The motion of galaxy clusters in inhomogeneous cosmologies

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Abstract

Lemaître–Tolman–Bondi inhomogeneous spacetimes can be used as a cosmological model to account for the type Ia supernova data. However, such models also give rise to large velocities of galaxy clusters with respect to the cosmic microwave background. These velocities can be measured using the kinematic Sunyaev–Zeldovich effect. This paper presents a calculation of galaxy cluster velocities as a function of redshift for such a model.

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1. Introduction

Observations of type Ia supernovae [1, 2], along with the usual assumption that the universe is homogeneous, indicate that the expansion of the universe is accelerating. This acceleration necessitates the presence of an exotic form of matter called dark energy, which can most easily be accounted for by a cosmological constant. However, because the scale of the cosmological constant is so small (∼10\(^{-120}\) in Planck units) it is natural to look for alternative explanations for the data. One such alternative explanation is to account for the supernova data using a cosmological model that is inhomogeneous [3, 4]. The simplest such models are the Lemaître–Tolman–Bondi (LTB) spacetimes [5–7]. These are spherically symmetric spacetimes with pressureless fluid (dust) as the matter. Models of this form can be constructed that account for the supernova data as well as the standard homogeneous ‘concordance’ cosmology does [8–12]. Thus, to test the difference between concordance cosmology and inhomogeneous cosmology, a different test is needed. Such a test is provided by the kinematic Sunyaev–Zeldovich (kSZ) effect [13]. This effect comes about when cosmic microwave background (CMB) photons scatter off the gas in a galaxy cluster. The spectrum of the scattered photons depends on the velocity of the galaxy cluster with respect to the microwave background, and thus an analysis of the CMB can be used to find this velocity. Thus, each LTB model must be compared not only to the supernova data but also to the kSZ data, and only those models that match both sets of data can be considered viable.
Analyses of the kSZ effect for particular LTB models have been performed [14, 15] as well as a general analysis for slightly inhomogeneous models [16]. Unfortunately, each LTB model is different, and so each must be separately subjected to the kSZ test (as well as the supernova test) to see if that model is viable. In this paper we will present a general method for finding the velocity of galaxy clusters in LTB models. We will then apply this method to the models treated in [12]. Section 2 presents the general method of calculating these velocities. Section 3 applies this method to the models of [12]. Conclusions are presented in section 4.

2. Calculating velocities

The metric in the LTB spacetimes takes the form

\[ ds^2 = -dt^2 + \left( \frac{r'}{1 + f} \right)^2 \, dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \]  

(1)

Here, the area radius \( r \) is not one of the coordinates but is instead a function of the coordinates \( t \) and \( \tilde{r} \). The function \( f \) is a function of \( \tilde{r} \). An overdot denotes derivative with respect to \( t \) while a prime denotes derivative with respect to \( \tilde{r} \). The LTB spacetime has a preferred time-like vector \( u^a = (\partial/\partial t) \) which is the four-velocity of the dust that makes up the matter in the spacetime. However, an LTB spacetime with a cosmic microwave background has another preferred vector, which we will call \( r^a \), the four-velocity of the CMB. Since the spacetime is spherically symmetric, there is also a preferred unit space-like vector \( r^a \) such that \( r^a + r^a \) is an outgoing null vector and \( r^a - r^a \) is an ingoing null vector. Expressed in terms of \( r^a \) and \( r^a \), the LTB four-velocity \( u^a \) takes the form

\[ u^a = \gamma (t^a + vr^a) \]  

(2)

where \( v \) is the velocity of the LTB observer with respect to the CMB and where \( \gamma = 1/\sqrt{1 - v^2} \). Now let \( n^a_- \) and \( n^a_+ \) be respectively ingoing and outgoing CMB photons each with the same frequency \( \omega \) as measured in the CMB rest frame. Then it follows that these vectors take the form

\[ n^a_- = \omega (t^a - r^a) \]  

(3)

\[ n^a_+ = \omega (t^a + r^a). \]  

(4)

Now define the quantity \( \alpha \) by

\[ \alpha = \frac{u^a n^a_+}{u^a n^a_-}. \]  

(5)

Then it follows that

\[ v = \frac{1 - \alpha}{1 + \alpha}. \]  

(6)

We restrict attention to times when the CMB has decoupled from the matter, so that \( n^a_\pm \) are null geodesics. Thus, to find the velocity of the galaxies with respect to the CMB, we need to find the behavior of null geodesics in LTB spacetimes.

To find the behavior of null geodesics, we first recall some facts about the LTB spacetimes. From the Einstein field equation, it follows that

\[ \dot{r}^2 = f + \frac{F}{r}. \]  

(7)
where the function $F$ depends only on $\tilde{r}$. The density is given by

$$\rho = \frac{F'}{8\pi r^2\tilde{r}'}.$$  
(8)

It is helpful to introduce the quantities $a$, $A$ and $B$ by

$$r = a\tilde{r}$$  
(9)

$$f = A\tilde{r}^2$$  
(10)

$$F = B\tilde{r}^3.$$  
(11)

Then equation (7) becomes

$$\dot{a}^2 = A + \frac{B}{a}$$  
(12)

whose solution is

$$t - t_0 = \int_0^a \frac{du}{\sqrt{A + \frac{B}{u}}}.$$  
(13)

Here, $t_0$ is a function of $\tilde{r}$ whose meaning is the time at which the shell of dust with coordinate $\tilde{r}$ shrinks to zero radius. To have a genuine big bang singularity (that is, one that occurs simultaneously for all the observers with four velocity $u^a$), we chose $t_0 = 0$. We use the coordinate freedom to change $\tilde{r}$ to any function of $\tilde{r}$ to set $B$ to the constant $4/9$. With this choice, the particular LTB model is then determined by the choice of the function $A(\tilde{r})$. In particular, when $A = 0$ the LTB solution becomes the standard dust FRW cosmology with $a = t^{2/3}$.

For any future-directed null geodesic $k^a$ define the quantity $\psi \equiv -ak^a u_a$. Then it follows that $k^a = -\left(\psi/a\right)\epsilon^a$, where

$$\epsilon^a = -\left(\frac{\partial}{\partial t}\right)^a + \epsilon \frac{\sqrt{1 + f}}{r'} \left(\frac{\partial}{\partial \tilde{r}}\right)^a.$$  
(14)

where $\epsilon = \pm 1$ and the sign of $\epsilon$ depends on whether $k^a$ is outgoing or ingoing. Note from the definition of $\psi$ that

$$\alpha = \frac{\psi_+}{\psi_-}$$  
(15)

where $\psi_+$ is $\psi$ corresponding to $n^a_+$ and correspondingly for $\psi_-$. It follows from the form of the LTB metric that

$$k^a k^b \nabla_a u_b = \frac{\psi^2}{a^2 r'}.$$  
(16)

Then using the geodesic equation we find

$$k^a \nabla_a \psi = -\frac{\psi^2}{a^2 r'} (-r' \dot{a} + a \dot{r} + \epsilon \sqrt{1 + f} a')$$  
(17)

from which it follows that

$$\epsilon^a \nabla_a \ln \psi = \frac{1}{ar} (F[a\dot{a}' - \dot{a} a'] + \epsilon \sqrt{1 + f} \dot{a}')$$  
(18)

we will use equation (18) to find the velocity $v$ of the dust relative to the CMB as follows: note that since we have chosen $t_0 = 0$ it follows that at early times the spacetimes become FRW. Note also that in FRW spacetimes $a' = 0$ and therefore $\psi$ is constant. For CMB photons with the average energy this constant, which we will call $\psi_0$ is the same for outgoing and ingoing
photons. Integrating equation (18) back from the point at which we want to calculate \( v \) to the big bang yields \( \ln \left( \frac{\psi}{\psi_0} \right) \) and since we do this for both outgoing and ingoing geodesics, we obtain both \( \frac{\psi_+}{\psi_0} \) and \( \frac{\psi_-}{\psi_0} \) which in turn allows us to calculate \( \frac{\psi_+}{\psi_-} \) and thus the velocity \( v \).

Note that in FRW \( a' = 0 \) and it therefore follows from equation (18) that \( \psi \) is constant along null geodesics. It then follows from equations (6) and (15) that \( v = 0 \) in FRW. In physical terms, in FRW the rest frame of the galaxies and the rest frame of the CMB are the same and therefore \( v \), which is the relative velocity between the galaxies and the CMB, vanishes. This conclusion remains unchanged in FRW with both dust and a cosmological constant. Note, however, that in the usual concordance model our universe is not FRW but rather FRW with perturbations that have grown to become a large scale structure which involves peculiar velocities of galaxies. Thus, the concordance model also predicts a nonzero result for kSZ measurements. But the radial component of these peculiar velocities is just as likely to be positive as negative; and as we will see, the magnitude of the peculiar velocity is considerably less than what one gets for \( v \) in the LTB models. So there is a clear difference in kSZ signature between the usual cosmology and the LTB models.

We will numerically integrate equation (18). However, in order to do that we will need to be able to evaluate all the quantities on the right-hand side of that equation at all points in the integration. As we will show, all that is needed is to know \( r \) and \( a \) and the rest of the quantities follow. However, \( r \) and \( a \) also change as the integration proceeds, so we need to know how they change. From equation (14) we find

\[
\ell a \nabla_a r = \epsilon \sqrt{1 + f - \dot{r}}
\]

(19)

\[
\ell a \nabla_a a = \epsilon \sqrt{1 + f + \frac{a'}{r'}} - \dot{a}.
\]

(20)

Note that an ingoing geodesic that goes through \( r = 0 \) becomes an outgoing geodesic, where this change is implemented as a change in the sign of \( \epsilon \). We must evolve the system of equations (18)–(20). To do this, we must be able to evaluate all quantities on the right-hand sides of these equations from just \( r \) and \( a \), that is, we must be able to find \( \tilde{r} \), \( f \), \( \dot{a} \), \( r' \), \( \dot{r}' \), \( a' \) and \( \tilde{r}' \). Given \( r \) and \( a \) we obtain \( \tilde{r} = r/a \) and thus we find all specified functions of \( \tilde{r} \) such as \( A \) and \( f \). From equation (12) we find

\[
\dot{a} = \sqrt{A + \frac{B}{a}}
\]

(21)

which then yields \( \tilde{r} \) since \( \dot{\tilde{r}} = \dot{a} \tilde{r} \). Differentiating equation (13) with respect to \( \tilde{r} \) and solving for \( a' \) yields

\[
a' = \dot{a} I A'
\]

(22)

where

\[
I = \frac{B}{2 A^{5/2}} \left[ s \left( 3 + \frac{Aa}{B} \right) - \frac{3}{2} \ln \left( \frac{1 + s}{1 - s} \right) \right]
\]

(23)

with \( s \equiv \sqrt{A/\dot{a}} \). This then yields \( r' \) through \( r' = a + \tilde{r}a' \). Finally, differentiating equation (21) with respect to \( \tilde{r} \) yields

\[
\dot{a}' = \frac{A'}{2} \left( \frac{1}{\tilde{a}^2} - \frac{BI}{a^2} \right)
\]

(24)

which in turn yields \( \dot{r}' = \dot{\tilde{r}} \tilde{r}' \).
What remains is then simply to find the initial values of $r$ and $a$ for each point in our past light cone, along with the corresponding values of the redshift $z$. As shown in [12] this can be done by integrating equations (19)–(20) from our current position, along with the evolution equation for the redshift

$$\ell^a \nabla_a z = \frac{\dot{r}'}{r'} (1 + z). \quad (25)$$

For this integration, the initial value of $r$ is zero, while the initial value of $a$ is given by

$$a = \frac{\Omega}{4} \left( \Omega_M^{-1} - 1 \right) \quad (26)$$

where $\Omega_M$ is the ratio of the density in matter to the critical density.

### 3. Results

In the models of [12] a choice of $\Omega_M$ is made, and the function $A$ takes the form

$$A = \frac{1}{1 + (c\tilde{r})^2} \quad (27)$$

where the constant $c$ is chosen for best fit with the supernova data. In particular, the two models considered are (i) $\Omega_M = 0.3$, which gives rise to $c = 8.5$, and (ii) $\Omega_M = 0.2$, which gives rise to $c = 5.1$. Each of these models fits the supernova data about as well as the standard concordance cosmology. Because $A \to 0$ at large $\tilde{r}$, these models are asymptotically $\Omega = 1$ FRW cosmologies with an underdensity in a region near the center. Note that the large value of $c$ means that in these models we live in an underdensity of fairly small size.

Using the method of the previous section, we calculate the velocity of the dust with respect to the CMB for each point on our past light cone. The results are shown in figures 1 and 2. Here, figure 1 is for the $\Omega_M = 0.3$ model and figure 2 is for the $\Omega_M = 0.2$ model. Also plotted on the figures are the results of observations of the kSZ effect for galaxy clusters [17–19].
The observations clearly rule out the $\Omega_{M} = 0.2$ model. In addition, the $\Omega_{M} = 0.3$ model does not provide a good fit to the observations, so this model may also be ruled out.

4. Conclusions

Any successful LTB model must pass both the supernova test and the kSZ test. Together these tests provide a powerful constraint that rules out the models of [12]. However, the models of [12] were chosen on the basis of simplicity and fitting the supernova test, so it is not surprising that one additional test suffices to rule them out. It is therefore possible that reasonable LTB models could be made that have fair agreement with both the supernova data and the kSZ data. However, as kSZ measurements improve in accuracy, this test will become more stringent and might eventually rule out all LTB models.

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