Sharp Hessian estimates for fully nonlinear elliptic equations under relaxed convexity assumptions, oblique boundary conditions and applications

by

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Abstract

We derive global $W^{2,p}$ estimates (with $n \leq p < \infty$) for viscosity solutions to fully nonlinear elliptic equations under relaxed structural assumptions on the governing operator that are weaker than convexity and oblique boundary conditions as follows:

$$\begin{cases}
F(D^2u, Du, u, x) = f(x) & \text{in } \Omega \\
\beta(x) \cdot Du(x) + \gamma(x)u(x) = g(x) & \text{on } \partial\Omega,
\end{cases}$$

for $f \in L^p(\Omega)$ and under appropriate assumptions on the data $\beta, \gamma, g$ and $\Omega \subset \mathbb{R}^n$. Our approach makes use of geometric tangential methods, which consist of importing “fine regularity estimates” from a limiting profile, i.e., the Recession operator, associated with the original second-order one via compactness and stability procedures. As a result, we pay special attention to the borderline scenario, i.e., $f \in \text{BMO}_p \supseteq L^\infty$. In such a setting, we prove that solutions enjoy BMO$_p$ type estimates for their second derivatives. Finally, as another application of our findings, we obtain Hessian estimates to obstacle-type problems under oblique boundary conditions and no convexity assumptions, which may have their own mathematical interest. A density result for a suitable class of viscosity solutions will also be addressed.

Keywords: Hessian estimates, fully nonlinear elliptic equations, oblique boundary conditions, relaxed convexity assumptions, obstacle type problems.

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1 Introduction

In this manuscript, we investigate global $W^{2,p}$ estimates (with $n \leq p < \infty$) for viscosity solutions to fully nonlinear elliptic equations of the form:

\[
\begin{cases}
F(D^2u, Du, u, x) = f(x) & \text{in } \Omega \\
B(x, u, Du) = g(x) & \text{on } \partial\Omega,
\end{cases}
\]

where

\[
B(x, u, Du) = \beta(x) \cdot Du(x) + \gamma(x)u(x)
\]

is the oblique boundary condition, $\Omega \subset \mathbb{R}^n$ a bounded domain with a regular boundary and data $f, g, \beta$ and $\gamma$ under appropriate conditions to be clarified soon.

Throughout this work, it is assumed that $F : \text{Sym}(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a second-order operator with uniformly elliptic structure, i.e., there exist ellipticity constants $0 < \lambda \leq \Lambda < \infty$ such that

\[
\lambda \|Y\| \leq F(X + Y, \varsigma, \varsigma, x) - F(X, \varsigma, \varsigma, x) \leq \Lambda \|Y\|
\]

for every $X, Y \in \text{Sym}(n)$ with $Y \geq 0$ (in the usual partial ordering on symmetric matrices) and $(\varsigma, \varsigma, x) \in \mathbb{R}^n \times \mathbb{R} \times \Omega$. Moreover, our approach does not impose any extra regularity assumption on nonlinearity $F$, e.g. concavity, convexity, or smoothness hypothesis (cf. [8], [9], [37] and [45] for classical and modern references).

To do this concept more precise, we introduce the notion of Recession operator, whose terminology comes from Giga-Sato’s work [21] on theory of Hamilton-Jacobi PDEs.

**Definition 1.1 (Recession profile).** We say that $F(X, \varsigma, s, x)$ is an asymptotically fully nonlinear elliptic operator if there exists a uniformly elliptic operator $F^\#(X, \varsigma, s, x)$, designated the Recession operator, such that

\[
F^\#(X, \varsigma, s, x) := \lim_{\tau \to 0^+} \tau \cdot F\left(\frac{1}{\tau} X, \varsigma, s, x\right)
\]

for all $X \in \text{Sym}(n), \varsigma \in \mathbb{R}^n, s \in \mathbb{R}$ and $x \in \Omega$.

For instance, by way of illustration, such a limiting profile $F^\#$ appears naturally in singularly perturbed free boundary problems ruled by fully nonlinear equations, whose Hessian of solutions blows-up through the phase transition of the model, i.e., $\partial\{u^\varepsilon > \varepsilon\}$, where $u^\varepsilon$ satisfies in the viscosity sense

\[
F(D^2u^\varepsilon, x) = Q_0(\varepsilon) \frac{1}{\varepsilon} \zeta\left(\frac{u^\varepsilon}{\varepsilon}\right).
\]

For such approximations, we have $0 < Q_0 \in C^0(\overline{\Omega}), 0 \leq \zeta \in C^\infty(\mathbb{R})$ with supp $\zeta = [0, 1]$. For this reason, in the above model, the limiting free boundary condition is governed by the $F^\#$ rather than $F$, i.e.,

\[
F^\#(z_0, \nabla u(z_0) \otimes \nabla u(z_0)) = 2T, \quad z_0 \in \partial\{u^\varepsilon > 0\}
\]

in some appropriate viscosity sense, for a certain total mass $T > 0$ (cf. [41] Section 6 for some enlightening example and details).
Furthermore, limit profiles such as \textbf{(Reces.)} also appears in higher order convergence rates in periodic homogenization of fully nonlinear uniformly parabolic Cauchy problems with rapidly oscillating initial data, as shown below:

\[
\begin{aligned}
\left\{ \frac{d}{dt} u^\varepsilon(x, t) &= \frac{1}{\varepsilon^2} F(\varepsilon^2 D^2 u^\varepsilon, x, t, \frac{z}{\varepsilon}, \frac{t}{\varepsilon}) &\text{in } \mathbb{R}^n \times (0, T) \\
u^\varepsilon(x, 0) &= g(x, \frac{z}{\varepsilon}) &\text{on } \mathbb{R}^n.
\end{aligned}
\]

In such a context

\[
\frac{1}{\varepsilon^2} F(\varepsilon^2 P, x, t, y, s) \to F^\varepsilon(P, x, t, y, s) \quad \text{as } \varepsilon \to 0^+,
\]

uniformly for all \((P, x, t, y, s) \in (\text{Sym}(n) \setminus \{O_{n \times n}\}) \times \mathbb{R}^n \times [0, T] \times T^n \times T\) (see [27]). As a result, there exists a unique function \(v : \mathbb{R}^n \times [0, T] \times T^n \times [0, \infty) \to \mathbb{R}\) such that \(v(x, t, \cdot, \cdot)\) is a viscosity solution to

\[
\begin{aligned}
\left\{ \frac{d}{dt} v(y, s) &= F^\varepsilon(D^2 y v, x, t, y, s) &\text{in } T^n \times (0, \infty) \\
v(x, t, y, 0) &= g(y, x) &\text{on } T^n.
\end{aligned}
\]

To obtain our sharp Hessian estimates, we will assume that \(F^\varepsilon\) enjoys good structural properties (e.g., convexity/concavity, or suitable \textit{a priori} estimates). Thus, using geometric tangential mechanisms, we may access good regularity estimates and, in a suitable manner, transfer such estimates to solutions of the original problem via compactness and stability processes.

It is important to emphasize that convexity assumptions with respect to second-order derivatives play an essential role in obtaining several regularity issues in fully nonlinear elliptic models, e.g., Calderón-Zygmund estimates [8], Schauder type estimates [9] and higher estimates in some nonlinear models [18], [28], [49] and [50], and free boundary problems [13], [14], [19], [22], [23], [24] and [42] just to mention a few. Nevertheless, ensuring that such regularity issues hold true for problems like \((1.1)\) driven by operators satisfying the structure \textbf{(Reces.)} is not an easy task. For instance, consider perturbations of Bellman operators of the form

\[
\begin{aligned}
\left\{ F(X, x, s, x) := \inf_{i \in A} \left( \sum_{i,j=1}^{n} a_{ij}^\varepsilon(x) X_{ij} + \sum_{i=1}^{n} b_i^\varepsilon(x) \lambda_i + c^\varepsilon(x) s \right) + \sum_{i=1}^{n} \arctan(1 + \lambda_i(X)) = f(x) &\text{in } \Omega \\
B(x, s, \cdot) = g(x) &\text{on } \partial \Omega,
\end{aligned}
\]

where \((\lambda_i(X))_{i=1}^{n}\) are eigenvalues of the matrix \(X \in \text{Sym}(n), b_i^\varepsilon, c^\varepsilon : \Omega \to \mathbb{R}\) are real functions for each \(i \in A\) (for \(A\) any set of index), and \((a_{ij}^\varepsilon(x))_{i,j=1}^{n} \subset C^{0,\alpha}(\Omega)\) have eigenvalues in \([\lambda, \Lambda]\) for each \(x \in \Omega\) and \(\varepsilon \in \mathcal{A}\). Thus, it is easy to check that such operators are neither concave nor convex. However, its Recession profile \textbf{(Reces.)} inherits good structure properties, i.e.,

\[
\begin{aligned}
F^\varepsilon(D^2 \mathfrak{h}, D\mathfrak{h}, \mathfrak{h}, x) := \inf_{i \in \hat{A}} \left( \sum_{i,j=1}^{n} a_{ij}^\varepsilon(x) \mathfrak{h}_{ij} \right) = 0 &\text{in } \Omega \\
B(x, \mathfrak{h}, D\mathfrak{h}) = g(x) &\text{on } \partial \Omega,
\end{aligned}
\]

which is a convex operator, thereby enjoying higher regularity estimates in the face of Evans-Krylov-Trudinger’s type theory (with oblique boundary conditions) addressed in [10] Theorem 2.4., [33] Theorem 1.3], [33] Theorem 5.4], [43] §8, [44] Theorem 3.3] and [38] Theorem 8.3] (see also [9] Chapter 6], [13], [28], [49] and [50] for local estimates), i.e., for some \(\alpha \in (0, 1)\) and \(\Omega' \subset \subset \Omega \cap \Gamma - \text{with } C^{1,\alpha} \supseteq \Gamma \subset \partial \Omega, \beta, \gamma, g \in C^{1,\alpha}(\bar{\Gamma})\), there holds

\[
\| \mathfrak{b} \|_{C^{2,\alpha}(\Gamma)} \leq C \left( n, \lambda, \Lambda, \alpha, \| a_{ij}^\varepsilon \|_{C^{0,\alpha}(\Omega)}, \| \beta \|_{C^{1,\alpha}(\Gamma)}, \| \gamma \|_{C^{1,\alpha}(\Gamma)}, \Omega', \| \mathfrak{b} \|_{L^\infty(\Omega)} + \| g \|_{C^{1,\alpha}(\Gamma)} \right).
\]

Heuristically, if the “limiting profile” in \textbf{(Reces.)} there exists point-wisely and fulfills a suitable \textit{a priori} estimate like \((1.3)\), then a \(W^{2,p}\) regularity theory to \((1.1)\) holds true up to the boundary via geometric tangential methods (cf. [6], [12] and [33]).

Therefore, one of the main purposes of this work is to establish sharp Hessian estimates for problems enjoying a regular asymptotically elliptic property as \textbf{(Reces.)} on the governing operator
in (1.1), as well as present a number of applications in related analysis themes and free boundary problems of the obstacle-type.

Historically, the concept of an asymptotically regular problem can be traced back to Chipot-Evans’ work [11] in the context of certain problems in calculus of variations. Since then, there has been an increasing research interest in this class of problems and their related topics, see [7], [29], [40], [46], [47] for an incomplete list of contributions. To the best of our knowledge, no research has been conducted on such boundary asymptotic qualitative estimates of general problems as (1.1), as well as their intrinsic connections and applications in the analysis of PDEs and free boundary problems.

1.1 Assumptions and main results

For a given \( r > 0 \), we denote by \( B_r(x) \) the ball of radius \( r \) centered at \( x = (x', x_n) \in \mathbb{R}^n \), where \( x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \). We write \( B_r = B_r(0) \) and \( B_r^+ = B_r \cap \mathbb{R}_+^n \). Also, we write \( T_r := \{(x', 0) \in \mathbb{R}^{n-1} : |x'| < r \} \) and \( T_r(x_0) := T_r + x_0 \) where \( x_0 \in \mathbb{R}^{n-1} \). Finally, Sym(\( n \)) will denote the set of all \( n \times n \) real symmetric matrices.

Throughout this manuscript, we will make the following assumptions:

(A1) (Structural conditions) We assume that \( F \in C^0(\text{Sym}(n), \mathbb{R}^n, \mathbb{R}, \Omega) \). Moreover, there are constants \( 0 < \lambda \leq \Lambda, \sigma \geq 0 \) and \( \xi \geq 0 \) such that

\[
\begin{align*}
\mathcal{P}_{\lambda, \Lambda}(X - Y) - \sigma |q_1 - q_2| - \xi |r_1 - r_2| & \leq F(X, q_1, r_1, x) - F(Y, q_2, r_2, x) \\
(1.4) & \leq \mathcal{P}_{\lambda, \Lambda}^+(X - Y) + \sigma |q_1 - q_2| + \xi |r_1 - r_2|
\end{align*}
\]

for all \( X, Y \in \text{Sym}(n) \), \( q_1, q_2 \in \mathbb{R}^n \), \( r_1, r_2 \in \mathbb{R} \), \( x \in \Omega \), where

\[
(1.5) \quad \mathcal{P}_{\lambda, \Lambda}^+(X) := \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i \quad \text{and} \quad \mathcal{P}_{\lambda, \Lambda}^-(X) := \Lambda \sum_{e_i < 0} e_i + \lambda \sum_{e_i > 0} e_i,
\]

are the Pucci’s extremal operators and \( e_i = e_i(X) \) (\( 1 \leq i \leq n \)) denote the eigenvalues of \( X \).

(A2) (Regularity of the data) The data satisfy \( f \in C^0(\Omega) \cap L^p(\Omega) \) for \( n \leq p < \infty \), \( g, \gamma \in C^0(\partial \Omega) \) with \( \gamma \leq 0 \) and \( \beta \in C^0(\partial \Omega; \mathbb{R}^n) \) with \( \| \beta \|_{L^\infty(\partial \Omega)} \leq 1 \) and there exists a positive constant \( \mu_0 \) such that \( \beta \cdot \tilde{n} \geq \mu_0 \), where \( \tilde{n} \) is the outward normal vector of \( \Omega \).

(A3) (Continuous coefficients in the \( L^p \)-average sense) According to [10] for a fixed \( x_0 \in \Omega \), we define the quantity:

\[
\psi_{F^I}(x; x_0) := \sup_{X \in \text{Sym}(n)} \frac{|F^I(X, 0, 0, x) - F^I(X, 0, 0, x_0)|}{\|X\| + 1},
\]

which measures the oscillation of the coefficients of \( F \) around \( x_0 \). Furthermore, for notation purposes, we shall often write \( \psi_F(x, 0) = \psi_F(x) \). Finally, the mapping \( \Omega \ni x \mapsto F^I(X, 0, 0, x) \) is assumed to be Hölder continuous function (in the \( L^p \)-average sense) for every \( X \in \text{Sym}(n) \). This means that, there exist universal constant \( 0 < \alpha \in (0, 1) \), \( \theta_0 > 0 \) and \( 0 < r_0 \leq 1 \) such that

\[
\left( \int_{\Omega \setminus B_r(x_0) \cap \Omega} \psi_{F^I}(x, x_0)^p \, dx \right)^{1/p} \leq \theta_0 r^\alpha
\]

for \( x_0 \in \overline{\Omega} \) and \( 0 < r \leq r_0 \).

\footnote{Throughout this work, a constant is said to be universal if it depends only on \( n, \lambda, \Lambda, p, \mu_0, \gamma \|_{C^{1, \alpha}(\partial \Omega)} \) and \( \|\beta\|_{C^{1, \alpha}(\partial \Omega)} \).}
(A4) \textbf{\textit{(C}^{1,1} \textit{a priori estimates)}} We will assume that the recession operator $F^\sharp$ there exists and fulfills a up-to-the boundary \textit{C}^{1,1} \textit{a priori estimates}, i.e., for $x_0 \in B_1^+$ and $g_0 \in \text{C}^{1, \alpha} (T_1)$ (for some $\alpha \in (0, 1)$), there exists a solution $h \in \text{C}^{1,1} (B_1^+) \cap C^0 (\overline{B}_1^+)$ of

\[
\begin{cases}
  F^\sharp (D^2 h, x_0) = 0 & \text{in } B_1^+ \\
  B (x, w, Dh) = g_0 (x) & \text{on } T_1,
\end{cases}
\]

such that

\[\| h \|_{C^{1,1} (\overline{B}_1^+)} \leq C_1 (\text{universal}) \left( \| h \|_{L^\infty (B_1^+)} + \| g_0 \|_{C^{1, \alpha} (T_1)} \right)\]

for some constant $C_1 > 0$.

From now on, an operator fulfilling (A1) will be referred to as a $(\lambda, \Lambda, \sigma, \xi)$--elliptic operator. Furthermore, we shall always assume the normalization condition: $F (C_{n \times n}, 0, 0, x) = 0$ for all $x \in \Omega$, which is not restrictive, because one can reduce the problem in order to check it.

We now present our first main result, which endures global $W^{2, p}$ estimates for viscosity solutions to asymptotically fully nonlinear equations.

**Theorem 1.2 (W^{2, p} estimates under oblique boundary conditions).** Let $n \leq p < \infty$, $\Omega \subset \mathbb{R}^n$, $\partial \Omega \in C^{2, \alpha}$, $\beta, \gamma, g \in C^{1, \alpha} (\partial \Omega)$ (for some $\alpha \in (0, 1)$) and $u$ be an $L^p$--viscosity solution of (1.1). Further, assume that structural assumptions (A1)-(A4) are in force. Then, there exists a constant $C = C (n, \lambda, \Lambda, p, \mu_0, \| \gamma \|_{C^{1, \alpha} (\partial \Omega)}, \| \beta \|_{C^{1, \alpha} (\partial \Omega)}, \| g \|_{C^{1, \alpha} (\partial \Omega)}) > 0$ such that $u \in W^{2, p} (\Omega)$.

Furthermore, the following estimate holds

\[\| u \|_{W^{2, p} (\Omega)} \leq C \cdot (\| u \|_{L^\infty (\Omega)} + \| f \|_{L^p (\Omega)} + \| g \|_{C^{1, \alpha} (\partial \Omega)}) \quad (1.6)\]

We also supply a sharp estimate in the scenario that $f$ exceeds the scope of $L^\infty$ functions. Precisely, our second result concerns BMO type \textit{a priori} estimates for second derivatives provided the RHS of (1.1) is a BMO function and $F^\sharp$ enjoys classical \textit{a priori} estimates.

In contrast to the assumptions on Section 2 we will assume, for the next Theorem, the following hypothesis on $F^\sharp$:

(A4)* \textbf{\textit{(Higher a priori estimates)}} $F^\sharp$ fulfills a $C^{2, \psi}$ \textit{a priori} estimate up to the boundary, for some $\psi \in (0, 1)$, i.e., for any $g_0 \in C^0 (B_1^+) \cap C^{1, \psi} (T_1)$, $\beta \in C^{1, \psi} (T_1)$, there exists $C_\ast > 0$, depending only on universal parameters, such that any viscosity solution of

\[
\begin{cases}
  F^\sharp (D^2 h, 0, 0, x) = 0 & \text{in } B_1^+ \\
  \beta \cdot Dh = g_0 (x) & \text{on } T_1
\end{cases}
\]

belongs to $C^{2, \psi} (B_1^+) \cap C^0 (\overline{B}_1^+)$ and fulfills

\[\| h \|_{C^{2, \psi} (\overline{B}_1^+)} \leq C_\ast \left( \| h \|_{L^\infty (B_1^+)} + \| g_0 \|_{C^{1, \psi} (T_1)} \right) \quad \forall \ 0 < \rho \ll 1.
\]

**Definition 1.3.** For a function $f \in L^1_{\text{loc}} (\mathbb{R}^n)$, $x_0 \in \Omega$ and $\rho > 0$, we define $(f)_{x_0, \rho}$ by

\[ (f)_{x_0, \rho} := \int_{B_\rho (x_0) \cap \Omega} f (x) dx. \]

Moreover, when $x_0 = 0$ we will denote $(f)_{\rho}$ by simplicity.
An $f \in L^1_{\text{loc}}(\Omega)$ is a function of $p$–bounded mean oscillation, in short $f \in p\text{-BMO}(\Omega)$, if

\begin{equation}
\|f\|_{p\text{-BMO}(\Omega)} := \sup_{B_r(x_0) \cap \Omega} \left( \frac{\int_{B_r(x_0) \cap \Omega} |f(x) - (f)_{x_0, r}|^p \, dx}{r^p} \right)^{\frac{1}{p}} < \infty
\end{equation}

for a constant $C > 0$ that is independent on $\rho$. As a consequence of the John–Nirenberg inequality, such a semi-norm is equivalent to the one in the classical BMO spaces (see [39] pp. 763-764]).

**Theorem 1.4** (Boundary $p$–BMO type estimates). Let $u$ be an $L^p$–viscosity solution to

\[
\begin{align*}
F(D^2u, Du, u, x) &= f(x) \quad \text{in} \quad B_1^+ \\
\beta \cdot Du &= g(x) \quad \text{on} \quad T_1,
\end{align*}
\]

where $f \in p\text{-BMO}(B_1^+ \cap L^p(B_1^+))$, for some $n \leq p \leq \infty$. Assume further that assumptions (A1)-(A3) and (A4)* are in force. Then, $D^2u \in p\text{-BMO}(B_1^+)$. Moreover, the following estimate holds

\begin{equation}
\|D^2u\|_{p\text{-BMO}(B_1^+)} \leq C \left( n, \lambda, \Lambda, p, C_*, \mu_0, \|\beta\|_{C^{1,\alpha}(\overline{T_1})} \left( \|u\|_{L^\infty(B_1^+)} + \|g\|_{C^{1,\alpha}(\overline{T_1})} + \|f\|_{p\text{-BMO}(B_1^+)} \right) \right).
\end{equation}

**Remark 1.5** ($C^{1,\Log\text{-Lip}}$ estimates). Since viscosity solutions to (1.1) have a Hessian control in the $L^p$–average sense, then by invoking the embedding result from [11] Lemma 1, we may establish $C^{1,\Log\text{-Lip}}$ type estimates: Let $h : B_1^+ \to \mathbb{R}$ be a function such that for all $1 \leq i, j \leq n$, locally $D_{ij}h \in BMO_p(B_1^+)$. Then, $h \in C^{1,\Log\text{-Lip}}(B_1^+)$. In other words,

\[
\sup_{\rho \in (0, 1)} \sup_{z \in \partial B_1} \sup_{\rho^2 \ln \rho^{-1} \leq |x - z|} \frac{|u(z) - u(x) + Du(z) \cdot (x - z)|}{\rho^2 \ln \rho^{-1}} \leq \tilde{C} \cdot \left( \|u\|_{L^\infty(\Omega)} + \|g\|_{C^{1,\alpha}(\partial \Omega)} + \|f\|_{p\text{-BMO}(\Omega)} \right),
\]

for some constant $\tilde{C} = \tilde{C}(n, \lambda, \Lambda, p, C_*, \mu_0, \|\beta\|_{C^{1,\alpha}(\overline{T_1})}, \|\partial \Omega\|_{C^{1,1}})$ (cf. [12] Theorem 5.2 for related boundary estimates). Finally, we must quote [25], where the authors prove essentially close-to-optimal assumptions on the $f(x, u)$, for which $C_{\text{loc}}^{1,1}$ regularity of $W^{2,p}$ solutions in the classical semi-linear equation $\Delta u = f(x, u)$ holds true.

We would like to mention that although our manuscript has been strongly motivated by recent works [2], [12] and [39], our approach has required a sort of non-trivial adaptation due to the presence of a non-homogeneous oblique boundary condition and the asymptotic nature of the limiting operator. In addition, unlike [22], our findings provide additional quantitative applications, such as BMO type estimates (see Section 4), $W^{2,p}$–regularity to obstacle-type problems with oblique boundary conditions (see Section 5), and density results for obstacle type problems (see Section 6) via tangential methods.

Additionally, our recession profiles encode more general operators than linear ones, as addressed in [52] (see also [6]). Our results, in particular, extend, in the oblique boundary scenario, the former results from [6], [39] and [18] (see also [12]), and to some extent, those from [6], [30], [31] and [52] by employing techniques tailored to the general framework of the fully nonlinear models under relaxed convexity assumptions and oblique boundary datum.

Finally, we also highlight that Theorems 1.2 and 1.4 can be extended to a more general class of models, including fully nonlinear parabolic PDEs. For sake of simplicity and clarity, we have decided to consider just the elliptic scenario in this paper. We pretend to study such issues in a forthcoming project.

### 1.2 Applications to obstacle type problems

In closing, we would like to highlight that boundary Hessian estimates are also useful in the context of obstacle type problems. Physically, the classical obstacle problem refers to the
There exists a modulus of continuity $\omega$ with oblique boundary conditions. We must quote [34] concerning gradient estimates for obstacle problems of elliptic models (under convex structure) with discontinuous coefficients and irregular obstacles. For our purpose, it is necessary to make the following assumptions:

(A5) There exists a modulus of continuity $\omega : [0, +\infty) \to [0, +\infty)$ with $\omega(0) = 0$, such that

$$F(X_1, q, r, x_1) - F(X_2, q, r, x_2) \leq \omega(|x_1 - x_2| + \alpha_0|x_1 - x_2|^2)$$

holds for any $x_1, x_2 \in \Omega$, $q \in \mathbb{R}^n$, $r \in \mathbb{R}$, $\alpha_0 > 0$, and $X_1, X_2 \in \text{Sym}(n)$ satisfying

$$-3\alpha_0 \begin{pmatrix} \text{Id}_n & 0 \\ 0 & \text{Id}_n \end{pmatrix} \leq \begin{pmatrix} X_2 & 0 \\ 0 & -X_1 \end{pmatrix} \leq 3\alpha_0 \begin{pmatrix} \text{Id}_n & -\text{Id}_n \\ -\text{Id}_n & \text{Id}_n \end{pmatrix},$$

where $\text{Id}_n$ is the identity matrix.

(A6) $F$ is a proper operator in the following sense:

$$d \cdot (r_2 - r_1) \leq F(X, q, r_1, x) - F(X, q, r_2, x),$$

for any $X \in \text{Sym}(n)$, $r_1, r_2 \in \mathbb{R}$, with $r_1 \leq r_2$, $x \in \Omega$, $q \in \mathbb{R}^n$, and some $d > 0$.

The previous assumptions are invoked to assure the validity of the Comparison Principle for oblique derivatives problems like (1.1) (see [10] Theorem 2.10 and [36] Theorem 7.17), ensuring the use of Perron’s Method for viscosity solutions (see Lieberman’s Book [36] Chapter 7.4 and 7.6).

Finally, our third result addresses a $W^{2,p}$-regularity theory for (1.9), that does not rely on any additional regularity assumptions on nonlinearity $F$. 

with $u - g \in H^1_0(\Omega)$.

We refer [4] for Calderón-Zygmund estimates for elliptic and parabolic obstacle problems in non-divergence form (under convex structure) with oblique boundary conditions. We must quote [34] concerning gradient estimates for obstacle problems of elliptic models (under convex/convex structure) with oblique boundary conditions.

Now, we would like to focus our attention to the recent work [3, Theorem 1.1.], where $W^{2,p}$ estimates for the obstacle problem with oblique boundary conditions

$$
\begin{align*}
\Delta u & \leq f \quad \text{in the weak sense in } \Omega \\
\Delta u & = f \quad \text{in the weak sense in } \{u > \varphi\} \\
u & \geq \varphi \quad \text{in } \Omega
\end{align*}
$$

for a given obstacle $\varphi \in W^{2,p}(\Omega)$ satisfying $\beta(x) \cdot Du(x) + \gamma(x)u(x) = g(x)$ a.e., on $\partial\Omega$, have been obtained in the case when $F$ is a convex operator and $\gamma = g \equiv 0$.

Therefore, motivated by above problem, we will investigate $W^{2,p}$ regularity estimates for obstacle-type problems (1.9) under an asymptotic convexity assumption (weaker than convexity) and non-homogeneous oblique conditions.

For our purpose, it is necessary to make the following assumptions:

\begin{align*}
F(D^2u, Du, u, x) & \leq f(x) \quad \text{in } \Omega \\
(F(D^2u, Du, u, x) - f(x))(u(x) - \varphi(x)) & = 0 \quad \text{in } \Omega \\
u(x) & \geq \varphi(x) \quad \text{in } \Omega \\
\beta(x) \cdot Du(x) + \gamma(x)u(x) & = g(x) \quad \text{on } \partial\Omega
\end{align*}

for a given obstacle $\varphi \in W^{2,p}(\Omega)$ satisfying $\beta(x) \cdot D\varphi(x) \geq g(x)$ a.e., on $\partial\Omega$, have been obtained in the case when $F$ is a convex operator and $\gamma = g \equiv 0$.

Finally, our third result addresses a $W^{2,p}$-regularity theory for (1.9), that does not rely on any additional regularity assumptions on nonlinearity $F$. 

7
where $C$ with Dirichlet boundary conditions:

$$F(D^2u, Du, u, x) \leq f(x) \quad \text{in } \Omega$$

$$(F(D^2u, Du, u, x) - f(x))(u(x) - \phi(x)) = 0 \quad \text{in } \Omega$$

$$u(x) \geq \phi(x) \quad \text{in } \Omega$$

$$u(x) = g(x) \quad \text{on } \partial \Omega$$

by invoking the related global $W^{2,p}$ regularity estimates addressed in [6] Theorem 2.7 and [12] Theorems 1.1 with the following estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C(\text{universal}) \cdot \left(\|f\|_{L^p(\Omega)} + \|\phi\|_{W^{2,p}(\Omega)} + \|g\|_{W^{2,p}(\partial \Omega)}\right) \quad \text{for } n < p < \infty.$$ 

Finally, we would like to mention the interesting manuscript [22], in which the author utilized the regularity (in the case $p = \infty$) to prove a conjecture on the geometry of the free and fixed boundaries in the setting of fully nonlinear obstacle problems (cf. also [23] for related regularity results).

1.3 Hessian estimates to models in non-divergence form: State-of-Art

Regularity estimates in Sobolev spaces for nonlinear models have established an important chapter in the modern history of the elliptic theory of viscosity solutions. Moreover, because of its non-variational nature, obtaining sharp integrability properties of solutions of such elliptic operators has been a demanding task throughout the last years. One of the most pressing issues was whether $W^{2,p}$ a priori estimates could be addressed for any fully nonlinear operator, thereby attracting notable efforts from the academic community over the last three decades or so, while remaining open to the advancement of modern techniques.

In this direction, Caffarelli proved in his seminal work [8] Theorem 7.1 that $C^0 -$ viscosity solutions to

$$F(D^2u, x) = f(x) \quad \text{in } B_1$$

satisfy an interior $W^{2,p}-$regularity for $n < p < \infty$, with the following estimate

$$\|u\|_{W^{2,p}(B_{\frac{1}{2}})} \leq C(n,p, \Lambda, \epsilon_0) \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right),$$

provided a small oscillation of $F(X, x)$ in the variable $x$ and $C^{1,1}$ a priori estimates for homogeneous equation with “frozen coefficients” $F(D^2w, x_0) = 0$ are in force. Furthermore, Caffarelli also demonstrated that for any $p < n$, there exists a fully nonlinear operator fulfilling the previous hypothesis, for which $W^{2,p}-$estimates fail to hold. In this regard, Dong-Kim’s work (citeDK91) is worth mentioning for notable examples of linear elliptic operators in non-divergence form with piecewise constant coefficients, whose solutions lack $W^{2,p}$ estimates.

A few years later, Caffarelli’s $W^{2,p}-$estimates were extended by Escauriaza in [17] Theorem 1 in the context of $L^n-$viscosity solutions to the range of exponents $n - \varepsilon_0 < p < \infty$ with $\varepsilon_0 = \varepsilon_0 \left(\frac{1}{2}, n\right) \in \left(\frac{1}{2}, n\right)$ (the Escauriaza’s constant), which provides the minimal range of integrability
for which the Harnack inequality (resp. Hölder regularity) holds for viscosity solutions to (1.11). Furthermore, when the boundary of the domain is smooth enough, namely $C^{1,1}$, a global $W^{2,p}$ estimate is obtained for $n-\varepsilon_0 < p < \infty$ by Winter in [51, Theorem 4.5].

In the aftermath, Byun-Lee-Palagachev in [5] extend the Escauriaza-Winter’s results by studying the global regularity of viscosity solutions to the following Dirichlet problem for a fully nonlinear uniformly elliptic equation:

\begin{equation}
\begin{aligned}
F(D^2u, Du, u, x) &= f(x) & \text{in } \Omega \\
\quad u(x) &= 0 & \text{on } \partial \Omega
\end{aligned}
\end{equation}

in weighted Lebesgue spaces. Indeed, under appropriate structural conditions, if $f \in L^p_0(\Omega)$ for $p > n-\varepsilon_0$ and $w \in A_{\frac{p}{p-n}}$ (Muckenhoupt classes), then the problem (1.12) admits a unique solution $u \in W^{2,p}_0(\Omega)$ such that

$$
\|u\|_{W^{2,p}_0(\Omega)} \leq C(\text{universal}) \|f\|_{L^p_0(\Omega)}.
$$

Recently, in a new landmark for this theory, Caffarelli’s local $W^{2,p}$ regularity estimates were extended by Pimentel-Teixeira in [39]. Precisely, the novelty with respect to the Caffarelli’s results is the concept of \textit{Recession} operator (a sort of tangent profile for $F$ at “infinity”, see [Reces.]). In this context, the authors relaxed the hypothesis of $C^{1,1}$ \textit{a priori} estimates for solutions of equations without dependence on $x$, by assuming that $F$ must be “convex or concave” only at the ends of $\text{Sym}(n)$. In this setting, the authors proved that any viscosity solution to

\begin{equation}
\begin{aligned}
F(D^2u) &= f(x) & \text{in } B_1, \\
\quad u(x) &= 0 & \text{on } \partial B_1
\end{aligned}
\end{equation}

where $f \in L^p(B_1) \cap C^0(B_1)$, fulfills $u \in W^{2,p}_{loc}(B_1)$ with an \textit{a priori} estimate

$$
\|u\|_{W^{2,p}(B_1)} \leq C(n, p, \lambda, \Lambda) \left( \|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)} \right) .
$$

We must quote similar global $W^{2,p}$-estimates independently proved by Byan et al. in [9, Theorems 2.5 and 2.7] for solutions to asymptotically linear operators with a zero Dirichlet boundary condition. Their approach is based on a proper transformation that converts a given asymptotically elliptic PDE to a suitable uniformly elliptic one. We also cite Silvestre-Teixeira’s work [48, Theorems 1 and 4] for an interesting survey on this theme in the setting of sharp gradient estimates.

In the sequel, global $W^{2,p}$, BMO$_p$ and $C^{1,\log,\text{Lip}}$ regularity estimates, in the context of asymptotic convex operators (see Definition (1.11), i.e.,

\begin{equation}
\begin{aligned}
F(D^2u, Du, u, x) &= f(x) & \text{in } \Omega \\
\quad u(x) &= g(x) & \text{on } \partial \Omega
\end{aligned}
\end{equation}

were addressed by da Silva-Ricarte in [12, Theorems 1.1, 1.2 and 5.2]. Precisely, they obtain

$$
\|u\|_{W^{2,p}(\Omega)} \leq C(n, p, \lambda, \Lambda, \|\partial \Omega\|_{C^{1,1}}) \left( \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|g\|_{W^{2,p}(\partial \Omega)} \right) .
$$

The proof of such estimates in [12] is based on the successful adaptation of compactness arguments inspired by the ideas from Pimentel-Teixeira [39], Silvestre-Teixeira [48], and Winter [51]. We also refer the reader to Lee’s work [32] for Hessian estimates for solutions of (1.13) in the framework of weighted Orlicz spaces.

Among other works on this research topic, we must cite the sequence of Krylov’s fundamental manuscripts [29], [30] and [31], which are associated with the existence of $L^p$-viscosity solutions under either relaxed or no convexity assumptions on $F$.

Now, concerning problems with oblique boundary conditions, Hessian estimates for viscosity solutions of elliptic equations like (1.1) has been regarded as an important issue in the study of
PDEs for a long time. The theory of this subject has also been improved over the last several decades. For general fully nonlinear problems like (1.1), the existence and uniqueness for viscosity solutions of oblique boundary problems was proved in [26] and [36].

Inspired by the ideas from Caffarelli’s fundamental work [8], Byan and Han in [2] establish global $W^{2,p}$–estimates for viscosity solutions of convex fully nonlinear elliptic equations with a homogeneous oblique derivative boundary condition. In this context, the nonlinearity is assumed to fulfill a certain asymptotic condition hold:

$$
\exists \text{ a relatively open set. A function } u \in C^{0,1}\text{ satisfies the following inequalities:}
$$

It is also worth highlighting that $C^{0,\alpha}$, $C^{1,\alpha}$ and $C^{2,\alpha}$ boundary regularity were addressed for the Neumann problem on flat boundaries by Milakis-Silvestre in [38]. Moreover, such results were recently extended to the general oblique scenario by Li-Zhang in [33].

Finally, we must refer to the work of Zhang et al. [52], which establishes $W^{2,p}$–regularity for viscosity solutions of fully nonlinear elliptic equations with a homogeneous oblique derivative boundary condition. In this context, the nonlinearity is assumed to fulfill a certain asymptotic regularity condition.

In conclusion, taking such historical advances into account, in this work, we aim to obtain a global $W^{2,p}$ estimates for a large class of nonlinear elliptic PDEs. To that end, we only assume that $F(X,\xi,s,x)$ is asymptotically elliptic, that is, we establish global Sobolev a priori estimates (suitable asymptotic structure of $F$ just at ”infinity” of $\text{Sym}(n)$ according to the Definition 1.1).

This paper is organized as follows: Section 2 contains the main notations and preliminary results on which we will work throughout this manuscript. In Section 2.1 we present the Modus Operandi, i.e., the tangential approximation mechanism used in order to relate the estimates coming from Recession operator $F^\sharp$ with its original counterpart $F$. Section 3 is devoted to proving Theorem 1.2. In Section 4 we establish the regularity estimates in the borderline setting, namely for p-BMO spaces. Finally, in the final sections, we establish the global $W^{2,p}$ estimate for obstacle problems with oblique boundary conditions, and a density result in a suitable class of viscosity solutions.

## 2 Preliminaries and auxiliary results

To (1.1), we introduce the appropriate notions of viscosity solutions.

**Definition 2.1 (C$^0$–viscosity solutions).** Let $F$ be a $(\lambda,\Lambda,\sigma,\xi)$–elliptic operator and $\Gamma \subset \partial \Omega$ a relatively open set. A function $u \in C^{0}(\overline{\Omega})$ is said a $C^0$–viscosity solution if the following condition hold:

a) for all $\forall \varphi \in C^2(\Omega \cup \Gamma)$ touching $u$ by above at $x_0 \in \Omega \cup \Gamma$,

$$
\begin{align*}
&F(D^2 \varphi(x_0), D\varphi(x_0), \varphi(x_0), x_0) \geq f(x_0) \quad \text{when } x_0 \in \Omega \\
&\quad \text{and } \quad B(x_0, u(x_0), D\varphi(x_0)) \geq g(x_0) \quad \text{at } x_0 \in \Gamma.
\end{align*}
$$

b) for all $\forall \varphi \in C^2(\Omega \cup \Gamma)$ touching $u$ by below at $x_0 \in \Omega \cup \Gamma$,

$$
\begin{align*}
&F(D^2 \varphi(x_0), D\varphi(x_0), \varphi(x_0), x_0) \leq f(x_0) \quad \text{when } x_0 \in \Omega \\
&\quad \text{and } \quad B(x_0, u(x_0), D\varphi(x_0)) \leq g(x_0) \quad \text{at } x_0 \in \Gamma.
\end{align*}
$$

**Definition 2.2 (L$^p$-viscosity solution).** Let $F$ be a $(\lambda,\Lambda,\sigma,\xi)$-elliptic operator, $p \geq n$ and $f \in L^p(\Omega)$. Assume that $F$ is continuous in $X$, $q$, $r$ and measurable in $x$. A function $u \in C^0(\overline{\Omega})$ is said an $L^p$-viscosity solution for (1.1) if the following assertions hold:
a) For all \( \varphi \in W^{2,p}(\Omega) \) touching \( u \) by above at \( x_0 \in \Omega \)
\[
\begin{cases}
F(D^2\varphi(x_0), D\varphi(x_0), \varphi(x_0), x_0) \geq f(x_0) & \text{when } x_0 \in \Omega \\
B(x_0, u(x_0), D\varphi(x_0)) \geq g(x_0) & \text{at } x_0 \in \partial \Omega.
\end{cases}
\]
b) For all \( \varphi \in W^{2,p}(\Omega) \) touching \( u \) by below at \( x_0 \in \Omega \)
\[
\begin{cases}
F(D^2\varphi(x_0), D\varphi(x_0), \varphi(x_0), x_0) \leq f(x_0) & \text{when } x_0 \in \Omega \\
B(x_0, u(x_0), D\varphi(x_0)) \leq g(x_0) & \text{at } x_0 \in \partial \Omega.
\end{cases}
\]

For convenience of notation, we define
\[
\mathcal{L}^\pm(u) := \mathcal{S}^\pm_{\lambda, \Lambda} (D^2u) \pm \sigma |Du|.
\]

**Definition 2.3.** We define the class \( \mathcal{S}(\lambda, \Lambda, \sigma, f) \) and \( \mathcal{S}^\pm(\lambda, \Lambda, \sigma, f) \) to be the set of all continuous functions \( u \) that satisfy \( \mathcal{L}^+(u) \geq f \), respectively \( \mathcal{L}^-(u) \leq f \) in the viscosity sense (see Definition 2.1). We define,
\[
\mathcal{S}(\lambda, \Lambda, \sigma, f) := \mathcal{S}(\lambda, \Lambda, \sigma, f) \cap \mathcal{S}(\lambda, \Lambda, \sigma, f) \quad \text{and} \quad \mathcal{S}^\pm(\lambda, \Lambda, \sigma, f) := \mathcal{S}(\lambda, \Lambda, \sigma, f) \cap \mathcal{S}(\lambda, \Lambda, \sigma, -f).
\]
Moreover, when \( \sigma = 0 \), we denote \( \mathcal{S}^\pm(\lambda, \Lambda, 0, f) \) just by \( \mathcal{S}^\pm(\lambda, \Lambda, f) \) (resp. by \( \mathcal{S}_-^\pm, \mathcal{S}, \mathcal{S} \)).

Now, we present a compactness result, whose proof can be found in [33, Theorem 1.1].

**Theorem 2.4 (\( C^{0,\alpha'} \) regularity).** Let \( u \in C^0(\Omega) \cap C^0(\Gamma) \) be satisfying
\[
\begin{cases}
\mathcal{S}(\lambda, \Lambda, f) & \text{in } \Omega \\
\mathcal{B}(x, u, Du) = g(x) & \text{on } \Gamma.
\end{cases}
\]
Then, for any \( \Omega' \subset \subset \Omega \cup \Gamma \), \( u \in C^{0,\alpha'}(\partial \Omega) \) and
\[
\|u\|_{C^{0,\alpha'}(\partial \Omega)} \leq C(n, \lambda, \Lambda, \mu_0, \|\sigma\|_{L^\infty(\Gamma)}, \Omega' \cup \Gamma \left( \|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Gamma)} \right)
\]
where \( \alpha' \in (0,1) \) depends on \( n, \lambda, \Lambda \) and \( \mu_0 \).

Next, we present the following Stability result (see [10, Theorem 3.8] for a proof).

**Lemma 2.5 (Stability Lemma).** For \( k \in \mathbb{N} \) let \( \Omega_k \subset \Omega_{k+1} \) be an increasing sequence of domains and \( \Omega := \bigcup_{k=1}^{\infty} \Omega_k \). Let \( p \geq n \) and \( F, F_k \) be \( (\lambda, \Lambda, \sigma, \xi) \)–elliptic operators. Assume \( f \in L^p(\Omega) \), \( f_k \in L^p(\Omega_k) \) and that \( u_k \in C^0(\Omega_k) \) are \( L^p \)–viscosity sub-solutions (resp. super-solutions) of
\[
F_k(D^2u_k, Du_k, u_k, x) = f_k(x) \quad \text{in } \Omega_k.
\]
Suppose that \( u_k \to u_\infty \) locally uniformly in \( \Omega \) and that for \( B_r(x_0) \subset \Omega \) and \( \varphi \in W^{2,p}(B_r(x_0)) \) we have
\[
(2.1) \quad \|(\tilde{g} - \tilde{g}_k)^+\|_{L^p(B_r(x_0))} \to 0 \quad \text{(resp. } \|(\tilde{g} - \tilde{g}_k)^-\|_{L^p(B_r(x_0))} \to 0),
\]
where \( \tilde{g}(x) := F(D^2\varphi, D\varphi, u, x) - f(x) \) and \( \tilde{g}_k(x) = F_k(D^2\varphi, D\varphi, u_k, x) - f_k(x) \). Then, \( u \) is an \( L^p \)–viscosity sub-solution (resp. super-solution) of
\[
F(D^2u, Du, u, x) = f(x) \quad \text{in } \Omega.
\]
Moreover, if \( F, f \) are continuous, then \( u \) is a \( C^0 \)–viscosity sub-solution (resp. super-solution) provided that \( (2.1) \) holds for all \( \varphi \in C^2(B_r(x_0)) \) test function.
The next result is an A.B.P. Maximum Principle (see [33] Theorem 2.1) for a proof.

**Lemma 2.6 (A.B.P. Maximum Principle).** Let $\Omega \subset B_1$ and $u \in C^0(\Omega)$ satisfying
\[
\begin{align*}
\{ & \begin{array}{l}
u \in S(\lambda, \Lambda, f) \quad \text{in} \quad \Omega \\
B(x, u, Du) = g(x) \quad \text{on} \quad \Gamma.
\end{array}
\end{align*}
\]
Suppose that exists $\varsigma \in \partial B_1$ such that $\beta \cdot \varsigma \geq \mu_0$ and $\gamma \leq 0$ in $\Gamma$. Then,
\[
\|u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\partial \Omega, \Gamma)} + C(n, \lambda, \Lambda, \mu_0)(\|g\|_{L^\infty(\Gamma)} + \|f\|_{L^\infty(\Omega)})
\]
In the sequel, we comment on the existence and uniqueness of viscosity solutions with oblique boundary conditions. For that purpose, we will assume the following condition on $F$:

**(SC)** There exists a modulus of continuity $\tilde{\omega}$, i.e., $\tilde{\omega}$ is nondecreasing with $\lim_{t \to 0} \tilde{\omega}(t) = 0$ and
\[
\psi_F(x, y) \leq \tilde{\omega}(|x - y|).
\]

Now, we will ensure the existence and uniqueness of viscosity solutions to the following problem:
\[
\begin{align*}
\{ & \begin{array}{l}
F(D^2u, x) = f(x) \quad \text{in} \quad \Omega, \\
B(x, u, Du) = g(x) \quad \text{on} \quad \Gamma, \\
u(x) = \varphi(x) \quad \text{on} \quad \partial \Omega \setminus \Gamma,
\end{array}
\end{align*}
\]
where $\Gamma$ is relatively open of $\partial \Omega$. The next proof follows the ideas from [33] Theorem 3.1 with minor modifications. For this purpose, we will omit it here.

**Theorem 2.7.** Suppose that $\Gamma \in C^2$ and $\beta \in C^2(\Gamma)$. Assume that $F$ satisfies (A1) and (SC). Let $u$ and $v$ be satisfying
\[
\begin{align*}
\{ & \begin{array}{l}
F(D^2u, x) \geq f_1(x) \quad \text{in} \quad \Omega, \\
B(x, u, Du) \geq g_1(x) \quad \text{on} \quad \Gamma, \\
u(x) = \varphi(x) \quad \text{on} \quad \partial \Omega \setminus \Gamma,
\end{array}
\end{align*}
\]
and
\[
\begin{align*}
\{ & \begin{array}{l}
F(D^2v, x) \leq f_2(x) \quad \text{in} \quad \Omega, \\
B(x, v, Dv) \leq g_2(x) \quad \text{on} \quad \Gamma.
\end{array}
\end{align*}
\]
Then,
\[
\begin{align*}
\{ & \begin{array}{l}
u - v \in S(\frac{\Lambda}{\lambda}, \Lambda, f_1 - f_2) \quad \text{in} \quad \Omega, \\
B(x, u - v, Du - Dv) \geq (g_1 - g_2)(x) \quad \text{on} \quad \Gamma.
\end{array}
\end{align*}
\]

**Theorem 2.8 (Uniqueness).** Let $\Gamma \in C^2$, $\beta \in C^2(\Gamma)$, $\gamma \leq 0$ and $\varphi \in C^0(\partial \Omega \setminus \Gamma)$. Suppose that there exists $\varsigma \in \partial B_1$ such that $\beta \cdot \varsigma \geq \mu_0$ on $\Gamma$. Assume that $F$ satisfies (A1). Then, there exists at most one viscosity solution of
\[
\begin{align*}
\{ & \begin{array}{l}
F(D^2u, x) = f(x) \quad \text{in} \quad \Omega, \\
B(x, u, Du) = g(x) \quad \text{on} \quad \Gamma, \\
u(x) = \varphi(x) \quad \text{on} \quad \partial \Omega \setminus \Gamma.
\end{array}
\end{align*}
\]

**Proof.** The proof is direct consequence of Theorem 2.7 with A.B.P. estimate (Lemma 2.6). \hfill $\Box$

**Theorem 2.9 (Existence).** Let $\Gamma \in C^2$, $\beta \in C^2(\Gamma)$, $\gamma \leq 0$ and $\varphi \in C^0(\partial \Omega \setminus \Gamma)$. Suppose that there exists $\varsigma \in \partial B_1$ such that $\beta \cdot \varsigma \geq \mu_0$ on $\Gamma$ and assume (SC). In addition, suppose that $\Omega$ satisfies an exterior cone condition at any $x \in \partial \Omega \setminus \Gamma$ and satisfies an exterior sphere condition at any $x \in \Gamma \cap (\partial \Omega \setminus \Gamma)$. Then, there exists a unique viscosity solution $u \in C^0(\Omega)$ of
\[
\begin{align*}
\{ & \begin{array}{l}
F(D^2u, x) = f(x) \quad \text{in} \quad \Omega, \\
B(x, u, Du) = g(x) \quad \text{on} \quad \Gamma, \\
u(x) = \varphi(x) \quad \text{on} \quad \partial \Omega \setminus \Gamma.
\end{array}
\end{align*}
\]
Proof. The proof follows the same lines as the one in [33, Theorem 3.3] with minor modifications. For instance, we must invoke Theorem 2.8 instead of [33, Theorem 3.2]. Other necessary modifications are the same made in the proof of Theorem 2.7.

Next result is a key tool in our tangential approach. Precisely, it describes how the point-wise convergence of $F_{\tau}$ to $F^\sigma$ takes place.

**Lemma 2.10.** Let $F$ be a uniformly elliptic operator and assume $F^\sigma$ there exists. Then, given $\epsilon > 0$ there exists $\tau_0(\lambda, \Lambda, \epsilon, \psi_{F^\sigma}) > 0$ such that, for every $\tau \in (0, \tau_0)$ there holds

$$\left| \tau F \left( \tau^{-1}X, 0, 0, x \right) - F^\sigma(X, 0, 0, x) \right| \leq \epsilon,$$

for every $X \in \text{Sym}(n)$.

**Proof.** The proof follows the same lines as the one in [38] (see also [39]). For this reason, we omit it here.

**Remark 2.11.** It is worth highlighting that the $W^{2,p}$ estimates from Theorem 1.2 depends not only on universal constants, but in effect on the “modulus of convergence” $F_{\tau} \to F^\sigma$. Precisely, by defining $\varsigma : (0, \infty) \to (0, \infty)$ as follows

$$\varsigma(\epsilon) := \sup_{X \in \text{Sym}(n)} \left\{ \frac{\left| \tau F \left( \tau^{-1}X, 0, 0, x \right) - F^\sigma(X, 0, 0, x) \right|}{1 + \|X\|}, \epsilon \to 0^+ \Rightarrow \varsigma(\epsilon) \to 0^+ \right\},$$

then the constant $C > 0$ appearing in up-to-the boundary $W^{2,p}$ a priori estimates (1.6) also depends on $\varsigma$.

### 2.1 Approximation device in geometric tangential analysis

The main result of this section provides a compactness method that will be used as a key point throughout the whole paper. As a matter of fact, we will show that if our equation is close enough to the homogeneous equation with constant coefficients, then our solution is sufficiently close to a solution of the homogeneous equation with frozen coefficients. At the core of our techniques is the notion of the recession operator (see Definition 1.1). The appropriate way to formalize this intuition is with an approximation lemma.

**Lemma 2.12 (Approximation Lemma).** Let $n \leq p < \infty$, $0 \leq \nu \leq 1$ and assume that (A1) - (A4) are in force. Then, for every $\delta > 0$, $\varphi \in C^0(\partial B_1(0', \nu))$ with $\|\varphi\|_{L^\infty(\partial B_1(0', \nu))} \leq c_1$ and $g \in C^{1,\alpha}(T_2)$ with $0 < \alpha < 1$ and $\|g\|_{C^{1,\alpha}(T_2)} \leq c_2$ for some $c_2 > 0$ there exist positive constants $\epsilon = \epsilon(\delta, n, p, \lambda, \Lambda, \gamma, \epsilon_0, \epsilon_1) < 1$ and $\tau_0 = \tau_0(\delta, n, \lambda, \Lambda, \mu_0, \epsilon_1, \epsilon_2) > 0$ such that, if

$$\max \left\{ |F_{\tau}(X, x) - F^\sigma(X, x)|, \|\psi_{F^\sigma}(\cdot, 0)\|_{L^{p}(B_1^+)}{\|f\|_{L^{p}(B_1^+)}} \right\} \leq \epsilon \quad \text{and} \quad \tau \leq \tau_0$$

then, any two $L^{p}$-viscosity solutions $u$ (normalized, i.e., $\|u\|_{L^{\infty}(B^+_1(0', \nu))} \leq 1$) and $b$ of

$$\begin{cases}
F_{\tau}(D^2 u, x) = f(x) & \text{in } B^{+}_{1}(0', \nu) \cap \mathbb{R}^{n}_+
B(x, u, Du) = g(x) & \text{on } B_{r}(0', \nu) \cap T_r\n u(x) = \varphi(x) & \text{on } \partial B_{r}(0', \nu) \cap \mathbb{R}^{n}_+
\end{cases}$$

and

$$\begin{cases}
F^\sigma(D^2 b, 0) = 0 & \text{in } B^{+}_{1}(0', \nu) \cap \mathbb{R}^{n}_+
B(x, b, Db) = g(x) & \text{on } B_{r}(0', \nu) \cap T_r\nb(x) = u(x) & \text{on } \partial B_{r}(0', \nu) \cap \mathbb{R}^{n}_+
\end{cases}$$

satisfy

$$\|u - b\|_{L^{\infty}(B^{+}_{1}(0', \nu))} \leq \delta.$$
Proof. We may assume, without loss of generality that \( r = 1 \). We will proceed with a \textit{reductio ad absurdum} argument. Thus, let us assume that the claim is not satisfied. Then, there exists \( \delta_0 > 0 \) and a sequence of functions \( (F_{\tau_j})_{j \in \mathbb{N}}, (F_{\tau_j}^2)_{j \in \mathbb{N}}, (u_j)_{j \in \mathbb{N}}, (\varphi_j)_{j \in \mathbb{N}}, \{g_j\}_{j \in \mathbb{N}} \) and \( (h_j)_{j \in \mathbb{N}} \) linked through

\[
\begin{align*}
F_{\tau_j}(D^2u_j, x) &= f_j(x) \quad \text{in } B_1(0', \nu_j) \cap \mathbb{R}^n_+ \\
B(x, u_j, Du_j) &= g_j(x) \quad \text{on } B_1(0', \nu_j) \cap T_1 \\
u_j(x) &= \varphi_j(x) \quad \text{on } \partial B_1(0', \nu) \cap \mathbb{R}^n_+ 
\end{align*}
\]

and

\[
\begin{align*}
F_{\tau_j}^2(D^2h_j, 0) &= 0 \quad \text{in } B_{\frac{R_0}{2}}(0', \nu_j) \cap \mathbb{R}^n_+ \\
B(x, h_j, Dh_j) &= g_j(x) \quad \text{on } B_{\frac{R_0}{2}}(0', \nu_j) \cap T_1 \\
h_j(x) &= u_j(x) \quad \text{on } \partial B_{\frac{R_0}{2}}(0', \nu) \cap \mathbb{R}^n_+. 
\end{align*}
\]

where \( \tau_j, ||\psi_{F_{\tau_j}}^j(\cdot, 0)||_{L^p(B^n_+)} \), \( ||f_j||_{L^p(B^n_+)} \) go to zero as \( j \to \infty \) and such that

\begin{equation}
||u_j - h_j||_{L^\infty(B_{\frac{R_0}{2}}(0', \nu) \cap \mathbb{R}^n_+)} > \delta_0.
\end{equation}

Here, \( \varphi_j \in C^0(\partial B_1(0', \nu_j)) \) and \( g_j \in C^{1, \alpha}(T_2) \) satisfy \( ||\varphi_j||_{L^\infty(\partial B_1(0', \nu_j))} \leq c_1 \) and \( ||g_j||_{C^{1, \alpha}(T_2)} \leq c_2 \), respectively. From Theorem 2.4 we have for all \( 0 < \rho < 1 \)

\begin{equation}
||u_j||_{C^{0, \alpha'}(B_{1 \rho}(0', \nu_j) \cap \mathbb{R}^n_+)} \leq C(n, \lambda, \Lambda, c_1, \lambda, c_2, \mu_0) \rho^{-\alpha'}
\end{equation}

for some \( \alpha' = \alpha'(n, \lambda, \Lambda, \mu_0) \) and sufficiently large \( j \). Suppose that there exists a number \( \nu_{\infty} \) and a subsequence \( \{\nu_{j_k}\} \) such that \( \nu_{j_k} \to \nu_{\infty} \) as \( k \to +\infty \). We can assume that such a subsequence is monotone. If \( \nu_{j_k} \) is decreasing, we can check that

\[
B_1(0', \nu_{j_k}) \cap \mathbb{R}^n_+ \subset B_1(0', \nu_{j_k}) \cap \mathbb{R}^n_+ 
\]

for any \( k \). Thus, we observe that

\begin{equation}
||u_{j_k}||_{C^{0, \alpha'}(B_{15/16}(0', \nu_{\infty}) \cap \mathbb{R}^n_+)} \leq C^{0}(c_1, c_2, n, \lambda, \Lambda, \mu_0)
\end{equation}

by using (2.3). On the other hand, if \( \nu_{j_k} \) is increasing, there exists a number \( k_0 \) such that

\[
B_{31/32}(0', \nu_{j_k}) \cap \mathbb{R}^n_+ \supset B_{15/16}(0', \nu_{\infty}) \cap \mathbb{R}^n_+ \quad \text{for } k \geq k_0.
\]

Then we can deduce once again (2.4) for some proper subsequence \( u_{j_k} \). Thus, according to Arzelà-Ascoli’s compactness criterium, there exist functions \( u_{\infty} \in C^0(\partial B_1(0', \nu_{\infty}) \cap \mathbb{R}^n_+) \), \( g_{\infty} \in C^0(\partial B_1^+) \) and subsequences such that \( u_{j_k} \to u_{\infty} \) in \( C^{0, \alpha'}(B_1^+) \) and \( u_{\infty} = g_{\infty} \) on \( B_{15/16}(0', \nu_{\infty}) \cap T_1 \).

Since the functions \( F_{\tau_j}^2(\cdot, 0) \to F_{\infty}^2(\cdot, 0) \) uniformly in compact sets of \( Sym(n) \) and for every \( \varphi \in C^2(\overline{B_1^+}) \),

\[
|F_{\tau_{j_k}}(D^2\varphi, x) - f_{j_k}(x) - F_{\infty}^2(D^2\varphi, 0)| \leq |F_{\tau_{j_k}}(D^2\varphi, x) - F_{j_k}^2(D^2\varphi, x)| + |f_{j_k}| + + |F_{j_k}^2(D^2\varphi, x) - F_{j_k}^2(D^2\varphi, 0)| + + |F_{j_k}^2(D^2\varphi, 0) - F_{\infty}^2(D^2\varphi, 0)| \leq |F_{\tau_{j_k}}(D^2\varphi, x) - F_{j_k}^2(D^2\varphi, x)| + |f_{j_k}| + + \psi_{F_{j_k}^2}(x, 0)(1 + |D^2\varphi|)
\]

then

\[
\lim_{k \to +\infty} ||F_{\tau_{j_k}}(D^2\varphi, x) - f_{j_k}(x) - F_{\infty}^2(D^2\varphi, 0)||_{L^p(B_{\nu}(x_0))} = 0.
\]
for any ball $B_r(x_0) \subset B_{15/16}(0, \nu_\infty) \cap \mathbb{R}^n_+$. Therefore, the Stability Lemma 2.5 ensures that $u_\infty$ satisfies

$$
\begin{align*}
F^2\frac{\partial}{\partial y} u_\infty, 0, 0 & = 0 \quad \text{in} \ B_{15/16}(0, \nu_\infty) \cap \mathbb{R}^n_+ \\
B(x, u_\infty, D u_\infty) & = g_\infty(x) \quad \text{on} \ B_{15/16}(0, \nu_\infty) \cap T_1
\end{align*}
$$

in the viscosity sense. Now, we consider $w_{j_k} := u_\infty - h_{j_k}$ for each $k$. Then, by Lemma 4.3, $w_{j_k}$ satisfies

$$
\begin{align*}
w_{j_k} & \in S(\frac{r}{2^n}, \Lambda, 0) \quad \text{in} \ B_{7/8}(0, \nu_\infty) \cap \mathbb{R}^n_+ \\
B(x, w_{j_k}, D w_{j_k}) & = g_\infty(x) - g_{j_k}(x) \quad \text{on} \ B_{7/8}(0, \nu_\infty) \cap T_1 \\
w_{j_k}(x) & = u_\infty(x) - u_{j_k}(x) \quad \text{on} \ \partial B_{7/8}(0, \nu_\infty) \cap \mathbb{R}^n_+.
\end{align*}
$$

Now, using Lemma 2.6, we observe that

$$
\|w_{j_k}\|_{L^\infty(B_{7/8}(0, \nu_\infty) \cap \mathbb{R}^n_+)} \leq \|u_\infty - u_{j_k}\|_{L^\infty(\partial B_{7/8}(0, \nu_\infty))} + \right.
C(n, \Lambda, \rho_0)\|g_{h_{j_k}} - g_{j_k}\|_{L^\infty(B_{7/8}(0, \nu_\infty) \cap T_1)} \to 0 \quad \text{as} \ k \to +\infty.
$$

This is, $w_{j_k}$ converges uniformly to zero. This implies that $h_{j_k}$ converges uniformly to $u_\infty$ in $B_{7/8}(0, \nu_\infty) \cap \mathbb{R}^n_+$, which contradicts (2.2) for $k \gg 1$.

\section{$W^{2,p}$ estimates: Proof of Theorem 1.2}

Before starting the proof of Theorem 1.2 we must introduce some useful terminologies: let

$$
Q_r^d(x_0) := \left(x_0 - \frac{r}{2}, x_0 + \frac{r}{2}\right)^d
$$

be the $d$–dimensional cube of side-length $r$ and center $x_0$. In the case $x_0 = 0$, we will write $Q_r^d$. Furthermore, if $d = n$ we will write $Q_r(0, 0)$. We will need later on the Calderón-Zygmund decomposition Lemma for cubes: Let $Q_1^{n-1} \times (0, 1)$ be measurable sets and $\delta \in (0, 1)$ such that

(a) $\mathcal{L}^n(A) \leq \delta$;

(b) If $Q$ is a dyadic cube such that $\mathcal{L}^n(A \cap Q) > \delta \mathcal{L}^n(Q)$, then $\tilde{Q} \subset B$.

Then, $\mathcal{L}^n(A) \leq \delta \mathcal{L}^n(B)$.

Remember that the Hardy-Littlewood maximal operator is defined as follows for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$:

$$
\mathcal{M}(f)(x) := \sup_{\rho > 0} \frac{1}{\mathcal{L}^n(B_r(x))} \int_{B_r(x)} |f(y)| dy.
$$

We say that $P_M$ is a paraboloid with opening $M > 0$ whenever $P_M(x) = \pm \frac{M}{2} |x|^2 + p_1 \cdot x + p_0$. Observe that such a paraboloid is a convex function in the case of “plus” sign and a concave function otherwise. Now, for $u \in C^0(\Omega), \Omega' \subset \Omega$ and $M > 0$ we define

$$
\mathcal{G}_M(u, \Omega') := \{x_0 \in \Omega'; \exists P_M \text{ concave paraboloid s. t. } P_M(x_0) = u(x_0), P_M(x) \leq u(x) \forall x \in \Omega'\}.
$$
and
\[ A_M(u, \Omega') := \Omega' \setminus G_M(u, \Omega'). \]

Similarly, by using convex paraboloid we define \( G_M(u, \Omega') \) and \( A_M(u, \Omega') \) and the sets
\[ G_M(u, \Omega') := \bar{G}_M(u, \Omega') \cap G_M(u, \Omega') \quad \text{and} \quad A_M(u, \Omega') := A_M(u, \Omega') \cap \bar{A}_M(u, \Omega'). \]
In addition, we define:
\[ \overline{\Theta}(u, \Omega', x) := \inf \{ M > 0; x \in \overline{G}_M(u, \Omega') \}. \]
We also can define \( \Theta(u, \Omega', x) \). Finally, we define:
\[ \Theta(u, \Omega', x) := \sup \{ \overline{\Theta}(u, \Omega', x), \bar{\Theta}(u, \Omega', x) \}. \]

The first step towards \( W^{2,p} \)-estimates up to the boundary are estimates for paraboloids on the boundary. Hence, in this section, we will prove an appropriated power decay on the boundary for \( L^n(G_t(u, \Omega)) \). For this end, we will need the following standard technical result (cf. [9]):

**Proposition 3.2.** Let \( 0 \leq h : \Omega \to \mathbb{R} \) be a measurable function, \( \mu_h(t) := L^n(\{ x \in \Omega : h(x) \geq t \}) \) its distribution function and \( \eta > 0 \), \( M \geq 1 \) constants. Then,
\[ h \in L^p(\Omega) \iff \sum_{j=1}^{\infty} M^{pj} \mu_h(\eta M^j) < \infty \]
for every \( p \in (0, +\infty) \). Particularly, there exists a constant \( C = C(n, \eta, M) \) such that
\[ C^{-1} \left( \sum_{j=1}^{\infty} M^{pj} \mu_h(\eta M^j) \right)^{\frac{1}{p}} \leq \| h \|_{L^p(\Omega)} \leq C \left( L^n(\Omega) + \sum_{j=1}^{\infty} M^{pj} \mu_h(\eta M^j) \right)^{\frac{1}{p}}. \]

Once we have such estimates for paraboloids on the boundary, the ideas will follow as the ones of Caffarelli’s original proof for the interior estimates (cf. [9]). In effect, consider the distribution function of \( \Theta \), i.e., \( \mu_{\Theta, \Omega'}(t) := L^n(\{ x \in \Omega' : \Theta(x) > t \}) \). Then, one check that
\[ \mu_{\Theta, \Omega'}(t) = L^n(A_t(u, \Omega')). \]
Moreover, as an application of Proposition 3.2 we have
\[ \Theta(u, \Omega', \cdot) \in L^p(\Omega') \iff \sum_{j=1}^{\infty} M^{pj} L^n(A_{\eta M^j(u, \Omega')}) < \infty. \]
Finally, from [9] Proposition 1.1] we conclude
\[ \| D^2 u \|_{L^p(\Omega')} \leq C(\eta, M, p) \left( L^n(\Omega) + \sum_{j=1}^{\infty} M^{pj} L^n(A_{\eta M^j(u, \Omega')}) \right). \]
Therefore, to obtain the desired \( W^{2,p} \)-estimates in \( \Omega' \) it is enough to prove the corresponding summability for \( \sum_{j=1}^{\infty} M^{pj} L^n(A_{\eta M^j(u, \Omega')}) \).

We gather only a few elements involved in the proof of Theorem 1.2 despite the fact that such results are well-known in the literature. We observe that such an \( a \ priori \) estimate is independent of further assumptions on the operator \( F \), and it follows merely from uniform ellipticity and the integrability of the source term. Thus, the proof is omitted in what follows. See [9] Lemma 7.8] and [21] Lemma 2.7] for details.
Assumption A4) and \( \| \mathbf{A}(x, \mathbf{u}) \|_{\mathbb{L}^{\infty}(\Omega)} \leq 1 \). Then, there exist universal constants \( C > 0 \) and \( \mu > 0 \) such that
\[
\| f \|_{\mathbb{L}^{\infty}(\mathbb{B}^+_{1,\tau}, \mathbb{R}^n)} \leq 1
\]
for any \( x_0 \in \mathbb{B}_9 \cap \mathbb{R}^n \) and \( t > 1 \).

Proposition 3.3 yields key information on the measure of \( \mathcal{G}(u, \Omega) \cap (\{Q_n^{n-1} \times (0,1) + x_0\}) \), provided
\[
\mathcal{G}(u, \Omega) \cap (\{Q_n^{n-1} \times (0,2) + x_0\}) \neq \emptyset.
\]

The proof of Theorem 1.2 will be split into two main steps: Firstly, we will investigate equations governed by operators without dependence on the function or gradient entry. In the sequel, we reduce the analysis for operators with full dependence. The first step towards such estimates is the following Proposition.

**Proposition 3.4.** Let \( u \) be a normalized viscosity solution of
\[
\begin{cases}
F(D^2 u, x) = f(x) & \text{in } \mathbb{B}^+_1, \\
B(x, u, Du) = g(x) & \text{on } T_1,
\end{cases}
\]
where \( \beta, \gamma, \delta \in C^{1,\alpha}(T_1) \) with \( \beta \cdot \nabla u \geq \mu_0 \) for some \( \mu_0 > 0 \), \( \gamma \leq 0 \) and \( f \in L^p(\mathbb{B}^+_1) \cap C^0(\mathbb{B}^+_1), \) for \( n \leq p < \infty \). Further, assume that assumptions (A1)-(A4) are in force. Then, there exist positive constants \( \psi_0 \) and \( r_0 \) depending on \( n, \lambda, \beta, \gamma, \delta \), such that if
\[
\left( \int_{\mathbb{B}^+_1(x_0) \cap \mathbb{B}^+_1} \psi_F(x, x_0)^p dx \right)^{\frac{1}{p}} \leq \psi_0
\]
for any \( x_0 \in \mathbb{B}^+_1 \) and \( r \in (0, r_0) \), then \( u \in W^{2,p}(\mathbb{B}^+_2) \) and
\[
\| u \|_{W^{2,p}(\mathbb{B}^+_2)} \leq C \left( \| u \|_{\mathbb{L}^{\infty}(\mathbb{B}^+_1)} + \| f \|_{L^p(\mathbb{B}^+_1)} + \| g \|_{C^{1,\alpha}(T_1)} \right),
\]
where \( C = C(n, \lambda, \beta, \gamma, \delta, p, \| \gamma \|_{C^{1,\alpha}(T_1)}, r_0 > 0) \).

In order to prove Proposition 3.4, we need a number of auxiliary results:

**Proposition 3.5.** Assume that assumptions (A1)-(A4) hold. Let \( \mathbb{B}^+_{1,\tau} \subset \Omega \subset \mathbb{R}^n \) and \( u \) be a viscosity solution of
\[
\begin{cases}
F_r(D^2 u, x) = f(x) & \text{in } \mathbb{B}^+_{1,\tau}, \\
B(x, u, Du) = g(x) & \text{on } T_{1,\tau},
\end{cases}
\]
Assume further that max \( \| f \|_{L^\infty(\mathbb{B}^+_{1,\tau})} \), \( \tau \) \( \leq \epsilon \). Finally, suppose for some \( \tilde{x}_0 \in \mathbb{B}_9 \cap \{ x_n \geq 0 \} \) the following
\[
\mathcal{G}(u, \Omega) \cap (\{Q_n^{n-1} \times (0,2) + \tilde{x}_0\}) \neq \emptyset.
\]
Then,
\[
\mathcal{L}^n (\mathcal{G}(u; \Omega) \cap (\{Q_n^{n-1} \times (0,1) + x_0\})) \geq 1 - \epsilon_0,
\]
where \( x_0 \in \mathbb{B}_9 \cap \{ x_n \geq 0 \}, M > 1 \) depends only on \( n, \lambda, \beta, \gamma, \delta \), \( \| \gamma \|_{C^{1,\alpha}(T_{1,\tau})} \), \( C_1 \) (from Assumption A4) and \( \| g \|_{C^{1,\alpha}(T_{1,\tau})} \), and \( \epsilon_0 \in (0, 1) \).
Proof. Consider $x_1 \in G_1(u, \Omega) \cap (Q_{12}^{n-1} \times (0, 2) + \bar{x}_0)$. Then, there exist paraboloids with an opening $t = 1$ touching $u$ at $x_1$ from above and below, this is,

$$-\frac{1}{2}|x - x_1|^2 \leq u(x) - \ell(x) \leq \frac{1}{2}|x - x_1|^2$$

for $x \in \Omega$ and an affine function $\ell$.

Now, we set

$$v(x) = \frac{u(x) - \ell(x)}{C_u},$$

where $C_u > 0$ is a dimensional constant selected (large enough) so that $\|v\|_{L^\infty(B_{12\sqrt{\pi}}^+)} \leq 1$ and

$$-|x|^2 \leq v(x) \leq |x|^2 \quad \text{in} \quad \Omega \setminus B_{12\sqrt{\pi}}^+.$$

Next, we observe that $v$ is a viscosity solution to

$$\begin{align*}
\tilde{F}_t(D^2v, x) &= \tilde{f}(x) \quad \text{in} \quad B_{12\sqrt{\pi}}^+; \\
B(x, v, Dv) &= C_u \left\{ g(x) - \beta \cdot D\ell - \gamma \ell \right\} \quad \text{on} \quad T_{12\sqrt{\pi}}.
\end{align*}$$

where

$$\tilde{F}_t(X, x) := \frac{1}{C_u} F_t(C_u x, x) \quad \text{and} \quad \tilde{f}(x) := \frac{1}{C_u} f(x)$$

Now, we consider $h$ the function $c$-close to $u$, coming from Lemma 2.12 this is, let $h \in C^{1,1}(B_{13\sqrt{\pi}}^+) \cap C^0(B_{13\sqrt{\pi}}^+) \subset \mathbb{R}$ be solution of

$$\begin{align*}
\tilde{F}_t^4(D^2h, 0) &= 0 \quad \text{in} \quad B_{13\sqrt{\pi}}^+; \\
B(x, h, Dh) &= \frac{1}{C_u} \left\{ g(x) - \beta(x) \cdot D\ell(x) - \gamma(x)\ell(x) \right\} \quad \text{on} \quad T_{13\sqrt{\pi}}.
\end{align*}$$

such that

$$\|v - h\|_{L^\infty(B_{13\sqrt{\pi}}^+)} \leq \hat{\delta} < 1$$

Notice that $\beta \cdot D\ell \in C^{1,\alpha}(T_{14\sqrt{\pi}})$ since $\beta \in C^{1,\alpha}(\overline{T}_{14\sqrt{\pi}})$ and $D\ell$ is a constant vector field. Hence, the A.B.P. Maximum Principle (Lemma 2.6) assures that

$$\begin{align*}
\|h\|_{L^\infty(B_{13\sqrt{\pi}}^+)} &\leq \|v\|_{L^\infty(B_{13\sqrt{\pi}}^+ \setminus T_{13\sqrt{\pi}})} + \frac{C}{C_u} \left[ \|g\|_{L^\infty(T_{13\sqrt{\pi}})} + \|D\ell\|_{L^\infty(T_{14\sqrt{\pi}})} + \|\gamma\|_{L^\infty(T_{13\sqrt{\pi}})} \right] \\
&\leq C(n, \|\ell\|_{L^\infty(T_{14\sqrt{\pi}})}, \|g\|_{L^\infty(T_{14\sqrt{\pi}})}, \|\gamma\|_{L^\infty(T_{13\sqrt{\pi}})}, \|D\ell\|_{L^\infty(T_{14\sqrt{\pi}})}, \|\gamma\|_{L^\infty(T_{13\sqrt{\pi}})}) \\
&:= \tilde{C}
\end{align*}$$

In consequence, condition (A4) ensures that

$$\|h\|_{C^{1,1}(B_{12\sqrt{\pi}}^+)} \leq C(C_1, \tilde{C}) \quad \text{for some} \quad N = N(C_1, \tilde{C}) > 1, \quad \text{and we extended} \quad h \mid_{B_{12\sqrt{\pi}}^+} \quad \text{(continuously) outside} \quad B_{12\sqrt{\pi}}^+ \quad \text{such that} \quad h = v \text{ outside} \quad B_{13\sqrt{\pi}}^+ \quad \text{and} \quad \|v - h\|_{L^\infty(\Omega)} = \|v - h\|_{L^\infty(B_{12\sqrt{\pi}}^+)}. \quad \text{For this reason,}

$$

$$

\|v - h\|_{L^\infty(\Omega)} \leq \tilde{C}(C_1, \tilde{C})$$

and

$$-(\tilde{C}(C_1, \tilde{C}) + |x|^2) \leq h(x) \leq \tilde{C}(C_1, \tilde{C}) + |x|^2 \quad \text{in} \quad \Omega \setminus B_{12\sqrt{\pi}}^+.$$
Therefore, there exists $M_0 = M_0(C_1, \hat{C}) \geq N > 1$, for which
\[
\mathcal{A}_{M_0}(b, \Omega) \cap ((Q_1^{n-1} \times (0, 1)) + x_0) = \emptyset.
\]
Summarizing,
\[
(3.1) \quad (Q_1^{n-1} \times (0, 1) + x_0) \subset \mathcal{G}_{M_0}(b, \Omega).
\]
Now, we define
\[
w(x) := \frac{1}{2C_5} \tilde{v}(x).
\]
Therefore, $w$ fulfills the assumptions of Proposition 3.1. Thus, we obtain for $t > 1$ the following
\[
\mathcal{L}^n \left( \mathcal{A}_t(w, \Omega) \cap ((Q_1^{n-1} \times (0, 1)) + x_0) \right) \leq C t^{-\mu} \quad \text{(for a } \mu \text{ universal)}.
\]
By using $\mathcal{A}_t(u) \subset \mathcal{A}_t(w) \cup \mathcal{A}_t(h)$ and (3.1) we conclude that
\[
\mathcal{L}^n \left( \mathcal{G}_t(u - h, \Omega) \cap ((Q_1^{n-1} \times (0, 1)) + x_0) \right) \geq 1 - C \epsilon^{-\mu}.
\]
Finally, we conclude that
\[
\mathcal{L}^n \left( \mathcal{G}_t(u - h, \Omega) \cap ((Q_1^{n-1} \times (0, 1)) + x_0) \right) \geq 1 - C \epsilon^{-\mu}.
\]
The proof is completed by choosing $\epsilon < 1$ in an appropriate manner and by setting $M = 2M_0$.

**Lemma 3.6.** Given $\epsilon_0 \in (0, 1)$ and let $u$ be a normalized viscosity solution to
\[
\begin{align*}
F_t(D^2u, x) &= f(x) \quad \text{in } B_{14\sqrt{\pi}}^+,
B(x, u, Du) &= g(x) \quad \text{on } T_{14\sqrt{\pi}}.
\end{align*}
\]
Assume that (A1)-(A4) hold true and that $f$ is extended by zero outside $B_{14\sqrt{\pi}}^+$. For $x \in B_{14\sqrt{\pi}}$, let
\[
\max \left\{ \tau, \|f\|_{L^n(B_{14\sqrt{\pi}})} \right\} \leq \epsilon
\]
for some $\epsilon > 0$ depending only on $n, \epsilon_0, \lambda, \Lambda, \mu_0, \alpha$. Then, for $k \in \mathbb{N} \setminus \{0\}$ we define
\[
\begin{align*}
\mathcal{A} &:= \mathcal{A}_{M^{k+1}}(u, B_{14\sqrt{\pi}}^+) \cap \left( Q_1^{n-1} \times (0, 1) \right)
\mathcal{B} &:= \left( \mathcal{A}_{M^k}(u, B_{14\sqrt{\pi}}^+) \cap \left( Q_1^{n-1} \times (0, 1) \right) \right) \cup \{ x \in Q_1^{n-1} \times (0, 1); M(f^n) \geq (C_0 M^k)^n \},
\end{align*}
\]
where $M = M(n, C_0) > 1$. Then,
\[
\mathcal{L}^n(\mathcal{A}) \leq \epsilon_0(n, \epsilon, \lambda, \Lambda) \mathcal{L}^n(\mathcal{B}).
\]

**Proof.** We will use Lemma 3.1. Observe that $\mathcal{A} \subset B \subset (Q_1^{n-1} \times (0, 1))$ and of the Proposition 3.1 we conclude that $\mathcal{L}^n(\mathcal{A}) \leq \delta < 1$ provided we choose $\delta = \epsilon_0$. Thus, it remains to show the following implication: for dyadic cubes $Q$
\[
\mathcal{L}^n(\mathcal{A} \cap Q) > \epsilon_0 \mathcal{L}^n(Q) \quad \Rightarrow \quad \tilde{Q} \subset B.
\]
For that purpose, assume that for some $i \geq 1$, $Q = \left( Q_1^{n-1} \times (0, \frac{1}{2^{i-1}}) \right) + x_0$ is a dyadic cube with predecessor $\tilde{Q} = \left( \frac{Q_1^{n-1}}{2^{i-1}} \times (0, \frac{1}{2^{i-1}}) \right) + \tilde{x}_0$. Now, we assume that $Q$ satisfies
\[
(3.2) \quad \mathcal{L}^n(\mathcal{A} \cap Q) = \mathcal{L}^n(\mathcal{A}_{M^{k+1}}(u, B_{14\sqrt{\pi}}^+) \cap Q) > \epsilon_0 \mathcal{L}^n(Q),
\]
however the inclusion $\tilde{Q} \subset B$ does not hold. In consequence, there must be $x_1 \in \tilde{Q} \setminus B$, i.e.,
\[
(3.3) \quad x_1 \in \tilde{Q} \cap G_M(u, B_{14\sqrt{\pi}}^+) \quad \text{and} \quad M(f^n)(x_1) < (C_0 M^k)^n.
\]
In the sequel, we must split the analysis into two cases:
In the sequel, we will establish the proof of Proposition 3.4.

Proposition 3.5 we obtain a contradiction to (3.2). This completes the proof of the Lemma.

proceed similarly to the first part of the proof. Finally, applying [39, Lemma 5.2] instead of

In this case, we conclude that $B_{\sqrt{n}2}$.

Thus,

Therefore, we have shown that the assumptions of Proposition 3.5 are true. Hence, it follows:

Now notice that $\tilde{\tau}$ fulfills $C^{1,1}$-estimates with the same constant as $F^\delta$. Moreover, from (3.3) we obtain

As a result, $\|\tilde{f}\|_{L^\infty(B_{\sqrt{n}2}^+)} \leq \epsilon$ provided we select $C_0$ small enough in (3.3). In addition, from (3.3) we conclude that

Furthermore, $|x_0 - \tilde{x}_0| \leq \frac{1}{2^7} \sqrt{n}$ implies $|2^i(\tilde{x}_0 - (x_0, 0))| < 9\sqrt{n}$.

Therefore, we have shown that the assumptions of Proposition 3.5 are true. Hence, it follows:

Thus,

which contradicts (3.2).

Case 2: If $|x_0 - (x_0', 0)| \geq \frac{1}{2^7} \sqrt{n}$.

In this case, we conclude that $B_{\sqrt{n}2} \supset \left( x_0 + \frac{1}{2^7} e_n \right) \subset B_{\sqrt{n}2}$, where $e_n$ is the $n$th unit vector of canonical base. Now, by defining the transformation: $T(y) := (x_0 + \frac{1}{2^7} e_n) + \frac{1}{2^7} y$, we proceed similarly to the first part of the proof. Finally, applying [39, Lemma 5.2] instead of Proposition 3.5 we obtain a contradiction to (3.2). This completes the proof of the Lemma.

In the sequel, we will establish the proof of Proposition 3.4.
Proof of Proposition 3.4. Fix $x_0 \in B_{1/2} \cap \{x_n \geq 0\}$. When $x_0 \in T_{1/2}$, $0 < r < \frac{1-\|x_0\|}{14\sqrt{n}}$ define:

$$\kappa := \frac{\epsilon r}{\epsilon r - 1 \|u\|_{L^\infty(B_{14r\sqrt{n}}(x_0))} + \|f\|_{L^n(B_{14r\sqrt{n}}(x_0))} + \epsilon r - 1 \|g\|_{C^{1,\alpha}(T_{14r\sqrt{n}}(x_0))}}$$

where the constant $\epsilon = \epsilon(n, \epsilon_0, \lambda, \Lambda, \mu_0, \alpha, \|\beta\|_{C^{1,\alpha}(T_{1/2})})$ is as the one in Proposition 3.5 and $\epsilon_0 \in (0, 1)$ will be determined a posteriori. Now, we define: $\tilde{u}(y) := \frac{r}{\kappa} u(x_0 + ry)$. Then, $\tilde{u}$ is a normalized viscosity solution to

\[
\begin{cases}
\tilde{F}(D^2\tilde{u}, y) = \tilde{f}(x) & \text{in } B^+_{14\sqrt{n}}, \\
\tilde{B}(y, \tilde{u}, D\tilde{u}) = \tilde{g}(x) & \text{on } T_{14\sqrt{n}},
\end{cases}
\]

where

\[
\begin{align*}
\tilde{F}(X, y) &:= \kappa F \left( \frac{1}{r} X, ry + x_0 \right) \\
\tilde{f}(y) &:= \kappa f(x_0 + ry) \\
\tilde{B}(y, s, \tilde{v}) &:= \tilde{\beta}(y) \cdot \tilde{v} + \tilde{\gamma}(y)s \\
\tilde{\beta}(y) &:= \beta(x_0 + ry) \\
\tilde{\gamma}(y) &:= r \gamma(x_0 + ry) \\
\tilde{g}(y) &:= \frac{r}{\kappa} g(x_0 + ry).
\end{align*}
\]

Hence, $\tilde{F}$ fulfills (A1)-(A4). Moreover, for all $n$ and $0 \leq \|\cdot\|_{L^n(B^+_{14\sqrt{n}})} \leq 1$,

(3.4) \[\|\tilde{f}\|_{L^n(B^+_{14\sqrt{n}})} = \frac{\kappa}{r} \|f\|_{L^n(B^+_{14\sqrt{n}})} \leq \epsilon \text{ and } \|\tilde{\beta}\|_{C^{1,\alpha}(T_{14\sqrt{n}})} \leq 1,\]

which ensures that the hypotheses of Lemma 3.6 are in force. Now, let $M > 0$ and $C_0 > 0$ be as in the Lemma 3.6 and choose $\epsilon_0 := \frac{1}{2Mr}$. Now, for $k \geq 0$ we define

\[
\begin{align*}
\alpha_k &:= \mathcal{L}^n \left( \mathcal{A}_{\beta_k}(\tilde{u}, B^+_{14\sqrt{n}}) \cap (Q_1^{n-1} \times (0, 1)) \right) \\
\beta_k &:= \mathcal{L}^n \left( \left\{ x \in (Q_1^{n-1} \times (0, 1)): \mathcal{M}(\tilde{f}^n)(x) \geq (C_0 M^k)^n \right\} \right).
\end{align*}
\]

As a result, Lemma 3.6 implies that $\alpha_{k+1} \leq \epsilon_0 (\alpha_k + \beta_k)$ and thus

(3.5) \[\alpha_k \leq \epsilon_0^k + \sum_{i=0}^{k-1} \epsilon_0^{i-k} \beta_i.\]

On the other hand, from assumption (A2), we have $\tilde{f}^n \in L_{(B^+_{14\sqrt{n}})},$ consequently $\mathcal{M}(\tilde{f}^n) \in L_{(B^+_{14\sqrt{n}})}$. Thus, by (3.4)

\[
\|\mathcal{M}(\tilde{f}^n)\|_{L_{(B^+_{14\sqrt{n}})}} \leq C(n, p)\|\tilde{f}^n\|_{L_{(B^+_{14\sqrt{n}})}} \leq C(n, p)\|\tilde{f}\|^n_{L_{(B^+_{14\sqrt{n}})}} \leq C.
\]

Therefore, by Proposition 3.2 we obtain

(3.6) \[\sum_{k=1}^{\infty} M^k \beta_k \leq C(n, p).\]

Finally, taking into account the choice of $\epsilon_0$, (3.5) and (3.6) we conclude that

\[
\sum_{k=1}^{\infty} M^k \alpha_k \leq \sum_{k=1}^{\infty} 2^{-k} \left( \sum_{k=0}^{\infty} M^k \beta_k \right) \cdot \left( \sum_{k=1}^{\infty} M^k \epsilon_0 \right) \leq C(n, p),
\]

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Thus, we can apply Proposition 3.4 and conclude that
\[ \|D^2u\|_{L^p(B_2^+(x_0))} \leq C(n, p, M) \]
and consequently
\[ \|D^2u\|_{L^p(B_1^+(x_0))} \leq C(n, \lambda, \Lambda, p, r) \left( \|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)} + \|g\|_{C^{1,\alpha}(T_1)} \right). \]

On the other hand, if \( x_0 \in B_{1/2}^+ \), we can use the result of interior estimates (cf. [39, Theorem 6.1], see also [12]). Finally, by combining interior and boundary estimates, we obtain the desired results by using a standard covering argument. This completes the proof of the Proposition.

\[ \square \]

**Corollary 3.7.** Let \( u \) be a bounded viscosity solution of
\[
\begin{align*}
F(D^2u, Du, u, x) &= f(x) \quad \text{in} \quad B_1^+, \\
B(x, u, Du) &= g(x) \quad \text{on} \quad T_1,
\end{align*}
\]
where \( \beta, \gamma, \alpha \in C^{1,\alpha}(T_1) \) with \( \beta \cdot \nabla \) \( u \geq \mu_0 \) for some \( \mu_0 > 0, \gamma \leq 0 \). Further, assume that (A1)-(A4) are in force. Then, there exists a constant \( C > 0 \) depending on \( n, \lambda, \Lambda, p, C_1 \), such that \( u \in W^{2,p}(B_1^+) \) and
\[ \|u\|_{W^{2,p}(B_1^+)} \leq C \cdot \left( \|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)} + \|g\|_{C^{1,\alpha}(T_1)} \right). \]

**Proof.** Notice that \( u \) is also a viscosity solution of
\[
\begin{align*}
\tilde{F}(D^2u, x) &= \tilde{f}(x) \quad \text{in} \quad B_1^+, \\
B(x, u, Du) &= g(x) \quad \text{on} \quad T_1,
\end{align*}
\]
where \( \tilde{F}(X, x) := F(X, 0, 0, x) \) and \( \tilde{f} \) is a function satisfying
\[ |\tilde{f}| \leq \sigma |Du| + \xi |u| + |f|. \]
Thus, we can apply Proposition 3.4 and conclude that
\[ \|u\|_{W^{2,p}(B_1^+)} \leq C \cdot \left( \|u\|_{L^\infty(B_1^+)} + \|\tilde{f}\|_{L^p(B_1^+)} + \|g\|_{C^{1,\alpha}(T_1)} \right). \]

Now, using the same reasoning as in [2, Corollary 4.5] we conclude that \( u \in W^{1,p}(B_1^+) \) and
\[ \|u\|_{W^{1,p}(B_1^+)} \leq C \cdot (\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)}). \]

Finally, by combining (3.7) and (3.8) we complete the desired estimate.

\[ \square \]

**Corollary 3.8.** Let \( u \) be a bounded \( L^p \)-viscosity solution of
\[
\begin{align*}
F(D^2u, Du, u, x) &= f(x) \quad \text{in} \quad B_1^+, \\
B(x, u, Du) &= g(x) \quad \text{on} \quad T_1,
\end{align*}
\]
where \( \beta, \gamma, \alpha \in C^{1,\alpha}(T_1) \) with \( \beta \cdot \nabla \) \( u \geq \mu_0 \) for some \( \mu_0 > 0, \gamma \leq 0 \) and \( f \in L^p(B_1^+) \), for \( n \leq p < \infty \). Further, assume that \( F^2 \) satisfies (A4) and \( f \) fulfills the condition (SC). Then, there exist positive constants \( \beta_0 = \beta_0(n, \lambda, \Lambda, p), r_0 = r_0(n, \lambda, \Lambda, p) \) and \( C = C(n, \lambda, \Lambda, p, r_0) > 0 \), such that if
\[ \left( \int_{B_{r_0}(x_0) \cap B_1^+} \psi_{F}(x, x_0)^p dx \right)^{1/p} \leq \psi_0 \]
for any \( x_0 \in B_1^+ \) and \( r \in (0, r_0) \), then \( u \in W^{2,p}(B_{1/2}^+) \) and
\[ \|u\|_{W^{2,p}(B_{1/2}^+)} \leq C \cdot \left( \|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)} + \|g\|_{C^{1,\alpha}(T_1)} \right). \]
Therefore, the assumptions of the Theorem 3.4 are in force. As a result, which by Lemma 2
Thus,
Theorem 4.3] for details). We will approximate functions:
Proof of Theorem 1.2.

Proof. It is sufficient to prove the result for equations with no dependence on $Du$ and $u$ (see [51
Theorem 4.3] for details). We will approximate $f$ in $L^p$ by functions $f_j \in C^\infty(\overline{B_1^+}) \cap L^p(B_1^+)$ such that $f_j \to f$ in $L^p(B_1^+)$, and also approximate $g$ by a sequence $(g_j)$ in $C^{1,\alpha}(T_1)$ with the property that $g_j \to g$ in $C^{1,\alpha}(T_1)$. By Theorems of Uniqueness and Existence (Theorem 2.8 and 2.9), there exists a sequence of functions $u_j \in C^0(B_1^+)$, which they are viscosity solutions of following family of PDEs:

\[
\begin{cases}
F(D^2 u_j, x) = f_j(x) & \text{in } B_1^+,
B(x, u_j, Du_j) = g_j(x) & \text{on } T_1,
\end{cases}
\]

Thus, $u_j(x) = u(x)$ on $\partial B_1^+ \setminus T_1$.

Therefore, the assumptions of the Theorem 3.4 are in force. As a result,

\[
\|u_j\|_{W^{2,p}(\overline{B_1^+/2})} \leq C(\text{universal}). \left(\|u_j\|_{L^\infty(B_1^+)} + \|f_j\|_{L^p(B_1^+)} + \|g_j\|_{C^{1,\alpha}(T_1)}\right),
\]

for a constant $C > 0$. Furthermore, a standard covering argument also yields $u_j \in W^{2,p}(B_1^+)$. From Lemma 2.6 $(u_j)_{j \in \mathbb{N}}$ is uniformly bounded in $W^{2,p}(B_1^+/2)$ for $\rho \in (0,1)$.

Once again, we can apply Lemma 2.6 and obtain

\[
\|u_j - u_k\|_{L^\infty(B_1^+)} \leq C(n, \lambda, \Lambda, \mu_0)(\|f_j - f_k\|_{L^p(B_1^+)} + \|g_j - g_k\|_{C^{1,\alpha}(T_1)}).
\]

Thus, $u_j \to u_\infty$ in $C^0(B_1^+)$. Moreover, since $(u_j)_{j \in \mathbb{N}}$ is bounded in $W^{2,p}(B_1^+/2)$ we obtain $u_j \to u_\infty$ weakly in $W^{2,p}(B_1^+/2)$. Thus,

\[
\|u_\infty\|_{W^{2,p}(\overline{B_1^+/2})} \leq C \cdot \left(\|u_\infty\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)} + \|g\|_{C^{1,\alpha}(T_1)}\right).
\]

Finally, stability results (see Lemma 2.5) ensure that $u_\infty$ is an $L^p$-viscosity solution to

\[
\begin{cases}
F(D^2 u_\infty, x) = f(x) & \text{in } B_1^+,
B(x, u_\infty, Du_\infty) = g(x) & \text{on } T_1,
\end{cases}
\]

Thus, $w := u_\infty - u$ fulfills

\[
\begin{cases}
w \in S_n(\frac{1}{2}, \Lambda, 0) & \text{in } B_1^+,
B(x, w, Dw) = 0 & \text{on } T_1,
w = 0 & \text{on } \partial B_1^+ \setminus T_1.
\end{cases}
\]

which by Lemma 2.6 we can conclude that $w = 0$ in $\overline{B_1^+} \setminus T_1$. By continuity, $w = 0$ in $\overline{B_1^+}$ which completes the proof.

Finally, we are now in position to prove the Theorem 1.2.

Proof of Theorem 1.2. Based on standard reasoning (cf. [51]) it is important to highlight that it is always possible to perform a change of variables in order to flatten the boundary. For this end, consider $x_0 \in \partial \Omega$. Since $\partial \Omega \in C^{2,\alpha}$ there exists a neighborhood of $x_0$, which we will label $\mathcal{V}(x_0)$ and a $C^{2,\alpha}$-diffeomorphism $\Phi: \mathcal{V}(x_0) \to B_1(0)$ such that $\Phi(x_0) = 0$ and $\Phi(\Omega \cap \mathcal{V}(x_0)) = B_1^+$. Now, for $\tilde{\varphi} \in W^{2,p}(B_1^+)$ we set $\varphi = \tilde{\varphi} \circ \Phi \in W^{2,p}(\mathcal{V}(x_0))$. Thus, we obtain that

\[
D\varphi = (D \tilde{\varphi} \circ \Phi) D\Phi \quad \text{and} \quad D^2 \varphi = D\Phi^t \cdot (D^2 \tilde{\varphi} \circ \Phi) \cdot D\Phi + ((D \tilde{\varphi} \circ \Phi) \partial_{ij} \Phi)_{1 \leq i, j \leq n}.
\]
Moreover, \( \tilde{u} := u \circ \Phi^{-1} \in C^{0}(B^+_1) \). Next, we observe that \( \tilde{u} \) is an \( L^p \)-viscosity solution to

\[
\begin{cases}
\tilde{F}(D^2\tilde{u}, D\tilde{u}, \tilde{u}, y) = \tilde{f}(x) & \text{in } B^+_1, \\
\tilde{B}(y, \tilde{u}, D\tilde{u}) = \tilde{g}(x) & \text{on } T_1.
\end{cases}
\]

where for \( y = \Phi^{-1}(x) \) we have

\[
\begin{align*}
\tilde{F}(D^2\tilde{v}, D\tilde{v}, x) & := F(D\Phi^1(y)D^2\tilde{v}D\Phi(y) + D\tilde{v}D^2\Phi(y), D\tilde{v}D\Phi(y), \tilde{u}, y) \\
\tilde{f}(x) & := f \circ \Phi^{-1}(x) \\
\tilde{B}(x, s, \tilde{v}) & := \beta(x) \cdot \tilde{v} + \gamma(x)s \\
\beta(x) & := (\beta \circ \Phi^{-1}) \cdot (D\Phi \circ \Phi^{-1})^t \\
\gamma(x) & := (\gamma \circ \Phi^{-1}) \cdot (D\Phi \circ \Phi^{-1})^t \\
\tilde{g}(x) & := g \circ \Phi^{-1}
\end{align*}
\]

Furthermore, note that \( \tilde{F}(X, \varsigma, \eta, y) = F(D\Phi^1(y) \cdot X \cdot D\Phi(y) + \varsigma D^2\Phi, \varsigma D\Phi(y), \eta, y) \) is a uniformly elliptic operator with ellipticity constants \( \lambda C(\Phi), \Lambda C(\Phi) \). Thus,

\[
\tilde{F}^2(X, \varsigma, \eta, x) = F^2(D\Phi^1(\Phi^{-1}(x)) \cdot X \cdot D\Phi^{-1}(x)) + \varsigma D^2\Phi(\Phi^{-1}(x)), 0, 0, \Phi^{-1}(x) \).
\]

In consequence, we conclude that \( \psi_{\phi}(x, x_0) \leq C(\Phi)\psi_{\phi^2}(x, x_0) \), ensuring that \( \tilde{F} \) falls into the assumptions of Corollary 3.8. This finishes the proof, as well as establishes the proof of the Theorem 1.2. \( \square \)

## 4 BMO type estimates: Proof of Theorem 1.4

This section will be devoted to establishing boundary BMO type estimates. Before proving Theorem 1.4 we will present some key results that play a crucial role in our strategy.

**Lemma 4.1 (Approximation Lemma II).** Let \( g \in C^{1,\psi}(T_1) \) such that \( \|g\|_{C^{1,\psi}(T_1)} \leq C_g \) and \( \beta \in C^{1,\psi}(T_1) \). Let \( F \) and \( F^\infty \) be \( \left( \frac{n}{p+1}, \Lambda \right) \)-elliptic operators. Given \( \delta_0 > 0 \), there exists \( \epsilon_0 = \epsilon_0(\delta_0, n, \lambda, \Lambda, C_g) < 1 \) such that, if

\[
\max \left\{ \frac{|F(X, x) - F^\infty(X, x_0)|}{\|X\| + 1}, \|f\|_{L^p(B^+_1) \cap \text{BMO}(B^+_1)} \right\} \leq \epsilon_0,
\]

then any two \( L^p \)-viscosity solutions \( u \) and \( v \) of

\[
\begin{cases}
F(D^2u, x) = f(x) & \text{in } B^+_1, \\
\beta \cdot Du = g(x) & \text{on } T_1,
\end{cases}
\]

and

\[
\begin{cases}
F^\infty(D^2\mathfrak{h}, x_0) = 0 & \text{in } B^+_1, \\
\beta \cdot D\mathfrak{h} = g(x) & \text{on } T_1,
\end{cases}
\]

satisfy

\[
\|u - \mathfrak{h}\|_{L^\infty(\delta_0^+)} \leq \delta_0.
\]

**Proof.** The proof includes the same lines as Lemma 2.12 along with minor changes. \( \square \)

**Lemma 4.2 (A quadratic approximation).** Under the assumptions from Lemma 4.1, assume that \( F^\infty \) satisfies (Ad). Let \( u \) be an \( L^p \)-viscosity solution to

\[
\begin{cases}
F(D^2u, x) = f(x) & \text{in } B^+_1, \\
\beta \cdot Du = g(x) & \text{on } T_1.
\end{cases}
\]

Then, there exist universal constants \( C_\mathfrak{h} > 0 \) and \( r_0 \in (0, 1/2) \), as well as a quadratic polynomial \( \mathfrak{P} \), with \( \|\mathfrak{P}\|_{L^\infty(B^+_1)} \leq C_\mathfrak{h} \) such that

\[
\sup_{B^+_r} |u(x) - \mathfrak{P}(x)| \leq r^2 \quad \text{for every } r \leq r_0.
\]
Proof. Firstly, take \( \hat{\delta}_0 \in (0,1) \), to be chosen a posteriori, and apply Lemma 4.1 to obtain \( \epsilon_0 > 0 \) and an \( L^p \)-viscosity solution to

\[
\begin{align*}
F^\infty(D^2 h, x) &= 0 \quad \text{in } B^+_1, \\
\beta \cdot D h &= g(x) \quad \text{on } T_1.
\end{align*}
\]

such that

\[
\sup_{B^+_1} |u - h| \leq \hat{\delta}_0.
\]

Remember that \( F^\infty \) fulfills assumption (A4)*, then \( h \) enjoys classical estimates. As a result, its Taylor’s expansion of second order, namely \( \mathcal{P} \), is well-defined. Furthermore, we have

\[
\sup_{B^+_\rho} |h - \mathcal{P}| \leq C_* \rho^{2+\psi} \quad \forall \, \rho \leq r_0.
\]

Now, we select \( 0 < r \ll 1 \) (small enough) in such a way that

\[
r \leq \min \left\{ r_0, \left( \frac{1}{10C_*} \right)^{\frac{1}{2}} \right\}.
\]

Thus,

\[
\sup_{B^+_\rho} |h - \mathcal{P}| \leq \frac{1}{10} r^2.
\]

On the other hand, we select \( \hat{\delta}_0 := \frac{1}{10} r^2 \). As a result, we obtain

\[
\sup_{B^+_\rho} |u - h| \leq \frac{1}{10} r^2.
\]

Finally, by combining the previous inequalities, we conclude that

\[
\sup_{B^+_\rho} |u - \mathcal{P}| \leq r^2.
\]

As a byproduct of the above result, we can yield a quadratic approximation at the recession level. The proof holds along with straightforward modifications in the Lemma 4.2. For this reason, we will omit it here.

Corollary 4.3. Assume the assumptions from Lemma 4.1. Then, there exist universal constants \( C_* > 0, \tau_0 > 0 \) and \( r > 0 \), such that if \( u \) is a (normalized) viscosity solution of

\[
\begin{align*}
F_\tau(D^2 u, x) &= f(x) \quad \text{in } B^+_1, \\
\beta(x) \cdot D u(x) &= g(x) \quad \text{on } T_1.
\end{align*}
\]

with \( \max \left\{ \tau, \| f \|_{-BMO(B^+_1)} \right\} \leq \tau_0 \), there exists a quadratic polynomial \( \mathcal{P} \), with \( \| \mathcal{P} \|_\infty \leq C_* \) satisfying

\[
\sup_{B^+_\rho} |u(x) - \mathcal{P}(x)| \leq r^2.
\]

We are already prepared to prove Theorem 1.4 in the face of such above auxiliary results.
Proof of Theorem 1.4. In fact, for $\kappa \in (0, 1)$ to be determined \textit{a posteriori}, we define $v(x) := \kappa u(x)$ such that $v$ is a (normalized) $L^p$–viscosity solution to

\[
\begin{aligned}
F_\tau(D^2v, x) &= \hat{f}(x) \quad \text{in } B_1^+,
\beta(x) \cdot Dv(x) &= \hat{g}(x) \quad \text{on } T_1.
\end{aligned}
\]

where $\tau := \kappa$, $\hat{f}(x) := \kappa f(x)$ and $\hat{g}(x) := \kappa g(x)$. Now, we are going to determine $\kappa$. This will be done in such a way that $\max \{\tau, \|\hat{f}\|_{p-BMO(B_1^+)}\} \leq \tau_0$, where $\tau_0$ comes from Corollary 4.3. Finally, we will establish the result for $v$, which will lead to the Theorem’s statement.

From now on, we will show that there exists a sequence of second order polynomials $(\mathcal{P}_k)_{k \in \mathbb{N}}$ such that

\[
\mathcal{P}_k(x) = \frac{1}{2} x^T \cdot A_k \cdot x + B_k \cdot x + C_k
\]

where $C_k > 0$ is a universal constant, $0 < r \ll 1$ comes from Lemma 4.2 and

\[
\sup_{B_1^+} |v(x) - \mathcal{P}_k(x)| \leq r^{2k}.
\]

The proof will follow by way of an induction process. Let us define $\mathcal{P}_0$ and $\mathcal{P}_{-1}$ to be $\mathcal{P}_0(x) = \mathcal{P}_{-1}(x) = \frac{1}{2} x^T \cdot X_0 \cdot x$, where $X_0 \in Sym(n)$ fulfills $F_1^2(X_0, x) = (\hat{f})_1$.

The first step of the argument, i.e., the case $k = 0$, is naturally verified. Now, suppose we have established the existence of such polynomials for $k = 0, 1, \ldots, j$. Then, we define the following auxiliary function $v_j : B_1^+ \rightarrow \mathbb{R}$ given by

\[
v_j(x) := \frac{(v - \mathcal{P}_j)(r^j x)}{r^{2j}},
\]

where we have by the induction assumption that $v_j$ is a (normalized) $L^p$–viscosity solution to

\[
\begin{aligned}
F_j(D^2v_j, x) &= \tilde{f}_j(x) \quad \text{in } B_1^+,
\beta(x) \cdot Dv_j(x) &= \tilde{g}_j(x) \quad \text{on } T_1.
\end{aligned}
\]

where

\[
\begin{aligned}
F_j(X, x) &= \tau F\left(\frac{1}{2}(X + A_j), r^j x\right),
\tilde{f}_j(x) &= \hat{f}(r^j x),
\tilde{g}_j(x) &= \hat{g}(r^j x) - \frac{1}{r} \tilde{\beta}(x) \cdot D\mathcal{P}_j(r^j x),
\tilde{\beta}(x) &= r^j \beta(r^j x)
\end{aligned}
\]

with

\[
\|f_j\|_{p-BMO(B_1^+)} = \sup_{0 < s \leq 1} \left( \int_{B_1^+} |f_j(x) - (f_j)_s|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \sup_{0 < s \leq 1} \left( \int_{B_{r^j}^+} |f(x) - (f)_s|^p \, dz \right)^{\frac{1}{p}}
\]

\[
\leq \|f\|_{p-BMO(B_1^+)} + \tau_0.
\]

Furthermore, since $F_1^2(A_j, x) = (\hat{f})_1$, the PDE with oblique boundary conditions

\[
\begin{aligned}
F_j(D^2\bar{u}, x) &= (\hat{f})_1 \quad \text{in } B_1^+,
\beta \cdot D\bar{u}(x) &= \hat{g}(x) \quad \text{on } T_1.
\end{aligned}
\]

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satisfies the same (up to the boundary) $C^{2, \psi}$ a priori estimates from the problem driven by $F$, and it is under the assumption of Corollary 4.3. Thus, there exists a quadratic polynomial $\tilde{P}$ with $\|\tilde{P}\|_{\infty} \leq C_{\#}$ such that

$$\sup_{B_{r}^{k+1}} |v(x) - \left[\tilde{P}(x) + r^{2k}\tilde{P}\left(\frac{x}{r}\right)\right]| \leq r^{2(k+1)}.
$$

Rewriting (4.3) back to the original domain yields

$$\sup_{B_{r}^{k+1}} \left|D^{2}v(z) - A_{k}\right| \leq C_{\#} r^{2k+1}.$$ 

Finally, by defining $\tilde{P}_{j+1}(x) := \tilde{P}_{j}(x) + r^{2j}\tilde{P}(r^{j}x)$ we check the $(k+1)^{th}$ step of induction. Furthermore, the required conditions (4.1) and (4.2) are satisfied. In conclusion, for $\rho > 0$, choose an integer $k$ such that $r^{k+1} \leq \rho \leq r^{k}$. Then, using Theorem 1.2 to $v_{k}$, we obtain

$$\sup_{\rho \in (0, 1/2)} \left(\int_{B_{\rho}^{k+1}} \left|D^{2}v(z) - A_{k}\right|^{p}dz\right)^{1/p} \leq C(\text{universal}).$$

Now, recall the general inequality:

$$\int_{B_{\rho}^{k+1}} \left|D^{2}v - \int_{B_{\rho}^{k+1}} D^{2}v dy\right|^{p} dx \leq 2\int_{B_{\rho}^{k+1}} \left|D^{2}v - A_{k}\right|^{p} dx.$$ 

In effect, by triangular inequality,

$$\left(\int_{B_{\rho}^{k+1}} \left|D^{2}v - \int_{B_{\rho}^{k+1}} D^{2}v dy\right|^{p} dx\right)^{1/p} \leq \left(\int_{B_{\rho}^{k+1}} \left|D^{2}v - A_{k}\right|^{p} dx\right)^{1/p} + \left(\int_{B_{\rho}^{k+1}} D^{2}v - A_{k} dy\right) \leq 2 \left(\int_{B_{\rho}^{k+1}} \left|D^{2}v - A_{k}\right|^{p} dx\right)^{1/p},$$

Therefore,

$$\|Dv\|_{p-BMO(\frac{B_{\rho}^{k+1}}{\rho^{k}})} := \sup_{\rho \in (0, 1/2)} \left(\int_{B_{\rho}^{k+1}} \left|D^{2}v - \int_{B_{\rho}^{k+1}} D^{2}v dy\right|^{p} dx\right)^{1/p} \leq C(\text{universal}),$$

thereby finishing the proof of the estimate (1.8).

5 Obstacle-type problems: Proof of Theorem 1.6

In this section, we will look at an important class of free boundary problems, namely the obstacle problem with oblique boundary datum (1.9) for a given obstacle $\phi \in W^{2,p}(\Omega)$ satisfying $B(x, u, D\phi) \geq g$ a.e. on $\partial\Omega$, and $F$ a uniformly elliptic operator fulfilling $F(O_{n,n}, \tilde{0}, 0, x) = 0$. 27
we conclude that the viscosity solution to (5.1). Therefore, by Schauder’s fixed point theorem, there exists $u$ such that

$$
\Psi_\varepsilon(s) \equiv 0 \text{ if } s \leq 0; \quad \Psi_\varepsilon(s) \equiv 1 \text{ if } s \geq \varepsilon, \\
0 \leq \Psi_\varepsilon(s) \leq 1 \text{ for any } s \in \mathbb{R}.
$$

and

$$
h(x) := f(x) - F(D^2\phi, D\phi, \phi, x).
$$

Now, we define $\hat{u}_\varepsilon(x) := h^+(x)\Psi_\varepsilon(u_\varepsilon - \phi) + f(x) - h^+(x)$. Then, note that, $h \in L^p(\Omega)$ with the estimate

$$
\|h\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|F(D^2\phi, D\phi, \phi, x)\|_{L^p(\Omega)} \\
\leq C \cdot (\|f\|_{L^p(\Omega)} + \|\phi\|_{W^{2,p}(\Omega)}),
$$

for some $C = C(n, \lambda, \sigma, \varepsilon, p) > 0$, since $F$ fulfills (A1). In a consequence, we have

$$
\|\hat{u}_\varepsilon\|_{L^p(\Omega)} \leq C(n, \lambda, \sigma, \varepsilon, p) (\|f\|_{L^p(\Omega)} + \|\phi\|_{W^{2,p}(\Omega)}).
$$

Next, we claim that the problem (5.1) admits a viscosity solution that enjoys \textit{a priori} estimates. In effect, according to Perron’s Method (see Lieberman’s Book [36, Theorem 7.19]) it follows that for each $v_0 \in L^p(\Omega)$ fixed, there exists a unique viscosity solution $u_\varepsilon \in W^{2,p}(\Omega)$ satisfying in the viscosity sense

$$
\left\{
\begin{array}{ll}
F(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x) = h^+(x)\Psi_\varepsilon(v_0 - \phi) + f(x) - h^+(x) & \text{in } \Omega \\
B(x, u_\varepsilon, Du_\varepsilon) = g(x) & \text{on } \partial\Omega,
\end{array}
\right.
$$

fulfilling (due to Theorem 1.2) the estimate

$$
\|u_\varepsilon\|_{W^{2,p}(\Omega)} \leq C \cdot (\|u_\varepsilon\|_{L^\infty(\Omega)} + \|\hat{f}_\varepsilon\|_{L^p(\Omega)} + \|g\|_{C^{1,\alpha}(\partial\Omega)}),
$$

for some $C = C(n, \lambda, \mu_0, \sigma, \omega, \|\beta\|_{C^2(\partial\Omega)}, |\gamma|, \|\xi\|_{C^2(\partial\Omega), \theta_0, \text{diam}(\Omega)}) > 0$. Moreover, as in (5.3), we have $\hat{f}_\varepsilon \in L^p(\Omega)$.

Now, from A.B.P. Maximum Principle (Lemma 2.6) we obtain

$$
\|u_\varepsilon\|_{W^{2,p}(\Omega)} \leq C \cdot (\|\hat{f}_\varepsilon\|_{L^p(\Omega)} + \|g\|_{C^{1,\alpha}(\partial\Omega)}) \\
\leq C \cdot (\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)} + \|g\|_{C^{1,\alpha}(\partial\Omega)}) \\
\leq \tilde{C} \cdot (\|f\|_{L^p(\Omega)} + \|g\|_{C^{1,\alpha}(\partial\Omega)} + \|\phi\|_{W^{2,p}(\Omega)}),
$$

Thus,

$$
\|u_\varepsilon\|_{W^{2,p}(\Omega)} \leq C_0 (\tilde{C}, n, \lambda, \mu_0, \sigma, \omega, \|\beta\|_{C^2(\partial\Omega)}, \theta_0, \text{diam}(\Omega), \|f\|_{L^p(\Omega)}, \|g\|_{C^{1,\alpha}(\partial\Omega)}, \|\phi\|_{W^{2,p}(\Omega)}),
$$

where $C_0 > 0$ is independent on $v_0$.

At this point, by defining the operator $T : L^p(\Omega) \to W^{2,p}(\Omega) \subset L^p(\Omega)$ given by $T(v_0) = u_\varepsilon$, we conclude that $T$ maps the $C_0$-ball (in $L^p(\Omega)$) into itself. Hence, $T$ is a compact operator. Therefore, by Schauder’s fixed point theorem, there exists $u_\varepsilon$ such that $T(u_\varepsilon) = u_\varepsilon$, which is a viscosity solution to (5.1).

Finally, we are in a position to establish our main result.
Proof of Theorem 1.6. From (5.5) we observe that
\[ \|u_\varepsilon\|_{W^{2,p}(\Omega)} \leq C \cdot (\|f\|_{L^p(\Omega)} + \|g\|_{C^{1,\alpha}(\partial\Omega)} + \|\phi\|_{W^{2,p}(\Omega)}) \]
for some \( C = C(n, \lambda, \alpha, p, \mu_0, \sigma, \xi, \|\beta\|_{C^2(\partial\Omega)}, \text{diam}(\Omega), \theta_0) \). This ensures that \( \{u_\varepsilon\}_{\varepsilon > 0} \) is uniformly bounded in \( W^{2,p}(\Omega) \). Hence, by standard compactness arguments, we can find a subsequence \( \{u_{\varepsilon_j}\}_{j \in \mathbb{N}} \) with \( \varepsilon_j \to 0 \) and a function \( u_\infty \in W^{2,p}(\Omega) \) such that
\[
\begin{cases}
u_{\varepsilon_j} \to u_\infty & \text{in } W^{2,p}(\Omega) \\
u_{\varepsilon_j} \to u_\infty & \text{in } C^{0,\alpha}(\Omega) \\
u_{\varepsilon_j} \to u_\infty & \text{in } C^{1,1-\frac{2}{p}}(\Omega) 
\end{cases}
\]
for some universal \( \alpha_0 \in (0,1) \).

Now, we assert that \( u_\infty \) is a viscosity solution of (1.9). In effect:

✓ Since \( \beta(x) \cdot Du_{\varepsilon_j}(x) + \gamma(x)u_{\varepsilon_j}(x) = g(x) \) and \( \{u_{\varepsilon_j}\}_{j \in \mathbb{N}} \) are uniformly bounded and equi-continuous on \( \partial\Omega \), then from (5.5) and Morrey embedding \( W^{2,p}(\Omega) \hookrightarrow C^{1,1-\frac{2}{p}} \), we have
\[
\beta(x) \cdot Du_\infty(x) + \gamma(x)u_\infty(x) = g(x) \quad \text{on } \partial\Omega
\]
in the viscosity sense (and point-wisely).

✓ On the other hand, from (5.1), we have
\[
F(D^2u_{\varepsilon_j}, Du_{\varepsilon_j}, u_{\varepsilon_j}, x) = h^+(x)\Psi_{\varepsilon_j}(u_{\varepsilon_j} - \phi) + f(x) - h^+(x) \
\leq f(x) \quad \text{in } \Omega \quad \text{for each } j \in \mathbb{N}.
\]
Thus, by applying Stability results (Lemma 2.5), by passing to the limit \( j \to +\infty \), we can conclude
\[
F(D^2u_\infty, Du_\infty, u_\infty, x) \leq f \quad \text{in } \Omega.
\]

✓ Next, we are going to prove that
\[
u_\infty \geq \phi \quad \text{in } \overline{\Omega}.
\]
Firstly, we see that \( \Psi_{\varepsilon_j}(u_{\varepsilon_j} - \phi) \equiv 0 \) on the set \( \mathcal{O}_j := \{x \in \overline{\Omega} : u_{\varepsilon_j}(x) < \phi(x)\} \). Note that, if \( \mathcal{O}_j = \emptyset \), we have nothing to prove. Thus, we suppose that \( \mathcal{O}_j \neq \emptyset \). Then,
\[
F(D^2u_{\varepsilon_j}, Du_{\varepsilon_j}, u_{\varepsilon_j}, x) = f(x) - h^+(x) \quad \text{for } x \in \mathcal{O}_j.
\]
Now, for each \( j \in \mathbb{N} \) note that \( \mathcal{O}_j \) is relatively open in \( \overline{\Omega} \), since \( u_{\varepsilon_j} \in C^0(\overline{\Omega}) \). Furthermore, from definition of \( h \) we have
\[
F(D^2\phi, D\phi, \phi, x) = f(x) - h(x) \geq F(D^2u_{\varepsilon_j}, Du_{\varepsilon_j}, u_{\varepsilon_j}, x) \quad \text{in } \mathcal{O}_j.
\]
Moreover, we have \( u_{\varepsilon_j} = \phi \) on \( \partial\mathcal{O}_j \setminus \partial\Omega \). Thus, from the Comparison Principle (see [10] Theorem 2.10 and [36] Theorem 7.17) we conclude that \( u_{\varepsilon_j} \geq \phi \) in \( \mathcal{O}_j \), which yields a contradiction. Therefore, \( \mathcal{O}_j = \emptyset \) and \( u_\infty \geq \phi \) in \( \overline{\Omega} \).

✓ Finally, it remains to show that
\[
F(D^2u_\infty, Du_\infty, u_\infty, x) = f(x) \quad \text{in } \{u_\infty > \phi\}
\]
in the viscosity sense. For this end, for each \( k \in \mathbb{N} \) we notice that
\[
h^+(x)\Psi_{\varepsilon_j}(u_{\varepsilon_j} - \phi) + f(x) - h^+(x) \to f(x) \quad \text{a.e. in } \{x \in \Omega : u_\infty(x) > \phi(x) + \frac{1}{k}\}.
\]
Therefore, via Stability results (Lemma 2.5) we conclude (in the viscosity sense) that
\[
F(D^2u_\infty, Du_\infty, u_\infty, x) = f(x) \quad \text{in } \{u_\infty > \phi\} = \bigcup_{k=1}^{\infty} \left\{u_\infty > \phi + \frac{1}{k}\right\} \quad \text{as } j \to +\infty,
\]
thereby proving the claim.
Finally, from (5.5) \( u_\infty \) satisfies the following regularity estimate
\[
\| u_\infty \|_{W^{2,p}(\Omega)} \leq \liminf_{j \to +\infty} \| u_{x_j} \|_{W^{2,p}(\Omega)} \\
\leq C(n, \lambda, \mu_0, \sigma, \omega, \| \beta \|_{C^2(\partial\Omega)}, \| \gamma \|_{C^2(\partial\Omega)}, \text{diam}(\Omega), \theta_0) \cdot \left( \| f \|_{L^p(\Omega)} + \| g \|_{C^{1,\alpha}(\partial\Omega)} + \| \phi \|_{W^{2,p}(\Omega)} \right)
\]
thereby finishing the proof of the Theorem. \( \square \)

As a consequence, we obtain the uniqueness of existing solutions.

**Corollary 5.1 (Uniqueness).** The viscosity solution found in the Theorem 1.6 is unique.

**Proof.** Indeed, let \( u \) and \( v \) be two viscosity solutions of (1.9). Assume that \( u \neq v \). Then, we may suppose without loss of generality that
\[
\mathcal{O}_2 = \{ v > u \} \neq \emptyset.
\]
Since \( v > u \geq \phi \) in \( \mathcal{O}_2 \), we obtain in the viscosity sense
\[
F(D^2v, Dv, v, x) = f(x) \quad \text{in} \quad \mathcal{O}_2.
\]
Hence, we conclude that
\[
\begin{cases}
F(D^2u, Du, u, x) \leq f(x) \leq F(D^2v, Dv, v, x) \quad \text{in} \quad \mathcal{O}_2 \\
B(x, u, Du) = g(x) = B(x, v, Dv) \quad \text{on} \quad \partial\mathcal{O}_2 \setminus \partial\Omega.
\end{cases}
\]
Therefore, according to Comparison Principle for problems with oblique boundary conditions [10, Theorem 2.10] and [36, Theorem 7.17], we conclude that \( u \geq v \) in \( \mathcal{O}_2 \) or not. However, this contradicts the definition of the set \( \mathcal{O}_2 \), thereby proving that \( u = v \). \( \square \)

### 6 Final conclusions: Density of viscosity solutions

In this final part, we will deliver another application to \( W^{2,p} \) regularity estimates.

**Theorem 6.1 (\( W^{2,p} \) density on the class \( C^0 \)-viscosity solutions).** Let \( u \) be a \( C^0 \)-viscosity solution of
\[
\begin{cases}
F(D^2u, x) = f(x) \quad \text{in} \quad B_1^{+} \\
B(x, u, Du) = g(x) \quad \text{on} \quad T_1,
\end{cases}
\]
where \( f \in L^p(B_1^{+}) \cap C^0(B_1^{+}) \) (for \( n \leq p < \infty \), \( \beta, \gamma, g \in C^{1,\alpha}(T_1) \) with \( \gamma \leq 0 \) and \( \beta \cdot \overrightarrow{n} \geq \mu_0 \) on \( T_1 \), for some \( \mu_0 > 0 \). Then, for any \( \delta > 0 \), there exists a sequence \( \{ u_j \}_{j=1}^{\infty} \subset W^{2,p}_{\text{loc}}(B_1^{+}) \cap \mathcal{S}(\Lambda - \delta, \Lambda + \delta, f) \) converging local uniformly to \( u \).

**Proof.** The proof follows some ideas from [39, Theorem 8.1] adapted to oblique boundary scenery. We will present the details to the reader for completeness. Firstly, we are going to build up such a desired sequence of operators \( F_j : \text{Sym}(n) \times B_1^{+} \to \mathbb{R} \) as follow: Given \( \delta > 0 \) we consider the Pucci Maximal operator
\[
\mathcal{L}_\delta(X) := \mathcal{P}_{\delta,\Lambda}(\Lambda + \delta)(X) = (\Lambda + \delta) \sum_{e_i > 0} e_i(X) + (\Lambda - \delta) \sum_{e_i < 0} e_i(X),
\]
where \( e_i(X) \) are the eigenvalues of matrix \( X \in \text{Sym}(n) \). Now, we define
\[
F_j : \text{Sym}(n) \times B_1^{+} \to \mathbb{R} \\
(X, x) \mapsto \max\{ F(X, x), \mathcal{L}_\delta(X) - C_j \},
\]
where \( C_j \) is a function of \( j \).

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for $(C_j)_{j \in \mathbb{N}}$ a sequence of divergence positive number (chosen precisely a posteriori). Thus, notice that $F_j$ is continuous (because it is the maximum between two continuous functions) and uniformly elliptic with ellipticity constants $\lambda - \delta \leq \Lambda + \delta$, as $F$ and $\mathcal{L}_3$ are. Now, taking into account the $(\lambda, \Lambda)$-ellipticity of $F$ there holds

\[ F(X, x) \geq \lambda \sum_{e_i > 0} e_i(X) + \Lambda \sum_{e_i < 0} e_i \]

\[ \geq \lambda \sum_{e_i > 0} e_i(X) - \Lambda ||X|| \]

\[ = \mathcal{L}_3(X) - (\lambda + \delta - \lambda) \sum_{e_i > 0} e_i - (\lambda - \delta) \sum_{e_i < 0} e_i - \Lambda ||X|| \]

\[ \geq \mathcal{L}_3(X) - (2\Lambda + \lambda + \delta) ||X|| \]

\[ \geq \mathcal{L}_3(X) - C_j, \quad \forall B_j \subset \text{Sym}(n), \quad \forall x \in B_1^+, \]

where we select $C_j := j(2\Lambda + \lambda + \delta)$. Hence, $F \equiv F_j$ in $B_1 \times B_1^+ \subset \text{Sym}(n) \times B_1^+$. On the other hand, we investigate the recession operator associated to $F_j$. For that purpose, for each $\mu > 0$ we have

\[ (F_j)_\mu(X, x) = \mu F_j(\mu^{-1}X, x) = \max\{F_\mu(X, x), L_\delta(X) - \mu C_j\}. \]

Since $F_\mu$ is $(\lambda, \Lambda)$-elliptic for any $\mu > 0$ we get

\[ F_\mu(X, x) \leq \Lambda \sum_{e_i > 0} e_i(X) + \lambda \sum_{e_i < 0} e_i(X) \]

\[ = \mathcal{L}_3(X) - \delta \sum_{e_i > 0} e_i(X) + \delta \sum_{e_i < 0} e_i(X) \]

\[ \leq \mathcal{L}_3(X) - \delta ||X|| \]

\[ \leq \mathcal{L}_3(X) - \mu C_j, \]

for all $X \in B_{\mu C_j} \subset \text{Sym}(n)$ and $x \in B_1^+$. Therefore, we conclude that $F_j(X, x) = \mathcal{L}_3(X) - C_j$ outside a ball of radius $\sim C_j$, then $F_j^\sharp = \mathcal{L}_3$. Now, notice that $F_j^\sharp$ fulfills $C^{1, 1}$ a priori estimates, i.e., given $g_0 \in C^{1, \alpha}(T_1)$ any viscosity solution to

\[
\begin{align*}
F_j^\sharp(D^2h, x_0) &= 0 \quad \text{in } B_1^+ \\
B(x, h, Dh) &= g_0(x) \quad \text{on } T_1
\end{align*}
\]

satisfies $h \in C^{1, 1}(\overline{B_{1/2}}^+)$ (see e.g., [33 Teorema 1.3]) with the following estimate:

\[
\|h\|_{C^{1, 1}(\overline{B_{1/2}}^+)} \leq C \cdot \left( \|h\|_{L^\infty(B_1^+)} + \|g\|_{C^{1, \alpha}(T_1)} \right).
\]

Therefore, statements of Proposition 3.4 are in force. Then, for each fixed $j \in \mathbb{N}$, any viscosity solution of

\[
\begin{align*}
F_j(D^2v, x) &= f(x) \quad \text{in } B_1^+ \\
B(x, v, Dv) &= g(x) \quad \text{on } T_1
\end{align*}
\]

enjoys $W^{2,p}$ regularity estimates. To be more specific, for each $j \in \mathbb{N}$ there exists a constant $\kappa_j > 0$ such that

\[
\|v\|_{W^{2,p}(\overline{B_{1/2}}^+)} \leq \kappa_j \cdot (\|v\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)} + \|g\|_{C^{1, \alpha}(T_1)}).
\]

Finally, we build up the desired sequence $(u_j)_{j \in \mathbb{N}}$ to be a viscosity solution to

\[
\begin{align*}
F_j(D^2u_j, x) &= f(x) \quad \text{in } B_1^+ \\
B(x, u_j, Du_j) &= g(x) \quad \text{on } T_1 \\
u(x) &= u_j(x) \quad \text{on } \partial B_1^+ \setminus T_1,
\end{align*}
\]
whose the existence does hold from Theorem 2.9. Furthermore, each $u_j \in W^{2,p}(B^+_1)$ and since operators $F_j$ are uniformly elliptic, it follow that $F_j(\cdot, x) \to F_0(\cdot, x)$ local uniformly in $\text{Sym}(n)$ for each $x \in B^+_1$. Furthermore, since $F_j \equiv F$ in $B_j \times B^+_1$ then $F_0 \equiv F$. In conclusion, using local $C^{0,\alpha}$ regularity and A.B.P. estimates (see Theorem 2.4 and Lemma 2.6 respectively) the sequence $(u_j)_{j \in \mathbb{N}}$ converge, up to a subsequence, local uniformly to $u_0$ in the $C^{0,\alpha}$-topology. In addition, according to Stability results (Lemma 2.5) $u_0$ is a viscosity solution of

$$\begin{cases}
F(D^2 u_0, x) = f(x) & \text{in } B^+_1 \\
B(x, u_0, Du_0) = g(x) & \text{on } T_1 \\
u(x) = u_0(x) & \text{on } \partial B^+_1 \setminus T_1.
\end{cases}$$

Finally, by taking $w = u_0 - u$, we can see that $w$ satisfies in the viscosity sense

$$\begin{cases}
w \in S(\lambda/n, \Lambda, 0) & \text{in } B^+_1 \\
B(x, w, Dw) = 0 & \text{on } T_1 \\
w = 0 & \text{on } \partial B^+_1 \setminus T_1.
\end{cases}$$

and, once again, by A.B.P. estimates (Lemma 2.6) we conclude that $w = 0$ in $B^+_1 \setminus T_1$. Therefore, $w \equiv 0$, i.e., $u = u_0$, thereby finishing the proof.

\[\square\]

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