ON THE ZERO DIVISOR GRAPHS OF THE RING OF LIPSCHITZ INTEGERS MODULO $n$.

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Abstract. This article studies the zero divisor graphs of the ring of Lipschitz integers modulo $n$. In particular we focus on the number of vertices, the diameter and the girth. We also give some results regarding the domination number of these graphs.

AMS 2010 Mathematics Subject Classification : 05C75; 11R52.
Keywords: Lipschitz Integers; Zero Divisor Graphs; Diameter; Girth.

1. Introduction

The idea of studying the interplay between ring-theoretic properties of a ring $R$ and graph-theoretic properties of a graph defined after it, is quite recent. It was first introduced for commutative rings by Beck in 1988 [7]. In Beck’s definition the vertices of the graph are the elements of the ring and two distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. Later, Anderson and Livingston [5] slightly modified this idea, considering only the non-zero zero divisors of $R$ as vertices of the graph with the same adjacency condition. Redmond [23] extended this notion of zero-divisor graph to noncommutative rings.

Given a noncommutative ring $R$, we define two different graphs associated to $R$. The directed zero divisor graph, $\Gamma(R)$, and the undirected zero divisor graph, $\overline{\Gamma}(R)$. Both graphs share the same vertex set, namely, the set $Z(R)^*$ of non-zero zero divisors of $R$. In $\Gamma(R)$, given two distinct vertices $x$ and $y$, there is a directed edge of the form $x \rightarrow y$ if and only if $xy = 0$. On the other hand, two distinct vertices $x$ and $y$ of $\overline{\Gamma}(R)$ are connected by an edge if and only if either $xy = 0$ or $yx = 0$. Several properties of zero divisor graphs of different general classes of rings are studied in [3, 4, 15, 23, 25].

Recall that, if $n > 1$ is a rational integer and $\langle n \rangle$ is the ideal in the Gaussian integers generated by $n$, then the factor ring $\mathbb{Z}[i]/\langle n \rangle$ is isomorphic to the Gaussian integers modulo $n$

$$\mathbb{Z}_n[i] := \{a + bi : a, b \in \mathbb{Z}_n\}.$$ 

The zero divisor graph of the ring of Gaussian integers modulo $n$ has recently received great attention [1, 2, 22].

The algebraic construction defined above for the Gaussian integers can be easily extended to the ring $\mathbb{Z}[i, j, k]$ of Lipschitz integer quaternions. Indeed, let again be $n > 1$ a rational integer and denote by $\langle n \rangle$ the principal ideal in $\mathbb{Z}[i, j, k]$ generated by $n$. Then, the factor ring $\mathbb{Z}[i, j, k]/\langle n \rangle$ is isomorphic to

$$\mathbb{Z}_n[i, j, k] := \{a + ib + cj + dk : a, b, c, d \in \mathbb{Z}_n\},$$

which is called the ring of Lipschitz quaternions modulo $n$. 

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The aim of this paper is to study the zero divisor graphs of the ring of Lipschitz quaternions modulo \( n \), both the directed \( \Gamma(\mathbb{Z}_n[i,j,k]) \) and the undirected \( \overline{\Gamma}(\mathbb{Z}_n[i,j,k]) \).

If \( n = p_1^{r_1} \ldots p_k^{r_k} \), is the prime power factorization of \( n \), the Chinese remainder theorem induces a natural isomorphism

\[
\mathbb{Z}_n[i,j,k] \cong \mathbb{Z}_{p_1^{r_1}}[i,j,k] \oplus \cdots \oplus \mathbb{Z}_{p_k^{r_k}}[i,j,k].
\]

Therefore, in order to study the structure of the rings \( \mathbb{Z}_n[i,j,k] \) we can restrict ourselves to the prime power case.

If \( p \) is an odd prime and \( l \) is a positive integer, then \( \mathbb{Z}_{p^l}[i,j,k] \) is isomorphic to the full matrix ring \( M_2(\mathbb{Z}_{p^l}) \) [13][23]. Consequently, for an odd positive integer \( n \) the ring \( \mathbb{Z}_n[i,j,k] \) is isomorphic to the matrix ring \( M_2(\mathbb{Z}_n) \). Hence, in this case we can use known results about the zero divisor graph of matrix rings over commutative rings.

Unfortunately, if \( n \) is even it is no longer true that \( \mathbb{Z}_n[i,j,k] \) is isomorphic to the matrix ring \( M_2(\mathbb{Z}_n) \). In fact, note that an element \( z = z_0 + z_1 i + z_2 j + z_3 k \in \mathbb{Z}_n[i,j,k] \) is a unit if and only if its norm \( \|z\| = z_0^2 + z_1^2 + z_2^2 + z_3^2 \) is a unit in \( \mathbb{Z}_n \). Since

\[
\|z + w\| = \|z\| + \|w\| + 2\text{Re}(zw),
\]

the sum of two units of \( \mathbb{Z}_n[i,j,k] \) is never a unit. This fact is clearly false in \( M_2(\mathbb{Z}_n) \) and the claim holds.

2. THE NUMBER OF VERTICES

Recall that both graphs \( \Gamma(\mathbb{Z}_n[i,j,k]) \) and \( \overline{\Gamma}(\mathbb{Z}_n[i,j,k]) \) share the same set of vertices. Namely, the non-zero zero divisors of \( \mathbb{Z}_n[i,j,k] \). Due to the isomorphism \[11\] we can focus on the case when \( n \) is a prime power.

For an odd prime power \( p^\alpha \), we have the isomorphism \( \mathbb{Z}_{p^\alpha}[i,j,k] \cong M_2(\mathbb{Z}_{p^\alpha}) \). Thus, it is enough to determine the number of non-zero zero divisors in the matrix ring \( M_2(\mathbb{Z}_{p^\alpha}) \). We do so in the following proposition.

**Proposition 1.** Let \( p \) be an odd prime number and \( \alpha \geq 1 \). Then, the number of zero divisors in \( M_2(\mathbb{Z}_{p^\alpha}) \) is

\[
p^{4\alpha-1} + p^{4\alpha-2} - p^{4\alpha-3}.
\]

**Proof.** Let us denote by \( \mathcal{N}(M_2(\mathbb{Z}_{p^\alpha})) \) the nilradical of \( M_2(\mathbb{Z}_{p^\alpha}) \). It is known [14] p.125] that \( \mathcal{N}(M_2(\mathbb{Z}_{p^\alpha})) = M_2(\mathcal{N}(\mathbb{Z}_{p^\alpha})) \) and recall that \( \mathcal{N}(\mathbb{Z}_{p^\alpha}) = \{a \in \mathbb{Z}_{p^\alpha} : \gcd(a,p) > 1\} \) is precisely the unique maximal ideal of the local ring \( \mathbb{Z}_{p^\alpha} \). This implies that the factor ring \( M_2(\mathbb{Z}_{p^\alpha})/\mathcal{N}(M_2(\mathbb{Z}_{p^\alpha})) \) is simple, because every ideal of the full matrix ring \( M_n(R) \) over a unital commutative ring \( R \) is of the form \( M_n(L) \) for some ideal \( L \) of \( R \).

Now, since every artinian simple ring is isomorphic to the full matrix ring over a division ring [17] p.39] and by the well-known Wedderburn’s little theorem a finite division ring is a field, we conclude that \( M_2(\mathbb{Z}_{p^\alpha})/\mathcal{N}(M_2(\mathbb{Z}_{p^\alpha})) \) is isomorphic to a full matrix ring over a finite field.

Moreover, since

\[
|M_2(\mathbb{Z}_{p^\alpha})/\mathcal{N}(M_2(\mathbb{Z}_{p^\alpha})))| = \frac{p^{2\alpha}}{p^{4\alpha-1}} = p^4,
\]

it follows that \( M_2(\mathbb{Z}_{p^\alpha})/\mathcal{N}(M_2(\mathbb{Z}_{p^\alpha})) \) is isomorphic to the full matrix ring \( M_2(\mathbb{Z}_p) \).
Finally, since every element of \( N(M_2(\mathbb{F}_p)) \) is a zero divisor and there are exactly \( p^3 + p^2 - p \) zero divisors in \( M_2(\mathbb{F}_p) \), it follows that the number of zero divisors in \( M_2(\mathbb{F}_p) \) is

\[
p^{4(\alpha - 1)}(p^3 + p^2 - p) = p^{4\alpha - 1} + p^{4\alpha - 2} - p^{4\alpha - 3},
\]
as claimed.

\[\square\]

**Proposition 2.** The number of vertices of the graph \( \Gamma(\mathbb{Z}_2[i, j, k]) \) is \( 2^{4t-1} - 1 \).

**Proof.** Note that an element \( z = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{Z}_2[i, j, k] \) is a unit if and only if its norm \( \|z\| = x_0^2 + x_1^2 + x_2^2 + x_3^2 \) is a unit in \( \mathbb{Z}_2 \). On the other hand, an element is a unit in \( \mathbb{Z}_2 \) if its reduction modulo 2 is a unit. So consider the composition of homomorphisms

\[
\mathbb{Z}_2[i, j, k] \xrightarrow{\|} \mathbb{Z}_2 \xrightarrow{\mod 2} \mathbb{Z}_2,
\]
the kernel of this composition if the set of zero divisors in \( \mathbb{Z}_2[i, j, k] \). Thus,

\[
\mathbb{Z}_2[i, j, k]/Z(\mathbb{Z}_2[i, j, k]) \cong \mathbb{Z}_2,
\]
where \( Z(\mathbb{Z}_2[i, j, k]) \) denotes the set of zero divisors of \( \mathbb{Z}_2[i, j, k] \). Therefore \( |\mathcal{U}(\mathbb{Z}_2[i, j, k])| = |Z(\mathbb{Z}_2[i, j, k])| = 2^{4t-1} - 1 \). Hence, there are exactly \( 2^{4t-1} - 1 \) non-zero zero divisors in \( \mathbb{Z}_2[i, j, k] \), which is the number of vertices of the graph \( \Gamma(\mathbb{Z}_2[i, j, k]) \) as claimed. \( \square \)

**Remark 2.** Observe that, as in Remark \( \boxed{1} \) the number of units in \( \mathbb{Z}_2[i, j, k] \) is \( 2^{4t-1} - 1 \).

Recall that, given a direct sum of rings \( R = R_1 \oplus \ldots \oplus R_k \), an element \( r \in R \) is a unit if and only if every projection of \( r \) in \( R_i \) is a unit in \( R_i \). Hence, if \( n = 2^{t} p_1^{\alpha_1} \ldots p_k^{\alpha_k} \) is the prime power decomposition of \( n \), isomorphism \( \boxed{1} \) together with Remarks \( \boxed{1} \) and \( \boxed{2} \) lead to:

\[
|\mathcal{U}(\mathbb{Z}_n[i, j, k])| = \begin{cases} 
\prod_{i=1}^{k} (p_i^{4\alpha_i} - p_i^{4\alpha_i-1} - p_i^{4\alpha_i-2} + p_i^{4\alpha_i-3}), & \text{if } t = 0; \\
2^{4t-1} \prod_{i=1}^{k} (p_i^{4\alpha_i} - p_i^{4\alpha_i-1} - p_i^{4\alpha_i-2} + p_i^{4\alpha_i-3}), & \text{if } t > 0.
\end{cases}
\]

As a consequence of the previous work we have the main result of this section.

**Theorem 1.** Let \( n = 2^{t} p_1^{\alpha_1} \ldots p_k^{\alpha_k} \) be the prime power decomposition of \( n \). Then, the number of vertices in the graph \( \Gamma(\mathbb{Z}_n[i, j, k]) \) or \( \Gamma(\mathbb{Z}_n[i, j, k]) \) is:

\[
|V(\Gamma(\mathbb{Z}_n[i, j, k]))| = \begin{cases} 
n^4 - \prod_{i=1}^{k} (p_i^{4\alpha_i} - p_i^{4\alpha_i-1} - p_i^{4\alpha_i-2} + p_i^{4\alpha_i-3}) - 1, & \text{if } t = 0; \\
n^4 - 2^{4t-1} \prod_{i=1}^{k} (p_i^{4\alpha_i} - p_i^{4\alpha_i-1} - p_i^{4\alpha_i-2} + p_i^{4\alpha_i-3}) - 1, & \text{if } t > 0.
\end{cases}
\]

**Proof.** Just apply the previous observation recalling that the non-zero elements of a finite ring are either units or zero-divisors. \( \square \)
3. The diameter

We recall that the distance between two distinct vertices \(a\) and \(b\) of a graph, denoted by \(d(a, b)\), is the length of the shortest path connecting them (the distance being infinity if no such path exists). The diameter of a graph \(G\), denoted by \(\text{diam}(G)\), is given by

\[
\text{diam}(G) = \sup\{d(a, b) : a, b \text{ distinct vertices of } G\}.
\]

Our objective in this section is to find the diameter of the directed zero divisor graph \(\Gamma(Z_n[i, j, k])\) and of the undirected zero divisor graph \(\overline{\Gamma}(Z_n[i, j, k])\).

Recall that \(Z_L(R)\) and \(Z_R(R)\) denote, respectively, the set of left and right zero divisors of \(R\). The following result was proved in [23].

**Theorem 2.** Let \(R\) be a noncommutative ring, with \(Z^*(R) \neq \emptyset\). Then \(\Gamma(R)\) is connected if and only if \(Z_L(R) = Z_R(R)\). If \(\Gamma(R)\) is connected, then \(\text{diam}(\Gamma(R)) \leq 3\).

Note that in any finite ring \(R\) we have \(Z_L(R) = Z_R(R)\). Hence, the previous theorem implies that the directed zero divisor graph \(\Gamma(Z_n[i, j, k])\) is connected and \(\text{diam}(\Gamma(Z_n[i, j, k])) \leq 3\). We will now see that, in many cases, equality holds. To do so, we first need a technical result involving the direct sum of finite unital noncommutative rings. A commutative version was established in [6].

**Lemma 3.** Let \(R = R_1 \oplus R_2\), where \(R_1\) and \(R_2\) are finite unital noncommutative rings. Then, \(\text{diam}(\Gamma(R)) = 3\).

**Proof.** First note that, since \(R_1\) and \(R_2\) are finite and noncommutative if follows by the Wedderburn’s little theorem that both \(R_1\) and \(R_2\) have nonzero zero divisors.

On the other hand, since \(R_1\) and \(R_2\) are rings with identity we can choose a unit \(u_1\) from \(R_1\) and a unit \(u_2\) from \(R_2\). Let \(x \in Z(R_1)^*\) and \(y \in Z(R_2)\) and consider the elements \((x, u_2), (u_1, y) \in Z(R)^*\). We will prove that the distance between the vertices \((x, u_2)\) and \((u_1, y)\) is 3. Indeed \((x, u_2)(u_1, y) = (xu_1, u_2y) \neq (0, 0)\). Hence \(d((x, u_2), (u_1, y)) > 1\). On the other hand if \((a, b) \in Z(R)^*\) satisfies

\[
(x, u_2)(a, b) = (a, b)(u_1, y) = (0, 0),
\]

then we have \(u_2b = 0\) and \(au_1 = 0\) implying \(a = b = 0\), a contradiction. Therefore, \(d((x, u_2), (u_1, y)) > 2\). Finally, using theorem 2 we get the result. \(\square\)

As a consequence, we have the following result.

**Proposition 3.** Let \(n\) be an integer divisible by at least two primes. Then,

\[
\text{diam}(\Gamma(Z_n[i, j, k])) = 3.
\]

**Proof.** It is enough to apply isomorphism (1) together with Lemma 3. \(\square\)

**Remark 3.** It is clear that \(\text{diam}(\Gamma(Z_n[i, j, k])) \leq \text{diam}(\Gamma(Z_n[i, j, k]))\). Now, if \(n\) is divisible by at least two primes, there exist vertices in \(\overline{\Gamma}(Z_n[i, j, k])\) that are not at distance 2. Hence, we obtain the equality \(\text{diam}(\Gamma(Z_n[i, j, k])) = \text{diam}(\Gamma(Z_n[i, j, k]))\) in this case.

Now, we must focus on the prime power case. We first look at the odd case, where the following technical lemma is useful [9 Lem. 4.2; Cor. 4.1].
Lemma 4. Let $R$ be a commutative ring. If every finite set of zero divisors from $R$ has a non-zero annihilator, then $\text{diam}(\Gamma(M_p(R))) = 2$. In particular, if $F$ is a field, then $\text{diam}(\Gamma(M_n(F))) = 2$.

Proposition 4. Let $t \geq 1$ and let $p$ be an odd prime. Then,
$$\text{diam}(\Gamma(Z_{p^t}[i,j,k])) = 2.$$ 

Proof. If $t = 1$, $Z_p$ is a field and the result follows from the second part Lemma 4. Now assume that $t > 1$. In this case the maximal ideal of the local ring $Z_{p^t}$ is the principal ideal generated by $p$. This maximal ideal is nilpotent with index of nilpotence $t$. Therefore, the element $p^{t-1}$ belongs to the annihilator of every zero divisor in $Z_{p^t}$ and Lemma 4 applies again. □

Remark 4. Again, $\text{diam}(\Gamma(Z_{p^t}[i,j,k])) \leq \text{diam}(\Gamma(Z_{p^t}[i,j,k]))$. Now, if $p$ is an odd prime, there exist vertices in $\Gamma(Z_{p^t}[i,j,k])$ that are not at distance 1. Hence, we obtain the equality $\text{diam}(\Gamma(Z_{p^t}[i,j,k])) = \text{diam}(\Gamma(Z_{p^t}[i,j,k]))$ in this case.

Finally, we turn to the $p = 2$ case. The following result computes the diameter of the undirected zero divisor graph.

Proposition 5. Let $t \geq 1$. Then $\text{diam}(\Gamma(Z_{2^t}[i,j,k])) = 2$.

Proof. The graph $\Gamma(Z_{2^t}[i,j,k])$ is not complete since the vertices $1 + i$ and $1 + j$ are not adjacent. On the other hand, from equation (1) if follows that the sum of two zero divisors in $Z_{p^t}[i,j,k]$ is a zero divisor. Hence, the set of all zero divisors is an ideal. Clearly, this ideal equals the nilradical $\mathcal{N}(Z_{2^t}[i,j,k])$. Since the nilradical is nilpotent, it follows that there is a vertex of $\Gamma(Z_{2^t}[i,j,k])$ adjacent to all others, and thus $\text{diam}(\Gamma(Z_{2^t}[i,j,k])) = 2$. □

Recall that a directed graph $G$ is called symmetric if, for every directed edge $x \to y$ that belongs to $G$, the corresponding reversed edge $y \to x$ also belongs to $G$. We are going to prove that the directed zero divisor graph $\Gamma(Z_{2^t}[i,j,k])$ is symmetric. As a consequence, it will follow that $\text{diam}(\Gamma(Z_{2^t}[i,j,k])) = \text{diam}(\Gamma(Z_{2^t}[i,j,k]))$.

A ring $R$ is called reversible [10] if, for every $a, b \in R$, $ab = 0$ implies that $ba = 0$. Clearly, a ring $R$ is reversible if and only if its directed zero divisor graph $\Gamma(R)$ is symmetric. Thus, to prove that $\Gamma(Z_{2^t}[i,j,k])$ is symmetric we will prove that $Z_{2^t}[i,j,k]$ is reversible. To do so, we need a series of technical lemmata.

Lemma 5. Let $w \in Z[i,j,k]$. If $\|w\| \equiv 0 \pmod{4}$, then either all the components of $w$ are even or all of them are odd.

Proof. Put $w = a_1 + a_2i + a_3j + a_4k$ and denote by $n_w := \text{card}\{i : a_i \text{ is even}\}$. Since $a_i^2 \equiv 0, 1 \pmod{4}$, it follows that $0 \equiv \|w\| \equiv n_w \pmod{4}$ and hence the result. □

Lemma 6. Let $w \in Z[i,j,k]$. If $\|w\| \equiv 0 \pmod{8}$, then all the components of $w$ are even.

Proof. Put $w = a_1 + a_2i + a_3j + a_4k$. Since $\|w\| = a_1^2 + a_2^2 + a_3^2 + a_4^2 \equiv 0 \pmod{4}$ the previous lemma implies that either all the components of $w$ are even or all of them are odd. Assume that all of them are odd and put $a_i = 2a_i' + 1$ for every $i$. Then, $\|w\| = a_1'^2 + a_2'^2 + a_3'^2 + a_4'^2 = 4(a_1'^2) + a_1'^2 + (a_2'^2) - a_2'^2 + a_3^2 + a_4^2 + (a_4'^2) + a_4 + 4$. Hence, $(a_1'^2) + a_1'^2 + (a_2'^2) + a_2'^2 + (a_3'^2) + a_3'+ (a_4'^2) + a_4') + 1 \equiv 0 \pmod{2}$. This is clearly a contradiction and the result follows. □
Proposition 6. Let \( w, z \in \mathbb{Z}[i, j, k] \). If \( wz \equiv 0 \pmod{2^s} \), then \( zw \equiv 0 \pmod{2^s} \). In other words, the ring \( \mathbb{Z}[i, j, k] \) is reversible.

Proof. We will proceed by induction on \( s \).

The case \( s = 1 \) is obvious since \( \mathbb{Z}[i, j, k]/2\mathbb{Z}[i, j, k] \) is trivially commutative.

Let us consider \( s = 2 \) and assume that \( wz \equiv 0 \pmod{4} \). Hence, \( \|w\|\|z\| = \|wz\| \equiv 0 \pmod{16} \). If \( \|w\| \equiv 0 \pmod{4} \) we can apply Lemma 6 to conclude that \( w = 2w' \) for some \( w' \in \mathbb{L} \). Hence (using the case \( s = 1 \)) we have, \( wz \equiv 0 \pmod{4} \leftrightarrow w'z \equiv 0 \pmod{4} \leftrightarrow zw \equiv 0 \pmod{4} \).

Now, assume that \( s > 2 \) and that \( wz \equiv 0 \pmod{2^s} \). In this case \( \|w\|\|z\| = \|wz\| \equiv 0 \pmod{2^{s+1}} \) and, since \( s > 2 \) it follows that either \( \|w\| \equiv 0 \pmod{8} \) or \( \|z\| \equiv 0 \pmod{8} \). If, for instance, \( \|w\| \equiv 0 \pmod{8} \) we apply Lemma 6 again to conclude that \( w = 2w' \) for some \( w' \in \mathbb{Z}[i, j, k] \) and we can proceed like in the previous paragraph. The same holds if \( \|z\| \equiv 0 \pmod{2^s} \) and the proof is complete. \( \square \)

Remark 5. The concept of symmetric ring was defined by Lambek in [18]: a ring \( R \) is symmetric if, for every \( a, b, c \in R \), \( abc = 0 \) implies that also \( acb = 0 \). It has sometimes been erroneously asserted (and even “proved”) that reversible and symmetric are equivalent conditions. If a unital ring is symmetric, then it is also reversible. But this is no longer true for non-unital rings, as illustrated by an example of Birkenmeier [8]. In the case of unital rings, the smallest known reversible non-symmetric ring was given in [20]. Namely, it is the group algebra \( \mathbb{F}_2Q_8 \) where \( Q_8 \) is the quaternion group. In [17] it was proved that this is in fact the smallest reversible group algebra over a field which is not symmetric. In [17] it was also confirmed that \( \mathbb{F}_2Q_8 \) is indeed the smallest reversible group ring which is not symmetric. Note that \( \mathbb{Z}_4[i, j, k] \) is a reversible ring due to Proposition 6 which is trivially non-symmetric. In fact, it is the smallest known ring with characteristic different from 2 with this property, having the same number of elements (256) as the aforementioned example \( \mathbb{F}_2Q_8 \).

The reversibility of \( \mathbb{Z}[i, j, k] \) implies that \( \text{diam}(\Gamma(\mathbb{Z}[i, j, k])) = \text{diam}(\Gamma(\mathbb{Z}[i, j, k])) \), so from all the previous work we obtain the following result.

Theorem 7. Let \( n \) be any integer. Then

\[
\text{diam}(\Gamma(\mathbb{Z}_n[i, j, k])) = \begin{cases} 
2, & \text{if } n \text{ is a prime power;} \\
3, & \text{otherwise.}
\end{cases}
\]

Recall that a graph \( G \) is complete provided every pair of distinct vertices is connected by a unique edge. In [2] Theorem 15 it was proved that the undirected zero divisor graph for the ring of Gaussian integers modulo \( n \), \( \overline{\Gamma(\mathbb{Z}_n[i]))} \), is complete if and only if \( n = q^2 \), where \( q \) is a rational prime such \( q \equiv 3 \pmod{4} \). In our case we have the following.

Corollary 1. The graph \( \overline{\Gamma(\mathbb{Z}_n[i, j, k])} \) is never complete.

Proof. The diameter of a complete graph is 1. Since this is not possible due to Theorem 7, the result follows. \( \square \)
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4. The girth

A cycle in a graph is a path starting and ending at the same vertex. The girth of $G$, denoted by $g(G)$, is the length of the shortest cycle contained in $G$. If the graph does not contain any cycle, its girth is defined to be infinity. All the previous concepts can be defined for directed graphs just considering directed paths.

Let us consider the directed zero divisor graph $\Gamma(\mathbb{Z}_n[i,j,k])$. If $n$ is even then $g(\Gamma(\mathbb{Z}_n[i,j,k])) = 2$, since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$ 

For $n = 2^t$ the ring $\mathbb{Z}_{2^t}[i,j,k]$ is reversible and consequently $g(\Gamma(\mathbb{Z}_n[i,j,k])) = 2$.

Now, we turn to the undirected case. It is clear that $\Gamma(\mathbb{Z}_n[i,j,k])$ is a simple graph; i.e., it does not contain loops and two vertices are not connected by more than one edge. Thus, it follows that $g(\Gamma(\mathbb{Z}_n[i,j,k])) \geq 3$. We will now see that equality holds.

To compute the girth of $\Gamma(\mathbb{Z}_n[i,j,k])$ for odd $n$, we recall the following result [9, Prop. 3.2]

**Proposition 7.** Let $R$ be a commutative ring. Then $g(\Gamma(M_n(R))) = 3$.

This proposition clearly implies that $g(\Gamma(\mathbb{Z}_n[i,j,k])) = 3$ because, for odd $n$, we have that $\mathbb{Z}_n[i,j,k] \cong M_2(\mathbb{Z}_n)$.

The case $n = 2^t$ is analyzed in the following result.

**Proposition 8.** Let $t \geq 1$. Then $g(\Gamma(\mathbb{Z}_{2^t}[i,j,k])) = 3$.

**Proof.** If $t = 1$, we have the cycle (see the previous remark) $(1+i) - (j+k) - (1+i+j+k) - (1+i)$, for instance. If $t = 2$, we have the cycle $2 - 2i - (2+2i) - 2$, for instance. Finally, if $t > 2$ we can consider the cycle $2t-1 - 2 - 2t-1i - 2t-1$. This proves the result. \qed

Finally, if we recall isomorphism (1), the previous discussion leads to the following.

**Theorem 8.** For every integer $n$, $g(\Gamma(\mathbb{Z}_n[i,j,k])) = 3$.

A graph $G$ is complete bipartite if its vertices can be partitioned into two subsets such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. In [2, Theorem 17] it was proved that the undirected zero divisor graph for the ring of Gaussian integers modulo $n$, $\Gamma(\mathbb{Z}_n[i])$, is complete bipartite if and only if $n = p^2$, where $p$ is a rational prime such $p \equiv 1 \pmod{4}$ or $n = q_1q_2$, with $q_1, q_2$ rational primes such that $q_1 \equiv q_2 \equiv 3 \pmod{4}$. In our case we have the following.

**Corollary 2.** The graph $\Gamma(\mathbb{Z}_n[i,j,k])$ is never complete bipartite.

**Proof.** The girth of a complete bipartite graph is 4. Since this is not possible due to Theorem 8 the result follows. \qed

5. The domination number

A dominating set for a graph $G$ is a subset of vertices $D$, such that every vertex not in $D$ is adjacent to at least one member of $D$. The domination number, denoted by $\gamma(G)$, is the number of vertices in a minimal dominating set.
The problem of determining the dominating number of an arbitrary graph is NP-complete \[\|1\|\]. Nevertheless, particular cases have been recently studied. In \[\|2\|\], for instance, the dominating number of the zero divisor graph of the ring of Gaussian integers modulo \(n\) was studied. In particular, the authors characterized the values of \(n\) for which the domination number of \(\Gamma(Z_n[i])\) is 1 or 2.

This section is devoted to study the domination number of the undirected zero divisor graph \(\Gamma(Z_2[i,j,k])\). The easiest case arises when \(n\) is a power of 2.

**Theorem 9.** The domination number of the undirected zero divisor graph \(\Gamma(Z_2[i,j,k])\) is 1 for every \(t \geq 1\).

**Proof.** Just observe that \(\{2^{t-1}(1 + i + j + k)\}\) is a dominating set of the graph \(\Gamma(Z_2[i,j,k])\) because \(2^{t-1}(1 + i + j + k)z = 0\) for every non-zero zero divisor \(z\). \(\square\)

The rest of the section will be devoted to study the case when \(n\) is an odd prime. In particular we will prove that, for an odd prime number \(p\), the domination number of \(\Gamma(Z_p[i,j,k])\) is \(p + 1\).

Let \(\mathcal{U}(M_2(Z_p))\) be the group of units in the matrix ring \(M_2(Z_p)\). We can consider a natural left group action on the vertices of \(\Gamma(M_2(Z_p))\) defined by \((U, X) \rightarrow UX\) from \(\mathcal{U}(M_2(Z_p)) \times Z(M_2(Z_p))^*\) to \(Z(M_2(Z_p))^*\).

In the following lemma we characterize the structure of the orbits under the natural left action.

**Lemma 10.** The distinct orbits of the regular left action on \(Z^*(M_2(Z_p))\) are

\[o(\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}), \text{ for some } a \in Z_p, \text{ and } o\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right).\]

**Proof.** Let \(A\) be an element of \(Z(M_2(Z_p))^*\). As is well-known from linear algebra performing a row operation on a matrix, with entries in a field, is equivalent to multiplying on the left by a suitable invertible matrix. Since by row operations we can reduce any matrix to an upper triangular matrix it follows that there is an invertible matrix \(U \in M_2(Z_p)\) such that

\[UA = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix},\]

for some \(\alpha, \beta, \gamma \in Z_p\).

Assume first that \(\alpha \neq 0\). Since \(A\) is singular it follows that \(\gamma = 0\). So, we have

\[\begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^{-1} \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \alpha^{-1}\beta \\ 0 & 0 \end{bmatrix},\]

and we get the result.

Now assume that \(\alpha = 0\). If \(\gamma \neq 0\), then we have

\[\begin{bmatrix} \gamma^{-1} & 0 \\ 0 & \gamma^{-1} \end{bmatrix}\begin{bmatrix} 0 & \beta \\ 0 & \gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\]

as desired. If \(\gamma = 0\), since \(A\) is not the null matrix it follows that \(\beta \neq 0\). Hence we have

\[\begin{bmatrix} \beta^{-1}\gamma & \beta^{-1} \\ \beta^{-1} & 0 \end{bmatrix}\begin{bmatrix} 0 & \beta \\ 0 & \gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\]

and the proof is completed. \(\square\)
Similarly, if we consider the right regular action of $\mathcal{U}(M_2(\mathbb{Z}_p))$ on the vertices of $\Gamma(M_2(\mathbb{Z}_p))$ defined by $(U, X) \rightarrow XU$, we have that the distinct orbits are

$$o\left(\begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}\right), \text{ for some } b \in \mathbb{Z}_p, \text{ and } o\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

On the other hand, for any ring $R$, if $u \in R$ is a unit and $x \in R$ is any element of $R$ we have $ann_r(ax) = ann_r(x)$ and $ann_l(xu) = ann_l(x)$. Consequently, the distinct right annihilators of a single element in the ring $M_2(\mathbb{Z}_p)$ are

(2) $\text{ann}_r\left(\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} -az & -aw \\ z & w \end{bmatrix} : z, w \in \mathbb{Z}_p \right\}$

and

(3) $\text{ann}_r\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \left\{ \begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix} : z, w \in \mathbb{Z}_p \right\}$

Since any vertex in the graph $\Gamma(M_2(\mathbb{Z}_p))$ belongs to a right annihilator, it follows that

$$D = \left\{ \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}_p \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a dominating set for the graph $\Gamma(M_2(\mathbb{Z}_p))$. Therefore, $p + 1$ is an upper bound for the domination number of the graph $\Gamma(M_2(\mathbb{Z}_p))$. We will prove that this upper bound is the exact domination number of $\Gamma(M_2(\mathbb{Z}_p))$.

Similarly, the distinct left annihilators are

(4) $\text{ann}_l\left(\begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} -bx & x \\ -by & y \end{bmatrix} : x, y \in \mathbb{Z}_p \right\}$

and

(5) $\text{ann}_l\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \left\{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} : x, y \in \mathbb{Z}_p \right\}$

Note that the intersection of two distinct right annihilators (or two left annihilators) contains only the zero element.

**Lemma 11.** The intersection of a right annihilator of type (2) or (3) with a left annihilator of type (4) or (5) is a set with $p$ elements.

**Proof.** First note that both a left annihilator and a right annihilator are subgroups, of order $p^2$, of the additive group $M_2(\mathbb{Z}_p)$. Since no right annihilator of type (2) equals a left annihilator of type (4), it follows that their intersection is subgroup of order $p$ or else the trivial subgroup.

The result is clearly true if the right annihilator is of type (3) or the left annihilator is of type (5). So, assume that the right annihilator is of type (2) and the left annihilator is of type (4). Note that, the cardinality of the intersection

$$\text{ann}_r\left(\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}\right) \cap \text{ann}_l\left(\begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}\right),$$

equals the cardinality of the solution set of the homogeneous system of linear equation

$$az - bx = 0; \ x + aw = 0; \ z + by = 0; \ w = y.$$ 
Since the coefficient matrix of this system is singular we get the result. \qed
Theorem 12. The domination number of the zero divisor graph \( \Gamma(\mathbb{Z}_p[i, j, k]) \), where \( p \) is an odd prime number, is \( p + 1 \).

Proof. Suppose the theorem were false. Then, we could find a dominating set \( D = \{D_1, \ldots, D_k\} \), where \( k < p + 1 \). Hence, there is a right annihilator \( \text{ann}_r(X) \) of type \([2]\) or \([3]\) that is not among the \( k \) right annihilators \( \text{ann}_r(D_1), \ldots, \text{ann}_r(D_k) \) and a left annihilator \( \text{ann}_l(Y) \) that is not among the \( k \) left annihilators \( \text{ann}_l(D_1), \ldots, \text{ann}_l(D_k) \). Since all the elements in the right annihilator \( \text{ann}_r(X) \) are vertices of the graph \( \Gamma(\mathbb{Z}_p[i, j, k]) \), it follows that

\[
\text{ann}_r(X) \subseteq \bigcap_{i=1}^{k} \text{ann}_l(D_i).
\]

Therefore, \( \text{ann}_r(X) \cap \text{ann}_l(Y) = \{0\} \), which contradicts lemma 11. \( \square \)

As a consequence of the previous result we can easily compute the domination number of \( \Gamma(\mathbb{Z}_n[i, j, k]) \) when \( n \) is an odd square-free integer.

Theorem 13. Let \( n = p_1 \cdots p_k \) with \( p_i \) prime for every \( i \). Then, the domination number of the zero divisor graph \( \Gamma(\mathbb{Z}_n[i, j, k]) \) is \( k + p_1 + \ldots + p_k \).

Proof. Let \( C_i := \{M_{i,1}, \ldots, M_{i,1+p_i}\} \) be the dominating set for the graph \( \Gamma(\mathbb{Z}_{p_i}[i, j, k]) \) given by Theorem 12. Now, it is easy to see that the set

\[
\bigcup_{i=1}^{k} \{(0, 0, \ldots, M_{i,1}, \ldots, 0, 0), \ldots, (0, 0, \ldots, M_{i,1+p_i}, \ldots, 0, 0, 0)\}
\]

is a minimal dominating set of \( \Gamma(\mathbb{Z}_n[i, j, k]) \) and hence the result. \( \square \)

In a similar way, we can proof the following result.

Theorem 14. Let \( n = 2^s p_1 \cdots p_k \) with \( p_i \) is prime for every \( i \) and \( s > 0 \). Then, the domination number of the zero divisor graph \( \Gamma(\mathbb{Z}_n[i, j, k]) \) is \( 1 + k + p_1 + \ldots + p_k \).

We end this section presenting two open problems:

1. For an odd prime number \( p \) and a positive integer \( t \), what is the domination number of \( \Gamma(\mathbb{Z}_{p^t}[i, j, k]) \)?
2. Let \( t \) be a positive integer and denote by \( \mathbb{F}_q \) the finite field of \( q = p^t \) elements. For a positive integer \( n > 2 \), what is the domination number of the zero divisor graph \( \Gamma(M_n(\mathbb{F}_q)) \)?

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