A new proof of the higher-order superintegrability of a noncentral oscillator with inversely quadratic nonlinearities

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Abstract

The superintegrability of a rational harmonic oscillator (non-central harmonic oscillator with rational ratio of frequencies) with non-linear “centrifugal” terms is studied. In the first part, the system is directly studied in the Euclidean plane; the existence of higher-order superintegrability (integrals of motion of higher order than 2 in the momenta) is proved by introducing a deformation in the quadratic complex equation of the linear system. The constants of motion of the nonlinear system are explicitly obtained. In the second part, the inverse problem is analyzed in the general case of n degrees of freedom; starting with a general Hamiltonian $H$, and introducing appropriate conditions for obtaining superintegrability, the particular “centrifugal” nonlinearities are obtained.

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1 Introduction

A superintegrable system is a system that is integrable (in the sense of Liouville-Arnold) and that, in addition to this, possesses more constants of motion than degrees of freedom. At this point we must note that the maximum number $N$ of functionally independent constants of motion for a system in a $d$-dimensional manifold is $N = d - 1$. Thus if a Lagrangian (or Hamiltonian) system has $n$ degrees of freedom then, as the phase space is $2n$-dimensional, we have that the maximum number of independent constants of motion is $N = 2n - 1$. There are three well known examples of this very particular class of systems, namely, the Kepler problem, the isotropic harmonic oscillator, and the non-isotropic oscillator with commensurable frequencies. The two-dimensional harmonic oscillator is a system trivially integrable since it can be considered as a kind of “direct sum” of two systems with one degree of freedom. If the oscillator is isotropic then it has the angular momentum as an additional integral of motion. If the oscillator is non-isotropic the angular momentum is not preserved as the potential is not central; nevertheless when the quotient of the two frequencies is a rational number then the system has another additional integral. Concerning the three-dimensional Kepler problem, it possesses not only the energy and the angular momentum as constants of motion, but also the Runge-Lenz vector; only five of these seven integrals are functionally independent since in this case the number of degrees of freedom is $n = 3$. In these three cases it is well known that all the orbits became closed for the case of bounded motions. This high degree of regularity (existence of periodic motions) is a consequence of the superintegrable character.

Fris et al [1] studied the two-dimensional Euclidean systems which admit separability in two different coordinate systems and obtained four families of potentials $V_r$, $r = a, b, c, d$, possessing three functionally independent integrals of the motion. A very important point is that these three constants of motion were linear or quadratic in the velocities (momenta). In fact, if we call superseparable a system that admits Hamilton-Jacobi separation of variables (Schrödinger in the quantum case) in more than one coordinate system, then quadratic superintegrability (i.e., superintegrability with linear or quadratic constants of motion) can be considered as a property arising from superseparability. The first two families

\[
V_a = \frac{1}{2} \omega_0^2 (x^2 + y^2) + \frac{k_2}{x^2} + \frac{k_3}{y^2}, \\
V_b = \frac{1}{2} \omega_0^2 (4x^2 + y^2) + k_2x + \frac{k_3}{y^2},
\]

can be considered as the more general deformations (with strengths $k_2, k_3$) of the 1:1 and 2:1 harmonic oscillators preserving quadratic superintegrability (the other two families, $V_c$ and $V_d$, were related with the Kepler problem). The superintegrability of $V_a$ was later on studied by Evans [2, 3] in the more general case of $n$ degrees of freedom.
A natural generalization of $V_a$ is given by the following potential

$$V_a(\omega_1, \omega_2) = \frac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2) + \frac{k_1}{x^2} + \frac{k_2}{y^2},$$

that contains a more general harmonic oscillator with anisotropy. This new potential is only separable in Cartesian coordinates and its superintegrability (if it exists) must be of higher-order (‘higher-order superintegrability’ means that some of the integrals of motion are polynomials in the momenta of order higher than 2). Therefore, the method of the multiple separability cannot be used and it must be studied by making use of a different approach. In fact, the superintegrability of this nonlinear system, for the case of rational ratio of the frequencies, was first proved by Rañada et al in [4] using the properties of the isotonic (or singular) oscillator and the Pinney-Ermakov equation [5] for obtaining a complex factorization. More recently Evans et al [6] and Rodriguez et al [7, 8] have also proved this property using the geometric formalism of dimensional reduction. It have been proved that certain nonlinear integrable systems may arise as reductions of very simple systems defined in higher-dimensional spaces (see e.g. [9, 10]). In this particular case the authors start with an harmonic oscillator in a higher-dimensional space and then the dimensional reduction introduces the nonlinearities but in a way that preserves the superintegrability. We note that this geometric method has been also used for proving the superintegrability of the Kepler–Coulomb system with nonlinear terms [8, 11, 12].

Now, we present a new method to prove the higher-order superintegrability of this nonlinear system. This new method, that is more straightforward than the previously known methods, is directly related with the approach presented in [4] (but without making use of the properties of the Pinney-Ermakov equation) and, at the same time, it is also related with some of the results obtained in [7, 8].

The plan of the article is as follows: in Sec. 2 we present the problem from the Lagrangian viewpoint and we study the superintegrability of the two dimensional potential $V_a(\omega_1, \omega_2)$ using the existence of a complex factorization as an approach. First we consider the linear harmonic oscillator and then we prove the superintegrability of the nonlinear system. The constants of motion of the nonlinear system are explicitly obtained. In Sec. 3 we analyze the general case of $n$ degrees of freedom. It is written by using the Hamiltonian approach and it has a more generic character. We start with a very general Hamiltonian $H$ and then we introduce the restrictions to obtain superintegrability (but without making use of the property of separability). The idea is that this more general approach could be used in future papers as a starting point for the search of other more general superintegrable systems. Finally in Sec. 4 we make some comments and present some open questions.
2 Complex factorization and superintegrability

2.1 Superintegrability of the linear Harmonic Oscillator

The two-dimensional harmonic oscillator

\[ L_{\text{HO}} = \frac{1}{2} (v_x^2 + v_y^2) - \frac{1}{2} (\omega_1^2 x^2 + \omega_2^2 y^2) \]

has the two partial (one-degree of freedom) energies, \( I_1 = E_x \) and \( I_2 = E_y \), as fundamental integrals. The superintegrability of the rational case, \( \omega_1 = n_x \omega_0, \omega_2 = n_y \omega_0 \), with integers \( n_x, n_y \), can be proved by making use of a complex formalism \cite{13, 14, 15}. Let \( K_i, i = x, y \), be the following two complex functions

\[ K_x = v_x + i n_x \omega_0 x, \quad K_y = v_y + i n_y \omega_0 y, \]

then we have the following time-evolution

\[ \frac{d}{dt} K_x = i n_x \omega_0 K_x, \quad \frac{d}{dt} K_y = i n_y \omega_0 K_y. \]

Thus, the functions \( K_{ij} \) defined as

\[ K_{ij} = (K_i)^i (K_j)^j, \quad i, j = x, y, \]

are constants of motion. The two real functions \( |K_{xx}|^2 \) and \( |K_{yy}|^2 \) are proportional to the energies \( E_x \) and \( E_y \) and concerning \( K_{xy} \), since it is a complex function, it determines not one but two real first integrals, \( \text{Im}(K_{xy}) \) and \( \text{Re}(K_{xy}) \). So, we have obtained four integrals but, since the system is two-dimensional, only three of them can be independent. We can choose \( I_1 = E_x, I_2 = E_y, \) and \( I_3 = \text{Im}(K_{xy}) \) as the set of fundamental constants of motion (the other constant \( I_4 = \text{Re}(K_{xy}) \) can be expressed as a function of \( E_x, E_y, \) and \( \text{Im}(K_{xy}) \)).

As an example, for the Isotropic case, \( \omega_1 = \omega_2 = \omega_0 \), we obtain

\[ I_3 \equiv \frac{1}{\omega_0} \text{Im}(K_{xy}) = xv_y - yv_x, \]

\[ I_4 \equiv \text{Re}(K_{xy}) = v_x v_y + \omega_0^2 xy, \]

(\( I_3 \) is the angular momentum and \( I_4 \) the component \( F_{xy} \) of the Fradkin tensor \cite{16}) and for the first non-isotropic case, \( \omega_1 = 2 \omega_0, \omega_2 = \omega_0, \) we arrive to

\[ I_3 \equiv \frac{1}{2 \omega_0} \text{Im}(K_{xy}) = (xv_y - yv_x)v_y - \omega_0^2 xy^2, \]

\[ I_4 \equiv \text{Re}(K_{xy}) = v_x v_y^2 + \omega_0^2 (4 xv_y - yv_x) y. \]

The integral \( I_3 \) of the 3:1 oscillator will be cubic and, in the general \( n_x : n_y \) case, the function \( I_3 \) will be a polynomial in the velocities (momenta) of degree \( n_x + n_y - 1 \).
2.2 Superintegrability of the nonlinear system

The technique presented in the previous section proves not only the super-integrability of the rational case but also the existence of a complex factorization for the additional constant of motion. Now we assume, as starting point of our approach, that if a new system can be obtained by a deformation of the harmonic oscillator then it must be also endowed with a similar property.

The analysis will proceed in two steps.

Step 1. The associated quadratic equation

Let \( K = a + ib \) be such that \( dK/dt = i\omega K \). Then the function \( K_2 \) defined as \( K_2 = K^2 = (a^2 - b^2) + 2i ab \) satisfies \( dK_2/dt = 2i \omega K_2 \). Thus the time evolution of the functions

\[
K_{2x} = (v_x^2 - n_x^2 \omega_0^2 x^2) + 2i n_x \omega_0 x v_x, \quad K_{2y} = (v_y^2 - n_y^2 \omega_0^2 y^2) + 2i n_y \omega_0 y v_y, 
\]

is given by

\[
\frac{d}{dt} K_{2x} = 2i n_x \omega_0 K_{2x}, \quad \frac{d}{dt} K_{2y} = 2i n_y \omega_0 K_{2y}, 
\]

and hence the complex functions \( K_{2ij} \) defined as

\[
K_{2ij} = (K_{2i})^{n_j} (K_{2j}^*)^{n_i}, \quad i, j = x, y, 
\]

are constants of motion.

The two complex functions, \( K_{ij} \) and \( K_{2ij} \), must be considered as two alternative ways to prove superintegrability but, of course, the first one is simpler than the quadratic. As an example we have \( |K_{2xx}| = |K_{xx}|^2 \) and \( |K_{2yy}| = |K_{yy}|^2 \). In the general \( n_x : n_y \) case, the functions \( I_3 = \text{Im}(K_{2xy}) \) and \( I_4 = \text{Re}(K_{2xy}) \) will be a polynomials in the velocities (momenta) of degree \( 2(n_x + n_y - 1) - 1 \) and \( 2(n_x + n_y) \). So, if the study is restricted to the harmonic oscillator, it is better to use the function \( K_{ij} \) since it leads to simpler expressions for the constants of the motion.

Step 2. Introducing a deformation in the “quadratic equation”

Let us now consider the following \((F, G)\)-dependent family of potentials

\[
V(n_x, n_y, F, G) = \frac{1}{2} \omega_0^2 (n_x^2 x^2 + n_y^2 y^2) + \frac{1}{2} F(x) + \frac{1}{2} G(y). 
\]

The problem is to determine which particular values of the functions \( F \) and \( G \) can preserve the existence of a complex factorization. At this point we assume that, in order to solve this problem, it is more convenient to use the quadratic equation since it seems as more ‘deformable’ than the linear one.
Let us denote by $A_j$ and $B_j$, $j = x, y$, the following functions

$$
A_x = v^2_x - n_x^2 \omega_0^2 x^2 + F(x), \quad B_x = 2n_x \omega_0 x v_x,
$$
$$
A_y = v^2_y - n_y^2 \omega_0^2 y^2 + G(y), \quad B_y = 2n_y \omega_0 y v_y.
$$

Then, if we require for the functions $A_j$ and $B_j$ a time-evolution of the form

$$
\frac{d}{dt} A_j = -2n_j \omega_0 B_j, \quad \frac{d}{dt} B_j = 2n_j \omega_0 A_j, \quad j = x, y,
$$

we arrive (we omit the details) to the first-order differential equations

$$
x F' + 2F = 0, \quad y G' + 2G = 0,
$$

with solutions $F = k_1/x^2$ and $G = k_2/y^2$ with arbitrary constants $k_1$ and $k_2$. Therefore, only if $F$ and $G$ have this particular structure, the complex functions $M_j$ defined as $M_j = A_j + i B_j$ play, in this nonlinear case, a similar role to the complex functions $K_j$ of the linear case.

**Proposition 1** Consider the non-linear potential

$$
V_{a}(n_x, n_y) = \frac{1}{2} \omega_0^2 (n_x^2 x^2 + n_y^2 y^2) + \frac{k_1}{2x^2} + \frac{k_2}{2y^2}
$$

representing an harmonic oscillator with rational ratio of frequencies, $\omega_1 = n_x \omega_0$, $\omega_2 = n_y \omega_0$, and inversely quadratic nonlinearities and let us denote by $M_j$, $j = x, y$, the following two complex functions

$$
M_x = \left(v^2_x - n_x^2 \omega_0^2 x^2 + \frac{k_1}{x^2}\right) + 2i n_x \omega_0 x v_x, \quad M_y = \left(v^2_y - n_y^2 \omega_0^2 y^2 + \frac{k_2}{y^2}\right) + 2i n_y \omega_0 y v_y.
$$

Then, the complex functions $M_{ij}$ defined as

$$
M_{ij} = (M_i)^{n_j} (M_j^*)^{n_i}, \quad i, j = x, y,
$$

are constants of the motion.

Firstly, the moduli of $M_x$ and $M_y$ are given by

$$
|M_x|^2 = 4(E_x^2 - k_1 n_x^2 \omega_0^2), \quad |M_y|^2 = 4(E_y^2 - k_2 n_y^2 \omega_0^2).
$$

The time-evolution of the functions $M_x$ and $M_y$ is given by

$$
\frac{d}{dt} M_x = 2i n_x \omega_0 M_x, \quad \frac{d}{dt} M_y = 2i n_y \omega_0 M_y.
$$
Thus we have
\[
\frac{d}{dt} M_{xy} = n_y (M_x)^{(n_y-1)} (M_y^*)^{n_x} \dot{M}_x + n_x (M_x)^{n_x} (M_y^*)^{(n_x-1)} \dot{M}_y^* \\
= (2i\omega_0)(n_y n_x - n_x n_y)(M_x)^{n_x} (M_y^*)^{n_y} = 0.
\]

As in the linear case, \(M_{xy}\) can be considered as coupling the two degrees of freedom.

Hence the potential \(V_a(n_x, n_y)\) is superintegrable for any rational value of the quotient \(\omega_2/\omega_1\). In the particular 1:1 case, \(\omega_1 = \omega_2 = \omega_0\), the potential reduces to the \(V_a\) potential
\[
V_a(1, 1) \equiv V_a = \frac{1}{2} \omega_0^2 (x^2 + y^2) + \frac{k_1}{2x^2} + \frac{k_2}{2y^2}
\]
(that is the only superseparable potential in this family) and in the general case, \(\omega_1 = n_x \omega_0, \omega_2 = n_y \omega_0\), the potential \(V_a(n_x, n_y)\) represents a generalized \(V_a\) potential with a non-isotropic \(n_x:n_y\) oscillator (note that in the 2:1 case, the potential \(V_a(2, 1)\) is different from the family \(V_0\)).

In the general \(n_x:n_y\) case, the functions \(\text{Re}(M_{xy})\) and \(\text{Im}(M_{xy})\) will be polynomials in the velocities (momenta) of degree \(2(n_x + n_y)\) and \(2(n_x + n_y) - 1\) respectively. The real part takes the form
\[
\text{Re}(M_{xy}) = 2^{(n_x+n_y)}(E_x)^{n_y} (E_y)^{n_x} + \lambda \omega_0^2 J_3,
\]
where \(\lambda\) is a numerical coefficient. Thus the additional constant of \(V_a(n_x, n_y)\) is in fact the function \(J_3\) that is of degree \(2(n_x + n_y - 1)\). If we denote by \(I_3\) the constant of motion of the associated linear system (of degree \(n_x + n_y - 1\)), then the additional third integral \(J_3\) can be written as follows
\[
J_3 = I_3^2 + k_1 J_3^{(10)} + k_2 J_3^{(11)} + k_1 k_2 J_3^{(2)} + \ldots + k_1^{n_x} J_3^{(0n_x)}.
\]
That is, the integral \(J_3\) of the non-linear system \((k_1 \neq 0, k_2 \neq 0)\) appears as a deformation, not of the function \(I_3\) itself, but of its square \(I_3^2\) (this property was already mentioned in [4]).

Next, we give the expressions of the constant \(J_3\) for the three first cases:

(i) Potential \(V_a(1, 1)\) corresponding to a central (isotropic) harmonic oscillator.
In this case we have \(\omega_1 = \omega_2 = \omega_0\) and the integral of motion \(I_3\) of the associated linear system is just the angular momentum, \(I_3 = (1/\omega_0) \text{Im}(K_{xy}) = xv_y - yv_x\). Then we have
\[
\text{Re}(M_{xy}) = 4E_x E_y - 2\omega_0^2 I_3,
\]
with \(J_3\) given by
\[
J_3 = I_3^2 + k_1 \left(\frac{y}{x}\right)^2 + k_2 \left(\frac{x}{y}\right)^2.
\]
(ii) Potential $V_a(2, 1)$ corresponding to a non-isotropic 2:1 oscillator with frequencies $\omega_1 = 2\omega_0$, $\omega_2 = \omega_0$.

If we denote by $I_3$ the integral of motion of the associated linear system

$$I_3 = \frac{1}{2\omega_0} \text{Im}(K_{xy}) = (xv_y - yv_x)v_y - \omega_0^2 xy^2,$$

then we obtain

$$\text{Re}(M_{xy}) = 8E_xE_y^2 - 8\omega_0^2 J_3,$$

with $J_3$ given by

$$J_3 = I_3^2 + k_1 \left( \frac{y^2}{x^2} \right) v_y^2 + \frac{k_2}{2y^2} (yv_x - 2xv_y)^2 + \frac{k_1 k_2}{2x^2} + k_2^2 \left( \frac{x^2}{y^2} \right).$$

(iii) Potential $V_a(3, 1)$ corresponding to a non-isotropic 3:1 oscillator with frequencies $\omega_1 = 3\omega_0$, $\omega_2 = \omega_0$.

If we denote by $I_3$ the constant of motion of the associated linear system

$$I_3 = \frac{1}{\omega_0} \text{Im}(K_{xy}) = 3(xv_y - yv_x)v_y^2 + \omega_0^2 (yv_x - 9xv_y)y^2,$$

then we obtain

$$\text{Re}(M_{xy}) = 16E_xE_y^3 - 2\omega_0^2 J_3,$$

with $J_3$ given by

$$J_3 = I_3^2 + k_1 J_3^{(10)} + k_2 J_3^{(01)} + k_1 k_2 J_3^{(1,1)} + k_2^2 J_3^{(1,2)} + k_1 k_2 J_3^{(0,2)} + k_2^3 J_3^{(0,3)},$$

with the functions $J_3^{(10)}$, $J_3^{(01)}$, $J_3^{(1,1)}$, $J_3^{(0,2)}$ and $J_3^{(0,3)}$, given by

$$J_3^{(10)} = \frac{y^2}{x^2} (3v_y^2 - \omega_0^2 y^2)^2,$$

$$J_3^{(01)} = \frac{3 y^2}{y^2} (2yv_x v_y - 3xv_y^2 + 3\omega_0^2 xy^2)^2,$$

$$J_3^{(1,1)} = \frac{12 v_y^2}{x^2},$$

$$J_3^{(0,2)} = \frac{3 y^4}{y^4} (3xv_y - yv_x)^2,$$

$$J_3^{(1,2)} = \frac{3 y^2}{y^2},$$

$$J_3^{(0,3)} = \frac{9 x^2}{y^6}.$$

Summarizing, within the $V_a(n_x, n_y)$ family, only in the particular isotropic 1:1 case the function $J_3$ is quadratic in the velocities and, because of this, only in this case the superintegrability arises from separability in two different coordinate systems. In all the remaining cases, $V_a(n_x, n_y)$ is a superintegrable but not superseparable potential.
3 Hamiltonian formalism and $n$ degrees of freedom

The previous section was directly focused on the two-dimensional potential $V_a(n_x, n_y)$. Now we present the study of the general case of $n$ degrees of freedom but using a rather different approach. The idea is to start with an integrable but very general Hamiltonian $H$ and then look for the properties to be satisfied by $H$ in order to admit superintegrability determined by appropriate complex functions $K_i$.

Let $H$ be the following Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V, \quad V = \sum_{i=1}^{n} V_i(x_i),$$

defined in a $2n$-dimensional phase space $T^*Q$ ($Q$ is the $n$-dimensional configuration space) endowed with the standard Poisson bracket

$$\{R, S\} = \sum_i \left( \frac{\partial R}{\partial x_i} \frac{\partial S}{\partial p_i} - \frac{\partial R}{\partial p_i} \frac{\partial S}{\partial x_i} \right).$$

It is clear that $H$ is integrable; now we set out the problem to determine the expressions of the functions $V_i(x_i)$ to admit superintegrability (but without making use of the property of separability).

Let us now define a set of $n$ functions (possibly complex) linear in the momenta

$$K_i = p_i + f_i(x_i), \quad i = 1, 2, \ldots, n,$$

(note that we have chosen the coefficient of $p_i$ equal to 1). Then, it is evident that

$$\{K_i, K_j\} = 0,$$

but

$$\{K_i, H\} = f'_i(x_i)p_i - V'_i(x_i) \neq 0,$$

(except in the trivial case $f_i(x_i) = \text{constant}$ and $V_i(x_i) = \text{constant}$). Our next step is to write the Hamiltonian in terms of the $K_i$ functions. Since we assume that these functions can be complex, we impose

$$H = \frac{1}{2} \sum_{i=1}^{n} K_i K_i^* = \frac{1}{2} \sum_{i=1}^{n} |K_i|^2$$

where $K_i^*$ is the complex conjugate of $K_i$. So we have

$$|K_i|^2 = p_i^2 + p_i(f_i(x_i) + f^*(x_i)) + |f(x_i)|^2.$$
If we assume that the Hamiltonian $H$ is quadratic in the momenta, and without linear terms, then the functions $f_i(x_i)$ should be pure imaginary. So we arrive to

$$K_i = p_i + i g_i(x_i), \quad V = \frac{1}{2} \sum_{i=1}^{n} g_i(x_i)^2,$$

with $g_i(x_i)$ real functions.

### 3.1 From integrability to superintegrability via the functions $K_i$

The Hamiltonian $H$ is trivially integrable, the $n$ functions $|K_i|^2$ are functionally independent constants of motion and the Hamiltonian is half their sum. In order to study its superintegrability, we make use of the tensor $T_{ij}$ defined by

$$T_{ij} = K_i K_j^* \quad \text{with} \quad H = \frac{1}{2} \sum_{i=1}^{n} T_{ii}.$$

Since the Poisson bracket of the functions $K_i$ with the Hamiltonian $H$ is now given by

$$\{K_i, H\} = g'_i(x_i)(i p_i - g_i(x_i)) = i g'_i(x_i)K_i,$$

then the off diagonal tensor components have the following Poisson bracket with the Hamiltonian

$$\{T_{ij}, H\} = \{K_i, H\} K_j^* + K_i \{K_j^*, H\} = i (g'_i(x_i) - g'_j(x_j)) T_{ij}.$$

Thus if the $g'_i(x_i)$ are a numerical constant (independent of $i$) then the functions $T_{ij}$ are constants of the motion. This happens with

$$g_i(x_i) = \omega_0 x_i, \quad i = 1, \ldots, n,$$

so we get the isotropic harmonic oscillator in $n$ dimensions which is certainly superintegrable:

$$H_1 = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \omega_0^2 \sum_{i=1}^{n} x_i^2.$$

### 3.2 Generalizing the Hamiltonian $H_1$

If we consider, instead of the tensor $T$, a new tensor $\tilde{T}$ with components

$$\tilde{T}_{ij} = K_i^{\ast n_j} (K_j^{\ast})^{n_i}, \quad i, j = 1, \ldots, n,$$
(where \( n_i \) are positive integers) we can repeat the arguments above and obtain

\[
\{ \tilde{T}_{ij}, H \} = n_j \{ K_i, H \} K_j^{n_j-1} (K_j^*)^{n_j} + n_i K_i^{n_i} \{ K_j^*, H \} (K_j^*)^{n_i-1}
\]

\[
= i(n_j g'_i(x_i) - n_i g'_j(x_j)) \tilde{T}_{ij}.
\]

Thus if the functions \( g_i(x_i) \) satisfy the relations

\[
n_j g'_i(x_i) - n_i g'_j(x_j) = 0, \quad i, j = 1, \ldots, n,
\]

then the functions \( \tilde{T}_{ij} \) (components of the new tensor \( \tilde{T} \)) are constants of the motion. In this way we obtain the Hamiltonian of the rational anisotropic harmonic oscillator (non-central harmonic oscillator with rational ratio of frequencies)

\[
H_2 = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \omega_0^2 \sum_{i=1}^{n} n_i^2 x_i^2
\]

which represents the most simple superintegrable generalization of the Hamiltonian \( H_1 \).

### 3.3 Generalizing the Hamiltonian \( H_2 \)

In order to obtain a superintegrable generalization of the Hamiltonian \( H_2 \), we first consider the squares of the functions \( K_i \)

\[
K_i^2 = p_i^2 - n_i^2 \omega_0^2 x_i^2 + 2i n_i \omega_0 x_i p_i, \quad i = 1, \ldots, n,
\]

satisfying

\[
\{ K_i^2, H_2 \} = \{ K_i, H_2 \} K_i + K_i \{ K_i, H_2 \} = 2i n_i \omega_0 K_i^2,
\]

and then we introduce a deformation of the real part

\[
M_i = p_i^2 - n_i^2 \omega_0^2 x_i^2 + h_i(x_i) + 2i n_i \omega_0 x_i p_i, \quad i = 1, \ldots, n.
\]

Now let us consider the following Hamiltonian

\[
H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \omega_0^2 \sum_{i=1}^{n} n_i^2 x_i^2 + \sum_{i=1}^{n} V_i(x_i),
\]

where \( V_i(x_i) \) are some functions to be determined by imposing that \( |M_i|^2 \) Poisson commutes with \( H \). We have

\[
\{ |M_i|^2, H \} = 2(h_i' - 2V_i') p_i^3 + 2 \left[ 2(h_i' - 2V_i') h_i - n_i^2 \omega_0^2 x_i^2 (4h_i + x_i h_i' + 2x_i V_i') \right] p_i,
\]

so that we arrive at

\[
h_i' - 2V_i' = 0, \quad 4h_i + x_i h_i' + 2x_i V_i' = 0,
\]
with solution
\[ V_i(x_i) = \frac{1}{2} h_i(x_i) = \frac{k_i}{2x_i^2}, \quad i = 1, \ldots, n, \]
(up to inessential additive constants). Hence, the above Hamiltonian \( H \) becomes
\[
H_3 = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \omega_0^2 \sum_{i=1}^{n} n_i^2 x_i^2 + \frac{1}{2} \sum_{i=1}^{n} \frac{k_i}{x_i^2},
\]
in such a way that the \( n \) functions \(|M_i|^2\) coincide with the square of the \( n \) energies \( E_i \) up to a constant
\[
|M_i|^2 = 4(E_i^2 - k_i n_i \omega_0^2), \quad E_i = \frac{1}{2} p_i^2 + \frac{1}{2} \omega_0^2 n_i^2 x_i^2 + \frac{k_i}{2x_i^2}.
\]
The important point is that the functions \( M_i \) satisfy
\[
\{M_i, H_3\} = 2 i n_i \omega_0 M_i.
\]
Hence, if we denote by \( M_{ij} \) the functions defined by the products
\[
M_{ij} = M_i^{n_j} (M_j^*)^{n_i} = \left( p_i^2 - n_i^2 \omega_0^2 x_i^2 + \frac{k_i}{x_i^2} + 2i n_i \omega_0 x_i p_i \right)^{n_j} \times \left( p_j^2 - n_j^2 \omega_0^2 x_j^2 + \frac{k_j}{x_j^2} - 2i n_j \omega_0 x_j p_j \right)^{n_i},
\]
then we have
\[
\{M_{ij}, H_3\} = 0, \quad i, j = 1, \ldots, n,
\]
what means that both the real part and the imaginary part of \( M_{ij} \) are constants of the motion for \( H_3 \) (the diagonal functions \( M_{ii} \) are real and, as we have seen, they correspond to the energies \( E_i \)). As the system has \( n \) degrees of freedom (and the phase space is \( 2n \)-dimensional) the maximum number of independent constants of motion is \( N = 2n - 1 \); so it is clear that not all of these quantities will be functionally independent but, nevertheless, we can extract from them a fundamental set of \( 2n - 1 \) functionally independent invariants. So we conclude that the Hamiltonian \( H_3 \) is superintegrable for all the values of the integer numbers \( n_i \) and for arbitrary values of the constants \( k_i \).

Next we prove the above statement (it is always possible to extract, from the large number of constants of motion, a fundamental set of \( 2n - 1 \) functionally independent functions) by following the same arguments used for the harmonic oscillator (isotropic and nonisotropic). We first recall the following two points

(i) In the isotropic \( n \)-dimensional case, the Fradkin tensor \( F \) \footnote{[16]} is represented by a symmetric \( n \)-dimensional matrix \( F_{ij} = p_i p_j + \omega_0^2 x_i x_j, \quad i, j = 1, 2, \ldots, n \), so that
it provides a total set of \((\frac{1}{2})n(n + 1)\) constants of motion. The integrability is consequence of the \(n\) diagonal entries \(F_{ii}\) (related with the energies \(E_i\) that are independent in a trivial way; so we have

\[
dF_{11} \wedge dF_{22} \wedge \ldots \wedge dF_{nn} \neq 0.
\]

For proving the superintegrability we recall that every nondiagonal function \(F_{ij}\) only depends of the four variables \((x_i, p_i, x_j, p_j)\). Thus we can add, for example, the \(n - 1\) entries \(F_{jj+1}\) of the upper-next-diagonal so that we obtain

\[
dF_{11} \wedge dF_{22} \wedge \ldots \wedge dF_{nn} \wedge dF_{12} \wedge dF_{23} \wedge \ldots \wedge dF_{n-1n} \neq 0.
\]

Hence, these \(N = 2n - 1\) constants of motion are functionally independent.

(ii) In the non-isotropic \(n\)-dimensional case, the method of the complex factorization \([13, 14]\) discussed in subsection (2.1) leads to a complex Hermitian \(n\)-dimensional matrix \(K_{ij}\), \(i, j = 1, 2, \ldots, n\), with \(K_{ij}\) only depending of the four variables \((x_i, p_i, x_j, p_j)\); so the property of independence

\[
dK_{11} \wedge dK_{22} \wedge \ldots \wedge dK_{nn} \wedge d(\text{Im } K_{12}) \wedge d(\text{Im } K_{23}) \wedge \ldots \wedge d(\text{Im } K_{n-1n}) \neq 0,
\]

is also true in this case (of course in this complex case it is also possible to choose the Real part of the functions \(K_{ij}\)).

Now, in the case of the nonlinear Hamiltonian \(H_3\), we have obtained a complex Hermitian \(n\)-dimensional matrix \(M_{ij}\), \(i, j = 1, 2, \ldots, n\), that can be considered as a nonlinear deformation of the matrix \(K_{ij}\). But, since \(M_{ij}\) only depends of the four variables \((x_i, p_i, x_j, p_j)\), we also have the following property

\[
dM_{11} \wedge dM_{22} \wedge \ldots \wedge dM_{nn} \wedge d(\text{Im } M_{12}) \wedge d(\text{Im } M_{23}) \wedge \ldots \wedge d(\text{Im } M_{n-1n}) \neq 0.
\]

Thus, we have proved the existence of a set of \(N = 2n - 1\) functionally independent constants of motion.

We close this section with an interesting property. If we return (for ease of notation) to two degrees of freedom, then in the case of the harmonic oscillator \((k_1 = 0, k_2 = 0)\), the constant of the motion \(I_4 = \text{Re}(K_{xy})\) can be obtained (up to a factor) as the Poisson bracket of \(I_3 = \text{Im}(K_{xy})\) with the energy \(E_x\) (for example, in the 1:1 case the component \(F_{xy}\) of the Fradkin tensor arises as the Poisson bracket of the angular momentum \(J\) with \(E_x\)). This property is preserved by the deformation \((k_1 \neq 0, k_2 \neq 0)\) and remains true for the nonlinear system.

**Proposition 2** The Poisson brackets of \(\text{Re}(M_{xy})\) and \(\text{Im}(M_{xy})\) with \(E_x\) are given by

\[
\{\text{Im}(M_{xy}), E_x\} = 2\omega_0 n_x n_y \text{Re}(M_{xy}), \\
\{\text{Re}(M_{xy}), E_x\} = -2\omega_0 n_x n_y \text{Im}(M_{xy}).
\]
Note that the second Poisson bracket means that $\text{Im}(M_{xy})$ (degree $2(n_x + n_y) - 1$) is just the Poisson bracket of $J_3$ with $E_x$

$$\{J_3, E_x\} = -\frac{2n_x n_y}{\lambda \omega_0} \text{Im}(M_{xy})$$

where $\lambda$ is a numerical coefficient.

4 Comments and open questions

As stated in the introduction the superintegrability of $V_a(n_x, n_y)$ was firstly proved in [4] and then in [6, 7, 8] by the use of different methods. Of course these methods are all correct (the dimensional reduction has been previously applied to the study of a certain number of integrable systems) but we think that the approach presented in this paper (deformation of the quadratic equation) has the great advantage of possessing a great level of elegance and simplicity. Moreover it is related with one of the more fundamental properties of the harmonic oscillator.

We note that although this method is rather different from the dimensional reduction they have in common some important points. In the geometric method [6, 7, 8], the authors start with a four dimensional harmonic oscillator $V_4 = \frac{1}{2} \sum_{a=1}^{4} \omega^2 n_a^2 s_a^2$ and then they obtain the nonlinear system by reducing the dimension from $n = 4$ to $n = 2$. In the method presented in this paper we also start with an harmonic oscillator but then we obtain the nonlinear system by a deformation of the complex quadratic equation. Thus, in both cases the starting point is the (linear) harmonic oscillator, which is superintegrable in any dimension.

Finally, it is natural to think that the introduction of the deformation (functions $h_i(x_i)$) in the quadratic equation can also be applied to equations of order higher than 2. Therefore, a natural generalization of this formalism would be the search of new superintegrable systems by introducing deformations in higher powers $K_i^m$ with $m > 2$ of the functions $K_i$. Note that we have restricted the study to deformations of the real part of the functions $K_i^2$; nevertheless in the more general case the deformations could be introduced also in the imaginary part of the complex functions. This question, as well as some other related problems (as the properties of the quantum version of this system) are open questions to be studied.

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