2 + 1 dimensional loop quantum cosmology of Bianchi I models

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We study the anisotropic Bianchi I loop quantum cosmology in 2+1 dimensions. Both the \(\bar{\mu}\) and \(\bar{\mu}'\) schemes are considered in the present paper and the following expected results are established: (i) the massless scalar field again play the role of emergent time variables and serves as an internal clock; (ii) By imposing the fundamental discreteness of length operator, the total Hamiltonian constraint is obtained and gives rise the evolution as a difference equation; and (iii) the exact solutions of Friedmann equation are constructed rigorously for both classical and effective level. The investigation extends the domain of validity of loop quantum cosmology to beyond the four dimensions.

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I. INTRODUCTION

Loop quantum cosmology (LQC)\(^1\), which applies principles of loop quantum gravity (LQG)\(^3\) to cosmological settings, has been developed as a symmetry-reduced model of the full theory of LQG, to implement and test main ideas of LQG. Many obscure aspects in LQG become transparent in LQC, since in LQC the mathematical structure is much simpler. Particularly, the space-flat \((k = 0)\) Friedmann-LeMaitre-Robertson-Walker (FLRW) model without cosmological constant \((\Lambda = 0)\) can be solved exactly if the scalar field is used as an internal clock already in the classical theory, prior to quantization, and works in a suitable representation\(^5\). In the exactly soluble LQC model, the big bang singularity is resolved, which is replaced by a big bounce; it is obtained the analytical expression of the upper bound of the energy density operator; furthermore, questions regarding the behavior of fluctuations and preservation of semi-classicality across the bounce can be answered in detail. More interestingly, these features are still valid in arbitrary spacetime dimensions\(^4\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)\(^\,\)"}
as our fundamental building blocks. Thus the vanishing behavior of the length scale factors $a_I$ remains [12]. However, in 2+1 dimensions, the spectrum of length operator is fundamentally discreet, thus all the singularity appeared in 2+1 dimensional classical Bianchi I model is very likely to be resolved. Thus in this paper, we will mimic the scheme proposed in [13], to study the quantization of Bianchi I model in 2+1 dimensions, instead of the physical 3+1 dimensions.

This paper is organized as follows: After a brief introduction in the beginning, we present the classical dynamics of 2+1 dimensional Bianchi I models from Hamiltonian framework in Section II. With this classical Hamiltonian dynamics, in section III we construct the corresponding loop quantum cosmology of 2+1 dimensional Bianchi I models. Then in sections IV and V we discuss the action of Hamiltonian operator and the properties of quantum dynamics respectively. Some conclusions are presented in last section.

II. CLASSICAL DYNAMICS OF 2+1 DIMENSIONAL BIANCHI I MODELS

A. The canonical pair of 2+1 dimensional Bianchi I models

Let us start by summarizing the classical dynamics of 2+1 dimensional Bianchi I models. Our spacetime manifold $M$ is topologically $\mathbb{R}^3$. We restrict ourselves to diagonal Bianchi I metrics, given in terms of the directional scale factors $a_I$ with $I = 1, 2$

$$ds^2 = -N^2dt^2 + a_1^2dx^1 + a_2^2dx^2,$$

where $N$ represents the lapse. Since we consider noncompact Bianchi I model and all fields are spatially homogeneous, we can introduce an elemental cell $V$ and restrict all integrations to it [23]. We choose elementary cell $V$ as its edges lie along the coordinate axis $x, y$. We fix a fiducial flat metric $a_{ab}$ with line element $ds^2 = dx^2 + dy^2$. We denote by $a_{ab}$ the determinant of this metric, by $L_I$ the lengths of the two edges of $V$ as measured by $a_{ab}$, and by $V = L_1L_2$ the volume of $V$ as measured by $a_{ab}$. Among the fiducial co-dyads compatible with $a_{ab}$, we select $\omega^I_a$, without generality, such that

$$a^\omega_a = D_a x^I, \quad \text{and} \quad a^{\omega}_a = 0. \quad (2)$$

With this fiducial structure at hand, we can now introduce the phase space of 2+1 dimensional Bianchi I models, which is reduced from the one of the full theory. In the full theory of 2+1 dimensional LQG [17], the phase space is spanned by a canonical pair $(A^I_a, E^a_I)$, with $A^I_a$ an $\mathfrak{su}(2)$ connection and $E^a_i = \delta_{ij}e^a_b e^b_i$ the momentum conjugate to $A^I_a$. And the symplectic structure is given by

$$\{A^I_a(\vec{x}), E^I_b(\vec{x'})\} = 8\pi G\gamma \delta^{ij}_a \delta(\vec{x}, \vec{x'}). \quad (3)$$

Because of Bianchi symmetry, the connections $A^I_a$ are reduced to 2 constants $c^I$, and the momenta $E^a_i$ are reduced to 2 constants $p_I$:

$$A^I_a = c^I(\Delta^I) - \omega^I_a, \quad \text{and} \quad A^2_a = 0; \quad (4)$$

$$E^2_i = p_1L_1V^{-1}e^b_\delta\delta^1_\omega_d, \quad \text{and} \quad E^3_3 = 0. \quad (5)$$

Note that there is no summation over $I$ and the repeated upper and lower $J$ is summed. The momentum variables $p_I$ are directly related to the scale factors $a_I$ as

$$p_1 = a_1L_2, \quad p_2 = a_2L_1. \quad (6)$$

As we will see below, the connections $c^I$ are directly related to time derivatives of the scale factors. The resulting non-vanishing Poisson brackets are given by

$$\{\rho, p_I\} = 8\pi G\gamma \delta^I_J, \quad (7)$$

where $\gamma$ is the Barbero-Immirzi parameter.

B. Evolution equations

Now we come to the constraints. Because we have restricted ourselves to diagonal metrics and fixed the internal gauge, the Gauss and the diffeomorphism constraints are identically satisfied. The Hamiltonian constraints can be obtained by restricting the spatial integrations to the fiducial cell $V$:

$$\mathcal{C} = \mathcal{C}_{gr} + \mathcal{C}_M = \int_V N(\mathcal{H}_{gr} + \mathcal{H}_M) d^2x, \quad (8)$$

where $N$ is the lapse and we will fix $N = 1$ in the following for simplicity. The gravitational and the matter parts of the constraint densities are given by

$$\mathcal{H}_{gr} = -\frac{E^a_iE^b_j}{16\pi G\gamma^2\sqrt{q}} F_{abc}^k \quad \delta^i_j F_{ab}^k \quad (9)$$

$$\mathcal{H}_M = \sqrt{q}\rho_M. \quad (10)$$

Here $F_{ab}^k$ is the curvature of connection $A^I_a$, and $\rho_M$ is the energy density of the matter fields. The constraint density in Eq. (9) is different from the one in full theory of 2+1 dimensional LQG, by considering the flat-space property of Bianchi I models. We consider a massless scalar field $T$ coupled to gravity, thus the Hamiltonian constraint, in terms of induced variables, is given by

$$\mathcal{C} = -\frac{c_1 c_2}{8\pi G\gamma^2} + \frac{p_T^2}{2p_1 p_2}, \quad (11)$$

where $p_T$ is the conjugate momentum of the scalar field $T$, and the matter energy density is given by $\rho_M = p_T^2/2V^2$. From the Hamiltonian constraint (11), we find that $p_T$ is a constant of motion and $T$ is a monotonic function of $t$. Thus scalar $T$ can be considered as the internal time. The Poisson brackets of the matter part
is given by \( \{T, p_T\} = 1 \). The time evolution of \( p_I \) thus yield

\[
\dot{p}_1 = \{p_1, C\} = -8\pi G\gamma \frac{\partial C}{\partial c_1} = \frac{c_2}{\gamma} \tag{12}
\]

\[
\dot{p}_2 = \{p_2, C\} = -8\pi G\gamma \frac{\partial C}{\partial c_2} = \frac{c_1}{\gamma} \tag{13}
\]

\[
\dot{c}_1 = \{c_1, C\} = 8\pi G\gamma \frac{\partial C}{\partial p_1} = -8\pi G\gamma p_2 \rho_M \tag{14}
\]

\[
\dot{c}_2 = \{c_2, C\} = 8\pi G\gamma \frac{\partial C}{\partial p_2} = -8\pi G\gamma p_1 \rho_M \tag{15}
\]

Combining Eqs. (10), (12) and (13), one gets

\[
c_I = \gamma \dot{a}_J L_I, \tag{16}
\]

with \( J \neq I \). Eq. (10) shows the relation of connections and time derivative of scale factors, as mentioned before.

Now we come to show that \( c_I p_I \) are constants of motion. From Eqs. (12)-(15), we have

\[
\frac{d}{dt}(c_I p_I) = -8\pi G\gamma \mathcal{C}. \tag{17}
\]

Thus both of \( c_I p_I \) are constants of motion, since their time derivatives are proportional to the total Hamiltonian constraint.

Next, let us introduce the directional Hubble parameters \( H_I \equiv \dot{a}_I/a_I = p_I/p_I \). Then using Eqs. (6) and (10), the vanishing of Hamiltonian constraint (11) can be written as

\[
H_1 H_2 = 8\pi G \rho_M, \tag{18}
\]

where \( \rho_M = p_T^2/2p_1^2 p_2^2 \) is the energy density of the matter field \( T \). From Eqs. (6) and (10), the directional Hubble parameter can be also related to \( c_I p_I \) by

\[
c_1 p_1 = \gamma V_o a^2 H_2, \quad c_2 p_2 = \gamma V_o a^2 H_1. \tag{19}
\]

Here \( a \equiv \sqrt{a_1 a_2} \) denotes the mean scale factor, which defines the mean Hubble parameter:

\[
H \equiv \frac{\dot{a}}{a} = \frac{1}{2}(H_1 + H_2). \tag{20}
\]

Squaring Eq. (20) and using Eq. (15), we obtain the generalized Friedmann equation for 2+1 dimensional Bianchi I models:

\[
H^2 = 8\pi G \rho_M + \frac{\Sigma^2}{a^4}. \tag{21}
\]

Here

\[
\Sigma^2 \equiv \frac{a^4}{4}(H_1 - H_2)^2 \tag{22}
\]

is the shear term, which can be reexpressed by

\[
\Sigma^2 = \frac{(c_1 p_1 - c_2 p_2)^2}{4\gamma^2 V_o^2}, \tag{23}
\]

using Eq. (19). This expression, together with Eq. (17), leads to the result that \( \Sigma^2 \) is a constant of motion:

\[
\frac{d}{dt}(\Sigma) = 0. \tag{24}
\]

For the isotropic case, \( \Sigma = 0 \) and Eq. (21) reduces to the usual Friedmann equation for 2+1 dimensional isotropic cosmology.

Now we come to consider the reflections \( \Pi_I \):

\[
\Pi_1(c_1, c_2) = (c_1, -c_2) \tag{25}
\]

\[
\Pi_1(p_1, p_2) = (-p_1, p_2). \tag{26}
\]

The action of \( \Pi_2 \) is given by replacement \( (1 \leftrightarrow 2) \). Under each of \( \Pi_I \), the Hamiltonian constraint (11) is left invariant. Therefore, in the classical theory, we can restrict ourselves to the positive octant \( p_I \geq 0 \), and dynamics in any other octant can be obtained by (combinations of) reflections \( \Pi_I \). We will see this reflection \( \Pi_I \) will play an important role in quantum theory.

Now we restrict ourselves to the positive octant \( p_I \geq 0 \), and solve the generalized Friedmann equation (21).

Using Eqs. (21) and (22), we have

\[
p_1 p_2 H_I = \sqrt{4\pi G p_T^2 + \Sigma^2 V_o^2} \pm \Sigma V_o. \tag{27}
\]

Note that \( H_I = \dot{p}_I/p_I \), hence we have

\[
p_1 p_2 = 2t \sqrt{4\pi G p_T^2 + \Sigma^2 V_o^2}. \tag{28}
\]

And consequently we have

\[
H_I = \frac{\dot{p}_I}{p_I} = \frac{1}{2t} \left( 1 \pm \frac{\Sigma V_o}{\sqrt{4\pi G p_T^2 + \Sigma^2 V_o^2}} \right), \tag{29}
\]

and its solutions are given by

\[
p_I(t) = p_I(0) e^{\kappa_I t}. \tag{30}
\]

where the Kasner exponents \( \kappa_I \) are given as

\[
\kappa_I = \frac{1 + 2 \sqrt{\Sigma V_o}}{\sqrt{16\pi G p_T^2 + 4\Sigma^2 V_o^2}}. \tag{31}
\]

From the Hamiltonian constraint (11), the solution of \( T \) is given by

\[
T = T_o + \frac{\kappa_T}{\sqrt{8\pi G}} \ln t, \tag{32}
\]

with

\[
\kappa_T = \frac{\sqrt{8\pi G p_T}}{\sqrt{16\pi G p_T^2 + 4\Sigma^2 V_o^2}}. \tag{33}
\]

Using (31) and (33), we have

\[
\kappa_1 + \kappa_2 = 1, \quad \kappa_1^2 + \kappa_2^2 + \kappa_T^2 = 1. \tag{34}
\]

The form of the solutions to 2+1 dimensional Bianchi I model coupled with a massless scalar is very like the one to the 3+1 Bianchi I model (12), as well as the ones to the
arbitrary dimensional Bianchi I models \[24\]. Combing Eqs. (30) and (32), \(p_I\) can be given as a function of \(T\):

\[
p_I(T) = p_I(T_0)e^{\sqrt{8\pi G/\kappa}(T-T_o)}.
\]  

(35)

From the expression of Kasner exponents \[31\], we have \(\kappa_1, \kappa_2 > 0\). Thus \(p_1, p_2\) tend towards zero simultaneously, so do the directional scale factors \(a_1, a_2\). This is different from the singularities in 3+1 Bianchi I model coupled with a massless scalar, where the scale factors in the three directions approach zero not always together, and can be classified into four types \[25\]. In 2+1 dimensional Bianchi I models, the singularity occurs when the energy density of the matter content plays an important role, and the anisotropic shear does not play a dominant role.

C. From the action of 2+1 dimensional Bianchi I models

The classical dynamics of 2+1 dimensional Bianchi I models mentioned here, including the Poisson brackets in Eq. (7) and the constraint in Eq. (11), is reduced from the Hamiltonian framework of full theory of 2+1 dimensional LQG. Equivalently, it can be also derived by Hamiltonian analysis of the action of Bianchi I models, which is in terms of scale factors. To show this, we start from the action of Bianchi I models:

\[
S_{\text{gr}} = \frac{V_o}{8\pi G} \int dt \sqrt{-g} R
\]

(36)

with \(\sqrt{-g} = N a_1 a_2\) the determinant of the metric given by Eq. (11),

\[
R = \sum_I \left( \frac{\dot{a}_I}{N^2 a_I} - \frac{\dot{N} a_I}{N^3 a_I} \right) + \frac{\dot{a}_1 \dot{a}_2}{N^2 a_1 a_2}
\]

(37)

the curvature scalar, and \(V_o = \int d^2 x\) the coordinate volume. The action in the equation above can be rewritten by

\[
S_{\text{gr}} = -\frac{V_o}{8\pi G} \int dt \frac{\dot{a}_1 \dot{a}_2}{N},
\]

(38)

up to boundary terms. By fixing the coordinate volume \(V_o = 1\) and the lapse \(N = 1\), the Lagrangian density is given by

\[
\mathcal{L}_{\text{gr}} = \frac{\dot{a}_1 \dot{a}_2}{8\pi G}.
\]

(39)

The conjugate momentum of \(a_I\) is given by

\[
\pi^i := \frac{\partial \mathcal{L}_{\text{gr}}}{\partial \dot{a}_i} = -\frac{\dot{a}_2}{8\pi G},
\]

(40)

The gravitational part of the Hamiltonian density of gravitational part is given by

\[
\mathcal{H}_{\text{gr}} = \dot{a}_1 \pi^1 - L_{\text{gr}} = -8\pi G \pi^1 a_2.
\]

(42)

Minimally coupled with a massless scalar field \(T\), the total Hamiltonian density is given by

\[
\mathcal{H} = -8\pi G a_1 a_2 \frac{\dot{p}_2}{2a_1 a_2} + \mathcal{H}_{\text{gr}}.
\]

(43)

Using the relations (7) and (10), the Poisson brackets (11) and the Hamiltonian density (42) are equivalent to Eqs. (7) and (11). Thus we can obtain the same generalized Friedmann equation (21).

III. 2+1 DIMENSIONAL DIMENSIONAL LOOP QUANTUM COSMOLOGY OF BIANCHI I MODELS

In the full theory of 2+1 dimensional LQG, the spectrum of one dimensional “area” operator is discrete \[2\] and has a non-zero minimal value \(\Delta\) \[17, 27, 28\]. We will use this minimal area \(\Delta\) to construct the curvature in 2+1 dimensional LQC. For concreteness, we consider a plaquette \(\Box_{IJ}\) and define curvature along this plaquette. Here the sides of \(\Box_{IJ}\) are along diagonal directions \(I, J\), and the length of the sides are \(\bar{\mu}_I L_I\) and \(\bar{\mu}_J L_J\) respectively measured by fiducial metric \(\gamma_{ab}\); the value of \(\bar{\mu}\) can be obtained from the minimal area \(\Delta\) by some certain scheme.

In 3+1 loop quantum gravity, the so-called “improved scheme” leads to very successful quantum dynamics in isotropic case \[29\], where \(\bar{\mu} \propto 1/|p|\). This scheme is extended to 3+1 Bianchi I models, in the literature in two different approaches. In Bianchi I models, there are three \(p_i\) and three \(\bar{\mu}_i\) (or \(\bar{\mu}'_i\)). One scheme assumes that \(\bar{\mu}_i \propto 1/|a_i|\) \[12, 30\], which suffers from fiducial cell scaling and serious problems \[14, 13\]. The other scheme, where \(\bar{\mu}'_i \propto 1/|a_i|\) \[13, 30, 31\], solves these problems. Here we will mimic the latter to set \(\bar{\mu}'_i \propto 1/|a_i|\) in 2+1 dimensional Bianchi I models. The former scheme is also discussed in appendix A.

A. Quantum Kinematics

The gravitational part of kinematical Hilbert space is given by \(\mathcal{H}_{\text{Kin}}^\text{gr} = L^2[\mathbb{R}^{\text{Bohr}}, \pi^a_{\text{Bohr}}]^{\otimes 2}\) with the orthonormal basis elements labeled by two real numbers \(|p_1, p_2\rangle\),

\[1\] The approach to singularity for the vacuum 2+1 dimensional Bianchi I model is different: one of the scale factors tends to zero while the other approaches to constant.

\[2\] In the full theory of 2+1 dimensional LQG, the spectrum of area operator is discrete, if the gauge group is \(SU(2)\), as we consider here. However, if non-compact group \(SO(2, 1)\) is considered instead, the spectrum of spacelike intervals is continuous \[24\].
where $\mathbb{R}_{\text{Bohr}}$ stands for the Bohr compactification of a real line. The kinematical scalar product is defined as

$$\langle p_1, p_2 | p_1', p_2' \rangle = \delta_{p_1, p_1'} \delta_{p_2, p_2'}.$$  \hfill (44)

Any state $|\Psi\rangle \in H_{\text{Kin}}^{gr}$ can be considered as a countable linear combination of this orthonormal basis as

$$|\Psi\rangle = \sum_{p_1, p_2} \Psi(p_1, p_2) |p_1, p_2\rangle \quad \text{with} \quad \sum_{p_1, p_2} |\Psi(p_1, p_2)|^2 < \infty.$$  \hfill (45)

The elemental operators on the gravitational part of kinematical space are momenta $\hat{p}_I$ and $SU(2)$ holonomies along edge $e_I$ in the diagonal direction $I$

$$\hat{h}^{I}_{I}(\mu) = \cos \frac{c_I \mu_I}{2} \mathbb{I} + 2 \sin \frac{c_I \mu_I}{2} \tau_I,$$  \hfill (46)

where $\mathbb{I}$ is the identity matrix, $\tau_I = \sigma_I/2i$ with $\sigma_i$ the Pauli matrices, and $\mu_I L_i$ is the length of the edge with respect to $a_{qab}$. The action of the elemental operators on the gravitational part of kinematical space $H_{\text{Kin}}^{gr}$ are given by

$$\hat{p}_I \Psi(p_1, p_2) = p_I \Psi(p_1, p_2),$$
and similarly for $\exp(i \mu_2 c_2)$.

Next, for the matter part, the scalar field $T$ is quantized as usual:

$$\hat{T} \Psi(p_1, p_2, T) = T \Psi(p_1, p_2, T),$$
$$\hat{\rho}_T \Psi(p_1, p_2, T) = -i \hbar \frac{d}{dT} \Psi(p_1, p_2, T),$$

where $\Psi(p_1, p_2, T) \in H_{\text{Kin}}^{gr} \otimes L^2(\mathbb{R}, dT)$. In the next subsection, we will introduce the Hamiltonian operator in terms of these elemental operators, acting on the total kinematical space $H_{\text{Kin}} = H_{\text{Kin}}^{gr} \otimes L^2(\mathbb{R}, dT)$.

### B. Construction of Hamiltonian operator

To construct the Hamiltonian constraint operator, we first express the curvature $F_{ab}^k$ in terms of holonomies:

$$F_{ab}^k = -2 \sum_{I, J} \text{Tr} \left( \frac{\hbar^{(I)}_{I,J} - 1}{\Delta_{I,J}} \right) \omega_a^I \omega_b^J.$$  \hfill (49)

Here $\hbar^{(I)}_{I,J} \equiv \hat{h}^{(\mu_I)}_{I,J} \hat{h}^{-1}_{I,J}$ is the holonomy around the plaquette $\square_{I,J}$, whose sides are along diagonal directions $I, J$ and have length $\mu_I L_I$ and $\mu_J L_J$ respectively measured by fiducial metric $a_{qab}$; $\Delta_{I,J} = \mu_I \mu_J$ is the area of the plaquette $\square_{I,J}$. And $\mu_I$ is given as

$$\mu_1 = \frac{\Delta}{|p_2|}, \quad \mu_2 = \frac{\Delta}{|p_1|}.$$  \hfill (50)

where $\Delta$ is the minimal 1-dimensional “area”. Using the formula (48) of curvature together with Thiemann’s trick in 2+1 dimensional LQG [17],

$$\frac{1}{2} \epsilon^{ijk} \epsilon_{ab} \epsilon_{d}^{ij} \frac{E_{a}^{d}}{E_{b}^{d}} \frac{E_{b}}{E_{a}} = \frac{1}{2(8 \pi G)^2} \epsilon_{ijk} \epsilon^{ab} \left\{ A_{a}^{d}, V \right\} \left\{ A_{b}^{d}, V \right\},$$

the operator of the gravitational part of the Hamiltonian constraint

$$\mathcal{C}_{gr} = - \frac{1}{16 \pi G^2} \int_{V} E_{a}^{d} E_{b}^{d} \epsilon_{ijk} F_{ab}^{k}$$  \hfill (51)

can be given by

$$\mathcal{C}_{gr} = \frac{2}{(8 \pi G)^3 \gamma^4} \sum_{I, J, K, L} \epsilon_{i j}^{I J} \epsilon_{K L}^{d} \text{Tr} \left( h_{I,J} h_{K}^{-1} \sqrt{V} \right) h_{L} \left( h_{K}^{-1}, \sqrt{V} \right),$$  \hfill (52)

where $h_{K}$ is short for $h_{h_{K}^k}$, the holonomy along edge $e_{K}$ in the diagonal direction $K$. If the Poisson brackets are replaced by commutators : $\{ , \} \rightarrow [ , ]/i \hbar$, and if the observables are replaced by the corresponding operators, we will obtain the operator of the gravitational part of the Hamiltonian constraint, up to factor orderings

$$\hat{\mathcal{C}}_{gr} = \frac{2}{(8 \pi G)^3 \gamma^4} \frac{1}{\hbar^2} \sum_{I \neq J, K \neq L} \sin(\mu_I c_I) \sin(\mu_J c_J) \frac{\mu_J}{\mu_I} \hat{A}_{K} \hat{A}_{L}$$  \hfill (53)

where

$$\hat{A}_{K} = (\hat{\mu}_{K}^2)^{-2} \left( \cos(\frac{\hat{\mu}_{K} C K}{2}) \sqrt{V} \sin(\frac{\hat{\mu}_{K} C K}{2}) - \sin(\frac{\hat{\mu}_{K} C K}{2}) \sqrt{V} \cos(\frac{\hat{\mu}_{K} C K}{2}) \right).$$  \hfill (54)
By expanding the summation over \( I \neq J, K \neq L \), we will find four terms of the same value. Thus
\[
\hat{C}_{gr} = \frac{8}{(8\pi G)^{3/4} \hbar^2} \sin(\mu'_1 c_1) \sin(\mu'_2 c_2) A_1 A_2.
\] (55)
The symmetrized operator is given by
\[
\hat{C}_{gr} = \sin((\mu'_1 c_1)) \hat{A} \sin(\mu'_2 c_2) + \sin(\mu'_2 c_2) \hat{A} \sin(\mu'_1 c_1),
\] (56)
with
\[
\hat{A} = \frac{4}{(8\pi G)^{3/4} \hbar^2} \hat{A}_1 \hat{A}_2.
\] (57)

Before going to study the action of the Hamiltonian operator in next subsection, we show the induced action of reflections \( \Pi_I \) on the classical phase space.

On the space of wave functions \( \Psi(p_1, p_2) \), the two reflections \( \Pi_I \) on the classical phase space have a natural induced action \( \Pi_I \), for example, \( \Pi_I \Psi(p_1, p_2) = \Psi(-p_1, p_2) \). We will assume that the wave function \( \Psi(p_1, p_2) \) is symmetric under the action \( \Pi_I \), which implies
\[
\Psi(p_1, p_2) = \Psi(|p_1|, |p_2|).
\] (58)

On the quantum operators \( \hat{O} \), the induced action of reflections \( \hat{\Pi}_I \) is given by
\[
\hat{O} \rightarrow (\hat{\Pi}_I \hat{O} \hat{\Pi}_I)^\dagger \Psi := \hat{\Pi}_I \hat{O} \hat{\Pi}_I \Psi.
\] (59)

Action of the reflections \( \hat{\Pi}_I \) on elemental operators is given by
\[
\hat{\Pi}_I \hat{p}_J \hat{\Pi}_I = s_{I,J} \hat{p}_J
\] (60)
\[
\hat{\Pi}_I \exp(\pm i\mu'_I c_I) \hat{\Pi}_I = \exp(\pm s_{I,J} i\mu'_J c_J)
\] (61)
with \( s_{I,J} = \pm 1 \) given by
\[
s_{I,J} = \begin{cases} 
-1 & \text{if } I = J; \\
1 & \text{if } I \neq J.
\end{cases}
\] (62)

For the matter part, both of \( \hat{T} \) and \( \hat{p}_T \) are invariant under reflections \( \hat{\Pi}_I \). From the action of reflections \( \hat{\Pi}_I \) on the elemental operators (60) and (61), we find the gravitational part of Hamiltonian operator (56) is reflection symmetric:
\[
\hat{\Pi}_I \hat{C}_{gr} \hat{\Pi}_I = \hat{C}_{gr},
\] (63)
just as in the classical theory. Therefore, its action is well defined on \( \mathcal{H}_{kin}^{gr} \).

IV. ACTION OF HAMILTONIAN OPERATOR

To see explicitly the action of the Hamiltonian operator (56), we introduce a new orthonormal basis \( |\lambda_1, \lambda_2\rangle \) in \( \mathcal{H}_{kin}^{gr} \) with
\[
\lambda_I = \frac{p_I}{\sqrt{8\pi |\gamma| \Delta \ell_P}}.
\] (64)

where \( \ell_P = G \hbar \) is the Planck length in 2+1 dimensions. The action of elemental operators on wave functions \( \Psi(\lambda_1, \lambda_2) \) is given by
\[
(p_I \Psi)(\lambda_1, \lambda_2) = \sqrt{8\pi |\gamma| \Delta \ell_P} \lambda_I \Psi(\lambda_1, \lambda_2),
\] (65)
\[
\exp(\pm i\mu'_I c_I) \Psi(\lambda_1, \lambda_2) = \Psi \left( \lambda_1 \mp \frac{1}{|\lambda_2|} \lambda_2 \right),
\] (66)
and similar for \( \exp(\pm i\mu'_I c_I) \). Equation (65) can be given by the expression (54) of \( \mu'_I \) and the Poisson brackets, which imply
\[
\exp(\pm i\mu'_I c_I) = \exp \left( \mp \frac{8\pi \gamma \Delta \ell_P}{|p_J|} \frac{d}{dp_J} \right),
\] (67)
\[
= \exp \left( \pm \frac{1}{|\lambda_J|} \frac{d}{d\lambda_J} \right) =: E_F^\pm
\] (68)
with \( J \neq I \).

To make the quantum dynamics easier to compare with that of the Friedmann models in 2+1 dimensions [9], we introduce the volume of the elementary cell \( V \) as one of the arguments of the wave function, mimicking [12]. Set
\[
v = 2\lambda_1 \lambda_2,
\] (69)
and use \( \lambda_1, v \) as the configuration variables in place of \( \lambda_1, \lambda_2 \), then the action of the volume of \( V \) on \( \Psi(\lambda_1, v) \) is given by
\[
\hat{V} \Psi(\lambda_1, v) = 4\pi |\gamma| \Delta \ell_P |v| \Psi(\lambda_1, v).
\] (70)

We now restrict the argument of \( \hat{C}_{gr} \) to the positive octant. The action of \( \hat{A}_I \) in equation (54) on \( \Psi(\lambda_1, v) \) is given by
\[
\hat{A}_I \Psi(\lambda_1, v) = \frac{4\pi |\gamma| \Delta \ell_P}{i \Delta} \lambda_I^2 (\sqrt{|v-1|} - \sqrt{|v+1|}) \Psi(\lambda_1, v),
\] (57)
with \( J \neq I \). Then the action of \( \hat{A} \) in equation (54) on \( \Psi(\lambda_1, v) \) is given by
\[
\hat{A} \Psi(\lambda_1, v) = -\frac{\pi \ell_P}{8\pi G \sqrt{\Delta}} v^2 (\sqrt{|v-1|} - \sqrt{|v+1|})^2 \Psi(\lambda_1, v).
\]
Now let us give the action of the gravitational part (56) of the Hamiltonian operator:
\[
\hat{C}_{gr} \Psi(\lambda_1, v) = f(v+2)(\Psi_4^+ - \Psi_0^-) - f(v-2)(\Psi_0^- + \Psi_4^-),
\] (71)
where
\[
f(v) = \frac{\pi \ell_P}{32\pi G \gamma \Delta} v^2 (\sqrt{|v-1|} - \sqrt{|v+1|})^2,
\]
and \( \Psi_0^\pm, \Psi_4^\pm \) are defined as follows:
\[
\Psi_0^\pm = \Psi \left( \frac{v \pm 2}{v} \lambda_1, v \pm 4 \right) + \Psi \left( \frac{v \pm 4}{v} \lambda_1, v \pm 4 \right),
\] (72)
\[
\Psi_4^\pm = \Psi \left( \frac{v \pm 2}{v} \lambda_1, v \right) + \Psi \left( \frac{v \pm 2}{v} \lambda_1, v \right).
\] (73)
Now we come to consider the operator of the matter part of the Hamiltonian constraint, whose classical expression is given by

$$C_M = \frac{\pi^2}{2V}.$$  \hfill (74)

When the inverse volume operator corresponding to $1/V$ is defined, three ambiguities appear \[92\]:

$$\frac{1}{V} = (\dot{V})^p \left( \frac{1}{2\alpha^p} \left| \dot{V} + \alpha s - |\dot{V} - \alpha s| \right| \right)^{\frac{1+p}{p}}, \hfill (75)$$

with $p > 0$, $\alpha > 0$, and $0 < s < 1$. These are all reasonable inverse volume states, since they annihilate zero volume states and approximate to $1/V$ for large $V$. The particular inverse volume operator in 2+1 dimensional FLRW model \[9\] is obtained by setting

$$\alpha = 4\pi\gamma\Delta\ell_P, \quad s = \frac{1}{4} \quad \text{and} \quad p = 2;$$

$$\hat{V}^{-1}\Psi(\lambda_1, v) = \frac{4|v|^2}{\pi\gamma\Delta\ell_P} \left| |v + 1|^{1/4} - |v - 1|^{1/4} \right|^4 \Psi(\lambda_1, v)$$

$$= \frac{B(v)}{4\pi\gamma\Delta\ell_P} \Psi(\lambda_1, v), \quad \hfill (76)$$

where $B(v) \equiv 16 \left| |v + 1|^{1/4} - |v - 1|^{1/4} \right|^4$ is different from the one defined in 2+1 dimensional FLRW model \[4\] by a factor of $4\pi\gamma\Delta\ell_P$. We will use this particular inverse volume operator in the following, to have the Hamiltonian operator comparable to the one of 2+1 dimensional FLRW model. Thus the action of the matter part of the Hamiltonian operator on wave functions $\Psi(\lambda_1, v; T) \in \mathcal{H}_{\text{Kin}} = \mathcal{H}_{\text{kin}}^\ell \otimes L^2(\mathbb{R}, dT)$ is given by

$$\hat{C}_M \Psi(\lambda_1, v; T) = -\frac{\hbar^2}{4\pi\gamma\Delta\ell_P} B(v) \partial_T^2 \Psi(\lambda_1, v; T). \quad \hfill (77)$$

Collecting the gravitational part (71) and the matter part (77) of the Hamiltonian operator, we can express the vanishing of the total Hamiltonian operator as

$$\partial_T^2 \Psi(\lambda_1, v; T) = [B(v)]^{-1} \left( C_+(v) \left( \Psi_+^+(T) - \Psi_0^+(T) \right) - C_-(v) \left( \Psi_0^-(T) + \Psi_-^-(T) \right) \right), \quad \hfill (78)$$

V. PROPERTIES OF THE QUANTUM DYNAMICS

A. Relation to the 2+1 dimensional LQC Friedmann dynamics

In classical theory, the Friedmann model can be reduced from the Bianchi I model by applying isotropic conditions $a_1 = a_2$ in 2+1 dimensional case or $a_1 = a_2 = a_3$ in 3+1 dimensional case. In 3+1 LQC, it is shown in \[13\] that there is a natural projection from a dense subspace of the physical Hilbert space of Bianchi I model to that of the Friedmann model. In this subsection, we will mimic the 3+1 projection to construct a 2+1 dimensional projection, which maps the Bianchi I Hamiltonian constraint (78) to that of 2+1 dimensional Friedmann model. The idea is to integrate out the extra, anisotropic degrees of freedom, which first appear in \[92\].

We define a projection $\mathcal{P}$ from states $\Psi(\lambda_1, v)$ of the 2+1 dimensional Bianchi I models to the states $\psi(v)$ of 2+1 dimensional Friedmann model of \[9\] as follows:

$$\Psi(\lambda_1, v) \rightarrow (\mathcal{P}\Psi)(v) = \sum_{\lambda_1} \Psi(\lambda_1, v) \equiv \psi(v). \quad \hfill (83)$$

Applying this projection map to the Hamiltonian operator (78), we find

$$\partial_T^2 \psi(v; T) = 2[B(v)]^{-1} \left( C_+(v) \left( \psi(v + 4; T) - \psi(v; T) \right) - C_-(v) \left( \psi(v - 4; T) + \psi(v; T) \right) \right), \quad \hfill (84)$$

which is the total Hamiltonian operator of 2+1 dimensional Friedmann model. This result shows there is a
simple and exact relation between quantum dynamics of Bianchi I model and Friedmann model.

B. Effective equations

Because of the complexity of Bianchi I model, and the set of our $\tilde{\mu}_i$, it is not easy to carry out the semi-classical analysis and derive the effective equation. In this subsection, we obtain effective equation by analog of the classical equation, replacing $c_1$ by $\sin(\tilde{\mu}_i'c_1)/\tilde{\mu}_i'$. The effective equation obtained in this approach shows to be the same with the one derived by semi-classical analysis in isotropic case \cite{21,22} and the $\tilde{\mu}_i' \propto 1/\sqrt{|p_i|}$ scheme of Bianchi I model \cite{12}.

The effective Hamiltonian constraint is given by the analog of classical form \cite{14}:

$$C_{\text{eff}}^{\text{eff}} = C_{\text{gr}}^{\text{eff}} + p_1p_2\rho_M,$$  \hspace{1cm} (85)

where

$$C_{\text{eff}}^{\text{eff}} = -\frac{p_1p_2}{8\pi G\gamma^2\Delta^2} \sin(\tilde{\mu}_i'c_1) \sin(\tilde{\mu}_2'c_2).$$  \hspace{1cm} (86)

The vanishing of the effective Hamiltonian constraint \cite{21} gives upper bound of the matter density:

$$\rho_M = \rho_c \sin(\tilde{\mu}_i'c_1) \sin(\tilde{\mu}_2'c_2) \leq \rho_c,$$  \hspace{1cm} (87)

where $\rho_c = 1/8\pi G\gamma^2\Delta^2$ is the maximal density. Note that the matter density becomes infinite at the big-bang singularity in the classical evolution, thus the upper-bounded density shows that the singularity is resolved in the effective theory. This is different from the case in 3+1 dimensions. Remind that in 3+1 dimensions, the planar collapse for Kasner-like solutions is not resolved by the quantum effect and thus the vanishing behavior of the length scale factors $a_I$ remains \cite{12}. In 2+1 dimensions, we only have one type of solution and the spectrum of length operator is discreet, thus all the singularity appeared in 2+1 dimensional classical Bianchi I model is resolved.

Using Eq. \cite{21}, we can go further to define the effective “directional” matter density,

$$\rho_I = \rho_c \sin^2(\tilde{\mu}_i'c_1),$$  \hspace{1cm} (88)

which is also bounded by the maximal density $\rho_c$ and

$$\rho_1\rho_2 = \rho_M^2.$$  \hspace{1cm} (89)

Effective equations are obtained via Poisson brackets

$$\dot{p}_1 = \{p_1, C_{\text{eff}}^{\text{eff}}\} = \frac{p_1}{\gamma \Delta} \cos(\tilde{\mu}_i'c_1) \sin(\tilde{\mu}_2'c_2),$$  \hspace{1cm} (90)

$$\dot{c}_1 = \frac{c_1p_2}{\gamma \Delta p_1} \sin(\tilde{\mu}_i'c_1) \cos(\tilde{\mu}_2'c_2) - 16\pi G\gamma^2 p_2 \rho_M$$

$$\Rightarrow \frac{c_1}{p_1} \dot{p}_2 - 16\pi G\gamma^2 p_2 \rho_M,$$  \hspace{1cm} (91)

where the vanishing of effective Hamiltonian constraint \cite{21} is used, and $\rho_2$, $\dot{c}_2$ can be obtained by replacement of $(1 \leftrightarrow 2)$.

The effective directional Hubble parameter is thus given by

$$H_I = \frac{\dot{p}_I}{p_I} = \frac{1}{\gamma \Delta} \cos(\tilde{\mu}_i'c_1) \sin(\tilde{\mu}_j'c_j),$$  \hspace{1cm} (92)

with $J \neq I$. Using definition of the shear in Eq. \cite{21} and the effective directional Hubble parameter in Eq. \cite{21}, the effective shear is given by

$$\Sigma = \frac{p_1p_2}{2\gamma \Delta V_\rho} \sin(\tilde{\mu}_1'c_1 - \tilde{\mu}_2'c_2),$$  \hspace{1cm} (93)

which implies the fact that the effective shear is finite throughout the evolution. However, in effective theory, the shear is no longer a constant of motion as it is in classical theory.

Using the effective directional Hubble parameter in Eq. \cite{21}, the effective mean Hubble parameter can be given by

$$H = \frac{1}{2}(H_1 + H_2) = \frac{1}{2\gamma \Delta} \sin(\tilde{\mu}_1'c_1 + \tilde{\mu}_2'c_2).$$  \hspace{1cm} (94)

Using Eqs. \cite{21}, \cite{21} and \cite{21}, the square of the effective Hubble parameter in Eq. \cite{21} can be given by

$$H^2 = 8\pi G\rho_M \left[ 1 - \frac{\rho_1}{\rho_c} \sqrt{1 - \frac{\rho_2}{\rho_c} + \frac{\Sigma^2}{\rho_M^2}} \right].$$  \hspace{1cm} (95)

For the isotropic case, the effective shear term vanishes, and $\rho_1 = \rho_2 = \sqrt{\rho_M}$, thus the effective generalized Friedmann equation \cite{21} will reduce to the usual Friedmann equation in 2+1 dimensional Friedmann model \cite{21}: $H^2 = 8\pi G\rho_M(1 - \rho_M/\rho_c)$. For the classical limit, $\tilde{\mu}_i'c_1 << 1$, we have $\sin(\tilde{\mu}_i'c_1) \rightarrow \tilde{\mu}_i'c_1$ and $\cos(\tilde{\mu}_i'c_1) \rightarrow 1$, thus the effective generalized Friedmann equation in \cite{21} will approximately become the classical generalized Friedmann equation \cite{21}. If we drop the higher order, Eq. \cite{21} becomes

$$H^2 = 8\pi G\rho_M \left[ 1 - \frac{\rho_M}{\rho_c} \right] + \frac{\Sigma^2}{\rho_M^2} \left( 1 - \frac{2\rho_M}{\rho_c} \right) + O((\tilde{\mu}_i'c_i)^4).$$  \hspace{1cm} (96)

Again we can see in the classical limit $\rho_M << \rho_c$, it will go to the generalized Friedmann equation \cite{21}. Vanishing of the Hubble parameter in Eq. \cite{21} will give the matter energy density at the bounce

$$\rho_{\text{bounce}} \approx \frac{\rho_c}{2} - \frac{\Sigma^2}{8\pi Ga^4} + \sqrt{\left(\frac{\rho_c}{2}\right)^2 + \left(\frac{\Sigma^2}{8\pi Ga^4}\right)^2},$$

which is bounded by $\rho_c$.

VI. CONCLUSIONS

This paper gives the detailed construction of loop quantum cosmology of Bianchi I models in 2+1 dimensions. Both the $\tilde{\mu}$ and $\tilde{\mu}'$ schemes which appeared in four dimensional case are successfully deployed. In order to
make this paper as compact as possible, we only discuss the more physical intuitively $\bar{\mu}$' scheme in the main text, while leave the discussion of $\mu$ scheme in the appendix. Our results show that the discreteness of the underlying quantum geometry of 2+1 dimensions again gives rise to a difference equation which represents the evolution of the three dimensional universe.

Meanwhile, we interestingly observe that in 2+1 dimensions also admitting some new features. More precisely, in 3+1 dimensions, the planar collapse for Kasner-like solutions is not resolved by the quantum effect and thus the vanishing behavior of the length scale factors $a_I$ remains. However, we only have one type of solution and note that in 2+1 dimensional case, the spectrum of length operator is discreet, thus all the singularity appeared in 2+1 dimensional classical Bianchi I model is resolved. Of course, the results in present paper is still quite preliminary and requires further investigations.

There are several possible extensions of our results. The first one is to generalize our result to higher dimensions, while the second one is more interesting and subtly. Namely, our results can serve as the first step of link LQC from LQG in 2+1 dimensions directly from 3+1 dimensions [35–46]. However, since efforts have been done towards this important direction [28], we only discuss the central argument of our results in present paper is still resolved. Of course, the results in present paper is still perfectly and note that in 2+1 dimensional case, the spectrum of length-operator is discreet, thus all the singularity appeared in 2+1 dimensional classical Bianchi I model is resolved.

In this scheme, the effective Hamiltonian constraint is given by

$$\mathcal{C}_{\text{eff}} = \mathcal{C}_{\text{eff}} + p_1 p_2 \rho_M,$$

where

$$\mathcal{C}_{\text{eff}} = -\frac{p_1 p_2}{8 \pi G \gamma^2 \Delta^2} \sin (\bar{\mu}_1 c_1) \sin (\bar{\mu}_2 c_2).$$

Although Eq. (A2) seems the same in form as Eq. (11), they are different, since they have different definition of $\bar{\mu}_I$. Since they are same in form, the vanishing of Eq. (A2) gives the same upper bound of the matter density:

$$\rho_M = \rho_c \sin (\bar{\mu}_1 c_1) \sin (\bar{\mu}_2 c_2) \leq \rho_c.$$  

The effective equations are different:

$$\dot{p}_1 = \frac{\gamma \Delta}{p_1} \cos (\bar{\mu}_1 c_1) \sin (\bar{\mu}_2 c_2)$$

$$\dot{c}_1 = \frac{c_1 p_2}{\gamma \Delta p_1} \cos (\bar{\mu}_1 c_1) \sin (\bar{\mu}_2 c_2) - 16 \pi G \gamma^2 p_2 \rho_M$$

$$\dot{c}_1 = \frac{c_1 p_1}{p_1} - 16 \pi G \gamma^2 p_2 \rho_M,$$

where the vanishing of the effective Hamiltonian constraint (A2) is used. We have shown in section II B that $c_1 p_1$ are constants of motion in classical theory. Their effective analogs in the scheme (A1) can be defined as

$$G_I := p_1 \frac{\sin (\bar{\mu}_1 c_1)}{\bar{\mu}_1} = \frac{p_1^2}{\Delta} \sin (\bar{\mu}_1 c_1),$$

with the time derivative

$$\frac{d}{dt} G_I = 0.$$  

Hence $G_I$ are constants of motion in effective theory. However, this form defined in the scheme [50] is not constant anymore. The vanishing of the effective Hamiltonian constraint (A2) gives

$$G_1 G_2 = 4 \pi G \gamma^2 p_1^2.$$  

Using Eqs. (A5) and (A7), we have

$$\frac{dp_1}{dT} = \frac{G_2}{\gamma p_T} \sqrt{p_1^2 - \frac{G_2^2 \Delta^2}{p_1^2}},$$

which gives the solution to the effective equations:

$$p_1 (T) = \sqrt{\frac{1}{2} \left( G^2 \Delta^2 e^{-2 \pi y \Delta^2 / (T-T_o)} + e^{2 \pi y / p_1^2 (T-T_o)} \right)},$$

and similarly for $p_2 (T)$.

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Appendix A: Alternate quantization

In this appendix, we consider the effective dynamics from an alternative scheme, which assumes that $\bar{\mu}_I \propto 1/\sqrt{|p_I|}$ in 3+1 Bianchi I model [12, 30]. For 2+1 dimensional Bianchi I model, $\bar{\mu}_I$ in this scheme is given as

$$\bar{\mu}_I = \frac{\Delta}{|p_I|}.$$  

In this scheme, the effective Hamiltonian constraint is given by

$$\mathcal{C}_{\text{eff}} = \mathcal{C}_{\text{eff}} + p_1 p_2 \rho_M,$$

where

$$\mathcal{C}_{\text{eff}} = -\frac{p_1 p_2}{8 \pi G \gamma^2 \Delta^2} \sin (\bar{\mu}_1 c_1) \sin (\bar{\mu}_2 c_2).$$

Although Eq. (A2) seems the same in form as Eq. (11), they are different, since they have different definition of $\bar{\mu}_I$. Since they are same in form, the vanishing of Eq. (A2) gives the same upper bound of the matter density:

$$\rho_M = \rho_c \sin (\bar{\mu}_1 c_1) \sin (\bar{\mu}_2 c_2) \leq \rho_c.$$  

The effective equations are different:

$$\dot{p}_1 = \frac{\gamma \Delta}{p_1} \cos (\bar{\mu}_1 c_1) \sin (\bar{\mu}_2 c_2)$$

$$\dot{c}_1 = \frac{c_1 p_2}{\gamma \Delta p_1} \cos (\bar{\mu}_1 c_1) \sin (\bar{\mu}_2 c_2) - 16 \pi G \gamma^2 p_2 \rho_M$$

$$\dot{c}_1 = \frac{c_1 p_1}{p_1} - 16 \pi G \gamma^2 p_2 \rho_M,$$

where the vanishing of the effective Hamiltonian constraint (A2) is used. We have shown in section II B that $c_1 p_1$ are constants of motion in classical theory. Their effective analogs in the scheme (A1) can be defined as

$$G_I := p_1 \frac{\sin (\bar{\mu}_1 c_1)}{\bar{\mu}_1} = \frac{p_1^2}{\Delta} \sin (\bar{\mu}_1 c_1),$$

with the time derivative

$$\frac{d}{dt} G_I = 0.$$  

Hence $G_I$ are constants of motion in effective theory. However, this form defined in the scheme [50] is not constant anymore. The vanishing of the effective Hamiltonian constraint (A2) gives

$$G_1 G_2 = 4 \pi G \gamma^2 p_1^2.$$  

Using Eqs. (A5) and (A7), we have

$$\frac{dp_1}{dT} = \frac{G_2}{\gamma p_T} \sqrt{p_1^2 - \frac{G_2^2 \Delta^2}{p_1^2}},$$

which gives the solution to the effective equations:

$$p_1 (T) = \sqrt{\frac{1}{2} \left( G^2 \Delta^2 e^{-2 \pi y \Delta^2 / (T-T_o)} + e^{2 \pi y / p_1^2 (T-T_o)} \right)},$$

and similarly for $p_2 (T)$.

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