Running Vacuum from Dynamical Spacetime Cosmology

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The Running Vacuum Model (RVM) has been a candidate to solve the tension between the Hubble constant from the early and the late universe data fit. However the model does consider a Lagrangian formulation directly. In this paper we formulate an action principle that approaches the RMV from the second type, with a scalar field model for the whole dark components. The dynamical space time vector field \(\chi_i\) is used as a Lagrange multiplier that forces the kinetic term of the scalar field to evaluates as the modified dark matter component from the RMV model. When we replace the vector field to a derivative of a scalar, the model predicts diffusion interactions between the dark components and the cosmological solution have a different correspondence to the RMV. All of these new solutions yield new cosmological scenarios that should be studied in detail in the future.

**INTRODUCTION**

Almost twenty years after the observational evidence of cosmic acceleration, the cause of this phenomenon, labeled as dark energy remains an open question which challenges the foundations of theoretical physics: The cosmological constant problem - why there is a large disagreement between the vacuum expectation value of the energy momentum tensor which comes from quantum field theory and the observable value of dark energy density [1, 2]. The simplest model of dark energy and dark matter is the ΛCDM that contains non-relativistic dark matter and cosmological constant.

Interaction between dark matter and dark energy was considered in many cases, such as [3]. Unification between dark energy and dark matter from an action principle were obtained from scalar fields [4–7] including Galileon cosmology [8] or Telleparallel modified theories of gravity [9–12]. A diffusive interaction between dark energy and dark matter was introduced in [13–16]. Interacting scenarios prove to be efficient in alleviating the two known tension of modern cosmology, namely the \(H_0\) [17–28]. Despite the extended investigation of interacting scenarios the choice of the interaction function remains unknown.

The Running Vacuum Model [29–37] is a good modified model for the cosmological background particularly because they can resolve the tensions. The main point of the RMV comes from Quantum Field Theory in curved spacetime, but here we formulate an action principle that approach the RMV at late times.

**THE RUNNING VACUUM MODEL**

For a homogeneous expanding universe, the RVM expects that the vacuum energy density and the gravitational coupling are functions of the cosmic time through the Hubble rate, assuming the canonical equation of state \(p_\Lambda = -\rho_\Lambda(H)\) for the vacuum energy density. The corresponding Friedamark matter equations (with the presence of radiation \(\rho_R\) and cold dark matter density \(\rho_m\)) read:

\[
3H^2 = 8\pi G(H) (\rho_m + \rho_r + \rho_\Lambda(H)),
\]

\[
3H^2 + 2\dot{H} = -8\pi G(H) (\rho_r - \rho_\Lambda(H)).
\]

The RVM structure for the dynamical vacuum energy assumes the expansion:

\[
\rho_\Lambda(H;\nu,\alpha) = \frac{3}{8\pi G} \left( \epsilon_0 + \nu H^2 + \frac{2}{3} \alpha H \right) + ..., \]

based on quantum corrections of QFT in curved spacetime [38]. The coefficients \(\nu\) and \(\alpha\) are dimensionless. For \(\nu = \alpha = 0\) we recover the cosmological constant.

The RVM suggests two types of models. Here we compare the DST cosmology with the second type of RVM, that assumes \(G =\)const. The main reason for that is the theories we introduce written as a background to Einstein-Hilbert action (Einstein frame), and therefore begin also from \(G =\)const. For alternative theories that couples the Einstein term into some scalar fields give a running Newtonian constant.

The conservation of the total energy momentum tensor gives the extended Friedmann matter equation:

\[
\rho = \Omega_\Lambda^{(0)} + \Omega_m^{(0)} a^{-3\xi} + \Omega_r^{(0)} a^{-4\xi'}. \]

where \(\Omega_i = \rho_i/\rho_0\) are the current cosmological parameters for matter and radiation. The new coupling constant read:

\[
\xi = \frac{1-\nu}{1-\alpha} \equiv 1 - \nu_{eff}, \quad \xi' = \frac{1-\nu}{1-\frac{4}{3}\alpha} \equiv 1 - \nu_{eff}'.
\]

The standard expressions for matter and radiation energy densities are recovered for \(\xi, \xi' \to 1\). The lack of an action principle for the RMV may be solved with DST formulation.
DYNAMICAL SPACE TIME THEORY

The conservation of energy can be derived from the time translation invariance principle. However using a Lagrange multiplier can derive the local conservation of an energy momentum tensor $\mathcal{T}^{\mu\nu}$. Let’s consider a 4 dimensional case where a conservation of a symmetric energy momentum tensor $\mathcal{T}^{\mu\nu}$ is imposed by introducing the term in the action \[39-41\]:

$$S(\chi) = \int d^4x \sqrt{-g} \chi_{\mu\nu} T^{\mu\nu} \tag{5}$$

where $\chi_{\mu\nu} = \partial_\nu \chi_\mu - \Gamma^\lambda_{\mu\nu} \chi_\lambda$. The vector field $\chi_\mu$ is a dynamical space time vector, because of the energy density of $T^{\mu\nu}$ is a canonically conjugated variable to $\chi_0$. In the metric formalism the variation with respect to $\chi_\mu$ gives a covariant conservation law:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{6}$$

From the variation of the action with respect to the metric, we get a conserved stress energy tensor $G^{\mu\nu}$ (in appropriate units) which is well known from Einstein equation:

$$G^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} [\mathcal{L}_\chi + \mathcal{L}_m], \quad \nabla_\mu G^{\mu\nu} = 0 \tag{7}$$

where $G^{\mu\nu}$ is Einstein tensor, $\mathcal{L}_\chi$ is the Lagrangian in (5) and $\mathcal{L}_m$ is an optional action that involve other contributions.

A particular case of the stress energy tensor with the form $\mathcal{T}^{\mu\nu} = L_1 g^{\mu\nu}$ corresponds to a modified measure theory. By substituting this stress energy tensor into the action itself, the determinant of the metric is cancelled:

$$\sqrt{-g} \phi_\mu L_1 = \partial_\mu (\sqrt{-g} \chi_\mu) L_1 = \Phi L_1 \tag{8}$$

where $\Phi = \partial_\mu (\sqrt{-g} \chi_\mu)$ is like a ”modified measure”. A variation with respect to the dynamical time vector field will give a constraint on $L_1$ to be a constant:

$$\partial_\alpha L_1 = 0 \quad \Rightarrow \quad L_1 = Const \tag{9}$$

This situation corresponds to the "Non-Canonical Volume-Forms" \[42, 43\] where in addition to the regular measure of integration in the action $\sqrt{-g}$ includes another measure of integration which is also a density and a total derivative.

RUNNING VACUUM WITH DYNAMICAL TIME

In order to obtain the Friedmann equations as the RVM predicts from the DST cosmology, the stress energy momentum tensor $\mathcal{T}^{\mu\nu}$ is chosen to be:

$$\mathcal{T}^{\mu\nu} = \frac{\lambda_1}{2} \phi^\alpha \phi^\nu - \frac{\lambda_2}{2} g^{\mu\nu} (\phi, \phi^\alpha) \tag{10}$$

$\lambda_1$ and $\lambda_2$ are arbitrary constants. The density and pressure resulting from $\mathcal{T}^{\mu\nu}$ are:

$$\rho(\chi) = \frac{\lambda_1}{2} \dot{\phi}^2, \quad p(\chi) = -\frac{\lambda_2}{2} \dot{\phi}^2 \tag{11}$$

where $\lambda := \lambda_1 + \lambda_2$. Together with action (5) we obtain the DST cosmological solution:

$$\mathcal{L} = \frac{1}{2} \dot{R} + \chi_{\mu\nu} T^{\mu\nu} - \frac{1}{2} g^{\alpha\beta} \phi_\alpha \phi_\beta - V(\phi) \tag{12}$$

The action depends on three different variables: the scalar field $\phi = \phi(t)$, the dynamical space time vector $\chi_\mu = (\chi_0(t), 0, 0, 0)$ and the metric:

$$ds^2 = -\mathcal{N}(t) dt^2 + a^2(dx^2 + dy^2 + dz^2), \tag{13}$$

where $a$ is the scale factor and the $\mathcal{N}(t)$ is the Lapse function, which in the equations of motion is gauged to be $\mathcal{N}(t) = 1$. In the Mini-Super-Space the action (5) reads:

$$\mathcal{L}_{M.S.S} = \frac{3a^3}{\mathcal{N}} \dot{\phi}^2 + \frac{3a^2}{\mathcal{N}^2} \dot{\chi}_0^2 + \frac{3a}{\mathcal{N}^3} \dot{\chi}_0 \dot{\phi} + \frac{3a^2}{\mathcal{N}^4} \dot{\phi}^2 + \frac{\lambda_1 a^3}{2N^4} + \frac{\lambda_2 a^3}{4N^4} \mathcal{N} \dot{\phi}^2 \tag{14}$$

The variation with respect the Dynamical Time Vector field $\chi_0$ yields:

$$\frac{3}{2} \lambda_1 H \dot{\phi} + \lambda \dot{\phi} = 0 \tag{15}$$

which is integrated to give:

$$\dot{\phi} = C_1 \dot{a}^{-3\lambda_1 / 2 \lambda}, \tag{16}$$

with an integration constant $C_1$. The second variation with respect to the scalar field $\phi$ gives:

$$2\lambda \left[3(\lambda - \lambda_1) \chi_0 \dot{H} + \lambda_1 \dot{\chi}_0\right] + \frac{9H^2}{2} \chi_0^3 + H \left(3(3\lambda - \lambda_1) \chi_0^2 - 3(\lambda + 1)\right) = 0 \tag{17}$$

The last variation, with respect to the metric, gives the stress energy tensor. The energy density and the pressure of the scalar field are:

$$\rho = \frac{1}{2} \phi^2 (H(9\lambda_1 - 6\lambda) \chi_0^2 - 2\lambda \chi_0 + 1) + \lambda \chi_0 \dot{\phi} + V(\phi) \tag{18a}$$

$$p = (\lambda - \lambda_1) \chi_0 \phi^2 + \frac{1}{2} \phi^2 (\lambda_1 \chi_0 - 1) - V(\phi) \tag{18b}$$

with the Friedmann equations:

$$\rho = 3H^2, \quad p = -3H^2 - 2\dot{H}. \tag{19}$$
In order to tack the evolution of the solution, we use the asymptotic solution: with a power law and exponential expansion.

We assume power law solution for the scale factor for large times \( a \sim t^\alpha \) with an asymptotically constant potential \( V = \text{const.} \). Using power law scale factor in Eq. (17), we get the solution for \( \chi_0 \) as:

\[
\chi_0(t) = \frac{t}{\lambda + 3\alpha(\lambda - \lambda_1)} + B_1 t^{\frac{3\alpha(\lambda_1+\lambda)}{2\lambda}} + B_2 t^{\frac{3\alpha(\lambda-\lambda_1)}{\lambda}}
\]

where \( B_1 \) and \( B_2 \) are integration constants. For large time, considering \( 2\lambda < 3\alpha(\lambda_1 + \lambda) \), the second and third terms become sub dominating, hence can be neglected. Therefore, the solution for \( \chi_0 \) simplifies to

\[
\chi_0(t) = \frac{t}{\lambda + 3\alpha(\lambda - \lambda_1)}
\]

Substituting the solutions for the derivative of \( \phi \) (16) and the solution of \( \chi_0 \) from Eq. (21) into the density equation (18a) giving:

\[
\rho = C_2 \frac{(\lambda + 3\alpha(\lambda_1 + 3\lambda))}{2(3\alpha(\lambda_1 - \lambda) + \lambda)} a^{-3\lambda_1/\lambda} + V.
\]

We can also obtain the same asymptotic behavior if we consider exponential scale factor given by \( a \sim e^{H_0 t} \). Similarly, solution for \( \chi_0 \) is given by

\[
\chi_0(t) = \frac{1}{3H_0(\lambda - \lambda_1)} + C_1 e^{-\frac{3H_0(\lambda_1+\lambda)}{2\lambda}} + C_2 e^{-\frac{3H_0(\lambda-\lambda_1)}{\lambda}}
\]

Once again for large times, one can neglect the last two terms, hence

\[
\chi_0(t) = \frac{1}{3H_0(\lambda - \lambda_1)}
\]

Substituting the above solutions into the density equation (18a), we get the expression:

\[
\rho = C_1 \frac{\lambda_1 + 3\lambda}{2(\lambda_1 - \lambda)} a^{-3\lambda_1/\lambda} + V
\]

which is similar to the power law expansion, but with different coupling constants.

Notice that for the case \( \lambda = \lambda_1 \) the solution should be different, but solved analytically and numerically in Ref. [40, 44]. The RVM energy density (from the second type) corresponds to the asymptotic solution of the DST cosmology for:

\[
\xi = \lambda_1/\lambda, \quad \xi' = 1
\]

for modifying the matter part and leave the radiation as an external field. However to obtain both densities from an action principle, we use the Diffusive action.

In order to assess the viability of the model, we try to obtain complete solution for this model and see how some physical quantities change versus the red-shift (z). As we know, the relation between the cosmic time derivative and the red-shift derivative reads as:

\[
\frac{d}{dt} = -(1 + z) H(z) \frac{d}{dz}
\]

Using the above relation in Eqs. (18a),(17),(15), we solve them numerically and obtain the evolution of cosmological parameters with redshift. In Figure 1 we see \( \Omega_\Lambda \) and \( \Omega_m \) for certain choice of the parameters.

### DIFFUSIVE EXTENSION

In order to break the conservation of \( T^{\mu\nu} \) as in the diffusion equation, the vector field \( \chi_\mu \) should be coupled in a mass like term in the action:

\[
S(\chi,A) = \int d^4x \sqrt{-g} \chi_\mu T^{\mu\nu} + \frac{\sigma}{2} \int d^4x \sqrt{-g} (\chi_\mu + \partial_\mu A)^2
\]

where \( A \) is a scalar field different from \( \phi \). From a variation with respect to the dynamical space time vector field \( \chi_\mu \) we obtain:

\[
\nabla_\nu T^{\mu\nu} = \sigma (\chi^\mu + \partial^\mu A) = f^\mu,
\]
In addition for the same theoretical reason we assume that $V(\phi) = \textbf{Const}$. Then variation with respect to the scalar field $\phi$ yields:

$$\frac{\lambda_1}{2} - \lambda_2)(\dot{\chi} + (1 - 3H\dot{\chi})\lambda_2 = \frac{\sigma_2}{\phi\alpha^3}$$

where $\sigma_2$ is another integration constant. Now from the stress energy momentum tensor the total energy density term is:

$$\rho = \frac{3}{2}H(\lambda_1 - 2\lambda_2)(\dot{\chi})^2 + \frac{1}{2}\dot{\phi}^2(1 - 2(\lambda_1 + \lambda_2)\dot{\chi}) + \chi\phi((\lambda_1 + \lambda_2)\ddot{\phi}) + V,$$

and the total pressure is:

$$p = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\lambda_1\dot{\chi}\dot{\phi}^2 + \lambda_2\chi\ddot{\phi} - V.$$ 

We aren’t able to find the exact solutions for the Einstein equation together with the equations for the scalar fields. So we are looking for asymptotic solutions.

We assume a power law solution for a large time $a \sim t^\alpha$. Then from Eq. (35) the solution for the scalar field $\phi$ derivative is:

$$\dot{\phi} = \sqrt{\frac{2\sigma_1}{3\alpha(\lambda_1 - \lambda_2) + \lambda_1 + \lambda_2}} t^{\frac{\lambda_1 - \lambda_2}{2\alpha}}$$

In this section we consider the following action:

$$\mathcal{L} = \frac{1}{2}\mathcal{R} + \chi_{,\mu}\chi^{,\nu} - \frac{1}{2}\phi^{,\mu}\phi_{,\mu} - V(\phi)$$

which contains a scalar field with potential $V(\phi)$.

There are three independent sets of equations of motions: $\chi$, $\phi$ and the metric $g_{\mu\nu}$.

In the Mini-Super-Space the action (33) reads:

$$\mathcal{L}_{M.S.S} = \frac{3a^2 \dot{a}}{N^2} - \frac{3a^2 \dot{\chi}^2}{2N^2} - \frac{3a^2 \dot{\phi}^2}{2N^2} + \frac{3aa^2}{N^2}$$

The variation with respect to the scalar $\chi$ gives:

$$(\lambda_1 + \lambda_2)\phi\ddot{\chi} + 3H\chi\phi' = \frac{\sigma_1}{\alpha^3},$$

where $\sigma_1$ is an integration constant. Then the solution for Eq. (36) is:

$$\phi^2 = \phi_0^2 a^{\frac{3\lambda_1}{\lambda_1 + \lambda_2}} + \frac{\sigma_1}{\lambda_1 + \lambda_2} a^{\frac{3\lambda_1}{\lambda_1 + \lambda_2}} \int_0^t ds a^{-\frac{3\lambda_1}{\lambda_1 + \lambda_2}}$$

where the current source reads: $f^\mu = \sigma(\chi^\mu + \partial^\mu A)$. From the variation with respect to the new scalar $A$ a covariant conservation of the current indeed emerges:

$$\nabla_\mu f^\mu = \sigma \nabla_\mu (\chi^\mu + \partial^\mu A) = 0$$

A particular case of diffusive energy theories is obtained when $\sigma \rightarrow \infty$. In this case, the contribution of the current $f_\mu$ in the equations of motion goes to zero and yields a constraint for the vector field being a gradient of the scalar:

$$f_\mu = \sigma(\chi_\mu + \partial_\mu A) = 0 \Rightarrow \chi_\mu = -\partial_\mu A$$

For the rest of the paper we use the notation $\chi$ for the scalar field which is coupled to the stress energy momentum tensor and not $A$ due to earlier publications. The theory (28) is reduced to a theory with higher derivatives:

$$S = -\int d^4x \sqrt{-g} \chi_{,\mu ;\nu} T^{\mu\nu}$$

The variation with respect to the scalar $A$ gives $\nabla_\mu \nabla_\nu T^{\mu\nu} = 0$ which corresponds to the variations (29) – (30). In the following paper we use the reduced theory with higher derivative in the action.

**SCALAR FIELD GRAVITY WITH DIFFUSIVE BEHAVIOR**

FIG. 2: Upper graph: Evolution of dark energy density parameter $\Omega_\Lambda$ with redshift for different choices of the parameters. Lower graph: Evolution of dark matter density parameter $\Omega_m$ with redshift for different choices of the parameters.
The solution for the scalar field $\chi$ is:

$$\dot{\chi} = \frac{2\lambda_2}{-6\alpha\lambda_2 + \lambda_1 - 2\lambda_2} t.$$  \hspace{0.5cm} (41)

By inserting the solutions (40) and (41) into Einstein equation we obtain:

$$\rho = \frac{\alpha_1}{a^3} + \frac{\alpha_2 t}{a^3} + V$$  \hspace{0.5cm} (42)

where the constants are:

$$\alpha_1 = \frac{18\alpha^2\lambda_2(2\lambda_2 - \lambda_1)}{2(\lambda_1 - 2\lambda_2(3\alpha + 1))} \hspace{0.5cm} (43)$$

$$\alpha_2 = \frac{(6\alpha + 2)\lambda_1\lambda_2 + 2(3\alpha + 1)(\lambda_2 - 1)\lambda_2 + \lambda_1}{2(\lambda_1 - 2\lambda_2(3\alpha + 1))} \hspace{0.5cm} (44)$$

For exponential solution, the asymptotic limit reads different. For exponential solution $a \sim e^{\delta t}$ in Eq. (36). Then we get:

$$\dot{\phi}^2 = \dot{\phi}_0^2 a^{-\frac{\nu_1}{1 + \alpha_1}} - \sigma_1 H_0 \frac{\lambda_1 + \lambda_2}{a^3} \hspace{0.5cm} (45)$$

if we impose $\frac{\nu_1}{1 + \alpha_1} > 0$. Then from Eq. (37) we get that the density is given by:

$$\rho = H_0(3\lambda_2 - 1)\sigma_1 \frac{\lambda_1 + \lambda_2}{6\lambda_2} \frac{1}{a^3} + V$$

$$+ \frac{1}{2} \dot{\phi}_0^2 (1 - 2\lambda_2) a^{-\frac{3\lambda_1}{1 + \alpha_1}}$$  \hspace{0.5cm} (46)

With this solution, the corresponding energy density for the RVM

$$\xi = 1, \hspace{0.3cm} \xi' = 3\lambda_1/(4\lambda)$$  \hspace{0.5cm} (47)

In this case, we modify the radiation part and obtain the matter field from one action. In both cases we can modify one part of the Friedmann matter equation. However to obtain the full RVM energy density from those theories is not possible, but still implies for the direction how to use Lagrange multipliers as the DST cosmology to obtain the successful RVM.

Like before, we obtain complete solution for this model and see how $\Omega_m$ and $\Omega_{\Lambda}$ change with the red-shift $z$. Using Eq. (27) in Eqs. (38), (37),(35), we solve them numerically and obtain the evolution of cosmological parameters with redshift. In Figure 2 we see $\Omega_{\Lambda}$ and $\Omega_m$ for certain choice of the parameters.

**DISCUSSION**

We know though CDM could be the simplest phenomenological explanation for the observed acceleration of the Universe, there still exist a disagreement between the predicted and observed value of lambda. In particular, we are still facing the crucial question whether $\Lambda$ is truly a fundamental constant or a mildly evolving dynamical variable.

It turns out that the $\Lambda = \text{const}$, despite being the simplest, may well not be the most favored one when compared with specific dynamical models of the vacuum energy. It also is unable to solve the tension related to the Hubble constant. Recently it has been shown the RVM are good modified model candidate to solve the Hubble tension. However, the model considers a Lagrangian formulation directly.

In this paper we obtain a candidate for the Running Vacuum Model formulated by an action principle that approaches the RMV from the second type asymptotically, without the requirement of dark components. We study this in dynamical space time vector model and also its diffusive extension. The scalar field model takes care of the behavior of the dark components. The kinetic term mimics the behavior of the dark matter and the potential terms acts like dark energy. Through the asymptotically analysis, we have found that the DST and its diffusive counterpart have a different correspondence to the RMV.

We have obtained analytical solutions for large times and have shown that the dynamical time does not always behave like cosmic time. In fact for large times, the dynamical time approaches a finite value which depends on the constants $\lambda_1$ and $\lambda_2$. We compare our resulting solutions with those of RVMs and obtain the correspondence. Thus we show how to use the Lagrange multipliers as the DST cosmology to obtain the successful RVM. We also obtain numerical solutions for the models and show the evolution of the cosmological parameters for different choices of the constants $\lambda_1$ and $\lambda_2$.

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