Halfspace type theorems for self-shrinkers in arbitrary codimension

Doan The Hieu and Nguyen Thi My Duyen
Department of Mathematics
College of Education, Hue University, Hue, Vietnam
dthieu@hueuni.edu.vn, ntmduyen@hueuni.edu.vn

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Abstract
In this paper, we generalize some halfspace type theorems for self-shrinkers of codimension 1 to the case of arbitrary codimension.

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1 Introduction
The halfspace theorem says that “There is no non-planar, complete, minimal surface properly immersed in a halfspace of $\mathbb{R}^3$. “ The theorem is due to Hoffman and Meeks. In fact they proved a stronger version, the strong halfspace theorem, “Two disjoint complete properly immersed minimal surfaces in $\mathbb{R}^3$ are planes” (see [14]).

The halfspace theorem is essentially a three-dimensional one. In $\mathbb{R}^n, n > 3$, the halfspace theorem is false because there are minimal Catenoids with bounded height.

Many generalizations of the theorem have been made by several authors, see [8], [9], [13], [19], [22], [23] and references therein.

The first halfspace theorem for self-shrinker in codimension 1 was proved in [20] based on the weighted parabolicity of self-shrinkers. A similar result in a more general setting was proved recently in [15].

Theorem 1 (Theorem 3 in [20]; Theorem 1.1 in [2]). Let $P$ be a hyperplane passing through the origin. The only properly immersed self-shrinker contained in one of the closed halfspace determined by $P$ is $\Sigma = P$.

In contrast with the case of minimal surfaces, the halfspace theorem for self-shrinkers holds true in any dimension. Moreover, one can consider a type of halfspace theorems for self-shrinker containing inside or outside a hypercylinder.

In 2016, Cavalcante and Espinar [2] showed some halfspace type theorems for self-shrinkers of codimension 1 including Theorem 1 with a different proof.
Theorem 2 (Theorem 1.2 in [2]). The only complete self-shrinker properly immersed in a closed cylinder $B^{k+1}(R) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$, for some $k \in \{1, \ldots, n\}$ and radius $R$, $R \leq \sqrt{2k}$, is the cylinder $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$.

Theorem 3 (Theorem 1.3 in [2]). The only complete self-shrinker properly immersed in an exterior closed cylinder $E^{k+1}(R) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$, for some $k \in \{1, \ldots, n\}$ and radius $R$, $R \geq \sqrt{2k}$, is the cylinder $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$. Here $E^{k+1}(R) = R^{k+1} - B^{k+1}(R)$.

In 2018, Vieira and Zhou [26] proved similar results, where spheres or balls center at the origin are replaced by ones with arbitrary centers and suitable radius. Recently, Imper, Pigola and Rimoldi [21] recovered Cavalcante and Espinar’s results with short proofs by using potential theoretic arguments.

The paper aims to generalize the above halfspace type results for codimension 1 self-shrinkers to the case of arbitrary codimension. The first step in our approach is somewhat similar to the one in [21] for codimension 1 but the use of maximal principle for weighted superharmonic functions together with the weighted parabolicity of self-shrinkers is replaced by an application of a divergence theorem (Theorem 5). In fact our proofs recovered some key formulas (9), (10), (13) that are due Colding-Minicozzi [7] for the case of codimension 1. Arezzo-Sun [1] observed that these formulas are also true for the case of arbitrary codimension.

We would like to thank Vieira, Rimoldi, Rosales for introducing us to their interesting works and the others for helpful comments and suggestions.

2 Preliminaries

In this paper, we use the following notations

1. $B^k(a, R)$, the $k$-ball with center $a$ and radius $R$;
2. $E^k(a, R) = \mathbb{R}^k - B^k(a, R)$, the complement of $B^k(a, R)$;
3. $S^k(a, R)$, the $k$-sphere with center $a$ and radius $R$;
4. $\overline{A}$, the closure of the set $A$.

For simple, when the center of spheres or balls is the origin we write $B^k(R)$, $E^k(R)$, $S^k(R)$.

2.1 Self-shrinkers

An $n$-dimensional submanifold $\Sigma$ immersed in $\mathbb{R}^m$, $m > n$, is called a self-shrinker for the mean curvature flow (MCF), if

$$H = -\frac{1}{2}X^N,$$

where $H$ is the mean curvature vector of $\Sigma$, $X$ is the position vector, and $X^N$ denotes the normal part of $X$.

Self-shrinkers are self-similar solutions to MCF and play an important role in the study of singularities of the flow. For more information about self-shrinkers as well as singularities, we refer the readers to [5], [9], [10], [17].
A complete self-shrinker $\Sigma^n$ in $\mathbb{R}^m$ is said to have polynomial volume growth if there exist constants $C_1$ and $d_1$ such that for all $R \geq 1$, there holds
\[
\text{Vol}(B^m(R) \cap \Sigma) \leq C_1 R^{d_1}.
\] (2)

In 2013, Cheng-Zhou [4] and Ding-Xin [10], proved that

“A complete non-compact properly immersed self-shrinker $\Sigma^n$ in $\mathbb{R}^m$, $m > n$, has Euclidean volume growth at most, i.e.

\[
\text{Vol}(B^m(R) \cap \Sigma) \leq CR^n
\]

for $R \geq 1$.”

2.2 Some typical examples

It is not hard to verify all of the followings are $n$-dimensional complete self-shrinkers in $\mathbb{R}^m$.

1. An $n$-plane passing through the origin.
2. $S^n(\sqrt{2n}) \subset \mathbb{R}^{n+1}$.
3. The cylinder $S^k(\sqrt{2n}) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$, $0 < k < n$.
4. $S^{n_1}(\sqrt{2n_1}) \times S^{n_2}(\sqrt{2n_2}) \times \ldots \times S^{n_k}(\sqrt{2n_k}) \subset \mathbb{R}^{n+1}$, $n_1 + n_2 + \ldots + n_k = n$.
5. $S^{n_1}(\sqrt{2n_1}) \times S^{n_2}(\sqrt{2n_2}) \times \ldots \times S^{n_k}(\sqrt{2n_k}) \times \mathbb{R}^p \subset \mathbb{R}^{n+1}$, $p \geq 1$ and $n_1 + n_2 + \ldots + n_k + p = n$.
6. $n$-dimensional complete minimal submanifolds of the sphere $S^{m-1}(\sqrt{2n})$ (see Theorem 4.1 in [1] or subsection 1.4 in [25]).

For some more well-known results about complete self-shrinkers, we refer the readers to [7], [17], [20] for the case of codimension 1 and [3], [24] for the case of arbitrary codimension.

2.3 Some calculations

In this subsection, we calculate the surface divergence of some vector fields that will be used in the proofs of the main results. The calculations are straightforward, but for the sake of completeness we present them here.

Let $e_1, e_2, \ldots, e_m$ be the coordinate vector fields for $\mathbb{R}^m$, $\Sigma^n$ be a complete self-shrinker in $\mathbb{R}^m$, $\{E_1, E_2, \ldots, E_n\}$ be an orthonormal basis for $T_X \Sigma$, $X = \sum_{i=1}^{m} x_i e_i$ be the position vector field and $u = \sum_{i=1}^{k+1} x_i e_i$, $k \leq m - 1$. We have the following lemma.

Lemma 4. 1.
\[
\text{div}_\Sigma X^T = n - \frac{1}{2} |X_N|^2.
\] (3)

2.
\[
\text{div}_\Sigma e_l^T = -\frac{1}{2} \langle X, e_l^N \rangle, \quad l = 1, 2, \ldots, m.
\] (4)

3.
\[
\text{div}_\Sigma x_i e_l^T = |e_l^T|^2 - \frac{1}{2} x_l \langle X, e_l^N \rangle, \quad l = 1, 2, \ldots, m.
\] (5)
\[ \text{div} \Sigma u^T = (k + 1) - \frac{1}{2} |u|^2 - \sum_{i=1}^{k+1} |e_i|^2. \]  

(6)

\[ \text{div} \Sigma \frac{1}{|u|} u^T = \frac{1}{|u|} \left[ k - \frac{1}{2} |u|^2 - \sum_{i=1}^{k+1} |e_i|^2 + \frac{|u|^2}{|u|^2} \right]. \]  

(7)

**Proof.** We use the summation convention.

1. We have
\[ \text{div} \Sigma X = n, \]
and
\[ \text{div} \Sigma X^N = \langle E_i, \nabla E_i X^N \rangle = \nabla E_i \langle E_i, X^N \rangle - \langle \nabla E_i E_i, X^N \rangle \]
\[ = \nabla E_i (0) - \langle (\nabla E_i E_i)^N, X \rangle = -\langle H, X \rangle = \frac{1}{2} |X|^2. \]
Therefore,
\[ \text{div} \Sigma X^T = n - \frac{1}{2} |X|^2. \]

2.
\[ \text{div} \Sigma e_i^T = \text{div} \Sigma e_i - \text{div} \Sigma e_i^N = 0 - \langle E_i, \nabla E_i e_i^N \rangle \]
\[ = \langle \nabla E_i E_i, e_i^N \rangle = \langle (\nabla E_i E_i)^N, e_i \rangle = \langle H, e_i \rangle \]
\[ = -\frac{1}{2} \langle X, e_i^N \rangle. \]

3.
\[ \text{div} \Sigma x_i e_i^T = \text{div} \Sigma x_i e_i - \text{div} \Sigma x_i e_i^N = |e_i|^2 - \langle E_i, \nabla E_i x_i e_i^N \rangle \]
\[ = |e_i|^2 + \langle (\nabla E_i E_i)^N, x_i e_i \rangle = |e_i|^2 + \langle H, x_i e_i \rangle \]
\[ = |e_i|^2 - \frac{1}{2} x_i \langle X, e_i^N \rangle. \]

4. For \( v \in T_p \Sigma, \)
\[ \nabla_v u = \pi_1(v) = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \ldots + \langle v, e_{k+1} \rangle e_{k+1}. \]
We have
\[ \text{div}_\Sigma(u) = \langle E_i, \nabla E_i u \rangle = \sum_{j=1}^{k+1} \sum_{i=1}^{n} \langle E_i, e_j \rangle^2 \]
\[ = \sum_{j=1}^{k+1} |e_j|^2 = (k + 1) - \sum_{j=1}^{k+1} |e_j|^2, \]
\[
\text{and } \quad \text{div}_\Sigma u^N = \langle E_i, \nabla E_i u^N \rangle = \nabla E_i \langle E_i, u^N \rangle - \langle \nabla E_i E_i, u^N \rangle = \nabla E_i (0) - \langle (\nabla E_i)^N, u \rangle = -\langle H, u \rangle = \frac{1}{2} |u^N|^2.
\]

Therefore,
\[
\text{div}_\Sigma u^T = (k + 1) - \frac{1}{2} |u^N|^2 - \sum_{i=1}^{k+1} |e_i^N|^2.
\]

5.
\[
\text{div}_\Sigma \frac{1}{|u|} u^T = \langle \nabla \frac{1}{|u|}, u^T \rangle + \frac{1}{|u|} \text{div} u^T
= -\frac{1}{|u|^2} |u^T|^2 + \frac{1}{|u|} \left( (k + 1) - \frac{1}{2} |u^N|^2 - \sum_{i=1}^{k+1} |e_i^N|^2 \right)
= \frac{1}{|u|} \left[ k - \frac{1}{2} |u^N|^2 - \sum_{i=1}^{k+1} |e_i^N|^2 + \frac{|u^N|^2}{|u|^2} \right].
\]

\[\Box\]

3 Results

In this section, \(\Sigma\) is assumed to be an \(n\)-dimensional complete (without boundary) self-shrinker properly immersed in \(\mathbb{R}^m, m > n\).

The condition of polynomial volume growth is essential for using an integral formula that is similar to the generalized divergence theorem for compact manifolds. We have the following theorem.

Theorem 5. Let \(F\) be a smooth tangent vector field on \(\Sigma\). For every \(X \in \Sigma\), if \(\left| \text{div}_\Sigma F(X) \right| \leq C_2 |X|^{d_2}\), where \(C_2\) is a positive constant and \(d_2\) is a positive integer, then
\[
\int_\Sigma \text{div}_\Sigma (e^{-\frac{\alpha^2}{2}} F) dV = 0. \quad (8)
\]

Proof. We only need to prove for the case \(\Sigma\) is non-compact. Since \(\Sigma\) is proper, \(\partial (B_R \cap \Sigma) \neq \emptyset\) when \(R\) is large enough. Since \(F\) is tangent to \(\Sigma\), the generalized divergence theorem for \(e^{-\frac{\alpha^2}{2}} F\) yields
\[
\int_{B_R \cap \Sigma} \text{div}_\Sigma (e^{-\frac{\alpha^2}{2}} F) dV = e^{-\frac{\alpha^2}{2}} \int_{\partial (B_R \cap \Sigma)} \langle F, \nu \rangle dA.
\]

Taking the limit when \(R \to \infty\), the theorem is proved because
\[
\lim_{R \to \infty} e^{-\frac{\alpha^2}{2}} \left| \int_{\partial (B_R \cap \Sigma)} \langle F, \nu \rangle dA \right| = \lim_{R \to \infty} e^{-\frac{\alpha^2}{2}} \left| \int_{B_R \cap \Sigma} \text{div}_\Sigma F dV \right|
\leq \lim_{R \to \infty} e^{-\frac{\alpha^2}{2}} C_2 |X|^{d_2} \int_{B_R \cap \Sigma} dV
\leq \lim_{R \to \infty} e^{-\frac{\alpha^2}{2}} C_1 C_2 R^{d_1 + d_2} = 0.
\]

\[\Box\]
Applying Theorem 5 with suitable choices of tangent vector fields $F$, we obtain the main results of the paper.

### 3.1 Half space type result w.r.t. hyperplanes

The following theorem says that $\Sigma$ intersects every hyperplane passing through the origin.

**Theorem 6.** Let $P$ be a hyperplane passing through the origin. If $\Sigma$ lies in a closed halfspace determined by $P$, then $\Sigma \subset P$.

**Proof.** Without loss of generality, we can suppose that $P$ is the hyperplane $x_m = 0$ and $\Sigma$ is in the closed half space $\{(x_1, x_2, \ldots, x_m) : x_m \geq 0\}$.

By (4),

$$\text{div}(e^{-\frac{\Delta}{2}} e_m) = e^{-\frac{\Delta}{2}} \text{div} e_m - e^{-\frac{\Delta}{2}} \frac{1}{2} \langle X, e_m \rangle$$

$$= -\frac{1}{2} e^{-\frac{\Delta}{2}} \left[ \langle X, e_m^N \rangle + \langle X, e_m^T \rangle \right]$$

$$= -\frac{1}{2} e^{-\frac{\Delta}{2}} x_m.$$

Then Theorem 5 applying for $F = e_m$ yields (see [7] for the case of codimension 1, also see [1])

$$\int_{\Sigma} e^{-\frac{\Delta}{2}} x_m dV = 0. \quad (9)$$

Therefore, $x_m = 0$, i.e. $\Sigma \subset P$.

**Remark 7.** If $n = m - 1$, then $\Sigma = P$ ([20], Theorem 3 ; [3], Theorem 1.1).

**Corollary 8.** If there exist $m - n$ orthonormal vectors $v_1, v_2, \ldots, v_{m-n}$ such that for $i = 1, 2, \ldots, m - n$, $\langle X, v_i \rangle$ does not change sign, then $\Sigma$ is an $n$-plane passing through the origin.

**Proof.** Without loss of generality, we can assume that $v_i = e_{n+i}$ if $\langle X, v_i \rangle \geq 0$ and $v_i = -e_{n+i}$ if $\langle X, v_i \rangle \leq 0$. The assumption guarantees that $\Sigma$ is in the closed halfspace $\{(x_1, x_2, \ldots, x_m) : x_{n+i} \geq 0, i = 1, 2, \ldots, m-n\}$. The proof is then followed by applying Theorem 5 in turn for $v_1, v_2, \ldots, v_{m-n}$.

Based on the Bernstein result for self-shrinkers of codimension 1, “An entire graphic self-shrinker must be a hyperplane passing through the origin” (see [11], [27], [13]), and with the same argument as in the proof of Corollary 8 we have the following.

**Corollary 9** (A Bernstein type theorem). Let $F : \mathbb{R}^n \to \mathbb{R}^{m-n}, F(x) = (f_1(x), f_2(x), \ldots, f_{m-n}(x))$ be a smooth function and $\Sigma = \{(x, F(x)) : x \in \mathbb{R}^n\}$ be its graph. If there exist at least $(m - n - 1)$ functions $f_i$ that do not change sign, then $\Sigma$ is an $n$-plane passing through the origin.
3.2 Self-shrinkers inside or outside a ball

The following theorem says that a complete properly immersed self-shrinker $\Sigma^n$ and $S^{m-1}(\sqrt{2n})$ must be intersected.

**Theorem 10.** If $\Sigma \subset E^m(\sqrt{2n})$ or $\Sigma \subset B^m(\sqrt{2n})$, then $\Sigma$ is compact and $\Sigma \subset S^{m-1}(\sqrt{2n})$, i.e. $\Sigma$ is a minimal submanifold of $S^{m-1}(\sqrt{2n})$. Moreover, if $n = m - 1$, then $\Sigma = S^n(\sqrt{2n})$.

**Proof.** By (3),

$$\text{div}_\Sigma (e^{-\frac{x^2}{4}} X^T) = e^{-\frac{x^2}{4}} \text{div}_\Sigma X^T - e^{-\frac{x^2}{4}} \frac{1}{2} \langle X, X^T \rangle$$

$$= e^{-\frac{x^2}{4}} (n - \frac{1}{2} |X|^2) - e^{-\frac{x^2}{4}} \frac{1}{2} |X^T|^2$$

$$= e^{-\frac{x^2}{4}} (n - \frac{1}{2} |X|^2).$$

Applying Theorem 5 with $F = X^T$ (see [7] for the case of codimension 1, also see [1]),

$$\int_\Sigma e^{-\frac{x^2}{4}} (n - \frac{1}{2} |X|^2) dV = 0. \quad (10)$$

If $\Sigma \subset E^m(\sqrt{2n})$ ($\Sigma \subset B^m(\sqrt{2n})$), then $2n - |X|^2 \leq 0$ ($2n - |X|^2 \geq 0$). By (10), it follows that $2n - |X|^2 = 0$, i.e. $\Sigma \subset S^{m-1}(\sqrt{2n})$. Since $\Sigma$ is proper, it must be compact.

The case of $n = m - 1$ is obvious.

The following theorem can be seen as an arbitrary codimension version of Theorem 1 in [26]. Here the proof is also applied for the case of self-shrinkers are outside of spheres.

**Theorem 11.** 1. Any complete self-shrinker $\Sigma^n$ properly immersed in $\mathbb{R}^m$, $m > n$, intersects all members of the collection $C$ given by

$$C := \{ S^{m-1}(a, \sqrt{2n} + |a|^2) : a \text{ is a vector in } \mathbb{R}^m \}.$$  

2. If the $\Sigma$ lies in $B^m(a, \sqrt{2n} + |a|^2)$ or in $\mathbb{R}^m - B^m(a, \sqrt{2n} + |a|^2)$ then $\Sigma \subset S^{m-1}(a, \sqrt{2n} + |a|^2)$. Moreover, if $n = m - 1$, then $\Sigma$ is the sphere $S^n(\sqrt{2n})$.

**Proof.** From (3), it follows that

$$\int_\Sigma e^{-\frac{x^2}{4}} (X, a) dV = 0. \quad (11)$$

Therefore, (10) and (11) yields

$$\int_\Sigma e^{-\frac{x^2}{4}} (|X - a|^2 - (2n + |a|^2)) dV = 0. \quad (12)$$

The theorem is proved easily by some arguments as in the proof of Theorem 10. Note that, for codimension 1 case, the sphere $S^n(a, \sqrt{2n} + |a|^2)$ is a self-shrinker if and only if $a = 0$. \qed
Remark 12. Theorem 5.1 in [12] shows another version of Theorem 10, where self-shrinkers are assumed to be parabolic instead of proper. And a different proof of Theorem 10, stated in terms of $\lambda$-self-shrinkers, was also done in [12] (Theorem 6.3).

3.3 Half space type results w. r. t. cylinders

Theorem 13 (Self-shrinker inside a hypercylinder). Let $k \in \{m-n, \ldots, m-2\}$, $p = m-k-1$ and $R = \sqrt{2(n-p)}$. If $\Sigma$ is inside the closed cylinder $B^{k+1}(R) \times \mathbb{R}^p$, then $\Sigma \subset S^k(R) \times \mathbb{R}^p$.

Proof. By (5)

$$\text{div}_\Sigma (e^{-\frac{X^2}{4}} x_i e_i^T) = e^{-\frac{X^2}{4}} \left[ \text{div}_\Sigma (x_i e_i^T) - \frac{1}{2} \langle X, x_i e_i^T \rangle \right]$$

$$= e^{-\frac{X^2}{4}} \left[ |e_i^T|^2 - \frac{1}{2} x_i \langle X, e_i^N \rangle - \frac{1}{2} x_i \langle X, e_i^T \rangle \right]$$

$$= e^{-\frac{X^2}{4}} \left[ |e_i^T|^2 - \frac{1}{2} x_i^2 \right].$$

Applying Theorem 5 with $F = x_i e_i^T$, we have (see [7] for the case of codimension 1, also see [1])

$$\int \Sigma e^{-\frac{X^2}{4}} x_i^2 dV = 2 \int \Sigma e^{-\frac{X^2}{4}} |e_i^T|^2 dV. \quad (13)$$

Let $\{e_1, e_2, \ldots, e_m\}$ be the standard basis in $\mathbb{R}^m$, where $\{e_1, e_2, \ldots, e_{k+1}\} \subset \mathbb{R}^{k+1}$ and $\{e_{k+2}, e_{k+3}, \ldots, e_m\} \subset \mathbb{R}^p$. Denote $X = (u, v)$, where $u \in \mathbb{R}^{k+1}, v \in \mathbb{R}^p$.

By (10) and (13), we get

$$\int \Sigma e^{-\frac{X^2}{4}} \left[ |X|^2 - 2n - \sum_{i=k+2}^m x_i^2 \right] dV = \int \Sigma e^{-\frac{X^2}{4}} \left[ |u|^2 - 2n \right] dV$$

$$= -2 \int \Sigma e^{-\frac{X^2}{4}} \sum_{i=k+2}^m |e_i^T|^2 dV.$$

Since $|e_i^T|^2 = 1 - |e_i^N|^2$, it follows that

$$\int \Sigma e^{-\frac{X^2}{4}} \left[ |u|^2 - R^2 \right] dV = 2 \int \Sigma e^{-\frac{X^2}{4}} \sum_{i=k+2}^m |e_i^N|^2 dV \geq 0.$$

The assumption that $\Sigma$ is inside the closed cylinder $B^{k+1}(R) \times \mathbb{R}^p$, means

$|u|^2 - R^2 \leq 0$.

Therefore,

$|u|^2 - R^2 = 0$,

i.e. $\Sigma \subset S^k(R) \times \mathbb{R}^p$. 

\qed
Remark 14. 1. We see in the above proof that \( e_i^N = 0 \), i.e. \( e_i = e_i^T \), \( i = k + 2, \ldots, m \). Therefore, \( \Sigma = \Gamma \times \mathbb{R}^p \), where \( \Gamma \subset S^k \) is an \((n - p)\)-dimensional self-shrinker, i.e. an \((n - p)\)-dimensional minimal submanifold of \( S^k \).

2. If \( n = m - 1 \), then \( \Sigma = S^k(\sqrt{2k}) \times \mathbb{R}^{n-k} \) ([2], Theorem 1.2).

Theorem 15 (Self-shrinker outside a hypercylinder). Let \( k \in \{1, \ldots, n\} \). If \( \Sigma \) is contained in \( E^{k+1}(\sqrt{2k}) \times \mathbb{R}^{m-k-1} \), then \( \Sigma \subset S^k(\sqrt{2k}) \times \mathbb{R}^{m-k-1} \).

Proof. Let \( u = \sum_{i=1}^{k+1} x_i e_i \). By (17)

\[
\text{div}_\Sigma (e^{-\frac{x^2}{|u|^2}} \frac{1}{|u|} u^T) = \left[ e^{-\frac{x^2}{|u|^2}} \text{div}_\Sigma (\frac{1}{|u|} u^T) - \frac{1}{2} (X, \frac{u^T}{|u|}) \right]
= e^{-\frac{x^2}{|u|^2}} \frac{1}{|u|} \left[ k - \frac{1}{2} |u|^2 - \sum_{i=1}^{k+1} |e_i^N|^2 + \frac{|u|^2}{|u|^2} \right].
\]

It is not hard to check that

\[
\sum_{i=1}^{k+1} |e_i^N|^2 \geq \frac{|u|^2}{|u|^2}.
\]

Indeed, we have

\[
|u|^2 = \left| \sum_{i=1}^{k+1} x_i e_i^N \right|^2 = \sum_{i=1}^{k+1} x_i^2 |e_i^N|^2 + 2 \sum_{i \neq j} x_i x_j \langle e_i^N, e_j^N \rangle
\leq \sum_{i=1}^{k+1} x_i^2 |e_i^N|^2 + \sum_{i \neq j} x_i^2 |e_j^N|^2
\leq \left( \sum_{i=1}^{k+1} x_i^2 \right) \left( \sum_{i=1}^{k+1} |e_i^N|^2 \right) = |u|^2 \left( \sum_{i=1}^{k+1} |e_i^N|^2 \right).
\]

Applying Theorem 5 with \( F = \frac{1}{|u|} u^T \),

\[
\int_\Sigma e^{-\frac{x^2}{|u|^2}} \frac{1}{|u|} (2k - |u|^2) \ dV \geq 0.
\] (14)

But the assumption that \( \Sigma \) is in \( E^{k+1}(\sqrt{2k}) \times \mathbb{R}^{m-k-1} \) means

\[
|u|^2 - 2k \geq 0.
\]

Therefore, \( |u|^2 - 2k = 0 \), i.e. \( \Sigma \subset S^k(\sqrt{2k}) \times \mathbb{R}^{m-k-1} \).

Remark 16. If \( n = m - 1 \), then \( \Sigma = S^k(\sqrt{2k}) \times \mathbb{R}^{n-k} \) ([3], Theorem 1.3).

With the same arguments as in the proof of Theorem 14, we have the following theorem (see Corollary 1, [26] for the case of codimension 1).
Theorem 17. 1. If the self-shrinker $\Sigma^n$ lies inside the closed cylinder 

$$B^{k+1}(a, \sqrt{2(n-p) + |a|^2}) \times \mathbb{R}^p,$$

where $a \in \mathbb{R}^{k+1}$, then $\Sigma \subset S^k(a, \sqrt{2(n-p) + |a|^2}) \times \mathbb{R}^p$. Moreover, if $n = m - 1$, then $\Sigma = S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$.

2. The self-shrinker cannot lie outside the closed cylinder 

$$B^{k+1}(a, \sqrt{2(k+1) + |a|^2}) \times \mathbb{R}^p,$$

for any vector $a$ in $\mathbb{R}^{k+1}$.

References

[1] C. Arezzo and J. Sun, Self-shrinkers for the mean curvature flow in arbitrary codimension, Math. Z. 274 (2013), no. 3-4, 993-1027.

[2] M. P. Cavalcante and J. M. Espinar, Halfspace type theorems for self-shrinkers, Bull. Lond. Math. Soc. 48 (2016), no. 2, 242-250.

[3] Q-M. Cheng and Y. Peng, Complete self-shrinkers of the mean curvature flow, Calc. Var. Partial Differential Equations 52 (2015), no. 3-4, 497-506.

[4] X. Cheng and D. Zhou, Volume estimate about shrinkers, Proc. Amer. Math. Soc. 141 (2013), no. 2, 687-696.

[5] T. H. Colding, and W. P. Minicozzi, II, A course in minimal surfaces, Graduate Studies in Mathematics, vol. 121, American Mathematical Society, Providence, RI, 2011.

[6] T. H. Colding, and W. P. Minicozzi, II, Smooth Compactness of Self-shrinkers, Comment. Math. Helv. 87 (2012), no. 2, 463-475.

[7] T. H. Colding, and W. P. Minicozzi, II, Generic mean curvature flow I: generic singularities, Ann. of Math. (2) 175 (2012), no. 2, 755-833.

[8] B. Daniel and L. Hauswirth, Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group. Proc. Lond. Math. Soc. (3) 98 (2009), no. 2, 445-470.

[9] B. Daniel, W. H. Meeks, III, and H. Rosenberg, Half-space theorems for minimal surfaces in $\mathrm{Nil}_3$ and $\mathrm{Sol}_3$, J. Differential Geom., 88 (2011), no. 1, 41-59.

[10] Q. Ding and Y. L. Xin, Volume growth, eigenvalue and compactness for self-shrinkers, Asian J. Math. 17 (2013), no. 3, 443-456.

[11] K. Ecker and G. Huisken, Mean Curvature Evolution of Entire Graphs, Ann. of Math. 130 (1989), 453-471.

[12] V. Gimeno and V. Palmer, Parabolicity, Brownian Exit Time and Properness of Solutions of the Direct and Inverse Mean Curvature Flow, J. Geom. Anal., 31 (2021), no. 1, 579-618.
[13] D.T. Hieu, *A weighted volume estimate and its application to Bernstein type theorems in Gauss space* Colloq. Math. **159** (2020), no. 1, 25–28.

[14] D. Hoffman and W. H. Meeks, III, *The strong halfspace theorem for minimal surfaces*, Invent. Math. **101** (1990), no. 2, 373-377.

[15] A. Hurtado, V. Palmer and V. Rosales, *Parabolicity criteria and characterization results for submanifolds of bounded mean curvature in model manifolds with weights*, Nonlinear Anal. **192** (2020), 111681, 32 pp.

[16] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differ. Geom. **31** (1990), no. 1, 285-299.

[17] G. Huisken, *Local and global behaviour of hypersurfaces moving by mean curvature*, In: Differential Geometry: Partial Differential Equations on Manifolds, Los Angeles, CA, 1990, (eds. R. Greene and S.-T. Yau), Proc. Sympos. Pure Math., 54, Amer. Math. Soc, Providence RI, 1993.

[18] L. Mazet, *A general halfspace theorem for constant mean curvature surfaces*, Amer. J. Math. **135** (2013) 801–834.

[19] B. Nelli and R. Sa Earp, *A halfspace theorem for mean curvature \( H = 1/2 \) surfaces in \( H^2 \times \mathbb{R} \),* J. Math. Anal. Appl. **365** (2010), no. 1, 167-170.

[20] S. Pigola and M. Rimoldi, *Complete self-shrinkers confined into some regions of the space*, Ann. Global Anal. Geom. **45** (2014) 47-65.

[21] D. Impera, S. Pigola, M. Rimoldi, *The Frankel property for self-shrinkers from the viewpoint of elliptic PDE’s*, J. Reine Angew. Math. **773** (2021), 1-20.

[22] L. Rodriguez and H. Rosenberg, *Half-space theorems for mean curvature one surfaces in hyperbolic space*, Proc. Amer. Math. Soc. **126** (1998), no. 9, 2755-2762.

[23] H. Rosenberg, F. Schultz and J. Spruck, *The halfspace property and entire minimal graphs in \( M \times \mathbb{R} \)*, J. Differential Geom. **95** (2013) 321-336.

[24] K. Smoczyk, *Self-shrinkers of the mean curvature flow in arbitrary codimension*, Int. Math. Res. Not. 2005, no. 48, 2983-3004.

[25] Y. Xin, *Minimal submanifolds and related topics..* Second edition of [MR2035469]. Nankai Tracts in Mathematics, 16. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2019. xvi+380 pp.

[26] M. Vieira and D. Zhou, *Geometric properties of self-shrinkers in cylinder shrinking Ricci solitons*, J. Geom. Anal. **28** (2018), no. 1, 170-189.

[27] L. Wang, *A Benstein type theorem for self-similar shrinkers*, Geom. Dedicata, **151** (2011), 297-303.