A Hybridizable Discontinuous Galerkin and Boundary Element coupling method for electromagnetic simulations

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Abstract. In this paper, the hybridizable discontinuous Galerkin (HDG) method is combined with the electric field integral equation (EFIE) for the numerical simulation of time-harmonic Maxwell’s equation. The key feature of HDG-EFIE is that the HDG method and the boundary element method (BEM) can be coupled naturally through the extra hybrid variable on the faces of elements, the combined method produces a linear system in term of the degrees of freedom of the extra hybrid variable only. A numerical solution was compared to the analytical solution and found to be in excellent agreement.

1. Introduction

Recently, hybridizable discontinuous Galerkin (HDG) [1] have gained much popularity in the numerical solution of electromagnetic scattering problems. HDG method preserves all of the benefits of the DG method, furthermore, by introducing an extra hybrid term on the boundary of the elements, the number of DOFs is drastically reduced. HDG is well prepared for problems with complex media, however, when the problem is restricted to unbounded domain, artificial boundary condition [2] must be imposed, which yields to a great increase of degrees of freedom (DOFs). In contrast, boundary element methods (BEM) [3] works extremely well for solving open scattering and radiation problems involving conducting surfaces but failed to solve complex structure models particularly those with inhomogeneous dielectric. If we want to take advantages of both HDG and BEM, especially for problems with interior finite subdomains enmeshed in exterior unbounded subdomains, it is a natural idea to find a way to couple HDG with BEM. HDG-BEM has been introduced and analyzed by Cockburn [4, 5], Zhixing Fu [6] and their collaborators, which continued the works in DG-BEM and started In Bustinza, Gatica and Sayas [7–9]. In [10] some work had been done for applying HDG-BEM to waveguide problems. Our purpose here is to study the consequences of the combined method on numerical investigation of the electromagnetic scattering problems.

The rest of this article is organized as follows. In section 2, we introduce the Maxwell’s equations and the electric field integral equation (EFIE). In section 3, we propose HDG-EFIE formulation. In Section 4, numerical results are illustrated in details. At last, section 5 in this paper provides some concluding remarks.

2. Problem statement and notations

2.1. Time-harmonic Maxwell’s equations

Let Ω be a bounded domain with a Lipschitz boundary Γ (the artificial boundary) and structure area Ωs ⊆ Ω with boundary Γm. Then we consider the time-harmonic Maxwell’s equations in 2d
The electric field is \( \mathbf{E} = E_z \) and magnetic fields are \( \mathbf{H} = (H_x, H_y) \), the imaginary unit is \( i \), the angular frequency is \( \omega \), \( \varepsilon \) is the relative permittivity and \( \mu \) is the relative permeability. \( \mathbf{E}^{\text{inc}}(\mathbf{r}) \) is the incident wave, \( \mathbf{r} \) and \( \mathbf{r}' \) denote the observation point and the source point, when they are on the same contour \( \Gamma \), the exterior branch of electric field \( \mathbf{E}(\mathbf{r}) \) satisfy the integral equation as follow with \( H_0^{(2)} \) is the modified Bessel function of order zero and \( \mathbf{n} \) is the unit normal vector on \( \Gamma \). To make connection between (1) and (2), we need the following boundary condition on \( \Gamma \),

\[
\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0, \quad \mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0.
\]

2.2. Notations
We use the notations in [1] to give the discretization for further illustration, A triangulation \( \mathcal{T}_h \) of domain \( \Omega \) is the union of \( K \) elements. \( \Gamma_h^b \) is the union of all interior interfaces of \( \mathcal{T}_h \) and \( \Gamma_h^b \) is the union of all the boundary interfaces of \( \mathcal{T}_h \), therefore \( \Gamma_h = \Gamma_h^i \cup \Gamma_h^b \). The polynomial space of degree \( k \) or less are restricted to the domain \( K \in \mathcal{T}_h \) is denoted by \( P_k(K) \). Similarly, \( P_k(F) \) is the space for \( F \in \Gamma_h \). The discrete spaces for a given \( p \geq 0 \) are defined as follow

\[
\begin{align*}
V_h^p := \{ v \in L^2(\Omega) : v|_K \in V^p(K), \forall K \in \mathcal{T}_h \} \\
V_h^p := \{ v \in (L^2(\Omega))^2 : v|_K \in V^p(K), \forall K \in \mathcal{T}_h \} \\
M_h^p := \{ v \in L^2(\Gamma_h) : v|_F \in M^p(F), \forall F \in \Gamma_h \} \\
Y_h^p := \{ v \in L^2(\Gamma_h^b) : v|_F \in Y^p(F), \forall F \in \Gamma_h^b \}
\end{align*}
\]

For \( F \in \Gamma_h^b \), we define jump \([ \cdot ]\) and average \( \{ \cdot \} \)

\[
\begin{align*}
\{v\}_F = \frac{1}{2}(v^+ + v^-), \\
\{v\}_F = \frac{1}{2}(v^+ + v^-), \\
[\mathbf{n} \times v]_F = n^+ \times v^+ + n^- \times v^-, \\
[\mathbf{v} \mathbf{t}]_F = v^+ \mathbf{t}^+ + v^- \mathbf{t}^-.
\end{align*}
\]

The scripts \( + \), \( - \) represent the values of the elements on the different side of \( F \), the values from the left side of \( F \) is \( + \), right side is \( - \). \( \mathbf{n}^\pm \) denote the outward unit norm vector and \( \mathbf{t}^\pm \) is the unit tangent vector to ensure that \( \mathbf{t}^+ \times \mathbf{n}^+ = 1 \) and \( \mathbf{t}^- \times \mathbf{n}^- = 1 \). When applied to boundary faces, these expressions become

\[
\begin{align*}
\{v\}_F = v^+, \\
\{v\}_F = v^+, \\
[\mathbf{n} \times v]_F = n^+ \times v^+, \\
[\mathbf{v} \mathbf{t}]_F = v^+ \mathbf{t}^+.
\end{align*}
\]

3. Hybridization principle
By applying Green’s formulas to (1) and take \( \mathbf{E}_h \) and \( \mathbf{H}_h \) into the boundary terms, a HDG method seeks an approximate solution for

\[
\begin{align*}
i \omega \mathbf{E} - \nabla \times \mathbf{H} &= 0, \\
i \omega \mu \mathbf{H} + \nabla \times \mathbf{E} &= 0. \quad (1)
\end{align*}
\]
Where we used the formulation in [11] to use a hybrid variable $\lambda_h \in M_h$ instead of $(E_h, H_h)$ as
\[
\hat{E}_h = \lambda_h, \quad \forall F \in \partial T_h,
\]
\[
\hat{H}_h = H_h + \tau(E_h - \lambda_h)t \quad F \in \partial K.
\]

The boundary condition for all the interior faces $F$ satisfied in (3), is
\[
\langle [n \times \hat{H}_h]_F, v \rangle_{\Gamma_h} = 0.
\]

To model the exterior equivalent problem, EFIE is employed. To write it into operator formulation, we use the single-layer potential $S$ and double-layer potential $D$ below
\[
S\varphi = \int_{\Gamma} G(\cdot, y)\varphi(y)d\Gamma(y)
\]
\[
D\psi = \int_{\Gamma} \partial_n(\cdot, y)G(\cdot, y)\psi(y)d\Gamma(y)
\]
Using them in (2), then the BEM seeks an approximate solution $E^b_h \in Y_h^b$ for
\[
\frac{1}{2} \langle \hat{E}_h, v \rangle_F - \langle D \cdot E^b_h, v \rangle_F + \langle S \cdot J_h, v \rangle_F = \langle E^{inc}, v \rangle_F
\]
while
\[
\hat{E}_h^b = \lambda_h, \quad \forall F \in \Gamma_h^b,
\]
\[
J_h = \frac{\partial E^b_h}{\partial n_h} = -n_h \times \nabla \times E^b_h = i\omega\mu(n_h \times H^b_h), \quad \forall F \in \Gamma_h^b.
\]

Finally, we are supposed to couple (3) with (4) and seek an approximate solution $(E_h, H_h, \lambda_h, E^b_h, J_h)$ for
\[
\langle i\omega E_h, v \rangle_{\Gamma_h} - \langle \nabla \times H_h, v \rangle_{\Gamma_h} + \langle \tau(E_h - \lambda_h), v \rangle_{\partial T_h} = 0
\]
\[
\langle i\omega \mu H_h, v \rangle_{\Gamma_h} + \langle E_h, \nabla \times v \rangle_{\Gamma_h} - \langle \lambda_h, n \times v \rangle_{\partial T_h} = 0
\]
\[
\langle [n \times \hat{H}_h], v \rangle_{\Gamma_h^t} = 0
\]
\[
i\omega \mu \langle (n \times \hat{H}_h), \eta \rangle_{\Gamma_h^t} + \langle J_h, \eta \rangle_{\Gamma_h^t} = 0
\]
\[
\langle \lambda_h, \eta \rangle_{\Gamma_h^t} - \langle E^b_h, \eta \rangle_{\Gamma_h^t} = 0
\]
\[
\frac{1}{2} \langle \lambda_h, \eta \rangle_{\Gamma_h^t} - \langle D \cdot E^b_h, \eta \rangle_{\Gamma_h^t} + \langle S \cdot J_h, \eta \rangle_{\Gamma_h^t} = \langle E^{inc}, \eta \rangle_{\Gamma_h^t}
\]

4. Numerical experiments
In this section we present some numerical results to demonstrate the performance of the proposed method. Two test cases are considered: scattering plane wave by a dielectric cylinder and a dielectric coated cylinder. The grids for discretization were generated by MATLAB’s PDE toolbox.

4.1. Plane wave scattering by a dielectric cylinder
The first model problem is the electromagnetic scattering of a plane wave by an infinite dielectric cylinder with radius $r = 1m$ and the relative permittivity $\varepsilon = 2.25$. The formulation of incident field is denoted by
\[
E^{inc}(\theta) = e^{-jkr(x\cos\theta + y\sin\theta)}.
\]
To demonstrate the influence of the angular frequency and mesh sizes to the relative discretization error, we choose the incident plane wave to be $\omega = 2k\pi$, and the mesh sizes is $h = al$ with $l = 0.1r$. The contour lines of $E_z$ and $H_x$ are presented in figure 1.

### Table 1. Numerical results for scattering of a plane wave by a dielectric cylinder

| $k$    | $a$    | $E_z$  | $H_x$  |
|--------|--------|--------|--------|
| 1      | 1/2    | 1.15e-2| 3.79e-2|
|        | 1      | 3.12e-2| 5.93e-2|
| $E$    | 1/2    | 2.80e-3| 9.80e-3|
|        | 1/4    | 6.98e-4| 2.50e-2|
| $H$    | 1/2    | 1.00e-3| 1.85e-2|
|        | 1/4    | 3.40e-4| 6.00e-3|

Table 2 summarized the relative error of $E_z$ and $H_x$ according to different mesh size and wave length from the numerical results, while figure 2 shows the contour lines. The radius of the cylinder is $r = 1$, the thickness of coating is $d = 0.2m$. The mesh sizes are $h = al$ with $l = 0.1m$. The relative permittivity of the coating material is 2.25, the incident frequency is $k_f$, with $f = 300$ MHz, $k = 1$, $a = 1/2$ in figure 1. RCS of the scattering is presented in figure 3.

### Table 2: Numerical results for scattering of a plane wave by a dielectric coated cylinder

| $k$    | $a$    | $E_z$  | $H_x$  |
|--------|--------|--------|--------|
| 1      | 1/2    | 2.10e-3| 9.40e-3|
|        | 1      | 1.59e-2| 2.64e-2|
| $E$    | 1/2    | 5.13e-4| 2.30e-3|
|        | 1/4    | 1.28e-4| 5.96e-4|
| $H$    | 1/2    | 5.20e-3| 8.10e-3|
|        | 1/4    | 1.80e-4| 2.60e-3|

4.2. **Plane wave scattering by a dielectric coated cylinder**

Table 2 summarized the relative error of $E_z$ and $H_x$ according to different mesh size and wave length from the numerical results, while figure 2 shows the contour lines. The radius of the cylinder is $r = 1$, the thickness of coating is $d = 0.2m$. The mesh sizes are $h = al$ with $l = 0.1m$. The relative permittivity of the coating material is 2.25, the incident frequency is $k_f$, with $f = 300$ MHz, $k = 1$, $a = 1/2$ in figure 1. RCS of the scattering is presented in figure 3.
Figure 2. Contour lines of a plane wave scattering by a dielectric coated cylinder

Figure 3: RCS of a plane wave scattering by a dielectric coated cylinder

5. Conclusion
Throughout this paper we conduct a feasibility study of HDG-BEM coupling method and give a numerical investigation of HDG-EFIE. The algorithm is applied to two-dimensional TM-polarized scattering problems with open boundary. We implemented the three test cases and these results yielded similar performance. It is observed in table 1 and 2 that the approximation accuracy of the combined method is controlled by both mesh sizes $a$ and wave number $k$, which is similar to the numerical results in [12]. Using HDG-$P_2$ and HDG-$P_3$ don’t improve convergence order. This may lead to that the performance of the proposed method is determined by the BEM part. Such high accuracy for far field scattering as shown in the presented hybrid method are currently under investigation for complex structure. The experimental results will hopefully serve as useful feedback information for improvements for more complex scattering problems.

6. References
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