Two-generated verbally closed subgroups of a free solvable group $G$ are retracts of $G$

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Abstract

We prove that every verbally closed two-generated subgroup of a free solvable group $G$ of a finite rank is a retract of $G$.

1 Introduction

Algebraically closed objects play an important part in modern algebra. In this paper we study verbally closed and algebraically closed subgroups of free solvable groups.

We first recall that a subgroup $H$ is called algebraically closed in a group $G$ if for every finite system of equations $S = \{E_i(x_1, \ldots, x_n, H) = 1 \mid i = 1, \ldots, m\}$ with constants from $H$ the following holds: if $S$ has a solution in $G$ then it has a solution in $H$. Then we recall that a subgroup $H$ of a group $G$ is called a retract of $G$, if there is a homomorphism (termed retraction) $\phi : G \to H$ which is identical on $H$. It is easy to show that every retract of $G$ is algebraically closed in $G$. Furthermore, if $G$ is finitely presented and $H$ is finitely generated then the converse is also true (see [1]). This result still holds for any finitely generated group $G$ which is equationally Noetherian. Recall that a group $G$ is called equationally Noetherian if for any $n$ every system of equations in $n$ variables with coefficients from $G$ is equivalent (has the same solution set in $G$) to some finite subsystem of itself (see [2], [3].

Let $F(X)$ be the free group of countably infinite rank with basis $X = \{x_1, x_2, \ldots, x_n, \ldots\}$. For $w = w(x_1, \ldots, x_n) \in F(X)$ and a group $G$ by $w[G]$ we denote the set of all $w$-elements in $G$, i.e., $w[G] = \{w(g_1, \ldots, g_n) \mid g_1, \ldots, g_n \in G\}$. The verbal subgroup $w(G)$ is the subgroup of $G$ generated by $w[G]$. The $w$-width $l_w(g) = l_w,G(g)$ of an element $g \in w(G)$ is the minimal natural number $n$ such that $g$ is a product of $n$ $w$-elements in $G$ or their inverses; the width of $w(G)$ is the supremum of widths of its elements. The first question about $w$-width goes back to the Ore’s paper [4] where he asked whether the $[x, y]$-width (the commutator width) of every element in a non-abelian finite simple group is equal to 1 (Ore Conjecture). The conjecture was established in [5]. Observing paper [6] and monographs [7], [8] present results about verbal width in groups.

Two important questions arise naturally for an extension $H \leq G$ and a given word $w \in F(X)$:

- when it is true that $w[H] = w[G] \cap H$?
- when $l_{w,G}(h) = l_{w,H}(h)$ for a given $h \in w(H)$?
To approach these questions Mysnikov and the author introduced in [1] a new notion of verbally closed subgroups.

**Definition 1.1.** A subgroup $H$ of $G$ is called *verbally closed* if for any word $w \in F(X)$ and element $h \in H$ equation $w(x_1, \ldots, x_n) = h$ has a solution in $G$ if and only if it has a solution in $H$, i.e., $w[H] = w[G] \cap H$ for every $w \in F(X)$.

**Theorem 1.2.** [1]. Let $F$ be a free group of a finite rank. Then for a subgroup $H$ of $F$ the following conditions are equivalent:

a) $H$ is a retract of $F$.

b) $H$ is a verbally closed subgroup of $F$.

c) $H$ is an algebraically closed subgroup of $F$.

This result clarifies the nature of verbally or algebraically closed subgroups in $F$. Surprisingly, the "weak" verbal closure operator in this case is as strong as the standard one.

Similar result is true for any free nilpotent group $N$ of a finite rank.

**Theorem 1.3.** [9]. Let $N$ be a free nilpotent group of a finite rank and class $c$. Then for a subgroup $H$ of $N$ the following conditions are equivalent:

a) $H$ is a retract of $N$.

b) $H$ is a verbally closed subgroup of $N$.

c) $H$ is an algebraically closed subgroup of $N$.

d) $H$ is a free factor of the free group $N$ in the variety $N_c$ of all nilpotent groups of the class $\leq c$.

# 2 Preliminaries

In this section we collect some known or simple facts on verbally or algebraically closed subgroups.

**Proposition 2.1.** [1]. Let $H \leq G$ be a group extension. Then the following holds:

1) If $H$ is a retract of $G$ then $H$ is algebraically closed in $G$.

2) Suppose $G$ is finitely presented and $H$ is finitely generated. Then $H$ is algebraically closed in $G$ if and only if $H$ is a retract.

3) Suppose $G$ is finitely generated relative to $H$ and $H$ is equationally Noetherian. Then $H$ is algebraically closed in $G$ if and only if $H$ is a retract.

Further in the paper $M_r$ denotes a free metabelian group of rank $r$ and $M_{rk}$ denotes a free metabelian nilpotent of class $k$ group of rank $r$.

The following statement was proved in [10] (see also [8]).
Let $G$ denote $M_2$ or $M_{rk}, k \geq 4$, with basis $z_1, z_2$. Let tuple $(g_1, g_2)$ is a solution of the equation

$$(x_2, x_1, x_1, x_2) = (z_2, z_1, z_1, z_2).$$

(1)

Then $g_i \equiv z_i^{+1} \pmod{G'}, i = 1, 2$.

**Lemma 2.3** ([9], Lemma 1.1). Let $G$ be a group and let $N$ be a verbal subgroup of $G$. If $H$ is a verbally closed subgroup of $G$ then its image $H_N$ is verbally closed in $G_N = G/N$.

**Proof.** Suppose that equation $w(x_1, ..., x_n) = h$ ($h \in H_N$) has a solution $(g_1, ..., g_n)$ in $G_N$. Let $(g_1', ..., g_n')$ be a preimage of this solution in $G$. Then $(g_1', ..., g_n')$ is a solution of the equation $w(x_1, ..., x_n) = f'$ in $G$. Here $f' = w(g_1', ..., g_n')$ is a preimage of $h$ in $G$. There also is a preimage $h'$ of $h$ in $H$. Then $f' = h'v, v \in N$. Let $v = v(x_{n+1}, ..., x_{n+m})$ be a corresponding to $N$ word that has value 1 in $G/N$. The equation $w(x_1, ..., x_n)v(x_{n+1}, ..., x_{n+m})^{-1} = h'$ is solvable in $G$. Then it has a solution in $H$. The image of the $n$ first components of this solution will be a solution of $w(x_1, ..., x_n) = h$ in $H_N$.

**Lemma 2.4.** Let $H = gp(g, f)$ be a two-generated verbally closed noncyclic subgroup of a free solvable group $S_{rd}(r, d \geq 2)$. Then the image $H$ in the abelianization $A_r = S_{rd}/S_{rd}'$ (that is a free abelian group of rank $r$) is a direct factor of rank 2, and $H$ is a free solvable group of rank 2 and class $d$.

**Proof.** By Lemma 2.3 $H$ is verbally closed in $A_r$. By Theorem 1.3 $H$ is a direct factor of $A_r$. Obviously, $H$ is non-trivial. We need to prove that $H$ is not cyclic. By induction on $d$, we can assume that the image of $H$ in $S_{rd−1}/S_{rd}'$ is generated by $g_i$'s of $g$ (so it is cyclic) and that $f \in S_{rd}'$. Let $f(z_1, ..., z_r)$ be an expression of $f$ in the free generators $z_1, ..., z_r$ of $S_{rd}$. The equation $f(x_1, ..., x_r) = f$ is solvable in $S_{rd}$. Hence, it has a solution $h_1, ..., h_r$ in $H$. Then we can write the components in the form $h_i = g_i^{t_i}, t_i \in Z, \alpha_i \in Z[g_i']$, $i = 1, ..., r$. Then $f(h_1, ..., h_r) = f(g_1^{t_1}, ..., g_r^{t_r})f^{\delta} = f^{\delta}$, where $\delta = \sum_{i=1}^{r} \alpha_i d_i(f)$ and $d_i(f)$ is a value of $i$-th partial Fox derivation in $Z[g_i']$. Details about Fox derivatives see in [5]. It is easy to compute $\delta$ directly without Fox derivatives. We can collect exponents of $f$ from the expression $f(h_1, ..., h_r)$. Note that $\delta$ belongs to the fundamental ideal of $Z[g_i']$, because $f(x_1, ..., x_r)$ is a commutator word. Then $f_1^{1−\delta} = 1$ and $1 − \delta \neq 0$. This is impossible because $S_{rd}'$ has no module torsion (see [5]).

By the G. Baumslag’s result [11] $H = gp(g, f)$ is free solvable group of rank 2 and class $d$ with basis $g, f$. Recall that Baumslag proved in [11] the following statement: A subgroup $H$ of a free solvable group $S$ is itself a free solvable group if and only if there exists a set $Y$ of generators of $H$ which freely generates, modulo some term of the derived series of $S$, a free abelian group. He also gave an obvious formula of the solvability class of $H$. See also [12].

**3 Description of verbally and algebraically closed cyclic subgroups of free solvable groups**

We start with the case of a cyclic subgroup.
Lemma 3.1. Let $S$ be a free solvable group of a finite rank $r$ and let $H = gp(h)$ be a cyclic subgroup of $S$ generated by a non-trivial element $h \in S$. Then the following conditions are equivalent:

1) $H$ is verbally closed in $S$;
2) $H$ is a retract of $S$;
3) the image of $h$ in the abelianization $A_r = S/S'$ (that is free abelian group of rank $r$) is primitive.

Proof. Let $\{z_1, \ldots, z_r\}$ be a basis of $S$. The element $h$ can be expressed uniquely in the form

$$h = z_1^{k_1} \ldots z_r^{k_r} h'(z_1, \ldots, z_r), \quad (2)$$

where $k_1, \ldots, k_r \in \mathbb{Z}$ and $h'(z_1, \ldots, z_r)$ is a product of commutators of words in $z_1, \ldots, z_r$ (a commutator word). Then $h' \in S'$.

To show that 1) $\rightarrow$ 3) assume that $h$ has a non primitive image in $A_r$, i.e., either $h \in S'$ or $gcd(k_1, \ldots, k_r) = d > 1$.

We first suppose that $h \in S'$, so $k_1 = \ldots = k_r = 0$. Replacing each $z_i$ by a variable $x_i$ in (2) one gets an equation $h = x_1^{k_1} \ldots x_r^{k_r} h'(x_1, \ldots, x_r)$, with $h$ as a constant from $H$, which has a solution in $S$. However, this equation does not have a solution in $H$, since $H$ is abelian, so $h'(h_1, \ldots, h_r) = 1$ for any $h_1, \ldots, h_r \in H$. This shows that $H$ is not verbally closed in $S$ - contradiction. So $h \notin S'$. Then in this case $gcd(k_1, \ldots, k_r) = d > 1$. The equation

$$h = x_1^{k_1} \ldots x_r^{k_r} h'(x_1, \ldots, x_r)$$

still has a solution in $S$, but for any $h_1, \ldots, h_r \in H$ one has

$$h_1^{k_1} \ldots h_r^{k_r} h'(h_1, \ldots, h_r) = h_1^{k_1} \ldots h_r^{k_r} = h_{ds} \neq h,$$

for some $s \in \mathbb{Z}$. Hence, the equation does not have a solution in $H$, so $H$ is not verbally closed - contradiction.

To show that 3) $\rightarrow$ 2) assume that $h$ is primitive in the abelianization of $S$. Then there are integers $t_1, \ldots, t_r$ such that $k_1t_1 + \ldots k rt_r = 1$. Now we define a homomorphism $\varphi : S \rightarrow H = gp(h)$ by putting $\varphi(z_i) = h_{i}$ for $i = 1, \ldots, r$. Since $H$ is abelian $\varphi(h') = 1$, so $\varphi(h) = h$ and $\varphi$ is a retraction. Hence $H$ is a retract, as claimed.

2) $\rightarrow$ 1) follows from Proposition 2.1 statement 1). \(\square\)

4 Description of verbally and algebraically closed two-generated subgroups of free metabelian groups

Further in the Section, $z_1, \ldots, z_r$ is denoted a basis of group $G$ that is $M_r$ or $M_{rk}$.

Theorem 4.1. Let $H = gp(g, f)$ be a two-generated subgroup of $G$ that is $M_r$ or $M_{rk}$, $r \geq 2, k \geq 4$. Then the following conditions are equivalent:

1) $H$ is a retract of $G$. 

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2) $H$ is a algebraically closed subgroup of $G_r$.

3) $H$ is an verbally closed subgroup of $G$.

Proof. The implications 1) $\rightarrow$ 2) $\rightarrow$ 3) are obvious. We are to prove that 3) $\rightarrow$ 1).

By Lemma 2.3 the image of $H$ in the abelianization $A_r = G/G'$ is verbally closed. By Lemma 2.4 this image is a direct factor of $A_r$ of rank 2. We can assume that $g = uz_1, uG'$ and $f = vz_2, vG'$. Let $g(z_1, ..., z_r)$ and $f(z_1, ..., z_r)$ be expressions of $g$ and $f$ respectively. Then an equation

$$(f(x_1, ..., x_r), g(x_1, ..., x_r), g(x_1, ..., x_r), f(x_1, ..., x_r)) = (f, g, g, f)$$

has a solution in $G$. Thus it has a solution in $H$. It means that there are elements $h_1, ..., h_r$ in $H$ such that

$$(f(h_1, ..., h_r), g(h_1, ..., h_r), g(h_1, ..., h_r), f(h_1, ..., h_r)) = (f, g, g, f).$$

Now we consider the case $G = M_r$. By the Baumslag’s theorem (see above) [14] it is an inner automorphism (see also [12]).

Recall that an element $u$ of a group $G$ is said to be test element if any endomorphism $\alpha : G \rightarrow G$ for which $\alpha(g) = g$ is an automorphism of $G$. Timoshenko proved in [13] that every non-trivial element of $M'_r$ is a test element for $M_2$. Hence the following map

$$g \mapsto g(h_1, ..., h_r), f \mapsto f(h_1, ..., h_r)$$

defines an automorphism of $H$. Let $\epsilon, \eta = 1$. Then this automorphism is identical $(mod H')$. By Bachmuth’s theorem [14] it is an inner automorphism (see also [12]). Then there is an element $v \in H$ for which

$$g(h_1, ..., h_r) = g^v, f(h_1, ..., h_r) = f^v.$$  \hspace{1cm} (7)

It follows that $(f, g, g, f)^v = (f, g, g, f)$. Then $v \in H'$ because the centralizer of any non-trivial element of the commutant $M'_r$ of $M_r$ is equal to $M'_r$. This statement was proved by Mal’cev in [12].

Define an endomorphism of $M_r$ by the map

$$\beta : z_i \mapsto h_i^{v^i}, i = 1, ..., r.$$ \hspace{1cm} (8)

The image $\beta(M_r)$ lies in $H$. Moreover,

$$\beta(g) = \beta(g(z_1, ..., z_r)) = g(h_1, ..., h_r)^{v^{-1}} = g,$$

$$\beta(f) = \beta(f(z_1, ..., z_r)) = f(h_1, ..., h_r)^{v^{-1}} = f.$$ \hspace{1cm} (9)

Thus $\beta$ is identical on $H$ and so $\beta$ is a retraction and $H$ is a retract.

Now let $(\epsilon, \eta) \neq (1, 1)$. As above the following map

$$g^\epsilon \mapsto g(h_1, ..., h_r), f^\epsilon \mapsto f(h_1, ..., h_r)$$ \hspace{1cm} (10)

defines an automorphism of $H$. A composition of this automorphism with itself is identical $(mod H')$ and we can finish our proof as above.

Let $G = M_{rk}$. This group is free in a nilpotent variety $N$. Then a pair of elements $g, f \in H$ which induce a pair of free generators $\bar{g}, \bar{f}$ in $A_r = M_{rk}/M'_{rk}$, generate a free $N$-factor in $M_{rk}$. This factor is obviously a retract of $M_{rk}$. \hspace{1cm} \square
5 Description of verbally and algebraically closed two-generated subgroups of free solvable groups

To prove the main result of this section in the case of group $S_{r^3}$ we will use a concrete example of a test element of $S_{23}$, that was constructed in \[15\]. The proof in the general case of group $S_{r^d}$ will be slightly different.

**Example 5.1.** Let $x, y$ be a basis of $S_{23}$. For any pair of positive integers $k$ and $l$ we denote by $z(k; l)(x, y)$ the element $(y, x; k, y; l-1)$, and by $w(k; l; m; n)(x, y)$ we denote the element $(z(k; l)(x, y), z(m; n)(x, y))$. Then

$$u(x, y) = w(3; 2; 1; 1)(x, y)w(2; 2; 1; 2)(x, y)$$

(11)

is a test element of $S_{23}$.

The following lemma was proved in \[15\].

**Lemma 5.2.** Let $\phi$ be an endomorphism of $S_{23}$ for which $\phi(u) \equiv u(\mod \gamma_8 S_{23})$ where $u$ is the element in Example 5.1. Then $\phi$ is an automorphism identical modulo $S'_{23}$. In this case $\phi$ is inner by theorem proved in \[16\] (see also \[17\]).

The following more general result was proved in \[18\].

**Lemma 5.3.** Let $S_{r^d}$ be a free solvable group of rank $2$ and class $d \geq 2$ with basis $z_1, z_2$, and let $v \in S^{'(d-1)}_{r^d}, v \neq 1$. Then there is a positive number $m$ such that element $u = u^{(1-z_1^m)(1-z_2^m)}$ is a test element. Moreover, for every automorphism $\phi$ if $\phi(u) = u$ then $\varphi = \phi^2$ is an inner automorphism.

**Theorem 5.4.** Let $H = gp(g, f)$ be a two-generated subgroup of $S_{r^d}, r, d \geq 2$. Then the following conditions are equivalent:

1) $H$ is a retract of $S_{r^d}$.

2) $H$ is an algebraically closed subgroup of $S_{r^d}$.

3) $H$ is a verbally closed subgroup of $S_{r^d}$.

**Proof.** The implications 1) $\rightarrow$ 2) $\rightarrow$ 3) are obvious. We are to prove that 3) $\rightarrow$ 1).

Case of group $S_{r^3}$. Let $z_1, ..., z_r$ be a basis of $S_{r^3}$. By Theorem 4.1 and Lemmas 2.2 and 2.3 we can assume that $g = vz_1, v \in S'_{r^3}$, and $f = wz_2, w \in S'_{r^3}$. Let $g(z_1, ..., z_r)$ and $f(z_1, ..., z_r)$ be two expressions of $g$ and $f$ respectively. Let $u$ be the element described by Example 5.1. Then equation

$$u(g(x_1, ..., x_r), f(x_1, ..., x_r)) = u(g, f)$$

(12)

has a solution in $S_{r^3}$. Hence it has a solution $h_1, ..., h_r$ in $H$. The map $z_i \mapsto h_i, i = 1, ..., r,$ defines an endomorphism $S_{r^3}$ with the image in $H$. It maps $g \rightarrow g(z_1, ..., z_r)$ to $g(h_1, ..., h_r)$, and $f \rightarrow f(z_1, ..., z_r)$ to $f(h_1, ..., h_r)$. By Lemma 5.2 the map $g \mapsto g(h_1, ..., h_r), f \mapsto f(h_1, ..., h_r)$ defines an inner automorphism of $H$. Then there is $t \in H$ for which

$$g(h_1, ..., h_r) = g^t, f(h_1, ..., h_r) = f^t.$$

(13)
It follows that \( u(g, f)^t = u(g, f) \). In any group \( S_{rd}, r, d \geq 2 \), the centralizer of any non-trivial element \( y \in S_{rd}^{(d-1)} \) coincides with \( S_{rd}^{(d-1)} \) [12]. Hence \( t \in H^{(2)} \).

Let \( \phi \) be an endomorphism of \( S_{rd} \) defined by the map: \( z_i \mapsto h_{t i}^{-1}, i = 1, \ldots, r \). The image \( \phi(S_{rd}) \) lies in \( H \), and

\[
\phi(g) = \phi(g(z_1, \ldots, z_r)) = g(h_1, \ldots, h_r)_{t i}^{-1} = g, \phi(f) = \phi(f(z_1, \ldots, z_r))(f(h_1, \ldots, h_r))_{t i}^{-1} = f. \quad (14)
\]

Hence \( \phi \) is a retraction, and \( H \) is a retract.

Case of group \( S_{rd}, d \geq 2 \). Now \( u \) is defined in Lemma 5.3. We repeat all the arguments as above until we get the map \( g \mapsto g(h_1, \ldots, h_r) \), \( f \mapsto f(h_1, \ldots, h_r) \) which defines an automorphism \( \phi \) of \( H \). Then \( \phi(S_{rd}) \subseteq H \). By Lemma 5.3 \( \phi^2 \) is an inner automorphism of \( H \). For some \( t \in H \) we have equalities (13). Then we finish proof as in the previous case.

6 Open problems

Problem 6.1. Is it true that for any subgroup \( H \) of any free solvable group \( S_{rd} \) of rank \( r \geq 2 \) and class \( d \geq 2 \) the following conditions are equivalent:

1) \( H \) is a retract of \( S_{rd} \),
2) \( H \) is a algebraically closed subgroup of \( S_{rd} \),
3) \( H \) is an verbally closed subgroup of \( S_{rd} \)?

It is likely that the answer to this question is negative. By Timoshenko’s theorem (see [18] or [19]) the test rank of \( S_{rd}, r, d \geq 2 \), is \( r - 1 \). Recall, that test rank of a group \( G \) is the minimal number of elements of \( G \) such that every endomorphism fixing any of this elements is automorphism. In the general case \( H \) does not contain a test element. Hence, the methods of this paper do not work completely.

Problem 6.2. Is it true that for any two-generated subgroup \( H \) of any free polynilpotent group \( P \) of rank \( r \geq 2 \) and class \( (c_1, c_2, \ldots, c_l) \), \( l \geq 1 \), the following conditions are equivalent:

1) \( H \) is a retract of \( P \),
2) \( H \) is a algebraically closed subgroup of \( P \),
3) \( H \) is an verbally closed subgroup of \( P \)?

By Timoshenko’s theorem [18] (see also [19]) any free solvable group \( S_{2d}, d \geq 2 \), contains test elements, all belong to \( S_{2d}^{(d-1)} \). For the affirmative solution to this problem sufficient to prove that any \( S_{2d} \) contains a test element with properties as in Lemma 5.2. Note that the reference to the statement in [18] (or [17]) in Lemma 5.2 works in the general case. Namely, any automorphism of \( S_{2d}, d \geq 2 \) identical mod \( S_{2d}^{(d-1)} \), is inner.

Note, that by Gupta and the second author’s theorem [20] (see also [19]) any free polynilpotent group of class \( (c_1, c_2, \ldots, c_l) \) for \( l \geq 1 \) has a test element. We need in analog of Lemma 2.4 to solve this problem affirmatively.
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