THE SECOND DIRAC EIGENVALUE OF A NEARLY PARALLEL G\(_2\) -MANIFOLD

THOMAS FRIEDRICH

Abstract. We investigate the second Dirac eigenvalue on Riemannian manifolds admitting a Killing spinor. In small dimensions the whole Dirac spectrum depends on special eigenvalues on functions and 1-forms. We compute and discuss the formulas in dimension \(n = 7\).

1. Introduction

Let \((M^n, g)\) be a compact Riemannian spin manifold admitting a Killing spinor \(\psi\),
\[
\nabla_X \psi = a \cdot X \cdot \psi, \quad a \in \mathbb{R}^1.
\]
The \(M^n\) is an Einstein space with non-negative scalar curvature \(R \geq 0\) and
\[
n^2 a^2 = \mu_1(D^2) = \frac{n}{4(n-1)} R
\]
is the smallest eigenvalues of the square of the Riemannian Dirac operator \(D\), see [6]. The question whether or not one can estimate the next eigenvalue \(\mu_2(D^2)\) has not yet been investigated for Dirac operators. Remark that in case of the Laplacian acting on functions of an Einstein space \(M^n \neq S^n\), there are lower estimates for small eigenvalues depending on the minimum of the sectional curvature, see [18], [19].

The Killing spinor of \(M^n\) allows us to construct out of it other spinor fields. We can use, for example, test spinors \(\psi^* = f \cdot \psi + c \cdot df \cdot \psi\), where \(f\) is an eigenfunction of the Laplace operator. In this way we obtain an upper bound depending on the first positive eigenvalue \(\lambda_1^0\) of the Laplace operator on functions. A more difficult question is to find lower bounds for \(\mu_2(D^2)\) on Riemannian spin manifolds with Killing spinors. We study the problem comparing the Dirac spectrum with the Laplace spectrum on functions and 1-forms. In more details we will discuss the question in dimension 7 and remark that this method should work in dimensions \(n = 5\) (5-dimensional Sasaki-Einstein manifolds) and \(n = 6\) (nearly Kähler manifolds), too. In dimension \(n = 3\) a similar question has been discussed recently by E.C. Kim, see [15].

2. Estimates for \(\mu_2(D^2)\) in arbitrary dimension

Consider an eigenspinor \(\psi^*\) of the Dirac operator
\[
D(\psi^*) = m \cdot \psi^*, \quad \int_{M^n} \langle \psi, \psi^* \rangle = 0
\]
being $L^2$-orthogonal to the Killing spinor $\psi$. Moreover, suppose that the function $f := \langle \psi, \psi^* \rangle$ is not identically zero. A direct computation yields the formula

$$\int_{M^n} \Delta(f) \cdot f = \left\{ m^2 + 2am + a^2(2n - n^2) \right\} \int_{M^n} f^2.$$ 

Thus we obtain the estimate

$$\lambda^0_1 \leq m^2 + 2am + a^2(2n - n^2),$$

or, respectively,

$$\sqrt{\lambda^0_1 + a^2(1-n)^2} - |a| \leq |m|.$$

Conversely, if $f$ is a non-trivial eigenfunction, $\Delta(f) = \lambda^0_1 f$, then the two numbers

$$m := -a \pm \sqrt{\lambda^0_1 + a^2(1-n)^2}$$

are eigenvalues of the Dirac operator. The corresponding eigenspinor is given by the formula

$$\psi^* := f \cdot \psi + \frac{1}{m + 2a - na} df \cdot \psi.$$

**Remark 2.1.** If $M^n \neq S^n$ is not isometric to the sphere, then the Lichnerowicz-Obata theorem estimates $\lambda^0_1$, 

$$\lambda^0_1 > \frac{R}{n-1} = 4a^2n.$$ 

Therefore our lower bound

$$\sqrt{\lambda^0_1 + a^2(1-n)^2} - |a| > \sqrt{\mu_1(D^2)} = |a| n$$

is greater than the smallest eigenvalue of the Dirac operator.

Let us summarize the result.

**Theorem 2.1.** Let $M^n \neq S^n$ be a compact Riemannian spin manifold with a Killing spinor $\psi$, $\nabla_X \psi = a \cdot X \cdot \psi$. Then the first eigenvalue of the square $D^2$ of the Dirac operator equals $\mu_1(D^2) = a^2n^2$. The numbers

$$\left( \pm \sqrt{\lambda^0_1 + a^2(1-n)^2} - |a| \right)^2$$

are eigenvalues of $D^2$, too. The second eigenvalue can be estimated by

$$a^2n^2 = \mu_1(D^2) < \mu_2(D^2) \leq \left( \sqrt{\lambda^0_1 + a^2(1-n)^2} - |a| \right)^2$$

Finally, if

$$a^2n^2 = \mu_1(D^2) < \mu(D^2) < \left( \sqrt{\lambda^0_1 + a^2(1-n)^2} - |a| \right)^2$$

is any “small” eigenvalue and $\psi^*$ the eigenspinor, then the inner product $\langle \psi , \psi^* \rangle$ vanishes identically.

Spinor fields $\psi^* = \eta \cdot \psi$ given as the Clifford product of the Killing spinor $\psi$ by a 1-form $\eta$ satisfy automatically the condition $\langle \psi , \psi^* \rangle = 0$. We compute the Dirac operator,

$$D(\psi^*) = (n-2) a \cdot \eta \cdot \psi + d\eta \cdot \psi + \delta \eta \cdot \psi.$$

A first application of the formula is the following
Proposition 2.1. If the Killing spinor $\psi$ is preserved by the Killing 1-form $\eta$, $\mathcal{L}_\eta(\psi) = 0$, then the spinor $\psi^* = \eta \cdot \psi$ is an eigenspinor,

$$D(\psi^*) = (n + 2) a \cdot \psi^* .$$

Proof. The 1-form $\eta$ is coclosed, $\delta \eta = 0$, and the formula for the Lie derivative

$$0 = \mathcal{L}_\eta(\psi) = \nabla_\eta \psi - \frac{1}{4} d\eta \cdot \psi = a \cdot \eta \cdot \psi - \frac{1}{4} d\eta \cdot \psi ,$$

see [4], yields the result. $\square$

More generally, suppose that $\psi^*$ is an eigenspinor. The eigenvalue equation $D(\psi^*) = m \cdot \psi^*$ reads as

$$\left\{ (n - 2) a - m \right\} \eta + d\eta \cdot \psi = 0 .$$

The latter equation implies that the 1-form has to be coclosed and an eigenform of the Hodge-Laplace operator $\Delta_1$. Indeed, we have

Lemma 2.1. Let $\eta$ be a 1-form and $0 \neq c$ a constant such that

$$(c \cdot \eta + d\eta) \cdot \psi = 0$$

holds. Then the 1-form is a divergence-free eigenform of $\Delta_1$,

$$\delta \eta = 0, \quad \Delta_1(\eta) = c (c - (2n - 6) a) \eta .$$

Furthermore, the conditions $\delta \eta = 0$ and $d\eta \cdot \psi = 0$ imply $\Delta_1(\eta) = 0$ (the case of $c = 0$).

Proof. Fix an orthonormal frame $e_1, \ldots, e_n$ on $M^n$. Differentiate the equation for $\eta$ again, use the Killing equation for $\psi$ and contract via the Clifford multiplication. Then we obtain

$$\sum_{i=1}^n \left\{ c e_i \cdot (\nabla_{e_i} \eta) + a \cdot c e_i \cdot \eta \cdot e_i + e_i \cdot (\nabla_{e_i} d\eta) + a e_i \cdot d\eta \cdot e_i \right\} \cdot \psi = 0 .$$

The algebraic relations in the Clifford algebra

$$\sum_{i=1}^n e_i \cdot \eta \cdot e_i = (n - 2) \eta, \quad \sum_{i=1}^n e_i \cdot d\eta \cdot e_i = (4 - n) d\eta ,$$

as well as the well known formulas

$$\delta \xi = - \sum_{i=1}^n e_i \cdot \nabla_{e_i} \xi, \quad d\xi = \sum_{i=1}^n e_i \wedge \nabla_{e_i} \xi ,$$

for any differential form $\xi$ yield

$$\sum_{i=1}^n e_i \cdot \nabla_{e_i} \eta = \delta \eta + d\eta , \quad \sum_{i=1}^n e_i \cdot \nabla_{e_i} d\eta = \delta d\eta + d d\eta .$$

Inserting the latter formulas we obtain

$$\left\{ c \delta \eta + (c + (4 - n) a) \cdot d\eta + (n - 2) a \cdot c \cdot \eta + \delta d\eta \right\} \cdot \psi = 0 .$$

We multiply by the spinor $\psi$ and obtain $c \delta \eta |\psi|^2 = 0$, i.e. the 1-form $\eta$ is coclosed, $\delta \eta = 0$. Finally we use again the equation we started with, $d\eta \cdot \psi = -c \eta \cdot \psi$. Then

$$\left\{ c (-c + (2n - 6) a) \cdot \eta + \delta d\eta \right\} \cdot \psi = 0 .$$
This is a Clifford product of a 1-form by a spinor. Consequently, the 1-form has to be trivial and the result follows.

Let us introduce the eigenvalues $0 < \Lambda_1 < \Lambda_2 < \ldots$ as numbers such that the problem
\[ \Delta_1(\eta) = \Lambda \eta, \quad \delta \eta = 0 \]
has a non-trivial solution. In general, if $\eta$ is 1-form on an $n$-dimensional manifold, then
\[ ||\nabla \eta||^2 \geq \frac{1}{2} ||d \eta||^2 + \frac{1}{n} ||\delta \eta||^2 \]
holds (see [10], page 270). Consider a coclosed eigenform $\Delta_1(\eta) = \Lambda \eta, \delta \eta = 0$ on an Einstein space. Then the latter inequality as well as the Weitzenböck formula for 1-forms imply the estimate
\[ \Lambda_1 \geq \frac{2R}{n} = 8(n-1)a^2 . \]
The existence of a non-trivial solution of the equation $(c \cdot \eta + d \eta) \cdot \psi = 0$ implies the inequality
\[ 8(n-1)a^2 \leq \Lambda_1 \leq c(c - (2n-6)a) . \]
The latter inequality estimates the absolute value of $c$,
\[ \sqrt{\Lambda_1} + a^2(n-3)^2 - (n-3)|a| \leq |c| . \]
Inserting $m = (n-2)a - c$, we obtain
\[ \sqrt{\Lambda_1} + a^2(n-3)^2 - |a| \leq |m| . \]
We summarize the result of the previous discussion.

**Theorem 2.2.** The spinor field $\psi^* = \eta \cdot \psi$ is an eigenspinor, $D(\psi^*) = m \psi^*$, if and only if
\[ \left\{ ((n-2)a - m) \eta + d \eta \right\} \cdot \psi = 0 . \]
In this case the 1-form $\eta$ is a coclosed eigenform of the Laplace operator, and the eigenvalue can be estimated by
\[ \sqrt{\Lambda_1} + a^2(n-3)^2 - |a| \leq |m| . \]
Since in dimension seven any real spinor field $\psi^*$ being orthogonal to the Killing spinor $\psi$ is defined by a unique 1-form $\eta$, we are able to estimate the second eigenvalue of $D^2$ on 7-dimensional manifolds $M^7$ with Killing spinors. However, it may happen that $M^7$ admits more than only one Killing spinor. Consequently, we have to investigate the equation $\left\{ ((n-2)a - m) \eta + d \eta \right\} \cdot \psi = 0$ for the 1-form $\eta$ in more detail.

**Example 2.1.** Let $M^5 \neq S^5$ be a Einstein-Sasaki manifold of dimension five. Then $a = \pm 1/2$ and $M^5$ admits exactly one $(\pm 1/2)$-Killing spinor. We obtain
\[ \frac{25}{4} = \mu_1(D^2) < \mu_2(D^2) \leq \left( \sqrt{\Lambda_1^0 + 4} - \frac{1}{2} \right)^2 . \]
If the dimension of the isometry group is at least two, then there exists a Killing vector field $X$ preserving the Killing spinor. The spinor field $\psi^* := X \cdot \psi$ solves the Dirac equation with eigenvalue $m = 7/2$ and we obtain the upper bound
\[ \mu_2(D^2) \leq \frac{49}{4} . \]
The first Laplace eigenvalue $\lambda_0^1 \geq 5$ has been computed for special families, see [11] and [14]. There are examples with a 2-dimensional isometry group and

$$\lambda_1^0 = \frac{33}{4}, \quad \mu_2(D^2) \leq 9.$$  

3. The Dirac spectrum of a nearly parallel $G_2$-manifold

A 7-dimensional, simply-connected Riemannian spin manifold admits a Killing spinor if and only if there is a (nearly)-parallel $G_2$-structure $\omega^3$ in the sense of [5],

$$d\omega^3 = -8 a \ast \omega^3, \quad \delta\omega^3 = 0.$$  

The 3-form $\omega^3$ is defined by the spinor in a unique way, see [9]. If $a = 0$, then $M^7$ has a parallel $G_2$-structure and compact examples are known, see [13]. If $a \neq 0$ and $M^7 \neq S^7$, then there are three types. Indeed, denote by $m_a$ the dimension of the space of all Killing spinors. Then $1 \leq m_a \leq 3$, $m_a = 0$ and $M^7$ is either a 3-Sasakian manifold ($m_a = 3$), a Sasaki-Einstein manifold ($m_a = 2$) or a proper nearly parallel manifold ($m_a = 1$), see [7], [8]. Compact examples for any type are known, see [3], [9].

The first positive eigenvalue of the Laplace operator is bounded by ($a \neq 0$)

$$\frac{R}{6} = 28 a^2 \leq \lambda_1^0 < \lambda_2^0 < \ldots$$  

and equality occurs if and only of $M^7$ is isometric to the sphere $S^7$ (Lichnerowicz-Obata theorem). The further invariants we need are the numbers

$$48 a^2 \leq \Lambda_1 \leq \lambda_{1,+}^1 < \lambda_{2,+}^1 < \ldots$$  

such that there exists a 1-form $\eta$ with

$$\Delta_1(\eta) = \lambda \eta, \quad \delta \eta = 0, \quad (4 a \pm \sqrt{16 a^2 + \lambda}) \eta = -\ast(d\eta \ast \omega^3).$$  

This set is contained in the spectrum of the Laplace operator on 1-forms. Moreover, we have $\Lambda_i \leq \lambda_{i,\pm}^1$.

**Remark 3.1.** The equation ($4 a \pm \sqrt{16 a^2 + \lambda}$) $d \ast \eta = -\ast(d\eta \ast \omega^3)$ implies $\delta \eta = 0$ and $\Delta_1(\eta) = \lambda \eta$. Indeed, since $d \ast \omega^3 = 0$ we obtain

$$(4 a \pm \sqrt{16 a^2 + \lambda}) d \ast \eta = -d \ast(d\eta \ast \omega^3) = 0.$$  

If $a = 0$ and $\lambda_{1,\pm}^1 = 0$, then the Ricci flat manifold $M^7$ admits a parallel vector field, in particular $b_1(M^7) > 0$ holds. Consequently, for simply-connected, compact and parallel $G_2$-manifolds, the numbers $\lambda_{i,\pm}^1 > 0$ are positive.

We formulate the main result of the section.

**Theorem 3.1.** Let $(M^7, g)$ be a compact Riemannian spin manifold with a Killing spinor. Then the spectrum of the Dirac operator consist of $(-7 a)$ and the following sequences:

$$-a \pm \sqrt{36 a^2 + \lambda_0^0}, \quad \lambda_i^0 \in \text{Spec}(\Delta_0).$$

$$a - \sqrt{16 a^2 + \lambda_{i,+}^1}, \quad a + \sqrt{16 a^2 + \lambda_{i,-}^1}, \quad i = 1, 2, \ldots.$$  

The formulas simplify in case of a parallel $G_2$-structure ($a = 0$).
Proof. The 7-dimensional spin representation is real. Therefore, we consider real spinor fields. The Killing spinor $\psi$ has constant length one,
\[ \nabla_X \psi = a \cdot X \cdot \psi, \quad D(\psi) = -7a \cdot \psi, \quad a \in \mathbb{R}^1. \]
It defines in a unique way a generic 3-form $\omega^3$ and the Killing equation as well as the link between $\psi$ and $\omega^3$ reads as
\[ d\omega^3 = -8a^* \omega^3, \quad \delta\omega^3 = 0, \quad \omega^3 \cdot \psi = -7 \cdot \psi, \]
for details see [9]. $M^7$ is an Einstein space,
\[ \text{Ric} = 24 \cdot a^2 \text{Id}, \quad R = 7 \cdot 24 \cdot a^2. \]
A purely algebraic computation in the 7-dimensional spin representation yields the following result.

**Lemma 3.1.** Let $\eta$ be a 1-form, $\sigma$ a 2-form and $c \in \mathbb{R}^1$ a real number. Then
\[ (\eta + \sigma + c) \cdot \psi = 0 \]
is equivalent to
\[ \eta = -\ast (\sigma \wedge \ast \omega^3) \quad \text{and} \quad c = 0. \]
Any real spinor field $\psi^*$ is given by a pair $(f, \eta)$ of a real-valued function $f$ and a 1-form $\eta$,
\[ \psi^* = f \cdot \psi + \eta \cdot \psi. \]
The eigenvalue equation $D(\psi^*) = m \psi^*$ is equivalent to
\[ \left( (-7af - mf + \delta\eta) + (df + (5a - m)\eta) + d\eta \right) \cdot \psi = 0. \]
By the algebraic Lemma 3.1, the Dirac equation reads now as
\[ (7a + m)f = \delta\eta, \quad df + (5a - m)\eta = -\ast (d\eta \wedge \ast \omega^3). \]
Moreover, we know that $m^2 \geq 49a^2$ holds, see [6]. We differentiate this system of first order partial differential equations again and we obtain the necessary condition
\[ \Delta_0(f) = (m - 5a)(7a + m)f, \]
i.e. the function $f$ is an eigenfunction of the Laplace operator.

Let us first discuss the case $m = -7a$. Then $f$ is constant and the eigenspace $E_{-7a}(D)$ of the Dirac operator coincides with the space of all Killing spinors with Killing number $a$, see [6]. This space becomes isomorphic to
\[ E_{-7a}(D) = \mathbb{R}^1 \cdot \psi \oplus \{ \eta : 12a \eta = -\ast (d\eta \wedge \ast \omega^3) \quad \text{and} \quad \delta\eta = 0 \}. \]
One needs the second equation only if $a = 0$.

Suppose from now on that $m \neq -7a$. Since $m^2 \geq 49a^2$ we have $(m - 5a)(7a + m) \neq 0$, too. The function $f$ is either non-trivial or $f \equiv 0$. If $f \neq 0$ then
\[ (m - 5a)(7a + m) = \lambda^0_i \in \text{Spec}(\Delta_0) \]
is a positive eigenvalue of the Laplace operator and
\[ m = -a \pm \sqrt{36a^2 + \lambda^0_i}. \]
Conversely, given a non-trivial function with \( \Delta_0(f) = \lambda_0^0 f \) and \( (m - 5a)(7a + m) = \lambda_0^0 \), then the pair

\[
(f, \eta) := (f, \frac{1}{m - 5a} df)
\]
defines a solution of the Dirac equation \( D(\psi^*) = m \psi^* \). Any other solution with the same function \( f \) is given by a 1-form \( \eta_1 \) being a solution of the system

\[
\delta \eta_1 = 0, \quad (5a - m)\eta_1 = -*(d\eta_1 \wedge \omega^3).
\]
The latter equations describe the solutions of the Dirac equation for \( f \equiv 0 \).

Let us summarize the result. The eigenspace \( E_m(D) \) consist of all pairs

\[
(f, \frac{1}{m - 5a} df + \eta)
\]
where \( f \equiv 0 \) or \( f \) is an eigenfunction of the Laplace operator

\[
\Delta_0(f) = (m - 5a)(7a + m)f = \lambda_0^0 f, \quad m = -a \pm \sqrt{36a^2 + \lambda_0^0}.
\]
and \( \eta \) is a special 1-eigenform,

\[
(5a - m)\eta = -*(d\eta \wedge \omega^3), \quad \Delta_1(\eta) = (3a + m)(m - 5a)\eta = \lambda_1^1 \eta, \quad m = a \pm \sqrt{16a^2 + \lambda_1^1}.
\]

**Corollary 3.1.** Let \((M^7, g)\) be a compact Riemannian spin manifold with a parallel spinor \( \psi \) and denote by \( \omega^3 \) the associated parallel \( G_2 \)-structure. Then \( \mu_1(D^2) = 0 \) and the second eigenvalues of the square of the Dirac operator is given by

\[
\mu_2(D^2) = \min(\lambda_1^0, \lambda_1^{1, +}, \lambda_1^{1, -}).
\]
\( \lambda_1^0 > 0 \) is the first positive eigenvalue of the Laplace operator on functions and \( \lambda_1^{1, \pm} \) are the first positive numbers such that

\[
\pm \sqrt{\lambda} \cdot \eta = -*(d\eta \wedge \omega^3)
\]

admits a non-trivial solution.

**Corollary 3.2.** If \( M^7 \) admits exactly one Killing spinor \( (m_a = 1, a > 0) \), then \( \lambda_1^0 > 28a^2 \), \( \lambda_1^{1, +} > 48a^2 \), and

\[
\mu_2(D^2) = \min\left(\left(\sqrt{36a^2 + \lambda_1^0} - a\right)^2, \left(\sqrt{16a^2 + \lambda_1^{1, +}} - a\right)^2, \left(\sqrt{16a^2 + \lambda_1^{1, -}} + a\right)^2\right)
\]

Moreover, we have \( \sqrt{36a^2 + \lambda_1^0} + a \geq 9a \) and \( -a - \sqrt{36a^2 + \lambda_1^0} \leq -9a \).

**Corollary 3.3.** If \( M^7 \) admits exactly one Killing spinor \( (m_a = 1, a > 0) \) and at least one non-trivial Killing vector field, then \( \lambda_1^0 > 28a^2 \), \( \lambda_1^{1, +} > 48a^2 \), \( \lambda_1^{1, -} = 48a^2 \). In particular,

\[
\mu_2(D^2) = \min\left(\left(\sqrt{36a^2 + \lambda_1^0} - a\right)^2, \left(\sqrt{16a^2 + \lambda_1^{1, +}} - a\right)^2, 81a^2\right).
\]

**Proof.** Denote by \( X \) the Killing vector field. Since the Killing spinor \( \psi \) is unique, we obtain

\[
0 = \mathcal{L}_X \psi = \nabla_X \psi - \frac{1}{4} dX \cdot \psi = a X \cdot \psi - \frac{1}{4} dX \cdot \psi,
\]
(see [4]). Then the spinor field \( \psi^* := X \cdot \psi \) is an eigenspinor, \( D^2(\psi^*) = 81a^2 \psi^* \). \( \square \)
**Example 3.1.** \(\text{SO}(5)/\text{SO}_r(3), \ N(k,l) = \text{SU}(3)/S^1_{k,l}\) and deformations of 3-Sasakian manifolds are examples with exactly one Killing spinor, see [9].

**Corollary 3.4.** If \(M^7 \neq S^7\) admits at least two Killing spinors, then \(\lambda_1^0 > 28a^2, \ \lambda_{1,2}^1 = 48a^2\) and

\[
\mu_2(D^2) = \min\left(\left(\sqrt{36a^2 + \lambda_1^0} - a\right)^2, \left(\sqrt{16a^2 + \lambda_{2,1}^1} - a\right)^2, \left(\sqrt{16a^2 + \lambda_{1,-1}^1} + a\right)^2\right)
\]

**Example 3.2.** Two or three Killing spinors occur if \(M^7\) is a Sasaki-Einstein or a 3-Sasaki manifold, see [8], [9].

### 4. The 7-dimensional Sasaki-Einstein case

A simply-connected Sasaki-Einstein manifold \(M^7 \neq S^7\) admits at least two Killing spinors with Killing number \(a = 1/2\), see [8]. The scalar curvature equals \(R = 42\) and \(\lambda_1^0 > 7\). The second Killing spinor \(\psi^* = \eta \cdot \psi\) is given by a 1-form \(\eta\) satisfying the equation

\[
6\eta = - \ast (d\eta \wedge \ast \omega^3)
\]

and we obtain

\[
\lambda_{1,+}^1 = 12, \ \lambda_{1,-}^1 \geq 12.
\]

If \(\lambda_{1,-}^1 = 12\), then the corresponding eigenform \(\eta_1\) is a solution of the Laplace equation

\[
\Delta_1(\eta_1) = 12\eta_1 = 2 \text{Ric}(\eta_1), \ \delta\eta_1 = 0,
\]

i.e. a Killing vector field (see [16]). Consequently, if the isometry group of the Sasaki-Einstein manifold \(M^7\) is one-dimensional, we have \(\lambda_{1,-}^1 > 12\). If the dimension of the isometry group is at least two, then there exists a Killing vector field \(X\) preserving the Killing spinor. Then \(\lambda_{1,-}^1 = 12\) and and we obtain the following

**Theorem 4.1.** Let \(M^7 \neq S^7\) be a compact and simply-connected Sasaki-Einstein manifold and suppose that the dimension of the isometry group is at least two. Then the second eigenvalue \(\mu_2(D^2)\) is given by

\[
\frac{49}{4} = \mu_1(D^2) < \mu_2(D^2) = \min\left(\left(\sqrt{9 + \lambda_1^0} - \frac{1}{2}\right)^2, \left(\sqrt{4 + \lambda_{2,1}^1} - \frac{1}{2}\right)^2, \left(\sqrt{81} - \frac{1}{2}\right)^2\right).
\]

**Example 4.1.** Let us discuss the case of 3-Sasakian manifolds (see [12], [3]). The isometry group is at least 3-dimensional. Moreover, there exists a spinor field \(\psi_0 = \eta_1 \cdot \psi\) such that

\[
D(\psi_0) = \frac{9}{2} \psi_0, \ \ |\psi_0| \equiv 1
\]

holds, see [1] and [17]. This spinor satisfies even a stronger equation, namely

\[
\nabla_X \psi_0 = \frac{1}{2}X \cdot \psi_0 \quad \text{if} \quad X \in T^v, \quad \nabla_X \psi_0 = -\frac{3}{2}X \cdot \psi_0 \quad \text{if} \quad X \in T^h.
\]

The numbers \(\lambda_{1,+}^1 = \lambda_{1,-}^1 = 12\) coincide. The solutions of the equations

\[
-2\eta = - \ast (d\eta \wedge \ast \omega^3), \quad \text{and} \quad 6\eta = - \ast (d\eta \wedge \ast \omega^3)
\]
can be seen directly. Indeed, consider the three contact structures $\eta_1, \eta_2, \eta_3$ of the 3-Sasakian manifold. Then

\begin{align*}
    d\eta_1 &= -2(\eta_{23} + \eta_{45} + \eta_{67}), \\
    d\eta_2 &= 2(\eta_{13} - \eta_{46} + \eta_{57}), \\
    d\eta_3 &= -2(\eta_{12} + \eta_{47} + \eta_{56}).
\end{align*}

and

\begin{align*}
    \omega^3 &:= \frac{1}{2}(\eta_1 \wedge d\eta_1 - \eta_2 \wedge d\eta_2 - \eta_3 \wedge d\eta_3) \\
    *\omega^3 &= -\frac{1}{8}(d\eta_1 \wedge d\eta_1 - d\eta_2 \wedge d\eta_2 - d\eta_3 \wedge d\eta_3)
\end{align*}

is one of the associated nearly parallel $G_2$-structures (see [1]). A purely algebraic computation yields the relations

\begin{align*}
    -2\eta_1 &= -*(d\eta_1 \wedge *\omega^3), & 6\eta_2 &= -* (d\eta_2 \wedge *\omega^3), & 6\eta_3 &= -* (d\eta_3 \wedge *\omega^3).
\end{align*}

Let us consider the regular case. The contact structure induces a SO(2)-action and the orbit space $X^6 := M^7/\text{SO}(2)$ is a 6-dimensional Kähler-Einstein orbifold with scalar curvature $\bar{R} = 48$, see [8], [3]. The projection $\pi : M^7 \to X^6$ is a Riemannian submersion with a totally geodesic fiber and commutes with the Laplacian $\Delta_0$ on functions. Suppose that there exists an invariant eigenfunction. It projects to an eigenfunction of $\Delta_0$ on the orbifold $X^6$. Consequently, the corresponding eigenvalue (not the multiplicity) of $M^7$ and $X^6$ coincides,

$$\lambda_0^0(M^7) = \lambda_0^0(X^6).$$

In this case we can apply an estimate proved by A. Lichnerowicz for $\lambda_0^0$ of a smooth Kähler-Einstein manifold (see [2], page 84)

$$\lambda_0^0(X^6) \geq \frac{\bar{R}}{3} = 16.$$ 

Consequently, the general estimate of $\mu_2(D^2)$ simplifies.

**Theorem 4.2.** Let $M^7 \neq S^7$ be a compact and simply-connected Sasaki-Einstein manifold. Suppose that the dimension of the isometry group is at least two, that there exists a SO(2)-invariant eigenfunction with eigenvalue $\lambda_0^0$ and suppose that $X^6 = M^7/\text{SO}(2)$ is smooth. Then the second eigenvalue $\mu_2(D^2)$ is given by

$$\frac{49}{4} = \mu_1(D^2) < \mu_2(D^2) = \min\left(\sqrt{\frac{4}{\lambda_2^1} + \lambda_2^1} - \frac{1}{2}, \frac{81}{4}\right).$$

**Remark 4.1.** The assumptions are satisfied if $M^7/\text{SO}(2)$ is smooth and there exists a subgroup $\text{SO}(2) \subset G \subset \text{Iso}(M^7)$ being isomorphic to $G = \text{SO}(3), \text{Spin}(3)$. Indeed, any irreducible, real G-representation admits a SO(2)-invariant vector.

**References**

[1] I. Agricola and Th. Friedrich, 3-Sasakian manifolds in dimension seven, their spinors and $G_2$-structures, J. Geom. Phys. 60 (2010), 326-332.

[2] W. Ballmann, Lectures on Kähler manifolds, ESI Lectures, EMS Publishing House 2006.

[3] C. Boyer and K. Galicki, Sasakian Geometry, Oxford Mathematical Monographs, Oxford Univ. Press, 2008.
10 J.-P. Bourguignon and P. Gauduchon, Spineurs, Operateurs de Dirac et Variation de Metriques, Comm. Math. Phys. 144 (1992), 581-599.

[5] M. Fernandez and A. Gray, Riemannian manifolds with structure group $G_2$, Annali di Math. Pura e Appl. 132 (1982), 19-45.

[6] Th. Friedrich, Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung, Math. Nachr. 97 (1980), 117-146.

[7] Th. Friedrich and I. Kath, Varieties riemanniennes compactes de dimension 7 admettant des spineurs de Killing, C.R. Acad. Sci Paris 307 Serie I (1988), 967-969.

[8] Th. Friedrich and I. Kath, Compact seven-dimensional manifolds with Killing spinors, Comm. Math. Phys. 133 (1990), 543-561.

[9] Th. Friedrich, I. Kath, A. Moroianu, and U. Semmelmann, On nearly parallel $G_2$-structures, J. Geom. Phys. 23 (1997), 256-286.

[10] S. Gallot and D. Meyer, Operateur de courbure et Laplacian de formes differentielles d’une variete Riemanienne, J. Math. Pures Appl. 54, (1975), 259 - 284.

[11] G.W. Gibbons, S.A. Hartnoll and Y. Yasui, Properties of some five dimensional Einstein metrics, Class. Quant. Grav. 21 (2004), 4697, hep-th/0407030v2.

[12] S. Ishihara and M. Konish, Differential geometry of fibred spaces , Kyoto 1973.

[13] D. Joyce, Compact manifolds with special holonomy, Oxford University Press, 2000.

[14] H. Kihara, M. Sakaguchi, and Y. Yasui, Scalar Laplacian on Sasaki-Einstein manifolds $Y^{p,q}$, Phys. Lett. B 621 (2005), 288-294; hep-th/0505259

[15] E.C. Kim, Estimates of small Dirac eigenvalues on 3-dimensional Sasakian manifolds, Diff. Geom. Appl. 28 (2010), 648 - 655.

[16] S. Kobayashi, Transformation groups in differential geometry, Springer-Verlag 1972.

[17] A. Moroianu, Sur les valeurs propres de l’opérateur de Dirac d’une variété spinorielle simplement connexe admettant une 3-structure de Sasaki, Stud. Cerc. Mat. 48 (1996), 85-88.

[18] U. Simon, Curvature bounds for the spectrum of a closed Einstein space, Can. J. Math. 30 (1978), 1087-1091.

[19] S. Tanno, On a lower bound of the second eigenvalue of the Laplacian on an Einstein space, Coll. Math. 39 (1978), 285-288.

THOMAS FRIEDRICH
INSTITUT FÜR MATHEMATIK
HUMBOLDT-UNIVERSITÄT ZU BERLIN
SITZ: WBC ADLERSHOF
D-10099 BERLIN, GERMANY
friedric@mathematik.hu-berlin.de