Almost Kähler metrics and pp-wave spacetimes

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Abstract
We establish a one-to-one correspondence between a class of strictly almost Kähler metrics on the one hand and Lorentzian pp-wave spacetimes on the other; the latter metrics are well known in general relativity, where they model radiation propagating at the speed of light. Specifically, we construct families of complete almost Kähler metrics by deforming pp-waves via their propagation wave vector. The almost Kähler metrics we obtain exist in all dimensions $2n \geq 4$, and are defined on both $\mathbb{R}^{2n}$ and $S^1 \times S^1 \times M$, where $M$ is any closed almost Kähler manifold; they are not warped products, they include noncompact examples with constant negative scalar curvature, and all of them have the property that their fundamental 2-forms are also co-closed with respect to the Lorentzian pp-wave metric. Finally, we further deepen this relationship between almost Kähler and Lorentzian geometry by utilizing Penrose’s “plane wave limit,” by which every spacetime has, locally, a pp-wave metric as a limit: using Penrose’s construction, we show that in all dimensions $2n \geq 4$, every Lorentzian metric admits, locally, an almost Kähler metric of this form as a limit.

Keywords pp-waves · Almost Kähler metrics · Plane wave limit

Mathematics Subject Classification 53C50 · 53C55

1 Introduction

A Riemannian metric $g$ is almost Kähler if there is an almost complex structure $J$ compatible with $g$ and if the corresponding 2-form $g(\cdot, J)$ is a symplectic form. What makes this setting “almost” Kähler is that $J$ itself need not be integrable, as it would be for any Kähler metric, i.e., $J$ need not give rise to an atlas of holomorphic coordinate
charts. By omitting complex geometry in this way, almost Kähler metrics thereby sit at the boundary between the symplectic and the Kähler categories; a very comprehensive survey of them can be found in [1]. In this paper, we shine a light on these metrics from a new direction, by showing that a class of them derive from—indeed, are in one-to-one correspondence with—a distinguished class of Lorentzian metrics, namely the so-called pp-wave spacetimes modeling radiation propagating at the speed of light (see, e.g., [2, Chapter 13]); among the many remarkable properties exhibited by pp-waves is the fact that many of them have vanishing curvature invariants and yet are not flat [7]. In saying that our almost Kähler metrics \( g \) “derive from” them, we mean that if \( h \) is a pp-wave metric, then \( g \) will be given by

\[
g := h + 2T^b \otimes T^b
\]

for a suitable choice of vector field \( T \) satisfying \( h(T, T) = -1 \) (in the parlance of Lorentzian geometry, a so-called timelike vector field; here \( T^b = h(T, \cdot) \) is the one-form \( h \)-metrically equivalent to \( T \)). In our setting, \( T \) will arise from deforming the propagation wave vector of the radiation being modeled by the pp-wave metric, and we will say that \( g \) is “dual” to the pp-wave metric \( h \). In general, any choice of timelike \( T \) will yield a Riemannian metric as above, but as we show in Sect. 3, our particular choice of \( T \) will always yield, not only an almost Kähler metric, but a complete one (our metrics are defined in both the compact and the noncompact settings, and therefore, completeness must be explicitly verified in the latter).

In the remainder of Introduction, we outline how our almost Kähler metrics compare with those already in the literature. To the best of our knowledge, the first noteworthy example of a strictly almost Kähler metric was exhibited by [26], a compact 4-manifold defined as a 2-torus bundle over a 2-torus; noncompact examples—other than the tangent bundles of certain Riemannian manifolds, which were known—came soon afterward in [28], which also contained compact examples by way of Thurston’s. Thurston’s example was generalized to higher dimensions in [6]. In dimension 4, where the Hodge star operator can be used, [17] obtained strictly almost Kähler metrics via deformations of scalar-flat Kähler metrics. Another class of examples, due to Bérard-Bergery, can be found on Einstein metrics \( M \), with positive Einstein constant, that admit certain Riemannian submersions \( P \to M \) with \( P \) a principal \( S^1 \)-bundle; see [3, Theorem 9.76, p. 255]. Finally, [16] showed the existence of strictly almost Kähler metrics on products \( S^1 \times S^1 \times M \) where \( M \) is any almost Kähler metric, including those with constant negative scalar curvature on the 6-torus; these metrics are different from our compact examples because those in [16] are warped products, whereas ours are not; furthermore, our construction also yields complete examples on \( \mathbb{R}^{2n} \). Finally, conformally flat examples of the form \( \mathbb{R}^n \times M \), where \( M \) is a (not necessarily complete) Riemannian manifold, were constructed in [5]. In comparison, the distinguishing feature of our examples is their relationship, not merely to pp-waves, but to Lorentzian geometry in general: we show that every (even-dimensional) Lorentzian manifold admits, locally, an almost Kähler metric in an appropriate limit, namely, the “plane wave limit” due to Penrose [22].

This paper is organized as follows. Section 2 gives a brief overview of pp-wave spacetimes, emphasizing their geometric and geodesic properties. Sections 3 and 4 introduce
the candidates for our almost Kähler metrics on $\mathbb{R}^{2n}$, showing that they are complete and computing their curvature; in Sect. 5 we demonstrate that they are, in fact, strictly almost Kähler metrics (Theorem 1) and that they have the added property that their fundamental 2-form is co-closed with respect to the Lorentzian pp-wave metric as well (Corollary 3). Section 6 then generalizes this construction to the compact setting, on manifolds of the form $S^1 \times S^1 \times M$, where $M$ is any closed almost Kähler manifold (Theorem 2). Finally, Sect. 7 establishes a deeper connection between almost Kähler and Lorentzian geometry, by showing that every (even-dimensional) spacetime admits, locally, an almost Kähler metric via Penrose’s plane wave limit mentioned above; this is codified in Theorem 3; we also include here an introduction to Penrose’s limit itself.

2 Brief overview of pp-waves

The class of pp-wave spacetimes have their origin in gravitational physics and have been intensely studied therein; see, e.g., [12, 25], and [2, Chapter 13]. The definition we give here is due to [15, 19]; it is in fact the more modern, coordinate-independent version of the “standard” definition appearing in the physics literature. Before stating it, recall that with respect to a Lorentzian metric $h$ (with index $-++\cdots+$), nonzero vectors $X$ divide into three types:

\[
\begin{cases}
\text{"spacelike" if } h(X, X) > 0, \\
\text{"timelike" if } h(X, X) < 0, \\
\text{"lightlike" if } h(X, X) = 0.
\end{cases}
\]

**Definition 1** ([15]) On a (compact or noncompact) manifold $M$, a Lorentzian metric $h$ is a **pp-wave** if it admits a globally defined lightlike vector field $V$ that is parallel, $\nabla V = 0$ ($\nabla$ is the Levi-Civita connection of $h$), and if its curvature endomorphism $R$ satisfies

\[ R(X, Y) = 0 \quad \text{for all } X, Y \in \Gamma(V^\perp). \tag{1} \]

If in addition $\nabla_X R = 0$ for all $X \in \Gamma(V^\perp)$, then $(M, h)$ is a **plane wave**.

Locally, such manifolds always take the following form, a special case of a class of coordinates known as “Walker coordinates” [27]:

**Theorem** ([19]) On any pp-wave $(M, h)$, there exist local coordinates $(v, u, x^3, \ldots, x^n)$ in which $V = \partial_v$ and

\[ h = H(u, x^3, \ldots, x^n) du^2 + 2dvdu + \sum_{i=3}^n (dx^i)^2 \tag{2} \]

for some smooth function $H(u, x^3, \ldots, x^n)$ independent of $v$. Furthermore, $(M, h)$ will be a plane wave if and only if $H$ is a quadratic polynomial in $x^3, \ldots, x^n$. If (2) exists globally on $\mathbb{R}^n$, then $(\mathbb{R}^n, h)$ is a standard pp-wave or a standard plane wave. The universal cover of a compact pp-wave is globally isometric to a standard pp-wave.

Although standard pp-waves on $\mathbb{R}^n$ are the most common, **compact** pp-waves exist also; e.g., on the $n$-torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$, as shown in [19]. The coordinates (2)
make pp-waves look deceptively simple; nevertheless, they are a remarkable class of Lorentzian metrics. Their defining feature is the parallel lightlike vector field $V$ which, in the local coordinates (2), is the gradient

$$V = \partial_v = \text{grad} u.$$ 

This vector field models the wave vector of a gravitational or electromagnetic wave propagating at the speed of light. Indeed, not only are plane waves solutions to the linearized Einstein equations, but, as shown in [11], any vacuum solution to the Einstein equations that possesses a non-homothetic conformal Killing vector field is either conformally flat or a pp-wave. In fact the “wave nature” of pp-waves manifests directly as follows: if for an arbitrary lightlike vector field $Z$ one imposes the geodesic condition $\nabla Z Z = 0$ locally in the coordinates (2), then in dimension 3 this reduces to an (inviscid) Burgers’ PDE, which is well known to describe wave motion and shock waves (in higher dimensions it becomes a coupled system of Burger’s PDEs). Their wave nature aside, our interest in pp-waves is due in particular to their curvature and their geodesic properties:

1. Curvature: For any choice of $H(u, x^3, \ldots, x^n)$, the metric (2) is always scalar-flat; in fact, something deeper is true: for certain choices of $H$, all curvature invariants of (2) vanish identically (see [7]), and yet the metric is not flat in general (a distinctly Lorentzian phenomenon). It is, however, almost Ricci-flat: in the coordinate basis \{\partial_v, \partial_u, \partial_3, \ldots, \partial_n\}, the only nonzero component of the Ricci tensor is

$$\text{Ric}(\partial_u, \partial_u) = -\frac{1}{2} \sum_{i=3}^{n} H_{ii},$$

where $H_{ii} := \frac{\partial^2 H}{\partial x^i \partial x^i}$. Thus, a pp-wave is Ricci-flat if and only if $H$ is harmonic in $x^3, \ldots, x^n$.

2. Geodesics: In local coordinates (2), the geodesic equations of motion are $n$ second-order ODEs in $v, u, x^3, \ldots, x^n$—but in fact they reduce to just $n - 2$ ODEs, namely

$$\ddot{x}^i = H_i(t, x^3(t), \ldots, x^n(t)), \quad i = x^3, \ldots, x^n,$$

where dots indicate derivatives taken with respect to the affine parameter $t$, which can in fact be scaled to equal the coordinate $u$. In other words, the geodesic equations of motion reduce entirely to a (generally time-dependent) Hamiltonian system—and the completeness of such systems is well understood; see, e.g., [10, 12, 13, 29]. Indeed, as we’ll show, (3) is ultimately the reason why the (Riemannian) almost Kähler metrics we construct on $\mathbb{R}^{2n}$ will be geodesically complete.

### 3 From pp-waves to complete Riemannian metrics

It is precisely these two features of pp-waves that make them ideally suited to construct distinguished Riemannian metrics, a process which we now describe. When a nowhere
vanishing timelike vector field $T$ is present, any Lorentzian metric $h$ has a Riemannian “dual” given by

\[ g := h + 2T^b \otimes T^b. \]  

(Here $T^b = h(T, \cdot)$ is the one-form $h$-metrically equivalent to $T$, and in fact $g(T, \cdot) = -h(T, \cdot)$; note that we are assuming for convenience here that $h(T, T) = -1$.) The relationship (4) is well known and has been studied extensively; see, e.g., [20] for a recent analysis which includes, among other things, curvature formulae. We start by making clear what will play the role of the vector field “bridge,” $T$, for us:

**Definition 2** Let $(M, h)$ be a compact or noncompact pp-wave. For a given unit timelike vector field $T$ on $(M, h)$, the Riemannian metric

\[ g := h + 2T^b \otimes T^b \]  

is said to be $T$-dual to $h$. On a standard pp-wave $(\mathbb{R}^n, h)$ in the coordinates (2), define the unit timelike vector field

\[ T := \frac{1}{2}(H + 1)\partial_v - \partial_u. \]  

For this choice of $T$, the corresponding Riemannian metric (5) will be called the standard Riemannian dual.

An important question is when this dual metric $g$ will be complete. One general criterion for the completeness of the Riemannian metric $g$ in (4) is the following: if $g(T, \cdot)$ is bounded on $TM$, then $g$ will be complete if $h$ is (this is a consequence of [9, Proposition 3.4]). In our case, it turns out that a direct analysis of the geodesic equations of (5), without recourse to $h$, is better. Indeed, it turns out the choice of (6) will always yields a complete $g$—a testament to the simplicity of the Hamiltonian geodesic equations (3) of the original pp-wave:

**Proposition 1** The standard Riemannian dual $g$ is complete for any choice of smooth $H(u, x^3, \ldots, x^n)$. Furthermore, $\partial_v$ is a constant length Killing vector field with respect to $g$.

**Proof** Setting $i = x^3, \ldots, x^n$, and using the coordinate components of $g$ in (9), its non-vanishing Christoffel symbols are

\[ \Gamma^u_{vi} = -2\Gamma^i_{vu} = H_i, \quad \Gamma^v_{vi} = \Gamma^i_{iu} = -\Gamma^u_{ui} = -\frac{1}{2}HH_i, \]

\[ \Gamma^v_{uu} = \frac{1}{2}Hu, \quad \Gamma^v_{ui} = -\frac{1}{4}(H^2 - 1)H_i. \]
with corresponding geodesic equations of motion
\[
\begin{align*}
\ddot{v} &= \frac{1}{2} (2\dot{v} + \dot{u} H) H \left( \sum_{i=3}^{n} H_i \dot{x}^i \right) - \frac{1}{2} \left( \sum_{i=3}^{n} H_i \ddot{x}^i \dot{u} \right) - \frac{1}{2} H_u \ddot{u}^2, \\
\ddot{u} &= -(2\dot{v} + \dot{u} H) \left( \sum_{i=3}^{n} H_i \dot{x}^i \right), \\
\ddot{x}^i &= \frac{1}{2} (2\dot{v} + \dot{u} H) H_i \dot{u}.
\end{align*}
\]
\[ (7) \]

The first step is to recognize that \( \ddot{v} \) is integrable by using the equation for \( \ddot{u} \):
\[
\ddot{v} = -\frac{1}{2} H \ddot{u} - \frac{1}{2} \left( \sum_{i=3}^{n} H_i \dot{x}^i \dot{u} \right) - \frac{1}{2} H_u \ddot{u}^2 = -\frac{1}{2} \frac{d}{dt} (H \dot{u}).
\]

This yields a constant of the motion,
\[
2\dot{v} + \dot{u} H = 2\dot{v}_0 + \dot{u}_0 H_0 := c
\]
(with “0” denoting the initial value), which in turn simplifies \((7)\):
\[
\begin{align*}
\ddot{v} &= \frac{\zeta}{2} H \left( \sum_{i=3}^{n} H_i \dot{x}^i \right) - \frac{1}{2} \left( \sum_{i=3}^{n} H_i \ddot{x}^i \dot{u} \right) - \frac{1}{2} H_u \ddot{u}^2, \\
\ddot{u} &= -c \sum_{i=3}^{n} H_i \dot{x}^i, \\
\ddot{x}^i &= \frac{\zeta}{2} H_i \dot{u}.
\end{align*}
\]

If \( H(u, x^3, \ldots, x^n) \) is a given smooth function for which the solutions \( x^3(t), \ldots, x^n(t), \)
\( u(t) \) exist for all \( t \in \mathbb{R} \), then so must \( v(t) \). Hence, we show the former, by observing
that \( \ddot{x}^3, \ldots, \ddot{x}^n, \ddot{u} \) combine to yield another constant of the motion; indeed, denoting
the initial values by the subscript “0” as above,
\[
2 \sum_{i=3}^{n} \ddot{x}^i \dot{x}^i = -\ddot{u} \dot{u} \quad \Rightarrow \quad \sum_{i=3}^{n} (\ddot{x}^i(t))^2 + \frac{1}{2} \dot{u}(t)^2 = \sum_{i=3}^{n} (\ddot{x}_0^i(t))^2 + \frac{1}{2} \dot{u}_0^2.
\]
\[ (8) \]

In particular, each \( |\dot{x}^i(t)| \leq \sqrt{c_2} \), in which case,
\[
|x^i(t)| - |x^i_0| \leq \left| \int_{t_0}^{t} \dot{x}^i(s) \, ds \right| \leq \int_{t_0}^{t} |\dot{x}^i(s)| \, ds \leq \sqrt{c_2} \int_{t_0}^{t} ds = \sqrt{c_2}(t - t_0),
\]
so that \( x^i(t) \) must be bounded over any compact interval \([t_0, t]\), likewise with \( u(t) \).
In particular, their maximal solutions must be global. Finally, that \( \partial_v \) is a constant
length Killing vector field follows because with respect to the (global) coordinate basis \( \{\partial_v, \partial_u, \partial_3, \ldots, \partial_n\} \), the \( g_{ij} \)'s are independent of \( v \):
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\[
(g_{ij}) = \begin{pmatrix}
2 & H & 0 & 0 & \cdots & 0 \\
H & \frac{1}{2}(1 + H^2) & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]

(9)

This completes the proof. \qed

4 The curvature of \( g \)

Note that (9) is not a warped product, though it is an example of so-called *semigeodesic coordinates* in Riemannian geometry; see, e.g., [18]. Note also that \( \partial_v \) is not parallel with respect to \( g \), as it was with respect to \( h \). So much for the geodesic completeness of these metrics; we now move on to computing their curvature; to do so, it is best to work, not in the coordinate basis \( \{ \partial_v, \partial_u, \partial_x^3, \ldots, \partial_x^n \} \), but rather with respect to the following globally defined \( g \)-orthonormal frame:

**Proposition 2** With respect to the orthonormal frame \( \{ T, X_3, \ldots, X_n, Z \} \) defined by

\[
T := \frac{1}{2}(H + 1)\partial_v - \partial_u, \quad X_i := \partial_i, \quad Z := \frac{1}{2}(H - 1)\partial_v - \partial_u,
\]

(10)

any standard Riemannian dual \( g \) in Proposition 1 has Ricci tensor

\[
Ric_g = \frac{1}{2} \begin{pmatrix}
\sum_{i=3}^n H_{ii} & H_{i3} & H_{i4} & \cdots & H_{in} & -\sum_{i=3}^n H_i^2 \\
-H_{33}^2 & -H_{34} & \cdots & -H_{3n} & -H_{3u} & \sum_{i=3}^n H_i^2 \\
-H_{43} & -H_{44}^2 & \cdots & -H_{4n} & -H_{4u} & -H_{4u} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-H_n & -H_{nu} & \cdots & -H_{nn} & -H_{nu} & \sum_{i=3}^n H_i^2
\end{pmatrix}.
\]

(11)

**Proof** Using the \( g \)-orthonormal frame (10), we compute the diagonal components of the Ricci tensor. Using the Christoffel symbols computed in Proposition 1, the covariant derivatives are

\[
\nabla_T T = -\nabla_Z Z = \sum_{i=3}^n \frac{H_i}{2} X_i, \quad \nabla_T Z = \nabla_Z T = \nabla_{X_i} X_j = 0,
\]

\[
\nabla_T X_i = -\nabla_{X_i} Z = -\frac{H_i}{2} T, \quad \nabla_Z X_i = -\nabla_{X_i} T = \frac{H_i}{2} Z.
\]
The components of the Riemann 4-tensor $R_m$ are now easily computed:

\[
\begin{align*}
R_m(Z, T, T, Z) &= g(\nabla_Z \nabla_T T - \nabla_T \nabla_Z T - \nabla_Z [Z, T] T, Z) = \sum_{i=3}^{2n} \left( \frac{H_i}{2} \right)^2, \\
R_m(Z, T, X_i, Z) &= g(\nabla_Z \nabla_T X_i - \nabla_T \nabla_Z X_i - \nabla_Z [T, Z] X_i, Z) = \frac{H_{i,0}}{2}, \\
R_m(Z, T, X_i, T) &= g(\nabla_Z \nabla_T X_i - \nabla_Z \nabla_T X_i - \nabla_Z [T, X_i] T, X_i) = -\frac{H_{i,0}}{2}, \\
R_m(X_i, T, Z, X_j) &= g(\nabla_{X_i} \nabla_T Z - \nabla_T \nabla_{X_i} Z - \nabla_{X_i} [Z, T] X_j, X_j) = -\frac{H_{i,j}}{2}, \\
R_m(T, X_i, X_j, T) &= g(\nabla_T \nabla_X X_j - \nabla_T \nabla_X X_j - \nabla_T [T, X_i] X_j, T) = \frac{H_{i,j}}{2} - \frac{H_{i,j}}{4}, \\
R_m(Z, X_i, X_j, Z) &= g(\nabla_Z \nabla_X X_j - \nabla_X \nabla_Z X_j - \nabla_Z [X_j, T] X_j, Z) = -\frac{H_{i,j}}{2} + \frac{H_{i,j}}{4}, \\
R_m(X_i, T, X_j, X_j) &= g(\nabla_{X_i} \nabla_T X_j - \nabla_T \nabla_{X_i} X_j - \nabla_{X_i} [X_i, T] X_j, X_j) = 0, \\
R_m(Z, X_i, X_j, X_j) &= g(\nabla_Z \nabla_X X_j - \nabla_X \nabla_Z X_j - \nabla_Z [X_i, Z] X_j, X_j) = 0, \\
R_m(T, Z, X_i, X_j) &= g(\nabla_T \nabla_Z X_i - \nabla_Z \nabla_T X_i - \nabla_T [Z, X_i] T, X_j) = 0, \\
R_m(X_i, X_j, X_k) &= g(\nabla_{X_i} \nabla_X X_j - \nabla_X \nabla_{X_i} X_j - \nabla_{X_i} [X_i, X_j] X_k, X_k) = 0, \\
R_m(T, X_i, X_j, X_k) &= g(\nabla_T \nabla_X X_j - \nabla_X \nabla_T X_j - \nabla_T [X_i, Z] X_j, X_k) = 0, \\
R_m(Z, X_i, X_j, X_k) &= g(\nabla_Z \nabla_X X_j - \nabla_X \nabla_Z X_j - \nabla_Z [X_i, X_j] X_k, X_k) = 0.
\end{align*}
\]

From these data, the Ricci tensor (11) easily follows. \(\square\)

These metrics are strongly controlled by their scalar curvature, somewhat reminiscent of the behavior of anti-self-dual 4-manifolds:

**Corollary 1** Let $(\mathbb{R}^n, g)$ be the standard Riemannian dual to a pp-wave metric. Then, the scalar curvature $\text{scal}_g$ is nonpositive and vanishes if and only if $g$ is flat.

**Proof** Since the frame (10) is orthonormal, the scalar curvature is the trace of the matrix (11):

\[
\text{scal}_g = \frac{1}{2} \sum_{i=3}^{n} H_i^2, \tag{12}
\]

which vanishes if and only if each $H_i := \frac{\partial H}{\partial x^i} = 0$. As Proposition 2 shows, this is the case if and only if $R_m = 0$. \(\square\)

The property “$\text{scal}_g \leq 0$ and flat if $\text{scal}_g = 0$” in Corollary 1 bears comparison to the following two facts from almost Kähler geometry: i) anti-self-dual almost Kähler metrics on oriented 4-manifolds must satisfy $\text{scal}_g \leq 0$, and are Kähler if and only if $\text{scal}_g = 0$; ii) in any even dimension, this is also true for any conformally flat almost Kähler metric (see [1, Proposition 1]). As Theorem 1 shows, our almost Kähler metrics have the same property.

### 5 Strictly almost Kähler metrics on $\mathbb{R}^{2n}$

First, recall that an *almost complex structure* $J$ is a smooth endomorphism of the tangent bundle satisfying $J^2 = -1$; it is *compatible* with a Riemannian metric $g$ if $g(J, J) = g$. Given such a pair, the 2-form $\omega := g(\cdot, J)$ is called the *fundamental 2-form.*
**Definition 3** A $2n$-dimensional Riemannian manifold $(M, g)$ is an *almost Kähler manifold* if there is an almost complex structure $J$ compatible with $g$ and such that the fundamental $2$-form $\omega := g(\cdot, J \cdot)$ is closed.

Note that if $J$ were in addition integrable, meaning that its Nijenhuis tensor

$$N_J(a, b) := [Ja, Jb] - J[Ja, b] - J[a, Jb] - [a, b]$$

vanished identically, then $(g, J)$ would be a Kähler manifold. (If $\omega$ is not necessarily closed, then $(g, J)$ is an *almost Hermitian manifold.*) In any case, we are now going to use Proposition 1 to construct explicit examples of complete almost Kähler metrics in all even dimensions $\geq 4$. Our almost complex structure $J$ will be defined with respect to the globally defined $g$-orthonormal basis $\{T, X_3, \ldots, X_{2n}, Z\}$ defined in (10):

$$JT := Z, \quad JX_3 := X_4, \quad \ldots, \quad JX_{2n-1} := X_{2n}. \quad (13)$$

This choice of $J$ yields many complete almost Kähler metrics, in both the compact and the noncompact setting; we begin with the latter:

**Theorem 1** Let $(\mathbb{R}^{2n}, g)$ be the standard Riemannian dual (9) to a pp-wave metric, with $n \geq 2$. Let $J$ be the almost complex structure defined via (13). Then, $(\mathbb{R}^{2n}, g, J)$ is a complete almost Kähler manifold with nonpositive scalar curvature, which is Kähler (in fact, flat) if and only if $g$ is scalar flat.

**Proof** That $J$ is compatible with $g$ follows by definition of $J$ in (13), together with the fact that $\{T, X_3, \ldots, X_{2n}, Z\}$ is a $g$-orthonormal basis. To check that the fundamental form $\omega = g(\cdot, J \cdot)$ is closed, we consider the dual basis $\{\tau, \theta^3, \ldots, \theta^{2n}, \zeta\}$, where

$$\tau := \frac{1}{2}(H - 1)du + dv, \quad \theta^i := dx^i, \quad \zeta := -\frac{1}{2}(H + 1)du - dv. \quad (14)$$

With respect to this dual basis,

$$\omega = \zeta \wedge \tau + \theta^4 \wedge \theta^3 + \cdots + \theta^{2n} \wedge \theta^{2n-1},$$

which is the standard symplectic form on $\mathbb{R}^{2n}$. Thus, $\omega$ is closed and $(g, J)$ is an almost Kähler metric. Finally, we determine when $g$ can be Kähler, i.e., when $J$ will be integrable. Given $JX_i = X_{i+1}$,

$$N_J(T, X_i) = [JT, JX_i] - J[JT, X_i] - J[T, JX_i] - [T, X_i]$$

$$= \frac{H_i + 1}{2}(T - Z) - \frac{H_i + 1}{2}J(T - Z) - \frac{H_i + 1}{2}J(T - Z) - \frac{H_i + 1}{2}(T - Z)$$

$$= \frac{1}{2}(H_i - H_{i+1})(T - Z) + \frac{1}{2}(H_i + H_{i+1})J(T - Z)$$

$$= 0 \iff H_i = H_{i+1} = 0. \quad (16)$$

But as shown in Corollary 1, each $H_i = 0$ if and only if $g$ is scalar flat, which is the case if and only if $g$ is flat. \qed
Corollary 2 Among the strictly almost Kähler metrics on $\mathbb{R}^{2n}$ in Theorem 1, there are left-invariant metrics (for the additive Lie group structure on $\mathbb{R}^{2n}$), namely, for any

$$H(u, x^3, \ldots, x^{2n}) := \varphi(u) + \sum_{i=3}^{2n} a_i x^i,$$

where $\varphi$ is any smooth function on $\mathbb{R}$ and where each $a_i \in \mathbb{R}$.

Proof This is a direct consequence of (12) in Corollary 1, and the fact that in this case the Lie brackets of the $g$-orthonormal basis $\{T, X_3, \ldots, X_{2n}, Z\}$,

$$[T, X_i] = [Z, X_i] = -\frac{H_i}{2} (T - Z), \quad [T, Z] = [X_i, X_j] = 0,$$

will have constant structure constants. \hfill \square

Recall the codifferential $\delta$ of the exterior derivative $d$; when applied to the fundamental 2-form $\omega$, it yields a 1-form $\delta \omega$ given by

$$\delta \omega := (\ast^{-1} \cdot d \cdot \ast) \omega,$$

where $\ast$ is the Hodge star operator with respect to $g$; $\delta$ is the adjoint of $d$ with respect to $g$, in the sense that $\int g(\delta \omega_1, \omega_2) d\text{vol} = \int g(\omega_1, d\omega_2) d\text{vol}$. Its action can equivalently be expressed as

$$\delta \omega(\cdot) = -\sum_i (\nabla_{E_i} \omega)(E_i, \cdot),$$

where $\nabla$ is the Levi-Civita connection of $g$ and where $\{E_1, \ldots, E_{2n}\}$ is any $g$-orthonormal basis (see, e.g., [23, p. 335]). We say that $\omega$ is co-closed if $\delta \omega = 0$, a condition that holds automatically for any almost Kähler metric $g$. What is interesting about the fundamental 2-forms of the almost Kähler metrics of Theorem 1 is that they are co-closed, not just with respect to $g$, but also with respect to the Lorentzian pp-wave metric $h$:

Corollary 3 The fundamental 2-forms of the almost Kähler metrics of Theorem 1 are also co-closed with respect to the Lorentzian pp-wave metrics (2) to which they are dual.

Proof The basis $\{T, X_3, \ldots, X_{2n}, Z\}$ given by (10) is also $h$-orthonormal, with $h(T, T) = -1$ the timelike direction. It is straightforward to show that the Hodge-duals with respect to $g$ and $h$ differ by a constant multiple of $dx^3 \wedge \cdots \wedge dx^{2n}$, so that being co-closed with respect to $g$ implies being co-closed with respect to $h$. (One can also verify this via (18), with the Levi-Civita connection $\nabla^a$ of $h$ being used in place of $\nabla$.)

\hfill \square
6 Compact strictly almost Kähler metrics

We now move to the compact setting, and show that our construction yields a large class of strictly almost Kähler metrics here as well; furthermore, since completeness comes “for free” here, we can considerably expand our collection of almost Kähler metrics by generalizing our notion of a pp-wave. The generalization we need is well known in the literature and consists of allowing the Euclidean “plane front” $\sum_{i=3}^{n}(dx^i)^2$ in (2) to be an arbitrary Riemannian metric; see [8] for a detailed study of such metrics. Our definition here is specifically tailored to the compact setting:

**Definition 4** (General plane-fronted waves; [8]) Let $\varphi, \theta$ denote the standard angular coordinates on $S^1 \times S^1$, and let $(M, g_R)$ be any closed Riemannian manifold. Let $H$ be an arbitrary smooth function on $S^1 \times M$; i.e., one that is independent of the first angular coordinate $\varphi$. Then, the Lorentzian metric $h$ defined on $S^1 \times S^1 \times M$ by

$$h := 2d\varphi d\theta + H d\theta^2 + g_R$$  \hspace{1cm} (19)

is a (compact) general plane-fronted wave.

Note that we are not requiring the curvature condition (1) to hold here, though certainly this can be arranged; e.g., by taking $M = S^1 \times \cdots \times S^1$ with its standard flat metric (see, e.g., [19, Example 1]). Although the metrics (19) are more general than pp-waves, observe that they still come equipped with a parallel lightlike vector field, namely, $\partial_\varphi = \text{grad} \theta$. In any case, almost Kähler metrics exist naturally on such spaces:

**Theorem 2** Let $(M, g_R)$ be a closed almost Kähler manifold and consider the Lorentzian metric $h$ (19) on $S^1 \times S^1 \times M$, with $H$ any smooth function on $S^1 \times M$; i.e., one that is independent of the first angular coordinate $\varphi$. With respect to the vector field

$$T := \frac{1}{2}(H + 1)\partial_\varphi - \partial_\theta,$$

let $g$ denote the Riemannian metric dual to $h$:

$$g := h + 2T^b \otimes T^b.$$  \hspace{1cm} (20)

Then, $g$ is an almost Kähler metric on $S^1 \times S^1 \times M$, which is not a warped product, and which is Kähler if and only if $H$ is constant on $M$ and $(M, g_R)$ is Kähler. Furthermore, the fundamental 2-form of $g$ is also co-closed with respect to the Lorentzian metric $h$.

**Proof** Let $J_R$ be the almost complex structure on $M$ compatible with $g_R$; let $\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}\}$ be any locally defined $g_R$-orthonormal frame on $M$ satisfying

$$J_R X_i = X_{n+i}, \quad i = 1, \ldots, n.$$
Then, $J_R$ extends naturally to a $g$-compatible almost complex structure $J$ on $S^1 \times S^1 \times M$ provided we define, as we did on $\mathbb{R}^{2n}$,

$$JT := Z, \quad Z := \frac{1}{2}(H - 1)\partial_\psi - \partial_\theta.$$

The analogue of (10) on $S^1 \times S^1 \times M$ is therefore the $g$-orthonormal frame \{ $T$, $X_1$, \ldots, $X_n$, $X_{n+1}$, \ldots, $X_{2n}$, $Z$ \}, whose Lie brackets are, analogously to (17),

$$[T, X_i] = [Z, X_i] = -\frac{X_i(H)}{2}(T - Z), \quad [T, Z] = 0. \quad (21)$$

To show that the fundamental 2-form $\omega := g(\cdot, J)$ is closed, we may proceed as we did in the non-compact case in Theorem 1, by observing that $\omega$ arises from the symplectic structures on $S^1 \times S^1$ and $M$, as

$$\omega = d\varphi \wedge d\theta + \omega_m,$$

where $\omega_m := g_k(\cdot, J_k)$; this makes it clear that $(g, J)$ will be almost Kähler if and only if $(g_k, J_k)$ itself is so. (Alternatively, one may verify that $d\omega = 0$ component-by-component, using the Lie brackets (21).) Finally, $g$ will be Kähler if and only if $N_j(T, X_i) = 0$ and $N_j(X_i, X_j) = 0$; the former occurs when $X_i(H) = 0$ as in (16), so that $H$ must be constant on $M$; the latter occurs when $(M, g_k)$ itself is Kähler. Finally, that $\omega$ is also co-closed with respect to the Lorentzian metric $h$ follows as it did in Corollary 3; namely, the Hodge-duals with respect to $g$ and $h$ differ by a constant multiple of $\omega_m$, so that being co-closed with respect to $g$ implies being co-closed with respect to $h$. \hfill \Box

We close with a three final remarks regarding our construction.

1. Taking $M$ to be the $2n$-torus, with $g_k$ its standard flat metric, furnishes a plane-fronted wave (19) (in fact, a pp-wave) and thus an almost Kähler metric (20) on the $(2n + 2)$-torus. As this holds for every choice of smooth function $H$ on $S^1 \times M$, we thus have uncountably many compact examples of almost Kähler metrics dual to pp-waves.

2. If the product $S^1 \times S^1 \times M$ has any non-even odd Betti numbers (e.g., for $M = S^1 \times S^3$), then not only will $g$ not be Kähler, but the manifold $S^1 \times S^1 \times M$ cannot admit any Kähler metric (see, e.g., [3, p. 84]).

3. Speaking of Kähler metrics, we can also realize our almost Kähler metrics as arising from deformations of them. This is done by generalizing our definition of pp-wave by allowing $H$ to be a function of $v$ (for $\mathbb{R}^{2n}$) or $\varphi$ (on $S^1 \times S^1 \times M$) as well. Then, the following is true:

   (i) The vector fields $\partial_v$ or $\partial_\varphi$ would no longer be Killing vector fields. One consequence of this is that the metric on $\mathbb{R}^{2n}$ may no longer be complete (Proposition 1 no longer applies). Let us therefore consider only the compact case, $S^1 \times S^1 \times M$. 

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It turns out that, even when $H$ is taken to be a function of all the coordinates $(\varphi, \theta, x^1, \ldots, x^{2n})$, the corresponding Riemannian dual is still an almost Kähler metric. Indeed, the only change that arises is to the Lie bracket $[T, Z]$ in (17), which now becomes

$$[T, Z] = -\frac{H_\varphi}{2}(T - Z).$$

But a “component-by-component” inspection of $d\omega$ reveals that $d\omega = 0$ will still hold, as before.

Thus, our family of strictly almost Kähler metrics is enlarged considerably by allowing $H$ to be an arbitrary smooth function defined on $S^1 \times S^1 \times M$. But this opens up a new possibility: there are now choices of $H$ for which the Riemannian dual will in fact be a non-flat Kähler metric. Indeed, the condition for the integrability of $J$, (16), still holds as before: we must have each $X_i(H) = 0$, for $i = 1, \ldots, 2n$. If we suppose this happens, then we are left with a function on $S^1 \times S^1 \times M$ of the form $H = H(\varphi, \theta)$ (as opposed to just $H = H(\theta)$, as before). But when $H = H(\varphi, \theta)$, the resulting metric on $S^1 \times S^1 \times M$ will split as a product of a generally non-flat Kähler surface $S^1 \times S^1$ and an almost Kähler metric on $M$ (compare (9)). We may therefore view our construction as also arising from a deformation of Kähler products.

7 Plane wave limits and almost Kähler geometry

In this section, we demonstrate how every even-dimensional Lorentzian metric admits, locally, a (Riemannian) strictly almost Kähler metric via an appropriate limit. This limit is the “plane wave limit” due to Penrose [22], itself an instance of a more general notion of the “limit of a spacetime” pioneered by Geroch [14]. In fact, the existence of Penrose’s limit is intimately connected with the fact, mentioned in Sect. 2, that all the curvature invariants of pp-waves vanish. We begin with a brief, self-contained presentation of Penrose’s construction, followed afterward by Theorem 3, which connects this limit to the existence of almost Kähler metrics. Penrose’s construction is as follows:

1. Given a Lorentzian metric $h$ and a lightlike gradient vector field $N$,

$$N \neq 0, \quad h(N, N) = 0, \quad N = \text{grad}_h f,$$

a so-called lightlike coordinate system $(x^0, x^1, x^2, \ldots, x^n)$ can be set up with respect to which $N = \frac{\partial}{\partial x^\sigma}$ and such that $h$ has the form
\[
(h_{ij}) := \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
1 & h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\
0 & h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\
0 & h_{31} & h_{32} & h_{33} & \cdots & h_{3n} \\
\vdots & \vdots & \vdots & \vdots \ddots & \vdots \\
0 & h_{n1} & h_{n2} & h_{n3} & \cdots & h_{nn}
\end{pmatrix}.
\]

See, e.g., [21, Proposition 7.14, p. 61] for a derivation. (Note that such an \(N\) always exists locally, since the eikonal equation \(h^{ij} f_i f_j = 0\) always admits nontrivial solutions locally.)

2. Now scale these coordinates by defining another coordinate system \((\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n)\) via the diffeomorphism \(\varphi\) given by

\[
(x^0, x^1, x^2, \ldots, x^n) \mapsto (x^0, \Omega^{-2}x^1, \Omega^{-1}x^2, \ldots, \Omega^{-1}x^n),
\]

where \(\Omega > 0\) is a constant.

3. Next, define a new metric \(h_\Omega\) in the new coordinates \((\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n)\) as follows:

\[
(h_\Omega)_{ij} := \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
1 & \Omega^2 h_{11} & \Omega h_{12} & \Omega h_{13} & \cdots & \Omega h_{1n} \\
0 & \Omega h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\
0 & \Omega h_{31} & h_{32} & h_{33} & \cdots & h_{3n} \\
\vdots & \vdots & \vdots & \vdots \ddots & \vdots \\
0 & \Omega h_{n1} & h_{n2} & h_{n3} & \cdots & h_{nn}
\end{pmatrix},
\]

defined in the coordinates \((\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n)\)

where each component \((h_\Omega)_{ij}\) is defined as follows:

\[
(h_\Omega)_{11}(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) := \Omega^2 h_{11}(\tilde{x}^0, \Omega^2 \tilde{x}^1, \Omega \tilde{x}^2, \ldots, \Omega \tilde{x}^n).
\]

\[
(h_\Omega)_{22}(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) := h_{22}(\tilde{x}^0, \Omega^2 \tilde{x}^1, \Omega \tilde{x}^2, \ldots, \Omega \tilde{x}^n),
\]

and similarly with the others. Note that as \(\Omega \to 0\),

\[
\lim_{\Omega \to 0} (h_\Omega)_{11} \overset{(23)}{=} 0 \cdot h_{11}(\tilde{x}^0, 0, 0, \ldots, 0) = 0,
\]

\[
\lim_{\Omega \to 0} (h_\Omega)_{22} \overset{(24)}{=} h_{22}(\tilde{x}^0, 0, 0, \ldots, 0),
\]
etc. The crucial fact is that the metric $h_{\Omega}$ is conformal to the pullback metric $(\varphi^{-1})^*h$; to see this, use the fact that

$$((\varphi^{-1})^*h)(\partial_{\tilde{x}^i}, \partial_{\tilde{x}^j})d\tilde{x}^i \otimes d\tilde{x}^j = h(\partial_{x^i}, \partial_{x^j})dx^i \otimes dx^j,$$

as well as (22), to obtain

$$dx^0 \otimes dx^1 = \Omega^2 d\tilde{x}^0 \otimes d\tilde{x}^1,$$
$$h_{11}(x^0, x^1, x^2, \ldots, x^n) dx^1 \otimes dx^1 \overset{(23)}{=} \Omega^2 (h_{\Omega})_{11}(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) d\tilde{x}^1 \otimes d\tilde{x}^1,$$
$$h_{12}(x^0, x^1, x^2, \ldots, x^n) dx^1 \otimes dx^2 = \Omega^2 (h_{\Omega})_{12}(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) d\tilde{x}^1 \otimes d\tilde{x}^2,$$
$$h_{22}(x^0, x^1, x^2, \ldots, x^n) dx^2 \otimes dx^2 \overset{(24)}{=} \Omega^2 (h_{\Omega})_{22}(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) d\tilde{x}^2 \otimes d\tilde{x}^2,$$

and so on, which clearly yields the relationship

$$(\varphi^{-1})^*h = \Omega^2 h_{\Omega}. \tag{25}$$

In particular, setting $\tilde{h} := (\varphi^{-1})^*h$, the homothety (25) means that the Levi-Civita connections of $h_{\Omega}$ and $\tilde{h}$ are equal: $\nabla^\Omega = \nabla^{\tilde{h}}$.

4. Finally, take the limit

$$h_{rw} := \lim_{\Omega \to 0} h_{\Omega} = \lim_{\Omega \to 0} \frac{(\varphi^{-1})^*h}{\Omega^2}.$$

This limit metric $h_{rw}$ is precisely

$$\begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & h_{22}(\tilde{x}^0, 0, \ldots, 0) & h_{23}(\tilde{x}^0, 0, \ldots, 0) & \cdots & h_{2n}(\tilde{x}^0, 0, \ldots, 0) \\
0 & 0 & h_{32}(\tilde{x}^0, 0, \ldots, 0) & h_{33}(\tilde{x}^0, 0, \ldots, 0) & \cdots & h_{3n}(\tilde{x}^0, 0, \ldots, 0) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & h_{n2}(\tilde{x}^0, 0, \ldots, 0) & h_{n3}(\tilde{x}^0, 0, \ldots, 0) & \cdots & h_{nn}(\tilde{x}^0, 0, \ldots, 0)
\end{pmatrix},$$

defined in the coordinates $(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n)$.

which is in fact a plane wave metric in so-called Rosen coordinates; i.e., an isometry exists between this metric and (2), with $H(u, x^3, \ldots, x^n)$ quadratic in the $x^i$’s (per the definition of plane wave). For the particulars of this isometry, consult, e.g., [4].

In Penrose’s own words, a neighborhood of the integral curve $\gamma$ of $N = \text{grad}_hf$ through the origin has been expanded “out to infinity,” with the metric homothetically scaled up at the same time, all while keeping $\gamma$ itself unaffected—effectively “zooming in” infinitesimally close to $\gamma$. Although this construction is local and clearly depends on the choice of $N$, certain properties of $h$ are preserved regardless of how the limit is taken; e.g., if $h$ is Einstein, then (every) $h_{rw}$ is Ricci flat, and if $h$ is locally conformally flat, then (every) $h_{rw}$ is so as well (see, e.g., [24]). Theorem 1 now provides an easy step to almost Kähler geometry:
Theorem 3  Let \((M, h)\) be a Lorentzian manifold of dimension \(2n \geq 4\). Locally about any point in \(M\), take the plane wave limit \(h_{mw}\) of \(h\), and express it in the coordinates (2). Then, the standard Riemannian dual (5)–(6) of \(h_{mw}\) admits an almost Kähler structure as in Theorem 1.

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