Periodic striped configurations in the large volume limit

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Abstract

We show striped pattern formation in the large volume limit for a class of generalized antiferromagnetic local/nonlocal interaction functionals in general dimension previously considered in [GR19; DR19; DR21a] and in [GLL06; GS16] in the discrete setting. In such a model the relative strength between the short range attractive term favouring pure phases and the long range repulsive term favouring oscillations is modulated by a parameter $\tau$. For $\tau < 0$ minimizers are trivial uniform states. It is conjectured that $\forall d \geq 2$ there exists $0 < \bar{\tau} \ll 1$ such that for all $0 < \tau \leq \bar{\tau}$ and for all $L > 0$ minimizers are striped/lamellar patterns. In [DR19] the authors prove the above for $L = 2kh^*_\tau$, where $k \in \mathbb{N}$ and $h^*_\tau$ is the optimal period of stripes for a given $0 < \tau \leq \bar{\tau}$. The purpose of this paper is to show the validity of the conjecture for generic $L$.

1 Introduction

In this paper we consider the following class of functionals.

For $d \geq 1$, $L > 0$, $\tau > 0$, $p \geq d + 2$, $\beta = p - d - 1$, $E \subset \mathbb{R}^d$ $[0, L]^d$-periodic set, $Q_L = [0, L]^d$, define

$$\mathcal{F}_{\tau, L}(E) = \frac{1}{L^d} \left( -\text{Per}_1(E, Q_L) + \int_{\mathbb{R}^d} K_\tau(\zeta) \left( \int_{\partial E \cap Q_L} |\nu^E(x)||\zeta_i|\,d\mathcal{H}^{d-1}(x) - \int_{Q_L} |\chi_E(x) - \chi_E(x + \zeta)|\,dx \right)\,d\zeta \right),$$

(1.1)

where $\text{Per}_1(E, Q_L) = \int_{\partial E \cap Q_L} ||\nu^E(x)||_1\,d\mathcal{H}^{d-1}(x)$ is the 1-perimeter of the set $E$ in the cube $Q_L$, defined through the 1-norm $||z||_1 = \sum_{i=1}^d |z_i|$, and $K_\tau(\zeta) = \tau^{-p/\beta} K_1(\zeta^{\tau^{-1/\beta}})$, where $K_1(\zeta) = \frac{1}{(||\zeta||_1+1)^{p}}$.

The functional (1.1) is obtained by suitably rescaling the local/nonlocal interaction functional

$$\bar{\mathcal{F}}_{J, L}(E) = J\text{Per}_1(E, [0, L]^d) - \int_{\mathbb{R}^d} \int_{Q_L} |\chi_E(x) - \chi_E(x + \zeta)|K_1(\zeta)\,dx\,d\zeta,$$

(1.2)

where $J = J_c - \tau$ and $J_c = \int_{\mathbb{R}^d} |\zeta_i|K_1(\zeta)\,d\zeta$ is a critical constant such that for $J > J_c$ minimizers of $\bar{\mathcal{F}}_{J, L}$ are trivial (i.e. $E = \emptyset$ or $E = \mathbb{R}^d$).

While for $\tau < 0$ minimizers are trivial, when $\tau = J_c - J$ is positive and small the competition between the short range attractive term of perimeter type and the long range repulsive term with power

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law interaction kernel in (1.2) causes the breaking of symmetry w.r.t. coordinate permutations and global minimizers are expected to be one-dimensional and periodic. Since the optimal period among periodic one-dimensional sets is of the order \( \tau^{-1/\beta} \), and the optimal energy scales like \( \tau^{(p-d)/\beta} \), in order to see striped patterns in a fixed box as \( \tau \to 0 \) it is convenient to rescale the functional (1.2) in such a way that optimal stripes have width and energy of order \( O(1) \), thus getting (1.1).

Showing symmetry breaking and pattern formation in more than one space dimensions for local/nonlocal models retaining symmetry w.r.t. some rotational group turns out to be a challenging problem, which up to now has been proved only in a few cases ([GS16; DR19; DR20; DKR19; DR21b; DR21a]) for functionals retaining symmetry w.r.t. coordinate permutations and symmetric domains.

Defining the family of one-dimensional sets of period \( L \) as

\[
C_L = \{ E \subset \mathbb{R}^d : \text{up to coordinate permutations } E = \hat{E} \times \mathbb{R}^{d-1} \text{ with } \hat{E} \subset \mathbb{R} \text{ L-periodic} \}
\]

(1.3)

and denoting the minimal energy attained by the functional in (1.1) on such sets as \( L \) varies as follows

\[
e_{\infty,\tau} = \inf_L \inf_{E \in C_L} F_{\tau,L}(E),
\]

(1.4)

in [DR19] the authors proved that there exists \( \bar{\tau} > 0 \) such that for every \( 0 < \tau \leq \bar{\tau} \) the minimal energy \( e_{\infty,\tau} \) is attained on periodic stripes with density \( 1/2 \) and period \( 2h_\tau^* \) for a unique \( h_\tau^* > 0 \) (the existence of a possibly non unique optimal period had been previously shown in [CO05; Mül93; RW03; GLL06]). By periodic stripes with density \( 1/2 \) and period \( 2h \) (simply called periodic unions of stripes of period \( 2h \) in [DR19]) we mean here sets which, up to permutations of coordinates and translations, are of the form

\[
E = \bigcup_{k \in \mathbb{Z}} [2kh, (2k+1)h) \times \mathbb{R}^{d-1}.
\]

(1.5)

Indeed, periodic stripes of density \( 1/2 \) and period \( 2h_\tau^* \) not only minimize \( F_{\tau,L} \) among one-dimensional sets in the class \( C_L \) when \( L = 2kh_\tau^* \), but they turn out to be the unique global minimizers of \( F_{\tau,L} \) among all \( [0, L)^d \)-periodic locally finite perimeter sets in \( \mathbb{R}^d \).

More precisely, in [DR19] the following result is proved

**Theorem 1.1.** Let \( d \geq 1, p \geq d+2 \) and \( h_\tau^* \) be the optimal stripes’ width for fixed \( \tau \). Then there exists \( \bar{\tau} > 0 \), such that for every \( 0 < \tau < \bar{\tau} \), one has that for every \( k \in \mathbb{N} \) and \( L = 2kh_\tau^* \), the minimizers \( E_\tau \) of \( F_{\tau,L} \) are optimal stripes of period \( 2h_\tau^* \) and density \( 1/2 \).

Hence, for every \( L = 2kh_\tau^* \) and \( 0 < \tau \leq \bar{\tau} \) with \( \bar{\tau} \) independent of \( k \), minimizers of \( F_{\tau,L} \) are, up to translations and permutation of coordinates, of the form (1.5).

A crucial point in the proof of Theorem 1.1 was that the large volume limit structure of minimizers (namely for a fixed \( 0 < \tau \ll 1 \) and then letting \( L \to +\infty \)) had to be performed on boxes whose sizes \( L \) are even multiples of the optimal width \( h_\tau^* \). On these boxes the energy of minimizers reaches the minimal value obtainable by periodic stripes. This allowed in [DR19] to bound from below the energy of the set \( E \) along one-dimensional slices in the different coordinate directions with the minimal energy density for periodic sets \( e_{\infty,\tau} \).

An important question, especially for the applications, that is left open in [DR19], is whether one can prove the above result for boxes of arbitrary size \( L \), not necessarily compatible with the optimal period. Indeed, in most applications the box size is predetermined by external factors.

In this paper we give a positive answer to the above question. More precisely, we prove the following
**Theorem 1.2.** Let $d \geq 1$, $p \geq d + 2$. Then there exist $\bar{\tau} > 0$ and $\bar{L} > 0$ such that for all $0 < \tau < \bar{\tau}$ and $L > 2\bar{L}$ the $L$-periodic minimizers $E_{\tau}$ of $F_{\tau,L}$ are optimal stripes of period $2h_{\tau,L}$ and density $1/2$, for some $h_{\tau,L} > 0$.

**Remark 1.3.** The range $0 < L \leq 2\bar{L}$ (more in general, $L$ smaller than any fixed constant) is contemplated in [DR19]/[Theorem 1.2]. Therefore Theorem 1.2 above completes the proof of the conjecture for all $L > 0$.

Among the many improvements, there are two main novelties in this paper:

- We are able to devise one-dimensional optimization estimates depending on the minimal energy relative to the length of the intervals where such estimates are performed. Instead in [DR19] global bounds involving the minimal energy density $e_{\infty,\tau}$ are used;

- As a consequence of length-dependent optimization estimates, on some intervals the energy could in principle be smaller the minimal energy density of the box $[0, L]^d$. We show that this is not the case, with a new argument that exploits the strict convexity of the energy density of optimal stripes w.r.t. their period. Such an argument enters in the proof of Theorem 1.2 in Section 5.3.

1.1 Scientific context

Patterns emerge at nanoscale level in several physical/chemical systems. Surprisingly similar patterns among which droplets or stripes/lamellae can be found in different systems, with different types of interactions. As pointed out in [SA95], the emergence of periodic regular structures is universally believed to stem from the competition between short range attractive and long range repulsive (SALR) interactions. Though observed in experiments and reproduced by simulations, a rigorous mathematical proof of pattern formation starting from symmetric functionals and domains in more than one space dimensions is available only in a very few cases. The main difficulties lie in the symmetry breaking phenomenon and in the nonlocality of the interactions.

Below we report a (non-exhaustive) series of contributions on periodic stripes formation for symmetric functionals and domains in suitable regimes. In other regimes of competition between the short-range attractive and long-range repulsive forces with small volume constraints, strong indications of the emergence of patterns consisting of isolated droplets have been provided (see e.g. [MK14; GMS13; GMS14; CS13]).

The one-dimensional setting is relatively well-understood. Periodicity of global minimizers is known to hold for convex or reflection positive repulsive kernels (see e.g. [Hub78; PU78; Ker99; Mül93; RW03; CO05; GLL06; GLL08; GLL09]).

In several space dimensions, the first characterization of ground states as periodic stripes was given in [GS16] for a discrete version of the functional (1.2) in the range of exponents $p > 2d$. In the continuous setting, breaking of symmetry w.r.t. permutation of coordinates for the functional (1.2) has been shown in [GR19] in the range of exponents $p > 2d$. The precise structure of minimizers of (1.2) (namely periodic stripes) in the wider range of exponents $p \geq d + 2$ was given in [DR19]. In [Ker21] such a characterization of minimizers was proved to hold also in a small open range of exponents below $d + 2$. We mention that in physical applications the power law interactions have exponents smaller than or equal to $d + 1$. This, together with the increased nonlocality of the problem, makes the problem of lowering the exponent of the kernel particularly interesting. In
it has been shown that also for repulsive kernels of screened-Coulomb (or Yukawa) type global minimizers are, in a suitable regime, periodic stripes. In [DKR19] the authors consider the diffuse interface version of the functional \( J \) in a finite periodic box and prove one-dimensionality and periodicity of minimizers. In [DR21b] the results in [DKR19] are proved to hold in the large volume limit on boxes whose sizes are even multiples of an optimal period. In [DR21a] a characterization of minimizers for the functional \( J \) under the imposition of an arbitrary volume constraint \( \alpha \in (0,1) \) was given. In the regime \( 0 < \tau \ll 1 \) and when \( L \gg 1 \) is an even multiple of the optimal period \( h_{\tau, \alpha}^* \) of simple periodic stripes with density \( \alpha \), minimizers among all \([0, L]^d\) periodic sets of density \( \alpha \) happen to be given by sets of the form

\[
E = \bigcup_{k \in \mathbb{Z}} [2kh_{\tau, \alpha}^*,(2k + 2\alpha)h_{\tau, \alpha}^*] \times \mathbb{R}^{d-1}.
\]  

Thus, even in the low density regime, stripes are the first type of pattern to emerge from the competition between attractive/repulsive forces in the range immediately below the critical constant \( J_c \).

Regarding further fields of interest for pattern formation under attractive/repulsive forces in competition, we mention the following. Evolution problems of gradient flow type related to functionals with attractive/repulsive nonlocal terms in competition, both in presence and in absence of diffusion, are also well studied (see e.g. [CCH14; CCP19; CDFLS11; CT20; Cra17; DRR20]). In particular, one would like to show stability of the gradient flows or of their deterministic particle approximations around configurations which are periodic or close to periodic states. Another interesting direction would be to extend our rigidity results to non-flat surfaces without interpenetration of matter as investigated for rod and plate theories in [KS14; LMP10; OR17].

## 2 Notation and preliminary results

Let \( d \geq 1 \). On \( \mathbb{R}^d \) let us denote by \( \langle \cdot, \cdot \rangle \) the Euclidean scalar product and by \( | \cdot | \) the Euclidean norm. Let \( e_1, \ldots, e_n \) be the canonical basis on \( \mathbb{R}^d \). We will often employ slicing arguments, for this reason we need definitions concerning the \( i \)-th component. For \( x \in \mathbb{R}^d \) let \( x_i := \langle x, e_i \rangle \) and \( x_i^+ := x - x_i e_i \). Let \( |x|_i = \sum_{i=1}^d |x_i| \) be the 1-norm and \( |x|_\infty = \max_i |x_i| \) the \( \infty \)-norm. While writing slicing formulas, with a slight abuse of notation we will sometimes identify \( x_i \in [0,L) \) with the point \( x_i e_i \in [0,L]^d \) and \( \{ x_i^+ : x \in [0,L]^d \} \) with \([0,L)^d \subset \mathbb{R}^{d-1} \) so that \( x_i^+ \in [0,L)^d \).

Whenever \( \Omega \subset \mathbb{R}^d \) is a measurable set, we denote by \( \mathcal{H}^{d-1}(\Omega) \) its \((d-1)\)-dimensional Hausdorff measure and by \(|\Omega|\) its Lebesgue measure.

Given a measure \( \mu \) on \( \mathbb{R}^d \), we denote by \(|\mu|\) its total variation.

We recall that a set \( E \subset \mathbb{R}^d \) is of (locally) finite perimeter if the distributional derivative of its characteristic function \( \chi_E \) is a (locally) finite measure. We denote by \( \partial E \) the reduced boundary of \( E \) and by \( \nu^E \) its exterior normal.

The anisotropic 1-perimeter of \( E \) is given by

\[
\text{Per}_1(E,[0,L)^d) := \int_{\partial E \cap [0,L)^d} \| \nu^E(x) \|_1 \, d\mathcal{H}^{d-1}(x)
\]

and, for \( i \in \{1, \ldots, d\} \)

\[
\text{Per}_{1i}(E,[0,L)^d) := \int_{\partial E \cap [0,L)^d} |\nu^E_i(x)| \, d\mathcal{H}^{d-1}(x),
\]

(2.1)
thus \( \text{Per}_1(E, [0, L]^d) = \sum_{i=1}^d \text{Per}_1(E, [0, L]^d) \).

For \( i \in \{1, \ldots, d\} \), we define the one-dimensional slices of \( E \subset \mathbb{R}^d \) in direction \( e_i \) by

\[
E_{x_i^+} := \{ s \in [0, L) : se_i + x_i^+ \in E \}.
\]

Whenever \( E \) is a set of locally finite perimeter, for a.e. \( x_i^+ \) its slice \( E_{x_i^+} \) is a set of locally finite perimeter in \( \mathbb{R} \) and the following slicing formula holds for every \( i \in \{1, \ldots, d\} \)

\[
\text{Per}_1(E, [0, L]^d) = \int_{\partial E \cap [0, L]^d} |\nu_i^E(x)| \, d\mathcal{H}^{d-1}(x) = \int_{[0, L)^{d-1}} \text{Per}(E_{x_i^+}, [0, L]) \, dx_i^+. \]

Consider \( E \subset \mathbb{R} \) a set of locally finite perimeter and \( s \in \partial E \) a point in the relative boundary of \( E \). We will denote by

\[
s^+ := \inf\{ t' \in \partial E, \text{ with } t' > s \} \\
s^- := \sup\{ t' \in \partial E, \text{ with } t' < s \}. \tag{2.2}
\]

We will also apply slicing on small cubes, depending on \( l \), around a point. Therefore we introduce the following notation. For \( r > 0 \) and \( x_i^+ \) we let \( Q_r(x_i^+) = \{ z_i^+ : \|z_i^+ - x_i^+\|_\infty \leq r \} \) or we think of \( x_i^+ \in [0, L)^{d-1} \) and \( Q_r(x_i^+) \) as a subset of \( \mathbb{R}^{d-1} \). We denote also by \( Q_r(t_i) \subset \mathbb{R} \) the interval of length \( r \) centred in \( t_i \).

From [GR19, DR19] we recall that, using the equality \( |\chi_E(x) - \chi_E(x + \zeta)| = |\chi_E(x - \zeta e_i) - \chi_E(x) - 2\zeta e_i| \chi_E(x + \zeta e_i) - \chi_E(x + \zeta)| \) and \( Q_L \)-periodicity, the following lower bounds hold.

\[
\mathcal{F}_{\tau, L}(E) \geq \frac{1}{L^d} \sum_{i=1}^d \text{Per}_1(E, [0, L]^d) + \frac{1}{L^d} \sum_{i=1}^d \left[ \int_{[0, L)^{d-1}} |\nu_i^E(x)| \, K_\tau(\zeta) \, d\zeta \, d\mathcal{H}^{d-1}(x) \right. \\
- \int_{[0, L)^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta e_i) - \chi_E(x)| \, K_\tau(\zeta) \, d\zeta \, dx \\
+ \frac{2}{d} \frac{1}{L^d} \sum_{i=1}^d \int_{[0, L)^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta e_i) - \chi_E(x)| \chi_E(x + \zeta^+) - \chi_E(x) \, K_\tau(\zeta) \, d\zeta \, dx. \tag{2.3}
\]

Notice that in \( \text{Eq. (2.3)} \) equality holds whenever the set \( E \) is a union of stripes. Thus, proving that unions of stripes with density 1/2 are the minimizers of the r.h.s. of \( \text{Eq. (2.3)} \) implies that they are the minimizers for \( \mathcal{F}_{\tau, L} \).

Let us define

\[
\tilde{K}_\tau(\zeta_i) = \int_{\mathbb{R}^{d-1}} K_\tau(\zeta_i) \, d\zeta_i^+. \tag{2.4}
\]

As in Section 7 of [DR19] we further decompose the r.h.s. of \( \text{Eq. (2.3)} \) as follows.

\[
- \frac{1}{L^d} \text{Per}_1(E, [0, L]^d) + \frac{1}{L^d} \left[ \int_{[0, L)^{d-1}} |\nu_i^E(x)| \, K_\tau(\zeta) \, d\zeta \, d\mathcal{H}^{d-1}(x) \right. \\
- \int_{[0, L)^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta e_i) - \chi_E(x)| \, K_\tau(\zeta) \, d\zeta \, dx \\
- \frac{1}{L^d} \int_{[0, L)^{d-1}} \sum_{s \in \partial E_{x_i^+} \cap [0, L]} r_{i,s}(E, t_i, s) \, dt_i^+. \tag{2.4}
\]

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where for $s \in \partial E_{i_t}^+$

$$r_{i,\tau}(E, t_{i_t}^+, s) := -1 + \int_\mathbb{R} |\zeta_i \tilde{K}_\tau(\zeta_i)| \, d\zeta_i - \int_{s}^{s^+} \int_0^{\infty} |\chi_{E_{i_t}^+}(u + \rho) - \chi_{E_{i_t}^+}(u)| \tilde{K}_\tau(\rho) \, d\rho \, du - \int_{s}^{s^+} \int_{-\infty}^{0} |\chi_{E_{i_t}^+}(u + \rho) - \chi_{E_{i_t}^+}(u)| \tilde{K}_\tau(\rho) \, d\rho \, du.$$  \hspace{1cm} (2.5)

and $s^- < s < s^+$ are as in (2.2).

Defining

$$f_E(t_{i_t}^+, t_i, \zeta_i^+, \zeta_i) := |\chi_E(t_i e_i + t_{i_t}^+ + \zeta_i e_i) - \chi_E(t_i e_i + t_{i_t}^+ + \zeta_i^+) - \chi_E(t_i e_i + t_{i_t}^+)|,$$

one has that

$$\frac{2}{L^d} \sum_{i=1}^{d} \int_{[0,L]^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta_i e_i) - \chi_E(x)||\chi_E(x + \zeta_i^+) - \chi_E(x)| \tilde{K}_\tau(\zeta) \, d\zeta \, dx =$$

$$= \frac{2}{L^d} \int_{[0,L]^d} \int_{\mathbb{R}^d} f_E(t_{i_t}^+, t_i, \zeta_i^+, \zeta_i) \tilde{K}_\tau(\zeta) \, d\zeta \, dt$$

$$= \frac{1}{L^d} \int_{[0,L]^{d-1}} \sum_{s \in \partial E_{i_t}^+ \cap [0,L]} v_{i,\tau}(E, t_{i_t}^+, t_i) \, dt_{i_t}^+ + \frac{1}{L^d} \int_{[0,L]^d} w_{i,\tau}(E, t_{i_t}^+, t_i) \, dt,$$  \hspace{1cm} (2.7)

where

$$w_{i,\tau}(E, t_{i_t}^+, t_i) = \frac{1}{d} \int_{\mathbb{R}^d} f_E(t_{i_t}^+, t_i, \zeta_i^+, \zeta_i) \tilde{K}_\tau(\zeta) \, d\zeta.$$  \hspace{1cm} (2.8)

and

$$v_{i,\tau}(E, t_{i_t}^+, s) = \frac{1}{2d} \int_{s^-}^{s^+} \int_{\mathbb{R}^d} f_E(t_{i_t}^+, u, \zeta_i^+, \zeta_i) \tilde{K}_\tau(\zeta) \, d\zeta \, du.$$  \hspace{1cm} (2.9)

Hence, putting together (2.4) and (2.7) one has the following decomposition

$$\mathcal{F}_{\tau,L}(E) \geq \frac{1}{L^d} \int_{[0,L]^{d-1}} \sum_{s \in \partial E_{i_t}^+ \cap [0,L]} r_{i,\tau}(E, t_{i_t}^+, s) \, dt_{i_t}^+$$

$$+ \frac{1}{L^d} \int_{[0,L]^{d-1}} \sum_{s \in \partial E_{i_t}^+ \cap [0,L]} v_{i,\tau}(E, t_{i_t}^+, s) \, dt_{i_t}^+$$

$$+ \frac{1}{L^d} \int_{[0,L]^d} w_{i,\tau}(E, t_{i_t}^+, t_i) \, dt.$$  \hspace{1cm} (2.10)

The term $r_{i,\tau}$ penalizes oscillations with high frequency in direction $e_i$, namely sets $E$ whose slices in direction $e_i$ have boundary points at small minimal distance (see Lemma 4.1). The term $v_{i,\tau}$ penalizes oscillations in direction $e_i$ whenever the neighbourhood of the point in $\partial E \cap Q_l(z)$ is close in $L^1$ to a stripe oriented along $e_j$ (see Proposition 4.5).
For every cube $Q_l(z)$, with $l < L$ and $z \in [0, L]^d$, define now the following localization of $\mathcal{F}_{\tau,L}$

$$
\bar{F}_{i,\tau}(E, Q_l(z)) := \frac{1}{l^d} \left[ \int_{Q_l(z)} \sum_{s \in \partial E_{t_i^+}} (v_{i,\tau}(E, t_i^+, s) + r_{i,\tau}(E, t_i^+, s)) \, dt_i^+ + \int_{Q_l(z)} w_{i,\tau}(E, t_i^+, t_i) \, dt_i^+ \right],
$$

$$
\bar{F}_{\tau}(E, Q_l(z)) := \sum_{i=1}^d \bar{F}_{i,\tau}(E, Q_l(z)).
$$

(2.11)

The following inequality holds:

$$
\mathcal{F}_{\tau,L}(E) \geq \frac{1}{L^d} \int_{[0,L]^d} \bar{F}_{\tau}(E, Q_l(z)) \, dz.
$$

(2.12)

Since in (2.12) equality holds for unions of stripes, in order to prove Theorem 1.2 one can reduce to show that the minimizers of its right hand side are periodic optimal stripes of density $1/2$ provided $\tau$ and $L$ satisfy the conditions of the theorem.

In the next definition we define a quantity which measures the $L^1$ distance of a set from being a union of stripes.

**Definition 2.1.** For every $\eta$ we denote by $A^i_\eta$ the family of all sets $F$ such that

(i) they are union of stripes oriented along the direction $e_i$

(ii) their connected components of the boundary are distant at least $\eta$.

We denote by

$$
D^i_\eta(E, Q) := \inf \left\{ \frac{1}{\text{vol}(Q)} \int_Q |\chi_E - \chi_F| : F \in A^i_\eta \right\} \quad \text{and} \quad D_\eta(E, Q) = \inf_i D^i_\eta(E, Q). \quad (2.13)
$$

Finally, we let $A_\eta := \cup_i A^i_\eta$.

We recall also the following properties of the functional defined in (2.13).

**Remark 2.2.** The distance function from the set of stripes satisfies the following properties.

(i) Let $E \subset \mathbb{R}^d$. Then the map $z \mapsto D_\eta(E, Q_l(z))$ is Lipschitz, with Lipschitz constant $C_d/l$, where $C_d$ is a constant depending only on the dimension $d$.

In particular, whenever $D_\eta(E, Q_l(z)) > \alpha$ and $D_\eta(E, Q_l(z')) < \beta$, then $|z - z'| > l(\alpha - \beta)/C_d$.

(ii) For every $\varepsilon$ there exists $\delta_0 = \delta_0(\varepsilon)$ such that for every $\delta \leq \delta_0$ whenever $D^i_\eta(E, Q_l(z)) \leq \delta$ and $D^i_\eta(E, Q_l(z)) \leq \delta$ with $i \neq j$ for some $\eta > 0$, it holds

$$
\min \left( |Q_l(z) \setminus E|, |E \cap Q_l(z)| \right) \leq \varepsilon.
$$
3 Key features of the one-dimensional problem

In this section we study properties of the energy functional $F_{\tau,L}$ on one-dimensional sets, namely sets belonging to the set $C_L$ defined in (1.3), and of its minimizers. The main result of this section, which will be used in the proof of Theorem 1.2 is Theorem 3.5.

As recalled in Section 1, letting $e_{\infty,\tau}$ be the minimal value obtained by $F_{\tau,L}$ on the class of sets $C_L$ as $L$ varies (see (1.4)) and

$$E_h = \bigcup_{k \in \mathbb{Z}} [2kh, (2k+1)h) \times \mathbb{R}^{d-1},$$

one has the following

**Theorem 3.1** ([DR19] Theorem 1.1). There exists $\tau_0 > 0$ such that for all $0 \leq \tau \leq \tau_0$ there exists a unique $h_\tau^* > 0$ such that

$$e_{\infty,\tau} = F_{\tau,2h_\tau^*}(E_h^*) = \inf_h F_{\tau,2h}(E_h).$$

(3.1)

The existence of at least one finite optimal period for the one-dimensional problem has been proved in [CO05; Müll93; RW03] using convexity arguments and in [GLL06] using reflection positivity techniques. The uniqueness of the optimal period for sufficiently small $\tau$ has been proved in [DR19] for a slightly more general class of potentials with power law scaling and reflection positivity properties.

Let now for simplicity of notation define

$$e_\tau(h) = F_{\tau,2h}(E_h).$$

(3.2)

In particular, by Theorem 3.1

$$e_{\infty,\tau} = e_\tau(h_\tau^*) = \inf_h e_\tau(h).$$

As computed in [GR19; Ker21], one has the following formula for the energy of stripes of width and distance $h$

$$e_\tau(h) = \frac{1}{h} + \frac{C(\tau^{1/\beta}/h)}{h^{q-1}},$$

(3.3)

where

$$C(s) = \frac{2C_1}{(q-1)(q-2)} \left\{ \sum_{k \geq 0} \frac{2}{(2k+1+s)^{q-2}} - \frac{2}{(2k+2+s)^{q-2}} \right\}$$

(3.4)

and

$$C_1 = \int_{\mathbb{R}^{d-1}} \frac{1}{(\|\xi\|_1+1)^p} \, d\xi.$$  

For a complete and detailed proof of (3.3) and (3.4) we refer to [GR19; Ker21].

Before stating and proving Theorem 3.5 we need a series of preliminary results.

**Lemma 3.2.** Let $0 < \varepsilon \ll 1$. There exists $\bar{\tau}_1 > 0$ and $0 < \bar{c}_1 < \bar{c}_2$, $\bar{c}_3 > 0$ such that for all $0 \leq \tau \leq \bar{\tau}_1$ and for all $h > 0$ such that

$$e_\tau(h) \leq e_0(h_0^*) + \varepsilon$$

(3.5)

it holds

$$\bar{c}_1 \leq h \leq \bar{c}_2$$

(3.6)

and

$$\partial_h^2 e_\tau(h) \geq \bar{c}_3$$

(3.7)
The above lemma implies that whenever for $\tau$ sufficiently small the energy of some periodic stripes of period $h$ is close to the minimal energy for $\tau = 0$ as in (3.5) (which we will see it is the case for the stripes of minimal period in Lemma 3.4), then $h$ must lie in some given interval (3.6) on which the function $e_{\tau}$ is strictly convex (3.7).

**Proof. Step 1** First of all, we prove Lemma 3.2 for $\tau = 0$.

By explicit computations, one can see that the unique minimum and stationary point of $e_0$ is given by

$$h_0^* = \left((q-1)C(0)\right)^{\frac{1}{q-2}}.$$

Moreover, from the explicit formulas (3.3) and (3.4) one can directly see that there exist $0 < \bar{c}_1 < \bar{c}_2$ such that

$$e_0(h) \leq e_0(h_0^*) + \frac{3}{2} \varepsilon \Rightarrow \bar{c}_1 \leq h \leq \bar{c}_2. \quad (3.8)$$

Indeed,

$$e_0(h_0^*) = -\frac{q-2}{(q-1)^{\frac{q-2}{q-2}}} = -\frac{q-2}{q-1}C(0)^{-\frac{1}{q-2}},$$

thus the left inequality in (3.8) becomes

$$-h^{q-2} \leq -C(0) - \left[\bar{C}C_0^{-\frac{1}{q-2}} - \frac{3}{2} \varepsilon \right]h^{q-1}. \quad (3.9)$$

Hence, the bounds in the r.h.s. of (3.8) follow from the inequalities $-h^{q-2} \leq -C(0)$ and $-h^{q-2} \leq -\left[\bar{C}C_0^{-\frac{1}{q-2}} - \frac{3}{2} \varepsilon \right]h^{q-1}$.

Moreover, choosing eventually $\bar{c}_1$ and $\bar{c}_2$ relative to a smaller $\varepsilon$ in (3.8), there exists $\bar{c}_3 > 2\bar{c}_3 > 0$ such that

$$\bar{c}_1 \leq h \leq \bar{c}_2 \Rightarrow \bar{c}_3 \geq \partial_h^2 e_0(h) \geq 2\bar{c}_3. \quad (3.10)$$

Indeed, one has that

$$\partial_h^2 e_0(h) = -\frac{2h^{q-2} + q(q-1)C(0)}{h^{q+1}}.$$ 

and

$$(h_0^*)^{q-2} = (q-1)C(0).$$

In particular,

$$\partial_h^2 e_0(h_0^*) = (q-1)^{-\frac{3}{q-2}}(q-2)C(0)^{-\frac{3}{q-2}}.$$

Hence also (3.10) holds.

**Step 2** Let us now estimate the difference between $e_{\tau}$ and $e_0$ and between their derivatives for small values of $\tau$. First of all we claim that there exist $\bar{c}_4 > 0$ and $\bar{\tau} > 0$ such that $\bar{c}_4 < h_0^*$ and for all $0 \leq \tau \leq \bar{\tau}$ one has that

$$e_{\tau}(h) < 0 \Rightarrow h \geq \bar{c}_4. \quad (3.11)$$
Moreover, by comparing the expression for \( C(\tau^{1/\beta}/h) \) with the one for \( C(0) \) one has that there exist \( \bar{c}_5, \bar{c}_6, \bar{c}_7 > 0 \) such that for all \( h \geq \bar{c}_4 \) it holds

\[
|e_\tau(h) - e_0(h)| \leq \bar{c}_5 \tau^{1/\beta}, \\
|\partial_h e_\tau(h) - \partial_h e_0(h)| \leq \bar{c}_6 \tau^{1/\beta}, \\
|\partial_h^2 e_\tau(h) - \partial_h^2 e_0(h)| \leq \bar{c}_7 \tau^{1/\beta}.
\]

While (3.11) and (3.12) follow from the formula for \( e_\tau(h) \), in the proof of (3.13) and (3.14) one uses the formulas

\[
\partial_h e_\tau(h) = \frac{1}{h^2} - (q - 1) \frac{C(\tau^{1/\beta}/h)}{h^q} - \left( \frac{\tau^{1/\beta}}{h} \right) \frac{\partial_s C(\tau^{1/\beta}/h)}{h^q} \\
\partial_h^2 e_\tau(h) = -\frac{2}{h^3} + q(q - 1) \frac{C(\tau^{1/\beta}/h)}{h^{q+1}} \\
+ (q - 1) \left( \frac{\tau^{1/\beta}}{h} \right) \frac{\partial_s C(\tau^{1/\beta}/h)}{h^{q+1}} + \left( \frac{\tau^{1/\beta}}{h} \right)^2 \frac{\partial_s^2 C(\tau^{1/\beta}/h)}{h^{q+1}} \\
+ q \left( \frac{\tau^{1/\beta}}{h} \right) \frac{\partial_s C(\tau^{1/\beta}/h)}{h^{q+1}}.
\]

**Step 3** We are now ready to prove (3.10). Let \( 0 < \varepsilon \ll 1 \) to be fixed later. By (3.11) and (3.12), there exists \( 0 < \bar{\tau} \) such that whenever \( 0 \leq \tau \leq \bar{\tau} \) and \( e_\tau(h) < 0 \) then \( |e_\tau(h) - e_0(h)| \leq \varepsilon/2 \). In particular,

\[
e_\tau(h) \leq e_0(h_0^*) + \varepsilon \quad \Rightarrow \quad e_0(h) \leq e_0(h_0^*) + \frac{3}{2} \varepsilon.
\]

Thus, by (3.8), one has that \( \bar{c}_1 \leq h \leq \bar{c}_2 \). Provided \( \varepsilon, \bar{\tau} \) are sufficiently small, one can assume that

\[
\bar{c}_1 \leq \bar{c}_1 < h < \bar{c}_2.
\]

Using (3.10) and (3.14) one obtains

\[
\partial_h^2 e_\tau(h) \geq \partial_h^2 e_0(h) - |\partial_h^2 e_\tau(h) - \partial_h^2 e_0(h)| \\
\geq 2\bar{c}_3 - \bar{c}_3 \tau^{1/\beta} \\
\geq \bar{c}_3 > 0
\]

In particular, one has the strict convexity of \( e_\tau \) on the region where the minimizers are concentrated, thus proving (3.7). \( \square \)

From [GR19] we recall also the following result.

**Theorem 3.3.** There exists \( C > 0 \) and \( \bar{\tau}_2 > 0 \) such that for every \( 0 \leq \tau < \bar{\tau}_2 \) and for every \( L > 0 \), the minimizers of \( F_{\tau,L} \) in the class \( C_L \) are periodic stripes of period \( h_{\tau,L} \) for some (possibly non-unique) \( h_{\tau,L} > 0 \) satisfying

\[
|h_{\tau,L} - h_{\tau}^*| \leq \frac{C}{L}.
\]
Lemma 3.4. One has the following:

1. \[ \lim_{L \to +\infty} e_\tau(h_{\tau,L}) = \inf_{L} \inf_{h \in \mathbb{L}/2\mathbb{N}} e_\tau(h) = e_\tau(h_\tau^*) \] (3.19)

2. For any \( \varepsilon > 0 \) and \( c > 0 \) there exists \( \bar{\tau}_3 > 0 \) such that for all \( 0 \leq \tau \leq \bar{\tau}_3 \) it holds
\[ e_\tau(h_\tau^*) \leq e_0(h_0^*) + c\varepsilon. \] (3.20)

As a direct consequence of Theorem 3.3 of Lemma 3.4 and Lemma 3.2 one obtains the following

Theorem 3.5. Let \( 0 < \varepsilon \ll 1 \). There exist \( \bar{\tau}_4 \leq \min\{\bar{\tau}_0, \ldots, \bar{\tau}_3\} \), \( \bar{L} > 0 \) such that for all \( 0 \leq \tau \leq \bar{\tau}_4 \) and for all \( L \geq \bar{L} \)
\[ \bar{c}_1 < h_{\tau,L} < \bar{c}_2 \quad \text{and} \quad \partial^2 h e_\tau(h_{\tau,L}) \geq \bar{c}_3 > 0. \] (3.21)
where \( \bar{c}_1, \bar{c}_2 \) and \( \bar{c}_3 \) are defined in Lemma 3.2.
Moreover, the following holds:
\[ |\partial h e_\tau(h_{\tau,L})| \leq \varepsilon, \quad |e_\tau(h_{\tau,L}) - e_\tau(h_\tau^*)| \leq \frac{\varepsilon C}{L}, \] (3.22)
where \( C \) is the constant appearing in (3.18).

4 Preliminary lemmas

In this section we collect a series of Lemmas and Propositions which will be used in the proof of Theorem 1.2. In the one-dimensional optimization Lemma 4.6 and in Lemma 4.8 we will have now to take into account the minimal energy density for periodic sets of period compatible with the length of the interval on which the optimization takes place.

At this aim we introduce the following notation: for all intervals \( I \subset \mathbb{R} \), we define
\[ h_\tau(I) = \arg\min\{e_\tau(h) : h \in |I|/(2\mathbb{N})\}. \]

In general, \( h_\tau(I) \) might contain different periods \( h \) giving all the same energy \( e_\tau(h) \). Whenever, with a slight abuse of notation, we will be speaking of \( h_\tau(I) \) as if it were a single period is because the properties of all the periods contained in \( h_\tau(I) \) are in that case equivalent (meaning that that all \( h \in h_\tau(I) \) have the same property).

We start with recalling the following lemma, corresponding to Remark 7.1 in [DR19]. The term \( r_{i,\tau} \) penalizes small sets \( E \) whose one-dimensional slices in direction \( e_i \) have boundary points which are close to each other. This is expressed quantitatively by the estimate (4.1).

Lemma 4.1. There exist \( \eta_0 > 0 \) and \( \tau_0 > 0 \) such that for every \( 0 < \tau < \tau_0 \), whenever \( E \subset \mathbb{R}^d \) and \( s^- < s < s^+ \in \partial E_{t_i^+} \) are three consecutive points satisfying \( \min(|s - s^-|, |s^+ - s|) < \eta_0 \), then \( r_{i,\tau}(E, t_i^+, s) > 0 \).

In particular, the following estimate holds
\[ r_{i,\tau}(E, t_i^+, s) \geq -1 + C_1 C_2 \min(|s - s^+|^{-\beta}, \tau^{-1}) + C_1 C_2 \min(|s - s^-|^{-\beta}, \tau^{-1}) \] (4.1)
where \( C_1 = \int_{\mathbb{R}^{d-1}} \frac{1}{\|(\xi, 0)\|^{q-1}} \, d\xi \) and \( C_2 = \frac{1}{(q-1)(q-2)} \).
Corollary 4.4. In particular, one has the following $\Gamma$-convergence result.

\[
\lim_{\tau \to 0} \int_{\mathbb{R}} |\rho| \mathcal{K}_\tau(\rho) \, d\rho - \int_{s^-}^{s^+} \int_0^{+\infty} |\chi_{E}(\rho + u) - \chi_{E}(u)| \mathcal{K}_\tau(\rho) \, d\rho \, du \\
- \int_{s^-}^{s^+} \int_{-\infty}^{0} |\chi_{E}(\rho + u) - \chi_{E}(u)| \mathcal{K}_\tau(\rho) \, d\rho \, du.
\]

The quantities defined in (2.5) and (4.2) are related via $r_{i,\tau}(E, t_{i}^\perp, s) = r_{\tau}(E_{t_{i}^\perp}, s)$.

In the next Lemma we recall Lemma 7.5 in [DR19], containing a lower bound for the first term of the decomposition (2.10) as $\tau \to 0$. Thanks to the inequality (4.3) the penalization of close boundary points for a family of sets $E_\tau$ is preserved in the limit as $\tau \to 0$.

**Lemma 4.2.** Let $E_0, \{E_\tau\} \subset \mathbb{R}$ be a family of sets of locally finite perimeter and $I \subset \mathbb{R}$ be an open bounded interval. Moreover, assume that $E_\tau \to E_0$ in $L^1(I)$. If we denote by $\{k_i^0, \ldots, k_{m_0}^0\} = \partial E_0 \cap I$, then

\[
\liminf_{\tau \to 0} \sum_{s \in \partial E_\tau} r_{\tau}(E_\tau, s) \geq \sum_{i=1}^{m_0-1} (-1 + C_1 C_2 |k_i^0 - k_{i+1}^0|^{-1}),
\]

where $r_{\tau}$ is defined in (4.2).

The next proposition contains the main symmetry breaking result at mesoscopic scale $l$: on a square of size $l$, if $\tau$ is sufficiently close to 0, a bound on the energy corresponds to a bound on the $L^1$-distance to the unions of stripes. It corresponds to Lemma 7.6 in [DR19]. In the limit as $\tau \to 0$, for any $p \geq d+2$ sets of bounded energy have to be exactly stripes, via a rigidity argument that uses (4.3) and a lower bound on the cross interaction term $w_{\tau}$.

**Proposition 4.3** (Local Rigidity). For every $M > 1, l, \delta > 0$, there exist $\tau_1 > 0$ and $\bar{\eta} > 0$ such that whenever $0 < \tau < \tau_1$ and $\bar{F}_\tau(E, Q_l(z)) < M$ for some $z \in [0, L]^d$ and $E \subset \mathbb{R}^d [0, L]^d$-periodic, with $L > l$, then it holds $D_{\eta}(E, Q_l(z)) \leq \delta$ for every $\eta < \bar{\eta}$. Moreover $\bar{\eta}$ can be chosen independently of $\delta$. Notice that $\tau_1$ and $\bar{\eta}$ are independent of $L$.

In particular, one has the following $\Gamma$-convergence result.

**Corollary 4.4.** Let $0 < \tau \ll 1$. One has that the following holds:

- Let $\{E_\tau\}$ be a sequence such that $\sup_\tau \bar{F}_\tau(E_\tau, Q_l(z)) < \infty$. Then as $\tau \to 0$ the sets $E_\tau$ converge in $L^1$ up to subsequences to some set $E_0$ of finite perimeter and

\[
\liminf_{\tau \to 0} \bar{F}_\tau(E_\tau, Q_l(z)) \geq \bar{F}_0(E_0, Q_l(z)).
\]

- For every set $E_0$ with $\bar{F}_0(E_0, Q_l(z)) < +\infty$, there exists a sequence $\{E_\tau\}$ converging in $L^1$ to $E_0$ as $\tau \to 0$ and such that

\[
\limsup_{\tau \to 0} \bar{F}_\tau(E_\tau, Q_l(z)) = \bar{F}_0(E_0, Q_l(z)).
\]
The following local stability proposition corresponds to Lemma 7.8 in [DR19]. Roughly speaking, it shows that whenever a set $E \subset \mathbb{R}^d$ is $L^1$-close to a set $S$ which is a union of stripes with boundaries orthogonal to $e_i$ in a certain cube, then it is not energetically convenient for the set $E$ to have non-straight boundaries in direction $e_j$ with $j \neq i$ (namely, to deviate from being exactly stripes with boundaries orthogonal to $e_i$) Indeed, in such a case either the local contribution given by $r_{i,\tau}$ or the one given by the cross interaction term $v_{i,\tau}$ are large.

**Proposition 4.5 (Local Stability).** Let $(t_i^+ + se_i) \in (\partial E) \cap [0,l]^d$, and $\tau_0$, $\tau_0$ as in Lemma 4.1. Then there exist $\tau_2 \leq \tau_0$ and $\varepsilon_2$ (independent of $l$) such that for every $0 < \tau < \tau_2$, and $0 < \varepsilon < \varepsilon_2$ the following holds: assume that

(a) $\min(|s - l|,|s|) > \eta_0$ (i.e. the boundary point $s$ in the slice of $E$ is sufficiently far from the boundary of the cube)

(b) $D_0^1(E,[0,l]^d) \leq \frac{C}{\varepsilon^2}$ for some $\eta > 0$ and with $j \neq i$ (i.e. $E \cap [0,l]^d$ is close to stripes with boundaries orthogonal to $e_j$ for some $j \neq i$)

Then

$$r_{i,\tau}(E,t_i^+,s) + v_{i,\tau}(E,t_i^+,s) \geq 0.$$ 

As shown in [Ker21], the above stability argument can be extended to all $p > d + 1$, provided $\tau$ is sufficiently small depending on $p$.

In the following lemma we show the main one-dimensional estimate needed in the proof of Lemma 4.8. Roughly speaking, it shows that up to an error term (i.e. the constant $C_0$ in (4.6)), the contribution of the one-dimensional term $r_\tau(E,s)$ on an interval $I$ is bounded from below by the minimal energy density for periodic sets of period $|I|$. In particular, the estimate below takes into account minimal energy relative to the length of the interval on which it is performed, thus differing from previous optimization estimates obtained in [DR19] in which the global minimum for the energy density over all possible periods was considered. Notice that it is valid for sufficiently large intervals. This will not be a restriction since we are interested in proving optimality of striped patterns in the large volume limit.

**Lemma 4.6.** There exists $C_0 > 0$ such that the following holds. Let $E \subset \mathbb{R}$ be a set of locally finite perimeter and $I \subset \mathbb{R}$ be an open interval such that $|I| > \bar{L}$ and $\bar{L}$ is as in Theorem 3.5. Let $r_\tau(E,s)$ be defined as in (4.2). Then for all $0 < \tau < \min\{\tau_0, \tilde{\tau}_1\}$, where $\tau_0$ is given in Lemma 4.1 and $\tilde{\tau}_1$ is given in Theorem 2.5, it holds

$$\sum_{s \in \partial E, s \in I} r_\tau(E,s) \geq |I|e_\tau(h_\tau(I)) - C_0. \quad (4.6)$$

**Proof.** Let us denote by $k_1 < \ldots < k_m$ the points of $\partial E \cap I$, and

$$k_0 = \sup\{s \in \partial E : s < k_1\} \quad \text{and} \quad k_{m+1} = \inf\{s \in \partial E : s > k_m\}$$

W.l.o.g. we may assume that $r_{\tau}(E,k_1) < 0$ and that $r_{\tau}(E,k_m) < 0$. We claim that if this is not the case one can consider $I' \subset I$ such that $r_{\tau}(E,k'_1) < 0$ and $r_{\tau}(E,k'_m) < 0$, where $k'_1, \ldots, k'_m$ are the points of $\partial E \cap I'$. Indeed, if estimate (4.6) holds for $I'$ then one has the following chain of inequalities

$$\sum_{s \in \partial E, s \in I} r_{\tau}(E,s) \geq \sum_{s \in \partial E, s \in I'} r_{\tau}(E,s) \geq e_\tau(h_\tau(I'))|I'| - C_0. \quad (4.7)$$
Moreover, by Theorem 3.5 there exists \( h \in h_\tau(I) \) for some interval \( I \) with \( |I| \geq \bar{L} \) satisfying \( e_\tau(h_\tau(I)) \leq e_\tau(h_\tau(I')). \) Identifying with a slight abuse of notation such \( h \) with \( h_\tau(I) \), by Theorem 3.3 and Theorem 3.5 one has that
\[
h_\tau(I), h_\tau(I) \in [\bar{c}_1, \bar{c}_2], |h_\tau(I) - h_\tau(I)| \leq \frac{C}{\min\{|I|, |I'\}} \tag{4.8}
\]
and
\[
\partial_h^2 e_\tau(h) \geq \bar{c}_3 > 0 \quad \text{on} \ [\bar{c}_1, \bar{c}_2]. \tag{4.9}
\]
Then one has that
\[
e_\tau(h_\tau(I'))|I'| \geq e_\tau(h_\tau(I))|I'| \geq e_\tau(h_\tau(I))|I'| + (h_\tau(I) - h_\tau(I))\partial_h e_\tau(h_\tau(I))|I'|
\]
\[
\geq e_\tau(h_\tau(I))|I| + (h_\tau(I) - h_\tau(I))\partial_h e_\tau(h_\tau(I))|I'|, \tag{4.10}
\]
where in the last inequality we used the fact that \( |I| \geq |I'| \) and that \( e_\tau(h_\tau(I)) < 0 \). Now observe that by formula (3.15) on the interval \([\bar{c}_1, \bar{c}_2]\) of Theorem 3.5 the function \( \partial_h e_\tau(h) \) is uniformly bounded, i.e. \( |\partial_h e_\tau(h)| \leq \bar{C} \). This fact together with (4.8) implies the following
\[
e_\tau(h_\tau(I'))|I'| \geq e_\tau(h_\tau(I))|I| - \frac{C}{\min\{|I|, |I'\}}\bar{C}|I'|
\]
\[
\geq e_\tau(h_\tau(I))|I| - \bar{C}C, \tag{4.11}
\]
where in passing from (4.11) to (4.12) we used the fact that \( |I'| < |I| \) and that \( I \) with optimal compatible period \( h_\tau(I) \) can be chosen to be such that \( |I| = |I'| + O(1) \).

Thus from (4.7) and (4.12) it follows that, eventually enlarging the constant \( \bar{C}_0 \) in (4.6), the main estimate (4.6) is valid also for the interval \( I \) whenever it is valid for \( I' \).

Because of Lemma 4.1 the fact that \( r_\tau(E, k_1) < 0 \) and \( r_\tau(E, k_m) < 0 \) implies that there exists \( \eta_0 > 0 \) (for all \( \tau \leq \tau_0 \)) such that
\[
\min(|k_1 - k_0|, |k_2 - k_1|, |k_{m-1} - k_m|, |k_{m+1} - k_m|) > \eta_0.
\]

We claim that
\[
\sum_{i=1}^m r_\tau(E, k_i) \geq \sum_{i=1}^m r_\tau(E', k_i) - \bar{C}_0 \tag{4.13}
\]
where \( E' \) is obtained by extending periodically \( E \) with the pattern contained in \( E \cap (k_1, k_m) \) and \( \bar{C}_0 = \bar{C}_0(\eta_0) > 0 \). The construction of \( E' \) can be done as follows: if \( m \) is odd we repeat periodically \( E \cap (k_1, k_m) \), and if \( m \) is even we repeat periodically \( (k_1 - \eta_0, k_m) \).

Thus we have constructed a set \( E' \) which is periodic of period \( k_m - k_1 \) or \( k_m - k_1 + \eta_0 \). Therefore, setting
\[
\sum_{i=1}^m r_\tau(E', k_i) \geq e_\tau(h_\tau(I)) - \bar{C}_0, \tag{4.14}
\]
where \( \bar{C}_0 = \bar{C}_0(\eta_0) \). Inequality (4.14) follows by definition of optimal energy density relative to the interval \( I = (k_1, k_m) \).
Inequality (4.14) combined with (4.13) yields (4.6).

To show (4.13), notice that the symmetric difference between $E$ and $E'$ satisfies

$$E \Delta E' \subset (-\infty, k_1 - \eta_0) \cup (k_m + \eta_0, +\infty),$$

where $\eta_0$ is the constant defined in Lemma 4.1. To obtain (4.13), we need to estimate $|\sum_{i=1}^m r_\tau(E, k_i) - \sum_{i=1}^m r_\tau(E', k_i)|$. Let

$$\sum_{i=1}^m r_\tau(E, k_i) - \sum_{i=1}^m r_\tau(E', k_i) = I_1 + I_2,$$

where

\begin{align*}
I_1 &= \sum_{i=0}^{m-1} \int_{k_i}^{k_{i+1}} \int_{0}^{+\infty} \left[ (s - |\chi_E(s + u) - \chi_E(u)|) - (s - |\chi_{E'}(s + u) - \chi_{E'}(u)|) \right] \hat{K}_\tau(s) \, ds \, du \\
I_2 &= \sum_{i=1}^m \int_{k_i}^{k_{i+1}} \int_{-\infty}^{0} \left[ (s - |\chi_E(s + u) - \chi_E(u)|) - (s - |\chi_{E'}(s + u) - \chi_{E'}(u)|) \right] \hat{K}_\tau(s) \, ds \, du.
\end{align*}

Thus by using the integrability of $\hat{K}$, we have that

$$|I_1| \leq \int_{k_0}^{k_m} \int_{0}^{+\infty} \chi_{E \Delta E'}(u + s) \hat{K}_\tau(s) \, ds \, du \leq \int_{k_0}^{k_m} \int_{k_m+\eta_0}^{+\infty} \hat{K}_\tau(u - v) \, dv \, du \leq \frac{C_0}{2},$$

where $C_0$ is a constant depending only on $\eta_0$. Similarly, $|I_2| \leq C_0/2$

Thus we have that

$$|\sum_{i=1}^m r_\tau(E, k_i) - \sum_{i=1}^m r_\tau(E', k_i)| \leq C_0.$$

The next lemma is the analogue of Lemma 7.11 in [DR19] and gives a lower bound on the energy in the case almost all the volume of $Q_\ell(z)$ is filled by $E$ or $E^c$ (this will be the case on the set $A_{-1}$ defined in (5.14)).

**Lemma 4.7.** Let $E$ be a set of locally finite perimeter such that $\min(|Q_\ell(z) \setminus E|, |E \cap Q_\ell(z)|) \leq \delta \ell^d$, for some $\delta > 0$. Then

$$\bar{F}_\tau(E, Q_\ell(z)) \geq \frac{\delta d}{\eta_0},$$

where $\eta_0$ is defined in Lemma 4.1.

The following lemma contains the main lower bounds of the complete functional along one-dimensional slices. It relies on the previous lemmas of this section. In our setting, namely aiming at proving striped pattern formation in the large volume limit along periodic boxes of arbitrary size, the optimal energy densities relative to intervals of different length have to be taken into account. In order to exploit the validity of the one-dimensional estimate of Lemma 4.6 the mesoscopic scale $\ell$ has to be sufficiently large.

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Lemma 4.8. Let $\varepsilon_2, \tau_2 > 0$ as in Lemma 4.5, $\bar{\tau}_4$ as in Theorem 3.3. Let $\delta = \varepsilon^d/(16l^d)$ with $0 < \varepsilon \leq \varepsilon_2$, $0 < \tau \leq \min\{\tau_2, \bar{\tau}_4\}$ and $C_0$ be the constant appearing in Lemma 4.6. Let $t_i^+ \in [0, L)^d$ and $\eta > 0$.

The following hold: there exists a constant $C_1$ independent of $l$ (but depending on the dimension and on $\tau_0$ as in Lemma 4.1) such that

(i) Let $J \subset \mathbb{R}$ an interval such that for every $s \in J$ one has that $D_j^l(E, Q_l(t_i^+ + se_i)) \leq \delta$ with $j \neq i$. Then

$$
\int_J \bar{F}_{i,\tau}(E, Q_l(t_i^+ + se_i)) \, ds \geq -\frac{C_1}{l}.
$$

Moreover, if $J = [0, L)$, then

$$
\int_J \bar{F}_{i,\tau}(E, Q_l(t_i^+ + se_i)) \, ds \geq 0.
$$

(ii) Let $J = (a, b) \subset \mathbb{R}$. If for $s = a$ and $s = b$ it holds $D_j^l(E, Q_l(t_i^+ + se_i)) \leq \delta$ with $j \neq i$, then setting

$$
J_t = (a + l/4, b - l/4) \text{ if } |b - a| > l,
$$

one has that

$$
\int_J \bar{F}_{i,\tau}(E, Q_l(t_i^+ + se_i)) \, ds \geq \left(|J| \varepsilon \tilde{\tau}(h_{\tau}(J_t)) - C_0\right) \chi_{(0, +\infty)}(|J| - l) - \frac{C_1}{l},
$$

otherwise

$$
\int_J \bar{F}_{i,\tau}(E, Q_l(t_i^+ + se_i)) \, ds \geq \left(|J| \varepsilon \tilde{\tau}(h_{\tau}(J_t)) - C_0\right) \chi_{(0, +\infty)}(|J| - l) - C_0 l.
$$

Moreover, if $J = [0, L)$, then

$$
\int_J \bar{F}_{i,\tau}(E, Q_l(t_i^+ + se_i)) \, ds \geq L \varepsilon \tilde{\tau}(h_{\tau, L}).
$$

Proof. The proof of (i) follows from Lemma 4.5 as in Lemma 7.9 in [DR19]. Let us now prove (ii). For simplicity of notation we assume that $J = (0, l')$.

One has that

$$
\int_J \bar{F}_{i,\tau}(E, Q_l(t_i^+, s)) \, ds \geq \frac{1}{l'^{d-1}} \int_{Q^+_l(t_i^+)} \sum_{s' \in \partial E_{r, t_i^+, l'}} \frac{|Q^+_l(s') \cap J|}{l'} \left(r_{i,\tau}(E, t_i^+, s') + v_{i,\tau}(E, t_i^+ + s')\right) \, dt'^{\perp}_i
$$

$$
= \frac{1}{l'^{d-1}} \int_{Q^+_l(t_i^+)} \sum_{s' \in \partial E_{r, t_i^+, l'}} \frac{|Q^+_l(s') \cap J|}{l'} \left(r_{i,\tau}(E, t_i^+, s') + v_{i,\tau}(E, t_i^+ + s')\right) \, dt'^{\perp}_i
$$

$$
+ \frac{1}{l'^{d-1}} \int_{Q^+_l(t_i^+)} \sum_{s' \in \partial E_{r, t_i^+, l'}} \frac{|Q^+_l(s') \cap J|}{l'} \left(r_{i,\tau}(E, t_i^+, s') + v_{i,\tau}(E, t_i^+ + s')\right) \, dt'^{\perp}_i
$$

(4.21)
Let us now show (4.19). If \( l' \leq l \), then \((t_i^{1+}, s') \in Q_l(t_i^{1+}, 0)\) or \((t_i^{1+}, s') \in Q_l(t_i^{1+}, l')\). If the condition \( D_{\eta}^j(E, Q_l(t_i^{1+}, 0)) \leq \delta \) or \( D_{\eta}^j(E, Q_l(t_i^{1+}, l')) \leq \delta \) is missing, then we will estimate \( r_{i,\tau}(E, t_i^{1+}, s') + v_{i,\tau}(E, t_i^{1+}, s') \) from below with \(-1\) whenever the neighbouring “jump” points are further than \( \eta_0 \), together with the fact that \( \frac{|Q_l(s') \cap J|}{l} \leq 1 \). Otherwise \( r_{i,\tau} + v_{i,\tau} \geq 0 \). Hence, the inequality (4.21) can be estimated by

\[
\frac{1}{l^{d-1}} \int_{Q_{l_i}^+(t_i^+)} \sum_{s' \in \partial E_{e_i}^{t_i^{1+}} \cap (l/4, l/4), \ s' \in (-\frac{1}{2}, \frac{1}{2})} \left| Q_l^i(s') \cap J \right| \left( r_{i,\tau}(E, t_i^{1+}, s') + v_{i,\tau}(E, t_i^{1+}, s') \right) dt_i^{1+} \geq -C_1 l.
\]

If instead \( l' > l > 2\bar{L} \) we have that, by Lemma 4.6

\[
\frac{1}{l^{d-1}} \int_{Q_{l_i}^+(t_i^+)} \sum_{s' \in \partial E_{e_i}^{t_i^{1+}} \cap (l/4, l'/4), \ s' \in (-\frac{1}{2}, \frac{1}{2})} r_{i,\tau}(E, t_i^{1+}, s') dt_i^{1+} \geq \frac{1}{l^{d-1}} \int_{Q_{l_i}^+(t_i^+)} |J_t| e_{\tau}(h_r(J_t)) dt_i^{1+} - C_0
\]

\[
= |J_t| e_{\tau}(h_r(J_t)) - C_0,
\]

where we used the fact that \( |J_t| = l' - l/2 > \bar{L} \). Hence

\[
\frac{1}{l^{d-1}} \int_{Q_{l_i}^+(t_i^+)} \sum_{s' \in \partial E_{e_i}^{t_i^{1+}} \cap (l/4, l'/4), \ s' \in (-\frac{1}{2}, \frac{1}{2})} r_{i,\tau}(E, t_i^{1+}, s') dt_i^{1+} \geq \left( |J_t| e_{\tau}(h_r(J_t)) - C_0 \right) \chi_{\{0,\infty\}}(|J| - l) - C_1 l,
\]

Thus, since \( \frac{3}{4} \leq \frac{|Q_l(s') \cap J|}{l} \leq 1 \) whenever \( s' \in (l/4, l'/4), \) (4.19) follows.

Let us now turn to the proof of (4.18). Given that \( D_{\eta}^j(E, Q_l(t_i^{1+}, 0)) \leq \delta \) and \( D_{\eta}^j(E, Q_l(t_i^{1+}, l')) \leq \delta \) for some \( j \neq i \), by Lemma 4.5 with \( \delta = \varepsilon^d/(16 l^d) \) we have that

\[
r_{i,\tau}(E, t_i^{1+}, s') + v_{i,\tau}(E, t_i^{1+}, s') \geq 0
\]

whenever \( \min(|s' - l' + 1/2|, |s' - l' - 1/2|) \geq \eta_0 \) and \( (t_i^{1+}, s') \in Q_l(t_i^{1+}, l') \) or \( \min(|s' + l/2|, |s' - l/2|) \geq \eta_0 \) and \( (t_i^{1+}, s') \in Q_l(t_i^{1+}, 0) \).

Fix \( t_i^{1+} \). Then

\[
\sum_{s' \in \partial E_{e_i}^{t_i^{1+}} \cap (l/4, l'/4), \ s' \in (-\frac{1}{2}, \frac{1}{2})} \left| Q_l^i(s') \cap J \right| \left( r_{i,\tau}(E, t_i^{1+}, s') + v_{i,\tau}(E, t_i^{1+}, s') \right) \geq
\]

\[
\sum_{s' \in \partial E_{e_i}^{t_i^{1+}} \cap (l/4, l'/4), \ s' \in (-\frac{1}{2}, \frac{1}{2})} \left| Q_l^i(s') \cap J \right| \left( r_{i,\tau}(E, t_i^{1+}, s') + v_{i,\tau}(E, t_i^{1+}, s') \right) \]

\[
+ \sum_{s' \in \partial E_{e_i}^{t_i^{1+}} \cap (l/4, l'/4), \ s' \in (-\frac{1}{2}, \frac{1}{2})} \left| Q_l^i(s') \cap J \right| \left( r_{i,\tau}(E, t_i^{1+}, s') + v_{i,\tau}(E, t_i^{1+}, s') \right)
\]

(4.22)
Thus by using (4.22), we have that the first term on the r.h.s. above is positive. To estimate the last term on the r.h.s. above we notice that $r_{i,\tau} \geq 0$ whenever the neighbouring points are closer than $\eta_0$ and otherwise $r_{i,\tau} \geq -1$. Moreover, given that $\frac{|Q'(s') \cap J|}{l} < \frac{\eta_0}{l}$ for $s' \in (-l/2,l/2) \cup (l'-l/2,l'+l/2)$, we have that the last term on the r.h.s. above can be bounded from below by $-C_1/l$. Finally integrating over $t_i^\perp$ we obtain that

$$\frac{1}{l^{d-1}} \int_{Q_i^+(t_2^\perp)} \sum_{s' \in \partial E_i^\perp} \frac{|Q_i'(s') \cap J|}{l} (r_{i,\tau}(E, t_i^\perp, s') + v_{i,\tau}(E, t_i^\perp, s')) \, dt_i^\perp \geq -\frac{C_1}{l}. $$

By using the above inequality in (4.21) and the fact that for every $s' \in (l/2, l' - l/2)$ it holds $|Q_i'(s') \cap J| = 1$, we have that

$$\int_{J} F_{i,\tau}(E, Q_i(t_i^\perp, s)) \, ds \geq \frac{1}{l^{d-1}} \int_{Q_i^+(t_2^\perp)} \sum_{s' \in \partial E_i^\perp} (r_{i,\tau}(E, t_i^\perp, s') + v_{i,\tau}(E, t_i^\perp, s')) \, dt_i^\perp - \frac{C_1}{l}.$$

To conclude the proof of (4.18), as for (4.19), we notice that

$$\frac{1}{l^{d-1}} \int_{Q_i^+(t_2^\perp)} \sum_{s' \in \partial E_i^\perp} r_{i,\tau}(E, t_i^\perp, s') \, dt_i^\perp \geq \left(|J_i| e_r(h_{\tau}(J_i)) - C_0\right) \chi_{(0,\infty)}(|J| - l),$$

where in the last inequality we have used Lemma 4.6 for $E = E_i^\perp$, $J_i = (l/4, l' - l/4)$ with $|J_i| = l' - l/2 > l/2 > \bar{L}$. Hence one gets (4.18).

The proof of (4.20) proceeds using the $L$-periodicity of the contributions. \qed

5 Proof of Theorem 1.2

5.1 Setting the parameters

The sets defined in the proof and the main estimates will depend on a set of parameters $l, \delta, \rho, M, \eta$ and $\tau$. Our aim now is to fix such parameters, making explicit their dependence on each other. We will refer to such choices during the proof of the main theorem.

1. We first fix $\eta_0, \tau_0$ as in Lemma 4.1.

2. Then we choose $0 < \varepsilon \ll 1$ such that

$$\varepsilon < -e_r(h_\tau^*)/4, \quad 2\varepsilon C < \frac{1}{2} \min \left\{ -\frac{e_r(h_\tau^*)}{2}, 1 \right\}, \quad (5.1)$$

where $C$ is as in Theorem 3.3 and we let $\bar{L} > 0, \bar{\tau}_4 > 0$ as in Theorem 3.5 such that moreover $C/\bar{L} < 1$. In particular, $\varepsilon C/\bar{L} < -e_r(h_\tau^*)/4$.  

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3. Let then \( l > 0 \) s.t.

\[
    l \geq \max \left\{ 2\bar{L}, \frac{\bar{C}}{-e_{\tau}(h_{\tau}^*)/4} \right\} \geq \max \left\{ 2\bar{L}, \frac{\bar{C}}{-e_{\tau}(h_{\tau}^*)/2 - \varepsilon C/L} \right\},
\]

(5.2)

where \( \bar{L}, \bar{C} \) are the constant appearing in Theorem 3.5, \( \varepsilon \) is as in (5.1) and \( \bar{C} \) is the constant appearing in (5.21).

4. We find the parameters \( \varepsilon_2 = \varepsilon_2(\eta_0, \tau_0) \) and \( \tau_2 = \tau_2(\eta_0, \tau_0) \) as in Proposition 4.5.

5. We consider then \( \bar{\varepsilon} \leq \varepsilon_2, \tau \leq \min\{\tau_2, \bar{\tau}_4\} \) as in Lemma 4.8. We define \( \delta \) as \( \delta = \frac{\varepsilon d}{16} \). Moreover, by choosing \( \bar{\varepsilon} \) sufficiently small we can additionally assume that

\[
    D_{\eta}(E, Q_{l}(z)) \leq \delta \text{ and } D_{\eta}(E, Q_{l}(z)) \leq \delta, \; i \neq j \implies \min\{|E \cap Q_{l}(z)|, |E^c \cap Q_{l}(z)|\} \leq l^{d-1}.
\]

(5.3)

The above follows from Remark 2.2 (ii).

6. By Remark 2.2 (i), we then fix

\[
    \rho \sim \delta l.
\]

(5.4)

in such a way that for any \( \eta \) the following holds

\[
    \forall z, z' \text{ s.t. } D_{\eta}(E, Q_{l}(z)) \geq \delta, \; |z - z'|_{\infty} \leq \rho \implies D_{\eta}(E, Q_{l}(z')) \geq \delta/2.
\]

(5.5)

7. Then we fix \( M \) such that

\[
    \frac{M\rho}{2d} > C_1 l + 1,
\]

(5.6)

where \( C_1 \) is the constant appearing in Lemma 4.8.

8. By applying Proposition 4.3, we obtain \( \bar{\eta} = \bar{\eta}(M, l) \) and \( \tau_1 = \tau_1(M, l, \delta/2) \). Thus we fix

\[
    0 < \eta < \bar{\eta}, \; \bar{\eta} = \bar{\eta}(M, l).
\]

(5.7)

9. Finally, we choose \( \bar{\tau} > 0 \) s.t.

\[
    \bar{\tau} < \tau_0, \; \tau_0 \text{ as in Lemma 4.1},
\]

(5.8)

\[
    \bar{\tau} < \bar{\tau}_4, \; \bar{\tau}_4 \text{ as in Theorem 3.5},
\]

(5.9)

\[
    \bar{\tau} < \tau_2, \tau_2 \text{ as in Proposition 4.5 and Lemma 4.8},
\]

(5.10)

\[
    \bar{\tau} < \tau_1, \tau_1 \text{ as in Proposition 4.3 depending on } M, l, \delta/2.
\]

(5.11)

By \([0, L)^d\)-periodicity of \( E \) we will denote by \([0, L)^d\) the cube of size \( L \) with the usual identification of the boundary.
5.2 Decomposition of \([0, L)^d\)

Now we perform a decomposition of \([0, L)^d\) into different sets according to the \(L^1\) closeness of the minimizer \(E\) to stripes orthogonal to the different coordinate axes. The construction of this decomposition, in comparison to the one introduced in [DR19, DR20, Ker21], has to take into account the boundary effects on the slices in direction \(e_i\) when close to stripes with boundaries orthogonal to \(e_i\).

Let us now consider any \(L > l \geq 2L\) as in Point 1. of Section 5.1. We will have that \([0, L)^d = A_{-1} \cup A_0 \cup (B \setminus B_l) \cup A_{1, l} \cup \ldots \cup A_{d, l}\) where

- \(A_{i, l}\) with \(i > 0\) is made of points \(z\) such that there is only one direction \(e_i\) such that \(E_i \cap Q_l(z)\) is close to stripes with boundaries orthogonal to \(e_i\).
- \(A_{-1}\) is a set of points \(z\) such that \(E_i \cap Q_l(z)\) is close both to stripes with boundaries orthogonal to \(e_i\) and to stripes with boundaries orthogonal to \(e_j\) for some \(i \neq j\). In particular, by Remark 2.2 (ii) one has that either \(|E_i \cap Q_l(z)| \ll l^d\) or \(|E_j \cap Q_l(z)| \ll l^d\).
- \(B \setminus B_l\) is a suitable set of points close to the boundaries of the sets \(A_{i, l}\) as \(i \in \{1, \ldots, d\}\).
- \(A_0\) is a set of points \(z\) where none of the above points is true, namely the set \(E\) is far from stripes in any direction.

The aim is to show that \(A_0 \cup A_{-1} \cup (B \setminus B_l) = \emptyset\) and that there exists only one \(A_{i, l}\) with \(i > 0\).

Let us first define the sets \(A_i\) for \(i \in \{-1, 0, 1, \ldots, d\}\).

We preliminarily define

\[
\tilde{A}_0 := \left\{ z \in [0, L)^d : D_\eta(E, Q_l(z)) \geq \delta \right\}.
\]

Hence, by the choice of \(\delta, M\) made in Section 5.1 and by Proposition 4.3 for every \(z \in \tilde{A}_0\) one has that \(\tilde{F}_r(E, Q_l(z)) > M\).

Let us denote by \(\tilde{A}_{-1}\) the set

\[
\tilde{A}_{-1} := \left\{ z \in [0, L)^d : \exists i, j \text{ with } i \neq j \text{ s.t. } D_\eta^i(E, Q_l(z)) \leq \delta, D_\eta^j(E, Q_l(z)) \leq \delta \right\}.
\]

Since \(\delta\) satisfies (5.3), when \(z \in \tilde{A}_{-1}\), then one has that \(\min(|E \cap Q_l(z)|, |Q_l(z) \setminus E|) \leq l^{d-1}\). Thus, using Lemma 4.7 with \(\delta = 1/l\), one has that

\[
\tilde{F}_r(E, Q_l(z)) \geq -\frac{d}{ln_0}.
\]

The sets \(\tilde{A}_0\) and \(\tilde{A}_{-1}\) can be enlarged while keeping analogous properties. Indeed, by the choice of \(\rho\) made in (5.4), (5.5) holds, namely for every \(z \in \tilde{A}_0\) and \(|z - z'|_\infty \leq \rho\) one has that \(D_\eta(E, Q_l(z')) > \delta/2\). Moreover, let now \(z'\) such that \(|z - z'|_\infty \leq 1\) with \(z \in \tilde{A}_{-1}\). It is not difficult to see that if \(|Q_l(z) \setminus E| \leq l^{d-1}\) then \(|Q_l(z') \setminus E| \lesssim l^{d-1}\). Thus from Lemma 4.7 one has that

\[
\tilde{F}_r(E, Q_l(z')) \geq -\frac{\tilde{C}_d}{ln_0}.
\]

(5.12)
The above observations motivate the following definitions

$$A_0 := \left\{ z' \in [0, L)^d : \exists z \in \bar{A}_0 \text{ with } |z - z'|_\infty \leq \rho \right\} \quad (5.13)$$

$$A_{-1} := \left\{ z' \in [0, L)^d : \exists z \in \bar{A}_{-1} \text{ with } |z - z'|_\infty \leq 1 \right\}, \quad (5.14)$$

By the choice of the parameters and the observations above, for every $z \in A_0$ one has that $\tilde{F}_r(E, Q_I(z)) > M$ and for every $z \in A_{-1}$, $\tilde{F}_r(E, Q_I(z)) \geq -\tilde{C}_d/(l\eta_0)$. Let us denote by $A := A_0 \cup A_{-1}$.

The set $[0, L)^d \setminus A$ has the following property: for every $z \in [0, L)^d \setminus A$, there exists $i \in \{1, \ldots, d\}$ such that $D^i(E, Q_I(z)) \leq \delta$ and for every $k \neq i$ one has that $D^k(E, Q_I(z)) > \delta$.

Given that $A$ is closed, we consider the connected components $C_1, \ldots, C_n$ of $[0, L)^d \setminus A$. The sets $C_i$ are path-wise connected. Moreover, given a connected component $C_j$ one has that there exists $i$ such that $D^i(E, Q_I(z)) \leq \delta$ for every $z \in C_j$ and for every $k \neq i$ one has that $D^k(E, Q_I(z)) > \delta$. We will say that $C_j$ is oriented along direction $e_i$ if there is a point in $z \in C_j$ such that $D^i(E, Q_I(z)) \leq \delta$.

Because of the above being oriented along direction $e_i$, $A_{-1}$ is closed for every $i > 0$, are open.

We observe the following

(a) The sets $A = A_{-1} \cup A_0, A_1, A_2, \ldots, A_d$ form a partition of $[0, L)^d$.

(b) The sets $A_{-1}, A_0$ are closed and $A_i, i > 0$, are open.

(c) For every $z \in A_i$, we have that $D^i(E, Q_I(z)) \leq \delta$.

(d) There exists $\rho$ (independent of $L, \tau$) such that if $z \in A_0$, then $\exists z'$ s.t. $Q_{\rho}(z') \subset A_0$ and $z \in Q_0(z')$. If $z \in A_{-1}$ then $\exists z'$ s.t. $Q_1(z') \subset A_{-1}$ and $z \in Q_1(z')$.

(e) For every $z \in A_i$ and $z' \in A_j$ one has that there exists a point $\tilde{z}$ in the segment connecting $z$ to $z'$ lying in $A_0 \cup A_{-1}$.

Let now $B = \bigcup_{i > 0} A_i, A = A_0 \cup A_{-1}$.

From conditions (b) and (e) above, $B_{t_i}^+$ is a finite union of intervals, each belonging to some $A_{i, t_i}^+$, $i \in \{1, \ldots, d\}$. Moreover, by (d), for every point that does not belong to $B_{n_i}^+$ there is a neighbourhood of fixed positive size that is not included in $B_{n_i}^+$. Let $\{I_1^j, \ldots, I_{n(J(t_i^+))}^j\}$ such that

$$\bigcup_{i=1}^{n(J(t_i^+))} I_i^j = A_{j, t_i^+} \quad \text{with } I_i^j \cap I_k^j = \emptyset \text{ whenever } j \neq i \text{ or } j = i \text{ and } \ell \neq k.$$ 

We can further assume that $I_i^j \subseteq I_{i+1}^j$, namely that for every $s \in I_i^j$ and $s' \in I_{i+1}^j$ it holds $s \leq s'$. By construction there exists $J_k \subset A_{i, t_i^+}$ such that $I_i^j \subseteq J_k \subseteq I_{i+1}^j$, for every $\ell, j$. We set $\bar{n}(i, t_i^+) = \sum_{j=1}^{d} n(j, t_i^+)$ to be the number of such disjoint intervals $J_k \subset A_{i, t_i^+}$. Whenever $J_k \cap A_{0, t_0^+} \neq \emptyset$, we have that $|J_k| > \rho$ and whenever $J_k \cap A_{-1, t_0^+} \neq \emptyset$ then $|J_k| > 1$.

Given $i \in \{1, \ldots, d\}, \ell \in \{1, \ldots, n(i, t_i^+)\}$ and $I_i^j = (a_i^j, b_i^j)$, define $I_{i, l} \equiv (a_i^j + l/4, b_i^j - l/4)$ whenever $|b_i^j - a_i^j| > l/2$ and $I_{i, l} = \emptyset$ otherwise. Set also $n(i, t_i^+, l) = n(i, t_i^+) - \#\{\ell : |I_{i, l}| < l/2\}$. In particular, for all $\ell \in n(i, t_i^+, l)$ one has that $|I_{i, l}| \geq l$. Then define

$$A_{i, t_i^+, l} = \bigcup_{\ell=1}^{n(i, t_i^+, l)} I_{i, l} \quad (5.15)$$
and
\[ A_{i,t} = \bigcup_{\{ t^+ \in [0,L) \}^{d-1}} A_{i,t^+,t}, \quad B_t = \bigcup_{i=1}^d A_{i,t}. \] (5.16)

Thus we get the partition \([0, L]^d = A_0 \cup A_{-1} \cup (B \setminus B_t) \cup A_{1,t} \cup \ldots \cup A_{d,t}.

5.3 Proof of Theorem 1.2

Step 1 First we show the following estimate
\[
\frac{1}{L^d} \int_{B_{t^+}} \tilde{F}_{i, \tau}(E, Q_t(t^+_i + se_i)) \, ds + \frac{1}{dL^d} \int_{A_{i,t^+}} \tilde{F}_\tau(E, Q_t(t^+_i + se_i)) \, ds \\
\geq \sum_{\ell \in n(i,t^+_i,t)} e_\tau(h_\tau(I_{\ell,i,t})) |I_{\ell,i,t}| - C_0 n(i,t^+_i,t) \frac{1}{L^d} - C(d, \eta_0) \frac{|A_{i,t^+}|}{L^d} + \# \{ A_{0,t^+} \cap \partial A_{i,t^+} \}. 
\] (5.17)

By the definitions given in Section 5.2, one has that
\[
\frac{1}{L^d} \int_{B_{t^+}} \tilde{F}_{i, \tau}(E, Q_t(t^+_i + se_i)) \, ds + \frac{1}{dL^d} \int_{A_{i,t^+}} \tilde{F}_\tau(E, Q_t(t^+_i + se_i)) \, ds \\
\geq \sum_{j=1}^d \sum_{t=1}^{n(j,t^+_j)} \frac{1}{L^d} \int_{I_{j,t}^+} \tilde{F}_{i, \tau}(E, Q_t(t^+_i + se_i)) \, ds + \frac{1}{dL^d} \sum_{t=1}^{n(t^+_i)} \int_{J_t} \tilde{F}_\tau(E, Q_t(t^+_i + se_i)) \, ds \\
\geq \frac{1}{L^d} \sum_{j=1}^d \sum_{t=1}^{n(j,t^+_j)} \left( \int_{I_{j,t}^+} \tilde{F}_{i, \tau}(E, Q_t(t^+_i + se_i)) \, ds + \frac{1}{2d} \int_{J_{k(j,i,t)-1} \cup J_{k(j,i,t)}} \tilde{F}_\tau(E, Q_t(t^+_i + se_i)) \, ds \right),
\]

where in the second inequality we have used the \([0, L]^d\)-periodicity and the convention \( J_1 := J_{n(t^+_i)}. \)

Let us first consider \( I_{1,t} \subset A_{i,t^+_i} \). By construction, we have that \( \partial I_{1,t} \subset A_{i,t^+_i} \).

If \( \partial I_{1,t} \subset A_{i,t^+_i} \), by using our choice of parameters we can apply (4.18) in Lemma 4.8 and obtain
\[
\frac{1}{L^d} \int_{I_{1,t}^+} \tilde{F}_{i, \tau}(E, Q_t(t^+_i + se_i)) \, ds \geq \frac{1}{L^d} \left[ \left( e_\tau(h_\tau(I_{1,t})) |I_{1,t}| - C_0 \right) \chi_{(0, +\infty)}(|I_{1,t}|) - \frac{C_1}{L^d} \right].
\]

If \( \partial I_{1,t} \cap A_{0,t^+_1} \neq \emptyset \), by using our choice of parameters, namely (5.2) and (5.10), we can apply (4.19) in Lemma 4.8 and obtain
\[
\frac{1}{L^d} \int_{I_{1,t}^+} \tilde{F}_{i, \tau}(E, Q_t(t^+_i + se_i)) \, ds \geq \frac{1}{L^d} \left[ \left( e_\tau(h_\tau(I_{1,t})) |I_{1,t}| - C_0 \right) \chi_{(0, +\infty)}(|I_{1,t}|) - \frac{C_1}{L^d} \right].
\]

On the other hand, if \( \partial I_{1,t} \cap A_{0,t^+_1} \neq \emptyset \), we have that either \( J_{k(i,t)} \cap A_{0,t^+_1} \neq \emptyset \) or \( J_{k(i,t)-1} \cap A_{0,t^+_1} \neq \emptyset \). Thus
\[
\frac{1}{2dL^d} \int_{J_{k(i,t)}-1} \tilde{F}_\tau(E, Q_t(t^+_i + se_i)) \, ds + \frac{1}{2dL^d} \int_{J_{k(i,t)}} \tilde{F}_\tau(E, Q_t(t^+_i + se_i)) \, ds \\
\geq \frac{M_\rho}{2dL^d} - \frac{|J_{k(i,t)}-1 \cap A_{-1,t^+_1}|}{2dL^d} \frac{\bar{C}_d}{dL^d} - \frac{|J_{k(i,t)} \cap A_{-1,t^+_1}|}{2dL^d} \frac{\bar{C}_d}{dL^d},
\]

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where $\tilde{C}_d$ is the constant in (5.12).

Since $M$ satisfies (5.6), in both cases $\partial I^*_t \subset A_{-1,t^*_t}$ or $\partial I^*_t \cap A_{0,t^*_t} \neq \emptyset$, we have that

$$\frac{1}{L^d} \int_{I^*_t} \tilde{F}_{i,\tau}(E, Q_l(t^*_i + s e_i)) \, ds + \frac{1}{2dL^d} \int_{I^*_t} \tilde{F}_r(E, Q_l(t^*_i + s e_i)) \, ds + \frac{1}{2dL^d} \int_{I^*_t} \tilde{F}_r(E, Q_l(t^*_i + s e_i)) \, ds \geq \left( \frac{e_{\tau} (h_{\tau}(I^*_t)) |I^*_t|}{L^d} - \frac{C_0}{L^d} \right) \chi_{0, +\infty}(|I^*_t| - l) + \frac{\# \{ A_{0,t^*_t} \cap \partial I^*_t \}}{L^d} - \frac{|J_{k(i,t)} \cap A_{-1,t^*_t}| \tilde{C}_d}{2dL^d} \right).$$

(5.18)

If $I^*_t \subset A_{j,t^*_t}$ with $j \neq i$ from Lemma 4.8 Point (i) it holds

$$\frac{1}{L^d} \int_{I^*_t} \tilde{F}_{i,\tau}(E, Q_l(t^*_i + s e_i)) \, ds \geq - \frac{C_1}{L^d}.$$

In general for every $J_k$ we have that

$$\frac{1}{dL^d} \int_{J_k} \tilde{F}_r(E, Q_l(t^*_i + s e_i)) \, ds \geq \frac{|J_k \cap A_{0,t^*_t}| M}{dL^d} - \frac{\tilde{C}_d}{dL^d} \frac{|J_k \cap A_{-1,t^*_t}| \eta_0}{L^d}.$$

For $I^*_t \subset A_{j,t^*_t}$ such that $(J_{k(j,t^*_t)} \cup J_{k(j,t^*_t) - 1}) \cap A_{0,t^*_t} \neq \emptyset$ with $j \neq i$, we have that

$$\frac{1}{L^d} \int_{I^*_t} \tilde{F}_{i,\tau}(E, Q_l(t^*_i + s e_i)) \, ds + \frac{1}{2dL^d} \int_{J_{k(j,t^*_t) - 1}} \tilde{F}_r(E, Q_l(t^*_i + s e_i)) \, ds + \frac{1}{2dL^d} \int_{J_{k(j,t^*_t)}} \tilde{F}_r(E, Q_l(t^*_i + s e_i)) \, ds \geq - \frac{C_1}{L^d} + \frac{M \rho}{2dL^d} - \frac{|J_{k(j,t^*_t)} \cap A_{-1,t^*_t}| \tilde{C}_d}{2dL^d} \right) - \frac{|J_{k(j,t^*_t)} \cap A_{-1,t^*_t}| \tilde{C}_d}{2dL^d} \right).$$

where the last inequality is true due to (5.6).

For $I^*_t \subset A_{j,t^*_t}$ such that $(J_{k(j,t^*_t)} \cup J_{k(j,t^*_t) - 1}) \subset A_{-1,t^*_t}$ with $j \neq i$, we have that

$$\frac{1}{L^d} \int_{I^*_t} \tilde{F}_{i,\tau}(E, Q_l(t^*_i + s e_i)) \, ds + \frac{1}{2dL^d} \int_{J_{k(j,t^*_t) - 1}} \tilde{F}_r(E, Q_l(t^*_i + s e_i)) \, ds + \frac{1}{2dL^d} \int_{J_{k(j,t^*_t)}} \tilde{F}_r(E, Q_l(t^*_i + s e_i)) \, ds \geq - \frac{C_1}{L^d} - \frac{|J_{k(j,t^*_t)} \cap A_{-1,t^*_t}| \tilde{C}_d}{2dL^d} \right) - \frac{|J_{k(j,t^*_t)} \cap A_{-1,t^*_t}| \tilde{C}_d}{2dL^d} \right).$$

where in the last inequality we have used that $|J_{k(j,t^*_t)} \cap A_{-1,t^*_t}| \geq 1$, $|J_{k(j,t^*_t)} \cap A_{-1,t^*_t}| \geq 1$.

Summing (5.18) and (5.19) over $j \in \{1, \ldots, d\}$, and taking

$$C(d, \eta_0) = \max \left( C_1, \frac{\tilde{C}_d}{\eta_0} \right),$$

(5.20)
one obtains (5.17) as desired.

**Step 2**

Our aim is to deduce from (5.17) the following lower bound

\[
\mathcal{F}_{r,L}(E) \geq \sum_{i=1}^{d} \int_{[0,L)^{d-1}} \sum_{\ell \in n(i,t_i^+,l)} e_r(h_r(I_{\ell,i})) \frac{|I_{\ell,i}^i|}{L} \, dt_i^+ - \frac{C}{lL^d |B_i^c|} + \sum_{i=1}^{d} \int_{[0,L)^{d-1}} \frac{\# \{ A_{0,i^+} \cap \partial A_{i,t_i^+} \}}{L^d} \, dt_i^+,
\]

where \( \tilde{C} = 2C_0 + dC(d, \eta_0) \).

Integrating (5.17) w.r.t. \( t_i^+ \in [0, L)^{d-1} \) one has that

\[
\int_{A_i} \bar{F}_{i,\tau}(E, Q_i(z)) \, dz + \frac{1}{dL^d} \int_{[0,L)^{d-1}} \bar{F}_r(E, Q_i(z)) \, dz \geq \int_{[0,L)^{d-1}} \sum_{\ell \in n(i,t_i^+,l)} e_r(h_r(I_{\ell,i})) \frac{|I_{\ell,i}^i|}{L} \, dt_i^+ - \frac{C_0}{L^d} \sum_{i=1}^{d} n(i, t_i^+, l) \, dt_i^+ - \frac{C(d, \eta_0)}{lL^d} |A| \]

\[
+ \int_{[0,L)^{d-1}} \frac{\# \{ A_{0,i^+} \cap \partial A_{i,t_i^+} \}}{L^d} \, dt_i^+.
\]

Summing the above over \( i \in \{1, \ldots, d\} \) and using the lower bound (2.12) together with the definition of the sets in the decomposition one obtains

\[
\mathcal{F}_{r,L}(E) \geq \sum_{i=1}^{d} \int_{[0,L)^{d-1}} \sum_{\ell \in n(i,t_i^+,l)} e_r(h_r(I_{\ell,i})) \frac{|I_{\ell,i}^i|}{L} \, dt_i^+ - \frac{C_0}{L^d} \sum_{i=1}^{d} n(i, t_i^+, l) \, dt_i^+ - \frac{dC(d, \eta_0)}{lL^d} |A| \]

\[
+ \sum_{i=1}^{d} \int_{[0,L)^{d-1}} \frac{\# \{ A_{0,i^+} \cap \partial A_{i,t_i^+} \}}{L^d} \, dt_i^+ \]

\[
\geq \sum_{i=1}^{d} \int_{[0,L)^{d-1}} \sum_{\ell \in n(i,t_i^+,l)} e_r(h_r(I_{\ell,i})) \frac{|I_{\ell,i}^i|}{L} \, dt_i^+ - \frac{C}{lL^d |B_i^c|} + \sum_{i=1}^{d} \int_{[0,L)^{d-1}} \frac{\# \{ A_{0,i^+} \cap \partial A_{i,t_i^+} \}}{L^d} \, dt_i^+,
\]

where in the last inequality we observed that \( |B \setminus B_i| \geq \frac{1}{2} \sum_{i=1}^{d} \int_{[0,L)^{d-1}} n(i, t_i^+, l) \, dt_i^+ \).

**Step 3**

Now let us assume that \( E \) is a minimizer, namely \( \mathcal{F}_{r,L}(E) = e_r(h_r,L) \), and that \( |B_i^c| \neq 0 \).

First of all, we claim that the following estimate holds: for all \( i = 1, \ldots, d \) and for all \( t_i^+ \in [0, L)^{d-1} \) it holds

\[
\sum_{k=1}^{n(i,t_i^+,l)} e_r(h_r(I_{\ell_k,i})) \frac{|I_{\ell_k,i}^i|}{|A_{i,t_i^+,l}|} \geq e_r(h_r([0, |A_{i,t_i^+,l}|])) - \frac{\varepsilon C}{|A_{i,t_i^+,l}|}.
\]

where \( \varepsilon \) satisfies the conditions in (5.1) and \( C \) is the constant appearing in Theorem 3.5. Indeed,
by minimality of $e_\tau(h^*_i) = e_{\infty, \tau}$, the fact that $\sum_{k=1}^{n(i,t^+_l)} |I_{k,i,t^+_l}| = |A_i,t^+_l|$ and by (3.22), one has that

$$\sum_{k=1}^{n(i,t^+_l)} e_\tau(h_\tau(I_{k,i,t^+_l})) \frac{|I_{k,i,t^+_l}|}{|A_i,t^+_l|} \geq e_\tau(h^*_i) \geq e_\tau([0, |A_i,t^+_l|]) - \frac{\varepsilon C}{|A_i,t^+_l|}.$$  

Using the fact that $F_\tau L^d = e(h_\tau,L)|B_1| + e(h_\tau,L)|B_\tau^c|$, inequality (5.21) rewrites as

$$\sum_{i=1}^{d} \int_{[0,L]^{d-1}} \left[ e_\tau(h_\tau,L)|A_i,t^+_l| - \sum_{\ell \in \partial A_i,t^+_l} e_\tau(h_\tau(I_{\ell,i,t^+_l})) |I_{\ell,i,t^+_l}| \right] dt^+_l \geq \varepsilon C \sum_{i=1}^{d} \left\{ \{ t^+_l \in [0,L]^{d-1} : A_i,t^+_l \neq [0,L] \} \right\}$$

$$+ \sum_{i=1}^{d} \int_{[0,L]^{d-1}} \left( -e_\tau(h_\tau,L) - \frac{C}{l} \right) |A_i,t^+_l| dt^+_l$$

$$+ \sum_{i=1}^{d} \int_{[0,L]^{d-1}} \# \{ A_0,t^+_l \cap \partial A_i,t^+_l \} dt^+_l.$$  

Using in the above the lower bound (5.24) one has that

$$\sum_{i=1}^{d} \int_{[0,L]^{d-1}} \left[ e_\tau(h_\tau,L) - e_\tau(h_\tau([0, |A_i,t^+_l|])) \right] |A_i,t^+_l| dt^+_l \geq -\varepsilon C \sum_{i=1}^{d} \left\{ \{ t^+_l \in [0,L]^{d-1} : A_i,t^+_l \neq [0,L] \} \right\}$$

$$+ \sum_{i=1}^{d} \int_{[0,L]^{d-1}} \left( -e_\tau(h_\tau,L) - \frac{C}{l} \right) |A_i,t^+_l| dt^+_l$$

$$+ \sum_{i=1}^{d} \int_{[0,L]^{d-1}} \# \{ A_0,t^+_l \cap \partial A_i,t^+_l \} dt^+_l.$$  

Since both $L$ and $|A_i,t^+_l|$ are greater than $\bar{L}$, by Theorem 3.5 one has that

$$\left| e_\tau(h_\tau,L) - e_\tau(h_\tau([0, |A_i,t^+_l|])) \right| \leq \frac{\varepsilon C}{|A_i,t^+_l|}.$$  

Hence, (5.26) and (5.27) imply that

$$\varepsilon C \sum_{i=1}^{d} \left\{ \{ t^+_l \in [0,L]^{d-1} : A_i,t^+_l \neq [0,L] \} \right\} \geq -\varepsilon C \sum_{i=1}^{d} \left\{ \{ t^+_l \in [0,L]^{d-1} : A_i,t^+_l \neq [0,L] \} \right\}$$

$$+ \sum_{i=1}^{d} \int_{[0,L]^{d-1}} \left( -e_\tau(h_\tau,L) - \frac{C}{l} \right) |A_i,t^+_l| dt^+_l$$

$$+ \sum_{i=1}^{d} \int_{[0,L]^{d-1}} \# \{ A_0,t^+_l \cap \partial A_i,t^+_l \} dt^+_l.$$  

By Theorem 3.5 the assumptions (5.1) on $\varepsilon$ and (5.2) on $l$ one has that

$$-e_\tau(h_\tau,L) - \frac{C}{l} \geq -e_\tau(h^*_i) - \frac{\varepsilon C}{L} - \frac{C}{l} \geq -\frac{3}{4} e_\tau(h^*_i) - \frac{C}{l} \geq -\frac{e_\tau(h^*_i)}{2} > 0.$$  

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Hence from (5.28) and (5.29) one obtains

\[
2\varepsilon C \sum_{i=1}^{d} \left| \{ t^1_i \in [0, L)^{d-1} : A_{i,t^1_i} \neq [0, L) \} \right| \geq -\frac{e_{\tau}(h^*)}{2} \sum_{i=1}^{d} \int_{[0,L)^{d-1}} |A_{i,t^1_i}| \, dt^1_i \\
+ \sum_{i=1}^{d} \int_{[0,L)^{d-1}} \#\{A_{0,t^1_i} \cap \partial A_{i,t^1_i} \} \, dt^1_i. 
\]  
(5.30)

Now notice that whenever for some \( t^1_i \) one has that \( A_{i,t^1_i} \neq [0, L) \) or equivalently \( A_{i,t^1_i} \neq \emptyset \), then either \( A_{0,t^1_i} \cap \partial A_{i,t^1_i} \neq \emptyset \) and \( \#\{A_{0,t^1_i} \cap \partial A_{i,t^1_i} \} \geq 1 \) or \( A_{-1,t^1_i} \neq \emptyset \) and in particular \( |A_{i,t^1_i}| \geq 1 \).

Thus

\[
2\varepsilon C \sum_{i=1}^{d} \left| \{ t^1_i \in [0, L)^{d-1} : A_{i,t^1_i} \neq [0, L) \} \right| \geq \sum_{i=1}^{d} \left[ -\frac{e_{\tau}(h^*)}{2} \left| \{ t^1_i \in [0, L)^{d-1} : A_{-1,t^1_i} \neq \emptyset \} \right| + \left| \{ t^1_i \in [0, L)^{d-1} : A_{0,t^1_i} \neq \emptyset \} \right| \right] \\
\geq \min \left\{ -\frac{e_{\tau}(h^*)}{2}, 1 \right\} \sum_{i=1}^{d} \left| \{ t^1_i \in [0, L)^{d-1} : A_{i,t^1_i} \neq [0, L) \} \right|. 
\]  
(5.31)

If \( |B_i^c| \neq 0 \), then \( \sum_{i=1}^{d} \left| \{ t^1_i \in [0, L)^{d-1} : A_{i,t^1_i} \neq [0, L) \} \right| \neq 0 \) and if \( \varepsilon \) satisfies the conditions in (5.1), then (5.31) is not satisfied, thus reaching a contradiction.

Hence, for any minimizer \( E \) it holds \( |B_i^c| = 0 \). In particular, \( |A| \leq |B_i^c| = 0 \) and thus by (e) there is just one \( A_i, i > 0 \) with \( |A_i| > 0 \).

We now claim that the fact that there is just one \( A_i, i > 0 \) with \( |A_i| > 0 \) proves the statement of Theorem 1.2.

Indeed, let us consider

\[
\frac{1}{L^d} \int_{[0,L)^d} \bar{F}_\tau(E,Q_\tau(z)) \, dz = \frac{1}{L^d} \int_{[0,L)^d} \bar{F}_{i,\tau}(E,Q_\tau(z)) \, dz \\
+ \frac{1}{L^d} \sum_{j \neq i} \int_{[0,L)^d} \bar{F}_{j,\tau}(E,Q_\tau(z)) \, dz
\]  
(5.32)

(5.33)

We apply now Lemma 4.8 with \( j = i \) and slice the cube \([0, L)^d\) in direction \( e_i \). From (4.16), one has that (5.33) is nonnegative and strictly positive unless the set \( E \) is a union of stripes with boundaries orthogonal to \( e_i \). On the other hand, from (4.20), one has the r.h.s. of (5.32) is minimized by a periodic union of stripes with boundaries orthogonal to \( e_i \) and with period \( 2h_{\tau,L} \) and density 1/2. Thus, periodic stripes of period \( 2h_{\tau,L} \) and density 1/2 are optimal.

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