Noncommutative resolutions using syzygies

Hailong Dao, Osamu Iyama, Srikanth B. Iyengar, Ryo Takahashi, Michael Wemyss and Yuji Yoshino

Abstract

Given a noether algebra with a noncommutative resolution, a general construction of new noncommutative resolutions is given. As an application, it is proved that any finite length module over a regular local or polynomial ring gives rise, via suitable syzygies, to a noncommutative resolution.

The focus of this article is on constructing endomorphism rings with finite global dimension. This problem has arisen in various contexts, including Auslander’s theory of representation dimension [1], Dlab and Ringel’s approach to quasi-hereditary algebras in Lie theory [4, 7], Rouquier’s dimension of triangulated categories [11], cluster tilting modules in Auslander–Reiten theory [9], and Van den Bergh’s noncommutative crepant resolutions in birational geometry [13].

For a noetherian ring $R$ which is not necessarily commutative, and a finitely generated faithful $R$-module $M$, the ring $\text{End}_R(M)$ is a noncommutative resolution (abbreviated to NCR) if its global dimension is finite; see [6]. When this happens, $M$ is said to give an NCR of $R$. In the result below, we give a method for constructing new NCRs from a given one. As usual $\Omega^cX$ denotes a $c$th syzygy module of $X$. We defer introducing other terminology and notation to the next section.

Theorem 1. Let $R$ be a noether algebra, and let $M, X \in \text{mod} R$. If $M$ is a $d$-torsionfree generator giving an NCR, and $\text{gldim} \text{End}_R(X)$ is finite, then for any integer $0 \leq c < \min\{d, \text{grade}_R X\}$, the following statements hold.

1. The $R$-module $M \oplus \Omega^cX$ is a $c$-torsionfree generator.
2. There is an inequality

$$\text{gldim} \text{End}_R(M \oplus \Omega^cX) \leq 2 \text{gldim} \text{End}_R(M) + \text{gldim} \text{End}_R(X) + 1.$$ 

In particular, $M \oplus \Omega^cX$ gives an NCR of $R$.

The statement of Theorem 1 is inspired by a recollement type inequality (1.1) that yields that the finiteness of global dimensions of $eAe$ and $A/(e)$ implies that of $A$, provided $pd_A(A/(e))$ is finite. The hypotheses in Theorem 1 enable us to apply this fact to $A = \text{End}_R(M \oplus \Omega^cX)$ and the idempotent $e \in A$ corresponding to the direct summand $M$.
A commutative ring is equicodimensional if every maximal ideal has the same height. Typical examples are polynomial rings over a field, and regular local rings. The following corollary generalizes, and is inspired by, a result of Buchweitz and Pham (unpublished article), who considered the case \( N = k \); see also [5, Corollary 5.2].

**Corollary 2.** Let \( R \) be an equicodimensional regular ring, and \( N \) a finite length \( R \)-module such that \( \text{gldim} \text{End}_R(N) \) is finite. Given nonnegative integers \( c_1, \ldots, c_n \) with \( c_i < \dim R \) for each \( i \), the \( R \)-module \( M := R \oplus \Omega^{c_1} N \oplus \cdots \oplus \Omega^{c_n} N \) satisfies

\[
\text{gldim} \text{End}_R(M) \leq 2^n \dim R + (2^n - 1)(\text{gldim} \text{End}_R(N) + 1).
\]

In particular, \( M \) gives an NCR of \( R \).

For any finite length \( R \)-module \( X \), there exists a finite length \( R \)-module \( Y \) such that \( \text{End}_R(X \oplus Y) \) has finite global dimension [8]. In the setting of the corollary, it follows that an NCR can be constructed using any finite length \( R \)-module.

In the definition of noncommutative resolution, it is sometimes required that the module be reflexive [12]. If \( \dim R \geq 3 \) in the setting of the corollary, then for any finite length \( R \)-module, by taking all \( c_i \geq 2 \) it can be ensured that the module giving the NCR is reflexive, but is not free.

**Terminology and Proofs**

Throughout, \( R \) will be a noether algebra, in the sense that it is finitely generated as a module over its centre, and the latter is a noetherian ring. Thus \( R \) is a noetherian ring, and for any \( M \) in \( \text{mod} R \), the category of finitely generated left \( R \)-modules, the ring \( \text{End}_R(M) \) is also a noether algebra, and hence noetherian.

The grade of \( M \in \text{mod} R \) is defined to be

\[
\text{grade}_R M = \inf \{ n \mid \text{Ext}^n_R(M, R) \neq 0 \}.
\]

When \( R \) is commutative, this is the length of a longest regular sequence in the annihilator of the \( R \)-module \( M \); see, for instance, [10, Theorem 16.7].

A finitely generated \( R \)-module \( M \) is \( d \)-torsionfree, for some positive integer \( d \), if

\[
\text{Ext}^i_R(\text{Tr} M, R) = 0 \quad \text{for } 1 \leq i \leq d,
\]

where \( \text{Tr} M \) is the Auslander transpose of \( M \); see [2]. This is equivalent to the condition that \( M \) is the \( d \)th syzygy of an \( R \)-module \( N \) satisfying \( \text{Ext}^i_R(N, R) = 0 \) for \( 1 \leq i \leq d \); see [2]. For example, if \( R \) is commutative and Gorenstein, any \((\dim R)\)-th syzygy module of a finitely generated module is \( d \)-torsionfree, for any \( d \).

Given \( R \)-modules \( X \) and \( Y \), we write \( \text{Hom}_R(X, Y) \) for the quotient of \( \text{Hom}_R(X, Y) \) by the abelian subgroup of morphisms factoring through projective \( R \)-modules.

**Lemma 3.** Let \( 0 \to X \to Y \to Z \to 0 \) be an exact sequence of \( R \)-modules. If an \( R \)-module \( W \) satisfies \( \text{Hom}_R(W, Z) = 0 \), then the following sequence is exact.

\[
0 \to \text{Hom}_R(W, X) \to \text{Hom}_R(W, Y) \to \text{Hom}_R(W, Z) \to 0.
\]

**Proof.** By hypothesis any morphism \( f : W \to Z \) factors as \( W \to P \xrightarrow{f'} Z \), where \( P \) is a projective \( R \)-module, and since \( f' \) lifts to \( Y \), so does \( f \). \( \square \)

As usual, we write \( \Omega X \) for a syzygy of \( X \).

**Lemma 4.** Let \( X \) and \( Y \) be finitely generated \( R \)-modules.
(1) If $\text{Ext}^1_R(X, R) = 0$, then there is an isomorphism
\[ \Omega: \text{Hom}_R(X, Y) \xrightarrow{\cong} \text{Hom}_R(\Omega X, \Omega Y). \]
(2) If $0 < c < \text{grade}_R X$ and $n \geq 1$, then $\text{Hom}_R(\Omega^c X, \Omega^{c+n} Y) = 0$.

Proof. Part (1) is clear and standard, for instance both sides are isomorphic to $\text{Ext}^1_R(X, \Omega Y)$ using short exact sequences $0 \to \Omega X \to F \to X \to 0$ and $0 \to \Omega Y \to G \to Y \to 0$. Part (2) follows for its hypotheses yield
\[ \text{Hom}_R(\Omega^c X, \Omega^{c+n} Y) \cong \text{Hom}_R(X, \Omega^n Y), \]
and the right-hand module is zero as $\text{Hom}_R(X, R) = 0$ implies $\text{Hom}_R(X, \Omega^n Y) = 0$, since $\Omega^n Y$ is a submodule of a projective $R$-module.

Proof of Theorem 1. Part (1): since $c < \text{grade}_R X$, $\Omega^c X$ is $c$-torsionfree, see [2]. For part (2), set $A := \text{End}_R(M \oplus \Omega^c X)$, and let $e \in A$ be the idempotent corresponding to the direct summand $M$. Then $eAe = \text{End}_R(M)$, so given the inequality
\[ \text{gldim} A \leq \text{gldim}(eAe) + \text{gldim} A/(e) + \text{pd}_A(A/(e)) + 1 \] (1.1)
proved in [3, Theorem 5.4], it remains to prove the two claims below.

Claim. There is an isomorphism of rings $A/(e) \cong \text{End}_R(X)$.

Indeed, first note that $A/(e) = \text{End}_R(\Omega^c X)/[M]$, where $[M]$ denotes the two-sided ideal of morphisms factoring through $add M$. This does not rely on any special properties of $M$ or of $X$.

Since $\text{grade}_R X \geq 1$, one has $\text{Hom}_R(X, R) = 0$, and this yields the equality below
\[ \text{End}_R(X) = \text{End}_R(X) \cong \text{End}_R(\Omega^c X), \]
while the isomorphism is obtained by repeated application of Lemma 4(1), noting that $c < \text{grade}_R X$. Therefore, to verify the claim, it is enough to prove $\text{End}_R(\Omega^c X)/[M] = \text{End}_R(\Omega^c X)$, that is, any endomorphism of $\Omega^c X$ factoring through $add M$ factors through $add R$.

Given morphisms $\Omega^c X \xrightarrow{f} M \xrightarrow{g} \Omega^c X$, the morphism $f$ factors through $add R$ by Lemma 4(2), since $M$ is a $d$th syzygy module and $d > c$. This completes the proof of the claim.

Claim. There is an inequality $\text{pd}_A(A/(e)) \leq \text{gldim} \text{End}_R(M)$.

Set $n := \text{gldim} \text{End}_R(M)$. Then, the $\text{End}_R(M)$-module $\text{Hom}_R(M, \Omega^c X)$ has a finite projective resolution
\[ 0 \to P_n \to \cdots \to P_0 \to \text{Hom}_R(M, \Omega^c X) \to 0. \] (1.2)
As $\text{Hom}_R(M, -)$: $\text{add}_R M \to \text{proj} \text{End}_R(M)$ is an equivalence, there is a sequence
\[ 0 \to M_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} M_0 \xrightarrow{f_0} \Omega^c X \to 0 \] (1.3)
of $R$-modules, with $M_j \in \text{add} M$ for all $j$, such that the induced sequence
\[ 0 \to \text{Hom}_R(M, M_n) \to \cdots \to \text{Hom}_R(M, M_0) \to \text{Hom}_R(M, \Omega^c X) \to 0 \]
is isomorphic to (1.2). Since $R \in \text{add} M$, the sequence (1.3) is exact.

To justify the claim, it suffices to prove that the induced complex
\[ 0 \to \text{Hom}_R(\Omega^c X, M_n) \to \cdots \to \text{Hom}_R(\Omega^c X, M_0) \xrightarrow{g} \text{Hom}_R(\Omega^c X, \Omega^c X) \] (1.4)
obtained from (1.3) is exact, and Cok$(g)$ is isomorphic to $\text{End}_R(\Omega^c X)/[M] \cong A/(e)$. For, then there is a projective resolution

$$0 \to \text{Hom}_R(M \oplus \Omega^c X, M_n) \to \cdots \to \text{Hom}_R(M \oplus \Omega^c X, M_0) \to \text{Hom}_R(M \oplus \Omega^c X, \Omega^c X) \to A/(e) \to 0$$

of the $A$-module $A/(e)$, as desired.

By construction, one obtains the exact sequence

$$\text{Hom}_R(\Omega^c X, M_0) \to \text{Hom}_R(\Omega^c X, \Omega^c X) \to \text{End}_R(\Omega^c X)/[M] \to 0.$$

This justifies the assertion about Cok$(g)$. As to the exactness, for each $0 \leq i \leq n$ set $K_i := \text{Im}(f_i)$, where $f_i$ are the maps in (1.3). Then there are exact sequences

$$0 \to K_{i+1} \to M_i \to K_i \to 0.$$

For each $i \geq 1$, using the fact that $M_i$ is $d$-torsionfree, and $K_0 = \Omega^c X$, it follows by induction that $K_i$ is a $(c+i)$-th syzygy. Lemma 4(2) then yields that $\text{Hom}_R(\Omega^c X, K_i) = 0$ for $i \geq 1$. By Lemma 3, one then obtains an exact sequence

$$0 \to \text{Hom}_R(\Omega^c X, K_{i+1}) \to \text{Hom}_R(\Omega^c X, M_i) \to \text{Hom}_R(\Omega^c X, K_i) \to 0.$$

Thus sequence (1.4) is exact, as desired. \hfill \Box

Recall that a commutative ring $R$ is regular if it is noetherian and every localization at a prime ideal has finite global dimension. When $R$ is further equicodimensional, the global dimension of $R$ is finite, since it equals dim $R$.

**Proof of Corollary 2.** Up to Morita equivalence, we can assume that

$$c_1 > c_2 > \cdots > c_{n-1} > c_n.$$

Set $M_0 = R$ and for each integer $1 \leq j \leq n$, set

$$M_j := R \oplus \Omega^{c_1} N \oplus \cdots \oplus \Omega^{c_j} N.$$

We prove, by an induction on $j$, that $M_j$ is $c_j$-torsionfree and that

$$\text{gldim End}_R(M_j) \leq 2^j \text{ dim } R + (2^j - 1)(\text{gldim End}_R(N) + 1).$$

The base case $j = 0$ is a tautology, for $R$ is regular and hence its global dimension equals dim $R$. Assume that the inequality holds for $j - 1$ for some integer $j \geq 1$.

For the induction step, set $M = M_{j-1}$, so that

$$M_j = M_{j-1} \oplus \Omega^{c_j} N.$$

Since $R$ is equicodimensional, grade$_R N = \text{dim } R$ and $M_{j-1}$ is $c_{j-1}$-torsionfree, Theorem 1 applies to yield that $M_j$ is $c_j$-torsionfree, and further that

$$\text{gldim End}_R(M_j) \leq 2 \text{ gldim End}_R(M_{j-1}) + \text{gldim End}_R(N) + 1.$$

Applying the induction hypothesis gives the desired upper bound for the global dimension of $\text{End}_R(M_j)$. \hfill \Box

The following examples illustrate that, without additional inequality on $c$ in Theorem 1, $M \oplus \Omega^c X$ need not give an NCR.

**Example 5.** Let $Q$ be the cycle of length two and $R$ the quotient of $\mathbb{C}Q$ by all paths of length three. Let $M = R \oplus S_1$ and $X = S_2$, where $S_1$ and $S_2$ are simple $R$-modules.
Then, by a direct calculation one gets \( \text{gldim} \text{End}_R(M) = 4 \) and \( \text{gldim} \text{End}_R(X) = 0 \), whilst \( \text{gldim} \text{End}_R(M \oplus X) = \infty \).

Here is an example from commutative algebra: Let \( R = \mathbb{C}[x, y]/(x^{2n} - y^2) \) with \( x, y \) commuting indeterminates, and \( n \geq 2 \) a positive integer. Then \( R \) is a curve singularity of type \( A_{2n-1} \). For \( M = R \oplus (R/(x^n - y)) \) and \( X = R/(x^n + y) \), one gets \( \text{gldim} \text{End}_R(M) = 3 \) (in fact \( M \) is 2-cluster tilting) and \( \text{gldim} \text{End}_R(X) = 1 \), whilst \( \text{gldim} \text{End}_R(M \oplus X) = \infty \).

Acknowledgements. This paper was written during the AIM SQuaRE on Cohen–Macaulay representations and categorical characterizations of singularities. We thank AIM for funding, and for their kind hospitality.

It is a pleasure to thank Ragnar-Olaf Buchweitz, Eleonore Faber, Colin Ingalls, and an anonymous referee for their comments on earlier versions of this manuscript.

References

1. M. Auslander, *Representation dimension of Artin algebras*, Lecture Notes (Queen Mary College, London, 1971).
2. M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society 94 (American Mathematical Society, Providence, RI, 1969). MR0259685.
3. M. Auslander, M. I. Platzeck and G. Todorov, ‘Homological theory of idempotent ideals’, *Trans. Amer. Math. Soc.* 332 (1992) 667–692. MR1052903 (92j:16008).
4. E. Cline, B. Parshall and L. Scott, ‘Finite-dimensional algebras and highest weight categories’, *J. reine angew. Math.* 391 (1988) 85–99. MR961165.
5. H. Dao, E. Faber and C. Ingalls, ‘Noncommutative (crepant) desingularizations and the global spectrum of commutative rings’, *Algebr. Represent. Theory* 18 (2015) 633–664. MR3557942.
6. H. Dao, O. Iyama, R. Takahashi and C. Vial, ‘Non-commutative resolutions and Grothendieck groups’, *J. Noncommut. Geom.* 9 (2015) 21–34. MR3337953.
7. V. Dlab and C. M. Ringel, ‘Every semiprimary ring is the endomorphism ring of a projective module over a quasihereditary ring’, *Proc. Amer. Math. Soc.* 107 (1989) 1–5. MR943793.
8. O. Iyama, ‘Finiteness of representation dimension’, *Proc. Amer. Math. Soc.* 131 (2003) 1011–1014. MR1948089.
9. O. Iyama, ‘Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories’, *Adv. Math.* 210 (2007) 22–50. MR2298819.
10. H. Matsumura, *Commutative ring theory*, 2nd edn, Cambridge Studies in Advanced Mathematics 8 (Cambridge University Press, Cambridge, 1989). (Translated from the Japanese by M. Reid.) MR1011461 (90i:13001).
11. R. Rouquier, ‘Dimensions of triangulated categories’, *J. K-Theory* 1 (2008) 193–256. MR2434186.
12. S. Špenko and M. Van den Bergh, ‘Non-commutative resolutions of quotient singularities for reductive groups’, *Invent. Math.* 210 (2017) 3–67.
13. M. Van den Bergh, ‘Non-commutative crepant resolutions’, *The legacy of Niels Henrik Abel* (eds O. A. Landal and R. Piene; Springer, Berlin, 2004) 749–770. MR2077594.

Hailong Dao  
Department of Mathematics  
University of Kansas  
Lawrence, KS 66045-7523  
USA  
hdao@ku.edu

Osamu Iyama  
Graduate School of Mathematics  
Nagoya University  
Chikusaku Nagoya 464-8602  
Japan  
iyama@math.nagoya-u.ac.jp

Srikanth B. Iyengar  
Department of Mathematics  
University of Utah  
Salt Lake City, UT 84112  
USA  
iyengar@math.utah.edu

Ryo Takahashi  
Graduate School of Mathematics  
Nagoya University  
Chikusaku Nagoya 464-8602  
Japan  
takahashi@math.nagoya-u.ac.jp
