Conformal Field Theory, Geometry, and Entropy

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In the context of the AdS/CFT correspondence, an explicit relation between the physical degrees of freedom of 2+1d gravity and the stress tensor of 1+1d conformal field theory is exhibited. Gravity encodes thermodynamic state variables of conformal field theory, but does not distinguish among different CFT states with the same expectation value for the stress tensor. Simply put, gravity is thermodynamics; gauge theory is statistical mechanics.

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1. Introduction

The thermodynamic character of gravity in the presence of black holes has led to a longstanding search for an underlying statistical mechanics. This quest has been plagued by a number of conceptual issues. In a local field theoretic approach to gravity, what is the meaning of equilibrium thermodynamics of black holes when macroscopic regions of the spacetime are out of causal contact – especially since the origin of the thermodynamics is the presence of the horizon itself? If one can localize field theoretic excitations in a finite region, where and of what nature are the set of field configurations that characterize the entropy?

While a definitive answer to these questions has not yet been found, the duality between anti-de Sitter (super)gravity and conformal field theory (CFT) conjectured \[1\] on the basis of recent advances in string theory may contain the key physical insights. The maximal scope of the conjecture posits that the full M/string theory in asymptotically anti-de Sitter spacetimes $\text{AdS}_p \times K$ is equivalent to a particular conformal field theory in $p-1$ spacetime dimensions. In this construction, a conventional quantum field theory (with positive norm Hilbert space and unitary evolution) – which is the infrared limit of some generalized gauge theory of brane dynamics – provides the underlying degrees of freedom. “Local” quantum fields coupled to gravity are highly composite operators \[2-4\] built from these gauge theory degrees of freedom; however, the extent to which excitations may be actually localized in $\text{AdS}_p \times K$ is not clear. The fact that the density of states of the dual description grows asymptotically like that of a lower dimensional field theory (rather than a ten- or eleven-dimensional M/string theory) militates against locality \[6\]. On the other hand, this density of states is compatible with the entropy of AdS-Schwarzschild black holes \[3,5\].

Gravity in 2+1 dimensions is a useful arena for the exploration of the relation between quantum black holes and thermodynamics. In the presence of a negative cosmological constant $\Lambda = -\frac{1}{\ell^2}$, 2+1 gravity admits the analogue of anti-de Sitter Schwarzschild black hole solutions \[7\], known as BTZ black holes. These solutions exhibit all the usual thermodynamic properties of black holes: their entropy is the horizon area in Planck units

$$S = \frac{2\pi r_+}{4G},$$  (1.1)

and they obey the first law

$$dE = TdS + \Omega dJ.$$   (1.2)
Here $E$ is the ADM energy, $T$ is the Hawking temperature (defined as $1/2\pi$ times the surface gravity $\kappa$), $J$ is the angular momentum, and $\Omega$ is the angular potential.

A useful property of pure 2+1 gravity is the absence of dynamical bulk degrees of freedom; the gauge freedom is sufficient to push all the dynamics onto the boundaries of the space – horizons, naked singularities (such as are produced by heavy point particle sources), and the timelike boundary at spatial infinity when $\Lambda < 0$. Thus, there is a clean separation between the gravitational sector, containing only global degrees of freedom; and any given matter sector, whose local bulk dynamics one wishes to couple to gravity.

For these reasons, we wish to concentrate on the particular example of $AdS_3$ gravity, because all the ingredients of the black hole puzzle are present in a very controlled setting. Both $AdS_3$ gravity and its proposed dual 1+1 dimensional conformal field theory are representations of the infinite-dimensional Virasoro algebra with central extension $[8,11]$

$$c = \frac{3\ell}{2G}.$$ (1.3)

The unitary CFT has an asymptotic density of states $[12]$

$$S = 2\pi[(cL_0/6)^{\frac{1}{2}} + (c\tilde{L}_0/6)^{\frac{1}{2}}];$$ (1.4)

this level density matches that of BTZ black holes $[7]$ with mass and spin determined by equating the Casimirs of the Virasoro representation

$$\ell M = L_0 + \tilde{L}_0, \quad J = L_0 - \tilde{L}_0.$$ (1.5)

Thus one expects that the states of the CFT are indeed the microstates responsible for the BTZ black hole entropy.

The point of view we will try to justify here is that 2+1 gravity is a collective field excitation of the underlying dual conformal field theory description $[11]$, constructed from the CFT stress tensor. Because gravity itself carries no local excitations, only the global geometric data of the spacetime appears in the construction. It is this global data (which can be thought of as a set of Noether charges $[8]$) that couples to thermodynamics. On the other hand, since it is constructed solely from the CFT stress tensor, gravity cannot distinguish among CFT states of the same energy and other charges. Indeed, as we shall see, the density of states of gravity is (1.4) with $c = c_{\text{eff}} = 1$ rather than (1.3). To summarize the situation in brief:
Gravity is thermodynamics; gauge theory (of branes) is statistical mechanics.

Since the classical gravity solution represents the typical microstate of the underlying CFT with the same global properties (or equivalently, an average over these microstates), one should not be asking gravity to provide an explanation of the entropy. It is quite possible that gravity itself will ultimately be understood as a thermodynamic phenomenon; for thoughts along these lines, see [13].

2. Gravity on $AdS_3 \times K$ and its CFT dual

We will focus for the moment on the pure gravity sector. In general, one is interested in gravity coupled to matter (a particular case of interest is the matter arising from string theory compactification). Later we will return to the the inclusion of matter into our considerations. The low-energy regime of gravity in any of these theories is described by $AdS_3$ Chern-Simons (super)gravity [16,17] with gauge group $SL(2,R)_L \times SL(2,R)_R$, which is the global conformal symmetry of the dual CFT. The convenient variables are the ‘gauge’ fields $A = \omega + e/\ell$ and $\tilde{A} = \omega - e/\ell$, in terms of which the action is

$$\frac{1}{16\pi G} \int e(R - 2\Lambda) = \frac{k}{4\pi} \left( \int (AdA + \frac{2}{3}A^3) - \int (\tilde{A}d\tilde{A} + \frac{2}{3}\tilde{A}^3) \right). \quad (2.1)$$

Here $\Lambda = -\ell^{-2}$, and $k = \ell/(4G)$; $\ell$ is the $AdS_3$ radius, and $G \sim \ell_{pl}$ is the 2+1d Planck scale.

It was shown by Brown and Henneaux [8] that the global conformal algebra extends to the full Virasoro algebra of diffeomorphisms which preserve the asymptotically anti-de Sitter form of the metric; the resulting algebra of gravitational Noether charges $L_n, \bar{L}_n$ has central charge $c = 6k = \frac{3\ell}{2G}$. In global coordinates where the metric takes the asymptotic form

$$ds^2 \sim \ell^2 \left( \frac{1}{r^2} dr^2 - r^2 du dv + \gamma_{uu}(du)^2 + \gamma_{vv}(dv)^2 + O(1/r) \right), \quad (2.2)$$

these diffeomorphisms are analytic reparametrizations of $u$ and $v$ ($\theta = \frac{1}{2}(u - v)$ is taken periodic: $\theta \sim \theta + 2\pi$). Here and below, all coordinates will be made dimensionless by referring

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1 There are claims in the literature [14] that this is already sufficient to explain 2+1d black hole entropy (although see [11,13]).

2 The supersymmetric completion is irrelevant for the issues arising in the present investigation.

3 In the particular case of bound states of D1- and D5-branes, $K = S^3 \times M$, and a candidate for the dual CFT is a resolution of $S^k(M)$ – the symmetric orbifold of $k = Q_1Q_5$ copies of $M$ [18].
them to the scale \( \ell \). Declaring that these diffeomorphisms are not allowed gauge symmetries imposes the boundary conditions that the connections \( A_\mu \sim G^{-1} \partial_\mu G \), \( \tilde{A}_\mu \sim \partial_\mu \tilde{G} \cdot \tilde{G}^{-1} \), with

\[
G = \begin{pmatrix} \sqrt{r} & 0 \\ 0 & \sqrt{r} \end{pmatrix} g(u) \\
\tilde{G} = \tilde{g}(v) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{r} \end{pmatrix} ;
\]

furthermore, \( g(u) \) and \( \tilde{g}(v) \) undergo a Borel-type (i.e., upper- or lower-triangular) restriction \([19]\) that reduces the corresponding affine \( SL(2, R) \) currents \( g^{-1} \partial_u g, \partial_v g \cdot g^{-1} \) to left and right Virasoro algebras with \( c = 6k \) \([20]\). Canonical expressions for the generating functions of these algebras are \((\alpha = \frac{1}{2} \sigma_3)\) \([21, 22]\)

\[
T_{uu} = \sum_n L_n e^{-inu} = \frac{k}{2} \text{Tr} \left( 2\alpha \partial_u A_u + A_u A_u \right) \\
T_{vv} = \sum_n L_n e^{-inv} = \frac{k}{2} \text{Tr} \left( 2\alpha \partial_v \tilde{A}_v + \tilde{A}_v \tilde{A}_v \right) .
\]

The trivial bulk dynamics of pure 2+1 gravity allows one to collapse its dynamics onto the boundary; one finds \([19]\) a Liouville action, and the generators \((2.4)\) are just the components of the Liouville stress tensor

\[
T_{uu}^{\text{liou}} = k(\partial_u \varphi \partial_u \varphi - \partial_u^2 \varphi) = k \gamma_{uu} \\
T_{vv}^{\text{liou}} = k(\partial_v \varphi \partial_v \varphi - \partial_v^2 \varphi) = k \gamma_{vv} .
\]

In principle, there may be several boundaries of the space: Timelike singularities corresponding to point particle sources; the timelike boundary at spatial infinity of anti-de Sitter space; and one may also wish to consider the horizon of a black hole as a boundary. The first two of these have sensible interpretations in the AdS/CFT correspondence, whereas the last does not; below, we will try to argue that the black hole horizon should not be taken as a boundary of spacetime.

In the AdS/CFT correspondence, the CFT dual to 2+1 AdS gravity is a representation of the same algebra of global symmetries, generated by the CFT stress tensor. At the semiclassical level, the symmetry generators of the two theories must match. Thus one is led to propose the identification of the Liouville field \( \varphi \) as a kind of collective coordinate of the dual CFT at large \( k \) via

\[
\langle T_{uu} \rangle_{\text{CFT}} = T_{uu}^{\text{liou}} .
\]
This identification is an example of the general correspondence between the sub-leading asymptotic behavior of a bulk field and the expectation value of an operator in the CFT. The relation (2.6) is not meant to imply that the underlying large $k$ CFT is entirely equivalent to Liouville theory; rather, we are simply making use of the fact that the current sector of any conformal field theory is universal, and that the Liouville action is the universal effective action that encodes the Virasoro Ward identities at the semiclassical level. One expects that the quantum fluctuations of the two theories are rather different.

Classical solutions to 2+1 gravity can be characterized by the holonomies of the two Chern-Simons connections $A$, $\tilde{A}$. Recall that holonomies in $SL(2, R)$ fall into three conjugacy classes:

1. Hyperbolic elements, conjugate to a dilation $g \sim \left( e^{2\pi \lambda} 0 \\ 0 e^{-2\pi \lambda} \right)$.
2. Parabolic elements, conjugate to a translation $g \sim \left( 1 2\pi a \\ 0 1 \right)$.
3. Elliptic elements, conjugate to a rotation $g \sim \left( \cos 2\pi \alpha \ \ \text{sin} 2\pi \alpha \\ -\text{sin} 2\pi \alpha \ \ \cos 2\pi \alpha \right)$.

Let us first consider several examples; then we will re-examine these solutions in the framework of Liouville/Virasoro theory.

*Particle* states: Classical particle sources in 2+1 gravity introduce conical timelike singularities in the geometry, with the deficit angle $\pi(\alpha + \tilde{\alpha})$ proportional to the mass of the particle. Around such sources, the connections $A$ and $\tilde{A}$ have elliptic holonomy $cos[2\pi \alpha]$ and $cos[2\pi \tilde{\alpha}]$, respectively. The exact classical solution for a single source is

\[
A = \frac{1}{2} \left( \begin{array}{c} \frac{dr}{(r^2 + \gamma^2)^{1/2}} \\ \frac{dr}{(r^2 + \tilde{\gamma}^2)^{1/2}} \end{array} \right) \left( \begin{array}{c} (r^2 + \gamma^2)^{1/2} du \\ (r^2 + \tilde{\gamma}^2)^{1/2} dv \end{array} \right),
\]

\[
\tilde{A} = \frac{1}{2} \left( \begin{array}{c} -\frac{dr}{(r^2 + \gamma^2)^{1/2}} \\ \frac{dr}{(r^2 + \tilde{\gamma}^2)^{1/2}} \end{array} \right) \left( \begin{array}{c} (r^2 + \gamma^2)^{1/2} du \\ (r^2 + \tilde{\gamma}^2)^{1/2} dv \end{array} \right),
\]

where $\gamma = 1 - \alpha$, and $\tilde{\gamma} = 1 - \tilde{\alpha}$. Such sources can be thought of as Wilson lines of the Chern-Simons gravity theory. Particles with spin have $\alpha \neq \tilde{\alpha}$. The connection (2.7) leads to a stress tensor (2.4) with $L_0 = -k^2 \gamma^2$; the vacuum $AdS_3$ corresponds to $\gamma = 1$.

A geometrical particle source adds energy $L_0 = -k^2 (\alpha - 1)$ above the $AdS_3$ vacuum; its ADM mass and spin are

\[
\ell M = L_0 + \tilde{L}_0 - M_0
\]

\[
J = L_0 - \tilde{L}_0
\]

(2.8)
relative to the AdS vacuum of ‘mass’ $\ell M_0 = -k/2$ and spin $J = 0$.

One can compare this geometrical source with the quanta of a massive field in AdS$_3$. The field solves the wave equation

$$\Delta = -(L_1 L_{-1} + L_{-1} L_1) + 2L_0^2 = \frac{m^2 \ell^2}{2}.$$  \hfill (2.9)

The eigenfunctions of the wave operator with a given mass form a discrete series representation of $SL(2, R)_L \times SL(2, R)_R$, with highest weight

$$L_0 = \tilde{L}_0 = \frac{1}{2} [1 + (m^2 \ell^2 + 1)^{\frac{1}{2}}].$$  \hfill (2.10)

For sufficiently large mass $m\ell \gg 1$ – so that the particle can be treated as a semiclassical source for its gravitational field – the particle created by this field adds an ADM energy $\delta M = 1 + (m^2 \ell^2 + 1)^{\frac{1}{2}} \sim m\ell$ to the AdS vacuum. Thus we see that, to leading order, the geometrical notion of mass in 2+1 gravity agrees with the kinematic mass in the wave equation: $m\ell \sim k\alpha$.

The 2+1 black hole: The connections are \cite{7}, \cite{22} ($\rho$ is a radial coordinate, asymptotic to $\log(r)$)

$$A = \frac{1}{2} \begin{pmatrix} d\rho & z_+ e^\rho du \\ z_+ e^{-\rho} du & d\rho \end{pmatrix} \quad \tilde{A} = \frac{1}{2} \begin{pmatrix} d\rho & z_- e^{-\rho} dv \\ z_- e^\rho dv & d\rho \end{pmatrix},$$  \hfill (2.11)

in terms of which the mass and spin are

$$\ell M = L_0 + \tilde{L}_0 = k(z_+^2 + z_-^2)$$

$$J = L_0 - \tilde{L}_0 = k(z_+^2 - z_-^2);$$  \hfill (2.12)

Geometrically, $r_\pm = \frac{1}{2}(z_+ \pm z_-)$ are the radii of the inner and outer horizons. The holonomies at constant $\rho$, $t$ are

$$\text{Tr}(\exp \oint A) = 2 \cosh(\pi z_+) \quad , \quad \text{Tr}(\exp \oint \tilde{A}) = 2 \cosh(\pi z_-).$$  \hfill (2.13)

These holonomies are in the hyperbolic conjugacy class of $SL(2, R)$ (similarly for $\exp[\oint \tilde{A}]$). The extremal limit arises by taking, say, $z_-$ to zero; then $\exp[\oint A]$ is hyperbolic, and $\exp[\oint \tilde{A}]$ is parabolic. The ‘double extreme’ black hole, with $M = J = 0$, has $z_+ = z_- = 0$, and both holonomies are parabolic; this solution is also reached as the limit of the particle solution \cite{2.7} where $\gamma, \tilde{\gamma} \to 0$. A solution with nontrivial expectation value for the $L_n$’s is
the exact ‘gravitational wave’ solution built by a simple modification of the double extreme black hole

\[ A \sim \frac{1}{2} \begin{pmatrix} \frac{dr}{r} & ru \\ \frac{1}{krT} & -dr/r \end{pmatrix} \quad \tilde{A} \sim \frac{1}{2} \begin{pmatrix} -dr/r & 0 \\ rdv & dr/r \end{pmatrix} \quad (2.14) \]

3. Liouville interpretation of the solutions

There is a Virasoro interpretation of each of these classes of monodromies as well. It is intimately related to the mathematics of the classical solutions of Liouville theory. Given a stress tensor \( T_{uu} \), one can construct two solutions \( \psi_1, \psi_2 \) to the Schrodinger equation with \( T_{uu} \) as the potential:

\[ (\partial_u^2 + T_{uu})\psi = 0 \quad (3.1) \]

Setting the Wronskian of the two solutions to one, the quantity \( F(u) = \psi_1/\psi_2 \) is a uniformizing coordinate for a Riemann surface, and

\[ \psi_1 = \frac{F}{\sqrt{\partial F}} \quad \psi_2 = \frac{1}{\sqrt{\partial F}} \quad (3.2) \]

The stress tensor is simply the Schwarzian derivative of \( F \):

\[ T_{uu} = -\frac{k}{2} \{ F, u \} = -\frac{k}{2} \left[ \frac{\partial^3 F}{\partial u^3} - 3 \left( \frac{\partial^2 F}{\partial u^2} \right)^2 \right] \quad (3.3) \]

Around the singularities of \( T_{uu} \) (for example, \( h(e^{iu} - e^{iu_0})^{-2} \) for a primary field), one has \( SL(2, R) \) monodromy

\[ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (u + 2\pi) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (u) \quad , \quad ad - bc = 1 \quad (3.4) \]

Finally, having solved for \( F(u) \) as well as its counterpart \( \tilde{F}(v) \) built from \( T_{vv} \), a classical solution to Liouville theory can be constructed as \( \exp[-\varphi_L] = \psi_1 \tilde{\psi}_2 - \psi_2 \tilde{\psi}_1 \) (equivalently, \( \varphi = \partial \log \psi_2 \) is a free field which is the Backlund transform of the Liouville field). The three different conjugacy classes of \( SL(2, R) \) give rise to three distinct classes of solutions to Liouville theory:

| SL(2, R) Conj. class | Uniformizing coord. | Riem. surf. feature | 2+1d interp. |
|----------------------|---------------------|--------------------|-------------|
| Elliptic             | \( F(u) = \tan[\alpha u] \) | conical singularity | ptcle in \( AdS_3 \) |
| Parabolic            | \( F(u) = au \)     | cusp singularity   | extreme BH   |
| Hyperbolic           | \( F(u) = \exp[\lambda u] \) | handle             | nonextreme BH |
The stress tensor is $T_{\mu\nu} = -\frac{k}{4} \alpha^2$ in the elliptic case; $T_{\mu\nu} = 0$ in the parabolic case; and $T_{\mu\nu} = \frac{k}{4} \lambda^2$ in the parabolic case. To be completely explicit, the asymptotic behavior of the Chern-Simons connection is determined in terms of the above data via (2.3), with $g(u)$ the Wronskian matrix of $\psi_1, \psi_2$.

There is a bound $m\ell < k/2$ on particle masses; an object of larger mass is a black hole. A ‘stringy exclusion principle’ has been proposed \cite{10} to explain the truncation of the BPS supergravity spectrum. This latter bound amounts to $m\ell < k$; it would seem, therefore, that the bound is inextricable from black hole physics, and that one should not expect to be able to see it in perturbative supergravity (or perturbative string theory, for that matter).

We see that to each classical solution of 2+1 gravity, the asymptotic behavior of the metric at infinity is associated uniquely with a classical Liouville field; in turn, this Liouville field is in one-to-one correspondence with the stress tensor data of the dual conformal field theory.

Of course, a given asymptotic behavior can match onto many interior solutions; for instance, all stationary multiparticle states with a given total ADM energy will asymptote to (2.7). We can qualitatively build such multiparticle states in gravity in the context of the AdS/CFT correspondence. Geometrically, particles are Wilson lines if they are heavy enough (and field wavepackets if they are light), and gravitational waves are distortions of the geometry which can be introduced separately at each source (by cutting out a solid cylinder at fixed small radius enclosing the source, performing a conformal transformation on the boundary of the cylinder, and gluing the geometry back together). On the CFT side, the operators related to matter fields in the low-energy bulk theory are scaling operators of small anomalous dimension (relative to $k/4$) \cite{4}. The primary state $|\alpha\rangle = O_{\alpha} |0\rangle$ and its $SL(2, R)$ descendants $L^m L^m |\alpha\rangle$ form a basis of modes of the matter field \cite{23, 26, 4, 5}, while the higher Virasoro raising operators ($L_{-n}, \bar{L}_{-n}, n \geq 2$) acting on these states

\footnote{Strictly speaking, all of the considerations so far have involved only the asymptotic behavior of the metric; one might wonder whether a metric which is asymptotic to the BTZ solution can match onto a nonsingular interior, say a 2+1 stellar equilibrium solution. The work of \cite{23} shows that such solutions may exist, as well as the existence of an upper bound on their mass. Thus, even though there is no long-range gravitational force, the ‘attraction’ of geodesics in AdS$_3$ causes any sufficiently high mass state to evolve to a black hole.}

\footnote{The mismatch between these two bounds might be due to the considerations of the previous footnote.}
amount adding a ‘gravitational wave’ along the lines of (2.14). In the semiclassical limit of large \( k \), with well-separated sources, one should be able to largely ignore the nonlinear effects of gravity and treat the sources as independent. The analogous CFT state would be

\[
\prod_\alpha \left( \prod_{\{n_i, \bar{n}_i\}} L_{-n_i} \bar{L}_{\bar{n}_i} O_\alpha \right) |0\rangle.
\] (3.5)

Once the black hole transition is reached at \( L_0 = \tilde{L}_0 = k/4 \), the states of this form are a highly redundant description of the Hilbert space (c.f. [4] for a discussion); this is a manifestation of the ‘holography’ (vastly reduced number of degrees of freedom relative to local quantum field theory) of the construction. We will return to this subject below.

The fact that Euclidean saddle points of the gravitational action encode the black hole density of states [27], in a manner that does not admit interpretation in terms of state counting, supports the present perspective – that gravity is thermodynamic in nature and should not know about the microphysics. Chern-Simons/Liouville theory codes the 2+1 black hole density of states as the action of a saddle point in the Euclidean domain [28, 22, 10, 29]. It is in this way that the classical central charge (1.3) of Liouville theory enters the discussion. The Euclidean continuation of the BTZ solution (2.11) is a solid torus (a disk times a circle). The parameters of (2.11) continue to \( z_+ = z_- = r_+ + i\alpha \) (i.e. \( r_- = i\alpha \)), and \( u = -v^* \). The boundary of this space is a two-dimensional Euclidean torus of modulus \( \tau = i/z_+ \). This periodicity is required in order that the coordinate map \( F(u) = \exp[z_+ u] \) (which determines the classical Liouville field) is single-valued under \( u \to u + 2\pi \tau \). Note that this coordinate map is related to the thermal nature of the corresponding CFT state; we will see shortly that it is directly related to the Cardy formula [12] for the density of states in conformal field theory. Equivalently, the map \( u \to F(u) \) in the CFT induces a Bogoliubov transformation on the field modes which generates a thermal density matrix, and thus relates the Minkowski and Rindler vacua in 1+1 dimensions. The inverse temperature generated by the transformation is \( \beta = 2\pi \Im \tau \).

The Liouville zero-mode momentum is \( \partial_u \varphi = z_+ \). The classical Liouville action on this torus is thus

\[
I_{\text{cl}} = \frac{k}{4\pi} \int d^2 u \left| \partial_u \varphi \right|^2 = \frac{k}{2} \cdot 2\pi \Im \tau \cdot |z_+|^2
\]

\[
= \frac{2\pi r_+}{8G} = \beta (M + \Omega J) - S
\] (3.6)

The last line is the standard Gibbons-Hawking result [27], specialized to 2+1 BTZ black holes (c.f. [28, 22, 10, 29]); here \( \Omega = -\alpha/r_+ \) is the angular potential, and the mass \( M \) and
angular momentum $J$ are given in (2.12).

There is a direct relation between the preceding determination of the gravitational entropy from Liouville theory and the Cardy formula for the density of states of a unitary CFT; both arise from the anomalous transformation law of the stress tensor (see equation (3.3))

$$T(w)dw^2 = T(z)dz^2 + \frac{c}{12}\{z,w\}dw^2.$$  (3.7)

We follow closely the analysis of [30]. The partition function of a conformal field theory is a section of a projective line bundle $E_c$ on the moduli space $\mathcal{M}_g$ of Riemann surfaces. The projective connection $\mathcal{A}$ is determined from the CFT stress tensor by integration against a Beltrami differential $\mu$

$$\frac{1}{2\pi i} \int d^2 z \ T(\bar{m}, m, z) \mu(z, \bar{z}) = Z^{-1}(\delta \mu, Z) .$$  (3.8)

Let $\mathcal{A} = 0$ in coordinates $w \sim w + m\tau + n$ on the torus. This coordinate chart does not extend to the $-1/\tau \to i\infty$ boundary of the moduli space $\mathcal{M}_1$; good coordinates there are $z = \exp[2\pi iw/\tau]$, $q = \exp[-2\pi i/\tau]$. This coordinate transformation is the same as that of the classical Liouville solution. The anomalous transformation law of the stress tensor (3.7) induces the transformation of the partition function [30] via (3.8)

$$\tilde{Z}(z, q) = (q\bar{q})^{c/24} Z(w, \tau) ,$$  (3.9)

and the LHS is regular as $q \to 0$. The asymptotic behavior of the density of states then follows by the saddle point approximation

$$\exp[S(h, \bar{h})] = \frac{1}{(2\pi i)^2} \int \frac{dq d\bar{q}}{q^{h+1}\bar{q}^{\bar{h}+1}} \tilde{Z}(w, \tau) \sim \int d\tau d\bar{\tau} \exp[-2\pi i(h\tau - \bar{h}\bar{\tau}) + 2\pi i\left(\frac{1}{\tau} - \frac{1}{\bar{\tau}}\right)\frac{c}{24}] \tilde{Z}(z, q)$$  (3.10)

$$\sim \exp[2\pi ((\frac{1}{6}ch)^{\frac{1}{2}} + (\frac{1}{6}c\bar{h})^{\frac{1}{2}})] \tilde{Z}(q = e^{-2\pi(24h/c)^{1/2}});$$

the factor $\tilde{Z}$ of the last line is slowly varying at its point of evaluation. With the identifications

$$h = \frac{1}{2}(\ell M + J) , \quad \beta = 2\pi \text{Im}\tau$$
$$\bar{h} = \frac{1}{2}(\ell M - J) , \quad \Omega = -\frac{\text{Im}\tau}{\text{Re}\tau} ,$$  (3.11)

one recovers the free energy as $F = M + \Omega J - S/\beta = \mathcal{I}_c/\beta$. Thus the saddle point that controls the high energy density of states of a unitary conformal field theory is the same as the Liouville saddle point (3.6).
4. Black hole thermodynamics

Does 2+1d gravity itself provide an accounting of BTZ black hole microstates? No. The dynamics of the Chern-Simons gravity theory is completely equivalent to that of the boundary Liouville theory; the bulk theory is pure gauge. As discussed in [11], the asymptotic black hole level density (1.4), where \( c = 6k \), is not the level density of the Liouville theory, whose spectrum is (1.4) with \( c_{\text{eff}} = 1 \) [31]. The point is that the Chern-Simons/Liouville theory is an effective description. As we see from (2.6), the gravitational degrees of freedom are collective coordinates of the underlying microphysics that capture the properties of the current sector of the CFT dynamics (i.e. the Verma module of the identity).

The Hilbert space of the Liouville theory has in it only the density of states of a single scalar field. The Liouville field precisely accounts for the asymptotic data of 2+1 gravity. The center-of-mass mode of the Liouville field measures the holonomy of the Chern-Simons connection, as shown above. The Liouville oscillators encode the ‘gravitational wave’ data – the non-constant modes of \( T_{uu}, T_{vv} \) (2.3) – that one can add to a given solution. An example of this is the wave solution (2.14). The set of oscillator states of a single free boson is in one-to one correspondence with the generators of the Virasoro algebra; this fact is the basis of the string no-ghost theorem (c.f. [32] and references therein). Thus \( c_{\text{eff}} = 1 \) describes the density of states of pure gravity.

Because the current sector is universal and couples to all states of the CFT/gravity theory, it takes into account all of the degrees of freedom of the microphysics; however, it cannot distinguish microstates with the same asymptotic metric (e.g. the same energy). The current sector is thus thermodynamic in character. The currents are the generating functions of Noether charges on the gravity side – precisely the objects that one couples to thermodynamics via the introduction of a set of conjugate potentials. One sees this explicitly in the way the saddle point action (3.6) measures the density of states. Extremizing the classical (Euclidean) gravitational action amounts to extremizing a thermodynamic function, and determines properties of the equilibrium state.

It is thus incorrect to try to count microstates by quantizing the bulk gravity/Liouville theory; gravity fluctuations are not given by Liouville fluctuations (these have nothing a priori to do with the fluctuations of the full CFT), although of course all pure stress tensor

\[ ^6 \text{A useful analogy might be fluid dynamics; it is in general incorrect to quantize the Euler equations (which are in any event highly nonlinear and nonrenormalizable), rather one quan-} \]

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correlators will agree as a consequence of the Virasoro Ward identities. The Liouville theory also characterizes the conformal properties of thermal states in the conformal field theory, as shown above.

Another class of calculations [14,15] attempts to localize gravitational entropy in a set of degrees of freedom on the black hole horizon. One problem with this field-theoretic approach is that gravitational entropy is universal (this is the main strength of the Brown-Henneaux/Strominger construction), whereas any attempt to localize microstates on the horizon depends strongly on nonuniversal properties of the theory – how much supersymmetry it has, what the matter content is, etc. In Lorentz signature, quantitative calculations of this sort have only been performed [14] in the pure gravity theory, and rely strongly on the fact that the dimension of the gauge group \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) in (2.1) is six. If one modifies the theory, for instance enlarging the gauge group to that of extended Chern-Simons supergravity, the prescription of [14] no longer reproduces the black hole entropy [11]. A number of other difficulties inherent to this approach have recently been elaborated [15]. Essentially, there is no canonically defined CFT on the horizon due to the many possible choices of boundary condition and ways of embedding the Virasoro algebra into the three-dimensional diffeomorphism group. Even if these difficulties could somehow be resolved, the addition of dynamical matter gives a further (divergent) contribution [33] to the entropy. This additional contribution is entanglement entropy of field modes on either side of the horizon, and is not identical to information content and thermodynamic entropy. Naively one would expect the entanglement entropy to depend on the number and kind of matter fields contributing to it. Each time one considers a different matter content, one faces the task of explaining why the black hole entropy remains unchanged.

The standard field-theoretic approach to quantum gravity and black holes takes locality as a given. One is then led to ascribe a physical reality to the black hole horizon as a boundary of causal contact, and a certain degree of separability or distinctness between degrees of freedom inside and outside this horizon. This assumption leads directly to the black hole information paradox (c.f. [34]). However, in the classical theory the position of the horizon is not locally defined (the local notion of apparent horizon is not directly related to signal propagation); and in the quantum theory, the position of the horizon tizes the underlying many-body problem and then introduces collective variables suitable for the long-wavelength limit of the quantum system. In other words, quantum fluid-dynamics is not necessarily quantum-fluid dynamics.
is not well-defined due to quantum fluctuations in the geometry. The very presence of a boundary at the horizon violates the reparametrization gauge constraints, since the horizon degrees of freedom that are counted in [14,15] are ‘would-be’ gauge transformations that are not symmetries. (On the other hand, the analogous gauge transformations on the boundary at infinity are self-consistently frozen because fluctuations are suppressed by the infinite volume of the asymptotic region; imposition of a fixed classical geometry is thus sensible.) The above discussion suggests one should interpret the horizon gauge transformations as analogues of the descendants of particle states [3,4], in which case they grow asymptotically as $c_{\text{eff}} = \mathcal{O}(1)$ rather than as $c = 6k$.

The recent constructions of black holes in M/string theory (in the context of both the Maldacena conjecture and Matrix theory) have the property that any attempt to concentrate too much energy in a given region (as measured by stationary observers from afar) results in nonlocality on that scale, e.g. in an interdependence of creation operators of modes in the low-energy effective field theory (c.f. [4] for a recent discussion in the context of the AdS/CFT correspondence). One sees this in matrix theory black holes [35], where the size of the horizon is the uncertainty bound on the quantum wavefunction of the matrix degrees of freedom; and in AdS-Schwarzschild black holes of the AdS/CFT correspondence, where the horizon scale corresponds to the thermal wavelength in the conformal field theory [1,3,4,5]. In either case, any observable is unavoidably entangled strongly with the black hole degrees of freedom at this scale; this is a reflection of the fact that, at the length scale $r_+$ characteristic of the black hole, the number of independent available degrees of freedom is much smaller than in local field theory. In particular, there is no separation of degrees of freedom ‘inside’ and ‘outside’ of a black hole, and therefore no sense to the erection of a boundary that separates the black hole interior from the rest of spacetime. The correct classical limit then has only the boundary at spatial infinity of anti-de Sitter space; this justifies the above treatment of the 2+1 gravity dynamics, where only the Liouville degrees of freedom at infinity were considered.

Most of our discussion has ignored the effects of matter fields coupled to 2+1 gravity. Even though gravity itself appears to lack the necessary degrees of freedom to account for black hole entropy, might it be possible to enumerate a set of gravitationally dressed matter fields that reproduce the correct answer? In the context of perturbative local field theory, this approach seems to lead to the divergent entanglement entropies mentioned above. A more promising approach [36] might be to enumerate a basis for the Hilbert
space of the dual CFT of the AdS/CFT correspondence, using the creation operators of matter particles \[10,26,4,5\] as in (3.3). However, precisely in the regime of energy levels of interest \(L_0 \sim k\) and above, the nature of the Hilbert space changes character, and the products of such operators become a highly overcomplete basis \[4\]; this is simply a restatement of ‘holography’ \[6\], or of the ‘stringy exclusion principle’ of \[10\]. Rather than having mode creation operators spanning a free algebra, as one supposes in local quantum field theory, the algebra satisfies additional relations. These relations are manifested in the operator product expansion of the dual CFT, which forces the creation modes of different particle states to be interdependent at next-to-leading order in the \(1/k\) expansion. The interdependence of the creation modes has dramatic effects; for example \[10\], the \(n^{th}\) power \((n > 2k)\) of a supergraviton creation operator is not an \(n\)-particle state (this concept is well-defined for BPS states), rather it is a current algebra descendant of a state with \(m < 2k\) particles. Because the energy of such a state is above the threshold to create a black hole, the dependencies among the creation modes appear to be intimately connected with the Bekenstein bound. We might call the additional constraints ‘black hole operator product relations’. As a consequence, it is not clear that one can assign to the generators of any proposed basis of ‘multiparticle states’ the same meaning which they have in the construction of dilute gases of multiparticle states about the AdS vacuum (discussed at the end of the last section).

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