ON UNIFORMLY RECURRENT MOTIONS OF TOPOLOGICAL SEMIGROUP ACTIONS

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ABSTRACT. Let $G \curvearrowright X$ be a topological action of a topological semigroup $G$ on a compact metric space $X$. We show in this paper that for any given point $x$ in $X$, the following two properties that both approximate to periodicity are equivalent to each other:

- For any neighborhood $U$ of $x$, the return times set $\{g \in G : gx \in U\}$ is syndetic of Furstenburg in $G$.
- Given any $\varepsilon > 0$, there exists a finite subset $K$ of $G$ such that for each $g$ in $G$, the $\varepsilon$-neighborhood of the orbit-arc $K[gx]$ contains the entire orbit $G[x]$.

This is a generalization of a classical theorem of Birkhoff for the case where $G = \mathbb{R}$ or $\mathbb{Z}$. In addition, a counterexample is constructed to this statement, while $X$ is merely a complete but not locally compact metric space.

1. Introduction. Given any topological Hausdorff space $G$ with a continuous multiplicative binary operation $\circ$ and any underlying topological space $X$, in this paper we study the uniform recurrence dynamics of a continuous left-action of $G$ on $X$ written as

$$G \curvearrowright X; (g, x) \mapsto gx.$$ 

$X$ is referred to as a topological $G$-space in this case. Somewhat improperly, we call each $g$ in $G$ a continuous transformation of the space $X$ to itself. For any subset $\Gamma$ of $G$ and any point $x$ in $X$, by $\Gamma[x]$ we denote the partial orbit of passing through the point $x$; i.e., $\Gamma[x] = \{gx : g \in \Gamma\}$.

Throughout this paper, $G$ is a topological semigroup with an identity element $e$.

1.1. Uniformly recurrent motions. First of all, we recall a basic notion needed later.

(F) A subset $S$ of the topological semigroup $G$ is called syndetic if there exists a compact subset $K$ of $G$ such that for each element $g$ in $G$, there is some element $k$ in $K$ with $kg \in S$; that is, $(Kg) \cap S \neq \emptyset$ for all $g \in G$; see, e.g., [2, Definition 1.7].

When $G$ is a topological group, $S$ is syndetic if and only if there is some compact set $K' \subset G$ such that $G = K'S$. See more in Section 2.
A point \( x \) of \( X \) is said to be periodic for \( G \acts X \) provided there exists a syndetic subset \( T \) of \( G \) such that \( T[x] = \{x\} \). The motion we will consider here is defined as follows, which generalizes the notion of periodic point of \( G \acts X \):

**Definition 1.1** ([3, 2]). A point \( x \) in \( X \) is said to be uniformly recurrent for \( G \acts X \), if for any neighborhood \( U \) of \( x \), the set of return times \( N(x, U) = \{g \in G : gx \in U\} \) is syndetic in \( G \) in the sense of (F).

We note that the phrases “uniformly recurrent point” and “uniformly recurrent motion” are used virtually interchangeably. Clearly this recurrence depends upon the topology of \( G \) and the strongest type of uniform recurrence occurs when \( G \) is provided with the discrete topology. It also should be noted here that if the syndetic set in Definition 1.1 is required to be a compact subset \( K \) of \( X \), then following the terminology of [1, 6], the motion we consider here is called a uniformly recurrent motion, and the latter is systematically studied in [1, 6].

In addition if \( x \in X \) is uniformly recurrent for \( G \acts X \), then so is each point \( y \) in the orbit \( G[x] \). Indeed, let \( g_0 \) be an arbitrary element of \( G \) and \( V \) an open set containing \( g_0x \); then there is an open set \( U \) containing \( x \) such that \( g_0U \subseteq V \). Put \( \Delta = g_0K \) where \( K \) is a compact subset of \( G \) corresponding \( N(x, U) = \{g : gx \in U\} \) as in Definition 1.1. Since \( (Khg_0) \cap N(x, U) \neq \emptyset \) and so \( (g_0Khg_0) \cap \{g_0N(x, U)\} \neq \emptyset \) for all \( h \in G \), there follows \( (\Delta h) \cap \{g : gy \in V\} \neq \emptyset \) where \( y = g_0x \).

It should be noted that in some literature like [3, 4], [5, Definition 3.38 and Remark 4.02-(1)], a uniformly recurrent point is sometimes called an almost periodic point (in the sense of von Neumann) for \( G \acts X \). However, this is much more weaker than the following one corresponding to the classical almost periodic function of Bohr:

**Definition 1.2.** Let \( G \acts X \) be a topological action of a semigroup \( G \) on a metric space \( X \) with metric \( d \). A point \( x \) in \( X \) is called an almost periodic motion for \( G \acts X \) in the sense of Bohr if for any \( \epsilon > 0 \) the \( \epsilon \)-periods set

\[
\mathbb{T}(\epsilon) := \{\tau \in G : d(gx, \tau gx) < \epsilon \ \forall g \in G\}
\]

is syndetic in \( G \) (cf. [8, Definition V8.01] for the situation of \( G = \mathbb{R} \)).

We would like to remain the phrase “almost periodic” for the case of Bohr.

Obviously all these motions approximate to periodicity of \( G \acts X \).

1.2. **Characterization of uniform recurrence.** When the underlying space \( X \) is compact, besides by the dynamics of minimality of the orbit closure \( G[x] \), we present in this paper another characterization by the topological structure of the orbit \( G[x] \).

**Theorem 1.3.** Let \( G \acts X \) be a topological action of a topological semigroup on a compact metric space \( X \). Then a point \( x \) in \( X \) is uniformly recurrent for \( G \acts X \) if and only if there exists, for any \( \epsilon > 0 \), a finite subset \( K_\epsilon \) of \( G \) such that for each \( g \in G \), the entire orbit \( G[x] \) lies in the \( \epsilon \)-neighborhood of the orbit-arc \( K_\epsilon[gx] \).

This result shows that if \( x \) is uniformly recurrent for \( G \acts X \), then there is a finite subset \( K_\epsilon \) of \( G \) such that any orbit-arc \( K_\epsilon[gx] \) approximates the entire orbit \( G[x] \) with a precision to within \( \epsilon \); in other words, a uniformly recurrent orbit looks like a periodic orbit up to an \( \epsilon \)-error. This is different from the point of view of Definition 1.1 that is by considering the “relative density” of the return times set of the orbit \( G[x] \).
From [5, Remark 4.02] it follows that $x$ in $X$ is uniformly recurrent for $G \curvearrowright X$ if and only if for any neighborhood $U$ of $x$ there is a compact subset $K$ of $G$ such that $G[x] \subseteq \bigcup_{k \in K} k^{-1}[U]$, where $k^{-1}[U]$ is thought of as the preimage of $U$ under the continuous transformation $k$ of $X$. Although $\bigcup_{k \in K} k^{-1}[U]$ is a neighborhood of the orbit-arc $K[x]$ containing $G[x]$, yet it is possibly very fat because $K$ depends upon the choice of $U$ without the equi-uniform continuity hypothesis. So this cannot capture the “almost periodic” behavior of a uniformly recurrent motion as what provided by our approximation in Theorem 1.3.

We note here that when $G = \mathbb{R}$ or $\mathbb{Z}$, this consequence follows from some classical theorems of G.D. Birkhoff; see, e.g., [8, Theorems V7.06, V7.07 and V7.09].

1.3. Outlines. The rest of this paper will be mainly devoted to proving Theorem 1.3. A counterexample will be constructed in Section 4.2 to the conclusion of Theorem 1.3, while $X$ is merely a complete but not locally compact metric space; see Proposition 4.6 below. In addition, we will show that the uniform recurrence is not equivalent to the almost periodicity of von Neumann by constructing a simple example; see Propositions 4.7 and 4.8 below in Section 4.3. Finally, we will end this paper with two open questions.

2. Lemmas and almost periodicity of von Neumann. Let $G$ be a topological semigroup and $X$ a general topological space. In this section, let $G \curvearrowright X$, defined by $(g, x) \mapsto gx$, be a topological action of $G$ from left on $X$.

If $G = \mathbb{Z}_+$ or $\mathbb{R}_+$, then $\omega(x)$ denotes the $\omega$-limit set consisting of all the forward limit points of the orbit $G[x]$ for any $x \in X$.

2.1. Preliminary lemmas. Recall that a subset $A$ of $X$ is said to be $G$-invariant if $g[A] \subseteq A$ for all $g \in G$; and a set $\Sigma \subseteq X$ is called $G$-minimal if it is nonempty, closed and $G$-invariant, and has no proper subset possessing these three properties.

In order to prove Theorem 1.3, we will need a minimal dynamics description of the uniform recurrence included in the following two lemmas, both of whom are due to Birkhoff for $G = \mathbb{Z}$ and Gottschalk for general cases.

Lemma 2.1 (See [3, Theorem 1], [5, Theorem 4.05] and [2, Theorem 1.15]). If $\Sigma \subseteq X$ is a compact $G$-minimal set, then every point $x$ in $\Sigma$ is uniformly recurrent for $G \curvearrowright X$.

Thus there always exists a uniformly recurrent point for $G \curvearrowright X$ whenever $X$ is a compact space by Zorn’s lemma. There is a partial converse to Lemma 2.1, for the group-action version of whom readers may see [5, Theorem 4.07].

Lemma 2.2 ([3]). Let $X$ be a regular topological space. If $x$ in $X$ is a uniformly recurrent point for $G \curvearrowright X$, then the orbit-closure $\overline{G[x]}$ is $G$-minimal.

Combining the above two lemmas, we can easily obtain the following characterization of uniform recurrence via minimality:

Corollary 2.3 ([3]). Let $X$ be a compact regular topological space. Then a point $x$ in $X$ is uniformly recurrent for $G \curvearrowright X$ if and only if $\overline{G[x]}$ is $G$-minimal.

Define a cyclic system by $x_0 \mapsto x_1 \mapsto x_2 \mapsto x_1$ with $G = \mathbb{N}$ where $x_0, x_1, x_2$ are distinct points; then $\overline{G[x_0]} = \{x_1, x_2\}$ is minimal, yet $x_0$ is not recurrent. Thus if $G$ does not contains an identity, then Corollary 2.3 is not necessarily to be true.

Let us consider the cyclic dynamical system $\mathbb{Z} \curvearrowright \mathbb{R}$, defined by $(n, x) \mapsto n + x$. Then although the closed orbit $\mathbb{Z}[0] = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is $\mathbb{Z}$-minimal, the
point \(0 \in \mathbb{R}\) is not recurrent for \(Z \curvearrowright \mathbb{R}\) (and the \(\omega\)-limit set \(\omega(0)\) and the \(\alpha\)-limit set \(\alpha(0)\) both are void). Thus the compactness condition in Lemma 2.1 cannot be released in general. On the other hand, is \(\overline{Gx}\) compact in Lemma 2.2? We shall improve the above two lemmas soon via considering a locally compact underlying space \(X\) instead of compact.

2.2. Almost periodicity of von Neumann. When \(G = \mathbb{Z}_+^d\) or \(\mathbb{R}_+^d\) where as usual \(\mathbb{Z}_+ = \{0, 1, 2, \ldots\}\) and \(\mathbb{R}_+ = [0, \infty)\) which are additive topological semigroups, the compactness can be replaced in Lemma 2.1 and obtained in Lemma 2.2 by the local compactness of \(X\) (Corollary 2.11 and Theorem 2.12 below).

In order to prove that, we need to invoke the following lemma which may follows easily from the proof of [3, Lemma 1].

**Lemma 2.4.** Let \(E\) be a closed set in \(X\) and \(U\) an open set in \(X\) such that \(\overline{U}\) is compact. If \(T\) is a subset of \(G\) such that for each finite set \(A \subset T\) there is a point \(y \in E \cap U\) with \(A[y] \subset X - U\), then there exists a point \(z \in E \cap U\) with \(T[z] \subset X - U\).

**Proof.** Let \(T\) denote the collection of all finite subsets of \(T\). If \(K \in T\), then define \(C_K\) to be the set of all points \(y \in E \cap U\) such that \(K[y] \subset X - U\). Now \(C_K\) is nonempty and closed for each \(K \in T\). Furthermore the collection \(\{C_K\}_{K \in T}\) has the finite intersection property. Hence \(\bigcap_{K \in T} C_K\) contains a point \(z \in E \cap U\) by the compactness of \(\overline{U}\). Clearly such a point \(z\) has the required property. \(\square\)

Slightly different from (F) and Definition 1.1 introduced in Section 1.1, we introduce the following:

**Definition 2.5 ([5, 4]).** A subset \(S\) of the topological semigroup \(G\) is said to be GH syndetic (GH is for Gottschalk-Hedlund) provided that there exists a compact subset \(K\) of \(G\) with \(G = K'S\), that is, \(G = \bigcup_{s \in S} K's\).

Furthermore, a point \(x\) in \(X\) is called almost periodic of von Neumann for \(G \curvearrowright X\) provided that for any neighborhood \(U\) of \(x\), the return time set \(\{g \in G: gx \in U\}\) is GH syndetic.

Since here \(G\) is only a semigroup, the uniform recurrence of Gottschalk and Furstenberg by Definition 1.1 and the almost periodicity of von Neumann are conceptually different. However they agree to each other in the group action and cyclic system cases.

**Lemma 2.6.** Let \(G = \mathbb{Z}_+^d\) or \(\mathbb{R}_+^d\). If a subset \(S\) of \(G\) is GH syndetic, then it is syndetic in the sense of (F).

**Proof.** Let \(S\) be GH syndetic in \(G\). Then there is a compact set \(K' \subset G\) such that \(G = K' + S = \bigcup_{s \in S} (K' + s)\). We can choose an integer \(\ell > 1\) such that the closed cube \(\Delta_\ell = [0, \ell]^d\) contains \(K'\). Thus for any given \(g \in G\), there exits an element \(s \in S\) with \(g \in \Delta_\ell + s\). Moreover, there must be some element \(s' \in S \cap (\Delta_{2\ell} + g)\). Hence, \(S \cap (\Delta_{2\ell} + g) \neq \emptyset\) for all \(g \in G\). This means that \(S\) is syndetic of (F). \(\square\)

**Corollary 2.7.** Let \(G = \mathbb{Z}_+^d\) or \(\mathbb{R}_+^d\). Then for \(G \curvearrowright X\), if a point \(x\) in \(X\) is almost periodic of von Neumann it is uniformly recurrent.

It should be noted that the converse statements of Lemma 2.6 and Corollary 2.7 both are false. For example, let \(G = \mathbb{R}_+^2\) and \(S = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 1\}\); then it is easy to check that \(S\) is syndetic in the sense of (F) in \(G\), but it is not GH syndetic in \(G\). See Proposition 4.7 for a dynamical system realization of this.

However, for the 1-dimensional action case we can obtain the following.
Proposition 2.8. Let \( G = \mathbb{Z}_+ \) or \( \mathbb{R}_+ \) and \( S \subset G \) with \( 0 \in S \). Then \( S \) is GH syndetic if and only if it is syndetic in the sense of (F). Hence for \( G \curvearrowright X \), a point \( x \) in \( X \) is uniformly recurrent if and only if \( x \) is almost periodic of von Neumann.

Proof. By Lemma 2.6, we only need to show the syndeticity of (F) implies that of GH. For that let \( \Delta = [0,\ell] \) such that \( S \cap (\Delta + g) \neq \emptyset \) for all \( g \in G \). Clearly, for \( K' = [0,2\ell] \), we have \( G = \bigcup_{s \in S} (K' + s) \). This proves Proposition 2.8. \( \square \)

Recall that a compact subset of a Hausdorff space is closed and the closure of a compact subset of a regular topological space is compact. The following result is a merit of almost periodicity of von Neumann comparing with the uniform recurrence.

Theorem 2.9 (See [5, Theorem 4.09] for group actions). Let \( G \curvearrowright X \) be a topological action of the topological semigroup \( G \) on a locally compact topological space which is either Hausdorff or regular. If \( x \) in \( X \) is an almost periodic motion of von Neumann, then the orbit-closure \( \overline{G[x]} \) is compact and it contains a uniformly recurrent point.

Proof. Let \( x \in X \) be almost periodic of von Neumann. Thus one can find a compact subset \( K' \) of \( G \) and a compact neighborhood \( U \) of \( x \) such that \( G[x] \subset K'[U] \). Since \( G \curvearrowright X \) is a topological action and \( K' \times U \) is a compact subset of the product space \( G \times X \), \( K'[U] \) is compact and then so is \( \overline{G[x]} \). This establishes Theorem 2.9. \( \square \)

Unfortunately, we are not able to show the minimality of \( \overline{G[x]} \) in general. By Proposition 2.8 and Theorem 2.9 we can easily obtain the following

Corollary 2.10. Let \( G = \mathbb{Z}_+ \) or \( \mathbb{R}_+ \) and suppose that \( X \) is a locally compact topological space which is either Hausdorff or regular. Then for any uniformly recurrent point \( x \) in \( X \) for \( G \curvearrowright X \), the orbit-closure \( \overline{G[x]} \) is compact.

In addition, By Lemmas 2.2 and 2.6 and Theorem 2.9 we can easily obtain the following

Corollary 2.11. Let \( G = \mathbb{Z}^d_+ \) or \( \mathbb{R}^d_+ \) and suppose that \( X \) is a locally compact regular topological space. Then the orbit-closure \( \overline{G[x]} \) is compact G-minimal for any almost periodic point \( x \in X \) of von Neumann, for \( G \curvearrowright X \).

The following sufficient and necessary conditions are the converses of Lemmas 2.1 and 2.2 for the cyclic cases, which is of interest because of the above translation system \( \mathbb{Z} \curvearrowright \mathbb{R} \).

Theorem 2.12. Let \( G \curvearrowright X \) be a topological action of the topological semigroup \( G = \mathbb{Z}_+ \) or \( \mathbb{R}_+ \) on a locally compact regular topological space \( X \). Then the following three properties are equivalent to each other:

1. A point \( x \) in \( X \) is uniformly recurrent.
2. \( \overline{G[x]} \) is compact G-minimal.
3. \( \overline{G[x]} \) is G-minimal.

Proof. The fact of (1) \( \Rightarrow \) (2) follows easily from Lemma 2.2 and Corollary 2.10. The fact of (2) \( \Rightarrow \) (3) is evident.

Finally, in order to prove (3) \( \Rightarrow \) (1), let \( \overline{G[x]} \) be G-minimal. Since \( (G \cap [1,\infty))[y] \) is G-invariant for any \( y \in \overline{G[x]} \), \( y \) is a recurrent point for \( G \curvearrowright X \); that is, \( y \) itself is an \( \omega \)-limit point of \( y \). Hence if \( x \) is not uniformly recurrent, then there exists a neighborhood \( U \) of \( x \) with \( U \) compact such that \( \{ g \in G : gx \in U \} \) is not syndetic of (F). Thus, taking \( T = G \cap [1,\infty) \) and \( E = \overline{G[x]} \), the hypotheses of Lemma 2.4 are
satisfied. The conclusion of Lemma 2.4 contradicts the minimality of $E$. Therefore (3) $\Rightarrow$ (1) holds. This proves Theorem 2.12.

We note that if $G \curvearrowright X$ is a group action on a locally compact space $X$ which is either Hausdorff or regular, then $G[x]$ is compact for any uniformly recurrent point $x \in X$ of $G \curvearrowright X$ (cf. [5, Theorem. 4.09] or Theorem 2.9). However, as far as our knowledge this is not available for a general semigroup action on a locally compact space until now. When $X$ is not locally compact, we shall constructed an explicit counterexample to this in Section 4.

In a word some differences of dynamics between topological semigroup actions and group actions may be essential. See, e.g., Proposition 4.8 below.

2.3. **Kronecker system.** Let $f$ be a continuous map of a topological space $Y$ into itself. A point $y \in Y$ is called a (forwardly) recurrent point for $f$ or for the dynamical system $(Y, f)$ if for any neighborhood $V$ around $y$, there exists $n \geq 1$ with $f^n(y) \in V$. See [2, Definition 1.1].

As a complement of Theorem 2.12, we can obtain the following.

**Lemma 2.13.** Let $G \curvearrowright X$ be a topological action of the topological group $G = \mathbb{Z}$ or $\mathbb{R}$ on a locally compact space $X$ which is either Hausdorff or regular. If $\Sigma$ is a $G$-minimal subset of $X$ and if every point $x \in \Sigma$ is forwardly recurrent, then $\Sigma$ is compact and each $x \in \Sigma$ is uniformly recurrent for $G \curvearrowright X$.

**Proof.** First of all, we assert that each point $x \in \Sigma$ is uniformly recurrent for the topological semigroup action $G_+ \curvearrowright X$, where $G_+ = G \cap [1, \infty)$. Assume the contrary; then there exists some point $x \in \Sigma$ satisfying that one can find an open neighborhood $U$, whose closure is compact, around $x$ and $0 < n_k < n'_k < \infty$ such that
\[
n'_k - n_k \to \infty \quad \text{as} \quad k \to \infty, \quad \{tx \mid n_k < t < n'_k, t \in G\} \cap U = \emptyset, \quad \text{and} \quad n_k x \in U.
\]
Since $U$ is compact, we may let $n_k x \to x^* \in \Sigma \cap U$ as $k \to \infty$ by instead considering a subnet of the net $(n_k x)_{k \in \mathbb{N}}$ in $U$ if necessary and hence
\[
\{tx^* \mid t \in G_+\} \cap U = \emptyset.
\]
Thus the $\omega$-limit set $\omega(x^*|G_+ \curvearrowright X) \subseteq \Sigma$ is nonempty and $G$-invariant because of the forward $G$-recurrence of $x^*$ and $\omega(x^*|G \curvearrowright X) = \omega(x^*|G_+ \curvearrowright X)$. This is a contradiction to the $G$-minimality of $\Sigma$.

Therefore, $\Sigma = G_+[x]$ is compact for every $x \in \Sigma$ by Corollary 2.10. Then the statement follows from Lemma 2.1.

As a consequence of this lemma, we can obtain the following result concerning topological group which appears in A. Weil [9, p. 96].

**Corollary 2.14.** Let $K$ be a locally compact topological group with the identity element $e$, which is either Hausdorff or regular, and $a \in K$. Let $L_a : K \to K$ be the left multiplication defined by $L_a x = ax$ for all $x \in K$. Then either

1. $\varphi : \mathbb{Z} \to K$ by $n \mapsto a^n$, where $a^0 = e$, is isomorphic homeomorphism of $\mathbb{Z}$ into $K$,

or

2. $\{a^n \mid n \in \mathbb{Z}\}$ is a compact $L_a$-minimal abelian subgroup of $K$.

In case of (2), $\varphi$ is almost periodic in the sense of Bohr if $K$ is Hausdorff.
Proof. If the identity element $e$ is neither forwardly recurrent nor backwardly recurrent for the system $(K, L_0)$, then by the definition of recurrent point it can easily be seen that the first case (1) holds.

Suppose now that $e$ is forwardly recurrent for $L_a$. It follows from homogeneity that $L_a$ is pointwise forwardly recurrent (i.e. all points of $K$ are forwardly recurrent for $L_a$) and that $\{a^n : n \in \mathbb{Z}\}$ is $L_a$-minimal. Then (2) follows from Lemma 2.13.

If $e$ is backwardly recurrent for $L_a$, then (2) can be similarly proved.

Finally in the case (2), we may assume $K = \{a^n : n \in \mathbb{Z}\}$. If the topological group $K$ is Hausdorff, then $K$ is a regular $T_1$-space and hence $K$ is a compact uniform space. Therefore $e$ is almost periodic of Bohr by Kronecker’s theorem (cf. [2, Theorem 1.9]).

This completes the proof of Corollary 2.14. 

3. Birkhoff recurrent motions of actions on uniform spaces. Based on the sufficient condition of Theorem 1.3, we shall introduce and characterize the Birkhoff recurrent motions for a topological action of a topological semigroup on a uniform space. This type of motion turns out to be different from the uniformly recurrent one when the underlying space $X$ is not locally compact. We shall prove this by explicitly exhibiting a counterexample in the next section.

3.1. Uniform spaces. First of all, we let $X$ be a uniform space; that is to say, let $X$ be a topological space provided with a system of indexed neighborhoods $\mathcal{U} = \{U_\lambda(x)\}_{x \in X, \lambda \in \Lambda}$, which is referred to as a uniformity structure of $X$, subject to the following conditions (A. Weil [9] or W. Gottschalk [3]):

1) If $x \in X$ and if $\lambda \in \Lambda$, then $x \in U_\lambda(x)$.
2) If $\alpha, \gamma \in \Lambda$, then there is some $\lambda \in \Lambda$ such that $U_\lambda(x) \subset U_\alpha(x) \cap U_\gamma(x)$ for all $x$ in $X$.
3) For any $\epsilon \in \Lambda$ there exists some $\beta = \beta(\epsilon) \in \Lambda$ such that whenever $x, y, z \in X$ with $x, y \in U_\beta(z)$, there follows $x \in U_\epsilon(y)$.

It is obvious that any metric space $(X, d)$ is a uniform space with a uniformity structure $\mathcal{U} = \{U_\lambda(x)\}_{x \in X, \lambda \in \mathbb{N}}$, where $U_n(x) = \{y \in X : d(x, y) < n^{-2}\}$. In addition, every compact Hausdorff space is a uniform space with a unique compatible uniformity structure.

For any set $A \subset X$ and any index $\epsilon \in \Lambda$, the $\epsilon$-neighborhood of $A$ in $(X, \mathcal{U})$ is defined as the set $U_\epsilon(A) = \bigcup_{x \in A} U_\epsilon(x)$.

From now on, let $G \curvearrowright X; (g, x) \mapsto gx$ be a topological action of a topological semigroup $G$ on the uniform space $(X, \mathcal{U})$ unless an explicit illustration.

3.2. Birkhoff recurrent motions. For our convenience we introduce the following notation, which is not via the point of view of studying the set of semigroup elements that applied to the given point bring it close to itself.

Definition 3.1. A point $x$ in $X$ is said to be Birkhoff recurrent for $G \curvearrowright X$, if for any uniformity index $\epsilon$ of $(X, \mathcal{U})$ one can find a compact subset $K$ of $G$ such that

$$G[x] \subset U_{\epsilon}(K[gy]) \quad \forall y \in G, \quad \text{or equivalently,} \quad \overline{G[x]} \subset U_{\epsilon}(K[y]) \quad \forall y \in \overline{G[x]}.$$ 

See [8, Definition V7.05] for the case where $X$ is a metric space and $G = \mathbb{R}$.

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2In fact, for any $y \in \Sigma_e := \{a^n : n \in \mathbb{Z}\}$, let $\Sigma_y$ be the $L_a$-orbit-closure of $y$. Define homomorphism $R_g : \Sigma_e \to \Sigma_y$ by $x \mapsto xy$. Since $y^{-1} \in \Sigma_e$, $e \in \Sigma_y$ and then $\Sigma_e \subseteq \Sigma_y \subseteq \Sigma_e$. 

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This means that for a Birkhoff recurrent point \( x \), any orbit arc \( K[\gamma x] \) approximates the entire orbit \( G[x] \) with a precision to within \( \varepsilon \). Clearly from the definition, if \( x \in X \) is Birkhoff recurrent for \( G \curvearrowright X \), then so is each point \( y \) in \( G[x] \).

Theorem 1.3 asserts that if \( X \) is a compact metric space, then a point \( x \in X \) is Birkhoff recurrent whenever \( x \) is uniformly recurrent for \( G \curvearrowright X \).

In the uniform space situation, we can reformulate the Bohr almost periodic motion as follows:

**Definition 1.2′.** A point \( x \in X \) is said to be **almost periodic of Bohr** for \( G \curvearrowright X \), if for an index \( \alpha \) belonging to \((X, \mathcal{U})\) the set
\[
T_{\alpha} := \{ \tau \in G : \tau y \in U_\alpha(y) \ \forall y \in G[x] \},
\]
or equivalently,
\[
T_{\alpha} := \{ \tau \in G : \tau y \in U_\alpha(y) \ \forall y \in \overline{G[x]} \},
\]
is syndetic of (F) in \( G \). In some classical literature this case was called “\( G \) restricted to \( \overline{G[x]} \) is almost periodic”; see, e.g., [7, p. 323] and [3, p. 635].

We remark that the essential difference between the Birkhoff motion and the Bohr motion is that in Definition 3.1, the “translation” \( K \) is uniform for \( y \in G[x] \). However, in Definition 1.2′, the “return times” \( T_{\alpha} \) is uniform for \( y \in G[x] \).

If here \( G \) is a topological group, then similar to [8, Theorem V8.02] an almost periodic motion of Bohr as in Definition 1.2′ is not only uniformly recurrent but also Birkhoff recurrent for \( G \curvearrowright X \). Indeed let \( \varepsilon \) be an arbitrary uniformity index and let \( K \) be a compact subset of \( G \) corresponding the syndetic set \( T_{\alpha} \) in Definition 1.2′ where \( \alpha = \beta(\varepsilon) \) is given as in (3); now for any \( g, h \in G \), there is some \( k \in K \) with \( \tau = k(gh^{-1}) \in T_{\alpha} \) and so \( \tau(hx) \in U_\alpha(hx) \); further \( kgx \in U_\alpha(hx) \) and \( hx \in U_\varepsilon(kgx) \); hence \( G[x] \subset U_\varepsilon(K[\gamma x]) \).

Although there is no the equi-continuity of \( G \curvearrowright X \) at the initial point \( x \in X \), comparing to [3, Theorem 1], we can obtain the following precompact property.

**Lemma 3.2.** Let \( x \in X \) be a Birkhoff recurrent point for \( G \curvearrowright X \). Then the orbit-closure \( \overline{G[x]} \) is totally bounded; more precisely, for any index \( \varepsilon \) belonging to \((X, \mathcal{U})\), there are finitely many elements \( k_1, \ldots, k_l \in G \) with \( \overline{G[x]} \subset U_\varepsilon(k_1x) \cup \cdots \cup U_\varepsilon(k_lx) \).

**Proof.** Let \( \varepsilon \) be an arbitrary index belonging to \((X, \mathcal{U})\). By (3) there exists an index \( \gamma \) belonging to \((X, \mathcal{U})\) so that \( y \in U_\gamma(b) \) and \( b \in U_\gamma(a) \) imply \( y \in U_\varepsilon(a) \).

Let \( \varepsilon_1 = \beta(\gamma) \) and \( \varepsilon_2 = \beta(\varepsilon_1) \) be defined as in (3). Then for any compact subset \( A \) of \( X \), the \( \varepsilon_1 \)-neighborhood of \( A \) in \((X, \mathcal{U})\) is contained in the \( \gamma \)-neighborhood of finitely many points of \( A \). Indeed, take \( \{ U_{\varepsilon_2}(a_1), \ldots, U_{\varepsilon_2}(a_l) \} \) a cover of \( A \) in \((X, \mathcal{U})\). Then for any \( y \in U_{\varepsilon_1}(A) \), there is some \( a \in A \) and \( 1 \leq i \leq l \) such that \( y \in U_{\varepsilon_2}(a) \) and \( \gamma \)-neighborhood \( a \in U_{\varepsilon_2}(a_1) \); hence \( a \in U_{\varepsilon_1}(a) \) and further \( y \in U_\varepsilon(a_1) \). Thus, \( U_{\varepsilon_1}(A) \subset U_\gamma(\{a_1, \ldots, a_l\}) \).

Now since \( x \) is Birkhoff recurrent, there is a compact set \( K \) in \( G \) such that \( G[x] \subset U_{\varepsilon_1}(K[x]) \). By replacing \( A \) with \( K[x] \), one can find elements \( k_1, \ldots, k_l \in K \) with \( G[x] \subset U_\gamma(\{k_1x, \ldots, k_lx\}) \) and further \( \overline{G[x]} \subset U_\varepsilon(\{k_1x, \ldots, k_lx\}) \).

The proof of Lemma 3.2 is thus complete.

\[ \square \]

We note that the converse statement of Lemma 3.2 is obviously false; for example, if \( X \) is compact, then all motions are totally bounded.
3.3. **Characterization of Birkhoff motions.** Paralleled to Lemma 2.1, the following is a generalization of Birkhoff’s theorem (cf. [8, Theorem V7.06] for the case of $G = \mathbb{R}$).

**Lemma 3.3.** Let $\Sigma$ be a compact minimal set for $G \curvearrowright X$. Then for any uniformity index $\epsilon$ belonging to $(X, \mathcal{U})$, there exists a finite subset $K$ of $G$ such that $\Sigma$ is contained in $U_\epsilon(K[x])$ for each $x \in \Sigma$.

**Proof.** Our argument is motivated by the proof of [2, Lemma 2.2]. Let $\Sigma$ be a compact $G$-minimal subset of $X$ for $G \curvearrowright X$. There is no loss of generality in assuming $\Sigma = X$.

To see this, let $\epsilon$ be an arbitrary uniformity index and let $\{U_\beta(y_i)\}_{i=1}^n$ be a finite covering of $X$, where $\beta = \beta(\epsilon)$ is an index given as in the condition 3) of the definition of the uniform space $(X, \mathcal{U})$. Then since $X$ is compact and $G$-minimal, for each $U_\beta(y_i)$ one can find a finite subset $\{g_{ij}\}_{j=1}^n$ of $G$ such that

$$X = \bigcup_{j=1}^n g_{ij}^{-1}U_\beta(y_i)$$

by [2, Lemma 1.14]. It is now clear that for the finite set $K = \{g_{ij}\}$, one can gain that for all $z, y \in X$,

$$y \in \bigcup_{k \in K} U_\epsilon(kz) = U_\epsilon(K[z]).$$

The proof of Lemma 3.3 is thus completed. \qed

This lemma implies that every point of a compact minimal set is a Birkhoff recurrent point for $G \curvearrowright X$. Note that the $K$ is uniform for $x \in \Sigma$ in this lemma.

The following is also a generalization of another theorem due also to Birkhoff (cf. [8, Theorem V7.07] for $G = \mathbb{R}$).

**Lemma 3.4.** Let $X$ be a uniform space which is regular. If $x$ in $X$ is Birkhoff recurrent for $G \curvearrowright X$, then $\overline{G[x]}$ is totally bounded and $G$-minimal.

**Proof.** Let $x \in X$ be a Birkhoff recurrent point for $G \curvearrowright X$; then $\overline{G[x]}$ is totally bounded by Lemma 3.2. Next let $\overline{G[x]}$ be not minimal for $G \curvearrowright X$. Then there is some point $y \in \overline{G[x]}$ such that $x \notin \overline{G[y]} \subset \overline{G[x]}$. Since $X$ is regular, one can find some index $\epsilon$ belonging to $(X, \mathcal{U})$ such that $\overline{G[y]} \cap U_\epsilon(x) = \emptyset$. Choosing an index $\beta = \beta(\epsilon)$ as in 3), by Definition 3.1 we can take a compact subset $K = K_\beta$ of $G$ with the property that $\overline{G[x]} \subset U_\beta(K[y])$. Then $x \in U_\beta(z)$ for some $z \in G[y]$, and hence this implies a contradiction $z \in U_\epsilon(x)$.

This thus proves Lemma 3.4. \qed

In the proof of Lemma 3.4, reasoning of the total boundedness does not need the regularity assumption of $X$, but the minimality of $\overline{G[x]}$ does need that.

Recall that a family $\mathcal{F}$ of subsets of $X$ is called a **filter** if the following three properties are satisfied:

(a) $\mathcal{F} \neq \emptyset$;

(b) $\mathcal{F}$ has the finite intersection property;

(c) if $F \in \mathcal{F}$ and $F \subset H$, then $H \in \mathcal{F}$.

A filter $\mathcal{F}$ is referred to as a **Cauchy filter** of the uniform space $(X, \mathcal{U})$ if for any uniformity index $\epsilon$ there is some point $x \in X$ and some $F \in \mathcal{F}$ such that $F \subset U_\epsilon(x)$. And a point $x$ is called a **limit point** of $\mathcal{F}$ if $U_\epsilon(x) \in \mathcal{F}$ for any index
$\varepsilon \in \Lambda$. Furthermore, the uniform space $(X, \mathcal{U})$ is said to be **complete** provided that every Cauchy filter has a limit point.

We can easily obtain the following from Lemmas 3.3 and 3.4:

**Corollary 3.5.** Let $G \ltimes X$ be a topological action of a topological semigroup on a complete Hausdorff uniform space $(X, \mathcal{U})$. Then $x \in X$ is Birkhoff recurrent for $G \ltimes X$ if and only if for any index $\varepsilon$ belonging to $(X, \mathcal{U})$ there is a finite subset $K$ of $G$ so that $G[x] \subset U_{\varepsilon}(K[y])$ for all $y \in G[x]$.

**Proof.** We need only to prove the necessity and then assume $x \in X$ is Birkhoff recurrent for $G \ltimes X$. Since a Hausdorff uniform space is completely regular, from Lemma 3.4 it follows that $G[x]$ is compact and $G$-minimal. The necessity then follows at once from Lemma 3.3. \qed

3.4. **Pointwise Birkhoff recurrent action.** We say that $G \ltimes X$ is **pointwise Birkhoff recurrent** provided that each point of $X$ is Birkhoff recurrent for $G \ltimes X$.

Recall that $X = \bigcup_{\lambda \in \Lambda} X_{\lambda}$ is called a **compact partition** of $X$ if $X_{\lambda}$ is compact and either $X_{\lambda} \cap X_{\lambda'} = \emptyset$ or $X_{\lambda} = X_{\lambda'}$ for any $\lambda, \lambda' \in \Lambda$.

Recall a basic fact that a complete metric space is compact if and only if it is totally bounded. Then the following corollary follows easily from Lemmas 3.3 and 3.4.

**Corollary 3.6.** Let $G \ltimes X$ be a topological action of a topological semigroup on a complete metric space $X$. Then $G \ltimes X$ is pointwise Birkhoff recurrent if and only if the collection of orbit-closures is a compact partition of $X$.

4. **Conclusion remarks and examples.** In this section, we shall complete the proof of Theorem 1.3 and prove the Birkhoff recurrence is, in noncompact situation, stronger than the uniform recurrence by constructing an explicit example. In addition, we will show that the uniform recurrence in the sense of Definition 1.1 is not equivalent to the almost periodicity in the sense of Definition 2.5.

4.1. **Proof of Theorem 1.3.** First of all, based on Lemmas 2.1, 2.2, 3.3, and 3.4, we can now readily prove Theorem 1.3 as follows:

**Theorem 4.1.** Let $G \ltimes X$ be a topological action of a topological semigroup $G$ on a compact Hausdorff space $X$ and $x \in X$. Then the following four properties are equivalent to each other:

1. The point $x$ is uniformly recurrent for $G \ltimes X$ as Definition 1.1.
2. $G[x]$ is compact and $G$-minimal.
3. The point $x$ is Birkhoff recurrent for $G \ltimes X$ as Definition 3.1.
4. For any index $\varepsilon$ belonging to $\mathcal{U}$ there exists a finite subset $K$ of $G$ such that $G[x] \subset U_{\varepsilon}(K[y])$ for all $y \in G[x]$.

**Proof.** Let $G$ be a topological semigroup, $X$ a compact Hausdorff (uniform) space with the uniformity structure $\mathcal{U}$, and let $G \ltimes X$ be a topological action. Then (1) $\Leftrightarrow$ (2) follows by Lemmas 2.1 and 2.2. And (2) $\Leftrightarrow$ (3) follows from Lemmas 3.3 and 3.4. Finally, (3) $\Leftrightarrow$ (4) follows from Corollary 3.5. This therefore proves Theorem 4.1. \qed

The following corollary is a generalization of [8, Theorem 7.09] from the case where $G = \mathbb{R}$ or $\mathbb{Z}$ and $X$ is compact to the case of a general group action on a locally compact space.
Corollary 4.2. Let $G \acts X$ be a topological action of topological group $G$ on a locally compact metric space $X$. Then a point $x$ in $X$ is uniformly recurrent for $G \acts X$ if and only if it is Birkhoff recurrent for $G \acts X$.

Proof. Let $x \in X$ be any given. If $x$ is Birkhoff recurrent for $G \acts X$, then by Definition 3.1, $x$ is also uniformly recurrent for $G \acts X$ in the sense of Definition 1.1. Indeed, for any $\varepsilon > 0$, let $K$ be given as in Definition 3.1; then for any $g \in G$, since $G[x] \subset U_\varepsilon(K[gx])$, we can find some $k \in K$ such that $d(x, gx) < \varepsilon$ and then $kg \in S = \{\tau \in G: \tau x \in B(x)\}$. This implies that $S$ is syndetic of (F).

Conversely let $x$ is uniformly recurrent for $G \acts X$; then from [5, Theorem 4.09] or Theorem 2.9 it follows that $\Sigma = \overline{G[x]}$ is compact $G$-minimal. Hence restricting to the subsystem $G \acts \Sigma$, Theorem 1.3 follows that $x$ is Birkhoff recurrent for $G \acts X$.

This completes the proof of Corollary 4.2. □

Corollary 4.3. Let $G \acts X$ be a topological action of the semigroup $\mathbb{Z}_+^d$ or $\mathbb{R}_+^d$ on a locally compact metric space $X$. Then if a point $x \in X$ is almost periodic of von Neumann for $G \acts X$ it is Birkhoff recurrent for $G \acts X$.

Proof. By Corollary 2.7 $\overline{G[x]}$ is $G$-minimal, and by Theorem 2.9 it is compact. Then the statement follows from Theorem 4.1. □

4.2. An example of uniformly recurrent non-minimal motion. Let the underlying space $X$ be only a complete metric space, not compact. Then Lemma 3.4 asserts that if $x \in X$ is Birkhoff recurrent for a semigroup topological action $G \acts X$, then the orbit-closure $\overline{G[x]}$ is compact. Naturally we ask that: If $x \in X$ is a uniformly recurrent point in the sense of Definition 1.1 for $G \acts X$, is $\overline{G[x]}$ compact? The answer is “Yes” if $X$ is locally compact and $G = \mathbb{Z}_+$ or $\mathbb{R}_+$ by Theorem 2.12. However, if $X$ is not locally compact, the answer is “No” as shown by an example constructed below. Thus these two kinds of motions are different in general.

Let $I = [0, 1]$ be the unit interval endowed with the discrete topology and define the cartesian product $X = I^\mathbb{Z}$ equipped with the standard metric:

$$d(x(\cdot), x'(\cdot)) = \inf \left\{ \frac{1}{n+1}: x(i) = x'(i) \text{ for } |i| < n \right\}.$$ 

Then $X$ is a complete metric space, but it is not locally compact in the sense of the topology induced by $d(\cdot, \cdot)$. We will consider the left-shift (noncompact) Bebutov system:

$$\mathbb{Z} \acts X; \ (n, x(\cdot)) \mapsto x(n + \cdot).$$

Next we will follow the procedure of constructing uniformly recurrent points in compact Bebutov systems introduced by Furstenberg [2, Section 1.5].

For any word $w \in I^\ell$ of length $\ell \geq 1$, we say $w$ occurs in $x(\cdot) \in X$ if there is some $n \in \mathbb{Z}$ with $w = (x(n), \ldots, x(n + \ell - 1))$.

The following criterion of uniform recurrence for $\mathbb{Z} \acts X$ follows directly from Definitions 1.1 and (F):

Lemma 4.4. A point $x(\cdot)$ in $X$ is uniformly recurrent for $\mathbb{Z} \acts X$ if and only if every word that occurs in $x(\cdot)$ occurs along a syndetic set in $\mathbb{Z}$.

We note that if instead $I$ is a finite alphabet, then Lemma 4.4 reduces to [2, Proposition 1.22].
Lemma 4.5. Let \(d_1, d_2, d_3, \ldots\) be a sequence of natural numbers with \(d_1 \geq 2\) such that \(d_k|d_{k+1}\) for all \(k \geq 1\). Then one can express \(\mathbb{Z}\) as a disjoint union of the infinite arithmetic progressions \(\{d_k \mathbb{Z} + a_k\}\) where the \(a_k\) are appropriately chosen:

\[
\mathbb{Z} = \bigcup_{k=1}^{\infty} (d_k \mathbb{Z} + a_k).
\]

Proof. (Cf. [2, p. 33]) Assuming that we have chosen \(a_1, \ldots, a_k\) we could let \(a_{k+1}\) be the integer with smallest absolute value not covered by \(d_1 \mathbb{Z} + a_1, \ldots, d_k \mathbb{Z} + a_k\). \(\square\)

Now we define a point \(x(\cdot) : \mathbb{Z} \to I\) in \(X\) by

\[
x(n) = \frac{1}{k}
\]

if and only if \(n \in d_k \mathbb{Z} + a_k\), for all \(n \in \mathbb{Z}\).

The following result shows that \(x\) is uniformly recurrent but not Birkhoff recurrent and further not almost periodic by constructing an explicit simple example.

Proposition 4.6. The point \(x(\cdot)\) is a uniformly recurrent point for \(\mathbb{Z} \acts X\), but the orbit-closure \(\overline{\{x(\cdot)\}}\) is not compact.

Proof. It is clear that if \([-N, N] \subset \bigcup_{k=1}^{N+1} (d_k \mathbb{Z} + a_k)\), then the \((2N+1)-\text{length}\) word \((x(-N), \ldots, x(N))\) is equal to \((x(-N + nd_1), \ldots, x(N + nd_1))\) for all \(n \in \mathbb{Z}\). From this we see that every word occurring in \(x(\cdot)\) occurs along a periodic sequence. By Lemma 4.4, the point \(x(\cdot)\) is uniformly recurrent for \(\mathbb{Z} \acts X\).

By Lemma 4.5, we may see that all the symbols \(\frac{1}{k}\) occur in \(x(\cdot)\). Then there exists a sequence of points \(\{x(n_k + \cdot)\}_{k=1}^{\infty}\) in \(\overline{\{x(\cdot)\}}\) such that \(x(n_k + 0) = \frac{1}{k}\). Obviously, \(d(x(n_k + \cdot), x(n_{k'} + \cdot)) = 1\) if \(k \neq k'\) and hence this sequence has no convergent subsequence. Therefore \(\overline{\{x(\cdot)\}}\) is not compact.

The proof of Proposition 4.6 is thus completed. \(\square\)

We note that if we take another metric on \(X\) as follows:

\[
D(x, x') = \sum 2^{-|n|} |x(n) - x'(n)|,
\]

then \(X\) is a compact metric space and so \(\overline{\{x(\cdot)\}}\) is compact and \(\mathbb{Z}\)-minimal by Theorem 1.3 and Lemma 3.4.

4.3. Uniform recurrence \(\not\Rightarrow\) almost periodicity of von Neumann. Let \(G\) be a topological semigroup and \(X\) a compact metric space. Then by Lemma 2.1 there always exists a uniformly recurrent point in the sense of Definition 1.1 for \(G \acts X\). Corollary 2.7 follows that if \(G\) is either \(\mathbb{Z}_+^4\) or \(\mathbb{R}_+^4\), then an almost periodic point of von Neumann (cf. Definition 2.5) is also a uniformly recurrent point for \(G \acts X\). Next we will prove that a uniform recurrent point is not necessarily to be an almost periodic point of von Neumann by constructing an explicit simple example.

Let \(X = [0,1]\) with the distance \(d(x, y) = |x - y|\). By \(C(X, X)\) we denote the set of all continuous maps of \(X\) to itself, which is equipped with the uniform convergence topology: \(\|f - g\| = \max_{x \in X} |f(x) - g(x)|\) for all \(f, g \in C(X, X)\). Then \((C(X, X), \| \cdot \|)\) is a complete metric space, and under the multiplicative binary operation \(fg = f \circ g\) of maps composition \(C(X, X)\) is a topological semigroup. It is easy to check that the action

\[
C(X, X) \acts X; \quad (g, x) \mapsto gx
\]

is jointly continuous with respect to \(g \in C(X, X)\) and \(x \in X\); that is, \(C(X, X) \acts X\) is a topological semigroup action.
Proposition 4.7. Let $T_1 x = \frac{x}{2}$ and $T_2 x = \frac{x+1}{2}$ for all $x \in X$. Let $G = [T_1, T_2]$ be the countable sub-semigroup of the topological semigroup $C(X, X)$, which is generated by the two elements $T_1$ and $T_2$ not necessarily containing the identity $I$. Then the point $0 \in X$ is uniformly recurrent of $(F)$ but not almost periodic of von Neumann for $G \curvearrowright X$.

Proof. First of all, we note that for any $\varepsilon > 0$ there exists an integer $N_\varepsilon > 0$ sufficiently large such that $0 \leq T_1^n x < \varepsilon$ and $1 - \varepsilon < T_2^n x \leq 1$ for any $n \geq N_\varepsilon$ and any $x \in X$.

Since $X$ is compact, by Zorn’s lemma and Lemma 2.1 there must be a point, say $a \in X$, which is uniformly recurrent for $G \curvearrowright X$. Then by $\{T_1^n a \mid n \geq N_\varepsilon\} \subset G[a]$ for any $\varepsilon > 0$, it follows that $0 \in G[a]$ and hence the point $0$ itself is uniformly recurrent for $G \curvearrowright X$.

Finally we assert that the point $0$ is not almost periodic in the sense of Definition 2.5. For that, let $\varepsilon > 0$ be sufficiently small and put $S = \{s \in G : s0 \leq \varepsilon\}$. To prove the assertion, it is sufficient to show that $S$ is not GH syndetic; that is, $G \neq \bigcup_{s \in S} Ks$ for any compact subset $K$ of the topological semigroup $G$. Indeed let $K \subset G$ be any given compact set. For any $(I \neq) s = T_{i_1} \cdots T_{i_k} \in S$ where $(i_1, \ldots, i_k) \in \{1, 2\}^k$, by $s0 \leq \varepsilon$ the element $s$ must have the form $s = T_1 T_{i_1} \cdots T_{i_k}$. Therefore we can find some number $\gamma$ with $0 < \gamma < 1$ such that $K[s0] \subseteq [0, \gamma]$ for all $s \in S$, since $K$ and $[0, 1/2]$ are compact and $gx < 1$ for all $g \in K$ and $0 \leq x \leq 1/2$. Now for any $n > N_\gamma$, the element $g = T_2^n$ in $G$ does not belong to $KS$. This proves our assertion.

Thus the proof of Proposition 4.7 is completed. \hfill \Box

This result also shows a syndetic set of $(F)$ does not need to be a GH syndetic set. In fact, we can similarly prove the following

Proposition 4.8. Let $G \curvearrowright X$ be the same dynamical system as in Proposition 4.7. Then there is no almost periodic point of von Neumann in $X$ for $G \curvearrowright X$.

This result and Theorem 1.3 show that for semigroup actions, Definition 1.1 is better than Definition 2.5.

4.4. Some open questions. We now conclude this paper with two open questions for our further study.

Question 4.9. Does the statement of Theorem 1.3 still hold if $G \curvearrowright X$ is a topological semigroup action on a locally compact metric space $X$?

The local compactness can not be further relaxed by Proposition 4.6.

Question 4.10. Does the almost periodicity of von Neumann imply the uniform recurrence of Definition 1.1 for a general topological semigroup action $G \curvearrowright X$?

The statement of Corollary 2.7 suggests that Question 4.10 could have a positive solution.

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