K3–Fibrations and Heterotic-Type II String Duality

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Abstract

We analyze the map between heterotic and type II N=2 supersymmetric string theories for certain two and three moduli examples found by Kachru and Vafa. The appearance of elliptic j-functions can be traced back to specializations of the Picard-Fuchs equations to systems for $K_3$ surfaces. For the three-moduli example we write the mirror maps and Yukawa couplings in the weak coupling limit in terms of j-functions; the expressions agree with those obtained in perturbative calculations in the heterotic string in an impressive way. We also discuss symmetries of the world-sheet instanton numbers in the type II theory, and interpret them in terms of S–duality of the non-perturbative heterotic string.
1. Introduction

In a very interesting recent paper [1], Kachru and Vafa provided concrete evidence
of the conjecture that the exact non-perturbative behavior of the heterotic string
compactified on $K_3 \times T_2$ is governed by certain Calabi-Yau (CY) manifolds [2],
and can effectively be described in terms of type II strings [3].

In particular, there are examples [1,4] of CY’s that, when taken as background of
type II theories, lead to prepotentials that reproduce certain perturbative corrections
of the heterotic theory in the weak string coupling limit (for non-zero coupling, one
expects new stringy non-perturbative phenomena [5] to become visible, analogous to
rigid N=2 Yang-Mills theory [6]). It is known from explicit heterotic string computa-
tions [4,8] that these corrections are given in terms of elliptic $j$-functions in the
$T_2$ moduli. That precisely such combinations of $j$-functions really do appear [5]
in the moduli spaces of certain CY manifolds, is highly suggestive, and at first rather
surprising.

One of the purposes of this letter is to gain insight in the origin of such modular
functions in the moduli spaces of certain Calabi-Yau’s. We will show that this can
be very simply understood in terms of specializations of Picard-Fuchs equations, and
more abstractly in terms of CY manifolds being elliptic or $K_3$ fibrations. This under-
standing opens up the possibility of a more systematic construction of CY manifolds
that describe the exact quantum theory of $N=2$ supersymmetric heterotic strings.
In particular, it also allows us to perform further non-trivial checks on the original
examples of Kachru and Vafa, by explicitly writing certain “Yukawa couplings”† in
terms of $j$-functions.

We will also briefly investigate the symmetry structure of certain models, link-
ing the symmetries of the CY instanton expansion to the perturbative and non-
perturbative $T$- and $S$-dualities of the quantum heterotic string. In particular, we
find evidence that in some models there is a symmetry of exchanging the heterotic
dilaton $S$ with a target space moduli field, $T$.

† What we mean are the triple derivatives of the prepotential, which have in the present context,
strictly speaking, the interpretation of anomalous magnetic moments.
2. Modular properties of certain Calabi-Yau moduli sub-spaces

The appearance of \( j \)-functions is the key for making the relationship of heterotic strings compactified on \( K_3 \times T_2 \) with type II compactifications evident [1]. Many of the examples of “dual” Calabi-Yau threefolds in [10,11] are actually elliptic fibrations over rational surfaces, or \( K_3 \) fibrations over rational curves. In this section we show how these properties lead to the crucial modular properties of the mirror map (and Yukawa couplings) in the weak string coupling domain.

In fact, \( K_3 \) fibrations are of natural interest for the conjectured duality between heterotic and type II compactifications, because they automatically give rise to prepotentials in the large complex structure limit of the form: \( F = sQ(t) + C(t) \) (where \( Q \) is quadratic polynomial and \( C \) is a cubic polynomial). That is, \( s \) is a good candidate for the heterotic dilaton.

Specifically, consider the model \( X_{12}(1,1,2,2,6)^{-252}, \) which is the first of the examples discussed in [1]. The defining polynomial is

\[
p(x) = x_1^{12} + x_2^{12} + x_3^6 + x_4^6 + x_5^2 - 12\psi x_1 x_2 x_3 x_4 x_5 - 2\phi x_1^6 x_2^6 ,
\]

and the weak coupling limit was identified in [1] with \( y \rightarrow 0 \) and \( x = \frac{2\phi}{(12\psi^2)} \), finite. In terms of these variables, the Picard-Fuchs operators look (\( \theta_x \equiv x \partial_x \), etc.):

\[
D_1 = \theta_x^2 (\theta_x - 2 \theta_y) - 8 (6 \theta_x + 5)(6 \theta_x + 3)(6 \theta_x + 1) \\
D_2 = \theta_y^2 - y (2 \theta_y - \theta_x + 1)(2 \theta_y - \theta_x).
\]

One way of understanding why a modular function appears in the \( y \rightarrow 0 \) limit is via the following two steps. First, the surface (1) is a \( K_3 \) fibration [3] in that there is a linear system \( |L| \) generated by the polynomial of degree one, whose divisors are described after the substitution \( x_1 = \lambda x_2 \) and the single-valued variable change \( \bar{x}_1 = x_1^2 \) as following family of degree 12 \( K_3 \) hypersurfaces in \( \mathbb{P}^{(1,1,1,3)} \):

\[
K : (1 + \lambda^{12} - 2\phi \lambda^6) \bar{x}_1^6 + x_3^6 + x_4^6 + x_5^2 - 12\psi \lambda \bar{x}_1 x_3 x_4 x_5 = 0
\]

As divisors in \(|L|\) are disjoint, \(|L| \cdot |L| = 0\) holds and thus the cubic intersection form has indeed the desired property. Moreover, taking the above limit \( \phi \rightarrow \infty \), \( \psi \sim \phi^{1/6} \bar{\psi} \)

\[\diamond\] We use the notation of [10], e.g. \( X_{d_1,\ldots,d_r}(w_1,\ldots,w_n) \) \( \chi_{h_{1,1}} \) is a complete intersection (hypersurface) of multi-degree \( d_1,\ldots,d_r \) in weighted projective space \( \mathbb{P}^{n-1}(w_1,\ldots,w_n) \) (if all \( w_i = 1 \) we omit them) with Euler number \( \chi \) and Betti number \( h_{1,1} \).

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and \( \lambda \sim \phi^{-1/6} \) all terms in (3) stay finite, and \( x = -\frac{2}{(12\phi^2)^3} \) can be identified as the canonical one-parameter deformation of \( \mathcal{K} \).

Second, there is strong evidence that one-modulus deformations of \( K_3 \) surfaces are intrinsically related to modular functions [12]. That is, it was observed in [13] that the \( W_3 \)-invariant of any single-modulus Picard-Fuchs operator of \( \mathcal{K} \) vanishes, and since \( W_2 \) transforms under coordinate changes \( z \to \zeta(z) \) as \( W_2 \to W_2 + \{\zeta, z\} \), it is possible to rewrite the PF operator, after gauging away \( W_2 \), as \( D = \partial_x^3 \). In order to implement this gauging, one needs to solve a Schwarzian differential equation of the form

\[
\{x, t_x\} = 2Q(x) (\partial_{t_x} x(t_x))^2
\]

for some \( Q(x) \). Its solution \( t_x(x) \) is given by a triangle function \( s(x) \), whose inverse yields a modular function that is automatically associated with some discrete subgroup of \( PSL(2, \mathbb{R}) \). (Equivalently, the mirror map \( x(q_x) \), where \( q_x \equiv e^{2\pi i t_x} \), is given by the ratio of two independent solutions of the associated PF-system.)

It was observed in [12] that in all examples investigated so far this subgroup is given by a subgroup of the modular group \( SL(2, \mathbb{Z}) \) (possibly together with some extra Atkin-Lehner involutions), and the authors conjectured this to be true for general one-modulus deformations of \( K_3 \) arising from orbifold constructions. In the present example, \( Q(x) = \frac{1-1968x+2654208x^2}{4x^2(1728x-1)^2} \), \( t_x = s\left(\frac{1}{2}, \frac{1}{3}; 0; j(q_x)\right) \), and this leads indeed to \( x = 1/j(q_x) \) (this feature of the mirror map for vanishing \( y \) was noticed first in [1]).

The occurrence of this kind of specialization to \( K_3 \) surfaces, with similar modular properties, is actually ubiquitous in the class of complete intersection (hypersurface) CY spaces. Consider, for example, the families

\[
A : \quad X_8(1, 1, 2, 2, 2) \rightarrow 86
B : \quad X_{6,4}(1, 1, 2, 2, 2, 2) \rightarrow 132
C : \quad X_{4,4,4}(1, 1, 2, 2, 2, 2, 2) \rightarrow 112
\]

and their associated PF systems, whose relevant parts are

\[
A : \quad D_1 = \theta_x^2 (2\theta_y - \theta_x) + 4x(4\theta_x + 3)(4\theta_x + 2)(4\theta_x + 1)
B : \quad D_1 = \theta_x^2 (2\theta_y - \theta_x) + 6x(2\theta_x + 1)(3\theta_x + 2)(3\theta_x + 1)
C : \quad D_1 = \theta_x^2 (2\theta_y - \theta_x) + 8x(2\theta_x + 1)^3.
\]

† More precisely, they conjectured the mirror maps to be given by Thompson series, which have an intrinsic relationship to modular functions and to the representation theory of the Conway-Norton groups.
Together with the first example (1), these examples represent selected one-modulus $K_3$ fibrations, and the PF operators (2), (3) effectively reduce under $y \to 0$ to the following list of $K_3$ operators [12]:

| $K_3$ family | PF operator | mod. group |
|--------------|--------------|------------|
| $K$          | $X_0(1,1,1,3)$ | $\theta^4 - 8x(6\theta + 5)(6\theta + 3)(6\theta + 1)$ | $\Gamma_0(1) \cong \Gamma$ |
| $K_A$        | $X_4$        | $\theta^3 - 4x(4\theta + 3)(4\theta + 2)(4\theta + 1)$ | $\Gamma_0(2)_+$ |
| $K_B$        | $X_{3,2}$    | $\theta^3 - 6x(2\theta + 1)(3\theta + 2)(3\theta + 1)$ | $\Gamma_0(3)_+$ |
| $K_C$        | $X_{2,2,2}$  | $\theta^3 - 8x(2\theta + 1)^3$ | $\Gamma_0(4)_+$ |

Model $A$ was briefly discussed in [1], where it was conjectured that the relevant modular group should be given by an extension of some $\Gamma_0(2^k)$; from the table we can infer that this is indeed true, with $k = 1$. The commensurability relations of the $K_3$ mirror maps $x(q_x)$ with the $j$-function were explicitly worked out in [12]:

$K : P(j, x) = 1 - jx = 0,$
$K_A : P(j, x) = 1 + 432x - jx + 62208x^2 + 207jx^2 + 2985984x^3$
$- 3456jx^3 + j^2x^3 = 0,$

etc.

For the first model we immediately recover $x = 1/j(q_x)$. Similarly, the mirror maps for the models $A, B, C$, when written in the form $1/x(q_x) - c$ (with $c = 104, 42, 24$), specialize to the Hauptmodul of $\Gamma_0(N)$ for $N = 2, 3, 4$, while $y(q_x = 0, q_y) = \frac{q_y}{(1+q_y)^2}$. They are given by certain Thompson series [14], which can be written in terms of modular functions as follows:

$A : x(q_x, q_y = 0) = \frac{16(\eta(q_x)\eta(q_x^2)^8}{(\eta^3 + \eta^3_0)^4}$
$B : x(q_x, q_y = 0) = \left(\frac{\eta^{12}(q_x)}{\eta^{12}(q_x^3)} + 729\frac{\eta^{12}(q_x^3)}{\eta^{12}(q_x)} + 54\right)^{-1}$
$C : x(q_x, q_y = 0) = \frac{\eta^{24}(q_x)\eta^{24}(q_x^4)}{\eta^{18}(q_x^2)}$

These expressions might be useful for further checks on the conjectured heterotic-type II string duality.

$\sharp \Gamma_0(N)_+$ denotes a group that in general includes certain Atkin-Lehner involutions besides $\Gamma_0(N)$; see [14] for details.
A simple generalization of (1) and example A would be to take $K_3$ hypersurfaces in weighted $\mathbb{P}^3$ of the form $X_d(1,w_3,w_4,w_5)$, and consider as CY the $K_3$ fibration $X_{2d}(1,1,2w_3,2w_4,2w_5)$. From the examples we have checked, it appears that the discriminant naturally is of the form

$$\Delta = \Delta(K_3)^2 + \ldots ,$$

where the dots denote terms which vanish in an appropriate (weak coupling) limit. There are 95 transversal families of such $K_3$ surfaces \[15\]; 31 of them with $w_1 = 1$ give rise to transversal Calabi-Yau configurations and are listed in the Appendix. It would be very interesting to investigate whether these CY manifolds describe non-perturbative quantum heterotic strings.

In fact, the surface $X_{24}(1,1,2,8,12)_{3}^{-480}$, which was studied too in \[1\], is precisely of this type. The defining polynomial is given by

$$p = x_1^2 + x_2^3 + x_3^{12} + x_4^{24} + x_5^{24} - 12\psi_0 x_1 x_2 x_3 x_4 x_5 - 2\psi_1 (x_3 x_4 x_5)^6 - \psi_2 (x_4 x_5)^{12}, \quad (7)$$

and variables that are appropriate near the point of maximal unipotent monodromy in the large complex structure limit are: $x = -\frac{2\psi_1}{1728^2 \psi_0^5}, y = \frac{1}{\psi_2}, z = -\frac{\psi_1}{4\psi_1}$. For $y \to 0$, the PF-system

$$\begin{align*}
\mathcal{D}_1 &= \theta_y (\theta_x - 2 \theta_z) - 12 x (6 \theta_x + 5) (6 \theta_x + 1) \\
\mathcal{D}_2 &= \theta_y^2 - y (2 \theta_y - \theta_z + 1) (2 \theta_y - \theta_z) \\
\mathcal{D}_3 &= \theta_z (\theta_z - 2 \theta_y) - z (2 \theta_z - \theta_x + 1) (2 \theta_z - \theta_x)
\end{align*} \quad (8)$$

degenerates to the two moduli system of the generic fiber, given by a $K_3$ family of type $X_{12}(1,1,4,6)$. Actually, this $K_3$ is in itself a elliptic fibration over $\mathbb{P}^1$ with generic fiber $X_6(1,2,3)$, as can be seen in an analogous way.

It is quite clear that elliptic fibrations lead very directly to modular functions. Specifically, we present below a table of elliptic curves $\mathcal{E}$, noticing that the present example corresponds to the last entry.

* For $K_3$ fibrations one can always choose a basis s.t. $\mathcal{F} = sQ(t) + C(t)$ and $\int c_2 \wedge s = 24$. Conversely given this topological data one has still to find the suitable projection map. For hypersurfaces in toric varieties there seem to be the combinatorical condition, that the $K_3$ polyhedron is embedded in the Calabi-Yau polyhedron.
Table 1: Families of elliptic curves \( E \), their Picard-Fuchs operators, commensurability relations of the mirror-map \( x(q) \) as defined in \([10,11]\) with the \( j \)-function, and the relevant modular subgroup of \( SL(2, \mathbb{Z}) \). In the first two cases the Hauptmodul of \( \Gamma(3) \) and \( \Gamma(2) \) is related to the mirror map by removing from \( 1/x \) the constant term. In the third case the commensurability polynomial is of genus one, meaning that one needs two generators to define the function field on \( E_3 \).

The PF operator of \( E_3 \) obviously represents the \( y, z \to 0 \) limit of (8). From the commensurability relation of the mirror map of \( E_3 \) with \( j(x) \) we can immediately see that the mirror map of \([11]\) in the limit \( z, y \to 0 \) is given by

\[
x(q_x) = \frac{2}{j(q_x) + \sqrt{j(q_x)(j(q_x) - 1728)}}.
\]

(9)

In addition, it follows from analyzing the corresponding degeneration limits of the PF-system that the mirror-maps \( y(q_y) \) (and \( z(q_z) \)) specialize to rational functions on the boundary of the moduli space, \( x = z = 0 \) (\( x = y = 0 \), resp.): \( y(q_y) = \frac{q_y}{(1+q_y)^2} \), \( z(q_z) = \frac{q_z}{(1+q_z)^2} \). We will use the solution \([3]\) below to provide further evidence in favor of the conjectured heterotic-type II string duality.

Moreover, we can infer from Table 1 that the mirror map \( x(q_x) \) (and \( y(q_y) \)) of the hypersurface of bidegree \((3, 3)\) in \( \mathbb{P}^1 \times \mathbb{P}^1 \), denoted \( X_{3|3}(1, 1, 1, 1, 1, 1)_{2,83}^{-162} \), of \([10]\) for \( y = 0 \) (\( x = 0 \)) is related to the Hauptmodul of \( \Gamma(3) \). Finally, we find that the mirror map \( X_{12}(1, 1, 1, 3, 6)_{3(1),165}^{-324} \) is related to the Hauptmodul of \( \Gamma(2) \) at the boundary \( y = z = 0 \).

† The problem of including the twisted sector in the analysis of the PF-system was recently solved in \([16]\).
3. The three-moduli example $X_{24}(1, 1, 2, 8, 12)$ revisited

We have seen how the appearance of elliptic functions in the mirror maps of Calabi–Yau compactifications can naturally be understood in terms of their special structure as fibrations, at least in the case of one modulus, where we could use the results known in the mathematical literature. Unfortunately, an analogous treatment for more than one modulus does not seem to exist. We will now show that the mirror map and Yukawa couplings of the three-moduli example of $X_{24}(1, 1, 2, 8, 12)$, can nevertheless be written in terms of elliptic functions in the weak coupling limit. This will provide a further impressive non–trivial check on the conjecture of equivalence of the corresponding $N=2$ heterotic and type II strings.

Following Kachru and Vafa [1], we identify $y \sim e^{-a S} \rightarrow 0$ with the weak coupling limit of the heterotic string theory. This identification is motivated by the fact that the discriminant locus of the mirror CY becomes a perfect square, representing the splitting [6] of the classical $SU(2)$ singularity into two branches in the quantum theory. Specifically, the discriminant is

$$\Delta = [ (1 - \tau)^2 - \eta \tau^2 ] \times [ (1 - \tau)^2 - \eta^2 \tau^4 ] \times [ 1 - \eta ] \equiv \Delta_1 \times \Delta_2 \times \Delta_3 , \quad (10)$$

where $\tau = 432x, \eta = 4y, \tau = 4z$. For $\eta \rightarrow 0$, $\Delta$ degenerates into quadratic factors that have the following significance with respect to gauge symmetry enhancements in the heterotic theory:

- $\Delta_1 = 0 : T = U \quad SU(2)$
- $\Delta_2 = 0 : T = U = i \quad SU(2) \times SU(2)$
- $T = U = \rho \quad SU(3)$.

The discriminant factor $\Delta_3$ has the interpretation of a strong-coupling singularity in the heterotic theory. The conjectured duality between the type II theory and the heterotic theory implies that the perturbative $SO(2, 2, \mathbb{Z})$ symmetry of the latter theory should be encoded in the former one. Indeed it turns out that in the limit $\eta \rightarrow 0$ the mirror map for $\tau$ and $\bar{\tau}$ can be written in terms of elliptic $j$-functions. More precisely, using (11) and the fact that $T$ and $U$ should enter symmetrically, we find:

$$\tau = q_1 + \sum_{m+n>1} a_{mn} q_1^m q_3^n = \mu \frac{j(T) + j(U) - \mu}{2 j(T) j(U) + \sqrt{j(T)(j(T) - \mu)} \sqrt{j(U)(j(U) - \mu)}}$$

$$\bar{\tau} = q_3 + \sum_{m+n>1} b_{mn} q_1^m q_3^n = \frac{(j(T) j(U) + \sqrt{j(T)(j(T) - \mu)} \sqrt{j(U)(j(U) - \mu)})^2}{j(T) j(U) (j(T) + j(U) - \mu)^2},$$

(11)
where \( \mu \equiv j(i) = 1728 \) and where we have defined \( q_1 \equiv q_T, q_3 \equiv q_U/q_T \) (we also have verified directly that this corresponds to solutions of the PF equations).

Although on the first glance complicated, eqs. (11) can be recognized as appropriate generalization of the various limits. That is, for the \( SU(2) \) enhanced line, \( T = U \), we find
\[
\bar{x} = \frac{\mu}{2j}, \quad \bar{z} = 1,
\]
and consequently \( \triangle_1 = 0 \). For the points of further enhancement we get
\[
T = U = i : \quad \bar{x} = \frac{1}{2}, \quad \text{and} \quad T = U = \rho : \quad \bar{x} = \infty,
\]
such that in addition \( \triangle_2 = 0 \). Moreover, in the limit \( U \to i\infty \) we recover (12):
\[
\bar{z} = 0, \quad \bar{x} = \frac{\mu}{2 j(T) + \sqrt{j(T)(j(T) - \mu)}}
\]
(and similarly for \( T \to i\infty \)). This equation also reflects the invariance of the defining polynomial (10) under a subgroup of general automorphisms: \( x_i \to x_i, \ i = 3, 4, 5, \ x_2 \to x_2 + a(x_3x_4x_5)^2, \ x_1 \to x_1 + b(x_3x_4x_5)^3 + cx_2x_3x_4x_5 \) that acts non-trivially on the moduli space as follows:
\[
\psi_0 \to i\psi_0, \psi_1 \to \psi_1 + 2\mu\psi_0^6, \psi_2 \to \psi_2
\]
and hence:
\[
\chi_1 \to \frac{\chi_1}{\chi_1 - 1} (\mathbb{Z}_2)
\]
on \( \chi_1 \equiv 1/\bar{x} \). Identifying the invariant expression with \( j(T) \) reproduces (12). Note also that the symmetry (13) exchanges the factors of the discriminant (11):
\[
\triangle_1 \to \bar{x}^4 \triangle_2, \quad \triangle_2 \to \frac{1}{(1 - \bar{x})^4} \triangle_2.
\]

The identifications (11) provide a further, highly non–trivial check on the conjectured string duality. For this purpose we need the translation of the Yukawa couplings (that were determined in [10]) in terms of \( S, T \) and \( U \), as well as the expression for the mirror map of the third Calabi–Yau modulus, \( \overline{y} \). It has the general form
\[
\overline{y} = q_2 \sum_{m+n \geq 1} q_1^m q_3^n + \mathcal{O}(q_2^2) \equiv q_sf_{\overline{y}}(q_1, q_3) + \mathcal{O}(q_2^2), \tag{14}
\]
\( \Diamond \) For the definition of the special coordinates \( t_i = 1/(2\pi i) \ln q_i \), see ref. [10].
with \( q_s = e^{-8\pi^2 S} \), where \( S \) will be identified with the (tree level) dilaton of the heterotic string. Then, using

\[
\partial_T \mathfrak{t} = -j_T(T) \frac{\sqrt{j(U)(j(U)-\mu)}}{\sqrt{j(T)(j(T)-\mu)}} \frac{1}{j(T) + j(U) - \mu} \mathfrak{t}(1 - \mathfrak{t})
\]

\[
\partial_T \mathfrak{y} = \mathfrak{y} \partial_T \ln f_y(q_1, q_3)
\]

\[
\partial_T \mathfrak{z} = -j_T(T) \mathfrak{z} \times \frac{\sqrt{j(T)(j(T) - \mu)}(j(T) + j(U) - \mu) - 2j_T(T) \sqrt{j(U)(j(U) - \mu)}(1 - \mathfrak{z})}{j(T) \sqrt{j(T)(j(T) - \mu)(j(T) + j(U) - \mu)}}
\]

\[
\partial_S \mathfrak{t} = \partial_S \mathfrak{z} = 0, \quad \partial_S \mathfrak{y} = -8\pi^2 \mathfrak{y},
\]

\( (j_T(T) \equiv \partial_T j(T)) \) and the analogous relations obtained by exchanging \( T \) and \( U \), the CY Yukawa couplings given in \[10\] when written in terms of \( S, T, U \) read:

\[
\tilde{K}_{SSS} = \tilde{K}_{SST} = \tilde{K}_{SSU} = \tilde{K}_{STT} = \tilde{K}_{SUU} = 0
\]

\[
\tilde{K}_{STU} = 1
\]

\[
\tilde{K}_{TTT} = \frac{i}{2\pi} \frac{E_4(T)E_4(U)E_6(U)(E_4(T)^3 - E_6(T)^2)}{E_4(U)^3E_6(T)^2 - E_4(T)^3E_6(U)^2}
\]

\[
= -\frac{1}{4\pi^2} \frac{j_T(T)^2j(U)(j(U) - \mu)}{(j(T) - j(U))j_T(T)(j(T) - \mu)j_U(U)}
\]

\[
\tilde{K}_{TTU} = -\frac{i}{2\pi} \frac{E_4(T)^2E_6(T)(E_4(U)^3 - E_6(U)^2)}{E_4(U)^3E_6(T)^2 - E_4(T)^3E_6(U)^2} + \frac{i}{2\pi} \partial_T \ln f_y(q_1, q_3)
\]

\[
= \frac{1}{4\pi^2} \frac{j_T(T)}{j(T) - j(U)} + \frac{i}{2\pi} \partial_T \ln f_y(q_1, q_3).
\]

Here \( E_{4,6} \) are the normalized Eisenstein series, and \( \tilde{K}_{ABC} = 1/(2\pi^2\omega_0)^2 K_{ABC} \), where \( \omega_0 = E_4(T)^{1/4}E_4(U)^{1/4} \) is the fundamental period and the transition from \( K_{ABC} \) to \( \tilde{K}_{ABC} \) corresponds to going to the canonical gauge. The expressions (13) must be compared with the results from perturbative string calculations performed in the heterotic theory \[7,8\]. We find indeed perfect agreement! Note also that corrections to \( K_{TTT} \) arising from the \( T \)-dependence of \( \mathfrak{y} \) cancel in a very non–trivial way. This is just as expected: for the coupling \( K_{TTT} \) we do know the exact expression from the calculations in the heterotic theory, while the corrections to \( K_{TTU} \) proportional to \( f_y(q_1, q_3) \) are not known (this coupling has been determined only to the leading order in \( T - U \)).
As was pointed out to us [17], the mirror map for $\bar{y}$ provides a further independent consistency check. In fact, since $\bar{y}$ is invariant under the CY monodromy group that contains the modular groups for $T$ and $U$, its logarithm should be identified with the modular invariant dilaton defined in [7]:

$$S_{inv} = S - \frac{1}{2} \partial_T \partial_U h^{(1)}(T, U) - \frac{1}{8\pi^2} \ln(j(T) - j(U)) + \text{const.} .$$  \hspace{1cm} (16)

Here $h^{(1)}(T, U)$ is the moduli dependent one-loop contribution to the holomorphic prepotential. Since $h^{(1)}(T, U)$ transforms as modular form of weight $(-2, -2)$ under $(T, U)$ duality transformations up to terms quadratic in $T$ and $U$ [7,8], $\partial_T \partial_U h^{(1)}(T, U)$ is a modular form of weight $(4, 4)$. Consistency of (16) and (14) then implies

$$\partial_T \partial_U \ln f_{\bar{y}}(q_1, q_3) = M(q_1, q_3) + \alpha \partial_T \partial_U \ln(j(T) - j(U)) ,$$  \hspace{1cm} (17)

where $M$ is a modular form of weight $(4, 4)$ that is regular in the fundamental domain, except for a fourth order pole at $T = U$. We indeed find that (17) is fulfilled, with $\alpha = 1$ and $M = 4\pi^2 \partial_U \tilde{K}_{TTT}$.

4. Duality symmetries

Having successfully met further non-trivial checks on the heterotic–type II duality for the model $X_{24}(1, 1, 2, 8, 12)$, we now address some questions beyond leading perturbation theory. Adopting the duality hypothesis, we know that important information about the non–perturbative S- and T-duality of the heterotic string must be encoded in the symmetries of the CY moduli space. Its monodromy properties have been explicitly worked out [9] for the two-moduli example $X_8(1, 1, 2, 2, 2)$, and can be straightforwardly determined for other examples as well, using known results for the periods. Another type of symmetries arise from the defining polynomial, such as the automorphisms (13). Alternatively, we may directly look for symmetries in the instanton expansions, which reflect some of the monodromy properties.

More specifically, analyzing various instanton expansions, we are lead to consider the following symmetries acting on $q_i$ :

\begin{align*}
a) \quad X_{24}(1, 1, 2, 8, 12) & \quad q_1 \rightarrow q_1 q_3 \quad q_3 \rightarrow 1/q_3 \quad (q_2 \equiv 0) \quad \bar{y} \quad T \leftrightarrow U \\
b) \quad X_8(1, 1, 2, 2, 2) & \quad q_3 \rightarrow q_2 q_3 \quad q_2 \rightarrow 1/q_2 \quad \bar{y} \quad ? \\
c) \quad X_{12}(1, 1, 2, 2, 6) & \quad q_1 \rightarrow q_1 q_2 \quad q_2 \rightarrow 1/q_2 \quad \bar{y} \quad ?
\end{align*}
We indicated in the last two columns the relevant modulus and the interpretation of the symmetry in terms of the dual heterotic string; evidence for what the question marks should stand for will be presented below.

Symmetry \( a \) is easily identified as the mirror symmetry of the heterotic string that exchanges the Kähler and complex structure moduli of the compactification torus, \( T_2 \). We did not found a simple generalization to \( q_2 \neq 0 \), possibly indicating a non-perturbative breaking of this symmetry. Furthermore \( \mathcal{T}|_{q_3=1} \neq 1 \) for \( q_2 \neq 0 \), that there is a shift of the singularity \( T = U \). However a clarification of these points is of course strongly connected to the precise relation between heterotic and CY moduli at all orders in \( q_2 \), an information which is beyond the present knowledge.

A consequence of the inversion symmetry \( q_3 \rightarrow 1/q_3 \) is that the mirror map has the property that powers of \( q_3 \) are accompanied by a sufficient number of powers of \( q_1 \). Translated into the heterotic string language, this means that no negative powers of \( q_T, q_U \) appear in the expansions of physical quantities such as the Yukawa couplings.‡ In fact, this property can be traced back to the special form of the basis vectors of the Mori cone. Furthermore, note that the only association of \( t_3 \) with the moduli \( T, U \in \mathbb{H}^+ \) of the heterotic string that is consistent with the symmetry \( q_3 \rightarrow 1/q_3 \), is given by: \( q_3 = q_T/q_U \). Finally, remember from section 2 that the mirror map for \( \mathcal{T} \) reduces to \( q_3/(1 + q_3)^2 \) in the limit \( q_1, q_2 \rightarrow 0 \).

A crucial observation is that these features are shared also by the other two symmetries, \( b \) and \( c \). However, these transformations are quite different from the point of view of the heterotic string, in that they involve \( q_2 \) that is related to the dilaton \( S \).

If we allow \( q_2 \) to have additional dependence on \( T \), and define for convenience \( q_2 = q_S q_T^2 \), the symmetry \( c \) translates to \( T \rightarrow (1 + \alpha)T + S', S' \rightarrow -(1 + \alpha)S' - \alpha(2 + \alpha)T \). Realizing the shift symmetries \( t \rightarrow t + 1 \) for \( t = t_1, t_2, S', T \), requires then \( \alpha \) to be an integer. Imposing in addition the reasonable condition that the positivity of the imaginary parts of \( S' \) and \( T \) (which are \( \sim g^{-2} \) and \( \sim R^2 \), respectively) is preserved, enforces \( \alpha = -1 \), and this implies: \( q_2 = q_S/q_T \). That is, the natural interpretation of the symmetry \( c \) is a symmetry under exchange of \( S \) and \( T \) (and similarly of \( S \) and \( U \) for \( b \)) ! Note also that a linear change of variables, \( S' \rightarrow S' + \alpha T \), does not alter the expressions for the Yukawa couplings, apart from a constant shift.

‡ More precisely, the mirror map for \( \mathcal{T} \) has an additional overall factor of \( q_3 \) that is compatible with (15).
Finally, we point out a remarkable property of the mirror maps $\overline{z}, \overline{y}$ in the models $X_{24}$ ($X_8, X_{12}$). That is,

$$\overline{z}_{X_{24}}|_{q_2=0} = 1, \quad \overline{y}_{X_{24}}|_{q_2=1} = 1, \quad \overline{y}_{X_8,X_{12}}|_{q_2=1} = 1,$$

are independent of $q_1$ (and of $q_3$ for the second equation). Comparing with the discriminants of these models, we see that the singularities $\overline{z} = 1$ (cf., $T = U$), $\overline{y} = 1$ (cf., strong coupling) are precisely at the fixed points of the symmetries $a)\cdots c)$. This means that the singularities correspond to $t_i = 0$ of the relevant moduli. Keeping in mind the relation $t_i = (\sum_i A_i^j \omega_j)/\omega_0$, where $\omega_i$ are the (generically independent) components of the CY period vector, we see that this implies the vanishing of certain linear combinations of the periods. Hence, according to the $N = 2$ mass formula [18], Bogomolnyi states with the appropriate quantum numbers should become massless at these points in the moduli space (provided they exist). The fact that these singularities are not of the simple conifold type indicates that the ideas described in [5] might apply in a more general context.

5. Conclusions

We have amassed further, and we think convincing, evidence in favor of the conjectured duality between heterotic strings compactified on $K_3 \times T_2$, and type II strings compactified on certain Calabi-Yau manifolds. We have also gained insight in the modular properties of certain CY theories in the weak coupling limit, and believe that by focusing on $K_3$ fibrations, many more examples can be systematically studied. We also analyzed the symmetry structure of some models, linking the symmetries of the CY instanton expansion to the perturbative and non-perturbative $T$- and $S$-dualities of the quantum heterotic string. We hope that the methods described in this paper can be developed further, allowing to make new, concrete and truly non-perturbative predictions for $N=2$ supersymmetric string theory.

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Appendix: Calabi-Yau manifolds that are $K_3$ fibrations

| $h_{1,1}$ | $h_{2,1}$ | $\chi$ | deg. | weights |
|-----------|-----------|--------|------|---------|
| 1         | 2         | 86     | 8    | (1, 1, 2, 2, 2) |
| 2         | 2         | 128    | 12   | (1, 1, 2, 2, 6) |
| 3         | 3         | 99     | 10   | (1, 1, 2, 2, 4) |
| 4         | 3         | 243    | 24   | (1, 1, 2, 8, 12) |
| 5         | 4         | 148    | 16   | (1, 1, 2, 4, 8) |
| 6         | 4         | 190    | 20   | (1, 1, 2, 6, 10) |
| 7         | 5         | 101    | 12   | (1, 1, 2, 4, 4) |
| 8         | 5         | 121    | 14   | (1, 1, 2, 4, 6) |
| 9         | 5         | 161    | 18   | (1, 1, 2, 6, 8) |
| 10        | 7         | 143    | 20   | (1, 1, 4, 4, 10) |
| 11        | 7         | 271    | 36   | (1, 1, 4, 12, 18) |
| 12        | 8         | 104    | 16   | (1, 1, 4, 4, 6) |
| 13        | 8         | 164    | 24   | (1, 1, 4, 6, 12) |
| 14        | 8         | 194    | 28   | (1, 1, 4, 8, 14) |
| 15        | 9         | 111    | 18   | (1, 1, 4, 6, 6) |
| 16        | 9         | 125    | 20   | (1, 1, 4, 6, 8) |
| 17        | 9         | 153    | 24   | (1, 1, 4, 8, 10) |
| 18        | 9         | 321    | 48   | (1, 1, 6, 16, 24) |
| 19        | 10        | 194    | 32   | (1, 1, 6, 8, 16) |
| 20        | 10        | 220    | 36   | (1, 1, 6, 10, 18) |
| 21        | 10        | 376    | 60   | (1, 1, 8, 20, 30) |
| 22        | 11        | 131    | 24   | (1, 1, 6, 8, 8) |
| 23        | 11        | 143    | 26   | (1, 1, 6, 8, 10) |
| 24        | 11        | 167    | 30   | (1, 1, 6, 10, 12) |
| 25        | 11        | 227    | 40   | (1, 1, 8, 10, 20) |
| 26        | 11        | 251    | 44   | (1, 1, 8, 12, 22) |
| 27        | 11        | 485    | 84   | (1, 1, 12, 28, 42) |
| 28        | 12        | 164    | 32   | (1, 1, 8, 10, 12) |
| 29        | 12        | 186    | 36   | (1, 1, 8, 12, 14) |
| 30        | 12        | 318    | 60   | (1, 1, 12, 16, 30) |
| 31        | 13        | 229    | 48   | (1, 1, 12, 16, 18) |

Table A.1: Simple hypersurfaces in weighted $\mathbb{P}^4$ which are $K_3$ fibrations.
Table A.2: Simple complete intersections Calabi-Yau spaces in weighted $\mathbb{P}^5$ which are $K_3$ fibrations.
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