We study the derivative expansion for the effective action in the framework of the Exact Renormalization Group for a single component scalar theory. By truncating the expansion to the first two terms, the potential $U_k$ and the kinetic coefficient $Z_k$, our analysis suggests that a set of coupled differential equations for these two functions can be established under certain smoothness conditions for the background field and that sharp and smooth cut-off give the same result. In addition we find that, differently from the case of the potential, a further expansion is needed to obtain the differential equation for $Z_k$, according to the relative weight between the kinetic and the potential terms. As a result, two different approximations to the $Z_k$ equation are obtained. Finally a numerical analysis of the coupled equations for $U_k$ and $Z_k$ is performed at the non-gaussian fixed point in $D < 4$ dimensions to determine the anomalous dimension of the field.

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to this equation, obtained by restricting the problem to analyzing the RG evolution of the constant (x-independent) part of the action, namely the Local Potential Approximation (LPA), has been shown to contain many interesting features. For instance the exactness of the LPA for $N \to \infty$ has been found in $[3]$ ($N$ is the number of scalar fields $\phi_i$), as well as the fixed point structure of the scalar theory has been recovered in the LPA $[4,8]$; furthermore, the typically non-perturbative feature of convexity of the effective potential is already contained in the LPA $[5,9,10]$.

In order to extract informations on the dynamics of the field, however, it is necessary to improve the LPA by including the effects of the field fluctuations. The natural extension of the LPA is the obtained by considering a Derivative Expansion of the action, that is by adding in the general form of the action terms with higher and higher number of derivatives of the fields. This expansion is motivated by the expectation that, for a sufficiently smooth background configuration, the action should be quasi-local. A set of two coupled equations for the local potential $U_k(\phi)$ ($k$ is the momentum scale at which the action $S_k$ is defined) and the coefficient $Z_k(\phi)$ of the lowest order derivative term, $\partial_k \phi \partial^\mu \phi$ has been deduced and studied by resorting to a smooth cut-off $[8,11]$ that allows a weighted integration of the various modes appearing in the original action. An equation for $Z_k(\phi)$ which allows to recover the lowest order perturbative anomalous dimension of the field in four dimensions has also been obtained $[12]$ from the WH equation by making use of a sharp cut-off to integrate out the ultraviolet modes.

In this paper we critically reconsider the set of coupled equations for $U_k(\phi)$ and $Z_k(\phi)$ and more generally discuss the derivative expansion in the framework of the WH equation. In particular, after a brief review in Sect.II of the WH method and the derivation of the LPA, in Sect. III motivated by the result of $[12]$ we check the reliability of the equations derived in that paper and evaluate the anomalous dimension of the $g_3 \phi^3$ theory in $D = 6$ dimensions to the order $O(g_3^2)$ and, afterwards, the anomalous dimension of the scalar theory at the Wilson-Fisher (WF) fixed point in $D < 4$ dimensions.

In Sect. IV we move to a critical revision of the derivation of this system of coupled differential equations for $U_k$ and $Z_k$. This gives us the possibility of reconsidering the longstanding problem which affects the implementation of the derivative expansion of the WH equation. As soon as a non-constant field is considered, the differential equation for $Z_k(\phi)$ (and, more generally, the equations for the higher order coefficients in the derivative expansion), is affected by the presence of some non-analytical terms that apparently spoil the very differential nature of the equation. The origin of these terms is analysed in detail and the derivation of the equation for $Z_k(\phi)$ in the momentum space is presented. We shall see that under a specific assumption of smoothness of the background field, it is possible to derive the differential equation and that the dangerous terms are actually negligible. The fate of these non-analytical terms when the sharp cut-off is replaced with a smooth one is also discussed and we shall show that for a sufficiently slow fluctuating background field, both sharp and smooth cut-off produce the same equations. In addition we show that the assumption about the relative weight between kinetic and potential terms in the action is crucial in the derivation itself of the $Z_k$ equation. Our conclusions are summarized in Sect.V.

II. LPA AND LOOP EXPANSION

Let us call $S_\Lambda[\Phi]$ the action of a scalar field $\Phi$ which contains all the Fourier components $\phi_q$ such that $0 < |q| < \Lambda$. For the sake of simplicity in this paper we shall only consider the single component scalar theory. Usually we want to describe a physical process as the scattering of two bosons with momenta $p_\mu$ such that $p << \Lambda$. As already mentioned the origin of the difficulties in perturbation theory is due the fact that we have to take into account the contributions to this process coming from all the Fourier modes between $p$ and $\Lambda$. Actually the action $S_\Lambda[\Phi]$, often called the wilsonian effective action at the scale $\Lambda$, is more appropriate to describe the physics of the system at scales $p \sim \Lambda$. At a scale $p << \Lambda$ it would seem more convenient to directly deal with an effective action $S_p[\phi]$ that describes the physical phenomena taking into account only those scales around $p$. The Renormalization Group Method provides this effective action. Let us define $S_p[\phi]$, in the Euclidean version of the theory, through the equation

$$e^{-\frac{i}{\hbar} S_p[\phi]} = \int [D\zeta] e^{-\frac{i}{\hbar} S_\Lambda[\phi+\zeta]}$$

(1)

where we have split the original field $\Phi(x)$ into a background field $\phi(x)$ containing only the modes between zero and $p$ and a fluctuation field $\zeta(x)$ containing those between $p$ and $\Lambda$.

$$\Phi(x) = \phi(x) + \zeta(x).$$

(2)

$S_p[\phi]$, called the wilsonian effective action at the scale $p$, contains, within its parameters, the effect of the interactions among the modes in the range $[p, \Lambda]$ and the modes below $p$ and therefore it is the effective action we were looking for. The difficult task here is to perform the integration over the high frequency modes $\zeta$. The above integral can
be approximated by means of the loop expansion which amounts to a series expansion in powers of $\hbar$. The one loop approximation gives the $O(\hbar)$ correction to the tree level result $S_\Lambda[\phi] = S\Lambda[\phi]$ and corresponds to a gaussian integral over the fluctuation $\zeta$ obtained expanding $S\Lambda[\phi + \zeta]$ up to $\zeta^2$ [13].

We may now pose the question in a different way and ask how does the wilsonian effective action $S_k$ evolve once we integrate the modes in the shell $[k - \delta k, k]$ with an infinitesimal $\delta k$ to get $S_k[\phi]$. In this case the gaussian approximation, i.e., the one loop result, becomes exact [3]. The problem of the evaluation of the effective action at the scale $k$ in Eq. (1) can then be turned into the problem of solving a first order differential equation for $S_k[\phi]$ with respect to the variable $k$. The action $S\Lambda[\phi]$ is the the boundary condition for this equation. This is the Wegner-Houghton method. Actually the problem of solving this differential equation for $S_k$ is as difficult as to perform the integral in Eq. (1) and we have to resort sooner or later to some approximation if we want to make any progress. As a first step let us go back to Eq. (1) which, replacing $\Lambda$ with $k$ and $p$ with $k - \delta k$, is the defining equation for $S_{k-\delta k}$ in terms of $S_k$

$$e^{-\frac{\hbar}{\pi}S_{k-\delta k}[\phi]} = \int[D\zeta]e^{-\frac{\hbar}{\pi}S_k[\phi + \zeta]}$$

(3)

Our second step is to insert in the above equation an ansatz for $S_k[\Phi]$. At this point we make the assumption that a good approximation to $S_k[\Phi]$ is provided by the derivative expansion ($D$ is the number of dimensions)

$$S_k[\Phi] = \int d^D x \left[ U_k(\Phi) + \frac{1}{2} Z_k(\Phi) \partial_\mu \Phi \partial^\mu \Phi + Y_k(\Phi)(\partial_\mu \Phi \partial^\mu \Phi)^2 + \cdots \right]$$

(4)

By inserting Eq. (1) into (3) we should be able to derive an infinite system of coupled differential equations for the coefficients functions $U_k, Z_k, Y_k, \cdots$.

The lowest order approximation in the derivative expansion is the LPA. Replacing in Eq. (1) $Z_k = 1, Y_k = 0, \cdots$, and then considering a constant background field $\phi(x) = \phi_0$ we get from Eq. (3) an evolution equation for $U_k(\phi_0)$ only

$$k \frac{\partial}{\partial k} U_k(\phi_0) = - \frac{\hbar k^D N_D}{2} \ln(k^2 + U_k''(\phi_0))$$

(5)

Here the $'$ means derivative with respect to $\phi_0$ and the result of the angular integration is $N_D = 2/((4\pi)^{D/2} \Gamma(D/2))$.

This equation has been found again and again (see for instance [4,9]) and the consensus on this equation is unanimous.

The equation above is a non perturbative evolution equation for $U_k$. Suppose that the potential $U_k(\phi_0)$ has a polynomial expansion and for the sake of simplicity let us require the $Z(2)$ symmetry $\phi_0 \to -\phi_0$

$$U_k(\phi_0) = g_0(k) + \frac{1}{2} g_2(k) \phi_0^2 + \frac{1}{4!} g_4(k) \phi_0^4 + \frac{1}{6!} g_6(k) \phi_0^6 + \frac{1}{8!} g_8(k) \phi_0^8 + \cdots$$

(6)

By taking $2n$ times ($n = 0, 1, 2, \cdots$) the derivatives of Eq. (3) with respect to $\phi_0$ at $\phi_0 = 0$ we get an infinite system of coupled equations for the coupling constants $g_{2n}(k)$:

$$k \frac{\partial}{\partial k} g_2(k) = - \frac{\hbar k^D N_D}{2} \frac{g_4(k)}{k^2 + g_2(k)}$$

$$k \frac{\partial}{\partial k} g_4(k) = - \frac{\hbar k^D N_D}{2} \left[ \frac{g_6(k)}{k^2 + g_2(k)} - 3 \frac{g_2^3(k)}{(k^2 + g_2(k))^2} \right]$$

$$k \frac{\partial}{\partial k} g_6(k) = - \frac{\hbar k^D N_D}{2} \left[ \frac{g_8(k)}{k^2 + g_2(k)} - 15 \frac{g_4(k)g_6(k)}{(k^2 + g_2(k))^2} + 30 \frac{g_2^4(k)}{(k^2 + g_2(k))^3} \right]$$

$$\cdots$$

(7)

A diagrammatic interpretation of the above equations is straightforward, provided one identifies $g_{2n}$, with the $2n$ external legs vertices and the denominators $(k^2 + g_2(k))^m$ with $m$ propagators joining the various vertices; in fact each r.h.s.in Eqs. (3) represents the sum of all the one loop diagrams with fixed number of external legs that can be arranged combining the $g_{2n}$. It is then clear that the above system is an approximation to the infinite set of Schwinger-Dyson equations for the Green’s functions at zero external momenta. Of course as it is true for the complete Schwinger-Dyson system, Eq. (3) is indeterminate. To make contact with perturbation theory, more precisely with the $\hbar$ expansion, we
can seek for solutions where each coupling constant is developed in an $\hbar$-power series $g_{2n} = g_{2n}^{(0)} + \hbar g_{2n}^{(1)} + \hbar^2 g_{2n}^{(2)} + \cdots$. As it is well known in this case the solution of the Schwinger-Dyson equations and a fortiori of our system becomes unique and it may be easily verified that the ultraviolet behavior of the coupling constants flow at $O(\hbar)$ coincides with the usual one-loop result. Let us consider for instance the one loop contribution to the perturbative $\beta$-function for the $\phi^4$ theory. It is recovered in the ultraviolet regime, $k^2 >> g_2$, by retaining in the r.h.s. the $O(\hbar^0)$ values of the couplings, that is their boundary values at $k = \Lambda$, $g_4(\Lambda) = g_4^{\beta}(\Lambda) = g_6(\Lambda) = g_6^{\beta}(\Lambda) = \cdots = 0$. However Eqs. (3) show that $g_6$ is $O(\bar{g}_4^\frac{1}{4})$ and then $g_6$ in the r.h.s. of the $\beta$-function of $g_4$ provides a $O(\bar{g}_4^\frac{1}{2})$ effect. It is then clear that higher loops perturbative contributions to the $\beta$-function are present in Eqs. (3) due to the fact that all vertices in the r.h.s. are $k$-dependent. It is also easy to realize that, in order to recover the full two-loop $g_4$ $\beta$-function, it is necessary to go beyond the LPA since the $g_6$ and $g_2$ contributions are not sufficient to get the complete $O(\bar{g}_4^\frac{1}{2})$ effect.

Going back to Eqs. (3), in order to better exploit the non-perturbative character of this system, a different kind of truncation is needed. A first step toward its non-perturbative analysis has been taken in [15]. However, due to the appearance of non-trivial saddle points in the renormalization group equation studied in that paper, the results should be taken quite cautiously. The first step beyond the LPA is discussed in the next section.

### III. DERIVATIVE EXPANSION AND ANOMALOUS DIMENSION

By turning on the scale and field dependence in $Z_k$ we allow for a non-trivial lowest order derivative term in the action (4), thus obtaining the first improvement to the LPA where it was set $Z = 1$. In this case we have to deal with two coupled equations for $U_k$ and $Z_k$ which can again be reduced to an infinite set of coupled equations if one assumes a polynomial expansion for both $U_k$ and $Z_k$.

In [12] a specific procedure has been carried out in order to determine a differential equation for $Z_k$ in the sharp cut-off limit and it has been shown that in $D = 4$ for the $O(N)$ symmetric $\phi^4$ theory, the perturbative anomalous dimension of the field to $O(\bar{g}_4^\frac{1}{4})$ can be obtained from that equation. However it was also noticed that, following a different and, in principle, correct procedure, a different equation for $Z_k$ is obtained. In practice the gaussian integration of the fluctuation $\zeta(x)$ in Eq. (3) yields a contribution to $S_{k-\delta k}$ proportional to $ln([\delta^2 S_k/\delta \Phi \delta \Phi]|_{\Phi = \bar{\Phi}})$ and both procedures require to expand the logarithm in order to derive the differential equation for $Z_k$. In one case, following the steps outlined in [11] for the one component theory, we split the logarithm (again ‘’ indicates derivation with respect to the field)

$$ln([\delta^2 S_k/\delta \Phi \delta \Phi]|_{\Phi = \bar{\Phi}}) = ln((Z_k)_{00} \partial_{\mu} \partial_{\mu} + (U_k)_{0}'' + \Delta)$$

where $(Z_k)_{00}$ and $(U_k)_{0}''$ are evaluated at the constant field configuration $\phi_0$ and $\Delta$ is the $x$ dependent part of the propagator; the expansion is performed taking $\Delta$ small with respect to the non-fluctuating part. In the other case, according to the procedure developed in [17],

$$ln([\delta^2 S_k/\delta \Phi \delta \Phi]|_{\Phi = \bar{\Phi}}) = ln(Z_k k^2 + U_k'' + Z_k (\partial_{\mu} \partial_{\mu} - k^2) + C)$$

where $Z_k k^2$ is added and subtracted and the remaining part of the propagator $C$ is proportional to derivatives of the field. In the latter case the expansion is made requiring that $C$ and the difference $Z_k (\partial_{\mu} \partial_{\mu} - k^2)$ are small. This is justified due to the field derivative term in $C$, which is small within the derivative expansion framework, and to the fact that $\partial_{\mu} \partial_{\mu} \delta(x-y)$ yields in the infinitesimal integration shell $|k-\delta k|$, a factor $k^2$ which makes the difference in Eq. (3) vanishing. It is not surprising that two expansions obtained for different choices of the expansion parameter, can be not straightforwardly comparable. As a consequence of the different nature of the expansions in Eqs. (4) and (6), we end up with two different equations for $Z_k$. We will come back to this point in the next Section.

The point of view chosen in [12] was to test the reliability of these expansions by comparing the values of the anomalous dimension of the field $\eta$ at the lowest non-vanishing perturbative order obtained from the equations with the result of the usual perturbative method. It turned out that the equation obtained from the expansion (3) provides a value of $\eta$ in agreement with perturbation theory in $D = 4$.

In order to support the result of [12] and to show that it is not an accidental coincidence, one can easily repeat the calculation of the anomalous dimension performed in [12] for the case of the cubic theory $\phi^3$ in $D = 6$. Obviously eqs. (6) and (1) are not consistent with the symmetry of the latter theory, but the complete equation for the potential (6) and the equation for $Z_k$, deduced in [12], are still valid because they do not require any assumption on the internal symmetry of $U_k$ and $Z_k$. The explicit form of the $Z_k$ equation as deduced from the expansion of Eq. (3) in $D$ dimensions is

$$A = Z_k k^2 + U_k'' + \bar{h} = 1$$
\[
\frac{k}{\partial_k} Z_k = -\frac{k^D N_D}{2} \left( \frac{Z_k''}{A} - \frac{2Z_k A'}{A^2} - \frac{Z_k^2 k^2}{DA^2} + \frac{2Z_k A^2}{3A^3} + \frac{8Z_k Z_k' A^2 k^2}{3DA^3} - \frac{2Z_k^2 A^2 k^2}{DA^4} \right)
\]

and, in order to get a perturbative estimate of the anomalous dimension, we replace, in the r.h.s. of Eq. (10), \(U_k\) and \(Z_k\) with the corresponding bare quantities, namely \(\bar{\pi}_0\beta^0/3!\) and \(1\) (we perform our calculation in the ultraviolet regime where the mass term in the potential can be neglected). The integration of Eq. (10), from \(\Lambda\) down to a generic value \(k < \Lambda\), is then straightforward yielding, in \(D = 6\), \(Z_k = \bar{\pi}_0^3/(384\pi^3)\ln(\Lambda/k)\). The anomalous dimension is immediately deduced from this \(\phi\)-independent expression of \(Z_k\)

\[
\eta = -k \frac{\partial}{\partial k} \log(Z_k) = \frac{\bar{\pi}_0^3}{384\pi^3}
\]

and it is in agreement with the perturbative diagrammatic computation \(13\). Actually the usual perturbative approach in this case is very simple since it involves a one-loop computation and, correspondingly, we need only one integration of Eq. (10) to get the answer, differently from the \(D = 4\) case where, on one side a two-loop diagram is to be evaluated and, on the other side, a two step integration of Eq. (10) is required \(12\) to determine the lowest (non-vanishing) perturbative contribution to \(\eta\). However, the above calculation is similar to the one performed for the \(\phi^4\) theory in four dimensions since in both cases the anomalous dimension is perturbatively expressed as expansion in powers of the marginal dimensionless coupling appearing in the potential. The running of these couplings with the scale \(k\) is only logarithmic thus justifying their replacement with the bare constant \(\bar{\pi}_0\) for \(D = 6\) and \(\bar{\pi}_4\) for \(D = 4\). The same argument holds for the field independent part of \(Z_k\) which is dimensionless and can be replaced with its bare value 1.

The further important step is to check whether this equation is able to reproduce the value of \(\eta\) at the non-gaussian (WF) fixed point which appears in the scalar theory below four dimensions.

In order to get informations at a fixed point it is convenient to express all dimensionful quantities in terms of the running scale \(k\) and rewrite the coupled equations (9,10) in terms of dimensionless variables which can be conveniently introduced through the relations \(t = \ln(k/\Lambda), x = k^{2 - D - \eta/2}(2/N_D)^{1/2}\phi, u(x,t) = 2k^{-D}U_k/N_D, z(x,t) = k^{\eta}Z_k\). In these relations powers of \(N_D/2\) have also been included in order to get a simpler form of the differential equations; indeed the constant \(\alpha = N_D/2\) disappears from Eqs. (9,10) after the replacement \(U_k \rightarrow \alpha U_k, \phi \rightarrow \sqrt{\alpha}\phi, Z_k \rightarrow Z_k\). Furthermore, since we are interested in carrying out a numerical analysis of the problem, instead of directly attacking Eq. (9) which contains a logarithm, we shall consider the corresponding equation for the derivative of the potential, as already performed in \(13\). Actually the presence of the logarithm causes a stiffness of our set of equations which is a source of many numerical drawbacks. Therefore, by rearranging the equations in terms of the derivative of the scaled potential \(f(x,t) = \partial u(x,t)/\partial x\) we finally get the two coupled differential equations \((a = z + f')\) and this time \(\partial f/\partial t = \partial z/\partial t = 0\) solutions of Eqs. (12,13). In the following part of this Section we shall consider only \(t\)-independent solutions of Eqs. (12,13).

As discussed in \(13\), Eq. (12) alone allows to determine the fixed point structure of the theory and specifically the appearance of the WF fixed point is shown for \(D < 4\). It is easy to check that the gaussian fixed point, corresponding to \(f = 0, z = const \neq 0, \eta = 0\), is a solution of (12,13) for generic \(D\), whereas the determination of the non-gaussian fixed point with the corresponding value of \(\eta\) requires a numerical analysis.

Eqs. (12,13) are solved by requiring the usual normalization of the kinetic term in the action and two constraints on \(f'\) and \(z'\) at \(x = 0\) which preserve the \(Z(2)\) symmetry \(\phi \rightarrow -\phi\)

\[
z(0) = 1 \quad f'(0) = 0 \quad z'(0) = 0
\]

As explained in \(13\), from eqs. (12,13), one can easily determine the asymptotic behavior of the solutions \(f(x)\) and \(z(x)\) for large \(x\), up to two (one for each solution) multiplicative constants, \(c_f, c_z\). Thus we have enough boundary conditions to integrate our equations and to determine the three unknown constants \(c_f, c_z\) and \(\eta\). In order to deal with a two point boundary problem, we have used, as also suggested in \(13\), the ‘shooting method’ embedded in a Newton-Raphson algorithm \(20\). The results shown below have been obtained by requiring that the difference between the constraints in (14) and the shooting variables is less than \(\delta = 10^{-7}\).
In $D = 3$ the value obtained for the anomalous dimension is $\eta = -0.071$, quite different from the world best determination $\eta = 0.035$, quoted in [1], or the value determined through the $\epsilon$-expansion to the order $O(\epsilon^3)$, $\eta = 0.037$ (see [13]). The result is very poor even if compared to the one obtained starting from the equation for $Z_k$ derived resorting to a smooth cut-off [3].

However it should be noticed that the situation is less bad if $D$ increases. This is illustrated by comparing the value obtained for $\eta$ at the WF fixed point in $D = 3.4$, $\eta = 9.74 \times 10^{-3}$ with the estimate of $\eta$ obtained from the $\epsilon$-expansion [13] to $O(\epsilon^3)$, $\eta = 10.70 \times 10^{-3}$, and $O(\epsilon^4)$, $\eta = 9.62 \times 10^{-3}$. In $D = 3.6$ we get $\eta = 4.30 \times 10^{-3}$ to be compared with the value from the $\epsilon$-expansion to $O(\epsilon^3)$, $\eta = 4.16 \times 10^{-3}$ and to $O(\epsilon^4)$, $\eta = 3.95 \times 10^{-3}$. In $D = 3.4$ and $D = 3.6$ the differences between the various estimates are below 10%. In Figs. 1, 2, 3 we show $f(x)$ and $z(x)$ obtained at the WF fixed point for $D = 3, 3.4, 3.6$.

All curves for $f(x)$ in Fig. 1 intersect the $x$-axis at non-vanishing values of $x$, which correspond to non-zero minima in the potentials; the curves $z(x)$ in $D = 3.4$ and $D = 3.6$ are plotted in Fig. 2; analogously to what is found in [3] for the corresponding variable in the smooth cut-off framework in $D = 3$, $z(x)$ has a maximum and then decreases for large $x$ (even the dashed curve, although less evidently, smoothly decreases after a maximum at about $x = 8$). Conversely, as shown in Fig. 3, $z(x)$ in $D = 3$ is increasing: no maximum has been found even enlarging the $x$ range to the limits allowed by the integration routine (in practice the upper limit can be pushed up to about $x = 12$). Moreover, close to $x = 0$, $z(x) < 1$ and the curve has a minimum which is not present in the curves in Fig. 2.

In order to justify this peculiar behavior, we notice that the leading terms within the brackets in the r.h.s. of Eq. (13) are the ones proportional to $z$ and not $z'$, namely $((2z\partial_x^2)/(3a^3) - (2z^2\partial_x^2)/(Da^4))$, that is, as long as $D$ is not close to 3, these two terms are dominant since $z >> z'$ but in $D = 3$ and in the ultraviolet limit, when $a \sim z \sim 1$, they practically cancel out and the behavior of Eq. (13) is sensibly modified.

In conclusion, only close to three dimensions, where the infrared effects become more and more important, Eqs. (12,13) fail to reproduce the known results about the anomalous dimension.

### IV. WH EQUATION AND NON-ANALYTICAL TERMS

We now move to another point which, in some sense, is preliminary to the analysis of the coupled equations for the various parameters entering the local action, namely the rise of some undesirable terms as soon as one goes beyond the LPA in the derivative expansion. The presence of non-analytical terms has been recognized since long time [2,21] in relation with the use of a sharp cut-off and recently reconsidered in [1,2,3]. Here, in order to get a clearer insight into this problem, we shall study the origin of these terms working out the various steps for the determination of the differential equation for $Z_k$.

Let us go back to Eq. (3) and (4). The LPA approximation corresponds to setting $Z_k = 1$, dropping $Y_k$ and all the others higher order derivative coefficients and finally restricting to a constant background $\phi = \phi_0$. In order to get the equation for $Z_k$, we have to release the condition $Z_k = 1$ and collect all terms proportional to $\partial_\mu \partial_\nu \phi$ in the r.h.s. of Eq. (3) and, to this purpose, it is necessary to retain a non-constant background $\phi(x) = \phi_0 + \varphi(x)$. Let us choose for the non-uniform component of the background $\varphi(x)$ a single mode with $|q| << k$ (the reason for this choice will become apparent later)

$$\varphi(x) = \frac{1}{V} \left\{ \varphi_q e^{iqx} + \varphi_{-q} e^{-iqx} \right\}$$  \hspace{1cm} (15)

Here $V$ is the volume factor. As before, the fluctuation $\zeta(x)$, which must be integrated out, contains Fourier modes only in the shell $|k - \delta k|, k]$

$$\zeta(x) = \frac{1}{\sqrt{V}} \sum_{|p|} \zeta_p e^{ipx}$$  \hspace{1cm} (16)

The square bracket in Eq. (16), $[p]$, is a reminder that the sum is restricted only to those values of $p$ such that $k - \delta k \leq |p| \leq k$.

Therefore the original field is split into three parts, $\Phi(x) = \phi_0 + \varphi(x) + \zeta(x)$. In order to pick up contributions up to the second derivative term $\partial_\mu \partial_\nu \Phi$, the functions $U_k(\phi_0 + \varphi(x) + \zeta(x))$ and $Z_k(\phi_0 + \varphi(x) + \zeta(x))$ must be expanded around $\phi_0$ up to $\varphi^2(x)$, and since the gaussian functional integration is exact, as mentioned before, we only need to consider terms in $\zeta^2(x)$ up to $O(\zeta^2(x))$. Due to the condition $q << k$ the linear terms in $\zeta(x)$ drop out after the spatial integration. Let us consider now the terms proportional to $\zeta^2(x)$. To illustrate the various steps made to derive the equation and the related problems we do not need to consider here all of them. It will be sufficient to consider for instance those coming from the expansion of the potential. Performing the spatial integration with the help of Eqs.
For instance in the first sum in Eq. (18), \( p \) in each sum is obtained by requiring that all quantities in square brackets are constrained into the shell \( G(15,16) \) (in the following we shall consider, with no loss of generality the four dimensional case to avoid any misunderstanding of notations, the \( n \) times derived potential is indicated here as \( U^{(n)}_k \)) we have

\[
\frac{U^{(2)}_k(\phi_0)}{2} \int d^4x \zeta^2(x) = \frac{U^{(3)}_k(\phi_0)}{2} \sum_{[p]} \zeta_p \zeta_{-p}
\]

\[
\frac{U^{(3)}_k(\phi_0)}{2} \int d^4x \varphi(x) \zeta^2(x) = \frac{U^{(3)}_k(\phi_0)}{2 \sqrt{V}} \left( \varphi_q \sum_{[p]} \zeta_p \zeta_{-p-q} + \varphi_{-q} \sum_{[p]} \zeta_p \zeta_{-p+q} \right)
\]

\[
\frac{U^{(4)}_k(\phi_0)}{4} \int d^4x \varphi^2(x) \zeta^2(x) = \frac{U^{(4)}_k(\phi_0)}{4V} \left( \varphi_q^2 \sum_{[p]} \zeta_p \zeta_{-p-2q} + 2 \varphi_q \varphi_{-q} \sum_{[p]} \zeta_p \zeta_{-p} + \varphi_{-q}^2 \sum_{[p]} \zeta_p \zeta_{-p+2q} \right)
\]

One more comment about the notation. All sums above are single summations over \( p \) and the range spanned by \( p \) in each sum is obtained by requiring that all quantities in square brackets are constrained into the shell \( k - \delta k, k \). For instance in the first sum in Eq. (18), \( [p] \) \([-p-q] \) indicates the double constraint \( (k - \delta k) \leq [p], \mid p - q \mid \leq k \), which in turn determines the values of \( p \) selected in the summation. Then, as soon as we turn on a non-uniform background \( \varphi(x) \), we end up with a deformation of the integration region. The new integration region for \( p \) is now given by the overlap of two shells with one of the two depending on \( q \). Only at \( q = 0 \) the two shells are both reduced to the original one. The reason for the appearance of the non-analytical terms has to be traced back to this boundary effect.

In order to deal with shorter expressions we limit ourselves to the case of a \( \phi \)-independent \( Z_k \). Then, with the help of Eqs. (17), (18), we can easily expand the action \( S_k[\phi_0 + \varphi + \zeta] \) around \( \phi_0 \) and perform the quadratic integration in \( \zeta \) obtaining \( h = 1 \)

\[
\exp \{ -S_k - \delta_k[\phi_0 + \varphi] \} = \exp \{ -S_k[\phi_0 + \varphi] \} \exp \left\{ \frac{1}{2} \sum_{[p]} \ln G^{-1}(p) \right\} \\
\times \left\{ 1 + \frac{U^{(3)}_k(\phi_0)}{4V} \varphi_q \varphi_{-q} \sum_{[p]} G(p)G(p + q) + \sum_{[p]} G(p)G(p - q) \right\}
\]

(20)

where we have introduced the propagator-like notation \( G^{-1}(p) = Z_k p^2 + U^{(2)}_k(\phi_0) + U^{(4)}_k(\phi_0) \varphi_q \varphi_{-q} / V \).

Since we are just interested in the \( Z_k \) evolution, we again neglect cubic and higher powers of \( \varphi \) and expand \( G(p), G(p + q), G(p - q) \) around \( q = 0 \) up to \( O(q^2) \), obtaining \( (A(p) = Z_k k^2 + U^{(2)}_k(\phi_0) ) \)

\[
S_k - \delta_k[\phi_0 + \varphi] = S_k[\phi_0 + \varphi] + \frac{1}{2} \sum_{[p]} \ln(A(p)) + \frac{U^{(4)}_k(\phi_0)}{2V} \varphi_q \varphi_{-q} \sum_{[p]} \frac{1}{A(p)} \\
- \frac{U^{(3)}_k(\phi_0)^2}{4V} \varphi_q \varphi_{-q} \left\{ \sum_{[p]} \sum_{[p+q]} \frac{1}{A^2(p)} \left[ 1 - \frac{q^2 Z_k}{A(p)} + \frac{4(p \cdot q)^2 Z_k^2}{A^2(p)} \right] \right\} \\
+ \sum_{[p]} \sum_{[p]} \frac{1}{A^2(p)} \left[ 1 - \frac{q^2 Z_k}{A(p)} + \frac{4(p \cdot q)^2 Z_k^2}{A^2(p)} \right]
\]

(21)

At this point if we just neglect the boundary problem and everywhere integrate the momentum \( p \) within the shell \([k - \delta k, k]\), from the terms in Eq. (21) not depending on \( \varphi \) we get

\[
k \frac{\partial}{\partial k} U_k(\phi_0) = -k^4 \frac{8}{16 \pi^2} \ln(Z_k k^2 + U^{(4)}_k(\phi_0))
\]

(22)
This equation replaces Eq. (9) in $D = 4$. From the terms proportional to $\varphi^2 \varphi^{-q}$, we get the second derivative with respect to $\phi_0$ of Eq. (22). This is simply because those terms come from an expansion of Eq. (22) evaluated at $\phi(x) = \phi_0 + \varphi(x)$ around $\phi_0$.

Finally if we collect the terms proportional to $q^2 \varphi^2 \varphi^{-q}$ we get the evolution equation for $Z_k$

$$k \frac{\partial}{\partial k} Z_k = -\frac{k^4}{16\pi^2} \left( \frac{Z_k U_k^{(3)}(\phi_0)^2}{A(k)^3} - \frac{Z_k^2 U_k^{(3)}(\phi_0)^2 k^2}{A(k)^4} \right)$$  \hspace{1cm} (23)

In the absence of the boundary problem and under the assumption of a field independent $Z_k$, (24) and (25) are the system of coupled differential equations for $U_k$ and $Z_k$. In the more general case of a field dependent $Z_k$ by following the same steps as before we find that the equation that replaces (23) is

$$k \frac{\partial}{\partial k} Z_k = -\frac{k^4}{16\pi^2} \left( \frac{Z''}{A} - \frac{2Z'A'}{A^2} - \frac{Z''}{4A^2} - \frac{Z_k A^2}{A^3} + \frac{Z_k k^2}{A^3} - \frac{Z_k^2 A^2 k^2}{A^4} \right)$$  \hspace{1cm} (24)

Several comments are in order. This equation has already been derived by following different steps in [24,25]. As already noticed in [12] it is substantially different from Eq.(10), namely both equations contain the same kind of terms but with different numerical coefficients. In the previous Section we have already explained that to derive Eq. (10), an expansion where the "kinetic" term $Z_k(\partial^2_x \Phi - k^2)$ is considered small w.r. to the "potential" term $Z_k k^2 + U_k$ is used. The expansion that leads to Eq. (24) is of a completely different nature. In this latter case the potential term is considered small w.r. to the kinetic term. It is then not surprising that we get different results. It is well known that no one of these expansions can be considered as definitely superior to the other. The choice of one of them is rather dictated by the physical problem under investigation.

It may be stressed at this point that the fact of having derived two evolution equations for $Z_k$, namely Eqs. (24) and (25), is not the consequence of having used two different definitions of this parameter (which is actually introduced by the derivative expansion (1) of the action whose evolution is determined by the integration in Eq. (3), with the boundary condition $Z_A = 1$). Rather, it is the consequence of having used two different approximations, which in turn are necessary since the full problem cannot be solved exactly. More specifically, as we have just seen, we need to insert explicitly a non-uniform (and slowly varying, so that higher derivatives terms in the derivative expansion can be safely neglected) background field to read the differential evolution equation for the coefficient $Z_k$. Now the fluctuation operator $\frac{\delta^2 S}{\delta \phi(x) \delta \phi(y)}$ is obviously no longer diagonal in momentum space. To determine then a differential equation for $Z_k$, we need a further expansion in which a piece of the propagator is considered small with respect to the rest of the propagator. In the derivation of Eqs. (10) and (24) above this additional expansions correspond respectively to the physical case where the kinetic energy is small with respect to the potential energy and viceversa.

The method exposed above to obtain Eq. (10) is different from the methods used respectively in [24] and [25]. Nevertheless the expansion adopted in the three cases, namely a small coupling expansion, is the same and this explains why these methods all lead to the same equation.

Finally Eq. (24) is obtained by neglecting the boundary effects due to the deformation of the integration shell. Let us now go back to Eq. (22) and work out explicitly the consequences of this boundary distortion. To illustrate this problem we just need to consider one term. Let us take for example the term

$$\frac{U_k^{(3)}(\phi_0)^2}{4V} \varphi^2 \varphi^{-q} \sum_{[\mathbf{p}]} \frac{1}{A^2(p)} \frac{Z_k q^2}{A(p)}$$  \hspace{1cm} (25)

To further simplify the point, let us write down explicitly the above indicated sum in $D = 1$ dimensions. The only additional complications that would appear when considering higher dimensions are some factors coming from the angular integrations. By trivially replacing the sums with the integrals we get as a result of the overlap of the two shells ($L$ is the volume of the system)

$$\sum_{[\mathbf{p}]} \frac{1}{A^2(p)} \frac{Z_k q^2}{A(p)} = Z_k q^2 \frac{L}{2\pi} \left\{ \int_{-\delta k + q}^{k} dp \frac{dp}{(Z_k^2 p^2 + U_k^{(2)}(\Phi))^3} + \int_{k - \delta k + q}^{\delta k - q} \frac{dp}{(Z_k^2 p^2 + U_k^{(2)}(\Phi))^3} \right\}$$  \hspace{1cm} (26)

It must be noted that the companion term in eq.(21), which differs from the one in Eq. (26) only for the integration limits, gives the same result and we do not get any cancellation of the term proportional to $q$. The above result
explains our previous statement that the appearance of non analytic terms in $q^2$ is simply due to the distortion of the integration domain induced by the non uniform background configuration.

In Eq. (24) a term proportional to $q$ appears which is not proportional to $\delta k$. This in turn seems to imply that the very possibility to establish a differential equation for $Z_k$ is in danger and actually the presence of such terms has been considered as a serious drawback that blocks the way to any application of the WH method beyond the LPA approximation. That consideration has convinced some authors that to overcome this problem a smooth cut-off rather than a sharp cut-off should be used. We shall come back to the smooth cut-off later.

The above derivation actually teaches us that there is no serious drawback as far as we correctly interpret our equations. The undesired non-analytic terms that appear on the r.h.s. of the defining equation for $S_{k-\delta k}$ and that are not contained in the original derivative expansion ansatz, always appear in combination with $\delta k$ as in the example above. This means that as long as we keep $q << \delta k$, that is as long as the non-uniformity of the background field is small compared to the width of the shell within which the modes are treated as independent, these terms can be coherently neglected and the equations above are the correct WH RG equations in this approximation. This is actually one important result of our analysis. The very possibility to implement the RG transformations as a system of differential equations for $U_k$ and $Z_k$ (i.e. to consider such a truncation to the infinite system of equations for all the coefficient functions that appear in the derivative expansion as a good approximation) is intimately related to the nature of the background field $\phi(x)$ that we consider. It has to be sufficiently smooth over the scale $\delta k$ otherwise the differential equations simply cease to be valid. The important point here is that not only we learn that it is possible and perfectly legitimate to establish differential equations that implement the infinitesimal RG transformation even beyond the LPA approximation, but we also obtain the limit of validity of these equations.

It is worth at this point to pause for a moment and review all the steps and approximations involved in the above derivation. First note that the wilsonian renormalization group transformation gives:

$$S_{k-\delta k}(\phi) = S_{k-\delta k}(\phi) + \delta k(\cdots) + O((\delta k)^2)$$

From the derivation of Eq. (23) it is clear that $\delta k$ is the momentum range above which the modes are considered as independent, that is any momentum function $f(p)$ that appears in the above equation is practically constant within this range. This in turn means that we can neglect the $O((\delta k)^2)$ terms or, in other words, that the above equation can be rewritten as a differential equation. This is what is meant when the above equation is referred as an "exact differential renormalization group equation" and this is also the meaning of the equivalent statement saying that the gaussian integration is "exact". Actually all that means that the $O((\delta k)^2)$ terms can be consistently neglected.

Next we approximate $S_k$ by a derivative expansion. We have seen that as soon as we go beyond the lowest order and allow for a non constant background field we encounter singularities that make the formal mathematical limit $\delta k \rightarrow 0$ ill defined. We have seen above that this is none of a problem as far as the scale at which singular terms appear is kept far from the "resolution scale" $\delta k$. This is nothing but the condition $q << \delta k$.

The two points above can be easily understood by considering the following hydrodynamical example. To give a differential form to the equation for, say, the density $\rho$ of a fluid we need two conditions to be satisfied. The infinitesimal volume $d^3x$ has to be "sufficiently small" so that the density as well as any other macroscopic quantity can be considered as constant within this volume. At the same time it has to be "sufficiently large" so that a macroscopic number of molecules is contained within this volume. Under the above conditions we can write the evolution equation for the density in a differential form because the $O((d^3x)^2)$ contributions can be neglected and because due to the second condition we can neglect the singularities coming from the molecular scale $a$, once our "resolution scale" $dx$ has been taken much bigger than $a : dx >> a$. The complete analogy between the approximations involved in the derivation of our equations and the hydrodynamical example should be, by now, clear.

Obviously the differential equation for $\rho$ obtained this way becomes less reliable and ultimately totally wrong when we want to describe physical phenomena whose resolution scale approaches the molecular scale $a$. The same warning applies to our case when the background field is not sufficiently smooth within $\delta k$, and this happens when $q \rightarrow \delta k$.

Let us now turn our attention to the smooth cut-off procedure. To illustrate the point we can again reconsider the previous example and implement the constraint in Eq. (23) by means of differences between theta functions in the following way

$$\sum_{[p]} \frac{1}{A^2(p) A(p)} Z_k q^2 \frac{L}{2\pi} \int_{-\infty}^{+\infty} dp \frac{\Theta_0(p^2, k^2, (k-\delta k)^2) \Theta_0((p+q)^2, k^2, (k-\delta k)^2)}{(Z_k p^2 + U_k^{(2)}(\phi_0))^3}$$

(28)

where we have defined $\Theta_0(p^2, k^2, (k-\delta k)^2) = \Theta(p^2 - (k-\delta k)^2) - \Theta(p^2 - k^2)$.

The effect of a smooth cut-off is implemented here by replacing the $\Theta_0(p^2, k^2, (k-\delta k)^2)$ function with a smoothened version $\Theta_{\epsilon}(p^2, k^2, (k-\delta k)^2)$ where $\epsilon$ has the dimension of a momentum and is a new cut-off that we introduce in the theory; in the limit $\epsilon \rightarrow 0$ it should give $\Theta_0$. It is easy to see that all the non-analytic terms disappear. In fact
by expanding the function $\Theta_\epsilon((p + q)^2, k^2, (k - \delta k)^2)$ around $p^2$ we get (from now on we omit $k$ and $k - \delta k$ in the argument of the $\Theta_\epsilon$ function)

$$\Theta_\epsilon((p + q)^2) = \Theta_\epsilon(p^2) + \frac{d\Theta_\epsilon(p^2)}{dp^2}(2pq + q^2) + \frac{1}{2} \frac{d^2\Theta_\epsilon(p^2)}{d(p^2)^2}(2pq + q^2)^2 + \cdots$$  \hspace{1cm} (29)

Being now the integration region symmetric all contributions from odd powers of $p$ vanish and consequently odd powers of $q$ are absent. As we shall see in a moment we have gained nothing from having made the non-analytic terms disappear. Let us focus on the function $\Theta_\epsilon$ and regard it as a function of $|p|$. The scale $\epsilon$ gives the size of the region over which $\Theta_\epsilon$ as a function of $|p|$ changes significantly from zero to one. This means that its first derivative with respect to $|p|$ is $O(\frac{1}{\epsilon})$ in the two regions around $k$ and $k - \delta k$, and zero everywhere else. Analogously its second derivative is $O(\frac{1}{\epsilon^2})$ in the same region and zero everywhere else, and so on.

We could expect that $\epsilon$ is introduced just as an intermediate step and that the final results are obtained after sending $\epsilon$ to zero and they are finite. The above result shows that this is not the case. As a consequence of the introduction of the smoothening cut-off $\epsilon$ we have generated $\frac{1}{\epsilon}$ divergences which have already been noticed in [22,23]. This could appear at first sight as a disaster and again we might wonder whether it is possible to establish differential equations for $U_k$ and $Z_k$ (and more generally for the coefficient functions of the derivative expansion) under these conditions. Actually we can repeat here the same kind of considerations that we have done in connection with the appearance of the non analytic terms. After performing the integration in the momentum $p$ we see that

i) the terms proportional $\Theta_\epsilon(p^2)$ give contributions $O(\delta k)$. This is because the integration region where the integrand is significantly different from zero is the shell $[k - \delta k, k]$;

ii) the terms proportional to $\frac{d\Theta_\epsilon(p^2)}{dp^2}$ give contributions proportional to $\left(\frac{1}{\epsilon}\right)q$, $\left(\frac{1}{\epsilon^2}\right)^3 q$, and so on. This is because the region over which the integrand is significantly different from zero has now a width of $\epsilon$ and we get then a factor $\epsilon$ coming from the width and a factor $\frac{1}{\epsilon}$ coming from the derivative;

iii) the terms proportional to $\frac{d^2\Theta_\epsilon(p^2)}{d(p^2)^2}$ give contributions proportional to $\left(\frac{1}{\epsilon}\right)^2 q$, $\left(\frac{1}{\epsilon^2}\right)^3 q$, $\left(\frac{1}{\epsilon^3}\right)^4 q$, $\left(\frac{1}{\epsilon^4}\right)^5 q$, $\left(\frac{1}{\epsilon^5}\right)^6 q$, and so on. This is because the region over which the integrand is significantly different from zero has a width $\epsilon$ as before but this time we get a contribution $O(\frac{1}{\epsilon^2})$ from the second derivative of $\Theta_\epsilon$.

Let us now comment on the above results. First we make the choice $\epsilon \sim q$, that is a very good choice for $\epsilon$ once we remember its role of smoothening parameter around the points $k$ and $k - \delta k$ and that we want to integrate essentially only the modes within the shell. It is now apparent that as long as we keep the scale $q << \delta k$ the dominant contributions come from the terms proportional to $\Theta_\epsilon(p^2)$ and all the contributions coming from the derivatives of $\Theta_\epsilon(p^2)$ can be coherently neglected and the final output is practically equivalent to the sharp cut-off one.

An important comment has to be made at this point. The particular choice of the smoothening function $\Theta_\epsilon$ that we have done above, more precisely the fact that we have chosen it to be an even function of $|p|$, is responsible for getting only even powers of $q$ in the above results, in other words we do not get non analytic terms in $q^2$. But this is absolutely irrelevant because what matters to establish the differential equations is the condition $q << \delta k$ irrespectively of the fact that we have odd or even powers of $q$. In fact had we chosen $\Theta_\epsilon$ to have an odd dependence on $|p|$ we would have also found odd powers of $q$. Nevertheless under the conditions $q << \delta k$ and $q << k$, we can equally well neglect all the terms apart from the first and again recover the differential equations.

What we have just seen proves that actually there is no conflict between the sharp and the smooth cut-off approach. These two procedures, once correctly implemented, give precisely the same results as it should be expected. We have also learned under which conditions the couple of differential equations for $U_k$ and $Z_k$ are valid.

Another additional comment peculiar to the smooth cut-off procedure has to be made. Due to the disappearance of the non-analytic terms there has been in recent years some preference for the smooth cut-off implementation of the differential RG transformations w.r. to the sharp cut-off. As we can easily see from the points (i) (ii) and (iii) a large value of $\epsilon$ looks even more efficient in suppressing the undesired terms and we could be led to the conclusion that we can accommodate such large values of $\epsilon$ within the smooth cut-off approach. Some results have been recently derived by making use of this apparent better flexibility of the smooth cut-off versus the sharp cut-off. Actually the condition of having a small $\epsilon$ (where small means not too big compared to $\delta k$) is necessary to be coherent with the very strategy of the RG method as exposed before. Substantially only the modes within the shell have to be integrated out. By considering larger values of $\epsilon$ we move toward the independent mode approximation. The conclusion is that those results that have been obtained within the framework of the smooth cut-off procedure with the help of cut-off functions whose typical width is of $O(k)$, where $k$ is the UV cut-off, have to be taken with a grain of salt and the improvement w.r. to the perturbative result (independent mode approximation) is not very much under control.
V. SUMMARY AND CONCLUSIONS

We have carefully analysed the derivative expansion for the effective action and shown how the exact renormalization group equations for the coefficient functions $U_k$ and $Z_k$ are obtained. The most important lesson we have learned is that the system of coupled differential equations for them can actually be established provided the background field around which the quantum fluctuations are integrated is sufficiently smooth. The width $\delta k$ in Fourier space measures the momentum range within which the modes are treated as independent, meaning that any function $f(p)$ that has to be integrated in the shell $[k - \delta k, k]$ is considered to be constant within it. The key point of the method is that this shell has to be on the one hand sufficiently small, i.e. $\delta k \ll k$, where $k$ is the UV cut-off, so that the feed-back of the higher energy modes on the lower ones is correctly taken into account. On the other hand, and this is the crucial point that we want to emphasize here, it has to be sufficiently large in such a way that within this range the background field can be considered practically flat, i.e. $q \ll \delta k$. Having made clear this point we have seen that the non-analytic terms are no more source of problems and the differential equations can be safely established under these conditions. We have also proven that the introduction of a smooth cut-off practically does not change our conclusions. In fact, under the conditions quoted above, sharp and smooth cut-off produce precisely the same results. A word of caution has to be said regarding the results that have been obtained within the smooth cut-off procedure by allowing the smoothening scale $\epsilon$ to take values $O(k)$. In this case the danger is that the width of the shell $\delta k$ gets infinitesimal w.r. to the smoothening scale which means that the modes are practically all treated as independent and then there is no control on the possible improvement on the perturbative results.

Another important point we have learned from the above analysis is that depending on the physical problem at hand different expansions can be envisaged and this lead to different equations. The typical situations illustrated are the one in which the kinetic energy term is considered small compared to the potential and the opposite one. These two cases respectively produce Eq.(10) and Eq.(24) above.

One third important result of our analysis is that ( see Sec. III) the study of the coupled system of equations for $U_k$ and $Z_k$ has shown that we are unable to reproduce the value of $\eta$ at the Wilson-Fisher fixed point in $D = 3$ dimensions. Our previous results teach us that the reason of this failure is not to be searched in an intrinsic weakness of the sharp cut-off procedure versus the smooth cut-off. Presumably the reason is that we enter a region where the equations themselves are no longer valid. An indication for that can be seen in the behaviour of $Z_k$. When it deviates from $Z_k = 1$, then less smooth configurations play an important role. We may expect that a better result could be obtained once some other coefficient functions of the derivative expansion are added and the new set of coupled differential equations derived.

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FIG. 1. The derivative of the WF fixed point potentials $f(x)$, in $D = 3$ (dot-dashed line), $D = 3.4$ (dashed line), $D = 3.6$ (solid line).
FIG. 2. $z(x)$, at the WF fixed point in $D = 3.4$ (solid line), $D = 3.6$ (dashed line).
FIG. 3. $z(x)$, at the WF fixed point in $D = 3$. 