Gauge couplings in four-dimensional Type I string orbifolds

I. Antoniadis\textsuperscript{a}, C. Bachas\textsuperscript{b} and E. Dudas\textsuperscript{c}

\textsuperscript{a} Centre de Physique Théorique\textsuperscript{†} Ecole Polytechnique, F-91128 Palaiseau, France
\textsuperscript{b} Laboratoire de Physique Théorique, ENS, F-75231 Paris, France
\textsuperscript{c} LPT\textsuperscript{‡}, Bât. 210, Univ. Paris-Sud, F-91405 Orsay, France

Abstract

We compute threshold effects to gauge couplings in four-dimensional $Z_N$ orientifold models of type I strings with $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetry, and study their dependence on the geometric moduli. We also compute the tree-level (disk) couplings of the open sector gauge fields to the twisted closed string moduli of the orbifold in various models and study their effects and that of the one-loop threshold corrections on gauge coupling unification. We interpret the results from the (supergravity) effective theory point of view and comment on the conjectured heterotic-type I duality.

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\textsuperscript{†}Unité mixte de recherche du CNRS (UMR 7644).
\textsuperscript{‡}Unité mixte de recherche du CNRS (UMR 8627).

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1. Introduction

The purpose of this paper is to study threshold corrections to gauge couplings in certain $\mathcal{N} = 2$ and $\mathcal{N} = 1$ type I orientifolds [1]-[9]. Such vacua are of obvious phenomenological interest, but for a long time they had received much less attention than their weakly-coupled heterotic counterparts. As has been, however, recognized recently, type I vacua offer some added flexibility for model building, and can exhibit several novel interesting features. In particular, since gauge interactions are localized on D-branes the tree-level relations between gauge couplings and the string, compactification and Planck scales are not universal [10, 11], making it easier to consider scenarios in which these scales are hierarchically-different [12]-[20]. The question of whether such scenarios can be reconciled with the apparent unification of the observed low-energy couplings was the main motivation for this work.

Type I vacua are furthermore conjectured to have dual heterotic descriptions in appropriate circumstances [21]-[23]. Thus, another motivation for the present work has been to further elucidate this duality in the most interesting $\mathcal{N} = 1$ context [4, 7], by a detailed comparison of threshold corrections on the two sides, even if they cannot be simultaneously weakly coupled in ten dimensions. Moreover, the knowledge of the moduli dependence of gauge couplings is required for any phenomenological application of type I string theory, and is intimately related to the problems of supersymmetry breaking and possible dynamical determination of the moduli vacuum expectation values (VEVs).

Threshold corrections to gauge couplings in heterotic vacua have been studied extensively in the past and general results have been obtained for orbifold and smooth manifold compactifications [24, 25]. Their comparison to the effective field theory is particularly instructive and reveals the existence of a universal non-holomorphic correction associated to the so called Green-Schwarz term [26, 27]. On the type I side on the other hand, previous study was focused on the $\mathcal{N} = 2$ orientifold based on the $T^4/Z^2$ orbifold [28, 29].
In this work, we first generalize the studies to other $\mathcal{N} = 2$ orientifolds obtained by toroidal compactifications of six-dimensional (6d) vacua and compute the general one-loop dependence of gauge couplings on the geometric moduli $T \sim R_1 R_2$ and $U \sim R_1 / R_2$ of the two-torus with radii $R_{1,2}$. For vanishing Wilson lines, the corrections are proportional to the beta function coefficients and therefore they amount changing the unification scale from the string to the compactification size. The latter can be much larger than the string scale in a weakly coupled theory, in which case it is more appropriate to identify it as the winding scale of the T-dual theory. This result allows two possibilities for unification. (i) When the two radii are different $R_1 > R_2$, there is a linear correction $R_1 / R_2$ \cite{17} that can be interpreted as a power-law evolution \cite{30, 15} between $1/R_1$ and $1/R_2$, leading to unification at $1/R_2$ \cite{15}. (ii) If the two radii are approximately equal $R_1 \simeq R_2$, there is only a logarithmic correction which could possibly accommodate a “conventional” unification scenario at energies much larger than the string scale \cite{17, 18}. Threshold corrections can alternatively be interpreted as tree-level dependence of bulk fields on the transverse space \cite{18}. These corrections are non-universal (group dependent) if tree-level couplings to bulk fields are different for the various gauge group factors. In fact, there is a tree-level (disk) dependence of gauge couplings on the twisted Neveu-Schwarz (NS) moduli associated to twists different than $Z_2$’s. This dependence is already present at the level of six-dimensional theory, where these moduli are part of tensor multiplets \cite{31, 32}. Although in some particular examples this new (tree-level) contribution is miraculously proportional to the (one-loop) beta-function coefficients, in which case the unification scale is arbitrary, in the generic case the couplings become free parameters and the unification is lost.

These results are generalized easily in the case of 4d compactifications with $\mathcal{N} = 1$ supersymmetry. The geometric moduli dependence of the one-loop threshold corrections receives contributions from the $\mathcal{N} = 2$ sectors only, controled by the $\mathcal{N} = 2$ beta-functions, in analogy with the heterotic case. On the other hand, $\mathcal{N} = 1$ sectors provide moduli independent contributions that can be interpreted as one-loop running up to the string
scale. As a result, if the string scale is low the only way to achieve unification is through the $\mathcal{N} = 2$ beta-functions. However, as in $\mathcal{N} = 2$ compactifications, there is a non-universal tree-level dependence of the gauge couplings on the twisted NS-NS moduli for twists different than $Z_2$'s \cite{33}, which apriori can destroy unification. In this case though, a new phenomenon appears due to the existence of anomalous $U(1)$'s. At the points of maximal gauge symmetry, the VEVs of the twisted moduli are fixed and unification is recovered up to one-loop level.

We also compare our results to the effective supergravity. Unlike the heterotic case, we find that compatibility requires the existence of several non-universal (gauge group dependent) Green-Schwarz terms associated to the twisted NS-NS moduli \cite{34}, which now belong into several linear multiplets. In the $\mathcal{N} = 1$ orientifold examples the vanishing of the anomalous $U(1)$'s D-terms at the points of maximal gauge symmetry, imply a vanishing VEV for the twisted moduli coupled to gauge fields, in the string (linear multiplet) basis. This implies in particular for the $Z_{\text{odd}}$ orbifolds that physical gauge couplings have no dependence on geometric moduli up to one-loop level and unification arises at the type I string scale\cite{35}. Moreover, this result raises doubts on the validity of the conjectured heterotic – type I dual pairs with $\mathcal{N} = 1$ supersymmetry.

The paper is organized as follows. In section 2, we recall open string propagation in constant magnetic fields, that provides the method we use for computing the corrections to gauge couplings. In section 3, we derive the moduli dependence to gauge couplings of generic 4d $\mathcal{N} = 2$ orientifolds obtained by toroidal compactification from six dimensions. In section 4, we study in detail gauge coupling in the context of the 4d $\mathcal{N} = 1$ $Z_{\text{odd}}$ ($Z_3$ and $Z_7$) orientifolds, while in section 5, we study the $\mathcal{N} = 1$ $Z_6'$ and we obtain general expressions valid for any orientifold. In section 6, we compute threshold corrections in

\footnote{This result was also obtained recently in ref. \cite{34}, while our work was in print.}

\footnote{This result seems to contradict the recent claim of ref. \cite{35}.}
models with spontaneous $\mathcal{N} = 4 \to \mathcal{N} = 2$ supersymmetry breaking. In section 7, we compare our results with the effective field theory and comment on heterotic – type I duality. In section 8, we discuss anomalous $U(1)$’s and implications to gauge coupling unification. Our conclusions are presented in section 9.

2. Background-field method

The one-loop diagrams of type-I string theory are the torus $\mathcal{T}$, the Klein bottle $\mathcal{K}$, the annulus $\mathcal{A}$, and the Möbius strip $\mathcal{M}$. The diagrams with boundaries ($\mathcal{A}$ and $\mathcal{M}$) describe in the direct channel the propagation of the open string degrees of freedom, including gauge bosons and charged matter fields. The other two diagrams ($\mathcal{T}$ and $\mathcal{K}$) have no boundaries coupling to Chan-Paton charges – they describe the propagation in the loop of closed-string degrees of freedom related to the gravitational sector of the theory. Since the $\mathcal{T}$ and $\mathcal{K}$ diagrams cannot couple to external open-string states, they do not contribute to the renormalization of couplings in the open sector of the theory. These diagrams will not be of interest to us in the present work.

We will here focus our attention to four-dimensional orientifolds obtained by orbifolding the six real (three complex) internal coordinates by the twist $\theta = (e^{2i\pi v_1}, e^{2i\pi v_2}, e^{2i\pi v_3})$, where $\mathbf{v} \equiv (v_1, v_2, v_3)$ is called the twist vector and where for a $Z_N$ orbifold $\theta^N = 1$. The two one-loop amplitudes of interest can be written generically as

$$A = -\frac{1}{2N} \sum_{k=0}^{N-1} \int_0^\infty \frac{dt}{t} \text{Str} \left( \theta^k q(p^\mu p_\mu + m^2)/2 \right) \equiv -\frac{1}{2N} \sum_{k=0}^{N-1} \int_0^\infty \frac{dt}{t} A^{(k)}(q),$$

$$M = -\frac{1}{2N} \sum_{k=0}^{N-1} \int_0^\infty \frac{dt}{t} \text{Str} \left( \Omega \theta^k q(p^\mu p_\mu + m^2)/2 \right) \equiv -\frac{1}{2N} \sum_{k=0}^{N-1} \int_0^\infty \frac{dt}{t} M^{(k)}(-q),$$

(2.1)

where $m^2$ is the mass squared operator in four dimensions, the Regge slope $\alpha' = 1/2$, the
modular parameter of the ‘doubling torus’ (from which $A$ and $M$ are obtained by a $Z_2$ identification) is given by

$$\tau = \frac{it}{2} \quad \text{for } A, \quad \text{and} \quad \tau = \frac{it}{2} + \frac{1}{2} \quad \text{for } M,$$

and finally $q = e^{-\pi t}$. In the Möbius amplitude $\Omega$ is the world-sheet involution operator which exchanges left and right moving excitations of the closed string and acts as a phase on the open string oscillators: $\Omega \alpha_m = \pm e^{i\pi m} \alpha_m$, with the upper plus sign for the NN coordinates and the lower minus sign for the DD coordinates ($N=$Neumann, $D=$Dirichlet).

The supertrace stands for a sum over all open-string states – this includes the sum over Chan-Paton states of the two endpoints and the integration over four-momenta,

$$Str = \left( \sum_{bos} - \sum_{ferm} \right) \int \frac{d^4p}{(2\pi)^4}.$$  \hspace{1cm} (2.3)

The action of a twist element $\theta^k$ on the $n$ possible Chan-Paton states is realized by $n \times n$ matrices $\gamma^k$. In the simplest cases we have $n = 32$, corresponding to the 32 D9-branes of the ten-dimensional theory. More generally however the four-dimensional orientifold may also contain D5-branes, corresponding to additional Chan-Paton states. Conversely $n$ and the rank of the gauge group can be reduced by turning on $B_{\mu\nu}$ backgrounds in the internal manifold \cite{36}. Tadpole consistency conditions \cite{37} severely constrain the Chan-Paton matrix $\gamma$, and hence also the gauge group and the charged matter content of the corresponding vacuum.

We will compute the one-loop corrections to the gauge couplings using the background-field method \cite{38,28}. To this end we turn on a magnetic field in the (for example) $x^1$ direction

$$F_{23} = BQ,$$  \hspace{1cm} (2.4)

where $x^0 \cdots x^3$ are the uncompactified spacetime dimensions, and $Q$ is an appropriately normalized generator of the gauge group. The effect of the magnetic field on the open-string spectrum \cite{39} is to shift the oscillator modes of the complex $X^2 + iX^3$ string coordinate by
an amount $\epsilon$, where
\[
\pi \epsilon = \arctan(\pi q_L B) + \arctan(\pi q_R B)
\]
(2.5)
and $q_{L(R)}$ is the eigenvalue of the gauge-group generator $Q$ acting on the Chan-Paton states at the left(right) endpoint of the open string. For notational simplicity, the dependence of $\epsilon$ on the Chan-Paton states will be left implicit in the sequel. The torus and Klein-bottle amplitudes are not affected by the magnetic field, while the annulus and the Möbius strip are obtained by making in (2.1) the replacements \cite{38, 39}
\[
p^\mu p_\mu \rightarrow -(p_0)^2 + (p_1)^2 + (2n + 1)\epsilon + 2\epsilon \Sigma_{23}
\]
(2.6)
where $\Sigma_{23}$ is the spin operator in the (23) direction, the integer $n = 0, 1, \cdots$ labels the Landau levels, and $(q_L + q_R)B/2\pi$ is the degeneracy of Landau levels per unit area.

The full one-loop vacuum energy has the weak-field expansion
\[
\Lambda(B) = \frac{1}{2}\left(T + \mathcal{K} + \mathcal{A}(B) + \mathcal{M}(B)\right)
\]
\[
\equiv \Lambda_0 + \frac{1}{2}\left(\frac{B}{2\pi}\right)^2 \Lambda_2 + \frac{1}{24}\left(\frac{B}{2\pi}\right)^4 \Lambda_4 + \cdots
\]
(2.7)
For supersymmetric compactifications the one-loop cosmological constant vanishes, $\Lambda_0 = 0$.

The term quadratic in the background field contains the one-loop threshold corrections. More precisely, if we choose $Q$ to be an appropriately normalized generator inside the $a$th factor of the gauge group, then the one-loop corrected gauge coupling for this factor reads
\[
\frac{4\pi^2}{g_a^2} \bigg|_{\text{one loop}} = \frac{4\pi^2}{g_a^2} \bigg|_{\text{tree}} + \Lambda_{2,a}
\]
(2.8)
The loop correction has the structure of an integral
\[
\Lambda_{2,a} = \int_0^\infty \frac{dt}{4t} B_a(t)
\]
(2.9)
3The factor groups in the examples of interest will have a natural embedding into $SO(32)$. Thus we will consider the $F_{\mu\nu}$’s as matrices in the fundamental representation of $SO(32)$, with generators normalized so that $\text{tr}Q^2 = 1$. We will write the Yang-Mills lagrangians as $\mathcal{L}_{YM} = \text{tr}_a F_{\mu\nu} F^{\mu\nu}/4g_a^2$ — with these conventions the tree-level gauge couplings of the various factors $G_a$ are all equal.
with the upper and lower limits corresponding, respectively, to the ultraviolet and the infrared regions in the open channel. As explained in references [28], [17] in the context of the $Z_2$ orientifold [4], the integral must converge in the ultraviolet limit if all the tadpoles have been cancelled globally, and provided that the background field has no component along an anomalous $U(1)$ factor. The potential infrared divergences, on the other hand, are due to massless charged particles circulating in the loop, so that

$$\lim_{t \to \infty} B_a(t) = b_a$$

(2.10)

is the $\beta$-function coefficient of the effective field theory at energies much lower than the last massive threshold.

The quartic term in the expansion (2.7) is quadratically divergent due to the on-shell exchange of massless closed-string modes [28], [22]. These include the dilaton, the graviton and, in the cases of interest to us here, the twisted NS-NS moduli fields $m_k$ of the orbifold. These twisted moduli have non-universal couplings to the gauge fields of the gauge group factor $G_a$,

$$\mathcal{L}_{YM} = \left( \frac{1}{4g_a^2} + \sum_{k=1}^{\lfloor (N-1)/2 \rfloor} \frac{s_{ak}}{16\pi^2} m_k \right) \text{tr}_a F_{\mu\nu} F^{\mu\nu},$$

(2.11)

as conjectured, using anomaly-cancellation arguments and supersymmetry, in ref. [31, 32, 33]. In eq. (2.11), the sum over $k$ goes up to the integer part of $(N - 1)/2$ which counts the number of independent twisted sectors of the orbifold. Note that the supersymmetric partners of the $m_k$ are RR axions, dual to antisymmetric two-index tensors (NS=Neveu-Schwarz, R=Ramond). By analyzing the divergences of $\Lambda_4$ we will calculate the coefficients $s_k$ explicitly, and confirm the conjecture of [33]. This is important for discussions of unification, since such twisted modes could give rise to non-universal shifts of the gauge coupling constants at tree level.

In order to analyze the divergences of $\Lambda_4$ we must reexpress the amplitude as an integral over the modulus $l$ in the transverse (closed-string) channel. This is related to the direct
channel modulus as follows,

\[ l = \frac{1}{t} \text{ for } A, \quad l = \frac{1}{4t} \text{ for } M. \]  

(2.12)

The elliptic functions in the integrands can be reexpressed in terms of \( l \) by using the modular transformations

\[ \tau = \frac{it}{2} \rightarrow -\frac{1}{\tau} = 2il \]  

(2.13)

for \( A \), and

\[ \tau = \frac{it}{2} + \frac{1}{2} \rightarrow -\frac{1}{\tau} \rightarrow \frac{1}{\tau} + 2 \rightarrow (\frac{1}{\tau} - 2)^{-1} = 2il - \frac{1}{2} \]  

(2.14)

for \( M \). After this change of variable the quartic term will take the form

\[ \Lambda_4 = -\frac{1}{4N} \sum_{k=0}^{N-1} \int_0^{\infty} \frac{dl}{l} \left\{ A_4^{(k)}(\bar{q}) + M_4^{(k)}(-\bar{q}) \right\}, \]  

(2.15)

where \( \bar{q} = e^{-4\pi l} \), and \( A_4^{(k)} \) and \( M_4^{(k)} \) are the coefficients in the Taylor expansion of the corresponding integrands at quartic order in \( B/2\pi \). These grow linearly at \( l \rightarrow \infty \), corresponding to a quadratic infrared divergence in the closed-string channel. The divergence in the untwisted \((k = 0)\) sector comes from the exchange of a graviton and dilaton and has been analyzed in [28, 22]. For even \( N \) the \( k = N/2 \) sector has no quadratic divergences, consistently with the fact that \( Z_2 \) twist fields do not couple to the Yang-Mills action. \footnote{This is obvious in six dimensions, where the \( Z_2 \) twist fields belong to hypermultiplets that do not couple to the vectors.}

For the remaining sectors \((k = 1, \cdots, \lfloor N/2 \rfloor)\) we will show that

\[ A_4^{(k)} + M_4^{(k)} = 3N\pi \ l \ s_k^2 + \cdots, \]  

(2.16)

where the dots stand for exponentially-suppressed terms and the precise values of the coefficients \( s_k \) depend on the model. This result is consistent with our interpretation of the corresponding divergence in \( \Lambda_4 \), as coming from the exchange of an on-shell twist field coupling to the background \( F_{\mu\nu} \) through equation (2.11). Notice that the closed-string propagator for a canonically normalized scalar is

\[ \Delta_{\text{closed}} = \frac{\pi}{2} \int_0^{\infty} dl \ e^{-\frac{2l}{2}(p^{\mu}p_{\mu}+M_{\text{closed}}^2)} , \]  

(2.17)
with \( l \) the modulus of the cylinder. The divergence of an on-shell propagator can thus be written formally as \( \pi f^\infty dl \).

3. \( \mathcal{N} = 2 \) supersymmetry: \( K3 \times T^2 \) orientifolds

We begin our discussion with six-dimensional \( Z_N \) models \([2, 3, 4]\), compactified further down to four dimensions on a two-torus. The twist vector for these models is of the form \( \mathbf{v} = (1/N, -1/N, 0) \). Assuming no antisymmetric tensor backgrounds, tadpole cancellation requires the presence of 32 D9 branes, and for even \( N \) of one set of 32 D5 branes. It also fixes the matrices \( \gamma_9 \) and \( \gamma_{\Omega,9} \) which represent the action of the orbifold twist \( \theta \) and the orientation reversal \( \Omega \) on Chan-Paton states in the D9 sector, as well as the corresponding ones in the D5-brane sector.\(^5\) The particular case of the \( Z_2 \) orientifold, \( N = 2 \), is the one analyzed previously in reference \([28]\). In this section we will extend the analysis to the other models of type A (using the language of \([3]\)), namely the \( Z_3, Z_4 \) and \( Z_6 \) orbifolds.

We will consider a background field living on the D9 branes of these models. To simplify the formulae we will further restrict our attention to the case where all D5-branes are located together at a fixed point of the orbifold, and there are no (99) Wilson lines. The amplitudes of interest are the (99) annulus and Möbius diagrams with insertion of a non-trivial twist \( \theta^k \) (\( k \neq 0 \)) and, in the presence of D5 branes, the (95) annulus with or without insertion of a twist. The (55) diagrams do not couple to the background field, while the untwisted (99) diagrams have effectively \( \mathcal{N} = 4 \) supersymmetries and thus do not contribute to the renormalization of gauge couplings. In the absence of a magnetic field the relevant amplitudes are (for \( k \neq 0 \))

\(^5\) Consistently with the group property the action of \( \theta^k \) and of \( \Omega \theta^k \) can be represented by the product matrices \( \gamma^k \) and \( \gamma^k \gamma_{\Omega} \) respectively.
and if the model has D5-branes (N even, all k)

\[ A^{(k)}_{99} = -\text{tr}(\gamma_9^k \otimes \gamma_9^k) \frac{\Gamma^{(2)}(t)}{4\pi^4 t^2} \sum_{\alpha,\beta=0,1/2} \frac{1}{2} \eta_{\alpha,\beta} \vartheta^{[\alpha]}_{[\beta]} \frac{\vartheta^{[\alpha]}_{[\beta]}}{\eta^6} \times (2 \sin \frac{\pi k}{N})^2 \left( \vartheta^{[\alpha]}_{[\beta+k/N]}\vartheta^{[\alpha]}_{[\beta-k/N]} \right), \]  

(3.1)

\[ M^{(k)}_{99} = \text{tr}(\gamma_9^{2k}) \frac{\Gamma^{(2)}(t)}{4\pi^4 t^2} \sum_{\alpha,\beta=0,1/2} \frac{1}{2} \eta_{\alpha,\beta} \vartheta^{[\alpha]}_{[\beta]} \frac{\vartheta^{[\alpha]}_{[\beta]}}{\eta^6} \times (2 \sin \frac{\pi k}{N})^2 \left( \vartheta^{[\alpha]}_{[\beta+k/N]}\vartheta^{[\alpha]}_{[\beta-k/N]} \right), \]  

(3.2)

In these expressions \( \eta_{\alpha,\beta} = (-1)^{2(\alpha+\beta+2\alpha\beta)} \) are the usual phases depending on the spin structures \((\alpha, \beta)\) and specifying the GSO projection. The definitions of the Jacobi functions \( \vartheta^{[\alpha]}_{[\beta]} \) and of the Dedekind function \( \eta \) are given in appendix A, while their argument \( [2,2] \) is here left implicit. The factor of 2 in front of the \((95)\) diagram counts the two orientations of the open string, and \( \Gamma^{(2)}(t) \) is the lattice sum over momenta along the untwisted two-torus.

Finally, the trace in the annulus amplitudes is over the tensor product of Chan-Paton states for the left and the right endpoints of the open string, while in the Möbius amplitude the left and right Chan-Paton charges are equal and we have used the identity \( \text{tr} \gamma^{2k} = \text{tr}(\gamma^{-1}_{109k}\gamma^{T}_{109k}) \) (see for instance [8, 9]).

The above vacuum amplitudes vanish of course by virtue of the space-time supersymmetry. Turning on the background magnetic field modifies these expressions as follows,

\[ A^{(k)}_{99}(B) = -iB \frac{\Gamma^{(2)}(t)}{4\pi^3 t} \sum_{\alpha,\beta=0,1/2} \frac{1}{2} \eta_{\alpha,\beta} \vartheta^{[\alpha]}_{[\beta]} \frac{\vartheta^{[\alpha]}_{[\beta]}}{\eta^3} \times (2 \sin \frac{\pi k}{N})^2 \left( \vartheta^{[\alpha]}_{[\beta+k/N]}\vartheta^{[\alpha]}_{[\beta-k/N]} \right), \]  

(3.3)

\[ M^{(k)}_{99}(B) = iB \frac{\Gamma^{(2)}(t)}{2\pi^3 t} \sum_{\alpha,\beta=0,1/2} \frac{1}{2} \eta_{\alpha,\beta} \vartheta^{[\alpha]}_{[\beta]} \frac{\vartheta^{[\alpha]}_{[\beta]}}{\eta^3} \times (2 \sin \frac{\pi k}{N})^2 \left( \vartheta^{[\alpha]}_{[\beta+k/N]}\vartheta^{[\alpha]}_{[\beta-k/N]} \right), \]  

(3.4)
and

\[
A^{(k)}_{95}(B) = iB \frac{\Gamma^{(2)}(t)}{2\pi^3 t} \sum_{\alpha,\beta=0,1/2} \frac{1}{2} \eta_{\alpha,\beta} \frac{\vartheta^{[\alpha]}_{[\beta]}}{\eta^3} \operatorname{tr} \left( (Q^{k}_9 \otimes \gamma^k_5) \frac{\vartheta^{[\alpha]}_{[\beta]}}{\vartheta^{[1/2]}_{[1/2]}} \right) 
\times \frac{\vartheta^{[\alpha+1/2]}_{[\beta+k/N]} \vartheta^{[\alpha+1/2]}_{[\beta-k/N]}}{\vartheta^{[0]}_{[1/2+k/N]} \vartheta^{[0]}_{[1/2-k/N]}}.
\] (3.5)

As previously, the \( \tau \) argument of the Jacobi functions \( \vartheta^{[\alpha]}_{[\beta]}(z|\tau) \) is implicit, while the argument \( z \) is only shown when it is non-zero. Note that the shift \( \epsilon \) of the oscillator frequencies depends on the charges of the left and the right string endpoint – the corresponding Jacobi functions have therefore been left inside the Chan-Paton trace. More explicitly in view of the definition (2.5) we have

\[
\pi \epsilon = \begin{cases} 
\arctan(\pi BQ) \otimes 1 + 1 \otimes \arctan(\pi BQ) & \text{in } A_{99}, \\
2 \arctan(\pi BQ) & \text{in } M_{99}, \\
\arctan(\pi BQ) & \text{in } A_{95}.
\end{cases}
\] (3.6)

The lattice sum and the orbifold partition function, on the other hand, are independent of the Chan-Paton states since we have assumed vanishing Wilson lines.

In order to compute threshold corrections, we must expand the above formulae to quadratic order in the background field. We need the following Taylor expansions

\[
\epsilon \simeq (q_L + q_R) B + o(B^3),
\]

\[
\vartheta_1(z) \simeq 2\pi \eta^3 z + o(z^3),
\]

\[
\vartheta_a(z) \simeq \vartheta_a + \frac{z^2}{2} \vartheta''_a + o(z^4) \text{ for } a = 2, 3, 4.
\] (3.7)

The integrands simplify enormously if one uses the modular identities (A8) of appendix A which reduce the entire string-oscillator sum to a number. As explained in [28], [40], [22] this is a consequence of \( \mathcal{N} = 2 \) supersymmetry: only short BPS multiplets can contribute corrections to the gauge couplings, and all open-string excitations are non-BPS. After some straightforward algebra the final result for the one-loop corrections (2.9) takes the form

\[
B_a(t) = b_a \times \Gamma^{(2)}_{\text{reg}}(t)
\] (3.8)
with
\[ b_a = -\frac{1}{N} \sum_{k=0}^{N-1} \left\{ 4 \sin^2 \frac{\pi k}{N} \left[ \text{tr}(Q_a^2 \gamma_9^k)\text{tr}\gamma_9^k - 2 \text{tr}(Q_a^2 \gamma_9^{2k})\right] - \text{tr}(Q_a^2 \gamma_9^k)\text{tr}\gamma_9^k \right\}. \] (3.9)

We have here used the obvious identity \( \text{tr}D \otimes E = \text{tr}D\text{tr}E \), and the fact that if \( Q \) does not have a component along an anomalous U(1), all the traces involving an odd power of \( Q \) are zero. Note also that \( Q \) must commute with \( \gamma_9 \) or else the corresponding gauge field would not have survived the orbifold projection.

The lattice sum in equation (3.8) has been regularized by the ‘Poisson-resummation prescription’ [22, 17]. Let us introduce the usual geometric modulus of the two-torus (which spans the dimensions 4 and 5),
\[ U = \frac{G_{45} + i\sqrt{G}}{G_{55}}, \] (3.10)
normalized so that \( \text{Im}U = R_4/R_5 \) and \( \sqrt{G} = R_4R_5 \) on a rectangular torus. The regularized Kaluza-Klein sum is
\[ \Gamma^{(2)}_{\text{reg}}(t) = \sum_{n_4,n_5} e^{-\pi t|n_4 + n_5 U|^2/(\sqrt{G}\text{Im}U)} - \frac{\pi \sqrt{G}}{t}. \] (3.11)

The subtraction term corresponds to the propagation of massless closed-string states in the transverse channel – it vanishes by global tadpole cancellation after adding all diagrams and imposing a homogeneous cutoff in the transverse proper time \( l \).\footnote{This amounts to different ultraviolet cutoffs in the direct channel for the annulus and the Möbius diagrams, see the relations (2.12).} We will explain how this happens in more detail in a minute. In the limit \( t \to \infty \) we find \( \Gamma^{(2)}_{\text{reg}} \to 1 \), so that \( b_a \) must the \( \beta \)-function coefficient of the four-dimensional theory.

We have checked that in all the models of [3] expression (3.9) coincides with the standard expression for the \( \mathcal{N} = 2 \) \( \beta \)-functions,
\[ b_a = 2 \sum_r T_a(r) - 2T_a(G), \] (3.12)
where the hypermultiplets transform in the representations $r$ of the gauge group $G_a$. The gauge groups, charged hypermultiplets, and the Chan-Paton matrix $\gamma_9$ for these models are summarized for convenience in table 1. Following the conventions of [3] we use a complex basis in which $\gamma_9$ is a diagonal matrix whose eigenvalues are phases occurring in complex-conjugate pairs. For instance for the $Z_3$ model

$$\gamma_9 = \text{diag} \left( e^{2\pi i/3} (8\text{ times}) \quad e^{-2\pi i/3} (8\text{ times}) \quad 1(16\text{ times}) \right).$$

In a self-explanatory notation we will write $\gamma_9 = \left( e^{2\pi i/3} I_8; I_8 \right)$ to denote this matrix. Note also that the twist matrix for the D5-branes can be chosen such that $\gamma_5 = e^{2\pi im/N} \gamma_9$ for any odd integer $m$ [3].

| Model | $\gamma_9$ | 99 Gauge Group | Charged Hypermultiplets |
|-------|------------|----------------|------------------------|
| $Z_2$ | $(iI_{16})$ | U(16)          | $2 \times 120; 16 \times 16$ |
| $Z_3$ | $(e^{\frac{2\pi i}{3}} I_8, I_8)$ | $U(8) \times SO(16)$ | $(28,1); (8,16)$ |
| $Z_4$ | $(e^{\frac{\pi i}{4}} I_8, e^{\frac{3\pi i}{4}} I_8)$ | $U(8) \times U(8)$ | $(28,1); (1,28); (8,8)$ |
| $Z_6$ | $(e^{\frac{\pi i}{6}} I_4, e^{\frac{5\pi i}{6}} I_4, iI_8)$ | $U(4)^2 \times U(8)$ | $(6,1,1); (1,6,1); (4,1,8); (1,4,8)$ |

Table 1: The K3 orientifolds of type A [3], their gauge groups living on D9-branes and corresponding charged hypermultiplets. Our notation for the Chan-Paton twist matrix is explained in the text.

The expressions (2.9) and (3.8) for the one-loop corrections to the gauge couplings, obtained at the special symmetric points in moduli space, can be in fact generalized easily to any $K3 \times T^2$ orientifold of type-I string theory. One needs only to replace

$$b_a \rightarrow 2 \sum_r T_a(r) e^{-\pi t M_r^2} - 2T_a(G)$$

(3.14)
in (3.8), with $M_r$ the masses of the hypermultiplets in six dimensions, and modify the Kaluza-Klein sum appropriately if there are non-vanishing Wilson lines on the two-torus. The potential $t \rightarrow 0$ divergence of the integral, and hence also the subtraction term in (3.11), are not affected by these ‘soft’ corrections.

Let us turn now to the ultraviolet divergences of the amplitudes (3.3–3.5). To investigate them we must reexpress the amplitudes as integrals over the proper time $l$ in the closed-string channel. Performing the sequences (2.13,2.14) of modular transformations, a Poisson resummation of the Kaluza-Klein sum, and using the well-known modular properties of the Dedekind and Jacobi functions (see appendix A) one finds after some algebra

\[
A^{(k)}_{99}(B) = -Bl \frac{W^{(2)}(l)}{16\pi^3} \sqrt{G} \sum_{\alpha,\beta=0,1/2} \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \left( (Q\gamma^k_9 \otimes \gamma^k_9 + Q^{-k}_9 \otimes Q^{k}_9) \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \right) \\
\times (2\sin \frac{\pi k}{N})^2 \frac{\vartheta[\frac{\beta+k}{\alpha}]}{\vartheta[\frac{1/2+k}{\alpha}]} \frac{\vartheta[\frac{\beta-k}{\alpha}]}{\vartheta[\frac{1/2-k}{\alpha}]} ,
\]

(3.15)

\[
M^{(k)}_{99}(B) = Bl \frac{W^{(2)}(4l)}{\pi^3} \sqrt{G} \sum_{\alpha,\beta=0,1/2} \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \left( Q\gamma^k_9 \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \right) \\
\times (2\sin \frac{\pi k}{N})^2 \frac{\vartheta[\frac{\alpha+2k}{\beta+k]} \vartheta[\frac{\alpha-2k}{\beta-k}]}{\vartheta[\frac{1/2+2k}{\beta+k} \vartheta[\frac{1/2-2k}{\beta-k}]} ,
\]

(3.16)

\[
A^{(k)}_{95}(B) = Bl \frac{W^{(2)}(l)}{8\pi^3} \sqrt{G} \sum_{\alpha,\beta=0,1/2} \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \left( (Q\gamma^k_9 \otimes \gamma^k_9) \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \frac{\vartheta[\frac{\beta}{\alpha}]}{\eta^3} \right) \\
\times \frac{\vartheta[\frac{\beta+k}{\alpha+1/2} \vartheta[\frac{\beta-k}{\alpha+1/2}]}{\vartheta[\frac{1/2+k}{\alpha + 1/2} \vartheta[\frac{1/2-k}{\alpha + 1/2}]} ,
\]

(3.17)

Taking the limit $l \rightarrow \infty$ yields the expressions

\[
\frac{1}{l} A^{(k)}_{99}(B) \simeq \frac{B \sqrt{G}}{2\pi^3} \sin^2 \frac{\pi k}{N} \left( (Q\gamma^k_9 \otimes \gamma^k_9 + Q^{-k}_9 \otimes Q^{k}_9) \frac{1}{\sin \pi \epsilon} \right) ,
\]

\[
\frac{1}{l} M^{(k)}_{99}(B) \simeq -\frac{8B \sqrt{G}}{\pi^3} \sin^2 \frac{\pi k}{N} \left( Q\gamma^k_9 \frac{1}{\sin (\pi \epsilon / 2)} \right) ,
\]

(3.18)

\[
\frac{1}{l} A^{(k)}_{95}(B) \simeq \frac{B \sqrt{G}}{4\pi^3} \sin \gamma^k_5 \left( Q\gamma^k_9 \frac{1}{\sin \pi \epsilon} \right) ,
\]
where we have separated inside the square brackets the contributions coming from the exchange of NS-NS closed-string states (proportional to the inverse sines) and those coming from RR states (which are proportional to the cotangents).

Out of the above three amplitudes only \( A_{99} \) contains ‘mixed terms’ with charge-operator insertions at both the left and the right cylinder boundaries. The \( A_{95} \) diagram has one of it boundaries stuck on D5-branes, which are blind to the (99) gauge groups, while the Möbius diagram has only a single boundary anyhow. These latter two diagrams combine with the ‘pure terms’ (all charge insertions on the same boundary) of \( A_{99} \) to give an infrared-finite expression in the closed-string channel. This is a consequence of tadpole cancellation, which guarantees that a zero-momentum massless particle cannot disappear into the vacuum in the absence of the magnetic-field background. This can be checked explicitly using the Chan-Paton matrices of table 1. For instance for the \( Z_3 \) model, there are no D5-branes and \( \text{tr} \gamma_9 = \text{tr} \gamma_9^2 = 8 \), which is precisely the condition for the sum of \( A_{99}^{(2k)} \) and \( M_{99}^{(k)} \) to be finite in the \( l \to \infty \) limit of the integration.

The leading non-cancelled divergences arise at quartic level from ‘mixed terms’ of the \( A_{99} \) amplitudes. They are due to the exchange of twist-field scalars transforming in tensor multiplets of \( N = 1 \) supersymmetry in six dimensions, and coupling to the Yang-Mills action as in (2.11). All models with the exception of \( Z_2 \) contain such tensor multiplets, localized at the fixed points of the orbifold [3]. Since the perturbative heterotic string has only a single tensor multiplet, only the \( Z_2 \) model has a perturbative heterotic dual, whose threshold calculation agrees with the results on the type-I side [23]. Note that supersymmetry does not allow the coupling (2.11) for hypermultiplets, so that in the \( Z_2 \) model the twisted cylinder amplitude is infrared finite [28].

Expanding out to quartic order the expressions (3.18) and using (2.7) we find

\[
\Lambda_4 = -\frac{12\pi^4 \sqrt{G}}{N} \sum_{k=1,k \neq N/2}^{N-1} \sin^2 \pi k v (\text{tr} Q^2 \gamma_k)^2 \int dl . \tag{3.19}
\]

The physical interpretation of the term (3.19) of the type \( (\text{tr} F^2)^2 \) is that twisted NS-NS
fields $m_k$ appear in the tree-level (disk) gauge kinetic function of the gauge group and generate at one-loop (tree-level in the transverse, closed string picture) a tadpole. By using the integral form (2.17) for the propagator of a canonically-normalized scalar, one can extract the couplings $s_k$ (2.11) for the various models. The result is:

$$s_k = \frac{2\pi^2}{G^{1/4}\sqrt{N\pi}} |\sin \pi k v| (\text{tr} Q^2 \gamma^k).  \tag{3.20}$$

Because of the Peccei-Quinn symmetries generated by the RR axions, these couplings must be linear (at the perturbative level) in $m_k$.

As previously discussed, these couplings are zero for the $Z_2$ example in Table 1 and for the corresponding sectors $k = N/2$ of the models $Z_4$ and $Z_6$. They are also zero for the $k = 1$ twisted moduli couplings to the $U(8)$ gauge factor in the $Z_6$ model, because $\text{tr} Q^2_{U(8)} \gamma_9 = 0$ in this case.

In one particular example ($Z_3$) these couplings are proportional to the 4d beta functions and therefore in this case the unification is preserved. Notice however that the twisted moduli fields are exact flat directions in the effective field theory, therefore by using the expression of the gauge couplings (2.11) we find that the unification scale is an arbitrary parameter in these models. In all the other examples, unification is lost because the couplings $s_k$ are not proportional to the 4d beta functions. We should also mention that some of the gauge couplings become strong for critical values of $m_k$, as it was first pointed out in [31].

---

7The gauge fields couple actually to linear combinations of twisted fields which in our conventions are canonically normalized.

8Strictly speaking, the metric $g_{kl}$ of $m_k$ should appear also in (3.20). The string result (3.20) is valid around $m_k = 0$, it therefore contains only the first term $g_{kl}(m) = \delta_{kl} + \cdots$ in a Taylor expansion.
4. \( \mathcal{N} = 1 \) supersymmetry: the \( Z_3 \) and \( Z_7 \) models

Our goal in this section is to compute the one-loop corrections to the gauge coupling constants coming from the \( \mathcal{N} = 1 \) sectors, in \( Z_N \) orbifolds with \( N \) a prime integer. Due to the absence of order two twist elements, these orientifolds have no 5-branes in the spectrum and are therefore the simplest 4d models with \( \mathcal{N} = 1 \) supersymmetry.

The annulus amplitude in the \( Z_N \) type I orientifolds with odd \( N \) can be written

\[
\mathcal{A} = \mathcal{A}_{N=4} - \frac{1}{2N} \sum_{k=1}^{N-1} \int_0^\infty \frac{dt}{t} \mathcal{A}^{(k)}(q)
\]

where \( \mathcal{A}_{N=4} \) is the contribution of the \( \mathcal{N} = 4 \) supersymmetric open spectrum, which does not contribute to the threshold corrections to the gauge couplings and \( \mathcal{A}^{(k)} \) is the contribution of the \( \gamma^k \equiv (\gamma)^k \) sectors given by

\[
\mathcal{A}^{(k)} = \frac{1}{8\pi^4 t^2} \sum_{\alpha,\beta=0,1/2} \frac{\eta_{\alpha,\beta}}{\eta^3} \prod_{i=1}^3 (-2 \sin \pi k v_i) \frac{\vartheta[\alpha]}{\vartheta[1/2+kv_i]} (\text{tr}\gamma^k)^2.
\]

The Möbius amplitude can be similarly written as in (4.1) by substituting \( \mathcal{A} \to \mathcal{M} \), with

\[
\mathcal{M}^{(k)} = -\frac{1}{8\pi^4 t^2} \sum_{\alpha,\beta=0,1/2} \frac{\eta_{\alpha,\beta}}{\eta^3} \prod_{i=1}^3 (-2 \sin \pi k v_i) \frac{\vartheta[\alpha]}{\vartheta[1/2+kv_i]} (\text{tr}\gamma^{2k}).
\]

Because of supersymmetry, the amplitudes (4.2), (4.3) vanish in the absence of the magnetic field.

In the presence of the background magnetic field \( B \), by using the modification (2.5), (2.6), the two amplitudes become

\[
\mathcal{A}^{(k)}(B) = \frac{iB}{8\pi^3 t^2} \text{tr} \left( (Q \gamma^k \otimes \gamma^k + \gamma^k \otimes Q \gamma^k) \sum_{\alpha,\beta=0,1/2} \frac{\vartheta[\alpha]}{\vartheta[1/2+kv_i]} \right) \prod_{i=1}^3 (-2 \sin \pi k v_i) \frac{\vartheta[\alpha]}{\vartheta[1/2+kv_i]},
\]

\[
\mathcal{M}^{(k)}(B) = \frac{iB}{4\pi^3 t^2} \text{tr} \left( Q \gamma^{2k} \sum_{\alpha,\beta=0,1/2} \frac{\vartheta[\alpha]}{\vartheta[1/2+kv_i]} \right) \prod_{i=1}^3 (-2 \sin \pi k v_i) \frac{\vartheta[\alpha]}{\vartheta[1/2+kv_i]}.
\]

In computing the threshold corrections, we are interested in the quadratic \( B^2 \) terms in a weak-field expansion. By using the identity \( \text{tr}Q\gamma^k = 0 \) for a nonabelian gauge factor and
after a straightforward algebra, we find \( A \equiv A_0 + B^2 A_2 + \cdots \), similarly for \( M \)

\[
A_2^{(k)} = -\frac{B^2}{32\pi^4} \text{tr}(Q^2 \gamma^k \otimes \gamma^k) \int \frac{dt}{t} \sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\eta^\prime[3]}{\eta^2} \prod_{i=1}^3 (-2 \sin \pi kv_i) \frac{\eta^\prime[1]}{\eta[1/2+kv_i]},
\]

\[
M_2^{(k)} = \frac{B^2}{16\pi^4} \text{tr}(Q^2 \gamma^{2k}) \int \frac{dt}{t} \sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\eta^\prime[3]}{\eta^2} \prod_{i=1}^3 (-2 \sin \pi kv_i) \frac{\eta^\prime[1]}{\eta[1/2+kv_i]} .
\]

It is useful in the following to use the modular identity formula, valid for \( v_1 + v_2 + v_3 = 0 \) (see Appendix A)

\[
\sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\eta^\prime[3]}{\eta^2} \prod_{i=1}^3 (-2 \sin \pi kv_i) \frac{\eta^\prime[1]}{\eta[1/2+kv_i]} = -2\pi \sum_{i=1}^3 \frac{\eta^\prime[1/2]}{\eta[1/2+kv_i]} .
\]

By using (4.6) into (2.1), (2.7) and (1.3), we arrive at the final result for the one-loop threshold corrections

\[
\frac{1}{g^2} = \frac{1}{g_0^2} \frac{1}{8\pi N} \sum_{k=1}^{N-1} \int \frac{dt}{t} \prod_{i=1}^3 (-2 \sin \pi kv_i) [(\text{tr}Q^2 \gamma^k)(\text{tr}\gamma^k) - 2(\text{tr}Q^2 \gamma^{2k})] \sum_{j=1}^3 \frac{\eta^\prime[1/2]}{\eta[1/2+kv_j]} .
\]

A first check of the formula (4.7) is by taking the infrared limit \( t \to \infty \), in which case the one-loop expression (4.7) must reproduce the one-loop running of the effective field theory, controlled by the renormalization group (RG) coefficients. In this limit we find

\[
\lim_{q \to 0} \frac{\eta^\prime[1/2]}{\eta[1/2-kv_j]} = \frac{\pi \cos(\pi kv_i)}{\sin(\pi kv_i)} .
\]

Let’s now check the result (4.7) in the \( Z_3 \) example case. The \( Z_3 \) four-dimensional \( \mathcal{N} = 1 \) type I orientifold \( \mathcal{O} \) is defined by the twist vector \( \mathbf{v} = (1/3, 1/3, -2/3) \). The tadpole consistency conditions ask for 32 D9 branes in the spectrum and fix the Chan-Paton matrix \( \gamma = \text{diag} (e^{2i\pi/3} I_{12}, I_4) \), where \( I_N \) is the \( N \times N \) identity matrix. The matrix \( \gamma \) then determines the gauge group to be \( SU(12) \times SO(8) \times U(1)_X \) (the \( U(1)_X \) factor is anomalous \( \mathcal{O} \), \( \mathcal{O} \)) and the charged matter fields are in the representations \( 3(12,8)_1 + 3(66,1)_{-2} \), where the subscripts denote the \( U(1)_X \) charges. This model contains only \( \mathcal{N} = 4 \) (\( \theta^0 \)) and \( \mathcal{N} = 1 \) (\( \theta^1, \theta^2 \)) sectors.
In this example we compute
\[
\left( \prod_{i=1}^{3} \sin \pi k v_i \right) \sum_{j=1}^{3} \frac{\cos(\pi k v_j)}{\sin(\pi k v_j)} = -\frac{9}{8},
\]
for \( k = 1, 2 \) and, for \( Z_3 \) we use \( \text{tr}\gamma = -4 \). We choose the generator \( Q \) such that, in a \( U(16) \) complex basis it reads
\[
Q_{SU(12)} = \frac{1}{2} \text{diag} (1, -1, 0^{14}) , \quad Q_{SO(8)} = Q_{U(4)} = \frac{1}{2} \text{diag} (0^{12}, 1, -1, 0, 0),
\]
and therefore \( \text{tr}Q^2\gamma^k = -1/2 \) for \( SU(12) \) and \( \text{tr}Q^2\gamma^k = +1 \) for the \( SO(8) \) gauge group factors. Finally, by cutting-off the integral in (4.7) by introducing the infrared (IR) regulator \( t \leq 1/\mu^2 \), we find the IR behaviour
\[
\frac{4\pi^2}{g^2_{SU(12)}} = \frac{4\pi^2}{g^2_{SU(12),0}} + \frac{9}{2} \ln \frac{\mu}{M_I}, \quad \frac{4\pi^2}{g^2_{SO(8)}} = \frac{4\pi^2}{g^2_{SO(8),0}} - 9 \ln \frac{\mu}{M_I}, \quad (4.11)
\]
which is indeed in agreement with the field theoretical RG coefficients \( b_1(SU(12)) = -9 \), \( b_2(SO(8)) = 18 \), where \( b\alpha = -3T_\alpha(G) + \Sigma_r T_\alpha(r) \). Notice that the corrections (4.7) are independent of the compactification radii, in analogy with the threshold corrections of \( \mathcal{N} = 1 \) sectors of heterotic models \( [29] \).

The second check of consistency of (4.7) is to go into the transverse channel and check that the UV divergences cancel. By using the modular transformation \( \mathcal{M} \) in (4.4), we find the amplitudes in the transverse channel
\[
\frac{1}{l}A^{(k)}(B) = \frac{iB}{8\pi^3} \text{tr} \left( (Q\gamma^k \otimes \gamma^k + \gamma^k \otimes Q\gamma^k) \sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\vartheta[\alpha]}{\vartheta[1/2]}(\epsilon) \right) \prod_{i=1}^{3} (-2 \sin \pi k v_i) \frac{\vartheta[\alpha+kv_i]}{\vartheta[1/2+kv_i]},
\]
\[
\frac{1}{l}M^{(k)}(B) = -\frac{iB}{\pi^3} \text{tr} \left( Q\gamma^{2k} \sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\vartheta[2\alpha]}{\vartheta[1/2]}(\frac{\pi}{2}) \right) \prod_{i=1}^{3} (-2 \sin \pi k v_i) \frac{\vartheta[\alpha+2kv_i]}{\vartheta[1/2+2kv_i]}. \quad (4.12)
\]
The UV behaviour of the above amplitudes can be easily worked out by taking the \( l \to \infty \) limit in (4.12). We find, by defining \( \eta_k = \text{sign}(\prod_{i=1}^{3} (\sin \pi k v_i)) \),
\[
\frac{1}{l}A^{(k)}(B) = -\frac{iB}{8\pi^3} \prod_{i=1}^{3} (-2 \sin \pi k v_i) \text{tr} \left( (Q\gamma^k \otimes \gamma^k + \gamma^k \otimes Q\gamma^k) [1 + i\eta_k(\frac{1}{\sin \pi \epsilon} - \cot \pi \epsilon)] \right),
\]
\[
\frac{1}{l}M^{(k)}(B) = \frac{iB}{\pi^3} \prod_{i=1}^{3} (-2 \sin \pi k v_i) \text{tr} \left( Q\gamma^{2k} [1 - i\eta_k(\frac{1}{\sin \pi \epsilon} - \cot \pi \epsilon)] \right). \quad (4.13)
\]
where, as in the previous section, we have explicitly displayed the contributions coming from the exchange of NS-NS and the RR states. By performing a small $B$ expansion in (4.13), it can be checked that the UV divergence in the $B^2$ term cancels, as expected, if the tadpole consistency condition is imposed. More explicitly, the quadratic UV divergence cancels provided
\[ \text{tr} \gamma^{2k} = \frac{4}{8 \prod_{i=1}^{3} \cos \pi k v_i}, \] (4.14)
equivalent to the usual tadpole condition for $Z_N$ odd orbifolds $\text{tr} \gamma^{2k} = 32 \prod_{i=1}^{3} \cos \pi k v_i$ if we use the equality $64(\prod_{i=1}^{3} \cos \pi k v_i)^2 = 1$. This last equality is a consequence of the fact that the number of fixed points for a $Z_N$ odd orbifold $N_k = 64(\prod_{i=1}^{3} \sin \pi k v_i)^2$ is independent of $k$, in particular $N_k = N_{2k}$.

An interesting phenomenon appears by expanding the above expressions (4.13) at order $B^4$. Indeed, by a straightforward computation we then find an UV divergence, equal to
\[ \Lambda_4 = -\frac{24 \pi^4}{N} \sum_{k=1}^{N-1} (\text{tr} Q^2 \gamma^k)^2 \prod_{i=1}^{3} |\sin \pi k v_i| \int dl, \] (4.15)
where the terms $(\text{tr} Q^4 \gamma^k)$ cancel exactly between the annulus and the Möbius. The interpretation of this term of the type $(\text{tr} F^2)^2$ is that twisted NS-NS fields $m_k$ (the blowing-up modes of the orbifold) appear at tree-level in the gauge kinetic function of the gauge group and generate at one-loop (tree-level in the transverse, closed string picture) a tadpole. Therefore, the tree-level gauge couplings (2.11) become
\[ \frac{4\pi^2}{g_{a,0}^4} = \frac{1}{\ell} + \sum_{k=1}^{N-1} s_{ak} m_k \] (4.16)
\[ = \frac{1}{\ell} + \sum_{k=1}^{N-1} \frac{8\pi^2}{\sqrt{2\pi N}} |\text{tr} Q^2 \gamma^k| \prod_{i=1}^{3} |\sin \pi k v_i|^{1/2} m_k, \] (4.17)
$\ell = e^{\phi_4} v^{-1/2}$, with $\phi_4$ the 4d dilaton and $v$ the volume of the 6d compact space in string units; its partner is the universal axion $a^{RR}$, dual to the untwisted RR antisymmetric
\[ \text{Note that going from eq. (4.15) to (4.17), there is a global sign ambiguity which cannot be fixed by our computation but does not affect our conclusions. This ambiguity propagates also in eqs. (4.20) and (5.6).} \]
The relation (4.17) confirms the result conjectured [33] on the basis of spacetime supersymmetry and anomaly considerations. Notice that, even if the one-loop threshold corrections in the IR (4.11) and the couplings (4.17) are separately non-universal (gauge-group dependent), remarkably enough the coefficients of the coupling of $m_k$ to the gauge fields are proportional to the beta function coefficients [41], shifting in a universal way the string scale for $Z_N$ odd orbifolds.

The anomalous $U(1)_X$ gauge factor has peculiar properties. First of all, notice that the result (4.17) applies to $U(1)_X$ as well. Moreover, by introducing a background magnetic field $B'$ for it, coupled to the gauge group generator $Q_{U(1)_X} \equiv Q_X$, where for example $Q_X = (1^{12}, 0^4)$ in the case of the $Z_3$ orientifold, we find a quadratic divergence

$$\frac{B'^2}{4N\pi^2} \sum_{k=1}^{N-1} \prod_{i=1}^3 |\sin \pi k v_i| (\text{tr} Q_X \gamma^k)^2 \int dl ,$$

which is physically interpreted as a mixing between the $U(1)_X$ gauge field and the Ramond-Ramond axions (antisymmetric tensors $C_{\mu\nu}$ in the string basis) in the tree-level lagrangian [42]. In order to identify the coupling, we use the (gauge-fixed) propagator

$$\Delta^{\mu\nu,\rho\sigma}(k^2) \equiv \langle C^{\mu\nu} C^{\rho\sigma} \rangle = \left( g_{\mu\rho} g^{\nu\sigma} - g_{\mu\sigma} g^{\nu\rho} \right) \frac{i}{k^2} ,$$

for the (RR) antisymmetric moduli. The resulting coupling at the orbifold point $m_k = 0$ is then

$$- \frac{1}{2\sqrt{2N\pi^3}} \sum_{k=1}^{N-1} \prod_{i=1}^3 |\sin \pi k v_i| \frac{1}{2} (-i \text{tr} Q_X \gamma^k) \epsilon_{\mu\nu\rho\sigma} C_{\mu\nu}^{k} F_{\rho\sigma}^{X} .$$

The $U(1)_X$ gauge boson becomes massive breaking spontaneously the symmetry, even for zero VEV’s of the twisted fields $m_k$. However, the corresponding global symmetry $U(1)_X$ remains unbroken, since the Fayet-Iliopoulos terms vanish in the orbifold limit $m_k = 0$ [44]. This property might be used in order to protect proton decay in low scale string models [13], [14], [15].

Notice that in the $Z_3$ case there is actually one linear (symmetric) combination of twisted moduli (out of the 27 blowing up modes) which couples to the gauge fields, which
is the same appearing in the mixing between the RR axions and the anomalous $U(1)$.

5. $\mathcal{N} = 1$ supersymmetry: $Z_6'$ and general models

The $Z_6'$ model is defined by the twist vector $\mathbf{v} = (1/6, -1/2, 1/3)$. The tadpole cancellation conditions ask for 32 D9 branes and one set of 32 D5 branes filling the third compact coordinate. The D5 branes are considered here to be all at the origin in the $(z_1, z_2)$ plane for simplicity, where $(z_1, z_2, z_3)$ denote the three complex compact coordinates. The solution for the Chan-Paton matrices reads

$$
\gamma_9 = \gamma_5 = \text{diag}\left(e^{i\pi/6}I_4, e^{5i\pi/6}I_4, iI_8\right).
$$

The gauge group of this model is $[U(4) \times U(4) \times U(8)]_9 \times [U(4) \times U(4) \times U(8)]_5$ and the charged matter representations are

99 or 55 : $(4, 4, 1) + (\bar{4}, \bar{4}, 1) + (4, 4, 1) + (6, 1, 1) + (1, 6, 1) +$

$$(1, 1, 28) + (1, 1, \bar{28}) + (1, 4, 8) + (\bar{4}, 1, \bar{8}) + (4, 1, \bar{8}) + (1, 4, 8),$$

59 : $(1, 4, 1; 1, 4, 1) + (4, 1, 1; 1, 1, 8) + (1, 1, 8; 4, 1, 1) +$

$$(\bar{4}, 1, 1; \bar{4}, 1, 1) + (1, \bar{4}, 1; 1, 1, 8) + (1, 1, 8; 1, \bar{4}, 1).$$

This model contains the $\mathcal{N} = 4$ sector $\theta^0$, $\mathcal{N} = 1$ sectors coming from $\theta, \theta^5$ and $\mathcal{N} = 2$ sectors coming from $\theta^2, \theta^3$ and $\theta^4$.

Our goal is the computation of the one-loop corrections to the gauge couplings coming from the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sectors. Then we will comment on the straightforward generalization of these formulas for a generic four-dimensional $\mathcal{N} = 1$ type I orientifold. For this purpose, we choose the D9 brane gauge group and therefore we introduce a background magnetic field coupled to the D9 branes. In this case the relevant amplitudes to consider are $A_{99}, A_{95}$ and $M_9$. The other choice of turning on a magnetic field in the D5 branes sector can be obtained easily from the previous one by T-duality.
The one-loop open string amplitudes of this model with and without the background magnetic field are displayed in the Appendix B. By collecting the results in Appendix B and performing a weak-coupling expansion similar to the one in section 3, we find

\[2N B_a(t) = - \sum_{k=1,5}^{3} \prod_{i=1}^{3} (-2 \sin \pi k v_i) [(\text{tr} Q_a^2 \gamma_9^k)(\tr \gamma_9^k) - 2(\text{tr} Q_a^2 \gamma_9^{2k})] \sum_{j=1}^{3} \frac{1}{\pi} \frac{\theta'_{[1/2-\text{kv}_j]}}{\theta'_{[1/2-\text{kv}_j]}}

-2 \sum_{k=2,4}^{3} 4 | \sin \pi k v_1 \sin \pi k v_3 [(\text{tr} Q_a^2 \gamma_9^k)(\tr \gamma_9^k) - 2(\text{tr} Q_a^2 \gamma_9^{2k})] \Gamma_1^{(2)}

+2\left\{32(\tr Q_a^2) - 4 \sin 3\pi v_1 \sin 3\pi v_2 [(\text{tr} Q_a^2 \gamma_9^3)(\tr \gamma_9^3) - 2(\text{tr} Q_a^2 \gamma_9^6)] \right\} \Gamma_3^{(2)}

+ \sum_{k=1,2,4,5} (\tr Q_a^2 \gamma_9^k)(\tr \gamma_9^k) \frac{2 \sin \pi k v_3}{\pi} \frac{\theta'_{[1/2-\text{kv}_3]}}{\theta'_{[1/2-\text{kv}_3]}} + \frac{2}{\pi} \frac{\theta'_{[1/2-\text{kv}_3]}}{\theta'_{[1/2-\text{kv}_3]}} \right) .

(5.3)

In putting the final result in the form (5.3), we used also the modular identities (A.9) in Appendix A. The contribution of the \( N = 2 \) sectors in (5.3) is easily identified with the terms containing the Kaluza-Klein momenta sums \( \Gamma_i^{(2)} \) along the compact direction \( z_i \).

Notice that the third line in (5.3) coming from the \( k = 3 \) sector can be identified with the threshold corrections in the \( Z_2 \) orbifold model [2] discussed in [28].

The generalization of the above-result to a generic \( N = 1 \) four-dimensional type I orientifold is straightforward. A general such model contains in the spectrum \( D5_i \) branes filling the 4d spacetime and a compact complex dimension \( z_i \). Different \( \theta^k \) sectors have \( N = 1 \) and \( N = 2 \) supersymmetry, each one having a contribution as in (5.3). The last line in (5.3) is replaced by the corresponding sum over \( 5_i \) brane contributions.

Taking the infrared limit in the above expression (5.3) allows to give a general formula for the beta function coefficient contributions of the various \( \theta^k \) sectors. By using the infrared limits (4.8) and (A.11) we find

\[b_a = \frac{4}{N} \sum_{k \neq N/2} [(\text{tr} Q_a^2 \gamma_9^k)(\tr \gamma_9^k) - 2(\text{tr} Q_a^2 \gamma_9^{2k})] (\prod_{i=1}^{3} \sin \pi k v_i) \sum_{j=1}^{3} \frac{\cos \pi k v_j}{\sin \pi k v_j}

+ \frac{1}{N} \sum_{i,k \neq N/2} (\tr Q_a^2 \gamma_9^k)(\tr \gamma_9^k) \cos \pi k v_i + \frac{24}{N} \tr Q_a^2 ,

(5.4)

where we considered here an \( Z_N \) orientifold, for simplicity and the last contribution in the
right-hand side of (5.4) comes from $A_{95}^{(0)}$ and $M_{99}^{(N/2)}$. The second line in (5.4) exist only for even $N$ orientifolds. In the particular case of the $Z_6'$ orientifold, by putting the generator $Q_a$ into different gauge algebra subgroups, it can be indeed checked that (5.4) reproduces the beta function coefficients of the effective field theory $b_1(SU(4)) = 9, b_1(SU(8)) = -6$.

It is also straightforward to study the general UV structure by taking the $l \to \infty$ limit of the various terms (B10)-(B13) in the transverse channel. The UV finiteness of (5.3) is then obtained by using the tadpole conditions $\text{tr} \gamma^k = \text{tr} \gamma^5 = 0$ for $k = 1, 3, 5$, $\text{tr} \gamma_9^2 = \text{tr} \gamma_5^2 = -8$, $\text{tr} \gamma_9^4 = \text{tr} \gamma_5^4 = 8$. By applying the same method as in the $Z_3$ orientifold case studied in the previous section, by expanding the transverse amplitudes to the order $B^4$ we find an UV divergence coming from the $\mathcal{N} = 1$ sectors $k = 1, 5$ and $\mathcal{N} = 2$ sectors $k = 2, 4$

$$\Lambda_4 = -\frac{3\pi^4}{N} \left[ \sum_{k=1,5}^3 (\text{tr} Q^2 a \gamma^k)^2 \prod_{i=1}^3 |2 \sin \pi k v_i| + \sum_{k=2,4} v_2 (\text{tr} Q^2 a \gamma^k)^2 \prod_{i=1,3} |2 \sin \pi k v_i| \right] \int dl , \quad (5.5)$$

where $v_2 = \sqrt{G_2/\alpha'}$ is the volume in string units of the second compact torus. The result (5.5) is interpreted as a tree-level modification of the gauge kinetic functions (4.16) with

$$\sum_k s_{ak} m_k = \sum_{k=1}^{[N/2]'} \frac{2\pi^2}{\sqrt{\pi N}} (\text{tr} Q^2 a \gamma^k)^2 \prod_{i=1}^3 |2 \sin \pi k v_i|^{1/2} m_k , k v_i \neq \text{integer}$$

$$\sum_k s_{ak} m_k = \sum_{k=1}^{[N/2]'} \frac{2\pi^2}{\sqrt{\pi N v_i}} (\text{tr} Q^2 a \gamma^k)^2 \prod_{i\neq i_0} |2 \sin \pi k v_i|^{1/2} m_k , k v_{i_0} = \text{integer} , \quad (5.6)$$

and the prime in the sum excludes the sectors $k$ with $2k v_i = \text{integer}$ for all $i = 1, 2, 3$, while $v_{i_0}$ is the volume of the $\mathcal{N} = 2$ complex torus $T_{i_0}$ in string units. The sectors $k$ excluded from the sum are associated to D5 branes and the corresponding twisted RR moduli are 4-forms in six dimensions; they belong to (neutral) hypermultiplets, which cannot couple to the kinetic terms of non-abelian gauge fields, by virtue of $\mathcal{N} = 1$ supersymmetry in 6d.

The remaining sectors fall in two categories: (i) $\mathcal{N} = 1$ sectors corresponding to nontrivial twists for all three planes with associated twisted moduli described by linear multiplets in 4d; (ii) $\mathcal{N} = 2$ sectors with no associated D5 branes and their corresponding (real) moduli belonging to tensor multiplets in six dimensions containing also their RR 2-form
counterpart. The moduli of both categories (i) and (ii) can generally couple to gauge fields according to (4.16).

Let us illustrate this general result to the case of $Z_6'$ orientifold. In this example, one has $N = 6$ in (5.6) and the prime in the sum excludes the $Z_2$ sector $k = 3$ associated to 32 D5 branes.

We now describe the threshold corrections coming from $\mathcal{N} = 2$ sectors in (5.3), using the results of section 3. These corrections depend on the geometric moduli $T_i, U_i$ defined as in eq. (3.10) for the three complex planes \[ S = a^{RR} + i \frac{\sqrt{G_1 G_2 G_3} M_6^6}{\lambda_I}, \quad U_i = \frac{G_i^{12} + i \sqrt{G_i} M_i^2}{G_i^{22}}, \quad T_i = b^{RR}_i + i \frac{\sqrt{G_i} M_i^2}{\lambda_I} \] (5.7) where $G_i$ is the metric on the torus $T_i$, related to the corresponding volume $v_i = \sqrt{G_i} M_i^2$ (see eq. (5.5)). Then the threshold corrections (2.9) in the direct (open string) channel are equal to

\[ \Lambda_{2,a} = \frac{1}{12} \sum_i b^{(N=2)}_{ai} = \frac{1}{12} \sum_i \left[ \int dt \sum_{(m_1^2, m_2^2)} \left[ 4 e^{\frac{t}{\sqrt{c_i} \text{Im} U_i} |m_1^2 + U_i m_2^2|^2} - e^{-\frac{t}{\sqrt{c_i} \text{Im} U_i} |m_1^2 + U_i m_2^2|^2} \right] \right], \] (5.8)

where $b^{(N=2)}_{ai}$ is the effective theory beta function coefficient of the corresponding $\mathcal{N} = 2$ sector.\[ By explicitly computing (5.8), we find the result\[ \Lambda_{2,a} = -\frac{1}{4} \sum_i b^{(N=2)}_{ai} \ln(\sqrt{G_i} \text{Im} U_i \mu^2) + \text{Im} f_a^{(1)} = \]

\[ -\frac{1}{4} \sum_i b^{(N=2)}_{ai} \ln \left[ \left( \frac{\text{Im} S}{\text{Im} T_j \text{Im} T_k} \right)^{1/2} \text{Im} U_i \frac{\mu^2}{M_i^2} \right] + \text{Im} f_a^{(1)}, \] (5.9)

with $j \neq k \neq i$ and where

\[ f_a^{(1)}(U) = -i \sum_i b^{(N=2)}_{ai} \ln \eta(U_i) . \] (5.10)

10 Notice that our definition of $b^{(N=2)}_{ai}$ differs from the definition of ref. [25] in the sense that ours represents the contribution of the ith $\mathcal{N} = 2$ sector to the total beta function and therefore equals $b^{(N=2)}_{ai}/\text{ind}$ in their notation.
The corrections (5.9) are similar with the heterotic ones in the $\text{Im}T_i \to \infty$ limit, taking into account that on the heterotic side the complex structure moduli have the same definition (5.7), while
\[ S = a + i \frac{\sqrt{G_1 G_2 G_3 M_H^6}}{\lambda_H^2}, \quad T_i = b_i + i \sqrt{G_i} M_H^2. \] (5.11)

Unlike the type I case, only the universal axion $a$ is described by a linear multiplet, the others fitting naturally into chiral multiplets.

Let's consider now the corrections given by an $\mathcal{N} = 2$ sector, depending on the complex torus of radii $R_{1,2}$. Notice that in the limit $R_1, R_2 \to \infty$ with $R_1/R_2 = \text{Im}U$ fixed, $\Lambda_2 \sim \ln(R_1 R_2 \mu^2)$, whereas in the limit $R_1 \to \infty$, $R_2$ fixed, the corrections are linearly divergent $\Lambda_2 \sim R_1/R_2$. These power-law corrections can be used for the phenomenological purpose of lowering the unification scale [15] in models with a low value of the string scale $M_I$ [13, 14]. Alternatively, if $R_1/R_2 \simeq 1$ the logarithmic running $\Lambda_2 \sim \ln(R_1 R_2 \mu^2)$ can also be used in order to achieve unification at a high Kaluza-Klein scale, even if the fundamental string scale has much lower values [17].

6. **Threshold corrections in models with spontaneous $\mathcal{N} = 4 \to \mathcal{N} = 2$ supersymmetry breaking**

Type I string models with a continous parameter which interpolates between models with different numbers of supersymmetries are an interesting framework for study threshold corrections and their dependence on the interpolating parameter. In particular we show that in the limit where the maximal supersymmetry is restored, there are no linear corrections in the corresponding radius. The dependence is logarithmic only, which could be an advantage, in the sense that the gauge hierarchy problem corresponding to these dimensions is improved [18].

$\mathcal{N} = 4 \to \mathcal{N} = 2$ Scherk-Schwarz breaking
This model can be described in the closed string sector, as a freely-acting orbifold $IIB/(-1)^m \mathcal{I}$, where $(-1)^m$ denotes the order-two shift $X^5 \rightarrow X^5 + \pi R_1$ and $\mathcal{I}$ denotes the inversion of the four internal coordinates $\mathcal{I}X^{6\ldots9} = -X^{6\ldots9}$ (a $R_1 \rightarrow 2R_1$ operation is required in order to go in the Scherk-Schwarz basis in which the amplitudes below are written). The resulting type I model [10] has $\mathcal{N} = 2$ supersymmetry in 4d and a gauge group $SO(N_1) \times SO(N_2)$ (with $N_1 + N_2 = 32$) originating from D9 branes and can be described as a Scherk-Schwarz deformation (a shift of the Kaluza-Klein modes $m_1$ with 1/2 unit along $R_1$) of the $\mathcal{N} = 4$ supersymmetric type I model with a Wilson line that breaks $SO(32)$ down to $SO(N_1) \times SO(N_2)$. $\mathcal{N} = 4$ supersymmetry is recovered in the $R_1 \rightarrow \infty$ limit. The open string massless spectrum contains, besides the adjoint $\mathcal{N} = 2$ vector multiplets $(N_1(N_1 - 1)/2, 1) + (1, N_2(N_2 - 1)/2)$, one hypermultiplet in the representation $(N_1, N_2)$.

The open string amplitudes for vanishing magnetic field are given by the expressions

\[
A = \frac{(tr1)^2}{8\pi^4 t^2} \sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\vartheta^4[\alpha]}{\eta^2} \Gamma^{(4)}(\Gamma_{m_1} + \Gamma_{m_1+1/2}) \Gamma_{m_2} + \\
\frac{(tr\gamma)^2}{8\pi^4 t^2} \sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\vartheta^2[\alpha]}{\eta^6} 4\vartheta[\alpha+1/2] \vartheta[\beta+1/2] \Gamma^{(4)}(\Gamma_{m_1} - \Gamma_{m_1+1/2}) \Gamma_{m_2} ,
\]

\[
M = -\frac{(tr1)}{8\pi^4 t^2} \sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\vartheta^4[\alpha]}{\eta^2} \Gamma^{(4)}(\Gamma_{m_1} + \Gamma_{m_1+1/2}) \Gamma_{m_2} - \\
\frac{(tr1)}{8\pi^4 t^2} \sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\vartheta^2[\alpha]}{\eta^6} 4\vartheta[\alpha+1/2] \vartheta[\beta+1/2] \Gamma^{(4)}(\Gamma_{m_1} - \Gamma_{m_1+1/2}) \Gamma_{m_2} ,
\]

where the action of the twist on the Chan-Paton degrees of freedom is $\gamma = diag (I_{N_1}, -I_{N_2})$. In (6.1), $\Gamma_{m_1} (\Gamma_{m_2})$ denotes the Kaluza-Klein momentum sum along $X^5 (X^4)$ and $\Gamma^{(4)}$ denotes the momentum sum along $T^4$.

The introduction of the background magnetic field is completely analogous to the cases studied in the previous sections. By using the relations $tr1 = N_1 + N_2$, $tr\gamma = N_1 - N_2$, $trQ^2\gamma = \pm 1$ (plus sign for $SO(N_1)$ and minus sign for $SO(N_2)$ gauge factors) and by using
the last modular identity in (A8), we find the threshold corrections

\[ B(t) = -2[\pm(N_1 - N_2) - 2(\Gamma_{m_1} - \Gamma_{m_1+1/2}) \Gamma_{m_2}] , \] (6.2)

which in the IR limit agree with the field theory beta-function coefficients \( b_1(SO(N_1)) = -2N_1 + 2N_2 + 4 \), \( b_2(SO(N_2)) = -2N_2 + 2N_1 + 4 \). Notice that, because of the remnant \( \mathcal{N} = 2 \) supersymmetry, the string oscillators decoupled in the final expression, in analogy with the \( \mathcal{N} = 2 \) sectors of orbifold models, as shown in [28] and in section 3.

These corrections, in analogy with eq. (5.8), can be written in a more compact way in the transverse channel

\[ \Lambda_2 = \frac{-\sqrt{G}}{2\pi} \int_\mu^2 \, dl \sum_{n_1,n_2} [1 - (-1)^{n_1}] \left( \pm(N_1 - N_2)e^{-\sqrt{G} |n_2 + U_{n_1}|^2} - 8e^{-4\sqrt{G} |n_2 + U_{n_1}|^2} \right) \]

\[ = \frac{1}{2} [\pm(N_1 - N_2) - 2] \ln[e^{-2\gamma_E} \mu^2 \sqrt{G} \text{Im}U] \left( \frac{\eta^3(U)}{2\theta_2(U)} \right)^2 - 2 \ln 2 , \] (6.3)

where \( U \) is the complex field corresponding to the \( R_1, R_2 \) torus, \( \sqrt{G} = R_1 R_2 \) and \( \gamma_E \) is the Euler number. In the \( R_2 \to \infty \) limit, \( R_1 \) fixed we get \( \Lambda_2 \sim R_2/R_1 \) and in the \( R_1, R_2 \to \infty \) limit we get \( \Lambda_2 \sim \ln(R_1 R_2) \), as expected. However, in the limit \( R_1 \to \infty, R_2 \) fixed, corresponding to the limit of the restoration of the full \( \mathcal{N} = 4 \) supersymmetry, we find just logarithmic corrections \( \Lambda_2 \sim \ln R_1 \), as in the dual heterotic models studied in [17]. The corresponding threshold corrections on the heterotic side agree with (6.3), as expected, in the \( T_H \to i\infty \) limit. Notice that in (6.3) the UV divergence is automatically zero and does not ask for the untwisted tadpole condition \( N_1 + N_2 = 32 \), in analogy with the previous examples, because the untwisted tadpole condition is related to the \( \mathcal{N} = 4 \) sector which does not contribute to the threshold corrections to the gauge couplings. The UV finiteness holds to all orders in the magnetic field \( B \), which means that no tree-level modification of the gauge kinetic function appears here, which is to be expected since there are no fixed points and therefore no twisted fields in a freely-acting orbifold.

\[ \mathcal{N} = 4 \to \mathcal{N} = 2 \text{ M-theory breaking} \]
This model is, in the closed string sector, the T-dual of the above one in the sense that it exchanges the momentum modes and the winding modes along $R_1$. The deformation shifts now the winding modes along $R_1$ with $1/2$ unit and $\mathcal{N} = 4$ supersymmetry is recovered in the $R_1 \to 0$ limit. The model, which can also be seen by duality arguments as M-theory compactified on $(T^4 \times S^1)/(\mathbb{Z}_2 \times \mathbb{Z}_2') \times S^2$, contains, by tadpole consistency conditions, 16 D9 branes and 16 D5 branes, with a gauge group $SO(16) \times SO(16)$. The massive spectrum of the model has $\mathcal{N} = 2$ supersymmetry, but the massless one has $\mathcal{N} = 4$ supersymmetry and consists of the vector multiplet in the adjoint representation of the gauge group. In order to distinguish the two gauge factors in the following, we write the gauge group as $SO(N) \times SO(D)$, where the tadpole conditions ask for $N = D = 16$.

The one-loop open string amplitudes for vanishing magnetic field read

$$A = A_{\mathcal{N}=\text{4}} + \frac{1}{2\pi^2 t^2} \sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\theta^2[\alpha] \theta^2[\alpha]}{\eta^{[\alpha]} N} \Gamma_{m_1+1/2} \Gamma_{m_2},$$

$$M = M_{\mathcal{N}=\text{4}} + \frac{1}{4\pi^4 t^2} \sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\theta^2[\alpha] 4\theta[\alpha] \theta[\alpha]}{\eta^{[\alpha]}} (N + D) \Gamma_{m_1+1} \Gamma_{m_2},$$

(6.4)

where the $\mathcal{N} = 4$ part is as usually irrelevant for our purposes. We choose to turn on a magnetic field in the $SO(N)$ gauge group factor. By following the same steps as in the previous sections we find the threshold corrections

$$B(t) = 2[-8\Gamma_{m_1+1} + D\Gamma_{m_1+1/2}] \Gamma_{m_2}$$

(6.5)

and therefore there are no IR divergences. This is in agreement with the vanishing of the beta function of the effective field theory. The threshold corrections can be quantitatively computed and the result is

$$\Lambda_2 = \frac{\sqrt{G}}{2\pi} \int_{0}^{\infty} dl \sum_{n_1,n_2} \left(-1\right)^{n_1} \frac{\theta_2(U)}{\eta(U)} \left[ e^{-\frac{\sqrt{G}}{\lambda} \left| n_2 + Un_1 \right|^2} - 16\left(-1\right)^{n_1} e^{-\frac{\sqrt{G}}{\lambda} \left| n_2 + Un_1 \right|^2} \right]$$

(6.6)

$$= \frac{D}{2} \ln[4\left| \frac{\theta_2(U)}{\eta(U)} \right|^2] + 4 \ln[4\left| \frac{\theta_2(U/2)}{\eta(U/2)} \right|^2].$$

In the limit of the restoration of $\mathcal{N} = 4$ supersymmetry $R_1 \to 0$, $R_2$ fixed, we find a result exponentially suppressed $\Lambda_2 \sim e^{-R_2/R_1}$. This model has the property that the tadpoles are
locally cancelled in the compact direction $R_1$. This can be understood by the fact that, in the transverse channel, the annulus and the Möbius amplitudes (6.4) have the same winding lattice sum $(-1)^nW_n$, while usually (in toroidal or orbifold compactifications without Wilson lines) the transverse annulus contains the full lattice $W_n$, while the transverse Möbius contains only the even windings lattice $W_{2n}$. Therefore not only the UV divergence in (6.10) coming from the $n = 0$ mode is cancelled once the tadpole condition $D = 16$ is imposed, but also simultaneously the contribution of the massive $n$ modes, which describe the position in the transverse (to the branes) space. The UV convergence holds at all orders, for the same physical reason as in the model of the previous subsection.

7. Effective field theory

In the remaining two sections we discuss how our string-theory results could fit (a) with the general expressions of supergravity and (b) with the conjectured $\mathcal{N} = 1$ heterotic/type I duality, although there is no regime in which the 10d string coupling is small on both sides. Thus, we should warn the reader that this discussion is only tentative. Our main results will be (a) that the one-loop truncated supergravity expressions can be fitted in the special cases of $Z_3$ and $Z'_6$ models and (b) that duality does not seem to work even in the $Z_3$ example where the perturbative spectra on the two sides agree. These points deserve definitely further study.

The starting point is the general (all-loop order) expression for the physical gauge couplings $g_a$ in locally supersymmetric field theories [18]:

$$\frac{4\pi^2}{g_a^2(\mu^2)} = \text{Im} f_a + \frac{b_a}{4} \ln \frac{M_P^2}{\mu^2} + \frac{c_a}{4} K + \frac{T_a(G)}{2} \ln g_a^{-2}(\mu^2) - \sum_r \frac{T_a(r)}{2} \ln \text{det} Z_{(r)}(\mu^2), \quad (7.1)$$

expressed in terms of the Wilsonian (holomorphic) gauge couplings $f_a$ and the wave-function normalization matrix $Z_{(r)}$ for the charged matter fields. Here, $M_P$ is the Planck mass, $K$ is the Kähler potential, $a$ denotes the gauge group factor and $r$ runs over the gauge
group representations with Dynkin index $T_a(r)$. The one-loop beta functions $b_a$ and the coefficients $c_a$ are given by

\[ b_a = \sum_r T_a(r) - 3T_a(G), \quad c_a = \sum_r T_a(r) - T_a(G). \tag{7.2} \]

Truncating the expression (7.1) to one-loop order requires the knowledge of the holomorphic gauge couplings $f_a$ at one-loop, while the Kähler potential and the wave-function normalization matrix need to be known only at tree-level. For simplicity, in the following we concentrate on the gauge couplings originating from D9 branes. The holomorphic gauge couplings and the Kähler potential can be generally expanded as

\[ f_a = S + s_{ak} M_k + f_a^{(1)}(U), \]

\[ K = -\ln(S - \bar{S}) + \hat{K}(M_k, T, U), \tag{7.3} \]

where $U(T)$ are the complex structure (Kähler class) moduli fields and $s_{ak}$ are numerical constants given by (5.6). An important point is that in the Wilsonian gauge kinetic function $f_a$ the fields $S, M_k$ form complex chiral multiplets and that $\text{Im}S, \text{Im}M_k$ are related to the linear multiplet fields $\ell, m_k$ (4.17), (4.16) in the string basis through a chiral-linear multiplet duality [26]. Notice that the twisted moduli $M_k$ can appear perturbatively only linearly in $f_a$ because of the Peccei-Quinn symmetries associated to the Ramond-Ramond axions. Similarly, the only possible dependence of $f_a$ on $T$ moduli is linear, but for D9 branes this dependence vanishes. As we saw in the preceding sections, all linear dependence appears only at the tree-level (disk diagram), while $f_a^{(1)}$ is the genus one correction (5.10). Inserting eqs. (7.2) and (7.3) into (7.1), we find

\[ \frac{4\pi^2}{g_a^2(\mu^2)^{1-\text{loop}}} = \text{Im}S + s_{ak}\text{Im}M_k + \frac{b_a}{4}(\ln \frac{M_k^2}{\mu^2} - \ln(S - \bar{S})) + \]

\[ \frac{1}{4} \left[ 4\text{Im}f_a^{(1)}(U) + c_a\hat{K} - 2\sum_r T_a(r) \ln \det Z_{(r)} + 2T_a(G) \ln(1 + s_{ak}\frac{\text{Im}M_k}{\text{Im}S}) \right]. \tag{7.4} \]

On the other hand, as we discussed in the previous sections, a direct one-loop type I string
computation in orbifold compactifications gives the (moduli-dependent) result

\[ \frac{4\pi^2}{g_a^2(\mu^2)^{1\text{-loop}}} = \frac{1}{\ell} + s_{ak}m_k + \]

\[ \frac{1}{4} \left[ 4\text{Im}f_a^{(1)}(U) + b_{a(N=1)}^i \ln \frac{M_H^2}{\mu^2} - \sum_i b_{ai(N=2)}^i \ln(\sqrt{G_i}\text{Im}U_i\mu^2) \right], \quad (7.5) \]

where the total beta function coefficient is a sum of contributions of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sectors: $b_a = b_{a(N=1)} + \sum_i b_{ai(N=2)}$.

Our goal here is to study the compatibility between the effective field theory result (7.4) and the string theory result (7.5) in the models $Z_3$ and $Z_6'$ discussed in the previous sections. We limit ourselves to the orbifold limit $m_k \to 0$, where the string result is really valid. It is instructive to review first the situation in the heterotic case. The corresponding effective field theory and the heterotic string expressions can be obtained by putting $s_{ak} = 0$ and $\text{Im}M_k = 0$ in (7.4) and (7.5), replacing $M_I$ by the heterotic string scale $M_H$ and allowing a dependence of the analytic one-loop corrections $f^{(1)}_i$ on the Kähler class moduli $T$ \cite{27}. This dependence turns out to be the sum $\text{Im}f^{(1)}_a(T) + \text{Im}f^{(1)}_a(U)$. Indeed, unlike in type I strings, in heterotic strings the $T$ moduli are not protected by Peccei-Quinn symmetries. More explicitly, the string expression becomes\footnote{Our notations are related to the function $\Delta_a$ of \cite{27} through the relation $\Delta_a = \text{Im}f^{(1)}_a - \sum_i b_{ai(N=2)}^i \ln(\text{Im}T_i\text{Im}U_i)$.}

\[ \frac{4\pi^2}{g_a^2(\mu^2)^{1\text{-loop}}} = \frac{k_a}{\ell} + \frac{1}{4} Y(T, U) + \]

\[ \frac{1}{4} \left[ 4\text{Im}f^{(1)}_a(T) + \text{Im}f^{(1)}_a(U) + b_a \ln \frac{M_H^2}{\mu^2} - \sum_i b_{ai(N=2)}^i \ln(\text{Im}T_i\text{Im}U_i) \right], \quad (7.6) \]

where $k_a$ are the Kac-Moody levels and $Y$ is a universal correction coming from the $\mathcal{N} = 2$ sectors (containing both the non-analytic and analytic pieces computed in the literature \cite{25, 26}). Notice first that the two terms inside the bracket multiplying $b_a$ in the first line of field theory expression (7.4) combine to form $\ln(M_H^2/\mu^2)$, which reproduces the corresponding term in the string expression (7.5). Furthermore, a case by case analysis
shows that the remaining two terms are equal to:

\[ c_a \hat{K} - 2 \sum_r T_a(r) \ln \det Z_r = \sum_i (\delta_{\text{GS}}^i - b_{ai}^{(N=2)}) \ln(\text{Im}T_i \text{Im}U_i), \]  

(7.7)

where \( \delta_{\text{GS}}^i \) are gauge group independent constants. This apparent difference (by a universal term) between the two results can be explained partly by the fact that the string result uses a linear multiplet for the dilaton \( L \), instead of a chiral one \( S \) used in the field theory expression. Indeed, the duality transformation that changes basis from the string to the supergravity framework, brings a universal term proportional to the one-loop correction to the moduli Kähler potential [26]. As a result, the non-analytic part in \( Y \) cancels out while the coefficients \( b_{ai}^{(N=2)} \) are shifted by the universal constants \( \delta_{\text{GS}}^i \) as in eq. (7.7).

We present in detail the explicit example of the \( Z_3 \) orbifold of the \( SO(32) \) heterotic string. The four-dimensional gauge group is \( SU(12) \times SO(8) \times U(1)_X \) (the \( U(1)_X \) factor is anomalous). The charged massless chiral multiplets are in the representations \( 3(12,8)_{-1} + 3(66,1)_{+2} \) from the untwisted sector and \( 27(1,1)_{-4} + 27(1,8)_{+2} \) from the twisted sector, where the subscripts denote the \( U(1)_X \) charge. In addition there are 9 neutral untwisted (Kähler class) moduli \( T_{ij} \). In the following, for simplicity we consider only the diagonal moduli \( T_{ii} \equiv T_i \). Since there are no \( N = 2 \) sectors, in the string expression (7.6) \( \Omega = 0, b_{ai}^{(N=2)} = 0 \) and there are no moduli dependent corrections to the Wilsonian couplings \( f^{(1)} = 0 \). On the other hand, in order to evaluate the field-theory expression (7.4), we use the results

\[ \hat{K} = -3 \sum_{i=1}^3 \ln(T_i - \bar{T}_i), \quad Z_{(12,8)} = Z_{(66,1)} = \frac{1}{\text{Im}T_i}, \quad Z_{(1,8)} = \frac{1}{[(\text{Im}T_1)(\text{Im}T_2)(\text{Im}T_3)]^{2/3}}, \]  

(7.8)

where the index \( i \) in the untwisted matter labels the three different representations. Notice that in (7.4), by using the definitions (7.2) and (7.8), we can write

\[ c_a \hat{K} - 2 \sum_r T_a(r) \ln \det Z_r = -\frac{1}{3} b_a \sum_i \ln(T_i - \bar{T}_i) + \frac{2}{3} \sum_{\text{twisted}} T_a(r) \sum_i \ln(T_i - \bar{T}_i), \]  

(7.9)

where the last sum in (7.9) is over the matter representations from the twisted sector. Moreover, by using the relation \( M_H^2 = M_P^2/(\text{Im}S)^2 \), we conclude that the string scale in
(7.4) cancels between the first line and the untwisted sector contribution of the second line. Therefore (7.4) becomes, for the two nonabelian gauge factors

\[
\frac{4\pi^2}{g^2_{SU(12)}} = \text{Im}S - \frac{b_{SU(12)}}{12} \sum_i \ln(\sqrt{G_i} \mu^2) ,
\]

\[
\frac{4\pi^2}{g^2_{SO(8)}} = \text{Im}S - \frac{b_{SO(8)}}{12} \sum_i \ln(\sqrt{G_i} \mu^2) + \frac{1}{6} \sum_{r \text{ twisted}} T_a(r) \sum_i \ln(T_i - \bar{T}_i) , \tag{7.10}
\]

where the beta function coefficients are \(b_{SU(12)} = -9\), \(b_{SO(8)} = 45\). As a result, we can rewrite the field theory expressions (7.10) as the (moduli independent) string theory expressions up to a universal contribution

\[
\frac{4\pi^2}{g^2_{SU(12)}} = \text{Im}S - \frac{b_{SU(12)}}{4} \ln(\frac{\mu^2}{M_H^2}) - \frac{b_{SU(12)}}{12} \ln(V M_H^6) ,
\]

\[
\frac{4\pi^2}{g^2_{SO(8)}} = \text{Im}S - \frac{b_{SO(8)}}{4} \ln(\frac{\mu^2}{M_H^2}) - \frac{b_{SU(12)}}{12} \ln(V M_H^6) , \tag{7.11}
\]

where \(V = \sqrt{G_1 G_2 G_3}\) is the (dimensionful) volume of the compact space. As discussed above, the universal term originates from the one-loop correction to the moduli metric, which translates into a correction to the gauge couplings after the change of basis from the string (linear multiplet) basis to the effective supergravity (chiral multiplet) basis

\[\frac{1}{\ell} = \text{Im}S - \sum_i \delta^i_{GS} \ln(T_i - \bar{T}_i) , \tag{7.12}\]

where \(\delta^i_{GS}\) are the so-called Green-Schwarz coefficients, which in this case are equal to \(\delta^1_{GS} = \delta^2_{GS} = \delta^3_{GS} = b_{SU(12)}/12\). Note that this phenomenon is absent in nonsingular (Calabi-Yau) blown-up \(Z_3\) compactifications where no twisted states are present in the spectrum. In this case, the last term in the second eq. in (7.10) is absent and the resulting expression coincides with the large volume limit of the string result [27]. Notice that the heterotic string scale disappears and the compactification volume plays the role of the unification scale.

We now turn to the type I case and start with the simplest \(Z_3\) orientifold example discussed in section 4. In this case, the string result is given by (7.3) with the couplings
of twisted moduli proportional to the beta functions \( s_a = c b_a \) \((a = SU(12), SO(8))\), \( f_a^{(N=2)} = 0 \) due to the absence of \( \mathcal{N} = 2 \) sectors, while \( b_a^{(N=1)} \) are identical to the full beta functions \( b_a \) given in (4.11). The result is

\[
\frac{4\pi^2}{g_{SU(12)}^2} = \frac{1}{l} + c b_{SU(12)} m - \frac{b_{SU(12)}}{4} \ln\left(\frac{\mu^2}{M_f^2}\right),
\]

\[
\frac{4\pi^2}{g_{SO(8)}^2} = \frac{1}{l} + c b_{SO(8)} m - \frac{b_{SO(8)}}{4} \ln\left(\frac{\mu^2}{M_f^2}\right). \tag{7.13}
\]

On the other hand, the field theory expression is given by (7.4), with the functions \( \hat{K} \) and \( Z_r \) as in (7.8) but with the representation \((1, 8_s)\) absent and where the complex moduli \( T_i \) have the type I definitions (5.7)). The final result is

\[
\frac{4\pi^2}{g_{SU(12)}^2} = \text{Im} M + c b_{SU(12)} \text{Im} M - \frac{b_{SU(12)}}{12} \sum_i \ln\left(\sqrt{G_i \mu^2}\right),
\]

\[
\frac{4\pi^2}{g_{SO(8)}^2} = \text{Im} M + c b_{SO(8)} \text{Im} M - \frac{b_{SO(8)}}{12} \sum_i \ln\left(\sqrt{G_i \mu^2}\right). \tag{7.14}
\]

Therefore, the one-loop string theory and the field theory results formally agree if the twisted moduli duality were

\[
m = \text{Im} M - \frac{1}{12c} \ln(V M_f^6) = \text{Im} M - \frac{1}{24c} \ln\left(\frac{(\text{Im} S)^3}{\text{Im} T_1 \text{Im} T_2 \text{Im} T_3}\right). \tag{7.15}
\]

This is a linear-chiral multiplet duality\(^{12}\) for the twisted moduli, analogous to (7.12), perturbatively valid around \( m_k = 0 \). The exact duality relation could depend on the complete lagrangian for the twisted moduli and on higher-order corrections that we do not discuss here. In analogy with the heterotic case, this result (7.13) is interpreted as a mixing between the twisted moduli \( M_k \) and the untwisted ones \( S, T_i \), which could in principle be checked by an explicit string computation.

The second, more involved, example, is the \( Z_6' \) orientifold. Let us compare again the string results obtained in section 5 and the effective supergravity. Using (5.6) and (5.10)

\(^{12}\)For a recent paper discussing duality in the case of several linear multiplets, see [43].
we find
\[
\frac{4\pi^2}{g_{SU(8)}^2} = \frac{1}{\ell} - 2s_2m_2
- \frac{1}{4} \left[ 24\text{Re}\ln\eta(U_2) + 4\ln\left(\frac{\mu^2}{M_1^2}\right) + 6\ln(\sqrt{G_2\text{Im}U_2\mu^2}) - 4\ln(\sqrt{G_3\mu^2}) \right],
\]
\[
\frac{4\pi^2}{g_{SU(4)}^2} = \frac{1}{\ell} + s_2m_2 + s_1m_1
- \frac{1}{4} \left[ -12\text{Re}\ln\eta(U_2) - 2\ln\left(\frac{\mu^2}{M_1^2}\right) - 3\ln(\sqrt{G_2\text{Im}U_2\mu^2}) - 4\ln(\sqrt{G_3\mu^2}) \right],
\]
where \(s_1, s_2\) are numerical coefficients computed in (5.6). On the other hand, the field theory relation (7.4) becomes
\[
\frac{4\pi^2}{g_{SU(8)}^2} = \text{Im}S - 2s_2M_2
- \frac{1}{4} \left[ 24\text{Re}\ln\eta(U_2) + 2\ln(\sqrt{G_1\mu^2}) + 6\ln(\sqrt{G_2\text{Im}U_2\mu^2}) - 2\ln(\sqrt{G_3\mu^2}) \right],
\]
\[
\frac{4\pi^2}{g_{SU(4)}^2} = \text{Im}S + s_2M_2 + s_1M_1
- \frac{1}{4} \left[ -12\text{Re}\ln\eta(U_2) - \ln(\sqrt{G_1\mu^2}) - 3\ln(\sqrt{G_2\text{Im}U_2\mu^2}) - 5\ln(\sqrt{G_3\mu^2}) \right].
\]
Therefore the string theory and the field theory results are compatible provided the following linear-chiral multiplet duality transformation is performed
\[
m_2 = \text{Im}M_2 + \frac{1}{4s_2}\ln(\sqrt{G_1G_3M_1^4}) = \text{Im}M_2 + \frac{1}{4s_2}\ln(\frac{\text{Im}S}{\text{Im}T_2}).
\]
even though there is no regime where both sides are weakly coupled and there are no BPS states to compare. The massless spectrum of the $Z_3$ case was described above (heterotic side) and in section 4 (type I side). On the heterotic side there is an anomalous gauge factor $U(1)_X$ which forces the twisted matter fields $(1,1)_{-4}$ to get a VEV, breaking $U(1)_X$ and giving superpotential masses to the 27 twisted charged fields $(1,8_8)_{+2}$. These VEV’s blow-up the heterotic orbifold singularities and the resulting heterotic massless spectrum coincides with the one of type I at the orbifold point $m_k = 0$. Notice that the $U(1)_X$ gauge field on the type I side becomes massive without the need of any scalar VEV, and thus leaving unbroken the global $U(1)_X$ symmetry which has a counterpart on the heterotic side. After this blowing-up procedure, the heterotic threshold corrections are given by the field theory expression (7.10, 7.11), that does not depend on the heterotic string scale, as explained in section 7. Comparing this expression with the type I one-loop threshold corrections at the orbifold point $m_k = 0$ (7.13) one finds a disagreement since the type I string scale appears explicitly. A possible explanation could be the existence of non-perturbative corrections, depending logarithmically on the string coupling, on one of the two sides which is necessarily strongly coupled. It is interesting to notice however that if the VEV of the type I twisted moduli were vanish in the chiral basis $M_k = 0$, the two one-loop results would match. In view of the relation (7.18), this would imply that the orbifold point should be unstable on the type I side as well, due to higher order corrections, which should induce a Fayet-Iliopoulos (FI) term for the anomalous $U(1)_X$ beyond one-loop, depending on the compactification radii. This possibility seems though unlikely, in view of the arguments of ref. [44].
8. Anomalous $U(1)$’s and gauge coupling unification

It is interesting to discuss in more detail the anomaly cancellation mechanism for the anomalous $U(1)$ factors in four-dimensional $\mathcal{N} = 1$ orientifold models\textsuperscript{13}. Let us start for simplicity with the $Z_3$ example discussed in section 4, where a linear symmetric combination $M$ of the 27 twisted moduli couples to the gauge fields

$$f_a = S + s_a M,$$

and the coefficients $s_a$ were computed in (4.17). The model contains a single anomalous $U(1)_X$ with the gauge generator $Q_X = (1^{12}, 0^4)$ in a complex $U(16)$ basis. Under a $U(1)_X$ gauge transformation with (superfield) parameter $\Lambda$, there are cubic gauge anomalies. The generalized Green-Schwarz mechanism requires a shift of the twisted moduli field combination $M$

$$V_X \rightarrow V_X + \frac{i}{2}(\Lambda - \bar{\Lambda}) , \ M \rightarrow M + \frac{1}{2} \epsilon \Lambda ,$$

such that the gauge-invariant combination appearing in the Kähler potential is $i(M - \bar{M}) - \epsilon V_X$. The mixed anomalies are cancelled provided the following condition holds

$$\frac{\epsilon}{4\pi^2} = \frac{C_{SU(12)}}{s_{SU(12)}} = \frac{C_{SO(8)}}{s_{SO(8)}} = \frac{C_{U(1)_X}}{s_{U(1)_X}}.$$ 

(8.3)

The value of $\epsilon$ was computed in (4.20)

$$\epsilon = \sqrt{\frac{2}{N\pi^3}} \sum_k \prod_{i=1}^3 |\sin \pi k v_i|^4 (-i \text{tr} Q_X \gamma^k)$$

and $(-i \text{tr} Q_X \gamma) = 12\sqrt{3}$. By using the values of the cubic anomalies $(C_{SU(12)}, C_{SO(8)}, C_{U(1)}) = (1/4\pi^2)(-18, 36, -432)$ and (4.17) it is straightforward to check (8.3), which is the direct check of the anomaly cancellation mechanism. By supersymmetry arguments, we can also write down the D-terms which encode the induced Fayet-Iliopoulos term

$$V_D = \frac{g^2}{2} \left( \sum_A X_A K_A \Phi^A + \epsilon K_M M_P^2 \right)^2 ,$$

(8.5)

\textsuperscript{13}More discussion can be found in the recent papers [49].
where $\Phi^A$ denotes the set of charged chiral fields of $U(1)_X$ charge $X_A$, $K_A = \partial K / \partial \Phi^A$ and analogously for $K_M$.

The above discussion generalizes easily in the case of more anomalous $U(1)_\alpha$ ($\alpha = 1 \cdots N_X$) and more linear combinations of twisted moduli fields $M_k$ coupling to gauge fields. In this case (8.1) becomes

$$f_a = S + \sum_k s_{ak} M_k ,$$

(8.6)

and (8.2) generalizes to

$$V_\alpha \to V_\alpha + \frac{i}{2}(\Lambda_\alpha - \bar{\Lambda}_\alpha) , \quad M_k \to M_k + \frac{1}{2} \epsilon_{ka} \Lambda_\alpha ,$$

(8.7)

in an obvious notation. Cancelation of gauge anomalies $\text{tr} X_\alpha Q_a^2$ described by the coefficients $C_{\alpha a}$ ask for the Green-Schwarz conditions

$$C_{\alpha a} = \frac{1}{4\pi^2} \sum_k s_{ak} \epsilon_{ka} ,$$

(8.8)

valid for each $\alpha, a$. The gauge-invariant field combination appearing in the Kähler potential is $i(M_k - \bar{M}_k) - \sum \epsilon_{ka} V_\alpha$ and generates, by supersymmetry, the D-terms

$$V_D = \sum_\alpha \frac{g_\alpha^2}{2} (\sum_A X_A^a K_\alpha \Phi^A + \sum_k \epsilon_{ka} \frac{\partial K}{\partial M_k} M_k^2)^2 .$$

(8.9)

In the above discussion, we neglected the additional complication of linear versus chiral multiplet that arises from the change of basis of the type (7.15). Although a detailed analysis is needed to be done in the presence of several linear multiplets, it appears that the gauge invariant combination entering in the Kähler potential involves the scalar of the linear multiplet $m_k$ instead of the chiral one $M_k$, as in the expression of gauge couplings. We will now show that, at least in the examples studied in this work, the linear combinations of twisted moduli appearing in gauge couplings are the same with the combinations entering in the anomalous $U(1)$ D-terms. Therefore, the vanishing of the latter at the point with maximal gauge symmetry determines the VEVs of the corresponding blowing-up directions and removes the twisted moduli dependence of gauge couplings.
In fact, in the $Z_3$ case there is one anomalous $U(1)$ with the corresponding FI term proportional to the symmetric combination of the 27 blowing-up modes, which also appears in the expression of gauge couplings, as discussed in section 4. Requiring that the non-abelian gauge group remains unbroken, the vanishing of the FI-term fixes the symmetric linear combination of the twisted moduli, removing the arbitrariness in the gauge couplings. At the one-loop level, this selects the orbifold point $m_k = 0$ (or equivalently $\text{Im} M_k = \frac{1}{12e} \ln(V M_I^0)$), implying that physical gauge couplings are moduli independent (up to one loop) and unify at the string scale. On the other hand, if there are higher order corrections that destabilize the orbifold vacuum and fix the twisted moduli VEVs at the point $M_k = 0$ as discussed in the previous section, the unification scale would be determined by the size of the compact space. This is an open important question that deserves further investigation.

In the $Z_6'$ orientifold, there are two linear combinations of twisted moduli fields entering into the expression of gauge couplings (4.16), (5.6). On the other hand, the model has two anomalous $U(1)$’s in each of the D9 and D5 brane sectors. A simple inspection shows that the vanishing of the corresponding FI terms (without breaking the non-abelian gauge symmetry) fixes both combinations appearing in the gauge couplings, removing again the arbitrariness. At the one-loop level, this selects as before the orbifold point $m_k = 0$. In this example, the $\mathcal{N} = 1$ sectors contribute to the running up to the type I string scale, while the $\mathcal{N} = 2$ sectors lead to threshold corrections depending on two compact tori, associated to the two $\mathcal{N} = 2$ sectors of the model ($\theta^2$ and $\theta^3$). The issue of higher loop corrections is similar to the previous ($Z_3$) example.

The situation simplifies in the case of freely acting type I orbifold compactifications [4]. The couplings of the gauge fields to the twisted moduli and the presence of anomalous $U(1)$’s is determined by the (large or small radius) limit where supersymmetry is restored. For instance, in all the examples studied in [4], in particular in the ones discussed in [14], it was assumed that the twisted moduli appearing in the gauge coupling are in the chiral basis $M_k$, with a non-vanishing VEV, leading to a logarithmic volume dependence in the gauge couplings.
section 6, there are no anomalous $U(1)$'s neither couplings to twisted moduli.

9. Conclusions and discussions

The primary goal of this paper was the study of threshold corrections to the gauge couplings in four-dimensional type I orientifolds. The method we use, developed in \cite{28,17}, consists in coupling a background magnetic field $B$ to the Chan-Paton charges of the open strings and computing the quadratic terms $B^2$ in a weak-field expansion. We find that, in the $\mathcal{N} = 2$ sectors of the orientifolds the string oscillators decouple and the result is entirely due to Kaluza-Klein modes of the complex two-tori. This is in agreement with the expectation that only BPS states contribute to the threshold corrections in these sectors \cite{40,28,17}. For a rectangular torus of radii $R_1, R_2$ these corrections are proportional to $\ln(R_1 R_2) + f(R_1/R_2)$, where the function $f$ diverges linearly $f \sim R_1/R_2$ in the $R_1 >> R_2$ limit. For phenomenological purposes, the linear term can be used in the accelerated unification scenario of \cite{15}, while the logarithmic correction can accommodate for a more traditional unification \cite{17}. In the $\mathcal{N} = 1$ sectors the string oscillators do not decouple and the corrections are independent of the compact volume in the orbifold limit. By identifying the string IR divergences with the effective theory running, a string formula for the one-loop beta coefficients of the effective field theory was derived in (5.4). We showed in section 6 that the dependence on the compact radii in models where supersymmetry is spontaneously broken by compactification \cite{9,46} is milder. In particular, in the Scherk-Schwarz model where the branes are parallel to the coordinate used to break supersymmetry $R_1 \to \infty$, the linear behaviour is absent. On the other hand, in the M-theory model where the branes are orthogonal to the breaking coordinate $R'_1 \to \infty$ (where $R'_1 = 1/R_1 M_I^2$ is the T-dual coordinate) the logarithmic corrections disappear as well and the thresholds are exponentially suppressed $e^{-R'_1 R_2 M_I^2}$ in the large radius limit, fact that could have interesting phenomenological implications.
We explicitly computed the UV divergences in $B^4$ in the one-loop open string amplitudes, interpreted in the closed string channel as the propagation of the massless twisted moduli $M_k$ coupling (at the disk level) to the gauge fields. By comparison of the two pictures, we derived the explicit form of these couplings \[(1.16), (4.17), (5.6)\], first discussed in [33]. It turns out that the twisted moduli of all sectors (except the sector $\theta^{N/2}$ for $N$ even) generically couple to the gauge fields, fact that was also justified by supersymmetry arguments in section 7. Similarly we can couple a background magnetic field $B'$ to the anomalous $U(1)$ factors. In this case, the $B'^2$ UV divergences in the open sector amplitudes allowed us to single out the mixing of the RR twisted moduli with the anomalous gauge fields \[(4.20)\]. Using these results, we discussed the Fayet-Iliopoulos terms for the anomalous U(1) factors in section 8, as well as the generalized Green-Schwarz mechanism [31] in section 7.

The method we used to obtain the above results for the D9 brane gauge groups can be applied in a straightforward way to the D5 branes gauge groups, as well, by coupling a background magnetic field to the Dirichlet strings. The difference compared to D9 branes is that the threshold corrections can depend only on the compact torus contained inside the D5 branes world-volume, instead of the three torii available for the D9 branes. Moreover, the couplings \[(5.6)\] of the twisted moduli to the D5 gauge fields exist only for D5 branes located at orbifold fixed points and are nonvanishing only for twisted moduli living in the fixed point where the D5 brane is located.

We also performed a comparison between the one-loop corrected string gauge couplings and the general field theory results [18, 27]. This was done explicitly in the examples of $Z_3$ and $Z'_6$ orientifolds. We found that the two results differ by gauge group dependent corrections, unlike the heterotic orbifolds, where the difference is given by a universal term. In the latter case, this difference is explained by a chiral-linear multiplet duality \[(7.12)\] for the dilaton multiplet, which involves the Green-Schwarz term related to the one-loop
correction to the Kähler potential. In type I orientifolds, the compatibility between the two results ask for specific chiral-linear multiplet duality relations (7.15), (7.18) for the twisted moduli, which are described by linear (chiral) multiplets $m_k (M_k)$ in the string (field theory) basis. This amounts to gauge group dependent corrections to the gauge couplings due to the non-universal tree-level (disk) couplings of the twisted moduli to the gauge fields. In analogy with the heterotic case, this requires loop corrections to the twisted moduli Kähler potential which would be interesting to be explicitly computed.

Our results on threshold corrections were used to discuss in section 7 heterotic - type I duality which was conjectured to hold for the $Z_3$ 4d vacuum. Comparing the threshold corrections of the two sides we find a disagreement which raises doubts on the perturbative validity of the conjectured duality.

We finish by discussing the implication of our results on the unification of gauge couplings. At the level of $\mathcal{N} = 2$ compactifications, in all orientifolds with the exception of $Z_2$, the tree-level (disk) gauge couplings depend linearly on the VEVs of twisted moduli which correspond to exact flat directions of the scalar potential. As a result, unification is in general lost, although it is preserved in special examples (such as $Z_3$ in 6d), where the coefficients of these couplings are proportional to the (4d) beta-functions. In the latter case, however, the unification scale is an arbitrary parameter, depending on the VEVs of the twisted moduli.

For $\mathcal{N} = 1$ orientifolds, the generic presence of anomalous $U(1)$’s fixes the VEVs of the linear combinations of the twisted moduli that appear in gauge couplings, at least in the examples we discussed in this work. At the point of maximal gauge symmetry, the dependence on twisted moduli of gauge couplings vanish (up to one-loop order), and one is left with the dependence on geometric moduli coming from the $\mathcal{N} = 2$ sectors. As a result, generically $\mathcal{N} = 1$ sectors contribute to the running up to the type I string scale, while the $\mathcal{N} = 2$ sectors run up to the corresponding compactification radii. For low scale
string models, our results imply that the only way to achieve gauge coupling unification is
by using the running controled by the $\mathcal{N} = 2$ beta-functions, that can be either power-like
or logarithmic.

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Appendix A

For the reader’s convenience we collect in this appendix the definitions, transformation properties and some identities among the modular functions that are used in the text. For a more extensive list see for instance [50]. The Dedekind function is defined by the usual product formula (with \( q = e^{2\pi i \tau} \))

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) .
\] (A1)

The Jacobi \( \vartheta \)-functions with general characteristic and arguments are

\[
\vartheta_{[\alpha \beta]}(z|\tau) = \sum_{n \in \mathbb{Z}} e^{i\pi \tau (n-\alpha)^2} e^{2\pi i (z-\beta)(n-\alpha)} .
\] (A2)

We give also the product formulae for the four special \( \vartheta \)-functions

\[
\begin{align*}
\vartheta_1(z|\tau) &\equiv \vartheta \left[ \frac{1}{2} \right] (z|\tau) = 2q^{1/8} \sin \pi z \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{2\pi i z})(1 - q^n e^{-2\pi i z}) \\
\vartheta_2(z|\tau) &\equiv \vartheta \left[ \frac{1}{2} \right] (z|\tau) = 2q^{1/8} \cos \pi z \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n e^{2\pi i z})(1 + q^n e^{-2\pi i z}) \\
\vartheta_3(z|\tau) &\equiv \vartheta \left[ 0 \right] (z|\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-1/2} e^{2\pi i z})(1 + q^{n-1/2} e^{-2\pi i z}) \\
\vartheta_4(z|\tau) &\equiv \vartheta \left[ \frac{1}{2} \right] (z|\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1/2} e^{2\pi i z})(1 - q^{n-1/2} e^{-2\pi i z})
\end{align*}
\] (A3)

The \( \vartheta_a \) for \( a = 2, 3, 4 \) are even functions of \( z \), while \( \vartheta_1 \) is an odd function whose first derivative at zero is

\[
\vartheta_1'(0) = 2\pi \eta^3 .
\] (A4)

The modular properties of these functions are described by

\[
\begin{align*}
\eta(\tau + 1) &= e^{i\pi/12} \eta(\tau) , \quad \vartheta \left[ \frac{\alpha}{\beta} \right] (z|\tau + 1) = e^{-i\pi \alpha (\alpha-1)} \vartheta \left[ \frac{\alpha}{\alpha + \beta - \frac{1}{2}} \right] (z|\tau) \\
\eta(-1/\tau) &= \sqrt{-i\tau} \eta(\tau) , \quad \vartheta \left[ \frac{\alpha}{\beta} \right] \left( \frac{z}{\tau} \right) = \sqrt{-i\tau} e^{2i\pi \alpha \beta + i\pi z^2/\tau} \vartheta \left[ \frac{\beta}{-\alpha} \right] (z|\tau) ,
\end{align*}
\] (A5)
A very useful identity is

$$\sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \vartheta \left[ \frac{\alpha}{\beta} \right] (z) \prod_{i=1}^{3} \vartheta \left[ \frac{\alpha}{\beta + v_i} \right] = -2 \vartheta_1 \left( -\frac{z}{2} \right) \vartheta_1 \left( \frac{z - v_1 + v_2 + v_3}{2} \right) \vartheta_1 \left( \frac{z + v_1 - v_2 + v_3}{2} \right) \vartheta_1 \left( \frac{z + v_1 + v_2 - v_3}{2} \right),$$

valid for $v_1 + v_2 + v_3 = 0$. By taking the second derivative of (A7) at zero argument it is easy to prove the following identities

$$\sum_{\alpha,\beta=0,1/2} (\alpha^2 \eta_{\alpha,\beta} \vartheta'' \left[ \frac{\alpha}{\beta} \right] \vartheta' \left[ \frac{\alpha}{\beta} \right] \vartheta \left[ \frac{\alpha}{\beta + k v_i} \right] \vartheta \left[ \frac{\alpha}{\beta + k v_j} \right] = -4\pi^2, \ k(v_i + v_j) = 1 (mod 2),$$

$$\sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\vartheta'' \left[ \frac{\alpha}{\beta} \right] \vartheta \left[ \frac{\alpha}{\beta} \right]}{\vartheta \left[ \frac{\alpha+1/2}{\beta + k v_i} \right] \vartheta \left[ \frac{\alpha+1/2}{\beta + k v_j} \right] = 4\pi^2,}$$

$$\sum_{\alpha,\beta=0,1/2} (\alpha^2 \eta_{\alpha,\beta} \vartheta'' \left[ \frac{\alpha}{\beta} \right] \vartheta' \left[ \frac{\alpha}{\beta} \right] \vartheta \left[ \frac{\alpha}{\beta + 1/2} \right] \vartheta \left[ \frac{\alpha}{\beta - 1/2} \right] = 4\pi^2,}$$

which help us to prove that the oscillator contributions to the threshold corrections decouple for the $\mathcal{N} = 2$ sectors $k = 2, 3, 4$ of the $Z_6'$ orientifold and in general for $\mathcal{N} = 2$ sectors of any four-dimensional type I orientifold.

In the contributions from the $\mathcal{N} = 1$ sectors coming from $\mathcal{A}_{95}$ in section 5, we used the following modular identity (also valid for $v_1 + v_2 + v_3 = 0$)

$$\sum_{\alpha,\beta=0,1/2} \eta_{\alpha,\beta} \frac{\vartheta'' \left[ \frac{\alpha}{\beta} \right] \vartheta \left[ \frac{\alpha}{\beta + k v_i} \right]}{\vartheta \left[ \frac{\alpha}{\beta + k v_i} \right] \vartheta \left[ \frac{\alpha}{\beta + k v_j} \right] \vartheta \left[ \frac{\alpha}{\beta + k v_i} \right] \vartheta \left[ \frac{\alpha}{\beta + k v_j} \right] = -2\pi \left\{ \frac{\vartheta' \left[ \frac{1/2}{1/2 + k v_i} \right]}{\vartheta \left[ \frac{1/2}{1/2 + k v_i} \right] + \vartheta' \left[ \frac{0}{1/2 - k v_i} \right] + \sum_{i=1}^2 \frac{\vartheta' \left[ \frac{1/2}{1/2 + k v_i} \right]}{\vartheta \left[ \frac{1/2}{1/2 + k v_i} \right]} \right\}.$$  

(A9)

In taking the infrared limit for the $Z_6'$ model, we need also the formula

$$\lim_{q \to 0} \frac{\vartheta' \left[ \frac{0}{1/2 - k v_i} \right]}{\vartheta \left[ \frac{0}{1/2 - k v_i} \right]} = 0.$$  

(A10)
Appendix B

The annulus amplitudes (for vanishing magnetic field) relevant for the computation of section 5 of the $Z_6'$ model are

\[
A^{(k)}_{99} = \frac{1}{8\pi^4 t^2} \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha]}{\eta^3} \prod_{i=1}^{3} (-2 \sin \pi k v_i) \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} (\text{tr} \gamma^k_9)^2, \quad k = 1, 5
\]

\[
A^{(k)}_{99} = \frac{1}{8\pi^4 t^2} \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha]}{\eta^6} \prod_{i=1}^{3} (2 \sin \pi k v_i) \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} (\text{tr} \gamma^k_9)^2 \Gamma^{(2)}_2, \quad k = 2, 4
\]

\[
A^{(3)}_{99} = \frac{1}{8\pi^4 t^2} \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha]}{\eta^6} \prod_{i=1}^{3} (2 \sin \pi k v_i) \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} (\text{tr} \gamma^k_9)^2 \Gamma^{(3)}_3, \quad (B1)
\]

for the NN (99) sector. The annulus amplitudes from the ND (95) sector are

\[
A^{(k)}_{95} = -\frac{1}{2\pi^4 t^2} \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha]}{\eta^3} \prod_{i=1}^{2} (2 \sin \pi k v_i) \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} (\text{tr} \gamma^k_9)^2 \Gamma^{(2)}_5, \quad k = 1, 2, 4, 5
\]

\[
A^{(k)}_{95} = \frac{1}{4\pi^4 t^2} \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha]}{\eta^6} \prod_{i=1}^{2} \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} (\text{tr} \gamma^k_9)^2 \Gamma^{(3)}_3, \quad k = 0, 3 \quad (B2)
\]

Similarly, the supersymmetric Möbius amplitudes from the 99 sector are

\[
M^{(k)}_{9} = -\frac{1}{8\pi^4 t^2} \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha]}{\eta^3} \prod_{i=1}^{3} (-2 \sin \pi k v_i) \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} (\text{tr} \gamma^k_9)^2, \quad k = 1, 5
\]

\[
M^{(k)}_{9} = -\frac{1}{8\pi^4 t^2} \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha]}{\eta^6} \prod_{i=1}^{3} (2 \sin \pi k v_i) \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} (\text{tr} \gamma^k_9)^2 \Gamma^{(2)}_2, \quad k = 2, 4
\]

\[
M^{(3)}_{9} = -\frac{1}{8\pi^4 t^2} \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha]}{\eta^6} \prod_{i=1}^{3} (2 \sin \pi k v_i) \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} (\text{tr} \gamma^k_9)^2 \Gamma^{(3)}_3. \quad (B3)
\]

In the presence of background magnetic field coupled to the D9 brane gauge group, the above annulus 99 amplitudes become

\[
A^{(k)}_{99} = -\frac{iB}{\pi^3 t} \text{tr} \left( (Q_{g^k} \otimes g^k + g^k \otimes Q_{g^k}) \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} \prod_{i=1}^{3} \sin \pi k v_i \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} \right) \Gamma^{(2)}_2 \quad (B4)
\]

for the $N = 1$ sectors $k = 1, 5$,

\[
A^{(k)}_{99} = \frac{iB}{2\pi^3 t} \text{tr} \left( (Q_{g^k} \otimes g^k + g^k \otimes Q_{g^k}) \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} \prod_{i=1}^{3} \sin \pi k v_i \frac{\vartheta[\frac{\alpha}{2}]_{1/2+kv_i}}{\vartheta[\frac{1}{2}]_{1/2+kv_i}} \right) \Gamma^{(2)}_2
\]
for the $\mathcal{N} = 2$ sectors $k = 2, 4$ and

$$A^{(3)}_{99} = \frac{IB}{2\pi^3 t} \text{tr} \left( (Q^{\gamma_9 \otimes \gamma_9 + \gamma_9 \otimes Q^{\gamma_9}}_{\alpha,\beta = 0,1/2} \frac{\partial^{[\alpha]}(\frac{i\epsilon t}{2})\partial^{[\beta+3\nu_t]}{\frac{1}{2}}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right) \prod_{i=1,2} \sin \pi 3v_i \frac{\psi^{[\alpha]}{\frac{1}{2}}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right)^{(2)}$$

for the $\mathcal{N} = 2$ sector $k = 3$. The annulus 95 amplitudes in the presence of the magnetic field become

$$A^{(k)}_{95} = -\frac{IB}{2\pi^3 t} \text{tr} \left( Q^{\gamma_9 \otimes \gamma_9}_{\alpha,\beta = 0,1/2} \frac{\partial^{[\alpha]}(\frac{i\epsilon t}{2})\partial^{[\beta+3\nu_t]}{\frac{1}{2}}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right) \prod_{i=1,2} \sin \pi 3v_i \frac{\psi^{[\alpha]}{\frac{1}{2}}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right)^{(2)}$$

for $k = 1, 2, 4, 5$ and

$$A^{(k)}_{95} = \frac{IB}{4\pi^3 t} \text{tr} \left( Q^{\gamma_9 \otimes \gamma_9}_{\alpha,\beta = 0,1/2} \frac{\partial^{[\alpha]}(\frac{i\epsilon t}{2})\partial^{[\beta+3\nu_t]}{\frac{1}{2}}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right) \prod_{i=1,2} \sin \pi 3v_i \frac{\psi^{[\alpha]}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right)^{(2)}$$

for $k = 0, 3$. Similarly, the Möbius amplitudes in the Neumann case become

$$M^{(k)}_{9} = \frac{IB}{\pi^3 t} \text{tr} \left( Q^{\gamma_9 \otimes \gamma_9}_{\alpha,\beta = 0,1/2} \frac{\partial^{[\alpha]}(\frac{i\epsilon t}{2})\partial^{[\beta+3\nu_t]}{\frac{1}{2}}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right) \prod_{i=1,3} \sin \pi 3v_i \frac{\psi^{[\alpha]}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right)^{(2)}$$

from the $\mathcal{N} = 1$ sectors $k = 1, 5$,

$$M^{(k)}_{9} = \frac{IB}{\pi^3 t} \text{tr} \left( Q^{\gamma_9 \otimes \gamma_9}_{\alpha,\beta = 0,1/2} \frac{\partial^{[\alpha]}(\frac{i\epsilon t}{2})\partial^{[\beta+3\nu_t]}{\frac{1}{2}}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right) \prod_{i=1,3} \sin \pi 3v_i \frac{\psi^{[\alpha]}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right)^{(2)}$$

from the $\mathcal{N} = 2$ sectors $k = 2, 4$ and

$$M^{(3)}_{9} = \frac{IB}{\pi^3 t} \text{tr} \left( Q^{\gamma_9 \otimes \gamma_9}_{\alpha,\beta = 0,1/2} \frac{\partial^{[\alpha]}(\frac{i\epsilon t}{2})\partial^{[\beta+3\nu_t]}{\frac{1}{2}}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right) \prod_{i=1,2} \sin \pi 3v_i \frac{\psi^{[\alpha]}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right)^{(2)}$$

from the $\mathcal{N} = 2$ sector $k = 3$.

We also display here the various amplitudes in the transverse channel, necessary in order to investigate the UV behaviour. The annulus amplitudes are

$$A^{(k)}_{99} = \frac{IB}{\pi^3 t} \text{tr} \left( (Q^{\gamma_9 \otimes \gamma_9 + \gamma_9 \otimes Q^{\gamma_9}}_{\alpha,\beta = 0,1/2} \frac{\partial^{[\alpha]}(\epsilon)\partial^{[\beta+3\nu_t]}{\frac{1}{2}}(\epsilon)}{\frac{1}{2}} \right) \prod_{i=1,3} \sin \pi 3v_i \frac{\psi^{[\alpha]}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right)^{(2)}$$

for the $\mathcal{N} = 1 k = 1, 5$ sectors,

$$A^{(k)}_{99} = \frac{Bv_2}{4\pi^3 t} \text{tr} \left( (Q^{\gamma_9 \otimes \gamma_9 + \gamma_9 \otimes Q^{\gamma_9}}_{\alpha,\beta = 0,1/2} \frac{\partial^{[\alpha]}(\epsilon)\partial^{[\beta+3\nu_t]}{\frac{1}{2}}(\epsilon)}{\frac{1}{2}} \right) \prod_{i=1,3} \sin \pi 3v_i \frac{\psi^{[\alpha]}(\frac{i\epsilon t}{2})\eta^3}{\frac{1}{2}} \right)^{(2)}$$
for the $N = 2$ sectors $k = 2, 4$ and

$$
\frac{1}{l} A_{99}^{(3)} = B v_3 \frac{4}{4 \pi^3} \text{tr} \left( (Q_{\gamma_9^3} \otimes \gamma_9^3 + \gamma_9^3 \otimes Q_{\gamma_9^3}) \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha][\epsilon]}{\vartheta[1/2][\epsilon]} \frac{\vartheta[\alpha + 3\nu_2]}{\vartheta[1/2 + 3\nu_2]} \right) \prod_{i=1,2} \sin \pi \nu_i \frac{\vartheta[\alpha + 3\nu_i]}{\vartheta[1/2 + 3\nu_i]}
$$

for the $N = 2$ sector $k = 3$, where $\nu_2(\nu_3)$ is the volume (in string units) of the second (third) compact torus. The corresponding Neumann-Dirichlet annulus amplitudes are

$$
\frac{1}{l} A_{95}^{(k)} = -i B \frac{2}{2 \pi^3} \text{tr} \left( Q_{\gamma_9^k} \otimes \gamma_9^k \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha][\epsilon]}{\vartheta[1/2][\epsilon]} \frac{\vartheta[\alpha + \nu_3]}{\vartheta[1/2 + \nu_3]} \right) \prod_{i=1,2} \frac{\vartheta[\alpha + \nu_i]}{\vartheta[1/2 + \nu_i]} W_{3}^{(2)}
$$

for the sectors $k = 1, 2, 4, 5$ and

$$
\frac{1}{l} A_{95}^{(k)} = -8 B \frac{2}{2 \pi^3} \text{tr} \left( Q_{\gamma_9^k} \otimes \gamma_9^k \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha][\epsilon]}{\vartheta[1/2][\epsilon]} \frac{\vartheta[\alpha + \nu_3]}{\vartheta[1/2 + \nu_3]} \right) \prod_{i=1,2} \frac{\vartheta[\alpha + \nu_i]}{\vartheta[1/2 + \nu_i]} W_{3}^{(2)}
$$

for $k = 0, 3$. The Neumann Möbius amplitudes read

$$
\frac{1}{l} M_{9}^{(k)} = 8i B \frac{2}{2 \pi^3} \text{tr} \left( Q_{\gamma_9^{2k}} \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha][\epsilon]}{\vartheta[1/2][\epsilon]} \frac{\vartheta[\alpha + 2\nu_i]}{\vartheta[1/2 + 2\nu_i]} \right) \prod_{i=1} \sin \pi \nu_i \frac{\vartheta[\alpha + 2\nu_i]}{\vartheta[1/2 + 2\nu_i]},
$$

for $k = 1, 5$,

$$
\frac{1}{l} M_{9}^{(k)} = -8i B \frac{2}{2 \pi^3} \text{tr} \left( Q_{\gamma_9^{2k}} \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha][\epsilon]}{\vartheta[1/2][\epsilon]} \frac{\vartheta[\alpha + 2\nu_i]}{\vartheta[1/2 + 2\nu_i]} \right) \prod_{i=1,3} \sin \pi \nu_i \frac{\vartheta[\alpha + 2\nu_i]}{\vartheta[1/2 + 2\nu_i]} W_{2}^{(2,e)}
$$

for $k = 2, 4$, where $W_{2}^{(2,e)}$ denote the even winding sum along the second compact complex coordinate and

$$
\frac{1}{l} M_{9}^{(k)} = -8i B \frac{2}{2 \pi^3} \text{tr} \left( Q_{\gamma_9^{6}} \sum_{\alpha, \beta = 0, 1/2} \eta_{\alpha, \beta} \frac{\vartheta[\alpha][\epsilon]}{\vartheta[1/2][\epsilon]} \frac{\vartheta[\alpha + 3\nu_i]}{\vartheta[1/2 + 3\nu_i]} \right) \prod_{i=1,2} \sin \pi \nu_i \frac{\vartheta[\alpha + 3\nu_i]}{\vartheta[1/2 + 3\nu_i]} W_{3}^{(2,e)}
$$

for $k = 3$.

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