INFINITELY MANY SYMMETRIC SOLUTIONS FOR
ANISOTROPIC PROBLEMS DRIVEN BY
NONHOMOGENEOUS OPERATORS

DUSAN D. REPOVŠ
Faculty of Education and Faculty of Mathematics and Physics
University of Ljubljana
Institute of Mathematics, Physics and Mechanics
SI-1000 Ljubljana, Slovenia

Dedicated to Professor Vicentiu Rădulescu for his 60th birthday

Abstract. We are concerned with the existence of infinitely many
radial symmetric solutions for a nonlinear stationary problem driven
by a new class of nonhomogeneous differential operators. Our proof
relies on the symmetric version of the mountain pass theorem.

1. Introduction

Given an even functional on an infinite-dimensional Banach space that fulfills
natural assumptions, the symmetric mountain pass lemma of P. Rabinowitz [18]
establishes the existence of an unbounded sequence of critical values. This re-
sult extends to a symmetric framework the initial version of the mountain pass
theorem due to A. Ambrosetti and P. Rabinowitz [1]. At the same time, the sym-
metric mountain pass theorem can be viewed as an extension of the Ljusternik-
Schnirelmann theorem in the framework of unbounded functionals defined on Ban-
ach spaces. As pointed out by H. Brezis and F. Browder [6], the mountain pass
theorem “extends ideas already present at Poincaré and Birkhoff”. We refer to
Y. Jabri [12] and P. Pucci and V. Rădulescu [17] for more details on the mountain
pass theorem and related applications.

We recall the original statement of the symmetric mountain pass theorem.

Theorem 1.1. Let $X$ be a real infinite-dimensional Banach space and $J \in \mathcal{C}^1(X, \mathbb{R})$
a functional satisfying the Palais-Smale condition and the following hypotheses:

(i) $J(0) = 0$ and there are constants $\rho, \alpha > 0$ such that $J_{|\partial B_\rho} \geq \alpha$;

(ii) $J$ is even; and

(iii) for all finite-dimensional subspaces $X_0 \subset X$, there exists $R = R(X_0) > 0$
such that

$$J(u) \leq 0 \quad \text{for all } u \in X_0 \setminus B_R(X_0).$$

Then $J$ has an unbounded sequence of critical values.

2010 Mathematics Subject Classification. Primary: 35J60; Secondary: 35A15,
35B38, 47H14, 58E05.

Key words and phrases. Anisotropic elliptic problem, nonhomogeneous differential operator, variable exponent, symmetric mountain pass theorem.
This result is an efficient tool for proving multiplicity properties in semilinear or quasilinear elliptic problems with odd nonlinearities and Dirichlet boundary condition. The standard application of Theorem 1.1 concerns the following boundary value problem (see Y. Jabri [12, pp. 122-124])

\[
\begin{cases}
-\Delta u = f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded domain with smooth boundary and \( f : \mathbb{R} \to \mathbb{R} \) is a Carathéodory function with the following properties:

(i) \( f \) is odd in \( u \), that is, \( f(x, -u) = -f(x, u) \);

(ii) there exists \( p \leq 2^* := 2N/(N - 2) \) such that \( f \) satisfies the growth condition

\[ |f(x, u)| \leq C(1 + |u|^{p-1}) \quad \text{a.e. } (x, u) \in \Omega \times \mathbb{R}; \]

(iii) there are constants \( \mu > 2 \) and \( r > 0 \) such that for almost every \( x \in \Omega \) and all \( |u| \geq r \)

\[ 0 < \mu F(x, u) \leq uf(x, u), \quad \text{where } F(x, u) := \int_0^u f(x, t)dt. \]

Under these hypotheses, Theorem 1.1 yields the existence of an unbounded sequence of solutions of problem (1).

The present paper was inspired by recent advances in the study of nonlinear stationary problems driven by nonhomogeneous differential operators. Important pioneering contributions to this field are due to T.C. Halsey [11] and V.V. Zhikov [20] who studied the behaviour of non-Newtonian electrorheological fluids and anisotropic materials. These models strongly rely on partial differential equations with variable exponent, which have been intensively studied in the last few decades. We refer to the recent monograph by V. Rădulescu and D. Repovš [19] for a comprehensive qualitative analysis of nonlinear PDEs with variable exponent by means of variational and topological methods. These problems (with possible lack of uniform convexity) are essentially described by the differential operator

\[ \Delta_{p(x)} u := \text{div} \left( |\nabla|^{p(x)-2} \nabla u \right), \]

which changes its growth properties according to the point. More precisely, the variable exponent \( p(x) \) describes the geometry of a material that is allowed to change its hardening exponent according to the point. Recently, I.H. Kim and Y.H. Kim [13] introduced a new class of nonhomogeneous differential operators, which extend the standard operators with variable exponent. We refer to S. Baraket, S. Chebbi, N. Chorfi, and V. Rădulescu [4] and N. Chorfi and V. Rădulescu [8] for contributions in this new abstract setting.

In order to introduce the problem studied in this paper and our main result, we need to recall some basic notions and properties. We refer to V. Rădulescu and D. Repovš [19], resp. I.H. Kim and Y.H. Kim [13] for more details.

2. Lebesgue and Sobolev spaces with variable exponent

Let

\[ C_+(\mathbb{R}^N) := \{ p : \mathbb{R}^N \to \mathbb{R}; \text{ continuous, } 2 \leq N < \inf_{x \in \mathbb{R}^N} p(x) \leq \sup_{x \in \mathbb{R}^N} p(x) < \infty \}. \]

If \( p \in C_+(\mathbb{R}^N) \), we set \( p^- := \inf_{x \in \mathbb{R}^N} p(x) \) and \( p^+ := \sup_{x \in \mathbb{R}^N} p(x) \).
Then for all $u$ holds:

$$L^p(x)(\mathbb{R}^N) := \left\{ u: \mathbb{R}^N \to \mathbb{R}; \; u \text{ is measurable and } \int_{\mathbb{R}^N} |u(x)|^{p(x)} \, dx < +\infty \right\}$$

and endowed with the norm

$$|u|_{p(x)} := \inf \left\{ \mu > 0; \; \int_{\mathbb{R}^N} \frac{|u(x)|^{p(x)}}{\mu} \, dx \leq 1 \right\}.$$

Let $L^{p'(x)}(\mathbb{R}^N)$ be the dual space of $L^{p(x)}(\mathbb{R}^N)$, where $1/p(x) + 1/p'(x) = 1$. Then for all $u \in L^{p(x)}(\mathbb{R}^N)$ and $v \in L^{p'(x)}(\mathbb{R}^N)$ the following Hölder-type inequality holds:

$$\int_{\mathbb{R}^N} u v \, dx \leq \left( \frac{1}{p(x)} + \frac{1}{p'(x)} \right) |u|_{p(x)} |v|_{p'(x)}.$$

Next, we define the corresponding Sobolev function space with variable exponent by

$$W^{1,p(x)}(\mathbb{R}^N) := \left\{ u \in L^{p(x)}(\mathbb{R}^N); \; |\nabla u| \in L^{p(x)}(\mathbb{R}^N) \right\}.$$ 

This space is endowed with the norm

$$\|u\|_{p(x)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

The critical Sobolev exponent of $p \in C_+(\mathbb{R}^N)$ is defined by

$$p^*(x) := \begin{cases} \frac{Np(x)}{N - p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

The function spaces with variable exponent have some striking properties, namely:

(i) If $p \in C_+(\mathbb{R}^N)$ and $p^* < \infty$, then the formula

$$\int_{\mathbb{R}^N} |u(x)|^p \, dx = p \int_0^\infty t^{p-1} \{ x \in \mathbb{R}^N; \; |u(x)| > t \} \, dt$$

has no variable exponent analogue.

(ii) Variable exponent Lebesgue spaces do not have the mean continuity property. More precisely, if $p$ is continuous and nonconstant in an open ball $B$, then there exists a function $u \in L^{p(x)}(B)$ such that $u(x + h) \notin L^{p(x)}(B)$ for all $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) The function spaces with variable exponent are never translation invariant. The use of convolution is also limited, for instance the Young inequality

$$|f * g|_{p(x)} \leq C \|f\|_{p(x)} \|g\|_{L^1}$$

holds if and only if $p$ is constant.

We refer to [19] for additional properties.

### 3. A new nonhomogeneous differential operator

Assume that $p \in C_+(\mathbb{R}^N)$ and consider the mapping $\phi: \mathbb{R}^N \times [0, \infty) \to [0, \infty)$ that satisfies the following conditions:

(H1) the function $\phi(\cdot, \xi)$ is measurable for all $\xi \geq 0$ and $\phi(x, \cdot)$ is locally absolutely continuous on $[0, \infty)$ for almost all $x \in \Omega$;
(H2) there exist \( a \in L^{p'}(\mathbb{R}^N) \) and \( b > 0 \) such that
\[
|\phi(x, |v|)v| \leq a(x) + b|v|^{p(x)-1}
\]
for almost all \( x \in \Omega \) and for all \( v \in \mathbb{R}^N \);

(H3) there exists \( c > 0 \) such that
\[
\phi(x, \xi) \geq c \xi^{p(x)-2}, \quad \phi(x, \xi) + \xi \frac{\partial \phi}{\partial \xi}(x, \xi) \geq c \xi^{p(x)-2}
\]
for almost all \( x \in \Omega \) and for all \( \xi > 0 \).

For \( \phi \) with the above properties we set
\[
A_0(x, t) := \int_0^t \phi(x, s)sds.
\]

Consider the functional \( A : W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R} \) defined by
\[
A(u) := \int_{\mathbb{R}^N} A_0(x, |\nabla u|)dx.
\]

Assume that hypotheses (H1) and (H2) hold. Then by [13, Lemma 3.2], the nonlinear operator \( A \) is of class \( C^1 \) and its Gâteaux derivative is given by
\[
\langle A'(u), v \rangle = \int_{\mathbb{R}^N} \phi(x, |\nabla u|)\nabla u \nabla v dx, \quad \text{for all } u, v \in W^{1,p(x)}(\mathbb{R}^N).
\]

Let us now assume that hypotheses (H1)–(H3) are fulfilled. By [13, Lemma 3.4], the operator \( A : W^{1,p(x)}(\mathbb{R}^N) \to W^{1,p'(x)}(\mathbb{R}^N) \) is strictly monotone and a mapping of type \((S_+)\), that is, if \( u_n \rightharpoonup u \) in \( W^{1,p(x)}(\mathbb{R}^N) \) as \( n \to \infty \) and \( \limsup_{n \to \infty} \langle A'(u_n) - A'(u), u_n - u \rangle \leq 0 \), then \( u_n \to u \) in \( W^{1,p(x)}(\mathbb{R}^N) \) as \( n \to \infty \).

The nonhomogeneous differential operator \( \text{div} \left( \phi(x, |\nabla u|) \nabla u \right) \), where \( \phi \) satisfies (H1)–(H3) was introduced in [13]. This operator generalizes the usual operators with variable exponent. For instance, if \( \phi(x, \xi) = \xi^{p(x)-2} \) then we obtain the standard \( p(x) \)-Laplace operator, that is, \( \Delta_{p(x)}u := \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) \). The new abstract setting includes the case \( \phi(x, \xi) = (1 + |\xi|^{p(x)})^{(p(x)-2)/2} \), which corresponds to the generalized mean curvature operator
\[
\text{div} \left[ (1 + |\nabla u|^{p(x)-2}/2)\nabla u \right].
\]

The capillarity equation corresponds to
\[
\phi(x, \xi) = \left( 1 + \frac{\xi^{p(x)}}{\sqrt{1 + \xi^{2p(x)}}} \right) \xi^{p(x)-2}, \quad x \in \Omega, \ \xi > 0,
\]
hence the corresponding capillary phenomenon is described by the differential operator
\[
\text{div} \left[ \left( 1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) |\nabla u|^{p(x)-2} \nabla u \right].
\]
4. The main result

Throughout this paper we shall assume that \( p \in C_+ (\mathbb{R}^N) \) and \( p \) is a radial function, that is, \( p (x) = p (|x|) \) for all \( x \in \mathbb{R}^N \). Let
\[
W_{\text{rad}}^{1,p(x)} (\mathbb{R}^N) := \{ u \in W^{1,p(x)} (\mathbb{R}^N) : u \text{ is radial} \}.
\]

We are concerned with the study of the following nonlinear problem
\[
(4) \quad - \text{div} \left( \phi (x, |\nabla u|) \nabla u \right) + \phi (x, |u|) u = V (x) f (u) \quad \text{in} \ \mathbb{R}^N,
\]
where the potential \( V : \mathbb{R}^N \to [0, +\infty) \) and the nonlinearity \( f \) satisfy the following hypotheses:
\[
(5) \quad V \in L^1 (\mathbb{R}^N) \cap L^\infty (\mathbb{R}^N) \text{ is radial and } \exists r_0 > 0 \text{ such that } \inf_{|x| \leq r_0} V (x) > 0,
\]
\[
(6) \quad f \text{ is odd and } \lim_{u \to 0} f (u) / |u|^{p^* - 1} = 0,
\]
and
\[
(7) \quad \text{there exist } \mu > \frac{bp^*}{c} \text{ and } M > 0 \text{ such that } 0 < \mu F (u) \leq uf (u) \text{ for all } |u| \geq M,
\]
where \( F (u) := \int_0^u f (t) dt \).

In this paper, due to the symmetry assumptions imposed to \( p \) and \( V \), we are looking for radial solutions of problem \((4)\).

We say that \( u \in W_{\text{rad}}^{1,p(x)} (\mathbb{R}^N) \setminus \{ 0 \} \) is a solution of problem \((4)\) if
\[
\int_{\mathbb{R}^N} \left[ \phi (x, |\nabla u|) \nabla u \cdot \nabla v + \phi (x, |u|) uv \right] dx = \int_{\mathbb{R}^N} V (x) f (u) v dx dx,
\]
for all \( v \in W_{\text{rad}}^{1,p(x)} (\mathbb{R}^N) \).

The energy functional associated to problem \((4)\) is \( \mathcal{E} : W^{1,p(x)} (\mathbb{R}^N) \to \mathbb{R} \) defined by
\[
\mathcal{E} (u) := \int_{\mathbb{R}^N} [A_0 (x, |\nabla u|) + A_0 (x, |u|)] dx - \int_{\mathbb{R}^N} V (x) F (u) dx.
\]

Let \( \mathcal{E}_0 \) denote \( \mathcal{E} \) restricted to the function space \( W_{\text{rad}}^{1,p(x)} (\mathbb{R}^N) \). By the isometric Palais principle [16] (see also [14, Theorem 1.50]), any critical point of \( \mathcal{E}_0 \) is also a critical point of \( \mathcal{E} \). This shows that finding radially symmetric solutions of problem \((4)\) reduces to finding the nontrivial critical points of the energy functional \( \mathcal{E}_0 \).

**Theorem 4.1.** Assume that hypotheses \((H1)-(H3)\), \((5)\), \((6)\), and \((7)\) are fulfilled. Then problem \((4)\) has infinitely many solutions.

As we shall see in the proof of this result, problem \((4)\) still has at least one (radially symmetric) solution, provided that the oddness symmetry hypothesis on \( f \) is removed.

Theorem 4.1 extends the pioneering multiplicity result of A. Ambrosetti and P. Rabinowitz [1, Theorem 3.13] in the following directions:
(i) the standard (linear) second order uniformly elliptic operator \( \Sigma_{i,j=1}^N (a_{ij} (x) u_{x_i})_{x_j} \) is replaced by the nonhomogeneous differential operator \( \text{div} (\phi (x, |\nabla u|) \nabla u) \);
(ii) our study is performed in the entire Euclidean space, instead of a bounded domain with smooth boundary. However, in our abstract setting, the lack of compactness of \( \mathbb{R}^N \) is compensated by the compactness of the embedding of the space \( W_{\text{rad}}^{1,p(x)} (\mathbb{R}^N) \) into \( L^\infty (\mathbb{R}^N) \), provided that \( N < p^- \leq p^+ < +\infty \).
We first check that $\mathcal{E}_0$ satisfies the geometric hypotheses of the mountain pass theorem.

**Step 1.** There exist positive constants $\rho$ and $\alpha$ such that $\mathcal{E}_0(u) \geq \alpha$ for all $u \in W^{1,p(x)}(\mathbb{R}^N)$ with $\|u\| = \rho$.

For $\rho \in (0,1)$ (to be prescribed later), we fix $u \in W^{1,p(x)}(\mathbb{R}^N)$ satisfying $\|u\|_{p(x)} = \rho$. By hypothesis (H3), we have

$$
\int_{\mathbb{R}^N} [A_0(x,\nabla u) + A_0(x,|u|)] \, dx \geq \frac{c}{p^+} \int_{\mathbb{R}^N} (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx.
$$

Next, we use relation (1.8) in [19, p. 11]. Thus, since $\|u\|_{p(x)} = \rho < 1$, we have

$$
\text{(8)} \quad \int_{\mathbb{R}^N} [A_0(x,\nabla u) + A_0(x,|u|)] \, dx \geq \frac{c}{p^+} \|u\|_{p(x)}^{p^+}.
$$

On the other hand, assumption (6) implies that $F(u)/|u|^{p^+} \to 0$ as $u \to 0$. Fix $\varepsilon > 0$. It follows that if $\rho > 0$ is small enough then

$$
\int_{\mathbb{R}^N} V(x) F(u) \, dx \leq \varepsilon \|V\|_{L^1} \|u\|_{L^{p^+}}^{p^+}.
$$

Since $W^{1,p(x)}(\mathbb{R}^N)$ is continuously embedded into $L^\infty(\mathbb{R}^N)$, we deduce that there exists $C > 0$ such that for all $u \in W^{1,p(x)}(\mathbb{R}^N)$ with $\|u\|_{p(x)} = \rho$ we have

$$
\text{(9)} \quad \int_{\mathbb{R}^N} V(x) F(u) \leq C \varepsilon \|u\|_{p(x)}^{p^+}.
$$

Combining relations (8) and (9), we obtain

$$
\mathcal{E}_0(u) \geq \frac{c}{p^+} \|u\|_{p(x)}^{p^+} - C \varepsilon \|u\|_{p(x)}^{p^+}
= \frac{c}{p^+} \rho^{p^+} - C \varepsilon \rho^{p^+}.
$$

Choosing $\varepsilon = c/(2Cp^+) > 0$, we have

$$
\mathcal{E}_0(u) \geq \frac{c}{2p^+} \rho^{p^+} =: \alpha > 0.
$$

**Step 2.** For all finite-dimensional subspaces $X_0 \subset W^{1,p(x)}(\mathbb{R}^N)$ there exists $R = R(X_0) > 0$ such that

$$
\mathcal{E}_0(u) \leq 0 \quad \text{for all } u \in X_0 \setminus B_R(X_0).
$$

We first claim that for all $w \in W^{1,p(x)}(\mathbb{R}^N)$ with $\|w\|_{p(x)} = 1$ there exists $\lambda(w) > 0$ such that

$$
\mathcal{E}_0(\lambda w) < 0 \quad \text{for all } \lambda \in \mathbb{R} \text{ with } |\lambda| \geq \lambda(w).
$$

We observe that hypothesis (7) implies that there are positive constants $C_1$ and $C_2$ such that

$$
f(u) \geq C_1 u^{\mu - 1} - C_2 \quad \text{for all } u \geq 0.
$$

Therefore

$$
F(u) \geq \frac{C_1}{\mu} u^\mu - C_2 u \quad \text{for all } u \geq 0.
$$

It follows that there exists $C_0 > 0$ such that

$$
\text{(11)} \quad F(u) \geq C_0 u^\mu \quad \text{for all } |u| \geq M.
$$
Fix $w \in W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N)$ with $\|w\|_{p(x)} = 1$ and $\lambda \in \mathbb{R}$ (with $|\lambda| > 1$).

In order to find an upper estimate for $\mathcal{E}_0(\lambda w)$, we first observe that using hypothesis (H2), we have

$$
(12) \quad \int_{\mathbb{R}^N} [A_0(x, |\lambda \nabla w|) + A_0(x, |\lambda w|)] dx \leq
$$

$$
|\lambda| \int_{\mathbb{R}^N} a(x)(|\nabla w| + |w|) dx + b \int_{\mathbb{R}^N} \left( \int_0^{[\lambda \nabla w]} s^{p(x) - 1} ds + \int_0^{[\lambda w]} s^{p(x) - 1} ds \right) dx \leq
$$

$$
|\lambda| \int_{\mathbb{R}^N} a(x)(|\nabla w| + |w|) dx + \frac{b|\lambda|^{p^*}}{p^*} \int_{\mathbb{R}^N} (|\nabla w|^{p(x)} + |w|^{p(x)}) dx =
$$

$$
\frac{b|\lambda|^{p^*}}{p^*} + C|\lambda|,
$$

where $C > 0$ is a constant depending only on $|a|_{p'(x)}$ and the best constant of the continuous embedding $W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^{p(x)}(\mathbb{R}^N)$.

By (11) we have

$$
(13) \quad \int_{|\lambda w| \geq M} V(x) F(\lambda w) dx \geq C_0 \int_{|\lambda w| \geq M} |\lambda|^\mu |w|^\mu dx = C_0 |\lambda|^{\mu} \int_{|\lambda w| \geq M} V(x)|w|^\mu dx.
$$

Since $F$ is bounded on the interval $[-M, M]$, there exists $C > 0$ such that $F(t) \geq -C$ for all $t \in [-M, M]$. It follows that

$$
(14) \quad \int_{|\lambda w| < M} V(x) F(\lambda w) dx \geq -C \int_{|\lambda w| < M} V(x) dx \geq -C \int_{\mathbb{R}^N} V(x) dx = -C \|V\|_{L^1}.
$$

Estimates (12), (13), and (14) imply

$$
\mathcal{E}_0(\lambda w) \leq \frac{b}{p^*} |\lambda|^{p^*} + C|\lambda| - C_0 |\lambda|^{\mu} \int_{|\lambda w| \geq M} V(x)|w|^\mu dx + C \|V\|_{L^1},
$$

hence $\mathcal{E}_0(\lambda w) \to -\infty$ as $|\lambda| \to \infty$. This proves our claim (10).

Next, since the space $W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N)$ is compactly embedded into $L^{\infty}(\mathbb{R}^N)$, we deduce that there exists $C > 0$ such that

$$
|w(x)| \leq C \quad \text{for all } w \in W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N) \text{ with } \|w\|_{p(x)} = 1.
$$

This fact implies that our initial claim (10) can be improved as follows: for all $w \in W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N)$ with $\|w\|_{p(x)} = 1$ there exist $\lambda(w) > 0$ and $\eta(w) > 0$ such that

$$
(15) \quad \mathcal{E}_0(\lambda z) < 0 \quad \forall |\lambda| \geq \lambda(w), \forall z \in W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N), \|z\|_{p(x)} = 1, \|z - w\| \leq \eta(w).
$$

Returning to Step 2, let $X_0 \subset W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N)$ be a finite-dimensional subspace. Thus the set $X_0 \cap \{w \in W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N); \|w\|_{p(x)} = 1\}$ is compact. Next, using (15), we deduce that there exists $\lambda_0 > 0$ depending only on $X_0$ such that

$$
\mathcal{E}_0(\lambda w) \leq 0 \quad \text{for all } |\lambda| \geq \lambda_0 \text{ and for all } w \in X_0, \|w\|_{p(x)} = 1.
$$

Choosing $R(X_0) = \lambda_0$, we obtain the statement contained in our Step 2.

**Step 3.** Any Palais-Smale sequence of $\mathcal{E}_0$ is bounded.

We recall that $(u_n) \subset W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N)$ is a Palais-Smale sequence of $\mathcal{E}_0$ if

$$
(16) \quad \mathcal{E}_0(u_n) = O(1) \quad \text{and} \quad \|\mathcal{E}_0'(u_n)\| = o(1) \quad \text{as } n \to \infty.
$$
We also recall that $\mathcal{E}_0$ satisfies the Palais-Smale condition if any Palais-Smale sequence $(u_n) \subset W^{1, p(x)}_\text{rad}(\mathbb{R}^N)$ of $\mathcal{E}_0$ is relatively compact.

Arguing by contradiction and normalizing, we assume that (up to a subsequence) $u_n = \lambda_n v_n$, where $\lambda_n = \|u_n\|_{p(x)} \to \infty$ and $\|v_n\|_{p(x)} = 1$. By (16) we deduce that (17)

$$o(\|u_n\|) = \mathcal{E}_0'(u_n), u_n) = \int_{\mathbb{R}^N} [\phi(x, |\nabla u_n|)|\nabla u_n|^2 + \phi(x, |u_n|)u_n^2]dx$$

$$\quad - \int_{|\lambda_n|v_n| \geq M} V(x) f(\lambda_n v_n)\lambda_n v_n dx - \int_{|\lambda_n|v_n| < M} V(x) f(\lambda_n v_n)\lambda_n v_n dx.$$

Using hypothesis (H2), relation (17) yields

$$\lambda_n o(1) \leq \lambda_n \int_{\mathbb{R}^N} a(x)(|\nabla v_n| + |v_n|)dx + b \int_{\mathbb{R}^N} \lambda_n^{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx$$

$$\quad - \int_{|\lambda_n|v_n| \geq M} V(x) f(\lambda_n v_n)\lambda_n v_n dx - \int_{|\lambda_n|v_n| < M} V(x) f(\lambda_n v_n)\lambda_n v_n dx.$$

It follows that (18)

$$\int_{|\lambda_n|v_n| \geq M} V(x) f(\lambda_n v_n)\lambda_n v_n dx \leq \lambda_n \int_{\mathbb{R}^N} a(x)(|\nabla v_n| + |v_n|)dx +$$

$$b \int_{\mathbb{R}^N} \lambda_n^{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx - \int_{|\lambda_n|v_n| < M} V(x) f(\lambda_n v_n)\lambda_n v_n dx + \lambda_n o(1).$$

By hypothesis (7), we deduce that (19)

$$\int_{|\lambda_n|v_n| \geq M} V(x) F(\lambda_n v_n)dx \leq \frac{1}{\mu} \int_{|\lambda_n|v_n| \geq M} V(x) f(\lambda_n v_n)\lambda_n v_n dx.$$

Using hypothesis (H3), we have (20)

$$\int_{\mathbb{R}^N} [A_0(x, |\nabla u_n|) + A_0(x, |u_n|)]dx =$$

$$\int_{\mathbb{R}^N} \left( \int_0^{\nabla u_n} s \phi(x, s)ds + \int_0^{u_n} s \phi(x, s)ds \right) dx \geq$$

$$c \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx = c \int_{\mathbb{R}^N} \frac{\lambda_n^{p(x)}}{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx \geq$$

$$\frac{c}{p^+} \int_{\mathbb{R}^N} \lambda_n^{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx.$$

Combining relations (19) and (20) in relationship with (16), we deduce

$$O(1) = \mathcal{E}_0(u_n) \geq \frac{c}{p^+} \int_{\mathbb{R}^N} \lambda_n^{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx$$

$$- \frac{1}{\mu} \int_{|\lambda_n|v_n| \geq M} V(x) f(\lambda_n v_n)\lambda_n v_n dx - \int_{|\lambda_n|v_n| < M} V(x) F(\lambda_n v_n)dx.$$
Using now the estimate established in (18), we obtain

\[ O(1) \geq \frac{c}{p^+} \int_{\mathbb{R}^N} \lambda_n^{p(x)} (|\nabla u_n|^{p(x)} + |v_n|^{p(x)}) dx - \frac{\lambda_n}{\mu} \int_{\mathbb{R}^N} a(x)(|\nabla u_n| + |v_n|) dx \\
- \frac{b}{\mu} \int_{\mathbb{R}^N} \lambda_n^{p(x)} (|\nabla v_n|^{p(x)} + |v_n|^{p(x)}) dx \\
+ \frac{1}{\mu} \int_{[\lambda_n|v_n| < M]} V(x) f(v_n) v_n dx - \int_{[\lambda_n|v_n| < M]} V(x) F(v_n) dx + \lambda_n o(1) \\
= \left( \frac{c}{p^+} - \frac{b}{\mu} \right) \int_{\mathbb{R}^N} \lambda_n^{p(x)} (|\nabla u_n|^{p(x)} + |v_n|^{p(x)}) dx \\
- \frac{\lambda_n}{\mu} \int_{\mathbb{R}^N} a(x)(|\nabla u_n| + |v_n|) dx \\
+ \frac{1}{\mu} \int_{[\lambda_n|v_n| < M]} V(x) f(v_n) v_n dx - \int_{[\lambda_n|v_n| < M]} V(x) F(v_n) dx + \lambda_n o(1). \]

Therefore

\[ O(1) = \mathcal{E}_0(u_n) \geq \left( \frac{c}{p^+} - \frac{b}{\mu} \right) \int_{\mathbb{R}^N} \lambda_n^{p(x)} (|\nabla u_n|^{p(x)} + |v_n|^{p(x)}) dx \\
- \frac{\lambda_n}{\mu} \int_{\mathbb{R}^N} a(x)(|\nabla u_n| + |v_n|) dx + \lambda_n o(1). \]

Combining hypothesis (7) with relation (21) and the fact that \( \lim_{n \to \infty} \lambda_n = +\infty \), we deduce that \( \mathcal{E}_0(u_n) \to +\infty \) as \( n \to \infty \), which contradicts (16). We prove that the Palais-Smale sequence \( (u_n) \) is bounded in \( W^{1,p(x)}(\mathbb{R}^N) \). We shall prove that in fact, the sequence converges strongly (up to a subsequence) in \( W^{1,p(x)}(\mathbb{R}^N) \). For this purpose, we apply some ideas developed by R. Filippucci, P. Pucci and V. Rădulescu [10, pp. 712-713].

**Step 4.** Any Palais-Smale sequence of \( \mathcal{E}_0 \) is relatively compact in \( W^{1,p(x)}(\mathbb{R}^N) \).

Let \( (u_n) \subset W^{1,p(x)}(\mathbb{R}^N) \) be an arbitrary Palais-Smale sequence of \( \mathcal{E}_0 \). By Step 3, \( (u_n) \) is bounded. Using the compact embedding \( W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N) \), we can assume that, up to a subsequence,

\[ u_n \to u \quad \text{in} \quad W^{1,p(x)}(\mathbb{R}^N) \]

\[ u_n \to u \quad \text{in} \quad L^\infty(\mathbb{R}^N). \]

By hypotheses (H2) and (H3) in conjunction with relations (22) and (23), we deduce that

\[ \int_{\mathbb{R}^N} \phi(x, |u_n|) u_n (u_n - u) dx - \int_{\mathbb{R}^N} \phi(x, |u|) u (u_n - u) dx \to 0 \quad \text{as} \quad n \to \infty \]

and

\[ \int_{\mathbb{R}^N} \phi(x, |\nabla u_n|) \nabla u_n \cdot \nabla (u_n - u) dx - \int_{\mathbb{R}^N} \phi(x, |\nabla u|) \nabla u \cdot \nabla (u_n - u) dx \to 0 \quad \text{as} \quad n \to \infty. \]

Recall that Proposition 3.3 in [13] established that under hypotheses (H1) and (H3) the following Simon-type inequality holds:

\[ \phi(x, |u|) (u - v) - \phi(x, |v|) (v - u) \geq 4^{1-p^+} c |u - v|^{p(x)} \quad \text{for all} \quad x \in \Omega. \]

Using this inequality, relations (24) and (25) imply that

\[ u_n \to u \quad \text{in} \quad W^{1,p(x)}(\mathbb{R}^N). \]
We conclude that any Palais-Smale sequence of $\mathcal{E}_0$ is relatively compact in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$, hence $\mathcal{E}_0$ satisfies the Palais-Smale condition.

Next, we observe that $\mathcal{E}_0$ is even and $\mathcal{E}_0(0) = 0$. The above steps show that $\mathcal{E}_0$ fulfills the hypotheses of Theorem 1.1. We deduce that problem (4) has infinitely many solutions in $W^{1,p(x)}_{rad}(\mathbb{R}^N)$. \hfill \Box

6. Epilogue

The methods developed in this paper can be applied to other recent classes of nonhomogeneous differential operators. For instance, the operator $\text{div} (\phi(x, |\nabla u|)\nabla u)$ that describes problem (4) can be replaced with $\text{div} [\phi(|\nabla u|)\nabla u]$, where for some $1 < p < q < N$, the function $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfies the following conditions:

1. $\phi(0) = 0$;
2. there exists $c_1 > 0$ such that $\phi(t) \geq c_1 t^{p/2}$ if $t \geq 1$ and $\phi(t) \geq c_1 t^{q/2}$ if $0 \leq t \leq 1$;
3. there exists $c_2 > 0$ such that $\phi(t) \leq c_2 t^{p/2}$ if $t \geq 1$ and $\phi(t) \leq c_2 t^{q/2}$ if $0 \leq t \leq 1$;
4. there exists $0 < \mu < 1/s$ such that $2t\phi'(t) \leq s\mu \phi(t)$ for all $t \geq 0$;
5. the mapping $t \mapsto \phi(t^2)$ is strictly convex.

This operator was introduced by A. Azzollini, P. d’Avenia, and A. Pomponio [3] and it is described by a potential with different growth near zero and at infinity (the double-power growth hypotheses). We refer to A. Azzollini [2] and N. Chorfi and V. Rădulescu [7] for recent contributions in connection with the abstract setting generated by this operator.

In a general framework, the presence of two variable exponents $p_1(x)$ and $p_2(x)$ dictates the geometry of a composite that changes its hardening exponent according to the point. Problems with nonstandard growth conditions of $(p,q)$-type have been initially studied by P. Marcellini [15] who was interested in the properties of the integral energy functional $\int_{\mathbb{R}^N} F(x, \nabla u) \, dx$, where $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies unbalanced polynomial growth conditions, namely

$$|\xi|^p \lesssim F(x, \xi) \lesssim |\xi|^q \quad \text{with} \quad 1 < p < q.$$ 

We believe that the main result of this paper can be extended to “unbalanced” anisotropic differential operators of the type

$$-\text{div} (\phi(x, |\nabla u|)\nabla u) - \text{div} (a(x)\psi(x, |\nabla u|)\nabla u)$$

and

$$-\text{div} (\phi(x, |\nabla u|)\nabla u) - \text{div} (a(x)\psi(x, |\nabla u|) \log(e + |x|)\nabla u).$$

This abstract setting is in close relationship with the recent contributions of G. Mingione et al. [5, 9], who studied non-autonomous problems with associated energies of the type

$$\int_{\mathbb{R}^N} [||\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)}] \, dx$$

and

$$\int_{\mathbb{R}^N} [||\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)} \log(e + |x|)] \, dx,$$

where $p_1(x) \leq p_2(x)$, $p_1 \neq p_2$, and $a(x) \geq 0$. 

Acknowledgements

This work was supported by the Slovenian Research Agency grants P1-0292, J1-8131, J1-7025, and N1-0064.

References

[1] (MR0370183) A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Functional Anal., 14 (1973), 349–381.

[2] (MR3431651) [10.1112/jlms/jdv050] A. Azzollini, Minimum action solutions for a quasilinear equation, J. Lond. Math. Soc., 92 (2015), 583–595.

[3] (MR3148112) [10.1007/s00526-012-0578-0] A. Azzollini, P. d’Avenia and A. Pomponio, Quasilinear elliptic equations in $\mathbb{R}^N$ via variational methods and Orlicz-Sobolev embeddings, Calc. Var. Partial Differential Equations, 49 (2014), 197–213.

[4] (MR3709043) [10.1515/ans-2016-6020] S. Baraket, S. Chebbi, N. Chorfi and V. Rădulescu, Non-autonomous eigenvalue problems with variable $(p_1,p_2)$-growth, Advanced Nonlinear Studies, 17 (2017), 781–792.

[5] (MR3570955) [10.1090/spmj/1392] P. Baroni, M. Colombo and G. Mingione, Non-autonomous functionals, borderline cases and related function classes, St. Petersburg Mathematical Journal, 27 (2016), 347–379.

[6] (MR1617413) [10.1006/aima.1997.1713] H. Brezis and F. Browder, Partial differential equations in the 20th century, Adv. Math., 135 (1998), 76–144.

[7] (MR3513973) N. Chorfi and V. Rădulescu, Standing wave solutions of a quasilinear degenerate Schroedinger equation with unbounded potential, Electronic Journal of the Qualitative Theory of Differential Equations, 37 (2016), 1–12.

[8] (MR3677858) [10.1016/j.jmaa.2017.05.007] N. Chorfi and V. Rădulescu, Small perturbations of elliptic problems with variable growth, Applied Mathematics Letters, 74 (2017), 167–173.

[9] (MR3360738) [10.1007/s00205-015-0859-9] M. Colombo and G. Mingione, Bounded minimisers of double phase variational integrals, Archive for Rational Mechanics and Analysis, 218 (2015), 219–273.

[10] (MR2424374) [10.1080/03605300701518208] R. Filippucci, P. Pucci and V. Rădulescu, Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions, Comm. Partial Differential Equations, 33 (2008), 706–717.

[11] T. C. Halsey, Electrorheological fluids, Science, 258 (1992), 761–766.

[12] (MR2012778) [10.1017/CBO978051146655] Y. Jabri, The Mountain Pass Theorem. Variants, Generalizations and Some Applications, Encyclopedia of Mathematics and its Applications, vol. 95, Cambridge University Press, Cambridge, 2003.

[13] (MR3336943) [10.1007/s00229-014-0718-2] I. H. Kim and Y. H. Kim, Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents, Manuscripta Math., 147 (2015), 169–191.

[14] (MR2683404) [10.1017/CBO9780511760631] A. Kristaly, V. Rădulescu and C. Varga, Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems,
Encyclopedia of Mathematics and its Applications, No. 136, Cambridge University Press, Cambridge, 384, 2010.

[15] (MR1094446) [10.1016/0022-0396(91)90158-6] P. Marcellini, Regularity and existence of solutions of elliptic equations with $(p,q)$-growth conditions, *J. Differential Equations*, **90** (1991), 1–30.

[16] (MR547524) [10.1007/BF01941322] R. Palais, The principle of symmetric criticality, *Commun. Math. Phys.*, **69** (1979), 19–30.

[17] (MR2742781) P. Pucci and V. Rădulescu, The impact of the mountain pass theory in nonlinear analysis: A mathematical survey, *Boll. Unione Mat. Ital.*, Series IX, **3** (2010), 543–582.

[18] (MR845785) [10.1090/cbms/065] P. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics, vol. 65, American Mathematical Society, Providence, RI, 1986.

[19] (MR3379920) [10.1201/b18601] V. Rădulescu and D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, CRC Press, Taylor & Francis Group, Boca Raton FL, 2015.

[20] (MR1209262) V. V. Zhikov, Lavrentiev phenomenon and homogenization for some variational problems, *C. R. Acad. Sci. Paris, Sér. I*, **316** (1993), 435–439.

E-mail address: dusan.repovs@guest.arnes.si