CONVERGENCE OF DEPTHS AND DEPTH-TRIMMED REGIONS

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Depth is a concept that measures the 'centrality' of a point in a given data cloud or in a given probability distribution. Every depth defines a family of so-called trimmed regions. For statistical applications it is desirable that with increasing sample size the empirical depth as well as the empirical trimmed regions converge almost surely to their population counterparts.

In this article the connections between different types of convergence are discussed. In particular, conditions are given under which the pointwise (resp. uniform) convergence of the data depth implies the pointwise (resp. compact) convergence of the trimmed regions in the Hausdorff metric as well as conditions under which the reverse implications hold. Further, it is shown that under relative weak conditions the pointwise convergence of the data depth (resp. trimmed regions) is equivalent to the uniform convergence of the data depth (resp. compact convergence of the trimmed regions).

1. Introduction. In recent years data depth has been increasingly studied and is more and more used in multivariate statistics. Applications of data depth in multivariate statistics include the construction of multivariate rank tests (Liu, 1992; Liu and Singh, 1993; Dyckerhoff, 2002), development of multivariate control charts (Liu, 1995), construction of confidence regions (Yeh and Singh, 1997), multivariate data analysis (Liu, Parelius and Singh, 1999), cluster analysis (Hoberg, 2000, 2003), outlier detection (Cramer, 2003), multivariate risk measurement (Cascos and Molchanov, 2007), classification (Mosler and Hoberg, 2006; Lange, Mosler and Mozharovskyi, 2014), and robust linear programming (Mosler and Bazovkin, 2014).

Data depth is a function which quantifies the 'centrality' of a point in a given probability distribution. Closely related to the notion of depth is the notion of central regions or depth-trimmed regions. Every depth defines a family of central regions in the following way. The $\alpha$-trimmed region consists of all points that have a depth of at least $\alpha$ w.r.t. a given distribution. This is the set of points that have a certain degree of centrality and thus, this set is also called the $\alpha$-trimmed region. Since every depth defines a family
of central regions and vice versa, the concepts of depth and central regions are in a sense equivalent.

Many depths have been proposed in the literature, e.g., the Mahalanobis depth (Mahalanobis, 1936), halfspace depth (Tukey, 1975), simplicial depth (Liu, 1988, 1990), majority depth (Singh, 1991), projection depth (Liu, 1992; Zuo and Serfling, 2000a; Zuo, 2003; based on a notion of outlyingness proposed by Stahel, 1981; Donoho, 1982), zonoid depth (Koshevoy and Mosler, 1997), weighted-mean depth (Dyckerhoff and Mosler, 2011, 2012) and others. These depths differ in many aspects, particularly in the shape of trimmed regions or the deepest point. However, they share certain properties which can be seen as desirable properties every depth should satisfy. We define a depth as a function that satisfies certain postulates which are stated in Dyckerhoff (2004). Slightly differing sets of postulates have been given in Liu (1990) and Zuo and Serfling (2000a).

In statistical applications the empirical depth, i.e., the depth w.r.t. the empirical measure defined by a sample $X_1, \ldots, X_n$, is used as an estimator for the depth w.r.t. the underlying distribution $P_X$. The same holds for the empirical depth-trimmed regions. So the question of almost sure convergence of the empirical quantities to their population counterparts is of crucial interest since it is equivalent to strong consistency of these estimators.

The convergence of depths and depth-trimmed regions has been extensively studied in the literature, e.g., for the the halfspace depth (Eddy, 1985; Donoho and Gasko, 1992; Nolan, 1992; Masse and Theodorescu, 1994; Masse, 2002, 2004), for the simplicial depth (Liu, 1990; Dümbgen, 1992), for the zonoid depth (Koshevoy and Mosler, 1997; Cascos and López-Díaz, 2016), for the $\alpha$-trimming (Casos and López-Díaz, 2008), for the projection depth (Zuo and Serfling, 2000b; Zuo, 2006), for general type D depth functions (Zuo and Serfling, 2000b), for generalized quantile functions defined by depth-trimmed regions (Serfling, 2002a), for weighted-mean trimmed regions (Dyckerhoff and Mosler, 2012).

Results for general depths (but mainly for elliptically contoured distributions) can be found in He and Wang (1997). A generalization of these results for unimodal distributions with uniformly bounded and positive everywhere density is given by (Kim, 2000). For general depths and without assumptions on the distributions first results on the connection between convergence of depths and convergence of trimmed regions have been established in Zuo and Serfling (2000b).

In the current paper we extend the results of Zuo and Serfling (2000b). In particular, we consider neither special depths nor special distributions. Instead, the results hold for all depths that satisfy the postulates of Dyckerhoff
Further we pose no restrictions on the considered distributions such as ellipticity or unimodality. In particular we answer the following questions: Under what conditions does pointwise (resp. uniform) convergence of the depth functions imply pointwise (resp. compact) convergence of the depth-trimmed regions and vice versa? Under what conditions does pointwise convergence of the depth functions (resp. trimmed regions) imply uniform convergence of the depth functions (resp. compact convergence of the trimmed regions)?

The paper is organized as follows. In Section 2 we define data depth in an abstract way as a function that satisfies a certain set of axioms. Section 3 contains two results on the continuity of the depth function and the trimmed regions. The main results on convergence of depths and depth-trimmed regions as well as some applications are then given in Section 4. The proofs of the main results are collected in Appendix A. Some important results on Hausdorff convergence of sets are stated in Appendix B.

In this paper we use the following notation. The complement of a set $A$ is denoted by $A^c$. Interior, closure, and boundary of a set $A$ are denoted by $\text{int} A$, $\text{cl} A$ and $\partial A$, respectively. The Hausdorff distance of two non-empty compact sets $A$ and $B$ is denoted by $\delta_H(A, B)$. The Hausdorff-limit of a sequence $(A_n)_{n \in \mathbb{N}}$ of non-empty compact sets is denoted by $\text{H-lim}_{n \to \infty} A_n$.

2. A general concept of data depth. We consider the depth of a point w.r.t. a probability distribution. Let $\mathcal{M}_0$ be the set of all probability measures on $(\mathbb{R}^d, \mathcal{B}^d)$, where $\mathcal{B}^d$ denotes the Borel $\sigma$-algebra on $\mathbb{R}^d$, and $\mathcal{M}$ a subset of $\mathcal{M}_0$. A depth assigns to each probability measure $P \in \mathcal{M}$ a real function $D(\cdot \mid P) : \mathbb{R}^d \to \mathbb{R}_+$, the so-called depth function w.r.t. $P$. The set of all points that have a depth of at least $\alpha$ is called the $\alpha$-trimmed region or $\alpha$-central region. The $\alpha$-trimmed region w.r.t. $P$ is denoted by $D_\alpha(P)$, i.e., $D_\alpha(P) = \{z \in \mathbb{R}^d \mid D(z \mid P) \geq \alpha\}$.

Often, the probability measure is the distribution $P_X$ of a $d$-variate random vector $X$. Since every probability measure $P$ on $(\mathbb{R}^d, \mathcal{B}^d)$ can be represented as the distribution of a $d$-variate random vector $X$, every statement about depths can either be expressed in terms of probability measures or in terms of random vectors.

We now state some axioms that every reasonable notion of depth should satisfy.

**D1: Affine invariance.** For every regular $d \times d$-Matrix $A$ and $b \in \mathbb{R}^d$ holds $D(z \mid P) = D(Az + b \mid P_{Ax+b})$, where $P_{Ax+b}$ denotes the image measure of $P$ under the transformation $x \mapsto Ax + b$.

**D2: Vanishing at infinity.** $\lim_{\|z\| \to \infty} D(z \mid P) = 0$. 
D3: Upper semicontinuity. For each $\alpha > 0$ the set $D_\alpha(P)$ is closed.

D4: Monotone on rays. For each $x_0$ of maximal depth and each $r \in \mathbb{R}^d$, $r \neq 0$, the function $\lambda \mapsto D(x_0 + \lambda r \, | \, P)$, $\lambda \geq 0$, is monotone decreasing.

D4′: Quasiconcavity. For every $\alpha \geq 0$ the set $D_\alpha(P)$ is convex.

The properties D1, D2, and D4 have been introduced by Liu (1990). A further set of axioms for a depth has been given by Zuo and Serfling (2000a). The main difference between their axioms and ours is that they do not require a depth to be upper semicontinuous. In addition, they require that for distributions having a properly defined unique center, the depth attains its maximum value at this center. However, for centrally symmetric distributions, this follows already from our axioms. For a discussion of these axioms, see e.g., Dyckerhoff (2004).

Definition 2.1. A mapping $D$, that assigns to each probability measure $P$ in a certain set $\mathcal{M}$ a function $D(\cdot \, | \, P) : \mathbb{R}^d \rightarrow \mathbb{R}$ and that satisfies the properties D1, D2, D3 and D4 is called depth. A depth that satisfies D4′ is called convex depth.

A depth always attains its maximum on $\mathbb{R}^d$. We denote this maximum depth by $\alpha_{\text{max}}(P) = \max\{D(z \, | \, P) \mid z \in \mathbb{R}^d\}$. A depth that has the same maximum depth for all probability measures $P$ is called a normed depth.

Properties D1 to D4 are formulated in terms of the depth itself. However, these properties can also be formulated in terms of the trimmed regions. We now state these equivalent properties.

R1: Affine equivariance. For every regular $d \times d$-matrix $A$ and $b \in \mathbb{R}^d$ holds $D_\alpha(P_{Ax+b}) = AD_\alpha(P) + b$.

R2: Boundedness. For every $\alpha > 0$ the region $D_\alpha(P)$ is bounded.

R3: Closedness. For every $\alpha > 0$ the region $D_\alpha(P)$ is closed.

R4: Starshapedness. If $x_0$ is contained in all non-empty regions $D_\alpha(P)$, then the non-empty regions $D_\alpha(P)$ are starshaped w.r.t. $x_0$.

R4′: Convexity. For every $\alpha > 0$ the region $D_\alpha(P)$ is convex.

In Dyckerhoff (2004) it has been shown that each of the statements D1, D2, D3, D4, D4′ is equivalent to the corresponding statement R1, R2, R3, R4, R4′. A further important property that is satisfied by the trimmed regions of a depth is left-continuity of the trimmed regions.

R5: Left-continuity. For every $\alpha > 0$ holds $D_\alpha(P) = \bigcap_{\beta < \alpha} D_\beta(P)$. Further, $\bigcap_{\alpha, \alpha \geq 0} D_\alpha(P) = \emptyset$ for every $P \in \mathcal{M}$.

R5 has been called intersection property in Dyckerhoff (2004).
In particular, it follows from R5 that the $\alpha$-trimmed regions are monotone decreasing in $\alpha$, i.e., for $0 \leq \alpha_1 \leq \alpha_2$ holds $D_{\alpha_2}(P) \subset D_{\alpha_1}(P)$.

At the beginning of this section we have started from a depth and defined the trimmed regions through the relation $D_{\alpha}(P) := \{z \mid D(z \mid P) \geq \alpha\}$. However, one can also start from a family of trimmed regions and define the associated depth by its trimmed regions. In Dyckerhoff (2004) it has been shown that a family $(Z_{\alpha}(P))_{\alpha > 0}$ of subsets of $\mathbb{R}^d$ that satisfies the properties R1 to R5 defines a depth in the sense of Definition 2.1 via

$$D(z \mid P) := \sup\{\alpha \mid z \in Z_{\alpha}(P)\}.$$  

If the sets $Z_{\alpha}(P)$ satisfy also R4', then $D$ is a convex depth.

The idea of generating a depth by a suitable family of nested regions goes back to Barnett (1976) and Eddy (1985). It has already been used by Koshevoy and Mosler (1997) to define the zonoid depth (see Example 2.3 below) and by Serfling (2002b) to define quantile functions. By the above construction the weighted-mean depth can be constructed from the weighted-mean trimmed regions defined in Dyckerhoff and Mosler (2011, 2012).

In the following we give some examples of depths which have already been proposed in the literature.

**Example 2.1.** The Mahalanobis depth (Mahalanobis, 1936) of a point $z$ is defined by

$$MD(z \mid P) = [1 + (z - \mu_P)^\prime \Sigma_P^{-1}(z - \mu_P)]^{-1},$$

where $\mu_P$ denotes the expectation of $P$ and $\Sigma_P$ the covariance matrix of $P$. It is well known that the Mahalanobis depth is a normed convex depth in the sense of Definition 2.1.

**Example 2.2.** The halfspace depth (Tukey, 1975) is defined by

$$HD(z \mid P) = \inf\{P(H) \mid H \text{ is a closed halfspace containing } z\}.$$  

The halfspace depth is a convex depth in the sense of Definition 2.1. A thorough discussion of the properties of the halfspace depth can be found in Rousseeuw and Ruts (1999).

**Example 2.3.** For a probability measure $P$ with finite first moments and $0 < \alpha \leq 1$ the $\alpha$-zonoid trimmed region $ZD_{\alpha}(P)$ is defined by

$$ZD_{\alpha}(P) = \left\{\int xg(x) \, dP \mid g : \mathbb{R}^d \to \left[0, \frac{1}{\alpha}\right], \text{measurable with } \int g(x) \, dP = 1\right\}.$$  

The zonoid depth is then defined by $ZD(z \mid P) = \sup\{\alpha \mid z \in ZD_{\alpha}(P)\}$. The zonoid depth has been introduced by Koshevoy and Mosler (1997). It is a
normed convex depth in the sense of Definition 2.1. The properties of the zonoid depth and the associated zonoid trimmed regions are discussed in Koshevoy and Mosler (1997); Mosler (2002).

**Example 2.4.** For a \( d \)-variate random vector \( X \), the weighted-mean regions \( \text{WMD}_\alpha(P_X) \) (Dyckerhoff and Mosler, 2011, 2012) are defined as the unique convex bodies whose support functions are given by

\[
h(p) = \int_0^1 Q_{p'X}(t) \, dr_\alpha(t), \quad p \in \mathbb{R}^d,
\]

where \( Q_{p'X} \) denotes the quantile function of \( p'X \) and \( r_\alpha \) is a suitable weighting function. The weighted-mean depth, defined by \( \text{WMD}(z \mid P_X) = \sup \{ \alpha \mid z \in \text{WMD}_\alpha(P_X) \} \), is a normed convex depth in the sense of Definition 2.1.

**Example 2.5.** The simplicial depth (Liu, 1988, 1990) and the majority depth (Singh, 1991) are no depths in the sense of Definition 2.1. The simplicial depth fails to satisfy D4 for discrete distributions (see the counterexample in Zuo and Serfling, 2000a). However, restricted to the class of angular symmetric distributions the simplicial depth is a depth in the sense of Definition 2.1. The majority depth does not satisfy D2.

3. Continuity of depths and trimmed regions. In this section we consider the continuity properties of depths and trimmed regions. By D3, any depth in the sense of Definition 2.1 is upper semicontinuous. The following theorem characterizes the depths that are even continuous.

**Theorem 3.1.** Let \( D \) be a depth. Then, the mapping \( z \mapsto D(z \mid P) \), \( z \in \mathbb{R}^d \), is continuous if and only if for all \( \beta > \alpha \) holds

\[
D_\beta(P) \subset \text{int } D_\alpha(P).
\]

The proofs of all theorems in Sections 3 and 4 can be found in Appendix A.

Next, we consider the question, under which conditions the trimmed regions are continuous. Since because of R2 and R3 the trimmed regions \( D_\alpha(P) \) are compact sets, we use the usual notion of convergence for such sets, i.e., convergence in the Hausdorff-metric or short Hausdorff-convergence. The definition as well as some important facts on Hausdorff convergence are given in Appendix B.

Since the \( \alpha \)-trimmed region is the intersection of all \( \beta \)-trimmed regions with \( \beta < \alpha \) (R5), it is not surprising that for every depth in the sense of
Definition 2.1 the mapping \( \alpha \mapsto D_{\alpha}(P) \) is left-continuous. Now, under which conditions is this mapping right-continuous, too? Intuitively, one has to demand, that the \( \beta \)-trimmed region is not much smaller than the \( \alpha \)-trimmed region, whenever \( \beta \) is not much larger than \( \alpha \). The following theorem shows that the trimmed regions are continuous in \( \alpha \) if and only if for every \( \alpha \) the \( \alpha \)-trimmed region is the closure of all points \( z \) with \( D(z) > \alpha \). This condition can be seen as some kind of strict monotonicity of the depth. If \( z \) is a point of depth \( \alpha \in (0, \alpha_{\text{max}}(P)) \) then each neighborhood of \( z \) contains points of depth larger than \( \alpha \). In other words, there is no neighborhood of \( z \) on which the depth is constant. Therefore, if for a probability measure \( P \) a depth satisfies this property, we will say that the depth is strictly monotone for \( P \).

Definition 3.1. Let \( D \) be a depth and \( P \) a probability measure. If for each \( \alpha \in (0, \alpha_{\text{max}}(P)) \) we have

\[
D_{\alpha}(P) = \text{cl} \{ z \in \mathbb{R}^d \mid D(z | P) > \alpha \},
\]

then \( D \) is said to be strictly monotone for \( P \).

Theorem 3.2. Let \( D \) be a depth. Then, the following assertions hold:

(i) The mapping \( \alpha \mapsto D_{\alpha}(P), 0 < \alpha \leq \alpha_{\text{max}}(P) \), is left-continuous w.r.t. the Hausdorff metric, i.e., for every sequence \( (\alpha_n)_{n \in \mathbb{N}} \) such that \( \alpha_n < \alpha_0 \) and \( \lim_{n \to \infty} \alpha_n = \alpha_0 \) we have \( \text{H-lim}_{n \to \infty} D_{\alpha_n}(P) = D_{\alpha_0}(P) \).

(ii) The mapping \( \alpha \mapsto D_{\alpha}(P), 0 < \alpha \leq \alpha_{\text{max}}(P) \), is continuous w.r.t. the Hausdorff metric if and only if \( D \) is strictly monotone for \( P \).

4. Convergence of depths and trimmed regions. In this section we consider the following problem. Let \( P_1, P_2, \ldots \) be a sequence of probability measures on \((\mathbb{R}^d, \mathcal{B}^d)\) and \( P \) a further probability measure on \((\mathbb{R}^d, \mathcal{B}^d)\). If the corresponding depth functions \( D(\cdot | P_n) \) converge to the depth function \( D(\cdot | P) \), what can be said about convergence of the corresponding trimmed regions? Of course the question can be posed also the other way: If the trimmed regions \( D_{\alpha}(P_n) \) converge to \( D_{\alpha}(P) \), what can be said about convergence of the depths? In this paper we make no assumptions on the sequence \( P_1, P_2, \ldots \) of probability measures. However, the main application of our results is the following situation: Let \( X \) be \( d \)-variate random vector with distribution \( P_X \) and \( X_1, X_2, \ldots \) a sequence of random vectors that are independent and identically distributed with distribution \( P_X \). Denote by \( P_n \) the empirical distribution on \( X_1, \ldots, X_n \), i.e., the distribution that assigns probability \( 1/n \) to each of these \( n \) points. The empirical depth \( D(\cdot | P_n) \) then constitutes an estimator for the population depth \( D(\cdot | P_X) \). In the
same way the *empirical trimmed regions* $D_\alpha(P_n)$ are estimators for the population trimmed region $D_\alpha(P_X)$. These estimators are strongly consistent if they converge with probability one to their population counterparts. We will comment on this situation later.

Intuitively, one would assume that if the empirical depths converge, the empirical trimmed regions will converge, too, and vice versa. We will show that in most circumstances this is true. However, there are also situations where the depths converge pointwise or even uniform but the corresponding trimmed regions do not converge for certain values of $\alpha$. Conversely, it may also happen that the trimmed regions do converge whereas the depths themselves do not. If one thinks deeper about this question, this is not surprising. Consider for example the sequence of empirical distribution functions. This sequence converges with probability one uniformly to the theoretical distribution function, whereas the sequence of empirical quantile functions need in general not converge for all $p$. The exact relationships between convergence of depths and convergence of corresponding trimmed regions are in fact far more complicated than this is the case for the relationship between convergence of empirical distribution and quantile functions.

We investigate the connections between the following notions of convergence for depths and depth-trimmed regions.

(PtwD) **Pointwise convergence of depths**
For all $z \in \mathbb{R}^d$ holds $\lim_{n \to \infty} D(z \mid P_n) = D(z \mid P)$.

(ComD) **Compact convergence of depths**
For every compact set $M \subset \mathbb{R}^d$ holds 
$$\lim_{n \to \infty} \sup_{z \in M} |D(z \mid P_n) - D(z \mid P)| = 0.$$ 

(UniD) **Uniform convergence of depths**
It holds 
$$\lim_{n \to \infty} \sup_{z \in \mathbb{R}^d} |D(z \mid P_n) - D(z \mid P)| = 0.$$ 

(PtwR) **Pointwise Hausdorff-convergence of trimmed regions**
For every $\alpha \in (0, \alpha_{\text{max}}(P))$ holds $\lim_{n \to \infty} D_\alpha(P_n) = D_\alpha(P)$.

(ComR) **Compact Hausdorff-convergence of trimmed regions**
For every compact interval $A \subset (0, \alpha_{\text{max}}(P))$ holds 
$$\lim_{n \to \infty} \sup_{\alpha \in A} \delta_H(D_\alpha(P_n), D_\alpha(P)) = 0.$$ 

A further complication results from the following observation. The Hausdorff distance between two sets is undefined whenever one of the sets is
empty. For this reason we have considered only values of \(\alpha\) in \((0, \alpha_{\max}(P))\) in (PtWR) and (ComR). For these values of \(\alpha\) at least \(D_{\alpha}(P_n)\) is non-empty. Of course it cannot be guaranteed that for these \(\alpha\)'s the trimmed regions \(D_{\alpha}(P_n)\) are non-empty for each \(n\), too. However, if we assume (PtWR) and (ComR), it is implicitly assumed that for sufficiently large \(n\) the distances \(\delta_H(D_{\alpha}(P_n), D_{\alpha}(P))\) and \(\sup_{\alpha \in A} \delta_H(D_{\alpha}(P_n), D_{\alpha}(P))\) are defined. For finitely many \(n\) these distances may be undefined.

Conversely it may occur that for every \(n\) the maximum depth \(\alpha_{\max}(P_n)\) w.r.t. \(P_n\) is greater than the maximum depth \(\alpha_{\max}(P)\) w.r.t. \(P\). In this case (PtWR) and (ComR) make assertions on the convergence of trimmed regions \(D_{\alpha}(P_n)\) for \(\alpha \in (0, \alpha_{\max}(P))\), but not on the ‘convergence’ of the trimmed regions \(D_{\alpha}(P_n)\) for \(\alpha > \alpha_{\max}(P)\). Therefore, we occasionally need a further condition that guarantees that the trimmed regions \(D_{\alpha}(P_n)\) converge for \(\alpha > \alpha_{\max}(P)\) to the empty set:

(RC) **Range condition** It holds \(\limsup_{n \to \infty} \alpha_{\max}(P_n) \leq \alpha_{\max}(P)\).

We will use this condition mostly in the following equivalent form:

(RC) **Range condition** For each \(\alpha > \alpha_{\max}(P)\) there exists \(N_\alpha \in \mathbb{N}\), such that \(D_{\alpha}(P_n) = \emptyset\) for all \(n \geq N_\alpha\).

The condition (RC) is trivially satisfied if the maximum depth is the same for all distributions, i.e., for normed depths. Examples of normed depths are the Mahalanobis depth and the zonoid depth. For both of these depths the point of maximum depth has always depth one.

**General assumption:** To avoid technical difficulties we assume in this section that the \(\alpha\)-trimmed regions have full dimension for \(0 < \alpha < \alpha_{\max}(P)\).

This condition is satisfied by the commonly used depths, unless the probability measure \(P\) is concentrated on a hyperplane.

We first prove a theorem that relates the pointwise convergence of depths to the set-theoretic limit of the trimmed regions. The limit inferior and the limit superior of a sequence of sets are defined by

\[
\liminf_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m \quad \text{resp.} \quad \limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.
\]

Clearly, the limit inferior is the set of all points that are eventually contained in all sets \(A_n\), the limit superior is the set of all points that are contained in infinitely many of the sets \(A_n\). Hence, \(\liminf_{n \to \infty} A_n \subseteq \limsup_{n \to \infty} A_n\). If \(\liminf_{n \to \infty} A_n = \limsup_{n \to \infty} A_n\), then the sequence \((A_n)_{n \in \mathbb{N}}\) is said to converge (in the set-theoretic sense) and one defines

\[
\lim_{n \to \infty} A_n := \liminf_{n \to \infty} A_n = \limsup_{n \to \infty} A_n.
\]
Theorem 4.1. The following two statements are equivalent:

(i) (PtwD)

(ii) For every $\alpha \geq 0$ holds

$$\{ z \in \mathbb{R}^d \mid D(z \mid P) > \alpha \} \subset \liminf_{n \to \infty} D_{\alpha}(P_n) \subset \limsup_{n \to \infty} D_{\alpha}(P_n) \subset D_{\alpha}(P).$$

Remark: Since the set $\{ z \in \mathbb{R}^d \mid D(z \mid P) > \alpha \}$ is non-empty for $\alpha \in (0, \alpha_{\text{max}}(P))$ the set $\liminf_{n \to \infty} D_{\alpha}(P_n)$ is non-empty as well. In particular, it follows from the preceding theorem that the trimmed regions $D_{\alpha}(P_n)$ are eventually non-empty for every $\alpha \in (0, \alpha_{\text{max}}(P))$.

The preceding theorem shows that in general the set-theoretic convergence of trimmed regions does not follow from the pointwise convergence of depths. This conclusion is justified only when $\{ z \in \mathbb{R}^d \mid D(z \mid P) > \alpha \} = D_{\alpha}(P)$. However, if the depth is continuous for $P$, this condition is never satisfied.

If one considers not the set-theoretic convergence but the Hausdorff convergence of trimmed regions the following theorem can be deduced. It says that for convex and strictly monotone depths the pointwise convergence of depths implies the Hausdorff convergence of the trimmed regions.

Theorem 4.2. Let $D$ be a convex depth and let $D$ be strictly monotone for $P$. Then, $(\text{PtwD}) \implies (\text{PtwR})$.

If $D_{\alpha_{\text{max}}(P)}(P)$ is a singleton and if the trimmed regions $D_{\alpha_{\text{max}}(P)}(P_n)$ are eventually non-empty, then $(\text{PtwR})$ holds also for $\alpha = \alpha_{\text{max}}(P)$.

As is shown in Example 2.1 in the supplement (Dyckerhoff, 2017), without the assumption of strict monotonicity the Hausdorff convergence of the trimmed regions can in general not be concluded.

The following theorem shows that for continuous convex depths the pointwise convergence of depths follows from the Hausdorff convergence of the trimmed regions.

Theorem 4.3. Let $D$ be a convex depth and let $D$ be continuous for $P$. Then, $[(\text{PtwR})$ and $(\text{RC})] \implies (\text{PtwD}).$

An example that shows that without the assumption of continuity the above theorem is in general false is given in Example 2.3 in the supplement (Dyckerhoff, 2017).

We will now study what are the implications of compact or uniform convergence of depths. On the one hand the following theorem shows that compact and uniform convergence of depths are in fact equivalent. On the other
hand it gives two conditions on the trimmed regions that are equivalent to the uniform convergence of depths. Essentially, these two conditions state that for sufficiently large \( n \) the empirical trimmed regions \( D_\alpha(P_n) \) lie between the trimmed regions \( D_{\alpha+\epsilon}(P) \) and \( D_{\alpha-\epsilon}(P) \). Further, it will be shown that – in contrast to the pointwise convergence – the uniform convergence implies the condition \((RC)\).

**Theorem 4.4.** The following statements are equivalent:

1. \((\text{UniD})\) (UniD)
2. \((\text{ComD})\) (ComD)
3. For every \( \alpha > 0 \) and every \( \epsilon > 0 \) there exists an \( N_{\alpha,\epsilon} \in \mathbb{N} \), such that
   \[
   D_{\beta+\epsilon}(P) \subset D_\beta(P_n) \subset D_{\beta-\epsilon}(P) \quad \text{for all } n \geq N_{\alpha,\epsilon} \text{ and } \beta \geq \alpha.
   \]
4. It holds \((\text{RC})\) and for every compact interval \( A \subset (0, \alpha_{\max}(P)) \) and \( \epsilon > 0 \) there exists an \( N_{A,\epsilon} \in \mathbb{N} \), such that
   \[
   D_{\beta+\epsilon}(P) \subset D_\beta(P_n) \subset D_{\beta-\epsilon}(P) \quad \text{for all } n \geq N_{A,\epsilon} \text{ and } \beta \in A.
   \]

The implication \((\text{ii}) \implies (\text{iii})\) in Theorem 4.4 was already proved in Theorem 4.1 in Zuo and Serfling (2000b).

The following theorem shows that for strictly monotone depths the compact convergence of the depth implies compact Hausdorff convergence of trimmed regions. As in Theorem 4.2, without the assumption of strict monotonicity this statement is in general not valid. This can be seen from Example 2.1 in the supplement (Dyckerhoff, 2017).

**Theorem 4.5.** Let \( D \) be strictly monotone for \( P \). Then, \((\text{ComD}) \implies (\text{ComR})\).

If \( D_{\alpha_{\max}(P)}(P) \) is a singleton and if the trimmed regions \( D_{\alpha_{\max}(P)}(P_n) \) are eventually non-empty, then \((\text{ComR})\) holds also for every compact interval \( A \subset (0, \alpha_{\max}(P)) \).

If the depth is convex and continuous for \( P \) and if the condition \((\text{RC})\) holds, the converse of the preceding theorem holds, i.e., compact Hausdorff convergence of the trimmed regions implies uniform convergence of the depths. Again, as in Theorem 4.3, without the assumption of continuity this does in general not hold, see Example 2.3 in the supplement Dyckerhoff (2017).

**Theorem 4.6.** Let \( D \) be a convex depth that is continuous for \( P \). Then, \([\text{(ComR) and (RC)}] \implies (\text{UniD})\).
We have seen in Theorem 4.4 that for depths compact and uniform convergence are equivalent. We will see in the next two theorems that under mild conditions even pointwise and compact convergence are equivalent. This holds for the convergence of depths as well as for the Hausdorff convergence of trimmed regions.

**Theorem 4.7.** Let $D$ be strictly monotone for $P$. Then, $(\text{PtwR}) \iff (\text{ComR})$.

If $D_{\alpha_{\max}}(P)$ is a singleton, then the open interval $(0, \alpha_{\max}(P))$ can be replaced by the half-open interval $(0, \alpha_{\max}(P)]$ in $(\text{PtwR})$ and $(\text{ComR})$.

The condition of strict monotonicity is crucial in the above theorem. Without strict monotonicity, Theorem 4.7 is not valid as can be seen from Example 2.2 in the supplement (Dyckerhoff, 2017).

We now state the result for convergence of depths. Here the depth has to be convex and continuous for $P$. Again, as in Theorems 4.3 and 4.6 the additional condition $(\text{RC})$ is needed to ensure that pointwise and uniform convergence are equivalent.

**Theorem 4.8.** Let $D$ be a convex depth that is continuous for $P$. Then, $[(\text{PtwD})$ and $(\text{RC})] \iff (\text{UniD})$.

Example 2.4 in the supplement (Dyckerhoff, 2017) shows that without the assumption of continuity the above theorem is in general false.

The connections between the different notions of convergence can be illustrated nicely by a diagram. The following figure shows the implications which result from the preceding theorems as well as the corresponding assumptions. For better clarity we have replaced the implication arrows by simple arrows.

**Remark:** The condition that the depth be strictly monotone for $P$ can also be seen as a continuity condition. In fact it follows from Theorem 3.2 that the mapping $\alpha \mapsto D_\alpha(P)$ is continuous in this case. Thus, one could as
well replace the condition ‘$D$ is strictly monotone for $P$’ by ‘the mapping $\alpha \mapsto D_\alpha(P)$ is continuous’.

For the important class of normed and convex depths one gets the following connections:

\[
\begin{align*}
(P_{twD})_{\text{strictly monotone}} & \quad \iff \quad (P_{twR})_{\text{strictly monotone}} \\
(P_{twD})_{\text{continuous}} & \quad \iff \quad (P_{twR})_{\text{continuous}} \\
(UniD)_{\text{strictly monotone}} & \quad \iff \quad (ComR)_{\text{continuous}}
\end{align*}
\]

A typical application of the above theorems arises when $X_1, X_2, \ldots$ is a sequence of $d$-variate random vectors, defined on a joint probability space $(\Omega, \mathcal{A}, P)$, that are independent and identically distributed with distribution $P_X$; in symbols $X_1, X_2, \ldots \overset{iid}{\sim} P_X$. Then, let $P_n$ be the empirical measure on $X_1, \ldots, X_n$, i.e., $P_n = 1/n \sum_{i=1}^n \varepsilon_{X_i}$, where $\varepsilon_{X_i}$ denotes the one-point measure on $X_i$. Note that $P_n$ is in fact a random measure since it depends on the concrete realizations $X_1(\omega), \ldots, X_n(\omega)$. It is well known that with probability one the empirical measures converges weakly to the distribution $P_X$. In this situation all of the above theorems have corollaries like the following.

**Corollary 4.1 (to Theorem 4.8).** Let $X_1, X_2, \ldots \overset{iid}{\sim} P_X$ and $P_n$ be the empirical measure on $X_1, \ldots, X_n$. Let further $D$ be a convex depth that is continuous for $P_X$. Then,

\[
| (P_{twD}) \text{ and } (RC) | \quad P\text{-almost surely} \iff (UniD) \quad P\text{-almost surely}.
\]

Analogous corollaries hold for all of the above theorems. We do not state them here to avoid unnecessary repetitions.

Here, one has to be careful to distinguish between the probability measure $P$ on the underlying probability space $(\Omega, \mathcal{A}, P)$ and the probability measure $P_X$ that is the distribution of each of the random variables $X_i$. The depth is computed w.r.t. the distribution $P_X$, whereas ‘$P$-almost surely’ refers to the measure $P$ of the underlying probability space.

We illustrate the application of the above results with some examples.

**Example 4.1 (Mahalanobis depth, see Example 2.1).** The Mahalanobis depth is a normed convex depth. It is continuous and strictly monotone for each $P$. From the strong law of large numbers follows (with probability one)
the pointwise convergence of the empirical Mahalanobis depth. Thus, with probability one, the empirical Mahalanobis depth converges uniformly to its population version and the empirical $\alpha$-trimmed regions converge compactly on $(0,1)$.

**Example 4.2** (Halfspace depth, see Example 2.2). The halfspace depth is a convex depth. It is continuous for distributions with density. Under some additional assumptions on $P$ (e.g., convex support) it is also strictly monotone. It is easy to show that with probability one the halfspace depth converges pointwise and the range condition is satisfied. Thus, under the above conditions, with probability one, the empirical halfspace depth converges uniformly to its population version and the empirical $\alpha$-trimmed regions converge compactly on $(0, \alpha_{\max}(P))$.

**Example 4.3** (Zonoid depth, see Example 2.3). The zonoid depth is a normed convex depth that is strictly monotone and continuous for distributions with density. It was shown in Mosler (2002) that, with probability one, the empirical zonoid regions converge pointwise to their population version. Thus, for distributions with density, the empirical zonoid depth converges uniformly to its population version and the empirical $\alpha$-trimmed regions converge compactly on $(0, 1)$.

**Example 4.4** (Asymmetric Mahalanobis depth). The asymmetric Mahalanobis depth (see Dyckerhoff, 2004) is defined by

$$
\text{AMD}(z \mid P) = \inf_{p \in S^{d-1}} \left[ 1 + \left( \frac{p'z - \mu_p'P}{\sigma_p'P} \right) \right]^{-1},
$$

where $\sigma_P^2$ denotes the upper semi-variance of $P$ and $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$. The asymmetric Mahalanobis depth is a normed convex depth. It is continuous and strictly monotone for each $P$. From the strong law of large numbers follows (with probability one) the pointwise convergence of the empirical asymmetric Mahalanobis depth. Thus, with probability one, the empirical asymmetric Mahalanobis depth converges uniformly to its population version and the empirical $\alpha$-trimmed regions converge compactly on $(0,1)$.

**Example 4.5** (Weighted-mean depth, see Example 2.1). The weighted-mean depth is a normed convex depth that is strictly monotone and continuous for distributions with density. Let $(X_n)$ be a sequence of random vectors with finite first moments that converges in distribution to a random vector $X$. It was shown in Dyckerhoff and Mosler (2012) that the weighted-mean
regions $WMD_\alpha(X_n)$ are pointwise Hausdorff convergent to $WMD_\alpha(X)$ provided the sequence $(X_n)$ is uniformly integrable. Thus, it follows from the above theorems that the weighted-mean regions are even compact convergent on $(0, 1]$. For distributions with density the associated depth functions converge uniformly on $\mathbb{R}^d$.

APPENDIX A: PROOFS OF THE MAIN THEOREMS

In this section we use the following notation: For a given depth $D$ and $n \in \mathbb{N}$ we denote the $\alpha$-trimmed region $D_\alpha(P_n)$ w.r.t. $P_n$ shortly by $D^n_\alpha$ and the depth $D(z | P_n)$ of a point $z \in \mathbb{R}^d$ w.r.t. $P_n$ shortly with $D^n_\alpha(z)$. We use the notation $D_\alpha$ for $D_\alpha(P)$ and $D(z)$ for $D(z | P)$ in the same way. In the same spirit we often write simply $\alpha_{\max}$ instead of $\alpha_{\max}(P)$.

Proof (of Theorem 3.1): Assume that $D_\beta \not\subset \text{int } D_\alpha$ for some $\beta > \alpha$. Then, there exists $z \in D_\beta$ that is contained in $\partial D_\alpha$. Obviously, $D_\alpha \neq \mathbb{R}^d$ in this case. Since $D_\alpha$ is closed, its complement is open and there exists a sequence $(z_n)_n \in \mathbb{N}$ in $D_\alpha^c$ converging to $z$. But $\limsup_{n \to \infty} D(z_n) \leq \alpha < \beta \leq D(z)$. Thus, the mapping $z \mapsto D(z)$ is not continuous.

Now assume that $D_\beta \subset \text{int } D_\alpha$ for all $\beta > \alpha$. We show that $D(\cdot | P)$ is lower semicontinuous, i.e., every set $\{z | D(z) > \alpha\}$ is open. For each $z_0 \in \{z | D(z) > \alpha\}$ there is $\gamma$ such that $D(z_0) =: \beta > \gamma > \alpha$. Since $D_\beta \subset \text{int } D_\gamma$, it follows $z_0 \in \text{int } D_\gamma$. Thus, there is a neighborhood $U$ of $z_0$ such that $U \subset D_\gamma \subset \{z | D(z) > \alpha\}$ which shows that $\{z | D(z) > \alpha\}$ is open. Hence, $D(\cdot | P)$ is lower semicontinuous. Since $D(\cdot | P)$ is also upper semicontinuous, $D(\cdot | P)$ is continuous. □

Proof (of Theorem 3.2): We start with Part (i). We assume w.l.o.g. that the sequence $(\alpha_n)_n \in \mathbb{N}$ is increasing. The sequence of sets $D_{\alpha_n}$ is decreasing and it follows from Proposition B.1 in Appendix B that

$$\lim_{n \to \infty} D_{\alpha_n} = \bigcap_{n=1}^{\infty} D_{\alpha_n} = D_{\alpha_0}.$$ 

To prove Part (ii) it suffices to show that the strict monotonicity for $P$ is equivalent to the right-continuity of the mapping $\alpha \mapsto D_\alpha$. This mapping is right continuous if and only if $\lim_{\alpha_n \to \infty} D_{\alpha_n} = D_{\alpha_0}$ for every sequence $(\alpha_n)_n \in \mathbb{N}$ that is decreasing to $\alpha_0 \in (0, \alpha_{\max}(P))$. Since the sequence of sets $D_{\alpha_n}$ is increasing it follows from Proposition B.1 in Appendix B that

$$\lim_{n \to \infty} D_{\alpha_n} = \text{cl} \left( \bigcup_{n=1}^{\infty} D_{\alpha_n} \right) = \text{cl} \left( \{z \in \mathbb{R}^d | D(z) > \alpha_0\} \right).$$
Thus, the mapping is right-continuous if and only if
\[
\text{cl}\left(\{z \in \mathbb{R}^d \mid D(z) > \alpha_0\}\right) = D_{\alpha_0}
\]
for each \(\alpha_0 \in (0, \alpha_{\text{max}}(P))\), i.e., if the depth is strictly monotone for \(P\). \(\square\)

**Proof (of Theorem 4.1):** We start with \((i) \implies (ii)\). Let \(z \in D_{\alpha+\epsilon}\), then \(D(z) \geq \alpha + \epsilon\). Since \(\lim_{n \to \infty} D^n(z) = D(z)\) for every \(\epsilon > 0\), there exists an \(N_\epsilon \in \mathbb{N}\) such that
\[
|D^n(z) - D(z)| < \epsilon \quad \text{for all } n \geq N_\epsilon.
\]
This implies
\[
D^n(z) > D(z) - \epsilon \geq \alpha + \epsilon - \epsilon = \alpha.
\]
Thus, \(z \in D^n_{\alpha}\) for all \(n \geq N_\epsilon\) and therefore \(z \in \liminf_{n \to \infty} D^n_{\alpha}\). This shows that
\[
(1) \quad D_{\alpha+\epsilon} \subset \liminf_{n \to \infty} D^n_{\alpha} \quad \text{for all } \epsilon > 0.
\]
Now we assume that \(z \notin D_{\alpha-\epsilon}\), i.e., \(D(z) < \alpha - \epsilon\). From \((i)\) follows that there exists \(N_\epsilon \in \mathbb{N}\), such that
\[
|D^n(z) - D(z)| < \epsilon \quad \text{for all } n \geq N_\epsilon.
\]
Therefore,
\[
D^n(z) < D(z) + \epsilon < \alpha - \epsilon + \epsilon = \alpha.
\]
Thus, \(z \in (D^n_{\alpha})^c\) for all \(n \geq N_\epsilon\), and therefore \(z \in \liminf_{n \to \infty} (D^n_{\alpha})^c = (\limsup_{n \to \infty} D^n_{\alpha})^c\). From this follows
\[
(2) \quad \limsup_{n \to \infty} D^n_{\alpha} \subset D_{\alpha-\epsilon} \quad \text{for all } \epsilon > 0.
\]
From the equations \((1)\) and \((2)\) follows
\[
\{z \mid D(z) > \alpha\} = \bigcup_{\epsilon > 0} D_{\alpha+\epsilon} \subset \liminf_{n \to \infty} D^n_{\alpha} \subset \limsup_{n \to \infty} D^n_{\alpha} \subset \bigcap_{\epsilon > 0} D_{\alpha-\epsilon} = D_{\alpha},
\]
as was to be shown.

We now prove the direction \((ii) \implies (i)\). Let \(z \in \mathbb{R}^d\) such that \(D(z) = \alpha\). We assume that the sequence \((D^n(z))_{n \in \mathbb{N}}\) does not converge to \(\alpha\). Then there is an \(\epsilon > 0\), such that \(|D^n(z) - D(z)| \geq \epsilon\) infinitely often. Thus we have \(D^n(z) \geq \alpha + \epsilon\) or \(D^n(z) \leq \alpha - \epsilon\) for infinitely many \(n\). In the first case \(z \in \limsup_{n \to \infty} D_{\alpha+\epsilon}\). From \((ii)\) follows \(z \in D_{\alpha+\epsilon}\), i.e., \(D(z) \geq \alpha + \epsilon\) in contradiction to \(D(z) = \alpha\). In the second case \(z \in \limsup_{n \to \infty} (D_{\alpha-\epsilon/2})^c =\)
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\( \lim_{n \to \infty} D_{\alpha - \epsilon/2} \). From (ii) follows \( z \notin \{ x \in \mathbb{R}^d \mid D(x) > \alpha - \epsilon/2 \} \), i.e., \( D(z) \leq \alpha - \epsilon/2 \), in contradiction to \( D(z) = \alpha \). \( \square \)

**Proof (of Theorem 4.2):** We show that (PtwD) implies that for every \( \alpha \in (0, \alpha_{\text{max}}) \) and for every \( M > 0 \) the equation

\[
\lim_{n \to \infty} \max\{ \delta(x, D^n_{\alpha}) \mid x \in D_{\alpha} \cap B(0, M) \} = 0
\]

as well as the equation

\[
\lim_{n \to \infty} \max\{ \delta(x, D_{\alpha}) \mid x \in D^n_{\alpha} \cap B(0, M) \} = 0
\]

hold. Since the trimmed regions \( D^n_{\alpha} \) are connected, it then follows from Theorem B.2 that \( H\lim_{n \to \infty} D^n_{\alpha} = D_{\alpha} \).

To show (3) we first show the slightly stronger assertion

\[
\lim_{n \to \infty} \max\{ \delta(x, D^n_{\alpha}) \mid x \in D_{\alpha} \} = 0
\]

Obviously, (5) implies (3). If (5) does not hold, then there exists \( \epsilon > 0 \) and a subsequence \( (x_{n_k})_{k \in \mathbb{N}} \) with \( x_{n_k} \in D_{\alpha} \) such that \( \delta(x_{n_k}, D^n_{\alpha}) > \epsilon \) for all \( k \in \mathbb{N} \). Since \( D_{\alpha} \) is compact, the sequence \( (x_{n_k})_{k \in \mathbb{N}} \) has a convergent subsequence. We therefore assume w.l.o.g. that the sequence \( (x_{n_k}) \) itself is convergent with \( \lim_{k \to \infty} x_{n_k} = x_0 \in D_{\alpha} \). For sufficiently large \( k \) we have \( \| x_0 - x_{n_k} \| < \frac{\epsilon}{2} \) and

\[
\delta(x_0, D^n_{\alpha}) \geq \delta(x_{n_k}, D^n_{\alpha}) - \| x_0 - x_{n_k} \| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2},
\]

i.e.,

\[
B\left(x_0, \frac{\epsilon}{2}\right) \cap D^n_{\alpha} = \emptyset.
\]

Since \( D_{\alpha} \) is the closure of all points with depth greater than \( \alpha \), there exists a point \( z \in B(x_0, \frac{\epsilon}{2}) \) with \( D(z) > \alpha \). It follows from Theorem 4.1 that \( z \in \lim\inf_{n \to \infty} D^n_{\alpha} \), i.e., the sets \( D^n_{\alpha} \) eventually contain \( z \). On the other hand \( z \notin D^n_{\alpha} \) for infinitely many \( k \), contradiction. Thus, (5) and therefore also (3) holds.

We now show that (4) holds for every \( M > 0 \). Assume that this is not the case. Then there is an \( M > 0 \) and an \( \epsilon > 0 \) as well as a sequence \( (x_{n_k})_{k \in \mathbb{N}} \) with \( \| x_{n_k} \| \leq M \) such that \( x_{n_k} \in D^n_{\alpha} \) and \( \delta(x_{n_k}, D_{\alpha}) > \epsilon \) for all \( k \in \mathbb{N} \). Since the sequence \( (x_{n_k})_{k \in \mathbb{N}} \) is bounded there is a convergent subsequence. Again, we assume w.l.o.g. that the sequence itself is convergent to a point \( x_0 \). From the continuity of the mapping \( x \mapsto \delta(x, D_{\alpha}) \) it follows that \( \delta(x_0, D_{\alpha}) \geq \epsilon \).

Since the interior of \( D_{\alpha} \) is non-empty, we can choose \( d \) points \( z_1, \ldots, z_d \) in
int $D_\alpha$ in such a way that $x_0$ and $z_1, \ldots, z_d$ are in general position. It is easy to show that $\lim_{k\to\infty} x_{nk} = x_0$ implies that

$$\text{H-lim}_{k\to\infty} S[x_{nk}, z_1, \ldots, z_d] = S[x_0, z_1, \ldots, z_d]$$

where $S[x_0, z_1, \ldots, z_d]$ denotes the simplex generated by the points $x_0, z_1, \ldots, z_d$. Since $D(z_i) > \alpha$ for $i = 1, \ldots, d$, (PtwD) implies that there exists $N$ such that $D^n(z_i) \geq \alpha$ for $i = 1, \ldots, d$ and $n \geq N$. Because of the convexity of the trimmed regions we also have $S[x_{nk}, z_1, \ldots, z_d] \subset D^n_{\alpha_k}$ for all $k$ with $n_k \geq N$. Now, let $z_0 \in \text{int} S[x_0, z_1, \ldots, z_d] \setminus D_\alpha$. Because of the Hausdorff convergence of the simplices, it follows from Corollary B.1 that there is a $K \in \mathbb{N}$ such that $z_0 \in D^n_{\alpha_k}$ for $k \geq K$. But then, $z_0 \in \limsup_{n \to \infty} D^n_{\alpha}$ in contradiction to $z_0 \notin D_\alpha$. Thus, (4) has to be valid and the Hausdorff convergence of the trimmed regions for $\alpha \in (0, \alpha_{\text{max}})$ is shown.

To show the second part of the theorem we assume w.l.o.g. that $D^n_{\alpha_{\text{max}}} \neq \emptyset$ for all $n \in \mathbb{N}$. According to Theorem 3.2 the mapping $\alpha \mapsto D_\alpha$ is left continuous on $(0, \alpha_{\text{max}}]$. Thus, there is an $\epsilon > 0$ and $\beta < \alpha_{\text{max}}$, such that $\delta_H(D_\beta, D_{\alpha_{\text{max}}}) < \frac{\epsilon}{6}$. From what has already been proven there exists $N \in \mathbb{N}$, such that $\delta_H(D^n_{\beta}, D_\beta) < \frac{\epsilon}{6}$ for all $n \geq N$. Because of $D^n_{\alpha_{\text{max}}}, \ H \subset D^n_{\beta}$ we have $\delta_H(D^n_{\alpha_{\text{max}}}, D^n_{\beta}) \leq \text{diam}(D^n_{\beta})$. It is easy to show that $\delta_H(A, B) < \epsilon$ implies that $\text{diam}(A) < \text{diam}(B) + 2\epsilon$. For $n \geq N$ we thus get

$$\delta_H(D^n_{\alpha_{\text{max}}}, D^n_{\beta}) \leq \text{diam}(D^n_{\beta}) < \text{diam}(D_\beta) + \frac{1}{3}\epsilon < \text{diam}(D_{\alpha_{\text{max}}}) + \frac{2}{3}\epsilon = \frac{2}{3}\epsilon,$$

since the diameter of a singleton is equal to zero. Therefore,

$$\delta_H(D^n_{\alpha_{\text{max}}}, D^n_{\alpha_{\text{max}}}) \leq \delta_H(D^n_{\alpha_{\text{max}}}, D^n_{\beta}) + \delta_H(D^n_{\beta}, D^n_{\alpha_{\text{max}}}) < \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}$$

for all $n \geq N$ and $\text{H-lim}_{n \to \infty} D^n_{\alpha_{\text{max}}} = D_{\alpha_{\text{max}}}$ is shown. \hfill \Box

**Proof (of Theorem 4.3):** Because of Theorem 4.1 we just have to show that for every $\alpha > 0$ holds:

$$\{z \in \mathbb{R}^d \mid D(z) > \alpha\} \subset \liminf_{n \to \infty} D^n_{\alpha} \subset \limsup_{n \to \infty} D^n_{\alpha} \subset D_\alpha.$$

We start to show $\{z \in \mathbb{R}^d \mid D(z) > \alpha\} \subset \liminf_{n \to \infty} D^n_{\alpha}$. If $\alpha \geq \alpha_{\text{max}}$, then $\{z \in \mathbb{R}^d \mid D(z) > \alpha\}$ is empty and the assertion is trivially satisfied. If $\alpha < \alpha_{\text{max}}$ and $D(z) > \alpha$ then because of the continuity of $D(\cdot | P)$ the point $z$ lies in the interior of $D_\alpha$. By assumption $\text{H-lim}_{n \to \infty} D^n_{\alpha} = D_\alpha$. From Corollary B.1 then follows that there is an $N$ such that $z \in D^n_{\alpha}$ for all $n \geq N$. But then, $z \in \liminf_{n \to \infty} D^n_{\alpha}$ as well and $\{z \in \mathbb{R}^d \mid D(z) > \alpha\} \subset \liminf_{n \to \infty} D^n_{\alpha}$. 


We next show $\limsup_{n \to \infty} D^n_\alpha \subset D_\alpha$. If $\alpha > \alpha_{\text{max}}$, then $D^n_\alpha = \emptyset$ for $n \geq N_\alpha$. Thus, $\limsup D^n_\alpha = \emptyset$ and the assertion is satisfied. Now, let $\alpha \leq \alpha_{\text{max}}$ and $D(z) < \alpha$. Then there is $\beta$ with $D(z) < \beta < \alpha$. Thus, $z$ lies in the complement of $D_\beta$. Again, it follows from Corollary B.1 that there is an $N$, such that $z \notin D^n_\beta$ for all $n \geq N$. Thus, $z \notin \limsup D^n_\beta$. Since $\limsup D^n_\alpha \subset \limsup D^n_\beta$ it follows that $z \notin \limsup D^n_\alpha$ and the assertion is proved. \qed

Proof (of Theorem 4.4): $(i) \implies (ii)$ is trivial.

We show $(ii) \implies (iii)$. Since every non-trivial depth assumes at least two values, there is $x_0 \in \mathbb{R}^d$ with $\alpha_0 := D(x_0) > 0$. We show the assertion w.l.o.g. for $0 < \alpha < \alpha_0$ and $\epsilon$ such that $\alpha - \epsilon > 0$ and $\alpha + \epsilon < \alpha_0$. In that case $D_{\alpha-\epsilon}$ is bounded and there is a compact set $M$ such that $D_{\alpha-\epsilon}$ is contained in the interior of $M$. Thus, there is $N_\epsilon$, such that

$$\sup_{x \in M} |D^n(x) - D(x)| \leq \epsilon \quad \text{for all } n \geq N_\epsilon.$$ 

Now let $n \geq N_\epsilon$ and $x \in D_{\beta+\epsilon} \cap M$ with $\beta \geq \alpha$. Then, $x \in M$ and it holds

$$|D^n(x) - D(x)| \leq \epsilon.$$ 

In particular,

$$D^n(x) \geq D(x) - \epsilon \geq (\beta + \epsilon) - \epsilon = \beta,$$

i.e., $x \in D^n_\beta$. Therefore, it is shown that $D_{\beta+\epsilon} \subset D^n_\beta$ for all $n \geq N_\epsilon$ and $\beta \geq \alpha$.

In the following let $n \geq N_\epsilon$. To show that $D^n_\beta \subset D_{\beta-\epsilon}$ for all $\beta \geq \alpha$, first note that

$$D^n(x) \leq D(x) + \epsilon \quad \text{for all } x \in M.$$ 

For $x \in M \setminus D_{\alpha-\epsilon}$ then holds

$$D^n(x) \leq D(x) + \epsilon < (\alpha - \epsilon) + \epsilon = \alpha$$

and therefore $M \setminus D_{\alpha-\epsilon} \subset (D^n_\alpha)^c$. The trimmed regions are star-shaped and therefore connected. Thus, $D^n_\alpha$ is either a subset of $D_{\alpha-\epsilon}$ or a subset of $M^c$. Assume that $D^n_\alpha \subset M^c$. From the choice of $\alpha$ it is clear that $x_0 \in D_{\alpha+\epsilon}$ and because of $D_{\alpha+\epsilon} \subset D^n_\alpha$ we have

$$x_0 \in D_{\alpha+\epsilon} \subset D^n_\alpha \subset M^c \subset (D_{\alpha-\epsilon})^c,$$

i.e., $D(x_0) < \alpha - \epsilon < \alpha_0$, contradiction. Thus, $D^n_\alpha \subset D_{\alpha-\epsilon}$ for all $n \geq N_\epsilon$.

Now let $\beta \geq \alpha$. Then, $D^n_\beta \subset D^n_\alpha \subset D_{\alpha-\epsilon} \subset M$. If $x \in D^n_\beta$, then

$$D(x) \geq D^n(x) - \epsilon \geq \beta - \epsilon,$$
i.e., \( x \in D_{\beta - \epsilon} \). Thus, it is shown that \( D^n_{\beta} \subset D_{\beta - \epsilon} \) for all \( n \geq N_\epsilon \) and \( \beta \geq \alpha \).

All in all we get

\[
D_{\beta + \epsilon} \subset D^n_{\beta} \subset D_{\beta - \epsilon} \quad \text{for all } n \geq N_\epsilon \text{ and } \beta \geq \alpha,
\]
as stated.

(iii) \( \implies \) (iv) is again trivial.

We now show (iv) \( \implies \) (i). We have to show that for every \( \epsilon > 0 \) there exists \( N_\epsilon \), such that

\[
\sup_{x \in \mathbb{R}^d} |D^n(x) - D(x)| \leq \epsilon \quad \text{for all } n \geq N_\epsilon.
\]

Let \( \epsilon > 0 \) be given. We choose \( A = [\epsilon/4, \alpha_{\max} - \epsilon/4] \). According to (iv) there exists \( N \in \mathbb{N} \), such that

\[
D_{\beta + \epsilon} \subset D^n_{\beta} \subset D_{\beta - \epsilon} \quad \text{for all } n \geq N \text{ and } \beta \in A,
\]
and \( D^n_{\alpha_{\max} + \epsilon} = \emptyset \) for all \( n \geq N \). In the following let \( n \geq N \).

**Case 1:** If \( x \in D_{\frac{\epsilon}{2}} \), then \( D(x) =: \gamma \geq \frac{\epsilon}{2} \). Thus, \( x \in D_{\frac{\epsilon}{2}} \) and because of the assumption also \( x \not\in D_{\gamma - \frac{\epsilon}{4}} \). Therefore,

\[
D^n(x) \geq \gamma - \frac{\epsilon}{4} = D(x) - \frac{\epsilon}{4}.
\]

To bound \( D^n(x) \) from above we distinguish two cases:

**Case 1a:** If \( \gamma < \alpha_{\max} - \frac{3}{4} \epsilon \), it follows from \( x \not\in D_{\gamma + \frac{\epsilon}{4}} \), that \( x \not\in D^n_{\gamma + \frac{\epsilon}{4}} \).

From this follows

\[
D^n(x) < \gamma + \frac{\epsilon}{2} = D(x) + \frac{\epsilon}{2}.
\]

**Case 1b:** If \( \gamma \geq \alpha_{\max} - \frac{3}{4} \epsilon \), then \( \gamma + \epsilon \geq \alpha_{\max} + \frac{\epsilon}{4} \) and thus \( D^n_{\gamma + \epsilon} \subset D^n_{\alpha_{\max} + \frac{\epsilon}{4}} = \emptyset \). It follows that

\[
D^n(x) < \gamma + \epsilon = D(x) + \epsilon.
\]

From Case 1 together with Cases 1a and 1b it follows

\[
D(x) - \frac{\epsilon}{4} \leq D^n(x) < D(x) + \epsilon \quad \text{for all } x \in D_{\frac{\epsilon}{2}}
\]

and therefore also

\[
\sup_{x \in D_{\frac{\epsilon}{2}}} |D^n(x) - D(x)| \leq \epsilon.
\]
Case 2: If \( x \notin D_{\frac{\epsilon}{2}} \), then \( D(x) < \frac{\epsilon}{2} \). Further, with \( \beta = \epsilon/4 \) it follows from equation (6) that \( x \notin D^n_{\frac{\beta}{4}} \), i.e., \( D^n(x) < \frac{3\epsilon}{4} \). From this we conclude

\[
|D^n(x) - D(x)| < \frac{3\epsilon}{4} \quad \text{for all } x \notin D_{\frac{\epsilon}{2}}.
\]

and

\[
\sup_{x \notin D_{\frac{\epsilon}{2}}} |D^n(x) - D(x)| \leq \frac{3\epsilon}{4}.
\]

From the two cases we finally get

\[
\sup_{x \in \mathbb{R}^d} |D^n(x) - D(x)| \leq \epsilon \quad \text{for all } n \geq N,
\]

as was to be shown. \( \square \)

Proof (of Theorem 4.5): Let \( [\alpha_1, \alpha_2] \subset (0, \alpha_{\text{max}}) \) and \( \epsilon > 0 \). Because of the strict monotonicity the mapping \( \alpha \mapsto D_\alpha \) is continuous. Since every continuous function on a compact set is uniformly continuous, this mapping is uniformly continuous on \( [\alpha_1/2, \alpha_{\text{max}}] \). Thus, there exists \( \gamma > 0 \), such that

\[
\delta_H(D_{\beta_1}, D_{\beta_2}) < \frac{\epsilon}{2} \quad \text{for all } \beta_1, \beta_2 \in [\alpha_1/2, \alpha_{\text{max}}] \text{ with } |\beta_1 - \beta_2| \leq 2\gamma.
\]

Assume w.l.o.g. that \( \gamma \) is so small, that \( \gamma < \alpha_1/2 \) and \( \alpha_2 \leq \alpha_{\text{max}} - \gamma \). Then it follows that

\[
\delta_H(D_{\beta}, D_{\beta+\gamma}) < \frac{\epsilon}{2} \quad \text{for all } \beta \in [\alpha_1, \alpha_{\text{max}} - \gamma].
\]

If (ComD) is satisfied then it follows from Theorem 4.4, Part (iii), that there exists \( N \in \mathbb{N} \), such that

\[
D_{\beta+\gamma} \subset D^n_{\beta} \subset D_{\beta-\gamma} \quad \text{for all } n \geq N \text{ and } \beta \geq \alpha_1.
\]

Trivially, \( D_{\beta+\gamma} \subset D_{\beta} \subset D_{\beta-\gamma} \) holds as well. Thus, for every \( n \geq N \) and \( \beta \in [\alpha_1, \alpha_{\text{max}} - \gamma] \):

\[
\delta_H(D^n_{\beta}, D_{\beta}) \leq \delta_H(D_{\beta+\gamma}, D_{\beta-\gamma}) < \frac{\epsilon}{2}.
\]

Therefore,

\[
\sup_{\alpha_1 \leq \beta \leq \alpha_2} \delta_H(D^n_{\beta}, D_{\beta}) \leq \frac{\epsilon}{2} \quad \text{for all } n \geq N,
\]

which implies (ComR).
For proving the second part of the Theorem, note that $D_{\beta+\gamma}$ will be empty, when $\beta > \alpha_{\text{max}} - \gamma$. Thus, for $\beta \in (\alpha_{\text{max}} - \gamma, \alpha_{\text{max}}]$ we only have

$$D_{\beta}^n \subset D_{\beta-\gamma} \quad \text{for all } n \geq N.$$  

Trivially, $D_{\beta} \subset D_{\beta-\gamma}$ holds as well. Now, if $D_{\alpha_{\text{max}}}$ is a singleton, i.e., $D_{\alpha_{\text{max}}} = \{x_0\}$, then

$$\delta_H(D_{\beta}^n, D_{\beta}) \leq \delta_H(D_{\beta}^n, D_{\alpha_{\text{max}}}) + \delta_H(D_{\alpha_{\text{max}}}, D_{\beta}) < \delta_H(D_{\beta}^n, \{x_0\}) + \frac{\epsilon}{2} \leq x \in D_{\beta-\gamma} \max_{x \in D_{\beta-\gamma}} \|x - x_0\| + \frac{\epsilon}{2} \leq \delta_H(D_{\beta-\gamma}, \{x_0\}) + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and the proof is finished.

**Proof (of Theorem 4.6):** We show that (ComR) together with (RC) implies Condition (iv) in Theorem 4.4. Thus, we have to show that for each $\epsilon > 0$ and every compact interval $A \subset (0, \alpha_{\text{max}})$ there exists $N_{A,\epsilon} \in \mathbb{N}$, such that

$$D_{\alpha+\epsilon} \subset D_{\alpha}^n \subset D_{\alpha-\epsilon} \quad \text{for all } n \geq N_{A,\epsilon} \text{ and } \alpha \in A.$$  

Let $A = [\alpha_1, \alpha_2]$ and $\epsilon$ be given. We assume w.l.o.g. that $\epsilon < \alpha_1$ and $\epsilon < \alpha_{\text{max}} - \alpha_2$. Since a continuous function, defined on a compact set, is uniformly continuous and since $D_{\alpha_{\text{max}}}^\beta$ is compact, the mapping $z \mapsto D(z|P)$ is uniformly continuous on $D_{\alpha_{\text{max}}}^\beta$. Thus, there is $\gamma > 0$, such that

$$|D(x) - D(y)| < \epsilon \quad \text{for all } x, y \in D_{\alpha_{\text{max}}} \text{ with } \|x - y\| < \delta.$$  

Further,

$$\min_{x \in \partial D_{\alpha+\epsilon}, y \in \partial D_{\alpha}} \|x - y\| \geq \gamma \quad \text{for all } \alpha \in [\alpha_1, \alpha_{\text{max}}].$$

This holds because if the above equation was not satisfied then there was $x \in \partial D_{\alpha_{\text{max}}}^\beta$ and $y \in \partial D_{\alpha}$ such that $\|x - y\| < \gamma$. Since $D(\cdot|P)$ is continuous this would imply $D(x) = \alpha - \epsilon$ and $D(y) = \alpha$. Therefore we would get

$$|D(x) - D(y)| = D(y) - D(x) = \alpha - (\alpha - \epsilon) = \epsilon$$

in contradiction to the uniform continuity.

Now, choose $N_{A,\epsilon}$ so large that $\delta_H(D_{\alpha}^n, D_{\alpha}) < \gamma$ for all $n \geq N_{A,\epsilon}$ and $\alpha \in A$. Since the trimmed regions are convex it follows from Proposition B.3 in the Appendix that

$$D_{\alpha_{\text{max}}}^n \subset D_{\alpha_{\text{max}}} \subset D_{\text{max}} - \epsilon \quad \text{for all } n \geq N_{A,\epsilon} \text{ and } \alpha \in A,$$
as was to be shown. □

In the proofs of Theorems 4.7 and 4.8 we make use of another notion of convergence, the so-called continuous convergence.

**Definition A.1.** Let \((X, \rho_X)\) and \((Y, \rho_Y)\) be metric spaces. A sequence \((f_n)_{n \in \mathbb{N}}\) of mappings from \(X\) to \(Y\) is said to converge continuously to \(f\) if for each \(x \in X\) and for each sequence \((x_n)_{n \in \mathbb{N}}\) such that \(\lim_{n \to \infty} x_n = x\) we have \(\lim_{n \to \infty} f_n(x_n) = f(x)\).

The following well-known result that connects continuous convergence and compact convergence will be useful in the proofs of Theorems 4.7 and 4.8.

**Proposition A.1.** Let \((X, \rho_X)\) and \((Y, \rho_Y)\) be metric spaces and \((f_n)_{n \in \mathbb{N}}\) be a sequence of functions from \(X\) to \(Y\):  

(i) If \(f_n\) converges continuously to \(f\), then \(f\) is continuous.

(ii) If the sequence \((f_n)\) converges continuously to \(f\), then \((f_n)\) is compact convergent to \(f\).

(iii) Let \(X\) be locally compact. If the sequence \((f_n)\) is compact convergent to \(f\) and \(f\) is continuous, then \((f_n)\) converges continuously to \(f\).

**Proof (of Theorem 4.7):** We show that under the given assumptions the sequence of mappings \(\alpha \mapsto D_\alpha^n, n \in \mathbb{N}\), is continuous convergent to the mapping \(\alpha \mapsto D_\alpha\), i.e., for every sequence \((\alpha_n)\) that is convergent to \(\alpha_0\) it holds that \(\text{H-lim}_{n \to \infty} D_{\alpha_n}^n = D_\alpha\). From Proposition A.1 above it then follows that the trimmed regions are compact convergent.

First assume that \(0 < \alpha_0 < \alpha_{\text{max}}\). Because of the strict monotonicity it follows from Theorem 3.2 that the mapping \(\alpha \mapsto D_\alpha\) is continuous. Thus, there is \(\gamma > 0\) such that \(\delta_H(D_{\alpha_0-\gamma}, D_{\alpha_0+\gamma}) < \epsilon/5\). Further, from the point-wise convergence of the trimmed regions it follows that there is an \(N_1 \in \mathbb{N}\) such that \(\delta_H(D_{\alpha_0-\gamma}, D_{\alpha_0-\gamma}^n) < \epsilon/5\) and \(\delta_H(D_{\alpha_0+\gamma}, D_{\alpha_0+\gamma}^n) < \epsilon/5\) for all \(n \geq N_1\). We conclude that for \(n \geq N_1\)

\[
\delta_H(D_{\alpha_0-\gamma}^n, D_{\alpha_0+\gamma}^n) \leq \delta_H(D_{\alpha_0-\gamma}, D_{\alpha_0-\gamma}) + \delta_H(D_{\alpha_0-\gamma}, D_{\alpha_0+\gamma}^n) + \delta_H(D_{\alpha_0+\gamma}, D_{\alpha_0+\gamma}^n) < \frac{3\epsilon}{5}.
\]

Since \((\alpha_n)\) converges to \(\alpha_0\) there is an \(N_2 \in \mathbb{N}\) such that \(|\alpha_n - \alpha_0| < \gamma\). For \(n \geq N_2\) the trimmed region \(D_{\alpha_n}^n\) lies between \(D_{\alpha_0-\gamma}^n\) and \(D_{\alpha_0+\gamma}^n\). Therefore

\[
\delta_H(D_{\alpha_n}^n, D_{\alpha_0-\gamma}^n) \leq \delta_H(D_{\alpha_0+\gamma}, D_{\alpha_0-\gamma}^n) < \frac{3\epsilon}{5}.
\]
for \( n \geq N_2 \). From this follows that for \( n \geq \max\{N_1, N_2\} \)
\[
\delta_H(D^n_{\alpha_n}, D_{\alpha_0}) \leq \delta_H(D^n_{\alpha_n}, D^n_{\alpha_0 - \gamma}) + \delta_H(D^n_{\alpha_0 - \gamma}, D_{\alpha_0 - \gamma}) + \delta_H(D_{\alpha_0 - \gamma}, D_{\alpha_0}) < \epsilon.
\]

Now assume that \( \alpha_0 = \alpha_{\max} \) and \( D_{\alpha_0} = \{x_0\} \) is a singleton. As above there is \( \gamma > 0 \) and \( N_1 \in \mathbb{N} \) such that \( \delta_H(D_{\alpha_0}, D_{\alpha_0 - \gamma}) < \epsilon/2 \) and \( \delta_H(D_{\alpha_0 - \gamma}, D^n_{\alpha_0 - \gamma}) < \epsilon/2 \) for \( n \geq N_1 \). Choose \( N_2 \in \mathbb{N} \) such that \( \alpha_n > \alpha_0 - \gamma \). Then, for \( n \geq N_2 \) the trimmed region \( D^n_{\alpha_n} \) is contained in \( D^n_{\alpha_0 - \gamma} \). For \( n \geq \max\{N_1, N_2\} \) we thus get
\[
\delta_H(D^n_{\alpha_n}, D_{\alpha_0}) \leq \delta_H(D^n_{\alpha_0 - \gamma}, D_{\alpha_0}) \leq \delta_H(D^n_{\alpha_0 - \gamma}, D_{\alpha_0 - \gamma}) + \delta_H(D_{\alpha_0 - \gamma}, D_{\alpha_0}) < \epsilon
\]
and the second part of the theorem is proved. \( \square \)

**Proof (of Theorem 4.8):** We show that under the assumptions the sequence \( (D^n)_{n \in \mathbb{N}} \) is continuous convergent to \( D \), i.e., for every sequence \( (z_n) \) that is convergent to \( z_0 \) it holds that \( \lim_{n \to \infty} D^n(z_n) = D(z_0) \). From Proposition A.1 above it then follows that \( (D^n) \) is compact and thus uniform convergent to \( D \).

We start with showing that for every \( \epsilon > 0 \) there is an \( N \in \mathbb{N} \) such that \( D^n(z_n) \geq D(z_0) - \epsilon \) for all \( n \geq N \). Let \( D(z_0) = \alpha \). Since \( D \) is continuous, \( z_0 \in \text{int} D_{\alpha - \epsilon/4} \). Then, there is \( \gamma > 0 \) such that \( B(z_0, \gamma) \subset \text{int} D_{\alpha - \epsilon/2} \). Choose \( d + 1 \) points \( x_1, \ldots, x_{d+1} \in B(z_0, \gamma) \) such that \( z_0 \in \text{int} S[x_1, \ldots, x_{d+1}] \), where \( S[x_1, \ldots, x_{d+1}] \) denotes the simplex generated by \( x_1, \ldots, x_{d+1} \).

From \( D(x_i) \geq \alpha - \frac{\epsilon}{2}, i = 1, \ldots, d + 1, \) and the pointwise convergence of the depths it follows that there exists \( N_1 \in \mathbb{N} \) such that \( D^n(x_i) \geq \alpha - \epsilon \) for all \( n \geq N_1 \) and \( i = 1, \ldots, d + 1 \). For \( n \geq N_1 \) we thus have \( x_i \in D^n_{\alpha - \frac{\epsilon}{2}}, i = 1, \ldots, d + 1 \). Since the trimmed regions of \( D \) are convex it follows that \( S[x_1, \ldots, x_{d+1}] \subset D^n_{\alpha - \epsilon} \) for \( n \geq N_1 \). Now, because of \( z_0 \in \text{int} S[x_1, \ldots, x_{d+1}] \) there is \( N_2 \in \mathbb{N} \) such that \( z_n \in S[x_1, \ldots, x_{d+1}] \) for all \( n \geq N_2 \). For \( n \geq \max\{N_1, N_2\} \) we therefore have \( z_n \in D^n_{\alpha - \epsilon}, \) i.e., \( D^n(z_n) \geq D(z_0) - \epsilon \).

It remains to show that there exists \( N \in \mathbb{N} \) such that \( D^n(z_n) \leq D(z_0) + \epsilon \) for all \( n \geq N \). Assume the contrary. Then there is a subsequence \( (n_k)_{k \in \mathbb{N}} \), such that
\[
D^{n_k}(z_{n_k}) > D(z_0) + \epsilon = \alpha + \epsilon \quad \text{for all } k \in \mathbb{N}.
\]

For \( \alpha = \alpha_{\max} \) this is a contradiction to (RC). If \( \alpha < \alpha_{\max} \) we assume w.l.o.g. that \( \alpha + \epsilon < \alpha_{\max} \). Since the trimmed regions are closed, their complements are open. Thus, there is \( \gamma > 0 \) such that \( B(z_0, \gamma) \cap D_{\alpha + \epsilon} = \emptyset \). For \( \beta \) with \( \alpha + \epsilon < \beta < \alpha_{\max} \) the trimmed region \( D_{\beta} \) is larger than \( D_{\alpha_{\max}} \) and the interior of \( D_{\beta} \) is non-empty. Choose \( d \) points \( x_1, \ldots, x_d \in D_{\beta} \) such that \( z_0, x_1, \ldots, x_d \) are in general position. Because of pointwise convergence there exists \( K_1 \in \mathbb{N} \) such that \( D^{n_k}(x_i) > \alpha + \epsilon \) for \( k \geq K_1 \) and \( i = 1, \ldots, d \). From the convexity
of the trimmed regions it follows that $S[z_{n_k}, x_1, \ldots, x_d] \subset D^\alpha_{\alpha + \epsilon}$ for $k \geq K_1$. It is easy to show that 

$$\text{H-lim}_{k \to \infty} S[z_{n_k}, x_1, \ldots, x_d] = S[z_0, x_1, \ldots, x_d].$$

Since $\text{int} S[z_0, x_1, \ldots, x_d] \cap B(z_0, \gamma)$ is non-empty there exists $z^* \in \text{int} S[z_{n_k}, x_1, \ldots, x_d] \cap B(z_0, \gamma)$. Because of the Hausdorff convergence of the simplices it follows from Corollary B.1 that there is $K_2 \in \mathbb{N}$ such that $z^* \in S[z_{n_k}, x_1, \ldots, x_d]$ for all $k \geq K_2$. Thus, $D^\alpha_k(z^*) \geq \alpha + \epsilon$ for $k \geq K = \max\{K_1, K_2\}$. From pointwise convergence of the depth it follows $D(z^*) \geq \alpha + \epsilon$ in contradiction to $B(z_0, \gamma) \cap D_{\alpha + \epsilon} = \emptyset$. Thus, the proof is finished.

**APPENDIX B: HAUSDORFF-CONVERGENCE**

In this section we state the definition of Hausdorff convergence as well as some important facts on this notion of convergence. Detailed studies of the notion of Hausdorff convergence can be found, e.g., in Klein and Thompson (1984) and Beer (1993).

The Euclidean distance between two points $x, y \in \mathbb{R}^d$ is given by $\delta(x, y) = \|x - y\|$. The distance between a point $x \in \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$ can then be defined by $\delta(x, A) = \inf_{y \in A} \delta(x, y)$. If $A$ is closed one can write min instead of inf. The set of all non-empty compact subsets of $\mathbb{R}^d$ is denoted by $\mathcal{K}_d^\infty$.

**Definition B.1** (Hausdorff distance). For $A, B \in \mathcal{K}_d^\infty$ the Hausdorff distance $\delta_H(A, B)$ is defined by

$$\delta_H(A, B) = \max\{\max_{x \in A} \delta(x, B), \max_{x \in B} \delta(x, A)\}.$$ 

For $A \subset \mathbb{R}^d$ and $\epsilon > 0$ the $\epsilon$-neighborhood $U_\epsilon(A)$ is given by $U_\epsilon(A) = \{x \in \mathbb{R}^d \mid \delta(x, A) < \epsilon\} = \bigcup_{x \in A} B(x, \epsilon)$. With this notation an equivalent definition of the Hausdorff distance is $\delta_H(A, B) = \inf\{\epsilon > 0 \mid A \subset U_\epsilon(B), B \subset U_\epsilon(A)\}$.

If one of the sets is a singleton, then $\delta_H(\{x_0\}, A) = \max_{x \in A} \delta(x, x_0)$.

If $A_1 \subset B \subset A_2$ then $\delta_H(A_1, B) \leq \delta_H(A_1, A_2)$ as well as $\delta_H(B, A_2) \leq \delta_H(A_1, A_2)$. If $A_1 \subset B_i \subset A_2$, $i = 1, 2$, then $\delta_H(B_1, B_2) \leq \delta_H(A_1, A_2)$.

The Hausdorff distance is a metric on $\mathcal{K}_d^\infty$. Thus, the pair $(\mathcal{K}_d^\infty, \delta_H)$ is a metric space. Therefore it is possible to define convergence of compact sets in the Hausdorff metric or short Hausdorff convergence.

**Definition B.2** (Hausdorff convergence). Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of non-empty compact subsets of $\mathbb{R}^d$. The sequence $(K_n)_{n \in \mathbb{N}}$ is said to be Hausdorff convergent to a set $K \in \mathcal{K}_d^\infty$ if $\lim_{n \to \infty} \delta_H(K_n, K) = 0$. In this case we write $\text{H-lim}_{n \to \infty} K_n = K$. 


Proposition B.1. If a sequence \((K_n)_{n \in \mathbb{N}}\) is decreasing, i.e., \(K_1 \supseteq K_2 \supseteq \ldots\), then the Hausdorff limit exists and is given by \(\text{H-lim}_{n \to \infty} K_n = \bigcap_{n=1}^{\infty} K_n\).

If a sequence \((K_n)_{n \in \mathbb{N}}\) is increasing, i.e., \(K_1 \subseteq K_2 \subseteq \ldots\), and the union of the sets is bounded, then the Hausdorff limit exists and is given by \(\text{H-lim}_{n \to \infty} K_n = \text{cl}(\bigcup_{n=1}^{\infty} K_n)\).

If the sets \(K_n, n \in \mathbb{N}\), are connected then the following criteria is useful.

Proposition B.2. Let \((K_n)_{n \in \mathbb{N}}\) be sequence of connected sets in \(\mathcal{K}_d^0\) and let \(K \in \mathcal{K}_d^0\). If for each \(M > 0\) it holds that
\[
\lim_{n \to \infty} \max\{\delta(x, K_n) \mid x \in K \cap B(0, M)\} = 0
\]
and
\[
\lim_{n \to \infty} \max\{\delta(x, K) \mid x \in K_n \cap B(0, M)\} = 0,
\]
then \(\text{H-lim}_{n \to \infty} K_n = K\).

The following proposition and its corollary show that for convex sets Hausdorff convergence behaves nicely.

Proposition B.3. If \(A_1, A_2\) are convex sets in \(\mathcal{K}_d^0\) such that \(A_1 \subset A_2\) and
\[
\min_{x \in \partial A_1, y \in \partial A_2} \delta(x, y) = \gamma > 0,
\]
then for every convex set \(B \in \mathcal{K}_d^0\) holds
\[
\delta_H(B, A_1) \leq \gamma \implies B \subset A_2,
\]
\[
\delta_H(B, A_2) \leq \gamma \implies A_1 \subset B.
\]

Corollary B.1. Let \((K_n)_{n \in \mathbb{N}}\) be a sequence of convex sets in \(\mathcal{K}_d^0\) with \(\text{H-lim}_{n \to \infty} K_n = K\), where \(K\) is a convex set in \(\mathcal{K}_d^0\). Then the following assertions hold:

(i) For every \(x \in \text{int} \ K\) there is an \(N \in \mathbb{N}\), such that \(x \in K_n\) for all \(n \geq N\).
(ii) For every \(x \in K^c\) there is an \(N \in \mathbb{N}\), such that \(x \notin K_n\) for all \(n \geq N\).
(iii) \(\text{int} \ K \subset \text{lim inf}_{n \to \infty} K_n \subset \text{lim sup}_{n \to \infty} K_n \subset K\).
CONVERGENCE OF DEPTHS

SUPPLEMENTARY MATERIAL

Supplement to “Convergence of depths and depth-trimmed regions” The supplement contains some examples that show that without the assumption ‘strictly monotone for $P$’ Theorems 4.2, 4.5, and 4.7 are in general false, and without the assumption ‘continuous for $P$’ Theorems 4.3, 4.6, and 4.8 are in general false.

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SUPPLEMENT TO “CONVERGENCE OF DEPTHS AND DEPTH-TRIMMED REGIONS”

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1. Introduction. In this supplement we present some examples to show that

(i) without the assumption ‘strictly monotone for $P$’ Theorems 4.2, 4.5, and 4.7 are in general false,

(ii) without the assumption ‘continuous for $P$’ Theorems 4.3, 4.6, and 4.8 are in general false.

2. Examples.

Example 2.1 (‘Strict monotonicity’ is needed in Theorems 4.2 and 4.5). Let $Q_1 = U([-3, -2] \cup [2, 3])$ be the uniform distribution on the union of the two intervals $[-3, -2]$ and $[3, 2]$. Let further $Q_2 = U([-1, 1])$. Now, for $n \in \mathbb{N}$ let

$$P_n = \frac{1}{2} \left( 1 + \frac{(-1)^n}{n} \right) Q_1 + \frac{1}{2} \left( 1 - \frac{(-1)^n}{n} \right) Q_2$$

and finally

$$P_0 = \frac{1}{2} Q_1 + \frac{1}{2} Q_2.$$

The following figure shows the functions $x \mapsto P_0((-\infty, x])$, $x \mapsto P_0([x, \infty))$ as well as the halfspace depth (in blue),

$$x \mapsto \text{HD}(x \mid P_0) = \min\{P_0((-\infty, x]), P_0([x, \infty))\}.$$
Note, that the halfspace depth is not strictly monotone for $P_0$, since

$$\text{HD} \left( \frac{1}{4} \right) \left( P_0 \right) = \left[ -2, 2 \right] \neq \left[ -1, 1 \right] = \text{cl} \left\{ z \in \mathbb{R} \mid \text{HD}(z \mid P_0) > \frac{1}{4} \right\}.$$ 

In the following figure the halfspace depth w.r.t. $P_0$ (blue line) as well as the halfspace depths w.r.t. $P_9$ (red line below blue line) and $P_{10}$ (red line above blue line) are drawn.

It is easy to see that

$$\text{HD}(z \mid P_n) \leq \text{HD}(z \mid P_0) \text{ for all } z \text{, if } n \text{ is odd},$$

$$\text{HD}(z \mid P_n) \geq \text{HD}(z \mid P_0) \text{ for all } z \text{, if } n \text{ is even}.$$

Further,

$$\sup_{z \in \mathbb{R}} \left| \text{HD}(z \mid P_n) - \text{HD}(z \mid P_0) \right| = \frac{1}{4n}.$$ 

Therefore,

$$\lim_{n \to \infty} \sup_{z \in \mathbb{R}} \left| \text{HD}(z \mid P_n) - \text{HD}(z \mid P_0) \right| = 0,$$

i.e., (UniD) and a-fortiori (ComD) and (PtwD) hold.

On the other hand we have

$$\text{HD}(1 \mid P_n) = \text{HD}(2 \mid P_n) < \frac{1}{4} = \text{HD}(2 \mid P_0) \text{ if } n \text{ is odd},$$

$$\text{HD}(1 \mid P_n) = \text{HD}(2 \mid P_n) > \frac{1}{4} = \text{HD}(2 \mid P_0) \text{ if } n \text{ is even}.$$ 

Therefore and because of symmetry of the depth functions, for $m$ odd and $n$ even holds

$$\text{HD}_{0.25}(P_n) \subset ( -1, 1 ) \subset \text{HD}_{0.25}(P_0) = [ -2, 2 ] \subset \text{HD}_{0.25}(P_n).$$

This shows that although the depth functions converge uniformly, the trimmed regions $\text{HD}_{0.25}(P_n)$ do not converge, neither in the Hausdorff sense nor in the set-theoretic sense. Thus, neither (ComR) nor (PtwR) hold. In other words, without the assumption of 'strict monotonicity for $P$', Theorems 4.2 and 4.5 do in general not hold.
Example 2.2 (‘Strict monotonicity’ is needed in Theorem 4.7). This example is similar to the previous one with the main difference that the alternating term \((-1)^n\) is missing. So, let \(Q_1 = U([-3, -2] \cup [2, 3])\) and \(Q_2 = U([-1, 1])\) as in Example 2.1. Now, for \(n \in \mathbb{N}\) let

\[
P_n = \frac{1}{2} \left(1 + \frac{1}{n}\right) Q_1 + \frac{1}{2} \left(1 - \frac{1}{n}\right) Q_2
\]

and finally \(P_0 = \frac{1}{2} Q_1 + \frac{1}{2} Q_2\) as in Example 2.1. In the following figure the halfspace depth w.r.t. \(P_0\) (blue line) as well as the halfspace depth w.r.t. \(P_{10}\) (red line above blue line) are shown.

As was shown in Example 2.1, the halfspace depth is not strictly monotone for \(P_0\).

Since in this example the depth function w.r.t. \(P_n\) lies always above the depth function w.r.t. \(P_0\), the oscillating behavior of the trimmed regions \(\text{HD}_{0.25}(P_n)\) in the previous example does not occur. Instead, for each \(\alpha\) (even for \(\alpha = 0.25\)) the trimmed regions \(\text{HD}_\alpha(P_n)\) converge to \(\text{HD}_\alpha(P_0)\) in the Hausdorff metric. Therefore (PtwR) holds.

However, (ComR) does not hold. This can be seen as follows. For \(n\) arbitrarily, choose \(\alpha_n\) such that

\[
\frac{1}{4} < \alpha_n < \frac{1}{4} \left(1 + \frac{1}{n}\right).
\]

Then,

\[
\text{HD}(1 \mid P_0) = \frac{1}{4} < \alpha_n < \frac{1}{4} \left(1 + \frac{1}{n}\right) = \text{HD}(2 \mid P_n).
\]

Therefore and because of symmetry of the depth function,

\[
\text{HD}_{\alpha_n}(P_0) \subset (-1, 1) \subset [-2, 2] \subset \text{HD}_{\alpha_n}(P_n).
\]

This shows that \(\delta_H(\text{HD}_{\alpha_n}(P_n), \text{HD}_{\alpha_n}(P_0)) \geq 1\). Since this holds for every \(n\), it follows

\[
\sup_{\alpha \in [0.1, 0.5]} \delta_H(D_\alpha(P_n), D_\alpha(P_0)) \geq 1 \quad \text{for all } n,
\]
i.e., there is a compact interval \( A \subset (0, \alpha_{\max}(P_0)) \) on which the trimmed regions do not converge uniformly. In other words, without the assumption of 'strict monotonicity for \( P \)', (PtwR) does not imply (ComR) and thus, Theorem 4.7 does in general not hold.

**Example 2.3** (‘Continuity’ is needed in Theorems 4.3 and 4.6). For \( n \in \mathbb{N}_0 \) define

\[
a_n = \begin{cases} 
1 + \frac{(-1)^n}{n+1}, & \text{if } n \in \mathbb{N}, \\
1 & \text{if } n = 0.
\end{cases}
\]

Now, let \( Q_{1,n} = U([-2, -a_n] \cup [a_n, 2]) \) be the uniform distribution on the union of the intervals \([-2, -a_n]\) and \([a_n, 2]\), \( Q_{2,n} = U([-a_n, a_n]) \) and \( Q_{3,n} = 0.5\delta_{-a_n} + 0.5\delta_{a_n} \), where \( \delta_x \) denotes the one-point measure on \( x \). For \( n \in \mathbb{N}_0 \) consider the probability measures

\[
P_n = 0.3Q_{1,n} + 0.3Q_{2,n} + 0.4Q_{3,n}.
\]

The following figure shows the functions \( x \mapsto P_0((-\infty, x]) \), \( x \mapsto P_0([x, \infty)) \) as well as the halfspace depth (in blue),

\[
x \mapsto \text{HD}(x | P_0) = \min\{P_0((-\infty, x]), P_0([x, \infty))\}.
\]

![Graph showing halfspace depth](image)

Obviously, the halfspace depth is not continuous for \( P_0 \).

In the following figure the halfspace depth w.r.t. \( P_0 \) (blue line) as well as the halfspace depths w.r.t. \( P_8 \) (red line above blue line) and \( P_9 \) (red line below blue line) are drawn.

![Graph showing halfspace depth](image)
It is easy to see that
\[
\text{HD}(z \mid P_n) \leq \text{HD}(x \mid P_0) \quad \text{for all } z, \text{ if } n \text{ is odd},
\]
\[
\text{HD}(z \mid P_n) \geq \text{HD}(x \mid P_0) \quad \text{for all } z, \text{ if } n \text{ is even}.
\]

Further,
\[
\sup_{\alpha \in [0, 1/2]} \delta_H(D_{\alpha}(P_n), D_{\alpha}(P_0)) = \frac{1}{n + 1}.
\]

Therefore,
\[
\lim_{n \to \infty} \sup_{\alpha \in [0, 1/2]} \delta_H(D_{\alpha}(P_n), D_{\alpha}(P_0)) = 0,
\]
i.e., (ComR) and a-fortiori (PtwR) hold.

On the other hand we have
\[
\text{HD}(1 \mid P_m) < 0.15 = \lim_{x \searrow 1} \text{HD}(x \mid P_0) < \text{HD}(1 \mid P_0) = 0.35 < \text{HD}(1 \mid P_n)
\]
for \(m\) odd and \(n\) even. This shows that although the trimmed regions converge uniformly, the depth functions HD(x | P_n) do not converge for x = 1. Thus, neither (UniD) nor (PtwD) hold. In other words, without the assumption of 'continuity for P', Theorems 4.3 and 4.6 do in general not hold.

**Example 2.4 ('Continuity' is needed in Theorem 4.8).** This example is similar to the previous one with the main difference that the alternating term \((-1)^n\) is missing. So, for \(n \in \mathbb{N}_0\) let
\[
a_n = \begin{cases} 
1 + \frac{1}{n + 1}, & \text{if } n \in \mathbb{N}, \\
1 & \text{if } n = 0.
\end{cases}
\]

Define \(Q_{1,n} = U([-2,-a_n] \cup [a_n,2])\), \(Q_{2,n} = U([-a_n,a_n])\) and \(Q_{3,n} = 0.5\epsilon_{-a_n} + 0.5\epsilon_{a_n}\) as in Example 2.3. Finally, for \(n \in \mathbb{N}_0\) consider the probability measures
\[
P_n = 0.3Q_{1,n} + 0.3Q_{2,n} + 0.4Q_{3,n}.
\]

As in Example 2.3, the halfspace depth is not continuous for \(P_0\).

In the following figure the halfspace depth w.r.t. \(P_0\) (blue line) as well as the halfspace depth w.r.t. \(P_8\) (red line above blue line) are shown.
Since in this example the depth function w.r.t. \( P_n \) lies always above the depth function w.r.t. \( P \), the oscillating behavior of the depth \( \text{HD}(1 \mid P_n) \) in the previous example does not occur. Instead, for each \( x \) (even for \( x = \pm 1 \)) the depth \( \text{HD}(x \mid P_n) \) converges to \( \text{HD}(x \mid P_0) \). Therefore, (PtwD) holds.

However, (UniD) does not hold. This can be seen as follows. For \( n \) arbitrarily, choose \( x_n \) such that \( 1 < x_n < a_n \). Since \( x_n \) lies between the jumps of \( \text{HD}(x \mid P_0) \) and \( \text{HD}(x \mid P_n) \) it follows

\[
\text{HD}(x_n \mid P_0) < 0.15 < 0.35 < \text{HD}(x_n \mid P_n).
\]

This shows that \( |\text{HD}(x_n \mid P_0) - \text{HD}(x_n \mid P_n)| > 0.2 \). Since this holds for every \( n \), it holds

\[
\sup_{z \in \mathbb{R}} |\text{HD}(z \mid P_0) - \text{HD}(z \mid P_n)| > 0.2 \quad \text{for all } n,
\]

i.e., the depth functions do not converge uniformly. In other words, without the assumption of 'continuity for \( P \)', (PtwD) does not imply (UniD) and thus, Theorem 4.8 does in general not hold.