Time-Dependent Hilbert Spaces, Geometric Phases, and General Covariance in Quantum Mechanics

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Abstract

We investigate consequences of allowing the Hilbert space of a quantum system to have a time-dependent metric. For a given possibly nonstationary quantum system, we show that the requirement of having a unitary Schrödinger time-evolution identifies the metric with a positive-definite (Ermakov-Lewis) dynamical invariant of the system. Therefore the geometric phases are determined by the metric. We construct a unitary map relating a given time-independent Hilbert space to the time-dependent Hilbert space defined by a positive-definite dynamical invariant. This map defines a transformation that changes the metric of the Hilbert space but leaves the Hamiltonian of the system invariant. We propose to identify this phenomenon with a quantum mechanical analogue of the principle of general covariance of General Relativity. We comment on the implications of this principle for geometrically equivalent quantum systems and investigate the underlying symmetry group.

1 Introduction

Much of our understanding of the universe relies on General Relativity (GR) and Quantum Mechanics (QM). Besides their overwhelming success in describing a variety of physical phenomena and their incredible predictive power, these theories do not actually have much in

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common. The lack of a consistent unification of GR and QM may be linked to their drastic differences in structure. This point of view underlines the significance of the study of the basic similarities between these theories and the search for their alternative formulations that make these similarities more transparent.

Unlike GR that is a fundamentally geometric theory, QM is algebraic in nature. Given the appeal of a geometric description, many researchers have attempted to formulate QM in a geometric language [1]. This has also been considered as a natural prerequisite for embedding QM into a nonlinear theory [2] that would share similar geometric features with GR and hopefully allow for their unification [3]. Another development that has led to great interest in exploring geometric features of QM is the discovery of the geometric phase and its implications [4].

QM is based on a linear vector space $H$ equipped with a (positive-definite) inner product such that the corresponding metric space is separable and complete. This makes $H$ a separable Hilbert space whose structure is unique. The situation is completely opposite in GR. GR is based on a 4-dimensional manifold $M$ whose structure is neither unique nor fixed by the postulates of the theory. In GR a concrete physical system corresponds to a solution of Einstein’s field equations. These identify a pseudo-Riemannian metric on $M$ which is however subject to active transformations associated with the diffeomorphisms of $M$ [5]. This is widely referred to as the principle of general covariance. It is probably the most important feature of GR and at the same time the very origin of almost all the difficulties associated with its quantization [6].

The purpose of this article is to reveal a previously unnoticed similarity of QM to GR, namely that one can consider formulating the description of a quantum system using a Hilbert space which does not have a fixed metric and that in doing so one is led to a symmetry of the system that is associated with certain transformations of the metric. This symmetry shares the basic properties of the diffeomorphism-invariance of GR and may be viewed as a quantum mechanical analogue of the principle of general covariance.¹

The article is organized as follows. In Section 2, we review some basic facts about unitary-equivalent Hilbert spaces. In Section 3, we outline a unitary quantum description of a given system using a time-dependent Hilbert space and show that the metric coincides with a positive-definite dynamical invariant. Furthermore, we construct a unitary operator that maps the initial time-independent Hilbert space to the time-dependent Hilbert space defined by a dynamical invariant and show that the Hamiltonian of the system is left invariant under this transformation of the metric. This signifies a symmetry of any quantum system that we identify as an analogue of the diffeomorphism-invariance of GR. In Section 4, we show that indeed the above symmetry

¹A quantum mechanical principle of general covariance has been advocated in [7] that is different from the one discussed in this paper.
is associated not with a particular quantum system but with classes of geometrically equivalent quantum systems. Here we also comment on the implications of our findings for a particular method of generalizing QM. Finally, in Section 5, we discuss the underlying symmetry group of the quantum general covariance.

In order to concentrate on the conceptual (rather than technical) issues, we will consider quantum systems with a finite-dimensional Hilbert space \( \mathcal{H} \). The infinite-dimensional case may be treated similarly.

2 Changing the Metric of the Hilbert Space

In the standard canonical formulation of QM, a quantum system is uniquely determined by a Hilbert space \( \mathcal{H} \) and a linear operator \( H : \mathcal{H} \to \mathcal{H} \) called the Hamiltonian. The (pure) states of the system are the rays in the Hilbert space (i.e., the elements of the projective Hilbert space) \[8\]. The physical observables are identified with the self-adjoint linear operators \( O : \mathcal{H} \to \mathcal{H} \) and the dynamics is governed by the Schrödinger equation

\[
\frac{i \hbar}{\partial t} \psi(t) = H \psi(t),
\]

where \( t \in \mathbb{R} \) denotes the time, \( \psi(t) \in \mathcal{H} \) is the evolving state vector, and the Hamiltonian \( H \) may be time-dependent.

Besides defining the dynamics of the system via the Schrödinger equation \[1\], the Hamiltonian \( H \) also determines the energy levels and definite-energy (eigen)states of the system. In order to distinguish the role of the Hamiltonian as the generator of dynamics and as an observable containing the information about the energy of the system, it is useful to introduce an energy observable \( \mathcal{E} : \mathcal{H} \to \mathcal{H} \) that coincides with the Hamiltonian, unless a time-dependent transformation is performed on the Hilbert space, \[9\].

The above description of QM relies on the choice of a fixed metric (inner product) on \( \mathcal{H} \). The observables (including \( \mathcal{E} \)) are required to be self-adjoint, so that their measured (eigen)values be real and their eigenvectors that represent the states of the system after a measurement be orthogonal. The Hamiltonian \( H \) is also assumed to be self-adjoint so that the time-evolution be unitary. This follows from the condition that the quantum system admits a probabilistic interpretation in which the total probability (namely unity) is preserved.

Clearly, the metric of the Hilbert space is not an observable quantity. It does not appear in the Schrödinger equation \[1\] either. Therefore, as far as the physical content of a quantum system is concerned, the choice of a metric on \( \mathcal{H} \) is not unique.

It is usually argued that demanding the self-adjointness of a sufficiently large number of independent observables (a complete or irreducible set of such) uniquely determines the metric
on the Hilbert space up to an irrelevant multiplicative constant. This point of view relies on a (quantization) scheme in which a complete set of observables may be identified with certain linear operators acting in a vector space of state vectors. The above requirement of self-adjointness then leads to an essentially fixed choice for the metric on this space. This however does not preclude the possibility of a simultaneous transformation of the metric, the Hamiltonian, and the observables in such a way that the transformed quantities yield an equivalent description of the same system. We will identify such a transformation with a symmetry of the system, if it leaves the Schrödinger operator $i\hbar \partial_t - H$ invariant. For a time-independent transformation this condition reduces to the invariance of the Hamiltonian $H$. The above definition of symmetry is slightly more general than the one that is usually adopted in quantum mechanics. The latter corresponds to imposing the additional condition that the symmetry transformations belong to the group $U(H)$ of the unitary transformations of the Hilbert space, i.e., in conventional approach to QM, a symmetry transformation is a metric-preserving linear operator that acts in the Hilbert space and commutes with the Schrödinger operator $i\hbar \partial_t - H$; in particular, a time-independent symmetry transformation is a metric-preserving linear operator acting in the Hilbert space and commuting with the Hamiltonian.

The apparent freedom in the choice of the metric on $H$ is, however, overshadowed by the fact that any two metrics on $H$ lead to the same Hilbert space structure. In particular, one may relate them by a unitary operator.

Now, consider a quantum system $S$ whose kinematical and dynamical aspects are respectively described by a Hilbert space $H_1$ and a Hamiltonian operator $H_1 : H_1 \to H_1$. Furthermore, let $H_2$ be another Hilbert space, such that $H_1$ and $H_2$ are unitarily equivalent, i.e., there is a (possibly time-dependent) unitary operator $U : H_1 \to H_2$. Then the system $S$ may also be described by the Hilbert space $H_2$ and the Hamiltonian operator

$$H_2 := U H_1 U^{-1} - i\hbar U \frac{d}{dt} U^{-1}. \tag{2}$$

The observables $O_2$ associated with the Hilbert space $H_2$ are related to the observables $O_1$ associated with $H_1$ according to

$$O_2 = U O_1 U^{-1}. \tag{3}$$

Relations (2) and (3) imply that $U$ maps the solutions $\psi_1(t)$ of the Schrödinger equation defined by the Hamiltonian $H_1$ to the solutions $\psi_2(t)$ of the Schrödinger equation defined by $H_2$, i.e., $\psi_2(t) = U\psi_1(t)$, and that it leaves the transition amplitudes between the energy states and

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A linear operator $U : H_1 \to H_2$ relating two Hilbert spaces $H_1$ and $H_2$ is said to be a unitary operator or an isometry if for all $\psi, \phi \in H_1$, $\langle U\psi, U\phi \rangle_2 = \langle \psi, \phi \rangle_1$, where $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively stand for the inner product of $H_1$ and $H_2$. 


the expectation values of the observables invariant. Therefore, the descriptions of $S$ in terms of $(\mathcal{H}_1, H_1)$ and $(\mathcal{H}_2, H_2)$ are equivalent. The fact that the energy operator $E_2 = U E_1 U^{-1}$ differs from the transformed Hamiltonian $H_2$ is actually necessary for the validity of this equivalence.\(^3\)

Next, suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are the Hilbert spaces obtained by endowing a vector space $\mathcal{V}$ with the inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively. Then there is a positive-definite operator \(^4\) $\eta : \mathcal{H}_1 \rightarrow \mathcal{H}_1$, called a metric, such that for all $\psi, \phi \in \mathcal{V}$, \([11]\),

$$\langle \psi, \phi \rangle_2 = \langle \psi, \eta \phi \rangle_1.$$ \(4\)

Because $\eta$ is a positive-definite operator, it has a unique positive-definite square root $\rho : \mathcal{H}_1 \rightarrow \mathcal{H}_1$. If we view $\rho^{-1}$ as an operator mapping $\mathcal{H}_1$ to $\mathcal{H}_2$, we can easily check that it is unitary: for all $\psi, \phi \in \mathcal{H}_1$,

$$\langle \rho^{-1} \psi, \rho^{-1} \phi \rangle_2 = \langle \rho^{-1} \psi, \eta \rho^{-1} \phi \rangle_1 = \langle \psi, \rho^{-1} \eta \rho^{-1} \phi \rangle_1 = \langle \psi, \phi \rangle_1.$$ 

Here and in what follows $\dagger$ denotes the adjoint of an operator viewed as acting in the Hilbert space $\mathcal{H}_1$, i.e., for a linear operator $L$ acting in $\mathcal{V}$, $L^\dagger$ is defined by $\langle \psi_1, L \psi_2 \rangle_1 = \langle L^\dagger \psi_1, \psi_2 \rangle_1$ for all $\psi_1, \phi_1 \in \mathcal{V}$.

This completes the demonstration of the unitary-equivalence of any two Hilbert spaces with the same vector space structure (dimension) \([12, 13]\). Its direct consequence is that one does not gain much by using different inner products. This is certainly true, if one restricts oneself to time-independent metric operators $\eta$ and the corresponding inner products.

3 Time-Dependent Hilbert Spaces and Dynamical Invariants

The need for formulating quantum mechanics on a Hilbert space with a time-dependent inner product arises in, for example, trying to develop a nonrelativistic quantum mechanics of a particle confined to move on an oscillating membrane or a particle subject to a time-dependent inhomogeneous gravitation field such as a gravitational wave.\(^5\) Nevertheless, to the author’s best

If $\mathcal{H}_1 = \mathcal{H}_2$, we can use any element of the group of all unitary operators acting in $\mathcal{H}_1$ to perform the above transformation of the Hamiltonian and the observables. By definition, these transformations leave the inner product (and metric) of the Hilbert space $\mathcal{H}_1$ invariant. They correspond to (time-dependent) quantum canonical transformations \([8]\).

A positive-definite operator is a positive invertible operator. Alternatively, it is a self-adjoint operator with a positive spectrum.

The same is true of studying the motion of a nonrelativistic particle confined to a box with moving walls. The standard approach to this problem is to make a unitary transformation that fixes the the Hilbert space but makes the potential time-dependent \([13]\). For further details and references see \([8]\).
knowledge, a comprehensive treatment of quantum mechanics with a time-dependent Hilbert space has not yet appeared. The subject has been considered within the context of canonical quantum gravity where the idea is to amend the Schrödinger equation with an extra term to compensate for the contribution of the metric operator that renders the time-evolution non-unitary even for a self-adjoint Hamiltonian \[15\]. This is a rather drastic departure from the ordinary QM. We shall pursue an alternative approach that allows for a unitary time-evolution without modifying the Schrödinger equation. It has its roots in a recent attempt to resolve some of the basic problems of quantum cosmology \[17\].

Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be the Hilbert spaces obtained by endowing a vector space \( \mathcal{V} \) with a time-independent inner product \( \langle \cdot, \cdot \rangle_1 \) and a time-dependent inner product \( \langle \cdot, \cdot \rangle_2 \), respectively. According to \[4\], \( \langle \cdot, \cdot \rangle_2 \) may be expressed in terms of \( \langle \cdot, \cdot \rangle_1 \) and a time-dependent metric operator \( \eta = \eta(t) \). Then a linear operator \( H : \mathcal{V} \to \mathcal{V} \) defines a unitary time-evolution in \( \mathcal{H}_2 \) according to the Schrödinger equation \[1\], if and only if for any pair of solutions \( \psi_2(t) \) and \( \phi_2(t) \) of \[1\] we have

\[
\frac{d}{dt} \langle \psi_2(t), \phi_2(t) \rangle_2 = 0.
\] (5)

Substituting \[4\] in this equation yields

\[
i\hbar \frac{d}{dt} \eta = H^\dagger \eta - \eta H.
\] (6)

In particular if \( H \) is self-adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle_1 \), then we obtain

\[
i\hbar \frac{d}{dt} \eta = [H, \eta].
\] (7)

This is the defining (Liouville-von Neumann) equation for a dynamical invariant \[18\] for a self-adjoint Hamiltonian \( H \) acting in \( \mathcal{H}_1 \). Equation \[6\] is its generalization for a non-self-adjoint Hamiltonian.\[7\]

Indeed, the most general solution of \[6\] is given by \[17\] \[19\]

\[
\eta(t) = U(t, t_0)^{-1} \eta_0 U(t, t_0)^{-1},
\] (8)

where

\[
U(t, t_0) := T e^{-\frac{i}{\hbar} \int_{t_0}^{t} H(t') dt'}
\] (9)

is the time-evolution operator for the Hamiltonian \( H \), \( T \) is the time-ordering operator, \( t_0 \in \mathbb{R} \) is an initial time, and \( \eta_0 : \mathcal{H}_1 \to \mathcal{H}_1 \) is a time-independent positive-definite operator. If we

\[\text{For a similar approach see } \[16\].\]

\[\text{Note however that the dynamical invariants obtained in this was are positive-definite operators.}\]
suppose that, as an operator acting in $\mathcal{H}_1$, $H$ is self-adjoint, i.e., $H^\dagger = H$, then (9) reduces to the following familiar relation for the dynamical invariants $[8]$

$$\eta(t) = U(t, t_0)\eta_0 U(t, t_0)^\dagger. \quad (10)$$

In this case the restriction of the positive-definiteness of $\eta_0$ is actually not significant. Given any dynamical invariant $I(t)$ for a self-adjoint Hamiltonian $H : \mathcal{H}_1 \to \mathcal{H}_1$, one can easily construct a positive definite invariant, e.g., if we use 1 to denote the identity operator for $\mathcal{H}_1$, then $I^2 + 1$ is a positive-definite invariant.

The requirement that $H$ be self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_1$ of $\mathcal{H}_1$ raises the following question. Given that we wish to formulate a quantum mechanics using the time-dependent Hilbert space $\mathcal{H}_2$, should not we require the Hamiltonian $H$ to be self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_2$? The answer to this question is actually negative. One can check that adopting any metric operator of the form $[8]$ ensures the unitarity of the time-evolution. Therefore, in principle, the Hamiltonian $H$ need not be self-adjoint with respect to $\langle \cdot, \cdot \rangle_2$. However, it turns out that requiring $H$ to be self-adjoint with respect to $\langle \cdot, \cdot \rangle_1$ does imply that it is also self-adjoint with respect to $\langle \cdot, \cdot \rangle_2$. To see this, we view $H$ as a self-adjoint Hamiltonian acting in the Hilbert space $\mathcal{H}_1$ and try to use the unitary map $\rho^{-1} = \eta^{-1/2}$ to obtain the transformed Hamiltonian $H_2$ acting in $\mathcal{H}_2$. In view of (10),

$$\rho(t)^{-1} = U(t, t_0)\eta_0^{-1/2}U(t, t_0)^\dagger. \quad (11)$$

Hence $\rho(t)^{-1}$ is also a positive-definite dynamical invariant. Substituting $H$ for $H_1$ and $\rho^{-1}$ for $\mathcal{U}$ in (2), we find the rather remarkable result

$$H_2 = H. \quad (12)$$

In other words, the transformation induced by the unitary operator $\rho^{-1}$ changes the metric of the Hilbert space into a time-dependent metric, but it leaves the Hamiltonian of the system invariant. Because by construction $\rho^{-1}$, viewed as mapping $\mathcal{H}_1$ onto $\mathcal{H}_2$, is a unitary operator and $H_1 = H$ is assumed to be self-adjoint with respect to $\langle \cdot, \cdot \rangle_1$, the Hamiltonian $H = H_2$ is also self-adjoint with respect to $\langle \cdot, \cdot \rangle_2$.

Unlike the Hamiltonian $H$, the observables and in particular the energy operator $\mathcal{E}$ do change under the unitary transformation induced by $\rho^{-1}$. However, their expectation values which are of physical significance remain invariant. As a result $\rho^{-1}$ is a genuine symmetry transformation for the physical system. It differs from the ordinary symmetry transformations — that are linked with the degeneracy structure of the Hamiltonian and realized in terms of the unitary transformations acting within a fixed Hilbert space $[8]$ — in that it changes the metric of the Hilbert space.
In a sense, the ordinary symmetries are the analogues of the passive coordinate transformations of GR and the metric-changing symmetries such as the ones induced by the invariants $\rho^{-1}$ are the analogues of the active diffeomorphisms of GR. In view of this analogy it is tempting to refer to the presence of the above-described metric-changing symmetries of a quantum system as a quantum mechanical principle of general covariance. The principle of general covariance of GR stems from the symmetry of the Einstein’s field equation (alternatively of the Hilbert-Einstein action) under active diffeomorphisms of the spacetime manifold. The quantum mechanical general covariance may also be viewed as a consequence of the invariance of the Schrödinger equation under the metric-changing symmetry transformations of the Hilbert space.

In summary, we identified a simple but generic symmetry of the quantum mechanical description of an arbitrary physical system. The corresponding symmetry transformations are defined by the positive-definite dynamical invariants. They change the metric of the Hilbert space but leave all the physical quantities associated with the system invariant. We propose to refer to this invariance or symmetry principle as the quantum mechanical general covariance.

### 4 Geometrically Equivalent Quantum Systems and Geometric Phases

Consider a pair of quantum systems $S_1$ and $S_2$ that are respectively described by the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and the Hamiltonians $H_1$ and $H_2$. Then, by definition, $S_1$ and $S_2$ are said to be geometrically equivalent if $\mathcal{H}_1 = \mathcal{H}_2$ and $S_1$ and $S_2$ share identical geometric phases for a complete set of initial states, alternatively they share a common nontrivial dynamical invariant [20]. As we argued in Section 3, one can always construct a nontrivial positive-definite dynamical invariant out of a given nontrivial invariant.  

Let $\eta$ be a common positive-definite dynamical invariant of a pair of geometrically equivalent quantum systems $S_1$ and $S_2$. Then the unitary transformation defined by $\rho^{-1} := \eta^{-1/2}$, that changes the fixed metric of the Hilbert space into the time-dependent metric $\eta$, leaves both the Hamiltonians $H_1$ and $H_2$ invariant. Therefore, the metric-changing symmetry-transformations actually signify the symmetries of the geometrically equivalent Hamiltonians. The latter differ by the ordinary symmetries of the invariant [20], i.e., $[H_1 - H_2, \eta] = 0$.

Next, we recall that the basic property of a dynamical invariant is that its eigenvalue problem is essentially equivalent to the time-dependent Schrödinger equation (1), in the sense that it has a complete set of eigenvectors that solve the Schrödinger equation (1). Therefore, any solution

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8Here by a nontrivial invariant, we mean a solution of (7) that is not a multiple of the identity operator.
of (1) may be expressed as a linear combination of a set of eigenvectors of the invariant $\mathbf{I}$. Specifically, suppose (for simplicity) that $\eta$ is a positive-definite dynamical invariant with a nondegenerate spectrum. Then it can be shown that the eigenvalues $\lambda_n$ of $\eta$ are constant and that the evolution operator associated with the Hamiltonian $H$ may be expressed in the form

$$U(t, t_0) = \sum_n e^{i\alpha_n(t,t_0)} |\lambda_n; t\rangle \langle \lambda_n; t_0|,$$

where $\{|\lambda_n; t\rangle\}$ is any complete set of orthonormal eigenvectors of $\eta$ and

$$\alpha_n(t, t_0) = \delta_n(t, t_0) + \gamma_n(t, t_0),$$

$$\delta_n(t, t_0) := -\frac{1}{\hbar} \int_{t_0}^{t} \langle \lambda_n; t'| H(t') | \lambda_n; t' \rangle \, dt',$$

$$\gamma_n(t, t_0) = i \int_{t_0}^{t} \langle \lambda_n; t'| \frac{d}{dt'} | \lambda_n; t' \rangle \, dt'.$$

The phase angles $\delta_n$ and $\gamma_n$ are known as the dynamical and geometric parts of the total phase angle $\alpha_n$. If for some $T \in \mathbb{R}$, $|\lambda_n; T + t_0\rangle = |\lambda_n; t_0\rangle$, then the initial state corresponding to the state vector $|\lambda_n; t_0\rangle$ will be a cyclic state with $\delta_n(t_0 + T, t_0)$ and $\gamma_n(t_0 + T, t_0)$ being the corresponding dynamical and (cyclic) geometric phase angles.

Identifying the positive-definite dynamical invariant $\eta$ with a metric on the Hilbert space reveals the curious fact that the geometric phases are determined by the metric. The latter also determines the evolving state in the projective Hilbert space. But it does not provide the information about the dynamical phase angles, and therefore falls short of fully determining the dynamics (specifically the evolution operator (13)) of the system. This is the key obstruction to formulating QM in terms of a dynamical metric $\eta(t)$ on the Hilbert space and the corresponding linear ‘field equation’ (7). Such a formulation requires, in addition, a mechanism to specify the dynamical phase angles $\delta_n$.

The observations described in the preceding paragraph may be viewed as grounds for a generalization of QM into a theory in which the dynamics of a physical system is determined by a time-dependent metric on the Hilbert space together with a prescription for specifying the dynamical angles $\delta_n$ and a ‘field equation’ for the metric that would be more general than (7). A potential candidate for the latter would be a master equation of the Lindblad type (21) that is used in modelling dissipation in QM. This would generalize (7) and (6) to

$$i\hbar \left[ \frac{d}{dt} \eta + D(\eta) \right] = H^\dagger \eta - \eta H,$$

where $D$ has the general form

$$D(\eta) = \frac{1}{2} \sum_{j=1}^{r} \left( [A_j^\dagger, A_j \eta] + [\eta A_j^\dagger, A_j] \right),$$
and $A_j$ are the Lindblad operators.  

If one adopts the master equation (17) to determine the metric $\eta$ — instead of (7) — and uses the same prescription to compute the dynamical phases appearing in (13) as in the ordinary QM, namely using Eq. (15), then one arrives at a generalization of the ordinary QM. The dynamics of the resulting theory is determined through the action of the time-evolution operator $U(t,t_0)$ according to $|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$, where $U(t,t_0)$ is given by (13) in which $|\lambda_n; t\rangle$ form a complete set of orthonormal eigenvectors of $\eta$ and the phase angles $\alpha_n(t,t_0)$ are given by (14) – (16). For a Hermitian Hamiltonian (in the original fixed metric on the Hilbert space $H$) $U(t,t_0)$ would again be unitary. Therefore, it can be linked to a Schrödinger equation with a Hermitian Hamiltonian:

$$H'(t) := i\hbar \left[ \frac{d}{dt} U(t,t_0) \right] U(t,t_0)^{-1}.$$  

However, as operators acting in the Hilbert space endowed with the metric $\eta$, $U(t,t_0)$ will not generally be unitary and $H'(t)$ will not be Hermitian. It is also not difficult to see that because of the dissipative term $D$ in (17) the quantum systems defined by $H'(t)$ in the fixed Hilbert space is not unitarily equivalent to the one defined by $H'(t)$ in the time-dependent system. The evolution defined by $U(t,t_0)$, as constructed above, will be generally nonunitary. This calls for a more detailed investigation of the implications and potential applications of this type of generalizations of QM that is based on a time-dependent metric.

5 Underlying Group Structure

Consider a quantum system $S$ with a fixed Hilbert space $\mathcal{H}$ and a Hamiltonian $H$. Suppose that $\eta_1$ and $\eta_2$ are a pair of positive-definite dynamical invariants of $H$, and $\mathcal{H}_i$ is the Hilbert space that has the same vector space structure as $\mathcal{H}$ but endowed with the inner product $\langle \cdot, \eta_i \cdot \rangle$, for $i \in \{1, 2\}$. Then $\eta_i^{-1/2}$ is the unitary operator mapping $\mathcal{H}$ onto $\mathcal{H}_i$ and consequently $\mathcal{U} := \eta_2^{-1/2} \eta_1^{1/2}$ is the unitary operator mapping $\mathcal{H}_1$ onto $\mathcal{H}_2$. Clearly, $H$ is invariant under all these transformations. However, the operator $\mathcal{U}$ viewed as an operator acting in $\mathcal{H}$ is not a positive-definite invariant of the Hamiltonian $H$, for although it satisfies the defining relation of an invariant it is not generally Hermitian with respect to the inner product on $\mathcal{H}$. This is easily seen if we use (10) to express $\mathcal{U}$ in the form

$$\mathcal{U}(t) = U(t,t_0)\eta_2(t_0)^{-1/2}\eta_1(t_0)^{1/2}U(t,t_0)^{-1}.$$  

Because $\eta_1$ and $\eta_2$ are determined by their initial values $\eta_1(t_0)$ and $\eta_2(t_0)$, we may view $U(t_0)$ as the transformation that maps the metric $\eta_1$ to $\eta_2$. As any pair of metrics may be
related in this way (while preserving the Hamiltonian), we may identify $U(t_0)$ with an element of the permutation group $S_M$ of the set $\mathcal{M}$ of all positive-definite operators $\eta_0$ acting in the Hilbert space $\mathcal{H}$. It is trivial to see that the converse is also true, i.e., any such permutation may be affected by an operator of the form $U(t_0)$. This shows that the principle of general covariance associated with the presence of metric-changing symmetries of a quantum system has the permutation group $S_M$ as its underlying symmetry group. This is the quantum mechanical analogue of the diffeomorphism group of spacetime in GR.

The main difference is that in GR even after moding out the diffeomorphism symmetry one still has a continuum infinity of possible geometries of the spacetime, whereas in QM moding out the above-mentioned permutation group symmetry one is left with a unique Hilbert space structure. The latter may be viewed as ‘fixing a gauge’ that corresponds to a particular metric on the Hilbert space. The conventional choice for the gauge (metric) is the usual time-independent one. But choosing a particular gauge does not destroy the gauge freedom. The main purpose of this article is to show that one may choose a gauge (a metric) that is time-dependent. This identifies it with a positive-definite dynamical invariant and sheds light on a variety of issues related to geometric phases, geometric descriptions of QM, and ways of generalizing it.

6 Conclusion

The uniqueness of the Hilbert space structure of the Hilbert space of a quantum system usually leads one to undermine the fact that the metric of the Hilbert space is not fixed by physical considerations. Allowing the metric to be dynamical reveals an interesting connection to the theory of dynamical invariants. It shows that the geometric phases may be identified as the properties of a time-dependent metric on the Hilbert space whereas the dynamical phases are not linked to such a manifestly geometric structure.

Another simple outcome of our analysis is the rather remarkable observation that one can indeed view density operators, that form positive-definite dynamical invariants, also as time-dependent metric operators.

The freedom in the choice of a metric among the set $\mathcal{M}$ of all positive-definite dynamical invariants of a given quantum system may be viewed as an albeit rather trivial quantum mechanical analogue of the principle of general covariance of GR. The role of the diffeomorphism group of GR is played by the permutation group of $\mathcal{M}$.

The Schrödinger dynamics of a given system may be formulated in terms of the Liouville-von Neumann equation $[17]$ for the metric, that determines the evolving states as well as the
corresponding geometric phases, and an additional prescription for computing the dynamical phases. This point of view may be used as the basis of a class of generalizations of QM in which the Liouville-von Neumann equation is replaced by a more general ‘field equation’ for the metric on the Hilbert space. A nonunitary class of examples are provided by the Lindblad’s master equation.

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