Spherical Harmonic Analysis in Terms of Unevenly Distributed Observation Points and Its Applications to Geomagnetic Data

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(Received October 30, 1995; Revised November 27, 1996; Accepted April 17, 1997)

Spherical harmonic expansion is affected by uneven distribution of observation points. A weighting scheme is proposed in order to reduce the effects. The weights of observation points are determined in order to homogenize the distribution. Spherical harmonic expansion of the synthetic geomagnetic data is demonstrated; the weighting works in terms of decrease of truncation error of the expansion. The method is applicable to find a new observation site and to prioritize present geomagnetic observatories.

1. Introduction

We usually desire the geophysical data observed at evenly distributed points. However, no measurements on the Earth's surface can provide such data because of geographies, politics, environments, and so on. A global analysis, for example, a spherical harmonic expansion of geomagnetic field, is affected by such measurements.

Standard least squares analysis without accounting for unevenly spaced points leads biased results (e.g., Snieder, 1993). In order to avoid the problem, we sometimes waste data observed at close locations (e.g., Yokoyama and Yukutake, 1991). Weighting of data is another way to consider an uneven distribution. One of weighting methods is found in stochastic inversion. This method uses a priori information and minimizes the variance of estimation error (Franklin, 1970; Jackson, 1972, 1979; Gubbins, 1983, 1984; Gubbins and Bloxham, 1985). Evaluation of the error in stochastic inversion depends on data, model, and a priori information of model parameters. In other words, estimated weights of data contain model error and misestimation of data characteristics.

We propose a method without the above-mentioned problem: weights of data are calculated independently of data and of model. We define distribution function determined by the positions of observation points, then calculate weights to homogenize the distribution. This leads two advantages: 1) We can evaluate observatory distribution in a straightforward way. This is contrasted to that stochastic inversion evaluates observatory distribution as a result of inversion. 2) The weighting is applicable to various kinds of data.

In the next section, we review linear inversion in order to compare our method to previous ones, especially to stochastic inversion. In Section 3, we explicitly consider the effects of unevenly distributed observation points, and propose a scheme to reduce the effects. In Sections 4 and 5, we apply the method to geomagnetic analysis, in which spherical harmonics analysis with the Gauss coefficients is essential (e.g., Langel, 1987). In Section 6, we introduce two applications of the method: one is a selection of a proper new observatory location. The other is prioritizing existing observatories. Our way, which is performed without data, is contrasted to previous works: Souriau and Woodhouse (1985) presented a distribution evaluation method designed especially for observation of earthquake. Langel et al. (1995) evaluated candidate sites using a kind of prediction error calculated from synthetic geomagnetic data.
2. Linear Inversion

We consider linear inversion expressed as (e.g., Aki and Richards, 1980)

\[ d = Gm + e \]  

where \( d \) is the \( N \)-dimensional data vector, \( m \) is the \( M \)-dimensional model (unknown parameters) vector, \( G \) is the \( N \times M \) matrix, and \( e \) is the \( N \)-dimensional vector of measurement error. The solution of Eq. (1) is expressed as

\[ m = G^{-1}d \]  

where the inverse \( G^{-1} \) is to be determined. We here use the notation of \( G^{-1} \) instead of \( G^{-1}(G^*G)^{-1}G^* \) for simplicity. Substituting Eq. (1) into Eq. (2), we obtain

\[ m = G^{-1}Gm + G^{-1}e. \]  

The matrix \( G^{-1}G \) is the resolution in the model space. We define

\[ d = Gm. \]  

Substituting Eq. (2) into Eq. (4), we obtain

\[ d = G G^{-1}d. \]  

The matrix \( GG^{-1} \) is the resolution in the data space.

In order to obtain a good solution, the resolution matrix should be close to the identity matrix \( I \). We first optimize the resolution in data space. We minimize

\[ |GG^{-1} - I|^2 \]  

to obtain

\[ G^{-1} = (G^*G)^{-1}G^* \]  

where the asterisk denotes conjugate transpose. The inverse given in Eq. (7) is equivalent to the least-squares inverse which is obtained by minimizing

\[ |d - Gm|^2. \]  

If the number of the data \( N \) is larger than that of the model \( M \), the inverse of the \( M \times M \) square matrix \( G^*G \) is expected to exist in the usual sense.

The data covariance matrix \( D \) is incorporated in the least-square inverse. Instead of Eq. (8), we minimize

\[ (d - Gm)^*D^{-1}(d - Gm) \]  

to obtain

\[ G^{-1} = (G^*D^{-1}G)^{-1}G^*D^{-1}, \]  

assuming that \( e \) and \( D^{-1} \) are uncorrelated.

In order to stabilize the solution, the damped least-squares inverse is obtained by minimizing

\[ |d - Gm|^2 + \lambda|m|^2 \]  

where \( \lambda \) is the damping parameter. We incorporate not only the data covariance \( D \) but the model covariance \( M \). We generalize Eq. (11) to

\[ (d - Gm)^*D^{-1}(d - Gm) + m^*M^{-1}m. \]
The stochastic inverse, which minimizes Eq. (12), is given by
\[ G^{-1} = \left( G^*D^{-1}G + M^{-1} \right)^{-1} G^*D^{-1}. \] (13)

The model covariance has been determined from \textit{a priori} information on the model (Jackson, 1979; Gubbins, 1983). The extension to the nonlinear case is given in Tarantola and Valette (1982), Jackson and Matsu’ura (1985), and Gubbins and Bloxham (1985).

We alternatively optimize the resolution in the model space. We minimize
\[ |G^{-1}G - I|^2 \] (14)
to obtain the inverse
\[ G^{-1} = G^*(GG^*)^{-1}. \] (15)

The inverse of the \( N \times N \) square matrix \( GG^* \) does not exist in the usual sense. The solution is obtained by minimizing the norm of the model given in the second term of Eq. (12) (Backus and Gilbert, 1967; Parker, 1977; Whaler and Gubbins, 1981; Shure \textit{et al.}, 1982).

In stead of Eq. (14), we may use (Backus and Gilbert, 1968, 1970)
\[ \text{Tr} \left[ (G^{-1}G - I)M(G^{-1}G - I)^* + G^{-1}DG^{-1*} \right] \] (16)
where ‘\text{Tr}’ denotes the trace. The inverse minimizing Eq. (16) is given by
\[ G^{-1} = MG^*(GMG^* + D)^{-1}. \] (17)

We can verify the equivalence between Eqs. (13) and (17), although the interpretation of \( M \) is different. In Backus and Gilbert (1970), the magnitude of \( M \) is determined from the ‘trade-off’ between the resolution and error, the first and second term of Eq. (16). However the terms have different dimensions and cannot be readily compared. It is therefore not necessary simple to determine the optimal point on the trade-off curve.

3. Spherical Harmonic Analysis

3.1 Distribution function

The distribution function of observation points is defined by
\[ X(\theta, \phi) = \sum_{i=1}^{N} a_i \delta(\theta - \theta_i)\delta(\phi - \phi_i) \] (18)
where \( \theta_i \) and \( \phi_i \) respectively denote the colatitude and longitude of \( i \)-th observation point, and \( N \) is the number of observation points. The function \( \delta \) is the Dirac’s delta function. The weight \( a_i \) is normalized by
\[ \sum_{i=1}^{N} a_i = N. \] (19)

The distribution function is expanded in terms of spherical harmonics as
\[ X(\theta, \phi) = \sum_{s=0}^{\infty} \sum_{l=-s}^{s} x_s^l Y_s^l(\theta, \phi) \] (20)
where the spherical harmonics \( Y_s^t(\theta, \phi) \) with degree \( s \) and order \( t \) are defined as

\[
Y_s^t(\theta, \phi) = (-1)^t \sqrt{\frac{(2s+1)(s-t)!}{4\pi(s+t)!}} \cos^t \phi P_s^t(\cos \theta)
\]

with the associated Legendre function \( P_s^t(\cos \theta) \). Utilizing the orthonormality of the spherical harmonics, Eqs. (18) and (20) give the expansion coefficient \( \chi_s^t \) as

\[
\chi_s^t = \int_0^{2\pi} \int_0^\pi \chi(\theta, \phi) Y_s^t*(\theta, \phi) \sin \theta d\theta d\phi
\]

\[
= \frac{4\pi}{N} \sum_{i=1}^{N} a_i Y_s^t*(\theta_i, \phi_i).
\]

We consider the sum

\[
\sum_{t=-s}^{s} |\chi_s^t|^2 = \left( \frac{4\pi}{N} \right)^2 \sum_{t=-s}^{s} \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j Y_s^t*(\theta_i, \phi_i) Y_s^t(\theta_j, \phi_j)
\]

\[
= \frac{4\pi}{N^2} (2s+1) \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j P_s(\cos \psi_{ij})
\]

\[
= \frac{4\pi}{N^2} (2s+1) \left[ \sum_{i=1}^{N} |a_i|^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j P_s(\cos \psi_{ij}) \right]
\]

where we use the addition theorem of spherical harmonics

\[
\sum_{t=-s}^{s} Y_s^t*(\theta_i, \phi_i) Y_s^t(\theta_j, \phi_j) = \frac{2s+1}{4\pi} P_s(\cos \psi_{ij})
\]

with

\[
\cos \psi_{ij} = \cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j \cos(\phi_i - \phi_j)
\]

and

\[
\cos \psi_{ii} = 1.
\]

If the distribution is assumed to be uniform \((a_i = 1)\), the second term in the brackets of Eq. (23) vanishes, and we obtain

\[
\frac{1}{2s+1} \sum_{t=-s}^{s} |\chi_s^t|^2 = \frac{4\pi}{N}.
\]

The magnitude of \( \chi_s^t \) is proportional to \( N^{-1/2} \).

We define

\[
\chi_s = \sqrt{\frac{N}{4\pi(2s+1)} \sum_{t=-s}^{s} |\chi_s^t|^2}
\]

which quantify the unevenness of the distribution. For a uniform distribution, the \( \chi_s \) reduces to unity. In the case of a systematic distribution, some \( \chi_s \) may become nearly zero because the second term in the bracket of Eq. (23) may be negative. This is also a kind of uniform distributions. Physical meanings of \( \chi_s < 1 \) and \( \chi_s = 1 \) are not different.
3.2 Evaluation of distribution

We show two examples of $\chi_s$ spectrum in order to show the nature of the distribution function. The example distributions are shown in Fig. 1. The first example is a random distribution of 158 points. The other is the same number of actual presently operating permanent geomagnetic observatories (Langel, 1987; Takeda et al., 1993); we chose the observatories whose annual mean values after 1985 are available.

The $\chi_s$ spectrum of the first case is shown in Fig. 2 with a solid line. Most $\chi_s$ are less than or equal to unity. This indicates an even distribution. The second case (a broken line in Fig. 2) has larger values of $\chi_s$ than two when $s = 1–6$. Such distribution affects spherical harmonic expansion of data unless the distribution is considered.
In order to evaluate the distribution numerically, we define unevenness of distribution as

$$\gamma = \sqrt{\frac{1}{L} \sum_{s=1}^{L} \chi_s^2},$$  

(29)

where $L$ is the maximum degree, set to 20 in the following calculations. The unevenness $\gamma$ becomes small as the distribution approaches evenness. The quantity $\gamma < 1$ is equivalent to $\gamma = 1$ from the same reason to $\chi_s$.

For example, the unevenness for $L = 20$ are 1.025 and 2.232 for the distributions of Figs. 1 (a) and (b), respectively. We calculated more 50 samples of random distribution. Mean and standard deviation of $\gamma$ of the samples are 1.005 and $4.111 \times 10^{-2}$, respectively. Because $\gamma$ of the distribution of Fig. 1 (b) beyonds the region of randomness, $1.005 + 4.111 \times 10^{-2}$, the distribution is far from even. For $s$ greater than 20, $\chi_s$ is mostly unity, so that relation of $\gamma$ between the distributions does not change even for $L \geq 20$.

### 3.3 Weighting on data

As demonstrated in the previous subsection, $\chi_s \leq 1$ indicates uniform distribution. We suppose that the minimum of $\sum_{s=1}^{\infty} |\chi_s|^2$ achieves the minimum of $\chi_s$. Therefore, in order to perform a uniform distribution, we impose the weights $a_i$ for the $i$-th observation point and minimize

$$\sum_{s=1}^{\infty} w(s) |\chi_s|^2$$

(30)

$$\sum_{s=1}^{\infty} w(s)$$

where $\chi_s$ is defined in Eq. (28), and $w(s)$ is a certain function of degree. Using Eqs. (23) and (28), Eq. (30) reduces to

$$\sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij}a_ia_j$$

(31)
with

\[ A_{ij} = \frac{\sum_{s=1}^{\infty} w(s) P_s(\cos \psi_{ij})}{N \sum_{s=1}^{\infty} w(s)}. \]  

(32)

The minimization leads

\[ \sum_{j=1}^{N} A_{ij} a_i = 0. \]  

(33)

Using the condition in Eq. (19), the solution of Eq. (33) becomes

\[ a_i = \frac{N \sum_{j=1}^{N} (A^{-1})_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{N} (A^{-1})_{ij}}. \]  

(34)

The matrix \( A \) corresponds to \( GMG^* \) of Eq. (17). The function \( w(s) \) (or equivalently \( M \)) can be selected objectively (Section 4). On the other hand, in stochastic inversion, the model covariance \( M \) is determined by \textit{a priori} information (e.g., Gubbins, 1983).

In order to stabilize the solution, instead of Eq. (31), we minimize

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} a_i a_j + \frac{\lambda}{N} \sum_{i=1}^{N} |a_i|^2, \]  

(35)

then we have

\[ a_i = \frac{N \sum_{j=1}^{N} \left( A + \frac{\lambda}{N} I \right)^{-1} \left( A + \frac{\lambda}{N} I \right)^{-1}}{\sum_{i=1}^{N} \sum_{j=1}^{N} \left( A + \frac{\lambda}{N} I \right)^{-1} \left( A + \frac{\lambda}{N} I \right)^{-1}}. \]  

(36)

The damping parameter \( \lambda \) can be determined from the trade-off between \( \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} a_i a_j \) and \( \frac{1}{N} \sum_{i=1}^{N} |a_i|^2 \). Both terms are normalized to become unity for uniform distribution, and the damping parameter is expected to be order of unity (Section 5). On the other hand, in Backus and Gilbert (1970), unnormalized terms should be compared.

We consider the spherical harmonic expansion of a function \( V(\theta, \phi) \),

\[ V(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} V_{\ell m} \ Y_{\ell m}^{m}(\theta, \phi). \]  

(37)

We define the weighted integral

\[ \hat{V}_{\ell m} = \int_{0}^{2\pi} \int_{0}^{\pi} \chi(\theta, \phi) V(\theta, \phi) \ Y_{\ell m}^{m}(\theta, \phi) \sin \theta d\theta d\phi, \]  

(38)
which can be determined from the observations. Substituting Eqs. (20) and (37) into Eq. (38), we have

\[
\hat{V}_\ell^m = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \sum_{m'=-\ell'}^{\ell'} \int_0^{2\pi} \int_0^\pi \chi_s^l V_{\ell'}^{m'} Y_s^l(\theta, \phi) Y_{\ell'}^{m'}(\theta, \phi) Y_{\ell'}^{m'}(\theta, \phi) \sin \theta d\theta d\phi. \quad (39)
\]

We may solve Eq. (39) to obtain spherical harmonic coefficients. We can verify that this procedure is related to the least-squares inverse. Using Eqs. (18), (37), and (38), we have

\[
\sum_{i=0}^{N} a_i V(\theta_i, \phi_i) Y_{\ell'}^{m'}(\theta_i, \phi_i) = \sum_{i=0}^{N} \sum_{\ell'=-\ell'}^{\ell'} \sum_{m'=-\ell'}^{\ell'} a_i V_{\ell'}^{m'} Y_{\ell'}^{m'}(\theta_i, \phi_i) Y_{\ell'}^{m'}(\theta_i, \phi_i). \quad (40)
\]

Equation (40) is equivalent to the least-squares inverse given in Eq. (10); the weights are related to the data covariance $D$.

The summation of $\ell'$ and $m'$ extends to infinity in Eqs. (39) and (40). Although effects of unevenness are corrected in the least-squares inverse, the correction is not complete because of the truncation. In other words, Eq. (39) shows the effects of the truncation on spherical harmonic expansion in terms of the distribution function. The integral given in Eq. (39) is Adams-Gaunt integral (e.g., Gaunt, 1929). The integral becomes zero under certain conditions (e.g., Bullard and Gellman, 1954). Degree $\ell$ of $V_{\ell}^{m}$ is limited in the region $|s - \ell| \leq \ell' \leq |s + \ell|$.

If the $\chi_s^l$ are larger for small $s$, the matrix of Eq. (39) (or $G^* D^{-1} G$ of Eq. (10)) has large components near the diagonal ($\ell \approx \ell'$). Unless weights are imposed, the coefficients of the spherical harmonic expansion with large angular order (near the truncation) are biased.

4. Geomagnetic Field Analysis

The magnetic field is given by the gradient of a scalar potential

\[
B = -\nabla V(r, \theta, \phi) \quad (41)
\]

with

\[
V(r, \theta, \phi) = R \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left( \frac{R}{r} \right)^{\ell+1} V_{\ell}^{m} Y_{\ell}^{m}(\theta, \phi) \quad (42)
\]

where $r$ is the radial distance and $R$ is the radius of the Earth. The components of internal magnetic field are expressed as

\[
\begin{align*}
B_r(r, \theta, \phi) &= \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} (\ell + 1) \left( \frac{R}{r} \right)^{\ell+2} V_{\ell}^{m} Y_{\ell}^{m}(\theta, \phi) \\
B_\theta(r, \theta, \phi) &= -\sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left( \frac{R}{r} \right)^{\ell+2} \frac{\partial V_{\ell}^{m}}{\partial \theta} Y_{\ell}^{m}(\theta, \phi) \\
B_\phi(r, \theta, \phi) &= \frac{1}{\sin \theta} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} m \left( \frac{R}{r} \right)^{\ell+2} V_{\ell}^{m} Y_{\ell}^{m}(\theta, \phi)
\end{align*}
\quad (43)
\]

Using the first of Eq. (43), the coefficients are given by

\[
V_{\ell}^{m} = \frac{1}{\ell + 1} \int_0^{2\pi} \int_0^\pi B_r(R, \theta, \phi) Y_{\ell}^{m*}(\theta, \phi) \sin \theta d\theta d\phi. \quad (44)
\]
The correspondence with least squares can be shown as in Subsection 3.3.

Using Eqs. (42) and the first of (43), we can relate the radial fields at the surface and at the core-mantle boundary (radius \( c \))

\[
B_r (R, \theta, \phi) = \int_0^{2\pi} \int_0^\pi K (R, \theta, \phi; c, \theta', \phi') B_r (c, \theta', \phi') \sin \theta' d\theta' d\phi'.
\]

(45)

The kernel is given by (Gubbins and Roberts, 1983)

\[
K (R, \theta, \phi; c, \theta', \phi') = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \xi^{\ell+2} Y^m_{\ell} (\theta, \phi) Y^{m*}_{\ell} (\theta', \phi')
\]

\[
= \frac{1}{4\pi Q^3} \xi^2 (1 - \xi^2)
\]

(46)

with

\[
Q = \sqrt{1 - 2\xi \cos \psi + \xi^2}.
\]

(47)

Here \( \xi = c/R \) and \( \psi \) is the angle between \((\theta, \phi)\) and \((\theta', \phi')\). In derivation of Eq. (46), we use formula in Eq. (24),

\[
\sum_{\ell=1}^{\infty} P_\ell (\cos \psi) \xi^\ell = \frac{1}{Q},
\]

(48)

and

\[
\sum_{\ell=1}^{\infty} (2\ell + 1) P_\ell (\cos \psi) \xi^\ell = \frac{1 - \xi^2}{Q^3}.
\]

(49)

We can verify

\[
\begin{align*}
K (R, \theta, \phi; c, \theta', \phi') &> 0 \\
\frac{\partial K (R, \theta, \phi; c, \theta', \phi')}{\partial \psi} &< 0
\end{align*}
\]

(50)

The kernel takes the maximum value at \( \psi = 0 \), and decreases monotonically. We note

\[
K (R, \theta, \phi; R, \theta', \phi')
\]

\[
= \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} Y^m_{\ell} (\theta, \phi) Y^{m*}_{\ell} (\theta', \phi')
\]

\[
= \frac{1}{\sin \theta} \delta (\theta - \theta') \delta (\phi - \phi').
\]

(51)

The second identity, which is well-known in the theory of spherical harmonics, is verified by Eq. (46);

\[
K (R, \theta, \phi; R, \theta', \phi') = 0
\]

(52)

unless

\[
\int_0^{2\pi} \int_0^\pi K (R, \theta, \phi; R, \theta', \phi') \sin \theta' d\theta' d\phi' = 1
\]

(53)

Equation (45) can be interpreted as follows. If we observe the radial field at \((R, \theta_i, \phi_i)\), the distribution of observations at the core-mantle boundary is given by \((R, \theta_i, \phi_i; c, \theta', \phi')\) for \( i = 1, 2, \cdots, N \). We define the unweighted distribution function at the core-mantle boundary

\[
\chi^{(r)} (\theta, \phi) = \frac{4\pi}{N} \sum_{i=1}^{N} \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \xi^s Y^s_{t} (\theta_i, \phi_i) Y^{s*}_{t} (\theta, \phi).
\]

(54)
The quantity defined in Eq. (28) is given by

$$\chi_r^{(r)} = \xi^s \sqrt{\frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathcal{P}_s(\cos \psi_{ij})}. \quad (55)$$

If we are interested in the geomagnetic field at the core-mantle boundary, it is natural to minimize

$$\sum_{s=1}^{\infty} \left| \chi_r^{(r)} \right|^2. \quad (56)$$

This is equivalent to setting in Eq. (30),

$$w(s) = \xi^{2s}. \quad (57)$$

The summation over $s$ can be performed analytically by Eq. (48).

The horizontal fields can be treated similarly. However, the difference is slight as far as we treat low degree harmonics. Therefore, we shall neglect the difference in the next section.

5. Numerical Experiments

We have made numerical experiments on the weighting scheme described in the previous section. We used the 80 geomagnetic observatories of Malin (1969) which are shown in Fig. 3 with solid circles. The weights were determined in order to reduce the effects of uneven distribution. Then, a spherical harmonic expansion of geomagnetic field was performed by using the weights.

5.1 Determination of weights

In order to determine the weights by minimizing Eq. (35), we must specify the function $w(s)$ and the damping parameter $\lambda$. Two kinds of $w(s)$ were used. We first used a simple function given by

$$w(s) = \begin{cases} 1 & (s \leq s_0) \\ 0 & (s > s_0) \end{cases} \quad (58)$$

where we set $s_0 = 4$; other choices of $s_0$ gave similar but less satisfactory results (selection of $s_0$ is demonstrated in Subsection 6.2).

We determined the damping parameter by the trade-off between the first and the second terms of Eq. (35). The determined parameter is 2 which equates the two terms. The calculated set of weights is shown in Fig. 3(a). The weights are large in regions where there are few observatories, and small in regions where observatories are concentrated, for example in Europe. The quantities $\chi_r$ calculated with and without the weights are shown in Fig. 4(a) with solid and broken lines, respectively. We note the quantities are reduced even for $s > s_0$, although we used Eq. (58).

We also used the second function given in Eq. (57) which is suitable for obtaining the magnetic field at the core-mantle boundary. The damping parameter is selected to be 6, and the results are given in Figs. 3(b) and 4(b). Comparing Figs. 3(a) and 3(b), the weighting for the latter is lighter than for the former. A homogeneous distribution at the core-mantle boundary can be attained more easily, because Eq. (46) is wider-spread than the delta function at the surface.

5.2 Spherical harmonic expansion of geomagnetic field

The geomagnetic Gauss coefficients of the DGRF 1985 with attended external coefficients were used as a test magnetic field. Using the coefficients, the three orthogonal components (see Eq. (43)) of the geomagnetic field were synthesized at each point shown in Fig. 3. Then, random noise of three levels was added to the synthesized data. The maximums of random noise are 0,
Fig. 3. Distribution of 80 geomagnetic observatories of Malin (1969) (solid circles) and calculated weights which reduce effects of unevenly distributed observatories using (a) the function of \( w(s) = 1 \) (\( s \leq 4 \)) and 0 (\( s > 4 \)), and (b) \( w(s) = \xi^{2s} \).

100, and 500 nT, assuming that they are anomaly biases (e.g., Langel, 1987). Spherical harmonic expansions were performed using the weights shown in Figs. 3(a) and 3(b) for each noise level. In the expansion, the maximum degree of the external field was taken as 1, and the maximum degree of the internal field, truncation level, was taken as \( T = 1-7 \).

We first performed the spherical harmonic expansion with the set of weights shown in Fig. 3(a). In order to evaluate estimation error, we calculated residuals, the root mean square of
Fig. 4. Spectrum of $\chi_s$ of homogenized distribution with (a) the function of $w(s) = 1$ ($s \leq 4$) and $0$ ($s > 4$), and (b) $w(s) = \xi^{2s}$. Solid and broken lines indicate the quantities with and without weighting.

the difference between the estimated and the originally given Gauss coefficients:

$$\sqrt{\frac{1}{T(T+2)+3} \times \left[ (q_1^0 - q_T^0)^2 + (\dot{q}_1^1 - q_T^1)^2 + (\ddot{s}_1^1 - s_T^1)^2 + \sum_{T=1}^{T} \left( (\dot{g}_T^0 - g_T^0)^2 + \sum_{m=1}^{T} \left( (\dot{h}_T^m - h_T^m)^2 - \right) \right) \right]}$$

(59)

where $q$, $s$, $g$, and $h$ denote the Gauss coefficients, and $^\sim$ denotes estimated quantity. Solid and broken lines in Fig. 5 indicate the residuals for the weighted and unweighted cases, respectively. A small difference is recognized among the results of the three noise levels: the residuals originate mostly from truncation error. For the weighted case, the residuals decrease rapidly, and small
Fig. 5. Residuals of spherical harmonic expansion of the geomagnetic field whose noise level is (a) 0 nT, (b) 100 nT, and (c) 500 nT. We use the function $w(s) = 1 \ (s \leq 4)$ and $0 \ (s > 4)$. Solid and broken lines indicate the quantities with and without weighting.
Fig. 6. Error covariance matrix of the Gauss coefficients estimated (a) without weighting and (b) with weighting. Elements of the matrices are in the order of $q^0_1, g_1^1, s_1^1, g_2^2, g_1^1, h_1^1, \ldots, g_4^4$, and $h_4^4$. 
Fig. 7. Error covariance matrix of the geomagnetic field at observation points estimated (a) without weighting and (b) with weighting. Elements of the matrices are ordered by colatitude, from north to south. Elements from 1 to 63 are of the Northern hemisphere and those from 64 to 80 are of the Southern hemisphere.
Fig. 8. Same as Fig. 5 except that we use the function $w(s) = \zeta^{2s}$, and residuals are calculated at the core-mantle boundary.
parameters are sufficient. This is important for actual applications, because the maximum degree of truncation is unknown.

We calculated error covariance matrix of the Gauss coefficients

$$\varepsilon_{ij}^{(g)} = (\hat{g} - g)_i (\hat{g} - g)_j$$

(60)

where $g$ is a vector with elements of $q_0^0, q_1^1, s_1^1, g_0^0, g_1^1, h_1^1, \ldots, g_4^4,$ and $h_4^4$. Figure 6 shows an example when the noise level is 100 nT and the truncation level $T = 4$. Higher degree coefficients of the weighted case are well determined than the unweighted case: truncation error is diminished.

Figure 7 shows an error covariance matrix of magnetic field

$$\varepsilon_{ij}^{(B)} = (\hat{B}_r - B_r)_i (\hat{B}_r - B_r)_j + (\hat{B}_\theta - B_\theta)_i (\hat{B}_\theta - B_\theta)_j + (\hat{B}_\phi - B_\phi)_i (\hat{B}_\phi - B_\phi)_j$$

(61)

of the same case to Fig. 6. Vectors $B_r, B_\theta, B_\phi$ are of magnetic field at observation points. The vector elements are sorted by colatitude, from north to south. As shown in Fig. 7, the error of weighted case is homogenized: large error of the observatories in southern hemisphere (elements 64–80) decrease, while the error in northern hemisphere (elements 1–63) increase.

We next used the set of weights shown in Fig. 3(b), which are intended to obtain the Gauss coefficients at the core-mantle boundary. Residuals were calculated for the coefficients extrapolated from the surface to the core-mantle boundary, assuming an electrically insulating mantle. The results in Fig. 8 shows that the weighting diminishes truncation error.

6. Applications

We introduce two applications of the proposed method in this section. The proposed weights are determined only from distribution of observatories. Hence the weights is useful to evaluate the distribution whose data are difficult to be obtain.

6.1 An application to determine position of an additional site

Suppose we are looking for a new site to locate an instrument for observation. The best place can be determined by calculating $\gamma$ in Eq. (29). We examined the case for the distribution shown in Fig. 1 (b). The distribution is of present geomagnetic observatories. We first prepared candidate distribution including existing points and a new point. The new point was taken on $5^\circ \times 5^\circ$ longitude-latitude mesh. Then, we calculated $\gamma$ for each candidate distribution.

From Eqs. (23) and (28), $\chi_s$ is proportional to $1/\sqrt{N}$. Adding a candidate site, $\chi_s$ of equivalent distribution becomes $\sqrt{158}/159$ times smaller (original distribution has 158 points). Hence, $\gamma$ of equivalent distribution is expected to be $2.232 \times \sqrt{158}/159 = 2.225$ ($\gamma$ of the original distribution is 2.232).

Calculated $\gamma$ is shown in Fig. 9. Addition of a new site to green-colored region (around $\gamma = 2.225$) in Fig. 9 does not change evenness of the distribution. A new site added in red-colored region makes the distribution worse. While, that in blue-colored region, which indicates the pacific and southern hemisphere, makes the distribution more even than the original.

The minimum $\gamma$ is 2.219. It is given for the site at $(-50^\circ, 240^\circ E)$; this site is the best for locating an additional observation point. Solving $2.232 \times \sqrt{158}/x = 2.219$, we estimate that 159.9 (= x) points of equivalent distribution can achieve the minimum $\gamma = 2.219$. This indicates that the site $(-50^\circ, 240^\circ E)$ worth almost twice ($\sim$159.9–158) of a site of the original distribution.

We note that the blue-colored region including the best site is close to the region that Langel et al. (1995) recommended for sea-bottom station. While they used geomagnetic data for the determination, we did not. This implies that observation location is much effective than characteristic of geomagnetic data and than a model. Though the determined location is singe in our case, we can recommend a group of locations as Langel et al. (1995) did. In this case,
Fig. 9. Validity of a new site indicated with $\gamma$. Addition of a site in red-colored region to the distribution of Fig. 1(b) makes the distribution worse. On the other hand, that in blue-colored region is effective.

Fig. 11. Estimated effect of weights $\gamma$, for damping parameter $\lambda = 10^{-1} - 10$ and truncation degree of weighting function $s_0 = 1-20$. 
Fig. 10. Distribution and weights of 77 geomagnetic observatories. Numbers indicate priority of the observatories smaller than 20.

all the $\gamma$ for combinations of candidate locations and existing stations should be calculated in a similar way.

6.2 An application to determine priority of data

We determined the priority for processing of geomagnetic observatory data using the weights. There is a project to construct a geomagnetic database of hourly mean values at the World Data Center-C2 for Geomagnetism of Kyoto University. They plan to collect the data between 1960 and 1980 as a first step. Though a part of the hourly mean geomagnetic data have been already offered by CD-ROM or on the Internet from the U. S. National Geophysical Data Center, those data are not enough for deriving the Gauss coefficients. More data published by books, micro films, and micro fishes are necessary. Because inputting the data into computer is performed mostly by hand, setting a priority for the data is important to optimize the effort.

We estimated how the stations contribute to the problem of harmonic expansion of the geomagnetic data. We selected 77 points according to two criteria: 1) data of the observatories are mostly continuous between 1960 and 1980, and 2) the geomagnetic latitude is less than 60°. The distribution of the selected points is shown in Fig. 10 with solid circles. Setting the weighting function as Eq. (58), $s_0 = 1-20$, and ranges of $\lambda = 10^{-1}-10$, we calculated weights for the observatories. The estimated effect of the weights, $\gamma$, for each set of $\lambda$ and $s_0$ is shown in Fig. 11. The minimum value of $\gamma = 1.28$ is obtained for several sets of $(s_0, \lambda)$. We selected the lowest degree $s_0 = 7$, and the corresponding $\lambda = 2.0$. This choice of $\lambda$ is consistent to the theory, in which $\lambda$ is normalized to unity (Eq. (35)). Choice of higher $s_0$ gives similar results, while choice of low $s_0$ makes calculation unstable. The weights obtained for $(s_0, \lambda) = (7, 2.0)$ are shown in Fig. 10 with bars. They are large in the area with a few observatories and small in dense area such as Europe. The priority of data to be input to the computer was determined according to the weights, and is shown in Fig. 10 with numbers.

7. Summary and Discussion

We proposed a two-step scheme for spherical harmonic expansion. We first determine weights for the observation points, and secondary apply the usual least-squares inverse.
An advantage of our method is that the weights are determined independently of a priori information on the magnetic field. Especially the function given in Eq. (57) is selected in order to perform a uniform distribution at the core-mantle boundary. In stochastic inversion, on the other hand, the same function has been used as model covariance which is derived from a priori information. Also the damping parameter of Eq. (35) is easily determined, because both terms of Eq. (35) are normalized.

The first step of the proposed method is quite convenient for evaluating data distribution objectively and numerically. It is easy to distinguish if a distribution is suitable for global measurement using the spectrum of $\chi^2_s$ and the unevenness $\gamma$.

The applicability of the method is demonstrated by the two applications. For example, in the case of locating a new observation point in the ocean, change of the location after the installation of instruments is quite difficult. Therefore, it is important to estimate the validity of the location before installation. Though the location is limited by various conditions, it is preferable to set the instrument at the point with small $\gamma$ as possible.

There also is the case when numbers of observatory data cannot be supported because of budget or other reasons. In such case, the priority derived from the locations of the observatories becomes one of the indicators in addition to quality of data, local environments, and so on.

Even though we did not mention in this paper, there is another case in which the weighting method is applicable: the method is applicable to global analysis of temporally continuous data. For example, when we need a time series of the Gauss coefficients of past decades, there is a problem of changes of the distribution of geomagnetic observatories. In such case, the method can diminish the effect of a poor distribution of data, and derive homogeneous time series.

We thank Mr. Y. Sugiyama for his assistance in computation. We also thank Profs. R. A. Langel, M. Kono, and H. Tsunakawa for helpful comments on the manuscript. An information regarding available geomagnetic data was provided from the World Data Center-C2 for geomagnetism, Kyoto University. A part of this research was supported by grant-in-aid of the Japanese Ministry of Education, Science and Culture.

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