Hirotaka’s problem 028

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**Introduction**

In (Majewski, 2020), the authors discuss some sangaku’s made by the Japanese mathematician Hirotaka Ebisui. Most of them only require a basic knowledge of plane Euclidean geometry to be solved. One of these is called problem HI 028. In what follows, we will give a proof and make some extra observations.

**The problem**

Given two perpendicular lines and two circles tangent to both lines. The circles are on the same side of one of the lines and on different sides of the other line.

![Diagram](Figure 1)

Here we have the circles c and c’ with midpoints G and D respectively and the perpendicular lines AE and AC. The circle c’ is tangent to AC in C and to AE in E. The circle c touches the line AC in F and the line AE in H. Note that HE is one of the internal common tangents of the circles c and c’ and FC one of the external common tangents. So one of the external common tangents is perpendicular to one of the internal common tangents.

We connect one of the tangent points on the internal common tangent of one of the circles with the tangent point of the external common tangent on the other circle, e.g. E with F and H with C. These lines intersect the circles in two other points: I and J. The objective is to prove that the lines IF and JC are parallel.

But there is more in the picture. We can also prove that the lines IC and FJ are perpendicular and that IJ is the other common external tangent of the circles. Moreover, if we indicate the other two intersection points of the lines IC and FJ
with the circles, we find the other internal common tangent, LM in the next figure. This internal common tangent is also perpendicular to the external common tangent IJ.

![Figure 2](image2.png)

We will try to prove these statements.

**Proof of the statements**

Let K be the intersection point of the lines HC and FE. First note that the angles $\angle KJC$ and $\angle CIF$ are equal to 45° since they are circumferential angles on the same arc as the 90° center angles $\angle EDC$ and $\angle HGF$.

![Figure 3](image3.png)
Since the triangles HCA and FEA are congruent (|AC| = |AE|, \( \angle C \hat{A} H = \angle E \hat{A} F = 90^\circ \), |AH| = |AF|), \( \angle F \hat{E} A = \angle H \hat{C} A = \angle A \hat{H} C = 90^\circ = \angle E \hat{H} K \). Hence, in triangle EKH the angle \( \angle E \hat{K} H = 90^\circ \).

Now this means that the angle \( \angle K \hat{C} J \) in triangle JKC is also 45°, as is \( \angle I \hat{K} F \) in triangle IKF. It then follows that \( \angle K \hat{I} F = \angle K \hat{C} J \), so JC \( \parallel \) IF.

This proves the statement made by Hirotaka and the statement that the lines IC and JF are perpendicular.

In order to see that IJ is the other common external tangent, we note that both triangles JKC and IKF are isosceles and right angled. The triangles KCF and KJI are then congruent since |KC| = |KJ|, the angles in K are opposite angles and |KF| = |KI|.

The angle between the lines IJ and JD is 90° because each of its parts is equal to the angle in C with the corresponding number (see figure 4): those with number one are 45°, the angles with number two are corresponding angles in the congruent triangles JKI and CKF and the angles numbered by 3 are the basic angles of the isosceles triangle JDC. Moreover, the sum of the angles in C equals 90° because FC is tangent in C to the circle \( c' \).

A similar reasoning can be made in the points I and F, but now we have to consider the sum of the angles numbered by 1 and 2 decreased with the angles numbered by 3. So we proved the second claim.

For the third claim, we first note that \( \angle M_2 = \angle M_3 = 90^\circ - \angle K \hat{L} M = 90^\circ - \angle N \hat{L} J = \hat{J}_2 = \hat{C}_2 \). Moreover, since the triangle MDC is isosceles, \( \angle M_1 = \hat{C}_1 + \hat{C}_3 \), which means that \( \angle M_1 + \angle M_2 = \hat{C}_1 + \hat{C}_3 + \hat{C}_2 = 90^\circ \). Hence LM is perpendicular to MD.
In a similar way, $\hat{L}_1 = \hat{F}_1 - \hat{F}_3$, which means that $\hat{L}_1 + \hat{L}_2 = \hat{F}_1 - \hat{F}_3 + \hat{F}_2 = 90^\circ$. Hence LM is perpendicular to GL. It follows that LM is the other common internal tangent of the two circles and that $\hat{L}_1 = \hat{J}_2$. 
Conclusion

As a conclusion, we can make the following statements. If two external touching circles have a perpendicular external and internal common tangent, then the other internal and external common tangent are also perpendicular. Moreover, the tangent points lie on two perpendicular lines and the lines connecting two outer tangent points of the same circle are parallel.

References

Majewski, M. C.-C. (2020, 09 01). The New Temple Geometry Problems in Hirotaka's Ebisui Files. from: atcm.mathandtech.org:
http://atcm.mathandtech.org/EP2010/invited/3052010_18118.pdf