Iterated integrals and unipotent periods on families of marked elliptic curves

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Abstract: We describe iterated integrals as unipotent periods on families of marked elliptic curves in terms of multiple zeta values and elliptic multiple zeta values.

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1. Introduction

Recently, iterated integrals on families of marked elliptic curves were well studied in the study of Feynman integrals (for example, see [19] and references therein) and of (nonabelian) unipotent extensions of elliptic period integrals (cf. [3, 8, 10, 16]). Considering results of of Brown [2] and Banks-Panzer-Pym [1] in the case of marked Riemann surfaces, it is natural to pose a conjecture that these integrals are explicitly described by multiple zeta values and elliptic multiple zeta values which are expressed as iterated integrals of $dz/z, dz/(1 - z)$ and of Eisenstein series respectively.

The aim of this paper is to make a contribution to this conjecture. More precisely, for maximally degenerating families of marked elliptic curves over $\mathbb{C}$, we describe these unipotent periods as computable perturbative expansion in canonical deformation parameters whose coefficients are generated over $\mathbb{Q}$ by multiple zeta values and elliptic multiple zeta values. In order to obtain this description, we use formal geometry over $\mathbb{Z}$ which provides arithmetic gluing processes of marked Riemann spheres and elliptic curves using Grothendieck's (formal) existence theorem [9, Section 5]. Note that this result be extended for curves of general genus by the theory of the universal Mumford curve given in [12].

In this paper, we first consider the universal marked Tate curve as an arithmetic universal family of marked elliptic curves, and construct the universal KZB connection on this curve by gluing the KZ (Knizhnik-Zamolodchikov) connection and its elliptic extension which are called the KZB (Knizhnik-Zamolodchikov-Bernard) connection. Therefore, we obtain universal unipotent periods which give isomorphisms between the Betti and de Rham tannakian fundamental groups for degenerating families of marked elliptic curves over $\mathbb{C}$. Then by studying the behavior of these periods under fusing moves on marked Tate curves, we show the main result stated above on the universal unipotent periods which is an extension of results of [2, 1] in the case of genus 0 to that of genus 1.
2. Universal marked Tate curve

2.1. Marked Tate curve. A (marked) curve is called degenerate if it is a stable (marked) curve and the normalization of its irreducible components are all projective (marked) lines. Then the dual graph $\Delta = (V, E, T)$ of a stable marked curve is a collection of 3 finite sets $V$ of vertices, $E$ of edges, $T$ of tails and 2 boundary maps

$$b : T \to V, \quad b : E \to (V \cup \{\text{unordered pairs of elements of } V\})$$

such that the geometric realization of $\Delta$ is connected and that $\Delta$ is stable, namely its each vertex has at least 3 branches. Denote by $\sharp X$ the number of elements of a finite set $X$, and call a (connected) stable graph $\Delta = (V, E, T)$ of $(g, n)$-type if $\text{rank}_\Z H_1(\Delta, \Z) = g$, $\sharp T = n$. Then under fixing a bijection $\nu : T \to \{1, \ldots, n\}$, which we call a numbering of $T$, $\Delta = (V, E, T)$ becomes the dual graph of a degenerate marked curve of genus $g$ such that each tail $h \in T$ corresponds to the $\nu(h)$th marked point. If $\Delta$ is trivalent, i.e. any vertex of $\Delta$ has just 3 branches, then a degenerate $\sharp T$-marked curve with dual graph $\Delta$ is maximally degenerate. An orientation of a stable graph $\Delta = (V, E, T)$ means giving an orientation of each $e \in E$. Under an orientation of $\Delta$, denote by $\pm E = \{e, -e \mid e \in E\}$ the set of oriented edges, and by $v_h$ the terminal vertex of $h \in \pm E$ (resp. the boundary vertex of $h \in T$). For each $h \in \pm E$, denote by $|h| \in E$ be the edge $h$ without orientation.

Let $\Delta = (V, E, T)$ be a stable graph. Fix an orientation of $\Delta$, and take a subset $E_\iota$ of $\pm E \cup T$ whose complement $E_\iota$ satisfies the condition that

$$\pm E \cap E_\iota \cap \{-h \mid h \in E_\iota\} = \emptyset,$$

and that $v_h \neq v_{h'}$ for any distinct $h, h' \in E_\iota$. We attach variables $x_h$ for $h \in E_\iota$ and $y_e = y_{-e}$ for $e \in E$ which are called moduli parameters and deformation parameters associated with $\Delta$. Let $R_\Delta$ be the $\Z$-algebra generated by $x_h$ ($h \in E_\iota$), $1/(x_e - x_{-e})$ ($e, -e \in E_\iota$) and $1/(x_h - x_{h'})$ ($h, h' \in E_\iota$ with $h \neq h'$ and $v_h = v_{h'}$), and let

$$A_\Delta = R_\Delta[[y_e (e \in E)]], \quad B_\Delta = A_\Delta \left[ \prod_{e \in E} y_e^{-1} \right].$$

For $h \in \pm E$, put

$$\phi_h = \left( \begin{array}{cc} x_h & x_{-h} \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & y_{-h} \end{array} \right) \left( \begin{array}{cc} x_h & x_{-h} \\ 1 & 1 \end{array} \right)^{-1},$$

$$= \frac{1}{x_h - x_{-h}} \left\{ \left( \begin{array}{cc} x_h & -x_h x_{-h} \\ 1 & -x_{-h} \end{array} \right) - \left( \begin{array}{cc} x_{-h} & -x_h x_{-h} \\ 1 & -x_{-h} \end{array} \right) y_h \right\},$$

where $x_h$ (resp. $x_{-h}$) means $\infty$ if $h$ (resp. $-h$) belongs to $E_\iota$. This gives an element of $\text{PGL}_2(B_\Delta) = GL_2(B_\Delta)/B_\Delta^\times$ denoted by the same symbol which satisfies

$$\frac{\phi_h(z) - x_h}{z - x_h} = y_h \frac{\phi_h(z) - x_{-h}}{z - x_{-h}} \ (z \in \mathbb{P}^1),$$

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where \( PGL_2 \) acts on \( \mathbb{P}^1 \) by linear fractional transformation. For any path \( \rho = h(1) \cdot h(2) \cdots h(l) \) in \( \Delta \) which is reduced in the sense that \( h(i) \neq -h(i+1) \), one can associate an element \( \rho^* \) of \( PGL_2(B_\Delta) \) having reduced expression \( \phi_{h(i)}\phi_{h(i-1)} \cdots \phi_{h(1)} \). Then it is shown in [11, Proposition 1.3] and its proof that there exist elements \( \alpha, \alpha' \in B_\Delta \) and \( \beta \in \left( \prod_{i=1} \mathbb{y}_h(i) \right) \cdot (A_\Delta)^* \) such that

\[
\frac{\rho^*(z) - \alpha}{z - \alpha} = \beta \rho^*(z) - \alpha' \quad (z \in \mathbb{P}^1).
\]

We call \( \alpha, \alpha' \) the attractive, repulsive fixed points of \( \rho^* \) respectively, and call \( \beta \) the multiplier of \( \rho^* \).

**Theorem 2.1.** Let \( \Delta = (V, E, T) \) be a stable graph of \( (1, n) \)-type. Then there exists a stable marked curve \( \mathcal{E}_\Delta \) of genus 1, called a \( n \)-marked Tate curve, over \( A_\Delta \) which satisfies the following properties:

1. **(P1)** The closed fiber \( \mathcal{E}_\Delta \otimes_{A_\Delta} R_\Delta \) of \( \mathcal{E}_\Delta \) obtained by substituting \( y_e = 0 \) \( (e \in E) \) becomes the degenerate marked curve over \( R_\Delta \) with dual graph \( \Delta \) which is obtained from the collection of \( P_v := \mathbb{P}^1_{R_\Delta} \) \( (v \in V) \) by identifying the points \( x_e \in P_{v_e} \) and \( x_{-e} \in P_{v_{-e}} \) \( (e \in E) \), where \( x_h \) denotes \( \infty \) if \( h \in E_\infty \).

2. **(P2)** \( \mathcal{E}_\Delta \) gives rise to a universal deformation of \( \mathcal{E}_\Delta \otimes_{A_\Delta} R_\Delta \) by the deformation parameters \( y_e \) \( (e \in E) \).

3. **(P3)** Taking \( x_h \) \( (h \in E) \) as complex numbers such that \( x_e \neq x_{-e} \) and that \( x_h \neq x_{h'} \) if \( h \neq h' \) and \( v_h = v_{h'} \), and \( y_e \) \( (e \in E) \) as sufficiently small nonzero complex numbers, \( \mathcal{E}_\Delta \) gives rise to a family of \( n \)-marked elliptic curves over \( \mathbb{C} \).

**Proof.** When \( n = 1, \sharp V = \sharp E = 1 \) and

\[
\{ x_h \mid h \in E \cup T \} = \{ 0, 1, \infty \},
\]

\( R_\Delta = \mathbb{Z} \) and \( A_\Delta = \mathbb{Z}[y_e] \), where \( E = \{ e \} \). Then it is shown by Tate (cf. [18, 17]) that \( \mathcal{E}_\Delta \) is given as the Tate (elliptic) curve over \( \mathbb{Z}[y_e] \) formally represented as \( \mathbb{G}_m / \langle y_e \rangle \) which becomes the quotient space of \( \mathbb{C}^* = \mathbb{C} - \{ 0 \} \) under the multiplication by a complex number \( y_e \) with \( |y_e| < 1 \). Therefore, if there is a loop in \( E \) which we denote by \( l \), then there exists a formal scheme \( \widetilde{\mathcal{E}}_\Delta \) over \( A_\Delta \) which is obtained from the Tate curve \( \mathbb{G}_m / \langle y_l \rangle \) and \( P_v = \mathbb{P}^1 \) \( (v \in V) \) with marked points \( x_t \) \( (t \in T \) with \( v_t = v) \) by gluing them with \( \phi_v \) \( (e \in E - \{ l \}) \). Since \( \widetilde{\mathcal{E}}_\Delta \otimes_{A_\Delta} R_\Delta \) has an ample divisor, by Grothendieck’s existence theorem, we have a required marked curve \( \mathcal{E}_\Delta \) as the algebraization of \( \widetilde{\mathcal{E}}_\Delta \), namely \( \mathcal{E}_\Delta \) is obtained as the formal completion of \( \widetilde{\mathcal{E}}_\Delta \) along its closed fiber.

When there is no loop in \( E \), as is shown by Ihara-Nakamura [13] in the case of general genus, by Grothendieck’s existence theorem, \( \mathcal{E}_\Delta \) is given as the algebraization of the formal scheme over \( A_\Delta \) which is obtained from \( P_v = \mathbb{P}^1 \) \( (v \in V) \) with marked points \( x_t \) \( (t \in T \) with \( v_t = v) \) by gluing them with \( \phi_v \) \( (e \in E) \). \( \square \)
2.2. Comparison of parameters. Let $\Delta_1 = (V_1, E_1, T_1)$ be a stable graph of $(1, n)$-type which is not trivalent. Then there exists a vertex $v_0 \in V_1$ which has at least 4 branches. Take two elements $h_1, h_2$ of $\pm E_1 \cup T_1$ such that $h_1 \neq h_2$ and $v_{h_1} = v_{h_2} = v_0$, and let $\Delta_2 = (V_2, E_2, T_2)$ be a stable graph obtained from $\Delta_1$ by replacing $v_0$ with an oriented (nonloop) edge $h_0$ such that $v_{h_1} = v_{h_2} = v_{h_0}$ and that $v_h = v_{-h_0}$ for any $h \in \pm E_1 \cup T_1 - \{h_1, h_2\}$ with $v_h = v_0$. Put $e_i = |h_i|$ for $i = 0, 1, 2$. Then we have the following identifications:

$$V_1 = V_2 - \{v_{-h_0}\} \text{ (in which } v_0 = v_{h_0}), \ E_1 = E_2 - \{e_0\}, \ T_1 = T_2,$$

and denote by $s_e$ the deformation parameters corresponding to $e \in E_2 = E_1 \cup \{e_0\}$.

**Theorem 2.2.**

1. The marked Tate curves $\mathcal{E}_{\Delta_1}, \mathcal{E}_{\Delta_2}$ associated with $\Delta_1, \Delta_2$ respectively are isomorphic over $R_{\Delta_2} [s_{e_0}^{-1}] [\{s_e \ (e \in E_2)\}]$, where

$$\frac{x_{h_1} - x_{h_2}}{s_{e_0}}, \frac{y_{e_1}}{s_{e_0} s_{e_i}} \ (i = 1, 2 \text{ with } h_i \notin T_1), \frac{y_{e}}{s_{e}} \ (e \in E_1 - \{e_1, e_2\})$$

belong to $(A_{\Delta_2})^\times$ if $h_1 \neq -h_2$, and

$$\frac{x_{h_1} - x_{h_2}}{s_{e_0}}, \frac{y_e}{s_e} \ (e \in E_1)$$

belong to $(A_{\Delta_2})^\times$ if $h_1 = -h_2$.

2. The assertion (1) holds in the category of complex geometry when $x_{h_1} - x_{h_2}, y_e$ and $s_e$ are taken to be sufficiently small complex numbers.

**Proof.** We prove the assertion (1). By [11, Lemma 1.2], $\phi_{-h_0}(t_{h_1}) - \phi_{-h_0}(t_{h_2})$ belongs to $s_{e_0} \cdot (A_{\Delta_2})^\times$, and hence $\mathcal{E}_{\Delta_2} \otimes A_{\Delta_2} R_{\Delta_2} [s_{e_0}^{-1}] [\{s_e \ (e \in E_2)\}]$ gives a universal deformation of a universal degenerate curve with dual graph $\Delta_1$. Then by the universality of marked Tate curves, there exists an injective homomorphism $A_{\Delta_1} \hookrightarrow R_{\Delta_2} [s_{e_0}^{-1}] [\{s_e\}]$ which gives rise to an isomorphism $\mathcal{E}_{\Delta_1} \cong \mathcal{E}_{\Delta_2}$. Denote by $t_h$ the moduli parameters of $\mathcal{E}_{\Delta_2}$ corresponding to $h \in \pm E_2 \cup T_2$. Then under this homomorphism,

$$(P_{v_{-h_0}}; \phi_{-h_0}(t_{h_1}), \phi_{-h_0}(t_{h_2}), t_h \ (v_h = v_{-h_0}, h \neq h_0))$$

$$\cong (P_{v_0}; x_{h_1}, x_{h_2}, x_h \ (v_h = v_0, h \neq h_1, h_2)),$$

and hence $x_{h_1} - x_{h_2} \in s_{e_0} \cdot (A_{\Delta_2})^\times$. Furthermore, when $h_1 \neq -h_2$, the deformation parameters of

$$(P_{v_{-h_0}}; \phi_{-h_0}(t_{h_1}), \phi_{-h_0}(t_{h_2}), t_h \ (v_h = v_{-h_0}, h \neq h_0))$$

corresponding to $h_i \cdot (-h_0) \ (i = 1, 2)$ are $y_{h_i}$, and hence by [11, Proposition 1.3], $y_{h_i} \in (s_{h_0} \cdot s_{h_i}) \cdot (A_{\Delta_2})^\times$. When $h_1 = -h_2$, the deformation parameters of

$$(P_{v_{-h_0}}; \phi_{-h_0}(t_{h_1}), \phi_{-h_0}(t_{h_2}), t_h \ (v_h = v_{-h_0}, h \neq h_0))$$
corresponding to \( h_0 \cdot h_1 \cdot (-h_0) \) is \( y_{h_1} \), and hence \( y_{h_1} \in s_{h_1} \cdot (A_{\Delta_2})^\times \). The assertion (2) follows from Theorem 2.1 (P3). \( \square \)

2.3. Universal marked Tate curve. For a positive integer \( n \), denote by \( \mathcal{M}_{1,n} \) the moduli stack over \( \mathbb{Z} \) of stable \( n \)-marked curves of genus 1, and by \( \mathcal{M}_{1,n} \) its substack classifying \( n \)-marked proper smooth curves of genus 1 \([6, 15, 14]\). Then by definition, there exits the universal stable \( n \)-marked curve \( E_{1,n} \) of genus 1 over \( \mathcal{M}_{1,n} \).

**Theorem 2.3.** Let \( \mathcal{D}_{1,n} \) be the closed substack of \( \mathcal{M}_{1,n} \) consisting of \( n \)-marked degenerate curves of genus 1, and denote by \( \mathcal{N}_{1,n} \) the formal completion of \( \mathcal{M}_{1,n} \) along \( \mathcal{D}_{1,n} \). Then there exists an algebraization \( \mathcal{N}_{1,n}^{\mathrm{alg}} \) of \( \mathcal{N}_{1,n} \), namely \( \mathcal{N}_{1,n}^{\mathrm{alg}} \) is a scheme containing \( \mathcal{D}_{1,n} \) as its closed subset such that \( \mathcal{N}_{1,n} \) is the formal completion of \( \mathcal{N}_{1,n}^{\mathrm{alg}} \) along \( \mathcal{D}_{1,n} \), and the fiber of \( \mathcal{E}_{1,n} \) over \( \mathcal{N}_{1,n} \) gives a universal family of \( n \)-marked Tate curves.

**Proof.** Let \( \Delta = (V,E,T) \) be a stable graph of \((1,n)\)-type, and take a system of coordinates on \( P_v = \mathbb{P}^1_{R_{\Delta}} \) \((v \in V)\) such that \( x_h = \infty \) \((h \in E_{\infty})\) and that \( \{0,1\} \subset P_v \) is contained in the set of points given by \( x_h \) \((h \in E_{\infty} \text{ with } v_h = v)\). Under this system of coordinates, one has the marked Tate curve \( \mathcal{E}_{\Delta} \) whose closed fiber \( \mathcal{E}_{\Delta} \otimes_{A_{\Delta}} R_{\Delta} \) gives a family of degenerate curves over the open subspace of

\[
S = \{ (p_h \in P_{v_h})_{h \in E_{\infty} \cup T} \mid p_h \neq p_{h'} \ (h \neq h', v_h = v_{h'}) \}
\]

defined as \( p_e \neq p_{-e} \) \((e \in E)\). Therefore, by taking another system of coordinates and comparing the associated marked Tate curves with the original \( \mathcal{E}_{\Delta} \) in a similar way to the proof of Theorem 2.2, \( \mathcal{E}_{\Delta} \) can be extended over the algebraization of the formal completion of \( S \) in \( \mathcal{M}_{1,n} \). Then by Theorem 2.1, marked Tate curves \( \mathcal{E}_{\Delta} \) for various stable graphs \( \Delta \) of \((1,n)\)-type are glued over \( \mathcal{N}_{1,n} \), and hence the assertion holds. \( \square \)

**Definition 2.3.** We call this universal family of \( n \)-marked Tate curves the universal \( n \)-marked Tate curve.

3. Unipotent periods of marked elliptic curves

3.1. Unipotent periods of Riemann surfaces. Fix nonnegative integers \( g, n \) such that \( 2g - 2 + n > 0 \). Let \( R^o \) be a (not necessarily compact) Riemann surface obtained from a compact Riemann surface \( R \) of genus \( g \) by removing \( n \) points. Then the category of unipotent local systems over \( \mathbb{C} \) on \( R^o \) is equivalent to that of vector bundles with nilpotent flat connection on \( R \) which have regular singularities at \( R - R^o \) with nilpotent residues (cf. \([4]\)). We consider the tannakian category of unipotent local systems over \( \mathbb{Q} \) on \( R^o \) with fiber functor obtained from taking the fiber over a (tangential) point \( x \in R^o \). Then its tannakian fundamental group \( \pi_1^{\mathrm{Be}}(R^o, x) \) is a profinite algebraic group over \( \mathbb{Q} \), and there exists a canonical homomorphism from the topological fundamental group \( \pi_1(R^o, x) \) into \( \pi_1^{\mathrm{Be}}(R^o, x) \).
For a subfield $K$ of $C$, let $C^o$ be a smooth curve obtained from a proper smooth curve $C$ of genus $g$ over $K$ by removing $n$ rational points over $K$. We consider the tannakian category of vector bundles with nilpotent flat connection over $K$ on $C$ which have regular singularities at $C - C^o$ with nilpotent residues with fiber functor obtained from taking the fiber over a $K$-rational (tangential) point $x \in C^o$. Then its tannakian fundamental group $\pi^1_{dR}(C^o, x)$ is a profinite algebraic group over $K$, and by the above categorical equivalence, there exists a canonical isomorphism

$$\pi^1_{dR}(C^o, x) \otimes_K C \cong \pi^1_{Be}((C^o)^{an}, x) \otimes_{\mathbb{Q}} C,$$

where $(C^o)^{an}$ denotes the Riemann surface associated with $C^o$. We call this isomorphism the unipotent period isomorphism which is described by monodromy representations of $\pi^1_{Be}((C^o)^{an}, x)$. Similarly, for (tangential) points $x, y$ on $C^o$, we have the associated monodromy of a nilpotent flat connection.

### 3.2. KZ connection and connection matrix.

A KZ connection is defined on $\mathbb{P}^1$ as a trivial bundle with flat connection which has regular singularities along $p_i \in \mathbb{P}^1$ with residue $X_i$ ($1 \leq i \leq m$), where $X_i$ are symbols satisfying $\sum_{i=1}^{m} X_i = 0$. When $m = 3$, $p_1 = 0$, $p_2 = 1$, $p_3 = \infty$, this connection is

$$df - f \left( X_0 \frac{dz}{z} + X_1 \frac{dz}{z-1} \right).$$

Then the associated monodromy from the tangential point $\vec{v}_0 = d/dz$ at 0 to $z$ with $|z| < 1$ becomes a noncommutative formal power series in $X_0, X_1$ whose coefficients are multiple polylogarithm functions

$$\text{Li}_{k_1, \ldots, k_l}(z) = \int_0^z \frac{dz}{1 - \frac{z}{k_1-1}} \frac{dz}{1 - \frac{z}{k_2-1}} \cdots \frac{dz}{1 - \frac{z}{k_l-1}},$$

of $z \in \mathbb{P}^1 - \{0, 1, \infty\}$ regularized at 0. Therefore,

$$\text{Li}_{k_1, \ldots, k_l}(z) = \sum_{0 < n_1 < \ldots < n_l} \frac{z^{n_l}}{n_1^{k_1} \cdots n_l^{k_l}}$$

are power series in $z$ over $\mathbb{Q}$.

If $z$ is replaced with the tangential point $-\vec{v}_1 = -d/dz$ at 1, then the associated monodromy is called the Drinfeld associator [5, 7] which is a noncommutative formal power series $\Phi(X_0, X_1)$ in $X_0, X_1$ with coefficients expressed by multiple zeta values:

$$\zeta(k_1, \ldots, k_l) = \sum_{0 < n_1 < \ldots < n_l} \frac{1}{n_1^{k_1} \cdots n_l^{k_l}} (k_l > 1).$$

The Drinfeld associator can be applied to calculating connection matrices for the KZ connection on the moduli space of a projective line with 4 marked points $0, 1, \infty$ and
When the associated connection has simple poles at $x = 0$ (resp. $x = 1$) with residue $X_0$ (resp. $X_1$), the connection matrix for the fusing move of $x \in (0, 1)$ between tangential points at 0 and 1 becomes $\Phi(X_0, X_1)$.

3.3. KZB connection on the Tate curve. We review results of Calaque-Enriquez-Etingof [3], Hain [10] and Luo [16] on the (universal) elliptic KZB connection following the formulation of [10]. Let $q$ be a variable, and denote by $E_q = \mathbb{G}_m/\langle q \rangle$ the Tate (elliptic) curve over $\mathbb{Z}[[q]]$. Then its closed fiber $E_0 = E_q \otimes_{\mathbb{Z}[[q]]} \mathbb{Z}[[q]]/\langle q \rangle$ minus the identity 1 is identified with $\mathbb{P}^1 - \{0, 1, \infty\}$. Denote by $\mathbb{Q}\langle\langle T, A \rangle\rangle$ of noncommutative formal power series over $\mathbb{Q}$ in the symbols $T, A$. Then under taking $\{T, A\}$ as a de Rham framing of the first cohomology group of $E_q$, Hain [10] constructs a vector bundle with flat connection, called the elliptic KZB connection, on $E_q$ whose fibers are identified with $\mathbb{Q}\langle\langle T, A \rangle\rangle$ such that the associated flat connection has regular singularity at 1 with residue $[T, A]$. Put $\tilde{v}_1 = d/dz$ at $1 \in E_q = \mathbb{G}_m/\langle q \rangle$. Then the monodromy of the elliptic KZB connection from $-\tilde{v}_1$ to $\tilde{v}_1$ in $E_q$ is called the Enriquez elliptic associator, and is represented as a noncommutative formal power series in $T, A$ whose coefficients are called elliptic multiple zeta values. The elliptic KZB connections is described in [10, 9.2], and it gives the above KZ connection on $E_0 - \{1\} = \mathbb{P}^1 - \{0, 1, \infty\}$ by composing with the homomorphism

$$\mathbb{Q}\langle\langle X_0, X_1, X_\infty \rangle\rangle/(X_0 + X_1 + X_\infty) \rightarrow \mathbb{Q}\langle\langle T, A \rangle\rangle$$

given by

$$X_0 \mapsto \left(\frac{T}{e^T - 1}\right) \cdot A, \quad X_1 \mapsto [T, A], \quad X_\infty = \left(\frac{T}{e^{-T} - 1}\right) \cdot A,$$

where $f(T, A) \cdot x = f(\text{ad}_T, \text{ad}_A) (x)$ for $f(T, A) \in \mathbb{Q}\langle\langle T, A \rangle\rangle$ (cf. [10, Section 18]).

3.4. KZB connection on marked Tate curves. Let $\Delta_0 = (V_0, E_0, T_0)$ be a stable graph with orientation of $(1, n)$-type which consists of two vertices $v_0, v_1$, one (nonloop) edge $e$ and loop $l$ with orientation such that $v_e = v_0, v_{e \pm l} = v_{-e} = v_1$. Then we call the associated marked Tate curve $\mathcal{E}_{\Delta_0}$ the basic marked Tate curve. We construct a vector bundle with flat connection $(\mathcal{V}_{v_0}, \mathcal{F}_{v_0})$ on $\mathcal{E}_{\Delta_0}$ whose each fiber is identified with the ring

$$\mathcal{A}_{\Delta_0} = \mathbb{Q}\langle\langle X_t, T_l, A_l \rangle\rangle$$

of noncommutative formal power series over $\mathbb{Q}$ in the symbols $X_t$ ($t \in T_0$) and $T_l, A_l$ satisfying the condition

$$\sum_{t \in T} X_t - [T_l, A_l] = 0.$$

Put

$$X_t = \left(\frac{T_l}{e^{2T_l} - 1}\right) \cdot A_l, \quad X_{-l} = \left(\frac{T_l}{e^{-T_l} - 1}\right) \cdot A_l,$$

and $X_e = -X_{-e} = -[T_l, A_l]$. Then for each vertex $v \in V_0$, the sum of $X_h$ for $h \in \pm E_0 \cup T_0$ with $v_h = v$ is equal to 0.
Theorem 3.1. There exists a vector bundle with flat connection \((V_{\Delta_0}, F_{\Delta_0})\) on \(E_{\Delta_0}\) which is constructed by gluing the KZ connection on \(P_v = \mathbb{P}^1 (v \in V_0)\) with regular singularities along \(x_h\) with residue \(X_h\) \((h \in \pm E_0 \cup T_0\) such that \(v_h = v_0)\), and the elliptic KZB connections on the Tate curves associated with \(l\).

Proof. Denote by \((V_{v_0}, F_{v_0})\) the vector bundle with flat connection on \(P_{v_0}\), where \(V_{v_0}\) is the trivial bundle with fiber \(A_{\Delta_0}\), and \(F_{v_0}\) has regular singularity along \(x_h\) with residue \(X_h\) for \(h \in \pm E_0 \cup T_0\) such that \(v_h = v_0\). As reviewed in 3.3, there exists the elliptic KZB connection \((V_{l}, F_{l})\) on the Tate elliptic curve \(E_{q_0} = \mathbb{G}_m / (q_i)\) over \(\mathbb{Z}[\{q_i\}]\), where fibers of \(V_{l}\) are identified with \(\mathbb{Q} \langle T_1, A_l \rangle \hookrightarrow A_{\Delta_0}\).

Let \(\xi_e \) (resp. \(\xi_{-e}\)) be tangential points over \(\mathbb{Z}\) at \(x_e \in P_{v_0} = \mathbb{P}^1\) (resp. the identity in \(E_{q_0}\)). Then the marked Tate curve \(E_{\Delta_0}\) can be obtained from \(P_{v_0} \cup E_q\) defined by \(\xi_e \cdot \xi_{-e} = y_e\), where \(y_e\) are variables, and hence \(E_{\Delta_0}\) is proper over the ring \(A_{\Delta_0}\) identified with

\[
\mathbb{Z} \left[ \frac{1}{x_h - x_{h'}} \right] (h, h' \in \{e\} \cup T_0 \text{ with } h \neq h') \left[ [q_i, y_e] \right].
\]

Since \(X_e \frac{d\xi_e}{\xi_e} = X_{-e} \frac{d\xi_{-e}}{\xi_{-e}}\), by gluing \((V_{v_0}, F_{v_0})\) and \((V_l, F_l)\) \((1 \leq i \leq g)\), we obtain a vector bundle with flat connection on the formal completion \(\widehat{E}_{\Delta_0}\) of \(E_{\Delta_0}\) along its closed subscheme \(E_{\Delta_0} \otimes A_{\Delta_0} (A_{\Delta_0}/I)\), where \(I\) is the ideal of \(A_{\Delta_0}\) generated by \(q_i, y_e\). Then by Grothendieck’s existence theorem, there exists the associated vector bundle with flat connection on \(E_{\Delta_0}\).

For each stable graph graph \(\Delta = (V, E, T)\) of \((1, n)\)-type with orientation, we attach symbols \(X_h^\Delta\) \((h \in \pm E \cup T)\) by the following rules:

- If \(\Delta = \Delta_0\), then \(X_h^\Delta = X_h\).
- For any \(v \in V\), the sum of \(X_h^\Delta\) \((h \in \pm E \cup T\) with \(v_h = v)\) is 0.
- Assume that \(\Delta_1 = (V_1, E_1, T_1)\) and \(\Delta_2 = (V_2, E_2, T_2)\) are given in 2.2. Then

\[
X_h^{\Delta_1} = X_h^{\Delta_2} + X_{h_i}^{\Delta_2} \quad \text{(i = 1, 2)}, \quad X_h^{\Delta_1} = X_h^{\Delta_2} \quad \text{(h \neq h_1, h_2)}
\]

if \(h_1 \neq -h_2\), and

\[
X_h^{\Delta_1} = X_h^{\Delta_2} \quad \text{(h \neq h_0)}
\]

if \(h_1 = -h_2\).

Theorem 3.2. The vector bundle with flat connection \((V_{\Delta_0}, F_{\Delta_0})\) on \(E_{\Delta_0}\) can be analytically continued to a vector bundle with flat connection on the universal \(n\)-marked Tate curve given in Definition 2.3. This restriction to the closed fiber of \(E_{\Delta}\) for a stable graph \(\Delta = (V, E, T)\) of \((1, n)\)-type gives the KZ connection on \(P_v (v \in V)\) having regular singularities along \(x_h\) with residue \(X_h^\Delta\) for \(h \in \pm E \cup T\) such that \(v_h = v\).
Proof. Let $\Delta_1 = (V_1, E_1, T_1)$, $\Delta_2 = (V_2, E_2, T_2)$ and $v_0 \in V_1, h_0 \in E_2$ be as in 2.2. Then $X^\Delta_{h_0} + X^\Delta_{-h_0} = 0$, and hence the KZ connection on $P_{v_0}$ is obtained from the KZ connections on $P_{v_{h_0}}, P_{v_{-h_0}}$ whose residues along $x_{h_0}, x_{-h_0}$ are $X^\Delta_{h_0}, X^\Delta_{-h_0}$ respectively by gluing $x_{h_0} \in P_{v_{h_0}}$ and $x_{-h_0} \in P_{v_{-h_0}}$. Therefore, by Theorem 2.20, $X^\Delta_{h} (h \in \pm E \cup T)$ and $X^\Delta_{h'} (h' \in \pm E' \cup T')$ satisfy the above rules. □

3.5. Unipotent periods of marked elliptic curves. Let $\Delta = (V, E, T)$ be a stable graph of $(1, n)$-type, and assume that $\Delta$ is trivalent. Then by taking coordinates on $P_v = \mathbb{P}^1 (v \in V)$ such that these points corresponding to $h \in \pm E \cup T$ with $v_h = v$ belong to $\{0, 1, \infty\}$, $A_\Delta = \mathbb{Z}[[y_e (e \in E)]]$. Furthermore, $\Delta$ can be obtained from the basic graph $\Delta_0$ by a combination of the alterations $\Delta_1 \leftrightarrow \Delta_2$ in 2.2 without shrinking the only one (oriented) loop $l$ in $\Delta_0$. Denote by $e_l$ the oriented edge in $\pm E$ obtained from $l$ under this operations, and put $E' = E - \{|e_l|\}$.

Definition 3.3. We define unipotent periods of $E_\Delta$ as the monodromies of $(V_\Delta, F_\Delta)$ between $\mathbb{Z}$-rational tangential points on a degenerating family of $n$-marked elliptic curves over $\mathbb{C}$ obtained from $E_\Delta$ as in 2.1 (P4). The unipotent periods are represented as noncommutative formal power series in $X^\Delta_e (e \in E' \cup T)$.

Theorem 3.4. Each coefficient of unipotent periods of $E_\Delta$ as noncommutative formal power series in $X^\Delta_e (e \in E' \cup T)$ is a formal power series in $y_e (e \in E')$ whose coefficients are generated over $\mathbb{Q}$ by positive powers of $\pi \sqrt{-1}$, multiple zeta values and elliptic multiple zeta values.

Proof. We will only prove the assertion in the case when $\Delta$ is obtained from a trivalent graph $\Delta_1$ with loop by a fusing move since one can prove the assertion in general cases by a similar method. Then $\Delta_1 = (V_1, E_1, T_1)$ has only one loop $l$ and nonloop edge $e_1$ with orientation such that $v_{e_1} = v_{-e_1}$ and that $\Delta$ is obtained from $\Delta_1$ by the fusing move $e_1 \to e'$ for certain edge $e'$ of $\Delta$.

First, we consider unipotent periods of the family of $n$-marked complex curve of genus 1 obtained from

$$(E_{\Delta_1})_0 = E_{\Delta_1} \otimes_{A_{\Delta_1}} A_{\Delta_1}/(y_{e_1})$$

which is a union of the Tate curve $\mathbb{G}_m/(y_l)$ and the universal rational curve $C_0$ with dual graph $\Delta_1 - \{l\}$. Then the unipotent periods for any rotation around $\mathbb{Z}$-rational tangential points at $y_e = 0 (e \in E_1)$ are integral powers of $\exp (\pi \sqrt{-1} X^\Delta_{e_1})$. Furthermore, the cross ratios associated with the maximal degeneration of $C_0$ (cf. [1, 2.4]) belong to $\mathbb{Z}[[y_e (e \in E_1 - \{l, e_1\})]]$. Therefore, the assertion on $(E_{\Delta_1})_0$ follows from [1, Theorem 2.20] and the definition of elliptic multiple zeta values reviewed in 3.3.

Second, we consider unipotent periods of the family of $n$-marked elliptic curves over $\mathbb{C}$ obtained from $E_\Delta$. This family is obtained by the fusing move $e_1 \to e'$ corresponding to $0 \to 1 - s_{e'}$ in $(0, 1) \subset \mathbb{R}$, where $s_{e'}$ can be regarded as a deformation parameter of $E_\Delta$ associated with $e'$ by Theorem 2.2. Then this fusing move is represented as the
monodromy along \((0, 1 - s_e')\) of the connection

\[ df - f \left( [T_l, A_l] \frac{dz}{z} + X_e' \frac{dz}{z - 1} \right), \]

and hence the associated coefficients are seen in 3.2 to be formal power series in \(s_e'\) whose coefficients are generated by multiple zeta values over \(\mathbb{Q}\). This fact together with the above assertion on \((E_{\Delta_1})_0\) imply the original assertion on \(E_{\Delta}\). □

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