New Symmetry Groups for Generalized Solutions of ODEs

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Abstract

It is shown for a simple ODE that it has many symmetry groups beyond its usual Lie group symmetries, when its generalized solutions are considered within the nowhere dense differential algebra of generalized functions.

0. Idea and Motivation

The standard Lie group theory applied to symmetries of solutions of PDEs, Olver [1-3], deals with classical, and specifically, $C^\infty$-smooth solutions of such equations. On the other hand, as is well known, and especially in the case of nonlinear PDEs, there is a major interest in solutions which are no longer classical, thus in particular, fail to be $C^\infty$-smooth, and instead are generalized solutions.

In Rosinger [2-13] a characterization and construction was given for the infinitely many differential algebras of generalized functions, each such algebra containing the Schwartz distributions. These algebras prove to be particularly suitable, among others, for finding generalized solutions to large classes of nonlinear PDEs, Rosinger [1,6-11,13,15], Oberguggenberger.
The construction of such algebras overcomes the celebrated and often misunderstood 1954 Schwartz impossibility of multiplying distributions. And in doing so, it shows that the multiplication of generalized functions, and in particular, of distributions, can naturally, and in fact, inevitably be done in infinitely many different ways. More precisely, in each of these infinitely many differential algebras of generalized functions, the multiplication of $C^\infty$-smooth functions is the same with their usual multiplication. However, the multiplication of functions which are less than $C^\infty$-smooth, and in particular, the multiplication of distributions or generalized functions can depend on the specific algebra in which the respective multiplication is performed.

As pointed out, Rosinger [9, pp. 1-9], at the root of the inevitability of having infinitely many such algebras one finds a rather simple conflict between multiplication, derivation and discontinuities.

It is precisely this natural and inevitable algebra dependent infinite branching of the multiplication of less than $C^\infty$-smooth functions, and specifically, distributions and generalized functions which gives the possibility to find new symmetry groups for generalized solutions of nonlinear PDEs. And as with the multiplications themselves, such new symmetry groups may depend on the differential algebras of generalized functions to which the respective generalized solutions belong.

In this paper we shall present new symmetry groups of generalized solutions for one of the simplest possible nontrivial ODEs, when these solutions are considered in the so called nowhere dense differential algebras of generalized functions, algebras introduced and used in Rosinger [1-14], see also Mallios & Rosinger [1].

1. Review of Lie Group Actions

Given a linear or nonlinear PDE

\begin{equation}
T(x, D) U(x) = 0, \quad x \in \Omega
\end{equation}

where $\Omega$ is a nonvoid open subset in $\mathbb{R}^n$, one of the major interests in Lie groups - according to Lie’s original aim - is in the study of the
symmetries of the solutions $U : \Omega \rightarrow \mathbb{R}$ of (1.1), which therefore, transform them into other solutions of (1.1).

For that purpose, one takes

$$(1.2) \quad M = \Omega \times \mathbb{R}$$

and finds the Lie groups $G$ corresponding to (1.1), as well as their actions

$$(1.3) \quad G \times M \ni (g, (x, u)) \mapsto g(x, u) \in M, \quad g \in G, \ x \in \Omega, \ u \in \mathbb{R}$$

which actions, when extended to the solutions $U \in C^\infty(\Omega, \mathbb{R})$ of (1.1), will transform them into solutions of the same equation.

Here of course in the standard theory, Bluman & Kumei, Olver [1-3], one only deals with classical, thus not generalized solutions. And in fact, all the way one assumes the $C^\infty$-smoothness of solutions.

The fact however is that, as is well known, large classes of important solutions of a whole variety of linear, and especially nonlinear PDEs of interest fail to be classical. And then the question arises :

Whether the standard Lie theory of symmetry of classical, that is, $C^\infty$-smooth solutions can be extended to generalized solutions as well?

A first comprehensive affirmative answer to that question was presented in Rosinger [1]. There, it was shown that for very large classes of linear and nonlinear PDEs in (1.1) and their generalized solutions, the following property holds :

- Whenever a Lie group transforms classical solutions of such an equation into other solutions of the same equation, that Lie group will also transform the generalized solutions of that equation into other generalized solutions of that equation.

This positive result was obtained by using only one among the infinitely many possible differential algebras of generalized functions.
Namely, all generalized solutions were elements of the *nowhere dense* algebras, the algebras first introduced and used in Rosinger [1-12,14].

Following the above positive result, a second question arises. Namely, there is a well known significant difference between the nature of classical, and on the other hand, generalized solutions, be they solutions of linear or nonlinear PDEs. And then one may naturally ask the question which, so far, has not been dealt with in the literature:

Given a linear or nonlinear PDE in (1.1), are there other groups associated with it, beyond the classical Lie groups, and which transform certain generalized solutions of that equation into other solutions of that equation?

In this paper - based on the mentioned infinite branching of the multiplication of generalized functions - we shall give an *affirmative* answer to that question.
Here we recall that, as seen in Rosinger [1-16], that infinite branching is already manifested in differential algebras of generalized functions on $\mathbb{R}$, that is, in the case of one single independent variable. Therefore, in finding new symmetry groups one may deal with ODEs, instead of PDEs.

Further, for convenience, one may also limit oneself to *projectable* Lie group actions, Olver [1-3], and their corresponding generalizations.

The full study of new symmetry groups for generalized solutions of PDEs, and the study of arbitrary, and not only projectable, such groups is to be undertaken elsewhere.

### 2. Projectable Lie Group Actions

Let us consider in more detail the Lie group actions (1.1), (1.2), namely

$$G \times M \longrightarrow M \quad (g, (x, u)) \longmapsto (\bar{x}, \bar{u}) = g(x, u) = (g_1(x, u), g_2(x, u))$$

where $x \in \Omega, \; u \in \mathbb{R}$ are the initial independent and dependent variables, respectively, while $\bar{x} \in \Omega, \; \bar{u} \in \mathbb{R}$ are, respectively, the trans-
formed independent and dependent variables. In other words

\[(2.2) \quad G \times M \ni (g, (x, u)) \mapsto \tilde{\tilde{x}} = g_1(x, u) \in \Omega\]
\[(2.2) \quad G \times M \ni (g, (x, u)) \mapsto \tilde{u} = g_2(x, u) \in \mathbb{R}\]

with \(g_1\) and \(g_2\) being \(C^\infty\)-smooth.

We note that, given \(g \in G\), in view of the Lie group axioms, it follows that the mapping

\[(2.3) \quad M \ni (x, u) \mapsto g(x, u) \in M\]

is a \(C^\infty\)-smooth diffeomorphism. Thus we have the injective group homomorphism

\[(2.4) \quad G \ni g \mapsto f_g \in \mathcal{Dif}f^\infty(M, M)\]

where \(f_g\) is defined by

\[(2.5) \quad M \ni (x, u) \mapsto f_g(x, u) = g(x, u) \in M\]

Here the noncommutative group structure on \(\mathcal{Dif}f^\infty(M, M)\) is defined by the usual composition of mappings, and thus the neutral element is \(e = id_M\), that is, the identity mapping of \(M\) onto itself.

In this way, in terms of the Euclidean domain \(M\), the group homomorphism (2.4) is but a group of smooth coordinate transforms, each of which has a smooth inverse.

Now, the Lie group actions (2.1), (2.2) are called projectable, Olver [1-3], if and only if \(g_1\) has the particular form of \(not\) depending on \(u\), namely

\[(2.6) \quad \tilde{x} = g_1(x, u) = g_1(x), \quad g \in G, \ (x, u) \in M\]

The advantage of such projectable Lie group actions (2.6) comes from the fact that, in view of (2.3), for \(g \in G\), we obtain the \(C^\infty\)-smooth diffeomorphism
Thus given any function $U : \Omega \rightarrow \mathbb{R}$, it is easy to define the group action $\tilde{U} = gU$ on $U$ for any group element $g \in G$, namely

$$(2.8) \quad \tilde{U}(x) = (gU)(x) = g_2(g_1^{-1}(x), U(g_1^{-1}(x))), \quad x \in \Omega$$

In this way projectable Lie group actions (2.1), (2.2), (2.6) can easily be extended to group actions on $C^\infty$-smooth functions

$$(2.9) \quad G \times C^\infty(\Omega, \mathbb{R}) \ni (g, U) \mapsto \tilde{U} = gU \in C^\infty(\Omega, \mathbb{R})$$

given by (2.8).

3. Nowhere Dense Differential Algebras of Generalized Functions

The differential algebras used in this paper, and called the nowhere dense algebras, are of the form

$$(3.1) \quad A_{nd}(\Omega) = (C^\infty(\Omega, \mathbb{R}))^N / \mathcal{I}_{nd}(\Omega)$$

where $\mathcal{I}_{nd}(\Omega)$ is a specially chosen, so called nowhere dense ideal in $(C^\infty(\Omega, \mathbb{R}))^N$, see (3.3) below.

Here we denoted by $(C^\infty(\Omega, \mathbb{R}))^N$ the set of all sequences $s = (s_0, s_1, s_2, \ldots)$ of functions $s_\nu \in C^\infty(\Omega, \mathbb{R})$. Clearly, just like $C^\infty(\Omega, \mathbb{R})$, so is $(C^\infty(\Omega, \mathbb{R}))^N$ an associative, commutative and unital algebra, when considered with the usual termwise operations on sequences of functions.

Let us here briefly comment on the reasons for the choice of the nowhere dense algebras. Such a comment may indeed be appropriate in view of the mentioned fact that there are infinitely many differential algebras of generalized functions to choose from.

From the start, let us note that, so far only two main types of such algebras have been used in a more consistent manner, although several other ones proved to be useful when dealing for instance with gener-
alized functions of the Dirac delta type, Rosinger [4-8,14].

Namely, the first to be introduced and systematically applied where the nowhere dense algebras, Rosinger [1-12,14], and recently, their natural extensions given by space-time foam algebras, Rosinger [13,15,16], Mallios & Rosinger [2]. The main feature of these space-time foam algebras is that they can easily handle not only singularities on closed, nowhere dense subsets, but also on arbitrarily large subsets, provided that the complementaries of such singularity sets are dense. For instance, in case $\Omega = \mathbb{R}$, the singularity set can be that of all irrational numbers, since its complementary, the rational numbers, is dense in $\mathbb{R}$. In this way, the space-time foam algebras can handle singularity sets which are dense, and which have a cardinal larger than that of the set of nonsingular points.

All these algebras are defined by conditions which only make use of the topology of their Euclidean domains of definition $\Omega \subseteq \mathbb{R}^n$. In particular, none of these algebras involve growth conditions in their definitions.

Since the mid 1980s, a second type of algebras introduced in Colombeau became popular among a number of analysts. These algebras make essential use in their definitions of polynomial type growth conditions.

It is well known, Rosinger [1-12,14], that the nowhere dense algebras $A_{nd}(\Omega)$ contain the Schwartz distributions, and in fact, contain the $C^\infty$-smooth functions as a differential subalgebra, namely

\begin{equation}
C^\infty(\Omega, \mathbb{R}) \subset \mathcal{D}'(\Omega) \subset A_{nd}(\Omega)
\end{equation}

The nowhere dense ideals $\mathcal{I}_{nd}(\Omega)$ are the set of all sequences of $C^\infty$-smooth functions $w = (w_0, w_1, w_2, \ldots) \in (C^\infty(\Omega))^\mathbb{N}$ which satisfy the condition
\[\exists \Gamma \subset \Omega, \Gamma \text{ closed, nowhere dense}:\]
\[
\forall \ x \in \Omega \setminus \Gamma:
\]
\[
(3.3) \quad \exists \ V \subset \Omega \setminus \Gamma, V \text{ neighbourhood of } x, \ \mu \in \mathbb{N}:
\]
\[
\forall \ \nu \in \mathbb{N}, \ \nu \geq \mu:
\]
\[
w_{\nu} = 0 \text{ on } V
\]
We note that with the termwise partial derivation \(D^p\), with \(p \in \mathbb{N}^n\), of sequences of functions, we have
\[
(3.4) \quad D^p \mathcal{I}_{nd}(\Omega) \subseteq \mathcal{I}_{nd}(\Omega)
\]
thus we can define the arbitrary partial derivatives for generalized functions in the nowhere dense algebras, by
\[
(3.5) \quad D^p : A_{nd}(\Omega) \rightarrow A_{nd}(\Omega)
\]
where
\[
(3.6) \quad A_{nd}(\Omega) \ni s + \mathcal{I}_{nd}(\Omega) \mapsto D^p s + \mathcal{I}_{nd}(\Omega) \in A_{nd}(\Omega)
\]
Finally, related to (3.2) we recall that the inclusion of differential algebras
\[
(3.7) \quad C^\infty(\Omega, \mathbb{R}) \subset A_{nd}(\Omega)
\]
takes place according to the mapping
\[
(3.8) \quad C^\infty(\Omega, \mathbb{R}) \ni \psi \mapsto (\psi, \psi, \psi, \ldots) + \mathcal{I}_{nd}(\Omega) \in A_{nd}(\Omega)
\]
It should be noted that the nowhere dense differential algebras of generalized functions \(A_{nd}(\Omega)\) where the first in the literature to contain the Schwartz distributions, Rosinger [4-7], and thus to overcome the 1954 Schwartz impossibility. Furthermore, these algebras proved to be useful in solving large classes of nonlinear PDEs, in abstract differential geometry, and in the first complete solution to Hilbert’s fifth
A main advantage of the nowhere dense differential algebras of generalized functions $A_{nd}(\Omega)$ comes from the fact that they are not defined by any sort of growth conditions, and instead, they only use the topology on the Euclidean domains $\Omega \subseteq \mathbb{R}^n$. That fact gives them a significant versatility in dealing with large classes of nonlinear operations and singularities. Further, it clearly distinguishes them from the later introduced Colombeau algebras which are defined by polynomial type growth conditions.

Also, the nowhere dense algebras $A_{nd}(\Omega)$ form a flabby sheaf, unlike the Colombeau, or many other algebras of generalized functions, or for that matter, the Schwartz distributions. And this flabbiness proves to have important applications, Malios & Rosinger [1].

In fact, owing to the structure of the nowhere dense ideals $\mathcal{I}_{nd}(\Omega)$, the nowhere dense algebras simply do not notice, are not sensitive to, or shall we say, jump over all singularities on closed, nowhere dense subsets of their domain of definition. And it should be noted that such a property in handling singularities is nontrivial, since closed nowhere dense subsets can have arbitrary large positive Lebesgue measure, Oxtoby.

Clearly, the case of such singularities on subsets $\Gamma$ of positive Lebesgue measure cannot be treated by the Schwartz distributions, and in particular, by Sobolev spaces. Equally, they cannot be treated by the generalized functions in the Colombeau algebras.

One of the immediate consequences of this treatment of any closed and nowhere dense singularity is the development in Mallios & Rosinger [1], which allows a far reaching extension of the de Rham cohomology, with the incorporation of singularities situated on the mentioned kind of arbitrary closed and nowhere dense subsets in Euclidean spaces.

An earlier consequence of this treatment of singularities was the global Cauchy-Kovalevskaiia theorem, Rosinger [8-10]. According to that theorem, arbitrary nonlinear systems of analytic PDEs have global generalized solutions on the whole of their domain of definition, solutions given by elements of the nowhere dense algebras. Furthermore, these
 GLOBAL generalized solutions are usual analytic functions on the whole of their domains of definition, except for closed nowhere dense subsets, which in addition, and if desired, can be chosen so as to have zero Lebesgue measure.

Such a type of very general nonlinear and global existence result has not been obtained in the Colombeau algebras of generalized functions, owing among others to the growth conditions which appear essentially in the definition of those algebras.

In Rosinger [7], a wide ranging and purely algebraic, namely, ring theoretic characterization was given for the first time for all possible differential algebras of generalized functions which, as in (3.2), contain the Schwartz distributions. Based on that characterization, the Colombeau algebras by necessity were shown to be a particular case, Rosinger [8,9], Grosser et.al. [p. 7], MR 92d:46098, Zbl. Math. 717 35001, MR 92d:46097, Bull. AMS vol.20, no.1, Jan 1989, 96-101, and MR 89g:35001.

4. An Example of ODE with Symmetry Groups Beyond Lie Symmetry Groups

Let us consider the simplest nontrivial ODEs which is of the form

\[ U'(x) = F(x), \quad x \in \Omega = \mathbb{R} \]

where \( F \in C^\infty(\Omega, \mathbb{R}) \). As is well known, Bluman & Kumei, the Lie group symmetry of (4.1) is given by the one dimensional, or one parameter action

\[ (4.2) \quad G \times M \ni (\epsilon, (x, u)) \mapsto (x, u + \epsilon) \in M \]

where \( G = (\mathbb{R}, +) \) is the additive group of real numbers, \( \epsilon \in G = \mathbb{R} \) is the one dimensional group parameter, while \( M = \Omega \times \mathbb{R} = \mathbb{R}^2 \).

This obviously is a projectable Lie group action, thus according to (2.9), it extends easily to an action on \( C^\infty \)-smooth functions, given by
\[(4.3) \quad \mathbb{R} \times C^\infty(\Omega, \mathbb{R}) \ni (\epsilon, U) \mapsto \tilde{U} = \epsilon U \in C^\infty(\Omega, \mathbb{R})\]

where

\[(4.4) \quad \tilde{U}(x) = U(x - \epsilon), \quad \epsilon \in \mathbb{R}, \ x \in \Omega\]

We shall now consider the ODE in (4.1) and its solutions - classical and generalized - within the nowhere dense algebra \(A_{nd}(\Omega)\). In this extended context, it turns out that the ODE in (4.1) has many generalized solutions. Furthermore, these generalized solution admit many symmetry groups in addition to (4.2).

Let us take any function \(\rho \in C^\infty(\Omega, \mathbb{R})\), such that

\[(4.5) \quad \rho = 1 \text{ on } (-\infty, -1] \cup [1, \infty), \quad \rho = 0 \text{ on } [-1/2, 1/2]\]

Then for any \(a, h \in \mathbb{R}\) we define a corresponding action

\[(4.6) \quad J_{a, h} : (C^\infty(\Omega, \mathbb{R}))^N \longrightarrow (C^\infty(\Omega, \mathbb{R}))^N\]

as follows. Given any sequence \(s = (s_0, s_1, s_2, \ldots ) \in (C^\infty(\Omega, \mathbb{R}))^N\), then

\[(4.7) \quad J_{a, h} s = (J_{a, h, 0} s_0, J_{a, h, 1} s_1, J_{a, h, 2} s_2, \ldots )\]

where

\[(4.8) \quad J_{a, h, \nu} s_{\nu}(x) = \begin{cases} \rho((\nu + 1)(x - a))s_{\nu}(x) & \text{if } x \leq a \\ \rho((\nu + 1)(x - a))(s_{\nu}(x) + h) & \text{if } x \geq a \end{cases}\]

for \(\nu \in \mathbb{N}, \ x \in \Omega\)

Thus we have for \(\nu \in \mathbb{N}, \ x \in \Omega\)
It follows that we have

\begin{equation}
\forall \ t, s \in (\mathcal{C}^\infty(\Omega, \mathbb{R}))^N : \\
t - s \in \mathcal{I}_{nd}(\Omega) \implies J_{a, h, t} - J_{a, h, s} \in \mathcal{I}_{nd}(\Omega)
\end{equation}

Indeed, according to (3.3) and the hypothesis, there exists a closed and nowhere dense subset \( \Gamma \subset \Omega \), such that for every \( x \in \Omega \setminus \Gamma \), there exists a neighbourhood \( V \subset \Omega \setminus \Gamma \) of \( x \) on which \( t_\nu = s_\nu \), for \( \nu \in \mathbb{N} \) large enough.

But \( \Gamma_a = \Gamma \cup \{a\} \) is still closed and nowhere dense in \( \Omega \). And it is easy to see that, for \( \Gamma_a \) and \( t - s \), the condition (3.3) holds.

It follows that we can define the action

\begin{equation}
J_{a, h} : A_{nd}(\Omega) \longrightarrow A_{nd}(\Omega)
\end{equation}

by

\begin{equation}
A_{nd}(\Omega) \ni s + \mathcal{I}_{nd}(\Omega) \mapsto J_{a, h} \ s + \mathcal{I}_{nd}(\Omega) \in A_{nd}(\Omega)
\end{equation}

The main point of interest is the following commutative group property of the action in (4.11), (4.12), namely

\begin{equation}
J_{a, h} \circ J_{a, k} = J_{a, h + k}, \quad h, \ k \in \mathbb{R}
\end{equation}

Indeed, in view of (4.9), we have for every sequence \( s = (s_0, s_1, s_2, \ldots) \in (\mathcal{C}^\infty(\Omega, \mathbb{R}))^N \) the relation

\[
(J_{a, h, \nu} (J_{a, k, \nu} s_\nu))(x) = s_\nu(x) \quad \text{if} \quad x \leq a - 1/(\nu + 1) \\
\quad s_\nu(x) + h + k \quad \text{if} \quad x \geq a + 1/(\nu + 1)
\]
Thus obviously $J_{a, h, \nu}(J_{a, k, \nu} s) - J_{a, h+k, \nu} s \in \mathcal{I}_{nd}(\Omega)$.

Finally, we can return to the ODE in (4.1). Let $U \in C^{\infty}(\Omega, \mathbb{R})$ be any classical solution of it. Then according to (3.6), (3.8), the generalized function

\[(4.14) \quad W = (U, U, U, \ldots) + \mathcal{I}_{nd}(\Omega) \in \mathcal{A}_{nd}(\Omega)\]

is a generalized solution of the ODE in (4.1), when this equation is considered in the nowhere dense algebra $\mathcal{A}_{nd}(\Omega)$.

If we take now any $a, h \in \mathbb{R}$ and apply the corresponding action $J_{a, h}$ in (4.11) to $W$, then clearly

\[(4.15) \quad J_{a, h}W \in \mathcal{A}_{nd}(\Omega) \setminus C^{\infty}(\Omega, \mathbb{R}), \quad h \in \mathbb{R}, \quad h \neq 0\]

yet it is easy to see that, with the derivation (3.6) in the nowhere dense algebras, we have

\[(4.16) \quad D(J_{a, h}W) = F\]

that is, $J_{a, h}W$ is a generalized solution of the ODE in (4.1) which is not classical when $h \neq 0$.

Finally, in view of the group property (4.13), it follows that the generalized solutions (4.15) of the ODE in (4.1) are transformed in generalized solution of the same equation.

It is important to note, however, that the transformations which turn generalized solution of the ODE in (4.1) into other generalized solution of that equation are far more numerous than those presented above. Indeed, let

\[(4.17) \quad A \subset \Omega \text{ be any discrete set}\]

and let be given any mapping $H$

\[(4.18) \quad A \ni a \mapsto h_a \in \mathbb{R}\]
Then following the above procedure, one can define an action

$$J_{A,H} : A_{nd}(\Omega) \rightarrow A_{nd}(\Omega)$$  \hspace{1cm} (4.19)

which will transform generalized solutions of the ODE in (4.1) into other generalized solutions of that equation. Furthermore, the composition (4.13) will extend in the case of actions (4.19) as follows. Let $A$ and $B$ be two discrete subsets of $\Omega$ and let $H : A \rightarrow \mathbb{R}$, $K : B \rightarrow \mathbb{R}$ any two mappings. Then

$$J_{A,H} \circ J_{B,K} = J_{C,L}$$  \hspace{1cm} (4.20)

where

$$C = A \cup B$$  \hspace{1cm} (4.21)

and $L : C \rightarrow \mathbb{R}$, such that

$$L(c) = \begin{cases} 
H(c) & \text{if } c \in A \setminus B \\
H(c) + K(c) & \text{if } c \in A \cap B \\
K(c) & \text{if } c \in B \setminus A 
\end{cases}$$  \hspace{1cm} (4.22)

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