Appearance of the prolate and the toroidal magnetic field dominated stars: Analytic approach

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Abstract

We have analyzed magnetized equilibrium states and shown a condition for the appearance of the prolate and the toroidal magnetic field-dominated stars using analytic approaches. Both observations and numerical stability analysis support that the magnetized star would have prolate and large internal toroidal magnetic fields. In this context, many investigations concerning magnetized equilibrium states have been tried to obtain the prolate and the toroidal dominant solutions, but many of them have failed to obtain such configurations. Since the Lorentz force is a cross-product of current density and magnetic field, the prolate-shaped configurations and the large toroidal magnetic fields in stars require a special relation between current density and the Lorentz force. We have analyzed simple analytical solutions and found that the prolate and the toroidal-dominant configuration require non-force-free toroidal current density that flows in the opposite direction with respect to the bulk current within the star. Such current density results in the Lorentz force which makes the stellar shape prolate. Satisfying this special relation between the current density and the Lorentz force is a key to the appearance of the prolate and the toroidal magnetic field-dominated magnetized star.

Key words: stars: magnetars — stars: magnetic fields — stars: rotation

1 Introduction

Anomalous X-ray pulsars and Soft-Gamma-ray-Repeaters (SGRs) are considered as special classes of neutron stars, i.e., magnetars (Thompson & Duncan 1995). According to observations of rotational periods and their time derivatives, the magnitudes of global dipole magnetic fields of magnetars reach about $10^{14}$–$15$ G. Recently, however, SGRs with weak dipole magnetic fields have been found (Rea et al. 2010, 2012). Their observational characteristics are very similar to those of ordinary SGRs but their global dipole magnetic fields are much weaker than those of ordinary magnetars. This might be explained by the possibility that such SGRs with small magnetic fields hide large toroidal magnetic fields under their surfaces and drive their activities by their internal toroidal magnetic energy (Rea et al. 2010). Recent X-ray observation of magnetar 4U 0142+61 also implies the presence of large toroidal magnetic fields and the possibility of a prolate-shaped neutron star (Makishima et al. 2014). By considering that possible growth of magnetic fields of magnetars occurs...
during protomagnetar phases, it could be said that strong differential rotation within protomagnetar would amplify their toroidal magnetic fields (Duncan & Thompson 1992; Spruit 2009). Therefore, it would be natural that some magnetars sustain large toroidal magnetic fields inside them.

The large toroidal fields are needed to stabilize the magnetic field according to the stability analyses. Stability analyses have shown that stars with purely poloidal fields or purely toroidal fields are unstable (Markey & Tayler 1973; Tayler 1973). Stable magnetized stars should have both poloidal and toroidal magnetic fields. Moreover, the toroidal magnetic field strengths of the stable magnetized stars have been considered to be comparable with those of poloidal components (Tayler 1980). However, we have not yet found the exact stability condition or stable magnetic field configurations, because it is too difficult to carry out stability analyses of stars with both poloidal and toroidal magnetic fields.

Nevertheless, stabilities of magnetic fields have been studied by performing dynamical simulations. Braithwaite and Spruit (2004) showed that twisted-torus magnetic-field structures are stable magnetic-field configurations on dynamical timescales. Stabilities of purely toroidal magnetic-field configurations or purely poloidal magnetic-field configurations have been studied in the Newtonian framework (Braithwaite 2006, 2007) and in the full general relativistic framework (Ciolfi et al. 2011; Kiuchi et al. 2011; Lasky et al. 2011; Ciolfi & Rezzolla 2012). Braithwaite (2009) and Duez, Braithwaite, and Mathis (2010) have found a stability criterion of the twisted-torus magnetic fields. It could be expressed as

$$\alpha \frac{M_t}{|W|} < \frac{M_p}{M} \leq 0.8,$$

(1)

where $M_t/|W|$ is the ratio of the total magnetic energy to the gravitational energy. $M_p/M$ is the ratio of the poloidal magnetic-field energy to the total magnetic-field energy. $\alpha$ is a certain dimensionless factor that is of the order of 10 for main-sequence stars and of the order $10^3$ for neutron stars. The ratio of $M_t/|W|$ is a small value ($\sim 10^{-5}$) even for magnetars. Therefore, the criterion becomes

$$0.2 \leq \frac{M_t}{M} \leq 0.99,$$

(2)

where $M_t$ is the toroidal magnetic-field energy. Stellar magnetic fields would be stable even for toroidal magnetic field-dominated configurations. Therefore, it is very natural that the toroidal magnetic-field strength of the stable stationary magnetized stars is comparable with or larger than those of poloidal component.

Until recently, however, almost all numerically obtained-equilibrium configurations for stationary and axisymmetric stars have only small fractions of toroidal magnetic fields, typically $M_t/M \sim 0.01$, even for twisted-torus magnetic-field configurations in the Newtonian gravity (Tomimura & Eriguchi 2005; Yoshida & Eriguchi 2006; Yoshida et al. 2006; Lander & Jones 2009; Lander et al. 2012; Fujisawa et al. 2012; Lander 2013, 2014; Bera & Bhattacharya 2014; Armaza et al. 2015), in general relativistic perturbative solutions (Ciolfi et al. 2009, 2010), and general relativistic nonperturbative solutions under both simplified relativistic gravity (Pili et al. 2014) and fully relativistic gravity (Uryû et al. 2014). None of them satisfy the stability criterion mentioned above.

On the other hand, there are several works which have successfully obtained the stationary states with strong toroidal magnetic fields by applying special boundary conditions. Glamadakis, Andersson, and Lander (2012) obtained strong toroidal magnetic-field models imposing surface currents on the stellar surface as their boundary condition. Duez and Mathis (2010) imposed the boundary condition that the magnetic flux on the stellar surface should vanish. Since the magnetic fluxes of their models are zero on the stellar surfaces, all the magnetic-field lines are confined within the stellar surfaces. They obtained configurations with strong toroidal magnetic fields which are essentially the same as those of classical works by Prendergast (1956), Woltjer (1959a, 1959b, 1960), and Wentzel (1960, 1961), and of recent general relativistic works by Ioka and Sasaki (2004) and Yoshida, Kiuchi, and Shibata (2012).

Very recently, have Fujisawa and Eriguchi (2013) found and shown that the strong toroidal magnetic fields within the stars require the non-force-free current or surface current which flows in the opposite direction with respect to the bulk current within the star. Such oppositely flowing currents can sustain large toroidal magnetic fields in magnetized stars. It is also very recently that Ciolfi and Rezzolla (2013) have obtained stationary states of twisted-torus magnetic-field structures with very strong toroidal magnetic fields using a special choice for the toroidal current. Their toroidal currents contain oppositely flowing current components and result in the large toroidal magnetic fields, although their paper does not explain the physical meanings of the appearances of such oppositely flowing toroidal currents. They also did not show clear conditions for the appearance of the toroidal magnetic-field dominated stars.

On the other hand, strong poloidal magnetic fields make the stellar shape an oblate one (e.g., Tomimura & Eriguchi 2005), but the strong toroidal magnetic field tends to make a stellar shape prolate (Haskell et al. 2008; Kiuchi & Yoshida 2008; Lander & Jones 2009; Ciolfi & Rezzolla 2013). Since the Lorentz force is a cross-product of the current density and the magnetic field, it requires a special relation between the magnetic fields and the current density. At the same
time, the large toroidal magnetic fields in stars also need a
special relation between current density and Lorentz force.
The oppositely flowing toroidal current density is a key to
revealing these relations and a condition for the appearance
of the toroidal magnetic field-dominated stars.

We analyze magnetic-field configurations and consider the
special relations in this paper. We find a condition for the appearance of the toroidal magnetic-field domi-
nated stars, which was not described in our previous work
(Fujisawa & Eriguchi 2013). In order to show the rela-
tions and condition clearly, simplified analytical models are
solved and we show examples of the prolate configura-
tions and the large toroidal magnetic fields within stars.
This paper is organized as follows. The formulation and
basic equations are shown in section 2. We present analytic
solutions and the relations. We also explain the important
role of oppositely flowing components of the \( \kappa \) currents for
the appearance of the prolate shapes and the presence of the
large toroidal magnetic fields using the relations in section 3.
Discussion and conclusions follow after (section 4). In
appendices 1 and 2 the deformation of stellar shape and the
gravitational potential perturbation are briefly summarized.

2 Formulation and basic equations

Stationary and axisymmetric magnetized barotropic stars
without rotation and meridional flows are analyzed in this
paper.

Some authors have claimed that no dynamically stable
barotropic magnetized stars exist (e.g., Mitchell et al. 2015),
but in our opinion their arguments should be applied only
to isentropic barotropes. The magnetized barotropic stars
are well-defined concepts apart from the thermal stability or
convective stability due to the entropy distributions and/or
due to the chemical composition distributions. Thus we
will investigate mechanical equilibrium states of tradition-
ally defined barotropes (e.g., Chandrasekhar & Prendergast
1956; Prendergast 1956) in this paper.

Since for such configurations the basic equations and
basic relations are shown, e.g., in Tomimura and Eriguchi
(2005) and Fujisawa and Eriguchi (2013), we show the
basic equations and basic relations briefly.

The stationary condition for the configurations men-
tioned above can be expressed as

\[
\int \frac{dp}{\rho} = -\phi + \int B d\Psi + C,
\]

(3)

where \( \rho \), \( p \), \( \phi \), and \( C \) are the density, the pressure, the
gravitational potential of the star, and an integral con-
stant, respectively. \( \Psi \) is the magnetic flux function. \( \mu \) is
an arbitrary function of \( \Psi \). The magnetic flux is governed by

\[
\Delta^* \Psi = -4\pi r \sin \theta \frac{j_c}{c},
\]

(4)

where

\[
\Delta^* = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \cos \theta \frac{\partial}{\partial \theta} - \frac{1}{r^2} \sin \theta \frac{\partial^2}{\partial \theta^2} \right),
\]

(5)

and \( j_c \) is a \( \psi \)-component, i.e., the toroidal component, of
the current density. The spherical coordinates \( (r, \theta, \varphi) \)
are used.

From the integrability condition of the equation of motion, the axisymmetry, and the stationarity, the fol-
lowing relations are derived:

\[
\frac{j}{c} = \frac{1}{4\pi} \frac{dk}{d\Psi} B + \rho r \sin \theta \mu(\Psi) e_\psi,
\]

(6)

\[
\kappa = \kappa(\Psi),
\]

(7)

where \( j \) and \( B \) are the current density and the magnetic
field, respectively, and \( \kappa \) is another arbitrary function of \( \Psi \). It would be helpful to note that the above relation for \( \kappa \) was
found by Mestel (1961) and Roxburgh (1966). Although \( \kappa(\Psi) \) is exactly a function of the magnetic-flux function only
in stationary and axisymmetric systems (Braithwaite 2009),
Braithwaite (2008) showed that the function \( \kappa(\Psi) \) during
the dynamical evolution of magnetized configurations is
nearly conserved even for non-axisymmetric systems.

Since the function \( \kappa \) and the \( \psi \)-component of the mag-
netic field \( B_\psi \) is related as

\[
\kappa(\Psi) = r \sin \theta B_\psi,
\]

(8)

the toroidal current density can be expressed as

\[
\frac{j_\psi}{c} = \frac{1}{4\pi} \frac{\kappa(\Psi)\kappa'(\Psi)}{r \sin \theta} + \rho r \sin \theta \mu(\Psi).
\]

(9)

Under our assumption that the magnetic-field energy is
small compared to the gravitational energy \( (M/|W|) < 10^{-5} \)
in this paper, the influence of the magnetic fields can be
treated as a small perturbation to a spherical star. There-
fore, we assume that the stellar configurations are
spherical and that the density profile depends only on \( r \),
i.e., \( \rho = \rho(r) \). For such situations, we can obtain analytical
solutions easily. Note that self-consistent approaches such
as Tomimura and Eriguchi (2005) would reveal some dif-
f erences in the magnetic-field solutions. In self-consistent
approaches, we need to calculate both magnetic fields and
matter equations iteratively. The stellar shape is no longer
spherical and the stellar configuration affects the magnetic-
field configuration. However, our result in this paper is
simple and might be important for both perturbative and self-consistent approaches.

3 Spherical models with weak magnetic fields

Our aim in this paper is analytically investigating the condition for the appearance of a toroidal magnetic field-dominated star. Note that our solutions of $\Psi$ themselves are classical and not new, but we use the solutions in order to show the special condition clearly.

3.1 Green’s function approach and analytic solutions

We follow mostly the formulation of classical works (Chandrasekhar & Prendergast 1956; Prendergast 1956; Wolter 1959a, 1959b, 1960; Wentzel 1960, 1961) and recent analytical works (Broderick & Narayan 2008; Duez & Mathis 2010; Fujisawa & Eriguchi 2013). In order to obtain analytical solutions, we choose the functional forms as follows:

$$\mu(\Psi) = \mu_0,$$

$$\kappa(\Psi) = \kappa_0 \Psi,$$  \hfill (10)

$$\kappa(\Psi) = \kappa_0 \Psi,$$  \hfill (11)

where $\mu_0$ and $\kappa_0$ are two constants. It should be noted that these functional forms always lead to non-zero surface currents unless magnetic fields are confined inside the star. The surface current induces a Lorentz force at the stellar surface (Lander & Jones 2012). It would be unphysical because the Lorentz force need to be balanced by other physics such as a crust of the neutron star (e.g., Fujisawa & Kisaka 2014). The models relying on a surface current might not be physically realistic. We emphasize that we are not asserting that surface currents themselves are necessarily significant in real stars. The surface current simply provides a mathematically convenient way of describing analytical solutions easily.

By using these functional forms, the toroidal current density can be expressed as

$$j_\phi = \frac{1}{c} \frac{\kappa_0^2 \Psi}{4\pi r \sin \theta} + \mu_0 \rho(r) r \sin \theta.$$  \hfill (12)

We name the first term (a current) “$j_\phi^0$ (force-free) term” and the second term (a current) “$j_\phi^0$ (non-force-free) term” (Fujisawa & Eriguchi 2013). Then, the equation for the magnetic flux becomes as follows:

$$\Delta^* \Psi + \kappa_0^2 \Psi = -4\pi \mu_0 \rho(r) r^2 \sin^2 \theta.$$  \hfill (13)

It should be noted that this is a linear equation for the magnetic flux function. If we impose the boundary condition $\Psi = 0$ at the center of the star, $\Psi$ is described as follows (Duez & Mathis 2010; Fujisawa & Eriguchi 2013):

$$\frac{\Psi}{\sin^2 \theta} = K \kappa_0 r j_1 (\kappa_0 r)$$

$$- 4\pi \mu_0 \kappa_0 \left[ r j_1 (\kappa_0 r) \int_{r}^{r_{c}} y_1 (\kappa_0 r') \rho (r') r'^3 dr' \right. + r y_1 (\kappa_0 r) \int_{0}^{r_{c}} i_1 (\kappa_0 r') \rho (r') r'^3 dr' \left. \right].$$  \hfill (14)

where we set the stellar radius $r_{c} = 1$ in this paper. $j_1$ and $y_1$ are the spherical Bessel functions of the first kind and the second kind, respectively, and $K$ is a coefficient which is determined by a boundary condition of $\Psi$ at the surface. According to the $\theta$-dependency of the inhomogeneous term, we search for solutions of the following form:

$$a(r) \sin^2 \theta \equiv \Psi(r, \theta).$$  \hfill (15)

Therefore we obtain the solution for $a(r)$ by imposing the boundary condition at the surface and integrating equation (14).

In this paper, we treat spherical polytropes with the polytropic indices $N = 0$ and $N = 1$. As for the configurations of the magnetic-fields, we choose two types: (1) closed-field models (e.g., Duez & Mathis 2010) and (2) open-field models (e.g., Broderick & Narayan 2008). For closed field models, since all magnetic-field lines are closed and confined within the star, the magnetic flux must vanish at the stellar surface as follows:

$$a(r_{c}) = 0.$$  \hfill (16)

For open-field models, since the poloidal magnetic-field lines must continue smoothly through the stellar surfaces to the outside, the boundary condition can be expressed as

$$a(r_{c}) = -\frac{da(r)}{dr} \bigg|_{r=r_{c}}.$$  \hfill (17)

The density profiles are

$$\rho(r) = \rho_{c}$$  \hfill (18)

for $N = 0$ polytrope, and

$$\rho(r) = \rho_{c} \frac{\sin(\pi r)}{\pi r}$$  \hfill (19)

for $N = 1$ polytrope, and $\rho_{c}$ is the central density. We can obtain four different analytical solutions according to four different situations. We name them $a_{0c}(r)$ ($N = 0$ with
closed fields), \(a_{00}(r)\) (\(N = 0\) with open fields), \(a_{1C}(r)\) (\(N = 1\) with closed fields) and \(a_{1O}(r)\) (\(N = 1\) with open fields).

Since the poloidal magnetic-field lines are continuous smoothly at the surfaces for open-field models, their external solutions \(a^{\text{ex}}(r)\) must be expressed as

\[
a^{\text{ex}}(r) = \frac{a(r_\ast)}{r}. \tag{20}
\]

It should be noted that the poloidal magnetic fields for closed-field models are discontinuous at the surfaces except for solutions with special values of \(\kappa_0\), i.e., eigen solutions with corresponding eigenvalues (Broderick & Narayan 2008; Duez & Mathis 2010; Fujisawa & Eriguchi 2013). Therefore, these non-eigen configurations have toroidal-surface currents. On the other hand, the toroidal magnetic fields for open-field models are always discontinuous because of the choice of the functional form of \(\kappa\) [equation (11)]. It implies that the open field configurations have non-zero poloidal surface currents.

For non-eigen solutions, the toroidal surface current density can be expressed as follows:

\[
j_{\psi,\text{surf}}(\theta) = \frac{1}{4\pi} \left( B_{0}^{\text{ex}} - B_{0}^{\text{in}} \right) \bigg|_{r=r_s} = \frac{1}{4\pi r_s \sin \theta} \left( \frac{\partial \Psi^{\text{ex}}}{\partial r} - \frac{\partial \Psi^{\text{in}}}{\partial r} \right) \bigg|_{r=r_s}
\]

\[
= \frac{\sin \theta}{4\pi r_s} \left( \frac{da^{\text{ex}}}{dr} - \frac{da^{\text{in}}}{dr} \right) \bigg|_{r=r_s} = j_0 \sin \theta, \tag{21}
\]

where the superscript \(\text{in}\) denotes an internal solution and \(j_0\) is a coefficient of the surface current density.

Analytic solutions are obtained by fixing a boundary condition and integrating equation (14). Four different inner solutions (\(0 \leq r \leq 1\)) can be obtained according to four different situations as follows:

\[
a_{0C}(r) = 4\pi \mu_0 \rho_c \left[ \sin(\kappa_0 r) - \kappa_0 r \cos(\kappa_0 r) \right] \left( \frac{r^2}{\kappa_0^2 \sin(\kappa_0 r) - \kappa_0 r \cos(\kappa_0 r)} \right), \tag{22}
\]

\[
a_{0O}(r) = 4\pi \mu_0 \rho_c \left[ \frac{3[\sin(\kappa_0 r) - \kappa_0 r \cos(\kappa_0 r)]}{\kappa_0^4 \sin(\kappa_0 r)} \right] \left( \frac{r^2}{\kappa_0^2 \sin(\kappa_0 r) - \kappa_0 r \cos(\kappa_0 r)} \right). \tag{23}
\]

\[
a_{1C}(r) = \frac{\mu_0 \rho_c}{r (\kappa_0^2 - \pi^2)^2} \left[ 8\pi \frac{\sin(\kappa_0 r) - \kappa_0 r \cos(\kappa_0 r)}{\sin \kappa_0 - \kappa_0 \cos \kappa_0} \right]
\]

\[
- \left[ (4\kappa_0^2 - 4\pi^2) r^2 + 8 \right] \sin(\pi r) + 8\pi r \cos(\pi r) \bigg]\right\} \tag{24}
\]

The open-field models \(a_{0O}\) and \(a_{1O}\) continue to the external solutions \((r \geq 1)\) expressed by equation (20). Here it would be helpful to explain several different kinds of characteristic solutions.

First, for \(a_{1C}(r)\) solutions there appears a singular solution at \(\kappa_0 = \pi\) (Haskell et al. 2008), while the solution \(a_{0C}\) is not singular at \(\kappa_0 = \pi\) (Fujisawa & Eriguchi 2013).

Secondly, although most solutions are accompanied by surface currents, some special solutions have no surface currents. We call such solutions without surface currents “eigen solutions” and the values of \(\kappa_0\) “eigenvalues.”

Thirdly, there appear many eigen solutions as the value of \(\kappa_0\) exceeds the first eigenvalue. We call those eigen solutions “higher-order eigen solutions” (see figures in Broderick & Narayan 2008; Duez & Mathis 2010; Yoshida et al. 2012). Those solutions appear when the value of \(\kappa_0\) exceeds the first eigenvalue of \(\kappa_0\) for each situation.

Fourthly, special solutions with different polytropic indices come to coincide with each other. In other words, those solutions do not depend on the matter distributions. As seen from the expression for the current density, the contribution from the \(\mu\) current term needs to disappear. It implies that those solutions are determined only by the \(\kappa\) current. Since the \(\kappa\) currents do not contribute to the Lorentz force, these solutions are called the “force-free solutions” (Wentzel 1961). The force-free solution is expressed by the following form:

\[
a_{ii}(r) = \frac{K}{\kappa_0 r} \left[ \sin(\kappa_0 r) - \kappa_0 r \cos(\kappa_0 r) \right]. \tag{26}
\]

\[
a_{1O}(r) = \frac{\mu_0 \rho_c}{r (\kappa_0^2 - \pi^2)^2} \left[ 4\pi - \kappa_0 r \cos(\kappa_0 r) \right]
\]

\[
- \left[ (4\kappa_0^2 - 4\pi^2) r^2 + 8 \right] \sin(\pi r) + 8\pi r \cos(\pi r) \bigg]\right\} \tag{24}
\]
Fig. 1. Distributions of the toroidal current density normalized by the maximum strength of $|\Psi_{\max}|$ are shown along the equatorial plane. Curves with different types denote the behaviors of the total toroidal current density, $j_\phi/c$, (thick solid line), the toroidal $\kappa_0$ current density (thin solid line), and the toroidal $\mu_0$ current density (thin dotted line). We set $\mu_0 = -1$ in order to plot these distributions. Left-hand panels show the profiles of solution $a_{1O}$ with $\kappa_0 = 1.0$ and 4.0 and right-hand panels show those of solution $a_{1C}$ with $\kappa_0 = 2.0$ and 7.0.

As seen in the upper panels in figure 1, the directions (signs) of the $\mu$ current, i.e., non-force-free current, and the $\kappa$ current, i.e., force-free current, are the same for solutions with smaller $\kappa_0$. By contrast, for solutions with larger $\kappa_0$ (lower panels in figure 1), the $\mu$ current flows oppositely to the $\kappa$ current (Fujisawa & Eriguchi 2013). Moreover, most of the $j_\phi/c$ (thick solid line) for solutions with $\kappa_0 = 4.0$ and $\kappa_0 = 7.0$ flows oppositely against the corresponding $\mu$ current. Since the sign of the total toroidal current determines the sign of the magnetic-flux function [see equation (4)], this implies that the sign of $\mu_0\Psi$ for the whole interior region changes at $\mu_0\Psi > 0$ to $\mu_0\Psi < 0$ at the force-free solutions. We calculate many solutions and confirm that the sign of $\mu_0\Psi$ for the whole interior region changes at the force-free solution. We call the current distribution for which $\mu_0\Psi < 0$ “oppositely flowing current.”

On the other hand, the surface toroidal currents in the closed-field models are always flowing oppositely to the total toroidal currents because of the zero-flux boundary-condition equation (16) and the form of the surface current equation (21).

### 3.2 Deep relation between the toroidal current and the poloidal deformations of stars

As the many previous works have pointed out, the toroidal magnetic fields tend to deform stellar shapes to prolate shapes, while the poloidal magnetic fields tend to deform them to oblate (Wentzel 1960, 1961; Ostriker & Gunn 1969; Mestel & Takhar 1972). These studies used only the magnetic fields in their formulations. The ideal magnetohydrodynamic system can be described by using only magnetic fields and one does not need to mention the electrical current density at all. In contrast, we consider both magnetic fields and current density in our calculation. Although these two approaches are equivalent, it is easier to interpret results physically in terms of the current density. This is the reason why we consider both magnetic fields and current density in this paper. As we have seen in subsection 3.1, the oppositely flowing toroidal current density ($\mu_0\Psi < 0$) plays a key role in the appearance of the large toroidal magnetic fields. The direction of the toroidal current seems to
relate to the stellar deformations because the Lorentz force is a cross-product of current density and magnetic field. We consider the relation between the toroidal current and the poloidal deformation of stars in this subsection.

In our analytic models, the Lorentz force \( L \) is expressed using the arbitrary function \( \mu(\Psi) \) as

\[
L = \left( \frac{i}{c} \times B \right) = \rho \nabla \int \mu(\Psi) d\Psi = \rho \mu(\Psi) \nabla \Psi
= \rho \mu_0 \frac{da}{dr} \sin^2 \theta e_r + 2 \rho \mu_0 \frac{r}{a} \sin \theta \cos \theta e_\phi.
\]  

(27)

Following Haskell et al. (2008), we consider the stellar quadrupole deformations of \( N = 1 \) polytropic stars. Haskell et al. (2008) calculated magnetic deformations of polytropic magnetized stars with poloidal and toroidal magnetic fields. Although they derived the general forms of the deformations [equations (64), (65), and (67) in their paper], they did not show their analytical expressions explicitly. They displayed only a few numerical results in table 1 in their paper. By contrast, we show the analytical solutions of the deformation in order to investigate the condition for the appearance of the toroidal magnetic field-dominated star.

We assume that the influence of the magnetic fields on the stellar structures are small and that their effects can be treated perturbatively. Due to the effects of the magnetic fields, a certain physical quantity \( X(r, \theta) \) is assumed to be expressed as

\[
X(r, \theta) = X(r) + \sum_{n=0}^{\infty} \delta X^{(n)}(r) P_n(\cos \theta),
\]  

(28)

where \( \delta X^{(n)} \) denotes a small change of the order of \( O(B^2) \) of the quantity \( X \) due to the Lorentz force. The angular dependencies are treated by the Legendre polynomial expansions and the coefficient of each Legendre polynomial is expressed as \( \delta X^{(n)}(r) \). This expansion is also applied to the Lorentz force as follows:

\[
L(r, \theta) = \sum_{n=0}^{\infty} L^{(n)}(r) P_n(\cos \theta).
\]  

(29)

From the perturbed equilibrium condition equations, the following relations can be derived:

\[
\frac{d\delta\rho^{(n)}}{dr} + \frac{\delta \phi^{(n)}}{dr} + \delta \rho^{(n)} \frac{d \phi^{(n)}}{dr} = L^{(n)}_r,
\]  

(30)

\[
\delta \rho^{(n)} + \rho \delta \phi^{(n)} = r L^{(n)}_{\theta}.
\]  

(31)

Since we are interested in the quadrupole deformation, we consider only \( n = 2 \) components of Lorentz force as follows:

\[
L^{(2)}_r = - \frac{2 \mu_0}{3} \frac{da}{dr},
\]

\[
L^{(2)}_\theta = \frac{2 \mu_0}{3} \frac{a(r)}{r},
\]

\[
_L^{(2)} = \frac{d}{dr} \left( \frac{L^{(2)}_r}{r^2} \right)
= \frac{2 \mu_0}{3} \frac{dp}{dr} a(r).
\]  

(32)

The change of the stellar surface to the order of the quadrupole term can be expressed as

\[
r_d(\theta) = r_s[1 + \varepsilon P_2(\cos \theta)] = r_s \left[ 1 + \frac{\varepsilon}{2} (3 \cos^2 \theta - 1) \right],
\]  

(33)

where \( r_d(\theta) \) denotes the deformed surface radius and \( \varepsilon \) is a small quantity which represents the fraction of the stellar surface along the pole. Following this expression, the stellar shape is prolate for \( \varepsilon > 0 \) and oblate for \( \varepsilon < 0 \).

### 3.2.1 Deformation of an \( N \neq 0 \) polytrope

By using these equations, the quadrupole change of the density is described by:

\[
\delta \rho^{(2)} = \left( \frac{dp}{dr} \delta \phi^{(2)} + L^{(2)}_r \right) \left( \frac{dp}{dr} \right)^{-1}.
\]  

(34)

Since the surface of the deformed star is defined by a set of points where the pressure vanishes, i.e.,

\[
p[r_d(\theta)] = \delta p(r_s) + \varepsilon r_s P_2(\cos \theta) \frac{dp}{dr} \bigg|_{r=r_s} = 0,
\]  

(35)

we can derive

\[
r_s \varepsilon \frac{dp}{dr} \bigg|_{r=r_s} + \delta \rho^{(2)} \bigg|_{r=r_s} = 0,
\]  

(36)

for polytropes with \( N \neq 0 \). For an \( N = 0 \) polytrope, this equation is reduced to the trivial relation \( 0 = 0 \) and so we will treat the \( N = 0 \) polytrope differently, as will be shown in the next sub-subsection.

Therefore, the quadrupole surface deformation \( \varepsilon \) for \( N \neq 0 \) is obtained by

\[
\varepsilon = - \left( \frac{dp}{dr} \right)^{-1} \delta \rho^{(2)} \bigg|_{r=r_s}.
\]  

(37)

It is clearly seen that, since \( (dp/dr) < 0 \) at the surface, the stellar deformation is prolate for \( \delta \rho^{(2)} > 0 \) and oblate
for $\delta \rho^{(2)} < 0$. In our situation, the explicit form of $\delta \rho^{(2)}$ can be expressed as

$$\delta \rho^{(2)} = \frac{d \rho}{dr} \left[ \frac{2 \mu_0}{3} a(r_s) + \delta \phi^{(2)}_\theta(r_s) \right] \left( \frac{d \phi_\theta}{dr} \right)^{-1} \bigg|_{r=r_s}, \quad (38)$$

and $\epsilon$ for $N \neq 0$ polytropes becomes as

$$\epsilon = -\left[ \frac{2 \mu_0}{3} a(r_s) + \delta \phi^{(2)}_\theta(r_s) \right] \left( \frac{d \phi_\theta}{dr} \right)^{-1} \bigg|_{r=r_s} = \rho \left( \frac{d \phi_\theta}{dr} \right)^{-1} \left[ \frac{2 \mu_0}{3} a(r_s) + \delta \phi^{(2)}_\theta(r_s) \right] \bigg|_{r=r_s}. \quad (39)$$

As shown in appendix 1 the gravitational change for an $N = 1$ polytrope can be obtained as

$$\delta \phi^{(2)}_\theta(x) = \frac{F^{(\rho)}(x)}{x^3} - \frac{1}{\pi^2} \frac{d F^{(\rho)}(x)}{dx} \bigg|_{x=\pi} j_2(x). \quad (40)$$

Thus for $x = \pi$, i.e., on the surface,

$$\delta \phi^{(2)}_\theta(\pi) = \frac{F^{(\rho)}(\pi)}{\pi^3} - \frac{3}{\pi^4} \frac{d F^{(\rho)}(\pi)}{dx} \bigg|_{x=\pi}, \quad (41)$$

where $j_2(\pi) = 3/\pi^2$ is used. The function $F^{(\rho)}(x)$ is defined in appendix 1. This is an analytic solution of the deformation.

Since the expression for the function $F^{(\rho)}$ is so complicated, the sign of the quantity [$\delta \phi^{(2)}_\theta(r_s) + 2\mu_0a(r_s)/3$] which determines the sign of the quantity $\epsilon$ is not clearly seen. In figure 2 we show the behavior of $-\delta \phi^{(2)}_\theta(r_s)$ and $-2\mu_0a(r_s)/3$ against the value of $\kappa_0$. As we have seen, the sign of $\mu_0\Psi$ changes at the force-free solution $\kappa_0 \approx 4.49$ for the closed model and $\approx \pi$ for the open model. As shown in this figure, the shape change from the effect due to the gravitational change is the same as that from the Lorentz term. Thus the sign of the quantity $\epsilon$ is essentially determined by the sign of the Lorentz term, i.e., the sign of the quantity $\mu_0a(r_s)$. Since $\rho(r)(dp/dr)_{r=r_s} < 0$, the stellar shape is oblate for $\mu_0\Psi(r_s, \theta) > 0$ for the whole interior region and prolate for $\mu_0\Psi(r_s, \theta) < 0$ for the whole interior region as far as the global poloidal magnetic field is dipole. Therefore, the direction of the deformation by Lorentz force is determined by the direction of the non-force-free current [$\mu$ current in equation (12)]. If the $\mu$ current flows oppositely to the magnetic flux ($\mu_0\Psi < 0$), the stellar shape is prolate. If the $\mu$ current flows in the same direction ($\mu_0\Psi > 0$), the stellar shape becomes an oblate one. This is a deep relation between the direction of the toroidal current and the poloidal deformations of stars.

In figure 3, the contours of $\Psi$ (dashed curves) and the directions of Lorentz force vectors (arrows) are displayed. It should be noted that directions of the Lorentz forces are totally opposite between the models with $\mu_0\Psi(r_s, \theta) > 0$ ($\kappa_0 = 1.0$ and 2.0) and those with $\mu_0\Psi(r_s, \theta) < 0$ ($\kappa_0 = 4.0$ and 7.0).

3.2.2 Deformation of an $N = 0$ polytrope

For an $N = 0$ polytrope, the gravitational change and the shape change are written as follows, as shown in appendix 2:

$$\delta \phi^{(2)}_\theta = -\frac{4}{5} \pi \mu_0 \rho_0 e r_s^2, \quad (42)$$

and

$$\epsilon = \left[ -\frac{4}{3} \pi \mu_0 \rho_0 e r_s^2 - \left( -\frac{4}{5} \pi \mu_0 e r_s^2 \right) \right]^{-1} \frac{2\mu_0}{3} a(r_s) = -\frac{5\mu_0}{4\pi \mu_0 e} a(r_s). \quad (43)$$

![Fig. 2](https://example.com/figure2.png)

**Fig. 2.** The values of $-2\mu_0a(x = \pi)/3$ (thin solid line) and $-\delta \phi^{(2)}_\theta(x = \pi)$ (thin dashed line) in the closed-field model (left-hand panel) and the open-field model (right-hand panel) are plotted. The thick vertical lines denotes the force-free limit. The toroidal current densities consist of oppositely flowing flows beyond the dashed thick vertical lines. We set $\mu_0 = -1$ and $\rho_0 = 1$ in order to plot these graphs.
3.3 Deep relation between the toroidal current and the strong toroidal magnetic fields

We found a relation between the oppositely flowing toroidal current density and the Lorentz force in the previous subsection. We consider a relation between the oppositely flowing toroidal current and the strong toroidal magnetic fields in this subsection.

In figure 4, the ratio of the toroidal magnetic field energy $M_t$ to the total magnetic field energy $M = M_p + M_t$ of each model is plotted for different situations. The solution becomes force-free at the point denoted by the vertical solid lines. The dashed vertical lines denote the critical values beyond which the oppositely flowing $\kappa$ current becomes the dominant component.

As seen in figure 4, $N = 0$ solutions and $N = 1$ solutions cross at $k_0 \sim 4.49$ and 7.73 for closed-field models and $k_0 = \pi$ and 2$\pi$ for open-field models, because the solutions at these points are force-free solutions as mentioned before. The energy ratio is $M_t/M \sim 0.5$ when the solutions are the first force-free configurations. Therefore the solutions are divided into two types at the force-free solution. The solution for which the $k_0$ value is smaller than force-free
\( \kappa_0 \) is the poloidal-dominant configuration, while the solution with larger \( \kappa_0 \) is the toroidal-dominant configuration. Since the sign of \( \mu_0 \Psi \) changes at the force-free solution, the solution is poloidal-dominant for \( \mu_0 \Psi(r, \theta) < 0 \) for the whole interior region (oppositely flowing current) and toroidal-dominant for \( \mu_0 \Psi(r, \theta) > 0 \) for the whole interior region. The oppositely flowing non-force-free current \([\mu_0 \Psi(r, \theta) < 0 \text{ for the whole interior region}]\) is required for large toroidal magnetic fields. This is a relation between the toroidal current density and the toroidal magnetic field.

### 3.4 A situation for the appearance of toroidal magnetic field-dominated configurations

As we have shown in previous parts of this paper, there are two deep relations between toroidal current, poloidal deformation, and strong toroidal magnetic field. One is a relation between the toroidal current and the poloidal deformation of stars, given in subsection 3.2. The other is a relation between the toroidal current and the strong toroidal magnetic fields. The important finding in this paper is that the appearance of oppositely flowing non-force-free current which fulfills the condition \( \mu_0 \Psi < 0 \) changes the stellar shape to prolate shape and makes the toroidal magnetic fields toroidal-dominant. Therefore, a well-known relation between toroidal-dominant magnetic fields and prolate shapes requires the oppositely flowing non-force-free toroidal current density. Although our result is very simple and natural, nobody has explicitly described that the oppositely flowing non-force-free current density makes the stellar shape prolate. It might be because almost all previous studies treated only magnetic fields and did not pay special attention to current density.

Consequently, we can conclude that a condition for the appearance of prolate configurations and the toroidal magnetic field-dominated configurations is that the arbitrary function \( \mu(\Psi) \) satisfies the condition

\[
\int \mu(\Psi)d\Psi < 0
\]  

(45)

for the whole interior region when the functional forms are equations (10) and (11). Although, to be exact, these analyses and conditions are valid within the present parameter settings, our results would be useful for more general situations. This might be naively seen from the contribution of the term \( \int \mu \Psi d\Psi \) in the stationary condition [equation (3)]. If this term is negative, it implies that the action of the Lorentz term is opposite to that of the centrifugal force which is expressed by the term \( \int \Omega(R)^2 R dR \) and is always positive. In other words, the magnetic forces or Lorentz forces act as if they are the “anti”-centrifugal forces and therefore shapes of stationary configurations become prolate (see also calculations in Fujisawa & Eriguchi 2014).

Although the condition presented in this paper might not be always correct, we could obtain the large toroidal magnetic fields by employing this criterion for more complicated calculations.

### 4 Discussion and summary

#### 4.1 Physical reason for the necessity of the appearance of \( \kappa \) currents to realize prolate configurations

In order to get configurations with prolate shapes, we need to include the “anti”-centrifugal effects or “anti”-centrifugal potentials. As is easily understood, the anti-centrifugal potentials should behave as decreasing functions from the symmetric axis, or, at least, they must contain decreasing branches which cover wide enough regions to result in effectively anticentrifugal actions.

For our formulation, the following properties are commonly found:

\[
\mu > 0 \rightarrow i_\psi^\mu > 0 \rightarrow \Psi > 0 \rightarrow \int \mu d\Psi > 0 ,
\]  

(46)

and

\[
\mu < 0 \rightarrow i_\psi^\mu < 0 \rightarrow \Psi < 0 \rightarrow \int \mu d\Psi > 0 .
\]  

(47)

In addition to these behaviors, for \( \Psi > 0 \) configurations the magnetic flux functions increase to the maximum values as the distance from the axis increases, and begin to decrease beyond the maximum point as follows:

\[
\frac{\partial \int \mu d\Psi}{\partial R} > 0 \quad \text{for} \quad R < R_{\text{max}},
\]  

(48)

\[
\frac{\partial \int \mu d\Psi}{\partial R} < 0 \quad \text{for} \quad R > R_{\text{max}},
\]  

(49)

where \( R_{\text{max}} \) is the location of the maximum point of the magnetic flux function \( \Psi \) (left-hand panel in figure 5).

For \( \Psi < 0 \) configurations, the magnetic flux functions decrease to the minimum values as the distance from the axis increases, and begin to increase beyond the minimum point:

\[
\frac{\partial \int \mu d\Psi}{\partial R} > 0 \quad \text{for} \quad R < R_{\text{min}},
\]  

(50)

\[
\frac{\partial \int \mu d\Psi}{\partial R} < 0 \quad \text{for} \quad R > R_{\text{min}},
\]  

(51)
Fig. 5. Distributions of the $\Psi$ (dashed line) and $\int \mu d\Psi/\partial R$ (solid line) of closed-field solutions are plotted. The left-hand panel shows the distributions with $\kappa_0 = 2.0$ and $\mu_0 = 1.0$ and the right-hand panel shows those with $\kappa_0 = 7.0$ and $\mu_0 = -1.0$.

where $R_{\text{min}}$ is the location of the minimum point of the magnetic flux function $\Psi$.

Although there exist decreasing branches for both situations, these decreasing branches cannot overcome the centrifugal effects due to the increasing branches. Therefore, the global configurations with purely $\psi$-currents would become oblate shapes.

From this consideration, the anticentrifugal forces could be realized if the following (necessary) conditions are fulfilled:

$$\mu > 0 \quad \text{and} \quad j_\psi < 0 \quad \text{and} \quad \Psi < 0 \quad \text{and} \quad \int \mu d\Psi < 0,$$

or

$$\mu < 0 \quad \text{and} \quad j_\psi > 0 \quad \text{and} \quad \Psi > 0 \quad \text{and} \quad \int \mu d\Psi < 0.$$  \hspace{1cm} (52)

These conditions could be realized only by including the $\kappa$-currents so that the following conditions are satisfied:

$$\mu > 0, \quad i_\psi < 0, \quad j_\psi > 0, \quad j_\psi = j_\kappa^c + j_\mu^c < 0, \quad \text{and} \quad \Psi < 0,$$

or

$$\mu < 0, \quad j_\psi > 0, \quad j_\psi < 0, \quad j_\psi = j_\kappa^c + j_\mu^c > 0, \quad \text{and} \quad \Psi > 0,$$

$$\frac{\partial \int \mu d\Psi}{\partial R} < 0 \quad \text{for} \quad R < R_{\text{min}},$$

$$\frac{\partial \int \mu d\Psi}{\partial R} > 0 \quad \text{for} \quad R > R_{\text{min}}.$$  \hspace{1cm} (59)

The right-hand panel in figure 5 shows the distributions of $\Psi$ and $\int \mu d\Psi/\partial R$ with $\kappa_0 = 7.0$ and $\mu = -1$. As seen in figures 1 and 5, the conditions mentioned above are satisfied undoubtedly. Therefore, the appearance of $\kappa$ currents $j_\kappa^c$ which are oppositely flowing with respect to the $\mu$ currents $j_\mu^c$ and at the same time have magnitudes large enough to overcome the $\mu$ currents are required to realize prolate shapes.

4.2 Twisted-torus configurations with large toroidal magnetic fields

Almost all investigations previously carried out for magnetized equilibrium states with twisted-torus magnetic fields have failed to obtain toroidal magnetic field-dominated ($M_t > M_p$) models. We have found that most models of these works do not satisfy the condition of equation (45) and the magnetized stellar shapes are oblate due to the $\mu$ current term. The $\kappa$ term in those works has been chosen as follows:

$$\kappa = \kappa_0 (\Psi - \Psi_{\text{max}})^{k_1 + 1} \Theta(\Psi - \Psi_{\text{max}}),$$

where $k_1$ is a constant $\Theta$ is the Heaviside step function and $\Psi_{\text{max}}$ is the maximum value of $\Psi$ on the last closed-field line within the star. Since the current density of this functional form vanishes at the stellar surface, there is no surface current and no exterior current density. This functional form was used by Tomimura and Eriguchi (2005) for the first time and results in the twisted-torus configurations. The same choice for $\kappa$ has been employed by many authors (e.g., Yoshida & Eriguchi 2006; Yoshida et al. 2006; Kiuchi & Kotake 2008; Lander & Jones 2009; Ciolfi et al. 2009, 2011; Fujisawa et al. 2012, 2013; Glampedakis et al. 2012; Lander et al. 2012;
Fujisawa & Eriguchi (2013; Lander 2013, 2014). While the functional form $\mu(\Psi) = \mu_0$ (constant) has been used in many investigations, Fujisawa, Yoshida, and Eriguchi (2012) and Fujisawa et al. (2013) used a different functional form:

$$\mu(\Psi) = \mu_0 (\Psi + \epsilon)^m.$$  \hspace{1cm} (61)

where $m$ and $\epsilon$ are positive constants. They have obtained highly localized poloidal magnetic-field configurations using this type of functional form. However, their works did not satisfy the condition of equation (45) and did not obtain models with large toroidal magnetic fields.

Recently, Ciolfi and Rezzolla (2013) adopted a perturbative approach and succeeded in obtaining magnetized equilibrium states with twisted-torus magnetic fields that had large toroidal fields. Their functional form of $\kappa$ is

$$\kappa(\Psi) = \kappa_0 \Psi(\Psi/\Psi_{\max} - 1) \Theta(\Psi/\Psi_{\max} - 1).$$  \hspace{1cm} (62)

On the other hand, their functional form of $\mu$ is

$$\mu(\Psi) = c_0 \left[ (1 - |\Psi/\Psi_{\max}|)^2 \Theta(1 - |\Psi/\Psi_{\max}|) - k \right] + X_0k(\Psi) \frac{dk(\Psi)}{d\Psi},$$  \hspace{1cm} (63)

where $c_0$, $\bar{k}$($>0$) and $X_0$ are constants. The toroidal magnetic field is confined within the last closed-field line in these functional forms. Outside the toroidal magnetic-field region, the function $\kappa$ vanishes and $\mu$ becomes

$$\mu(\Psi) = c_0 \left[ (1 - |\Psi/\Psi_{\max}|)^2 - \bar{k} \right].$$  \hspace{1cm} (64)

Since the first term and the second term are positive and negative, respectively, this function with larger $\bar{k}$ tends to satisfy the condition of equation (45). As Ciolfi and Rezzolla (2013) noted, larger values of $\bar{k}$ result in larger energy ratios $\mathcal{M}_t/\mathcal{M}$. As the value of $k$ increases, the energy ratio $\mathcal{M}_t/\mathcal{M}$ increases and the stellar shape becomes more prolate in general (see table 1 in Ciolfi & Rezzolla 2013). However, they assumed that the magnetic field configuration is purely dipole but their functional forms and toroidal current density distribution are far from dipole (see the bottom panels of figure 2 in Ciolfi & Rezzolla 2013). Nonperturbative studies with higher-order components were unable to reproduce their results and found contradictory results (Bucciantini et al. 2015). We need to calculate magnetic-field configurations with higher-order components for large toroidal models by using nonperturbative methods in the future.

The condition of equation (45) itself is valid when a star is barotropic. However, the relation between oppositely flowing toroidal current density and prolate shape is very simple and natural when a star is nonbarotropic. Therefore, this condition is also useful for recent perturbative nonbarotropic solutions (Mastrano et al. 2011; Mastrano & Melatos 2012; Akgün et al. 2013; Yoshida 2013). We also need to investigate nonperturbative, nonbarotropic magnetized equilibrium states in the future.

4.3 Summary

In this paper we have obtained four analytic solutions with both open and closed magnetic fields for spherical polytropes with weak magnetic fields.

Using the obtained solutions, we have discussed the situations for which the prolate equilibrium states and the toroidal magnetic field-dominated configurations appear. The main finding in this paper is that the appearance of the prolate shapes and the toroidal magnetic field-dominated states are accompanied by the appearance of oppositely flowing $\kappa$ currents with respect to the $\mu$ current. This situation seems to be related to the condition for the nonforce-free toroidal current contribution, i.e., $\int \mu(\Psi)d\Psi$, in the stationary state condition equation (3).

Although the appearance of prolate shapes and the occurrence of toroidal magnetic field-dominated states cannot be defined quantitatively, the rough qualitative idea about them can be determined by checking the sign of the magnetic field potential, i.e., the quantity $\int \mu(\Psi)d\Psi$.

Of course, the analytic solutions obtained in this paper have been derived under very restricted assumptions. However, as explained in the Discussion, the concept of the “anti”-centrifugal actions due to the magnetic potentials would be applied to more general situations for the magnetic fields.

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Appendix 1. Change of the gravitational potential for an $N = 1$ polytrope

The gravitational potential perturbation for an $N = 1$ polytrope is governed by the quadrupole component of Poisson’s equation under two boundary conditions ($\delta \phi_k^{(2)}$) is regular at $r = 0$ and continues the external solution smoothly at $r = r_0$:

$$\frac{d^2 \delta \phi_k^{(2)}}{dr^2} + \frac{2}{r} \frac{d \delta \phi_k^{(2)}}{dr} - \frac{6}{r^3} \delta \phi_k^{(2)} = 4\pi G\delta \rho^{(2)}. \hspace{1cm} (A1)$$
Considering the density perturbation expressed by equation (34), this equation can be rewritten as

\[
\frac{d^2 \delta \phi_s^{(2)}}{dr^2} + \frac{2}{r} \frac{d \delta \phi_s^{(2)}}{dr} + \left( \pi^2 - \frac{6}{r^2} \right) \delta \phi_s^{(2)} = 4\pi G \left( \frac{d \phi_s}{dr} \right)^{-1} L^{(2)}(r). \tag{A2}
\]

By introducing the new variable \( x = \pi r \), the left-hand side of the equation is reduced to

\[
\frac{d^2 \delta \phi_s^{(2)}}{dx^2} + \frac{2}{x} \frac{d \delta \phi_s^{(2)}}{dx} + \left( 1 - \frac{6}{x^2} \right) \delta \phi_s^{(2)} = 4\pi G \left( \frac{d \phi_s}{dx} \right)^{-1} L^{(2)}(r). \tag{A3}
\]

The solution to this equation can be obtained by taking the boundary conditions into account as follows:

\[
\delta \phi_s^{(2)}(x) = \frac{F^{(p)}(x)}{x^3} - \frac{1}{\pi^2} \frac{dF^{(p)}(\pi)}{dx} \bigg|_{x=\pi} j_1(x), \tag{A4}
\]

where

\[
F^{(p)}(x) = \frac{2}{3} \mu_0 A_1 \frac{\pi^2}{\kappa_0^2 (\pi^2 - \kappa_0^2)^2} \left\{ 6\pi^2 \kappa_0^2 + \left( \pi^2 - 3\kappa_0^2 \right) \kappa_0^2 x^2 \right\} \sin \left( \frac{\kappa_0}{\pi} x \right) \]
\[ - \frac{\kappa_0}{\pi} x \left[ 6\pi^2 \kappa_0^2 + \left( \pi^2 - 3\kappa_0^2 \right) \kappa_0^2 x^2 \right] \cos \left( \frac{\kappa_0}{\pi} x \right) \]
\[ - \frac{2}{3} \mu_0 A_2 \left[ \frac{1}{2} x^4 \sin x + \frac{1}{6} \left( \frac{\kappa_0^2}{\pi^2} - 1 \right) x^5 \cos x \right]. \tag{A5}
\]

Here the coefficients \( A_1 \) and \( A_2 \) are defined as

\[
A_1 = \frac{8\pi^2 \mu_0 \rho_c}{(\kappa_0^2 - \pi^2)^2} \frac{1}{(\sin \kappa_0 - \kappa_0 \cos \kappa_0)}, \tag{A6}
\]

\[
A_2 = \frac{4\pi \mu_0 \rho_c}{(\kappa_0^2 - \pi^2)^2}. \tag{A7}
\]

for \( N = 1 \) closed configurations and

\[
A_1 = - \frac{4\pi^2 \mu_0 \rho_c}{\kappa_0^2 (\kappa_0^2 - \pi^2)} \frac{1}{\sin \kappa_0}, \tag{A8}
\]

\[
A_2 = \frac{4\pi \mu_0 \rho_c}{(\kappa_0^2 - \pi^2)^2}. \tag{A9}
\]

for \( N = 1 \) open configurations.

### Appendix 2. Surface change for an \( N = 0 \) polytrope

The change of the gravitational potential due to the change of the surface, i.e., \( \sigma_s P_2(\cos \theta) \), can be obtained by

\[
\delta \phi_s^{(2)}(r) = -4\pi G \rho_0 \int_0^\pi d\theta' P_2(\cos \theta') \int_{r_1}^{r_{(1+\epsilon)}P_s} dr' r'^2 \frac{L^{(2)}(r')}{r'^3} = -\frac{4\pi G \rho_0}{5} r^2 \epsilon. \tag{A10}
\]

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