Holographic Glueballs and Infrared Wall Driven by Dilaton

Kazuo Ghoroku, Kouki Kubo, Tomoki Taminato, and Fumihiko Toyoda

1 Fukuoka Institute of Technology, Wajiro, Higashi-ku
Fukuoka 811-0295, Japan
2 Department of Physics, Kyushu University, Hakozaki, Higashi-ku
Fukuoka 812-8581, Japan
3 Faculty of Humanity-Oriented Science and Engineering, Kinki University,
Iizuka 820-8555, Japan

Abstract

We study glueballs in the holographic gauge theories, supersymmetric and non-supersymmetric cases, which are given by the type IIB superstring solutions with non-trivial dilaton. In both cases, the dilaton is responsible for the linear potential between the quark and anti-quark, then we could see the meson spectra. On the other hand, the glueball spectra are found for the non-supersymmetric case, but not for the supersymmetric case. We find that we need a sharp wall, which corresponds to an infrared cutoff, in order to obtain the glueballs. In the non-supersymmetric case, the quantized glueballs are actually observed due to the existence of such a wall driven by the dilaton. We could see the Regge behavior of the higher spin glueball states, and the slope of the glueball trajectory is half of the flavor meson’s one.
1 Introduction

Since the holographic approach is a powerful method to study the non-perturbative properties of the strong coupling gauge theories \[1\ 2\ 3\], various attempts to examine the properties of quantum chromo dynamics (QCD) have been performed. Among them, an interesting approach is to study the mass spectra of glueballs. The discrete mass spectra for small spin states are obtained in terms of the normalizable Kaluza-Klein modes of quantum fluctuations of bulk fields \[4\]-\[12\].

Previously we have studied the open strings (flavored mesons) \[14\] in the background with two simple dilaton configurations, supersymmetric \[15\ 16\] (SUSY) and non-supersymmetric (non-SUSY) \[17\] versions. These configurations are obtained as the solutions of type IIB supergravity with five form field flux, dilaton (and axion for SUSY version). In both solutions, the non-trivial dilaton provides the gauge condensate \(\langle F^2 \rangle\), and it leads to the tension \(\tau_M\) of the linear potential between the quark and anti-quark being proportional to \(\sqrt{\langle F^2 \rangle}\) \[18\]. Then we could obtain the mass spectra for the flavor mesons. Further, the Regge behavior for their higher spin states has been obtained in terms of rotating open-string configurations for the Nambu-Goto action \[14\].

Here, the analysis is extended to the glueballs (closed strings) in the two dilatonic backgrounds mentioned above in order to make clear the role of the dilaton or the gauge condensate furthermore. We firstly show the existence of a kind of potential wall for strings in the non-SUSY bulk, but it is absent in the SUSY case. Then we find the importance of this wall to realize the glueball spectra through the following two analyses.

Firstly, the classical configuration is studied for the closed spinning strings as performed in other cases \[13\ 20\ 21\ 22\ 23\ 24\ 25\]. Supposing the folded configuration, we find the Regge behavior for both SUSY and non-SUSY. However, in the SUSY case, the stable configuration is found at the horizon \(r = 0\) where a metric singularity is observed for \(g_{rr}(r)\) with respect to the radial coordinate \(r\). This singularity implies that the quantum fluctuation of this configuration in the direction of \(r\) is suppressed, then we expect the absence of the glueball obtained from the bulk fields.

This point is assured through the second glueball analysis, in which the glueballs are studied as the quantum fluctuations of bulk fields. Namely, while the glueballs are observed in the non-SUSY case, but not in the SUSY case for various bulk fields. Both analyses are therefore compatible. As a result, it can be said that the SUSY background considered here is not enough to realize the glueballs in spite of the fact that the quark confinement is realized in this case.

In the non-SUSY background, however, the bulk curvature has a (naked) singularity at \(r = r_0\). Then, we should perform the holographic analysis in the region of \(r\) outside of
this singularity. In the case of D4 brane model [4, 5], the singularity (at $r = 0$) is covered by the event horizon $r_h$, and the holographic analysis is restricted to $r \geq r_h (> 0)$. In the non-SUSY background considered here, such a parameter like $r_h$ is absent in the metric, however, we observe that strings and branes are prevented to arrive at $r_0$ due to the wall mentioned above. For example, we find that static open-strings are blocked at $r_m (> r_0)$, and then the rotating closed strings are trapped at $r_m$, where the metric is non-singular.

Another type of non-SUSY background solution, which also has a naked singularity at $r = 0$, has been proposed by Constable and Myers [6], and then it has been used to study the meson spectrum [26]. In this case also, a similar potential wall can be observed, and then This wall prevents any physical fields in the bulk from approaching to the naked singularity. However, in this case, glueballs coming from the graviton and dilaton were absent [6] in spite of the wall. Then, in this point, the dual theory of [6] is different from our non-SUSY case since glueballs for all fluctuations are observed in our case.

The outline of this paper is as follows. In the next section, the bulk solutions for our holographic model are given, then the wall of the gravitational potential is shown and examined for strings. In the section 3, glueballs are studied by solving the Nambu-Goto action, and then the role of the potential wall is shown. Then, in the next section, the glueball spectra are given by solving the equations of motion of the bulk field fluctuations. The results are compared with the lattice simulation and other calculations in the section 5. The summary and discussions are given in the final section.

2 Bulk Background

Here we give the ingredient of our holographic model for confining Yang-Mills theory to study glueballs. We consider 10D IIB model retaining the dilaton $\Phi$, axion $\chi$ and self-dual five form field strength $F_{(5)}$. The action is given as

$$S_{(10)} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left( R - \frac{1}{2} (\partial \Phi)^2 + \frac{1}{2} e^{2\Phi} (\partial \chi)^2 - \frac{1}{4 \cdot 5!} F_{(5)}^2 \right), \quad (2.1)$$

where other fields are consistently set to zero, and $\chi$ is Wick rotated [27]. Under the Freund-Rubin ansatz for $F_{(5)}$, $F_{\mu_1 \cdots \mu_5} = -\sqrt{\Lambda}/2 \, \epsilon_{\mu_1 \cdots \mu_5}$, and for the 10D metric as $M_5 \times S^5$ or $ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{kl}dx^k dx^l$, the solution given below has been found [15, 16]. Where $(\mu, \nu) = 0 \sim 4$ and $(k, l) = 5 \sim 9$. The five dimensional part ($M_5$) of the solution is obtained by solving the following reduced 5D action,

$$S_{(5)} = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left( R + 3\Lambda - \frac{1}{2} (\partial \Phi)^2 + \frac{1}{2} e^{2\Phi} (\partial \chi)^2 \right), \quad (2.2)$$

2
which is written in the Einstein frame. And the corresponding equations of motion are given as

\[ R_{MN} = \frac{1}{2} \left( \partial_M \Phi \partial_N \Phi - e^{2\Phi} \partial_M \chi \partial_N \chi \right) - \Lambda g_{MN} \]  

\( (2.3) \)

\[ \frac{1}{\sqrt{-g}} \partial_M \left( \sqrt{-g} g^{MN} \partial_N \Phi \right) = -e^{2\Phi} g^{MN} \partial_M \chi \partial_N \chi , \]  

\( (2.4) \)

\[ \partial_M \left( \sqrt{-g} e^{2\Phi} g^{MN} \partial_N \chi \right) = 0 \]  

\( (2.5) \)

The bulk solutions are obtained under the ansatz for the metric,

\[ ds^2_{10} = G_{MN} dX^M dX^N = e^{\Phi/2} g_{MN} dX^M dX^N \]  

\( (2.6) \)

where \( G_{MN} \) denotes the string (Einstein) frame metric and \( M, N = 0 \sim 9 \) and \( R = \sqrt{\Lambda}/2 = (4\pi g_s N_c \alpha')^{1/4} = (\lambda \alpha'^2)^{1/4} \) and \( \lambda = 4\pi g_s N_c \) denotes the 'tHooft coupling. We consider the following two simple solutions which are dual to the confining YM theory.

(i) Supersymmetric solution

In order to reserve supersymmetry, the solution is obtained under the ansatz,

\[ \chi = -e^{-\Phi} + \chi_0 . \]  

\( (2.7) \)

Then, we obtain

\[ e^\Phi = 1 + \frac{q}{r^4} , \quad A = 1 , \]  

\( (2.8) \)

where the dilaton is set as \( e^\Phi = 1 \) at \( r \to \infty \), and the parameter \( q \) corresponds to the vacuum expectation value (VEV) of gauge fields strength \([16]\) of the dual theory. Then this solution is dual to the four dimensional \( \mathcal{N}=4 \) SYM theory with a constant gauge condensate. Due to this condensate, the supersymmetry is reduced to \( \mathcal{N}=2 \) and then the conformal invariance is lost since the dilaton is non-trivial as given above. As a result, the theory is in the quark confinement phase since we find a linear rising potential between quark and anti-quark with the tension \( \sqrt{q}/(2\pi\alpha' R^2) \) \([15,16,18]\).

Furthermore, we can see that the space-time is regular at any point. In the ultraviolet limit, \( r \to \infty \), the dilaton part \( e^\phi \) approaches to one and the metric \( (2.6) \) is reduced to \( AdS_5 \times S^5 \). On the other hand, the dilaton part \( e^\phi \) diverges in the infrared limit \( r \to 0 \), so that one may expect a singularity at \( r = 0 \). However there is no such a singular behavior. This is assured by rewriting the metric \( (2.6) \) in terms of new coordinate \( z \), where \( z = R^2/r \). Then we obtain

\[ ds^2_{10} = e^{\Phi/2} \frac{R^2}{z^2} \left( -dt^2 + (dx^i)^2 + dz^2 + z^2 d\Omega_5^2 \right) . \]  

\( (2.9) \)
In the infrared limit $z \to \infty$, we have
\[
e^{\Phi/2} \frac{R^2}{z^2} = R^2 \sqrt{\frac{q}{R^8} + \frac{1}{z^4}} \sim \frac{\sqrt{q}}{R^2}.
\]

Therefore we find 10D flat space time in this limit and no singular point \[15, 16\].

Here we notice the following point. The left-hand side of Eq. (2.10) is expressed as
\[
e^{\Phi/2} \frac{R^2}{z^2} = \sqrt{|G_{tt}|} G_{ii} \equiv Q(z),
\]
where $G_{ii}$ expresses one of the metric of the three space component and is not summed up. $Q(z)$ has a minimum at $z = \infty$ ($r = 0$) and the value of the minimum is finite. This is the condition to reproduce the area law of the Wilson loop [28]. The minimum value of $Q(z)$ is proportional to the tension of the linear potential between the quark and the anti-quark. From this observation, we can assure that the background given here leads to the confinement of the dual gauge theory.

(ii) non-Supersymmetric solution

As for the non-supersymmetric case, the solution is given by retaining only the dilaton, namely for $\chi = 0$, then the supersymmetry is lost in this case. The solution is obtained as [17]
\[
A(r) = \left(1 - \left(\frac{r_0}{r}\right)^8\right)^{1/4}, \quad e^\Phi = \left(\frac{(r/r_0)^4 + 1}{(r/r_0)^4 - 1}\right)^{\sqrt{3}/2}, \quad \chi = 0.
\]

This configuration leads to curvature singularity at the horizon $r = r_0$. So we cannot extend our analysis up to this horizon where some terms like higher powers of curvatures or non-trivial RR fields would be needed to make smooth the singularity. This point is an open problem here.

The confinement property of the dual theory for this solution is assured as above through the factor $Q = \sqrt{|G_{tt}|} G_{ii}$. In this case, it is given as
\[
Q(r) = e^{\Phi/2} \frac{r^2}{R^2} A^2(r) = \frac{2^{1+\sqrt{3}/2}}{\left(1 - \left(\frac{r_m}{r}\right)^4\right)^{1+\sqrt{3}/2}}.
\]

This diverges at $r = r_0$, then rapidly decreases with increasing $r$ near $r_0$ (see Fig. 11). On the other hand, for large $r$, $e^{\Phi/2}$ and $A^2(r)$ approach to one, then $Q(r)$ increases with $r$ like $r^2$. These implies that $Q(r)$ has a minimum at a point $r = r_m (> r_0)$. Actually, from
\[
\left.\frac{\partial Q(r)}{\partial r}\right|_{r_m} = \left.\frac{2e^{\Phi/2}}{R^2 r^2 A^2} \left(r^8 - \sqrt{6}r_0^4 r^4 + r_0^8\right)\right|_{r_m} = 0,
\]

4
\( r_m \) and the minimum value \( Q(r_m) \) are obtained as follows

\[
r_m = \left( \frac{\sqrt{6} + \sqrt{2}}{2} \right)^{1/4} r_0 \approx 1.18 \ r_0, \quad Q(r_m) \approx 2.40 \left( \frac{r_0}{R} \right)^2.
\] (2.15)

Fig. 1: The gravitational potentials \( R^2 Q(r) \) for \( r_0 = 1 \) is shown for non-supersymmetric solution, and for supersymmetric case with \( q = \sqrt{6} r_0^{1/4} \). The minimum at about \( r = 1.18 \ r_0 \) and the steep potential wall near \( r_0 \) are seen for non-supersymmetric case. The curve \( c (= r^2) \) represent the potential for the case of \( \Phi = \text{constant} \), namely for no gauge condensate.

(iii) Wall for strings

As shown below, the static open strings could approach to \( r_m \), but they are blocked at \( r_m \) since infinite energy is needed to arrive at this point. In the following, we examine more details of this phenomenon.

The two-dimensional world-sheet coordinates of open string are set as \( (\tau, \sigma) = (t, x(r)) \), and the Nambu-Goto action for the open string is given as

\[
S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det G_{ab}}
\]

\[
= -\frac{1}{2\pi\alpha'} \int dt dx \sqrt{-G_{tt} (G_{xx} + G_{rr} (\partial_x r)^2)},
\] (2.16, 2.17)

where \( G_{ab} = G_{MN} \partial_a X^M \partial_b X^N \). Then, the energy is given as

\[
E = \frac{2}{2\pi\alpha'} \int_0^{L/2} dx \sqrt{-G_{tt} (G_{xx} + G_{rr} (\partial_x r)^2)},
\] (2.18)
where $L$ represents the distance between the end point of the open U-shaped string at the boundary $r = r_{\text{max}}$. The classical solution has been solved as U-shaped form in the $x$-$r$ plane. At the bottom of the U-shape string, $(x, r) = (0, r_{\text{min}})$, $\partial_x r = 0$. Therefore, $r = r_{\text{min}}$ is the minimum point of the open string of $r$. Solving the equation of motion for $r$, we get the following relation

$$\frac{Q^2}{\sqrt{Q^2 - G_{tt}G_{rr}(\partial_x r)^2}} = h, \quad (2.19)$$

where $Q^2$ is defined above and $h$ represents an integration constant. Then, this relation becomes

$$\partial_x r = \pm \sqrt{H \left( \frac{Q^2}{h^2} - 1 \right)}, \quad H = \frac{G_{xx}}{G_{rr}}. \quad (2.20)$$

Then we can take as

$$Q(r_{\text{min}}) = h. \quad (2.21)$$

As a result, $L$ and $E$ are given as

$$\frac{L}{2} = \int_{r_{\text{min}}}^{r_{\text{max}}} dr \frac{1}{\sqrt{H \left( \frac{Q^2}{h^2} - 1 \right)}}, \quad (2.22)$$

$$E = \frac{1}{\pi \alpha'} \int_0^{L/2} dx \frac{Q^2}{h}. \quad (2.23)$$

For the background (2.12), $L$ is estimated as

$$\frac{L}{2} = \left( \int_{r_{\text{min}}}^{r_{\text{min}}+\epsilon} dr + \int_{r_{\text{min}}+\epsilon}^{r_{\text{max}}} dr \right) \frac{1}{\sqrt{H \left( \frac{Q^2}{h^2} - 1 \right)}}, \quad (2.24)$$

where $\epsilon$ is a small finite number. The latter integral is finite since the integrand is finite in the region of $(r_m <) r_{\text{min}} + \epsilon < r < r_{\text{max}}$. Then, the former integral is estimated in the limit of $r_{\text{min}} \to r_m$ as follows\[^1\]

$$I_1 \equiv \lim_{r_{\text{min}} \to r_m} \int_{r_{\text{min}}}^{r_{\text{min}}+\epsilon} dr \frac{1}{\sqrt{H \left( \frac{Q^2}{h^2} - 1 \right)}} = \frac{Q}{\sqrt{\frac{1}{2} H (Q^2)'' + H' (Q^2)'}} \bigg|_{r=r_m}$$

$$\times \left\{ \log \left( \epsilon + p + \sqrt{\epsilon^2 + 2p\epsilon} \right) - \log (p) \right\}, \quad (2.25)$$

where the right hand side is expanded near $r_{\text{min}} = r_m$, and

$$p = \frac{1}{(Q^2)'' / (Q^2)' + 2H'/H} \bigg|_{r=r_m}. \quad (2.26)$$

\[^1\]The details of the calculation are given in Appendix A.
Here, for non-supersymmetric case, we notice that \( p \sim \left( \frac{Q^2}{q^2} \right) \bigg|_{r=r_m} = 0 \) and the prefactors is finite. Then we find

\[
L \big|_{r \rightarrow r_m} \rightarrow \infty .
\]  

(2.27)

Then the open strings cannot exceed \( r_m \) due to the infinite energy cost since the energy is proportional to the length of the string as shown below.

As for the energy \( E \), it would be estimated from (2.23) at its minimum according to the action principle as follows,

\[
E \approx \frac{Q(r_m)}{\pi \alpha'} \int_0^{L/2} dx = \frac{Q(r_m)L}{2\pi \alpha'} .
\]  

(2.28)

This implies that the tension of the linear potential between a quark and an anti-quark is given for the non-supersymmetric case as

\[
\tau = \frac{Q(r_m)}{2\pi \alpha'} .
\]  

(2.29)

This result guarantees the confinement of quarks.

Here we give the following comments related to the above calculation\(^2\).

i) The same phenomenon is seen for the supersymmetric case by replacing \( r = r_m \) by \( r = 0 \). In this sense, the wall is receded to the limit of \( r = 0 \), where \( Q(r) \) defined above takes its finite minimum value \( Q(0) = \sqrt{q/R^2} \) due to the non-vanishing \( q \). Then we could find linear potential also in the supersymmetric case \([15, 16, 18]\].

ii) In the case of the Witten’s D4 model, we also find this behavior of the string stretching at the event horizon. In this case, however, the origin of this behavior is reduced to the property \( H = 0 \) at the blackhole horizon. In this sense, the mechanism of the confinement in D4 model would be different from the our non-SUSY model. We show its details in the appendix A.

iii) It is possible that a string could pass \( r_m \) and approach to \( r_0 \) when it is pushed from a point \( r_i(> r_m) \) toward \( r_m \) with definite energy and velocity \([29]\). However, it can never touch \( r_0 \) since an infinite energy is necessary to arrive there.

iv) Another time dependent (moving) string is considered in the next section (Sec.4), namely the rotating closed string as a glueball state. Its stable state with a finite energy and angular momentum is found at \( r_m \), then it does not move from \( r_m \) to the larger nor smaller \( r \).

\(^2\)We also find more roles of the infrared wall for other classical configurations, for instance, D7-brane embedding as flavor-brane and D5-brane wrapped on \( S^5 \) as baryon vertex. Since its analyses are far from our purpose, we give the details in Appendix B.
3 Glueballs as Rotating closed string

Flavored mesons are given by a open string with two end points are on the D7 brane. On the other hand, the glueballs with higher spin would be represente d by rotating closed strings in the bulk. Such a rotating string is formulated as follows.

In the following analysis we adopt the coordinate \( z = R^2/r \), then the metric (2.6) is written as follows

\[
\begin{align*}
\text{ds}_{10}^2 &= e^{\Phi/2} \frac{R^2}{z^2} \left\{ A^2(z) \left( -dt^2 + (dx^i)^2 \right) + dz^2 + z^2 d\Omega_5^2 \right\}. \\
\end{align*}
\]

(3.1)

Further, the metric for the string which rotates around the \( x_3 \) axis is given by cylindrical polar coordinates as,

\[
\begin{align*}
\text{ds}^2 &= e^{\Phi/2} \frac{R^2}{z^2} \left( A^2(z) \left( -dt^2 + d\rho^2 + \rho^2 d\tilde{\theta}^2 + dx_3^2 \right) + dz^2 \right). \\
\end{align*}
\]

(3.2)

Taking the string world sheet as \( (\tau, \sigma) = (t, z) \) and the ansatz, \( \rho = \rho(z) \) and \( \tilde{\theta} = \omega t \), the induced metric is given as

\[
\begin{align*}
G_{\tau\tau} &= e^{\Phi/2} \frac{R^2}{z^2} A^2(z) \left( -1 + \omega^2 \rho^2 \right), \\
G_{\sigma\sigma} &= e^{\Phi/2} \frac{R^2}{z^2} \left( A^2(z) \rho^2 + 1 \right),
\end{align*}
\]

(3.3)

where prime denotes the derivative with respect to \( z \). Then we have

\[
S_{\text{string}} = \int dt \mathcal{L} = -\frac{1}{2\pi \alpha'} \int dt dz e^{\Phi/2} A^2(z) \frac{R^2}{z^2} \sqrt{1 + \omega^2 \rho^2} \left( \rho^2 + A^{-2}(z) \right). 
\]

(3.4)

From this, the spin \( J_s \) and the energy \( E_s \) of this string are given as

\[
\begin{align*}
J_s &= \frac{\partial \mathcal{L}}{\partial \omega} = \frac{1}{2\pi \alpha'} \int dz e^{\Phi/2} A^2(z) \frac{R^2}{z^2} \omega \rho^2 \sqrt{\frac{\rho^2 + A^{-2}(z)}{1 - \omega^2 \rho^2}}, \\
E_s &= \omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L} = \frac{1}{2\pi \alpha'} \int dz e^{\Phi/2} A^2(z) \frac{R^2}{z^2} \sqrt{\frac{\rho^2 + A^{-2}(z)}{1 - \omega^2 \rho^2}}.
\end{align*}
\]

(3.5)

(3.6)

These are estimated by giving appropriate solutions for the corresponding strings.

Equations of motion for folded strings

In order to solve the string equation, it is convenient to use the reparametrization invariant formalism since the configuration of a solution is given by a continuous curve. So the solution can be expressed by one parameter, \( s \) or \( \sigma \) as given in [14]. The Lagrangian is written in terms of \( s \) as

\[
\mathcal{L} = -\frac{1}{2\pi \alpha'} \int ds \tilde{L} = -\frac{1}{2\pi \alpha'} \int dz e^{\Phi/2} A^2(z) \frac{R^2}{z^2} \sqrt{1 - \omega^2 \rho^2} \left( \dot{\rho}^2 + \dot{z}^2 A^{-2}(z) \right).
\]

(3.7)
where dot denotes the derivative with respect to \( s \). \( s_i(= 0) \) and \( s_f(= 2\pi) \) are defined as

\[
\begin{align*}
z(s_i) &= z(s_f), & \dot{z}(s_i) &= \dot{z}(s_f) = 0. \\
\rho(s_i) &= \rho(s_f), & \dot{\rho}(s_i) &= \dot{\rho}(s_f) = 0.
\end{align*}
\]

These equations mean that the end point of the string is smoothly connected since we consider closed string solutions.

The equations of motion to be solved are obtained by introducing the canonical momentum as,

\[
\begin{align*}
p_{\rho} &= \frac{\partial \tilde{L}}{\partial \dot{\rho}}, & p_z &= \frac{\partial \tilde{L}}{\partial \dot{z}},
\end{align*}
\]

we have the Hamiltonian

\[
H = 2\tilde{H}/\Delta, \quad \Delta = \frac{F}{\sqrt{\dot{\rho}^2 + \dot{z}^2A^{-2}(z)}},
\]

\[
F = e^{\Phi/2}A^2(z)\frac{R^2}{z^2}\sqrt{1 - \omega^2\rho^2},
\]

\[
\tilde{H} = \frac{1}{2}\left(p_{\rho}^2 + p_z^2A^2(z) - F^2\right).
\]

Then the Hamilton equations are obtained from \( \tilde{H} \) instead of \( H \) for the simplicity,

\[
\begin{align*}
\dot{\rho} &= p_{\rho}, & \dot{z} &= p_zA^2(z), \\
\dot{p}_{\rho} &= -\omega^2\rho Q^2(z), & \dot{p}_z &= -p_z^2A(z)\frac{\partial A(z)}{\partial z} + \frac{1}{2}\left(1 - \omega^2\rho^2\right)\frac{\partial Q^2(z)}{\partial z},
\end{align*}
\]

and

\[
Q^2(z) = e^{\Phi}A^4(z)\frac{R^4}{z^4}.
\]

### 3.1 Solution in the SUSY background

We solve above equations for the closed string in the supersymmetric background, \((2.8)\), by imposing the ansatz,

\[
z = z_m,
\]

where \( z_m \) is a constant. This satisfies the above boundary condition \((3.8)\) of course. Then, from the second Eq.\((3.15)\), we find

\[
z_m = \infty,
\]

which means \( r_m = R^2/z_m = 0 \).
Alternative way to obtain this solution is as follows. First, rewrite the Eq. (3.7) by using ansatz (3.17) as follows

$$ L = -\frac{1}{2\pi\alpha'} Q(z_m) \int d\rho \sqrt{1 - \omega^2 \rho^2} , $$

where $Q(z)$ is given in the above (3.16). Then solving this with respect to $z_m$, we find it as the minimum point of $Q(z)$. This point is already discussed above in the previous section.

As for $\rho$, from the remaining equations we find

$$ \rho = \frac{1}{\omega} \sin \left( \sqrt{\frac{q}{R^2}} \omega s \right) . $$

(3.20)

We use this simple solution in the followings.

**Regge behavior**

The spin and the energy of this closed string configuration are given by using the above equations (3.5) and (3.6) as

$$ J_s = \frac{1}{2\alpha'\omega^2} \int_{-1/\omega}^{1/\omega} d\rho e^{\Phi/2} R^2 \sqrt{\frac{1 + (\partial z/\partial \rho)^2}{1 - \omega^2 \rho^2}} , $$

$$ E_s = \frac{1}{2\alpha'\omega^2} \int_{-1/\omega}^{1/\omega} d\rho e^{\Phi/2} R^2 \sqrt{\frac{1 + (\partial z/\partial \rho)^2}{1 - \omega^2 \rho^2}} . $$

(3.21)

(3.22)

Substituting the above closed string solution, we find

$$ J_s = \frac{1}{2\alpha'\omega^2} \frac{\sqrt{q}}{R^2} , \quad E_s = \frac{1}{\alpha'\omega} \frac{\sqrt{q}}{R^2} . $$

(3.23)

Then we obtain

$$ J_s = \alpha'_{\text{glueball}} E_s^2 , \quad \alpha'_{\text{glueball}} = \frac{1}{2} \frac{\alpha' R^2}{\sqrt{q}} = \frac{1}{2} \frac{\alpha'}{Q(z_m)} . $$

(3.24)

Here we notice that

$$ \alpha'_{\text{glueball}} = \frac{1}{2} \alpha'_{\text{meson}} $$

(3.25)

where $\alpha'_{\text{meson}}$ represents the slope parameter of the flavored mesons [14].

**Problems of SUSY solutions**

We notice here that the above solution is pulled down up to $r = 0$ by the gravitational attractive force. However, we find that $g_{rr}$ becomes infinite at this point. Then
it leads to a difficulty when we consider quantum fluctuations around this classical configuration. We can expand the action around the above classical solution as

\[
\mathcal{L} \equiv e^{\Phi/2} \sqrt{-g} = \sqrt{q g_0} \left( 1 + \frac{1}{2 g_0} \left\{ - (\rho')^2 \delta r^2 + \left( 1 - \omega^2 \rho^2 \right) \delta r' \right\} + \cdots \right),
\]

where dot and prime denote the derivative with respect to \( \tau \) and \( s \). The ellipsis represent other fluctuations and higher order terms. Then the coefficient of the quadratic terms of \( \delta r \) in \( \mathcal{L} \) diverges like \( 1/r_m^4 \) for \( r_m = 0 \). This implies that \( \delta r \) must be suppressed, then the quantum fluctuation of the closed string configuration given here cannot spread in the radial direction.

This point is the defect of the present supersymmetric model. Consider the zero size limit of this closed string solution, then it corresponds to a point particle in the bulk \([10]\). It is dual to the glueball operator of 4D Yang-Mills theory. However, this fluctuation could not propagate in the bulk. As shown below, in the present case, we actually cannot find glueball spectra through the fluctuations of the bulk fields in the supersymmetric bulk background. This indicates that we must improve the background configuration such that the classical configuration of a closed string allows the quantum fluctuation in the radial direction. One realization is given in the non-supersymmetric case as shown below.

### 3.2 Solution for non-SUSY background

For the non-supersymmetric background solution \((2.12)\), by solving the equation of motions \((3.14)\) and \((3.15)\), we find \( z_m \) as follows

\[
R^2/z_m = r_m \approx 1.18 \ r_0.
\]

This is the same result with the one given in \((2.15)\) since the same equation is solved. Namely it is obtained as the minimum of \( Q = \sqrt{|G_{tt}(r)|G_{ii}(r)} \). Then the closed strings are trapped at \( z_m \) and separated out of the singular point \( r_0 \). This fact is different from the case of Witten’s D4-brane background because the closed strings are not trapped at the horizon \( U = U_{KK} \) in the background, and the strings continue to drop up to \( U = 0 \) through the horizon \( U_{KK} \).

As for \( \rho(s) \), we find

\[
\rho = \frac{1}{\omega} \sin \left( Q(z_m) \omega s \right) .
\]

While, in the supersymmetric case, the metric divergence has appeared at \( r_m \), there is no such a metric divergence at \( r_m \) in the present case, since \( r_m > r_0 > 0 \).
We should notice here that $\rho$ is finite in spite of the fact that the string stays at $r_m$. In the case of the open strings discussed above, the length becomes infinite when the string approaches to $r_m$. There is no contradiction between the two results since the closed string in the present case is rotating. In general, moving string could pass the point $r_m$ if it has enough energy to climb the wall as seen in [29].

**Regge behavior**

The spin and the energy of this closed string configuration are estimated by using the equations (3.5) and (3.6) as in the supersymmetric case. Then we have the result,

$$J_s = \alpha'_{\text{glueball}} E_s^2, \quad \alpha'_{\text{glueball}} = \frac{1}{2} \alpha' \frac{1}{Q_m}.$$

Here we notice that $Q_m/(2\pi\alpha')$ represent the tension of the quark and anti-quark linear potential obtained for the non-supersymmetric model used here, then we also find

$$\alpha'_{\text{glueball}} = \frac{1}{2} \alpha'_{\text{meson}}$$

for mesons with large spin.

## 4 Glueballs from bulk field fluctuations

### 4.1 non-Supersymmetric case

For non-supersymmetric background, we find the classical stable configuration of glueballs corresponding to the large quantum number state. And quantum fluctuations can be added them to see the corrections to the Regge behavior obtained above. For zero size limit of the classical string, namely the point particle case, we study the corresponding glueball state by solving the field equation of the quantum fluctuation of the bulk fields as given below.

**Graviton $2^{++}$**: As for the glueball spectrum, many attempts have been made by solving the linearized field equations of bulk field fluctuations in the given background. Here we consider the field equation of the traceless and transverse component of the metric fluctuation, which is denoted by $h_{ij}$. Its linearized equation is given in the Einstein frame metric as

$$\frac{1}{\sqrt{-g}} \partial_M \left( \sqrt{-g} g^{MN} \partial_N h_{ij} \right) = 0,$$

where we use $z$ instead of $r$ and assumed as $h_{ij} = h_{ij}(x^0, x^i, z)$, then $M, N$ are the five dimensional ($(x^0, x^i, z)$) suffices. This equation is equivalent to the massless scalar

$^{3}$In the string frame metric case, this equation is written as $\frac{1}{\sqrt{-g}} \partial_M \left( \sqrt{-ge^{-2\Phi}} g^{MN} \partial_N h_{ij} \right) = 0$ as given in [5].
field equation. As shown in [11], this equation is common to \(2^{++}, 1^{++}\) and the one of the non-active dilaton \(0^{++}\), which are dual to the glueball of \(F_{\mu\nu}F^{\mu\nu}\). While it is usually used to derive the type IIA theory, the NS-NS part is common with the one of the type IIB theory. Then the masses of these three spin states degenerate. However, we are considering non-trivial dilaton background configuration, then the above Eq.(4.1) is used for the graviton fluctuation, the glueball of \(2^{++}\) state.

By setting as \(h_{ij} = p_{ij}e^{ikx}\phi(z)\) and \(-k^2 = m^2\), we get

\[
\partial_z^2 \phi + g_z(z)\partial_z \phi + \frac{m^2}{A^2} \phi = 0, \quad (4.2)
\]

\[
g_z(z) = \partial_z \left(\log \left(\frac{(R/z)^3}{A^4}\right)\right) = -\frac{3}{z} + 4\frac{\partial_z A}{A}. \quad (4.3)
\]

Here we notice

\[
A = \left(1 - \left(\frac{z}{z_0}\right)^8\right)^{1/4} \quad (4.4)
\]

where \(z_0 = R^2/r_0\). Then we see that the equation (4.2) has 10 regular singularities at \(z = 0, \infty\) and the points of \(\left(\frac{z}{z_0}\right)^8 = 1\). We therefore try to find the eigenfunctions in the region of \(0 \leq z \leq z_0\) through WKB approximation [5, 8] by changing the variable from \(z\) to \(y\) which is defined as

\[
z = \frac{z_0}{1 + e^y}, \quad z_0 = \frac{R^2}{r_0} \quad (4.5)
\]

where \(y\) is defined in the region of \(-\infty < y < \infty\).

Then the equation (4.2) and (4.3) are rewritten as

\[
\partial_y^2 \phi + g_2(y)\partial_y \phi + \frac{m^2}{A^2} \frac{z_0^2 e^{2y}}{(1 + e^y)^4} \phi = 0, \quad (4.6)
\]

\[
g_2(y) = \frac{5e^y}{1 + e^y} - 1 + 4\frac{\partial_y A}{A}. \quad (4.7)
\]

In order to perform the WKB approximation, we rewrite the wave function as \(\phi = e^{-\frac{1}{2} \int dg_2(y)} f(y)\), then we obtain

\[
-\partial_y^2 f + V(y) f = 0, \quad V = \frac{1}{4} g_2^2 + \frac{1}{2} \partial_y g_2 - \frac{m^2}{A^2} \frac{z_0^2 e^{2y}}{(1 + e^y)^4}. \quad (4.8)
\]

This is the one dimensional Schrödinger equation form with the potential \(V\) and the zero energy eigenvalue. For an appropriate mass \(m\), we can see that \(V\) has two turning points, \(y_1\) and \(y_2(> y_1)\), to give [5]

\[
\int_{y_1}^{y_2} \sqrt{-V} dy = \left(n + \frac{1}{2}\right) \pi \quad (4.9)
\]

\[\text{Here active means that the dilaton background solution is nontrivial as in the present case.}\]

\[\text{5 \textit{p}_{ij} denotes projection operator onto the traceless and transverse components.}\]
with integer \( n \). From this equation we obtain the discrete glueball mass \( m_n \), where \( n \) denotes the node number of the eigenfunction. The potential for the zero node is shown in the Fig.[2]

Here we should give the following comments.

- We notice here that the two turning points found above are finite. This fact is understood as follows. The "Schrödinger" potential \( V(y) \) given in (4.8) is expanded for small \( e^y(\equiv x) \) as

\[
V(y) = \frac{x}{4} - \frac{x^{3/2}}{2\sqrt{2}}(mz_0)^2 + \frac{77x^2}{16} + O(x^{5/2}). \tag{4.10}
\]

This implies that \( V(y) \) change sign near \( x = 0 \) (at about \( x \sim 1/(2m^4z_0^4) \)). On the other hand, at large \( x \), we have

\[
V(y) = 4 - \frac{15}{2x} + \left( \frac{45}{4} - (mz_0)^2 \right) \frac{1}{x^2} + O(1/x^3). \tag{4.11}
\]

Then, \( V(y) \) approaches to \( 4(>0) \) at large \( x \). Therefore, there are two turning points at finite \( y \) or \( x \). This point is very important since the turning point in the smaller side of \( y \) is found at \( y = -\infty \) in the D4 model [5] and Constable-Myers model [6]. In the Fig. 2 the potential \( V(y) \) is shown for the zero node state, and we actually have the two finite turning points in our case as, \( y^+ = 0.02065 \) and \( y^- = -6.526 \).

- In the case of D4 model, the point \( y = -\infty \) corresponds to the event horizon of the bulk black hole background. This point, fortunately, is not a singular point of the supergravity background, then it would be meaningful to impose a boundary condition as a turning point for the WKB approximation. However, for the Constable-Myers model, this point \( y = -\infty \) is at the naked singularity of the bulk background, so the authors of [6] concluded as that the glueballs of \( 2^{++} \) and \( 0^{++} \) cannot be seen in their model due to this reason.

- In our model, the turning points are far from \( y = -\infty \), where the naked singularity exists, then we can perform the WKB analysis without worrying on this point. This would be reduced to the fact the singular point is at \( r = r_0 \), which is finite, and the wall, which push out the classical string configurations and D-branes, also at the outside of the singularity. We suppose that, due to this wall, the quantum wave-function is also confined in a finite range, then we could find discrete spectrum as shown below in terms of the WKB approximation.

**Axion** \( 0^{-+} \): Since the axion and three form field strengths, which couple to the axion, are non-active in the present background, then the equation of motion for the axion fluctuation is obtained directly from the bulk action as

\[
\frac{1}{\sqrt{-g}} \partial_M \left( \sqrt{-g} g^{MN} e^{2\Phi} \partial_N \partial \chi \right) = 0, \tag{4.12}
\]
As above, we rewrite this equation using the following form, \( \chi = e^{ikx}e^{-\frac{1}{2}\int dy g_\chi(y)f_\chi(y)} \), as follows

\[-\partial_y^2 f_\chi + V_\chi(r)f_\chi = 0, \quad V_\chi = \frac{1}{4}g_\chi^2 + \frac{1}{2}g_\chi' - \frac{m^2 z_0^2 e^{2y}}{A^2 (1 + ey)^4}. \tag{4.13}\]

where

\[g_\chi(y) = -g_2(y) + 2\partial_y \Phi \tag{4.14}\]

For \( m = 3.05 \text{GeV} \) and \( z_0 = 2.0 \text{GeV}^{-1} \), the values of the potential \( V_\chi \) is shown in the Fig. 2. As shown in this figure, for the case of the axion, we could find two turning point (zero-point) for large enough value of \( m \), then the WKB method is useful as in the graviton case.

![V vs y](image-url)

**Fig. 2:** The Schrödinger potentials \( V(z) \) with \( z_0 = 2 \) (GeV\(^{-1}\)) are shown for (a) graviton \( m = 2.182(2.15) \) (GeV), (c) axion \( m = 3.05(2.25) \) (GeV), and (b) dilaton \( m = 1.207(1.47) \) (GeV) cases. The values in the parenthesis are the data of the lattice simulation [33, 34, 35].

**Dilaton \( 0^{++} \):**

In the present case, the dilaton is an active scalar, namely it has a classical configuration. Then its fluctuation \( \phi \) mixes with the scalar component of the graviton. Expanding the metric in terms of the scalar \( \psi \) and traceless transverse part \( h^{TT}_{ij} \) as

\[h_{ij} = 2\eta_{ij}\psi + h^{TT}_{ij} + \cdots. \tag{4.15}\]

Then the equation of motion of the scalar mode \( \zeta \), which is invariant under the general
coordinate transformation, is given as \[36\]

\[
\partial_y^2 \zeta + g_\zeta(y) \partial_y \zeta + \frac{m^2}{A^2} \frac{z_0^2 e^{2y}}{(1 + e^y)^4} \zeta = 0, \tag{4.16}
\]

\[
g_\zeta(y) = g_2(y) + \frac{2 \partial_y B}{B}, \tag{4.17}
\]

\[
\zeta = \psi - \phi B, \quad B = \frac{\partial_y \Phi}{\partial_y \left( \log \left( \frac{R(1+e^y)}{z_0} \right) \right)}. \tag{4.18}
\]

As above, we rewrite this equation by using \( \zeta = e^{ikx} e^{-\frac{1}{2} \int dy g_\zeta(y) f_\zeta(y) } \), as follows

\[- \partial_y^2 f_\zeta(r) + V_\zeta(r) f_\zeta = 0, \quad V_\zeta = \frac{1}{4} g_\zeta^2 + \frac{1}{2} \partial_y g_\zeta - \frac{m^2}{A^2} \frac{z_0^2 e^{2y}}{(1 + e^y)^4}. \tag{4.19}\]

Then we can perform the WKB analysis as above. The behavior of the potential \( V_\zeta \) is similar to the one of the graviton as seen from Fig. 2. However, due to a slight difference of the potentials leads to the difference of the eigen masses for the graviton and the dilaton as shown in the Table 1. In any case, the infinite series of the radial excitations for the three states are observed in the case of the non-supersymmetric background solution.

### 4.2 Supersymmetric case

In the supersymmetric case, we cannot find glueball state from the fluctuation mode of the bulk fields since there is no normalizable wave function with definite four dimensional mass eigenvalue.

**Graviton 2++**;

Firstly, this is shown for the graviton fluctuation. Its equation is given by \[11\], but the metric is used for the supersymmetric solution. In this case, there is no restriction to the variable \( r \) and we can consider the whole range, \( 0 \leq z < \infty \). The equation for the graviton is given by setting as above, \( h_{ij} = p_{ij} e^{ikx} \phi_s(z) \) and \(-k^2 = m^2\), then we get

\[
\partial_z^2 \phi_s - \frac{3}{z} \partial_z \phi_s + m^2 \phi_s = 0. \tag{4.20}
\]

This is solved as

\[
\phi_s(z) = \frac{z^2}{z_0^2} (C_1 J_2(mz) + C_2 N_2(mz)) \tag{4.21}
\]

where \( C_{1,2} \) are arbitrary constant and \( m \) denotes the glueball mass. \( J_n(x) \) and \( N_n(x) \) are the first kind and second kind Bessel functions. It is easily assured that this solution is not normalizable since

\[
\int_0^\infty \frac{dz}{z^3} |\phi_s(z)|^2 \tag{4.22}
\]
is divergent. This is because of that there is no infrared cutoff or wall in this case.

From the viewpoint of the one-dimensional Schrödinger equation, we can see that the potential has no two turning points. Actually, for the graviton we have

\[ V_{gr} = \frac{15}{4z^2} - m^2. \]  

(4.23)

This potential has only one zero point for any \( m \), then we could not obtain any mass state by the WKB approximation by using this potential.

**Dilaton and Axion 0±+;**

In the present case, both the dilaton and the axion have its classical solution. Then their fluctuations mix with the gravitational ones. Here we consider the mass eigenmodes of the two scalar fluctuations in a special gauge, where they decouple from each others. This is performed as follows.

At first, set the fluctuations, \( h_{MN}, \delta \Phi, \) and \( \delta \chi \), of each field as,

\[ g_{MN} = a^2(z)(\eta_{MN} + h_{MN}), \quad \Phi = \bar{\Phi} + \delta \Phi, \quad \chi = \bar{\chi} + \delta \chi \]  

(4.24)

where \( \bar{\Phi} \) and \( \bar{\chi} \) denote classical solutions,

\[ \bar{\chi} = -e^{\bar{\Phi}} + \chi_0, \quad e^{\bar{\Phi}} = 1 + \tilde{q} z^4, \quad \tilde{q} = q/R^8. \]  

(4.25)

Then, from (2.4) and (2.5), we obtain

\[ \frac{1}{a^3} \partial_M \left( a^3 \partial^M \delta \Phi + a^3 \left( \frac{1}{2} h_{MN} \right) \partial_N \bar{\Phi} \right) = \]

\[ - \left( \frac{1}{2} h_{\eta MN} - h^{MN} \right) \partial_M \bar{\Phi} \partial_N \bar{\Phi} - 2 \left( e^{\bar{\Phi}} \partial_M \delta \chi + \delta \Phi \partial_M \bar{\Phi} \right) \partial^M \bar{\Phi} \]  

(4.26)

\[ \partial_M \left( a^3 e^{2\Phi} \eta^{MN} \partial_N \delta \chi + a^3 \left( \frac{1}{2} h_{\eta MN} - h^{MN} + 2 \delta \Phi \eta^{MN} \right) e^{\Phi} \partial_N \bar{\Phi} \right) = 0 \]  

(4.27)

where

\[ h = h^M_M \]

and the suffixes \( M, N \) are raised and lowered by \( \eta_{MN} \) or \( \eta^{MN} \).

Here we take the following gauge conditions,

\[ \frac{1}{2} h - h^{zz} + 2 \delta \Phi = 0 \]  

(4.28)

\[ \partial_\mu h^{\mu z} - 2 e^{\Phi} \partial_z \delta \chi = 0 \]  

(4.29)

and

\[ f_\mu = 0, \]  

(4.30)
where \( f_\mu = 0 \) is defined as
\[
h_{MN} = \left( 2h_{\mu\nu} + (\partial_\mu f_\nu + \partial_\nu f_\mu) + 2\eta_{\mu\nu}\psi + 2\partial_\mu \partial_\nu E \frac{B_\mu + \partial_\mu C}{2\xi} \right),
\]
(4.31)
where we used the same setting with [37]. Then, (4.28) and (4.29) are rewritten in terms of the fields in (4.31) as
\[
2\Psi - \xi + \partial_\alpha^2 E + 2\delta\Phi = 0,
\]
(4.32)
\[
\partial_\mu^2 C + 2e^{\delta\Phi} \partial_\xi \delta\chi = 0.
\]
(4.33)
As a result, the gauge is completely fixed by the above three conditions.

Then the equations (4.26) and (4.27) are rewritten as
\[
\delta\Phi'' + \frac{a'}{a} \delta\Phi' + \left(m_\Phi^2 - 4(\Phi')^2\right) \delta\Phi = 0
\]
(4.34)
\[
\delta\chi'' + \left(3\frac{a'}{a} + 4\Phi'\right) \delta\chi' + m_\chi^2 \delta\chi = 0
\]
(4.35)
where prime denotes the derivative with respect to \( z \) and
\[
\partial_\mu^2 \delta\Phi = m_\Phi^2 \delta\Phi
\]
\[
\partial_\mu^2 \delta\chi = m_\chi^2 \delta\chi
\]

Then the equation (4.34) is rewritten by using
\[
\delta\Phi = e^{ikx} e^{-\frac{i}{2} \int dz g_\phi f_\phi(z)}
\]
as
\[
-f'_\phi'' + V_\phi(z) f_\phi = 0,
\]
(4.36)
\[
V_\phi(z) = \frac{1}{4} g_\phi^2 + \frac{1}{2} g_\phi' - m_\phi^2 - 4(\Phi')^2
\]
(4.37)
\[
g_\phi = 3\frac{a'}{a} = -3\frac{1}{z}
\]
(4.38)
And by using
\[
\delta\chi = e^{ikx} e^{-\frac{i}{2} \int dz g_\chi f_\chi(z)}
\]
(4.35) is rewritten as
\[
-f''_\chi + V_\chi(z) f_\chi = 0,
\]
(4.39)
\[
V_\chi(z) = \frac{1}{4} g_\chi^2 + \frac{1}{2} g_\chi' - m_\chi^2
\]
(4.40)
\[
g_\chi = 3\frac{a'}{a} + 4\Phi'
\]
(4.41)
The typical potentials are shown for both cases in the Fig. 3. As for the dilaton, the potential has a deep negative minimum, but the second zero point does not appear at the large z side. This situation is therefore similar to the one of the graviton, in which case there is no minimum. Namely only one zero point is observed for any $m_\phi$. Then we can not find any glueball state with a finite mass.

As for the case of axion, on the other hand, there seems to be a possibility of glueball’s existence due to the minimum of the potential and two zero points, say $(z_1, z_2)$ (see Fig. 3), which are seen in the Fig. 3. However, this minimum is not deep enough to produce a glueball state. We examined the value of

$$\int_{z_1}^{z_2} \sqrt{-V_\chi(z)} \, dz$$

for various parameter ranges of $q$ and $m_\chi$, but it is too small and does not satisfy the condition needed for the WKB bound state or (4.9).

As a result, the two scalar modes also have no glueball state as the graviton. This fact can be related to the metric singularity as mentioned above. Due to this singularity, the fluctuation of the classical closed string configuration cannot spread in the bulk. There might be several directions to remove this difficulty. One easy way is to introduce an artificial cutoff for the coordinate $r$. Another would be a modification of the model to our non-supersymmetric case as a simple example, which is shown above.
5 Numerical results for the glueball mass

The glueball mass depends only on the parameter $r_0/R^2$ in our model. We show our results in the Fig. 4 and in the Table 1. 

Fig. 4: **Left:** Numerical results of our the glueball mass for $r_0/R^2 = 0.5$. $n$ denotes the node number of the states. **Right:** The mass ratio, (our calculation)/(the one of D4 model [11]), for $J^{PC} = 2^{++}$ spectra.

Table 1: The glueball masses for $r_0/R^2 = 0.5$ (GeV). The column WKB shows our result of WKB calculations in the unit of GeV. $J^{PC}_n$ denotes spin ($J$), charge conjugation ($C$), parity ($P$), and node number ($n$) of the corresponding wave-functions respectively.

| $J^{PC}_n$ | WKB | $J^{PC}_n$ | WKB | $J^{PC}_n$ | WKB |
|------------|-----|------------|-----|------------|-----|
| $2^{++}_0$ | 2.176 | $0^{+}_0$ | 3.049 | $0^{++}_0$ | 1.207 |
| $2^{++}_1$ | 3.689 | $0^{+}_1$ | 4.673 | $0^{++}_1$ | 3.390 |
| $2^{++}_2$ | 5.181 | $0^{+}_2$ | 6.221 | $0^{++}_2$ | 4.981 |
| $2^{++}_3$ | 6.668 | $0^{+}_3$ | 7.743 | $0^{++}_3$ | 6.516 |
| $2^{++}_4$ | 8.154 | $0^{+}_4$ | 9.251 | $0^{++}_4$ | 8.031 |
| $2^{++}_5$ | 9.639 | $0^{+}_5$ | 10.752 | $0^{++}_5$ | 9.535 |

They are obtained by using $M_0(2^{++}) = 2.176$, which is given here as an average of the lattice simulation [35, 33, 34] and used as an input data. To use this value as an input is equivalent to fix the parameter of our model as

$$\frac{r_0}{R^2} = 0.50 \text{ GeV}. \quad (5.1)$$

Our results approximately reproduce the other data of the lattice simulation given as [35] $M_0(0^{++}) = 1.475$ and $M_0(0^{-+}) = 2.25$ (GeV) for the lowest modes. The masses of the exited state with higher node are also shown. Those one of $2^{++}$ are compared with the results obtained in a different holographic model [11], and we could find that
they are almost equal each other. For other spin states, which are not shown here, the ratios for those spectra are similarly near one.

On the other hand, we know another simulation result for the gauge condensate, \( \langle F_{\mu\nu} F^{\mu\nu} \rangle \), which is given as \( \frac{\lambda}{4\pi^2} \langle F_{\mu\nu} F^{\mu\nu} \rangle = 0.14\text{GeV}^4 \). (5.2)

When this is used, the parameter \( r_0/R^2 \) can be determined independently of the data for any glueball mass. In our model, there are three independent parameters, \( r_0 \), \( R \), and \( \lambda \). They are fixed as follows:

The value of \( r_m \), which is defined as the minimum of the gravitational potential \( Q(r) \), is given as

\[
\frac{R^2}{z_m} = r_m = 1.18r_0
\]

with

\[
Q(r_m) \equiv Q_m = 2.40 \left( \frac{r_0}{R} \right)^2 .
\]

Then the meson Regge slope parameter is written in terms of this \( Q_m \) as

\[
\alpha'_{Meson} = \frac{\alpha'}{Q_m} = \frac{\sqrt{\lambda}}{2.40} \left( \frac{\alpha'}{r_0} \right)^2 .
\]

Next, expanding the dilaton as

\[
e^\Phi = 1 + 6\frac{r_0^4}{r^4} + \cdots ,
\]

we find

\[
q = \sqrt{6}r_0^4 = \pi^2 \langle F_{\mu\nu} F^{\mu\nu} \rangle \lambda \alpha'^4 .
\]

Then, using (5.2) we obtain

\[
\frac{r_0}{\alpha'} = 2.17\text{GeV} .
\]

And, from (5.4) we find

\[
\alpha'_{Meson} = 0.088\sqrt{\lambda} \ (\text{GeV}^{-2}) .
\]

Then, finally we get

\[
\frac{r_0}{R^2} = \frac{r_0}{\alpha'\sqrt{\lambda}} = \frac{0.191}{\alpha'_{Meson}} \sim 0.218 \ \text{GeV} .
\]

When we respect this result, we find that the glueball masses are half of the one obtained by using (5.1)
Both results (5.1) and (5.10) are obtained by using the lattice simulations as the input in our analysis, and they are not compatible. If both the two lattice results are correct, this implies that our model is so simple that we could not reproduce well the lattice results since the number of the parameters of our theory may be too small. We should add other freedom in our theory, for example other bulk field condensations should be considered, but it is not our present scope to discuss this point.

6 Summary and Discussion

We have studied the role of the dilaton field being played in the holographic hadron physics, especially for the glueballs. Here we have studied two dilaton configurations, for supersymmetric and non-supersymmetric cases. In both cases, the dilaton contains the condensate of the gauge field strength, \( \langle F_{\mu\nu} F^{\mu\nu} \rangle \), which determines the properties of the vacuum of the Yang-Mills theory. This condensate is intimately related to the quark confinement and the tension of the linear potential between the quark and the anti-quark. This has been assured through the study of the Wilson-loop and classical string configurations obtained as solutions of the Nambu-Goto action.

The analysis is extended here to the glueballs (closed string), and we find the Regge behavior through the classical solutions for the folded closed string case. The result shows the slope of the glueball trajectory is the half of one of the flavor mesons which are given by the open strings. This relation is expected from the configurations of the folded closed string which has two times length of the extended part of the long open string. This behavior is seen both in the supersymmetric and non-symmetric cases.

In the supersymmetric case, however, the stable closed string (classical configuration) is found at the horizon of the background \((r = 0)\) where the metric singularity is seen in the radial coordinate \(r\) direction. This implies that the fluctuation of the closed string in the \(r\) direction should be suppressed. Then we cannot expect the quantum fluctuation mode of the glueballs in the bulk which extends to the direction \(r\). Actually, this point is assured by solving directly the equations of motion of the quantum fluctuations of the bulk fields. As expected, we cannot find any glueball state in this case.

In order to evade the metric singularity mentioned above, it would be needed to introduce an appropriate infrared cutoff in the theory. Although it is easy to introduce it by hand as in the hard wall model [39, 40], instead, we move to the non-supersymmetric solution to find a wall which supports the glueball states.

For non-supersymmetric case, the bulk curvature is singular at \(r_0\), but there is no metric singularity at the position, \(r = r_m (> r_0)\), where the classical closed-string is obtained. The reason why the classical solution is trapped at this point is that this point is the minimum point of the gravitational potential for the strings. As for the open string, its energy increases with increasing angular momentum or spin. As a
result, it grows long and the prolonged part approaches to \( r = r_m \). However it cannot go over this point \( r_m \) even if the energy becomes infinite. In this sense, the static strings are blocked there due to the wall.

Furthermore, we need probe branes, D7 branes for flavored quarks and D5 branes for supersymmetric background, they are also blocked by the same wall and cannot penetrate into the region \( r < r_m \). Namely their embedded configurations are stabilized in the region of \( r > r_m \).

As for the small spin states, they are expressed by the quantum fluctuations on the probe brane for the flavor mesons or in the bulk for the glueballs. In the case of the flavor mesons, their mass gap is given by the quark mass \( m_q = w(\infty) \) for the supersymmetric case. However, in the non-supersymmetric case, the wall generates an infrared cutoff \( w(0)(> r_m) \) due to the finite chiral condensate even if \( m_q = 0 \). So we could find flavored mesons with finite mass in the non-supersymmetric case even if \( m_q = 0 \).

In the case of the glueballs, on the other hand, the quarks are not contained in the state, then there is no mass scale to give a mass gap for the supersymmetric case. Then we cannot find any normalizable wave-functions of the glueball wave equation. On the other hand, in the non-supersymmetric case, there appears a wall near \( r_0 \), where infinite high potential wall stands for the strings. This implies that we should find the solutions of the glueball wave-equations by restricting the dynamical region of the wave function to \( r_0 < r \). This procedure is also adopted in the Witten model, in which the bulk configuration has a horizon coming from the bulk black hole geometry. So this point is not a singular point of the curvature, but metric singularity is generated at this point. In order to evade this singularity, the region of the radial variable is restricted above the horizon by introducing an appropriate change of the variable. This procedure is equivalent to introduce an infrared cutoff, which provides a mass gap for the glueball.

In this case, we find discrete spectra of glueballs in the non-supersymmetric case. It is possible to adjust the parameters of the theory consistently with the lattice-simulation results for the glueball masses. On the other hand, we know the lattice data for the gauge condensate, \( \langle F_{\mu\nu}F^{\mu\nu} \rangle \). When we respect this data, however, we find about half values of the glueball mass which are given in the lattice simulation. How to reconcile these two lattice-results with our model is an open problem here.

As for the confining theory proposed by Klebanov and Strassler\cite{kls}, it is supersymmetric and the potential \( Q(\tau) \) has a finite minimum at \( \tau = 0 \), then there is no infrared cutoff as in our non-supersymmetric case. However, in the case of KS model, there is no metric singularity at \( \tau = 0 \) so the quantum corrections around this minimum point are calculable for the classical closed-string solutions. Reflecting this fact, the glueball

\footnote{In the model of Klebanov and Strassler, the variable \( \tau \) corresponds to our \( r \).}
spectra are obtained without introducing any infrared cutoff. The dual gauge theory of this model is however different from our’s since the gauge condensate is not present in this case. It would be an interesting problem to study the relation to the dual theory of our model. This would be an open problem here.

Acknowledgments

K. Ghoroku thanks to Hirofumi Kubo for useful discussions in several parts of the contents.

A The evaluation of the U-shaped string length

Here we show that the energy configuration of U-shaped open-string, which corresponds to Wilson loop in the dual gauge theory, is infinite at \( r = r_m \). To show that, we evaluate the first integral in (2.24) by changing integral variable as \( \epsilon \equiv r - r_m \) and setting \( \delta \equiv r_{min} - r_m \). Relations between variables are illustrated in Fig. 5. Expanding

\[
\begin{align*}
\epsilon &\quad \text{measures a interval from } r_m \text{ to } r \text{ and it can take a value from bottom point } \delta \text{ to some finite value } \bar{\epsilon} \\
\end{align*}
\]

Fig. 5: The relations between variables. \( \epsilon \) measures a interval from \( r_m \) to \( r \) and it can take a value from bottom point \( \delta \) to some finite value \( \bar{\epsilon} \)

the function in the root in denominator of this integrand around \( r = r_m \) up to second
order respect to $\epsilon$ or $\delta$, it can be evaluated as follows,

$$I(\bar{\epsilon}, \delta) \equiv \int_\delta^\bar{\epsilon} d\epsilon \frac{Q(r_{\min})}{\sqrt{H(r)(Q^2(r) - Q^2(r_{\min}))}}$$
$$= \int_\delta^\bar{\epsilon} d\epsilon \frac{Q(r_{\min})}{\sqrt{a\epsilon^2 + b\epsilon + c}}$$
$$= \int_\delta^\bar{\epsilon} d\epsilon \frac{Q(r_{\min})}{\sqrt{a(\epsilon + \frac{b}{2a})^2 - \frac{b^2}{4a} + c}}$$
$$= \frac{Q(r_{\min})}{\sqrt{a}} \int_\delta^\bar{\epsilon} d\epsilon \frac{1}{\sqrt{(\epsilon + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a}}}$$

(A. 1)

where $Q, H$ is same definition as sec.2 and

$$a = \left\{ \frac{1}{2} \left( Q^2 \right)'' H + \left( Q^2 \right)' H' \right\}_{r=r_m},$$
$$b = \left\{ \left( Q^2 \right)' H - \left( Q^2 \right)' H' \delta \right\}_{r=r_m},$$
$$c = \left\{ - \left( Q^2 \right)' H\delta - \frac{1}{2} \left( Q^2 \right)'' H\delta^2 \right\}_{r=r_m}.$$

We can calculate integral (A. 1) by changing integration variable as $\bar{\epsilon} = \epsilon + \frac{b}{2a}$, and then result is

$$I(\bar{\epsilon}, \delta) = \frac{Q(r_{\min})}{\sqrt{a}} \left\{ \log \left( \bar{\epsilon} + \frac{b}{2a} + \sqrt{\bar{\epsilon}^2 + \frac{b}{a} \bar{\epsilon} + \frac{c}{a}} \right) - \log \left( \delta + \frac{b}{2a} + \sqrt{\delta^2 + \frac{b}{a} \delta + \frac{c}{a}} \right) \right\}.$$

(A. 2)

In the limit of $\delta \to 0$, this expression becomes

$$I(\bar{\epsilon}, 0) = \frac{Q(r_{\min})}{\sqrt{a}} \left\{ \log \left( \bar{\epsilon} + p + \sqrt{\bar{\epsilon}^2 + 2p\bar{\epsilon}} \right) - \log (p) \right\}$$

(A. 3)

and

$$p \equiv \frac{b}{2a} = \frac{(Q^2)' H}{(Q^2)'' H + 2(Q^2)' H'} \bigg|_{r=r_m}$$
$$= \frac{1}{(Q^2)'' / (Q^2)' + 2H'/H} \bigg|_{r=r_m}.$$

(A. 4)

If the prefactor of (A. 3) is finite, we can estimate the divergency of this expression by estimating $p = b/(2a)$. There might be several ways setting this variable to zero.
Our non-SUSY model achieves this due to \((Q^2)' = 0\), whereas Witten’s D4 model accomplish this due to \(H/H' = 0\). We will see these in the rest of this appendix.

First, we see about our non-SUSY model. Since our model has the same metric in \(t\) and \(x\) direction, then \((Q^2)' = 0\) means \(G_{ii}'(r_m) = 0\) because of \(G'_{ii}(r_m) = 0\). We can easily confirm that the only divergent ingredient is \((Q^2)'\) and others \(((Q^2)'', H, H')\) are finite at \(r_m\), therefore in (A. 3) the second term is logarithmic divergent and its coefficient is finite.

Second, Witten’s model has the metric displayed below,

\[
|G_{tt}| = G_{ii} = \left(\frac{U}{R}\right)^{3/2}, \quad G_{UU} = \left(\frac{R}{U}\right)^{3/2} \frac{1}{f(U)}, \quad f(U) = 1 - \left(\frac{U_{KK}}{U}\right)^3.
\]

In this model, there is the only divergent ingredient,

\[
H = \frac{G_{ii}}{G_{UU}} = \left(\frac{U}{R}\right)^3 f(U) = \frac{U^3 - U_{KK}^3}{R^3}.
\]

Then the second term is logarithmic divergent and its coefficient is finite in (A. 3) again, although the reason of divergence is not the same with the case of our non-SUSY model. In this case, the cause of the divergence is reduced to \(H(U_{KK}) = 0\) or equally to \(G_{UU}(U_{KK}) = +\infty\). Since it means that \(G_{ii}(U_{KK})/G_{UU}(U_{KK}) = 0\), the measure of \(i\) direction is infinitely small compared to the one of \(U\) direction.

### B More on Wall

In this appendix, we demonstrate that the infrared wall which suggested in sec.2 for classical open string configuration also arise in other classical configurations. The analysis is performed for two sorts of classical D-brane configuration, that is, embedded D7 brane and D5 brane.

#### B.1 Wall for D7 brane

Here, we consider the role of the wall in the D7 embedding. The embedding procedure is briefly reviewed. The world-volum of D7-brane is set by rewriting the extra six dimensional part of (2.6) as

\[
\frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 = \frac{R^2}{r^2} (dr^2 + r^2 d\Omega_5^2) = \frac{R^2}{r^2} \left(\sum_{i=4}^{9} (dX^i)^2\right) \quad \text{(B.1)}
\]

\[
= \frac{R^2}{r^2} \left( d\rho^2 + \rho^2 d\Omega_3^2 + \sum_{i=8}^{9} (dX^i)^2\right). \quad \text{(B.2)}
\]
The worldvolume coordinates $\xi^M (M = 0 \sim 7)$ of D7-brane are taken as $x^\mu \ (\mu = 0, \cdots, 3)$ and $(\rho, S^3)$, then its induced metric is expressed as

$$ds_8^2 = e^{\Phi/2} \left( \frac{r^2}{R^2} A^2(r) \eta_{\mu \nu} dx^\mu dx^\nu + \frac{R^2}{r^2} \left( 1 + w'(\rho)^2 \right) d\rho^2 + \rho^2 d\Omega_3^2 \right),$$  \hspace{1cm} (B.3)

where $r^2 = \rho^2 + w(\rho)^2$ and $w' = \partial_\rho w$. Here the D7-brane is embedded under the ansatz,

$$(X^8)^2 + (X^9)^2 = w(\rho)^2 \ 	ext{.}$$  \hspace{1cm} (B.4)

We can set the solution of $w(\rho)$ as $(X^8, X^9) = (w(\rho), 0)$ since the background is symmetric under the rotation on $X^8$-$X^9$ plane. Then the DBI action is expressed as

$$S_{D7} = -\tau_7 \int d^8\xi A(r)^{\frac{4}{3}} \rho^3 \sqrt{1 + w'(\rho)^2},$$  \hspace{1cm} (B.5)

where $\tau_7$ denotes the tension of D7-brane.

The equation of motion for $w(\rho)$ has been solved as

$$w(\rho) = m_q + \frac{C}{\rho^2} + \cdots, \ \rho \to \infty,$$  \hspace{1cm} (B.6)

where $m_q$ and $C$ denotes the current quark mass and the chiral condensate $\langle \bar{\psi}\psi \rangle$ respectively as known from the dictionary of the AdS/CFT correspondence. In the non-supersymmetric case, we notice that the chiral symmetry is spontaneously broken \[18\], namely $C > 0$ for $m_q = 0$.

In the supersymmetric case, we give a comment on the D7 brane embedding. In this case, the Chern-Simons term is added to the action in our model due to the existence of non-trivial zero-form field, namely the axion. In this case, the lowest energy embedding is given by \[18\]

$$w = m_q = \text{constant}. \hspace{1cm} (B.7)$$

Here, however, we concentrate on another quantity $w(0)$ of the solution for various quark mass $m_q$. The value of $w(0)$ represents the lowest value of $r$ on the embedded D7 brane. In a sense, therefore, $w(0)$ corresponds to the infrared cutoff, then it would determine the mass scale of the theory, for example the meson mass. In the supersymmetric case, the meson mass is actually given as \[19\]

$$M = \frac{2m_q}{R^2} \sqrt{(n + 1)(n + 2)} \hspace{1cm} (B.8)$$

for radial $(n)$ exited spectra. This is interpreted as the reflection of a finite infrared cutoff scale $m_q = w(0)$. Then the mass spectra would disappear in the limit of $m_q \to 0$ in this case. The similar phenomenon is seen for the glueballs as shown below.

---

\[7\] Since an argument for this solution has recently been given in \[30\], we give the details of the derivation of this result in the appendix \[C\]. In the appendix, we can see that the results given in \[18\] would not be altered.

\[8\] In this case, $q = 0$, but the situation is similar to the case of $q > 0$, since $q$ does not provide any infrared cutoff as seen above. The explicit $q$-dependence is examined in \[12\].
Fig. 6: The typical solutions of D7-brane embedding in the non-supersymmetric background for \( r_0 = 1 \). The broken line represents the singular point at \( r = r_0 \) and the dotted line represents \( r = r_m \). The solution of \( w(0) = w_1 \) corresponds to massless quark at UV boundary. We can see that the relation \( w(0) > r_m \) is satisfied for any quark mass.

On the other hand, in the non-supersymmetric case, we can assure that the finite value of \( w(0)(> r_m) \) is seen even if \( m_q = 0 \) due to the positive chiral condensate. In the Fig. 6 typical examples of the solution are shown. As seen from this figure, we find the limiting value of \( w(0)(\geq w_1) \) as \( w_1 = 1.2318 r_0 \) for \( R = 1 \). This is the reflection of the wall, which blocks the classical D7 brane configuration, which is restricted to the region \( r \geq w_1 \). The meson with small spin is obtained by the quantum fluctuations of the fields on the D7 brane, and they are also blocked by this infrared wall for any quark mass \( m_q \). Since the nearest point of D7 brane is cut at \( w_1 \) outside of \( r_m \) even if \( m_q \) is zero, the meson mass obtained from the D7 brane fluctuation get a finite mass due to this mass gap.

### B.2 Wall for D5 brane; Baryon vertex

Furthermore, we can see that the D5 brane, which is introduced as the vertex of baryons [31], is also blocked by the wall. The D5-brane wraps \( S^5 \) in the bulk \( M^5 \times S^5 \), and its action is given as [32]

\[
S_{D5} = -\tau_5 \int d^6\xi e^{-\Phi} \sqrt{-\det(G_{MN}\partial_a X^M \partial_b X^N + \tilde{F}_{ab})} + \tau_5 \int A_{(1)} \wedge G_{(5)} ,
\]

where \( \tau_5 \) denotes D5-brane tension, \( \tilde{F}_{ab} = 2\pi\alpha'F_{ab} \), \( A_{(1)} \) is the U(1) gauge field on D5-brane, and \( G_{(5)} \) represents the 5-form self-dual field strength of stacked D3-branes.

After performing the Legendre transformation with respect to the gauge field \( A_t \) to eliminate itself, the action is rewritten as [32]

\[
U = \frac{N}{3\pi^2\alpha'} \int d\theta \sqrt{|G_{tt}|G_{rr}(r'^2 + r^2)} \sqrt{D(\theta)^2 + \sin^2 \theta} ,
\]

where \( N/(3\pi^2\alpha') = T_5\Omega_4 R^4 \). The equation of motion for \( r(\theta) \) of the lagrangian (B.10)
is obtained as
\[
\partial_\theta \left( \frac{\sqrt{U(r)} \sqrt{V(\nu, \theta)}}{\sqrt{r^2 + r^2}} \right) - \frac{1}{2} \frac{\partial U(r)}{\partial r} (r^2 + r^2) + U(r) r \right) \sqrt{U(r)} \sqrt{r^2 + r^2} = 0 ,
\]
where we define \( U(r) \equiv |G_{tt}|G_{rr} \). In general, the lowest energy configuration is the point-like solution for \( r \)-direction given by \( r(\theta) = r_b \), where \( r_b \) is a constant. Assuming that \( U(r_b) \neq 0 \) and \( r_b \neq 0 \), the value of \( r_b \) is obtained as follows:
\[
U'(r_b) r_b + 2U(r_b) = 0 .
\]
In the non-SUSY background (2.6), we find
\[
r_b = \left( \sqrt{6} + \sqrt{5} \right)^{1/4} r_0 \approx 1.47 r_0 .
\]
Then the classical configuration of the vertex is also trapped at the point above \( r_m = 1.18 r_0 \).

\section{D7 brane embedding and eight form}

The supersymmetric solutions for the axion \( \chi \) and dilaton \( \Phi \) used here are obtained under the ansatz \cite{15, 16},
\[
\chi = -e^{-\Phi} + \chi_0 ,
\]
which is necessary to obtain supersymmetric solutions. The metric is expressed as
\[
ds_{10}^2 = e^{\Phi/2} \left\{ \frac{r^2}{R^2} \left( -dt^2 + (dx^i)^2 \right) + \frac{R^2}{r^2} \left( d\eta^2 + \eta^2 d\Omega_3^2 + (dX^8)^2 + (dX^9)^2 \right) \right\} .
\]
In this coordinate, the solution is obtained as \( \Phi = \Phi(r) \) and \( r^2 = \eta^2 + (X^8)^2 + (X^9)^2 \), so the nine form field strength dual to the \( \chi \) is given as \( F_{(9)} = *d\chi \) and
\[
F_{(9)} = g^{\eta\eta} \sqrt{-g} (\partial_\eta \chi \ dt \wedge d\bar{x} \wedge d\Omega_3 \wedge dX^8 \wedge dX^9 + \partial_8 \chi \ dt \wedge d\bar{x} \wedge d\eta \wedge d\Omega_3 \wedge dX^9)
\]
\[
- \partial_9 \chi \ dt \wedge d\bar{x} \wedge d\eta \wedge d\Omega_3 \wedge dX^8)
\]
\[
\equiv g^{\eta\eta} \sqrt{-g} (\partial_\eta \chi \ d_\eta + \partial_8 \chi \ d_{\bar{x}} - \partial_9 \chi \ d_{\bar{x}})
\]
where we defined as \( \partial_8 = \frac{\partial}{\partial X^8} \) and the outer product as
\[
d_{\bar{x}} \equiv dt \wedge d\bar{x} \wedge d\Omega_3 \wedge dX^8 \wedge dX^9
\]
etc. Here we notice \( g^{\eta\eta} = (r/R)^2 \) and \( \epsilon_{\xi\bar{y}\phi_3\bar{x}9} = 1 \).

Introducing the eight form in the form
\[
C_{(8)} = f_8(\eta, X^8, X^9) d_{\bar{x}} \bar{x} + f_9(\eta, X^8, X^9) d_{\bar{x}} \bar{x} + g_\eta(\eta, X^8, X^9) d_{\bar{x}} \bar{x}
\]
\[
= 29
\]
the nine form field strength is also obtained by $F_{(9)} = dC_{(8)}$ as

$$F_{(9)} = \partial_\eta f_8 dX^9 + \partial_8 f_8 d\eta + \partial_\eta f_9 d\chi^8 - \partial_8 f_9 d\eta + \partial_8 g_\eta d\chi^9 + \partial_\eta g_\eta d\chi^8 .$$  \hfill (C.7)

Comparing Eqs. (C.3) and (C.7), we find

$$\partial_9 f_8 - \partial_8 f_9 = g^m \sqrt{-g} \partial_\eta \chi ,$$  \hfill (C.8)
$$\partial_9 g_\eta + \partial_\eta f_9 = g^m \sqrt{-g} \partial_8 \chi ,$$  \hfill (C.9)
$$\partial_8 g_\eta + \partial_\eta f_8 = - g^m \sqrt{-g} \partial_9 \chi .$$  \hfill (C.10)

Noticing

$$g^m \sqrt{-g} = \sqrt{\epsilon_3} \eta^3 e^{2\Phi} ,$$  \hfill (C.11)

where $\epsilon_3$ denotes the metric of $S^3$ part, and using (2.7) and $e^\Phi = 1 + q/r^4$ we obtain

$$g^m \sqrt{-g} \partial_8 \chi = \sqrt{\epsilon_3} \eta^3 e^{2\Phi} \partial_8 \chi = -4 \sqrt{\epsilon_3} \eta^3 \frac{gX^8}{r^6} .$$  \hfill (C.12)

Then $f_9$ is obtained by solving (C.9)

$$f_9 = \sqrt{\epsilon_3} q X^8 \left( \frac{1}{r^2} + \frac{\eta^2}{r^4} \right) + C_9 ,$$  \hfill (C.13)

where

$$C_9 = - \int d\eta \, \partial_9 g_\eta .$$  \hfill (C.14)

In this indefinite integration with respect to $\eta$, we can add an arbitrary function of $X^8$ and $X^9$. We consider that it is included in $C_9$ here. Similarly, we obtain $f_8$ from (C.10) as

$$f_8 = - \sqrt{\epsilon_3} q X^9 \left( \frac{1}{r^2} + \frac{\eta^2}{r^4} \right) + C_8 ,$$  \hfill (C.15)

$$C_8 = - \int d\eta \, \partial_8 g_\eta .$$  \hfill (C.16)

Then the remaining Eq. (C.8) is rewritten by using (C.15) and (C.13) as

$$\partial_9 C_8 - \partial_8 C_9 = 0 .$$  \hfill (C.17)

While this gives a constraint on the arbitrary functions of $X^8$ and $X^9$ added to $C_8$ and $C_9$, it is however independent of $g_\eta$ since we can see

$$\partial_9 C_8 - \partial_8 C_9 = - \partial_9 \int d\eta \, \partial_8 g_\eta + \partial_8 \int d\eta \, \partial_9 g_\eta = 0 ,$$  \hfill (C.18)

where we ignored the added arbitrary functions of $X^8$ and $X^9$. In other words, we cannot get any constraint for $g_\eta$, which is the main part of the Chern-Simons term of the D7 brane action. Here, it is determined as follows.
The pull backed eight form fields are written for our world volume of D7 brane as
\[ C_{[8]} \equiv \tilde{g}_{(8)} d\tilde{X}^8 \tilde{X}^9 = (g_\eta - f_8 \dot{X}^8 - f_9 \dot{X}^9) d\tilde{X}^8 \tilde{X}^9 , \] (C.19)
where the dot denotes the derivative with respect to \( \eta \), for example \( \dot{X}^8 = \partial_\eta X^8 \), and the action of the D7 brane is given as
\[ S_{D7} = -\tau_7 \int d\xi^8 \left( e^{-\Phi} \sqrt{-G} + \tilde{g}_{(8)} \right) . \] (C.20)
Here \( G \) denotes the determinant of the induced metric of D7 brane, which is taken as
\[ ds_8^2 = e^{\Phi/2} \left\{ \frac{r^2}{R^2} \left( -dt^2 + (dx^i)^2 \right) + \frac{R^2}{r^2} \left( 1 + (\partial_\eta X^8(\eta))^2 + (\partial_\eta X^9(\eta))^2 \right) d\eta^2 \\
+ \eta^2 d\Omega_3^2 + (dX^8)^2 + (dX^9)^2 \right\} , \] (C.21)
where we notice \( r^2 = \eta^2 + X_8^2 + X_9^2 \). Then \( G \) is given as,
\[ \sqrt{-G} = \sqrt{\epsilon_3 \eta^3} e^{2\Phi} \sqrt{1 + (\partial_\eta X^8(\eta))^2 + (\partial_\eta X^9(\eta))^2} . \] (C.22)
In the above action, the Chern-Simons part \( \tilde{g}_{(8)} \) is given as
\[ \tilde{g}_{(8)} = g_\eta - f_8 \dot{X}^8 - f_9 \dot{X}^9 \]
\[ = g_\eta + q \left( \frac{1}{r^2} + \frac{\eta^2}{r^4} \right) \left( X^9 \dot{X}^8 - X^8 \dot{X}^9 \right) - C_8 \dot{X}^8 - C_9 \dot{X}^9 , \] (C.23)
where however \( g_\eta, C_8, \) and \( C_9 \) are not given explicitly since they are not determined. These undetermined functions would be given by appropriate boundary conditions of the system.

Our purpose is to find an embedding solution of the D7 brane. We try it by an ansatz to obtain a simple solution. Consider the parametrization,
\[ X^8 = w(\eta) \cos \theta , \quad X^9 = w(\eta) \sin \theta , \] (C.25)
where \( \theta \) is a constant and independent of \( \eta \). The embedding is given by the profile of \( w(\eta) \). In general, the solution has the following asymptotic form at \( \eta \to \infty \)
\[ w(\eta) = m_q + \frac{c}{\eta^2} + \cdots \] (C.26)
where \( m_q \) and \( c \) represent the current quark mass and the chiral order parameter, \( \langle \bar{\psi} \psi \rangle \) with the quark field \( \psi \). Further simplification is done by setting \( \theta = 0 \) or \( \theta = \pi/2 \). Here we take \( \theta = 0 \), then
\[ \tilde{g}_{(8)} = g_\eta - C_8 \dot{w} . \] (C.27)
This is rewritten by the partial integration with respect to $\eta$ as
\[
\tilde{g}(8) = g_\eta + w C_8 = g_\eta - w \partial_w g_\eta \equiv \Omega_3 \tilde{g}(8) .
\] (C.28)

where $\Omega_3$ is the volume of $S^3$ and we used $C_8 = -\int d\eta \partial_w g_\eta$. Then the effective D7 brane action is written as
\[
S_{D7} = -\tau_7 \Omega_3 \int dx^4 d\eta \left( \eta^3 e^\Phi \sqrt{1 + (\dot{w})^2} + \tilde{g}(8) \right) .
\] (C.29)

For simplicity, we assume as $g_\eta = g_\eta(\eta, w(\eta))$, then $\tilde{g}(8) = \tilde{g}(8)(\eta, w(\eta))$. Form (C.29), the equation of motion of $w$ is obtained as
\[
\eta^3 \partial_\eta \left( e^\Phi \sqrt{1 + (\dot{w})^2} \right) = -\partial_w \tilde{g}(8) .
\] (C.30)

Here we demand that there should exist a supersymmetric embedding, namely a constant $w$ (or $\dot{w} = 0$) is the solution. This implies our previous Chern-Simon term \[18\]
\[
\tilde{g}(8) = -\eta^3 e^\Phi ,
\] (C.31)

where we have neglected $w$-independent terms. We should say that this result is not unique of course. There are other setting of $\tilde{g}(8)$ which does not allow the supersymmetric solution of $w$. In such cases, however, we should find some dynamical origin of supersymmetry breaking in the dual theory. It may be an interesting problem, but it is opened here.

References

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B 428, 105 (1998) [hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [hep-th/9802150].

[4] H. Ooguri, H. Robins and J. Tannenhauser, “Glueballs and their Kaluza-Klein cousins,” Phys. Lett. B 437, 77 (1998) [hep-th/9806171].

[5] J. A. Minahan, “Glueball mass spectra and other issues for supergravity duals of QCD models,” JHEP 9901, 020 (1999) [hep-th/9811156].
[6] N. R. Constable and R. C. Myers, “Exotic scalar states in the AdS / CFT correspondence,” JHEP 9911, 020 (1999) [hep-th/9905081].

[7] N. R. Constable and R. C. Myers, “Spin two glueballs, positive energy theorems and the AdS / CFT correspondence,” JHEP 9910, 037 (1999) [hep-th/9908175].

[8] C. Csaki, H. Ooguri, Y. Oz and J. Terning, “Glueball mass spectrum from supergravity,” JHEP 9901, 017 (1999) [hep-th/9806021].

[9] E. Caceres and R. Hernandez, “Glueball masses for the deformed conifold theory,” Phys. Lett. B 504, 64 (2001) [hep-th/0011204].

[10] J. M. Pons, J. G. Russo and P. Talavera, “Semiclassical string spectrum in a string model dual to large N QCD,” Nucl. Phys. B 700, 71 (2004) [hep-th/0406266].

[11] R. C. Brower, S. D. Mathur and C. -ITan, “Glueball spectrum for QCD from AdS supergravity duality,” Nucl. Phys. B 587, 249 (2000) [hep-th/0003115].

[12] U. Gursoy, E. Kiritsis and F. Nitti, “Exploring improved holographic theories for QCD: Part II,” JHEP 0802, 019 (2008) [arXiv:0707.1349 [hep-th]].

[13] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A Semiclassical limit of the gauge / string correspondence,” Nucl. Phys. B 636, 99 (2002) [hep-th/0204051].

[14] K. Ghoroku, T. Taminato and F. Toyoda, “Holographic Approach to Regge Trajectory and Rotating D5 brane,” JHEP 1105, 006 (2011) [arXiv:1103.2428 [hep-th]].

[15] A. Kehagias and K. Sfetsos, “On asymptotic freedom and confinement from type IIB supergravity,” Phys. Lett. B 456, 22 (1999) [hep-th/9903091].

[16] H. Liu and A. A. Tseytlin, “D3-brane D instanton configuration and N=4 superYM theory in constant selfdual background,” Nucl. Phys. B 553, 231 (1999) [hep-th/9903091].

[17] A. Kehagias and K. Sfetsos, Phys. Lett. B 454, 270 (1999) [hep-th/9902125]; S. S. Gubser, [hep-th/9902155]; S. Nojiri and S. D. Odintsov, Phys. Lett. B 449, 39 (1999) [hep-th/9812017]; K. Ghoroku, M. Tachibana and N. Uekusa, Phys. Rev. D 68, 125002 (2003) [hep-th/0304051].

[18] K. Ghoroku and M. Yahiro, “Chiral symmetry breaking driven by dilaton,” Phys. Lett. B 604, 235 (2004) [hep-th/0408040].

[19] M. Kruczenski, D. Mateos, R. C. Myers and D. J. Winters, “Meson spectroscopy in AdS / CFT with flavor,” JHEP 0307, 049 (2003) [arXiv:hep-th/0304032 [hep-th]].

[20] L. A. Pando Zayas, J. Sonnenschein and D. Vaman, “Regge trajectories revisited in the gauge / string correspondence,” Nucl. Phys. B 682, 3 (2004) [hep-th/0311190].
[21] F. Bigazzi, A. L. Cotrone, L. Martucci and L. A. Pando Zayas, “Wilson loop, Regge trajectory and hadron masses in a Yang-Mills theory from semiclassical strings,” Phys. Rev. D 71, 066002 (2005) [hep-th/0409205].

[22] M. Kruczenski, L. A. Pando Zayas, J. Sonnenschein and D. Vaman, “Regge trajectories for mesons in the holographic dual of large-N(c) QCD,” JHEP 0506, 046 (2005) [hep-th/0410035].

[23] A. Paredes and P. Talavera, “Multiflavor excited mesons from the fifth dimension,” Nucl. Phys. B 713, 438 (2005) [hep-th/0412260].

[24] I. Kirsch and D. Vaman, “The D3 / D7 background and flavor dependence of Regge trajectories,” Phys. Rev. D 72, 026007 (2005) [hep-th/0505164].

[25] M. Huang, Q. -S. Yan and Y. Yang, “Confront Holographic QCD with Regge Trajectories,” Eur. Phys. J. C 66, 187 (2010) [arXiv:0710.0988 [hep-ph]].

[26] J. Babington, J. Erdmenger, N. J. Evans, Z. Guralnik and I. Kirsch, “Chiral symmetry breaking and pions in nonsupersymmetric gauge / gravity duals,” Phys. Rev. D 69, 066007 (2004) [hep-th/0306018].

[27] G. W. Gibbons, M. B. Green and M. J. Perry, “Instantons and seven-branes in type IIB superstring theory,” Phys. Lett. B 370, 37 (1996) [hep-th/9511080].

[28] Y. Kinar, E. Schreiber and J. Sonnenschein, “Q anti-Q potential from strings in curved space-time: Classical results,” Nucl. Phys. B 566, 103 (2000) [hep-th/9811192].

[29] N. Evans, J. French, K. Jensen and E. Threlfall, “Hadronization at the AdS wall,” Phys. Rev. D 81, 066004 (2010) [arXiv:0908.0407 [hep-th]].

[30] B. Gwak, M. Kim, B. -H. Lee, Y. Seo and S. -J. Sin, “Phases of a holographic QCD with gluon condensation at finite Temperature and Density,” arXiv:1105.2872 [hep-th].

[31] E. Witten, “Baryons and branes in anti-de Sitter space,” JHEP 9807, 006 (1998) [hep-th/9805112].

[32] C. G. Callan, Jr., A. Guijosa and K. G. Savvidy, “Baryons and string creation from the five-brane world volume action,” Nucl. Phys. B 547, 127 (1999) [hep-th/9810092].

[33] C. J. Morningstar and M. J. Peardon, “The Glueball spectrum from an anisotropic lattice study,” Phys. Rev. D 60, 034509 (1999) [hep-lat/9901004].
[34] Y. Chen, A. Alexandru, S. J. Dong, T. Draper, I. Horvath, F. X. Lee, K. F. Liu and N. Mathur et al., “Glueball spectrum and matrix elements on anisotropic lattices,” Phys. Rev. D 73, 014516 (2006) [hep-lat/0510074].

[35] H. B. Meyer, “Glueball regge trajectories,” [hep-lat/0508002].

[36] E. Kiritsis and F. Nitti, “On massless 4D gravitons from asymptotically AdS(5) space-times,” Nucl. Phys. B 772, 67 (2007) [hep-th/0611344].

[37] M. Giovannini [Institute for Theoretical Physics, Lausanne University Collaboration], “Gauge invariant fluctuations of scalar branes,” Phys. Rev. D 64, 064023 (2001) [hep-th/0106041].

[38] M. D’Elia, A. Di Giacomo and E. Meggiolaro, “Field strength correlators in full QCD,” Phys. Lett. B 408, 315 (1997) [hep-lat/9705032].

[39] J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, “QCD and a holographic model of hadrons,” Phys. Rev. Lett. 95, 261602 (2005) [hep-ph/0501128].

[40] G. F. de Teramond and S. J. Brodsky, “Hadronic spectrum of a holographic dual of QCD,” Phys. Rev. Lett. 94, 201601 (2005) [hep-th/0501022]; S. J. Brodsky and G. F. de Teramond, “Hadron spectroscopy and wavefunctions in QCD and the AdS/CFT correspondence,” AIP Conf. Proc. 814, 108 (2006) [hep-ph/0510240].

[41] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities,” JHEP 0008, 052 (2000) [arXiv:hep-th/0007191 [hep-th]].

[42] I. H. Brevik, K. Ghoroku and A. Nakamura, “Meson mass and confinement force driven by dilaton,” Int. J. Mod. Phys. D 15, 57 (2006) [hep-th/0505057].