The Lcm$(1, 2, \ldots, n)$ as a Product of Sine Values
Sampled Over the Points in Farey Sequences

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Abstract
Some easily proven trigonometric formulae are given. They lead to a shorter, alternate proof of a formula of G. Martin.

1 A formula for the lcm of \{1,2,...,n\}

Recently Greg Martin [4] derived an interesting formula for the least common multiple of \{1, 2, \ldots, n\}. In this paper, we give an exposition of the proof in terms of the sine function.

Let us first agree on some notation. We write LCM$(n)$ for lcm \{1, 2, \ldots, n\}. \(F(n)\) will denote the Farey sequence of order \(n\), that is the set of all reduced fractions in the interval [0,1] whose denominators are \(n\) or less, arranged in increasing order. We write \(k \perp n\) if \(k, n\) are integers and relatively prime, following Donald E. Knuth [1]. We say an integer \(n\) is not a prime power if at least two different primes divide \(n\) and write \(n = p^\alpha\) in this case; we will always assume \(\alpha\) an integer \(> 0\). Also note that we write \(\prod P\) instead of \((\prod P)^2\) to avoid big brackets around products.

Martin first proves
\[
\prod_{0 \leq k < n \atop k \perp n} \frac{\Gamma^2 \left( \frac{k}{n} \right)}{2\pi} = \begin{cases} 1, & \text{if } n \neq p^\alpha; \\ \frac{1}{p}, & \text{if } n = p^\alpha. \end{cases} \quad (n \geq 2) \tag{1}
\]

By multiplying for \(n = 2, 3, \ldots\), he derives
\[
\text{LCM}(n) = \prod_{r \in F(n) \atop 0 < r < 1} \frac{2\pi}{\Gamma^2(r)}. \tag{2}
\]

We observe that if \(r\) is a member of \(F(n)\) then \(1 - r\) is also a member. By the reflection formula of the Gamma function, we can trade in two Gamma evaluations for one sine evaluation \(\sin(\pi z) = \pi/(\Gamma(z)\Gamma(1-z))\). This way Martin’s formula becomes the simpler
\[
\text{LCM}(n) = \frac{1}{2} \prod_{r \in F(n) \atop 0 < r \leq 1/2} 2\sin(\pi r) \quad (n \geq 2). \tag{3}
\]
In a discussion in the newsgroup de.sci.mathematik Jutta Gut [3] observed that the left hand sides of (3) are equal for \( n \) and \( n - 1 \) if \( n \) is not a prime power; we may add the observation that for \( n > 2 \) the quotient of both equals \( p \) if \( n = p^\alpha \).

Equivalently, we may apply the reflexion formula to Martin’s theorem (1) directly. This immediately gives

\[
\prod_{0 < k < n \atop k \perp n} 2 \sin \frac{\pi k}{n} = \begin{cases} 
1, & \text{if } n \neq p^\alpha; \\
p, & \text{if } n = p^\alpha.
\end{cases} \quad (n \geq 0)
\] (4)

For \( n > 2 \) the range of the product can be reduced to \( 1 \leq k \leq \lfloor n/2 \rfloor \) provided the product is raised to the square as \( \sin \left( \frac{\pi}{n} k \right) = \sin \left( \frac{\pi}{n} (n - k) \right) \).

The interest we noted during these discussions motivates us to present an alternative: we first prove (4), and then derive (2) from it.

2 Sines of roots of unity

For a short proof of (4), we recall two well-known facts.

**Fact 1** is from elementary geometry: if the arc between two points on the unit circle has length \( \theta \), then the length of the chord between them is \( 2 \sin(\theta/2) \). Applying this to the points 1 and \( \exp \left( \frac{2k\pi i}{n} \right) \) where \( -n/2 \leq k \leq n/2 \) gives

\[
1 - \exp \left( \frac{2k\pi i}{n} \right) = 2 \sin \frac{k\pi}{n}.
\] (5)

**Fact 2** is about cyclotomic polynomials, which we denote by \( \Phi_n \). It can be found in many standard texts on algebra; see, for example, [2], p.280, Exercise 4. It says: if a prime \( p \) divides \( n \), then \( \Phi_{np}(X) = \Phi_n(X^p) \); if \( p \) does not divide \( n \), then \( \Phi_{np}(X) = \Phi_n(X^p) / \Phi_n(X) \). Plugging in \( X = 1 \) and using induction gives for all \( n > 2 \)

\[
\Phi_n(1) = \prod_{-n/2 \leq k \leq n/2 \atop k \perp n} \left( 1 - \exp \left( \frac{2k\pi i}{n} \right) \right) = \begin{cases} 
1, & \text{if } n \neq p^\alpha; \\
p, & \text{if } n = p^\alpha.
\end{cases}
\] (6)

**Proof of (4).** The formula holds for \( n = 2 \). Taking absolute values in (6), the \( k \)-th and \(-k\)-th factor become equal. Using (5) gives (4).

Using \( (X^n - 1)/(X - 1) \) instead of a cyclotomic polynomial, the same method has been used recently on Planet Math [5] to give a concise proof of

\[
\frac{n}{2^{n-1}} = \prod_{0 < k < n} \sin \frac{\pi k}{n}.
\] (7)

Clearly this can also be written as

\[
n = \prod_{0 < k < n \atop k \perp n} 2 \sin \frac{\pi k}{n} \prod_{0 < k < n \atop k \perp n} 2 \sin \frac{\pi k}{n}.
\] (8)
From (8) and (4) follows the counterpart of (4).

$$\prod_{0 < k < n, k \perp n} 2 \sin \frac{\pi k}{n} = \begin{cases} \frac{n}{p}, & \text{if } n = p^\alpha; \\ n, & \text{if } n \neq p^\alpha. \end{cases} \quad (n \geq 1)$$  

(9)

These relations lead to a complementary form of Martin’s identities (1) and (2). Let $$\delta(n) = \{ d : d \mid n \text{ and } 0 < d < n \}$$ denote the set of proper divisors of $$n \geq 0$$ and define $$\text{LCM}(n) = \text{lcm}(\delta(n))$$ if $$\delta(n)$$ is not empty, otherwise 1. Then

$$\text{LCM}(n) = \prod_{k \perp n, 0 < k < n} 2 \sin \frac{\pi k}{n} \quad (n \geq 1).$$  

(10)

Applying the reflection formula of the $$\Gamma$$ function this can be rewritten as

$$\prod_{0 < k < n} \frac{2\pi}{\Gamma^2 \left( \frac{k}{n} \right)} = \text{LCM}(n) \quad (n \geq 1).$$  

(11)

The sequences LCM(n) and LCM(n) are indexed in Sloane’s Online Encyclopedia of Integer Sequences as A003418 and A048671 respectively.

3 Cosines of roots of unity

The same method works for cosines instead of sines. Our first, geometric fact then says that the chord between the points $$-1$$ and $$\exp(2\pi ik/n)$$ has length $$2 \cos(\pi k/n)$$, such that in the next step the cyclotomic polynomials must be evaluated at $$-1$$.

Let $$\tilde{\epsilon}_n(k) = 1 + \exp(2\pi ik/n)$$, then by induction it is easily proved from the recursion formulas that

$$\Phi_n(-1) = \prod_{0 < k < n, k \perp n} -\tilde{\epsilon}_n(k) = \begin{cases} p, & \text{if } n = 2p^\alpha; \\ 1, & \text{otherwise}. \end{cases} \quad (n > 2)$$

The zero factor $$\Phi_2(-1)$$ makes results boring, and we will avoid it in what follows. Since $$\tilde{\epsilon}_n(k)\tilde{\epsilon}_n(n - k) = (2 \cos \pi k/n)^2$$ we get

$$\prod_{0 < k < n, k \perp n} 2 \left| \cos \frac{\pi k}{n} \right| = \begin{cases} p, & \text{if } n = 2p^\alpha; \\ 1, & \text{otherwise}. \end{cases} \quad (n > 2)$$  

(12)

Multiplying (12) for all denominators below a given bound we obtain a similar result for Farey sequences as before:

$$\prod_{r \in F(n), 0 < r < 1/2} 2 \cos (\pi r) = \text{LCM}(n/2).$$

In the case of cosines, too, the method can be applied easily to $$(X^n - 1)/(X - 1)$$ instead of a cyclotomic polynomial and gives

$$\prod_{1 \leq k \leq \lfloor n/2 \rfloor} 2 \cos \left( \frac{\pi k}{n} \right) = \begin{cases} 1, & \text{if } n \text{ odd} ; \\ 0, & \text{if } n \text{ even}. \end{cases}$$  

(13)

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Again, the zeroes come from a zero factor \( k = n/2 \); the product of the other factors is \( \frac{1}{2}n \) as can be seen by using \( \frac{X^n-1}{(X-1)(X+1)} = \sum_{m=0}^{(n/2)-1} X^{2m} \) for even \( n \).

4 The multiplication theorem revisited

Let us conclude with a formula that involves the gamma function again. We could construct it by converting our results on products of sines back to products of gammas (using the reflexion formula), but give another method here.

The multiplication theorem of Gauss states
\[
\prod_{0 \leq k \leq m-1} \Gamma \left( z + \frac{k}{m} \right) = (2\pi)^{\frac{1}{2}(m-1)} m^{(1/2-mz)} \Gamma (mz) .
\]
The substitution \( m \leftarrow \phi(n) + 1 \), \( z \leftarrow \frac{1}{\phi(n)+1} \), leads to
\[
\sqrt{\phi(n) + 1} \prod_{0 < k \leq \phi(n)+1} \Gamma \left( \frac{k}{\phi(n) + 1} \right) = (2\pi)^{\phi(n)} . (14)
\]
On the other hand, if \( \phi(n) \) denotes Euler’s totient function and dividing (2) for consecutive values \( n \) and \( n - 1 \), we immediately see that if \( n \) is not a prime power, then also
\[
\prod_{0 \leq k < n} \Gamma \left( \frac{k}{n} \right) = (2\pi)^{\frac{1}{2}\phi(n)} .
\]
Abbreviating \( N = \phi(n) + 1 \) and equating the left hand sides, we arrive at
\[
\prod_{0 < k < n} \Gamma \left( \frac{k}{n} \right) = \sqrt{N} \prod_{0 < k < N} \Gamma \left( \frac{k}{N} \right) \quad (n \neq p^a) .
\]
It would be interesting to know whether there is some natural direct proof of this formula that does not use sines and the reflexion formula.

References

[1] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley, 1989.

[2] Serge Lang. *Algebra*. Revised third edition. Springer, 2002.

[3] Jutta Gut. Comment in newsgroup de.sci.mathematik. 2009-08-11.

[4] Greg Martin. A product of gamma function values at fractions with the same denominator. *arXiv:0907.4384v1 [math.CA]*, 2009.

[5] PlanetMath Online Mathematics Encyclopedia. Trigonometric identity involving product of sines of roots of unity. 2009-08-22. Version 8.

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