Effective field theories on solitons of generic shapes

Sven Bjarke Gudnason\textsuperscript{1,2}, Muneto Nitta\textsuperscript{3}

\textsuperscript{1}Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China
\textsuperscript{2}Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden
\textsuperscript{3}Department of Physics, and Research and Education Center for Natural Sciences, Keio University, Hiyoshi 4-1-1, Yokohama, Kanagawa 223-8521, Japan

Abstract

A class of effective field theories for moduli or collective coordinates on solitons of generic shapes is constructed. As an illustration, we consider effective field theories living on solitons in the O(4) non-linear sigma model with higher-derivative terms.

Keywords: Effective field theory, Solitons

1. Introduction

Effective field theory is one of the most useful tools available to date. Even the standard model, although renormalizable in its present formulation, may also be just an effective theory of Nature where possible supersymmetric and/or grand unified extensions have been integrated out. For particles of accessible energies, we can neglect gravity and consider particles on flat space as a(n extremely) good approximation. This is just a consequence of the separation of scales between the particle mass and energy versus the scale of gravity, i.e. the Planck mass. Light fields do not only exist in all of spacetime but are sometimes confined to certain subspaces. For solitons hosting moduli, there is again a situation where separation of scales can be exploited; namely the mass of the soliton versus massless or light moduli. Effective field theories for moduli have been constructed for many kinds of solitons, but very often only in cases where the soliton has a simple, flat or straight shape. As examples, the effective actions for monopole moduli [1], domain-wall moduli [2, 3, 4, 5] and for orientational moduli of non-Abelian strings [6, 4, 5, 7] have been constructed. When solitons are particle-like such as monopoles this can describe the low-energy dynamics of the solitons in a compact way as geodesics of moduli spaces [1], while for solitons being extended objects such as domain walls or vortices, this describes field theories on their world-volume, as in the case of D-branes in string theory or more general branes. Solitons can, however, generically possess much more complicated shapes.

In this Letter we construct a first attempt of effective field theories in principle applicable to solitons of generic shapes and apply it to a class of models possessing soliton solutions of flat, spherical, cylindrical and toroidal shapes.

2. General considerations

Here we will consider a generalized framework where we expand a set of fields in eigenmodes as [2]

\[ \Phi^a = \sum_n \mathcal{M}_n(e_n) \zeta^a_n(e_n), \]

where \( \zeta^a_n \) are eigenfunctions, \( \mathcal{M}_n \) are moduli fields, while \( e_n \) and \( e_i \) are sets of vectors in transverse (world-volume) dimensions (\( \alpha = 0, 1, \ldots, t + c \)) and codimensions (\( i = t + 1, \ldots, t + c \)), respectively, of a soliton of a generic shape; see Fig. 1. For simplicity we consider only flat space in this Letter and we have made a decomposition of directions (locally) as \( \mathbb{R}^{d,1} = \mathbb{R}^{t,1} \times \mathbb{R}^c \), where the \( d = c + t \) spatial dimensions are split into \( c \) codimensions and \( t \) transverse dimensions.

The kinetic term in the underlying theory will give rise to a kinetic term for the moduli as

\[ \int_{e_i} |\nabla_{\mu} \Phi|^2 \geq |\nabla_{e_i} \mathcal{M}_n|^2 \int_{e_i} |\zeta_n|^2 \sim \frac{1}{M^c} |\nabla_{e_i} \mathcal{M}_n|^2, \]

where \( M \) is a characteristic mass of the soliton system and \( \mu \) are all spacetime indices. For higher-order derivative terms, one similarly obtains e.g.,

\[ \int_{e_i} |\nabla_{\mu} \Phi|^2 |\nabla_{\nu} \Phi|^2 \geq |\nabla_{e_i} \mathcal{M}_n|^2 \int_{e_i} |\nabla_{e_i} \zeta_n|^2 |\zeta_n|^2 \sim \frac{1}{M^{c-2}} |\nabla_{e_i} \mathcal{M}_n|^2. \]
We will now consider the soliton of the type which is described by a codimension-one field $\zeta(e_3)$ and two moduli $M_{1,2}(e_1,e_2)$, where the condensate field is a function of the direction spanned by the vector $e_3$. We will work in the limit where the moduli are functions of two orthogonal directions $e_1$ and $e_2$. We will consider the soliton of the type which is described by a codimension-one field $\zeta(e_3)$ and two moduli $M_{1,2}(e_1,e_2)$, where the condensate field is a function of the direction spanned by the vector $e_3$. We will work in the limit where the moduli are functions of two orthogonal directions $e_1$ and $e_2$.
\( \mathbf{e}_3 \). For concreteness we will parametrize the non-linear sigma-model field, \( \mathbf{n} \), as
\[
\mathbf{n} = (b \sin f, \cos f),
\]
where \( b \) are scalar fields of a unit 3-vector \((b \cdot b = 1)\) describing two moduli and is a function only of the orthogonal directions to the field \( f \), i.e. \( b(e_1, e_2) \). The domain solution also possesses a position modulus, which we will not take into account in this Letter. Taking the Lagrangian solution also possesses a position modulus, which we will not take into account in this Letter. Taking the Lagrangian
\[
l^2V = -\frac{1}{2}m_3^2n_3^2 + \frac{1}{2}m^2(1 - n_3^2),
\]
and integrating over the codimension spanned by \( e_3 \), we get
\[
-\mathcal{L}^2_{2} = \frac{a_{2,0}a'_{\alpha,\beta}}{M} \partial_\alpha b \cdot \partial_\beta b,
\]
\[
-\mathcal{L}^4_{4} = \frac{a_{2,2}a'_{\alpha,\beta'}}{M} \partial_\alpha b \cdot \partial_\beta b + \frac{a_{4,0}a_{4,0}^*}{2M} \partial_\alpha b \times \partial_\beta b \partial_\gamma b \partial_\gamma b,
\]
\[
-\mathcal{L}^6_{6} = \frac{a_{4,2}a_{4,2}^*}{M} \partial_\alpha b \times \partial_\beta b \partial_\gamma b \partial_\gamma b,
\]
where we have defined the dimensionless constants as follows
\[
a_{k,\ell}^{\alpha_1...\alpha_n} = \frac{M}{2} \int_e \sqrt{\det h} \sin f \left( \frac{\partial_\ell f}{M} \right) \kappa_{\alpha_1...\alpha_n},
\]
where \( h_{\alpha \beta} \) is the inverse induced metric on the surface of the host soliton. Note that to leading order which we consider here, the induced metric is diagonal.

The least surprising result is the kinetic term giving back the kinetic term for the moduli with the normalization constant of the effective Lagrangian being \( 1/M \). The Skyrme term gives also back a Skyrme term for the moduli, but in addition it induces again a kinetic term for the moduli, however enhanced by a factor of \( M^2 \). Finally and perhaps most interestingly, the sixth-order derivative term induces only the (baby-)Skyrme term for the moduli and nothing else.

Let us comment on leaving out the position modulus from the low-energy effective action. The position moduli are first of all not as interesting as the orientational moduli, in the physical context we have in mind here. Second, once the host soliton is curved, the position moduli acquire a mass of the order of the curvature scale and are thus subleading with respect to the orientational part. Let us however remark that the higher-order corrections coming from the kinetic term generically induce a mixing term between the position and orientation moduli at the fourth order in derivatives. These higher-order corrections can, however, be systematically calculated using the approach of Ref. [7].

The domain wall can host a so-called baby-Skyrmion [8], which can be identified with a Skyrmion in the bulk [9] (lower dimensional analogues of this correspondence can be found in Ref. [12]).

Using a scaling argument [13], we can estimate the size of the baby-Skyrmion on the flat domain wall as \( 1/L \sim \sqrt{m_3/M} \), and as \( M \gg m_3 \) it is always relatively large in units of \( 1/m \) (this estimate holds also for vanishing \( c_4 \)). If \( c_6 \) is very small or vanishes, the size estimate of the baby-Skyrmion on the flat domain wall in the large-\( M \) limit becomes \( 1/L \sim \sqrt{m_3/M} \), and so is independent of the thickness of the host domain wall.

3.1.1. Example 1: flat domain wall
The sine-Gordon kink is an exact solution to the equations of motion derived from the Lagrangian density [5] with the field parametrization [9]
\[
f = 2 \arctan (\pm M x / \sqrt{c_2}).
\]
This solution is exact when the moduli, \( b \) are (any) constants. Due to separation of scales, we can still use this soliton shape as a good approximation even when the moduli do possess dynamics. The full solutions have been obtained in Ref. [11].
A vast simplification in this case is that the induced metric is just the flat metric, so the effective Lagrangian coefficients, $a$, do not depend on the Greek indices. We can thus evaluate the coefficients in the effective Lagrangian

$$a_{k,\ell} = \frac{1}{2} c_2 (1 - \ell)^{1/2} \int_{-\Delta M/\sqrt{2}}^{\Delta M/\sqrt{2}} dy \sech^{k+\ell} y. \quad (15)$$

If we define, $a_{k+\ell} \equiv a_{k,\ell} c_2^{(\ell-1)/2}$, we have

$$a_{k+\ell} = \frac{1}{2} \sinh y \ 2F_1 \left( \frac{1}{2}, \frac{1 + k + \ell}{2}, \frac{3}{2}, -\sinh^2 y \right) \bigg|_{\Delta M/\sqrt{2}},$$

where $2F_1$ is a hypergeometric function. If we take the limit of $\Delta M \to \infty$, the constants are $a_2 = 1$, $a_4 = 2/3$, and $a_6 = 8/15$. $a_{2n} = \sqrt{n} \Gamma(n)/\Gamma(n+1/2)$, with $n \in \mathbb{Z}_{>0}$ a positive integer, takes value in $[0,1]$ and is monotonically decreasing with $n$. For the flat domain wall case, we can finally write down the effective Lagrangian density

$$-\mathcal{L}^{\text{eff}} = \left( \frac{c_2 a_2 m}{M} + \frac{c_2 a_4 M}{\sqrt{c_2} m^2} \right) m (\partial_\alpha \mathbf{b})^2$$

$$+ \left( \sqrt{c_2} c_4 a_4 m + \frac{c_6 a_6 M}{\sqrt{c_2} m^2} \right) \frac{1}{m} (\partial_\alpha \mathbf{b} \times \partial_\beta \mathbf{b})^2$$

$$- \sqrt{c_2} a_2 m^2 \frac{m^2}{M} b_3^2, \quad (16)$$

where the world-volume directions are $\{e_\alpha\} = \{y, z\}$.

### 3.1.2. Example 2: spherical domain wall

This is the first non-flat example. Let us begin with a word of caution. Our construction focuses on the effective description of the moduli on a soliton of a given shape. It does not guarantee stationarity or even existence of the host soliton. These questions are a separate issue and should be addressed carefully and independently. A further issue is that not all topological sectors of the moduli are available in the effective theory. In this case of a spherical domain wall; the moduli have to live in the topological charge-one sector. Full solutions have been constructed in the literature [14].

Here we will simply assume the form of the spherical domain wall with size $R$ and construct the effective theory that would live on such an object. $R$ will however be a function of the parameters in the theory as it is actually determined dynamically. The induced inverse metric is $\{h^{rr}, h^{\theta \theta}, h^{\phi \phi}\} = \{1, R^2/r^2, R^2/r^2 \sin^2 \theta\}$, and $\sqrt{\det h} = (r/R)^2 \sin \theta$, where we have rescaled the induced metric so that it is dimensionless. The constants of the effective Lagrangian can be determined from [12]. Finally, we can write the effective Lagrangian in this case

$$-\mathcal{L}^{\text{eff}} = m \sin \theta \left[ c_{2m} R a_{2,0,0} + \frac{c_4}{m R} a_{2,2,2} \right]$$

$$\times \left( (\partial_\alpha \mathbf{b})^2 + \frac{1}{\sin^2 \theta} (\partial_\alpha \mathbf{b})^2 \right)$$

$$+ \frac{m}{\sin \theta} \left[ \frac{c_4}{m R} a_{4,0,-2} + \frac{2 e_6}{(m R)^2} a_{4,2,-2} \right]$$

$$\times (\partial_\alpha \mathbf{b} \times \partial_\beta \mathbf{b})^2 - (m R)^2 R m_2^2 \sin \theta a_{2,0,2} b_5^2, \quad (17)$$

where the integration measure now is simply $\mathcal{dbd}\phi$ and the following dimensionless constants have been defined

$$a_{k,\ell,n} \equiv \frac{1}{2} \int dy \ y^n \sin^k f(y)(\partial_\alpha f)^j, \quad y \equiv \frac{r}{R}. \quad (18)$$

In this example we will not contemplate taking any limits, as the system is somewhat complicated. The size of the sphere, $R$, is dynamically determined and is a function of the other parameters in the theory, e.g. $c_2, c_4, e_6, m, M$ and so on.

#### 3.2. Codimension-two case

The final type of soliton we will consider is of codimension two and is described by two fields, $f(e_2, e_3), g(e_2, e_3)$ and a single modulus $M(e_1)$, where the modulus lives in a single dimension only (plus time). We will parametrize the non-linear sigma-model field, $n$, as

$$n = \{\sin f \cos g, \sin f \sin g, \mathbf{b} \cos f\}, \quad (19)$$

where $\mathbf{b}$ are scalar fields of a unit 2-vector ($\mathbf{b} \cdot \mathbf{b} = 1$) describing an $S^1$ modulus and is a function orthogonal to both the fields of the host soliton $f, g$, i.e. $\mathbf{b}(e_1)$. We take again the Lagrangian densities [68], choose the potential

$$m^2 V = \frac{1}{2} m_2^2 b_5^2 + m^2 V_{\text{vortex}}, \quad (20)$$

where $p_3 = 1, 2$ and integrate over the codimensions $e_2$ and $e_3$ to obtain

$$-\mathcal{L}^{\text{eff}}_2 = \frac{a_{2,0,0}^\alpha}{M^2} \partial_\alpha \mathbf{b} \cdot \partial_\alpha \mathbf{b}, \quad (21)$$

$$-\mathcal{L}^{\text{eff}}_4 = \frac{a_{2,1,0}^\alpha}{M^3} \partial_\alpha \mathbf{b} \cdot \partial_\alpha \mathbf{b}, \quad (22)$$

$$-\mathcal{L}^{\text{eff}}_6 = 2a_{2,0,2}^\alpha M^2 \partial_\alpha \mathbf{b} \cdot \partial_\alpha \mathbf{b}, \quad (23)$$

$$m^2 V^{\text{eff}} = -\frac{a_{2,0,0}^\alpha M^2 b_5^2}{M^2}, \quad (24)$$

which are all Lagrangians for a free (massive) theory for the modulus. The non-trivial part however is the content of the coefficients

$$a_{p,k,l}^{\alpha_1 \alpha_2 \ldots \alpha_n \alpha'_{n'}} \equiv \frac{M^{2-2k-2l}}{2}$$

$$\times \int_{e_2, e_3} \sqrt{\det h} \ \cos^p f \left[ (\nabla e_1, f)^2 + \sin^2 f (\nabla e_1, g)^2 \right]^k$$

$$\times \left[ \sin f e^e_1 \nabla e_1, f \nabla e_1, g \right]^{\ell} h^{\alpha_1 \alpha_2} \ldots h^{\alpha_n \alpha'_{n'}}. \quad (25)$$

4
Let us first put together the pieces of the effective Lagrangian density

\[-\mathcal{L}^{\text{eff}} = \left( \frac{c_{4}a_{2,0,0} \rho^{2}}{M^{2}} + c_{4}a_{2,1,0} \rho \right) \partial_{\alpha} b \cdot \partial_{\alpha} b - \frac{a_{p_{3},0} \rho m^{2} m^{2}}{M^{2}} \partial_{p_{3}}. \tag{26} \]

The first term \([21]\) gives just a kinetic term for the modulus with a standard prefactor, while the second term \([22]\) gives the kinetic term but with a relative enhancement by a factor proportional to the kinetic energy of the host soliton. Finally, and perhaps most interestingly, the last term \([23]\) gives again a kinetic term for the modulus, but with a prefactor proportional to the baby-Skyrmion charge of the orthogonal \(S^{2}\) to the \(S^{1}\) where the modulus lives. For some host solitons, this charge may vanish of course.

Let us again comment on the higher-order corrections from the lower-order terms due to integration out of massive modes. The first correction will be a fourth-order term from the lower-order terms due to integration out of mas-

sive modes. The first correction will be a fourth-order term

4. Discussion

We have developed a basic framework for calculating effective actions on, in principle, generic solitons. The advantage of this approach is the simplified theories for the types of objects a host soliton can host. The disadvantage is that stability and existence should be carefully examined separately. In this work we did not consider the translational type of moduli, which is related with the shape itself of the soliton as this is a somewhat more complicated problem; we thus leave this part for future developments. Needless to say, one can calculate higher-order classical corrections to these effective actions and finally one may consider also quantum corrections etc. A different generalization that would be interesting to consider for GUT-scale solitons is to take the curvature of spacetime into account; this can be seen as the next-level geometric backreaction.

Acknowledgments

The work of MN is supported in part by Grant-in-Aid for Scientific Research (No. 25400268) and by the “Topological Quantum Phenomena” Grant-in-Aid for Scientific Research on Innovative Areas (No. 25103720) from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. SBG thanks Keio University for hospitality where this work was carried out. SBG also thanks the Recruitment Program of High-end Foreign Experts for support.

References

[1] N. S. Manton, Phys. Lett. B 110, 54 (1982); M. F. Atiyah and N. J. Hitchin, Phys. Lett. A 107, 21 (1985); G. W. Gibbons and N. S. Manton, Phys. Lett. B 356, 32 (1995) [hep-th/9506052].

[2] B. Chibisov and M. A. Shifman, Phys. Rev. D 56, 7990 (1997) [Erratum-ibid. D 58, 109901 (1998)] [hep-th/9706141].

[3] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Lett. B 93, 161601 (2004) [hep-th/0404198]; Phys. Rev. D 70, 125014 (2004) [hep-th/0404194].

[4] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A 39, R315 (2006) [hep-th/0602170].

[5] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D 73, 125008 (2006) [hep-th/0602289].

[6] A. Henana and D. Tong, JHEP 0307, 037 (2003) [hep-th/0306150]; R. Anzei, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B 673, 187 (2003) [hep-th/0307287]; M. Shifman and A. Yung, Phys. Rev. D 70, 045004 (2004) [hep-th/0403019]; A. Hanany and D. Tong, JHEP 0404, 066 (2004) [hep-th/0403158]; Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D 71, 065018 (2005) [hep-th/0504152]; A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D 71, 045010 (2005) [hep-th/0412082]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. 96, 161601 (2006) [hep-th/0511088]; M. Eto, K. Konishi, G. Marmorkin, M. Nitta, K. Ohashi, W. Vinci and N. Yokoi, Phys. Rev. D 74, 065021 (2006) [hep-th/0607079]; S. B. Gudnason, Y. Jiang and K. Konishi, JHEP 1008, 012 (2010) [arXiv:1007.5116 [hep-th]].

[7] T. Fujimori, G. Marmorkin, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D 82, 065005 (2010) [arXiv:1002.4580 [hep-th]]; M. Eto, T. Fujimori, M. Nitta, K. Ohashi and N. Sakai, Prog. Theor. Phys. 128, 67 (2012) [arXiv:1206.0773 [hep-th]].

[8] B. M. A. Piette, B. J. Schroers and W. J. Zakrzewski, Z. Phys. C 65, 165 (1995) [arXiv:hep-th/9406160]; Nucl. Phys. B 439, 205 (1995) [arXiv:hep-ph/9410256].
[9] T. H. R. Skyrme, Proc. Roy. Soc. Lond. A 260, 127 (1961); Nucl. Phys. 31, 556 (1962).

[10] M. Nitta, Phys. Rev. D 87, 025013 (2013) [arXiv:1210.2233 [hep-th]]; Nucl. Phys. B 872, 62 (2013) [arXiv:1211.4916 [hep-th]].

[11] S. B. Gudnason and M. Nitta, Phys. Rev. D 89, 085022 (2014) [arXiv:1403.1245 [hep-th]].

[12] J. Garaud and E. Babaev, Phys. Rev. B 86, 060514 (2012) [arXiv:1201.2946 [cond-mat.supr-con]]; M. Nitta, Phys. Rev. D 86, 125004 (2012) [arXiv:1207.6958 [hep-th]]; M. Kobayashi and M. Nitta, Phys. Rev. D 87, 085003 (2013) [arXiv:1302.0989 [hep-th]]; P. Jennings and P. Sutcliffe, J. Phys. A 46, 465401 (2013) [arXiv:1305.2869 [hep-th]].

[13] G. H. Derrick, J. Math. Phys. 5, 1252 (1964).

[14] S. B. Gudnason and M. Nitta, Phys. Rev. D 89, 025012 (2014) [arXiv:1311.4454 [hep-th]].

[15] R. L. Davis and E. P. S. Shellard, Phys. Lett. B 209, 485 (1988); Nucl. Phys. B 323, 209 (1989); E. Radu and M. S. Volkov, Phys. Rept. 468, 101 (2008) [arXiv:0804.1357 [hep-th]].

[16] J. Garaud, E. Radu and M. S. Volkov, Phys. Rev. Lett. 111, 171602 (2013) [arXiv:1303.3041 [hep-th]].