I present a unified picture of center vortices and Abelian monopoles. Both appear as local gauge ambiguities in the Laplacian Center Gauge. This gauge is constructed for a general $SU(N)$ theory. Numerical evidence is presented, for $SU(2)$ and $SU(3)$, that the projected $Z_N$ theory confines with a string tension similar to the non-Abelian one.

1 Motivation and technical problem

The road traveled by physicists in their efforts to identify the effective, InfraRed degrees of freedom of QCD is far from straight. It bifurcates in many directions, most of which are still under exploration. Currently, the two most popular effective descriptions of confinement are in terms of Abelian monopoles\cite{1,2} and center vortices\cite{3,4}. I want to show that these two descriptions can be unified: these two branches of the road merge together, perhaps indicating that we are traveling towards a piece of Truth.

The study of center vortices, first proposed by Mack\cite{4} and by 't Hooft\cite{3}, has been revived by Greensite and collaborators. They project the $SU(2)$ lattice gauge theory to a $Z_2$ gauge theory, by partially fixing the gauge to Maximal Center Gauge, defined as the gauge in which

$$\sum_{x,\mu} |Tr U_\mu(x)|^2 \text{ maximum}.$$  \hspace{1cm} (1)

In this gauge, the center-projected links are $z_\mu(x) \equiv \text{sign}(Tr U_\mu(x))$. This $Z_2$ theory has defects corresponding to plaquettes taking value $-1$. Greensite et al\cite{5} showed that the string tension $\sigma$ given by these defects closely matches that of the original $SU(2)$ theory. Being skeptical about this, I investigated with M. D’Elia the coset theory, made of positive-trace links $U'_\mu(x) \equiv z_\mu(x) U_\mu(x)$. This theory has more short-range disorder than the original one, but carries

\footnote{presented by Ph. de Forcrand.}
no center vortices. Could it be that $\{U'_\mu(x)\}$ would show long-range order, and thus not confine? To my surprise, we found that in $\{U'_\mu(x)\}$: (i) $\sigma = 0$; (ii) $\langle \hat{\psi} \psi \rangle = 0$; (iii) $Q_{\text{top}} = 0$. Removal of center vortices causes deconfinement, chiral symmetry restoration and suppression of topological excitations. All non-perturbative properties disappear. Therefore, center vortices must carry the non-perturbative degrees of freedom.

There are two difficulties with this conclusion. First, a great deal of numerical evidence has been accumulated which ties confinement with Abelian monopoles instead of center vortices. Secondly, the above findings may depend on the choice of local maximum in (1). It is the purpose of this talk to resolve these two difficulties, as outlined already in [7].

The second problem is shared by the Abelian projection, which also proceeds by gauge fixing via the iterative, local maximization of a gauge functional. But in the Maximal Center Gauge, this problem can be acute. As shown in [8], the local maximum reached after starting from Landau gauge leads to a very small density of center vortices, which actually do not confine. Following this severe warning, some studies try to obtain the center-vortex properties of the global maximum of (1) by taking the highest among $m$ local maxima, and extrapolating to $m \to \infty$. One feature underlines the difficulty of this approach: the extrapolated value for the global maximum of (1) falls below the measured value obtained by the procedure of [8].

To illustrate why this technical problem is so hard to resolve, let us consider a toy example. Take a 1-dimensional ring of $U(1)$ links $\{e^{i\theta_i}, i = 1, \ldots, N\}$ such that the gauge invariant loop which they form is $-1$: $\prod_N e^{i\theta_i} = -1$. For such a system, the global maximum of (1), corresponding to Maximal Center Gauge, is obtained when $\theta_{i_0} = \pi$ for one link $i_0$, and $\theta_i = 0 \forall i \neq i_0$. The “kink” $\theta_{i_0} = \pi$ can be placed anywhere, giving rise to an $N$-fold degeneracy. The gauge-fixing functional (1) then takes value $N$. Let us now fix this system to Landau gauge, defined as the gauge which maximizes $\sum N \text{Re} \text{Tr} U_i$. This is achieved when $\theta_i = \pi/N$. All link angles are small and there is no sign of a kink. The Center Gauge functional (1) then takes the value

$$\sum_N |\text{Tr} U_i|^2 = N \cos \frac{2\pi}{N} \approx N - \frac{2\pi^2}{N} \quad (2)$$

One can check that Landau gauge represents a local maximum of (1); we see that the difference between this local maximum and the global one is vanishingly small as $N \to \infty$.

This case is not just a toy example. An Abelian loop having a phase of $\pi$ is precisely the signature of an $SU(2)$ center vortex, as illustrated in Fig.1. It
is therefore essential to address the technical problem of gauge-fixing in order to properly identify the center vortices.

2 Unambiguous gauge-fixing

The crucial observation is that, since a center vortex gives the fundamental Wilson loop a factor $-1 = \exp(i\pi\sigma_3)$, it will give the adjoint Wilson loop a factor $\exp(2i\pi\dot{\sigma}_3)$. Therefore, as one describes a small loop around point $x_0$ in Fig.1, the adjoint gauge field winds by $2\pi$ in color space. Thus, the center vortex at $x_0$ will appear as a gauge singularity of the adjoint field. More formally, one can observe that center vortices correspond to non-trivial topological classes of $\Pi_1(SU(N)/Z_N)$, so that Wilson loops in the “center-blind” adjoint representation are the proper objects to identify them.

A second observation motivates our gauge-fixing approach. If the gauge field $A^{ad}_{\mu}$ diverges like $\frac{1}{r}\dot{\sigma}_3$ near $x_0$, then so will the covariant derivative $D^{ad}_{\mu} = \partial_\mu + igA^{ad}_{\mu}$. This will force the vector $(D^{ad}_{\mu}\vec{v})(x)$ to orient itself along $\dot{\sigma}_3$ when $x$ approaches $x_0$, for any smoothly varying vector $\vec{v}(x)$. In particular, this statement applies to all the eigenvectors of any adjoint covariant derivative operator. In the vicinity of a thin center vortex similar to the “prototype” of Fig.1, these eigenvectors will become collimated in color space. In reality, the singularity is smoothed out: the center vortex is no longer “thin”, but
acquires a finite size. Collinearity of eigenvectors still takes place within the vortex core, but at an exact location which depends on the particular choice of covariant operator and eigenvectors.

This collinearity property can be used to detect center vortices on the lattice, without actually performing any gauge fixing or center projection. For simplicity, we take as covariant operator the adjoint Laplacian, discretized in the simplest way:

$$\Delta^{adj}_{xy} = \sum_{\pm \mu} \hat{U}_{\pm \mu}(x) \delta_{y,x \pm \mu} - 2d \delta_{y,x} ,$$

where \(\hat{U}_{ab} = \frac{1}{2} Tr(\tau_a U \tau_b U^\dagger)\) is the link in the adjoint representation, and the \(\tau\)'s are the generators of SU\((N)\). The signal for an SU\((2)\) center vortex is the parallelism in color space of two eigenvectors. Of course, this will never occur exactly at a lattice site, and interpolation is necessary. To reduce interpolation ambiguities, we choose the two lowest-lying eigenvectors, which are the least sensitive to UV fluctuations.

Because we want to study the projected Z\(_N\) theory, we proceed with gauge fixing. We can also make use of the eigenvectors of the Laplacian for that purpose instead of maximizing \([10]\). Since the Laplacian is covariant, a local gauge transformation \(\Omega(x)\), which transforms the gauge links \(U_{\mu}(x)\) into \(\Omega(x) U_{\mu}(x) \Omega^\dagger(x + \hat{\mu})\), will also transform the eigenvector \(v(x)\) into \(\Omega(x) v(x)\). Following Ref. [10], we can then fix the gauge uniquely by specifying the orientation of \(\Omega(x) v(x)\) in color space at each point \(x\). This kind of gauge was called Laplacian gauge in [10]. We generalize it below to the adjoint representation. When the adjoint SU\((N)/Z_N\) field is fully gauge-fixed, a remnant center gauge freedom \(Z_N\) subsists, and we can look at the projected \(Z_N\) gauge theory.

Gauge fixing proceeds in two steps. **Step 1.** At each point \(x\), the lowest-lying eigenvector \(v^{(1)}(x)\) has \((N^2 - 1)\) real color components. Let us form the hermitian matrix

$$\Phi^{(1)} = \sum_{a} v^{(1)}_a \tau_a$$

and apply the gauge transformation \(\Omega(x)\) which diagonalizes \(\Phi^{(1)}\) at every \(x\). After this step, \(v^{(1)}\) will be rotated to lie along the diagonal generators of SU\((N)\): it will be parallel to \(\sigma_3\) in the case of SU\((2)\), or lie in the plane \((\tau_3, \tau_8)\) for SU\((3)\). This gauge-fixing is not complete: in the SU\((2)\) case, an arbitrary rotation \(e^{i \theta \sigma_3}\) can still be performed. More generally, after one fixes some arbitrary ordering for the eigenvalues of \(\Phi^{(1)}\), there remains an Abelian
$U(1)^{N-1}$ gauge freedom. In other words, we have reduced the gauge symmetry to its maximal Cartan subgroup. This gauge was proposed for $SU(2)$ by A. van der Sijs, who called it Laplacian Abelian Gauge, as an unambiguous substitute for Maximal Abelian Gauge\cite{1} it achieves the same purpose but has no difficulty with gauge copies coming from local maxima. Here it appears as a natural intermediate step, which exhausts the information available from the lowest-lying eigenvector.

**Step 2.** Using the next eigenvector rotated by $\Omega(x)$, $\tilde{v}^{(2)}(x) = \tilde{\Omega}(x)v^{(2)}(x)$, we form the matrix

$$\Phi^{(2)} \equiv \sum_{a} \tilde{v}^{(2)}_{a} \tau_{a}.$$  \hspace{1cm} (5)

An Abelian rotation will leave the diagonal part of $\Phi^{(2)}$ invariant. Therefore, we complete the gauge fixing by enforcing $(N-1)$ constraints on off-diagonal elements, which we choose to be on the sub-diagonal. Specifically, for $SU(2)$ we require that $\tilde{v}^{(2)}$ lie in the positive $(\sigma_{1}, \sigma_{3})$ half-plane: $Tr\Phi^{(2)}\sigma_{2} = 0$, $Tr\Phi^{(2)}\sigma_{1} > 0$. Similarly, for $SU(3)$ we require $Tr\Phi^{(2)}\tau_{2} = Tr\Phi^{(2)}\tau_{7} = 0$, $Tr\Phi^{(2)}\tau_{1} > 0$, $Tr\Phi^{(2)}\tau_{6} > 0$. This 2-step procedure fixes the gauge completely, up to local $Z_{N}$ transformations which leave the adjoint links and the Laplacian eigenvectors unchanged.

### 3 Local gauge ambiguities

Laplacian Center Gauge is unambiguous: no matter what the starting point on the gauge orbit, the gauge-fixed configuration will be the same (up to a global gauge transformation). For some exceptional gauge fields, the smallest eigenvalues $\lambda_{1}, \lambda_{2}$ may be degenerate, leading to a continuous family of possible gauge-fixed solutions, which can legitimately be called Gribov copies. But this situation never occurs in practice.

What occurs for a generic gauge field, however, is that the gauge transformation becomes ill-defined at some point(s) $x$. These local gauge ambiguities prevent a complete gauge fixing at $x$; they locally enlarge the gauge symmetry beyond $Z_{N}$. As we shall now see, these gauge defects are center vortices and monopoles.

One of the constraints of Step 2 is automatically satisfied if the associated complex matrix element is zero. In that case the Abelian gauge rotation cannot be fixed, and the remaining gauge symmetry is enlarged from $Z_{N}$ to $U(1)$. Specifically, for $SU(2)$ this occurs when $\tilde{v}^{(2)}_{1} = \tilde{v}^{(2)}_{2} = 0$, so that $\tilde{v}^{(2)}$ lies along the color direction $\sigma_{3}$ just like $v^{(1)}$. A $U(1)$ gauge ambiguity occurs whenever $v^{(1)}$ and $v^{(2)}$ are collinear. One can check that a small Wilson loop
around such a point will acquire a phase $e^{i2\pi \sigma_3}$ in the adjoint representation, or $e^{i\pi \sigma_3} = -1$ in the fundamental. We have recovered the statement, made at the beginning of Section 2, that center vortices can be detected by the collinearity of eigenvectors. Collinearity implies $v^{(2)}_1 = v^{(2)}_2 = v^{(2)}_3 = v^{(1)}_1$, so that 2 constraints must be satisfied. Center vortices have codimension 2, forming closed surfaces in 4$d$. In the case of $SU(3)$, ambiguities happen when $\tilde{v}^{(2)}_1 = \tilde{v}^{(2)}_2 = 0$, or $\tilde{v}^{(2)}_6 = \tilde{v}^{(2)}_7 = 0$.

Local ambiguities can already arise at Step 1. For $SU(2)$, this happens whenever $|v^{(1)}(x)| = 0$, yielding 3 constraints. These defects have codimension 3, forming closed loops in 4$d$. Along such lines the Abelian projection is ill-defined. ’t Hooft followed by has shown that these defects can be identified with monopole worldlines. In the case of $SU(3)$, the analysis is more subtle. Whenever two eigenvalues of $\Phi^{(1)}$ become degenerate, their ordering becomes ambiguous. This enlarges the remaining gauge freedom from $U(1)^{N-1}$ to $U(1)^{N-2} \times SU(2)$. Again, these defects can be identified with codimension-3 monopole worldlines.

Now an intimate connection appears between monopoles and center vortices. The latter correspond to $v^{(1)}$ and $v^{(2)}$ being collinear. The two vectors can be parallel or anti-parallel. To go from one situation to the other along the center vortex surface, one must cross a line where $|v^{(1)}| = 0$ or $|v^{(2)}| = 0$. Thus,
monopole worldlines always separate patches of center vortices with oppositely oriented eigenvectors. This situation is schematically described in Fig. 2. One can interpret center vortices as the worldsheet of half Dirac strings between two monopoles. This explains the numerical finding that almost all monopoles are attached to two center vortices.

4 Numerical findings

We have applied the gauge-fixing procedure described above to SU(2) and SU(3) ensembles. The projected $Z_2$ or $Z_3$ ensembles are constructed by replacing each link by the center element whose trace is the closest. To estimate the string tension in the projected theory, we measure the Creutz ratios

$$\chi_{R,R} \equiv -\ln\frac{\langle W_{R,R} \rangle}{\langle W_{R+1,R+1} \rangle} / \langle W_{R,R+1} \rangle^2,$$

where $\langle W_{R,T} \rangle$ is the expectation value of an $R$ by $T$ Wilson loop.

We first performed a check of the usual Direct Maximal Center gauge approach. As illustrated in Figs. 3 and 4, the Creutz ratios in the projected $Z_2$ theory appear to grow with distance. This trend can already be noticed in Fig. 2 of Ref. It has also been observed in the projected $U(1)$ theory obtained after Maximal Abelian gauge-fixing. It may depend on the details of the iterative gauge-fixing algorithm. In any case, increasing Creutz ratios signal a lack of positivity of the transfer matrix. Positivity is of course not guaranteed after a non-local procedure such as center gauge-fixing and projection. We mention this as a potential problem of the iterative gauge-fixing approach, in addition to that of local maxima. As a final “nail in the coffin”, we should
add that Laplacian Center gauge fixing is cheaper: on a $16^4$ lattice, using the package ARPACK\(^1\) to find the necessary eigenvectors, the work performed to fix the gauge is equivalent to about 50 Monte Carlo sweeps for $SU(2)$, and 300 to 500 for $SU(3)$; this is far less than required by the iterative approach.

Figs. 3 to 5 show the Creutz ratios in the projected theory after Laplacian Center gauge fixing, for $SU(2)$ ($\beta = 2.4$ and 2.5) and $SU(3)$ ($\beta = 6$). As the distance increases, the Creutz ratios decrease rapidly then stabilize. The rapid decrease indicates the presence of many close pairs of vortices, quite unlike the Direct Maximum Center gauge results. Indeed, the vortex density is about a factor of 2 greater. This is in line with the increased monopole density observed in Laplacian Abelian gauge\(^11\). The plateau reached by the Creutz ratios appears close to the full $SU(2)$ or $SU(3)$ string tension. A more extensive numerical study is under completion\(^12\).

The significance of the rough agreement between the $Z_N$ and $SU(N)$ string tensions should not be exaggerated. Our measurements are taken at some finite, rather large value of the lattice spacing. We could choose another gauge, defined for example via a different discretization of the Laplacian, with higher derivative terms. We should expect the $Z_N$ Creutz ratios to be sensitive to this choice of gauge. It is only in the continuum limit that we must recover gauge independence and agreement between the $Z_N$ and $SU(N)$ string tensions.
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