Anisotropic gravity solutions in AdS/CMT

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Abstract

We have constructed gravity solutions by breaking the Lorentzian symmetry to its subgroup, which means there is Galilean symmetry but without the rotational and boost invariance. This solution shows anisotropic behavior along both the temporal and spatial directions as well as among the spatial directions and more interestingly, it displays the precise scaling symmetry required for metric as well as the form fields. From the field theory point of view, it describes a theory which respects the scaling symmetry, \( t \rightarrow \lambda^{z_1} t, \ x \rightarrow \lambda^{z_2} t, \ y \rightarrow \lambda y, \) for \( z_1 \neq z_2, \) as well as the translational symmetry associated to both time and space directions, which means we have found a non-rotational but Lifshitz-like fixed points from the dual field theory point of view. We also discuss the minimum number of generators required to see the appearance of such Lifshitz points. In 1 + 1 dimensional field theory, it is 3 and for 2 + 1 dimensional field theory, the number is 4.

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1 Introduction and motivation

After the recent advancement of gauge-gravity duality conjecture [1], [2] and [3], it has been becoming very interesting to apply this duality conjecture to somewhat realistic situations that is in condensed matter theories, so as to understand more about it, in the sense to have a better understanding of quantum phase diagram [9] and [13], at the strong coupling limit. In this context, it has been proposed in [4] that there exists a gravity solution for the Lifshitz-like fixed point and latter it was generalized to gravity solution showing only the temporal scale invariance in [5] and to solutions in arbitrary space time dimensions using the p-form fields in [6]. In particular, the solution in $d$ space time dimension is constructed so as to have a global $O(d-2)$ symmetry and in [7] a proper holographic renormalization prescription is initiated to compute the correlation functions and some other studies in [8].

It is interesting to recall that there are many important studies has been done in the context of quantum phase transitions using global rotational symmetry in field space. For example the super fluid-insulator transition in $\phi^4$ theory with $O(2)$ symmetry in 2+1 space time theory, but with a Lorentzian symmetry, more appropriately with a relativistic conformal invariance [10] and [11], in arbitrary dimensions in [12] and some other studies related to quantum critical points in [14], [16], [17] and [18].

The signature of quantum phase transition can be attributed to, from [9], as the appearance of non-analyticity in the ground state energy as a function of a dimensionless parameter $g$ at $g = g_c$ and at zero temperature. However, due to the unavoidable thermal fluctuations the system is studied around $|g - g_c|$ and at finite temperature, $T$. The mathematical definition for it is, by finding the quantum fluctuations to energy, $\Delta$, at zero $T$ as a function of the parameter $g$

$$\Delta \sim J|g - g_c|^{z\nu},$$

where $J$ is the energy scale of a characteristic microscopic coupling and $z\nu$ is the universal critical exponent. The other important ingredient is the characteristic correlation length $\xi$, which is related to $\Delta$ as

$$\Delta \sim \xi^{-z}.$$  

It is important to note that both $\nu$ and $z$ are positive as $\xi$ diverges at the critical point and $\Delta$ vanishes at the critical point $g_c$.

The Lifshitz point is described as a multicritical point on the curve describing the $\lambda$ line of the second order phase transition [14], which separates ordered phases. For example, in a magnetic systems the Lifshitz point is a triple point where paramagnetic, ferromagnetic and helicoidal phases meet [15].

Let us recall few definitions from [19], for a scaling transformation under which the correlation function $C(x_i; t)$ obeys

$$C(\lambda x_i; \lambda^z t) = \lambda^{-2z}C(x_i; t),$$

2
where \( \delta \) is the scaling dimension and \( z \) is called the anisotropy exponent or the dynamical exponent in equilibrium phase transition, especially for systems having \( z \neq 1 \) is called as strongly anisotropy critical systems. Examples of where the strongly anisotropy arises are (a) systems exhibiting percolation [20] and (b) systems showing the appearance of Lifshitz points [14].

For \( z = 1 \), we have already witnessed the examples that has the features of being recognized as the quantum critical points, these are the \( AdS_4 \) space times which respects the full conformal algebra. Whereas for \( z = 2 \), there were systems constructed very recently, which has the reduced symmetry group and is called as the Schrödinger group. It is interesting to note that the critical points of \( z = 1 \) theories are Lorentz invariants whereas for higher values of \( z \), the theory should necessarily break the Lorentz symmetry.

There is also a lot of other activity in a somewhat related area starting from [22] and [23], where the symmetry is enhanced than that we study here and is called as Schrödinger group. It contains the generators for spatial translations, time translations, rotations, boosts, number operator and dilatation operators. Latter this solution was embedded in string theory in [24],[25] and [26] using either the Null Melvin Twist [37], [38] or the TsT symmetry prescriptions [39] and this is applied further in [27] for the Sakai-Sugimoto model and some other examples in [28], also in [30] and there are some other related but earlier studies in [31],[32] and some recent studies in [33]-[36]. Also some aspects of non-relativistic hydrodynamics is studied in [29].

In this paper we have generated some other interesting solutions in the bulk by taking gravity and form-fields as the relevant degrees of freedom with Chern-Simon interaction. In particular, in 2+1 dimension by taking gravity and a 2-form field strength, we have constructed solution that exhibits the presence of a dynamical exponent \( z \) and this solution has the continuous symmetries of time translation, space translation and scaling symmetry. The discrete symmetries are time and space reflections. This solution is generated using a combination of both electric and magnetic 2-from flux. In this solution, we can have only temporal scaling symmetry as well as both temporal and spatial scaling symmetry.

In 3+1 dimension, we generate solutions that possess time translation, spatial translation and scaling symmetry but without any rotational or Galilean transformation, the analogue of boost symmetry. The discrete symmetries are the same as in 2+1 dimension. More interestingly, this solution has got two non-trivial exponents, \( z_1 \) and \( z_2 \). This solution is constructed using gravity, a 2-from field strength and two 3-form field strengths, which includes a massive 2-form potential.

The solutions that we have constructed in this paper uses an effective action in the bulk which tells us that we can have quantum critical points starting from 1+1 field theory and to see this we do not need to have the full symmetry group i.e. either the Lorentzian conformal symmetry or the Schrödinger group, with a lots of generators, instead a minimum of 3 generators for this particular space time, time translation, spatial translation and scaling symmetry, can generate the required dual solution.

The plan of the paper is in section 2, we shall study the 2+1 dimensional bulk theory and in section 3, the 3+1 dimensional theory and finally conclude in section 4.
2 1+1 dim field theory systems

In 2+1 dimensional bulk theory the solution, recalling from [6],

\[ ds^2 = L^2[-r^{2a}dt^2 + dx^2 + \frac{dr^2}{r^2}], \quad F_2 = \frac{AL^2}{r^{1-a}}dr \wedge dt, \quad A^2 = \frac{2a^2}{L^2}, \quad \Lambda = -\frac{a^2}{2L^2}, \quad (4) \]

which only shows the temporal scale invariance and is one of the allowed solution at zero temperature containing gravity and \( F_2 \) from as the non-trivial degrees of freedom. The other possible solution, more appropriately with these degrees of freedom, in 2+1 space time dimension can be realized by taking the action

\[ S = \frac{1}{2\kappa^2} \int dxdrdt \left[ \sqrt{-g} \left( R - 2\Lambda - \frac{F_2^2}{4} - \frac{H_2^2}{4} \right) - c \epsilon^{MNP} A_M \partial_N B_P \right] \quad (5) \]

with the ansatz to fluxes as

\[ F_2 = dA = A_1 L^2 r^{a-1} dr \wedge dt, \quad H_2 = dB = A_2 L^2 r^{b-1} dr \wedge dx \quad (6) \]

gives the solution as

\[ ds^2 = L^2[-r^{2a}dt^2 + r^{2b}dx^2 + \frac{dr^2}{r^2}], \quad A_2 = \frac{2b(a-b)}{L^2}, \quad A_1 = \frac{2a(a-b)}{L^2}, \quad c^2 L^2 = ab, \quad \Lambda = -\frac{a^2 + b^2}{2L^2}, \quad (7) \]

where \( b \neq 0 \) and \( a \geq b > 0 \).

By doing a change of coordinates, and defining \( z := \frac{a}{b} \), the metric becomes

\[ ds^2 = L^2[-\rho^{2z}dt^2 + \rho^2dx^2 + \frac{d\rho^2}{\rho^2}] \quad (8) \]

The generators that follows from eq(8) are

\[ H = -i\partial_t, \quad P_x = -i\partial_x, \quad D = -i[zt\partial_t + x\partial_x + \rho\partial_\rho] \quad (9) \]

and the algebra it satisfy

\[ [D, P_x] = i P_x, \quad [D, H] = iz H \quad (10) \]
2.1 Dimensions and Correlators

In this subsection, we would like to consider a massive scalar field of mass, $m$ that is moving in the Euclidean background of eq(8) and from which we shall generalize the gauge gravity duality \[40\] and present the dictionary. In due course we shall determine the dimensions of the operators that are dual to the bulk massive scalar field and present the conditions when and which of the dimensions will become relevant.

The equations of motion of the scalar field, $\phi$, for a special value to the exponent, $z = 2$ in $u = 1/r$, coordinate system

$$\partial_u^2 \phi - \frac{2}{u} \partial_u \phi - \left[ u^2 w^2 + k^2 + \frac{m^2 L^2}{u^2} \right] \phi = 0$$  \hspace{1cm} (11)

Asymptotically the operator $\Delta$, dual to massive scalar field $\phi$ satisfies

$$\Delta(\Delta - 3) = m^2 L^2, \quad \Delta_{\pm} = \frac{3}{2} \pm \sqrt{\frac{9}{4} + m^2 L^2}. \hspace{1cm} (12)$$

The part of the generalized structure to gauge gravity correspondence follows from requiring the finiteness of the Euclidean action and is that if the mass of scalar field stays above the following bound

$$(m L)^2 > -\frac{5}{4} \hspace{1cm} (13)$$

then only the $\Delta_+$ branch is allowed as the only possible solution. If the mass satisfies the following condition

$$-\frac{9}{4} < (m L)^2 < -\frac{5}{4} \hspace{1cm} (14)$$

then both the $\Delta_+$ and $\Delta_-$ branches are allowed. The analogue of Breitenlohner-Freedman bound \[41\] is

$$(m L)^2 > -\frac{9}{4}, \hspace{1cm} (15)$$

which says if the mass of the scalar field stays below this bound then there is an instability in the system.

For computation of the two point correlation function involving operators that are dual to the scalar field requires the form of the Green’s function $G(u, k)$, which is related to the source $\phi(0, k)$

$$\phi(u, k) = G(u, k) \phi(0, k). \hspace{1cm} (16)$$

Note that $k = (w, k)$ and the solution to the wave equation satisfied by the scalar field is

$$G(u, k) = c_1 \frac{2^{\frac{1}{2}} \sqrt{9 + 4m^2 L^2}}{4w} \times e^{-\frac{w^2}{2}} \times u^{\frac{3 + \sqrt{9 + 4m^2 L^2}}{2}} \times U\left(\frac{k^2 + (2 + \sqrt{9 + 4m^2 L^2})w}{4w}, \frac{2 + \sqrt{9 + 4m^2 L^2}}{2}, uw^2\right) \hspace{1cm} (17)$$
where \( U(a, b, z) \) is the confluent hypergeometric function of the second kind and \( c_1 \) is the normalization constant that need to be determined by imposing the following condition

\[
G(u \to \epsilon, k, w) = 1
\]  

(18)

From, now onwards we shall be considering only the massless case and in this case the properly normalized propagator

\[
G(u, k) = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{k^2 + 5w}{4w}\right)e^{-\frac{w^2}{4w}} U\left(\frac{k^2 - w}{4w}, -\frac{1}{2} wu^2\right)
\]  

(19)

From the Euclidean action of the scalar field, upon using the equations of motion results

\[
S = \int dk dw \phi(0, -k) \mathcal{F}(k) \phi(0, k),
\]  

(20)

where the flux factor is

\[
\mathcal{F}(k) = -2 \sqrt{g g^{uu}} \partial_u G(u, k)|_\epsilon^\infty.
\]  

(21)

In order to calculate the flux factor, we need the expansion of the Green’s function and its derivative up to quartic order in \( u \)

\[
G(u, k) = 1 - \frac{k^2 u^2}{2} + \frac{8}{3} \frac{w^3 \Gamma\left(\frac{k^2 + 5w}{4w}\right)}{\Gamma\left(\frac{k^2 - w}{4w}\right)} u^3 + \frac{1}{8} \frac{2w^2 - k^4}{u^4} + \mathcal{O}(u^5),
\]

\[
\partial_u G(u, k) = -k^2 u + \frac{8}{3} \frac{w^3 \Gamma\left(\frac{5w + k^2}{4w}\right)}{\Gamma\left(\frac{k^2 - w}{4w}\right)} u^2 + \frac{2w^2 - k^4}{2} u^3 + \frac{4}{3} \frac{k^2 w^3 \Gamma\left(\frac{5w + k^2}{4w}\right)}{\Gamma\left(\frac{k^2 - w}{4w}\right)} u^4 + \mathcal{O}(u^5),
\]

\[
\sqrt{g g^{uu}} = \frac{L}{u^2}
\]  

(22)

Now including everything and dropping the divergent terms in the \( \epsilon \to 0 \) limit, gives us the correlator

\[
< \mathcal{O}(-k) \mathcal{O}(k) > = \mathcal{F}(k) = -16Lw^{\frac{3}{2}} \frac{\Gamma\left(\frac{k^2 + 5w}{4w}\right)}{\Gamma\left(\frac{k^2 - w}{4w}\right)}
\]  

(23)

This structure of the 2-point correlator matches exactly as is found in [7], which means upon addition of proper counter terms to the action eq(20) will help us to cancel the irrelevant divergent terms.

3 3+1 dim: Symmetries

Let us do some studies of 3 + 1 dimensional examples which has got generators associated to time translations (\( H \)), spatial translations (\( P_x \), \( P_y \)) and dilatation generator, (\( D \)). The dilatation
generator associated to the scaling symmetry that we have has a very specific structure and it’s in the notation of [5]

\[ t \rightarrow \lambda^a t, \quad x \rightarrow \lambda^b x, \quad y \rightarrow \lambda^c y, \quad r \rightarrow \frac{r}{\lambda}. \]  

(24)

The metric that shows the above symmetries can have a structure

\[ ds^2 = L^2[-r^{2a}dt^2 + r^{2b}dx^2 + r^{2c}dy^2 + \frac{dr^2}{r^2}], \]  

(25)

where \( L \) is the size of the space time and the exponents \( a, b, c \) can take any real values. Let us do the following coordinate transformation for \( c \neq 0 \)

\[ r^c = \rho, \quad (t, x, y) \rightarrow \frac{1}{c}(t, x, y), \quad L \rightarrow L c \]  

(26)

and the resulting metric for \( z_1 := \frac{a}{c} \) and \( z_2 := \frac{b}{c} \)

\[ ds^2 = L^2[-\rho^{2z_1}dt^2 + \rho^{2z_2}dx^2 + \rho^2dy^2 + \frac{d\rho^2}{\rho^2}] \]  

(27)

Using this form of metric, the explicit structure of the generators in generic case, \( z_1 \neq z_2 \), are

\[ H = -i\partial_t, \quad P_x = -i\partial_x, \quad P_y = -i\partial_y, \quad D = -i[z_1 t\partial_t + z_2 x\partial_x + y\partial_y + \rho\partial_\rho] \]  

(28)

and these generators obey the following algebra

\[ [D, P_x] = iz_2 P_x, \quad [D, P_y] = iP_y, \quad [D, H] = iz_1 H. \]  

(29)

From the algebra or from the metric it follows that there exist two distinct nonequivalent exponents \( z_1 \) and \( z_2 \). For a special value, namely \( z_1 = z_2 = 1 \), the algebra is enhanced to that of the \( SO(2, 3) \) of \( AdS_4 \) spacetime. It is interesting to note that these exponents can take any real values as long as they do not break any consistency of the theory e.g. the unitarity or the geodesically incompleteness of the space time.

**3.1 Constraints**

It looks like that if the dual field theory have an action like

\[ S = \frac{1}{2} \int dt dx dy [-(\partial_t \chi)^\alpha + K_1 (\partial_x^2 \chi)^\beta + K_2 (\partial_y^2 \chi)^\gamma], \]  

(30)

where \( \chi \) is the field in the field theory with \( K_1 \) and \( K_2 \) are two constants and describes a plane containing fixed points. Roughly, speaking, there may be two constants from the dual field theory point of view as we have introduced the asymmetry between the spatial directions \( x \) and \( y \).
Now if we demand that the above action eq(30), respects the scaling symmetry eq(24), where the field $\chi$ transforms trivially, implies that the $\alpha$, $\beta$, $\gamma$ should take the following values
\[
\alpha = 1 + \frac{z_2}{z_1}, \quad \beta = \frac{1}{2} + \frac{z_1}{2z_2}, \quad \gamma = \frac{1 + z_1 + z_2}{2}.
\] (31)

Now, it is trivial to see that if we want the action eq(30) to be quadratic in fields, $\chi$ implies $z_1 = 2$ and $z_2 = 1^2$.

Let us study under what condition the field theory can be non-unitary? The way to see non-unitary behavior is to have the property of 2-point correlation function of operators that increases with their distance of separation
\[
< O(x)O(0) > \sim \frac{1}{|x|^{2\Delta}},
\] (32)
where $\Delta$ is the dimension of the operator $O(0)$, which is assumed to take only negative values.

For a minimally coupled scalar field, $\phi(x)$, of mass $m$ moving in the Euclidean background of
\[
ds^2 = L^2[-r^{2a}dt^2 + r^{2b}dx^2 + r^{2c}dy^2 + \frac{dr^2}{r^2}]
\] (33)
obeys an equation
\[
\partial_\mu^2 \phi + \frac{(1-a-b-c)}{u} \partial_\mu \phi - \left[u^{2(a-1)}\omega^2 + u^{2(b-1)}k_x^2 + u^{2(c-1)}k_y^2 + \frac{m^2L^2}{u^2}\right] \phi = 0,
\] (34)
where $u = 1/r$. The structure of the generalized form of the field theory and gravity correspondence can be understood by finding the relation between the mass $m$ of the field, in this case a scalar field $\phi$, with the dimension of the operator $\Delta$, that it is associated to in the dual field theory.

In the case of interest i.e. for the massive scalar field the relation is
\[
\Delta^2 - (a + b + c)\Delta - m^2L^2 = 0,
\]
\[
\Delta_{\pm} = \frac{(a + b + c)}{2} \pm \sqrt{\left(\frac{(a + b + c)}{2}\right)^2 + m^2L^2},
\] (35)
where $\Delta_{\pm}$ are the two roots and $\Delta_+ > \Delta_-$. The analogue of Breitenlohner-Freedman bound [41] is
\[
(mL)^2 > -\left(\frac{(a + b + c)}{2}\right)^2.
\] (36)

\[\text{In this context, it is probably correct to say that for a field theory which is a unitary theory should have a dual bulk space time solution which obeys the constraints of being a real solution with negative cosmological constant, imposes the restrictions that we may not have a solution which shows only spatial scale invariance.}\]
In order to see the situation, when the 2-point correlation function between the operators dual to scalar field increases, i.e. when the conformal dimension $\Delta$ becomes negative. The easiest way is to take the $\Delta_-$ branch and

$$\Delta_- = \frac{(a + b + c)}{2} - \sqrt{\left(\frac{(a + b + c)}{2}\right)^2 + m^2 L^2} \tag{37}$$

from which it follows that if

$$a + b + c < 0, \tag{38}$$

then it is trivial to see that the dimension of operators is negative, as long as, the object under the square-root is positive i.e.

$$\left(\frac{(a + b + c)}{2}\right)^2 + (mL)^2 > 0. \tag{39}$$

From which it follows that the 2-point correlation function increases with their distance of separation for the total negative sum of the exponents and to recall that the $\Delta_-$ branch comes into picture when the mass of the scalar field obeys

$$-(\frac{a + b + c}{2})^2 < (mL)^2 < 1 - \left(\frac{(a + b + c)}{2}\right)^2 \tag{40}$$

this condition. In fact, in this range of the $m^2$ both $\Delta_+$ and $\Delta_-$ branch comes into picture.

However, if the mass of the scalar field obeys

$$m^2 L^2 > 1 - \left(\frac{(a + b + c)}{2}\right)^2 \tag{41}$$

then only $\Delta_+$ branch comes into picture and to see under what condition can $\Delta_+$ becomes negative, so as to have the non-unitary feature of the 2-point correlation function ?

Generically, it does not becomes negative, however if $\left(\frac{a + b + c}{2}\right)$ dominates the term that comes under the square-root of $\Delta_+$, as it is always positive then $\Delta_+$ can becomes negative, i.e. if

$$a + b + c < 0, \quad \text{and} \quad \left(\frac{(a + b + c)}{2}\right) > \sqrt{\left(\frac{(a + b + c)}{2}\right)^2 + m^2 L^2}, \tag{42}$$

these conditions are full filled.

So, it just follows that if the sum of the exponents, $a + b + c$, is negative then the theory is not unitary and it says that even if one or two of the exponents are negative and if the total sum of the exponents is not negative then the theory is still unitary.

But this conclusion somehow does not make sense. Let us consider a situation where we take all the exponents as same and negative then we go over to $AdS_4$ spacetime, which is a unitary theory and contradicts with the conclusion of having a non-unitary theory for $a + b + c < 0$. 
So the proper way to find the dimension of the operators is to go over to the \( \rho \)-coordinate system with the exponents \( z_1 \) and \( z_2 \). However, for the computation of finding new solutions, the \( r \)-coordinate system is probably better and also to analyze solutions with only temporal scale invariance.

The easiest way to get the expression of the dimension of the massive scalar field is to start from eq(35) and define \( \Delta_c \) as the new \( \Delta \) and \( L_c \) as the new size parameter \( L \) and this is meaningful only when \( c \neq 0 \). It gives

\[
\Delta^2 - (1 + z_1 + z_2)\Delta - m^2 L^2 = 0,
\]

\[
\Delta_{\pm} = \frac{(1 + z_1 + z_2)}{2} \pm \sqrt{\left(\frac{1 + z_1 + z_2}{2}\right)^2 + m^2 L^2},
\]  

(43)

Now the previous analysis of getting the negative dimension translates to

\[
1 + z_1 + z_2 < 0.
\]  

(44)

So, we see that negativity of one of the exponent do not necessarily make the sum \( 1 + z_1 + z_2 \) to become negative, which means the correlation function of massive scalar fields do not increases with their distance of separation even if one of the exponent becomes negative.

However, if we make one of exponents negative, say \( z_1 \), then the time coordinate shrinks to zero at \( \rho \to \infty \) i.e. at the boundary and this says that the space time is not geodesically complete at the boundary and the same is true for the other exponent \( z_2 \).

### 3.2 The models in the bulk theory

In order to discuss the complete anisotropy in a 3+1 dimensional theory between the temporal and spatial coordinates as well as the anisotropy among the spatial coordinates, we need to consider an action which has the following degrees of freedom: metric, a two-form massive potential, a 2-form field strength and a 3-form field strength. In our discussion we have included the 4-form flux for completeness but it is not required and its absence is not going to change anything as roughly in 3+1 dimension, it is dual to a scalar field. Note that the cosmological constant is already there in the effective gravitational action.

The form of p-form fluxes need to be taken comes in a very specific way such that we need to have an anisotropy at the end of the calculation. Typically, the \( B_2 \) and \( F_2 \) form objects should be extended in a such a way that they have one leg in common and two legs for the \( H_3 \) and \( F_3 \) form field strength objects.

Let us look at a model in 4 dimensional theory, with an action

\[
S = \frac{1}{2\kappa^2} \int \left[ \sqrt{-g} \left( R - 2\Lambda - \frac{H_3^2}{12} - \frac{m_0^2}{2} B_2 - \frac{F_2^2}{12} - \frac{F_2^2}{4} - \frac{F_2^2}{24}\right) - c_1 \epsilon^{M_1} M_2 M_3 M_4 A_{M_1} M_2 F_{M_3} M_4 \right],
\]  

(45)
where $c_1$ is the topological coupling. The equations of motion that follows from it are
\[
\partial_M (\sqrt{-g} F^{M}{}_{M}{}_{A}) + \frac{2}{3} \epsilon^{M_1 M_2 M_3 M_4} c_1 F_{M_1 M_2 M_3} = 0, \\
\partial_M (\sqrt{-g} H^{M}{}_{M}{}_{A}) - 2m_0^2 \sqrt{-g} B^{M}{}_{M}{}_{A} = 0, \\
\partial_M (\sqrt{-g} F^{M}{}_{M}{}_{M}{}_{A}) - 2c_1 \epsilon^{M_1 M_2 M_3 M_4} F_{M_1 M_2 M_3} = 0,
\]
\[
R_{MN} - \frac{1}{2} g_{MN} R + \frac{1}{2} g_{MN} \left[ 2\Lambda + \frac{H_3}{12} + \frac{m_0^2}{2} B_2^2 + \frac{F_3^2}{12} + \frac{F_2^2}{4} + \frac{F_4^2}{24} \right] - \frac{1}{4} H_{MKL} H_{MN} = 0
\]
\[
\frac{1}{2} F_{MM} F_{N}{}^{M}{}_{A} - m_0^2 B_{MM} B_{N}{}^{M}{}_{A} - \frac{1}{4} F_{MM} F_{M}{}^{M}{}_{A} - \frac{1}{6} F_{MM} F_{M}{}^{M}{}_{A} F_{M}{}^{M}{}_{A} = 0
\]

Using an ansatz of the following kind for the metric and form fields
\[
ds^2 = L^2 \left[ -r^{2a} dt^2 + r^{2b} dx^2 + r^{2c} dy^2 + \frac{dr^2}{r^2} \right]
\]
\[
B_2 = A_2 L^2 r^{a+b} dt \wedge dx, \quad H_3 = A_2 (a + b) L^2 r^{a+b-1} dr \wedge dt \wedge dx,
\]
\[
F_2 = A_1 L^2 r^{a-1} dr \wedge dt, \quad F_3 = B L^3 r^{b+c-1} dr \wedge dx \wedge dy,
\]
\[
F_4 = f_0 L^4 r^{a+b+c-1} dr \wedge dt \wedge dx \wedge dy
\]

where $A_1$, $A_2$ and $B$ are constants. Solving the equations of motion of the $F_2$ and $F_3$ fluxes gives
\[
c_1 L = \frac{(b + c)}{4B} A_1 = \frac{ab}{4A_1},
\]

with the solutions to topological couplings $c_1$ as
\[
16c_1^2 L^2 = a(b + c),
\]

and the solution to $B_2$ equations of motion gives
\[
2m_0^2 L^2 = c(a + b).
\]

The equations of motion that results from the metric components are
\[
4(b^2 + bc + c^2) + (4\Lambda + 2f_0^2 + B^2 + A_1^2) L^2 + A_2^2 [(a + b)^2 + 2m_0^2 L^2] = 0,
\]
\[
4(a^2 + ac + c^2) + (4\Lambda + 2f_0^2 - B^2 - A_1^2) L^2 + A_2^2 [(a + b)^2 + 2m_0^2 L^2] = 0,
\]
\[
4(b^2 + ba + a^2) + (4\Lambda + 2f_0^2 - B^2 - A_1^2) L^2 - A_2^2 [(a + b)^2 + 2m_0^2 L^2] = 0,
\]
\[
4(ba + bc + ca) + (4\Lambda + 2f_0^2 - B^2 + A_1^2) L^2 + A_2^2 [(a + b)^2 - 2m_0^2 L^2] = 0
\]

Solving the equations of eq(51), using eq(50) gives
\[
A_1^2 = \frac{2a(a - b)}{L^2}, \quad B^2 = \frac{2(a - b)(b + c)}{L^2}, \quad \Lambda = -\frac{a^2 + ab + 2b^2 + bc + c^2 + f_0^2 L^2}{2L^2},
\]
\[
A_2^2 = \frac{2(b - c)}{(a + b)}
\]
It just says that in order to have a real solution, we should have a constraint as
\[ a \geq b \geq c > 0, \quad (53) \]
which means the exponents \( a, b \) and \( c \) has to be positive.

Let us take another choice to the ansatz for metric and form fields, instead of taking the ansatz as eq(47) and if we take
\[
\begin{align*}
\text{ds}^2 &= L^2[-r^{2a}dt^2 + r^{2b}dx^2 + r^{2c}dy^2 + \frac{dr^2}{r^2}], \\
B_2 &= A_2L^2r^{b+c}dx \wedge dy, \quad H_3 = A_2(b+c)L^2r^{b+c-1}dr \wedge dx \wedge dy, \\
F_2 &= A_1L^2r^{-c}dr \wedge dx, \quad F_3 = BL^3r^{a+c-1}dr \wedge dt \wedge dy, \\
F_4 &= f_0L^4r^{a+b+c-1}dr \wedge dt \wedge dx \wedge dy \quad (54)
\end{align*}
\]
then the solutions to topological coupling, \( c_1 \) and \( m^2_0 \) as well as to \( A_1, A_2, B \) and \( \Lambda \) are
\[
\begin{align*}
16c_1^2L^2 &= b(a+c), \quad 2m^2_0L^2 = a(b+c), \quad A_1^2 = \frac{2b(a-b)}{L^2}, \quad A_2^2 = \frac{2(a-c)}{(b+c)}, \\
B^2 &= 2\frac{(b-c)(a+c)}{L^2}, \quad \Lambda = -\frac{a^2 + b^2 + ac + bc + 2c^2 + f_0^2L^2}{2L^2} \quad (55)
\end{align*}
\]
This solution makes sense only when
\[ a \geq c \geq b > 0, \quad (56) \]
which means all the exponents are positive and are real. Now we can do a simple renaming of \( b \to c \) and generate another solution. But, the consistent way to do so is to take the following ansatz for metric and form fields
\[
\begin{align*}
\text{ds}^2 &= L^2[-r^{2a}dt^2 + r^{2b}dx^2 + r^{2c}dy^2 + \frac{dr^2}{r^2}], \\
B_2 &= A_2L^2r^{b+c}dx \wedge dy, \quad H_3 = A_2(b+c)L^2r^{b+c-1}dr \wedge dx \wedge dy, \\
F_2 &= A_1L^2r^{-c}dr \wedge dx, \quad F_3 = BL^3r^{a+c-1}dr \wedge dt \wedge dy, \\
F_4 &= f_0L^4r^{a+b+c-1}dr \wedge dt \wedge dx \wedge dy \quad (57)
\end{align*}
\]
then the solutions to topological coupling, \( c_1 \) and \( m^2_0 \) as well as to \( A_1, A_2, B \) and \( \Lambda \) are
\[
\begin{align*}
16c_1^2L^2 &= c(a+b), \quad 2m^2_0L^2 = a(b+c), \quad A_1^2 = \frac{2c(a-c)}{L^2}, \quad A_2^2 = \frac{2(a-b)}{(b+c)}, \\
B^2 &= 2\frac{(b-c)(a+b)}{L^2}, \quad \Lambda = -\frac{a^2 + c^2 + ab + bc + 2b^2 + f_0^2L^2}{2L^2} \quad (58)
\end{align*}
\]
This solution makes sense only when
\[ a \geq b \geq c > 0. \] (59)

So, we have generated several solutions with the desired anisotropy among the coordinates.

4 Conclusion and outlook

We have constructed models in 2+1 dimensional as well as 3+1 dimensional theories in the bulk, which displays a complete anisotropy in not only space and time coordinates but also among the spatial coordinates for the 3+1 dimensional examples. In particular, the exponents for the latter model comes in a specific order that is \( z_1 \geq z_2 \).

If we recall from the brief review article [13], that in one spatial dimension there exists an example of Tomonaga-Luttinger (TL) liquid which does show the quantum phase transition. The studies that we have done in section 2, shows that probably we are moving in the proper direction even though we are not in a stage to directly map the gravity solution of eq(7) to that of the TL liquid. First the TL liquid is a relativistic model and is studied using conformal field theory, whereas here we use a non-relativistic system and with fewer symmetries like time translation, spatial translation and scaling symmetry.

Second, what plays the role of the dimensionless coupling \( g \), from the bulk theory point of view? We cannot take the Newton’s constant to play that role either in 2+1 dimension or 3+1 dimensional, as it is a dimensional object or \( L \).

It is certainly very interesting to know the full criterion required to find either the Lifshitz point or the percolation point, so that we can have a better way to compute and predict things, from the bulk point of view.

The way we have approached to find the gravity dual is by demanding that the field theory system that we have should possess the minimal symmetry such as time translation, spatial translation and more importantly, the scaling symmetry. Under the latter symmetry we want that there should be an asymmetry in the way the time and the spatial coordinate transforms also an asymmetry between the spatial coordinates. For example, in 2+1 dimensional bulk theory the scaling symmetry for the temporal and spatial coordinate

\[ t \rightarrow \lambda^z t, \quad x \rightarrow \lambda x, \quad \rho \rightarrow \frac{\rho}{\lambda} \] (60)

In 3+1 dimensions or higher dimensions, we can construct models for which there is not any asymmetry in the scaling behavior of the spatial coordinates like

\[ t \rightarrow \lambda^z t, \quad x_i \rightarrow \lambda x_i, \quad \rho \rightarrow \frac{\rho}{\lambda}, \] (61)

where \( i \) takes the number of spatial directions. Note, in this case we have increased the number generators, simply due to the fact that we can have rotations among these spatial coordinates, \( x_i \).
However, in 3+1 dimension, if we want a symmetry under scaling
\[ t \rightarrow \lambda^{z_1} t, \quad x \rightarrow \lambda^{z_2} x, \quad y \rightarrow \lambda y, \quad \rho \rightarrow \frac{\rho}{\lambda}, \]
then we do not need to increase the number of generators beyond four and we saw examples of this kind.

So, to summarize this paragraph, we can have a field theoretic system with a dual bulk theory description with the minimum number of symmetries like time translation, spatial translation and scaling with an anisotropy in temporal and spatial coordinates (in 2+1 dimension) as well as among all the field theory coordinates (typically in 3+1 dimension) may gives rise to the appearance of Lifshitz point.

It is certainly very interesting to find the behavior of the ground state as we know that at the critical point, the theory is scale invariant and if the ground state of these systems shows either the full conformal symmetry or the Schrödinger group, then these theories are being said in [21] as conformal quantum critical points and is suggested that a 2 + 1 dimensional field theory showing all these at the quantum conformal critical point must have zero resistance to shear stress in the two dimensional plane, which is interesting to check from dual bulk theory side.

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