FREE LOOP SPACE HOMOLOGY OF $(n - 1)$-CONNECTED $2n$-MANIFOLDS

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Abstract. We compute the free loop space homology of $(n - 1)$-connected $2n$-manifolds $M$ for integer and field coefficients under some mild restrictions. In addition, we give partial information about the action of the BV-operator for rational coefficients.

1. Introduction

Let $\mathcal{L}X = \text{map}(S^1, X)$ denote the free loop space on $X$. This space comes equipped with an action $\nu : S^1 \times \mathcal{L}X \rightarrow \mathcal{L}X$ that rotates loops, and an induced degree 1 homomorphism

$$\Delta : H_*(\mathcal{L}X) \rightarrow H_{*+1}(\mathcal{L}X)$$

known as the BV-operator, defined by setting $\Delta(a) = \nu_*([S^1] \otimes a)$. In addition to this, Chas and Sullivan [4] constructed a pairing

$$H_p(\mathcal{L}X) \otimes H_q(\mathcal{L}X) \rightarrow H_{p+q-d}(\mathcal{L}X)$$

on a closed oriented $d$-manifold $X$ that (together with the BV-operator) turns the shifted homology $H_*(\mathcal{L}X) = H_{*+d}(\mathcal{L}X)$ into a Batalin-Vilkovisky (BV)-algebra.

These Batalin-Vilkovisky algebras have been computed in only a few special cases. One of the more general results to date (due to Felix and Thomas [7]) states that over a field $F$ of characteristic zero and 1-connected $X$, $H_*(\mathcal{L}X; F)$ is isomorphic to a BV-algebra structure defined on the Hochschild cohomology $HH^*(C^*(X), C^*(X))$. Unfortunately, this theorem is generally not true for fields with nonzero characteristic [13]. Beyond these results, the BV-algebra over various coefficient rings has been completely determined for spheres [5, 18, 13], certain Stiefel manifolds [17], Lie groups [10], and projective spaces [21, 15, 5, 20, 9], using a mixture of techniques ranging from homotopy theoretic to geometric, or using the well-known relations to Hochschild cohomology.

In this paper we focus on the free loop space homology of highly connected $2n$-manifolds, together with the action of the BV-operator. The coefficient ring $R$ for homology and cohomology is assumed to be either any field, or the integers $\mathbb{Z}$, but we suppress it from notation most of the time. Fix $n \geq 2$, $M$ a $(n - 1)$-connected, closed, oriented $2n$-manifold with $H^n(M)$ of rank $m \geq 1$. Let

$$C = [c_{ij} = \langle a_i \cup a_j, [M] \rangle]$$

be the $m \times m$ matrix for the intersection form $H^n(M) \times H^n(M) \rightarrow \mathbb{Z}$ with respect to some choice of basis $\{a_1, \ldots, a_m\}$ for $H^n(M)$ (we use the same notation for the dual basis of $H^n(M)$). This form is nonsingular, symmetric when $n$ is even, and skew-symmetric when $n$ is odd.

We will describe some algebraic constructs before stating the main theorem. Denote $H^n(M)$ and $H^{2n}(M) \cong \mathbb{Z}$ by the free graded modules $R$-modules $A = R\{a_1, \ldots, a_m\}$ and $K = R\{[M]\}$, and the desuspension of $A$ by $V = R\{u_1, \ldots, u_m\}$ with $|u_i| = n - 1$. Let

$$T(V) = R \oplus \bigoplus_{i \geq 1} V^\otimes i$$

Key words and phrases. string topology, free loop space, four manifolds.
be the free tensor algebra generated by $V$, and $I$ be the two-sided ideal of the tensor algebra $T(V)$ generated by the following degree 2 element

$$
\chi = \sum_{i<j} c_{ij} [u_i, u_j] + \sum_i c_i u_i^2,
$$

where $[x, y] = xy - (-1)^{|x||y|}yx$ denotes the graded Lie bracket in $T(V)$. Take the quotient algebra

$$
U = \frac{T(V)}{I}
$$

and the degree $-1$ maps of graded $R$-modules $d: A \otimes U \to U$ and $d': K \otimes U \to A \otimes U$, which are given for any $y \in U$ by the formulas

$$
d(a_i \otimes y) = [u_i, y] \\
d'(\{M\} \otimes y) = \sum_{i,j} c_{ij} (a_j \otimes [u_i, y]).
$$

If we apply the Jacobi identity to the summands $c_{ij} (a_j \otimes [u_i, y])$ in $d \circ d'(y)$ for $i < j$ (keeping in mind that $c_{ij} = (-1)^{n} c_{ji}$, $[u_i, [u_i, y]] = [u_i^2, y]$, and that products with $\chi$ are identified with zero in $U$), we see that $\text{Im} \ d' \subseteq \ker d$, so we obtain a chain complex

$$
0 \to K \otimes U \xrightarrow{d'} A \otimes U \xrightarrow{d} U \to 0.
$$

Now take the homology of this chain complex. That is, take the following graded $R$-modules:

$$
\mathcal{Q} = \frac{U}{\text{Im} \ d}, \quad \mathcal{W} = \frac{\ker d}{\text{Im} \ d'}, \quad \mathcal{Z} = \ker d'.
$$

One can think of $\mathcal{W}$ by first taking the $R$-submodule $W'$ of $\Sigma^{-1}A \otimes T(V) \subseteq T(V)$ generated by elements that are invariant modulo $I$ under graded cyclic permutations, that is, invariant after projecting to $U$. Then $\mathcal{W}$ is the projection of $\Sigma W'$ onto $(A \otimes U)/\text{Im} \ d'$.

Our main result says that the homology of this chain complex is the integral free loop space homology of $M$ under some dimension assumptions:

**Theorem 1.1.** Suppose $n \geq 2$, $n \neq 2, 4, 8$, and $m \geq 1$. Then there exists an isomorphism of graded $R$-modules

$$
H_*(\mathcal{L}M) \cong \mathcal{Q} \oplus \mathcal{W} \oplus \mathcal{Z}.
$$

The restriction away from 2,4, and 8 traces back to an argument that we use to determine $H_*(\Omega M)$, which does not apply to situation where there are cup product squares equal to the fundamental class $[M]$, or $-[M]$. The failure of a degree placement argument to compute certain differentials is another reason that we restrict away from $n = 2$.

We determine the action of the BV-operator on $H_*(\mathcal{L}M; \mathbb{Q})$, in a sense, up-to-abelianization of $U$ when $n > 3$ is odd. Consider the graded abelianization map $T(V) \xrightarrow{\eta} S(V)$, where $S(V)$ is the free graded symmetric algebra generated by $V$. Since $\eta(\chi) = 0$, $\eta$ factors through $U \xrightarrow{\eta} S(V)$. Also, consider the maps $A \otimes U \xrightarrow{1_A \otimes \eta} A \otimes S(V)$ and $K \otimes U \xrightarrow{1_K \otimes \eta} K \otimes S(V)$. Since $(1_A \otimes \eta) \circ d' = 0$ and $\eta \circ d = 0$, then $\eta$ and these two maps induce abelianization maps

$$
\mathcal{Q} \xrightarrow{\eta} S(V), \\
\mathcal{W} \xrightarrow{\eta W} A \otimes S(V), \\
\mathcal{Z} \xrightarrow{\eta Z} K \otimes S(V).
$$
Theorem 1.2. Let $n > 3$ be odd. The BV operator $\Delta: H_*(\mathcal{L}M; \mathbb{Q}) \to H_{*+1}(\mathcal{L}M; \mathbb{Q})$ satisfies $\Delta(\mathcal{Q}) \subseteq \mathcal{W}$ and $\Delta(\mathcal{W}) \subseteq \mathcal{Z}$, and $\Delta(\mathcal{Z}) = \{0\}$. Moreover, the composite $\mathcal{Q} \xrightarrow{\eta_w} \mathcal{W} \xrightarrow{\Delta} A \otimes S(V)$ is given by
\[
\eta_w \circ \Delta(1 \otimes (u_{i_1} \cdots u_{i_k})) = \sum_{j=1}^k a_j \otimes (u_{i_1} \cdots u_{i_{j-1}} u_{i_{j+1}} \cdots u_{i_k}),
\]
and $\mathcal{W} \xrightarrow{\Delta} \mathcal{Z} \xrightarrow{\eta_z} K \otimes S(V)$ is the restriction to $\ker d$ of the map $(A \otimes U)/\text{Im} d' \to S(A) \otimes S(V)$ given by
\[
a_i \otimes (u_{i_1} \cdots u_{i_k}) \mapsto \sum_{j=1}^k a_i a_j \otimes (u_{i_1} \cdots u_{i_{j-1}} u_{i_{j+1}} \cdots u_{i_k}),
\]
where $[M] \in K$ is identified with $(\sum_{i \leq j} c_{i,j} a_i a_j) \in S(A)$.

2. Fibrations with $H$-space Fibers

We begin with a few preliminaries that will be used in the proof of Theorem 1.1. Take a fibration sequence $F \xrightarrow{i} X \xrightarrow{f} B$ with $B$ simply-connected, and let
\[
\mathcal{E} = \{\mathcal{E}^r, \delta^r\}
\]
be the homology Serre spectral sequence for fibration $f$. The fibration $f$ is said to be principal if $F$ is a homotopy associative $H$-space, and if there is a left action $F \times X \to X$ fitting into a homotopy commutative square:

\[
\begin{array}{ccc}
F \times F & \xrightarrow{1 \times i} & F \times X \\
mult. & & \\
\downarrow & & \downarrow \\
F & \xrightarrow{i} & X
\end{array}
\]

($\text{mult.}$ denotes the $H$-space multiplication on $F$). A result of Moore [13] tells us that $\mathcal{E}$ inherits the the structure of a left $H_*(F)$-module, which means there is a left action
\[
H_*(F) \otimes \mathcal{E}^r_{i,j} \to \mathcal{E}^r_{i,j+1}
\]
reducing to the Pontrjagin multiplication on $\mathcal{E}^2_{i,j} \cong H_*(F)$, and differentials respect this action. Thus much of the effort in computing differentials is reduced to determining those emanating from the degree 0 horizontal line in the spectral sequence.

For a more generic fibration $f$, the induced homotopy fibration sequence
\[
\Omega B \xrightarrow{\partial} F \xrightarrow{i} X
\]
is in all cases principal. The $H$-space structure on $\Omega B$ is taken as the one defined by composing loops, and the action $\Omega B \times F \xrightarrow{\partial} F$ here is defined by applying the homotopy lifting property to loops in $B$. Since fibrations are characterized by satisfying the homotopy lifting property, in some rough sense we might expect any information gained from the left-$H_*(\Omega B)$ structure on $H_*(F)$ to have some bearing on the spectral sequence for $f$. This was exploited by McCleary in [12], using a result of Brown [3] and Shih [16], to give a computation of the free loop space homology of certain low rank Stiefel manifolds. The aim of the following proposition is to strengthen the technique used therein under the condition that $F$ is a homotopy associative $H$-space, the gain here being that one can to do away with an assumption about certain elements being trangressive. We let $\mathcal{E} = \{\mathcal{E}^r, \delta^r\}$ denote the homology Serre spectral sequence for the path-loop fibration sequence $\Omega B \xrightarrow{\xi} \mathcal{P}B \xrightarrow{\eta_B} B$. 


Proposition 2.1. Suppose $H_*(B)$, $H_*(\Omega B)$ are torsion free, and $F$ is a homotopy associative $H$-space. Given $z \in H_*(B)$, suppose $d^r(z \otimes 1) = 0 \in E^2_{\ast,*}$ for $2 \leq s < r$,

\[ d^r(z \otimes 1) = \sum_i x_i \otimes v_i, \]

and $z \otimes y \in E^2_{\ast,*}$ survives to $E^r_{\ast,*}$. Then for every $y \in H_*(F)$ and $2 \leq s < r$, $d^r(z \otimes y) = 0 \in E^r_{\ast,*}$ and

\[ d^r(z \otimes y) = \sum_i x_i \otimes \theta_i(v_i \otimes y). \]

Here, for clarity, we use $\otimes$ to indicate tensors in $H_*(\Omega B \times F) \cong H_*(\Omega B) \otimes H_*(F)$.

Proof. First recall the following well-known property (which is essentially the homotopy lifting property in disguise). Let $P^{ev_0,f} \subseteq \text{map}([0,1],B) \times X$ be the pullback of $X \overset{f}{\rightarrow} B$ and the evaluation map $\text{map}([0,1],B) \overset{ev_0}{\rightarrow} B$ given by $ev_0(\omega) = \omega(0)$. Now consider the map $\bar{f}: \text{map}([0,1],X) \rightarrow P^{ev_0,f}$ defined by $\bar{f}(\omega) = (f \circ \omega, \omega(0))$. Then a surjection $f$ is a fibration if and only if there exists a map $g:P^{ev_0,f} \rightarrow \text{map}([0,1],X)$ such that $\bar{f} = 1:P^{ev_0,f} \rightarrow P^{ev_0,f}$.

Take the inclusion $\phi:PB \times F \rightarrow P^{ev_0,f}$ given by $\phi(\omega,a) = (\omega,a)$, and take the the composite

\[ \hat{\theta}:(PB \times F) \overset{\phi}{\rightarrow} P^{ev_0,f} \overset{g}{\rightarrow} \text{map}([0,1],X) \overset{ev_1}{\rightarrow} X. \]

Let the fibration sequence

\[ (1) \quad \Omega B \times F \overset{c \times 1}{\rightarrow} PB \times F \overset{ev_1 \times \pi}{\rightarrow} B \times * \]

be the product of the path-loop fibration sequence $\Omega B \overset{c}{\rightarrow} PB \overset{ev_1}{\rightarrow} B$ and the trivial fibration sequence $F \overset{1}{\rightarrow} F \overset{1}{\rightarrow} *$. Both the path-loop fibration and the trivial fibration are principal, and $\Omega B \times F$ is a homotopy associative $H$-space, since both $F$ and $\Omega B$ are. Then this product fibration is a principal fibration sequence, as is apparent in the following commutative diagram

\[ \begin{array}{cccc}
(\Omega B \times F) \times (\Omega B \times F) & (1 \times 1) \times (c \times 1) & (\Omega B \times F) \times (PB \times F) \\
\downarrow 1 \times T \times 1 & \downarrow (1 \times c) \times (1 \times 1) & \downarrow 1 \times T \times 1 \\
(\Omega B \times \Omega B) \times (F \times F) & (1 \times c) \times (1 \times 1) & (\Omega B \times PB) \times (F \times F) \\
\downarrow \text{mult.} \times \text{mult.} & \downarrow \psi_0 \times \text{mult.} & \downarrow \psi_0 \times \text{mult.} \\
\Omega B \times F & c \times 1 & PB \times F.
\end{array} \]

Here $T$ is the transposition map, the bottom square is the product of the squares that commute due to the trivial fibration and path-loop fibration being principal. The right vertical composite defines the action $\psi$ of $\Omega B \times F$ on $PB \times F$, where $\psi_0$ is the action for the path-loop fibration, given by composing a based loop with a based path at the basepoint.

Consider the commutative diagram of fibration sequences

\[ (2) \quad \begin{array}{ccc}
\Omega B & PB & B \\
\downarrow 1 \times c & \downarrow 1 \times c & \downarrow = \downarrow 1 \times c \times 1 \\
\Omega B \times F & PB \times F & B \times *.
\end{array} \]

We let $\hat{E} = \{ \hat{E}', \hat{d}' \}$ be the homology Serre spectral sequence for the fibration sequence \[H\], and

\[ \gamma: E \rightarrow \hat{E} \]
the morphism of spectral sequences induced by diagram \( \mathcal{B} \). One can also easily check that the following diagram of fibration sequences commutes:

\[
\begin{array}{ccc}
\Omega B \times F & \xrightarrow{c \times 1} & PB \times F \\
\downarrow \theta & & \downarrow \delta \\
F & \xrightarrow{i} & X & \xrightarrow{f} & B,
\end{array}
\]

with the action \( \theta \) constructed as the restriction of \( \delta \) to the subspace \( \Omega B \times F \), which is the reason for the left-most commutative square. Let

\[ \zeta : \hat{E} \rightarrow \mathcal{E} \]

be the morphism of spectral sequences induced by this diagram.

The element \( z \otimes (1 \otimes y) \in \hat{E}^2_{*,*} \) survives to \( \hat{E}^r_{*,*} \) as follows. Inductively, assume that it has survived to \( \hat{E}^r_{*,*} \) for some \( 2 \leq s < r \). Since our assumption is that \( z \otimes y \in \mathcal{E}^2_{*,*} \) survives to \( \mathcal{E}^r_{*,*} \), \( z \otimes y \) is not in the image of any differential \( \delta^r \) for \( 2 \leq s < r \). Since \( z \otimes y = \zeta^r(z \otimes (1 \otimes y)) \), by naturality \( z \otimes (1 \otimes y) \) is also not in the image of any differential \( \delta^s \). Now using the fact that the bottom fibration sequence in diagram \( \mathcal{B} \) is principal, and that \( d^s(z \otimes 1) = 0 \in \hat{E}^s_{*,*} \) for \( 2 \leq s < r \), in \( \hat{E}^s_{*,*} \) we have

\[ \hat{d}^s(z \otimes (1 \otimes y)) = (1 \otimes (1 \otimes y))\hat{d}^s(z \otimes (1 \otimes 1)) \]

\[ = (1 \otimes (1 \otimes y))\hat{d}^s(\gamma^s(z \otimes 1)) \]

\[ = (1 \otimes (1 \otimes y))\gamma^s(d^s(z \otimes 1)) = 0, \]

which implies \( z \otimes (1 \otimes y) \) survives to \( \hat{E}^{s+1}_{*,*} \). This completes the induction.

Finally, in \( \hat{E}^r_{*,*} \) we have the formula

\[ \hat{d}^r(z \otimes (1 \otimes y)) = (1 \otimes (1 \otimes y))\hat{d}^r(z \otimes (1 \otimes 1)) \]

\[ = (1 \otimes (1 \otimes y))\gamma^r(d^r(z \otimes 1)) \]

\[ = (1 \otimes (1 \otimes y))\gamma^r(\sum_i x_i \otimes v_i) \]

\[ = (1 \otimes (1 \otimes y))\sum_i (x_i \otimes (v_i \otimes 1)) \]

\[ = \sum_i x_i \otimes (v_i \otimes y), \]

which we can use to obtain the following

\[ \delta^r(z \otimes y) = \delta^r(\zeta^r(z \otimes (1 \otimes y))) \]

\[ = \zeta^r(\hat{d}^r(z \otimes (1 \otimes y))) \]

\[ = \zeta^r\left(\sum_i x_i \otimes (v_i \otimes y)\right) \]

\[ = \sum_i x_i \otimes \theta_*(v_i \otimes y). \]

Similarly, \( \delta^s(z \otimes y) = 0 \) for \( 2 \leq s < r \).

Proposition \( \mathcal{B} \) still manages to hold if \( F \) is not an \( H \)-space, as long as there is a map \( G \xrightarrow{f} F \) with \( G \) an \( H \)-space, and we restrict \( y \in \text{Im } f_* \subset H_*(F) \) in the statement of the proposition. One replaces the fibration sequence \( \mathcal{B} \) in the proof with \( \Omega B \times G \xrightarrow{c \times 1} PB \times G \xrightarrow{c \times 1 \times s} B \times \ast \), and composes it with diagram \( \mathcal{B} \) using the map \( f \). Generally, if \( F \) is not an \( H \)-space, the proposition holds when \( z \) is transgressive, which is the result due to Brown and Shih.
We now turn our focus towards the free loop space fibration sequence
\[(4) \quad \Omega B \xrightarrow{\vartheta} LB \xrightarrow{ev_1} B.\]
The map $\vartheta$ is the canonical inclusion $\Omega B \subseteq LB$, and $ev_1$ is the evaluation map $ev_1(\omega) = \omega(1)$. The homology Serre spectral sequence for this fibration sequence will be denoted by
\[\mathcal{E} = \{E^r, \delta^r\},\]
and as before $E = \{E^r, d^r\}$ is the homology Serre spectral sequence for the path-loop fibration sequence of $B$.

Some basic properties of the free loop space fibration are as follows. The map $LB \xrightarrow{ev_1} B$ has a section $B \xrightarrow{s} LB$ defined by mapping a point $b \in B$ to the constant loop at $b$, which implies the connecting map $\vartheta$ for the induced principal homotopy fibration $\Omega B \xrightarrow{\vartheta} \Omega B \xrightarrow{s} LB$ is null homotopic. The associated left action
\[\theta : \Omega B \times \Omega B \rightarrow \Omega B\]
is given by the formula
\[\theta(\omega, \lambda) = \omega^{-1} \cdot \lambda \cdot \omega\]
for any $\omega, \lambda \in \Omega B$. If $v \in H_*(\Omega B)$ is primitive, then for any $y \in H_*(\Omega B)$ one has the formula
\[\theta_*(v \circ y) = (-1)^{|v||y|} yv = -[v, y],\]
where the multiplication on $H_*(\Omega B)$ is the Pontrjagin multiplication induced by loop composition on $\Omega B$. The proof of these facts can be found in [12] (for example).

Combining these properties with Propositions 2.1, we gain the following description of the differentials in the spectral sequence $\mathcal{E}$.

**Proposition 2.2.** Suppose $H_*(B)$ and $H_*(\Omega B)$ are torsion free, and $B$ is 1-connected. Fix $z \in H_*(B)$. Suppose $d^s(z \otimes 1) = 0 \in E^s_{*,0}$ for $2 \leq s < r$, and
\[d^r(z \otimes 1) = \sum_i x_i \otimes v_i.\]
such that each $v_i$ is primitive. Then for every $y \in H_*(\Omega B)$ and $2 \leq s < r$, we have $\delta^s(z \otimes y) = 0 \in E^s_{*,*}$, and
\[\delta^r(z \otimes y) = -\sum_i x_i \otimes [v_i, y].\]

While it may not be apparent at first, there are instances where this formula will fail to give us enough information to determine some of the higher differentials. For example, if one found themselves in the situation where $\delta^s(z \otimes y) = 0$ for $s \leq r$ and $d^r(z \otimes y) \neq 0$, then $z \otimes y \in E^r_{*,*}$ survives to the $\mathcal{E}^{r+1}$ page, while $z \otimes y$ is not an element in $E^{r+1}_{*,*}$. In such case $\delta^s(z \otimes y)$ remains mysterious when $s > r$. An example where this situation happens in practice is the omitted 4-manifold case in Theorem 1.1.
3. Based Loop Space Homology

Returning to our manifold $M$ in the introduction, in this section we consider the Hopf algebra $H_*(\Omega M)$. This is the last piece in the puzzle required to prove Theorem 1.1. By Poincaré duality the only nonzero reduced homology groups of $M$ are in degrees $n$ and $2n$. This implies $M$ has a cell decomposition given by attaching an $n$-cell to an $n$-fold wedge of $n$-spheres $\vee_m S^n \simeq M - *$.

Very generally, if a space $Y$ is formed by attaching a $k$-cell to a space $X$ via an attaching map $S^{k-1} \xrightarrow{\alpha} X$, and $\alpha'$ is its adjoint, the composite with the looped inclusion $S^{k-2} \xrightarrow{\alpha'} \Omega X \xrightarrow{1_{\Omega}} \Omega Y$ is nullhomotopic, so one obtains a factorization of Hopf algebras through Hopf algebra maps

$$H_*(\Omega X; R)/I \xrightarrow{\theta} H_*(\Omega Y; R),$$

where $I$ is the two-sided ideal generated by $\alpha'(\{S^{k-2}\}) \in H_{k-2}(\Omega X; R)$. The problem of determining the conditions under which $\theta$ is a Hopf algebra isomorphism is part of what is known as the cell-attachment problem. One of these, the inert condition, states somewhat surprisingly that $\theta$ is a Hopf algebra isomorphism when $R$ is a field if and only if $(\Omega i)_*$ is a surjection (\cite{11, 8, 6}). Here we select $k = 2n$, $Y \simeq M$, and $X \simeq M - *$, and use the inert condition to prove the following:

**Proposition 3.1.** Suppose $n \geq 2$, $n \neq 2, 4, 8$, and $m \geq 1$.

(i) There is an isomorphism of Hopf algebras (free as $R$-modules)

$$H_*(\Omega M) \cong \frac{T(V)}{I}$$

where $V = R\{u_1, \ldots, u_m\}$, $|u_i| = n - 1$.

(ii) The element $\alpha'_*(\{S^{2n-2}\})$ generating the two-sided ideal $I$ is given by

$$\alpha'_*(\{S^{2n-2}\}) = \sum_{i,j} c_{ij} [u_j, u_i] + \sum_i c_i u_i^2.$$

Proof of part (i). In $\Omega M$ is shown to be a homotopy retract of $\Omega(M - *)$ when $n \neq 2, 4, 8$. Therefore $(\Omega i)_*$ is a split epimorphism, so we obtain an isomorphism $H_*(\Omega M; F) \cong H_*(\Omega(M - *); F)/I$ for any field $F$. Moreover, since $M - *$ is homotopy equivalent to $\vee_m S^n$, the $\mathbb{Z}$-module $H_*(\Omega(M - *); \mathbb{Z}) \cong T(V)$ is torsion-free. Therefore $H_*(\Omega M; \mathbb{Z})$ is torsion-free, and the Hopf algebra isomorphism holds for $R = \mathbb{Z}$ as well.

Proof of part (ii). We will write $u_j = (\Omega i)_*(u_j) \in H_{n-1}(\Omega M)$, and take $u_j$ to be the transgression of $a_j \in H_n(M)$.

Since the elements $u_1, \ldots, u_m$ in $H_{n-1}(\Omega(M - *))$ are primitive, and there are no monomials of length greater than 2 in degree $2n - 2$, the elements $u_i^2$ and $[u_j, u_i]$ form a basis for the primitives in $H_{2n-2}(\Omega(M - *))$. Now $\alpha'_*(\{S^{2n-2}\})$ is primitive since $[S^{2n-2}]$ is primitive, so we can set

$$\langle \alpha'_* \rangle_* \{S^{2n-2}\} \cong \sum_{i,j} c'_{ij} [u_i, u_j] + \sum_i c''_i u_i^2$$

for some integers $c''_{ij}$.

Consider the homology Serre spectral sequence $E$ for the path-loop fibration of $M$, with

$$E^2_{*,*} = H_*(M) \otimes H_*(\Omega M).$$
On the second page of spectral sequences we have the formula
\[ d_{2n}(a_j \otimes u_i) = d_{2n}(a_j \otimes 1)(1 \otimes u_i) + (a_j \otimes 1)d_{2n}(1 \otimes u_i) = (a_j \otimes 1)(a_i \otimes 1) = c_{ij}([M]^n \otimes 1), \]
so dualizing back to the homology spectral sequence gives us
\[ d^{2n}([M] \otimes 1) = \sum_{i,j} c_{ij}(a_j \otimes u_i). \]

Take \( \bar{E} \) to be the homology Serre spectral sequence for the path-loop fibration of \( M - * \), and consider the morphism of spectral sequences \( \gamma: E \rightarrow \bar{E} \) induced by the inclusion \( (M - *) \rightarrow M \). On the second page of spectral sequences \( \gamma_2 \) maps \( 1 \otimes u_i \) to \( 1 \otimes u_i \) and \( a_i \otimes 1 \) to \( a_i \otimes 1 \), and \( \bar{E}_{2n,2n-1} \rightarrow \bar{E}_{2n,2n-1} \) is an isomorphism for \( 2 \leq r \leq 2n \).

By part (i) of the theorem (and preceding discussion), \( (\alpha')_*([S^{2n-2}]) \) generates the kernel of \( (\Omega i)_*: H_{2n-2}(\Omega(M - *)) \rightarrow H_{2n-2}(\Omega M) \), so \( 1 \otimes (\alpha'_*)[S^{n-2}] \) generates the kernel of \( \gamma_2: E^{2n}_0,2n-2 \rightarrow E^{0,2n-2}_2 \). Since \( \gamma_r: E^r_{i,j} \rightarrow E^r_{i,j} \) is an isomorphism for \( i < 2n, j < 2n - 2 \), and all \( r \), then in fact \( 1 \otimes (\alpha'_*)[S^{2n-2}] \) generates the kernel of the map \( \bar{E}_{0,2n-2}^{r} \rightarrow \bar{E}_{0,2n-2}^{r} \) for \( 2 \leq r \leq 2n \).

Take the element
\[ \zeta'' = \sum_{i,j} c_{ij}''(a_j \otimes u_i - a_i \otimes u_j) \]
in \( E^{2n}_{2n,2n-1} \), for \( 2 \leq r \leq 2n \). Then
\[ \gamma_2(\zeta'') = \sum_{i,j} c_{ij}''(a_j \otimes u_i - a_i \otimes u_j), \]
and in \( E^{2n}_{0,2n-2} \) we have
\[ 1 \otimes (\alpha'_*)[S^{2n-2}] = \sum_{i,j} c_{ij}''(1 \otimes [u_i, u_j]) = d^{2n}(\zeta''). \]

Since \( E^{2n}_{2n,0} = \{0\} \), the differential \( d^{2n}: E^{2n}_{2n,2n-1} \rightarrow E^{2n}_{0,2n-2} \) is an isomorphism. Since \( E^{2n}_{2n,2n-1} \rightarrow E^{2n}_{0,2n-2} \) is an isomorphism, and \( 1 \otimes (\alpha'_*)[S^{2n-2}] \) generates the kernel of \( E^{2n}_{0,2n-2} \rightarrow E^{2n}_{0,2n-2} \), by naturality we see that the kernel of the differential \( E^{2n}_{2n,2n-1} \rightarrow E^{2n}_{0,2n-2} \) is generated by \( \gamma_2(\zeta'') \). In particular, we may project \( \gamma_2(\zeta'') \) down to \( E^{0,2n}_r \).

Let
\[ \mathcal{I} = \text{Im } d^{2n}: E^{2n}_{2n,0} \rightarrow E^{2n}_{2n,2n-1} \]
\[ \mathcal{K} = \ker d^{2n}: E^{2n}_{2n,2n-1} \rightarrow E^{2n}_{0,2n-2}. \]

As we saw above, \( \mathcal{I} \) is generated by \( d^{2n}([M] \otimes 1) \), and \( \gamma_2(\zeta'') \) generates \( \mathcal{K} \). But the short exact sequence
\[ 0 \rightarrow E^{2n}_{2n,0} \rightarrow E^{2n}_{2n,2n-1} \rightarrow E^{2n}_{0,2n-2} \rightarrow 0 \]
implies \( \mathcal{I} \subseteq \mathcal{K} \). Therefore \( d^{2n}([M] \otimes 1) = \pm \gamma_2(\zeta'') \). Now comparing coefficients in equations (6) and (7), the result follows.

\[ \square \]

4. Proof of Theorem 1.1

We now have everything required to prove Theorem 1.1 via a routine Serre spectral sequence argument. Let \( E = \{E^r, \delta^r\} \) be the homology Serre spectral sequence for the free loop space fibration sequence
\[ \Omega M \xrightarrow{\partial} LM \xrightarrow{e_{\Omega M}} M. \]
By Proposition 3.1 we have an isomorphism $H_*(\Omega M) \cong U = T(V)/I$ of Hopf algebras, which are free as $R$-modules. So we start with an isomorphism of free $R$-modules

$$E^2_{i,j} \cong R\{1, a_1, \ldots, a_m, [M]\} \otimes U.$$  

By Proposition 2.2

$$\delta^n(a_i \otimes y) = -1 \otimes [u_i, y],$$

and using part (ii) of Proposition 3.1

$$\delta^n([M] \otimes y) = - \sum_{i,j} c_{ij}(a_j \otimes [u_i, y]).$$

Therefore $E^{2n}_{0,*} \cong Q$, $E_{n,*}^{\infty} \cong E_{n,*}^{2n} \cong W$, and $E^{2n}_{2n,*} \cong Z$, while all other entries in the spectral sequence are zero. But since the nonzero elements in $Q$ and $Z$ are concentrated in degrees $k(n-1)$ and $2n+k(n-1)$ respectively, the differentials $\delta^{2n}$ are zero for degree placement reasons whenever $n > 2$. Thus these isomorphisms carry over to the infinity page, that is,

$$E^\infty_{i,j} \cong E^\infty_{0,*} \oplus E^\infty_{n,*} \oplus E^\infty_{2n,*} \cong Q \oplus W \oplus Z.$$

Generally, one has torsion here when $R = \mathbb{Z}$ (or at least in $Q$, and possibly $W$), so we must consider a potential extension problem. Once again placement reasons allow us to skirt around the issue.

From the construction of the homology Serre spectral sequence there are increasing filtrations $F_{i,j} = F_i H_j(\mathcal{L}M) \subseteq H_j(\mathcal{L}M)$ such that $F_{k,k} = H_k(\mathcal{L}M)$, $F_{i,j} = 0$ for $i < 0$, and

$$E^\infty_{i,j} \cong \frac{F_{i,j}}{F_{i-1,j}}.$$  

Since the nonzero elements in $Q$, $W$, and $Z$ are in degrees $k(n-1)$, $n+k(n-1)$, and $2n+k(n-1)$, $Q$, $W$, and $Z$ pairwise have no nonzero elements in the same degrees when $n > 3$. Since $F_{n-1,*} = F_{0,*} = Q$, we have $F_{n-1,n+k(n-1)} = \{0\}$, and we see that $F_{n,*} \cong F_{0,*} \oplus E^\infty_{n,*} \cong Q \oplus W$. Then $F_{2n-1,2n+k(n-1)} = F_{n,2n+k(n-1)} = \{0\}$, so $F_{2n,*} \cong F_{n,*} \oplus E^\infty_{2n,*}$, and we have

$$E^\infty_{2n,*} \cong F_{2n,*} = H_*(\mathcal{L}M)$$

whenever $n > 3$.

When $n = 3$, the common nonzero degrees shared between any pair of these three modules are of the form $2(k+3)$, and these are only between $Q$ and $Z$. But since $Z$ is torsion-free and $Q = F_{0,*}$ is at the bottom filtration, there are no extension issues here either.

5. Eilenberg-Maclane Spaces and the BV-operator

We will need some information about the action of the BV-operator on products of Eilenberg-Maclane spaces in the proof Theorem 1.2. The approach we take here is similar to the one taken by Hepworth in [10] to compute the BV-operator for Lie groups, which we begin this section with by recalling. Fix $R$ to be a principal ideal domain, and $X$ (homotopy type of a CW-complex) a path-connected topological group with multiplication $X \times X \xrightarrow{\mu} X$. This makes $\mathcal{L}X$ into topological group with multiplication $\mathcal{L}X \times \mathcal{L}X \xrightarrow{\mu} \mathcal{L}X$ defined by point-wise multiplication of loops $(\omega \cdot \gamma)(t) = \omega(t) \cdot \gamma(t)$. There is a well-known homeomorphism

$$h : X \times \Omega X \longrightarrow \mathcal{L}X$$

$$h(x, \omega) = x \cdot \omega.$$
with inverse \( h^{-1}: \mathcal{L}X \to X \times \Omega X \) given by \( h^{-1}(\omega) = (\omega(0), \omega(0)^{-1} \cdot \omega) \), where \( x \cdot \omega \) is the loop defined at each point by \( (x \cdot \omega)(t) = x \cdot \omega(t) \). These homeomorphisms are equivariant with respect to our action \( S^1 \times \mathcal{L}X \xrightarrow{\nu} \mathcal{L}X \), and the action

\[
\tilde{\nu}: S^1 \times X \times \Omega X \to X \times \Omega X
\]
defined by the formula

\[
\tilde{\nu}(t, x, \omega) = h^{-1} \circ \nu(t, x, \omega) = (x \cdot \omega_t(0), (x \cdot \omega_t(0))^{-1} \cdot x \cdot \omega_t)
\]
where \( \omega_t(s) = \omega(t)(s) = \omega(s + t) \). On homology we have a commutative square

\[
\begin{array}{ccc}
H_*(X \times \Omega X; R) & \xrightarrow{h_*} & H_*(\mathcal{L}X; R) \\
\Delta & \downarrow & \Delta \\
H_{*+1}(X \times \Omega X; R) & \xrightarrow{h_*} & H_{*+1}(\mathcal{L}X; R),
\end{array}
\]

where \( \Delta(e) = \tilde{\nu}_*([S^1] \otimes e) \). Clearly, after transposing \( X \) and \( S^1 \), \( \tilde{\nu} \) is the composite

\[
X \times (S^1 \times \Omega X) \xrightarrow{1 \times \Delta} X \times (S^1 \times \Omega X) \times (S^1 \times \Omega X) \xrightarrow{1 \times e \times e \times \phi} (X \times X) \times \Omega X \xrightarrow{\mu} X \times \Omega X,
\]

with \( e: S^1 \times \Omega X \to X \) the evaluation map \( e(t, \omega) = \omega(t) = \omega_t(0) \), and \( \phi : S^1 \times \Omega X \to \Omega X \) defined by \( \phi(t, \omega) = \omega_t(0)^{-1} \cdot \omega_t \). Thus, if \( H_*(\Omega X; R) \) is a free \( R \)-module, so that (for simplicity) the cross product \( H_*(X; R) \otimes H_*(\Omega X; R) \xrightarrow{\phi} H_*(X \times \Omega X; R) \) is an isomorphism, and the coproduct on an element \( b \in H_*(\Omega X; R) \) has the form \( \Delta_*(b) = \sum_i d_i \otimes e_i \), then \( \Delta \) satisfies

\[
(-1)^{|a|} \Delta(a \otimes b) = \sum_i (-1)^{|d_i|} \left(a(e_*([S^1] \otimes d_i)) \otimes \phi_*(1 \otimes e_i)\right) + \sum_i \left(a(e_*([S^1] \otimes e_i)) \otimes \phi_*(1 \otimes d_i)\right)
\]

where \( e: H_* (\Omega X; R) \to R \) is the augmentation. To complete this formula one needs to determine the maps \( \phi_* \) and \( e_* \). This latter map defines the homology suspension \( \sigma : H_*(\Omega X; R) \to H_{*+1}(X; R) \), \( \sigma(a) = e_*([S^1] \otimes a) \), which satisfies the formula

\[
\sigma(ab) = \sigma(a)e(b) + e(a)\sigma(b)
\]

for any product \( ab \in H_*(\Omega X; R) \) induced by the loop composition multiplication on \( \Omega X \). In particular, \( \sigma \) is zero on decomposable elements. One can derive this formula by observing that the following diagram commutes

\[
\begin{array}{ccc}
(S^1 \times S^1) \times (\Omega X \times \Omega X) & \xrightarrow{1 \times T \times 1} & (S^1 \times \Omega X) \times (S^1 \times \Omega X) \xrightarrow{e \times e \times \nu} X \times X \\
\Delta \times 1 \times 1 & \downarrow & \Delta \times 1 \times \mu \\
S^1 \times (\Omega X \times \Omega X) & \xrightarrow{1 \times \Omega \mu} & S^1 \times \Omega X \xrightarrow{e \nu} X,
\end{array}
\]

and that point-wise multiplication of based loops \( \Omega \mu \) on \( \Omega X \) is homotopy commutative and homotopic to the loop composition multiplication on \( \Omega X \) (this is a mapping space analogue of Theorem 5.21, Chapter III in [13]). Alternatively, it is a consequence of the Homology Suspension Theorem (13, Chapter VIII). The map \( \kappa(a) = \phi_*([S^1] \otimes a) \) is a bit more mysterious. At the very least, when \( \mu \) is commutative one obtains an analogous commutative diagram for \( \phi \) together with a derivation formula \( \kappa(ab) = \kappa(a)b + a \kappa(b) \), while for the case of compact Lie groups, \( \kappa \) is trivial since \( H_*(\Omega X) \) is concentrated even degrees. We consider the case where \( X \) is an Eilenberg-MacLane space \( K(R, n) \). These can be taken to be commutative topological spaces, and we may write \( K(G, n-1) = \Omega K(G, n) \).
with commutative multiplication induced by the one on $K(R, n)$, which by the way is homotopic to the loop composition multiplication.

**Proposition 5.1.** Let $J$ be the image of the cross product $H_\ast(K(R, n-1); R) \otimes H_\ast(K(R, n); R) \xrightarrow{x} H_\ast(K(R, n-1) \times K(R, n); R)$ (which is injective by the Künneth formula). Suppose the coproduct on $a \in H_\ast(K(R, n-1); R)$ is in the image of the cross product, that is, if of the form $\Delta_\ast(b) = \sum_i d_i \times e_i$. Then with respect to the isomorphism $h_\ast$, the BV-operator is given on $a \times b \in J \subseteq H_\ast(\mathcal{L}K(R, n); R)$ by the formula

$$\Delta(a \times b) = (-1)^{[a]} \sum_i (a(\rho_i((S^1) \otimes d_i)) \times e_i),$$

where $\Sigma(K(R, n-1) \xrightarrow{\Delta} K(R, n)$ is a classifying map for $1 \otimes \iota_{n-1} \in H^\ast(S^1 \otimes K(R, n-1); R) \cong H^\ast(S^1; R) \otimes H^\ast(K(R, n-1); R)$, and $\iota_{n-1} \in H^{n-1}(K(R, n-1); R)$ is the fundamental class.

**Proof.** Since our map $S^1 \times K(R, n-1) \xrightarrow{\bar{\phi}} K(R, n-1)$ restricts to the identity on the right factor, $\phi^*(\iota_{n-1}) = 1 \otimes \iota_{n-1}$, or in other words, $\phi$ is a classifying map of the cohomology class $1 \otimes \iota_{n-1} \in H^{n-1}(S^1 \times K(R, n-1); R)$. The projection map onto the right factor $S^1 \times K(R, n-1) \xrightarrow{\iota} K(R, n-1)$ is also a classifying map for $1 \otimes \iota_{n-1}$. Since cohomology classes are in one-to-one correspondence with the homotopy classes of the classifying maps representing them, $\phi$ must be homotopic to $\ast \times 1$. Therefore $\phi_*([S^1] \otimes d) = 0$ for any $d$.

Next, recall the suspension isomorphism $H_{n-1}(K(R, n-1); R) \xrightarrow{\Sigma} H_n(\Sigma K(R, n-1); R)$, sending $a \mapsto [S^1] \otimes a$, factors as the composite

$$H_{n-1}(K(R, n-1); R) \xrightarrow{\Sigma} [K(R, n-1), K(R, n-1)] \xrightarrow{\Sigma} [\Sigma K(R, n-1), K(R, n)]$$

where the last map is the adjoint isomorphism. Since the evaluation map $S^1 \times K(R, n-1) \xrightarrow{ev} K(R, n)$ restricts to the constant map on both the left and right factors, it factors as the composite

$$ev: S^1 \times K(R, n-1) \xrightarrow{\text{quotient}} \Sigma K(R, n-1) \xrightarrow{ev'} K(R, n),$$

where the last map $ev'$ (also known as the evaluation map in the literature) is the adjoint of the identity map $K(R, n-1) \xrightarrow{1} K(R, n-1)$. Since the identity is a classifying map of $\iota_{n-1}$, by the above factorization of the suspension, its adjoint $ev'$ is a classifying map of $[S^1] \otimes \iota_{n-1}$. The proposition now follows using equation \(\text{S}\).

\[\square\]

The BV-operator has a very clean form on decomposable elements when we take our multiplication on $H_\ast(\mathcal{L}X)$ to be the one induced by point-wise multiplication of loops $\mathcal{L}\mu$ (instead of the multiplication $(\Omega \mu \times \mu) \circ (1 \times T \times 1)$ based on each coordinate of $\Omega X \times X \simeq \mathcal{L}X$). Tamanoi gave a derivation formula with respect to this product

$$\Delta(ab) = \Delta(a)b + (-1)^{|a|}a\Delta(b),$$

which is a straightforward consequence of the following commutative diagram

$$\begin{array}{cccc}
(S^1 \times S^1) \times (\mathcal{L}X \times \mathcal{L}X) & \xrightarrow{1 \times \mathcal{L} \times 1} & (S^1 \times \mathcal{L}X) \times (S^1 \times \mathcal{L}X) & \xrightarrow{\nu \times} & \mathcal{L}X \times \mathcal{L}X \\
\Delta \times 1 \times 1 & & & \mathcal{L} \mu & \\
S^1 \times (\mathcal{L}X \times \mathcal{L}X) & \xrightarrow{1 \times \mathcal{L} \mu} & S^1 \times \mathcal{L}X & \xrightarrow{\nu} & \mathcal{L}X.
\end{array}$$

Both multiplications on $\mathcal{L}X$ are equal when the multiplication on $X$ is commutative. Since this is the case for $K(R, n)$, our formula in Proposition 5.1 satisfies

$$(-1)^{|b||c|} \Delta(ac \times bd) = \Delta((a \times b)(c \times d)) = \Delta(a \times b)(c \times d) + (-1)^{|a|+|b|}(a \times b)\Delta(c \times d)$$

\[\text{(10)}\]
The derivation formula can also be used to determine how the BV-operator interacts with the cross-product, as we see in the following:

**Proposition 5.2.** Let $X = X_1 \times \cdots \times X_k$ be a product of topological groups $(X_i, \mu_i)$. Then the BV-operator for $\mathcal{L}X \cong \mathcal{L}X_1 \times \cdots \times \mathcal{L}X_k$ satisfies

$$\Delta(a_1 \times \cdots \times a_k) = \sum_i (-1)^{k_i} (a_1 \times \cdots \times \Delta(a_i) \times \cdots \times a_k)$$

for $a_i \in H_\ast(\mathcal{L}X_i)$, where $k_i = \sum_{j=1}^{i-1} |a_j|$ and $k_1 = 0$. 

**Proof.** It suffices to prove the statement for length-2 products $X = X_1 \times X_2$. One can then iterate to obtain the general formula. Since the inclusion of the left factor $\mathcal{L}X_1 \xrightarrow{1 \times \ast} \mathcal{L}X_1 \times \mathcal{L}X_2$ induces the map on homology sending $a \mapsto a \times 1$ for any $a$, by naturality of the BV-operator we have $\Delta(a_1 \times 1) = (\mathbb{I} \times \ast) \ast (\Delta(a_1)) = \Delta(a_1) \times 1$. Similarly, $\Delta(1 \times a_2) = 1 \times \Delta(a_2)$. Since $X$ is a topological group with multiplication $\mu$ defined by the composite $X \times X \xrightarrow{1 \times T \times 1} (X_1 \times X_1) \times (X_2 \times X_2) \xrightarrow{\mu_1 \times \mu_2} X$, the point-wise loop multiplication $\mathcal{L}\mu$ is the composite

$$\mathcal{L}X \times \mathcal{L}X \xrightarrow{\varepsilon} (\mathcal{L}X_1 \times \mathcal{L}X_2) \times (\mathcal{L}X_1 \times \mathcal{L}X_2) \xrightarrow{1 \times T \times 1} (\mathcal{L}X_1 \times \mathcal{L}X_1) \times (\mathcal{L}X_2 \times \mathcal{L}X_2) \xrightarrow{\mathcal{L}\mu_1 \times \mathcal{L}\mu_2} \mathcal{L}X.$$

Therefore $(a_1 \times 1)(1 \times a_2) = a_1 \times a_2$ with respect to this induced product, and by the derivation formula we have

$$\Delta(a_1) = \Delta(a_1),$$

$$\Delta(a_2) = \Delta(a_2),$$

$$\Delta(a_1 \times a_2) = \Delta(a_1)(1 \times a_2) + (-1)^{|a_1|} (a_1 \times 1) \Delta(a_2) = \Delta(a_1) \times a_2 + (-1)^{|a_1|} a_1 \times \Delta(a_2).$$

In the case of rational coefficients, a simply connected $H$-space $X$ has a rational decomposition $X_\mathbb{Q} \cong \prod_i K(\mathbb{Q}, n_i)$, and the classifying maps $\Sigma K(\mathbb{Q}, n_i - 1) \rightarrow K(\mathbb{Q}, n_i)$ can be identified with the Freudenthal suspension $S^n_{\mathbb{Q}} \rightarrow \Omega \Sigma S^n_{\mathbb{Q}}$ in the $n_i$ even case, and evaluation $\Sigma \Omega S^n_{\mathbb{Q}} \rightarrow S^n_{\mathbb{Q}}$ in the odd case. We see then that the action of $\Delta$ on $H_\ast(\mathcal{L}X; \mathbb{Q})$ with respect to the algebra structure induced by the group multiplication on $\prod_i K(\mathbb{Q}, n_i)$ can be determined by applying Propositions 5.2 and 5.1.

This technique can still be used to obtain some useful information for more general coefficients. Suppose $H_\ast(X; R)$ is free as an $R$-module, and $a \in H_n(X; R)$ is an indecomposable element in the Hopf algebra $H_\ast(X; R)$. Then the cohomology dual $\check{a} \in H^n(X; R)$ of $a$ is a primitive element in the dual Hopf algebra $H^\ast(X; R)$, the classifying map $X \xrightarrow{e} K(R, n)$ of $\check{a}$ is an $H$-map, and moreover it is natural with respect to the homeomorphism $h$. That is, the following squares commute up to homotopy

\[
\begin{array}{ccc}
X \times X & \xrightarrow{\text{exc}} & K(R, n) \times K(R, n) \\
\mu \downarrow & & \downarrow \text{mult.} \\
X & \xrightarrow{e} & K(R, n)
\end{array}
\quad
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\text{exc}} & K(R, n) \times \Omega K(R, n) \\
\mu \downarrow & & \downarrow \text{mult.} \\
X \Omega X & \xrightarrow{\check{e}} & \Omega K(R, n)
\end{array}
\]
The proof of commutativity is as follows. For degree reasons, the fundamental class \( \iota_n \) satisfies 
\[
(\text{mult.})^*(\iota_n) = (\iota_n \times 1 + 1 \times \iota_n),
\]
so we have \( (c \times c)^* \circ (\text{mult.})^*(\iota_n) = \hat{a} \otimes 1 + 1 \otimes \hat{a} \). Likewise, since \( \hat{a} \) is primitive, \( \mu^* \circ c^*(\iota_n) = \mu^*(\hat{a}) = \hat{a} \otimes 1 + 1 \otimes \hat{a} \). Thus both the composites in the first square are classifying maps of \( \hat{a} \otimes 1 + 1 \otimes \hat{a} \), meaning they are homotopic. This gives the first square. To obtain the second square, let \( H: (X \times X) \times I \to K(R, n) \) be a choice of homotopy between the composites in the first square. Define the homotopy \( G: (X \times X) \times I \to LK(R, n) \) by \( G(x, \omega, t) = \omega_x \cdot t \), where \( \omega_x: S^1 \to X \) is the loop given by \( \omega_x(s) = H(x, \omega(s), t) \). Then \( G \) defines a homotopy between the two composites in the second square. As a consequence of these diagrams, \( Lc_\ast \) is an algebra map with respect to the algebra structure induced by the isomorphisms \( \mu \) and \( \eta \) in \( \Omega \) and \( \mu \) in \( \Omega \).

Now suppose \( n \) is odd, \( a \) is trangressive, and \( \tau(a) \in H_{n-1}(\Omega X; R) \) is its trangression. Since \( c_\ast \) maps \( a \) to the homology dual \( \hat{i}_n \) of \( \iota_n \), and \( \hat{i}_n \) is trangressive onto \( \tau(\hat{i}_n) = \hat{i}_{n-1} \), the homology dual of the fundamental class of \( \Omega K(R, n) = K(R, n-1) \), we have \( (\Omega c_\ast)(\tau(a)) = \hat{i}_{n-1} \). Then \( (\Omega c_\ast)(\Delta(v \otimes \tau(a))) = \Delta((\Omega c_\ast)(v \otimes \tau(a))) = \Delta(c_\ast(v) \otimes \hat{i}_{n-1}) = (-1)^{|v|}(c_\ast(v) \otimes \hat{i}_{n-1}) \times 1 \) by Proposition 5.1 and applying the derivation formula \( [10] \) inductively,
\[
(\Omega c_\ast)(\Delta(v \otimes \tau(a)^k)) = (\Delta(c_\ast(v) \otimes \hat{i}_{n-1}^k) = k(-1)^{|v|}(c_\ast(v) \otimes \hat{i}_{n-1}^{k-1}).
\]

Since \( (\Omega c_\ast)(v a \otimes \tau(a)^{k-1}) = (c_\ast(v) \otimes \hat{i}_{n-1}) \times \hat{i}_{n-1}^{k-1} \), if we assume \( \tau(a)^{k-1} \) generates \( H_{(k-1)(n-1)}(\Omega X; R) \), then
\[
\Delta(v \otimes \tau(a)^k) = k(-1)^{|v|}(v a \otimes \tau(a)^{k-1}).
\]

For example, if we take \( R = \mathbb{Z}_p \) for \( p \) an odd prime, \( X = S^n(p) \) as a \( p \)-localized sphere (which is an \( H \)-space for \( n \) odd \([11]\)), and \( a = \beta \), then this formula completely determines the action of \( \Delta \) on \( H(\mathbb{L}S^n; \mathbb{Z}_p) \cong H(\mathbb{L}X; \mathbb{Z}_p) \). This is a somewhat different approach for spheres than the one taken by Westerland in \([18]\), and Menichi in \([12]\).

6. Proof of Theorem 1.2

For degree placement reasons, it is clear that \( \Delta(Q) \subseteq W, \Delta(W) \subseteq Z, \) and \( \Delta(Z) = \{0\} \) when \( n > 3 \). Consider the composite
\[
f: M \xrightarrow{\Delta} \prod_{i=1}^m M \xrightarrow{\Pi_i f} \prod_{i=1}^m K(\mathbb{Q}, n) = P,
\]
where \( f_i \) is the classifying map of the generator \( a_i \in H^n(M; \mathbb{Q}) \). Let \( \iota_i \in H_n(K(\mathbb{Q}, n); \mathbb{Q}) \) denote the homology dual of the fundamental class for the \( i \)th factor, and \( \hat{i}_i \in H_{n-1}(K(\mathbb{Q}, n-1); \mathbb{Q}) \) the corresponding trangression. Let \( W = \mathbb{Q}\{\iota_1, \ldots, \iota_m\} \) and \( \hat{W} = \mathbb{Q}\{\hat{i}_1, \ldots, \hat{i}_m\} \).

Since \( n \) is odd, \( H_*(K(\mathbb{Q}, n); \mathbb{Q}) \cong \Lambda\{\iota_i\}, \) \( H_*(K(\mathbb{Q}, n-1); \mathbb{Q}) \cong \mathbb{Q}\{\hat{i}_i\}, \) \( f \) induces the injection \( H_*(M; \mathbb{Q}) \cong V \otimes K \longrightarrow \Lambda\{\hat{W}\}, \) mapping \( a_i \mapsto \iota_i \) and \( [M] \mapsto \beta = \sum_{ij} c_{ij} \iota_i \iota_j \), and \( \Omega f \) induces the algebra map \( \mathbb{Q}\{\hat{W}\} \cong S(V), \) mapping \( u_i \mapsto \hat{i}_i. \)

Consider the morphism of rational homology Serre spectral sequences \( E \to E \) induced by the map of free loop space fibrations
\[
\begin{array}{ccc}
\Omega M & \xrightarrow{\psi_1} & M \\
\downarrow{\Omega f} & & \downarrow{f} \\
\Omega P & \xrightarrow{\psi_1} & P.
\end{array}
\]
The spectral sequence \( E \) for the bottom fibration collapses since the total space is a topological group with section. On the infinity page
\[
H_*(\mathbb{L}P; \mathbb{Q}) \cong H_*(P; \mathbb{Q}) \otimes H_*(\Omega P; \mathbb{Q}) \cong \bigoplus_{i=0}^m E^n_{i,i,*},
\]
and $\phi^\infty$ restricts to the maps $Q \xrightarrow{\eta_q} Q[\bar{W}] \cong E^\infty_{0,*}$, $W \xrightarrow{\eta_w} W \otimes Q[\bar{W}] \cong E^\infty_{n,*}$, and $Z \xrightarrow{\eta_z} Q\{\beta\} \otimes Q[\bar{W}] \subseteq E^\infty_{2n,*}$ (note $W \cong V$, $Q\{\beta\} \cong K$, and $Q[\bar{W}] \cong S(V)$ in the introduction).

Let $F$ be the filtration of $H_*(\mathcal{L}P;Q)$ associated with the spectral sequence $E$. Notice $E^\infty_{n,*} \cong F_{n,n+*}/Q[\bar{W}]$, and $Q[\bar{W}]$ is concentrated in degrees $k(n-1)$, while $W$ is concentrated in degrees $n+k(n-1)$, which are never equal when $n \geq 3$, so they do not share any nonzero elements in the same degree. Similarly, $E^\infty_{2n,*} \cong F_{2n,2n+*}/F_{n,2n+*}$, $F_{n,*} \cong Q[\bar{W}] \oplus (W \otimes Q[\bar{W}])$ is concentrated in degrees $k(n-1)$ and $n+k(n-1)$, and $Z$ is concentrated in degrees $2n+k(n-1)$, which are never equal when $n > 3$. Therefore, with respect to our isomorphism $H_*(\mathcal{L}M;Q) \cong \bar{Q} \oplus W \oplus Z$, $(\mathcal{L}f)_*$ restricts to the maps $\eta_q$, $\eta_w$, and $\eta_z$ on each summand.

The action of $\Delta$ on $H_*(\mathcal{L}K(n,n-1);Q)$ is given by $\Delta(1 \otimes i^k_i) = k(i_i \otimes i_i^{k-1})$ and $\Delta(a \otimes i_i) = 0$ when $|\alpha| > 0$. This follows from Proposition 5.1 and iterating formula (10). Alternatively, it follows from [18, 13]. Now by Proposition 5.2

$$\Delta(a \otimes i_1^{k_1} \cdots i_m^{k_m}) = \sum_{i=1}^{m} k_i (a_i \otimes i_1^{k_1} \cdots i_i^{k_i} \cdots i_m^{k_m}) \subseteq W \otimes Q[\bar{W}] \cong A \otimes S(V)$$

for any integers $k_i \geq 0$. Since for any $q \in Q$, we have $\Delta(q) \in W$,

$$\Delta \circ \eta_q (q) = \Delta \circ (\mathcal{L}f)_* (q) = (\mathcal{L}f)_* \circ \Delta (q) = \eta_w \circ \Delta (q),$$

we obtain the formula for the composite $Q \xrightarrow{\Delta} W \xrightarrow{\eta_w} A \otimes S(V)$. Similarly we obtain the formula for the composite $W \xrightarrow{\Delta} Z \xrightarrow{\eta_z} K \otimes S(V)$.

7. **Acknowledgements**

The second author was supported by a Leibniz-Fellowship from Mathematik-Forschungsinstitut-Oberwolfach and an Invitation to the Max-Planck-Institut für Mathematik in Bonn. Both authors are grateful to the MFO’s hospitality to let them spend some time together to work on this project, and Professor Jie Wu and John McCleary for their helpful comments.

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