DYNAMICS OF FINITE-MULTIVALUED TRANSFORMATIONS

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ABSTRACT
We consider a transformation of a normalized measure space such that the image of any point is a finite set. We call such transformation $m$-transformation. In this case the orbit of any point looks like a tree. In the study of $m$-transformations we are interested in the properties of the trees.

An $m$-transformation generates a stochastic kernel and a new measure. Using these objects, we introduce analogies of some main concepts of ergodic theory: ergodicity, Koopman and Frobenius-Perron operators etc. We prove ergodic theorems and consider examples. We also indicate possible applications to fractal geometry and give a generalization of our construction. Some results which have analogies in the classical ergodic theory are proved using standard methods (see [1], [6]). Other results, for instance Theorem 2 and Example 5, have no analogies.

1 Main definitions and examples

Throughout the paper $(X, \mathcal{B}, \mu)$ denotes a normalized measure space. Let $m$ be a positive integer.

Definition 1 We call a multivalued transformation $S : X \to X$ an $m$-transformation if $1 \leq |S(x)| \leq m$ for any $x \in X$, where $|A|$ is just a number of elements in $A$.

Let

$$S_{k,l}^{-1}(B) \equiv \{ x \in X : |S(x)| = k, |S(x) \cap B| = l \},$$

where $B \subset X$ and $k, l \in \mathbb{N}$. Note that sets $S_{k,l}^{-1}(B)$ are pairwise disjoint for the fixed $B$.

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Definition 2 The $m$-transformation $S : X \to X$ is measurable if $S_{k,l}^{-1}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$ and $k, l \in \mathbb{N}$.

Let $K : X \times \mathcal{B} \to \mathbb{R}^+$ be the function
\[
K(x, B) \equiv \frac{1}{|S(x)|} \sum_{y \in S(x)} \chi_B(y).
\]
For each $x \in X$, $K(x, \cdot) : \mathcal{B} \to \mathbb{R}^+$ is a normalized measure and for each $B \in \mathcal{B}$, $K(\cdot, B) : X \to \mathbb{R}^+$ is measurable by the Definition 2. Therefore $K$ is a stochastic kernel that describes the $m$-transformation $S$. We will use $K$ as a tool for proving some results. For a more complete study of stochastic kernels the reader is referred to [5].

For any measurable $m$-transformation $S$ we define a new measure $S\mu$ on $(X,\mathcal{B},\mu)$
\[
S\mu(B) \equiv \int_X K(x, B) \, d\mu = \sum_{k=1}^{m} \sum_{l=1}^{k} \frac{l}{k} \mu(S_{k,l}^{-1}(B)).
\]

Definition 3 We say the measurable $m$-transformation $S : X \to X$ preserves measure $\mu$ or that $\mu$ is $S$-invariant if $S\mu = \mu$.

Definition 4 Let the $m$-transformation $S : X \to X$ preserve measure $\mu$. The quadruple $(X,\mathcal{B},\mu,S)$ is called an $m$-dynamical system.

The next proposition gives a number of examples of $m$-dynamical systems.

Proposition 1 Let $\{S_i\}_1^k$ be a finite collection of the $\mu$-preserving $m_i$-transformations of $(X,\mathcal{B},\mu)$ and let $S(x) = \bigcup_{i=1}^{k} S_i(x)$ be measurable. Let $K, K_i$ be the stochastic kernels that generates $S, S_i$ correspondently. If for any $B \in \mathcal{B}$
\[
K(x, B) = \frac{1}{k} \sum_{i=1}^{k} K_i(x, B)
\]
for almost all $x \in X$, then $S$ is $\mu$-preserving.

\begin{itemize}
\item For any measurable $B$ we have
\[
S\mu(B) = \int_X K(x, B) \, d\mu = \frac{1}{k} \sum_{i=1}^{k} \int_X P_i(x, B) \, d\mu = \mu(B).
\]
\end{itemize}

In the following examples $\lambda$ denotes the Lebesgue measure on $[0,1]$. 

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Example 1 Let $S : [0, 1] \to [0, 1]$ be defined by $S(x) = \{x, 1 - x\}$. Then $S$ is $\lambda$-preserving.

Example 2 Let $S : [0, 1] \to [0, 1]$ be defined by

$$S(x) = \begin{cases} 
2x, 1 - 2x, & x \in [0, \frac{1}{2}] \\
2x - 1, & x \in (\frac{1}{2}, 1]. 
\end{cases}$$

Then $S$ is $\lambda$-preserving.

The following example shows that not every $\lambda$-preserving $m$-transformation is union of $\lambda$-preserving transformations.

Example 3 Let $S : [0, 1] \to [0, 1]$ be defined by

$$S(x) = \begin{cases} 
\frac{3}{2}x, & x \in [0, \frac{1}{3}) \\
\frac{3}{2}x, \frac{3}{2}x - \frac{1}{2}, & x \in [\frac{1}{3}, \frac{2}{3}] \\
\frac{3}{2}x - \frac{1}{2}, & x \in (\frac{2}{3}, 1]. 
\end{cases}$$

Then $S$ is $\lambda$-preserving, but $S$ can not be represented as union of $\lambda$-preserving transformations.

Assume $S(x) = \bigcup_{i=1}^{k} S_i(x)$, where $S_i$ are the $\lambda$-preserving transformations. Then there are a measurable set $B \subset [\frac{1}{3}, \frac{2}{3}]$ of positive measure and transformation $S_i$ (for instance $S_1$), such that $S_1(B) \subset [0, \frac{1}{2}]$. We have

$$\lambda(S_1^{-1}(S_1(B))) = \lambda(B \cup (B - \frac{1}{3})) = 2\lambda(B) \text{ and } \lambda(S_1(B)) = \frac{3}{2}\lambda(B).$$

Since $S_1$ is the $\lambda$-preserving transformation, $\lambda(S_1(B)) = \lambda(B) = 0.$

Example 4 Let $S : [0, 1] \to [0, 1]$ be defined by

$$S(x) = \begin{cases} 
2x, 1 - 2x, & x \in [0, \frac{1}{2}] \\
2x - 1, & x \in (\frac{1}{2}, 1]. 
\end{cases}$$

Then $S$ isn’t $\lambda$-preserving.
For instance,
\[
S\lambda([0, \frac{1}{2}]) = \frac{2}{3}\lambda([0, \frac{1}{2}]) + \frac{1}{2}\lambda([\frac{1}{2}, \frac{3}{4}]) = \frac{11}{24} \neq \lambda([0, \frac{1}{2}]).
\]

Nevertheless, we can represent \(S\) as the union of the \(\lambda\)-preserving transformations \(S_1(x) = x\) and \(S_2\) from Example 2. Of course, (1) does not hold true.

Let \(S^{-1}(B) = \{x \in X : S(x) \cap B \neq \emptyset\}\) denote the full preimage of \(B\).

**Definition 5** A measurable \(m\)-transformation \(S : X \to X\) is said to be **nonsingular** if for any \(B \in \mathcal{B}\) such that \(\mu(B) = 0\), we have \(\mu(S^{-1}(B)) = 0\), i.e., \(S\mu \ll \mu\).

## 2 Recurrence and ergodic theorems

Let \(S : X \to X\) be an \(m\)-transformation. The \(n\)-th iterate of \(S\) is denoted by \(S^n\). The **tree** at \(x_0 \in X\) is the set \(\{x \in X : x \in S^n(x_0)\text{ for some } n \geq 0\}\).

Any sequence \(x_0, x_1, x_2, \ldots\) with \(x_{n+1} \in S(x_n)\) for all \(n \geq 0\) is called an **orbit** of \(x_0\).

In the study of \(m\)-dynamical systems, we are interested in properties of the trees. For example, in the recurrence of trees of \(S\), i.e., the property that if the tree in \(x\) starts in a specified set, some orbits of \(x\) return to that set infinitely many times.

**Proposition 2** Let \(S\) be a nonsingular \(m\)-transformation on \((X, \mathcal{B}, \mu)\) and let \(\mu(A) \leq \mu(S^{-1}(A))\) for any \(A \in \mathcal{B}\). If \(\mu(B) > 0\), then for almost all \(x \in B\) there is an orbit of \(x\) that returns infinitely often to \(B\).

Let \(B\) be a measurable set with \(\mu(B) > 0\), and let us define the set \(A\) of points that never return to \(B\), i.e.,\( A = \{x \in B : S^n(x) \cap B = \emptyset\text{ for all } n \geq 1\} = B \setminus \bigcup_{n=1}^{\infty} S^{-n}(B)\). Consider a collection of sets

\[
A_1 = A \cup S^{-1}(A), \quad A_i = A \cup S^{-1}(A_{i-1}), \quad i \geq 2.
\]

It is clear that \(A \cap S^{-1}(A_{i-1}) = \emptyset\). Hence

\[
\mu(A_i) = \mu(A) + \mu(S^{-1}(A_{i-1})) \geq \mu(A) + \mu(A_{i-1}) \geq \ldots \geq (i + 1)\mu(A).
\]
Therefore, \( \mu(A) = 0 \). Since \( \mu \) is nonsingular, \( \mu(S^{-n}(A)) = 0 \) for any \( n \geq 0 \). This gives \( \mu(B \setminus \bigcup_n S^{-n}(A)) = \mu(B) \), and for any \( x \in B \setminus \bigcup_n S^{-n}(A) \) there exists an orbit of \( x \) that return infinitely often to \( B \).

If \( S \) is measure preserving, then we have an analogue of Poincare’s Recurrence Theorem.

**Corollary 1** Let \( S \) be a measure-preserving \( m \)-transformation on \((X, \mathcal{B}, \mu)\). If \( \mu(B) > 0 \), then for almost all \( x \in B \) there is an orbit of \( x \) that returns infinitely often to \( B \).

\[
\triangleright \text{Note that } S\mu \ll \mu \text{ and for any measurable } A \\
\mu(A) = S\mu(A) = \sum_{k=1}^{m} \sum_{l=1}^{k} \frac{1}{k} \mu(S_{k,l}^{-1}(A)) \leq \mu(S^{-1}(A)) \ . \]

Example 1 shows there are orbits that do not return to \( B \). If \( B = [0, \frac{1}{2}) \), then for any \( x \in B \) the orbit \( \{ x, 1-x, 1-x, ... \} \) doesn’t return to \( B \).

For any nonsingular \( m \)-transformation \( S \) and function \( f \) on \( X \) we define a new function \( Uf \) on \( X \) by the equality

\[
Uf(x) \equiv \int_X f \, dK(x, \cdot) = \frac{1}{|S(x)|} \sum_{y \in S(x)} f(y) 
\]

**Proposition 3** If \( S \) is a nonsingular \( m \)-transformation and \( f \) is a real-valued measurable function on \( X \), then

\[
\int_X f \, dS\mu = \int_X Uf \, d\mu 
\]

in the sense that if one of these integrals exists then so does the other and the two are equal.

\[
\triangleright \text{We first show that } Uf \text{ is measurable. Given any } \alpha \in \mathbb{R} \text{ consider an increasing sequence of rational numbers } \alpha_1 < \ldots < \alpha_k, \text{ where } k \leq m \text{ and } \sum_{i=1}^{k} \alpha_i < k\alpha. \text{ Then the set} \\
B_{\alpha_1, \ldots, \alpha_k} = S^{-1}(f^{-1}(-\infty, \alpha_1]) \cap S^{-1}(f^{-1}(\alpha_1, \alpha_2]) \cap \ldots \cap S^{-1}(f^{-1}(\alpha_{k-1}, \alpha_k])
\]
is measurable. Taking the union of \( B_{\alpha_1, \ldots, \alpha_k} \) for all possible \( k \leq m \) and \( \alpha_1, \ldots, \alpha_k \) we conclude that the set \( \{ x : (Uf)(x) < \alpha \} \) is measurable.
When \( f = \chi_B \) is the characteristic function of \( B \in \mathcal{B} \),
\[
\int_X \chi_B \, d\mu = S\mu(B)
\]
and
\[
\int_X U\chi_B \, d\mu = \int_X \frac{1}{|S(x)|} \sum_{y \in S(x)} \chi_B(y) \, d\mu = \int_X \sum_{k=1}^m \sum_{l=0}^k \chi(S^{-1}_k(B)) \, d\mu = S\mu(B).
\]

Since \( U \) is a linear operator, the formula is also true for simple functions. If \( f \) is a nonnegative measurable function, then \( f \) is the \( S\mu \)-pointwise limit of an increasing sequence of simple functions \( f_i \), and the result follows from the fact that \( Uf \) is the \( \mu \)-pointwise limit of the increasing sequence of functions \( Uf_i \) and Monotone Converges Theorem. Finally, any measurable function \( f \) can be written as the difference \( f = f^+ - f^- \) of two nonnegative measurable functions, so the formula is true in general. ▶

**Corollary 2** Let \( S : X \to X \) be a measurable m-transformation on \((X, \mathcal{B}, \mu)\). Then \( S \) is \( \mu \)-preserving if and only if
\[
\int_X f \, d\mu = \int_X Uf \, d\mu
\]
for any \( f \in L^1 \).

▶ This follows from the Proposition above and from the equality
\[
\mu(B) = \int_X U\chi_B \, d\mu = \int_X \left( \int_X \chi_B \, dK(x, \cdot) \right) \, d\mu = \int_X K(x, B) \, d\mu = S\mu(B). \]

**Proposition 4** Let \( S : X \to X \) be a \( \mu \)-preserving m-transformation on \((X, \mathcal{B}, \mu)\). Then the positive linear operator \( U \) is a contraction on \( L^p \) for any \( 1 \leq p \leq \infty \).

▶ It is easily seen that \( U \) is a contraction on \( L^\infty \). By Jensen inequality \( |Uf|^p \leq U|f|^p \) for any \( p \geq 1 \) and \( f \in L^p \) (see [5], Chapter 1, Lemma 7.4 for a more general statement). Then
\[
\|Uf\|_p^p = \int_X |Uf|^p \, d\mu \leq \int_X |f|^p \, d\mu = \int_X |f|_p^p \, d\mu = \|f\|_p^p.
\]
For a function $f$ on $X$ and an $m$-transformation $S : X \to X$, we define the averages

$$A_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} U^k f, \quad n = 1, 2, \ldots .$$

From the Birkhoff Ergodic Theorem for Markov operators (see [4] for the details) and from the Proposition above we get the following theorem.

**Theorem 1** Suppose $S : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure preserving $m$-transformation and $f \in L^1$. Then there exists a function $f^* \in L^1$ such that

$$A_n(f) \to f^*, \mu \text{-a.e.}$$

Furthermore, $Uf^* = f^* \mu$-a.e. and $\int_X f^* \, d\mu = \int_X f \, d\mu$.

**Corollary 3** Let $1 \leq p < \infty$ and let $S$ be a measure preserving $m$-transformation on $(X, \mathcal{B}, \mu)$. If $f \in L^p$, then there exists $f^* \in L^p$ such that $Uf^* = f^* \mu$-a.e. and $\|f^* - A_n(f)\|_p \to 0$ as $n \to \infty$.

Let us fix $1 \leq p \leq \infty$ and $f \in L^p$. Since $\|A_n(f)\|_p \leq \|f\|_p$, we have by Fatou’s lemma,

$$\int_X |f^*|^p \, d\mu \leq \liminf_{n \to +\infty} \int_X |A_n(f)|^p \, d\mu \leq \int_X |f|^p \, d\mu .$$

Hence, the operator $L : L^p \to L^p$ defined by $L(f) = f^*$ is a contraction on $L^p$. By Theorem 1, $\|f^* - A_n(f)\|_p \to 0$ as $n \to \infty$ for any bounded function $f \in L^p$. Let $f \in L^p$ be a function, not necessarily bounded. For any $\varepsilon > 0$ we can find a bounded function $f_B \in L^p$ such that $\|f - f_B\|_p < \varepsilon$. Then, since $L$ is a contraction on $L^p$, we have

$$\|f^* - A_n(f)\|_p \leq \|f_B^* - A_n(f_B)\|_p + \|A_n(f - f_B)\|_p + \|(f - f_B)^*\|_p ,$$

which can be made arbitrarily small. 

3 Ergodicity

Assume $Uf = f$ for some measurable function $f$. It is very important to know condition on $S$ under that $f$ is constant.
Definition 6 We call a nonsingular $m$-transformation $S$ ergodic if for any $B \in \mathcal{B}$, such that $B \setminus S^{-1}(B) = B^c \setminus S^{-1}(B^c) = \emptyset$, $\mu(B) = 0$ or $\mu(B^c) = 0$.

It is obvious that if $S$ is the union of $\mu$-preserving $m$-transformations (see Proposition 11), one of which is not ergodic, then $S$ is not ergodic.

Theorem 2 The following three statements are equivalent for any nonsingular $m$-transformation $S : X \to X$.

1. $S$ is ergodic

2. for any $B \in \mathcal{B}$, such that $\mu(B \setminus S^{-1}(B)) = \mu(B^c \setminus S^{-1}(B^c)) = 0$, $\mu(B) = 0$ or $\mu(B^c) = 0$.

3. for any disjoint sets $B_1, B_2 \in \mathcal{B}$, such that $\mu(B_1 \setminus S^{-1}(B_1)) = \mu(B_2 \setminus S^{-1}(B_2)) = 0$, $\mu(B_1) = 0$ or $\mu(B_2) = 0$.

We see at once that (3)$\Rightarrow$(1).

(1)$\Rightarrow$(2) Suppose $S$ is ergodic and $B \in \mathcal{B}$, such that $\mu(B \setminus S^{-1}(B)) = \mu(B^c \setminus S^{-1}(B^c)) = 0$. Let $A_1 = (B \cap S^{-1}(B)) \cup (B^c \setminus S^{-1}(B^c))$, $A_i = A_{i-1} \cap S^{-1}(A_{i-1})$ for $i \geq 2$, and $A = \cap_{i=1}^{\infty} A_i$. We have $A_1 \supset A_2 \supset \ldots$ and
\[ A_{i-1} \setminus A_i \subset S^{-1}(A_{i-2} \setminus A_{i-1}) \subset \ldots \subset S^{-i+2}(A_1 \setminus A_2) \subset S^{-i+1}(B \setminus S^{-1}(B)) \]

Therefore, $\mu(A \Delta B) = 0$. Let $x \in A$, then there is at least one point in $S(x)$ that belongs to infinite many of $A_i$. This gives $A \subset S^{-1}(A)$.

Let $C_1 = A^c$, $C_i = C_{i-1} \cap S^{-1}(C_{i-1})$ for $i \geq 2$, and $C = \cap_{i=1}^{\infty} C_i$. We have $C_1 \supset C_2 \supset \ldots$ and
\[ C_{i-1} \setminus C_i \subset \ldots \subset S^{-i+2}(C_1 \setminus C_2) \subset S^{-i+1}(B^c \setminus S^{-1}(B^c)) \cup S^{-i+2}(B \setminus A) \]

Therefore, $\mu(C \Delta B^c) = 0$. Let $x \in C$, then there is at least one point in $S(x)$ that belongs to infinite many of $C_i$. This gives $C \subset S^{-1}(C)$. Moreover,
\[ C^c = A \cup C_1 \setminus C \subset S^{-1}(A) \cup S^{-1}(C_1 \setminus C) \cup S^{-1}(A) = S^{-1}(C^c) \]

We conclude from the ergodicity of $S$ that $\mu(B^c) = \mu(C) = 0$ or $\mu(B) = \mu(C^c) = 0$.

(2)$\Rightarrow$(3) Suppose (2) holds true and let $B_1, B_2 \in \mathcal{B}$ be the disjoint sets, such that $\mu(B_1 \setminus S^{-1}(B_1)) = \mu(B_2 \setminus S^{-1}(B_2)) = 0$. Let $C_1 = B_1^c$, $C_i = C_{i-1} \setminus S^{-1}(C_{i-1})$ for $i \geq 2$, and $C = \cap_{i=1}^{\infty} C_i$. We have $C_1 \supset C_2 \supset \ldots$ and
\[ \mu(B_2 \backslash C_i) = 0. \] Therefore \( \mu(C) \geq \mu(B_2) \). Let \( x \in C_i \), then there is at least one point in \( S(x) \) that belongs to infinite many of \( C_i \). This gives \( C \subset S^{-1}(C) \). Moreover \( \mu(C^c \backslash S^{-1}(C^c)) = 0 \) and \( \mu(C^c) \geq \mu(B_1) \). By assumption \( \mu(C) = 0 \) or \( \mu(C^c) = 0 \). This finishes the proof. \( \blacksquare \)

**Example 5** We will prove the ergodity of

\[ S(x) = \begin{cases} 
2x, 1 - 2x, & x \in [0, \frac{1}{2}] \\
2x - 1, & x \in (\frac{1}{2}, 1].
\end{cases} \]

Let \( B \subset S^{-1}(B) \) and \( B^c \subset S^{-1}(B^c) \) \( (2) \).

Set \( A_1 = \{x : \{x, 1 - x\} \subset B\}, A_2 = \{x : \{x, 1 - x\} \subset B^c\} \) and \( A_3 = (A_1 \cup A_2)^c \).

Let \( x \in A_1 \). By \( (2) \)

\[ \frac{1 + x}{2} \in B, \quad \frac{2 - x}{2} \in B^c, \quad \frac{1 - x}{2} \in B, \quad \frac{x}{2} \in B. \]

Therefore \( \bar{S}^{-1}(A_1) \subset A_1 \), where \( \bar{S} \) is the well known ergodic single-valued transformation \( \bar{S}(x) = 2x \pmod{1}, x \in [0, 1] \). By ergodicity of \( \bar{S} \), \( \lambda(A_1) = 0 \) or \( \lambda(A_1) = 1 \). Similarly, \( \lambda(A_2) = 0 \) or \( \lambda(A_2) = 1 \).

Since \( \lambda(A_1) = 1 \) leads to \( \lambda(B^c) = 0 \) and \( \lambda(A_2) = 1 \) leads to \( \lambda(B) = 0 \), we need only consider

\[ \lambda(A_3) = 1. \] \( (3) \)

Let \( x \in B \). By \( (2) \) and \( (3) \)

\[ \frac{1 + x}{2} \in B, \quad \frac{2 - x}{2} \in B^c \ \text{a.s.}, \quad \frac{1 - x}{2} \in B^c \ \text{a.s.}, \quad \frac{x}{2} \in B \ \text{a.s.} \]

Therefore \( \lambda(\bar{S}^{-1}(B) \backslash B) \). By ergodicity of \( \bar{S} \), \( \lambda(B) = 0 \) or \( \lambda(B) = 1 \). \( \blacksquare \)

**Example 6** The 2-transformation \( S : [0, 1] \to [0, 1] \)

\[ S(x) = \begin{cases} 
\{\frac{3}{2}x\}, & x \in [0, \frac{1}{3}) \\
\{\frac{3}{2}x, \frac{3}{2}x - \frac{1}{2}\}, & x \in [\frac{1}{3}, \frac{2}{3}] \\
\{\frac{3}{2}x - \frac{1}{2}\}, & x \in (\frac{2}{3}, 1].
\end{cases} \]

is not ergodic.
Proposition 5 Let $S$ be ergodic. If $f$ is measurable and $(Uf)(x) = f(x)$ a.e., then $f$ is constant a.e.

- For each $r \in \mathbb{R}, E_r = \{x \in X : (Uf)(x) = f(x) > r \}$ is measurable. Then $E_r \subseteq S^{-1}(E_r)$ and $E_r^c \subseteq S^{-1}(E_r^c)$, hence $E_r$ has measure 0 or 1. But if $f$ is not constant a.e., there exists an $r \in \mathbb{R}$ such that $0 < \mu(E_r) < 1$. Therefore $f$ must be constant a.e.

Corollary 4 If a measure preserving $m$-transformation $S$ is ergodic and $f \in \mathcal{L}^1$, then the limit of the averages $f^* = \int_X f \, d\mu$ is constant a.e. Thus, if $\mu(B) > 0$, then for almost all $x \in X$ there is a orbit of $x$ that returns infinitely often to $B$.

- We conclude from Theorem 1 and from Proposition 5 that $f^* = \int_X f \, d\mu$. To prove the second statement we consider $f = \chi_B$ and apply Corollary 1.

Corollary 5 Let measure preserving $m$-transformation $S$ be ergodic and $\mu(S^{-1}(X)) < 1$, i.e., the set $\{x \in X : |S(x)| \geq 2 \}$ has positive measure. If $\mu(B) > 0$, then for almost all $x \in X$ there are uncountable many orbits of $x$ that return infinitely often to $B$.

- We just apply the corollary above to the sets $B$ and $(S_{11}^{-1}(X))^c$.

Corollary 6 Let $S$ be a measure preserving ergodic $m$-transformation and $f \in \mathcal{L}^1$ such that $f(x) \geq f(y)(f(x) \leq f(y))$, for any $y \in S(x)$. Then $f$ is constant a.e.

- We have $Uf \leq f$, hence the limit of averages $f^* \leq f$. By Corollary 4 $f = f^*$ is constant a.e.

### 4 The Frobenius-Perron operator

Assume that a nonsingular $m$-transformation $S : X \to X$ on a normalized measure space is given. We define an operator $P : \mathcal{L}^1 \to \mathcal{L}^1$ in two steps.
1. Let \( f \in \mathcal{L}^1 \) and \( f \geq 0 \). Write
\[
\nu(B) = \int_X f(x)K(x,B) \, d\mu.
\]
Then, by the Radon-Nikodym Theorem, there exists a unique element in \( \mathcal{L}^1 \), which we denoted by \( Pf \), such that
\[
\nu(B) = \int_B Pf \, d\mu.
\]

2. Now let \( f \in \mathcal{L}^1 \) be arbitrary, not necessarily nonnegative. Write \( f = f^+-f^- \) and define \( Pf = Pf^+-Pf^- \). From this definition we have
\[
\int_B Pf \, d\mu = \int_X f^+(x)K(x,B) \, d\mu - \int_X f^-(x)K(x,B) \, d\mu
\]
or, more completely,
\[
\int_B Pf \, d\mu = \int_X f(x)K(x,B) \, d\mu. \tag{4}
\]

**Definition 7** If \( S : X \to X \) is a nonsingular \( m \)-transformation the unique operator \( P : \mathcal{L}^1 \to \mathcal{L}^1 \) defined by equation (4) is called the *Frobenius-Perron operator* corresponding to \( S \).

It is straightforward to show that \( P \) is a positive linear operator and
\[
\int_X Pf \, d\mu = \int_X f \, d\mu.
\]

**Proposition 6** If \( f \in \mathcal{L}^1 \) and \( g \in \mathcal{L}^\infty \), then \( \langle Pf, g \rangle = \langle f, Ug \rangle \), i.e.,
\[
\int_X (Pf) \cdot g \, d\mu = \int_X f \cdot (Ug) \, d\mu. \tag{5}
\]
Let $B$ be a measurable subset of $X$ and $g = \chi_B$. Then the left hand side of (5) is
\[ \int_B Pf \, d\mu = \int_X f(x)K(x,B) \, d\mu \]
and the right hand side is
\[ \int_X f \cdot (U\chi_B) \, d\mu = \int_X f \cdot \left( \int_X \chi_B \, dK(x,\cdot) \right) \, d\mu = \int_X f(x)K(x,B) \, d\mu . \]

Hence (5) is verified for characteristic functions. Since the linear combinations of characteristic functions are dense in $\mathcal{L}^\infty$, (5) holds for all $f \in \mathcal{L}^1$ and $g \in \mathcal{L}^\infty$. ▴

The following proposition says that a density $f_*$ is a fixed point of $P$ if and only if it is a density of a $S$-invariant measure $\nu$, absolutely continuous with respect to a measure $\mu$.

**Proposition 7** Let $S : X \rightarrow X$ be nonsingular and let $f_* \in \mathcal{L}^1$ be a density function on $(X,\mathcal{B},\mu)$. Then $Pf_* = f_*$ a.e., if and only if the measure $\nu = f_* \cdot \mu$, defined by $\nu(B) = \int_B f_* \, d\mu$, is $S$-invariant.

Let $B \subset X$ be a measurable. Then
\[ S\nu(B) = \int_X K(x,B) \, d\nu = \int_X f_*(x)K(x,B) \, d\mu = \int_B Pf_* \, d\mu . \]

On the other hand
\[ \nu(B) = \int_B f_* \, d\mu . \]

**Proposition 8** Let $S : X \rightarrow X$ be a nonsingular $m$-transformation and $P$ the associated Frobenius-Perron operator. Assume that an $f \geq 0$, $f \in \mathcal{L}^1$ is given. Then
\[ \text{supp } f \subset S^{-1}(\text{supp } Pf) \text{ a.s.} \]

By the definition of the Frobenius-Perron operator, we have $Pf(x) = 0$ a.e. on $B$ implies that $f(x) = 0$ for a.a. $x \in S^{-1}(B)$. Now setting $B = (\text{supp } f)^c$, we have $Pf(x) = 0$ for a.a. $x \in B$ and, consequently, $f(x) = 0$ for a.a. $x \in S^{-1}(B)$, which means that $\text{supp } f \subset (S^{-1}(B))^c$. Since $(S^{-1}(B))^c \subset S^{-1}(B^c)$ a.s., this completes the proof. ▴
Proposition 9 Let $S : X \to X$ be a nonsingular $m$-transformation and $P$ the associated Frobenius-Perron operator. If $S$ is ergodic, then there is at most one stationary density $f_*$ of $P$.

Assume that $S$ is ergodic and that $f_1$ and $f_2$ are different stationary densities of $P$. Set $g = f_1 - f_2$, so that $Pg = g$. Since $P$ is a Markov operator, $g^+$ and $g^-$ are both stationary densities of $P$. By assumption, $f_1$ and $f_2$ are not only different but are also densities we have $g^+ \not\equiv 0$ and $g^- \not\equiv 0$. Set

$$B_1 = \text{supp } g^+ \quad \text{and} \quad B_2 = \text{supp } g^- .$$

It is evident that $B_1$ and $B_2$ are disjoint sets and both have positive measure. By Proposition 8 we have

$$B_1 \subset S^{-1}(B_1) \quad \text{a.s.} \quad \text{and} \quad B_2 \subset S^{-1}(B_2) \quad \text{a.s.}$$

But, from Theorem 2 it follows that $\mu(B_1) = 0$ or $\mu(B_2) = 0$. ◻

5 Applications and generalization

We now apply the method of $m$-transformation to the intersection of two middle-$\beta$ Cantor sets (see [8] and the references given there).

Let $\alpha \in [\frac{1}{3}, \frac{2}{3}]$ and $\psi_1(x) = \alpha x$, $\psi_1(x) = \alpha x + 1 - \alpha$ be a contracting similarity maps on $I = [0,1]$ endowed with Lebesgue measure $\lambda$. There is a unique compact set $C_\alpha \subset I$ which satisfies the set equation

$$C_\alpha = \psi_1(C_\alpha) \cup \psi_2(C_\alpha) .$$

It is easily checked that $C_\alpha$ is the middle-$\beta$ Cantor set for $\beta = 1 - 2\alpha$. Let $x \in I$ and $f(x)$ denotes the Hausdorff dimension of the set $C_\alpha \cap (C_\alpha + x)$. Let $B_{ij} = \psi_i(C_\alpha) \cap \psi_j(C_\alpha + x)$, $i, j = 1, 2$. From the construction of $C_\alpha$ it follows that $B_{12} = \emptyset$,

$$\text{dim}_H B_{11} = \text{dim}_H B_{22} = \begin{cases} f(\frac{x}{\alpha}) , & 0 \leq x \leq \alpha \\ 0 , & \alpha < x \leq 1 \end{cases}$$

and

$$\text{dim}_H B_{21} = \begin{cases} 0 , & 0 \leq x < 1 - 2\alpha \\ f(-\frac{x}{\alpha} + \frac{1}{\alpha} - 1) , & 1 - 2\alpha \leq x < 1 - \alpha \\ f(\frac{x}{\alpha} - \frac{1}{\alpha} + 1) , & 1 - \alpha \leq x \leq 1 . \end{cases}$$
Since \( C_\alpha \cap (C_\alpha + x) = B_{11} \cup B_{21} \cup B_{22} \), we have
\[
f(x) = \max \{ \dim H_{ij} : i, j = 1, 2 \} = \max \{ f(y) : y \in S(x) \},
\]
where
\[
S(x) = \begin{cases} 
\{ \frac{x}{\alpha} \}, & 0 \leq x < 1 - 2\alpha \\
\{ \frac{x}{\alpha}, -\frac{x}{\alpha} + \frac{1}{\alpha} - 1 \}, & 1 - 2\alpha \leq x \leq \alpha \\
\{ -\frac{x}{\alpha} + \frac{1}{\alpha} - 1 \}, & \alpha < x \leq 1 - \alpha \\
\{ \frac{x}{\alpha} - \frac{1}{\alpha} + 1 \}, & 1 - \alpha < x \leq 1
\end{cases}
\]
(see Examples 2 and 5 under \( \alpha = \frac{1}{2} \)).

Using Leibniz’s rule, we find the Frobenius-Perron operator corresponding to \( S \):
\[
(Pf)(x) = \begin{cases} 
\alpha(f(1 - \alpha - \alpha x) + f(1 - \alpha + \alpha x) + f(\alpha x)), & 0 \leq x < \frac{1}{\alpha} - 2 \\
\alpha(f(1 - \alpha - \alpha x) + \frac{1}{2}f(1 - \alpha + \alpha x) + \frac{1}{2}f(\alpha x)), & \frac{1}{\alpha} - 2 \leq x \leq 1.
\end{cases}
\]
Assume there exist a stable point \( f_\ast \) of \( P \). Then by Proposition 4 the measure \( \mu = f_\ast \cdot \lambda \) is \( S \)-invariant. If in addition \( S : (I, \mathcal{B}, \mu) \to (I, \mathcal{B}, \mu) \) is ergodic, then by (6) and Corollary 6 \( f \) is constant \( \mu \)-a.e. The same method works in case of the intersection of two arbitrary self-similar sets.

Using \( m \)-transformations we can develop a new approach to the self-similar sets with overlaps (see [2], [7]). Let \( \psi_1, \ldots, \psi_m \) be contracting similarity maps on \( \mathbb{R}^n \), and let \( X = \bigcup_{i=1}^m \psi_i(X) \) be an attractor of the iterated function system. Given normalized measure \( \mu \) on \( X \) we consider \( m \)-transformation of \( X 
\]
Assume, using the Frobenius-Perron operator corresponding \( S \), we have found \( S \)-invariant ergodic measure on \( X \). This measure gives us an interesting information about \( X \). For instance, if the conditions of Corollary 3 hold true, we see that a.a. points of \( X \) have uncountable many of addresses (see [3] for details).

From these examples we see, that the main problem of the investigation is to find an \( S \)-invariant ergodic measure. To decide this problem we propose a following generalization of an \( m \)-transformation.
Given $m$-transformation $S$ on a normalized measure space $(X, B, \mu)$ we consider a collection of pairs \( \{S_i, \alpha_i\}_{i=1}^{m} \), where \( S_i : X \rightarrow X \) are the single-valued measurable transformations such that \( S(x) = \cup_{i=1}^{m} S_i(x) \) for any \( x \in X \), and \( \alpha_i : X \rightarrow [0, 1] \) are the measurable functions such that \( \sum_{i=1}^{m} \alpha_i(x) = 1 \) for any \( x \in X \). Let us consider the stochastic kernel

\[
K(x, B) = \sum_{i=1}^{m} \alpha_i(x) \chi_B(S_i(x))
\]

and a new measure on $X$

\[
S\mu(B) \equiv \int_X K(x, B) \, d\mu.
\]

If we choose $S_i$ and $\alpha_i$ such that $S\mu = \mu$, we can employ the results of this paper to the measure preserving transformation $S$.

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