Derived equivalence of holomorphic symplectic manifolds

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Abstract. We use twisted Fourier-Mukai transforms to study the relation between an abelian fibration on a holomorphic symplectic manifold and its dual fibration. Our reasoning leads to an equivalence between the derived category of coherent sheaves on one space and the derived category of twisted sheaves on the other space.

1. Introduction

Fourier-Mukai transforms were introduced by Mukai [Mu1] in the early 80s, and have recently been instrumental in the understanding of several important problems in higher dimensional algebraic geometry (for example, see Bridgeland, King, and Reid [BrKR] or Bridgeland [Br2]). The original example of Mukai related the derived categories of an abelian variety and its dual. In this paper we investigate a relative version of this equivalence using the twisted Fourier-Mukai transforms of Căldăraru [Ca]; this leads to (twisted) derived equivalences for holomorphic symplectic manifolds.

The main idea is to begin with a holomorphic symplectic manifold which is fibred by abelian varieties (which we call an abelian fibration), construct its dual fibration, and then relate the derived category of coherent sheaves on the original manifold to the derived category of twisted sheaves on the dual fibration. The twisting comes from a gerbe which arises as the natural obstruction to extending the fibrewise equivalence to a family.

The motivation for studying this problem came from a desire to understand the relation between different abelian fibrations. An elliptic K3 surface can be deformed, through elliptic surfaces, to an elliptic K3 surface with a section. For a higher dimensional abelian fibration X we’d like to construct a fibration X° which is locally isomorphic to X and which admits a section (so that X is a torsor over X°). We’d also like X° to be a deformation of X, though the example in Subsection 5.4 indicates that this may not always be the case. Elliptic K3 surfaces which admit sections are easily classified via their Weierstraß models, and the hope is that in higher dimensions one may be able to classify (holomorphic symplectic) abelian fibrations which admits sections (see Sawon [Sa1]).

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Section 2 contains some preliminaries on holomorphic gerbes, twisted sheaves, and derived categories. In Section 3 we review Fourier-Mukai transforms, their twisted version, and some results important for their application. In Section 4 we discuss fibrations by abelian varieties on holomorphic symplectic manifolds. We then describe how one can construct a dual fibration and a derived equivalence coming from a twisted Fourier-Mukai transform. Singular fibres provide a significant technical obstacle: even in the ‘nicest’ cases our results depend on several as-yet unverified assumptions (these are explained in the text; one involves extending an autoduality result to compactified Jacobians, the other involves extending a Fourier-Mukai result to the twisted case). In the final section we present examples of abelian fibrations and their duals. The author intends to present the first of these examples in complete rigour in a future article \cite{Sa2}. The remaining examples are meant to illustrate some interesting behaviour that warrants further investigation.

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### 2. Preliminaries

**2.1. Holomorphic gerbes.** In trying to extend certain local sheaves to global objects we will come up against obstructions in the form of holomorphic gerbes. Gerbes are higher dimensional analogues of line bundles, as we now explain. Let \( M \) be a complex manifold with Stein cover \( \{ U_i \} \). An element in \( H^1(M, \mathcal{O}^*) \) is an equivalence class of holomorphic line bundles on \( M \). A holomorphic gerbe is the geometric object whose equivalence classes are elements of \( H^2(M, \mathcal{O}^*) \).

**Definition 2.1.** A (holomorphic) gerbe is a collection \( \{ L_{ij} \} \) of line bundles on two-fold intersections \( U_{ij} := U_i \cap U_j \) such that

1. \( L_{ii} = \mathcal{O}_{U_i} \),
2. \( L_{ji} = L_{ij}^{-1} \),
3. \( L_{ijk} := L_{ij} \otimes L_{jk} \otimes L_{ki} \) has a given trivialization,
4. the trivialization of \( L_{ijk} \otimes L_{jkl}^{-1} \otimes L_{kl} \otimes L_{lij}^{-1} \) induced by the given trivializations of \( L_{ijk} \) is the canonical one.

**Remark 2.2.** Since we chose our cover to be Stein, each line bundle \( L_{ij} \) is trivializable. If we fix trivializations of each \( L_{ij} \), this gives a trivialization of \( L_{ijk} \), but this may differ by a non-zero holomorphic function

\[
\beta_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_{U_{ijk}}^*)
\]

from the trivialization given in the definition. The fourth condition then states that \( \delta \beta \) is trivial, so that \( \beta \) is a cocycle representing a class in \( H^2(M, \mathcal{O}^*) \).

**Definition 2.3.** The isomorphism class of a gerbe is the class \([\beta] \in H^2(M, \mathcal{O}^*)\).

We will usually abuse notation by writing simply \( \beta \) for \([\beta] \).

**Remark 2.4.** Isomorphism of gerbes can be defined in the expected way, by taking into account different choices of trivialization of \( L_{ij} \) and refinement of the
cover (for a more detailed discussion see Hitchin [Hi]). There is a bijection between the set of all gerbes isomorphic to a given gerbe and the set of all cocycles representing $[\beta]$.

In this article, all our constructions can be made to depend only on the isomorphism class of the gerbe involved, not the gerbe itself. We will often abuse terminology by referring to the isomorphism class simply as a gerbe.

**Example 2.5.** The exponential exact sequence 

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0$$

induces the long exact sequence 

$$\ldots \rightarrow \mathcal{H}^2(M, \mathbb{Z}) \rightarrow \mathcal{H}^2(M, \mathcal{O}) \rightarrow \mathcal{H}^2(M, \mathcal{O}^*) \rightarrow \mathcal{H}^3(M, \mathbb{Z}) \rightarrow \ldots$$

On a Calabi-Yau three-fold $\mathcal{H}^2(M, \mathcal{O})$ vanishes and therefore $\mathcal{H}^2(M, \mathcal{O}^*)$ injects into the lattice $\mathcal{H}^3(M, \mathbb{Z})$ and is discrete. We say that each topological gerbe has a unique holomorphic structure.

On a K3 surface $\mathcal{H}^2(M, \mathcal{O}) = \mathbb{C}$ and $\mathcal{H}^3(M, \mathbb{Z})$ vanishes. Therefore $\mathcal{H}^2(M, \mathcal{O}^*)$ is a quotient of $\mathbb{C}$ (possibly non-Hausdorff, depending on the rank of the image of $\mathcal{H}^2(M, \mathbb{Z})$). We say that there are no non-trivial topological gerbes, only non-trivial holomorphic structures.

As explained by Căldăraru [Ca], a gerbe can also be described as a sheaf of Azumaya algebras over $M$. However, this applies only to gerbes whose class in $\mathcal{H}^2(M, \mathcal{O}^*)$ is torsion, a restriction that we’d prefer to avoid.

**2.2. Twisted sheaves.** Suppose we are given a gerbe, denoted $\beta$, on the complex manifold $M$ with corresponding trivializations

$$\beta_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_{U_{ijk}}).$$

There is a natural notion of sheaves twisted by $\beta$.

**Definition 2.6.** A $\beta$-twisted sheaf on $M$, denoted $\mathcal{F}$, is a collection $\{\mathcal{F}_i\}$ of local coherent sheaves (on $U_i$) and isomorphisms $\phi_{ij} : \mathcal{F}_i \rightarrow \mathcal{F}_j$ satisfying

1. $\phi_{ji} = \phi_{ij}^{-1}$,
2. $\phi_{ki} \circ \phi_{jk} \circ \phi_{ij} = \beta_{ijk}\text{Id}$.

**Remark 2.7.** The second condition means that the composition on the left, which is an automorphism of $\mathcal{F}_i$ restricted to $U_{ijk}$, is given by $\beta_{ijk}$ times the identity isomorphism.

If there exists a locally free $\beta$-twisted sheaf of rank $n$, then one can show that $\beta$ must be torsion of order dividing $n$. In this case, when $\beta$ is torsion, twisted sheaves can also be describes as sheaves of modules over the corresponding sheaf of Azumaya algebras (see Căldăraru [Ca]).

The space of gerbes forms a group which we write multiplicatively. We can also pull back a gerbe $\beta$ by a map $f : N \rightarrow M$ to get a gerbe $f^*\beta$ on $N$.

**Lemma 2.8 (Căldăraru [Ca]).** The pull-back of a $\beta$-twisted sheaf $\mathcal{F}$ on $M$ by $f : N \rightarrow M$ is a $f^*\beta$-twisted sheaf $f^*\mathcal{F}$ on $N$. The tensor product of a $\beta_1$-twisted sheaf $\mathcal{F}_1$ and a $\beta_2$-twisted sheaf $\mathcal{F}_2$ on $M$ is a $\beta_1 \otimes \beta_2$-twisted sheaf $\mathcal{F}_1 \otimes \mathcal{F}_2$ on $M$. In particular, if $\beta_2 = \beta_1^{-1}$, the inverse gerbe, then $\mathcal{F}_1 \otimes \mathcal{F}_2$ is untwisted, i.e. a genuine sheaf on $M$. 

Strictly speaking, the tensor product makes sense for locally free sheaves; for more general sheaves we should pass to the derived category and use derived tensor product, which we shall do in the next subsection.

2.3. Derived categories. The basic idea behind working with derived categories is that an object in an abelian category should be identified with all its resolutions. This ensures that certain functors behave nicely. For example, exactness of sequences of coherent sheaves is not preserved under tensor product or push-forward, but if we restrict to projective or injective sheaves (respectively) then exactness is preserved. In this case, rather than working with the abelian category of coherent sheaves, we should take its derived category so that we can replace sheaves by projective or injective resolutions. See Thomas [T] or page 143 of Gelfand and Manin [GM] for further motivation behind the derived category.

Definition 2.9. Let $C$ be an abelian category (basically, the set of morphisms between any two objects is an abelian group, and kernels and cokernels exist; see Definition 4.1 in [T]). Examples include the category Coh($M$) of coherent sheaves on a complex manifold $M$, or in the presence of a gerbe $\beta$, the category Coh($M, \beta$) of $\beta$-twisted sheaves.

Let Kom($C$) be the category of chain complexes over $C$. A quasi-isomorphism is a morphism between two complexes which induces an isomorphism on the level of cohomology.

The derived category $D(C)$ of $C$ is the category whose objects are chain complexes over $C$ and whose morphisms are usual morphisms of chain complexes, plus formal inverses of quasi-isomorphisms. Thus in the derived category, quasi-isomorphic objects become isomorphic. The bounded derived category is given by considering only bounded chain complexes.

In this article we will use the bounded derived categories of Coh($M$) and Coh($M, \beta$) on $M$, which we denote by $D^{b}_{coh}(M)$ and $D^{b}_{coh}(M, \beta)$ respectively.

The derived category is an additive category, but not abelian as kernels and cokernels do not exist. Instead, given a morphism $B^\bullet \to C^\bullet$ of chain complexes, there exists an “exact triangle”

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

where $A^\bullet[1]$ is the complex $A^\bullet$ shifted one place to the right. It is these exact triangles which are preserved under tensor product and push-forward, and which induce long exact sequences in cohomology. Thus derived categories are examples of triangulated categories, and when we talk about functors between derived categories we shall always mean functors which preserve exact triangles.

The derived category $D^{b}_{coh}(M)$ of a projective variety $M$ has recently emerged as an important tool in studying the birational geometry of $M$. It contains a great deal of information about $M$. Indeed Bondal and Orlov [BO] proved that if $M$ has ample canonical or anti-canonical bundle, it can be recovered up to isomorphism from its derived category. This is no longer the case when the canonical bundle is trivial. In the case of K3 surfaces, Orlov [O] showed that two K3 surfaces have equivalent derived categories if and only if there is a Hodge isometry between their transcendental lattices.

In this article we deal with holomorphic symplectic manifolds, higher dimensional analogues of K3 surfaces. Our aim is to find pairs of non-isomorphic manifolds with equivalent derived categories; more accurately, the derived category of
one manifold should be equivalent to the twisted derived category of the other (so this is a not quite a direct analogue of the K3 case, as studied by Orlov).

3. Fourier-Mukai transforms

3.1. Untwisted. Fourier-Mukai transforms were first introduced in [Mu1] as a way of producing equivalences of derived categories of non-isomorphic varieties. Recently there have been some striking applications of these methods, including Bridgeland, King, and Reid’s “derived” McKay correspondence [BrKKR] and Bridgeland’s construction of three-fold flops [Br2].

Let $M$ be a fine moduli space of stable sheaves on some projective variety $X$. In other words, $M$ parametrizes a complete family of stable sheaves on $X$ and there exists a universal sheaf $U$ on $X \times M$. Denote by $\pi_X$ and $\pi_M$ the projections to $X$ and $M$ respectively.

**Definition 3.1 (Mukai [Mu1]).** We call the functor

$$\Phi^U_{M \to X} : D^{b}_{\text{coh}}(M) \to D^{b}_{\text{coh}}(X)$$

$$\mathcal{E}^\bullet \mapsto R\pi_X^*(U \otimes \pi_M^*\mathcal{E}^\bullet)$$

an integral transform. If it is an equivalence of categories we call it a Fourier-Mukai transform.

**Remark 3.2.** Defining this operation on the category of coherent sheaves would be unsatisfactory, as it would not preserve exactness. Instead we use derived categories, and derived functors which preserve exact triangles.

**Remark 3.3.** The dual sheaf $U^\vee$ allows us to define a functor

$$\Phi^{U^\vee}_{X \to M} : D^{b}_{\text{coh}}(X) \to D^{b}_{\text{coh}}(M),$$

which is the inverse equivalence in the case of a Fourier-Mukai transform. Note that the dimensions of $X$ and $M$ must agree if their derived categories are equivalent.

**Example 3.4.** Let $X$ be an elliptic curve $E$, and let $M$ the dual elliptic curve $\hat{E}$, which parametrizes degree zero line bundles on $E$. The Poincaré line bundle $P$ is a universal bundle on $E \times \hat{E}$. In higher dimensions, we could let $X$ be an abelian variety and $M$ its dual. Once again the Poincaré line bundle is a universal bundle. Mukai [Mu1] showed these universal bundles induce equivalences of derived categories.

Let $\mathcal{O}_m$ be the skyscraper sheaf supported at $m \in M$. Then $\mathcal{U}_m := \Phi^U_{M \to X} \mathcal{O}_m$ is the sheaf on $X$ which the point $m \in M$ parametrizes. The skyscraper sheaves on $M$ ‘span’ the derived category and in some sense are ‘orthonormal’ with respect to the Ext$^\bullet$-pairing. If the integral transform $\Phi^U_{M \to X}$ is an equivalence it should preserve these properties. Based on these ideas, the following theorem was proved by Bridgeland, extending work of Mukai, Bondal, and Orlov.

**Theorem 3.5 (Bridgeland [Br1]).** Assume that $M$ has the same dimension as $X$, and both are smooth varieties. The functor $\Phi^U_{M \to X}$ is an equivalence of categories if and only if the following criteria are satisfied:

1. for all $m \in M$, $\mathcal{U}_m$ is a simple sheaf, i.e.

   $$\Hom(\mathcal{U}_m, \mathcal{U}_m) = \mathbb{C},$$

   and $\mathcal{U}_m \otimes K_X = \mathcal{U}_m$ where $K_X$ is the canonical bundle of $X$,
(2) for all \( m_1 \neq m_2 \in M \) and all integers \( i \),
\[
\text{Ext}^i_X(U_{m_1}, U_{m_2}) = 0.
\]

**Example 3.6.** In the previous example, when \( X \) is an abelian variety and \( M \) its dual, \( U_m \) is a line bundle and
\[
\text{Ext}^i_X(U_{m_1}, U_{m_2}) = 0.
\]
vanishes for \( m_1 \neq m_2 \) by standard results on the cohomology of line bundles on abelian varieties (see chapter 3 of Birkenhake and Lange [BL]). The first criterion is also easy to verify, and hence the integral transform is indeed an equivalence.

In general the second criterion can be more difficult to verify. However, Bridgeland and Maciocia proved the following theorem, which says it is enough to check the condition on a set of sufficiently large codimension.

**Theorem 3.7 (Bridgeland-Maciocia [BrM]).** Assume that \( M \) has the same dimension \( n \) as \( X \), and that \( X \) is a smooth variety. Suppose that the following criteria are satisfied:

1. for all \( m \in M \), \( U_m \) is a simple sheaf and \( U_m \otimes K_X = U_m \),
2. for all \( m_1 \neq m_2 \in M \)

\[
\text{Hom}_X(U_{m_1}, U_{m_2}) = 0,
\]
and the closed subscheme
\[
\Gamma(U) := \{(m_1, m_2) \in M \times M | \text{Ext}^i_X(U_{m_1}, U_{m_2}) \neq 0 \text{ for some } i \in \mathbb{Z}\}
\]
of \( M \times M \) has dimension at most \( n + 1 \).

Then in fact \( \Gamma(U) \) is the diagonal, \( M \) is also smooth, and \( \Phi^d_M \rightarrow X \) is an equivalence of categories.

**Remark 3.8.** If the dimension \( n \) is greater than two, it can also be difficult to show directly that \( M \) is smooth; but the theorem takes care of that.

### 3.2. Twisted

In [Ca] Căldărușan generalized the ideas of the previous subsection to twisted sheaves by introducing twisted Fourier-Mukai transforms. Before giving the definition, let us explain how twisted sheaves naturally arise in this context.

Perhaps the clearest motivation comes from considering relative moduli spaces. Suppose that \( p_X : X \rightarrow B \) is a proper fibration and let \( M \) be an irreducible component of the relative moduli space of stable sheaves on \( X \). In other words, each point of \( M \) corresponds to a stable sheaf supported on a fibre of \( X \rightarrow B \). Thus \( M \) is also fibre over \( B \), and we write \( p_M : M \rightarrow B \).

**Example 3.9.** Let \( X \rightarrow B \) be an elliptic fibration and let \( M \) be its compactified relative Jacobian \( \overline{\text{Jac}(X/B)} \) (see D’Souza [DS], Altman-Iarrobino-Kleiman [AIK], or Rego [R]). Then \( M \) parametrizes torsion-free rank one degree zero sheaves on the fibres of \( X \). Since these sheaves are rank one, a destabilizing sheaf cannot exist, and hence \( M \) is a component of the moduli space of stable sheaves on \( X \) (we use Simpson’s definition of stability and construction of the moduli space [Si]).

To construct \( M \) in this example, smooth fibres of \( X \) are replaced by their dual elliptic curves. Of course elliptic curves are self-dual, but the global structure of \( M \) may also differ from \( X \). For instance, \( M \) has a canonical section (given by the trivial sheaf on each fibre of \( X \)) whereas \( X \) may not admit any global section.
Let $t \in B$, and consider the fibres $X_t$ and $M_t$. One can show that provided $X_t$ is reduced (it need not be irreducible) then $M_t$ is a moduli space of stable sheaves on $X_t$. This may not be the case if $X_t$ is non-reduced because $M_t$ might include ‘fat’ sheaves whose scheme-theoretic support is non-reduced. We assume that $M_t$ is a fine moduli space of sheaves on $X_t$, so that a universal sheaf exists, and moreover that these fibrewise universal sheaves fit together locally to give a local universal sheaf over an open subset in $B$ containing $t$.

Example 3.10. In the previous example, the Poincaré line bundle is a universal sheaf for any smooth fibre of $X_t$ and its dual $M_t$. One can extend it over a neighbourhood $B_t$ of $t \in B$.

Now assume that $B$ is covered by open sets $B_t$ such that there exists a local universal sheaf $U_t$ over each $X \times M_t$, where $M_t := p^{-1}_M(B_t)$. More precisely, the local universal sheaf is, a priori, really a sheaf over $X_t \times_B M_t$; but we can extend it (by zero) to $X \times_B M_t$ and then push it forward by the inclusion of the fibre-product in $X \times M_t$. On the intersection $X \times M_{ij}$ of $X \times M_i$ and $X \times M_j$, the restrictions of the two sheaves $U_i$ and $U_j$ are universal sheaves for the same moduli problem. Hence they are isomorphic and there exists a line bundle $L_{ij}$ on $M_{ij}$ such that

$$U_i|_{X \times M_{ij}} = \pi^*_M L_{ij} \otimes U_j|_{X \times M_{ij}}.$$  

The fibre of $L_{ij}$ over a point $m \in M_{ij}$ is identified with the set of morphisms from $U_i|_{X \times m}$ to $U_j|_{X \times m}$. These sheaves are both isomorphic to $U_m$, and thus we have implicitly used the fact that $M$ parametrizes stable, and hence simple sheaves on $X$; i.e. the fibres of $L_{ij}$ are isomorphic to $C$.

Theorem 3.11 (Căldăraru [Ca]). In the situation described above, the collection $\{L_{ij}\}$ defines a gerbe on $M$. The class $\beta \in H^2(M, \mathcal{O}^*)$ of this gerbe represents the obstruction to patching the local universal sheaves $U_i$ together; i.e. there exists a global universal sheaf on $X \times M$ if and only if $\beta = 0$.

Example 3.12. For an elliptic fibration $X \rightarrow B$ and its compactified relative Jacobian $M := \text{Jac}(X/B)$ one can show that the following are equivalent (provided the singular fibres are well enough behaved):

1. $X$ is isomorphic to $M$,
2. $X$ admits a global section,
3. there is a global universal sheaf on $X \times M$,
4. $\beta$ vanishes.

The first two statements are equivalent because $X$ is a torsor over $M$. The last two statements are equivalent by the previous theorem. For the remaining equivalence, one shows that a local section of $X \rightarrow B$ can be used to locally extend the Poincaré bundle, thereby producing a local universal bundle (see Căldăraru [Ca] for details).

Next observe that the collection $\{U_i\}$ of local universal sheaves forms a $\pi^*_M \beta$-twisted sheaf on $X \times M$, which we denote simply by $U$. So in the twisted case, the failure of the existence of a universal sheaf is precisely controlled by $\beta$, and moreover we have instead a twisted universal sheaf $U$. We can also construct integral transforms with twisted sheaves.

Definition 3.13. If the functor

$$\Phi_M^{U} : \mathcal{D}_{\text{coh}}^b(M, \beta^{-1}) \rightarrow \mathcal{D}_{\text{coh}}^b(X)$$


\[ \mathcal{E}^\bullet \mapsto R\pi_{X^\ast}(U \otimes \pi_M^\ast \mathcal{E}^\bullet) \]
is an equivalence of categories we call it a twisted Fourier-Mukai transform.

**Remark 3.14.** In the definition, \( \mathcal{E}^\bullet \) denotes a complex of \( \beta^{-1} \)-twisted sheaves on \( M \). Therefore \( \pi_M^\ast \mathcal{E}^\bullet \) is a complex of \( \beta^{-1} \)-twisted sheaves on \( X \times M \), and \( U \otimes \pi_M^\ast \mathcal{E}^\bullet \) is an untwisted sheaf on \( X \times M \). The final push-forward takes us to the derived category of \( X \).

Căldăraru \([\text{Ca}]\) generalized Bridgeland’s criteria for when an integral transform is an equivalence (Theorem \([\text{Ca}]\)) to the twisted case. Skyscraper sheaves can always be regarded as twisted sheaves, so defining \( U_m := \Phi_{M \to X}^\ast \mathcal{O}_m \) still makes perfect sense in the twisted case. The criteria are then precisely the same as in the untwisted case, and we will not repeat them here.

**Example 3.15.** If \( \beta \) is non-zero in Example \([\text{Ka}]\), then provided the singular fibres are well enough behaved, we can apply the criteria and hence show that

\[ \Phi_{M \to X}^\ast : D^b_coh(M, \beta^{-1}) \to D^b_coh(X) \]
is an equivalence. Various examples where \( X \) is an elliptic Calabi-Yau three-fold are discussed in Căldăraru \([\text{Ca}]\). Dealing with the singular fibres is one of the biggest difficulties, so usually one begins with an example whose singular fibres are fairly mild.

### 4. Holomorphic symplectic manifolds

#### 4.1. Abelian fibrations.

**Definition 4.1.** Let \( X \) be a compact Kähler manifold. We call \( X \) a holomorphic symplectic manifold if it admits a closed non-degenerate holomorphic two-form

\[ \sigma \in H^0(X, \Lambda^2 T^\ast) = H^{2,0}(X) \]

(\( \sigma \) is the holomorphic analogue of a symplectic structure). Note that the canonical bundle \( K_X \) is trivialized by \( \sigma \wedge n \), where \( n \) is half the dimension of \( X \). If \( X \) is simply connected and \( H^{2,0}(X) = \mathbb{C} \) is generated by \( \sigma \) then we say \( X \) is irreducible.

**Remark 4.2.** By Bogomolov’s decomposition theorem a holomorphic symplectic manifold has a finite unramified cover which is a product of complex tori and irreducible holomorphic symplectic manifolds. The latter are therefore the **building blocks** of the theory, and we will always mean ‘irreducible holomorphic symplectic’ when we write simply ‘holomorphic symplectic’, unless stated otherwise.

In two dimensions the only (irreducible) holomorphic symplectic manifolds are K3 surfaces. A proper morphism from a K3 surface to a curve is necessarily an elliptic fibration, with base \( \mathbb{P}^1 \) and generic fibre an elliptic curve. In higher dimensions we have the following theorem of Matsushita.

**Theorem 4.3 (Matsushita \([\text{Mat1, Mat2}]\)).** Let \( X \) be a projective irreducible holomorphic symplectic manifold of dimension \( 2n \). Let \( p_X : X \to B \) be a proper surjective projective morphism, with connected generic fibre, and with projective base \( B \) of dimension strictly between zero and \( 2n \). Then

1. the generic fibre is a (holomorphic) Lagrangian abelian variety of dimension \( n \),
(2) the base is Fano with the same Hodge numbers as $\mathbb{P}^n$.

It follows from the second statement and the classification of surfaces that the base must be $\mathbb{P}^2$ when $n = 2$.

**Definition 4.4.** If there exists a morphism $p_X : X \to B$ as in the theorem, we call $X$ an abelian fibration. In this case $X$ is also known as an algebraically complete integrable system.

A K3 surface is elliptic if and only if it contains a non-trivial divisor with square zero (with respect to intersection pairing). Similarly, in higher dimensions we expect that $X$ is an abelian fibration if and only if it contains a divisor with some special properties. Presently this is only conjectural (see Sawon [Sa1]). Moreover, we don’t yet know whether every holomorphic symplectic manifold can be deformed to an abelian fibration.

**4.2. Dual fibrations.** Now we come to the heart of this article. Given an abelian fibration, our aim is to construct its dual fibration and then relate it, via a twisted Fourier-Mukai transform, to the original fibration. In this subsection we outline the general theory, which is a higher dimensional analogue of the elliptic fibrations discussed in Examples 3.12 and 3.15. In the next section we will present some examples that indicate what kind of behaviour we can expect.

Let $p_X : X \to B$ be an abelian fibration of dimension $2n$. We will assume that all the fibres are geometrically integral (i.e. reduced and irreducible).

**Definition 4.5.** The compactified relative Picard scheme $\text{Pic}^0(X/B)$ of $X$ is the moduli space parametrizing torsion-free rank one sheaves of degree zero (vanishing first Chern class) supported on the fibres of $X$. Its construction, which is due to Altman and Kleiman [AK], already requires that the fibres of $X$ be geometrically integral.

Let $M$ be the compactified relative Picard scheme $\text{Pic}^0(X/B)$ of $X$. Then $M$ parametrizes stable sheaves: they are pure-dimension with geometrically integral support, so there can be no destabilizing subsheaf (we are using Simpson’s definition of stability [Si]).

A priori we don’t know very much about $M$. It need not be a smooth space. There is a projection $p_M : M \to B$, though we cannot expect this to be flat in general, as we now explain. If $X_t$ is a smooth fibre of $X$, $t \in B$, then clearly the corresponding fibre $M_t$ of $M$ is simply the dual abelian variety. More generally, if $X_t$ is a singular fibre one can show that $M_t$ parametrizes sheaves on $X$ of the form $\iota_*\mathcal{E}$, where $\iota : X_t \hookrightarrow X$ is the inclusion of the fibre and $\mathcal{E}$ is a stable rank one degree zero sheaf on $X_t$. In particular, $M_t$ is always isomorphic to a moduli space of stable sheaves on $X_t$ (our singular fibres are geometrically integral, though in fact this appears to be true provided $X_t$ contains no non-reduced components).

This gives us a precise description of the singular fibres of $M$, but unfortunately moduli spaces of stable sheaves on singular varieties have not been extensively studied. Undesirable behaviour cannot automatically be ruled out: for example, the singular fibres of $M$ need not be irreducible and they may contain components of dimension greater than $n$. One example we can try to understand is the (compactified) Jacobian of a degeneration of a curve.
Example 4.6. Let $C$ be a singular curve contained in a surface. Assume that $C$ has at worst double point singularities (for example, nodes, cusps, or tacnodes). Altman, Kleiman, and Iarrobino [ATK] showed that the compactified Jacobian $J := \text{Jac} C$ of such a curve is geometrically integral; it is a degeneration of an abelian variety. We can then apply Altman and Kleiman’s construction to $J$, and hence obtain the compactified Picard scheme $\text{Pic} J$. A recent theorem of Esteves, Gagné, and Kleiman [EGK] gives an isomorphism of uncompactified Picard schemes
\[ \text{Pic}^0 C \cong \text{Pic}^0 J. \]
It is expected [E] that this isomorphism will extend to the compactifications
\[ \text{Pic}^0 C \cong \text{Pic}^0 J. \]
The left hand side is just $J$ itself, up to isomorphism.

In terms of our abelian fibration $X$ and its dual fibration $M$, this means that if the singular fibre $X_t$ can be identified as the compactified Jacobian of a mildly singular curve then the corresponding singular fibre $M_t$, which is $\text{Pic}^0 (X_t)$, will be isomorphic to $X_t$. In particular $M_t$ is geometrically integral and has dimension $n$, so $M \to B$ will be flat. Singular fibres like this will arise in the examples of the next section.

The example shows that in some special cases we can acquire an exact understanding of the singular fibres of $M$, so we will proceed under the assumption that the singular fibres of $M$ are sufficiently well-behaved. In fact, we’d like $M$ to be locally isomorphic to $X$ as a fibration over $B$. However, this is too much to ask for in general: if the fibres of $X$ are not principally polarized then even a smooth fibre $X_t$ of $X$ need not be isomorphic to the corresponding fibre $M_t$ of $M$. This problem does not arise with elliptic fibres, which are always principally polarized. The remedy is to define $p_0 : X^0 \to B$ to be the compactified relative Picard scheme $\text{Pic}^0 (M/B)$ of $p_M : M \to B$. For a smooth fibre $X_t$ of $X$, the corresponding fibre $X^0_t$ of $X^0$ is the double dual of, and hence canonically isomorphic to, $X_t$. We will assume the same is true for singular fibres (this should be true when $X_t$ is a mild degeneration of a Jacobian, as in the last example).

Thus $p_X : X \to B$ and $p_0 : X^0 \to B$ are locally isomorphic as fibrations. Since the compactified relative Picard scheme over $B$ always has a canonical section, $X$ is therefore a torsor over $X^0$.

Lemma 4.7. The set of torsors $X$ over $X^0$ is one-to-one correspondence with the sheaf cohomology $H^1 (B, X^0)$, where by $X^0$ we really mean the sheaf of local sections of $X^0 \to B$.

Proof. Choose an open cover (in the analytic topology) $\{ B_i \}$ of $B$ such that $X$ and $X^0$ are isomorphic fibrations over each $B_i$, i.e. there exists local isomorphisms $f_i : X_i := p_X^{-1} (B_i) \to X^0_i := p_0^{-1} (B_i)$ over $B_i$. On the overlap $X^0_{ij} := p_0^{-1} (B_{ij})$ the composition $f_j \circ f_i^{-1}$ gives an automorphism $\alpha_{ij}$ of the fibration $X^0_{ij} \to B_{ij}$, which is given by a translation in each fibre. Since we have a basepoint in each fibre ($X^0$ has a global section), the family of translations $\alpha_{ij}$ is equivalent to a section of $X^0_{ij} \to B_{ij}$. These local sections form a 1-cocycle $\alpha \in H^1 (B, X^0)$. Conversely, choosing a representative of $\alpha$ gives
us the families of translations $\alpha_{ij}$ on overlaps needed to glue the local fibrations $X^0_i \to B_i$ together to obtain $X$. □

**Lemma 4.8.** To each torsor $X$ over $X^0$ we can assign a gerbe $\beta \in H^2(M, \mathcal{O}^*)$ on $M$. This is the same gerbe that arises as the obstruction to the existence of a global universal sheaf on $X \times M$.

**Proof.** We have seen how to construct $\alpha \in H^1(B, X^0)$ from $X$. Taking a representative, we obtain local sections $\alpha_{ij}$ of $X^0_i$; but each point of $X^0_i$ represents a rank one degree zero sheaf (generically a line bundle) supported on a fibre of $M \to B$, so the local section $\alpha_{ij}$ gives a line bundle $L_{ij}$ on $M_{ij} := p^{-1}_M(B_{ij})$. One can show that this collection of line bundles forms a gerbe $\beta \in H^2(M, \mathcal{O}^*)$, which does not depend on our choice of representative of $\alpha$.

To relate this to the gerbe arising as an obstruction in Theorem 3.11, we first consider $X^0$ as a moduli space on $M$. On an abelian variety and its dual, the Poincaré line bundle is a universal sheaf. Since both $X^0$ and $M$ have global sections, these Poincaré line bundles fit together to give a global universal sheaf on $M \times X^0$. Restricting to

$$M \times X^0_i \cong M \times X_i,$$

gives a bundle whose support is contained in $M_i \times X_i$. Interchanging the factors, and extending by zero, gives a local universal sheaf on $X \times M_i$. Note that we have implicitly used the relation between the Poincaré bundle of an abelian variety and the Poincaré bundle of its dual (see Exercise 16 on page 45 of Birkenhake and Lange [BL]). Finally, it is not difficult to see that these local universal sheaves no longer patch together to give a global sheaf on $X \times M$; moreover they differ on overlaps precisely by tensoring with $L_{ij}$. □

**Remark 4.9.** The lemma implies that there is a map

$$H^1(B, X^0) \to H^2(M, \mathcal{O}^*).$$

Our next theorem says that this is an inclusion, but it will not be an isomorphism in general because the gerbes we obtain from torsors $X$ over $X^0$ have a somewhat specialized form.

Now we come to our main result, the generalization of Example 3.12 to abelian fibrations.

**Theorem 4.10.** Let $X \to B$ be an abelian fibration, $M$ the compactified relative Picard scheme of $X$, and $X^0$ the compactified relative Picard scheme of $M$. Assume that for all $t \in B$ the fibres $X^0_t$ and $X_t$ are isomorphic. Let $\beta \in H^2(M, \mathcal{O}^*)$ be the gerbe associated to the torsor $X$ over $X^0$, as in the previous lemma. Then the following are equivalent:

1. $X$ is isomorphic to $X^0$,
2. $X$ admits a global section,
3. there is a global universal sheaf on $X \times M$,
4. $\beta$ vanishes.

**Proof.** The first two statements are equivalent because $X$ is a torsor over $X^0$. The last two statements are equivalent by the previous lemma and Theorem 3.11. The remaining equivalence is shown using the same method as in the elliptic case: namely one shows that a local section of $X \to B$ can be used to locally extend the Poincaré bundle, thereby producing a local universal bundle. □
Finally we want to generalize Example 3.15 and show that we have an equivalence of derived categories for abelian fibrations. So assume we are in the same situation as in Theorem 4.10, and that $\beta$ is non-zero. We want to use Căldăraru’s generalization of Bridgeland’s criteria to show that the twisted universal sheaf $\mathcal{U}$ induces an equivalence

$$\Phi_U : D^b_{\text{coh}}(M, \beta^{-1}) \to D^b_{\text{coh}}(X).$$

For all $m \in M$, $\mathcal{U}_m$ is a stable and hence simple sheaf on $X$. Moreover $X$ is holomorphic symplectic, with trivial canonical bundle, and thus $\mathcal{U}_m \otimes K_X = \mathcal{U}_m$. The first criterion is therefore satisfied, and it remains to show that if $m_1 \neq m_2 \in M$ then

$$\text{Ext}_X^i(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0$$

for all integers $i$. We can consider various cases:

1. if $m_1$ and $m_2$ belong to different fibres of $M$, then $\mathcal{U}_{m_1}$ and $\mathcal{U}_{m_2}$ are supported on different fibres of $X$, and the Ext-groups must vanish,

2. if $m_1$ and $m_2$ belong to the same smooth fibre of $M$, then $\mathcal{U}_{m_1}$ and $\mathcal{U}_{m_2}$ are (push-forwards of) distinct line bundles $L_1$ and $L_2$ on the same smooth fibre $X_t$ of $X$; in this case all cohomology groups $H^i(X_t, L_1^* \otimes L_2)$ vanish (see Chapter 3 of Birkenhake and Lange [BL]), the normal bundle to the fibre is holomorphically trivial, and the Koszul spectral sequence then shows that the Ext-groups also vanish (as in Subsection 7.2 of Bridgeland and Maciocia [BrM]),

3. if $m_1$ and $m_2$ belong to the same singular fibre of $M$, it is more difficult to proceed with this calculation.

For elliptic fibrations, a direct approach to the third case is feasible. For abelian fibrations, we instead want to apply Bridgeland and Maciocia’s Theorem 3.7. Since the proof of that theorem uses only local arguments, the result should still be valid in the twisted case. We need to bound the dimension of $\Gamma(\mathcal{U})$. Let $X$ have dimension $2n$. Singular fibres occur over a codimension one subset of the base $B$, and the fibres themselves have dimension $n$, so the pairs $(m_1, m_2)$ corresponding to case three above are contained in a subset of $M \times M$ of dimension $3n - 1$. The theorem requires this to be at most one larger than the dimension of $X$, and therefore $n$ can be at most two.

Using these arguments, we can expect to produce equivalences

$$\Phi_U : D^b_{\text{coh}}(M, \beta^{-1}) \to D^b_{\text{coh}}(X),$$

for holomorphic symplectic four-folds $X$ fibred by abelian surfaces, under some additional hypotheses. In higher dimensions the same equivalence ought to hold, but there are some additional technical difficulties yet to be overcome. We will discuss some examples of abelian fibrations in the next section.

5. Examples

5.1. The Hilbert scheme of two points on a K3.

Definition 5.1. Let $S$ be a K3 surface. The Hilbert scheme of two points on $S$ is the space

$$S^{[2]} := \text{Blow}_\Delta(S \times S)/\mathbb{Z}_2$$

obtained by blowing up the diagonal in $S \times S$ and quotienting by the involution interchanging the two factors.
Fujiki showed that the four-fold $S^2$ has a holomorphic symplectic structure; this was the first example of a higher dimensional irreducible holomorphic symplectic manifold. We will be interested in certain deformations of $S^2$ which are abelian fibrations, i.e. fibred over $\mathbb{P}^2$ by abelian surfaces.

**Example 5.2.** Let $S$ be a double cover of the plane $(\mathbb{P}^2)^\vee$ ramified over a generic sextic. The linear system of lines in $(\mathbb{P}^2)^\vee$ is simply the dual plane $\mathbb{P}^2$. Pulling these lines back to $S$ gives a complete linear system of genus two curves on $S$. Denote the family of these curves by $C \rightarrow \mathbb{P}^2$. One can show that all of these curves are irreducible (they develop at worst two nodes or a cusp), and we can therefore consider the compactified relative Jacobian $Z^d := \overline{\text{Jac}}(C/\mathbb{P}^2)$, and more generally the compactified degree $d$ relative Picard scheme $Z_d := \overline{\text{Pic}}^d(C/\mathbb{P}^2)$. Each space $Z^d$ is clearly a fibration

$$\overline{\text{Pic}}^d \hookrightarrow Z^d \xrightarrow{\pi} \mathbb{P}^2$$

with generic fibre a smooth abelian surface.

Now given a degree $d$ line bundle (or more generally, a torsion-free rank one sheaf) on a curve $C$ in the linear system $\mathbb{P}^2$, we can take the push-forward by the inclusion $C \hookrightarrow S$. This gives a (torsion) sheaf on $S$, which moreover is stable. We can therefore regard $Z^d$ as an irreducible component of the moduli space of stable sheaves on $S$. Mukai showed that these moduli spaces are smooth and admit holomorphic symplectic forms, and hence $Z^d$ is an irreducible holomorphic symplectic four-fold and an abelian fibration. Moreover, it is a deformation of the Hilbert scheme $S^2$ (see Yoshioka, for instance).

We summarize some further properties of the four-folds $Z^d$, which hold for all $d \in \mathbb{Z}$ (these statements are mostly straight-forward, and will be proved in Sawon):

- all fibres of $Z^d$ are irreducible,
- $Z^d$ is locally isomorphic to $Z^{d+1}$ as a fibration over $\mathbb{P}^2$,
- $Z^0 \rightarrow \mathbb{P}^2$ admits a global section, given by taking the trivial degree zero line bundle on each curve,
- $Z^1 \rightarrow \mathbb{P}^2$ admits a birational two-valued section,
- $Z^d$ is globally isomorphic to $Z^{d+2}$, the isomorphism given by tensoring each degree $d$ line bundle by the canonical bundle of the genus two curve; thus $Z^d$ admits a global section if $d$ is even, and a birational two-valued section if $d$ is odd,
- $Z^1$ is not isomorphic to $Z^0$, as they have different weight-two Hodge structures; thus $Z^d$ does not admit a section if $d$ is odd.

Thus we essentially have just two different spaces, with $Z^1$ a torsor over $Z^0$, and $Z^1$ corresponds to a two-torsion element $\alpha \in H^1(\mathbb{P}^2, Z^0)$. These spaces were investigated by Markushevich; he showed that a holomorphic symplectic four-fold which is also the compactified relative Jacobian of a family of curves must be isomorphic to $Z^0$. Our approach will be slightly different, as we next want to interpret $Z^0$ as a moduli space of sheaves on the *four-fold* $Z^1$, so that we can set up a twisted Fourier-Mukai transform between them.
Theorem 5.3. Under a mild assumption (explained in the proof below), \( Z^0 \) is the compactified relative Picard scheme \( \overline{\text{Pic}}^d(Z^1/\mathbb{P}^2) \) of \( Z^1 \to \mathbb{P}^2 \).

Proof. The fibres of \( Z^1 \) are compactified Jacobians of surficial curves with at worst double point singularities. As explained in Example 4.6, we expect that the autoduality result of Esteves, Gagné, and Kleiman [EGK] should extend to compactified Picard schemes, and thus the fibres of \( \overline{\text{Pic}}^d(Z^1/\mathbb{P}^2) \) will be isomorphic to the corresponding fibres of \( Z^1 \), and of \( Z^0 \). Thus \( \overline{\text{Pic}}^d(Z^1/\mathbb{P}^2) \) will be a torsor over \( Z^0 \); since they both admit global sections they must be isomorphic. \( \square \)

Remark 5.4. Thus \( Z^0 \) is the dual fibration of \( Z^1 \). We also obtain the smoothness of the compactified relative Picard scheme of \( Z^1 \). This is difficult to prove directly, as the compactified relative Picard scheme is a moduli space of sheaves on a four-fold, and moduli spaces on higher dimensional varieties are poorly understood.

There exists a gerbe \( \beta \in H^2(Z^0, \mathcal{O}^*) \) and a twisted universal sheaf \( \mathcal{U} \) on \( Z^1 \times Z^0 \). Since \( Z^0 \) and \( Z^1 \) are not isomorphic, \( \beta \) is non-trivial by Theorem 4.10.

Theorem 5.5. Assuming that the theorem of Bridgeland and Maciocia (Theorem 3.7) extends to the twisted case, we obtain an equivalence of derived categories

\[
\Phi^d_{Z^0 \to Z^1} : \mathcal{D}_{\text{coh}}(Z^0, \beta^{-1}) \to \mathcal{D}_{\text{coh}}(Z^1).
\]

Proof. This follows from the arguments at the end of Subsection 4.2. In this example the degeneracy locus is a curve in \( \mathbb{P}^2 \), so pairs of points \((m_1, m_2)\) in the same singular fibre contribute a five dimensional component to \( \Gamma(U) \subset Z^0 \times Z^0 \). Since \( \dim Z^1 + 1 = 5 \), \( \Gamma(U) \) is not too large. \( \square \)

5.2. The Hilbert scheme of more than two points on a K3. Generalizing the example of the last subsection, the Hilbert scheme \( S^{[n]} \) of \( n \) points on a K3 surface \( S \) is a minimal desingularization of the symmetric product \( \text{Sym}^n S \). Beauville [Be] showed that \( S^{[n]} \) is an irreducible holomorphic symplectic manifold of dimension \( 2n \).

Example 5.6. Let \( S \) be a genus \( g \) K3 surface, \( g > 2 \), i.e. \( S \) contains a smooth genus \( g \) curve \( C \). Assume that \( S \) is otherwise generic. The curve moves in a \( g \) dimensional linear system; denote this family of curves by \( C \to \mathbb{P}^g \). One can show that all the curves are irreducible, and hence we can consider the compactified degree \( d \) relative Picard scheme \( Z^d := \overline{\text{Pic}}^d(C/\mathbb{P}^g) \). These give fibrations

\[
\overline{\text{Pic}}^d \hookrightarrow Z^d \to \mathbb{P}^g
\]

whose generic fibres are \( g \) dimensional abelian varieties.

As in the previous subsection, the spaces \( Z^d \) can also be interpreted as irreducible components of the Mukai moduli space of stable sheaves on \( S \) [Mu2]. Thus they are smooth irreducible holomorphic symplectic manifolds. They have dimension \( 2g \) and are deformation equivalent to the Hilbert scheme \( S^{[g]} \) (see [Y]), which can therefore be deformed to an abelian fibration.

We can generalize some of the statements of the previous subsection:
• all fibres of $Z^d$ are irreducible,
• $Z^d$ is locally isomorphic to $Z^{d+1}$ as a fibration over $\mathbb{P}^g$,
• $Z^0 \to \mathbb{P}^g$ admits a global section, given by taking the trivial degree zero line bundle on each curve,
• $Z^d$ is globally isomorphic to $Z^{d+2g-2}$, the isomorphism given by tensoring each degree $d$ line bundle by the canonical bundle of the genus $g$ curve; thus $Z^d$ admits a global section if $d$ is a multiple of $2g - 2$.

Thus $Z^1, Z^2, \ldots$, and $Z^{2g-1}$ are all torsors over $Z^0$, and there are corresponding elements $\alpha_1, \alpha_2, \ldots$, and $\alpha_{2g-1} \in H^1(\mathbb{P}^g, Z^0)$. These $2g - 1$ spaces won’t all be different; for example, there are isomorphisms

$$Z^{2g-1} \cong Z^{-1} \cong Z^1$$

where the second isomorphism is given by taking a line bundle of degree $-1$ to its dual. Nevertheless, it should be possible to show that we obtain at least some spaces which are not isomorphic to $Z^0$ (for instance, by looking at their weight-two Hodge structures). Let $Z^i$ be such a space.

**Conjecture 5.1.** The space $Z^0$ is the compactified relative Picard scheme $\overline{\text{Pic}}(Z^i/\mathbb{P}^g)$ of $Z^i \to \mathbb{P}^g$. Hence there is a (non-trivial) gerbe $\beta \in H^2(Z^0, \mathcal{O}^*)$ and a twisted universal sheaf $\mathcal{U}$ on $Z^i \times Z^0$, which induces an equivalence of derived categories

$$\Phi_{Z^0 \to Z^i} : \mathcal{D}_{\text{coh}}^b(Z^0, \beta^{-1}) \to \mathcal{D}_{\text{coh}}^b(Z^i).$$

**Remark 5.7.** We first need to look carefully at the curves in the family $\mathcal{C} \to \mathbb{P}^g$ to see what kind of singularities they can acquire; the first statement should then follow from a compact version of the autoduality result of Esteves, Gagné, and Kleiman [EGK], as in the proof of Theorem 5.3.

To show that we get an equivalence we need the twisted version of Bridgeland and Maciocia’s theorem, as in the proof of Theorem 5.5. However, this is not enough since the degeneracy locus is codimension one in $\mathbb{P}^g$, so the dimension of $\Gamma(\mathcal{U})$ is

$$(g - 1) + 2\dim(\text{fibre of } Z^0) = 3g - 1 > 2g + 1 = \dim Z^i + 1$$
as $g > 2$. To make $\Gamma(\mathcal{U})$ smaller, we need to show directly that

$$\text{Ext}_Z^j(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0$$

for all integers $j$ in some of the cases when $m_1$ and $m_2$ belong to the same singular fibre of $Z^0$. This can be done: the generic point $m_1$ on a singular fibre of $Z^0$ parametrizes a locally free sheaf on the corresponding singular fibre of $Z^i$, and then some perseverance with the standard machinery of homological algebra shows that the above Ext-groups vanish (the same can be done when $m_2$ parametrizes a locally free sheaf). This leaves the case when $m_1$ and $m_2$ both parametrize non-locally free sheaves, so both $m_1$ and $m_2$ are codimension one in the fibre of $Z^0$. This gives

$$\dim \Gamma(\mathcal{U}) = 3g - 3 \leq 2g + 1 = \dim Z^i + 1$$

provided $g \leq 4$. We expect the equivalence should still exist when $g > 4$, but new techniques must be developed to verify this.
5.3. The ten dimensional example of O’Grady. In the previous subsection we considered a K3 surface containing a genus $g$ curve, which is otherwise generic. The K3 surface $S$ of Section 5.1 (the double cover of the plane) contains a genus five curve $E$, the pull-back of a conic from the plane; however, it is not otherwise generic as it also contains a genus two curve. So we don’t expect the conjecture of the previous subsection to apply. Nevertheless, this example is still of interest as we now explain.

The curve $E$ moves in a five dimensional linear system $\mathbb{P}^5$; we denote the family of curves by $E \to \mathbb{P}^5$. These curves are certainly not all irreducible: for example, the family includes the pull-backs from the plane of singular conics (pairs of lines). The family also includes non-reduced curves (pull-backs of double lines). Consequently it does not make immediate sense to define the compactified relative Picard schemes of $E \to \mathbb{P}^5$. However, we can define $Y^k$ to be a component of the Mukai moduli space of stable sheaves on $S$; its generic element will be a torsion sheaf obtained by pushing forward a degree $k$ line bundle on a smooth curve in the family $E \to \mathbb{P}^5$.

We now consider two cases:

1. when $k = 2m + 1$ is odd $Y^{2m+1}$ is smooth and compact, and deformation equivalent to $S[5]$: $Y^{2m+1}$ can therefore be regarded as a smooth compactification of $\text{Pic}^{2m+1}(E/\mathbb{P}^5)$,

2. when $k = 2m$ is even $Y^{2m}$ is non-compact; we can compactify to $\overline{Y}^{2m}$ by adding semi-stable sheaves but this introduces singularities.

By Hironaka’s theorem $\overline{Y}^{2m}$ can be desingularized, but to find a holomorphic symplectic resolution is more difficult.

Recently O’Grady \cite{OG} produced a new ten dimensional irreducible holomorphic symplectic manifold by showing that a similar singular moduli space of semi-stable sheaves $M$ admits a symplectic desingularization $\tilde{M}$. In fact the singular spaces $M$ and $\overline{Y}^6$ are birational, and the structure of their singularities agree, at least locally \cite{Le}. So we expect that there exists a similar symplectic desingularization $\tilde{Y}^6$ of $\overline{Y}^6$, which would be birational to O’Grady’s example $\tilde{M}$ (and hence also deformation equivalent to it, by a result of Huybrechts \cite{Hu}). The space $\tilde{Y}^6$ could be regarded as a smooth compactification of $\text{Pic}^{6}(E/\mathbb{P}^5)$.

Fix now a specific example from the first case: $Y^5$ for instance. For a smooth genus five curve in the family $E \to \mathbb{P}^5$, the fibres $\text{Pic}^5$ and $\text{Pic}^6$ of $Y^5 \to \mathbb{P}^5$ and $\overline{Y}^6 \to \mathbb{P}^5$ are isomorphic smooth abelian varieties. Denote by $U^5 \subset Y^5$ and $U^6 \subset \overline{Y}^6$ the Zariski open subsets consisting of the union of these smooth fibres; then in fact $U^5$ and $U^6$ are locally isomorphic fibrations, but globally we expect a twist.

We summarize this in the following diagram:

$$
\begin{array}{cccc}
S[5] & \overset{\text{deform}}{\rightarrow} & Y^5 & \overset{\text{deform}}{\rightarrow} & \tilde{M} \\
\cup & \quad & \cup & \quad & \\
U^5 & \overset{\text{twist}}{\rightarrow} & U^6
\end{array}
$$

The Hilbert scheme $S[5]$ and O’Grady’s example $\tilde{M}$ are not deformation equivalent as their second Betti numbers differ; therefore $Y^5$ and $\overline{Y}^6$ cannot be deformation equivalent. It is therefore interesting that large subsets of them both (i.e. $U^5$ and $U^6$) are so closely related.
The hope of constructing a twisted Fourier-Mukai transform relating $Y^5$ and $\tilde{Y}^6$ (and thereby $S^{[5]}$ and $\tilde{M}$) seems remote, as their singular fibres are quite different: they could contain different numbers of irreducible components, for instance. Nevertheless, some interesting questions still arise: can one define a dual fibration of $Y^5$ in a sensible way? It should agree with $Y^0$ over smooth fibres; how is $Y^0$ related to $Y^6$? Cho, Miyaoka, and Shepherd-Barron [CMS-B] described how to construct an abelian fibration $X^0$ with a section from an abelian fibration $X$ with only stable fibres, such that $X$ is a torsor over $X^0$; over smooth fibres $X^0$ is just $\text{Pic}^0(\text{Pic}^0(X/B))$, which is then completed by adding the same singular fibres as occur in $X$. In our case it is not clear that $Y^5$ does not contain non-reduced, and hence unstable, fibres (which might occur over non-reduced curves in the family $E \to P^5$).

5.4. The generalized Kummer four-fold. We return to irreducible holomorphic symplectic four-folds. Currently just two examples are known: the Hilbert scheme of two points on a K3 surface, that we saw in Subsection 5.1, and the generalized Kummer four-fold $K_4$. Beginning with an abelian surface $A$, the Hilbert scheme $A^{[3]}$ of three points on $A$ is a minimal resolution of the symmetric product $\text{Sym}^3 A$. It is holomorphic symplectic but not irreducible. However, there is a composition of maps

$$A^{[3]} \to \text{Sym}^3 A \to A$$

(the second map is addition of the three points in $A$) and all fibres of this composition are isomorphic, irreducible holomorphic symplectic four-folds, which we denote by $K_4$ (see Beauville [Be]).

Example 5.8. Suppose that $A$ is polarized by a smooth genus four curve $C$ (this is a type $(1,3)$ polarization). The curve $C$ moves in a two dimensional linear system $P^2$. The induced map $A \to (P^2)^\circ$ is six-to-one, ramified over a degree eighteen curve (see Lange and Sernesi [LS]), and the family $C \to P^2$ of curves are pull-back of lines. If $A$ is chosen to be otherwise generic, then the curves in this family are all irreducible and we can define the compactified relative Picard schemes

$$Z^k := \overline{\text{Pic}^k(C/P^2)},$$

which are six-folds.

For each $k$, there is a map $\pi_k : Z^k \to A$ which is just the Albanese map of $Z^k$.

The kernel $Y^k$ is a smooth four-fold, and it was shown by Debarre [D] that $Y^k$ is deformation equivalent to $K_4$. Restricting $\pi_k$ to a fibre of $Z^k$ gives a surjective map from a principally polarized four dimensional abelian variety (or a degeneration thereof) to the $(1,3)$ polarized abelian surface $A$; the kernel of this map is thus a $(1,3)$ polarized abelian surface (or degeneration thereof). It follows that $Y^k$ is an abelian fibration over $P^2$ whose generic fibre is a $(1,3)$ polarized abelian surface.

Unlike the examples we have already seen, whose generic fibres were principally polarized Jacobians, the fibres of $Y^k$ will not be self-dual. Assuming that the fibres of $Y^k$ are irreducible, we can define the compactified relative Picard scheme

$$P := \overline{\text{Pic}^0(Y^k/P^2)}.$$  

Since this is the dual fibration, even around smooth fibres it won’t be isomorphic to $Y^k$, only isogeneous. Numerous questions now arise: is $P$ a deformation of $K_4$? Is it even smooth or holomorphic symplectic?

If $P$ is a deformation of $K_4$, then we observe that the exponential long exact sequence gives

$$H^2(P, O) \to H^2(P, O^*) \to H^3(P, Z) \to 0.$$
where the first term is $C$ and the third term is $\mathbb{Z}^{B_8}$. Thus the space of gerbes $H^2(P, O^*)$ contains an infinite number of connected components. Can we construct torsors $Y_\beta$ for all gerbes $\beta$? If so, and if $\beta_1$ and $\beta_2$ belong to different connected components of $H^2(P, O^*)$, then it would appear that $Y_{\beta_1}$ cannot be deformed through abelian fibrations to $Y_{\beta_2}$. Whether $Y_{\beta_1}$ and $Y_{\beta_2}$ are related through any deformation at all is another interesting question.

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