WEAK GALERKIN MIXED FINITE ELEMENT METHODS FOR PARABOLIC EQUATIONS WITH MEMORY

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Abstract. We develop a semidiscrete and a backward Euler fully discrete weak Galerkin mixed finite element method for a parabolic differential equation with memory. The optimal order error estimates in both $\|\cdot\|$ and $L^2$ norms are established based on a generalized elliptic projection. In the numerical experiments, the equation is solved by the weak Galerkin schemes with spaces $\{[P_k(T)]^2, P_k(e), P_{k+1}(T)\}$ for $k = 0$ and the numerical convergence rates confirm the theoretical results.

1. Introduction. Weak Galerkin mixed finite element method (WG-MFEM), which can be viewed as an extension of weak Galerkin finite element method [7], was first proposed by Wang and Ye for the second-order elliptic equations [8]. WG-MFEM can provide accurate numerical approximations for both the primary and flux variables just as the existing mixed finite element methods. In particular, based on a use of weak divergence, WG-MFEM is designed to be more flexible in using discontinuous piecewise polynomials on finite element partitions with arbitrary shape of polygons/polyhedra. Later on, WG-MFEM was successfully applied to the biharmonic equations [3]. In [4], a hybridized formulation for the WG-MFEM was proposed which can reduce significantly the discrete linear system. For second-order elliptic equations with Robin boundary condition, Zhang et al. studied the WG-MFEM by proving inf-sup condition with a suitable norm and testing a series of numerical examples [11]. Furthermore, WG-MFEMs for heat equations are developed in [12] by Zhou et al. However, study on WG-FEMs for the parabolic equations with memory is still limited.

Compared to the traditional heat equation, the parabolic differential equation with integral term is more suitable in modeling the heat conduction in materials with memory [10], which is also known as parabolic integro-differential equation. Due to the integral term, the analysis and implementation of numerical schemes for parabolic integro-differential equation are more complex accordingly. In order to obtain both the primary and flux unknowns, several mixed finite element methods have been developed for the generalized equation in recent years [1, 2, 5, 6, 13, 14, 15]. In this paper, we consider the initial-boundary value problem for linear
parabolic equation with memory term
\[ \begin{cases} u_t - \nabla \cdot \left( A\nabla u + \int_0^t B\nabla u d\tau \right) = f, & \text{in } \Omega, \ t \in J, \\ u = 0, & \text{on } \partial \Omega, \ t \in J, \\ u(\cdot, 0) = u_0, & \text{in } \Omega, \end{cases} \]  
(1)

where \( \Omega \subset \mathbb{R}^2 \) is a bounded convex polygonal domain with boundary \( \partial \Omega \), the time interval \( J = (0, T] \) with \( T > 0 \), \( A = A(x) \) and \( B = B(x) \) are sufficiently smooth functions. In view of the accuracy and the flexibility of the finite element partition of WG-MFEM, we present WG-MFEM approaches for the linear parabolic integro-differential problem (1).

The main goal of this paper is to extend the WG-MFEM to the parabolic integro-differential equation with the aid of discrete weak divergence operator. We will prove the existence and uniqueness of the WG-MFEM solutions. In the error analysis, to obtain the optimal convergence order of the WG-MFEMs, we will propose a generalized weak Galerkin mixed elliptic projection corresponding the mixed variational form of the integro-differential equation.

The rest of paper is organized as follows. In Section 2, we present a semidiscrete and a backward Euler fully discrete WG-MFEM for the parabolic integro-differential problem (1). In Section 3, some useful lemmas are listed. We prove the existence and uniqueness of the WG-MFEM approximations in Section 4. An appropriate generalized weak Galerkin mixed elliptic projection is introduced in Section 5. Optimal error estimates in both \( ||| \cdot ||| \) and \( L^2 \) norms of the semidiscrete scheme and the fully discrete scheme are established in Section 6. In Section 7, numerical experiments are carried out for two examples to verify the theoretical results. Finally, we draw our conclusions in Section 8.

2. Weak Galerkin mixed finite element schemes. In the parabolic integro-differential problem (1), we define a flux variable
\[ q = -A\nabla u - \int_0^t B\nabla u d\tau \]
and let
\[ \alpha = A^{-1}, \beta = \alpha B, \gamma = -\nabla \beta. \]
Then problem (1) can be rewritten as the following system of first order partial differential equations in \( \Omega \times J \)
\[ \begin{cases} u_t + \nabla \cdot q = f, \\ \alpha q + \nabla u + \int_0^t \nabla (\beta u) d\tau + \int_0^t \gamma ud\tau = 0, \end{cases} \]  
(2)
with boundary and initial conditions
\[ u = 0, \text{ on } \partial \Omega, \ t \in J, \]
\[ u(\cdot, 0) = u_0, \text{ in } \Omega. \]  
(3)
(4)
The variational weak formulation of this system is to seek \( \{ q, u \} : J \rightarrow H(\text{div}; \Omega) \times L^2(\Omega) \) such that for any \( t \in J \),
\[ \begin{cases} (u_t, w) + (\nabla \cdot q, w) = (f, w), & \forall w \in L^2(\Omega), \\ (\alpha q, v) - \left( u + \int_0^t \beta ud\tau, \nabla \cdot v \right) + \left( \int_0^t \gamma ud\tau, v \right) = 0, & \forall v \in H(\text{div}; \Omega), \\ u(\cdot, 0) = u_0, \end{cases} \]  
(5)
where $(\cdot, \cdot)$ is the standard $L^2$-inner product.

We recall some notations and definitions for the WG-MFEMs used in [8] and [11]. Let $\mathcal{T}_h = \{T\}$ be a quasi-uniform triangulation of $\Omega$ with $h = \max_{T \in \mathcal{T}_h} \text{diam}(T)$. Denote by $\mathcal{E}_h$ the collection of all edges in $\mathcal{T}_h$ and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega$. For any element $T \in \mathcal{T}_h$, we denote the space of weak vector-valued function on $T$ by

$$M(T) = \{ \mathbf{v} = (\mathbf{v}_0, \mathbf{v}_b) : \mathbf{v}_0 \in [L^2(T)]^2, \mathbf{v}_b \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\partial T) \},$$

where $\mathbf{n}$ is the unit outward normal direction of $T$ on $\partial T$. Let $M = \prod_{T \in \mathcal{T}_h} M(T)$ be the weak vector-valued function space on $\mathcal{T}_h$, and $V$ be a subspace of $M$ as

$$V = \{ \mathbf{v} \in M : \exists \mathbf{q} \in H(\text{div}, \Omega) \text{ s.t. } \mathbf{q}|_{\partial T} \cdot \mathbf{n} = \mathbf{v}_b|_{\partial T} \cdot \mathbf{n}, \forall T \in \mathcal{T}_h \}. \quad (7)$$

For a non-negative integer $k$, we denote $P_k(T)$ the set of polynomials on $T$ with degree no more than $k$. By using a combination of sets $\{[P_k(T)]^2, P_k(e), P_{k+1}(T)\}$, we define a finite element space $W_h$ as

$$W_h = \{ w \in L^2(\Omega) : w|_T \in P_{k+1}(T), \forall T \in \mathcal{T}_h \}, \quad (8)$$

and a finite element space $V_h \subset V$ as

$$V_h = \{ \mathbf{v} = (\mathbf{v}_0, \mathbf{v}_b) : \mathbf{v}_0|_T \in [P_k(T)]^2, \mathbf{v}_b|_e = v_b \mathbf{n}_e, \mathbf{v}_b \in P_k(e), \forall T \in \mathcal{T}_h, \forall e \in \mathcal{E}_h \}, \quad (9)$$

where the unit vector $\mathbf{n}_e$ is normal to $e$ for $e \in \mathcal{E}_h^0$, and outward normal to $e$ for $e \in \mathcal{E}_h \setminus \mathcal{E}_h^0$. Then we introduce the discrete weak divergence operator $\nabla_{w,k}$ as an approximation to $\nabla \cdot$ in finite element spaces [11]. For each $\mathbf{v} \in V$, the discrete weak divergence $(\nabla_{w,k} \cdot v)|_T \in P_k(T)$ on element $T$ is determined by

$$(\nabla_{w,k} \cdot v, \phi)_T = -\langle \mathbf{v}_0, \nabla \phi \rangle_T + \langle \mathbf{v}_b \cdot \mathbf{n}, \phi \rangle_{\partial T}, \quad \forall \phi \in P_k(T). \quad (10)$$

Thus we have the discrete weak divergence operator $\nabla_{w,k+1} : V_h \rightarrow W_h$.

Considering the variational form (5), we define bilinear operators as

$$a(\mathbf{q}, \mathbf{v}) = (\alpha \mathbf{q}_0, \mathbf{v}_0), \quad \mathbf{q}, \mathbf{v} \in V_h, \quad (11)$$

$$b(\mathbf{v}, u) = (\nabla_{w,k+1} \cdot \mathbf{v}, u), \quad \mathbf{v} \in V_h, u \in W_h. \quad (12)$$

To ensure the weak Galerkin mixed finite element scheme has a unique solution, a stabilized form $a_s(\cdot, \cdot)$ is defined by

$$a_s(\mathbf{q}, \mathbf{v}) = a(\mathbf{q}, \mathbf{v}) + s(\mathbf{q}, \mathbf{v}), \quad \mathbf{q}, \mathbf{v} \in V_h, \quad (13)$$

where

$$s(\mathbf{q}, \mathbf{v}) = \sum_{T \in \mathcal{T}_h} h_T (\langle (\mathbf{q}_0 - \mathbf{q}_b) \cdot \mathbf{n}, (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n} \rangle_{\partial T}). \quad (14)$$

Now we get a semidiscrete weak Galerkin mixed finite element method (SWG-MFEM) for problem (2-4) by seeking $\{q_h, u_h\} : J \rightarrow V_h \times W_h$ such that for any $t \in J$,

\[
\begin{aligned}
& (u_{h,t}, w) + b(q_h, w) = (f, w), & \forall w \in W_h, \\
& a_s(q_h, v) - b(v, u_h + \int_0^t \beta u_h d\tau) + \left( \int_0^t \gamma u_h d\tau, v \right) = 0, & \forall v \in V_h, \\
& q_h(0) = R_0 q(\cdot, 0), \quad u_h(\cdot, 0) = E_h u_0,
\end{aligned}
\]  

with appropriately chosen initial values $R_0 q(\cdot, 0)$ and $E_h u(\cdot, 0)$ to be defined in (51).
Let $\Delta t = T/N$ with positive integer $N$ and $t_n = n\Delta t$ for $n = 0, 1, \cdots, N$. We also obtain a backward Euler fully discrete weak Galerkin mixed finite element method (FWG-MFEM): find $\{q^n_h, u^n_h\} \in V_h \times W_h$ for $n = 1, 2, \cdots, N$ such that

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\delta_t u^n_h (w) + b(q^n_h, w) = (f (\cdot, t_m), w), \\
\alpha_s(q^n_h, v) - (v, u^n_h) - \Delta t \sum_{i=0}^{n-1} (b(v, u^i_h) - (\gamma u^i_h, v)) = 0, \\
q^n_h = R_h q^s(\cdot, 0), \\
u^0_h = E_h u_0,
\end{array} \right. \\
&\forall w \in W_h,
\end{aligned}
\]

where $\delta_t u^n_h = (u^n_h - u^{n-1}_h)/\Delta t$.

3. $L^2$ projections and some lemmas. For any $v = \{v_0, v_h n\} \in V$, $v_h \in L^2(\Omega)$, we denote $Q_h : V \rightarrow V_h$, such that

\[
Q_h v = \{Q_0 v_0, (Q_h v_n)n\}, \quad \forall \tau \in T_h,
\]

where $Q_0$ is the $L^2$ projection from $[L^2(T)]^2$ to $[P_k(T)]^2$ and $Q_h$ is the $L^2$ projection from $L^2(e)$ onto $P_h(e)$. Let $Q_{n,h}$ be the $L^2$ projection from $L^2(\Omega)$ onto $W_h$. In addition, we introduce a new norm $|| \cdot ||$ in finite element space $V_h$ and a norm $|| \cdot ||_{1,h}$ in the space $W_h + H^1(\Omega)$, which are defined by [11]

\[
|\|v\|^2 = \sum_{T \in T_h} (v_0, v_0) + s(v, v), \quad v \in V_h
\]

\[
|\|w\|^2 = \sum_{T \in T_h} |\|\nabla w\|^2 + \sum_{e \in E_h} h^{-1}_e |\|Q_0[w]\|^2, \quad w \in W_h + H^1(\Omega),
\]

where $[w]$ means the jump of $w$ on edge $e$. It is easy to prove that $a_s(\cdot, \cdot)$ is bounded and coercive, i.e., there exist positive constants $C_1, C_2$ such that for any $q_h, v \in V_h$

\[
|a_s(q, v)| \leq C_1 |\|q\| |\|v\|, \\
C_2 |\|v\|^2 \leq a_s(v, v).
\]

We recall some useful lemmas in [8] and [9].

Lemma 3.1. [8] The bilinear form $b(\cdot, \cdot)$ is bounded in $V_h \times W_h$ and satisfies the inf-sup condition, i.e., for any $v \in V_h, w \in W_h$,

\[
\sup_{v \in V_h,|\|v\|\neq 0} \frac{|b(v, w)|}{|\|v\|} \geq C_4 |\|w\|_{1,h},
\]

where $C_3, C_4$ are positive constants independent of $h$.

Lemma 3.2. [8] For any $q \in [H^1(\Omega)]^2$, $w \in W_h$, we have

\[
b(Q_h q, w) = (\nabla \cdot q, w) - \sum_{T \in T_h} (q \cdot n - Q_0(q \cdot n), w)_{\partial T}.
\]

Lemma 3.3. [8] For any $v = \{v_0, v_h\} \in V_h, w \in H^1(\Omega)$, we have

\[
b(v, Q_h w) = -(v_0, \nabla_h w) + \sum_{T \in T_h} ((v_0 - v_h) \cdot n, w - Q_h w)_{\partial T} + (v_h \cdot n, w)_{\partial \Omega},
\]

where $\nabla_h w$ is the gradient of $w$ taken element-by-element.

Lemma 3.4. [8] Let $q \in [H^{k+1}(\Omega)]^2$. For $0 \leq s \leq k$, we have

\[
|\|q - Q_h q\| \leq h^{s+1}|\|q\|_{s+1}.
\]
Lemma 3.5. [8] Let \( u \in H^{k+2}(\Omega) \) and \( q \in [H^{k+1}(\Omega)]^2 \) be two smooth functions on \( \Omega \). Then we have
\[
\sum_{T \in T_h} |(v_0 - v_b) \cdot n, Q_h u - u|_{\partial T} \leq C h^{k+1} \|u\|_{s+2} \|v\|, \quad 0 \leq s \leq k, \tag{27}
\]
\[
\sum_{T \in T_h} |(q \cdot n - \beta_b(q \cdot n), \varphi)|_{\partial T} \leq C h^{k+1} \|q\|_{s+1} \|\nabla_h \varphi\|, \quad 0 \leq s \leq k, \tag{28}
\]
for all \( v \in V \) and \( \varphi \in \prod_{T \in T_h} H^1(T) \).

Lemma 3.6. [8] Let \( (q_u) \) be the solution of (1). Assume that \( u \in H^{k+2}(\Omega) \), \( q \in [H^{(k+1)}(\Omega)]^2 \), \( \Psi \in [H^1(\Omega)]^2 \) and \( \phi \in H^2(\Omega) \). Then we have
\[
|a_s(q - Q_b q, Q_h \Psi)| \leq C h^{k+2} \|q\|_{k+1} \|\Psi\|_1, \tag{29}
\]
\[
\sum_{T \in T_h} |(Q_0 \Psi \cdot n - Q_b(\Psi \cdot n), Q_h u - u)|_{\partial T} \leq C h^{k+2} \|u\|_{k+2} \|\Psi\|_1, \tag{30}
\]
\[
\sum_{T \in T_h} |(q \cdot n - Q_b(q \cdot n), Q_h \phi - \phi)|_{\partial T} \leq C h^{k+2} \|q\|_{k+1} \|\phi\|_2. \tag{31}
\]

Lemma 3.7. [9] For all \( v \in [H^1(\Omega)]^2 \), we have
\[
\nabla_{w,k+1} \cdot (Q_h v) = Q_h (\nabla \cdot v). \tag{32}
\]

4. Existence and uniqueness of numerical schemes. In this section, we prove in Theorem 4.1 and 4.2 that the two numerical schemes defined in (15) and (16) are uniquely solvable.

Theorem 4.1. The SWG-MFEM (15) has a unique solution \( \{q_h, u_h\} : J \rightarrow V_h \times W_h \).

Proof. We only need to prove that the following system of the corresponding homogenous equations of (15) has only trivial solution:
\[
(u_{h,t}, w) + b(q_h, w) = 0, \quad \forall w \in W_h, \tag{33}
\]
\[
a_s(q_h, v) - b(v, u_h + \int_0^t \beta u_h d\tau) - \left( \int_0^t \gamma u_h d\tau, v \right) = 0, \quad \forall v \in V_h, \tag{34}
\]
\[
q_h(\cdot, 0) = 0, \quad u_h(\cdot, 0) = 0, \tag{35}
\]
for \( t \in J \). First, we write (34) as
\[
b(v, u_h) = a_s(q_h, v) - b(v, \int_0^t \beta u_h d\tau) - \left( \int_0^t \gamma u_h d\tau, v \right). \tag{36}
\]
It follows from the boundedness of \( a_s(\cdot, \cdot) \) and Lemma 3.1 that
\[
\|u_h\|_{1,h} \cdot \|v\| \leq C \left( \|q_h\|_{1,h} \cdot \|v\| + \int_0^t \|u_h\|_{1,h} d\tau \cdot \|v\| \right). \tag{37}
\]
Due to Gronwall inequality, we have
\[
\|u_h\|_{1,h} \leq C \|q_h\|. \tag{38}
\]
Let \( w = u_h \) in (33) and \( v = q_h \) in (34), then
\[
a_s(q_h, q_h) + (u_{h,t}, u_h) = b(q_h, \int_0^t \beta u_h d\tau) - \left( \int_0^t \gamma u_h d\tau, q_h \right), \quad t \in J. \tag{39}
\]
Similarly, we consider the following homogenous linear system

\begin{equation}
\text{Proof.}
\end{equation}

Theorem 4.2. The FWG-MFEM (16) has a unique solution \( \{ q_h^n, u_h^n \} \in V_h \times W_h \) for \( n = 1, 2, \cdots, N \).

\begin{proof}
Similarly, we consider the following homogenous linear system

\begin{equation}
(\delta_t u_h^n, w) + b(q_h^n, w) = 0, \quad \forall w \in W_h, \quad (43)
\end{equation}

\begin{equation}
a_s(q_h^n, v) - b(v, u_h^n) - \Delta t \sum_{i=1}^{n-1} b(v, \beta u_h^i) + \Delta t \sum_{i=0}^{n-1} (\gamma u_h^i, v) = 0, \quad \forall v \in V_h, \quad (44)
\end{equation}

with \( q_h^0 = 0, \quad u_h^0 = 0, \quad (45) \)

for \( 1 \leq n \leq N \). Let \( w = u_h^n \) in (43) and \( v = q_h^n \) in (44). Then we have

\begin{equation}
a_s(q_h^n, q_h^n) + (\delta_t u_h^n, u_h^n) = \Delta t \sum_{i=1}^{n-1} b(q_h^n, \beta u_h^i) - \Delta t \sum_{i=0}^{n-1} (\gamma u_h^i, q_h^n), \quad 1 \leq n \leq N. \quad (46)
\end{equation}

Notice that

\begin{equation}
(\delta_t u_h^n, u_h^n) = \frac{1}{\Delta t}(u_h^n - u_h^{n-1}, u_h^0) \geq \frac{1}{2\Delta t}||u_h^n||^2 - ||u_h^{n-1}||^2, \quad 1 \leq n \leq N. \quad (47)
\end{equation}

From (21) and (46), we estimate that

\begin{equation}
C_2||q_h^n||^2 + \frac{1}{2\Delta t}(||u_h^n||^2 - ||u_h^{n-1}||^2) \leq C \Delta t \sum_{i=1}^{n-1} ||u_h^i||_V^2 + \frac{C_2}{2}||q_h^n||^2 \leq C \Delta t \sum_{i=1}^{n-1} ||q_h^i||^2 + \frac{C_2}{2}||q_h^n||^2 \quad (48)
\end{equation}

Adding the equation above with \( n \) from 1 to \( m \) (\( 1 < m \leq N \)) with \( u_h^0 = 0 \), we arrive at

\begin{equation}
C_2 \Delta t \sum_{n=1}^{m} ||q_h^n||^2 + ||u_h^m||^2 \leq C(\Delta t)^2 \sum_{n=1}^{m} \sum_{i=1}^{n-1} ||q_h^i||^2. \quad (49)
\end{equation}

Thus we get

\begin{equation}
\Delta t \sum_{n=1}^{m} ||q_h^n||^2 \leq 0, \quad 1 < m \leq N, \quad (50)
\end{equation}

By using (21) (38) and Lemma 3.1, we arrive at

\begin{equation}
C_2||q_h^n||^2 + \frac{1}{2\Delta t}||u_h^n||^2 \leq C \int_0^t ||u_h^n||_V^2 + C_2 \sum_{i=1}^{n-1} ||q_h^i||^2, \quad t \in J. \quad (40)
\end{equation}

Integrating \((40)\) from 0 to \( t \) with \( u_h(t, 0) = 0 \), then we get

\begin{equation}
\frac{C_2}{2} \int_0^t ||q_h^n||^2 d\tau + \frac{1}{2} ||u_h^n||^2 \leq C \int_0^t \int_0^t ||q_h(\zeta)||^2 d\zeta d\tau. \quad (41)
\end{equation}

By using Gronwall inequality again, we have

\begin{equation}
\int_0^t ||q_h^n||^2 d\tau \leq 0, \quad (42)
\end{equation}

which implies that \( q_h = 0 \) and \( u_h = 0 \) for any \( t \) in \( J \). Therefore, The SWG-MFEM (15) has a unique solution.

\end{proof}

with the discrete Gronwall inequality. This implies that $q^n_h = 0$ and $u^n_h = 0$ for $1 \leq n \leq N$. Therefore, The FWG-MFEM (16) has a unique solution. 

5. A generalized weak Galerkin mixed elliptic projection. In the study of finite element methods of differential equations of evolution, an elliptic projection associated the problems is usually introduced. The elliptic projection usually plays a key role in obtaining the optimal convergence order of finite element approximations. Since the integro-differential problem has a integral term, the projections for finite element methods solving differential equations (e.g. heat equation) will not work. Assume that $q$ and $u$ are the solutions of variational form (5). Here we introduce a generalized weak Galerkin mixed elliptic projection for the WG-MFEMs applicable to the mixed system (5) by defining a map $\{R_h q, E_h u\} : J \rightarrow V_h \times W_h$ such that

$$\begin{cases}
    b(R_h q, w) = (\hat{f}, w), \\
    a_s(R_h q, v) - b(v, E_h u + \int_0^t \beta E_h u d\tau) + \left( \int_0^t \gamma E_h u d\tau, v \right) = l(v), \\
    R_h q(\cdot, 0) = Q_h q(\cdot, 0), \quad E_h u(\cdot, 0) = Q_h u_0,
\end{cases}$$

for any $w \in W_h, v \in V_h, t \in J$, where $\hat{f} = \nabla \cdot q$ and

$$l(v) = (\alpha q + \nabla u + \int_0^t \nabla(\beta u) d\tau + \int_0^t \gamma u d\tau, v_0).$$

First, we prove the existence and uniqueness of the solution (51). It suffices to show that the associated homogeneous system has only trivial solution, which is

$$\begin{cases}
    b(R_h q, w) = 0, \\
    a_s(R_h q, v) - b(v, E_h u + \int_0^t \beta E_h u d\tau) + \left( \int_0^t \gamma E_h u d\tau, v \right) = 0,
\end{cases}$$

for any $w \in W_h, v \in V_h$. We choose $w = \nabla \cdot R_h q$ and $v = R_h q$ in (52), then get

$$\begin{align*}
    C_2 \| R_h q \|^2 &\leq a_s(R_h q, R_h q) - \left( \int_0^t \gamma E_h u d\tau, R_h q \right) \\
    &\leq C \int_0^t \| E_h u \|_{1,h} d\tau \| R_h q \|,
\end{align*}$$

which leads to

$$\| R_h q \| \leq C \int_0^t \| E_h u \|_{1,h} d\tau. \quad (54)$$

From (20) and Lemma 3.1, we have

$$\| E_h u \|_{1,h} \cdot \| v \| \leq | b(v, E_h u) |$$

$$= \left| a_s(R_h q, v) - b(v, \int_0^t \beta E_h u d\tau) + \left( \int_0^t \gamma E_h u d\tau, v \right) \right|$$

$$\leq C \left( \| R_h q \| + \int_0^t \| E_h u \|_{1,h} d\tau \right) \cdot \| v \|.$$  

Then we arrive at

$$\| E_h u \|_{1,h} \leq C \left( \| R_h q \| + \int_0^t \| E_h u \|_{1,h} d\tau \right).$$  

(56)
A combination of (54) and (56) gives
\[ \|E_hu\|_{1,h} \leq C \int_0^t \|E_hu\|_{1,h} d\tau. \] (57)

According to Gronwall inequality and inequality (54), we have \( \|E_hu\|_{1,h} = 0 \), \( |||R_hq||| = 0 \), respectively. Therefore, the system (51) is uniquely solvable. This shows that the generalized weak Galerkin mixed elliptic projection is well defined.

Next, we derive the error equations of \( R_hq \) and \( E_hu \). Set
\[ \tilde{\xi} = R_hq - Q_hq, \]
\[ \tilde{\eta} = E_hu - Q_hu, \]
\[ \tilde{\theta} = E_hu - u = \tilde{\eta} + Q_hu - u. \]

On one hand, applying Lemma 3.2 to the first equation of (51), we have
\[ b(R_hq - Q_hq, w) = \sum_{T \in T_h} (q \cdot n - Q_h(q \cdot n), w)_{\partial T}. \] (58)

On the other hand, we have \( (\alpha q, v_0) = a_s(q, v) \) from the definition of \( s(\cdot, \cdot) \). Notice that \( Q_h(\beta u) = \beta Q_hu \). According to Lemma 3.3, we obtain
\[ l(v) = \left( \alpha q + \nabla u + \int_0^t \nabla(\beta u) d\tau + \int_0^t \gamma ud\tau, v_0 \right) \]
\[ = a_s(Q_hq, v) - a_s(Q_hq - q, v) - b(v, Q_hu + \int_0^t \beta Q_hud\tau) \]
\[ + \sum_{T \in T_h} ((v_0 - v_b) \cdot n, u - Q_hu + \int_0^t \beta(u - Q_hu)d\tau)_{\partial T} \]
\[ + \left( \int_0^t \gamma u, v_0 \right). \] (59)

Combining (59) with the second equation of (51), we have
\[ a_s(R_hq - Q_hq, v) - b(v, E_hu - Q_hu + \int_0^t \beta(E_hu - Q_hu)d\tau) \]
\[ + \left( \int_0^t \gamma(E_hu - u) d\tau, v_0 \right) + a_s(Q_hq - q, v) \]
\[ - \sum_{T \in T_h} \langle(v_0 - v_b) \cdot n, u - Q_hu + \int_0^t \beta(u - Q_hu)d\tau \rangle_{\partial T} = 0. \] (60)

According to (58) and (60), we rewrite (51) as
\[
\begin{cases}
\{ b(\tilde{\xi}, w) = \sum_{T \in T_h} \langle q \cdot n - Q_b(q \cdot n), w \rangle_{\partial T}, \\
a_s(\tilde{\xi}, v) - b\left(v, \tilde{\eta} + \int_0^t \beta \tilde{\eta} d\tau \right) + \left( \int_0^t \gamma \tilde{\theta} d\tau, v_0 \right) = m(v),
\end{cases}
\] (61)

for any \( w \in W_h, v \in V_h, t \in J \), where
\[ m(v) = -a_s(Q_hq - q, v) + \sum_{T \in T_h} \langle(v_0 - v_b) \cdot n, u - Q_hu + \int_0^t \beta(u - Q_hu)d\tau \rangle_{\partial T}. \]
Theorem 5.1. Assume that \( \mathbf{q} \in L^\infty[0, T; [H^{k+1}(\Omega)]^2] \), \( u \in L^\infty[0, T; H^{k+2}(\Omega)] \) are the solutions of (5) and \( \{R_h \mathbf{q}, E_h u\} \) satisfies system (51), then we have

\[
\|\tilde{\xi}\| + \|\tilde{\eta}\|_1,h \leq C h^{k+1} \left( \|\mathbf{q}\|_{k+1} + \|u\|_{k+2} + \int_0^t \|\mathbf{q}\|_{k+1} + \|u\|_{k+2} d\tau \right). \tag{62}
\]

Proof. Similar to the derivation of (37), it follows from the second equation of (61) that

\[
\|\tilde{\eta}\|_1,h \cdot \|\mathbf{v}\| \leq C \left( \|\tilde{\xi}\| + \int_0^t \|\tilde{\eta}\|_1,h d\tau \cdot \|\mathbf{v}\| + \int_0^t \|\tilde{\eta}\| d\tau \cdot \|\mathbf{v}\| + |m(\mathbf{v})| \right). \tag{63}
\]

From Lemma 3.4 and 3.5, we get

\[
|m(\mathbf{v})| \leq C h^{k+1} \left( \|\mathbf{q}\|_{k+1} + \|u\|_{k+2} + \int_0^t \|u\|_{k+2} d\tau \right) \|\mathbf{v}\|. \tag{64}
\]

Notice that \( \tilde{\theta} = \tilde{\eta} + Q_h u - u \). Combining (63) and (64), we have

\[
\|\tilde{\eta}\|_1,h \leq C \left\{ \|\tilde{\xi}\| + \int_0^t \|\tilde{\eta}\|_1,h d\tau + h^{k+1} \left( \|\mathbf{q}\|_{k+1} + \|u\|_{k+2} + \int_0^t \|u\|_{k+2} d\tau \right) \right\}. \tag{65}
\]

Using Gronwall inequality, we have

\[
\|\tilde{\eta}\|_1,h \leq C \left\{ \|\tilde{\xi}\| + h^{k+1} \left( \|\mathbf{q}\|_{k+1} + \|u\|_{k+2} + \int_0^t \|u\|_{k+2} d\tau \right) \right\}. \tag{66}
\]

Substitute \( w = \tilde{\eta} \) and \( v = \tilde{\xi} \) into (61), then

\[
a_s(\tilde{\xi}, \tilde{\eta}) = b(\tilde{\xi}, \int_0^t \tilde{\eta} d\tau) - \left( \int_0^t \gamma \tilde{\theta} d\tau, \tilde{\xi}_0 \right) + \sum_{T \in T_h} \langle \mathbf{q} \cdot \mathbf{n} - Q_h (\mathbf{q} \cdot \mathbf{n}), \tilde{\eta} \rangle_{\partial T} + m(\tilde{\xi}). \tag{67}
\]

Using equations (21) (64), Lemma 3.1 and Lemma 3.5, we obtain

\[
C_2 \|\tilde{\xi}\|^2 \leq C \|\tilde{\xi}\| \int_0^t \|\tilde{\eta}\|_1,h d\tau + C h^{k+1} \|\mathbf{q}\|_{k+1} \|\tilde{\eta}\|_1,h
\]

\[
+ C h^{k+1} \left( \|\mathbf{q}\|_{k+1} + \|u\|_{k+2} + \int_0^t \|u\|_{k+2} d\tau \right) \cdot \|\tilde{\xi}\|. \tag{68}
\]

From \( \varepsilon \)-inequality and (66), we have

\[
\|\tilde{\xi}\|^2 \leq C h^{2(k+1)} \left( \|\mathbf{q}\|_{k+1}^2 + \|u\|_{k+2}^2 + \int_0^t \|\mathbf{q}\|_{k+1}^2 + \|u\|_{k+2}^2 d\tau \right). \tag{69}
\]

Thus, combining (66) and the inequality above, we obtain

\[
\|\tilde{\xi}\| + \|\tilde{\eta}\|_1,h \leq C h^{k+1} \left( \|\mathbf{q}\|_{k+1} + \|u\|_{k+2} + \int_0^t \|\mathbf{q}\|_{k+1} + \|u\|_{k+2} d\tau \right). \tag{70}
\]

Q.E.D.

Theorem 5.2. Assume that \( \mathbf{q} \in L^\infty[0, T; [H^{k+1}(\Omega)]^2] \), \( u \in L^\infty[0, T; H^{k+2}(\Omega)] \) are the solutions of (5) and \( \{R_h \mathbf{q}, E_h u\} \) satisfies system (51), then we have

\[
\|\tilde{\eta}\| \leq C h^{k+2} \left( \|\mathbf{q}\|_{k+1} + \|u\|_{k+2} + \int_0^t \|\mathbf{q}\|_{k+1} + \|u\|_{k+2} d\tau \right). \tag{71}
\]
Proof. For $\tilde{\eta} \in L^2(\Omega)$, we denote $(\Psi, \phi) \in [H^1(\Omega)]^2 \times H^2(\Omega)$ the solution of
\begin{align}
\begin{cases}
\alpha \Psi + \nabla \phi = 0, & \text{in } \Omega, \\
\nabla \cdot \Psi = \tilde{\eta}, & \text{in } \Omega, \\
\phi = 0, & \text{on } \partial \Omega.
\end{cases}
\end{align}
(72)
Then
$$\|\phi\|_2 + \|\Psi\|_1 \leq C \|	ilde{\eta}\|.\quad (73)$$
We test the second equation of (72) against $\tilde{\eta}$. It follows from Lemma 3.2 that
$$(\tilde{\eta}, \tilde{\eta}) = (\nabla \cdot \Psi, \tilde{\eta}) = b(Q_h \Psi, \tilde{\eta}) + \sum_{T \in T_h} (\Psi \cdot \mathbf{n} - Q_b(\Psi \cdot \mathbf{n}), \tilde{\eta})_{\partial T}.\quad (74)$$
Choosing $\mathbf{v} = Q_h \Psi$ in the second equation of (61) and substituting it into (74), we get
$$\begin{align}
(\tilde{\eta}, \tilde{\eta}) &= a_s(\tilde{\xi}, Q_h \Psi) - b\left(Q_h \Psi, \int_0^t \beta \tilde{\eta} d\tau\right) + \left(\int_0^t \gamma \tilde{\eta} d\tau, Q_0 \Psi\right) + a_s(Q_h \mathbf{q} - \mathbf{q}, Q_h \Psi) \\
&\quad + \sum_{T \in T_h} \left(\langle Q_0 \Psi \cdot \mathbf{n} - Q_b(\Psi \cdot \mathbf{n}), u - Q_h u + \int_0^t \beta (u - Q_h u) d\tau\rangle_{\partial T} - \langle \Psi \cdot \mathbf{n} - Q_b(\Psi \cdot \mathbf{n}), \tilde{\eta} \rangle_{\partial T}\right) \\
&\quad = : I + II + III + IV + V + VI.
\end{align}\quad (75)$$
First, we rewrite the first term $I$ as
$$I = a_s(\tilde{\xi}, Q_h \Psi) = a_s(\tilde{\xi}, Q_h \Psi - \Psi) + a_s(\tilde{\xi}, \Psi).\quad (76)$$
Using (20) and Lemma 3.4, we get
$$|a_s(\tilde{\xi}, Q_h \Psi - \Psi)| \leq C \|	ilde{\xi}\| \cdot \|Q_h \Psi - \Psi\| \leq C h \|\tilde{\xi}\| \cdot \|\Psi\|_1,\quad (77)$$
and
$$a_s(\tilde{\xi}, \Psi) = a(\tilde{\xi}, \Psi) = a(\tilde{\xi}, \tilde{\xi}_0) = - (\nabla \phi, \tilde{\xi}_0).\quad (78)$$
Notice that $\phi \in H^1(\Omega)$ and $\phi = 0$ on $\partial \Omega$. From the first equation of (61), Lemma 3.3, Lemma 3.6 and Lemma 3.5 with $s = 0$, we have
$$\begin{align}
(\nabla \phi, \tilde{\xi}_0)
&= - b(\tilde{\xi}, Q_h \phi) + \sum_{T \in T_h} \langle (\tilde{\xi}_0 - \tilde{\xi}_h) \cdot \mathbf{n}, \phi - Q_h \phi \rangle_{\partial T} \\
&= - \sum_{T \in T_h} \langle \mathbf{q} \cdot \mathbf{n} - Q_0(\mathbf{q} \cdot \mathbf{n}), Q_h \phi \rangle_{\partial T} + \sum_{T \in T_h} \langle (\tilde{\xi}_0 - \tilde{\xi}_h) \cdot \mathbf{n}, \phi - Q_h \phi \rangle_{\partial T} \\
&= \sum_{T \in T_h} \langle \mathbf{q} \cdot \mathbf{n} - Q_b(\mathbf{q} \cdot \mathbf{n}), \phi - Q_h \phi \rangle_{\partial T} + \sum_{T \in T_h} \langle (\tilde{\xi}_0 - \tilde{\xi}_h) \cdot \mathbf{n}, \phi - Q_h \phi \rangle_{\partial T} \\
&\leq C h^{k+2} \|\mathbf{q}\|_{k+1} + h \|\tilde{\xi}\| \|\phi\|_2.
\end{align}\quad (79)$$
Combining (75)-(78), we get
$$|I| \leq C (h^{k+2} \|\mathbf{q}\|_{k+1} + h \|\tilde{\xi}\| \|\phi\|_2 + Ch \|\tilde{\xi}\| \cdot \|\Psi\|_1.\quad (79)$$
Next, from Lemma 3.7, we have

\[ |II| = \left| -b(Q_h \Psi, \int_0^t \beta \tilde{\eta} d\tau) \right| \]

\[ \leq C \int_0^t \| \tilde{\eta} \| d\tau \| \nabla_w \cdot Q_h \Psi \| \]

\[ = C \int_0^t \| \tilde{\eta} \| d\tau \| Q_h (\nabla \cdot \Psi) \| \]

\[ \leq C \int_0^t \| \tilde{\eta} \| d\tau \| \Psi \|_1, \tag{80} \]

and

\[ |III| = \left| \left( \int_0^t \gamma \tilde{\eta} d\tau, Q_0 \Psi \right) \right| \]

\[ = \left| \int_0^t \gamma (\tilde{\eta} + Q_h u - u) d\tau, Q_0 \Psi \right| \]

\[ \leq C \int_0^t \| \tilde{\eta} \| d\tau \| Q_0 \Psi \| + C \int_0^t \| Q_h u - u \| d\tau \| Q_0 \Psi \| \]

\[ \leq C \int_0^t \| \tilde{\eta} \| d\tau \| \Psi \|_1 + C h^{k+2} \int_0^t \| u \|_{k+2} d\tau \| \Psi \|_1. \tag{81} \]

By Lemma 3.5 and 3.6, we get

\[ |IV| = |a_s(Q_h q - q, Q_h \Psi)| \leq C h^{k+2} \| q \|_{k+1} \| \Psi \|_1, \tag{82} \]

\[ |V| = \sum_{T \in T_h} \langle (Q_0 \Psi) \cdot n - Q_0 (\Psi \cdot n), u - Q_h u + \int_0^t \beta (u - Q_h u) d\tau \rangle_{\partial T} \]

\[ \leq C h^{k+2} \| u \|_{k+2} \int_0^t \| u \|_{k+2} d\tau \| \Psi \|_1, \tag{83} \]

\[ |VI| = \sum_{T \in T_h} \langle \Psi \cdot n - Q_h (\Psi \cdot n), \tilde{\eta} \rangle_{\partial T} \leq C h \| \nabla_h \tilde{\eta} \| \| \Psi \|_1. \tag{84} \]

In summary, we have

\[ (\tilde{\eta}, \tilde{\eta}) \leq C(k^{k+2} \| q \|_{k+1} + h \| \xi \|_2) \| \phi \|_2 + C h \| \xi \| \| \Psi \|_1 \]

\[ + C h^{k+2} \| u \|_{k+2} \int_0^t \| u \|_{k+2} d\tau \| \Psi \|_1 \]

\[ + C \int_0^t \| \tilde{\eta} \| d\tau \| \Psi \|_1 + C h \| \nabla_h \tilde{\eta} \| \| \Psi \|_1. \tag{85} \]

From (73), we rewrite (85) as

\[ \| \tilde{\eta} \| \leq C h \| \xi \|_2 + \| \nabla_h \tilde{\eta} \| \]

\[ + C h^{k+2} \left( \| q \|_{k+1} + \| u \|_{k+2} \int_0^t \| u \|_{k+2} d\tau \right) + C \int_0^t \| \tilde{\eta} \| d\tau. \tag{86} \]

It follows from Gronwall inequality and Theorem 5.1 that

\[ \| \tilde{\eta} \| \leq C h^{k+2} \left( \| q \|_{k+1} + \| u \|_{k+2} \int_0^t \| u \|_{k+2} + \| q \|_{k+1} d\tau \right). \tag{87} \]

This completes the proof. \qed
We take the derivatives with respect to $t$ of the results in Theorem 5.1 and 5.2 respectively. Then the estimates of $\|\tilde{\xi}_t\|$ and $\|\tilde{\eta}_t\|$ are obtained in the following theorem.

**Theorem 5.3.** Assume that $q \in W^{1,\infty}[0,T;[H^{k+1}(\Omega)]^2]$, $u \in W^{1,\infty}[0,T;H^{k+2}(\Omega)]$ and \{${R_h q, E_h u}$\} satisfies system (51), then we have

$$\|\tilde{\xi}_t\| \leq C h^{k+1} \left( \|q_t\|_{k+1} + \|u_t\|_{k+2} + \int_0^t \|q_t\|_{k+1} + \|u_t\|_{k+2} d\tau \right),$$

(88)

$$\|\tilde{\eta}_t\| \leq C h^{k+2} \left( \|q_t\|_{k+1} + \|u_t\|_{k+2} + \int_0^t \|q_t\|_{k+1} + \|u_t\|_{k+2} d\tau \right).$$

(89)

6. **Error estimates.** In this section, we establish the optimal order error estimates in the $L^2$ and $\|\cdot\|$ norms for SWG-MFEM (15) and FWG-MFEM (16) in Section 2 respectively.

For SWG-MFEM (15), we denote

$$\xi = q_h - R_h q, \quad \tilde{\xi} = R_h q - Q_h q,$$

$$\eta = u_h - E_h u, \quad \tilde{\eta} = E_h u - Q_h u, \quad t \in J.$$

For FWG-MFEM (16), we denote

$$\xi^n = q_h^n - R_h q^n(\cdot, t_n), \quad \tilde{\xi}^n = R_h q^n(\cdot, t_n) - Q_h q^n(\cdot, t_n),$$

$$\eta^n = u_h^n - E_h u^n(\cdot, t_n), \quad \tilde{\eta}^n = E_h u^n(\cdot, t_n) - Q_h u^n(\cdot, t_n), \quad 1 \leq n \leq N.$$

Our main goal is to bound $q_h - Q_h q, u_h - Q_h u$. In view of Theorem 5.1 and 5.2, all we need to analyze are $\xi, \eta$ for SWG-MFEM and $\xi^n, \eta^n$ for FWG-MFEM. Theoretical results are presented in Theorem 6.1 and 6.2 respectively.

**Theorem 6.1.** Let \{${q, u}$\} and \{${q_h, u_h}$\} be the solutions of system (2-4) and semidiscrete weak Galerkin scheme (15), respectively. If $q \in W^{1,\infty}[0,T;[H^{k+1}(\Omega)]^2], u \in W^{1,\infty}[0,T;H^{k+2}(\Omega)]$, then there exists a constant $C$ such that

$$\|u_h - Q_h u\|^2 \leq C h^{2(k+2)} \left( \|q\|^2_{k+1} + \|u\|^2_{k+2} + \|q_t\|^2_{k+1} + \|u_t\|^2_{k+2} \right)$$

(90)

$$+ C h^{2(k+2)} \left( \int_0^t \left( \|q_t\|^2_{k+1} + \|u_t\|^2_{k+2} \right) d\tau + \int_0^t \left( \|q_t\|^2_{k+1} + \|u_t\|^2_{k+2} \right) d\tau \right),$$

$$\|q_h - Q_h q\|^2 \leq C h^{2(k+1)} \left( \|q\|^2_{k+1} + \|u\|^2_{k+2} + \int_0^t \left( \|q_t\|^2_{k+1} + \|u_t\|^2_{k+2} \right) d\tau \right)$$

(91)

$$+ C h^{2(k+2)} \left( \|q_t\|^2_{k+1} + \|u_t\|^2_{k+2} + \int_0^t \left( \|q_t\|^2_{k+1} + \|u_t\|^2_{k+2} \right) d\tau \right),$$

for $t \in J$.

**Proof.** Choose $w \in W_h$ in the first equation of (5) and notice that $(Q_h u_t, w) = (u_t, w)$, we obtain

$$\langle Q_h u_t, w \rangle + (\nabla \cdot q, w) = (f, w), \quad t \in J.$$  

(92)

Subtracting (92) from the first equation of semidiscrete scheme (15) and considering (51), we get

$$\langle u_{ht}, w \rangle - \langle Q_h u_t, w \rangle + b(q_h, w) - b(R_h q, w) = 0, \quad t \in J, \quad \text{(93)}$$

for $t \in J$. 

We set $w = q_h$, $w = u_h$, $w = R_h q$, $w = E_h u$. Then the above inequality gives

$$\langle \xi_t, w \rangle + \langle \tilde{\xi}_t, w \rangle = \langle \xi, w \rangle + \langle \tilde{\xi}, w \rangle - \langle \tilde{\xi}_t, w \rangle$$

for $t \in J$. 

We integrate the above equality over $[0, t]$ for $t \in J$ and use the above results, the desired estimate follows.
i.e.,
\[(\eta_t + \tilde{\eta}, w) + b(\xi, w) = 0, \quad t \in J.\] (94)
Choose \(v = \{v_0, v_b\} \in V_h\) in the second equation of the (5), then subtract it from the second equation of (15), we get
\[a_s(\xi, v) - b \left( v, \eta + \int_0^t \beta \eta d\tau \right) + \left( \int_0^t \gamma \eta d\tau, v \right) = 0, \quad \forall v \in V_h, \quad t \in J.\] (95)
By Adding (94) and (95) together with \(w = \eta, \ v = \xi\), we arrive at
\[a_s(\xi, \xi) + (\eta_t, \eta) = b \left( \xi, \int_0^t \beta \eta d\tau \right) - \left( \int_0^t \gamma \eta d\tau, \xi \right) - (\tilde{\eta}, \eta), \quad t \in J.\] (96)
Notice the (21) and Lemma 3.1, we have
\[C_2 \|\xi\|^2 + \frac{1}{2} \frac{d}{dt} \|\eta\|^2 \leq C_2 \|\xi\|^2 + C \int_0^t \|\eta\|_1 h d\tau + \frac{1}{2} \|\bar{\eta}\|^2 + \frac{1}{2} \|\eta\|^2.\] (97)
By (38), We integrate (97) from 0 to \(t\) to arrive at
\[\|\eta\|^2 \leq \int_0^t \|\bar{\eta}\|^2 d\tau + \int_0^t \|\eta\|^2 d\tau, \quad t \in J,\] (98)
where \(\|\eta(\cdot, 0)\| = \|u_h(\cdot, 0) - E_h u(\cdot, 0)\| = 0\). It follows from Gronwall inequality and Theorem 5.3 that
\[\|\eta\|^2 \leq C \int_0^t \|\bar{\eta}\|^2 d\tau
\leq Ch^{2(k+2)} \left( \|q_k\|^2_{k+1} + \|u_t\|^2_{k+2} + \int_0^t \|q_t\|^2_{k+1} + \|u_t\|^2_{k+2} d\tau \right), \quad t \in J.\] (99)
By taking a derivative of (95) with respect to \(t\), we get
\[a_s(\xi_t, v) - b(v, \eta_t + \beta \eta) + (\gamma \eta, v) = 0, \quad \forall v \in V_h, \quad t \in J.\] (100)
Choose \(w = \eta_t\) in (94) and \(v = \xi\) in (100) respectively. By adding them together, we have
\[a_s(\xi_t, \xi) + (\eta_t, \eta_t) = b(\xi, \beta \eta) - (\gamma \eta, \xi) - (\tilde{\eta}, \eta_t), \quad t \in J.\] (101)
Hence
\[\frac{1}{2} \frac{d}{dt} a_s(\xi, \xi) + \|\eta_t\|^2 \leq C \|\xi\| \cdot \|\eta\|_1 h + \|\bar{\eta}\| \cdot \|\eta_t\|, \quad t \in J.\] (102)
Integrating (102) from 0 to \(t\) and using (38), we get
\[C_2 \|\xi\|^2 + 2 \int_0^t \|\eta_t\|^2 d\tau
\leq C \int_0^t \|\xi\|^2 d\tau + \int_0^t \|\bar{\eta}\|^2 d\tau + \int_0^t \|\eta_t\|^2 d\tau, \quad t \in J,\] (103)
where \(\xi(\cdot, 0) = q_h(\cdot, 0) - R_h q(\cdot, 0) = 0\). Finally, using Gronwall inequality and Theorem 5.3, we have
\[\|\xi\|^2 \leq C \int_0^t \|\bar{\eta}\|^2 d\tau
\leq Ch^{2(k+2)} \left( \|q_k\|^2_{k+1} + \|u_t\|^2_{k+2} + \int_0^t \|q_t\|^2_{k+1} d\tau + \int_0^t \|u_t\|^2_{k+2} d\tau \right).\] (104)
for \( t \in J \). This completes the proof. \( \square \)

**Theorem 6.2.** Let \( \{q, u\} \) and \( \{q^n, u^n\} \) be the solutions of system (2-4) and semidiscrete weak Galerkin scheme (16), respectively. If \( q \in H^{1, \infty}[0, T; H^{k+1}(\Omega)] \), \( u \in W^{1, \infty}[0, T; H^{k+1}(\Omega)] \), \( u_t \in L^{\infty}[0, T; L^2(\Omega)] \), then there exists a positive constant \( C \) independent of \( h \) and \( \Delta t \) such that

\[
\|u^n_h - Q_h u(\cdot, t_n)\|^2 \\
\leq C h^{2(k+2)} \left[ \|u\|^2 H^2 + \|q\|^2_k + \int_0^{t_n} (\|u\|^2_{k+2} + \|q\|^2_{k+1}) \, ds \right] \\
+ \int_0^{t_n} (\|q_t\|^2_k + \|u_t\|^2_{k+2}) \, d\tau + C(\Delta t)^2 \left[ \int_0^{t_n} \|u_{tt}\|^2 \, d\tau + \|u_t\|^2_{L^2(0, T; H^{k+2})} \right],
\]

for \( 1 \leq n \leq N \).

**Proof.** We choose \( w \in W_h \) in the first equation of (5) at \( t = t_n \) to get

\[
(u_t(\cdot, t_n), w) + (\nabla \cdot q(\cdot, t_n), w) = (f(\cdot, t_n), w), \quad 1 \leq n \leq N.
\]

(107)

Subtracting (107) from the first equation of the problem (16), we have from (51) that

\[
(\delta_t u^n_h - Q_h u_t(t_n), w) + b(q^n_h - R_h q(t_n), w) = 0, \quad 1 \leq n \leq N.
\]

Notice that \( (u_t(\cdot, t_n), w) = (Q_h u_t(\cdot, t_n), w) \). By adding and subtracting \( \delta_t Q_h u_t(t_n) \) in the equation (108), we write (108) as

\[
(\delta_t \eta^n - \delta_t \eta^n, w) + b(\xi^n, w) = (z^n, w), \quad 1 \leq n \leq N,
\]

(109)

where \( z_n = Q_h u_t(t_n) - \delta_t Q_h u_t(t_n) \). Similar to the deduction of (95), we get

\[
a_s(\xi^n, \nabla v) - b(\xi^n, \eta^n) \\
= \Delta t \sum_{i=0}^{n-1} b(\mathbf{v}, \beta u^n_i) - \Delta t \sum_{i=0}^{n-1} (\gamma u^n_i, \mathbf{v}) - b \left( \mathbf{v}, \int_0^{t_n} \beta \mathbf{E}_h u(\tau) \, d\tau \right) \\
+ \left( \int_0^{t_n} \gamma \mathbf{E}_h u(\tau) \, d\tau, \mathbf{v}_0 \right) \\
= \Delta t \sum_{i=0}^{n-1} b(\mathbf{v}, \beta \mathbf{E}_h u(t_i)) - b \left( \mathbf{v}, \int_0^{t_n} \beta \mathbf{E}_h u(\tau) \, d\tau \right)
\]
\[
- \Delta t \sum_{i=0}^{n-1} (\gamma E_h u(t_i), \nu) + \left( \int_0^{t_n} \gamma E_h u(\tau) d\tau, v_0 \right) \\
+ \Delta t \sum_{i=0}^{n-1} b(v, \beta u_h - \beta E_h u(t_i)) - \Delta t \sum_{i=0}^{n-1} (\gamma u_h^i - \gamma E_h u(t_i), v)
\]

for \(1 \leq n \leq N\). We add (109) and (110) together with \(w = \eta^n, v = \xi^n\) to arrive at

\[
(z^n, \eta^n) - (z^n, \eta^{n-1}) + \Delta t \sum_{i=0}^{n-1} b(\xi^n, \beta E_h u(t_i)) - b(\xi^n, \int_0^{t_n} \beta E_h u(\tau) d\tau)
\]

\[
- \Delta t \sum_{i=0}^{n-1} (\gamma E_h u(t_i), \xi^n) + \left( \int_0^{t_n} \gamma E_h u(\tau) d\tau, \xi^n_0 \right)
\]

\[
+ \Delta t \sum_{i=0}^{n-1} b(\xi^n, \beta u_h^i - \beta E_h u(t_i)) - \Delta t \sum_{i=0}^{n-1} (\gamma u_h^i - \gamma E_h u(t_i), \xi^n_0),
\]

for \(1 \leq n \leq N\). We estimate each term on both side of the equation above as follows

\[
(\delta_i \eta^n, \eta^n) = \frac{1}{\Delta t} (\eta^n - \eta^{n-1}, \eta^n) \geq \frac{1}{2 \Delta t} (||\eta^n||^2 - ||\eta^{n-1}||^2),
\]

\[a_s(\xi^n, \xi^n) \geq C_2 ||\xi^n||^2,\]

\[(z^n, \eta^n) \leq C ||z^n||^2 + \frac{1}{4} ||\eta^n||^2
\]

\[
= C \left\| Q_h \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} (\tau - t_{n-1}) u_{tt}(\tau) d\tau \right) \right\| + \frac{1}{4} ||\eta^n||^2
\]

\[
\leq C \Delta t \int_{t_{n-1}}^{t_n} ||u_{tt}(\tau)||^2 d\tau + \frac{1}{4} ||\eta^n||^2,
\]

\[-(\delta_i \tilde{\eta}^n, \eta^n) \leq \frac{1}{(\Delta t)^2} \left\| \int_{t_{n-1}}^{t_n} \tilde{\eta}_h d\tau \right\|^2 + \frac{1}{4} ||\eta^n||^2
\]

\[
\leq \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} ||\tilde{\eta}_h||^2 d\tau + \frac{1}{4} ||\eta^n||^2.
\]

\[
\Delta t \sum_{i=0}^{n-1} b(\xi^n, \beta E_h u(t_i)) - b(\xi^n, \int_0^{t_n} \beta E_h u(\cdot, \tau) d\tau)
\]

\[
= b(\xi^n, \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\beta E_h u(\cdot, t_i) - \beta E_h u(\cdot, \tau)) d\tau)
\]

\[
= b\left( \xi^n, \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \frac{d}{d\zeta} (\beta E_h u(\cdot, \zeta)) d\zeta d\tau \right)
\]

\[
\leq C \Delta t \|u_t\|_{L^\infty(0,T;H^{k+2})} \cdot ||\xi^n||
\]

\[
\leq C(\Delta t)^2 \|u_t\|_{L^\infty(0,T;H^{k+2})}^2 + \frac{C_2}{8} ||\xi^n||^2.
\]
Similarly, we also have
\[
\Delta t \sum_{i=0}^{n-1} \langle \gamma E_h u(t_i), \xi^n_i \rangle - \left( \int_0^{t_n} \gamma E_h u(\tau) d\tau, \xi^n_0 \right) 
\leq C(\Delta t)^2 \| u_t \|_{L^\infty([0,T;H^{s+2})}^2 + \frac{C_2}{8} \| \xi^n \|^2.
\] (116)

By Lemma 3.1 and (38), we obtain
\[
\Delta t \sum_{i=0}^{n-1} b(\xi^n_i, \beta u_h - \beta E_h u(t_i)) \leq C \Delta t \sum_{i=0}^{n-1} \| \eta^i \|_{1,\alpha} \cdot \| \xi^n \|.
\] (117)
\[
\leq C(\Delta t)^2 \sum_{i=0}^{n-1} \| \xi^i \|^2 + \frac{C_2}{8} \| \xi^n \|^2.
\]

Thus
\[
\Delta t \sum_{i=0}^{n-1} (\gamma u_h^i - \gamma E_h u(t_i), \xi^n_i) \leq C(\Delta t)^2 \sum_{i=0}^{n-1} \| \xi^i \|^2 + \frac{C_2}{8} \| \xi^n \|^2. \quad (118)
\]

We substitute the above inequality into (111) and multiply it by 2\Delta t, then sum over \( n \) to obtain
\[
\| \eta^n \|^2 + C_2 \sum_{i=0}^n \| \xi^i \|^2 
\leq C(\Delta t)^2 \int_0^{t_n} \| u_{tt} \|^2 d\tau + \Delta t \sum_{i=0}^n \| \eta^i \|^2 
\]
\[
+ C \int_0^{t_n} \| \bar{\eta}_t \|^2 d\tau + C(\Delta t)^2 \| u_t \|_{L^\infty([0,T;H^{s+2})}^2 + C(\Delta t)^2 \sum_{j=1}^{j-1} \sum_{i=0}^{j-1} \| \xi^i \|^2,
\] (119)

where \( \eta^0 = 0 \). Using discrete Gronwall inequality and Theorem 5.3, we obtain
\[
\| \eta^n \|^2 \leq C h^{2k+4} \int_0^{t_n} (\| q_i \|^2_{k+1} + \| u_t \|^2_{k+2}) d\tau 
\]
\[
+ C(\Delta t)^2 \left( \int_0^{t_n} \| u_{tt} \|^2 d\tau + \| u_t \|_{L^\infty([0,T;H^{s+2})}^2 \right). \quad (120)
\]

Next, if the backward difference operator \( \delta_t \) is applied to (110), we can get
\[
a_s(\delta_t \eta^n, v) - b(v, \delta_t \eta^n) = \delta_t \left( \Delta t \sum_{i=0}^{n-1} b(v, \beta u_h^i) - \int_0^{t_n} b(v, \beta E_h u(\tau)) d\tau \right)
\]
\[
+ \delta_t \left( \int_0^{t_n} (\gamma E_h u(\tau), v_0) d\tau - \Delta t \sum_{i=0}^{n-1} (\gamma u_h^i, v) \right), \quad (121)
\]

for \( 1 \leq n \leq N \). Choosing \( w = \delta_t \eta^n \) in (109) and \( v = \xi^n \) in (121), we arrive at
\[
(\delta_t \eta^n, \delta_t \eta^n) + a_s(\delta_t \xi^n, \xi^n) 
= (z^n, \delta_t \eta^n) - (\delta_t \rho^n, \delta_t \eta^n)
= \delta_t \left( \Delta t \sum_{i=0}^{n-1} b(\xi^n_i, \beta u_h^i) - \int_0^{t_n} b(\xi^n, \beta E_h u(\tau)) d\tau \right) 
\]
The source term $f$ of the integro-differential problem (1) is $\varepsilon e$ angular mesh shown in Figure 1. Denote $7.

Numerical experiments. In each example, we set $\Omega = (0,1)$ and take $k(16)$ by solving the integro-differential problems with two examples is tested. We according to Theorem 5.1 and triangle inequality, the proof is completed.

Notice that $\xi = 0$. Then we substitute them into (111), multiply it by $2\Delta t$ and sum over $n$ to obtain

$$
||\xi^n||^2 \leq C(\Delta t)\int_0^{t_n} ||u(t)||^2 d\tau + \int_0^{t_n} ||\eta||^2 d\tau
+ 2\left(\Delta t \sum_{i=0}^{n-1} b(\xi^n, \beta u^i) - \int_0^{t_n} b(\xi^n, \beta E_h u(\tau)) d\tau\right)
+ 2\left(\int_0^{t_n} (\gamma E_h u(\tau, \xi^n)) d\tau - \Delta t \sum_{i=0}^{n-1} (\gamma u^i, \xi^n_0)\right), \quad 1 \leq n \leq N.
$$

Then we substitute them into (111), multiply it by $2\Delta t$ and sum over $n$ to obtain

$$
||\xi^n||^2 \leq C(\Delta t)\int_0^{t_n} ||u(t)||^2 d\tau + \int_0^{t_n} ||\eta||^2 d\tau
+ 2\left(\Delta t \sum_{i=0}^{n-1} b(\xi^n, \beta u^i) - \int_0^{t_n} b(\xi^n, \beta E_h u(\tau)) d\tau\right)
+ 2\left(\int_0^{t_n} (\gamma E_h u(\tau, \xi^n)) d\tau - \Delta t \sum_{i=0}^{n-1} (\gamma u^i, \xi^n_0)\right), \quad 1 \leq n \leq N.
$$

Notice that $\xi^0 = 0$. By applying the conclusion of (115) and (116), we finally get

$$
||\xi^n||^2 \leq C(h^{2(k+2)} \int_0^{t_n} (||q||_{k+1}^2 + ||u||_{k+2}^2)d\tau
+ C(\Delta t)\left(\int_0^{t_n} ||u(t)||^2 d\tau + ||u||_{L^\infty(0,T;H^{k+2})}^2\right).
$$

According to Theorem 5.1 and triangle inequality, the proof is completed. □

7. Numerical experiments. In this section, the convergence rate of FWG-MFEM (16) by solving the integro-differential problems with two examples is tested. We take $k = 0$ in the discretization of the weak Galerkin mixed finite element spaces. In each example, we set $\Omega = (0,1) \times (0,1)$, $J = (0,1]$ and use the uniform triangular mesh shown in Figure 1. Denote $e_h = q_h^N - Q_h q(t, N) = \{e_0, e_h\}$ and $\varepsilon_h = u_h^N - Q_h u(\tau, \cdot)$, which are measured by the following norms:

$$
||e_h||^2 = \sum_{T \in T_h} \left(\int_T |e_0|^2 dT + h_T \int_{\partial T} |(e_0 - e_h) \cdot n|^2 ds\right),
$$

$$
||\varepsilon_h||^2 = \sum_{T \in T_h} \int_T |\varepsilon_h|^2 dT.
$$

The source term $f$ can be obtained by substituting the exact $u$ into the integro-differential equation.

In the first example, we set the coefficient $A = B = 1$ and the exact solution of the integro-differential problem (1)

$$
u = t^2 xy(1 - x)(1 - y).$$
In the second example, we set variable coefficients $A = 1 + x^2 + 2y^2$, $B = 1 + 2x^2 + y^2$, and the exact solution

$$u = e^{-t} \sin(\pi x) \sin(\pi y).$$

Table 1-2 report the errors and convergence rate of the FWG-MFEM for solving the parabolic integro-differential equation in both examples with $k = 0$ respectively. As we can see, the weak Galerkin mixed finite element method has good performance for solving the parabolic integro-differential equations. For $k = 0$, the $|| \cdot ||$-norm for $e_h$ and $\cdot \cdot$-norm for $e_h$ are of order $O(\Delta t + h)$ and $O(\Delta t + h^2)$ respectively. This verifies our theoretical analysis in Section 6.

### Table 1. Error behaviors of FWG-MFEM for the first example with $\Delta t = 4h^2$

| $h$   | $||e_h||$         | order $\approx$ | $||\varepsilon_h||$ | order $\approx$ |
|-------|-------------------|-----------------|----------------------|-----------------|
| $2^{-3}$ | 4.8132e-002      | -               | 1.7834e-003          | -               |
| $2^{-4}$ | 2.3657e-002      | 1.0247          | 4.3564e-004          | 2.0334          |
| $2^{-5}$ | 1.1823e-002      | 1.0007          | 1.0872e-004          | 2.0026          |
| $2^{-6}$ | 5.9209e-003      | 0.9977          | 2.7312e-005          | 1.9929          |
| $2^{-7}$ | 2.9583e-003      | 1.0010          | 6.8160e-006          | 2.0026          |

### Table 2. Error behaviors of FWG-MFEM for the second example with $\Delta t = 4h^2$

| $h$   | $||e_h||$         | order $\approx$ | $||\varepsilon_h||$ | order $\approx$ |
|-------|-------------------|-----------------|----------------------|-----------------|
| $2^{-3}$ | 1.0576e-000      | -               | 2.1024e-002          | -               |
| $2^{-4}$ | 5.1613e-001      | 1.0350          | 4.8434e-003          | 2.1177          |
| $2^{-5}$ | 2.5868e-001      | 0.9966          | 1.2043e-003          | 2.0081          |
| $2^{-6}$ | 1.2929e-001      | 1.0006          | 3.0093e-004          | 2.0007          |
| $2^{-7}$ | 6.4627e-002      | 1.0004          | 7.5140e-005          | 2.0018          |
8. Conclusions. In consideration of the practical requirement of accurate approximations to both primary and flux variables in the parabolic differential equations with memory term, we developed the semidiscrete and the backward Euler fully discrete weak Galerkin mixed finite element method with the aid of discrete weak divergence operator. The existence and uniqueness of the two weak Galerkin schemes were proved. We proposed a generalized weak Galerkin mixed elliptic projection corresponding the mixed variational form of the integro-differential equation, based on which, the optimal convergence order of the semidiscrete and fully discrete weak Galerkin mixed finite element methods in both \(|\|·\|\|\) and \(L^2\) norms were established. The numerical experiments verified the accuracy of the weak Galerkin schemes studied in this paper.

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