Bases of tensor products and geometric Satake correspondence

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To the memory of C. S. Seshadri

Abstract

The geometric Satake correspondence can be regarded as a geometric construction of the rational representations of a complex connected reductive group $G$. In their study of this correspondence, Mirković and Vilonen introduced algebraic cycles that provide a linear basis in each irreducible representation. Generalizing this construction, Goncharov and Shen define a linear basis in each tensor product of irreducible representations. We investigate these bases and show that they share many properties with the dual canonical bases of Lusztig.

1 Introduction

Let $G$ be a connected reductive group over the field of complex numbers, endowed with a Borel subgroup and a maximal torus. Let $\Lambda^+$ be the set of dominant weights relative to these data. Given $\lambda \in \Lambda^+$, denote by $V(\lambda)$ the irreducible rational representation of $G$ with highest weight $\lambda$. Given a finite sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ in $\Lambda^+$, define

$$V(\lambda) = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n).$$

A construction due to Mirković and Vilonen [39] in the context of the geometric Satake correspondence endows the spaces $V(\lambda)$ with linear bases. Specifically, $V(\lambda)$ is identified with the intersection homology of a parabolic affine Schubert variety, and Mirković and Vilonen’s basis is the datum of a family $Z(\lambda)$ of algebraic cycles in this variety. In [22], Goncharov and Shen extend this construction to the tensor products $V(\lambda)$. In the present paper, we investigate the properties of these bases, which we call MV bases.

We show that MV bases are compatible with the isotypical filtration of the representations $V(\lambda)$. They are also compatible with the action of the Chevalley generators of the Lie algebra $\mathfrak{g}$ of $G$, in the sense that the leading terms of this action define on $Z(\lambda)$ the structure of a
crystal in the sense of Kashiwara; this result is due to Braverman and Gaitsgory [8] in the case \( n = 1 \). With the help of the path model [33], we prove that there is a natural crystal isomorphism

\[
\mathcal{Z}(\lambda) \cong \mathcal{Z}(\lambda_1) \otimes \cdots \otimes \mathcal{Z}(\lambda_n)
\]

(\text{Braverman and Gaitsgory implicitly treated the case } n = 2). We study the transition matrix between the MV basis of a tensor product \( V(\lambda) \) and the tensor product of the MV bases of the different factors \( V(\lambda_1), \ldots, V(\lambda_n) \). Using the fusion product in the sense of Beilinson and Drinfeld [5], we explain that the entries of this transition matrix can be computed as intersection multiplicities. As a consequence, the transition matrix is unitriangular with nonnegative integral entries.

The properties stated above are analogues of results obtained by Lusztig about the dual canonical bases. To be specific, Lusztig [37] defines a notion of based module (module endowed with a basis enjoying certain specific properties) over the quantized enveloping algebra \( U_v(\mathfrak{g}) \) and shows the following facts:

- A simple module over \( U_v(\mathfrak{g}) \), endowed with its canonical basis, is a based module.
- The tensor product of finitely many based modules can be endowed with a basis that makes it a based module. This basis is constructed from the tensor product of the bases of the factors by adding corrective terms in an unitriangular fashion.
- The basis of a based module is compatible with the decreasing isotypical filtration of the module. Each subquotient in this filtration, endowed with the induced basis, is isomorphic as a based module to the direct sum of copies of a simple module endowed with its canonical basis.

The dual canonical bases for the representations \( V(\lambda) \) (see [14]) can then be defined by dualizing Lusztig’s construction and specializing the quantum parameter at \( v = 1 \).

Because of its compatibility with the isotypical filtration, the dual canonical basis of a tensor product \( V(\lambda) \) yields a linear basis of the invariant subspace \( V(\lambda)^G \). Just as well, the MV basis provides a linear basis (sometimes called the Satake basis) of \( V(\lambda)^G \). The Satake basis and the dual canonical basis of \( V(\lambda)^G \) generally differ (the paper [12] provides a counterexample); nonetheless we show that the Satake basis enjoys the first two items in Khovanov and Kuperberg’s list of properties for the dual canonical basis [30]. In particular, after restriction to the invariant subspaces, the signed permutation

\[
V(\lambda_1) \otimes V(\lambda_2) \otimes \cdots \otimes V(\lambda_n) \xrightarrow{\zeta} V(\lambda_2) \otimes \cdots \otimes V(\lambda_n) \otimes V(\lambda_1)
\]

maps the Satake basis of the domain to that of the codomain.
As explained in [2], the MV bases of the irreducible representations $V(\lambda)$ can be glued together to produce a basis of the algebra $\mathbb{C}[N]$ of regular functions on the unipotent radical $N$ of $B$. Of particular interest would be any relation with the cluster algebra structure on $\mathbb{C}[N]$ [6, 21]. The methods developed in the present paper allow for explicit computations; for instance we show that the cluster monomials attached to certain seeds belong to the MV basis. However $\mathbb{C}[N]$ is not of finite cluster type in general; in other words cluster monomials do not span the whole space. In type $D_4$ we compare the MV basis element with the dual canonical and the dual semicanonical elements at the first position not covered by cluster monomials.

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2 Mirković-Vilonen cycles and bases

In the whole paper $G$ is a connected reductive group over $\mathbb{C}$, endowed with a Borel subgroup $B$ and a maximal torus $T \subseteq B$. We denote by $\Lambda$ the character lattice of $T$, by $\Phi \subseteq \Lambda$ the root system of $(G, T)$, by $\Phi^\vee$ the coroot system, and by $W$ the Weyl group. The datum of $B$ determines a set of positive root in $\Phi$; we endow $\Lambda$ with the dominance order $\leq$ relative to this and denote the cone of dominant weights by $\Lambda^+$. We denote by $\rho : \Lambda \to \mathbb{Q}$ the half-sum of the positive coroots; so $\langle \rho, \alpha \rangle = 1$ for each simple root $\alpha$. The Langlands dual of $G$ is the connected reductive group $G^\vee$ over $\mathbb{C}$ built from the dual torus $T^\vee = \Lambda \otimes_{\mathbb{Z}} \mathbb{G}_m$ and the root system $\Phi^\vee$; the positive coroots define a Borel subgroup $B^\vee \subseteq G^\vee$.

2.1 Recollection on the geometric Satake correspondence

The geometric Satake correspondence was devised by Lusztig [36] and given its definitive shape by Beilinson and Drinfeld [5] and Mirković and Vilonen [39]. Additional references for the material exposed in this section are [46] and [3].

As recalled in the introduction of [41], loop groups appear under several guises across mathematics: there is the differential-geometric variant, the algebraic-geometric one, etc. We adopt the framework of Lie theory and Kac-Moody groups [31]. For instance, though the affine Grassmannian is a (generally non-reduced) ind-scheme, we will only look at its complex ind-variety
Let $O = \mathbb{C}[z]$ be the ring of formal power series in $z$ with complex coefficients and let $K = \mathbb{C}((z))$ be the fraction field of $O$. The affine Grassmannian of the Langlands dual $G^{\vee}$ is the homogeneous space $Gr = G^{\vee}(K)/G^{\vee}(O)$. This space, a partial flag variety for an affine Kac-Moody group, is endowed with the structure of an ind-variety.

Each weight $\lambda \in \Lambda$ gives a point $z^{\lambda}$ in $T^{\vee}(K)$, whose image in $Gr$ is denoted by $L_{\lambda}$. The $G^{\vee}(O)$-orbit through $L_{\lambda}$ in $Gr$ is denoted by $Gr^{\lambda}$; this is a smooth connected simply-connected variety of dimension $2\rho(\lambda)$. The Cartan decomposition implies that

$$Gr = \bigsqcup_{\lambda \in \Lambda^+} Gr^{\lambda}; \quad \text{moreover} \quad Gr^{\lambda} = \bigsqcup_{\mu \in \Lambda^+, \mu \leq \lambda} Gr^{\mu}.$$ 

Let $\text{Perv}(Gr)$ be the category of $G^{\vee}(O)$-equivariant perverse sheaves on $Gr$ (for the middle perversity) supported on finitely many $G^{\vee}(O)$-orbits, with coefficients in $\mathbb{C}$. This is an abelian semisimple category; the simple objects in $\text{Perv}(Gr)$ are the intersection cohomology sheaves

$$I_{\lambda} = IC(Gr^{\lambda}, \mathbb{C}).$$

(By convention, IC sheaves are shifted so as to be perverse.)

Let $\theta \in \Lambda$ be a dominant and regular weight. The embedding

$$\mathbb{C}^\times \xrightarrow{\theta} T^{\vee}(\mathbb{C}) \to G^{\vee}(K)$$

gives rise to an action of $\mathbb{C}^\times$ on $Gr$. For each $\mu \in \Lambda$, the point $L_{\mu}$ is fixed by this action; we denote its stable and unstable sets by

$$S_{\mu} = \left\{ x \in Gr \mid \lim_{c \to 0} \theta(c) \cdot x = L_{\mu} \right\} \quad \text{and} \quad T_{\mu} = \left\{ x \in Gr \mid \lim_{c \to \infty} \theta(c) \cdot x = L_{\mu} \right\},$$

and denote the inclusion maps by $s_{\mu} : S_{\mu} \to Gr$ and $t_{\mu} : T_{\mu} \to Gr$.

Given $\mu \in \Lambda$ and $\mathcal{A} \in \text{Perv}(Gr)$, Mirković and Vilonen identify the homology groups

$$H_c(S_{\mu}, (s_{\mu})^\ast \mathcal{A}) \quad \text{and} \quad H(T_{\mu}, (t_{\mu})_1^{\ast} \mathcal{A})$$

via Braden’s hyperbolic localization, show that these groups are concentrated in degree $2\rho(\mu)$, and define

$$F_{\mu}(\mathcal{A}) = H^{2\rho(\mu)}(T_{\mu}, (t_{\mu})_1^{\ast} \mathcal{A}) \quad \text{and} \quad F(\mathcal{A}) = \bigoplus_{\mu \in \Lambda} F_{\mu}(\mathcal{A}).$$
Then $F$ is an exact and faithful functor from $\text{Perv}(\text{Gr})$ to the category of finite dimensional $\Lambda$-graded $C$-vector spaces. Mirković and Vilonen prove that $F$ induces an equivalence $\overline{F}$ from $\text{Perv}(\text{Gr})$ to the category $\text{Rep}(G)$ of finite dimensional rational representations of $G$, the $\Lambda$-gradation on $F(\mathcal{A})$ giving rise to the decomposition of $\overline{F}(\mathcal{A})$ into weight subspaces. In the course of the proof, it is shown that $\overline{F}$ maps $\mathcal{I}_\lambda$ to the irreducible highest weight representation $V(\lambda)$.

The map $G^\vee(K) \to \text{Gr}$ is a principal $G^\vee(O)$-bundle. From the $G^\vee(O)$-space $\text{Gr}$, we form the associated bundle 

$$\text{Gr}_2 = G^\vee(K) \times^{G^\vee(O)} \text{Gr}.$$ 

This space is called the 2-fold convolution variety and has the structure of an ind-variety. The action of $G^\vee(K)$ on $\text{Gr}$ defines a map $m : \text{Gr}_2 \to \text{Gr}$ of ind-varieties. Let $p : G^\vee(K) \to \text{Gr}$ and $q : G^\vee(K) \times \text{Gr} \to \text{Gr}_2$ be the quotient maps. Given two equivariant perverse sheaves $\mathcal{A}_1$ and $\mathcal{A}_2$ on $\text{Gr}$, there is a unique equivariant perverse sheaf $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ on $\text{Gr}_2$ such that 

$$p^* \mathcal{A}_1 \boxtimes \mathcal{A}_2 = q^* (\mathcal{A}_1 \boxtimes \mathcal{A}_2)$$

in the equivariant derived category of constructible sheaves on $G^\vee(K) \times \text{Gr}$. We then define the convolution product of $\mathcal{A}_1$ and $\mathcal{A}_2$ to be 

$$\mathcal{A}_1 * \mathcal{A}_2 = m_* (\mathcal{A}_1 \boxtimes \mathcal{A}_2).$$

Using Beilinson and Drinfeld’s fusion product, one defines associativity and commutativity constraints and obtains a monoidal structure on $\text{Perv}(\text{Gr})$. Then $F$ is a tensor functor; in particular, the fusion product imparts an explicit identification of $\Lambda$-graded vector spaces 

$$F(\mathcal{A}_1 * \mathcal{A}_2) \cong F(\mathcal{A}_1) \otimes F(\mathcal{A}_2)$$

for any $(\mathcal{A}_1, \mathcal{A}_2) \in \text{Perv}(\text{Gr})^2$.

### 2.2 Mirković-Vilonen cycles

In this paper we study tensor products $V(\lambda) = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n)$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a sequence of dominant weights. Accordingly, we want to consider convolution products 

$$\mathcal{I}_\lambda = \mathcal{I}_{\lambda_1} \ast \cdots \ast \mathcal{I}_{\lambda_n}$$

and therefore need the $n$-fold convolution variety 

$$\text{Gr}_n = \underbrace{G^\vee(K) \times^{G^\vee(O)} \cdots \times^{G^\vee(O)}}_{n \text{ factors}} G^\vee(K) / G^\vee(O).$$
As customary, we denote elements in $\text{Gr}_n$ as classes $[g_1, \ldots, g_n]$ of tuples of elements in $G^\vee(K)$. Plainly $\text{Gr}_1 = \text{Gr}$; in consequence, we write the quotient map $G^\vee(K) \to \text{Gr}$ as $g \mapsto [g]$. We define a map $m_n : \text{Gr}_n \to \text{Gr}$ by setting $m_n([g_1, \ldots, g_n]) = [g_1 \ldots g_n]$.

Given $G^\vee(O)$-stable subsets $K_1, \ldots, K_n$ of $\text{Gr}$, we define

$$K_1 \times \cdots \times K_n = \{ [g_1, \ldots, g_n] \in \text{Gr}_n \mid [g_1] \in K_1, \ldots, [g_n] \in K_n \}.$$ 

Alternatively, $K_1 \times \cdots \times K_n$ can be defined as

$$\widehat{K}_1 \times G^\vee(O) \times \cdots \times G^\vee(O) / G^\vee(O)$$

where each $\widehat{K}_j$ is the preimage of $K_j$ under the quotient map $G^\vee(K) \to \text{Gr}$.

For $\lambda = (\lambda_1, \ldots, \lambda_n)$ in $(\Lambda^+)^n$, we set

$$\text{Gr}_n^\lambda = \text{Gr}^{\lambda_1} \times \cdots \times \text{Gr}^{\lambda_n} \quad \text{and} \quad |\lambda| = \lambda_1 + \cdots + \lambda_n.$$ 

Viewing $\text{Gr}_n^\lambda$ as an iterated fibration with base $\text{Gr}^{\lambda_1}$ and successive fibers $\text{Gr}^{\lambda_2}, \ldots, \text{Gr}^{\lambda_n}$, we infer that it is a smooth connected simply-connected variety of dimension $2\rho(|\lambda|)$; also

$$\text{Gr}_n = \bigsqcup_{\lambda \in (\Lambda^+)^n} \text{Gr}_n^\lambda \quad \text{and} \quad \overline{\text{Gr}}_n^\lambda = \text{Gr}^{\lambda_1} \times \cdots \times \text{Gr}^{\lambda_n} = \bigsqcup_{\mu \in (\Lambda^+)^n, \mu_1 \leq \lambda_1, \ldots, \mu_n \leq \lambda_n} \text{Gr}_n^\mu.$$

**Proposition 2.1** Let $\lambda \in (\Lambda^+)^n$. Then $\mathcal{I}_\lambda$ is the direct image by $m_n$ of the intersection cohomology sheaf of $\text{Gr}_n^\lambda$ with trivial local system, to wit

$$\mathcal{I}_\lambda = (m_n)_* \text{IC}(\overline{\text{Gr}}_n^\lambda, \mathbb{C}),$$

and the cohomology sheaves $\mathcal{H}^k \text{IC}(\overline{\text{Gr}}_n^\lambda, \mathbb{C})$ vanish unless $k$ and $2\rho(|\lambda|)$ have the same parity.

**Proof.** We content ourselves with the case $n = 2$; the proof is the same in the general case but requires more notation. Working out the technicalities explained in [3], §1.16.4, we get

$$\text{IC}(\overline{\text{Gr}}_2^{(\lambda_1, \lambda_2)}, \mathbb{C}) = \text{IC}(\overline{\text{Gr}}^{\lambda_1}, \mathbb{C}) \boxtimes \text{IC}(\overline{\text{Gr}}^{\lambda_2}, \mathbb{C}).$$

Applying $(m_2)_*$ then gives the announced equality, while the parity property follows from [36], sect. 11. □
A key argument in Mirković and Vilonen’s proof of the geometric Satake correspondence is the fact that for any \( \lambda \in \Lambda^+ \) and \( \mu \in \Lambda \), all the irreducible components of \( \text{Gr}^{\lambda} \cap S_\mu \) (respectively, \( \text{Gr}^{\lambda} \cap T_\mu \)) have dimension \( \rho(\lambda + \mu) \) (respectively, \( \rho(\lambda - \mu) \)) ([39], Theorem 3.2). We need a similar result for the intersections \( \text{Gr}^{\lambda}_n \cap (m_n)^{-1}(S_\mu) \) and \( \text{Gr}^{\lambda}_n \cap (m_n)^{-1}(T_\mu) \) inside the \( n \)-fold convolution variety.

Let \( N^\vee \) be the unipotent radical of \( B^\vee \). Then \( S_\mu \) is the \( N^\vee(K) \)-orbit through \( L_\mu \); this well-known fact follows from the easily proved inclusion \( N^\vee(K) L_\mu \subseteq S_\mu \) and the Iwasawa decomposition

\[
G^\vee(K) = \bigsqcup_{\mu \in \Lambda} N^\vee(K) z_\mu G^\vee(O).
\]  

(1)

We record that \( S_\mu = (N^\vee(K) z_\mu) / N^\vee(O) \) and that for each \( \lambda \in \Lambda^+ \), the action of the connected subgroup \( N^\vee(O) \) leaves stable the intersection \( \text{Gr}^{\lambda} \cap S_\mu \), hence leaves stable each irreducible component of this intersection.

The construction of the \( n \)-fold convolution variety is functorial in the group \( G^\vee \); applied to the inclusion \( B^\vee \rightarrow G^\vee \), this remark provides a natural map

\[
\Psi : \bigsqcup_{(\mu_1, \ldots, \mu_n) \in \Lambda^n} (N^\vee(K) z^{\mu_1}) \times^{N^\vee(O)} \cdots \times^{N^\vee(O)} (N^\vee(K) z^{\mu_n}) / N^\vee(O) \rightarrow \text{Gr}_n.
\]

Using (1), we easily see that \( \Psi \) is bijective.

Given weights \( \mu_1, \ldots, \mu_n \) and \( N^\vee(O) \)-stable subsets \( Z_1 \subseteq S_{\mu_1}, \ldots, Z_n \subseteq S_{\mu_n} \), we define

\[
Z_1 \times \cdots \times Z_n = \overline{Z}_1 \times^{N^\vee(O)} \cdots \times^{N^\vee(O)} \overline{Z}_n / N^\vee(O)
\]

where each \( \overline{Z}_j \) is the preimage of \( Z_j \) under the quotient map \( N^\vee(K) z^{\mu_j} \rightarrow S_{\mu_j} \). With this notation, the bijectivity of \( \Psi \) implies that

\[
\text{Gr}_n = \bigsqcup_{(\mu_1, \ldots, \mu_n) \in \Lambda^n} \Psi(S_{\mu_1} \times \cdots \times S_{\mu_n}).
\]

If \( Z_1, \ldots, Z_n \) are varieties, then \( Z_1 \times \cdots \times Z_n \) is an iterated fibration with basis \( Z_1 \) and successive fibers \( Z_2, \ldots, Z_n \) and \( \Psi \) induces an homeomorphism from \( Z_1 \times \cdots \times Z_n \) onto its image.

We use the symbol \( \text{Irr}(-) \) to designate the set of irreducible components of its argument. For \( \lambda \in \Lambda^+, \lambda \in (\Lambda^+)^n \), and \( \mu \in \Lambda \), we define

\[
*_\mathcal{Z}(\lambda)_\mu = \text{Irr}(\text{Gr}^{\lambda} \cap S_\mu) \quad \text{and} \quad *_\mathcal{Z}(\lambda)_\mu = \text{Irr}(\text{Gr}^{\lambda}_n \cap (m_n)^{-1}(S_\mu)).
\]
Proposition 2.2 Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) in \((\Lambda^+)^n\) and let \( \mu \in \Lambda \).

(i) All the irreducible components of \( \overline{\text{Gr}^\lambda_n} \cap (m_n)^{-1}(S_\mu) \) have dimension \( \rho(|\lambda| + \mu) \).

(ii) The map \( (Z_1, \ldots, Z_n) \mapsto \Psi(Z_1 \times \cdots \times Z_n) \) induces a bijection

\[
\bigsqcup_{(\mu_1,\ldots,\mu_n) \in \Lambda^n \atop \mu_1+\cdots+\mu_n=\mu} *\mathcal{Z}(\lambda_1)_{\mu_1} \times \cdots \times *\mathcal{Z}(\lambda_n)_{\mu_n} \cong \star\mathcal{Z}(\lambda)_{\mu}.
\]

(The bar above \( \Psi(Z_1 \times \cdots \times Z_n) \) means closure in \((m_n)^{-1}(S_\mu)\).)

Proof. One easily checks that \( (m_n \circ \Psi)(S_{\mu_1} \times \cdots \times S_{\mu_n}) \subseteq S_{\mu_1+\cdots+\mu_n} \), whence

\[
(m_n)^{-1}(S_\mu) = \bigsqcup_{(\mu_1,\ldots,\mu_n) \in \Lambda^n \atop \mu_1+\cdots+\mu_n=\mu} \Psi(S_{\mu_1} \times \cdots \times S_{\mu_n})
\]

for any \( \mu \in \Lambda \). Adding \( \lambda = (\lambda_1, \ldots, \lambda_n) \) to the mix, we see that

\[
\overline{\text{Gr}^\lambda_n} \cap (m_n)^{-1}(S_\mu) = \bigsqcup_{(\mu_1,\ldots,\mu_n) \in \Lambda^n \atop \mu_1+\cdots+\mu_n=\mu} \Psi\left(\left(\overline{\text{Gr}^{\lambda_1}_n} \cap S_{\mu_1}\right) \times \cdots \times \left(\overline{\text{Gr}^{\lambda_n}_n} \cap S_{\mu_n}\right)\right)
\]

is the disjoint union over \((\mu_1,\ldots,\mu_n)\) of an iterated fibration with base \( \overline{\text{Gr}^{\lambda_1}_n} \cap S_{\mu_1} \) and successive fibers \( \overline{\text{Gr}^{\lambda_2}_n} \cap S_{\mu_2}, \ldots, \overline{\text{Gr}^{\lambda_n}_n} \cap S_{\mu_n} \). Statement (i) follows directly from this observation and Mirković and Vilonen’s dimension estimates. Statement (ii) requires the additional argument that the spaces \( \overline{\text{Gr}^{\lambda_j}_n} \cap S_{\mu_j} \) are contractible (because of the \( C^* \)-action), so the monodromy is necessarily trivial. \( \Box \)

For \( \lambda \in \Lambda^+, \lambda \in (\Lambda^+)^n \), and \( \mu \in \Lambda \), we similarly define

\[
*\mathcal{Z}(\lambda)_{\mu} = \text{Irr}\left(\overline{\text{Gr}^\lambda_n} \cap T_\mu\right) \quad \text{and} \quad \star\mathcal{Z}(\lambda)_{\mu} = \text{Irr}\left(\overline{\text{Gr}^\lambda_n} \cap (m_n)^{-1}(T_\mu)\right).
\]

Then all cycles in \( \star\mathcal{Z}(\lambda)_{\mu} \) have dimension \( \rho(|\lambda| - \mu) \) and there is a natural bijection

\[
\bigsqcup_{(\mu_1,\ldots,\mu_n) \in \Lambda^n \atop \mu_1+\cdots+\mu_n=\mu} \star\mathcal{Z}(\lambda_1)_{\mu_1} \times \cdots \times \star\mathcal{Z}(\lambda_n)_{\mu_n} \cong \star\mathcal{Z}(\lambda)_{\mu}.
\]

(2)

Elements in \( \star\mathcal{Z}(\lambda)_{\mu}, \mathcal{Z}(\lambda)_{\mu}, *\mathcal{Z}(\lambda)_{\mu} \) or \( \mathcal{Z}(\lambda)_{\mu} \) are called Mirković-Vilonen (MV) cycles. For future use, we note that the map \( Z \mapsto Z \cap \text{Gr}^\lambda_n \) provides a bijection from \( \mathcal{Z}(\lambda)_{\mu} \) onto the set of irreducible components of \( \text{Gr}^\lambda_n \cap (m_n)^{-1}(T_\mu) \). (Each \( Z \in \mathcal{Z}(\lambda)_{\mu} \) meets the open subset \( \text{Gr}^\lambda_n \) of \( \overline{\text{Gr}^\lambda_n} \), because the dimension of \( (\text{Gr}^\lambda_n \setminus \text{Gr}^\lambda_n) \cap (m_n)^{-1}(T_\mu) \) is smaller than the dimension of \( Z \).)
2.3 Mirković-Vilonen bases

Following Goncharov and Shen ([22], sect. 2.4), we now define the MV basis of the tensor product representations

\[ V(\lambda) = F(\mathcal{I}_\lambda) = \bigoplus_{\mu \in \Lambda} F_\mu(\mathcal{I}_\lambda). \]

Let \( \lambda \in (\Lambda^+)^n \) and let \( \mu \in \Lambda \). By base change in the Cartesian square

\[
\begin{array}{ccc}
\overline{\text{Gr}}^\lambda_n \cap (m_n)^{-1}(T_\mu) & \xrightarrow{f} & \overline{\text{Gr}}^\lambda_n \\
T_\mu & \xrightarrow{t_\mu} & \text{Gr}
\end{array}
\]

we compute

\[ F_\mu(\mathcal{I}_\lambda) = H^{2\rho(\mu)}\left( T_\mu, (t_\mu)^! (m_n)_* \IC\left( \overline{\text{Gr}}^\lambda_n, \mathbb{C} \right) \right) = H^{2\rho(\mu)}\left( \overline{\text{Gr}}^\lambda_n \cap (m_n)^{-1}(T_\mu), f^! \IC\left( \overline{\text{Gr}}^\lambda_n, \mathbb{C} \right) \right). \]

Let \( j : \text{Gr}^\lambda_n \to \text{Gr}^\lambda_n \) and \( g : \text{Gr}^\lambda_n \cap (m_n)^{-1}(T_\mu) \to \text{Gr}^\lambda_n \) be the inclusion maps. We can then look at the sequence of maps

\[ F_\mu(\mathcal{I}_\lambda) \to H^{2\rho(\mu)}\left( \text{Gr}^\lambda_n \cap (m_n)^{-1}(T_\mu), f^! j_* j^* \IC\left( \overline{\text{Gr}}^\lambda_n, \mathbb{C} \right) \right) \]

\[ = H^{2\rho(\mu)}\left( \text{Gr}^\lambda_n \cap (m_n)^{-1}(T_\mu), g^! \mathbb{C}_{\text{Gr}^\lambda_n}[\dim \text{Gr}^\lambda_n] \right) \]

\[ \xrightarrow{\cap[\text{Gr}^\lambda_n]} H^{2\rho(\lambda|-\mu)}_{BM}(\text{Gr}^\lambda_n \cap (m_n)^{-1}(T_\mu)). \]

Here the first two maps carry out the restriction to \( \text{Gr}^\lambda_n \) (technically, an adjunction followed by a base change) and the last map is the Alexander duality.*

We claim that these maps are isomorphisms. For the Alexander duality, this comes from the smoothness of \( \text{Gr}^\lambda_n \). For the restriction, consider a stratum \( \text{Gr}^\eta_n \subseteq \text{Gr}^\lambda_n \) with \( \eta \neq \lambda \); denoting the inclusion map by \( i \) and using the perversity condition, the parity property in Proposition 2.1, and the dimension estimate for \( \text{Gr}^\eta_n \cap (m_n)^{-1}(T_\mu) \), one checks that

\[ H^k\left( \text{Gr}^\lambda_n \cap (m_n)^{-1}(T_\mu), f^! i_* i^! \IC\left( \overline{\text{Gr}}^\lambda_n, \mathbb{C} \right) \right) \]

vanishes if \( k < 2\rho(\mu) + 2 \); therefore the stratum \( \text{Gr}^\eta_n \) does not contribute to \( F_\mu(\mathcal{I}_\lambda) \).

*Specifically, the generalization presented in [15], sect. 19.1, equation (3) or in [24], Theorem IX.4.7.
To sum up there is a natural isomorphism
\[ F_\mu(\mathcal{J}_\lambda) \cong H_{2\rho(\lambda|-\mu)}^{BM} \left( \text{Gr}_{\lambda}^n \cap (m_n)^{-1}(T_\mu) \right). \] (3)

The irreducible components \( \text{Gr}_{\lambda}^n \cap (m_n)^{-1}(T_\mu) \) have all dimension \( \rho(|\lambda| - \mu) \) and their fundamental classes provide a basis of the Borel-Moore homology group above. Gathering these bases for all weights \( \mu \in \Lambda \) produces what we call the MV basis of \( V(\lambda) \).

3. **L-perfect bases**

In this section we consider a general setup, which captures properties shared by both the MV bases and the dual canonical bases. As before, \( G \) is a connected reductive group over \( \mathbb{C} \) endowed with a Borel subgroup \( B \) and a maximal torus \( T \subseteq B \), \( \Lambda \) is the character lattice of \( T \), \( \Phi \) and \( \Phi^\vee \) are the root and coroot systems of \( (G,T) \). We denote by \( \{ \alpha_i \mid i \in I \} \) the set of simple roots defined by \( B \), and by \( \{ \alpha_i^\vee \mid i \in I \} \) the set of simple coroots. We endow \( \Lambda \) with the dominance order \( \leq \), and denote the cone of dominant weights by \( \Lambda^+ \). We regard the Weyl group \( W \) as a subgroup of \( \text{Aut}(\Lambda) \); for \( i \in I \), we denote by \( s_i \) the simple reflection along the root \( \alpha_i \). When needed, we choose simple root vectors \( e_i \) and \( f_i \) of weights \( \pm \alpha_i \) in the Lie algebra of \( G \) in such a way that \([e_i, f_i] = -\alpha_i^\vee\).

3.1 **Semi-normal crystals**

We start by recalling the following definitions due to Kashiwara [28]. A semi-normal crystal is a set \( B \) endowed with a map \( \text{wt} : B \to \Lambda \) and, for each “color” \( i \in I \), with a partition into a collection of finite oriented strings. This latter structure is recorded by the datum of operators
\[ \tilde{e}_i : B \to B \cup \{0\} \quad \text{and} \quad \tilde{f}_i : B \to B \cup \{0\} \]
which move an element of \( b \in B \) upwards and downwards, respectively, along the string of color \( i \) to which \( b \) belongs. The special value 0 is assigned to \( \tilde{e}_i(b) \) or \( \tilde{f}_i(b) \) when \( b \) is at the upper or lower end of a string of color \( i \). For convenience one usually further sets \( \tilde{e}_i(0) = \tilde{f}_i(0) = 0 \).

The position of \( b \) in its string of color \( i \) is recorded by functions \( \varepsilon_i \) and \( \varphi_i \) defined as follows:
\[ \varepsilon_i(b) = \max\{ n \geq 0 \mid \tilde{e}_i(b) \neq 0 \}, \quad \varphi_i(b) = \max\{ n \geq 0 \mid \tilde{f}_i(b) \neq 0 \}. \]

Two compatibility conditions between the weight map \( \text{wt} \) and the datum of the partitions into oriented strings are required: first, \( \text{wt}(b) \) increases by \( \alpha_i \) when \( b \) moves upwards the string of color \( i \) to which it belongs; second, \( \varphi_i(b) - \varepsilon_i(b) = \langle \alpha_i^\vee, \text{wt}(b) \rangle \) for any \( b \in B \) and any \( i \in I \).
These conditions imply that the image of a string of color \( i \) by the map \( wt \) is stable under the action of the simple reflection \( s_i \). As a consequence, the set \( \{ wt(b) \mid b \in B \} \) is stable under the action of the Weyl group \( W \).

The direct sum of two semi-normal crystals \( B_1 \) and \( B_2 \) is defined to be just the disjoint union of the underlying sets. The tensor product \( B_1 \otimes B_2 \) is defined to be the Cartesian product of the sets endowed with the maps given in [28], §7.3. Notably, for each color, the strings in \( B_1 \otimes B_2 \) are created from the strings contained in \( B_1 \) and in \( B_2 \) by the process illustrated by the picture below.

3.2 \( L \)-perfect bases

To a subset \( J \subseteq I \) we attach the standard Levi subgroup \( M_J \) of \( G \), the cone

\[
\Lambda_J^+ = \{ \lambda \in \Lambda \mid \forall j \in J, \langle \alpha_J^* \lambda \rangle \geq 0 \}
\]

of dominant weights for \( M_J \), and the \( J \)-dominance order \( \leq_J \) on \( \Lambda \) defined by

\[
\mu \leq_J \lambda \iff \lambda - \mu \in \text{span}_N \{ \alpha_j \mid j \in J \}.
\]

Given \( \lambda \in \Lambda_J^+ \) we denote by \( V_J(\lambda) \) the irreducible rational representation of \( M_J \) with highest weight \( \lambda \). Given a finite sequence \( \lambda = (\lambda_1, \ldots, \lambda_n) \) in \( \Lambda_J^+ \) we define

\[
V_J(\lambda) = V_J(\lambda_1) \otimes \cdots \otimes V_J(\lambda_n).
\]

For \( J = I \) we recover the conventions previously used by dropping the decoration \( J \) in the notation \( \Lambda_J^+ \), \( \leq_J \) or \( V_J(\lambda) \).

Let \( V \) be a rational representation of \( G \). With respect to the action of \( M_J \) the space \( V \) can be uniquely written as a direct sum of isotypical components

\[
V = \bigoplus_{\mu \in \Lambda_J^+} V_{J, \mu}
\]
where $V_{J,\mu}$ is the sum of all subrepresentations of $\text{res}^G_{M_J}(V)$ isomorphic to $V_J(\mu)$. We define

$$V_{J,\leq_J \mu} = \bigoplus_{\nu \in \Lambda^+_J \atop \nu \leq_J \mu} V_{J,\nu}.$$ 

We say that a linear basis $B$ of $V$ is $L$-perfect\(^\dagger\) if for each $J \subseteq I$ and each $\mu \in \Lambda^+_J$:

(P1) The subspace $V_{J,\leq_J \mu}$ is spanned by a subset of $B$.

(P2) The induced basis on the quotient $V_{J,\leq_J \mu}/V_{J,\lt_J \mu} \cong V_{J,\mu}$ is compatible with a decomposition of the isotypical component as a direct sum of irreducible representations.

Taking $J = \emptyset$, we see that an $L$-perfect basis of $V$ consists of weight vectors (note that $\leq_{\emptyset}$ is the trivial order on $\Lambda$). Now let $i \in I$, and for each nonnegative integer $\ell$, define

$$V_{(i),\leq \ell} = \bigoplus_{\mu \in \Lambda \atop 0 \leq \langle \alpha_i^\vee, \mu \rangle \leq \ell} V_{(i),\mu},$$

the sum of all irreducible subrepresentations of $\text{res}^G_{M_{(i)}}(V)$ of dimension at most $\ell + 1$. If $B$ satisfies the conditions (P1) and (P2) for $J = \{i\}$, then $V_{(i),\leq \ell}$ is spanned by $B \cap V_{(i),\leq \ell}$ and the induced basis on the quotient $V_{(i),\leq \ell}/V_{(i),\leq \ell-1}$ is compatible with a decomposition as a direct sum of irreducible representations. Therefore $(B \cap V_{(i),\leq \ell}) \setminus (B \cap V_{(i),\leq \ell-1})$ decomposes as the disjoint union of oriented strings of length $\ell$, in such a way that the simple root vector $e_i$ or $f_i$ acts on a basis vector of $V_{(i),\leq \ell}/V_{(i),\leq \ell-1}$ by moving it upwards or downwards along the string that contains it, up to a scalar.

We can sum up the discussion in the previous paragraph as follows: if $B$ satisfies (P1) and (P2) for all $J$ of cardinality $\leq 1$, then $B$ is endowed with the structure of crystal and is perfect in the sense of Berenstein and Kazhdan ([7], Definition 5.30).

**Lemma 3.1** Let $B$ be an $L$-perfect basis of a rational representation $V$ of $G$ and let $B' \subseteq B$. Assume that the linear space $V'$ spanned by $B'$ is a subrepresentation of $V$. Then $B'$ is an $L$-perfect basis of $V'$ and (the image of) $B \setminus B'$ is an $L$-perfect basis of the quotient $V/V'$.

\(^\dagger\) $L$ stands for Levi. This notion of $L$-perfect basis appears unnamed (and in a dual form) in Braverman and Gaitsgory’s paper ([8], sect. 4.3).
Proof. Let $J \subseteq I$ and $\mu \in \Lambda^+_J$. Then $V'_{J,\mu} = V' \cap V_{J,\mu}$ and $V'_{J,\leq \mu} = V' \cap V_{J,\leq \mu}$. Now both spaces $V'$ and $V_{J,\leq \mu}$ are spanned by a subset of the basis $B$, so their intersection $V'_{J,\leq \mu}$ is spanned by a subset of $B$, namely $B' \cap V_{J,\leq \mu}$.

Let $C$ be the image of $(B \cap V_{J,\leq \mu}) \setminus (B \cap V_{J,\leq \mu})$ in the quotient $V_{J,\leq \mu}/V_{J,\leq \mu} \cong V_{J,\mu}$. Then $C$ can be viewed as a basis of $V_{J,\mu}$ and it can be partitioned into disjoint subsets $C_1, \ldots, C_n$ so that each $C_k$ spans an irreducible subrepresentation. By construction, the subspace $V'_{J,\mu}$ is spanned by a subset $C' \subseteq C$. Each subset $C_k$ can be either contained in $C'$ or disjoint from $C'$, depending whether the subrepresentation that it spans is contained in $V'_{J,\mu}$ or meets trivially $V'_{J,\mu}$. Therefore $C'$ is the disjoint union of some $C_k$; in other words $C'$ is compatible with a decomposition of $V'_{J,\mu}$ as a direct sum of irreducible subrepresentations.

Thus, $B'$ satisfies both conditions (P1) and (P2), and is therefore an $L$-perfect basis of $V'$. The proof that $B \setminus B'$ yields an $L$-perfect basis of the quotient $V/V'$ rests on similar arguments and is left to the reader. □

Under the assumptions of Lemma 3.1, the subset $B'$ is a subcrystal of $B$; in other words, the crystal structure on $B$ is the direct sum of the crystal structures on $B'$ and $B \setminus B'$.

**Proposition 3.2** Let $V$ be a rational representation of $G$. Up to isomorphism, the crystal of an $L$-perfect basis of $V$ depends only on $V$, and not on the basis.

Proof. Let $B$ be an $L$-perfect basis of $V$. The conditions imposed on $B$ with the choice $J = I$ imply the existence in $V$ of a composition series compatible with $B$. By Lemma 3.1, the crystal $B$ is the direct sum of the crystals of the $L$-perfect bases induced by $B$ on the subquotients. It thus suffices to prove the desired uniqueness property in the particular case where $V$ is an irreducible representation, which in fact is just Theorem 5.37 in [7]. □

In particular, the crystal of an $L$-perfect basis of an irreducible representation $V(\lambda)$ is unique. We use henceforth the notation $B(\lambda)$ for the associated crystal.

**Remark 3.3.** The crystal $B(\lambda)$ of an irreducible representation $V(\lambda)$ was introduced by Kashiwara in the context of representations of quantum groups. The definition via crystallization at $q = 0$ and the definition via the combinatorics of $L$-perfect bases yield the same crystal; this follows from [26], sect. 5.

Fortunately this nice little theory is not empty. As mentioned in the introduction, any tensor product of irreducible representations has an $L$-perfect basis, namely its dual canonical basis. Another example for a $L$-perfect basis: it can be shown, in the case where $G$ is simply laced, that the dual semicanonical basis of an irreducible representation is $L$-perfect.
Theorem 3.4 The MV basis of a tensor product of irreducible representations is $L$-perfect.

The end of sect. 3 is devoted to the proof of this result. The case of an irreducible representation is Proposition 4.1 in [8]. The proof for an arbitrary tensor product follows the same lines; it is only sketched in loc. cit., and we add quite a few details to Braverman and Gaitsgory’s exposition.

3.3 Geometric Satake and restriction to a standard Levi subgroup

Consider a subset $J \subseteq I$. In this section we recall Beilinson and Drinfeld’s geometric construction of the restriction functor $\text{res}_{M,J}^G$ ([5], sect. 5.3). Additional details can be found in [38], sect. 8.6 and [3], sect. 1.15.

Define the root and coroot systems

$$\Phi_J = \Phi \cap \text{span}_\mathbb{Z}\{\alpha_j \mid j \in J\}$$
$$\Phi_J^\vee = \Phi^\vee \cap \text{span}_\mathbb{Z}\{\alpha_j^\vee \mid j \in J\}$$

and denote by $\rho_J : \Lambda \to \mathbb{Q}$ the half-sum of the positive coroots in $\Phi_J^\vee$. Then $\rho - \rho_J$ vanishes on $\mathbb{Z}\Phi_J$ so induces a linear form $\rho_{I,J} : \Lambda/\mathbb{Z}\Phi_J \to \mathbb{Q}$.

To $J$ we also attach the standard Levi subgroup $M_J^\vee$ of $G^\vee$. Choose a dominant $\theta_J \in \Lambda$ such that $\langle \alpha_J^\vee, \theta_J \rangle = 0$ for each $j \in J$ and $\langle \alpha_i^\vee, \theta_J \rangle > 0$ for each $i \in I \setminus J$. The embedding

$$\mathbb{C}^\times \xrightarrow{\theta_J} T^\vee(\mathbb{C}) \to G^\vee(\mathbb{C})$$

gives rise to an action of $\mathbb{C}^\times$ on $\text{Gr}$. Then the set $\text{Gr}_J$ of fixed points under this action can be identified with the affine Grassmannian for $M_J^\vee$. We denote by $\text{Perv}(\text{Gr}_J)$ the category of $M_J^\vee(\mathcal{O})$-equivariant sheaves on $\text{Gr}_J$ supported on finitely many $M_J^\vee(\mathcal{O})$-orbits.

Let $\zeta \in \Lambda/\mathbb{Z}\Phi_J$ be a coset; then all the points $L_\mu$ for $\mu \in \zeta$ belong to the same connected component of $\text{Gr}_J$, which we denote by $\text{Gr}_{J,\zeta}$. The map $\zeta \mapsto \text{Gr}_{J,\zeta}$ is a bijection from $\Lambda/\mathbb{Z}\Phi_J$ onto $\pi_0(\text{Gr}_J)$. We denote the stable and unstable sets of $\text{Gr}_{J,\zeta}$ with respect to the $\mathbb{C}^\times$-action by $\text{Gr}_{J,\zeta}^+$ and $\text{Gr}_{J,\zeta}^-$ and form the diagram

$$\text{Gr}_{J,\zeta}^+ \xleftarrow{s^J_\zeta} \text{Gr} \xrightarrow{t^J_\zeta} \text{Gr}_{J,\zeta}^- \xleftarrow{p^J_\zeta} \text{Gr}_{J,\zeta} \xrightarrow{q^J_\zeta} \text{Gr}_{J,\zeta}$$

(4)
where \( s^I_\zeta \) and \( t^I_\zeta \) are the inclusion maps and the maps \( p_{J,\zeta} \) and \( q_{J,\zeta} \) are defined by
\[
p_{J,\zeta}(x) = \lim_{c \to 0} \theta_J(c) \cdot x \quad \text{and} \quad q_{J,\zeta}(x) = \lim_{c \to \infty} \theta_J(c) \cdot x.
\]

Given \( \zeta \in \Lambda/\mathbb{Z}\Phi_J \) and \( A \in \text{Perv}(\text{Gr}) \), Beilinson and Drinfeld identify the two sheaves
\[
(q_{J,\zeta})_*(t^I_\zeta)^1 A \quad \text{and} \quad (p_{J,\zeta})_!(s^I_\zeta)^* A
\]
on \( \text{Gr}_{J,\zeta} \) via Braden’s hyperbolic localization and show that they live in perverse degree \( 2\rho_{I,J}(\zeta) \). Then they define a functor \( r^I_J : \text{Perv}(\text{Gr}) \to \text{Perv}(\text{Gr}_J) \) by
\[
r^I_J(\mathcal{A}) = \bigoplus_{\zeta \in \Lambda/\mathbb{Z}\Phi_J} (q_{J,\zeta})_*(t^I_\zeta)^1 \mathcal{A}[2\rho_{I,J}(\zeta)].
\]

For \( \mu \in \Lambda \), let \( T_{J,\mu} \) be the analog of the unstable subset \( T_\mu \) for the affine Grassmannian \( \text{Gr}_J \). Let \( \zeta \) be the coset of \( \mu \) modulo \( \mathbb{Z}\Phi_J \) and let \( t_{J,\mu} : T_{J,\mu} \to \text{Gr}_{J,\zeta} \) be the inclusion map. Using the Iwasawa decomposition, one checks that
\[
T_\mu = (q_{J,\zeta})^{-1}(T_{J,\mu}). \tag{5}
\]
Performing base change in the Cartesian square
\[
\begin{array}{ccc}
T_\mu & \xrightarrow{t_\mu} & \text{Gr}_{J,\zeta} \\
\downarrow & & \downarrow \text{\footnotesize\( q_{J,\zeta} \)} \\
T_{J,\mu} & \xrightarrow{t_{J,\mu}} & \text{Gr}_{J,\zeta}
\end{array}
\]
we obtain, for any sheaf \( \mathcal{A} \in \text{Perv}(\text{Gr}) \), a canonical isomorphism
\[
H^{2\rho(\mu)}(T_\mu, (t_\mu)^1 \mathcal{A}) \cong H^{2\rho_J(\mu)}(T_{J,\mu}, (t_{J,\mu})^1 r^I_J(\mathcal{A})). \tag{6}
\]

For \( \mathcal{B} \in \text{Perv}(\text{Gr}_J) \), define
\[
F_{J,\mu}(\mathcal{B}) = H^{2\rho_J(\mu)}(T_{J,\mu}, (t_{J,\mu})^1 \mathcal{B}) \quad \text{and} \quad F_J(\mathcal{B}) = \bigoplus_{\mu \in \Lambda} F_{J,\mu}(\mathcal{B}).
\]
Then (6) can be rewritten as \( F_\mu = F_{J,\mu} \circ r^I_J \). This equality can be refined in the following statement: the functor \( F_J \) induces an equivalence \( F_J \) from \( \text{Perv}(\text{Gr}_J) \) to the category \( \text{Rep}(M_J) \) of finite dimensional rational representations of \( M_J \) and the following diagram commutes.
\[
\begin{array}{ccc}
\text{Perv}(\text{Gr}) & \xrightarrow{r^I_J} & \text{Perv}(\text{Gr}_J) \\
\downarrow \text{\footnotesize\( F_J \)} & & \downarrow \text{\footnotesize\( \text{res}^G_{M_J} \)} \\
\text{Rep}(G) & \underset{\text{\footnotesize\( \text{res}^G_{M_J} \)}}{\xrightarrow{F_J}} & \text{Rep}(M_J)
\end{array}
\]
3.4 The $J$-decomposition of an MV cycle

We fix a subset $J \subseteq I$. We denote by $P_J^{-\vee}$ the parabolic subgroup of $G^\vee$ containing $M_J^\vee$ and the negative root subgroups.

The group $P_J^{-\vee}(K)$ certainly acts on $\text{Gr}$; it also acts on $\text{Gr}_J$ via the quotient morphism $P_J^{-\vee}(K) \to M_J^\vee(K)$. Given $\mu \in \Lambda_J^+$, we denote by $\text{Gr}_\mu$ the orbit of $L_\mu$ under the action of $M_J^\vee(O)$ (or $P_J^{-\vee}(O)$) on $\text{Gr}_J$. Noting that
\[
\lim_{a \to \infty} \theta_J(a) g \theta_J(a)^{-1} = 1
\]
for all $g$ in the unipotent radical of $P_J^{-\vee}$, we see that for any $\zeta \in \Lambda_J/\mathbb{Z}\Phi_J$, the connected component $\text{Gr}_{J,\zeta}$ of $\text{Gr}_J$ and the unstable subset $\text{Gr}_{J,\zeta}$ in $\text{Gr}$ are stable under the action of $P_J^{-\vee}(O)$ and that the map $q_{J,\zeta}$ is equivariant.

Let $\lambda \in (\Lambda^+)^n$, let $\mu \in \Lambda_J^+$ and let $\zeta$ be the coset of $\mu$ modulo $\mathbb{Z}\Phi_J$. We consider the following diagram.

\[
\begin{array}{ccc}
(m_n)^{-1}(\text{Gr}_{-\zeta}) & \xrightarrow{m_n} & \text{Gr}_{-\zeta} \\
\downarrow & & \downarrow \\
\text{Gr}_n & \xrightarrow{m_n} & \text{Gr}
\end{array}
\]

The group $G^\vee(K)$ acts on $\text{Gr}_n$ by left multiplication on the first factor and the action of the subgroup $G^\vee(O)$ leaves $\text{Gr}_n$ stable. Let $H$ be the stabilizer of $L_\mu$ with respect to the action of $P_J^{-\vee}(O)$ on $\text{Gr}_J$; it acts on $E = \text{Gr}_n \cap (q_{J,\zeta} \circ m_n)^{-1}(L_\mu)$. Since $q_{J,\zeta} \circ m_n$ is equivariant under the action of $P_J^{-\vee}(O)$, we can make the identification
\[
P_J^{-\vee}(O) \times^H E \cong \text{Gr}_n \cap (q_{J,\zeta} \circ m_n)^{-1}(\text{Gr}_\mu)
\]
where the left vertical arrow is the projection along the first factor. We thereby see that the right vertical arrow is a locally trivial fibration.

In particular, all the fibers $\text{Gr}_n \cap (q_{J,\zeta} \circ m_n)^{-1}(x)$ with $x \in \text{Gr}_\mu$ are isomorphic varieties. Remembering that $(q_{J,\zeta})^{-1}(L_\mu) \subseteq T_\mu$, we find the following bound for their dimension:
\[
\dim(\text{Gr}_n \cap (q_{J,\zeta} \circ m_n)^{-1}(x)) = \dim E \leq \dim(\text{Gr}_n \cap (m_n)^{-1}(T_\mu)) = \rho(|\lambda| - \mu).
\]
\[
\dim \left( \text{Gr}_n^\lambda \cap (q_{J, \zeta} \circ m_n)^{-1}(\text{Gr}_J^\mu) \right) \leq \dim \text{Gr}_J^\mu + \rho(|\lambda| - \mu) = 2\rho_J(\mu) + \rho(|\lambda| - \mu). \quad (7)
\]

Since \(\text{Gr}_J^\mu\) is connected and simply-connected, the fibration induces a bijection between the set of irreducible components of \(\text{Gr}_n^\lambda \cap (q_{J, \zeta} \circ m_n)^{-1}(\text{Gr}_J^\mu)\) and the set of irreducible components of any fiber \(\text{Gr}_n^\lambda \cap (q_{J, \zeta} \circ m_n)^{-1}(x)\).

We define
\[
\mathcal{Z}^J(\lambda)_\mu = \left\{ Z \in \text{Irr} \left( \text{Gr}_n^\lambda \cap (q_{J, \zeta} \circ m_n)^{-1}(\text{Gr}_J^\mu) \right) \mid \dim Z = 2\rho_J(\mu) + \rho(|\lambda| - \mu) \right\}.
\]

For \(\nu \in \Lambda\), we define
\[
\mathcal{Z}_J(\mu)_\nu = \text{Irr} \left( \text{Gr}_J^\mu \cap T_{J, \nu} \right);
\]

as we saw in sect. 2.2, the map \(Z \mapsto Z \cap \text{Gr}_J^\mu\) is a bijection from \(\mathcal{Z}_J(\mu)_\nu\) onto the set of irreducible components of \(\text{Gr}_J^\mu \cap T_{J, \nu}\).

Fix now \(\lambda \in (\Lambda^+)^n\) and \(\nu \in \Lambda\). Following Braverman and Gaitsgory’s method, we define a bijection
\[
\mathcal{Z}^J(\lambda)_\mu \simeq \bigsqcup_{\mu \in \Lambda_J^+} \mathcal{Z}^J(\lambda)_\mu \times \mathcal{Z}_J(\mu)_\nu.
\]

The union above is in fact restricted to those weights \(\mu\) such that \(\mu - \nu \in \mathcal{Z}_J\), for otherwise \(\mathcal{Z}_J(\mu)_\nu\) is empty. Let \(\zeta\) denote the coset of \(\nu\) modulo \(\mathcal{Z}_J\).

First choose \(\mu \in \Lambda_J^+ \cap \zeta\) and a pair \((Z^J, Z_J) \in \mathcal{Z}_J^J(\lambda)_\mu \times \mathcal{Z}_J(\mu)_\nu\). Using (5) and the fibration above, we see that \(Z^J \cap (q_{J, \zeta} \circ m_n)^{-1}(Z_J \cap \text{Gr}_J^\mu)\) is an irreducible subset of
\[
\text{Gr}_n^\lambda \cap (q_{J, \zeta} \circ m_n)^{-1}(T_{J, \nu}) = \text{Gr}_n^\lambda \cap (m_n)^{-1}(T_{\nu})
\]
of dimension
\[
\dim Z^J - \dim \text{Gr}_J^\mu + \dim Z_J = \rho(|\lambda| - \mu) + \rho_J(\mu - \nu) = \rho(|\lambda| - \nu).
\]

Therefore there is a unique \(Z \in \mathcal{Z}^J(\lambda)_\nu\) that contains \(Z^J \cap (q_{J, \zeta} \circ m_n)^{-1}(Z_J \cap \text{Gr}_J^\mu)\) as a dense subset.

Conversely, start from \(Z \in \mathcal{Z}^J(\lambda)_\nu\). Then \(Z \subseteq T_{\nu} \subseteq \text{Gr}_{J, \zeta}^\nu\). We can thus partition \(Z\) into locally closed subsets as follows.
\[
Z = \bigsqcup_{\mu \in \Lambda_J^+ \cap \zeta} \left( Z \cap (q_{J, \zeta} \circ m_n)^{-1}(\text{Gr}_J^\mu) \right)
\]
Since $Z$ is irreducible, there is a unique $\mu \in \Lambda^+_J \cap \zeta$ such that $Z \cap (q_{J,\zeta} \circ m_n)^{-1}(\text{Gr}^\mu_J)$ is open dense in $Z$. That subset is certainly irreducible, hence contained in an irreducible component $Z^J$ of $\text{Gr}^\mu_J \cap (q_{J,\zeta} \circ m_n)^{-1}(\text{Gr}^\mu_J)$. Also, $q_{J,\zeta} \circ m_n$ maps $Z \cap (q_{J,\zeta} \circ m_n)^{-1}(\text{Gr}^\mu_J)$ to an irreducible subset of $\text{Gr}^\mu_J \cap T_{J,\nu}$, which in turn is contained in an irreducible component $Z_J \in Z_J(\lambda)$. Then

$$Z \cap (q_{J,\zeta} \circ m_n)^{-1}(\text{Gr}^\mu_J) \subseteq Z^J \cap (q_{J,\zeta} \circ m_n)^{-1}(Z_J).$$

The left-hand side has dimension $\rho(|\lambda| - \nu)$ and the right-hand side has dimension

$$\dim Z^J - 2\rho_J(\mu) + \dim Z_J = \dim Z^J - 2\rho_J(\mu) + \rho(\mu - \nu);$$

combining with the bound (7) we get

$$\dim Z^J = \rho(|\lambda| - \mu) + 2\rho_J(\mu)$$

and therefore $Z^J \in Z^J(\lambda)_\mu$.

These two constructions define mutually inverse bijections; in particular,

$$Z^J \cap (q_{J,\zeta} \circ m_n)^{-1}(Z_J \cap \text{Gr}^\mu_J) = Z \cap (q_{J,\zeta} \circ m_n)^{-1}(\text{Gr}^\mu_J).$$

We record that to each MV cycle $Z \in Z^J(\lambda)_\nu$ is assigned a weight $\mu \in \Lambda^+_J$ characterized by the conditions

$$(q_{J,\zeta} \circ m_n)(Z) \subseteq \text{Gr}^\mu_J \quad \text{and} \quad (q_{J,\zeta} \circ m_n)(Z) \cap \text{Gr}^\mu_J \neq \emptyset;$$

in the sequel this weight will be denoted by $\mu_J(Z)$.

### 3.5 MV bases are $L$-perfect

We now give the proof of Theorem 3.4, properly speaking. We fix a positive integer $n$ and a tuple $\lambda \in (\Lambda^+_J)^n$. We need two ingredients besides the constructions explained in sects. 2.3 and 3.3.

(A) Take $\mathcal{A} \in \text{Perv}(\text{Gr})$ and write the sheaf

$$\mathcal{B} = r^J_J(\mathcal{A}) = \bigoplus_{\zeta \in \Lambda_J \cap Z_{\Phi,J}} \left( q_{J,\zeta} \circ (t^J_{\zeta})^* \mathcal{A}|2\rho_{J,\zeta}(\zeta) \right)$$

in $\text{Perv}(\text{Gr}_J)$ as a direct sum of isotypical components

$$\mathcal{B} = \bigoplus_{\mu \in \Lambda^+_J} \text{IC}(\text{Gr}^\mu_J, \mathcal{L}_\mu).$$
The local systems $\mathcal{L}_\mu$ on $\text{Gr}_J^\mu$ that appear in (8) can be expressed as $\mathcal{L}_\mu = \mathcal{H}^h \mathcal{B}$ where $h : \text{Gr}_J^\mu \rightarrow \text{Gr}_J$ is the inclusion map and $k = -\dim \text{Gr}_J^\mu = -2\rho_J(\mu)$. With $e : \{x\} \rightarrow \text{Gr}_J^\mu$ the inclusion of a point and $\zeta$ the coset of $\mu$ modulo $\mathbb{Z}\Phi_J$, the fiber of $\mathcal{L}_\mu$ is

$$(\mathcal{L}_\mu)_x \cong e^! \mathcal{L}_\mu[2 \dim \text{Gr}_J^\mu] \cong H^{2\rho_J(\mu)}(\{x\}, e^! h^! \mathcal{B}).$$

For the specific case

$$\mathcal{A} = \mathcal{A}_\lambda = (m_n)_* \text{IC}(\overline{\text{Gr}_n^\lambda}, \underline{\zeta}),$$

noting the equality $\rho_J(\mu) + \rho_{I, J}(\zeta) = \rho(\mu)$, we get

$$(\mathcal{L}_\mu)_x \cong H^{2\rho(\mu)}((q_{I, \zeta} \circ m_n)^{-1}(x), f^! \text{IC}(\overline{\text{Gr}_n^\lambda}, \underline{\zeta}))$$

where $f$ is the injection map depicted in the Cartesian diagram below.

![Cartesian diagram](#)

The same reasoning as in sect. 2.3 proves that only the stratum $\text{Gr}_n^\lambda$ contributes to this cohomology group. Denoting by $g : \text{Gr}_n^\lambda \cap (q_{J, \zeta} \circ m_n)^{-1}(x) \rightarrow \text{Gr}_n^\lambda$ the inclusion map, this observation leads to an isomorphism

$$(\mathcal{L}_\mu)_x = H^{2\rho_J(\mu) + 2\rho_{I, J}(\zeta)}(\overline{\text{Gr}_n^\lambda \cap (q_{J, \zeta} \circ m_n)^{-1}(x)}, g^! \underline{\text{IC}}(\overline{\text{Gr}_n^\lambda}, \underline{\zeta}))[\dim \text{Gr}_n^\lambda] \cong H_{BM}^2(\mathbb{Z}) \circ \text{Gr}_n^\lambda \cap (q_{J, \zeta} \circ m_n)^{-1}(x)).$$

Thus the local systems $\mathcal{L}_\mu$ appearing in (8) have a natural basis, namely the set $\mathcal{I}^J(\lambda)_\mu$.

We record the following consequence of this discussion: given $(\lambda, \mu, \nu) \in (\Lambda^+)^n \times \Lambda^+ \times \Lambda$ such that $\mu - \nu \in \mathbb{Z}\Phi_J$, we have

$$\dim H^{2\rho_J(\nu)}(T_{J, \nu}, (t_{J, \nu})^! \text{IC}(\overline{\text{Gr}_J^\nu}, \underline{\zeta})) = \text{rank} \mathcal{L}_\mu \times \dim H^{2\rho_J(\nu)}(T_{J, \nu}, (t_{J, \nu})^! \text{IC}(\overline{\text{Gr}_J^\nu}, \underline{\zeta}))$$

$$= \text{Card} \mathcal{I}^J(\lambda)_\mu \times \text{Card} \mathcal{I}^J(\mu)_\nu$$

$$= \text{Card}\{Z \in \mathcal{I}(\lambda)_\nu \mid \mu_J(Z) = \mu\}.$$  \hspace{1cm} (9)
(B) Now let us start with a sheaf $\mathcal{B}$ in $\text{Perv}(\text{Gr}_J)$ and a weight $\mu \in \Lambda^+_J$. Let us denote by $i : \text{Gr}^\mu_J \to \text{Gr}_J$ the inclusion map. By [4], Amplification 1.4.17.1, the largest subobject of $\mathcal{B}$ in $\text{Perv}(\text{Gr}_J)$ supported on $\text{Gr}^\mu_J$ is $\mathcal{B}_{\leq \mu} = p_{\tau \leq 0} i_* i^! \mathcal{B}$, where $p_{\tau \leq 0}$ is the truncation functor for the perverse $t$-structure. From the distinguished triangle

$$p_{\tau \leq 0} i_* i^! \mathcal{B} \to i_* i^! \mathcal{B} \to p_{\tau > 0} i_* i^! \mathcal{B} \to$$

in the bounded derived category of constructible sheaves on $\text{Gr}_J$, we deduce the long exact sequence

$$H^{2p_J(\nu)-1}(T_{J,\nu}, (t_{J,\nu})^! p_{\tau > 0} i_* i^! \mathcal{B}) \to H^{2p_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^! p_{\tau \leq 0} i_* i^! \mathcal{B})$$

$$\to H^{2p_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^! i_* i^! \mathcal{B}) \to H^{2p_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^! p_{\tau > 0} i_* i^! \mathcal{B}).$$

Theorem 3.5 in [39] implies that the two extrem terms vanish, and therefore

$$F_{J,\nu}(\mathcal{B}_{\leq \mu}) = H^{2p_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^! i_* i^! \mathcal{B}).$$

Let us patch all these pieces together. We take $(\Lambda, \mu, \nu) \in (\Lambda^+_J)^n \times \Lambda^+_J \times \Lambda$ such that $\mu$ and $\nu$ belong to the same coset $\zeta \in \mathbb{Z}/\mathbb{Z} \Phi_J$ and we consider

$$\mathcal{I}_\lambda = (m_n), \text{IC} \left( \text{Gr}^\lambda_n \mathbb{C} \right) \quad \text{and} \quad \mathcal{B} = r_{J,\nu}(\mathcal{I}_\lambda).$$

Composing the isomorphisms given in (6) and (3), we get

$$H^{2p_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^! \mathcal{B}) \cong H^{2p_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^! \mathcal{I}_\lambda) \cong H^{BM}_{2p(\lambda) - \nu}(\text{Gr}^\lambda_n \cap (m_n)^{-1}(T_{\nu})). \quad (10)$$

To save space we set $S = (q_{J,\zeta})^{-1}(\text{Gr}^\mu_J) \cap T_{\nu}$ and denote by $s : S \to T_{\nu}$ the inclusion map. Chasing in the three-dimensional figure

$$\begin{array}{c}
\text{Gr}^\mu_J \quad \text{Gr}_J \quad \text{Gr}^\lambda_n \\
\downarrow \quad \downarrow \quad \downarrow \\
T_{J,\nu} \quad T_{\nu} \quad \text{Gr}^\lambda_n \cap (m_n)^{-1}(T_{\nu}) \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{Gr}^\mu_J \cap T_{J,\nu} \quad \text{Gr}^\mu_J \quad S \quad \text{Gr}^\lambda_n \cap (m_n)^{-1}(S)
\end{array}$$

\begin{equation}
\begin{array}{c}
\text{Gr}^\mu_J \quad \text{Gr}_J \quad \text{Gr}^\lambda_n \\
\downarrow \quad \downarrow \quad \downarrow \\
T_{J,\nu} \quad T_{\nu} \quad \text{Gr}^\lambda_n \cap (m_n)^{-1}(T_{\nu}) \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{Gr}^\mu_J \cap T_{J,\nu} \quad \text{Gr}^\mu_J \quad S \quad \text{Gr}^\lambda_n \cap (m_n)^{-1}(S)
\end{array}
\end{equation}
we complete (10) in the following commutative diagram.

\[
\begin{array}{ccc}
H^{2\rho_J}(T_{J,\nu}, (t_{J,\nu})^! i_* i^! \mathcal{B}) & \cong & H^{2\rho(J)}(S, (t_{\nu})^! i_* i^! \mathcal{F}_\lambda) \\
\downarrow & & \downarrow \\
H^{2\rho_J}(T_{J,\nu}, (t_{J,\nu})^! \mathcal{B}) & \cong & H^{2\rho(J)}(T_\nu, (t_\nu)^! i_* i^! \mathcal{F}_\lambda) \\
\end{array}
\]

As explained in (B), the left vertical arrow of this diagram is the inclusion map

\[
F_{J,\nu}(\mathcal{B}_{\leq J}) \rightarrow F_{J,\nu}(\mathcal{B}).
\]

If an MV cycle \( Z \in \mathcal{Z}(\lambda) \) satisfies \( \mu(J(Z)) \leq J \mu \), then it is contained in \((m_n)^{-1}(S)\), so the fundamental class of \( Z \cap Gr^\lambda_n \) belongs to \( F_{J,\nu}(\mathcal{B}_{\leq J}) \). Looking at equation (9), we see that there are just enough such MV cycles to span this subspace. Going through the geometric Satake correspondence, we conclude that the MV basis of \( V(\lambda) \) satisfies the condition (P1) for being \( L \)-perfect.

Eying now to the condition (P2), we consider the diagram below, consisting of inclusion maps.

\[
\begin{array}{ccc}
\overline{Gr^\mu_J} \setminus Gr^\mu_J & \xrightarrow{f} & \overline{Gr^\mu_J} \\
\downarrow & & \downarrow \\
Gr^\mu_J & \xleftarrow{g} & Gr^\mu_J \\
\downarrow h & & \downarrow \\
Gr^\mu_J & & \\
\end{array}
\]

For any sheaf \( \mathcal{F} \in \text{Perv}(Gr_J) \) supported on \( \overline{Gr^\mu_J} \setminus Gr^\mu_J \), we have

\[
\text{Hom}(\mathcal{F}, p_{\tau \leq 0}(h_* h^! \mathcal{B})) = \text{Hom}(\mathcal{F}, h_* h^! \mathcal{B}) = \text{Hom}(h^! \mathcal{F}, h^! \mathcal{B}) = 0
\]

in the bounded derived category of constructible sheaves over \( Gr_J \) (the first two equalities by adjunction, the last one because \( h^! \mathcal{F} = 0 \)). Since \( h_* h^! \mathcal{B} \) is concentrated in nonnegative perverse degrees ([4], Proposition 1.4.16), the sheaf \( p_{\tau \leq 0}(h_* h^! \mathcal{B}) \) is perverse, and from the semisimplicity of \( \text{Perv}(Gr_J) \) we conclude that

\[
\text{Hom}(p_{\tau \leq 0}(h_* h^! \mathcal{B}), \mathcal{F}) = 0.
\]

Again, in the distinguished triangle

\[
i_* f_* f^! i^! \mathcal{B} \rightarrow i_* i^! \mathcal{B} \rightarrow i_* g_* g^! i^! \mathcal{B} \rightarrow
\]
all sheaves are concentrated in nonnegative perverse degrees. Denoting by $pH^1$ the first homology group for the perverse $t$-structure, we obtain the exact sequence

$$0 \to ^p\tau_{\leq 0}(i_*f_!*i^!\mathcal{B}) \to ^p\tau_{\leq 0}(i_*i^!\mathcal{B}) \to ^p\tau_{\leq 0}(h_*h^!\mathcal{B}) \to ^pH^1(i_*f_!*i^!\mathcal{B}).$$

The perverse sheaf on the right is supported on $\text{Gr}^\mu_j \setminus \text{Gr}^\mu_j$, so the right arrow is zero by the previous step. The resulting short exact sequence can be identified with

$$0 \to \mathcal{B}_{<j\mu} \to \mathcal{B}_{<j\mu} \to \mathcal{B}_{<j\mu}/\mathcal{B}_{<j\mu} \to 0.$$ 

With the same arguments as in the point (B) above, we deduce that

$$F_{J,\nu}(\mathcal{B}_{<j\mu}/\mathcal{B}_{<j\mu}) = F_{J,\nu}(^p\tau_{\leq 0} h_*h^!\mathcal{B}) = H^{2\rho_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^! h_*h^!\mathcal{B}).$$

In (11), we replace $\text{Gr}^\mu_j$ by $\text{Gr}^\mu_j$; the same chasing as before now leads to the isomorphism

$$H^{2\rho_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^! h_*h^!\mathcal{B}) \cong H^{BM}_2(\text{Gr}_n \cap (q_{J,\zeta} \circ m_n)^{-1}(\text{Gr}^\mu_j) \cap (m_n)^{-1}(T_{\nu})).$$

(12)

Now the point (A) at the beginning of this section explains that $\mathcal{R}^k h^!\mathcal{B}$, where $k = -2\rho_J(\mu)$, is the local system $\mathcal{L}_\mu$ and that it comes with a natural basis, namely $\mathcal{Z}_{J}(\lambda)$. This basis induces a decomposition of $\mathcal{B}_{<j\mu}/\mathcal{B}_{<j\mu}$ into a sum of simple objects in $\text{Perv}^\text{res}(\text{Gr}_J)$. On the one hand, this decomposition can be followed through the geometric Satake correspondence, where it gives a decomposition of the subquotient of the isotypical filtration of $\text{res}_{M_j}^G V(\lambda)$ into a direct sum of irreducible representations. On the other hand, it can also be tracked through the isomorphism (12):

$$F_{J,\nu}(\mathcal{B}_{<j\mu}/\mathcal{B}_{<j\mu}) \cong \bigoplus_{\lambda \in \mathcal{Z}_{J}(\lambda)_\mu} H^{BM}_2(\text{Gr}_n \cap (q_{\lambda,\zeta} \circ m_n)^{-1}(\text{Gr}^\mu_j) \cap (m_n)^{-1}(T_{\nu})).$$

(13)

From sect. 3.4, we see that the irreducible components of

$$\text{Gr}_n^\lambda \cap (q_{J,\zeta} \circ m_n)^{-1}(\text{Gr}^\mu_j) \cap (m_n)^{-1}(T_{\nu})$$

of dimension $\rho(\lambda - \nu)$ are the cycles $Z^J \cap (q_{J,\zeta} \circ m_n)^{-1}(Z_J \cap \text{Gr}^\mu_j)$, with $(Z^J, Z_J) \in \mathcal{Z}_{J}(\lambda)_\mu \times \mathcal{Z}_{J}(\mu)_\nu$. The basis of the right-hand side of (12) afforded by the fundamental classes of these irreducible components is thus compatible with the decomposition (13). Therefore, the MV basis of $V(\lambda)$ satisfies the condition (P2) for being $L$-perfect.

The proof of Theorem 3.4 is now complete.

**Remark 3.5.** The proof establishes that the MV basis of $V(\lambda)$ satisfies a stronger property than (P2): there exists an isomorphism of the isotypical component $V(\lambda)_{J,\mu}$ with a direct sum of copies of the irreducible representation $V_J(\mu)$ such that the induced basis on $V(\lambda)_{J,\mu}$ matches the direct sum of the MV bases of the summands.
### 3.6 Crystal structure on MV cycles

Let $\lambda \in (\Lambda^+)^n$. The MV basis of $V(\lambda)$ defined in sect. 2.3 is indexed by

$$\mathcal{Z}(\lambda) = \bigcup_{\nu \in \Lambda} \mathcal{Z}(\lambda)_\nu$$

and is $L$-perfect. Thus, the set $\mathcal{Z}(\lambda)$ is endowed with the structure of a crystal, as explained in sect. 3.2. Obviously the weight of an MV cycle $Z \in \mathcal{Z}(\lambda)_\nu$ is simply $\text{wt}(Z) = \nu$. The aim of this section is to characterize the action on $\mathcal{Z}(\lambda)$ of the operators $\tilde{e}_i$ and $\tilde{f}_i$.

In semisimple rank 1, one can give an explicit analytical description of the MV cycles, as follows.

**Proposition 3.6** Assume that $G$ has semisimple rank 1 and denote by $\alpha$ and $\alpha^\vee$ the positive root and coroot. Let $y : G_\alpha \to G^\vee$ be the additive one-parameter subgroup for the root $-\alpha^\vee$. Let $(\mu, \nu) \in \Lambda^+ \times \Lambda$ and set $r = \langle \alpha^\vee, \mu \rangle$. Then $\text{Gr}^{\mu} \cap T_\nu$ is nonempty if and only if there exists $p \in \{0, 1, \ldots, r\}$ such that $\nu = \mu - p\alpha$; in this case, the map $a \mapsto y(az^{p-r})L_\nu$ induces an isomorphism of algebraic varieties

$$O/z^pO \xrightarrow{\sim} \text{Gr}^{\mu} \cap T_\nu$$

so $\text{Gr}^{\mu} \cap T_\nu$ is an affine space of dimension $p$ and $\mathcal{Z}(\mu)_\nu$ is a singleton.

We skip the proof since this proposition is well-known; compare for instance with [1], Proposition 3.10. We can now describe the crystal structure on $\mathcal{Z}(\lambda)$, which extends [1], Proposition 4.2.

**Proposition 3.7** Let $(\lambda, \nu) \in (\Lambda^+)^n \times \Lambda$, let $i \in I$ and let $Z \in \mathcal{Z}(\lambda)_\nu$.

(i) We have $\text{wt}(Z) = \nu$, $\varepsilon_i(Z) = \frac{1}{2}\langle \alpha_i^\vee, \mu_{\{i\}}(Z) - \nu \rangle$ and $\varphi_i(Z) = \frac{1}{2}\langle \alpha_i^\vee, \mu_{\{i\}}(Z) + \nu \rangle$.

(ii) Let $Y \in \mathcal{Z}(\lambda)_{\nu + \alpha_i}$. Then $Y = \tilde{e}_iZ$ if and only if $Y \subseteq \overline{Z}$ and $\mu_{\{i\}}(Y) = \mu_{\{i\}}(Z)$.

**Proof.** Let $\lambda$, $\nu$, $i$, $Z$ as in the statement and set $\mu = \mu_{\{i\}}(Z)$. By definition, the MV cycles $\tilde{e}_iZ$ and $\tilde{f}_iZ$ (if nonzero) are obtained by letting the Chevalley generators $e_i$ and $f_i$ act on (the basis element indexed by) $Z$ in the appropriate subquotient of the isotypical filtration of $\text{res}_{M_{\{i\}}}^{G_\mu} V(\lambda)$. According to (13), this entails that

$$\mu_{\{i\}}(Z) = \mu_{\{i\}}(\tilde{e}_iZ) = \mu_{\{i\}}(\tilde{f}_iZ)$$

and $Z^{(i)} = (\tilde{e}_iZ)^{(i)} = (\tilde{f}_iZ)^{(i)}$. 


In addition, \( Z_{(i)}(\mu_{\nu+\alpha_i}) \) and \( Z_{(i)}(\mu_{\nu-\alpha_i}) \) are empty or singletons, and the MV cycles \( (\tilde{e}_iZ)_{(i)} \) and \( (\tilde{f}_iZ)_{(i)} \) in the affine Grassmannian \( Gr_{(i)} \) are uniquely determined by weight considerations.

The statements can be deduced from these facts by using the explicit description provided by Proposition 3.6 and the construction of the map \( (Z^J, Z_J) \mapsto Z \) in sect. 3.4. \( \square \)

4 The path model and MV cycles

In the previous section, we defined the structure of a crystal on the set \( \mathcal{Z}(\mathbf{\lambda}) \). In this section, we turn to Littelmann’s path model [33] to study this structure. This combinatorial device can be used to effectively assemble MV cycles. Our construction is inspired by the results presented in [17] but is more flexible, for it relaxes the restriction to minimal galleries.

In this paper, piecewise linear means continuous piecewise linear. We keep the notation set up in the header of sect. 3.

4.1 Recollections on the path model

Let \( \Lambda_\mathbb{R} = \Lambda \otimes \mathbb{Z} \otimes \mathbb{R} \) be the real space spanned by the weight lattice and let \( \Lambda_\mathbb{R}^+ \) be the dominant cone inside \( \Lambda_\mathbb{R} \).

A path is a piecewise linear map \( \pi : [0, 1] \to \Lambda_\mathbb{R} \) such that \( \pi(0) = 0 \) and \( \pi(1) \in \Lambda \). We denote by \( \overline{\Pi} \) the set of all paths. The concatenation \( \pi * \eta \) of two paths is defined in the usual way:

\[
\pi * \eta(t) = \begin{cases} 
\pi(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\
\eta(2t-1) + \pi(1) & \text{for } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

In [33], the third author associates to each simple root \( \alpha \) of \( \Phi \) a pair \( (e_\alpha, f_\alpha) \) of “root operators” from \( \overline{\Pi} \) to \( \overline{\Pi} \sqcup \{0\} \) and shows that the construction yields a semi-normal crystal structure on \( \overline{\Pi} \). Here the weight map is given by \( \text{wt}(\pi) = \pi(1) \). To agree with the notation in sect. 3.1, we will write \( \tilde{e}_i \) and \( \tilde{f}_i \) instead of \( e_\alpha \) and \( f_\alpha \) for each \( i \in I \).

Let \( \ell : [0, 1] \to \mathbb{R} \) be a piecewise linear function. We say that \( p \in \mathbb{R} \) is a local absolute minimum of \( \ell \) if there exists a compact interval \( [a, b] \subseteq [0, 1] \) over which \( \ell \) takes the value \( p \), and there exists an \( \epsilon > 0 \) such that \( \ell(x) > p \) for all \( x \in (a - \epsilon, a) \cap [0, 1] \) and all \( x \in (b, b + \epsilon) \cap [0, 1] \).

Given \( \pi \in \overline{\Pi} \), we denote by \( A\pi \) the set of all paths \( \eta \in \overline{\Pi} \) that can be obtained from \( \pi \) by applying a finite sequence of root operators \( \tilde{e}_i \) or \( \tilde{f}_i \). We say that a path \( \pi \in \overline{\Pi} \) is integral if for each \( \eta \in A\pi \) and each \( i \in I \), all local absolute minima of the function \( t \mapsto \langle \alpha_i^\vee, \eta(t) \rangle \) are integers.
We denote the set of all integral paths by $\Pi$. Obviously, $\Pi$ is a subcrystal of $\tilde{\Pi}$. Moreover, the root operators have a simpler form on $\Pi$, since integral paths need only to be cut into three parts: the initial part is left invariant, the second part is reflected, and the third part is translated. Specifically, given $(\pi, \eta) \in \Pi^2$ and $i \in I$, we have $\eta = \tilde{e}_i \pi$ if and only if there exist a negative integer $p \in \mathbb{Z}$ and two reals $a$ and $b$ with $0 \leq a < b \leq 1$, such that the function $t \mapsto \langle \alpha_i, \pi(t) \rangle$ is weekly decreasing on $[a, b]$, and for each $t \in [0, 1]$:

- if $t \leq a$, then $\langle \alpha_i, \pi(t) \rangle \geq p + 1$ and $\eta(t) = \pi(t)$;
- if $t = a$, then $\langle \alpha_i, \pi(t) \rangle = p + 1$;
- if $a < t < b$, then $p \leq \langle \alpha_i, \pi(t) \rangle < p + 1$ and $\eta(t) = \pi(t) - (\langle \alpha_i, \pi(t) \rangle - p - 1) \alpha_i$;
- if $t = b$, then $\langle \alpha_i, \pi(t) \rangle = p$;
- if $t \geq b$, then $\langle \alpha_i, \pi(t) \rangle \geq p$ and $\eta(t) = \pi(t) + \alpha_i$.

We say that an integral path $\pi \in \Pi$ is dominant if its image is contained in $\Lambda^+_\mathbb{R}$.

Remark 4.1. Let $\Gamma$ be the group of all strictly increasing piecewise linear maps from $[0, 1]$ onto itself, the product being the composition of functions. The group acts on the set of all paths by right composition: $\pi \mapsto \pi \circ \gamma$ for a path $\pi$ and $\gamma \in \Gamma$. We say that $\pi \circ \gamma$ is obtained from $\pi$ by a piecewise linear reparameterization. Visibly, the set $\Pi$ of integral paths is stable under this action, the weight map $\text{wt}$ is invariant, and the root operators are equivariant. We can thus safely consider all our previous constructions modulo this action. In the sequel we sometimes implicitly assume that this quotient has been performed, i.e. among the possible parameterizations we choose one which is appropriate for the application in view.

The first two items in the following proposition ensure that there is an abundance of integral paths.

**Proposition 4.2**

(i) A dominant path $\pi$ is integral as soon as for each $i \in I$, the function $t \mapsto \langle \alpha_i, \pi(t) \rangle$ is weakly increasing.

(ii) The set $\Pi$ is stable under concatenation of paths and the map $\pi \otimes \eta \mapsto \pi \ast \eta$ is a strict morphism of crystals from $\Pi \otimes \Pi$ to $\Pi$.

(iii) Let $\pi \in \Pi$. Then $A\pi$ contains a unique dominant path $\eta$ and is isomorphic as a crystal to $B(\text{wt}(\eta))$. 

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Proof. For $\nu \in \Lambda_{\mathbb{R}}$ let $\pi_\nu$ be the map $[0,1] \to \Lambda_{\mathbb{R}}$, $t \mapsto t \nu$. A path in Proposition 4.2 (i) is of the form $\pi = \pi_{\nu_1} \ast \ldots \ast \pi_{\mu_s}$, where $\nu_1, \ldots, \nu_s \in \Lambda_{\mathbb{R}}^+$ are dominant and $\nu_1 + \cdots + \nu_s \in \Lambda^+$. Such a path can be approximated by a rational dominant path, i.e. a path $\eta = \pi_{\mu_1} \ast \ldots \ast \pi_{\mu_s}$ such that $\mu_1, \ldots, \mu_s \in \Lambda_{\mathbb{Q}}^+$ and $\mu_1 + \cdots + \mu_s = \nu$. By Lemma 4.4 below, a rational dominant path can be approximated by a locally integral concatenation ([35], Definition 5.3). These paths are integral by loc. cit., Lemma 5.6 and Proposition 5.9. The integrality property in (i) follows now by the continuity of the root operators loc. cit., property (v) CONTINUITY.

It remains to prove the other two statements. The endpoint of a path is by definition an element of the lattice, so the concatenation of integral paths is an integral path. Moreover, by Lemma 6.12 in loc. cit., concatenation defines a strict morphism of crystals $\Pi \otimes \Pi \to \Pi$. This shows (ii). Statement (iii) follows by Lemma 6.11 in loc. cit. □

For a rational dominant path $\pi = \pi_{\mu_1} \ast \ldots \ast \pi_{\mu_s}$, the property of being a locally integral concatenation is equivalent to:

\((\ast)\) For all $j = 1, \ldots, s$, the affine line passing through $\mu_1 + \cdots + \mu_{j-1}$ and $\mu_1 + \cdots + \mu_j$ meets at least two lattice points.

An equivalent formulation: the affine line meets at least one rational point and one lattice point.

To prove the approximation property used in the proof above, we need the following elementary geometric construction. We fix a scalar product $\langle \cdot, \cdot \rangle$ on $\Lambda_{\mathbb{R}}$ and let $d(\cdot, \cdot)$ be the corresponding distance function. We fix a basis $\mathbb{B}$ of $\Lambda$ and let $\mathbb{I}_1 \subset \Lambda_{\mathbb{R}}$ be the associated unit cube, i.e. the set of points in $\Lambda_{\mathbb{R}}$ which can be written as a linear combination of $\mathbb{B}$ with coefficients in the interval $[0,1]$. Let $M$ be the maximal distance between two points in $\mathbb{I}_1$. Let $P \in \Lambda_{\mathbb{R}}^+$ be a dominant rational point and let $S(P,1)$ be the sphere with center $P$ and radius 1. Let $g$ be a ray starting in $P$ and let $g_1$ be the intersection point of this ray with the sphere $S(P,1)$.

Lemma 4.3 One can find for any $\epsilon > 0$ a ray $f$ starting in $P$ such that $(f \setminus \{ P \}) \cap \Lambda \neq \emptyset$, and for $\{ f_1 \} = f \cap S(P,1)$ we have $d(g_1, f_1) < \epsilon$.

Proof. Parametrize $g$ by $g(t) = P + t(g_1 - P)$ for $t \geq 0$. Choose $t_2 \gg 0$ and pick $\lambda \in \Lambda$ such that $g(t_2) \in \lambda + \mathbb{I}_1$. Let $f$ be the ray starting in $P$ passing through $\lambda$. Let $f_1$ be the intersection point of this ray with $S(P,1)$; then $f(t) = P + t(f_1 - P)$ for $t \geq 0$ is a parameterization of $f$. Set $g_2 = g(t_2)$ and $f_2 = f(t_2)$. Noting that $d(P, f_2) = t_2 = d(P, g_2)$ and using the triangular inequality, we get

$$d(f_2, \lambda) = |d(P, f_2) - d(P, \lambda)| = |d(P, g_2) - d(P, \lambda)| \leq d(g_2, \lambda) \leq M$$

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whence \(d(g_2, f_2) \leq 2M\). By the intercept theorem \(d(f_1, g_1)/d(g_2, f_2) = 1/t_2\), and hence \(d(f_1, g_1) \leq (2M)/t_2\). For \(t_2\) large enough we obtain \(d(g_1, f_1) < \epsilon\). \(\square\)

**Lemma 4.4** A dominant rational path can be approximated by a locally integral concatenation.

*Proof.* Let \(\pi = \pi_{\lambda_1} \cdots \pi_{\lambda_s}\) be a dominant rational path ending in \(\lambda \in \Lambda^+\). We define the support of an element \(\mu \in \Lambda^+_R\) as the subset of simple roots such that \(\langle \alpha^\vee, \mu \rangle \neq 0\). We can assume that the support of each \(\lambda_j\) is the same as the support of \(\lambda\); otherwise we approximate \(\pi\) by a path we get by slightly perturbing \(\lambda_1, \ldots, \lambda_s\), for instance by replacing \(\lambda_i\) by \(\lambda_i + \epsilon(\lambda/s - \lambda_i)\) for some rational \(0 < \epsilon \ll 1\). We can also assume the support of \(\lambda\) is \(I\), otherwise we work within the subspace \(\bigcap_{i \in I \setminus \text{supp}(\lambda)} (\ker \alpha_i^\vee)\).

Under these assumptions, small perturbations \(\lambda_1', \ldots, \lambda_{s-1}'\) of the directions \(\lambda_1, \ldots, \lambda_{s-1}\) remain dominant, and so does \(\lambda_s' = \lambda - (\lambda_1' + \cdots + \lambda_{s-1}')\). By Lemma 4.3, one can perturb in such a way that the new path \(\eta := \pi_{\lambda_1'} \cdots \pi_{\lambda_s'}\) is a rational dominant path and the first \(s - 1\) line segments of \(\eta\) satisfy the affine line condition (*). The last line segment of \(\eta\) meets the lattice point \(\lambda\) and a rational point, and thus satisfies the affine line condition (*) too. Hence \(\eta\) is a locally integral concatenation, approximating the dominant rational path \(\pi\). \(\square\)

### 4.2 From paths to MV cycles

We need additional terminology before we proceed to the main construction of this section.

To each coroot \(\alpha^\vee \in \Phi^\vee\) corresponds an additive one-parameter subgroup \(x_{\alpha^\vee} : \mathbb{G}_a \to G^\vee\). Given additionally an integer \(p \in \mathbb{Z}\), we define a map

\[
x_{(\alpha^\vee, p)} : \mathbb{C} \to G^\vee(\mathcal{K}), \quad a \mapsto x_{\alpha^\vee}(az^p).
\]

An affine coroot is a pair \((\alpha^\vee, p)\) consisting of a coroot \(\alpha^\vee \in \Phi^\vee\) and an integer \(p \in \mathbb{Z}\). The direction of an affine coroot \((\alpha^\vee, p)\) is \(\alpha^\vee\). An affine coroot is said to be positive if its direction is so. We denote the set of affine coroots by \(\Phi_a^\vee\) and the set of positive affine coroots by \(\Phi_a^{\vee+}\).

To an affine coroot \(\beta\), besides the map \(x_\beta\) defined above, we attach an hyperplan \(H_\beta\) and a negative closed half-space \(H_\beta^-\) in \(\Lambda^+_R\) as follows:

\[
H_{(\alpha^\vee, p)} = \{x \in \Lambda^+_R \mid \langle \alpha^\vee, x \rangle = p\}, \quad H_{(\alpha^\vee, p)}^- = \{x \in \Lambda^+_R \mid \langle \alpha^\vee, x \rangle \leq p\}.
\]

Let \(s_\beta\) be the reflection across the hyperplane \(H_\beta\); concretely

\[
s_{(\alpha^\vee, p)}(x) = x - (\langle \alpha^\vee, x \rangle - p) \alpha
\]

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for any $x \in \Lambda_\mathbb{R}$. In addition, we denote by $\tau_\lambda$ the translation $x \mapsto x + \lambda$ by the element $\lambda \in \Lambda$. The subgroup of $\text{Aut}(\Lambda_\mathbb{R})$ generated by all the reflections $s_\beta$ is the affine Weyl group $\tilde{W}_a$; when we add the translations $\tau_\lambda$, we obtain the extended affine Weyl group $\tilde{W}_a$. Then $\tau_\lambda \in W_a$ if and only if $\lambda \in \mathbb{Z}\Phi$.

The group $\tilde{W}_a$ acts on the set $\Phi_a^\vee$ of affine roots: one demands that $w(H_\beta) = H_{w\beta}$ for each element $w \in \tilde{W}_a$ and each affine coroot $\beta \in \Phi_a^\vee$. Then for each $\beta \in \Phi_a^\vee$ and each $\lambda \in \Lambda$, we have $x_{\tau_\lambda\beta}(a) = z^\lambda x_\beta(a) z^{-\lambda}$ for all $a \in \mathbb{C}$.

We denote by $\mathcal{H}$ the arrangement formed by the hyperplanes $H_\beta$, where $\beta \in \Phi_a^\vee$. It divides the vector space $\Lambda_\mathbb{R}$ into faces. The closure of a face is the disjoint union of faces of smaller dimension. Endowed with the set of all faces, $\Lambda_\mathbb{R}$ becomes a polysimplicial complex, called the affine Coxeter complex.

For each face $\mathfrak{f}$ of the affine Coxeter complex, we denote by $N^\vee(\mathfrak{f})$ the subgroup of $N^\vee(\mathcal{K})$ generated by the elements $x_{\alpha^\vee}(az^p)$, where $a \in \mathcal{O}$ and $(\alpha^\vee, p)$ is a positive affine coroot such that $\mathfrak{f} \subseteq H_{(\alpha^\vee, p)}$. We note that $N^\vee(\tau_\mathfrak{f}) = z^\lambda N^\vee(\mathfrak{f}) z^{-\lambda}$ for each face $\mathfrak{f}$ and each $\lambda \in \Lambda$ and that $N^\vee(\mathfrak{f}) \subseteq N^\vee(\mathcal{O})$ if $\mathfrak{f} \subseteq \Lambda_\mathbb{R}^+$.

For $x \in \Lambda_\mathbb{R}$, we denote by $\mathfrak{f}_x$ the face in the affine Coxeter complex that contains the point $x$. We use the symbol $\prod'$ to denote the restricted product of groups, consisting of families involving only finitely many nontrivial terms. Also, remember the notation introduced in sect. 2.2 about the $n$-fold convolution variety $\text{Gr}_n$.

With these conventions, given $(\pi_1, \ldots, \pi_n) \in \Pi^n$, we define $\mathcal{Z}(\pi_1 \otimes \cdots \otimes \pi_n)$ as the subset of $\text{Gr}_n$ of all elements

$$
\left[ \left( \prod_{t_1 \in [0,1]} v_{1,t_1} \right)^{z^{\text{wt}(\pi_1)}}, \ldots, \left( \prod_{t_n \in [0,1]} v_{n,t_n} \right)^{z^{\text{wt}(\pi_n)}} \right]
$$

with

$$
((v_{1,t_1}), \ldots, (v_{n,t_n})) \in \prod_{t_1 \in [0,1]} N^\vee(\mathfrak{f}_{\pi_1(t_1)}) \times \cdots \times \prod_{t_n \in [0,1]} N^\vee(\mathfrak{f}_{\pi_n(t_n)}).
$$

The resort to the restricted infinite products is here merely cosmetic; we could as well use finite products, since the paths $\pi_j$ meet only finitely many faces. The point that matters is to compute products of elements in the various groups $N^\vee(\mathfrak{f}_{\pi_j(t)})$ in the order indicated by the paths.

**Proposition 4.5** Let $(\pi_1, \ldots, \pi_n) \in \Pi^n$.

(i) The set $\mathcal{Z}(\pi_1 \otimes \cdots \otimes \pi_n)$ is stable under left multiplication by $N^\vee(\mathcal{O})$.  

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(ii) Let \( \mu = \text{wt}(\pi_1) + \cdots + \text{wt}(\pi_n) \); then the set \( \hat{Z}(\pi_1 \otimes \cdots \otimes \pi_n) \) is an irreducible constructible subset of \( (m_n)^{-1}(S_\mu) \).

(iii) We have
\[
\hat{Z}(\pi_1 \cdots \pi_n) = \Psi\left(\hat{Z}(\pi_1) \times \cdots \times \hat{Z}(\pi_n)\right)
\]
and
\[
\hat{Z}(\pi_1 \cdots \pi_n) = m_n\left(\hat{Z}(\pi_1 \otimes \cdots \otimes \pi_n)\right).
\]

(iv) Let \( i \in I \) and compute \( \eta_1 \otimes \cdots \otimes \eta_n = \tilde{e}_i(\pi_1 \otimes \cdots \otimes \pi_n) \) in the crystal \( \Pi^{\otimes n} \), assuming that this operation is doable. Then \( \hat{Z}(\pi_1 \otimes \cdots \pi_n) \) is contained in the closure of \( \hat{Z}(\eta_1 \otimes \cdots \otimes \eta_n) \) in \( \text{Gr}_n \).

**Proof.** Assertion (i) is a direct consequence of the equality \( N^\vee(f_{\pi_1(0)}) = N^\vee(f_0) = N^\vee(O) \).

Assertion (ii) comes from general principles once we have replaced the restricted infinite product by a finite one.

The first equation in (iii) is tautological. In the second one, we view the concatenation \( \pi = \pi_1 \cdots \pi_n \) as a map from \([0, n] \) to \( \Lambda_{\mathbb{R}} \), each path \( \pi_1, \ldots, \pi_n \) being travelled at nominal speed. For \( j \in \{1, \ldots, n\} \), we set \( \nu_{j-1} = \text{wt}(\pi_1) + \cdots + \text{wt}(\pi_{j-1}) \); then \( \pi(t+(j-1)) = \nu_{j-1} + \pi_j(t) \) for all \( t \in [0, 1] \), and accordingly \( N^\vee(f_{\pi(t+(j-1))}) = z^{\nu_{j-1}} N^\vee(f_{\pi_j(t)}) z^{-\nu_{j-1}} \). A banal calculation then yields the desired result.

The proof of assertion (iv) is much more involved. We defer its presentation to sect. 4.5. \( \square \)

For \( (\pi_1, \ldots, \pi_n) \in \Pi^n \), we denote by \( \hat{Z}(\pi_1 \otimes \cdots \otimes \pi_n) \) the closure of \( \hat{Z}(\pi_1 \otimes \cdots \otimes \pi_n) \) in \( (m_n)^{-1}(S_\mu) \), where \( \mu = \text{wt}(\pi_1) + \cdots + \text{wt}(\pi_n) \).

**Theorem 4.6** Let \( (\pi_1, \ldots, \pi_n) \in \Pi^n \), set \( \mu = \text{wt}(\pi_1) + \cdots + \text{wt}(\pi_n) \) and for \( j \in \{1, \ldots, n\} \), let \( \lambda_j \) be the weight of the unique dominant path in \( A\pi_j \). Then \( \hat{Z}(\pi_1 \otimes \cdots \otimes \pi_n) \) is an MV cycle; specifically \( \hat{Z}(\pi_1 \otimes \cdots \otimes \pi_n) \in \hat{Z}(\lambda_1, \ldots, \lambda_n)_\mu \).

**Proof.** We start with the particular case \( n = 1 \). Let \( \pi \in \Pi \), let \( \eta \) be the unique dominant path in \( A\pi \), set \( \lambda = \text{wt}(\eta) \) and \( \mu = \text{wt}(\pi) \), and set \( p = 2\rho(\lambda) \) and \( k = \rho(\lambda + \mu) \). By Proposition 4.2 (iii), the crystal \( A\pi \) is isomorphic to \( B(\lambda) \), so it contains a unique lowest weight element \( \xi \). Then \( \pi \) can be reached by applying a sequence of root operators \( \tilde{f}_i \) to \( \eta \) or by applying a sequence of root operators \( \tilde{e}_i \) to \( \xi \). Thus, there exists a finite sequence \( (\pi_0, \ldots, \pi_p) \) of elements in \( A\pi \) such that \( \pi_0 = \xi \), \( \pi_k = \pi \), \( \pi_p = \eta \) and such that each \( \pi_{j+1} \) is obtained from \( \pi_j \) by applying a root operator \( \tilde{e}_i \).
Since \(\eta\) is dominant, each face \(f_\eta(t)\) is contained in \(\Lambda^\perp_+\), so each group \(N^\vee(f_\eta(t))\) is contained in \(N^\vee(O)\). Then by construction \(Z(\eta) \subseteq N^\vee(O) L_\lambda\), and therefore \(Z(\eta)\) (the closure of \(Z(\eta)\) in \(\text{Gr}\)) is contained in \(\text{Gr}^\lambda\). Also, Proposition 4.5 (iv) implies that
\[
Z(\eta) \subseteq Z(\pi_1) \subseteq \cdots \subseteq Z(\pi_p).
\]
These inclusions are strict because \(Z(\pi_j)\) is contained in the closure of \(\text{Gr}^\lambda \cap S_{\text{wt}(\pi)}\), which is disjoint from \(S_{\text{wt}(\pi_{j+1})}\) by [39], Proposition 3.1 (a), while \(Z(\pi_{j+1})\) is contained in \(S_{\text{wt}(\pi_{j+1})}\). Thus (14) is a strictly increasing chain of closed irreducible subsets of \(\text{Gr}^\lambda\). As \(\text{Gr}^\lambda\) has dimension \(p\), we see that each \(Z(\pi_j)\) has dimension \(j\).

In particular, \(Z(\pi_1)\) has dimension \(k\). But \(Z(\pi)\) is locally closed, because it is a closed subset of \(\text{Gr}^\lambda\), so \(Z(\pi)\) has dimension \(k\). At this point, we know that \(Z(\pi)\) is a closed irreducible subset of \(\text{Gr}^\lambda \cap S_\mu\) of dimension \(k = \rho(\lambda + \mu)\). Therefore \(Z(\pi)\) belongs to \(\text{Z}(\lambda)\).

The reasoning above establishes the case \(n = 1\) of the Theorem. The general case then follows from Propositions 2.2 (ii) and 4.5 (iii). □

### 4.3 A more economical definition

For the proofs in the following sections, it will be convenient to have a more economical presentation of the sets \(Z(\pi_1 \otimes \cdots \otimes \pi_n)\). We need a few additional pieces of notation.

When \(\mathfrak{f}\) and \(\mathfrak{f}'\) are two faces of the affine Coxeter complex such that \(\mathfrak{f}\) is contained in the closure \(\overline{\mathfrak{f}}\) of \(\mathfrak{f}'\), we denote by \(\Phi^+_{a}(\mathfrak{f}, \mathfrak{f}')\) the set of all positive affine coroots \(\beta\) such that \(\mathfrak{f} \subseteq H_\beta\) and \(\mathfrak{f}' \not\subseteq H_\beta\). We denote by \(\mathcal{N}^\vee(\mathfrak{f}, \mathfrak{f}')\) the subgroup of \(\mathcal{N}^\vee(K)\) generated by the elements \(x_\beta(a)\) with \(\beta \in \Phi^+_{a}(\mathfrak{f}, \mathfrak{f}')\) and \(a \in \mathbb{C}\).

The following result is Proposition 19 (ii) in [1].

**Lemma 4.7** Let \(\mathfrak{f}\) and \(\mathfrak{f}'\) be two faces of the affine Coxeter complex such that \(\mathfrak{f} \subseteq \overline{\mathfrak{f}}\). Then \(N^\vee(\mathfrak{f})\) is the bicrossed product \(\mathcal{N}^\vee(\mathfrak{f}, \mathfrak{f}') \bowtie N^\vee(\mathfrak{f}')\); in other words, the product induces a bijection
\[
\mathcal{N}^\vee(\mathfrak{f}, \mathfrak{f}') \times N^\vee(\mathfrak{f}') \cong N^\vee(\mathfrak{f})
\]
In addition, the map
\[
\mathbb{C}^{\Phi^+_{a}(\mathfrak{f}, \mathfrak{f}')} \to \mathcal{N}^\vee(\mathfrak{f}, \mathfrak{f}'), \quad (a_\beta) \mapsto \prod_{\beta \in \Phi^+_{a}(\mathfrak{f}, \mathfrak{f}')} x_\beta(a_\beta)
\]
is bijective, whichever order on \(\Phi^+_{a}(\mathfrak{f}, \mathfrak{f}')\) is used to compute the product.
Given $\pi \in \Pi$ and $t \in [0, 1]$, we denote by $f_\pi(t+0)$ the face in the affine Coxeter complex that contains the points $\pi(t+h)$ for all small enough $h > 0$. Obviously, its closure meets, hence contains, the face $f_\pi(t)$. We set $\Phi_\pi^{\vee, +}(\pi, t) = \Phi_\pi^{\vee, +}(f_\pi(t), f_\pi(t+0))$ and $\N^\vee(\pi, t) = \N^\vee(f_\pi(t), f_\pi(t+0))$.

Concretely, $\Phi_\pi^{\vee, +}(\pi, t)$ is the set of all $\beta \in \Phi_\pi^{\vee, +}$ such that $\pi$ quits the half-space $H^-_\beta$ at time $t$ and $\N^\vee(\pi, t)$ is the subgroup of $\N^\vee(K)$ generated by the elements $x_\beta(a)$ with $\beta \in \Phi_\pi^{\vee, +}(\pi, t)$ and $a \in \C$. Note that $\Phi_\pi^{\vee, +}(\pi, t)$ is empty save for finitely many $t$.

**Proposition 4.8** Let $(\pi_1, \ldots, \pi_n) \in \Pi^n$. Then $\Z(\pi_1 \otimes \cdots \otimes \pi_n)$ is the set of all elements

$$\left[ \left( \prod_{t_1 \in [0,1]} v_{1,t_1} \right)^{\text{wt}(\pi_1)}, \ldots, \left( \prod_{t_n \in [0,1]} v_{n,t_n} \right)^{\text{wt}(\pi_n)} \right]$$

with

$$(v_{1,t_1}, \ldots, v_{n,t_n}) \in \prod_{t_1 \in [0,1]} \N^\vee(\pi_1, t_1) \times \cdots \times \prod_{t_n \in [0,1]} \N^\vee(\pi_n, t_n).$$

**Proof.** Let $\pi \in \Pi$. Let $(t_1, \ldots, t_m)$ be the ordered list of all elements $t \in [0,1]$ such that $\Phi_\pi^{\vee, +}(\pi, t) \neq \emptyset$ and set $t_{m+1} = 1$.

Pick $\ell \in \{1, \ldots, m\}$; between the times $t_\ell$ and $t_{\ell+1}$, the path $\pi$ never quits the half-space $H^-_\beta$ of a positive affine coroot $\beta$; as a consequence, the map $t \mapsto N^\vee(f_\pi(t))$ is non-decreasing on the interval $[t_\ell, t_{\ell+1}]$. This map is also non-decreasing on the interval $[0, t_1]$ if $t_1 > 0$. However when $t$ goes past a point $t_\ell$, the group $\N^\vee(f_\pi(t))$ slims down by dropping the subgroup $\N^\vee(\pi, t_\ell)$, in the sense brought by the bicrossed product $\N^\vee(f_\pi(t_\ell)) = \N^\vee(\pi, t_\ell) \triangleleft \N^\vee(f_\pi(t_{\ell+0}))$.

It follows that for any family $(u_t)$ in $\prod_{t \in [t_\ell, t_{\ell+1}]} \N^\vee(f_\pi(t))$, there exists $(v, u) \in \N^\vee(\pi, t_\ell) \times \N^\vee(f_\pi(t_{\ell+1}))$ such that $\prod_{t \in [t_\ell, t_{\ell+1}]} u_t = vu$. To see this, one decomposes $u_t$ as $vu'$ according to the bicrossed product and one defines $u$ as the product of $u'$ and of the $u_t$ for $t \in [t_\ell, t_{\ell+1}]$. Assembling these pieces (and an analogous statement over the interval $[0, t_1]$ if $t_1 > 0$) from left to right, and noting that $\N^\vee(f_\pi(1))$ stabilizes $L_{\text{wt}(\pi)}$, we deduce that $\Z(\pi)$ is the image of the map

$$\prod_{t \in [0,1]} \N^\vee(\pi, t) \rightarrow \text{Gr}, \quad (v_t) \mapsto \left( \prod_{t \in [0,1]} v_t \right)_{L_{\text{wt}(\pi)}}.$$ 

This proves our statement in the case of just one path. The general case then follows from Proposition 4.5 (iii). □
4.4 Isomorphisms of crystals

In the previous section we explained how to build elements in $\mathcal{X}(\lambda)_\mu$, while in sect. 3 we were dealing with MV cycles in $\mathcal{X}(\lambda)$. This clumsiness is due to a mismatch between the definition of the path model and the conventions in [39] and [2]. To mitigate the disagreement, we define a crystal structure on

$$\mathcal{X}(\lambda) = \bigsqcup_{\mu \in \Lambda} \mathcal{X}(\lambda)_\mu.$$  

Remember the setup of sect. 3.3: we consider a subset $J \subseteq I$, define an action of $\mathbb{C}^\times$ on $\Gr$ given by a special dominant weight $\theta_J$, and get the diagram (4). Now let $(\lambda, \nu) \in (\Lambda^+)^n \times \Lambda$, let $\zeta$ be the coset of $\nu$ modulo $\mathbb{Z}\Phi_J$, and let $Z \in \mathcal{X}(\lambda)_\nu$. Then $m_n(Z) \subseteq S_\nu \subseteq \Gr_{J,\zeta}$ and there is a unique weight $\mu \in \Lambda_J^+$ characterized by the conditions

$$(p_{J,\zeta} \circ m_n)(Z) \subseteq \Gr_{J}^\mu \quad \text{and} \quad (p_{J,\zeta} \circ m_n)(Z) \cap \Gr_J^\mu \neq \emptyset.$$  

We denote this weight by $\mu_J(Z)$.

By analogy with Proposition 3.7, we can then claim the existence of a crystal structure on $\mathcal{X}(\lambda)$ such that for all $\nu \in \Lambda$, $i \in I$ and $Z \in \mathcal{X}(\lambda)_\nu$:

- We have $\wt(Z) = \nu$, $\epsilon_i(Z) = \frac{1}{2}(\alpha_i^\vee, \mu_i | Z) - \nu$ and $\varphi_i(Z) = \frac{1}{2}(\alpha_i^\vee, \mu_i | Z) + \nu$.
- Let $Y \in \mathcal{X}(\lambda)_\nu + \alpha_i$. Then $Y = \tilde{e}_i Z$ if and only if $\tilde{Y} \supseteq Z$ and $\mu_i(Y) = \mu_i(Z)$.

**Theorem 4.9** Let $(\lambda_1, \ldots, \lambda_n) \in (\Lambda^+)^n$, and for each $j \in \{1, \ldots, n\}$ choose a subcrystal $\Pi_j$ of $\Pi$ isomorphic to $B(\lambda_j)$. Then the map $(\pi_1, \ldots, \pi_n) \mapsto \mathbb{Z}^{\pi_1 \otimes \cdots \otimes \pi_n}$ is an isomorphism of crystals

$$\Pi_1 \otimes \cdots \otimes \Pi_n \simeq \mathcal{X}(\lambda_1, \ldots, \lambda_n).$$

**Proof.** Let $i \in I$ and let $(\pi_1, \ldots, \pi_n) \in \Pi_1 \times \cdots \times \Pi_n$. Set $\nu = \wt(\pi_1) + \cdots + \wt(\pi_n)$, let $\zeta$ be the coset of $\nu$ modulo $\mathbb{Z}\alpha_i$, and set $\pi = \pi_1 \cdots \pi_n$,

$$p = \min\{ \langle \alpha_i^\vee, \pi(t) \rangle \mid t \in [0, 1] \} \quad \text{and} \quad q = \langle \alpha_i^\vee, \nu \rangle = \langle \alpha_i^\vee, \pi(1) \rangle.$$  

For any $a \in \mathbb{C}[z, z^{-1}]$ and any positive coroot $\alpha^\vee$, we have

$$\lim_{c \to 0} \theta_{\{i\}}(c) \cdot x_{\alpha^\vee}(a) \cdot \theta_{\{i\}}(c)^{-1} = \begin{cases} x_{\alpha^\vee}(a) & \text{if } \alpha^\vee = \alpha_i^\vee, \\ 1 & \text{otherwise.} \end{cases}$$
Using Proposition 4.8, we see that \( p_{\{i\}, \zeta}(Z(\pi)) \) is the set of all elements of the form

\[
\lim_{c \to 0} \prod_{t \in [0,1]} \left( \prod_{\beta \in \Phi_{+}^{\vee}} \theta_{\{i\}}(c) x_{\beta}(a_{t,\beta}) \theta_{\{i\}}(c)^{-1} \right) L_{\nu}
\]

where \( a_{t,\beta} \) are complex numbers. All factors in the product disappear in the limit \( c \to 0 \), except those for the affine roots \( \beta \) of direction \( \alpha_{i}^{\vee} \). Let \((\alpha_{i}^{\vee}, p_{1}), \ldots, (\alpha_{i}^{\vee}, p_{s}) \) be these affine roots. Since the function \( t \mapsto \langle \alpha_{i}^{\vee}, \pi(t) \rangle \) assumes the value \( p \) and thereafter reaches the value \( q \), the path \( \pi \) must, at some point, quit each half-space \( H_{-(\alpha_{i}^{\vee}, p)} \), \( H_{-(\alpha_{i}^{\vee}, p+1)} \), \ldots, \( H_{-(\alpha_{i}^{\vee}, q-1)} \), so

\[
\{p, p+1, \ldots, q-1\} \subseteq \{p_{1}, \ldots, p_{s}\} \subseteq \{p, p+1, \ldots\}.
\]

We conclude that

\[
p_{\{i\}, \zeta}(Z(\pi)) = \{x_{\alpha_{i}^{\vee}}(az^{p}) L_{\nu} \mid a \in \mathcal{O}/z^{q-p}\mathcal{O}\}.
\]

Proposition 4.5 (iii) and a variant of Proposition 3.6 then yield

\[
(p_{\{i\}, \zeta} \circ m_{n})(Z(\pi_{1} \otimes \cdots \otimes \pi_{n})) = p_{\{i\}, \zeta}(Z(\pi)) = \text{Gr}_{\{i\}, \nu}^{\mu} \cap S_{\{i\}, \nu}
\]

where \( \mu = \nu - p\alpha_{i} \). Thus,

\[
\mu_{\{i\}}(Z(\pi_{1} \otimes \cdots \otimes \pi_{n})) = \nu - p\alpha_{i}
\]

and

\[
\varepsilon_{i}(Z(\pi_{1} \otimes \cdots \otimes \pi_{n})) = -p \quad \text{and} \quad \varphi_{i}(Z(\pi_{1} \otimes \cdots \otimes \pi_{n})) = q - p.
\]

These latter equations show that the map \((\pi_{1}, \ldots, \pi_{n}) \mapsto Z(\pi_{1} \otimes \cdots \otimes \pi_{n})\) is compatible with the functions \( \varepsilon_{i} \) and \( \varphi_{i} \).

Now compute

\[
\eta_{1} \otimes \cdots \otimes \eta_{n} = \tilde{e}_{i}(\pi_{1} \otimes \cdots \otimes \pi_{n})
\]

in the crystal \( \Pi_{1} \otimes \cdots \otimes \Pi_{n} \), assuming this operation to be doable. By Proposition 4.5 (iv),

\[
Z(\eta_{1} \otimes \cdots \otimes \eta_{n}) \supseteq Z(\pi_{1} \otimes \cdots \otimes \pi_{n}).
\]

Let \( \eta = \eta_{1} \ast \cdots \ast \eta_{n} \). Then \( \eta = \tilde{e}_{i} \pi \) by Proposition 4.2 (ii), and therefore

\[
\text{wt}(\eta) = \nu + \alpha_{i} \quad \text{and} \quad \min \{ \langle \alpha_{i}^{\vee}, \eta(t) \rangle \mid t \in [0,1] \} = p + 1.
\]

Repeating the arguments above, we get

\[
\mu_{\{i\}}(Z(\eta_{1} \otimes \cdots \otimes \eta_{n})) = (\nu + \alpha_{i}) - (p + 1)\alpha_{i} = \nu - p\alpha_{i}.
\]
Together with (15) and (16), this gives
\[ Z(\eta_1 \otimes \cdots \otimes \eta_n) = \tilde{e}_i Z(\pi_1 \otimes \cdots \otimes \pi_n). \]

We conclude that the map \((\pi_1, \ldots, \pi_n) \mapsto Z(\pi_1 \otimes \cdots \otimes \pi_n)\) has the required compatibility with the operations \(\tilde{e}_i\). □

**Corollary 4.10** Let \((\lambda_1, \ldots, \lambda_n) \in (\Lambda^+)^n\). Then the map \((Z_1, \ldots, Z_n) \mapsto \Psi(Z_1 \otimes \cdots \otimes Z_n)\) from Proposition 2.2 (ii) is an isomorphism of crystals
\[ \star \mathcal{Z}(\lambda_1) \otimes \cdots \otimes \star \mathcal{Z}(\lambda_n) \cong \star \mathcal{Z}(\lambda_1, \ldots, \lambda_n). \]

The crystals \(\mathcal{Z}(\lambda)\) enjoy a factorization property analogous to Corollary 4.10; **one must however use the opposite tensor product on crystals.**

**Remark 4.11.** The plactic algebra [32] is an algebraic combinatorial tool invented by Lascoux and Schützenberger long before the notion of a crystal basis of a representation was introduced. Loosely speaking, for a complex reductive algebraic group, the plactic algebra is the algebra having as basis the union \(\bigcup_{\lambda \in \Lambda^+} B(\lambda)\) of the crystal bases \(B(\lambda)\) for all irreducible representations, the product being given by the tensor product of crystals. For \(G = SL_n(\mathbb{C})\), Lascoux and Schützenberger give a description of such an algebra in terms of the word algebra modulo the Knuth relations, and it was shown later that this algebra is isomorphic to the one given by the crystal basis. A combinatorial Lascoux-Schützenberger type description for the other types was given in [34]; this description uses the path model.

It is natural to ask whether it is possible to do the same with MV cycles: endow the set of all MV cycles for all dominant weights \(\lambda \in \Lambda^+\) with the structure of a crystal and define (in a geometric way) a multiplication on the cycles which mimics the plactic algebra. For \(G = SL_n(\mathbb{C})\), a positive answer was given in [18]. This approach was adapted to the symplectic case in [44].

The results in this section can be naturally viewed as a generalization of [18] to arbitrary connected reductive groups. Using [34] and Proposition 4.2, one can use the set \(\Pi\) to construct the plactic algebra so that it has as basis equivalence classes (generalized Knuth relations) of elements in \(\Pi\). The sets \(Z(\pi_1 \otimes \cdots \otimes \pi_n)\) (Proposition 4.5) replace in the general setting the Białynicki-Birula cells in [18]. By combining Proposition 4.5 (iii) and Theorem 4.9, we see that the closure of \(m_n(Z(\pi_1 \otimes \cdots \otimes \pi_n))\) is an MV cycle which depends only on the class of the path \(\pi_1 \ast \cdots \ast \pi_n\) modulo the generalized Knuth relations. In particular, the main result of [18] follows as a special case.
A different approach to this problem was taken by Xiao and Zhu [45]. They define a set of ‘elementary Littelmann paths’, modeled over minuscule or quasi-minuscule representations, use the methods from [40] to assign an MV cycle to each concatenation of elementary Littelmann paths, and show that the resulting map factorizes through the generalized Knuth relations.

4.5 Proof of Proposition 4.5 (iv)

This section can be skipped without substantial loss for the appreciation of our main storyline. We follow the same method as in [1], proof of Proposition 5.11.

The group $G^\vee(\mathbb{C})$ is generated by elements $x_{\alpha^\vee}(a)$ and $c^\lambda$, where $(a, \alpha^\vee) \in \mathbb{C} \times \Phi^\vee$ and $(c, \lambda) \in \mathbb{C}^\times \times \Lambda$, which obey the following relations:

- For any $(a, \alpha^\vee) \in \mathbb{C} \times \Phi^\vee$ and any $(c, \lambda) \in \mathbb{C}^\times \times \Lambda$,
  \[ c^\lambda x_{\alpha^\vee}(a) c^{-\lambda} = x_{\alpha^\vee}(c^{(\alpha^\vee, \lambda)}a). \]

- Given two linearly independent elements $\alpha^\vee$ and $\beta^\vee$ in $\Phi^\vee$, there exist constants $C_{i,j}$ such that
  \[ x_{\alpha^\vee}(a) x_{\beta^\vee}(b) x_{\alpha^\vee}(a)^{-1} x_{\beta^\vee}(b)^{-1} = \prod_{(i,j)} x_{i\alpha^\vee+j\beta^\vee}(C_{i,j} a^i b^j) \]  \hspace{1cm} (17)
  for any $(a, b) \in \mathbb{C}^2$. The product in the right-hand side is taken over all pairs of positive integers $(i, j)$ for which $i\alpha^\vee + j\beta^\vee \in \Phi^\vee$, in order of increasing $i + j$.

Further, the one-parameter subgroups $x_{\alpha^\vee}$ can be normalized so that for any root $\alpha \in \Phi$:

- For any $(a, b) \in \mathbb{C}^2$ such that $1 - ab \neq 0$,
  \[ x_{\alpha^\vee}(a) x_{-\alpha^\vee}(b) = x_{-\alpha^\vee}(b/(1 - ab)) (1 - ab)^{\alpha} x_{\alpha^\vee}(a/(1 - ab)). \]  \hspace{1cm} (18)

- There exists an element $\overline{s_\alpha} \in G^\vee(\mathbb{C})$ such that for any $a \in \mathbb{C}^\times$,
  \[ x_{\alpha^\vee}(a) x_{-\alpha^\vee}(a^{-1}) x_{\alpha^\vee}(a) = x_{-\alpha^\vee}(a^{-1}) x_{\alpha^\vee}(a) x_{-\alpha^\vee}(a^{-1}) = a^{\alpha} \overline{s_\alpha} = \overline{s_\alpha} a^{-\alpha}. \]  \hspace{1cm} (19)

This element $\overline{s_\alpha}$ lifts in the normalizer of $T^\vee(\mathbb{C})$ the reflection $s_\alpha \in W$ along the root $\alpha$. All the above relations also hold for scalars $b, c$ in $\mathcal{K}$, provided of course that we regard them in $G^\vee(\mathcal{K})$.

The Chevalley commutation relation (17) implies the following easy lemma.
Lemma 4.12 Let $\mathfrak{f}$ be a face of the affine Coxeter complex, let $(\alpha^\vee, p)$ and $(\beta^\vee, q)$ be two positive affine coroots, and let $(a, b) \in \mathcal{O}^2$. Assume that $\alpha^\vee$ is simple, that $\alpha^\vee \neq \beta^\vee$, and that

$$\mathfrak{f} \subseteq H^-_{(\mathfrak{f}, -p, -\alpha^\vee)} \cap H^-_{(\mathfrak{f}, -\beta^\vee, q)}.$$ 

Then

$$x^-_{\alpha^\vee}(az^{-p}) x^\vee_{\beta^\vee}(bz^q) x^-_{\alpha^\vee}(-az^{-p}) \in N^\vee(\mathfrak{f}).$$

Proof. We consider the situation set forth in the statement of the lemma. Using (17), we write

$$x^-_{\alpha^\vee}(az^{-p}) x^\vee_{\beta^\vee}(bz^q) x^-_{\alpha^\vee}(-az^{-p}) = \prod_{(i,j)} x^-_{i\alpha^\vee + j\beta^\vee}(C_{i,j} a^i b^j z^{-ip+jq})$$

where the product in the right-hand side is taken over all pairs of positive integers $(i, j)$ for which $-i\alpha^\vee + j\beta^\vee$ is a coroot.

Consider such a pair $(i, j)$. In view of our assumptions, the coroot $-i\alpha^\vee + j\beta^\vee$ is necessarily positive. Moreover for any $x \in \mathfrak{f}$ we have

$$\langle -i\alpha^\vee + j\beta^\vee, x \rangle = i\langle -\alpha^\vee, x \rangle + j\langle \beta^\vee, x \rangle \leq i(-p) + jq,$$

so $\mathfrak{f} \subseteq H^-_{-i\alpha^\vee + j\beta^\vee, -ip+jq}$. It follows that the right-hand side of (20) lies in $\mathcal{N}^\vee(\mathfrak{f})$, which readily implies the statement. □

Given $g \in N^\vee(\mathcal{K})$, there is a unique tuple $(a_i) \in \mathcal{K}^I$ such that

$$g \equiv \prod_{i \in I} x^-_{\alpha^\vee}(a_i) \mod (N^\vee(\mathcal{K}), N^\vee(\mathcal{K}));$$

looking at a specific $i \in I$, we denote by $a_{i,p}(g)$ the coefficient of $z^p$ in the Laurent series $a_i$. This procedure defines a morphism of groups $a_{i,p} : N^\vee(\mathcal{K}) \to \mathbb{C}$ for each pair $(i, p) \in I \times \mathbb{Z}$.

Lemma 4.13 Let $\pi \in \Pi$ and let $(t_1, \ldots, t_m)$ be the ordered list of all elements $t \in [0, 1]$ such that $\Phi^\vee_{\alpha^\vee}(\pi, t) \neq \emptyset$. Set $t_{m+1} = 1$. Let $i \in I$ and set

$$p = \min\{\langle \alpha^\vee_i, \pi(t) \rangle \mid t \in [0, 1]\}.$$ 

Let $r \in \{1, \ldots, m + 1\}$ and let $(v_{\ell}) \in \prod_{\ell=r}^m \mathcal{N}^\vee(\pi, t_{\ell}).$
(i) Let \( r^+ \) be the smallest element in

\[
\{ \ell \in \{r, \ldots, m\} \mid (a_i^\ell, p) \in \Phi_a^{\vee}(\pi, t_\ell) \},
\]

contingent on this set to be nonempty. Then for any \( u \in N^\vee(\ell_\pi(t_\pi)) \) there exists \( (v'_\ell) \in \prod_{\ell=r}^m N^\vee(\pi, t_\ell) \) such that

\[
v'_r \cdots v'_m L_{wt(\pi)} = u v_r \cdots v_m L_{wt(\pi)}
\]

and

\[
a_{i,p}(v'_\ell) = \begin{cases} a_{i,p}(u) + a_{i,p}(v_\ell) & \text{if } \ell = r^+, \\ a_{i,p}(v_\ell) & \text{for all other } \ell \in \{r, \ldots, m\} \end{cases}
\]

(ii) For any \( c \in 1 + z \mathcal{O} \) and any \( \lambda \in \Lambda \), there exists \( (v'_\ell) \in \prod_{\ell=r}^m N^\vee(\pi, t_\ell) \) such that

\[
v'_r \cdots v'_m L_{wt(\pi)} = c^\lambda v_r \cdots v_m L_{wt(\pi)}
\]

and \( a_{i,p}(v'_\ell) = a_{i,p}(v_\ell) \) for all \( \ell \in \{r, \ldots, m\} \).

(iii) For any \( b \in \mathbb{C} \) not in

\[
\{0\} \cup \{ a_{i,p}(v_r) + \cdots + a_{i,p}(v_\ell) \mid \ell \in \{r, \ldots, m\} \},
\]

there exists \( (v'_\ell) \in \prod_{\ell=r}^m N^\vee(\pi, t_\ell) \) such that

\[
v'_r \cdots v'_m L_{wt(\pi)} = x_{(-\alpha'_r, -p)}(1/b) v_r \cdots v_m L_{wt(\pi)}.
\]

Proof. The lemma is trivial for \( r = m + 1 \). Proceeding by decreasing induction, we choose \( r \in \{1, \ldots, m\} \), assume that statements (i), (ii) and (iii) hold for \( r + 1 \), and show that they also hold for \( r \). We recall (see the proof of Proposition 4.8) that

\[
N^\vee(\ell_\pi(t_\pi)) = \mathcal{N}^\vee(\pi, t_\pi) \bowtie N^\vee(\ell_\pi(t_\pi+1)) \quad \text{and} \quad N^\vee(\ell_\pi(t_\pi+1)) \subseteq N^\vee(\ell_\pi(t_\pi+1)).
\]

Let \( (v_\ell) \in \prod_{\ell=r}^m \mathcal{N}^\vee(\pi, t_\ell) \).

We start with (i). Let \( u \in N^\vee(\ell_\pi(t_\pi)) \). We can write \( uv_r \in N^\vee(\ell_\pi(t_\pi)) \) as a product \( u'v'_r \) with \( (v'_r, u') \in \mathcal{N}^\vee(\pi, t_\pi) \times N^\vee(\ell_\pi(\pi+1)) \). Then

\[
a_{i,p}(u) + a_{i,p}(v_\ell) = a_{i,p}(v'_r) + a_{i,p}(u').
\]

Noting that \( u' \in N^\vee(\ell_\pi(t_\pi+1)) \), we make use of the inductive assumption: there exists \( (v'_\ell) \in \prod_{\ell=r+1}^m \mathcal{N}^\vee(\pi, t_\ell) \) such that

\[
v'_{r+1} \cdots v'_m L_{wt(\pi)} = u' v_{r+1} \cdots v_m L_{wt(\pi)}.
\]
We distinguish two cases. If $\alpha_i \cdot p \in \Phi_{\alpha_i}^\vee \Gamma \Rightarrow H \Phi_{\alpha_i}^\vee \Gamma$, then $f_{\pi(t_r+0)} \not\subseteq H_{\alpha_i}^\vee \Gamma$, whence $a_{i,p}(u') = 0$; also $r^+ = r$ in this case. If $(\alpha_i^\vee, p) \notin \Phi_{\alpha_i}^\vee \Gamma \Rightarrow H_{\alpha_i}^\vee \Gamma$, then $a_{i,p}(v_r) = a_{i,p}(v_r') = 0$; here $r^+ = (r + 1)^+$. In both cases, routine checks conclude the proof of (i).

We now turn to statement (ii). Let $c \in 1 + zO$ and let $\lambda \in \Lambda$. One easily checks that any subgroup of the form $N^\vee(f)$, in particular $N^\vee(f_{\pi(t_r)})$, is stable under conjugation by $c^\lambda$. Additionally, for any $v \in N^\vee(f_{\pi(t_r)})$, when we write

$$v \equiv \prod_{i \in I} x_{\alpha_i^\vee}(a_i) \mod (N^\vee(K), N^\vee(K)),$$

the Laurent series $a_i$ has valuation at least $p$; this series is multiplied by $c^{(\alpha_i^\vee, \lambda)}$ when one conjugates $v$ by $c^\lambda$; looking at the coefficient of $z^p$ then gives $a_{i,p}(v) = a_{i,p}(c^{\lambda} v c^{-\lambda})$.

Write $c^\lambda v_r c^{-\lambda} \in N^\vee(f_{\pi(t_r)})$ as a product $v_r' u$ with $(v_r', u) \in \mathcal{N}^\vee(\pi, t_r) \times N^\vee(f_{\pi(t_r+0)})$. Then

$$a_{i,p}(v_r) = a_{i,p}(c^\lambda v_r c^{-\lambda}) = a_{i,p}(v_r') + a_{i,p}(u).$$

By induction, there exists $v'_r \equiv \prod_{i=r+1}^m N^\vee(\pi, t_i)$ such that

$$v'_{r+1} \cdots v'_m L_{wt(\pi)} = uc^\lambda v_{r+1} \cdots v_m L_{wt(\pi)},$$

and

$$a_{i,p}(v'_r) = \begin{cases} a_{i,p}(u) + a_{i,p}(v_r) & \text{if } \ell = (r + 1)^+, \\ a_{i,p}(v_r) & \text{for all other } \ell \in \{r + 1, \ldots, m\}. \end{cases}$$

Again we distinguish two cases. If $(\alpha_i^\vee, p) \in \Phi_{\alpha_i}^\vee \Gamma \Rightarrow H \Phi_{\alpha_i}^\vee \Gamma$ and therefore $a_{i,p}(u) = 0$. If $(\alpha_i^\vee, p) \notin \Phi_{\alpha_i}^\vee \Gamma \Rightarrow H \Phi_{\alpha_i}^\vee \Gamma$, then $a_{i,p}(v_r) = a_{i,p}(v_r') = 0$ and anew $a_{i,p}(u) = 0$. Thus, $a_{i,p}(u) = 0$ holds unconditionally, which concludes the proof of (ii).

Lastly, let us deal with statement (iii). We distinguish three cases.

Suppose first that $(\alpha_i^\vee, p) \in \Phi_{\alpha_i}^\vee \Gamma \Rightarrow H \Phi_{\alpha_i}^\vee \Gamma$. We write $v_r = x_{(\alpha_i^\vee, p)}(a) \hat{v}_r$ where $a = a_{i,p}(v_r)$ and $\hat{v}_r$ is a product of elements $x_{\beta}(a_{\beta})$ with $\beta \in \Phi_{\alpha_i}^\vee \Gamma \setminus \{(\alpha_i^\vee, p)\}$ and $a_{\beta} \in \mathbb{C}$. From (18) we get

$$x_{(-\alpha_i^\vee, -p)}(1/b) x_{(\alpha_i^\vee, p)}(a) = (1 - a/b)^{-a_i} x_{(\alpha_i^\vee, p)}(a(1 - a/b)) x_{(-\alpha_i^\vee, -p)}(1/(b - a)).$$
By Lemma 4.12,
\[ x(\alpha, p) \cdot v_r \cdot x(\alpha, -p) \cdot (-1/b) \]

belongs to \( N^\vee(f(p)) \); we write it as a product \( v_r' \cdot u \) with \( (v_r', u) \in N^\vee(\pi, t_r) \times N^\vee(f(p)) \).

By induction, there exists \( v'_r \in \prod_{i=r+1}^m N^\vee(\pi, t_i) \) such that
\[ v'_{r+1} \cdots v'_m \cdot L_{\text{wt}(\pi)} = u \cdot x(\alpha, -p) \cdot (1/b) \cdot v_{r+1} \cdots v_m \cdot L_{\text{wt}(\pi)}. \]

Then
\[ x(\alpha, -p) \cdot (1/b) \cdot v_r \cdots v_m \cdot L_{\text{wt}(\pi)} = (1 - a/b)^{-\alpha} \cdot [x(\alpha, p) \cdot (a(1 - a/b)) \cdot v'_r] \cdot v'_{r+1} \cdots v'_m \cdot L_{\text{wt}(\pi)}. \]

Denoting the element between square brackets above by \( v'_r \), we get the desired expression, up to the inconsequential left multiplication by \( (1 - a/b)^{-\alpha} \).

Suppose now that there exists \( q > p \) such that \( (\alpha, q) \in \Phi_a^\vee(\pi, t_r) \); then \( a_{i, p}^r(v_r) = 0 \). We write \( v_r = x(\alpha, q) \cdot v_r \) where \( a \in \mathbb{C} \) and \( \tilde{v}_r \) is a product of elements \( x(\beta, a_{i, p}) \) with \( \beta \in \Phi_a^\vee(\pi, t_r) \setminus \{ (\alpha, q) \} \) and \( a_{i, p}^r \in \mathbb{C} \). Let \( c \) be a square root in \( 1 + t \mathcal{O} \) of \( 1 - (a/b) \cdot t^{q-p} \). From (18) we get
\[ x(\alpha, -p) \cdot (1/b) \cdot x(\alpha, q) \cdot (a) \cdot x(\alpha, -p) \cdot (1/b) \cdot c^{-\alpha}. \]

By Lemma 4.12,
\[ x(\alpha, -p) \cdot (1/b) \cdot (c^{-\alpha} \cdot v_r \cdot c^\alpha) \cdot x(\alpha, -p) \cdot (-1/b) \]

belongs to \( N^\vee(f(p)) \); we write it as a product \( v_r' \cdot u \) with \( (v_r', u) \in N^\vee(\pi, t_r) \times N^\vee(f(p)) \).

By induction, there exists \( v'_r \in \prod_{i=r+1}^m N^\vee(\pi, t_i) \) such that
\[ v'_{r+1} \cdots v'_m \cdot L_{\text{wt}(\pi)} = u \cdot x(\alpha, -p) \cdot (1/b) \cdot v_{r+1} \cdots v_m \cdot L_{\text{wt}(\pi)}. \]

Then
\[ x(\alpha, -p) \cdot (1/b) \cdot v_r \cdots v_m \cdot L_{\text{wt}(\pi)} = c^{-\alpha} \cdot v_{r+1} \cdots v_m \cdot L_{\text{wt}(\pi)}. \]

Denoting the element between square brackets above by \( v'_r \), we get the desired expression, up to the inopportune left multiplication by \( c^{-\alpha} \); the latter can however be wiped off by a further application of the inductive assumption.

Last, suppose that no affine coroot of direction \( \alpha \) occurs in \( \Phi_a^\vee(\pi, t_r) \); then \( a_{i, p}^r(v_r) = 0 \). By Lemma 4.12,
\[ x(\alpha, -p) \cdot (1/b) \cdot v_r \cdot x(\alpha, -p) \cdot (-1/b) \]

belongs to \( N^\vee(f(p)) \); we write it as a product \( v_r' \cdot u \) with \( (v_r', u) \in N^\vee(\pi, t_r) \times N^\vee(f(p)) \).

By induction, there exists \( v'_r \in \prod_{i=r+1}^m N^\vee(\pi, t_i) \) such that
\[ v'_{r+1} \cdots v'_m \cdot L_{\text{wt}(\pi)} = u \cdot x(\alpha, -p) \cdot (1/b) \cdot v_{r+1} \cdots v_m \cdot L_{\text{wt}(\pi)}. \]
Then
\[ x(\alpha_i^\gamma, -p)(1/b) v_r \cdots v_m L_{wt}(\pi) = v'_r v'_{r+1} \cdots v'_m L_{wt}(\pi), \]
as desired, which concludes the proof of (iii). ∎

Let us now consider \( i \in I \) and two integral paths \( \pi \) and \( \eta \) related by the equation \( \eta = \bar{\epsilon}_i \pi \). We denote by \( p \) the minimum of the function \( t \mapsto \langle \alpha_i^\gamma, \pi(t) \rangle \) over the interval \([0, 1]\) and by \( a \) and \( b \) the two points in time where \( \pi \) is bent to produce \( \eta \). Noting that the conditions spelled out in sect. 4.1 do not uniquely determine \( b \), we choose it to be the largest possible: either \( b = 1 \) or \( \langle \alpha_i^\gamma, \pi(b + h) \rangle > p \) for all small enough \( h > 0 \).

Let \((t_1, \ldots, t_m)\) be the ordered list of all elements in \([0, 1]\) such that \( \Phi_{\alpha_i^\gamma}^\gamma(t, \pi) \neq \emptyset \). We set \( t_{m+1} = 1 \). The set \( \Phi_{\alpha_i^\gamma}^\gamma(t, \pi, a) \) may be empty; if this happens, we insert \( a \) in the list \((t_1, \ldots, t_m)\), for it will simplify the notation hereafter. On the contrary, the above condition imposed on \( b \) ensures that either \( b = 1 \) or \( \langle \alpha_i^\gamma, p \rangle \in \Phi_{\alpha_i^\gamma}^\gamma(t, \pi) \), so \( b \) automatically appears in the list \((t_1, \ldots, t_{m+1})\). We denote by \( r \) and \( s \) the indices in \([1, \ldots, m + 1]\) such that \( a = t_r \) and \( b = t_s \); by design \( t_r = a < t_{r+1} \leq t_s = b \).

**Lemma 4.14** In the setting described in the two paragraphs above, let \((v_\ell) \in \prod_{\ell=1}^m N^\vee(\pi, t_\ell)\). Assume that \( a_{i,p}(v_s) + \cdots + a_{i,p}(v_\ell) \neq 0 \) for each \( \ell \in \{s, \ldots, m\} \). Then for any \( h \in \mathbb{C}^\times \), one can construct \((w_\ell) \in \prod_{\ell=1}^m N^\vee(\eta, t_\ell)\) such that
\[ v_1 \cdots v_{r-1} x(\alpha_i^\gamma, -p-1)(h) v_r \cdots v_m L_{wt}(\pi) = w_1 \cdots w_m L_{wt}(\eta). \]

**Proof.** Let \((v_\ell)\) be as in the statement and let \( h \in \mathbb{C}^\times \). We set
\[ A = v_1 \cdots v_{r-1} \quad \text{and} \quad B = v_r \cdots v_m. \]

We note that \( f_{\pi(t_r)} \subseteq H(\alpha_i^\gamma, p+1) \), so \( x(\alpha_i^\gamma, p+1)(-1/h) \in N^\vee(f_{\pi(t_r)}) \).

Using Lemma 4.13 (i), we find \((v'_r, \ldots, v'_m) \in \prod_{\ell=r}^m N^\vee(\pi, t_\ell)\) such that
\[ x(\alpha_i^\gamma, p+1)(-1/h) B L_{wt}(\pi) = v'_r \cdots v'_m L_{wt}(\pi) \]
and \( a_{i,p}(v'_\ell) = a_{i,p}(v_\ell) \) for all \( \ell \in \{r, \ldots, m\} \). We set \( c = a_{i,p}(v_s) \) and write \( v'_s = x(\alpha_i^\gamma, p)(c) \bar{v}'_s \); then \( \bar{v}'_s \in N^\vee(\pi, t_s) \) and \( a_{i,p}(\bar{v}'_s) = 0 \). We also set
\[ C = v'_r \cdots v'_{s-1} \quad \text{and} \quad D = \bar{v}'_s v'_{s+1} \cdots v'_m. \]

Using Lemma 4.13 (iii), we find \((\bar{v}'_s, v''_{s+1}, \ldots, v''_m) \in \prod_{\ell=s}^m N^\vee(\pi, t_\ell)\) such that
\[ x(\alpha_i^\gamma, -p)(-1/c) D L_{wt}(\pi) = \bar{v}'_s v''_{s+1} \cdots v''_m L_{wt}(\pi). \]
Then
\[ A x_{(-\alpha_i^\gamma, -p-1)}(h) B_{\text{wt}(\pi)} = AKCEF L_{\text{wt}(\pi)}. \] (21)

Observing that

\[ \Phi_a^{\vee, +}(\eta, t_\ell) = \begin{cases} 
\Phi_a^{\vee, +}(\pi, t_\ell) & \text{if } 1 \leq \ell < r, \\
\{(\alpha_i^\gamma, p + 1)\} \sqcup s(\alpha_i^\gamma, p+1)(\Phi_a^{\vee, +}(\pi, t_\ell)) & \text{if } \ell = r, \\
s(\alpha_i^\gamma, p+1)(\Phi_a^{\vee, +}(\pi, t_\ell)) & \text{if } r < \ell < s, \\
\tau_{\alpha_i}(\Phi_a^{\vee, +}(\pi, t_\ell)) & \text{if } s \leq \ell \leq m,
\end{cases} \]

we check that the sequence

\[
\left(v_1, \ldots, v_{r-1}, x_{(\alpha_i^\gamma, p+1)}(-h) \left(z^{(p+1)\alpha_i\text{wt}_i}\right) v'_r \left(z^{(p+1)\alpha_i\text{wt}_i}\right)^{-1},
\right. \\
\left(z^{(p+1)\alpha_i\text{wt}_i}\right) v'_{r+1} \left(z^{(p+1)\alpha_i\text{wt}_i}\right)^{-1}, \ldots, 
\left(z^{(p+1)\alpha_i\text{wt}_i}\right) v'_{s-1} \left(z^{(p+1)\alpha_i\text{wt}_i}\right)^{-1},
\]

\[
\left(z^{\alpha_i^\gamma} x_{(\alpha_i^\gamma, p)}(-c) v''_s z^{-\alpha_i}, z^{\alpha_i^\gamma} v''_{s+1} z^{-\alpha_i}, \ldots, z^{\alpha_i^\gamma} v''_m z^{-\alpha_i}\right)
\]

belongs to \(\prod_{\ell=1}^m N^{\vee}(\eta, t_\ell)\). In addition, the product of the elements in this sequence is

\[
A x_{(\alpha_i^\gamma, p+1)}(-h) \left(z^{(p+1)\alpha_i\text{wt}_i}\right) C \left(z^{(p+1)\alpha_i\text{wt}_i}\right)^{-1} z^{\alpha_i} F z^{-\alpha_i}.
\]

We now apply two transformations to the sequence (22): we conjugate the last \(m-s+1\) terms by \((-c)^{-\alpha_i}\), and we conjugate the last \(m-r+1\) by \(h^{-\alpha_i}\). The resulting sequence, denoted by \((w_\ell)\), still belongs to \(\prod_{\ell=1}^m N^{\vee}(\eta, t_\ell)\), because all our constructions are \(T^{\vee}(\mathbb{C})\)-equivariant.

Observing that

\[ K = h^{-\alpha_i} x_{(\alpha_i^\gamma, p+1)}(-h) \left(z^{(p+1)\alpha_i\text{wt}_i}\right) \quad \text{and} \quad E = \left(z^{(p+1)\alpha_i\text{wt}_i}\right)^{-1} (-c)^{-\alpha_i} z^{\alpha_i}\]

(see equation (19)), we obtain

\[ w_1 \cdots w_m = AKCEF z^{-\alpha_i} (-ch)^{\alpha_i}, \]

and a comparison with (21) yields

\[
A x_{(-\alpha_i^\gamma, -p-1)}(h) B_{\text{wt}(\pi)} = AKCEF z^{-\alpha_i} L_{\text{wt}(\eta)} = w_1 \cdots w_m L_{\text{wt}(\eta)},
\]

as desired. \(\square\)
We can now prove Proposition 4.5 (iv). We consider the situation
\[ \eta_1 \otimes \cdots \otimes \eta_n = \tilde{\epsilon}_1(\pi_1 \otimes \cdots \otimes \pi_n) \]
in the crystal \( \Pi^\otimes n \), and our aim is to show that \( \tilde{\mathcal{Z}}(\pi_1 \otimes \cdots \otimes \pi_n) \) is contained in the closure of \( \mathcal{Z}(\eta_1 \otimes \cdots \otimes \eta_n) \) in \( \text{Gr}_n \).

As in the proof of Proposition 4.5 (iii), we regard the concatenation \( \pi = \pi_1 \ast \cdots \ast \pi_n \) as a map from \([0,n]\) to \( \Lambda_\mathbb{R} \), each path \( \pi_1, \ldots, \pi_n \) being travelled at nominal speed, and ditto for \( \eta = \eta_1 \ast \cdots \ast \eta_n \). Thus, for each \( j \in \{1, \ldots, n\} \) the restriction of \( \pi \) to the interval \([j-1,j]\) is \( \pi_j \), up to the obvious shifts in time and space.

By Proposition 4.2 (ii), we have \( \eta = \tilde{\epsilon}_j \pi \). We denote by \( a \) and \( b \) the two points in time where \( \pi \) is bent to produce \( \eta \). Let \((t_1, \ldots, t_m)\) be the ordered list of all elements in \([0,n]\) such that \( \Phi_a^\vee+(\pi,t) \neq \emptyset \). We insert \( a \) in this list if it does not already appear there. We set \( t_0 = 0 \) and \( t_{m+1} = n \). We denote by \( r \) and \( s \) the indices in \( \{1, \ldots, m+1\} \) such that \( a = t_r \) and \( b = t_s \).

There is a unique integer \( k \in \{1, \ldots, n\} \) such that \( a \) and \( b \) both belong to \([k-1,k]\); plainly, \( \eta_k = \tilde{\epsilon}_k \pi_k \) and \( \eta_j = \pi_j \) for all \( j \in \{1, \ldots, n\} \setminus \{k\} \). We record that \( \eta_1 \ast \cdots \ast \eta_j = \tilde{\epsilon}_i(\pi_1 \ast \cdots \ast \pi_j) \) if \( j \in \{k, \ldots, n\} \).

For \( j \in \{1, \ldots, n\} \), we set \( \nu_j = \text{wt}(\pi_1) + \cdots + \text{wt}(\pi_j) \) and denote by \( m_j \) the largest element \( \ell \in \{0, \ldots, m\} \) such that \( t_\ell \in [0, j[ \). Then \( \tilde{\mathcal{Z}}(\pi_1 \otimes \cdots \otimes \pi_n) \) is the set of all elements
\[
\left[ \prod_{\ell=1}^{m_1} v_\ell \right] z^{\nu_1}, z^{-\nu_1} \left[ \prod_{\ell=m_1+1}^{m_2} v_\ell \right] z^{\nu_2}, \ldots, z^{-\nu_{n-1}} \left[ \prod_{\ell=m_{n-1}+1}^{m_n} v_\ell \right] z^{\nu_n} \tag{23}\]
with \((v_\ell) \in \prod_{\ell=1}^{m} \mathcal{A}^\vee(\pi, t_\ell)\).

Now assume that \((v_\ell)\) is chosen so that \( a_\ell p(v_s) + \cdots + a_\ell p(v_\ell) \neq 0 \) for each \( \ell \in \{s, \ldots, m\} \) and pick \( h \in \mathbb{C}^\times \). Lemma 4.14 provides us with a sequence \((w_\ell) \in \prod_{\ell=1}^{m} \mathcal{A}^\vee(\eta, t_\ell)\) such that
\[ v_1 \cdots v_{r-1} x(-\alpha_1, -p-1)(h) v_r \cdots v_m L_{\text{wt}(\pi)} = w_1 \cdots w_m L_{\text{wt}(\eta)}. \]

However \((w_\ell)\) satisfies more equations: for \( j \in \{1, \ldots, n\} \), we have
\[
\begin{cases} 
  v_1 \cdots v_{m_j} L_{\nu_j} = w_1 \cdots w_{m_j} L_{\nu_j} & \text{if } j < k, \\
  v_1 \cdots v_{r-1} x(-\alpha_1, -p-1)(h) v_r \cdots v_{m_j} L_{\nu_j} = w_1 \cdots w_{m_j} L_{\nu_j + \alpha_1} & \text{if } j \geq k, 
\end{cases} \tag{24}
\]
in the first case because \( w_\ell = v_\ell \) for all \( \ell \in \{1, \ldots, m_{k-1}\} \), in the second case because Lemma 4.14 would have returned the subsequence \((w_\ell)_{1 \leq \ell \leq m_j}\) if we had fed it with the paths \( \pi_1 \ast \cdots \ast \pi_j \) and \( \eta_1 \ast \cdots \ast \eta_j \) and the datum \((v_\ell)_{1 \leq \ell \leq m_j}\) and \( h \).
The system (24) translates to a single equation in $\text{Gr}_n$, which manifests that the element obtained by inserting $x_{(-\alpha_\gamma', -p-1)}(h)$ just before $v_r$ in (23) belongs to $\mathbb{Z}(\eta_1 \otimes \cdots \otimes \eta_n)$. Letting $h$ tend to 0, we conclude that (23) lies in the closure of this set. To be sure, this conclusion has been reached under the assumption that $a_{i,p}(v_s) + \cdots + a_{i,p}(v_\ell) \neq 0$ for each $\ell \in \{s, \ldots, m\}$, but this restriction can be removed by a small perturbation of $a_{i,p}(v_s)$.

Thus, Proposition 4.5 (iv) is, at last, fully proven.

5 Comparison with the tensor product basis

We keep the notation from sect. 2. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ in $(\Lambda^+)^n$. The tensor product $V(\lambda)$ can be endowed on the one hand with its MV basis (sect. 2.3), on the other hand with the tensor product of the MV bases of the factors $V(\lambda_1), \ldots, V(\lambda_n)$. In this section, we compare these two bases through the explicit identification

$$F(I_{\lambda_1} \ast \cdots \ast I_{\lambda_n}) \cong F(I_{\lambda_1}) \otimes \cdots \otimes F(I_{\lambda_n})$$

afforded by Beilinson and Drinfeld's fusion product. We show that the transition matrix is upper unitriangular and that its entries are intersection multiplicities. The order relation needed to convey the triangularity involves the inclusion of cycles.

5.1 Deformations

The Beilinson-Drinfeld Grassmannian $\mathcal{G}^{\text{BD}}_n$ is a relative version of the affine Grassmannian where the base is the space of effective divisors on a smooth curve. The choice of the affine line amply satisfies our needs and offers three advantages: there is a natural global coordinate on $\mathbb{A}^1$, every $G$-torsor on $\mathbb{A}^1$ is trivializable, and the monodromy of any local system is trivial. Rather than looking for more generality, we will pragmatically stick with this choice. Consistent with sect. 2, the coordinate on $\mathbb{A}^1$ is denoted by $z$.

Formally, the Beilinson-Drinfeld Grassmannian $\mathcal{G}^{\text{BD}}_n$ is defined as the functor on the category of commutative $\mathbb{C}$-algebras that assigns to an algebra $R$ the set of isomorphism classes of triples $(x_1, \ldots, x_n; F, \beta)$, where $(x_1, \ldots, x_n) \in \mathbb{A}^n(R)$, $F$ is a $G'$-torsor over $\mathbb{A}^1_R$ and $\beta$ is a trivialization of $F$ away from the points $x_1, \ldots, x_n$ ([5], sect. 5.3.10; [43], Definition 3.3; [46], Definition 3.1.1). We denote by $\pi : \mathcal{G}^{\text{BD}}_n \to \mathbb{A}^n$ the morphism to the base, which forgets $F$ and $\beta$. It is known that $\mathcal{G}^{\text{BD}}_n$ is representable by an ind-scheme and that $\pi$ is ind-proper.

We are only interested in the set of $\mathbb{C}$-points, endowed with its ind-variety structure. Using a trivialization of $F$, we can thus adopt the following simplified definition: $\mathcal{G}^{\text{BD}}_n$ is the set of
pairs \((x_1, \ldots, x_n; [\beta])\), where \((x_1, \ldots, x_n) \in \mathbb{C}^n\) and \([\beta]\) belongs to the homogenenous space

\[ G^\vee(\mathbb{C}[z, (z - x_1)^{-1}, \ldots, (z - x_n)^{-1}]) / G^\vee(\mathbb{C}[z]) \]

This set is endowed with the structure of an ind-variety.

**Example 5.1.** ([5], Remark in sect. 5.3.10.) We consider the case \(G^\vee = \text{GL}_N\). Here the datum of \([\beta]\) is equivalent to the datum of the \(\mathbb{C}[z]\)-lattice \(\beta(L_0)\) in \(\mathbb{C}(z)^N\), where \(L_0 = \mathbb{C}[z]^N\) is the standard lattice. Let us write \(x\) for the point \((x_1, \ldots, x_n)\) and set \(f_x = (z - x_1) \cdots (z - x_n)\); then a lattice \(L\) is of this form \(\beta(L_0)\) if and only if there exists a positive integer \(k\) such that \(f_x^{-k}L_0 \subseteq L \subseteq f_x^{-k}L_0\). For each positive integer \(k\), define \((G_{r_n}^{\text{BD}})_k\) to be the subset of \(G_{r_n}^{\text{BD}}\) consisting of all pairs \((x; L)\) with \(f_x^kL_0 \subseteq L \subseteq f_x^{-k}L_0\). We identify \(\mathbb{C}[z]/(f_x^k)\) with the vector space \(V\) of polynomials of degree strictly less than \(2kn\), and subsequently identify \(L_0/f_x^{2k}L_0\) with \(V^N\). The space \((G_{r_n}^{\text{BD}})_k\) can then be realized as a Zariski-closed subset of

\[ \mathbb{C}^n \times \bigcup_{d=0}^{2knN} \mathbb{G}_d(V^N) \]

where \(\mathbb{G}_d(V^N)\) denotes the Grassmannian of \(d\)-planes in \(V^N\). In this way, \(G_{r_n}^{\text{BD}}\) is the inductive limit of a system of algebraic varieties and closed embeddings, in other words, an ind-variety.

We also want to deform the \(n\)-fold convolution variety \(G_{r_n}\). Accordingly, we define \(G_{r_n}\) as the set of pairs \((x_1, \ldots, x_n; [\beta_1, \ldots, \beta_n])\), where \((x_1, \ldots, x_n) \in \mathbb{C}^n\) and \([\beta_1, \ldots, \beta_n]\) belongs to

\[ G^\vee(\mathbb{C}[z, (z - x_1)^{-1}] \times G^\vee(\mathbb{C}[z]) \cdots \times G^\vee(\mathbb{C}[z]) G^\vee(\mathbb{C}[z, (z - x_n)^{-1}]) / G^\vee(\mathbb{C}[z]) \]

(see [43], Definition 3.8, or [46], (3.1.21)). This set \(G_{r_n}\) is endowed with the structure of an ind-variety; it comes with a map \(m_n : G_{r_n} \rightarrow G_{r_n}^{\text{BD}}\) defined by

\[ m_n(x_1, \ldots, x_n; [\beta_1, \ldots, \beta_n]) = (x_1, \ldots, x_n; [\beta_1 \cdots \beta_n]). \]

**Example 5.2.** We again consider the case \(G^\vee = \text{GL}_N\). Then an element in \(G_{r_n}\) is the datum of a point \((x_1, \ldots, x_n) \in \mathbb{C}^n\) and a sequence \((L_1, \ldots, L_n)\) of \(\mathbb{C}[z]\)-lattices in \(\mathbb{C}(z)^N\) for which there exists a positive integer \(k\) such that

\[ (z - x_j)^k L_{j-1} \subseteq L_j \subseteq (z - x_j)^{-k} L_{j-1} \]

for all \(j \in \{1, \ldots, n\}\); here again \(L_0 = \mathbb{C}[z]^N\) is the standard lattice and \(L_j = (\beta_1 \cdots \beta_j)(L_0)\).
In the above example, we can partition $\mathcal{G}_n$ into cells by specifying the relative positions of the pairs of lattices $(L_{j-1}, L_j)$ in terms of invariant factors. This construction can be generalized to an arbitrary group $G$ as follows: given $\lambda = (\lambda_1, \ldots, \lambda_n)$ in $(\Lambda^+)^n$, we define $\mathcal{G}_{r_n}^\lambda$ as the subset of $\mathcal{G}_n$ consisting of all pairs $(x_1, \ldots, x_n; [\beta_1, \ldots, \beta_n])$ with

$$\beta_j \in G^\vee(\mathbb{C}[z]) (z - x_j)^{\lambda_j} G^\vee(\mathbb{C}[z])$$

for $j \in \{1, \ldots, n\}$. The Cartan decomposition

$$G^\vee(\mathbb{C}[z, (z - x_j)^{-1}]) = \bigsqcup_{\lambda_j \in \Lambda^+} G^\vee(\mathbb{C}[z]) (z - x_j)^{\lambda_j} G^\vee(\mathbb{C}[z])$$

yields

$$\mathcal{G}_n = \bigsqcup_{\lambda \in (\Lambda^+)^n} \mathcal{G}_{r_n}^\lambda$$

and it can be checked that

$$\mathcal{G}_{r_n}^\lambda = \bigsqcup_{\mu \in (\Lambda^+)^n} \mathcal{G}_{r_n}^\mu, \quad (25)$$

In addition, the maps $(x_1, \ldots, x_j; [\beta_1, \ldots, \beta_j]) \mapsto (x_1, \ldots, x_{j-1}; [\beta_1, \ldots, \beta_{j-1}])$ exhibit $\mathcal{G}_{r_n}^\lambda$ as the total space of an iterated fibration with base $\mathcal{G}_1^\lambda_1$ and successive fibers $\mathcal{G}_1^\lambda_2$, $\ldots$, $\mathcal{G}_1^\lambda_n$; it follows that $\mathcal{G}_{r_n}^\lambda$ is a smooth connected variety of dimension $2\rho(\lambda) + n$.

Let us now investigate the fibers of the map $\pi \circ m_n : \mathcal{G}_n \to \mathbb{C}^n$. Given $x \in \mathbb{C}$, we set $O_x = \mathbb{C}[z - x]$ and $K_x = \mathbb{C}((z - x))$; thus $K_x$ is the completion of $\mathbb{C}(z)$ at the place defined by $x$. We identify $O$ and $K$ with $O_x$ and $K_x$ by means of the map $z \mapsto z - x$.

We fix $x = (x_1, \ldots, x_n)$ in $\mathbb{C}^n$. Let supp($\mathbf{x}$) be the set of values $y \in \mathbb{C}$ that appear in the tuple $\mathbf{x}$. For $y \in$ supp($\mathbf{x}$), denote by $m_y$ the number of indices $j \in \{1, \ldots, n\}$ such that $x_j = y$ and choose an increasing sequence $(p_0 = 0, p_1, p_2, \ldots, p_{m_y} = n)$ in a way that each interval $[p_{k-1} + 1, p_k]$ contains exactly one index $j$ such that $x_j = y$. For $\beta = [\beta_1, \ldots, \beta_n]$ in the fiber

$$(\mathcal{G}_n)_x = G^\vee(\mathbb{C}[z, (z - x_1)^{-1}]) \times G^\vee(\mathbb{C}[z]) \times \cdots \times G^\vee(\mathbb{C}[z]) \times G^\vee(\mathbb{C}[z, (z - x_n)^{-1}]) / G^\vee(\mathbb{C}[z]),$$

we define $\Theta(\beta)_y$ as the point $[(\beta_1 \cdots \beta_{p_1}), (\beta_{p_1+1} \cdots \beta_{p_2}), \ldots, (\beta_{p_{m_y}+1} \cdots \beta_n)]$ in

$$(G^\vee(K_y) \times G^\vee(O_y) \cdots G^\vee(K_y) / G^\vee(O_y)) \cong \mathcal{G}_{m_y}$$

(note that $\Theta(\beta)_y$ does not depend on this choice, because $\beta_j \in G^\vee(O_y)$ if $x_j \neq y$).
Proposition 5.3 The map $\beta \mapsto (\Theta(\beta)_y)$ is a bijection

$$(\mathcal{G}r_n)_x \cong \prod_{y \in \text{supp}(x)} \text{Gr}_{m_y}. \quad (\mathcal{G}r_n)$$

Proof. Combining the Iwasawa decomposition (1) with the easily proven equality

$$N^\vee(K_x) = N^\vee(C[z, (z - x)^{-1}]) N^\vee(O_x), \quad (26)$$

we obtain the well-known equality

$$G^\vee(K_x) = G^\vee(C[z, (z - x)^{-1}]) G^\vee(O_x),$$

for each $x \in \mathbb{C}$.

The case $n = 1$ of the proposition is banal. Assume that $n \geq 2$, and for $y \in \text{supp}(x)$, pick $\gamma_y \in \text{Gr}_{m_y}$. Set $x' = (x_1, \ldots, x_{n-1})$ and $m = m_{x_n}$, write $\gamma_{x_n} = [\gamma_1, \ldots, \gamma_m]$. Reasoning by induction on $n$, we know that there is a unique $\beta' = [\beta_1, \ldots, \beta_{n-1}]$ in $(\mathcal{G}r_{n-1})_{x'}$ such that

$$\Theta(\beta')_y = \begin{cases} 
\gamma_y & \text{if } x_n \neq y, \\
[\gamma_1, \ldots, \gamma_m] & \text{if } x_n = y.
\end{cases}$$

The elements $\gamma_1, \ldots, \gamma_m$ belong to $G^\vee(K)$, which we identify to $G^\vee(K_{x_n})$. We choose $\beta_n \in G^\vee(C[z, (z - x_n)^{-1}])$ such that

$$(\beta_1 \ldots \beta_{n-1})^{-1}([\gamma_1 \ldots \gamma_m]) \in \beta_n G^\vee(O_{x_n}).$$

Then $[\beta_1, \ldots, \beta_{n-1}, \beta_n]$ is the unique element $\beta$ in $(\mathcal{G}r_n)_x$ such that $\Theta(\beta)_y = \gamma_y$ for all $y$. □

5.2 Global cycles

Recall our notation $N^\vee$ for the unipotent radical of $B^\vee$. For $\mu \in \Lambda$ and $x \in \mathbb{C}$, we define

$$\tilde{S}_{\mu|x} = (z - x)^\mu N^\vee(C[z, (z - x)^{-1}]) = N^\vee(C[z, (z - x)^{-1}]) (z - x)^\mu.$$
Equation (26) expresses that the natural map
\[ N^\vee(C[z, (z-x)^{-1}]) / N^\vee(C[z]) \to N^\vee(K_x)/N^\vee(O_x) \]
is bijective; composing with the natural map \( N^\vee(K)/N^\vee(O) \to \text{Gr} \), we obtain, after left multiplication by \((z-x)^\mu\), a bijection
\[ \tilde{S}_{\mu|x} / N^\vee(C[z]) \cong S_{\mu}. \]
For \((\mu_1, \ldots, \mu_n) \in \Lambda^n\), let \( S_{\mu_1} \times \cdots \times S_{\mu_n} \) be the set of all pairs \((x_1, \ldots, x_n; \beta_1, \ldots, \beta_n)\) with \((x_1, \ldots, x_n) \in \mathbb{C}^n\) and \([\beta_1, \ldots, \beta_n] \in \tilde{S}_{\mu_1|x_1} \times N^\vee(C[z]) \cdots \times N^\vee(C[z]) \tilde{S}_{\mu_n|x_n} / N^\vee(C[z])\).

Rewriting the Iwasawa decomposition as
\[ G^\vee(C[z, (z-x)^{-1}]) = \bigsqcup_{\mu \in \Lambda} N^\vee(C[z, (z-x)^{-1}]) (z-x)^\mu G^\vee(C[z]), \]
we then see that the natural map
\[ \Psi : \bigsqcup_{(\mu_1, \ldots, \mu_n) \in \Lambda^n} S_{\mu_1} \times \cdots \times S_{\mu_n} \to \text{Gr}_n \]
is bijective. Here \(\Psi\) is regarded as the calligraphic variant of the letter \(\Psi\) used in sect. 2.2; these two glyphs may be hard to distinguish, but hopefully this choice will not lead to any confusion.

More generally, given \((\mu_1, \ldots, \mu_n) \in \Lambda^n\) and \( N^\vee(O)-\text{stable subsets} Z_1 \subseteq S_{\mu_1}, \ldots, Z_n \subseteq S_{\mu_n},\) we define \( Z_1 \times \cdots \times Z_n \) to be the subset of all pairs \((x_1, \ldots, x_n; \beta_1, \ldots, \beta_n)\) with \((x_1, \ldots, x_n) \in \mathbb{C}^n\) and \([\beta_1, \ldots, \beta_n] \in \tilde{Z}_{1|x_1} \times N^\vee(C[z]) \cdots \times N^\vee(C[z]) \tilde{Z}_{n|x_n} / N^\vee(C[z])\)
where each \(\tilde{Z}_{j|x_j}\) is the preimage of \(Z_j\) under the map \(\tilde{S}_{\mu_j|x_j} \to S_{\mu_j}\).

For \(\mu \in \Lambda\), we define
\[ \tilde{S}_\mu = \bigsqcup_{(\mu_1, \ldots, \mu_n) \in \Lambda^n, \mu_1 + \cdots + \mu_n = \mu} \Psi(S_{\mu_1} \times \cdots \times S_{\mu_n}). \]

**Proposition 5.4** Let \(\lambda = (\lambda_1, \ldots, \lambda_n)\) in \((\Lambda^+)^n\) and let \(\mu \in \Lambda\).

(i) All the irreducible components of \(\text{Gr}_n^\lambda \cap \tilde{S}_\mu\) have dimension \(\rho(|\lambda| + \mu) + n\).
(ii) The map \((Z_1, \ldots, Z_n) \mapsto \Psi(Z_1 \propto \cdots \propto Z_n)\) induces a bijection

\[
\bigsqcup_{(\mu_1, \ldots, \mu_n) \in \Lambda^n \atop \mu_1 + \cdots + \mu_n = \mu} \ast \mathfrak{Z}(\lambda_1)_{\mu_1} \times \cdots \times \ast \mathfrak{Z}(\lambda_n)_{\mu_n} \cong \text{Irr}\left(\overline{Gr^\lambda_n \cap \dot{S}_\mu}\right).
\]

(The bar above \(\Psi(Z_1 \propto \cdots \propto Z_n)\) means closure in \(\dot{S}_\mu\).)

Proof. Let \((\mu_1, \ldots, \mu_n) \in \Lambda^n\) be such that \(\mu_1 + \cdots + \mu_n = \mu\) and let \((Z_1, \ldots, Z_n) \in \ast \mathfrak{Z}(\lambda_1)_{\mu_1} \times \cdots \times \ast \mathfrak{Z}(\lambda_n)_{\mu_n}\). Then the set \(\Psi(Z_1 \propto \cdots \propto Z_n)\) is irreducible. By Proposition 5.3 and its proof, the fiber of this set over a point \(x \in \mathbb{C}^n\) is isomorphic to the product, over all \(y \in \text{supp}(x)\), of cycles

\[
\Psi(Z_{j_1} \propto \cdots \propto Z_{j_m}) \subseteq \text{Gr}_m
\]

where \(j_1, \ldots, j_m\) are the indices \(j \in \{1, \ldots, n\}\) such that \(x_j = y\). We remark that if we set \(\lambda_y = (\lambda_{j_1}, \ldots, \lambda_{j_m})\) and \(\mu_y = \mu_{j_1} + \cdots + \mu_{j_m}\), then this cycle belongs to \(\ast \mathfrak{Z}(\lambda_y)_{\mu_y}\). By Proposition 2.2 (i), the dimension of the fiber of \(\Psi(Z_1 \propto \cdots \propto Z_n)\) over \(x\) is therefore

\[
\sum_{y \in \text{supp}(x)} \rho(|\lambda_y| + \mu_y) = \rho(|\lambda| + \mu)
\]

and we conclude that \(\Psi(Z_1 \propto \cdots \propto Z_n)\) has dimension \(\rho(|\lambda| + \mu) + n\).

To finish the proof, we observe that these sets \(\Psi(Z_1 \propto \cdots \propto Z_n)\) cover \(\overline{Gr^\lambda_n \cap \dot{S}_\mu}\) and are not redundant. \(\square\)

Our MV bases are defined with the help of the unstable subsets \(T_\mu\) instead of the stable subsets \(S_\mu\). We can easily adapt the constructions of this subsection to this case by replacing the Borel subgroup \(B^\lor\) with its opposite with respect to \(T^\lor\), and replacing similarly its unipotent radical \(N^\lor\). We shall do this while keeping the notation \(\propto\) and \(\Psi\). Note that when we replace \(\dot{S}_\mu\) by

\[
\dot{T}_\mu = \bigcup_{(\mu_1, \ldots, \mu_n) \in \Lambda^n \atop \mu_1 + \cdots + \mu_n = \mu} \Psi(T_{\mu_1} \propto \cdots \propto T_{\mu_n})
\]

in Proposition 5.4, \(\rho(|\lambda| + \mu) + n\) must be replaced by \(\rho(|\lambda| - \mu) + n\) and the sets \(\ast \mathfrak{Z}(\lambda_j)_{\mu_j}\) must be replaced by their unstarred counterparts.
5.3 The fusion product

For any $x \in \mathbb{C}$, the fibers of $Gr_{BD}^n$ and $Gr_n$ over $(x, \ldots, x)$ are isomorphic to $Gr$ and $Gr_n$, respectively. Thus,

$$Gr_{BD}^n \mid_\Delta \cong \Delta \times Gr \quad \text{and} \quad Gr_n \mid_\Delta \cong \Delta \times Gr_n,$$

where $\Delta$ is the small diagonal, defined as the image of the map $x \mapsto (x, \ldots, x)$ from $\mathbb{C}$ to $\mathbb{C}^n$.

In the other extreme, the morphism $m_n : Gr_n \to Gr_{BD}^n$ is an isomorphism after restriction to the open locus $U \subseteq \mathbb{C}^n$ of points with pairwise different coordinates ([46], Lemma 3.1.23), and by Proposition 5.3, $Gr_n \mid_U$ is isomorphic to $U \times (Gr)^n$. We define maps $\tau$, $i$, $j$ and $\zeta$ according to the diagram below.

Let $\lambda \in (\Lambda^+)^n$ and $\mu \in \Lambda$, set

$$\mathcal{B}(\lambda) = IC\left(\Gr^\lambda_n, \mathbb{C}\right), \quad d = \dim \Gr^\lambda_n = 2\rho(|\lambda|) + n, \quad k = 2\rho(\mu) - n$$

and denote the inclusion $\hat{\nu} \to Gr_n$ by $\hat{i}_\mu$. The next statement is due to Mirković and Vilonen.

**Proposition 5.5**

(i) There are natural isomorphisms

$$i^! \mathcal{B}(\lambda)[n] \cong \tau^! IC\left(\Gr^\lambda_n, \mathbb{C}\right)$$

and

$$j^! \mathcal{B}(\lambda)[n] \cong \zeta^! \left( IC\left(\Gr^\mu_1, \mathbb{C}\right) \boxtimes \cdots \boxtimes IC\left(\Gr^n, \mathbb{C}\right) \right).$$

(ii) Each cohomology sheaf of $(\pi \circ m_n)_* \mathcal{B}(\lambda)$ is a local system on $\mathbb{C}^n$.

(iii) The complex of sheaves $(\pi \circ m_n \circ \hat{i}_\mu)_* (\hat{i}_\mu)^! \mathcal{B}(\lambda)$ is concentrated in degree $k$ and its $k$-th cohomology sheaf is a local system on $\mathbb{C}^n$. 

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Proof. To prove statement (i), one follows the reasoning in [3], sect. 1.7.5, noting that $B(\lambda)$ and $IC(Gr^\lambda_n, \mathbb{C})$ are the sheaves denoted by $(\tau^\circ s_{\lambda_1}) \boxtimes \cdots \boxtimes (\tau^\circ s_{\lambda_n})$ and $s_{\lambda_1} \boxtimes \cdots \boxtimes s_{\lambda_n}$ in loc. cit. Statement (ii) is [39], (6.4). Statement (iii) is contained in the proof of [39], Proposition 6.4, up to a base change in the Cartesian square

\[
\begin{array}{ccc}
\hat{T}_\mu & \xrightarrow{i_\mu} & Gr_n \\
\downarrow & & \downarrow m_n \\
T_\mu(\mathbb{A}^n) & \xrightarrow{k_\mu} & Gr^BD_n
\end{array}
\]

\[\square\]

Combining Propositions 2.1 and 5.5 (i), we see that the total cohomology of the stalk of the complex $(\pi \circ m_n)_* B(\lambda)$ identifies with $F(s_{\lambda_1})$ at any point in $\Delta$, and with $F(s_{\lambda_1}) \otimes \cdots \otimes F(s_{\lambda_n})$ at any point in $U$. Statement (ii) in Proposition 5.5 thus provides the identification

\[F(s_{\lambda}) \cong F(s_{\lambda_1}) \otimes \cdots \otimes F(s_{\lambda_n})\]

required to compare the two bases of $V(\lambda)$. Statement (iii) further identifies the weight spaces

\[F_{\mu}(s_{\lambda}) \cong \bigoplus_{(\mu_1,\ldots,\mu_n) \in \Lambda^n \atop \mu_1 + \cdots + \mu_n = \mu} F_{\mu_1}(s_{\lambda_1}) \otimes \cdots \otimes F_{\mu_n}(s_{\lambda_n}).\]

5.4 Intersection multiplicities

We keep the setup introduced in the previous section, in particular $\lambda \in (\Lambda^+)^n$, $\mu \in \Lambda$, $i_\mu : \hat{T}_\mu \to Gr_n$, $B(\lambda) = IC(Gr^\lambda_n, \mathbb{C})$, $d = \dim Gr^\lambda_n = 2\rho(|\lambda|) + n$, $k = 2\rho(\mu) - n$.

In addition, we denote by

\[L_\mu(\lambda) = \mathcal{H}^k (\pi \circ m_n \circ i_\mu)_* (i_\mu)\; B(\lambda)\]

the local system appearing in Proposition 5.5 (iii).
For each point \( x \in \mathbb{C}^n \), we define maps as indicated below

\[
\begin{array}{cccc}
(Gr_n^\lambda \cap \hat{T}_\mu)_x & \xrightarrow{g'} & (Gr_n^\lambda \cap \hat{T}_\mu)_x & \xrightarrow{g} \to \{x\} \\
\end{array}
\]

where for instance \((Gr_n^\lambda)_x\) is the fiber of \(Gr_n^\lambda\) over \(x\). (The notation \(i\) and \(j\) does not designate the same maps as in the previous subsection.) We then construct the following diagram, referred to as (\(\bigodot\)) in the sequel.

\[
\begin{array}{cccc}
H^k(\hat{T}_\mu, (i_\mu)^! B(\lambda)) & \xrightarrow{\cong} & H^{k+d}(Gr_n^\lambda \cap \hat{T}_\mu, g^! \mathcal{G}_{Gr_n^\lambda}) & \xrightarrow{\cap[Gr_n^\lambda]} \to H_{d-k}^{BM}(Gr_n^\lambda \cap \hat{T}_\mu) \\
\bigodot & & \bigodot & \bigodot \downarrow \bigodot \downarrow \\
H^k((\hat{T}_\mu)_x, (i_\mu)^! i^* B(\lambda)) & \xrightarrow{\cong} & H^{k+d}(Gr_n^\lambda \cap \hat{T}_\mu)_x, g^! \mathcal{G}_{Gr_n^\lambda}) & \xrightarrow{\cap[Gr_n^\lambda]} \to H_{d-k-2n}^{BM}(Gr_n^\lambda \cap \hat{T}_\mu)_x \\
\end{array}
\]

The left vertical arrow in (\(\bigodot\)) is the restriction of the cohomology with support in \(\hat{T}_\mu\) from \(Gr_n\) to \((Gr_n)_x\); in other words, it is the image by the functor \(H^k(\hat{T}_\mu, (i_\mu)^! -)\) of the adjunction morphism \(\mathcal{B}(\lambda) \to i_* i^* \mathcal{B}(\lambda)\). Lemma 5.7 below implies that it is an isomorphism. Likewise, the middle vertical arrow is the restriction from \(Gr_n^\lambda\) to \((Gr_n^\lambda)_x\), afforded by the adjunction morphism \(j^* \mathcal{B}(\lambda) \to h_* h^* j^* \mathcal{B}(\lambda)\).

On the top line, the left arrow is the restriction from \(Gr_n\) to \(Gr_n^\lambda\), fulfilled by the adjunction morphism \(\mathcal{B}(\lambda) \to j_* j^* \mathcal{B}(\lambda) = j_* \mathcal{G}_{Gr_n}[d]\). On the bottom line, it is the restriction from \((Gr_n)_x\) to \((Gr_n^\lambda)_x\), achieved by \(i^* \mathcal{B}(\lambda) \to (j')_*(j')^* i^* \mathcal{B}(\lambda)\). Mirković and Vilonen’s argument (reproduced in sect. 2.3) shows that these two arrows are isomorphisms.

The two paths around the left square in (\(\bigodot\)) are two different expressions for the restriction from \(Gr_n\) to \((Gr_n^\lambda)_x\); therefore this square commutes.

In both lines of (\(\bigodot\)) the right arrow is Alexander duality; we note that \(H_{d-k}^{BM}(Gr_n^\lambda \cap \hat{T}_\mu)\) and \(H_{d-k-2n}^{BM}((Gr_n^\lambda \cap \hat{T}_\mu)_x)\) are the top-dimensional Borel-Moore homology groups.

The map \(h\) is a regular embedding of codimension \(n\). Its orientation class (generalized Thom
class) is an element 
\[ u_x \in H^{2n}\left( (\text{Gr}_n^\lambda)_x \cap G r_n^\lambda \right). \]

The right vertical arrow in (\bigtriangledown) is the cap product with 
\[ g^* u_x \in H^{2n}\left( (\text{Gr}_n^\lambda \cap T_\mu)_x \cap g^! C G r_n^\lambda \cap T_\mu \right), \]
the restriction of \( u_x \) to \( \text{Gr}_n^\lambda \cap T_\mu \).

**Lemma 5.6** In the diagram (\bigtriangledown), the square on the right commutes.

**Proof.** Applying formula IX.4.9 in [24], we get 
\[ u_x \in \left[ (\text{Gr}_n^\lambda) \right]_X. \]

Formula (8) in [15], sect. 19.1 (or formula IX.3.4 in [24]) asserts that given a topological manifold \( X \) and inclusions of closed subsets \( a : A \to X \) and \( b : B \to X \), for any 
\[ \alpha \in H^*(A, a^\!* \mathbb{C}_X), \quad \beta \in H^*(B, b^\!* \mathbb{C}_X) \quad \text{and} \quad C \in H^{BM}_*(X) \]
one has 
\[ (b^* \alpha) \cap (\beta \cap C) = (\alpha \cup \beta) \cap C. \quad (27) \]
Using the six operations formalism, one checks without much trouble that this result is also valid if \( A \) and \( B \) are only locally closed.

Now pick 
\[ \xi \in H^{k+d}\left( \text{Gr}_n^\lambda \cap T_\mu, g^! C G r_n^\lambda \right). \]

Applying (27) twice and using that \( u_x \) has even degree, we compute 
\[ (h^* \xi) \cap \left( u_x \cap [\text{Gr}_n^\lambda] \right) = (\xi \cup u_x) \cap [\text{Gr}_n^\lambda] = (u_x \cup \xi) \cap [\text{Gr}_n^\lambda] = (g^* u_x) \cap \left( \xi \cap [\text{Gr}_n^\lambda] \right). \]

This equality means precisely that \( \xi \) has the same image under the two paths in (\bigtriangledown) that circumscribe the square on the right. \( \square \)

**Lemma 5.7** There are natural isomorphisms 
\[ H^k\left( \hat{T}_\mu, (i_\mu)^\!* \mathcal{B}(\lambda) \right) \cong H^0\left( \mathbb{C}^n, \mathcal{L}_\mu(\lambda) \right) \quad \text{and} \quad H^k\left( (\hat{T}_\mu)_x, (i_\mu^\!* \mathcal{B}(\lambda))_x \right) \cong \left( \mathcal{L}_\mu(\lambda) \right)_x \]
and the left vertical arrow in (\bigtriangledown) is the stalk map \( H^0\left( \mathbb{C}^n, \mathcal{L}_\mu(\lambda) \right) \to \left( \mathcal{L}_\mu(\lambda) \right)_x. \)
Proof. The first isomorphism is

\[ H^0(\mathbb{C}^n, \mathcal{L}_\mu(\lambda)) = H^k(\mathbb{C}^n, (\pi \circ m_n)_* (i_\mu)_* (i_\mu)^! \mathcal{B}(\lambda)) = H^k(\mathcal{T}_\mu, (i_\mu)^! \mathcal{B}(\lambda)). \]

The second one requires the notion of a universally locally acyclic complex (see [9], sect. 5.1). Specifically, \( \mathcal{B}(\lambda) \) is \((\pi \circ m_n)\)-ULA ([42], proof of Proposition IV.3.4, or [43], Lemma 3.20), so there is an isomorphism

\[ i^* \mathcal{B}(\lambda) \to i^! \mathcal{B}(\lambda)[2n]. \]

Then

\[ H^k((\mathcal{T}_\mu)_x, (i_\mu)^! i^* \mathcal{B}(\lambda)) = H^k((\mathcal{T}_\mu)_x, (i_\mu)^! i^! \mathcal{B}(\lambda)[2n]) \]

\[ = H^k(x, (\pi \circ m_n)_* (i_\mu)_* (i_\mu)^! i^! \mathcal{B}(\lambda)[2n]) \]

\[ = H^k(x, (\pi \circ m_n)_* (i_\mu)_* i^! (i_\mu)^! \mathcal{B}(\lambda)[2n]) \]

\[ = H^k(x, (i_0)_* (\pi \circ m_n)_* (i_\mu)_* (i_\mu)^! \mathcal{B}(\lambda)[2n]), \]

the last step being proper base change. Now \((\pi \circ m_n)_* (i_\mu)_* (i_\mu)^! \mathcal{B}(\lambda)\) is the local system \( \mathcal{L}_\mu(\lambda) \) shifted by \(-k\), and therefore

\[ H^k((\mathcal{T}_\mu)_x, (i_\mu)_! i^* \mathcal{B}(\lambda)) = H^0(\{x\}, (i_0)^! \mathcal{L}_\mu(\lambda)[2n]) = H^0(\{x\}, (i_0)^* \mathcal{L}_\mu(\lambda)) = (\mathcal{L}_\mu(\lambda))_x \]

as announced. \( \square \)

By Proposition 5.4, the irreducible components of \( \mathcal{G}_{\lambda_n} \cap \mathcal{T}_\mu \) are all top-dimensional and can be indexed by

\[ \bigsqcup_{(\mu_1, \ldots, \mu_n) \in \Lambda^n} \mathcal{Z}(\lambda_1)_{\mu_1} \times \cdots \times \mathcal{Z}(\lambda_n)_{\mu_n}; \quad (28) \]

namely, to a tuple \( Z = (Z_1, \ldots, Z_n) \) is assigned the component

\[ \mathcal{X}(Z) = \overline{\prod(Z_1 \times \cdots \times Z_n)} \cap \mathcal{G}_{\lambda_n}, \]

the bar denoting closure in \( \mathcal{T}_\mu \). From now on, to lighten the writing, we will substitute \( \mathcal{Z}(\lambda)_\mu \) for the cumbersome compound (28), using implicitly the bijection (2).

The proof of Proposition 5.4 shows that for any \( x \in \mathbb{C}^n \), the irreducible components of the fiber \( \mathcal{G}_{\lambda_n} \cap \mathcal{T}_\mu)_x \) have all the same dimension and can be indexed by \( \mathcal{Z}(\lambda)_\mu \). Let us look more closely at two particular cases.

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If $x \in \mathbb{C}^n$ lies in the open locus $U$ of points with pairwise different coordinates, then, under the bijection $(Gr_n)_x \simeq (Gr)^n$ from Proposition 5.3, the irreducible components of $(Gr^\lambda_n \cap \check{T}_\mu)_x$ are identified with the sets

$$X(Z)_x \simeq (Z_1 \cap Gr^{\lambda_1}) \times \cdots \times (Z_n \cap Gr^{\lambda_n})$$

with $Z = (Z_1, \ldots, Z_n)$ in $\mathcal{Z}(\lambda)_\mu$.

On the other hand, recalling that an element $Z \in \mathcal{Z}(\lambda)_\mu$ is a subset of $Gr^\lambda_n$, we may consider the preimage $\mathcal{Y}(Z)$ of $\Delta \times (Z \cap Gr^\lambda_n)$ under the isomorphism $Gr_n|_\Delta \simeq \Delta \times Gr_n$. Then for any $x \in \Delta$, the irreducible components of the fiber $(Gr^\lambda_n \cap \check{T}_\mu)_x$ are the sets $\mathcal{Y}(Z)_x$ for $Z \in \mathcal{Z}(\lambda)_\mu$.

Let us introduce a last piece of notation before stating the next theorem. In sect. 2.3, we explained the construction of the MV basis of the $\mu$-weight space of $V(\lambda)$; this basis is in bijection with $\mathcal{Z}(\lambda)_\mu$ and we denote by $\langle Z \rangle$ the element indexed by $Z$. On the other hand, given $Z = (Z_1, \ldots, Z_n)$ in $\mathcal{Z}(\lambda_1) \times \cdots \times \mathcal{Z}(\lambda_n)$, we can look at $\langle \langle Z \rangle \rangle = \langle Z_1 \rangle \otimes \cdots \otimes \langle Z_n \rangle$, another element in $V(\lambda)$.

**Theorem 5.8** Let $(Z', Z'') \in (\mathcal{Z}(\lambda)_\mu)^2$. The coefficient $a_{Z', Z''}$ in the expansion

$$\langle \langle Z'' \rangle \rangle = \sum_{Z \in \mathcal{Z}(\lambda)_\mu} a_{Z, Z'} \langle Z \rangle$$

is the multiplicity of $\mathcal{Y}(Z')$ in the intersection product $\mathcal{X}(Z'') \cdot (Gr^\lambda_n)|_\Delta$ computed in the ambient space $Gr^\lambda_n$.

**Proof.** Taking into account Lemma 5.7, the diagram (\bigcirc) can be rewritten as follows.

$$\begin{array}{ccc}
H^0(\mathbb{C}^n, \mathcal{L}_\mu(\lambda)) & \simeq & H^{BM}_{\text{top}}(Gr^\lambda_n \cap \check{T}_\mu) \\
\simeq & & \downarrow (g^* u_x) \cap \\
(\mathcal{L}_\mu(\lambda))_x & \simeq & H^{BM}_{\text{top}}((Gr^\lambda_n \cap \check{T}_\mu)_x)
\end{array}$$

The fundamental classes of the irreducible components of $Gr^\lambda_n \cap \check{T}_\mu$ and $(Gr^\lambda_n \cap \check{T}_\mu)_x$ provide bases of the two Borel-Moore homology groups, both indexed by $\mathcal{Z}(\lambda)_\mu$. In these bases, the right vertical arrow can be regarded as a matrix, say $Q_x$. This matrix can be computed by intersection theory: applying Theorem 19.2 in [15], we see that if $x \in U$ (respectively, $x \in \Delta$), then the entry in $Q_x$ at position $(Z', Z'')$ is the multiplicity of $\mathcal{X}(Z')_x$ (respectively, $\mathcal{Y}(Z')_x$) in the intersection product $\mathcal{X}(Z') \cdot (Gr^\lambda_n)_x$.
computed in the ambient space $\mathcal{G}r_n^\lambda$. Because (29) holds steadily over the open set $U$, we see that $Q_{xU}$ is just the identity matrix.

According to the discussion at the end of sect. 5.3, the geometric Satake correspondence identifies $V(\lambda)_\mu$ with each fiber of the local system $\mathcal{L}^\mu_\lambda$. The basis element $\langle \mathbf{Z} \rangle$ is the fundamental class of $\mathcal{X}(\mathbf{Z})_\mathbf{x}$ when $\mathbf{x} \in \Delta$, and the basis element $\langle \langle \mathbf{Z} \rangle \rangle$ is the fundamental class of $\mathcal{X}(\mathbf{Z})_\mathbf{x}$ when $\mathbf{x} \in U$. Therefore, the coefficient $a_{\mathbf{Z};\mathbf{Z}''}$ in the statement of the theorem is the entry at position $(\mathbf{Z}',\mathbf{Z}'')$ in the product $Q_{x\Delta} \times (Q_{xU})^{-1}$, for any choice of $(x\Delta,xU) \in \Delta \times U$. □

In particular, the entries $a_{\mathbf{Z};\mathbf{Z}''}$ of the transition matrix between our two bases are nonnegative integers.

**Proposition 5.9** In the setup of Theorem 5.8, the diagonal entry $a_{\mathbf{Z};\mathbf{Z}''}$ is equal to one.

**Proof.** Write $\mathbf{Z}'' = (Z_1, \ldots, Z_n)$ in $\mathcal{L}^\lambda(\lambda_1) \times \cdots \times \mathcal{L}^\lambda(\lambda_n)$. By the slice theorem applied to the quotient map $G^\vee(\mathbb{C}[z,z^{-1}]) \to \mathcal{G}r$ (or, in this concrete situation, using Remark 15 and Corollary 5 in [17]), we can find, for each $j \in \{1, \ldots, n\}$, an affine variety $U_j$ and a map $\phi_j : U_j \to G^\vee(\mathbb{C}[z,z^{-1}])$ such that $u \mapsto [\phi_j(u)]$ sends $U_j$ isomorphically to an open subset of $\mathcal{G}r^\lambda_j$ which meets $Z_j$.

For $x \in \mathbb{C}$ and $u \in U_j$, let $\phi_j(u)|_x$ denote the result of substituting $z - x$ for $z$ in $\phi_j(u)$. We can then define an open embedding $\Phi$ as on the diagram

\[
\begin{array}{ccc}
\mathbb{C}^n \times (U_1 \times \cdots \times U_n) & \xrightarrow{\Phi} & \mathcal{G}r_n^\lambda \\
\downarrow & & \downarrow \pi \circ m \\
\mathbb{C}^n & \xrightarrow{\Phi} & \mathbb{C}^n 
\end{array}
\]

by setting

$$
\Phi(x_1, \ldots, x_n; u_1, \ldots, u_n) = (x_1, \ldots, x_n; [\phi_1(u_1)|_{x_1}, \ldots, \phi_n(u_n)|_{x_n}]).
$$

Since intersection multiplicities are of local nature, $a_{\mathbf{Z}';\mathbf{Z}''}$ can be computed after restriction to the image of $\Phi$, where the situation is that of a trivial bundle. □

### 5.5 An example

It is possible to put coordinates on $\mathcal{G}r_n^\lambda$ and to effectively compute the intersection multiplicities mentioned in Theorem 5.8. In this section, we look at the case of the group $G = \text{SL}_3$. We
adopt the usual description \( \Lambda = (\mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2 \oplus \mathbb{Z}\varepsilon_3)/\mathbb{Z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \) of the weight lattice, so that \( V(\varepsilon_1) \) is the defining representation of \( G \) and \( V(-\varepsilon_3) \) is its dual.

We consider the sequence of dominant weights \( \lambda = (\varepsilon_1, -\varepsilon_3) \). The basic MV cycles are \( Z_i = \overline{\text{Gr}^{\varepsilon_1} \cap T_{\varepsilon_i}} \) and \( Z_{-i} = \overline{\text{Gr}^{-\varepsilon_3} \cap T_{-\varepsilon_i}} \) for \( i \in \{1, 2, 3\} \), and with this notation

\[ \mathcal{Z}(\lambda) = \{(Z_i, Z_{-j}) \mid (i, j) \in \{1, 2, 3\}^2\}. \]

To abbreviate, we set \( Z_{i,-j} = (Z_i, Z_{-j}) \). For weight reasons, \( \langle \langle Z_{i,-j} \rangle \rangle = \langle Z_{i,-j} \rangle \) if \( i \neq j \). The rest of the transition matrix between the two bases is given as follows.

\[
\begin{aligned}
\langle \langle Z_{1,-1} \rangle \rangle &= \langle Z_{1,-1} \rangle \\
\langle \langle Z_{2,-2} \rangle \rangle &= \langle Z_{2,-2} \rangle + \langle Z_{1,-1} \rangle \\
\langle \langle Z_{3,-3} \rangle \rangle &= \langle Z_{3,-3} \rangle + \langle Z_{2,-2} \rangle
\end{aligned}
\]

From these relations, we get \( \langle Z_{3,-3} \rangle = \langle Z_{3,-3} \rangle - \langle Z_{2,-2} \rangle + \langle Z_{1,-1} \rangle \). This allows to check that \( \langle Z_{3,-3} \rangle \) is \( G \)-invariant, which in truth is a consequence of the compatibility of the MV basis of \( V(\lambda) \) with the isotypical filtration (Theorem 3.4).

As an example, let us sketch out a computation which justifies that \( \langle Z_{1,-1} \rangle \) appears with coefficient one in \( \langle Z_{2,-2} \rangle \). We consider two charts on \( \text{Gr}^\lambda \), both with \( \mathbb{C}^6 \) as domain:

\[
\begin{align*}
\phi_1 : (x_1, x_2, a, b, c, d) &\mapsto \left( x_1, x_2; \begin{bmatrix} z - x_1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ c & z - x_2 & 0 \\ d & 0 & z - x_2 \end{bmatrix} \right), \\
\phi_2 : (x_1, x_2, a', b', c', d') &\mapsto \left( x_1, x_2; \begin{bmatrix} 1 & 0 & 0 \\ a' & z - x_1 & b' \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} z - x_2 & c' & 0 \\ 0 & 1 & 0 \\ 0 & d' & z - x_2 \end{bmatrix} \right).
\end{align*}
\]

(The matrices here belong to the group \( \text{PGL}_3(\mathbb{C}[z - x_1, z - x_2]) \). One easily computes the transition map between these two charts:

\[
a' = 1/a, \quad b' = -b/a, \quad c' = -a(ac + bd + x_2 - x_1), \quad d' = -ad.
\]

In the chart \( \phi_1 \), the cycle \( \mathcal{Y}(Z_{1,-1}) \) is defined by the equations \( a = b = x_2 - x_1 = 0 \). In the chart \( \phi_2 \), the cycle \( \mathcal{X}(Z_{2,-2}) \) is defined by the equations \( b' = c' = 0 \). Thus, the ideals in \( R = \mathbb{C}[x_1, x_2, a, b, c, d] \) of the subvarieties

\[ V = \phi_1^{-1}(\mathcal{Y}(Z_{1,-1})) \quad \text{and} \quad X = \phi_1^{-1}(\mathcal{X}(Z_{2,-2})) \]

are respectively

\[ p = (a, b, x_2 - x_1) \quad \text{and} \quad q = (b, ac + x_2 - x_1). \]
Since \( q \subseteq p \), we have \( V \subseteq X \); in fact, \( V \) is a subvariety of \( X \) of codimension one. The local ring \( A = \mathcal{O}_{V,X} \) of \( X \) along \( V \) is the localization of \( R/q \) at the ideal \( p/q \). Observing that \( c \) is not in \( p \), we see that its image in \( A \) is invertible, and then that \( x_2 - x_1 \) generates the maximal ideal of \( A \). As a consequence, the order of vanishing of \( x_2 - x_1 \) along \( V \) (see [15], sect. 1.2) is equal to one. By definition, this is the multiplicity of \( Y(Z_{1,-1}) \) in the intersection product \( \mathcal{X}(Z_{2,-2}) \cdot \mathcal{G}_{\lambda_2}^\Delta \).

### 5.6 Factorizations

A nice feature of the Beilinson-Drinfeld Grassmannian is its so-called factorizable structure (see for instance [42], Proposition II.1.13). On the other side of the geometric Satake equivalence, this corresponds to associativity properties of partial tensor products.

Let \( n = (n_1, \ldots, n_r) \) be a composition of \( n \) in \( r \) parts. We define the partial diagonal

\[
\Delta_n = \{ (x_1, \ldots, x_r) \mid (x_1, \ldots, x_r) \in \mathbb{C}^r \}.
\]

We write \( \lambda \) as a concatenation \( (\lambda(1), \ldots, \lambda(r)) \), where each \( \lambda(j) \) belongs to \( (\Lambda^+)^{n_j} \), and similarly we write each \( Z \in \mathcal{X}(\lambda)_\mu \) as \( (Z(1), \ldots, Z(r)) \) with \( Z(j) \in \mathcal{X}(\lambda(j)) \). Then

\[
V(\lambda) = V(\lambda(1)) \otimes \cdots \otimes V(\lambda(r)) \quad \text{and} \quad \langle Z(j) \rangle \in V(\lambda(j)).
\]

Further, define

\[
\mathcal{X}(Z, n) = \Psi(Z_1 \otimes \cdots \otimes Z_n)|_{\Delta_n} \cap \mathcal{G}_{\lambda_n}^\Delta
\]

where the bar means closure in \( (\mathcal{T}_n)|_{\Delta_n} \). These \( \mathcal{X}(Z, n) \) generalize the set \( \mathcal{X}(Z) \) defined in sect. 5.4, as the latter corresponds to the composition \((1, \ldots, 1)\).

Theorem 5.8 can then be extended to this context in a straightforward fashion, as demonstrated by the following statement.

**Proposition 5.10** Let \((Z', Z'') \in (\mathcal{X}(\lambda)_\mu)^2\). The coefficient \( b_{Z', Z''} \) in the expansion

\[
\langle Z'_1 \rangle \otimes \cdots \otimes \langle Z'_{r_1} \rangle = \sum_{Z \in \mathcal{X}(\lambda)_\mu} b_{Z, Z''} \langle Z \rangle
\]

is the multiplicity of \( Y(Z') \) in the intersection product \( \mathcal{X}(Z'', n) \cdot (\mathcal{G}_{\lambda_n}^\Delta)|_{\Delta_n} \) computed in the ambient space \( \mathcal{G}_{\lambda_n}^\Delta|_{\Delta_n} \).

The proof does not require any new ingredient and is left to the reader.
5.7 Triangularity

In this section, we show that the transition matrix described in Theorem 5.8 is unitriangular with respect to an adequate order on $\mathcal{F}(\lambda)_{\mu}$.

**Proposition 5.11** Let $(\mu_1, \ldots, \mu_n)$ and $(\nu_1, \ldots, \nu_n)$ in $\Lambda^n$ and let $S$ be a stratum for the ind-structure of $\mathcal{G}r_n$. If $\Psi(T_{\mu_1} \otimes \cdots \otimes T_{\mu_n})$ meets the closure of $S \cap \Psi(T_{\nu_1} \otimes \cdots \otimes T_{\nu_n})$, then

$$\nu_1 \geq \mu_1, \quad \nu_1 + \nu_2 \geq \mu_1 + \mu_2, \quad \ldots, \quad \nu_1 + \cdots + \nu_n \geq \mu_1 + \cdots + \mu_n.$$ 

**Proof.** Given a tuple $\zeta = (\zeta_1, \ldots, \zeta_n)$ in $(\Lambda/\mathcal{F})^n$, we set

$$\mathcal{G}r_{n, \zeta} = \bigcup_{\lambda \in (\Lambda^+)^n} \mathcal{G}r^n_{\lambda}.$$ 

From equation (25), we deduce that each $\mathcal{G}r_{n, \zeta}$ is closed and connected in the ind-topology; as these subsets form a finite partition of the space $\mathcal{G}r_n$, they are its connected components. We easily verify that a subset of the form $\Psi(T_{\mu_1} \otimes \cdots \otimes T_{\mu_n})$ is contained in $\mathcal{G}r_{n, \zeta}$ if each $\zeta_j$ is the coset of $\mu_j$ modulo $\mathcal{F}$. Therefore, a necessary condition for $\Psi(T_{\nu_1} \otimes \cdots \otimes T_{\nu_n})$ to meet the closure of $S \cap \Psi(T_{\mu_1} \otimes \cdots \otimes T_{\mu_n})$ is that $\mu_j - \nu_j \in \mathcal{F}$ for each $j \in \{1, \ldots, n\}$.

Let $\lambda^\vee \in \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})$ be a dominant integral weight for the group $G^\vee$ and let $V$ be the finite dimensional irreducible representation of $G^\vee$ of highest weight $\lambda^\vee$. Then $G^\vee(\mathbb{C}(z))$ acts on $V \otimes \mathbb{C}(z)$. The standard lattice $L_0 = V \otimes \mathbb{C}[z]$ is left stable by $G^\vee(\mathbb{C}[z])$.

We choose a nonzero linear form $p : V \rightarrow \mathbb{C}$ that vanishes on all weight subspaces of $V$ but the highest weight subspace. Extending the scalars, we regard $p$ as a linear form $V \otimes \mathbb{C}(z) \rightarrow \mathbb{C}(z)$.

For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, we set $f_x(z) = (z - x_1) \cdots (z - x_n)$. Let $S$ be a stratum for the ind-structure of $\mathcal{G}r_n$. There exists a positive integer $k$ such that $f_x^k L_0 \subseteq \beta_1 \cdots \beta_n(L_0) \subseteq f_x^{-k} L_0$ for each $(x_1, \ldots, x_n; [\beta_1, \ldots, \beta_n]) \in S$.

Now we take $(\mu_1, \ldots, \mu_n) \in \Lambda^n$ and $(x_1, \ldots, x_n; [\beta_1, \ldots, \beta_n])$ in $S \cap \Psi(T_{\mu_1} \otimes \cdots \otimes T_{\mu_n})$. Then $p(\beta_1 \cdots \beta_n(L_0))$ is the fractional ideal

$$(z - x_1)^{(\lambda^\vee, \mu_1)} \cdots (z - x_n)^{(\lambda^\vee, \mu_n)} \mathbb{C}[z],$$

and therefore

$$\dim(p(\beta_1 \cdots \beta_n(L_0)) / f_x^k \mathbb{C}[z]) = kn - \langle \lambda^\vee, \mu_1 + \cdots + \mu_n \rangle.$$
If the point \((x_1, \ldots, x_n; [\beta_1, \ldots, \beta_n])\) degenerates to 
\[
(y_1, \ldots, y_n; [\gamma_1, \ldots, \gamma_n]) \in \Psi(T_{\nu_1} \otimes \cdots \otimes T_{\nu_n}),
\]
then
\[
\dim(p(\gamma_1 \cdots \gamma_n(L_0))/f_y^k \mathbb{C}[z]) \leq \dim(p(\beta_1 \cdots \beta_n(L_0))/f_x^k \mathbb{C}[z])
\]
which translates to
\[
\langle \lambda^\vee, \nu_1 + \cdots + \nu_n \rangle \geq \langle \lambda^\vee, \mu_1 + \cdots + \mu_n \rangle.
\]
This inequation holds for any dominant coweight \(\lambda^\vee\), hence \(\nu_1 + \cdots + \nu_n \geq \mu_1 + \cdots + \mu_n\).
This proves the last inequality among those announced in the statement. The other ones can be shown in a similar way, by taking the image under the obvious truncation map \(\mathcal{G}r_n \rightarrow \mathcal{G}r_j\) for each \(j \in \{1, \ldots, n\}\). 

**Corollary 5.12** Adopt the setup of Theorem 5.8. Let \((\mu_1, \ldots, \mu_n)\) and \((\nu_1, \ldots, \nu_n)\) in \(\Lambda^n\) be such that \(Z' \in \mathcal{F}(\lambda_1)_{\nu_1} \times \cdots \times \mathcal{F}(\lambda_n)_{\nu_n}\) and \(Z'' \in \mathcal{F}(\lambda_1)_{\mu_1} \times \cdots \times \mathcal{F}(\lambda_n)_{\mu_n}\). A necessary condition for \(a_{Z';Z''} \neq 0\) is that
\[
\nu_1 \geq \mu_1, \quad \nu_1 + \nu_2 \geq \mu_1 + \mu_2, \quad \ldots, \quad \nu_1 + \cdots + \nu_{n-1} \geq \mu_1 + \cdots + \mu_{n-1}.
\]
We can obtain more stringent conditions regarding the transition matrix by looking at the associativity properties from sect. 5.6. The sharpest result is obtained with a composition \((n_1, n_2)\) of \(n\) in two parts. Accordingly, we write \(\lambda\) as a concatenation \(\lambda(1), \lambda(2)\) and similarly write each \(Z \in \mathcal{F}(\lambda)\) as \((Z(1), Z(2))\). Here \(Z(1)\) is an element in \(\mathcal{F}(\lambda_1) \times \cdots \times \mathcal{F}(\lambda_{n_1})\), but owing to the bijection (2) it can also be regarded as a cycle in \(\mathcal{G}r_{n_1}^{\lambda(1)}\).

**Theorem 5.13** Let \((Z', Z'') \in (\mathcal{F}(\lambda)_{\mu})^2\). Consider the expansion
\[
\langle Z''(1) \rangle \otimes \langle Z''(2) \rangle = \sum_{Z \in \mathcal{F}(\lambda)_{\mu}} b_{Z;Z''} \langle Z \rangle.
\]
If \(b_{Z';Z''} \neq 0\), then either \(Z' = Z''\) or \(Z'(1) \subseteq \overline{Z''(1)}\) as cycles in \(\mathcal{G}r_{n_1}^{\lambda(1)}\). In addition, \(b_{Z';Z''} = 1\).

**Proof.** Let \(Z' = (Z'_1, \ldots, Z'_n)\) and \(Z'' = (Z''_1, \ldots, Z''_n)\) in \(\mathcal{F}(\lambda)_{\mu}\).
For \(j \in \{1, \ldots, n\}\), let \(\mu_j\) be the weight such that \(Z''_j \in \mathcal{F}(\lambda_j)_{\mu_j}\). Using the gallery models from [17] (or Theorem 4.6 and Proposition 4.8 above), we find a nonnegative integer \(d_j\) and
construct a map \( \phi_j : \mathbb{C}^{d_j} \rightarrow N^{-\mathcal{N}}(\mathbb{C}[z, z^{-1}]) \) such that \( \{ [\phi_j(a) z^{\mu_j}] \mid a \in \mathbb{C}^{d_j} \} \) is a dense subset of \( Z''_j \). Then
\[
\phi : (x; a_1, \ldots, a_n) \mapsto \left( x; [\phi_1(a_1)|_{x_1} (z - x_1)^{\mu_1}, \ldots, \phi_n(a_n)|_{x_n} (z - x_n)^{\mu_n}] \right)
\]
maps \( \mathbb{C}^n \times \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n} \) onto a dense subset of \( \Psi(Z''_1 \times \cdots \times Z''_n) \), where \( \phi_j(a_j)|_{x_j} \) means the result of substituting \( z - x_j \) for \( z \) in \( \phi_j(a_j) \).

Assume that \( b_{Z';Z''} \neq 0 \). By Proposition 5.10, \( \mathcal{V}(Z') \) is contained in \( \mathcal{X}(Z''_1, (n_1, n_2)) \), hence in the closure of \( \Psi(Z''_1 \times \cdots \times Z''_n)|_{\Delta(n_1, n_2)} \).

Take a point in \( Z'(1) \cap \text{Gr}^{\lambda(1)}_{n_1} \), written as \([g_1, \ldots, g_{n_1}]\) where each \( g_j \) is in \( G^\nu(\mathbb{C}[z, z^{-1}]) \). We can complete this datum to get an element
\[
\Gamma = (0, \ldots, 0; [g_1, \ldots, g_{n_1}])
\]
of \( \mathcal{V}(Z') \). Working in the analytic topology for expositional simplicity, we see that \( \Gamma \) is the limit of a sequence \( \left( \phi(x_p; a_{1,p}, \ldots, a_{n,p}) \right)_{p \in \mathbb{N}} \) with \( x_p \in \Delta(n_1, n_2) \) and \( (a_{1,p}, \ldots, a_{n,p}) \in \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n} \).

We write
\[
x_p = \left( x_{1,p}, \ldots, x_{1,p}, x_{2,p}, \ldots, x_{2,p} \right) \quad \text{with of course} \quad \lim_{p \to \infty} x_{1,p} = \lim_{p \to \infty} x_{2,p} = 0. \quad (30)
\]

Then
\[
[g_1, \ldots, g_{n_1}] = \lim_{p \to \infty} \left[ \phi_1(a_{1,p}) z^{\mu_1}, \ldots, \phi_{n_1}(a_{1,n_1}) z^{\mu_{n_1}} \right]_{x_{1,p}} = \lim_{p \to \infty} \left[ \phi_1(a_{1,p}) z^{\mu_1}, \ldots, \phi_{n_1}(a_{1,n_1}) z^{\mu_{n_1}} \right]
\]
is the limit of a sequence of points in \( Z''(1) \). Therefore \( Z'(1) \cap \text{Gr}^{\lambda(1)}_{n_1} \subseteq \overline{Z''(1)} \), whence the inclusion \( Z'(1) \subseteq \overline{Z''(1)} \).

In addition to \( b_{Z';Z''} \neq 0 \), assume that the latter inclusion is an equality. Then \( Z'(1) = Z''(1) \) because these two MV cycles are irreducible components of the same \( \text{Gr}^{\lambda(1)}_{n_1} \cap (m_{n_1})^{-1}(T_{\mu(1)}) \), with indeed \( \mu(1) = \mu_1 + \cdots + \mu_{n_1} \). We regard \( Z'(2) \) and \( Z''(2) \) as cycles in \( \text{Gr}^{\lambda(2)}_{n_2} \). Take a point in \( Z'(2) \cap \text{Gr}^{\lambda(2)}_{n_2} \), written as \([g_{n_1+1}, \ldots, g_n]\) where each \( g_j \) is in \( G^\nu(\mathbb{C}[z, z^{-1}]) \). We can then look at the element
\[
\Gamma = (0, \ldots, 0; [z^{\mu_1}, \ldots, z^{\mu_{n_1}}, g_{n_1+1}, \ldots, g_n])
\]

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of \( \mathcal{Y}(Z') \). Again \( \Gamma \) is the limit of a sequence \((\phi(x_p; a_{1,p}, \ldots, a_{n,p}))_{p \in \mathbb{N}} \) with \( x_p \in \Delta_{(n_1, n_2)} \) and \((a_{1,p}, \ldots, a_{n,p}) \in \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n} \). We set
\[
B_p = z^{-\mu(1)} \phi_1(a_{1,p}) z^{\mu_1} \cdots \phi_{n_1}(a_{n_1,p}) z^{\mu_{n_1}}.
\]
Writing again (30), we have
\[
L_{\mu(1)} = \lim_{p \to \infty} [z^{\mu(1)} B_p]_{x_1,p} = \lim_{p \to \infty} [z^{\mu(1)} B_p]
\]
and
\[
z^{\mu(1)}[g_{n_1+1}, \ldots, g_n] = \lim_{p \to \infty} (z^{\mu(1)} B_p)_{x_1,p} [\phi_{n_1+1}(a_{n_1+1,p}) z^{\mu_{n_1+1}}, \ldots, \phi_n(a_{n,p}) z^{\mu_n}]_{x_2,p}.
\]
Let \( K \) be the kernel of the evaluation map \( N^{-\vee}(\mathbb{C}[z^{-1}]) \to N^{-\vee}(\mathbb{C}) \) at \( z = \infty \). The multiplication induces a bijection
\[
K \times N^{-\vee}(\mathbb{C}[z]) \xrightarrow{\sim} N^{-\vee}(\mathbb{C}[z, z^{-1}]).
\]
We decompose \( B_p \) as a product \( B_{-p}B_{+p} \) according to this bijection. Using (31) and identifying the ind-variety \( T_0 \) with \( K \), we obtain that \( B_{-p} \) tends to one when \( p \) goes to infinity. Inserting this information in (32), we obtain
\[
[g_{n_1+1}, \ldots, g_n] = \lim_{p \to \infty} B_{+p} [\phi_{n_1+1}(a_{n_1+1,p}) z^{\mu_{n_1+1}}, \ldots, \phi_n(a_{n,p}) z^{\mu_n}],
\]
so \([g_{n_1+1}, \ldots, g_n]\) is the limit of a sequence of points in \( Z''_{(2)} \). We conclude that \( Z'_{(2)} \subseteq \overline{Z''_{(2)}} \), and since these two cycles have the same dimension, that actually \( Z'_{(2)} = \overline{Z''_{(2)}} \).

To sum up: if \( b_{Z',Z''} \neq 0 \), then \( Z'_{(1)} \subseteq \overline{Z''_{(1)}} \), and in case of equality \( Z'_{(1)} = \overline{Z''_{(1)}} \), we additionally have \( Z'_{(2)} = Z''_{(2)} \). This proves the first statement in the theorem. The second one is proved in the same manner as Proposition 5.9. \( \Box \)

**Remark 5.14.** Using Theorem 5.13, one easily sharpens Corollary 5.12: with the notation of the latter, if \( a_{Z',Z''} \neq 0 \), then either \( Z' = Z'' \) or one of the displayed inequalities is strict. The proof is left to the reader.

**Application to standard monomial theory.**

Let \( \lambda \in \Lambda^+ \) and let \( \ell \subseteq V(\lambda)^* \) be the line spanned by the highest weight vectors. The group \( G \) acts on the projective space \( \mathbb{P}(V(\lambda)^*) \); let \( Q \) be the stabilizer of \( \ell \), a parabolic subgroup of \( G \). The map \( g \mapsto g\ell \) induces an embedding of the partial flag variety \( X = G/Q \) in \( \mathbb{P}(V(\lambda)^*) \). We
denote by $\mathcal{L}$ the pull-back of the line bundle $\mathcal{O}(1)$ by this embedding; then the homogeneous coordinate ring of $X$ is

$$R_\lambda = \bigoplus_{m \geq 0} H^0(X, \mathcal{L}^\otimes m);$$

here $H^0(X, \mathcal{L}^\otimes m)$ is isomorphic to $V(m\lambda)$ and the multiplication in $R_\lambda$ is given by the projection onto the Cartan component

$$V(m\lambda) \otimes V(n\lambda) \to V((m+n)\lambda).$$

The algebra $R_\lambda$ is endowed with an MV basis, obtained by gathering the MV bases of the summands $V(m\lambda)$. Each MV cycle $Z \in \mathcal{Z}(\lambda)$ defines a basis element $\langle Z \rangle \in V(\lambda)$. Given an $m$-tuple $Z = (Z_1, \ldots, Z_m)$ of elements of $\mathcal{Z}(\lambda)$, the product $\langle Z_1 \rangle \cdots \langle Z_m \rangle$ in the algebra $R_\lambda$ is the image of $\langle \langle Z \rangle \rangle = \langle Z_1 \rangle \otimes \cdots \otimes \langle Z_m \rangle$ under the projection $V(\lambda)^\otimes m \to V(m\lambda)$. This product is called standard if $Z$ lies in the Cartan component of the crystal $\mathcal{Z}(\lambda)^\otimes m$.

Remark 3.5 implies that the MV basis element $\langle Z \rangle \in V(\lambda)^\otimes m$ goes, under the projection $V(\lambda)^\otimes m \to V(m\lambda)$, either to an element in the MV basis of $V(m\lambda)$ or to 0, depending on whether $Z$ lies or not in the Cartan component of $\mathcal{Z}(\lambda)^\otimes m$.

Using Corollary 5.12 and Remark 5.14, we can then endow, for each degree $m$, the Cartan component of $\mathcal{Z}(\lambda)^\otimes m$ with an order, so that the transition matrix expressing the standard monomials in the MV basis of $R_\lambda$ is unitriangular. In particular, the standard monomials form a basis for the algebra $R_\lambda$ too, and straightening laws can be obtained from Theorem 5.8.

The dual of the MV basis is compatible with the Demazure modules contained in $V(m\lambda)^*$; this property is recorded as Remark 2.6 (ii) in [2] but the crux of the argument is due to Kashiwara [27]. This implies that for any Schubert variety $Y \subseteq X$, the kernel of the restriction map

$$\bigoplus_{m \geq 0} H^0(X, \mathcal{L}^\otimes m) \to \bigoplus_{m \geq 0} H^0(Y, \mathcal{L}^\otimes m)$$

is spanned by a subset of the MV basis of $R_\lambda$. Therefore the homogeneous coordinate ring of $Y$ is also endowed with an MV basis.

These observations suggest that the MV basis could be a relevant tool for the study of the standard monomial theory.

### 5.8 A conjectural symmetry

Recall the notation set up in sects. 3.1–3.2. Given $\lambda \in \Lambda^+$, we set $\lambda^* = -w_0\lambda$, where as usual $w_0$ denotes the longest element in the Weyl group $W$. As is well known, there exists a
unique bijection $\sigma : B(\lambda) \to B(\lambda^*)$ which for each $i \in I$ exchanges the actions of $\tilde{e}_i$ and $\tilde{f}_i$.

In our context, we will regard $\sigma$ as a bijection $\mathcal{Z}(\lambda) \to \mathcal{Z}(\lambda^*)$ and may define it by means of Lemma 2.1 (e) in [33] and Theorem 4.9.

Now let $n \geq 1$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ in $(\Lambda^+)^n$. We set $\lambda^* = (\lambda_n^*, \ldots, \lambda_1^*)$ and define a bijection $\sigma : \mathcal{Z}(\lambda_1) \times \cdots \times \mathcal{Z}(\lambda_n) \to \mathcal{Z}(\lambda_n^*) \times \cdots \times \mathcal{Z}(\lambda_1^*)$ by $\sigma(Z_1, \ldots, Z_n) = (\sigma(Z_n), \ldots, \sigma(Z_1))$. (Using the same symbol $\sigma$ to denote different bijections is certainly abusive, but adding extra indices to disambiguate would overload the notation without clear benefit.) The Cartesian products above are in fact tensor product of crystals, and here again $\sigma$ exchanges the actions of $\tilde{e}_i$ and $\tilde{f}_i$ for each $i \in I$ ([23], Theorem 2).

Let $\mu \in \Lambda$ and choose $(Z', Z'') \in (\mathcal{Z}(\lambda))_\mu^2$; we then obtain $\sigma(Z')$ and $\sigma(Z'')$ in $\mathcal{Z}(\lambda^*)_{-\mu}$. Recall the notation introduced in Theorem 5.8 to denote the entries of the transition matrix between the two bases of $V(\lambda)$ and adopt a similar notation as regards $V(\lambda^*)$.

**Conjecture 5.15.** The equality $a_{Z', Z''} = a_{\sigma(Z'), \sigma(Z'')}$ holds.

According to [11], this conjecture is true in type $A_1$. Its general validity would have two interesting consequences.

Firstly, one could then strengthen Theorem 5.13; indeed $b_{Z', Z''} \neq 0$ would imply not only $Z'_{(1)} \subset \overline{Z''_{(1)}}$, but also $\sigma(Z'_{(2)}) \subset \sigma(Z''_{(2)})$, restoring the symmetry between the two tensor factors.

Secondly, the MV basis of an irreducible representation $V(\lambda)$ would then satisfy the analogue of [37], Proposition 21.1.2. In fact, one easily verifies that the MV basis enjoys this property if $\lambda$ is minuscule or quasi-minuscule; our conjecture would allow to deduce the general case by taking suitable tensor products, mimicking the strategy of proof from [40].

### 6 The basis on the invariant subspace

Let $n \geq 1$ and let $\lambda \in (\Lambda^+)^n$. The MV basis of $V(\lambda)$ is compatible with the isotypical filtration, hence provides a basis of the invariant subspace $V(\lambda)^G$, called the Satake basis in [12]. In this section we study two properties of this basis.
6.1 Cyclic permutations

Let us write \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and consider the rotated sequence \( \lambda^{[1]} = (\lambda_2, \ldots, \lambda_n, \lambda_1) \). Thus,

\[ V(\lambda) = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n) \quad \text{and} \quad V(\lambda^{[1]}) = V(\lambda_2) \otimes \cdots \otimes V(\lambda_n) \otimes V(\lambda_1). \]

The signed cyclic permutation

\[ x_1 \otimes \cdots \otimes x_n \mapsto (-1)^{2\rho(\lambda_1)} x_2 \otimes \cdots \otimes x_n \otimes x_1, \]

defines an isomorphism of \( G \)-modules \( R : V(\lambda) \to V(\lambda^{[1]}) \). In particular, \( R \) induces a linear bijection between the invariant subspaces.

**Theorem 6.1** The signed cyclic permutation \( R \) maps the Satake basis of \( V(\lambda)^G \) to the Satake basis of \( V(\lambda^{[1]})^G \).

Theorem 6.1 replicates a similar result for the dual canonical basis due to Lusztig ([37], 28.2.9), and our proof below mirrors Lusztig’s argument. It has been proved by Fontaine, Kamnitzer and Kuperberg in the case where all the weights \( \lambda_j \) are minuscule ([12], Theorem 4.5). The bijection induced by \( R \) between the two Satake bases has a nice interpretation, both in terms of crystals (see [13]) and in terms of cluster combinatorics (see [22], sect. 2.1.6).

The rest of this section is devoted to the proof of Theorem 6.1.

As in sect. 3, we denote by \( \{\alpha_i \mid i \in I\} \) the set of simple roots and choose simple root vectors \( e_i \) and \( f_i \) in the Lie algebra of \( G \) of weights \( \pm \alpha_i \) such that \( [e_i, f_i] = -\alpha_i^\vee \). The Weyl group \( W \) is generated by the simple reflections \( s_i \) and contains a longest element \( w_0 \).

Given \( \lambda \in \Lambda^+ \) and \( w \in W \), we can pick a reduced word \( (i_1, \ldots, i_\ell) \) of \( w \) and form the product of divided powers

\[ \theta(w, \lambda) = f_{i_1}^{(n_1)} \cdots f_{i_\ell}^{(n_\ell)}, \quad \text{where} \quad n_j = \langle \alpha_{i_j}^\vee, s_{i_{j+1}} \cdots s_{i_\ell} \lambda \rangle. \]

This element does not depend on the choice of \( (i_1, \ldots, i_\ell) \) ([37], Proposition 28.1.2), which legitimizes the notation. We note that \( \theta(w_0, \lambda) \) acts on \( V(\lambda) \) by mapping highest weight vectors to lowest weight vectors.

We set \( \lambda = \lambda_1 \), the first element in the sequence \( \lambda \). With the notation of sect. 5.4, the highest and lowest weight elements in the MV basis of \( V(\lambda) \) are

\[ v_\lambda = \langle \{L_\lambda\} \rangle \quad \text{and} \quad v_{w_0 \lambda} = \langle \text{Gr}^\lambda \rangle. \]
Under suitable normalizations in the geometric Satake equivalence, these two elements are related by \( v_{w_0 \lambda} = \theta(w_0, \lambda) \cdot v_\lambda \) (see [2], Theorem 5.2 and Remark 2.10). We define \( \xi \) to be the linear form on \( V(\lambda) \) such that \( \langle \xi, v_\lambda \rangle = 1 \) and that vanishes on all weight subspaces of weight different from \( \lambda \). Similarly, we define \( \eta \) to be the linear form on \( V(\lambda) \) such that \( \langle \eta, v_{w_0 \lambda} \rangle = 1 \) and that vanishes on all weight subspaces of weight different from \( w_0 \lambda \).

Let \( M \) be a representation of \( G \). The assignment \( v \otimes m \mapsto (-1)^{2\rho(\lambda)} m \otimes v \) defines an isomorphism \( P : V(\lambda) \otimes M \to M \otimes V(\lambda) \).

We set \( \lambda^* = -w_0 \lambda \). Let \( M^\circ \) be the isotypical component of \( M \) corresponding to the highest weight \( \lambda^* \), namely, the sum of all subrepresentations isomorphic to \( V(\lambda^*) \). Given a weight \( \mu \in \Lambda \), we denote by \( M_\mu \) the corresponding weight subspace of \( M \) and set \( M_\mu^\circ = M^\circ \cap M_\mu \). Then \( M_\mu^\circ \) is the set of all vectors in \( M_{\lambda^*} \) that are annihilated by all the root vectors \( e_i \) and \( M_{w_0 \lambda^*}^\circ \) is the set of all vectors in \( M_{w_0 \lambda^*} \) that are annihilated by all the root vectors \( f_i \).

**Lemma 6.2** The following diagram commutes and consists of linear bijections.

\[
\begin{array}{ccc}
(V(\lambda) \otimes M)^G & \xrightarrow{P} & (M \otimes V(\lambda))^G \\
\eta \otimes \text{id}_M & \downarrow & \text{id}_M \otimes \xi \\
M_\lambda^\circ & \xrightarrow{\theta(w_0, \lambda^*)} & M_{w_0 \lambda^*}^\circ
\end{array}
\]

\((33)\)

**Proof.** By additivity, we can reduce to the case where \( M \) is a simple representation. If \( M \) is not isomorphic to the dual of \( V(\lambda) \), then all four spaces are zero and the statement is banal. We therefore assume that \( M \cong V(\lambda^*) \); in this case, all four spaces are one dimensional.

Let \( m_{\lambda^*} \) be a highest weight vector in \( M \) and set \( m_{w_0 \lambda^*} = \theta(w_0, \lambda^*) \cdot m_{\lambda^*} \). There exists a unique \( G \)-invariant bilinear form \( \Phi : V(\lambda) \times M \to \mathbb{C} \) such that \( \Phi(v_{w_0 \lambda}, m_{\lambda^*}) = 1 \). This form \( \Phi \) is non-degenerate and a standard computation gives \( \Phi(v_\lambda, m_{w_0 \lambda^*}) = (-1)^{2\rho(\lambda)} \).

The assignment \( v \otimes m \mapsto \Phi(v, ?) m \) defines a \( G \)-equivariant isomorphism \( V(\lambda) \otimes M \to \text{End}(M) \). The preimage \( x \) of \( \text{id}_M \) by this bijection spans the vector space \( (V(\lambda) \otimes M)^G \). By construction, \( (\eta \otimes \text{id}_M)(x) = m_{\lambda^*} \) and \( (\xi \otimes \text{id}_M)(x) = (-1)^{2\rho(\lambda)} m_{w_0 \lambda^*} \). Thus, both paths around the diagram map \( x \) to \( m_{w_0 \lambda^*} \). \( \square \)

We take \( M = V(\lambda_2) \otimes \cdots \otimes V(\lambda_n) \). We define \( M^\bullet \) to be the step in the isotypical filtration of \( M \) where the component \( M^\circ \) is appended to smaller ones. There is a natural quotient map \( p : M^\bullet \to M^\circ \).

We set \( M = \mathcal{L}(\lambda_2) \times \cdots \times \mathcal{L}(\lambda_n) \). Using the notation introduced in sect. 5.4, the MV basis of \( M \) consists of elements \( \langle Z \rangle \) for \( Z \in M \). Let \( M^\bullet \) be the set of all \( Z \in M \) such that \( \langle Z \rangle \in M^\bullet \);
since MV bases are $L$-perfect, $\{ (Z) \mid Z \in \mathcal{M}^\bullet \}$ is a basis of $M^\bullet$. Let $\mathcal{M}^\circ$ be the set of all $Z \in \mathcal{M}^\bullet$ such that $(Z) \notin \ker \rho$; then $\{ p((Z)) \mid Z \in \mathcal{M}^\circ \}$ is a basis of $M^\circ$. In consequence, each weight subspace of $M^\circ$ is endowed with a basis.

As a crystal, $\mathcal{M}$ decomposes as the disjoint union (direct sum) of its connected components, and $\mathcal{M}^\circ$ is the union of the connected components of $\mathcal{M}$ that are isomorphic to $\mathcal{Z}(\lambda^*)$. For each connected component $\mathcal{C} \subseteq \mathcal{M}^\circ$, the subspace of $M^\circ$ spanned by $B_{\mathcal{C}} = \{ p((Z)) \mid Z \in \mathcal{C} \}$ is a subrepresentation isomorphic to $V(\lambda^*)$, and by Remark 3.5, $B_{\mathcal{C}}$ identifies with the MV basis of $V(\lambda^*)$. The action of $\theta(w_{0}, \lambda^*)$ therefore maps the highest weight element in $B_{\mathcal{C}}$ to the lowest element in $B_{\mathcal{C}}$. We conclude that the bottom horizontal arrow in (33) maps the basis of $M_{\lambda^*}^\circ$ to the basis of $M_{w_{0}\lambda^*}^\circ$.

Each element in the MV basis of $V(\lambda^{[1]}) = M \otimes V(\lambda)$ is of the form $(Z)$, with $Z$ in $\mathcal{Z}(\lambda^{[1]}) = \mathcal{M} \otimes \mathcal{Z}(\lambda)$. Let $V(\lambda) \neq \lambda$ be the sum of all the weight subspaces of $V(\lambda)$ other than the higher weight subspace. Theorem 5.13 implies that for each $Z_{(1)} \in \mathcal{M}$,

$$\langle Z_{(1)} \rangle \otimes \langle \{ L_{\lambda} \} \rangle \equiv \langle \{ (Z_{(1)}, L_{\lambda}) \} \rangle \pmod{M \otimes V(\lambda) \neq \lambda}.$$

Thus, for $Z_{(1)} \in \mathcal{M}$ and $Z = (Z_{(1)}, \{ L_{\lambda} \})$, we have $(id_{M} \otimes \xi)(\langle Z \rangle) = \langle Z_{(1)} \rangle$.

As evidenced by the crystal structure on $\mathcal{M} \otimes \mathcal{Z}(\lambda)$, the Satake basis of the space $(M \otimes V(\lambda))^G$ consists of the vectors $\langle Z \rangle$ for the pairs $Z = (Z_{(1)}, \{ L_{\lambda} \})$ such that $Z_{(1)} \in \mathcal{M}_{\lambda^*}^\circ_{\lambda}$. Noting that $\langle Z_{(1)} \rangle \in M_{w_{0}\lambda^*}^\circ$ for those $Z_{(1)}$, we conclude that the right vertical arrow in (33) maps basis elements to basis elements.

Similarly, the left vertical arrow in (33) maps the Satake basis of $(V(\lambda) \otimes M)^G$ to the basis of $M_{\lambda^*}^\circ$. Lemma 6.2 then concludes the proof of Theorem 6.1.

### 6.2 Tensor product with an invariant element

Let $(n', n'')$ be a composition of $n$ in two parts. Correspondingly, we write $\lambda \in (\Lambda^+)^{n}$ as a concatenation $\langle \lambda', \lambda'' \rangle$ and view each element in $\mathcal{Z}(\lambda)$ as a pair $(Z', Z'') \in \mathcal{Z}(\lambda') \times \mathcal{Z}(\lambda'')$.

The following proposition implies that the Satake basis of the invariant subspace of $V(\lambda)$ satisfies the second item in Khovanov and Kuperberg’s list of properties for the dual canonical basis (see the introduction of [30]).

**Proposition 6.3** Let $(Z', Z'') \in \mathcal{Z}(\lambda)$. If $\langle Z' \rangle \in V(\lambda')^G$, then $\langle Z' \rangle \otimes \langle Z'' \rangle = \langle (Z', Z'') \rangle$.

**Proof.** Let $Z' \in \mathcal{Z}(\lambda')$. Recall the map $m_{\mu} : \text{Gr}_{\mu} \rightarrow \text{Gr}$ defined in sect. 2.2 and the notation $\mu_I$ from sect. 3.4 and set $\mu = \mu_I(Z')$. Then $m_{\mu}(Z') \subseteq \text{Gr}_{\mu}$ and $\langle Z' \rangle$ appear in the isotypical filtration of $V(\lambda')$ at the step where the component of type $V(\mu)$ is appended.
If $\langle Z' \rangle \in V(\lambda')^G$, then $\mu = 0$, accordingly $\overline{\text{Gr}^p} = \{L_0\}$, and as a result

$$\overline{Z} \subseteq (m_{n'})^{-1}(\{L_0\}) \subseteq (m_{n'})^{-1}(T_0).$$

This implies that no MV cycle in $\mathcal{Z}(\lambda')$ can be strictly contained in $\overline{Z'}$. (Such a cycle would be contained in $(m_{n'})^{-1}(T_0)$, so would be an irreducible component of $\overline{\text{Gr}^p} \cap (m_{n'})^{-1}(T_0)$, and would end up having dimension $\rho(|\lambda'|)$, the same as $Z'$.) The desired result now directly follows from Theorem 5.13. □

7 Applications to the MV basis of $\mathbb{C}[N]$

We adopt the notation set up in the preamble of sect. 3. Let $N$ be the unipotent radical of the Borel subgroup $B$ and let $\mathbb{C}[N]$ be the algebra of regular functions on $N$. At the expense of an isogeny, which does not alter $N$, we can assume that $G$ is simply-connected.

For each dominant weight $\lambda \in \Lambda^+$, we can choose a highest weight vector $v_\lambda$ in the representation $V(\lambda)$ and define the linear form $v_\lambda^* : V(\lambda) \to \mathbb{C}$ such that $\langle v_\lambda^*, v_\lambda \rangle = 1$ and $\langle v_\lambda^*, v \rangle = 0$ for all weight vectors $v$ of weight other than $\lambda$. This yields an embedding $\Psi_\lambda : v \mapsto \langle v_\lambda^*, v \rangle$ of $V(\lambda)$ into $\mathbb{C}[N]$, where $\langle v_\lambda^*, v \rangle$ stands for the function $n \mapsto \langle v_\lambda^*, nv \rangle$. The MV bases of the representations $V(\lambda)$ can be transported to $\mathbb{C}[N]$ through these maps $\Psi_\lambda$, and they glue together to form a basis of $\mathbb{C}[N]$, which we call the MV basis of $\mathbb{C}[N]$ (see [2]).

The algebra $\mathbb{C}[N]$ comes with several remarkable bases: the MV basis, subject of our current investigation, but also the dual canonical basis of Lusztig/upper global basis of Kashiwara, and (in simply laced type) the dual semicanonical basis. The theory of cluster algebras was developed in order to compute effectively these bases (or at least, the dual canonical basis). Concretely, the cluster structure of $\mathbb{C}[N]$ allows to define specific elements, called cluster monomials, which are linearly independent and easily amenable to calculations. It is known that both the dual canonical and the dual semicanonical bases contain all the cluster monomials [20, 25], but also that these bases differ (except when cluster monomials span $\mathbb{C}[N]$).

The methods developed in sect. 5 allow to effectively compute products of elements of the MV basis of $\mathbb{C}[N]$. This allows us to prove that this basis contains quite a few cluster monomials (Proposition 7.2) and that it generally differs from both the dual canonical and the dual semicanonical bases (Proposition 7.3).
7.1 Cluster monomials

As explained in [21], each reduced word \((i_1, \ldots, i_\ell)\) of the longest element \(w_0\) in the Weyl group \(W\) yields a standard seed of the cluster structure of \(\mathbb{C}[N]\). The main result of this section, Proposition 7.2, claims that sometimes the cluster monomials built from the variables of a standard seed belong to the MV basis of \(\mathbb{C}[N]\).

Set \(t^\vee = \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{R})\) and let \(C = \{ x \in t^\vee \ | \ \forall i \in I, \langle x, \alpha_i \rangle > 0 \}\) be the Weyl chamber in \(t^\vee\). We consider the following condition about a reduced word \((i_1, \ldots, i_\ell)\):

\[
(A) \text{ There exist } x_1 \in s_{i_1}(C), x_2 \in (s_{i_1}s_{i_2})(C), \ldots, x_\ell \in (s_{i_1} \cdots s_{i_\ell})(C) \text{ such that } x_k - x_{k+1} \in C \text{ for each } k \in \{1, \ldots, \ell - 1\}.
\]

For instance, choose \((x, y) \in C^2\) in such a way that the straight line joining \(x\) to \(-y\) avoids all the two-codimensional faces of the Weyl fan in \(t^\vee\). List in order the chambers successively crossed by this line: \(C, s_{i_1}C, (s_{i_1}s_{i_2})(C), \ldots\). The word \((i_1, i_2, \ldots)\) produced in this manner is then reduced and obviously satisfies condition \((A)\).

Let \(Q \subseteq \Lambda\) be the root lattice. We denote by \(Q_+\) the positive cone in \(Q\) with respect to the dominance order; in other words, \(Q_+\) is the set of all linear combinations of the simple roots \(\alpha_i\) with non-negative integral coefficients. We set \(Q_- = -Q_+\).

**Lemma 7.1** Let \((i_1, \ldots, i_\ell)\) be a reduced word, set \(w_k = s_{i_1} \cdots s_{i_k}\) for \(k \in \{1, \ldots, \ell\}\), and let \((\nu_1, \ldots, \nu_\ell) \in w_1(Q_-) \times \cdots \times w_\ell(Q_-)\). Assume that \(\nu_1 + \cdots + \nu_\ell \in Q_+\) for all \(k \in \{1, \ldots, \ell - 1\}\), that \(\nu_1 + \cdots + \nu_\ell = 0\), and that \((i_1, \ldots, i_\ell)\) satisfies condition \((A)\). Then \(\nu_1 = \cdots = \nu_\ell = 0\).

**Proof.** We set \(\mu_0 = 0\) and \(\mu_k = \nu_1 + \cdots + \nu_k\) for \(k \in \{1, \ldots, \ell\}\). We pick \(x_1, \ldots, x_\ell\) as stated in condition \((A)\). Then

\[
\sum_{k=1}^{\ell} \langle x_k, \nu_k \rangle = \sum_{k=1}^{\ell} \langle x_k, \mu_k - \mu_{k-1} \rangle = \sum_{k=1}^{\ell - 1} \langle x_k - x_{k+1}, \mu_k \rangle.
\]

From \(x_k \in w_k(C)\) and \(\nu_k \in w_k(Q_-)\), we deduce that \(\langle x_k, \nu_k \rangle \leq 0\) for each \(k \in \{1, \ldots, \ell\}\). On the other hand, from \(x_k - x_{k+1} \in C\) and \(\mu_k \in Q_+\), we deduce that \(\langle x_k - x_{k+1}, \mu_k \rangle \geq 0\) for each \(k \in \{1, \ldots, \ell - 1\}\). We conclude that each \(\langle x_k, \nu_k \rangle\) is indeed zero, which implies \(\nu_k = 0\). \(\square\)

In sect. 6.1, we defined, for each \((\lambda, w) \in \Lambda^+ \times W\), a product \(\theta(\lambda, w)\) of divided powers of the root vectors \(f_i\). We can then set \(v_{w\lambda} = \theta(\lambda, w) \cdot v_{\lambda}\); this is a vector of weight \(w\lambda\) in \(V(\lambda)\). We define \(\Delta_{\lambda,ww} = \Psi_\lambda(v_{w\lambda})\), usually called a flag minor if \(\lambda\) is minuscule. We denote by \(\{\varpi_i \ | \ i \in I\}\) the set of fundamental weights.

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**Proposition 7.2** Let \((i_1, \ldots, i_\ell)\) be a reduced word and define \(x_k = \Delta_{\omega_{i_k}, s_{i_1} \cdots s_k \omega_{i_k}}\) for each \(k \in \{1, \ldots, \ell\}\). If \((i_1, \ldots, i_\ell)\) satisfies condition (A), then any monomial in \(x_1, \ldots, x_\ell\) belongs to the MV basis of \(\mathbb{C}[N]\).

**Proof.** We choose \(\lambda = (\lambda_1, \ldots, \lambda_\ell)\) in \((\Lambda^+)^\ell\). For \(k \in \{1, \ldots, \ell\}\), we set \(w_k = s_{i_1} \cdots s_{i_k}\). The extremal weight vector \(v_{w_k \lambda_k} \in V(\lambda_k)\) belongs to the MV basis ([2], Remark 2.10 and Theorem 5.2), so \(v_{w_k \lambda_k} = \langle Z_k \rangle\) where \(Z_k\) is the cycle \(\mathbb{G}_{\nu_k}^{\lambda_k} \cap T_{w_k \lambda_k}\). We set \(\mu = w_1 \lambda_1 + \cdots + w_\ell \lambda_\ell\) and \(Z = (Z_1, \ldots, Z_\ell)\). We adopt the convention of sect. 5.4 and regard \(Z\) as an element of \(\mathcal{Z}(\lambda)_\mu\); then \(\langle Z \rangle = v_{w_1 \lambda_1} \otimes \cdots \otimes v_{w_\ell \lambda_\ell}\).

Let us expand this element on the MV basis of \(V(\lambda)\). As in Theorem 5.8, we write

\[
\langle Z \rangle = \sum_{Z' \in \mathcal{Z}(\lambda)_\mu} a_{Z'; Z} \langle Z' \rangle. \tag{34}
\]

Suppose \(Z' \in \mathcal{Z}(\lambda)_\mu\) satisfies \(a_{Z'; Z} \not= 0\). Let \((\nu_1, \ldots, \nu_\ell) \in \Lambda^\ell\) be the tuple of weights such that \(Z' \in \mathcal{Z}(\lambda_1)_{\nu_1} \times \cdots \times \mathcal{Z}(\lambda_\ell)_{\nu_\ell}\). For each \(k \in \{1, \ldots, \ell\}\), we have \(\mathcal{Z}(\lambda_k)_{\nu_k} \not= \emptyset\), so \(w_k^{-1} \nu_k\) is a weight of \(V(\lambda_k)\), whence \(\nu_k - w_k \lambda_k \in w_k(\mathbb{Q}_+)\). From \(\nu_1 + \cdots + \nu_\ell = \mu\), we deduce that \((\nu_1 - w_1 \lambda_1) + \cdots + (\nu_\ell - w_\ell \lambda_\ell) = 0\). And by Corollary 5.12, we get

\[
(\nu_1 - w_1 \lambda_1) + \cdots + (\nu_k - w_k \lambda_k) \in \mathbb{Q}_+
\]

for each \(k \in \{1, \ldots, \ell - 1\}\). Then, assuming that \((i_1, \ldots, i_\ell)\) satisfies condition (A) and applying Lemma 7.1, we obtain \(\nu_k = w_k \lambda_k\) for each \(k \in \{1, \ldots, \ell - 1\}\). In other words, none of the inequalities given in Corollary 5.12 is strict. By Remark 5.14, this forces \(Z' = Z\). Thus, the expansion (34) contains a single term, namely \(\langle Z \rangle\).

Set \(\lambda = \lambda_1 + \cdots + \lambda_\ell\) and let \(p : V(\lambda) \to V(\lambda)\) be the unique morphism that maps \(v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_\ell}\) to \(v_{\lambda}\). Noting that \(p\) is the quotient map to the top factor in the isotypical filtration of \(V(\lambda)\) and applying Remark 3.5, we obtain that \(p(\langle Z \rangle)\) belongs to the MV basis of \(V(\lambda)\). From the equality \(v_{w_1 \lambda_1} \otimes \cdots \otimes v_{w_\ell \lambda_\ell} = \langle Z \rangle\), we deduce that

\[
\Delta_{\lambda_1, w_1 \lambda_1} \cdots \Delta_{\lambda_\ell, w_\ell \lambda_\ell} = \langle v_{\lambda}^*, p(\langle Z \rangle) \rangle = \Psi_{\lambda}(p(\langle Z \rangle))
\]

belongs to the MV basis of \(\mathbb{C}[N]\). The claim in the proposition is the particular case where each \(\lambda_k\) is a multiple of \(\omega_{i_k}\). □

**7.2 A computation in type \(D_4\)**

In [29], Kashiwara and Saito found an example in type \(A_5\) where the singular support of a simple perverse sheaf related to the canonical basis is not irreducible. Looking again at this
situation, Geiß, Leclerc and Schröer [19] computed the dual canonical and dual semicanonical elements and found that they were different. They also observed that a similar situation occurs in type $D_4$. In [2], Kamnitzer, Knutson and the first author observed that in both situations $A_5$ and $D_4$, the MV basis is a third basis, different from the two other ones.

Let us have a closer look at the $D_4$ case. As usual, we label the vertices of the Dynkin diagram from 1 to 4, with 2 for the central node. Our three bases are indexed by the crystal $B(\infty)$: given $b \in B(\infty)$, we denote the corresponding dual semicanonical basis element by $C(b)$, the dual canonical basis element by $C'(b)$, and the MV basis element by $C''(b)$. Calling $b_0$ the highest weight element in $B(\infty)$, we set

$$b_1 = (\tilde{f}_2(\tilde{f}_1\tilde{f}_3\tilde{f}_4)\tilde{f}_2)^2 b_0 \quad \text{and} \quad b_{12} = (\tilde{f}_2)^2 (\tilde{f}_1\tilde{f}_3\tilde{f}_4)^2 (\tilde{f}_2)^2 b_0.$$  

**Proposition 7.3** The basis elements are related by the equations

$$C(b_{12}) = C''(b_{12}) + 2C(b_1) \quad \text{and} \quad C'(b_{12}) = C''(b_{12}) + C(b_1).$$

The proof is given in [2], sect. 2.7, except for one justification left to the present paper. We here fill the gap.

The fundamental weight $\varpi_2$ is the highest root $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$. The crystal of the representation $V(\varpi_2)$ (the adjoint representation) is pictured below. Highest weights are towards the left, vertices are represented as keys $p \atop q$ with $p, q \in \{1, 2, 3, 4, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, and operators $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}_3$ and $\tilde{f}_4$ are indicated by dashed, solid, dotted and dash-dotted arrows, respectively.

If we endow the weight lattice $\Lambda$ with its usual basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, then the weight of the element $\begin{array}{c} p \\ q \end{array}$ is simply $\varepsilon_p + \varepsilon_q$, with the convention that $\varepsilon_{\bar{i}} = -\varepsilon_i$ for $i \in \{1, 2, 3, 4\}$. The crystal contains four elements of weight zero, namely $\begin{array}{c} 2 \\ \bar{2} \end{array}$, $\begin{array}{c} 3 \\ \bar{3} \end{array}$, $\begin{array}{c} 4 \\ \bar{4} \end{array}$ and $\begin{array}{c} \bar{3} \\ 4 \end{array}$. 

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We set $\lambda = (\varpi_2, \varpi_2)$ and look at the tensor square $V(\lambda) = V(\varpi_2)^{\otimes 2}$. As in sect. 5.4, its MV basis consists of symbols $\langle \mathbf{Z} \rangle$, where $\mathbf{Z} = (Z_1, Z_2)$ is a pair in $\mathcal{Z}(\varpi_2) \times \mathcal{Z}(\varpi_2)$. In addition, $V(\lambda)$ is endowed with the tensor product basis. In the matter of notation, we simply use the keys $\varpi_2$ to denote MV cycles, taking advantage that $\mathcal{Z}(\varpi_2)$ is isomorphic to the crystal pictured above.

We claim that

$$\left\langle \frac{2}{3} \right\rangle \otimes \left\langle \frac{3}{4} \right\rangle = 2 \left( \left\langle \frac{1}{2}, \frac{3}{4} \right\rangle + \left\langle \frac{1}{2}, \frac{1}{4} \right\rangle + \left\langle \frac{1}{3}, \frac{3}{4} \right\rangle + \left\langle \frac{1}{3}, \frac{1}{4} \right\rangle \right) + \left( \left\langle \frac{1}{3}, \frac{1}{4} \right\rangle + \left\langle \frac{2}{4}, \frac{4}{3} \right\rangle + \left\langle \frac{2}{4}, \frac{3}{3} \right\rangle + \left\langle \frac{3}{4}, \frac{3}{3} \right\rangle \right).$$

(35)

Let $p : V(\varpi_2)^{\otimes 2} \to V(2\varpi_2)$ be the unique morphism that maps $v_{\varpi_2} \otimes v_{\varpi_2}$ to $v_{2\varpi_2}$. Applying $\Psi_{2\varpi_2} \circ p$ to the equality (35), we obtain the equation

$$C''(b_{13}) C''(b_{14}) = 2 C''(b_1) + \sum_{i=2}^{8} C''(b_i) + C''(b_{12})$$

asserted without proof in [2]. Thus, establishing (35) will complete the proof of Proposition 7.3. Actually, an inspection of the proof in loc. cit. reveals that it is enough to justify that the coefficient in front of $\left( \left\langle \frac{1}{2}, \frac{3}{4} \right\rangle \right)$ is strictly larger than one.

We will use Theorem 5.8 to prove this fact. Here the group $G^\vee$ is $SO_8$. For $(i, j) \in \{1, \ldots, 8\}$, we denote by $E_{i,j}$ the matrix of size $8 \times 8$ with zeros everywhere except for a one at position $(i, j)$.

For each coroot $\alpha^\vee \in \Phi^\vee$, we define a subgroup $x_{\alpha^\vee} : \mathbb{C} \to G^\vee$ by the following formulas, where $I$ is the identity matrix, $a \in \mathbb{C}$, and $i, j$ are elements in $\{1, 2, 3, 4\}$ such that $i < j$.

$$x_{(\varepsilon_i - \varepsilon_j)^\vee}(a) = I + a(E_{i,j} - E_{9-j,9-i}) \quad x_{(\varepsilon_i + \varepsilon_j)^\vee}(a) = I + a(E_{i,9-j} - E_{j,9-i})$$
$$x_{(\varepsilon_j - \varepsilon_i)^\vee}(a) = I + a(E_{9-i,9-j} - E_{j,i}) \quad x_{(-\varepsilon_i - \varepsilon_j)^\vee}(a) = I + a(E_{9-i,j} - E_{9,j-i})$$

For each root $\alpha$, we define a map $\chi_\alpha : \mathbb{C}^{10} \to G^\vee(\mathbb{C}[z, z^{-1}])$ by the formula

$$\chi_\alpha(a)(z) = \left( \prod_{k=1}^{8} x_{\beta_k^\vee}(a_k) \right) x_{\alpha^\vee}(a_9 + za_{10}) \ z^\alpha$$

where $a$ stands for the tuple $(a_1, \ldots, a_{10}) \in \mathbb{C}^{10}$ and where $\beta_1^\vee, \ldots, \beta_8^\vee$ are the coroots $\beta^\vee$ such that $\langle \beta^\vee, \alpha \rangle = 1$. We specify the enumeration in our cases of interest as follows.
We now define two charts on $\mathcal{G}^2$, both with $\mathbb{C}^{22}$ as domain:

\[\phi_1 : (x_1, x_2, a, b) \mapsto (x_1, x_2; \left[\chi_{e_1+e_2}(a)(z - x_1), \chi_{e_1-e_2}(b)(z - x_2)\right]),\]

\[\phi_2 : (x_1, x_2, a', b') \mapsto (x_1, x_2; \left[\chi_{e_2-e_3}(a')(z - x_1), \chi_{e_3-e_2}(b')(z - x_2)\right]).\]

One can then compute the transition map between these two charts. (The calculations were actually carried out with the help of the computer algebra system SINGULAR [10].) One finds the variables $a'_1, \ldots, b'_{10}$ as rational functions in $x_2 - x_1, a_1, \ldots, b_{10}$. We denote by $f$ the l.c.m. of the denominators.

Recall the notation used in sect. 5.4. In the chart $\phi_1$, the cycle $\mathcal{Y}\left(\frac{1}{2}, \frac{7}{11}\right)$ is defined by the equations $a_1 = \cdots = a_{10} = x_2 - x_1 = 0$, so the ideal in $R = \mathbb{C}[x_1, x_2, a_1, \ldots, a_{10}, b_1, \ldots, b_{10}]$ of

\[V = \phi_1^{-1}(\mathcal{Y}\left(\frac{1}{2}, \frac{7}{11}\right))\]

is

\[p = (a_1, \ldots, a_{10}, x_2 - x_1).\]

In the chart $\phi_2$, the cycle $\mathcal{X}\left(\frac{2}{11}, \frac{3}{7}\right)$ is defined by the equations $a'_2 = a'_3 = a'_4 = a'_8 = a'_9 = a'_{10} = b'_2 = b'_5 = b'_6 = b'_7 = 0$. Since the zero locus of $f$ contains the locus where the transition map between the charts is not defined, the ideal $q$ of the subvariety

\[X = \phi_1^{-1}(\mathcal{X}\left(\frac{2}{11}, \frac{3}{7}\right))\]

is the preimage in $R$ of the ideal $q_f = (a'_2, a'_3, a'_4, a'_5, a'_6, a'_7, b'_2, b'_3, b'_4, b'_5)$ of the localized ring $R_f$. SINGULAR gives the following expression:

\[q = (a_1a_4 + a_2a_3, a_1a_6 - a_2a_5, a_3a_6 + a_4a_5, a_1a_7 - a_3a_5, a_2a_7 + a_4a_5, a_1a_8 - a_4a_5, a_2a_8 - a_4a_6, a_3a_8 - a_4a_7, a_5a_8 + a_6a_7, a_9, a_{10}, a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1, a_5b_4 + a_6b_3 + a_7b_2 + a_8b_1, a_3b_6 + a_4b_5 + a_7b_2 + a_8b_1 - (x_2 - x_1), a_2b_7 - a_3b_6 + a_6b_3 - a_7b_2, a_1b_8 + a_3b_6 + a_5b_4 + a_7b_2, a_2b_8 - a_4b_6 + a_6b_4 - a_8b_2, a_3b_8 - a_4b_7 + a_7b_4 - a_8b_3).\]
We observe that $q \subseteq p$, hence $V \subseteq X$.

Let $a$ and $x$ be two indeterminates. Let $B$ be the field $\mathbb{C}(x, b_1, \ldots, b_{10})$. Extract the last seven equations from $q$ and remove the term $x_2 - x_1$ in the third one: we then deal with seven linear equations with coefficients in $B$ in the eight variables $a_1, \ldots, a_8$. This system has a non-zero solution $(c_1, \ldots, c_8) \in B^8$. We can then define an algebra morphism $u : R/q \rightarrow B[a]/(a^2)$ by

$$u(x_1) = u(x_2) = x, \quad u(b_i) = b_i \quad \text{for } i \in \{1, \ldots, 10\},$$

$$u(a_9) = u(a_{10}) = 0, \quad u(a_i) = c_ia \quad \text{for } i \in \{1, \ldots, 8\}.$$

The ring $B[a]/(a^2)$ is local with maximal ideal $(a)$ and the preimage of this ideal by $u$ is the ideal $p/q$ of $R/q$.

Let $A$ be the localization of $R/q$ at $p/q$. Then $u$ extends to an algebra morphism $\pi : A \rightarrow B[a]/(a^2)$. By construction, the kernel of $\pi$ contains $x_2 - x_1$ but not all $a_1, \ldots, a_8$. Therefore $x_2 - x_1$ does not generate the maximal ideal of $A$. Since $A$ is the local ring $\mathcal{O}_{V,X}$ of $X$ along $V$, this means that the order of vanishing of $x_2 - x_1$ along $V$ is larger than one. In other words, the multiplicity of $\mathcal{Y}\left(\frac{1}{2}, \frac{3}{4}\right)$ in the intersection product $\mathcal{X}\left(\frac{2}{3}, \frac{3}{2}\right) \cdot \mathcal{G}_{\pi^2}|_\Delta$ is larger than one.

Applying Theorem 5.8, we conclude that in (35) the coefficient in front of $\left\langle \left(\frac{1}{2}, \frac{3}{4}\right) \right\rangle$ is strictly larger than one.

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