Inequalities for semi-stable surface fibrations, and strictly maximal Higgs fields

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Introduction

The aim of this note is twofold. First, I want to place some of the results from [L-Z] in the context of "classical" surface theory (slope inequalities for fibrations, the Bogomolov-Miyaoka-Yau inequalities etc.). The second goal is to simplify the proof of the main new inequality of loc. cit. I hope that the way I set up the machinery is more understandable to surface specialists. For that reason the arguments from loc. cit. concerning stable and semi-stable curves and their moduli spaces have been explained in somewhat greater detail.

Let me now proceed to discuss the set-up in [L-Z]. Start with a surface fibration $f : X \rightarrow B$ in semi-stable genus $g$ curves; one wants to know when for such a fibration the classical Arakelov inequality (see (13) below) is attained. This is known to be a Hodge theoretic question and is closely linked to the existence of special curves in the moduli space of genus $g$ curves (see the Corollary at the end of this Introduction).

Indeed, it happens precisely when the corresponding log-Hodge bundle underlies a strictly maximal Higgs field. To explain what this means, note that $R^1 f_* \mathbb{Z}$ admits a variation of weight 1 Hodge structure at least over the complement of the critical locus of $f$. In fact, it can be extended over those critical points over which the fiber is a union of smooth curves: the corresponding generalized jacobian is then an abelian variety. Let $\Sigma \subset B$ be the set of the remaining critical points. The Hodge bundle $F^1 \subset R^1 f_* \mathbb{C} \otimes \mathcal{O}_{B-\Sigma}$ admits a Deligne extension $H^{1,0} \subset H$ over all of $B$. The Higgs bundle is the associated graded $H^{1,0} \oplus H^{0,1}$. The Gauss-Manin connection induces the
Higgs field $\sigma : H^{1,0} \to H^{0,1} \otimes \Omega^1(\log \Sigma)$. For a generalized polarized weight 1 variation of Hodge structure, $\ker(\sigma)$ is direct summand of the Higgs bundle; it is the maximal (unitary) locally constant subsystem. The Higgs field is called maximal if $\sigma$ induces an isomorphism on the remaining part. It is called strictly maximal if, moreover, $\ker(\sigma) = 0$.

Although the Jacobian of a curve is irreducible as a polarized abelian variety, it may be isogeneous to a product of abelian varieties. If this happens in a family, the Higgs field splits although the polarized variation is indecomposable. In particular, even for families of curves, a Higgs field may be maximal without being strictly maximal.

I can now state the result from [L-Z] I alluded to:

**Theorem** (= Theorem 4.1). Let $f : X \to B$ be a relatively minimal semi-stable genus $g$ surface fibration with $g \geq 2$. Assume that the corresponding Higgs bundle is strictly maximal, i.e., $f$ reaches the Arakelov bound $\deg(f_\ast \omega_{X/B}) = \frac{1}{2}g \cdot \deg \Omega^1_B(\log(\Sigma))$, where $\Sigma \subset B$ is the set of critical values for the associated jacobian fibration. Then $g \leq 4$.

The proof I give below uses some of the same ingredients of loc. cit. However, it has been simplified by using the strengthening of the Bogomolov-Miyaoka-Yau inequality which follows from Yau’s approach (see [Ch-Y, Ko]) instead of Miyaoka’s version [Miy] which does not tell what happens in case of equality. The original proof had to make up for this default by an elaborate covering trick.

Let me finish this introduction by recalling the main application of the above theorem:

**Corollary.** No Shimura curve is generically contained in the moduli space of curves of genus $g \geq 5$ if its associated family has strictly maximal Higgs bundle.

As explained in [L-Z], this bound is sharp. Also, if one allows maximal instead of strictly maximal Higgs fields, the best one can hope for is $g \geq 8$. In loc. cit. this bound is shown for the hyperelliptic locus.

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1 Proportionality Deviation for Semi-Stable Fibrations

Let $X$ be a minimal compact complex algebraic surface and suppose that

$$f : X \to B, \quad \text{genus } B = b,$$

is a fibration in genus $g \geq 2$ semi-stable curves. Denote the fibre over $s$ by $X_s$; $\text{Jac}(X_s)$ denotes the generalized Jacobian of $X_s$. As usual, put

$$\omega_{X/B} = K_X \otimes f^*K_B^{-1}, \quad \text{the relative dualizing sheaf.}$$

Recall [B-P-H-V, Theorem III, 18.2] that a relatively minimal fibration in genus $\geq 2$ curves $f : X \to B$ is isotrivial if and only if $\deg f_*\omega_{X/B} = 0$. So, if $f$ is not isotrivial, one can introduce the slope

$$\lambda(f) := \frac{\omega^2_{X/C}}{\deg(f_*\omega_{X/C})}. \quad (2)$$

Recall also that $f$ is called a Kodaira fibration if, moreover, $f$ is smooth. In that case, the period map for $f$ sends $B$ to a compact curve in the moduli space $M_g$ of genus $g$ curves. For a Kodaira fibration the base genus $b \geq 2$. See [B-P-H-V, V. 14].

Relations between the various numerical invariants of a surface fibration are gathered in the next Lemma.

**Lemma 1.1.** For a relatively minimal genus $g$ surface fibration $f : X \to B$, $g(B) = b$, one has:

1. $\deg(f_*\omega_{X/B}) = \frac{1}{12}(c^2_1(X) + e(X)) - (b - 1)(g - 1)$.
2. $c^2_1(X) = \omega^2_{X/B} + 8(g - 1)(b - 1)$.
3. $e(X) = 4(g - 1)(b - 1) + \sum_{s \in B} \delta_s$, where

$$\delta_s = e(X_s) - e(X_{\text{gen}}) \geq 0, \quad e(X_{\text{gen}}) = 2(1 - g).$$

4. $12 \deg f_*\omega_{X/B} = \omega^2_{X/B} + \sum_s \delta_s$. 

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1 (i.e. there are no $(-1)$-curves in the fibres)
2 The fibres need not be semi-stable.
Suppose, moreover, that $f$ is non-isotrivial and $g \geq 2$. Then

$$4 - \frac{4}{g} \leq \lambda(f) \leq 12.$$  \hspace{3em} (3)

The leftmost inequality is Xiao’s slope inequality. The right hand inequality becomes an equality, $\lambda(f) = 12$ if and only if $f$ is a Kodaira fibration.

**Proof:** (i) is a direct consequence of Riemann-Roch and the Leray spectral sequence.

(ii) follows from the definition of $\omega_{X/B}$.

(iii) See [B-P-H-V, Prop. III, 11.4].

(iv) Follows directly from (i),(ii), (iii).

(v) See [Xiao]. The remaining assertion about the upper bound of $\lambda(f)$ follows immediately from (iv).

Motivated by the Bogomolov-Miyaoka-Yau inequality $c_1^2(X) \leq 3c_2(X)$ (see [B-P-H-V, Theorem VII, 4.1]), introduce the proportionality deviation

$$\beta(X) := 3c_2(X) - c_1^2(X).$$  \hspace{3em} (4)

by loc. cit. $\beta \geq 0$ with equality if and only if $X$ is a ball quotient.

**Corollary 1.2.** In the situation of (1), one has

$$\beta(X) = 4(b - 1)(g - 1) - \omega_{X/B}^2 + 3 \sum_{s \in B} \delta_s$$

The invariants $\delta_s$ are easy to identify:

**Lemma 1.3.** Suppose $f : X \to B$ is a semi-stable genus $g$ fibration. Then

$$\delta_s = \# \text{(double points of } X_s).$$

**Proof:** The Euler number of any normal crossing curve $C = \sum_{\alpha=1}^{N+1} C_\alpha$ can be calculated from the so-called dual graph $\Gamma_C$ as follows. A vertex $\alpha \in \Gamma_C$ corresponds to an irreducible component $C_\alpha$. It is labelled by the genus $g_\alpha$ of the normalization of $C_\alpha$; two intersecting components give a corresponding edge and other double points give circular edges. So the total number of edges
equals the total number $\delta_C$ of double points. Using the additive property of the Euler number one finds $e(C) = \sum (2 - 2g_\alpha) - \delta_C$, and so

$$e(X_s) = \sum (2 - 2g_\alpha) - \delta_s$$  \hfill (5)

On the other hand, $p_a(X_s)$ is independent of $s \in B$ since by the adjunction formula $2p_a(X_s) - 2 = K_X \cdot X_s + X_s^2$ which does not depend on $s$. For an irreducible component $C_\alpha$ of $X_s$ one has $p_a(C_\alpha) = g_\alpha + \delta_\alpha$ with $\delta_\alpha$ the number of double points of $C_\alpha$; the number $\sum_{\alpha<\beta} C_\alpha C_\beta$ is the number $\delta_s'$ of those double points of $X_s$ that are intersections of two components. So

$$e(X_{\text{gen}}) = 2 - 2g = -(K_X \cdot X_s + X_s^2)$$

$$= -\sum_\alpha (K_X \cdot C_\alpha + C_\alpha^2) - 2 \sum_{\alpha<\beta} C_\alpha \cdot C_\beta$$

$$= \sum (2 - 2g_\alpha) - 2(\sum_{\delta_s} \delta_\alpha + \delta_s').$$

Comparing this with (5) shows that $e(X_s) - e(X_{\text{gen}}) = \delta_s$. \hfill \Box

2 \hspace{1em} \textbf{Refined Bogomolov-Yau Inequality}

I shall explain a refinement of the Bogomolov-Yau inequality due to R. Kobayashi [Ko], and Cheng-Yau [Ch-Y]. An algebraic proof has been given by Miyaoka [Miy], but his proof cannot tell what happens in case of equality.

Some preparations are needed. First, recall that if $X$ is a minimal general type surface, its canonical bundle is nef and big, but not necessarily ample due the presence of so-called A-D-E curve configurations which contract to rational double points in the minimal model. Such singularity $p$ is a quotient singularity: its germ is of the form $U_p/G_p$ where $U_p$ is smooth and $G_p$ a finite group.

The refinement gives an inequality valid for the logarithmic Chern classes for a Zariski-open $X - C$ where $C \subset X$ is a normal crossing curve. Recall that the logarithmic Chern invariants as defined as follows:

$$c_2(X, C) := c_2(X) - e(C);$$

$$c_1^2(X, C) := c_1(\Omega_X^1(\log C))^2 = c_1^2(X) + 2K_X \cdot C + C^2;$$

$$c_2(X, C) := c_2(X) - e(C);$$

$$c_1^2(X, C) := c_1(\Omega_X^1(\log C))^2 = c_1^2(X) + 2K_X \cdot C + C^2;$$
and introduce

\[ \beta(X, C) := 3c_2(X) - c_1^2(X) - \left(3\epsilon(C) + 2KC + C^2\right). \]

One also needs a modification of these invariants when \( X - C \) has A-D-E configurations. One introduces for such a configuration \( R_p \), contracting to the singular point \( p \) the following invariant which is needed to correct \( c_2(X, C) \):

\[ \beta(p) = 3\left( e(R_p) - \frac{1}{|G_p|}\right) > 0. \] (6)

Now I can state the promised strengthening of the Bogomolov-Miyaoka-Yau inequality which results from Yau’s techniques:

**Theorem 2.1** ([Ko Thm 2], [Miy Ch-Y]). Let \( X \) be a surface of general type, \( C \) a normal crossing divisor such that \( K_X + C \) is nef. Let \( R_j, j = 1, \ldots, \) be the A-D-E configurations in \( X - C \) and let \( X' \) be the normal surface obtained after contracting the \( R_j \) to a singularity \( p_j \). Then

\[ \beta(X, C) \geq \sum_j \beta(p_j) \] (7)

and equality holds if and only if \( X' - C = \Gamma\backslash B^2 \), the quotient of the 2-ball \( B^2 \) by a discrete subgroup \( \Gamma \subset \text{PSU}(2,1) \) acting freely except over the singularities of \( X' \) where \( \Gamma \) acts with isolated fixed points.

**Remark 2.2.** (1) The excess invariant \( \beta(C) \) is additive in the sense for disjoint curves \( C \) and \( D \) one has \( \beta(C \coprod D) = \beta(C) + \beta(D) \).

(2) The result applies to any semi-stable curve \( C \) on \( X \) since for all components \( C_i \) of \( C \) with \( p_a(C_i) > 0 \), one has \( (K + C) \cdot C_i > KC_i + C_i^2 = 2p_a(C) - 2 \geq 0 \) while a smooth rational component \( C_i \) of \( C \) must meet the union \( D_i \) of the other components in at least 2 points (by definition of semi-stability) so that \( (K + C) \cdot C_i = K \cdot C_i + D_i \cdot C_i + C_i \cdot C_i \geq 2 + (-2) \geq 0 \). So \( K_X + C \) is nef indeed. See also [Sak, Theorem 7.4].

In particular, if \( C = X_s \) a possibly singular fibre of a semi-stable fibration in curves of genus \( g \) one finds:

\[ \beta(X_s) = 3\delta_s + 2(1 - g). \] (8)
This can indeed be positive and in that case gives an amelioration of the Bogomolov-Miyaoka-Yau inequality.

(3) Let $C = E_s$, an elliptic tail of a fibre $X_s$ as in (2), i.e., $X_s = E_s + R$, $E_s \cdot R = 1$, $E_s$ a smooth elliptic curve. Then

$$\beta(E_s) = -E_s^2.$$  \hspace{1cm} (9)

From this remark one deduces:

**Corollary 2.3.** Let $X$ be a compact complex surface of general type,

$$f : X \to B$$

a semi-stable fibration of genus $g$ curves. Let $\Sigma_c \subset B$, the set points $s$ for which $\text{Jac}(X_s)$ is, smooth compact (i.e. the components of $X_s$ are smooth) and let $\Sigma \subset B$ be the remaining set of critical points for $f$. Finally, let $E$ be the union of all elliptic tails and set

$$C = E + \coprod_{s \in \Sigma} X_s,$$

Then

$$\omega_{X/B}^2 \leq 2(g - 1) \deg \Omega^1_B(\log(\Sigma)) + \sum_{s \in \Sigma} 3\delta_s + E^2.$$  \hspace{1cm} (10)

Moreover, equality holds precisely when $\beta(X, C) = 0$.

**Proof:** First of all, by Theorem 2.1 and Eqn. (8), (9) one has

$$\beta(X, C) = \beta(X) - \sum_{s \in \Sigma} 3\delta_s + E^2 + 2(g - 1) \cdot |\Sigma| \geq 0,$$

So, by Cor. 1.2, setting $b = \text{genus}(B)$, the inequality

$$4(b - 1)(g - 1) - \omega_{X/B}^2 + \sum_{s \in B - \Sigma} 3\delta_s + E^2 + 2(g - 1) \cdot |\Sigma| \geq 0,$$

holds, which is equivalent to the stated inequality. \qed
3 Moriwaki’s Slope Inequality

There is a stronger lower bound, the slope inequality which is due to Moriwaki. It is a consequence of certain enumerative properties of cycles on the compactified moduli space $\overline{M}_g$. To explain it, let me introduce some more notation. The moduli space $\overline{M}_g$ has at its boundary the irreducible divisors $\Delta_j$, $j = 0, \ldots, \lfloor \frac{g}{2} \rfloor$ where the stable genus $g$ curve $C$ belongs to $\Delta_0$ if $C$ is irreducible and to $\Delta_j$, $j > 0$, if it is of the form $C = C_1 + C_2$ where $p_a(C_1) = j$, $p_a(C_2) = g - j$. The double point $C_1 \cap C_2$ is then called a double point of type $j$. The double points of an irreducible curve are called of type 0.

If $p_f : B \to \overline{M}_g$ is the period map for $f$, the curve $p_f(B)$ meets the divisor $\Delta_j$ exactly in the points $s \in B$ over which there is a singular fibre with a double point of type $j$. Let $\delta_j(f)$ be the total number of such points. Moriwaki’s inequality [Mor, Theorem D] reads:

\[
(8g + 4) \deg f_*\omega_{X/B} \geq g\delta_0(f) + 4 \sum_{j=1}^{[\frac{g}{2}]} j(g - j)\delta_j(f). \tag{11}
\]

To see that this is a refinement of the slope inequality, observe that by Lemma 1.3 $\sum_{s \in B} \delta_s$ is the total number of double points $= \sum_{j=0}^{[\frac{g}{2}]} \delta_j(f)$ and so the right hand side of Moriwaki’s inequality reads

\[
g(\sum_{s \in B} \delta_s) + \sum_{j=1}^{[\frac{g}{2}]} (4j(g - j) - g) \delta_j(f) = g(12 \deg f_*\omega_{X/B} - \omega^2_{X/B}) + \sum_{j=1}^{[\frac{g}{2}]} (4j(g - j) - g) \delta_j(f).
\]

Note that the second line uses Lemma 1.3(iv). Dividing the inequality (11) by $g \cdot \deg f_*\omega_{X/B}$ then indeed leads to the sharpening of (3):

\[
\lambda(f) \geq 4 - \frac{4}{g} \frac{1}{\deg f_*\omega_{X/B}} \left[ \sum_{j=1}^{[\frac{g}{2}]} \left( \frac{4j(g - j)}{g} - 1 \right) \delta_j(f) \right] \tag{12}
\]

4 Application of the Slope Inequality to Arakelov Maximal Fibrations

Recall [Fal] the Arakelov inequality

\[
\deg(f_*\omega_{X/B}) \leq \frac{1}{2} g \cdot \deg \Omega_B^1(\log(\Sigma)). \tag{13}
\]
Suppose that equality holds. Then one calls $f$ strictly Arakelov–maximal.

**Theorem 4.1.** For a non-isotrivial strictly Arakelov maximal genus $g \geq 2$ fibration $f : X \to B$ of semi-stable curves one has $g \leq 4$.

**Proof:** Step 1. One first has to argue that $X$ is of general type. Since $g \geq 2$, if $b \geq 2$ this would follow from the Iitaka-inequality [B-P-H-V, Theorem 18.4]. So one may assume that $b = 0$ or $b = 1$. If $b = 1$, from Lemma 1.1 one gets $c_1^2 = \omega_{X/B}^2 > 0$ because of (2). Now Riemann-Roch on $B$ for $f_* \omega_X^\otimes n$ yields for the plurigenera

$$P_n(X) \geq n \deg(f_* \omega_{X/B}) + n(2b - 2) + 1 - b.$$  

So, if $b = 1$, the Kodaira dimension is 1 or 2. Also, because of (2) $c_1^2(X) = \omega_{X/B}^2 > 0$. Then, by the Enriques-Kodaira classification [B-P-H-V, Table 10] $X$ must be of general type.

Next, let $b = 0$. One may assume that $g \geq 5$. The Arakelov equality reads

$$\deg(f_* \omega_{X/C}) = \frac{1}{2} g(-2 + \# \Sigma) \geq 3$$

and so, as before, the Kodaira dimension of $X$ is 1 or 2. By Xiao’s slope inequality (3), $\omega_{X/B}^2 \geq \left( 4 - \frac{4}{g} \right) \deg f_* \omega_{X/B} \geq 10$ and so, by Lemma 1.1 one has $c_1^2(X) > 0$ and so, as before, by classification (loc. cit.), the Kodaira dimension has to be 2.

Step 2: Kodaira fibrations cannot be Arakelov maximal. Indeed, $\omega_{X/B}^2 = 12 \deg f_* \omega_{X/B} = 12g(b - 1)$ while, by Cor. 2.3 $\omega_{X/B}^2 \leq 4(g - 1)(b - 1) < 12g(b - 1)$. It follows that at least one $\delta_i(f)$ is non-zero.

Step 3: The case where $\delta_j(f) > 0$ for some $j > 0$. Since equality is assumed to hold in (13), one may substitute $\deg(f_* \omega_{X/C}) = \frac{1}{2} g \cdot \deg \Omega^1_B(\log(\Delta))$ in the expression for the slope and so (10) now becomes

$$\lambda(f) \leq 4 - \frac{4}{g} - \frac{1}{\deg f_* \omega_{X/B}} \left[ \sum_{s \in \Sigma_c} 3\delta_s + E^2 \right].$$

Now apply Eqn. (12). One gets

$$\sum_{j=1}^{[\frac{g}{2}]} \left( \frac{4j(g - j)}{g} - 1 \right) \delta_j(f) - 3 \sum_{s \in \Sigma_c} \delta_s - E^2 \leq 0 \quad (14)$$

with equality if $\beta(X, C) = 0$. Rewrite this as a sum of terms involving $\delta_j(f)$. Three terms involve double points of type 1: the first sum which contributes
3 − \frac{4}{g}$, the second which contributes $−3$, and, finally the elliptic tail $E_s$ which contributes $−E_s^2 ≥ 1$. It follows that the total coefficient of $\delta_1(f)$ is at least $1 − \frac{4}{g}$. For the other double points, the first term contributes at least $7 − \frac{16}{g}$ and the second $−3$ (if $s ∈ Σ_c$) or $0$ (if $s ∈ Σ$). Hence, in total, one gets:

\[
0 \geq \sum_{j=1}^{\lfloor \frac{g}{2} \rfloor} \left( \frac{4j(g-j)}{g} - 1 \right) \delta_j(f) - 3 \sum_{s ∈ Σ_c} \delta_s - E^2
\]

\[
\geq (1 - \frac{4}{g}) \cdot \delta_1(f) + \left[ 4 - \frac{16}{g} \right] \cdot \sum_{j ≥ 0} \delta_j(f)
\]

It follows that $g ≤ 4$.

Step 4: The case $\delta_j(f) = 0$ for $j > 0$. I shall argue that this case does not occur, thereby completing the proof. Note that there are no elliptic tails as well and (14) is an equality so that $\beta(X, C) = 0$. This implies that the right hand side in the inequality (7) is also zero and so $X' = X$. In particular, $X − C$ is a locally symmetric space of rank 1. Let $X^*$ be its Baily-Borel compactification. It consists of $X − C$ together with some zero-dimensional boundary components. Since $X^*$ is the minimal compactification, there is a holomorphic map $X → X^*$ which necessarily must contract the curves from $C$, a non-empty union of semi-stable fibers, each consisting of an irreducible curve of arithmetic genus $g$ plus, possibly some rational $(-2)$-curves. However, the genus $g$ curves in these fibres all have self-intersection $0$ and hence cannot be blown down. This contradiction shows that this case does not occur.

\[\square\]

\textbf{References}

[B-P-H-V] Barth, W., C. Peters, K. Hulek and A. van de Ven: Compact Complex Surfaces (second enlarged edition) Springer Verlag (2004)

[Fal] Faltings, G.: Arakelov’s theorem for abelian varieties, Inv. Math. \textbf{73} (1983) 337–348

[Ch-Y] Cheng, S. Y. and S.-T. Yau: Inequality between Chern numbers of singular Kähler surfaces and characterizations of orbit space of
discrete group of $SU(2,1)$, Contemporary Math., 49, World Sci. Publishing, Singapore, (1986), 31–43

[Ko] Kobayashi, R.: Einstein-Kaehler metrics on open algebraic surfaces of general type, Tohoku Math. J. 37 (1985) 43–77

[L-Z] Lu, X. and K. Zuo: On Shimura curves in the Torelli locus of curves. http://arxiv.org/abs/1311.5858

[Miy] Miyaoka, M.: The maximal number of quotient singularities on surfaces with given numerical invariants. Math. Ann. 268 (1984) 159–171

[Mor] Moriwaki, : Relative Bogomolov’s inequality and the cone of positive divisors on the moduli space of stable curves, J. Am. Math. Soc. 11 (19989) 569–600

[Sak] Sakai, F.: Semistable curves on algebraic surfaces and logarithmic pluricanonical maps. Math. Ann. 254 (1980) 89—120.

[Xiao] Xiao, G.: Fibered algebraic surfaces with low slope, Math. Ann. 276 (1987) 125–148