Spherically Symmetric solutions on a cosmological dynamical background with BSSN equations

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Abstract. We give a summary of the work presented in [1]. We expose a numerical method for the study of cosmological problems in spherical symmetry in full General Relativity. The stability of the code close to the origin is made possible through the use of the Partially Implicit Runge-Kutta (PIRK) algorithm described in [2]. We demonstrate the stability and convergence properties and give a simple application to the evolution of the Lemaître-Tolman-Bondi spacetime. This work is a generalisation of the study given in [3] performed on an asymptotically flat background.

1. Introduction
A definitive theory of the evolution of the Universe should explain how small portions of spacetime can collapse and lead to the formation of structures on an expanding background. Such processes have been studied in many ways which all make simplifying assumptions departing from General Relativity. On the other hand, recent progresses in the field of Numerical Relativity have allowed to solve many problems on Minkowskian background. The first step towards solving the complete collapse of structures is to assume spherical symmetry. This involves the need to regularise operators of the form $1/r^p$ close to the origin of coordinates. The PIRK methods are an easy way to solve the instability problems [2].

In the following sections, we start by recalling the BSSN formalism. This gives us the opportunity to define the variables used throughout this work. We then illustrate the stability and convergence properties of the code by studying the gauge dynamics on a de Sitter background and proceed to reproduce the dynamics of the LTB solution. This describes the expanding spacetime around a spherically distributed dust density profile hence its interest for both cosmology and numerical relativity.

2. Formalism
The metric ansatz is

$$ds^2 = -((\alpha - \beta^2)dt^2 + 2\beta dt dx + \psi^4 a^2(t)(\hat{a} dr^2 + \hat{b} r^2 d\Omega^2)).$$

(1)

The only change compared to the usual BSSN formulation lies in the presence of the homogeneous scale factor $a(t)$ in the explicit form of the conformal metric. This scalar function describes the...
cosmological background. The extrinsic curvature defined as $K_{ij} := -\frac{1}{2} \mathcal{L}_n \gamma_{ij}$, with $\gamma_{ij}$ being the induced 3-metric can be decomposed in trace $K$ and conformally-scaled trace-free part $\hat{A}_{ij}$ as
\begin{equation}
K_{ij} = \frac{1}{3} \gamma_{ij} K + \psi^4 a^2 \hat{A}_{ij}.
\end{equation}
In spherical symmetry, $\hat{A}_{ij}$ has only two non zero components, $A_\alpha := \hat{A}_r^\alpha$ and $A_\beta := \hat{A}_\theta^\beta$. Since $\hat{A}_{ij}$ must be traceless, one further has $A_a + 2A_b = 0$. The last dynamical variable of the BSSN formalism is
\begin{equation}
\Delta^r = \frac{1}{a} \left[ \frac{\partial_r \hat{a}}{2\hat{a}} - \frac{\partial_r \hat{b}}{\hat{b}} - \frac{2}{r} \left( 1 - \frac{\hat{a}}{\hat{b}} \right) \right].
\end{equation}
Throughout this work, we have used $\beta = 0$. We thus omit this variables in the evolution equations given below.
\begin{align*}
\partial_t \hat{a} &= -2\alpha \hat{a} A_a, \\
\partial_t \hat{b} &= -2\alpha \hat{b} A_b, \\
\partial_t \psi &= -\frac{1}{6} \alpha \psi K - \frac{1}{2} \frac{\dot{\alpha}}{a} \psi, \\
\partial_t K &= -\nabla^2 \alpha + \alpha (A_a^2 + 2A_b^2 + \frac{1}{3} K^2) + 4\pi \alpha (E + S_a + 2S_b), \\
\partial_t A_a &= -\left( \nabla^r \nabla_r \alpha - \frac{1}{3} \nabla^2 \alpha \right) + \alpha \left( R^r_r - \frac{1}{3} K \right) + \alpha K A_a - \frac{16}{3} \pi \alpha (S_a - S_b), \\
\partial_t \Delta^r &= -\frac{2}{a} \left( A_a \partial_r \alpha + \alpha \partial_r A_a \right) + 2\alpha \left( A_a \Delta^r - \frac{2}{r \hat{b}} (A_a - A_b) \right) \\
&\quad + \frac{\xi \alpha}{a} \left[ \partial_r A_a - \frac{2}{3} \partial_r K + 6A_a \frac{\partial_r \psi}{\psi} + (A_a - A_b) \left( \frac{2}{r} + \frac{\partial_r \hat{b}}{\hat{b}} \right) - 8\pi j_r \right],
\end{align*}
\begin{align*}
The strong hyperbolicity of these equations is ensured by choosing $\xi > 1/2$. Following [3], we take $\xi = 2$ which is the standard choice. The functions $E$, $j_r$, $S_a$ and $S_b$ are built from the various projections of the stress-energy tensor (see [1]). The Hamiltonian and Momentum constraints read
\begin{align}
\mathcal{H} &\equiv R - (A_a^2 + 2A_b^2) + \frac{2}{3} K^2 - 16\pi E = 0, \\
\mathcal{M}^r &\equiv \partial_r A_a - \frac{2}{3} \partial_r K + 6A_a \frac{\partial_r \psi}{\psi} \\
&\quad + (A_a - A_b) \left( \frac{2}{r} + \frac{\partial_r \hat{b}}{\hat{b}} \right) - 8\pi j_r = 0.
\end{align}
The dynamics of the background Universe is described by the usual Friedmann and acceleration equations
\begin{align}
\frac{1}{\alpha_{bkg}^2} \left( \frac{\dot{a}}{a} \right)^2 &= \frac{8\pi}{3} \rho_{bkg}, \\
\frac{1}{\alpha_{bkg}^2} \frac{\ddot{a}}{a} - \frac{\dot{a}}{a} \frac{\dot{\alpha}_{bkg}}{\alpha_{bkg}} &= \frac{8\pi}{6} (\rho_{bkg} + 3p_{bkg}).
\end{align}
The homogeneous background density, $\rho_{bkg}$, and pressure, $p_{bkg}$, are functions of time. $\alpha_{bkg}$ is the homogeneous lapse function.
The adjustment of boundary conditions is important in this work as the background is dynamical. We use a generalised form of dissipative boundary conditions

\[ \partial_t f = \partial_t f_{\text{bkg}} - v \partial_r f - \frac{v}{r}(f - f_{\text{bkg}}), \]  

with \( f_{\text{bkg}} \), the homogeneous background value of the field \( f \) and \( v \), its characteristic velocity.

The Numerical solution of the above equations are obtained by using the PIRK evolution scheme (see [2] for details) along with a fourth order finite difference discretisation of the spatial derivative operators. No numerical dissipation is needed.

3. Gauge Dynamics

We investigate the stability and convergence properties of the code with the study of pure gauge dynamics. We look at the evolution of a lapse pulse on a De Sitter Universe background with a constant energy density mimicking a cosmological constant with \( \Lambda = 8\pi \rho_{\text{bkg}} \) and \( p_{\text{bkg}} = -\rho_{\text{bkg}} \). The initial lapse pulse is

\[ \alpha(t = 0) = \alpha_{0,\text{bkg}}^0 + \frac{\alpha_0 r_0^2}{1 + r^2} \left[ e^{-(r-r_0)^2} + e^{-(r+r_0)^2} \right]. \]  

Fixing \( \alpha_{0,\text{bkg}}^0 = 1 \), the Friedmann equation gives \( E = \rho_{\text{bkg}} = \frac{3}{8\pi} H_0^2 \). The initial conditions

\[ \dot{a}(t = 0) = \dot{b}(t = 0) = \psi(t = 0) = 1, \]  

\[ K = 3H_0, \ A_a = 0. \]  

then solve satisfy the constraint equations. The evolution is performed in the harmonic slicing with zero shift:

\[ \partial_t \alpha = -\alpha^2 K. \]  

We set and \( \alpha_0 = 0.01 \) and \( H_0 = 0.01 \), that is well in the exponential regime of the cosmic expansion for time scales up to \( t \sim 10 \). The Courant-Friedrichs-Lewy (CFL) factor for simulations in this section is \( \Delta t/\Delta r = 0.25 \). The spatial resolution is \( \Delta r = 0.05 \). Fig. 1 shows the evolution of the lapse function. This should be compared with Fig. 2 showing the evolution of its asymptotic cosmological value.

**Figure 1.** Evolution of a pure gauge pulse in time on a de Sitter background. The asymptotic value of \( \alpha \) gets rescaled during the evolution.

**Figure 2.** Evolution of the scale factor \( a(t) \) (upper panel) and background lapse function \( \alpha_{\text{bkg}} \) (lower panel).
The convergence and stability of the code are illustrated on Fig. 3 showing the Hamiltonian constraint violation for different values of the resolution. The rescaling of the error demonstrates the second-order convergence of the method. Similarly to the results of [3] the hamiltonian constraint violation is maximum at the centre of coordinates. Yet the error remains bounded and the central value of all fields remains controlled.

**Figure 3.** Value of the hamiltonian constraint in pure gauge dynamics for 3 different resolutions at $t = 10$. The rescaling of the curves shows the good agreement with the expected second-order convergence of the numerical method. The inner panel shows that the convergence regime is not yet reached for $\Delta r = 0.1$. These results were obtained for $H_0 = 0.01$ and $\Delta t/\Delta r = 0.25$.

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**4. Lemaître-Tolman-Bondi Solution**

The LTB solution describes a spherically symmetric dynamical spacetime filled with dust. The analytical expression of its metric is usually given in the geodesic gauge $\alpha = 1$. The line element is

$$ds^2 = -dt^2 + \frac{a_\|^2(t, r)}{1 + 2E_{\text{LTB}}(r)} + a_\bot^2(t, r)r^2d\Omega^2,$$

with $a_\| = \partial_r(r a_\bot)$ and where $E_{\text{LTB}}(r)$ is a free function. The evolution equations for the metric component are generalisation of the Friedmann equations. Those are given in [1] along with the evolution equation for an initial distribution of matter. Reproducing the LTB solution starts with solving the initial data for such initial distribution. We set

$$\dot{a}(t = 0) = \dot{b}(t = 0) = 1, \quad K(t = 0) = -3H_0,$$

$$A_a(t = 0) = A_b(t = 0) = 0,$$

$$E(r, t = 0) = [1 + \delta_m(r)] \rho_{\text{bkg}}^0,$$

with $\rho_{\text{bkg}}^0 = \rho_{\text{bkg}}(t = 0)$ and $\delta_m(r)$ some profile in the constraint equations. Upon using the Friedmann equation, the Hamiltonian constraint reduces to

$$\partial_r^2\psi + \frac{2}{r}\partial_r\psi = 16\pi \rho_{\text{bkg}}^0 \delta_m^0(r) a_0^2 \psi^5.$$

This can be solved as a boundary value problem. The boundary conditions are $\partial_r\psi|_{r=0} = 0$ and $\psi \rightarrow 1$ for $r \rightarrow \infty$. Once the solution is found, this is used to write the initial values of $a_\bot$ and $a_\|$ and their time derivatives.
Fig. 4 shows the good agreement between the solutions obtained with the LTB equations and the evolution using BSSN variables at early times. The maximum of the relative difference between both solution is of the order $\sim 10^{-5}$. The agreement stays equally valid at later time. Fig. 5 shows the comparison between the background scale factor and the central expansion of spacetime. The results presented here were obtained with $H_0 = 0.1$, $\Delta r = 0.1$. The CFL factor is $\Delta t/\Delta r = 0.5$.

Figure 4. Metric components $\gamma_{rr}$ (top curve) and $\gamma_{\theta\theta}/r^2$ (bottom curve). The plain lines is the result of the LTB equations. The dots are the result of the BSSN equations.

Figure 5. Evolution of the background scale factor (top curve) compared with the central expansion factor (bottom line).

5. Conclusion
We have shown how to apply the BSSN formalism to deal with the evolution of spherically symmetric solutions on a cosmological background. The validity of the numerical method has been proven by studying pure gauge dynamics and we have applied it to study the evolution of a simple dust distribution thus recovering the usual LTB solution. Further directions include the study of spacetime less restrictive than LTB. The application of the method here described in the case of a background Universe filled with a scalar field of Dark Energy is underway as of the moment of the writing of the present proceeding.

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