Analysis of Vehicle Routing Problem in Presence of Noisy Channels

Nishikanta Mohanty,1, * Christoher Ferrie,1, † and Bikash K. Behera2, ‡

1 Centre for Quantum Software and Information, University of Technology Sydney, Ultimo, NSW 2007, Australia
2 Bikash’s Quantum (OPC) Private Limited, Balindi, Mohanpur, 741246, Nadia, West Bengal, India

Vehicle routing problem (VRP) is an NP-hard optimization problem that has been an interest of research for decades in science and industry. The objective is to plan routes of vehicles to deliver a fixed number of customers with optimal efficiency. Classical tools and methods provide good approximations to reach the optimal global solution. Quantum computing and quantum machine learning provide a new approach to solving combinatorial optimization problems faster due to inherent speedups of quantum effects. Many solutions of VRP are offered across different quantum computing platforms using hybrid algorithms such as quantum approximate optimization algorithm and quadratic unconstrained binary optimization. Quantum computers such as IBM-Q experience limited distance covered by k vehicles constitutes the optimal route. This problem can be referred as a generalization of the classical traveling salesman problem (TSP)10,11, where now a group of k salesmen has to serve collectively (n - 1) locations, such that each location is served exactly once. In most real-world applications, the VRP problem is generally augmented by constraints, such as vehicle capacity or limited coverage time. Thus, many classical and quantum solutions have been proposed to arrive at a solution efficiently. Current quantum approaches for solving VRP includes QAOA16, QUBO17,18, Quantum Annealing19–21 and VQE.

However, in this project, we aim to look at this classic problem in a different light. Here we explore the existing solutions and try to find out if adding certain bias to an adapted quantum solution improves the overall solution. In the process, we analyze the effect of various noise channels on an existing yet variable ANSATZ that is developed as a solution to VRP. We apply amplitude damping, bit-flip, phase-flip, bit-phase-flip and depolarising noise channels to VRP circuit and analyze the effects and consolidate our findings.

The paper is organized as follows. Sec. II discusses the fundamental mathematical concepts such as combinatorial optimization (CO), adiabatic computation (AC), QAOA and Ising model. Sec. III discusses the formulation of VRP using the concepts discussed in previous Section. Then Sec. IV covers the basic building blocks of circuits to solve VRP. Sec. V covers the building of ANSATZ for VRP. Finally, Sec. VI covers the effects of applying noise models on the VRP circuit. We conclude...
by comparing the effects of various noise models on the VRP circuit and future directions of research.

II. MATHEMATICAL BACKGROUND

The fundamental concepts of solving routing problems lie in dealing with techniques/procedures to solve combinatorial optimization problems. This is followed by converting the mathematical models to a quantum equivalent mathematical model for formulating the objective function. The solution of the objective function is often achieved by maximization or minimization of the function. Under this section, we outline the concepts with the following heads

A. Combinatorial Optimization

A classical combinatorial optimization (CO) problem is the task of finding an optimal object from a finite set of objects. Exhaustive search is not practical in finding the optimal object due to very high number of objects. The only approach that exists is if the set of solutions are discrete or can be reduced to discrete with the aim to find the optimal solution. Hence CO problems are maximization or minimization problems with ‘s’ input strings in some set ‘S’ and m clauses with (S greater than equal to m). Each clause takes as input a string and returns a value. The total cost function of a string is the sum over the m clauses. If we denote the input string as z and clauses as $C_\alpha$, then the total cost function is written as

$$C(z) = \sum_{\alpha=1}^{m} C_\alpha (z)$$  \hspace{1cm} (1)

The goal is to find $\exists \in S$ such that $C(\exists) \geq C(z)$ for all $z \in S$ (in the case of minimization, $C(\exists) \leq C(z)$ for all $z \in S$). Here $\exists$ need not be unique. For simplification, restrictions are put on clauses and input strings where clauses are restricted to integers 0 and 1, whereas input strings are restricted to a binary representation of integers 0 through $2^n-1$. Hence z can be written as $z=z_0z_1z_2 \ldots z_{n-1}$ for $z_i \in \{0,1\}$. Also, considering only maximization problems, the minimization problems can be studied as

$$C'(z) = \sum_{k=0}^{m-1} C'_\alpha (z) = \sum_{k=0}^{m-1} (1 - C_\alpha (z)) = m - \sum_{k=0}^{m-1} C_\alpha (z) = m - C(z)$$  \hspace{1cm} (2)

B. Adiabatic Quantum Computation

Adiabatic quantum computation (AQC) is a theoretical framework of a quantum computer proposed by Farhi et al., who proposed QAQ in 2000. The principle behind AQC is the adiabatic theorem, which states that if a Hamiltonian is altered slowly enough, a system in the ground state of that Hamiltonian will remain in the ground state. Hamiltonian represents the energy state of a system. In AQC, we deal with two Hamiltonians: the driver Hamiltonian ($H_d$) and the problem Hamiltonian ($H_p$). The driver Hamiltonian ($H_d$) is a ground state of a system that is easy to prepare, whereas the problem Hamiltonian ($H_p$) is the desired state we are interested, and is obtained as a ground state after evolution. The time required is determined by the minimal energy difference between the interpolating Hamiltonian’s two lowest states. In general, we are unable to assess this disparity. In other words, we begin with an easy-to-prepare ground state (i.e., the ground state of ($H_d$)) and hope to end up with the quantum state we want (i.e., the ground state of ($H_p$)), which is usually significantly challenging to prepare. Mathematically, we have a function $s(t)$ on $[0,T]$ where $s(0) = 0$ and $s(T) = 1$. $T$ is the value of time set high enough for the adiabatic theorem to hold. We define the Hamiltonian, $H(t) = (1-s(t))H_d + s(t)H_p$. By the adiabatic theorem, given an appropriate $s(t)$, we stay in the ground state of this $H(t)$ during the entire interval $[0,T]$. Therefore, we can see that at time $t = 0$, we are in the ground state of ($H_d$) and by time $t = T$, we are in the desired ground state of ($H_p$). The time evolution under this time-dependent Hamiltonian involves an integral that is hard to evaluate:

$$U(t) = \tau exp \left\{ -\frac{i}{\hbar} \int_0^t H(T)dT \right\}$$  \hspace{1cm} (3)

Totterization technique can be used to evaluate this Hamiltonian. We discretize $U(T)$ into intervals of $\delta t$ small enough that the Hamiltonian is approximately constant over each interval. This allows us to use the much cleaner formula for the time-dependent Hamiltonian. If $U(b,a)$ represents time evolution from time a to time b,

$$U(T,0) = U(T,T-\delta t)U(T-\delta t,T-2\delta t) \cdots U(\delta t,0) = \prod_{j=1}^{p} U(j\delta t,(j-1)\delta t) \approx \prod_{j=1}^{p} e^{-iH(j\delta t)\delta t}$$  \hspace{1cm} (4)

Where the approximation improves as $p$ gets larger (or, equivalently, as $\delta t$ gets smaller) and we have chosen $\delta t$ to be in units of $\hbar$. Now using the approximation $e^{i(A+B)x} = e^{iAx}e^{iBx} + O(x^2)$ and adding Hamiltonian
gives the solution or the approximate solution of the expectation value of the cost function is a superposition state of all input qubits. The ex-

This AQC can be approximated by letting the system evolve under $H_P$ for some small $s(j\delta t)$ and then $H_D$ for some small $(1-s(j\delta t))\delta t$ and unitaries can be constructed for these operations $U= e^{-i\alpha H_D \delta t}$. Where $\alpha$ is some number in the interval $[0, 1]$, and that incorporates the scaling due to $s(j\delta t)$. AQC forms the theoretical basis of variational quantum algorithm QAOA, which is discussed briefly in ongoing section.

C. QAOA

Quantum Approximate Optimization Algorithm aka QAOA is a variational algorithm proposed by Farhi et al. in 2014\textsuperscript{5,29}. This algorithm relies on the framework of adiabatic quantum computation. It is an algorithm that uses both classical and quantum techniques thus, it is a hybrid algorithm. In the previous section, we touch upon the quantum adiabatic computation where we move from the eigenstate of driver Hamiltonian to that of the eigenstate of problem Hamiltonian. The problem Hamiltonian can be written as

$$C(z) = \sum_{\alpha=1}^{m} C_\alpha(z) |z\rangle$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (6)

We know that the maximum energy eigenstate of $C$ is the solution to the combinatorial optimization problem. Similarly, for driver Hamiltonian we use

$$B = \sum_{j=1}^{n} \sigma_j^x$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (7)

where $\sigma_j^x$ is the $\sigma^x$ Pauli operator on bit $z_j$ also know as the mixing operator. Let us also define $U_c(\gamma) = e^{-i\gamma C_z}$ and $U_B(\beta) = e^{-i\beta B}$ which lets the system evolve under $C$ for some $\gamma$ amount of time and under $B$ for some $\beta$ amount of time, respectively. Basically, QAOA constructs a state $|\beta, \gamma\rangle = e^{-i\beta B} e^{-i\beta C} \ldots e^{-i\beta B} e^{-i\beta C} e^{-i\gamma C} |s\rangle$, where $|s\rangle$ is a superposition state of all input qubits. The expectation value of the cost function $\sum_{\alpha=1}^{m} \langle \beta, \gamma | C | \beta, \gamma \rangle$ gives the solution or the approximate solution of the problem\textsuperscript{27,30}.

D. Ising Model

The ISING model was proposed by Lenz (1920) is a mathematical model for ferromagnetism in statistical mechanics\textsuperscript{32}. The model is made up of discrete variables that represent magnetic dipole moments of atomic “spins” in one of two states (+1 or 1). The spins are arranged in a graph, typically a lattice (in which the local structure repeats periodically in all directions), which allows each spin to interact with its neighbors. Neighboring spins that agree have less energy than those that disagree; the system tends to the lowest energy, but heat disrupts this tendency, allowing alternative structural phases to emerge. As a simplified model of reality, the model enables for the identification of phase transitions\textsuperscript{33}. The total energy of the spins is described by the following Hamiltonian

$$H_C = -\sum_{\langle i,j \rangle} J_{ij} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^z$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (8)

Where $J_{ij}$ represents the interaction between $i$ and $j$, which are adjacent spins. “$h$” represents an external magnetic field. If $J$ is positive, the ground state at $h = 0$ is a ferromagnet. If $J$ is negative, the ground state at $h = 0$ is an anti-ferromagnet for a bipartite lattice. Hence for simplification and in the context of this document, we can write the Hamiltonian as

$$H_C = -\sum_{\langle i,j \rangle} J_{ij} \sigma_i^{(i)} \sigma_j^{(i)} - h_i \sigma_i^{(i)}$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (9)

Here $\sigma_z$ and $\sigma_x$ represent Pauli $z$ and $x$ operator. And for simplification, we can consider the following conditions, $J_{ij} = \pm 1$ to ferromagnetic ($J_{ij} > 0$), $h = 0$ assuming no external influence on spin. Thus we can rewrite the Hamiltonian as following

$$H_C = -\sum_{\langle i,j \rangle} J_{ij} \sigma_i^{(i)} \sigma_j^{(j)} = -\sum_{\langle i,j \rangle} Z_i Z_j$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (10)

III. MODELLING VRP IN QUANTUM

To find a solution to the vehicle routing problem, we can map the cost function to an Ising Hamiltonian $H_C$\textsuperscript{35}. The minimization of Ising Hamiltonian $H_C$ gives the solution to the problem. To begin, let us consider an arbitrary connected graph of $n$ vertices and $n-1$ edges. Also, let us assume we have to route or vehicle among two vertices of the graph which are not adjacent. To formulate this, let us consider a binary decision variable $x_{ij}$ who has a value 1 if there exists an edge between $i$ and $j$ for edge weight $w_{ij} > 0$ else, the value is 0. Now to represent VRP problem, we need $n \times (n-1)$ decision variables. For every edge from $i \rightarrow j$ we define two sets of nodes source [$i$] and target [$j$]. The set source [$i$] contains the nodes $j$ to which $i$ sends an edge $j \in$ source[$i$]. The set target [$j$] contains the nodes $i$ to which $i$ sends an edge $i \in$ target[$j$]. We define VRP as follows
VRP\( (n, k) = \min_{\{x_{ij}\}_{i \to j \in \{0, 1\}}} \sum_{i \to j} w_{ij} x_{ij} \) \( (11) \)

Where \( k \) is number of vehicles and \( n \) is total number of locations, but considering the starting location as 0th location or Depot \( D \) we have \( n - 1 \) locations for vehicles to travel. This is of course subject to the following constraints,

\[
\begin{align*}
\sum_{j \in \text{source}[i]} x_{ij} &= 1, \forall i \in \{1, \ldots, n-1\} \\
\sum_{j \in \text{target}[i]} x_{ji} &= 1, \forall i \in \{1, \ldots, n-1\} \\
\sum_{j \in \text{source}[0]} x_{0j} &= k \\
\sum_{j \in \text{target}[0]} x_{j0} &= k 
\end{align*}
\( (12) \)

From the above set of constraints, the first two impose the restriction that each node must be visited only once by the delivering vehicle. The last two constraints enforce the restriction that the vehicle must return to the depot after delivering the goods. For the VRP equation and constraints, we can form the Hamiltonian of VRP as follows\(^{13} \)

\[
H_{VRP} = H_A + H_B + H_C + H_D + H_E
\]

\[
H_A = \sum_{i \to j} w_{ij} x_{ij}
\]

\[
H_B = A \left( \sum_{i \in 1, \ldots, n-1} \left( 1 - \sum_{j \in \text{source}[i]} x_{ij} \right)^2 \right)
\]

\[
H_C = A \left( \sum_{i \in 1, \ldots, n-1} \left( 1 - \sum_{j \in \text{target}[i]} x_{ji} \right)^2 \right)
\]

\[
H_D = A \left( k - \sum_{j \in \text{source}[0]} x_{0j} \right)^2
\]

\[
H_E = A \left( k - \sum_{j \in \text{target}[0]} x_{j0} \right)^2
\] \( (13) \)

\( \quad A > 0 \) is a constant. The set of all binary decision variables \( x_{ij} \) can be represented in vector form as

\[
\vec{d} = [x_{(0, 1)}, x_{(0, 2)}, \ldots, x_{(1, 0)}, x_{(1, 2)}, \ldots, x_{(n-1, n-2)}]^T
\] \( (14) \)

Using the above vector, we can define two additional vectors for each node (in the beginning of the section we defined two sets for source and target nodes so two vectors will represent) \( \vec{z}^S_{[i]} \) and \( \vec{z}^T_{[i]} \).

\[
\vec{z}^S_{[i]} = \{ x_{ij} = 1, x_{kj} = 0, k \neq i, \forall j, k \in \{0, \ldots, n-1\} \}
\]

\[
\vec{z}^T_{[i]} = \{ x_{ji} = 1, x_{jk} = 0, k \neq i, \forall j, k \in \{0, \ldots, n-1\} \}
\] \( (15) \)

The above vectors will assist in the formulation of the QUBO model of VRP. In general, the QUBO model\(^{17, 34-36} \) of a connected graph \( G = (N, V) \) is defined as,
\[ f(x)_{QUBO} = \min_{x \in \{0,1\}^{(N \times V)}} x^T Q x + g^T x + c \quad (16) \]

where, \( Q = \) quadratic coefficient or the edge weight, \( g = \) linear coefficient or the node weight and \( c = \) constant. \( J = \) matrix of all ones, \( I = \) identity matrix, \( e_0 = [1, 0, \ldots, 0]^T \). From the above Eqs, we can expand the Eq. 16 to form the ISING Hamiltonian of VRP\(^{18,37}\).

\[ H_{ISING} = -\sum_i \sum_{j < i} J_{ij} s_is_j - \sum_i h_is_i + d \quad (17) \]

Here \( S_{ij} \in \{-1, 1\} \) represents the spin variables which is transformed from our original binary decision variable \( x_{ij} = (s_{ij} + 1)/2 \). The terms \( J_{ij} , h_i , d \) are defined as follows,

\[ \begin{align*}
J_{ij} &= -\frac{Q_{ij}}{2}, \; \forall \; i < j \\
h_i &= \frac{g_i}{2} + \sum_{j < i} \frac{Q_{ij}}{4} + \sum_j \frac{Q_{ji}}{4} \\
d &= c + \sum_i \frac{g_i}{2} + \sum_{i < j} \frac{Q_{ij}}{4}
\end{align*} \quad (18) \]

IV. ANALYSIS AND CIRCUIT BUILDING

In this section, we analyze all the previous equations and come up with the circuit using the IBM gate model which we implement using Qiskit framework. For any arbitrary VRP problem using qubits, we begin with the state of \( |+\rangle^{\otimes (n-1)} \) the ground state of \( H_{mixer} \) by applying the Hadamard to all qubits initialized as zero state, and we prepare the following state.

\[ |\beta, \gamma\rangle = e^{-iH_{mixer}} |+\rangle \otimes |\beta, \gamma\rangle \]

(19)

The energy \( E \) of the state \( |\beta, \gamma\rangle \) is calculated by expectation of \( H_{cost} \) as follows.

\[ E = \langle \beta, \gamma | \hat{H}_{cost} | \beta, \gamma \rangle \quad (20) \]

By Ising model we can write \( H_{cost} \) as

\[ \hat{H}_{cost} = -\sum_i \sum_{j < i} J_{ij} \sigma_i^z \sigma_j^z - \sum_i h_i \sigma_i^z - d \quad (21) \]

Thus for a single term of state \( |\beta, \gamma\rangle \) as \( |\beta_0, \gamma_0\rangle \) as, \( e^{-iH_{mixer}} |\beta_0, \gamma_0\rangle \) the first term \( H_{cost} \) can be expanded to following

\[ e^{iJ_{0,0}}, |\gamma_0\rangle = (\cos J_{0,0} I + i \sin J_{0,0} \beta \sigma_i) |\beta, \gamma_0\rangle \]

\[ = \begin{bmatrix} e^{iJ_{0,0}} & 0 & 0 & 0 \\ 0 & e^{-iJ_{0,0}} & 0 & 0 \\ 0 & 0 & e^{-iJ_{0,0}} & 0 \\ 0 & 0 & 0 & e^{iJ_{0,0}} \end{bmatrix} \]

(22)

Applying \( CNOT \) gate on before and after the above matrix ‘A’ we can swap the diagonal elements

\[ CNOT(A)CNOT = \begin{bmatrix} e^{iJ_{0,0}} & 0 & 0 & 0 \\ 0 & e^{-iJ_{0,0}} & 0 & 0 \\ 0 & 0 & e^{iJ_{0,0}} & 0 \\ 0 & 0 & 0 & e^{-iJ_{0,0}} \end{bmatrix} \]

(23)

Observing the upper and lower blocks of matrix we can rewrite,

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} e^{iJ_{0,0}} & 0 \\ 0 & e^{-iJ_{0,0}} \end{bmatrix} = I \otimes e^{iJ_{0,0}} \begin{bmatrix} 1 & 0 \\ 0 & e^{-2iJ_{0,0}} \end{bmatrix} \]

(24)

is a phase gate. Looking at the 2\text{nd} term \( H_{cost} \) we get,

\[ H_{cost} = \sum_i h_i \sigma_i^z \]

\[ e^{ih_i \sigma_i} = \cos h_i \gamma_0 I + i \sin h_i \gamma_0 \sigma_i \]

\[ = \cos h_i \gamma_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i \sin h_i \gamma_0 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]

\[ = \begin{bmatrix} e^{ih_i \gamma_0} & 0 \\ 0 & e^{-ih_i \gamma_0} \end{bmatrix} \]

(25)

\[ e^{ih_i \gamma_0} \]

is a phase gate.

Fig. 3 depicts the basic circuit with two qubits along with gate selections for \( H_{cost} \), the full circuit generated by Qiskit in Sec. V.

V. VQE SIMULATION OF VRP

We construct the VRP circuit (Fig. 4) using the above Eqs and create the Hamiltonians for 3 city and 4 city scenarios. Since we need \( n(n-1) \) qubits we end up Hamiltonians and circuits with 6 and 12 qubits only. Beyond
4 cities, it is not possible for simulating in a classical desktop computer due to memory limitations. We create the ANSATZ using a complete set of random weights between 0 and 1 and run it across various VQE optimizers available in IBM Qiskit framework namely ‘COBYLA’, ‘L_BFGS_B’, ‘SPSA’, and ‘SLSQP’. We try running the circuit up to 5 layers across all the optimizers and obtain the following results. From the plots (Fig. 5), we can observe that COBYLA is clearly the best performing optimizer followed by SPSA, ‘L_BFGS_B’ and finally ‘SLSQP’. Since COBYLA obtains the lowest energy values among the 4 optimizers, COBYLA for simulation of VQE algorithms could prove advantageous. SPSA closely follows COBYLA in energy values though not the exact same values. To analyse further, we pass the circuit with various noise models available in Qiskit which are described in the further sections.

VI. NOISE MODEL SIMULATION OF VQE

In quantum noisy environment, a pure input state will be transformed into a mixed state, which is more effective to be represented by density operator. In our case considering a 6-qubit entangled state, $|\psi\rangle_{q_1,q_2,q_3,q_4,q_5,q_6}$ is a pure state where as the density matrix can be defined as $\rho = |\psi\rangle_{q_1,q_2,q_3,q_4,q_5,q_6}\langle\psi|_{q_1,q_2,q_3,q_4,q_5,q_6}$. After the implementation of noise model, the density matrix takes the following form,

$$\xi_r(\rho) = \sum_m (E_{m}^{\rho_0})(E_{m}^{\rho_1})...(E_{m}^{\rho_5})\rho \times (E_{m}^{x_{\rho_0}})(E_{m}^{x_{\rho_1}})...(E_{m}^{x_{\rho_5}})$$

where $r \in \{A, B, W, F, D\}$. The elements of the noise channels are described as follows, $A =$Amplitude damping noise, $B =$Bit-flip noise, $W =$Phase-flip noise, $F =$Bit-phase-flip noise, $D =$Depolarising noise. In our case, we apply these noise channels to our VRP circuit and ANSATZ which is variable based on the number of qubits (6 or 12) and layers (1 to 5). For simulation purpose, we choose two optimizers ‘COBYLA’ and ‘L_BFGS_B’ as two have different characteristics in terms of VQE performance also we restricted the noise probability to 0.5 as we believe complete noisy environments inside quantum computers are improbable. In preceding subsections we discuss about the noise channels and operators we experimented on our VRP circuit.

A. Amplitude Damping

The energy dissipation is a consequence of interaction of quantum system with an amplitude damping channel. In simple terms, a quantum system gaining or losing energy from or to its environment is described as change in its amplitude or amplitude damping. If $\kappa_A$ is the probability of gain or loss of amplitude or decoherence rate; the Kraus operators of amplitude damping channel can be described as follows (Eq. 27),

$$E_0^A = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1 - \kappa_A} \end{bmatrix}$$

$$E_1^A = \sqrt{\kappa_A} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

In our experiment, as per Eq. 27 we vary the noise probability from 0 to 0.5 and observe the energy values of VQE. As we execute the circuit with 1 to 4 layers with two of chosen optimizers ‘COBYLA’ and ‘L_BFGS_B’, we get the characteristics (Fig. 6).

As we look at the figures of amplitude damping noise simulation (Fig. 6), we observe that global minimum for both optimizers ‘COBYLA’ and ‘L_BFGS_B’ fluctuates a lot at lower number of qubits (6 qubits). For COBYLA, global minimum decreases gradually with the lowest energy at 4th layer while for ‘L_BFGS_B’ the fluctuation is consistent with the lowest being on the 4th layer. The characteristic however changes to more of consistent values for higher number of qubits (12 qubits). For ‘L_BFGS_B’, the values across the layers are close with minimum being at 3rd layer. Similarly, for ‘COBYLA’ the values of global minimums are consistent except the 3rd layer. We can therefore conclude that for higher number of qubit circuits, we can achieve global minimum with lesser number of layers.

B. Bitflip Noise

Bitflip noise is characterised by random bitflip errors with probability $0 \leq \kappa_B \leq 1$ (say for 0) while the other probability will be $1 - \kappa_B$ (say for 1). Thus, the Kraus operators of bitflip noise channel can be described as Eq. 28,

$$E_0^B = \sqrt{1 - \kappa_B}I$$

$$E_1^B = \sqrt{\kappa_B}X = \sqrt{\kappa_B} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

### Table I. VQE Simulation of amplitude damping channel for 6 and 12 qubits with 4 layers involving optimizers COBYLA and ‘L_BFGS_B’.

| Qubits | Layers | energy cost COBYLA | energy L_BFGS_B |
|--------|--------|---------------------|-----------------|
| 6      | 1      | -15.8838458         | -7.010115818    |
| 6      | 2      | -16.39073411        | -11.47507553    |
| 6      | 3      | -19.68135723        | -8.34777234     |
| 6      | 4      | -20.87440305        | -11.87197559    |
| 12     | 1      | -65.04439038        | -33.3599923     |
| 12     | 2      | -62.25281911        | -29.66629302    |
| 12     | 3      | -75.06834159        | -34.48916126    |
| 12     | 4      | -64.0657809         | -32.52414461    |
FIG. 4. VRP circuit showing gate selections generated by Qiskit. This is a one layer circuit of 6 qubits. The circuits include both terms $H_{\text{cost}}$ and $H_{\text{mixer}}$: while the first term is already depicted in Fig. 3, the 2nd term $H_{\text{mixer}}$ is represented by just another phase $U$ gate at the last layer.

In our experiment, we vary the noise probability as per Eq. 28 from 0 to 0.5 and observe the energy values of VQE. As we executed the circuit with 1 to 4 layers with two of chosen optimizers ‘COBYLA’ and ‘L_BFGS_B’, we get the characteristics described in Fig. 7.

As we look at the figures of bitflip noise simulation (Fig. 7) and Table II, we observe that global minimum for both optimizers ‘COBYLA’ and ‘L_BFGS_B’ remain fairly consistent at lower number of qubits (6 qubits). For COBYLA global minimum with the lowest energy at the 3rd layer while for ‘L_BFGS_B’ the global minimum with the lowest being on the 2nd layer. The characteristic however changes to more of inconsistent values for higher number of qubits (12 qubit). For ‘L_BFGS_B’ the values across the layers are quite close with minimum being at the 4th layer. Similarly, for ‘COBYLA’ the values of global minimums are quite inconsistent with minimum on 1st layer itself. We can therefore conclude that for higher and lower number of qubit circuits affected by bitflip noise, we can achieve global minimum with lesser number of layers.
FIG. 5. Plot illustrating the circuit simulation of VRP with 5 layers using various optimizers (COBYLA, ‘L_BFGS_B’, SLSQP, SPSA). The plot consists of two separate graphs depicting simulation output of 6 qubit and 12 qubit circuits respectively. Each plot in turn consists of four lines indicating energy values for separate optimizers. The minimum value at each Layer is represented in pairs with (layer, minimum energy) format.

FIG. 6. Plot illustrating output of amplitude damping simulation of VRP circuit with 4 layers using optimizers COBYLA and ‘L_BFGS_B’. The plot consists of four separate graphs depicting simulation output of 6 qubit and 12 qubit circuits respectively for each optimizer. For simplification, the minimum energy value of each layer from 1-4 has been presented in format (layer, minimum energy). Multiple coloured lines represent energy characteristics of each layer across optimizers (COBYLA, ‘L_BFGS_B’) and qubit count (6 qubit and 12 qubit). The values of the plot have been summarized in the Table I.

C. Phase Flip Noise

Phaseflip noise alters the phase parameter of quantum system without exchange of energy\(^{38,39}\). The decoherence rate or the phaseflip noise parameter also follows the distribution \(0 \leq \kappa_w \leq 1\), thus the Kraus operators of phaseflip noise channel can be defined as Eq. 29,

\[
E_W^0 = \sqrt{1 - \kappa_w} I
\]

\[
E_W^1 = \sqrt{\kappa_w} Z = \sqrt{\kappa_w} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]  

(29)

In our experiment, we vary the noise probability as per Eq. 29 from 0 to 0.5 and observe the energy values of VQE. As we execute the circuit with 1 to 4 layers with
FIG. 7. Plot illustrating output of BitFlip simulation of VRP circuit with 4 layers using optimizers COBYLA, ‘L_BFGS_B’. The plot consists of four separate graphs depicting simulation output of 6 qubit and 12 qubit circuits respectively for each optimizer. For simplification the minimum energy value of each layer from 1-4 has been presented in format (layer, minimum energy). Multiple coloured lines represent energy characteristics of each layer across optimizers (COBYLA, ‘L_BFGS_B’) and qubit count (6 qubit and 12 qubit). The values of the plot are summarized in Table II.

| Qubits | Layers | energy cost COBYLA | energy L_BFGS_B |
|--------|--------|--------------------|-----------------|
| 6      | 1      | -16.1513373508335  | -7.966428184354938 |
| 6      | 2      | -18.32163823503105 | -9.06122357865426 |
| 6      | 3      | -18.649710638298    | -9.019122357865426 |
| 6      | 4      | -17.232214541650    | -9.613217089450552 |
| 12     | 1      | -68.96015396042787  | -31.0110501411043 |
| 12     | 2      | -47.89237353743253  | -23.2258234296239 |
| 12     | 3      | -52.9856067336116   | -31.67115539842111 |
| 12     | 4      | -68.78296781241127  | -35.27741464628871 |

TABLE III. VQE simulation of phaseflip noise channel for 6 and 12 qubits with 4 layers involving optimizers COBYLA and L_BFGS_B.

| Qubits | Layers | energy cost COBYLA | energy L_BFGS_B |
|--------|--------|--------------------|-----------------|
| 6      | 1      | -4.735460381020471 | -4.51787125530202 |
| 6      | 2      | -6.44141627037905  | -4.41513213286429 |
| 6      | 3      | -3.7564555093166   | -3.1323934951265 |
| 6      | 4      | -6.85588724560632  | -4.29905109460053 |
| 12     | 1      | -20.38256478845435 | -10.95799437272029 |
| 12     | 2      | -21.54003593605693 | -10.68890486221641 |
| 12     | 3      | -21.6674993885087  | -11.8585357539583 |
| 12     | 4      | -24.29659403214718 | -18.7512901037579 |

D. Bit-Phase Flip Noise

Bit-phase flip noise channel is characterised by combination of random bitflip errors along with change in phase information of the quantum system without loss of energy. Similar to other noise channels, the decoherence rate or the combined probability of bit-phase flip error follows the distribution $0 \leq \kappa_F \leq 1$. The Kraus operator of the bit-phase flip channel could be given by Eq. 30,

$$
E^F_0 = \sqrt{1 - \kappa_F} I \\
E^F_1 = \sqrt{\kappa_F} Y = \sqrt{\kappa_F} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}
$$  (30)

In our experiment, we vary the noise probability as per Eq. 30 from 0 to 0.5 and observe the energy values of VQE. As we execute the circuit with 1 to 4 layers with two of chosen optimizer ‘COBYLA’ and ‘LBFGSB’ we get the characteristics.

As we look at the figures of phaseflip noise simulation from Fig. 8 and table III, we observe that global minimum for both optimizers ‘COBYLA’ and ‘L_BFGS_B’ remains fairly consistent at lower number of qubits (6 qubits). For ‘COBYLA’ global minimum fluctuates in a very narrow range with the lowest energy at the 4th layer while for ‘L_BFGS_B’ the fluctuation is in a very narrow range with the lowest being on 1st layer. The characteristic however changes for higher number of qubits (12 qubits). For ‘L_BFGS_B’ the values across the layers are quite close with minimum being at 4th layer. Similarly, for ‘COBYLA’ the values of global minimums are quite consistent except the 4th layer. We can therefore conclude that for suitable phase flip noise parameter we can achieve global minimum for VQE circuits with lesser number of layers.
FIG. 8. Plot illustrating output of Phaseflip simulation of VRP circuit with 4 layers using optimizer COBYLA, L_BFGS_B. The plot consists of four separate graphs depicting simulation output of 6 qubit and 12 qubit circuits respectively for each optimizer. For simplification the minimum energy value of each layer from 1-4 has been presented in format (layer, minimum energy). Multiple coloured lines represent energy characteristic of each layer across optimizer (COBYLA, L_BFGS_B) and qubit count (6 qubit and 12 qubit). The values of the plot has been summarized in Table III.

| Qubits | Layers | Energy Cost COBYLA | Energy Cost L_BFGS_B |
|--------|--------|-------------------|---------------------|
| 6      | 1      | -1.664534227258453 | -1.051335123998299  |
| 6      | 2      | -2.426489119219003 | -1.110646113705209  |
| 6      | 3      | -2.062970150047617 | -1.205352371463018  |
| 6      | 4      | -2.365264871779612 | -1.175696876609562  |
| 12     | 1      | -5.292070888042392 | -4.179033202171581  |
| 12     | 2      | -7.779784254700785 | -5.849307202627475  |
| 12     | 3      | -7.303861393746143 | -3.270714093676236  |
| 12     | 4      | -6.69305386265159  | -5.331770985991372  |

TABLE IV. VQE simulation of bit-phaseflip noise channel for 6 and 12 qubits with 4 layers involving optimizers COBYLA and L_BFGS_B.

two of chosen optimizers ‘COBYLA’ and ‘L_BFGS_B’, we get the characteristics as given in Fig. 9.

As we look at the figures of bit-phase-flip noise simulation (Fig. 9 and Table IV), we observe that global minimum for both optimizers ‘COBYLA’ and ‘L_BFGS_B’ remains fairly consistent at lower/higher number of qubits (6/12 qubit). For COBYLA global Minimum with the lowest energy at 2nd layer while for L_BFGS_B the global minimum with the lowest being on 3rd layer. The characteristic however changes to more of inconsistent values (over a narrow range) for higher number of qubits (12 qubit). For L_BFGS_B and COBYLA, the values across the layers are close with minimum being at 2nd layer. We can therefore conclude that for VQE circuits with suitable value of bit-phase-flip noise, we can achieve global minimum with the lesser number of layers.

E. Depolarizing Noise

A depolarizing noise channel is characterized by a maximally mixed state of $1/d$ for a $d$-level quantum system or left untouched. The decoherence rate or the depolarization noise probability follows the distribution $0 \leq \kappa_D \leq 1$. Considering 3 Pauli operators $X, Y, Z$ our quantum system will have a mixed state of $\kappa_D/3^{38,39}$. We define the Kraus operators as follows,

$$
E_D^0 = \sqrt{1 - \kappa_D} I \\
E_D^1 = \sqrt{\frac{\kappa_D}{3}} X = \sqrt{\frac{\kappa_D}{3}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
E_D^2 = \sqrt{\frac{\kappa_D}{3}} Y = \sqrt{\frac{\kappa_D}{3}} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
E_D^3 = \sqrt{\frac{\kappa_D}{3}} Z = \sqrt{\frac{\kappa_D}{3}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
$$

(31)

In our experiment, we vary the noise probability from 0 to 0.5 and observed the energy values of VQE. As we execute the circuit with 1 to 4 layers with two of chosen optimizers COBYLA and ‘L_BFGS_B’, we get the characteristics Fig. 10.

As we look at the figures of depolarizing noise simulation (Fig. 10 and Table V), we observe that global minimum for both optimizers COBYLA and ‘L_BFGS_B’ remains fairly consistent at lower/higher number of qubits (6/12 qubit). For COBYLA and L_BFGS_B global minimum with lowest energy is achieved at 2nd layer for both higher and lower number of qubits circuits. The characteristic however changes to more of inconsistent values
We can therefore conclude that for VQE circuits with suitable value of depolarising noise we can achieve global minimum with lesser number of layers.

VII. CONCLUSION

As we compare all the above noise models \(\{A, B, W, F, D\}\) we notice that with noise added to circuits the global minima are reached at much lower energy values than without noise. COBYLA performs better than other optimizers by reaching significantly lower energy values. Along with these, we have observed that fluctuations among global minima across the layers

for \(\text{L}_B\text{FGS}_B\) in higher number of qubits (12 qubit).
TABLE V. VQE simulation of depolarising noise channel for 6 and 12 qubits with 4 layers involving optimizers COBYLA and ‘L_BFGS_B’.

| Qubits | Layers | energy cost COBYLA | energy cost L_BFGS_B |
|--------|--------|--------------------|----------------------|
| 6      | 1      | -7.97872220559296  | -5.55759693889606    |
| 6      | 2      | -8.09472802587516  | -6.51372645832148    |
| 6      | 3      | -8.08719946982046  | -5.41907108726278    |
| 6      | 4      | -8.07515372075307  | -6.50619787942831    |
| 12     | 1      | -29.34188337801883 | -22.65198817359857   |
| 12     | 2      | -30.02397262573487 | -23.60811769302392   |
| 12     | 3      | -29.9727758292617  | -15.77237278107185   |
| 12     | 4      | -29.9592268472541  | -19.78962247843698   |

are quite consistent if we ignore an occasional dip in one layer in either higher or lower qubits circuit. These characteristics therefore indicate that controlled noise values could help in reducing the circuit depth and help algorithms perform efficiently. The above observations could be applied to more complex circuits and further investigations could be made.

The work done so far in this paper has presented an interesting avenue in solving and improving VRP. Though quantum noise is considered as a problematic aspect in today’s NISQ devices, yet limited amount of controlled noise could be beneficial in attending solutions of VRP. Besides that we have observed that solution characteristics differ significantly by considering various optimizers. As in this paper, we have considered COBYLA and ‘L_BFGS_B’; there is still room for study for the other two optimizers, SLSQP and SPSA. We aim to explore similar experiments with quantum annealing systems such as D WAVE, Fujitsu, and IONQ etc. With the use of cloud platforms such as AMAZON Braket and Starangeworks; we believe it is possible to create larger Hamiltonian and hence a higher qubit solution and its noise characteristics could be explored.

ACKNOWLEDGMENT

The authors are grateful to IMB QE team for developing Qiskit platform and accessing simulators for performing experiments.

* Nishikanta.M.Mohanty@student.uts.edu.au
† christopher.ferrie@uts.edu.au
‡ bikas.riki@gmail.com
1 A. Montanaro, Quantum algorithms: an overview, npj Quantum Inf. 2, 15023 (2016).
2 Chapter: 3 Quantum Algorithms and Applications, https://www.nap.edu/read/25196/chapter/5#63
3 Quantum Algorithm Zoo, https://quantumalgorithmzoo.org/#ONML
4 J. Ahljiijth et al., Quantum Algorithm Implementations for Beginners, arxiv:1804.03719
5 E. Farhi, J. Goldstone, and S. Gutmann, The Quantum Approximate Optimization Algorithm, arXiv:1411.4028v1
6 E. Farhi, J. Goldstone, and S. Gutmann, Quantum Computation by Adiabatic Evolution, arXiv:quant-ph/0001106v1
7 L. K. Grover, A fast quantum mechanical algorithm for database search, arXiv:quant-ph/9605043v3
8 V. R. Dasari, M. S. Im, and L. Beshaj, Solving machine learning optimization problems using quantum computers, Proc. Dis. Tech. Inf. Sci. 11419, 1-10 (2020).
9 Vehicle routing problem https://en.wikipedia.org/wiki/Vehicle_routing_problem
10 S. Harwood, C. Gambella, D. Trence, A. Simonetto, D. Bernal, and D. Greenberg, Formulating and Solving Routing Problems on Quantum Computers, TQE.2021.3049230
11 K. Srinivasan, S. Satyajit, B. K. Behera, and P. K. Panigrahi, Efficient quantum algorithm for solving traveling salesman problem: An IBM quantum experience, arXiv:1805.10928
12 S. Feld, C. Roch, T. Gabor, C. Seidel, F. Neukart, I. Galter, W. Maurer, and C. Linnhoff-Popien, A Hybrid Solution Method for the Capacitated Vehicle Routing Problem Using a Quantum Annealer, Front. ICT 6, 1-13 (2019).
13 U. Azad, B. K. Behera, and P. K. Panigrahi, Solving Vehicle Routing Problem Using Quantum Approximate Optimization Algorithm, arXiv:2002.01351v1
14 A. Nowbargh, B. Behera, and P. Panigrahi, A Quantum Approach for Solving Vehicle Routing Problem: An IBM Quantum Experience. 10.13140/RG.2.2.18975.30887.
15 C. Papalitsas, T. Andronikos, K. Giannakis, G. Theocharopoulos, and S. Fanarioti, A QUBO Model for the Traveling Salesman Problem with Time Windows, MDPI:a12110224
16 E. Farhi, J. Goldstone, and S. Gutmann, Solving Vehicle Routing Problem Using Quantum Approximate Optimization Algorithm, arXiv:2002.01351v1
17 G. Kochenberger, J. K. Hao, and F. Glover, and M. Lewis, and Z. Lu, and H. Wang, and Y. Wang, The unconstrained binary quadratic programming problem: a survey, J. Comb. Optin. 28, 58-81 (2014).
18 F. Glover, and G. Kochenberger, and M. Ma, and Y. Du, Quantum Bridge Analytics II: QUBO-Plus, network optimization and combinatorial chaining for asset exchange, s10288-020-00464-9
19 A. Crispin, and A Syrichas, Quantum Annealing Algorithm for Vehicle Scheduling, SMC.2013.601
20 H. Irie, G. Wongsaisarnsin, M. Terabe, A. Miki, and S. Taguchi, Quantum Annealing of Vehicle Routing Problem with Time, State and Capacity, In: Feld S., Linhoff-Popien C. (eds) Quantum Technology and Optimization Problems. QTOP Lecture Notes in Computer Science, 11413. Springer (2019).
21 M. Sato, H. Watanabe, Y. Musha, and A. Utsunomiya, Application of digital annealer for faster combinatorial optimization, FSTJ 55, 45-51 (2019).
22 A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency, JCSS: ISBN:9783540443896 (2003).
Combinatorial optimization, https://en.wikipedia.org/wiki/Combinatorial_optimization

W. J. Cook, W. H. Cunningham, W. R. Pulleyblank, A. Schrijver, Combinatorial Optimization. Wiley. ISBN 0-471-55894-X

T. Albash, and D. A. Lidar, Adiabatic quantum computation, Rev. Mod. Phys. 90, 1539-0756 (2018).

Adiabatic quantum computation, https://en.wikipedia.org/wiki/Adiabatic_quantum_computation

R. Hoque, The Quantum Approximate Optimization Algorithm https://ryanhoque.github.io/data/QAOA.pdf

S. Yin et al., Adiabatic Quantum Simulation Using Trotterization. 2018. eprint:arXiv:1805.11568

E. Farhi, J. Goldstone, and S. Gutmann, The Quantum Approximate Optimization Algorithm Needs to See the Whole Graph: A Typical Case, arXiv:2004.09002v1

L. Zhou, S.T. Wang, S. Choi, H. Pichler, M. D. Lukin, Quantum Approximate Optimization Algorithm: Performance, Mechanism, and Implementation on Near-Term Devices, physrevx.10.021067

S. Wald, Thermalisation and Relaxation of Quantum Systems , RG.2.2.25169.63842

Ising Model, https://en.wikipedia.org/wiki/Ising_model

A. Lucas, Ising formulations of many NP problems, fphy.2014.00005

Quadratic unconstrained binary optimization, https://en.wikipedia.org/wiki/Quadratic_unconstrained_binary_optimization

F. Glover, G. Kochenberger, and Y. Du, A Tutorial on Formulating and Using QUBO Models, ARXIV:1811.11538

G. G. Guerreschi, Solving Quadratic Unconstrained Binary Optimization with divide-and-conquer and quantum algorithms, arXiv:2101.07813

Difference between BQM, Ising, and QUBO problems, https://support.dwavesys.com/hc/enus/community/posts/360017439853-Difference-between-BQM-Ising-and-QUBO-problems.

R.G. Zhou, Y.N. Zhang, R. Xu, C. Qian and I. Hou , Asymmetric Bidirectional Controlled Teleportation by Using Nine-Qubit Entangled State in Noisy Environment, IEEE Access, 7, 75247-75264 (2019)

S. Ahadpour and F. Mirmasoudi, The role of noisy channels in quantum teleportation, Rev. mex. fis., Vol. 66, 378-387 (2020)

Amazon Braket Service, https://aws.amazon.com/braket/

Strange works https://strangeworks.com/

Quantum future - quantum present, https://www.fujitsu.com/au/imagesgig5/Digital_Annealer_case_study.pdf