Data Analysis Recipes:
Products of multivariate Gaussians in Bayesian inferences

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Abstract: A product of two Gaussians—or normal distributions—is another Gaussian. That’s a valuable and useful fact! Here we use it to derive a refactoring of a common product of multivariate Gaussians: The product of a Gaussian likelihood times a Gaussian prior, where some or all of those parameters enter the likelihood only in the mean and only linearly. That is, a linear, Gaussian, Bayesian model. This product of a likelihood times a prior pdf can be refactored into a product of a marginalized likelihood (or a Bayesian evidence) times a posterior pdf, where (in this case) both of these are also Gaussian. The means and variance tensors of the refactored Gaussians are straightforward to obtain as closed-form expressions; here we deliver these expressions, with discussion. The closed-form expressions can be used to speed up and improve the precision of inferences that contain linear parameters with Gaussian priors. We connect these methods to inferences that arise frequently in physics and astronomy.

If all you want is the answer, the question is posed and answered at the beginning of Section 3. We show two toy examples, in the form of worked exercises, in Section 4. The solutions, discussion, and exercises in this Note are aimed at someone who is already familiar with the basic ideas of Bayesian inference and probability.

1 Inferences with linear parameters

It is common in physics, astronomy, engineering, machine learning, and many other fields that likelihood functions (probabilities of data given parameters) are chosen to be Gaussian (or normal\(^2\)): One reason is that a likelihood function is basically a noise model, and it is often case that the noise is treated as Gaussian. This assumption for the likelihood function is accurate when the noise model has benefitted from the central limit theorem. This is true, for example, when the noise is thermal, or when the noise is shot noise and the numbers (numbers of detected photons or other particles) are large. Another reason that the likelihood function is often treated as Gaussian is that Gaussians are generally tractable: Many computations we like to perform on Gaussians, like integrals and derivatives and optimizations, have closed-form solutions. Even when we

\(^2\)In this Note, we obey physics and astronomy conventions and refer to the normal pdf as the Gaussian pdf. We realize that it is irresponsible to name things after people when those same things also have generic, descriptive names. On the other hand, “normal” isn’t the finest name either. Perhaps Gaussian pdfs should be called “central” since they are produced by the central limit theorem. Anyway, we will continue with the name “Gaussian” despite our own reservations. We apologize to our reader.
also be written as a pdf factorization:

\[ p(y, \theta | \mathcal{M}) = p(y | \theta, \mathcal{M}) p(\theta | \mathcal{M}) = p(\theta | y, \mathcal{M}) p(y | \mathcal{M}) \]  

where \( p(y, \theta | \mathcal{M}) \) is the joint probability of data \( y \) and parameters \( \theta \) given your model assumptions and hyper parameters (symbolized jointly as \( \mathcal{M} \)),\(^5\) \( p(y | \theta, \mathcal{M}) \) is the likelihood, or probability of data \( y \) given parameters (and assumptions), \( p(\theta | \mathcal{M}) \) is the prior pdf for the parameters \( \theta \), \( p(\theta | y, \mathcal{M}) \) is the posterior pdf for the parameters \( \theta \) given the data, and \( p(y | \mathcal{M}) \) is the pdf for the data, marginalizing out all of the linear parameters (hereafter, we refer to this as the marginal likelihood\(^7\)).

If the likelihood is Gaussian, and the expectation of the data depends linearly on the parameters, and if we choose the prior pdf to also be Gaussian, then all the other pdfs (the joint, the posterior, and the marginalized likelihood) all become Gaussian too. The main point of this Note is that the means and variances of these five Gaussians are all related by simple, closed-form expressions, given below. One consequence of this math is that if you have a Gaussian likelihood function, and if you have a subset of parameters that are linearly related to the expectation of the data, then you can obtain both the posterior pdf \( p(\theta | y, \mathcal{M}) \) and the marginalized likelihood \( p(y | \mathcal{M}) \) with closed-form transformations of the means and variances of the likelihood and prior pdf.

A currently popular data-analysis context in which Gaussian likelihoods are multiplied by Gaussian priors is Gaussian processes (GPs), which is a kind of non-parametric fitting in which a kernel function sets the flexibility of a data-driven model. A full discussion of GPs is beyond don’t use the closed-form solutions, there are many contexts in which Gaussians lead to convex optimizations, providing guarantees to resulting inferences.

It is also common in physics and astronomy that models for data include parameters such that the expectation value for the data (in, say, a set of repeated experiments) is linearly proportional to some subset of the parameters. This is true, for example, when we fit a histogram of Large Hadron Collider events affected by the Higgs boson,\(^3\) where the expected number of counts in each energy bin is proportional to a linear combination of the amplitudes of various backgrounds and some coupling to the Higgs. Another linear-parameter context, for example, arises when we fit for the radial-velocity variation of a star in response to a faint, orbiting companion.\(^4\) In this problem, the expectation of the radial-velocity measurements depends linearly on the binary system velocity and some combination of masses and system inclination (with respect to the line of sight). In both of these cases, there are both linear parameters (like the amplitudes) and nonlinear parameters (like the mass of the Higgs, or the orbital period of the binary-star system). In what follows, we will spend our energies on the linear parameters, though our work on them is in service of learning the nonlinear parameters too, of course.

In Bayesian inference contexts, the “models” to which we are referring are expressions for likelihood functions and prior pdfs; these are the things that will be Gaussians here. Bayes’ theorem is often written as expressions for likelihood functions and prior pdfs; these are the formalisms that lead to convex optimizations, providing guarantees to resulting inferences.

3See, for example, Aad et al. (2012), and Chatrchyan et al. (2012), and references therein.

4See, for an example in our own work, Price-Whelan et al. (2017, 2020). That project and those papers would have been impossible without the speed-ups provided by the expressions derived in this Note. Indeed, the writing of this Note was motivated by the work presented in those papers. (Okay full disclosure: It was motivated in part by a mistake made by DWH in one of those papers!)

5For a tutorial on probability factorizations, see Hogg (2012).

6In this Note, we typeset different mathematical objects according to their mathematical or transformation properties. We typeset vectors (which are column vectors) as \( \mathbf{a}, \mathbf{b}, \mathbf{\theta} \), we typeset variance tensors (which in this case are square, non-negative semi-definite matrices) as \( \mathbf{C}, \mathbf{\Lambda} \), we typeset other matrices (which will in general be non-square) as \( \mathbf{M}, \mathbf{U} \), and we typeset blobs or unstructured collections of information as \( \mathcal{M}, \mathcal{X} \). Related to this typography is an implicit terminology: We distinguish variance tensors from matrices. This distinction is somewhat arbitrary, but the strong constraints on the variance tensors (non-negative, real eigenvalues) make them special beasts, with special geometric properties, like that they can be used as metrics in their respective vector spaces.

7In the case that the problem has no parameters other than the linear parameters \( \theta \), this term, \( p(y | \mathcal{M}) \), is sometimes called the Bayesian evidence or the fully marginalized likelihood.

8We like the free book by Rasmussen and Williams (2005).
the scope of this Note, but excellent discussions abound. The math below can be applied in many GP contexts. Indeed, most linear model fits of the kind we describe below can be translated into the language of GPs, because any noise process that delivers both a prior pdf and a likelihood with Gaussian form is technically identical to a (probably non-stationary) GP. We leave that translation as an exercise to the ambitious reader.

## 2 Marginalization by refactorization

Imagine that we are doing an inference using data \( y \) (which is a \( N \)-dimensional vector, say). We are trying to learn linear parameters \( \theta \) (a \( K \)-dimensional vector) and also nonlinear parameters \( \mathbf{x} \) (an arbitrary vector, list, or blob).\(^9\) Whether we are Bayesian or frequentist, the inference is based on a likelihood function, or probability for the data given parameters

\[
\text{likelihood: } p(y | \theta, \mathbf{x}) .
\]

(2)

Now let’s imagine that the parameters \( \theta \) are either nuisance parameters, or else easily marginalized, so we want to marginalize them out. This will leave us with a lower-dimensional marginalized likelihood function

\[
\text{marginalized likelihood: } p(y | \mathbf{x}) .
\]

(3)

That’s good, but the marginalization comes at a cost: We have to become Bayesian, and we have to choose a prior

\[
\text{prior on nuisance parameters: } p(\theta | \mathbf{x}) .
\]

(4)

This is the basis for the claim\(^{11}\) that Bayesian inference requires a likelihood function, and priors on the nuisance parameters. It does not require a prior on everything, contrary to some statements in the literature.\(^{12}\) We have said “\( p(\theta | \mathbf{x}) \)” because this prior pdf may depend on the nonlinear parameters \( \mathbf{x} \), but it certainly doesn’t have to. Armed with the likelihood and prior—if you want it—you can construct the posterior pdf for the linear parameters

\[
\text{posterior for nuisance parameters: } p(\theta | y, \mathbf{x}) .
\]

(5)

To perform a marginalization of the likelihood, we have two choices. We can either do an integral:

\[
p(y | \mathbf{x}) = \int p(y | \theta, \mathbf{x}) p(\theta | \mathbf{x}) d\theta ,
\]

(6)

where the integral is implicitly over the entire domain of the linear parameters \( \theta \) (or the entire support of the prior). Or we can re-factorize the expression using Bayes’ theorem:

\[
p(y | \theta, \mathbf{x}) p(\theta | \mathbf{x}) = p(\theta | y, \mathbf{x}) p(y | \mathbf{x}) .
\]

(7)

That is, in certain magical circumstances it is possible to do this re-factorization without explicitly doing any integral. When this is true, the marginalization is sometimes far easier than the relevant integral.

\(^9\) If you want a cheat sheet, we come close to performing this translation in Luger et al. (2017).

\(^{10}\) Here, \( \mathbf{x} \) represents the nonlinear parameters and assumptions or hyper parameters. That is, it contains everything on which the linear model is conditioned, including not just nonlinear parameters but also investigator choices. Note our subjectivism here!

\(^{11}\) A claim that perhaps hasn’t been made clearly yet, but will eventually be by at least one of these authors.

\(^{12}\) It is very common for papers or projects with Bayesian approaches to claim that the goal of Bayesian inference is to create posterior pdfs. That isn’t correct. Different Bayesian inferences have different objectives. The fundamental point of Bayesian inference is that consistently held beliefs obey the rules of probability. That, in turn, says that if you want to communicate to others things useful to the updating of their beliefs, you want to communicate about your likelihood. Your posterior pdf isn’t all that useful to them!
The point of this Note is that this magical circumstance arises when the two probability distributions—the likelihood and the prior—are both Gaussian in form, and when the model is linear over the parameters we would like to marginalize over. In detail we will assume

1. the likelihood \( p(y \mid \theta, \mathcal{X}) \) is a Gaussian in \( y \),
2. the prior \( p(\theta \mid \mathcal{X}) \) is a Gaussian in \( \theta \),
3. the mean of the likelihood Gaussian depends linearly on the linear parameters \( \theta \), and
4. the linear parameters \( \theta \) don’t enter the likelihood anywhere other than in the mean.

In equations, this becomes:

\[
p(y \mid \theta, \mathcal{X}) = \mathcal{N}(y \mid M \cdot \theta, C) \tag{8}
\]

\[
p(\theta \mid \mathcal{X}) = \mathcal{N}(\theta \mid \mu, \Lambda) \tag{9}
\]

\[
\mathcal{N}(x \mid m, V) \equiv \frac{1}{\sqrt{|2\pi V|^{1/2}}} \exp\left(-\frac{1}{2} |x - m|^2 \cdot V^{-1} \cdot |x - m| \right) , \tag{10}
\]

where \( \mathcal{N}(x \mid m, V) \) is the multivariate Gaussian pdf\(^{13}\) for a vector \( x \) given a mean vector \( m \) and a variance tensor \( V \), \( M \) is a \( N \times K \) rectangular design matrix (which depends, in general, on the nonlinear parameters \( \mathcal{X} \)), \( C \) is a \( N \times N \) covariance matrix of uncertainties for the data (diagonal if the data dimensions are independent). That is, the likelihood is a Gaussian with a mean that depends linearly on the parameters \( \theta \), and \( \mu \) and \( \Lambda \) are the \( K \)-vector mean and \( K \times K \) variance tensor for the Gaussian prior.

In this incredibly restrictive—but also surprisingly common—situation, the re-factored pdfs \( p(\theta \mid y, \mathcal{X}) \) (the posterior for the linear parameters, conditioned on the nonlinear parameters in \( \mathcal{X} \)) and \( p(y \mid \mathcal{X}) \) (the marginalized likelihood, similarly conditioned) will also both be Gaussian. We will solve this problem for general multivariate Gaussians in spaces of different dimensionality (and units) but the one-dimensional case is illustrated in Figure 1. Obtaining the specific form for the general Gaussian product is the object of this Note.

### 3 Products of two Gaussians

On the internets, there are many documents, slide decks, and videos that explain products of Gaussians in terms of other Gaussians.\(^{14}\) The vast majority of these consider either the univariate case (where the data \( y \) and the parameter \( \theta \) are both simple scalars, which is not useful for our science cases), or the same-dimension case (where the data \( y \) and the parameter vector \( \theta \) are the same length, which never occurs in our applications). Here we solve this problem in the general case:\(^{15}\) The inputs are multivariate (vectors) and the two Gaussians we are multiplying live in spaces of different dimensions. That is, we solve the following problem:

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\(^{13}\)Check out what we did with the “2\(\pi\)” in the determinant in equation (10): We wrote an expression for the multivariate Gaussian that never makes any reference to the dimension \( d \) of the \( x \)-space. Most expressions in the literature have a pesky \( d/2 \) in them, which is ugly and implies some need for code or equations to know the dimension explicitly, even though all the terms (determinant, inner product) are coordinate-free scalar forms. If you use the expression as we have written it here, you never have to explicitly access the dimensions.

\(^{14}\)Two good examples are Roweis (1999), and Petersen and Pedersen (2012). The closest—that we know of—to a discussion with the generality of what is shown here is perhaps our own previous contribution Luger et al. (2017).

\(^{15}\)We solve this general case, but we are not claiming priority in any sense: This mathematics has been understood for many many decades or even centuries. This Note is a pedagogical contribution, not a research contribution.
Problem: Find $K$-vector $a$, $K \times K$ variance tensor $A$, $N$-vector $b$, and $N \times N$ variance tensor $B$ such that
\[
\mathcal{N}(y \mid M \cdot \theta, C)\mathcal{N}(\theta \mid \mu, \Lambda) = \mathcal{N}(\theta \mid a, A)\mathcal{N}(y \mid b, B),
\]
and such that $a$, $A$, $b$, and $B$ don’t depend on $\theta$ at all. Note that $y$ is a $N$-vector, $M$ is a $N \times K$ matrix, $\theta$ is a $K$-vector, $C$ is a $N \times N$ non-negative semi-definite variance tensor, $\mu$ is a $K$-vector, and $\Lambda$ is a $K \times K$ non-negative semi-definite variance tensor.

Solution:
\[
A^{-1} = \Lambda^{-1} + M^T \cdot C^{-1} \cdot M \quad (12)
\]
\[
a = A \cdot (\Lambda^{-1} \cdot \mu + M^T \cdot C^{-1} \cdot y) \quad (13)
\]
\[
B = C + M \cdot \Lambda \cdot M^T \quad (14)
\]
\[
b = M \cdot \mu. \quad (15)
\]

This is the complete solution to the problem, and constitutes the main point of this Note. For completeness, we will give some discussion!

Proof: The two sides of equation (11) are identical if two things hold. The first thing is that the determinant products must be equal:
\[
||C|| \cdot ||A|| = ||A|| \cdot ||B||,
\]
because the determinants are involved in the normalizations of the functions. This equality of determinant products follows straightforwardly from the matrix determinant lemma
\[
||Q + U \cdot V^T|| = ||I + V^T \cdot Q^{-1} \cdot U|| \cdot ||Q||,
\]
where $U$ and $V$ can be rectangular, and $I$ is the correct-sized identity matrix. This identity implies that
\[
||A^{-1}|| = ||I + M^T \cdot C^{-1} \cdot M \cdot \Lambda|| \cdot ||A^{-1}|| \quad (18)
\]
\[
||B|| = ||I + M^T \cdot C^{-1} \cdot M \cdot \Lambda|| \cdot ||C|| \quad (19)
\]
where we had to apply the identity twice to get the $||A^{-1}||$ expression. We can ratio these as follows to prove this first thing:
\[
\frac{||A|| \cdot ||B||}{||A^{-1}||} = \frac{||C||}{||\Lambda^{-1}||} \quad (20)
\]

The second thing required for the proof is that the quadratic scalar form
\[
[y - M \cdot \theta]^T \cdot C^{-1} \cdot [y - M \cdot \theta] + [\theta - \mu]^T \cdot \Lambda^{-1} \cdot [\theta - \mu] \quad (21)
\]
must equal the quadratic scalar form
\[
[\theta - a]^T \cdot A^{-1} \cdot [\theta - a] + [y - b]^T \cdot B^{-1} \cdot [y - b] \quad (22)
\]
because these quadratic scalar forms appear in the exponents in the functions. This equality follows from straightforward expansion of all the quadratic forms, plus some use of the matrix inversion lemma

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16See, for example, Wikipedia contributors (2020a), and Harville (2011).

17This useful lemma is also called the Woodbury matrix identity. See also Wikipedia contributors (2020b), and Harville (2011).
which gives an expression for the inverse $B^{-1}$ of the marginalized likelihood variance:

$$B^{-1} = C^{-1} - M \cdot [A^{-1} + M^T \cdot C^{-1} \cdot M]^{-1} \cdot M^T \cdot C^{-1} \cdot M$$

After that it’s just a lot of grinding through matrix expressions.\footnote{We leave this grinding to the avid reader. For guidance, it might help to realize that there are terms that contain $\theta^0 \cdot \ldots \cdot \theta^0 \cdot y^0 \cdot \ldots \cdot \theta^0 \cdot \mu$, and $\mu^1 \cdot \ldots \cdot \mu$. If you expand out each of these five kinds of terms, each of the five should lead to an independent-ish equality.}

**Solution notes:** In principle we found this factorization by expanding the quadratic in (21) and then completing the square. Of course we didn’t really; we used arguments (which physicists love) called detailed balance: We required that the terms that look like $\theta^0 \cdot Q \cdot \theta$ were equal between the LHS (21) and the RHS (22), and then all the terms that look like $\mu^T \cdot S \cdot \mu$, and so on. It turns out you don’t have to consider them all to get the right solution.

There is an alternative derivation or proof involving the canonical form for the multivariate Gaussian. This form is

$$N(x \mid m, V) = \exp \left( -\frac{1}{2} x^T \cdot H \cdot x + \eta^T \cdot x - \frac{1}{2} \xi \right)$$

$$H = V^{-1} ; \quad \eta = V^{-1} \cdot m ; \quad \xi = \ln \|2\pi V\| + \eta^T \cdot V \cdot \eta$$

In the canonical form, many products and other manipulations become simpler, so it is worth trying this route if you get stuck when manipulating Gaussian expressions.

Because the matrix $M$ is not square, it has no inverse. And because this is a physics problem, $M$ has units (which are the units of $d y / d \theta$). It’s beautiful in the solution that $M$ and $M^T$ appear only where the units make sense. They make sense because the units of $C^{-1}$ are inverse data-squared (where $y$ is the data vector) and the units of $A$ are parameter-squared and the units of $M$ are data over parameters. And they are all different sizes.

If you remember the Bayesian context around equation (1) and the Bayesian discussion thereafter, the Gaussian $N(\theta \mid a, A)$ is the posterior pdf for the linear parameters $\theta$, and the Gaussian $N(y \mid b, B)$ is the marginalized likelihood, marginalizing out the linear parameters $\theta$. This marginalization is usually thought of as being an integral, like the one given in equation (6). How are these linear-algebra expressions in any sense “doing this integral”? The answer is: That integral is a correlation of two Gaussians,\footnote{Astronomers like to say that it is the “convolution” of two Gaussians, but it is really the correlation of two Gaussians. The differences between convolution and correlation are minimal, though, and we aren’t sticklers.} and the correlation of two Gaussians delivers a new Gaussian with a shifted mean that is wider than either of the original two Gaussians. This factorization does, indeed, deliver the correct marginalization integral.

Continuing along these lines, various parts of the solution are highly interpretable in terms of the objects of Bayesian inference. For example, because the term $p(\theta \mid a, A)$ is the conditional posterior pdf\footnote{We say “conditional” here because it is conditioned on nonlinear parameters $\lambda$. The vector $a$ and variance tensor $A$ will depend on the nonlinear parameters $\lambda$ through the design matrix $M$.} for the linear parameters $\theta$, the vector $a$ is the maximum a posteriori (or MAP) value for the parameter vector $\theta$. It is found by inverse-variance-weighted combinations of the data and the prior. In some projects, posterior pdfs or MAP parameter values are the goal (although we don’t think they often should be). The variance tensor $A$ is the posterior

\[ [Q + U \cdot S \cdot V]^{-1} = Q^{-1} - Q^{-1} \cdot U \cdot [S^{-1} + V^T \cdot Q^{-1} \cdot U]^{-1} \cdot V^T \cdot Q^{-1} \]
variance in the parameter space. It is strictly smaller (in eigenvalues or determinant) than either the prior variance $\Lambda$ or the parameter-space data-noise variance $[M^T \cdot C^{-1} \cdot M]^{-1}$. The vector $b$ is the prior-optimal (maximum a priori) value for the data $y$. It is the most probable data vector (prior to seeing any data), and also the prior expectation for the data, under the prior pdf. The variance tensor $B$ is the prior variance expanded out to the data space, and including the noise variance in the data. It is strictly larger than both the data noise variance $C$ and the data-space prior variance $M \cdot \Lambda \cdot M^T$.

**Implementation notes:** The solution gives an expression for the variance tensor $B$, but note that when you actually evaluate the pdfs you probably need to have either the inverse of $B$, or else an operator that computes the product of the inverse and vectors, as in $B^{-1} \cdot y$ and the same for $b$. To get the inverse of the tensor $B$ stably, you might want to use the matrix inversion lemma (23) given above. This is often useful because you often know or are given the data inverse variance tensor $C^{-1}$ for the noise, and the prior variance inverse $\Lambda^{-1}$, and the lemma manipulates these into the answer without any heavy linear algebra. The lemma saves you the most time and precision when the parameter size $K$ is much smaller than the data size $N$ (or vice versa); that is, when $M$ is “very non-square”.

We also give the general advice that one should avoid taking an explicit numerical inverse (unless you know the inverse exactly in closed form, as you do for, say, diagonal tensors). In your code, it is typically stabler to use a `solve()` function instead of the `inv()` function. The reason is that the code operation `inv(B)` returns the best possible inverse to machine precision (if you are lucky), but what you really want instead is the best possible product of that inverse times a vector. So, in general, `solve(B,y)` will deliver more precise results than the mathematically equivalent `dot(inv(B),y)`.

The expressions in equation (11) do not require that the variance tensors $C$, $\Lambda$, $A$, $B$ be positive definite; they only require that they be non-negative semi-definite. That means that they can have zero eigenvalues. As can their inverses $C^{-1}$, $\Lambda^{-1}$, $A^{-1}$, $B^{-1}$. If either of these might happen in your problem—like if your prior freezes the parameters to a subspace of the $\theta$-space, which would lead to a zero eigenvalue in $\Lambda$, or if a data point is unmeasured or missing, which would lead to a zero eigenvalue in $C^{-1}$—you might have to think about how you implement the linear algebra operations to be zero-safe.\(^{22}\)

**Simplification: single multiplicative scaling** One interesting case is when $K = 1$, so the design matrix in fact reduces to a model vector $m$, multiplied by a scalar $\theta$, and $\mu$ and $\Lambda$ are now scalars as well:

$$\
\mathcal{N}(y | \theta m, C), \mathcal{N}(\theta | \mu, \Lambda) = \mathcal{N}(\theta | a, A), \mathcal{N}(y | b, B) \ .
$$

(27)

This can arise if one wants to multiplicatively scale a model to the data. $a$ would then correspond to the maximum a posteriori value of the multiplicative scaling to fit $y$ with $m$. In this case, the previous equations are simplified and no longer involve many matrix operations. It’s a nice exercise to simplify the solution above for this scalar case.

\(^{22}\) A completely zero-safe implementation is somewhat challenging, but one comment to make is that if, say, $A$ contains a zero eigenvalue, then there is a direction in the parameter space (the $\theta$-space) in which the variance vanishes. This means that all valid parameter combinations lie on a linear subspace of the full $K$-dimensional space. All parameter combinations that wander off the subspace get strictly zero probability or negative infinities in the log. If your inference is valid, it will probably be the case that the vectors at which you want to evaluate always lie in the non-zero subspace. It makes sense, then, in this case, to work in a representation in which it is easy to enforce or ensure that. This usually involves some kind of coordinate transformation or rotation or projection. Doing this correctly is beyond the scope of this Note.
**Special case: wide prior** Another interesting case that often arises in inferences is the use of an improper (infinitely wide) prior on the parameters $\theta$. Rather than ignoring the prior pdf on the left-hand side of equation (11), which is technically incorrect, the correct posterior pdf can be obtained by taking the limit $\Lambda^{-1} \to 0$ in the fiducial results derived above. It is perhaps worth noting that in the improper-prior case, the posterior can still be fine, but the marginalized likelihood will make no sense (it will technically vanish).

**Generalization: product of many Gaussians** A case that arises in some applications is that the likelihood is made of multiple Gaussian terms, each of which is a different linear combination of the linear parameters $\theta$. That is, there are $J$ data vectors $y_j$, each of which has size or length $N_j$, and each of which has an expectation set linearly by the parameters $\theta$ but through a different design matrix $M_j$.

Provided that the different data vectors $y_j$ are independently “observed” (that is, they have independent noise draws with noise variance tensors $C_j$), the total likelihood is just the product of the individual-data-vector likelihoods. An example of this case arises in astronomy, for example, when considering radial velocity measurements of a star taken with different instruments that may have systematic offsets between their velocity zero-points.

In principle we could work around this problem—reduce it to the previously solved problem—by forming a large vector $y$ which is the concatenation of all the individual data vectors $y_j$, and a large design matrix $M$ which is the concatenation of all the individual design matrices $M_j$, and a large total covariance matrix $C$ which is a block diagonal matrix containing the noise variance tensors $C_j$ on the diagonal blocks. We could then apply the result of the single-data-vector problem above. However, this can result in significant unnecessary computation, and it is hard to write the answer in a simple form. Instead we can take advantage of the separability of the likelihoods, and write the following generalized problem statement:

Find $K$-vector $a$, $K \times K$ variance tensor $A$, $J$ vectors $b_j$ (each of which is a different length $N_j$), and $J$ variance tensors $B_j$ (each of which is a different size $N_j \times N_j$) such that

$$N(\theta | \mu, A) \prod_{j=1}^{J} N(y_j | M_j \cdot \theta, C_j) = N(\theta | a, A) \prod_{j=1}^{J} N(y_j | b_j, B_j) ,$$

and such that $a$, $A$, all the $b_j$, and all the $B_j$ don’t depend on $\theta$ at all. Note that $\theta$ is a $K$-vector, $\mu$ is a $K$-vector, $A$ is a $K \times K$ non-negative semi-definite variance tensor, each $y_j$ is an $N_j$-vector, each $M_j$ is a $N_j \times K$ matrix, and each $C_j$ is a $N_j \times N_j$ non-negative semi-definite variance tensor.

One way to solve this problem is to write all Gaussians in their canonical form, then separate the elements that depend on $\theta$ and on the individual $y_j$. The result can be written as an iteration over data vectors $y_j$:

initialize: $A_0^{-1} = \Lambda^{-1}$; $a_0 = \mu$; $x_0 = \Lambda^{-1} \cdot \mu$

$$\text{(28)}$$
iterate: \[ B_j = C_j + M_j \cdot A_{j-1} \cdot M_j^T \] \[ b_j = M_j \cdot a_{j-1} \] \[ A_j^{-1} = A_{j-1}^{-1} + M_j^T \cdot C_j^{-1} \cdot M_j \] \[ x_j = x_{j-1} + M_j^T \cdot C_j^{-1} \cdot y_j \] \[ a_j = A_j \cdot x_j \] finish: \[ A = A_j \] \[ a = a_j . \] (30) (31) (32) (33) (34) (35) (36)

The solution is an iteration because you can think of adding each new data set \( y_j \) sequentially, with the prior for set \( j \) being the posterior from set \( j - 1 \). The way this solution is written is unpleasant, because the specific values you get for the vectors \( b_j \) and tensors \( B_j \) depend on the order in which you insert the data. But—and very importantly for the rules of Bayesian inference—the posterior mean \( a \) and variance \( A \) do not depend on the order!\(^{23}\)

4 Worked Examples

When working with a probabilistic model that meets the strong requirements imposed above (Gaussians everywhere; expectations linear in parameters), the identities described in this Note have practical uses: (1) To simplify the posterior pdf of your model (which makes generating samples or computing integrals far simpler), and (2) to reduce the dimensionality of your model (by enabling closed-form marginalizations over linear parameters). Reducing the dimensionality of your parameter-space will in general improve convergence of Markov Chain Monte Carlo\(^{24}\) (MCMC) sampling methods, or enable alternate sampling methods (for example, rejection sampling) that may be intractable when the parameter dimensionality is large: These two benefits also typically make inference procedures (like sampling) far faster. Here, we demonstrate the utility of the identities shown above with two worked exercises.

Exercise 1: A fully linear model: We observe a set of data \((x_i, y_i)\) (indexed by \(i\)) with known, Gaussian uncertainties in \(y\), \(\sigma_y\), and no uncertainty in \(x\). The parametric model we will use for these data is a quadratic polynomial,

\[ f(x; \alpha, \beta, \gamma) = \alpha x^2 + \beta x + \gamma \] (37)

and we assume we have Gaussian prior pdfs on all of the \(K = 3\) linear parameters \((\alpha, \beta, \gamma)\),

\[ p(\alpha) = \mathcal{N}(\alpha | \mu_\alpha, \sigma_\alpha) \] (38)
\[ p(\beta) = \mathcal{N}(\beta | \mu_\beta, \sigma_\beta) \] (39)
\[ p(\gamma) = \mathcal{N}(\gamma | \mu_\gamma, \sigma_\gamma) \] . (40)

While this example may seem overly simple or contrived, quadratic models are occasionally useful in astronomy and physics, for example, when

\[ x \quad y \quad \sigma_y \]
\[-0.6 \quad 12.2 \quad 0.8 \]
\[ 2.0 \quad 4.1 \quad 3.2 \]
\[ 2.7 \quad 0.9 \quad 3.3 \]
\[ 3.6 \quad -15.0 \quad 3.9 \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Top: Data generated with “true” parameters 
\((\alpha, \beta, \gamma) = (3.21, 2.44, 14.82)\). Bottom: The solution to Exercise 1. The data points (black markers) show the data from the table. The MAP parameter values were found using equation (13). All data files and solution notebooks are available via Zenodo (Hogg et al. 2020).}
\end{figure}

\(^{23}\)It is literally part of the fundamental justification of Bayesian inference that the knowledge you eventually have (your final beliefs) does not depend on the order in which you observed the data. This is one of the axioms or inputs to the theorems that underlie the consistency of Bayesian reasoning.

\(^{24}\)We have also written a tutorial on MCMC in this series (Hogg and Foreman-Mackey 2018).

\(^{25}\)Fitting second-order polynomials has been shown to be great for centroiding peaks in astronomy contexts. See, for example, Vakili and Hogg (2016), and Teague and Foreman-Mackey (2018).
centroiding a peak, and polynomial models are often used to capture smooth trends in data.

The data table shown in Figure 2 contains \( N = 4 \) data points, \((x_i, y_i, \sigma_{y_i})\), generated using this quadratic model. Assuming values for the prior means, \( \mu \), and prior variance tensor, \( \Lambda \),

\[
\mu^T = (\mu_\alpha, \mu_\beta, \mu_\gamma) = (1, 3, 9) \tag{41}
\]
\[
\Lambda = \begin{pmatrix} 5^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 8^2 \end{pmatrix} \tag{42}
\]
compute the MAP parameter values \( a^T = (\alpha_{\text{MAP}}, \beta_{\text{MAP}}, \gamma_{\text{MAP}}) \). Plot the data (with error bars) and over-plot the model evaluated at the MAP parameter values. Generate 4096 posterior samples of the linear parameters. Over-plot a shaded region showing the 68 percent credible region for the model, estimated using these samples.

Solution: Given the assumptions and prior parameter values above, the design matrix, \( M \), is

\[
M = \begin{pmatrix} 0.36 & -0.6 & 1 \\ 4.0 & 2.0 & 1 \\ 7.29 & 2.7 & 1 \\ 12.96 & 3.6 & 1 \end{pmatrix} \tag{43}
\]

By plugging in to equation (13), we find MAP parameter values for the linear parameters

\[
a^T = (\alpha_{\text{MAP}}, \beta_{\text{MAP}}, \gamma_{\text{MAP}}) = (3.61, 1.98, 14.26) \tag{44}
\]

Figure 2 shows the data (black points), the model computed with the MAP parameter values (blue line), and the 68-percent credible region (shaded blue region) estimated using posterior samples generated from \( \mathcal{N}(\theta | a, A) \). The companion IPython notebook (Exercise1.ipynb) contains the full solution.

Exercise 2: A model with a nonlinear parameter: We observe a set of data \((x_i, y_i)\) (indexed by \(i\)) with known, Gaussian uncertainties in \(y\), \(\sigma_{y_i}\), and no uncertainty in \(x\). The parametric model we will use for these data is a generalized sinusoid with a constant offset,

\[
f(x; \alpha, \beta, \gamma, \omega) = \alpha \cos(\omega x) + \beta \sin(\omega x) + \gamma \tag{45}
\]
and we again assume we have Gaussian prior pdfs on all of the linear parameters \((\alpha, \beta, \gamma)\). Models like this (a periodic model with both linear and nonlinear parameters) are common in astronomy, especially in the context of asteroseismology, light curve analysis, and radial velocity variations from massive companions (binary star systems or exoplanets).

For this setup, we can no longer analytically express the posterior pdf because of the nonlinear parameter \(\omega\), but we can compute the marginal likelihood (marginalizing over the linear parameters) conditioned on the frequency \(\omega\).

\[
\begin{array}{ccc}
  x & y & \sigma_y \\
  -1.2 & 11.2 & 0.2 \\
  1.3 & 16.1 & 0.2 \\
  3.1 & 10.2 & 0.3 \\
  4.1 & 13.5 & 0.3 \\
\end{array}
\]

Figure 3: Top: Data generated with “true” parameters \((\alpha, \beta, \gamma, \omega) = (3.21, 2.44, 13.6, 1.27)\). Middle and Bottom: The solution to Exercise 2. All data files and solution notebooks are available via Zenodo (Hogg et al. 2020).
The posterior pdf over $\omega$ will be extremely multimodal. Don’t fire up standard MCMC! Try using rejection sampling instead: Generate a dense prior sampling in the nonlinear parameter $\omega$, evaluate the marginalized likelihood at each sample in $\omega$, and use this to reject prior samples.

The table shown in Figure 3 contains $N = 4$ data points, $(x_i, y_i, \sigma_{y_i})$, generated with this sinusoid model. Assuming values for the prior means, $\mu$, and prior variance tensor, $\Lambda$,

$$\mu^T = (\mu_\alpha, \mu_\beta, \mu_\gamma) = (0, 0, 0) \quad (46)$$

$$\Lambda = \begin{pmatrix} 5^2 & 0 & 0 \\ 0 & 5^2 & 0 \\ 0 & 0 & 10^2 \end{pmatrix} \quad (47)$$

write a function to compute the vectors and matrices we need for the linear parameters (the design matrix and components $a, A, b, B$ of the factorization) at a given value of the frequency $\omega$. Assuming a prior on $\omega$ that is uniform in $\ln \omega$ over the domain $(0.1, 100)$,

$$p(\omega) \propto \frac{1}{\omega} \quad (48)$$

evaluate the log-marginal likelihood $\ln p(y | \omega)$ and add to the log-frequency prior $\ln p(\omega)$ over a grid of 16,384 frequencies $\omega$ between $(0.1, 100)$. Plot both the marginal likelihood (not log!) $p(y | \omega)$ and the posterior pdf $p(y | \omega) p(\omega)$ as a function of this frequency grid.

Generate 512 posterior samples$^{26}$ in the full set of parameters $(\alpha, \beta, \gamma, \omega)$. Make a scatter plot showing a 2D projection of these samples in $(\alpha, \ln \omega)$. Plot the data, and over-plot 64 models (equation 45) computed using a fair subset of these posterior samples. The companion IPython notebook (Exercise2.ipynb) contains the full solution.

$^{26}$The posterior pdf over $\omega$ will be extremely multimodal. Don’t fire up standard MCMC! Try using rejection sampling instead: Generate a dense prior sampling in the nonlinear parameter $\omega$, evaluate the marginalized likelihood at each sample in $\omega$, and use this to reject prior samples.
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