On the equation \( n_1 n_2 = n_3 n_4 \) restricted to factor closed sets

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Abstract. We study the number of solutions \( N(B, F) \) of the diophantine equation \( n_1 n_2 = n_3 n_4 \), where \( 1 \leq n_i \leq B, 1 \leq i \leq 4 \) and \( n_1, n_2, n_3, n_4 \in F \). We assume that \( F \) is factor closed, in short \( F \) is an FC set. By this we mean that \( d[k] \neq 0 \) for all \( k \in F \); in particular \( 1 \in F \). By definition, this notion extends to sets \( F \) formed with not necessarily distinct elements, for instance the set \( M_k(B), B \geq 1 \), of all possible products \( m_1 \ldots m_k \) obtained by taking \( 1 \leq m_i \leq B \), \( 1 \leq i \leq k \). We refer to Haukkanen, Wang and Sillanpää \( \cite{2} \) (see also references therein) concerning this notion and extensions, also Weber \( \cite{9} \). Typical examples of FC sets are naturally intervals \([1, B]\), the set of divisors of an integer, the set of squarefree integers less than \( B \), the multiplicative semi-group generated by a given set of integers, the trace \( F \cap [1, B] \) of an FC set \( F \).

Let \( N(B, F) \) denote the number of integers solutions of the restricted equation
\[
(1.2) \quad n_1 n_2 = n_3 n_4,
\]
where the unknowns satisfy \( 1 \leq n_1 \leq B, 1 \leq n_3 \leq B, n_2, n_4 \in F \).

It is of interest to observe that the initial equation is just a particular case of equation \((1.2)\). This can be generalized. Let \( k \geq 1 \) and let \( N_k(B) \) denote the number of solutions of the equation
\[
(1.3) \quad n_1 \ldots n_{k+1} = n_{k+2} \ldots n_{2(k+1)},
\]
with unknowns verifying \( 1 \leq n_i \leq B, 1 \leq i \leq 2(k+1) \). One sees that solving equation \((1.2)\) amounts to solving equation \((1.3)\) with \( F = M_k(B) \).

We also consider equation \((1.2)\) with the constraint that all unknowns must belong to \( F \), and denote by \( N(F, F) \) the corresponding number of solutions.

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We use the approach developed in [7] to establish the following results. Given two numbers \( n \) and \( m \), let \( n \lor m = \max(n, m) \).

**Proposition 1.1.** For any FC set \( F \) and any integer \( B \geq 1 \), we have

\[
N(B, F) = \sum_{n, m \in F} \left( \sum_{\gcd(n, m) = 1} 1 \right) \left\lfloor \frac{B}{n \lor m} \right\rfloor.
\]

If \( F \subset [1, B] \), then

\[
N(B, F) \leq 2B^2 \sum_{n \in F} \frac{1}{n} + B^2,
\]

\[
N(B, F) \geq \sum_{\mu, \nu \in F} \gcd(\mu, \nu),
\]

\[
N(F, F) = \sum_{n, m \in F} \left( \sum_{n, m \in F} \frac{1}{n^2} \right)^2.
\]

**Remark 1.2.** For the case \( F = [1, B] \), we recover that \( N(B) \leq CB^2 \log B \). We also recall that ([8, Exercise 59])

\[
(1.4) \sum_{1 \leq \gcd(\mu, \nu) \leq B} \gcd(\mu, \nu) = \frac{6\pi}{\pi^2} B^2 \log B + \mathcal{O}(B^2).
\]

We deduce

**Theorem 1.3.** Suppose that \( F = \{m_1m_2 : 1 \leq m_1, m_2 \leq [B^\alpha]\} \) with \( \alpha \in [0, 1/2] \). We have

\[
N(B, F) \leq \left( 1 + \frac{12}{\pi^2} + 2 \sum_{m \in F} \frac{1}{m} \right) B^{1+2\alpha}.
\]

Let us consider the following typical example of a FC set. Let

\[
(1.5) \quad F = \left\{m = p_{\varepsilon_1}^{\varepsilon_1} \ldots p_{\varepsilon_k}^{\varepsilon_k} : \varepsilon_j \in \{0, 1\}, 1 \leq j \leq k\right\},
\]

where \( p_1 < \ldots < p_k \) are prime numbers. Recall that \( \omega(n) = \#\{p \text{ prime} : p | n\} \) is the prime divisor function, and that \( \omega(1) = 0 \).

**Theorem 1.4.** For any positive integer \( B \) and FC set of the type (1.5),

\[
N(B, F) \leq C B^{2k} \prod_{i=1}^{k} \left( 1 + \frac{1}{4p_i} \right) \min \left( B, \left( \frac{5}{4} \right)^k \right)
\]

\[
+ C 2^k \prod_{i=1}^{k} \left( 1 + \frac{1}{3p_i} \right) \min \left( B \left( \frac{3}{2} \right)^k, 2^k \right).
\]

Here \( C \) is a universal constant.

Also,

\[
N(B, F) \geq 2^k B + 2^k B \sum_{n \in F} \frac{2^{\omega(n)}}{n}.
\]

Further, if \( F \subset [1, B] \),

\[
N(B, F) \geq 2^k B + 2^k B \left( \prod_{j=1}^{k} \left( 1 + \frac{1}{2p_j} \right) \right).
\]
In the course of the proof, we show a better but less explicit result (see (1.13)), from which it follows that when the $p_i$ are all large, then

$$(1.6) \quad N(B, F) \sim B^k \sum_{n,m \in F \cap [1,B]} \frac{1}{2^{\omega(m)2^{\omega(n)}}(n \lor m)}.$$  

See Remark 4.3.

The paper is organized as follows. In the next section, some preparatory lemmas are established. In Sections 3 and 4 we prove Theorems 1.3 and 1.4 respectively. In Section 5.2, we conclude with a remark concerning equation (1.3), and give an elementary proof of an almost optimal upper bound of $N_k(B)$. We also suggest a possible extension of Theorem 1.4.

### 2. Proof of Proposition 1.1

Equation (1.2) means that $n_1 n = n_3 m$, where $n_2 = dn$, $n_4 = dm$, $d = \gcd(n_2, n_4)$ and $\gcd(n,m) = 1$. Since $F$ is factor closed, we necessarily have that $n, m \in F$.

Now given $n, m \in F$ fixed such that $\gcd(n, m) = 1$, the number of integers $n_2, n_4 \in F$ such that $n_2 = dn$, $n_4 = dm$ for some $d \geq 1$, is obviously equal to

$$\sum_{\frac{n_2,n_4 \in F}{\gcd(n_2,n_4) = 1}} 1.$$  

Further, the solutions to the equation $n_1 n = n_3 m$ in the unknowns $1 \leq n_1 \leq B$, $1 \leq n_3 \leq B$ are trivially $n_1 = \lambda m$, $n_3 = \lambda n$, with $1 \leq \lambda \leq \lfloor \frac{B}{n \lor m} \rfloor$ if $n \lor m \leq B$, and there is no solution otherwise.

Hence the number of solutions in the unknowns $n_1, n_3 \in [1, B]$, $n_2, n_4 \in F$ verifying $\frac{n_2}{n_3} = \frac{n_4}{n_2} = \frac{m}{m}$, is

$$(2.1) \quad \left( \sum_{\frac{n_2,n_4 \in F}{\gcd(n_2,n_4) = 1}} 1 \right) \left\lfloor \frac{B}{n \lor m} \right\rfloor.$$  

Note that when $F = [1,B]$, this simplifies and one gets $\left\lfloor \frac{B}{n \lor m} \right\rfloor^2$, which for $n = m = 1$ reduces to $B^2$. Also (see [7], (4)),

$$N(B) = \sum_{1 \leq n,m \leq B} \left\lfloor \frac{B}{n \lor m} \right\rfloor^2 = B^2 + 2 \sum_{1 \leq n,m \leq B} \left\lfloor \frac{B}{n \lor m} \right\rfloor.$$  

In our case we get

$$(2.2) \quad N(B, F) = \sum_{\frac{n,m \in F}{\gcd(n,m) = 1}} \left( \sum_{\frac{n_2,n_4 \in F}{\gcd(n_2,n_4) = 1}} 1 \right) \left\lfloor \frac{B}{n \lor m} \right\rfloor.$$  

If $F \subset [1,B]$, we have the obvious bound

$$\sum_{\frac{n_2,n_4 \in F}{\gcd(n_2,n_4) = 1}} 1 \leq \left\lfloor \frac{B}{n \lor m} \right\rfloor,$$  

and so,

$$N(B, F) \leq \sum_{\frac{n,m \in F}{\gcd(n,m) = 1}} \left\lfloor \frac{B}{n \lor m} \right\rfloor^2 = B^2 + 2 \sum_{\frac{n,m \in F}{n < m}} \left\lfloor \frac{B}{m} \right\rfloor^2.$$  

On the equation $n_1 n_2 = n_3 n_4$ restricted to FC sets
Plainly,
\[
\sum_{\frac{n \cdot m \in F}{\gcd(n, m) = 1}} \left| \frac{B}{m} \right|^2 \leq B^2 \sum_{m \in F} \frac{1}{m^2} \sum_{\frac{n \in F}{\gcd(n, m) = 1}} 1 \leq B^2 \sum_{m \in F} \frac{\phi(m)}{m^2} \leq B^2 \sum_{m \in F} \frac{1}{m},
\]

since \(\phi(m) \leq m\). Hence the claimed bound.

Further,
\[
(2.3) \quad N(F, F) = \sum_{\frac{n \cdot m \in F}{\gcd(n, m) = 1}} \left( \sum_{\frac{n \cdot m \in F}{\gcd(n, m) = 1}} 1 \right)^2.
\]

Next we prove the lower bound for \(N(B, F)\) in Proposition 11. Let \((\nu, \mu) \in F^2\) and write \(\mu = \gcd(\nu, \mu) m, \nu = \gcd(\nu, \mu) n\) with \(\gcd(n, m) = 1\). Associate to \((\nu, \mu)\) the set \(c(\nu, \mu) = \{(nm), \nu, (an), \mu) \leq d \leq \gcd(\nu, \mu)\}\). These quadruples provide \(\gcd(\nu, \mu)\) solutions of the restricted equation (1.2), since obviously \(dn\nu = dm\mu\) and \(d \max(m, n) \leq \max(\nu, \mu) \leq B\).

Naturally if \((\mu', \nu') \in F^2\) and \((\mu', \nu') \neq (\mu, \nu)\), then \(c(\nu', \mu') \cap c(\nu, \mu) = \emptyset\). Thus,
\[
N(B, F) \geq \sum_{\frac{n \cdot m \in F}{\gcd(n, m) = 1}} \gcd(\mu, \nu).
\]

Remark 2.1. Let \(F\) as in (1.3). Then,
\[
(2.4) \quad \sum_{\frac{n \cdot m \in F}{\gcd(n, m) = 1}} \gcd(\mu, \nu) = 3^k \prod_{i=1}^{k} \left(1 + \frac{1}{3} p_i\right).
\]

Further,
\[
(2.5) \quad N(F, F) = 6^k.
\]

Indeed, let \(a \in F, n, m \in F\) with \(\gcd(n, m) = a\). Thus \(m = m_1 a, n = n_1 a\) with \(\gcd(n_1, m_1) = 1\). Because of the choice of \(F\), we moreover have that \(\gcd(n_1, a) = 1 = \gcd(m_1, a)\). Thus
\[
\sum_{\frac{n \cdot m \in F}{\gcd(n, m) = 1}} \gcd(\mu, \nu) = \sum_{a \in F} a \sum_{\frac{a \cdot m_1 \in F}{\gcd(a, m_1) = 1}} \sum_{\frac{a \cdot n_1 \in F}{\gcd(a, n_1) = 1}} 1 = \sum_{a \in F} a \sum_{\frac{a \cdot m_1 \in F}{\gcd(a, m_1) = 1}} 2^{k-\omega(a)-\omega(m_1)}
\]
\[
= 2^k \sum_{a \in F} a 2^{\omega(a)} \sum_{j=0}^{k-\omega(a)} C_{k-\omega(a)}^j 3^{-j} = 2^k \sum_{a \in F} a 2^{\omega(a)} \left(\frac{3}{2}\right)^{k-\omega(a)}
\]
\[
= 3^k \sum_{a \in F} a \left(\frac{1}{3}\right)^{\omega(a)} = 3^k \prod_{i=1}^{k} \left(1 + \frac{1}{3} p_i\right).
\]

Further
\[
N(F, F) = \sum_{\frac{n \cdot m \in F}{\gcd(n, m) = 1}} \left( \sum_{\frac{n \cdot m \in F}{\gcd(n, m) = 1}} 1 \right)^2
\]
\[
= \sum_{\frac{n \cdot m \in F}{\gcd(n, m) = 1}} 4^{k-\omega(m)-\omega(n)} = 5^k \sum_{m \in F} 5^{-\omega(m)} = 6^k.
\]

3. Proof of Theorem 1.3

By (2.2), we have
\[
N(B, F) = \#(F) B + 2 \sum_{\frac{n \cdot m \in F}{\gcd(n, m) = 1, n < m}} \left| \frac{B}{m} \right| \sum_{\frac{n \cdot m \in F}{\gcd(n, m) = 1, n < m}} 1 := \left(\#F\right) B + 2 \mathcal{B}_0.
\]
We get

\[ \mathcal{B}_0 = \left( \sum_{1 < m \leq B^\alpha} \frac{B}{m} \right) \sum_{\gcd(n,m)=1} \sum_{n_2, n_4 \in F \atop n_2 \cdot n_4 = \frac{n}{2}} 1 =: \mathcal{B}_{01} + \mathcal{B}_{02}. \]

Let us consider the first sum above, \( \mathcal{B}_{01} \).

\[ \mathcal{B}_{01} = \sum_{1 < m \leq B^\alpha} \left\lfloor \frac{B}{m} \right\rfloor \sum_{n_2, n_4 \in F \atop \gcd(n,m)=1, n_2 \cdot n_4 = \frac{n}{2}} \frac{\phi(m)}{m} \]

\[ = \frac{6}{\pi^2} B^{1+2\alpha} + O(B^{1+\alpha} \log B), \]

where \( \sum_{n \leq x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + O(\log x) \) is used.

Now let us estimate \( \mathcal{B}_{02} \).

\[ \mathcal{B}_{02} \leq \sum_{B^{\alpha} < m \leq B^{2\alpha}} \frac{B^{2\alpha}}{m} \sum_{n_2, n_4 \in F \atop \gcd(n,m)=1, n_2 \cdot n_4 = \frac{n}{2}} \frac{1}{m} \leq B^{1+2\alpha} \sum_{B^{\alpha} < m \leq B^{2\alpha}} \frac{\phi(m)}{m^2} \]

Putting together the above estimates, we have

\[ N(B, F) \leq \left( 1 + \frac{12}{\pi^2} + 2 \sum_{B^{\alpha} < m \leq B^{2\alpha}} \frac{1}{m} \right) B^{1+2\alpha}. \]

4. Proof of Theorem 1.4

By Lemma 1.1

\[ (4.1) \quad N(B, F) = \sum_{n,m \in F \atop \gcd(n,m)=1} \left( \sum_{n_2, n_4 \in F \atop \frac{n_2 \cdot n_4}{n} = \frac{m}{2}} 1 \right) \left\lfloor \frac{B}{n \lor m} \right\rfloor. \]

Let \( m \in F \). Given an integer \( a \geq 2 \), we define \( \langle a \rangle = \{ p : p | a \} \). Recall that \( \omega(n) \) denotes the prime divisor function, and \( \omega(1) = 0 \). Consider for \( n, m \in F \) with \( \gcd(n,m) = 1 \), the sum

\[ \sum_{\frac{n_2 \cdot n_4}{n} = \frac{m}{2}} 1. \]

Then \( \frac{n_4}{n_2} = \frac{m}{2} \) gives rise to solutions \( n_4 = \lambda m, n_2 = \lambda n \). As \( \lambda, \lambda m, \lambda n \in F \), it follows by definition of \( F \) that \( \langle \lambda \rangle \cap \langle m \rangle = \langle \lambda \rangle \cap \langle n \rangle = \emptyset \). Otherwise, if for instance some \( p_j \) verifies \( p_j \in \langle \lambda \rangle \cap \langle m \rangle \), then \( p_j^2 | \lambda m = n_4 \), which is impossible. Thus \( \langle \lambda \rangle \subseteq \{ p_1, \ldots, p_k \} - \langle m \rangle - \langle n \rangle \). Conversely any subset \( A \) of it provides a suitable \( \lambda \) with \( \langle \lambda \rangle = A \). And so we have

\[ \sum_{\frac{n_2 \cdot n_4}{n} = \frac{m}{2}} 1 = 2^{k-\omega(m)-\omega(n)}. \]

for all \( n, m \in F \) with \( \gcd(n,m) = 1 \).

Inserting this into (4.1) we get,

\[ (4.2) \quad N(B, F) = \sum_{n,m \in F \atop \gcd(n,m)=1} 2^{k-\omega(m)-\omega(n)} \left\lfloor \frac{B}{n \lor m} \right\rfloor. \]
First consider the lower bound. We have
\[ N(B, F) \geq 2^k \sum_{n \in F, n \leq B} 2^{-\omega(n)} \left\lfloor \frac{B}{n} \right\rfloor. \]
Observe that for \( X \neq 0 \),
\[ \sum_{d \in F} \frac{1}{d} X^{-\omega(d)} = \prod_{j=1}^{k} \left( 1 + \frac{1}{Xp_j} \right). \]
Thus if \( F \subset [1, B] \),
\[ N(B, F) \geq 2^k B \left( \prod_{j=1}^{k} \left( 1 + \frac{1}{2p_j} \right) \right), \]
which proves the lower bound.

Next consider the upper bound for \( N(B, F) \). We have
\[ N(B, F) = B 2^k \sum_{n, m \in F, \gcd(n, m) = 1, \frac{m}{n} \leq B} \frac{1}{2^{\omega(m)} 2^{\omega(n)} (n \lor m)} \]
\[ + 2^k \mathcal{O} \left( \sum_{n, m \in F, \gcd(n, m) = 1, \frac{m}{n} \leq B} \frac{1}{2^{\omega(m)} 2^{\omega(n)}} \right), \]
(4.4)

Put
\[ Y = \sum_{n, m \in F, \gcd(n, m) = 1, \frac{m}{n} \leq B} \frac{1}{2^{\omega(m)} 2^{\omega(n)} (n \lor m)}, \quad Y_0 = \sum_{n, m \in F, \gcd(n, m) = 1, \frac{m}{n} \leq B} \frac{1}{2^{\omega(m)} 2^{\omega(n)}}. \]

We thus start with the formula
\[ N(B, F) = B 2^k Y + 2^k \mathcal{O}(Y_0). \]
(4.5)

We note that
\[ Y = 1 + 2 \sum_{n \leq B, \frac{n}{m} \leq B} \frac{1}{m 2^{\omega(m)}} \sum_{n \in F, \gcd(n, m) = 1, n \leq m} \frac{1}{2^{\omega(n)}}. \]

The presence of the order relation “<” on \( F \), a set of squarefree numbers, in the summation index, makes that sum not easy to manipulate. We cannot bound \( Y \) directly and will thus proceed differently. We first note the relation
\[ \frac{1}{n \lor m} = \frac{1}{\sqrt{n}} \frac{1}{\sqrt{m}} \left( \frac{n \land m}{n \lor m} \right)^{1/2}. \]
(4.6)

Now as \( e^{-|\theta|} = \int_{\mathbb{R}} e^{i\theta t} \frac{dt}{\pi(t^2 + 1)} \), it follows that
\[ \left( \frac{n}{m} \right)^s = \int_{\mathbb{R}} \frac{1}{n^{-ist} m^{ist}} \frac{dt}{\pi(t^2 + 1)} \quad (m \geq n). \]
(4.7)

Take \( s = 1/2 \). We get
\[ \frac{1}{n \lor m} = \frac{1}{\sqrt{n}} \frac{1}{\sqrt{m}} \int_{\mathbb{R}} \frac{1}{n^{-it/2} m^{it/2}} \frac{dt}{\pi(t^2 + 1)}. \]
Recall that \( \mu \) denotes the Möbius function and that
\[ \sum_{d \mid n} \mu(d) = \delta(n) := \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases} \]
(4.8)
By Lemma 2.4 in [10],

So that,

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By Lemma 2.4 in [10],

for any real \( s \geq 0 \). In our case \( s = 1/2 \). Now by Lemma 2.5 in [10],

for any \( s > 0 \) and complex numbers \( x_j, j = 1, \ldots, N \).

Therefore,

– If \( d = 1 \), we have

We note that

Further,

\[
\sum_{\nu \in F, \nu \leq B}^{1} \frac{1}{4^{\omega} \nu} = \sum_{\nu \in F, \omega(\nu) = y}^{1} 4^{-y} = \sum_{y=0}^{\nu} C_k y 4^{-y} = \left( \frac{5}{4} \right)^{k}.
\]
Thus
\[\int_{\mathbb{R}} \left| \frac{1}{n_1^{1/2}} \right|^{2} \frac{dt}{\pi(t^2 + 1)} \leq \min \left( B \prod_{i=1}^{r} \left( 1 + \frac{1}{4p_i} \right), \left( \frac{5}{4} \right)^k \right). \]

- If \( d > 1 \), \( d \in F \), then similarly,
\[
\int_{\mathbb{R}} \left| \sum_{\nu \in F, \nu \leq B} \frac{1}{\sqrt{n_1 \nu^{1/2}}} \right|^{2} \frac{dt}{\pi(t^2 + 1)}
= \frac{1}{2^{2\omega(d)}d} \int_{\mathbb{R}} \left| \sum_{\nu \leq B/d} \frac{1}{\sqrt{n_1 \nu^{1/2}}} \right|^{2} \frac{dt}{\pi(t^2 + 1)}
\leq \frac{C}{2^{2\omega(d)}d} \sum_{\nu \leq B/d} \frac{1}{\sqrt{n_1 \nu^{1/2}}} = \frac{C}{2^{2\omega(d)}d} \sum_{\nu \leq B/d} \frac{1}{\sqrt{n_1 \nu^{1/2}}}.
\]

We also note that
\[
\sum_{\nu \leq B/d, \gcd(\nu, d) = 1} \frac{1}{\sqrt{n_1 \nu^{1/2}}} = \frac{B}{d} \sum_{\nu \leq B/d} \frac{1}{\sqrt{n_1 \nu^{1/2}}} \leq \frac{B}{d} \prod_{1 \leq i \leq k} \left( 1 + \frac{1}{4p_i} \right).
\]

Next,
\[
\sum_{\nu \leq B/d, \gcd(\nu, d) = 1} \frac{1}{\sqrt{n_1 \nu^{1/2}}} = \sum_{\nu \leq B/d} \frac{1}{\sqrt{n_1 \nu^{1/2}}} = \sum_{\nu \leq B/d} 4^{-\nu} = \sum_{\nu \leq B/d} \frac{1}{\sqrt{n_1 \nu^{1/2}}} = \left( \frac{5}{4} \right)^k \left( \frac{4}{5} \right)^{\omega(d)}.
\]

Therefore,
\[
\int_{\mathbb{R}} \left| \sum_{\nu \leq B/d} \frac{1}{\sqrt{n_1 \nu^{1/2}}} \right|^{2} \frac{dt}{\pi(t^2 + 1)}
\leq \frac{C}{4^{\omega(d)}d} \min \left( \frac{B}{d} \prod_{1 \leq i \leq k} \left( 1 + \frac{1}{4p_i} \right), \left( \frac{5}{4} \right)^k \left( \frac{4}{5} \right)^{\omega(d)} \right).
\]

Consequently,
\[
\sum_{d \not\in F, d \leq 1} |\mu(d)| \int_{\mathbb{R}} \left| \sum_{\nu \leq B/d} \frac{1}{\sqrt{n_1 \nu^{1/2}}} \right|^{2} \frac{dt}{\pi(t^2 + 1)}
\leq \sum_{d \not\in F, d \leq 1} \frac{C}{4^{\omega(d)}d} \min \left( \frac{B}{d} \prod_{1 \leq i \leq k} \left( 1 + \frac{1}{4p_i} \right), \left( \frac{5}{4} \right)^k \left( \frac{4}{5} \right)^{\omega(d)} \right).
\]

On the one hand,
\[
\sum_{d \not\in F, d \leq 1} \frac{1}{4^{\omega(d)}d} \left( \frac{5}{4} \right)^k \left( \frac{4}{5} \right)^{\omega(d)} = \left( \frac{5}{4} \right)^k \sum_{d \not\in F, d \leq 1} \frac{1}{4^{\omega(d)}d} \leq \left( \frac{5}{4} \right)^k \prod_{i=1}^{r} \left( 1 + \frac{1}{5p_i} \right).
\]

On the other hand, using the definition of \( F \),
\[
B \sum_{d \not\in F, d \leq 1} \frac{1}{4^{\omega(d)}d} \prod_{1 \leq i \leq k} \left( 1 + \frac{1}{4p_i} \right)
\]
\[
\varepsilon(F) = \prod_{i=1}^{k} \left(1 + \frac{1}{4p_i} \right) - 1.
\]

Thus
\[
\sum_{d \in F, d \neq 1} |\mu(d)| \int_{\mathbb{R}} \left| \sum_{n \in F, d\mid n, n \leq B} 2^{\omega(n)} \sqrt{n \ln 1/2} \right|^2 \frac{dt}{\pi(t^2 + 1)} 
\leq C \min \left( B \prod_{i=1}^{k} \left(1 + \frac{1}{4p_i} \right) \varepsilon(F), \left(\frac{5}{4}\right)^k \prod_{i=1}^{k} \left(1 + \frac{1}{5p_i} \right) \right),
\]

whence,
\[
\left| Y - \sum_{n, m \in F \cap [1, B]} \frac{1}{2^{\omega(m)} 2^{\omega(n)} (n \land m)} \right| 
\leq C \prod_{i=1}^{k} \left(1 + \frac{1}{4p_i} \right) \min \left( B \varepsilon(F), \left(\frac{5}{4}\right)^k \right).
\]

Using now \((4.10)\) we obtain the bound,
\[
Y \leq \min \left( B \prod_{i=1}^{k} \left(1 + \frac{1}{4p_i} \right), \left(\frac{5}{4}\right)^k \right) + C \prod_{i=1}^{k} \left(1 + \frac{1}{4p_i} \right) \min \left( B \varepsilon(F), \left(\frac{5}{4}\right)^k \right)
\leq C \prod_{i=1}^{k} \left(1 + \frac{1}{4p_i} \right) \min \left( B, \left(\frac{5}{4}\right)^k \right),
\]
since \(\varepsilon(F) \leq C\) uniformly in \(F\). Now plainly,
\[
\sum_{(n, m) \in F, n, m \leq 1} \frac{1}{2^{\omega(m) + \omega(n)}} = \sum_{m \in F} \frac{1}{2^{\omega(m)}} \sum_{(n, m) \in F} \frac{1}{2^{\omega(n)}} = \sum_{m \in F} \frac{1}{2^{\omega(m)}} \left(\frac{3}{2}\right)^{k-\omega(m)} = \left(\frac{3}{2}\right)^k \sum_{m \in F} \frac{1}{2^{\omega(m)}} = \left(\frac{3}{2}\right)^k \left(\frac{4}{3}\right)^k = 2^k.
\]

Also
\[
\sum_{n, m \in F, \ \gcd(n, m) = 1 \ (n \land m) \leq B} \frac{1}{2^{\omega(n) + \omega(m)}} \leq B \sum_{n, m \in F, \ \gcd(n, m) = 1 \ (m \land n) \leq B} \frac{1}{m 2^{\omega(m)} 2^{\omega(n)}} = B \sum_{m \in F} \frac{1}{m 2^{\omega(m)} \sum_{n \in F, \ \gcd(m, n) = 1} \frac{1}{2^{\omega(n)}}}
\leq B \sum_{m \in F} \frac{1}{m 2^{\omega(m)} \left(\frac{3}{2}\right)^{k-\omega(m)}} = B \left(\frac{3}{2}\right)^k \sum_{m \in F} \frac{1}{m 3^{\omega(m)}} = B \left(\frac{3}{2}\right)^k \prod_{i=1}^{k} \left(1 + \frac{1}{3p_i} \right).
\]
Thus

\[ Y_0 \leq \min \left( B \left( \frac{3}{2} \right)^k \prod_{i=1}^{k} (1 + \frac{1}{3p_i}), 2^k \right). \]

As by (4.1), \( N(B, F) = B 2^k Y + 2^k \mathcal{O}(Y_0) \), we get,

\[
\begin{align*}
& \left| N(B, F) - B 2^k \sum_{n,m \in F \cap [1, B]} \frac{1}{2^{\omega(m)2^{\omega(n)}(n \lor m)}} \right| \\
\leq & \ 2^k \mathcal{O} \left( \min \left( B \prod_{i=1}^{k} \left( 1 + \frac{1}{4p_i} \right) (1 + \varepsilon(F)), \left( \frac{5}{4} \right)^k \right) \right) \\
& + C B 2^k \prod_{i=1}^{k} \left( 1 + \frac{1}{4p_i} \right) \min \left( B \varepsilon(F), \left( \frac{5}{4} \right)^k \right).
\end{align*}
\]

(4.13)

Indeed,

\[
\begin{align*}
& \left| N(B, F) - B 2^k \sum_{n,m \in F \cap [1, B]} \frac{1}{2^{\omega(m)2^{\omega(n)}(n \lor m)}} \right| \\
\leq & \ \left| N(B, F) - B 2^k Y \right| + B 2^k Y - \sum_{n,m \in F \cap [1, B]} \frac{1}{2^{\omega(m)2^{\omega(n)}(n \lor m)}} \\
\leq & \ 2^k \mathcal{O}(Y_0) + C B 2^k \prod_{i=1}^{k} \left( 1 + \frac{1}{4p_i} \right) \min \left( B \varepsilon(F), \left( \frac{5}{4} \right)^k \right) \\
\leq & \ 2^k \mathcal{O} \left( \min \left( B \left( \frac{3}{2} \right)^k \prod_{i=1}^{k} \left( 1 + \frac{1}{3p_i} \right), 2^k \right) \right) \\
& + C B 2^k \prod_{i=1}^{k} \left( 1 + \frac{1}{4p_i} \right) \min \left( B \varepsilon(F), \left( \frac{5}{4} \right)^k \right).
\end{align*}
\]

Also

\[
\begin{align*}
N(B, F) \leq & \ B 2^k \min \left( B \prod_{i=1}^{r} \left( 1 + \frac{1}{4p_i} \right), \left( \frac{5}{4} \right)^k \right) + \\
& + 2^k \mathcal{O} \left( \min \left( B \left( \frac{3}{2} \right)^k \prod_{i=1}^{k} \left( 1 + \frac{1}{3p_i} \right), 2^k \right) \right) \\
& + C B 2^k \prod_{i=1}^{k} \left( 1 + \frac{1}{4p_i} \right) \min \left( B \varepsilon(F), \left( \frac{5}{4} \right)^k \right) \\
\leq & \ C B 2^k \prod_{i=1}^{k} \left( 1 + \frac{1}{4p_i} \right) \min \left( B , \left( \frac{5}{4} \right)^k \right) + \\
& + C 2^k \prod_{i=1}^{k} \left( 1 + \frac{1}{3p_i} \right) \min \left( B \left( \frac{3}{2} \right)^k, 2^k \right).
\end{align*}
\]

(4.14)

Remark 4.1. When the \( p_i \) are all large, then \( \varepsilon(F) \) becomes small and we see with (4.13) that

\[
N(B, F) \sim B 2^k \sum_{n,m \in F \cap [1, B]} \frac{1}{2^{\omega(m)2^{\omega(n)}(n \lor m)}}.
\]

(4.15)

Remark 4.2. By Weierstrass’ inequality, if \( 0 < a_k < 1 \) and \( \sum_{k=1}^{n} a_k < 1 \), then

\[
1 + \sum_{k=1}^{n} a_k < \prod_{k=1}^{n} \left( 1 + a_k \right) < \frac{1}{1 - \sum_{k=1}^{n} a_k}.
\]
See Mitrinović [3] 3.2.37(3)]. Thus if \( \sum_{i=1}^{k} \frac{1}{p_i^2 + p_i} \leq 2 \),
\[
\varepsilon(F) = \prod_{i=1}^{k} \left( 1 + \frac{1}{(4p_i^2 + 4p_i)} \right) - 1 \leq \frac{\sum_{i=1}^{k} \frac{1}{(4p_i^2 + 4p_i)}}{1 - \sum_{i=1}^{k} \frac{1}{(4p_i^2 + 4p_i)}} \leq 2 \sum_{i=1}^{k} \frac{1}{p_i^2 + p_i}.
\]

5. Concluding Remarks.

5.1. A remark concerning Equation [1,3] Granville and Soundarajan (unpublished) proved using contour integral representation the following estimate
\[
N_k(B) \sim c(k) B^{k+1} (\log B)^{k^2},
\]
where the constant \( c(k) \) depends on \( k \) only. We show here that the following almost optimal upper bound
\[
N_k(B) \ll_k B^{k+1} (\log B)^{k^2 + 2k - 2}, \quad (k \geq 1),
\]
can be proved quite elementarily. It is an interesting question to know whether the approach we propose can be used to remove the extra term \( 2k - 2 \).

Let \( d_k(n) \) denote the Piltz divisor function counting the number of ways to write \( n \) as a product of \( k \) factors. We will use the fact that \( d_k \) is sub-multiplicative: \( d_k(nm) \leq d_k(n)d_k(m) \), for all \( n, m \geq 1 \). This follows from the formula ([6], p. 40)
\[
d_k(n) = \prod_{p \mid n} C_{v_p(n) + k - 1} = \prod_{p \mid n} \left( \frac{v_p(n) + j}{j} \right),
\]
where \( v_p(n) \) is the \( p \)-valuation of \( n \), i.e. \( p^{v_p(n)} \mid n \) and \( v_p(1) = 0 \).

Proof of (5.2). Applying Lemma 1.1 with \( F = M_k(B) \) gives,
\[
N_k(B) = \sum_{g_{d(n,m)}=1}^{B} \left( \sum_{\frac{B}{m} \mid m} 1 \right).
\]
Thus
\[
N_k(B) = B\#(M_k(B)) + 2 \sum_{g_{d(n,m)}=1}^{B} \left( \sum_{\frac{B}{m} \mid m} 1 \right)
\]
\[
= B\#(M_k(B)) + 2Bk.
\]
We note that \( \frac{m_1}{m_2} = \frac{m}{m} \) for some \( m_1, m_2 \in M_k(B) \), means that \( mm_1 = c = nm_1, nm \mid c \) and \( nm \leq c \leq nB^k \). Thus
\[
\sum_{m_1, m_2 \in M_k(B)} \frac{1}{m_1} = \sum_{m_1, m_2 \in M_k(B)} \sum_{c=nm}^{nB^k} 1 = \sum_{m_1, m_2 \in M_k(B)} \sum_{c=nm}^{nB^k} 1 = \sum_{m_1, m_2 \in M_k(B)} \sum_{c=nm}^{nB^k} 1.
\]
(5.3) \( (c = jmn) \)
\[
\sum_{m_1, m_2 \in M_k(B)} \frac{1}{m_1} = \sum_{j=1}^{[B^k/m]} \left( \sum_{m_1, m_2 \in M_k(B)} 1 \right) \left( \sum_{m_1, m_2 \in M_k(B)} 1 \right),
\]
and so,
\[
\sum_{m_1, m_2 \in M_k(B)} \frac{1}{m_1} \leq \sum_{j=1}^{[B^k/m]} d_k(jn)d_k jm \leq d_k(n)d_k (m) \sum_{j=1}^{[B^k/m]} d_k(j)^2
\]
(5.4) \[ \ll_k \frac{B^k}{m} (\log B)^{k^2-1} d_k(n)d_k(m), \]

where we have used sub-multiplicativity of \(d_k\) and the estimate \(\sum_{m \leq x} d_k^2(m) = (C_k + o(1)) x \log^{k^2-1} x\). See [3] (9.33) for instance. Thus

(5.5) \[ B_k \ll_k B^{k+1} (\log B)^{k^2-1} \sum_{2 \leq m \leq B} \sum_{n < m, \gcd(n,m) = 1} \frac{d_k(n)d_k(m)}{m^2}. \]

Now, plainly

\[ \sum_{2 \leq m \leq B} \sum_{n < m, \gcd(n,m) = 1} \frac{d_k(n)d_k(m)}{m^2} \leq \sum_{2 \leq m \leq B} \frac{d_k(m)}{m^2} \sum_{n < m} d_k(n) \ll_k (\log B)^{k-1} \sum_{2 \leq m \leq B} \frac{d_k(m)}{m} \ll_k (\log B)^{2k-1}, \]

since \(\sum_{m \leq x} d_k(m) \sim C_k x (\log x)^{k-1}\), (see notably Theorem 14.9 in [3]), and further that \(\sum_{n \leq x} \frac{d_k(n)}{n} \sim C_k (\log x)^k\). This along with (5.5) implies

(5.6) \[ B_k \ll_k B^{k+1} (\log B)^{k^2+2k-2}. \]

By combining and since \#(\(M_k(B)\)) = \(B^k\),

\[ N_k(B) \ll_k B^{k+1} (\log B)^{k^2+2k-2}. \]

5.2. Problem. Consider Equation (1.2) with \(F = \{n \leq B : n \text{ squarefree}\}\). For the study of this very interesting case, part of the proof of Theorem 1.3 can probably still be used. This will be investigated elsewhere.

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