CORRELATIONS OF NEARBY LEVELS INDUCED BY A RANDOM POTENTIAL

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Abstract

We consider a Hamiltonian $H$ which is the sum of a deterministic part $H_0$ and of a random potential $V$. For finite $N \times N$ matrices, following a method introduced by Kazakov, we derive a representation of the correlation functions in terms of contour integrals over a finite number of variables. This allows one to analyse the level correlations, whereas the standard methods of random matrix theory, such as the method of orthogonal polynomials, are not available for such cases. At short distance we recover, for an arbitrary $H_0$, an oscillating behavior for the connected two-level correlation.

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1 Introduction

Let us first recall the results for the correlations between two eigenvalues for the simple unitary ensemble, in which the full Hamiltonian is treated as random. In the simplest Gaussian ensemble one considers $N \times N$ random Hermitian matrices $H$ with probability distribution

$$P(H) = \frac{1}{Z} \exp\left(-\frac{N}{2} \text{Tr} H^2\right)$$ (1.1)

Two kinds of universal correlations between eigenvalues are known to be present for such problems: a) a short-distance universal oscillatory behavior; b) a finite distance universality of smoothed correlations.

Let us review these two properties. The density of eigenvalues and the two-level correlation function are defined as

$$\rho(\lambda) = \langle \frac{1}{N} \text{Tr} \delta(\lambda - H) \rangle$$ (1.2)

and

$$\rho^{(2)}(\lambda, \mu) = \langle \frac{1}{N} \text{Tr} \delta(\lambda - H) \frac{1}{N} \text{Tr} \delta(\mu - H) \rangle$$ (1.3)

The correlation function, when $\lambda$ and $\mu$ are arbitrary, has a complicated, non-universal, oscillatory behavior. It simplifies and is independent of the probability distribution of $H$, when

a) $\lambda - \mu$ is small, $N$ is large, and the scaling variable

$$x = \pi N(\lambda - \mu)\rho\left(\frac{1}{2}(\lambda + \mu)\right)$$ (1.4)

is held finite. Then one finds [1]

$$\rho^{(2)}(\lambda, \mu) \approx \frac{1}{N} \delta(\lambda - \mu) \rho(\lambda) - \rho(\lambda)\rho(\mu) \frac{\sin^2 x}{x^2}$$

$$\approx \frac{1}{N} \delta(\lambda - \mu) \rho(\lambda) - \frac{1}{\pi^2 N^2} \frac{\sin^2[\pi N(\lambda - \mu)\rho(\frac{\lambda + \mu}{2})]}{(\lambda - \mu)^2}$$ (1.5)

b) Away from this short-distance region, for arbitrary $\lambda$ and $\mu$, the correlations simplify only if one "smooths" the oscillations. This is what one
usually does, if one lets N go to infinity first in the resolvent, before returning
to the real axis. The result, which is known to be universal, is [2]
\[
\rho^{(2)}_c(\lambda, \mu) = -\frac{1}{2N^2\pi^2} \frac{1}{(\lambda - \mu)^2} \frac{(a^2 - \lambda \mu)}{[(a^2 - \lambda^2)(a^2 - \mu^2)]^{1/2}}
\]
(1.6)
where a is an end point of the support.

There are many equivalent derivations of the property b). They are based
either on orthogonal polynomials [2], or on summing over planar diagrams [3, 4], or solving an integral equation [3, 5]; however the property a) is known
only through the orthogonal polynomials approach [2]. For the generalization
that we have in view in this article, in which the "unperturbed" part of the
Hamiltonian is deterministic, if again for b) a diagrammatic approach still
works [3, 4, 7, 8], we are not aware of any method which would allow us
to study whether a) still holds. To this effect we shall generalize a method,
introduced by Kazakov [9], to the study of correlation functions. It consists
of introducing an external matrix source. It leads to a representation of the
correlation function in terms of contour integrals over two variables for finite
N. We have used already this representation in a previous paper devoted to
random matrices made of complex blocks, i.e. the Laguerre ensemble [10],
but there we have let the source go to zero at the end of the calculation.
Keeping this source finite allows one to deal with an arbitrary deterministic
$H_0$. We shall illustrate it here when the random potential $V$ belongs to the
simple Gaussian unitary ensemble.

2 An external matrix source: deterministic
plus random hamiltonian

We consider a Hamiltonian $H = H_0 + V$, where $H_0$ is deterministic and $V$ is
a random $N \times N$ matrix. The Gaussian distribution $P$ is given by

\[
P(H) = \frac{1}{Z} e^{-\frac{N}{2} \text{Tr} V^2}
\]

\[
= \frac{1}{Z} e^{-\frac{N}{2} \text{Tr}(H^2 - 2H_0H + H_0^2)}
\]
(2.1)

We are thus simply dealing with a Gaussian unitary ensemble modified
by a matrix source $A = -H_0$. Up to a factor the probability distribution for
H is thus
\[ P_A(H) = \frac{1}{Z_A} \exp\left( -\frac{N}{2} \text{Tr} H^2 - N \text{Tr} AH \right) \] (2.2)

Let us first show how one deals with the density of states \( \rho(\lambda) \). It is the Fourier transform of the average "evolution" operator
\[ U_A(t) = \langle \frac{1}{N} \text{Tr} e^{itH} \rangle \] (2.3)

and \( \rho(\lambda) \) is
\[ \rho(\lambda) = \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{-it\lambda} U_A(t) \] (2.4)

We integrate first over the unitary matrix \( \omega \) which diagonalizes \( H \), and without loss of generality we may assume that \( A \) is a diagonal matrix with eigenvalues \( (a_1, \cdots, a_N) \). This is done by the well-known Itzykson-Zuber integral [1],
\[ \int d\omega \exp(\text{Tr} A \omega B \omega^\dagger) = \text{det}(\exp(a_i b_j)) / \Delta(A) \Delta(B) \] (2.5)

where \( \Delta(A) \) is the Van der Monde determinant constructed with the eigenvalues of \( A \):
\[ \Delta(A) = \prod_{i<j} (a_i - a_j) \] (2.6)

We are then led to
\[ U_A(t) = \frac{1}{Z_A \Delta(A)} \frac{1}{N} \sum_{a_1}^{N} \int dr_1 \cdots dr_N e^{itr_a} \Delta(r_1, \cdots, r_N) \times \exp\left( -\frac{N}{2} \sum r_i^2 - N \sum a_i r_i \right) \] (2.7)

The normalization is
\[ U_A(0) = 1 \] (2.8)

The integration over the \( r_i \) may be done easily, if we note that
\[ \int dr_1 \cdots dr_N \Delta(r_1, \cdots, r_N) e^{\frac{N}{2} \sum r_i^2 - N \sum b_i r_i} = \Delta(b_1, \cdots, b_N) e^{\frac{N}{2} \sum b_i^2} \] (2.9)
If we use this, with
\[ b_i = a_i - \frac{it}{\sqrt{N}} \delta_{\alpha,i} \]  
we obtain
\[ U_A(t) = \frac{1}{N} \sum_{\alpha=1}^{N} \prod_{\gamma \neq \alpha} \left( \frac{a_{\alpha} - a_{\gamma} - \frac{it}{\sqrt{N}}}{a_{\alpha} - a_{\gamma}} \right) e^{-\frac{t^2}{N} - it a_{\alpha}} \]  
(2.11)

The sum over \( N \) terms may be replaced by a contour-integral in the complex \( u \) plane,
\[ U_A(t) = -\frac{1}{it} \oint \frac{du}{2\pi i} \prod_{\gamma=1}^{N} \left( \frac{u - a_{\gamma} - \frac{it}{\sqrt{N}}}{u - a_{\gamma}} \right) e^{-itu - \frac{t^2}{2N}} \]  
(2.12)

The contour of integration encloses all the eigenvalues \( a_{\gamma} \). If we let all the \( a_{\gamma} \) go to zero, we obtain
\[ U_0(t) = -\frac{1}{it} e^{-\frac{t^2}{2N}} \oint \frac{du}{2\pi i} e^{-itu} (1 - \frac{it}{N u})^N \]  
(2.13)

From this exact representation for finite \( N \), it is immediate to recover all the well-known properties, the semi-circle law, or the more subtle edge behavior of the density of states. Since the result is similar to the Laguerre case that we have discussed in an earlier paper \[10\], we shall not discuss it here.

For the two-level correlation function, \( \rho^{(2)}(\lambda, \mu) \) is obtained from the Fourier transform \( U_A(t_1, t_2) \),
\[ \rho^{(2)}(\lambda, \mu) = \int \int dt_1 dt_2 (2\pi)^2 e^{-it_1 \lambda - it_2 \mu} U_A(t_1, t_2) \]  
(2.14)

where \( U_A(t_1, t_2) \) is
\[ U_A(t_1, t_2) = < \frac{1}{N} Tr e^{it_1 H} \frac{1}{N} Tr e^{it_2 H} > \]  
(2.15)

The normalization conditions are
\[ U_A(t_1, t_2) = U_A(t_2, t_1) \]
\[ U_A(t_1, 0) = U_A(t_1) \]
\[ U_A(0) = 1 \]  
(2.16)

Dealing with \( U_A(t_1, t_2) \) is also simple; we have exposed the technique in more details in the almost similar problem of the Laguerre ensemble\[10\]. After
performing the Itzykson-Zuber integral over the unitary group as in \((2.5)\), we obtain through the same procedure,

\[
U_A^{(2)}(t_1, t_2) = \frac{1}{N^2} \sum_{\alpha_1, \alpha_2=1}^N \prod_{i=1}^N dr_i \frac{\Delta(r)}{\Delta(A)} e^{-N \sum(\frac{1}{2} r_i^2 + t_1 r_{\alpha_1} + t_2 r_{\alpha_2})} \tag{2.17}
\]

After integration over the \(r_i\), we obtain

\[
U_A(t_1, t_2) = \frac{1}{N^2} \sum_{\alpha_1, \alpha_2} \prod_{i<j} (a_i - a_j - \frac{it_1}{N}(\delta_{i,\alpha_1} - \delta_{j,\alpha_1}) - \frac{it_2}{N}(\delta_{i,\alpha_2} - \delta_{j,\alpha_2})) \prod_{i<j} (a_i - a_j) \times e^{-it_1 a_{\alpha_1} - it_2 a_{\alpha_2} - \frac{t_1^2}{2N} - \frac{t_2^2}{2N} - \frac{it_1}{N} \delta_{\alpha_1, \alpha_2}} \tag{2.18}
\]

The terms of this double sum in which \(\alpha_1 = \alpha_2\) are written as a single contour integral and their sum is simply \(\frac{1}{N} U_A(t_1 + t_2)\) of \((2.11)\). The Fourier transform of this term becomes

\[
\frac{1}{N(2\pi)^2} \int \int dt_1 dt_2 e^{-it_1 \lambda - it_2 \mu} U_A(t_1 + t_2) = \frac{1}{N} \delta(\lambda - \mu) \rho(\lambda) \tag{2.19}
\]

The remaining part, after the subtraction of the disconnected part, becomes

\[
U_A(t_1, t_2) = -\frac{1}{N^2} \oint \frac{du dv}{(2\pi i)^2} e^{-\frac{u^2}{2N} - \frac{v^2}{2N} - it_1 u - it_2 v} \frac{1}{(u - v)^2} \prod_{\gamma=1}^N (1 - \frac{it_1}{N(u - a_{\gamma})})(1 - \frac{it_2}{N(v - a_{\gamma})}) \tag{2.20}
\]

where the contours are taken around \(u = a_{\gamma}\) and \(v = a_{\gamma}\). If we include also the contour-integration around the pole, \(v = u - \frac{it_1}{N}\), this gives precisely the term \(U_A(t_1 + t_2)\) of \((2.19)\), which contributes to the delta-function part. This coincidence had already been noticed for the Laguerre ensemble \([10]\). We are now in position to study the various properties of this random matrix problem with an arbitrary source \(A\).

3 Large N limit of the density of states

For arbitrary \(A\), the density of state \(\rho(\lambda)\) was first found, in the large \(N\) limit, by Pastur\([12]\). The result may be easily recovered by summing diagrams. Indeed in the large \(N\) limit the leading diagrams are planar, and
the one-particle Green function is the sum of \"rainbow\" diagrams. It follows immediately that the self-energy is proportional to the Green function itself in the large N limit \cite{3}, and this leads at once to Pastur’s result. From the contour-integral representation (2.14), let us show first how to recover this result. The average resolvent \( G(z) \) is written in terms of the evolution operator as

\[
G(z) = \frac{1}{N} \text{Tr} \frac{1}{z - H} >
\]

\[
= i \int_{0}^{+\infty} dt e^{-itz} U_A(t)
\] (3.1)

We substitute (2.12) for \( U_A(t) \) and replace the product

\[
\prod_{\gamma=1}^{N} \left( 1 - \frac{it}{N(u - a_{\gamma})} \right) = \exp \sum_{\gamma=1}^{N} \log \left( 1 - \frac{it}{N(u - a_{\gamma})} \right)
\] (3.2)

by its leading term in the large N limit, namely

\[
\exp(-it\sum_{\gamma=1}^{N} \frac{1}{N} \frac{1}{u - a_{\gamma}})
\] (3.3)

If we define the density of states of the external matrix A

\[
\rho_0(a) = \frac{1}{N} \sum_{\alpha=1}^{N} \delta(a - a_{\alpha})
\] (3.4)

we may write this expression as

\[
\exp(-it \int da \rho_0(a) \frac{1}{u - a})
\] (3.5)

Note that the \"unperturbed\" resolvent

\[
G_0(z) = \frac{1}{N} \text{Tr} \frac{1}{z - H_0} >
\] (3.6)

is related to \( \rho_0 \) by

\[
G_0(z) = \int da \rho_0(a) \frac{1}{u + a}
\] (3.7)
since $H_0 = -A$. We obtain then easily
\[
\frac{\partial G}{\partial z} = \oint \frac{du}{2\pi i u + G_0(u) - z} \tag{3.8}
\]

We have now to specify the contour of integration in the complex $u$-plane. It surrounds all the eigenvalues of $H_0$ and we have to determine the location of the zeroes of the denominator with respect to this contour. Let us return to the discrete form for the equation
\[
u + G_0(u) = z \tag{3.9}
\]
i.e.
\[
u + \frac{1}{N} \sum_{i=1}^{N} \frac{1}{u - \epsilon_i} = z \tag{3.10}
\]
which possesses $(N + 1)$ real or complex roots in the $u$-plane. For $z$ real and large, $N$ of these roots are close to the $\epsilon_i$ and one, which will be denoted $\hat{u}(z)$, goes to infinity with $z$ as
\[
\hat{u}(z) = z - \frac{1}{z} + O\left(\frac{1}{z^2}\right) \tag{3.11}
\]

Therefore, for large $z$, the contour encloses all the roots of (3.10) except $\hat{u}(z)$. When $z$ decreases the contour should not be crossed by any other root of the equation, therefore it is defined by the requirement that only one root remains at its exterior. Therefore it is easier to calculate the integral (3.8) by taking the residues of the singularities outside of the contour, rather than the $N$ poles enclosed by this contour. There are two of them outside; one is $\hat{u}(z)$ and the other one is at infinity (since for large $u$, $G_0(u)$ vanishes). Taking these two singularities we obtain
\[
\frac{\partial G}{\partial z} = 1 - \frac{1}{1 + \frac{dG_0}{d\hat{u}(z)}} = 1 - \frac{d\hat{u}(z)}{dz} \tag{3.12}
\]
The integration gives
\[
G(z) = z - \hat{u}(z) \tag{3.13}
\]
(there is no integration constant since \( G(z) \) vanishes for \( z \) large; note that it does behave as it should as \( \frac{1}{z} \) for \( z \) large). This combined with (3.9) gives Pastur’s self-consistent relation

\[
G(z) = G_0(z - G(z))
\]  

(3.14)

## 4 Universal correlations

In the integral representation (2.20) we may neglect the terms \( t^2/N \) in the large \( N \) limit and replace the products as in (3.5). This gives the large \( N \)-limit of \( U_A^{(2)}(t_1, t_2) \) as

\[
U_A^{(2)}(t_1, t_2) = -\frac{1}{N^2} \int \frac{dudv}{(2\pi i)^2} \frac{1}{(u - v)^2} e^{-it_1(u + \int \rho_0(a) da)} - it_2(v + \int \rho_0(a) da) \quad (4.1)
\]

Noting that

\[
\frac{\partial^2}{\partial z_1 \partial z_2} \ln[\hat{u}(z_1) - \hat{u}(z_2)] = \frac{1}{(\hat{u}(z_1) - \hat{u}(z_2))^2} \frac{d\hat{u}}{dz_1} \frac{d\hat{u}}{dz_2} \quad (4.2)
\]

we obtain, through identical steps, the connected two-particle Green function

\[
G_c^{(2)}(z_1, z_2) = \frac{1}{N} \text{tr} \frac{1}{z_1 - H} \frac{1}{N} \text{tr} \frac{1}{z_2 - H} > c
\]

\[
= -\frac{1}{N^2} \frac{\partial^2}{\partial z_1 \partial z_2} \ln[\hat{u}(z_1) - \hat{u}(z_2)] \quad (4.3)
\]

This result was derived earlier by diagrammatic methods [3], and was used to show that the singularity of the correlations, obtained when \( z_1 \) and \( z_2 \) approach the real axis with opposite imaginary parts, is universal.

However if we want to study the correlation function in the short-distance limit, we cannot use the resolvent any more (since we need to let the imaginary parts of \( z_1, z_2 \) go to zero before \( N \) goes to infinity).

Returning then to (2.20), and making the shifts, \( t_1 \rightarrow t_1 - iuN \), and \( t_2 \rightarrow t_2 - ivN \), the two-level correlation function is remarkably factorized since

\[
\rho_c(\lambda_1, \lambda_2) = \int \frac{dt_1}{2\pi} \int \frac{dv}{2\pi i} \prod_{\gamma=1}^N \frac{a_{\gamma} + it_1}{v + \frac{iu}{N}} e^{-\frac{Nv^2}{2} - \frac{1}{2N} - it_1\lambda_1 - Nv\lambda_2}
\]
\[
\times \int \frac{dt}{2\pi} \oint \frac{du}{2\pi i} \prod_{\gamma=1}^{N} \frac{a_{\gamma} + it}{u - a_{\gamma}} \frac{1}{u + \frac{it}{N}} e^{-\frac{N}{2}u^2 - \frac{t^2}{2N} - itu\lambda_2 - Nu\lambda_1} \\
= -K_{N}(\lambda_1, \lambda_2)K_{N}(\lambda_2, \lambda_1)
\]

This kernel \(K_{N}(\lambda_1, \lambda_2)\) is further simplified by the shift \(t_1 \to t + ivN\),

\[
K_{N}(\lambda_1, \lambda_2) = \int \frac{dt}{2\pi} \oint \frac{dv}{2\pi i} \prod_{\gamma=1}^{N} (1 - \frac{it}{N(v - a_{\gamma})})e^{-\frac{t^2}{2N} - ivt - it\lambda_1 + Nu(\lambda_1 - \lambda_2)}
\]

Note that \(K_{N}(\lambda_1, \lambda_1)\) reduces to the density of states. We replace again the product in (4.4) by its large N-limit, neglect \(\frac{t^2}{N}\) and integrate over \(t\), leading to

\[
\frac{\partial K_{N}}{\partial \lambda_1} = \frac{1}{\pi} \text{Im} \oint \frac{du}{2\pi i} \frac{1}{u + \frac{G_0(u) - \lambda_1 + i\epsilon}{u}} e^{-uy}
\]

with \(y = N(\lambda_1 - \lambda_2)\). Therefore

\[
\frac{\partial K_{N}}{\partial \lambda_1} = \frac{1}{\pi} \text{Im} \frac{d\hat{u}}{d\lambda_1} e^{-y\hat{u}(\lambda_1 - i\epsilon)}
= -\frac{1}{\pi y} \frac{\partial}{\partial \lambda_1} \text{Im} \left(e^{-y\hat{u}(\lambda_1 - i\epsilon)}\right)
\]

Since, from (3.13),

\[
\hat{u}(\lambda_1 - i\epsilon) = \lambda_1 - \text{Re}G(\lambda_1) - i\pi\rho(\lambda_1)
\]

we obtain

\[
K_{N}(\lambda_1, \lambda_2) = -\frac{1}{\pi y} e^{-y[\lambda_1 - \text{Re}G(\lambda_1)]/\sin[y\rho(\lambda_1)]}
\]

Repeating this calculation for \(K_{N}(\lambda_2, \lambda_1)\) we end up, in the large N, finite \(y\) limit, with

\[
\rho_c(\lambda_1, \lambda_2) = -\frac{1}{\pi y^2} \sin^2[\pi y(\lambda_1 + \lambda_2)/2]
\]

Note that this result is independent of \(H_0\) (apart from the scale factor present in the density of states). In the case in which \(H_0\) vanishes it is also independent of the probability distribution of \(V[2]\). It is natural to conjecture that this remains true for \(H_0\) non-zero as well, but we do not know of any method to prove it.
5 Conclusion

The exact expressions for finite $N$ of the correlation functions in terms of contour integrals, have allowed us to study the short-distance correlations for an arbitrary unperturbed Hamiltonian. We are not aware of any other method which could be used for solving this problem. The result is, as expected, a universal short distance behaviour, which depends on $H_0$ only through scale factors.
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