On uniformly $S$-Artinian rings and modules

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Abstract

Let $R$ be a commutative ring with identity and $S$ a multiplicative subset of $R$. An $R$-module $M$ is said to be a uniformly $S$-Artinian ($u$-$S$-Artinian for abbreviation) module if there is $s \in S$ such that any descending chain of submodules of $M$ is $S$-stationary with respect to $s$. $u$-$S$-Artinian modules are characterized in terms of ($S$-MIN)-conditions and $u$-$S$-cofinite properties. We call a ring $R$ is a $u$-$S$-Artinian ring if $R$ itself is a $u$-$S$-Artinian module, and then show that any $u$-$S$-semisimple ring is $u$-$S$-Artinian. It is proved that a ring $R$ is $u$-$S$-Artinian if and only if $R$ is $u$-$S$-Noetherian, the $u$-$S$-Jacobson radical $\text{Jac}_S(R)$ of $R$ is $S$-nilpotent and $R/\text{Jac}_S(R)$ is a $u$-$S/\text{Jac}_S(R)$-semisimple ring. Besides, some examples are given to distinguish Artinian rings, $u$-$S$-Artinian rings and $S$-Artinian rings.

Key Words: $u$-$S$-Artinian ring; $u$-$S$-Artinian module; $u$-$S$-Noetherian ring; $u$-$S$-semisimple ring; $u$-$S$-Jacobson radical.

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1. Introduction

Throughout this article, all rings are commutative rings with identity and all modules are unitary. A subset $S$ of $R$ is called a multiplicative subset of $R$ if $1 \in S$ and $s_1s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$. Early in 2002, Anderson and Dumitrescu \cite{2} introduced the so-called $S$-Noetherian ring $R$, in which for any ideal $I$ of $R$, there exists a finitely generated ideal $K$ of $R$ such that $sI \subseteq K \subseteq I$ for some $s \in S$. Note that Cohen’s Theorem, Eakin-Nagata Theorem and Hilbert Basis Theorem for $S$-Noetherian rings are also given in \cite{2}. The notion of $S$-Noetherian rings provides a good direction for $S$-generalizations of other classical rings (see \cite{1, 3, 6, 7, 9} for example). However, it is often difficult to study these $S$-generalizations of classical rings via a module-theoretic approach. The essential difficulty is that the selected element $s \in S$ is often not “uniform” in their definitions. To overcome this difficulty for Noetherian properties, Qi et al. \cite{13} recently introduced the notions of uniformly
S-Noetherian rings which are S-Noetherian rings such that s is independent on I in the definition of S-Noetherian rings. They also introduced the notion of $u$-S-injective modules and then characterized uniformly S-Noetherian rings in terms of $u$-S-injective modules. Some other uniform S-versions of rings and modules, such as semisimple rings, von Neumann regular rings, projective modules and flat modules are introduced and studied by the named authors and coauthors in [16, 17].

In 2020, Sengelen et al. [14] introduced the notions of S-Artinian rings for which any descending chain of ideals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m \supseteq \cdots$ of $R$ satisfies S-stationary condition, i.e., then there exist $s \in S$ and $k \in \mathbb{Z}^+$ such that $sI_k \subseteq I_n$ for all $n \geq k$. In the definition of S-Artinian rings, it is easy to see that although the element $s \in S$ is independent on $n$ but it is certainly dependent on the given descending chain of ideals. Recently, Özen et al. [12] extended the notion of S-Artinian rings to that of S-Artinian modules by replacing descending chains of ideals to these of submodules. And then, Omid et al. [11] introduced the notion of weakly S-Artinian modules, for which every descending chain $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_m \supseteq \cdots$ of submodules of $M$ is weakly S-stationary, i.e., there exists $k \in \mathbb{Z}^+$ such that for each $n \geq k$, $s_nN_k \subseteq N_n$ for some $s_n \in S$. In the definition of weakly S-Artinian modules, it is easy to see the element $s_n$ is dependent both on $n$ and the descending chain of ideals. So the notion of weakly S-Artinian modules is certainly a “weak” version of that of S-Artinian modules. In this article, we introduced and study the “uniform” S-version of Artinian rings and modules (we call them $u$-S-Artinian rings and modules) such that the element $s$ given in the definition of S-Artinian rings and modules is both independent on $n$ and the descending chain of ideals or submodules. Obviously, we have the following implications:

\[
\text{Artinian rings} \implies \text{u-S-Artinian rings} \implies \text{S-Artinian rings}
\]

But neither of implications can be reversed (see Example 2.7 and Example 3.4). Denote by $\text{Jac}_S(R)$ the u-S-Jacobson radical of a ring $R$ (see Definition 4.2). Then it is also worth to mention that a ring $R$ is u-S-Artinian if and only if $R$ is u-S-Noetherian, the u-S-Jacobson radical $\text{Jac}_S(R)$ of $R$ is S-nilpotent and $R/\text{Jac}_S(R)$ is a u-S/$\text{Jac}_S(R)$-semisimple ring (see Theorem 4.9).

The related notions of uniformly S-torsion theory originally emerged in [16], and we give a quick review below. An $R$-module $T$ is called u-S-torsion (with respect to $s$) provided that there exists $s \in S$ such that $sT = 0$. A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of $R$-modules is called a short u-S-exact sequence (with respect to $s$), if $s\ker(f) = s\coker(f) = 0$, $s\ker(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \ker(g)$ for some $s \in S$. An $R$-homomorphism $f : M \to N$ is a u-S-monomorphism (resp., u-S-epimorphism,
$u$-$S$-isomorphism) (with respect to $s$) provided $\ker(f)$ is (resp., $\coker(f)$ is, both $\ker(f)$ and $\coker(f)$ are) $u$-$S$-torsion (with respect to $s$). Recall from [17] an $R$-module $P$ is called $u$-$S$-projective provided that the induced sequence

$$0 \to \text{Hom}_R(P, A) \to \text{Hom}_R(P, B) \to \text{Hom}_R(P, C) \to 0$$

is $u$-$S$-exact for any $u$-$S$-exact sequence $0 \to A \to B \to C \to 0$. Suppose $M$ and $N$ are $R$-modules. We say $M$ is $u$-$S$-isomorphic to $N$ if there exists a $u$-$S$-isomorphism $f : M \to N$. A family $\mathcal{C}$ of $R$-modules is said to be closed under $u$-$S$-isomorphisms if $M$ is $u$-$S$-isomorphic to $N$ and $M$ is in $\mathcal{C}$, then $N$ is also in $\mathcal{C}$. Note that the class of $u$-$S$-projective modules is closed under $u$-$S$-isomorphisms. One can deduce from the following [17, Lemma 2.1] that the existence of $u$-$S$-isomorphisms of two $R$-modules is actually an equivalence relationship.

2. uniformly $S$-Artinian modules

Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. Suppose $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ is a descending chain of submodules of $M$. The family $\{M_i\}_{i \in \mathbb{Z}^+}$ is said to be $S$-stationary (with respect to $s$) if there exists $s \in S$ and $k \in \mathbb{Z}^+$ such that $sM_k \subseteq M_n$ for every $n \geq k$. And $M$ is called an $S$-Artinian module if each descending chain of submodules $\{M_i\}_{i \in \mathbb{Z}^+}$ of $M$ is $S$-stationary (see [12, definition 1]). Note that in the definition of $S$-Artinian module, the element $s$ is dependent on the given descending chain of submodules. The main purpose of this section is to introduce and study a “uniform” version of $S$-Artinian modules.

**Definition 2.1.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. An $R$-module $M$ is called a $u$-$S$-Artinian (abbreviates uniformly $S$-Artinian) module (with respect to $s$) provided that there exists $s \in S$ such that each descending chain $\{M_i\}_{i \in \mathbb{Z}^+}$ of submodules of $M$ is $S$-stationary with respect to $s$.

Trivially, if $0 \in S$, then every $R$-module is $u$-$S$-Artinian. If $S_1 \subseteq S_2$ are multiplicative subsets of $R$ and $M$ is $u$-$S_1$-Artinian, then $M$ is obviously $u$-$S_2$-Artinian. Note that Artinian modules are exactly $u$-$\{1\}$-Artinian modules. So all Artinian modules are $u$-$S$-Artinian modules for any multiplicative set $S$. Next we give a $u$-$S$-Artinian module which is not Artinian.

**Example 2.2.** Let $R = \mathbb{Z}$ be the ring of integers, $p$ a prime in $R$ and $M = \mathbb{Z}_p[[x]]$ the set of all formal power series over $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$. Set $S = \{p^n \mid n \in \mathbb{N}\}$. Then $M$ is not Artinian since the descending chain

$$\langle x \rangle \supseteq \langle x^2 \rangle \supseteq \cdots \supseteq \langle x^n \rangle \supseteq \cdots$$
is not stationary. However, since $M$ is obviously $u$-$S$-torsion, $M$ is a $u$-$S$-Artinian module.

Let $S$ be a multiplicative subset of $R$. We always denote by $S^* = \{ r \in R \mid rt \in S$ for some $t \in R \}$ and call it to be the saturation of $S$. A multiplicative set $S$ is said to be saturated if $S = S^*$. Trivially, we have $S \subseteq S^*$ for all multiplicative subset $S$ of $R$.

**Proposition 2.3.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. Let $S^*$ be the saturation of $S$. Then $M$ is a $u$-$S$-Artinian $R$-module if and only if $M$ is a $u$-$S^*$-Artinian $R$-module.

**Proof.** Suppose $M$ is a $u$-$S$-Artinian $R$-module. Then $M$ is trivially a $u$-$S^*$-Artinian $R$-module since $S \subseteq S^*$. Now, suppose $M$ is a $u$-$S^*$-Artinian $R$-module. Then there is $r \in S^*$ such that each descending chain of submodules $\{M_i\}_{i \in \mathbb{Z}^+}$ of $M$ is $S^*$-stationary with respect to $r$, i.e., there exits $k \in \mathbb{Z}^+$ such that $rM_k \subseteq M_n$ for each $n \geq k$. Since $r \in S^*$, $rt \in S$ for some $t \in R$. Note that $rtM_k \subseteq rM_k \subseteq M_n$. Hence $M$ is a $u$-$S$-Artinian $R$-module.

Let $R$ be a ring, $M$ an $R$-module and $S$ a multiplicative subset of $R$. For any $s \in S$, there is a multiplicative subset $S_s = \{1, s, s^2, \ldots \}$ of $S$. We denote by $M_s$ the localization of $M$ at $S_s$. Note that $M_s \cong M \otimes_R R_s$.

**Lemma 2.4.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. If $M$ is a $u$-$S$-Artinian $R$-module, then there exists an element $s \in S$ such that $M_s$ is an Artinian $R_s$-module.

**Proof.** Let $s$ be an element in $S$ such that each family of descending chain of submodules $\{M_i\}_{i \in \mathbb{Z}^+}$ of $M$ is $S$-stationary with respect to $s$. Let $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ be a descending chain of $R_s$-submodules of $M_s$. Considering the natural homomorphism $f : M \to M_s$, we have a a descending chain of submodules of $M$ as follows:

$$f^{-1}(M_1) \supseteq f^{-1}(M_2) \supseteq \cdots \supseteq f^{-1}(M_n) \supseteq \cdots$$

There is a $k \in \mathbb{Z}^+$ such that $s f^{-1}(M_k) = f^{-1}(sM_k) = f^{-1}(M_k) \subseteq f^{-1}(M_n)$ for each $n \geq k$ since $M_k$ is an $R_s$-module. Hence $M_k = M_n$ for each $n \geq k$. Consequently, $M_s$ is an Artinian $R_s$-module.

**Remark 2.5.** The converse of Lemma 2.4 also does not hold in general. Let $R = k[[x]]$ the formal power series ring over a field $k$. Let $S = \{1, x, x^2, \cdots \}$. Then $R_S$ is a field, and so is an Artinian $R_S$-module. However, $R$ is not a $u$-$S$-Artinian $R$-module as $R$ is not Artinian (see Proposition 3.2).
Proposition 2.6. Let $R$ be a ring, $S$ a finite multiplicative subset of $R$ and $M$ an $R$-module. Then $M$ is a $u$-$S$-Artinian module if and only if $M$ is an $S$-Artinian module.

Proof. If $M$ is a $u$-$S$-Artinian module, then trivially $M$ is $S$-Artinian. Suppose $S = \{s_1, ..., s_n\}$ and set $s = s_1 \cdots s_n$. Suppose $M$ is an $S$-Artinian module and $\{M_i\}_{i \in \mathbb{Z}^+}$ a descending chain of submodules of $M$. Then there exist $s_i \in S$ and $k \in \mathbb{Z}^+$ such that $s_i M_k \subseteq M_n$ for each $n \geq k$. So $s M_k \subseteq s_i M_k \subseteq M_n$ for each $i = 1, \cdots n$. Hence $M$ is a $u$-$S$-Artinian module.

However, the following example shows $S$-Artinian modules are not $u$-$S$-Artinian modules in general.

Example 2.7. Let $R$ be a valuation domain whose valuation group is $G = \prod_{\aleph_0} \mathbb{Z}$ the Hahn product $\aleph_0$-copies of $\mathbb{Z}$ with lexicographic order, where $\aleph_0$ is an uncountable regular cardinal (see [3] for example). Let $S = R \setminus \{0\}$ the set of all nonzero elements of $R$. Then $R$ itself is an $S$-Artinian $R$-module but not $u$-$S$-Artinian.

Proof. First, we claim that $R$ is an $S$-Artinian $R$-module. Indeed, let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ is a descending chain of ideals of $R$. We may assume that each $I_i$ is not equal to 0. Then for each $I_i$, there exist an nonzero element $r_i \in I_i$ such that $v(r_i) \in G$. Moreover, we can assume all $r_i$ satisfy $v(r_i) \leq v(r_{i+1})$. Since $\aleph_0$ is an uncountable regular cardinal, $\lim_{\rightarrow} v(r_i) < \aleph_0$. So there is an element $x \in G$ such that $x \geq v(r_i)$ for each $i$. Suppose $v(s) = x$. Then $0 \neq s \in \bigcap_{i=1}^{\infty} I_i$. Hence $s I_k \subseteq I_n$ for each $n \geq k$. Consequently, $R$ is an $S$-Artinian $R$-module. Now, we claim that $R$ is not a $u$-$S$-Artinian $R$-module. On contrary, suppose $R$ is $u$-$S$-Artinian. Then, by Lemma 2.4 there is an $s \in S$ such that $R_s$ is an artinian domain, which is exactly a field. Let $r$ be a nonzero element such that $v(r) > nv(s)$ for all non-negative integer $n$. Then $r$ is not a unit in $R_s$, which is a contradiction. Consequently, $R$ is not a $u$-$S$-Artinian $R$-module. (Note that it can also be easily deduced by Proposition 3.2) □

Lemma 2.8. Let $R$ be a ring and $S$ a multiplicative subset of $R$. Let $M$ and $N$ be $R$-modules. If $M$ is $u$-$S$-isomorphic to $N$, then $M$ is $u$-$S$-Artinian if and only if $N$ is $u$-$S$-Artinian.
Proof. Let $M$ be $u$-$S$-Artinian with respect to $s \in S$ and $f : M \to N$ a $u$-$S$-isomorphism. Suppose $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ is a descending chain of submodules of $N$. Then $f^{-1}(N_1) \supseteq f^{-1}(N_2) \supseteq \cdots \supseteq f^{-1}(N_n) \supseteq \cdots$ is a descending chain of submodules of $N$. So there is $k \in \mathbb{Z}^+$ such that $sf^{-1}(N_k) \subset f^{-1}(N_n)$ for any $n \geq k$. Hence $M$ is $u$-$S$-Artinian. Suppose $N$ is $u$-$S$-Artinian. Then by [7] Lemma 2.1, there is a $u$-$S$-isomorphism $g : N \to M$. So we can show $M$ is $u$-$S$-Artinian similarly. \hfill \Box

**Proposition 2.9.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Let $0 \to A \to B \to C \to 0$ be an $S$-exact sequence. Then $B$ is $u$-$S$-Artinian if and only if $A$ and $C$ are $u$-$S$-Artinian. Consequently, a finite direct sum $\bigoplus_{i=1}^{n} M_i$ is $u$-$S$-Artinian if and only if each $M_i$ is $u$-$S$-Artinian ($i = 1, \cdots, n$).

**Proof.** We can assume that $0 \to A \to B \to C \to 0$ is an exact sequence by Lemma 2.8. If $B$ is $u$-$S$-Artinian, then it is easy to verify $A$ and $C$ are $u$-$S$-Artinian. Now suppose $A$ and $C$ are $u$-$S$-Artinian. Let $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$ be a descending chain of submodules of $B$. Then

$$B_1 \cap A \supseteq B_2 \cap A \supseteq \cdots \supseteq B_n \cap A \supseteq \cdots$$

is a descending chain of submodules of $A$, and

$$(B_1 + A)/A \supseteq (B_2 + A)/A \supseteq \cdots \supseteq (B_n + A)/A \supseteq \cdots$$

is a descending chain of submodules of $C \cong B/A$. So there is $s_1, s_2 \in S$ and $k \in \mathbb{Z}^+$ such that $s_1 B_k \cap A \subseteq B_n \cap A$ and $s_2 (B_k + A) \subseteq B_n + A$ for any $n \geq k$. Then one can verify that $s_1 s_2 B_k \subseteq B_n$ for any $n \geq k$. Hence, $B$ is $u$-$S$-Artinian. \hfill \Box

Let $\mathfrak{p}$ be a prime ideal of $R$. We say an $R$-module $M$ is $u$-$\mathfrak{p}$-Artinian provided that $M$ is $u$-$(R \setminus \mathfrak{p})$-Artinian. The next result gives a local characterization of Artinian modules.

**Proposition 2.10.** Let $R$ be a ring and $M$ an $R$-module. Then the following statements are equivalent:

1. $M$ is Artinian;
2. $M$ is $u$-$\mathfrak{p}$-Artinian for any $\mathfrak{p} \in \text{Spec}(R)$;
3. $M$ is $u$-$\mathfrak{m}$-Artinian for any $\mathfrak{m} \in \text{Max}(R)$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ be a descending chain of submodules of $M$. Then, for each $\mathfrak{m} \in \text{Max}(R)$, there exist $s_m \notin \mathfrak{m}$ and $k_m \in \mathbb{Z}^+$ such that $s_m M_{k_m} \subseteq M_n$ for each $n \geq k_m$. Since $R$ is generated by $\left\{ s_m \mid \mathfrak{m} \in \text{Max}(R) \right\}$. So
there is a finite subset \( \{s_{m_1}, \ldots, s_{m_t}\} \) that generates \( R \). Let \( k = \max\{k_{m_1}, \ldots, k_{m_t}\} \). Then \( M_k = \langle s_{m_1}, \ldots, s_{m_t} \rangle M_k \subseteq \sum_{i=1}^{t}(s_{m_i}M_{k_{m_i}}) \subseteq M_n \) for all \( n \geq k \). Hence \( M \) is Artinian.

Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). Recall from [12], an \( R \)-module \( M \) is said to be \( S \)-cofinite (called finitely \( S \)-cogenerated in [12, Definition 3]) if for each nonempty family of submodules \( \{M_i\}_{i \in \Delta} \) of \( M \), \( \bigcap_{i \in \Delta} M_i = 0 \) implies that \( s(\bigcap_{i \in \Delta'} M_i) = 0 \) for some \( s \in S \) and a finite subset \( \Delta' \subseteq \Delta \). If \( S = \{1\} \), then \( S \)-cofinite modules are exactly the classical cofinite modules.

**Definition 2.11.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). An \( R \)-module \( M \) is called \( u \)-\( S \)-cofinite (with respect to \( s \)) if there is an \( s \in S \) such that for each nonempty family of submodules \( \{M_i\}_{i \in \Delta} \) of \( M \), \( \bigcap_{i \in \Delta} M_i = 0 \) implies that \( s(\bigcap_{i \in \Delta'} M_i) = 0 \) for a finite subset \( \Delta' \subseteq \Delta \).

Obviously, cofinite \( R \)-modules are \( u \)-\( S \)-cofinite, and \( u \)-\( S \)-cofinite \( R \)-modules is \( S \)-cofinite.

**Proposition 2.12.** Let \( R \) be a ring, \( M \) an \( R \)-module and \( S \) a multiplicative subset of \( R \). Then the following statements hold.

1. If \( S_1 \subseteq S_2 \) are multiplicative subsets of \( R \) and \( M \) is \( u \)-\( S_1 \)-cofinite. Then \( M \) is \( u \)-\( S_2 \)-cofinite.
2. Suppose \( S^* \) is the saturation of \( S \). Then \( M \) is \( u \)-\( S \)-cofinite if and only if \( M \) is \( u \)-\( S^* \)-cofinite.

**Proof.** (1) is trivial.

(2) If \( M \) is \( u \)-\( S \)-cofinite, then \( M \) is also \( u \)-\( S^* \)-cofinite by (1). Suppose \( M \) is \( u \)-\( S^* \)-cofinite. Then there is an \( r \in S^* \) such that for each nonempty family of submodules \( \{M_i\}_{i \in \Delta} \) of \( M \), \( \bigcap_{i \in \Delta} M_i = 0 \) implies that \( r(\bigcap_{i \in \Delta'} M_i) = 0 \) for a finite subset \( \Delta' \subseteq \Delta \). Let \( s := rt \in S \) for some \( t \in R \). Then \( sr(\bigcap_{i \in \Delta'} M_i) = s0 = 0 \). Hence \( M \) is \( u \)-\( S \)-cofinite. \( \square \)

Let \( \mathfrak{p} \) be a prime ideal of \( R \). We say an \( R \)-module \( M \) is \( u \)-\( \mathfrak{p} \)-cofinite provided that \( M \) is \( u \)-(\( R \setminus \mathfrak{p} \))-cofinite. The next result gives a local characterization of cofinite modules.

**Proposition 2.13.** Let \( R \) be a ring and \( M \) an \( R \)-module. Then the following statements are equivalent:

1. \( M \) is cofinite;
2. \( M \) is \( u \)-\( \mathfrak{p} \)-cofinite for any \( \mathfrak{p} \in \text{Spec}(R) \);
(3) $M$ is $u$-$m$-cofinite for any $m \in \text{Max}(R)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let $\{M_i\}_{i \in \Delta}$ be a family of submodules of $M$ such that $\bigcap_{i \in \Delta} M_i = 0$. Then, for each $m \in \text{Max}(R)$, there exist $s_m \not\subseteq m$ and $k_m \in \mathbb{Z}^+$ such that $s_m(\bigcap_{i \in \Delta_m} M_i) = 0$ for a finite subset $\Delta_m' \subseteq \Delta$. Since $R$ can be generated by a finite subset $\{s_m, \ldots, s_{m_t}\}$, let $\Delta_m' = \{s_m, \ldots, s_{m_t}\}$. Then $\bigcap_{i \in \Delta_m'} M_i = \langle s_m, \ldots, s_{m_t} \rangle(\bigcap_{i \in \Delta_m'} M_i) \subseteq \sum_{i=1}^{t}(s_{m_i} \bigcap_{i \in \Delta_m'} M_i) = 0$. Hence, $M$ is cofinite. □

Definition 2.14. Let $\mathcal{N}$ be a nonempty family of submodules of $M$. Then $N \in \mathcal{N}$ is called an $S$-minimal element of $\mathcal{N}$ with respect to $s$ if whenever $N' \subseteq N$ for some $N \in \mathcal{N}$ then $sN \not\subseteq N'$. We say $M$ satisfies $(S\text{-MIN})$-condition with respect to $s$ if every nonempty family of submodules of $M$ has an $S$-minimal element with respect to $s$.

It follows from [4, Proposition 10.10] that an $R$-module $M$ is Artinian if and only if every factory $M/N$ is finitely cogenerated, if and only if $M$ satisfies (MIN)-Condition. Recently, Özen extended this result to $S$-Artinian rings in [12, Theorem 3]. Now we give a uniform $S$-version of [4, Proposition 10.10].

Theorem 2.15. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. Let $s \in S$. Then the following statements are equivalent:

1. $M$ is a $u$-$S$-Artinian module with respect to $s$;
2. $M$ satisfies $(S\text{-MIN})$-condition with respect to $s$;
3. For any nonempty family $\{N_i\}_{i \in \Gamma}$ of submodules of $M$, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $s \bigcap_{i \in \Gamma_0} N_i \subseteq \bigcap_{i \in \Gamma} N_i$;
4. Every factor module $M/N$ is $u$-$S$-cofinite with respect to $s$.

Proof. (1) $\Rightarrow$ (2) : Suppose that $M$ is a $u$-$S$-Artinian module with respect to $s$. Let $\mathcal{N}$ be a nonempty set of submodules of $M$. On contrary, suppose $\mathcal{N}$ has no $S$-minimal element of $\mathcal{N}$ with respect to $s$. Take $N_1 \in \mathcal{N}$, and then there exists $N_2 \in \mathcal{N}$ such that $N_1 \supseteq N_2$ and $sN_1 \not\subseteq N_2$. Iterating these steps, we can obtain a descending chain $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ such that $sN_k \not\subseteq N_{k+1}$ for any $k$. This implies $M$ is not a $u$-$S$-Artinian module with respect to $s$, which is a contradiction.

(2) $\Rightarrow$ (3) : Let $\{N_i\}_{i \in \Gamma}$ be a nonempty family of submodules of $M$. Set $N = \bigcap_{i \in \Gamma} N_i$. Let $\mathcal{A}$ be the set of all intersections of finitely many $N_i$. Then each $N_i \in \mathcal{A}$,
and so $\mathcal{A}$ is nonempty. So there is an $S$-minimal element, say $A = \bigcap_{i \in \Gamma_0} N_i$, of $\mathcal{A}$. It is clear that $N \subseteq A$. For each $i \in \Gamma$, we have $sA \subseteq A \cap N_i \subseteq N_i$ by the $S$-minimality of $A$ with respect to $s$. So $sA \subseteq \bigcap_{i \in \Gamma} N_i = N$.

(3) $\Rightarrow$ (1): Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ be a descending chain of submodules of $M$. Then there is positive inter $k$ such that $sN_k = s \bigcap_{i=1}^{k} N_i \subseteq \bigcap_{i=1}^{\infty} N_i$. So $sN_k \subseteq N_n$ for any $n \geq k$.

(3) $\Rightarrow$ (4): Suppose $\bigcap_{i \in \Gamma} N_i/N = 0$ for some family of submodules $\{N_i/N\}_{i \in \Gamma}$ of $M/N$. Then $\bigcap_{i \in \Gamma} N_i = N$. Set $\mathcal{A} = \{ \bigcap_{i \in \Gamma'} N_i \mid \Gamma' \subseteq \Gamma \text{ is a finite subset } \}$. By (3), $\mathcal{A}$ has an $S$-minimal element with respect to $s$, say $M_0 = \bigcap_{i \in \Gamma} N_i$ for some finite subset $\Gamma_0 \subseteq \Gamma$. So, for any $k \in \Gamma - \Gamma_0$, we have $sM_0 \subseteq M_0 \cap N_k$. Thus $sN \subseteq M_0 \cap (\bigcap_{k \in \Gamma - \Gamma_0} N_k) \subseteq \bigcap_{k \in \Gamma} N_k = N$. Consequently, $M/N$ is $u$-$S$-cofinite with respect to $s$.

(4) $\Rightarrow$ (1): Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ be a descending chain of submodules of $M$. Set $N = \bigcap_{i=1}^{\infty} N_i$. By assumption, $M/N$ is is $u$-$S$-cofinite with respect to $s$. Note that $\bigcap_{i=1}^{k} N_i/N = 0$ in $M/N$. Then there is a positive integer $k$ such that $\bigcap_{i=1}^{k} N_i/N = 0$ by (4). So $sN_k \subseteq N_n$ for all $n \geq k$. So $M$ is a $u$-$S$-Artinian module with respect to $s$. $\square$

3. Basic properties of $u$-$S$-Artinian rings

Recall from [14, Definition 2.1] that a ring $R$ is called an $S$-Artinian ring if any descending chain of ideals $\{I_i\}_{i \in \mathbb{Z}^+}$ of $R$ is $S$-stationary with respect to some $s \in S$. Note that the $s$ is determined by the descending chain of ideals in the definition of $S$-Artinian rings. Now we introduce a “uniform” version of $S$-Artinian rings.

**Definition 3.1.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Then $R$ is called a $u$-$S$-Artinian (abbreviates uniformly $S$-Artinian) ring (with respect to $s$) provided that $R$ is a $u$-$S$-Artinian $R$-module (with respect to $s$), that is, there exists $s \in S$ such that each descending chain $\{I_i\}_{i \in \mathbb{Z}^+}$ of ideals of $R$ is $S$-stationary with respect to $s$.

Since $u$-$S$-Artinian rings are $u$-$S$-Artinian modules over themselves, the results in Secton 1 also hold for $u$-$S$-Artinian rings. Specially, $u$-$S$-Artinian rings are $S$-Artinian. However, $S$-Artinian rings are not $u$-$S$-Artinian in general. A counterexample was given in Example 2.7. If $0 \in S$, then every ring $R$ is $u$-$S$-Artinian. So
Artinian rings are not \( u\)-\( S\)-Artinian in general. A multiplicative set \( S \) is said to be regular if every element in \( S \) is a non-zero-divisor. The following Proposition shows that \( u\)-\( S\)-Artinian rings are exactly Artinian for any regular multiplicative set \( S \).

**Proposition 3.2.** Let \( R \) be a ring and \( S \) a regular multiplicative subset of \( R \). If \( R \) is a \( u\)-\( S\)-Artinian ring, then \( R \) is an Artinian ring.

**Proof.** Let \( s \) be an element in \( S \). Consider the descending chain \( Rs \subseteq Rs^2 \subseteq \cdots \) of ideals of \( R \). Then there exists \( k \) such that \( sRs^k \subseteq Rs^n \) for any \( n \geq k \). In particular, we have \( s^{k+1} = rs^{k+2} \) for some \( r \in R \). Since \( s \) is a non-zero-divisor, we have \( 1 = rs \), and thus \( s \) is a unit. So \( R \) is an Artinian ring. \( \square \)

In order to give a non-trivial \( u\)-\( S\)-Artinian ring which is not Artinian, we consider the direct product of \( u\)-\( S\)-Artinian rings.

**Proposition 3.3.** Let \( R = R_1 \times R_2 \) be direct product of rings \( R_1 \) and \( R_2 \) and \( S = S_1 \times S_2 \) a direct product of multiplicative subsets of \( R_1 \) and \( R_2 \). Then \( R \) is a \( u\)-\( S\)-Artinian ring if and only if \( R_i \) is a \( u\)-\( S_i\)-Artinian ring for each \( i = 1, 2 \).

**Proof.** Suppose \( R \) is a \( u\)-\( S\)-Artinian ring with respect to \( s_1 \times s_2 \). Let \( \{I^1_i\}_{i \in \mathbb{Z}^+} \) be a descending chain of ideals of \( R_1 \). Then \( \{I^1_i \times 0\}_{i \in \mathbb{Z}^+} \) be a descending chain of ideals of \( R \). Then there exists an integer \( k \) such that \((s_1 \times s_2)(I^1_k \times 0) \subseteq I^1_n \times 0 \) for all \( n \geq k \). Hence \( s_1I^1_k \subseteq I^1_n \) for all \( n \geq k \). So \( R_1 \) is a \( u\)-\( S_1\)-Artinian ring with respect to \( s_1 \). Similarly, \( R_2 \) is a \( u\)-\( S_2\)-Artinian ring with respect to \( s_2 \). On the other hand, suppose \( R_i \) is a \( u\)-\( S_i\)-Artinian ring with respect to \( s_i \) for each \( i = 1, 2 \). Let \( \{I_i = I^1_i \times I^2_i\}_{i \in \mathbb{Z}^+} \) be a descending chain of ideals of \( R \). Then there exists an integer \( k_i \) such that \( s_iI^1_{k_i} \subseteq I^1_n \) for all \( n \geq k_i \). Set \( k = \max\{k_1, k_2\} \). Then \((s_1 \times s_2)I_k = s_1I^1_k \times s_2I^2_k \subseteq I^1_n \times I^2_n = I_n \) for all \( n \geq k \). So \( R \) is a \( u\)-\( S\)-Artinian ring. \( \square \)

The promised non-Artinian \( u\)-\( S\)-Artinian rings are given as follows.

**Example 3.4.** Let \( R = R_1 \times R_2 \) be a product of \( R_1 \) and \( R_2 \), where \( R_1 \) is an Artinian ring while \( R_2 \) is not Artinian. Set \( S = \{1\} \times \{1, 0\} \). Then \( R \) is \( u\)-\( S\)-Artinian rings but not Artinian by Proposition 3.3.

Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). Recall from [17] that an \( R \)-module \( M \) is called \( u\)-\( S\)-semisimple (with respect to \( s \)) provided that any \( u\)-\( S\)-short exact sequence \( 0 \to A \overset{f}{\to} M \overset{g}{\to} C \to 0 \) is \( u\)-\( S\)-split (with respect to \( s \)), i.e., there exists an \( R \)-homomorphism \( h : B \to A \) such that \( h \circ f = s \text{Id}_A \) for some \( s \in S \). A ring \( R \) is called a \( u\)-\( S\)-semisimple ring if every free \( R \)-module is \( u\)-\( S\)-semisimple.
The rest of this section is devoted to show any \( u-S \)-semisimple ring is \( u-S \)-Artinian. First, we introduce the notion of \( u-S \)-simple modules.

**Definition 3.5.** An \( R \)-module \( M \) is said to be \( u-S \)-simple (with respect to \( s \)) provided that \( M \) is not \( u-S \)-torsion with respect to some \( s \in S \), and any proper submodule of \( M \) is \( u-S \)-torsion with respect to \( s \).

Since any proper submodule of a \( u-S \)-simple \( R \)-module is \( u-S \)-torsion, we have any \( u-S \)-simple \( R \)-module is \( u-S \)-semisimple by [17] Lemma 2.1. Moreover, we have the following result.

**Proposition 3.6.** Suppose \( M \) is a \( u-S \)-simple \( R \)-module. Then \( M^{(\mathfrak{R})} \) is a \( u-S \)-semisimple \( R \)-module for any ordinal \( \mathfrak{R} \).

**Proof.** Suppose \( M \) is a \( u-S \)-simple \( R \)-module with respect to \( s \) and \( N \) is a submodule of \( M^{(\mathfrak{R})} \) such that \( M^{(\mathfrak{R})}/N \) is not \( u-S \)-torsion with respect to \( s \). Set \( \Gamma = \{ \alpha \subseteq \mathfrak{R} \mid s(N \cap M^{(\alpha)}) = 0 \} \). For each \( i \in \mathfrak{R} \), we set \( M^i \) to be the \( i \)-th component of \( M^{(\mathfrak{R})} \). Then we claim there is \( i \in \mathfrak{R} \) such that \( s(N \cap M^i) = 0 \), and hence \( \Gamma \) is not empty. Indeed, on contrary, suppose \( N \cap M^i \) is not \( u-S \)-torsion for any \( i \in \mathfrak{R} \). Since \( M \) is \( u-S \)-simple with respect to \( s \), then \( N \cap M^i = M^i \cong M \), and hence \( N = M^{(\mathfrak{R})} \) which is a contradiction. Let \( \Lambda \) be a chain in \( \Gamma \). Then \( s(N \cap \bigcup_{\alpha \in \Lambda} M^{(\alpha)}) = \bigcup_{\alpha \in \Lambda} s(N \cap M^{(\alpha)}) = 0 \). So \( \Lambda \) has a upper bound. By Zorn Lemma, there is a maximal element, say \( \beta \), in \( \Gamma \). Set \( L = M^{(\beta)} \). We claim that \( M^{(\mathfrak{R})} \) is \( u-S \)-isomorphic to \( N + L \). Otherwise, since \( M^j \cong M \) is \( u-S \)-simple, there is an \( M^j \not\subseteq N + L \) for some \( j \in \mathfrak{R} \). Hence \( N \cap M^{(\beta \cup \{ j \})} \) is \( u-S \)-torsion with respect to \( s \), which contradicts the maximality of \( \beta \). Since \( N \cap L \) is \( u-S \)-torsion, \( N \) is \( u-S \)-isomorphic to \( N/(N \cap L) \) and \( (N + L)/(N \cap L) \) is \( u-S \)-isomorphic to \( (N + L) \) which is also \( u-S \)-isomorphic to \( M^{(\mathfrak{R})} \). Considering the split monomorphism \( g : N/(N \cap L) \to (N + L)/(N \cap L) \), we have the natural embedding map \( N \to M \) is also a \( u-S \)-split monomorphism.

**Theorem 3.7.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). Suppose \( R \) is a \( u-S \)-semisimple ring. Then \( R \) is a \( u-S \)-artinian ring.

**Proof.** Suppose \( R \) is a \( u-S \)-semisimple ring. Let \( \mathfrak{R} \) be a cardinal greater than \( 2^{\sharp(R)} \cdot \aleph_0 \), where \( \sharp(R) \) is the cardinal of \( R \). Then the free \( R \)-module \( R^{(\mathfrak{R})} \) is \( u-S \)-semisimple with respect to some \( s \in S \). And so every subquotient of \( R^{(\mathfrak{R})} \) is also \( u-S \)-semisimple with respect to some \( s \in S \). Let \( I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots \) be a descending chain of ideals of \( R \). Note that there are at most \( 2^{\sharp(R)} \cdot \aleph_0 \) such chains. Set \( R = I_0 \). Consider the exact sequence \( \xi_i : 0 \to I_i \to I_{i-1} \to I_{i-1}/I_i \to 0 \) for any positive integer \( i \). Since \( R \) is a \( u-S \)-semisimple ring, then each \( I_{i-1}/I_i \) are \( u-S \)-projective by [17] Theorem
3.5]. So, by Corollary 2.10, each \( \xi_i \) is \( u \)-\( S \)-split with respect to \( s \). Hence, by Lemma 2.4, there is a \( u \)-\( S \)-isomorphism \( f_i : I_{i-1} \to I_i \oplus I_{i-1}/I_i \) with respect to \( s \) for each \( i \). So there are \( u \)-\( S \)-isomorphisms

\[
R \xrightarrow{f_1} I_1 \bigoplus R/I_1 \xrightarrow{f_2 \oplus \text{Id}} I_2 \bigoplus I_1/I_2 \bigoplus R/I_1 \to \cdots \xrightarrow{f_k \oplus \text{Id}} I_k \bigoplus \left( \bigoplus_{i=0}^{k} I_i/I_{i-1} \right) \to \cdots
\]

Assume \( f(1) \in \bigoplus_{i=1}^{k} I_{i-1}/I_i \subseteq \bigoplus_{i=1}^{\infty} I_{i-1}/I_i \) where \( f = \lim_{\to k}(f_k \oplus \text{Id}) \circ \cdots \circ f_1 \). Then \( R \) is \( u \)-\( S \)-isomorphic to \( \bigoplus_{i=0}^{k} I_i/I_{i-1} \) with respect to \( s \). And so \( I_k \) is \( u \)-\( S \)-torsion with respect to \( s \). Hence \( sI_k/I_n = 0 \), i.e., \( sI_k \subseteq I_n \) for all \( n \geq k \). So \( R \) is a \( u \)-\( S \)-artinian ring.

**Corollary 3.8.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). Suppose \( R \) is a \( u \)-\( S \)-semisimple ring. Then any \( S \)-finite \( R \)-module is \( u \)-\( S \)-artinian.

**Proof.** Since the class of \( u \)-\( S \)-artinian modules is closed under \( u \)-\( S \)-isomorphisms, we just need to show any finitely generated \( R \)-module is \( u \)-\( S \)-artinian, which can easily be deduced by Proposition 2.9 and Theorem 3.7. \( \square \)

### 4. A Characterization of \( u \)-\( S \)-Artinian Rings

It is well-known that a ring \( R \) is Artinian if and only if \( R \) is a Noetherian ring with its Jacobson radical \( \text{Jac}(R) \) nilpotent and \( R/\text{Jac}(R) \) a semisimple ring (see [15, Theorem 4.1.10]). In this section, we will give a “uniform” \( S \)-version of this result. We begin with the notion of \( u \)-\( S \)-maximal submodules and the \( u \)-\( S \)-Jacobson radical of a given \( R \)-module.

**Definition 4.1.** Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \) and \( s \in S \). Then a submodule \( N \) is said to be \( u \)-\( S \)-maximal in an \( R \)-module \( M \) with respect to \( s \) provided that

1. \( M/N \) is not \( u \)-\( S \)-torsion with respect to \( s \);
2. if \( N \nsubseteq H \subseteq M \), then \( M/H \) is \( u \)-\( S \)-torsion with respect to \( s \).

Note that an \( R \)-module \( N \) is a \( u \)-\( S \)-maximal submodule of \( M \) (with respect to \( s \)) if and only if \( M/N \) is \( u \)-\( S \)-simple (with respect to \( s \)). If \( M \) does not have any \( u \)-\( S \)-maximal submodule with respect to \( s \), then we denote by \( \text{Jac}_s(M) = M \). Otherwise, we denote by \( \text{Jac}_s(M) \) the intersection of all \( u \)-\( S \)-maximal submodules of \( M \) with respect to \( s \). The submodule \( \text{Jac}_s(M) \) of \( M \) is called the \( u \)-\( S \)-Jacobson radical of \( M \) with respect to \( s \).
Definition 4.2. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. Then the $u$-$S$-Jacobson radical of $M$ is defined to be $\text{Jac}_{S}(M) = \bigcap_{s \in S} \text{Jac}_s(M)$ under the above notions.

First, we have the following obversion.

Lemma 4.3. Let $M$ be an $R$-module. Then

$$\text{Jac}_s(M/\text{Jac}_s(M)) = 0, \text{ and } \text{Jac}_S(M/\text{Jac}_S(M)) = 0.$$  

Proof. We just note that an $R$-module $N/\text{Jac}_s(M) \subseteq M/\text{Jac}_s(M)$ is $u$-$S$-maximal with respect to $s$ if and only if $N + \text{Jac}_s(M) \subseteq M$ is $u$-$S$-maximal with respect to $s$.  

Proposition 4.4. Suppose $M$ is a $u$-$S$-Artinian $R$-module with $\text{Jac}_S(M)$ $u$-$S$-torsion. Then there exists $T \subseteq M$ such that $sT = 0$ and $M/T \subseteq \bigoplus_{i=1}^n S_i$ where each $S_i$ is $u$-$S$-simple with respect to $s$ for some $s \in S$. Consequently, $M$ is a $u$-$S$-Noetherian $R$-module.

Proof. If $M$ has no $u$-$S$-maximal submodule, then we may assume $T = \text{Jac}_S(M) = M$ is $u$-$S$-torsion. So the assertion trivially holds. Now suppose $M$ has a $u$-$S$-maximal submodules. Then the intersection of all $u$-$S$-maximal submodules of $M$ is $u$-$S$-torsion. Since $M$ is $u$-$S$-Artinian, then there exists a finite family of $u$-$S$-maximal submodules, say $\{M_1, \ldots, M_n\}$, of $M$ such that $T := \bigcap_{i=1}^n M_i$ is $u$-$S$-torsion with respect to some $t \in S$ by Theorem 2.15. Note that $M/T$ is a submodule of $\bigoplus_{i=1}^n M_i$, where $M_i$ is $u$-$S$-simple with respect to some $s_i \in S$. Set $s = ts_1 \cdots s_n$. Then each $S_i := M_i$ is $u$-$S$-simple with respect to $s$. One can easily check that $M/T$, as a submodule of $\bigoplus_{i=1}^n M_i$, is $u$-$S$-Noetherian with respect to $s$. Thus $M$ is $u$-$S$-Noetherian with respect to $s$.  

Proposition 4.5. Suppose $R$ is a $u$-$S$-Artinian ring. Then $R/\text{Jac}_S(R)$ is a $u$-$S/\text{Jac}_S(R)$-semisimple ring.

Proof. Write $J = \text{Jac}_S(R)$, $\overline{R} = R/J$ and $\overline{S} = S/\text{Jac}_S(R)$. Since $R$ is a $u$-$S$-Artinian ring, $\overline{R}$ is a $u$-$S$-Artinian $R$-module by Proposition 2.9. By Lemma 4.3, $\text{Jac}_S(\overline{R}) = 0$. It follows from by Proposition 4.4 that there exists an element $s \in S$ and an submodule $T$ of $\overline{R}$ such that $sT = 0$ and $\overline{R}/T \subseteq \bigoplus_{i=1}^n S_i$ with each $S_i$ $u$-$S$-simple with respect to $s$. Now let $F = \overline{R}^{(n)}$ be a free $\overline{R}$-module with $n$ an arbitrary cardinal. Then there is a short exact sequence $0 \rightarrow T^{(n)} \rightarrow \overline{R}^{(n)} \rightarrow (\overline{R}/T)^{(n)} \rightarrow 0$. 

13
By Proposition 4.4, $(\overline{R}/T)^{(n)}$ is a submodule of $(\bigoplus_{i=1}^{t} S_i)^{(n)} = (\bigoplus_{i=1}^{t} (S_i)^{(n)}$ which is $u$-$S$-semisimple by Proposition 3.6. Hence $(\overline{R}/T)^{(n)}$ is also $u$-$S$-semisimple by [17, Proposition 3.3]. Since $sT^{(n)} = 0$, $(\overline{R})^{(n)}$ is a $u$-$S$-semisimple $R$-module. So $(\overline{R})^{(n)}$ is actually a $u$-$\overline{S}$-semisimple $\overline{R}$-module, that is, $\overline{R}$ is a $u$-$\overline{S}$-semisimple ring. \hfill \Box

**Lemma 4.6.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $S^*$ the saturation of $S$. Suppose $r \in R$ and $s \in S$. If $r - s \in \text{Jac}_S(R)$, then $r \in S^*$.

**Proof.** Assume on contrary $r \notin S^*$. Then we claim $R/Rr$ is not $u$-$S$-torsion. Indeed, if $t(R/Rr) = 0$ for some $t \in S$. Then $t = rr'$ for some $r' \in R$. So $r \in S^*$, which is a contradiction. We also claim that there exists a $u$-$S$-maximal ideal $I$ of $R$ such that $r \in I$. Indeed, let $\Lambda$ be the set of ideals $J$ of $R$ that contains $r$ satisfying $R/J$ is not $u$-$S$-torsion. One can check that the union of any ascending chain in $\Lambda$ is also in $\Lambda$. So there is a maximal element $I$ in $\Lambda$ by Zorn Lemma. Hence $I$ is a $u$-$S$-maximal ideal of $R$. Since $\text{Jac}_S(R) \subseteq I$, we have $s \in I$. Then $s(R/I) = 0$, which is a contradiction. So $r \in S^*$.

**Proposition 4.7.** (Nakayama Lemma for $S$-finite modules) Let $R$ be a ring, $I \subseteq \text{Jac}_S(R)$, $S$ a multiplicative subset of $R$ and $M$ an $S$-finite $R$-module. If $sM \subseteq IM$ for some $s \in S$, then $M$ is $u$-$S$-torsion.

**Proof.** Let $F$ be a finitely generated submodule, say generated by $\{m_1, \cdots, m_n\}$, of $M$ satisfying $s'M \subseteq F$ and $sM \subseteq IM$ for some $s', s \in S$. Then $ss'F \subseteq IF$. So we can assume $M$ itself is generated by $\{m_1, \cdots, m_n\}$. Then we have $a := s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n = 0$ where $a_i \in I^i$ by [10, Theorem 2.1]. Note that $aM = 0$ and $a - s^n \in I$. By Lemma 4.6, $a \in S^*$, that is, there is $r \in R$ such that $ar \in S$. Hence $arM = 0$, and so $M$ is $u$-$S$-torsion. \hfill \Box

Let $I$ be an ideal of $R$. we say $I$ is $S$-nilpotent if $sI^k = 0$ for some integer $k$ and $s \in S$.

**Proposition 4.8.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Suppose $R$ is a $u$-$S$-Artinian ring. Then $\text{Jac}_S(R)$ is $S$-nilpotent.

**Proof.** Suppose $R$ is a $u$-$S$-Artinian ring with respect to some $t \in S$. Write $J = \text{Jac}_S(R)$. Consider the descending chain

$$J \supseteq J^2 \supseteq \cdots \supseteq J^n \supseteq J^{n+1} \supseteq \cdots$$

Then there exists an integer $k$ such that $tJ^k \subseteq J^n$ for some $n \geq k$. We claim that $sJ^k = 0$ for some $s \in S$. On contrary, set $\Gamma = \{I \subseteq R \mid sJ^k I \neq 0 \text{ for all } s \in S\}$. Since $R \in \Gamma$, $\Gamma$ is non-empty. So there is an $S$-minimal element $I$ in $\Gamma$ by Theorem
Let \( x \in I \) such that \( sJ^k x \neq 0 \) for all \( s \in S \). Then \( 0 \neq stJ^k x \subseteq sJ^{k+1} x \). So \( sJ^{k+1} x \neq 0 \) for any \( s \in S \). Hence \( Jx \in \Gamma \). Since \( Jx \subseteq Rx \subseteq I \), there exists \( s_1 \in S \) such that \( s_1 Rx \subseteq s_1 I \subseteq Jx \subseteq Rx \) by the \( S \)-minimality of \( I \). So there exists \( s_2 \in S \) such that \( s_2 Rx = 0 \) by Proposition \( 4.7 \) which contradicts \( sJ^k x \neq 0 \) for all \( s \in S \).

Recently, Qi et al. [13, Definition 2.1] introduced the notion of \( u \)-\( S \)-Noetherian rings. A ring \( R \) is called a \( u \)-\( S \)-Noetherian (abbreviates uniformly \( S \)-Noetherian) ring provided there exists an element \( s \in S \) such that for any ideal \( I \) of \( R \), \( sI \subseteq K \) for some finitely generated sub-ideal \( K \) of \( I \). Finally, we will show the promised result.

**Theorem 4.9.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). Then the following statements are equivalent:

1. \( R \) is a \( u \)-\( S \)-Artinian ring;
2. \( R \) is a \( u \)-\( S \)-Noetherian ring, \( \text{Jac}_S(R) \) is \( S \)-nilpotent and \( R/\text{Jac}_S(R) \) is a \( u \)-\( S/\text{Jac}_S(R) \)-semisimple ring.

*Proof.* (1) \( \Rightarrow \) (2) Suppose \( R \) is a \( u \)-\( S \)-Artinian ring with respect to some \( s \in S \). We just need to prove \( R \) is \( u \)-\( S \)-Noetherian because the other two statements are showed in Proposition \( 4.8 \) and Proposition \( 4.5 \) respectively. Write \( J = \text{Jac}_S(R) \). By Proposition \( 4.8 \) there exits an integer \( m \) such that \( tJ^m = 0 \) for some \( t \in S \). We will show \( R \) is \( u \)-\( S \)-Noetherian by induction on \( m \). Let \( m = 1 \). It follows by Proposition \( 4.4 \) that \( R \) is \( u \)-\( S \)-Noetherian. Now, let \( m > 1 \). Set \( \overline{R} = R/J^{m-1} \). Then \( \overline{R} \) is also \( u \)-\( S \)-Artinian by Proposition \( 2.9 \). Note that \( \text{Jac}_S(\overline{R}) = J/J^{m-1} \). So \( \text{Jac}_S(\overline{R}) \) is also \( u \)-\( S \)-torsion. Hence \( \overline{R} \) is \( u \)-\( S \)-Noetherian by induction. Since \( tJ^m = 0 \), \( tJ^{m-1} \) can also be seen as an ideal of \( R/J \). Since \( R/J \) is a \( u \)-\( S/\text{Jac}_S(R) \)-semisimple ring, \( R/J \) is also \( u \)-\( S \)-Noetherian by [17, Corollary 3.6]. So \( tJ^{m-1} \), and hence \( J^{m-1} \), are both \( u \)-\( S \)-Noetherian \( R \)-modules since \( J^{m-1} \) is \( u \)-\( S \)-isomorphic to \( tJ^{m-1} \). Considering the exact sequence \( 0 \to J^{m-1} \to R \to R/J^{m-1} \to 0 \), we have \( R \) is also \( u \)-\( S \)-Noetherian by [13, Lemma 2.12].

(2) \( \Rightarrow \) (1) Write \( J = \text{Jac}_S(R) \). We may assume \( R \) is \( u \)-\( S \)-Noetherian with respect to \( s \) such that \( sJ^m = 0 \) and \( R/J \) is a \( u \)-\( S \)-semisimple \( R \)-module with respect to \( s \). We claim that \( J \) is a \( u \)-\( S \)-Artinian \( R \)-module. Since \( sJ^{m-1} \) is an \( S \)-finite \( R/J \)-module and \( R/J \) is a \( u \)-\( S \)-semisimple ring, we have \( sJ^{m-1} \) is \( u \)-\( S \)-Artinian by Corollary \( 3.8 \). Consider the sequence \( 0 \to sJ^{m-1} \to sJ^{m-2} \to sJ^{m-2}/sJ^{m-1} \to 0 \). Since \( sJ^{m-2}/sJ^{m-1} \) is an \( S \)-finite \( R/J \)-module, we have \( sJ^{m-2}/sJ^{m-1} \) is \( u \)-\( S \)-Artinian by
Corollary 3.8 Since $sJ^{m-1}$ is $u$-$S$-Artinian, $sJ^{m-2}$ is also $u$-$S$-Artinian by Proposition 2.9 Iterating these steps, we have $sJ$ is also $u$-$S$-Artinian. So $J$ is also $u$-$S$-Artinian since $J$ is $u$-$S$-isomorphic to $sJ$.

Let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of ideals of $R$. Consider the following natural commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & I_i \cap J & \rightarrow & I_i & \rightarrow & (I_i + J)/J & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & I_{i+1} \cap J & \rightarrow & I_{i+1} & \rightarrow & (I_{i+1} + J)/J & \rightarrow & 0.
\end{array}
\]

Since $J$ is $u$-$S$-Artinian, there is an integer $k_1$ such that $s(I_{k_1} \cap J) \subseteq I_n \cap J$ for $n \geq k_1$. Since $R/J$ is a $u$-$S/J$-semisimple ring, $R/J$ is also a $u$-$S/J$-Artinian ring by Theorem 3.7. Hence there is an integer $k_2$ such that $s((I_{k_2} + J)/J) \subseteq (I_n + J)/J$ for $n \geq k_2$. Taking $k = \max\{k_1, k_2\}$, we can easily deduce $sI_k \subseteq I_n$ for $n \geq k$. Hence $R$ is a $u$-$S$-Artinian ring. □

Let $R$ be a commutative ring and $M$ an $R$-module. Then the idealization of $R$ by $M$, denoted by $R(+)M$, is equal to $R \bigoplus M$ as $R$-modules with coordinate-wise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. It is easy to verify that $R(+)M$ is a commutative ring with identity $(1, 0)$ (see [5] for more details). Note that there is a natural exact sequence of $R(+)M$-modules:

\[0 \rightarrow 0(+)M \rightarrow R(+)M \xrightarrow{\pi} R \rightarrow 0.\]

Let $S$ be a multiplicative subset of $R$. Then it is easy to verify that $S(+)M = \{(s, m)|s \in S, m \in M\}$ is a multiplicative subset of $R(+)M$. Now, we give a $u$-$S$-Artinian property on idealizations.

Corollary 4.10. Let $R$ be a commutative ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. Then $R(+)M$ is a $u$-$S(+)M$-Artinian ring if and only if $R$ is a $u$-$S$-Artinian ring and $M$ is an $S$-finite $R$-module.

Proof. Suppose $R(+)M$ is a $u$-$S(+)M$-Artinian ring. Then $R \cong R(+)M/0(+)M$ is a $u$-$S$-Artinian ring essentially by Proposition 2.9. By Theorem 4.9, $R(+)M$ is a $u$-$S(+)M$-Noetherian ring. So $0(+)M$ is an $S(+)M$-finite ideal of $R(+)M$, which implies that $M$ is an $S$-finite $R$-module.

Suppose $R$ is a $u$-$S$-Artinian ring and $M$ is an $S$-finite $R$-module. Then $M$ is $u$-$S$-Artinian $R$-module by Proposition 2.9. Let $I^*: I_1 \supseteq I_2 \supseteq \cdots$ be an descending chain of ideals of $R(+)M$. Then there is an descending chain of ideals of $R$: $\pi(I^*): \pi(I_1) \supseteq \pi(I_2) \supseteq \cdots$, where $\pi: R(+)M \rightarrow R$ is the natural epimorphism. Thus there is an element $s' \in S$ which is independent of $I^*$ satisfying that there exists $k' \in \mathbb{Z}^+$
such that \( s' \pi(I_{k'}) \subseteq \pi(I_n) \) for any \( n \geq k' \). Similarly, \( I^* \cap 0(+)_M : I_1 \cap 0(+)_M \supseteq I_2 \cap 0(+)_M \supseteq \cdots \) is an descending chain of sub-ideals of \( 0(+)_M \) which is equivalent to a descending chain of submodules of \( M \). So there is an element \( s'' \in S \) satisfying that there exists \( k'' \in \mathbb{Z}^+ \) such that \( s''(I_{k''} \cap 0(+)_M) \subseteq I_n \cap 0(+)_M \) for any \( n \geq k'' \). Let \( k = \max(k', k'') \) and \( n \geq k \). Consider the following natural commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & I_n \cap 0(+)_M & \longrightarrow & I_n & \longrightarrow & \pi(I_n) & \longrightarrow & 0 \\
& & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & I_k \cap 0(+)_M & \longrightarrow & I_k & \longrightarrow & \pi(I_k) & \longrightarrow & 0.
\end{array}
\]

Set \( s = s's'' \). Then we have \( sI_k \subseteq I_n \) for any \( n \geq k \). So \( R(+)M \) is a \( u\)-\( S(+)M \)-Artinian ring. \( \square \)

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