G-INDEX, TOPOLOGICAL DYNAMICS
AND THE MARKER PROPERTY

BY

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Dedicated to Professor Benjamin Weiss on the occasion of his 80th birthday

ABSTRACT

Given an action of a finite group $G$ on a topological space, we can define its index. The $G$-index roughly measures a size of the given $G$-space. We explore connections between the $G$-index theory and topological dynamics. For a fixed-point free dynamical system, we study the $\mathbb{Z}_p$-index of the set of $p$-periodic points. We find that its growth is at most linear in $p$. As an application, we construct a free dynamical system which does not have the marker property. This solves a problem which has been open for several years.

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1. Introduction

1.1. Background on the $\mathbb{Z}_p$-index. When a finite group $G$ acts on a topological space, we can define its index, which roughly measures a “size” of the given $G$-space. The $G$-index theory is a generalization of the Borsuk–Ulam theorem, and it has several striking applications to combinatorics [Mat08].

The purpose of this paper is to exhibit new applications of the $G$-index to dynamical systems theory. In particular, we prove the following theorem. The terminology of dynamical systems will be explained in §1.2 and §1.3.

**Theorem 1.1:** There exists a free dynamical system which does not have the marker property.

This solves a problem which has been open for several years. We give a background of this theorem in §1.3.

First we introduce basic definitions of the $G$-index theory. Our presentation follows a book of Matoušek [Mat08, Sections 6.1 and 6.2]. In this paper, we consider only the case of $G = \mathbb{Z}/p\mathbb{Z}$ for prime numbers $p$. We set

$$\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}.$$ 

A pair $(X, T)$ is called a $\mathbb{Z}_p$-space if $X$ is a topological space and $T : X \to X$ is a homeomorphism satisfying $T^p = \text{id}$. We often omit $T$ from the notation and simply say “$X$ is a $\mathbb{Z}_p$-space”.

Let $(X, T)$ be a $\mathbb{Z}_p$-space. It is said to be free if

$$T^a x \neq x$$

for any $1 \leq a \leq p - 1$ and any $x \in X$. Since $p$ is a prime number, this is equivalent to the condition that $T x \neq x$ for any $x \in X$. Let $n \geq 0$ be an integer. A free $\mathbb{Z}_p$-space $(X, T)$ is called an $E_n \mathbb{Z}_p$-space if it satisfies the following:

- $X$ is an $n$-dimensional finite simplicial complex.
- $T : X \to X$ is a simplicial map, i.e., mapping each simplex to a simplex affinely.
- $X$ is $(n - 1)$-connected, i.e., the $k$-th homotopy group $\pi_k(X)$ vanishes for all $0 \leq k \leq n - 1$. 


For example, the discrete space \( \mathbb{Z}_p \) with the natural \( \mathbb{Z}_p \) action is an \( E_0 \mathbb{Z}_p \)-space. (We consider that \( \mathbb{Z}_p \) is \((-1)\)-connected.) In general, the join\(^1\) of \((n+1)\)-copies of \( \mathbb{Z}_p \) is an \( E_n \mathbb{Z}_p \)-space:

\[
E_n \mathbb{Z}_p = \underbrace{\mathbb{Z}_p \ast \mathbb{Z}_p \ast \cdots \ast \mathbb{Z}_p}_{(n+1)\text{-times}}.
\]

Here \( \mathbb{Z}_p \) acts on each component of the join simultaneously. (If a finite simplicial complex \( X \) is \( n \)-connected and a finite simplicial complex \( Y \) is \( m \)-connected, then the join \( X \ast Y \) is \((n+m+2)\)-connected [Mat08, 4.4.3 Proposition].)

There also exist other models of \( E_n \mathbb{Z}_p \)-spaces. For example ([Mat08, pp. 149–150])

\[
(1.1) \quad E_n \mathbb{Z}_p = \begin{cases} 
S^n & \text{(when } n \text{ is odd)} \\
S^{n-1} \ast \mathbb{Z}_p & \text{(when } n \text{ is even)}
\end{cases}
\]

So \( E_n \mathbb{Z}_p \)-space is not unique. However, these spaces are “essentially” unique for our purpose because any \( E_n \mathbb{Z}_p \)-space equivariantly and continuously maps to any other ([Mat08, 6.2.2 Lemma]).

Let \((X,T)\) be a free \( \mathbb{Z}_p \)-space. We define

\[
\text{ind}_p(X,T) = \min \{ n \geq 0 \mid \exists f : X \to E_n \mathbb{Z}_p : \text{\( \mathbb{Z}_p \)-equivariant continuous mapping} \}. 
\]

We set \( \text{ind}_p(X,T) = \infty \) if there is no equivariant continuous map \( f : X \to E_n \mathbb{Z}_p \) for any \( n \geq 0 \). We often abbreviate \( \text{ind}_p(X,T) \) as \( \text{ind}_p X \). We have \( \text{ind}_p E_n \mathbb{Z}_p = n \) ([Mat08, 6.2.5 Theorem]). This can be seen as a generalization of the Borsuk–Ulam theorem. We use the convention that if \( X \) is empty then \( \text{ind}_p X = -1 \).

For a (non-empty) topological space \( X \), we define its connectivity \( \text{conn}(X) \) as the smallest integer \( n \geq -1 \) satisfying \( \pi_{n+1}(X) \neq 0 \). We set \( \text{conn}(\emptyset) := -2 \). It is known ([Mat08, 6.2.4 Proposition]) that for a \( \mathbb{Z}_p \)-space \((X,T)\)

\[
(1.2) \quad \text{ind}_p(X,T) \geq \text{conn}(X) + 1.
\]

\(^1\) For topological spaces \( X \) and \( Y \), the join \( X \ast Y \) is defined by \( X \ast Y = [0,1] \times X \times Y / \sim \), where the equivalence relation is given by

\[
(0,x,y) \sim (0,x',y'), \quad (1,x,y) \sim (1,x',y)
\]

for any \( x,x' \in X \) and \( y,y' \in Y \). The equivalence class of \((t,x,y)\) is usually written as \((1-t)x \oplus ty\). When \( X \) and \( Y \) are \( \mathbb{Z}_p \)-spaces, the join \( X \ast Y \) also becomes a \( \mathbb{Z}_p \)-space. The \( \mathbb{Z}_p \) action on the join \( X \ast Y \) is given by the simultaneous actions on the components \( X \) and \( Y \). We use the convention that if \( Y = \emptyset \) then \( X \ast Y = X \).
1.2. DYNAMICAL SYSTEMS AND $\mathbb{Z}_p$-INDEX. Next we introduce basic definitions of dynamical systems and state our first main result. A pair $(X, T)$ is called a dynamical system if $X$ is a compact metrizable space and $T : X \to X$ is a homeomorphism.\(^2\) We sometimes omit $T$ from the notation and simply say “$X$ is a dynamical system” when the action $T$ can be understood in the context.

Let $(X, T)$ be a dynamical system. For a natural number $n = 1, 2, 3, \ldots$, we define $P_n(X, T)$ as the set of $n$-periodic points:

$$P_n(X, T) := \{ x \in X \mid T^n x = x \}.$$ 

We often abbreviate $P_n(X, T)$ as $P_n(X)$.

A dynamical system is said to be free if it has no periodic points, namely $P_n(X, T) = \emptyset$ for all $n \geq 1$. A dynamical system is said to be fixed-point free if it has no fixed points, namely $P_1(X, T) = \emptyset$.

Suppose a dynamical system $(X, T)$ has no fixed-points. For each prime number $p$, we consider a $\mathbb{Z}_p$-space

$$(P_p(X, T), T).$$

Since $(X, T)$ has no fixed points, this is a free $\mathbb{Z}_p$-space. So we can consider its $\mathbb{Z}_p$-index, and we get a sequence

$$\text{ind}_p(P_p(X, T), T) \quad (p = 2, 3, 5, 7, 11, 13, 17, \ldots).$$

Now we ask a question: What kind of sequence can appear? Is there any restriction?

Indeed there is a restriction. Its growth is at most linear in $p$:

**THEOREM 1.2:** Let $(X, T)$ be a fixed-point free dynamical system. Then $\text{ind}_p P_p(X)$ is finite for all prime numbers $p$, and the sequence

$$\text{ind}_p P_p(X) \quad (p = 2, 3, 5, 7, 11, \ldots)$$

has at most linear growth. Namely there exists a positive number $C$ satisfying

$$\text{ind}_p P_p(X) < C \cdot p$$

for all prime numbers $p$.

We have

$$\text{conn} P_p(X) \leq \text{ind}_p P_p(X) - 1$$

by (1.2). So we get the next corollary.

\(^2\) Notice that we assume the compactness of $X$. This is essential for our results.
**Corollary 1.3:** Let \((X, T)\) be a fixed-point free dynamical system. Then the connectivity of \(P_p(X)\) grows at most linearly in \(p\).

1.3. **The Marker Property.** Theorem 1.2 indicates that we can detect a hidden structure of dynamical systems by using \(\mathbb{Z}_p\)-index theory. But is there any application of this new structure? Surprisingly, it has an application to a problem which seems to have nothing to do with \(\mathbb{Z}_p\)-index theory.

A dynamical system \((X, T)\) is said to have the **marker property** if for any natural number \(N\) there exists an open set \(U \subseteq X\) satisfying the following two conditions.

- No point in \(U\) returns to \(U\) within \(N\) steps, namely
  \[ U \cap T^{-n}U = \emptyset \quad \text{for all } 1 \leq n \leq N. \]

- Every orbit of \(T\) has non-empty intersection with \(U\), namely
  \[ X = \bigcup_{n \in \mathbb{Z}} T^{-n}U. \]

For example, infinite minimal dynamical systems\(^3\) and their extensions have the marker property. The marker property was first implicitly used by Lindenstrauss [Lin99, proof of Lemma 3.3]. Gutman [Gut15, Gut17] explicitly defined it and introduced the terminology “marker property”. He also investigated several properties of this notion.

It is easy to see that if a dynamical system \((X, T)\) has the marker property, then it is free. Indeed, suppose it has a periodic point \(x\) of period \(N \geq 1\). Take an open set \(U \subseteq X\) satisfying \(U \cap T^{-n}U = \emptyset\) for all \(1 \leq n \leq N\) and \(X = \bigcup_{n \in \mathbb{Z}} T^{-n}U\). Then \(y := T^n(x) \in U\) for some \(n\). The point \(y\) is also an \(N\)-periodic point. But \(y = T^N(y) \notin U\). This is a contradiction.

The marker property has been intensively used in the context of mean dimension theory [Lin99, Gut15, Gut17, GLT16, GQT19, LT19, GT20, Tsu20]. Mean dimension is a topological invariant of dynamical systems which counts the number of parameters of dynamical systems per iterate. It was introduced by Gromov [Gro99]. See also the paper of Lindenstrauss–Weiss [LW00]. We will not use mean dimension in the main body of the paper. So we do not provide the precise definition.

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\(^3\) A dynamical system \((X, T)\) is called an **infinite minimal system** if \(X\) is an infinite set and every orbit of \(T\) is dense in \(X\). For example, an irrational rotation of the circle is an infinite minimal system.
Here are two samples of theorems using the marker property in their assumptions ([Lin99, Proposition 6.14] and [GQT19, Main Theorem 1]):

**Theorem 1.4 ([Lin99]):** Suppose a dynamical system $(X, T)$ has the marker property. Then $(X, T)$ has zero mean dimension if and only if it is isomorphic to an inverse limit of finite entropy dynamical systems.

**Theorem 1.5 ([GQT19]):** Suppose a dynamical system $(X, T)$ has the marker property. Let $N$ be a natural number. If $(X, T)$ has mean dimension smaller than $N/2$ then it can be embedded in the shift action on the infinite-dimensional cube

$$([0, 1]^N) \times \cdots \times [0, 1]^N \times [0, 1]^N \times \cdots.$$ 

As we remarked above, if a dynamical system has the marker property then it is free. It is very natural to ask whether the converse holds or not:

**Problem 1.6:** Does every free dynamical system have the marker property?

Besides its naturalness, this question is also important for better understanding the range of applicability of various theorems using the marker property, such as the above Theorems 1.4 and 1.5.

Problem 1.6 was stated by Gutman ([Gut15, Problem 5.4] and [Gut17, Problem 3.4]). The origin of the question goes back to the paper of Lindenstrauss [Lin99]. In that paper he introduced a certain topological analogue of the Rokhlin tower lemma ([Lin99, Lemma 3.3]). This Rokhlin-type lemma is an equivalent form of the marker property. After proving the lemma for extensions of infinite minimal systems, he wrote [Lin99, p. 234, lines 1 and 2]:

In this form, it does not seem that this Rokhlin-type Lemma can be extended to a more general setup.

So he more or less conjectured a negative answer to Problem 1.6. Probably many serious readers of [Lin99] had an interest in the question.

Problem 1.6 looks an innocent question. However, unexpectedly, it turns out to be difficult. The main reason is it has an infinite-dimensional nature. Gutman [Gut15, Theorem 6.1] proved that every finite-dimensional free dynamical system has the marker property. Here a dynamical system $(X, T)$ is said

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4 [Lin99, Proposition 6.14] was stated only for extensions of infinite minimal systems. But actually the proof is valid for all dynamical systems with the marker property.

5 We will review this in Lemma 5.1 below.
to be **finite-dimensional** if the topological dimension (Lebesgue covering dimension) of $X$ is finite. So the main objects in the study of Problem 1.6 are infinite-dimensional dynamical systems. The problem is difficult simply because our current technology for understanding infinite-dimensional systems is very limited. Probably the most important discovery of the paper is that $\mathbb{Z}_p$-index theory provides a new tool for studying infinite-dimensional dynamical systems.

As we already mentioned in §1.1, the answer to Problem 1.6 is no. This is our second main result.

**Theorem 1.7** (= Theorem 1.1): *There exists a free dynamical system which does not have the marker property.*

The relation between this theorem and Theorem 1.2 in §1.2 is roughly as follows. In the proof of Theorem 1.2 we introduce a certain universal fixed-point free dynamical system $\mathcal{X}(N,\delta)$, which is larger than the given $(X,T)$ from the viewpoint of $\mathbb{Z}_p$-index theory. It turns out that this universal system also becomes a basic building block for constructing a free dynamical system which does not have the marker property.

### 2. Proof of Theorem 1.2

In this section we introduce dynamical systems $\mathcal{X}(N,\delta)$ which have a certain universal property. Theorem 1.2 immediately follows from the universality of $\mathcal{X}(N,\delta)$. The universality will be also a crucial ingredient for the proof of Theorem 1.1 later.

#### 2.1. $\varepsilon$-embedding

This subsection is a preparation. Let $(X,d)$ and $(Y,d')$ be metric spaces. We assume that $X$ is compact. Let $\varepsilon > 0$. A continuous map $f : X \to Y$ is called an **$\varepsilon$-embedding** if for all $y \in Y$

$$\text{diam } f^{-1}(y) < \varepsilon.$$ 

Since we assume the compactness of $X$, if $f$ is an $\varepsilon$-embedding then there exists $\delta > 0$ such that for any two points $x_1, x_2 \in X$

$$d'(f(x_1), f(x_2)) < \delta \implies d(x_1, x_2) < \varepsilon.$$ 

**Lemma 2.1:** Let $\varepsilon > 0$ and let $(X,d)$ be a compact metric space. There are a natural number $N$ and an $\varepsilon$-embedding from $X$ to the $N$-dimensional cube $[0,1]^N$. 
Proof. We can assume diam $X \leq 1$ (by using some scale change). We take open balls $B_1, \ldots, B_N$ of radius $\varepsilon/2$ covering $X$. Let $x_1, \ldots, x_N$ be the centers of $B_1, \ldots, B_N$ respectively. We define

$$f : X \to [0, 1]^N, \quad x \mapsto (d(x, x_1), d(x, x_2), \ldots, d(x, x_N)).$$

This is an $\varepsilon$-embedding. Indeed, suppose $x, y \in X$ satisfy $f(x) = f(y)$. Suppose, say, $x \in B_1$. Then $d(y, x_1) = d(x, x_1) < \varepsilon/2$ and hence $y \in B_1$. Since $B_1$ is an $\varepsilon/2$-ball, we get

$$d(x, y) < \varepsilon.$$

2.2. Universal fixed-point free dynamical systems. Let $N$ be a natural number. We consider the infinite product of the copies of the $N$-dimensional cube:

$$([0, 1]^N)^\mathbb{Z} = \cdots [0, 1]^N \times [0, 1]^N \times [0, 1]^N \times \cdots.$$

This becomes a dynamical system under the shift map

$$\sigma : ([0, 1]^N)^\mathbb{Z} \to ([0, 1]^N)^\mathbb{Z}, \quad (x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}.$$

Let $\delta > 0$. We define a subsystem $\mathcal{X}(N, \delta)$ of $([0, 1]^N)^\mathbb{Z}$ by

$$\mathcal{X}(N, \delta) := \{(x_n)_{n \in \mathbb{Z}} \mid \forall n \in \mathbb{Z} : |x_n - x_{n+1}| \geq \delta\}.$$

The pair $(\mathcal{X}(N, \delta), \sigma)$ becomes a dynamical system. (We often omit $\sigma$ from the notation and simply write “the dynamical system $\mathcal{X}(N, \delta)$”.) This system is the central object of the paper. It immediately follows from the definition that $\mathcal{X}(N, \delta)$ has no fixed points:

$$P_1(\mathcal{X}(N, \delta), \sigma) = \emptyset.$$

$\mathcal{X}(N, \delta)$ has the following universality property.

Theorem 2.2 (Universality of $\mathcal{X}(N, \delta)$): Let $(X, T)$ be a fixed-point free dynamical system. There are a natural number $N$ and a positive number $\delta$ such that there exists an equivariant continuous map from $X$ to $\mathcal{X}(N, \delta)$.

Proof. Let $d$ be a metric on $X$ compatible with the topology. Since $(X, T)$ has no fixed points, there is a positive number $\varepsilon$ satisfying

$$\forall x \in X : d(x, Tx) > \varepsilon.$$
By Lemma 2.1, there is an \( \varepsilon \)-embedding \( f : X \to [0, 1]^N \) for some natural number \( N \). Since \( X \) is compact, we can find \( \delta > 0 \) such that for any two points \( x, y \in X \)

\[
|f(x) - f(y)| < \delta \implies d(x, y) < \varepsilon.
\]

Then for all \( x \in X \)

\[
(2.1) \quad |f(x) - f(Tx)| \geq \delta.
\]

We define \( F : X \to ([0, 1]^N)^\mathbb{Z} \) by

\[
F(x) = (f(T^n x))_{n \in \mathbb{Z}}.
\]

Now \( F \) is continuous and equivariant, i.e., \( F \circ T = \sigma \circ F \). By the above (2.1), \( F(x) \in \mathcal{X}(N, \delta) \) for all \( x \in X \). Thus \( F \) is an equivariant continuous map from \( X \) to \( \mathcal{X}(N, \delta) \).

Although the above proof is very simple, Theorem 2.2 is a key result of the paper.

2.3. \( P_p(\mathcal{X}(N, \delta)) \) AND THE PROOF OF THEOREM 1.2. Let \( N \) be a natural number and \( \delta \) a positive number. For a prime number \( p \), we study the set of \( p \)-periodic points of the dynamical system \( \mathcal{X}(N, \delta) \). From the definition, \( P_p(\mathcal{X}(N, \delta)) \) is naturally identified with

\[
(2.2) \quad \{(x_0, x_1, \ldots, x_{p-1}) \in ([0, 1]^N)^\mathbb{Z} \mid \forall n \in \mathbb{Z}_p : |x_n - x_{n+1}| \geq \delta\}.
\]

(Here the condition \( |x_n - x_{n+1}| \geq \delta \) becomes \( |x_{p-1} - x_0| \geq \delta \) when \( n = p - 1 \).

The group \( \mathbb{Z}_p \) acts on this space by the cyclic shift:

\[
(x_0, x_1, x_2, \ldots, x_{p-1}) \mapsto (x_1, x_2, \ldots, x_{p-1}, x_0).
\]

Also \( P_p(\mathcal{X}(N, \delta)) \) is a free \( \mathbb{Z}_p \)-space. From (2.2),

\[
P_p(\mathcal{X}(N, \delta)) \subset (\mathbb{R}^N)^\mathbb{Z}_p \setminus \{(x, x, \ldots, x) \mid x \in \mathbb{R}^N\}.
\]
The right-hand side is $\mathbb{Z}_p$-equivariantly homotopic\(^6\) to the $(Np - N - 1)$-dimensional sphere

\[(2.3) \quad \{ (x_0, x_1, \ldots, x_{p-1}) \in (\mathbb{R}^N)^{\mathbb{Z}_p} | x_0 + x_1 + \cdots + x_{p-1} = 0, (x_0, x_1, \ldots, x_{p-1}) = 1 \}. \]

This is an $E_{Np - N - 1} \mathbb{Z}_p$-space. (See (1.1).) So its $\mathbb{Z}_p$-index is equal to $Np - N - 1$. Therefore

$$\text{ind}_p P_p(\mathcal{X}(N, \delta)) \leq Np - N - 1.$$ 

Now we are ready to prove Theorem 1.2. For the convenience of readers, we write the statement again:

**Theorem 2.3** (\(=\) Theorem 1.2): Let $(X, T)$ be a fixed-point free dynamical system. Then $\text{ind}_p P_p(X)$ has at most linear growth in $p$.

**Proof.** From Theorem 2.2, there is an equivariant continuous map

$$f : X \to \mathcal{X}(N, \delta)$$

for some $N \geq 1$ and $\delta > 0$. Let $p$ be any prime number. Restricting $f$ to the set of $p$-periodic points, we get a $\mathbb{Z}_p$-equivariant continuous map

$$f : P_p(X) \to P_p(\mathcal{X}(N, \delta)).$$

Then

$$\text{ind}_p P_p(X) \leq \text{ind}_p P_p(\mathcal{X}(N, \delta)) \leq Np - N - 1. \tag*{$\blacksquare$}$$

\(^6\) For a point $(x_0, x_1, \ldots, x_{p-1}) \in (\mathbb{R}^N)^{\mathbb{Z}_p} \setminus \{(x, x, \ldots, x) | x \in \mathbb{R}^N\}$, we set

$$y = (x_0 + x_1 + \cdots + x_{p-1})/p$$

and define

$$f(x_0, x_1, \ldots, x_{p-1}) = \frac{(x_0 - y, x_1 - y, \ldots, x_{p-1} - y)}{|(x_0 - y, x_1 - y, \ldots, x_{p-1} - y)|}.$$ 

This is a point in the sphere (2.3). Then

$$(1 - t)(x_0, x_1, \ldots, x_{p-1}) + tf(x_0, x_1, \ldots, x_{p-1}), \quad (0 \leq t \leq 1)$$

provides a $\mathbb{Z}_p$-equivariant strong deformation retraction from

$$(\mathbb{R}^N)^{\mathbb{Z}_p} \setminus \{(x, x, \ldots, x) | x \in \mathbb{R}^N\}$$

to the sphere (2.3).
3. The coindex of free $\mathbb{Z}_p$-spaces

The following three sections are preparations for constructing a free dynamical system which does not have the marker property. In this section we introduce the coindex of free $\mathbb{Z}_p$-spaces and study its basic properties. Most of the results in this section are certainly well-known (for example, see [Mat08, p. 99]). But we provide full proofs for completeness.

Throughout this section we assume that $p$ is a prime number. Let $(X, T)$ be a free $\mathbb{Z}_p$-space. We define its coindex by

$$\text{coind}_p(X, T) := \max \{ n \geq 0 \mid \exists f : E_n\mathbb{Z}_p \to X : \text{equivariant continuous map} \}.$$

We often abbreviate $\text{coind}_p(X, T)$ as $\text{coind}_p X$. We use the convention that the coindex of the empty set is $-1$.

It is known that if there exists an equivariant continuous map from $E_m\mathbb{Z}_p$ to $E_n\mathbb{Z}_p$ then $m \leq n$ ([Mat08, 6.2.5 Theorem]). So we always have

$$\text{coind}_p(X, T) \leq \text{ind}_p(X, T).$$

We have

$$\text{coind}_p E_n\mathbb{Z}_p = n.$$

The reasons why we use $\text{coind}_p X$ instead of $\text{ind}_p X$ are the properties (2) and (3) of the next proposition. (See also Remark 3.2 below.)

**Proposition 3.1** (Basic properties of coindex): Let $X$ and $Y$ be free $\mathbb{Z}_p$-spaces.

1. If there exists an equivariant continuous map $f : X \to Y$, then

$$\text{coind}_p X \leq \text{coind}_p Y.$$

2. The product $X \times Y$ also becomes a free $\mathbb{Z}_p$-space. ($\mathbb{Z}_p$ acts on each component simultaneously.) Its coindex is given by

$$\text{coind}_p(X \times Y) = \min(\text{coind}_p X, \text{coind}_p Y).$$

3. The join $X \ast Y$ is also a free $\mathbb{Z}_p$-space, and we have

$$\text{coind}_p(X \ast Y) \geq \text{coind}_p X + \text{coind}_p Y + 1.$$
Proof. (1) Suppose $\text{coind}_p X \geq n$. Then there exists an equivariant continuous map $g : E_n \mathbb{Z}_p \to X$. The composition $f \circ g : E_n \mathbb{Z}_p \to Y$ is also equivariant and continuous. So $\text{coind}_p Y \geq n$.

(2) The projections from $X \times Y$ to each of the factors $X$ and $Y$ are equivariant continuous maps. So, from (1), we get

$$\text{coind}_p (X \times Y) \geq \min(\text{coind}_p X, \text{coind}_p Y).$$

Suppose we are given equivariant continuous maps

$$f : E_m \mathbb{Z}_p \to X, \quad g : E_n \mathbb{Z}_p \to Y.$$

Assume $m \leq n$. Then there exists an equivariant continuous map

$$h : E_m \mathbb{Z}_p \to E_n \mathbb{Z}_p.$$

(This is obvious if we use the models $E_m \mathbb{Z}_p = (\mathbb{Z}_p)^{(m+1)} \subset E_n \mathbb{Z}_p = (\mathbb{Z}_p)^{(n+1)}$.) Then

$$E_m \mathbb{Z}_p \to X \times Y, \quad u \mapsto (f(u), g(h(u)))$$

is an equivariant continuous map. So $m \leq \text{coind}_p (X \times Y)$. Thus

$$\text{coind}_p (X \times Y) \geq \min(\text{coind}_p X, \text{coind}_p Y).$$

(3) Suppose we are given equivariant continuous maps

$$f : E_m \mathbb{Z}_p \to X, \quad g : E_n \mathbb{Z}_p \to Y.$$

Then the join of the maps

$$f \ast g : E_m \mathbb{Z}_p \ast E_n \mathbb{Z}_p \to X \ast Y,$$

$$(1 - t)u \oplus tv \mapsto (1 - t)f(u) \oplus tg(v)$$

is also an equivariant continuous map. If we use the model $E_n \mathbb{Z}_p = (\mathbb{Z}_p)^{(n+1)}$, then

$$E_m \mathbb{Z}_p \ast E_n \mathbb{Z}_p = (\mathbb{Z}_p)^{(m+n+2)} = E_{m+n+1} \mathbb{Z}_p.$$

Therefore $f \ast g$ is an equivariant continuous map from $E_{m+n+1} \mathbb{Z}_p$ to $X \ast Y$. So

$$\text{coind}_p (X \ast Y) \geq m + n + 1.$$

Thus

$$\text{coind}_p (X \ast Y) \geq \text{coind}_p X + \text{coind}_p Y + 1.$$
Remark 3.2: The above proposition shows basic properties of the $\mathbb{Z}_p$-coindex. Here we briefly mention the corresponding properties of the $\mathbb{Z}_p$-index [Mat08, 6.2.4 Proposition]: Let $X$ and $Y$ be free $\mathbb{Z}_p$-spaces.

(1) If there exists an equivariant continuous map $f : X \to Y$ then

$$\text{ind}_p X \leq \text{ind}_p Y.$$  

(2) $\text{ind}_p (X \times Y) \leq \min(\text{ind}_p X, \text{ind}_p Y)$.

(3) $\text{ind}_p (X * Y) \leq \text{ind}_p X + \text{ind}_p Y + 1$.

Notice the different behaviors of the index and the coindex under the operations of product and join. The coindex is more suitable for our application.

Recall that a free $\mathbb{Z}_p$-space $(X, T)$ is called an $E_n\mathbb{Z}_p$-space if $X$ is an $n$-dimensional and $(n - 1)$-connected finite simplicial complex with a simplicial map $T : X \to X$. It follows from this definition that if $(X, T)$ is an $E_n\mathbb{Z}_p$-space then, for any natural number $1 \leq a \leq p - 1$, the $\mathbb{Z}_p$-space $(X, T^a)$ is also an $E_n\mathbb{Z}_p$-space. We use this fact in the proof of the next lemma.

Lemma 3.3: Let $(X, T)$ be a free $\mathbb{Z}_p$-space. For any natural number $1 \leq a \leq p - 1$

$$\text{coind}_p (X, T^a) = \text{coind}_p (X, T).$$

Proof. Let $(E_n\mathbb{Z}_p, S)$ be an $E_n\mathbb{Z}_p$-space ($n \geq 0$), and suppose we are given an equivariant continuous map $f : (E_n\mathbb{Z}_p, S) \to (X, T)$. Then the same map $f$ also gives an equivariant continuous map from $(E_n\mathbb{Z}_p, S^a)$ to $(X, T^a)$. Since $(E_n\mathbb{Z}_p, S^a)$ is also an $E_n\mathbb{Z}_p$-space, we get

$$\text{coind}_p (X, T^a) \geq n.$$  

Thus

$$\text{coind}_p (X, T^a) \geq \text{coind}_p (X, T).$$

Take a natural number $1 \leq b \leq p - 1$ with $ab \equiv 1(\text{mod } p)$. Applying the same argument to $(X, T^a)$, we have

$$\text{coind}_p (X, (T^a)^b) \geq \text{coind}_p (X, T^a).$$

Since $T^{ab} = T$, the left-hand side is equal to $\text{coind}_p (X, T)$. Thus

$$\text{coind}_p (X, T) \geq \text{coind}_p (X, T^a).$$
Let \( N \) be a natural number. We consider the product of \( p \) copies of the \( N \)-dimensional cube:

\[ ([0, 1]^N)^{\mathbb{Z}_p} = \underbrace{[0, 1]^N \times [0, 1]^N \times \cdots \times [0, 1]^N}_{p \text{ times}}. \]

This becomes a (non-free) \( \mathbb{Z}_p \)-space under the cyclic shift:

\[ \sigma : ([0, 1]^N)^{\mathbb{Z}_p} \to ([0, 1]^N)^{\mathbb{Z}_p}, \quad (x_0, x_1, x_2, \ldots, x_{p-1}) \mapsto (x_1, x_2, \ldots, x_{p-1}, x_0). \]

**Corollary 3.4:** Let \( 1 \leq m \leq p-1 \) be a natural number and let \( \delta \) be a positive number. We consider the following two \( \mathbb{Z}_p \)-subspaces of \( ([0, 1]^N)^{\mathbb{Z}_p} \):

\[
X := \{ (x_0, x_1, \ldots, x_{p-1}) \in ([0, 1]^N)^{\mathbb{Z}_p} \mid \forall n \in \mathbb{Z}_p : |x_n - x_{n+1}| \geq \delta \},
\]

\[
Y := \{ (x_0, x_1, \ldots, x_{p-1}) \in ([0, 1]^N)^{\mathbb{Z}_p} \mid \forall n \in \mathbb{Z}_p : |x_n - x_{n+m}| \geq \delta \}.
\]

(Here \( X \) is the same with the space \( P_p(X(N, \delta)) \) studied in §2.3.) Then \((X, \sigma)\) and \((Y, \sigma)\) are free \( \mathbb{Z}_p \)-spaces, and

\[ \text{coind}_p(X, \sigma) = \text{coind}_p(Y, \sigma). \]

**Proof.** We can immediately see from the definition that \((X, \sigma)\) and \((Y, \sigma)\) are free. Take a natural number \( 1 \leq l \leq p-1 \) with \( lm \equiv 1 \pmod{p} \). We define continuous maps \( f : X \to Y \) and \( g : Y \to X \) by

\[
f(x_0, x_1, \ldots, x_{p-1}) = (x_0, x_l, x_{2l}, \ldots, x_{l(p-1)}),
\]

\[
g(y_0, y_1, \ldots, y_{p-1}) = (y_0, y_m, y_{2m}, \ldots, y_{m(p-1)}).
\]

More precisely,

\[
f((x_n)_{n \in \mathbb{Z}_p}) = (x_{ln})_{n \in \mathbb{Z}_p}, \quad g((y_n)_{n \in \mathbb{Z}_p}) = (y_{mn})_{n \in \mathbb{Z}_p}.
\]

Then

\[ f \circ g = \text{id}, \quad g \circ f = \text{id}. \]

Moreover

\[ f \circ \sigma = \sigma^m \circ f, \quad g \circ \sigma^m = \sigma \circ g. \]

Therefore the \( \mathbb{Z}_p \)-space \((X, \sigma)\) is isomorphic to \((Y, \sigma^m)\). Then

\[ \text{coind}_p(X, \sigma) = \text{coind}_p(Y, \sigma^m) = \text{coind}_p(Y, \sigma). \]

The last equality follows from Lemma 3.3.  \[\blacksquare\]
4. Properties of $X(N,\delta)$ and its variants

In this section we further study a property of the dynamical system $X(N,\delta)$ introduced in §2.2. We also introduce its variants.

4.1. Join of dynamical systems. Let $(X,T)$ and $(Y,S)$ be dynamical systems. We consider their join:

$$(X \ast Y, T \ast S).$$

As usual, this is defined by

$$X \ast Y = [0,1] \times X \times Y / \sim,$$

where $(0,x,y) \sim (0,x',y)$, $(1,x,y) \sim (1,x',y)$,

$$T \ast S((1-t)x \oplus ty) = (1-t)T(x) \oplus tS(y).$$

Here $(1-t)x \oplus ty$ denotes the equivalence class of $(t,x,y)$. The join $(X \ast Y, T \ast S)$ is also a dynamical system. For any natural number $n$

$$P_n(X \ast Y) = P_n(X) \ast P_n(Y).$$

In particular, if $(X,T)$ and $(Y,S)$ are both fixed-point free, then their join is also fixed-point free. Let $p$ be a prime number. Applying Proposition 3.1 (3) to

$$P_p(X \ast Y) = P_p(X) \ast P_p(Y),$$

we have

$$\text{coind}_p P_p(X \ast Y) \geq \text{coind}_p P_p(X) + \text{coind}_p P_p(Y) + 1.$$  

We are going to use this when $Y$ is a symbolic subshift.

Consider the full-shift on the alphabet $\{1,2,3\}$:

$$\{1,2,3\} = \{1,2,3\} \times \{1,2,3\} \times \{1,2,3\} \times \cdots,$$

$$\sigma : \{1,2,3\} \to \{1,2,3\}, \quad \sigma ((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}.$$  

The pair $((1,2,3),\sigma)$ is a dynamical system. We define its subsystem $\Sigma$ by

$$\Sigma = \{(x_n)_{n \in \mathbb{Z}} \in \{1,2,3\} \mid \forall n \in \mathbb{Z} : x_n \neq x_{n+1}\}.$$  

LEMMA 4.1: $(\Sigma,\sigma)$ has no fixed points. For any $m \geq 2$, $P_m(\Sigma) \neq \emptyset$.

Proof. $P_1(\Sigma) = \emptyset$ is obvious. Let $m \geq 2$. We are going to prove that $\Sigma$ has $m$-periodic points. When $m$ is an even number, the point

$$\ldots 121212 \ldots$$
is an $m$-periodic point. Let $m = 2l + 1$ be odd with $l \geq 1$. Define a word $u$ of length $m$ by
\[ u = 1212 \ldots 123. \]
Then the point 
\[ \ldots uuu \ldots \]
is an $m$-periodic point.

**Corollary 4.2:** Let $(X, T)$ be a fixed-point free dynamical system. There are a natural number $N$ and a positive number $\delta$ such that for all prime numbers $p$
\[ \coind_p P_p(\mathcal{X}(N, \delta)) \geq \coind_p P_p(X) + 1. \]

**Proof.** Let $(\Sigma, \sigma)$ be the symbolic subshift introduced in the above. We consider the join
\[ (X \star \Sigma, T \star \sigma). \]
This is fixed-point free. By (4.1)
\[ \coind_p P_p(X \star \Sigma) \geq \coind_p P_p(X) + \coind_p P_p(\Sigma) + 1. \]
By Lemma 4.1, $P_p(\Sigma)$ is a non-empty finite set with a free $\mathbb{Z}_p$-action. So it is an $E_0\mathbb{Z}_p$-space. (Recall that the discrete space $\mathbb{Z}_p$ is an $E_0\mathbb{Z}_p$-space.) Hence
\[ \coind_p P_p(\Sigma) = 0. \]
Therefore
\[ \coind_p P_p(X \star \Sigma) \geq \coind_p P_p(X) + 1. \]
Now we apply Theorem 2.2 (the universality of $\mathcal{X}(N, \delta)$) to the join $X \star \Sigma$. There are $N \geq 1$ and $\delta > 0$ such that there exists an equivariant continuous map
\[ f : X \star \Sigma \to \mathcal{X}(N, \delta). \]
Let $p$ be any prime number. Restricting $f$ to the set of $p$-periodic points, we have a $\mathbb{Z}_p$-equivariant continuous map
\[ f : P_p(X \star \Sigma) \to P_p(\mathcal{X}(N, \delta)). \]
By Proposition 3.1 (1)
\[ \coind_p P_p(\mathcal{X}(N, \delta)) \geq \coind_p P_p(X \star \Sigma) \geq \coind_p P_p(X) + 1. \]
4.2. **Variants of \( \mathcal{X}(N, \delta) \).** Let \( N \) be a natural number. We consider

\[
([0,1]^N)^\mathbb{Z} = \cdots \times [0,1]^N \times [0,1]^N \times [0,1]^N \times \cdots
\]

This becomes a dynamical system under the shift

\[
\sigma((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}.
\]

Let \( \delta \) be a positive number and \( m \) a natural number. We define a subsystem of \( (([0,1]^N)^\mathbb{Z}, \sigma) \) by

\[
\mathcal{X}_m(N, \delta) := \{(x_n)_{n \in \mathbb{Z}} \mid \forall n \in \mathbb{Z} : |x_n - x_{n+m}| \geq \delta\}.
\]

This is a fixed-point free dynamical system. When \( m = 1 \), it coincides with the system \( \mathcal{X}(N, \delta) \):

\[
\mathcal{X}_1(N, \delta) = \mathcal{X}(N, \delta).
\]

**Lemma 4.3:** The dynamical system \( (\mathcal{X}_m(N, \delta), \sigma) \) has no \( m \)-periodic points. For any prime number \( p > m \),

\[
\text{coind}_p P_p(\mathcal{X}_m(N, \delta)) = \text{coind}_p P_p(\mathcal{X}(N, \delta)).
\]

**Proof.** Obviously \( P_m(\mathcal{X}_m(N, \delta)) = \emptyset \). The set \( P_p(\mathcal{X}_m(N, \delta)) \) is identified with

\[
\{(x_0, x_1, \ldots, x_{p-1}) \in ([0,1]^N)^\mathbb{Z}_p \mid \forall n \in \mathbb{Z}_p : |x_n - x_{n+m}| \geq \delta\}.
\]

From Corollary 3.4, for prime numbers \( p > m \)

\[
\text{coind}_p P_p(\mathcal{X}_m(N, \delta)) = \text{coind}_p P_p(\mathcal{X}(N, \delta)).
\]

5. **Lindenstrauss’ lemma and its consequence**

In this section we review a topological analogue of the Rokhlin tower lemma introduced by Lindenstrauss [Lin99]. We also study its consequence.

5.1. **Lindenstrauss’ lemma.** As we mentioned in §1.3, Lindenstrauss [Lin99, Lemma 3.3] introduced a topological dynamics version of the Rokhlin tower lemma in ergodic theory. Here is that lemma:

**Lemma 5.1** ([Lin99]): Let \( (X, T) \) be a dynamical system having the marker property. For any natural number \( N \) there is a continuous function \( \varphi : X \to \mathbb{R} \) such that the set

\[
E := \{x \in X \mid \varphi(Tx) \neq \varphi(x) + 1\}
\]

satisfies \( E \cap T^{-n}E = \emptyset \) for all \( 1 \leq n \leq N \).
Proof. We explain Lindenstrauss’ ingenious proof for completeness.\(^7\) From the

definition of the marker property in §1.3, there is an open set \(U \subset X\) satisfying

\[
U \cap T^{-n}U = \emptyset \ (\forall 1 \leq n \leq N), \quad X = \bigcup_{n \in \mathbb{Z}} T^n U.
\]

Since \(X\) is compact, there is a natural number \(M \geq N\) satisfying

\[
X = \bigcup_{n=0}^{M} T^n U.
\]

Then we can find a compact set \(K \subset U\) satisfying\(^8\)

\[
X = \bigcup_{n=0}^{M} T^n K.
\]

Take a continuous function \(w : X \to [0,1]\) satisfying

\[
w(x) = 1 \quad (\forall x \in K), \quad \text{supp } w \subset U.
\]

We consider a **Markovian random walk** on \(X\) defined by

- When a particle is at a point \(x\), the random walk ends with probability \(w(x)\) and it moves to \(T^{-1}x\) with probability \((1 - w(x))\).

Since \(X = \bigcup_{n=0}^{M} T^n K\) and \(w = 1\) on \(K\), this random walk stops or enters the
set \(\{w = 1\}\) (and stops) within \(M\) steps.

---

\(^7\) [Lin99, Lemma 3.3] was stated only for extensions of infinite minimal systems. But the
proof is valid for all dynamical systems having the marker property.

\(^8\) This can be seen as follows. Let \(d\) be a metric on \(X\). Set \(U_n = T^n U\). Since \(X = \bigcup_{n=0}^{M} U_n\),
we have

\[
\max_{0 \leq n \leq M} d(x, U_n^c) > 0
\]

for all \(x \in X\). Since \(X\) is compact, there is \(\delta > 0\) satisfying

\[
\max_{0 \leq n \leq M} d(x, U_n^c) \geq \delta
\]

for all \(x \in X\). We define a compact set \(K_n \subset U_n\) by \(K_n = \{x \mid d(x, U_n^c) \geq \delta\}\). Then
\(X = K_0 \cup K_1 \cup \cdots \cup K_M\). Set

\[
K := K_0 \cup T^{-1}K_1 \cup T^{-2}K_2 \cup \cdots \cup T^{-M}K_M.
\]

This \(K\) satisfies the requirement.
For $x \in X$, we define $\varphi(x)$ as the expected number of steps in the random walk starting at $x$. The explicit formula is as follows:\footnote{Readers can easily understand this by checking the cases of some small $M$. For example, if $M = 1$ then 
$\varphi(x) = 1 - w(x)$. If $M = 2$ then 
$\varphi(x) = (1 - w(x))w(T^{-1}x) + 2(1 - w(x))(1 - w(T^{-1}x))w(T^{-2}x)$.}

$$\varphi(x) = \sum_{n=1}^{M} n \cdot \left( \prod_{k=0}^{n-1} (1 - w(T^{-k}x)) \right) \cdot w(T^{-n}x).$$

Now we have defined a continuous function $\varphi : X \to \mathbb{R}$. We are going to prove that the set $E = \{ x \mid \varphi(Tx) \neq \varphi(x) + 1 \}$ satisfies the requirement. Take a point $x \notin U$. Then $w(x) = 0$ and hence the random walk at $x$ moves to $T^{-1}x$ with probability one. So $\varphi(x) = \varphi(T^{-1}x) + 1$. Therefore we have

$$E \subset T^{-1}U.$$ 

Then for $1 \leq n \leq N$

$$E \cap T^{-n}E \subset T^{-1}U \cap T^{n-1}U = T^{-1}(U \cap T^{-n}U) = \emptyset.$$  

Remark 5.2: As is pointed out by Gutman [Gut15, Theorem 7.3], the existence of a continuous function $\varphi$ stated in Lindenstrauss’ lemma is indeed equivalent to the marker property. Namely, a dynamical system $(X,T)$ has the marker property if and only if for any natural number $N$ there is a continuous function $\varphi : X \to \mathbb{R}$ such that the set $E = \{ x \in X \mid \varphi(Tx) \neq \varphi(x) + 1 \}$ satisfies $E \cap T^{-n}E = \emptyset$ for all $1 \leq n \leq N$. So Lindenstrauss’ lemma provides an equivalent condition for the marker property.

5.2. Dynamical systems $\mathcal{Y}$ and $\mathcal{Z}$. In this subsection we introduce two dynamical systems related to the marker property. We consider a circle $\mathbb{R}/2\mathbb{Z}$. (Here we use $\mathbb{R}/2\mathbb{Z}$ instead of more natural $\mathbb{R}/\mathbb{Z}$ for later convenience.) We define a metric $\rho$ on it by

$$\rho(x, y) := \min_{n \in \mathbb{Z}} |x - y - 2n|.$$ 

The diameter of $(\mathbb{R}/2\mathbb{Z}, \rho)$ is one. (The distance between antipodal points is equal to one, e.g., $\rho(0,1) = 1$.)
We consider the infinite product of the copies of $\mathbb{R}/2\mathbb{Z}$:

$$(\mathbb{R}/2\mathbb{Z})^\mathbb{Z} = \cdots \times \mathbb{R}/2\mathbb{Z} \times \mathbb{R}/2\mathbb{Z} \times \mathbb{R}/2\mathbb{Z} \times \cdots .$$

This becomes a dynamical system under the shift map

$$\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}.$$

We introduce two subsystems of $((\mathbb{R}/2\mathbb{Z})^\mathbb{Z}, \sigma)$ by

$$\mathcal{Y} := \{ (x_n)_{n\in\mathbb{Z}} \in (\mathbb{R}/2\mathbb{Z})^\mathbb{Z} | \forall n \in \mathbb{Z} : \max(\rho(x_n, x_{n+1}), \rho(x_{n+1}, x_{n+2})) = 1 \},$$

$$\mathcal{Z} := \{ (x_n)_{n\in\mathbb{Z}} \in (\mathbb{R}/2\mathbb{Z})^\mathbb{Z} | \forall n \in \mathbb{Z} : \max(\rho(x_n, x_{n+1}), \rho(x_{n+1}, x_{n+2})) \geq \frac{1}{2} \}.$$

We have $\mathcal{Y} \subset \mathcal{Z}$. They are both fixed-point free dynamical systems.

The dynamical system $(\mathcal{Y}, \sigma)$ is related to the marker property by the next lemma. (The dynamical system $(\mathcal{Z}, \sigma)$ will be used in the next section.)

**Lemma 5.3:** Let $(X, T)$ be a dynamical system having the marker property. Then there is an equivariant continuous map from $X$ to $\mathcal{Y}$.

**Proof.** We use Lindenstrauss’ lemma (Lemma 5.1) with $N = 1$. Then there is a continuous function $\varphi : X \to \mathbb{R}$ such that the set

$$E := \{ x \in X | \varphi(Tx) \neq \varphi(x) + 1 \}$$

satisfies $E \cap T^{-1}E = \emptyset$.

Consider the composition of $\varphi$ and the natural projection $\mathbb{R} \to \mathbb{R}/2\mathbb{R}$. We also denote it as $\varphi : X \to \mathbb{R}/2\mathbb{Z}$.

**Claim 5.4:** For any $x \in X$

$$\max(\rho(\varphi(x), \varphi(Tx)), \rho(\varphi(Tx), \varphi(T^2x))) = 1.$$ 

Indeed, if $x \notin E$ then $\varphi(Tx) = \varphi(x) + 1$ and hence $\rho(\varphi(x), \varphi(Tx)) = 1$. If $x \in E$ then $Tx \notin E$. So $\varphi(T^2x) = \varphi(Tx) + 1$ and $\rho(\varphi(Tx), \varphi(T^2x)) = 1$. This proves the claim.

We define an equivariant continuous map $f : X \to (\mathbb{R}/2\mathbb{R})^\mathbb{Z}$ by

$$f(x) := (\varphi(T^nx))_{n\in\mathbb{Z}}.$$

Take any $x \in X$ and $n \in \mathbb{Z}$. Applying the above Claim 5.4 to the point $T^nx$, we get

$$\max(\rho(\varphi(T^nx), \varphi(T^{n+1}x)), \rho(\varphi(T^{n+1}x), \varphi(T^{n+2}x))) = 1.$$ 

Therefore $f(x) \in \mathcal{Y}$. So $f$ is an equivariant continuous map from $X$ to $\mathcal{Y}$. □
6. Existence of a free dynamical system which does not have the marker property: Proof of Theorem 1.1

In this section we combine all the preparations and construct a free dynamical system which does not have the marker property.

6.1. FROM AN INFINITE PRODUCT TO A FINITE PRODUCT. Here we introduce a trick to reduce a problem on an infinite product to one on a finite product. Recall that we have introduced two dynamical systems \( \mathcal{Y} \) and \( \mathcal{Z} \) in §5.2.

**Lemma 6.1:** Let \( (X_m, T_m) \ (m \in \mathbb{N}) \) be a countable number of dynamical systems. Consider their product.\(^\text{10}\)

\[
\left( \prod_{m=1}^{\infty} X_m, \prod_{m=1}^{\infty} T_m \right).
\]

Suppose there is an equivariant continuous map

\[
f : \prod_{m=1}^{\infty} X_m \to \mathcal{Y}.
\]

Then there are a natural number \( M \) and an equivariant continuous map

\[
g : X_1 \times X_2 \times \cdots \times X_M \to \mathcal{Z}.
\]

**Proof.** Set

\[
(\mathcal{X}, T) := \left( \prod_{m=1}^{\infty} X_m, \prod_{m=1}^{\infty} T_m \right).
\]

For each natural number \( M \) we set

\[
(\mathcal{X}_M, T_M) := \left( \prod_{m=1}^{M} X_m, \prod_{m=1}^{M} T_m \right).
\]

We denote by \( \pi_M : \mathcal{X} \to \mathcal{X}_M \) the natural projection.

---

\(^{10}\) This is also a dynamical system. The map \( \prod_{m=1}^{\infty} T_m \) is defined by

\[
\left( \prod_{m=1}^{\infty} T_m \right)(x_1, x_2, x_3, \ldots) = (T_1 x_1, T_2 x_2, T_3 x_3, \ldots).
Let $\varphi: X \to \mathbb{R}/2\mathbb{Z}$ be the 0-th component of $f: X \to Y \subset \mathbb{R}/2\mathbb{Z}$. We have

$$f(x) = (\varphi(T^n x))_{n \in \mathbb{Z}}.$$  

Since $X$ is compact, $\varphi$ is uniformly continuous. Hence we can choose a natural number $M$ such that for any two points $x, y \in X$

$$(6.1) \quad \pi_M(x) = \pi_M(y) \implies \rho(\varphi(x), \varphi(y)) < \frac{1}{4}.$$  

Fix a point $u \in X_{M+1} \times X_{M+2} \times X_{M+3} \times \cdots$. We define a continuous map $\psi: X_M \to \mathbb{R}/2\mathbb{Z}$ by

$$\psi(x) := \varphi(x, u).$$

We also define an equivariant continuous map $g: X_M \to \mathbb{R}/2\mathbb{Z}$ by

$$g(x) := (\psi(T^n_M x))_{n \in \mathbb{Z}} = (\varphi(T^n_M x, u))_{n \in \mathbb{Z}}.$$  

We need to show $g(x) \in \mathbb{Z}$.

For any $x \in X_M$ and $n \in \mathbb{Z}$

$$\pi_M(T^n_M x, u) = \pi_M(T^n (x, u)) = T^n_M x.$$  

By (6.1),

$$(6.2) \quad \rho(\psi(T^n_M x), \varphi(T^n(x, u))) = \rho(\varphi(T^n_M x, u), \varphi(T^n(x, u))) < \frac{1}{4}.$$  

**Claim 6.2:** For any $x \in X_M$

$$\max\{\rho(\psi(x), \psi(T_M x)), \rho(\psi(T_M x), \psi(T^2_M x))\} \geq \frac{1}{2}.$$  

Indeed, since $f(x, u) \in Y$, there is $k \in \{0, 1\}$ satisfying

$$\rho(\varphi(T^k(x, u)), \varphi(T^{k+1}(x, u))) = 1.$$  

Using (6.2) for $n = k$ and $n = k + 1$,

$$\rho(\psi(T^n_M x), \varphi(T^n(x, u))) < \frac{1}{4}, \quad \rho(\psi(T^{n+1}_M x), \varphi(T^{n+1}(x, u))) < \frac{1}{4}.$$  

By the triangle inequality

$$\rho(\psi(T^n_M x), \psi(T^{n+1}_M x)) \geq \rho(\varphi(T^n(x, u)), \varphi(T^{n+1}(x, u)))$$

$$- \rho(\psi(T^n_M x), \varphi(T^n(x, u))) - \rho(\psi(T^{n+1}_M x), \varphi(T^{n+1}(x, u)))$$

$$> 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}.$$  

This proves the above claim.
Take any point \( x \in X_M \) and any integer \( n \). We apply Claim 6.2 to the point \( T_M^n x \) and get

\[
\max\{\rho(\psi(T_M^n x), \psi(T_M^{n+1} x)), \rho(\psi(T_M^n x), \psi(T_M^{n+2} x))\} \geq \frac{1}{2}.
\]

This shows that \( g(x) \in \mathcal{Z} \). So \( g \) is an equivariant continuous map from \( X_M \) to \( \mathcal{Z} \).

6.2. PROOF OF THEOREM 1.1. Now we are ready to prove Theorem 1.1. We write the statement again.

**Theorem 6.3 (= Theorem 1.1):** There exists a free dynamical system which does not have the marker property.

**Proof.** Recall that the dynamical system \( \mathcal{Z} \) (introduced in §5.2) is fixed-point free. Applying Corollary 4.2 to \( \mathcal{Z} \), we can choose a natural number \( N \) and a positive number \( \delta \) such that for all prime numbers \( p \)

\[
(6.3) \quad \coind_p P_p(\mathcal{X}(N, \delta)) \geq \coind_p P_p(\mathcal{Z}) + 1.
\]

Now we consider the dynamical systems \( \mathcal{X}_m(N, \delta) \) \((m \in \mathbb{N})\) introduced in §4.2. (The parameters \( N \) and \( \delta \) have been fixed by the condition (6.3).) For simplicity of the notation, we set \( \mathcal{X}_m := \mathcal{X}_m(N, \delta) \). We define a dynamical system \( X \) as the product of \( \mathcal{X}_m \):

\[
X := \prod_{m=1}^{\infty} \mathcal{X}_m.
\]

We are going to prove that \( X \) is free and does not have the marker property.

For each natural number \( m \), the system \( \mathcal{X}_m \) has no \( m \)-periodic points (Lemma 4.3). So \( X \) has no \( m \)-periodic points. Since \( m \) is arbitrary, \( X \) is free.

Suppose \( X \) has the marker property. From Lemma 5.3, there is an equivariant continuous map \( f : X \to \mathcal{Y} \).

Applying Lemma 6.1 to the infinite product \( X = \prod_{m=1}^{\infty} \mathcal{X}_m \), there are a natural number \( M \) and an equivariant continuous map

\[
g : \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_M \to \mathcal{Z}.
\]

Take a prime number \( p \) larger than \( M \), and restrict the map \( g \) to the set of \( p \)-periodic points:

\[
g : P_p(\mathcal{X}_1) \times P_p(\mathcal{X}_2) \times \cdots \times P_p(\mathcal{X}_M) \to P_p(\mathcal{Z}).
\]
This is a $\mathbb{Z}_p$-equivariant continuous map. So by Proposition 3.1 (1)
\begin{align}
(6.4) \quad \text{coind}_p \{ P_p(X_1) \times P_p(X_2) \times \cdots \times P_p(X_M) \} \leq \text{coind}_p P_p(Z).
\end{align}
By Proposition 3.1 (2), the left-hand side is equal to
\[ \min\{ \text{coind}_p P_p(X_1), \text{coind}_p P_p(X_2), \ldots, \text{coind}_p P_p(X_M) \}. \]
By Lemma 4.3, for all $1 \leq m \leq M$
\[ \text{coind}_p P_p(X_m) = \text{coind}_p P_p(\mathcal{X}(N, \delta)). \]
Hence
\[ \min\{ \text{coind}_p P_p(X_1), \text{coind}_p P_p(X_2), \ldots, \text{coind}_p P_p(X_M) \} = \text{coind}_p P_p(\mathcal{X}(N, \delta)). \]
Therefore the inequality (6.4) becomes
\[ \text{coind}_p P_p(\mathcal{X}(N, \delta)) \leq \text{coind}_p P_p(Z). \]
However, from (6.3)
\[ \text{coind}_p P_p(\mathcal{X}(N, \delta)) \geq \text{coind}_p P_p(Z) + 1. \]
This is a contradiction. Thus $X$ does not have the marker property. \hfill \qed

Remark 6.4: By investigating the above arguments a bit more closely, we can actually prove the following stronger statement: Let $N \geq 1$ and $\delta > 0$, and set
\[ (X, T) := \prod_{m=1}^{\infty} (\mathcal{X}_m(N, \delta), \text{shift}). \]
This is a free dynamical system. If $N$ is sufficiently large and $\delta$ is sufficiently small, then there is no open set $U \subset X$ satisfying
\[ U \cap T^{-1}U = \emptyset, \quad X = \bigcup_{n \in \mathbb{Z}} T^{-n}U. \]

7. Open problems

This paper is just a starting point for investigating applications of the $G$-index to dynamical systems theory. There are definitely many open problems and new phenomena to be explored. Here we mention just a few questions directly related to the main results of the paper.\footnote{After the initial version of this paper was submitted to the journal, some progress was made about the questions posed in this section. We will briefly explain it in Remark 7.7 below.}
Theorem 1.2 states that, for any fixed-point free dynamical system \((X, T)\), the sequence
\[
\text{ind}_p P_p(X) \quad (p = 2, 3, 5, 7, 11, 13, 17, \ldots)
\]
grows at most linearly in \(p\). However, we do not know whether there is an example which actually has linear growth. Namely

**Problem 7.1:** Is there a fixed-point free dynamical system \((X, T)\) satisfying
\[
\text{ind}_p P_p(X) \geq Cp
\]
for some positive constant \(C\) and all sufficiently large prime numbers \(p\)? We can ask the same question for \(\text{coind}_p P_p(X)\) and \(\text{conn} P_p(X)\).

The authors have spent a lot of time trying to solve this problem. But we have not succeeded. It seems better to study the following simpler question before attacking Problem 7.1.

**Problem 7.2:** Is there a fixed-point free dynamical system \((X, T)\) such that the sequence
\[
\text{ind}_p P_p(X) \quad (p = 2, 3, 5, 7, 11, 13, 17, \ldots)
\]
is unbounded? We can ask the same question for \(\text{coind}_p P_p(X)\) and \(\text{conn} P_p(X)\).\(^\text{12}\)

The above two problems might look abstract. But indeed they are concrete questions. By the universality of \(\mathcal{X}(N, \delta)\) in Theorem 2.2, the questions (concerning the \(\mathbb{Z}_p\)-index and coindex)\(^\text{13}\) reduce to the case of \(X = \mathcal{X}(N, \delta)\). So they are more or less equivalent to the following concrete problem.

**Problem 7.3:** Let \(N\) be a natural number and \(\delta\) a positive number. For prime numbers \(p\), estimate the \(\mathbb{Z}_p\)-index (or \(\mathbb{Z}_p\)-coindex) of
\[
P_p(\mathcal{X}(N, \delta)) = \{(x_0, x_1, \ldots, x_{p-1}) \in ([0, 1]^N)_{\mathbb{Z}_p} \mid \forall n \in \mathbb{Z}_p : |x_n - x_{n+1}| \geq \delta\}.
\]
Notice that we are mainly interested in the asymptotic behavior as \(p \to \infty\) (while \(N\) and \(\delta\) are fixed).

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\(^{12}\) See Remark 7.7 below for recent progress.  
\(^{13}\) The question about \(\text{conn} P_p(X)\) does not reduce to the case of \(X = \mathcal{X}(N, \delta)\) because the connectivity does not necessarily increase under morphisms.
We also propose some problems on the marker property. Theorem 1.1 shows that periodic-point freeness alone does not imply the marker property. So we need an additional condition besides the freeness for guaranteeing the marker property. The following conjecture seems plausible.

**Conjecture 7.4:** If a free dynamical system has zero mean dimension, then it has the marker property.

Indeed this is equivalent to the following conjecture of Lindenstrauss:

**Conjecture 7.5 (Lindenstrauss):** If a free dynamical system has zero mean dimension, then it has the small boundary property.

The small boundary property is a dynamical analogue of totally disconnectedness introduced by Lindenstrauss–Weiss [LW00, Section 5]. (Here we do not provide the precise definition.) Lindenstrauss [Lin99, Theorem 6.2] proved that if a zero mean dimensional dynamical system has the marker property, then it has the small boundary property. On the other hand, Gutman (in a private communication) proved that if a free dynamical system has the small boundary property, then it has the marker property. So Conjectures 7.4 and 7.5 are equivalent.

We notice that the dynamical system \( \prod_{m=1}^{\infty} X_m(N, \delta) \) constructed in §6.2 has infinite mean dimension. It seems difficult to construct a finite mean dimensional example by our current method. So we propose:

**Problem 7.6:** Construct a finite mean dimensional free dynamical system which does not have the marker property.\(^{14}\)

Finally we would like to mention a curious observation. Let \( Z \) be the dynamical system introduced in §5.2. If Conjecture 7.4 is true, then the sequence

\[
\text{coind}_p P_p(Z) \quad (p = 2, 3, 5, 7, 11, \ldots)
\]

must be unbounded. (In particular, the answer to Problem 7.2 on the unboundedness of \( \text{ind}_p P_p(X) \) and \( \text{coind}_p P_p(X) \) becomes “yes” if Conjecture 7.4 is true.)

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\(^{14}\) This problem was affirmatively solved by [Shi21] after the initial version of the paper was submitted to the journal. See Remark 7.7 below.
On the contrary, suppose there is a natural number $K$ satisfying
\[ \coind_p P_p(Z) \leq K \]
for all prime numbers $p$.

For each natural number $m$, we define a symbolic subshift $\Sigma_m \subset \{1, 2, 3\}^\mathbb{Z}$ by
\[ \Sigma_m = \{(x_n)_{n \in \mathbb{Z}} \in \{1, 2, 3\}^\mathbb{Z} \mid \forall n \in \mathbb{Z} : x_n \neq x_{n+m}\}. \]
It is easy to check that $\Sigma_m$ has no $m$-periodic point and $P_p(\Sigma_m) \neq \emptyset$ for all prime numbers $p > m$. We define a dynamical system $X_m$ by
\[ X_m = (\Sigma_m)^{(K+2)} \quad \text{(the join of the $(K+2)$ copies of $\Sigma_m$)}. \]

The system $X_m$ has no $m$-periodic point. For prime numbers $p > m$, it follows from Proposition 3.1 (3) that
\[ (7.1) \quad \coind_p P_p(X_m) = \coind_p P_p(\Sigma_m)^{(K+2)} \geq K + 1 > \coind_p P_p(Z). \]

Now we consider the infinite product $\prod_{m=1}^\infty X_m$. This is a free dynamical system. By applying the argument of §6.2, we can prove that $\prod_{m=1}^\infty X_m$ does not have the marker property. Indeed, suppose $\prod_{m=1}^\infty X_m$ has the marker property. By Lemma 5.3 there is an equivariant continuous map $f : \prod_{m=1}^\infty X_m \to \mathcal{Y}$. By Lemma 6.1, for some $M \geq 1$, there is an equivariant continuous map $g : \prod_{m=1}^M X_m \to \mathcal{Z}$. Take a prime number $p > M$ and restrict $g$ to the set of $p$-periodic points:
\[ g : \prod_{m=1}^M P_p(X_m) \to P_p(Z). \]
Then we get
\[ \min_{1 \leq m \leq M} \coind_p P_p(X_m) \leq \coind_p P_p(Z). \]
This contradicts the above (7.1).

On the other hand, each $X_m$ is finite-dimensional (and hence has zero mean dimension). Then, by [Lin99, p. 231, (2.2)], the product $\prod_{m=1}^\infty X_m$ has zero mean dimension. If Conjecture 7.4 is true, then $\prod_{m=1}^\infty X_m$ has the marker property. This is a contradiction.

Conjecture 7.4 (equivalently, Conjecture 7.5) is very difficult for our current technology. However, the above system $\prod_{m=1}^\infty X_m$ has a very special form (i.e., infinite product of finite-dimensional systems). So we might be able to prove by our current technology that it has the marker property.
Remark 7.7: After the initial version of this paper was submitted to the journal, Shi [Shi21] solved Problem 7.6 affirmatively. Moreover, he [Shi21, Theorem 1.2] proved that for any positive number \( \eta \) there exists a free dynamical system of mean dimension smaller than \( \eta \) which does not have the marker property. His proof replaced our infinite product construction in §6.2 with a clever construction of an inverse limit and used the \( \mathbb{Z}_p \)-coindex.

By using the method of [Shi21], Shi–Tsukamoto [ST21] proved (without assuming Conjecture 7.4) that

\[
\lim_{p \to \infty} \text{coind}_p P_p(Z) = \infty.
\]

This answers the index and coindex parts of Problem 7.2 affirmatively. The connectivity part of Problem 7.2 remains open.

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