Pertfect matching and zero-sum 3-magic labeling

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Abstract
A mapping $l : E(G) \rightarrow A$, where $A$ is an abelian group which written additively, is called a labeling of the graph $G$. For every positive integer $h \geq 2$, a graph $G$ is said to be zero-sum $h$-magic if there is an edge labeling $l$ from $E(G)$ into $\mathbb{Z}_h \setminus \{0\}$ such that $s(v) = \sum_{uv \in E(G)} l(uv) = 0$ for every vertex $v \in V(G)$. In 2014, Saieed Akbari, Farhad Rahmati and Sanaz Zare conjectured that every 5-regular graph admits a zero-sum 3-magic labeling. In this paper, we obtained that every 5-regular graph with every edge contains in a triangle must have a perfect matching, and admits a zero-sum 3-magic labeling, which partially confirms this conjecture.

Keywords: 5-regular graph; zero-sum 3-magic labeling; perfect matching

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1. Introduction
Graphs considered here are all connected, finite and undirected with multiple-edges and loops permitted. The symbols $k'(G)$, $\Delta(G)$ and $\delta(G)$ denotes the edge-connectivity, maximum degree and minimum degree of $G$ respectively. The union of graphs $G_1, \ldots, G_k$, written $G_1 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^k V(G_i)$ and edge set $\bigcup_{i=1}^k E(G_i)$. A 3-cycle is also known as a triangle. If every vertex in a graph has the same degree $r$ then this graph is referred to as a $r$-regular graph. An odd (or even) component of a graph is a connected component of odd (or even) order, and an odd (or even) vertex is a vertex with odd (or even) degree. A matching $M$ in $G$ is a set of independent edges, and $|M|$ denotes the number of edges in $M$. A factor of a graph $G$ is a spanning subgraph of $G$. A $k$-factor of $G$ is a factor of $G$ that is $k$-regular. Thus a 1-factor of $G$ is a matching that saturates all vertices of $G$, and is called a perfect matching of $G$. A mapping $l : E(G) \rightarrow A$, where $A$ is an abelian group which written additively, is called a labeling
of the graph $G$. Given a labeling $l$ of the graph $G$, the symbol $s(v)$, which represents the sum of the labels of edges incident with $v$, is defined to be $s(v) = \sum_{uv \in E(G)} l(uv)$, where $v \in V(G)$. For every positive integer $h \geq 2$, a graph $G$ is said to be zero-sum $h$-magic if there is an edge labeling from $E(G)$ into $\mathbb{Z}_h\setminus\{0\}$ such that $s(v) = 0$ for every vertex $v \in V(G)$. Perfect matching is an important parameter in graph theory and has been received lots of research focusing on the characterization of it. In case of the regular graphs, although Peterson proved that

**Lemma 1.1.** Every bridgeless 3-regular graph has a perfect matching.

but little is known about the 5-regular graphs. In this paper we will provide some characterizations for 5-regular graph containing perfect matchings. The reason for considering this problem is due to the following conjecture proposed by Saieed Akbari, Farhad Rahmati and Sanaz Zare.

**Conjecture 1.** Every 5-regular graph admits a zero-sum 3-magic labeling

In this paper, we give an affirmative answer to this conjecture for some classes of 5-regular graphs. The following lemma is essential in the present paper.

**Lemma 1.2.** A graph $G$ has a perfect matching if and only if $q(G-S) \leq |G|$ for all $S \subseteq V(G)$, where $q(G-S)$ denotes the number of odd components of $G-S$.

If a graph $G$ has no perfect matching, then there must exist $S \subseteq VG$ with $q(G-S) > |G|$. Such a set $S$ is called an antifactor set of $G$.

Then we will prove the theorem:

**Theorem 1.3.** Every 5-regular graph in which each edge is within a triangle has a perfect matching

According to Theorem 1.3 every 5-regular graph in which each edge is within a triangle has a perfect matching. We suppose that the edge set of the perfect maching denoted by $EM$, we make a labeling $l : E(EM) \rightarrow 2$, and $E(E(G) - EM) \rightarrow 1$. It is easily to see it is a zero-sum 3-magic-labeling of 5-regular graph in which each edge is within a triangle has a perfect matching.

Throughout the rest of this article, we will prove theorem 1.3. The following definition will be useful for our prove:
Definition 1.1. For a graph G, for any vertex set S of G, and any odd component Q_i of G − S, consider the connection part of G(S, Q_i) which has odd edges in E(S, Q_i), its edge induced subgraphs vertex induced subgraph in G, We call it an odd component induced subgraph of Q_i, denoted by P_{ij} (where i, j means the jth odd component induced subgraph of the ith odd component).

As we can see in Figure 1, the vertex set of P_{ij} is \{v_1, v_2, v_3, v_4\}, and the edge set of P_{ij} is \{v_1v_2, v_1v_3, v_2v_4, v_3v_4\}.

Definition 1.2. For any vertex set S, and any vertex v ∈ S which is connected to d odd components in G − S, and if moving v to any one of the odd components which is connected with v, then for the new vertex set S’ we have q(G − S’) = q(G − S) − k and |S’| = |S| − k, then we call the vertex v a movable vertex.

Definition 1.3. For any vertex set S, any vertex pair (v, w)(v, w ∈ S) and (v, w) are connected to d odd components in G − S, and if moving (v, w) to any of the odd components which are connected with them, then for the new vertex set S’ we have q(G − S’) = q(G − S) − k and |S’| = |S| − k, then we call the vertex pair (v, w) a movable vertex pair.

Definition 1.4. For any vertex set S, consider one of the odd component of G − S, if it has an odd components induced subgraph P_{ij}, there exist a j such that there is a movable vertex or a movable vertex pair in P_{ij}, then we call this odd component the first type of odd component, otherwise known as a second type of odd component.
Definition 1.5. For a given vertex set $S$, an odd component of $G - S$ and its odd components induced subgraph $P_{ij}$, a substructure is called a 2-claw structure, if its vertex set is the vertex $\{v, v_1, v_2\} \in S \cap V(P_{ij})$, and each vertex connects 3 different odd components. And the degree of one of the vertex is 3 in $P_{ij}$, and the other 2 vertexs’ degree are 2 in $P_{ij}$, which means $d_{P_{ij}}(v) = 3$ and $d_{P_{ij}}(v_i) = 2$ ($i = 1, 2$).

Terminologies and notations not defined here can be found in [4] for graph theory in general and in [5] for special topics of factors and factorizations.
2. The proof of Theorem 1.3

In order to prove our theorem, we also need some lemmas.

**Lemma 2.1.** For any vertex set $S$, every odd component of $G - S$ has at least one odd component induced subgraph.

*Proof.* For any vertex set $S$, and any odd component $Q_i$ of $G - S$, we assume $|E(S, Q_i)| = d$, the number of vertices in $Q_i$ is $p$. Since graph $G$ is a 5-regular graph, according to the definition of the odd component, we know that $p$ is odd. Consider the sum of the degrees of each vertex in $Q_i$, denoted by $q$. According to the vertex degree theorem, we know that $q$ is even, and according to the definition of $G$, $p$ is odd, and we can easily see:

$$q = 5p - d$$

So $d$ is also odd. Therefore, the sum of the edges of each connected portion of $G(S, Q_i)$ is odd, so $E(S, Q_i)$ has at least one connected subgraph having an odd number of edges, so its edge induced subgraph’s vertex induced subgraph in $G$ is an odd component induced subgraph. $\square$

**Lemma 2.2.** For every $P_{ij}$ there exist at least one vertex $v \in S \cap V(P_{ij})$, such that $d_{P_{ij}}(v) \geq 3$.

*Proof.* Assume that there is no $v$ that satisfies the condition. For any vertex $v$ in $S \cap V(P_{ij})$, we have $d_{P_{ij}}(v) \leq 2$. Since each edge is in a triangle, there is no vertex in $S \cap V(P_{ij})$ such that $d_{P_{ij}}(v) = 1$ holds. Therefore, if for any vertex $v$ in $S \cap V(P_{ij})$, $d_{P_{ij}}(v) = 2$ is satisfied, this makes each connected subgraph of $G(S, Q_i)$ a triangle, and the sum of the edges of each connected portion of $G(S, Q_i)$ is even, a contradict. $\square$

**Lemma 2.3.** If a vertex $v$ in $S$ is connected with $d$ odd components, then $v$ is a movable vertex ( where $d \leq 2$).

*Proof.* For any vertex $v$ in $S$, if $v$ is connected with only one odd component, denoted by $Q_i$, then we move $v$ to $Q_i$, and the new vertex set we get denoted as $S_1$, obviously $|S_1| = |S| - 1$, considering the number of odd components $q(G - S_1)$, in this case, obviously there is no change in the other odd components except $Q_i$, and $Q_i$ originally has an odd number of vertices, and now it has a new vertex, the number of the vertex in $Q_i$ is changed to an
even number, so \( Q_i \) is not an odd component any more, that is, \( q(G - S_1) = q(G - S) - 1 \), so \( v \) is a movable vertex.

If \( v \) is connected with 2 odd components, denoted by \( Q_i, Q_j \), then we move \( v \) to any odd component of \( Q_i, Q_j \), lets assume that \( v \) moves to \( Q_i \), and the new vertex set we get denoted as \( S_1 \). Obviously that \( |S_1| = |S| - 1 \), considering the number of odd components \( q(G - S_1) \), obviously there is no change in the other odd components except \( Q_i, Q_j \), and because \( v \) moves to \( Q_i \), and \( v \) is connected with \( Q_j \), so at this time \( Q_i, Q_j \) and \( v \) become a connected component, and the number of its vertices \( |Q_i| + |Q_j| + 1 \) is odd, so the number of odd components is reduced by one, that is, \( q(G - S_1) = q(G - S) - 1 \), so \( v \) is a movable vertex.

Similar but a little different (if \( d \leq 3 \) it is the same situation with Lemma 2.3) we can get the next Lemma for the vertex pair:

\textbf{Lemma 2.4.} If a vertex pair \((v, w)\) in \( S \) is connected with \( d \) odd components, then \((v, w)\) is a movable vertex pair (where \( d = 3 \)).

\textit{Proof.} We assume the 3 odd components connect with the vertex pair \((v_1, v_2)\) are \( Q_i, Q_j \) and \( Q_k \), and we move the vertex pair \((v_1, v_2)\) to any of the odd components of \( Q_i, Q_j \) and \( Q_k \). We supposed to move \((v_1, v_2)\) to \( Q_i \), and the new vertex set we get denoted as \( S_1 \). Obviously \( |S_1| = |S| - 2 \). Consider the number of odd components \( q(G - S_1) \), obviously there is no change in the other odd components except \( Q_i, Q_j \) and \( Q_k \). Since the vertex pair \((v_1, v_2)\) move to \( Q_i \), and the vertex pair \((v_1, v_2)\) are connected with \( Q_j \) and \( Q_k \), so the component \( Q_i, Q_j \) and \( Q_k \) and the vertex pair \((v_1, v_2)\) become a connected component, and the number of its vertices \( |Q_i| + |Q_j| + |Q_k| + 2 \) is odd, so the number of odd components is reduced by two, that is, \( q(G - S_1) = q(G - S) - 2 \), therefore, the vertex pair \((v_1, v_2)\) is a movable vertex pair. \(\square\)

\textbf{Lemma 2.5.} If \( Q_i \) is a second type of odd component, then for any \( j \), \( P_{ij} \) has one unique structure, which is 2-claw structure.

\textit{Proof.} Given a 5-regular graph \( G \) in which each edge is within a triangle, take any subset \( S \) from \( V(G) \). According to Lemma 2.1, every odd component \( Q_i \) has an odd component induced subgraph \( P_{ij} \). Lets consider the vertex \( v \) which has maximum degree in \( V(P_{ij}) \cap \)
S, according to Lemma 2.2, we have:

\[ 3 \leq d_{P_{ij}}(v) \leq 5 \]

Now, we have three cases:

**Case 1.** \( d_{P_{ij}}(v) = 5 \)

In this case, since graph \( G \) is a 5-regular graph, all the edges connected with \( v \) are in \( P_{ij} \), so there is only one odd component connected with \( v \), which can be known by Lemma 2.3, where \( v \) is a movable vertex, and \( Q_i \) is a first type of odd component.

**Case 2.** \( d_{P_{ij}}(v) = 4 \)

In this case, since graph \( G \) is a 5-regular graph, only one of the edges which connect with \( v \) is not in \( P_{ij} \), so there are at most 2 odd components connected with \( v \). We can know from Lemma 2.3 that \( v \) is a movable vertex, and \( Q_i \) is a first type of odd component.

**Case 3.** \( d_{P_{ij}}(v) = 3 \)

In this case, consider about the number of edges in \( E(S) \cap E(P_{ij}) \), we have 4 subcases:

**Subcase 3.1.** \( v \) has 0 edge in \( E(S) \cap E(P_{ij}) \)

In this subcase, since graph \( G \) is a 5-regular graph, and \( v \) has 3 edges in \( P_{ij} \), so at most 3 odd components are connected with \( v \). If \( v \) connect with 2 other different odd components \( Q_t, Q_s \), and we supposed \( e_{Q_t}(v) \) connect with \( Q_t \), and \( e_{Q_s}(v) \) connect with \( Q_s \). Let's consider about \( e_{Q_t}(v) \): according to the definition of \( G \), \( e_{Q_t}(v) \) must be an edge of a triangle, thus there must exist another edge connect with \( v \) in this triangle we call it \( e_t \), if \( e_t = e_{Q_s}(v) \) then we will have \( Q_t \) is connected with \( Q_s \) it is a contradiction, because \( Q_t, Q_s \) are different odd components. For the same reason, \( e_t \notin E(P_{ij}) \), so \( e_t \) is the 6th edge which is connected with \( v \), a contradiction.

Thus in this case \( v \) at most connect with 2 different odd components (\( Q_i \) and another different odd component), according to Lemma 2.3, \( v \) is a movable vertex, and \( Q_i \) is a first type of odd component.

**Subcase 3.2.** \( v \) has 1 edge in \( E(S) \cap E(P_{ij}) \)
Figure 3. $d_{P_i}(v) = 3$ and $v$ has 0 edge in $E(S) \cap E(P_{ij})$

In this subcase, $v$ has 2 edges in $E(Q_i, S)$. The vertices called $v_1$, $v_2$, if $v_i \in Q_i$ and $v_3$ is adjacency with $v$. And the vertex called $v_3$, if $v_3$ is adjacency with $v$ and $v_3 \in S$.

We can see that $v$ already has 3 neighbors in $V(P_{ij})$, and $v$ already connect with one odd component($Q_i$), for $G$ is 5-regular, so $v$ at most connect with 2 other odd components. If $v$ connect with $k$ other odd components( besize $Q_i$), which $k = 0, 1$, according to Lemma 2.3, $v$ is a movable vertex, and $Q_i$ is a first type of odd component.

Or else, $v$ connect with 2 other odd components $Q_t$, $Q_s$. We call the vertex $v_4$, $v_5$, such that $v_4$ connect with $Q_t$ and $v_5$ connect with $Q_s$. Because of the definition of $G$, $v_4$ must be in a triangle, so the vertex set of this triangle is $\{v, v_4, v_q\}$. And $v_q$ is a neighbor of $v$, but $v$ already has 5 neighbors. So $v_q \notin \{v_i|i = 1, 2, 4, 5\}$, for $Q_i$, $Q_t$, $Q_s$ are different odd components, so any two of $Q_i$, $Q_t$, $Q_s$ are not connected. Thus $v_q = v_3$. For the same reason, $v_3$ is connected with $Q_s$. Now we consider about $v_3$. Because $v_3 \in V(P_{ij})$, there exist at least one edge which join $v_3$ is in $E(P_{ij}, S)$. Thus we know 4 neighbors of $v_3$, which are $v$, $v_4$, $v_5$ and a vertex in $Q_i$, and for the pair of vertex $(v, v_3)$, it connect 3 odd components, if the last neighbor of $v_3$ is not in another odd component, according to Lemma 2.4 $(v, v_3)$ is a movable vertex pair of $Q_i$. Then $Q_i$ is a first type of odd component.

Or else, we assume the last neighbor of $v_3$ is in another odd component, we call it $v_p$, and we call the edge $e$, which join $v_p$ and $v_3$. According to the definition of $G$, $e$ is in a triangle, so $v_3$ must has an other neighbor which adjacency with $v_p$, for the reason we
Figure 4. \( d_{P_{ij}}(v) = 3 \) and \( v \) has 1 edge in \( E(S) \cap E(P_{ij}) \)

have already said this vertex can’t be \( v_4, v_5 \) or the vertex in \( Q_i \), but the last vertex which adjacency to \( v_3 \) is \( v \), which means \( v \) has 6 neighbors, a contradiction.

Thus the pair of vertex \((v, v_3)\) at most connect with 3 odd components. According to Lemma 2.3 and Lemma 2.4, either \( v \) is a movable vertex or \((v, v_3)\) is a movable vertex pair of \( Q_i \), in conclusion, \( Q_i \) is a first type of odd component.

Subcase 3.3. \( v \) has 2 edge in \( E(S) \cap E(P_{ij}) \)

In this subcase, we call the 2 vertices \( v_1, v_2 \) which is the neighbor of \( v \) and in \( S \cap V(P_{ij}) \). If one of the vertex in \( \{v, v_1, v_2\} \) connect with \( k \) other odd components \( (k \leq 1) \), then according to Lemma 2.3, \( v_i \) is a movable vertex, and \( Q_i \) is a first type of odd component.

Or else, there are 2 odd components different from \( Q_i \) connect with \( v \), there are at most 3 odd components different from \( Q_i \) connect with any one of \( v_1, v_2 \). Since \( G \) is 5-regular.

We assume that there exist a vertex \( v_i \in \{v_1, v_2\} \), which connect with 3 odd components different from \( Q_i \), and the edges join \( v_i \) and the odd comptents denoted by \( e_1, e_2, e_3 \). There is only one edge joins \( v_i \) in \( S \), we denote it by \( e_4 \), and \( v_i \) is connected with \( Q_i \), the edge denotes as \( e_5 \). Since each edge is in a triangle, there will be an edge \( e_j \) connect with any one of \( e_1, e_2, \) and \( e_3 \). According to the previous discussion \( e_j \neq e_i \ (i = 1, 2, 3, 5) \). Therefore \( e_j = e_4 \), because \( j \in \{1, 2, 3\} \), so \( v \) needs 3 neighbors which is not in \( P_{ij} \), which means \( v \) has 6 neighbors, a contradiction.
In conclusion, in this subcase, if one of the vertex in \(\{v, v_1, v_2\}\) connect with \(k\) other odd components \((k \leq 1)\), then \(Q_i\) is a first type of odd component. Or each of the vertex in \(\{v, v_1, v_2\}\) connect with 2 other odd components, according to the definition it is a second type of odd component.

**Subcase 3.4.** \(v\) has 3 edge in \(E(S) \cap E(P_{ij})\).

In this subcase, it means \(E(v) \cap E(P_{ij}) = \emptyset\), this contradicts the definition of \(P_{ij}\). so there dont exist a vertex which has 3 edges in \(E(S)\).

In other words, an odd component \(Q_i\) is a second type of odd component if and only if for any of its odd component induced subgraph \(P_{ij}\), the vertex \(v\) which has maximum degree in \(V(P_{ij}) \cap S\) has 2 neighbors in \(V(P_{ij}) \cap S\) and each of them connect 3 different odd components, which means a 2-claw structure. \(\square\)

Next lets prove Theorem 1.3

For a given graph \(G\) that satisfies the condition, we will explain below that \(G\) must have a perfect matching. Let's take any vertex set \(S\). According to Lemma 2.5, we know that for any odd component \(Q_i\), if it is a first type of odd component, then there exist an odd component induced subgraph \(P_{ij}\), so that \(P_{ij}\) has a movable vertex or a movable vertex pair.

We put this vertex or pair into \(Q_i\), we assume that we put \(k_1\) vertices and \(k_2\) pairs out of \(S\), we have \(q(G-S) - k_1 - 2k_2\) odd components denoted by \(q_k\) and we call the set \(S_k\), which is the subset of \(S\) and move out \(k_1, 2k_2\) vertexs, thus there are \(|S| - k_1 - 2k_2\) vertices in \(S_k\).
Lets consider the $S_k$, which already take $k_1$, $2k_2$ vertexs out of $S$. For any odd component of $G - S_k$, denoted by $Q_i$, and $Q_i$ is a second type of odd component, we just consider one of its odd component induced subgraph $P_{ij}$, it has at least 3 neighbors $\{v_1, v_2, v_3\}$ in $S_k$, which means $E(Q_i, S_k) \geq 3$, therefore all the second type of odd components need at least $3q(G - S_k)$ edges.

On the other hand, for any 2-claw structure of the odd component $Q_i$, we denote the 3 vertexs which in the vertex set $S_k$ as $v_{ij}$ ($j = 1, 2, 3$), and by definition we know for each $v_{ij}$ ($j = 1, 2, 3$) is connected with 3 different odd components.

The set of all vertexs $v_{ij}$ in the 2-claw structure, denoted by $S_p$, since the edges between the 2-claw structures and $S_k$ are provided by vertexs in $S_p$, and each odd component exists at least one 2-claw structure, so there is $3|S_p| \geq 3q_k$. According to the definition of $S_p$, which has $|S_k| \geq |S_p|$, so:

$$3|S_k| \geq 3|S_p| \geq 3q(G - S_k)$$

Then,

$$|S_k| \geq q(G - S_k)$$

So,

$$|S| \geq q(G - S)$$

In graph $G$, due to the arbitrariness of $S$, we have $|S| \geq q(G - S)$ for all vertices sets $S$, there exist a perfect matching in $G$ from Tutte’s theorem.
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