Photon-added Coherent States in Parametric Down-conversion

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Abstract

Photon-added coherent states have been realized in optical parametric down-conversion by Zavatta et al [Science 306 (2004) 660-662]. In this report, it is established that the states generated in the process are ideal photon-added coherent states. It is shown that the scheme can generate higher order photon-added coherent states. A comparative study of the down-conversion process and atom-cavity interaction in generating the photon-added coherent states is presented.

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1 Introduction

Quantum-classical divide continues to be enigmatic. Good experiments are necessary to improve our understanding of the issue. One way is to generate and study states that interpolate between the classical and quantum domains. Coherent state is considered to be a classical state in the sense that the Glauber-Sudarshan function is an admissible phase-space probability density distribution, i.e., non-negative on the entire phase space. This feature is retained by coherent states of arbitrary amplitude. Additionally, coherent states exhibit minimum fluctuations (uncertainties) in their amplitudes and phases. States obtained by the action of harmonic oscillator creation operator on the coherent states do not admit non-negative phase-space distributions. Such states are classified as nonclassical. Photon-added coherent states (PACS) is defined as $|\alpha, m\rangle = \hat{a}^m |\alpha\rangle$ (unnormalized), where $|\alpha\rangle$ is a coherent state of amplitude $\alpha$, $\hat{a}$ is the creation operator for the field and $m$ is a nonnegative integer [1]. The state $|\alpha, m\rangle$ is said to be a PACS of order $m$ and amplitude $\alpha$. Unlike the coherent states $|\alpha\rangle$, the PACS $|\alpha, m\rangle$ exhibits nonclassical features such as squeezing, sub-Poissonian statistics, etc [1]. Hence, the PACS is considered as a state that interpolates between the classical and nonclassical states. While the action of the creation operator on a coherent state leads to a nonclassical state, the action of the annihilation operator $\hat{a}$ on a coherent state does not change the state. Thus, experimentally realizing the action of the creation and annihilation operators on a state is a way of testing the fundamental commutation relation $[\hat{a}, \hat{a}^\dagger] = I$ [2, 3, 4]. Moreover, photon addition to any state, not necessarily the coherent state, of light is being viewed as a way of introducing nonclassicality [3].

Generation of the PACS is possible in the cavity-atom interaction [1], conditional measurements of beam-splitter output [5], etc. Single-photon-added coherent states (SPACSs) have been experimentally realized in an all-optical scheme employing a nonlinear medium [2]. It has been shown that the higher order PACS $|\alpha, m\rangle$ corresponding to $m > 1$, can be realized in this scheme [6]. It may be added that the recent suggestion for tailoring the interaction in optomechanical systems (micro-resonator interacting with laser light) by proper detuning can be used to generate PACS in the optomechanical domain [7]. A common feature of the aforementioned systems is that they are all interacting, bipartite systems. During evolution, the two subsystems are entangled. This enables to make suitable conditional measurements on one of the subsystems so that the other subsystem is pre-
pared in a PACS. Except for the cavity-atom scheme, the other proposals are based on bipartite, coupled oscillators. For instance, in the optomechanical scheme, the resonator mode is an oscillator and the laser field is another oscillator. Interaction between the oscillators can be tailored by detuning. In particular, the following forms of interactions are possible [8]:

\[ H_n \propto \hat{a}^{\dagger} \hat{b}^{\dagger} + \hat{a} \hat{b}, \]  
\[ H_p \propto \hat{a}^{\dagger} \hat{b} + \hat{a} \hat{b}^{\dagger}. \]  

In the optomechanical case, the interactions \( H_n \) and \( H_p \) correspond respectively to negative detuning when the resonator frequency is smaller than the laser frequency and positive detuning wherein the laser frequency is smaller than the resonator frequency. The operators \( \hat{a}^{\dagger} \) and \( \hat{b}^{\dagger} \) are the creation operators of the quantized optical field and the resonator mode respectively. The corresponding annihilation operators are \( \hat{a} \) and \( \hat{b} \) respectively. The Hamiltonian \( H_n \) describes the opto-mechanical equivalent of the all-optical system in the experimental scheme of Zavatta et al. Similar interaction Hamiltonians arise in the context of optically coupled nano-resonators [9] and optical parametric amplifiers [8].

In this report, a comparative study of two processes, namely, parametric downconversion and atom-cavity interaction, that can generate PACS is presented. By expressing the time-evolved states in a suitable non-orthogonal basis, it is established that the later method generates ideal PACS, a feature that is not present in the atom-cavity scheme. Further, the parametric downconversion itself is shown to be capable of generating ideal \( m \)-photon-added coherent state (MPACS), without requiring higher order processes.

2 Cavity-atom scheme

In the cavity-atom scheme, a two-level atom interacts with the electromagnetic field in a single mode cavity. The two levels of the atom are \( |g\rangle \) and \( |e\rangle \) respectively. Interaction between the two-level atom and the field mode of the cavity is described by the Jaynes-Cummings Hamiltonian [10]

\[ \hat{H}_{JC} = \hbar \beta \left[ \hat{a}^{\dagger} |g\rangle \langle e| + \hat{a} |e\rangle \langle g| \right]. \]  

Here \( \hat{a}^{\dagger} \) and \( \hat{a} \) are respectively the creation and annihilation operators for the quantized field in the cavity. The transition operators \( |g\rangle \langle e| \) and \( |e\rangle \langle g| \)
are respectively the lowering and raising operators for the atom. The coupling strength between the atom and the cavity field is characterized by the coupling constant $\beta$. The initial state of the system is $|\alpha\rangle|e\rangle$, i.e., the state of cavity field is the coherent state $|\alpha\rangle$ and the atom is in the excited state $|e\rangle$. For short times, the evolution operator $\exp\left[-it\hat{H}_{JC}/\hbar\right]$ can be truncated to first order in $\beta t$. In this approximation, the state of the system at time $t$ is

$$|\psi(t)\rangle_{\text{app}} \approx |\alpha\rangle|e\rangle - i\beta t \hat{a}^{\dagger}|\alpha\rangle|g\rangle.$$ (4)

The approximate final state $|\psi(t)\rangle_{\text{app}}$ is an entangled state of the cavity field and the atom. If the atom is detected in its ground state $|g\rangle$, the cavity field is the SPACS $\hat{a}^{\dagger}|\alpha\rangle$ (not normalized), whose amplitude is same as that of the initial coherent state $|\alpha\rangle$. The suitability of this approximation depends on the interaction duration $t$ and the coupling constant $\beta$. To assess the nature of the approximation in a better way, the complete time-evolved state is required. To this end, the evolution operator is expanded as a power series in $\hat{H}_{JC}$. Using the series expression for the evolution operator, the time-evolved state is expressed as

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \frac{\tau^{2n}}{(2n)!} (\hat{a} \hat{a}^{\dagger})^n |\alpha\rangle|e\rangle + \sum_{n=0}^{\infty} \frac{\tau^{2n+1}}{(2n+1)!} (\hat{a} \hat{a}^{\dagger})^n \hat{a} \hat{a}^{\dagger}|\alpha\rangle|g\rangle.$$ (5)

where $\tau = -i\beta t$. The states $|e\rangle$ and $|g\rangle$ are orthogonal to each other. Therefore, if the atom is in the ground state $|g\rangle$ on exit from the cavity, the state of the field in the cavity is

$$|\alpha, \tau\rangle\rangle = \sum_{n=0}^{\infty} \frac{\tau^{2n+1}}{(2n+1)!} (\hat{a} \hat{a}^{\dagger})^n \hat{a} |\alpha\rangle.$$ (6)

The symbol $|\ldots\rangle\rangle$ denotes the state of the cavity field. Using the identities[11],

$$(\hat{a}^{\dagger} \hat{a})^n = \sum_{k=1}^{n} S(n, k) \hat{a}^{\dagger k} \hat{a}^k,$$ (7)

$$\hat{a}^{\dagger k} \hat{a}^k \hat{a}^{\dagger} = k \hat{a}^{\dagger k} \hat{a}^k - 1 + \hat{a}^{\dagger k+1} \hat{a}^k,$$ (8)

where $S(n, k)$ are the Stirling numbers of the second kind[12], the expression for $|\alpha, \tau\rangle\rangle$ is recast as

$$|\alpha, \tau\rangle\rangle = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\tau^{2n+1}}{(2n+1)!} B(n, m, \alpha) |\alpha, m\rangle.$$ (9)
Here,

\[
B(n, m, \alpha) = \alpha^{m-1} \left[ mS(n, m) + S(n, m-1)\alpha \right] \sqrt{m!} L_m(-|\alpha|^2)
\]

and \( |\alpha, m\rangle = \hat{a}^m |\alpha\rangle \), are the MPACS (unnormalized) of amplitude \( \alpha \). The function \( L_m(-|\alpha|^2) \) is the Laguerre function of order \( m \)\(^{12}\). The above result implies that the cavity field is a superposition of various MPACS. The MPACS of different orders but of same amplitude are linearly independent and non-orthogonal. Hence, the superposition coefficients in Eq. 9 cannot be interpreted as probability amplitudes.

On detecting the atom in its ground state, the cavity field is expected to be in the SPACS \( |\alpha, 1\rangle \). If it is indeed the case, it is not enough that the overlap between the states \( |\alpha, \tau\rangle \rangle \) and \( |\alpha, 1\rangle \) is nearly unity. It is required that \( |\langle\langle \alpha, \tau | \alpha, m\rangle\rangle | \approx |\langle \alpha, 1 | \alpha, m\rangle| \) for all \( m \) to ensure that the states generated are indeed the PACS of amplitude \( \alpha \). In Fig. 1, the variation of overlap as the interaction duration increases is shown. The coupling constant \( \beta \) is chosen to be 2\( \pi \) MHz. The respective overlap functions of the cavity state \( |\alpha, \tau\rangle \rangle \) with the MPACS of order \( m = 1, 2 \) and 3 are shown. The initial coherent state is of amplitude \( \alpha = 0.8 \) and it is expected that the scheme generates the SPACS of amplitude \( \alpha = 0.8 \). As expected, for short interaction times the overlap between the cavity state \( |\alpha, \tau\rangle \rangle \) and the SPACS \( |\alpha, 1\rangle \) remains close to unity and continues to be substantial (> 0.9) even when the interaction duration extends to 30\( \mu s \). However, the overlap of the cavity state with higher order PACS are much smaller than the required values. For instance, the overlap with \( |\alpha, 2\rangle \) (dashed line in Fig. 1) falls to 0.3 from the short time value of 0.74 if the interaction duration extends to 30\( \mu s \). Similarly, overlap with \( |\alpha, 3\rangle \) (dotted curve in Fig. 1) decreases rapidly with the increase of interaction time. In short, as the interaction time becomes longer, the state of the cavity field differs significantly from the expected SPACS \( |\alpha, 1\rangle \).

3 Coupled oscillators scheme

In the coupled oscillators scheme, the bipartite system is composed of two oscillators which interact. The two oscillator systems could be the two modes of the electromagnetic field or the field mode of a microcavity and a laser field or two coupled microresonators, etc. In this work, the two oscillators are referred as \( a \)-mode and \( b \)-mode respectively. The creation and annihilation operators for the \( a \)-mode are \( \hat{a}^\dagger \) and \( \hat{a} \) respectively. The corresponding
operators for the \( b \)-mode are \( \hat{b}^\dagger \) and \( \hat{b} \) respectively. The Hamiltonian describing the interaction between the two modes in this bipartite system is

\[
\hat{H}_n = \bar{\hbar}\lambda \left( \hat{a}^\dagger \hat{b}^\dagger + \hat{a} \hat{b} \right),
\]

where \( \lambda \) is the coupling constant. For short times, the corresponding evolution operator \( \exp(-i\lambda t \hat{H}_n/\bar{\hbar}) \) is truncated to \( 1 - i\lambda t \hat{H}_n \). If the initial state of the two modes is \( |\alpha\rangle|0\rangle \), then the evolved state is a superposition of \( |\alpha\rangle|0\rangle \) and \( |\alpha, 1\rangle|1\rangle \). On detecting the \( b \)-mode in the one-photon state \( |1\rangle \), the \( a \)-mode is prepared in the SPACS \( |\tilde{\alpha}, 1\rangle \).

In order to know how well the generated state approximates the SPACS, it is necessary to solve for the dynamics without making any approximation. This is facilitated by the fact that the operators \( \hat{a}^\dagger \hat{b}^\dagger, \hat{a} \hat{b} \) and \( \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1 \) provide a realization of the generators of the SU(1,1) algebra. Consequently, the evolution operator \( \exp(-it \hat{H}_n/\bar{\hbar}) \) admits the following factorization \( \exp(-i\lambda t \hat{a}^\dagger \hat{b}^\dagger + \hat{a} \hat{b}) = \exp(u \hat{a}^\dagger \hat{b}^\dagger) \exp[v(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1)] \exp(w \hat{a} \hat{b}) \).

With \( \lambda t = r \exp(i\phi) \), the constants \( u, v \) and \( w \) are \( \tanh r, -\log \cosh r \) and \( -\tanh r \) respectively. This factorized form of the of the evolution operator is used to obtain the state of the bipartite system at time \( t \). If the initial state is \( |\alpha\rangle|0\rangle \), the state \( |\chi\rangle \) of the coupled oscillators at time \( t \) is

\[
|\chi\rangle = \exp\left(\frac{-|\alpha|^2 \tanh^2 r}{2 \cosh r}\right) \sum_{n=0}^{\infty} \frac{(-i \exp(i\phi) \tanh r)^n}{\sqrt{n!}} \hat{a}^\dagger^n |\tilde{\alpha}\rangle |n\rangle.
\]

In the above expression, the state \( |\tilde{\alpha}\rangle = |\alpha/\cosh r\rangle \) is a coherent state of amplitude \( \alpha/\cosh r \). The state \( |\chi\rangle \) is an entangled state of the two modes. The Fock states of the \( b \)-mode appearing in the expression for \( |\chi\rangle \) are orthogonal to each other while the states \( \hat{a}^\dagger^n |\tilde{\alpha}\rangle \) of the \( a \)-mode are not. The orthogonality of the states of the \( b \)-mode renders it possible to make conditional measurements so that the state of the other mode is the PACS. In particular, if \( b \)-mode is in the one-photon state \( |1\rangle \), the state of the \( a \)-mode is the SPACS \( |\tilde{\alpha}, 1\rangle = \hat{a}^\dagger |\tilde{\alpha}\rangle \). More generally, if \( b \)-mode is measured to be the number state \( |m\rangle \), then state of the \( a \)-mode is \( \hat{a}^\dagger m |\tilde{\alpha}\rangle \) which is MPACS. If photon losses due to absorption and other dissipative mechanisms are absent or negligible and the interaction duration is sufficiently longer so that \( r = |\lambda| t \gg 1 \), then the amplitude \( \tilde{\alpha} \) of the state of the \( a \)-mode becomes nearly zero as \( \cosh r \) becomes large. In this limit, the state of the \( a \)-mode is
very close to the number state $|m\rangle$.

A major difference between the states generated in the cavity-atom interaction and those generated in the coupled oscillators is worth mentioning. In the later scheme, the state of the $a$-mode is precisely the SPACS of amplitude of $\alpha/cosh r$ if the $b$-mode is in its first excited state. In general, if the $b$-mode is detected to be in the Fock state $|m\rangle$, the state of the $a$-mode is the MPACS $|\tilde{\alpha}, m\rangle$. In the atom-cavity case, the state of the cavity has contributions from the PACS of all orders of amplitude $\alpha$. This superposition is never an ideal PACS.

The overlap between the SPACS $|\tilde{\alpha}, 1\rangle$ generated in the process of down-conversion and the expected SPACS $|\tilde{\alpha}, 1\rangle$ is

$$|\langle \alpha, 1 |\tilde{\alpha}, 1 \rangle|^2 = \left[ 1 + \frac{|\alpha|^2}{1 + |\alpha|^2} \right] \exp \left[ -|\alpha|^2 (1 - \frac{1}{cosh^2 r}) \right]. \quad (13)$$

The overlap between saturates at $\exp(-|\alpha|^2)/(1 + |\alpha|^2)$ as $t \to \infty$. Hence, in this scheme too the overlap of the generated state with SPACS of amplitude $\alpha$ falls with interaction duration. However, as noted previously, the states generated are indeed PACS of suitably scaled amplitude.

The coefficient in the expression for $|\chi\rangle$ is the probability amplitude for realizing the state $|\tilde{\alpha}, m\rangle|m\rangle$. Hence, the relevant probability is

$$P_m = |\langle \chi |a^{+m}|\tilde{\alpha}\rangle| m\rangle|^2 = \frac{exp(-|\alpha|^2 tanh^2 r)}{cosh^2 r} \frac{tan^2 m \ t L_m}{cosh^2 r} \int \frac{|\alpha|^2}{cosh^2 r} \exp \left[ -|\alpha|^2 (1 - \frac{1}{cosh^2 r}) \right]. \quad (14)$$

Since different Fock states of the $b$-mode are orthogonal to each other, the probability $P_m$ is the probability of realizing the MPACS $a^{+m}|\alpha\rangle$. Though this probability decreases with increasing $m$, there is a finite probability of detecting the states corresponding to higher values of $m$. In practical terms, this would mean that more experimental runs will be required. Nevertheless, the method of Zavatta et al can generate ideal MPACS.

4 Summary

States generated in the parametric downconversion process are ideal photon-added coherent states. The amplitude of the photon-added coherent state
generated in the process is smaller in magnitude compared to the amplitude of the initial seed coherent state. If the initial coherent state is of amplitude \( \cosh(\lambda \tau)\alpha \), the photon-added coherent state generated is of amplitude \( \alpha \). This relation fixes the amplitude of the initial coherent state in terms of the interaction duration (\( \tau \)), coupling constant (\( \lambda \)) and the required amplitude for the photon-added coherent state. The process is capable of generating ideal \( m \)-photon-added coherent states, though the probability of generation falls with increasing \( m \). In contrast, the interaction between a two-level atom and a cavity field in a coherent state does not generate ideal photon-added coherent state of any amplitude. In the atom-cavity scheme, higher order processes are required to generate higher order photon-added coherent states. Typically, \( m \)-photon processes are necessary for generating \( m \)-photon-added coherent states.

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Figure 1: Temporal evolution of overlap $|\langle \alpha, m|\alpha, \tau \rangle|^2$ of the state of the cavity and the MPACS is shown for $m$=1 (continuous), 2 (dash) and 3 (dot). The initial state is a coherent state of amplitude $\alpha = 0.8$. 