Remarks on replica diagonal collective field condensations in SYK

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Abstract: In the Sachdev-Ye-Kitaev model with generic order \( q \geq 4 \) random couplings, we compute the critical temperature relating the Majorana fermions high temperature perturbative vacuum to the vacuum where the replica diagonal collective field \( G(\tau, \tau') \) condenses. We study, by a finite temperature diagrammatic analysis, the effective action of an auxiliary Hubbard-Stratonovich bilocal field related to \( G(\tau, \tau') \) in the large \( N \) limit. Subtleties that arise in switching from the operatorial to the functional integral representation of the SYK thermal partition function are also discussed.

Keywords: 1/N Expansion, Field Theories in Low Dimensions, Disordered Systems
1 Introduction

The Sachdev-Ye-Kitaev model (SYK) has recently driven intense activity both in the condensed matter [1–5] and in the high energy communities [6–9]. It is a disordered quantum mechanics of $N$ Majorana fermions with the remarkable property of being exactly solvable in the large $N$ limit, even in the strongly coupled regime. As a solvable many-body system it provides a tool for modelling the behavior of strange metals [3, 4]. Interestingly enough, its peculiar properties transformed SYK in a good playground for studying black hole physics [6, 7].

Large $N$ solvability is due to the dominance of a set of Feynman diagrams, the so-called melonic diagrams. The model has also a convenient master fields reformulation, in terms of two bilinear in the Majorana degrees of freedom and bilocal collective fields $G(\tau, \tau')$ and $\Sigma(\tau, \tau')$, that are evocative of a bulk description from the holographic point of view. In the deep low energy limit SYK exhibits non-trivial conformal invariance, which is spontaneously broken by condensation of the collective fields. Slightly away from the conformal point, the residual of conformal invariance is also broken explicitly and the time reparametrization mode ceases to be a zero mode of the action and becomes the soft mode of an effective local Schwarzian action. Properties of the soft mode correspond holographically [9–14] to properties of two dimensional Jackiw-Teitelboim gravity [15, 16], which is a two dimensional reduction of a higher dimensions nearly extremal black hole solution. The
spectrum of SYK contains, besides the soft mode, sets of operators that acquire an $O(1)$ anomalous dimension even with spin larger then two [8, 17, 18], which is unexpected both from a $\alpha' \to 0$ limit in string theory and from Vasiliev theory. It is still an open problem to find a description of the SYK bulk dual beyond the soft mode. Another important feature is that SYK is maximally chaotic for Liapunov behavior of certain out-of-time order correlators [6, 7, 9], with Liapunov exponent that saturates the chaos bound [19]. This is precisely what one expects in models that are holographically dual to a gravity black hole solution [20]. A relevant point about SYK is the possibility of performing numerical analysis, at least up to a certain number of fermions, to get, holographically, hints about non perturbative quantum gravity black holes regimes. By numerical analysis a remarkable connection to Random Matrix theory emerge [21]. Another interesting question is whether, due to the presence of disorder, SYK had a spin glass phase at low temperatures, eventually related to replica symmetry breaking. Ref. [22] argues for absence of a spin-glass phase at low temperatures. However, real replicas, not in the sense of the replica trick, seem to play a relevant role in explaining what is observed numerically for the SYK spectral form factor in certain regimes [23].

SYK is asymptotically free in the UV, therefore, at high enough temperatures, one expects the model to be in its perturbative fermion vacuum. However, as mentioned before, the model can be described by two collective bilocal master fields $G(\tau, \tau')$ and $\Sigma(\tau, \tau')$, whose action develops a conformal symmetry in the deep infrared. Conformal symmetry is broken by the master fields condensation, therefore one expects to find a critical temperature where a phase transition occurs, relating the high temperature fermion perturbative vacuum to a vacuum where the collective fields condense.

In this paper we study such a phase transition and compute the condensation critical temperature. These results are achieved by the following steps, we introduce a Hubbard-Stratonovich (HS) auxiliary field related to the collective field $G(\tau, \tau')$ in the large $N$ limit. We then perform a finite temperature perturbative analysis of the disorder averaged replica SYK action, by computing self-energy corrections to the bare mass of the HS field. The dominant term, for large $N$, of this self-energy detects the critical condensation temperature for the collective field $G(\tau, \tau')$, that occurs when the HS auxiliary field squared mass becomes negative. We report details for SYK with $q = 4$ all to all random interactions and, afterwards, generalize to generic even $q > 4$.

The organization of the paper is the following. We begin in section 2.1 to recall basics of the SYK model, including functional integral representation of the replica ensemble thermal partition function and its average over disorder. In section 2.2 we then focus on SYK model with order $q = 4$ random couplings and describe a Hubbard-Stratonovich transformation on the replica ensemble SYK functional integral that connects the HS field to the $G(\tau, \tau')$ collective field in the large $N$ limit. We proceed in section 2.3 by writing the finite temperature action in the space of Matsubara frequencies, and compute the leading order Feynman diagrams that correct the free two-point function of the HS auxiliary field. In section 2.4 we compute the critical temperature for condensation of the replica diagonal part of collective bilocal field $G(\tau, \tau')$. In section 3 we consider the SYK model with generic even $q \geq 4$ random couplings and report the main steps that lead to the computation of
the critical temperature. Finally, in section 4 we make some remarks on the possibility in SYK at very low temperatures of replica symmetry breaking and the existence of spin glass phases, both from the perspective of the diagrammatic analysis developed in this paper and from results appeared recently in the literature. Subtleties that arise in switching from the operatorial to the functional integral representation of the SYK thermal partition function by coherent states methods are discussed in appendix A.

2 Computation of the critical temperature for the condensation of the replica diagonal collective field

2.1 The SYK model and the functional integral representation for the replica thermal partition function

SYK is a disordered quantum mechanics of $N$ Majorana spinors \[ \hat{\psi}_i, \, i = 1, \ldots, N \]

\[
\left\{ \hat{\psi}_i, \hat{\psi}_j \right\} = \delta_{ij}, \tag{2.1}
\]

with a $q \geq 4$ (even) body all to all random interactions Hamiltonian

\[
\mathcal{H}_q = -\frac{(q)!}{q!} \sum_{i_1, \ldots, i_q=1}^{N} J_{i_1, \ldots, i_q} \hat{\psi}_{i_1} \ldots \hat{\psi}_{i_q}. \tag{2.2}
\]

The couplings $J_{i_1, \ldots, i_q}$ are totally antisymmetric, independent and identically distributed random variables. Each of them is picked from a Gaussian probability density with zero mean $\langle J_{i_1, \ldots, i_q} \rangle = 0$ and variance $\langle J_{i_1, \ldots, i_q}^2 \rangle = (q-1)! \frac{J^2}{N^{q-1}}$. Brackets $\langle \cdot \rangle$ denote average over the disorder. In the following we concentrate first on the $q = 4$ case,

\[
H := \mathcal{H}_4 = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl} \hat{\psi}_i \hat{\psi}_j \hat{\psi}_k \hat{\psi}_l. \tag{2.3}
\]

From a statistical mechanics perspective, SYK presents a remarkable resemblance with certain models of quantum spin glasses, where the classical spin variables are promoted to second-quantization bosonic field operators (see, for example, \[25\]). In this sense, it could be viewed as an extension of such models to fermionic degrees of freedom, although it is not yet clear to what extent.

The thermal equilibrium partition function at temperature $T = 1/\beta$

\[
Z = \text{Tr} \left( e^{-\beta H} \right) = \sum_n \langle n | e^{-\beta H} | n \rangle, \tag{2.4}
\]

by using for the basis $|n\rangle$ a suitable complete set of coherent states (see appendix A for details), can be re-written in the following functional integral representation

\[
Z = \int \mathcal{D}[\psi] \exp \left\{ - \int_0^{\beta} d\tau \left[ \sum_{i=1}^{N} \psi_i(\tau) \partial_\tau \psi_i(\tau) - H[\psi(\tau)] \right] \right\} \tag{2.5}
\]
where $\psi_i(\tau)$ is a Grassmannian field with antiperiodic boundary conditions along the imaginary time circle $\psi(\tau + \beta) = -\psi(\tau)$ and $D[\psi_i]$ the corresponding Berezin integrations. In order to compute the ensemble-averaged free energy, we consider $m$ replicas of the system by assigning to the fermion fields $\psi_i^a(\tau)$ a replica index $a = 1, \ldots, m$. The partition function for the replica ensemble is

$$Z^m = \int D[\psi_i^a] \exp\left\{-\sum_{a=1}^m \int_0^\beta d\tau \left[ \sum_{i=1}^N \psi_i^a(\tau) \partial_\tau \psi_i^a(\tau) - \frac{1}{4!} \sum_{i,j,k,l=1}^N J_{ijkl} \psi_i^a(\tau) \psi_j^a(\tau) \psi_k^a(\tau) \psi_l^a(\tau) \right] \right\}. \quad (2.6)$$

The average of $Z^m$ over the disorder induces an interaction between different replicas

$$\langle Z^m \rangle = \int D[\psi_i^a] \exp\left\{-\sum_{a=1}^m \int_0^\beta d\tau \psi_i^a(\tau) \partial_\tau \psi_i^a(\tau) + \frac{J^2}{8N^3} \sum_{a,b=1}^m \int_0^\beta d\tau d\tau' \left( \psi_i^a(\tau) \psi_i^b(\tau') \right)^4 \right\}. \quad (2.7)$$

In the above expression the repeated flavor fermion index $i = 1, \ldots, N$ is intended to be summed.

### 2.2 Hubbard-Stratonovich transformation

For every pair of replica indices $(a, b)$ we introduce a Hubbard-Stratonovich bilocal auxiliary field $\sigma_{ab}(\tau, \tau')$ with the effect of halving the power of the quartic term in the collective bilinear field in (2.7)

$$e^{+\frac{J^2}{8N^3} \int_0^\beta d\tau d\tau' \left[ \psi_i^a(\tau) \psi_i^b(\tau') \right]^4} = \mathcal{N} \int D[\sigma_{ab}] e^{-\frac{J^2}{2} \int_0^\beta d\tau d\tau' \left\{ N[\sigma_{ab}(\tau, \tau')]^2 - \frac{1}{4} \left[ \psi_i^a(\tau) \psi_i^b(\tau') \right]^2 \sigma_{ab}(\tau, \tau') \right\}}, \quad (2.8)$$

where $\mathcal{N}$ is an irrelevant normalization constant. The disorder average replica thermal partition function is now given by

$$\langle Z^m \rangle = \int D[\psi_i^a] D[\sigma_{ab}] \exp\left\{-\sum_{a=1}^m \int_0^\beta d\tau \psi_i^a(\tau) \partial_\tau \psi_i^a(\tau) - \frac{N J^2}{2} \sum_{a,b=1}^m \int_0^\beta d\tau d\tau' \left[ \sigma_{ab}(\tau, \tau') \right]^2 \right. \right.$$

$$\left. + \frac{J^2}{2N} \sum_{a,b=1}^m \int_0^\beta d\tau d\tau' \left[ \psi_i^a(\tau) \psi_i^b(\tau') \right]^2 \sigma_{ab}(\tau, \tau') \right\}. \quad (2.9)$$

The Lagrangian in terms of the auxiliary bilocal field and the collective fermions quartic composite field reads

$$\mathcal{L} = \sum_{a=1}^m \psi_i^a(\tau) \partial_\tau \psi_i^a(\tau) + \frac{N J^2}{2} \sum_{a,b=1}^m \sigma_{ab}(\tau, \tau') - \frac{N J^2}{2} \sum_{a,b=1}^m \left[ \frac{1}{N} \psi_i^a(\tau) \psi_i^b(\tau') \right]^2 \sigma_{ab}(\tau, \tau'), \quad (2.10)$$
In the large $N$ limit, the saddle point equation for the auxiliary field

$$
\sigma_{ab}(\tau, \tau') = \frac{1}{2} \left[ \frac{1}{N} \psi_i^a(\tau) \psi_i^b(\tau') \right]^2 = \frac{1}{2} \left[ G^{ab}(\tau, \tau') \right]^2,
$$

(2.11)

relates it to the collective bilocal field $G(\tau, \tau')$.

From eq. (2.10) we extract the effective action for the replica diagonal term at fixed value of the replica index $a$

$$
S_a = \int_0^\beta d\tau \psi_i^a(\tau) \partial_\tau \psi_i^a(\tau) + \frac{NJ^2}{2} \int_0^\beta d\tau d\tau' \sigma_{aa}^2(\tau, \tau') - \frac{J^2}{2N} \int_0^\beta d\tau d\tau' \psi_i^a(\tau) \psi_i^a(\tau') \psi_i^a(\tau') \sigma_{aa}(\tau, \tau').
$$

(2.12)

As discussed in section 4, replica off-diagonal interaction terms $a \neq b$ in (2.10) give rise to contributions to the $\sigma_{aa}$ self-energy that are suppressed with respect to the ones given by (2.12) in the large $N$ limit.

### 2.3 Matsubara frequencies representation and leading order Feynman diagrams

We expand the fields in terms of their Fourier modes and their Matsubara frequencies

$$
\psi_i(\tau) = \sum_{\omega} \psi_{i,\omega} e^{i\omega \tau} = \sum_{\nu \in \mathbb{Z}} \psi_{i,\nu} e^{i\pi(\nu + \frac{1}{2}) \tau}
$$

(2.13)

$$
\sigma(\tau, \tau') = \sum_{\omega, \omega'} \sigma_{\omega,\omega'} e^{i(\omega \tau + \omega' \tau')} = \sum_{(n, n') \in \mathbb{Z}^2} \sigma_{n, n'} e^{i\pi(n \tau + n' \tau')},
$$

(2.14)

where from now on we will omit in the notation the fixed value of the replica index $a$.

The action for a replica diagonal mode $\sigma(\tau, \tau')$ coupled to $N$ Majorana fermions $\psi_i(\tau)$ (2.12) in frequency space reads

$$
S_a = i\beta \sum_i^N \sum_{\omega} \omega \psi_{i,\omega} \psi_{i,-\omega} + \frac{N(\beta J)^2}{2} \sum_{\omega, \omega'} \sigma_{\omega,\omega'} \sigma_{-\omega,-\omega'} - \frac{(\beta J)^2}{2N} \sum_{i \neq j}^N \sum_{\omega_1, \omega_2, \omega_3, \omega_4} \psi_{i,\omega_1} \psi_{i,\omega_3} \psi_{j,\omega_2} \psi_{j,\omega_4} \sigma_{\omega,\omega'} \delta(\omega_1 + \omega_2 + \omega) \delta(\omega_1' + \omega_2' + \omega').
$$

(2.15)

In order to study the replica diagonal collective field condensation, we focus on the constant mode $\sigma_0$ of the auxiliary field frequencies expansion (2.14), whose dynamics is govern by the $\omega = \omega'$ part of the above action

$$
S_a = -i\beta \sum_i^N \sum_{\omega} \omega \psi_{i,\omega} \psi_{i,-\omega} + \frac{N(\beta J)^2}{2} \sigma_0^2 - \frac{(\beta J)^2}{2N} \sum_{i \neq j}^N \sum_{\omega, \omega'} \psi_{i,\omega} \psi_{i,\omega'} \psi_{j,-\omega} \psi_{j,-\omega'} \sigma_0.
$$

(2.16)

The static field $\sigma_0$ has a bare dimensionless (in $\beta$ units) mass $m^{(0)} = \beta J \sqrt{N}$. The collective field condenses below the temperature where radiative corrections make its squared mass
to become negative. The self-energy is encoded by irreducible Feynman diagrams contributions to the two-point function for the static field. From (2.15) the relevant Feynman rules are

\[
\begin{align*}
\int \frac{d\omega}{\omega} \frac{\delta_{ij} \delta_{ab}}{-i \beta \omega} &= \frac{i \delta_{ij} \delta_{ab}}{2\pi (\nu + \frac{1}{2})} = D^{ij,ab} \quad \text{(fermion propagator)} \quad (2.17) \\
\int \frac{d\omega}{\omega} \frac{\delta_{ac} \delta_{bd}}{-i \beta \omega} &= \frac{i \delta_{ac} \delta_{bd}}{2\pi (\nu + \frac{1}{2})} \quad \text{(auxiliary field propagator)} \quad (2.18) \\
\int \frac{d\omega}{\omega} \frac{\delta_{i_1 a_1 i_2 a_2}}{-i \beta \omega} &= \frac{i \delta_{i_1 a_1} \delta_{i_2 a_2}}{2\pi (\nu + \frac{1}{2})} \quad \text{(interaction vertex)} \quad (2.19)
\end{align*}
\]

where \(D^{ij,ab}\) is the propagator for the Fermi field, \(D^{ab,cd}_0\) is the propagator for the diagonal auxiliary field (as a graph represented by a double line notation), and \(V\) is the rule for the vertex

\[
V_{\omega, \omega_1, \omega_2, \omega_1, \omega_1, \omega_2, \omega_2, \omega_2} = -\frac{(\beta J)^2}{2N} \delta_{a_1 a_2} \delta_{b_1 b_2} \delta_{\omega, \omega_1 + \omega_2} \delta_{\omega, \omega_1 + \omega_2} \delta_{\omega, \omega_1 + \omega_2}, \quad (2.20)
\]

where to each of the two lines of the auxiliary field a pair of fermion lines are attached. Note that, because of the non-local nature of the model, (2.19) has to be considered as a single vertex. The corresponding rules for the action (2.16), where only the zero mode of the auxiliary field appears, can be simply obtained from the above ones evaluating the double lines at zero frequency.

The leading contribution to the \(\sigma_0\) self-energy is given by the following diagram

![Diagram](attachment:diagram.png)

which amounts to the following leading order in \(N\) contributions to the static field self-energy \(\Sigma\)

\[
\Sigma = \left( \frac{(\beta J)^2}{2N} \right)^2 N(N - 1)^2 \left( \frac{i}{2\pi} \right)^4 \sum_{\nu \in \mathbb{Z}} \frac{1}{(\nu + 1/2)^4} = \frac{(\beta J)^4}{4} N c_4 + O(1), \quad (2.21)
\]

where

\[
c_4 := \frac{15}{8\pi^4} \zeta(4) = \frac{1}{48} \quad (2.22)
\]
and

$$\zeta(4) := \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad (2.23)$$

is the Riemann zeta-function.

All the other Feynman diagrams, such as

are suppressed in the large $N$ limit w.r.t. the first diagram considered above.

2.4 Critical temperature for the condensation of the collective bilocal field $G(\tau, \tau')$

As summarized in the previous paragraph, the bare dimensionless mass of the auxiliary field $\sigma_0$ is $m^{(0)} = \beta J \sqrt{N}$, while the leading $N$ contribution to its self-energy $\Sigma$ is given in eq. (2.21). Therefore in the $N \to \infty$ limit, $(N \gg (\beta J)^2)$, the squared mass of the auxiliary field goes as

$$m^2 \sim N(\beta J)^2 - \Sigma \sim N(\beta J)^2 \left(1 - \frac{(\beta J)^2}{4} c_k\right), \quad (2.24)$$

which gives a condensation of the collective field below the following critical temperature

$$T \leq T^*_c := \frac{J}{8\sqrt{3}} \sim 0.072 J. \quad (2.25)$$

Interestingly, in a model related to SYK, the complex SYK model, there is a similar phase where the bilocal collective field condenses [26]. In that model, at least for large enough masses, a phase transition occurs too where the collective field condensate becomes unstable. The largest value of the temperature at which this arises is $T_c \sim 0.069 J$, which is very close to the value in (2.25).
3 $T_c$ for generic even $q \geq 4$

We switch now to the generic $q \geq 4$ (even) SYK with Hamiltonian given in (2.2) and report the main steps that lead to the computation of the collective field $G(\tau, \tau')$ critical temperature condensation.

After the introduction of the Hubbard-Stratonovich auxiliary field, the replicated Lagrangian is now given by

$$\mathcal{L} = \sum_{a=1}^{m} \bar{\psi}_a^i(\tau) \partial_\tau \psi_a^i(\tau) + \frac{N J^2}{2} \sum_{a,b=1}^{m} \sigma_{ab}^2(\tau, \tau') + (i) \frac{q}{\sqrt{q}} \frac{N J^2}{2} \sum_{a,b=1}^{m} \left( \frac{1}{N} \bar{\psi}_a^i(\tau) \psi_b^i(\tau') \right)^{q/2} \sigma_{ab}(\tau, \tau').$$

The large $N$ saddle point equation for the auxiliary field connects it to the collective field $G(\tau, \tau')$

$$\sigma_{ab}(\tau, \tau') = -(i) \frac{q}{\sqrt{q}} \left[ \frac{1}{N} \bar{\psi}_a^i(\tau) \psi_b^i(\tau') \right]^{q/2} = -(i) \frac{q}{\sqrt{q}} \left[ G_{ab}(\tau, \tau') \right]^{q/2}.$$ (3.2)

From (3.1) we pick the replica diagonal term at fixed replica index $a$, since replica off-diagonal $a \neq b$ interaction terms give rise to contributions suppressed for large $N$ w.r.t. the replica diagonal interaction vertex in (3.1) (see section 4 for a discussion on this point). The effective action suitable in order to study the replica diagonal collective field $G_{aa}(\tau, \tau')$ condensation critical temperature in the large $N$ limit is then

$$S_a = \int_0^\beta d\tau \bar{\psi}_a^i(\tau) \partial_\tau \psi_a^i(\tau) + \frac{N J^2}{2} \int_0^\beta d\tau d\tau' \sigma_{aa}^2(\tau, \tau')$$

$$+ (i) \frac{q}{\sqrt{q}} \frac{J^2}{N^{q/2-1}} \int_0^\beta d\tau d\tau' \bar{\psi}_1^a(\tau) \psi_1^a(\tau') \cdots \psi_{q/2}^a(\tau) \psi_{q/2}^a(\tau') \sigma_{aa}(\tau, \tau').$$ (3.3)

By expanding fields in Fourier modes as in (2.14), one gets the following frequency space action for the dynamics of the replica diagonal constant mode $\sigma_0^{aa}$ of the auxiliary field and the same replica fermion fields $\psi_a^i(\tau)$

$$S_a = -i\beta \sum_\omega \omega \psi_{1,\omega}^i \psi_{1,-\omega} + \frac{N(\beta J)^2}{2} \sigma_0^2$$

$$+ (i) \frac{q}{\sqrt{q}} \frac{(\beta J)^2}{N^{q/2-1}} \sum_{\omega_1, \omega_2, \ldots, \omega_{q/2}} \psi_{1,\omega_1}^i \psi_{1,\omega_2}^i \cdots \psi_{1,\omega_{q/2}}^i \omega_{q/2} \delta(\omega_1 + \cdots + \omega_{q/2}) \delta(\omega_1' + \cdots + \omega_{q/2}') \sigma_0,$$ (3.4)

where, as in the discussion of the $q = 4$ in the previous paragraph, we omit the fixed value
of the replica index \( a \). The relevant Feynman rules of the theory are

\[
i, a \quad \omega_{\nu} \quad j, b = D^{ij,ab}_{\omega_{\nu}} = \frac{\delta_{ij} \delta_{ab}}{-i\beta \omega_{\nu}} = \frac{i \delta_{ij} \delta_{ab}}{2\pi(\nu + \frac{1}{2})}, \quad \nu_i \in \mathbb{Z}
\]

\[
D_0^{ab,cd} = \frac{\delta_{ac} \delta_{bd}}{N(\beta J)^2}
\]

\[
\text{(3.5)}
\]

\[
V_{\omega_{\mu}, \omega_{\nu_1}, \ldots, \omega_{\nu_{q/2}}, \omega_{\mu_1}, \ldots, \omega_{\mu_{q/2}} = (i)^{\frac{q}{2}} \frac{(\beta J)^2}{\sqrt{q} N\frac{q}{2} - 1} \epsilon_{i_1, \ldots, i_{q/2}} \prod_{k=1}^{q/2} \left( \delta_{i_k, j \pi(k)} \right) \delta_{aa} \delta_{bb} \cdot \delta(\omega_{\nu_1} + \cdots + \omega_{\nu_{q/2}}) \delta(\omega_{\mu_1} + \cdots + \omega_{\mu_{q/2}})}
\]

\[
\text{(3.6)}
\]

where \( S_{\frac{q}{2}} \) is the Symmetric group of order \( \frac{q}{2} \), \( D^{ij,ab}_{\omega_{\nu}} \) is the propagator for the Fermi field mode of flavor \( i \) and frequency \( \omega_{\nu} \), \( D_0^{ab,cd} \) is the propagator for the diagonal auxiliary field \( \sigma \) (as a graph represented by a double line notation), and \( V \) is the the vertex, where to each of the two lines of the auxiliary field \( q/2 \) fermion lines are now attached. The totally antisymmetric Levi-Civita tensor \( \epsilon_{i_1, \ldots, i_{q/2}} \) in the rule for the vertex \( V \) encodes the constraint that all the fermion flavors attached to a given vertex have to be different. The overall sign given by the Levi-Civita antisymmetric tensor is irrelevant for this computation.

The leading order in \( N \) diagram that contributes to the \( \sigma_0 \) self-energy is

\[
\text{All the other diagrams are } N \text{ suppressed w.r.t. above diagram. As an example, the following diagram is } O(1/N) \text{ w.r.t. previous one:}
\]
The leading contribution to the self-energy $\Sigma(q)$ for generic $q \geq 4$ even is given by

$$
\Sigma(q) = \left( i \frac{2}{\sqrt{q}} \frac{\beta J^2}{N^{q-1}} \right)^2 N^q - (i)^q c_q = (\beta J)^4 N \frac{c_q}{q},
$$

where the first factor arises from the squared vertex $V$, while the $N^q$ factor arises by counting Fermion flavors (we omit subleading in $N$ terms from the Levi-Civita constraint in the vertex). The numerical coefficient $c_q$ includes the contribution from the following multiple convergent series over Matsubara frequencies

$$
c_q = \frac{1}{\beta q} \sum_{\omega \in \left( \frac{2\pi}{\beta} \mathbb{Z} + \frac{1}{2} \right)_{q-1}} \frac{\delta \left( \omega_i + \omega_j + \cdots + \omega_j^q \right)}{\omega_i^2 \omega_j \cdots \omega_j^q \omega_k \cdots \omega_k^q}
$$

The above result for the leading order self-energy $\Sigma(q)$ leads to the following result for the squared mass of the auxiliary field $\sigma_0$ for generic $q$

$$
m^2(q) \sim (\beta J)^2 N - \Sigma(q) = (\beta J)^2 N \left[ 1 - (\beta J)^2 \frac{c_q}{q} \right].
$$

which leads to the following critical temperature $T_c$ for the collective field $G(\tau, \tau')$ condensation in the generic SYK model with $q \geq 4$ even all to all random coupling

$$
T_c \sim J \sqrt{\frac{c_q}{q}}
$$

which can be trusted in the regime $(\beta J)^2 \ll N^{q-1}$. 

4 Remarks on the condensation of replica off diagonal collective field and spin glass phases

An interesting question is whether in SYK there is a critical temperature where replica off-diagonal components of the collective field $G_{ab}(\tau, \tau')$ for $a \neq b$ condense. This would be a necessary condition for the existence in SYK of a spin glass phase.\footnote{At this level one should think about the possibility of a replica symmetry breaking (RSB). This can occur by condensation of off-diagonal elements to at least two distinct values. Another possibility for having a RSB it would be a condensation of at least two diagonal elements to two distinct values. While [22] argues for impossibility in SYK of condensation of replica off-diagonal elements, their analysis does not rule out diagonal RSB.} In this paragraph we
make some comments about this issue. Firstly, from the point of view of the diagrammatic analysis we carried on in this paper, we notice that a condensation of $G_{ab}(\tau, \tau')$ for $a \neq b$ could be seen by computing the large $N$ leading contributions to the self-energy of the static part of the auxiliary field $\sigma_{ab}$ with $a \neq b$. Since the fermion propagator is diagonal in replica indices, when $a \neq b$ the interaction vertex does not admit fermion propagators connecting two different replica lines. This implies that corrections to the $\sigma_{ab}$ self-energy in the $a \neq b$ case are suppressed in the large $N$ limit w.r.t. to the replica diagonal case. One can check that, for $q = 4$, the large $N$ leading contribution to the two point function for $\sigma_{ab}$ for $a \neq b$ is given by the following diagram

which is $1/N$ suppressed w.r.t. the leading contribution to the diagonal $\sigma_{aa}$ two point function given in (2.21). A naive computation of the leading correction to the bare mass of $\sigma_{ab}$ for $a \neq b$ would give a critical temperature for its condensate of order $O(1/N)$. However, this conclusion would be incorrect since, as it was shown in (2.25), the diagonal $G_{aa}(\tau, \tau')$ condenses well before, at a critical temperature $T_c = O(J)$. This implies that a computation for the condensation of $G_{ab}$ with $a \neq b$ cannot be consistently performed on the perturbative Majorana fermion vacuum, but it has to be carried on the vacuum where the diagonal $G_{aa}(\tau, \tau')$ has condensed. Such a computation has been carried on in [22] in the deep infrared limit, where $\langle G_{aa}(\tau, \tau') \rangle$ can be approximated by its conformal analytic expression. It comes out an instability for the static $G_{ab}$ for $a \neq b$ for $T_c \sim Je^{-\sqrt{2\pi N}}$, a result which lies outside the regime of validity of the perturbative computation $\beta J \ll N$. Then, by using results of [27], [22] rules out the condensation of $G_{ab}$ for $a \neq b$ even for $\beta J \gg N$.

A Subtleties in switching from an operatorial to a functional integral representation of the SYK thermal partition function

The procedure of constructing the fermionic Fock space via complexification of the Majorana fermions, needed to obtain the functional representation of the partition function, is usually performed (see, for example, [28]) through the halving of the degrees of freedom, introducing the canonical fermion operators

$$
\hat{\chi}_j = \frac{1}{\sqrt{2}} \left( \hat{\psi}_{2j-1} - i \hat{\psi}_{2j} \right), \quad \hat{\chi}^\dagger_j = \frac{1}{\sqrt{2}} \left( \hat{\psi}_{2j-1} + i \hat{\psi}_{2j} \right), \quad j = 1, \cdots, N/2. \quad (A.1)
$$

This path does not lead to the representation (2.5), unless implying some sort of normal-ordering in the original Hamiltonian with respect to the canonical operators. Indeed, the
expansion on coherent states need the Hamiltonian to be in an ordered form: if this form has to be obtained by hand, it contains all the interactions between $q - p$ fermions, with all $p$'s even. A simple way to see this is to take the minimal value $N = 4$, for which

$$H = J_{1234} \hat{\psi}_1 \hat{\psi}_2 \hat{\psi}_3 \hat{\psi}_4$$

(A.2)

with no summation implied. Introducing the canonical operators

$$\hat{\chi} = \frac{\hat{\psi}_1 - i \hat{\psi}_2}{\sqrt{2}}, \quad \hat{\eta} = \frac{\hat{\psi}_3 - i \hat{\psi}_4}{\sqrt{2}}$$

(A.3)

the Hamiltonian is, in normal-ordered form,

$$H = \frac{\hbar^2}{4} J_{1234} \left(-4 \hat{\chi} \hat{\eta}^\dagger \hat{\eta} \hat{\chi} - 2 \hat{\eta} \hat{\eta}^\dagger - 2 \hat{\chi} \hat{\chi}^\dagger + 1\right)$$

(A.4)

Introducing, on each time-slice of the Fock space, a basis of coherent states, defined by the property

$$\hat{\chi} |\chi_t, \eta_t\rangle = \chi_t |\chi_t, \eta_t\rangle, \quad \hat{\eta} |\chi_t, \eta_t\rangle = \eta_t |\chi_t, \eta_t\rangle$$

(A.5)

where $\chi_t, \eta_t$ are Grassmann symbols, the trace in (2.4) can be evaluated explicitly with a Lie-Trotter product, obtaining

$$\langle \eta_t, \chi_t | (1 - aH) | \chi_{t+a}, \eta_{t+a}\rangle \langle \eta_t, \chi_t | \chi_t, \eta_t \rangle = e^{\left[\chi_t \chi_{t+a} + \eta_t \eta_{t+a} - \frac{\hbar^2}{4} J_{1234} \left(-4 \chi_t \eta_t \chi_{t+a} - 2 \eta_t \eta_{t+a} - 2 \chi_t \chi_{t+a+1}\right)\right]} + O(a)$$

(A.6)

where $a$ is the lattice spacing to be sent to zero. Defining the Majorana Grassmann variables in the same way as the Majorana operators, the result is

$$H(\psi_{1t}, \psi_{2t}, \psi_{3t}, \psi_{4t}) = J_{1234} \left(\psi_{1t} \psi_{2t} \psi_{3t} \psi_{4t} - \frac{i}{2} \psi_{1t} \psi_{2t} - \frac{i}{2} \psi_{3t} \psi_{4t} + 1\right).$$

(A.7)

In no way the quadratic terms can be dropped, unless the starting Hamiltonian was already normal-ordered with respect to the canonical creation and annihilation operators, in such a way that

$$:H: = J_{1234} \hat{\chi} \hat{\eta}^\dagger \hat{\eta}^\dagger \hat{\chi}$$

(A.8)

An alternative way to find a functional representation, as explained in [24], consists in doubling the Majorana fermions, by supposing that the theory contains extra $N$ free Majorana degrees of freedom (and thus not appearing in the Hamiltonian) from which the complex fermions can be defined as

$$\hat{\psi}_j = \frac{1}{\sqrt{2}} (\hat{\chi}_j - i \hat{\chi}_{j+N}), \quad \hat{\psi}_j^\dagger = \frac{1}{\sqrt{2}} (\hat{\chi}_j + i \hat{\chi}_{j+N}), \quad j = 1, \cdots, N.$$

(A.9)

In this way, the antisymmetry of the couplings $J_{ijkl}$ ensures the resulting Hamiltonian for complex fermions to be already normal-ordered, and so producing only the couplings between $q$ fermions in the action appearing in (2.5). The added $N$ free fermions do not interact and can be integrated out, producing inessential numerical factors.
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