M. Martinis and N. Perković

Is exponential metric a natural space-time metric of Newtonian gravity?

Received: date / Accepted: date

Abstract We show how Newtonian gravity with effective (actually observed) masses, obeying the mass-energy relation of special relativity, can explain all observations used to test General relativity. Dynamics of a gravitationally coupled binary system is considered in detail, and the effective masses of constituents are determined. Interpreting our results in terms of motion in a curved space-time background, we are led, using the Lagrangian formalism, to consider the exponential metric as a natural space-time metric of Newtonian gravity.

Keywords Newtonian gravity · Exponential metric · Perihelion precession · Gravitational light deflection

PACS 04.20.Cv · 04.20.Fy
1 Introduction

Newtonian dynamics introduced into scientific terminology the concepts of inertial and gravitational mass, which for Newton were mutually proportional quantities. The exact equality \( m = m_g \) between inertial and gravitational mass was postulated by Einstein as the Principle of Equivalence upon which the General Theory of Relativity was founded. Modern physical theories widely use two other concepts of mass, the invariant bare or proper mass \( m_0 \) and the observer dependent effective mass \( m_g \). The effective mass can be viewed as a "dressed" bare mass due to its interaction with the surrounding medium, the space-time background for example.

In this paper, we reconsider the motion of a point-like objects through a gravitational background. The interaction with the background is described by the Newtonian-like gravity force with effective (actually observed) masses, obeying the Einstein’s mass-energy relation, \( E = mc^2 \). The connection between bare and effective mass is then given by \( m = m_0 dt/\tau = m_g dt_g/\tau_g \), where \( t \) and \( t_g \) are the observer time and the gravitational time, respectively, while \( \tau \) denotes the proper time of the moving object. The gravitational time is the time which shows the clock locally attached to the gravitational field.

2 Gravitational two body problem

In classical Newtonian mechanics the gravitational two body problem with inertial (bare) masses is exactly solvable in an analytical form. However, these solutions fail to explain observed facts such as the motion of planetary perihelion, the starlight deflection around the Sun, and the gravitational red shift.

We shall show now that Newtonian gravity with effective masses, obeying the mass-energy relation \( E = mc^2 \) can give us the satisfactory explanations to all the observations used to test General relativity, without invoking the field equations of General relativity.

We begin by considering an isolated, gravitationally coupled, binary system with point-like bare masses \( M_0 \) and \( m_0 << M_0 \). In the rest frame of \( M_0 \), the gravitational field around \( M_0 \) is approximately static, spherically symmetric, and isotropic.

In this frame the orbital motion of \( m_0 \) is described by the Newtonian-like gravity force

\[
F_g = -GM_0m_g/r^2 = E_g du/dr
\]

where now \( E_g = m_g c^2 \) defines the effective mass of \( m_0 \) which is actually observed from \( M_0 \). Since the gravitational field around \( M_0 \) is static and spherically symmetric, the \( m_g \) will depend only on the radial distance from \( M_0 \). Therefore, the \( E_g \) changes according to the well known rule

\[
dE_g = drF_g = E_g du,
\]

with an obvious solution

\[
m_g = m_0 e^u,
\]
where \( u = GM_0/rc^2 = R_g/r \) is the gravitational dimensionless potential of the mass \( M_0 \). By observing that the relativistic energy of \( m_0 \) can be written as \( E = mc^2 = m_0c^2/\sqrt{1 - \beta^2_g} = m_gc^2dt_g/d\tau \), we easily find the form of \( d\tau \) as
\[
d\tau = dt_g\sqrt{1 - \beta^2_g} = dt e^{-u}\sqrt{1 - e^{4u}\beta^2}, \tag{4}
\]
where \( \beta^2_g = e^{4u}\beta^2 \) with \( \beta^2 = \sqrt{1 - \beta^2_g} \). Notice, that \( dt_g = e^{-u} \) and \( dr_g = e^u dr \) are time and space intervals measured in the coordinate frame locally attached to the gravitational field. The connection with the structure of the space-time background is obtained from \( ds = cdt = \sqrt{g_{\mu\nu}dx^\mu dx^\nu} \). We find that the background space-time metric is isotropic and exponential:
\[
\begin{align*}
g_{00} &= e^{-2u}, \\
g_{ii} &= -e^{-2u}, \\
g_{0j} &= 0.
\end{align*} \tag{5}
\]
It is now easy to find the Lagrangian and the Hamiltonian of the system. From \( Ldt = -m_0c\sqrt{g_{\mu\nu}dx^\mu dx^\nu} \), it follows that
\[
L = -m_0c^2e^{-u}\sqrt{1 - e^{4u}\beta^2}, \tag{6}
\]
and from \( H = \psi p_i - L \), we find the form of the Hamiltonian
\[
H = e^{-u}\sqrt{p_ip_i + m_0^2c^4} = e^{-2u}E, \tag{7}
\]
where \( p_i = \frac{\partial L}{\partial \dot{x}_i} = e^{2u}p_i \) denotes the canonical 3-momenta, and \( p^i = \psi E/c^2 \) with \( \psi = dr^i/dt \).

The orbital motion of \( m_0 \) is best analyzed in the polar coordinate frame, \((r, \theta, \phi)\). There are two constants of motion \( H \) and \( L \) which are obtained from the Lagrangian equation of motion. They express the fact that the gravitational field is static \((H)\) and spherically symmetric \((L)\). The conservation of \( L \) enable us to consider the equatorial motion, \((\theta = \pi/2)\), of \( m_0 \) only, given by
\[
\left(\frac{d\phi}{ds}\right)^2 = (h^2 - e^{-2u} - e^{-4u}\frac{l^2_{\phi}}{r^2}) \tag{8}
\]
where \( h = H/m_0c^2 \) and \( l_{\phi} = L/m_0c = e^{2u}r^2d\phi/ds \) are now those two constants of motion.

In the \((u, \phi)\)-plane, we have to solve two equivalent differential equations
\[
u'^2 = (h^2e^{2u} - 1)e^{2u}/l^2 \tag{9}
\]
and
\[
u'' + u = e^{2u}/l^2 + 2(u'^2 + u^2) \tag{10}
\]
where \( u' = du/d\phi \) and \( l = l_{\phi}/R_g \).

The exact analytic solution of these two equations is not yet feasible, but the required accuracy of an approximate solution can be achieved by expanding the right hand side of \( u'^2 \) in a truncated power series in the variable \( u \ll 1: \)
\[
u'^2 = 2[2\epsilon + (4\epsilon + 1)u + (8\epsilon + 3)u^2 + ...]/l^2 - u^2. \tag{11}
\]
where \(2\epsilon = h^2 - 1 < 0\) for a bound motion.

In the region where \(|\epsilon|\) and \(u\) are very small, a close graphical inspection of the right hand side of \(u''\) shows that already a quadratic polynomial is a very good approximation to \(u''\) and the results we are going to derive.

The equations to be analyzed now are of the form

\[
u'' + u = \frac{1}{l^2}
\]

and

\[
u'' + (1 - 6 \frac{u}{l^2}) u = \frac{1}{l^2}. \tag{13}
\]

The classical (non-relativistic) solution is obtained if the relativistic term \(6/u^2\) is neglected in (13). We are then left with the equation

\[
u'' + u = \frac{1}{l^2} \tag{14}
\]

which has the standard Newtonian solution

\[
u(\varphi) = \frac{1}{l^2}(1 + e \cos \varphi) \tag{15}
\]

This particular solution is designed so that \(u(0) = u_1 = u_{\text{max}}\) and \(u(\pi) = u_2 = u_{\text{min}}\). The eccentricity \(e\) of the elliptical orbit is \(e < 1\), and is given by

\[
e = \frac{u_1 - u_2}{u_1 + u_2} = \sqrt{1 + 2\epsilon l^2}. \tag{16}
\]

We note that the classical solution also obeys the relation \(u'(0) = u' (\pi) = 0\). This relation would not be obeyed if relativistic corrections were included, namely in this case \(u'(\pi) \neq 0\).

3 Motion of the perihelion

We show now that (13) for \(e < 1\) describes an ellipse with moving perihelion. The solution of (13) can be written in the following form

\[
u(\varphi) = \frac{1}{l^2}(1 + e_r \cos(\varphi \sqrt{1 - 6l^{-2}})) \tag{17}
\]

where \(l_r^2 = l^2 - 6\), and \(e_r = \sqrt{1 + 2\epsilon l_r^2}\). The turning points of the orbit are defined by \(\nu'(\varphi) = 0\). They are found at \(\varphi = \varphi_n = (n - 1)\pi / \sqrt{1 - 6l^{-2}}\), where \(n = 1, 2, 3, \ldots\). The angle by which the perihelion of the elliptical orbit is shifted in the direction of motion per one revolution is given by

\[
\Delta \varphi = 2(\varphi_{n+1} - \varphi_n - \pi) = 2\pi \left( \frac{1}{\sqrt{1 - 6l^{-2}}} - 1 \right) \approx \frac{6\pi}{l^2}. \tag{18}
\]

Here we can write, with sufficient accuracy, \(l = (c/\omega)\sqrt{1 - e^2}\), where \(a\) is the major semi-axis and \(\omega = 2\pi/T\) with \(T\) being the period of revolution.
The final and well known expression for the perihelion shift per period is obtained
\[
\Delta \phi = 24\pi \frac{a^2}{c^2 T^2 (1 - e^2)}.
\] (19)

In the classical non-relativistic limit the motion of perihelion is not predicted, and \(\Delta \phi_{\text{classical}} = 0\).

Attempts to explain the motion of the planetary perihelion, using only special relativity and modified by hand inertial masses in Newtonian gravity, date back to 1917 and earlier years [5].

4 Gravitational light deflection

The gravitational light deflection around the Sun was one of the crucial tests of General relativity in 1919. By allowing the light pulse to have a mass that moves with the velocity of light, the calculations using Newtonian gravity gave only half a value that was observed for the light deflection angle. According to special relativity, light pulse has a vanishing bare (rest) mass. Light motion through the static and spherically symmetric gravitational field is described by the Hamiltonian,
\[
H = c \sqrt{p_i p_i},
\]
which is obtained from (7) by setting \(m_0 = 0\). In this limit, \(l \to \infty\), but in such a way that \(h/l \approx u_1\) remains constant. The light trajectories are then determined by solving the following two differential equations:
\[
u'' + \nu = 2(u^2 + u^2) \approx 2u_1^2[1 + 4(u - u_1) + 8(u - u_1)^2 + ...]\] (20)
and
\[
u'' + \nu = 2(u^2 + u^2) \approx 2u_1^2[1 + 4(u - u_1) + 8(u - u_1)^2 + ...].\] (21)

where \(u_1 = R_g/r_{\text{min}}\) and \(r_{\text{min}}\) is the closest distance that the light pulse gets to the source of gravity field \(M_0\). The appropriate coordinate frame for studying shapes of light trajectories is \(u(\varphi) = u(-\varphi), u(0) = u_1, u'(0) = 0\) and \(u(\pi/2 + \Delta \varphi/2) = 0\). Here, \(\Delta \varphi\) is by definition the deflection angle for light in a gravitational field. In the weak field limit, \(u \ll 1\), it is sufficient to study the equation
\[
u'' + \nu = 2u_1^2.\] (22)

in order to determine the shape of the light trajectory. In our coordinate frame, the solution of (21) is \(u(\varphi) = 2u_1 + B \cos \varphi\), where \(B = u_1(1 - 2u_1)\). The condition \(u(\pi/2 + \Delta \varphi/2) = 0\) yields to \(\sin(\Delta \varphi/2) = 2u_1/(1 - 2u_1)\) from which we get, to lowest order in \(\Delta \varphi\) and \(u_1\), the final expression for the deflecting angle of light in a gravitational field:
\[
\Delta \varphi = 4u_1 = \frac{4GM_0}{c^2 r_{\text{min}}},\] (23)
This result is found by using Newtonian gravity with effective masses obeying the mass-energy relation. Classical calculations, using Newtonian gravity with bare masses, would lead to the equation $u'' + u = u_1^2$ for the light trajectory. This equation predicts a deflection angle $\Delta \varphi_{\text{classical}} = 2u_1$ that is half the value that was observed.

5 Conclusion

In this paper we studied the implications of interpreting the Newtonian gravity force in terms of actually observed effective masses obeying the mass-energy relation of special relativity. We claimed that the gravitational attraction should be acting between all mass-equivalent energies. Using the Lagrangian formalism, the corresponding space-time background metric is found to be exponential. This metric was first introduced by Yilmaz [6] in an attempt to modify the Einstein’s field equations of General relativity. However, the Yilmaz theory was immediately sharply criticized [7] on various grounds [8] as being ill defined and because it does not predict existence of black-holes [9]. In our approach, however, the exponential metric arises as a consequence of introducing an observer in the process of measuring the gravitational attraction by means of the Newtonian gravity with effective masses. We claim that the proper (bare) mass $m_0$ of a body in a gravitational field will always be observed as an effective mass $m_g = m_0 e^u$. For example, on the Earth’s surface, the ratio $m_g / m_0$ is very close to unity, or more precisely $m_g / m_0 - 1 \approx 7 \times 10^{-10}$.

The exponential metric belongs to a large family of alternative theories of gravity, which all agree with General relativity to a first order in $u$.

In this paper, we have also demonstrated how Newtonian gravity with effective masses explains the motion of perihelion in binary systems, and the gravitational deflection of light rays. The gravitational red-shift can also be explained by our theory [10] from the relation

$$\frac{E_g(2) - E_g(1)}{E_g(1)} = e^{u(2) - u(1)} - 1 \approx u(2) - u(1) + ...$$

This result agrees both with the prediction of General relativity and with observations.

Acknowledgements This work was supported by the Croatian Ministry of Science, Education and Sport, Project No.098-0982930-2900.

References

1. P. G. Roll, R. Krotkov and R. H. Dicke, The equivalence of inertial and passive gravitational mass, Annals of Physics, 26, 442 (1964).
2. A. Einstein, The Meaning of Relativity, Princeton University Press (2005).
3. L.B. Okun, The concept of mass, Physics Today, June 1989, 31.
4. M. B. Pinto, Introducing the notion of bare and effective mass via Newton’s second law of motion, Eur J. Phys. 28, 171 (2007)
5. L. Silverstein, The motion of the perihelion of Mercury deduced from the classical theory of relativity, Monthly Notices of the Royal Astronomical Society 77, 503 (1917).
6. H. Yilmaz, Phys. Rev. 111 (1958) 1417; Annals of Physics 101 (1976) 413
7. C. W. Misner, “Yilmaz Cancels Newton, Nuovo Cimento B 114, 1079 (1999); C.O. Alley, H. Yilmaz, Refutation of C. W. Misner’s claims, Nuovo Cimento B 114, 1087 (1999).
8. M. Ibison, Cosmological test of the Yilmaz theory of gravity, Class. quantum gravity 23 557 (2006).
9. S. L. Robertson, X-ray novae, event horizons, and the exponential metric, Astrophys. J. 515, 355 (1999).
10. M. Martinis, N. Perković, On the Gravitational Energy Shift for matter waves, http://arxiv.org/abs/1004.0826