External geometry of $p$-minimal surfaces

Tkachev Vladimir G.

Abstract. A surface $\mathcal{M}$ is called $p$-minimal if one of the coordinate functions is $p$-harmonic in the inner metric. We show that in the two-dimensional case the Gaussian map of such surfaces is quasiconformal. In the case when the surface is a tube we study the geometrical structure of such surfaces. In particular, we establish the second order differential inequality for the form of the sections of $\mathcal{M}$ which generalizes the known ones in the minimal surfaces theory.

1. Introduction

1.1. Let $\mathcal{M} = (M; x)$ be a surface given by a $C^2$-immersion $x : M \to \mathbb{R}^{n+1}$ of $n$-dimensional orientable noncompact manifold $M$.

Definition 1. A surface $\mathcal{M}$ is said to be minimal if its mean curvature vector $H(m) \equiv 0$.

The well-known property of the minimal immersions in the Euclidean space is harmonicity of their coordinate functions. Moreover, if one coordinate function of an immersion is harmonic then all coordinates satisfy this property and the immersion is minimal. On the other hand, for $n = 2$ this condition yields the fact that the Gauss map of such surfaces is conformal [14].

The natural question arises: what happens if we replace the requirement of harmonicity by $p$-harmonicity?

Definition 2. For a fixed $p > 1$ a surface $\mathcal{M}$ is said to be $p$-minimal if one of the coordinate functions is $p$-harmonic with respect to the inner metric of $\mathcal{M}$. Other words, there exists a direction $e \in \mathbb{R}^n$ such that

$$\Delta_p f \equiv \text{div}|\nabla f|^p - 2\nabla f = 0,$$

where $f(m) = \langle x(m), e \rangle$ and $\nabla$ is the covariant derivative on $M$.

One easily can be shown that $p$-harmonicity of one coordinate function can't be extended on the others provided $p \neq 2$. This means that $\mathbb{R}^{n+1}$ is equipped with a specified direction $e$. This kind of asymmetry is typical for the Minkowski spaces $\mathbb{R}^{n+1}_1$ with time-axis $Oe$. Another example is the tubular minimal surfaces (see the definition below) which are Euclidean analogous of the relative strings in nuclear physics.

The paper was supported by RFRF, project 93-011-176.
We should also mention that equation (1) is of great importance in the non-linear potential theory [2] and the elliptic type PDE’s [7], [4]. On the other hand, it is closely linked to quasiregular mappings (see [3] for detailed information).

In the first part of our paper we discuss the basic facts of $p$-minimal surfaces. In particular, we show that the Gauss map of a two-dimensional $p$-minimal surface is $K(p)$-quasiconformal. In section 3 we develop a new method for studying the shape of $p$-minimal tubes. We establish also the estimates for the sizes of cross-sections of such surfaces which provides us by information about the evolution of a $p$-minimal tube in the ‘time-direction’.

It is also the aim of this paper to demonstrate the properties which are common for tubular minimal and $p$-minimal surfaces. We only briefly discuss the non-parametric case for $p$-minimal surfaces. The general theory of $p$-minimal surfaces and some examples will be given in a forthcoming paper.

2. Preliminary properties of $p$-minimal surfaces

2.1. We have noticed above that the case $p = 2$ corresponds to the minimal surfaces. To clarify the geometrical meaning of (1) for arbitrary $p$ we denote by $k_e(m)$ the curvature of $\mathcal{M}$ in $e$-direction (i.e. the curvature of the section of $\mathcal{M}$ by 2-plane spanning on $e$ and the unit normal $\nu$ to $\mathcal{M}$ at a point $m$).

**Proposition 1.** Let $m$ be a noncritical point, i.e. $e \wedge \nu(m) \neq 0$. Then

\[ H(m) = -(p - 2)k_e(m). \]

**Proof.** Really, let $\nabla$ and $\nabla$ denote the standart covariant derivatives in $\mathbb{R}^{n+1}$ and $\mathcal{M}$ respectively. Then

\[ \nabla f(m) = (\nabla x(m), e)^T = e^T, \]

where $e^T$ is the projection of $e$ onto the tangent space to the surface $\mathcal{M}$ at a point $m$. It follows from the assumptions of the proposition that $|e^T| \neq 0$ or, it is the same $|\nabla f(m)| \neq 0$. Thus, for any tangent vector $X$ we obtain

\[ \nabla_X |\nabla f| = \nabla_X |e^T| = \frac{\nabla_X e^T, e^T}{|e^T|} = \frac{\nabla_X (-e^T), e^T}{|e^T|} = \langle e, \nu \rangle \langle A(X), \frac{e^T}{|e^T|} \rangle. \]

Here $A$ is the Weingarten map of $\mathcal{M}$ and $e^\perp$ is the projection of $e$ onto the normal space to $\mathcal{M}$. By virtue of the symmetry of $A$ we conclude that

\[ \nabla |\nabla f| = \langle e, \nu \rangle A(\tau), \]

where $\tau = e^T/|e^T|$ is well defined at $m$. After substituting (3) into (1) we have

\[ \Delta_p f = \text{div}(||\nabla f||^{p-2} \nabla f) = (p - 2)||\nabla f||^{p-3} \langle \nabla f, \nabla (||\nabla f||) \rangle + \]

\[ +||\nabla f||^{p-2} \Delta f = ||\nabla f||^{p-4} \langle e, \nu \rangle (|e^T|^2 \Delta f + (p - 2)\langle A(e^T), e^T \rangle). \]
Finally, the definition of $k_e(m)$ together with the well known connection between the mean curvature $H(m)$ and the inner Laplacian [6]: $\Delta f(m) = H(m)\langle e, \nu \rangle$ yield from (4)

(5) $\Delta_p f = |\nabla f|^{p-4}\langle e, \nu \rangle|e^\top|^2 \left( H(m) + (p-2)k_e(m) \right)$,

everywhere in the regular part $M_0 \equiv \{ m \in M : |e^\top(m)| \neq 0 \}$.

We assume now that equality (2) doesn’t hold at some noncritical point $m_1 \in M_0$. Then in view of (5) and (1), $\langle e, \nu(m_1) \rangle \equiv 0$, and by continuity of the expression in parentheses in (5), the last identity holds everywhere in some neighbourhood $\Omega(m_1)$. Thus, in $\Omega(m_1) \cap M_0$ the coordinate function $f(m) = \langle e, x \rangle$ is constant and, it follows that $A \equiv 0$ in $\Omega(m_1)$. But this conclusion trivially yields validity of (2) which contradicts our assumption.

The following assertion is an immediate consequence of the Meusnier theorem.

**Corollary 1.** The mean curvature $H$ of a $p$-minimal surface $M$ and the mean curvature $h$ of the section $\Sigma(\tau)$ linked by

(6) $h(m) = -\frac{p-1}{\omega}k_e(m) = \frac{p-1}{p-2} \frac{H(m)}{\omega}$

where $\omega = \langle \nu_m, e \rangle$.

We use further the auxiliary assertion which clarifies the local structure of a $p$-minimal surface near a critical point. We notice that this property has not an analogue in minimal surfaces theory.

**Lemma 1.** Let $M$ be a $p$-minimal surface given as a graph of $C^2$-function $f(x)$ defined in a domain $G \subset \mathbb{R}^n$. Let $x_0 \in G$ be a critical point of $f(x)$, i.e. $\nabla f(x_0) = 0$. Then the Hessian $\nabla^2 f$ is degenerate. Other words, $x_0$ is a planar point.

**Proof.** To prove this assertion we rewrite (2) in a more suitable way. In the local coordinates we have the following formulas for the mean curvature $H(m)$ and the Laplace-Beltrami operator $\Delta$ respectively

$$H(m) = \frac{1}{g^{3/2}} \sum_{i,j=1}^n (g\delta_{ij} - \nabla_i f \nabla_j f) \nabla_i^2 f,$$

(7) $$\Delta u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \nabla_j (g^{ij} \sqrt{g} \nabla_j u),$$

where $\nabla_i$ denotes the covariant derivative along the coordinate vector $e_i$, $g^{ij}$ is the inverse matrix to the metric tensor $g_{ij} = \delta_{ij} + \nabla_i f \nabla_j f$ and $g = \det \| g_{ij} \|^2$. Hence, we obtain from (1) and (4)

(8) $$g|\nabla f|^2 \text{tr} \nabla^2 f + \sum_{i,s=1}^n (p-2 - |\nabla f|^2) \nabla_i f \nabla_s f \nabla_{is}^2 f = 0,$$
\( \nabla^2 f \) is the Hessian of \( f(x) \) and the trace \( \text{tr} \nabla^2 f \) is equal to the euclidean Laplace operator in \( \mathbb{R}^n \). Write \( a_{ij} = \nabla^2_{ij}(x_0) \) and \( A = |\!|a_{ij}\!|\| \). Then for an appropriate choice of \( \varepsilon > 0 \) and every vector \( y \in \mathbb{R}^n \) such that \( |y| < \varepsilon \) we have

\[
\nabla_k f(x_0 + y) = \sum_{i=1}^{n} a_{ki} y_i + o(|y|),
\]

and

\[
|\nabla f(x_0 + y)|^2 = O(|y|^2).
\]

Substituting these relations in (8), we arrive at

\[
\sum_{k,l,s=1}^{n} \sum_{i,j=1}^{n} (a_{ki}a_{kj}\text{tr}A + (p-2)a_{li}a_{sj}a_{ls})y_iy_j = o(|y|^2).
\]

Taking into account validity of the last equality for all sufficiently small \( y \in \mathbb{R}^n \) we obtain a matrix equation

\[
A^2(I \text{tr}A + (p-2)A) = 0,
\]

where \( I \) is the unit matrix. By virtue of symmetry of the Hessian \( A \) we can choose the orthonormal basis of \( \mathbb{R}^n \) consisting of the eigenvectors of \( A \). Namely, \( A \) takes a diagonal form \( \lambda_i \delta_{ij} \) and from (9) we have for \( i : 1 \leq i \leq n \),

\[
\lambda_i \left( \lambda_i (p-2) + \text{tr}A \right) = 0.
\]

We see from the last identity that all non-zero eigenvalues \( \lambda_i \) must be equal to \( -(p-2)^{-1} \text{tr}A \). Let \( \lambda_1, \ldots, \lambda_k \) be all such numbers. Then after summing we obtain

\[
\text{tr}A = \sum_{i=1}^{k} \lambda_i = -\frac{k}{p-2} \text{tr}A.
\]

On the other hand \( \text{tr}A = k\lambda_1 \neq 0 \). It follows from (10) that \( p = 2 - k \), where \( k \geq 1 \) is a positive integer. But it contradicts with \( p > 1 \) and hence, all of \( \lambda_i \) are zeroes. Now the theorem follows from the standard properties of symmetric matrices.  \( \square \)

2.2. Given a surface \( \mathcal{M} \) in \( \mathbb{R}^3 \) we denote by \( \gamma(m) : \mathcal{M} \to S^2 \) the standard Gauss map. A result of Gauss state that, if the surface is minimal that map is conformal. Here we extend this property on \( p \)-minimal surfaces. First we remind

**Definition 3.** ([1], [2]) A map \( F : M_1 \to M_2 \) of two smooth Riemannian manifolds \( M_1 \) \( M_2 \) is called a *quasiconformal map* if the Jacobian \( \det d_x F \) doesn’t change the sign on \( M_1 \) and for almost every \( x \in M_1 \),

\[
\max |d_x F(E)| \leq K_m \min |d_x F(E)|
\]

where \( \min \) and \( \max \) are given over all unit tangent vectors \( E \) of \( T_x M_1 \). The number \( K = \max_{m \in M_1} K_m \) is called the *distortion coefficient* of \( F \).
Theorem 1. Let $\mathcal{M}$ be a two-dimensional $p$-minimal surface in $\mathbb{R}^3$. Then the Gauss map is $K(p)$-quasiconformal map with the distortion coefficient

$$K(p) = \max\{p - 1; 1/(p - 1)\}.$$ (12)

**Proof.** We notice that the tangent spaces $T_m M$ to $M$ and $T_{\gamma(m)} S^2$ to the unit sphere $S^2$ can be regarded as canonical isomorphic ones. Really, we identify the vector $A(E)$ with $d\gamma_m(E)$, where $d\gamma_m$ is the differential of the Gauss map at $m$. We specify a point $m \in M$ and choose the orthonormal basis $E_1, E_2$ of the tangent space $T_m M$ which diagonalizes $A$, i.e.

$$A(E_i) = \lambda_i E_i.$$ 

Here $\lambda_1, \lambda_2$ are the principal curvatures of $\mathcal{M}$ at $m$. Without loss of generality we can arrange that $|e^T(m)| \neq 0$. Really Lemma 1 yields that the homomorphism $A$ is the identical zero and (11) is trivial.

Let us denote $\tau = e^T/|e^T|$. Then for some angle $\psi \in [0; 2\pi]$,

$$\tau = E_1 \cos \psi + E_2 \sin \psi,$$

and by the Meusnier theorem we have

$$\langle A \tau, \tau \rangle = \lambda_1 \cos^2 \psi + \lambda_2 \sin^2 \psi = -\frac{1}{p-2}(\lambda_1 + \lambda_2).$$

Hence

$$\lambda_1 = -\lambda_2 \frac{1 + (p-2) \sin^2 \psi}{1 + (p-2) \cos^2 \psi}.$$ 

It is a direct consequence of the last identity that the Jacobian $\det(d_m \gamma) = \lambda_1 \lambda_2$ must be negative. Using standard facts of the quadratic forms theory allowed us to conclude that the distortion coefficient of $\gamma$ at a point $m$ is less or equal to

$$K_m = \max_{\psi} \{q; \frac{1}{q}\}, \quad q = \frac{1 + (p-2) \sin^2 \psi}{1 + (p-2) \cos^2 \psi}.$$ 

Then varying $\psi$ we obtain the required maximum of $K_m$. 

L. Simon in [15] established that every entire two-dimensional nonparametric surface with quasiconformal Gauss map must be plane. As a consequence of this result we obtain a version of the well-known Bernstein theorem.

**Corollary 2.** Let $\mathcal{M}$ be an entire $p$-minimal graph in $\mathbb{R}^3$. Then $\mathcal{M}$ is a plane.

**Remark.** As follows from [10] that every minimal $n$-dimensional graph $\mathcal{M}$ in $\mathbb{R}^{n+1}$ has parabolic conformal type. Other words, every compact on $\mathcal{M}$ have zero $n$-capacity. In these papers we apply the quasiconformal mapping theory to minimal surfaces. The methods used there allows us to conclude that the similar property holds for $p$-minimal graphs also if $p \geq n$. These facts together with Corollary 2 make very likely to be true the following:
Conjecture. Let $\mathcal{M}$ be an entire $p$-minimal graph given over the whole $\mathbb{R}^n$. If $p \geq n$ then $\mathcal{M}$ is a hyperplane.

3. Tubular $p$-minimal hypersurfaces

3.1. In this section we deals with tubular type $p$-minimal surfaces. This class of surfaces in twodimensional case was involved by J. C. C. Nitsche [13] and have been studied by V. M. Miklyukov [9] in highdimensional situation.

Definition 4. We say that a surface $\mathcal{M}$ is a tube with the projection interval $\tau(\mathcal{M}) \subset Ox_{n+1}$, if

1. for any $\tau \in \tau(\mathcal{M})$ the sections $\Sigma_\tau = x(M) \cap \Pi_\tau = \{x \in \mathbb{R}_{1}^{n+1} : x_{n+1} = \tau\}$ by hyperplanes $\Pi_\tau = \{x \in \mathbb{R}_{1}^{n+1} : x_{n+1} = \tau\}$ are not empty compact sets;
2. for $\tau', \tau'' \in \tau(\mathcal{M})$ any part of $\mathcal{M}$ situated between two different $\Pi_{\tau'}$ and $\Pi_{\tau''}$ is a compact set.

If $\tau(\mathcal{M})$ is an infinite interval we call the surface to be an infinite tube. Otherwise, we call a length of $\tau(\mathcal{M})$ the life-time of $\mathcal{M}$.

Let

$$
\rho(\tau) = \max_{m \in \Sigma(\tau)} \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}.
$$

It follows from the results of [12], [11], [5], that every $n$-dimensional minimal tube of arbitrary codimension satisfies the following differential inequality

$$
\rho(\tau)\rho''(\tau) \geq (n - 1)(1 + \rho'(\tau)^2),
$$

which is crucial for all theory of minimal tubes. As a consequence every minimal tube for $n \geq 3$ contains in a slab between two parallel planes. Hence, there are not many-dimensional infinite minimal tubes. In contrast, the twodimensional case essentially differs from the highdimensional one: there are tubes of finite life-time as well as infinite tubes. Moreover, we shown in [18] that the life-time in the first case is derived by the angle between the full-flow vector of a minimal tube and the time-axe.

Lemma 2. Let $V$ be a convex compact in $\mathbb{R}^n$ and $W$ be a compact such that $W \setminus V \neq \emptyset$. Then there exists a closed ball $B \subset \mathbb{R}^n$ such that

$$
W \subset B
$$

and

$$
\partial B \cap (W \setminus V) \neq \emptyset.
$$

Proof. The distance function $f(x) = \text{dist}(x, V)$ is continuous on $\mathbb{R}^n$. It follows from the conditions of the lemma that this function attains the maximum value on $W$ at some point $a \in W$ and $d = f(a) > 0$. On the other hand, by virtue of convexity of $V$ there exist the unique point $b \in \partial V$ such that $f(a) = \|b - a\|$. 
We choose a new coordinate system of $\mathbb{R}^n$ with the origin at $a$, the first coordinate vector

$$e_1 = \frac{b - a}{d}$$

and the others $e_2, \ldots, e_n$ to be based on an orthonormal system together with $e_1$. Then, the hyperplane given by $x_1 = d$ is one of support to $V$ at $a$. It follows from the triangle inequality that $W$ contains in halfspace $\{x_1 \geq 0\}$ and $V$ in $\{x_1 \geq d\}$.

Given positive $h$ and $R$ we specify an open ball $B(R, h) = \{x \in \mathbb{R}^n : (x_1 + R)^2 + x_2^2 + \ldots x_n^2 < (R + h)^2\}$.

By our choice and compactness of $V$, given a positive $\varepsilon$ there exists $R > 0$ such that $V$ contains in the ball $B(R, \varepsilon)$.

Suppose $\varepsilon = d/2$ and $R_0$ is the corresponding radius. Then the definition of $d$ yields that $a \notin B(R_0, d/2)$, however the greater ball $B(R_0, 3d/2)$ contains $V$ as well as $W$. Let $\delta_0$ is the minimum over all $\delta \in (0; d)$ such that

$$W \subset \overline{B(R_0, d/2 + \delta)}.$$ 

Then $a \in \partial B$, where $B = B(R_0, d/2 + \delta_0)$ and $V \cap B = \emptyset$.

Corollary 3 (Maximum Principle). Let $\mathcal{M} = (M, x)$ be an immersed compact $p$-minimal hypersurface in $\mathbb{R}^{n+1}$ with nonempty boundary $\partial M$. Then

$$\text{conv } x(\partial M) = \text{conv } x(M),$$

where $\text{conv } E$ is the convex hull of $E$.

**Proof.** Let us denote $\Omega = \text{conv } x(\partial M)$ and assume that (16) fails. Then it implies $x(M) \setminus \Omega \neq \emptyset$. By Lemma 2 we can find the closed ball $B$ such that $x(M) \subset B$ and there exists a point $m \in \text{int } M$, $x(m) \in \partial B$. We choose the neighbourhood $\mathcal{O}$ of $m$ such that the restriction of $x$ on $\mathcal{O}$ is embedding. The further arguments will be local and we can arrange that $\mathcal{M} = x(\mathcal{O})$ without loss of generality.

Because of the choice of $B$, the tangent spaces to $\mathcal{M}$ and $\partial B$ at $x(m)$ coincide. Moreover, $\mathcal{M} \subset B$ and the standart comparison principle for touching surfaces gives the following inequality

$$\lambda_i \geq \frac{1}{R},$$

where $\lambda_i$ are the principal curvatures of $\mathcal{M}$ at $m$ with respect to the inward normal of $\partial B$ and $R$ is the radius of $B$.

We now turn to identity (2). By the definition of $k_e(m)$ there exists a system of positive numbers $\alpha_i \leq 1$ such that

$$\sum_{i=1}^{n} \alpha_i = 1$$
and

\[ k_n(m) = \sum_{i=1}^{n} \alpha_i \lambda_i \]

It follows from these relations, (17) and (2) that

\[ 0 = \sum_{i=1}^{n} \lambda_i (1 + (p - 2) \alpha_i) \geq \frac{n + p - 2}{R} > 0. \]

This contradiction proves Corollary 3.

\[ \square \]

3.2. Further we use the Minkowski operations. Namely, given \( A, B \subseteq \mathbb{R}^n \) the notations \( A \oplus B \) and \( \lambda A \) reserved for the sets \( \{ x = a + b : a \in A, b \in B \} \) and \( \{ x = \lambda a : a \in A \} \).

**Definition 5.** A family of convex sets \( \{ \Omega(\tau) : \tau \in [\alpha, \beta] \} \) is called \([8]\) convex if for arbitrary \( \tau_1 < \tau_2 \) from interval \([\alpha; \beta]\) and a nonnegative \( t \leq 1 \) one holds

\[ \Omega(\tau_1 t + \tau_2 (1 - t)) \subseteq t\Omega(\tau_1) \oplus (1 - t)\Omega(\tau_2). \]

Let \( \mathcal{M} \) be an \( n \)-dimensional \( p \)-minimal tube in \( \mathbb{R}^{n+1} \). Let us denote by \( \Omega(\tau) \) the projection of the convex hull of the section \( \Sigma(\tau) \) onto the hyperplane \( \Pi_0 = \{ x_{n+1} = 0 \} \). Then

\[ \text{conv } \Sigma(\tau) = \tau e_{n+1} \oplus \Omega(\tau). \]

**Theorem 2.** The family \( \{ \Omega(\tau) : \tau \in \tau(\mathcal{M}) \} \) is convex.

**Proof.** We specify \( \tau_1 < \tau_2 \) from interval \( \tau(\mathcal{M}) \) and \( t \in [0; 1] \). Let \( H \) be the slab \( \{ x : x_{n+1} \in (\tau_1; \tau_2) \} \) and \( M' = x^{-1}(H \cap x(\mathcal{M})) \). Then Corollary 3 gives

\[ V = \text{conv}(\Sigma(\tau_1) \cup \Sigma(\tau_2)) = \text{conv } x(M'). \]

Let \( \tau_0 = t\tau_1 + (1 - t)\tau_2 \). Then \( \Sigma(\tau_0) \subseteq V \) and by the definition of the convex hull we conclude that \( \text{conv } \Sigma(\tau_0) \subseteq V \).

We choose an arbitrary \( z \in \Omega(\tau_0) \). Then \( y = z + \tau_0 e_{n+1} \in \Pi(\tau_0) \cap V \), and there exist \( y_i \in \text{conv } \Sigma(\tau_i) \) and \( \lambda \in [0; 1] \) such that

\[ y = \lambda y_1 + (1 - \lambda) y_2. \]

Then decomposition of \( y_i = z_i + \tau_i e_{n+1} \) for certain \( z_i \in \Omega(\tau_i) \) and (18) give

\[ z = \lambda z_1 + (1 - \lambda) z_2, \quad \tau_0 = \lambda \tau_1 + (1 - \lambda) \tau_2. \]

Hence, \( \lambda = t \) and it follows \( z \in t\Omega(\tau_1) \oplus (1 - t)\Omega(\tau_2) \) as required.

\[ \square \]

The following assertion gives a sample of applications of the last result.

**Corollary 4.** Let \( R(\tau) \) be the radius of the least ball which contains \( \Sigma(\tau) \) (further we call such a ball to be circumscribed near \( \Sigma(\tau) \)). Then \( R(\tau) \) is a convex function.
Proof. We denote by $B(\tau)$ the projection of the circumscribed near $\Sigma(\tau)$ ball onto $\Pi_0$. Then, by virtue of convexity of $B(\tau)$ we have $B(\tau) \supset \Omega(\tau)$, and Theorem 2 yields for arbitrary $t \in [0; 1]$

$$\Omega(\tau_0) \subset t\Omega(\tau_1) \oplus \overline{t}\Omega(\tau_2) \subset tB(\tau_1) \oplus \overline{t}B(\tau_2) = B_0,$$

where $\tau_0 = \tau_1 t + \tau_2 (1-t)$. By the definition, $R(\tau_0) \leq R_0$, where $R_0$ is the radius of $B_0$. On the other hand, $R_0 = tR(\tau_1) + \overline{t}R(\tau_2)$ and we obtain the required inequality

$$R(\tau_1 t + \tau_2 (1-t)) \leq R(\tau_1) + \overline{t}R(\tau_2).$$

\[ \square \]

3.3. Now we study the structure of $\Sigma(\tau)$ in more details. This requires further delicate information not only about $R(\tau)$ but about the curve of the centers of the balls $B(\tau)$ as well. Let us denote by $\xi(\tau)$ the center of $B(\tau)$. We remind without proof the well known extremal property of $B(\tau)$ (see [8], Theorem 7.5).

Lemma 3. Let $E$ be a closed subset of $\mathbb{R}^n$ and $B(E)$ the circumscribed near $E$ ball with the center $\xi$. Then for all unit vectors $y \in \mathbb{R}^n$ there exists $b \in \partial E \cap \partial B(E)$ such that

(19) $$\langle b - \xi, y \rangle \geq 0.$$  

We denote by

$$\sigma(E) = \min_{y \in S^{n-1}} \max_{b \in \partial B \cap E} \frac{\langle b - \xi, y \rangle}{R},$$

where $B$ is the circumscribed ball near a compact $E$, $R$ is the radius and $\xi$ is the center of $B$. It follows from (19) that $0 \leq \sigma(E) \leq 1$. Moreover, one easy to see that $\sigma(E) = 0$ if and only if the intersection of the boundary sphere $S = \partial B$ with $F$ lies in some equatorial semisphere of $S$.

Theorem 3. Let $\mathcal{M}$ be a $p$-minimal tube in $\mathbb{R}^{n+1}$ such that

(20) $$\sigma(\Sigma(\tau)) \geq \epsilon > 0, \quad \forall \tau \in \tau(\mathcal{M}).$$

Then $\xi(\tau)$ is a $\delta$-convex curve of $\tau$. Other words, any coordinate function $\xi_k(\tau)$ admits the composition

$$\xi_k(\tau) = \varphi_k(\tau) - \psi_k(\tau),$$

with $\varphi_k(\tau), \psi_k(\tau)$ to be convex functions.

Proof. We consider $\tau_1, \tau_2$ from $\tau(\mathcal{M})$ and $t \in [0; 1]$. Let us denote by $B(\tau_i) = B_i(\xi(\tau_i), R_i)$ the corresponding circumscribed near $\Sigma(\tau_i)$ balls. As above we have for $\tau_0 = t\tau_1 + \overline{t}\tau_2$

$$\Omega(\tau_0) \subset tB(\tau_1) \oplus \overline{t}B(\tau_2),$$

In force of Lemma 3 we can find $y \in \partial B(\tau_0) \cap \Sigma(\tau_0)$ such that

$$\langle y - \xi(\tau_0), \xi(\tau_0) - \xi_0 \rangle \geq \epsilon |y - \xi(\tau_0)| \cdot |\xi(\tau_0) - \xi_0|,$$
where $\xi_0 = t\xi(\tau_1) + \tau\xi(\tau_2)$. Hence,

$$|y - \xi_0|^2 = |(y - \xi(\tau_0)) + (\xi(\tau_0) - \xi_0)|^2 \geq$$

and taking into account that $|y - \xi(\tau_0)| = R(\tau_0)$ and $|y - \xi_0| \leq R_0$ we obtain

$$|\xi(\tau_0) - \xi_0|^2 + 2\epsilon|y - \xi_0| \cdot |\xi(\tau_0) - \xi_0| + (R_0^2 - R_0^2) \leq 0,$$

and as a consequence,

$$|\xi(\tau_0) - \xi_0| \leq \frac{R_0^2 - R_0^2(\tau_0)}{\epsilon(R(\tau_0) + R_0)} = \frac{1}{\epsilon}(R_0 - R^2(\tau_0)).$$

By Corollary 4 we have $R_0 \geq R(\tau_0)$ and from (21),

$$|\xi(\tau_0) - \xi_0| \leq \frac{R_0^2 - R_0^2(\tau_0)}{\epsilon(R(\tau_0) + R_0)} = \frac{1}{\epsilon}(R_0 - R(\tau_0)).$$

We consider the coordinate function $\xi_k(\tau) = \langle \xi(\tau), e_k \rangle$. Then (22) yields

$$t\xi_k(\tau_1) + \tau\xi_k(\tau_2) - \xi_k(\tau_0) \leq \frac{1}{\epsilon}(R(\tau_1) + \tau R(\tau_2) - R(\tau_0)).$$

This inequality means that the difference $\psi(\tau) = \epsilon^{-1}R(\tau) - \xi_k(\tau)$ is convex. Therefore, by Corollary 4 we obtain the required decomposition of $\xi_k(\tau)$ into difference of two convex functions

$$\xi_k(\tau) = \frac{1}{\epsilon}R(\tau) - \psi(\tau)$$

and the lemma is proved. \(\square\)

**Theorem 4.** Let $M$ be a p-minimal surface with assumption (20) and $\beta = (n - 1)/(p - 1)$. Then $R(\tau)$ and $\xi(\tau)$ satisfy the differential inequality

$$R(\tau)R''(\tau) \geq \beta(1 + R'(\tau)^2) + |\xi'(\tau)|^2 \min\{\beta; 1\}$$

almost everywhere in $\tau(M)$.

**Proof.** Convexity of a function provides existence a.e. of the second differential (see [8] or [2], Theorem 5.3). It follows from Corollary 4, Theorem 3 that $R(\tau)$ as well as $\xi_k(\tau)$ have the second differentials almost everywhere in $\tau(M)$. We denote by $\tau'(M)$ the set of full measure where the second differentials of $R(\tau)$ and $\xi_k(\tau)$, $1 \leq k \leq n + 1$ do exist.

Let $S^{n-1}$ be the unit sphere in $\Pi_0 \sim \mathbb{R}^n$ endowed by the standart metric. We consider the hypersurface $M_0$ given by

$$w(\theta, \tau) = \xi(\tau) + R(\tau)\theta + \tau e_{n+1} : S^{n-1} \times \mathbb{R} \to \mathbb{R}^{n+1}$$

where $\theta \in S^{n-1}$. We have shown in [17] that for such a surface the curvature $k_{e,M_0}$ in $e$-direction can be calculated by

$$k_{e,M_0}(\theta, \tau) = \frac{\omega^3}{R(\tau)} [R(\tau)R''(\tau) + R(\tau)\langle \xi''(\tau), \phi \rangle + \langle \xi'(\tau), \theta \rangle^2 - |\xi'|^2]$$
where
\[ \omega^2 = \langle \nu_m, e \rangle^2 = \frac{1}{1 + \left( R'(\tau) + \langle \theta, \xi'(\tau) \rangle \right)^2}. \]

By the definition of functions \( R(\tau) \) and \( \xi(\tau) \) we conclude that the surface \( \mathcal{M} \) contains inside of \( \mathcal{M}_0 \) in the sense that \( \Sigma(\tau) \) is a subset of \( \Pi(\tau) \cap \mathcal{M}_0 \) for all \( \tau \in \tau(\mathcal{M}) \).

Let us consider an arbitrary \( \tau \in \tau'(\mathcal{M}) \) and \( E = \Omega(\tau) \cap \partial B(\tau) \). The surfaces \( \mathcal{M} \) and \( \mathcal{M}_0 \) have the common outward normal \( \nu_m \) at \( m = y \oplus \tau e_{n+1} \) for every \( y \in E \) (we mean by \( \text{outward} \) the normal which is directed out from the inside of \( B(\tau) \)). Let \( \mathcal{O} \) be the neighbourhood of \( m \) where \( x(\cdot) \) is an embedding. It is a consequence of the definition of \( \mathcal{M}_0 \) that \( \nu_m \wedge e_{n+1} \neq 0 \). We denote by \( \gamma(\tau) \) and \( \gamma_0(\tau) \) the sections of \( x(M) \) and \( M_0 \) by the two-plane spanning on \( \nu_m \) and \( e_{n+1} \). Then the comparison principle for touching surfaces yields
\[ k_{e_{\cdot}M}(m) \leq k_{e_{\cdot}M_0}(m). \]

We write \( h(m) \) and \( h_0(m) \) for the mean curvatures at \( m \) of the sections \( \Sigma(\tau) \) and \( \Pi(\tau) \cap \mathcal{M}_0 = \xi(\tau) \oplus \tau e_{n+1} \oplus B(\tau) \) with respect to their common outward normal. Then the comparison principle arrive at the inequality
\[ h(m) \leq h_0(m) = -\frac{n-1}{R(\tau)}, \]
and after (6)
\[ -\frac{n-1}{\omega} k_c(m) \leq -\frac{n-1}{R(\tau)}. \]

By (24) we obtain after simplification
\[ R(\tau)R''(\tau) - \beta(1 + R'(\tau)^2) \geq (\beta - 1)(\xi'(\tau), \theta)^2 + |\xi'|^2 + (\theta, y), \]
where \( y = 2\beta R'(\tau)\xi'(\tau) - R(\tau\xi''(\tau)) \). Thus, Lemma 3 to be applied to the vector \( y \) provides \( b \in E \) such that \( \langle b - \xi(\tau), y \rangle \geq 0 \). We take
\[ \theta_0 = \frac{b - \xi(\tau)}{R(\tau)} \]
and it follows from (25)
\[ R(\tau)R''(\tau) - \beta(1 + R'(\tau)^2) \geq (\beta - 1)(\xi'(\tau), \theta_0)^2 + |\xi'|^2 + (\theta, y) \geq |\xi'(\tau)|^2 \min\{\beta; 1\}, \]
and the theorem is proved completely.

\[ \square \]

**Remark.** Finally, we notice that the quantity \( R(\tau) \) measures the size of the section \( \Sigma(\tau) \) instead of the distance this section from a fixed line in the previous inequalities (13). Moreover, in the base case \( p = 2 \) the established inequality (23) is more strong then (13).

On the other hand, \( \delta \)-convex functions belong to the class \( W_{1,\text{loc}}^2(\tau(\mathcal{M})) \); that is, has a second-order generalized derivative that is a measure (see [2], Chapter 2,
§4.10, Corollary). This allows to proceed the integration of (23) to comletion in the standard way [12]:

**Corollary 5.** Let $\mathcal{M}$ be a $p$-minimal tube, $\dim\mathcal{M} = n > p > 1$. Then has finite life-time length $\tau(\mathcal{M})$. Moreover,

$$\text{length} \tau(\mathcal{M}) \leq 2c_\beta r(\mathcal{M}), \quad \beta = \frac{n-1}{p-1}$$

where

$$r(\mathcal{M}) \equiv \min_{\tau \in \tau(\mathcal{M})} R(\tau) > 0$$

and

$$c_\beta = \int_0^{+\infty} \frac{dt}{(1+t^{2\beta})^{1/2}}.$$

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