Hamiltonian analysis of non-chiral Plebanski theory and its generalizations

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Abstract

We consider the non-chiral, full Lorentz group-based Plebanski formulation of general relativity in its version that utilizes the Lagrange multiplier field \(\Phi\) with ‘internal’ indices. The Hamiltonian analysis of this version of the theory turns out to be simpler than in the previously considered in the literature version with \(\Phi\) carrying spacetime indices. We then extend the Hamiltonian analysis to a more general class of theories whose action contains scalar invariants constructed from \(\Phi\). Such theories have recently been considered in the context of unification of gravity with other forces. We show that these more general theories have six additional propagating degrees of freedom as compared to general relativity. This has not been appreciated in the literature treating them as being not much different from GR.

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1. Introduction

The original Plebanski formulation of general relativity [1], see also [2], is chiral, i.e., based on the self- anti-self-dual split of the Lie algebra of the Lorentz group. A formulation using the same key ideas but based on the full Lorentz group has been considered e.g. in [3]. In this non-chiral, full Lorentz group-based version, it was later generalized in [4] to general relativity in an arbitrary number of spacetime dimensions. This paper also observed that when one works with the full Lorentz group there are two classically equivalent but distinct Plebanski-type formulations. Namely, one formulation uses the Lagrange multiplier field with only ‘internal’ indices (in the case of four spacetime dimensions, this is the theory considered in [3]), while the other formulation uses spacetime indices. It is this latter version of the full Lorentz group Plebanski formulation that has been mainly considered as the starting point of the so-called spin foam quantization, due to the fact that its discretization leads to the so-called simplicity.
constraints most naturally. The Hamiltonian analysis of this version of \textit{SO}(4) Plebanski theory has been carried out in [5, 6].

In this short paper, we revisit the Hamiltonian analysis of the non-chiral, full Lorentz group-based Plebanski formulation of general relativity, and perform the analysis of the version [3] of the theory with the internal index Lagrange multiplier field. The analysis turns out to be simpler than that in [5], which is one of our motivations for writing it down.

However, the main purpose of this paper is to extend the Hamiltonian treatment to a more general class of theories, which is as follows. Generalizing the self-dual Plebanski theory, paper [7] by one of the present authors proposed a class of modified gravity theories with the action including scalar invariants constructed from Plebanski’s Lagrange multiplier field. Paper [8] later combined these ideas with the earlier ideas of Peldan [9] on ‘unification’ by extension of the internal gauge group. It considered the following theory based on a gauge group \( G \):

\[
S = \int_{M} \left[ B^A \wedge F^A - \frac{1}{2} \Phi^{AB} B^A \wedge B^B + \frac{g}{2} \Phi^{CD} \Phi^{CD} B^A \wedge B^A \right], \tag{1}
\]

where the upper case Latin letters from the beginning of the alphabet are the Lie algebra ones, \( B^A \) is a 2-form field, \( F^A \) is the curvature of a \( G \)-connection \( A^A \), and \( \Phi^{AB} \) is the ‘Lagrange multiplier’ field that is required to be traceless, \( \Phi^{AA} = 0 \). The author argued that it can be interpreted as a \( G/SO(4) \) gauge theory coupled to gravity. In particular, it was implied that the theory (1) based on the gauge group \( SO(4) \) is gravity (possibly modified).

The main purpose of this paper is to elucidate the nature of this theory for \( G \) being the Lorentz group. We find, somewhat surprisingly, that, unlike the modified theories [7] based on the self-dual Plebanski gravity, the non-chiral theory (1) contains many more propagating degrees of freedom (DOF) as compared to general relativity. It may still be possible to interpret it as a gravitational theory, but before such an interpretation can be possible one has to face a very difficult question of why the additional propagating degrees of freedom predicted by it are not observed. We do not attempt to develop an interpretation for such a theory (1) in this short paper, our main aim being just to point out that the theory is much farther from the Plebanski’s version of general relativity than one might naively expect.

Our analysis applies not just to (1), but to a more general class of theories of the type proposed in [7] that are parametrized by a single scalar function of the ‘Lagrange multiplier’ field \( \Phi \). Their action is given by

\[
S = \int_{M} \left[ B^A \wedge F^A - \frac{1}{2} (\Phi^{AB} - \Lambda(\Phi)^{AB}) B^A \wedge B^B \right], \tag{2}
\]

where \( \Lambda(\Phi) \) is an arbitrary \( G \)-invariant scalar function of the traceless ‘internal’ tensor \( \Phi^{AB} \), and \( g^{AB} \) is an invariant metric on the Lie algebra of \( G \). The case considered in [8] corresponds to \( \Lambda(\Phi) = g^{AB} \Phi^{AB} \). In this paper we shall study the case of the Lorentz group only, but the case of an arbitrary gauge group can be treated along the same lines. Let us also note that the case \( G \) being the Lorentz group and \( \Lambda = \text{const} \) is just the non-chiral Plebanski theory [3] equivalent to general relativity. Thus, the main result of this paper is that the non-chiral Plebanski theory \( \Lambda = \text{const} \) is a very degenerate member of a much more general class of theories (2), with a generic theory from this class having six more DOF than the \( \Lambda = \text{const} \) one.

The organization of this paper is as follows. In section 2, we perform the Hamiltonian analysis of non-chiral Plebanski theory in its version using the Lagrange multiplier field with internal indices. In section 3, we repeat the analysis for the class of the Lorentz group-based generalized theories (2) and show that they contain six more propagating DOF as compared to the case \( \Lambda = \text{const} \) that gives general relativity.
Our conventions and notations are as follows. We consider simultaneously two signatures that are distinguished by the parameter \( \sigma = \pm 1 \): it is positive in the case of the Riemannian signature when the gauge group is \( G = SO(4) \) and negative in the Lorentzian case when \( G = SO(3, 1) \). In both cases we obtain similar results. In particular, all the results about the structure of the phase space of the theory and the number of propagating DOF do not depend on the signature. We use Greek letters for spacetime indices, small Latin letters from the middle of the alphabet for spatial indices, capital Latin letters from the middle of the alphabet for internal vector indices, \( I, J, \ldots \in \{1, 2, 3, 4\} \), and small Latin letters from the beginning of the alphabet as \( so(3) \) indices, \( a, b, \ldots \in \{1, 2, 3, 4\} \). Symmetrization and antisymmetrization of indices are denoted by \((\cdots)\) and \([\cdots]\) correspondingly and both are defined with the weight \( 1/2 \). The antisymmetric tensor \( \varepsilon^{IJKL} \) is normalized such that \( \varepsilon^{1234} = 1 \) and the internal indices are lowered and raised with the metric \( g_{IJ,KL} = \text{diag}(\sigma, 1, 1, 1, 1) \). As a result, one obtains that \( \varepsilon^{IJKL} \varepsilon^{IJKL} = \sigma 4! \). A metric \( g^{IJ,KL} \) on the Lie algebra is defined as \( g^{IJ,KL} = (1/2)(g_{IK}g_{JL} - g_{JK}g_{IL}) \). The structure constants of \( G \) are denoted by \( f^{IJ}_{KL,MN} \).

2. Canonical analysis of Plebanski theory

We consider the non-chiral Plebanski action for general relativity with a cosmological constant, in its version due to [3] with the Lagrange multiplier field with internal indices:

\[
S_{Pl}[A, B, \psi] = \frac{1}{2} \int_M d^4x \varepsilon^{\mu\nu\rho\sigma} \left( g^{IJ,KL} B^{IJ}_{\mu\nu} F_{\rho\sigma}^{KL} + \frac{1}{2} (\psi_{IJ,KL} + \Lambda \varepsilon^{IJ}_{KL}) B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} \right). \tag{3}
\]

In this expression, \( \psi_{IJ,KL} = \psi_{[IJ][KL]} \) is an ‘internal’ tensor, playing the role of a field of Lagrange multipliers; see on this below. For this reason we shall refer to it as a ‘Lagrange multiplier’ field, even in the generalized case considered in the following section, where its Lagrange multiplier role is lost. The field \( \psi_{IJ,KL} \) must be symmetric under the exchange of the pair \([IJ]\) with \([KL]\) and is required to satisfy the following tracelessness condition:

\[
\varepsilon^{IJKL} \psi_{IJ,KL} = 0. \tag{4}
\]

The action (3) represents general relativity with the cosmological constant \( \Lambda \) as the topological \( BF \) theory with additional constraints (the ‘simplicity’ constraints generated when one varies the action with respect to \( \psi_{IJ,KL} \)) on the 2-form \( B \) ensuring that it comes from a frame field.

The Hamiltonian formulation of this system is obtained as follows. First, the action is rewritten as

\[
S_{Pl} = \int dt \int_{\Sigma} d^3x \left( \tilde{P}_{IJ}^i \partial_i A_{IJ}^i - H \right), \tag{5}
\]

where we have introduced the momentum conjugate to the connection field:

\[
\tilde{P}_{IJ}^i = g^{IJ,KL} \varepsilon^{ijk} B_{jkl}^{KL}.
\]

and the canonical Hamiltonian is given by

\[
-H = A_{IJ}^i D_i \tilde{P}_{IJ}^i + \frac{1}{2} \left( g_{IJ,KL} \varepsilon^{ijk} F_{jkl}^{KL} + (\psi_{IJ,KL} + \Lambda \varepsilon^{IJ}_{KL}) \tilde{P}_{IJKL}^i \right), \tag{7}
\]

where we have integrated by parts to get the first term.

The main difference between the version of the theory (3) and that analyzed in [5] with the Lagrange multiplier field carrying spacetime indices is that the variables \( B_{0i}^{IJ} \) appear in the Hamiltonian form of the action (5) linearly, while in the other version of the theory they appear quadratically. Because of this, work [5] introduces momenta conjugate to \( B_{0i}^{IJ} \), which complicates the analysis. There is no need for this complication in the case analyzed here.
Thus, the variables $A_{IJ}^0$, $B_{0i}^{IJ}$ and $\varphi_{IJKL}$ have vanishing conjugate momenta and therefore play the role of Lagrange multipliers. The variation with respect to these variables generates the following conditions:

$$G_{IJ} \equiv D_i \tilde{P}_{IJ} = \partial_0 \tilde{P}_{IJ} + \int_{IJKL} A_{IJ}^K \tilde{P}_{MN}^i \approx 0,$$

$$C_i^J \equiv \delta_{IJ}^{[k} F_{j]K} + \left( \varphi_{IKL} + \Lambda_{IJ}^{KL} \right) \tilde{P}_{KL}^i \approx 0,$$

$$\Phi_{IJKL} \equiv B_{0i}^{IJ} \tilde{P}_{KL}^i - \frac{\sigma}{4} \epsilon_{ijkl} \varphi_{IJKL} \approx 0,$$

where we used the following definition of the four-dimensional volume:

$$V = \frac{1}{24} \epsilon^{\mu\nu\rho\sigma} \epsilon_{IJKL} B_{0i}^{IJ} B_{0j}^{KL} = \frac{1}{6} \epsilon^{IJ} B_{0i}^{KL} \tilde{P}_{KLM}^i.$$

In the following, we shall assume that the volume is non-vanishing.

Now, as usual, the constraints (8) are just generators of the internal gauge rotations. To disentangle the structure of the other constraints, we note that some of them can be interpreted as equations fixing the Lagrange multipliers. To see this, let us first concentrate on the conditions (9) and split them into several components. A convenient way to do this is to note that the quantities $\tilde{P}_{IJ}^i$ and $B_{0i}^{IJ}$ form an independent basis in the Lie algebra. This follows from the condition (10) and our assumption that the volume $V$ is non-zero. Therefore, we can use $\tilde{P}_{IJ}^i$ and $B_{0i}^{IJ}$ to trade the Lie algebra indices $[IJ]$ for the 3D space indices $i, j$. We start by considering the following combinations:

$$\mathcal{H}_i \equiv -\frac{1}{2} \delta^{i[j} \tilde{P}_{j]}_{IJ} C_k^i \epsilon_{KLM} = -\tilde{P}_{IJ}^j F_{ij}^j \approx 0,$$

$$\mathcal{H}_0 \equiv B_{0i}^{IJ} C_i^J = \delta_{IJ}^{[k} B_{0j}^{IJ} F_{j]}^k + 6 \Lambda V + \varphi_{IJKL} \Phi_{IJKL} \approx 0.$$ 

Note that these are just the anti-symmetric part of the projection onto $\tilde{P}_{IJ}^i$ and the trace part of the projection onto $B_{0i}^{IJ}$. These combinations do not depend on the Lagrange multipliers (besides the last term that can be weakly dropped) and thus generate primary constraints. The other independent combinations are the symmetric part of the projection onto $\tilde{P}_{IJ}^i$ and the trace-free part of the projection onto $B_{0i}^{IJ}$:

$$g^{[ij} \tilde{P}_{ij}^{k]} C_k^l \epsilon_{KLM} = \delta_{ij}^{[k} \tilde{P}_{k]}_{IJ} F_{ij}^j + \left( \varphi_{IJKL} + \Lambda_{ij}^{KL} \right) \tilde{P}_{ij}^k F_{kL}^j \approx 0,$$

$$B_{0i}^{IJ} C_i^J = -\frac{1}{3} \delta_i^{[j} B_{0j}^{IJ} C_k^i = \epsilon_{IJ}^{[kl} B_{0k}^{IJ} F_{l]}^k + \left( \delta_i^{[j} B_{0j}^{IJ} \right) F_{k]}^k + \left( \varphi_{IJKL} + \Lambda_{ij}^{KL} \right) \left( B_{0i}^{IJ} \tilde{P}_{KL}^i - \frac{1}{3} \delta_i^{[j} B_{0j}^{IKL} \right) \approx 0.$$ 

Here we have in total $6 + 8 = 14$ equations that can be interpreted as those for the components of the Lagrange multiplier field $\varphi_{IJKL}$ of which there is 20. Indeed, the equations allow us to find the $(\tilde{P}^i \varphi \tilde{P}_i)$ components, of which there is 6, as well as the traceless part of the $(\tilde{P}^i \varphi B_{0i})$ components, of which there is 8. The remaining six components of the Lagrange multiplier field $\varphi$ are those corresponding to contractions $(B_{0i} \varphi B_{0i})$.

Next we turn to the conditions (10). To deal with these equations, it will be convenient to introduce the following notations:

$$hh^{ij} = \frac{\sigma}{2} g^{ij} \tilde{P}_{ij}^i \tilde{P}_{KL}^j, \quad h = \det h_{ij},$$

$$\Lambda^i = -\frac{\sigma}{2h} \epsilon_{ijkl} h_{lij} B_{0k}^{ij} \tilde{P}_{KL}^j, \quad \Lambda = \frac{\Lambda^i}{h}.$$


and
\[ \Phi^{ij} = -\frac{\sigma}{2} \varepsilon^{ijkl} \tilde{P}_{ij} \tilde{P}_{kl}. \]

Let us now note that instead of using \( \tilde{P}^i \), \( B_{0i} \) as the basis in the Lie algebra, we may as well use the quantities \( P^i \) together with its Hodge dual. Projecting the conditions (10) on \( \tilde{P}^i \) and its dual, after some simple algebra, we get the following two equations:
\[ B_{0i}^{ij} + \frac{\sigma}{2\hbar} h_{ij} B_{0k}^{IJKL} \tilde{P}_{jk}^{IJKM} B_{0k}^{MN} = \frac{1}{4\hbar} \hbar_{ij} \varepsilon^{IJKL} \tilde{P}_{KL}^{ij} = 0, \]
\[ \sigma \varepsilon_{IJKL} B_{0j}^{KL} \Phi^{ij} = \frac{1}{2} \varepsilon_{IJKL} \varepsilon^{IJKM} B_{0j}^{KL} \tilde{P}_{JMN}^{i} + \hbar_{ij} \tilde{P}_{IJ}^{ij} = 0. \]

Contracting the first of these equations with \( \tilde{P}_{ij} \), one finds that
\[ h_{(ik} B_{0j)}^{ij} = -\frac{\sigma}{4\hbar} \hbar_{ik} \hbar_{ij} \Phi^{kl}. \]

Using this and (17) in (19), one obtains
\[ B_{0i}^{ij} = \frac{1}{4\hbar} \hbar_{ij} \varepsilon^{IJKL} \tilde{P}_{ij}^{IJKM} = \frac{1}{2} \varepsilon_{ij} \hbar_{ij} \tilde{P}_{ij}^{ij} + \frac{1}{8\hbar} \hbar_{ij} \tilde{P}_{ij}^{ij} \Phi^{ij}. \]

Substituting this result into equation (20), one finds that it reduces to the condition independent of \( B_{0i} \):
\[ \Phi^{ij} = 0. \]

Thus, we have shown that 20 equations (10) give 6 primary constraints (23) and allow us to find 14 out of 18 components of \( B_{0i}^{ij} \) via formula (22). The remaining components of these Lagrange multipliers are given by lapse \( \hbar_{ij} \) and shift \( \Pi^{ij} \), which are left undetermined in (22).

Substituting the obtained results for the Lagrange multipliers into the Hamiltonian, one obtains
\[ -H = A_{ij}^{IJ} \mathcal{G}_{ij} + H_0 = A_{ij}^{IJ} \mathcal{G}_{ij} + \hbar_{ij} \mathcal{H}_i + \hbar_{ij} \mathcal{H} + \lambda_{ij} \Phi^{ij}, \]
where
\[ \mathcal{H} = \frac{1}{4} \hbar_{ij} \varepsilon^{ikl} \varepsilon^{J} \varepsilon_{KL} \mathcal{P}_{ij}^{J} - 6\Lambda_{ij}, \]
and \( \lambda_{ij} \) is some complicated matrix which will not play any role in the following. The only important fact for us is that it contains six remaining undetermined components of the Lagrange multiplier field \( \mathcal{G}_{IJKL} \), i.e., those corresponding to the projections \( (B_{0i} \mathcal{G}_{0j}) \).

At this point, we get exactly the same system as that obtained in the covariant canonical formulation of the Hilbert–Palatini action in [11] (see also [6]). This allows the results on the constraint analysis to be borrowed from this work. One finds that the primary constraints \( \mathcal{G}_{ij}, \mathcal{H}_i, \mathcal{H} \) do not generate any further conditions, whereas \( \Phi^{ij} \) give rise to six secondary constraints:
\[ \Psi^{ij} = \varepsilon_{IJKL} \varepsilon^{ijkl} h_{km} \tilde{P}_{ij}^{ij} D_k \tilde{P}_{KL}^{ij} \approx 0. \]

The condition of the conservation of \( \Psi^{ij} \) then generates a new constraint. Since the covariant derivative in \( \Psi^{ij} \) contains the connection, the commutator of the two constraints (23) and (26)

\[ 3 \text{ In fact, in the Riemannian case, equation (20) has two additional solutions (as can be seen from equation (39) for } \Lambda_{ij} = 0: \Phi^{ij} = \pm 2\hbar h^{ij}. \text{ These are equivalent to conditions that } \tilde{P}_{ij} \text{ is (anti-) self-dual. Thus, these solutions of the simplicity constraints reproduce the (anti-) self-dual sector of Euclidean general relativity. It is interesting that these sectors are contained in the non-chiral } SO(4) \text{ Plebanski formulation without any need to introduce the Immirzi parameter [10].} \]
is non-vanishing. As a result, the tertiary constraint gets a contribution from the last term in the Hamiltonian (24) proportional to $\lambda_{ij}$. As we mentioned, the latter contains the remaining unknown components of $\varphi_{IJKL}$. Thus, the role of the tertiary constraint is simply to fix these last six components of the Lagrange multiplier field.

Due to the non-vanishing commutator, the constraints $\Phi^{ij}$ and $\Psi^{ij}$ are of second class, and thus can be imposed strongly provided the symplectic structure was replaced by that given by the Dirac bracket. The other constraints $G^I$, $H_i$, $H$ are of first class. Their physical meaning is that $G^I$ generates Lorentz gauge transformations, whereas the other constraints are responsible for the spatial and temporal diffeomorphisms, correspondingly.

The arising structure of the phase space is then as follows. The kinematical phase space is that of pairs $(\tilde{P}_{IJ}, A^{IJ})$, with the configuration space—the space of $G$-connections—being $3 \times 6 = 18$ dimensional. We have gauge symmetries as well as diffeomorphisms acting on this space, with the action generated by the first-class constraint each of which reduces the dimension of the configuration space by one. This leaves us with $18 - 6 - 4 = 8$ dimensional configuration space. On top of this, we have 6+6 second-class constraints, each of which reduces the dimension of the phase space by one, thus leaving us with a two-dimensional configuration physical space, which describes the two propagating DOF of general relativity.

3. Canonical analysis of generalized Plebanski theory

We now consider a more general class of theories described in the introduction, where the cosmological constant $\Lambda_1$ is replaced by a generic function of the Lagrange multipliers $\varphi_{IJKL}:

$$
S_{\text{gPl}}[A, B, \varphi] = \frac{1}{2} \int \mathcal{M} d^4 x \tilde{\epsilon}^{\mu \nu \rho \sigma} \left[ g_{IJKL} A^{IJ}_{\mu \nu} F_{KL}^{\rho \sigma} + \frac{1}{2} (\varphi_{IJKL} + \Lambda(\varphi)\epsilon_{IJKL}) B_{\mu \nu}^{IJ} B_{\rho \sigma}^{KL} \right].
$$

(27)

Now the action depends on the ‘Lagrange multiplier’ fields $\varphi^{IJKL}$ non-linearly. As is standard in this situation, to facilitate the canonical analysis, it is convenient to introduce the momenta conjugate to these fields. Thus, we add to the action the following terms:

$$
\int \mathcal{M} d^4 x \left[ \psi_{IJKL} \partial_t \varphi_{IJKL} + \lambda_{IJKL} \psi_{IJKL} \right].
$$

(28)

The first term introduces momenta conjugated to $\varphi_{IJKL}$ which makes them dynamical fields. The second term imposes constraints that the momenta are vanishing, which returns us to the original action.

Splitting the time and space coordinates brings the action into the form

$$
S_{\text{gPl}} = \int dt \int_\Sigma d^3 x \left( \tilde{P}_{ij}^{I} \partial_t A_{ij}^{IJ} + \psi^{IJKL} \partial_t \varphi_{IJKL} - H \right),
$$

(29)

with the canonical Hamiltonian being

$$
-H = A_{ij}^{IJ} D_i \tilde{P}_{ij}^{I} + B_{\mu \nu}^{IJ} \left( g_{IJKL} \epsilon^{ijk} F_{jk}^{KL} + (\varphi_{IJKL} + \Lambda(\varphi)\epsilon_{IJKL}) \tilde{P}_{KL}^{ij} \right) \lambda_{IJKL}.
$$

(30)

The variables $A_{ij}^{IJ}, B_{\mu \nu}^{IJ}$ and $\lambda_{IJKL}$ have vanishing conjugated momenta and therefore play the role of Lagrange multipliers. A variation with respect to these variables generates the following conditions:

$$
G_{ij} = D_i \tilde{P}_{ij} \approx 0,
$$

(31)

$$
C_{ij}^{I} = g_{IJKL} \epsilon^{ijk} F_{jk}^{KL} + (\varphi_{IJKL} + \Lambda(\varphi)\epsilon_{IJKL}) \tilde{P}_{KL}^{ij} \approx 0,
$$

(32)
\[ \psi_{\ijkl} \approx 0. \] (33)

As before, the conditions (31) do not involve the Lagrange multipliers and thus give primary constraints. However, they do not yet give generators of gauge transformations for all the fields, as they do not act on the ‘Lagrange multiplier’ fields \( \phi_{\ijkl} \). Thus, it is convenient to shift them by adding a linear combination of the constraints (33):

\[ \tilde{\mathcal{G}}_{\ij} = \mathcal{G}_{\ij} - 2 f_{\ijkl \mn pq} \psi_{\mn pq} \phi_{\ijkl}. \] (34)

This shift amounts in a simple redefinition of the Lagrange multipliers

\[ \tilde{\lambda}_{\ijkl} = \lambda_{\ijkl} - 2 f_{\ijkl \mn pq} \phi_{\ijkl}. \] (35)

The new constraints (34) generate gauge transformations of all the variables, including \( \phi \) and \( \psi \). This is convenient as, since the Hamiltonian is a gauge scalar, \( \tilde{\mathcal{G}}_{\ij} \) are stable under its action and no secondary constraints get produced.

Next we turn to the constraints (33). Commuting them with the Hamiltonian, we get additional conditions:

\[ \Phi_{\ijkl} \equiv B_{\ijkl}^0 \overset{\psi}{\sim} \tilde{\psi}_{\ijkl} = -\frac{\sqrt{2}}{4} \left( \sigma \epsilon_{\ijkl} - 24 \Lambda_{\ijkl}^1 \right) \approx 0, \] (36)

where we have introduced

\[ \Lambda_{\ijkl}^1 := \frac{\partial \Lambda(\psi)}{\partial \psi_{\ijkl}}. \] (37)

The conditions (36) are what replace (10) in the case of usual Plebanski theory.

We can analyze the consequences of (36) using the same procedure and the same notations (16)–(18) as in section 2. Here the first step is to find an expression for the Lagrange multipliers \( B_{\ijkl}^0 \). One finds

\[ B_{\ijkl}^0 = \frac{1}{8 \hbar} N_{\ijkl} \left( \epsilon_{\ijkl} - 24 \Lambda_{\ijkl}^1 \right) \tilde{\psi}_{\ijkl} - \frac{1}{2} \epsilon_{\ijkl} \tilde{\psi}_{\ijkl} + \frac{1}{8 \hbar} N_{\ijkl} h_{ij} \tilde{\psi}_{\ijkl} \left( \Phi_{\ijkl}^i + 12 \tilde{\psi}_{\ijkl} \Lambda_{\ijkl}^1 \tilde{\psi}_{\ijkl} \right). \] (38)

Unlike the case of usual Plebanski theory, this now explicitly depends on the ‘Lagrange multipliers’ \( \phi_{\ijkl} \). As in the case of the usual Plebanski theory, the obtained expression for the quantities \( B_{\ijkl}^0 \) leaves four of them (the lapse and the shift) undetermined. Thus, to find them we have utilized 18 \(- 4 = 14\) out of 20 constraints (36), leaving 6 additional constraints whose meaning is to be clarified.

Using the same procedure that led to the simplicity constraints (23) we now get the following six additional constraints:

\[ X_{\ij} \equiv \Phi_{\ij} - \frac{\sigma}{4 \hbar^2} \Phi_{\ijkl} h_{\kl} \Phi_{\mn} h_{\mn} \Phi_{\ij} \]

\[ + 12 \left( \star \tilde{\psi}_{\ijkl} + \frac{1}{2 \hbar} \Phi_{\ijkl}^i \tilde{\psi}_{\ijkl} \right) \Lambda_{\ijkl}^1 (\psi) \left( \star \tilde{\psi}_{\kl} + \frac{1}{2 \hbar} \Phi_{\klm}^i h_{\mn} \tilde{\psi}_{\kl}^n \right) \]

\[ = \frac{1}{2} \left( \star \tilde{\psi}_{\ijkl} + \frac{1}{2 \hbar} \Phi_{\ijkl}^i \tilde{\psi}_{\ijkl} \right) \left( \epsilon_{\ijkl} + 24 \Lambda_{\ijkl}^1 (\psi) \right) \left( \star \tilde{\psi}_{\kl} + \frac{1}{2 \hbar} \Phi_{\klm}^i h_{\mn} \tilde{\psi}_{\kl}^n \right) \approx 0. \] (39)

Unlike the case of usual Plebanski theory analyzed in the previous section, the constraints \( X_{\ij} \) now explicitly depend on \( \phi_{\ijkl} \). We will see that it is this fact that eventually results in the theory having more propagating DOF.
Applying now the stabilization procedure to $X^{ij}$, one finds further conditions:

$$Y^{ij} \equiv 12 \lambda^{i j k l} \left( \mathbf{P}^{j} + \frac{1}{2 \hbar} \Phi^{j} h^{kl} \mathbf{P}^{k} \right) \Lambda^{j k l, m n p q}_{(2)} \left( \mathbf{P}^{n} + \frac{1}{2 \hbar} \Phi^{n} h_{m n} \mathbf{P}_{m} \right) + \{ X^{ij}, C_{1 i j} \} B^{i j k l}_{0 k} \approx 0. \quad (40)$$

Let us leave for the moment these new conditions and turn to the equations (32). Assuming that the independent components of $C^{ij}_{1 j}$ are exhausted by contraction with $\mathbf{P}^{j}$, the trace part of $B_{0 j}$ and the traceless part of $\mathbf{P}^{j}$, we split them into four parts as follows:

$$\mathcal{H}_{i j} \equiv - \frac{1}{2} g^{i j k l} \epsilon_{i j k} \mathbf{P}^{k} \mathbf{C}_{k l} = - \mathbf{P}^{i j} F_{i j} \approx 0. \quad (41)$$

$$\mathcal{H}_{0 j} \equiv B_{0 i}^{i j} C^{i j} = g^{i j k l} B_{0 i}^{i j} F^{k l}_{j} + 6 \left( \Lambda(\psi) - \psi_{i j k l} \Lambda_{(i j k l)}^{(1)} \right) + \psi_{j k l} \Phi^{j k l} \approx 0. \quad (42)$$

$$C^{ij} = g^{i j k l} \mathbf{P}^{i j} C_{k l} = e^{i j k l} \mathbf{P}^{i j} F^{k l}_{j} + \left( \psi_{i j k l} + \Lambda(\psi) e^{i j k l} \right) \mathbf{P}^{i j} \approx 0. \quad (43)$$

$$C^{i j}_{0} \equiv C^{i j} \star \tilde{D}^{i j} - 1 \frac{1}{3} h^{i j} h_{k l} C^{k l} \star \tilde{D}^{i j} \approx 0. \quad (44)$$

All of these conditions are primary constraints. Note that the dependence of the constraint $\mathcal{H}_{0 j}$ on the fields $\psi^{i j k l}$ is that of the Legendre transform of the function $\Lambda(\psi)$; the phenomenon also observed in the case of self-dual theory in [12]. The stabilization procedure applied to the last two constraints produces further conditions which can be written as

$$\tilde{\lambda}_{i j k l} \left( \mathbf{P}^{i j} \mathbf{P}_{k l} - 2 \sigma \Phi^{i j} \Lambda^{j k l}_{(1)} \right) + \{ C^{i j}, A^{i j}_{0} \} G_{i j} + D_{0} = 0. \quad (45)$$

$$\left( \delta_{i j}^{k l} - \frac{1}{3} h^{i j} h_{k l} \right) \mathbf{P}^{i j} \lambda^{i j k l} \star \tilde{D}_{k l} + \{ C^{i j}, A^{i j}_{0} \} G_{i j} + D_{0} = 0. \quad (46)$$

which gives in total $6 + 8 = 14$ conditions. For non-vanishing $\Lambda_{(2)}$, these conditions together with 6 conditions (40) allow one to find all 20 Lagrange multipliers $\lambda_{i j k l}$. Thus, the secondary constraints (40), (45) and (46) do not contain constraints on canonical variables, and do not generate any further conditions. On the other hand, a set of constraints $X^{ij}$, $C^{ij}$, $C^{i j}_{0}$ allows one to find all the components of the ‘Lagrange multiplier’ field $\psi^{i j k l}$. They are of second class because they do not commute with $\psi^{i j k l}$.

All this is in contrast to what was happening in the case of the usual Plebanski theory, where the constraints $X^{ij}$ were $\psi^{i j k l}$ independent, and thus gave constraints on the phase space variables $\mathbf{P}^{i j}$. Their commutator with the Hamiltonian resulted in secondary second-class constraints, and only the condition that those secondary second-class constraints are preserved under the evolution allowed one to determine the remaining six components of the Lagrange multiplier field $\psi^{i j k l}$. In the case of generalized theory we are now considering, the situation is simpler, in spite of the seeming complexity of all these equations. Indeed, all the constraints are now simply equations allowing us to determine the Lagrange multipliers $\psi^{i j k l}$ and $\lambda^{i j k l}$ and do not generate any constraints on the other phase-space variables. In particular, the stabilization procedure finishes one step earlier than in the case of the usual Plebanski theory.

It remains to consider the constraints $\mathcal{H}_{i j}$ and $\mathcal{H}_{0 j}$. For the first set of constraints $D_{i j}$, it is possible to shift them by means of other constraints in such a way that they become generators
of spatial diffeomorphisms and thus stable under the time evolution. For this we define
\[ \mathcal{D}_i \equiv H_i + A_i^{IJ} \hat{G}_{IJ} - \psi_{IJKL} D_i \psi_{IJKL} = H_i + A_i^{IJ} \hat{G}_{IJ} - \psi_{IJKL} \partial_i \psi_{IJKL}. \] (47)
The constraint \( \mathcal{H}_0 \) is replaced by the full Hamiltonian with the Lagrange multipliers \( \psi_{IJKL} \) fixed by the previous equations
\[ \mathcal{D}_0 \equiv H(\psi = \psi(A, \hat{P})). \] (48)
The structure of the arising phase space is then as follows. Solving all second-class constraints and conditions on the Lagrange multipliers, one determines all of the components of the fields \( \psi_{IJKL}, \lambda_{IJKL} \). In addition, the momentum conjugate to \( \psi_{IJKL} \) is zero. The reduced phase space is then parametrized by pairs \((\hat{P}_{IJ}, A_i^{IJ})\) with a set of first-class constraints \( \mathcal{G}_{IJ}, \mathcal{D}_i, \mathcal{D}_0 \) acting on it. This is similar to what we have seen in the case of the usual Plebanski theory, but the key difference now is that there are no additional second-class constraints on the phase-space variables. The dimension of the physical configuration space is then \( 18 - 6 - 4 = 8 \), which is the two DOF available in the usual Plebanski theory plus additional six propagating DOF.

As we have seen, the question of the number of DOF described by the theory (27) crucially depends on the properties of the matrix of second derivatives \( \Lambda^{IJKL, MNPQ} \) of the function \( \Lambda(\psi) \). Our result about the number of DOF certainly applies to the case of the quadratic such function considered in [8], as the matrix of second derivatives in this case is just the identity matrix in the appropriate space. It would be interesting to know if there are some other choices of \( \Lambda(\psi) \) (apart from the ‘trivial’ constant function) that lead to theories with two propagating DOF. More generally, it would be interesting to characterize the ‘landscape’ of functions \( \Lambda(\psi) \) in terms of the number of DOF that the corresponding theory would produce. We leave this interesting problem to future research. Another interesting problem is to find an interpretation of these additional degrees of freedom.

Let us conclude by reiterating our main message: a general theory from the class (27) is very far from the usual Plebanski theory, as it contains many more propagating DOF. Whether such a more non-trivial theory can be meaningfully interpreted as a gravity theory only the future can tell.

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