Boundary scattering in the $SU(N)$ principal chiral model on the half-line with conjugating boundary conditions

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Abstract

We investigate the $SU(N)$ Principal Chiral Model on a half-line with a particular set of boundary conditions (BCs). In previous work these BCs have been shown to correspond to boundary scattering matrices ($K$-matrices) which are representation conjugating and whose matrix structure corresponds to one of the symmetric spaces $SU(N)/SO(N)$ or $SU(N)/Sp(N)$. Starting from the bulk particle spectrum and the $K$-matrix for a particle in the vector representation we construct $K$-matrices for particles in higher rank representations scattering off the boundary. We then perform an analysis of the physical strip pole structure and provide a complete set of boundary Coleman-Thun mechanisms for those poles which do not correspond to particles coupling to the boundary. We find that the model has no non-trivial boundary states.

1 The bulk PCM and boundary scattering in the vector representation

The principal chiral field, $g(x,t)$, takes values in a compact Lie group $G$. Its dynamics are governed by the lagrangian

$$\mathcal{L} = \frac{1}{2} \text{Tr} \left( \partial_\mu g^{-1} \partial^\mu g \right).$$

In the bulk model the spacetime coordinates $x, t$ are allowed to take any values in the reals, $-\infty < x, t < \infty$. 

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The model is classically integrable [1, 2, 3] and this classical integrability is believed to extend to the quantum level, allowing the techniques of 1+1 dimensional integrable QFT to be applied. The exact $S$-matrices describing two-particle interactions were calculated for various cases by Ogievetsky, Reshetikhin and Wiegmann [4].

In this paper we shall be concerned with the particular case $G = SU(N)$, for which the $S$-matrix describing the scattering of two vector particles can be written

$$S^{PCM}_{(1,1)}(\theta) = X_{(1,1)}(\theta) \left( S_{(1,1)}(\theta)_L \otimes S_{(1,1)}(\theta)_R \right).$$

Here $X_{(1,1)}(\theta) = (4)$ is the CDD factor, where

$$\langle y \rangle = \frac{\sinh \left( \frac{\theta}{2} + \frac{i\pi y}{4N} \right)}{\sinh \left( \frac{\theta}{2} - \frac{i\pi y}{4N} \right)},$$

and $S_{(1,1)}(\theta)_{L,R}$ are left and right copies of the minimal $S$-matrix

$$S_{(1,1)}(\theta) = \sigma(\theta) \mathbb{P} \left( P^S_2 - [4] P^A_2 \right).$$

The scalar prefactor $\sigma(\theta)$ is given by

$$\sigma(\theta) = \frac{\Gamma \left( \frac{\theta}{2\pi} + \frac{1}{2} \right) \Gamma \left( \frac{-\theta}{2\pi} \right)}{\Gamma \left( \frac{-\theta}{2\pi} + \frac{1}{2} \right) \Gamma \left( \frac{\theta}{2\pi} \right)},$$

whilst the square brackets denote the function

$$[y] = \frac{2N\theta + i\pi y}{2N\theta - i\pi y}.$$

The $S$-matrix (1.4) acts on the tensor product of two copies of the vector representation space: $\mathbb{P}$ is the transposition operator acting on the two spaces, whilst $P^S_{2,A}$ are the projectors onto the symmetric and antisymmetric subspaces of the product space. The variable $\theta$ is the rapidity difference between the two incoming multiplets; for a discussion of 1+1 dimensional factorisable scattering theory the reader is directed to [5].

1.1 The spectrum of bound states

The bound state spectrum of the bulk model was described in [4]; here we shall recall some of the features of the particles and their interactions which will prove useful later.

Firstly, there are $N-1$ particle types, which transform as the fundamental representations (that is the completely antisymmetric representations) of $SU(N)$. We label the particle type transforming as the $n$th fundamental representation by $n$. Thus a particle of type $n$ is conjugate to one of type $N-n$. 
If we set the mass scale $\mathcal{M}$ to be such that the mass of the vector particle is given by

$$\mathcal{M}_1 = \mathcal{M} \sin \left( \frac{\pi}{N} \right)$$

then the mass of the $n$th particle is given by

$$\mathcal{M}_n = \mathcal{M} \sin \left( \frac{n\pi}{N} \right).$$

This mass formula follows from the fact that the particle interactions are such that particles of type $n$ and $m$ (for $n+m \leq N$) scattering at rapidity difference $\theta = \frac{(n+m)\pi}{N}$ form a particle of type $n+m$ as a bound state.

1.2 The half-line model: scattering of the vector particle

We now turn our attention to the PCM on the half-line: the space coordinate is restricted to the range $-\infty < x \leq 0$. In a previous paper [6] various boundary conditions were found which are believed to preserve the quantum integrability of the model with boundary.

The $S$-matrix description for the model must now be extended to include scattering off the boundary. It was established in [6] that certain classical boundary conditions classified by the symmetric spaces

$$\frac{SU(N)}{SO(N)} \quad \text{and} \quad \frac{SU(N)}{Sp(N)}$$

correspond to representation conjugating $K$-matrices. These $K$-matrices, describing the scattering of a vector particle off the boundary, are given by

$$K_{1}^{PCM}(\theta) = Y_1(\theta) \left( K_1(\theta)_L \otimes K_1(\theta)_R \right).$$

The CDD factor is given by

$$Y_1(\theta) = (\gamma N + 2)(\delta N + 4),$$

where the four choices $\gamma, \delta = 1, 3$ each provide a suitable PCM $K$-matrix for the vector particle: in each of the four cases the $K$-matrix has no poles in the physical strip ($\text{Im}(\theta) \in [0, \frac{\pi}{2}]$). We shall find in section 3 that one of the four choices is preferred by the boundary interactions of the higher rank particles. $K_1(\theta)_L, R$ are left and right copies of the minimal $K$-matrices

$$K_1(\theta) = \rho(\theta)E$$

where the matrix part, $E$, is unitary; it is symmetric in the case of $SU(N)/SO(N)$ and antisymmetric in the case of $SU(N)/Sp(N)$. The scalar prefactor is given by

$$\rho(\theta) = \frac{\Gamma \left( \frac{\theta}{2\pi + \frac{1}{4}} \right) \Gamma \left( \frac{-\frac{\theta}{2\pi} + \frac{\alpha}{4} + \frac{1}{2N}}{\frac{2}{4} + \frac{1}{2N}} \right)}{\Gamma \left( \frac{-\theta}{2\pi} + \frac{\alpha}{4} + \frac{1}{2N} \right) \Gamma \left( \frac{\theta}{2\pi + \frac{1}{4}} \right)}.$$
where $\alpha = 1$ for $SU(N)/SO(N)$ and $\alpha = 3$ for $SU(N)/Sp(N)$.

Before going on to consider the scattering of higher rank bulk particles off the ground state, we note that because neither of these PCM $K$-matrices has any poles on the physical strip the bulk vector particle cannot bind to the boundary with either choice of BC.

## 2 Higher rank particle boundary scattering

We calculate the $K$-matrices for higher rank particles by fusion. We shall illustrate the procedure, in some detail, with the $n = 2$ particle before proceeding to the general case. Diagrammatically this calculation is

\[
\begin{align*}
\theta & \quad = \\
\theta + \frac{i\pi}{N} & \quad \quad \quad \theta - \frac{i\pi}{N}
\end{align*}
\]

Thus, we have the following expression for the second rank particle minimal $K$-matrix

\[
K_2(\theta) = S_{(1,\bar{1})}(2\theta) (I \otimes K_1(\theta + \frac{i\pi}{N})) S_{(1,1)}(\theta + i\pi N) (I \otimes K_1(\theta - \frac{i\pi}{N})).
\] (2.1)

We note that since the reflection matrix is representation conjugating it is $S_{(1,\bar{1})}(2\theta)$ that we require. This is obtained as

\[
S_{(1,\bar{1})}(\theta) = \omega(i\pi - \theta) \left( - \frac{N(i\pi - \theta)}{2\pi} \right) \rho(\theta + \frac{i\pi}{N}) \rho(\theta - \frac{i\pi}{N} - i\pi N) \left( - \frac{N(i\pi - \theta)}{2\pi} \right).
\] (2.2)

where we have introduced a diagrammatic notation to make the subsequent calculations more transparent: we represent the $N \times N$ identity matrix by a line, $\ldash$. We denote the matrix part, $E$, of the vector particle $K$-matrix by $\rightarrow$. Matrix multiplication is given by concatenation of the lines and the scattering order goes from right to left. The lines are implicitly directed, with conjugation of the particle reversing the line direction, but we can ignore this here.

Substituting in for all the minimal $S$- and $K$-matrices we obtain

\[
K_2(\theta) = \omega(\frac{2i\pi}{N}) \omega(i\pi - 2\theta) \rho(\theta + \frac{i\pi}{N}) \rho(\theta - \frac{i\pi}{N}) \left( - \frac{N(i\pi - \theta)}{2\pi} \right) \left( - \frac{N(i\pi - \theta)}{2\pi} \right) \left( - \frac{N(i\pi - \theta)}{2\pi} \right).
\] (2.3)
At this point the cases $SU(N)/SO(N)$ and $SU(N)/Sp(N)$ diverge: we need to consider the matrix parts separately. Firstly we consider $SU(N)/SO(N)$, for which $(\sigma)^t = \sigma$. We have
\[
\begin{pmatrix} -X \end{pmatrix} \begin{pmatrix} -N(i\pi-2\theta) \end{pmatrix} = \frac{N(i\pi-2\theta)}{4\pi} \begin{pmatrix} -X \end{pmatrix} = \frac{N(i\pi-2\theta)}{i\pi} P_2,
\]
where we have set
\[
P_2 = \frac{1}{2} \begin{pmatrix} -X \end{pmatrix}.
\]
This satisfies $P_2(P_2)^\dagger = P_2^A$ (we recall that $E$ is unitary).

Moving on to the case $SU(N)/Sp(N)$, for which $(\sigma)^t = -(\sigma)$, we have
\[
\begin{pmatrix} -X \end{pmatrix} \begin{pmatrix} -N(i\pi-2\theta) \end{pmatrix} = 2 \begin{pmatrix} \end{pmatrix} + \frac{N(i\pi-2\theta)}{2\pi} \begin{pmatrix} -X \end{pmatrix} = \frac{N(i\pi-2\theta)}{i\pi}(P_2^{(2)} - [N] P_2^{(0)}),
\]
where we have set
\[
P_2^{(2)} = \frac{1}{2} \begin{pmatrix} -X \end{pmatrix} + \frac{1}{N} \begin{pmatrix} \end{pmatrix}, \quad P_2^{(0)} = \frac{1}{N} \begin{pmatrix} \end{pmatrix}.
\]
Setting $(\sigma)^t = \sigma$, these satisfy (recall $=\sigma$)
\[
P_2^{(2)}(P_2^{(2)})^\dagger = \frac{1}{2} \begin{pmatrix} -X \end{pmatrix} - \frac{1}{N} \begin{pmatrix} \end{pmatrix}, \quad P_2^{(0)}(P_2^{(0)})^\dagger = \frac{1}{N} \begin{pmatrix} \end{pmatrix},
\]
which are orthogonal projectors.

In both cases, $SU(N)/SO(N)$ and $SU(N)/Sp(N)$, we obtain the scalar prefactor
\[
\rho_2(\theta) = (N - \frac{2N\theta}{i\pi})\omega(\frac{2i\pi}{N})\omega(i\pi-2\theta)\rho(\theta+i\pi N)\rho(\theta-i\pi N).
\]
Thus we have calculated the second rank particle minimal $K$-matrices
\[
K_2(\theta) = \rho_2(\theta)P_2 \quad SU(N)/SO(N),
\]
\[
K_2(\theta) = \rho_2(\theta)(P_2^{(2)} - [N] P_2^{(0)}) \quad SU(N)/Sp(N).
\]
The CDD factor is given by
\[
Y_2(\theta) = \text{Res}_{v=\frac{2\pi}{N}} X_{(1,1)}(v) Y_1(\theta+i\pi N)X_{(1,1)}(i\pi-2\theta)Y_1(\theta-i\pi N),
\]
and so we have the PCM $K$-matrices
\[
K_2^{PCM}(\theta) = Y_2(\theta) \left( K_2(\theta)_L \otimes K_2(\theta)_R \right).
\]
2.1 The \( n \)th rank particle \( K \)-matrices

Having considered the calculation of the second rank particle \( K \)-matrices, we now proceed to the general \( n \)th rank case. We consider the diagram

\[
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\includegraphics[width=0.4\textwidth]{diagram.png}
\end{array}

\end{array}
\]

Algebraically we have

\[
K_n(\theta) = S_{(1,n-1)}(\frac{n\pi}{N}) \left( I^{\otimes (n-1)} \otimes K_{1}(\theta + \frac{(n-1)i\pi}{N}) \right)
\]

\[
\tilde{S}_{(n-1,1)}(2\theta + \frac{(n-2)i\pi}{N}) \left( I \otimes \tilde{K}_{n-1}(\theta - \frac{i\pi}{N}) \right),
\]

where the relations

\[
S_{(n-1,1)}(\phi) = \Omega(n-1) \tilde{S}_{(n-1,1)}(\phi), \quad K_{n-1}(\phi) = \Omega(n-1) \tilde{K}_{n-1}(\phi)
\]

and

\[
\Omega(n) = \prod_{l=1}^{n-1} \prod_{k=1}^{l} (k+1) \omega(\frac{2k\pi}{N})
\]

define the quantities \( \tilde{S} \) and \( \tilde{K} \). (Note that we keep track of these \( \theta \)-independent factors, \( \Omega \), for the sake of completeness rather than necessity.)

In order to calculate \( K_n(\theta) \) from the expression above we again turn to our diagrammatic notation. We can then express the \( S \)-matrix \( \tilde{S}_{(1,n-1)}(\theta) \) as

\[
\tilde{S}_{(1,n-1)}(\theta) = \psi_{n-1}(\theta)^{\bigotimes n-1} \left( \begin{array}{c}
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\end{array}\right),
\]

where

\[
\psi_{n-1}(\theta) = \left( \prod_{k=1}^{n-1} \omega(\theta - \frac{n\pi}{N} + 2k\pi) \right) \left( \prod_{k=2}^{n-1} (\frac{N\theta}{2\pi} - \frac{n}{2} + k) \right)
\]

and \( P_{n-1}^A \) is the projector onto the completely antisymmetric subspace of the \( n-1 \)-fold tensor product of vector representations (the \( n-1 \)th fundamental representation of \( SU(N) \)). Setting \( \theta = \frac{n\pi}{N} \) we obtain

\[
\tilde{S}_{(1,n-1)}(\frac{n\pi}{N}) = n\psi_{n-1}(\frac{n\pi}{N}) P_{n}^A \quad \implies \quad S_{(1,n-1)}(\frac{n\pi}{N}) = \Omega(n) P_{n}^A,
\]

(2.17)
which follows from \( n\Omega(n-1)\psi_{n-1}(\frac{ni\pi}{N}) = \Omega(n) \). We note that the above expression for \( \tilde{S}_{(1,n-1)}(\theta) \) is obtained by fusing bulk vector particles together. We will not prove the result here (it was established in [4]), but remark that we have provided enough details for the interested reader to assemble an inductive proof fairly straightforwardly.

In order to obtain \( \tilde{S}_{(n-1,1)}(\theta) \) we perform the usual crossing operation, obtaining

\[
\tilde{S}_{(n-1,1)}(\theta) = \psi_{n-1}(i\pi-\theta)
\]

Substituting into our expression for \( K_n(\theta) \) we have

\[
K_n(\theta) = \Omega(n)\rho(\theta + \frac{(n-1)i\pi}{N})\bar{\rho}_{n-1}(\theta - \frac{i\pi}{N})\psi_{n-1}(i\pi-2\theta + \frac{(2-n)i\pi}{N})
\]

where we have split the \( n-1 \) rank particle \( K \)-matrix into a scalar prefactor and a matrix part,

\[
\tilde{K}_{n-1}(\theta) = \bar{\rho}_{n-1}(\theta)X_{n-1}(\theta).
\]

We again have to consider the two cases separately, as we did for \( n = 2 \); firstly we calculate the minimal \( K \)-matrix for \( SU(N)/SO(N) \).

### 2.2 Calculating \( K_n(\theta) \) for \( SU(N)/SO(N) \)

In the case of \( SU(N)/SO(N) \) we shall find that the matrix part of the minimal \( K \)-matrix, \( X_n(\theta) \), is constant. We recall that \( X_1(\theta) = E (= P_1) \) and \( X_2(\theta) = P_2 \) and we take as an induction hypothesis \( X_{n-1}(\theta) = P_{n-1} \) (with \( P_{n-1} \) the obvious generalisation of \( P_1 \) and \( P_2 \); thus \( P_{n-1}(P_{n-1})^\dagger = P_{n-1}^A \)). Substituting into our expression for \( K_n(\theta) \) we have

\[
K_n(\theta) = \Omega(n)\rho(\theta + \frac{(n-1)i\pi}{N})\bar{\rho}_{n-1}(\theta - \frac{i\pi}{N})\psi_{n-1}(i\pi-2\theta + \frac{(2-n)i\pi}{N})
\]
The $P_{n-1}^A$ factor contracts with the $P_n^A$ factor, leaving us to consider a sum of terms whose matrix parts are of the form

\[\begin{pmatrix}
& & \\
& & \\
& & \\
& & \\
& & \\
\end{pmatrix}
\]

Since $\sim$ is symmetric it is clear that all these will give zero except the contribution from

\[\begin{pmatrix}
& & \\
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\end{pmatrix}
= (-1)^{n-1} \rho_n.
\]

Thus, we have

\[K_n(\theta) = \Omega(n) \rho(\theta + \frac{(n-1)i\pi}{N}) \bar{\tilde{\rho}}_{n-1}(\theta - \frac{i\pi}{N}) \psi_{n-1}(i\pi - 2\theta + \frac{(2-n)i\pi}{N})(\frac{N}{2} - \frac{N\theta}{i\pi} - n + 2)P_n. \quad (2.22)\]

From this we read off the scalar prefactor

\[\rho_n(\theta) = \Omega(n) \rho(\theta + \frac{(n-1)i\pi}{N}) \bar{\tilde{\rho}}_{n-1}(\theta - \frac{i\pi}{N}) \psi_{n-1}(i\pi - 2\theta + \frac{(2-n)i\pi}{N})(\frac{N}{2} - \frac{N\theta}{i\pi} - n + 2), \quad (2.23)\]

whilst the matrix part is $X_n(\theta) = P_n$. We have proven the conjectured form of $K_n(\theta)$ by induction and established a recurrence relation for the scalar prefactor, $\rho_n(\theta)$. We turn our attention now to the analogous calculation for $SU(N)/Sp(N)$.

2.3 Calculating $K_n(\theta)$ for $SU(N)/Sp(N)$

For the $SU(N)/Sp(N)$ case we appeal to the boundary tensor product graph method [7], which gives the matrix part of the minimal $K$-matrix as

\[X_n(\theta) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \prod_{k=1}^{r} [N - 2n + 4k] P_n^{(n-2r)}, \quad (2.24)\]

where $P_n^{(n-2r)}(P_n^{(n-2s)})^\dagger$ are the orthogonal projectors corresponding to the irreducible representations of the embedded $Sp(N)$. We note the relation $P_n^{(n-2r)}(P_n^{(n-2s)})^\dagger = 0$ for $r \neq s$, which allows us to write

\[K_n(\theta)(P_n^{(n)})^\dagger = \rho_n(\theta)P_n^{(n)}(P_n^{(n)})^\dagger. \quad (2.25)\]
If we are to simplify our expression for $K_n(\theta)$ (2.19) we will need to know how the operators $P_{n-1}^{(n-2r)},$ in $K_{n-1}(\theta - \frac{i\pi}{N}),$ relate to the operators $P_n^{(n-2s)}$. The relation which allows us to make progress is $(E \otimes P_{n-1}^{(n-1)})(P_n^{(n)})^\dagger = P_n^{(n)}(P_n^{(n)})^\dagger,$ then we have

$$K_n(\theta)(P_n^{(n)})^\dagger = \Omega(n)\rho(\theta + \frac{(n-1)i\pi}{N})\tilde{\rho}_{n-1}(\theta - \frac{i\pi}{N})\psi_{n-1}(i\pi - 2\theta + \frac{(2-n)i\pi}{N})$$

The $P_{n-1}^A$ factor contracts with the $P_n^A$ factor, leaving us to consider a sum of terms whose matrix parts are of the form

The orthogonal property, $(P_n^{(n-2r)})^\dagger P_n^{(n)} = 0$ for $r \neq 0,$ guarantees that the only such term not equal to zero is

$$K_n(\theta)(P_n^{(n)})^\dagger = \Omega(n)\rho(\theta + \frac{(n-1)i\pi}{N})\tilde{\rho}_{n-1}(\theta - \frac{i\pi}{N})\psi_{n-1}(i\pi - 2\theta + \frac{(2-n)i\pi}{N})(\frac{N}{2} - \frac{N\theta}{i\pi} - n)P_n^{(n)}(P_n^{(n)})^\dagger.$$

Comparing this with the expression (2.25) we obtain

$$\rho_n(\theta) = \Omega(n)\rho(\theta + \frac{(n-1)i\pi}{N})\tilde{\rho}_{n-1}(\theta - \frac{i\pi}{N})\psi_{n-1}(i\pi - 2\theta + \frac{(2-n)i\pi}{N})(\frac{N}{2} - \frac{N\theta}{i\pi} - n),$$

which is exactly the recurrence we had for the $SU(N)/SO(N)$ case.

In order to calculate fully the minimal $K$-matrices for the two cases, $SU(N)/SO(N)$ and $SU(N)/Sp(N),$ we need to use the above recurrence relation to calculate an expression for $\rho_n(\theta)$ in terms of basic quantities. Starting from the recurrence

$$\tilde{\rho}_n(\theta) = \rho(\theta + \frac{(n-1)i\pi}{N})\tilde{\rho}_{n-1}(\theta - \frac{i\pi}{N})\psi_{n-1}(i\pi - 2\theta + \frac{(2-n)i\pi}{N})(\frac{N}{2} - \frac{N\theta}{i\pi} - n),$$
we can prove by induction that the following expression for $\rho_n(\theta)$ holds:

$$\rho_n(\theta) = \Omega(n) \prod_{k=1}^{n} \rho(\theta + \frac{(2k-n-1)i\pi}{N}) \prod_{k=2-n}^{n-2} \eta(i\pi - 2k\pi + \frac{2k\pi}{N})^{\frac{n-|k|}{2}}$$

(2.30)

where

$$\eta(\theta) = \frac{N\theta}{2\pi} \omega(\theta).$$

(2.31)

We note that the procedure we have used to calculate the minimal $K$-matrices is valid only for $n \leq \frac{N}{2}$. We can obtain the boundary scattering description for the particles of rank $> \frac{N}{2}$ by recalling that they are the conjugates of particles of rank $< \frac{N}{2}$ and fusing together conjugate vector particles. The minimal $K$-matrix for the conjugate vector particle is given by [6]

$$K_1(\theta) = \rho(\theta) E^1.$$  

(2.32)

Thus, more generally, we can exchange the $K$-matrix of a particle and that of its conjugate particle by hermitian conjugation of the $P_n$ or $P_{n-2r}$ operators.

### 2.4 The PCM $K$-matrices

Now that we have calculated the minimal $K$-matrices for the cases $SU(N)/SO(N)$ and $SU(N)/Sp(N)$ we need to determine the CDD factors (which will have the same functional form, in terms of the basic vector particle CDD quantities, for both cases). We have the following recurrence for the CDD factors

$$Y_n(\theta) = \underbrace{Res_{\phi=\frac{ni\pi}{N}} \left( X_{(1,n-1)}(\phi) \right)}_{Y_{n-1}(\theta)} Y_{1}(\theta + \frac{(n-1)i\pi}{N}) \bar{X}_{n-1,1}(2\theta + \frac{(n-2)i\pi}{N}) \bar{Y}_{n-1}(\theta - \frac{i\pi}{N}),$$

(2.33)

where

$$\bar{Y}_{n-1}(\theta) = \frac{Y_{n-1}(\theta)}{Res_{\phi=\frac{(n-1)i\pi}{N}} \left( X_{(1,n-2)}(\phi) \right)}, \quad \bar{X}_{n-1,1}(\theta) = \frac{X_{n-1,1}(\theta)}{Res_{\phi=\frac{(n-1)i\pi}{N}} \left( X_{(1,n-2)}(\phi) \right)}.$$ 

(2.34)

From this we can prove inductively the result

$$Y_n(\theta) = \underbrace{Res_{\phi=\frac{ni\pi}{N}} \left( X_{(1,n-1)}(\phi) \right)}_{Y_{n-1}(\theta)} \prod_{k=1}^{n} Y_{1}(\theta + \frac{(2k-n-1)i\pi}{N}) \prod_{k=2-n}^{n-2} \left( X_{(1,1)}(2\theta + \frac{2k\pi}{N}) \right)^{\frac{n-|k|}{2}}.$$ 

(2.35)

Having calculated the complete PCM $K$-matrices we can move on to analyse their pole structures.
The physical strip pole structure of the PCM $K$-matrices and boundary Coleman-Thun mechanisms

We recall the structure of the PCM $K$-matrices

$$K_{n}^{PCM}(\theta) = Y_{n}(\theta) \left( K_{n}(\theta)_{L} \otimes K_{n}(\theta)_{R} \right). \quad (3.1)$$

Determining the physical strip poles of these $K$-matrices involves collecting together all the pole and zero locations of the constituent parts, remembering that any poles and zeroes in the minimal $K$-matrices, $K_{n}(\theta)$, will appear with their order doubled as there are two copies present in the above expression. We present just the results here as the method is only a matter of meticulous accounting.

In section 1.2 we gave four possible CDD factors for the vector particle $K$-matrix

$$Y_{1}(\theta) = (\gamma N + 2)(\delta N + 4), \quad (3.2)$$

where $\gamma, \delta = 1, 3$ were the four possibilities. The vector particle scattering indicates no preference between these four choices. The higher rank particle scattering, however, singles out a preferred choice in each of the cases $SU(N)/SO(N)$ and $SU(N)/Sp(N)$.

For $SU(N)/SO(N)$ we find that $\gamma = 3, \delta = 1$ is the preferred choice. The other three possibilities lead to physical strip poles which are either inconsistent (for example a pole at $\theta = \frac{i\pi}{2}$ when $K_{n}(\frac{i\pi}{2})$ does not project onto a scalar representation subspace of $Sp(N)$) or undesirable. With the preferred choice $K_{n}^{PCM}(\theta)$ has no physical strip poles, and only has the following simple zeroes on the physical strip

$$\theta = \frac{i\pi}{2}, \theta = \frac{i\pi}{2} + \frac{(1-n)i\pi}{N}, \theta = \frac{i\pi}{2} - \frac{ni\pi}{N} \quad \text{(not present when } n = \frac{N}{2}). \quad (3.3)$$

Since there are no poles on the physical strip in $K_{n}^{PCM}(\theta)$ for any $n \leq \frac{N}{2}$, we conclude that no boundary bound states can be formed. We have a consistent set of $K$-matrices describing boundary interactions in the case of boundary conditions corresponding to the symmetric space $SU(N)/SO(N)$ with no boundary bound states.

For $SU(N)/Sp(N)$ we find that $\gamma = 3, \delta = 1$ is again allowed and produces the, rather trivial, structure described above. More interestingly, $\gamma = \delta = 1$ is also allowed (again the other two possibilities lead to inconsistent/undesirable physical strip poles) and this we take as the preferred choice. With this choice $K_{n}^{PCM}(\theta)$ has the following poles and zeroes which lie on the physical strip

$$\theta = \frac{i\pi}{2} \quad \text{(n odd only)}, \quad \theta = \frac{i\pi}{2} + \frac{(1-n)i\pi}{N} \quad \text{simple zeroes},$$

$$\theta = \frac{i\pi}{2} \quad \text{(n even only)} \quad \text{simple pole}, \quad (3.4)$$

$$\theta = \frac{i\pi}{2} + \frac{(2-n)i\pi}{N}, \theta = \frac{i\pi}{2} + \frac{(4-n)i\pi}{N}, \ldots, \theta = \begin{cases} \frac{i\pi}{2} - \frac{2i\pi}{N} & n \text{ even} \\ \frac{i\pi}{2} - \frac{i\pi}{N} & n \text{ odd} \end{cases} \quad \text{double poles.}$$
We note that at rapidity $\theta = \frac{i\pi}{2} + \frac{(2r-n)i\pi}{N}$ for $r = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ the minimal $K$-matrix projects onto the subspace associated with 

$$P_n^{(n-2r)} \oplus P_n^{(n-2r-2)} \oplus \ldots \oplus P_n^{(b)}$$

where $b = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$. \hfill (3.5)

### 3.1 Boundary Coleman-Thun mechanisms for $SU(N)/Sp(N)$

Having established the physical strip pole structure we can examine it to see what bulk-boundary couplings are required for a consistent description of the boundary interactions.

Firstly we note that $K_n^{PCM}(\theta)$ has a simple pole at $\theta = \frac{i\pi}{2}$ when $n$ is even. Since the boundary is in the ground state, appealing to lemma 2 of a paper by Mattsson and Dorey [8], there can be no explanation for such simple poles other than a coupling of the even rank bulk particles to the boundary at this rapidity. The minimal $K$-matrix contains only a contribution from the projector $P_n^{(0)}$ at $\theta = \frac{i\pi}{2}$ and so projects onto the embedded $Sp(N)$ scalar representation subspace. This is consistent with such a coupling of the bulk particles to the boundary and we conclude that the even rank particles couple to the boundary at $\theta = \frac{i\pi}{2}$. Diagrammatically we have

![Diagram](image-url)

We are now in a position to explain all the other physical strip poles, namely the double poles at $\theta = \frac{i\pi}{2} + \frac{(2r-n)i\pi}{N}$ for $r = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$, by boundary Coleman-Thun mechanisms. We consider the diagram

![Diagram](image-url)

which provides a valid bCTm for the double pole in $K_n^{PCM}(\theta)$ at $\theta = \frac{i\pi}{2} + \frac{(2r-n)i\pi}{N}$ as the diagram is second order. We recall that the order of a diagram is given by

$$\text{order} = \#\text{internal edges} - 2\#\text{closed loops}$$ \hfill (3.6)

(there is no contribution from $K_{n-2r}^{PCM}(\frac{i\pi}{2} - \frac{m\pi}{N})$ to the order as this is finite, non-zero).
The diagram we have drawn is valid for all $2 \leq n < \frac{N}{2}$. In the case $n = \frac{N}{2}$ it becomes

\[
\begin{array}{c}
\frac{N}{2} \\
2r \\
\frac{N}{2} - 2r
\end{array}
\]

which is still second order. Since we are dealing with representation conjugating $K$-matrices a factor $K^{PCM}_{\frac{N}{2}-2r}(0)$ must be present in the above or the diagram would not be consistent. We allow this since we can make the $\frac{N}{2} - 2r$ bulk particle arbitrarily close to the boundary.

Thus we have constructed a consistent picture of boundary interactions in which all physical strip poles are explained by the coupling of all even rank bulk states to the boundary at rapidity $\theta = \frac{i\pi}{2}$.

\section*{4 Conclusions}

We have investigated the scattering of particles off the boundary in the $SU(N)$ Principal Chiral Model on a half-line with conjugating boundary conditions. We have constructed boundary scattering matrices for all the bulk particles in the two cases where the BCs correspond to the symmetric spaces $SU(N)/SO(N)$ and $SU(N)/Sp(N)$. Having examined the physical strip pole structures of these $K$-matrices, we have concluded that none of the bulk particles couple to the boundary in the case where the BCs correspond to $SU(N)/SO(N)$. Taking BCs corresponding to $SU(N)/Sp(N)$ we have found that all even rank bulk particles couple to the boundary at rapidity $\theta = \frac{i\pi}{2}$, whilst all other physical strip poles in the $K$-matrices can be explained by boundary Coleman-Thun mechanisms involving just these bulk-boundary couplings.

Thus we have revealed that there are no boundary bound states in either case of conjugating boundary conditions. This is in contrast to the model with non-conjugating boundary conditions, which appears to have a very rich boundary spectrum and is the subject of current study by the author.

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