We construct finite element subspaces of the space of symmetric tensors with square-integrable divergence on a three-dimensional domain. These spaces can be used to approximate the stress field in the classical Hellinger–Reissner mixed formulation of the elasticity equations, when standard discontinuous finite element spaces are used to approximate the displacement field. These finite element spaces are defined with respect to an arbitrary simplicial triangulation of the domain, and there is one for each positive value of the polynomial degree used for the displacements. For each degree, these provide a stable finite element discretization. The construction of the spaces is closely tied to discretizations of the elasticity complex, and can be viewed as the three-dimensional analogue of the triangular element family for plane elasticity previously proposed by Arnold and Winther.

1. Introduction

The classical Lagrange finite element spaces provide natural simplicial finite element discretizations of the Sobolev space $H^1$. Similarly, various finite element spaces derived in the theory of mixed finite elements, such as the Raviart–Thomas and Nedelec spaces, provide the natural finite element discretizations of the spaces $H(\text{div})$ and $H(\text{curl})$. (These statements are made precise and treated in a uniform framework of the finite element exterior calculus in [9].) In this paper we consider the finite element discretization of the space $H(\text{div}, \Omega; S)$ consisting of square-integrable symmetric tensors (or, given a choice of coordinates, symmetric matrix fields) with square-integrable divergence. In the classical Hellinger–Reissner mixed formulation of the elasticity equations, the stress is sought in $H(\text{div}, \Omega; S)$ and the displacement in $L^2(\Omega; \mathbb{R}^n)$. The natural discretization of the latter space is evident—piecewise polynomial of some degree without interelement continuity constraints—but the development of an appropriate finite element subspace of $H(\text{div}, \Omega; S)$ to use with these is a long-standing and challenging problem. For plane elasticity, the known stable mixed finite element methods have mostly involved composite elements for the stress [9, 15, 16, 21]. To avoid these, other authors have modified the standard mixed variational formulation of elasticity to a formulation that uses general, rather than symmetric, tensors for the stress, with the symmetry imposed weakly; see [2, 5, 7, 17, 18, 19, 20]. Not until 2002 was a stable non-composite finite method for the classical mixed formulation of plane elasticity found [10]. This work can be seen as answering the question “what are
the natural finite element discretizations of $H(\text{div}, \Omega; S)$" in the case of two dimensions. In this paper we address this question for three dimensions.

Up until recently, there were no mixed finite elements for the Hellinger–Reissner formulation in three dimensions known to be stable. In [1], a partial analogue of the lowest order element in [10] was proposed and shown to be stable. Here we will derive the full analogue of the results of [10]. We construct a family of finite element subspaces of $H(\text{div}, \Omega; S)$ which, when used to discretize the stress in elasticity along with the obvious discontinuous piecewise polynomial discretization for the displacement, provide stable mixed finite elements for the Hellinger–Reissner principle. As in two dimensions, these spaces lead to a finite element subcomplex of the elasticity complex, related to it by commuting diagrams. The elasticity complex reveals a close connection between the finite element discretization of $H(\text{div}, \Omega; S)$ and finite element discretization of the space $H(\text{curl} \ curl^*, \Omega; S)$, involving a second order operator. A key step in our analysis is the identification of the conditions which are necessary for a piecewise smooth matrix field to belong to this space. These conditions are derived in Section 5 below.

We recall the standard mixed formulation for the elasticity equations. Let $\Omega$ be a contractible polyhedral domain in $\mathbb{R}^3$, occupied by a linearly elastic body $\varepsilon$ which is clamped on the boundary $\partial \Omega$, and let $S$ and $u$ denote the stress and displacement fields engendered by a force $f$ acting on the body. The matrix field $S$ and the vector field $u$ can be characterized as the unique critical point of the Hellinger-Reissner functional

$$\mathcal{J}(T, v) = \int_{\Omega} \left( \frac{1}{2} A T : T + \text{div} T \cdot v - f \cdot v \right) \, dx$$

over the space $H(\text{div}, \Omega; S) \times L^2(\Omega; \mathbb{R}^3)$. Here, $S$ is the six dimensional space of symmetric matrices and $S : T$ denotes the Frobenius product on $S$. The given compliance tensor $A = A(x) : S \to S$ is symmetric, and bounded and positive definite uniformly with respect to $x \in \Omega$. The divergence operator, div, is applied to a matrix field by taking the divergence of each row. Hence, this operator maps the space $H(\text{div}, \Omega; S)$ into $L^2(\Omega; \mathbb{R}^3)$.

A mixed finite element method determines an approximate stress field $S_h$ and an approximate displacement field $u_h$ as the unique critical point $(S_h, u_h)$ of the Hellinger–Reissner functional in a finite element space $\Sigma_h \times V_h \subset H(\text{div}, \Omega; S) \times L^2(\Omega; \mathbb{R}^3)$, where $h$ denotes the mesh size. Equivalently, $(S_h, u_h) \in \Sigma_h \times V_h$ solves the saddle point system

$$\int_{\Omega} (AS_h : T + \text{div} T \cdot u_h + \text{div} S_h \cdot v) \, dx = \int_{\Omega} f v \, dx, \quad (T, v) \in \Sigma_h \times V_h.$$

(1.1)

To ensure that the discrete system has a unique solution and that it provides a good approximation of the true solution the finite dimensional spaces $\Sigma_h$ and $V_h$ must satisfy the stability conditions from the theory of mixed finite element methods, see [11, 12]. As is well known, see for example [10], the following two conditions are sufficient:

- $\text{div} \Sigma_h \subset V_h$.
- There exists a linear operator $\Pi_h : H^1(\Omega; S) \to \Sigma_h$, bounded in $L(H^1; L^2)$ uniformly with respect to $h$, and such that $\text{div} \Pi_h S = \Pi_h^\prime \text{div} S$ for all $S \in H^1(\Omega; S)$, where $\Pi_h^\prime : L^2(\Omega; \mathbb{R}^3) \to V_h$ denotes the $L^2$-projection.

As mentioned above, the construction of finite element spaces which fulfill these two conditions has proved to be surprisingly hard. In this paper, we will derive a
family of finite element spaces $\Sigma_h$ and $V_h$ based on tetrahedral meshes, and show that they satisfy these two stability conditions. There is one member of the family for each polynomial degree $k \geq 1$. The space $V_h$ for the displacements is simply the space of all piecewise polynomial vector fields of degree at most $k$. In the lowest order case, $k = 1$, the space $\Sigma_h$ contains the full space of quadratic polynomials on each element, augmented by divergence-free polynomials of degree 3 and 4.

The local dimension of $\Sigma_h$ is 162, or 27 per component of stress on average. The analogous space in two dimensions, derived in [10], was of local dimension 24 (8 per component).

The complexity of the elements may very well limit their practical significance. However, we believe that the determination of the natural discretization of space $H(\text{div}, \Omega; \mathbb{S})$ provides important insight, both into the obstacles to the derivation of simpler methods, and for the design of alternative procedures, such as nonconforming methods.

This paper is organized as follows. After giving some preliminaries remarks in Section 2 we present the lowest order element and establish its properties in Section 3. A family of higher order elements is then presented in Section 4. Key to the analysis of these elements is the description of the polynomial space of symmetric matrix fields with vanishing divergence and vanishing normal traces on the boundary of a simplex $K$. The dimension of this space is derived in Section 7 based on preliminary results derived in Sections 5 and 6. Furthermore, an explicit basis for this space, necessary for the computational procedure, is also given in the lowest order cases. Finally, in Section 8 we summarize the results of our construction by presenting a discrete analogue of the elasticity complex.

2. Notation and preliminaries

We begin with some basic notation. If $K \subset \mathbb{R}^3$ is a tetrahedron, then $\Delta_2(K)$ denotes the set of the four 2–dimensional faces of $K$, $\Delta_1(K)$ the set of the six 1–dimensional edges, and $\Delta_0(K)$ the set of the four vertices. Furthermore, $\Delta(K)$ is the set of all subsimplices of $K$ (of dimension 0, 1, 2 or 3).

We let $\mathbb{M}$ be the space of $3 \times 3$ real matrices, and $\mathbb{S}$ and $\mathbb{K}$ the subspaces of symmetric and skew symmetric matrices, respectively. The operators $\text{sym} : \mathbb{M} \to \mathbb{S}$ and $\text{skw} : \mathbb{M} \to \mathbb{K}$ denote the symmetric and skew symmetric parts, respectively. Note that an element of the space $\mathbb{K}$ can be identified with its axial vector in $\mathbb{R}^3$ given by the map $\text{vec} : \mathbb{K} \to \mathbb{R}^3$:

$$\text{vec} \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

i.e., $\text{vec}^{-1}(v)w = v \times w$ for any vectors $v$ and $w$.

For any vector space $X$, we let $L^2(\Omega; X)$ be the space of square-integrable vector fields on $\Omega$ with values in $X$. For our purposes, $X$ will usually either be $\mathbb{R}$, $\mathbb{R}^3$, or $\mathbb{M}$, or some subspace of one of these. In the case $X = \mathbb{R}$, we will simply write $L^2(\Omega)$. The corresponding Sobolev space of order $k$, i.e., the subspace of $L^2(\Omega; X)$ consisting of functions with all partial derivatives of order less than or equal to $k$ in $L^2(\Omega; X)$, is denoted $H^k(\Omega; X)$, and its norm by $\| \cdot \|_k$. The space $H(\text{div}, \Omega; \mathbb{S})$ is defined by

$$H(\text{div}, \Omega; \mathbb{S}) = \{ T \in L^2(\Omega; \mathbb{S}) \mid \text{div} T \in L^2(\Omega; \mathbb{R}^3) \},$$
where the divergence of a matrix field is the vector field obtained by applying the divergence operator row–wise. For a vector field $v : \Omega \to \mathbb{R}^3$, $\text{grad} v$ is the matrix field with rows the gradient of each component, and the symmetric gradient, $\epsilon(v)$, is given by $\epsilon(v) = \text{sym grad} v$. Furthermore,

$$\text{curl} v = -2 \text{vec skw grad} v = \begin{pmatrix} \partial_3 v_2 - \partial_2 v_3 \\ -\partial_1 v_1 + \partial_3 v_1 \\ \partial_2 v_1 - \partial_1 v_2 \end{pmatrix}.$$ 

If we consider a linear coordinate transformation of the form $x = B \hat{x} + b$, with corresponding vector fields $v$ and $\hat{v}$ related by $v(x) = (B')^{-1} \hat{v}(\hat{x})$, then we have

$$\text{skw grad}_x v = (B')^{-1} (\text{skw grad}_\hat{x} \hat{v}) B^{-1}.$$ 

Here $B'$ denotes the transpose of $B$. In particular, if $B$ is orthogonal, i.e., $B'B = I$, then $v = B \hat{v}$ and

$$\text{skw grad}_x v = B (\text{skw grad}_\hat{x} \hat{v}) B'.$$

As for the divergence and the gradient operator, the operator curl acts on a matrix field by applying the ordinary curl operator to each row of the matrix. The operator curl$^\ast$ is the corresponding operator obtained by taking the curl of each column. Alternatively, we have curl$^\ast T = (\text{curl} T')'$ for any matrix field $T$. The second order operator curl$^2$ maps symmetric matrix fields into symmetric matrix fields. Let $\Xi : M \to M$ be the algebraic operator $\Xi T = T' - \text{tr}(T) I$, where $I$ is the identity matrix. Then $\Xi$ is invertible with $\Xi^{-1} T = T' - \text{tr}(T) I/2$. The following identities are useful:

$$\text{vec skw curl} T = -\frac{1}{2} \text{div} \Xi T, \quad T \in C^\infty(\Omega, M),$$

$$\text{curl} T = \Xi \text{grad} \text{vec} T, \quad T \in C^\infty(\Omega, K),$$

$$\text{tr} \text{curl} T = -2 \text{div} \text{vec skw} T, \quad T \in C^\infty(\Omega, M).$$

These formulas can be verified directly, but they are also consequences of the discussions given in [8, 9], cf. Section 4 of [8] or Section 11 of [9].

For $K \subset \mathbb{R}^3$ we let $P_k(K; \mathbb{X})$ be the space of polynomials of degree $k$, defined on $K$ and with values in $\mathbb{X}$. We write $P_k$ or $P_k(K)$ for $P_k(K; \mathbb{R})$. The de Rham complex has a polynomial analogue of the form

$$\mathbb{R} \xrightarrow{\text{grad}} P_{k+3} \xrightarrow{\text{curl}} P_{k+2}(K; \mathbb{R}^3) \xrightarrow{\text{div}} P_{k+1}(K; \mathbb{R}^3) \xrightarrow{\text{div}} P_k \to 0.$$ 

In fact, this complex is an exact sequence [9].

In recent years differential complexes have come to play a significant role in the design of mixed finite element methods [3, 10, 8, 9]. For the equations of elasticity, the relevant differential complex is the elasticity complex. In three space dimensions, the elasticity complex takes the form

$$\mathbb{T} \xrightarrow{\text{div}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega; \mathbb{S}) \xrightarrow{\text{curl}^\ast} C^\infty(\Omega; \mathbb{S}) \xrightarrow{\text{div}} C^\infty(\Omega; \mathbb{R}^3) \to 0,$$

where $\mathbb{T}$ is the six-dimensional space of infinitesimal rigid motions, i.e., the space of linear polynomial functions of the form $x \mapsto a + b \times x$ for some $a, b \in \mathbb{R}^3$. It is straightforward to verify that the elasticity complex is a complex, i.e., the composition of two successive operators is zero. In fact, if the domain $\Omega$ is contractible, then the elasticity complex is an exact sequence; see [8, 9].
An analogous complex with less smoothness is

\[ T \mapsto H^1(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl curl}^*} H(\text{curl curl}^*, \Omega; \mathbb{S}) \xrightarrow{\text{curl curl}^*} H(\text{div}, \Omega; \mathbb{S}) \xrightarrow{\text{div}} L^2(\Omega; \mathbb{R}^3) \to 0, \]

where \( H(\text{curl curl}^*, \Omega; \mathbb{S}) = \{ S \in L^2(\Omega; \mathbb{S}) \mid \text{curl curl}^* S \in L^2(\Omega; \mathbb{S}) \} \).

There is also a polynomial analogue of the elasticity complex. Let \( K \subset \mathbb{R}^3 \) be tetrahedron and \( k \geq 0 \). The polynomial elasticity complex is given by

\[(2.6) \quad T \mapsto \mathcal{P}_{k+4}(K; \mathbb{R}^3) \xrightarrow{\text{curl curl}^*} \mathcal{P}_{k+3}(K; \mathbb{S}) \xrightarrow{\text{curl curl}^*} \mathcal{P}_{k+1}(K; \mathbb{S}) \xrightarrow{\text{div}} \mathcal{P}_k(K; \mathbb{R}^3) \to 0.\]

This complex is an exact sequence. To prove the exactness, we first show that if \( S \) is a matrix field in \( \mathcal{P}_{k+3}(\Omega; \mathbb{S}) \), with \( \text{curl curl}^* S = 0 \), then \( S = \epsilon(u) \) for \( u = (u_1, u_2, u_3)^T \in \mathcal{P}_{k+4}(\Omega; \mathbb{R}^3) \). Clearly \( S = \epsilon(u) \) for \( u \in C^\infty(K; \mathbb{R}^3) \). It is enough to show that all second derivatives of \( u_k, k = 1, 2, 3 \), are in \( \mathcal{P}_{k+2}(\Omega; \mathbb{R}) \). This follows from the identity \( \partial_{ij} u_k = \partial_i \epsilon(u)_{jk} + \partial_j \epsilon(u)_{ik} - \partial_k \epsilon(u)_{ij} \). Next, we show that if \( S \in \mathcal{P}_{k+1}(\Omega; \mathbb{S}) \), and \( \text{div} S = 0 \), then \( S = \text{curl curl}^* T \) for some \( T \in \mathcal{P}_{k+3}(\Omega; \mathbb{S}) \).

First we observe that since \( \text{div} S = 0 \) it follows from the fact that (2.6) is exact that \( S = \text{curl} U \) for some \( U \in \mathcal{P}_{k+2}(K; \mathbb{M}) \). Furthermore, since \( S \) is symmetric it follows from (2.2) that \( \text{div} \Xi U = 0 \), and as a consequence, using (2.5) once more, we obtain that \( \Xi U = \text{curl} T \) for some \( T \in \mathcal{P}_{k+3}(K; \mathbb{M}) \), or \( S = \text{curl} U = \epsilon(\Xi^{-1} \text{curl} T) \).

However, by (2.3) we have \( \epsilon(\Xi^{-1} \text{curl} T) = \text{curl} \text{grad} \text{vec} \text{skw} T = \text{curl} \text{grad} \text{vec} \text{skw} T \). Hence, we can take \( T \in \mathcal{P}_{k+3}(K; \mathbb{S}) \). Finally, we observe that if \( T \) is symmetric, then (2.4) implies that \( \text{tr} \text{curl} T = 0 \), and therefore \( S = \epsilon(\Xi^{-1} \text{curl} T) = \text{curl} \text{curl}^* T \).

To establish the surjectivity of the last map, one can use the fact that \( \dim \mathcal{P}_k = (k+1)(k+2)(k+3)/6 \) to verify that the alternating sum of the dimensions of the spaces in the sequence is zero. The same arguments show that (2.6) is exact for \( k = -1, -2, \) or \(-3 \), if \( \mathcal{P}_j \) is interpreted as the zero space for \( j < 0 \).

Let \( \{ \mathcal{T}_h \} \) denote a family of triangulations of \( \Omega \) by tetrahedra with diameter bounded by \( h \). We assume that the intersection of any two tetrahedra in \( \mathcal{T}_h \) is either empty or a common subsimplex of each. The family \( \{ \mathcal{T}_h \} \) is also assumed to be shape regular in the sense that the ratio of the radii of the circumscribed and inscribed spheres of all the tetrahedra can be bounded by a fixed constant. Furthermore, we will use the notation \( \Delta_j(\mathcal{T}_h) \), for \( j = 0, 1, 2 \), to denote the set of vertices, edges, and faces, respectively, associated with the mesh \( \mathcal{T}_h \). In Section 3 we will define a family of finite element spaces \( \Sigma_h \subset H(\text{div}, \Omega; \mathbb{S}) \) and \( \mathcal{V}_h \subset L^2(\Omega; \mathbb{R}^3) \) for the elasticity problem consisting of piecewise polynomial spaces with respect to \( \mathcal{T}_h \) of arbitrarily high polynomial order. However, we will first consider the lowest order case of this family in Section 3 below. All our spaces will have the property that \( \text{div} \Sigma_h \subset \mathcal{V}_h \). Furthermore, we will identify a corresponding projection operator \( \Pi_h : H^1(\Omega; \mathbb{S}) \to \Sigma_h \) satisfying the commutativity relation

\[(2.7) \quad \text{div} \Pi_h T = \Pi_h^\text{V} \text{div} T, \quad T \in H^1(\Omega; \mathbb{S}), \]

and the bound

\[(2.8) \quad \|\Pi_h T\|_0 \leq C\|T\|_1, \quad T \in H^1(\Omega; \mathbb{S}), \]

with constant \( C \) independent of \( h \). Here \( \Pi_h^\text{V} : L^2(\Omega; \mathbb{R}^3) \to \mathcal{V}_h \) is the \( L^2 \) projection. It is a consequence of the general error bounds derived in [13], cf. also [10], that the properties above imply that \( (\Sigma_h, \mathcal{V}_h) \) is a stable pair of elements for the
we obtain that each edge for both faces meeting the edge. By the third set of degrees of freedom
 equals 0 at the vertices, the second set of degrees of freedom imply that
 holds, with a constant c independent of h. Here (S, u) is the unique critical point of the
 Hellinger-Reissner functional over \( H(\text{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^2) \) and \((S_h, u_h) \in \Sigma_h \times V_h\) the corresponding finite element solution. In addition, \( \text{div} S_h = \Pi_h^\tau \text{div} S \).

3. The lowest order element

We first describe the restriction of the lowest order spaces \( \Sigma_h \) and \( V_h \) to a single tetrahedron \( K \in \mathcal{T}_h \). Define

\[
\Sigma_K = \{ T \in \mathcal{P}_4(K; \mathbb{S}) \mid \text{div} T \in \mathcal{P}_1(K; \mathbb{R}^3) \}, \quad V_K = \mathcal{P}_1(K; \mathbb{R}^3).
\]

The space \( V_K \) has dimension 12 and a complete set of degrees of freedom is given by the zero and first order moments with respect to \( K \). The space \( \Sigma_K \) has dimension at least 162 since the dimension of \( \mathcal{P}_4(K; \mathbb{S}) \) is 210 and the condition \( \text{div} T \in \mathcal{P}_1(K; \mathbb{R}^3) \) represents 48 linear constraints. We will show that \( \dim \Sigma_K = 162 \) by exhibiting 162 degrees of freedom which determine the elements uniquely. Define

\[
\mathcal{M}(K) = \{ T \in \mathcal{P}_4(K; \mathbb{S}) \mid \text{div} T = 0, \ Tn = 0 \text{ on } \partial K \}.
\]

It will follow from Theorem 2.2 below that the dimension of \( \mathcal{M}(K) \) is 6. Furthermore, in Section 7 we will also give an explicit basis for this space. Using this basis, we can state the 162 degrees of freedom for the space \( \Sigma_K \).

**Lemma 3.1.** A matrix field \( T \in \Sigma_K \) is uniquely determined by the following degrees of freedom

1. \( \text{the values of } T \text{ at the vertices of } K, \ 4 \times 6 = 24 \text{ degrees of freedom} \),
2. \( \text{for each edge } e \in \Delta_1(K) \text{ with unit tangent vector } s \text{ and linearly independent normal vectors } n_- \text{ and } n_+, \text{ the constant, linear and quadratic moments over } e \text{ of } s’Tn_-, s’Tn_+, n’_Tn_-, n’_Tn_+, n’_Tn_+, \ 6 \times 3 \times 5 = 90 \text{ degrees of freedom} \),
3. \( \text{for each face } f \in \Delta_2(K), \text{ with normal } n, \text{ the constant and linear moments over } f \text{ of } Tn, \ 4 \times 3 \times 3 = 36 \text{ degrees of freedom} \),
4. \( \text{the average of } T \text{ over } K, \ 6 \text{ degrees of freedom} \),
5. \( \text{the value of the moments } \int_K T : U \ dx, \ U \in \mathcal{M}(K), \ 6 \text{ degrees of freedom} \).

**Proof.** We assume that all degrees of freedom vanish and show that \( T = 0 \). Since \( T = 0 \) at the vertices, the second set of degrees of freedom imply that \( Tn = 0 \) on each edge for both faces meeting the edge. By the third set of degrees of freedom we obtain that \( Tn = 0 \) on each face of \( K \). For \( v = \text{div } T \in \mathcal{P}_1(K, \mathbb{R}^3) \) we have

\[
\int_K v^2 \ dx = -\int_K T : \epsilon(v) \ dx + \int_{\partial K} Tn \cdot v \ dx_f = -\int_K T : \epsilon v \ dx = 0
\]

by the fourth set of degrees of freedom. Here and below, \( dx_f \) denotes the surface measure on \( \partial K \). We conclude that \( \text{div} T = 0 \), and, by the last set of degrees of freedom, that \( T = 0 \). \( \square \)

We now describe the finite element spaces on the triangulation \( \mathcal{T}_h \). We denote by \( V_h \) the space of vector fields which belong to \( V_K \) for each \( K \in \mathcal{T}_h \) and by \( \Sigma_h \) the space of matrix fields which belong piecewise to \( \Sigma_K \), and with the continuity
conditions induced by the degrees of freedom. In particular, for \( T \in \Sigma_h \), the normal components \( Tn \) are continuous across all faces \( f \in \Delta_2(\mathcal{T}_h) \) and, hence, \( \Sigma_h \subset H(\text{div}; \Omega; \mathbb{S}) \). In addition, if \( T \in \Sigma_h \), \( e \in \Delta_1(\mathcal{T}_h) \), and \( s \) and \( n \) are vectors which are tangential and normal to \( e \), respectively, then \( s'Tn \) is continuous on \( e \).

It remains to define an interpolation operator \( \Pi_h : H^1(\Omega; \mathbb{S}) \to \Sigma_h \) which satisfies (2.7) and (2.8). Because of the vertex and edge degrees of freedom, the canonical interpolation operator for \( \Sigma_h \), \( \Pi_h^C \), defined directly from the degrees of freedom, is not bounded on \( H^1(\Omega; \mathbb{S}) \). In order to overcome this difficulty we introduce the operator \( \Pi_h^0 : H^1(\Omega; \mathbb{S}) \to \Sigma_h \) defined from the degrees of freedom above, but where the vertex and edge degrees of freedom are set equal to zero, i.e., we have

\[
\Pi_h^0 T(x) = 0, \quad x \in \Delta_0(\mathcal{T}_h),
\]

\[
\int_e \Pi_h^0 T n \cdot v \, ds = 0, \quad e \in \Delta_1(\mathcal{T}_h), \ v \in \mathcal{P}_2(e; \mathbb{R}^3), \ n \in e^\perp,
\]

\[
\int_f (T - \Pi_h^0 T) n \cdot v \, dx_f = 0, \quad f \in \Delta_2(\mathcal{T}_h), \ v \in \mathcal{P}_1(f; \mathbb{R}^3),
\]

\[
\int_K (T - \Pi_h^0 T) \, dx = 0, \quad K \in \mathcal{T}_h,
\]

\[
\int_K (T - \Pi_h^0 T) : U \, dx = 0, \quad K \in \mathcal{T}_h, \ U \in \mathcal{M}(K).
\]

The commutativity property (2.7) for \( \Pi_h^0 \) follows from (3.3) and (3.4) since

\[
\int_K (\text{div} \Pi_h^0 T - \text{div} T) \cdot v \, dx = -\int_K (\Pi_h^0 T - T) : \epsilon(v) \, dx + \int_{\partial K} (\Pi_h^0 T - T)n \cdot v \, dx_f = 0.
\]

The uniform boundedness (2.8) can be seen from a standard scaling argument using the matrix Piola transform. Let \( \hat{K} \) be a fixed reference tetrahedron and \( F = F_K : \hat{K} \to K \) be an affine isomorphism of the form \( F\hat{x} = B\hat{x} + b \). Given a matrix field \( \hat{T} : \hat{K} \to \mathbb{S} \), define \( T : K \to \mathbb{S} \) by the matrix Piola transform \( T(x) = BT(\hat{x})B^T \), with \( x = F\hat{x} \). Using \( \text{div} T(x) = B \text{div} \hat{T}(\hat{x}) \), it is easy to verify that \( T \in \Sigma_K \) if and only if \( \hat{T} \in \Sigma_{\hat{K}} \). Furthermore, as in [10], a scaling argument can be used to verify the uniform boundedness condition (2.8) for the operator \( \Pi_h^0 \). We can therefore conclude that the operator \( \Pi_h^0 \) satisfies the two conditions (2.7) and (2.8). However, the operator \( \Pi_h^0 \) lacks good approximation properties. Therefore, in order to obtain error estimates from the general bounds (2.9) and (2.10) a more accurate interpolation operator is needed.

Consider the modified interpolation operator \( \Pi_h : H^1(\Omega; \mathbb{S}) \to \Sigma_h \) of the form

\[
\Pi_h = \Pi_h^0 (I - R_h) + R_h,
\]

where \( R_h : L^2(\Omega; \mathbb{S}) \to \Sigma_h \) is the Clément operator onto the continuous piecewise quadratic subspace of \( \Sigma_h \) [13]. This operator satisfies the bounds

\[
\|R_h T - T\|_j \leq ch^{m-j} \|T\|_m, \quad 0 \leq j \leq 1, \quad j \leq m \leq 3.
\]

As a consequence of this bound, and the boundedness (2.8) of \( \Pi_h^0 \), we obtain the estimate

\[
\|\Pi_h T - T\|_0 \leq c h^m \|T\|_m, \quad 1 \leq m \leq 3
\]

for the interpolation error. Furthermore, since \( \Pi_h^0 \) satisfies (2.7) and \( \Pi_h^0 \text{div} R_h = \text{div} R_h \), we conclude that \( \Pi_h \) satisfies (2.7).
We also recall that the projection operator $\Pi_h^V : L^2(\Omega; \mathbb{R}^3) \rightarrow V_h$ satisfies the error estimate
\[(3.8) \quad \| \Pi_h^V v - v \|_0 \leq c h^m \| v \|_m, \quad 0 \leq m \leq 2.\]
The estimates (3.7) and (3.8), combined with the basic error bounds (2.9) and (2.11), and the fact that $\text{div} \, S_h = \Pi_h^V \text{div} \, S$, imply the following error estimates for the finite element method generated by $\Sigma_h \times V_h$.

**Theorem 3.2.** Let $(S, u)$ denote the unique critical point of the Hellinger-Reissner functional over $H(\text{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^2)$ and let $(S_h, u_h)$ be the unique critical point over $\Sigma_h \times V_h$. Then
\[
\| S - S_h \|_0 \leq c h^m \| S \|_m, \quad 1 \leq m \leq 3,
\]
\[
\| \text{div} \, S - \text{div} \, S_h \|_0 \leq c h^m \| \text{div} \, S \|_m, \quad 0 \leq m \leq 2,
\]
\[
\| u - u_h \|_0 \leq c h^m \| u \|_{m+1}, \quad 1 \leq m \leq 2.
\]

**Remark.** In general it seems not possible to lower the polynomial degree of the stress space $\Sigma_h$ introduced above. However, as in [10], a minor simplification is possible. On each tetrahedron $K \in T_h$ we take the restricted displacement space $V_K$ to be the rigid motions $T \in T$. The proof of the fact that these degrees of freedom are unisolvent for $\tilde{\Sigma}_h$ by (4) in Lemma 3.1. The proof of the fact that these degrees of freedom are unisolvent for $\Sigma_K$ follows by a simple modification of the proof of Lemma 3.1 above. Just observe that if $v = \text{div} \, T \in T$ then $\epsilon(v) = 0$. However, the simplified element is less accurate, since the stress space lacks some quadratics, and the displacement space some linears. Instead of the error estimates given in Theorem 3.2 we obtain at most $O(h^2)$ convergence for $\| S - S_h \|_0$ and at most first order convergence for $\| \text{div} (S - S_h) \|_0$ and $\| u - u_h \|_0$.

4. A FAMILY OF HIGHER ORDER ELEMENTS

In this section we describe a family of stable element pairs, one for each degree $k \geq 1$. The lowest order case $k = 1$ is the one treated above. We first describe the elements on a single tetrahedron. Define
\[
\Sigma_K = \{ T \in \mathcal{P}_{k+3}(K; \mathbb{S}) \mid \text{div} \, T \in \mathcal{P}_k(K; \mathbb{R}^3) \}, \quad V_K = \mathcal{P}_k(K; \mathbb{R}^3).
\]
Then
\[
\dim V_K = 3 \binom{k+3}{3} = \frac{(k+3)(k+2)(k+1)}{2},
\]
\[
\dim \Sigma_K \geq d_k := \dim \mathcal{P}_{k+3}(K; \mathbb{S}) - [\dim \mathcal{P}_{k+2}(T; \mathbb{R}^3) - \dim \mathcal{P}_k(T; \mathbb{R}^3)]
\]
\[
= 6 \binom{k+6}{3} - 3 \binom{k+5}{3} + 3 \binom{k+3}{3} = k^3 + 12k^2 + 56k + 93.
\]
Notice that the space $\epsilon(\mathcal{P}_k(K; \mathbb{R}^3))$ has dimension $(k+3)(k+2)(k+1)/2 - 6$. Analogous to the lowest order case, we define the space
\[
\mathcal{M}_k(K) = \{ T \in \mathcal{P}_k(K; \mathbb{S}) \mid \text{div} \, T = 0, \quad T n = 0 \quad \text{on} \ \partial K \}.
\]
We will prove in Section 7, Theorem 7.2, that \( \dim M_k(K) \) is \((k+2)(k-2)(k-3)/2\) for \( k \geq 4 \).

The degrees of freedom for \( V_h \) are the moments of degree less than or equal to \( k \) with respect to \( K \). A unisolvent set of degrees of freedom for \( \Sigma_K \) are given by

1. the values of \( T \) at the vertices of \( K \), \( 4 \times 6 = 24 \) degrees of freedom,
2. for each edge \( e \in \Delta_1(K) \) with unit tangent vector \( s \) and linearly independent normal vectors \( n_- \) and \( n_+ \), the moments of degree at most \( k+1 \) over \( e \) of \( s'Tn_- \), \( s'Tn_+ \), \( n'_-Tn_- \), \( n'_+Tn_+ \), \( 6 \times (k+2) \times 5 = 30k+60 \) degrees of freedom,
3. for each \( f \in \Delta_2(K) \) with normal vector \( n \), the moments of degree at most \( k \) over \( f \) for \( Tn \), \( 3 \times 4 \times (k+2)(k+1)/2 = 6k^2+18k+12 \) degrees of freedom,
4. \( \int_K T : U \, dx \), \( U \in \mathcal{E}(V_h) \), \( (k+3)(k+2)(k+1)/2 - 6 \) degrees of freedom,
5. \( \int_K T : U \, dx \), \( U \in M_{k+3}(K) \), \( (k+5)(k+1)/2 \) degrees of freedom.

The proof that this set of functionals is unisolvent for the space \( \Sigma_K \) is almost identical to the lowest order case, and it is easily checked that their numbers sum up to \( d_k \). Hence, we have shown that \( \dim \Sigma_k = d_k \). Furthermore, in Section 7 we will give an explicit basis for the space \( M_{k}(K) \) when \( k = 4 \) and \( k = 5 \).

The finite element space \( V_h \subset L^2(\Omega; \mathbb{R}^3) \) consists of all vector fields which belong to \( \mathcal{P}_k(K; \mathbb{R}^3) \) for each \( K \in \mathcal{T}_h \), while the corresponding stress space \( \Sigma_k \) is the space of matrix fields which belong piecewise to \( \Sigma_K \), and with the continuity conditions induced by the degrees of freedom. In particular, this implies that the normal components \( Tn \), for \( T \in \Sigma_k \), are continuous over all faces in \( \Delta_2(T_h) \). Hence, as in the lowest order case we have that \( \Sigma_h \subset H(\text{div}, \Omega; \mathbb{S}) \).

The \( L^2 \) projection \( \Pi^h \) onto \( V_h \) satisfies the estimate

\[
\| \Pi^h v - v \|_0 \leq ch^m \| v \|_m, \quad 0 \leq m \leq k + 1.
\]

We also introduce the Clément interpolant \( R_h : L^2(\Omega; \mathbb{S}) \to \Sigma_h \) defined as the \( \mathbb{S} \)-valued version of the standard scalar Clément interpolator into continuous piecewise polynomials of order \( k+1 \). Hence, the operator \( R_h \) satisfies

\[
\| R_h T - T \|_j \leq ch^{m-j} \| T \|_m, \quad 0 \leq j \leq 1, \; j \leq m \leq k + 2.
\]

Furthermore, we define the modified canonical interpolation operator \( \Pi^h \) by (3.6), where the operator \( \Pi^0_h \) is defined in complete analogy with the lowest order case, by setting the degrees freedom associated with \( \Delta_0(T_h) \) and \( \Delta_1(T_h) \) equal to zero. Then the operator \( \Pi^h \) satisfies (2.7) and (2.8), and the error bound

\[
\| \Pi^h T - T \|_0 \leq ch^m \| T \|_m, \quad 1 \leq m \leq k + 2.
\]

As above, the interpolation estimates (4.1) and (4.2), and the error bounds (2.9) and (2.11), leads to the following error estimates.

**Theorem 4.1.** Let \( (S, u) \) denote the unique critical point of the Hellinger-Reissner functional over \( H(\text{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^2) \) and let \( (S_h, u_h) \) be the unique critical point over \( \Sigma_h \times V_h \). Then

\[
\| S - S_h \|_0 \leq ch^m \| S \|_m, \quad 1 \leq m \leq k + 2,
\]

\[
\| \text{div} S - \text{div} S_h \|_0 \leq ch^m \| \text{div} S \|_m, \quad 0 \leq m \leq k + 1,
\]

\[
\| u - u_h \|_0 \leq ch^m \| u \|_{m+1}, \quad 1 \leq m \leq k + 1.
\]
5. Some properties of vector fields and matrix fields

In order to perform a more detailed analysis of the spaces $\mathcal{M}_k(K)$ and the finite element spaces $\Sigma_h$ we will need some basic properties of vector fields and matrix fields. These properties will be reviewed in the present section.

5.1. Identities for vector fields. Let $n$ denote a fixed unit vector in $\mathbb{R}^3$, $P_n = nn'$ the orthogonal projection onto $\mathbb{R}^n$, $f = n^\perp$ the plane orthogonal to $n$, and $Q_n = I - P_n$ is the orthogonal projection onto $f$. Furthermore, set

$$C_n = \begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix} = -\text{vec}^{-1} n,$$

so that $C_nv = v \times n$. The following identities are easily checked:

$$C_n' = -C_n, \quad C_n^2 = -Q_n, \quad C_nP_n = 0, \quad C_nQ_n = Q_nC_n = C_n.$$

For any vector field $v = (v_1, v_2, v_3)'$ in $\mathbb{R}^3$, we obviously have $v = P_nv + Q_nv$ and

$$\text{curl} v = \text{curl} P_nv + \text{curl} Q_nv = P_n\text{curl}P_nv + P_n\text{curl}Q_nv + Q_n\text{curl}P_nv + Q_n\text{curl}Q_nv.$$

It is elementary to verify that

$$P_n\text{curl}P_nv = 0, \quad Q_n\text{curl}Q_nv = C_n\frac{\partial v}{\partial n},$$

if $n = e_3 = (0,0,1)'$. In view of the transformation formula (2.1), these identities hold for an arbitrary unit vector $n$. Furthermore, we define

$$\text{rot}_f v = P_n\text{curl}P_nv = P_n\text{curl}Q_nv = -(\text{div} C_nv)n, \quad \text{curl}_f v = Q_n\text{curl}P_nv.$$

With this notation we obtain the decomposition

$$\text{curl} v = \text{rot}_f v + \text{curl}_f v + C_n\frac{\partial v}{\partial n}.$$  \hspace{1cm} (5.1)

We also define the tangential gradient $\text{grad}_f \phi = Q_n \text{grad} \phi$, for a scalar field $\phi$. For a vector field $v$, $\text{grad}_f v = (\text{grad} v)Q_n$ is the matrix field with rows equal to the tangential gradients of the components of $v$, and we let

$$\epsilon_f(v) = \frac{1}{2}\{\text{grad}_f(Q_nv) + [\text{grad}_f(Q_nv)]'\} = Q_n \epsilon(v)Q_n$$

be the tangential part of the symmetric gradient. Note that the definitions of $\text{grad}_f v$, $\epsilon_f(v)$, $\text{curl}_f v$, and $\text{rot}_f v$ do not depend on the choice of the unit normal $n$ to $f$. The identity

$$\text{curl}_f v = -C_n \text{grad}(n'v) = -C_n \text{grad}_f(n'v)$$  \hspace{1cm} (5.3)

can be easily verified in the special case $n = e_3$, and holds in general.

5.2. Identities for matrix fields. We extend the operators curl, curl$_f$ and rot$_f$ to act on and yield $3 \times 3$ matrix fields by applying the vector operations row-wise. More precisely, $\text{rot}_f S = (\text{curl} S)P_n = (\text{curl} SQ_n)P_n$ and $\text{curl}_f S = (\text{curl} SP_n)Q_n$. We notice that, for any constant matrix $A$, $\text{curl} AS = A\text{curl} S$. We also recall that $\text{curl}' S = (\text{curl} S')'$ is the corresponding operator obtained by applying the curl operation to each column. The corresponding column operators rot$^c_f$ and curl$^c_f$ are defined similarly, i.e., $\text{rot}_f^c S = P_n\text{curl}' S = P_n\text{curl}' Q_n S$ and $\text{curl}_f^c S = Q_n\text{curl}' P_n S$. For a given row vector $v$, $\text{grad}_f v = (\text{grad}_f v')' = Q_n(\text{grad} v')'$, which is the matrix whose columns are the tangential gradients of the components of $v$. 

We now extend the decomposition (5.2) to curl and curl*. It is easy to see that 
$C_n S$ results in $C_n$ applied to the columns of $S$, while $-SC_n$ is $C_n$ applied row-wise. 
It follows that

$$
(5.4) \quad \text{curl} \, S = \text{curl}_f \, S + \partial S \partial n C_n,
$$

and

$$
(5.5) \quad \text{curl}^* \, S = \text{curl}^*_f \, S + \partial n \, S + C_n \partial S \partial n,
$$

where $\partial S \partial n$ is obtained by taking the directional derivative of each component. 
Furthermore, the identities

$$
(5.6) \quad \text{curl} \, f = (\text{grad} \, f) \, n \, C_n, \quad \text{curl}^*_f \, S = -C_n \text{grad}^* \, n' \, S
$$

are just matrix analogues of (5.3). Note also that from the definitions of the 
operators rot$^*_f$ and rot$^*$ we have

$$
(5.7) \quad P_n(\text{curl} \, \text{curl}^* \, S) P_n = P_n(\text{rot} \, \text{curl}^* \, S) = \text{rot} \, (P_n \, \text{curl}^* \, S)
$$

$$
(5.8) \quad = \text{rot} \, \text{rot}^*_f \, S = \text{rot} \, \text{rot}^*_f (Q_n S Q_n).
$$

We will also need exact sequences relating spaces of functions defined on a two 
dimensional space. Let $f = n^+$. In analogy with (2.10) the following two dimensional 
complexes are exact:

$$
(5.9) \quad T_f \hookrightarrow \mathcal{P}_{k+3}(f; Q_n \mathbb{R}^3) \xleftarrow{\epsilon_f} \mathcal{P}_{k+2}(f; Q_n S Q_n) \xrightarrow{\text{rot} \, \text{rot}^*_f} \mathcal{P}_k(f; \mathbb{R}^n) \rightarrow 0,
$$

$$
(5.10) \quad \mathcal{P}_1(f; \mathbb{R}) \hookrightarrow \mathcal{P}_{k+3}(f; \mathbb{R}) \xleftarrow{\text{grad}_f \, \text{grad}^*_f} \mathcal{P}_{k+1}(f; Q_n S Q_n) \xrightarrow{\text{rot} \, \text{rot}^*_f} \mathcal{P}_k(f; Q_n S P_n) \rightarrow 0.
$$

Here, $T_f$ is the 3-dimensional space of vector fields on $f$ of the form $v(x) = Q_n w(x)$
for some $w \in T$.

5.3. Integration by parts. Above we discussed the operators $P_n$, $Q_n$, and $C_n$
with respect to a fixed linear space $f$ with a unit normal vector $n$. If $\Omega$ is a bounded
subset of $\mathbb{R}^3$, with a piecewise smooth boundary $\partial \Omega$, we define these operators on
$\partial \Omega$ with respect to the tangent space $f = f(x)$ and the outer unit normal vector
$n = n(x)$ for $x \in \partial \Omega$. Furthermore, the tangential differential operators grad$^*$, 
$\text{curl}_f$, rot$^*_f$, $\epsilon_f$ and so on are then defined pointwise on $\partial \Omega$.

If $v$ is a smooth vector field on $\Omega$, then

$$
\int_{\Omega} v \cdot \text{curl} \, w \, dx = \int_{\Omega} \text{curl} \, v \cdot w \, dx - \int_{\partial \Omega} C_n v \cdot w \, dx_f,
$$

where $n$ denotes the outward unit normal to $\Omega$, and $dx_f$ is the surface measure on
$\partial \Omega$. Moreover, if $v$ and $w$ are smooth vector fields on $\partial \Omega$, and $f$ is a piecewise
smooth submanifold, then we have

$$
\int_{f} \text{rot} \, v \cdot w \, dx_f = \int_{f} v \cdot \text{curl} \, w \, dx_f - \int_{\partial f} (s') (v) (n' \, w) \, ds,
$$

where $ds$ denotes the arc length measure on $\partial f$.

We next extend the previous integration by parts formula to matrix fields. Let
$S$ and $T$ be two smooth matrix fields on $\mathbb{R}^3$ not necessarily symmetric. We have

$$
(5.11) \quad \int_{\Omega} S : \text{curl} \, T \, dx = \int_{\Omega} \text{curl} \, S : T \, dx + \int_{\partial \Omega} S C_n : T \, dx_f
$$

$$
= \int_{\Omega} \text{curl} \, S : T \, dx + \int_{\partial \Omega} S C_n : T \, dx_f
$$
Lemma 5.1.

By parts. \( \Omega \)

Given a symmetric matrix field \( S \) define \( \Lambda_f(S) : f \rightarrow Q_nSQ_n \) by

\[
\Lambda_f(S) = 2 \epsilon_f(Sn) - Q_n \partial_n SQ_n,
\]

where \( \partial_n S := \partial S / \partial n \). Hence, \( \Lambda_f(S) \) is a symmetric matrix field defined on \( f \). If \( T = \epsilon(v) \), where \( v \) is a vector field then we have

\[
2 \epsilon_f(Tn) = \text{grad}_f \text{grad}_f^*(n'v) + Q_n \partial_n \epsilon(v)Q_n.
\]

Hence, we obtain that

\[
(5.13) \quad \Lambda_f(\epsilon(v)) = \text{grad}_f \text{grad}_f^*(n'v).
\]

The tangential–normal components of the matrix field \( \text{curl} \text{curl}^* S \) on \( f \) can be expressed in terms of \( \Lambda_f(S) \). Indeed, by the definition of the operator \( \text{rot}_f \) and \((5.5)\), we have

\[
C_n(\text{curl} \text{curl}^* S)P_n = C_n \text{rot}_f \text{curl}^* S = \text{rot}_f C_n(\text{curl}_f^* S + \text{rot}_f^* S + C_n \partial_n S).
\]

However, \( C_n \text{rot}_f^* S = 0 \) and, by \((5.5)\), \( C_n \text{curl}_f^* S = Q_n \text{grad}_f^* n'S \). Hence,

\[
(5.14) \quad C_n(\text{curl} \text{curl}^* S)P_n = \text{rot}_f Q_n(\text{grad}_f^* n'S - \partial_n S)Q_n = \text{rot}_f \Lambda_f(S).
\]

The next lemma indicates how the operator \( \Lambda_f \) arises when integrating \( \text{curl} \text{curl}^* \) by parts.

**Lemma 5.1.** Let \( S \) and \( T \) be two smooth matrix fields, with \( S \) symmetric, on \( \Omega \), where \( \Omega \) is a bounded subset of \( \mathbb{R}^3 \) with a piecewise smooth boundary \( \partial \Omega \). Then

\[
\int_{\Omega} S : \text{curl} \text{curl}^* T \, dx = \int_{\Omega} \text{curl}^* \text{curl} S : T \, dx - \int_{\partial \Omega} C_n S C_n : \frac{\partial T}{\partial n} \, dx_f
\]

\[
- \int_{\partial \Omega} [C_n \text{rot}_f S - (\text{rot}_f^* S)C_n + C_n \Lambda_f(S)C_n] : T \, dx_f.
\]

**Proof.** Since \( \partial \Omega \) has no boundary, it follows from \((5.12)\) that

\[
(5.15) \quad \int_{\partial \Omega} \text{rot}_f S : T \, dx_f = \int_{\partial \Omega} S : \text{curl}_f T \, dx_f.
\]

Then using \((5.4)\) and the antisymmetry of \( C_n \) we obtain

\[
\begin{align*}
\int_{\partial \Omega} S : \text{curl} T \, dx_f &= \int_{\partial \Omega} S : \text{curl}_f T \, dx_f + \int_{\partial \Omega} S : \text{rot}_f T \, dx_f - \int_{\partial \Omega} S : \frac{\partial T}{\partial n} C_n \, dx_f \\
&= \int_{\partial \Omega} (\text{rot}_f + \text{curl}_f) S : T \, dx_f + \int_{\partial \Omega} S C_n : \frac{\partial T}{\partial n} \, dx_f.
\end{align*}
\]

(So far we have not used the assumption that \( S \) is symmetric.) Next, from \((5.11)\) we get

\[
\begin{align*}
\int_{\Omega} S : \text{curl} \text{curl}^* T \, dx &= \int_{\Omega} \text{curl} S : (\text{curl} T')' \, dx + \int_{\partial \Omega} S C_n : (\text{curl} T')' \, dx \\
&= \int_{\Omega} (\text{curl} S)' : \text{curl} T' \, dx - \int_{\partial \Omega} C_n S' : \text{curl} T' \, dx_f
\end{align*}
\]
and
\[ \int_{\Omega} (\text{curl } S)' : \text{curl} T' \, dx = \int_{\Omega} \text{curl}(\text{curl } S)' : T' \, dx + \int_{\partial\Omega} (\text{curl } S)' C_n : T' \, dx_f. \]
Since \( [\text{curl}(\text{curl } S)]' = \text{curl}^* \text{curl } S \) we obtain the identity
\[ \int_{\Omega} \text{curl}^* \text{curl } S : T \, dx = \int_{\Omega} S : \text{curl} \text{curl}^* T \, dx + \int_{\partial\Omega} C_n \text{curl } S : T \, dx_f + \int_{\partial\Omega} C_n S : \text{curl} T' \, dx_f. \]
Using (5.16), with \( S \) replaced by \( C_n S \), we can rewrite the last term as
\[ \int_{\partial\Omega} C_n S : \text{curl} T' \, dx_f = \int_{\partial\Omega} C_n (\text{curl} f + \text{rot } f) S : T' \, dx_f + \int_{\partial\Omega} C_n SC_n : \frac{\partial T'}{\partial n} \, dx_f \]
\[ = - \int_{\partial\Omega} [((\text{curl} f^* + \text{rot}^* f)S] C_n : T \, dx_f + \int_{\partial\Omega} C_n SC_n : \frac{\partial T'}{\partial n} \, dx_f. \]
Thus we obtain
\[ \int_{\Omega} S : \text{curl} \text{curl}^* T \, dx = \int_{\Omega} \text{curl}^* \text{curl } S : T \, dx - \int_{\partial\Omega} C_n SC_n : \frac{\partial T'}{\partial n} \, dx_f \]
\[ - \int_{\partial\Omega} C_n \text{curl } S : T \, dx_f + \int_{\partial\Omega} [((\text{curl} f^* + \text{rot}^* f)S] C_n : T \, dx_f. \]
In order to see that (5.17) is equivalent to the desired identity it is enough to show that
\[ (5.17) \quad C_n \text{curl } S - (\text{curl} f^* S) C_n = C_n \Lambda f(S) C_n + C_n \text{rot } f \, S. \]
However, from (5.6) it follows that \( C_n \text{curl } f^* S - (\text{curl} f^* S) C_n = 2C_n \epsilon(S) C_n \), and hence (5.18) follows from (5.4). \( \square \)

The lemma can be used to determine when a symmetric matrix field \( S \) on \( \Omega \) which is piecewise smooth with respect to a given triangulation belongs to the space \( H(\text{curl} \text{curl}^*, \Omega; \mathbb{S}) \). This holds if and only if \( \text{curl} \text{curl}^* S \) defined piecewise coincides with \( \text{curl} \text{curl}^* S \) defined in the sense of distributions. Integrating against a smooth test function and using the lemma, we find that a necessary condition is that \( C_n SC_n \), or equivalently \( Q_n Q_n \), is continuous across element faces. In this case the quantities \( C_n \text{rot } f \, S \) and \( (\text{rot} f^* S) C_n \) are also continuous. Again invoking the lemma again, we obtain the following result.

**Theorem 5.2.** Assume that \( S \in L^2(\Omega; \mathbb{S}) \) is piecewise smooth with respect to a triangulation \( T \) of \( \Omega \). Then \( S \in H(\text{curl} \text{curl}^*, \Omega; \mathbb{S}) \) if and only if \( Q_n Q_n \) and \( \Lambda f(S) \) are continuous across each face in \( \Delta_2(T) \).

6. **POLYNOMIAL MATRIX FIELDS ON A SINGLE TETRAHEDRON**

In order to study the space \( \Sigma_h \subset H(\text{div} ; \Omega; \mathbb{S}) \) introduced above, we will need to study finite element subspaces \( \Theta_h \) of \( H(\text{curl} \text{curl}^*, \Omega; \mathbb{S}) \). The discussion in the present section will be restricted to a single tetrahedron \( K \), but based on the results derived here we will define the space \( \Theta_h \) in the final section of the paper. In fact, we will present a complete discrete elasticity complex of the form
\[ T \mapsto W_h \xrightarrow{\text{curl} \text{curl}^*} \Theta_h \xrightarrow{\text{curl} \text{curl}^*} \Sigma_h \xrightarrow{\text{div}} V_h \to 0, \]
where $W_h \subset \mathcal{H}^1(\Omega; \mathbb{R}^3)$ and $\Theta_h \subset H(\text{curl}\text{curl}^*, \Omega; \mathbb{S})$ are piecewise polynomial spaces with respect to the triangulation $T_h$.

Let $K \subset \mathbb{R}^3$ be a fixed tetrahedron and define the polynomial space

$$
(6.1) \quad \mathcal{N}_k = \mathcal{N}_k(K) = \{ S \in \mathcal{P}_k(K; \mathbb{S}) \mid Q_n S Q_n = \Lambda_f(S) = 0, \ f \in \Delta_2(K) \}.
$$

Most of the discussion in this section is devoted to computing the dimension and a basis for this space. However, first we need some additional notation.

If $f \in \Delta_2(K)$, we denote by $h_f$ the perpendicular distance from the opposite vertex to $f$ and by $n = n_f$ the outward normal vector to $f$. If $e$ is an edge, we let $s = s_e$ denote one of the unit vectors parallel to $e$. When the edge $e$ belongs to the face $f$, we write $m = m_{e,f}$ for the unit vector in $f$, normal to $e$, pointing from $e$ into $f$. See Figure 1. When the notations $f_+$ and $f_-$ are used to denote two faces, the corresponding normals will be denoted $n_+$ and $n_-$, and the perpendicular distances $h_+$ and $h_-$, respectively. The notations $m_+$ and $m_-$ will also be used to denote $m_{e,f_+}$ and $m_{e,f_-}$ where $e$ is the edge common to $f_+$ and $f_-.$

**Figure 1.** The $(n_f, s_e, m_{e,f})$ coordinate system for a face $f$ and edge $e$ of the tetrahedron $K$.

The barycentric coordinates on $K$ will be labelled by the faces. That is, they are $\lambda_f \in \mathcal{P}_1(K; \mathbb{R})$ determined by $\lambda_f \equiv 0$ on $f$ and $\sum_f \lambda_f \equiv 1$ on $K$. We recall that $\text{grad} \, \lambda_f = -n_f/h_f$. Let $g \in \Delta(K)$ be a face of dimension $m$ with vertices $x_{i_0}, x_{i_1}, \ldots, x_{i_m}$. For $m = 0$, $g$ is a vertex, for $m = 1$, $g$ is an edge, and so on. We define the bubble functions $b_g = \lambda_{f_{i_0}} \lambda_{f_{i_1}} \cdots \lambda_{f_{i_m}}$, where $f_{i_k}$ is the face opposite vertex $x_{i_k}$. For $d > 0$, $\mathcal{P}^d_g(K) = \text{span}\{ \lambda_{f_{i_0}}^{j_{i_0}} \lambda_{f_{i_1}}^{j_{i_1}} \cdots \lambda_{f_{i_m}}^{j_{i_m}} \mid j_{i_0} + \cdots + j_{i_m} = d \}$, so that $\dim \mathcal{P}^d_g(K) = \dim \mathcal{P}_d(g)$. Note that if $x$ is a vertex opposite face $f$, then $b_x = \lambda_f$ and $\mathcal{P}^d_x(K) = \mathbb{R} \lambda_f$, while $b_K = \prod_f \lambda_f$ and $\mathcal{P}^d_K = \mathcal{P}_d$. For a given face $f_0 \in \Delta_2(K)$, $b_{f_0} = b_K/\lambda_{f_0} = \prod_{f \neq f_0} \lambda_f$, and for a given edge $e \in \Delta_1(K)$, $b_e = b_K/(\lambda_{f_+} \lambda_{f_-})$, where $f_+$ and $f_-$ are the faces containing $e$.

The monomials of degree $k$ in the barycentric coordinates form a basis for $\mathcal{P}_k(K)$, and by grouping together terms according to which coordinates enter the monomial, we can uniquely represent any $p \in \mathcal{P}_k$ as

$$
(6.2) \quad p = \sum_{g \in \Delta(K)} b_g p_g, \quad p_g \in \mathcal{P}^g_{k-1-\dim g}(K).
$$

The standard Lagrangian degrees of freedom for $p \in \mathcal{P}_k(K)$ are the values of $p$ at the vertices, the moments of $p$ of degree at most $k-2$ on each of the edges of $K$, the moments of degree at most $k-3$ on each of the faces, and the moments of degree at most $k-4$ on $K$. From the vertex values of $p$ we may determine the polynomials
In fact, the transformation $e$ is a rotation in the plane orthogonal to $e$. Therefore, if we can show that $g$ clearly, if $e$ is then spanned by the matrix $G_e = n_- n'_+ + n'_+ n_-$. We note that $G_e s = 0$ and $m'_- G_e m_- = m'_+ G_e m_+ = 0$.

**Lemma 6.1.** For $k \geq 0$, the dimension of the space

$$N^g_k = N^g_k(K) := \{ S \in \mathcal{P}_k(K; \mathbb{S}) \mid Q_n SQ_n = 0, \quad f \in \Delta_2(K) \}$$

is $(k + 1)k(k - 1)$.

**Proof.** For a subsimplex $g \in \Delta(K)$, let us first define

$$N^g = \{ S \in \mathbb{S} \mid Q_n SQ_n = 0 \text{ for all faces containing } g \}.$$ 

Clearly, if $g = K$ then $\dim N^g = 6$, if $g \in \Delta_2(K)$ then $\dim N^g = 3$, and if $g \in \Delta_0(K)$ then $\dim N^g = 0$. Finally, if $g \in \Delta_1(K)$, i.e., $g = e$ is an edge, then $\dim N^g = 1$. In fact, the space $N^e$ is then spanned by the matrix $G_e$ introduced above.

If $S \in N^g_k$ then from (6.2) we obtain the representation

$$S = \sum_{g \in \Delta(K)} b_g S_g, \quad S_g \in \mathcal{P}^g_{k-1-\dim_g}(K; N^g).$$

As a consequence

$$\dim N^g_k = \sum_{g \in \Delta(K)} \dim \mathcal{P}^g_{k-1-\dim_g}(K; N^g)$$

$$= 6(k - 1) + 6(k - 1)(k - 2) + (k - 1)(k - 2)(k - 3) = (k + 1)k(k - 1).$$

Before we are able to compute $\dim N^g_k$, we need to establish several lemmas.

**Lemma 6.2.** If $S \in N^g_k$, $S$ is zero on each edge.

**Proof.** As above let $e = f_- \cap f_+ \in \Delta_1(K)$, with $f_-, f_+ \in \Delta_2(K)$. Then

$$m'_- n_+ = m'_+ n_- < 0.$$ 

In fact, the transformation

$$\left( \begin{array}{c} m_- \\ n_- \end{array} \right) \rightarrow \left( \begin{array}{c} n_+ \\ m_+ \end{array} \right)$$

is a rotation in the plane orthogonal to $e$.

Let $S \in N^g_k$. Since $N^g_k \subset N^g_0$ we know that $S|_e = \rho G_e$, where $\rho \in \mathcal{P}(e)$. Therefore, if we can show that

$$m'_+ S n_+ + m'_- S n_- = 0$$

on $e$, then $\rho(m'_- n_+ + m'_+ n_-) = 2pn'_+ n_- = 0$, and, as a consequence, $S$ is zero on $e$. It therefore suffices to show (6.5).

On $e$, we must have $s' \Lambda_f(S)m = 0$, i.e.,

$$\partial_s (m'_+ Sn) + \partial_m (s'Sn) - \partial_n (s'Sm) = 0,$$
where \( f \) is either \( f_- \) or \( f_+ \). By adding this property for the two faces we obtain
\[
- \partial_s (m'_+, S_{n_+} + m'_-, S_{n_-})
\]
\[
= [\partial_{m_-}(s'S_{n_-}) - \partial_{n_-}(s'S_{m_-})] - [\partial_{n_+}(s'S_{m_+}) - \partial_{m_+}(s'S_{n_+})].
\]
However, the right hand side here is zero as a consequence of the fact that the transformation (6.4) is a rotation. In fact, this property implies that
\[
\partial_{m_-}(v'n_-) - \partial_{n_-}(v'm_-) = \partial_{n_+}(v'm_+) - \partial_{m_+}(v'n_+)
\]
for any smooth vector field \( v \) on \( K \). Hence, we can conclude that \( m'_-, S_{n_+} + m'_-, S_{n_-} \) is a constant along \( e \), and since it is zero at the vertices, (6.3) holds. \( \square \)

Lemma 6.3. Let
\[
\mathcal{N}_{k, \partial K} := \{ U \in \mathcal{N}^0_k | U = \sum_{f \in \Delta_3(K)} b_f U_f, U_f \in \mathcal{P}_{k-3}^f(K; \mathbb{S}) \text{ and } \Lambda_f(U)|_e = 0 \text{ for each face } f \text{ and each edge } e \text{ of } f \}.
\]
Then \( \dim \mathcal{N}_{k, \partial K} = 6(k^2 - 6k + 10) \).

Proof. If \( U = \sum_f b_f U_f \), then \( U \in \mathcal{N}^0_k \) if and only if each coefficient \( U_f \in \mathcal{P}_{k-3}^f(K; \mathbb{S}) \) satisfies
\[
Q_n U_f Q_n = 0 \quad \text{on } f.
\]
Hence, this property is assumed to hold. We have \( \Lambda_f(U) = 0 \) on an edge \( e \subset f \) if and only if the three terms \( s'\Lambda_f(U)s, s'\Lambda_f(U)m \) and \( m'\Lambda_f(U)m \) vanish there. For any fixed unit vector \( t \) and \( e \in \Delta_3(K) \), we have
\[
(6.6) \quad \partial_t U = \sum_f (\partial_t b_f) U_f = -b_k \left( \frac{t'n_+}{h_+} U_{n_+} + \frac{t'n_-}{h_-} U_{n_-} \right) \quad \text{on } e,
\]
where \( f_- \) and \( f_+ \) are the two faces meeting the edge \( e \) and we have used that \( \text{grad } b_f = -n_+ b_e / h_+ \) on \( e \).

Recall that \( s'\Lambda_f(U)s = 2 \partial_n(s'Un) - \partial_n(s'Us) \). Since \( U = 0 \) on \( e \),
\[
s'\Lambda_f(U)s = -\partial_n(s'Us) \quad \text{on } e.
\]
However, since \( Q_n U_f Q_n = 0 \) on the face \( f \), we have \( s'U_{f_+} s = s'U_{f_+} s = 0 \) on \( e \). By (6.4), with \( t = n \), we conclude that \( s'\Lambda_f(U)s = 0 \) on \( e \).

Next, similar considerations for \( s'\Lambda_f(U)m = \partial_n(m'Un) + \partial_m(s'Un) - \partial_n(s'Um) \) give
\[
s'\Lambda_{f_+}(U)m_+ = \partial_{m_+}(s'Un_+) - \partial_{n_+}(s'Um_+) \quad \text{on } e.
\]
Furthermore, from (6.6) we have
\[
\partial_{m_+}(s'Un_+) = -m'_+ n_- b_e \cdot \frac{s'U_{f_+}n_+}{h_-}
\]
on \( e \), and, using \( s'U_{f_-} m_- = s'U_{f_+} m_+ = 0 \), we obtain
\[
\partial_{n_+}(s'Um_+) = -b_e \cdot \frac{s'U_{f_+}m_+}{h_+} = -m'_+ n_- b_e \cdot \frac{s'U_{f_+}n_-}{h_+}.
\]
It follows that
\[
s'\Lambda_{f_+}(U)m_+ = m'_+ n_- b_e \cdot \left( \frac{s'U_{f_+}n_-}{h_-} - \frac{s'U_{f_+}n_+}{h_+} \right) \quad \text{on } e.
\]
We have therefore shown that $s'\Lambda_f(U)m = 0$ on all edges if and only if,

$$\frac{s'U_{f_-}n_-}{h_+} = \frac{s'U_{f_+}n_+}{h_-}$$

(6.7)

on all edges of $K$.

Finally, we consider $m'\Lambda_f(U)m = 2\partial_m(m'U_n) - \partial_n(m'Um)$. Using the fact that both $m'_-U_{f_-}m_-$ and $m'_+U_{f_+}m_+$ vanish on $e$, we obtain from (6.6) that, on $e$,

$$\partial_n m'_+Um_+ = -\frac{b_e}{h_+} m'_+U_{f_+}m_+$$

$$\quad = -\frac{b_e}{h_+} [(m'_+)^2 n'_-U_{f_-}n_- + 2m'_-_m_-m'_-U_{f_-}n_-],$$

so

$$m'_+\Lambda_f(U)m_+ = m'_+_n_-b_e[m'_+_n_- \frac{n'_-U_{f_-}n_-}{h_+} - 2(\frac{m'_+_U_{f_+}n_+}{h_-} - \frac{m'_-U_{f_-}n_-}{h_+})].$$

Hence, $m'_+\Lambda_f(U)m_+$ vanishes on $e$ if and only if,

$$n'_-U_{f_-}n_- = \frac{2h_+}{m'_+_n_-}(\frac{m'_+U_{f_+}n_+}{h_-} - \frac{m'_-_m_-m'_-U_{f_-}n_-}{h_+})$$
on $e$.

Note that this condition is not symmetric in $f_-$ and $f_+$. We thus obtain two conditions for each edge $e$.

Since $U_f \in \mathcal{P}_{k-3}(K)$, with $Q_nU_fQ_n = 0$ on $f$, it follows that $U_f$ is uniquely determined by the vector field $v_f := U_f n \in \mathcal{P}_{k-3}(f; \mathbb{R}^3)$. The analysis above shows that $U = \sum_f b_fU_f \in \mathcal{N}_{k, \partial K}$ if and only if these vector fields satisfy

(A) $\frac{s'v_{f_+}}{h_+} = \frac{s'v_{f_+}}{h_-}$ on $e$, and

(B) $n'_-v_{f_-} = \frac{2h_+}{m'_+_n_-}(\frac{m'_+v_{f_+}}{h_-} - \frac{m'_-_m_-m'_-v_{f_-}}{h_+})$ on $e$,

whenever an edge $e$ is shared by faces $f_-$ and $f_+$. Therefore, there is an isomorphism between $\mathcal{N}_{k, \partial K}$ and

$$\{ (v_f) \in \prod_{f \in \Delta_3(K)} \mathcal{P}_{k-3}(f; \mathbb{R}^3) \mid \text{the } v_f \text{ satisfy (A) and (B)} \}.$$ 

(6.8)

To compute the dimension of the space (6.8) we consider the relations (A) and (B) at a fixed vertex $x$ of $K$. If $(v_f)$ is an element of the space (6.8) define $z \in \mathbb{R}^3$ by

$$s'z = h_f s'v_f(x)$$

(6.9)

for $s$ chosen as tangents to each edge $e$ meeting $x$, and where $f$ is a face meeting $e$. Note that the vector $z$ is well-defined as a consequence of condition (A), and that for each face $f$ containing $x$ we have $h_f t'v_f(x) = t'z$ for all vectors $t$ which are tangential to the face $f$. Using the expansion $n_- = [m_+ - (m'_+m_-)/m'_+n_-]$, we can then rewrite condition (B) at the vertex $x$ as

$$h_f n'_-v_f(x) = 2n'_fx.$$ 

(6.10)

From this discussion we can conclude that the dimension of the space (6.8), and hence $\dim \mathcal{N}_{k, \partial K}$, is at least $6(k^2 - 6k + 10)$. To see this observe that

$$\dim \prod_f \mathcal{P}_{k-3}(f, \mathbb{R}^3) = 6(k - 2)(k - 1).$$
Furthermore, the conditions (A) and (B) represent a total of $6 \cdot 3 \cdot (k - 4) = 18(k - 4)$ constraints in the interior of the edges and $4 \cdot 6 = 24$ constraints at the vertices. Since $6(k - 2)(k - 1) - 18(k - 4) - 24 = 6(k^2 - 6k + 10)$, this is a lower bound for $\dim N_{k,\partial K}$.

We complete the proof by showing that elements of the space (6.8) are determined by $6(k^2 - 6k + 10)$ degrees of freedom, in fact by degrees of freedom corresponding to the space

$$
\prod_{x \in \Delta_0(K)} \mathbb{R}^3 \times \prod_{e \in \Delta_1(K)} \mathcal{P}_{k-5}(e; \mathbb{R}^3) \times \prod_{f \in \Delta_2(K)} \mathcal{P}_{k-6}(f; \mathbb{R}^3).
$$

To see this, for each vertex $x$ pick a vector $z = z(x) \in \mathbb{R}^3$ and choose $v_f(x)$ such that the relations (6.9) and (6.10) hold for all faces meeting $x$. This determines the vectors $v_f(x)$ for all vertices $x \in \partial f$. We then define $h_f s' v_f$ on each edge by the standard interior degrees of freedom, and $m' v_f$ with respect to both faces meeting $e$ are determined similarly. These degrees of freedom on the edges correspond to the space $\prod_e \mathcal{P}_{k-5}(e; \mathbb{R}^3)$. The normal components $n'_f v_f$ are determined on each edge by condition (B). Finally, we must apply the interior degrees of freedom to $v_f$ on $f$. We conclude that elements of the space $N_{k,\partial K}$ are determined by $12 + 18 \dim \mathcal{P}_{k-5}(e) + 12 \dim \mathcal{P}_{k-6}(f) = 6(k^2 - 6k + 10)$ degrees of freedom. □

For $U \in N_{k,\partial K}$ and $f \in \Delta_2(K)$, $\lambda_f(U)$ is a polynomial vanishing on $\partial f$, and so the quotient $\lambda_f(U)/b_f$ is a polynomial. We define $T_f : N_{k,\partial K} \rightarrow \mathcal{P}_{k-4}(f, Q_nS_{Q_n})$ by

$$
T_f(U) = -h_f \lambda_f(U)/b_f.
$$

**Lemma 6.4.** If $f_-, f_+ \in \Delta_2(K)$, $e = f_- \cap f_+$, and $s$ is a unit vector parallel to $e$, then

$$
s'T_{f_-}(U)s = s'T_{f_+}(U)s \quad \text{on } e, \quad U \in N_{k,\partial K}.
$$

**Proof.** First we show that

$$
\partial_{m_+} s' \lambda_{f_+}(U)s = \partial_{m_-} s' \lambda_{f_-}(U)s \quad \text{on } e.
$$

Recall that $\partial_{m_+} b_{f_+} = -m'_+ n_- b_e/h_-$, $\partial_{m_-} b_{f_-} = -m'_- n_+ b_e/h_+$ on $e$ and $m'_+ n_- = m'_- n_+$. We have on an edge $e$, $\partial_{m'} s' \lambda_f(U)s = 2 \delta_e \partial_{m'} s' U n - \partial_{m'} \partial_{n'} s' U s$. Using (6.6) we obtain

$$
\partial_{m_+} s' U n_+ = -b_e \frac{m'_+ n_-}{h_-} s' U f_+ n_+,
$$

which is symmetric in $f_-$ and $f_+$ as a consequence of (6.7) and $m'_+ n_- = m'_- n_+$.

The identity (6.11) will follow if we show that $\partial_{m_-} \partial_{n_+} s' U s = \partial_{m_-} \partial_{n_-} s' U s$. Consider first the term

$$
V = b_{f_-} U f_+ + b_{f_+} U f_-.
$$

Since $Q_n U f Q_n = 0$ for $f = f_-, f_+$ and $\grad b_{f_-} = -b_{f_-} / h_+ n_+$ on $f_+$ we derive that at the edge $e$

$$
\partial_{m_+} \partial_{n_+} s' V s = -b_e \left[ \frac{1}{h_+} s' \partial_{m_+} U f_- s + \frac{m'_+ n_-}{h_-} s' \partial_{n_+} U f_+ s \right]
$$

$$
= -(m'_+ n_-) b_e \left[ \frac{1}{h_+} s' \partial_{n_-} U f_- s + \frac{1}{h_-} s' \partial_{n_+} U f_+ s \right],
$$

and this expression is symmetric in $f_-$ and $f_+$. Finally, consider terms of the form $W = b_f U f$, where $f$ is neither $f_-$ nor $f_+$. In this case $b_f = \lambda_f \chi_{f_+} \lambda$, where $\lambda$ is
Theorem 6.6. For $Q$ such that $T$ moments from $N(6.12)$ of freedom on $K$. Similarly, for the interior degrees of freedom on each face we let $Q \in \mathbb{S}$ Then we define $S$. Let $h \lambda s'Ufs$. This is again symmetric in $f_-$ and $f_+$. We have therefore established (6.11).

Proof. By (6.11), the left-hand side is unchanged if we interchange the subscripts + and −, whenever $e = f_+ \cap f_-, f_-, f_+ \in \Delta_2(K)$. Then there exist an $S \in \mathcal{P}_k(K; \mathbb{S})$ such that $QnSQn = Tf$ for all $f \in \Delta_2(K)$.

Proof. We will define $S \in \mathcal{P}_k(K; \mathbb{S})$ by first specifying its vertex values, then specifying its moments of degree at most $k-2$ on the edges, then its moments of degree at most $k-3$ on faces, and then the moments of degree at most $k-4$ over the interior of $K$.

Let $x$ be a vertex. We define the matrix $S(x) \in \mathbb{S}$ by specifying the values $s'_iS(x)s_j$ where the $s_i$ are the tangents to the edges $e_i$ meeting at $x$ (and so the $s_i$ form a basis for $\mathbb{R}^3$). Namely we take $s'_iS(x)s_j = s'_iTfsj$ with $f$ the face containing $e_i$ and $e_j$. If $i = j$ there are two possible choices of the face $f$, but they give the same result by assumption.

For the interior degrees of freedom on an edge $e$ we use the basis $s, m_-, m_+$ of $\mathbb{R}^3$, and let $T_e \in \mathcal{P}_k(e; \mathbb{S})$ be given by

$$s'T_e = s'Tf_+s = s'Tf_+s, \quad s'T_em_- = s'Tf_+m_-, \quad s'T_em_+ = s'Tf_+m_+,$$

$$m'_T_em_- = m'_Tf_+m_-, \quad m'_T_em_+ = m'_Tf_+m_+,$$

Then we define $S|_e$ by

$$\int_e (S - T_e)V ds = 0, \quad V \in \mathcal{P}_{k-2}(e; \mathbb{S}).$$

Similarly, for the interior degrees of freedom on each face we let $QnSQn$ inherit the moments from $T_f$, while the data for $SP_n$ is taken to be zero. The interior degrees of freedom on $K$ are all taken to be zero. □

As a consequence of the two previous lemmas, there is a map

(6.12) $\mathcal{N}_k,\partial K \rightarrow \mathcal{P}_{k-4}(K; \mathbb{S}), \quad U \mapsto S(U),$

such that $QnS(U)Qn = Tf(U)$ for all faces $f \in \Delta_2(K)$.

We are finally ready to compute the dimension of the space $\mathcal{N}_k$ defined in (6.1).

Theorem 6.6. For $k \geq 3$, the dimension of the space $\mathcal{N}_k$ is $k(k^2 - 6k + 11)$.

Proof. Let $S \in \mathcal{N}_k$. By Lemma 5.2, $S$ must be zero on each edge and so can be written $S = \sum_f b_fS_f + bKS_K$, where $S_f \in \mathcal{P}_{k-3}(K; \mathbb{S})$ and $S_K \in \mathcal{P}_{k-4}(K; \mathbb{S})$. Now $\varepsilon_f(bKS_Kn_f)$ vanishes on $f$ since $bK$ does, while

$$\partial_n(bKQnS_KQn) = (\partial_n(bK))QnS_KQn = -\frac{b_f}{h_f}QnS_KQn \quad \text{on} \quad f.$$
Thus
\begin{equation}
(6.13) \quad \Lambda_f(b_K S_K) = \frac{b_f}{h_f} Q_n S_K Q_n.
\end{equation}

In particular, \( \Lambda_f(b_K S_K) \) vanishes on \( \partial f \). It follows that if \( S \in N_k \) and we define \( U = \sum_f b_f S_f \) then \( U \in N_{k, \partial K} \). Therefore, the map \((U, S_K) \mapsto U + b_K S_K\) defines an isomorphism from
\[ \{(U, S_K) \in N_{k, \partial K} \mid S_K Q_n Q_n = T_f(U) \} \]
ono onto \( N_k \).

Finally, note that a matrix field of the form \( b_K V, V \in \mathcal{P}_{k-4}(K; \mathbb{S}) \), belongs to \( N_k \) if and only if \( V \in N_{k-4}^0 \). Therefore, using the map \((6.12)\), the mapping
\[ N_{k, \partial K} \times N_{k-4}^0 \rightarrow N_k, \quad (U, V) \mapsto (U, S(U) + V), \]
is an isomorphism. It follows that \( \dim N_k = \dim N_{k, \partial K} + \dim N_{k-4}^0 \) and using Lemma 6.3 and Lemma 6.4 we get \( \dim N_k = 6(k^2 - 6k + 10) + (k - 3)(k - 4)(k - 5) = k(k^2 - 6k + 11) \). \hfill \square

7. The space of divergence-free matrix fields with vanishing normal traces

Recall that the space
\[ \mathcal{M}_k = \mathcal{M}_k(K) = \{ S \in \mathcal{P}_k(K; \mathbb{S}) \mid \text{div} S = 0 \quad \text{on} \quad K, \quad P_n S = 0, \quad f \in \Delta_2(K) \} \]
appears in the degrees of freedom for the finite element space \( \Sigma_h \subset H(\text{div}, \Omega; \mathbb{S}) \) introduced in Sections 3 and 4. Therefore, a derivation of the dimension of this space is fundamental for our theory, while a construction of a (dual) basis for the space \( \mathcal{M}_k \) is necessary for the implementation of the method. The dimension formula will be a simple consequence of the following lemma, in which \( \mathcal{P}_{k+3}^0(K; \mathbb{R}^3) := \{ v \in \mathcal{P}_{k+3}(K; \mathbb{R}^3) \mid v \equiv 0 \quad \text{on} \quad \partial K \} = b_K \mathcal{P}_{k+1}(K; \mathbb{R}^3) \).

**Lemma 7.1.**

1. The operator curl curl* maps \( N_{k+2}(K) \) onto \( \mathcal{M}_k(K) \).
2. \( \{ T \in N_{k+2} \mid \text{curl curl*} T = 0 \} = \epsilon \mathcal{P}_{k+3}^0(K; \mathbb{R}^3) \).
3. The following sequence is exact:
\begin{equation}
(7.1) \quad 0 \rightarrow \mathcal{P}_{k+3}^0(K; \mathbb{R}^3) \rightarrow N_{k+2}(K) \xrightarrow{\text{curl curl*}} \mathcal{M}_k(K) \rightarrow 0.
\end{equation}

**Proof.** It follows directly from (5.7) and (5.14) that \( \text{curl curl*} N_{k+2} \subset \mathcal{M}_k \). Hence, to prove the first statement we need only show that \( \mathcal{M}_k \subset \text{curl curl*} N_{k+2} \). Let \( S \in \mathcal{M}_k \). Since \( \text{div} S = 0 \), it follows from the exactness of the complex (2.6) that there is a \( T \in \mathcal{P}_{k+2}(K; \mathbb{S}) \) such that \( S = \text{curl curl*} T \). The proof will be completed by constructing a vector field \( u \in \mathcal{P}_{k+3}(K; \mathbb{R}^3) \) such that
\begin{equation}
(7.2) \quad Q_n(T - \epsilon(u)) Q_n = 0, \quad \Lambda_f(T - \epsilon(u)) = 0, \quad \text{on each face } f.
\end{equation}

Note that since \( S \in \mathcal{M}_k \) it follows from (5.7) that
\[ \text{rot}_f \text{rot}_f^* Q_n T Q_n = P_n(\text{curl curl*} T) P_n = P_n S P_n = 0 \quad \text{on each face } f. \]

Hence, from the exact sequence (5.9) we conclude that for each face \( f \in \Delta_2(K) \) there is a vector field \( v_f \in \mathcal{P}_{k+3}(f; \mathbb{R}^3) \), with \( P_n v_f = 0 \), such that \( Q_n T Q_n = \epsilon_f(v_f) \). The vector fields \( v_f \) are uniquely determined up to a 2D rigid motion and hence we may normalize them so that \( \int s' v_f ds = 0 \) on each edge \( e \subset f \).

Since \( \partial_s(s' v_{f_-}) = s'T s = \partial_s(s' v_{f_+}) \) on each edge, we obtain that \( P_n v_{f_-} = P_n v_{f_+} \).
on each edge \( e = f_+ \cap f_- \). As a consequence, there is \( v \in \mathcal{P}_{k+3}(K; \mathbb{R}^3) \) such that \( Q_n v Q_n = v_f \) on each face \( f \). Then \( Q_n \epsilon(v) Q_n = \epsilon_f(v_f) = Q_n T Q_n \), i.e., \( Q_n U Q_n = 0 \) on each face \( f \), where \( U = T - \epsilon(v) \). This implies, in particular, that \( U s \) and \( \text{grad}(s' U s) \) vanish on each edge \( e \in \Delta_1(K) \). Therefore,

\[
(7.3) \quad s' \Lambda_f(U)s = 2 \partial_s(n' U s) - \partial_s(s'U s) = 0 \quad \text{on} \, \partial f
\]

for each face \( f \).

Next, observe that, by (7.14), \( \text{rot}_f \Lambda_f(U) = C_n S P_n = 0 \). Hence, (7.10) implies that there is a scalar field \( q_f \in \mathcal{P}_{k+4}(f; \mathbb{R}) \), uniquely determined up to a linear function on \( f \), such that \( \Lambda_f(U) = \text{grad}_f \text{grad}_f^* q_f \). On each edge \( e \in \partial f \), we have by (7.3) that \( 0 = s' \Lambda_f(U)s = \partial_s^2 q_f \). It follows that we can assume that \( q_f \equiv 0 \) on \( \partial f \). Hence, there exists another vector field \( w \) in \( \mathcal{P}_{k+3}(K; \mathbb{R}^3) \) such that \( Q_n w = 0 \) and \( n'w = q_f \) on each face. Recall by (7.13) that \( \Lambda_f(\epsilon(w)) = \text{grad}_f \text{grad}_f^* q_f = \Lambda_f(U) \). Hence, if we let \( u = v + w \), then the relation (7.2) holds. This proves the first statement.

We now prove the second statement. If \( T = \epsilon(v) \) for some \( v \in \mathcal{P}^0_{k+3}(K; \mathbb{R}^3) \), then \( \text{curl} \text{curl}^* T = 0 \), and, by (7.13),

\[
(7.4) \quad Q_n T Q_n = \epsilon_f(Q_n v), \quad \Lambda_f(T) = \text{grad}_f \text{grad}_f^* (n'v) \quad \text{on} \, \partial f
\]

for each face \( f \). Since \( v \) vanishes on \( f \), the right hand sides of these equations vanish, and so \( T \) belongs to \( \mathcal{N}_{k+2}(K) \). Conversely, if \( T \in \mathcal{N}_{k+2}(K) \), then, by the exactness of the sequence (2.6), \( T = \epsilon(v) \) for some \( v \in \mathcal{P}_{k+3}(K; \mathbb{R}^3) \), which is determined uniquely if we require that

\[
(7.5) \quad \int_e s'v \, ds = 0, \quad e \in \Delta_1(K).
\]

(The functionals \( v \mapsto \int_e s'v \, ds, \, e \in \Delta_1(K) \), form a set of degrees of freedom for \( T \), the null space of \( \epsilon \).) From the first equation in (7.4) and (7.5), we find that \( Q_n v \) vanishes on each face \( f \). Therefore the entire vector \( v \) vanishes on each edge \( e \). Using the second equation in (7.4), we see that \( n'v \) vanishes on each face as well, so \( v \in \mathcal{P}^0_{k+3}(K; \mathbb{R}^3) \). This completes the proof of the second statement.

The third statement is an immediate consequence of the first two and the fact that \( \mathcal{T} \cap \mathcal{P}^0_{k+3}(K; \mathbb{R}^3) = 0 \). \( \Box \)

**Theorem 7.2.** For \( k \geq 4 \) the space \( \mathcal{M}_k(K) \) has dimension \((k+2)(k-2)(k-3)/2\).

**Proof.** Using first the short exact sequence in the lemma and then the dimension formula in Theorem 6.6 we get

\[
\dim \mathcal{M}_k = \dim \mathcal{N}_{k+2} - \dim \epsilon(\mathcal{P}^0_{k+3}(K; \mathbb{R}^3)) = (k+2)(k^2 - 2k + 3) - \dim \mathcal{P}_{k-1}(K; \mathbb{R}^3) = (k+2)(k-2)(k-3)/2. \quad \Box
\]

To conclude this section, we construct a basis for the space \( \mathcal{M}_k(K) \) for \( k = 4 \) and \( k = 5 \). (Alternatively a basis could be constructed for any \( k \) using computational algebra software.) For this we use the following lemma, similar to Lemma 7.1.

**Lemma 7.3.**

1. The operator \( \text{curl} \text{curl}^* \) maps \( b_K \mathcal{N}_{k-2}^{0}(K) \) onto \( \mathcal{M}_k(K) \).
2. \( \{ T \in b_k \mathcal{N}_{k-2}^0 \mid \text{curl} \text{curl}^* T = 0 \} = \epsilon(b_k^2 \mathcal{P}_{k-5}(K; \mathbb{R}^3)). \)
3. The following sequence is exact:

\[
(7.6) \quad 0 \to b_k^2 \mathcal{P}_{k-5}(K; \mathbb{R}^3) \to b_K \mathcal{N}_{k-2}^0(K) \xrightarrow{\text{curl} \text{curl}^*} \mathcal{M}_k(K) \to 0.
\]
Proof. Note that \( b_K \mathcal{N}_k^0 \subset \mathcal{N}_k^2 \) by (6.13), and so \( \text{curl curl}^* b_K \mathcal{N}_k^0 \subset \mathcal{M}_k(K) \).

First we prove 2. Let \( w \in \mathcal{P}_{k-5}(K; \mathbb{R}^3) \). By the Leibniz rule
\[
\epsilon (b_K^2 w) = b_K^2 \epsilon (w) + b_K (\text{grad} b_K) w' + w (\text{grad} b_K)' \cdot \nabla.
\]
Clearly \( b_K \epsilon (w) \in \mathcal{N}_k^0 \), and, recalling that \( \text{grad} b_K = -b_f n_f / h_f \), we see that \( (\text{grad} b_K) w' + w (\text{grad} b_K)' \in \mathcal{N}_k^0 \). Thus \( \epsilon (b_K^2 w) \in b_K \mathcal{N}_k^0 \), giving the inclusion \( \supset \). Conversely, if \( T \in b_K \mathcal{N}_k^0 \) with \( \text{curl curl}^* T = 0 \), then, by Lemma 7.1, \( T = \epsilon (b_K v) \) for some \( v \in \mathcal{P}_{k-1}(K; \mathbb{R}^3) \) and we need to show that \( v = 0 \) on \( \partial K \). Using the Leibniz rule and the fact that \( T \) vanishes on \( \partial K \), we get that \( nv' + vn' \) vanishes on each face. We conclude that \( v \) vanishes on the face, using the elementary identity \( v = (I + Q_e) (nv' + vn') n/2 \).

It follows that
\[
\dim[\text{curl curl}^* (b_K \mathcal{N}_k^0)] = \dim \mathcal{N}_k^0 - \dim \mathcal{P}_{k-5}(K; \mathbb{R}^3) = \dim \mathcal{M}_k,
\]
where we have used Lemma 6.1 and Theorem 7.2. The exactness of (7.6), and so also the first statement of the lemma, follows.

Thus for \( k = 4 \), \( \text{curl curl}^* \) is injective on \( b_K \mathcal{N}_3^0 \), and so a basis for \( \mathcal{M}_4 = \text{curl curl}^* (b_K \mathcal{N}_3^0) \) is computable directly from a basis for \( \mathcal{N}_3^0 \), which may be obtained directly from (6.3).

Now let \( k = 5 \). The map \( \text{curl curl}^* \) is not injective on \( b_K \mathcal{N}_3^0 \), but has a kernel of dimension 3. In this case, the representation (6.3) presents an arbitrary element \( S \in \mathcal{N}_3^0 \) as
\[
S = \sum_{e \in \Delta_2(K)} b_e S_e + \sum_{f \in \Delta_2(K)} b_f S_f, \quad S_e \in \mathcal{P}_1(K; \mathbb{R}^3), \quad S_f \in \mathcal{N}_1.
\]
Fix a particular face \( f_0 \in \Delta_2(K) \) and define \( \mathcal{N}_3^{00} \) as the subspace of \( S \in \mathcal{N}_3^0 \) for which \( S_{f_0} = 0 \) in this representation, clearly a subspace of codimension 3. We claim that \( \text{curl curl}^* \) is injective on the space \( b_K \mathcal{N}_3^{00} \), and hence a basis for \( \mathcal{M}_5 \) can be computed from a corresponding basis of \( \mathcal{N}_3^{00} \). The injectivity follows since if \( w \in \mathbb{R}^3 \), with \( \epsilon (b_K^2 w) \in b_K \mathcal{N}_3^{00} \), then we get \( w = 0 \) arguing as in the proof of Lemma 7.3.

8. A Discrete Elasticity Complex

The results of Section 7 above completes the description of the finite element spaces \( V_h \) and \( \Sigma_h \), introduced in Sections 4 and 5, and therefore also of the finite element method (1.1). However, as already indicated in the beginning of Section 6 there are more structures hidden in the construction above. In fact, the spaces \( V_h \) and \( \Sigma_h \) are constituents of a discrete elasticity complex of the form
\[
\begin{align*}
T & \hookrightarrow W_h \xrightarrow{\epsilon} \Theta_h \xrightarrow{\text{curl curl}^*} \Sigma_h \xrightarrow{\text{div}} V_h \rightarrow 0, \\
\mathcal{N}_0^V & \subset H^1(\Omega; \mathbb{R}^3) \quad \text{and} \quad \Theta_h \subset H(\text{curl curl}^*, \Omega; \mathbb{S}) \quad \text{are piecewise polynomial spaces with respect to the triangulation} \ T_h. \\
\text{Furthermore, there exist interpolation operators} \ \Pi_h \ W, \Pi_h \Theta, \Pi_h \Sigma \quad \text{and} \quad \Pi_h V \quad \text{such that the diagram}
\end{align*}
\]
\[
\begin{array}{ccccccc}
T & \hookrightarrow & C^\infty(\Omega; \mathbb{R}^3) & \xrightarrow{\epsilon} & C^\infty(\Omega; \mathbb{S}) & \xrightarrow{\text{curl curl}^*} & C^\infty(\Omega; \mathbb{S}) & \xrightarrow{\text{div}} & C^\infty(\Omega; \mathbb{R}^3) & \rightarrow & 0 \\
\Pi_h W & \hookrightarrow & \Pi_h \epsilon \Theta_h & \xrightarrow{\text{curl curl}^*} & \Pi_h \Sigma_h & \xrightarrow{\text{div}} & \Pi_h V & \rightarrow & 0
\end{array}
\]
Next we will show that the degrees of freedom associated with the subsimplexes of \( f \) imply that
\[
\Lambda \in L^2(\Omega; \mathbb{R}^3) \text{ is composed of piecewise linears.}
\]

We will first describe the corresponding space \( \Theta_h \subset H(\text{curl curl}^*, \Omega; \mathbb{S}) \). Locally on each tetrahedron this space consists of functions in \( \mathcal{P}_6(K; \mathbb{S}) \), which is a space of dimension 504. In order to specify the degrees of freedom on a tetrahedron \( K \in T_h \) we introduce the polynomial space
\[
E(f) = \{ v \in \mathcal{P}_7(f; \mathbb{R}) \mid v|_{\partial f} = 0, \int_f \partial_n v \, ds = 0, e \in \Delta_1(f) \}
\]
for each face \( f \in \Delta_2(K) \). The dimension of \( E(f) \) is 12. Also for each \( e \in \Delta_1(K) \) define the operator \( \Gamma_e : C^\infty(K; \mathbb{S}) \rightarrow C^\infty(e; e^+) \) by \( \Gamma_e(S) = 2\partial_e Q_s(s) - \text{grad}_e \cdot s'Ss \).

Note that if \( f \in \Delta_2(K) \), with \( n = n_f \), and \( e \in \Delta_1(f) \), then \( n^* \Gamma_e(S) = s'^*\Lambda_f(S)s \) on \( e \).

The 504 degrees of freedom used to define the finite element space \( \Theta_h \) are the following:

1. \( S \) and \( \text{curl curl}^* S \) at each vertex, 48 degrees of freedom,
2. \( \int_e s'Ss \, v \, ds, v \in \mathcal{P}_4(e; \mathbb{R}), e \in \Delta_1(K) \), 30 degrees of freedom,
3. \( \int_e \Gamma_e(S) \cdot v \, ds, v \in \mathcal{P}_5(e; \mathbb{Q}_s \mathbb{R}^3), e \in \Delta_1(K) \), 72 degrees of freedom,
4. \( \int_e Q_s(\text{curl curl}^* S) : W \, ds, W \in \mathcal{P}_2(e; \mathbb{Q}_s \mathbb{S}), e \in \Delta_1(K) \), 90 degrees of freedom,
5. \( \int_f \text{rot}_e^* P_s Q_s \, ds, e \in \Delta_1(K) \), 6 degrees of freedom,
6. \( \int_f Q_s Q_n : \epsilon_f(v) \, dx_f, v \in \text{by}_f \mathcal{P}_4(f; \mathbb{Q}_n \mathbb{R}^3), f \in \Delta_2(K) \), 120 degrees of freedom,
7. \( \int_e (s'^* \Lambda_f(S)m) \, v \, ds, v \in \mathcal{P}_4(e; \mathbb{R}) \) with \( \int_e v \, ds = 0, f \in \Delta_2(K), e \in \Delta_1(f) \), 12 degrees of freedom,
8. \( \int_f \Lambda_f(S) \, dx_f, f \in \Delta_2(K) \), 12 degrees of freedom,
9. \( \int_f \text{grad}_f^* s'v \, dx_f, v \in E(f), f \in \Delta_2(K) \), 48 degrees of freedom,
10. \( \int_K \text{curl curl}^* S : T \, dx, T \in \mathcal{M}_4(K) \), 6 degrees of freedom.
11. \( \int_K S : \epsilon(v) \, dx, v \in \mathcal{P}_0(K; \mathbb{R}^3) \), 60 degrees of freedom.

These degrees of freedom are unisolvant for the space \( \mathcal{P}_6(K; \mathbb{S}) \). This is in fact a consequence of the following result.

**Lemma 8.1.** Let \( f \in \Delta_2(K) \) be fixed and assume that \( S \in \mathcal{P}_6(K; \mathbb{S}) \) with all the degrees of freedom (1)–(9) associated to all \( g \in \Delta(f) \) equal to zero. Then \( Q_s Q_n \) and \( \Lambda_f(S) \) are identically zero on the face \( f \).

**Proof.** Let \( f \in \Delta_2(K) \) be fixed, and assume that \( S \in \mathcal{P}_6(K; \mathbb{S}) \) has all degrees of freedom associated with the subsimplexes of \( f \) equal to zero. We start by observing that the degrees of freedom (1)–(4) implies that
\[
(8.3) \quad s'Ss = 0, \quad Q_s \text{curl curl}^* S = 0, \quad \Gamma_e(S) = 0, \quad \text{on } e \in \Delta_1(f).
\]
Next we will show that
\[
(8.4) \quad \text{rot}_f \text{rot}_f^* Q_n Q_n = P_n(\text{curl curl}^* S)P_n = 0
\]
on \( f \). Note that by (8.3) this quantity vanishes on \( \partial f \), i.e., \( \text{rot} f \text{rot}^* f Q_n S Q_n \in b_P (f; \mathbb{R} P_n) \). Furthermore, using (5.12), for each \( U \in P_1 (f; \mathbb{R} P_n) \)
\[
\int f \text{rot} f \text{rot}^* f Q_n S Q_n : U \, dx_f = - \int \partial_f [\text{rot} f (S s) \cdot (U n) + (S s) \cdot \text{curl} f (U n)] \, ds.
\]
Also, by (5.1) we obtain \( \text{rot} f (S s) = - [\partial_s (s' C_n S s) + \partial_m (m' C_n S s)] n \). However, the fact that \( m' \Gamma_c (S) = 0 \) on \( e \in \Delta_1 (f) \) implies that \( \partial_m s' S s = 2 \partial_s m' S s \) or \( \partial_m m' C_n S s = - 2 \partial_s s' C_n S s \). Therefore, we can conclude that \( \text{rot} f (S s) = \partial_s (s' C_n S s) \) on \( e \in \Delta_1 (f) \). On the other hand, since \( s' S s = 0 \) on \( e \) we have from (5.3) that
\[
(S s) \cdot \text{curl} f (U n) = - (S s) \cdot [C_n \text{grad} (n' U n)]
\]
\[
= (C_n S s) \cdot \text{grad} (n' U n) = (s' C_n S s) \partial_s (n' U n).
\]
Since \( S \) is zero at each vertex of \( f \) we can therefore conclude that
\[
\int f \text{rot} f \text{rot}^* f Q_n S Q_n : U \, dx_f = - \int \partial_s [(s' C_n S s) (n' U n)] \, ds = 0
\]
for all \( U \in P_1 (f; \mathbb{R} P_n) \), and hence (8.4) follows. As a consequence, we obtain from (5.9) that \( Q_n S Q_n = \epsilon_f (u) \) for a suitable \( u \in P_7 (f; Q_n \mathbb{R}^3) \), where \( u \) can be chosen such that \( \int e s' u \, ds = 0 \) for each edge \( e \in \Delta_1 (f) \). Therefore, since \( s' S s = \partial_s (s' u) = 0 \) on each \( e \in \Delta_1 (f) \), we conclude that \( s' u = 0 \) on \( \partial f \), and that \( u = 0 \) at each vertex. Furthermore, on each edge \( e \in \Delta_1 (f) \)
\[
0 = m' \Gamma_c (S) = 2 \partial_s (m' S s) - \partial_m (s' S s) = \partial_s^2 (m' u),
\]
and therefore \( u = 0 \) on \( \partial f \). Hence, \( u \) vanishes on \( f \) by the degrees of freedom (6), and so does \( Q_n S Q_n \).

Next we will show that \( \text{rot} f \Lambda_f (S) = 0 \) on \( f \). By (5.14) we have that \( \text{rot} f \Lambda_f (S) = C_n (\text{curl} \text{curl}^* S ) P_n \), and therefore (8.3) implies that \( \text{rot} f \Lambda_f (S) \) is zero on \( \partial f \). Hence, it is enough to show that
\[
\int f \text{rot} f \Lambda_f (S) : V \, dx_f = 0, \quad V \in P_1 (f; Q_n S P_n).
\]
Furthermore, by the degrees of freedom (8) we obtain that
\[
\int f \text{rot} f \Lambda_f (S) : V \, dx_f = \int \partial_f [\Lambda_f (S) s] \cdot (V n) \, ds = \int [m' \Lambda_f (S) s] \cdot (m' V n) \, ds,
\]
for any \( V \in P_1 (f; Q_n S P_n) \), where the last identity holds since \( s' \Lambda_f (S) s = n' \Gamma_c (S) = 0 \) on \( e \in \Delta_1 (f) \). From (7) above, this will be zero if we can show that \( \int e m' \Lambda_f (S) s \, ds \) vanishes for all \( e \in \Delta_1 (f) \). However, \( m' \Lambda_f (S) s = \partial_s (m' S n) + \partial_m (s' S n) - \partial_n (s' S m) \). Therefore the degrees of freedom (5) imply that \( \int e m' \Lambda_f (S) s \, ds = \int [ \partial_s (m' S n) \, ds = 0 \). We therefore conclude that \( \text{rot} f \Lambda_f (S) \) is zero on \( f \), and from (5.10) we can conclude that \( \Lambda_f (S) = \text{grad} f \text{grad}^* f v \) for a suitable scalar field \( v \in P_7 (f; \mathbb{R}) \), where we assume that \( v \) is chosen to be zero at each vertex. Furthermore, the fact that \( 0 = n' \Gamma_c (S) = s' \Lambda_f (S) s = \partial_s^2 (m' u) \) implies that \( v \equiv 0 \) on \( \partial f \). In particular, \( \text{grad} f v \equiv 0 \) at each vertex. Therefore, \( v \in E (f) \) since (7) implies that \( 0 = \int e s (s' \Lambda_f (S) m) \, ds = \int e \partial_s \partial_m v \, ds = - \int e \partial_m v \, ds \), and, as a consequence of (9), \( v = 0 \), and so is \( \Lambda_f (S) \).

The lemma above implies that the set of functionals (1)–(11) are unsolvent for the space \( P_0 (K; S) \). This follows since if all the functionals (1)–(11) are zero then
\( S \in \mathcal{N}_6 \) by this lemma, and hence \( \text{curl curl}^* S \in \mathcal{M}_4 \). By the degrees of freedom (10) \( \text{curl curl}^* S = 0 \), and by Lemma 7.1 and the degrees of freedom (11), \( S = 0 \).

The finite element space \( \Theta_h \) is defined as all functions which belongs to \( \mathcal{P}_6(K; \mathbb{S}) \) for all \( K \in \mathcal{T}_h \) and with the continuity conditions induced by the degrees of freedom. Hence, as a consequence of Lemma 8.1, the variables \( Q_n \) and \( \Lambda_f \) are continuous for all \( S \in \Theta_h \) and \( f \in \Delta_2(T_h) \), and therefore Theorem 5.2 implies that \( \Theta_h \subset H(\text{curl curl}^*, \Omega; \mathbb{S}) \). The degrees of freedom (1)–(11) also define a canonical interpolation operator \( \Pi_h^\Theta : C^\infty(\Omega; \mathbb{S}) \rightarrow \Theta_h \) by requiring that it reproduces all the functionals.

Finally, we need to describe the finite element space \( W_h \subset H^1(\Omega; \mathbb{R}^3) \). Locally on each \( K \in \mathcal{T}_h \) this space is taken to be \( \mathcal{P}_7(K; \mathbb{R}^3) \), which is a space of dimension 360. A vector field \( w \in \mathcal{P}_7(K; \mathbb{R}^3) \) is uniquely determined by:

1. the values of \( w \) and its first order derivatives at each vertex, 48 degrees of freedom,
2. \( \int_f w \cdot q \, ds \), \( q \in \mathcal{P}_3(e; \mathbb{R}^3) \), \( e \in \Delta_1(K) \), 72 degrees of freedom,
3. \( \int_f \epsilon_f Q_n w : \epsilon_f (q) \, dx_f \), \( q \in b_f \mathcal{P}_4(f; Q_n \mathbb{R}^3) \), \( f \in \Delta_2(K) \), 120 degrees of freedom,
4. \( \int_m \partial_m (n' w) \, ds \), \( f \in \Delta_2(K) \), \( e \in \Delta_1(f) \), 12 degrees of freedom
5. \( \int_f \text{grad}_f \text{grad}_f^j (n' w) : \text{grad}_f \text{grad}_f^j (q) \, dx_f \), \( q \in E(f) \), \( f \in \Delta_2(K) \), 48 degrees of freedom,
6. \( \int_K \epsilon(w) \cdot \epsilon(q) \, dx \) for all \( q \in \mathcal{P}_9(K; \mathbb{R}^3) \), 60 degrees of freedom.

It is rather straightforward to check that this set of functionals is unisolvent for the space \( \mathcal{P}_7(K; \mathbb{R}^3) \). The finite element space \( W_h \) is then defined as the set of functions which are locally in \( \mathcal{P}_7(K; \mathbb{R}^3) \) and with the continuity induced by these degrees of freedom. Hence, all \( w \in W_h \) are continuous across faces \( f \) in \( \Delta_2(T_h) \), and they are \( C^1 \) at the vertices. The associated interpolation operator \( \Pi_h^W : C^\infty(\Omega; \mathbb{R}^3) \rightarrow W_h \) will reproduce all degrees of freedom given by (1)–(6).

By using the degrees of freedom for the discrete spaces, one may also verify that the diagram (8.2) commutes. This verification is tedious, and we will drop most of the details. However, we will illustrate what needs to be done. For example, as part of the verification of the relation \( \Pi_h^\Theta \circ \text{curl curl}^* = \text{curl curl}^* \circ \Pi_h^\Theta \) we must show that for \( T = (I - \Pi_h^\Theta) S \), \( S \in C^\infty(\Omega; \mathbb{S}) \), we have (cf. degrees of freedom (3) for \( \Sigma_h \))

\[
\int_f P_n(\text{curl curl}^* T) P_n : U \, dx_f = \int_f \text{rot}_f^* \text{rot}_f \, Q_n T Q_n : U \, dx_f = 0,
\]

\[
\int_f C_n(\text{curl curl}^* T) P_n : V \, dx_f = \int_f \text{rot}_f \Lambda_f (T) : V \, dx_f = 0,
\]

for \( U \in \mathcal{P}_1(f; \mathbb{R} P_n) \), \( V \in \mathcal{P}_1(f; Q_n S P_h) \), and all \( f \in \Delta_2(T_h) \). However, the verifications of these identities are almost identical to arguments leading to the formulas (8.3) and (8.0) in the proof of Lemma 8.1 above. The two identities above show that the degrees of freedom (3) for \( \Sigma_h \) are zero for functions of the form \( (\Pi_h^\Theta \circ \text{curl curl}^* - \text{curl curl}^* \circ \Pi_h^\Theta) S \in \Sigma_h \). Similar, but simpler, arguments can be used to show that the other degrees of freedom for \( \Sigma_h \) are zero. Hence, the identity \( \Pi_h^\Theta \circ \text{curl curl}^* = \text{curl curl}^* \circ \Pi_h^\Theta \) follows, and corresponding verifications can be done for the identity \( \Pi_h^\Theta \circ \epsilon = \epsilon \circ \Pi_h^W \).
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