Stability analysis of uncertain structures with uncertain-but-bounded parameters

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Abstract. This paper focuses on stability analysis of uncertain structures with uncertain-but-bounded parameters. For uncertain structures, the standard and geometry stiffness matrices are both uncertain. With non-negative decomposition of a matrix, the standard and geometry stiffness matrices are both expressed as the combination of the uncertain structural parameters. Then, the approach acquiring the critical buckling load vector of an uncertain structure is presented. A numerical example was used to show the effects of the proposed method.

1. Introduction
The stability of a body may be defined as a measure of its tendency to persist in some state under a disturbing influence of an external agency which encourages the body to abandon that state [1].

The world is an uncertain world. Uncertainty is the absolute property while certainty is the relative property. Thus, it is necessary to take uncertainties into account for engineering design. In engineering field, uncertainties can be divided into three categories: physical aspect, material aspect and geometrical aspect. For physical aspect, for example, the mass and the density have some uncertainties. For material aspect, for instance, there are some uncertainties exist in Young’s modulus, yield stress and Poisson ratio. For geometrical aspect, the measured dimensions such as length, width and height, are with some errors under common conditions. Deterministic stability theory [2] is not applicable to structures with uncertain parameters any longer since the predictions are not accurate and may lead to the dangerous structural design.

A good way to describe these uncertain structural parameters including uncertain loads is interval. An interval contains a lower bound and an upper bound which reflects more available information than a single point. The mathematical model describing this kind of uncertainties called non-probabilistic model. Qiu et al. [3-7], Ben-Haim and Elishakoff [8], Elishakoff et al. [9-10], Pantelides [11], Mullen and Muhanna [12] and Chen and Yang [13], developed different interval analysis methods for analysing structures with uncertain-but-bounded parameters. In their interval analysis methods, only values of the lower and upper bounds for uncertain-but-bounded parameters are needed, the structural responses which are in the form of intervals can be finally obtained.

In this study, with the non-negative decomposition of the standard and geometry stiffness matrices of a structure, a novel approach for solving the critical buckling load of uncertain structures with uncertain-but-bounded parameters is proposed.
2. Stability problem of an uncertain structure

The buckling eigenvalue problem with uncertainties can be written as follows

\[(K(b) - \lambda K_g(b))u_i = 0 \quad (i = 1, 2, \ldots, n)\]  

where \(K\) is the global standard stiffness matrix and \(K_g\) is the global geometric stiffness matrix.

The structural parameter vector can be expressed as

\[b \leq b \leq \bar{b} \quad \text{or} \quad b_j \leq b_j \leq \bar{b}_j, \quad j = 1, 2, \ldots, m\]  

where \(b = (b_j)\) is the lower bound vector of the structural parameter vector \(b\) and \(\bar{b} = (\bar{b}_j)\) is the upper bound vector. Using interval mathematics and interval analysis theory [14], Eq. (2) should be rewritten as

\[b \in b' \quad \text{or} \quad b_j \in b'_j, \quad j = 1, 2, \ldots, m\]  

where \(b'\) is the interval structural parameter vector, \(b'_j, \quad j = 1, 2, \ldots, m\), are the components of the interval vector.

The eigenvalues in Eq. (1) with satisfying constrain conditions (2) can be expressed in set form as

\[\Gamma = \{\lambda : \lambda \in \mathbb{R}, (K(b) - \lambda K_g(b))u = 0, \quad b \in b'\}\]  

Then, the interval of the buckling eigenvalue \(\lambda_i^l\) should be determined

\[\Gamma \subset \lambda_i^l = [\lambda_i^l, \lambda_i^u] = \left[\hat{\lambda}_i, \bar{\lambda}_i\right], \quad i = 1, 2, \ldots, n\]  

where

\[\hat{\lambda}_i = \min_{b \in b'} \lambda_i(K(b), K_g(b)) \quad \text{and} \quad \bar{\lambda}_i = \max_{b \in b'} \lambda_i(K(b), K_g(b)).\]  

The \(i\)-th buckling eigenvalue can be acquired through

\[\lambda_i(K(b), K_g(b)) = \min_{\Phi_i \subset \mathbb{R}^n} \max_{u \in \Phi_i \setminus \{0\}} \left\{ \frac{u^T K(b)u}{u^T K_g(b)u} \right\}\]  

where \(\Phi_i \subset \mathbb{R}^n\) is an arbitrary \(i\)-dimensional subspace [15].

After acquiring the interval buckling eigenvalues \(\lambda_i^l = [\hat{\lambda}_i, \bar{\lambda}_i]\) \((i = 1, 2, \ldots, n)\), we may easily find out the critical buckling load vector \(p_{cr}\) of the structure with uncertain-but-bounded parameters as

\[p_{cr} = -2 \min_{1 \leq i \leq n} \hat{\lambda}_i\]  

where \(\hat{\lambda}_{\min} = \min_{1 \leq i \leq n} \hat{\lambda}_i\), \(p\) is defined as the reference load vector, the minimum lower bound \(\hat{\lambda}_{\min}\) should be used for searching the corresponding critical buckling load vector \(p_{cr}\) to ensure the structural safety.

3. The approach for determining the interval buckling eigenvalues

The non-negative decomposition of the standard and geometric stiffness matrices are required for determining the interval buckling eigenvalues.
It is assumed that the global standard stiffness matrix has \( k \) structural parameters \( b_1, b_2, \ldots, b_k \) and the global geometric stiffness matrix has \( m - k \) (\( m > k \)) structural parameters \( b_{k+1}, b_{k+2}, \ldots, b_m \), the non-negative decomposition of the both matrices are

\[
K = K_0 + \sum_{j=1}^{k} b_j K_j = K_0 + b_1 K_1 + b_2 K_2 + \cdots + b_k K_k
\]

\[
K_g = K_0 + \sum_{j=k+1}^{m} b_j K'_j = K_0 + b_{k+1} K'_{k+1} + b_{k+2} K'_{k+2} + \cdots + b_m K'_m
\]

(10)

where \( K_j \) and \( K'_j \) are positive definite and the parameters \( b_j \) are positive. This decomposition is called the non-negative decomposition of a matrix. Such decompositions arise naturally in practical engineering application. For instance, in finite element analysis of structures, \( K_j \) and \( K'_j \) can be considered as the elemental standard and geometric stiffness matrices with respect to the structural parameter \( b_j \).

Further, combine the parameter vectors of standard and geometric stiffness matrices as \( b = (b_1, b_2, \ldots, b_m) \), Eq.(10) should be expressed as

\[
K(b) = K_0 + \sum_{j=1}^{m} b_j K_j = K_0 + b_1 K_1 + b_2 K_2 + \cdots + b_m K_m
\]

\[
K_g(b) = K_0 + \sum_{j=1}^{m} b'_j K'_j = K_0 + b_{k+1} K'_{k+1} + b_{k+2} K'_{k+2} + \cdots + b_m K'_m
\]

(11)

where the matrices \( K_{k+1}, K_{k+2}, \ldots, K_m \) and \( K'_{k}, K'_{k+2}, \ldots, K'_k \) are null matrices.

It is assumed that the parameter interval vector is given by \( b^I \). Then, the set of boundary vector of the interval vector \( b^I \) of the uncertain parameters is

\[
\hat{b} = \{ b : b \in b^I, \ b = (b_j), \ \text{and} \ \hat{b}_j = \bar{b}_j \ or \ \check{b}_j = \underline{b}_j; \ j = 1, 2, \ldots, m \}\.
\]

(12)

Since the set of boundary vectors are the extreme vectors of the parameter interval vector, the element number should be \( 2^m \), where \( m \) is the length of the vector \( b \).

With the non-negative decomposition of the standard and geometric stiffness matrices, the \( i \)-th buckling eigenvalue \( \lambda_i \) can be obtained. \( \lambda_i \) is contained in an interval \( \lambda_i \in \lambda_i^I = [\underline{\lambda}_i, \bar{\lambda}_i] \), where the lower bound and upper bound of the interval buckling eigenvalues are expressed

\[
\underline{\lambda}_i = \min \{ \lambda_i(\{K(b), K_g(b)\}) : b \in \hat{b} \}, \ \bar{\lambda}_i = \max \{ \lambda_i(\{K(b), K_g(b)\}) : b \in \hat{b} \}
\]

(13)

From Eq. (8), it is known that buckling eigenvalues are the function of structural parameter vector \( b \) noted as \( f(b) \), with non-negative decomposition (Eq.(11)), we have

\[
f(b) = \frac{u^T K(b) u}{u^T K_g(b) u} = \frac{\kappa_0 + \sum_{j=1}^{m} \kappa_j b_j}{\mu_0 + \sum_{j=1}^{m} \mu_j b_j}
\]

(14)

where \( \kappa_h = u^T K_h u \geq 0 \) and \( \mu_h = u^T K'_h u \geq 0 \) (\( h = 0, 1, \ldots, m \)).

Since the parameters \( b_j \) are positive. Both numerator and denominator of \( f(b) \) are monotone functions of \( b \) and there is no turning point included in \( f(b) \). It is evident that the \( f(b) \) must reach its
maximum/minimum values when the parameters $b_j$ are maximum/minimum, which equivlents $b \in \hat{b}$.

Thus, Eq.(13) holds on.

4. Numerical examples

![Figure 1. A simple portal frame](image)

This example considers the simple portal frame shown in Figure 1, where the lengths of the three elements are $L_1 = L_3 = 1\, \text{m}, L_2 = 0.8\, \text{m}$.

Two cases for the plane frame with uncertain-but-bounded structural parameters will be discussed to compare the interval buckling eigenvalues obtained by the presented parameter vertex solution method and the conventional stability theory.

**Case 1.** In this case, the moments of inertia of the elements are deterministic parameters given by,

$$ I_1 = 2 \times 10^{-8} \, \text{m}^4, \quad I_2 = 1.5 \times 10^{-8} \, \text{m}^4, \quad I_3 = 5 \times 10^{-8} \, \text{m}^4 $$

The Young’s moduli of the elements are uncertain-but-bounded parameters and their interval values are

$$ E_1^I = [195, 205] \, \text{GN/m}^2, \quad E_2^I = [190, 210] \, \text{GN/m}^2, \quad E_3^I = [185, 215] \, \text{GN/m}^2 $$

Table 1 shows the upper and lower bounds of the buckling eigenvalues calculated by the proposed parameter vertex solution method. The critical loads $P_{cr}$ calculated by the parameter vertex solution method is $44417.41 \, \text{N}$. If the structure is deterministic or the structural parameters are constant, that is $E_1 = E_2 = E_3 = 200 \, \text{GN/m}^2$, in this case, the critical load obtained by the conventional stability theory is $46845.94 \, \text{N}$.

**Case 2.** In this case, the Young’s moduli and the moments of inertia of the elements are all taken as uncertain-but-bounded parameters. Their interval values are

$$ E_1^I = [195, 205] \, \text{GN/m}^2, \quad E_2^I = [190, 210] \, \text{GN/m}^2, \quad E_3^I = [185, 215] \, \text{GN/m}^2 $$

$$ I_1^I = [1.99 \times 10^{-8}, 2.01 \times 10^{-8}] \, \text{m}^4, \quad I_2^I = [1.48 \times 10^{-8}, 1.52 \times 10^{-8}] \, \text{m}^4, \quad I_3^I = [4.97 \times 10^{-8}, 5.03 \times 10^{-8}] \, \text{m}^4 $$

Table 2 shows the upper and lower bounds of the eigenvalues, which are calculated by the proposed parameter vertex solution method. The critical loads calculated by the parameter vertex solution method is $44084.38 \, \text{N}$. If the deterministic structure is taken into account, that is $I_1 = 2 \times 10^{-8} \, \text{m}^4$, $I_2 = 1.5 \times 10^{-8} \, \text{m}^4$, $I_3 = 5 \times 10^{-8} \, \text{m}^4$ and $E_1 = E_2 = E_3 = 200 \, \text{GN/m}^2$ in this case, the critical load obtained by the conventional stability theory is $46845.94 \, \text{N}$.

Table 1. Interval Buckling Eigenvalues for the simple portal frame: Case 1

| $\lambda_1$   | $\lambda_2$   |
|---------------|---------------|
| 4.441741E+04 | 4.925817E+04 |
| 2.513402E+05 | 2.752646E+05 |
Table 2. Interval Buckling Eigenvalues for the simple portal frame: Case 2

| $\lambda_i$ | $\hat{\lambda}_i$ | $\tilde{\lambda}_i$ |
|-------------|---------------------|---------------------|
| $\lambda_1$ | 4.408438E+04        | 4.962646E+04        |
| $\lambda_2$ | 2.491112E+05        | 2.777497E+05        |
| $\lambda_3$ | 4.755705E+05        | 5.482878E+05        |

It is found out that the critical buckling load $P_{cr}$ acquired by the proposed parameter vertex solution method is smaller than that from the conventional stability theory. The results indicate that more conservative critical buckling loads can be obtained by the present method for structures with uncertain parameters.

5. Conclusions

In this paper, the stability analysis of uncertain structures with uncertain-but-bounded parameters is proposed. By using the non-negative decomposition of the standard and geometric stiffness matrices, the approach for determining the interval buckling eigenvalues is presented. The presented method is capable of dealing with the stability problem of uncertain structures. A numerical example of a simple portal frame was used to show the effectiveness of the presented method. Results indicate that the proposed approach is capable of providing us with safer critical buckling load than the conventional stability theory.

In conclusion, the presented method is a practical and effective for dealing with the stability problem of uncertain structures.

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