Logics for Rough Concept Analysis

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Abstract. Taking an algebraic perspective on the basic structures of Rough Concept Analysis as the starting point, in this paper we introduce some varieties of lattices expanded with normal modal operators which can be regarded as the natural rough algebra counterparts of certain subclasses of rough formal contexts, and introduce proper display calculi for the logics associated with these varieties which are sound, complete, conservative and with uniform cut elimination and subformula property. These calculi modularly extend the multi-type calculi for rough algebras to a ‘nondistributive’ (i.e. general lattice-based) setting.

Keywords: Rough Set Theory · Formal Concept Analysis · modal logic · lattice-based logics · algebras for rough sets · structural proof theory.

1 Introduction

This paper continues a line of investigation started in \cite{9} and aimed at introducing sequent calculi for the logics of varieties of ‘rough algebras’, introduced and discussed in \cite{1,19}. The ‘rough algebras’ considered in the present paper are nondistributive (i.e. general lattice-based) generalizations of those of \cite{19}; specifically, they are varieties of lattices expanded with normal modal operators, natural examples of which arise in connection with (certain subclasses of) rough formal contexts, introduced by Kent in \cite{15} as the basic notion of Rough Concept Analysis (RCA), a synthesis of Rough Set Theory \cite{18} and Formal Concept Analysis \cite{7}. The core idea of Kent’s approach is to use a given indiscernibility relation \(E\) on the objects of a formal context \((A,X,I)\) to generate \(E\)-definable approximations \(R\) and \(S\) of the relation \(I\) such that \(S \subseteq I \subseteq R\). The starting point of our approach is that \(R\) and \(S\) can be used to generate tuples of adjoint normal modal operators \(\langle S \rangle \vdash [S]\) and \(\langle R \rangle \vdash [R]\). We identify conditions under which \([S]\) and \(\langle R \rangle\) are interior operators and \([R]\) and \(\langle S \rangle\) are closure operators. This provides the basic algebraic framework, which we axiomatically extend so as to define ‘nondistributive’ counterparts of the varieties introduced in \cite{19}. 

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From an algebraic perspective, it is interesting to observe that, unlike \( \langle S \rangle \) and \([S]\), the modal operators \( \langle R \rangle \) and \([R]\) play the reverse roles they usually have in rough set theory: namely, \([R]\), being an inflationary map, plays naturally the role of the closure operator providing the upper approximation of a given formal concept, and similarly \( \langle R \rangle \), being a deflationary map, plays the role of the interior operator, providing the lower approximation of a given formal concept.

From a proof-theoretic perspective, these properties make it possible to introduce a modular generalization of the multi-type approach taken in [9] to endow the logics of ‘rough algebras’ with analytic calculi, so as to adapt it to a ‘nondistributive’ propositional base. For the sake of introducing the structural counterparts of the lattice connectives \( \land \) and \( \lor \) (the reasons for which are explained below), our basic calculus does not have the display property, since the usual display rules for \( \land \) and \( \lor \) are not sound in the general lattice setting. However, the cut elimination and subformula property for the calculi defined in Section 6 can be straightforwardly verified by appealing to the meta-theorem of [5]. Another interesting departure from the calculi of [9] concerns the counterparts of the IA3 condition, which in the present paper comes in two variants: the lower (strict), and the upper (lax). The inequality corresponding to the lower variant of IA3, which was analytic in the presence of distributivity, is not analytic inductive in the absence of distributivity (cf. [12, Definition 55]). However, the inequality corresponding to the upper variant of IA3 is analytic inductive, and hence can be captured in terms of an analytic structural rule.

## 2 Preliminaries

The purpose of this section, which is based on [3 Appendix] and [2 and [17] Sections 2.3 and 2.4], is to briefly recall the basic notions of the theory of enriched formal contexts (cf. Definition 2) while introducing the notation which will be used throughout the paper. For any relation \( T \subseteq U \times V \), and any \( U' \subseteq U \) and \( V' \subseteq V \), let

\[
T^{(0)}[V'] := \{ u \mid \forall v \in V' \Rightarrow uTv \} \quad T^{(1)}[U'] := \{ v \mid \forall u \in U' \Rightarrow uTv \}.
\]

It can be easily verified that \( U' \subseteq T^{(0)}[V'] \) if \( V' \subseteq T^{(1)}[U'] \), that \( V_1 \subseteq V_2 \subseteq V \) (resp. \( U_1 \subseteq U_2 \subseteq U \)) implies that \( T^{(0)}[V_2] \subseteq T^{(0)}[V_1] \) (resp. \( T^{(1)}[U_2] \subseteq T^{(1)}[U_1] \)), and \( S \subseteq T \subseteq U \times V \) implies that \( S^{(0)}[V'] \subseteq T^{(0)}[V'] \) and \( S^{(1)}[U'] \subseteq T^{(1)}[U'] \) for all \( V' \subseteq V \) and \( U' \subseteq U \).

**Formal contexts, or polarities**, are structures \( \mathbb{P} = (A, X, I) \) such that \( A \) and \( X \) are sets, and \( I \subseteq A \times X \) is a binary relation. Intuitively, formal contexts can be understood as abstract representations of databases [7], so that \( A \) represents a collection of objects, \( X \) as a collection of features, and for any object \( a \) and feature \( x \), the tuple \( (a, x) \) belongs to \( I \) exactly when object \( a \) has feature \( x \). In what follows, we use \( a, b \) (resp. \( x, y \)) for elements of \( A \) (resp. \( X \)), and \( B \) (resp. \( Y \)) for subsets of \( A \) (resp. of \( X \)).

As is well known, for every formal context \( \mathbb{P} = (A, X, I) \), the pair of maps

\[
(\cdot)^\uparrow : \mathcal{P}(A) \to \mathcal{P}(X) \quad \text{and} \quad (\cdot)^\downarrow : \mathcal{P}(X) \to \mathcal{P}(A),
\]

respectively defined by the assignments \( B^\uparrow := I^{(1)}[B] \) and \( Y^\downarrow := I^{(0)}[Y] \), form a Galois connection and hence induce the closure operators \( (\cdot)^\uparrow \) and \( (\cdot)^\downarrow \) on \( \mathcal{P}(A) \) and on \( \mathcal{P}(X) \).
such that $R$ is a formal context, and $R$ motivates the following definitions:

**Definition 1.** For every formal context $\mathbb{P} = (A, X, I)$, a formal concept of $\mathbb{P}$ is a pair $c = (B, Y)$ such that $B \subseteq A$, $Y \subseteq X$, and $B^\uparrow = Y$ and $Y^\downarrow = B$. The set $B$ is the extension of $c$, which we will sometimes denote $[[c]]$, and $Y$ is the intension of $c$, sometimes denoted $\langle c \rangle$. Let $L(\mathbb{P})$ denote the set of the formal concepts of $\mathbb{P}$. Then the concept lattice of $\mathbb{P}$ is the complete lattice

$$\mathbb{P}^+ := (L(\mathbb{P}), \wedge, \vee),$$

where for every $X \subseteq L(\mathbb{P})$,

$$\wedge X := (\bigcap_{c \in X}[[c]], (\bigcap_{c \in X}\langle c \rangle)^\downarrow) \quad \text{and} \quad \vee X := (\bigcap_{c \in X}[[c]], (\bigcap_{c \in X}\langle c \rangle)^\uparrow).$$

Then clearly, $\top^\mathbb{P} := \wedge \emptyset = (A, A^\uparrow)$ and $\bot^\mathbb{P} := \vee \emptyset = (X^\downarrow, X)$, and the partial order underlying this lattice structure is defined as follows: for any $c, d \in L(\mathbb{P})$,

$$c \leq d \iff [[c]] \subseteq [[d]] \quad \text{iff} \quad [[d]] \subseteq [[c]].$$

**Theorem 1.** (Birkhoff’s theorem, main theorem of FCA) Any complete lattice $\mathbb{L}$ is isomorphic to the concept lattice $\mathbb{P}^+$ of some formal context $\mathbb{P}$.

**Definition 2.** An enriched formal context is a tuple $\mathbb{F} = (\mathbb{P}, R_\square, R_\diamond)$ such that $\mathbb{P} = (A, X, I)$ is a formal context, and $R_\square \subseteq A \times X$ and $R_\diamond \subseteq X \times A$ are $I$-compatible relations, that is, $R_\square^{(0)}[x]$ (resp. $R_\square^{(0)}[a]$) and $R_\diamond^{(1)}[a]$ (resp. $R_\diamond^{(1)}[x]$) are Galois-stable for all $x \in X$ and $a \in A$. The complex algebra of $\mathbb{F}$ is

$$\mathbb{F}^+ := (\mathbb{P}^+, [R_\square], (R_\diamond)),
\qquad$$

where $\mathbb{P}^+$ is the concept lattice of $\mathbb{P}$, and $[R_\square]$ and $(R_\diamond)$ are unary operations on $\mathbb{P}^+$ defined as follows: for every $c \in \mathbb{P}^+$,

$$[R_\square]c := (R_\square^{(0)}[[c]], (R_\square^{(0)}[[c]])^\downarrow) \quad \text{and} \quad (R_\diamond)c := ((R_\diamond^{(0)}[[c]])^\uparrow, (R_\diamond^{(0)}[[c]])^\downarrow).$$

Since $R_\square$ and $R_\diamond$ are $I$-compatible, $[R_\square], (R_\diamond), [R_\diamond^{-1}], (R_\square^{-1}) : \mathbb{P}^+ \to \mathbb{P}^+$ are well-defined.

**Lemma 1.** (cf. [12] Lemma 3) For any enriched formal context $\mathbb{F} = (\mathbb{P}, R_\square, R_\diamond)$, the algebra $\mathbb{F}^+ = (\mathbb{P}^+, [R_\square], (R_\diamond))$ is a complete lattice expanded with normal modal operators such that $[R_\square]$ is completely meet-preserving and $(R_\diamond)$ is completely join-preserving.

**Definition 3.** For any formal context $\mathbb{P} = (A, X, I)$ and any $I$-compatible relations $R, T \subseteq A \times X$, the composition $R; T \subseteq A \times X$ is defined as follows: for any $a \in A$ and $x \in X$,

$$(R; T)^{(1)}[a] = R^{(1)}[T^{(0)}[a]]$$

or equivalently

$$(R; T)^{(0)}[x] = R^{(0)}[T^{(1)}[x]].$$

When $B = \{a\}$ (resp. $Y = \{x\}$) we write $a^{1\downarrow}$ for $[a]^{1\downarrow}$ (resp. $x^{1\uparrow}$ for $\{x\}^{1\uparrow}$).
3 Motivation: Kent’s Rough Concept Analysis

Below, we report on the basic definitions and constructions in Rough Concept Analysis [15], cast in the notational conventions of Section 2.

Rough formal contexts (abbreviated as $Rfc$) are tuples $\mathcal{G} = (\mathcal{P}, E)$ such that $\mathcal{P} = (A, X, I)$ is a polarity (cf. Section 2), and $E \subseteq A \times A$ is an equivalence relation (the indiscernibility relation between objects). For every $a \in A$ we let $(a)_E := \{ b \in A \mid a E b \}$. The relation $E$ induces two relations $R, S \subseteq A \times I$ approximating $I$, defined as follows: for every $a \in A$ and $x \in X$,

$$aRx \text{ iff } bIx \text{ for some } b \in (a)_E; \quad aSx \text{ iff } bIx \text{ for all } b \in (a)_E.$$  

(1)

By definition, $R,S$ are $E$-definable (i.e. $R^{(0)}[x] = \bigcup_{aRx} (a)_E$ and $S^{(0)}[x] = \bigcup_{aSx} (a)_E$ for any $x \in X$), and $E$ being reflexive immediately implies that

**Lemma 2.** For any $Rfc \mathcal{G} = (\mathcal{P}, E)$, if $R$ and $S$ are defined as in (1), then

$$S \subseteq I \quad \text{and} \quad I \subseteq R.$$  

(2)

Intuitively, we can think of $R$ as the lax version of $I$ determined by $E$, and $S$ as its strict version determined by $E$. Following the methodology introduced in [4] and applied in [23] to introduce a polarity-based semantics for the modal logics of formal concepts, under the assumption that $R$ and $S$ are $I$-compatible (cf. Definition 2), the relations $R$ and $S$ can be used to define normal modal operators $[R], (R), [S], (S)$ on $\mathbb{P}^+$ defined as follows: for any $c \in \mathbb{P}^+$,

$$[[[R]c]] := R^{(0)}[[c]] = \{ a \in A \mid \forall x (x \in [c] \Rightarrow aRx) \}$$  

(3)

$$[[[S]c]] := S^{(0)}[[c]] = \{ a \in A \mid \forall x (x \in [c] \Rightarrow aSx) \}.$$  

(4)

That is, the members of $[R]c$ are exactly those objects that satisfy (possibly by proxy of some object equivalent to them) all features in the description of $c$, while the members of $[S]c$ are exactly those objects that not only satisfy all features in the description of $c$, but that ‘force’ all their equivalents to also satisfy them. The assumption that $S \subseteq I$ implies that $[[[S]c]] = S^{(0)}[[c]] \subseteq I^{(0)}[[c]] = [[c]]$, hence $[S]c$ is a sub-concept of $c$. The assumption that $I \subseteq R$ implies that $[[c]] = I^{(0)}[[c]] \subseteq R^{(0)}[[c]] = [[[R]c]]$, hence $[R]c$ is a super-concept of $c$. Moreover, for any $c \in \mathbb{P}^+$,

$$[[[R]c]] := R^{(1)}[[c]] = \{ x \in X \mid \forall a (a \in [c] \Rightarrow aRx) \}$$  

(5)

$$[[[S]c]] := S^{(1)}[[c]] = \{ x \in X \mid \forall a (a \in [c] \Rightarrow aSx) \}.$$  

(6)

That is, $(R)c$ is the concept described by those features shared not only by each member of $c$ but also by their equivalents, while $(S)c$ is the concept described by the common features of those members of $c$ which ‘force’ each of their equivalents to share them. The assumption that $I \subseteq R$ implies that $[[c]] = I^{(1)}[[c]] \subseteq S^{(1)}[[c]] = [[(R)c]]$, and hence $(R)c$ is a sub-concept of $c$. The assumption that $S \subseteq I$ implies that $[[S]c] = S^{(1)}[[c]] \subseteq I^{(1)}[[c]] = [[c]]$, and hence $(S)c$ is a super-concept of $c$. Summing up the discussion
above, we have verified that the conditions \( I \subseteq R \) and \( S \subseteq I \) imply that the following sequents of the modal logic of formal concepts are valid on Kent’s basic structures:

\[
\Box_{I} \phi \vdash \phi \quad \phi \vdash \Box_{I} \phi \quad \phi \vdash \Diamond_{I} \phi \quad \phi \vdash \Box_{I} \Diamond_{I} \phi,
\]

where \( \Box_{I} \) is interpreted as \([S], \Box_{I} \) as \([R], \Diamond_{I} \) as \([S)\) and \(\Diamond_{I} \) as \([R)\). Translated algebraically, these conditions say that \( \Box_{I} \) and \(\Diamond_{I} \) are deflationary, as interior operators are, \(\Diamond_{I} \) and \(\Box_{I} \) are inflationary, as closure operators are. Hence, it is natural to ask under which conditions they (i.e. their semantic interpretations) are indeed closure/interior operators. The next definition and lemma provide answers to this question.

**Definition 4.** An Rfc \( \mathcal{G} = (\mathcal{P}, E) \) is amenable if \( E, R \) and \( S \) (defined as in \((1)\)) are \( I \)-compatible.

**Lemma 3.** For any amenable Rfc \( \mathcal{G} = (\mathcal{P}, E) \), if and \( R \) and \( S \) are defined as in \((1)\), then

\[
R; R \subseteq R \quad \text{and} \quad S \subseteq S; S.
\]

**Proof.** Let \( x \in X \). To show that \( R^{(0)}[I^{(1)}[R^{(0)}[x]]] \subseteq R^{(0)}[x] \), let \( a \in R^{(0)}[I^{(1)}[R^{(0)}[x]]] \), which implies that \( I^{(0)}[R^{(1)}[a]] \subseteq I^{(0)}[I^{(1)}[R^{(0)}[x]]] = R^{(0)}[x] \), the last equality holding since \( R \) is \( I \)-compatible by assumption. Moreover, \( I \subseteq R \) (cf. Lemma \((2)\)) implies that \( I^{(1)}[a] \subseteq R^{(1)}[a] \), which implies that \( I^{(0)}[R^{(1)}[a]] \subseteq I^{(0)}[I^{(1)}[a]] \subseteq \langle a \rangle_{E} \), the last inclusion holding since \( E \) is \( I \)-compatible by assumption. Hence, \( I^{(0)}[R^{(1)}[a]] \subseteq R^{(0)}[x] \cap \langle a \rangle_{E} \). Suppose for contradiction that \( a \notin R^{(0)}[x] \). By the \( E \)-definability of \( R \), this is equivalent to \( R^{(0)}[x] \cap \langle a \rangle_{E} = \emptyset \). Hence \( I^{(0)}[R^{(1)}[a]] \subseteq \emptyset \), from which it follows that \( R^{(1)}[a] = I^{(1)}[I^{(0)}[R^{(1)}[a]]] = I^{(1)}[\emptyset] = X \). Hence, \( x \in R^{(1)}[a] \), i.e. \( a \in R^{(0)}[x] \).

Let \( x \in X \). To show that \( S^{(0)}[x] \subseteq S^{(0)}[I^{(1)}[S^{(0)}[x]]] \), assume that \( a \in S^{(0)}[x] \). Since \( S \) is \( E \)-definable by construction, this is equivalent to \( \langle a \rangle_{E} \subseteq S^{(0)}[x] \). To show that \( a \in S^{(0)}[I^{(1)}[S^{(0)}[x]]] \), we need to show that \( b/Iy \) for any \( b \in \langle a \rangle_{E} \) and any \( y \in I^{(1)}[S^{(0)}[x]] \). Let \( y \in I^{(1)}[S^{(0)}[x]] \). Hence, by definition, \( b/Iy \) for every \( b' \in S^{(0)}[x] \). Since \( \langle a \rangle_{E} \subseteq S^{(0)}[x] \), this implies that \( b/Iy \) for any \( b \in \langle a \rangle_{E} \), as required.

By the general theory developed in \((4)\) and applied to enriched formal contexts in \((17)\) Proposition 5], properties \((5)\) guarantee that the following sequents of the modal logic of formal concepts are also valid on amenable Rfc’s:

\[
\Box a \phi \vdash a \Box a \phi \quad \Box a \Box a \phi \vdash \Box a \phi \quad \Box a \Diamond a \phi \vdash \Diamond a \phi \quad \Diamond a \phi \vdash \Box a \Diamond a \phi.
\]

Finally, again by \((17)\) Proposition 5], the fact that by construction \( \Box a \) and \(\Diamond a \) (resp. \(\Box a \) and \(\Diamond a \)) are interpreted by operations defined in terms of the same relation guarantees the validity of the following sequents on amenable Rfc’s:

\[
\phi \vdash \Box a \Diamond a \phi \quad \Box a \Diamond a \phi \vdash \phi \quad \phi \vdash \Box a \Diamond a \phi \quad \Diamond a \Box a \phi \vdash \phi.
\]

Axioms \((7), (9)\) and \((10)\) constitute the starting point and motivation for the proof-theoretic investigation of the logics associated to varieties of algebraic structures which can be understood as abstractions of amenable Rfc’s. We define these varieties in the next section.

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7 The assumption that \( E \) is \( I \)-compatible does not follow from \( R \) and \( S \) being \( I \)-compatible. Let \( \mathcal{G} = (\mathcal{P}, \text{Id}_{A}) \) for any polarity \( \mathcal{P} \) such that not all singleton sets of objects are Galois-stable. Hence \( E = \text{Id}_{A} \) is not \( I \)-compatible. However, if \( E = \text{Id}_{A} \), then \( R = S = I \) are \( I \)-compatible.
4 Kent algebras

In the present section, we introduce basic Kent algebras (and the variety of abstract Kent algebras (aKa) to which they naturally belong), as algebraic generalizations of amenable Rfc’s, and then introduce some subvarieties of aKas in the style of [19].

Definition 5. A basic Kent algebra is a structure $\mathcal{A} = (\mathcal{L}, \square_s, \diamond_s, \square_t, \diamond_t)$ such that $\mathcal{L}$ is a complete lattice, and $\square_s, \diamond_s, \square_t, \diamond_t$ are unary operations on $\mathcal{L}$ such that for all $a, b \in \mathcal{L}$,
\begin{align*}
\diamond_s a \leq b & \text{ iff } a \leq \square_s b \quad \text{and} \quad \diamond_t a \leq b & \text{ iff } a \leq \square_t b, \tag{11}
\end{align*}
and for any $a \in \mathcal{L}$,
\begin{align*}
\square_s a \leq a & \quad a \leq \diamond_s a & \quad a \leq \square_t a & \quad \diamond_t a \leq a \tag{12}
\end{align*}
\begin{align*}
\square_s a \leq \square_s \diamond_s a & \quad \diamond_s \diamond_s a \leq \diamond_s a & \quad \square_t \square_t a \leq \square_t a & \quad \diamond_t a \leq \diamond_t \diamond_t a \quad \tag{13}
\end{align*}
We let $\text{K}^+$ denote the class of basic Kent algebras.

From (11) it follows that, in basic Kent algebras, $\square_s$ and $\square_t$ are completely meet-preserving, $\diamond_s$ and $\diamond_t$ are completely join-preserving. For any amenable Rfc $\mathcal{G} = (\mathcal{P}, E)$, if $R$ and $S$ are defined as in (1), then
\begin{align*}
\mathcal{G}^+ := (\mathcal{P}^+, [S], \langle S \rangle, [R], \langle R \rangle)
\end{align*}
where $\mathcal{P}^+$ is the concept lattice of the formal context $\mathcal{P}$ and $[S], \langle S \rangle, [R], \langle R \rangle$ are defined as in (5)–(6). The following proposition is an immediate consequence of [17, Proposition 5], using Lemmas [2] and [3] and the fact that $[R]$ and $\langle R \rangle$ (resp. $[S]$ and $\langle S \rangle$) are defined using the same relation.

Proposition 1. If $\mathcal{G} = (\mathcal{P}, E)$ is an amenable Rfc, then $\mathcal{G}^+$ is a basic Kent algebra.

The natural variety containing basic Kent algebras is defined as follows.

Definition 6. An abstract Kent algebra (aKa) is a structure $\mathcal{A} = (\mathcal{L}, \square_s, \diamond_s, \square_t, \diamond_t)$ such that $\mathcal{L}$ is a lattice, and $\square_s, \diamond_s, \square_t, \diamond_t$ are unary operations on $\mathcal{L}$ validating (11), (12), and (13). We let $\text{K}$ denote the class of abstract Kent algebras.

From (11) it follows that, in aKas, $\square_s$ and $\square_t$ are finitely meet-preserving, $\diamond_s$ and $\diamond_t$ are finitely join-preserving.

Lemma 4. For any aKa $\mathcal{A} = (\mathcal{L}, \square_s, \diamond_s, \square_t, \diamond_t)$ and every $a \in \mathcal{L}$,
\begin{align*}
\square_s a \lor \diamond_t a \leq \square_t a \land \diamond_s a. \tag{14}
\end{align*}
\begin{align*}
a \leq \square_s \diamond_s a & \quad \diamond_s \diamond_s a \leq a & \quad a \leq \square_t \diamond_t a & \quad \diamond_t \diamond_t a \leq a \tag{15}
\end{align*}
\begin{align*}
\square_s a \leq \square_s \diamond_s a & \quad \diamond_s \diamond_s a \leq \diamond_s a & \quad \square_t a \leq \square_t \diamond_t a & \quad \diamond_t a \leq \square_t \diamond_t a. \tag{16}
\end{align*}
\begin{align*}
\diamond_s \diamond_s a \leq \square_s a & \quad \diamond_s a \leq \square_s \diamond_s a & \quad \diamond_t \diamond_t a \leq \square_t a & \quad \diamond_t a \leq \square_t \diamond_t a. \tag{17}
\end{align*}
Proof. The inequalities in (15) are straightforward consequences of (11). The inequalities in (14) and (16) follow from (12) and (15), using the transitivity of the order. The inequalities in (17) follow from those in (13) using (11).

Conditions (17) define the ‘Kent algebra’ counterparts of topological quasi Boolean algebras 5 (tqBa5) [19]. In the next definition, we introduce ‘Kent algebra’ counterparts of some other varieties considered in [12], and also varieties characterized by interaction axioms between lax and strict connectives which follow the pattern of the 5-axioms in rough algebras.

Definition 7. An aKa $\mathcal{A}$ as above is an aKa5′ if for any $a \in \mathbb{L}$,

\[
\begin{align*}
\Diamond \alpha & \leq \Box \Diamond \alpha \\
\Diamond \Box \alpha & \leq \Diamond \Box \alpha \\
\Box \alpha & \leq \Box \Box \alpha \\
\Box \Diamond \alpha & \leq \Box \Diamond \alpha
\end{align*}
\]  
(18)

is a K-IA3, if for any $a, b \in \mathbb{L}$,

\[
\Box \alpha \leq \Box b \text{ and } \Diamond \alpha \leq \Diamond b \implies a \leq b,
\]
(19)

and is a K-IA3r if for any $a, b \in \mathbb{L}$,

\[
\Box r \alpha \leq \Box \alpha \text{ and } \Diamond r \alpha \leq \Diamond \alpha \implies a \leq b.
\]
(20)

Interestingly, the third and fourth inequality in (18) are not analytic inductive (cf. [12] Definition 55); however, they are equivalent to analytic inductive inequalities in the multi-type language of the heterogeneous algebras discussed in the next section.

5 Multi-type presentation of Kent algebras

Similarly to what holds for rough algebras (cf. [9] Section 3), since the modal operations of any aKa $\mathcal{A} = (\mathbb{L}, \Box, \Diamond, \Box r, \Diamond r)$ are either interior operators or closure operators, each of them factorizes into a pair of adjoint normal modal operators which are retraction or co-retractions, as illustrated in the following table:

| $\Box _c$ | $\Box _f$ | $\Diamond _f$ | $\Box _C$ | $\Diamond _C$ | $\Diamond _S$ |
|----------|----------|------------|----------|----------|----------|
| $\Box _c : S_f \rightarrow L$ | $\Box _f : L \rightarrow S_f$ | $\Diamond _f : L \rightarrow S_f$ | $\Diamond _C : L \rightarrow C_f$ | $\Diamond _S : S_f \rightarrow L$ |
| $\Box _C : L \rightarrow C_f$ | $\Diamond _C : C_f \rightarrow L$ | $\Diamond _S : L \rightarrow S_f$ | $\Diamond _S : S_f \rightarrow L$ |

where $S_f := \Box _f [L]$, $S_C := \Diamond _C [L]$, $L_C := \Box _C [L]$, and $L_f := \Diamond _f [L]$, and such that for all $\alpha \in S_f$, $\delta \in S_C$, $a \in L$, $\pi \in L_f$, $\sigma \in L_C$,

\[
\begin{align*}
\Diamond \alpha \leq a & \text{ if } \alpha \leq \Box _f a \\
\Diamond _C a \leq \delta & \text{ iff } a \leq \Diamond _C \delta \\
\Diamond _C a \leq \pi & \text{ iff } a \leq \Box _C \pi \\
\Diamond _S \sigma \leq a & \text{ iff } \sigma \leq \Diamond _S a.
\end{align*}
\]  
(21)

Again similarly to what observed in [9], the lattice structure of $L$ can be exported to each of the sets $S_f, S_C, L_C$ and $L_f$ via the corresponding pair of modal operators as follows.

Definition 8. For any aKa $\mathcal{A}$, the strict interior kernel $S_f = (S_f, \cup_f, \cap_f, r_f, i_f)$ and the strict closure kernel $S_C = (S_C, \cup_C, \cap_C, t_C, f_C)$ are such that, for all $\alpha, \beta \in S_f$, and all $\delta, \gamma \in S_C$. 

\boldsymbol{Proof.}
\[\begin{align*}
\alpha \cup_I \beta & := \bigcirc_I (\alpha \vee \beta) \\
\alpha \cap_I \beta & := \bigcirc_I (\alpha \wedge \beta) \\
t_I & := \bigcirc_I \top, \quad t_I := \bigcirc_I \bot \\
\end{align*}\]

The lax interior kernel \(L_I = (L_I, \cup_I, \cap_I, 1_I, 0_I)\) and the lax closure kernel \(L_C = (L_C, \cup_C, \cap_C, 1_C, 0_C)\) are such that, for all \(\pi, \xi, \sigma, \tau \in L_C\),

\[\begin{align*}
\pi \cup_I \xi := & \circ_I (\bigcirc_I \pi \vee \bigcirc_I \xi) \\
\pi \cap_I \xi := & \circ_I (\bigcirc_I \pi \wedge \bigcirc_I \xi) \\
1_I := & \circ_I \top, \quad 0_I := \circ_I \bot \\
\end{align*}\]

Similarly to what observed in [9], it is easy to verify that the algebras defined above are lattices, and the operations indicated with a circle (either black or white) are lattice homomorphisms (i.e., are both normal box-type and normal diamond-type operators). The construction above justifies the following definition of class of heterogeneous algebras equivalent to aKas:

**Definition 9.** A heterogeneous aKa (haKa) is a tuple

\[H = (L, S_I, S_C, L_I, L_C, \bullet_I, \mathbf{L}, \bigcirc_C, \bigcirc_I, \circ_I, \lambda_I, \cdot_C, C)\]

such that:

- **H1** \(L, S_I, S_C, L_I, L_C\) are bounded lattices;
- **H2** \(\circ_I : S_I \hookrightarrow L, \quad \circ_C : S_C \hookrightarrow L, \quad \bigcirc_I : L \twoheadrightarrow L_I, \quad \bigcirc_C : L \twoheadrightarrow L_C\) are lattice homomorphisms;
- **H3** \(\circ_I \ast \mathbf{L}, \quad \bigcirc_C \ast \bigcirc_C, \quad \mathbf{C} \ast \circ_C, \quad \circ_C \ast \circ_C, \quad \circ_I \ast \bigcirc_I, \quad \bigcirc_I \ast \circ_I\);
- **H4** \(\mathbf{L} \ast \mathbf{L}, \quad \circ_I \circ_C = \circ_C \circ_I \ast \mathbf{C}, \quad \circ_C \circ_C = \circ_C \circ_C \ast \mathbf{C}, \quad \circ_I \ast \circ_I = \circ_I \ast \circ_I\).

The haKas corresponding to the varieties of Definition [9] are defined as follows:

| Algebra | Acronym | Conditions |
|---------|---------|------------|
| heterogeneous aKa5' | haKa5' | \(\circ_I \pi \leq \circ_I \mathbf{L}, \quad \circ_I \mathbf{L} \leq \circ_I \pi, \quad \circ_C \circ_C \sigma \leq \circ_C \sigma, \quad \circ_I \alpha \leq \circ_I \mathbf{L}, \quad \circ_I \mathbf{L} \leq \circ_I \alpha, \quad \circ_C \circ_C \delta \leq \circ_C \delta\) |
| heterogeneous K-IA3' | hK-IA3' | \(\mathbf{L} \subseteq \mathbf{L}, \quad \circ_I a \leq \circ_I b \quad \text{and} \quad \circ_I \circ_C b \quad \text{imply} \quad a \leq b\) |
| heterogeneous K-IA32' | hK-IA32' | \(\mathbf{L} \subseteq \mathbf{L}, \quad \circ_I a \leq \circ_I b \quad \text{and} \quad \circ_I \circ_C a \leq \circ_I \circ_C b \quad \text{imply} \quad a \leq b\) |

Notice that the inequalities defining haKa5' are all analytic inductive. A heterogeneous algebra \(H\) is perfect if every lattice in the signature of \(H\) is perfect (cf. [4], Definition 1.8), and every homomorphism (resp. hemimorphism) in the signature of \(H\) is a complete homomorphism (resp. hemimorphism).

Similarly to what discussed in [9] Section 3], one can readily show that the classes of haKas defined above correspond to the varieties defined in Section 4. That is, for any aKa \(A\) one can define its corresponding haKa \(A^*\) using the factorizations described at the beginning of the present section and Definition 8 and conversely, given a haKa \(H\), one can define its corresponding aKa \(H_+\) by endowing its first domain \(L\) with modal operations defined by taking the appropriate compositions of pairs of heterogeneous maps of \(H\). Then, for every \(X \in \{\text{aKa}, \text{aKa5'}, \text{K-IA3}, \text{K-IA32}\}\), letting \(\mathbb{H}\) denote its corresponding class of heterogeneous algebras, the following holds:

---

8 Condition H3 implies that \(\mathbf{L} : L \rightarrow S_I\) and \(\mathbf{C} : L \rightarrow S_C\) are \(\wedge\)-hemimorphisms and \(\bigcirc_C : L \rightarrow S_C\) and \(\circ_C : L \rightarrow S_C\) are \(\vee\)-hemimorphisms; condition H4 implies that the black connectives are surjective and the white ones are injective.
Proposition 2.  
1. If $A \in \mathbb{K}$, then $A^+ \in \mathbb{HK}$;  
2. If $\mathbb{H} \in \mathbb{HK}$, then $\mathbb{H}_+ \in \mathbb{K}$;  
3. $A \cong (A^+)^+$ and $\mathbb{H} \cong (\mathbb{H}_+)^+$.  
4. The isomorphisms of the previous item restrict to perfect members of $\mathbb{K}$ and $\mathbb{HK}$.  
5. If $A \in \mathbb{K}$, then $A^0 \cong ((A^+)^0)^+$ and if $\mathbb{H} \in \mathbb{HK}$, then $\mathbb{H}^0 \cong ((\mathbb{H}_+)^0)^+$.

6 Multi-type calculi for the logics of Kent algebras

In the present section, we introduce the multi-type calculi associated with each class of algebras $K \in \{aKa, aKaS', K-IA3\}$. The language of these logics matches the language of haKas, and is built up from structural and operational (i.e. logical) connectives. Each structural connective is denoted by decorating its corresponding logical connective with $\wedge$ (resp. $\wedge$ or $\wedge$). In what follows, we will adopt the convention that unary connectives bind more strongly than binary ones.

\[
\begin{align*}
\text{general lattice } L &::= p \mid \top \mid \bot \mid \alpha \mid \sigma \mid \triple \mid \sqcap \mid \sqcup \mid \diamond \mid A \land A \mid A \lor A \\
X &::= A \mid \bot \mid \top \mid \sqcap I \mid \sqcup I \mid \top I \mid \bot I \mid \sqcap I \mid \sqcup I \\
\text{strict-interior kernel } S_1 &::= \alpha \mid \sigma \mid \top \mid \bot \mid \diamond \mid A \mid \sigma A \\
\text{lax-interior kernel } L_1 &::= \pi \mid \alpha A \\
\text{strict-closure kernel } S_C &::= \delta \mid \sigma \mid \top \mid \bot \mid \diamond \mid \sigma A \\
\text{lax-closure kernel } L_C &::= \sigma \mid \alpha A \\
\delta &::= \delta \mid \sigma \mid \top \mid \bot \mid \diamond \mid A \mid \sigma A \\
\sigma &::= \sigma \mid \alpha \mid \top \mid \bot \mid \diamond \mid A \mid \sigma A \\
\text{Interpretation of structural connectives as their logical counterparts.} &::= \top \mid \bot \mid \top \mid \bot \\
\end{align*}
\]

1. structural and operational pure L-type connectives:

| structural operations | $\top$ | $\bot$ | $\land$ | $\lor$ |
|-----------------------|-------|-------|--------|--------|
| logical operations    | $\top$ | $\bot$ | $\land$ | $\lor$ |

2. structural and operational pure $S_1$-type and $S_C$-type connectives:

| structural operations | $\lor$ | $\land$ | $\top$ | $\bot$ |
|-----------------------|-------|--------|-------|--------|
| logical operations    | $\lor$ | $\land$ | $\top$ | $\bot$ |

3. structural and operational pure L-type and $L_C$-type connectives:

| structural operations | $\lor$ | $\land$ | $\top$ | $\bot$ |
|-----------------------|-------|--------|-------|--------|
| logical operations    | $\lor$ | $\land$ | $\top$ | $\bot$ |

4. structural and operational multi-type strict connectives:

The connectives which appear in a grey cell in the synoptic tables will only be included in the present language at the structural level.
5. structural and operational multi-type lax connectives:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{types} & L \rightarrow S_I & L \rightarrow S_C & S_I \rightarrow L & S_C \rightarrow L \\
\hline
\text{structural operations} & 
\begin{array}{cccc}
\Diamond_I & \Box_I & \Diamond_C & \Box_C \\
\delta_I & \delta_I & \delta_C & \delta_C \\
\end{array} \\
\hline
\text{logical operations} & 
\begin{array}{cccc}
\Diamond_I & \Box_I & \Diamond_C & \Box_C \\
\circ_I & \circ_I & \circ_C & \circ_C \\
\end{array} \\
\hline
\end{array}
\]

In what follows, we will use \(x, y, z\) as structural variables of arbitrary types, \(a, b, c\) as term variables of arbitrary types.

The calculus D.AKA consists of the following axiom and rules.

- Identity and Cut:

\[
\begin{array}{c}
\text{Id.} \\
\begin{array}{c}
p \vdash p \\
\end{array} \\
\begin{array}{c}
\frac{x \vdash a \quad a \vdash y}{x \vdash y} \quad \text{Cut}
\end{array}
\end{array}
\]

- Multi-type display rules (we omit the display rules capturing the adjunctions \(\Diamond_I \dashv \Box_I \dashv \Box_I\) and \(\Diamond_I \dashv \Box_I \dashv \Box_I\)):

\[
\begin{array}{c}
\text{Id.} \\
\begin{array}{c}
p \vdash p \\
\end{array} \\
\begin{array}{c}
\frac{x \vdash a \quad a \vdash y}{x \vdash y} \quad \text{Cut}
\end{array}
\end{array}
\]

- Multi-type structural rules for strict-kernel operators:

\[
\begin{array}{c}
\text{Id.} \\
\begin{array}{c}
p \vdash p \\
\end{array} \\
\begin{array}{c}
\frac{x \vdash a \quad a \vdash y}{x \vdash y} \quad \text{Cut}
\end{array}
\end{array}
\]

- Multi-type structural rules for lax-kernel operators:

\[
\begin{array}{c}
\text{Id.} \\
\begin{array}{c}
p \vdash p \\
\end{array} \\
\begin{array}{c}
\frac{x \vdash a \quad a \vdash y}{x \vdash y} \quad \text{Cut}
\end{array}
\end{array}
\]

- Multi-type structural rules for the correspondence between kernels:

\[
\begin{array}{c}
\text{Id.} \\
\begin{array}{c}
p \vdash p \\
\end{array} \\
\begin{array}{c}
\frac{x \vdash a \quad a \vdash y}{x \vdash y} \quad \text{Cut}
\end{array}
\end{array}
\]
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- Logical rules for multi-type connectives related to strict kernels:

\[ \begin{align*}
\diamondsuit_I A \vdash \Gamma & \quad \quad X \vdash A \\
\diamondsuit_I A \vdash \Gamma & \quad \quad X \vdash \diamondsuit_I A \\
A \vdash X & \quad \quad A \vdash \Box C A \\
\Box C A \vdash X & \quad \quad X \vdash \Box C A \\
\end{align*} \]

- Logical rules for multi-type connectives related to lax kernels:

\[ \begin{align*}
\check{\diamondsuit}_I \pi \vdash X & \quad \quad \Pi \vdash \pi \\
\check{\diamondsuit}_I \pi \vdash X & \quad \quad \check{\diamondsuit}_I \Pi \vdash \diamondsuit_I \pi \\
X \vdash \check{\diamondsuit}_C \sigma & \quad \quad X \vdash \check{\diamondsuit}_C \Sigma \\
\boxdot C \sigma \vdash \Sigma & \quad \quad \Sigma \vdash \check{\diamondsuit}_C A \\
\end{align*} \]

- Logical rules for lattice connectives:

\[ \begin{align*}
\tau \vdash X & \quad \quad \tau \vdash T \\
\tau \vdash T & \quad \quad \tau \vdash \bot \\
\bot \vdash \bot & \quad \quad X \vdash \bot \\
\end{align*} \]

The proper display calculi for the subvarieties of aK\(\alpha\) discussed in Section 4 are obtained by adding the following rules:

| Logic | Calculus | Rules |
|-------|----------|-------|
| H.aK\(\alpha\) | D.aK\(\alpha\) | \(\check{\diamondsuit}_I \Pi \vdash X\) |
|  |  | \(X \vdash \Box C \Sigma\) |
|  |  | \(\check{\diamondsuit}_C \sigma \vdash \Sigma\) |
|  |  | \(\Sigma \vdash \check{\diamondsuit}_C A\) |
|  |  | \(\Box C \sigma \vdash \Sigma\) |
|  |  | \(\Sigma \vdash \check{\diamondsuit}_C A\) |
| K-IA3f | D.K-IA3f | \(X \vdash \Box C \check{\diamondsuit}_I Y\) |
|  |  | \(\check{\diamondsuit}_C \bullet C X \vdash Y\) |
|  |  | \(X \vdash Y\) |

These calculi enjoy the properties of soundness, completeness, conservativity, cut elimination and subformula property the verification of which is standard and follows from the general theory of proper display calculi (cf. [14][10][13][11][16][20][6]). These verifications are discussed in the appendix.

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A Properties

Throughout this section, we let \( K \in \{aK_a, aK_a^5, K\text{-}IA_3\ell\} \), and \( \text{HK} \) the class of heterogeneous algebras corresponding to \( K \). Further, we let \( \text{D.K} \) denote the multi-type calculus for the logic \( H.K \) canonically associated with \( K \).

A.1 Soundness for perfect HK algebras

The verification of the soundness of the rules of \( \text{D.K} \) w.r.t. the semantics of perfect elements of \( \text{HK} \) (see Definition 9) is analogous to that of many other multi-type calculi (cf. [14,10,13,11,16,20,6]). Here we only discuss the soundness of the rule \( k\text{-}ia3\ell \). By definition, the following quasi-inequality is valid on every \( K\text{-IA}_3\ell \):

\[
\Box_I a \leq \Box_I b \text{ and } \Diamond_I a \leq \Diamond_I b \text{ imply } a \leq b.
\]

This quasi-inequality equivalently translates into the multi-type language as follows:

\[
\Box_C \bullet_C a \leq \Box_C \bullet_C b \text{ and } \Diamond_I \bullet_I a \leq \Diamond_I \bullet_I b \text{ imply } a \leq b.
\]

By adjunction, the quasi-inequality above can be equivalently rewritten as follows:

\[
\Diamond_C \bullet_C \Box_C \bullet_C a \leq b \text{ and } a \leq \Box_I \bullet_I \Diamond_I \bullet_I b \text{ imply } a \leq b,
\]

which, thanks to a well known property of adjoint maps, simplifies as:

\[
\Diamond_C \bullet_C a \leq b \text{ and } a \leq \Box_I \bullet_I b \text{ imply } a \leq b.
\]

Hence, the quasi-inequality above is equivalent to the following inequality:

\[
a \land \Box_I \bullet_I b \leq \Diamond_C \bullet_C a \lor b.
\]

The inequality above is analytic inductive (cf. [12, Definition 55]), and hence running ALBA on this inequality produces:

\[
\forall a \forall b [a \land \Box_I \bullet_I b \leq \Diamond_C \bullet_C a \lor b] \\
\text{iff } \forall p \forall q \forall a \forall b [(p \leq a \land \Box_I \bullet_I b \land \Diamond_C \bullet_C a \lor b \leq q) \Rightarrow p \leq q] \\
\text{iff } \forall p \forall q \forall a \forall b [(p \leq a \land p \leq \Box_I \bullet_I b \land b \leq q \land \Diamond_C \bullet_C a \leq q) \Rightarrow p \leq q] \\
\text{iff } \forall p \forall q [(p \leq \Box_I \bullet_I q \land \Diamond_C \bullet_C p \leq q) \Rightarrow p \leq q].
\]

The last quasi-inequality above is the semantic translation of the rule \( k\text{-}ia3\ell \):

\[
\begin{array}{c}
X \vdash \Diamond_I \bullet_I Y \\
\hat{\Diamond_C \bullet_C X} \vdash Y
\end{array}
\]

which we then proved to be sound on every perfect heterogeneous \( K\text{-IA}_3\ell \), by the soundness of the ALBA steps. Likewise, the defining condition of \( K\text{-IA}_3\ell \) translates into the inequality

\[
a \land \circ_C \bullet_C b \leq \circ_I \bullet_I a \lor b,
\]

which, however, is not analytic inductive, and hence it cannot be transformed into an analytic rule via ALBA.
A.2 Completeness

Let \( A^* \vdash B^* \) be the translation of any sequent \( A \vdash B \) in the language of H.K into the language of D.K induced by the correspondence between K and HK described in Section 5.

**Proposition 3.** For every H.K-derivable sequent \( A \vdash B \), the sequent \( A^* \vdash B^* \) is derivable in D.K.

Below we provide the multi-type translations of the single-type sequents corresponding to inequalities (11). All of them are derivable in D.AKA by logical introduction rules and display rules.

\[
\Diamond_\tau A \vdash A \quad \Diamond_\tau \Box_\tau A \vdash \Box_\tau A \quad \Diamond_\tau \Box_\tau A \vdash \Box_\tau A \quad \Diamond_\tau \Box_\tau A \vdash \Box_\tau A
\]

Below we provide the multi-type translations of the single-type sequents corresponding to inequalities (11) and (13), respectively. All of them are derivable in D.AKA by logical introduction rules and display rules.

\[
\square_\tau A \vdash A \quad \Diamond_\tau \Box_\tau A \vdash \Box_\tau A \quad A \vdash \Box_\tau A \quad \Diamond_\tau \Box_\tau A \vdash \Box_\tau A \quad \Diamond_\tau \Box_\tau A \vdash \Box_\tau A \quad \Diamond_\tau \Box_\tau A \vdash \Box_\tau A
\]

Below we provide the multi-type translation of the single-type sequents corresponding to inequalities (13). All of them are derivable in D.AKA5.

\[
\Diamond_\tau A \vdash \Box_\tau A \quad \Diamond_\tau \Box_\tau A \vdash \Box_\tau A \quad \Diamond_\tau \Box_\tau A \vdash \Box_\tau A \quad \Diamond_\tau \Box_\tau A \vdash \Box_\tau A
\]

Below we provide the multi-type translations of the single-type rules corresponding to quasi-inequality (20), respectively.

\[
\Diamond_\tau A \vdash \Box_\tau A \quad \Box_\tau A \vdash \Box_\tau A \quad \Box_\tau A \vdash \Box_\tau A
\]

Below, we derive (20). Firstly, \( A \land \Box_\tau \circ_\tau B \vdash \Box_\tau \circ_\tau A \lor B \) is derivable via \( k-ia3_\tau \) by means of the following derivation \( D \):

\[
\begin{align*}
B \vdash B & \quad A \vdash A \\
B \vdash \Box_\tau A \lor B & \quad A \land \Box_\tau \circ_\tau B \vdash A \\
B \vdash \Box_\tau A \lor B & \quad \Box_\tau A \land \Box_\tau A \lor B \vdash A \\
\Box_\tau A \land \Box_\tau A \lor B \vdash \Box_\tau A \lor B & \quad \Box_\tau A \land \Box_\tau A \lor B \vdash \Box_\tau A \lor B \\
\Box_\tau A \land \Box_\tau A \lor B \vdash \Box_\tau A \lor B & \quad \Box_\tau A \land \Box_\tau A \lor B \vdash \Box_\tau A \lor B \\
\Box_\tau A \land \Box_\tau A \lor B \vdash \Box_\tau A \lor B & \quad \Box_\tau A \land \Box_\tau A \lor B \vdash \Box_\tau A \lor B
\end{align*}
\]
Assuming $\Diamond_I \bowtie I \cdot B$ and $\square_c \cdot C \cdot A \vdash \square_c \cdot C \cdot B$, we derive $A \vdash B$ via cut as follows:

$$
\begin{array}{c}
\text{Cut} \\
\text{Consequence relation arising from the perfect members of class } (a), \text{ assumptions of [5, Theorem 4.1]. All of them except C and reflect the translation. Let } A \text{ so that the semantic consequence relations arising from each type of structures preserve} \\
\text{on the rules. Condition C that } A \text{ satisfying the rules. Condition C that}\end{array}
$$

$$
\begin{array}{c}
\text{A.3 Conservativity} \\
\text{To argue that D.K is conservative w.r.t. H.K, we follow the standard proof strategy discussed in [128]. We need to show that, for all formulas A and B in the language of H.K, if } A^r \vdash B^r \text{ is a D.K-derivable sequent, then } A \vdash B \text{ is derivable in H.K. This claim can be proved using the following facts: (a) The rules of D.K are sound w.r.t. perfect members of HK (cf. Section A.1); (b) H.K is complete w.r.t. the class of perfect algebraics in k; (c) A perfect element of K is equivalently presented as a perfect member of HK so that the semantic consequence relations arising from each type of structures preserve and reflect the translation. Let } A, B \text{ be as above. If } A^r \vdash B^r \text{ is D.K-derivable, then by} \\
\text{this implies that } \equiv_K A^r \vdash B^r. \text{ By (c), this implies that } \equiv_K A \vdash B, \text{ where } \equiv_K \text{ denotes the semantic consequence relation arising from the perfect members of class K. By (b), this implies that } A \vdash B \text{ is derivable in H.K, as required.}
\end{array}
$$

$$
\begin{array}{c}
\text{A.4 Cut elimination and subformula property} \\
\text{Cut elimination and subformula property for each D.K are obtained by verifying the} \\
\text{assumptions of [5 Theorem 4.1]. All of them except C’ require checking that reduction steps can be performed for} \\
every application of cut in which both cut-formulas are principal, which either remove \\
\text{the original cut altogether or replace it by one or more cuts on formulas of strictly lower} \\
\text{complexity. In what follows, we only show C’ for some heterogeneous connectives.}
\end{array}
$$

$$
\begin{array}{c}
\text{C} \\
\text{A \vdash B} \text{ preserves the class of perfect algebras in K, as required.}
\end{array}
$$

The remaining cases are analogous.