoint interiors such that their union is the convex hull \( CH(P) \) of \( P \) and that no point in \( P \) lies in the interior of any polygon in \( \Pi \). A geometric graph with vertex set \( P \) is a graph \( G \), drawn in the plane in such a way that every edge is a straight line segment with ends in \( P \).

Let \( \Pi \) be a convex decomposition of \( P \). We denote by \( G(\Pi) \) the skeleton graph of \( \Pi \), that is the plane geometric graph with vertex set \( P \) in which the edges are the sides of all polygons in \( \Pi \). An edge \( e \) of \( \Pi \) is an interior edge if \( e \) is not an edge of the boundary of \( CH(P) \).

An interior edge \( e = uv \) of \( \Pi \) is deletable if the geometric graph \( G(\Pi) - e \), obtained from \( G(\Pi) \) by deleting the edge \( e \), is the skeleton graph of a convex decomposition of \( P \). Neumann-Lara et al. [6] proved that if a convex decomposition \( \Pi \) of a set \( P \) of \( n \) points consists of more that \( \frac{3n-2k}{2} \) polygons, where \( k \) is the number of vertices of \( CH(P) \), then \( \Pi \) has at least one deletable edge.

An interior edge \( e = uv \) of \( \Pi \) is contractible from \( u \) to \( v \) if the geometric graph \( G(\Pi)/\mathbf{uv} = (G(\Pi) - \{x_1u,x_2u,\ldots,x_mu,uv\}) + \{x_1v,x_2v,\ldots,x_mv\} \) is a skeleton graph of a convex decomposition of \( P \setminus \{u\} \), where \( x_1, x_2, \ldots, x_m \) are the remaining vertices of \( G(\Pi) \) which are adjacent to \( u \).

A simple convex deformation of \( \Pi \) is a convex decomposition \( \Pi' \) obtained from \( \Pi \) by moving a single point \( x \) along a straight line segment, together with all the edges incident with \( x \), in such a way that at each stage we have a convex decomposition of the corresponding set of points. Deformations of plane graphs have been studied by several authors, both theoretically and algorithmically, see for instance [3, 4, 7] and [1, 2, 5], respectively.

Let \( P_1 \) and \( P_2 \) be sets of \( n \geq 3 \) points in general position in the plane. A convex decomposition \( \Pi_1 \) of \( P_1 \) and a convex decomposition \( \Pi_2 \) of \( P_2 \) are isomorphic if there is an isomorphism of \( G(\Pi_1) \) onto \( G(\Pi_2) \), as abstract plane graphs, such that the boundaries of \( CH(P_1) \) and \( CH(P_2) \) correspond to each other with the same orientation.

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Partially supported by Conacyt, México.
Thomassen [7] proved that if $\Pi_1$ and $\Pi_2$ are isomorphic convex decompositions, then $\Pi_2$ can be obtained from $\Pi_1$ by a finite sequence of simple convex deformations. As a tool, Thomassen proved that if $\Pi$ is a convex decomposition with at least two polygons, then there is an isomorphic convex decomposition $\Pi'$ that can be obtained from $\Pi$ by a finite number of simple convex deformations that preserve the boundary and such that $\Pi'$ contains either a deletable edge or a contractible edge. In this note we prove that every convex decomposition $\Pi$ with at least two polygons contains an edge which is deletable or contractible. Furthermore, if $P$ contains at least one interior point, then $\Pi$ contains a contractible edge.

2. Preliminary results

Let $\Pi$ be a convex decomposition of $P$ containing no deletable edges. For every interior edge $e$ of $G(\Pi)$, the graph $G(\Pi) - e$ has an internal face $Q_e$ which is not convex and at least one end of $e$ is a reflex vertex of $Q_e$.

We define an abstract directed graph $\overrightarrow{G(\Pi)}$ with vertex set $P$ in which $\overrightarrow{uv} \in A(\overrightarrow{G(\Pi)})$ if and only if $u$ is a reflex vertex of $Q_{uv}$. Notice that for each interior edge $uv$ of $G(\Pi)$, the directed graph $\overrightarrow{G(\Pi)}$ contains at least one of the arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$ (see Fig. 1).

Remark 1.

1. The outdegree of every vertex $u$ of $\overrightarrow{G(\Pi)}$ is at most 3.
2. The outdegree of every vertex $u$ in the boundary of $CH(P)$ is 0.
3. An interior vertex $u$ of $\Pi$ has outdegree 3 in $\overrightarrow{G(\Pi)}$ if and only if $u$ has degree 3 in $G(\Pi)$.
4. If $\overrightarrow{uw}, \overrightarrow{vu} \in A(\overrightarrow{G(\Pi)})$, then $uv$ and $uw$ lie in a common face of $G(\Pi)$.

For two points $\alpha$ and $\beta$ in the plane, we denote by $r(\alpha\beta)$ the ray, with origin $\alpha$, that contains the segment $\alpha\beta$.

Lemma 2. An edge $uv$ of $\Pi$ is not contractible from $u$ to $v$ if and only if there are edges $yx$ and $xu$, lying in a common face of $G(\Pi)$ that contains vertex $u$, such that the ray $r(yx)$ meets the edge $uv$ at point $u_t$, with $u \neq u_t \neq v$, and that the triangular region defined by $x, u_t$ and $u$ contains no point of $P$ in its interior.

Proof. It is easy to see that the existence of such edges $yx$ and $xu$ implies that the edge $uv$ cannot be contracted from $u$ to $v$; we proceed to prove the sufficiency part of the
lemma. Let $uv$ be an interior edge of $\Pi$ with $u$ not lying in the boundary of $CH(\Pi)$ and let $x_1, x_2, \ldots, x_m$ be the remaining vertices of $G(\Pi)$ which are adjacent to $u$. We contract the edge $uv$ in a continuous way as follows: Slide the point $u$ along the ray $r(uv)$, together with the edges $x_1u, x_2u, \ldots, x_mu$ (see Fig. 2).

If $uv$ is not contractible from $u$ to $v$, then either the transformed graph $T(G(\Pi))$ becomes non planar or one of its faces becomes non convex. This implies that we must reach a point $u_t = u + t(v - u)$, with $0 < t < 1$, such that there are two edges $yx_i$ and $x_iu_t$ lying in a common face, which become collinear in $T(G(\Pi))$ (see Fig. 3).

Notice that two or more different pairs of edges $yx_i, x_iu_t$ and $yx_j, x_ju_t$ may become collinear simultaneously; in such a case we may choose any of those pairs and proceed with the proof.

The triangular region defined by $x_i, u_t$ and $u$ is the region swept by the edge $x_iu$, $0 \leq s \leq t$ and therefore it contains no point of $P$ in its interior. The lemma follows since the edges $yx_i$ and $x_iu$ lie in a common face of $G(\Pi)$ and the ray $r(yx_i)$ meets the edge $uv$ at the point $u_t$. □

Let $N$ denote the set of arcs $\vec{uv}$ of $G(\Pi)$ such that the edge $uv$ is not contractible from $u$ to $v$ in $\Pi$. For each $\vec{uv} \in N$ let $y = y_{uv}$, $x = x_{uv}$ and $u_t$ be as in Lemma 2. Since the edges $y_{uv}x_{uv}$ and $x_{uv}u$ lie in a common face of $G(\Pi)$ and the triangular region, defined by $x_{uv}, u_t$ and $u$, contains no point of $P$ in its interior, the geometric graph $G(\Pi) - x_{uv}u$ contains a face $Q_{x_{uv}u}$ in which $x_{uv}$ is a reflex vertex and therefore $\vec{x_{uv}u} \in A(G(\Pi))$.

This defines a function $f : N \rightarrow A(G(\Pi))$ given by $f(\vec{uv}) = \vec{x_{uv}u}$.

Notice that the arcs $f(\vec{uv})$ and $\vec{uv}$ form a directed path in $G(\Pi)$ with length 2 and middle vertex $u$. This implies that if $f(\vec{uv}_1) = f(\vec{uv}_2)$, then $u_1 = u_2$. Moreover, if $uv_1, uv_2$ and $uv_3$ are distinct arcs such that $f(\vec{uv}_1) = f(\vec{uv}_2) = f(\vec{uv}_3) = \vec{xu}$, then $u$ is adjacent in $G(\Pi)$ to $v_1, v_2, v_3$ and to $x$, which is not possible by Remark 1, since $u$ has outdegree 3 in $G(\Pi)$. It follows that there are no three arcs in $N$ with the same image under the function $f$ and therefore $|\text{Im}(f)| = |N| - |U|$, where $U$ is the set of points $u$ of $P$ for which there is a pair of arcs $\vec{uv}, \vec{uv} \in N$ such that $f(\vec{uv}) = f(\vec{uv})$. 

![Figure 2. Contracting an edge uv continuously.](image-url)
Figure 3. Edges $yx$ and $xu_1$ become collinear.

**Lemma 3.** Let $\Pi$ be a convex decomposition of $P$ with no deletable edges. If $U \neq \emptyset$, then there is a function $g : U \rightarrow A\left(\overrightarrow{G(\Pi)}\right)$ such that for each $u \in U$, $g(u)$ is not in the image of the function $f$.

**Proof.** Let $u \in U$ and let $v, w$ and $x = x_{uv} = x_{uw}$ be points in $P$ such that $f(\overrightarrow{uv}) = f(\overrightarrow{uw}) = \overrightarrow{xu}$. If $u$ has degree larger than 3 in $G(\Pi)$, let $z \notin \{v, w, x\}$ be such that $uz$ is an edge of $G(\Pi)$. By Remark 1, the outdegree of $u$ in $\overrightarrow{G(\Pi)}$ is at most 2, therefore $\overrightarrow{zu}$ is not an arc of $\overrightarrow{G(\Pi)}$. It follows that $\overrightarrow{xu}$ must be an arc of $\overrightarrow{G(\Pi)}$. In this case $g(u) = \overrightarrow{xu} \notin \text{Im}(f)$ since $z \neq x$ and $\overrightarrow{xu}$ is the unique arc in $\text{Im}(f)$ that ends at $u$.

If $u$ has degree 3 in $G(\Pi)$, then $u$ has outdegree 3 in $\overrightarrow{G(\Pi)}$, by Remark 1 and therefore $\overrightarrow{xu}$ is an arc $\overrightarrow{G(\Pi)}$. We claim that in this case $g(u) = \overrightarrow{xu} \notin \text{Im}(f)$. Let $l_{ux}$ denote the line containing the edge $ux$, and let $y_{uv}$ and $y_{uw}$ be points in $P$ such that the rays $r(y_{uv}x)$ and $r(y_{uw}x)$ intersect the edges $uv$ and $uw$, respectively.

Without loss of generality we assume that $l_{ux}$ is a vertical line such that $v$ and $y_{uw}$ lie to the left of $l_{ux}$ and $w$ and $y_{uv}$ lie to the right of $l_{ux}$ (see Fig. 4). Clearly the angles
\[ \angle y_u x u \text{ and } \angle y_w x u \text{ are smaller than } \pi, \text{ it is easy to see that } \angle y_u w x u \text{ is also smaller than } \pi. \]

Therefore if \( xz \) is an edge of \( \Pi \) with \( z \notin \{u, y_u, y_w\} \), then \( \overrightarrow{xz} \) is not an arc of \( \overrightarrow{G(\Pi)} \). This implies that if \( \overrightarrow{ux} \in \text{Im}(f) \), then \( \overrightarrow{ux} = f(\overrightarrow{xy_u}) \) or \( \overrightarrow{ux} = f(\overrightarrow{xy_w}) \) since \( f(\overrightarrow{d}) \) and \( \overrightarrow{d} \) form a directed path of length 2 for each arc \( \overrightarrow{d} \in N \).

Suppose \( \overrightarrow{ux} = f(\overrightarrow{xy_u}) \). By the definition of \( f \), there is an edge \( y_{xy_u} u \) such that the ray \( r(y_{xy_u} u) \) intersects the edge \( x y_{yu} \). Since \( v \) and \( w \) are the only vertices different from \( x \) which are adjacent to \( u \) in \( G(\Pi) \), one of them must be the vertex \( y_{xy_u} \). Since both edges \( uw \) and \( x y_{yu} \) lie in the right halfplane defined by \( l_{ux} \) then \( r(uw) \) cannot intersect the edge \( x y_{yu} \) and therefore \( y_{xy_u} \neq w \). Finally, since \( r(y_{yu} x) \) intersects the edge \( uv \), \( r(vu) \) cannot intersect the edge \( x y_{yu} \). Therefore \( \overrightarrow{ux} \neq f(\overrightarrow{xy_u}) \); analogously \( \overrightarrow{ux} \neq f(\overrightarrow{xy_w}) \). □

3. Main results

In this section we prove our main results.

**Theorem 4.** Let \( P \) be a set of points in general position in the plane. If \( \Pi \) is a convex decomposition of \( P \) consisting of more than one polygon, then either \( \Pi \) contains a deletable edge or \( \Pi \) contains a contractible edge.

**Proof.** Assume the result is false and \( \Pi \) contains no deletable edges and no contractible edges. Define the directed graph \( \overrightarrow{G(\Pi)} \) as in the previous section, notice that \( A(\overrightarrow{G(\Pi)}) \neq \emptyset \) since \( \Pi \) contains at least two polygons. Since \( \Pi \) contains no contractible edges, \( N = A(\overrightarrow{G(\Pi)}) \).

Let \( B = B(\overrightarrow{G(\Pi)}) \) be the set of arcs of \( \overrightarrow{G(\Pi)} \) of the form \( \overrightarrow{uw} \), with \( w \) in the boundary of \( CH(P) \), and let \( \overrightarrow{uw} \in B \). By Remark 1, \( w \) has outdegree 0 in \( \overrightarrow{G(\Pi)} \) which implies \( \overrightarrow{uw} \notin \text{Im}(f) \).
If $U = \emptyset$, then $\text{Im} \left( f \right) \subset A \left( G(\Pi') \right) \setminus B$, therefore $|N| = |\text{Im} \left( f \right)| \leq |A \left( G(\Pi) \right) \setminus B| \leq |A \left( G(\Pi) \right)| - 3$, which is not possible since $\Pi$ contains no deletable edges and $|B| \geq 3$.

And if $U \neq \emptyset$, by Lemma 3 no arc in $\text{Im} \left( g \right)$ lies in $\text{Im} \left( f \right)$, therefore $\text{Im} \left( f \right) \subset A \left( G(\Pi) \right) \setminus \left( \text{Im} \left( g \right) \cup B \right)$. In this case $|\text{Im} \left( f \right)| \leq |A \left( G(\Pi) \right)| - |\text{Im} \left( g \right)| - |B|$, since $g(u) \notin B$. This is a contradiction since $A \left( G(\Pi) \right) = N$, $|\text{Im} \left( g \right)| = |U|$, $|B| \geq 3$ and $|\text{Im} \left( f \right)| = |N| - |U|$. \hfill \Box

**Corollary 5.** Let $\Pi$ be a convex decomposition of a set of points $P$ in general position in the plane. If $P$ contains at least one interior point, then $\Pi$ contains at least one contractible edge.

**Proof.** Let $\Pi'$ be a convex decomposition of $P$ obtained from $\Pi$ by removing deletable edges, one at a time, until no such edges remain, and let $G(\Pi')$ be the corresponding directed abstract graph. Since $P$ contains an interior point, $\Pi'$ contains at least one interior edge.

By Theorem 4, there is an arc $\bar{uv} \in A \left( G(\Pi') \right)$ such that $uv$ is contractible from $u$ to $v$ in $\Pi'$. If $uv$ is not contractible in $\Pi$, then by Lemma 1 there are edges $yx$ and $xu$ lying in a common face of $G(\Pi)$ such that the ray $r(yx)$ meets the edge $uv$ at an interior point $u$, and that the triangular region $yu,u$ contains no point of $P$ in its interior. This implies that the geometric graph $G(\Pi) - xu$ contains a face $Q_x$ in which $x$ is a reflex vertex and therefore $xu$ is not deletable in $\Pi$ and $\overline{xu}$ is an arc of $G(\Pi)$.

Let $R$ be the face of $G(\Pi)$ which contains both edges $yx$ and $xu$. Since $\Pi'$ is obtained from $\Pi$ by deleting edges but no points, then there is a face $R'$ of $G(\Pi')$ which contains the edge $xu$ and the region bounded by $R$, let $y' \in P$ be such that $y'x$ is an edge of $R'$. Notice that $y' \neq y$ otherwise $uv$ could not be a contractible edge of $\Pi'$ because the ray $r(y'x)$ meets the edge $uv$ at the point $u$ (Fig. 8 left). Nevertheless, since the face $R'$ contains the edge $xu$ and the region bounded by $R$, the ray $r(y'x)$ also meets the edge $uv$ at an interior point $u'$ (Fig. 8 right) which again is a contradiction. \hfill \Box

**Corollary 6.** Let $\Pi$ be a convex decomposition of a set of points $P$ in general position in the plane and $Q$ be the set of points in the boundary of $CH(P)$. There is a sequence $P = P_0, P_1, \ldots, P_m = Q$ of subsets of $P$, and a sequence $\Pi_0, \Pi_1, \ldots, \Pi_m$ of convex decompositions of $P_0, P_1, \ldots, P_m$, respectively, such that $\Pi_0 = \Pi$, $\Pi_m$ consists of the boundary of $CH(P)$ and for $i = 0, 1, \ldots, k$, $\Pi_{i+1}$ is obtained from $\Pi_i$ by contracting an edge and for $i = k + 1, k + 2, \ldots, m - 1$, $\Pi_{i+1}$ is obtained from $\Pi_i$ by deleting an edge.

**Proof.** By Corollary 5, if $P_i$ contains interior points, then $\Pi_i$ has a contractible edge. If $P_i$ contains no interior points, then each interior edge of $\Pi_i$ is a deletable edge. \hfill \Box
Figure 5. Left: Ray $r(yx)$ meets edge $uv$ at the point $u_t$. Right: Ray $r(y'x)$ meets edge $uv$ at an interior point $u_t'$.

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