Cosmic structures in Ricci-inverse theories of gravity

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Abstract. We discuss a no-go theorem for the novel Ricci-inverse theory of gravity. Referring to a static spherically symmetric matter distribution embedded in a de Sitter cosmology, we prove that it is impossible to have a stable Sub-Horizon non-relativistic Weak-Field limit in any model proposed in previous works to avoid some cosmological and inflationary instabilities. We discuss possible ways to circumvent the issue, and point out a machinery to build new stable models under Sub-Horizon non-relativistic Weak-Field limit. Such models exhibit full consistency with predictions of General Relativity on small scales.
1 Introduction

Modeling the current accelerated expansion of our Universe represents one of the biggest challenges of modern cosmology [1–3]. Due to the well known cosmological constant problems burning the ΛCDM model [4], the issue has led to the exploration of theories of Modified Gravity (MG) alternative to General Relativity (GR) [5–7].

A very novel class of fourth-order promising MG models is provided by the so-called Ricci-inverse gravity framework [8], where the Einstein-Hilbert action is generalized with some function of the Ricci scalar $R$ and the anticurvature scalar $A$, defined as the trace of the Ricci-inverse tensor $A_{\mu\nu}$,

$$A_{\mu\sigma}R_{\sigma\nu} = \delta_{\mu\nu}.$$  \hspace{1cm} (1.1)

The Ricci-inverse theory has been shown to admit both a cosmological and an inflationary no-go theorems: actions that contain terms linear in any positive or negative power of $A$ are ruled out as dark energy candidates [8], whereas it is impossible to have a stable isotropic inflation in any linear combination of $R$ with $A$ and $A^2$ [9, 10].

As a first step, some trivial non-linear terms have been proposed to circumvent the cosmological no-go theorem proposed in [8]. However, an in-depth investigation must be considered since generic non-linear combinations of $R$ and $A$ are expected to contain ghosts or other instabilities [11, 12].

Motivated by such considerations, in this work we will deal with a third version of no-go theorem focused on the stability and the consistency of the Ricci-inverse theory within the predictions of GR on small scales. In particular, referring to a static spherically symmetric matter distribution embedded in a de Sitter cosmology, our main results are the following: (i) it is impossible to have a stable Sub-Horizon non-relativistic Weak-Field limit in any linear combination of $R$ with $A$ and $A^2$ [10] or in any non-linear term proposed in [8] to circumvent the cosmological no-go theorem; (ii) we point out some non-linear terms of $A$ and $R$ whose combination avoids any instability under the Sub-Horizon non-relativistic Weak-Field perspective; (iii) requiring stability, such combination is fully consistent with predictions of GR,
exhibiting the impossibility of testing signatures of Ricci-inverse theories using astrophysical objects such as stars and galaxy clusters \[13–15\].

To achieve our results, the paper is organized as follows. In section 2 we briefly introduce the full Ricci-inverse theory and exhibit the related covariant field equations. In the following section 3 we discuss the de Sitter background, whereas in section 4 we derive the perturbed equations for a static spherically symmetric matter source. Afterwards, we discuss the related Weak-Field limit. We show in section 5 that a general no-go theorem prevents divergences and ghosts in any linear combination of $R$ with $A$ and $A^2$, which consequently is ruled out as cosmological candidate. Section 6 is fully dedicated to discuss possible ways to circumvent our no-go theorem, and in particular we point out a stable action that is fully consistent with predictions of GR. Finally, section 7 summarizes our results.

We use the metric signature $(-, +, +, +)$ and we set the reduced Planck mass to unity. Greek indices run from 0 to 3.

## 2 The Ricci-inverse theory

Let us consider the full action for the Ricci-inverse theory of gravity \[8\]

$$S = \int d^4x \sqrt{-g} \left[ f(R, A) + \mathcal{L}_m \right],$$

(2.1)

where $g$ is the determinant of the metric $g_{\mu\nu}$, and $\mathcal{L}_m$ is the matter Lagrangian, that we assume coupled with the metric only. The arbitrary function $f(R, A)$ depends on the Ricci scalar $R$ and the ant curvature scalar $A \equiv g_{\mu\nu} A^{\mu\nu}$.

By differentiating Eq. (2.1), the covariant equation of motion with respect to the metric field $g_{\mu\nu}$ is

$$\delta S/\delta g_{\mu\nu} = 0,$$

whose expression is \[8\]

$$f_R R^{\mu\nu} - f_A A^{\mu\nu} - \frac{1}{2} f g^{\mu\nu} + g^{\rho\mu} \nabla_\rho f A_{A_\alpha} A^{\nu_\sigma} - \frac{1}{2} \nabla_\alpha f A_{A_\sigma} A^{\mu_\rho} - \frac{1}{2} \nabla_\sigma f_A A^{A_{\alpha}} A^{\beta_\sigma} - \nabla^{\mu} \nabla_\nu f_R + g^{\mu\nu} \nabla^{A_\alpha} \nabla_A f_R = T^{\mu\nu},$$

(2.2)

where the energy-momentum tensor $T_{\mu\nu}$ is defined as,

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \delta(\sqrt{-g}\mathcal{L}_m) \delta g_{\mu\nu},$$

(2.3)

In our notation the $\nabla_\mu$ symbol denotes the covariant derivative, whereas the subscripts $A$ and $R$ stand for partial derivatives, e.g. $f_{AAR} \equiv \partial R \partial A \partial A f$.

## 3 Ricci-inverse in de Sitter background

Consider a spatially flat de Sitter cosmological background. Using Friedmann Lemaître Robertson Walker (FLRW) coordinates $(\tau, \rho, \theta, \phi)$, the metric can be written as

$$ds^2 = -d\tau^2 + e^{2H\tau} (d\rho^2 + \rho^2 d\Omega^2_2),$$

(3.1)

where $H$ is the constant Hubble expansion rate and $d\Omega^2_2$ the solid angle-element.
Using our metric (3.1), the resulting background Ricci scalar and anticurvature scalar are, respectively
\[
R^{(0)} = 12H^2 \quad A^{(0)} = \frac{4}{3}H^{-2}.
\] (3.2)
Since all the terms with derivatives vanish, we take the trace of the equation of motion (2.2), and we finally get, in vacuum [8]
\[
18f_{R}^{(0)}H^2 - 2H^{-2}f_{A}^{(0)} - 3f^{(0)} = 0,
\] (3.3)
where we introduced the background notation \( f^{(0)} \equiv f_{|R^{(0)},A^{(0)}} \) to indicate the evaluation w.r.t background.

The following sections will consider cosmic structures described in spherical Schwarzschild-like coordinates \((t, r, \theta, \phi)\), that can be obtained from the FLRW ones by performing the transformation [13]
\[
\tau(t, r) = t + \frac{1}{2H} \ln (1 - H^2r^2) \quad \rho(t, r) = \frac{re^{-Ht}}{\sqrt{1 - H^2r^2}},
\] (3.4)
with the assumption \( 1 - H^2r^2 \geq 0 \). Writing down the metric (3.1) in Schwarzschild-like coordinates (3.4), it is easily find that the de Sitter background can be rewritten as
\[
ds^2_{(0)} = -(1 - H^2r^2)dt^2 + \frac{dr^2}{1 - H^2r^2} + r^2d\Omega^2_{2}.
\] (3.5)

4 Static spherically symmetric matter distribution in Ricci-inverse gravity

Let us embed a static and spherically symmetric structure into the de Sitter cosmological background (3.1). Such source affects the background spacetime, which in spherical Schwarzschild-like coordinates becomes
\[
ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2d\Omega^2_{2},
\] (4.1)
with \( \nu(r) \) and \( \lambda(r) \) two radial-dependent metric potentials.

Writing down \( R \) and \( A \) in terms of the spherical metric (4.1), it is easy to find
\[
R = (\xi_1 + r\xi_2 - r\xi_3)e^{-\lambda}r^{-2} \quad A = (4r\xi_1^{-1} + \xi_2^{-1} - \xi_3^{-1})re^{\lambda},
\] (4.2)
with \( \xi_1 \equiv r(\lambda' - \nu') + 2(e^\lambda - 1) \), and \( \xi_2 \equiv \lambda - \frac{1}{4}(\nu'^2 - \nu'\lambda' + 2\nu'') \), and \( \xi_3 \equiv \nu + \frac{1}{4}(\nu'^2 - \nu'\lambda' + 2\nu'') \). In our notation the ' symbol denotes derivation w.r.t. to the radial coordinate \( r \). We see that \( A \) is singular when \( \xi_1 \) or \( \xi_2 \) or \( \xi_3 \) vanish. This means that if a solution passes through any one of these conditions, it develops a singularity that rules out the model.

Assuming a spherical symmetric perfect fluid configuration with energy density \( \varepsilon(r) \) and pressure \( P(r) \), the energy-momentum tensor (2.3) has expression
\[
T^{\nu}_{\nu} \equiv \text{diag}\{-\varepsilon(r), P(r), P(r), P(r)\},
\] (4.3)
Substituting the explicit expressions (4.1) and (4.3) inside Eq. (2.2), one gets that the relevant equations of motion are the \( t-t \) and \( \theta-\theta \) components, which are linear combinations of derivatives of \( f \) (namely \( f_i = f_j f, f_A, f_R, \ldots \)), multiplied by polynomials in \( r \) and derivatives of metric potentials \( (P_i, Q_i) \):
\[
\varepsilon e^{-\nu} = \sum f_i P_i \quad P r^{-2} = \sum f_i Q_i.
\] (4.4)
We do not report the too long expressions for \( P_i \) and \( Q_i \).
4.1 Sub-Horizon non-relativistic Weak-Field limit

The consistency of MG theories with predictions of GR can be tested on small scales by considering the so-called Sub-Horizon non-relativistic Weak-Field limit. As a starting point, let us perturb the metric potentials about their cosmological values

$$\nu(r) \sim \nu^{(0)}(r) + \delta \nu(r) \quad \lambda(r) \sim \lambda^{(0)}(r) + \delta \lambda(r),$$  \hspace{1cm} (4.5)

with $\delta \nu \ll \nu^{(0)}$ and $\delta \lambda \ll \lambda^{(0)}$. As $r$ approaches to the de Sitter horizon, $\delta \lambda$ and $\delta \nu$ vanish and the background de Sitter metric (3.5) becomes dominant.

Considering such decompositions, the Ricci and the anticurvature scalars can be rewritten as follows

$$R \sim R^{(0)} + \delta R \quad A \sim A^{(0)} + \delta A,$$  \hspace{1cm} (4.6)

where the background expressions $R^{(0)}$ and $A^{(0)}$ are given in Eq. (3.2), while

$$\delta R = \delta \nu''(H^2 r^2 - 1) + \frac{5}{2} \delta \nu' \frac{H^2 r^2 - 2}{r} - \delta \lambda' \frac{3H^2 r^2 - 2}{r} - 2\delta \lambda \frac{6H^2 r^2 - 1}{r^2},$$  \hspace{1cm} (4.7)

$$\delta A = -\frac{1}{9} \delta \nu'' \frac{H^2 r^2 - 1}{H^4} - \frac{1}{9} \delta \nu' \frac{5H^2 r^2 - 2}{H^4 r} + \frac{1}{9} \delta \lambda' \frac{3H^2 r^2 - 2}{H^4 r} + \frac{2}{9} \delta \lambda \frac{6H^2 r^2 - 1}{H^4 r^2}.$$  \hspace{1cm} (4.8)

Consequently, the scalar function $f(R,A)$ can be decomposed as $f \sim f^{(0)} + \delta f$ where

$$\delta f = f_R^{(0)} \delta R + f_A^{(0)} \delta A.$$  \hspace{1cm} (4.9)

The same approach is extended to any derivative of $f$.

Supposing a mass distribution of the matter source

$$M(r) \equiv 4 \pi \int_0^r s^2 \varepsilon(s) ds \quad M(r \to +\infty) \equiv M,$$  \hspace{1cm} (4.10)

the Sub-Horizon non-relativistic Weak-Field limit is operatively achieved by considering sequentially: (i) the Weak-Field condition $\delta \nu' \sim \delta \lambda \sim M/r \ll 1$, (ii) the sub-horizon condition $r \equiv Hr \ll 1$, (iii) the non-relativistic Newtonian condition $P \ll \varepsilon$. In such limit, the metric potentials $\delta \nu$ and $\delta \lambda$ can be related to the Newtonian potential and curvature perturbations as follows

$$\Phi'(r) = \frac{\delta \nu'(r)}{2} \quad \Psi'(r) = \frac{\delta \lambda(r)}{2r}.$$  \hspace{1cm} (4.11)

In the case of GR, it is well known that $\Phi_{GR}' = \Psi_{GR}' = M/r^2$.

Let us apply the procedure to our theory. Starting with the Weak-Field condition, after onerous computations, the field equations (4.4) can be rewritten as

$$\varepsilon(1 - x^2)^{-1} = \left\{ 3x^4 f_R^{(0)} - \frac{1}{2} x^2 f_A^{(0)} - \frac{1}{2} x^2 f_\lambda^{(0)} \right\} \left\{ r^2 (x^2 - 1) x^2 \right\}^{-1}
+ \frac{1}{81} \delta \nu'' \left\{ 5r^8 (5 - 6x^2) f_A^{(0)} - 3r^6 (37x^2 - 27)x^2 f_\lambda^{(0)} - 54r^4 (x^2 - 1)x^4 f_\lambda^{(0)}
+ 405(2x^2 - 1)x^8 f_\lambda^{(0)} \right\} \left\{ r^2 (x^2 - 1) x^6 \right\}^{-1}
+ \frac{1}{81} \delta \nu' \left\{ 2r^8 f_A^{(0)} - 9r^6 x^4 f_\lambda^{(0)} - 54r^4 (5x^2 - 2)x^4 f_\lambda^{(0)} + 324(5x^2 - 1)x^8 f_\lambda^{(0)} \right\} \left\{ r^2 (x^2 - 1) x^8 \right\}^{-1}
- \frac{1}{81} \delta \lambda' \left\{ 4r^8 (-9x^4 + 5x^2 + 1) f_A^{(0)} + 3r^6 (-40x^4 + 17x^2 + 2)x^4 f_\lambda^{(0)} - 54r^4 (3x^2 - 2)x^6 f_\lambda^{(0)} \right\} \left\{ r^2 (x^2 - 1) x^8 \right\}^{-1}$$
\[
+ 81r^2(x^2 - 1)x^8 f_A^{(0)} - 486(4x^2 - 1)x^{10} f_{RR}^{(0)} \bigg\} \bigg\{ r^2(x^2 - 1)x^8 \bigg\}^{-1}
\]
\[
+ \frac{1}{81} \delta \lambda \bigg\{ 2r^8(-18x^4 + 3x^2 + 2)f_A^{(0)} + 3r^6(-27x^4 + 5x^2 + 2)x^2 f_A^{(0)} + 108r^4(6x^2 - 1)x^6 f_A^{(0)}
\]
\[
+ 81r^2(3x^2 - 1)x^8 f_{RR}^{(0)} - 162(18x^2 - 1)x^{10} f_{RR}^{(0)} \bigg\} \bigg\{ r^2(x^2 - 1)x^8 \bigg\}^{-1}
\]
\[
+ \frac{1}{81} \delta \nu \bigg\{ 2r^4 f_A^{(0)} + 3r^2 f_A^{(0)} x^2 - 18x^4 f_R^{(0)} \bigg\} \bigg\{ r^2(x^2 - 1)x^8 \bigg\}^{-1}
\]
\[
+ \frac{1}{81} \delta \nu'' \bigg\{ -4r^8(3x^2 - 1)f_A^{(0)} - 3r^6(13x^2 - 4)x^2 f_A^{(0)} + 81x^{10} f_{RR}^{(0)} \bigg\} r^{-1} x^{-8}
\]
\[
+ \frac{1}{81} \delta \nu'' \bigg\{ r^8(27x^4 - 28x^2 + 2)f_A^{(0)} + 3r^6(21x^4 - 20x^2 + 1)x^2 f_A^{(0)}
\]
\[
- 81(3x^2 - 2)x^{10} f_{RR}^{(0)} \bigg\} \bigg\{ r^2(x^2 - 1)x^8 \bigg\}^{-1}
\]
\[
- \frac{1}{81} \delta \nu'''' \bigg\{ \bigg( r^2 f_A^{(0)} + 3x^2 f_A^{(0)} \bigg) (x^2 - 1) \bigg\} x^{-8}
\]
\[
+ \frac{1}{81} \delta \lambda m^5 \bigg\{ r^2(3x^2 - 2)f_A^{(0)} + 3(2x^2 - 1)x^2 f_A^{(0)} \bigg\} x^{-8},
\]
(4.12)

and

\[
Pr^{-2} = \frac{1}{3} f_A x^2 - \frac{1}{2} f r^{-2} + 3x^2 f_R^{(0)} r^{-4}
\]
\[
+ \frac{1}{102} \delta \nu'' \bigg\{ 4r^8(-15x^4 + 9x^2 + 1)f_A^{(0)} - 3r^6(45x^4 - 37x^2 + 2)x^2 f_A^{(0)} - 108r^4(4x^2 - 1)x^6 f_A^{(0)}
\]
\[
- 81r^2(x^2 - 1)x^8 f_A^{(0)} + 324(5x^4 - 6x^2 + 1)x^8 f_{RR}^{(0)} \bigg\} r^{-4} x^{-8}
\]
\[
- \frac{1}{102} \delta \nu'' \bigg\{ 2r^8(3x^2 + 2)f_A^{(0)} + 3r^6(10x^4 + x^2 - 2)x^2 f_A^{(0)} + 108r^4(5x^2 - 2)x^6 f_A^{(0)}
\]
\[
+ 81r^2(4x^2 - 1)x^8 f_A^{(0)} + 162(-20x^4 + 9x^2 + 2)x^8 f_{RR}^{(0)} \bigg\} r^{-5} x^{-8}
\]
\[
- \frac{1}{102} \delta \lambda' \bigg\{ 2r^8(-36x^4 + 5x^2 + 4)f_A^{(0)} + 3r^6(-80x^4 + 35x^2 + 6)x^2 f_A^{(0)}
\]
\[
- 108r^4(3x^2 - 2)x^6 f_A^{(0)} - 81r^2(2x^2 - 1)x^8 f_A^{(0)} + 486(8x^2 - 7)x^{10} f_{RR}^{(0)} \bigg\} r^{-5} x^{-8}
\]
\[
+ \frac{1}{81} \delta \lambda'' \bigg\{ 2r^8(-18x^4 + 3x^2 + 4)f_A^{(0)} + 3r^6(-27x^4 + 12x^2 + 2)x^2 f_A^{(0)} + 108r^4(6x^2 - 1)x^6 f_A^{(0)}
\]
\[
+ 243r^2 x^{10} f_{RR}^{(0)} + 162(-18x^4 + x^2 + 2)x^8 f_{RR}^{(0)} \bigg\} r^{-6} x^{-8}
\]
\[
+ \frac{1}{81} \delta \nu'' \bigg\{ -2r^8(4x^4 - 5x^2 + 1)f_A^{(0)} - r^6(14x^4 - 19x^2 + 5)x^2 f_A^{(0)}
\]
\[
+ 54(x^4 - 2x^2 + 1)x^8 f_{RR}^{(0)} \bigg\} r^{-3} x^{-8}
\]
\[
+ \frac{1}{81} \delta \lambda'' \bigg\{ r^8(27x^2 - 25)f_A^{(0)} + 3r^6(21x^4 - 22x^2 + 2)f_A^{(0)} - 81(3x^4 - 5x^2 + 2)x^6 f_{RR}^{(0)} \bigg\} r^{-4} x^{-6}
\]
\[
- \frac{1}{102} \delta \nu'''' \bigg\{ (2r^2 f_A^{(0)} + 3x^2 f_A^{(0)}) (x^4 - 2x^2 + 1) \bigg\} x^{-8}
\]
\[
+ \frac{1}{102} \delta \lambda'''' \bigg\{ 2r^8(3x^2 - 5x^2 + 2)f_A^{(0)} + 3(4x^4 - 7x^2 + 3)x^2 f_A^{(0)} \bigg\} x^{-8}.
\]
(4.13)

Maintaining any parametrization for \( f(R, A) \), Eqs. (4.12) and (4.13) generally exhibit two manifest kind of instabilities, namely (i) divergences when the Sub-Horizon limit \( x \to 0 \) is imposed, and (ii) ghosts induced by terms depending on higher-order derivatives of \( \delta \nu \) and \( \delta \lambda \). The only possibility to escape this evidence is to find, if there exists, a fine-tuned set of \( f(R, A) \) functions that can exorcize both divergences and ghosts.

- 5 -
Before speculating about workarounds, we show that it is impossible to have a stable 
Sub-Horizon non-relativistic Weak-Field limit in any linear combination of $R$ with $A$ and $A^2$ [10] or in any non-linear term proposed to circumvent the cosmological no-go theorem proposed in [8].

5 A no-go theorem

In this section we consider the simple case

$$f(R, A) = R + kA + \ell A^2,$$  \hspace{1cm} (5.1)

with $k, \ell$ constants. The background Eq. (3.3), solved w.r.t. the parameter $k$, yields

$$k = -\frac{16\ell^6 + 27x^6}{9r^4x^2}.$$  \hspace{1cm} (5.2)

Substituting the above parametrization (5.1) into the simplified Eqs. (4.12), (4.13), and after using the background relation (5.2), one gets

$$\varepsilon(1 - x^2)^{-1} = -\frac{1}{243} \delta'' \left\{ 476\ell^6 x^2 - 366\ell^6 - 999x^6 + 729x^6 \right\} (x^2 - 1) x^6 \varepsilon^{-1}
- \frac{1}{243} \delta' \left\{ 8\ell^6 x^2 - 4\ell^6 - 27x^6 \right\} (x^2 - 1) x^6 \varepsilon^{-1}
+ \frac{1}{243} \delta \left\{ 536\ell^6 x^4 - 256\ell^6 x^2 - 40\ell^6 - 837x^{10} + 216x^6 + 54x^6 \right\} (x^2 - 1) x^8 \varepsilon^{-1}
- \frac{2}{243} \delta \left\{ 216\ell^6 x^4 - 38\ell^6 x^2 - 20\ell^6 - 729x^{10} + 189x^6 + 27x^6 \right\} (x^2 - 1) x^8 \varepsilon^{-1}
- \frac{1}{243} \delta'''' (176\ell^6 x^2 - 56\ell^6 - 351x^8 + 108x^6) x^{-8}
+ \frac{1}{243} \delta'''' (330\ell^6 x^4 - 328\ell^6 x^2 + 20\ell^6 - 567x^{10} + 540x^8 - 27x^6) (x^2 - 1) x^{-8}
- \frac{1}{243} \delta'''' (14\ell^6 - 27x^6) (x^2 - 1) x^{-8}
+ \frac{1}{243} \delta'''' (34\ell^6 x^2 - 20\ell^6 - 54x^8 + 27x^6) x^{-8},$$  \hspace{1cm} (5.3)

and

$$P r^{-2} = -\frac{1}{243} \delta'''' \left\{ 360\ell^6 x^4 - 256\ell^6 x^2 - 4\ell^6 - 486x^{10} + 378x^8 - 27x^6 \right\} r^{-2} x^{-8}
- \frac{1}{243} \delta''' \left\{ 40\ell^6 x^4 + 22\ell^6 x^2 + 4\ell^6 + 351x^{10} - 135x^8 + 27x^6 \right\} r^{-3} x^{-8}
+ \frac{1}{243} \delta'' \left\{ 536\ell^6 x^4 - 170\ell^6 x^2 - 48\ell^6 - 837x^{10} + 351x^8 + 81x^6 \right\} r^{-3} x^{-8}
- \frac{2}{243} \delta' \left\{ 216\ell^6 x^4 - 66\ell^6 x^2 - 32\ell^6 - 729x^{10} + 162x^6 + 27x^6 \right\} r^{-4} x^{-8}
- \frac{1}{243} \delta'' \left\{ 256\ell^6 x^2 - 76\ell^6 - 378x^8 + 135x^6 \right\} (x^2 - 1) r^{-1} x^{-8}
+ \frac{1}{243} \delta'' \left\{ 330\ell^6 x^4 - 326\ell^6 x^2 + 16\ell^6 - 567x^{10} + 594x^8 - 54x^6 \right\} r^{-2} x^{-8}
- \frac{1}{243} \delta'' \left\{ 20\ell^6 - 27x^6 \right\} (x^2 - 1)^2 x^{-8}
+ \frac{1}{243} \delta'' \left\{ 68\ell^6 x^2 - 48\ell^6 - 108x^8 + 81x^6 \right\} (x^2 - 1)r^{-1} x^{-8}.$$  \hspace{1cm} (5.4)
Analyzing qualitatively the system (5.3) and (5.4), we see that, any value of \( \ell \neq 0 \) we choose, no divergence can be cured as \( x \) approaches to zero. The same problem emerges when we try to put to zero higher order derivatives terms. As a consequence, this fact rules out the realization (5.1) of the theory. Through the same analysis of the linearized system, we find the same critical qualitative behavior if we substitute (5.1) with the following profiles, proposed by [8] to prove or circumvent a cosmological no-go theorem for Ricci-inverse theories:

\[
\begin{align*}
  f &= R + \frac{\alpha}{A} & f &= R + \alpha R^2 A & f &= R + \alpha Re^{-\beta(RA)^2} & f &= \frac{R}{1 + \alpha RA},
\end{align*}
\]

with \( \alpha, \beta \) dimensionless constants. This demonstrates that all these models are not good Ricci-inverse candidates to support a stable existence of static spherically symmetric matter distributions.

### 6 Circumventing the no-go theorem

We discuss how to avoid the no-go theorem of section 5. Our reasoning begins noticing that if \( f_A^{(0)} \sim x^3 \) and \( f_{AA}^{(0)} \sim x^5 \), all divergences are ruled out from Eqs. (4.12) and (4.13) as \( x \) approaches to 0. Since \( A^{(0)} \sim x^{-2} \), the following model is suggested:

\[
f(R, A) = R + \frac{k}{A^2} - 2\Lambda, \tag{6.1}
\]

with \( k, \Lambda \) constant parameters.

Again, the background Eq. (3.3), solved w.r.t. the parameter \( \Lambda \), yields

\[
\Lambda = 3x^2r^{-2}, \tag{6.2}
\]

which computed inside Eqs. (4.12), (4.13), as expected, leads to the divergence-free perturbed equations of motion

\[
\begin{align*}
  \varepsilon (1 - x^2)^{-1} &= \frac{1}{128} \delta \nu'' k \left\{ 58x^2 - 33 \right\} x^2 r^{-2}(x^2 - 1)^{-1} + \frac{3}{64} \delta \nu' k \left\{ 2x^2 + 1 \right\} x^2 r^{-3}(x^2 - 1)^{-1} \\
  &\quad - \frac{1}{32} \delta \lambda' \left\{ 13kx^4 - 2kx^2 + k - 32r^2x^2 + 32r^2 \right\} r^{-3}(x^2 - 1)^{-1} + \frac{1}{128} \delta \lambda'' k \left\{ x^2 - 2 \right\} r^{-1} \\
  &\quad - \frac{1}{64} \delta \lambda \left\{ kx^2 - 2k - 192r^2x^2 + 64r^2 \right\} r^{-4}(x^2 - 1)^{-1} + \frac{1}{32} \delta \nu''' k \left\{ 4x^2 - 1 \right\} r^{-1} \\
  &\quad - \frac{1}{128} \delta \lambda'' k \left\{ 3x^4 + 4x^2 - 2 \right\} r^{-2}(x^2 - 1)^{-1} + \frac{1}{128} \delta \nu''' k \left\{ x^2 - 1 \right\}, \tag{6.3}
\end{align*}
\]

and

\[
\begin{align*}
  Pr^{-2} &= -\frac{1}{128} \delta \nu'' \left\{ 10kx^2 - 5k + 32r^2x^2 - 32r^2 \right\} r^{-4} + \frac{1}{64} \delta \lambda \left\{ -15kx^2 + 8k + 192r^2x^2 \right\} r^{-6} \\
  &\quad - \frac{1}{128} \delta \nu' \left\{ -20kx^4 + 7kx^2 + 10k + 256r^2x^2 - 64r^2 \right\} r^{-5} - \frac{1}{128} \delta \nu''' k \left\{ x^2 - 1 \right\} r^{-2} \\
  &\quad + \frac{1}{128} \delta \lambda' \left\{ -52kx^4 + 55kx^2 + 128r^2x^2 - 64r^2 \right\} r^{-5} + \frac{1}{128} \delta \lambda''' k \left\{ x^2 - 1 \right\} x^2 r^{-3} \\
  &\quad - \frac{1}{128} \delta \nu''' k \left\{ (x^2 - 1)(8x^2 + 1) \right\} r^{-3} - \frac{1}{128} \delta \lambda'' k \left\{ 3x^4 - 13x^2 + 8 \right\} r^{-4}. \tag{6.4}
\end{align*}
\]

By construction, imposing the Sub-Horizon limit \( x \to 0 \) we get

\[
\varepsilon = \frac{1}{128} \delta \lambda' \left\{ k + 32r^2 \right\} r^{-3} - \frac{1}{64} \delta \lambda \left\{ k - 32r^2 \right\} r^{-4}
\]
and
\[ P r^{-2} = \frac{1}{64} \delta \nu'' \left\{ 5k + 32r^2 \right\} r^{-4} - \frac{1}{64} \delta \nu' \left\{ 5k - 32r^2 \right\} r^{-5} \]
\[ - \frac{1}{8} \delta \lambda' r^{-3} + \frac{1}{8} \delta \lambda k r^{-6} + \frac{1}{128} \delta \nu''' k r^{-3} - \frac{1}{16} \delta \lambda'' k r^{-4} - \frac{1}{128} \delta \nu''' k r^{-2}. \] (6.6)

This result confirms that ghosts are still present, and no value for \( k \neq 0 \) seems to cure them.

In the following discussion we suggest a mechanism for the generation of ghost-free models for perturbed Ricci-inverse Eqs. (4.12), (4.13). In fact, repeating the same procedure as in the previous case (6.1), it should be proved that generally the same divergence-free behavior of Eqs. (6.5), (6.6) can be reproduced when we set
\[ f(R, A) = R + k R^4 i A^{2+i} - 2\Lambda \quad i \in \mathbb{Z}. \] (6.7)

Starting from this evidence, being Eqs. (4.12) and (4.13) linear in \( f(R, A) \), one can build a divergence-free model considering a parametrized linear combination of functional independent profiles like (6.7). This ensures that then, if possible, we can avoid ghosts by fine-tuning the free parameters in such a way that higher-order derivatives in \( \delta \nu, \delta \lambda \) are zero.

Here we present a simple example of what just discussed to show that this is exactly what happens. Consider the model
\[ f(R, A) = R + \frac{\ell_1}{6} A^2 + \frac{\ell_2}{12} R^3 A + \frac{\ell_3}{12} R^3 A + \frac{\ell_4}{12} R A^2 - 2\Lambda, \] (6.8)
with \( \ell_1, \ell_2, \ell_3, \ell_4 \) constant parameters. Referring to Eqs. (4.12), (4.13), we compute the profile (6.8) and substitute the background Eq. (3.3). Due to their divergence-free nature, the resulting Sub-Horizon Weak-Field equations are
\[ \varepsilon = \frac{1}{1536} \delta \lambda \left\{ 8\ell_1 + 1536r^2 + 16384 + 9\ell_4 \right\} r^{-3} - \frac{1}{1536} \delta \lambda \left\{ 8\ell_1 - 1536r^2 + 16384\ell_3 + 9\ell_4 \right\} r^{-4} \]
\[ - \frac{1}{3072} \delta \nu''' \left\{ 96\ell_2 + \ell_1 - 4096\ell_3 \right\} r^{-1} - \frac{1}{3072} \delta \lambda'' \left\{ 8\ell_1 + 16384\ell_3 + 9\ell_4 \right\} r^{-2} \]
\[ - \frac{1}{768} \delta \nu''' \left\{ 96\ell_2 + \ell_1 - 4096\ell_3 \right\} - \frac{1}{3072} \delta \lambda'' \left\{ 8\ell_1 + 16384\ell_3 + 9\ell_4 \right\} r^{-1}, \] (6.9)
and
\[ Pr^{-2} = \frac{1}{1536} \delta \nu'' \left\{ 768\ell_2 + 20\ell_1 + 768r^2 + 16384\ell_3 + 15\ell_4 \right\} r^{-4} \]
\[ - \frac{1}{1536} \delta \nu' \left\{ 768\ell_2 + 20\ell_1 - 768r^2 + 16384\ell_3 + 15\ell_4 \right\} r^{-5} \]
\[ + \frac{1}{3072} \delta \lambda' \left\{ 768\ell_2 - 1536r^2 - 49152\ell_3 - 9\ell_4 \right\} r^{-5} + \frac{1}{768} \delta \lambda' \left\{ 384\ell_2 + 16\ell_1 + 32768\ell_3 + 15\ell_4 \right\} r^{-6} \]
\[ + \frac{1}{3072} \delta \nu''' \left\{ 768\ell_2 + 4\ell_1 - 16384\ell_3 - 3\ell_4 \right\} r^{-3} - \frac{1}{3072} \delta \lambda'' \left\{ 1536\ell_2 + 32\ell_1 + 16384\ell_3 + 21\ell_4 \right\} r^{-4} \]
\[ - \frac{1}{6144} \delta \nu''' \left\{ 8\ell_1 + 16384\ell_3 + 9\ell_4 \right\} r^{-2} - \frac{1}{2048} \delta \lambda'' \left\{ 256\ell_2 - 16384\ell_3 - 3\ell_4 \right\} r^{-3}. \] (6.10)

We see that higher-order derivative terms proportional to \( \delta \nu''', \delta \nu'', \delta \lambda'' \) are now weighted by the presence of \( \ell_1, \ell_2, \ell_3, \ell_4 \) coefficients. As a consequence, it is easy to find that higher-order derivatives disappear in both the equations if simultaneously
\[ \begin{cases} \ell_1 + 96\ell_2 - 4096\ell_3 = 0 \\
256\ell_2 - 16384\ell_3 - 3\ell_4 = 0 \\
4\ell_1 + 768\ell_2 - 16384\ell_3 - 3\ell_4 = 0 \end{cases}. \] (6.11)
The resolution of this system of equations provides $\ell_1 = -128\ell_2$ and $\ell_3 = -\frac{1}{128}\ell_2$ and $\ell_4 = 128\ell_2$, that inserted into Eqs. (6.9) and (6.9) yeld

$$\delta\lambda' r + \delta\lambda = r^2\varepsilon \quad \delta\nu'' r + \delta\nu' - \delta\lambda' = 2Pr.$$  \hspace{1cm} (6.12)

Integrating the first equation and imposing the non-relativistic limit $P \ll \varepsilon$, it is simple to finally obtain from definitions (4.11) the Newtonian potential and curvature perturbations

$$\frac{d\Phi(r)}{dr} = \frac{M(r)}{r^2} \quad \frac{d\Psi(r)}{dr} = \frac{M(r)}{r^2}.$$  \hspace{1cm} (6.13)

Such result never seems obtained before for Ricci-inverse theories and it is completely consistent with GR at low scales.

7 Discussion and conclusions

We have analyzed the novel Ricci-inverse MG theory within the physical context of a non-relativistic static and spherically symmetric cosmic structure embedded into a de Sitter cosmology. Taking into account the Sub-Horizon non-relativistic Weak-Field limit, we found that field equations generally exhibit two manifest kind of instabilities, namely (i) divergences when the Sub-Horizon limit is imposed, and (ii) ghosts induced by terms depending on higher-order derivatives of metric potentials perturbations. Under this perspective, we have been able to prove a new version of no-go theorem applied to small scales. Our result rules out all the Ricci-inverse models proposed in [8] and [10] to verify or circumvent cosmological and inflationary no-go theorems. A further investigation led us to discuss possible ways out of the theorem and point out a machinery to build stable models. We then showed that such models are fully consistent with GR predictions on small scales. We leave for future studies a general contextualization of our approach and the search of related new cosmological and astrophysical phenomenology.

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