A Problem of Hsiang-Palais-Terng on Isoparametric Submanifolds

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Abstract
We solve the problem raised by Hsiang, Palais and Terng in [HPT]:
Is it possible to have an isoparametric foliation on $\mathbb{R}^{52}$ whose marked
Dynkin diagram is of type $D_4$ and with all multiplicities uniformly
equal to 4?

2000 Mathematical Subject classification: 53C42 (14M15; 57R20)
Key words and phrases: Isoparametric submanifolds; Vector bundles;
Characteristic classes.

1 Introduction
We assume familiarity with the notations, terminology and results developed
by Hsiang, Palais and Terng in [HPT].

Associated to an isoparametric submanifold $M$ in an Euclidean $n$-space
$\mathbb{R}^n$, there is a Weyl group with Dynkin diagram marked with multiplicities.
In this paper we show

Theorem. There is no isoparametric submanifold in $\mathbb{R}^{52}$ whose marked
Dynkin diagram is of type $D_4$ and with all multiplicities uniformly equal to 4.

This result solves Problem 1 raised by Hsiang, Palais and Terng in [HPT].
As was pointed out by the authors of [HPT], it implies also that

Corollary. There is no isoparametric submanifolds whose marked Dynkin
diagrams with uniform multiplicity 4 of $D_k$-type, $k > 5$ or $E_k$-type, $k = 6, 7, 8$. 

1
2 Isoparametric submanifolds of type $D_4$

Assume throughout this section that $M \subset \mathbb{R}^n$ is an irreducible isoparametric submanifold with uniform even multiplicity $m$ of $D_4$-type. With this assumption we have

$$\dim M = 12m$$ and $n = 12m + 4$.

The inner product on $\mathbb{R}^n$ will be denoted by $(\cdot, \cdot)$.

Fix a base point $a \in M$ and let $P \subset \mathbb{R}^n$ be the normal plane to $M$ at $a$. It is a subvector space with $\dim P = 4$.

Let $\Lambda$ be the focal set of the embedding $M \subset \mathbb{R}^n$. $P$ intersects $\Lambda$ at 12 linear hyperplanes (through the origin in $P$)

$$\Lambda \cap P = L_1 \cup \cdots \cup L_{12}.$$  

The reflection $\sigma_i$ of $P$ in the $L_i$, $1 \leq i \leq 12$, generate the Weyl group $W$ of type $D_4$, considered as a subgroup of the isometries of $P$.

It follows that we can furnish $P$ with an orthonormal basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ so that the set $\Phi = \{\alpha_i \in P \mid 1 \leq i \leq 12\}$ of positive roots of $W$ relative to $a \in P$ is given and ordered by (cf. [Hu, p.64])

$$\begin{align*}
\alpha_1 &= \varepsilon_1 - \varepsilon_2; & \alpha_2 &= \varepsilon_2 - \varepsilon_3; & \alpha_3 &= \varepsilon_3 - \varepsilon_4; \\
\alpha_4 &= \varepsilon_1 - \varepsilon_3; & \alpha_5 &= \varepsilon_2 - \varepsilon_4; & \alpha_6 &= \varepsilon_1 - \varepsilon_4; \\
\alpha_7 &= \varepsilon_2 + \varepsilon_1; & \alpha_8 &= \varepsilon_3 + \varepsilon_2; & \alpha_9 &= \varepsilon_4 + \varepsilon_3; \\
\alpha_{10} &= \varepsilon_3 + \varepsilon_1; & \alpha_{11} &= \varepsilon_4 + \varepsilon_2; & \alpha_{12} &= \varepsilon_4 + \varepsilon_1.
\end{align*}$$

Consequently, we order the planes $L_i$ by the requirement that each $\alpha_i$ is normal to $L_i$, $1 \leq i \leq 12$. As results we have

$$(2-1) \quad \alpha_1, \alpha_2, \alpha_3 \text{ and } \alpha_9 \text{ form the set of simple roots relative to } a, \text{ and the corresponding Cartan matrix is}$$

$$\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & -1 \\
0 & -1 & 2 & 0 \\
0 & -1 & 0 & 2
\end{pmatrix}; \quad \beta_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}.$$  

$$(2-2) \quad \text{The group } W \text{ is generated } \sigma_1, \sigma_2, \sigma_3 \text{ and } \sigma_9, \text{ whose actions on } P \text{ are given respectively by}$$

$$\begin{align*}
\sigma_1 : \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} &\rightarrow \{\varepsilon_2, \varepsilon_1, \varepsilon_3, \varepsilon_4\}, \\
\sigma_2 : \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} &\rightarrow \{\varepsilon_1, \varepsilon_3, \varepsilon_2, \varepsilon_4\}, \\
\sigma_3 : \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} &\rightarrow \{\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_3\}, \\
\sigma_9 : \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} &\rightarrow \{\varepsilon_1, \varepsilon_2, -\varepsilon_4, -\varepsilon_3\};
\end{align*}$$

$$(2-3) \quad \text{Let } b = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \in P. \text{ The subgroup } W_b \text{ of } W \text{ that fixes } b \text{ is generated by } \sigma_1, \sigma_2 \text{ and } \sigma_3;$$

Using (2-2) one verifies directly that

$$(2-4) \quad \text{the set simple roots can be expressed in term of the } W\text{-action on } \alpha_1 \text{ as}$$

$$\alpha_2 = \sigma_1 \sigma_2 (\alpha_1); \quad \alpha_3 = \sigma_2 \sigma_1 \sigma_3 \sigma_2 (\alpha_1); \quad \alpha_4 = \sigma_2 (\alpha_1);$$
\[ \alpha_5 = \sigma_1 \sigma_3 \sigma_2(\alpha_1); \quad \alpha_6 = \sigma_3 \sigma_2(\alpha_1); \]
\[ \alpha_7 = \sigma_2 \sigma_3 \sigma_2(\alpha_1); \quad \alpha_8 = \sigma_1 \sigma_3 \sigma_2 \sigma_2(\alpha_1); \quad \alpha_9 = \sigma_2 \sigma_1 \sigma_2 \sigma_2(\alpha_1); \]
\[ \alpha_{10} = \sigma_3 \sigma_2 \sigma_2(\alpha_1); \quad \alpha_{11} = \sigma_1 \sigma_2 \sigma_2(\alpha_1); \quad \alpha_{12} = \sigma_2 \sigma_2(\alpha_1). \]

**Remark 1.** (2-1) implies the following geometric facts. The planes \( L_i \) partition \( P \) into \( \left| W \right| = 2^3 \cdot 4! \) convex open hulls (called Weyl chambers), and the base point \( a \) is contained in the one \( \Omega \) bounded by the \( L_i, 1 \leq i \leq 3 \) and \( L_9 \). In (2-3) the point \( b \) lies on the edge \( L_1 \cap L_2 \cap L_3 \) of \( \Omega \). \( \Box \)

Let \( M_b \subset \mathbb{R}^n \) be the focal manifold parallel to \( M \) through \( b \) [HPT]. We have a smooth projection \( \pi : M \twoheadrightarrow M_b \) whose fiber over \( b \in M_b \) is denoted by \( F \).

Recall from [HPT] that the tangent bundle \( TM \) of \( M \) has a canonical splitting as the orthogonal direct sum of 12 subbundles
\[ TM = E_{\alpha_1} \oplus \cdots \oplus E_{\alpha_{12}}, \dim \mathbb{R} E_{\alpha_i} = m, \]
in which \( E_i \) is the curvature distribution of \( M \) relative to the root \( \alpha_i, 1 \leq i \leq 12 \). From [HPT] we have

**Lemma 1.** Let \( TN \) be the tangent bundle of a smooth manifold \( N \). Then

1. the subbundle \( \oplus_{1 \leq i \leq 6} E_{\alpha_i} \) of \( TM \) restricts to \( TF \);
2. the induced bundle \( \pi^* TM_b \) agrees with \( \oplus_{7 \leq i \leq 12} E_{\alpha_i} \). \( \Box \)

### 3 The cohomology of the fibration \( \pi : M \to M_b \)

Let \( b_i \in H_m(M; \mathbb{Z}), 1 \leq i \leq 12, \) be the homology class of the leaf sphere \( S_i(a) \subset M \) of the intergrable bundle \( E_{\alpha_i} \) through \( a \in M \), and let \( d_i \in H^m(M; \mathbb{Z}) \) be the Euler class of \( E_{\alpha_i} \).

**Lemma 2.** The Kronecker pairing \( <, > : H^m(M; \mathbb{Z}) \otimes H_m(M; \mathbb{Z}) \to \mathbb{Z} \) can be expressed in term of the inner product \( (, ) \) on \( P \) as
\[ < d_i, b_j > = 2 (\alpha_i, \alpha_j), 1 \leq i, j \leq 12. \Box \]

**Remark 2.** Let \( G \) be a compact connected semi-simple Lie group with a fixed maximal torus \( T \) and Weyl group \( W \). Fix a regular point \( a \) in the Cartan subalgebra \( L(T) \) of the Lie algebra \( L(G) \) corresponding to \( T \). The orbit of the adjoint action of \( G \) on \( L(G) \) through \( a \) yields an embedding \( G/T \to L(G) \) which defines the flag manifold \( G/T \) as an isoparametric submanifold in \( L(G) \) with associated Weyl group \( W \) and with equal multiplicities \( m = 2 \).

In this case Lemma 2 has its generality due to Bott and Samelson [BS], and the numbers \( 2 (\alpha_i, \alpha_j) \) are the Cartan numbers of \( G \) (only 0, \( \pm 1, \pm 2, \pm 3 \) can occur). \( \Box \)
Since the roots $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_9$ form a set of simple roots, the classes $b_1, b_2, b_3$ and $b_9$ constitute an additive basis of $H_m(M; \mathbb{Z})$. Since the $M$ is $m - 1$ connected, we specify a basis $\omega_1, \omega_2, \omega_3, \omega_9$ of $H^m(M; \mathbb{Z})$ (in term of Kronecker pairing) as

$$< b_i, \omega_j >= \delta_{ij}, i, j = 1, 2, 3, 9.$$  

It follows from Lemma 2 that

**Lemma 3.**  

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_9 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_9 \end{pmatrix}. \square$$

In term of the $\omega_i$ we introduce in $H^m(M; \mathbb{Z})$ the classes $t_i, 1 \leq i \leq 4$, by the relation

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_9 \end{pmatrix}. \ (3-1)$$

Conversely,

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}. \ (3-2)$$

The Weyl group $W$ (acting as isometries of $P$) has the effect to permute roots (cf. (2-3)). On the other hand, $W$ acts also smoothly on $M$ [HPT], hence acts as automorphisms of the cohomology (resp. homology) of $M$.

For an $w \in W$ we write $w^*$ (resp. $w_*$) for the induced action on the cohomology (resp. homology).

**Lemma 4.** With respect to the $\mathbb{Q}$ basis $t_1, t_2, t_3, t_4$ of $H^2(M; \mathbb{Q})$, the action of $W$ on $H^2(M; \mathbb{Q})$ is given by

$$\sigma_i^1: \{t_1, t_2, t_3, t_4\} \rightarrow \{t_2, t_1, t_3, t_4\},$$

$$\sigma_i^2: \{t_1, t_2, t_3, t_4\} \rightarrow \{t_1, t_3, t_2, t_4\},$$

$$\sigma_3^2: \{t_1, t_2, t_3, t_4\} \rightarrow \{t_1, t_2, t_3, t_4\},$$

$$\sigma_3^3: \{t_1, t_2, t_3, t_4\} \rightarrow \{t_1, t_2, t_3, t_4\}.$$  

**Proof.** Let $i, j, k = 1, 2, 3, 9$. In term of the Cartan matrix (2-1), the action of $\sigma_i$ on the $\mathbb{Z}$-basis $b_1, b_2, b_3, b_9$ has been determined in [HPT] as

$$\sigma_i(b_j) = b_j + \beta_{ij}b_i \quad (3-3)$$

(ef. the proof of 6.11. Corollary in [HPT]). By the naturality of Kronecker pairing

$$< \sigma_i^1(\omega_k), b_i >= < \omega_k, \sigma_i^2(b_j) >= \delta_{kj} + \beta_{ij}\delta_{ki},$$

we get from (3-3) that

4
\( (3-4) \quad \sigma_i^*(\omega_k) = \begin{cases} \omega_k & \text{if } i \neq k; \\ \omega_k - (\beta_{k1}\omega_1 + \beta_{k2}\omega_2 + \beta_{k3}\omega_3 + \beta_{k4}\omega_4) & \text{if } i = k. \end{cases} \)

With \((\beta_{ij})\) being given explicitly in (2-1), combining (3-1), (3-4) with (3-2) verifies Lemma 4. \(\square\)

Let the algebra \(Q[t_1, t_2, t_3, t_4]\) of polynomials in the variables \(t_1, t_2, t_3, t_4\) be graded by \(\deg(t_i) = m, 1 \leq i \leq 4\). Let \(e_i \in Q[t_1, t_2, t_3, t_4]\) (resp. \(\theta_i \in Q[t_1, t_2, t_3, t_4]\)) be the \(i^{th}\) elementary symmetric functions in \(t_1, t_2, t_3, t_4\) (resp. in \(t_1^2, t_2^2, t_3^2, t_4^2\)), \(1 \leq i \leq 4\). We note that the \(\theta_i\) can be written as a polynomial in the \(e_i\)

\[\theta_1 = e_1^2 - 2e_2; \quad \theta_2 = e_2^2 - 2e_1 e_3 + 2e_4; \quad \theta_3 = e_3^2 - 2e_2 e_4.\]

**Lemma 5.** In term of generator-relations the rational cohomology of \(M\) is given by

\[H^*(M; Q) = Q[t_1, t_2, t_3, t_4]/Q^+[\theta_1, \theta_2, \theta_3, e_4].\]

Further, the induced homomorphism \(\pi^*: H^*(M_b; Q) \to H^*(M; Q)\) maps the algebra \(H^*(M_b; Q)\) isomorphically onto the subalgebra \(Q[e_1, e_2, e_3]/Q^+[\theta_1, \theta_2, \theta_3]\).

**Proof.** It were essentially shown in [HPT, 6.12. Theorem; 6.14. Theorem] that for any \(Q\) basis \(y_1, y_2, y_3, y_4\) of \(H^m(M; Q)\) one has the grade preserving \(W\)-isomorphisms

\[H^*(M; Q) = Q[y_1, y_2, y_3, y_4]/Q^+[y_1, y_2, y_3, y_4]^W; \quad H^*(M_b; Q) = Q[y_1, y_2, y_3, y_4]^W_b/Q^+[y_1, y_2, y_3, y_4]^W_b,\]

where the \(W\)-action on the \(Q\)-algebra \(Q[y_1, y_2, y_3, y_4]\) is induced from the \(W\)-action on the \(Q\)-vector space \(H^m(M; Q) = \text{span}_Q\{y_1, y_2, y_3, y_4\}\) and where \(Q[y_1, y_2, y_3, y_4]^W_b\) (resp. \(Q^+[y_1, y_2, y_3, y_4]^W\)) is the subalgebra of \(W_b\)-invariant polynomials (resp. \(W\)-invariant polynomials in positive degrees).

Since the transition matrix from \(t_1, t_2, t_3, t_4\) to the \(\mathbb{Z}\)-basis \(\omega_1, \omega_2, \omega_3, \omega_9\) of \(H^m(M; \mathbb{Z})\) is non-singular by (3-1), the \(t_i\) constitute a basis for \(H^m(M; Q)\). Moreover, it follows from Lemma 4 that

\[Q[t_1, t_2, t_3, t_4]^W_b = Q[e_1, e_2, e_3, e_4], \quad Q[t_1, t_2, t_3, t_4]^W = Q[\theta_1, \theta_2, \theta_3, e_4].\]

This completes the proof. \(\square\)

**Remark 3.** If \(G = SO(2n)\) (the special orthogonal group of rank \(2n\)), the embedding \(G/T \to L(G)\) considered in Remark 2 gives rise to an isoparametric submanifold \(M = SO(2n)/T\) in the Lie algebra \(L(SO(2n))\) which is of \(D_n\)-type with equal multiplicities \(m = 2\). The corresponding \(M_b\) is known as the Grassmannian of complex structures on \(\mathbb{R}^{2n}\) [D]. In this case Borel computed the algebras \(H^*(M; Q)\) and \(H^*(M_b; Q)\) in [B] which are compatible with Lemma 5. \(\square\)
4 Computation in the Pontrijagin classes

Turn to the case $m = 4$ concerned by our Theorem. Denote by $p_1(\xi) \in H^4(X;\mathbb{Z})$ for the first Pontrijagin class of a real vector bundle $\xi$ over a topological space $X$.

Lemma 6. For an $w \in W$ and an $\alpha \in \Phi$ one has
$$w^*(p_1(E_\alpha)) = p_1(E_{w^{-1}(\alpha)}).$$
In particular, (2-3) implies that
$$H\mathbb{Z}(\alpha)\implies H\mathbb{Z}(\alpha).$$

Proof. In term of the $W$-action on the set of roots, the induced bundle
$$w^*(E_\alpha) \in E_{w^{-1}(\alpha)}$$
(cf. [HTP, 1.6]). Lemma 6 comes now from the naturality of Pontrijagin classes and from (2-4).□

Lemma 7. $p_1(E_{\alpha_1}) = k(t_1 + t_2 - t_3 - t_4)$ for some $k \in \mathbb{Q}$.

Proof. In view of Lemma 4 we can assume that
$p_1(E_{\alpha_1}) = k_1 t_1 + \cdots + k_4 t_4,$ $k_i \in \mathbb{Q}.$

Since the restricted bundle $E_{\alpha_1} | S_1(a)$ is the tangent bundle of the 4-sphere
$S_1(a)$ and therefore is stably trivial, we have
$$< p_1(E_{\alpha_1}), b_1 > = k_1 - k_2 = 0.$$

That is
$$p_1(E_{\alpha_1}) = k t_1 + k t_2 + k t_3 + k t_4.$$

Since the actions of the $\alpha_i^*$, $i = 1, 2, 3, 9$, on the $t_j$ are known by Lemma 4,
combining (4-1) with the relations in Lemma 6 yields
$$p_1(E_{\alpha_2}) = k_3 t_1 + k_3 t_2 + k_3 t_3 + k_4 t_4;$$
$$p_1(E_{\alpha_3}) = k_3 t_1 + k_4 t_2 + k_3 t_3 + k_4 t_4;$$
$$p_1(E_{\alpha_4}) = k t_1 + k t_2 + k t_3 + k t_4;$$
$$p_1(E_{\alpha_5}) = k_3 t_1 + k t_2 + k_4 t_3 + k t_4;$$
$$p_1(E_{\alpha_6}) = k t_1 + k t_2 + k t_3 + k t_4;$$
$$p_1(E_{\alpha_7}) = k t_1 - k t_2 + k t_3 - k t_4;$$
$$p_1(E_{\alpha_8}) = k_3 t_1 + k t_2 - k t_3 - k t_4;$$
$$p_1(E_{\alpha_9}) = k_3 t_1 - k t_2 + k t_3 - k t_4;$$
$$p_1(E_{\alpha_{10}}) = k t_1 + k t_2 - t_3 - k t_4;$$
$$p_1(E_{\alpha_{11}}) = k_3 t_1 + k t_2 - k t_3 - k t_4;$$
$$p_1(E_{\alpha_{12}}) = k t_1 + k t_2 - k t_3 - k t_4.$$

Since the tangent bundle of any isoparametric submanifold is stably trivial, we get from $TM = E_{\alpha_1} \oplus \cdots \oplus E_{\alpha_{12}}$ that
Comparing the coefficients of \( t_1 \) on both sides of the equation turns out \( k_3 = -k \). Substituting this in (4-2) gives rise to, in particular, that

\[
\begin{align*}
  p_1(E_{\alpha_7}) &= kt_1 - kt_2 - kt_3 - k_4t_4; \\
  p_1(E_{\alpha_8}) &= -kt_1 + kt_2 - kt_3 - k_4t_4; \\
  p_1(E_{\alpha_9}) &= -kt_1 - k_4t_2 + kt_3 - kt_4; \\
  p_1(E_{\alpha_{10}}) &= kt_1 - kt_2 - kt_3 - k_4t_4; \\
  p_1(E_{\alpha_{11}}) &= -kt_1 + kt_2 - k_4t_3 - kt_4; \\
  p_1(E_{\alpha_{12}}) &= kt_1 - kt_2 - k_4t_3 - kt_4.
\end{align*}
\]

Finally, since

\[
p_1(E_{\alpha_7}) + \cdots + p_1(E_{\alpha_{12}}) = \pi^* p_1(TM_b) \in \text{Im } [\pi^* : H^*(M_b; \mathbb{Q}) \to H^*(M; \mathbb{Q})]
\]

by (2) of Lemma 1, it must be symmetric in \( t_1, t_2, t_3, t_4 \) by (2) of Lemma 4. Consequently, \( k_4 = -k \). This completes the proof of Lemma 6. □

We emphasis what we actually need in the next result.

**Lemma 8.** If \( M \subset \mathbb{R}^{52} \) is an irreducible isoparametric submanifold with uniform multiplicity 4 of \( D_4 \)-type, there exists a 4-plane bundle \( \xi \) over \( M \) whose Euler and the first Pontrjagin classes are respectively

\[
\begin{align*}
e(\xi) &= 2\omega_1 - \omega_2; \\
p_1(\xi) &= 2k(\omega_2 - \omega_4)
\end{align*}
\]

for some \( k \in \mathbb{Z} \).

**Proof.** Take \( \xi = E_{\alpha_1} \). Then \( e(\xi) = d_1 = 2\omega_1 - \omega_2 \) by Lemma 3 and

\[
p_1(\xi) = k (t_1 + t_2 - t_3 - t_4) \text{ (by Lemma 7)} = 2k(\omega_2 - \omega_9) \text{ (by (3-1))}
\]

for some \( k \in \mathbb{Q} \). Moreover we must have \( k \in \mathbb{Z} \) since

1. \( M \) is 3 connected and the classes \( \omega_1, \omega_2, \omega_3, \omega_9 \) constitute an additive basis for \( H^4(M; \mathbb{Z}) \); and since
2. the first Pontrjagin class of any vector bundle over a 3 connected CW-complex is an integer class and is divisible by 2 [LD]. □

## 5 A topological constraint on isoparametric submanifolds with equal multiplicity 4

Let \( Vec^m(S^n) \) be the set of isomorphism classes of Euclidean \( m \)-vector bundles over the \( n \)-sphere \( S^n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \cdots + x_{n+1}^2 = 1 \} \).

If \( n = m = 4 \) we introduce the map \( f : Vec^4(S^4) \to \mathbb{Z} \oplus \mathbb{Z} \) by

\[
f(\xi) = (\langle e(\xi), [S^4] \rangle, \langle p_1(\xi), [S^4] \rangle),
\]

where

1. \([S^4] \in H_4(S^4; \mathbb{Z}) = \mathbb{Z}\) is a fixed orientation class;
2. \(\langle, \rangle\) is the Kronecker pairing between cohomology and homology;

and where
(3) \(e(\xi)\) and \(p_1(\xi)\) are respectively the Euler and the first Pontrijagin classes of \(\xi \in Vect^4(S^4)\).

**Example** (cf. [MS, p.246]). Let \(\tau \in Vect^4(S^4)\) be the tangent bundle of \(S^4\), and let \(\gamma \in Vect^4(S^4)\) be the real reduction of the quaternionic line bundle over \(HP^1 = S^4\) (1-dimensional quaternionic projective space). Then \(f(\tau) = (2,0)\); \(f(\gamma) = (1,-2)\).

Our theorem will follow directly from Lemma 8 and the next result that improves Lemma 20.10 in [MS].

**Lemma 9.** \(f\) fits in the short exact sequence
\[
0 \to Vect^4(S^4) \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{g} \mathbb{Z}_4 \to 0,
\]
where \(g(a,b) \equiv (2a-b) \mod 4\).

**Proof of the Theorem.** Assume that there exists an irreducible isoparametric submanifold \(M \subset R^{52}\) with uniform multiplicity 4 of \(D_4\)-type. Let \(\xi\) be a 4-plane bundle over \(M\) whose Euler and the first Pontrijagin classes are given as that in (4-4).

Let \(\xi_1, \xi_2 \in Vect^4(S^4)\) be obtained respectively by restricting \(\xi\) to the leaf spheres \(S_2(a), S_9(a)\) (cf. section 3). By the naturality of characteristic classes we get from (4-4) that
\[
f(\xi_1) = (-1, 2k); \quad f(\xi_2) = (0, -2k),
\]
where the integer \(k\) must satisfy the congruences
\[
-2 \equiv 2k \mod 4; \quad 0 \equiv -2k \mod 4.
\]
by Lemma 9. The proof is done by the obvious contradiction. □

It suffices now to justify Lemma 9.

Let \(SO(m)\) be the special orthogonal group of rank \(m\) and denote by \(\pi_r(X)\) the \(r\)-homotopy group of a topological space \(X\). The Steenrod correspondence is the map \(s : Vect^m(S^n) \to \pi_{n-1}(SO(m))\) defined by
\[
s(\xi) = \text{the homotopy class of a clutching function } S^{n-1} \to SO(m) \text{ of } \xi.
\]
In [S, §18], Steenrod showed that

**Lemma 10.** \(s\) is a one-to-one correspondence. □

It follows that \(Vect^m(S^n)\) has a group structure so that \(s\) is a group isomorphism. We clarify this structure in Lemma 11.

Fix a base point \(s_0 = (1,0,\cdots,0) \in S^n\). For two \(\xi, \eta \in Vect^m(S^n)\) write \(\xi \lor \eta\) for the \(m\)-bundle over \(S^n \lor S^n\) (one point union of two \(S^n\) over \(s_0\)) whose restriction to the first (resp. the second) sphere agrees with \(\xi\) (resp. \(\eta\)). Define the addition \(+ : Vect^m(S^n) \times Vect^m(S^n) \to Vect^m(S^n)\) and the inverse \(- : Vect^m(S^n) \to Vect^m(S^n)\) operations by the rules
\[
\xi + \eta = \mu^*(\xi \lor \eta) \quad \text{and} \quad -\xi = \nu^*\xi,
\]

\[8\]
where \( \xi, \eta \in Vect^m(S^n) \), \( \mu : S^n \to S^n \vee S^n \) is the map that pinches the equator \( x \in S^n, x_{n+1} = 0 \) to the base point \( s_0 \) and where \( \nu : S^n \to S^n \) is the restriction of the reflection of \( \mathbb{R}^{n+1} \) in the hyperplane \( x_{n+1} = 0 \). It is straightforward to see that

**Lemma 11.** With respect to the operations \(+\) and \(−\)

1. \( Vect^m(S^n) \) is an abelian group with zero \( \varepsilon^m \), the trivial \( m \)-bundle over \( S^n \) and
2. the maps \( s \) and \( f \) are homomorphisms. \( \square \)

We are ready to show Lemma 9.

**The proof of Lemma 9.** It is essentially shown by Milnor [MS, p.245] that \( f \) is injective and satisfies \( \text{Im} \, f \supseteq \text{Ker} \, g \). It suffices to show that \( \text{Im} \, f \subseteq \text{Ker} \, g \).

Assume on the contrary that there is a \( \xi \in Vect^4(S^4) \) such that \( f(\xi) = (a, b) \) with \( 2a - b = 4k + i, i = 1, 2, 3 \). Moreover, one must has \( i = 2 \) since the first Pontrijagin class of any vector bundle over \( S^4 \) is divisible by 2. That is

(5.1) \( f(\xi) = (a, 2a - 4k - 2) \)

Using the group operations in \( Vect^4(S^4) \) we form the class

\( \hat{\xi} = -\xi - (a - 2k - 1)\gamma + (a - k)\tau, \)

where \( \tau \) and \( \gamma \) were given in the Example. By the additivity of \( f \) we get from the Example and (5.1) that

(5.2) \( f(\hat{\xi}) = (1, 0). \)

Consider the following diagram

\[
\begin{array}{ccc}
0 & \to & \pi_4(S^4) \xrightarrow{\partial} \pi_3(SO(4)) = Vect^4(S^4) \xrightarrow{i_*} \pi_3(SO(5)) = Vect^5(S^4) \to 0 \\
& & \downarrow p_1 \swarrow \uparrow p_1 \\
& & H^4(S^4; \mathbb{Z})
\end{array}
\]

in which

1. the top row is a section in the homotopy exact sequence of the fibration \( SO(4) \hookrightarrow SO(5) \to S^4 \) (cf. [Wh, p.196]);
2. \( p_1 \) assigns a bundle with its first Pontrijagin class;
3. via the Steenrod isomorphism, the homomorphism \( i_* \) induced by the fibre inclusion corresponds to the operation \( \zeta \to \zeta \oplus \varepsilon^1 \), where \( \oplus \) means Whitney sum and where \( \varepsilon^1 \) is the trivial 1-bundle over \( S^4 \); and
4. the map triangle commutes by the stability of Pontrijagin classes.

From the Bott-periodicity we have \( \pi_3(SO(5)) = \mathbb{Z} \). It is also known that \( p_1 : Vect^5(S^4) \to H^4(S^4; \mathbb{Z}) \) is surjective onto the subgroup \( 2H^4(S^4; \mathbb{Z}) \subset H^4(S^4; \mathbb{Z}) \). Summarizing we have

(5) \( p_1 : Vect^5(S^4) \to H^4(S^4; \mathbb{Z}) \) is injective.

From (5.2) we find that \( p_1(\hat{\xi}) = 0 \). As a result (4) and (5) imply that \( i_*(\hat{\xi}) = 0 \). From the exactness of the top sequence one concludes
\[ \hat{\xi} = k \partial(\iota_4) \text{ for some } k \in \mathbb{Z}, \]
where \( \iota_4 \in \pi_4(S^4) = \mathbb{Z} \) is the class of identity map. Since \( \partial(\iota_4) = \tau \) (cf. [Wh, p.196]) we have \( f(\hat{\xi}) = (2k, 0) \) by the Example. This contradiction to (5-2) completes the proof. □

**Remark 4.** The proof of Lemma 9 indicates that the bundles \( \tau \) and \( \gamma \) in the Example generate the group \( Vect^4(S^4) \). These two bundles were used by Milnor in [MS, p.247] to illustrate his original construction of different differential structures on the 7-sphere in 1956 [M]. □

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