A consistent, local coordinate formulation of covariant Hamiltonian field theory is presented. Whereas the covariant canonical field equations are equivalent to the Euler-Lagrange field equations, the covariant canonical transformation theory offers more general means for defining mappings that preserve the form of the field equations than the usual Lagrangian description. It is proved that Poisson brackets, Lagrange brackets, and canonical 2-forms exist that are invariant under canonical transformations of the fields. The technique to derive transformation rules for the fields from generating functions is demonstrated by means of various examples. In particular, it is shown that the infinitesimal canonical transformation furnishes the most general form of Noether’s theorem. We furthermore specify the generating function of an infinitesimal space-time step that conforms to the field equations.

Keywords: Field theory; Hamiltonian density; covariant.

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1. Introduction

Relativistic field theories and gauge theories are commonly formulated on the basis of a Lagrangian density $L[1234]$. The space-time evolution of the fields is obtained by integrating the Euler-Lagrange field equations that follow from the four-dimensional representation of Hamilton’s action principle. A characteristic feature of this approach is that the four independent variables of space and time are treated on equal footing, which automatically ensures the description to be relativistically correct. This is reflected by the fact that the Lagrangian density $L$ depends — apart from a possible explicit dependence on the four space-time coordinates $x^\mu$ — on the set of fields $\phi_I$ and evenly on all four derivatives $\partial_\mu \phi_I$ of those fields with
respect to the space-time coordinates, i.e. \( \mathcal{L} = \mathcal{L}(\phi_I, \partial_\mu \phi_I, x^\mu) \). Herein, the index "I" enumerates the individual fields that are involved in the given physical system.

When the transition to a Hamiltonian description is made in textbooks, the equal footing of the space-time coordinates is abandoned\(^{5,11,2}\). In these presentations, the Hamiltonian density \( \mathcal{H} \) is defined to depend on the set of scalar fields \( \phi_I \) and on one set of conjugate scalar fields \( \pi_I \) that counterpart the time derivatives of the \( \phi_I \). Keeping the dependencies on the three spatial derivatives \( \partial_i \phi_I \) of the fields \( \phi_I \), the functional dependence of the Hamiltonian is then defined as \( \mathcal{H} = \mathcal{H}(\phi_I, \pi_I, \partial_\mu \phi_I, x^\mu) \). The canonical field equations then emerge as time derivatives of the scalar fields \( \phi_I \) and \( \pi_I \). In other words, the time variable is singled out of the set of independent space-time variables. While this formulation is doubtlessly valid and obviously works for the purpose pursued in these presentations, it closes the door to a full-fledged Hamiltonian field theory. In particular, it appears to be impossible to formulate a theory of canonical transformations on the basis of this particular definition of a Hamiltonian density.

On the other hand, numerous papers were published that formulate a covariant Hamiltonian description of field theories where — similar to the Lagrangian formalism — the four independent variables of space-time are treated equally. These papers are generally based on the pioneering works of De Donder\(^6\) and Weyl\(^7\). The key point of this approach is that the Hamiltonian density \( \mathcal{H} \) is now defined to depend on a set of conjugate 4-vector fields \( \pi_I^\mu \) that counterbalance the four derivatives \( \partial_\mu \phi_I \) of the Lagrangian density \( \mathcal{L} \), so that \( \mathcal{H} = \mathcal{H}(\phi_I, \pi_I^\mu, x^\mu) \). Corresponding to the Euler-Lagrange equations of field theory, the canonical field equations then take on a symmetric form with respect to the four independent variables of space-time. This approach is commonly referred to as “multisymplectic” or “polysymplectic field theory”, thereby labelling the covariant extension of the symplectic geometry of the conventional Hamiltonian theory\(^8\). Mathematically, the phase space of multisymplectic Hamiltonian field theory is defined within modern differential geometry in the language of “jet bundles”\(^16,17\).

Obviously, this theory has not yet found its way into mainstream textbooks. One reason for this is that the differential geometry approach to covariant Hamiltonian field theory is far from being straightforward and raises mathematical issues that are not yet clarified (see, for instance, the discussion in Ref.\(^12\)). Furthermore, the approach is obviously not unique — there exist various options to define geometric objects such as Poisson brackets\(^8\). As a consequence, any discussion of the matter is unavoidably shifted into the realm of mathematics.

With the present paper, we do not pursue the differential geometry path but provide a local coordinate treatise of De Donder and Weyl’s covariant Hamiltonian field theory. The local description enables us to keep the mathematics on the level of tensor calculus. Nevertheless, the description is chart-independent and thus applies to all local coordinate systems. With this property, our description is sufficiently general from the point of view of physics. Similar to textbooks on Lagrangian gauge
theories, we maintain a close tie to physics throughout the paper.

Our paper is organized as follows. In Sec. 2 we give a brief review of De Donder and Weyl’s approach to Hamiltonian field theory in order to render the paper self-contained and to clarify notation. After reviewing the covariant canonical field equations, we evince the Hamiltonian density $\mathcal{H}$ to represent the eigenvalue of the energy-momentum tensor and discuss the non-uniqueness of the field vector $\pi^I\mu$.

The main benefit of the covariant Hamiltonian approach is that it enables us to formulate a consistent theory of covariant canonical transformations. This is demonstrated in Sec. 3. Strictly imitating the point mechanics’ approach\(^{18,19}\), we set up the transformation rules on the basis of a generating function by requiring the variational principle to be maintained\(^{20}\). In contrast to point mechanics, the generating function $F_1^\mu$ now emerges in our approach as a 4-vector function. We recover a characteristic feature of canonical transformations by deriving the symmetry relations of the mutual partial derivatives of original and transformed fields. By means of covariant Legendre transformations, we show that equivalent transformation rules are obtained from generating functions $F_2^\mu$, $F_3^\mu$, and $F_4^\mu$. Very importantly, each of these generating functions gives rise to a specific set of symmetry relations of original and transformed fields.

The symmetry relations set the stage for proving that 4-vectors of Poisson and Lagrange brackets exist that are invariant with respect to canonical transformations of the fields. We furthermore show that each vector component of our definition of a $(1,2)$-tensor, i.e. a “4-vector of 2-forms $\omega^{\mu\nu}$ is invariant under canonical transformations — which establishes Liouville’s theorem of canonical field theory. We conclude this section deriving the field theory versions of the Jacobi identity, Poisson’s theorem, and the Hamilton-Jacobi equation. Similar to point mechanics, the action function $S^\mu$ of the Hamilton-Jacobi equation is shown to represent a generating function $S^\mu \equiv F_2^\mu$ that is associated with the particular canonical transformation that maps the given Hamiltonian into an identically vanishing Hamiltonian.

In Sec. 4 examples of Hamiltonian densities are reviewed and their pertaining field equations are derived. As the relativistic invariance of the resulting fields equations is ensured if the Hamiltonian density $\mathcal{H}$ is a Lorentz scalar, various equations of relativistic quantum field theory are demonstrated to embody, in fact, canonical field equations. In particular, the Hamiltonian density engendering the Klein-Gordon equation manifests itself as the covariant field theory analog of the harmonic oscillator Hamiltonian of point mechanics.

Section 5 starts sketching simple examples of canonical transformations of Hamiltonian systems. Similar to the case of classical point mechanics, the main advantage of the Hamiltonian over the Lagrangian description is that the canonical transformation approach is not restricted to the class of point transformations, i.e., to cases where the transformed fields $\phi_I'$ only depend on the original fields $\phi_I$. The most general formulation of Noether’s theorem is, therefore, obtained from a general infinitesimal canonical transformation. As an application of this theorem, we show that an invariance with respect to a shift in a space-time coordinate leads
to a corresponding conserved current that is given by the pertaining column vector of the energy-momentum tensor.

By specifying its generating function, we furthermore show that an infinitesimal step in space-time which conforms to the canonical field equations itself establishes a canonical transformation. Similar to the corresponding time-step transformation of point mechanics, the generating function is mainly determined by the system’s Hamiltonian density. It is precisely this canonical transformation which ensures that a Hamiltonian system remains a Hamiltonian system in the course of its space-time evolution. The existence of this canonical transformation is thus crucial for the entire approach to be consistent.

As canonical transformations establish mappings of one physical system into another, canonically equivalent system, it is remarkable that Higg’s mechanism of spontaneous symmetry breaking can be formulated in terms of a canonical transformation. This is shown in Sec. 5.8 We close our treatise with a discussion of the generating function of a non-Abelian gauge transformation. With Appendix B we add an excursion to differential geometry by providing a geometrical representation of the canonical field equations.

2. Covariant Hamiltonian density

2.1. Covariant canonical field equations

The transition from particle dynamics to the dynamics of a continuous system is based on the assumption that a continuum limit exists for the given physical problem. This limit is defined by letting the number of particles involved in the system increase over all bounds while letting their masses and distances go to zero.

In this limit, the information on the location of individual particles is replaced by the value of a smooth function \( \phi(x^\mu) \) that is given at a spatial location \( x^1, x^2, x^3 \) at time \( t \equiv x^0 \). The differentiable function \( \phi(x^\mu) \) is called a field. In this notation, the index \( \mu \) runs from 0 to 3, hence distinguishes the four independent variables of space-time \( x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z) \), and \( \dot{x}_\mu \equiv (\dot{x}_0, \dot{x}_1, \dot{x}_2, \dot{x}_3) \equiv (c\dot{t}, \dot{x}, \dot{y}, \dot{z}) \). We furthermore assume that the given physical problem can be described in terms of \( I = 1, \ldots, N \) — possibly interacting — scalar fields \( \phi_I(x^\mu) \), with the index “\( I \)” enumerating the individual fields. In order to clearly distinguish scalar quantities from vector quantities, we denote the latter with boldface letters. Throughout our paper, the summation convention is used. This means that whenever a pair of the same upper and lower indices appears on one side of an equation, this index is to be summed over. If no confusion can arise, we omit the indices in the argument list of functions in order to avoid the number of indices to proliferate.

The Lagrangian description of the dynamics of a continuous system is based on the Lagrangian density function \( \mathcal{L} \) that is supposed to carry the complete information on the given physical system. In a first-order field theory, the Lagrangian density \( \mathcal{L} \) is defined to depend on the \( \phi_I \), possibly on the vector of independent variables \( x^\mu \), and on the four first derivatives of the fields \( \phi_I \) with respect to the
independent variables, i.e., on the 1-forms
\[ \partial_\mu \phi_I \equiv (\partial_t \phi_I, \partial_x \phi_I, \partial_y \phi_I, \partial_z \phi_I). \]

The Euler-Lagrange field equations are then obtained as the zero of the variation \( \delta S \) of the action integral
\[ S = \int L(\phi_I, \partial_\mu \phi_I, x^\mu) \, dx \]  

as
\[ \frac{\partial}{\partial x^\alpha} \frac{\partial L}{\partial (\partial_\alpha \phi_I)} - \frac{\partial L}{\partial \phi_I} = 0. \]  

To derive the equivalent covariant Hamiltonian description of continuum dynamics, we first define for each field \( \phi_I(x^\nu) \) a 4-vector of conjugate momentum fields \( \pi^{I\mu}(x^\nu) \). Its components are given by
\[ \pi^{I\mu} = \frac{\partial L}{\partial (\partial_\mu \phi_I)} \equiv \frac{\partial L}{\partial \left( \partial_{\alpha} \phi_I \right)} \]  

The 4-vector \( \pi^{I\mu} \) is thus induced by the Lagrangian \( L \) as the dual counterpart of the 1-form \( \partial_\mu \phi_I \). For the entire set of \( N \) scalar fields \( \phi_I(x^\nu) \), this establishes a set of \( N \) conjugate 4-vector fields. With this definition of the 4-vectors of canonical momenta \( \pi^{I\mu}(x^\nu) \), we can now define the Hamiltonian density \( H(\phi_I, \pi^{I\mu}, x^\mu) \) as the covariant Legendre transform of the Lagrangian density \( L(\phi_I, \partial_\mu \phi_I, x^\mu) \)
\[ H(\phi_I, \pi^{I\mu}, x^\mu) = \pi^{I\alpha} \frac{\partial \phi_I}{\partial x^\alpha} - L(\phi_I, \partial_\mu \phi_I, x^\mu). \]  

At this point we suppose that \( L \) is regular, hence that for each index \( I \) the Hesse matrices \( (\partial^2 L / \partial (\partial_\mu \phi_I) \partial (\partial_\nu \phi_I)) \) are non-singular. This ensures that \( H \) takes over the complete information on the given dynamical system from \( L \) by means of the Legendre transformation. The definition of \( H \) by Eq. (4) is referred to in literature as the “De Donder-Weyl” Hamiltonian density.

Obviously, the dependencies of \( H \) and \( L \) on the \( \phi_I \) and the \( x^\mu \) only differ by a sign,
\[ \frac{\partial H}{\partial \phi_I} = - \frac{\partial L}{\partial \phi_I}, \quad \frac{\partial H}{\partial x^\mu} |_{\text{expl}} = - \frac{\partial L}{\partial x^\mu} |_{\text{expl}}. \]  

These variables do not take part in the Legendre transformation of Eqs. (3), (4).

With regard to this transformation, the Hamiltonian density \( H \) is, therefore, to be considered as a function of the \( \pi^{I\mu} \) only, and, correspondingly, the Lagrangian density \( L \) as a function of the \( \partial_\mu \phi_I \) only. In order to derive the canonical field equations, we calculate from Eq. (4) the partial derivative of \( H \) with respect to \( \pi^{I\mu} \),
\[ \frac{\partial H}{\partial \pi^{I\mu}} = \delta^I_J \delta_\mu^\alpha \frac{\partial \phi_J}{\partial x^\alpha} + \pi^{J\alpha} \frac{\partial (\partial_\alpha \phi_J)}{\partial \pi^{I\mu}} - \frac{\partial L}{\partial (\partial_\alpha \phi_I)} \frac{\partial (\partial_\alpha \phi_J)}{\partial \pi^{I\mu}} = \frac{\partial \phi_I}{\partial x^\mu}. \]
According to the definition of $\pi^{\mu I}$ in Eq. (3), the second and the third terms on the right hand side cancel. In conjunction with the Euler-Lagrange equation, we obtain the set of covariant canonical field equations finally as
\begin{equation}
\frac{\partial H}{\partial \pi^{\mu I}} = \frac{\partial \phi_I}{\partial x^\mu}, \quad \frac{\partial H}{\partial \phi_I} = -\frac{\partial \pi^{\mu I}}{\partial x^\alpha} \delta_{\alpha I}.
\end{equation}
This pair of first-order partial differential equations is equivalent to the set of second-order differential equations of Eq. (2). We observe that in this formulation of the canonical field equations all coordinates of space-time appear symmetrically — similar to the Lagrangian formulation of Eq. (2). Provided that the Lagrangian density $L$ is a Lorentz scalar, the dynamics of the fields is invariant with respect to Lorentz transformations. The covariant Legendre transformation (4) passes this property to the Hamiltonian density $H$. It thus ensures a priori the relativistic invariance of the fields that emerge as integrals of the canonical field equations if $L$ — and hence $H$ — represents a Lorentz scalar.

### 2.2. Energy-Momentum Tensor

In the Lagrangian description, the energy-momentum tensor $T^\nu_{\mu}$ is defined as the following mixed second rank tensor
\begin{equation}
T^\nu_{\mu} = \frac{\partial L}{\partial (\partial_{\nu} \phi_I)} \frac{\partial \phi_I}{\partial x^\mu} - L \delta^\nu_{\mu}.
\end{equation}

With the definition (3) of the conjugate momentum fields $\pi^{\mu I}$, and the Hamiltonian density of Eq. (4), the energy-momentum tensor (6) is equivalently expressed as
\begin{equation}
T^\nu_{\mu} = H \delta^\nu_{\mu} + \pi^{1 \nu} \frac{\partial \phi_1}{\partial x^\mu} - \delta^\nu_{\mu} \pi^{1 \alpha} \frac{\partial \phi_1}{\partial x^\alpha}.
\end{equation}

The inner product of the mixed tensors $T^\nu_{\mu}$ of Eq. (7) with the 1-form $\partial_{\nu} \phi_I$ yields
\begin{equation}
T^\nu_{\mu} \frac{\partial \phi_I}{\partial x^\nu} = H \delta^\nu_{\mu} \frac{\partial \phi_I}{\partial x^\nu} + \pi^{1 \nu} \frac{\partial \phi_1}{\partial x^\nu} \frac{\partial \phi_I}{\partial x^\mu} - \pi^{1 \alpha} \frac{\partial \phi_1}{\partial x^\alpha} \delta^\nu_{\mu} \frac{\partial \phi_I}{\partial x^\alpha},
\end{equation}

hence
\begin{equation}
(T^\nu_{\mu} - H \delta^\nu_{\mu}) \frac{\partial \phi_I}{\partial x^\nu} = 0.
\end{equation}

We can similarly set up the inner product of $T^\nu_{\mu}$ with the vector $\pi^{1 \mu}$
\begin{equation}
T^\nu_{\mu} \pi^{1 \mu} = H \delta^\nu_{\mu} \pi^{1 \mu} + \pi^{1 \nu} \pi^{1 \mu} \frac{\partial \phi_1}{\partial x^\mu} - \delta^\nu_{\mu} \pi^{1 \alpha} \pi^{1 \mu} \frac{\partial \phi_1}{\partial x^\alpha},
\end{equation}

hence
\begin{equation}
(T^\nu_{\mu} - H \delta^\nu_{\mu}) \pi^{1 \mu} = 0.
\end{equation}

This shows that the De Donder-Weyl Hamiltonian density $H$ constitutes the eigenvalue of the energy-momentum tensor $T^\nu_{\mu}$ with eigenvectors $\partial_{\nu} \phi_I$ and $\pi^{1 \mu}$. By identifying $H$ as the eigenvalue of the energy-momentum tensor, we obtain a clear
interpretation of the physical meaning of the De Donder-Weyl Hamiltonian density $\mathcal{H}$.

An important property of the energy-momentum tensor is revealed by calculating the divergence $\partial T_{\alpha}^\mu / \partial x^\alpha$. From the definition (7), we find

$$
\begin{align*}
\frac{\partial T_{\alpha}^\mu}{\partial x^\alpha} &= \delta_\alpha^\mu \left( \frac{\partial \mathcal{H}}{\partial \phi_I} \frac{\partial \phi_I}{\partial x^\alpha} + \frac{\partial \mathcal{H}}{\partial \pi_I^{\alpha\beta}} \frac{\partial \pi_I^{\alpha\beta}}{\partial x^\alpha} + \frac{\partial \mathcal{H}}{\partial x^\alpha} \right)_{\text{expl.}} + \frac{\partial \pi_I^{\alpha\beta}}{\partial x^\alpha} \frac{\partial \phi_I}{\partial x^\beta} \\
&\quad + \pi_I^{\alpha\beta} \frac{\partial^2 \phi_I}{\partial x^\mu \partial x^\alpha} - \delta^\mu_\alpha \left( \frac{\partial \pi_I^{\alpha\beta}}{\partial x^\alpha} \frac{\partial \phi_I}{\partial x^\beta} + \pi_I^{\alpha\beta} \frac{\partial^2 \phi_I}{\partial x^\alpha \partial x^\beta} \right)
\end{align*}
$$

Inserting the canonical field equations (5), this becomes

$$
\frac{\partial T_{\alpha}^\mu}{\partial x^\alpha} = \left. \frac{\partial H}{\partial x^\mu} \right|_{\text{expl.}},
$$

(8)

If the Hamiltonian density $\mathcal{H}$ does not explicitly depend on the independent variable $x^\mu$, then $\mathcal{H}$ is obviously invariant with respect to a shift of the reference system along the $x^\mu$ axis. Then, the components of the $\mu$-th column of the energy-momentum tensor satisfy the continuity equation

$$
\frac{\partial T_{\mu}^\alpha}{\partial x^\alpha} = 0 \iff \left. \frac{\partial H}{\partial x^\mu} \right|_{\text{expl.}} = 0.
$$

Using the definition (7) of the energy-momentum tensor, we infer from Eq. (8)

$$
\frac{\partial T_{\mu}^\alpha}{\partial x^\alpha} = \left. \frac{\partial T_{\mu}^\alpha}{\partial x^\alpha} \right|_{\text{expl.}}.
$$

Based on the four independent variables $x^\mu$ of space-time, this divergence relation for the energy-momentum tensor constitutes the counterpart to the relation $dH/dt = \partial H/\partial t$ of the time derivatives of the Hamiltonian function of point mechanics. Yet, such a relation does not exist in general for the Hamiltonian density $\mathcal{H}$ of field theory. As we easily convince ourselves, the derivative of $\mathcal{H}$ with respect to $x^\mu$ is not uniquely determined by its explicit dependence on $x^\mu$

$$
\begin{align*}
\frac{\partial H}{\partial x^\mu} &= \left. \frac{\partial H}{\partial x^\mu} \right|_{\text{expl.}} + \frac{\partial H}{\partial \phi_I} \frac{\partial \phi_I}{\partial x^\mu} + \frac{\partial H}{\partial \pi_I^{\alpha\beta}} \frac{\partial \pi_I^{\alpha\beta}}{\partial x^\mu} \\
&= \left. \frac{\partial H}{\partial x^\mu} \right|_{\text{expl.}} + \frac{\partial \phi_I}{\partial x^\mu} \frac{\partial \pi_I^{\alpha\beta}}{\partial x^\mu} - \frac{\partial \phi_I}{\partial x^\alpha} \frac{\partial \pi_I^{\alpha\beta}}{\partial x^\beta} \\
&= \left. \frac{\partial H}{\partial x^\mu} \right|_{\text{expl.}} + \left( \frac{\partial \pi_I^{\alpha\beta}}{\partial x^\mu} - \delta_\alpha^\mu \frac{\partial \pi_I^{\alpha\beta}}{\partial x^\beta} \right) \frac{\partial \phi_I}{\partial x^\alpha}.
\end{align*}
$$

(9)

Owing to the fact that the number of independent variables is greater than one, the two rightmost terms of Eq. (9) constitute a sum. In contrast to the case of point mechanics, these terms generally do not cancel by virtue of the canonical field equations.
2.3. Non-uniqueness of the conjugate vector fields $\pi^I{}^\mu$

From the right hand side of the second canonical field equation (5) we observe that the dependence of the Hamiltonian density $H$ on $\phi$ only determines the divergence of the conjugate vector field $\pi^I{}^\mu$. Vice-versa, the canonical field equations are invariant with regard to all transformations of the mixed tensor $(\partial \pi^I{}^\mu / \partial x^\nu)$ that preserve its trace. The expression $\partial H / \partial \phi$ thus only quantifies the change of the flux of $\pi^I{}^\mu$ through an infinitesimal space-time volume around a space-time location $x^\nu$. The vector field $\pi^I{}^\mu$ itself is, therefore, only determined up to a vector field $\eta^I{}^\mu(x')$ that leaves its divergence invariant

$$\pi^I{}^\mu \rightarrow \pi^I{}^\mu = \pi^I{}^\mu - \eta^I{}^\mu.$$  \hspace{1cm} (10)

This is obviously the case if

$$\frac{\partial \eta^I{}^\alpha}{\partial x^\alpha} = 0.$$ \hspace{1cm} (11)

With this condition fulfilled, we are allowed to subtract a field $\eta^I{}^\mu(x')$ from $\pi^I{}^\mu(x')$ without changing the canonical field equations (5), hence the description of the dynamics of the given system. We will show in Sec. 5.2 that the transition (10) can be conceived as a canonical transformation of the given Hamiltonian system.

In the Lagrangian formalism, the transition (10) corresponds to the transformation

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} - \eta^I{}^\alpha(x) \frac{\partial \phi_I(x)}{\partial x^\alpha},$$

which leaves — under the condition (11) — the Euler-Lagrange equations (2) invariant.

The Hamiltonian density $\mathcal{H}'$, expressed as a function of $\pi_e$, is obtained from the Legendre transformation

$$\mathcal{H}'(\phi, \pi_e, x) = \pi^I_e \frac{\partial \phi_I}{\partial x^\alpha} - \mathcal{L}'(\phi, \partial \phi, x)$$

$$= \pi^I_e \frac{\partial \phi_I}{\partial x^\alpha} - \eta^I{}^\alpha \frac{\partial \phi_I}{\partial x^\alpha} - \mathcal{L}' + \eta^I{}^\alpha \frac{\partial \phi_I}{\partial x^\alpha}$$

$$= \mathcal{H}(\phi, \pi, x).$$

The value of the Hamiltonian density $\mathcal{H}$ thus remains invariant under the action of the shifting transformation (10), (11). This means for the canonical field equations (5)

$$\frac{\partial \mathcal{H}'}{\partial \pi^I_e} = \frac{\partial \mathcal{H}'}{\partial \pi^I} = \frac{\partial \phi_I}{\partial x^\nu}, \quad \frac{\partial \mathcal{H}'}{\partial \phi_I} = \frac{\partial \mathcal{H}}{\partial \phi_I} = \frac{\partial \pi^I_e}{\partial x^\alpha}, \quad \frac{\partial \mathcal{H}'}{\partial x^\alpha} \big|_{\text{expl.}} = \frac{\partial \mathcal{H}}{\partial x^\alpha} \big|_{\text{expl.}}.$$

Thus, both momentum fields $\pi^I_e$ and $\pi^I$ equivalently describe the same physical system. In other words, we can switch from $\pi^I$ to $\pi^I_e = \pi^I - \eta^I$ with $\partial \eta^I / \partial x^\alpha = 0$ without changing the physical description of the given system.
3. Canonical transformations in covariant Hamiltonian field theory

3.1. Generating functions of type $F_1(\phi, \phi', x)$

Similar to the canonical formalism of point mechanics, we call a transformation of the fields $(\phi, \pi) \mapsto (\phi', \pi')$ canonical if the form of the variational principle that is based on the action integral (1) is maintained,

$$\delta \int_R \left( \pi^{I\alpha} \frac{\partial \phi_I}{\partial x^\alpha} - \mathcal{H}(\phi, \pi, x) \right) d^4x = \delta \int_R \left( \pi'^{I\alpha'} \frac{\partial \phi_I'}{\partial x^{\alpha'}} - \mathcal{H}'(\phi', \pi', x) \right) d^4x. \quad (12)$$

Equation (12) tells us that the integrands may differ by the divergence of a vector field $F^{\mu}_1$, whose variation vanishes on the boundary $\partial R$ of the integration region $R$ within space-time

$$\delta \int_R \frac{\partial F^{\alpha}_1}{\partial x^{\alpha}} d^4x = \delta \oint_{\partial R} F^{\alpha}_1 dS^{\alpha} = 0.$$

The immediate consequence of the form invariance of the variational principle is the form invariance of the covariant canonical field equations (5)

$$\frac{\partial \mathcal{H}'}{\partial \pi^{I\mu}} = \frac{\partial \phi_I'}{\partial x^\mu}, \quad \frac{\partial \mathcal{H}'}{\partial \phi_I'} = -\frac{\partial \pi^{I\alpha'}}{\partial x^\alpha}. \quad (13)$$

For the integrands of Eq. (12) — hence for the Lagrangian densities $\mathcal{L}$ and $\mathcal{L}'$ — we thus obtain the condition

$$\mathcal{L} = \mathcal{L}' + \frac{\partial F^{\alpha}_1}{\partial x^{\alpha}}$$

$$\pi^{I\alpha} \frac{\partial \phi_I}{\partial x^\alpha} - \mathcal{H}(\phi, \pi, x) = \pi'^{I\alpha'} \frac{\partial \phi_I'}{\partial x^{\alpha'}} - \mathcal{H}'(\phi', \pi', x) + \frac{\partial F^{\alpha}_1}{\partial x^{\alpha}}. \quad (14)$$

With the definition $F^{\mu}_1 \equiv F^{\mu}_1(\phi, \phi', x)$, we restrict ourselves to a function of exactly those arguments that now enter into transformation rules for the transition from the original to the new fields. The divergence of $F^{\mu}_1$ writes, explicitly,

$$\frac{\partial F^{\alpha}_1}{\partial x^{\alpha}} = \frac{\partial F^{\alpha}_1}{\partial \phi_I} \frac{\partial \phi_I}{\partial x^\alpha} + \frac{\partial F^{\alpha}_1}{\partial \phi_I'} \frac{\partial \phi_I'}{\partial x^\alpha} + \frac{\partial F^{\alpha}_1}{\partial x^{\alpha}} \bigg|_{\text{expl}}. \quad (15)$$

The rightmost term denotes the sum over the explicit dependence of the generating function $F^{\mu}_1$ on the $x^\alpha$. Comparing the coefficients of Eqs. (14) and (15), we find the local coordinate representation of the field transformation rules that are induced by the generating function $F^{\mu}_1$

$$\pi^{I\mu} = \frac{\partial F^{\mu}_1}{\partial \phi_I}, \quad \pi'^{I\mu'} = -\frac{\partial F^{\mu}_1}{\partial \phi_I'}, \quad \mathcal{H}' = \mathcal{H} + \frac{\partial F^{\alpha}_1}{\partial x^{\alpha}} \bigg|_{\text{expl}}. \quad (16)$$

The transformation rule for the Hamiltonian density implies that summation over $\alpha$ is to be performed. In contrast to the transformation rule for the Lagrangian density $\mathcal{L}$ of Eq. (14), the rule for the Hamiltonian density is determined only by the explicit dependence of the generating function $F^{\mu}_1$ on the $x^\mu$. 

Differentiating the transformation rule for $\pi^{I\mu}$ with respect to $\phi_J'$, and the rule for $\pi^{I\mu'}$ with respect to $\phi_I$, we obtain a symmetry relation between original and transformed fields

$$\frac{\partial \pi^{I\mu}}{\partial \phi_J'} = \frac{\partial^2 F^\mu_1}{\partial \phi_I \partial \phi_J'} = -\frac{\partial \pi^{J\mu'}}{\partial \phi_I}.$$  \hfill (17)

The emerging of symmetry relations is a characteristic feature of canonical transformations. As the symmetry relation directly follows from the second derivatives of the generating function, it does not apply for arbitrary transformations of the fields that do not follow from generating functions.

### 3.2. Generating functions of type $F_2^\phi(\phi, \pi', x)$

The generating function of a canonical transformation can alternatively be expressed in terms of a function of the original fields $\phi_I$ and of the new conjugate fields $\pi^{I\mu'}$. To derive the pertaining transformation rules, we perform the covariant Legendre transformation

$$F^\mu_2(\phi, \pi', x) = F^\mu_1(\phi, \phi', x) + \phi_I' \pi^{I\mu'}, \quad \pi^{I\mu'} = -\frac{\partial F^\mu_1}{\partial \phi_I'}.$$  \hfill (18)

By definition, the functions $F^\mu_1$ and $F^\mu_2$ agree with respect to their $\phi_I$ and $x^\mu$ dependencies

$$\frac{\partial F^\mu_2}{\partial \phi_I} = \pi^{I\mu}, \quad \frac{\partial F^\mu_2}{\partial x^\alpha}_{\text{expl}} = \frac{\partial F^\mu_1}{\partial x^\alpha}_{\text{expl}} = H' - H.$$

These two $F^\mu_2$-related transformation rules thus coincide with the respective rules derived previously from $F^\mu_1$. In other words, the variables $\phi_I$ and $x^\mu$ do not take part in the Legendre transformation from Eq. (13). We must, therefore, conceive $F^\mu_1$ as a function of the $\phi_I'$ only, and, correspondingly, $F^\mu_2$ as a function of $\pi^{I\mu'}$ only.

The new transformation rule thus follows from the derivative of $F^\mu_2$ with respect to $\pi^{I\mu'}$,

$$\frac{\partial F^\mu_2}{\partial \pi^{J\nu'}} = \frac{\partial F^\mu_2}{\partial \phi_I} + \phi_I' \frac{\partial \pi^{I\mu'}}{\partial \phi_J'} + \pi^{I\mu'} \frac{\partial \phi_I'}{\partial \phi_J'} = -\pi^{I\mu'} \frac{\partial \phi_I'}{\partial \phi_J'} + \phi_I' \delta^I_J \delta^\mu_{\nu'} + \pi^{I\mu'} \frac{\partial \phi_I'}{\partial \phi_J'} = \phi_{I'} \delta^\mu_{\nu'}.$$

We thus end up with set of transformation rules

$$\pi^{I\mu} = \frac{\partial F^\mu_2}{\partial \phi_I'}, \quad \phi_{I'} \delta^\mu_{\nu'} = \frac{\partial F^\mu_2}{\partial \pi^{J\nu'}}, \quad H' = H + \frac{\partial F^\alpha_1}{\partial x^\alpha}_{\text{expl}},$$  \hfill (19)

which is equivalent to the set (16) by virtue of the Legendre transformation (18) if $\partial^2 F^\mu_2/\partial \phi_I \partial \phi_{I'} \neq 0$ for all indices “$\mu$”, “$I$”, and “$J$”. From the second partial derivatives of $F^\mu_2$ one immediately derives the symmetry relation

$$\frac{\partial \pi^{I\mu}}{\partial \pi^{J\nu'}} = \frac{\partial^2 F^\mu_2}{\partial \phi_I \partial \phi_J'} = \frac{\partial \phi_{I'}}{\partial \phi_I} \delta^\mu_{\nu'}.$$  \hfill (20)
3.3. Generating functions of type $F_3(\phi', \pi, x)$

By means of the Legendre transformation

$$F_3^\mu(\phi', \pi, x) = F_1^\mu(\phi, \phi', x) - \phi_I \pi^I \mu, \quad \pi^{I \mu} = \frac{\partial F_3^\mu}{\partial \phi_I}, \quad (21)$$

the generating function of a canonical transformation can be converted into a function of the new fields $\phi_I'$ and the original conjugate fields $\pi^{I \mu}$. The functions $F_1^\mu$ and $F_3^\mu$ agree in their dependencies on $\phi_I'$ and $x^\mu$,

$$\frac{\partial F_3^\mu}{\partial \pi^{J \nu}} = \frac{\partial F_1^\mu}{\partial \pi^{J \nu}}, \quad \frac{\partial F_3^\alpha}{\partial x^\alpha} \bigg|_{\text{expl}} = \frac{\partial F_1^\alpha}{\partial x^\alpha} \bigg|_{\text{expl}} = \mathcal{H}' - \mathcal{H}. \quad (22)$$

Consequently, the pertaining transformation rules agree with those of Eq. (16). The new rule follows from the dependence of $F_3^\mu$ on the $\pi^{J \nu}$:

$$\frac{\partial F_3^\mu}{\partial \phi_I'} = \frac{\partial F_1^\mu}{\partial \phi_I'} = -\pi^{I \mu}', \quad \frac{\partial F_3^\alpha}{\partial x^\alpha} \bigg|_{\text{expl}} = \frac{\partial F_1^\alpha}{\partial x^\alpha} \bigg|_{\text{expl}} = \mathcal{H}' - \mathcal{H}. \quad (23)$$

For $\frac{\partial^2 F_1^\mu}{\partial \phi_I \partial \phi_J} \neq 0$, we thus get a third set of equivalent transformation rules,

$$\pi^{I \mu}' = -\frac{\partial F_4^\mu}{\partial \phi_I'}, \quad \phi_I \delta^\mu_{J'} = -\frac{\partial F_3^\mu}{\partial \pi^{J \nu}}, \quad \mathcal{H}' = \mathcal{H} + \frac{\partial F_3^\alpha}{\partial x^\alpha} \bigg|_{\text{expl}}. \quad (22)$$

The pertaining symmetry relation between original and transformed fields emerging from $F_3^\mu$ writes

$$\frac{\partial \pi^{I \mu'}}{\partial \pi^{J \nu}} = -\frac{\partial^2 F_3^\mu}{\partial \phi_I \partial \pi^{J \nu}} = \frac{\partial \phi_I}{\partial \phi_I'} \delta^\mu_{J'}. \quad (23)$$

3.4. Generating functions of type $F_4(\pi, \pi', x)$

Finally, by means of the Legendre transformation

$$F_4^\mu(\pi, \pi', x) = F_3^\mu(\phi', \pi, x) + \phi_I' \pi^{I \mu}, \quad \pi^{I \mu'} = -\frac{\partial F_3^\mu}{\partial \phi_I'}, \quad (24)$$

we may express the generating function of a canonical transformation as a function of both the original and the transformed conjugate fields $\pi^{I \mu}, \pi^{I \mu'}$. The functions $F_4^\mu$ and $F_3^\mu$ agree in their dependencies on the $\pi^{I \mu}$ and $x^\mu$,

$$\frac{\partial F_4^\mu}{\partial \pi^{I \nu}} = -\phi_I \delta^\mu_{J'}, \quad \frac{\partial F_3^\alpha}{\partial x^\alpha} \bigg|_{\text{expl}} = \frac{\partial F_3^\alpha}{\partial x^\alpha} \bigg|_{\text{expl}} = \mathcal{H}' - \mathcal{H}. \quad (22)$$
The related pair of transformation rules thus corresponds to that of Eq. (22). The new rule follows from the dependence of $F_4^{\mu}$ on the $\pi^J \nu$,

$$\frac{\partial F_4^{\mu}}{\partial \pi^J \nu} = \frac{\partial F_4^{\mu}}{\partial \phi^J \nu} + \frac{\partial \pi^J \nu}{\partial \pi^J \nu} \frac{\partial \phi^J \nu}{\partial \pi^J \nu} = \frac{\partial F_4^{\mu}}{\partial \pi^J \nu} = \phi^J \nu \delta^\mu_{\nu}.$$

Under the condition that $\partial^2 F_4^{\mu}/\partial \phi^J \nu = 0$, we thus get a fourth set of equivalent transformation rules

$$\phi^J \nu \delta^\mu_{\nu} = \frac{\partial F_4^{\mu}}{\partial \pi^J \nu}, \quad \phi^J \nu \delta^\mu_{\nu} = -\frac{\partial F_4^{\mu}}{\partial \pi^J \nu}, \quad \mathcal{H}' = \mathcal{H} + \left. \frac{\partial F_4^{\mu}}{\partial x^\alpha} \right|_{\text{expl}}. \quad (25)$$

The subsequent symmetry relation between original and transformed fields that is associated with $F_4^{\mu}$ follows as

$$\frac{\partial \phi^J}{\partial \pi^J \nu} \delta^\mu_{\nu} = -\frac{\partial^2 F_4^{\mu}}{\partial \pi^J \nu} = -\frac{\partial \phi^J}{\partial \pi^J \nu} \delta^\mu_{\nu}. \quad (26)$$

For the particular cases $\alpha = \beta = \mu$, this means

$$\frac{\partial \phi^J}{\partial \pi^J \nu} = -\frac{\partial \phi^J}{\partial \pi^J \nu}. \quad (27)$$

With regard to Eq. (27), we observe that the symmetry relation (17) similarly depicts only the particular cases $\alpha = \beta = \mu$. Making use of the complete set of symmetry relations, we show in Appendix A that — in analogy to Eq. (26) — the general form of Eq. (17) is given by

$$\frac{\partial \pi^J \nu}{\partial \phi^J \nu} \delta^\mu_{\nu} = -\frac{\partial \pi^J \nu}{\partial \phi^J \nu} \delta^\mu_{\nu}. \quad (28)$$

### 3.5. Consistency check of the canonical transformation rules

As a test of consistency of the canonical transformation rules derived in the preceding four sections, we now rederive the rules obtained from the generating function $F_4^{\mu}$ from a Legendre transformation of $F_4^{\mu}$. Both generating functions are related by

$$F_4^{\mu}(\phi, \phi', \mathbf{x}) = F_4^{\mu}(\pi, \pi', \mathbf{x}) + \phi^J \pi^J \nu - \phi^J \nu \pi^J \nu', \quad \phi^J \nu \delta^\mu_{\nu} = \frac{\partial F_4^{\mu}}{\partial \pi^J \nu}, \quad \phi^J \nu \delta^\mu_{\nu} = -\frac{\partial F_4^{\mu}}{\partial \pi^J \nu}. \quad (29)$$

In this case, the generating functions $F_4^{\mu}$ and $F_4^{\mu}$ only agree in their explicit dependence on $x^\mu$. This involves the common transformation rule

$$\left. \frac{\partial F_4^{\mu}}{\partial x^\alpha} \right|_{\text{expl}} = \left. \frac{\partial F_4^{\mu}}{\partial x^\alpha} \right|_{\text{expl}} = \mathcal{H}' - \mathcal{H}.$$
In the actual case, we thus transform at once two field variables \( \phi_I, \phi_{I'} \) and \( \pi^{I \mu}, \pi^{I' \mu} \).

The transformation rules associated with \( F_1^\mu \) follow from its dependencies on both \( \phi_I \) and \( \phi_{I'} \) according to

\[
\frac{\partial F_1^\mu}{\partial \phi_I} + \frac{\partial F_1^\mu}{\partial \phi_{I'}} \frac{\partial \phi_{I'}}{\partial \phi_I} = \frac{\partial F_4^\mu}{\partial \pi^{J \alpha}} \frac{\partial \pi^{J \alpha}}{\partial \phi_I} + \frac{\partial F_4^\mu}{\partial \pi^{J' \alpha'}} \frac{\partial \pi^{J' \alpha'}}{\partial \phi_I} + \pi^{I \mu} + \phi_J \frac{\partial \pi^{J \mu}}{\partial \phi_I} - \pi^{J \mu} \frac{\partial \phi_J}{\partial \phi_I} - \phi_J' \frac{\partial \pi^{J' \mu}}{\partial \phi_I} + \pi^{J' \mu} \frac{\partial \phi_{I'}}{\partial \phi_I}.
\]

Comparing the coefficients, we encounter the transformation rules

\[
\pi^{I \mu} = \frac{\partial F_1^\mu}{\partial \phi_I}, \quad \pi^{I' \mu} = -\frac{\partial F_1^\mu}{\partial \phi_{I'}}.
\]

As expected, the rules obtained previously in Eq. (16) are recovered. The same result follows if we differentiate \( F_1^\mu \) with respect to \( \phi_{I'} \).

### 3.6. Poisson brackets, Lagrange brackets

For a system with given Hamiltonian density \( \mathcal{H}(\phi, \pi, x) \), and for two differentiable functions \( f(\phi, \pi, x), g(\phi, \pi, x) \) of the fields \( \phi_I, \pi^{I \mu} \) and the independent variables \( x^\mu \), we define the \( \mu \)-th component of the Poisson bracket of \( f \) and \( g \) as follows

\[
[f, g]_{\phi, \pi}^{\mu} = \frac{\partial f}{\partial \phi_I} \frac{\partial g}{\partial \pi^{I \mu}} - \frac{\partial f}{\partial \pi^{I \mu}} \frac{\partial g}{\partial \phi_I}.
\]

(29)

With this definition, the four Poisson brackets \( [f, g]_{\phi, \pi}^{\mu} \) constitute the components of a dual 4-vector, i.e., a 1-form. Obviously, the Poisson bracket (29) satisfies the following algebraic rules

\[
[f, g]_{\phi, \pi}^{\mu} = -[g, f]_{\phi, \pi}^{\mu},
\]

\[
[c f, g]_{\phi, \pi}^{\mu} = c[f, g]_{\phi, \pi}^{\mu}, \quad c \in \mathbb{R}
\]

\[
[f, g]_{\phi, \pi}^{\mu} + [h, g]_{\phi, \pi}^{\mu} = [f + h, g]_{\phi, \pi}^{\mu}.
\]
The fundamental Lagrange brackets then emerge as

$$\{f, g\}_{\phi, \pi^\mu} = \frac{\partial f}{\partial \phi_I} \frac{\partial}{\partial \pi^\mu_I} (g h) - \frac{\partial g}{\partial \phi_I} \frac{\partial f}{\partial \pi^\mu_I} (h g)$$

$$= \frac{\partial f}{\partial \phi_I} \left( h \frac{\partial g}{\partial \pi^\mu_I} + g \frac{\partial h}{\partial \pi^\mu_I} \right) - \frac{\partial g}{\partial \phi_I} \left( g \frac{\partial h}{\partial \phi_I} + h \frac{\partial g}{\partial \phi_I} \right)$$

$$= h \left( \frac{\partial f}{\partial \phi_I} \frac{\partial g}{\partial \pi^\mu_I} - \frac{\partial f}{\partial \pi^\mu_I} \frac{\partial g}{\partial \phi_I} \right) + g \left( \frac{\partial f}{\partial \phi_I} \frac{\partial h}{\partial \phi_I} - \frac{\partial f}{\partial \phi_I} \frac{\partial h}{\partial \pi^\mu_I} \right)$$

$$= \{ f, g \}_{\phi, \pi^\mu} + h \{ f, h \}_{\phi, \pi^\mu}$$

For an arbitrary differentiable function \( f(\phi_I, \pi^I, \pi) \) of the field variables, we can, in particular, set up the Poisson brackets with the canonical fields \( \phi_I \) and \( \pi^I \).

As the individual field variables \( \phi_I \) and \( \pi^I \) are independent by assumption, we immediately get

$$\{ \phi_I, f \}_{\phi, \pi^\mu} = \frac{\partial \phi_I}{\partial \phi_J} \frac{\partial f}{\partial \phi_J} - \frac{\partial \phi_I}{\partial \pi^\mu} \frac{\partial f}{\partial \pi^\mu} = 0,$$

$$\{ \pi^I, f \}_{\phi, \pi^\mu} = \frac{\partial \pi^I}{\partial \phi_J} \frac{\partial f}{\partial \phi_J} - \frac{\partial \pi^I}{\partial \pi^\mu} \frac{\partial f}{\partial \pi^\mu} = \{ \pi^I, f \},$$

$$\{ \pi^I, \pi^J \}_{\phi, \pi^\mu} = \frac{\partial \pi^I}{\partial \phi_J} \frac{\partial \pi^J}{\partial \phi_J} - \frac{\partial \pi^I}{\partial \pi^\mu} \frac{\partial \pi^J}{\partial \pi^\mu} = \{ \pi^I, \pi^J \}.$$
3.7. Canonical invariance of Poisson and Lagrange brackets

In the first instance, we will show that the fundamental Poisson brackets are invariant under canonical transformations, hence that the relations (30) equally apply for canonically transformed fields \( \phi_I' \) and \( \pi_I' \). Making use of the symmetry relations \( (20), (23), (27), \) and \( (28) \), we get

\[
\begin{align*}
\left[ \phi_I', \phi_J' \right]_{\phi, \pi} &= \frac{\partial \phi_I'}{\partial \phi_K} \frac{\partial \phi_J'}{\partial \pi_K} - \frac{\partial \phi_I'}{\partial \pi_K} \frac{\partial \phi_J'}{\partial \phi_K} \\
&= - \frac{\partial \phi_I'}{\partial \phi_K} \frac{\partial \phi_J'}{\partial \pi_K} \\
&= - \frac{\partial \phi_I'}{\partial \phi_K} \frac{\partial \phi_J'}{\partial \pi_K} \\
&= \frac{\partial \phi_I'}{\partial \phi_K} \frac{\partial \phi_J'}{\partial \pi_K} \\
&= \frac{\partial \phi_I'}{\partial \phi_K} \frac{\partial \phi_J'}{\partial \pi_K} = 0
\end{align*}
\]

\( (33) \)

\[
\begin{align*}
\left[ \phi_I', \pi_J' \right]_{\phi, \pi} &= \frac{\partial \phi_I'}{\partial \phi_K} \frac{\partial \pi_J'}{\partial \pi_K} - \frac{\partial \phi_I'}{\partial \pi_K} \frac{\partial \pi_J'}{\partial \phi_K} \\
&= \frac{\partial \phi_I'}{\partial \phi_K} \frac{\partial \pi_J'}{\partial \pi_K} \\
&= \frac{\partial \phi_I'}{\partial \phi_K} \frac{\partial \pi_J'}{\partial \pi_K} \\
&= \frac{\partial \phi_I'}{\partial \phi_K} \frac{\partial \pi_J'}{\partial \pi_K} \\
&= \frac{\partial \phi_I'}{\partial \phi_K} \frac{\partial \pi_J'}{\partial \pi_K} = 0
\end{align*}
\]

\( (34) \)

\[
\begin{align*}
\left[ \pi_I'^{\alpha}, \pi_J'^{\beta} \right]_{\phi, \pi} &= \frac{\partial \pi_I'^{\alpha}}{\partial \phi_K} \frac{\partial \pi_J'^{\beta}}{\partial \pi_K} - \frac{\partial \pi_I'^{\alpha}}{\partial \pi_K} \frac{\partial \pi_J'^{\beta}}{\partial \phi_K} \\
&= \frac{\partial \pi_I'^{\alpha}}{\partial \phi_K} \frac{\partial \pi_J'^{\beta}}{\partial \pi_K} \\
&= \frac{\partial \pi_I'^{\alpha}}{\partial \phi_K} \frac{\partial \pi_J'^{\beta}}{\partial \pi_K} \\
&= \frac{\partial \pi_I'^{\alpha}}{\partial \phi_K} \frac{\partial \pi_J'^{\beta}}{\partial \pi_K} \\
&= \frac{\partial \pi_I'^{\alpha}}{\partial \phi_K} \frac{\partial \pi_J'^{\beta}}{\partial \pi_K} = 0
\end{align*}
\]

\( (35) \)

The Poisson bracket of two arbitrary differentiable functions \( f(\phi, \pi, x) \) and \( g(\phi, \pi, x) \), as defined by Eq. \( (29) \), can now be expanded in terms of transformed fields \( \phi_I' \) and \( \pi_I' \). For a general transformation \( (\phi, \pi) \leftrightarrow (\phi', \pi') \), we have

\[
\begin{align*}
\left[ f, g \right]_{\phi, \pi} &= \frac{\partial f}{\partial \phi_K} \frac{\partial g}{\partial \pi_K} - \frac{\partial f}{\partial \pi_K} \frac{\partial g}{\partial \phi_K} \\
&= \left( \frac{\partial f}{\partial \phi_K} + \frac{\partial f}{\partial \pi_K} \frac{\partial \phi_K}{\partial \phi_K} \right) \left( \frac{\partial g}{\partial \phi_K} - \frac{\partial g}{\partial \pi_K} \frac{\partial \phi_K}{\partial \phi_K} \right) \left( \frac{\partial g}{\partial \phi_K} + \frac{\partial g}{\partial \pi_K} \frac{\partial \phi_K}{\partial \pi_K} \right) \\
&- \left( \frac{\partial f}{\partial \phi_K} + \frac{\partial f}{\partial \pi_K} \frac{\partial \phi_K}{\partial \pi_K} \right) \left( \frac{\partial g}{\partial \phi_K} - \frac{\partial g}{\partial \pi_K} \frac{\partial \phi_K}{\partial \pi_K} \right) \left( \frac{\partial g}{\partial \phi_K} + \frac{\partial g}{\partial \pi_K} \frac{\partial \phi_K}{\partial \phi_K} \right)
\end{align*}
\]
After working out the multiplications, we can recollect all products so as to form fundamental Poisson brackets
\[
[f, g]_{\phi, \pi} = \left( \frac{\partial f}{\partial \phi_{I'}} \frac{\partial g}{\partial \pi_{J'}^\alpha} - \frac{\partial f}{\partial \pi_{I'}^{1\alpha}} \frac{\partial g}{\partial \phi_{J'}} \right) [\phi_{I'}, \pi_{J'}^{1\alpha}]_{\phi, \pi} + \frac{\partial f}{\partial \pi_{I'}^{1\alpha}} \frac{\partial g}{\partial \pi_{J'}^{1\beta}} [\pi_{I'}^{1\alpha}, \pi_{J'}^{1\beta}]_{\phi, \pi}
\]

For the special case that the transformation is canonical, the equations (33), (34), and (35) for the fundamental Poisson brackets apply. We then get
\[
[f, g]_{\phi, \pi} = \left( \frac{\partial f}{\partial \phi_{I'}} \frac{\partial g}{\partial \pi_{J'}^\alpha} - \frac{\partial f}{\partial \pi_{I'}^{1\alpha}} \frac{\partial g}{\partial \phi_{J'}} \right) \delta_{\alpha}^\mu \delta_{\beta}^\mu = [f, g]_{\phi', \pi'}.
\]

We thus abbreviate in the following the index notation of the Poisson bracket by writing
\[
[f, g]_{\phi, \pi} \equiv [f, g]_{\phi', \pi'},
\]

as the brackets do not depend on the underlying set of canonical field variables \(\phi_{I'}, \pi_{I'}^{1\mu}\).

The proof of the canonical invariance of the fundamental Lagrange brackets is based on the symmetry relations (17), (20), (23), and (26). Explicitly, we have

\[
\{\phi_{I'}, \phi_{J'}^\beta\}_{\phi, \pi} = \frac{\partial \phi_K}{\partial \phi_{I'}} \frac{\partial \pi_{I'}^{1\mu}}{\partial \phi_{J'}^\beta} \frac{\partial \phi_K}{\partial \phi_{I'}} \frac{\partial \pi_{I'}^{1\mu}}{\partial \phi_{J'}^\beta} - \frac{\partial \phi_K}{\partial \phi_{I'}} \frac{\partial \pi_{I'}^{1\mu}}{\partial \phi_{J'}^\beta} \frac{\partial \phi_K}{\partial \phi_{I'}} \frac{\partial \pi_{I'}^{1\mu}}{\partial \phi_{J'}^\beta} \delta_{\beta}^\mu = 0 = \{\phi_{I'}, \phi_{J'}^\beta\}_{\phi, \pi}
\]

\[
\{\phi_{I'}, \pi_{J'}^{1\alpha}\}_{\phi, \pi} = \frac{\partial \phi_K}{\partial \phi_{I'}} \frac{\partial \pi_{I'}^{1\mu}}{\partial \pi_{J'}^{1\alpha}} \frac{\partial \phi_K}{\partial \phi_{I'}} \frac{\partial \pi_{I'}^{1\mu}}{\partial \pi_{J'}^{1\alpha}} - \frac{\partial \phi_K}{\partial \phi_{I'}} \frac{\partial \pi_{I'}^{1\mu}}{\partial \pi_{J'}^{1\alpha}} \frac{\partial \phi_K}{\partial \phi_{I'}} \frac{\partial \pi_{I'}^{1\mu}}{\partial \pi_{J'}^{1\alpha}} \delta_{\alpha}^\mu = \frac{\partial \phi_K}{\partial \phi_{I'}} \frac{\partial \phi_{I'}}{\partial \phi_{I'}} \frac{\partial \phi_K}{\partial \phi_{I'}} \frac{\partial \phi_K}{\partial \phi_{I'}} \delta_{\alpha}^\mu \delta_{\alpha}^\mu = \{\pi_{I\alpha'}, \pi_{J\beta'}\}_{\phi, \pi}
\]

\[
\{\pi_{I\alpha'}, \pi_{J\beta'}\}_{\phi, \pi} = \frac{\partial \phi_K}{\partial \pi_{I\alpha'}} \frac{\partial \pi_{I\beta'}}{\partial \pi_{J\beta'}} \frac{\partial \phi_K}{\partial \pi_{I\alpha'}} \frac{\partial \pi_{I\beta'}}{\partial \pi_{J\beta'}} - \frac{\partial \phi_K}{\partial \pi_{I\alpha'}} \frac{\partial \pi_{I\beta'}}{\partial \pi_{J\beta'}} \frac{\partial \phi_K}{\partial \pi_{I\alpha'}} \frac{\partial \pi_{I\beta'}}{\partial \pi_{J\beta'}} \delta_{\beta}^\mu \delta_{\beta}^\mu = \left( \frac{\partial \phi_K}{\partial \pi_{I\alpha'}} \frac{\partial \phi_{I'}}{\partial \phi_{I'}} \frac{\partial \phi_K}{\partial \pi_{I\alpha'}} \frac{\partial \phi_K}{\partial \pi_{I\alpha'}} \delta_{\beta}^\mu \delta_{\beta}^\mu \right)
\]

\[
= \frac{\partial \phi_{I'}}{\partial \pi_{I\alpha'}} \delta_{\beta}^\mu = 0 = \{\pi_{I\alpha'}, \pi_{J\beta'}\}_{\phi, \pi}
\]

\[
(37)
\]

\[
(38)
\]

\[
(39)
\]
The Lagrange bracket (31) of two arbitrary differentiable functions $f(\phi, \pi_I, x)$ and $g(\phi, \pi_I, x)$ can now be expressed in terms of transformed fields $\phi_I', \pi_{I'}$

$$\{f, g\}^{\phi', \pi'} = \frac{\partial \phi_K}{\partial f} \frac{\partial \pi_K^\mu}{\partial g} - \frac{\partial \pi_K^\mu}{\partial f} \frac{\partial \phi_K}{\partial g}$$

$$= \left( \frac{\partial \phi_{I'}}{\partial f} \frac{\partial \phi_{I'}}{\partial g} + \frac{\partial \phi_K}{\partial f} \frac{\partial \pi_{I'}^\alpha}{\partial g} \right) \left( \frac{\partial \pi_K^\mu}{\partial \phi_{I'}} + \frac{\partial \pi_{I'}^\beta}{\partial \phi_{J'}} \right)$$

$$+ \left( \frac{\partial \pi_K^\mu}{\partial \phi_{I'}} + \frac{\partial \pi_{I'}^\alpha}{\partial \phi_{J'}} \right) \left( \frac{\partial \phi_{I'}}{\partial f} \frac{\partial \phi_{I'}}{\partial g} + \frac{\partial \phi_K}{\partial f} \frac{\partial \pi_{I'}^\beta}{\partial g} \right).$$

Multiplication and regathering the terms to form fundamental Lagrange brackets yields

$$\{f, g\}^{\phi, \pi} = \frac{\partial \phi_{I'}}{\partial f} \frac{\partial \phi_{I'}}{\partial g} \{\phi', \phi'\}^{\phi', \pi'} + \frac{\partial \pi_{I'}^\alpha}{\partial f} \frac{\partial \pi_{I'}^\beta}{\partial g} \{\pi_{I'}, \pi_{I'}\}^{\phi', \pi'}$$

$$+ \frac{\partial \phi_{I'}}{\partial f} \frac{\partial \pi_{I'}^\beta}{\partial g} \{\phi', \pi_{I'}\}^{\phi', \pi'} - \frac{\partial \pi_{I'}^\alpha}{\partial f} \frac{\partial \phi_{I'}}{\partial g} \{\phi_{I'}, \pi_{I'}\}^{\phi', \pi'}.$$

For canonical transformations, we can make use of the relations (37), (38), and (39) for the fundamental Lagrange brackets. We thus obtain

$$\{f, g\}^{\phi, \pi} = \frac{\partial \phi_{I'}}{\partial f} \frac{\partial \pi_{I'}^\beta}{\partial g} \delta_{I'}^\alpha \delta_{I'}^\beta - \frac{\partial \pi_{I'}^\alpha}{\partial f} \frac{\partial \phi_{I'}}{\partial g} \delta_{I'}^\alpha \delta_{I'}^\beta$$

$$= \frac{\partial \phi_{I'}}{\partial f} \frac{\partial \pi_{I'}^\mu}{\partial g} - \frac{\partial \pi_{I'}^\mu}{\partial f} \frac{\partial \phi_{I'}}{\partial g}$$

$$= \{f, g\}^{\phi, \pi}.$$

The notation of the Lagrange brackets (31) can thus be simplified as well. In the following, we denote these brackets as $\{f, g\}''$ since their value does not depend on the particular set of canonical field variables $\phi_I, \pi_I^\mu$.

### 3.8. Liouville’s theorem of covariant Hamiltonian field theory

For general transformations $(\phi, \pi) \mapsto (\phi', \pi')$ of the scalar fields $\phi_I$ and the pertaining conjugate vector fields $\pi_I^\mu$, the transformation of the 2-form $\omega^\mu = d\phi_I \wedge d\pi_I^\mu$ is
determined by

\[
\begin{align*}
\text{d}\phi_I \wedge \text{d}\pi^{I\mu} &= \left( \frac{\partial \phi_I}{\partial \phi_{J'}} \text{d}\phi_{J'} + \frac{\partial \phi_I}{\partial \pi^{J\alpha'}} \text{d}\pi^{J\alpha'} \right) \wedge \left( \frac{\partial \pi^{I\mu}}{\partial \phi_{K'}} \text{d}\phi_{K'} + \frac{\partial \pi^{I\mu}}{\partial \pi^{K\beta'}} \text{d}\pi^{K\beta'} \right) \\
&= \frac{\partial \phi_I}{\partial \phi_{J'}} \frac{\partial \pi^{I\mu}}{\partial \phi_{K'}} \text{d}\phi_{J'} \wedge \text{d}\phi_{K'} + \frac{\partial \phi_I}{\partial \pi^{J\alpha'}} \frac{\partial \pi^{I\mu}}{\partial \pi^{K\beta'}} \text{d}\pi^{J\alpha'} \wedge \text{d}\pi^{K\beta'} \\
&\quad + \frac{\partial \phi_I}{\partial \phi_{J'}} \frac{\partial \pi^{I\mu}}{\partial \pi^{K\beta'}} - \frac{\partial \phi_I}{\partial \pi^{J\alpha'}} \frac{\partial \pi^{I\mu}}{\partial \phi_{J'}} \text{d}\phi_{J'} \wedge \text{d}\pi^{K\beta'} \\
&\quad + \frac{\partial \phi_I}{\partial \phi_{J'}} \frac{\partial \pi^{I\mu}}{\partial \phi_{K'}} - \frac{\partial \phi_I}{\partial \pi^{J\alpha'}} \frac{\partial \pi^{I\mu}}{\partial \pi^{K\beta'}} \text{d}\phi_{J'} \wedge \text{d}\phi_{K'}.
\end{align*}
\]

The terms in parentheses can be expressed as Lagrange brackets

\[
\begin{align*}
\text{d}\phi_I \wedge \text{d}\pi^{I\mu} &= \frac{1}{2} \{ \phi^{J'}, \phi^{K'} \}^\mu \text{d}\phi_{J'} \wedge \text{d}\phi_{K'} + \frac{1}{2} \{ \pi^{J\alpha'}, \pi^{K\beta'} \}^\mu \text{d}\pi^{J\alpha'} \wedge \text{d}\pi^{K\beta'} \\
&\quad + \{ \phi^{J'}, \pi^{K\alpha'} \}^\mu \text{d}\phi_{J'} \wedge \text{d}\pi^{K\alpha'}.
\end{align*}
\]

If the transformation of the fields is \textit{canonical}, then we can apply the transformation rules for the fundamental Lagrange brackets of Eqs. (37), (38), and (39). Then, the transformation of the 2-form \(\omega^\mu\) simplifies to

\[
\omega^\mu \equiv \text{d}\phi_I \wedge \text{d}\pi^{I\mu} = \delta^I_K \delta^\mu_\alpha \text{d}\phi_{J'} \wedge \text{d}\phi_{K'} = \text{d}\phi_I \wedge \text{d}\pi^{I\mu} \equiv \omega^\mu.
\]

The 2-forms \(\omega^\mu\) are thus \textit{invariant} under canonical transformations. We may thus refer to the \(\omega^\mu\) as \textit{canonical} 2-forms.

### 3.9. Jacobi’s identity and Poisson’s theorem in canonical field theory

In order to derive the canonical field theory analog of Jacobi’s identity of point mechanics, we let \(f(\phi, \pi, x)\), \(g(\phi, \pi, x)\), and \(h(\phi, \pi, x)\) denote arbitrary differentiable functions of the canonical fields. The sum of the three cyclicly permuted nested Poisson brackets be denoted by \(a_{\mu\nu}\),

\[
a_{\mu\nu} = \left[ f, [g, h]_{\nu} \right]_{\mu} + \left[ h, [f, g]_{\mu} \right]_{\nu} + \left[ g, [h, f]_{\mu} \right]_{\nu}.
\]

We will now show that the \(a_{\mu\nu}\) are the components of an anti-symmetric (0, 2) tensor, hence that

\[
a_{\mu\nu} + a_{\nu\mu} = 0.
\]

Writing Eq. (41) explicitly, we get a sum of 24 terms, each of them consisting of a triple product of \textit{two} first-order derivatives and \textit{one} second-order derivative of the
functions $f$, $g$, and $h$

\[ a_{\mu\nu} = \frac{\partial f}{\partial \phi_j} \frac{\partial}{\partial \phi_1} \left( \frac{\partial g}{\partial \phi_i} \frac{\partial h}{\partial \phi_1} \left( \frac{\partial g}{\partial \phi_1} \frac{\partial h}{\partial \phi_i} \right) \right) - \frac{\partial f}{\partial \pi_j} \frac{\partial}{\partial \pi_1} \left( \frac{\partial g}{\partial \phi_1} \frac{\partial h}{\partial \phi_i} \left( \frac{\partial g}{\partial \phi_i} \frac{\partial h}{\partial \phi_1} \right) \right) + \frac{\partial h}{\partial \phi_j} \frac{\partial}{\partial \phi_1} \left( \frac{\partial f}{\partial \phi_i} \frac{\partial g}{\partial \phi_1} \left( \frac{\partial f}{\partial \phi_i} \frac{\partial g}{\partial \phi_1} \right) \right) - \frac{\partial h}{\partial \pi_j} \frac{\partial}{\partial \pi_1} \left( \frac{\partial f}{\partial \phi_i} \frac{\partial g}{\partial \phi_1} \left( \frac{\partial f}{\partial \phi_i} \frac{\partial g}{\partial \phi_1} \right) \right) + \frac{\partial g}{\partial \phi_j} \frac{\partial}{\partial \phi_1} \left( \frac{\partial h}{\partial \phi_i} \frac{\partial f}{\partial \phi_1} \left( \frac{\partial h}{\partial \phi_i} \frac{\partial f}{\partial \phi_1} \right) \right) \]

\[ a_{\mu\nu} = \frac{\partial h}{\partial \phi_1} \frac{\partial g}{\partial \phi_1} \frac{\partial^2 f}{\partial \phi_1 \partial \phi_1} - \frac{\partial h}{\partial \phi_1} \frac{\partial g}{\partial \phi_1} \frac{\partial^2 f}{\partial \phi_1 \partial \phi_1} \]

\[ a_{\mu\nu} = \frac{\partial h}{\partial \phi_1} \frac{\partial g}{\partial \phi_1} \frac{\partial^2 f}{\partial \phi_1 \partial \phi_1} - \frac{\partial h}{\partial \phi_1} \frac{\partial g}{\partial \phi_1} \frac{\partial^2 f}{\partial \phi_1 \partial \phi_1} \]

The proof can be simplified making use of the fact that the terms of $a_{\mu\nu}$ from Eq. (41) emerge as cyclic permutations of the functions $f$, $g$, and $h$. With regard to the explicit form of Eq. (41) from above it suffices to show that Eq. (42) is fulfilled for all terms containing second derivatives of, for instance, $f$, $g$, $h$

\[ a_{\mu\nu} = \frac{\partial h}{\partial \phi_j} \frac{\partial g}{\partial \phi_j} \frac{\partial^2 f}{\partial \phi_j \partial \phi_j} - \frac{\partial h}{\partial \phi_j} \frac{\partial g}{\partial \phi_j} \frac{\partial^2 f}{\partial \phi_j \partial \phi_j} \]

Resorting and interchanging the sequence of differentiations yields

\[ a_{\mu\nu} = -\frac{\partial h}{\partial \phi_j} \frac{\partial g}{\partial \phi_j} \frac{\partial^2 f}{\partial \phi_j \partial \phi_j} + \frac{\partial h}{\partial \phi_j} \frac{\partial g}{\partial \phi_j} \frac{\partial^2 f}{\partial \phi_j \partial \phi_j} \]

Mutually renaming the formal summation indices $I$ and $J$, the right hand sides of Eqs. (43) and (44) differ only by the sign and the interchange of the indices $\mu$ and $\nu$. Thereby, Eq. (42) is proved.
Poisson’s theorem in the realm of canonical field theory is based on the identity

$$ \frac{\partial}{\partial x^\nu} [f, g]_\mu = \left[ \frac{\partial f}{\partial x^\nu}, g \right]_\mu + \left[ f, \frac{\partial g}{\partial x^\nu} \right]_\mu. \quad (45) $$

In contrast to point mechanics, this identity is most easily proved directly, i.e., without referring to the Jacobi identity (42).

From the definition (29) of the Poisson brackets, we conclude for two arbitrary differentiable functions $f(\phi, \pi, x)$ and $g(\phi, \pi, x)$

$$ \frac{\partial}{\partial x^\nu} [f, g]_\mu = \frac{\partial f}{\partial \pi^\mu} \left( \frac{\partial g}{\partial \phi_1} \frac{\partial \phi_1}{\partial x^\nu} - \frac{\partial g}{\partial \phi_1} \frac{\partial \phi_1}{\partial x^\nu} \right) + \frac{\partial f}{\partial \pi^\mu} \left( \frac{\partial g}{\partial \phi_1} \frac{\partial \phi_1}{\partial x^\nu} - \frac{\partial g}{\partial \phi_1} \frac{\partial \phi_1}{\partial x^\nu} \right) $$

$$ = \frac{\partial g}{\partial \pi^\mu} \left( \frac{\partial^2 f}{\partial \phi_1^2} \frac{\partial \phi_1}{\partial x^\nu} + \frac{\partial^2 f}{\partial \phi_1 \partial \pi^\alpha} \frac{\partial \pi^\alpha}{\partial x^\nu} + \frac{\partial^2 f}{\partial \pi^\mu \partial x^\nu} \right) + \frac{\partial f}{\partial \pi^\mu} \left( \frac{\partial^2 g}{\partial \phi_1^2} \frac{\partial \phi_1}{\partial x^\nu} + \frac{\partial^2 g}{\partial \phi_1 \partial \pi^\alpha} \frac{\partial \pi^\alpha}{\partial x^\nu} + \frac{\partial^2 g}{\partial \pi^\mu \partial x^\nu} \right) $$

$$ = \frac{\partial f}{\partial \pi^\mu} \left( \frac{\partial f}{\partial \phi_1} \frac{\partial \phi_1}{\partial x^\nu} + \frac{\partial f}{\partial \pi^\alpha} \frac{\partial \pi^\alpha}{\partial x^\nu} - \frac{\partial f}{\partial \phi_1} \frac{\partial \phi_1}{\partial x^\nu} - \frac{\partial f}{\partial \phi_1} \frac{\partial \phi_1}{\partial x^\nu} \right) $$

$$ = \left[ \frac{\partial f}{\partial x^\nu}, g \right]_\mu + \left[ f, \frac{\partial g}{\partial x^\nu} \right]_\mu.
Provided that both the first derivative $\partial / \partial x^\nu$ as well as the two second derivatives $\partial^2 / \partial \phi_I \partial x^\nu$ and $\partial^2 / \partial \pi_{I\mu} \partial x^\nu$ vanish for both functions $f$ and $g$, then the first derivative with respect to $x^\nu$ of the Poisson bracket $[f, g]_\mu$ also vanishes

$$\frac{\partial f}{\partial x^\nu} = 0, \quad \frac{\partial g}{\partial x^\nu} = 0, \quad \frac{\partial^2 f}{\partial \phi_I \partial x^\nu} = 0, \quad \frac{\partial^2 f}{\partial \pi_{I\mu} \partial x^\nu} = 0 = \Rightarrow \frac{\partial}{\partial x^\nu} [f, g]_\mu = 0.$$  

This establishes Poisson’s theorem for canonical field theory.

### 3.10. Hamilton-Jacobi equation

In the realm of canonical field theory, we can set up the Hamilton-Jacobi equation as follows: we look for a generating function $F_1(\phi, \phi', x)$ of a canonical transformation that maps a given Hamiltonian density $H$ into a transformed density that vanishes identically, $H' \equiv 0$. In the transformed system, all partial derivatives of $H'$ thus vanish as well — and hence the derivatives of all fields $\phi_I'(x), \pi_I'(x)$ with respect to the system’s independent variables $x^\mu$,

$$\frac{\partial H'}{\partial \phi_I'} = 0 = \frac{\partial \pi_{I\mu}'}{\partial x^\mu}, \quad \frac{\partial H'}{\partial \pi_{I\mu}'} = 0 = \frac{\partial \phi_I'}{\partial x^\mu}.$$

According to the transformation rules (16) that arise from a generating function of type $S \equiv F_1(\phi, \phi', x)$, this means for a given Hamiltonian density $H$

$$H(\phi, \pi, x) + \frac{\partial S_\alpha}{\partial x^\alpha}_{\text{expl}} = 0.$$  

In conjunction with the transformation rule $\pi_{I\mu} = \partial S_\alpha / \partial \phi_I$, we may subsequently set up the Hamilton-Jacobi equation as a partial differential equation for the 4-vector function $S$

$$\mathcal{H}(\phi, \frac{\partial S}{\partial \phi}, x) + \frac{\partial S_\alpha}{\partial x^\alpha}_{\text{expl}} = 0.$$  

This equation illustrates that the generating function $S$ defines exactly that particular canonical transformation which maps the space-time state of the system into its fixed initial state

$$\phi_I' = \phi_I(0) = \text{const.}, \quad \pi_{I\mu}' = \pi_{I\mu}(0) = \text{const.}$$

The inverse transformation then defines the mapping of the system’s initial state into its actual state in space-time.

As a result of the fact that $H'$ as well as all $\partial \phi_I' / \partial x^\mu$ vanish, the divergence of $S(\phi, \phi', x)$ simplifies to

$$\frac{\partial S_\alpha}{\partial x^\alpha} = \frac{\partial S_\alpha}{\partial \phi_I} \frac{\partial \phi_I}{\partial x^\alpha} + \frac{\partial S_\alpha}{\partial x^\alpha}_{\text{expl}}$$

$$= \pi_{I\alpha} \frac{\partial \phi_I}{\partial x^\alpha} - \mathcal{H}$$

$$= \mathcal{L}.$$
This equation coincides with the transformation rule \( \text{(14)} \) of the Lagrangians for the particular case \( L' = 0 \). The 4-vector function \( S \) thus embodies the field theory analogue of Hamilton’s principal function \( S, dS/dt = L \) of point mechanics.

4. Examples for Hamiltonian densities in covariant field theory

4.1. Ginzburg-Landau Hamiltonian density

We consider the scalar field \( \phi(x,t) \) whose Lagrangian density \( L \) is given by

\[
L(\phi, \partial_t \phi, \partial_x \phi) = \frac{1}{2} \left[ (\partial_t \phi)^2 - v^2 (\partial_x \phi)^2 \right] + \lambda (\phi^2 - 1)^2.
\]

Herein, \( v \) and \( \lambda \) are supposed to denote constant quantities. The particular Euler-Lagrange equation for this Lagrangian density simplifies the general form of Eq. \( \text{(2)} \) to

\[
\frac{\partial}{\partial t} \frac{\partial L}{\partial (\partial_t \phi)} + \frac{\partial}{\partial x} \frac{\partial L}{\partial (\partial_x \phi)} - \frac{\partial L}{\partial \phi} = 0.
\]

The resulting field equation is

\[
\frac{\partial^2 \phi}{\partial t^2} - v^2 \frac{\partial^2 \phi}{\partial x^2} - 4 \lambda \phi (\phi^2 - 1) = 0.
\]

In order to derive the equivalent Hamiltonian representation, we first define the conjugate momentum fields from \( L \)

\[
\pi_t(x,t) = \frac{\partial L}{\partial (\partial_t \phi)} = \frac{\partial \phi}{\partial t}, \quad \pi_x(x,t) = \frac{\partial L}{\partial (\partial_x \phi)} = -v^2 \frac{\partial \phi}{\partial x}.
\]

The Hamiltonian density \( H \) now follows as the Legendre transform of the Lagrangian density \( L \)

\[
H(\phi, \pi_t, \pi_x) = \pi_t \frac{\partial \phi}{\partial t} + \pi_x \frac{\partial \phi}{\partial x} - L(\phi, \partial_t \phi, \partial_x \phi).
\]

The Ginzburg-Landau Hamiltonian density \( H \) is thus given by

\[
H(\phi, \pi_t, \pi_x) = \frac{1}{2} \left[ \pi_t^2 - \frac{1}{v^2} \pi_x^2 \right] - \lambda (\phi^2 - 1)^2.
\]

The canonical field equations for the density \( H \) of Eq. \( \text{(48)} \) are

\[
\frac{\partial H}{\partial \pi_t} = \frac{\partial \phi}{\partial t}, \quad \frac{\partial H}{\partial \pi_x} = \frac{\partial \phi}{\partial x}, \quad \frac{\partial H}{\partial \phi} = -\frac{\partial \pi_t}{\partial t} - \frac{\partial \pi_x}{\partial x},
\]

from which we derive the following set coupled first order equations

\[
\pi_t = \frac{\partial \phi}{\partial t}, \quad \pi_x = -v^2 \frac{\partial \phi}{\partial x}, \quad \frac{\partial \pi_t}{\partial t} + \frac{\partial \pi_x}{\partial x} - 4 \lambda \phi (\phi^2 - 1) = 0.
\]

As usual, the canonical field equations for the scalar field \( \phi(x,t) \) just reproduce the definition of the momentum fields \( \pi_t \) and \( \pi_x \) from the Lagrangian density \( L \).

By inserting \( \pi_t \) and \( \pi_x \) into the second field equation the coupled set of first order field equations is converted into a single second order equation for \( \phi(x,t) \):

\[
\frac{\partial^2 \phi}{\partial t^2} - v^2 \frac{\partial^2 \phi}{\partial x^2} - 4 \lambda \phi (\phi^2 - 1) = 0,
\]

which coincides with Eq. \( \text{(47)} \), as expected.
4.2. “Natural” Hamiltonian density

A general Hamiltonian system with a quadratic momentum dependence is often referred to as a “natural”

\[ \mathcal{H} = \frac{1}{2} \pi^\alpha \pi_\alpha + W(\phi, x). \]

We note that this Hamiltonian density \( \mathcal{H} \) resembles the Hamiltonian function \( H \) of a conservative Hamiltonian system of classical particle mechanics, which is given by \( H = T + V \) as the sum of kinetic energy \( T \) and potential energy \( V \). The first set of canonical field equations then follows as

\[
\frac{\partial \phi}{\partial x^\mu} = \frac{\partial \mathcal{H}}{\partial \pi^\mu} = \frac{1}{2} \pi_\mu + \frac{1}{2} \pi^\alpha \frac{\partial \pi_\alpha}{\partial x^\mu} = \frac{1}{2} \pi_\mu + \frac{1}{2} \pi_\alpha \delta^\alpha_\mu = \pi_\mu.
\]

(49)

The second canonical field equation writes for the present Hamiltonian density

\[
\frac{\partial \mathcal{H}}{\partial \phi} = \frac{\partial W}{\partial \phi} = -\frac{\partial \pi^\alpha}{\partial x^\alpha} = -\frac{\partial \pi_\alpha}{\partial x^\alpha}.
\]

Inserting the momentum fields \( \pi^\mu, \pi_\mu \) we again end up with a second order equation for the scalar field \( \phi(x) \)

\[
\frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\alpha} \phi(x) + \frac{\partial W(\phi, x)}{\partial \phi} = 0.
\]

For a “harmonic” potential

\[ W(\phi, x) = \frac{1}{2} V(x) \phi^2, \]

we immediately obtain the Klein-Gordon equation

\[
\left[ \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\alpha} + V(x) \right] \phi(x) = 0.
\]

(50)

Equation (50) is thus the field equation pertaining to the Klein-Gordon Hamiltonian density

\[ \mathcal{H}_{\text{KG}} = \frac{1}{2} \pi^\alpha \pi_\alpha + \frac{1}{2} V(x) \phi^2. \]

In this regard, the Klein-Gordon equation is nothing else as the field theory analog of the equation of motion of the harmonic oscillator of point mechanics. For the constant potential factor

\[ V(x) = \left( \frac{mc}{\hbar} \right)^2 \Rightarrow \mathcal{H}_{\text{KG}} = \frac{1}{2} \pi^\alpha \pi_\alpha + \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 \phi^2, \]

(51)

we obtain the particular Klein-Gordon equation which describes in relativistic quantum field theory the dynamics of a free particle of zero spin and mass \( m \)

\[
\left[ \frac{\partial^2}{\partial x_\alpha \partial x^\alpha} + \left( \frac{mc}{\hbar} \right)^2 \right] \phi(x) = 0.
\]
Regarding the first canonical field equation (49) that follows from the “natural” Hamiltonian density, we observe that the momentum fields $\pi^\mu(x)$, $\pi_\mu(x)$ coincide with the partial derivatives of the scalar field $\phi(x)$. This reminds of the method of “canonical quantization”, where the transition from the classical mechanics is made by replacing the canonical momenta $p^\mu$ with corresponding operators $\hat{p}^\mu$ that are supposed to act on a complex-valued “wave function” $\phi(x)$. In position representation, these operators are

$$\hat{p}^\mu = i\hbar \frac{\partial}{\partial x^\mu}, \quad \hat{p}_\mu = i\hbar \frac{\partial}{\partial x^\mu}. \quad (52)$$

Solved for the conjugate momentum fields $\pi^\mu$ of covariant field theory, this yields the connection of the $\pi^\mu$ to the operator notation of quantum mechanics for all “natural” Hamiltonian densities

$$\pi^\mu(x) \equiv -\frac{i}{\hbar} \hat{p}^\mu \phi(x), \quad \pi_\mu(x) \equiv -\frac{i}{\hbar} \hat{p}_\mu \phi(x).$$

In the usual quantum mechanics’ formulation, the Klein-Gordon equation is derived by replacing according to Eq. (52) the physical quantities momentum and energy in the relativistic energy-momentum relation

$$p_\mu p^\mu = m^2 c^2$$

by the corresponding operators $p^\mu \to \hat{p}^\mu$ and letting the resulting operator act on a wave function $\phi(x)$. Obviously, this yields exactly the same field equation that we obtain in covariant field theory for the “harmonic” Hamiltonian density from Eq. (51).

4.3. Klein-Gordon Hamiltonian density for complex fields

We first consider the Klein-Gordon Lagrangian density $L_{KG}$ for a complex scalar field $\phi$ (see, for instance, Ref. 1):

$$L_{KG}(\phi, \phi^*, \partial^\mu \phi, \partial_{\mu} \phi^*) = (\hbar c \partial_\alpha \phi^*)(\hbar c \partial^\alpha \phi) - (mc^2 \phi^*)(mc^2 \phi). \quad (53)$$

Herein $\phi^*$ denotes complex conjugate field of $\phi$. Both quantities are to be treated as independent. The Euler-Lagrange equations (2) for $\phi$ and $\phi^*$ follow from this Lagrangian density as

$$\frac{\partial^2}{\partial x_\alpha \partial x^\alpha} \phi^* = -\left(\frac{mc}{\hbar}\right)^2 \phi^*, \quad \frac{\partial^2}{\partial x_\alpha \partial x^\alpha} \phi = -\left(\frac{mc}{\hbar}\right)^2 \phi. \quad (54)$$

As a prerequisite for deriving the corresponding Hamiltonian density $H_{KG}$ we must first define from $L_{KG}$ the conjugate momentum fields,

$$\pi_\mu = \frac{\partial L_{KG}}{\partial (\partial^\mu \phi)} = \hbar^2 c^2 \frac{\partial \phi^*}{\partial x^\mu}, \quad \pi_\mu^* = \frac{\partial L_{KG}}{\partial (\partial_{\mu} \phi^*)} = \hbar^2 c^2 \frac{\partial \phi}{\partial x^\mu}.$$

The Hamiltonian density $H$ then follows again as the Legendre transform of the Lagrangian density

$$H(\pi_\mu, \pi_\mu^*, \phi_I, \phi_I^*) = \pi_\mu \frac{\partial \phi_I}{\partial x^\alpha} + \pi_\mu^* \frac{\partial \phi_I^*}{\partial x^\alpha} - L.$$
The Klein-Gordon Hamiltonian density $H_{\text{KG}}$ is thus given by

$$H_{\text{KG}}(\pi_\mu, \pi^{\mu*}, \phi, \phi^*) = \frac{1}{\hbar^2 c^2} \pi_\mu \pi^{\mu*} + (mc^2)^2 \phi \phi^*. \quad (55)$$

For the Hamiltonian density (55), the canonical field equations (5) provide the following set of coupled first order partial differential equations

$$\frac{\partial H_{\text{KG}}}{\partial \pi_\mu} = \frac{\partial \phi}{\partial x_\mu} = \frac{1}{\hbar^2 c^2} \pi_\mu, \quad \frac{\partial H_{\text{KG}}}{\partial \pi^{\mu*}} = \frac{\partial \phi^*}{\partial x_\mu} = \frac{1}{\hbar^2 c^2} \pi^{\mu*}$$

Again, the canonical field equations for the scalar fields $\phi$ and $\phi^*$ coincide with the definitions of the momentum fields $\pi_\mu$ and $\pi^{\mu*}$ from the Lagrangian density $L_{\text{KG}}$. Eliminating the $\pi_\mu$, $\pi^{\mu*}$ from the canonical field equations then yields the Euler-Lagrange equations of Eq. (54).

For complex fields, the energy-momentum tensor in the Lagrangian formalism is defined analogously to the real case of Eq. (6)

$$T_{\nu \mu} = \frac{\partial L}{\partial (\partial_\nu \phi_I)} \frac{\partial \phi_I}{\partial x_\mu} - \frac{\partial L}{\partial \phi_I} \frac{\partial \phi_I}{\partial x_\mu} - \frac{\partial \pi_\mu}{\partial \pi_{\alpha}} \frac{\partial \phi_I}{\partial x_\mu}$$

Expressed by means of the complex Hamiltonian density $H$, this means

$$T_{\nu \mu}^\nu = \delta_\nu^\nu H + \pi^{\nu \mu} \frac{\partial \phi_I}{\partial x_\mu} - \delta_\nu^\nu \pi^{\nu \mu} \frac{\partial \phi_I}{\partial x_\mu} + \pi^{\nu \mu} \frac{\partial \phi_I}{\partial x_\mu}$$

For the Klein-Gordon Hamiltonian density $H_{\text{KG}}$ from Eq. (55), we thus get the particular energy-momentum tensor $T_{\nu \mu, \text{KG}}$

$$T_{\nu \mu, \text{KG}} = \frac{1}{\hbar^2 c^2} \left( \pi_\mu \pi^{\mu*} + \pi_\mu \pi^{\mu*} - \delta_\mu^\nu \pi_\mu \pi^{\mu*} \right) + \delta_\mu^\nu (mc^2)^2 \phi \phi^*.$$

4.4. Maxwell’s equations as canonical field equations

The Lagrangian density $L_{\text{M}}$ of the electromagnetic field is given by

$$L_{\text{M}}(A, \partial A, x) = -\frac{1}{4} \int j_\mu f^{\mu \nu} - \frac{4\pi}{c} j^\mu(x) A_\mu, \quad j_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \quad (57)$$

Herein, the four components $A_\mu$ of the 4-potential $A_\mu$ now take the place of the fields $\phi_I \equiv \phi_\mu \equiv A_\mu$ in the notation used so far. The Lagrangian density (57) thus entails a set of four Euler-Lagrange equations, i.e., an equation for each component $A_\mu$. The source vector $j^\mu = (c\rho, j_x, j_y, j_z)$ denotes the 4-vector of electric currents combining the usual current density vector $(j_x, j_y, j_z)$ of configuration space with the charge density $\rho$. In this notation, the Euler-Lagrange equations (2) take on the form,

$$\frac{\partial}{\partial x^\mu} \frac{\partial L_{\text{M}}}{\partial (\partial_\nu A_\mu)} - \frac{\partial L_{\text{M}}}{\partial A_\mu} = 0, \quad \mu = 0, \ldots, 3. \quad (58)$$
With $\mathcal{L}_M$ from Eq. (57), we obtain directly
\[
\frac{\partial f^{\mu
u}}{\partial x^{\nu}} + \frac{4\pi}{c} j^\mu = 0.
\] (59)

This is the tensor form of the inhomogeneous Maxwell equation. In order to formulate the equivalent Hamiltonian description, we first define, according to Eq. (3), the tensor field components $\pi^{\mu\nu}$ as the conjugate objects of vector components $A^\mu$,
\[
\pi^{\mu\nu}(x) = \frac{\partial \mathcal{L}_M}{\partial (\partial_x A_\mu)}.
\] (60)

With the particular Lagrangian density \([57]\), this means, in detail,
\[
\pi^{\lambda\alpha} = -\frac{1}{2} \left( \frac{\partial f^{\mu\nu}}{\partial (\partial_\alpha A_\lambda)} f^{\mu\nu} + \frac{\partial f^{\mu\nu}}{\partial (\partial_\alpha A_\lambda)} f^{\mu\nu} \right)
= -\frac{1}{2} \left( \frac{\partial f^{\mu\nu}}{\partial (\partial_\alpha A_\lambda)} f^{\mu\nu} + \frac{\partial f^{\mu\nu}}{\partial (\partial_\alpha A_\lambda)} f^{\mu\nu} \right)
= -\frac{1}{2} f^{\mu\nu} \frac{\partial f^{\mu\nu}}{\partial (\partial_\alpha A_\lambda)}
= -\frac{1}{2} f^{\mu\nu} \frac{\partial}{\partial (\partial_\alpha A_\lambda)} \left[ \left( \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right) \right]
= -\frac{1}{2} f^{\mu\nu} \left( \delta^\alpha_{\nu} \delta^\lambda_{\mu} - \delta^\lambda_{\nu} \delta^\alpha_{\mu} \right)
= -\frac{1}{2} \left( f^{\alpha\lambda} - f^{\lambda\alpha} \right)
= f^{\lambda\alpha} = \frac{\partial A^\alpha}{\partial x_\lambda} - \frac{\partial A^\lambda}{\partial x_\alpha}.
\]

The tensor $\pi^{\mu\nu}$ thus coincides with the electromagnetic field tensor $f^{\mu\nu}$, defined in Eq. (57). Corresponding to Eq. (4), we obtain the Hamiltonian density $\mathcal{H}_M$ as the Legendre-transformed Lagrangian density $\mathcal{L}_M$
\[
\mathcal{H}_M(A, \pi, x) = \pi^{\mu\nu} \frac{\partial A_\mu}{\partial x_\nu} - \mathcal{L}_M(A, \partial A, x).
\]

The double sum $\pi^{\mu\nu} \frac{\partial A_\mu}{\partial x_\nu}$ can be expressed in terms of the Lagrangian expression $f^{\mu\nu} f_{\mu\nu}$,
\[
f^{\mu\nu} f_{\mu\nu} = \left( \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right) \left( \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right)
= 2 \frac{\partial A_\mu}{\partial x_\nu} \frac{\partial A_\mu}{\partial x_\nu} - 2 \frac{\partial A_\mu}{\partial x_\nu} \frac{\partial A_\nu}{\partial x_\mu} = -2 \frac{\partial A_\mu}{\partial x_\nu} \left( \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right)
= -2 \pi^{\mu\nu} \frac{\partial A_\mu}{\partial x_\nu}.
\] (61)

Because of $\pi^{\mu\nu} = f^{\mu\nu}$, the Hamiltonian density $\mathcal{H}_M$ of the electromagnetic field is then obtained as
\[
\mathcal{H}_M(A, \pi, x) = -\frac{1}{2} \pi^{\mu\nu} \pi^{\mu\nu} + \frac{4\pi}{c} j^\mu(x) A_\mu, \quad \pi^{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}.
\] (62)
The first canonical field equation of Eqs. (5) follows from the derivative of the Hamiltonian density (62) with respect to \( \pi^\lambda_\alpha \)

\[
\frac{\partial A_\lambda}{\partial x^\alpha} = \frac{\partial H_M}{\partial \pi^\lambda_\alpha} = \frac{1}{2} \pi^\mu_\nu \frac{\partial \pi^\mu_\nu}{\partial \pi^\lambda_\alpha} - \frac{1}{2} \pi^\mu_\nu \frac{\partial \pi^\nu_\mu}{\partial \pi^\lambda_\alpha} = \frac{1}{2} \pi^\mu_\mu \frac{\partial \pi^\mu_\nu}{\partial \pi^\lambda_\alpha} - \frac{1}{2} \pi^\mu_\nu \frac{\partial \pi^\nu_\mu}{\partial \pi^\lambda_\alpha} = -\frac{1}{2} \pi^\mu_\nu \frac{\partial \pi^\nu_\mu}{\partial \pi^\lambda_\alpha} + \frac{1}{2} \pi^\mu_\nu \delta^\nu_\alpha.
\]

Interchanging the indices, we likewise get

\[
\frac{\partial A_\alpha}{\partial x^\lambda} = -\frac{1}{2} \pi^\alpha_\lambda.
\]

Making use of the antisymmetry of the tensor \( \pi^\mu_\nu \), the two preceding field equations can be combined to yield

\[
\frac{\partial A_\alpha}{\partial x^\lambda} - \frac{\partial A_\lambda}{\partial x^\alpha} = \pi^\lambda_\alpha. \tag{63}
\]

Similar to the previous examples, the first field equation reproduces the definition of the conjugate tensor field \( \pi^\mu_\nu \) from the Lagrangian density \( L_M \).

The second canonical field equation of Eqs. (5) is obtained calculating the derivative of the Hamiltonian density (62) with respect to \( A_\lambda \)

\[
-\frac{\partial \pi^\lambda_\alpha}{\partial x^\alpha} = \frac{\partial H_M}{\partial A_\lambda} = \frac{4 \pi}{c} j^\lambda.
\]

For the Maxwell Hamiltonian density (62), the second field equation is thus given by

\[
\frac{\partial \pi^\lambda_\alpha}{\partial x^\alpha} + \frac{4 \pi}{c} j^\lambda = 0, \tag{64}
\]

which agrees, as expected, with the corresponding Euler-Lagrange equation (59) because of \( \pi^{\mu\nu} = f^{\mu\nu} \).

### 4.5. The Proca Hamiltonian density

In relativistic quantum field theory, the dynamics of particles of spin 1 and mass \( m \) is derived from the Proca Lagrangian density \( L_P \),

\[
L_P = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} + \frac{1}{4} \Omega^2 A^\mu A_\mu, \quad f_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}, \quad \Omega = \frac{mc}{\hbar}. \tag{65}
\]

We observe that the kinetic term of \( L_P \) agrees with that of the Lagrangian density \( L_M \) of the electromagnetic field of Eq. (57). Therefore, the Euler-Lagrange equations read similar to those of Eq. (59)

\[
\frac{\partial f^{\mu\nu}}{\partial x^\nu} - \Omega^2 A^\mu = 0. \tag{66}
\]
The transition to the corresponding Hamilton description is performed by defining the momentum field tensors $\Pi^{\mu\nu}$ on the basis of the actual Lagrangian $L_P$ by

$$\Pi^{\mu\nu} = \frac{\partial L_P}{\partial (\partial_\nu A_\mu)}, \quad \Pi_{\mu\nu} = \frac{\partial L_P}{\partial (\partial^\nu A^\mu)}.$$  

Similar to the preceding section, we conclude

$$\Pi^{\mu\nu} = f^{\mu\nu}, \quad \Pi_{\mu\nu} = f_{\mu\nu}.$$  

With the Legendre transformation

$$\mathcal{H}_P = \Pi^{\mu\nu} \frac{\partial A_\mu}{\partial x^\nu} - L_P,$$

we obtain the Proca Hamiltonian density by following the path of Eq. (61)

$$\mathcal{H}_P = -\frac{1}{4} \Pi^{\mu\nu} \Pi_{\mu\nu} - \frac{1}{2} \Omega^2 A^\mu A_\mu.$$  

The canonical field equations emerge as

$$\frac{\partial \mathcal{H}_P}{\partial \Pi^{\mu\nu}} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} = \Pi_{\mu\nu},$$  

$$\frac{\partial \mathcal{H}_P}{\partial A_\mu} = -\frac{\partial \Pi^{\mu\nu}}{\partial x^\nu} = -\Omega^2 A^\mu.$$  

By means of eliminating $\Pi^{\mu\nu}$, this coupled set of pairs of first order equations can be converted into second order equations for the vector field $A^\mu$,

$$\frac{\partial}{\partial x^\nu} \left( \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \right) + \Omega^2 A^\mu = 0.$$  

As expected, this equation coincides with the Euler-Lagrange equation (66).

### 4.6. Canonical field equations of a coupled Klein-Gordon-Maxwell system

The Lagrangian density $L_{KGM}$ of a complex Klein-Gordon field $\phi$ that couples minimally to an electromagnetic 4-vector potential $A^\mu$ is given by

$$L_{KGM} = \left( \frac{\partial \phi}{\partial x^\mu} + iqA_\mu \phi \right) \left( \frac{\partial \phi^*}{\partial x_\mu} - iqA^\mu \phi^* \right) + \Omega^2 \phi^* \phi - \frac{1}{4} f^{\mu\nu} f_{\mu\nu}.$$  

The components $f_{\mu\nu}$ of the electromagnetic field tensor are defined in Eq. (57). The conjugate fields of $\phi$ and $A^\mu$ are defined from the Lagrangian $L_{KGM}$ by

$$\pi^\nu = \frac{\partial L_{KGM}}{\partial (\partial_\nu \phi)} = \frac{\partial \phi^*}{\partial x_\nu} - iqA^\nu \phi^*,$$

$$\pi^*_\nu = \frac{\partial L_{KGM}}{\partial (\partial^\nu \phi^*)} = \frac{\partial \phi}{\partial x_\nu} + iqA_\nu \phi,$$

$$\Pi^{\mu\nu} = \frac{\partial L_{KGM}}{\partial (\partial_\nu A_\mu)} = f^{\mu\nu}.$$
The corresponding Hamiltonian density $H_{\text{KGM}}$ is now obtained as the Legendre transform of $L_{\text{KGM}}$,

$$H_{\text{KGM}} = \Pi^{\mu\nu} \frac{\partial A_\mu}{\partial x^\nu} + \pi^\mu \frac{\partial \phi}{\partial x^\mu} + \pi^{\star \mu} \frac{\partial \phi^{\star}}{\partial x^\mu} - L_{\text{KGM}}.$$ 

To obtain the canonical form of $H_{\text{KGM}}$, all partial derivatives of the fields $\phi$ and $A^\mu$ must be replaced by the conjugate fields $\pi^\mu$ and $\Pi^{\mu\nu}$, respectively,

$$H_{\text{KGM}} = \pi^\mu \pi^{\star \mu} + iqA_\mu (\pi^{\star \mu} \phi^{\star} - \pi^{\mu} \phi) - \Omega^2 \phi^{\star} \phi - \frac{1}{2} \Pi^{\mu\nu} \Pi_{\mu\nu}. \quad (69)$$

As shown in Sect. 4.4, the derivative of the Hamiltonian density $H_{\text{KGM}}$ with respect to $\Pi^{\mu\nu}$ yields the canonical equation

$$\Pi_{\nu\mu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}.$$ 

From the derivatives of $H_{\text{KGM}}$ with respect to the $\pi^\mu$ and $\pi^{\star \mu}$, the following canonical field equations arise

$$\frac{\partial H_{\text{KGM}}}{\partial \pi^\mu} = \pi^{\star \mu} - iqA_\mu \phi = \frac{\partial \phi}{\partial x^\mu},$$

$$\frac{\partial H_{\text{KGM}}}{\partial \pi^{\star \mu}} = \pi^\mu + iqA_\mu \phi^{\star} = \frac{\partial \phi^{\star}}{\partial x^\mu}.$$ 

The third group of canonical field equations emerges from the derivatives of $H_{\text{KGM}}$ with respect to the $A_\mu$, and with respect to the $\phi$, $\phi^{\star}$ as

$$\frac{\partial H_{\text{KGM}}}{\partial A_\mu} = iq \left( \phi^{\star} \pi^{\mu \star} - \phi \pi^\mu \right) - \frac{\partial \Pi_{\mu \alpha}}{\partial x^\alpha},$$

$$\frac{\partial H_{\text{KGM}}}{\partial \phi} = -\Omega^2 \phi^{\star} - iqA_\alpha \pi^{\alpha} = -\frac{\partial \pi^{\alpha}}{\partial x^\alpha},$$

$$\frac{\partial H_{\text{KGM}}}{\partial \phi^{\star}} = -\Omega^2 \phi^{\star} + iqA_\alpha \pi^{\alpha} = -\frac{\partial \pi^{\star \alpha}}{\partial x^\alpha}.$$ 

By eliminating the conjugate fields $\Pi^{\mu\nu}$ and $\pi^\mu$, these field equations can be rewritten as second order partial differential equations, corresponding to those that follow from the Euler-Lagrange equations for the Lagrangian density $L_{\text{KGM}}$

$$\frac{\partial^2 \phi}{\partial x^\alpha \partial x^\alpha} - (\Omega^2 + q^2 A_\alpha A^\alpha) \phi^{\star} - 2iqA_\alpha \frac{\partial \phi}{\partial x^\alpha} - iq\phi \frac{\partial A^\alpha}{\partial x^\alpha} = 0,$$

$$\frac{\partial^2 \phi^{\star}}{\partial x^\alpha \partial x^\alpha} - (\Omega^2 + q^2 A_\alpha A^\alpha) \phi^{\star} + 2iqA_\alpha \frac{\partial \phi}{\partial x^\alpha} + iq\phi \frac{\partial A^\alpha}{\partial x^\alpha} = 0,$$

with the $J^\mu$ being defined by

$$J^\mu = -iq \left[ \phi^{\star} \left( \frac{\partial \phi}{\partial x^\mu} + iqA^\mu \phi \right) - \phi \left( \frac{\partial \phi^{\star}}{\partial x^\mu} - iqA^\mu \phi^{\star} \right) \right].$$
4.7. The Dirac Hamiltonian density

The dynamics of particles of spin $\frac{1}{2}$ having mass $m$ is described by the Dirac Lagrangian density $\mathcal{L}_D$. Introducing anticommutating $4 \times 4$ matrices $\gamma_i$, $i = 1, \ldots, 4$ and spin $\frac{1}{2}$ fields $\psi$, the Dirac Lagrangian density is given by

$$\mathcal{L}_D = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi, \quad (70)$$

wherein

$$\bar{\psi} \equiv \psi^\dagger \gamma^0.$$ 

In the following we show some fundamental relations among $\gamma$ matrices:

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$\gamma^0 = \gamma^0 \gamma^0 = 1$$

$$\gamma^\mu \gamma^\nu = 4$$

$$[\gamma^\mu, \gamma^\nu] \equiv \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu = -2i\sigma^{\mu\nu} \quad (71)$$

Note that in Eq. (70) the derivative acts on $\psi$ on the right. The Dirac Lagrangian density $\mathcal{L}_D$ can be symmetrized using the aforementioned relations of $\gamma$ matrices and by combining the Lagrangian density Eq. (70) with its adjoint, which leads to

$$\mathcal{L}_D = \frac{i}{2} (\bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi - m \bar{\psi} \psi). \quad (72)$$

The resulting Euler-Lagrange equations are identical to those derived from Eq. (70),

$$i \gamma^\mu \partial_\mu \psi - m \psi = 0$$

$$i \partial_\mu \bar{\psi} \gamma^\mu + m \bar{\psi} = 0. \quad (73)$$

Since the Wronskian determinant vanishes,

$$\det \left[ \frac{\partial^2 \mathcal{L}_D}{\partial (\partial_\mu \psi) \partial (\partial_\nu \psi)} \right] = 0, \quad (74)$$

the corresponding Legendre transformation of the Lagrangian density Eq. (72) is irregular. A term including the derivatives $\partial_\mu \psi$ and $\partial_\nu \psi$ enters the Lagrangian density with a prefactor of dimension mass$^{-1}$. As has been shown e.g. by Gasiorowicz, one can construct a divergence-free term of this structure, leading to invariant equations of motion and a regular Legendre transformation. The additional term is given by

$$\partial_\mu \bar{\psi} \sigma^{\mu\nu} \partial_\nu \psi,$$

corresponding to the divergence of $\bar{\psi} \sigma^{\mu\nu} \partial_\nu \psi$:

$$d \bar{\psi} \sigma^{\mu\nu} \partial_\nu \psi = \left[ \frac{\partial}{\partial x^\mu} (\bar{\psi} \sigma^{\mu\nu} \partial_\nu \psi) \right] = \frac{\partial}{\partial x^\mu} \left[ \bar{\psi} \gamma^\mu \partial_\mu \psi + (\partial_\mu \bar{\psi}) \frac{\partial}{\partial \psi} (\partial_\nu \psi) \right]$$

$$= \partial_\mu \bar{\psi} \sigma^{\mu\nu} \partial_\nu \psi + (\partial_\mu \partial_\nu \bar{\psi}) \bar{\psi} \sigma^{\mu\nu} \partial_\nu \psi = \partial_\mu \bar{\psi} \sigma^{\mu\nu} \partial_\nu \psi = \partial_\mu \bar{\psi} \sigma^{\mu\nu} \partial_\nu \psi$$
Note that \((\partial_\mu \partial_\nu \psi) \bar{\psi} \sigma^{\mu\nu}\) vanishes, since this term is summed over a symmetric and antisymmetric part. One obtains the equivalent Lagrangian density

\[
\mathcal{L}'_D = \frac{i}{2} \left( \bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi \right) - i\lambda \partial_\mu \bar{\psi} \sigma^{\mu\nu} \partial_\nu \psi. \tag{75}
\]

The canonical momenta follow as

\[
\pi^\mu = -\frac{\partial \mathcal{L}'_D}{\partial (\partial_\mu \psi)} = -\frac{i}{2} \bar{\psi} \gamma^\mu \psi - i\lambda \sigma^{\mu\nu} \partial_\nu \psi,
\]

\[
\bar{\pi}^\mu = \frac{\partial \mathcal{L}'_D}{\partial (\partial_\mu \bar{\psi})} = \frac{i}{2} \bar{\psi} \gamma^\mu \psi - i\lambda \bar{\psi} \sigma^{\mu\nu} \partial_\nu \psi. \tag{76}
\]

In order to Legendre transform the Lagrangian density, it is useful to express \(\partial_\mu \psi\) and \(\partial_\mu \bar{\psi}\) in terms of \(\bar{\pi}^\mu\) and \(\pi^\mu\). For the "inversion" of Eq. (76) the matrix

\[
\tau^\mu_{\nu\lambda} = \frac{2}{3} \sigma^{\mu\nu} - \frac{1}{3} \sigma^{\mu\lambda}
\]

is introduced which obeys the following relations:

\[
\gamma^\mu \tau^\mu_{\nu\lambda} = \tau^\nu_{\mu\lambda},
\]

\[
\tau^\mu_{\nu\alpha} \sigma^{\alpha\beta} = \delta^\mu_{\beta} \tau^\nu_{\mu\lambda} = \delta^\lambda_{\mu}.
\]

The terms \(\tau^\mu_{\nu\lambda} \pi^\mu\) and \(\bar{\pi}^\mu \tau^\mu_{\nu\lambda}\) lead to

\[
\partial_\nu \psi = \frac{i}{\lambda} \tau^\nu_{\mu\lambda} \pi^\mu + \frac{i}{6\lambda} \gamma^\nu \psi,
\]

\[
\partial_\nu \bar{\psi} = \frac{i}{\lambda} \bar{\pi}^\nu_{\mu\lambda} \tau^\mu_{\nu\lambda} - \frac{i}{6\lambda} \bar{\psi} \gamma^\nu \psi. \tag{77}
\]

Therefore the Legendre transformation can be fulfilled using

\[
\mathcal{H}_D = \bar{\pi}^\nu \partial_\nu \psi + \partial_\nu \bar{\psi} \pi^\nu - \mathcal{L}'_D.
\]

The terms arising in the expansion of this equation can be simplified using Eqs. (71), leading to the Dirac Hamiltonian density of the form

\[
\mathcal{H}_D = \frac{1}{\lambda} \left( i\bar{\pi}^\nu \tau^\mu_{\nu\lambda} \pi^\lambda + i\bar{\pi}^\nu \gamma^\nu \psi - \frac{1}{6} \bar{\psi} \gamma^\nu \pi^\nu + \frac{1}{3} \bar{\psi} \psi \right) + m\bar{\psi} \psi. \tag{78}
\]

The canonical equations of motion

\[
\partial_\mu \psi = \frac{\partial \mathcal{H}_D}{\partial \pi^\mu} = \frac{i}{\lambda} \left( \tau^\nu_{\mu\lambda} \pi^\lambda + \frac{1}{6} \gamma^\nu \psi \right),
\]

\[
\partial_\mu \bar{\psi} = \frac{\partial \mathcal{H}_D}{\partial \bar{\pi}^\mu} = \frac{i}{\lambda} \left( \bar{\pi}^\nu \tau^\mu_{\nu\lambda} - \frac{1}{6} \bar{\psi} \gamma^\nu \psi \right).
\]
correspond to the definition of the canonical momenta, see also Eq. (76) and Eq. (77). The other canonical equations of motion are given by

\[ \partial_{\mu} \pi^\mu = -\frac{\partial H_D}{\partial \psi} = \frac{1}{3\lambda} \left( \frac{i}{2} \gamma_{\nu} \pi^\nu - \psi \right) - m\psi \]  

(79)

\[ \partial_{\mu} \bar{\pi}^\mu = -\frac{\partial H_D}{\partial \bar{\psi}} = -\frac{1}{3\lambda} \left( \frac{i}{2} \bar{\pi}^\nu \gamma_{\nu} + \bar{\psi} \right) - m\bar{\psi}. \]

(80)

In order to show the equivalence of these equations of motion to those derived in the formalism of Euler-Lagrange, we express the canonical momenta through \( \psi \) and \( \partial_{\mu} \psi \), thereby using Eq. (76). The relations of \( \gamma \)-matrices Eqs. (71) then yield Eqs. (73), as shall be demonstrated for \( \partial_{\mu} \pi^\mu \)

\[ \partial_{\mu} \pi^\mu = \frac{i}{6\lambda} \gamma_{\nu} \left( -\frac{i}{2} \gamma^\nu \psi - i\lambda \sigma^{\mu\nu} \partial_{\nu} \psi \right) - \frac{1}{3\lambda} \psi - m\psi \]

\[ = \frac{1}{6} \gamma_{\nu} \sigma^{\mu\nu} \partial_{\mu} \psi - m\psi \]

\[ = \frac{i}{12} \left[ 4\gamma^\mu - \gamma_{\nu} \left( 2g^{\mu\nu} - \gamma^\nu \gamma^\mu \right) \right] \partial_{\mu} \psi - m\psi \]

\[ = \frac{i}{2} \gamma^\mu \partial_{\mu} \psi - m\psi \]

\[ \partial_{\mu} \pi^\mu = \partial_{\mu} \left( -\frac{i}{2} \gamma^\mu \psi - i\lambda \sigma^{\mu\nu} \partial_{\nu} \psi \right) \]

\[ = -\frac{i}{2} \partial_{\mu} \gamma^\mu \psi - i\lambda \partial_{\mu} \sigma^{\mu\nu} \partial_{\nu} \psi \]

\[ \Rightarrow i\gamma^\mu \partial_{\mu} \psi - m\psi = 0. \]

It should be mentioned that this section is similar to the derivation of the Dirac Hamiltonian density in Ref. [22]. However, the results of this section are worked out here in order to present a consistent and thorough study of covariant Hamiltonian field theory.

4.8. Hamiltonian density for a SU(2) gauge theory

The Lagrangian density \( \mathcal{L}_{YM} \) of a SU(2) Yang-Mills gauge theory consisting of a complex doublet \( \phi \) of scalar fields, the coupling constant \( g \) and SU(2) gauge fields \( A_{\mu}^a (a = 1, 2, 3) \) is given by (see e.g. Ref. [23])

\[ \mathcal{L}_{YM} = \left[ \left( \partial_{\mu} - ig \frac{\gamma}{2} A_{\mu} \right) \phi \right]\dagger \left[ \left( \partial_{\mu} - ig \frac{\gamma}{2} A_{\mu} \right) \phi \right] - V(\phi\dagger\phi) - \frac{1}{4} f_{\alpha\beta}^a f_{\mu\nu}^a. \]  

(81)

This Lagrangian density is invariant under space-time-dependent SU(2) gauge transformations. In the following the Hamiltonian density corresponding to Eq. (81) shall be derived and in section [53] we will present the generating function of an infinitesimal local SU(2) gauge transformation. The Hamiltonian density will be
obtained by a Legendre transformation without restriction to a particular gauge. The resulting Hamiltonian density is therefore different from those given in Ref. [22], where a gauge \( A_0 = 0, a = 1, 2, 3 \) has been chosen and only gauge fields have been taken into account.

Making use of the Levi-Civita tensor \( \epsilon_{abc} \), the following relations and definitions for Hermitian Pauli matrices \( \tau_a \) and covariant derivative \( D^\mu \) hold:

\[
\vec{\tau} = (\tau_1, \tau_2, \tau_3) \\
\left[ \frac{\tau_a}{2}, \frac{\tau_b}{2} \right] = i\epsilon_{abc} \frac{\tau^c}{2} \quad a, b, c = 1, 2, 3 \\
\phi = \left( \phi_1 \phi_2 \right) \\
\vec{A}^\mu = (A_1^\mu, A_2^\mu, A_3^\mu) \\
D^\mu = \left( \partial^\mu - ig \frac{\vec{\tau}}{2} \vec{A}^\mu \right) \\
f_a^{\mu\nu} = \partial^\mu A_\nu^a - \partial^\nu A_\mu^a + g\epsilon_{abc} A_\mu^b A^c_v \\
f_a^{\mu\nu} = -f_a^{\nu\mu} \quad (82)
\]

The momenta conjugate to \( \phi \) are given by

\[
\pi^\mu = \frac{\partial L_{YM}}{\partial (\partial_\mu \phi)} = \left( \partial^\mu - ig \frac{\vec{\tau}}{2} \vec{A}^\mu \right) \phi = \partial^\mu \phi + ig \vec{\tau} \frac{\vec{A}^\mu}{2} \\
\pi^{\mu\nu} = \frac{\partial L_{YM}}{\partial (\partial_\nu \phi^\dagger)} = \left( \partial^\mu - ig \frac{\vec{\tau}}{2} \vec{A}^\mu \right) \phi = \partial^\mu \phi - ig \vec{\tau} \frac{\vec{A}^\mu}{2} \phi. \quad (83)
\]

Note that \( \pi^\mu \) has the form of a \((1 \times 2)\) matrix in SU(2) parameter space, while \( \pi^{\mu\nu} \) takes on the form of a \(2 \times 1\) matrix. In analogy to quantum electrodynamics, an Abelian gauge theory, we obtain the conjugate momentum field tensors

\[
\Pi_\alpha^\lambda = \frac{\partial L_{YM}}{\partial (\partial_\alpha A_\lambda^\mu)} = f_\alpha^\lambda, \quad (84)
\]

as can be shown in the following way:

\[
\Pi_\alpha^\lambda = \frac{\partial}{\partial (\partial_\alpha A_\lambda^\mu)} \left[ -\frac{1}{4} \left( \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha \right) \left( \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right) - \frac{1}{4} \left( g\epsilon_{abc} A_\mu^a A^c_v \left( \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right) g\epsilon_{abc} A_\mu^b A_\lambda^c \right) \right] \\
= \frac{\partial}{\partial (\partial_\alpha A_\lambda^\mu)} \left[ -\frac{1}{4} \left( \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha \right) \left( \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right) - \frac{1}{2} g\epsilon_{abc} A_\mu^a A^c_v \left( \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right) \right] \quad (85)
\]

Note that here the term without \( \partial^\mu A^\nu \) contribution has been omitted. In analogy to the conjugate momentum tensor of electrodynamics, the first line in Eq. (85) can
be written as $\partial^\lambda A^\mu_\alpha - \partial^\alpha A^\lambda_\mu$. Furthermore one obtains

$$\Pi^{\lambda\alpha}_a = \partial^\lambda A^\mu_a - \partial^\alpha A^\lambda_a - \frac{1}{2} g \epsilon_{abc} A^b\nu A^{c\nu} \left( \delta^\lambda_a \delta^\nu_{\mu} - \delta^\alpha_a \delta^\nu_{\mu} \right)$$

$$= \partial^\lambda A^\mu_a - \partial^\alpha A^\lambda_a - \frac{1}{2} g \epsilon_{abc} (A^{b\alpha} A^{c\lambda} - A^{b\lambda} A^{c\alpha})$$

$$= \partial^\lambda A^\mu_a - \partial^\alpha A^\lambda_a + \frac{1}{2} g \epsilon_{abc} (A^{b\lambda} A^{c\alpha} + A^{c\alpha} A^{b\lambda})$$

$$= f^{\lambda\alpha}_a.$$

The Hamiltonian density $\mathcal{H}_{YM}$ follows as the Legendre transform of the Lagrangian density $\mathcal{L}_{YM}$:

$$\mathcal{H}_{YM} = \Pi^\mu_a \partial_\mu A^a \phi + \partial_\mu A^a \phi^\dagger + \Pi^\mu_a \partial_\mu A^a - \mathcal{L}_{YM}$$

Using Eq. (83) and Eq. (84) to express the Lagrangian density in terms of the conjugate momenta yields

$$\mathcal{L}_{YM} = \pi^\mu \pi^\dagger_\mu - V (\phi^\dagger \phi) - \frac{1}{4} \Pi^\mu_a \Pi^a_{\mu\nu}.$$
The equations of motion for scalar fields and the corresponding conjugate momenta are derived in the following way

\[
\frac{\mathcal{H}_{YM}}{\partial \pi^\nu} = \pi^\dagger_\nu + \frac{ig}{2} \vec{A}_\nu \phi = \partial_\nu \phi
\]

\[
\frac{\mathcal{H}_{YM}}{\partial \pi^\dagger_\nu} = \pi^\nu - ig\phi^\dagger \frac{\vec{A}_\nu}{2} = \partial_\nu \phi^\dagger
\]

\[
\frac{\mathcal{H}_{YM}}{\partial \phi} = ig\pi^\nu \frac{\vec{A}_\nu}{2} + \frac{\partial}{\partial \phi} V \left( \phi^\dagger \phi \right) = -\frac{\partial \pi^\nu}{\partial x_\nu}
\]

\[
\frac{\mathcal{H}_{YM}}{\partial \phi^\dagger} = -ig\frac{\vec{A}_\nu \pi^\nu}{2} + \frac{\partial}{\partial \phi^\dagger} V \left( \phi^\dagger \phi \right) = -\frac{\partial \pi^\nu}{\partial x_\nu}.
\]

As for the equations of motion for gauge fields and the corresponding momentum tensors we obtain

\[
\frac{\mathcal{H}_{YM}}{\partial \Pi_{\mu \nu}^{\lambda}} = \partial_\nu A_\mu^\lambda - \partial_\mu A_\nu^\lambda = \Pi_{a \mu \nu}^\lambda - g \epsilon^{abc} A_b^\nu A_c^\mu = \partial \Pi_{a \mu \nu}^{\lambda} - g \epsilon^{abc} A_b^\nu A_c^\mu = -\frac{\partial L_{YM}}{\partial x_\nu}
\]

It is worth noting that the equation of motion Eq. (87) follows from

\[
\frac{\mathcal{H}_{YM}}{\partial \Pi_{a \mu}^{\lambda}} = \frac{\partial L_{YM}}{\partial (\partial_\nu A_\mu^\lambda)} - \frac{\partial L_{YM}}{\partial (\partial_\nu A_\mu^\lambda)} = \frac{1}{2} g \left( \epsilon^{abc} \Pi_{a \mu}^{\lambda} A_b^\nu A_c^\mu + \epsilon^{abc} \Pi_{a \mu}^{\lambda} A_b^\nu A_c^\mu \right)
\]

Moreover the term proportional to \( \Pi_{b \mu \nu}^{\lambda} \) in the equation of motion Eq. (88) can be derived as

\[
\frac{\partial}{\partial x_\lambda} \left( \frac{1}{2} g \epsilon^{abc} \Pi_{a \mu}^{\lambda} A_b^\nu A_c^\mu \right) = \frac{1}{2} g \epsilon^{abc} \Pi_{a \mu}^{\lambda} \left( \epsilon_{d \nu}^{\lambda} \delta_{d \mu} A_c^\nu + \epsilon_{d \nu}^{\lambda} \delta_{d \mu} A_b^\nu \right)
\]

5. Examples of canonical transformations in covariant Hamiltonian field theory

5.1. Point transformation

Canonical transformations for which the transformed fields \( \phi_\nu \) only depend on the original fields \( \phi_\lambda \), and possibly on the independent variables \( x_\mu \), but not on the original conjugate fields \( \pi_\lambda \) are referred to as point transformations. The generic
form of a 4-vector generating function $F_2$ that defines such transformations has the components

$$F_2^\mu(\phi, \pi', x) = f_J(\phi, x) \pi^{J\mu'}.$$ 

Herein, $f_J = f_J(\phi, x)$ denotes a set of differentiable but otherwise arbitrary functions. According to the general rules (19) for generating functions of type $F_2$, the transformed field $\phi_{I'}$ follows as

$$\phi_{I'} = \partial x^{J\alpha} \partial \phi_{I'} = f_J(\phi, x) \delta^J_I \delta^\mu_{\nu}.$$ 

The complete set of transformation rules is then

$$\pi_{I\mu} = \pi'^{J\mu} \partial f_J \partial \phi_I, \quad \phi_{I'} = f_I(\phi, x), \quad H' = H + \pi'^{J\alpha} \partial f_J \partial x^\alpha_{\text{expl}}.$$ 

As a trivial example of a point transformation, we consider the generating function of the identical transformation

$$F_2^\mu(\phi, \pi') = \phi_J \pi^{J\mu'}.$$ 

The pertaining transformation rules (89) for the particular case $f_J(\phi) = \phi_J$ are, viz,

$$\pi_{I\mu} = \pi'^{J\mu}, \quad \phi_{I'} = \phi_I, \quad H' = H.$$ 

The existence of a neutral element is a necessary condition for the set of canonical transformations to form a group.

### 5.2. Canonical shift of the conjugate momentum vector field $\pi'$

The generator of a canonical transformation that shifts the conjugate 4-vector field $\pi'(x)$ can be defined in terms of a function of type $F_3(\phi', \pi, x)$ as

$$F_3^\mu = -\phi_I \left( \pi^{I\mu} + \eta^{I\mu} \right).$$

Herein, the 4-vector fields $\eta^I = \eta^I(x)$ are supposed to denote arbitrary functions of the $x^\mu$. The general transformation rules (22) simplify for this particular generating function to

$$\pi'^{I\mu} = \pi_{I\mu} + \eta^{I\mu} \quad \phi_{I'} = \phi_I \quad H' = H - \phi_{I'} \frac{\partial \eta^{I\alpha}}{\partial x^\alpha_{\text{expl}}},$$

hence

$$\pi'^{I\mu} = \pi_{I\mu} + \eta^{I\mu}, \quad \phi_{I'} = \phi_I, \quad H' = H - \phi_{I'} \frac{\partial \eta^{I\alpha}}{\partial x^\alpha_{\text{expl}}}. $$

(92)
Provided that the divergence of the fields $\eta_I(x)$ vanishes, then the Hamiltonian density $H$ is conserved

$$\frac{\partial \eta_I^\alpha}{\partial x^\alpha} = 0 \implies H' = H. \quad (93)$$

This means for the canonical field equations (9) that the vector field $\pi_I(x)$ conjugate to $\phi_I(x)$ is only determined up to a “shifting” field $\eta_I(x)$ that conforms to the condition (93),

$$\frac{\partial H}{\partial \pi_I^\alpha'} = \frac{\partial \phi_I}{\partial x^\mu}, \quad \frac{\partial H}{\partial \phi_I} = -\frac{\partial \pi_I^\alpha}{\partial x^\alpha}.$$

5.3. **Local and global gauge transformation of the field $\phi_I$**

A phase transformation of the field $\phi_I(x)$ of the form

$$\phi_I(x) \rightarrow \phi_I'(x) = \phi_I(x) e^{i\theta(x)} \quad (94)$$

is commonly called a “local gauge transformation”. We can conceive this as a point transformation that is generated by a 4-vector function of type $F_2$

$$F_2^\mu(\phi, \pi', x) = \phi_I \pi_I^\nu e^{i\theta(x)}. \quad (95)$$

The pertaining transformation rules follow directly from the general rules of Eqs. (19)

$$\phi_I' = \phi_I e^{i\theta(x)}, \quad \pi_I'^\nu = \pi_I^\nu e^{-i\theta(x)}, \quad H' = H + i \phi_I \pi_I^\alpha \frac{\partial \theta(x)}{\partial x^\alpha}. \quad (96)$$

In the particular case that $\theta$ does not depend on the $x^\mu$, hence if $\theta = \text{const.}$, then the gauge transformation is referred to as “global”. In that case, the generating function (95) itself does no longer explicitly depend on the $x^\mu$. The Hamiltonian density is thus always conserved under global gauge transformations $\phi_I(x) \rightarrow \phi_I(x) e^{i\theta}$,

$$\phi_I' = \phi_I e^{i\theta}, \quad \pi_I'^\nu = \pi_I^\nu e^{-i\theta}, \quad H' = H.$$

5.4. **Infinitesimal canonical transformation, generalized Noether theorem**

The generating function $F_2^\mu$ of an *infinitesimal* canonical transformation differs from that of an *identical* transformation (93) by a small quantity $\delta x^\alpha g_0^\alpha(\phi, \pi, x)$

$$F_2^\mu(\phi, \pi', x) = \phi_I \pi_I^\nu' + \delta x^\alpha g_0^\alpha(\phi, \pi, x). \quad (96)$$

To first order in the $\delta x^\alpha$, the subsequent transformation rules (19) are

$$\pi_I'^\nu = \pi_I^\nu - \delta x^\alpha \frac{\partial g_0^\alpha}{\partial \phi_I}, \quad \phi_I' = \phi_I + \delta x^\alpha \frac{\partial g_0^\alpha}{\partial \pi_I^\nu}, \quad H' = H + \delta x^\alpha \frac{\partial g_0^\beta}{\partial x^\beta}. \quad \text{expl}, \quad (97)$$

$$\delta \pi_I'^\nu = -\delta x^\alpha \frac{\partial g_0^\alpha}{\partial \phi_I}, \quad \delta \phi_I' = \delta x^\alpha \frac{\partial g_0^\alpha}{\partial \pi_I^\nu}, \quad \delta H = \delta x^\alpha \frac{\partial g_0^\beta}{\partial x^\beta}. \quad \text{expl}. \quad (97)$$
In order to derive Noether’s theorem, we additionally need the transformation rule for the partial derivative $\partial \phi_1 / \partial x^\nu$, which we derive from the rule for $\phi_1$ from Eq. (97) by calculating the divergence
\[
\frac{\partial \phi_1}{\partial x^\nu} \delta^\nu_\nu = \frac{\partial \phi_1}{\partial x^\nu} \delta^\nu_\nu + \delta x^\alpha \frac{\partial}{\partial x^\mu} \left( \frac{\partial g^\alpha_1}{\partial \pi^\nu} \right),
\]
thus
\[
\frac{\partial \phi_1}{\partial x^\nu} = \frac{\partial \phi_1}{\partial x^\nu} + \delta x^\alpha \frac{\partial}{\partial x^\mu} \left( \frac{\partial g^\alpha_1}{\partial \pi^\nu} \right).
\]
We furthermore need to calculate the divergence of the characteristic function $g^\alpha_\beta$ of the generating function (96). With the transformation rules (97), the divergence reads
\[
\delta x^\alpha \frac{\partial g^\beta_\alpha}{\partial x^\beta} = \delta x^\alpha \left( \frac{\partial g^\beta_\alpha}{\partial \phi_1} \frac{\partial \phi_1}{\partial x^\beta} + \frac{\partial g^\beta_\alpha}{\partial \pi^\gamma} \frac{\partial \pi^\gamma}{\partial x^\beta} \right|_{\text{expl}} \right)
\]
\[
= -\delta x^\alpha \frac{\partial \phi_1}{\partial x^\beta} + \delta x^\alpha \frac{\partial \pi^\gamma}{\partial x^\beta} + \delta H.
\]
As we are interested in symmetries that evolve in the course of the system’s space-time evolution, the canonical field equations (5) can be inserted to yield
\[
\delta x^\alpha \frac{\partial g^\beta_\alpha}{\partial x^\beta} = -\frac{\partial H}{\partial \pi^\beta} \delta x^\alpha \frac{\partial \phi_1}{\partial x^\beta} - \frac{\partial H}{\partial \phi_1} \delta \phi_1 + \frac{\partial H}{\partial \phi_1} \delta \phi_1 + \frac{\partial H}{\partial \pi^\beta} \delta x^\alpha \frac{\partial H}{\partial x^\alpha} \right|_{\text{expl}}
\]
\[
= \delta x^\alpha \frac{\partial H}{\partial x^\alpha} \right|_{\text{expl}},
\]
Therefore,
\[
\frac{\partial g^\beta_\alpha}{\partial x^\beta} \right|_{\text{expl}} = \frac{\partial H}{\partial x^\alpha} \right|_{\text{expl}} - \frac{\partial g^\beta_\alpha}{\partial \phi_1} \frac{\partial \phi_1}{\partial x^\beta} - \frac{\partial g^\beta_\alpha}{\partial \pi^\beta} \frac{\partial \pi^\gamma}{\partial x^\beta} \right|_{\text{expl}}.
\]
Noether’s theorem can now be derived by calculating the change of the Lagrangian density $L$ that is induced by the infinitesimal canonical transformation (97). As the transformation is supposed to be canonical, original and transformed Lagrangian densities, $L$ and $L'$, can only differ be a divergence $\delta x^\alpha \partial f^\beta_\alpha / \partial x^\beta$,
\[
\delta L \equiv L' - L = \delta x^\alpha \frac{\partial f^\beta_\alpha}{\partial x^\beta}.
\]
This means in the Hamiltonian description
\[
\pi^\beta \frac{\partial \phi_1}{\partial x^\beta} - H = \pi^\beta \frac{\partial \phi_1}{\partial x^\beta} - H' - \delta x^\alpha \frac{\partial f^\beta_\alpha}{\partial x^\beta}.
\]
The primed quantities in the preceding equation is now expressed in terms of the unprimed ones according to the transformation rules (97)
\[
\pi^\beta \frac{\partial \phi_1}{\partial x^\beta} = \left( \pi^\beta - \delta x^\alpha \frac{\partial g^\beta_\alpha}{\partial \phi_1} \right) \left( \frac{\partial \phi_1}{\partial x^\beta} + \delta x^\alpha \frac{\partial}{\partial x^\mu} \left[ \frac{\partial g^\alpha_1}{\partial \pi^\mu} \right] \right) - \delta x^\alpha \frac{\partial g^\beta_\alpha}{\partial x^\beta} \right|_{\text{expl}} - \delta x^\alpha \frac{\partial f^\beta_\alpha}{\partial x^\beta}.
\]
The terms not depending on $\delta x^\alpha$ cancel. As the $\delta x^\alpha$ are supposed to be independent, the first-order terms in $\delta x^\alpha$ entail the set of equations

$$
\pi^\beta I \frac{\partial}{\partial x^\mu} \left( \frac{\partial g^\alpha_\mu}{\partial \pi^I} \right) - \frac{\partial g^\alpha_\mu}{\partial \phi_I} \frac{\partial}{\partial x^\beta} = \frac{\partial f^\beta_\alpha}{\partial x^\beta}_{\text{expl}}.
$$

Inserting Eq. (99), this writes, equivalently

$$
\pi^\mu I \frac{\partial}{\partial x^\beta} \left( \frac{\partial g^\beta_\mu}{\partial \pi^I} \right) + \frac{\partial g^\beta_\mu}{\partial \pi^I} \frac{\partial}{\partial x^\beta} = \frac{\partial H}{\partial x^\alpha}_{\text{expl}} + \frac{\partial f^\beta_\alpha}{\partial x^\beta}.
$$

The sum on the left hand side can now be written as a divergence,

$$
\frac{\partial}{\partial x^\beta} \left( \pi^I \frac{\partial g^\beta_\mu}{\partial \pi^I} - f^\beta_\alpha \right) = \frac{\partial H}{\partial x^\alpha}_{\text{expl}},
$$

(101)

This is the generalized Noether theorem of classical field theory in the Hamiltonian formulation. The theorem thus consists of the continuity equation that emerges if we relate the characteristic function $g^I_\mu$ of (96) with the change of the Lagrangian density $\mathcal{L}$ that is induced by the infinitesimal canonical transformation (96). If we apply an infinitesimal canonical transformation with characteristic function $g^I_\mu$ to a given Hamiltonian system $\mathcal{H}$, then the related change $\delta \mathcal{L}$ of the Lagrangian density $\mathcal{L}$ is determined by functions $f^\beta_\alpha$. For the four vectors of the 4-current densities $j_\alpha$ ("Noether currents")

$$
j^\beta_\alpha (\phi, \pi, x) = \pi^I \frac{\partial g^\beta_\mu}{\partial \pi^I} - f^\beta_\alpha (\phi, \pi, x)
$$

(102)

we have a set of four equations

$$
\frac{\partial j^\beta_\alpha}{\partial x^\alpha} = \frac{\partial H}{\partial x^\alpha}_{\text{expl}},
$$

(103)

which each represents a continuity equation for the Noether current $j_\alpha$ if the Hamiltonian density $\mathcal{H}$ does not explicitly depend on the respective $x^\alpha$. It is, of course, not assured a priori that for a given function $g^I_\mu$ in the generator (97) analytical functions $f^\beta_\alpha (\phi, \pi, x)$ exist that satisfy Eq. (101). If, however, functions $f^\beta_\alpha$ of the canonical fields $\phi_I$, $\pi^I$ exist, such that

$$
\delta \mathcal{L} = \delta x^\alpha \frac{\partial f^\beta_\alpha}{\partial x^\beta},
$$

holds under transformation (97), then the infinitesimal canonical transformation generated by Eq. (96) represents a symmetry transformation of the given system. In that case, the continuity equation (103) holds for the 4-currents $j_\alpha$, defined by Eq. (102).

We can express the Noether currents (102) alternatively in terms of the variation of the fields $\phi_I$. Inserting the transformation rule of Eq. (97) into Eq. (102), we get

$$
\delta x^\alpha j^\beta_\alpha = \pi^I \delta \phi_I - \delta x^\alpha f^\beta_\alpha.
$$

(104)
Defining functions $\psi_{I\alpha}$ by
\[ \delta \phi_I = \delta x^\alpha \psi_{I\alpha}(\phi, \pi, x), \]
then we can write Eq. (104) separately for each component $\alpha$ because of the independence of the $\delta x^\alpha$. We thus get an alternative formulation of Noether’s theorem of Eqs. (102), (103),
\[ \frac{\partial j^\beta_\alpha}{\partial x^\beta} = \frac{\partial H}{\partial x^\alpha} \bigg|_{\text{expl}}, \quad j^\beta_\alpha = \pi^{I\beta} \psi_{I\alpha}(\phi, \pi, x) - f^\beta_\alpha(\phi, \pi, x). \]

In the particular case that the Lagrange density $L$ remains invariant ($\delta L = 0$) under the infinitesimal canonical transformation (97), we have $\partial f^\beta_\alpha / \partial x^\beta = 0$. We may then set $f^\beta_\alpha = 0$, as otherwise the $f^\beta_\alpha$ would be trivial contributions to the current components $j_\alpha$. Then, Noether’s theorem takes on the simple form
\[ \frac{\partial j^\beta_\alpha}{\partial x^\beta} = \frac{\partial H}{\partial x^\alpha} \bigg|_{\text{expl}}, \quad j^\beta_\alpha(\phi, \pi, x) = \pi^{I\beta} \frac{\partial g^\beta_\alpha}{\partial \pi^{I\gamma}}, \quad (105) \]
The conventional form of Noether’s is recovered for the special case of an infinitesimal point transformation. The latter is associated with a characteristic function $g^\mu_\nu$ in the generator (96) that depends linearly on the $\pi^{I\nu}$
\[ g^\mu_\nu(\phi, \pi, x) = \pi^{I\nu} \psi_{I\mu}(\phi, x), \quad (106) \]
wherein each $\psi_{I\mu}$ denotes an arbitrary function of the $\phi_I$ and the independent variables $x$. The $\psi_{I\mu}$ then determine the variation of the fields $\phi_I$ according to the transformation rule from Eq. (106)
\[ \delta \phi_I \equiv \phi_I' - \phi_I = \delta x^\alpha \psi_{I\alpha}(\phi, x). \]
The special Noether theorem — as it emerges similarly from the Lagrangian formalism — thus reads
\[ \frac{\partial j^\beta_\alpha}{\partial x^\beta} = \frac{\partial H}{\partial x^\alpha} \bigg|_{\text{expl}}, \quad j^\beta_\alpha(\phi, \pi, x) = \pi^{I\beta} \psi_{I\alpha}(\phi, x) - f^\beta_\alpha(\phi, \pi, x). \quad (107) \]
We note that due to the restriction to a generating function (96) with particular characteristic function $g^\mu_\nu$ from Eq. (106), the Noether theorem of Eq. (107) cannot cover, in general, all symmetries of a given system. The reason is that the point transformation defined by Eq. (106) is not the most general transformation that conserves the variational principle of Eq. (12). Symmetries of Hamiltonian (Lagrangian) systems that cannot be represented by point transformations, are referred to in literature as “non-Noether symmetries”. Yet, such symmetries can always be expressed in terms of the generalized form of Noether’s theorem from Eqs. (102), (103).
5.4.1. Example: shift of reference system in space-time

As a simple example of an application of Noether’s theorem, we now determine the continuity equation that emerges if a given system is invariant with respect to a shift $\delta \mathbf{x}$ in space-time

$$\mathbf{x}' = \mathbf{x} + \delta \mathbf{x}.$$  

The related change of the Lagrangian density $\mathcal{L}$ is expressed by Eq. (100),

$$\delta \mathcal{L} \equiv \mathcal{L}' - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x^\alpha} \delta x^\alpha = \frac{\partial f^\beta_\alpha}{\partial x^\beta} \delta x^\alpha.$$  

As the $\delta x^\alpha$ do not depend on each other, we get separately an equation for each component

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{\partial f^\beta_\alpha}{\partial x^\beta}.$$  

A shift of the reference system in space-time entails a variation of the fields $\phi_I$,  

$$\delta \phi_I \equiv \phi_I' - \phi_I = \frac{\partial \phi_I}{\partial x^\alpha} \delta x^\alpha.$$  

As the transformed fields $\phi_I'$ only depend on the original fields $\phi_I$, we are dealing with a point transformation. According to Eq. (106), the components $g^\nu_\mu$ of the generating function (96) follow as

$$\psi_I^\mu(\phi, \mathbf{x}) = \frac{\partial \phi_I}{\partial x^\mu}, \quad g^\nu_\mu(\phi, \pi, \mathbf{x}) = \pi^I_\nu \frac{\partial \phi_I}{\partial x^\mu}.$$  

For each index $\alpha$, we can now set up Noether’s theorem from Eq. (101),

$$\frac{\partial}{\partial x^\beta} \left( \pi^{I_\mu} \frac{\partial \phi_I}{\partial x^\mu} - \frac{\partial \mathcal{L}}{\partial x^\alpha} \delta^\beta_\alpha \mathcal{H} \right) = \frac{\partial \mathcal{H}}{\partial x^\alpha}.$$  

Inserting the actual $g^\beta_\alpha$ and replacing the Lagrangian density $\mathcal{L}$ by a Hamiltonian density $\mathcal{H}$ according to Eq. (4) yields

$$\frac{\partial}{\partial x^\beta} \left( \pi^{I_\mu} \frac{\partial \phi_I}{\partial x^\mu} - \frac{\partial \mathcal{L}}{\partial x^\alpha} \delta^\beta_\alpha \mathcal{H} \right) = \frac{\partial \mathcal{H}}{\partial x^\alpha}.$$  

With the terms in parentheses on the left hand side, Noether’s theorem obviously provides the components of the energy-momentum tensor $T^\beta_\alpha$ from Eq. (7)

$$\frac{\partial T^\beta_\alpha}{\partial x^\beta} = \frac{\partial \mathcal{H}}{\partial x^\alpha}.$$  

If the Hamiltonian density $\mathcal{H}$ does not explicitly depend on $x^\mu$, then the system is invariant with respect to a shift of the independent variable $x^\mu$. We then get a continuity equation for the related Noether 4-current $j^\mu$,

$$\frac{\partial \mathcal{H}}{\partial x^\mu} = 0 \iff \frac{\partial T^\beta_\mu}{\partial x^\beta} = 0, \quad T^\beta_\mu = \pi^{I_\mu} \frac{\partial \phi_I}{\partial x^\mu} - \delta^\beta_\mu \pi^I_\gamma \frac{\partial \phi_I}{\partial x^\gamma} + \delta^3_\mu \mathcal{H}.$$  

For $\frac{\partial \mathcal{H}}{\partial x^\mu} = 0$, the four components of the conserved Noether current are thus given by the $\mu$-th column of the energy-momentum tensor $T^\nu_\mu$. 

5.5. Canonical transformation inducing an infinitesimal space-time step

We consider the following generating function $F^{\mu}_{2}$ of an infinitesimal canonical transformation:

$$F^{\mu}_{2}(\phi, \pi', x) = \phi_I^{'} \pi^{I\mu'} + \delta x^\alpha \left[ \delta^{\mu}_\alpha \mathcal{H} + \pi^{I\mu} \frac{\partial \phi_I}{\partial x^\alpha} - \phi_I \frac{\partial \pi^{I\mu}}{\partial x^\alpha} - \delta^{\mu}_\alpha \left( \pi^{I\beta} \frac{\partial \phi_I}{\partial x^\beta} - \phi_I \frac{\partial \pi^{I\mu}}{\partial x^\beta} \right) - \delta^{\mu}_\alpha \pi^{I\beta} \frac{\partial \phi_I}{\partial x^\beta} \right].$$

(109)

In order to illustrate this generating function, we imagine for a moment a system with only one independent variable, $t$. As a consequence, only one conjugate field $\pi^I$ could exist for each $\phi_I$. In that system, the last six terms of Eq. (109) would obviously cancel, hence, the generating function $F_2$ would simplify to

$$F_2(\phi_I, \pi^I, t) = \phi_I^{'} \pi^I + \mathcal{H} \delta t.$$

We recognize this function from point mechanics as the generator of the infinitesimal canonical transformation that shifts an arbitrary Hamiltonian system along an infinitesimal time step $\delta t$.

Alternatively, we can express the generating function (109) in terms of the energy-momentum tensor from Eq. (11)

$$F^{\mu}_{2}(\phi, \pi', x) = \phi_I^{'} \pi^{I\mu'} + \delta x^\alpha \left[ T^{\mu}_{\alpha} - \phi_I \left( \pi^{I\beta} \frac{\partial \phi_I}{\partial x^\beta} - \frac{\partial \pi^{I\beta}}{\partial x^\beta} \right) + x^\beta \left( \frac{\partial \pi^{I\mu}}{\partial x^\alpha} \frac{\partial \phi_I}{\partial x^\beta} - \frac{\partial \pi^{I\mu}}{\partial x^\beta} \frac{\partial \phi_I}{\partial x^\alpha} \right) \right].$$

We observe that it is essentially the energy-momentum tensor (7) that determines the infinitesimal space-time step transformation.

Applying the general transformation rules (19) for generating functions of type $F_2$ to the generator from Eq. (109), then — similar to the preceding example — only terms of first order in $\delta x^\mu$ need to be taken into account. The derivative of $F^{\mu}_{2}$ with respect to $\phi_I$ yields

$$\pi^{I\mu} = \frac{\partial F^{\mu}_{2}}{\partial \phi_I} = \pi^{I\mu'} + \delta x^\alpha \left( \delta^{\mu}_\alpha \frac{\partial \mathcal{H}}{\partial \phi_I} - \pi^{I\beta} \frac{\partial \phi_I}{\partial x^\beta} + \delta^{\mu}_\alpha \frac{\partial \pi^{I\beta}}{\partial x^\beta} \right)$$

$$= \pi^{I\mu'} + \delta x^\alpha \left( -\delta^{\mu}_\alpha \frac{\partial \phi_I}{\partial x^\beta} - \frac{\partial \pi^{I\beta}}{\partial x^\beta} + \delta^{\mu}_\alpha \frac{\partial \pi^{I\beta}}{\partial x^\beta} \right)$$

$$= \pi^{I\mu'} - \frac{\partial \pi^{I\mu}}{\partial x^\alpha} \delta x^\alpha.$$

This means for $\delta \pi^{I\mu} \equiv \pi^{I\mu'} - \pi^{I\mu}$

$$\delta \pi^{I\mu} = \frac{\partial \pi^{I\mu}}{\partial x^\alpha} \delta x^\alpha.$$

(110)
To first order, the general transformation rule (19) for the field $\phi_I$ takes on the particular form for the actual generating function (109):

$$
\phi_I' \delta_\mu^\nu = \frac{\partial F_2}{\partial \pi^{\mu \nu}} = \phi_I \delta_\mu^\nu + \delta x^\alpha \left( \delta_\alpha^\mu \frac{\partial H}{\partial x^\alpha} - \delta_\nu^\mu \frac{\partial \phi_I}{\partial x^\alpha} \right)
$$

$$
= \phi_I \delta_\mu^\nu + \delta x^\alpha \left( \delta_\alpha^\mu \frac{\partial \phi_I}{\partial x^\alpha} + \delta_\nu^\mu \frac{\partial \phi_I}{\partial x^\alpha} - \delta_\alpha^\mu \frac{\partial \phi_I}{\partial x^\alpha} \right)
$$

$$
= \phi_I \delta_\mu^\nu + \delta x^\alpha \frac{\partial \phi_I}{\partial x^\alpha} \delta x^\alpha,
$$

hence with $\delta \phi_I \equiv \phi_I' - \phi_I$

$$
\delta \phi_I = \frac{\partial \phi_I}{\partial x^\alpha} \delta x^\alpha.
$$

The transformation rule $\delta \mathcal{H} \equiv \mathcal{H}' - \mathcal{H}$ for the Hamiltonian density finally follows from the explicit dependence of the generating function on the $x^\nu$:

$$
\delta \mathcal{H} = \frac{\partial F_2}{\partial x^\mu} \bigg|_{\text{expl}} = \delta x^\alpha \left[ \delta_\alpha^\mu \frac{\partial \mathcal{H}}{\partial x^\alpha} + \delta_\beta^\mu \left( \frac{\partial \pi^{\beta \mu}}{\partial x^\alpha} \frac{\partial \phi_I}{\partial x^\alpha} - \frac{\partial \pi^{\beta \mu}}{\partial x^\alpha} \frac{\partial \phi_I}{\partial x^\alpha} \right) \right]
$$

$$
= \delta x^\alpha \left[ \frac{\partial \mathcal{H}}{\partial x^\alpha} + \frac{\partial \pi^{\alpha \mu}}{\partial x^\alpha} \frac{\partial \phi_I}{\partial x^\alpha} - \frac{\partial \pi^{\alpha \mu}}{\partial x^\alpha} \frac{\partial \phi_I}{\partial x^\alpha} \right]
$$

$$
= \delta x^\alpha \left[ \frac{\partial \mathcal{H}}{\partial x^\alpha} + \frac{\partial \mathcal{H}}{\partial x^\alpha} \frac{\partial \phi_I}{\partial x^\alpha} + \frac{\partial \mathcal{H}}{\partial x^\alpha} \frac{\partial \phi_I}{\partial x^\alpha} \right]
$$

$$
= \frac{\partial \mathcal{H}}{\partial x^\alpha} \delta x^\alpha.
$$

Summarizing, we infer from the transformation rules (110), (111), and (112) that the generating function (109) defines the particular canonical transformation that infinitesimally shifts a given system in space-time in accordance with the canonical field equations (5). As such a canonical transformation can be repeated an arbitrary number of times, we can induce that a transformation along finite steps in space-time is also canonical. We thus have the important result the space-time evolution of a system that is governed by a Hamiltonian density itself constitutes a canonical transformation. As canonical transformations map Hamiltonian systems into Hamiltonian systems, it is ensured that each Hamiltonian system remains so in the course of its space-time evolution.

5.6. Lorentz gauge as a canonical point transformation of the Maxwell Hamiltonian density

The Hamiltonian density $\mathcal{H}_M$ of the electromagnetic field was derived in Sec. 4.4. The correlation of the conjugate fields $\pi_{\mu \nu}$ with the 4-vector potential $\mathbf{A}$ is determined by the first field equation (63) as the generalized rotation of $\mathbf{A}$. This means,
on the other hand, that the correlation between $A$ and the $\pi_{\mu\nu}$ is not unique. Defining a transformed vector potential $A'$ according to

$$A_{\mu}' = A_{\mu} + \frac{\partial \chi(x)}{\partial x^\mu}, \quad (113)$$

with $\chi = \chi(x)$ an arbitrary differentiable function, we find

$$\pi_{\mu\nu}' = \frac{\partial A_{\nu}'}{\partial x^\mu} - \frac{\partial A_{\mu}'}{\partial x^\nu} = \frac{\partial A_{\nu}}{\partial x^\mu} \frac{\partial \chi}{\partial x^\nu} + \frac{\partial^2 \chi}{\partial x^\mu \partial x^\nu} = \pi_{\mu\nu}. \quad (114)$$

We will now show that the gauge transformation (113) can be regarded as a canonical point transformation, whose generating function $F_{\nu}^{\alpha}$ is given by

$$F_{\nu}^{\alpha}(A, \pi, x) = \left( A_{\lambda} + \frac{\partial \chi(x)}{\partial x^\lambda} \right) \pi_{\alpha}. \quad (115)$$

In the notation of this example, the general transformation rules (19) are rewritten as

$$\pi_{\mu\nu}' = \frac{\partial F_{\nu}^{\alpha}}{\partial A_{\mu}}, \quad A_{\mu}' = A_{\mu} + \epsilon \frac{\partial F_{\nu}^{\alpha}}{\partial \pi_{\alpha}} \quad (116)$$

which yield for the particular generating function of Eq. (115) the transformation prescriptions

$$\pi_{\mu\nu}' = \frac{\partial A_{\lambda'}}{\partial A_{\mu}} \pi_{\lambda\nu} = \delta_{\lambda}^{\mu} \pi_{\lambda\nu} = \pi_{\mu\nu}'$$

$$A_{\mu}' = A_{\mu} + \epsilon \frac{\partial \chi(x)}{\partial x^\mu} \quad (116)$$

$$\mathcal{H}' = \mathcal{H} + \frac{\partial F_{\nu}^{\alpha}}{\partial x^\alpha} \pi_{\alpha} = \mathcal{H}. \quad (116)$$

The canonical transformation rules coincide with the correlations of Eqs. (113) and (114) defining the Lorentz gauge. The last equation holds because of the antisymmetry of the canonical momentum tensor $\pi_{\lambda\alpha} = -\pi_{\alpha\lambda}$. Thus, the value of the Hamiltonian density (62) is invariant under the Lorentz gauge.

In order to determine the conserved “Noether current” that is associated with the canonical point transformation generated by $F_{\nu}^{\alpha}$ from Eq. (115), we need the generator of the corresponding infinitesimal canonical point transformation,

$$F_{\nu}^{\alpha}(A, \pi', x) = A_{\lambda} \pi_{\lambda\nu}' + \epsilon g'(\pi, x), \quad g'(\pi, x) = \pi_{\lambda\nu} \frac{\partial \chi(x)}{\partial x^\lambda}. \quad (117)$$

From the pertaining transformation rules

$$\pi_{\mu\nu}' = \pi_{\mu\nu}, \quad A_{\mu}' = A_{\mu} + \epsilon \frac{\partial \chi(x)}{\partial x^\mu}, \quad \mathcal{H}' = \mathcal{H},$$

we directly find that $\mathcal{L}$ is also maintained, which means that $\partial f^\beta / \partial x^\beta = 0$, hence $f^\beta = 0$ according to Eq. (100). In a source-free region, we have $j(x) = 0$, hence
\[ \frac{\partial H}{\partial x^\mu} \bigg|_{\text{expl}} = 0 \] for all \( \mu \). The Noether theorem for point transformations from Eq. (107) then directly yields the conserved 4-current \( j^\mu_N \)

\[ j^\mu_N(\pi, \chi) = \pi^\lambda \frac{\partial \chi}{\partial x^\lambda} = \left( \frac{\partial A^\mu}{\partial x^\chi} - \frac{\partial A^\chi}{\partial x^\mu} \right) \frac{\partial \chi}{\partial x^\lambda}. \]

We verify that \( j^\mu_N \) is indeed the conserved Noether current by calculating its divergence

\[ \frac{\partial j^\mu_N}{\partial x^\alpha} = \pi^\lambda \frac{\partial^2 \chi}{\partial x^\lambda \partial x^\alpha} + \frac{\partial \pi^\lambda}{\partial x^\alpha} \frac{\partial \chi}{\partial x^\lambda}. \]  

(118)

For the Hamiltonian density \( H_M \) of the electromagnetic field from Eq. (62), the tensor \( \pi^{\mu \nu} \) of canonical momenta is antisymmetric. The first term on the right hand side of Eq. (118) thus vanishes from symmetry considerations. In source-free regions, the canonical field equation from Eq. (59) is

\[ \frac{\partial \pi^\lambda}{\partial x^\alpha} = 0. \]

Therefore, the second term on the right hand side of Eq. (118) also vanishes.

5.7. Extended gauge transformation of the coupled Klein-Gordon-Maxwell field, local gauge invariance

The Hamiltonian density \( H_{KGM} \) of a complex Klein-Gordon field that couples minimally to an electromagnetic 4-vector potential \( A \) was introduced in Sec. 4.6 by Eq. (69). We now define for this Hamiltonian density an “extended gauge transformation” by means of the generating function

\[ F^\mu_2 = \phi \pi^\mu e^{-iq\chi(x)} + \phi^* \pi^\mu^* e^{iq\chi(x)} + \left( A^\mu + \frac{\partial \chi(x)}{\partial x^\mu} \right) \Pi^\mu\nu. \]  

(119)

Because of the explicit dependence of \( \chi \) on \( x \), this generator (119) defines a local gauge transformation. The general transformation rules (19), (116) read for the present generating function:

\[ \Pi^\lambda\nu = \Pi^\lambda\mu, \quad A^\mu = A^\mu + \frac{\partial \chi(x)}{\partial x^\mu}, \]

\[ \pi^\mu = \pi^\mu e^{iq\chi(x)}, \quad \phi' = \phi e^{-iq\chi(x)}, \]

\[ \pi^\mu^* = \pi^\mu e^{-iq\chi(x)}, \quad \phi'^* = \phi^* e^{iq\chi(x)}, \]

\[ \mathcal{H}' = \mathcal{H} + i q \left( \phi^* \pi^\mu^* - \phi \pi^\mu \right) \frac{\partial \chi(x)}{\partial x^\alpha}. \]

In the transformation rule for the Hamiltonian density, the term \( \Pi^\alpha \nu \frac{\partial^2 \chi}{\partial x^\nu \partial x^\alpha} \) vanishes because as \( \Pi^\alpha \nu \) is antisymmetric. The gauge-transformed Hamiltonian density \( H'_{KGM} \) is now obtained by inserting the transformation rules into the Hamiltonian density \( H_{KGM} \) of Eq. (69):

\[ H'_{KGM} = \pi^\mu^* \pi^\mu + i q A^\mu \left( \pi^\mu^* \phi' - \pi^\mu \phi' \right) - \Omega^2 \phi'^* \phi' - \frac{1}{4} \Pi^\alpha \nu \Pi^\mu\nu. \]
We observe that the Hamiltonian density (69) is form invariant under the canonical transformation generated by \( F_2 \) from Eq. (119).

In order to derive the conserved Noether current that is associated with this symmetry transformation, we first set up the generating function of the infinitesimal canonical transformation corresponding to (119)

\[ F_2^\mu = \phi \pi^{\mu'} + \phi^* \pi^{\mu''} + A_\nu \Pi^{\nu \mu'} + \epsilon (g_1^{\mu} + g_2^{\mu} + g_3^{\mu}) , \]

with the characteristic functions \( g_{1,2,3}^{\mu} \) given by:

\[
\begin{align*}
g_1^{\mu} &= -iq \phi \pi^\mu (x) , \\
g_2^{\mu} &= iq \phi^* \pi^\mu (x) , \\
g_3^{\mu} &= \frac{\partial \chi (x)}{\partial x^\nu} \Pi^{\nu \mu}.
\end{align*}
\]

The subsequent transformation rules are

\[
\begin{align*}
\Pi^{\lambda \mu'} &= \Pi^{\lambda \mu} , \\
A_\mu' &= A_\mu + \epsilon \frac{\partial \chi (x)}{\partial x^\mu} , \\
\pi^{\mu'} &= \pi^\mu (1 + \epsilon iq \chi (x)) , \\
\phi' &= \phi (1 - \epsilon iq \chi (x)) , \\
\phi'^* &= \phi^* (1 + \epsilon iq \chi (x)) ,
\end{align*}
\]

For a conserved Lagrangian density, we have \( j^\beta = 0 \). The Noether theorem from Eq. (107) now directly yields the conserved Noether current \( j_N \)

\[
j_N^\mu = g_1^{\mu} + g_2^{\mu} + g_3^{\mu}
\]

hence for the present case

\[
j_N^\mu = iq \chi (x) \left( \phi^* \pi^{\mu'} - \phi \pi^\mu \right) + \Pi^{\nu \mu} \frac{\partial \chi (x)}{\partial x^\nu}.
\]

By direct calculation, we again verify that \( \partial j_N^\alpha / \partial x^\alpha = 0 \).

5.8. Spontaneous breaking of gauge symmetry, Higgs mechanism

In order to present the Higgs mechanism in the context of the canonical transformation approach, we consider the Hamiltonian density

\[
\mathcal{H}_H = \pi^* \pi^{\mu} + iqA^{\mu} (\pi^* \phi^* - \pi \phi) - \Omega^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2 - \frac{\lambda}{4} \Pi^{\mu \nu} \Pi_{\mu \nu}.
\]

This Hamiltonian density differs from the density (69) of Sec. 4.6 by an additional fourth order potential term. As all terms of the form \( \phi^* \phi \) are invariant with respect to local gauge transformations generated by Eq. (119), the Hamiltonian density (120) is also form invariant. The potential \( U(\phi) \) of the Hamiltonian density (120)

\[
U(\phi) = -\Omega^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2
\]
has a minimum

$$(\phi^* \phi)_{\text{min}} = \frac{\Omega^2}{\lambda^2}.\)$$

Thus, $\phi_{\text{min}}$ lies on a circle in the complex plane

$$\phi_{\text{min}} = \frac{\Omega}{\lambda} e^{i\omega}, \quad 0 \leq \omega \leq 2\pi.$$

We now want to express the Hamiltonian density from Eq. (120) in terms of the shifted potential $\phi'$

$$\phi' = \phi - \phi_{\text{min}}, \quad \frac{\partial \phi'}{\partial x^\mu} = \frac{\partial \phi}{\partial x^\mu}.$$

Because of $\phi_{\text{min}} = \text{const.}$, the derivatives of $\phi$ with respect to the $x^\mu$ must be unchanged under this transformation. Being defined by the generating function

$$F_2^\mu = \left( \pi^\mu - \frac{i}{\lambda} q A^\mu e^{-i\omega} \right) \left( \phi - \frac{\Omega}{\lambda} e^{i\omega} \right),$$

this transformation is actually canonical. As the transformation only affects the fields $\phi$ and $\pi^\mu$, the other fields, $A_\mu$ and $\Pi^{\mu\nu}$ that are contained in the Hamiltonian density (120) must be treated as constant parameters. The transformation rules following from Eq. (121) are

$$\pi'^\mu = \pi^\mu + \frac{\Omega}{\lambda} q A^\mu e^{-i\omega}, \quad \pi'^{\ast\mu} = \pi^{\ast\mu} - \frac{i}{\lambda} q A^\mu e^{i\omega},$$

$$\phi' = \phi - \frac{\Omega}{\lambda} e^{i\omega}, \quad \phi'^* = \phi^* - \frac{\Omega}{\lambda} e^{-i\omega}.$$

As the generating function (121) does not explicitly depend on the $x^\mu$, we conclude that $\mathcal{H}'_H = \mathcal{H}_H$. The transformed Hamiltonian density $\mathcal{H}'_H$ is thus obtained by expressing the unprimed fields in terms of the primed ones,

$$\mathcal{H}'_H = \pi'^{\ast\mu} \pi'^\mu + \frac{1}{4} \left[ \Omega \left( \phi'^* e^{i\omega} + \phi' e^{-i\omega} \right) + \lambda \phi'^* \phi' \right]^2 - \frac{1}{4} \Pi^{\mu\nu} \Pi_{\mu\nu} - \frac{\Omega^2}{\lambda^2} q^2 A^\mu A_\mu$$

$$+ i q A^\mu \left( \pi^{\ast\mu} \phi' - \pi'^{\ast\mu} \phi' \right) - \frac{\Omega}{\lambda} i A^\mu A_\mu \left( \phi'^* e^{i\omega} + \phi' e^{-i\omega} \right) - \frac{\Omega^4}{2 \lambda^2}. \quad (122)$$

We verify that the transformation does not change the derivatives of $\phi$, as requested,

$$\frac{\partial \phi}{\partial x^\mu} = \frac{\partial \mathcal{H}_H}{\partial \pi^\mu} = \pi^{\ast\mu} - i q A_\mu \phi$$

$$= \pi'^{\ast\mu} + \frac{i}{\lambda} q A^\mu e^{i\omega} - i q A_\mu \left( \phi' + \frac{\Omega}{\lambda} e^{i\omega} \right) = \pi'^{\ast\mu} - i q A_\mu \phi'$$

$$= \frac{\partial \mathcal{H}'_H}{\partial \pi'^\mu} = \frac{\partial \phi'}{\partial x^\mu}.$$
representation of $\mathcal{H}'_H$:

$$\mathcal{H}'_H = \pi_{1,\mu}^\prime \pi_1^{\prime\mu} + \pi_{2,\mu}^\prime \pi_2^{\prime\mu} + \frac{1}{2} \left[ 2\Omega \left( \phi_1' \cos \omega + \phi_2' \sin \omega \right) + \lambda \left( \phi_1'^2 + \phi_2'^2 \right) \right]^2 - \frac{1}{4} \Pi^{\mu\nu} \Pi_{\mu\nu} - \frac{\Omega^2}{\lambda^2} q^2 A_\mu A_\mu$$

$$+ 2q A_\mu (\pi_{2,\mu}^\prime \phi_1' + \pi_{1,\mu}^\prime \phi_2') - \frac{2\Omega}{\lambda} q^2 A_\mu (\phi_1' \cos \omega + \phi_2' \sin \omega) - \frac{\Omega^4}{2\lambda^2}. \tag{123}$$

We now observe that the massless gauge field $A_\mu$ that is contained in the original Hamiltonian density from Eq. (120) now appears as massive field through the term $(q \Omega/\lambda)^2 A_\mu A_\mu = (\phi_1^2 + \phi_2^2)_{\text{min}} q^2 A_\mu A_\mu$ in the transformed Hamiltonian density from Eq. (123) — independently from the gauge of $\phi$ and the angle $\omega$. This term thus originates in the shift of the reference system of $\phi$. The amount of the emerging mass depends on the depth of the potential’s minimum, hence from the underlying potential model.

Because of the gauge freedom of the original Hamiltonian density (120), we may gauge it to yield $\phi_2 \equiv 0, \partial \phi_2 / \partial x^\mu \equiv 0$. Under this gauge, we have $\pi_{2,\mu}^\prime = -q A_\mu \phi_1'$, which eliminates the unphysical coupling terms $A_\mu \partial \phi_1'/\partial x^\mu$ that are contained in the Hamiltonian density (123). The transformed Hamiltonian density from Eq. (123) then simplifies to, setting $\omega = 0$

$$\mathcal{H}'_H = \pi_{1,\mu}^\prime \pi_1^{\prime\mu} + 2\Omega^2 \phi_1'^2 + 2\lambda \phi_1'^3 + \frac{1}{2} \lambda^2 \phi_1'^4 - \frac{1}{4} \Pi^{\mu\nu} \Pi_{\mu\nu} - \frac{\Omega^2}{\lambda^2} q^2 A_\mu A_\mu$$

$$- 2q^2 A_\mu A_\mu \phi_1'^2 - \frac{2\Omega}{\lambda} q^2 A_\mu A_\mu \phi_1' - \frac{\Omega^4}{2\lambda^2}. \tag{124}$$

The physical system that is described by the Hamiltonian density (124) emerged by means of a canonical transformation from the original density (120). Therefore, both systems are canonically equivalent. In the transformed system, we only consider the real part $\phi_1$ of the complex field $\phi$. The corresponding degree of freedom now finds itself in the mass of the massive vector field $A_\mu$. This transformation of two massive scalar fields $\phi_{1,2}$ that interact with a massless vector field $A_\mu$ into a single massive scalar field $\phi_1'$ plus one now massive vector field $A_\mu$ is commonly referred to as Higgs mechanism.

### 5.9. SU(2) gauge transformation as a canonical transformation

Provided that the parameters for a local SU(2) transformation are given by $\vec{\tilde{\theta}}(x) = (\theta_1, \theta_2, \theta_3)$, the infinitesimal generating function that defines the transformation of the scalar fields has the form

$$F^\mu_2 = \left(1 - \frac{i}{2} \vec{\tau} \vec{\tilde{\theta}} \right) \pi^{1,\mu} \phi_1,$$

with the Pauli matrices from Eqs. (82). The general transformation rules from Eqs. (19) then yield the following particular equations for the transformation of
scalar fields and conjugate momenta

$$\phi_I' \delta^\mu_\nu = \frac{\partial F^\mu_{2}}{\partial \pi^\mu_\nu} = \left(1 - \frac{i}{2} \vec{\tau} \vec{\theta}\right) \phi_I \delta^\mu_\nu$$

$$\Rightarrow \phi_I' = \left(1 - \frac{i}{2} \vec{\tau} \vec{\theta}\right) \phi_I$$

$$\pi^I\mu = \frac{\partial F^\mu_{2}}{\partial \phi_I} = \left(1 - \frac{i}{2} \vec{\tau} \vec{\theta}\right) \pi^I\mu'.$$

For an infinitesimal SU(2) transformation of vector fields, we find up a generating function of the form

$$F^\mu_2 = \left(A^a_\nu + \epsilon^{abc} \theta_b A_{c\nu} - \frac{1}{g} \partial_\nu \theta^a\right) \Pi^\nu_\mu.$$

Consequently, the generating function for an infinitesimal SU(2) gauge transformation is

$$F^\mu_2 = \left(1 - \frac{i}{2} \vec{\tau} \vec{\theta}\right) \pi^I\mu' \phi_I + \left(A^a_\nu + \epsilon^{abc} \theta_b A_{c\nu} - \frac{1}{g} \partial_\nu \theta^a\right) \Pi^\nu_\mu.$$

This generator induces the following transformation rules for the gauge fields and the tensors of conjugate momentum fields

$$A^a_\nu \delta^\mu_\lambda = \frac{\partial F^\mu_2}{\partial \Pi^\nu_\mu} = \left(A^a_\nu + \epsilon^{abc} \theta_b A_{c\nu} - g^{-1} \partial_\nu \theta^a\right) \delta^\mu_\lambda$$

$$\Rightarrow A^a_\nu' = A^a_\nu + \epsilon^{abc} \theta_b A_{c\nu} - g^{-1} \partial_\nu \theta^a$$

$$\Pi^\mu_\nu = \frac{\partial F^\mu_2}{\partial A^a_\nu} = \Pi^\mu_\nu' - \epsilon^{abc} \delta^\mu_\nu \Pi^\mu_\nu'.$$

Both, the gauge fields as well as the momentum field tensors are thus mapped like a triplet under a SU(2) transformation. In the Abelian case of the electromagnetic field, all terms with $\epsilon$ contributions vanish. The non-Abelian gauge fields thus carry a charge under SU(2) so that self-interactions take place. The corresponding self-coupling terms of the gauge fields can thus be deduced from the Lagrangian density (81), considering the structure of $f^{\alpha\nu}_{\mu}$ in (82), or, alternatively, from the Hamiltonian density (86) considering Eq. (84).

6. Conclusions

With the present paper, we have worked out a consistent local coordinate description of the foundations of covariant Hamiltonian field theory. The essential feature of this field theory is to define for each scalar field $\phi_I$ a 4-vector $\pi^I$ of conjugate fields. Similar to classical Hamiltonian point dynamics, these fields, $\phi_I$ and the four $\pi^I\mu$, are treated as independent variables. All mappings $(\phi_I, \pi^I) \to (\phi_I', \pi^I')$ of these fields that preserve the Hamiltonian structure of the given dynamical system are referred to as canonical transformations. The local coordinate description enables us to explicitly formulate the field transformation rules for canonical transformations as derivatives of generating functions. In contrast to the scalar generating
functions of Hamiltonian point dynamics, the generating functions in the realm of Hamiltonian field theory are now 4-vectors. Similarly, Poisson brackets, Lagrange brackets, as well as the canonical 2-forms now equally emerge as components of vector quantities. This emerging of 4-vector quantities in place of the scalar quantities in Hamiltonian point dynamics reflects the transition to four independent variables in Lorentz-invariant field theories.

In the usual Lagrangian description that is based on a covariant Lagrangian density $L$, mappings of the fields $\phi_I$ and their four partial derivatives $\partial_\mu \phi_I$ are formulated in terms of point transformations — which constitute a subgroup of the superordinated group of canonical transformations. Thus, all point transformations in the Lagrangian formalism can be reformulated as canonical point transformations in the Hamiltonian description. This was demonstrated in our example section, where several well-known symmetry transformations of Lagrangian densities $L$ were reformulated as canonical point transformations of corresponding Hamiltonian densities $H$.

Yet, the converse is not true. There exist canonical transformations of the fields that cannot be expressed as point transformations in the Lagrangian formalism. We provided two examples of canonical transformations like that, namely the general infinitesimal canonical transformation yielding the generalized Noether theorem, and the canonical transformation inducing an infinitesimal space-time step that conforms to the canonical field equations. Compared to the Lagrangian description, the canonical transformation formalism of covariant Hamiltonian field theory thus allows to define more general classes of gauge transformations in relativistic quantum field theories.

Appendix A. Proof of the symmetry relation from Eq. (28)

From the transformation rules for the generating functions $F_{1,2,3,4}$, the symmetry relations (17), (20), (23), and (26) were derived, viz,

$$\frac{\partial \phi_I}{\partial \phi_{I'}} \delta^\mu = \frac{\partial \pi^{I\mu'}}{\partial \pi^{I\beta}}, \quad \frac{\partial \phi_I}{\partial \phi_{I'}} \delta^\mu = -\frac{\partial \phi_{J'}}{\partial \phi_{I'}} \delta^\mu, \quad \frac{\partial \pi^{I\alpha}}{\partial \phi_{I'}} = -\frac{\partial \phi_{I'}}{\partial \phi_{I}} \delta^\mu.
$$

The third symmetry relation represents the particular case $\mu = \alpha = \beta$ of the general relation

$$\frac{\partial \pi^{I\alpha}}{\partial \phi_{I'}} \delta^\beta = -\frac{\partial \pi^{I\beta'}}{\partial \phi_{I'}} \delta^\mu.
$$

To prove this, we first consider the following identities that hold for the derivatives of the fields that emerge from a general, not necessarily canonical transformation
The transformation is canonical if

\[ \phi_I = \phi_I(\phi, \pi), \pi^I = \pi^I(\phi, \pi) \]

and

\[ \pi^{I\mu} = \pi^{I\mu}(\phi, \pi) \]:

\[
\begin{align*}
\delta_I^J &= \frac{\partial \phi_I}{\partial \phi_J} = \frac{\partial \phi_I}{\partial \phi_{K'}} \frac{\partial \phi_{K'}}{\partial \phi_J} + \frac{\partial \phi_I}{\partial \pi^{K\alpha'}} \frac{\partial \pi^{K\alpha'}}{\partial \phi_J} \\
0 &= \frac{\partial \phi_I}{\partial \pi^{J\nu}} = \frac{\partial \phi_I}{\partial \phi_{K'}} \frac{\partial \phi_{K'}}{\partial \pi^{J\nu}} + \frac{\partial \phi_I}{\partial \pi^{K\alpha'}} \frac{\partial \pi^{K\alpha'}}{\partial \pi^{J\nu}} \\
0 &= \frac{\partial \pi^{I\mu}}{\partial \phi_J} = \frac{\partial \pi^{I\mu}}{\partial \phi_{K'}} \frac{\partial \phi_{K'}}{\partial \phi_J} + \frac{\partial \pi^{I\mu}}{\partial \phi_{K'}} \frac{\partial \phi_{K'}}{\partial \phi_J} \\
\delta_I^J \delta^\mu_J &= \frac{\partial \pi^{I\mu}}{\partial \pi^{J\nu}} = \frac{\partial \pi^{I\mu}}{\partial \phi_{K'}} \frac{\partial \phi_{K'}}{\partial \pi^{J\nu}} + \frac{\partial \pi^{I\mu}}{\partial \phi_{K'}} \frac{\partial \phi_{K'}}{\partial \pi^{J\nu}}.
\end{align*}
\]

We express these identities in matrix form as

\[
\begin{pmatrix}
\delta_I^J \\
\delta^I_J \delta^\mu_J
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial \phi_I}{\partial \phi_{K'}} & \frac{\partial \phi_I}{\partial \pi^{K\alpha'}} \\
\frac{\partial \pi^{I\mu}}{\partial \phi_{K'}} & \frac{\partial \pi^{I\mu}}{\partial \phi_{K'}}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \phi_{K'}}{\partial \phi_J} & \frac{\partial \phi_{K'}}{\partial \phi_J} \\
\frac{\partial \pi^{K\alpha'}}{\partial \phi_J} & \frac{\partial \pi^{K\alpha'}}{\partial \phi_J}
\end{pmatrix}.
\]

Both sides of this equation are now multiplied from the right by the matrix

\[
\begin{pmatrix}
\frac{\partial \pi^{M\beta'}}{\partial \pi^{J\nu}} \\
\frac{\partial \pi^{M\beta'}}{\partial \phi_J}
\end{pmatrix}.
\]

We thus get

\[
\begin{pmatrix}
\frac{\partial \pi^{M\beta'}}{\partial \pi^{J\nu}} - \frac{\partial \phi_M}{\partial \pi^{J\nu}} \\
- \frac{\partial \pi^{M\beta'}}{\partial \phi_J} \delta^\mu_J \delta^\nu_J + \frac{\partial \phi_M}{\partial \phi_J}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial \phi_I}{\partial \phi_{K'}} & \frac{\partial \phi_I}{\partial \pi^{K\alpha'}} \\
\frac{\partial \pi^{I\mu}}{\partial \phi_{K'}} & \frac{\partial \pi^{I\mu}}{\partial \phi_{K'}}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \phi_{K'}}{\partial \phi_J} & \frac{\partial \phi_{K'}}{\partial \phi_J} \\
\frac{\partial \pi^{K\alpha'}}{\partial \phi_J} & \frac{\partial \pi^{K\alpha'}}{\partial \phi_J}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \pi^{M\beta'}}{\partial \pi^{J\nu}} \\
\frac{\partial \pi^{M\beta'}}{\partial \phi_J}
\end{pmatrix}.
\]

If the transformation is canonical, then we can insert the identities for the fundamental Poisson brackets from Eqs. (33) and (34):

\[
\begin{pmatrix}
\frac{\partial \pi^{M\beta'}}{\partial \pi^{J\nu}} - \frac{\partial \phi_M}{\partial \pi^{J\nu}} \\
- \frac{\partial \pi^{M\beta'}}{\partial \phi_J} \delta^\mu_J \delta^\nu_J + \frac{\partial \phi_M}{\partial \phi_J}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial \phi_I}{\partial \phi_{K'}} & \frac{\partial \phi_I}{\partial \pi^{K\alpha'}} \\
\frac{\partial \pi^{I\mu}}{\partial \phi_{K'}} & \frac{\partial \pi^{I\mu}}{\partial \phi_{K'}}
\end{pmatrix}
\begin{pmatrix}
\delta^K_I \delta^\mu_J \delta^\nu_J \\
\frac{\partial \pi^{K\alpha'}}{\partial \phi_J}
\end{pmatrix}.
\]
The matrix products on the right hand side is
\[
\begin{pmatrix}
\frac{\partial \pi^M}{\partial \pi^{1\nu}} & \frac{\partial \phi_I}{\partial \pi^{1\nu}} \\
\frac{\partial \pi^M}{\partial \phi_I} & \frac{\partial \phi_I}{\partial \phi_I}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial \phi_I}{\partial \phi_I} \delta^\beta_\nu + \frac{\partial \phi_I}{\partial \pi^{K\alpha}} \left[ \pi^{K\alpha'}, \pi^{M\beta'} \right]_\nu \frac{\partial \phi_I}{\partial \pi^{M\nu}} \\
\frac{\partial \phi_I}{\partial \phi_I} \delta^\beta_\nu + \frac{\partial \phi_I}{\partial \pi^{M\nu}} \left[ \pi^{K\alpha'}, \pi^{M\beta'} \right]_\nu \frac{\partial \phi_I}{\partial \pi^{M\nu}}
\end{pmatrix}.
\]

By comparing the individual matrix components, we obtain the following four relations:
\[
\begin{align*}
\frac{\partial \pi^{M\beta'}}{\partial \pi^{1\nu}} &= \frac{\partial \phi_I}{\partial \phi_I} \delta^\beta_\nu + \frac{\partial \phi_I}{\partial \pi^{K\alpha}} \left[ \pi^{K\alpha'}, \pi^{M\beta'} \right]_\nu, \\
\frac{\partial \phi_I}{\partial \phi_I} &= \frac{\partial \phi_I}{\partial \pi^{M\nu}}, \\
\frac{\partial \pi^{M\beta'}}{\partial \phi_I} \delta^\mu_\nu &= \frac{\partial \phi_I}{\partial \pi^{1\nu}} \delta^\beta_\nu + \frac{\partial \phi_I}{\partial \pi^{K\alpha}} \left[ \pi^{K\alpha'}, \pi^{M\beta'} \right]_\nu, \\
\frac{\partial \phi_I}{\partial \phi_I} \delta^\mu_\nu &= \frac{\partial \phi_I}{\partial \pi^{M\nu}}.
\end{align*}
\]  

Comparing now Eq. (A.1a) with the symmetry relation from Eq. (23), we conclude
\[
\frac{\partial \phi_I}{\partial \pi^{K\alpha'}} \left[ \pi^{K\alpha'}, \pi^{M\beta'} \right]_\nu = 0.
\]

Consequently, the inner product of this expression with \(\partial \phi_{I'}/\partial \phi_I\) must also vanish,
\[
\frac{\partial \phi_{I'}}{\partial \phi_I} \frac{\partial \phi_I}{\partial \pi^{K\alpha'}} \left[ \pi^{K\alpha'}, \pi^{M\beta'} \right]_\nu = 0.
\]

Because of \(\partial \phi_{I'}/\partial \pi^{K\alpha'} = 0\), this equation is equivalent to
\[
\frac{\partial \phi_{I'}}{\partial \phi_I} \frac{\partial \phi_I}{\partial \pi^{1\nu}} \left[ \pi^{K\alpha'}, \pi^{M\beta'} \right]_\nu = 0.
\]

As this equation must hold for \textit{arbitrary} coefficients \(\partial \phi_{I'}/\partial \pi^{1\nu}\), it must hold separately for each index \(\mu\),
\[
\frac{\partial \phi_{I'}}{\partial \phi_I} \frac{\partial \phi_I}{\partial \pi^{1\nu}} \left[ \pi^{K\alpha'}, \pi^{M\beta'} \right]_\nu = 0.
\]

In conjunction with Eq. (A.1a), this proves the assertion.

Appendix B. Geometric representation of the field equations

Let \(f = f(\phi_I, \partial_\mu \phi)\) denote a function depending on the fields \(\phi_I\) and on the 1-form \(\partial_\mu \phi\) that is constituted by their first derivatives. Of course, the dynamics of the fields \(\phi_I\) is supposed to follow from the Euler-Lagrange field equations (2). The change of \(f\) due to a change of the \(\phi_I\) and the \(\partial_\mu \phi_I\) is then
\[
\frac{\partial f}{\partial x^\mu} = \Delta^\mu_\nu f, \quad \Delta^\mu_\nu = \frac{\partial f_I}{\partial x^\mu} \frac{\partial}{\partial \phi_I} + \frac{\partial^2 f_I}{\partial x^\mu \partial x^\alpha} \frac{\partial}{\partial (\partial_\alpha \phi_I)}.
\]
We refer to $\Delta^\xi_\mu$ as the Lagrangian vector field. Acting on the function $f$, this is identical to the Lie derivative of $f$ with respect to the vector field $\Delta^\xi_\mu$

$$\Delta^\xi_\mu f \equiv \mathbf{L}_{\Delta^\xi_\mu} f.$$ 

Herein $\mathbf{L}_{\Delta^\xi_\mu}$, $\mu = 0, \ldots, 3$ denotes the four Lie operators that act of the function $f$.

We now define the Liouville 1-forms $\theta^\mu$ in local coordinates by

$$\theta^\mu = \frac{\partial L}{\partial (\partial_\mu \phi_I)} \ d\phi_I.$$ 

The Lie derivative of this 1-form along the vector field $\Delta^\xi_\mu$, followed by a summation over $\mu$ yields another 1-form,

$$\mathbf{L}_{\Delta^\xi_\mu} \theta^\mu = \Delta^\xi_\mu \ lbracket \frac{\partial^2 L}{\partial (\partial_\mu \phi_I) \partial (\partial_\nu \phi_J)} \ d(\partial_\nu \phi_J) \wedge d\phi_I + \frac{\partial^2 L}{\partial (\partial_\mu \phi_I) \partial \phi_J} \ d\phi_J \wedge d\phi_I \rbracket + \frac{\partial L}{\partial (\partial_\mu \phi_I)} d(\partial_\mu \phi_I)$$

Inserting the Euler-Lagrange field equation (2), we finally get

$$\mathbf{L}_{\Delta^\xi_\mu} \theta^\mu = \frac{\partial L}{\partial \phi_I} d\phi_I + \frac{\partial L}{\partial (\partial_\mu \phi_I)} d(\partial_\mu \phi_I)$$

The 1-form equation

$$\mathbf{L}_{\Delta^\xi_\mu} \theta^\mu = d\mathbf{L}$$

is thus the geometric representation of the Euler-Lagrange field equation (2). The Lie derivative of the 1-Form $\theta^\mu$ along the $\mu$ component $\Delta^\xi_\mu$ of the Lagrangian vector field, summed over all $\mu = 0, \ldots, 3$, equals the exterior derivative $d\mathbf{L}$ of the Lagrangian density $\mathbf{L}$. All three constituents of this equation, namely, the operators $\mathbf{L}_{\Delta^\xi_\mu}$ of the Lie derivative along the Lagrangian vector field $\Delta^\xi_\mu$, the 1-forms $\theta^\mu$, and the exterior derivative of the Lagrangian density $\mathbf{L}$ show up as geometric quantities, hence without any reference to a local coordinate system.

In order to formulate the corresponding geometric representation of the covariant canonical field equations (5), we define the Hamiltonian vector field $\Delta^\xi_\mu$ as the Legendre transformed vector field $\Delta^\xi_\mu$,

$$\Delta^\xi_\mu = \frac{\partial \phi_I}{\partial x^\mu} \frac{\partial}{\partial \phi_I} + \frac{\partial \pi^{I\alpha}}{\partial x^\mu} \frac{\partial}{\partial \pi^{I\alpha}}.$$ 

(B.2)
In terms of the conjugate fields $\pi^I{_{\mu}}$, we rewrite the 1-form $\theta^\mu$ as

$$\theta^\mu = \pi^I{_{\mu}} \, d\phi^I.$$  \hfill (B.3)

The 2-form $\omega^\mu$ can now be defined as the negative exterior derivative of the 1-form $\theta^\mu$

$$\omega^\mu = -d\theta^\mu = -d(\pi^I{_{\mu}} \, d\phi^I),$$

hence as the wedge product

$$\omega^\mu = -d\pi^I{_{\mu}} \wedge d\phi^I = d\phi^I \wedge d\pi^I{_{\mu}}.$$

We calculate the inner product $\Delta^\mathcal{H}{_{\mu}} \cdot \omega^\mu$, i.e., the 1-form that emerges from contracting the 2-forms $\omega^\mu$ with the vector fields $\Delta^\mathcal{H}{_{\mu}}$,

$$\Delta^\mathcal{H}{_{\mu}} \cdot \omega^\mu = \Delta^\mathcal{H}{_{\mu}} \cdot (d\phi^I \wedge d\pi^I{_{\mu}}) = (\Delta^\mathcal{H}{_{\mu}} \cdot d\phi^I) \, d\pi^I{_{\mu}} - (\Delta^\mathcal{H}{_{\mu}} \cdot d\pi^I{_{\mu}}) \, d\phi^I = \frac{\partial \phi^I}{\partial \pi^I{_{\mu}}} \, d\pi^I{_{\mu}} - \frac{\partial \pi^I{_{\mu}}}{\partial x^\alpha} \, d\phi^I.$$

Inserting the canonical field equations \[5\] yields

$$\Delta^\mathcal{H}{_{\mu}} \cdot \omega^\mu = \frac{\partial \mathcal{H}}{\partial \pi^I{_{\mu}}} \, d\pi^I{_{\mu}} + \frac{\partial \mathcal{H}}{\partial \phi^I} \, d\phi^I = d\mathcal{H}.$$  \hfill (B.4)

The 1-form equation

$$\Delta^\mathcal{H}{_{\mu}} \cdot \omega^\mu = d\mathcal{H}$$

thus embodies the geometric representation of the covariant canonical field equations \[5\]. The contraction of the Hamiltonian vector fields $\Delta^\mathcal{H}{_{\mu}}$ with the 2-forms $\omega^\mu$ is equal to the exterior derivative $d\mathcal{H}$ of the Hamiltonian density $\mathcal{H}$.

We can now furthermore calculate the sum of the Lie derivatives $L_{\Delta^\mathcal{H}{_{\mu}}}$ of the 1-forms $\theta^\mu$ along the pertaining Hamiltonian vector fields $\Delta^\mathcal{H}{_{\mu}}$,

$$L_{\Delta^\mathcal{H}{_{\mu}}} \theta^\mu = d \left( \Delta^\mathcal{H}{_{\mu}} \cdot \theta^\mu \right) + \Delta^\mathcal{H}{_{\mu}} \cdot d\theta^\mu = d \left( \pi^I{_{\mu}} \frac{\partial \phi^I}{\partial x^\mu} - \mathcal{H} \right) = d\mathcal{L}.$$

We immediately conclude that the sum of the Lie derivatives of the 2-forms $\omega^\mu$ must vanish

$$L_{\Delta^\mathcal{H}{_{\mu}}} \omega^\mu = -L_{\Delta^\mathcal{H}{_{\mu}}} (d\theta^\mu) = -dL_{\Delta^\mathcal{H}{_{\mu}}} \theta^\mu = -dd\mathcal{L} = 0.$$

The sum of the 2-forms $\omega^\mu$ along the fluxes that are generated by the $\Delta^\mathcal{H}{_{\mu}}$ is thus invariant.

Vice versa, if $L_{\Delta^\mathcal{H}{_{\mu}}} \omega^\mu = 0$ holds, then we have, because of $d\omega^\mu = 0$

$$L_{\Delta^\mathcal{H}{_{\mu}}} \omega^\mu = d(\Delta^\mathcal{H}{_{\mu}} \cdot \omega^\mu).$$
The 2-form $\mathbf{L}_{\Delta H} \omega^\mu = 0$ is thus closed and also exact, according to Poincaré’s Lemma. Therefore, we can always find a function $H$ with $dH = \Delta H \omega^\mu$. 

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