Localization computation of one-point disk invariants of projective Calabi-Yau complete intersections

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Abstract

We define one-point disk invariants of a smooth projective Calabi-Yau (CY) complete intersection (CI) in the presence of an anti-holomorphic involution via localization. We show that these invariants are rational numbers and obtain a formula for them which confirms, in particular, a conjecture by Jinzenji-Shimizu [JS, Conjecture 1].

1 The one-point disk mirror theorem

The problem of defining and computing disk invariants with respect to a Lagrangian was and is the subject of much research in both mathematics and theoretical physics. The problem of defining them is solved under certain constraints in [So], [Cho], and [Ge1]. The disk invariants of projective CY CI threefolds with respect to a Lagrangian defined by the fixed locus of an anti-holomorphic involution were computed in [PSoW] and [Sh].

In this paper, we define one-point disk invariants of a smooth projective CY CI in the presence of an anti-holomorphic involution via localization as suggested by [Wi, (4.7),(4.8)] and [JS]. Thus, our invariants are sums over graphs of rational functions in torus weights; see Section 2.3. We prove that these invariants are actually rational numbers and obtain a formula for them. In the case when the CI is a hypersurface, our theorem confirms a conjecture by Jinzenji-Shimizu; see [JS] Conjecture 1 and the Theorem below.

Throughout this paper, we consider the anti-holomorphic involution

$$\Omega : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{n-1}, \quad \Omega([z_1, z_2, \ldots, z_n]) = \begin{cases} [\bar{z}_2, \bar{z}_1, \ldots, \bar{z}_n, \bar{z}_{n-1}] & \text{if } 2 | n; \\ [\bar{z}_2, \bar{z}_1, \ldots, \bar{z}_{n-1}, \bar{z}_{n-2}, \bar{z}_n] & \text{if } 2 \not| n. \end{cases}$$

Let \( \mathbf{a} = (a_1, a_2, \ldots, a_l) \) be a tuple of positive odd integers and

\[ X_\mathbf{a} \subset \mathbb{P}^{n-1} \]

a smooth CI of multi-degree \( \mathbf{a} \) preserved by \( \Omega \) (such as the Fermat quintic threefold). We assume that \( X_\mathbf{a} \) is CY, i.e.

$$\sum_{k=1}^{l} a_k = n$$

and that \( X_\mathbf{a} \) has odd dimension, i.e.

$$n - l \in \mathbb{Z}^{>0}$$

is even.
For each positive odd integer $d$, we denote by $N^{\text{disk}}_{1,d}$ the degree $d$ one-point disk invariant of $(X_a, \Omega)$ described by Definition 2.1 below. The invariant $N^{\text{disk}}_{1,d}$ will be expressed in terms of explicit formal power series in the variables $q^l$ and $q$ respectively, which we now define. Let

$$
\tau_a(q) \equiv 2 \sum_{d \in \mathbb{Z}^>0 \atop d \text{ odd}} q^d \frac{l \prod_{k=1}^{l} (a_k d)!}{(d!)^n} \in \mathbb{Q}[[q^l]].
$$

The remaining formal power series necessary for our formula occur in the closed genus 0 and 1 mirror formulas in [Z1, PoZ, Z2, Po]. These are encoded by

$$
F(w, q) \equiv \sum_{d=0}^\infty q^d \frac{l \prod_{k=1}^{l} (a_k w + r)}{\prod_{r=1}^{d} (w + r)^n}.
$$

This is a formal power series in $q$ with constant term 1 whose coefficients are rational functions in $w$ which are regular at $w = 0$. As in [ZaZ], we denote the subgroup of all such power series by $\mathcal{P}$ and define

$$
\mathbf{M} : \mathcal{P} \to \mathcal{P} \quad \text{by} \quad \mathbf{M} H(w, q) \equiv 1 + \frac{q}{w dq} \left( \frac{H(w, q)}{H(0, q)} \right).
$$

If $p \in \mathbb{Z}^>0$, let

$$
I_p(q) \equiv \mathbf{M}^p F(0, q) \in 1 + q \cdot \mathbb{Q}[[q]].
$$

Let

$$
J(q) \equiv \frac{1}{I_0(q)} \left\{ \sum_{d=1}^\infty q^d \frac{l \prod_{k=1}^{l} (a_k d)!}{(d!)^n} \left( \sum_{k=1}^{l} \sum_{r=d+1}^{a_k d} \frac{a_k}{r} \right) \right\} \in q \cdot \mathbb{Q}[[q]] \quad \text{and} \quad Q \equiv q e^{J(q)}.
$$

The map $q \to Q$ is a change of variables called the mirror map.

As is usually done in Gromov-Witten theory, we package the one-point disk invariants $N^{\text{disk}}_{1,d}$ of Definition 2.1 below into a generating function in the formal variable $Q^l$:

$$
Z^{\text{disk}}_1(Q) \equiv \sum_{d \in \mathbb{Z}^>0 \atop d \text{ odd}} Q^l N^{\text{disk}}_{1,d}.
$$

\textbf{Theorem.} The one-point disk invariants $N^{\text{disk}}_{1,d}$ of $(X_a, \Omega)$ as described by Definition 2.1 below are rational numbers. They are explicitly computed by

$$
Z^{\text{disk}}_1(Q) = \frac{2^{\frac{a_l + 2}{2}}}{I_{a_l + 2}^{-1}(q)} \frac{d}{dq} \left\{ \frac{1}{I_{a_l + 2}^{-1}(q)} \frac{d}{dq} \left\{ \frac{1}{I_{a_l + 2}^{-1}(q)} \frac{d}{dq} \left\{ \cdots \frac{d}{dq} \left\{ \frac{\tau_a(q)}{I_0(q)} \right\} \right\} \right\} \right\},
$$

where $Q$ and $q$ are related by the mirror map $Q = q e^{J(q)}$ in (1.1).
The above **Theorem** implies [JS, Conjecture 1] in the case when $X_a$ is a hypersurface. In the case of the quintic threefold $X_{(5)} \subset \mathbb{P}^4$, the mirror formula (1.3) recovers [PSoW, Theorem 1] as can be seen using the divisor equation and $Q \frac{d}{dq} = \frac{q}{r(q)} \frac{d}{dq}$.

Gromov-Witten (GW) invariants of a smooth projective variety $X$ are typically defined by integrating certain cohomology classes against a virtual fundamental class of a moduli space of stable maps. See [MirSym, Section 26.2] for the case of the closed invariants and [Ge1] for the case of the disk invariants. If $X$ is a CI in $\mathbb{P}^{n-1}$, then genus 0 Gromov-Witten invariants of $X$ are related to those of $\mathbb{P}^{n-1}$ via an Euler class formula which allows computation via equivariant localization. See [MirSym, Theorem 26.1.1] for the case of the closed invariants and [PSoW, Theorem 3] for the case of the disk invariants of $X_{(5)} \subset \mathbb{P}^4$. The following questions regarding the invariants $N_{1,d}^{\text{disk}}$ in Definition 2.1 below remain open - in the general case - as far as the author is aware:

- When and how is it possible to define these invariants as an integral over the virtual fundamental class of a moduli space of stable maps?
- Is there an Euler class formula as suggested by (2.3), [Ge2, Corollary 1.10, Remark 1.11], generalizing [PSoW, Theorem 3] to higher dimensions?

### 1.1 Acknowledgments

I thank Aleksey Zinger for bringing [JS] to my attention and for collaborating on [PoZ]. The present paper is an application of one of the mirror formulas in [PoZ, Theorem 6] and an extension of [PoZ, Lemma 6.2].

I thank Johannes Walcher for independently pointing out [JS].

### 2 The one-point disk invariants

#### 2.1 On what the invariants should be

This section explains non-rigorously what the invariants should be providing the motivation for Section 2.3 where we actually define them. The idea appears in [JS] and in the rich research work on disk invariants preceding it. The notation used in this section will not be used in the remaining sections, unless re-defined.

For each positive odd integer $d$, we denote by $\mathcal{M}_{0,1}(X_a,d)^{\Omega}$ the moduli space of stable one-point degree $d$ doubled disk maps to $X_a$. An element in this space is an equivalence class $[(\Sigma, c), f, z_1, c(z_1)]$ where

- $(\Sigma, c)$ is a genus 0 nodal curve together with an anti-holomorphic involution $c: \Sigma \rightarrow \Sigma$ so that $\Sigma^c \equiv \{ p \in \Sigma : c(p) = p \}$ is a chain of circles;
- $\Omega \circ f = f \circ c$;
- $z_1 \in \Sigma - \Sigma^c$;

1 If $n - l - 2 = 0$, then (1.3) should be read $Z_{1,d}^{\text{disk}}(Q) = \frac{\tau(a)}{\tau(0)}$.
2 See Appendix A for the correspondence between the relevant notation in [JS] and our notation.
• The tuple \((\Sigma, f, z_1, c(z_1))\) is a degree \(d\) two-point genus 0 stable map and so it determines an element in \(\mathcal{M}_{0,2}(X, d)\).

If \(\Sigma/c\) consists of only one circle, then \(\Sigma/c\) is a genus 0 nodal curve with a disk attached to it. Since \(X\) is CY, the virtual real dimension of \(\mathcal{M}_{0,1}(X, d)^{\Omega}\) should be \(n - l - 2\).

There is a natural evaluation map

\[
ev_1 : \mathcal{M}_{0,1}(X, d)^{\Omega} \to X, \quad \ev_1[(\Sigma, c), f, z_1, c(z_1)] \equiv f(z_1).
\]

With \(H \in H^2(\mathbb{P}^{n-1}; \mathbb{Z})\) denoting the hyperplane class on \(\mathbb{P}^{n-1}\), the one-point disk invariant that we will compute should be of the form

\[
2 \int_{[\mathcal{M}_{0,1}(X, d)^{\Omega}]}^{vir} \ev_1^*H^{n-l-2}. \tag{2.1}
\]

The number 2 in (2.1) comes from the fact that a stable one-point degree \(d\) doubled disk map corresponds to two one-point disk maps, by restricting the doubled map to either the “upper or the lower half” of its domain.

As in the case of closed genus 0 GW invariants [BDPP Section 2.1.2] and in that of closed reduced genus 1 GW invariants [LiZ Theorem 1.1], we hope that the “one-point disk invariants” of \((X, \Omega)\) are related to those of the ambient space \(\mathbb{P}^{n-1}\) via an Euler class formula. This is indeed the case for disk invariants without marked points of the quintic threefold; see [PSDW Theorem 3].

We denote by \(e(V_d^\Omega)\) the “Euler class of the bundle”

\[
V_d^\Omega \equiv \mathcal{M}_{0,1}^1(\mathcal{L}, d) \to \mathcal{M}_{0,1}(\mathbb{P}^{n-1}, d),
\]

where

\[
\mathcal{L} \equiv \bigoplus_{k=1}^{l} \mathcal{O}_{\mathbb{P}^{n-1}}(a_k) \to \mathbb{P}^{n-1}. \tag{2.2}
\]

Thus, we expect that (2.1) should equal an expression of the form

\[
2 \int_{[\mathcal{M}_{0,1}(\mathbb{P}^{n-1}, d)^{\Omega}]}^{vir} e(V_d^\Omega)ev_1^*H^{n-l-2}. \tag{2.3}
\]

### 2.2 Setup and closed genus 0 generating functions

In this section, we briefly recall the setup in [PoZ Sections 1.1,1.2,3] that we need for our graph-sum definition.

In the remaining part of this paper, all cohomology groups are with rational coefficients.

We write \(\mathcal{M}_{0,2}(\mathbb{P}^{n-1}, d)\) for the moduli space of stable degree \(d\) maps into \(\mathbb{P}^{n-1}\) from genus 0 curves with 2 marked points and

\[
\ev_i : \mathcal{M}_{0,2}(\mathbb{P}^{n-1}, d) \to \mathbb{P}^{n-1}
\]

for the evaluation map at the \(i\)-th marked point. Denote by

\[
\mathcal{V}_a \to \mathcal{M}_{0,2}(\mathbb{P}^{n-1}, d)
\]
the vector bundle corresponding to the locally free sheaf

\[ \pi_* \ev^* \mathcal{L} \to \overline{M}_{0,2}(\mathbb{P}^{n-1}, d), \]

where \( \mathcal{L} \to \mathbb{P}^{n-1} \) is the vector bundle giving \( X_a \); see (2.2). For each \( i = 1, 2 \), there is a well-defined bundle map

\[ \tilde{e}v_i : \mathcal{V}_a \to \ev_i^* \mathcal{L}, \quad \tilde{e}v_i([\mathcal{C}, f; \xi]) = [\xi(z_i(C))], \]

where \( z_i(C) \) is the \( i \)-th marked point of \( C \). Let

\[ \mathcal{V}'_a \equiv \ker \tilde{e}v_1 \to \overline{M}_{0,2}(\mathbb{P}^{n-1}, d) \quad \text{and} \quad \mathcal{V}''_a \equiv \ker \tilde{e}v_2 \to \overline{M}_{0,2}(\mathbb{P}^{n-1}, d). \]

We denote by \( \mathbb{T} \) the complex \( n \)-torus. The group cohomology of \( \mathbb{T} \) is

\[ H^*_\mathbb{T} = \mathbb{Q}[\alpha_1, \alpha_2, \ldots, \alpha_n], \]

where \( \alpha_i \equiv \pi_i^* c_1(\gamma^*) \), \( \gamma \to \mathbb{P}^\infty \) is the tautological line bundle, and \( \pi_i : (\mathbb{P}^\infty)^n \to \mathbb{P}^\infty \) is the projection to the \( i \)-th component. We denote by

\[ \mathbb{Q}_\alpha \equiv \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \]

its field of fractions.

We denote the equivariant \( \mathbb{Q} \)-cohomology of a topological space \( M \) with a \( \mathbb{T} \)-action by \( H^*_\mathbb{T}(M) \). If the \( \mathbb{T} \)-action on \( M \) lifts to an action on a complex vector bundle \( V \to M \), let \( e(V) \in H^*_\mathbb{T}(M) \) denote the equivariant Euler class of \( V \).

We consider the standard action of \( \mathbb{T} \) on \( \mathbb{C}^n \),

\[ (t_1, t_2, \ldots, t_n) \cdot (z_1, z_2, \ldots, z_n) \equiv (t_1z_1, t_2z_2, \ldots, t_nz_n). \]

This action naturally induces actions on \( \mathbb{P}^{n-1} \) and the tautological line bundle \( \gamma \to \mathbb{P}^{n-1} \), on \( \overline{M}_{0,2}(\mathbb{P}^{n-1}, d) \) and \( \mathcal{V}_a, \mathcal{V}'_a, \mathcal{V}''_a \), and on the universal cotangent line bundles. We denote the equivariant Euler class of the universal cotangent line bundle for the \( i \)-th marked point by \( \psi_i \).

We denote by

\[ x \equiv e(\gamma^*) \in H^*_\mathbb{T}(\mathbb{P}^{n-1}) \]

the equivariant hyperplane class. The equivariant cohomology of \( \mathbb{P}^{n-1} \) is given by

\[ H^*_\mathbb{T}(\mathbb{P}^{n-1}) = \mathbb{Q}[x, \alpha_1, \ldots, \alpha_n]/(x - \alpha_1) \cdots (x - \alpha_n). \]

With \( p = \frac{n-l-2}{2} \) we define, following [PoZ],

\[ Z_p(x, h, Q) \equiv x^{l+p} + \sum_{d=1}^{\infty} Q^d \ev_1^* \left[ \frac{e(V''_a)ev_2^*x^{l+p}}{h-\psi_1} \right] \in (H^*_\mathbb{T}(\mathbb{P}^{n-1}))[[h^{-1}, Q]]. \tag{2.4} \]

### 2.3 The graph-sum definition

In this section we define the one-point disk invariants \( N_{1,d}^{\text{disk}} \) of \( (X_a, \Omega) \) via localization with respect to the \( \mathbb{T}^m \)-action in (2.6) below. Our definition is motivated by the Localization Theorem [ABo, GraPa], by [PSoW, PoZ Lemma 3.1], (2.3), and the idea of “breaking” a localization graph at a distinguished vertex [Gi].
For each $i = 1, 2, \ldots, n$, let
\[ \phi_i \equiv \prod_{k \neq i} (x - \alpha_k) \in H^*_T(\mathbb{P}^{n-1}). \]

We denote by
\[ P_1 \equiv [1, 0, 0, \ldots, 0], \quad P_2 \equiv [0, 1, 0, \ldots, 0], \quad P_n \equiv [0, 0, \ldots, 0, 1] \in \mathbb{P}^{n-1} \]
the $T$-fixed points in $\mathbb{P}^{n-1}$. If $\eta \in H^*_T(\overline{\mathbb{M}}_{0,2}(\mathbb{P}^{n-1}, d))$, then
\[ (ev_1)_*\eta|_{P_1} = \int_{\mathbb{P}^{n-1}} \phi_i (ev_1)_*\eta = \int_{\overline{\mathbb{M}}_{0,2}(\mathbb{P}^{n-1}, d)} \eta ev_1^* \phi_i. \]

Denote by $m$ the integer part of $n/2$ and by $T^m$ the complex $m$-torus. The embedding
\[ \iota: T^m \to \mathbb{T}, \quad (u_1, u_2, \ldots, u_m) \to \begin{cases} (u_1, u_1^{-1}, \ldots, u_m, u_m^{-1}) & \text{if } n = 2m, \\ (u_1, u_1^{-1}, \ldots, u_m, u_m^{-1}, 1) & \text{if } n = 2m + 1, \end{cases} \]
induces a $T^m$-action on $\mathbb{P}^{n-1}$.

Let $\lambda_1, \ldots, \lambda_m$ be the weights of the standard representation of $T^m$ on $\mathbb{C}^m$. It follows that
\[ (\alpha_1, \ldots, \alpha_n)|_{T^m} = \begin{cases} (\lambda_1, -\lambda_1, \ldots, \lambda_m, -\lambda_m), & \text{if } n = 2m; \\ (\lambda_1, -\lambda_1, \ldots, \lambda_m, -\lambda_m, 0), & \text{if } n = 2m + 1. \end{cases} \]

In [PoZ, Section 6.1], $D_{i, \gamma} \in \mathbb{Q}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ denotes the contribution of the half-edge disk map whose doubled corresponds to a cover of the line through $P_i$ and $\Omega(P_i)$ without a marked point to the degree $\gamma$ disk invariants of the CY CI threefold. For each $1 \leq i \leq 2m$ and each positive odd integer $\gamma$, let
\[ D_{1, i, \gamma} \equiv \frac{D_{i, \gamma}}{\gamma} \equiv \frac{\prod_{k=1}^{i} (a_k \gamma)!!}{\gamma \prod_{k=1}^{n} \prod_{s \text{ odd}} \left( \frac{\alpha_i}{\gamma} - \frac{\alpha_k}{\gamma} \right)^{s}} \prod_{(k,s) \neq (i,\gamma)}^{s} \left( \frac{\alpha_i}{\gamma} \right)^{\frac{n+1}{2}}, \]
where the above rational expression in $\alpha$ should be viewed in $\mathbb{Q}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ via \((2.7)\).

If $F \subset \overline{\mathbb{M}}_{0,2}(\mathbb{P}^{n-1}, r)$ is a $T^m$-fixed component, we denote by $NF$ its normal bundle.

For each positive odd integer $d$, let
\[ N_{1,d}^{disk} \equiv 2 \sum_{1 \leq i \leq 2m, \gamma \in \mathbb{Z}^+ \text{ even odd}} \int_F \frac{\mathcal{E}(\gamma^a) \psi^e \phi_i \chi^{n-2} (h - \psi/2) e(NF)}{\mathcal{D}_{1, i, \gamma}|_{h = \frac{2m}{\gamma}} \in \mathbb{Q}(\lambda_1, \lambda_2, \ldots, \lambda_m)}, \]
where we set the summand equal to $\frac{n-2}{2} D_{1, i, \gamma}$ if $r = 0$ and the sum is to be taken in $\mathbb{Q}(\lambda_1, \ldots, \lambda_m)$ via \((2.7)\).
In this section we recall parts of [PoZ, Theorem 6] that we need for the proof of the Theorem.

3 Proof of the Theorem

3.1 Closed genus 0 mirror theorem

In this section we recall parts of [PoZ] Theorem 6 that we need for the proof of the Theorem.

As in [PoZ], let

\[ Y(x, h, q) = \sum_{d=0}^{\infty} q^d \prod_{k=1}^{\infty} \prod_{r=1}^{l} (a_k x + rh) \in \mathbb{Q}(x, h)[[q]]. \] (3.1)

Following [PoZ] Section 3.1, we define \( \mathcal{D}^p Y_0(x, h, q) \) inductively by

\[ \mathcal{D}^0 Y_0(x, h, q) \equiv Y_0(x, h, q) = \frac{x}{I_0(q)} \frac{d}{dq} Y(x, h, q), \]
\[ \mathcal{D}^p Y_0(x, h, q) \equiv \frac{1}{I_p(q)} \left\{ x + hq \frac{d}{dq} \right\} \mathcal{D}^{p-1} Y_0(x, h, q) \quad \forall \ p \geq 1. \] (3.2)

By [PoZ] Theorem 6, there exist \( \tilde{C}_p(q) \in \mathbb{Q}[\alpha_1, \ldots, \alpha_n][[q]] \) and \( C_1(q) \in \mathbb{Q}[[q]] \) such that

\[ Z_p(x, h, Q) = e^{-J(q)} \tilde{e}^{-C_1(q)} \sum_{r=0}^{p} \sum_{s=0}^{p-r} \tilde{C}_{p,s}(q) h^{p-r-s} \mathcal{D}^s Y_0(x, h, q), \] (3.3)

where \( p = \frac{n-l-2}{2} \), \( Q \) and \( q \) are related by the mirror map \( Q = qe^J(q) \) in (1.1), and \( \sigma_1 \equiv \sum_{i=1}^{n} \alpha_i \).
3.2 The proof

In this section we prove the Theorem using (3.3) and an extension of [PoZ] Lemma 6.2.

Throughout this section, we consider only the $\mathbb{T}^m$-action on $\mathbb{P}^{n-1}$ defined by (2.6) and so in the computation below the $\mathbb{T}$-weights $\alpha_1, \ldots, \alpha_n$ should be expressed in terms of the $\mathbb{T}^m$-weights $\lambda_1, \ldots, \lambda_m$ via (2.7). We denote by

\[
\mathfrak{R}_{z=z_0} f(z)
\]

the residue of $f$ at $z_0$.

By (2.7), $\sigma_1 = 0$. This together with Definition 2.1 and (3.3) shows that

\[
Z_1^{\text{disk}}(Q) = 2 \sum_{1 \leq i \leq 2m} \sum_{\gamma \in \mathbb{Z}^\gamma > 0} q^{\gamma} \frac{1}{\alpha_i^1} \frac{1}{\alpha_i^1} \left\{ \sum_{t \geq \gamma} \lambda_{i,t} q^{\gamma} \right\} \left( q^{\gamma} \right) \left( D_1, \gamma \right)\]

(3.4)

where $Q = qe^{i(q)}$.

By (3.1) and (3.2),

\[
\mathcal{Y}_0(\alpha_i, h, q) \bigg|_{h = \frac{2\alpha_i}{\gamma}} = \frac{\alpha_i^1}{I_0(q)} \sum_{d=0}^{\infty} q^d \left( \alpha_i^1 \right) \frac{n^d}{\prod_{k=1}^{n^d} (a_k + (\gamma r + 2d))!!} \frac{\prod_{k=1}^{l} (a_k + \gamma + 2d)!!}{\prod_{k=1}^{n^d} (a_k + \gamma + 2d)!!} \left( s^{\alpha_i^1} - \alpha_k \right).
\]

(3.5)

By (2.8) and (3.5), for each integer $p \geq 0$,

\[
I_0(q) \sum_{1 \leq i \leq 2m} \sum_{\gamma \in \mathbb{Z}^\gamma > 0} q^{\gamma} \frac{1}{\alpha_i^1} \frac{1}{\alpha_i^1} \left\{ \sum_{t \geq \gamma} \lambda_{i,t} q^{\gamma} \right\} \left( q^{\gamma} \right) \left( D_1, \gamma \right)\]

(3.6)

\[
= 2^p \sum_{1 \leq i \leq 2m} \sum_{\gamma \in \mathbb{Z}^\gamma > 0} \sum_{t \geq \gamma} q^{\gamma} \left( \alpha_i^1 \right) \frac{n^d}{\prod_{k=1}^{n^d} (a_k + \gamma + 2d)!!} \frac{\prod_{k=1}^{l} (a_k + \gamma + 2d)!!}{\prod_{k=1}^{n^d} (a_k + \gamma + 2d)!!} \left( s^{\alpha_i^1} - \alpha_k \right).
\]

(3.6)

The last equality above follows from the residue theorem on $\mathbb{P}^1$.

Note that if $S(x, h, q) \in Q(\alpha, x, h)[[q]]$, then

\[
q^{\gamma} \left\{ \alpha_i + hq \frac{d}{dq} \right\} S(\alpha, h, q) \bigg|_{h = \frac{2\alpha_i}{\gamma}} = hq \frac{d}{dq} \left\{ q^{\gamma} S(\alpha_i, h, q) \right\} \bigg|_{h = \frac{2\alpha_i}{\gamma}}.
\]

(3.7)
By (3.7) and (3.2), whenever \( s \in \mathbb{Z}^+ \),
\[
I_s(q) q^{\alpha_i} \frac{d}{dq} \left\{ q^2 h^{s+1} \mathcal{D}^{s-1} \mathcal{Y}_0(\alpha, h, q) \right|_{h=2\alpha_i} \right\}. 
\] (3.8)

Using the \( 0 \leq p \leq \frac{n-l-2}{2} - 1 \) cases of (3.6), (3.8), and induction on \( s \), we obtain
\[
\sum_{1 \leq i \leq 2m} \sum_{\gamma \in \mathbb{Z}^+} \frac{q^2 h^{s} \mathcal{D}^{s} \mathcal{Y}_0(\alpha, h, q)}{\alpha_i^2} \left|_{h=2\alpha_i} \right. = 0 \quad \text{if} \quad p + s \leq \frac{n-l-2}{2} - 1. 
\] (3.9)

Using the \( p = \frac{n-l-2}{2} \) case of (3.6), (3.8), and induction on \( s \) with \( 0 \leq s \leq \frac{n-l-2}{2} \), we obtain
\[
2 \sum_{1 \leq i \leq 2m} \sum_{\gamma \in \mathbb{Z}^+} \frac{q^2 h^{s} \mathcal{D}^{s} \mathcal{Y}_0(\alpha, h, q)}{\alpha_i^2} \left|_{h=2\alpha_i} \right. = 2 \frac{n-l-2}{2} \sum_{1 \leq i \leq 2m} \sum_{\gamma \in \mathbb{Z}^+} \frac{1}{\alpha_i^2} \mathcal{D}^{s} \mathcal{Y}_0(\alpha, h, q) \left|_{h=2\alpha_i} \right. 
\] (3.10)

The Theorem follows from (3.4), (3.9), and the \( s = \frac{n-l-2}{2} \) case of (3.10).

A Correspondence between notation in [JS] and our notation

While in [JS], the target space is \( M_N^k = \{ X^k_1 + X^k_2 + \ldots + X^k_N = 0 \} \subset \mathbb{P}^{N-1} \), our target space is \( X_a \subset \mathbb{P}^{n-1} \).

| JS                                      | our notation          |
|-----------------------------------------|-----------------------|
| \( e^x \)                               | \( q \)               |
| \( I_p^{k,k}(e^x) \)                    | \( I_p(q) \)          |
| \( e^{t(x)} \)                          | \( q e^{t(q)} \)      |
| \( F^k_0(x) \)                          | right-hand side of (1.3) |
| \( h \)                                 | \( H \) in Section 2.1 |
| \( (O_h \mathcal{O}_{k+1})_{disk,2d-1} \) | \( N_{disk,1,2d-1} \) |
| \( \tau_k(x) \)                        | \( \tau_a(q) \)       |

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References

[ABo] M. Atiyah and R. Bott, The moment map and equivariant cohomology, Topology 23 (1984), 1–28.

3The \( s = 0 \) case of the right-hand side of (3.10) is \( 2 \frac{n-l-2}{2} \tau_a(q) \).
[BDPP]  G. Bini, C. de Cocini, M. Polito, and C. Procesi, On the work of Givental relative to mirror symmetry, Appunti dei Corsi Tenuti da Docenti della Scuola, Scuola Normale Superiore, Pisa, 1998.

[Cho] C.-H., Cho, Counting real $J$-holomorphic discs and spheres in dimension four and six, J. Korean Math. Soc. 45 (2008), no. 5, 1427–1442.

[Ge1] P. Georgieva, Orientability of moduli spaces and open Gromov-Witten invariants, Ph.D. thesis, Stanford University, 2011.

[Ge2] P. Georgieva, The orientability problem in open Gromov-Witten theory, math/1207.5471.

[Gi] A. Givental, Equivariant Gromov-Witten invariants, IMRN (1996), no. 13, 613–663.

[GraPa] T. Graber and R. Pandharipande, Localization of virtual classes, Invent. Math. 135 (1999), no. 2, 487–518.

[JS] M. Jinzenji and M. Shimizu, Open virtual structure constants and mirror computation of open Gromov-Witten invariants of projective hypersurfaces, math/1108.4766.

[LiZ] J. Li and A. Zinger, On the genus-one Gromov-Witten invariants of complete intersections, J. Differential Geom. 82 (2009), no. 3, 641–690.

[MirSym] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, Mirror Symmetry, Clay Math. Inst., AMS, 2003.

[Po] A. Popa, The genus one Gromov-Witten Invariants of Calabi-Yau complete intersections, Trans. AMS 365 (2013), no. 3, 1149–1181.

[PoZ] A. Popa and A. Zinger, Mirror symmetry for closed, open, and unoriented Gromov-Witten invariants, math/1010.1946.

[PSoW] R. Pandharipande, J. Solomon and J. Walcher, Disk enumeration on the quintic 3-fold, J. AMS 21 (2008), 1169-1209.

[Sh] V. Shende, One point disc descendants of complete intersections, in preparation.

[So] J. Solomon, Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions, math/0606429.

[W] J. Walcher, Opening mirror symmetry on the quintic, Comm. Math. Phys. 276 (2007), no. 3, 671–689.

[ZaZ] D. Zagier and A. Zinger, Some properties of hypergeometric series associated with mirror symmetry, Modular Forms and String Duality, 163–177, Fields Inst. Commun. 54, AMS, 2008.

[Z1] A. Zinger, Genus zero two-point hyperplane integrals in Gromov-Witten theory, Comm. Analysis Geom. 17 (2010), no. 5, 1–45.

[Z2] A. Zinger, The reduced genus-one Gromov-Witten invariants of Calabi-Yau hypersurfaces, J. Amer. Math. Soc. 22 (2009), no. 3, 691–737.