Abelian subgroups of $\text{Out}(F_n)$

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Abstract

We classify abelian subgroups of $\text{Out}(F_n)$ up to finite index in an algorithmic and computationally friendly way. A process called disintegration is used to canonically decompose a single rotationless element $\phi$ into a composition of finitely many elements and then use these elements to generate an abelian subgroup $A(\phi)$ that contains $\phi$. The main theorem is that up to finite index every abelian subgroup is realized by this construction. As an application we classify, up to finite index, abelian subgroups of $\text{Out}(F_n)$ and of $\text{IA}_n$ with maximal rank.

Contents

1 Introduction 2
2 Background 5
3 Rotationless Abelian Subgroups 14
4 Generic Elements of rotationless abelian subgroups 20
5 $A(\phi)$ 23
6 Disintegrating $\phi$ 25
7 Finite Index 32
8 Abelian Subgroups of Maximal Rank 42

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1 Introduction

In this paper we classify abelian subgroups of $\text{Out}(F_n)$ up to finite index in an algorithmic and computationally friendly way. There are two steps. The first is to construct an abelian subgroup $D(\phi)$ from a given $\phi \in \text{Out}(F_n)$ by a process that we call disintegration. The subgroup $D(\phi)$ is very well understood in terms of relative train track maps and has natural coordinates that embed it into some $\mathbb{Z}^M$. The second step is to prove the following theorem.

**Theorem 7.2.** For every abelian subgroup $A$ of $\text{Out}(F_n)$ there exists $\phi \in A$ such that $A \cap D(\phi)$ has finite index in $A$.

To motivate the disintegration process, consider a pure element $\mu$ of the mapping class group $\text{MCG}(S)$ of a compact oriented surface $S$. By the Thurston classification theorem [Thu88], there is a decomposition of $S$ into subsurfaces $S_l$, some of which are annuli and the rest of which have negative Euler characteristic, and there is a homeomorphism $h : S \to S$ representing $\mu$, called a normal form for $\mu$, that preserves each $S_l$. If $S_l$ is an annulus then $h|S_l$ is a non-trivial Dehn twist. If $S_l$ has negative Euler characteristic then $h|S_l$ is either the identity or is pseudo-Anosov. In all cases, $h|\partial S_l$ is the identity.

We may assume that the $S_l$'s are numbered so that $h|S_l$ is the identity if and only if $l > M$ for some $M$. For each $M$-tuple of integers $a = (a_1, \ldots, a_M)$ let $h_a : S \to S$ be the homeomorphism that agrees with $h^{a_l}$ on $S_l$ for $1 \leq l \leq M$ and is the identity on the remaining $S_l$'s. Then $h_a$ is a normal form for an element $\mu_a \in \text{MCG}(S)$ and we define $D(\mu)$ to be the subgroup consisting of all such $\mu_a$. It is easy to check that $\mu_a \to a$ defines an isomorphism between $D(\mu)$ and $\mathbb{Z}^M$.

An element $\phi$ of $\text{Out}(F_n)$ has finite sets of natural invariants on which it acts by permutation. If these actions are trivial then we say that $\phi$ is rotationless; complete details can be found in section 3. This property is similar to being pure, which is defined as acting trivially on $H_1(F_n, \mathbb{Z}_3)$. Abelian subgroups are both virtually rotationless and virtually pure. The latter is obvious and the former follows from Corollary 3.11. We work in the rotationless category since it is more natural for our constructions.

Suppose that $\phi$ is a rotationless element of $\text{Out}(F_n)$. The analog of a normal form $h : S \to S$ is a relative train track map $f : G \to G$, which is a particularly nice homotopy equivalence of a marked graph that represents $\phi$ in the sense that the outer automorphism of $\pi_1(G)$ that it induces is identified with $\phi$ by the marking. There is an associated maximal filtration $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ by $f$-invariant subgraphs. The $i^{th}$ stratum $H_i$ is the closure of $G_i \setminus G_{i-1}$. The exact properties satisfied by $f : G \to G$ and $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ are detailed in section 2.
As a first attempt to mimic the construction of $D(\mu)$, let $X_1, \ldots, X_M$ be the strata that are not fixed by $f$, let $a = (a_1, \ldots, a_M)$ be an $M$-tuple of non-negative integers and define $f_a$ to agree with $f^{a_i}$ on $X_i$ and to be the identity on the subgraph of edges fixed by $f$. Although it is not obvious, $f_a : G \to G$ is a homotopy equivalence (see Lemma 6.5) and so defines an element $\phi_a \in \text{Out}(F_n)$.

Without some restrictions on $a$ however, the subgroup generated by the $\phi_a$’s need not be abelian. In the following examples, we do not distinguish between a homotopy equivalence of the rose and the outer automorphism that it represents.

**Example 1.1.** Let $G$ be the graph with one vertex and with edges labelled $A, B$ and $C$. Define $f : G \to G$ by

\[
A \mapsto A \quad B \mapsto BA \quad C \mapsto CB.
\]

Let $X_1 = \{B\}$ and $X_2 = \{C\}$ and $a = (m, n)$. Then

\[
f_{(m, n)} \circ f(C) = f_{(m, n)}(CB) = f^n(C)f^m(B)
\]

and

\[
f \circ f_{(m, n)}(C) = f(f^n(C)) = f^n(f(C)) = f^n(CB) = f^n(C)f^n(B).
\]

This shows that $f_{(m, n)}$ commutes with $f = f_{(1, 1)}$ if and only if $m = n$.

The underlying problem is that strata are not invariant. It does not matter that the path $f(B)$ crosses $A$ since $A$ is fixed by $f$. The lack of commutativity stems from the fact that $f(C)$ crosses $B$.

To address this problem we enlarge the $X_i$’s to be unions of strata. It is not necessary to choose the $X_i$’s to be fully invariant (i.e. to satisfy $f(X_i) \subset X_i$) but they must be *almost invariant* as made precise in Definition 6.3.

The next example illustrates a more complicated relation on the coordinates of $a$ that is needed to insure that the $f_a$’s commute.

**Example 1.2.** Let $G$ be the graph with one vertex and with edges labelled $A, B, C, D$ and $E$. Define $f : G \to G$ by

\[
A \mapsto A \quad B \mapsto BA^2 \quad C \mapsto CB \quad D \mapsto DA^5 \quad E \mapsto DCB
\]

where $\bar{B}$ is $B$ with its orientation reversed. Let $X_1 = \{B, C\}, X_2 = \{D\}$ and $X_3 = \{E\}$ and let $a = (m, n, p)$. Then

\[
f \circ f_a(D) = f(f^p(D)) = f^p(f(D)) = f^p(DC\bar{B}) = f^p(D)f^p(C\bar{B})
\]

and

\[
f_a \circ f(D) = f_a(DCB) = f^p(D)f_a(CB).
\]

If $f$ commutes with $f_a$ then

\[
f^p(C\bar{B}) = f_a(C\bar{B}).
\]

Thus $CA^{3p}\bar{B} = CA^{5n-2m}\bar{B}$ and $3p = 5n - 2m$. One can check that the converse holds as well. Namely if we require that $a$ be an element of the linear subspace of $\mathbb{Z}^3 = \{(m, n, p)\}$ defined by $3p = 5n - 2m$ then the $\phi_a$’s commute.
The path $CB$ of Example 1.2 is quasi-exceptional as defined in section 6. When the image of an edge in $X_k$ contains a quasi-exceptional path with initial edge in $X_i$ and terminal edge in $X_j$ then there is an induced relation between the $i^{th}$, $j^{th}$ and $k^{th}$ coefficient of $a$. These define a linear subspace of $\mathbb{Z}^M$. The non-negative $M$-tuples that lie in this subspace are said to be admissible. The map $a \rightarrow \phi_a$ on admissible $M$-tuples extends to an injective homomorphism of the full linear subspace and we define the image of this subspace to be $D(\phi)$.

The mapping class group version of Theorem 7.2 is a straightforward consequence of two easily proved, well known facts. The first (see for example Corollary 5.2 of [FHPa]) is that the subsurfaces $S_l$ can be chosen independently of $\mu \in A$. The second (see for example Lemma 2.10 of [FHPb]) is that an abelian subgroup containing a pseudo-Anosov element is virtually cyclic.

The proof for $\text{Out}(F_n)$ is considerably harder. This is due, in part, to the fact that disintegration in $\text{Out}(F_n)$ is a more complicated operation, as illustrated by the examples, than it is $\text{MCG}(S)$. Another factor is that, unlike normal forms in the mapping class group, relative train track maps representing an element $\phi \in \text{Out}(F_n)$ are not unique. No matter how canonical a construction is with respect to a particular $f : G \rightarrow G$, one must still check the extent to which it is independent of the choice of $f : G \rightarrow G$. The most technically difficult argument in this paper (section 7) is a proof that the rank of the admissible linear subspace of $\mathbb{Z}^M$ described above depends only on $\phi$ and not the choice of $f : G \rightarrow G$.

Recall that $IA_n$ is the subgroup of $\text{Out}(F_n)$ consisting of elements that act as the identity on $H_1(F_n)$. As an application of Theorem 7.2 we classify, up to finite index, abelian subgroups of $\text{Out}(F_n)$ and of $IA_n$ with maximal rank. The exact statements appear as Proposition 8.6 and Proposition 8.7. Roughly speaking, we prove that if $D(\phi)$ has maximal rank in $\text{Out}(F_n)$ then $f : G \rightarrow G$ has $2n - 3$ strata, each of which is either a single linear edge or is exponentially growing and is closely related to a pseudo-Anosov homeomorphism of a four times punctured sphere. If $D(\phi)$ has maximal rank in $IA_n$ then $f : G \rightarrow G$ has $2n - 4$ such strata and pointwise fixes a rank two subgraph.

From an algebraic point of view, the natural abelian subgroup associated to an element $\phi \in \text{Out}(F_n)$ is the center $Z(C(\phi))$ of the centralizer $C(\phi)$ of $\phi$ which can also be described as the intersection of all maximal (with respect to inclusion) abelian subgroups that contain $\phi$. In our context it is natural to look at the weak center $WZ(C(\phi))$ of $C(\phi)$ defined as the subgroup of elements that commute with an iterate of each element of $C(\phi)$. The following result is a step toward an algorithmic construction of $Z(C(\phi))$. Every abelian subgroup $A$ has a finite index subgroup $A_R$, all of whose elements are rotationless.

**Theorem 6.18.** $D_R(\phi) \subset WZ(C(\phi))$ for all rotationless $\phi$.

In section 9 we apply this theorem to give algebraic characterizations of certain maximal rank abelian subgroups of $\text{Out}(F_n)$ and $IA_n$. This characterization is needed
in the calculation of the commensurator group of Out($F_n$) [FHa].

In section 3 we define what it means for $\phi \in \text{Out}(F_n)$ to be rotationless, prove that the rotationless elements of any abelian subgroup $A$ form a finite index subgroup $A_R$ and consider lifts of $A_R$ from Out($F_n$) to Aut($F_n$). These lifts are essential to our approach and are similar to ones used in [BFH04].

In section 4 we define a natural embedding of $A_R$ into a lattice in Euclidean space and say what it means for an element of $A_R$ to be generic with respect to this embedding.

In section 5 we associate an abelian subgroup $A(\phi)$ to each rotationless $\phi$ and prove that if $\phi$ is generic in $A_R$ then $A_R \subset A(\phi)$. We also prove (Corollary 5.5) that $A(\phi) \subset WZ(C(\phi))$.

In section 6, we define $D(\phi)$ and prove (Corollary 6.16) that $D_R(\phi) \subset A(\phi)$, thereby completing the proof of Theorem 6.18.

In section 7 we prove (Theorem 7.1) that $D_R(\phi)$ has finite index in $A(\phi)$ by reconciling the normal forms point of view used to define $D(\phi)$ with the ‘action on $\partial F_n$’ point of view used to define $A(\phi)$. Theorem 7.2 is an immediate consequence of this result and the fact, mentioned above, that $A_R \subset A(\phi)$ for generic $\phi \in A$.

We make use of several important results from [FHb], including the Recognition Theorem and the existence of relative train track maps that are especially well suited to disintegrating an element and forming $D(\phi)$. Section 2 reviews this and other relevant material and sets notation for the paper.

2 Background

Fix $n \geq 2$ and let $F_n$ be the free group of rank $n$. Denote the automorphism group of $F_n$ by Aut($F_n$), the group of inner automorphisms of $F_n$ by Inn($F_n$) and the group of outer automorphisms of $F_n$ by Out($F_n$) = Aut($F_n$)/Inn($F_n$). We follow the convention that elements of Aut($F_n$) are denoted by upper case Greek letters and that the same Greek letter in lower case denotes the corresponding element of Out($F_n$). Thus $\Phi \in \text{Aut}(F_n)$ represents $\phi \in \text{Out}(F_n)$.

Marked Graphs and Outer Automorphisms Identify $F_n$ with $\pi_1(R_n,*)$ where $R_n$ is the rose with one vertex $*$ and $n$ edges. A marked graph $G$ is a graph of rank $n$ without valence one vertices, equipped with a homotopy equivalence $m : R_n \to G$ called a marking. Letting $b = m(*) \in G$, the marking determines an identification of $F_n$ with $\pi_1(G,b)$.

A homotopy equivalence $f : G \to G$ and a path $\sigma$ from $b$ to $f(b)$ determines an automorphism of $\pi_1(G,b)$ and hence an element of Aut($F_n$). As the homotopy class of $\sigma$ varies, the automorphism ranges over all representatives of the associated outer automorphism $\phi$. We say that $f : G \to G$ represents $\phi$. We always assume that the restriction of $f$ to any edge is an immersion.
**Paths, Circuits and Edge Paths** Let $\Gamma$ be the universal cover of a marked graph $G$ and let $pr : \Gamma \to G$ be the covering projection. A proper map $\tilde{\sigma} : J \to \Gamma$ with domain a (possibly infinite) interval $J$ will be called a path in $\Gamma$ if it is an embedding or if $J$ is finite and the image is a single point; in the latter case we say that $\tilde{\sigma}$ is a trivial path. If $J$ is finite, then $\tilde{\sigma} : J \to \Gamma$ is homotopic rel endpoints to a unique (possibly trivial) path $[\tilde{\sigma}]$; we say that $[\tilde{\sigma}]$ is obtained from $\tilde{\sigma}$ by tightening. If $f : \Gamma \to \Gamma$ is a lift of a homotopy equivalence $f : G \to G$, we denote $[f(\tilde{\sigma})]$ by $f_\#(\tilde{\sigma})$.

We will not distinguish between paths in $\Gamma$ that differ only by an orientation preserving change of parametrization. Thus we are interested in the oriented image of $\tilde{\sigma}$ and not $\tilde{\sigma}$ itself. If the domain of $\tilde{\sigma}$ is finite, then the image of $\tilde{\sigma}$ has a natural decomposition as a concatenation $E_1 \tilde{E}_2 \ldots \tilde{E}_{k-1} \tilde{E}_k$ where $\tilde{E}_i$, $1 < i < k$, is an edge of $\Gamma$, $E_1$ is the terminal segment of an edge and $\tilde{E}_k$ is the initial segment of an edge. If the endpoints of the image of $\tilde{\sigma}$ are vertices, then $\tilde{E}_1$ and $\tilde{E}_k$ are full edges. The sequence $\tilde{E}_1 \tilde{E}_2 \ldots \tilde{E}_k$ is called the edge path associated to $\tilde{\sigma}$. This notation extends naturally to the case that the interval of domain is half-infinite or bi-infinite. In the former case, an edge path has the form $\tilde{E}_1 \tilde{E}_2 \ldots$ or $\tilde{E}_2 \tilde{E}_1$ and in the latter case has the form $\ldots \tilde{E}_{-1} \tilde{E}_0 \tilde{E}_1 \tilde{E}_2 \ldots$.

A path in $G$ is the composition of the projection map $pr$ with a path in $\Gamma$. Thus a map $\sigma : J \to G$ with domain a (possibly infinite) interval will be called a path if it is an immersion or if $J$ is finite and the image is a single point; paths of the latter type are said to be trivial. If $J$ is finite, then $\sigma : J \to G$ is homotopic rel endpoints to a unique (possibly trivial) path $[\sigma]$; we say that $[\sigma]$ is obtained from $\sigma$ by tightening. For any lift $\tilde{\sigma} : J \to \Gamma$ of $\sigma$, $[\tilde{\sigma}] = pr[\tilde{\sigma}]$. We denote $[f(\sigma)]$ by $f_\#(\sigma)$. We do not distinguish between paths in $G$ that differ by an orientation preserving change of parametrization. The edge path associated to $\sigma$ is the projected image of the edge path associated to a path $\tilde{\sigma}$. Thus the edge path associated to a path with finite domain has the form $E_1 E_2 \ldots E_{k-1} E_k$ where $E_i$, $1 < i < k$, is an edge of $G$, $E_1$ is the terminal segment of an edge and $E_k$ is the initial segment of an edge. We will identify paths with their associated edge paths whenever it is convenient.

We reserve the word circuit for an immersion $\sigma : S^1 \to G$. Any homotopically non-trivial map $\sigma : S^1 \to G$ is homotopic to a unique circuit $[\sigma]$. As was the case with paths, we do not distinguish between circuits that differ only by an orientation preserving change of parametrization and we identify a circuit $\sigma$ with a cyclically ordered edge path $E_1 E_2 \ldots E_k$.

A path or circuit crosses or contains an edge if that edge occurs in the associated edge path. For any path $\sigma$ in $G$ define $\tilde{\sigma}$ to be '$\sigma$ with its orientation reversed'. For notational simplicity, we sometimes refer to the inverse of $\tilde{\sigma}$ by $\tilde{\sigma}^{-1}$.

A decomposition of a path or circuit into subpaths is a splitting for $f : G \to G$ and is denoted $\sigma = \ldots \sigma_1 \cdot \sigma_2 \ldots$ if $f^k(\sigma) = \ldots f^k(\sigma_1) f^k(\sigma_2) \ldots$ for all $k \geq 0$. In other words, a decomposition of $\sigma$ into subpaths $\sigma_i$ is a splitting if one can tighten the image of $\sigma$ under any iterate of $f_\#$ by tightening the images of the $\sigma_i$’s.

A path $\sigma$ is a periodic Nielsen path if $f^k_\#(\sigma) = \sigma$ for some $k \geq 1$. The minimal
such $k$ is the period of $\sigma$ and if $k = 1$ then $\sigma$ is a Nielsen path. A (periodic) Nielsen path is indivisible if it does not decompose as a concatenation of non-trivial (periodic) Nielsen subpaths. A path is primitive if it is not multiple of a simpler path.

**Automorphisms and Lifts** Section 1 of [GJLL98] and section 2.1 of [BFH04] are good sources for facts that we record below without specific references. The universal cover $\Gamma$ of a marked graph $G$ with marking $m : R_n \to G$, is a simplicial tree. We always assume that a base point $\tilde{b} \in \Gamma$ projecting to $b = m(*) \in G$ has been chosen, thereby defining an action of $F_n$ on $\Gamma$. The set of ends $E(\Gamma)$ of $\Gamma$ is naturally identified with the boundary $\partial F_n$ of $F_n$ and we make implicit use of this identification throughout the paper.

Each $c \in F_n$ acts by a covering translation $T_c : \Gamma \to \Gamma$ and each $T_c$ induces a homeomorphism $\hat{T}_c : \partial F_n \to \partial F_n$ that fixes two points, a sink $T_c^+$ and a source $T_c^-$. The line in $\Gamma$ whose ends converge to $T_c^-$ and $T_c^+$ is called the axis of $T_c$ and is denoted $A_c$. The image of $A_c$ in $G$ is the circuit corresponding to the conjugacy class of $c$.

If $f : G \to G$ represents $\phi \in \text{Out}(F_n)$ then the choice of a path $\sigma$ from $b$ to $f(b)$ determines both an automorphism representing $\phi$ and a lift of $f$ to $\Gamma$. This defines a bijection between the set of lifts $\tilde{f} : \Gamma \to \Gamma$ of $f : G \to G$ and the set of automorphisms $\Phi : F_n \to F_n$ representing $\phi$. Equivalently, this bijection is defined by $\tilde{f}T_c = T_{\phi(c)}\tilde{f}$ for all $c \in F_n$. We say that $\tilde{f}$ corresponds to $\Phi$ or is determined by $\Phi$ and vice versa. Under the identification of $E(\Gamma)$ with $\partial F_n$, a lift $\tilde{f}$ determines a homeomorphism $\hat{f}$ of $\partial F_n$. An automorphism $\Phi$ also determines a homeomorphism $\hat{\Phi}$ of $\partial F_n$ and $\hat{f} = \hat{\Phi}$ if and only if $\tilde{f}$ corresponds to $\Phi$. In particular, $\hat{i}_c = \hat{T}_c$ for all $c \in F_n$ where $i_c(w) = cwc^{-1}$ is the inner automorphism of $F_n$ determined by $c$. We use the notation $\hat{f}$ and $\hat{\Phi}$ interchangeably depending on the context.

We are particularly interested in the dynamics of $\hat{f} = \hat{\Phi}$. The following two lemmas are contained in Lemma 2.3 and Lemma 2.4 of [BFH04] and in Proposition 1.1 of [GJLL98].

**Lemma 2.1.** Assume that $\tilde{f} : \Gamma \to \Gamma$ corresponds to $\Phi \in \text{Aut}(F_n)$. Then the following are equivalent:

(i) $c \in \text{Fix}(\Phi)$.

(ii) $T_c$ commutes with $\tilde{f}$.

(iii) $\hat{T}_c$ commutes with $\hat{f}$.

(iv) $\text{Fix}(\hat{T}_c) \subset \text{Fix}(\hat{f}) = \text{Fix}(\hat{\Phi})$.

(v) $\text{Fix}(\hat{f}) = \text{Fix}(\hat{\Phi})$ is $\hat{T}_c$-invariant.

**Lemma 2.2.** Assume that $\tilde{f} : \Gamma \to \Gamma$ corresponds to $\Phi \in \text{Aut}(F_n)$ and that $\text{Fix}(\hat{\Phi}) \subset \partial F_n$ contains at least three points. Denote $\text{Fix}(\hat{\Phi})$ by $F$. Then
(i) $\partial F$ is naturally identified with the closure of $\{T_c^\pm : T_c \in T(\Phi)\}$ in $\partial F_n$. None of these points is isolated in $\text{Fix}(\hat{\Phi})$.

(ii) Each point in $\text{Fix}(\hat{\Phi}) \setminus \partial F$ is isolated and is either an attractor or a repeller for the action of $\hat{\Phi}$.

(iii) There are only finitely many $F$-orbits in $\text{Fix}(\hat{\Phi}) \setminus \partial F$.

**Lines and Laminations** Unoriented bi-infinite paths in $G$ or its universal cover $\Gamma$ are called *lines*. There is a bijection between lines in $\Gamma$ and unordered pairs of distinct elements of $\partial F_n$, the latter being the endpoints of the former. The advantage of the $\partial F_n$ description is that it allows us to work with abstract lines that are realized in any $\Gamma$ but are not tied to any particular $\Gamma$.

A closed set of lines in $G$ or an equivariant closed set of lines in $\Gamma$ is called a *lamination* and the lines that compose it are called *leaves*. If $\Lambda$ is a lamination in $G$ then we denote its pre-image in $\Gamma$ by $\hat{\Lambda}$ and vice-versa.

Suppose that $f : G \to G$ represents $\phi$ and that $\tilde{f}$ is a lift of $f$. If $\tilde{\gamma}$ is a line in $\Gamma$ with endpoints $P$ and $Q$, then there is a bounded homotopy from $\tilde{f}(\tilde{\gamma})$ to the line $\tilde{f}_\#(\gamma)$ with endpoints $\tilde{f}(P)$ and $\tilde{f}(Q)$. This defines an action $\tilde{f}_\#$ of $\tilde{f}$ on lines in $\Gamma$. If $\Phi \in \text{Aut}(F_n)$ corresponds to $\tilde{f}$ then $\hat{\Phi}_\# = \tilde{f}_\#$ is described on abstract lines by $(P,Q) \mapsto (\hat{\Phi}(P),\hat{\Phi}(Q))$. There is an induced action $\hat{\phi}_\#$ of $\hat{\phi}$ on lines in $G$ and in particular on laminations in $G$. The stabilizer $\text{Stab}(\Lambda)$ of a lamination $\Lambda$ is the subgroup of elements of $\text{Out}(F_n)$ that preserve $\Lambda$.

A point $P \in \partial F_n$ determines a lamination $\Lambda(P)$, called the *accumulation set of $P$*, as follows. Let $\Gamma$ be the universal cover of a marked graph $G$ and let $\tilde{R}$ be any ray in $\Gamma$ converging to $P$. A line $\tilde{\sigma} \subset \Gamma$ belongs to $\hat{\Lambda}(P)$ if every finite subpath of $\tilde{\sigma}$ is contained in some translate of $\tilde{R}$. Since any two rays converging to $P$ have a common infinite end, this definition is independent of the choice of $\tilde{R}$. The bounded cancellation lemma implies (cf. Lemma 3.1.4 of [BFH00]) that this definition is independent of the choice of $G$ and $\Gamma$ and that $\hat{\Phi}_\#(\hat{\Lambda}(P)) = \hat{\Lambda}(\hat{\Phi}(P))$. In particular, if $P \in \text{Fix}(\hat{\Phi})$ then $\Lambda(P)$ is $\phi_\#$-invariant.

For each $\phi \in \text{Out}(F_n)$ there is an associated $\phi$-invariant finite set $\mathcal{L}(\phi)$ of laminations called the set of *attracting laminations for $\phi$*. For each $\Lambda \in \mathcal{L}(\phi)$ there is an *expansion factor homomorphism* $\mathcal{P}_\Lambda$ defined on $\text{Stab}(\Lambda)$ and with image a discrete subgroup of $\mathbb{R}$. Each $\Lambda \in \mathcal{L}(\phi)$ has birecurrent leaves called *generic leaves*. See section 3.3 of [BFH00] for complete details.

**The Recognition Theorem ([FHb])** The set of non-repelling fixed [resp. periodic] points of $\hat{\Phi}$ is denoted by $\text{Fix}_N(\hat{\Phi})$ [resp. $\text{Per}_N(\hat{\Phi})$].

**Definition 2.3.** If the cardinality of $\text{Fix}_N(\hat{\Phi})$ is greater than two or if $\text{Fix}_N(\hat{\Phi})$ is a pair of points that does not cobound either some axis $T_c$ or a generic leaf of an
element of $\mathcal{L}(\phi)$ then $\Phi$ is a principal automorphism and we write $\Phi \in P(\phi)$. The corresponding lift of $f$ is called a principal lift.

There is a natural equivalence relation on automorphisms defined by $\Phi_1 \sim c \Phi_2 i_c^{-1}$ for some $c \in F_n$. There are only finitely many such equivalence classes of principal automorphisms - see Remark 3.9 of [FHb].

**Definition 2.4.** An outer automorphism $\phi$ is forward rotationless if $\text{Fix}_N(\hat{\Phi}) = \text{Per}_N(\hat{\Phi})$ for all $\Phi \in P(\phi)$ and if for each $k \geq 1$, $\Phi \mapsto \Phi^k$ defines a bijection between $P(\phi)$ and $P(\phi^k)$. Our standing assumption is that $n \geq 2$. For notational convenience we say that the identity element of $\text{Out}(F_1)$ is forward rotationless.

Every $\phi \in \text{Out}(F_n)$ has a forward rotationless iterate $\phi^k$ by Corollary 3.30 of [FHb]. As an illustration of the utility of this property, and for convenient reference, we recall Lemma 3.29 of [FHb].

**Lemma 2.5.** The following properties hold for each forward rotationless $\phi \in \text{Out}(F_n)$.

1. Each periodic conjugacy class is fixed and each representative of that conjugacy class is fixed by some principal automorphism representing $\phi$.

2. Each $\Lambda \in \mathcal{L}(\phi)$ is $\phi$-invariant.

3. A free factor that is invariant under an iterate of $\phi$ is $\phi$-invariant.

Several of our constructions are motivated by the following theorem from [FHb]. We also use this theorem directly to prove that $A(\phi)$ is abelian.

**Theorem 2.6.** (Recognition Theorem) Suppose that $\phi, \psi \in \text{Out}(F_n)$ are forward rotationless and that

1. $\text{PF}_\Lambda(\phi) = \text{PF}_\Lambda(\psi)$, for all $\Lambda \in \mathcal{L}(\phi) = \mathcal{L}(\psi)$.

2. there is bijection $h : P(\phi) \rightarrow P(\psi)$ such that:

   (i) (fixed sets preserved) $\text{Fix}_N(\hat{\Phi}) = \text{Fix}_N(\hat{h(\Phi)})$

   (ii) (twist coordinates preserved) If $w \in \text{Fix}(\Phi)$ is primitive and $\Phi, i_w d \Phi \in P(\phi)$, then $h(i_w d \Phi) = i_w h(\Phi)$.

**Remark 2.7.** In the special case that $\phi$ is realized as an element of $\text{MCG}(S)$, a $w$ that occurs in item 2-(ii) represents a reducing curve and $d$ is the degree of Dehn twisting about that reducing curve. See also the discussion of ‘axes’ at the end of this section.

We include the following result for easy reference.

**Lemma 2.8.** The following properties hold for all $\Phi$ representing $\phi$ and $\Psi$ representing $\psi$. 
1. $\text{Fix}(\psi\phi\psi^{-1}) = \psi(\text{Fix}(\phi))$ and $\text{Fix}_N(\psi\phi\psi^{-1}) = \psi(\text{Fix}_N(\phi))$.

2. Conjugation by $\psi$ defines a bijection $i_\psi: P(\phi) \mapsto P(\psi\phi\psi^{-1})$ that preserves equivalence classes. The induced bijection on the set of equivalence classes depends only on $\psi$ and not on the choice of $\psi$.

Proof. (1) is standard and easily checked; it implies that $i_\psi: P(\phi) \mapsto P(\psi\phi\psi^{-1})$ is a bijection. The rest of (2) follows from $\psi(i_c\phi i_c^{-1})\psi^{-1} = i_{\psi(c)}\psi\phi\psi^{-1}i_{\psi(c)}$ and $(i_d\psi)(\phi)(i_d\psi)^{-1} = i_d(\psi\phi\psi^{-1})i_d^{-1}$. \hfill \qed

**Free Factor Systems** The conjugacy class of a free factor $F^i$ of $F_n$ is denoted $[[F^i]]$. If $F^1, \ldots, F^k$ are non-trivial free factors and if $F^1 \ast \ldots \ast F^k$ is a free factor then we say that the collection $\{[[F^1]],\ldots,[[F^k]]\}$ is a free factor system. For example, if $G$ is a marked graph and $G_r \subset G$ is a subgraph with non-contractible components $C_1, \ldots, C_k$ then the conjugacy class $[[\pi_1(C_i)]]$ of the fundamental group of $C_i$ is well defined and the collection of these conjugacy classes is a free factor system denoted $\mathcal{F}(G_r)$.

The image of a free factor $F$ under an element of $\text{Out}(F_n)$ is a free factor. This induces an action of $\text{Out}(F_n)$ on the set of free free systems. We sometimes say that a free factor is $\phi$-invariant when we really mean that its conjugacy class is $\phi$-invariant. If $[[F]]$ is $\phi$-invariant then $F$ is $\Phi$-invariant for some automorphism $\Phi$ representing $\phi$ and $\Phi|F$ determines a well defined element $\phi|F$ of $\text{Out}(F)$.

The conjugacy class $[a] \in F_n$ is carried by $[[F^i]]$ if $F^i$ contains a representative of $[a]$. Sometimes we say that $a$ is carried by $F^i$ when we really mean that $[a]$ is carried by $[[F^i]]$. If $G$ is a marked graph and $G_r$ is a subgraph of $G$ such that $[[F^i]] = \mathcal{F}(G_r)$ then $[a]$ is carried by $[[F^i]]$ if and only if the circuit in $G$ that represents $[a]$ is contained in $G_r$. We say that an abstract line $\ell$ is carried by $[[F^i]]$ if its realization in $G$ is contained in $G_r$ for some, and hence any, $G$ and $G_r$ as above. Equivalently, $\ell$ is the limit of periodic lines corresponding to $[c_i]$ where each $[c_i]$ is carried by $[[F^i]]$. A collection $W$ of abstract lines and conjugacy classes in $F_n$ is carried by a free factor system $\mathcal{F} = \{[[F^1]],\ldots,[[F^k]]\}$ if each element of $W$ is carried by some $F^i$.

There is a partial order $\sqsubseteq$ on free factor systems generated by inclusion. More precisely, $[[F^1]] \sqsubseteq [[F^2]]$ if $F^1$ is conjugate to a free factor of $F^2$ and $\mathcal{F}_1 \sqsubseteq \mathcal{F}_2$ if for each $[[F^i]] \in \mathcal{F}_1$ there exists $[[F^j]] \in \mathcal{F}_2$ such that $[[F^i]] \sqsubseteq [[F^j]]$.

The complexity of a free factor system is defined on page 531 of [BFH00]. We include the following results for easy reference. The first is Corollary 2.6.5. of [BFH00]. The second is an immediate consequence of the uniqueness of $\mathcal{F}(W)$.

**Lemma 2.9.** For any collection $W$ of abstract lines there is a unique free factor system $\mathcal{F}(W)$ of minimal complexity that carries every element of $W$. If $W$ is a single element then $\mathcal{F}(W)$ has a single element.

**Corollary 2.10.** If a collection $W$ of abstract lines and conjugacy classes in $F_n$ is $\phi$-invariant then $\mathcal{F}(W)$ is $\phi$-invariant.
Further details on free factor systems can be found in section 2.6 of [BFH00].

**Relative Train Track Maps** We assume some familiarity with the basic definitions of relative train track maps. Complete details can be found in [FHb] and [BFH00].

Suppose that \( f : G \to G \) is a relative train track map defined with respect to a maximal filtration \( \emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G \). A path or circuit has height \( r \) if it is contained in \( G_r \) but not \( G_{r-1} \). A lamination has height \( r \) if each leaf in its realization in \( G \) has height at most \( r \) and some leaf has height \( r \). The \( r \)th stratum \( H_r \) is defined to be the closure of \( G_r \setminus G_{r-1} \). If \( f(H_r) \subset G_{r-1} \) then \( H_r \) is called a zero stratum; all other strata have irreducible transition matrices and are said to be irreducible. If \( H_r \) is irreducible and if the Perron-Frobenius eigenvalue of the transition matrix for \( H_r \) is greater than one, then \( H_r \) is exponentially growing or simply EG. All other irreducible strata are non-EG or simply NEG.

A direction \( d \) at \( x \in G \) is the germ of an initial segment of an oriented edge (or partial edge if \( x \) is not a vertex) based at \( x \). There is an \( f \)-induced map \( Df \) on directions and we say that \( d \) is a periodic direction if it is periodic under the action of \( Df \); if the period is one then \( d \) is a fixed direction. Thus the direction determined by an oriented edge \( E \) is fixed if and only if \( E \) is the initial edge of \( f(E) \).

A turn is an unordered pair of directions with a common base point. The turn is nondegenerate if is defined by distinct directions and is degenerate otherwise. If \( E_1 E_2 \ldots E_{k-1} E_k \) is the edge path associated to a path \( \sigma \), then we say that \( \sigma \) contains the turns \( (E_i, E_{i+1}) \) for \( 1 \leq i \leq k - 1 \). A turn is illegal with respect to \( f : G \to G \) if its image under some iterate of \( Df \) is degenerate; a turn is legal if it is not illegal. A path or circuit \( \sigma \subset G \) is legal if it contains only legal turns. If \( \sigma \subset G_r \) does not contain any illegal turns in \( H_r \), meaning that both directions correspond to edges of \( H_r \), then \( \sigma \) is \( r \)-legal. It is immediate from the definitions that \( Df \) maps legal turns to legal turns and that the restriction of \( f \) to a legal path is an immersion.

A non-trivial path in a zero stratum \( H_i \) whose endpoints belong to EG strata is called a connecting path.

Suppose that \( H_i \) and \( H_j \) are distinct NEG strata consisting of single edges \( E_i \) and \( E_j \), that \( w \) is a primitive Nielsen path and that \( f(E_i) = E_i w^{d_i} \) and \( f(E_j) = E_j w^{d_j} \) for some \( d_i, d_j > 0 \). Then a path of the form \( E_i w^p \bar{E}_j \) is called an exceptional path of height \( \max(i,j) \) or just an exceptional path if the height is not relevant. The set of exceptional paths of height \( i \) is invariant under the action of \( f_\# \).

**Definition 2.11.** A non-trivial path or circuit \( \sigma \) is completely split if it has a splitting, called a complete splitting, into subpaths, each of which is either a single edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path or a connecting path that is maximal in the sense that it cannot be extended to a larger connecting path in \( \sigma \).

**Definition 2.12.** A relative train track map is completely split if
1. \( f(E) \) is completely split for each edge \( E \) in an irreducible stratum.

2. \( f|\sigma \) is an immersion and \( f(\sigma) \) is completely split for each connecting path \( \sigma \).

**Remark 2.13.** For \( f : G \to G \) satisfying the conclusions of Theorem 2.17 below, a completely split path or circuit has a unique complete splitting by Lemma 4.14 of [FHb].

**Definition 2.14.** A periodic vertex \( w \) that does not satisfy one of the following two conditions is principal.

- \( w \) is the only element of \( \text{Per}(f) \) in its Nielsen class and there are exactly two periodic directions at \( w \), both of which are contained in the same EG stratum.

- \( w \) is contained in a component \( C \) of \( \text{Per}(f) \) that is topologically a circle and each point in \( C \) has exactly two periodic directions.

We also say that a lift of a principal vertex to the universal cover is a principal vertex.

**Remark 2.15.** It is immediate from the definition that the initial endpoint of an NEG edge is a principal vertex. By Lemma 3.18 of [FHb] every EG stratum \( H_r \) contains a principal vertex that is the basepoint for a periodic direction in \( H_r \).

**Definition 2.16.** If the endpoints of all indivisible periodic Nielsen paths are vertices and if each principal vertex and each periodic direction at a principal vertex has period one then we say that \( f : G \to G \) is forward rotationless.

Proposition 3.28 of [FHb] states that for relative train track maps satisfying certain mild assumptions the two notions of forward rotationless coincide. Namely \( f : G \to G \) is forward rotationless if and only if \( \phi \) is forward rotationless. Assuming that \( f : G \to G \) is forward rotationless, Corollary 3.21 and Lemma 3.26 of [FHb] imply that \( \tilde{f} \) is a principal lift if and only if some (and hence every) element of \( \text{Fix}(\tilde{f}) \) is a principal vertex.

A vertex in \( G \) is an attaching vertex if it belongs to a non-contractible component of \( G_r \) and is the endpoint of an edge in \( H_s \) for \( s > r \). We recall Theorem 4.6 of [FHb].

**Theorem 2.17.** Every forward rotationless \( \phi \in \text{Out}(F_n) \) is represented by a forward rotationless completely split relative train track map \( f : G \to G \). If \( \mathcal{F} \) is a \( \phi \)-invariant free factor system, then one may choose the associated \( f \)-invariant filtration \( \emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G \) so that \( \mathcal{F} = \mathcal{F}(G_l) \) for some filtration element \( G_l \). Moreover,

(V) Each attaching vertex is principal (and hence fixed).

(NEG) Each non-fixed NEG stratum \( H_r \) is a single edge \( E_r \) oriented so that \( f(E_r) = E_r \cdot u_r \) for some nontrivial closed path \( u_r \subset G_{r-1} \). The initial vertex of \( E_r \) is principal.
If $u_r$ is a non-trivial Nielsen path then $H_r$ and $E_r$ are said to be linear and we sometimes write $u_r = w_r d_r$ where $w_r$ is primitive.

(L) If $E_r$ and $E_s$ are distinct linear edges and if $w_r$ and $w_s$ agree as unoriented loops, then $w_r = w_s$ and $d_r \neq d_s$.

(N) Every periodic Nielsen path has period one. The endpoints of each indivisible Nielsen path $\sigma$ are vertices. For each $EG$ stratum $H_r$ there is at most one $\sigma$ of height $r$. If $\sigma$ has height $r$ and if $H_r$ is not $EG$ then $H_r$ is linear and $\sigma = E_r w_r^k \bar{E}_r$ for some $k \neq 0$.

(Per) The vertices in any non-trivial component $C$ of $Per(f)$ are principal. In particular $C \subset Fix(f)$. If $C$ is contractible and contains an edge in $H_r$, $1 \leq r \leq N$, then some vertex of $C$ has valence at least two in $G_{r-1}$.

(Z) $H_i$ is a zero stratum if and only if it is a contractible component of $G_i$. If $H_j$ is the first irreducible stratum following a zero stratum then $H_j$ is $EG$ and all components of $G_j$ are non-contractible. If $H_i$ is a zero stratum then $f|H_i$ is an immersion. If the link of a vertex $v$ is contained in a zero stratum $H_i$ then $v$ has valence at least three in $H_i$.

We assume throughout the remainder of this paper that $f : G \to G$ satisfies the conclusions of Theorem 2.17.

**Iterating an Edge**  We make frequent use of isolated points in $Fix_N(\hat{f})$ for principal lifts $\tilde{f}$. We quote two results that we refer to several times for the reader’s convenience. The first is a combination of Lemma 3.25 and Lemma 4.19 of [FHb]. The second is Lemma 4.21 of that same paper.

**Lemma 2.18.** Assume that $f : G \to G$ satisfies the conclusions of Theorem 2.17. The following properties hold for every principal lift $\tilde{f} : \Gamma \to \Gamma$.

1. If $\tilde{v} \in Fix(\tilde{f})$ and a non-fixed edge $\tilde{E}$ determines a fixed direction at $\tilde{v}$, then $\tilde{E} \subset \tilde{f}_#(\tilde{E}) \subset \tilde{f}_#^2(\tilde{E}) \subset \ldots$ is an increasing sequence of paths whose union is a ray $\tilde{R}$ that converges to some $P \in Fix_N(\hat{f})$ and whose interior is fixed point free. If $\tilde{E}$ is a lift of an edge in an $EG$ stratum then accumulation set of $P$ is the element in $\mathcal{L}(\phi)$ corresponding to that stratum.

2. For every isolated $P \in Fix_N(\hat{f})$ there exists $\tilde{E}$ and $\tilde{R}$ as in (1) that converges to $P$.

If $\tilde{E}$ and $P$ are as in Lemma 2.18 then we say that $\tilde{E}$ *iterates to* $P$ and that $P$ is associated to $\tilde{E}$.
Lemma 2.19. Suppose that \( \psi \in \text{Out}(F_n) \) is forward rotationless and that \( P \in \text{Fix}_N(\hat{\Psi}) \) for some \( \Psi \in \text{P}(\psi) \). Suppose further that \( \Lambda \) is an attracting lamination for some element of \( \text{Out}(F_n) \), that \( \Lambda \) is \( \psi \)-invariant and that \( \Lambda \) is contained in the accumulation set of \( P \). Then \( \text{PF}_\Lambda(\psi) \geq 0 \) and \( \text{PF}_\Lambda(\psi) > 0 \) if and only if \( P \) is isolated in \( \text{Fix}_N(\hat{\Psi}) \).

Axes Assume that \( \phi \) is forward rotationless and that \( f : G \to G \) is as in Theorem 2.17. Following the notation of [BFH04] we say that an unoriented conjugacy class \( \mu \) of a primitive element of \( F_n \) is an axis for \( \phi \) if for some (and hence any) representative \( c \in F_n \) there exist distinct \( \Phi_1, \Phi_2 \in \text{P}(\phi) \) that fix \( c \). Equivalently \( \text{Fix}_N(\Phi_1) \cap \text{Fix}_N(\Phi_2) \) is the endpoint set of the axis \( A_c \) for \( T_c \). The number of distinct elements of \( \text{P}(\phi) \) that fix \( c \) is called the multiplicity of \( \mu \). It is a consequence of Lemma 2.20 below that both the number of axes and the multiplicity of each axis is finite.

Lemma 2.5 implies that the oriented conjugacy class of \( c \) is \( \phi \)-invariant. By Lemmas 4.1.4 and 4.2.6 of [BFH00], the circuit \( \gamma \) representing \( c \) splits into a concatenation of periodic, and hence fixed, Nielsen paths. There is an induced decomposition of \( A_c \) into subpaths \( \tilde{\alpha}_i \) that project to either fixed edges or indivisible Nielsen paths. The lift \( \tilde{f}_0 : \Gamma \to \Gamma \) that fixes the endpoints of each \( \tilde{\alpha}_i \) is a principal lift and commutes with \( T_c \). We say that \( \tilde{f}_0 \) and the corresponding \( \Phi_0 \in \text{P}(\phi) \) are the base lift and base principal automorphism associated to \( \mu \) and the choices of \( T_c \) and \( f : G \to G \). (If \( \mu \) is not represented by a basis element then \( \Phi_0 \) is independent of the choice of \( f : G \to G \) but otherwise it is not; see Example 6.9 for ramifications of this fact.) Remark 2.13 implies that \( \tilde{f}_0 \) is the only lift that commutes with \( T_c \) and has fixed points in \( A_c \).

We recall Lemma 4.23 of [FHb].

Lemma 2.20. Assume notation as above and that \( f : G \to G \) satisfies the conclusions of Theorem 2.17. There is a bijection between principal lifts [principal automorphisms] \( \tilde{f}_j \neq \tilde{f}_0 \) [respectively \( \Phi_j \neq \Phi_0 \in \text{P}(\phi) \)] that commute with \( T_c \) [fix c] and the linear edges \( \{E_j\} \) with \( w_j \) representing \( \mu \). Moreover, if \( f(E_j) = E_j w_j^{d_j} \) then \( \tilde{f}_j = T_c^{d_j} \tilde{f}_0 \left[ \Phi_j = \hat{\alpha}_j^d(\Phi_0) \right] \).

3 Rotationless Abelian Subgroups

The Recognition Theorem is stated purely in terms of \( \phi \) and its forward iterates. No condition on \( \phi^{-1} \) is required. In the context of abelian subgroups, it is more natural to give \( \phi \) and \( \phi^{-1} \) equal footing.

Definition 3.1. \( \text{P}^\pm(\phi) = \text{P}(\phi) \cup \text{P}(\phi^{-1})^{-1} \). An outer automorphism \( \phi \) is rotationless if \( \text{Fix}(\hat{\Phi}) = \text{Per}(\hat{\Phi}) \) for all \( \Phi \in \text{P}^\pm(\phi) \) and if for each \( k \geq 1 \), \( \Phi \mapsto \Phi^k \) defines a bijection (see Remark 3.2) between \( \text{P}^\pm(\phi) \) and \( \text{P}^\pm(\phi^k) \). A subgroup of \( \text{Out}(F_n) \) is rotationless if each of its elements is.
Remark 3.2. There is no loss in replacing the assumption that \( \Phi \mapsto \Phi^k \) defines a bijection with the a priori weaker assumption that \( \Phi \mapsto \Phi^k \) defines a surjection. Indeed if \( \Phi \mapsto \Phi^k \) is not injective then there exist distinct \( \Phi_1, \Phi_2 \in \mathcal{P}^\pm(\phi) \) and \( k \geq 1 \) such that \( \Phi_1^k = \Phi_2^k \). This contradicts the fact that \( \Phi_2 \Phi_1^{-1} \) is a non-trivial covering translation and the fact that \( \text{Fix}(\Phi_2 \Phi_1^{-1}) \) contains \( \text{Fix}(\hat{\Phi}_1) = \text{Fix}(\hat{\Phi}_1^k) = \text{Fix}(\hat{\Phi}_2) \) and so contains at least three points.

The natural guess is that \( \phi \) is rotationless if and only if \( \phi \) and \( \phi^{-1} \) are forward rotationless. The following lemma and corollary fall short of proving this but is sufficient for our needs.

Lemma 3.3.  
1. If \( \phi \) is rotationless then \( \phi \) and \( \phi^{-1} \) are forward rotationless.
2. If \( \phi \) and \( \phi^{-1} \) are forward rotationless and \((*)\) is satisfied for \( \theta = \phi \) and \( \theta = \phi^{-1} \) then \( \phi \) is rotationless.

\((*)\) For all \( \Theta \in \mathcal{P}(\theta) \), the set of repelling periodic points for \( \hat{\Theta} \) is not a period two orbit that is the endpoint set of a lift of a generic leaf \( \gamma \) of an element of \( \mathcal{L}(\theta^{-1}) \).

Proof. Assume that \( \phi \) is rotationless. For \( k > 0 \), each element of \( \mathcal{P}(\phi^k) \) has the form \( \Phi^k \) where \( \text{Fix}(\hat{\Phi}) = \text{Per}(\hat{\Phi}) \) and hence \( \text{Fix}_N(\hat{\Phi}) = \text{Per}_N(\hat{\Phi}^k) \). Thus \( \Phi \in \mathcal{P}(\phi) \) proving that \( \phi \) is forward rotationless. The symmetric argument showing that \( \phi^{-1} \) is forward rotationless completes the proof of (1).

Assume now that the hypotheses of (2) are satisfied, that \( k \geq 1 \) and that \( \Phi_k \in \mathcal{P}(\phi^k) \). The plus and minus cases are symmetric so we may assume that \( \Phi_k \in \mathcal{P}(\phi^k) \). Since \( \phi \) is forward rotationless, \( \Phi_k = \Phi^k \) for some \( \Phi \in \mathcal{P}(\phi) \) satisfying \( \text{Fix}_N(\hat{\Phi}) = \text{Per}_N(\hat{\Phi}) \). To prove that \( \text{Fix}(\hat{\Phi}) = \text{Per}(\hat{\Phi}) \) it suffices to show that all periodic repelling points for \( \hat{\Phi} \) have period one. Since \( \phi^{-1} \) is forward rotationless, the only way this could fail would be if the repelling set is a period two orbit and if \( \Phi^2 \notin \mathcal{P}(\phi^{-1}) \). This possibility is ruled out by \((*)\). \(\square\)

Corollary 3.4. If \( \phi \) and \( \phi^{-1} \) are forward rotationless then \( \phi^2 \) is rotationless. There exists \( k > 0 \) so that \( \phi^{2k} \) is rotationless for every \( \phi \in \text{Out}(F_n) \).

Proof. The first statement follows from Lemma 3.3. The second follows from the first and from the fact (Corollary 4.26 of [FHb]) that there is a uniform \( k > 0 \) such that \( \phi^k \) are forward rotationless for all \( \phi \). \(\square\)

Example 3.5. Let \( G \) be the graph with one vertex \( v \) and edges labelled \( A, B \) and \( C \). Let \( f : G \to G \) be the homotopy equivalence defined by 
\[
A \mapsto B^3A \\
B \mapsto C^3B \\
C \mapsto (B^3A)^3C.
\]

The directions at \( v \) determined by \( \bar{A}, \bar{B} \) and \( \bar{C} \) are fixed by \( Df \) and those determined by \( B \) and \( C \) are interchanged by \( Df \). Thus \( f \) is not rotationless and the outer
automorphism $\phi$ that it determined is neither forward rotationless nor rotationless. The map $f$ factors as $f_3 f_2 f_1$ where $f_1$ fixes $A$ and $B$ and $f_1(C) = A^3 C$, $f_2$ fixes $B$ and $C$ and $f_2(B) = C^3 B$ and $f_3$ fixes $A$ and $B$ and $f_3(A) = B^3 A$. It is easy to check that each of these homotopy equivalence determines a rotationless element of $\text{Out}(F_n)$. This shows that the composition of rotationless elements need not be rotationless. Obviously, $\phi$ induces the identity on $H_1(G, \mathbb{Z}_3)$ and so illustrates that not every such element is rotationless.

**Lemma 3.6.** If $\phi$ is rotationless and $\Phi \in P(\phi)$ then $\text{Fix}_N(\hat{\Phi}^{-1}) \neq \emptyset$.

**Proof.** Choose $f : G \to G$ representing $\phi^{-1}$ and let $\hat{f} : \Gamma \to \Gamma$ be the lift corresponding to $\Phi^{-1}$. It suffices to show that $\text{Fix}_N(\hat{f}^k) \neq \emptyset$ for some $k \geq 1$. This follows from Lemma 3.23 of [FHb] if $\text{Fix}(\hat{f}) = \emptyset$ and from Lemma 3.26 of [FHb] otherwise. □

Abelian subgroups of $\text{Out}(F_n)$ are finitely generated [BL94]. Thus given any generating set for an abelian subgroup, there is a finite subset which also generates. At the end of this section (Corollary 3.11) we prove that an abelian subgroup $A$ of $\text{Out}(F_n)$ that is generated by rotationless elements, is rotationless.

Many of our arguments proceed by induction on the cardinality of a given set of rotationless generators.

**Lemma 3.7.** If $\phi$ is rotationless and $F$ is a $\phi$-invariant free factor of rank at least two then $\phi|F$ is rotationless.

**Proof.** This follows from the definitions and the fact that an element of $P^\pm(\phi|F)$ extends to an element of $P^\pm(\phi)$. □

We produce lifts of an abelian subgroup of $\text{Out}(F_n)$ to $\text{Aut}(F_n)$ that is generated by rotationless elements via the following definition and lemma.

**Definition 3.8.** A set $X \subset \partial F_n$ with at least three points is a principal set for a subgroup $A$ of $\text{Out}(F_n)$ if each $\psi \in A$ is represented by $\Psi \in \text{Aut}(F_n)$ satisfying $X \subset \text{Fix}(\Psi)$ and if this necessarily unique $\Psi$ is an element of $P^\pm(\psi)$. The assignment $\psi \mapsto \Psi$ is a lift of $A$ from $\text{Out}(F_n)$ to $\text{Aut}(F_n)$.

**Lemma 3.9.** Suppose that $A$ is an abelian subgroup of $\text{Out}(F_n)$ that is generated by rotationless elements, that $\phi \in A$ is rotationless and that $\Phi \in P^\pm(\phi)$.

1. If $\text{Fix}(\Phi)$ has rank zero then $\text{Fix}(\hat{\Phi})$ is a principal set for $A$.
2. If $\text{Fix}(\Phi)$ has rank one with generator $c$ and if $P$ is an isolated point in $\text{Fix}(\hat{\Phi})$ then $\{P, T^\pm_c\}$ is a principal set for $A$.
3. If $\text{Fix}(\Phi)$ has rank at least two then $\partial \text{Fix}(\Phi)$ contains at least one principal set $X$ for $A$ and one can choose $X$ to contain $T^\pm_c$ for any given $A$-invariant $[c]$ for $c \in \text{Fix}(\Phi)$. Moreover, for every isolated point $P$ in $\text{Fix}(\hat{\Phi})$ there is a principal set $Y$ for $A$ that contains $P$ and at least two elements of $\partial \text{Fix}(\Phi)$.

16
In particular, $\text{Fix}(\hat{\Phi})$ contains at least one principal set for $A$ and every isolated point in $\text{Fix}(\hat{\Phi})$ is contained in a principal set. If $s:A \to \text{Out}(F_n)$ is the lift determined by a principal set contained in $\text{Fix}(\hat{\Phi})$ then $s(\phi) = \Phi$.

**Proof.** Let $\psi$ be an element of a a finite rotationless generating set $S$ for $A$ and let $\Psi$ represent $\psi$. Lemma 2.8 implies that conjugation by $\Psi$ defines a permutation of the finite set of equivalence classes in $P^\pm(\phi)$. Choose $k > 0$ so that the permutation induced by $\Psi^k$ is trivial. Then $\Psi^k\Phi\Psi^{-k} = i_c\Phi i_c^{-1}$ for some $c \in F_n$ and $\Psi_k := i_c^{-1}\Psi^k$ commutes with $\Phi$. In particular, $\mathbb{F} := \text{Fix}(\hat{\Phi})$ is $\Psi_k$-invariant.

Assume at first that $\mathbb{F}$ has rank zero. By Lemma 2.2, $\text{Fix}(\hat{\Phi})$ is a finite union of attractors and repellers and by Lemma 3.6 there is at least one of each. Since $\Phi \in P^\pm(\phi)$, there are at least three points in $\text{Fix}(\hat{\Phi})$.

We claim that if $\Theta$ represents $\theta \in A$ and if $\text{Fix}(\hat{\Phi}) \subset \text{Fix}(\hat{\Theta})$ then $\Theta \in P^\pm(\theta)$. If $\text{Fix}(\hat{\Theta})$ contains at least five points then this is obvious. After replacing $\theta$ with its inverse if necessary, there are two potentially bad cases. The first is that $\text{Fix}(\Theta)$ has exactly one repelling point and exactly two attracting points and that the attractors bound a lift $\hat{\gamma}$ of a generic leaf of some $\Lambda \in \mathcal{L}(\theta)$. Since the endpoints of $\hat{\gamma}$ are isolated fixed points of $\hat{\Phi}$, $\Lambda \in \mathcal{L}(\phi) \cup \mathcal{L}(\phi^{-1})$ by Lemma 2.19. After replacing $\phi$ with its inverse if necessary, we may assume that $\Lambda \in \mathcal{L}(\phi)$ and that the endpoints of $\hat{\gamma}$ are attractors for $\Phi$. Since $\text{Fix}(\hat{\Phi})$ contains only three points and by Lemma 3.6 has at least one $\hat{\Phi}$-repeller, this contradicts the assumption that $\Phi \in P^\pm(\phi)$.

The other bad possibility is that $\text{Fix}(\hat{\Theta})$ is a four point set with two repelling points that bound a lift of a leaf of an element of $\mathcal{L}(\theta^{-1})$ and two attracting points that bound a lift of a leaf of an element of $\mathcal{L}(\theta)$. As in the previous case, this description also applies to $\Phi$ in contradiction to the assumption that $\Phi \in P^\pm(\phi)$.

This completes the proof that $\Theta \in P^\pm(\theta)$.

After replacing $\Psi_k$ with an iterate, we may assume that $\text{Fix}(\hat{\Phi}) \subset \text{Fix}(\hat{\Psi}_k)$ and hence that $\Psi_k \in P^\pm(\psi^k)$. Since $\psi$ is rotationless, there exists $\Psi \in P^\pm(\psi)$ with $\text{Fix}(\hat{\Phi}) \subset \text{Fix}(\hat{\Psi})$. As this holds for every element of $S$, we have proved (1).

Suppose next that $\mathbb{F}$ has rank one with generator $c$ and that $P$ is an isolated point in $\text{Fix}(\hat{\Phi})$. Lemma 2.2 implies that there are only finitely many $i_c$-orbits of isolated points in $\text{Fix}(\hat{\Phi})$. After increasing $k$ if necessary, we may assume that $c \in \text{Fix}(\Psi_k)$ and that $\Psi_k$ preserves each such $i_c$-orbit. In particular, $\Psi_k(P) = \hat{T}_c^q(P)$ for some $q$. Let $\Psi_k' := i_c^{-q}\Psi_k$. Then $\{T_c^\pm, P\} \subset \text{Fix}(\hat{\Psi}_k')$ and $\Psi_k' \in P^\pm(\psi)$. Since $\psi$ is rotationless, there exists $\Psi \in P^\pm(\psi)$ such that $\{T_c^\pm, P\} \subset \text{Fix}(\hat{\Psi})$. As this holds for every element of $S$, it follows that for each $\theta \in A$ there exists $\Theta$ such that $\{T_c^\pm, P\} \subset \text{Fix}(\hat{\Theta})$. In this case it is obvious that $\Theta \in P^\pm(\theta)$. This completes the proof of (2).

We turn next to the moreover part of (3). Assume that $P$ is an isolated point in $\text{Fix}(\hat{\Phi})$. As in the rank one case, the fact that there are only finitely many $i_c$-orbits of isolated points in $\text{Fix}(\hat{\Phi})$ allows us to choose $\Psi_k^*$ representing an iterate $\psi^k$ of $\psi$ such that $P \in \text{Fix}(\Psi_k^*)$ and such that $\mathbb{F}$ is $\Psi_k^*$-invariant. We claim that $\Psi_k^* \in P^\pm(\psi^k)$. Assuming without loss that $\text{Fix}(\Psi_k^*|\mathbb{F})$ is finite, Lemma 3.6 implies,
after replacing \( \Psi_k \) by an iterate if necessary, that \( \text{Fix}(\Psi_k|_F) \) has at least one non-attractor \( Q_- \) and one non-repeller \( Q_+ \). Lemma 2.19 implies that \( Q_+ \) and \( Q_- \) do not cobound a lift of a generic leaf of an attracting lamination. (This method for proving that a pair of points do not cobound a lift of a generic leaf of an attracting lamination is used implicitly throughout the rest of the proof.) Generic leaves of an attracting lamination are birecurrent and so either have both endpoints in \( \partial F \) or neither endpoint in \( \partial F \). Thus \( P \) and \( Q_{\pm} \) do not cobound a lift of a generic leaf of an attracting lamination. (This method for proving that a pair of points do not cobound a lift of a generic leaf of an attracting lamination is used implicitly throughout the rest of the proof.)

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We have shown that if \( S = \{ \psi_1, \ldots, \psi_K \} \) then for all \( 1 \leq j \leq K \) there exists \( \psi_j \in \mathbb{P}^\pm(\psi) \) such that \( P \in \text{Fix}(\hat{\psi}_j) \) and such that \( F \) is \( \psi_j \)-invariant. Since \( P \) is not fixed by any covering translation, the \( \psi_j \)'s commute. By Lemma 2.19 applied \( \psi \) and \( Y \). To this end, let \( Y_j = (\bigcap_{i=1}^j \text{Fix}(\hat{\psi}_i)) \cap \partial F \) and let \( I_j \) be the statement that \( Y_j \) either contains three points or contains two points that do not cobound a lift of a generic leaf of any attracting lamination. As noted above, \( P \) and an element of \( \partial F \) can not cobound a generic leaf of an attracting lamination. Thus \( I_K \) completes the proof of the moreover part of (3).

\( I_1 \) follows from Lemma 3.6 applied to \( \Psi_1|_F \). Assume that \( I_{j-1} \) holds. \( Y_{j-1} \) is \( \psi_j \)-invariant. If \( Y_{j-1} \) is finite then it is fixed by an iterate of \( \hat{\psi}_j \) and hence by \( \hat{\psi}_j \). If \( Y_{j-1} \) contains \( T_b^\pm \) for some unique primitive unoriented \( b \) then \( T_b^\pm \) is fixed by an iterate of \( \hat{\psi}_j \) and hence by \( \hat{\psi}_j \). In either case \( I_j \) holds. In the remaining case \( \bigcap_{i=1}^{j-1} \text{Fix}(\psi_j) \) intersects \( F \) in a subgroup \( F_{j-1} \) of rank at least two and \( I_j \) follows from Lemma 3.6 applied \( \hat{\psi}_j|_{F_{j-1}} \), keeping in mind that \( \text{Fix}(\hat{\psi}_j|_{F_{j-1}}) \subset Y_j \). This completes the induction step and so proves \( I_K \).

It remains to prove the main statement of (3). We argue by induction on the cardinality \( K \) of a given rotationless generating set for \( A \). If \( K = 1 \) and \( S = \{ \psi \} \) then there exists \( \psi \in \mathbb{P}^\pm(\psi) \) such that \( \text{Fix}(\hat{\psi}) = \text{Fix}(\hat{\phi}) \) and \( \text{Fix}(\hat{\psi}) \) is obviously a principal set for \( A \). We now assume that \( K \geq 2 \) and that (3) holds for subgroups that are generated by fewer than \( K \) rotationless elements.

The defining property of \( \Psi_k \) is that it commutes with \( \Phi \). We may therefore replace our current \( \Psi_k \) with any lift of any iterate of \( \psi \) that preserves \( F \). By Lemma 5.2 of [BFH04] or Proposition 1.5 of [LL00], there is such a lift, still called \( \Psi_k \), such that \( \Psi_k|_F \in \mathbb{P}^\pm(\psi_k|_F) \); moreover if \( c \in \text{Fix}(\hat{\psi}) \) is \( A \)-invariant then we may choose \( \Psi_k \) so that \( c \in \text{Fix}(\Psi_k) \). Since \( \psi \) is rotationless, there exists \( \psi \in \mathbb{P}^\pm(\psi) \) such that \( \text{Fix}(\hat{\psi}_k) = \text{Fix}(\hat{\psi}_k) \). Thus \( \text{Fix}(\hat{\psi}) \cap \text{Fix}(\hat{\phi}) \) contains at least three points which implies that \( \psi \) commutes with \( \Phi \). To summarize, we have \( \psi \in \mathbb{P}^\pm(\psi) \) that preserves \( F \) and such that \( \psi|_F \in \mathbb{P}^\pm(\psi|_F) \); if \( c \in \text{Fix}(\hat{\phi}) \) is \( A \)-invariant then we may assume that \( c \in \text{Fix}(\psi) \). As each \( \Psi \) preserves \( F \), it follows that \( \mathbb{P}^\pm(\psi) \) is \( A \)-invariant.

Let \( A' = \mathbb{A}|_F \), let \( \psi'|_F \) and let \( \Psi' = \Psi|_F \). A principal set for \( A' \) is also a principal set for \( A \) because an automorphism of \( F \) representing \( \theta|_F \in A' \) extends
Lemma 3.10. An abelian subgroup $A$ that is generated by rotationless elements is torsion free.

Proof. If $\theta \in A$ is a torsion element then it is represented by a finite order homeomorphism $f' : G' \to G'$ of a marked graph $G'$. Suppose that $X$ is a principal set for $A$ and that $P_1, P_2, P_3 \in X$. There is a lift $\tilde{f}' : \Gamma' \to \Gamma'$ such that each $P_i \in \text{Fix}(\tilde{f}')$. The line $L_{12}$ with endpoints $P_1$ and $P_2$ and the line $L_{13}$ with endpoints $P_1$ and $P_3$ are $\tilde{f}'$-invariant and since $\tilde{f}'$ is a homeomorphism they are $f'$-invariant. The intersection $L_{12} \cap L_{13}$ is an $f'$-invariant ray and so is contained in $\text{Fix}(f')$. It follows that $L_{12} \subset \text{Fix}(f')$ and that the image of $L_{12}$ in $G'$ is contained in $\text{Fix}(f')$. It therefore suffices to show that every edge of $G'$ is crossed by at least one such line.

For any set $Y \subset \partial F_n$, let $C_Y$ be the set of bi-infinite lines cobounded by pairs of elements of $Y$. Let $W_A = \cup C_X$ where the union is over all principal sets $X$ for $A$ and let $\mathcal{F}$ be the smallest free factor system that carries $W_A$. It suffices to show that $\mathcal{F} = \{[[F_n]]\}$. The proof of this assertion is by induction on the cardinality $K$ of a given rotationless generating set $S$ for $A$.

Assume to the contrary that $\mathcal{F}$ is proper and choose $\psi \in S$. There is a homotopy equivalence $f : G \to G$ representing $\psi$ as in Theorem 2.17 in which $\mathcal{F}$ is realized as a filtration element $G_r$. Lemma 3.25(2) implies that each $\Lambda \in \mathcal{L}(\psi)$ is the accumulation set of an isolated point in $\text{Fix}_N(\Psi)$ for some $\Psi \in \mathcal{P}(\psi)$. By Lemma 3.9, $\Lambda$ is carried by $\mathcal{F}$. Thus each stratum above $G_r$ is NEG. Items (NEG) and (PER) of Theorem 2.17 imply that every edge $E$ of $G \setminus G_r$ has an orientation so that its initial vertex is principal and so that its initial direction is fixed. Choose a lift $\tilde{E}$ of $E$ and a principal lift $\tilde{f} : \Gamma \to \Gamma$ that fixes the initial direction determined by $\tilde{E}$. There is a ray that begins with $\tilde{E}$ and converges to a point in $\text{Fix}_{\Lambda}(\tilde{f})$. This follows from Lemma 2.18 if $E$ is not a fixed edge and from Lemma 3.26 of [FHb] otherwise. Let $\Psi$ be the principal automorphism corresponding to $\tilde{f}$. It suffices to show that each element of $C_{\text{Fix}(\Psi)}$ is carried by $\mathcal{F}$. This is obvious if $K = 1$. We have now proved the basis step of our induction argument and may assume that $K > 1$ and that $\mathcal{F} = \{[[F_n]]\}$ when $A$ has a rotationless generating set with fewer than $K$ elements.

If $\text{Fix}(\Psi)$ has rank zero then $\text{Fix}(\Psi)$ is contained in a principal set for $A$ by Lemma 3.9(1) and $C_{\text{Fix}(\Psi)}$ is carried by $\mathcal{F}$. If $\text{Fix}(\Psi)$ has rank one with generator $c$ then Lemma 3.9(2) implies that the line connecting $P$ to $T_c^+$ is carried by $\mathcal{F}$ for each $P \in \text{Fix}(\Psi)$. It follows that the line connecting any two points of $\text{Fix}(\Psi)$ is carried by $\mathcal{F}$.
We may therefore assume that \( \text{Fix}(\Psi) \) has rank at least two. Let us show that \( \text{Fix}(\Psi) \) is carried by \( F \). The inductive hypothesis and the fact that \( A \mid \text{Fix}(\Psi) \) has a generating set with fewer than \( K \) elements implies that no proper free factor system of \( \text{Fix}(\Psi) \) carries \( W_{A \mid \text{Fix}(\Psi)} \). The Kurosh subgroup theorem therefore implies that any free factor system of \( F_n \) that carries \( W_{A \mid \text{Fix}(\Psi)} \) also carries all of \( \text{Fix}(\Psi) \). Since \( W_{A \mid \text{Fix}(\Psi)} \subset W_A \) we conclude that \( \text{Fix}(\Psi) \) is carried by \( F \).

Lemma 3.9(3) implies that for each \( P \in \text{Fix}(\hat{\Psi}) \) there exists \( Q \in \partial \text{Fix}(\Psi) \) so that the line connecting \( P \) to \( Q \) is carried by \( F \). Since the line connecting any two points in \( \partial \text{Fix}(\Psi) \) is carried by \( F \) it follows that the line connecting any two points in \( \text{Fix}(\hat{\Psi}) \) is carried by \( F \).

Corollary 3.11. An abelian subgroup \( A \) that is generated by rotationless elements is rotationless.

Proof. Suppose that \( \phi \in A \), that \( k > 1 \) and that \( \Phi_k \in \mathbb{P}^{\pm}(\phi^k) \). Choose \( m \geq 1 \) so that \( \phi^{km} \) is rotationless. By Lemma 3.9 there is a principal set \( X \) for \( A \) with \( X \subset \text{Fix}(\hat{\Phi}_k^m) \). Let \( s : A \to \text{Aut}(F_n) \) be the lift determined by \( X \) and let \( \Phi = s(\phi) \). Then \( \Phi^{km} = s(\phi^k)^m = \Phi_k^m \) and so \( \Phi^k = s(\phi_k) = \Phi_k \) by Lemma 3.10. To complete the proof it suffices by Remark 3.2 to show that \( \text{Fix}(\hat{\Phi}) = \text{Fix}(\Phi^{km}) \). Let \( F = \text{Fix}(\Phi^{km}) \) and note that \( F \) is \( s(A) \)-invariant. Lemma 3.10 implies that \( \Phi \) is uniquely characterized by \( \Phi^{km} \) and hence that \( \Phi \) is independent of the choice of \( X \). Parts (1) and (2) of Lemma 3.9 therefore imply that \( \text{Fix}(\Phi) \) contains each isolated point in \( \text{Fix}(\Phi_k^m) \) and contains \( \partial F \) if \( F \) has rank less than two. If \( F \) has rank at least two then \( F \subset \text{Fix}(\Phi) \) by Lemma 3.10 applied to \( \phi \mid F \).

Corollary 3.12. For each abelian subgroup \( A \) of \( \text{Out}(F_n) \), the set of rotationless elements is a rotationless subgroup \( A_R \) that has finite index in \( A \).

Proof. This is an immediate corollary of Corollary 3.11 and the fact that every element of \( A \) has a rotationless iterate.

4 Generic Elements of rotationless abelian subgroups

In this section we define an embedding of a given rotationless abelian subgroup \( A \) into an integer lattice \( \mathbb{Z}^N \) and say what it means for an element of \( A \) to be generic with respect to this embedding.

Definition 4.1. Suppose that \( X_1 \) and \( X_2 \) are principal sets for \( A \) that define distinct lifts \( s_1 \) and \( s_2 \) of \( A \) to \( \text{Aut}(F_n) \) and that \( T^\pm \in X_1 \cap X_2 \). Then \( s_2(\psi) = \text{Id}_{\psi} \cdot s_1(\psi) \) for all \( \psi \in A \) and some \( d(\psi) \in \mathbb{Z} \); the assignment \( \psi \mapsto d(\psi) \) defines a homomorphism that we call the comparison homomorphism \( \omega : A \to \mathbb{Z} \) determined by \( X_1 \) and \( X_2 \).
Lemma 4.2. For any rotationless abelian subgroup $A$ there are only finitely many comparison homomorphisms $\omega : A \to \mathbb{Z}$.

Proof. Distinct comparison homomorphisms must disagree on some basis element of $A$ so we can restrict attention to those comparison homomorphisms that disagree on a single element $\psi \in A$. If $\omega$ is defined with respect to $X_1, X_2$ and $c$ then $[c]_u$, the unoriented conjugacy class of $c$, is an axis of $\psi$. As $\psi$ only has finitely many axes, we may restrict attention to those comparison homomorphisms that are defined with respect to the same $[c]_u$. If $a \in F_n$ and $\omega'$ is defined with respect to $\hat{i}_aX_1, \hat{i}_aX_2$ and $i_a(c)$ then $\omega' = \omega$. We may therefore restrict attention to comparison homomorphisms that are defined with respect to the same $c$. The number of such comparison homomorphisms is bounded by the multiplicity of $[c]_u$ as an axis for $\phi$ by Lemma 2.20.

Lemma 4.3. If $A$ is a rotationless abelian subgroup then $L(A) = \bigcup_{\phi \in A} L(\phi)$ is a finite collection of $A$-invariant laminations.

Proof. Let $\{\psi_1, \ldots, \psi_K\}$ be a rotationless basis for $A$. If $L(\phi) = \{\Lambda_1, \ldots, \Lambda_q\}$ and $F(\Lambda_i)$ is the smallest free factor that carries $\Lambda_i$ then the $F(\Lambda_i)$’s are distinct by Lemma 3.2.4 of [BFH00]. Each $\psi_j$ permutes the $\Lambda_i$’s by Lemma 3.1.6 of [BFH00] and so permutes the $F(\Lambda_i)$’s by Lemma 2.10. Since $\psi_j$ is rotationless, each $F(\Lambda_i)$, and hence each $\Lambda_i$, is $\psi_j$-invariant by Lemma 2.5. This proves that $\Lambda_i$ is $A$-invariant and hence that $PF_{\Lambda_i}$ is defined on $A$. Each $PF_{\Lambda_i}$ must be non-zero when applied to some $\psi_j$ and by Lemma 3.3.1 of [BFH00] this is equivalent to $\Lambda \in L(\psi_j) \cup L(\psi_j^{-1})$, which is a finite set.

Definition 4.4. For each $\Lambda \in L(A)$, we say that $PF_{\Lambda}|A$ is the expansion factor homomorphism for $A$ determined by $\Lambda$. Let $N$ be the number of distinct comparison and expansion factor homomorphisms for $A$. Define $\Omega : A \to \mathbb{Z}^N$ to be the product of these homomorphisms. We say that $\Omega$ is the coordinate homomorphism for $A$ and that each comparison homomorphism and expansion factor homomorphism is a coordinate of $\Omega$.

Lemma 4.5. If $A$ is a rotationless abelian subgroup then $\Omega : A \to \mathbb{Z}^N$ is injective.

Proof. Given non-trivial $\theta \in A$, choose $f : G \to G$ and $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ representing $\theta$ as in Theorem 2.17 and let $H_l$ be the lowest non-fixed irreducible stratum. If $H_l$ is EG then $PF_{\Lambda}(\theta) \neq 0$ for the attracting lamination $\Lambda \in L(\theta)$ associated to $H_l$. Otherwise $H_l$ is a single edge $E$ and $f(E) = E \cdot u$ where $u \subset G_{l-1}$ is a loop that is fixed by $f$. Lemma 2.20 implies that there are distinct principal lifts $\Theta_1$ and $\Theta_2$ of $\theta$ that fix the primitive element $c \in F_n$ determined by $u$. Thus $\Theta_2 = T^d\Theta_1$ for some $d \neq 0$. By Lemma 3.9 there exists principal sets $X_1 \subset Fix(\Theta_1)$ and $X_2 \subset Fix(\Theta_2)$ that contain $c$. These determine a comparison homomorphism $\omega$ such that $\omega(\theta) = d$. We have shown that some coordinate of $\Omega(\theta) \neq 0$ and since $\theta$ was arbitrary, $\Omega$ is injective.
Definition 4.6. Assume that $A$ is a rotationless abelian subgroup and that $\Omega : A \to \mathbb{Z}^N$ is its coordinate homomorphism. Then $\phi \in A$ is generic if all coordinates of $\Omega(\phi)$ are non-zero.

Remark 4.7. $\phi$ is generic in $A$ if and only if $L(\phi) = L(A)$ and \( \phi \) has the same axes and multiplicity as $A$.

Lemma 4.8. Every rotationless abelian subgroup $A$ has a basis of generic elements.

Proof. Given a basis $\psi_1, \ldots, \psi_K$ for $A$ and $\theta \in A$ let $NZ(\theta) \subset \{1, \ldots, N\}$ be the non-zero coordinates of $\Omega(\theta)$. For all but finitely many positive integers $a_2$, $NZ(\psi_1^2) = NZ(\psi_1) \cup NZ(\psi_2)$. Inductively choose positive integers $a_i$ for $i > 1$ so that $\Psi'_1 := \Psi_1\Psi_2^{a_2} \cdots \Psi_K^{a_K}$ satisfies $NZ(\psi_1') = \cup_{i=1}^K NZ(\psi_i) = \{1, \ldots, N\}$. Replacing $\psi_1$ with $\psi_1'$ produces a new basis in which the first element is generic. Repeating this step $K$ times produces the desired basis or just replace $\psi_2$ with $\psi_2\psi_1^{bg}$ and so on.

A principal set $X$ for $A$ determines a lift of $A$ to $\text{Aut}(F_n)$. If $X_1 \subset X_2$ are principal sets for $A$ then $X_1$ and $X_2$ determine the same lift. It therefore makes sense to consider principal sets that are maximal with respect to inclusion.

Lemma 4.9. If $\phi \in A$ is generic then \{Fix($\hat{\phi}$) : $\Phi \in P^\pm(\phi)$\} is the set of maximal (with respect to inclusion) principal sets for $A$.

Proof. Each principal set $X'$ for $A$ determines a lift $s : A \to \text{Aut}(F_n)$. If $\Phi \in P^\pm(\phi)$ and $\text{Fix}(\hat{\Phi}) \subset X'$ then $s(\phi) = \Phi$ and $X' \subset \text{Fix}(\hat{\Phi})$. This proves that $\text{Fix}(\Phi)$ is a maximal principal set if it is a principal set. It therefore suffices to show that each $\text{Fix}(\hat{\Phi})$ is a principal set.

If $\mathbb{F} := \text{Fix}(\hat{\Phi})$ has rank zero then $\text{Fix}(\hat{\Phi})$ is a principal set by Lemma 3.9(1). If $\mathbb{F}$ has rank one with generator $c$ and with isolated points $P,Q \in \text{Fix}(\hat{\Phi})$ then by Lemma 3.9(2) there is a maximal principal set $X_P$ that contains $P$ and $T_c^\pm$ and a maximal principal set $X_Q$ that contains contains $Q$ and $T_c^\pm$. If $X_P \neq X_Q$ then the comparison homomorphism that they determine evaluates to zero on $\phi$ since $\Phi_P = \Phi_Q = \Phi$ in contradiction to the assumption that $\phi$ is generic. Thus $X_P = X_Q$. Since $P$ and $Q$ are arbitrary, $X_P = \text{Fix}(\hat{\Phi})$.

Suppose finally that $\mathbb{F}$ has rank at least two. We claim that $A|\mathbb{F}$ is trivial. If not, let $\Omega'$ be the homomorphism defined on $A|\mathbb{F}$ as the product of expansion factor and comparison homomorphisms that occur for $A|\mathbb{F}$. Each coordinate $\omega'$ of $\Omega'$ extends to a coordinate $\omega$ of $\Omega$. Since $\phi|\mathbb{F}$ is the identity, $\omega(\phi) = 0$ in contradiction to the assumption that $\phi$ is generic. Thus $A|\mathbb{F}$ is trivial and $\partial\mathbb{F}$ is contained in a maximal principal set $X$ for $A$.

By Lemma 3.9(3), each isolated point $P$ in $\text{Fix}(\hat{\Phi})$ is contained in a maximal principal set $X_P$ whose intersection $Y$ with $\partial\mathbb{F}$ contains at least two points. If $X_P \neq X$ then $Y$ has exactly two points and in fact equals $\{T_b^\pm\}$ for some $b \in F_n$ since every lift of the identity outer automorphism is an inner automorphism. The comparison homomorphism $\omega$ determined by $X_P$ and $X$ evaluates to zero on $\phi$ in contradiction to the
assumption that $\phi$ is generic. Thus $X_P = X$ for all isolated points $P$ and $\text{Fix}(\hat{\Phi}) = X$ as desired. □

It is an immediate corollary, that from the point of view of fixed points of principal lifts, generic elements are indistinguishable.

**Corollary 4.10.** For any generic $\phi, \psi \in A$ there is a bijection $h : P^{\pm}(\phi) \to P^{\pm}(\psi)$ such that $\text{Fix}(\hat{\Phi}) = \text{Fix}(\hat{h}(\Phi))$ for all $\Phi \in P^{\pm}(\phi)$.

## 5 $A(\phi)$

The data required in the Recognition theorem has both qualitative and quantitative components. If we fix the qualitative part and allow the quantitative part to vary then we generate an abelian group that is naturally associated to the outer automorphism being considered. This section contains a formal treatment of this observation. A more computational friendly approach in terms of relative train track maps is given in the next section.

**Definition 5.1.** Assume that $\phi$ is rotationless. $A(\phi)$ is the subgroup of $\text{Out}(F_n)$ generated by rotationless elements $\theta$ for which there is a bijection $\hat{h} : P^{\pm}(\phi) \to P^{\pm}(\theta)$ satisfying $\text{Fix}(\hat{h}(\Phi)) = \text{Fix}(\hat{\Phi})$ for all $\hat{\Phi} \in P(\phi)$.

**Remark 5.2.** It is an immediate consequence of the definitions that $A(\phi) = A(\phi^k)$ for all $k \neq 0$ for all rotationless $\phi$.

**Lemma 5.3.** If $A$ is a rotationless abelian subgroup and $\phi$ is generic in $A$, then $A \subset A(\phi)$.

**Proof.** Lemma 4.8 and Corollary 4.10 imply that there is a generating set of $A$ that is contained in $A(\phi)$. □

To prove that $A(\phi)$ is abelian we appeal to the following characterization of the rotationless elements in the centralizer $C(\phi)$ of $\phi$.

**Lemma 5.4.** If $\phi, \psi \in \text{Out}(F_n)$ are rotationless, then $\psi \in C(\phi)$ if and only if all for $\Phi \in P^{\pm}(\phi)$:

1. There exists $\Psi \in P^{\pm}(\psi)$ such that $\text{Fix}(\hat{\Psi})$ is $\hat{\Psi}$-invariant.
2. If $P \in \text{Fix}(\hat{\Phi})$ is isolated then one may choose $\Psi$ in (1) such that $P \in \text{Fix}(\hat{\Psi})$.
3. If $a \in \text{Fix}(\Phi)$ and $[a]_u$ is an axis of $\phi$ then one may choose $\Psi$ in (1) such that $a \in \text{Fix}(\hat{\Psi})$.

Moreover, if $\psi \in C(\phi)$ and $\Psi$ is as in (1) then $\Psi$ commutes with $\Phi$. 

23
Proof. If \( \psi \in C(\phi) \), let \( A = \langle \phi, \psi \rangle \). Lemma 3.9 implies that for each \( \Phi \in P^\pm(\phi) \), there is a principal set \( X \) for \( A \) whose associated lift \( s : A \to \text{Aut}(F_n) \) satisfies \( s(\phi) = \Phi \). Then \( s(\psi) \in P^\pm(\psi) \) commutes with \( \Phi \) and \((\Phi - 1)\) is satisfied. \( (\Phi - 2) \) follows from Lemma 3.9. If \([a]_u \) is an axis of \( \phi \) then \([a]_u \) is \( \psi^k \)-invariant for some \( k > 0 \) and so is \( \psi \)-invariant by Lemma 2.5. Items (2) and (3) of Lemma 3.9 allow us to choose \( X \) to contain \( T^\pm_a \) which implies \((\Phi - 3)\). This completes the only if direction of the lemma.

For the if direction, we assume that \( \psi \) satisfies the three items, define \( \phi' := \psi \phi \psi^{-1} \) and prove that \( \phi' = \phi \) by applying the Recognition Theorem.

For each \( \Phi \in P(\phi) \) choose \( \Psi_1 \) satisfying \((\Phi - 1)\) and define \( \Phi' = \Psi_1 \Phi \Psi_1^{-1} \in P(\phi') \).

If \( \Psi_2 \) also satisfies \((\Phi - 1)\) then \( \Psi_2 = \Psi_1 i_x \) where \( \text{Fix}(\hat{\Phi}) \) is \( \hat{i}_x \)-invariant. By Lemma 2.1, \( x \in \text{Fix}(\hat{\Phi}) \). Thus \( \Psi_2 \Phi \Psi_2^{-1} = \Psi_1 i_x \Phi i_x^{-1} \Psi_1^{-1} = \Psi_1 \Phi \Psi_1^{-1} \) and \( \Phi' \) is independent of the choice of \( \Psi_1 \).

We denote \( \Phi \mapsto \Phi' \) by \( h : P(\phi) \to P(\phi') \) and note that \( \text{Fix}(h(\Phi)) = \hat{\Psi}_1(\text{Fix}(\hat{\Phi})) = \hat{\Psi}_1(\text{Fix}(\hat{\Phi})) = \hat{\text{Fix}}_N(h(\Phi)) = \hat{\text{Fix}}_N(\hat{\Phi}) \).

In particular, \( h \) is injective. If \( \Phi \) is replaced by \( i_c \Phi i_c^{-1} \) then \( \Phi \) is replaced by \( i_c \Phi i_c^{-1} \). Thus the restriction of \( h \) to an equivalence class in \( P(\phi) \) is a bijection onto an equivalence class in \( P(\phi') \). Lemma 2.8(2) implies that \( P(\phi) \) and \( P(\phi') \) have the same number of equivalence classes and hence that \( h \) is a bijection.

Suppose that \( \Phi_1 \in P(\phi) \), that \( a \in \text{Fix}(\Phi_1) \) is primitive and that \( \Phi_2 := i_a^d \Phi_1 \in P(\phi) \) for some \( d \neq 0 \). Then \([a]_u \) is an axis for \( \phi \) and by \((\Phi_1 - 3)\) and \((\Phi_2 - 3)\) we may choose \( \Psi_1 \) for \( \Phi_1 \) and \( \Psi_2 \) for \( \Phi_2 \) to fix \( a \). Thus \( \Psi_2 = i_a^m \Psi_1 \) for some \( m \) and \( \Phi_2 = i_a^m \Psi_1 i_a^d \Phi_1 \Psi_1^{-1} i_a^{-m} = i_a^d \Psi_1 \Phi_1 \Psi_1^{-1} = i_a^d \Phi_1' \) which proves that \( h \) satisfies Theorem 2.6-2(ii).

By Lemma 2.18, for each \( \Lambda \in \mathcal{L}(\phi) \) there exists \( \Phi \in P(\phi) \) and an isolated point \( P \in \text{Fix}_N(\hat{\Phi}) \) whose accumulation set equals \( \Lambda \). By \((\Phi - 2)\), we may assume that \( P \) is \( \hat{\Psi}_1 \)-invariant and hence that \( \Lambda \) is \( \psi \)-invariant. It follows that \( \Lambda \) is \( \phi' \)-invariant and that \( \text{PF}_A(\phi') = \text{PF}_A(\phi) \). Theorem 2.6 implies that \( \phi = \phi' \) and since \( \text{Fix}_N(\Phi') = \text{Fix}_N(\Phi) \), \( \Phi = \Phi' \), which proves that \( \Phi \) commutes with \( \Phi \).

We denote the center of a group \( H \) by \( Z(H) \) and define the weak center \( WZ(H) \) to be the subgroup of \( H \) consisting of elements that commute with some iterate of each element of \( H \).

**Corollary 5.5.** If \( \phi \in \text{Out}(F_n) \) is rotationless then \( A(\phi) \) is an abelian subgroup of \( C(\phi) \). Moreover, each element of \( A(\phi) \) commutes with each rotationless element of \( C(\phi) \) and so \( A(\phi) \subset WZ(C(\phi)) \).

**Proof.** Lemma 5.4 implies that \( \theta \in C(\phi) \) for each \( \theta \) in the defining generating set of \( A(\phi) \) and that \( C(\phi) \) and \( C(\theta) \) contains the same rotationless elements. The corollary follows.

**Remark 5.6.** In general, \( A(\phi) \) is not contained in the center of \( C(\phi) \). For example, if \( n = 2k \) and \( \Phi \in P^\pm(\phi) \) commutes with an order two automorphism \( \Theta \) that interchanges the free factor generated by the first \( k \) elements in a basis with the free factor.
generated by the last \( k \) elements of that basis, then \( \mathcal{A}(\phi) \) will contain elements that do not commute with \( \theta \).

It is natural to ask if \( \phi \) is generic in \( \mathcal{A}(\phi) \).

**Lemma 5.7.** If \( \phi \) is rotationless then \( \phi \) is generic in \( \mathcal{A}(\phi) \).

**Proof.** We must show that if \( \omega \) is a coordinate of \( \Omega : \mathcal{A}(\phi) \to \mathbb{Z}^N \) then \( \omega(\phi) \neq 0 \). Choose an element \( \theta \) of the defining generating set for \( \mathcal{A}(\phi) \) such that \( \omega(\theta) \neq 0 \). If \( \omega = PF_{\Lambda} \) then, after replacing \( \theta \) with \( \theta^{-1} \) if necessary, \( \Lambda \in \mathcal{L}(\theta) \). By Lemmas 2.15 and 2.18, there exist \( \Theta \in \mathcal{P}(\theta) \) and an isolated point \( P \in \text{Fix}_N(\Theta) \) whose accumulation set is \( \Lambda \). After replacing \( \phi \) with \( \phi^{-1} \) if necessary, there exists \( \Phi \in \mathcal{P}(\phi) \) such that \( \text{Fix}(\Phi) = \text{Fix}(\Theta) \) and such that \( P \) is an isolated point in \( \text{Fix}_N(\Psi) \). Lemma 2.19 implies that \( \omega(\phi) \neq 0 \).

If \( \omega \) is a comparison homomorphism determined by lifts \( s, t : \mathcal{A}(\phi) \to \text{Aut}(F_n) \) then \( s(\theta) \neq t(\theta) \). Thus \( \text{Fix}(\widehat{s}(\phi)) = \text{Fix}(\widehat{s}(\theta)) \neq \text{Fix}(\widehat{t}(\theta)) = \text{Fix}(\widehat{t}(\phi)) \) which implies that \( \omega(\phi) \neq 0 \). \( \square \)

# 6 Disintegrating \( \phi \)

We have reduced the study of rotationless abelian subgroups of \( \text{Out}(F_n) \), and so of abelian subgroups of \( \text{Out}(F_n) \) up to finite index, to the study of \( \mathcal{A}(\phi) \) for rotationless \( \phi \in \text{Out}(F_n) \). In this section we construct the subgroup \( \mathcal{D}(\phi) \) of \( \mathcal{A}(\phi) \) described in the introduction. In section 7 we show that \( \mathcal{D}(\phi) \) has finite index in \( \mathcal{A}(\phi) \).

Choose \( f : G \to G \) representing \( \phi \) as in Theorem 2.17. We will need a coarsening of the complete splitting of a path. For each axis \( \mu \) of \( \phi \) there exists a primitive closed Nielsen path \( w \) and edges \( \{E_i\} \) as in Lemma 2.20 such that \( f(E_i) = E_i \cdot w^{d_i} \); we say that these edges are associated to \( \mu \) and that \( d_i \) is the exponent of \( E_i \). For distinct \( E_i \) and \( E_j \) associated to \( \mu \), paths of the form \( E_i w^s \tilde{E}_j \) are said to belong to the same quasi-exceptional family or to be quasi-exceptional. By assumption \( d_i \neq d_j \). If \( d_i \) and \( d_j \) have the same sign then \( E_i w^s \tilde{E}_j \) is an exceptional path but otherwise it is not.

Assume that \( \sigma = \sigma_1 \ldots \sigma_s \) is the unique complete splitting of \( \sigma \). If \( \sigma_{ab} := \sigma_a \ldots \sigma_b \) is quasi-exceptional then we say that \( \sigma_{ab} \) is a QE-subpath of \( \sigma \).

**Lemma 6.1.** For any completely split path \( \sigma \), distinct QE-subpaths of \( \sigma \) have disjoint interiors.

**Proof.** Suppose that \( \sigma = \sigma_1 \ldots \sigma_s \) is the complete splitting of \( \sigma \) and that there exist \( 1 \leq a < b \leq s \) and \( 1 \leq a \leq c < d \leq s \) such that \( \sigma_{ab} := \sigma_a \ldots \sigma_b \) and \( \sigma_{cd} := \sigma_c \ldots \sigma_d \) are distinct quasi-exceptional paths. We must show that \( c > b \).

Since \( \sigma_{ab} \) is quasi-exceptional, \( \sigma_a \) and \( \sigma_b \) are linear edges and \( \sigma_{a+1} \ldots \sigma_{b-1} \) is a Nielsen path. Each \( \sigma_l, a < l < b \) must be a Nielsen path, which implies that \( c \geq b \). Since \( \tilde{E}_j \) is the not an initial segment of any quasi-exceptional path, \( c > b \). \( \square \)
Definition 6.2. The QE-splitting of a completely split path $\sigma$ is the coarsening of the complete splitting of $\sigma$ obtained by declaring each QE-subpath to be a single element. Thus the QE-splitting is a splitting into single edges, connecting subpaths Nielsen paths and quasi-exceptional paths. These subpaths are the terms of the QE-splitting.

Definition 6.3. For a stratum $H_\alpha$ whose edges are not fixed by $f$, we let $A_\alpha \subset H_\alpha$ denote an edge if the stratum $H_\alpha$ is irreducible and a connecting path if $H_\alpha$ is a zero stratum. The rule

$$H_i \sim H_j \text{ if there exist } A_i \subset H_i \text{ and } A_j \subset H_j \text{ such that } A_j \text{ occurs as a term in the QE-splitting of } f(A_i).$$

generates an equivalence relation on those strata on which $f$ is not the identity. The equivalence classes $X_1, \ldots, X_M$ are called almost invariant subgraphs.

For each $M$-tuple $a$ of non-negative integers, define $f_a : G \to G$ by

$$f_a(E) = \begin{cases} f_a^\#(E) & \text{for each edge } E \subset X_i \\ E & \text{for each edge } E \text{ that is fixed by } f \end{cases}$$

Remark 6.4. If $H_i$ is a zero stratum and $H_j$ is the first irreducible stratum above $H_i$ then $H_i$ and $H_j$ belong to the same almost invariant subgraph. This follows from the fact that every edge in $H_i$ is contained in a connecting path in $H_j$ that is in the image of either an edge in $H_j$ or a connecting path in some zero stratum between $H_i$ and $H_j$.

Lemma 6.5. $f_a : G \to G$ is a homotopy equivalence for all $a$.

Proof. Let $NI$ be the number of irreducible strata in the filtration and for each $0 \leq m \leq NI$, let $G_i(m)$ be the smallest filtration element containing the first $m$ irreducible strata. We will prove by induction that each $f_a|G_i(m)$ is a homotopy equivalence.

Since $H_1$ is never a zero stratum, $i(1) = 1$. If $G_1$ is not a single edge fixed by $f$, then every edge in $G_1$ is contained in a single almost invariant subgraph $X_i$. Thus $f_a|G_1$ is either the identity or is homotopic to $f^a|G_1$; in either case it is a homotopy equivalence.

We assume now that $f_a|G_i(m)$ is a homotopy equivalence. Define $g_1 : G_i(m+1) \to G_i(m+1)$ on edges by

$$g_1(E) = \begin{cases} f_a(E) & \text{if } E \subset G_i(m) \\ E & \text{if } E \subset G_i(m+1) \setminus G_i(m) \end{cases}$$

Every vertex in $G_i(m)$ whose link is not entirely contained in $G_i(m)$ is an attaching vertex (see Theorem 2.18(V)) and so is fixed by $f$. This guarantees that $g_1$ is well defined. It is easy to check that $g_1$ is a homotopy equivalence. If the edges of $H_i(m+1)$ are fixed by $f$, then $g_1 = f_a|G_i(m+1)$ and we are done.
If $f \mid H_{i(m+1)}$ is not the identity, then Remark 6.4 implies that the edges in $G_{i(m+1)} \setminus G_{i(m)}$ are contained in a single almost invariant subgraph, say $X_k$. Define $g_2 : G_{i(m+1)} \rightarrow G_{i(m+1)}$ on edges by

$$g_2(E) = \begin{cases} f^a_k(E) & \text{if } E \subset G_{i(m)} \\ E & \text{if } E \subset G_{i(m+1)} \setminus G_{i(m)} \end{cases}$$

and $g_3 : G_{i(m+1)} \rightarrow G_{i(m+1)}$ on edges by

$$g_3(E) = \begin{cases} E & \text{if } E \subset G_{i(m)} \\ f^a_k(E) & \text{if } E \subset G_{i(m+1)} \setminus G_{i(m)} \end{cases}$$

Then $g_2$ is a homotopy equivalence and $f^a_k \mid G_{i(m+1)} = g_3g_2$. Each component of $G_{i(m+1)}$ is non-contractible by item (Z) of Theorem 2.17, so $f^a_k \mid G_{i(m+1)}$ is a homotopy equivalence. It follows that $g_3$, and hence also $f_a \mid G_{i(m+1)} = g_3g_1$ is a homotopy equivalence.

Almost invariant subgraphs are defined without reference to the quasi-exceptional paths in the QE-splitting of edge images. The next definition brings these into the discussion.

**Definition 6.6.** If $\{X_1, \ldots, X_M\}$ are the almost invariant subgraphs of $f : G \rightarrow G$ then an $M$-tuple $a = (a_1, \ldots, a_M)$ of non-negative integers is admissible if for all axes $\mu$, whenever:

- $X_s$ contains an edge $E_i$ associated to $\mu$ with exponent $d_i$
- $X_t$ contains an edge $E_j$ associated to $\mu$ with exponent $d_j$
- $X_r$ contains an edge $E_k$ such that some term of the QE-splitting of $f(E_k)$ is in the same quasi-exceptional family as $E_r E_j$

then $a_r(d_i - d_j) = a_s d_i - a_t d_j$.

**Example 6.7.** Suppose that $G$ is the rose with edges $E_1, E_2, E_3$ and $E_4$ and that $f : G \rightarrow G$ is defined by $E_1 \mapsto E_1$, $E_2 \mapsto E_2 E_1^2$, $E_3 \mapsto E_3 E_1$ and $E_4 \mapsto E_4 E_3 E_3 E_2$. Then $M = 2$ with $X_1$ having the single edge $E_2$ and $X_2$ consisting of $E_3$ and $E_4$. The pair $(a_1, a_2)$ is admissible if $a_2 = 2a_2 - a_1$ or equivalently $a_2 = a_1$. Thus $f_a = f^{a_1}$ for each admissible $a$.

**Definition 6.8.** Each $f_a$ determines an element $\phi_a \in \text{Out}(F_n)$ and also an element $[f_a]$ in the semigroup of homotopy equivalences of $G$ that respect the filtration modulo homotopy relative to the set of vertices of $G$. Define $D(\phi) = \langle \phi_a : a \text{ is admissible} \rangle$. Both $\phi_a$ and $D(\phi)$ depend on the choice of $f : G \rightarrow G$; see Example 6.9 below. Since we work with a single $f : G \rightarrow G$ throughout the paper and since $D(\phi)$ is well defined up to finite index by Theorem 7.1, we suppress this dependence in the notation.

27
Example 6.9. Let $G$ be the rose with edges $E_1, E_2$ and $E_3$. Define $f_1 : G \to G$ by

$$E_1 \mapsto E_1 \quad E_2 \mapsto E_1 E_2 \quad E_3 \mapsto E_1^2 E_3 E_1$$

and $f_2 : G \to G$ by

$$E_1 \mapsto E_1 \quad E_2 \mapsto E_2 E_1 \quad E_3 \mapsto E_1 E_3 E_1^2.$$ 

These maps differ by $i_{E_1}$ and so determine the same element $\phi \in \text{Out}(F_n)$. The homotopy equivalence of $G$ that fixes $E_1$ and $E_3$ and maps $E_2$ to $E_2 E_1$ represents an element of $D(\phi)$ if $f_2$ is used but not if $f_1$ is used.

Lemma 6.10. Suppose that $a$ is admissible and that $\sigma$ is a path in $G$.

1. If $\sigma$ is Nielsen path for $f$ then $\sigma$ is a Nielsen path for $f_a$.

2. If $\sigma$ is quasi-exceptional and if some path in the same quasi-exceptional family as $\sigma$ occurs as a term in the QE-splitting of $f(E)$ for some edge $E$ in $X_k$ then $(f_a)_\#(\sigma) = f_a^\#(\sigma)$.

Proof. The proof is by induction on the height $r$ of $\sigma$. In the context of (1), we may assume that $\sigma$ is either indivisible or a single fixed edge. $G_1$ is either a single fixed edge or is contained in a single almost invariant subgraph. Thus $f_a|G_1$ is either the identity or an iterate of $f|G_1$. In either case (1) is obvious for $\sigma \subset G_1$. Since $G_1$ does not contain any quasi-exceptional paths, the lemma holds for $\sigma \subset G_1$. We assume now that $r \geq 2$, that the lemma holds for paths in $G_{r-1}$ and that $\sigma$ has height $r$ and is either an indivisible Nielsen path or a quasi-exceptional path. Property (N) of Theorem 2.17 implies that $H_r$ is either EG or linear.

Let $X_\alpha$ be the almost invariant subgraph containing $H_r$. If $H_r$ is linear then it is a single edge $E_r$ and $f(E_r) = E_r w^{a_{\alpha}}$ for some non-trivial Nielsen path $w$. If $\sigma$ is an indivisible Nielsen path, then $\sigma = E_r w^p \hat{E}_r$ for some integer $p$. By the inductive hypothesis, $(f_a)_\#(w) = w$ so

$$(f_a)_\#(\sigma) = [(E_r w^{a_{\alpha} d_r}) w^p (\hat{w}_{\alpha} d_r \hat{E}_r)] = E_r w^p \hat{E}_r = \sigma.$$ 

If $\sigma$ is as in (2), then up to a reversal of orientation, $\sigma = E_r w^p \hat{E}_j$ for some edge $E_j$ associated to the same axis as $E_r$. Let $X_i$ be the almost invariant subgraph containing $E_j$. Since $a$ is admissible, $a_k (d_r - d_j) = a_s d_r - a_t d_j$.

Thus
\[(f_a)_\#(\sigma) = [f^a_\#(E_r)(f_a(w))^p f^{a_1}_\#(\bar{E}_j)]\]
\[= [E_r w^{a_d} w^p w^{a_d} E_j]\]
\[= [E_r w^{a_d - a d_j} + p \bar{E}_j]\]
\[= [E_r w^{a_k(d_j - d_j)} + p E_j]\]
\[= [E_r w^{a_d} w^p w^{a_d} E_j]\]
\[= [f^{a_k}_\#(E_r)(f^{a_k}_\#(w))^p f^{a_k}_\#(\bar{E}_j)]\]
\[= f^{a_k}_\#(\sigma).\]

Suppose now that \(H_r\) is EG. There are no quasi-exceptional paths of height \(r\) so \(\sigma\) is an indivisible Nielsen path of height \(r\). By item (2) of Lemma 5.11 of \([BH92]\), \(\sigma = \alpha \beta\) where \(\alpha\) and \(\beta\) are \(r\)-legal paths that begin and end with edges in \(H_r\). It suffices to show that \((f_a)_\#(\alpha) = f^a_\#(\alpha)\) and \((f_a)_\#(\beta) = f^a_\#(\beta)\). The argument is the same for both \(\alpha\) and \(\beta\). If \(E_\alpha\) is the initial edge of \(\alpha\), then there exists \(m \geq 0\) such that \(\alpha \subset f^m_\#(E_\alpha)\). The terms in the quasi-exceptional splitting of \(f^m_\#(E_\alpha)\) are edges and connecting paths in \(X_s\), Nielsen paths in \(G_{r-1}\) or quasi-exceptional paths in \(G_{r-1}\). Since \(\alpha\) begins and ends with an edge in \(H_r\), the quasi-exceptional splitting of \(f^m_\#(E_\alpha)\) restricts to a quasi-exceptional splitting of \(\alpha\). By definition and by the inductive hypothesis, \((f_a)_\#\) equals \(f^a_\#\) on all four types of subpath. Thus \((f_a)_\#\) equals \(f^a_\#\) on \(\alpha\) as desired. \(\square\)

**Corollary 6.11.** For \(1 \leq s \leq M\), let \(P_s\) be the set of completely split paths whose quasi-exceptional splittings are composed of: (i) edges and connecting paths in \(X_s\); (ii) indivisible Nielsen paths; and (iii) quasi-exceptional paths in the same quasi-exceptional family as a term in the QE-splitting of \(f(E)\) for some edge \(E\) in \(X_s\). Then \(P_s\) is preserved by both \(f_\#\) and \((f_a)_\#\) and moreover \((f_a)_\#(\sigma) = f^a_\#(\sigma)\) for all \(\sigma \in P_s\).

**Proof.** This is an immediate consequence of Lemma 6.10, the definition of \(X_s\) and the definition of \(f_a\). \(\square\)

**Corollary 6.12.** For each admissible \(a\) and \(b\), \([f_a][f_b] = [f_b][f_a] = [f_{a+b}]\). In particular, \(D(\phi)\) is abelian.

**Proof.** It suffices to check that \((f_{a+b})_\#(E) = (f_a)_\#(f_b)_\#(E)\) for each edge \(E\). If \(E\) is fixed by \(f\), then \(E\) is also fixed by \(f_a, f_b\) and \(f_{a+b}\). Suppose that \(E \subset X_k\). Then \((f_{a+b})_\#(E) = f^{(a+b)}_\#(E) = f^{a_k+b_k}_\#(E) = f^{a_k}_\#(f^{b_k}_\#(E)) = (f_a)_\#(f^{b_k}_\#(E)) = (f_a)_\#(f_b)_\#(E)\) where the next to the last equality comes from Corollary 6.11. \(\square\)
Definition 6.13. An admissible \( a \) is *generic* if each \( a_i > 0 \) and if whenever \( E_i \in X_r \) and \( E_j \in X_s \) are distinct linear edges associated to the same axis, then \( a_d d_i \neq a_s d_j \) where \( d_i \) and \( d_j \) are the exponents of \( E_i \) and \( E_j \) respectively.

Lemma 6.14. If \( a \) is generic then \( f_a : G \to G \) satisfies the conclusions of Theorem 2.17 and \( f_a \) has the same principal vertices and Nielsen paths as \( f \).

*Proof.* Corollary 6.11 implies that \( f_a \) is a completely split relative train track map for \( \phi_a \) with respect to the filtration \( \emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G \), that \( f_a \) has the same principal vertices as \( f \) and that \( f_a \) satisfies all of the items listed in the statement of Theorem 2.17 except perhaps for (L) and (N). Property (L) follows from the genericity assumption and the observation that if \( E_j \in X_s \) is a linear edge for \( f \) with exponent \( d_j \) then \( E_j \) is a linear edge for \( f_a \) with exponent \( a_s d_j \).

We show below that if \( \rho \) is an indivisible Nielsen path for \( f_a \) then it is a Nielsen path for \( f \). Combined with Lemma 6.10(1), this proves that \( f \) and \( f_a \) have the same Nielsen paths and hence that (N) is satisfied. Corollary 6.11 then implies that \( f_a \) is forward rotationless and completes the proof. (Nielsen paths are relevant to this because it is part of the definition of forward rotationless that the endpoints of all indivisible periodic Nielsen paths be vertices.)

Suppose then that \( \rho \) is an indivisible Nielsen path for \( f_a \). Let \( i \) be the height of \( \rho \) and let \( X_i \) be the almost invariant subgraph that contains \( H_i \). If \( H_i \) is EG then by Lemma 5.11 of [BH92] \( \rho = \alpha \beta \) where \( \alpha \) and \( \beta \) are \( i \)-legal paths for \( f_a \) that begin and end in \( H_i \). Let \( E_\alpha \) be the edge whose interior contains an initial segment of \( \alpha \). If the initial endpoint \( x \) of \( \alpha \) is a vertex let \( \alpha' = \alpha \); otherwise \( \alpha' \) is the extension of \( \alpha \) that contains all of \( E_\alpha \). Choose \( k \geq 1 \) so that \( (f_a^k)_\#(E_\alpha) \) contains \( \alpha \). Since both \( E_\alpha \) and the terminal edge of \( \alpha \) are edges of height \( i \) (see Theorem 2.18(N)), the quasi-exceptional splitting of \( (f_a^k)_\#(E_\alpha) \) restricts to a quasi-exceptional splitting of \( \alpha' \).

Corollary 6.11 implies that \( (f_a^k)_\#(\alpha') = f_a^a(\alpha') \) and since \( \alpha \) and \( \alpha' \) are \( i \)-legal it follows that \( (f_a^k)_\#(\alpha) = (f_a^k)_\#(\alpha) \). The analogous argument applies to \( \beta \) and we conclude that \( \rho \) is an indivisible periodic Nielsen path for \( f \), and so by (N), an indivisible Nielsen path for \( f \).

Suppose next that \( H_i \) is a single NEG edge \( E_i \). Choose lifts \( \tilde{\rho} \) and \( \tilde{f_a} \) such that \( \tilde{f_a} \) fixes the endpoints of \( \tilde{\rho} \). Let \( \tilde{f} \) be the lift of \( f \) that fixes the initial endpoint and direction determined by \( \tilde{\rho} \). By Lemma 2.18 there is a ray \( \tilde{R}_1 \) with the same initial vertex and direction as \( \tilde{\rho} \) and satisfying the following properties.

- \( \text{Fix}(\tilde{f}) \cap \tilde{R}_1 \) is the initial endpoint of \( \tilde{R}_1 \).
- If \( \tilde{R}_1 = \tilde{\tau}_1 \cdot \tilde{\tau}_2 \cdot \ldots \) is the quasi-exceptional splitting of \( \tilde{R}_1 \) and if \( \tilde{x}_l \) is the terminal endpoint of \( \tilde{\tau}_l \) then \( f(\tilde{x}_l) = \tilde{x}_k \) for some \( k > l \) and \( Df \) maps the turn taken by \( \tilde{R}_1 \) at \( \tilde{x}_l \) to the turn taken by \( \tilde{R}_1 \) at \( \tilde{x}_k \).
- \( \tilde{R}_1 \) converges to some \( P_1 \in \text{Fix}_N(\tilde{f}) \).
We claim that these three items also hold with \( f \) replaced by \( f_b \). The second and third items follow from Corollary 6.11. If the first item fails then a fixed point in the interior of \( \hat{R}_1 \) must be in the interior of some \( \hat{\tau}_l \) that is not a connecting subpath and so there exists an initial subpath \( \hat{\mu} \) of \( \hat{\tau}_l \) such that \((f_a)^{\#}(\hat{\mu})\) is trivial. But no such \( \hat{\mu} \) can exist. This follows from Corollary 6.11 if \( \tau_l \) is a single edge and is easy to check by inspection if \( \tau_l \) is an exceptional path or a Nielsen path. This completes the proof of the claim. We now know that \( \hat{R}_1 \) does not contain the terminal endpoint of \( \hat{\rho} \).

Define \( \hat{R}_2 \) and \( P_2 \) similarly using the initial vertex and direction of \( \hat{\rho}^{-1} \). If \( P_1 \neq P_2 \) let \( \hat{L}_{12} \) be the line connecting \( P_1 \) to \( P_2 \). Then \( \hat{L}_{12} \) is contained in \( \hat{R}_1 \cup \hat{\rho} \cup \hat{R}_2 \) and does not contain the endpoints of \( \hat{\rho} \), which are also the endpoints of \( \hat{R}_1 \) and \( \hat{R}_2 \). It follows that \( \hat{L}_{12} \cap \text{Fix}(f_a) = \emptyset \) which contradicts (Lemma 3.15 of [FHb]) the fact that the two endpoints of \( L_{12} \) are attracting. We conclude that \( P_1 = P_2 \).

If \( P_1 \neq T_b^{\pm} \) for some \( b \in F_n \), then there is a unique \( \hat{f} \) that fixes \( P_1 \). In this case, the lifts of \( f \) that fix the initial and terminal endpoints of \( \hat{\rho} \) are equal and \( \rho \) is a Nielsen path for \( f \). We may therefore assume that \( P_1 = T_b^{\pm} \) in which case \( E_i \) and \( E_j \) are linear edges associated to the same axis for \( f \) and \( \rho \) is exceptional for \( f \). Property (L) for \( f_a \) implies that \( E_i = E_j \) and hence that \( \rho \) is a Nielsen path for \( f \).

We now relate \( D(\phi) \) to \( A(\phi) \), using the correspondence between principal lifts of relative train track maps and principal automorphisms.

**Corollary 6.15.** For each generic \( a \) there is a bijection \( h : P(\phi) \to P(\phi_a) \) such that \( \text{Fix}_N(h(\hat{\Phi})) = \text{Fix}_N(\hat{\Phi}) \) for all \( \hat{\Phi} \in P(\phi) \). If \( \hat{f} \) corresponds to \( \Phi \) and \( \hat{f}_a \) corresponds to \( h(\hat{\Phi}) \) then \( \text{Fix}(\hat{f}) = \text{Fix}(\hat{f}_a) \).

**Proof.** By Lemma 6.14, \( f \) and \( f_a \) have the same Nielsen classes of principal vertices. There is an induced bijection \( h \) between principal lifts of \( f_a \) and principal lifts of \( f \); if \( \hat{f}_a = h(\hat{f}) \) then \( \text{Fix}(\hat{f}) = \text{Fix}(\hat{f}_a) \). Lemma 2.2 implies that \( \text{Fix}_N(\hat{f}) \) and \( \text{Fix}_N(\hat{f}_a) \) have the same non-isolated points. Lemma 2.18 and Corollary 6.11 imply that \( \text{Fix}_N(\hat{f}) \) and \( \text{Fix}_N(\hat{f}_a) \) have the same isolated points.

Let \( D_R(\phi) \) be the finite index rotationless subgroup of \( D(\phi) \) given by Corollary 3.12.

**Corollary 6.16.** \( D_R(\phi) \) is contained in \( A(\phi) \) and is generated by elements of the form \( \phi_a \) with \( a \) generic.

**Proof.** Corollary 6.15 implies that if \( \phi_a \in D_R(\phi) \) is generic then \( \phi_a \in A(\phi) \). It therefore suffices to find a generating set \( S \) for \( D_R(\phi) \) in which each element has this form. Let \( S' = \{ \phi_a \} \) be any generating set for \( D_R(\phi) \). If \( I \) is the \( M \)-tuple with 1’s in each coordinate then \( \phi_I = \phi \) is generic and represented by \( f_I = f \). Corollary 6.12 implies that if \( k \) is sufficiently large then \( \phi^k \phi_a \) is represented by \( f_b \) where \( b = a + kI \) is projectively close to \( I \) and so is generic. Thus \( S = \{ \phi, \phi^k \phi_a : \phi_a \in S' \} \) is the desired generating set for \( D_R(\phi) \).
The definition of $D(\phi)$ is not symmetric in $\phi$ and $\phi^{-1}$ leaving open the following questions.

**Question 6.17.** Is each element of $D(\phi)$ rotationless? Is $D(\phi) = D(\phi^{-1})$?

**Theorem 6.18.** $D_R(\phi) \subset WZ(C(\phi))$ for all rotationless $\phi$.

*Proof.* $D_R(\phi) \subset A(\phi) \subset WZ(C(\phi))$ by Corollary 6.16 and Corollary 5.5.

### 7 Finite Index

Our goal in this section is to prove

**Theorem 7.1.** $D_R(\phi)$ has finite index in $A(\phi)$ for all rotationless $\phi$.

Before turning to the proof of Theorem 7.1 we use it to prove one of our main results.

**Theorem 7.2.** For every abelian subgroup $A$ of $\text{Out}(F_n)$ there exists $\phi \in A$ such that $A \cap D(\phi)$ has finite index in $A$.

*Proof.* Corollary 3.12 and Lemma 5.3 imply that $A \cap A(\phi)$ has finite index in $A$ for each generic $\phi \in A_R$. Theorem 7.1 therefore completes the proof.

Choose once and for all $f : G \to G$ representing $\phi$ as in Theorem 2.17.

We set notation for the linear edges associated to an axis $c_u$ of $\phi$ as follows. If $[c]_u$ has multiplicity $m + 1$ then there is a primitive closed path $w$ whose circuit represents $c$ and for $1 \leq j \leq m$, there are linear edges $E_j$ and distinct non-zero integers $d_j$ such that $f(E_j) = E_j \cdot w^{d_j}$. Choose a lift $\tilde{E}_j$ whose terminal endpoint is in the axis $A_c \subset \Gamma$. Following Lemma 2.20, the principal lift of $f$ that fixes the initial endpoint of $\tilde{E}_j$ is denoted $\tilde{f}_j$ and the associated principal automorphism is denoted $\Phi_j$; both $\tilde{f}_j$ and $\Phi_j$ are independent of the choice of $\tilde{E}_j$. By Lemma 4.9 and Lemma 5.7, $\text{Fix}(\tilde{\Phi}_j)$ is a maximal principal set $X_j$. The lift $s_j$ of $A(\phi)$ to $\text{Aut}(F_n)$ determined by $X_j$ satisfies $s_j(\phi) = \Phi_j$. The principal lift of $f$ that fixes the terminal endpoint of $\tilde{E}_j$ is denoted $f_0$, its associated principal automorphism is denoted $\Phi_0$, the maximal principal set $\text{Fix}(\tilde{\Phi}_0)$ is denoted $X_0$ and the lift to $\text{Aut}(F_n)$ determined by $X_0$ is denoted $s_0$. The automorphisms $\Phi_0, \ldots, \Phi_m$ are the only elements of $P(\phi)$ that commute with $T_c$ (Lemma 2.20).

For $1 \leq j \neq k \leq m$, let $\omega_{c,j}$ be the comparison homomorphism determined by $X_0$ and $X_j$ and let $\omega_{c,j,k}$ be the comparison homomorphism determined by $X_j$ and $X_k$. Thus $\omega_{c,j,k} = \omega_{c,j} - \omega_{c,k}$. There is an obvious bijection between the $\omega_{c,j}$'s and the linear edges $E_j$ associated to $c$. There is also a bijection between the $\omega_{c,j,k}$'s and the families of quasi-exceptional paths $E_j w^* E_k$ associated to to $c$. We make use these bijections without further notice.

For each $\Lambda \in \mathcal{L}(\phi)$ let $\omega_{\Lambda} = \text{PF}_\Lambda | A(\phi)$. We also identify $\Lambda$ with $\omega_{\Lambda}$ when convenient.
We define a new homomorphism $\Omega^\phi : \mathcal{A}(\phi) \to \mathbb{Z}^K$ whose coordinates are in one to one correspondence with the the linear and EG strata of $f : G \to G$ by removing extraneous coordinates from $\Omega : \mathcal{A}(\phi) \to \mathbb{Z}^N$.

**Definition 7.3.** $\Omega^\phi : \mathcal{A}(\phi) \to \mathbb{Z}^K$ is the product of the $\omega_{c,j}$'s and the $\omega_{\Lambda}$'s as $[c]_u$ varies over the axes of $\phi$ and as $\Lambda$ varies over $\mathcal{L}(\phi)$.

**Lemma 7.4.** $\Omega^\phi : \mathcal{A}(\phi) \to \mathbb{Z}^K$ is injective.

*Proof.* The coordinates of $\Omega^\phi$ are coordinates of the injective homomorphism $\Omega$. It therefore suffices to assume that $\omega(\psi) \neq 0$ for a coordinate $\omega$ of $\Omega$ and prove that the image of $\psi$ under some coordinate of $\Omega^\phi$ is non-zero. There is no loss in assuming that $\omega$ is not a coordinate of $\Omega^\phi$ and is therefore either some $\omega_{c,j,k}$ or $\omega_\Lambda$ for some $\Lambda \in \mathcal{L}(\phi^{-1})$. In the former case, $\omega_{c,j}(\psi) \neq 0$ or $\omega_{c,k}(\psi) \neq 0$ and we are done. In the latter case, Lemma 5.7 implies that $\Lambda \in \mathcal{L}(\psi) \cup \mathcal{L}(\psi^{-1})$. By Lemma 3.2.4 of [BFH00] there is a unique $\Lambda' \neq \Lambda \in \mathcal{L}(\phi) \cup \mathcal{L}(\phi^{-1})$ such that $\Lambda$ and $\Lambda'$ are carried by the same minimal rank free factor; moreover, $\Lambda' \in \mathcal{L}(\phi)$. Similarly, there is a unique $\Lambda'' \neq \Lambda \in \mathcal{L}(\psi) \cup \mathcal{L}(\psi^{-1})$ such that $\Lambda$ and $\Lambda''$ are carried by the same minimal rank free factor. Since $\phi$ is generic, $\omega_{\Lambda''}(\phi) \neq 0$ which implies that $\Lambda'' \in \mathcal{L}(\phi) \cup \mathcal{L}(\phi^{-1})$ and hence that $\Lambda' = \Lambda''$. Thus $\omega_{\Lambda''}$ is a coordinate of $\Omega^\phi$ and $\omega_{\Lambda''}(\psi) \neq 0$. \hfill $\square$

**Lemma 7.5.** If a coordinate $\omega$ of $\Omega^\phi$ corresponds to a stratum in the almost invariant subgraph $X_s$ then $\omega(\phi_a) = a_s \omega(\phi)$ for all $\phi_a \in \mathcal{A}(\phi)$.

*Proof.* We may assume by Corollary 6.16 that $a$ is generic. If $\omega = \omega_\Lambda$ then the lemma follows from Corollary 6.11 and the definition of the expansion factor homomorphism. Suppose then that $\omega = \omega_{c,j}$. Lemma 6.15 implies that $s_j(\phi_a)$ corresponds to the principal lift of $f_a$ that fixes the initial endpoint of $\tilde{E}_j$ and $s_0(\phi_a)$ corresponds to the principal lift of $f_a$ that fixes the terminal endpoint of $\tilde{E}_j$. Since $f_a(\tilde{E}_j) = E_j \cdot w^{a_s d_j}$ we have $\omega_{c,j}(\phi_a) = a_s d_j$. \hfill $\square$

**Corollary 7.6.** The rank of $D_R(\phi)$ is equal to the rank of the sublattice $L$ of $\mathbb{Z}^M$ generated by the admissible $M$-tuples for $f : G \to G$.

*Proof.* Let $\rho : L \to \mathcal{A}(\phi)$ be the homomorphism determined by $a \to \phi_a$. It suffices to show that $\rho$ is injective and for this it suffices to show that $\Omega^\phi \rho$ is injective. The lattice $L$ contains the $M$-tuple $I$, all of whose coordinates are 1. Given $x, y \in L$ there exists $k \geq 0$ so that $x + kI$ and $y + kI$ are admissible. Lemma 7.4 and Lemma 7.5 imply that $\Omega^\phi \rho(x + kI) \neq \Omega^\phi \rho(y + kI)$ and hence that $\Omega^\phi \rho(x) \neq \Omega^\phi \rho(y)$. \hfill $\square$

We now come to our main technical proposition, a generalization of Lemma 2.19. Before proving it we show that it implies Theorem 7.1. (The process of iterating an edge is discussed in section 2.)
Proposition 7.7. Suppose that \( E \) is a nonlinear edge of \( G \), that \( k > 0 \) and that \( \mu \) is a term in the quasi-exceptional splitting of \( f^k(E) \) that is either a linear edge, an EG edge or a quasi-exceptional subpath. Suppose further that \( f \) is a lift of \( \phi \) and let \( \tilde{E} \) be the homomorphism associated to \( \mu \) and let \( s \) be the lift of \( \mathcal{A}(\phi) \) to \( \text{Aut}(F_n) \) determined by the maximal principal set \( \text{Fix}(\hat{f}) \). Then the following are equivalent for all \( \psi \in \mathcal{A}(\phi) \).

1. \( P \) is isolated in \( \text{Fix}(\hat{s}(\psi)) \)
2. \( \omega(\psi) \neq 0. \)

Proof of Theorem 7.1 Each linear or EG stratum \( H_i \) determines a coordinate of \( \Omega^{\phi} \) that we denote \( \omega_i \). For all \( \psi \in \mathcal{A}(\phi) \), define \( a_i(\psi) = \omega_i(\psi)/\omega_i(\phi) \).

We first observe that there is a virtual basis \( \{\psi_i\} \) for \( \mathcal{A}(\phi) \), meaning that it is a basis of a finite index subgroup of \( \mathcal{A}(\phi) \), such that \( a_i(\psi_i) \) is a positive integer for all \( i \) and \( l \). To construct \( \{\psi_i\} \), start with any basis \( \{\eta_i\} \) of \( \mathcal{A}(\phi) \). Choose \( m \geq 1 \) so that each \( a_i(\eta_i^m) \) is an integer. Then \( \{\eta_i^m\} \) is a virtual basis and for all but at most one value of \( s \), the set obtained from \( \{\eta_i^m\} \) by replacing \( \eta_i^m \) with \( \psi_1 = \phi^s \eta_i^m \) is also a virtual basis for \( \mathcal{A}(\phi) \). If \( s \) is sufficiently large then \( a_i(\psi_1) \) is a positive integer. Repeat this, focusing on the second basis element and so on to arrive at the desired virtual basis.

In what follows we restrict to a single \( \psi_1 \) so we refer to \( \psi_1 \) simply as \( \psi \) and to \( a_i(\psi_1) \) simply as \( a_i \).

We show next that if \( H_i \) and \( H_j \) are linear or EG strata that belong to the same almost invariant subgraph then \( a_i = a_j \). Define \( \theta = \psi\phi^{-a_i} \). Then \( \omega_j(\theta) = \omega_j(\psi) - a_i \omega_j(\phi) \) and \( \omega_1(\theta) = \omega_1(\psi) - a_i \omega_1(\phi) = 0 \) so it suffices to show that \( \omega_j(\theta) = 0 \).

As a first case, suppose that \( H_i \) is EG, that some, and hence every, edge in \( H_j \) occurs as a term in the quasi-exceptional splitting of an iterate of some, and hence every, edge in \( H_i \). By Remark 2.15 there is an edge \( E_i \) in \( H_i \) whose initial vertex is principal and whose initial direction is fixed. Choose a lift \( \tilde{E}_i \), let \( \tilde{f} \) be the principal lift that fixes the initial endpoint of \( \tilde{E}_i \), let \( P \in \text{Fix}(\tilde{f}) \) be the terminal endpoint of the ray obtained by iterating \( \tilde{E}_i \) by \( \tilde{f} \) and let \( s : \mathcal{A}(\phi) \to \text{Aut}(F_n) \) be the lift determined by the maximal principal set \( \text{Fix}(\hat{f}) \). Proposition 7.7 with \( E = \mu = E_i \) and the assumption that \( \omega_i(\theta) = 0 \) imply that \( P \) is not isolated in \( \text{Fix}(s(\theta)) \). A second application of Proposition 7.7, this time with \( E = E_i \) and \( \mu \) an edge in \( H_j \) implies that \( \omega_j(\theta) = 0 \).

As a second case, suppose that there is a non-linear NEG edge \( E_l \) and that edges \( E_i \) of \( H_i \) and \( E_j \) of \( H_j \) occur as terms in the quasi-exceptional splitting of an iterate of \( E_l \). Define \( P \in \text{Fix}(\tilde{f}) \) and \( s : \mathcal{A}(\phi) \to \text{Aut}(F_n) \) as in the previous case using a lift of \( E_l \) instead of a lift of \( E_i \). As in the previous case Proposition 7.7 can be applied with \( E = E_l \) and with either \( \mu = E_i \) or \( \mu = E_j \). Thus \( \omega_i(\theta) = 0 \) if and only if \( \omega_j(\theta) = 0 \) as desired.
The equivalence relation on strata that defines almost invariant subgraphs is generated by these two cases so we have shown that the \( a_i \)'s determine a well defined \( M \)-tuple \( \tilde{a} = (\tilde{a_1}, \ldots, \tilde{a_M}) \) with one \( \tilde{a_s} \) for each almost invariant subgraph \( X_s \). To show that \( \tilde{a} \) is admissible, assume that \( E_i \in X_s, E_j \in X_l \) and \( E_k \in X_r \) are as in Definition 6.6. As in the previous cases, we may assume that the initial vertex of \( E_k \) is principal and the initial direction of \( E_k \) is fixed.

Define \( \eta = \psi \circ \phi^{−a_k} \). As in the previous cases, Proposition 7.7 can be applied with \( E = E_k \) and with either \( \mu = E_k \) or \( \mu \) equal to an element in the quasi-exceptional family determined by \( E_i E_j \). Since \( \omega_k(\eta) = \omega_k(\psi) - a_k \omega_k(\phi) = 0 \), it follows that \( 0 = \omega_{c,i,j}(\eta) = \omega_{c,i,j}(\psi) - a_k \omega_{c,i,j}(\phi) \). Keeping in mind that \( \hat{a}_r = a_k \), we have

\[
\omega_{c,i,j}(\psi) = \hat{a}_r \omega_{c,i,j}(\phi).
\]

Combining this with

\[
\hat{a}_r(d_i - d_j) = \hat{a}_r(\omega_i(\phi) - \omega_j(\phi)) = \hat{a}_r(\omega_{c,i,j}(\phi))
\]

and

\[
\omega_{c,i,j}(\psi) = \omega_i(\psi) - \omega_j(\psi) = \hat{a}_s \omega_i(\phi) - \hat{a}_t \omega_j(\phi) = \hat{a}_s d_i - \hat{a}_t d_j
\]

proves that \( \hat{a} \) is admissible. Choose \( K \geq 1 \) so that \( \phi^K_\hat{a} = \phi_{Ka} \) is rotationless. Corollary 6.16 implies that \( \phi^K_\hat{a} \in \mathcal{A}(\phi) \) and Lemma 7.4 then implies that \( \psi^K = \phi^K_\hat{a} \in \mathcal{D}_R(\phi) \). Thus \( \{\psi^K_1\} \subset \mathcal{D}_R(\phi) \) is a virtual basis for \( \mathcal{A}(\phi) \).

The remainder of the section is devoted to the proof of Proposition 7.7. For motivation we consider the proof as it applies to a simple example.

**Example 7.8.** Suppose that \( G = R_3 \) with edges \( A, B \) and \( C \) and that \( f : G \to G \) representing \( \phi \) is defined by \( A \mapsto A, B \mapsto BA \) and \( C \mapsto CB \).

Let \( T_A \) be the covering translation corresponding to \( A \) and let \( \tilde{B} \) be a lift of \( B \) with terminal endpoint in the axis of \( T_A \). Denote the principal lifts of \( f \) that fix the initial and terminal endpoints of \( \tilde{B} \) by \( \tilde{f}_- \) and \( \tilde{f}_+ \) respectively. The fixed point sets \( X_\pm \) of \( \tilde{f}_\pm \) are maximal principal sets for \( \mathcal{A}(\phi) \) and so determine lifts \( s_\pm : \mathcal{A}(\phi) \to \text{Aut}(F_n) \) such that \( X_\pm \subset \text{Fix}(s_\pm(\psi)) \) for all \( \psi \in \mathcal{A}(\phi) \). The coordinate homomorphism \( \omega \) corresponding to \( B \) satisfies \( \omega(\psi) = 0 \) if and only if \( s_+(\psi) = s_-(\psi) \). Note that \( T_A^\pm \) is contained in both \( X_+ \) and \( X_- \).

Choose a lift \( \tilde{C} \) of \( C \) and let \( \tilde{f} \) be the principal lift that fixes its initial endpoint. Iterating \( \tilde{E} \) by \( \tilde{f} \) produces a ray \( \tilde{R} \) that converges to some \( P \in \text{Fix}(\tilde{f}) \) and that projects to an \( f \)-invariant ray \( R = CBBABA^2 \ldots BA^lBA^{l+1}BA^{l+2} \ldots \). The maximal principal set \( \text{Fix}(\tilde{f}) \) determines a lift \( s : \mathcal{A}(\phi) \to \text{Aut}(F_n) \). Denote the subpath \( BBABA^2 \) of \( R \) that follows the initial \( C \) by \( \sigma_0 \) and the subpath \( f_\#(\sigma_0) = BA^lBA^{l+1}BA^{l+2} \) of \( R \) by \( \sigma_l \). There are lifts \( \tilde{\sigma}_l \subset \tilde{R} \) of \( \sigma_l, l \to \infty \), that are cofinal in \( \tilde{R} \) and so limit on \( P \).

There are also lifts \( \tilde{\delta}_l \) of \( \sigma_l \) for which \( \tilde{B} \) is the edge that projects to the middle \( B \) in \( \sigma_l \). The endpoints of \( \tilde{\delta}_l \) are denoted \( \tilde{x}_l \) and \( \tilde{y}_l \). The path connecting \( \tilde{x}_l \) to the initial
endpoint of $\tilde{B}$ is a lift of $BA^l$ and the path connecting the terminal endpoint of $\tilde{B}$ to $T_A^{-l}\tilde{y}_l$ is a lift of $ABA^{l+2}$. Thus $\tilde{x}_l \to Q_- \in X_- \setminus T_A^\pm$ and $T_A^{-l}\tilde{y}_l \to Q_+ \in X_+ \setminus T_A^\pm$. The line connecting $Q_-$ to $Q_+$ projects to $A^\inftyBABABA^\infty$.

Choose $g : G' \to G'$ representing $\psi$ as in Theorem 2.17. The lift $\tilde{g} : \Gamma' \to \Gamma'$ corresponding to $s(\psi)$ satisfies $P \in \text{Fix}(\tilde{g})$. For simplicity, we suppress the equivariant map that identifies $\Gamma$ with $\Gamma'$.

If $P$ is not isolated in $\text{Fix}(\tilde{g})$ then Lemma 2.2 implies that $\tilde{g}$ moves the endpoints of $\tilde{\sigma}_t$ by an amount that is bounded independently of $l$. Since $\tilde{\sigma}_t$ is a translate of $\tilde{\sigma}_l$ there is a lift $\tilde{g}_l$ of $g$ that moves $\tilde{x}_l$ and $\tilde{y}_l$ by a uniformly bounded amount, say $\kappa$. In Lemma 7.10 below we show that under these circumstances, $\tilde{g}_l$ commutes with $T_A$. Since there is a lift of $g$ that commutes with $T_A$ and fixes $Q_-$, it follows that $\tilde{g}_l(Q_-) = T_A^d_l(Q_-)$. If $d_l \neq 0$ and $x_l$ is sufficiently close to $Q_-$ then the distance between $x_l$ and $\tilde{g}_l(\tilde{x}_l)$ would be greater than $\kappa$ which is a contradiction. Thus $d_l = 0$ and $Q_- \in \text{Fix}(\tilde{g}_l)$ for all sufficiently large $l$. A second consequence of the fact that $\tilde{g}_l$ commutes with $T_A$ is that $\tilde{g}_l$ moves $T_A^{-l}\tilde{y}_l$ by a uniformly bounded amount. Arguing as in the previous case we conclude that $Q_+ \in \text{Fix}(\tilde{g}_l)$ for all sufficiently large $l$. For these $l$, $\text{Fix}(\tilde{g}_l)$ intersects both $X_+$ and $X_-$ in at least three points which implies that $\tilde{g}_l$ is the lift associated to both $s_-(\psi)$ and $s_+(\psi)$ and hence that $\omega(\psi) = 0$.

If $P$ is isolated in $\text{Fix}(\tilde{g})$ then by Lemma 2.18 there is an edge $\tilde{E}'$ of $\Gamma'$ that iterates toward $P$ under the action of $\tilde{g}$. The ray $\tilde{R}'$ connecting $\tilde{E}'$ to $P$ eventually agrees with $\tilde{R}$ and so contains $\tilde{\sigma}_l$ for large $l$. Lemma 7.12 below states, roughly speaking, that since iterating $\tilde{E}'$ by $g$ produces segments of the form $BA^l\tilde{B}$ for arbitrarily large $l$, it must be that $g_\#(BAB) = BA^kB$ for some $k > 0$. This implies that $A^\inftyBABABA^\infty$ is not $g_\#$-invariant and hence that the lifts of $g'$ corresponding to $s_-(\psi)$ and to $s_+(\psi)$ are distinct. Equivalently, $\omega(\psi) \neq 0$.

We now turn to the formal proof.

**Remark 7.9.** For the following lemmas it is useful to recall that if the circuits representing $[b]$ and $[c]$ have edge length $L_b$ and $L_c$ and if $A_b \cap A_c$ has edge length at least $L_b + L_c$ then $T_c$ commutes with $T_b$ because the initial endpoint $\tilde{x}$ of $A_b \cap A_c$ satisfies $T_b T_c(x) = T_c T_b(\tilde{x})$. It follows that $A_b = A_c$ and that $T_b = T_c^\pm$.

**Lemma 7.10.** Suppose that $\psi \in \text{Out}(F_n)$ is rotationless and that $g : G' \to G'$ represents $\psi$ and satisfies the conclusions of Theorem 2.17. Then for any primitive covering translation $T_c$ of the universal cover $\Gamma'$ of $G'$, there exists $K > 0$ with the following property. If $\tau \subset G'$ is a Nielsen path for $g$ and $\tilde{\tau} \subset \Gamma'$ is a lift whose intersection with the axis $A_c$ of $T_c$ contains at least $K$ edges, then the lift $\tilde{g}$ that fixes the endpoints of $\tilde{\tau}$ commutes with $T_c$.

**Proof.** Choose $L$ greater than the number of edges in each of the following:

1. the loop in $G'$ that represents $c$
2. each of the loops in $G'$ representing an axis of $\psi$
(3) any indivisible Nielsen path associated to an EG stratum for $g : G' \to G'$.

There is a decomposition $\tau = \tau_1 \cdot \ldots \cdot \tau_N$ into subpaths $\tau_i$ that are either fixed edges or indivisible Nielsen paths. The endpoints of the $\tau_i$’s are fixed by $\tilde{g}$. There is no loss in assuming that each $\tau_i$ intersects $A_c$ in at least an edge.

If $N \geq L$ then by (1), there exist $\tilde{\tau}_i$ with initial endpoint $\tilde{x}$ and $\tilde{\tau}_j$ with initial endpoint $T^i_c(\tilde{x})$ for some $i \neq 0$. Thus $\tilde{g}T^i_c(\tilde{x}) = T^i_c(\tilde{x}) = T^i_c\tilde{g}(\tilde{x})$. Since lifts of a map that agree on a point are identical, $\tilde{g}T^i_c = T^i_c\tilde{g}$. It follows that $\tilde{g}$ fixes $T^\pm_c$ which then implies that $\tilde{g}$ commutes with $T_c$.

We may therefore assume $N < L$. In fact we may assume that $N = 1$ : if $K$ works in this case then $(L + 2)K$ works in the general case. If $\tau$ is a fixed edge then $K = 2$ vacuously works. We may therefore assume that $\tau$ is indivisible.

Let $K = 2L + 2$. We may assume by (3) that $\tau$ is not associated to an EG stratum and so by Theorem 2.17(N), $\tilde{\tau} = \tilde{E}_i \tilde{w}^p \tilde{E}_j^{-1}$ for some linear edge $E_i$ satisfying $f(E_i) = E_i w^k$, where $w$ represents an axis $\mu$ of $\psi$ and therefore has fewer than $L$ edges. There is an axis $A_b$ for a primitive $b \in F_n$ that contains $\tilde{w}^p$ and whose projection into $G$ is the loop determined by $w$. Remark 7.9 and our choice of $K$ imply that $\hat{g} = T^\pm_c$. It is obvious that $\tilde{g}$ commutes with $T_b$ so $\tilde{g}$ also commutes with $T_c$. \hfill \Box

Suppose that $E_i$ is a linear edge and that $f(E_i) = E_i w^k$. If either $E_i$ or a quasi-exceptional path $E_i w^* \tilde{E}_j$ occurs as a term in the quasi-exceptional splitting of some $f^m_\#(\sigma)$ then $f^m_\#(\sigma)$ contains subpaths of the form $w^k$ where $k \to \pm \infty$ as $m \to \infty$. This is essentially the only way that such paths develop under iteration. Lemma 7.12 below is an application of this observation stated in the way that it is applied in the proof of Proposition 7.7.

We use $EL(\cdot)$ to denote edge length of a path or circuit. By extension, for $c \in F_n$, we use $EL(c)$ to denote the edge length of the circuit representing $[c]$.

We isolate the following observation for easy reference.

**Lemma 7.11.** Suppose that $g : G' \to G'$ satisfies the conclusions of Theorem 2.17 and that $\tau \subset G'$ is a completely split path such that $EL(g^m_\#(\tau))$ is not uniformly bounded. Then for all $L > 0$ there exists $M > 0$ so that for all $m \geq M$, $EL(g^m_\#(\tau)) > 2L$ and the initial and terminal subpaths of $g^m_\#(\tau)$ with edge length $L$ are independent of $m$.

**Proof.** The proof is by induction on the height $r$ of $\tau$. The $r = 0$ case is vacuous so we may assume that the lemma holds for paths of height less than $r$. By symmetry it is sufficient to show that $EL(g^m_\#(\tau)) \to \infty$ and that initial segment of $g^m_\#(\tau)$ with edge length $L$ stabilizes under iteration.

Let $\tau = \tau_1 \cdot \ldots \cdot \tau_s$ be the complete splitting of $\tau$ and let $\tau_i$ be the first term such that $EL(g^m_\#(\tau_i))$ is not uniformly bounded. The terms preceding $\tau_i$, if any, are Nielsen paths or pre-Nielsen connecting paths. Their iterates stabilize so there is no loss in truncating $\tau$ by removing them. We may therefore assume that $i = 1$. It now suffices to show that $EL(g^m_\#(\tau_1)) \to \infty$ and that initial segment of $g^m_\#(\tau_1)$ with edge length
L stabilizes under iteration. If \( \tau_1 \) is a connecting path this follows by induction on \( r \). The remaining cases are that \( \tau_1 \) is a non-fixed edge in an irreducible stratum or a quasi-exceptional path and the result is clear in both these cases.

The following lemma is a case-by-case analysis of the occurrence of long periodic segments in iterates of a single path. The basic observation is that once a periodic segment reaches a certain length it continues to get longer under further iteration.

**Lemma 7.12.** Suppose that \( g : G' \to G' \) satisfies the conclusions of Theorem 2.17, that \( c \in F_n \) is primitive and that \( \tilde{g} : \Gamma' \to \Gamma' \) is a lift of \( g \) that commutes with \( \Gamma \). Then for all completely split paths \( \sigma \subset G' \), there exists \( L_{\sigma} > 0 \) so that if \( m \geq 1 \) and \( \tilde{\rho}_m \) is a lift of \( \rho_m = g_m^w(\sigma) \) such that \( EL(\tilde{\rho}_m \cap A_c) > L_{\sigma} \) then \( E\tilde{\rho}_m(\tilde{\rho}_m \cap A_c) > EL(\tilde{\rho}_m \cap A_c) \).

**Proof.** Lemma 2.1 implies that the circuit corresponding to \( c \) is \( g_\# \)-invariant and hence that \( A_c \) decomposes as a concatenation of subpaths that project to \( g \)-fixed edges and indivisible Nielsen paths for \( g \). After composing \( \tilde{g} \) with an iterate of \( \Gamma \) if necessary, we may assume that these subpaths are \( \tilde{g} \)-Nielsen paths. The endpoints of these paths are called **splitting vertices** and their union is the set of \( \tilde{g} \)-fixed vertices in \( A_c \).

The proof is by induction on the height \( r \) of \( \sigma \). The induction statement is enhanced to include the following property: if \( EL(\tilde{\rho}_m \cap A_c) > L_{\sigma} \) and if \( \tilde{\rho}_m \cap A_c \) contains an endpoint \( \tilde{v} \) of \( \tilde{\rho}_m \) then \( \tilde{v} \) is a splitting vertex.

In certain cases we will show that \( A_c \cap \tilde{\rho}_m \) is uniformly bounded, meaning that it is bounded independently of \( m \). One then chooses \( L_{\sigma} \) greater than that bound. The \( r = 0 \) case is vacuously true so we may assume that the inductive statement holds for all paths of height less than \( r \).

Assume for now that there is only one term in the QE-splitting of \( \sigma \). There are five cases, two of which are immediate. If \( \sigma \) is a Nielsen path then \( A_c(\tilde{\rho}_m) \) is uniformly bounded and we are done. If \( \sigma \) is a connecting path then we let \( \sigma = o_{\tilde{g}_\#}(\sigma) \) where the latter exists by the inductive hypothesis and the fact that \( g_\#(\sigma) \) has height less than \( r \).

If \( \sigma \) is a linear edge \( E \) then \( \rho_m = Ew_{dm} \) for some Nielsen path \( w \) that forms a primitive circuit and some \( d > 0 \). Let \( L_{\sigma} = EL(c) + EL(w) \). If \( EL(\tilde{\rho}_m \cap A_c) > L_{\sigma} \) then by Remark 7.9 there is a lift \( \tilde{w} \) of \( w \) such that \( \tilde{\rho}_m \cap A_c = \tilde{w}_{dm} \) contains all of \( \tilde{\rho}_m \) but the initial edge and \( \tilde{g}_\#(\tilde{\rho}_m) \cap A_c = \tilde{w}_{dm} \). Since \( w \) is a Nielsen path and \( \tilde{w} \) is a fundamental domain of \( A_c \) the endpoints of \( \tilde{w} \) are splitting vertices.

If \( \sigma \) is an exceptional path \( E_iw^pE_j \) where \( g(E_i) = E_iw^{d_i} \) and \( g(E_j) = E_jw^{d_j} \), the proof is similar to the linear case and we can use the same value of \( L_{\sigma} \). If \( EL(\tilde{\rho}_m \cap A_c) > L_{\sigma} \) then there is a lift \( \tilde{w} \) of \( w \) such that \( \tilde{\rho}_m \cap A_c = \tilde{w}_{dm(d_i-d_j)+p} \) contains all of \( \tilde{\rho}_m \) but the initial and terminal edges and \( \tilde{g}_\#(\tilde{\rho}_m) \cap A_c = \tilde{w}_{dm(d_i-d_j)+p} \). In this case the endpoints of \( \tilde{\rho}_m \) are not contained in \( A_c \).

The fifth and hardest case is that \( \sigma \) is a single edge \( E \) in a non-linear irreducible stratum \( H_\tau \). If the height of \( A_c \) is greater than \( r \) then \( A_c \cap \tilde{\rho}_m \) has uniformly bounded
length. We may therefore assume that $A_c$ has height at most $r$. We consider the EG and NEG subcases separately.

Suppose that $H_r$ is EG. If $A_c$ has height $r$ then it has an illegal turn in the $r$-stratum which implies that $EL(A_c \cap \tilde{\rho}_m) < EL(c)$. We may therefore assume $A_c$ has height less than $r$. In particular, endpoints of $\tilde{\rho}_m$ are not contained in $A_c$. For each edge $E'$ of $H_r$ there is a coarsening of the $QE$-splitting of $g(E')$ into an alternating concatenation of subpaths in $H_r$ subpaths in $G_{r-1}$. Let $\{\mu_j\}$ be the set of paths of $G_{r-1}$ that occur as subpaths in this decomposition as $E'$ varies over all edges of $H_r$. The path $g^m(E)$ also splits as an alternating concatenation of subpaths in $H_r$ and subpaths in $G_{r-1}$; each of the subpaths in $G_{r-1}$ equals $g^m(\nu_j)$ for some $\nu_j$ and some $0 \leq l \leq m$. We may therefore choose $L_a = \max\{L_{\mu_j}\}$.

Finally, suppose that $H_r$ is non-linear and NEG. There is a path $u \subset G_{r-1}$ such that $g^m(E) = E \cdot u \cdot g^m(u) \cdot \ldots \cdot g^m(E)$ for all $m$ and such $EL(g^m(u)) \to \infty$. We may assume without loss that $A_c$ has height less than $r$ and hence that $\tilde{\rho}_m \cap A_c$ projects into $u \cdot g^m(u) \cdot \ldots \cdot g^m(u)$. We claim that if $r$ is sufficiently large, say $r > R$, then the projection of $\tilde{\rho}_m \cap A_c$ does not contain $g^m(u)$ for any $m$. Assume the claim for now. If $EL(\tilde{\rho}_m \cap A_c) > EL(u \cdot g^m(u) \cdot \ldots \cdot g^m(u))$ then the projection of $\tilde{\rho}_m \cap A_c$ is contained in $g^m(u) \cdot g^m(u) = g^m(u \cdot g^m(u))$ for some $q$. We may therefore choose $L_\sigma$ to be the maximum of $EL(u \cdot g^m(u) \cdot \ldots \cdot g^m(u))$ and $L_{u \cdot g^m(u)}$.

The claim is obvious if $u$ and $A_c$ have the same height, say $t$, so assume that this is the case. The claim is also obvious if the maximal length of a subpath of $g^m(u)$ with height less than $t$ goes to infinity with $q$. We may therefore assume that the number of height $t$ edges in $g^m(u)$ is unbounded. Thus $H_t$ is EG and $g^m(u)$ contains $t$-legal subpaths of length greater than $EL(c)$ for all sufficiently large $r$. Since no such subpath is contained in $A_c$ this completes the proof of the claim and so also the induction step when there is only one term in the $QE$-splitting of $\sigma$.

Assume now that $\sigma = \sigma_1 \cdot \ldots \cdot \sigma_s$ is the $QE$-splitting of $\sigma$ and that $s > 1$. Let $L_1 = \max\{L_{\sigma_i}\}$. By Lemma 7.11 there exists $M > 0$ so that for all $m > M$ and all $\sigma_i$, either $g^m(\sigma_i)$ is independent of $m$ or $EL(g^m(\sigma_i)) > 2L_1$ and the initial and terminal segments of $g^m(\sigma_i)$ with edge length $L_1$ are independent of $m$. The former corresponds to $\sigma_i$ being a Nielsen path or a pre-Nielsen connecting path and the latter to all remaining cases. Choose $L_a > sL_1$ so that $EL(g^m(\sigma)) < L_a$ for all $m \leq M$.

Denote $g^m(\sigma_i)$ by $\rho_{i,m}$ and write $\tilde{\rho}_m = \tilde{\rho}_{1,m} \cdot \ldots \cdot \tilde{\rho}_{s,m}$. If $EL(A_c \cap \tilde{\rho}) \geq L_a$ then $EL(A_c \cap \tilde{\rho}_i,m) \geq L_{\sigma_i}$ for some $1 \leq i \leq s$. Thus $EL(g^m(A_c \cap \tilde{\rho}_{i,m})) > EL(A_c \cap \tilde{\rho}_{i,m})$. If $A_c \cap \tilde{\rho} \subset \tilde{\rho}_{i,m}$ we are done. Otherwise we may assume that $A_c \cap \tilde{\rho}_{i+1,m}$ is a non-trivial initial segment of $\tilde{\rho}_{i+1,m}$ that begins at a splitting vertex of $A_c$. If $\rho_{i+1,m}$ is a Nielsen path or a pre-Nielsen connecting path then $g^m(A_c \cap \tilde{\rho}_{i+1,m}) = A_c \cap \tilde{\rho}_{i+1,m}$. This same equality holds if $EL(A_c \cap \tilde{\rho}_{i+1,m}) \leq L_1$ by our choice of $M$. Finally, if $EL(A_c \cap \tilde{\rho}_{i+1,m}) > L_1$ then $EL(g^m(A_c \cap \tilde{\rho}_{i+1,m})) > EL(A_c \cap \tilde{\rho}_{i+1,m})$. This completes the proof if $A_c \cap \tilde{\rho}_m \subset \tilde{\rho}_{i,m} \tilde{\rho}_{i+1,m}$. Iterating this argument completes the proof in general. \qed
We need one more lemma before proving the main proposition.

**Lemma 7.13.** Suppose that \( g : G' \to G' \) satisfies the conclusions of Theorem 2.17, that \( \sigma \) is a completely split non-Nielsen path for \( g \) and that \( \tilde{\sigma} \subset \Gamma' \) is a lift of \( \sigma \) with endpoints at vertices \( \tilde{x} \) and \( \tilde{y} \). If \( \tilde{g}' : \Gamma' \to \Gamma' \) fixes \( \tilde{x} \) then \( \lim_{k \to \infty} \tilde{g}'(\tilde{y}) \to Q \) for some \( Q \in \text{Fix}_N(\tilde{g}) \).

**Proof.** There is no loss in assuming that \( \sigma \) is either a single non-fixed edge or an exceptional path \( E\tau E' \). In the former case, the lemma follows from Lemma 2.18. In the latter case, \( Q \) is an endpoint of the axis of a covering translation corresponding to \( \tau \). \( \square \)

**Proof of Proposition 7.7** The case that \( \mu \) is an EG edge follows from Lemma 2.19. In the remaining cases there is an axis \( [c]_u \) associated to \( \mu \) and we let \( T_c, \Phi_0, \{ \Phi_i \}, \{ E_i \} \) and \( \{ d_i \} \) be as in Lemma 2.20. Thus \( \mu \) is either \( E_j \) for some \( j \) or an element of the quasi-exceptional family determined by \( E_j E_j' \) for some \( j \) and \( j' \).

Letting \( \tilde{u} \) be the path such that \( \tilde{f}(\tilde{E}) = \tilde{E} \cdot \tilde{u} \), we have \( \tilde{R} = \tilde{E} \cdot \tilde{R}_0 \) where \( \tilde{R}_0 = \tilde{u} \cdot \tilde{f}_\#(\tilde{u}) \cdot \tilde{f}_\#^2(\tilde{u}) \ldots \). Since \( \tilde{E} \) is not linear, \( \tilde{\mu} \) occurs infinitely often as a term in the \( QE \)-splitting of \( \tilde{R}_0 \), where we do not distinguish between elements of the same quasi-exceptional family of subpaths. There is a completely split subpath \( \tilde{\sigma}_0 \) of \( \tilde{R}_0 \) and a coarsening \( \tilde{\sigma}_0 = \tilde{\tau}_1 \cdot \tilde{\mu} \cdot \tilde{\tau}_2 \) of the \( \tilde{QE} \)-splitting of \( \sigma_0 \) where \( \tilde{\mu} \) is a lift of \( \mu \) and where \( \tilde{\tau}_1 \) and \( \tilde{\tau}_2 \) are not Nielsen paths. Denote the initial and terminal endpoints of \( \tilde{\sigma}_0 \) by \( \tilde{a}_0 \) and \( \tilde{b}_0 \) and for \( l \geq 1 \), let \( \tilde{\sigma}_l = \tilde{f}_\#^l(\tilde{\sigma}_0), \tilde{a}_l = \tilde{f}_\#^l(\tilde{a}_0), \text{ and } \tilde{b}_l = \tilde{f}_\#^l(\tilde{b}_0) \). Then

1. \( \tilde{\sigma}_1 \subset \tilde{R}_0 \) and \( \tilde{\sigma}_l \to P \).

Let \( \tilde{f}_j \) be the lift of \( f \) corresponding to \( \Phi_j \) and let \( \tilde{E}_j \) be a lift of \( E_j \) whose initial endpoint is fixed by \( \tilde{f}_j \) and whose terminal endpoint is contained in \( A_c \). There is a covering translation \( S_0 : \Gamma \to \Gamma \) such that \( \tilde{E}_j \) is the initial edge of \( S_0(\tilde{\mu}) \). Let \( \tilde{\delta}_0 = S_0(\tilde{\sigma}_0) \). For \( l \geq 1 \), let \( S_l : \Gamma \to \Gamma \) be the covering translation such that \( \tilde{f}_j^l S_0 = S_j^l \tilde{f}_j^l \), let \( \tilde{\delta}_l = S_l(\tilde{\sigma}_l) \) and let \( \tilde{x}_l \) and \( \tilde{y}_l \) be the endpoints of \( \tilde{\delta}_l \). It is immediate that

2. \( \tilde{E}_j \subset \tilde{\delta}_l \).
3. \( \tilde{\delta}_l = \tilde{f}_\#^l(\tilde{\delta}_0) \).
4. the length of \( \tilde{\delta}_l \cap A_c \) goes to infinity with \( l \).

Lemma 7.13 applied to \( \tilde{f}_j \) and \( S_0(\tilde{\tau}_1) \) implies that

5. \( \tilde{x}_l \to Q_- \in \text{Fix}_N(\tilde{\Phi}_j) \setminus \{ T_c^\pm \} \).

If \( \mu \) corresponds to \( E_j \), let \( m = d_j \) and \( t = 0 \). If \( \mu \) corresponds to \( E_j E_j' \), let \( m = d_j - d_j' \) and \( t = j' \). Then the terminal endpoint of \( S_0(\tilde{\mu}) \) is fixed by \( T_c^{-m} \tilde{f}_j \). Lemma 7.13 applied to \( T_c^{-m} \tilde{f}_j \) and \( S_0(\tilde{\tau}_2) \) implies that

40
The maximal principal sets $s_g$ be lifts of $\Psi_P$. It suffices to show that $\phi$ hold and each $\sigma$ that for all sufficiently large $l$ represents a lift of $\psi$ to the universal cover $\Gamma'$ corresponding to $\Psi = s(\psi)$, $\Psi_j = s_j(\psi)$ and $\Psi_t = s_t(\psi)$ respectively. The following are equivalent.

- $\omega(\psi) = 0$.
- $\Psi_j = \Psi_t$.
- $Q_+ \in \text{Fix}(\tilde{\Psi}_j)$.

It suffices to show that $P$ is isolated in $\text{Fix}(\tilde{\Psi})$ if and only if $Q_+ \notin \text{Fix}(\tilde{\Psi}_j)$.

To compare points in $\Gamma$ and $\Gamma'$, choose an equivariant map $h : \Gamma \rightarrow \Gamma'$ that preserves the markings; equivalently, when $\partial \Gamma$ and $\partial \Gamma'$ are identified with $\partial F_n$ then $h : \partial \Gamma \rightarrow \partial \Gamma'$ is the identity. Let $C$ be the bounded cancellation constant for $h : \Gamma \rightarrow \Gamma'$ and let $\hat{R}' = h_{\#}(R)$. We use prime notation for covering translations and axes of $\Gamma'$. Thus $S'_l : \Gamma' \rightarrow \Gamma'$ is the covering translation such that $S'_l h = h S_l$. Denote $h(\tilde{a}_l), h(b_l)$ and the path that they bound by $\tilde{a}'_l, \tilde{b}'_l$ and $\tilde{\sigma}'_l$. Let $\tilde{x}'_l = S'_l(\tilde{a}'_l) = h S_l(\tilde{a}_l) = h S_l(\tilde{a}_l) = h(x_j)$, let $\tilde{y}'_l = S'_l(\tilde{b}'_l) = h(\tilde{y}_l)$ and let $\delta'_l = S'_l(\tilde{\sigma}'_l) = h_{\#}(\delta_l)$ be the path connecting $\tilde{x}'_l$ to $\tilde{y}'_l$. We have

1. $\tilde{\sigma}'_l$ is $C$-close to $R'$ and $\tilde{\sigma}'_l \rightarrow P$.
2. the length of $\tilde{\sigma}'_l \cap A'_c$ goes to infinity with $l$.
3. $\tilde{x}'_l \rightarrow Q_- \in \text{Fix}(\tilde{\Psi}_j) \setminus \{T'^+\}$.
4. $T'^+_c \rightarrow Q_+ \in \text{Fix}(\tilde{\Psi}_t) \setminus \{T'^+_c\}$.

If $P$ is not isolated in $\text{Fix}(\tilde{\Psi})$ then Lemma 2.2 implies, after increasing $C$ if necessary, that $\tilde{a}'_l$ and $\tilde{b}'_l$ are $C$-close to $\text{Fix}(\tilde{g})$ for all sufficiently large $l$. After replacing $\tilde{a}'_l$ and $\tilde{b}'_l$ with $C$-close elements of $\text{Fix}(\tilde{g})$, replacing $\sigma'_l$ with the path connecting the new values of $\tilde{a}'_l$ and $\tilde{b}'_l$, and replacing $C$ by $2C$, properties (1'), (4'), (5') and (6') still hold and each $\sigma'_l$ is a Nielsen path for $g$. Since $\tilde{\sigma}'_l$ is a lift of $\sigma'_l$, Lemma 7.10 implies that for all sufficiently large $l$, the lift of $g$ that fixes $\tilde{x}'_l$ and $\tilde{y}'_l$ commutes with $T'_c$ and so equals $T'^d_l g_j$ for some $d_l$. Since $Q_- \in \text{Fix}(\tilde{g}_j)$ there is a neighborhood of $Q_-$ in $\Gamma'$ that is disjoint from $\text{Fix}(T'^m_l g_j)$ for all $m \neq 0$. Since $\tilde{x}'_l \rightarrow Q_-$, it follows that $d_l = 0$ and hence that $\tilde{y}'_l \in \text{Fix}(\tilde{g}_j)$ for all sufficiently large $l$. Since $\text{Fix}(\tilde{g}_j)$ is $T'_c$-invariant, $T'^m_l \tilde{y}'_l \in \text{Fix}(\tilde{g}_j)$ and so $Q_+ \in \text{Fix}(\tilde{g}_j)$ as desired.

Suppose then that $P$ is isolated in $\text{Fix}(\tilde{\Psi})$. After replacing $\tilde{a}'_l$ and $\tilde{b}'_l$ by their nearest points in $\hat{R}'$, we may assume that $\tilde{\sigma}'_l \subset R'$ and that properties (1'), (4'), (5') and (6')
still hold. Lemma 2.18 implies that there is a non-linear edge $\tilde{E}'$ that iterates toward $P$ under the action of $\tilde{g}$. Denoting $g^m_\tilde{g}(E')$ by $\rho_m$ we have that for all sufficiently large $l$ there exists $m > 0$ such that $\sigma'_l$ is a subpath of $\rho_m$. There is a lift $\tilde{\rho}_m$ of $\rho_m$ that contains $\tilde{g}'$ and so has endpoints $\partial_{\pm}\tilde{\rho}_m$ such that $\partial_-\tilde{\rho}_m \to Q_-$ and $T_r^{-u_l}\partial_+\tilde{\rho}_m \to Q_+$. The former implies that for sufficiently large $m$, the initial endpoints of $\tilde{\rho}_m \cap A'_c$ and $\tilde{g}_j(\tilde{\rho}_m) \cap A'_c$ are equal and the latter implies that if $Q_+ \notin \text{Fix}(\tilde{g}_j) = \text{Fix}(\hat{\Psi}_j)$ then the terminal endpoints of $\tilde{\rho}_m \cap A'_c$ and $\tilde{g}_j(\tilde{\rho}_m) \cap A'_c$ are equal. On the other hand, $\tilde{\rho}_m \cap A'_c$ and $\tilde{g}_j(\tilde{\rho}_m) \cap A'_c$ have different lengths by Lemma 7.12 so we conclude that $Q_+ \notin \text{Fix}(\Psi_j)$. \hfill \Box

8 Abelian Subgroups of Maximal Rank

By Theorem 7.2, all abelian subgroups are realized, up to finite index, as subgroups of some $D_R(\phi)$. In this section we describe those $\phi$ for which $D_R(\phi)$ has maximal rank. As usual, $\phi$ is represented by a relative train track map $f : G \to G$ and filtration $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ satisfying the conclusions of Theorem 2.17.

For the simplest example, start with $G_2$ having one vertex $v_1$, two edges $E_1$ and $E_2$ and with $f$ defined by $f(E_1) = E_1$ and $f(E_2) = E_2E_1^{m_1}$ for some $m_1 \in \mathbb{Z}$. For $k = 1, \ldots, n - 2$, add pairs of linear edges, $E_{2k+1}$ and $E_{2k+2}$, initiating at a new common vertex $v_{k+1}$, terminating at $v_1$ and satisfying $f(E_j) = E_jE_1^{m_j}$ for distinct $m_j$. Thus $G$ has $2n - 3$ linear edges and the resulting $D_R(\phi)$ has rank $2n - 3$, which is known [CV86] to be maximal. In this example all edges terminate at the same vertex and there is only one axis, but this is just for simplicity. One could, for example, take the terminal vertex of $E_5$ equal to $v_2$ and define $f(E_5) = E_5w_5$ where $w_5$ is a closed Nielsen path based at $v_2$. Similar modifications can be done to the other edges as well.

Another simple modification is to redefine $f|G_2$ so that $G_2$ is a single EG stratum with Nielsen path $\rho$ and redefine $f$ on the other edges to be linear with axis represented by $\rho$. We may view the original example as being built over a Dehn twist of the punctured torus and this modification as being built over a pseudo-Anosov homeomorphism of the punctured torus.

A perhaps more surprising example of a maximal rank abelian subgroup is constructed as follows. Let $S$ be the genus zero surface with four boundary components $\beta_1, \ldots, \beta_4$ and let $h : S \to S$ be a homeomorphism that represents a pseudo-Anosov mapping class and that pointwise fixes each $\beta_m$. Let $A$ be an annulus with boundary components $\alpha_1$ and $\alpha_2$ and with its central circle labeled $\alpha_3$. Define $D_{jk} : A \to A$ to be the homeomorphism that restricts to a Dehn twist of order $j$ on the subannulus bounded by $\alpha_1$ and $\alpha_3$ and to a Dehn twist of order $k$ on the subannulus bounded by $\alpha_2$ and $\alpha_3$. Finally, define $Y = S \cup A' \sim \sim \sim$ where $\sim$ identifies $\alpha_m$ to $\beta_m$ for $1 \leq m \leq 3$. The homeomorphisms $g_{ij} : Y \to Y$ induced by $h^i$ and $D_{jk}$ for $i, j, k \in \mathbb{Z}$ define a rank three abelian subgroup $\mathcal{A}'$. The fundamental group of $Y$ is a free group of rank three and the image of $\mathcal{A}'$ in $\text{Out}(F_3)$ is an abelian subgroup $\mathcal{A}$ of maximal rank.
We present a slight generalization of this example in terms of relative train tracks as follows.

**Example 8.1.** Suppose that $G$ is a rank three marked graph with vertices $v_1, \ldots, v_4$, that $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_4 = G$ is a filtration and that $f : G \to G$ is a relative train track map such that

- $G_1$ is a single fixed edge $E_1$ with both ends attached to $v_1$.
- for $m = 2, 3$, $H_m$ is a single edge $E_m$ with terminal endpoint $v_1$ and initial endpoint $v_m$; $f(E_m) = E_mE_1^{d_m}$ where $d_2$ and $d_3$ are distinct non-zero integers.
- $H_4$ is an EG stratum with three edges, one connecting $v_4$ to $v_l$ for each $l = 1, 2, 3$; for each edge $E$ of $H_4$, $f(E)$ is a concatenation of edges in $H_4$ and Nielsen paths of the form $E^*_1, E_2E_1^*E_2$ and $E_3E_1^*E_3$.

Then $f$ determines an element $\phi \in \text{Out}(F_3)$ such that $D_R(\phi)$ has rank three. The example described above using a four times punctured sphere is a special case of this construction with $* = \pm 1$. In general, $H_4$ is not a geometric stratum in the sense of [BFH00].

We think of the strata $H_2 \cup H_3 \cup H_4$ in Example 8.1 as being a single unit added on to the lower filtration element, which in this case is a single circle. If the lower filtration element has higher rank then we have the option of adding an additional linear edge. In the geometric case this amounts to Dehn twisting on three of components of the four times punctured sphere instead of just two. We formalize this as follows, where the acronym FPS is chosen to remind the reader of the four times punctured sphere.

**Notation 8.2.** We say that $H_{l+1} \cup H_{l+2} \cup H_{l+3}$ is a partial FPS subgraph if

- (1) There are (not necessarily distinct) closed Nielsen paths $\alpha_1, \alpha_2 \subset G_l$.
- (2) For $j = 1, 2$ the stratum $H_{l+j}$ is a single linear edge $E_{l+j}$ such that $f(E_{l+j}) = E_{l+j}^{d_j}$ for some non-zero $d_j$. The initial endpoints $v_{l+j}$ of $E_{l+j}$ are distinct and are not contained in $G_l$. (Equivalently, $G_{l+2}$ deformation retracts to $G_l$.)
- (3a) $H_{l+3}$ is EG.
- (3b) $H_{l+3} \cap G_{l+2} = \{v_{l+1}, v_{l+2}, v\}$ for some vertex $v \in G_l$.
- (3c) $H_{l+3}$ is either a pair of arcs joined at a common endpoint or a triad, which is three arcs joined at a common endpoint. whose valence three vertex is not contained in $G_{l+2}$. For each edge $E$ of $H_{l+3}$ the edge path $f(E)$ is a concatenation of edges in $H_{l+3}$, Nielsen paths of the form $E_{l+j}^{\alpha_j}E_{l+j}$ for $j = 1, 2$ and Nielsen paths in $G_l$.

**Remark 8.3.** In Example 8.1, $H_2 \cup H_3 \cup H_4$ is partial FPS subgraph.
Notation 8.4. We say that $H_{l+1} \cup \cdots \cup H_{l+4}$ is an \textit{FPS subgraph} if

(1) There are (not necessarily distinct) closed Nielsen paths $\alpha_1, \alpha_2, \alpha_3 \subset G_l$.

(2) For $j = 1, 2, 3$ the stratum $H_{l+j}$ is a single linear edge $E_{l+j}$ such that $f(E_{l+j}) = E_{l+j}\alpha_j d_j$ for some non-zero $d_j$. The initial endpoints $v_{l+j}$ of $E_{l+j}$ are distinct and are not contained in $G_l$. (Equivalently, $G_{l+3}$ deformation retracts to $G_l$.)

(3a) $H_{l+4}$ is EG.

(3b) $H_{l+4} \cap G_{l+2} = \{v_{l+1}, v_{l+2}, v_{l+3}\}$.

(3c) $H_{l+4}$ is either a pair of arcs joined at a common endpoint or a triad whose valence three vertex is not contained in $G_{l+3}$. For each edge $E$ of $H_{l+4}$ the edge path $f(E)$ is a concatenation of edges in $H_{l+4}$ and Nielsen paths of the form $E_{l+j}\alpha_j^* E_{l+j}$ for $j = 1, 2, 3$.

Remark 8.5. If $H_{l+1} \cup H_{l+2} \cup H_{l+3}$ is a partial FPS subgraph then $\chi(G_l) - \chi(G_{l+3}) = 2$. If $H_{l+1} \cup \cdots \cup H_{l+4}$ is a FPS subgraph then $\chi(G_l) - \chi(G_{l+4}) = 2$.

We can now state the main results of this section. We assume the existence of $f : G \to G$ satisfying the conclusions of Theorem 2.17 applied without reference to a particular $F$ and satisfying the additional condition that there are no non-trivial invariant forests. This is always possible by Remark 4.7 of [FHb]. Partial FPS subgraphs arise in the proof of the propositions but not in their statements.

Proposition 8.6. Suppose that $\phi \in \text{Out}(F_n)$ is rotationless, that $D_R(\phi)$ has rank $2n - 3$ and that $\phi$ is represented by $f : G \to G$ and $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$ as in Theorem 2.17. Suppose further that $f : G \to G$ has no non-trivial invariant forests. Then after reordering the filtration if necessary, there are $1 \leq l_1 < \cdots < l_K \leq N$ such that

(A) $G_{l_i}$ either:

1. has rank two and is a single $EG$ stratum.
2. has rank two and consists of two edges $E_1, E_2$ where $f(E_1) = E_1$ and $f(E_2) = E_2 E_i^m$ for some $m \geq 1$.
3. has rank three and $f|G_{l_i}$ is as in Example 8.1.

(B) for $m > 1$, $\bigcup_{j=l_m+1}^{l_{m+1}} H_j$ is either

1. a pair of linear edges with a common initial vertex that is not contained in $G_{l_m}$ or
2. an FPS subgraph.
There is an analogous result for abelian subgroups of the subgroup $IA_n$ of $Out(F_n)$ consisting of elements that act trivially in homology.

**Proposition 8.7.** Suppose that $\phi \in Out(F_n)$ is rotationless, that $D_R(\phi) \subset IA_n$ has rank $2n-4$ and that $\phi$ is represented by $f : G \to G$ and $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ as in Theorem 2.17. Suppose further that $f : G \to G$ has no non-trivial invariant forests. Then after reordering the filtration if necessary, there are $1 \leq l_1 < \cdots < l_K \leq N$ such that

(A) $l_1 = 2$ and $G_2$ is connected, has rank two and is contained in $Fix(f)$.

(B) for $m > 1$, $\bigcup_{j=l_m+1}^{l_{m+1}} H_j$ is either

1. a pair of linear edges with homologically trivial axes and a common initial vertex that is not contained in $G_{l_m}$.
2. an FPS subgraph with homologically trivial axes.

Recall that one uses the $QE$-splitting of the $f$-image of edges of $G$ to define almost invariant subgraphs $X_1, \ldots, X_M$ of $G$ and that if $a_i$ is a non-negative integer assigned to $X_i$ then $(a_1, \ldots, a_M)$ is admissible (Definition 6.6) if it satisfies certain linear relations involving three of the $a_i$'s. The rank of $D_R(\phi)$ is equal to the rank of the subspace of $\mathbb{R}^M$ generated by the admissible $M$-tuples for $f : G \to G$.

For induction purposes, it is useful to consider admissible sequences of $f|G_j$ for each filtration element $G_j$. Let $M_j$ be the number of almost invariant subgraphs for $f|G_j$ and let $R_j$ be the rank of the subspace of $\mathbb{R}^M$ generated by the admissible $M_j$-tuples defined with respect to $f|G_j$. Each almost invariant subgraph for $f|G_j$ determines a relation on $M_j$-tuples by inclusion. Thus every admissible $M_j$-tuple for $f|G_j$ ‘restricts’ to an admissible $M_j$-tuple for $f|G_j$. The only almost invariant subgraph of $f|G_j$ that can contain more than one almost invariant subgraph for $f|G_j$ is the one that contains $H_{j+1}$. Amalgamating almost invariant subgraphs of $f|G_j$ into a single almost invariant subgraph for $f|G_{j+1}$ can be viewed as a finite set of new relations. All other new relations involve the almost invariant subgraph containing $H_{j+1}$.

To consolidate ideas and as a warm up we prove a simple estimate on the $R_j$'s. A path $\sigma$ is pre-Nielsen if it is not a Nielsen path but $f_k^\#(\sigma)$ is a Nielsen path for some $k \geq 1$.

In the following lemmas $G$ is as in Propositions 8.6 and 8.7.

**Lemma 8.8.** If $G$ is as in Propositions 8.6 or 8.7 then the following hold for $r < s$ and $\Delta R = R_s - R_r$.

1. If $H_s$ is a fixed edge and $r = s - 1$ then $\Delta R = 0$.
2. If $H_s$ is a linear edge and $r = s - 1$ then $\Delta R = 1$. 

45
3. If \( H_s \) is a non-linear NEG edge and \( r = s - 1 \) then \( \Delta R \leq 0 \).

4. If \( H_r \) is irreducible, \( H_s \) is EG and if all strata between \( H_r \) and \( H_s \) are zero strata then \( \Delta R \leq 1 \) with equality if and only if for each edge \( E \) of \( H_s \) the terms in the QE-splitting of \( f(E) \) are either edges in \( H_s \), pre-Nielsen connecting paths or Nielsen paths in \( G_r \).

**Proof.** (1) is obvious since there are no new almost invariant subgraphs and no new relations. For the remaining items it is clear that \( \Delta R \leq 1 \) since there is at most one new almost invariant subgraph. \( \Delta R = 1 \) when \( G_s \setminus G_r \) is an entire almost invariant subgraph for \( f|G_s \) and when that almost invariant subgraph is not part of any relation. Equivalently, for each edge \( E \) of \( H_s \) the terms in the QE-splitting of \( f(E) \) are either edges in \( H_s \), pre-Nielsen connecting paths or Nielsen paths in \( G_r \). (2), (3) and (4) follow immediately.

We now come to the main step in the proofs of Proposition 8.6 and Proposition 8.7.

**Lemma 8.9.** Assume that \( G \) is as in Propositions 8.6 or 8.7, that \( r \leq u < s \) and that the following conditions are satisfied.

1. \( G_r \) and \( G_s \) have no valence one vertices.

2. For \( r < l \leq u \), \( H_l \) is a single edge \( E_l \) whose terminal vertex is in \( G_r \) and whose initial vertex has valence one in \( G_u \).

3. For \( u < l < s \), \( H_l \) is a zero stratum.

4. \( H_s \) is an EG stratum.

Let \( H_{rs} = \bigcup_{l=r+1}^{s} H_l \), let \( \Delta R = R_s - R_r \) and let \( \Delta \chi = \chi(G_r) - \chi(G_s) \). If there is a vertex \( v \in G_r \) and a fixed direction at \( v \) determined by an edge of \( H_{rs} \) let \( \delta = 1 \); otherwise \( \delta = 0 \).

Then

\[ \Delta R \leq 2\Delta \chi - \delta. \]

Moreover, if the inequality is an equality then one of the following holds:

(a) \( H_{rs} \) is an FPS subgraph and \( \delta = 0 \).

(b) \( H_{rs} \) is a partial FPS subgraph and \( \delta = 1 \).

**Proof.** If \( H_s \) is disjoint from \( G_u \) then \( s = r + 1 \) and \( H_s \) is a component of \( G_s \). In this case, \( \delta = 0 \), \( \Delta R = 1 \), \( \Delta \chi \geq 1 \) and the lemma is clear. We assume for the remainder of the proof that \( H_s \), and hence each component of \( H_s \), has non-empty intersection with \( G_u \).

Item (4) and Corollary 3.2.2 of [BFH00] imply that

\[ \Delta \chi \geq 2. \]
Let $H_{us} = \bigcup_{t=u+1}^{s} H_t$. Thus $G_s = G_u \cup H_{us}$. Denote $G_u \cap H_{us}$ by $V$, the cardinality of $V$ by $V$ and the number of components in $H_{us}$ by $C_{us}$. Then
\[
\Delta \chi \geq V - C_{us}
\]
with equality if and only if each component of $H_{us}$ is contractible and
\[
\Delta \chi \geq C_{us}
\]
with equality if and only if each component of $H_{us}$ is topologically either an arc whose interior is disjoint from $G_u$ or a loop that intersects $G_u$ in a single point. Thus
\[
2\Delta \chi \geq V
\]
with equality if and only if each component of $H_{us}$ is topologically a single edge $E$ linear. The initial endpoints $V_L$ of the edges in $E_L$ have valence one in $G_u$. We denote the cardinality of $V_L$ by $V_L$. Lemma 8.8 implies that
\[
\Delta R \leq V_L + 1.
\]
Note also that if $V = V_L$ then $\delta = 0$. Thus $V - \delta \geq V_L$ and
\[
\Delta R \leq V_L + 1 \leq V + 1 - \delta.
\]
If $C_{us} = 1$ then $\Delta \chi \geq V - C_{us} = V - 1$ and $\Delta \chi \geq 2$ imply that
\[
2\Delta \chi \geq V + 1
\]
with equality only if $\Delta \chi = 2$ and $V = 3$. Thus
\[
2\Delta \chi - \Delta R - \delta \geq (V + 1) - (V + 1 - \delta) - \delta = 0
\]
with equality only if $\Delta \chi = 2$, $V = 3$ and $\Delta R + \delta = 4$. To complete the proof in the $C_{us} = 1$ case, assume that equality holds. In the $\delta = 0$ case, $3 = V \geq V_L \geq \Delta R - 1 = 3$ so $V = V_L = 3$ and $\Delta R = 4$. Items (1) and (2) of Notation 8.4 follow from the fact that $V = V_L = 3$. Since $\Delta \chi = V - C_{us}$, $H_{us}$ is contractible; being attached to $G_u$
in three places, it is topologically either a triad or a pair of arcs joined at a point. In the former case the unique valence three vertex of \( H_{us} \) must be the base point of both a legal turn in \( H_s \) and an illegal turn in \( H_s \) and so is disjoint from \( G_{s-1} \). The elements of \( \mathcal{V}_L \) are fixed points and so are not contained in any zero strata. The remaining vertices in \( H_s \), if any, have valence two and have their links entirely contained in \( H_s \). There is no loss in erasing these vertices. Once this done, there are no zero strata in \( H_{us} \) so \( s = u + 1 = r + 4 \). Items (3a) and (3b) are immediate and (3c) follows from Lemma 8.8(4) and Theorem 2.17-(N). We have now verified that equality in the \( \delta = 0 \) case corresponds to case (a) of the lemma. In the \( \delta = 1 \) case, \( 3 = V > V_L > \Delta R - 1 = 2 \). The same argument shows that this corresponds to case (b). This completes the proof when \( C_{us} = 1 \).

For \( C_{us} > 1 \) we need another estimate on \( \Delta R \), namely

\[
C_{us} > 1 \Rightarrow \Delta R + \delta \leq V + 2 - C_{us}.
\]

From this, the lemma is completed by

\[
2\Delta \chi - \Delta R - \delta > V - (V + 2 - C_{us}) = C_{us} - 2 \geq 0.
\]

It remains therefore to prove the estimate and for this we show that there are enough relations between the almost invariant subgraph that contains \( H_{us} \) and the almost invariant subgraphs determined by the linear edges corresponding to \( \mathcal{V}_L \).

Let \( X_1, \ldots, X_d \) be the components of \( H_{us} \) that are disjoint from \( G_r \) and intersect \( V \) in a subset of \( \mathcal{V}_L \). Define a graph \( Z \) with one vertex \( z_p \) for each \( X_p \) and one additional vertex \( z \) representing \( H_s \cup G_r \). There is at most one edge connecting any pair of vertices. The edges of \( Z \) are defined as follows. Suppose that \( \mu \) is an edge of \( H_s \) or a connecting path in \( H_{us} \) and that there is a term \( \nu \subset G_u \) in the QE-splitting of \( f(\mu) \) that has exactly one endpoint in \( X_p \). After reversing the orientation on \( \nu \) if necessary, the initial edge of \( \nu \) is a linear edge \( E'_p \subset X_p \) and either \( \nu = E'_p \) or \( \nu \) is a quasi-exceptional subpath. If \( \nu = E'_p \) then \( E'_p \) belongs to the same almost invariant subgraph as \( H_s \). If \( \nu \) is quasi-exceptional and the terminal edge of \( \nu \) is contained in \( G_r \) then there is a linear relation between the coefficients associated to the almost invariant subgraph containing \( E'_p \), the almost invariant subgraph containing \( H_s \) and the almost invariant subgraph containing a stratum of \( G_r \). In both of these cases, \( Z \) has exactly one edge connecting \( z_p \) to \( z \). Otherwise, \( \nu \) is quasi-exceptional and the terminal edge of \( \nu \) is some \( E'_q \subset X_q \). In this case, there is a linear relation between the coefficients associated to the almost invariant subgraph containing \( E'_p \), the almost invariant subgraph containing \( H_s \) and the almost invariant subgraph containing \( E'_q \); \( Z \) has exactly one edge connecting \( z_p \) to \( z_q \).

Let \( a \) be an admissible \( M_s \)-tuple. If there is an edge connecting \( z_p \) to \( z \) then the coordinate of \( a \) corresponding to the almost invariant subgraph containing \( E_p^* \) is
determined by the coordinates of \( a \) corresponding to the almost invariant subgraph containing \( H_s \) and to the almost invariant subgraphs containing the strata of \( G_r \). Thus one does not need to count both \( E'_p \) and \( H_s \) when estimating \( \Delta_i R \). Similarly, if \( z_p \) and \( z_q \) belong to the same component of \( Z \) then the coordinate of \( a \) corresponding to the almost invariant subgraph containing \( E'_p \) is determined by the coordinates of \( a \) corresponding to the almost invariant subgraph containing \( H_s \) and to the almost invariant subgraph containing \( E'_q \). Thus one does not need to count both \( E'_p \) and \( E'_q \) when estimating \( \Delta R \). In both cases each edge of \( Z \) allows us to improve our estimate

\[
\Delta R \leq V_L + 1 \text{ by lowering the right hand side by one.}
\]

Let \( |Z| \) be the number of edges in \( Z \) and note that \( V - V_L \geq C_{us} - d \). Thus

\[
\Delta R + \delta \leq V_L + 1 - |Z| + \delta
\]

\[
= V - (V - V_L) - |Z| + 1 + \delta
\]

\[
\leq V - C_{us} + d - |Z| + 1 + \delta
\]

and it suffices to show that \( d + \delta - |Z| \leq 1 \).

Proof of Theorem 8.6 After reordering the strata of the filtration we may assume that there exists \( 1 \leq k < K \) and \( k = l_0 < l_1 < \cdots < l_K = N \) such that

- \( H_j \) is a non-contractible component of \( G_j \) if and only if \( j \leq k \). In particular, 
  \( G_k \) has no valence one vertices.

and such that the following hold for all \( 1 \leq i \leq K \):

- \( G_{l_i} \) has no valence one vertices. In particular, \( G'_{l_i} \) does not deformation retract to \( G'_{l_{i-1}} \).
- If \( l_{i-1} \leq j < l_i \) and \( H_j \) is irreducible then \( G_j \) deformation retracts to \( G_{l_{i-1}} \).
For $1 \leq i \leq K$, let $\Delta_i R = R_i - R_{i-1}$, let $\Delta_i \chi = \chi(G_{l_{i-1}}) - \chi(G_i)$, let $\hat{H}_i = \bigcup_{j=i-1+1}^i H_j$ and let $\delta_i = 1$ if there is a vertex $v \in G_{l_{i-1}}$ and a fixed direction at $v$ determined by an edge of $\hat{H}_i$ and $\delta_i = 0$ otherwise.

The following sublemma is an easy extension of Lemma 8.9. We separate it out of the proof for easy reference.

**Sublemma 8.10.** Assume notation as above. For $1 \leq i \leq K$,

$$\Delta_i R \leq 2\Delta_i \chi - \delta_i$$

with equality only if one of the following holds.

(a) $\hat{H}_i$ is an FPS subgraph and $\delta_i = 0$.

(b) $\hat{H}_i$ is a partial FPS subgraph and $\delta_i = 1$.

(c) $\hat{H}_i$ is a single linear edge and $\delta_i = 1$.

(d) $\hat{H}_i$ is a pair of linear edges with a common initial vertex and $\delta_i = 0$.

**Proof.** If $\hat{H}_i$ contains an EG stratum $H_j$ then $j = l_i$ because $G_j$ does not deformation retract to $G_{l_{i-1}}$. In this case, the sublemma follows from Lemma 8.9. If no stratum $H_j$ of $\hat{H}_i$ is EG then each $H_j$ is a single edge $E_j$ and $l_i$ equals $l_{i-1} + 1$ or $l_{i-1} + 2$. In both cases $\Delta_i \chi = 1$. If $l_i = l_{i-1} + 1$ then $\delta_i = 1$; if $l_i = l_{i-1} + 2$ then $\delta_i = 0$. Lemma 8.8 implies that if $l_i = l_{i-1} + 1$ then $\Delta_i R \leq 1$ with equality corresponding to (c) and that if $l_i = l_{i-1} + 2$ then $\Delta_i R \leq 2$ with equality corresponding to (d). \hfill \Box

The sublemma implies that

$$R_K - R_k = \sum_{i=1}^K \Delta_i R \leq \sum_{i=1}^K (2\Delta_i \chi - \delta_i) = 2\chi_k - 2\chi_K - \sum_{i=1}^K \delta_i.$$ 

Denote $|\chi(G_k)|$ by $c$. Since each component of $G_k$ is a single stratum and the restriction of $f$ to a rank one component of $G_k$ is the identity,

$$R_k \leq c$$

with equality if and only each component of $G_k$ has rank one or two. Thus

$$2n - 3 - R_k \leq 2(n - 1 - c) - \sum_{i=1}^K \delta_i \leq 2n - 2 - 2c - \sum_{i=1}^K \delta_i$$

50
implies that
\[ c \leq 2c - R_k \leq 1 - \sum_{i=1}^{K} \delta_i. \]

If \( c = 1 \) then each \( \delta_i = 0 \), which implies by Theorem 2.17-(PER) that no component of \( G_k \) has rank one. It follows that \( G_k = G_1 \) has rank two. Moreover, each inequality in the above displayed equations are equalities. The proposition in this case now follows from the sublemma.

If \( c = 0 \) then there is at most one non-zero \( \delta_i \). Together these imply that \( G_k = G_1 \) has rank one and that there is exactly one non-zero \( \delta_i \). If \( \delta_1 = 1 \) then the sublemma completes the proof; see Remark 8.3. Suppose then that \( \delta_i = 1 \) for \( i > 2 \). We will modify \( f: G \rightarrow G \), arranging that \( \delta_1 = 1 \) for the new homotopy equivalence.

Let \( v \) and \( E_1 \) be the unique vertex and edge in \( G_1 \) and let \( E_i \) be the edge in \( H_i \) that determines a fixed direction pointing out of \( G_1 \). Since \( \delta_1 = 0 \), \( G_2 \) is a single edge \( E_2 \) satisfying \( f(E_2) = E_2E_1^{d_2} \) for some \( d_2 \neq 0 \). The link \( L(G,v) \) of \( v \) in \( G \) consists of both ends of \( E_1 \), the initial end of \( E_i \) and the terminal ends of some linear edges \( E_j \), including \( E_2 \). For each such \( E_j \) there is a non-trivial closed path \( u_j \subset G_{j-1} \) such that \( f(E_j) = E_ju_j \).

Create a new graph \( G' \) by replacing \( v \) with a pair of vertices \( v_1 \) and \( v_2 \), adding a new edge \( E_0 \) with one endpoint at \( v_1 \) and the other at \( v_2 \) and partitioning \( L(G,v) \) into \( L(G',v_1) \cup L(G',v_2) \) as follows. Both ends of \( E_1 \) belong to \( L(G',v_1) \). The initial endpoint of \( E_i \) is in \( L(G',v_2) \). If the terminal end of \( E_j \) is contained in \( L(G,v) \) then it is assigned to \( L(G',v_1) \) if \( u_j = E_1 \) and to \( L(G',v_2) \) otherwise.

The map \( f': G' \rightarrow G' \) induced by \( f: G \rightarrow G \) satisfies the conclusions of Theorem 2.17 but has an invariant forest, namely the single edge \( E_0 \). Orient \( E_0 \) so that \( v_1 \) is its terminal edge and define \( u_0 \) to be the trivial path at \( v_1 \). With the exception of \( E_1 \), each element of \( L(G,v_1) \) is the terminal endpoint of an edge \( E_k \) satisfying \( f(E_k) = E_kE_1^{d_k} \). Define a new homotopy equivalence \( g: G' \rightarrow G' \) by replacing \( d_k \) with \( d_k - d_2 \). Note that \( f' \) and \( g \) are freely homotopic and so represent the same element of \( \text{Out}(F_n) \). We have changed the invariant forest from \( E_0 \) to \( E_2 \). Finally, modify \( g \) by collapsing \( E_2 \) to a point. We are now back to the case that \( \delta_2 = 1 \).

**Proof of Theorem 8.7** The proof is a variation on that of Theorem 8.6. No changes are required in the proof up through the sublemma so we do not repeat that here. The rest of the proof follows.

The sublemma implies that
\[ R_K - R_k = \sum_{i=1}^{K} \Delta_i R \leq \sum_{i=1}^{K} (2\Delta_i - \chi - \delta_i) = 2\chi - 2\chi_K - \sum_{i=1}^{K} \delta_i. \]

Denote \(|\chi(G_k)|\) by \( c \). Since each component of \( G_k \) is a single stratum and the restriction of \( f \) to a rank one component of \( G_k \) is the identity,
\[ R_k \leq c \]
with equality if and only each component of $G_k$ has rank one or two. Thus

$$2n - 4 - R_k \leq 2(n - 1 - c) - \sum_{i=1}^{K} \delta_i$$

$$= 2n - 2 - 2c - \sum_{i=1}^{K} \delta_i$$

which implies that

$$c \leq 2c - R_k \leq 2 - \sum_{i=1}^{K} \delta_i.$$  

If some component of $G_k$ has rank three then $R_k < c$ and the last displayed inequality is strict $c < 2 - \sum_{i=1}^{K} \delta_i \leq 2$ which is impossible. Thus each component of $G_k$ has rank one or two. Since IA$_2$ is trivial [Nie24], no component of $G_k$ can have rank two. (Rank two fixed subgraphs exist but they are not composed of a single stratum.)

We may therefore assume that $G_k$ is a union of fixed circles. Each of these circles represents a non-trivial homology class and so can not be the axis associated to any linear edge. We claim that $f$ restricts to the identity on the component $C$ of $G_{l_1}$ that has rank greater than one. If not, then $C$ is obtained from a fixed circle by adding one EG stratum and perhaps some zero strata. In this case, $R_{l_1} \leq 1$ while $C$ has rank at least three by Lemma 3.22 of [BFH00]. It follows that $2\chi(G_{l_1}) - R_{l_1} \geq 3$. The sublemma then implies that $2\chi(G_K) - R_K \geq 3$ which contradicts the fact that $\chi(G_K) = n - 1$ and $R_K = 2n - 4$. This verifies the claim. This same argument proves that $C$ has rank two and that $k = 1$. The sublemma completes the proof.  

9 Two Families of Abelian Subgroups

We now return to the simplest examples of maximal rank abelian subgroups, those that are rotationless, that have linear growth and that have only one axis. We prove that these subgroups and their standard generators can be characterized using only algebraic (as opposed to dynamical systems) properties. These results are needed in the calculation [FHa] of the commensurator of $\text{Out}(F_n)$.

We begin by relating the rank of $\mathcal{A}(\psi)$ to the dynamical properties of $\psi$ in a special case.

**Lemma 9.1.** Suppose that $A$ is a maximal rank rotationless abelian subgroup of $\text{Out}(F_n)$ or IA$_n$, that $\psi \in A$ and that $\mathcal{A}(\psi)$ has rank one. Then either $\mathcal{L}(\psi)$ has exactly one element and $\psi$ has no axes or $\mathcal{L}(\psi) = \emptyset$ and $\psi$ has exactly one axis and that axis has multiplicity one.
Proof. It suffices to show that \( \psi \) has a representative \( g : G' \rightarrow G' \) with exactly one non-fixed stratum.

By Lemma 5.3 there exists a rotationless \( \phi \) so that \( A \subset \mathcal{A}(\phi) \). Choose \( f : G \rightarrow G \) and \( \emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G \) that represent \( \phi \) as in Theorem 2.17. Theorem 7.1 implies that \( D_R(\phi) \cap D_R(\phi^{-1}) \) is finite index in \( \mathcal{A}(\phi) = \mathcal{A}(\phi^{-1}) \). After replacing \( \psi \) with an iterate, we may assume that \( \psi \in D_R(\phi) \cap D_R(\phi^{-1}) \).

Proposition 8.6 and Proposition 8.7 imply that if \( \Lambda \in \mathcal{L}(\phi) \) corresponds to an EG stratum \( H \), then both ends of every leaf of \( \Lambda \) intersect \( H \), infinitely often. By Lemma 3.1.15 of [BFH00] each leaf of \( \Lambda \) is dense in \( \Lambda \). In other words \( \Lambda \) is minimal. The symmetric argument applied to \( \phi^{-1} \) shows that every element of \( \mathcal{L}(\phi) \cup \mathcal{L}(\phi^{-1}) \), and hence (Lemma 5.7) every element of \( \mathcal{L}(\psi) \), is minimal.

We next prove that there is no proper free factor system \( \mathcal{F} \) that carries each element of \( \mathcal{L}(\phi) \cup \mathcal{L}(\phi^{-1}) \) and each \( \phi \)-invariant conjugacy class by assuming that there is such an \( \mathcal{F} \) and arguing to a contradiction. By Proposition 2.17 there exists \( f' : G' \rightarrow G' \) and \( \emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G \) representing \( \phi \) such that \( \mathcal{F} = \mathcal{F}(G_r) \) for some \( G_r \). Let \( p : G \rightarrow G' \) be the quotient map that collapses each invariant tree to a point. The homotopy equivalence \( f' : G' \rightarrow G' \) induced by \( f : G \rightarrow G \) satisfies the conclusions of Theorem 2.17 and has no invariant forests. Proposition 8.6 and Proposition 8.7 imply that \( p(G_r) = G' \). Equivalently, \( f'(G \setminus G_r) \) is the identity. But this implies that \( D_R(\mathcal{F}\mathcal{F}) \) has the same rank as \( D_R(\phi) \), which contradicts the fact that the maximal rank of an abelian subgroup of \( \text{Out}(F_n) \) [resp. \( \text{IA}_n \)] is strictly larger than of a proper free factor system of \( \text{Out}(F_n) \) [resp. \( \text{IA}_n \)].

Since \( \mathcal{L}(\psi) \subset \mathcal{L}(\phi) \cup \mathcal{L}(\phi^{-1}) \) and since each \( \phi \)-invariant conjugacy class is also \( \psi \)-invariant (Lemma 6.10), no proper free factor system carries each element of \( \mathcal{L}(\psi) \) and each \( \psi \)-invariant conjugacy class. These two facts, the minimality of elements of \( \mathcal{L}(\psi) \) and the absence of a free factor system as above, imply that each non-fixed stratum in a representative \( g : G' \rightarrow G' \) of \( \psi \) is either linear or EG and that the rank of \( D_R(\psi) \) is equal to the number of non-fixed strata of \( g : G' \rightarrow G' \). As this also equals the rank of \( \mathcal{A}(\psi) \) the lemma follows. \( \square \)

Let \( G \) be the the rose with \( n - 2 \) of its edges subdivided into two edges. Thus there are edges \( E_1, \ldots, E_{2n-2} \) and vertices \( v_1, \ldots, v_{n-1} \) with \( v_1 \) the terminal vertex of all edges and the initial edges of \( E_1 \) and \( E_2 \) and with \( v_k \) the initial vertex of \( E_{2k-1} \) and \( E_{2k} \) for \( 2 \leq k \leq n - 1 \).

For \( 1 \leq i \leq 2n - 3 \), define \( f_i : G \rightarrow G \) by \( E_{i+1} \mapsto E_{i+1}E_1 \). Choose a basis \( x_1, \ldots, x_n \) for \( F_n \) and a marking on \( G \) that identifies \( x_j \) with the \( j \)-th loop of \( G \). The elements \( \eta_i \in \text{Out}(F_n) \) determined by \( f_i \) are a basis for an abelian subgroup \( A_1 \) of rank \( 2n - 3 \). If \( i = 2k - 2 \) for \( k \geq 2 \) then \( \tilde{\eta}_i \) is defined by \( x_k \mapsto x_kx_1 \). If \( i = 2k - 1 \) for \( k \geq 2 \) then \( \tilde{\eta}_i \) is defined by \( x_k \mapsto x_1x_k \). The remaining element \( \tilde{\eta}_i \) is defined by \( x_2 \mapsto x_2x_1 \). Borrowing notation from [FHa] we say that \( A_1 \) is the type \( E \) subgroup associated to the basis \( x_1, \ldots, x_n \) and that \( \eta_1, \ldots, \eta_{2n-3} \) are its standard generators.

Remark 9.2. It is not hard to check (see for example Lemma 2.13 of [FHa]) that
η_1 is conjugate to each η_j and to η_jη_l if \{j, l\} \neq \{2k, 2k + 1\}. Corollary 5.5 and Lemma 4.5 of [FHa] imply that \mathcal{A}(η_1) has rank one. This explains the hypothesis in the next lemma.

**Lemma 9.3.** Suppose that φ_1, \ldots, φ_{2n-3} are a basis for an abelian subgroup \(\mathcal{A}\) of \(\text{Out}(F_n)\), \(n \geq 3\), that each \(\mathcal{A}(φ_j)\) has rank one and that \(\mathcal{A}(φ_jφ_l)\) has rank one if \{j, l\} \neq \{2k, 2k + 1\}. Then there is a basis \(x_1, \ldots, x_n\) for \(F_n\), standard generators \(η_j\) of the type E subgroup associated to this basis, and \(s, t > 0\) so that \(φ_j^s = η_j^t\).

**Proof.** After replacing each \(φ_j\) with \(φ_j^s\) for some fixed \(s \geq 1\) we may assume by Lemma 5.3 that there exists a rotationless \(θ \in \mathcal{A}\) so that each \(φ_j \in \mathcal{A}(θ)\). Choose \(f : G \to G\) representing \(θ\) as in Theorem 2.17. The coordinates of \(Ω^θ : \mathcal{A}(θ) \to \mathbb{Z}^{2n-3}\) (Definition 7.3) are in one to one correspondence with the non-fixed irreducible strata of \(f : G \to G\) representing \(θ\) and so correspond to linear edges and EG strata as described in Proposition 8.6.

For each \(φ_j\) there exists \(φ_j' \neq φ_j\) such that \(\mathcal{A}(φ_jφ_j')\) has rank one.

Suppose that \(ψ \in \mathcal{A}(θ)\), that \(ω_i\) is a coordinate of \(Ω^θ\) and that \(ω_i(ψ) \neq 0\). If \(ω_i = PF_Λ\) then \(Λ \in \mathcal{L}(θ) \cup \mathcal{L}(θ^{-1})\) by Corollary 3.3.1 of [BFH00]. If \(ω_i\) corresponds to a linear edge with associated axis \([c]_u\) then \([c]_u\) is an axis for \(ψ\); if \(ω_r\) also corresponds to a linear edge with associated axis \([c]_u\) and if \(ω_i(ψ) \neq ω_r(ψ)\) then \([c]_u\) is an axis for \(ψ\) with multiplicity greater than one. Lemma 9.1 therefore implies that for each \(φ_j\) the coordinates of \(Ω^θ(φ_j)\) takes on a single non-zero value and that if more than one coordinate takes this value then all such coordinates come from linear edges associated to the same axis. The same holds true for the coordinates of \(Ω^θ(φ_jφ_j')\).

Suppose that \(ω_i = PF_Λ\) and that \(ω_i(φ_j) \neq 0\). At least one of \(ω_i(φ_j)\) or \(ω(φ_jφ_j')\) is non-zero, say \(ω_i(φ_j')\). Then \(Ω^θ(φ_j)\) and \(Ω^θ(φ_j')\) are contained in a cyclic subgroup of \(\mathbb{Z}^{2n-3}\) in contradiction to the fact that \(φ_j\) and \(φ_j'\) generate a rank two subgroup and the injectivity of \(Ω^θ\). We conclude that each coordinate of \(Ω^θ\) corresponds to a linear edge of \(f : G \to G\).

Minor variations on this argument show that all linear edges correspond to the same axis, that only one coordinate of \(Ω^θ(φ_j)\) can be non-zero and that the non-zero value \(t\) that is taken is independent of \(j\). The details are left to the reader. The lemma now follows from the explicit description of \(f : G \to G\) given by Proposition 8.6 and the definition of \(Ω^θ\).

There is an analogous result for \(IA_n\). For the model subgroup, we use the same marked graph \(G\) as in the definition of type E subgroups. Choose a closed path in \(G_2\) based at \(v_1\) that forms a circuit and determines a trivial element of homology. For \(1 \leq i \leq 2n - 4\) define \(f_i : G \to G\) by \(E_{i+2} \mapsto E_{i+2}w\). The elements \(μ_{i,w} \in \text{Out}(F_n)\) determined by \(f_i\) are a basis for an abelian subgroup \(A_w\) of \(IA_n\) with rank \(2n - 4\). We think of \(w\) as both a path in \(G_2\) and an element of the free factor \(⟨x_1, x_2⟩\). If \(i = 2k - 5\) then \(μ_{i,w}\) is defined by \(x_k \mapsto x_kw\) and if \(i = 2k - 4\) then \(η_i\) is defined by \(x_k \mapsto \bar{w}x_k\). Borrowing notation from [FHa] we say that \(A_w\) is the type C subgroup
associated to \( w \) and to the basis \( x_1, \ldots, x_n \) and that \( \mu_{1,w}, \ldots, \mu_{2n-4,w} \) are its standard generators.

**Lemma 9.4.** Suppose that \( \phi_1, \ldots, \phi_{2n-4} \) are a basis for an abelian subgroup of \( \text{IA}_n \), \( n \geq 4 \), that each \( \mathcal{A}(\phi_j) \) has rank one and that \( \mathcal{A}(\phi_j \phi_l) \) has rank one if \( \{j, l\} \neq \{2k, 2k + 1\} \). Then there exists a basis \( x_1, \ldots, x_n \) for \( F_n \), a homologically trivial element \( w \in \langle x_1, x_2 \rangle \) and standard generators \( \eta_j \) of the type C subgroup associated to \( w \) and this basis, and \( s, t > 0 \) so that \( \phi_j^s = \eta_j^t \).

**Proof.** We have assumed that \( n \geq 4 \) so that for all \( \phi_j \) there exists \( \phi_{j'} \) such that \( \mathcal{A}(\phi_j \phi_{j'}) \) has rank one. Otherwise the proof of Lemma 9.3 carries over to this context without modification, \( w \) representing the unique axis of the elements \( \phi_j \).

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