Complexity of triangulations of the projective space

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Abstract

It is known that any two triangulations of a compact 3–manifold are related by finite sequences of certain local transformations. We prove here an upper bound for the length of a shortest transformation sequence relating any two triangulations of the 3–dimensional projective space, in terms of the number of tetrahedra.

MSC–class: 57Q25; 57Q15.

Key words: Projective space, local transformation, triangulation, normal surface.

1 Introduction

By the “Hauptvermutung”, that was proven by Moise [10], any two triangulations $T_1$ and $T_2$ of a compact 3–manifold $M$ have a common subdivision. This allows to show that $T_1$ and $T_2$ are related by finite sequences of certain local transformations of triangulations, e.g., stellar subdivisions [1] or elementary shellings [11]. These results do not provide explicit constructions of transformation sequences and do not yield a recognition algorithm for $M$. In this paper, we construct transformation sequences for triangulations of the 3–dimensional projective space $\mathbb{P}^3$. We consider the following local transformations, that generalise stellar subdivisions.

Definition 1 Let $T$ and $\tilde{T}$ be PL–triangulations of a closed PL–manifold, and let $e$ be an edge of $T$ with $\partial e = \{a, b\}$. Suppose that $\tilde{T}$ is obtained from $T$ by removing the open star of $e$ and identifying $a \ast \sigma$ with $b \ast \sigma$ for any simplex $\sigma$ in the link of $e$. Then $\tilde{T}$ is the result of the edge contraction of $T$ along $e$, and $T$ is the result an edge expansion of $\tilde{T}$ along $e$.

In general, there are edges of $T$ along which a contraction is impossible. This is the case, e.g., if the edge is part of an edge path of length 3 that does not bound a 2–simplex of $T$. It is easy to see that any PL–triangulation admits only a finite number of edge expansions.

Let $d(T_1, T_2)$ be the length of a shortest sequence of edge contractions and expansions relating two triangulations $T_1$ and $T_2$ of a closed 3–manifold $M$.

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The aim of this paper is to provide an upper bound for \( d(T_1, T_2) \) when \( M \) is homeomorphic to \( \mathbb{P}^3 \), as stated in Theorem 1 below. The proof is partially based on our work on the 3–sphere (see [7]–[9]). This paper is thought of as a first step towards a study of more general 3–dimensional manifolds, e.g., (atoroidal) Haken manifolds. A generalisation to all compact 3–manifolds, which would solve the algorithmic classification problem for compact 3–manifolds, is out of reach, as yet.

**Theorem 1** Any two triangulations of \( \mathbb{P}^3 \) with at most \( t \) tetrahedra are related by a sequence of less than \( 2^{27000t^2} \) edge contractions and expansions.

The constant factor in the exponent is certainly not optimal. According to the examples in [8], concerning the minimal number of edge expansions needed to transform a triangulation of \( S^3 \) into a polytopal triangulation, we believe that the bound in Theorem 1 can not be replaced by a subexponential bound.

We outline the proof of Theorem 1. It is based on Haken’s normal surface theory, Barnette’s work [2] on irreducible triangulations of the projective plane, and our techniques in [7] and [8]. Let \( T_1 \) be a triangulation of \( \mathbb{P}^3 \). By means of normal surface theory, we construct a certain projective plane \( P \subset \mathbb{P}^3 \). The complement of a regular neighbourhood of \( P \) is a ball. This allows to apply techniques of [7] and [8], yielding a sequence of edge contractions and expansions relating \( T_1 \) with a triangulation \( T_P \) that depends on the choice of \( P \). The next step is to simplify \( T_P \) by a certain series of edge contractions. In that way we transform \( T_1 \) via \( T_P \) into one of two standard triangulations of \( \mathbb{P}^3 \), corresponding to the two irreducible triangulations of the projective plane found by Barnette [2]. In the last step in the proof of Theorem 1 we relate the two standard triangulations of \( \mathbb{P}^3 \) by an explicit sequence of edge contractions and expansions. In conclusion, we construct transformation sequences that relate any two triangulations of \( \mathbb{P}^3 \) via the two standard triangulations. The bound stated in Theorem 1 follows from a complexity analysis of the construction.

The paper is organised as follows. In Section 2.1 we briefly outline the results of normal surface theory used in our proofs. Section 2.2 recalls a lemma from [8] on the construction of sequences of edge contractions. We prove the existence of \( P \) in Section 3.1. Section 3.2 is devoted to the transformation into \( T_P \), along the lines of [7]–[9]. Section 3.3 is concerned with the transformation into one of the two standard triangulations. A transformation sequence relating the two standard triangulations is finally given in Section 3.4.

## 2 Prerequisites

In this section, we collect some results used in the proof of Theorem 1. We denote the number of connected components of a topological space \( X \) by \( \#(X) \). For a tame subset \( Y \subset X \), we denote an open regular neighbourhood of \( Y \) in \( X \) by \( U(Y) \).

### 2.1 Normal surfaces

Let \( M \) be a closed 3–manifold with a cellular decomposition \( \mathcal{Z} \), so that the closure of any open cell of \( \mathcal{Z} \) is homeomorphic to a closed ball. Its \( k \)–skeleton is denoted by \( \mathcal{Z}^k \). In our applications, \( \mathcal{Z} \) is a triangulation or is dual to a
triangulation. A normal isotopy with respect to $Z$ is an ambient isotopy of $M$ that fixes each cell of $Z$ set-wise.

**Definition 2** Let $c$ be a closed 2–cell of $Z$ and let $\gamma \subset c$ be a closed embedded arc with $\gamma \cap \partial c = \partial \gamma$, disjoint from the vertices of $c$. If $\gamma$ connects two different edges in the boundary of $c$ then $\gamma$ is a normal arc. Otherwise it is a return.

**Definition 3** Let $S \subset M$ be a closed embedded surface transversal to $Z$. We call $S$ normal with respect to $Z$, if $S \cap Z^2$ is a union of normal arcs, $S \setminus Z^2$ is a disjoint union of discs, and the boundary of any connected component of $S \setminus Z^2$ meets any edge of $Z$ at most once.

When it is clear from the context, we do not specify with respect to which cellular decomposition a surface is normal. In the rest of this section, we focus on normal surfaces with respect to a triangulation $T$ of $M$. It is well known that normal surfaces in a triangulated 3–manifold are built from copies of so-called normal triangles and normal squares (see Figure 1).

![Figure 1: Normal triangles and squares](image)

**Theorem 2** Let $T$ be a triangulation of $M$ with $n$ tetrahedra. Let $S \subset M$ be a normal surface comprising more than $10n$ two-sided connected components. Then two connected components of $S$ are normally isotopic.

This result is originally due to Kneser and proven, e.g., in Lemma 4 of [3]. We use this result to show that certain iterative constructions of normal surfaces stop after a finite number of steps.

Under a technical condition (see [6] for details), one can define the sum $S_1 + S_2$ of two normal surfaces $S_1, S_2 \subset M$. The sum is a normal surface and is determined up to normal isotopy by the condition $(S_1 + S_2) \cap T^1 = (S_1 \cup S_2) \cap T^1$. The Euler characteristic is additive, $\chi(S_1 + S_2) = \chi(S_1) + \chi(S_2)$. Haken has shown that with this notion of a sum, the set of normal surfaces in $M$ with respect to $T$ is isomorphic to a subgroupoid of the semi group $G$ of integer points in a rational convex cone, the so-called Haken cone. The semi group $G$ is additively generated by a finite set of elements, that can be constructed by means of integer programming. The set of normal surfaces is finitely generated as well, by the following result. For an embedded surface $S \subset M$ that is in general position with respect to $T$, denote $\|S\| = \#(S \cap T^1)$. 

3
Theorem 3  Let $N \subset M \setminus U(T^0)$ be a sub-$3$–manifold whose boundary is a normal surface. There is a system $F_1, \ldots, F_q \subset N$ of normal surfaces such that

$$\|F_i\| < \|\partial N\| \cdot 2^{18n}$$

for $i = 1, \ldots, q$, and any normal surface $F \subset N$ can be expressed as a sum $F = \sum_{i=1}^{q} k_i F_i$ with non-negative integers $k_1, \ldots, k_q$.

The surfaces $F_1, \ldots, F_q$ are called fundamental surfaces in $N$. The preceding theorem is Theorem 3 of [7], that is formulated in a slightly more general setting, namely for so-called 2–normal surfaces. One can prove slightly better bounds in our special situation, but this concerns only the constant 18. Actually it follows from [5] that the bound in Theorem 3 cannot be replaced by a subexponential bound. In the case $N = M \setminus U(T^0)$, the existence and constructibility of fundamental surfaces is classical in Haken theory [6], and bounds for $\|F_i\|$ were obtained in [4].

2.2 Edge contractions

We recall in this subsection a lemma yielding sequences of edge contractions and expansions. Let $M$ be a closed $3$–manifold. A cellular decomposition of $M$ is simple, if any vertex is adjacent to exactly four edges, and any edge is adjacent to exactly three $2$–cells (counted with multiplicity). A cellular decomposition is regular if the closure of any open $k$–cell is homeomorphic to a $k$–ball ($k = 1, \ldots, 3$).

Let $C_1$ be a simple regular cellular decomposition. In general, $C_1$ is not dual to a triangulation, as multiple edges might occur. However, the barycentric subdivision $C'_1$ of $C_1$ is a triangulation of $M$. By the next lemma, that is proven in [8], the deletion of an appropriate 2–cell of $C_1$ gives rise to a sequence of contractions of $C'_1$.

**Lemma 1** Let $C_1, C_2$ be two simple regular cellular decompositions of $M$, so that $C'_2 \setminus C'_1$ is an open $2$–cell of $C_1$ with $k$ vertices in its boundary. Then $C'_2$ is obtained from $C'_1$ by a series of $4k + 2$ contractions.

3 Transforming triangulations

The four parts of this section form the proof of Theorem 1 as outlined in the introduction. We fix the following notations. Let $C$ be a cellular decomposition of $P^3$ that is dual to a triangulation, and let $T$ be its barycentric subdivision. Let $n$ be the number of tetrahedra of $T$. We consider $C^2$ as a subset of $T^2$.

3.1 Fundamental projective plane

It is well known (see [4]) that there is a fundamental projective plane with respect to $T$. We prove here that we can choose it so that additionally it is normal with respect to $C$. This technical condition is needed to make the techniques of [8] work (see next subsection).
Lemma 2 Among the fundamental surfaces with respect to $T$ in $\mathbb{P}^3 \setminus U(T^0)$, there is a projective plane $P$ that is normal with respect to $C$.

Proof Choose an embedded projective plane $P \subset \mathbb{P}^3$ so that the triple

$$(\#(P \cap C^1), \#(P \cap T^1), \#(P \cap T^2))$$

is minimal in lexicographic order. We first prove that $P$ is normal with respect to $T$ and $C$, by a modification of standard techniques (compare [4]). Since $\#(P \cap C^1)$ is minimal, $P \cap C^2$ contains no returns. Removing a return in $P \cap T^2$ does not increase $\#(P \cap C^1)$, thus there is no return in $P \cap T^2$ by minimality of $\#(P \cap T^1)$. The minimality of $\#(P \cap T^2)$ excludes circles in $(P \cap T^2) \setminus T^1$, since cutting-and-pasting along such a circle does not increase $\#(P \cap C^1)$ and $\#(P \cap T^1)$.

Assume that there is a circle $\gamma$ in $(P \cap C^2) \setminus C^1$, contained in a 2–cell $c$ of $C$. By the preceding paragraph, we know that $\gamma$ is a union of normal arcs with respect to $T$, and $\gamma$ bounds a disc in $c$ containing a vertex of $T$ (the barycenter of $c$). Therefore $\#(\gamma \cap T^1) \geq 6$. Let $D \subset c$ be the disc bounded by $\gamma$. Since $P$ is incompressible in $\mathbb{P}^3$, there is a disc $D' \subset P$ bounded by $\gamma$. We replace $D'$ by a parallel copy $D''$ of $D$ with $D'' \cap D' = \emptyset$. We can choose $D''$ so that $\#(D'' \cap T^1) = 1$, namely intersecting an edge of $T$ that connects the barycenter of $c$ with the barycenter of a 3–cell of $C$. Hence, the cut-and-paste operation replacing $D'$ by $D''$ decreases $\#(P \cap T^1)$ by at least 5, without increasing $\#(P \cap C^1)$. So by minimality of $\#(P \cap T^1)$, there is no circle in $(P \cap C^2) \setminus C^1$.

By the preceding paragraph, any boundary component of a connected component of $P \setminus C^2$ meets $C^1$. Thus the connected component of $P \setminus C^2$ are discs, since otherwise one can decrease $\#(P \cap C^1)$ by a cut-and-paste operation. If the boundary of a connected component $P \setminus C^2$ intersects some edge of $C$ at least twice, then there is a closed embedded disc $D \subset \mathbb{P}^3$ so that $D \cap C^2 = \partial D \cap C^1$ is an arc contained in the interior of an edge of $C$, $D \cap P \subset \partial D$, and $\partial D \subset P \cup C^1$. Sliding $P$ across $D$ decreases $\#(P \cap C^1)$. The corresponding operations, applied to connected components of $P \setminus T^2$, decreases $\#(P \cap T^1)$ without increasing $\#(P \cap C^1)$. In conclusion, the minimality of $(\#(P \cap C^1), \#(P \cap T^1), \#(P \cap T^2))$ implies that the boundary of any connected component of $P \setminus C^2$ (resp. $P \setminus T^2$) meets any edge of $C$ (resp. of $T$) at most once. Hence $P$ is normal both with respect to $T$ and to $C$.

We represent $P$ as a sum $F_1 + \cdots + F_k$ of fundamental surfaces with respect to $T$. We can assume that none of $F_1, \ldots, F_k$ is a 2–sphere (see [4]). Since the Euler characterstic is additive under the addition of normal surfaces, one summand (say, $F_1$) has positive Euler characteristic, thus, is a projective plane. Since $C^1 \subset T^1$, we have $\#(F_1 \cap C^1) \leq \#(P \cap C^1)$ and $\#(F_1 \cap T^1) \leq \#(P \cap T^1)$, and $\#(F_1 \cap T^2) \leq \#(P \cap T^2) = 1$, since $P$ and $F_1$ are connected normal surfaces. Thus, the choice of $P$ implies $P = F_1$, i.e., $P$ is fundamental with respect to $T$ in $\mathbb{P}^3 \setminus U(T^0)$.

3.2 Transformation into $T_P$

For any projective plane $P$ as in Lemma 2 we define a 2–dimensional polyhedron

$$Q_P = (C^2 \cap U(P)) \cup \partial U(P).$$
Since $P$ is normal with respect to $C$ and $\mathbb{P}^3 \setminus U(P)$ is a ball, $Q_P$ is the 2–skeleton of a simple cellular decomposition of $\mathbb{P}^3$, which we denote by $C_P$. Any 2–cell of $C_P$ is contained in the boundary of two different 3–cells. Thus $C_P$ is regular, and the barycentric subdivision $T_P = C_P'$ of $C_P$ is a triangulation of $\mathbb{P}^3$. The aim of this subsection is to relate $T$ with $T_P$ by a sequence of edge contractions and expansions.

We outline the construction of the transformation sequence. All ingredients are taken from \cite{7}–\cite{9}, it is only needed to adapt it to the present situation. Since $B = \mathbb{P}^3 \setminus U(P)$ is a ball, there is an embedding $H: S^2 \times [0,1] \rightarrow B$ so that $H(S^2 \times 0) = \partial U(x)$ for some vertex $x \in C^0 \subset T^0$, and $H(S^2 \times 1) = \partial B = \partial U(P)$. Let $c(H,T^1)$ (resp. $c(H,C^i)$) be the number of parameters $\xi \in [0,1]$ for which the surface $H_\xi = H(S^2 \times \xi)$ is not in general position to $T^i$ (resp. $C^i$), for $i = 1,2$. We assume that $c(H,T^1)$ is minimal. An analysis of the Rubinstein–Thompson algorithm as in \cite{7} yields an upper bound for $c(H,C^i)$ in terms of the number $n$ of tetrahedra of $T$. Techniques from \cite{8} allow to bound $c(H,C^2)$ as well. To any surface $H_\xi$ that is in general position to $C^2$, we define an embedded 2–complex $Q_\xi \subset \mathbb{P}^3$ that is the 2–skeleton of a simple regular cellular decomposition of $\mathbb{P}^3$, and $Q_1 = Q_P$. If $H_{\xi_0}$ is not transversal to $C^2$ for some $\xi_0 \in [0,1]$ and $\epsilon > 0$ is sufficiently small, the complex $Q_{\xi_0 + \epsilon}$ is related to $Q_{\xi_0}$, by isotopy and a bounded number of deletions and insertions of 2–cells. An application of Lemma \ref{lem:boundary} yields a transformation of $T$ into $T_P$ by a bounded number of edge contractions and expansions.

In the following lemma, “normal” shall mean “normal with respect to $T$”. The estimate for $c(H,T^1)$ is based on the construction of a so-called maximal normal sphere system. This is a system $\Sigma \subset \mathbb{P}^3$ of disjoint normal 2–spheres that are pairwise not normally isotopic, so that any normal 2–sphere in $\mathbb{P}^3 \setminus \Sigma$ is normally isotopic to a connected component of $\Sigma$.

**Lemma 3.** There is a maximal normal sphere system $\Sigma \subset \mathbb{P}^3 \setminus U(P)$ with at most $10n$ connected components and $\|\Sigma\| < 2^{18n}n^2$.

**Proof.** We construct $\Sigma$ iteratively. Define $\Sigma_1 = \partial U(P \cup T^0)$. Since $\partial U(T^0)$ meets each edge of $\Sigma$ exactly twice and $\Sigma$ has at most $2n$ edges, we have $\|\partial U(T^0)| \leq 4n$. The projective plane $P$ is fundamental in $\mathbb{P}^3 \setminus U(T^0)$. Thus, by Theorem \ref{thm:rubinstein} and since $n \geq 24$, we have

$$\|\Sigma_1\| < 4n + 4n \cdot 2^{18n} < 2^{19n}.$$ 

For $i \geq 1$, suppose that there is a connected component $N_i$ of $\mathbb{P}^3 \setminus U(\Sigma_i)$ and a normal 2–sphere $S \subset N_i$ that is not normally isotopic to a connected component of $\Sigma_i$. It follows that $N_i$ is not a regular neighbourhood of $P$ or of a vertex of $T$. We choose $S$ so that $\|S\|$ is minimal.

Assume that $S$ can be represented as a sum $S_1 + S_2$ of non-empty normal surfaces in $N_i$. Since the Euler characteristic is additive and since there is no embedded projective plane in the 3–ball $B = \mathbb{P}^3 \setminus U(P)$, one of the summands, say $S_1$, is a sphere. It is not normally isotopic to a component of $\Sigma_i$, since otherwise $S_1 + S_2$ would be the disjoint union of $S_1$ and $S_2$, thus would not be a sphere. We obtain a contradiction to the choice of $S$, since $\|S_1\| < \|S\|$. Thus $S$ is fundamental in $N_i$.

We define $\Sigma_{i+1} = \Sigma_i \cup S$. By Theorem \ref{thm:rubinstein} and since $\|\partial N_i\| \leq \|\Sigma_i\|$, we have $\|\Sigma_{i+1}\| \leq \|\Sigma_i\| + \|\Sigma_i\| \cdot 2^{18n}$. The iteration stops after at most $10n$ steps, by
Theorem 2. Thus, we end with a maximal system \( \Sigma \) of normal 2–spheres with
\[
\| \Sigma \| < \| \Sigma_1 \| \cdot (2^{18n})^{10n-1} < 2^{181n^2}
\]
and at most 10n connected components. \( \square \)

**Lemma 4** One can transform \( \mathcal{T} \) into \( \mathcal{T}_p \) by a sequence of less than \( 2^{184n^2} \) edge contractions and expansions.

**Proof** This lemma is a variant of Theorem 3 in [9]. We give here an outline of the proof, all details can be found in [8] and [9]. As in Lemma 34 of [9], there is an embedding \( H: S^2 \times [0,1] \to \mathbb{P}^3 \setminus U(P) \) in general position to \( T^1 \) so that \( H(S^2 \times 0) = \partial U(x) \) for some \( x \in C^0 \subset T^0 \), \( H(S^2 \times 1) = 2P \), and \( c(H, T^1) < \#(\Sigma) \cdot \| \Sigma \| \cdot 2^{18n} \). Since \( C^1 \subset T^1 \), by Lemma 34 and since \( n \geq 24 \), it follows
\[
c(H, C^1) < c(H, T^1) < \#(\Sigma) \cdot \| \Sigma \| \cdot 2^{18n} < (10n) \cdot 2^{181n^2+18n} < 2^{182n^2}.
\]

We denote \( H_\xi = H(S^2 \times \{ \xi \}) \) for \( \xi \in [0,1] \). By the choice of \( P \) in Lemma 2, both \( H_1 = 2P \) and \( H_0 = \partial U(x) \) are normal with respect to \( C \). In particular, for any 3–cell \( X \) of \( C \), any connected component of \( H_0 \cap \partial X \) (resp. of \( H_1 \cap \partial X \)) bounds a disc in \( H_0 \cap X \) (resp. in \( H_1 \cap X \)). Therefore \( H \) satisfies the technical assumptions in Subsection 3.1 of [8]. Hence by Lemmas 9 and 10 of [8], we can choose \( H \) so that
\[
c(H, C^2) \leq 1 + \chi (C^2 \cap U(P)) - \chi (C^2) + \chi (C^0) + c(H, C^1)
< 1 - \frac{1}{2} \#(P \cap C^1) + 10n + 2^{182n^2}
< 2^{183n^2}.
\]

For \( \xi \in [0,1] \), let \( B^+(\xi) \) be the connected component of \( \mathbb{P}^3 \setminus H_\xi \) that contains \( P \). We define
\[
Q_\xi = H_\xi \cup (C^2 \cap B^+(\xi)).
\]
If \( H_\xi \) is in general position to \( C^2 \) then, by Lemma 13 in [8], \( Q_\xi \) is the 2–skeleton of a simple regular cellular decomposition of \( \mathbb{P}^3 \), whose barycentric subdivision is a triangulation \( T_\xi \). Let \( \xi_0 \in [0,1] \) so that \( H_{\xi_0} \) is not in general position with respect to \( C^2 \), and let \( \epsilon > 0 \) be sufficiently small. It is shown after Lemma 13 in [8] how one can transform \( Q_{\xi_0-\epsilon} \) into \( Q_{\xi_0+\epsilon} \). In the first step, depending on the type of non-transversality of \( H_{\xi_0} \), one adds two 2–cells to \( Q_{\xi_0-\epsilon} \) with 2 vertices in the boundary, or one adds one 2–cell with at most 5 vertices, or one deletes from \( Q_{\xi_0-\epsilon} \) a 2–cell with 3 vertices. In the second step, one deletes a 2–cell with at most 4 vertices. By an application of Lemma 14, it follows that \( T_{\xi_0-\epsilon} \) can be transformed into \( T_{\xi_0+\epsilon} \), be a sequence of at most 22 + 18 edge expansions or contractions.

The complex \( Q_0 \) is obtained from \( C^2 \) by insertion of a triangular 2–cell. Hence one can transform \( \mathcal{T} = C^1 \) into \( T_0 \) by 14 edge expansion. Furthermore, we have \( T_1 = T_p \). In conclusion, we obtain a sequence of less than
\[
14 + (22 + 18) \cdot c(H, C^2) < 2^{184n^2}
\]
contractions and expansions relating \( \mathcal{T} \) with \( T_p \). \( \square \)
3.3 Standard triangulations

Let $P$, $C_P$ and $T_P$ be as in the preceding subsection. Let $Z$ be a simple cellular decomposition of $P$ that is dual to a triangulation. We lift it along the fibres of the $I$–bundle $U(P)$ over $P$, and obtain a simple cellular decomposition of $U(P)$, so that each $d$–cell in $\partial U(P)$ corresponds to a $d$–cell of $Z$, and the intersection of a $d$–cell in $U(P)$ with $P$ is a $(d-1)$–cell of $Z$, for $d = 1, \ldots, 3$. Since $\mathbb{P}^3 \setminus U(P)$ is a ball, we obtain a simple cellular decomposition of $\mathbb{P}^3$. It is easy to see that its barycentric decomposition is a triangulation. We denote this triangulation by $T(Z)$.

If there is an edge $e$ in $Z^1$ so that $Z^1 \setminus e$ is the 1–skeleton of a cellular decomposition of $P$ that is dual to a triangulation, then we replace $Z$ by this simpler cellular decomposition. We iterate this process until it stops at a simple cellular decomposition $\hat{Z}$. It was proven by Barnette [2] that $\hat{Z}$ is one of the two decompositions $Z_1, Z_2$ depicted in Figure 2 (opposite points in the boundary of the discs are identified to obtain the projective plane $P$).

**Lemma 5** The triangulation $T_P$ is related to one of the two standard triangulations $T(Z_1)$ and $T(Z_2)$ by a sequence of less than $2^{20n}$ edge contractions.

**Proof** Let $Z$ be the simple cellular decomposition of $P$ induced by $C$. We have $T(Z) = T_P$. The deletion of edges of $Z$ corresponds to the deletion of 2–cells of $C_P$. Each of these 2–cells has four vertices in its boundary. Hence the deletion of one edge gives rise to 18 edge contractions of $T_P$, by Lemma 1. Since $Z$ has $\frac{3}{2} \cdot \#(P \cap C^1) \leq \frac{3}{2} \|P\| < 2^{19n}$ edges, $T_P$ is related to $T(\hat{Z})$ by a sequence of less then $18 \cdot 2^{19n} < 2^{20n}$ edge contractions.

3.4 Proof of Theorem 1

Let $T_1$ and $T_2$ be two triangulations of $\mathbb{P}^3$ with at most $t$ tetrahedra. They are related to its barycentric subdivision $T'_1, T'_2$ by a sequence of at most $5t$...
edge expansions. Since $T_1'$ and $T_2'$ have at most $12t$ tetrahedra, it follows from Lemmas 4 and 5 that $T_1'$ and $T_2'$ are related to one of the standard triangulations $T(Z_1)$ and $T(Z_2)$ by sequences of less than

$$2^{184 \cdot (12t)^2} + 2^{20 \cdot 12t} < 2^{26500 t^2}$$

edge expansions and contractions.

![Figure 3: Transforming $Z_1$ into $Z_2$](image)

Theorem 1.

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