Quantum cosmology from three different perspectives

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Abstract

Our review is devoted to three promising research lines in quantum cosmology and the physics of the early universe. The nonperturbative renormalization programme is making encouraging progress that we here assess from the point of view of cosmological applications: Lagrangian and Hamiltonian form of pure gravity with variable $G$ and $\Lambda$; power-law inflation for pure gravity; an accelerating universe without dark energy. In perturbative quantum cosmology, on the other hand, diffeomorphism-invariant boundary conditions lead naturally to a singularity-free one-loop wave function of the Universe. Last, but not least, in the braneworld picture one discovers the novel concept of cosmological wave function of the bulk space-time. Its impact on quantum cosmology and singularity avoidance is still, to a large extent, unexplored.
I. FOREWORD

In this brief review we focus on three (among the many) peculiar aspects of modern quantum cosmology, with the hope of leading quickly the reader towards open research problems. No completeness can be achieved in such a short presentation, and we therefore apologize in advance with the colleagues who might not find a proper acknowledgment of their work.

II. QUANTUM COSMOLOGY VIA FUNCTIONAL INTEGRALS

The familiar formulation of quantum cosmology via functional integrals relies upon the pioneering work of Misner [1], Hartle and Hawking [2]. The main idea of the functional-integral approach is to build in-out amplitudes following Feynman: the amplitude to go from a metric $g_1$ and a (matter) field configuration $\phi_1$ on a spacelike surface $S_1$ to a metric $g_2$ and a (matter) field configuration $\phi_2$ on a spacelike surface $S_2$ is (formally) expressed as the functional integral of the exponential of $i$ times the action, supplemented by gauge-fixing and ghost terms [3], taken over all metrics and (matter) fields matching the given boundary data on $S_1$ and $S_2$. In order to obtain a well-defined prescription, in-out amplitudes are sometimes written first as Euclidean functional integrals, but severe technical problems occur: integration measure over all four-geometries with their topologies and unboundedness from below of the Euclidean action among the many [4].

III. HARTLE–HAWKING QUANTUM STATE

In a cosmological setting, one therefore arrives at the Hartle–Hawking quantum state [2]. According to these authors, the quantum state of the Universe [5] can be expressed by an Euclidean functional integral over compact four-geometries matching the boundary data on the surface $S_2$, while the three-surface $S_1$ shrinks to a point (hence the name “no boundary proposal”). One can therefore derive, in principle, all we know about cosmology from a choice of boundary conditions [6], including formation of structure [7], coupling to matter fields [8], inflationary solutions [9], supersymmetric models [10, 11].
IV. RENORMALIZATION-GROUP APPROACH

Recent progress relies instead on a completely different approach: one builds a scale-dependent effective action $\Gamma(k)$ for quantum Einstein gravity, which is ruled by the renormalization-group (hereafter RG) equation. If $\Gamma(k)$ equals the classical Einstein–Hilbert action at the ultraviolet cut-off scale $\kappa$, one uses the RG equation to evaluate $\Gamma(k)$ for $k < \kappa$, and then sends $k \to 0$ and $\kappa \to \infty$. The continuum limit as $\kappa \to \infty$ should exist after renormalizing finitely many parameters in the action, and is taken at a non-Gaussian fixed point of the RG-flow [12]. Over the years, strong evidence has been obtained in favour of the new ultraviolet fixed point, regardless of the truncation used [13]. The plot in [13] shows part of theory space of the Einstein–Hilbert truncation with its RG flow. The arrows therein point in the direction of decreasing values of $k$ [14]. The flow is dominated by a non-Gaussian fixed point in the first quadrant and a trivial one at the origin [14].

V. COSMOLOGICAL APPLICATIONS

After investigating the RG-improved equations for self-interacting scalar fields coupled to gravity in a FLRW Universe [15], we have improved the action principle itself, building the Lagrangian and Hamiltonian formalism with variable $G$ and $\Lambda$ treated as dynamical variables [16]. The latter point is substantially innovative, since all other investigations in the literature treated $G$ and $\Lambda$ as external parameters at the very best, but not as dynamical variables with an Euler–Lagrange equation for $G$.

VI. POWER-LAW INFLATION FOR PURE GRAVITY

Unlike models where only the Einstein equations are RG-improved, our framework allows for a non-trivial dynamics of the scale factor even in the absence of coupling to a matter field. Indeed, if in the pure-gravity case we look for power-law solutions of the Euler–Lagrange equations of the type [16, 17]

$$a(t) = A t^\alpha, \ G(t) = g_* \frac{t^2}{\xi^2}, \ \Lambda(t) = \lambda_* \frac{\xi^2}{t^2},$$

we find for example, in a spatially flat FLRW Universe, that $A$ is undetermined, while

$$\alpha = \frac{1}{6} \left(3 \pm \sqrt{9 + 12 \xi^2 \lambda_*} \right).$$

(2)
Our modified Lagrangian \[16\] allows therefore for power-law inflation in pure-gravity models, unlike all previous models in the literature \[13, 18, 19\].

VII. INFRARED FIXED POINT

The derivation of an infrared fixed point is not on a footing as firm as the evidence in favour of an ultraviolet fixed point \[13\]. Nevertheless, on assuming its existence, we have linearized the RG-flow and, after evaluating the critical exponents, we have found how the infrared fixed point would be approached \[20\]. We have also obtained a smooth transition between FLRW cosmology and the observed accelerated expansion of the universe \[20\].

VIII. PERTURBATIVE QUANTUM COSMOLOGY

Perturbative quantum cosmology studies instead the first quantum corrections to the underlying classical dynamics. In particular, one-loop effects can be evaluated after imposing gauge-invariant boundary conditions, according to the recipe for imposing gauge-invariant boundary conditions in quantum field theory \[21\]. On denoting by \(\pi^i_j\) a projector acting on the gauge fields \(\varphi^j\), by \(P^\alpha(\varphi)\) and \(\psi_\beta\) the gauge-fixing functionals and ghost fields, respectively, such boundary conditions read as

\[
\left[\pi^i_j \, \varphi^j\right]_{\partial M} = 0, \quad (3)
\]

\[
\left[P^\alpha(\varphi)\right]_{\partial M} = 0, \quad (4)
\]

\[
\left[\psi_\beta\right]_{\partial M} = 0. \quad (5)
\]

IX. SINGULARITY AVOIDANCE AT ONE LOOP?

For pure gravity, one-loop quantum cosmology in the limit of small three-geometry \[22\] describes a vanishing probability of reaching the singularity at the origin (of the Euclidean four-ball) only with diffeomorphism-invariant boundary conditions \[23, 24\], which are a particular case of the previous scheme. All other sets of boundary conditions lead instead to a divergent one-loop wave function \[22, 24, 25\].
X. PECULIAR PROPERTY OF THE FOUR-BALL?

We stress that we do not require a vanishing one-loop wave function. We rather find it, on the Euclidean four-ball, as a consequence of diffeomorphism-invariant boundary conditions. Peculiar cancellations occur on the Euclidean four-ball, and the spectral (also called generalized) \( \zeta \)-function remains regular at the origin \([23, 24]\), despite the lack of strong ellipticity of the boundary-value problem \([21]\).

XI. TOWARDS BRANE-WORLD QUANTUM COSMOLOGY

In the braneworld picture, branes are timelike surfaces with metric \( g_{\alpha\beta} \) embedded into bulk space-time with metric \( G_{AB} \). The action functional can be taken to be the sum of a four-dimensional (brane) and five-dimensional (bulk) contribution, i.e. \([26]\)

\[
S = S_4[g_{\alpha\beta}(x)] + S_5[G_{AB}(X)].
\]

In general, there exist vector fields \( R_B, R_\nu \) on the space of histories such that

\[
R_B S_5 = 0, \quad R_\nu S_4 = 0,
\]

with Lie brackets given by

\[
[R_B, R_D] = C^A_{BD} R_A, \quad [R_\mu, R_\nu] = C^\lambda_{\mu\nu} R_\lambda.
\]

The components of the vector fields \( R_B \) and \( R_\nu \) generate five-dimensional and four-dimensional diffeomorphisms, respectively, while the bulk and brane ghost operators read \([26]\) (with \( F^A \) and \( \chi^\mu \) the bulk and brane gauge-fixing functionals, respectively)

\[
Q^A_B \equiv R_B F^A = F^A_{,a} R^a_B,
\]

\[
J^\mu_\nu \equiv R_\nu \chi^\mu = \chi^\mu_{,i} R^i_\nu.
\]

On denoting by \( S_A \) and \( T^B \) the bulk ghost fields, the cosmological wave function of the bulk space-time can be written as \([26]\)

\[
\psi_{\text{Bulk}} = \int_{G_{AB}[\partial M]} \mu(G_{AB}, S, T) e^{iS_5},
\]

with \( \mu(G_{AB}, S, T) \) a suitable measure functional, while

\[
\tilde{S}_5 = S_5[G_{AB}] + \frac{1}{2} F^A \omega_{AB} F^B + S_A Q^A_B T^B.
\]
XII. BRANEWORLD EFFECTIVE ACTION

The braneworld effective action $\Gamma$ can (in principle) be obtained from the formula \[26\]

$$e^{i\Gamma} = \int \nu(g_{\alpha\beta}, \rho, \sigma) e^{i\tilde{S}_4} \psi_{\text{Bulk}},$$

where $\nu(g_{\alpha\beta}, \rho, \sigma)$ is a suitable measure functional over brane metrics and brane ghost fields, while \[26\]

$$\tilde{S}_4 = S_4 + \frac{1}{2} \chi^\mu C_{\mu\nu} \chi^\nu + \rho_\mu J^\mu_\nu \sigma^\nu.$$ \[14\]

Recent developments in this respect can be found in \[27\], where the authors lay the foundations for a systematic application of the background-field method to the braneworld picture.

XIII. SELECTED OPEN PROBLEMS

In our opinion, it is of crucial importance to work at least on the following unsettled issues:

(i) Can one prove in a rigorous way that an infrared fixed point occurs in the nonperturbative approach?

(ii) Can the spectral cancellations found in \[23, 24\] survive the choice of curved backgrounds with boundary?

(iii) Is braneworld quantum cosmology one-loop singularity free?

Hopefully, the years to come will shed some light on these open problems, and we also hope that quantum cosmology will make a closer contact with the rich world of observational cosmology.

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