LINEARIZED ESTIMATE OF THE BACKWARD ERROR FOR THE EQUALITY
CONSTRAINED INDEFINITE LEAST SQUARES PROBLEM

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Abstract. In this note, we concentrate on the backward error of the equality constrained indefinite least squares problem. For the normwise backward error of the equality constrained indefinite least square problem, we adopt the linearization method to derive the tight estimate for the exact backward normwise error. The numerical examples show that the linearization estimate is effective for the normwise backward errors.

Key words. Indefinite least squares, the equality constrained indefinite least squares problem, normwise backward error, linearization estimate.

AMS subject classifications. 65F99, 65G99.

1. Introduction. The indefinite least squares (ILS) problem [1, 3] is given by:

\[
\text{ILS} : \quad \min_x (b - Ax)^\top \Sigma_{pq} (b - Ax),
\]

where \( A^\top \) is the transpose of \( A \), \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), \( m \geq n \) and the signature matrix

\[
\Sigma_{pq} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad p + q = m.
\]

The ILS (1.1) has applications in the total least squares problem [21] and \( H^\infty \)-smoothing in optimization [8, 19] see references and therein. The equality constrain indefinite linear least square problem (ILSE) was first proposed by Bojanczyk et al. in [2], which is a generalization of ILS. Suppose \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), \( B \in \mathbb{R}^{s \times n} \), \( d \in \mathbb{R}^s \), \( m \geq n \), and the signature matrix \( \Sigma_{pq} \) is defined by (1.2). The ILSE has the form

\[
\text{ILSE} : \quad \min_x (b - Ax)^\top \Sigma_{pq} (b - Ax) \quad \text{subject to} \quad Bx = d.
\]

The existence and uniqueness of the solution to ILSE is given in [2], i.e.,

\[
\text{rank}(B) = s, \quad x^\top (A^\top \Sigma_{pq} A)x > 0,
\]

where \( x \in \mathcal{N}(B) \) and \( \mathcal{N}(B) \) denotes the null space of \( B \). The rank condition guarantees there exists a solution to the equality constrain in (1.3), while the second one in (1.4), which means that \( A^\top \Sigma_{pq} A \) is positive definite on \( \mathcal{N}(B) \), ensures that the uniqueness of a solution to the ILSE problem. When (1.4) is satisfied, the unique solution \( x \) to the ILSE problem (1.3) can be determined by the following normal equation

\[
A^\top \Sigma_{pq}(b - Ax) = B^\top \xi, \quad Bx = d,
\]
where $\xi$ is a vector of Lagrange multipliers. On the other hand, the augmented system also defines the unique solution $x$ as follows

$$(1.6) \quad Ax := \begin{bmatrix} 0 & 0 & B^\top \\ 0 & \Sigma_{pq} & A \\ B^\top & A^\top & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ s \\ x \end{bmatrix} = \begin{bmatrix} d \\ b \\ 0 \end{bmatrix} := \mathbf{b},$$

where $s = \Sigma_{pq}r$, $r$ is the residual vector $r = b - Ax$ and $\lambda = -\xi$. As pointed in [2], when (1.4) holds, the coefficient matrix $A$ in (1.6) is invertible. For the numerical algorithms and theory for ILSE, we refer to the papers [17, 13, 14, 18] and etc.

Backward error analysis is important in numerical linear algebra, which can help us to examine the stability of numerical algorithms in matrix computation. Moreover, backward error can be used as the basis of effective stopping criteria for the iterative method for large scale problems. The concept of backward error can be traced to Wilkinson and others, see [Page 33] for details. Many researchers had concentrated on the backward error analysis for the linear least squares problem [20, 10, 22, 12, 6, 7], the scale total least squares (STLS) problem [4], and the equality constrained least squares (LSE) problem and the least squares problem over a sphere (LSS) [5, 16]. Since the formulae and bounds for backward errors for least squares problems are expensive to evaluate, the linearization estimate for them was proposed; see for [4, 6, 9, 15] and references therein. To our best knowledge, there are no works on the normwise backward error for ILSE. In this paper, we will introduce the normwise backward error for ILSE and derive its linearization estimate.

The paper is organized as follows. We define the normwise backward error for ILSE and derive its linearization estimate in Section 2. We do some numerical examples to show the effectiveness of the proposed linearization estimate for the normwise backward error in Section 3. At end, in Section 4, concluding remarks are drawn.

2. Main results. In this section, we will focus on the linearization estimate for the normwise backward error for ILSE. Assume that we have the computed solution $y$ to (1.3). There exits matrices and vectors $E, F, f$ and $g$, which are the perturbations on $A, B, b$ and $d$ respectively, such that the computed solution $y$ is the exact solution of the following perturbed ILSE problem

$$(2.7) \quad \min_{z} (b + f - (A + E)z)^\top \Sigma_{pq}(b + f - (A + E)z), \text{ subject to } (B + F)y = d + g.$$

There may have many possible perturbations satisfying (2.7). Thus the following perturbation set 

$$S_{\text{ILSE}}(y) = \{(E, f, F, g) \mid (A + E)^\top \Sigma_{pq}(b + f - (A + E)y) = (B + F)^\top \xi, (B + F)y = d + g\},$$

is introduced, where $\xi$ is the vector given in (1.5). Therefore the normwise backward error for $y$ can be defined as follows:

$$(2.8) \quad \mu_{\text{ILSE}} = \min \left\| \begin{bmatrix} E \\ \theta_{2}F \\ \theta_{3}g \end{bmatrix} \right\|_{F},$$

where $\| \cdot \|_{F}$ is Frobenius norm, $(E, f, F, g) \in S_{\text{ILSE}}(y)$ and $\theta_{1}, \theta_{2}$ and $\theta_{3}$ are positive parameters to give the weights to $f, F$ and $g$, respectively. However it seems that it is difficult to derive the explicit expression of $\mu_{\text{ILSE}}$ because of the non-linearity of (2.7) with respect to the perturbations of $E, F, f$ and $g$. In the following we will deduce the linearize estimate for $\mu_{\text{ILSE}}$ via linearizing (2.7) by dropping the higher order terms of the perturbations $E, F, f$ and $g$ in (2.7).
First, we rewrite (2.7) as follows:

\[
J(\xi) \begin{bmatrix} \text{vec}(E) \\ \theta_1 f \\ \theta_2 \text{vec}(F) \\ \theta_3 g \end{bmatrix} = \begin{bmatrix} B^T \xi - A^T \Sigma_{pq} t_y \\ d - By \end{bmatrix} - \begin{bmatrix} E^T \Sigma_{pq} [E \ \theta_1 f \ [y]^{1-y} ] \\ 0 \end{bmatrix},
\]

where \( \text{vec}(A) \) stacks the columns of \( A \) one by one,

\[
J(\xi) = \begin{bmatrix} I_n \otimes (r_y^T \Sigma_{pq}) - A^T \Sigma_{pq} (y^T \otimes I_m) & \theta_1^{-1} A^T \Sigma_{pq} & -\theta_2^{-1} (I_n \otimes \xi^T) \\ 0 & 0 & \theta_2^{-1} (y^T \otimes I_s) \end{bmatrix} - \begin{bmatrix} -\theta_3^{-1} I_s \end{bmatrix},
\]

the symbol \( \otimes \) is Kronecker product, \( I_n \) denotes the \( n \times n \) identity matrix and \( r_y = b - Ay \). Suppose \( r_y \neq 0 \), it is easy to verify that for any vector \( \xi \in \mathbb{R}^s \), the matrix \( J(\xi) \) is full rank. Let

\[
\tau(\xi) = \| J(\xi) \|_2, \quad \rho(\xi) = \left\| J(\xi)^\dagger \begin{bmatrix} B^T \xi - A^T \Sigma_{pq} t_y \\ d - By \end{bmatrix} \right\|_2,
\]

where \( \| \cdot \|_2 \) is the spectral norm of a matrix or 2-norm of a vector, and \( A^\dagger \) is Moore-Penrose inverse of \( A \). From the equation below

\[
\| J(\xi) \|_2 \leq \tau_0 := \left\| \begin{bmatrix} I_n \otimes (r_y^T \Sigma_{pq}) - A^T \Sigma_{pq} (y^T \otimes I_m) & \theta_1^{-1} A^T \Sigma_{pq} & 0 \\ 0 & 0 & -\theta_3^{-1} I_s \end{bmatrix}^\dagger \right\|_2,
\]

we know that \( \rho(\xi) \) is continuous with respect to \( \xi \). We define the linearized estimate

\[
(2.11) \quad \rho = \min_{\xi} \rho(\xi)
\]

for \( \mu_{\text{ILSE}} \). In the following theorem, we prove that \( \rho \) is an upper bound for \( \mu_{\text{ILSE}} \).

**Theorem 2.1.** If \( 4\tau_0 \rho \sqrt{\theta_1^{-2} + \|y\|_2^2} < 1 \), we have \( \mu_{\text{ILSE}} < 2\rho \).

**Proof.** Suppose \( \xi_0 \in \mathbb{R}^s \) such that \( \rho = \rho(\xi_0) \). Consider the following nonlinear system:

\[
J(\xi_0) \begin{bmatrix} \text{vec}(E) \\ \theta_1 f \\ \theta_2 \text{vec}(F) \\ \theta_3 g \end{bmatrix} = \begin{bmatrix} B^T \xi_0 - A^T \Sigma_{pq} t_y \\ d - By \end{bmatrix} - \begin{bmatrix} E^T \Sigma_{pq} [E \ \theta_1 f \ [y]^{1-y} ] \\ 0 \end{bmatrix},
\]

and the mapping \( \Gamma : \mathbb{R}^{(n+1)(m+s)} \rightarrow \mathbb{R}^{(n+1)(m+s)} \) defined by

\[
\Gamma \begin{bmatrix} \text{vec}(E) \\ \theta_1 f \\ \theta_2 \text{vec}(F) \\ \theta_3 g \end{bmatrix} = J(\xi_0)^\dagger \begin{bmatrix} B^T \xi_0 - A^T \Sigma_{pq} t_y \\ d - By \end{bmatrix} - J(\xi_0)^\dagger \begin{bmatrix} E^T \Sigma_{pq} [E \ \theta_1 f \ [y]^{1-y} ] \\ 0 \end{bmatrix}.
\]

From \( J(\xi_0)J(\xi_0)^\dagger = I \), we know that any fixed point of \( \Gamma \) is a solution to (2.12). Let

\[
\rho_1 = \frac{2\rho}{1 + \sqrt{1 - 4\theta_1^{-2} + \|y\|^2_2 \tau_0 \rho}}, \quad S_2 = \left\{ z \in \mathbb{R}^{(n+1)(m+s)} \mid \|z\|_2 \leq \rho_1 \right\},
\]
then $S_2$ is a convex and closed set of $\mathbb{R}^{(n+1)(m+s)}$. Moreover, for arbitrary $z = \begin{bmatrix} \text{vec}(E) \\ \theta_1 f \\ \theta_2 \text{vec}(F) \\ \theta_3 g \end{bmatrix} \in S_2$, we can deduce that

$$||\Gamma z||_2 \leq \rho + \tau_0 \sqrt{\theta_1^{-2} ||y||_2^2} ||z||_2 \leq \rho_1,$$

which means that the continuous mapping $\Gamma$ maps $S_2$ to $S_2$. From Brouwer fixed point principle, the mapping $\Gamma$ has a fixed point in $S_2$, then we prove that

$$\mu_{\text{ILSE}} \leq \rho_1 \leq 2\rho.$$ 

In the following, we will consider how to estimate $\rho$, because it is not easy to derive the explicit expression for $\rho$. From (2.10), we arrive at

$$\rho(\xi) \leq \tau_0 \left\| \begin{bmatrix} B^T \xi_0 - A^T \Sigma_{pq} r_y \\ d - B y \end{bmatrix} \right\|_2.$$ 

Apparently, the minimal value of the upper bound in (2.13) is attainable at

$$\xi_1 = (B^T)^{1} A^T \Sigma_{pq} r_y.$$ 

We have $\rho \leq \rho(\xi_1)$. From the above deduction, if $4\tau_0 \rho \sqrt{\theta_1^{-2} + ||y||_2^2} < 1$, it is not difficult to see that

$$\mu_{\text{ILSE}} < 2\rho(\xi_1).$$

On the other hand, we find the lower bound for $\mu_{\text{ILSE}}$ in the following theorem.

**Theorem 2.2.** If $r_y \neq 0$, then

$$\mu_{\text{ILSE}} \geq \frac{2\rho}{1 + \sqrt{1 + 4\tau_0 \sqrt{\theta_1^{-2} + ||y||_2^2}}}.$$ 

**Proof.** For the following nonlinear system:

$$J(\xi)^T J(\xi) \begin{bmatrix} \text{vec}(E) \\ \theta_1 f \\ \theta_2 \text{vec}(F) \\ \theta_3 g \end{bmatrix} = J(\xi)^T \begin{bmatrix} B^T \xi_0 - A^T \Sigma_{pq} r_y \\ d - B y \end{bmatrix} = J(\xi)^T \begin{bmatrix} E^T \Sigma_{pq} [E \theta_1 f] \begin{bmatrix} -y \\ 0 \end{bmatrix} \theta_1^{-1} \end{bmatrix},$$

we know that any solution to (2.9) is also a solution to (2.15). Because $J(\xi)$ is full row rank, any solution to (2.15) is a solution to (2.9). From $||J(\xi)^T J(\xi)||_2 = 1$ and (2.15), we can prove that

$$\rho(\xi) \leq \lambda + \tau_0 \sqrt{\theta_1^{-2} + ||y||_2^2} \lambda^2,$$

where $\lambda = \left\| \begin{bmatrix} E \\ \theta_2 F \\ \theta_3 g \end{bmatrix} \right\|_F$. Then, we deduce that

$$\lambda \geq \frac{2\rho(\xi)}{1 + \sqrt{1 + 4\tau_0 \sqrt{\theta_1^{-2} + ||y||_2^2}}}.$$
Because the function $f(t) = \frac{2t}{(1 + \sqrt{1 + 4\tau_0 \theta_1^{-2} + \|y\|_2^2})}$ is increasing with respect to $t (t \geq 0)$, we prove this theorem.

Combing Theorems 2.1 and 2.2, we have the following corollary.

**Corollary 2.3.** If $4\tau_0 \theta_1^{-2} + \|y\|_2^2 < 1$, then $\frac{2\rho}{1+\sqrt{2}} \leq \mu_{\text{ILSE}} \leq 2\rho$.

The above corollary indicates that when $\rho$ is small enough, then $\rho$ is a good estimation for $\mu_{\text{ILSE}}$. On the contrary, the next result shows that if $\rho$ is not small, then $\rho$ cannot be a good approximate solution.

**Theorem 2.4.** Suppose $y$ is an approximation solution to (1.3), and $x_{\text{ILSE}}$ is its exact solution, then the following inequality
\[
\|x_{\text{ILSE}} - y\|_2 \geq \frac{1}{\|\begin{bmatrix} A^\top \Sigma_{pq}A \\ B \end{bmatrix}\|_2} \left\| \begin{bmatrix} B\xi_1 - A^\top \Sigma_{pq}r_y \\ d - By \end{bmatrix} \right\|_2
\]

holds.

**Proof.** Since $x_{\text{ILSE}}$ is the exact solution to (1.3), there exits a vector $\xi_2 \in \mathbb{R}^s$ such that
\[
A^\top \Sigma_{pq}(b - Ax_{\text{ILSE}}) = B^\top \xi_2, \quad Bx_{\text{ILSE}} = d.
\]
Let $r_1 = B^\top \xi_2 - A^\top \Sigma_{pq}r_y$, and $r_2 = d - By$, then
\[
r_1 = A^\top \Sigma_{pq}A(y - x_{\text{ILSE}}), \quad r_2 = -B(y - x_{\text{ILSE}}).
\]
Then
\[
\left\| \begin{bmatrix} A^\top \Sigma_{pq}A \\ B \end{bmatrix}\right\|_2 \|y - x_{\text{ILSE}}\|_2 \geq \left\| \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \right\|_2 \geq \left\| \begin{bmatrix} B\xi_1 - A^\top \Sigma_{pq}r_y \\ d - By \end{bmatrix} \right\|_2,
\]
which completes the proof of this theorem.

Next, we analyze $\tau_0$ in (2.10). From the definition of $\tau_0$, we have the following result.

**Theorem 2.5.** With the notations above, we have $\tau_0 = \max \{\theta_3, \alpha^{-1}\}$, where
\[
\alpha = \sigma_{\text{min}} \left( \left[ I_n \otimes (r_y^\top \Sigma_{pq}) - A^\top \Sigma_{pq}(y^\top \otimes I_m) \right] \left[ \begin{array}{c} \theta_1^{-1}A^\top \Sigma_{pq} \end{array} \right] \right).
\]

The next theorem gives a lower bound of $\alpha$ and thus an upper bound of $\tau_0$.

**Theorem 2.6.** If $r_y \neq 0$, then $\alpha \geq \frac{\|r_y\|_2}{\sqrt{1 + \theta_1^2\|y\|_2^2}}$.

**Proof.** Noting $\Sigma_{pq}^2 = I_m$, since
\[
\left[ I_n \otimes (r_y^\top \Sigma_{pq}) - A^\top \Sigma_{pq}(y^\top \otimes I_m) \right] \left[ \begin{array}{c} \theta_1^{-1}A^\top \Sigma_{pq} \end{array} \right] = (\theta_1^{-2} + \|y\|_2^2) \left\{ (A - r_y y_0)^\top (A - r_y y_0) + \frac{\|r_y\|_2^2}{\theta_1^{-2} + \|y\|_2^2} I_n - (r_y y_0)^\top (r_y y_0) \right\} \\
\geq (\theta_1^{-2} + \|y\|_2^2) \left\{ \frac{\|r_y\|_2^2}{\theta_1^{-2} + \|y\|_2^2} I_n - (r_y y_0)^\top (r_y y_0) \right\},
\]

(here for two symmetric semi-positive matrix \(M\) and \(N\), \(M \geq N\) means that \(M - N\) is still semi positive), where \(y_0 = [1/(\theta_1^{-2} + \|y\|_2^2)]y\), we have
\[
\alpha^2 \geq (\theta_1^{-2} + \|y\|_2^2) \left\{ \frac{\|r_y\|_2^2}{\theta_1^{-2} + \|y\|_2^2} - \|r_y\|_2^2 \|y_0\|_2^2 \right\} = \frac{\|r_y\|_2^2}{1 + \theta_1^{-2} \|y\|_2^2}.
\]

3. Numerical examples. In this section we will test the effectiveness of the linearization estimate \(\rho\) for the normwise backward error \(\mu_{\text{ILSE}}\) of ILSE (1.3). All the computations are carried out using MATLAB 8.1 with the machine precision \(\varepsilon = 2.2 \times 10^{-16}\).

We adopt the method in [17] to construct the data. Let the matrix \(A\), given \(\kappa_A\), be generated as \(A = QDU\), where \(Q \in \mathbb{R}^{m \times m}\) is a \(\Sigma_{pq}\)-orthogonal matrix, i.e., such that \(Q^\top \Sigma_{pq} Q = \Sigma_{pq}\). \(D \in \mathbb{R}^{m \times n}\) is a diagonal matrix with decreasing diagonal values geometrically distributed between 1 and \(\kappa_A\), and \(U \in \mathbb{R}^{n \times n}\) is a random orthogonal matrix generated by the function \texttt{gallery}('qmult', \(n\)). Furthermore, \(A\) is normalized such that \(\|A\|_2 = 1\). The matrix \(B \in \mathbb{R}^{n \times n}\), given its condition number \(\kappa_B\), is formed by using MATLAB routine \(B = \texttt{gallery}('randsvd', [s, n], \kappa_B)\) with \(\|B\|_2 = 1\) and its singular values are geometrically distributed between 1 and \(1/\kappa_B\). We construct the random vectors \(b\) and \(d\) which are satisfied the standard Gaussian distribution for ILSE (1.3). For all the experiments, we choose \(n = 50\), \(s = 20\), \(p = 60\), \(q = 40\). For each generated data, we compute the solution via the augmented system (1.6). For the perturbations, we generate them as
\[
\Delta A = \varepsilon \cdot \Delta A_1, \quad \Delta B = \varepsilon \cdot \Delta B_1, \quad \Delta b = \varepsilon \cdot \Delta b_1 \cdot \|b\|_2, \quad \Delta d = \varepsilon \cdot \Delta d_1 \cdot \|d\|_2
\]
where each components of \(\Delta A_1 \in \mathbb{R}^{m \times n}\), \(\Delta B_1 \in \mathbb{R}^{s \times n}\), \(\Delta b_1 \in \mathbb{R}^m\) and \(\Delta d_1 \in \mathbb{R}^s\) satisfy the standard Gaussian distribution. Let the computed solution \(y\) be computed via solving the corresponding augmented system to the following perturbed ILSE problem
\[
\min \ ((b + \Delta b) - (A + \Delta A)y)^\top \Sigma_{pq} ((b + \Delta b) - (A + \Delta A)y), \quad \text{subject to} \quad (B + \Delta B)y = d + \Delta d.
\]

For the computed solution \(y\), its normwise backward error \(\mu_{\text{ILSE}}\) is defined by (2.8), and its linearization estimate \(\rho\) for the normwise backward error \(\mu_{\text{ILSE}}\) is given by (2.11). Because there is no explicit expression for \(\rho\), we use \(\rho(\xi_1)\) where \(x_1\) is given by (2.14) to approximate \(\mu_{\text{ILSE}}\). We always use the common choice \(\theta_1 = \theta_2 = \theta_3 = 1\) in (2.8). There is no explicit expression for the normwise backward error \(\mu_{\text{ILSE}}\). Since the perturbations \(\Delta A\), \(\Delta B\), \(\Delta b\) and \(\Delta d\) are known in advance, we can calculate the following quantity \(\mu_1\) to approximate \(\mu_{\text{ILSE}}\):
\[
\mu_1 = \left\| \begin{bmatrix} \Delta A & \Delta b \\ \Delta B & \Delta d \end{bmatrix} \right\|_F,
\]
and compare \(\mu_1\) with the linearization estimate \(\rho(\xi_1)\) to show the effectiveness of \(\rho(\xi_1)\). From the definition of the normwise backward error \(\mu\) defined in (2.8), it is easy to see that \(\mu \leq \mu_1\). Note that \(\mu\) may be much smaller that \(\mu_1\) because \(\mu\) is the smallest perturbation magnitude over the set of all perturbations \(\mathcal{S}_{\text{ILSE}}\). We test different choices of the perturbations magnitude \(\varepsilon\) and the parameter \(\kappa_A\), \(\kappa_B\). We report the numerical values of \(\mu_1\), \(\rho(\xi_1)\) and the residual norms \(\gamma\) and \(\tilde{\gamma}\) corresponding to the original and perturbed augmented system in Table 1.

From Table 1, it is observed that the residual norms \(\gamma\) and \(\tilde{\gamma}\) are always small regardless of different choices of \(\varepsilon\), \(\kappa_A\) and \(\kappa_B\). Thus the solutions \(x\) and \(y\) are acceptable in the sense of the residual norms for
the augmented system. The differences between $\mu_1$ and $\rho(\xi_1)$ are not too big. Most values of $\mu_1$ are one hundredfold of the corresponding of $\rho(\xi_1)$. However, we cannot conclude that $\rho(\xi_1)$ gives a bad estimation for $\mu_{\text{ILSE}}$ because $\mu_{\text{ILSE}}$ is the smallest perturbation magnitude to let the computed solution $y$ be the exact solution of the perturbed ILSE mathematically. The values of $\rho(\xi_1)$ are coincided with the perturbation magnitude $\varepsilon$, which indicates the linearized estimation $\rho(\xi_1)$ is effective.

4. Concluding Remarks. In this paper we studied the linearization estimate for the normwise backward error of the equality constrained indefinite least squares problem. The explicit sub-optimal linearization
estimate is given. We tested the derived sub-optimal linearization estimate through numerical examples, which showed that it is reliable and effective.

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