High Fidelity Quantum Gates for Trapped Ions under Micromotion

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Two or three dimensional Paul traps can confine a large number of ions forming a Wigner crystal, which would provide an ideal architecture for scalable quantum computation except for the micromotion, an issue that is widely believed to be the killer for high fidelity quantum gates. Surprisingly, here we show that the micromotion is not an obstacle at all for design of high fidelity quantum gates, even though the magnitude of the micromotion is significantly beyond the requirement of the Lamb-Dicke condition. Through exact solution of the quantum Mathieu equations, we demonstrate the principle of the gate design under micromotion using two ions in a quadrupole Paul trap as an example. The proposed micromotion quantum gates can be extended to the many ion case, paving a new way for scalable trapped ion quantum computation.

PACS numbers: 03.67.Lx, 03.67.Ac, 37.10.Ty

Trapped ions constitute one of the most promising systems for realization of quantum computation [1]. All the quantum information processing experiments so far are done in linear Paul traps, where the ions form a one-dimensional (1D) crystal along the trap axis [1-3]. In this configuration, the external radio-frequency (r.f.) Paul trap can be well approximated by a static trapping potential with negligible micromotion, which is believed to be critical for design of high fidelity quantum gates. However, in term of scalability, the linear configuration is not the optimal one for realization of large scale quantum computation: first, the number of ions in a linear trap is limited [3]; and second, the linear configuration is not convenient for realization of fault-tolerant quantum computation. The effective qubit coupling in a large ion chain is dominated by the dipole interaction, which is only good for short-range quantum gates because of its fast decay with distance. In a linear chain with short range quantum gates, the error threshold for fault tolerance is very tough and hard to be met experimentally [1-7].

From a scalability point of view, two (2D) or three dimensional (3D) Paul traps would be much better for quantum computation compared with a linear chain. In a 2D or 3D trap, one can hold a large number of qubits with a high error threshold for fault tolerance, in the range of a percent level, even with just the nearest neighbor quantum gates [7]. Thousands to millions of ions have been successfully trapped to form 2D or 3D Wigner crystals in a Paul trap [8]. However, there is a critical problem to use this system for quantum computation, i.e., the micromotion issue. In the 2D or 3D configuration, micromotion cannot be compensated, and the magnitude of micromotion for each ion can be significantly beyond the optical wavelength (i.e., outside of the Lamb-Dicke region). As the micromotion is from the driving force of the Paul trap, it cannot be laser cooled. The messy and large-magnitude micromotion well beyond the Lamb-Dicke condition is believed to be a critical hurdle for design of entangling quantum gate operations in this architecture.

In this paper, we show that the micromotion surprisingly is not an obstacle at all for design of high-fidelity quantum gates. When the ions form a crystal in a time-dependent Paul trap, they will be described by a set of Mathieu equations. We solve exactly the quantum Mathieu equations in general with an inhomogeneous driving term and find that the micromotion is dominated by a well-defined classical trajectory with no quantum fluctuation. This large classical motion is far outside of the Lamb-Dicke region, however, it does not lead to infidelity of quantum gates if it is appropriately taken into account in the gate design. The quantum part of the Mathieu equation is described by the secular mode with a micromotion correction to its mode function. This part of motion still satisfies the Lamb-Dicke condition at the Doppler temperature, which is routine to achieve for experiments. We use two ions in a quadrupole trap, which have large micromotion, as an example to show the principle of the gate design, and give the explicit gate scheme both in the slow and the fast gate regions using multi-segment laser pulses [1-10], with the intrinsic gate infidelity arbitrarily approaching zero under large micromotion. We finally give a brief discussion of the general procedure of the gate design under micromotion, which in principle can work for any number of ions, with important implication for large-scale quantum computation.

To illustrate the general feature of micromotion in a Paul trap and the principle of the gate design under micromotion, we consider a three-dimensional (3D) anisotropic quadrupole trap with a time dependent potential \( \Phi(x, y, z) = (U_0 + V_0 \cos(\Omega_T t)) \left( \frac{x^2 + y^2 - 2z^2}{a_0^2} \right) \equiv \alpha(t)(x^2 + y^2 - 2z^2) \) from an electric field oscillating at the r.f. \( \Omega_T \), where \( U_0, V_0 \) are voltages for the d.c. and a.c. components and \( a_0 \) characterizes the size of the trap. We choose a positive \( U_0 \) to reduce the effective trap strength along the \( z \) direction so that the two ions align along the \( z \)-axis. Since the motions in different directions do not
coordinate operators couple to each other under quadratic expansion, we focus our attention on the z direction. The total potential energy of two ions (each with charge e and mass m) is

$$V(z_1, z_2) = -2e\alpha(t) \left( z_1^2 + z_2^2 \right) + \frac{e^2}{4\pi\epsilon_0 |z_1 - z_2|}. \quad (1)$$

Define center-of-mass (CM) coordinate $u_{cm} = (z_1 + z_2)/2$ and relative coordinate $u_r = z_1 - z_2$. Without loss of generality, we assume $u_r > 0$ and its average $\bar{u}_r = u_0$. We assume the magnitude of the ion motion is significantly less than the ion separation, which is always true for the ions in a crystal phase. The Coulomb interaction can then be expanded around the average distance $\bar{u}_r$ up to the second order of $|u_r - u_0|$. Under this expansion, the total Hamiltonian $H = p_{cm}^2/4m + p_r^2/m + V(z_1, z_2)$ is quadratic (although time-dependent) in terms of the coordinate operators $u_{cm}, u_r$ and the corresponding momentum operators $p_{cm} = p_1 + p_2$, $p_r = (p_1 - p_2)/2$. The Heisenberg equations under this Hamiltonian $H$ yield the following quantum Mathieu equations respectively for the coordinate operators $u_{cm}$ and $u_r$

$$\frac{d^2 u_{cm}}{dt^2} + (a_{cm} - 2q_{cm} \cos (2\xi)) u_{cm} = 0 \quad (2)$$

$$\frac{d^2 u_r}{dt^2} + (a_r - 2q_r \cos (2\xi)) u_r = f_0 \quad (3)$$

where the dimensionless parameters $a_{cm} = -16eU_0 / (m\Omega^2 |\Omega_T2|)$, $a_r = a_{cm} + 4e^2 / (\pi\epsilon_0 mu^2 |\Omega_T2|)$, $q_{cm} = q_r = 8eV_0 / (m\Omega^2 |\Omega_T2|)$ and the dimensionless time $\xi = \Omega_T t/2$. The driving term $f_0 = 6e^2 / (\pi\epsilon_0 mu^2 |\Omega_T2|)$. The quantum operators $u_{cm}$ and $u_r$ satisfy the same form of the Mathieu equations (except for the driving term $f_0$) as for the classical variables. As these equations are linear, we can use the solutions known for the classical Mathieu equation to construct a quantum solution that takes into account of the quantum fluctuation.

It is well known that the solution to the classical Mathieu equation $\frac{d^2 v}{dt^2} + (a - 2q \cos (2\xi)) v = 0$ is a combination of Mathieu sine $S(a, \xi)$ and Mathieu cosine $C(a, \xi)$ functions, which reduce to the conventional sine and cosine functions when micromotion is neglected [11]. The solution to a homogeneous quantum Mathieu equation $\frac{d^2 \hat{u}}{dt^2} + (a - 2q \cos (2\xi)) \hat{u} = 0$ can be described using the reference oscillator technique [12]. From the classical solution $v$ and the quantum operator $\hat{u}$, one can introduce the following annihilation operator of a reference oscillator (remember that $\xi = \Omega_T t/2$ is the dimensionless time)

$$\hat{u}(t) = \sqrt{m / 2\hbar\omega} \left( v(t) \hat{a}(t) - \hat{v}(t) \hat{a}^\dagger(t) \right), \quad (4)$$

where $\omega$ is a normalization constant typically taken as the secular motion frequency of the corresponding Mathieu equation. In addition, we impose the initial condition for $v(t)$ with $v(t)|_{t=0} = 1$ and $\dot{v}(t)|_{t=0} = i\omega$.

The position operator $\hat{u}(t)$ and its conjugate momentum $\hat{p}(t) \equiv \hat{m} \hat{u}(t)$ satisfy the commutator $[\hat{u}(t), \hat{p}(t)] = i\hbar$. From the above definition, one can easily check that

$$\frac{d}{dt} \hat{u}(t) \propto v \frac{d^2}{dt^2} \hat{u}(t) - \hat{u} \frac{d^2}{dt^2} v = 0,$$

so $\hat{u}(t) \equiv \hat{a}$ is a constant of motion. Furthermore, $\hat{a}$ satisfies the standard commutator

$$[\hat{a}, \hat{a}^\dagger] = (m/2\hbar\omega)(i\hbar/m) (v(t)v^*(t) - v^*(t)v(t))|_{t=0} = 1.$$

When micromotion is neglected, $v(t) = e^{i\omega t}$ and $\hat{a}$ reduces to the annihilation operator of a harmonic oscillator. In the presence of micromotion, $v(t) = C(a, q, \xi) + iS(a, q, \xi)$. The solution to the position operator $\hat{u}$ takes the form

$$\hat{u}(t) = u_0 (v^*(t)\hat{a} + v(t)\hat{a}^\dagger) \quad (5)$$

where $u_0 \equiv \sqrt{\hbar/2m\omega}$ is the oscillation length.

The above solution gives a complete description of the center-of-mass motion with the operator

$$u_{cm}(t) = u_{0cm} (v^*(t)\hat{u}_{cm} + v(t)\hat{u}_{cm}^\dagger), \quad (6)$$

where $u_{0cm} \equiv \sqrt{\hbar/4m\omega_{cm}}$ and $\omega_{cm}$ is the secular frequency of the center of mass mode. The relative motion $u_r$ satisfies the inhomogeneous quantum Mathieu equation (3). To solve it, we let $u_r = u_r^0 + \bar{u}_r$, where $u_r^0$ is an operator that inherits the commutators for $u_r$ and satisfies the homogeneous quantum Mathieu equation and $\bar{u}_r$ is a classical variable corresponding to a special solution of the Mathieu equation $\frac{d^2 \bar{u}_r}{dt^2} + (a_0 - 2q_0 \cos (2\xi)) \bar{u}_r = f_0$. The special solution $\bar{u}_r$ can be found through the series expansion $\bar{u}_r = f_0 \sum_{n=0}^{\infty} c_n \cos (2n\xi)$, where the expansion coefficients $c_n$ satisfy the recursion relations $a_0 c_0 - q_0 c_1 = 1$ and $c_n = D_n (c_{n-1} + c_{n+1} + c_0 \delta_{n1})$ for $n \geq 1$ with $D_n \equiv \omega_r / (4n^2 - a_0)$. When $a_0 < 1$ and $q_0 \ll 1$, which is typically true under real experimental configurations, $c_n$ rapidly decays to zero with $|c_{n+1}/c_n| \approx q_0/4(n+1)^2$ and we can keep only the first few terms in the expansion and obtain an approximate analytical expression for $\bar{u}_r [13]$. The complete solution of $u_r$ is then given by

$$u_r(t) = u_{0r} (v^*(t)\hat{a}_r + v(t)\hat{a}_r^\dagger) + \bar{u}_r(t), \quad (7)$$

where $u_{0r} \equiv \sqrt{\hbar/m\omega_r}$ and $\omega_r$ is the secular frequency of the relative mode.

Now we show how to design high fidelity quantum gates under micromotion. To perform the controlled phase flip (CPF) gate, we apply laser induced spin dependent force on the ions, with the interaction Hamiltonian described by [10]

$$H = \sum_{j=1}^{2} \hbar \Omega_j \cos (k_\delta z_j + \mu_\delta t + \phi_j) \sigma_j^z. \quad (8)$$

where $k_\delta$ is the wave vector difference of the two Raman beams along the z direction, $\mu_\delta$ is the two-photon Raman detuning, $\Omega_j$ (real) is the Raman Rabi frequency.
for the ion $j$, and $\phi_j$ is the corresponding initial phase. In terms of the normal modes, the position operators $z_j = u_{cm} - (1)^j u_{t}/2$, where $u_{cm}$, $u$ are given by Eqs. (6) and (7). We introduce three Lamb-Dicke parameters, $\eta_{cm} \equiv k_{\beta} u_{cm}$ for the CM mode, $\eta_{t} \equiv k_{\beta} u_{t}/2$ for the relative mode, and $\eta_{\text{nm}} \equiv k_{\beta} \eta_{t}/2$ for pure micromotion. Under typical experimental configurations, $\eta_{cm} \sim \eta_{t} \ll 1$.

The parameter $\eta_{\text{nm}}$ is a classical variable that oscillates rapidly with time by multiples of the micromotion frequency $\Omega_{\tau}$. In Fig. 1(a), we show a typical trajectory of $\eta_{\text{nm}}(t)$. The magnitude of variation of $\eta_{\text{nm}}$ is considerably larger than 1. The function from the micromotion. The magnitude of $v_{cm}(t)$ is bounded by a constant slightly larger than 1. The function $v_{cm}(t)$ is very similar, but except that $\omega_{cm}$ is replaced by $\omega_{\tau}$. From this consideration of parameters, we can expand the term $\cos(k_{\beta} z_j + \mu_\delta t + \phi_j)$ with small parameters $\eta_{cm}, \eta_{t}$, but $\eta_{\text{nm}}$ is a big term which needs to be treated exactly. After the expansion, to leading order in $\eta_{cm}$ and $\eta_{t}$, the Hamiltonian $H$ takes the form

$$H \approx -[\chi_1(t)\sigma^z_1 + \chi_2(t)\sigma^z_2] \hat{f}_{cm} - [\chi_1(t)\sigma^z_1 - \chi_2(t)\sigma^z_2] \hat{f}_{\tau},$$

where we have defined

$$\hat{f}_\mu \equiv \eta_{\mu} \left( v^*_\mu(t)\hat{a}_\mu + v_\mu(t)\hat{a}^\dagger_\mu \right), \quad \chi_j(t) \equiv \hbar \Omega_j \sin \left[ \mu_\delta t + \phi_j - (-1)^j \eta_{\text{nm}}(t) \right],$$

where the subscript $\mu = \text{cm, \tau}$ and $j = 1, 2$. In Eq. (9), we have dropped the term $\cos(\mu_\delta t + \phi_j \pm \eta_{\text{nm}})$ which induces single-bit phase shift but is irrelevant for the CPF gate. The evolution operator at the gate time $\tau$ generated by the Hamiltonian $H$ can be expressed as

$$U(\tau) = D_{\text{cm}}(\alpha_{cm})D_{\tau}(\alpha_\tau) \exp \left[ i(\gamma_{\tau} - \gamma_{cm})\sigma^z_1 \sigma^z_2 \right] \hat{f}_{cm},$$

where the displacement operator $D_{\mu}(\alpha_{\mu}) \equiv \exp(\alpha_{\mu}\hat{a}^\dagger_{\mu} - \alpha^*_\mu\hat{a}_{\mu})$ ($\mu = \text{cm, \tau}$). Let $\gamma_{\mu} = 1$ for $\mu = \text{cm}$ and $\gamma_{\mu} = -1$ for $\mu = \tau$. The displacement $\alpha_{\mu}$ and the accumulated phase $\gamma_{\mu}$ have the following expression

$$\alpha_{\mu} = i\eta_{\mu} \int_0^\tau (\chi_1(t)\sigma^z_1 + j_\mu \chi_2(t)\sigma^z_2) u_{\mu}(t) \, dt,$$

$$\gamma_{\mu} = i(\eta_{\mu})^2 \int_0^\tau \int_0^t \int_0^{t_2} S [\chi_1 \chi_2] \text{Im} \left[ u_{\mu}(t_1) u^*_\mu(t_2) \right] \, dt \, dt_2,$$

where $S [\chi_1 \chi_2] \equiv \chi_1(t_1) \chi_2(t_2) + \chi_1(t_2) \chi_2(t_1)$. To realize the CPF gate, we require $\alpha_{\mu} = 0$ and $\gamma_{\tau} - \gamma_{cm} = \pi/4$. The integrals $\alpha_{\mu}$ can be evaluated semi-analytically [13] or purely numerically. We normally take $\Omega_1 = \Omega_2 \equiv \Omega$. Note that even in this case $\chi_1(t_1) \neq \chi_2(t_2)$ with the micromotion term $\eta_{\text{nm}}(t)$. This is different from the case of a static trap. From Eq. (12), we see that

$\alpha_{\mu} = 0$ for a fixed $\mu$ gives two complex and thus four real constraints. With excitation of $N$ motional modes, the total number of (real) constraints to realize the CPF gate is therefore $4N + 1$ (the condition $\gamma_{\tau} - \gamma_{cm} = \pi/4$ gives one constraint). To satisfy these constraints, we divide the Rabi frequency $\Omega(t)$ ($0 \leq t \leq \tau$) into $m$ equal-time segments, and take a constant $\Omega_{\delta} (\delta = 1, 2, \ldots, m)$ for the $\beta$th segment. This kind of modulation can be conveniently done through an acoustic optical modulator in experiments [14]. The Rabi frequencies are our control parameters. For the two ion case, under fixed detuning $\mu_\delta$ and gate time $\tau$, in general we can find a solution for the CPF gate with $m = 9$ segments. For some specific detuning $\mu_\delta$ very close to a secular mode frequency, off-resonant excitations become negligible and a solution is possible under one segment of pulse by tuning of the gate time $\tau$, which corresponds to the case of the Sørensen-Mølmer gate [3] generalized to include the micromotion correction.

To characterize the quality of the gate, we use the fidelity $F \equiv \text{tr}_\mu \left[ \rho_\mu \left| \Psi_0 \right\rangle U_{\text{CPF}}^\dagger U(\tau) \left| \Psi_0 \right\rangle \right]^2$, defined as the overlap of the evolution operator $U(\tau)$ with the perfect one $U_{\text{CPF}} \equiv e^{i\pi \sigma^z_1 \sigma^z_2/4}$ under the initial state $\left| \Psi_0 \right\rangle$ for the ion spins and the thermal state $\rho_\mu$ for the phonon modes. In our calculation, without loss of generality, we take $\left| \Psi_0 \right\rangle = (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)/2$ and assume the Doppler temperature $T_D$ for all the phonon modes. For any given detuning $\mu_\delta$ and gate time $\tau$, we optimize the control parameters $\Omega_{\delta} (\delta = 1, 2, \ldots, m)$ to get the maximum fidelity $F$. In Fig. 2, we show the gate fidelity as a function of gate time for $\mu_\delta = 0.95 \omega_{cm}$ (close to a secular frequency) by applying a single segment laser pulse of a constant Rabi frequency $\Omega$. In the figure, the dashed line corresponds to the result in a static harmonic trap.
with the same secular frequencies but no micromotion. If we take into account the micromotion contribution but do not change the gate design, the result is described by the dash-dot line, with a low fidelity about only 50%. When we optimize the gate design (optimize $\Omega_\beta$) including the micromotion correction, the gate fidelity is represented by the solid line, which approaches the optimal fidelity achievable in a static trap. The gate infidelity $\delta F = 1 - F$ approaches $2 \times 10^{-3}$ at the optimal gate time $\tau = 20.005 T_z$, where $T_z = 2\pi/\omega_{cm}$.

By applying 9 segments of laser pulses with optimized $\Omega_\beta$ ($\beta = 1, 2, \cdots, 9$), the gate fidelity $F$ can attain the unity at arbitrary detuning $\mu_\beta$ for the two ion case. As an example, In Fig. 3(a), we show the optimized solution of $\Omega_\beta$ (blue lines) at an arbitrarily chosen detuning $\mu_\beta = 1.4 \omega_{cm}$. For comparison, the red lines represent the solution of $\Omega_\beta$ in a static harmonic trap with otherwise the same parameters. The maximum magnitude of $|\Omega_\beta|$ significantly increases in the presence of micromotion. This is understandable as fast oscillations of the micromotion tend to lower the effective Rabi frequencies. In Fig. 3(b), we show the maximum magnitude of $|\Omega_\beta|$ as a function of the gate time $\tau$. Compared with the solution in a static harmonic trap, the maximum $|\Omega_\beta|$ in general needs to increase by about an order of magnitude under micromotion.

In conclusion, we demonstrate that arbitrarily high fidelity quantum gates can be achieved under large micromotion. The demonstration in this paper uses the example of two ions in a quadrupole trap, which has the micromotion magnitude significantly beyond the Lamb-Dicke limit. Apparently, the idea here is applicable to the many ion case. For a system of $N$ ions in any dimension, as long as the ions crystallize, each ion has an average equilibrium position. We can then expand the Coulomb potential around these equilibrium positions. Under the r.f. Paul trap and the Coulomb interaction, the motion of the ions can then be described by a set of coupled time-dependent Mathieu equations. Using the technique in this paper, we can solve the motional dynamics and optimize the gate design that explicitly takes into account all the micromotion contributions. The gate design technique under micromotion proposed in this paper solves a major obstacle for high fidelity quantum computation in real r.f. traps beyond the 1D limitation and opens a new way for scalable quantum computation based on large 2D or 3D trap-ion crystals in Paul traps.

Acknowledgments. This work was supported by the NBRPC (973 Program) 2011CBA00300 (2011CBA00302), the IARPA MUSIQC program, the ARO and the AFOSR MURI programs, and the DARPA OLE program.

Appendix: Supplementary Material

In this appendix, we show in detail how to solve the driven Mathieu equation and give an approximate treatment of the motional integrals.
SOLUTION OF DRIVEN MATHEIU EQUATION

We show in detail how to solve the Mathieu equation with a constant drive term.

\[ \frac{d^2u}{dt^2} + (a - 2q \cos(2\xi))u = f_0 \]

Let us assume that \( u(\xi) = f_0 \sum_{n=0}^{\infty} c_n \cos(2n\xi) \) and insert it into the equation. After re-organization, we get

\[ ac_0 - qc_1 + \sum_{n=1}^{\infty} [(a - 4n^2)c_n - q(c_{n-1} + c_{n+1}) - qc_0\delta_{n,1}] \cos(2nt) = 1. \]

Defining \( D_n \equiv (a - 4n^2)/q \), we have the following set of linear equations

\[ ac_0 - qc_1 = 1 \]
\[ c_n - \frac{1}{D_n}(c_{n-1} + c_{n+1} + c_0\delta_{n,1}) = 0. \]

In matrix form,

\[
\begin{pmatrix}
\frac{a - q}{D_1} & 0 & \cdots & 0 \\
0 & \frac{1}{D_2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \frac{1}{D_{n-1}} & 0 \\
0 & \cdots & \cdots & \frac{1}{D_n}
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
0
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}. \tag{15}
\]

The factor \( 1/D_n \) decreases very fast as \( n \) increases and we can truncate the expansion of \( u(\xi) \) at a small \( n \). Numerically we observe that typically keeping up to \( c_2 \) already gives enough accuracy. We can thus get a very accurate analytical expression

\[ c_0 \approx \frac{64 + a(a - 20) - q^2}{(32 - 3a)q^2 + a(a - 4)(a - 16)}, \]
\[ c_1 \approx \frac{2(a - 16)q}{(32 - 3a)q^2 + a(a - 4)(a - 16)}, \]
\[ c_2 \approx \frac{2q^2}{(32 - 3a)q^2 + a(a - 4)(a - 16)}. \]

For the example in the main text, \( a_r = -0.0388 \) and \( q_r = 0.283 \), we have \( c_0 = 1132.8 \) and \( u_r(\xi) = f_0 c_0 \left[ 1 - 0.14 \cos(2\xi) + 0.0025 \cos(4\xi) + \cdots \right] \).

The micromotion corrected equilibrium position is \( f_0 c_0 \) and should be identified with \( u_0 \) around which we expand the Coulomb potential in the first place. Thus we should determine them self-consistently. Taking the relative motion in the manuscript as an example, since both \( a_r \equiv \frac{16eU_0}{md^2_{\mathcal{F}} + \frac{4\pi^2}{\pi_{\mathcal{F}}m_{\mathcal{F}}^2}\pi_{\mathcal{F}}} \) and \( f_0 \equiv \frac{6e^2}{\pi_{\mathcal{F}}m_{\mathcal{F}}^2}\pi_{\mathcal{F}} \) are functions of \( u_0 \), then the self-consistent equation

\[ u_0 = f_0 c_0 \approx f_0 \frac{64 + a_r(a_r - 20) - q_r^2}{(32 - 3a_r)q_r^2 + a_r(a_r - 4)(a_r - 16)} \]

gives the correct \( u_0 \). With the iterative method it typically takes only a few iterations to converge to the correct value when starting from a proper initial value of \( u_0 \).

TWO-STAGE TIME INTEGRAL

Here we offer an approximate treatment of motional integrals. We notice that the secular frequency \( \omega \) and the micromotion frequency \( \Omega \) are well separated, i.e. \( \omega \ll \Omega \). This means quantities with characteristic frequency \( \omega \) or
below stay constant within one period of micromotion. So we can perform the time integral in two steps: we first integrate over one period of the micromotion, obtaining a slowly varying integrand, which we then integrate again. By doing this we will show that the dominant effect of micromotion is to modulate the effective Rabi frequency. Notice that the integrals \( \int_0^\tau \chi(t)u(t) \, dt \) can be reduced to the form below (ignoring micromotion frequencies \( n\Omega \pm \omega \) with \( n \geq 2 \))

\[
I = \int_0^\tau \sin (a_0(t) + a_1(t) \cos(\Omega t + \phi(t))) (b_0(t) + b_1(t) \cos (\Omega t + \varphi(t)))
\]

where \( a_0(t), \, a_1(t), \, b_0(t), \, b_1(t) , \phi(t) \) and \( \varphi(t) \) are all real slowly varying functions within one period of micromotion \( \frac{2\pi}{\Omega} \). The above integral can be further broken into two parts, \( I_1 \) and \( I_2 \), where

\[
I_1 \approx \int_0^\tau dt \frac{\Omega}{2\pi} \int_{-\pi}^{\pi+2\pi/\Omega} dt_1 \sin (a_0(t) + a_1(t) \cos(\Omega t + \phi)) b_0(t)
\]

\[
= \int_0^\tau dt \frac{1}{2\pi} \int_{-\pi}^{\pi} dt' \sin (a_0(t) + a_1(t) \cos(t')) b_0(t)
\]

\[
= \text{Im} \left[ \int_0^\tau dt \exp(i a_0(t)) \frac{1}{2\pi} \int_{-\pi}^{\pi} dt' \exp(i a_1 \cos(t')) b_0(t) \right]
\]

\[
= \int_0^\tau dt \sin (a_0(t)) b_0(t) J_0(a_1(t))
\]

and

\[
I_2 = \int_0^\tau dt \sin (a_0(t) + a_1(t) \cos(\Omega t + \phi)) b_1(t) \cos (\Omega t + \varphi(t))
\]

\[
\approx \int_0^\tau dt \frac{\Omega}{2\pi} \int_{-\pi}^{\pi+2\pi/\Omega} dt_1 \sin (a_0(t) + a_1(t) \cos(\Omega t + \phi)) b_1(t) \cos (\Omega t + \varphi)
\]

\[
= \int_0^\tau dt \cos (a_0(t)) \cos(\varphi - \phi) J_1(a_1(t))
\]

where \( J_0 \) and \( J_1 \) denote the Bessel functions. In both cases, the micromotion gives rise to slowly varying modulation factors, \( J_0(a_1(t)) \) and \( \cos(\varphi - \phi)J_1(a_1(t)) \). Moreover in \( I_2 \) the phase of the original integrand is also shifted, \( \sin(a_0(t)) \rightarrow \cos(a_0(t)) \). For the actual experimental system, the term \( I_2 \) contributes much less than the \( I_1 \) to the target integral \( I \), due to the much smaller coefficient of the micromotion component than that of the secular component in \( v(t) \). So in leading order, micromotion reduces the laser Rabi frequency seen by the ion by a factor on the order of \( J_0(a_1(t)) \).

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