GRavitational Collapse of Fluid Bodies
And Cosmic Censorship: Analytic Insights

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Abstract

The present analytical understanding on the nature of the singularities which form
at the endstate of gravitational collapse of massive fluid bodies (“stars”) is reviewed.
Special emphasis is devoted to the issue of physical reasonability of the models.

1 Introduction

The investigation on the “final fate” of gravitational collapse of initially regular distributions
of matter is one of the most active field of research in contemporary general relativity. It is,
indeed, known that under fairly general hypothesis solutions of the Einstein equations with
“physically reasonable” matter can develop into singularities [1]. The key problem that still
remains unsolved is the nature of such singularities. The main open issue is whether the
singularities, which arise as the end point of collapse, can actually be observed.

Roger Penrose [2], was the first to propose the idea, known as cosmic censorship con-
jecture: does there exist a cosmic censor who forbids the occurrence of naked singularities,
clothing each one in an absolute event horizon? This conjecture can be formulated in a
“strong” sense (in a “reasonable” spacetime we cannot have a naked singularity) or in a
weak sense (even if such singularities occur they are safely hidden behind an event horizon,
and therefore cannot communicate with far-away observers). Since Penrose’s proposal there
have been various attempts to prove the conjecture (see [3] and references therein). Unfor-
tunately, no such attempts have been successful so far. As a consequence, the research in
this field turned to more tractable objectives. In particular, one would like to understand
what happens in simple systems, like spherically symmetric ones (interestingly enough, even
this apparently innocuous problem is far from being completely solved, although, as we shall
see, a general pattern does seem to arise).

Our aim here is to overview only models which have a clear physical interpretation.
Therefore, we shall require satisfaction of the weak energy condition as well as existence of
a singularity free initial data surface. Moreover, we shall take into consideration solutions of
the Einstein field equations which are physically meaningful in terms of a (phenomenological)
equation of state of the matter. In this respect it is worth mentioning that material schemes
having a well defined microscopical interpretation, like the Vlasov-Einstein system, would be

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closer to such a requisite \cite{4}. However, very little is known on the dynamics of such models (a numerical investigation has been recently carried out \cite{3}).

In presence of excellent general reviews on gravitational collapse and cosmic censorship \cite{6, 7, 8, 9, 10, 11}, we have focused our attention on a specific issue namely, the investigation of analytical models describing the gravitational collapse of massive stars. Therefore we are not going to address here many related important topics. These include Vaidya spacetimes \cite{4}, radiation shells (see \cite{9} and references therein), gravitational collapse of scalar fields \cite{12, 13}, critical behaviour in numerical relativity \cite{14}, stability of Cauchy horizon in Reissner-Nordström spacetimes \cite{15}, the Hoop conjecture \cite{16, 17} among others \cite{6, 10, 18, 19}.

## 2 Einstein equations for spherical collapse

What is known analytically in gravitational collapse is essentially restricted to spherical symmetry (one exception is given by the Szekeres “quasi-spherical” spacetimes \cite{20}, but the results \cite{21, 22} are very similar to those holding in Tolman-Bondi models – See Section 4.2.4). Therefore we discuss the mathematical structure of the Einstein field equations describing collapse of a deformable body only in the spherically symmetric case. For perfect fluids this structure is well known \cite{23}; we present here a more general case which takes into account anisotropic materials as well \cite{24}. Also, we consider only non-dissipative processes since very little is known in the dissipative case \cite{25}.

The general, spherically symmetric, non-static line element in comoving coordinates $t, r, \theta, \varphi$ can be written in terms of three functions $\nu, \eta, Y$ of $r$ and $t$:

$$ds^2 = -e^{2\nu}dt^2 + \eta^{-1}dr^2 + Y^2(d\theta^2 + \sin^2 \theta d\varphi^2) .$$

(1)

Throughout this paper, we shall assume that the collapsing body is “materially spherically symmetric” in the sense that all the physical “observables” do not depend on angles. The matter density of the material (baryon number density) is then given by

$$\rho = \rho_0(r) Y^{-2} \sqrt{\eta} ,$$

(2)

where $\rho_0(r)$ is an arbitrary (positive) function. As in any theory of continuous media, to choose a specific material one has to specify the internal energy $\epsilon$. This function depends on the parameters characterizing the state of strain of the body (for a recent review of relativistic elasticity theory see \cite{26}). It can be shown that such parameters can be identified with $Y$ and $\eta$ in the comoving frame (in other words, any deformation is described “gravitationally”). Therefore, we introduce as equation of state of the material a positive function

$$\epsilon = \psi(r, Y, \eta) .$$

Here the explicit dependence on $r$ takes into account possible inhomogeneities.

The energy-momentum tensor can be readily calculated and the result is a diagonal tensor of the form $T^\mu_\nu = \text{diag}(-\epsilon, \Sigma, \Pi, \Pi)$. Here the stress-strain relations (i.e. the relations giving the radial stress $\Sigma$ and the tangential stress $\Pi$ in terms of the constitutive function) are given by

$$\Sigma = 2\eta \frac{\partial \psi}{\partial \eta} - \psi ,$$

(3)

$$\Pi = \frac{1}{2} Y \frac{\partial \psi}{\partial Y} - \psi .$$

(4)
We shall always use the word stress rather than pressure since both \( \Sigma \) and \( \Pi \) can be in principle negative (tensions) without violating the energy conditions.

Different materials correspond to different choices of the function \( \psi \). It is, however, worth mentioning that the material scheme most widely used in astrophysical applications is the perfect fluid, which can be characterized as a material whose function of state depends on the number density only (\( \epsilon = \dot{\psi}(\rho) \)). In this case both stresses coincide:

\[
\Pi = \Sigma = \rho \frac{d\tilde{\psi}}{d\rho} - \tilde{\psi} := p ,
\]

where \( p \) is the isotropic pressure. Two particular cases are worth mentioning in this scheme. One is that of linear pressure-density relation (\( p = k\epsilon \)); the corresponding equation of state is \( \dot{\psi}(\rho) = \rho(1 + A\rho^k) \) where \( A \) is a constant. The other one is the dust model for which \( p = 0 \). In this case the energy is distributed proportionally to the mass (\( \ddot{\psi}(\rho) = \rho \)).

As soon as one allows anisotropy to occur, other interesting models appear (see [27] for a review on the role of anisotropy in relativistic astrophysics). Recently, a particular anisotropic model has been singled out [24, 28] (for previous investigations on this kind of models see references in [28]). In this model, one assumes that the radial stress identically vanishes. The key role is played by the equation (3) which shows that the dependence of \( \psi \) on \( \eta \) must be a multiplicative dependence from \( \sqrt{\eta} \) only. Therefore, materials with vanishing radial stresses can be characterized, using equation (2), via equations of state of the form

\[
\psi(r, \eta, Y) = \rho h(r, Y),
\]

where \( h \) is a positive, but otherwise arbitrary, function.

Once an equation of state has been chosen, the Einstein field equations become a closed system; in spherical symmetry there are three independent equations for the three variables \( \nu, \eta \) and \( Y \). It has proven, however, to be very useful to write the field equations as a system of four differential equations. This is done by introducing the mass function [23, 29] defined as

\[
m(r, t) = \frac{Y}{2} \left( 1 - Y^2\eta + \dot{Y}^2 e^{-2\nu} \right),
\]

where a dash and a dot denote derivatives with respect to \( r \) and \( t \) respectively. The mass function is arbitrary (positive) and allows us to write the field equations in the following form (four compatible equations for three variables)

\[
m' = 4\pi \epsilon Y^2 Y',
\]

\[
\dot{m} = -4\pi \Sigma Y^2 \dot{Y} ,
\]

\[
Y' \dot{\eta} = -2\eta(\dot{Y}' - \dot{Y} \nu') ,
\]

\[
\Sigma' = - (\epsilon + \Sigma) \nu' - 2(\Sigma - \Pi)(Y'/Y) .
\]

### 3 Physical reasonability and initial data

It is easy to produce new “solutions” of the Einstein field equations in “matter”. Indeed, just pick up a metric at will and claim that this is a “solution” referring to the calculated energy momentum tensor. Of course, what one has to do to remove the quotation marks from the above statements is to check the physical reasonability of the results.
First of all, the weak energy condition must be imposed on the energy momentum tensor. This condition requires \( T_{\mu\nu}u^\mu u^\nu \geq 0 \) for any non spacelike \( u^\mu \) and implies, besides positivity of \( \epsilon \), non-negativeness of \( \epsilon + \Sigma \) and \( \epsilon + \Pi \). Due to equations (3) and (4), such conditions are equivalent to differential inequalities on the function \( \psi \), namely, \( \partial \psi / \partial \eta \geq 0 \) and \( \partial \psi / \partial Y \leq 0 \).

Imposing the weak energy condition *per se* does not assure physical reasonability, since there is no guarantee that the energy momentum tensor can be deduced from a field theoretic description of matter. What is needed is the satisfaction of a suitable equation of state. We shall require, in addition, the equation of state to be locally stable \(^{24}\) (this last requirement could be relaxed in presence of rotationally-induced stress). The local stability condition requires the (local) equilibrium state of the material to be unstrained. In the “comoving picture” this amounts to say that the energy must have an absolute minimum at the flat-space values of the metric.

Let us collect the functions describing admissible equations of state in spherical symmetry in a set

\[
\Psi = \left\{ \psi \in C^2(\mathbb{R}^3_+, \mathbb{R}_+) : \psi(r, r, 1) = \min \psi(r, Y, \eta), \frac{\partial \psi}{\partial \eta} \geq 0, \frac{\partial \psi}{\partial Y} \leq 0 \right\}.
\]

We shall always assume that the value of \( \psi \) at the minimum (which in general can be a function of \( r \)) has been rescaled to unity (this can be done without loss of generality).

A solution of the Einstein field equations describes the collapse of an initially regular distribution of matter only if the spacetime admits a spacelike hypersurface \((t = 0 \text{ say})\) which carries *regular* initial data. This means that the metric, its inverse, and the second fundamental form all have to be continuous at \( t = 0 \).

On the initial hypersurface we use the scaling freedom of the \( r \) coordinate to set \( Y(r, 0) = r \). We call a set of initial data *complete* if it is minimal, in the sense that no part of its content can be gauged away by a coordinate transformation. We will now prove that a complete set of initial data for equations (7)-(10), at fixed equation of state, is composed by a pair of functions. Physically, such functions describe the initial distribution of energy density \( \epsilon_0 = \epsilon(r, 0) \) and of velocity \( V_0 = e^{-\nu(r, 0)}Y(r, 0) \). It is sometimes convenient to parameterize these two distributions in terms of two other functions \( \{F(r), f(r)\} \) where \( F(r) = m(r, 0) \) is the initial distribution of mass and \( f(r) \) is called “energy function”. The relationship between the two sets \( \{F, f\} \) and \( \{\epsilon_0, V_0\} \) is given by the following formulae

\[
F' = 4\pi r^2 \epsilon_0, \quad f = V_0^2 - 2F/r. \tag{11}
\]

The function \( F \) has to be non-negative with \( \epsilon_0(0) = \lim_{r \to 0} F(r)/r^3 \) finite and non-vanishing, while \( f \) has to be greater than \(-1\) to preserve the signature of the metric (see equation (11) below) with \( \lim_{r \to 0} f(r) = 0 \).

The proof that only two arbitrary functions of \( r \) are a complete set of initial data at fixed equation of state is implicitly contained in many papers on spherical collapse (see e.g. \(^{23, 30}\)). It seems, however, that a complete proof has never been published in details, so we take this occasion to give it. We denote by a subscript, the initial value of each quantity appearing in the Einstein field equations. We know \( F = m_0, V_0 = e^{-\nu_0}Y_0 \). Since \( Y_0 = r \) and \( \psi \) is a known function of \( r, Y \) and \( \eta \), its initial value \( \psi_0 \) is also known - as well as the initial values of the stresses due to formulae (3) and (4) - as a function of \( r \) and \( \eta_0 (\psi_0 = \psi(r, r, \eta_0)) \).

Evaluation of equation (3) at \( t = 0 \) now gives \( \psi_0 = F'/(4\pi r^2) \), from which the value of \( \eta_0 \) can be extracted algebraically. At this point, the remaining three field equations can be used to evaluate the remaining data, i.e., \( m_0, \nu_0 \) and \( \eta_0 \):
\[
m_0 = -4\pi \Sigma_0 r^2 V_0, \quad \nu'_0 = -\frac{\Sigma'_0}{\psi_0 + \Sigma_0} - 2 \frac{\Sigma_0 - \Pi_0}{r(\psi_0 + \Sigma_0)}, \quad \eta_0 = -2\eta_0 e^{\nu_0} V'_0. \tag{12}
\]

This completes the proof.

In what follows, we consider only solutions which can be interpreted as models of collapsing stars, i.e., isolated objects rather than “universes”. This is possible only if the metric matches smoothly with the Schwarzschild vacuum solution (the matching between two metrics is smooth if both the first and the second fundamental form are continuous on the matching surface).

### 4 Classification and nature of singularities

To understand the collapsing scenarios as well as the nature of the singularities, one would like to analyse exact solutions of the Einstein field equations. However, it goes without saying that the non-linearity of such equations makes them essentially untractable, even in spherical symmetry, without further simplifying hypotheses (like e.g. vanishing shear or acceleration, see [31, 32]). In view of such difficulties, one would like to extract information from the Einstein equations without solving them completely.

The mathematical structure of spherical collapse discussed in the previous Section shows that there is a one-to-one correspondence between solutions and choices of triplets \((s, \text{say})\) of functions \(s = \{F(r), f(r), \psi(r, \eta, Y)\}\) (we are, of course, identifying solutions modulo gauge transformations). We denote the set of solutions parameterized in this way by \(S\). Since in this parameterization we have already taken into account regularity as well as physical admissibility, the whole physical content of the cosmic censorship problem can be translated in the mathematical terms of predicting the endstate of any choice of \(s \in S\).

#### 4.1 Non-singular solutions.

The existence of non-singular, non-static solutions of the Einstein field equations is not forbidden by the singularity theorems (we refer the reader to [33] for a recent review on singularity theorems and further references) and a gravitational collapse can lead to a bouncing back, at a finite area radius, without singularity formation. This phenomenon can be recursive, producing an eternally oscillating, globally regular solution [34, 35, 36]. Speaking very roughly, one can prepare regular initial data in such a way that the region of possible trapped surface formation is disconnected from the data. In this way the remaining hypothesis of the singularity theorems can be satisfied without singularity formation (see also [24]).

Much more exotic than the oscillating solutions, are the globally regular solutions that describe non-singular blackholes [33]. These are matter objects “sitting” inside their Schwarzschild radii. Their finite extension replaces the central singularity with a matter-filled, non-singular region. In this case trapped surfaces are obviously present and therefore the strong energy condition must be violated (a simple argument shows that also the dominant energy condition is necessarily violated). In any case, such strange objects are not ruled out by general relativity as far as only the weak energy condition and an equation of state are required [37, 38, 39]. However, as far as we are aware, it is not known if a fully dynamic solution exists which could eventually lead to this exotic end state.
4.2 Singular solutions

We now move to the case in which a singularity is formed in the future of a regular initial data set. The singularities of spherically symmetric matter filled spacetimes can be recognized from divergence of the energy density and curvature scalars, like e.g. the Kretschmann scalar $R^\mu\nu\rho\sigma R_{\mu\nu\rho\sigma}$. Essentially, these singularities can be of two kinds. We shall call shell crossing singularities those at which $Y'$ vanishes ($Y \neq 0$), and shell focusing singularities those at which $Y$ vanishes. To the two kind of singularities correspond two curves $t_{sc}(r)$ and $t_{f}(r)$ in the $r-t$ plane, defined by $Y'(r, t_{sc}(r)) = 0$ and by $Y(r, t_{f}(r)) = 0$ respectively. Physically, the shell crossing curve gives the time at which two neighbouring shells of matter intersect each other, while the shell focusing curve identifies at which time the shell labeled $r$ “crushes to zero size”. Of these two kinds of singularities, the one of physical interest is obviously that occurring first in the sense that, at fixed $r$, one has $t_{f} > t_{sc}$ or vice versa. It would be very interesting, therefore, to carry out a study of the field equations in order to obtain conditions for shell crossing avoidance in spherical spacetimes. This study has been up to now carried out only for dust spacetimes [40, 41, 42].

From the point of view of censorship, the nature of a singularity in an asymptotically flat, initially regular spacetime can be one of the following. First of all, a singularity can be space-like, like e.g. the Schwarzschild singularity or the singularity occurring at the endstate of the collapse of a Oppenheimer–Snyder dust cloud (see Section 4.2.2 below). These singularities lie in the future of all possible observers and therefore are strongly censored (i.e. allowed by strong cosmic censorship). If a singularity is not strongly censored, then it is naked, i.e. visible to some observer. However, two different cases can occur, namely the singularity can be locally or globally naked. A singularity is locally naked if light signals can emerge from it but fall back without reaching any asymptotic observer. A singularity of this kind will be visible only to observers who have crossed the horizon; therefore, the weak cosmic censorship holds for such endstates. An example of a spacetime containing a locally naked singularity is provided by the Kerr spacetime with mass greater than the angular momentum per unit mass (or by the Reissner–Nordström spacetime with mass greater than charge). Finally, a singularity is globally naked if light-rays emerging from it can reach an asymptotic observer.

4.2.1 Shell crossing singularities.

The first explicit example showing formation of a naked singularity was found as a shell crossing singularity in a spherical dust cloud [13]. It can be shown that these singularities are timelike and are always locally naked.

Some definitions have been proposed to put singularities “in the order of increasing seriousness” [14]. Essentially, what is done is to check the behaviour of the invariants of the Riemann tensor in the approach to the singularity. According to such criteria, the shell crossing singularities turn out to be “weak” at least as compared with the shell focusing singularities. This “weakness’ is considered by some authors as an hint of a possible extension of the spacetime [15]. However, in spite of their “weakness”, there is at the moment no available general proof of extendibility of spacetimes through a shell cross (although some encouraging results exist, see [16]). The unique exception is a paper by Papapetrou and Hamou [17]. In this paper the authors claim to have explicitly found the extension in the case of “degenerate” shell crossing singularities, i.e. when the curve $t_{sc}(r)$ degenerates to an “instant of time” $t_{sc} = T = \text{const}$. In this case, it is easy to check that the crossing happens at a “point” $r = \text{const}.$ rather than at a “3-space” and this is the key to their treatment.
However, some results of this paper are unclear from the physical point of view.

What is actually available is only a continuous extension of shell crossing singularities exists in the dust case [41]. We shall show this in a slightly more general case. Integrating equation (9) formally with respect to time we can write

$$\eta^{-1} = \frac{Y^2}{1 + f} \Omega^2,$$

where $f$ is the energy function and $\Omega(r, t) = \exp \left( - \int_{t_0}^t \nu \frac{dY}{Y} d\tilde{t} \right)$. Changing variable from $r$ to $Y = Y(r, t)$ one gets the metric

$$ds^2 = \frac{\Omega^2}{1 + f} \left[ (\dot{Y}^2 - (1 + f)\Omega^{-2} e^{2\nu}) dt^2 - 2\dot{Y}dYdt + dY^2 \right] + Y^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

If $\Omega$ is finite and non-vanishing at $t = t_{sc}(r)$, the above metric is continuous with continuous inverse at such surface. This happens if $\nu'$ vanishes (dust case) or, more generally, if $\nu'$ goes to zero at least as $Y'$ at the shell cross (this happens, for instance, in the case of vanishing radial stresses).

In a recent paper [48], Szekeres and Lun have shown that there exist a system of coordinates in which the metric is of class $C^1$ as $t \to t_{sc}^*$. However, again, this result per se does not show extendibility of the spacetime (see also [49]).

4.2.2 Non-central shell focusing singularities

It is important to distinguish the central shell focusing singularity, i.e. that occurring at $r = 0$, from the other focusing singularities, since in many cases it is easy to prove that non-central singularities are censored.

A necessary condition for the visibility of a “point” $r$ is that the condition $1 - 2 \eta^2 / Y(r, t) > 0$, implying absence of trapped surfaces (see next Sub–section), is satisfied. Since the apparent horizon is the boundary of the region containing trapped surfaces, the above condition implies that the time of formation of the apparent horizon $t_{ah}(r)$, defined as $2m(r, t_{ah}(r)) = Y(r, t_{ah}(r))$, must not be before $t_f(r)$, i.e. $t_{ah}(r) \geq t_f(r)$. Now suppose $m(r, t_f(r))$ to be different from zero (as we have seen, this is not the case at the central singularity where $m$ has to vanish as $r^3$). Then $1 - 2m(r, t) / Y(r, t)$ goes to minus infinity as $t$ tends to $t_f(r)$ so that the singularity is covered. This shows that all the naked shell focusing singularities are necessarily massless, in the sense that $m$ has to vanish there [51] . It follows that any non-central singularity will certainly be censored if the mass is an increasing function with respect to $t$. Now equation (5) gives $\dot{m} = -4\pi \Sigma Y^2 \dot{Y}$ and, since $\dot{Y} < 0$ during collapse, we conclude that non-central singularities are always covered if the radial stress $\Sigma$ is non-negative [51] (in particular, all non-central singularities occurring in dust as well as in models with only tangential stress are covered, since the mass does not depend on time in this case). In the presence of radial tensions, the question is still open. It is known that a perfect fluid with $p = k \epsilon$ exhibits naked non-central singularities if $k < -1/3$ [51].

4.2.3 Central singularities: the root equation

The first explicit examples of the formation of a naked shell focusing singularity were provided by Eardley [52], Eardley and Smarr [53] and by Christodoulou [54]. Since then the techniques to study the nature of the singularities in spherically symmetric spacetimes have
been developed by many authors (see references in [42]) and finally settled up by Dwivedi and Joshi [55].

The key idea is the following: if the singularity is visible, at least locally, there must exist light signals coming out from it. Therefore, by investigating the behaviour of radial null geodesics near the singularity, one can try to find out if outgoing null curves meet the singularity in the past. On such radial null geodesics, the derivative of \( Y(r,t) \) reads

\[
\frac{dY}{dr} = Y' + \eta^{-1/2}Ye^{-\nu}.
\]

Using (8) and (13) in the above equation we obtain

\[
\frac{dY}{dr} = Y' \left[ 1 - \sqrt{1 + \frac{\Omega^2}{1 + f} \left( \frac{2m}{Y} - 1 \right)} \right],
\]

(14)

where \( Y' \) has to be understood as a known function of \( Y \) and \( r \). If the singularity is naked, equation (14) must have at least one solution with definite, outgoing tangent at \( r = 0 \), i.e. a solution of the kind \( Y = X_0 r^\alpha \) where \( \alpha > 1 \) and \( X_0 \) is a positive constant. Clearly, this behaviour is possible only if the necessary condition \( 1 - 2m/Y > 0 \) is satisfied. Indeed \( Y' \) is equal to one, and therefore positive, on the initial data surface. If no shell crossing occurs, it remains positive, so that the right hand side of equation (14) cannot remain positive if \( 2m/Y - 1 \) changes sign.

Once the necessary condition is satisfied, one has to check if both \( X_0 \) and \( \alpha \) exist such that the solution of equation (14) is of the specified form near the singularity. On using L’Hospital rule we have

\[
X_0 = \lim_{r \to 0} \left( \frac{1}{\alpha r^{\alpha-1}} \frac{dY}{dr} \right)_{Y=X_0r^\alpha}.
\]

(15)

Using again (14), this equation becomes an algebraic relation for \( X_0 \) at fixed \( \alpha \). If a positive definite \( X_0 \) exists the singularity is naked.

4.2.4 The dust case

The general exact solution of the Einstein field equations is known in the most simple case of vanishing pressure (dust) [29, 56, 57]. In this case, from equation (10), one gets \( \nu = 0 \) (more precisely, \( e^\nu \) is an arbitrary function of \( t \) only which can be rescaled to unity without loss of generality). Then \( \Omega = 0 \) and it follows from equation (13) that \( \eta^{-1} = Y'^2/(1 + f) \).

The mass is constant in time \( (m = F(r)) \) due to equation (8) with \( \Sigma = 0 \). Therefore (8) can be written as a Kepler-like equation \( Y'^2 = f + 2F/Y \), which is integrable in parametric form for \( f \neq 0 \) and in closed form for \( f = 0 \). Finally, the density can be read off from (7) as

\[
\epsilon = F'/(4\pi Y^2 Y').
\]

A great effort has been paid to understand the nature of the central singularity in this solution [11, 12, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62] and we now know the complete spectrum of possible endstates of the dust evolution in dependence of the initial data. We recall here what happens in the case of marginally bound solutions \( (f = 0) \) [62] since it is sufficiently general to illustrate a tendency and simple enough to be recalled in a few lines.

For marginally bound dust, the solutions \( s = \{F,1,1\} \) can be uniquely characterized by the expansion of the function \( F(r) \) at \( r = 0 \) or, and that is the same, by the expansion of \( \epsilon_0 = F'/4\pi r^2 \). Using this expansion in the root equation, it is not difficult to check the following results:
Figure 1: Penrose diagrams for collapsing dust clouds. The dotted line represents the center of the star, and the bold jagged line represents the singularity. The continuous line connecting $i^-$ to the singularity represents the boundary of the collapsing cloud. Three different situations can occur, depending on the choice of initial data: (a) “generalized” Oppenheimer-Snyder collapse: the singularity is covered. (b) locally naked singularity (c) globally naked singularity.

- If the first non–vanishing term corresponds to $n = 1$ or $n = 2$ equation (15) always has a real positive root: the singularity is naked;
- If the first non–vanishing term is $n = 3$ the root equation reads
  \[ 2x^4 + x^3 + \xi x - \xi = 0 , \]  
  where $x^2 = X$ and $\xi = F_3/F_0^{5/2}$. From the theory of quartic one can show that this equation admits a real positive root if $\xi < \xi_c = -(26 + 15\sqrt{3})/2$. Therefore, $\xi_c$ is a critical parameter: at $\xi = \xi_c$ a “phase transition” occurs and the endstate of collapse turns from a naked singularity to a blackhole.
- If $n > 3$ the limit in equation (14) diverges: the singularity is covered. In particular, this case contains the solution first discovered by Oppenheimer and Snyder [58] describing a homogeneous dust cloud.

The naked singularities mentioned above are locally naked. It can, however, be shown that if locally naked singularities occur in dust spacetimes, then spacetimes containing globally naked singularities can be build up from these by matching procedures. The Penrose diagrams corresponding to the three different cases are shown in Figure 1.

The above results can be extended to the general case of collapsing dust clouds, so that the final fate of the dust solutions $s = \{F, f, 1\}$ is completely known. The final fate depends on a parameter which is a combination of coefficients of the expansions of $F$ and $f$ near $r = 0$, and a structure similar to that of marginally bound collapse arises (see [42] for details).

4.2.5 Vanishing radial stresses

Recently, the general solution for spherically symmetric dust has been extended to the case in which only the radial stress vanishes [24, 28]. The solution can be reduced to quadratures using a system of coordinates first introduced by Ori [63] for charged dust. One of the new coordinate is the mass $m$ which is constant in time (due to (8) with $\Sigma = 0$), the other coordinate is the “area radius” $Y$. In such coordinates the metric reads
\[
\begin{align*}
 ds^2 &= -\Gamma^2 \left(1 - \frac{2m}{Y}\right) \, dm^2 + 2\sqrt{1 + f} \frac{\Gamma}{hu} \, dY \, dm - \frac{1}{u^2} \, dY^2 + Y^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \quad (17)
\end{align*}
\]

where
\[
 u = -\sqrt{-1 + \frac{2m}{Y} + \frac{1 + f}{h^2}}, \quad \Gamma = g(m) + \int \frac{h}{u^2 \sqrt{1 + f}} \frac{\partial u}{\partial m} \, dY, \quad (18)
\]

and the function \( g(m) \) is arbitrary.

The problem of understanding the nature of the singularities for such solutions is essentially still open. It is, indeed, possible to write the root equation in explicit form, but this equation contains a sort of “non-locality” due to the integral entering in the definition of \( \Gamma \). As a result only a few special cases have been analysed so far [28, 64, 65].

Among the solutions with tangential stresses a particularly interesting one is the Einstein cluster [66, 67, 68]. This is a spherically symmetric cluster of rotating particles. The motion of the particles is sustained by the angular momentum \( L \) whose average effect is to introduce a non-vanishing tangential stress in the energy-momentum tensor. The corresponding equation of state has the form \( h(m, Y) = \sqrt{1 + L^2(m)/Y^2} \). Therefore, a solution is uniquely identified by the choice of three arbitrary functions of \( m \) only, namely \( F, f \) and \( L^2 \) (for \( L = 0 \) one recovers dust). It turns out that the final state “at fixed dust background” (i.e. for fixed \( F, f \)) depends on the expansion of \( L^2 \) near \( m = 0 \) (\( L \approx \beta m^y, \) say) [63, 71]. Considering, for simplicity, the marginally bound case, one finds that for \( 4/3 < y < 2 \) either the singularity does not form (the system bounces back) or a blackhole is formed. The threshold of naked singularity formation lies at \( y = 2 \), where a 2-parameter structure very similar to that of dust occurs. At \( y = 7/3 \) a sort of transition takes place and the evolution of the model is such that only the critical branch is changed, un-covering a part of the blackhole region in the corresponding dust spacetime; the non-critical branch is the same as in dust spacetimes. Finally for \( y > 7/3 \) the evolution always leads to the same end state of the corresponding dust solution.

### 4.2.6 Self-similar collapse

A spherically symmetric spacetime is self-similar if it admits an homothetic vector \( \xi \), i.e. a vector satisfying \( \mathcal{L}_\xi g_{\mu\nu} = 2g_{\mu\nu} \). In the comoving frame the dimensionless variables \( \nu, \lambda \) and \( Y/r \) depend only on the “similarity variable” \( z := r/t \), and the Einstein field equations become ordinary differential equations (we refer the reader to [71] for a complete treatment of self-similar solutions). Being governed by ordinary differential equations, self-similar spherical collapse can be analyzed with the powerful techniques of dynamical systems theory [72].

The analysis of the singularities forming in self-similar spacetimes has been done by many authors for different equations of state, like dust [52, 53, 74, 75, 76, 77], barotropic perfect fluids [78, 79], radiation (Vaidya) shells [80, 81, 82] and in general cases [83, 84].

The picture arising resembles the dust case in the sense that both naked singularities and blackholes can form depending on the values of the parameters characterizing the solution. A thoroughly review of these and other features of self-similar solutions can be found in [73]; here we limit ourself to stress that naked singularities exist in self-similar solutions with pressure, thereby showing that the phenomenon of naked singularities formation cannot be considered as an artifact of dust (i.e. vanishing stresses) solutions.
4.2.7 General stresses

The problem of predicting the final fate of an initially regular distribution of matter supported by an arbitrary distribution of stress-energy (including e.g. the case of isotropic perfect fluids but also anisotropic crystalline structures which are thought to form at extremely high densities), is still open even in the spherically symmetric case. First of all, one has to take into account the fact that there is a high degree of uncertainty in the properties of the equation of state at very high densities. Recently, Christodoulou [85] initiated the analysis of a simple model composed by a dust (“soft”) phase for energy density below a certain value \( \bar{\varepsilon} \) and a stiff (“hard”) phase for \( \varepsilon > \bar{\varepsilon} \) (in the hard phase the pressure is given by \( p = \varepsilon - \bar{\varepsilon} \)).

Although the details of the collapse in presence of general matter fields are still largely unknown, it is very unlikely that the “embarrassing” examples of naked singularity formation like those occurring in dust can eventually be eliminated with the “addition” of stresses. Indeed, it is reasonable to think that a sector of naked singularities exists in the choice of initial data for any fixed equation of state [50]. The main issues that have, therefore, to be addressed are the genericity and the stability of naked singularity formation.

Both the above italicized terms have a somewhat intuitive meaning that is, however, difficult to express in mathematical terms. Regarding genericity, one can mean that the set of initial data leading to naked singularities is not of measure zero. For instance, it has been shown [12] that among the dust solutions \( s = \{F, f, 1\} \) naked singularities are generic in the sense that at a fixed density profile \( F \), one can always choose energy functions \( f \) leading to blackholes or naked singularities. A generalization of such a result would be that the naked singularities are generic - in this specific sense - at fixed, but arbitrary, equation of state \( \psi \) (there is some convincing evidence for this, see [30]).

The issue of stability is even more delicate than that of genericity. Indeed, any exact solution of a physical theory must survive to small but arbitrary perturbations in order to serve as a candidate for describing nature. In the case of exact solutions of the Einstein field equations, the notion of stability is very delicate due to to the gauge invariance of the theory with respect to spacetime diffeomorphisms. Recently, some evidence of stability of dust naked singularities against perturbations has been obtained [86].

5 Discussion: a picnic on the side of the road

It is well known that, if the mass of a collapsing object does not fall below the neutron star limit (\( \sim 3M_\odot \)), no physical process is able to produce enough pressure to balance the gravitational pull so that continued gravitational collapse must occur. It is widely believed that the final state of this process is a blackhole. However, what general relativity actually predicts, in the cases which have been analysed so far, is that either a blackhole or a naked singularity is formed, depending on the initial distribution of density and velocity and on the constitutive nature of the collapsing matter. One may raise the objection that most of the known analytical results could be an artifact of spherical symmetry. However, from the numerical point of view some evidence that this is not the case is coming up. Therefore, as singularity theorems showed that the singularities occurring in collapse are generic and not any artifact of symmetry, a similar situation may hold for the nature of singularities as well.

One may at this point ask if and when, a cosmic censorship theorem holds in nature. An answer to such a question could be in the negative. However, it remains to understand
the physics underlying the end state of gravitational collapse with respect to the choice of initial data at fixed matter model. In a famous novel [87] (that inspired the film Stalker by A. Tarkovsky) a short visit of extraterrestrial life on the earth occurs. The gap between the two civilizations is so high that human beings are, with respect to the “garbage” left by the visitors, like ants exploring what remains on the side of the road after a human picnic. Something they find is useful, something useless, something even dangerous, but anyway everything is obscure and difficult to understand, looking like the weak shadow of a wonderful abyss of knowledge. Our present cosmic censorship understanding resembles this situation. Indeed, we are getting a variety of mathematical hints with somewhat obscure physical meanings. For instance: the condition on $\xi$ recalled in Section 4.2.4, the constraints arising in the gravitational collapse of Einstein clusters, the dimensionless numbers arising in Choptuik’s numerical results [14], the condition on the radiation flux which arises in Vaidya collapse [3]. Such “numbers” should presumably be the remnant weak shadow of a general theorem when its hypotheses are enormously restricted by the choice of the equation of state and of the adopted symmetries.

To get rid of this puzzle appears to be one of the most exciting objectives of future research in classical relativity theory.

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